Progress in Nonlinear Differential Equations and Their Applications

Variational and Topological Methods in the Study of Nonlinear Phenomena

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Birkhäuser
Boston • Basel • Berlin

V. Benci
Dipartimento di Matematica Applicata
Universita di Pisa
56126 Pisa, Italy

G. Cerami
Dipartimento di Matematica
ed Applicazioni
Universita di Palermo
90123 Palermo, Italy

M. Degiovanni
Dipartimento di Matematica e Fisica
Universita Cattolica del Sacro Cuore
25121 Brescia, Italy

D. Fortunato
Dipartimento Interuniversitario
di Matematica
Universita di Bari
70125 Bari, Italy

F. Giannoni
Dipartimento di Matematica e Fisica
Universita di Camerino
62032 Camerino, Italy

A. M. Micheletti
Dipartimento di Matematica
Applicata
Universita di Pisa
56126 Pisa, Italy

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Preface

The articles in this volume are an outgrowth of an international conference entitled *Variational and Topological Methods in the Study of Nonlinear Phenomena*, held in Pisa in January-February 2000. Under the framework of the research project *Differential Equations and the Calculus of Variations*, the conference was organized to celebrate the 60th birthday of Antonio Marino, one of the leaders of the research group and a significant contributor to the mathematical activity in this area of nonlinear analysis.

The volume highlights recent advances in the field of nonlinear functional analysis and its applications to nonlinear partial and ordinary differential equations, with particular emphasis on variational and topological methods. A broad range of topics is covered, including: concentration phenomena in PDEs, variational methods with applications to PDEs and physics, periodic solutions of ODEs, computational aspects in topological methods, and mathematical models in biology.

Though well-differentiated, the topics covered are unified through a common perspective and approach. Unique to the work are several chapters on computational aspects and applications to biology, not usually found with such basic studies on PDEs and ODEs. The volume is an excellent reference text for researchers and graduate students in the above mentioned fields.

Contributors are M. Clapp, M.J. Esteban, P. Felmer, A. Ioffe, W. Marzantowicz, M. Mrozek, M. Musso, R. Ortega, P. Pilarczyk, M. del Pino, E. Séré, E. Schwartzman, P. Sintzoff, R. Turner, and M. Willem.

Vieri Benci
Giovanna Cerami
Marco Degiovanni
Dino Fortunato
Fabio Giannoni
Anna Maria Micheletti
October 2001

Morse Indices at Mountain Pass Orbits of Symmetric Functionals

M. Clapp

Dedicated to Antonio Marino

ABSTRACT We consider functionals that are invariant under the action of an arbitrary group of symmetries and have the mountain pass geometry, and introduce a suitable notion of genus which allows computation of lower bounds for the Morse indices of critical orbits at corresponding minimax values. Via a Borsuk–Ulam type property, we relate these critical values to those which are relevant for obtaining multiplicity results for perturbed symmetric problems. This allows us to obtain good estimates for their growth, which are useful in applications, and extends previous results of Bahri and Lions and Tanaka for even functionals to more general group actions.

1 Introduction

This work was motivated by the problem of existence of multiple solutions of perturbed symmetric functionals. Multiplicity results for perturbations of even functionals were first obtained in the early eighties by Bahri and Berestycki [2], Struwe [24] and Rabinowitz [21], and further developed by many others, e.g., [3], [25], [17], [9], [7], [8]. Results for more general group actions have been given in [11], [13]. All of these results require appropriate estimates of the growth of some critical values of the unperturbed symmetric problem. These critical values are given by a topological invariant, originally introduced by Krasnoselskii [19], which captures some relevant non-symmetric properties of symmetric sets. This invariant has been extended to more general group actions and to the mountain pass setting in [11], [13]. We call it the equivariant capacity.

For even functionals with constraints Bahri and Lions [3] showed that one can obtain good estimates of the growth of those critical values in terms of lower bounds for the Morse indices of critical points given by the

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dual topological invariant: the Krasnoselskii genus (refered to as "cogenus" in [3]). A similar result was obtained by Tanaka [25] for even functionals having the mountain pass geometry. He computed the Morse index of dual critical values obtained via a finite-dimensional approximation procedure.

Here we consider arbitrary symmetries given by the action of an arbitrary compact Lie group G. We introduce a suitable notion of relative equivariant genus, which provides critical values for the equivariant mountain pass situation directly (without going through a finite-dimensional approximation), and which allows one to compute lower bounds for the equivariant Morse indices of the corresponding critical orbits using only simple methods of equivariant topology.

As for even maps, one can compare the minimax values given by the equivariant genus with those given by the equivariant capacity, provided that the group G satisfies some Borsuk–Ulam type property. This will be the case if G is a torus, a p-torus or a cyclic p-group [12]. So for these groups one does obtain appropriate estimates of those critical values relevant to perturbed symmetric problems. Applications to systems of elliptic differential equations may be found in [13].

This paper is organized as follows: In Section 1 we recall some basic topological notions and fix notation. In Section 2 we introduce and discuss the notions of equivariant capacity and equivariant genus in a mountain pass setting, and give a comparison result between the corresponding minimax values. Section 3 is devoted to computing lower bounds for the Morse indices of critical orbits related to these values. We summarize with some final remarks in Section 4.

2 Preliminaries

We start by recalling some basic notions of equivariant topology. A detailed account may be found in [16]. Let G be a compact Lie group. A *G-space* is a topological space Y with a continuous action of G. A *G-map* is a continuous map $\phi : Y \to Z$ which preserves the G-action, i.e., $\phi(gy) = g\phi(y)$. A G-map ϕ which is a homeomorphism is called a *G-homeomorphism.* We denote by $Gy := \{gy \in Y : g \in G\}$ the *G-orbit of* y, and by $G_y := \{g \in G : gy = y\}$ the *isotropy subgroup of* y. Observe that Gy is G-homeomorphic to G/G_y. We denote by $Y^G := \{y \in Y : gy = y \text{ for all } g \in G\}$ the *fixed point set of* Y.

By a *G-pair* (Y, A) we mean a G-space Y together with a G-invariant subspace A of Y, and by a *G-map (of pairs)* $\phi : (Y, A) \to (Z, B)$ we mean a G-map $\phi : Y \to Z$ which maps A into B. Two such G-maps ϕ, ψ are *G-homotopic*, denoted by

$$\phi \overset{G}{\simeq} \psi : (Y, A) \to (Z, B),$$

if there exists a *G-homotopy (of pairs)* $\Theta : (Y \times [0,1], A \times [0,1]) \to (Z, B)$ from ϕ to ψ, i.e., for each $t \in [0,1]$, the map

$$\Theta_t : (Y, A) \to (Z, B), \qquad \Theta_t(y) := \Theta(y,t),$$

is a G-map (of pairs), $\Theta_0 = \phi$ and $\Theta_1 = \psi$. A G-map $\phi : (Y, A) \to (Z, B)$ is a *G-homotopy equivalence* if there is a G-map $\eta : (Z, B) \to (Y, A)$ such that

$$\eta \circ \phi \overset{G}{\simeq} id : (Y, A) \to (Y, A) \quad \text{and} \quad \phi \circ \eta \overset{G}{\simeq} id : (Z, B) \to (Z, B).$$

We denote

$$(Y, A) \times [0,1] := (Y \times [0,1], A \times [0,1]).$$

The *join* $Y * Z$ of two non-empty G-spaces Y and Z is the quotient space of $Y \times [0,1] \times Z$ obtained by identifying $(y, 0, z)$ with $(y, 0, z')$ and $(y, 1, z)$ with $(y', 1, z)$ for all $y, y' \in Y$, $z, z' \in Z$. It has a natural G-action given by $g(y,t,z) = (gy, t, gz)$. We define $Y * \emptyset := Y =: \emptyset * Y$. If $f_i : Y_i \to Z_i$, $i = 1, 2$, are G-maps then

$$f_1 * f_2 : Y_1 * Y_2 \to Z_1 * Z_2$$

is the G-map given by $(y_1, t, y_2) \mapsto (f_1(y_1), t, f_2(y_2))$.

We denote by

$$E_k G := \underbrace{G * \cdots * G}_{k \text{ times}}$$

the *n-fold join of* G. This is a free G-space. If $G = \mathbb{Z}/2$, then $E_k(\mathbb{Z}/2)$ is ($\mathbb{Z}/2$-homeomorphic to) the unit sphere $\mathbb{S}^{k-1} \subset \mathbb{R}^k$ with the antipodal action. If $G = \mathbb{S}^1$, then $E_k(\mathbb{S}^1)$ is ($\mathbb{S}^1$-homeomorphic to) the unit sphere $\mathbb{S}^{2k-1} \subset \mathbb{C}^k$ with the action given by multiplication on each coordinate. If B is a topological space with the trivial G-action, then the fixed point set of $B * E_k G$ is precisely B. We denote

$$(Y, A) * E_k G := (Y * E_k G, A * E_k G).$$

3 Equivariant capacity and equivariant genus

Let X be a G-Hilbert space with norm $\| \ \|$ and such that $\dim X = \infty$ and $\dim(X^G) < \infty$. Let $\Phi : X \to \mathbb{R}$ be a G-invariant, C^1-functional which satisfies the Palais–Smale condition:

(PS) Every sequence (u_k) in X such that $\Phi(u_k) \to c$ and $\|\Phi'(u_k)\| \to 0$ has a convergent subsequence, and the two mountain pass conditions:

($\mathbf{MP}_1$) $\sup_{B^G} \Phi = 0$ and there is a sequence of finite-dimensional G-linear subspaces of X

$$X_0 := X^G \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

with $\dim X_n \to \infty$, and a non-decreasing sequence of real numbers

$$1 = R_0 \leq R_1 \leq \cdots \leq R_n \leq \cdots$$

such that $\Phi(u) \leq 0$ if $u \in X_n$ and $\|u\| \geq R_n$,

($\mathbf{MP}_2$) There exist $r, a > 0$ such that $\Phi(u) \geq a$ for all $u \in X_1^{\perp}$, $\|u\| = r$.

We shall define two topological invariants that give rise to critical values: The relative equivariant capacity and the relative equivariant genus.

Let $B := \{u \in X : \|u\| \leq 1\}$ and $S := \{u \in X : \|u\| = 1\}$ be the unit ball and the unit sphere in X respectively, and let $B^G := B \cap X^G$ and $S^G := S \cap X^G$ be the unit ball and the unit sphere in the fixed point space X^G. Let

$$D := \bigcup_{n=0}^{\infty} \{u \in X_n : \|u\| \geq R_n\} .$$

Definition 3.1. For every G-invariant subset Y of X such that $Y \supset D \cup X^G$, we define the *G-capacity $\kappa_G(Y)$ of Y* to be the greatest number $k \geq 0$ such that there exists a G-map

$$\sigma : (B^G, S^G) * E_k G \to (Y, D)$$

whose restriction to the fixed point sphere $\sigma \mid S^G : S^G \simeq D^G$ is a homotopy equivalence. If such a map exists for all k we shall say that $\kappa_G(Y) = \infty$.

For example, if $G = \mathbb{Z}/2 = \{1, -1\}$ we consider the representations $X^G \oplus \mathbb{R}^k$, where $\mathbb{Z}/2$ acts trivially on X^G and by multiplication on $\mathbb{R}^k$. Then $\kappa_G(Y)$ is the greatest number $k \geq 0$ such that there exists an odd map $\sigma : B(X^G \oplus \mathbb{R}^k) \to Y$ from the unit ball in $X^G \oplus \mathbb{R}^k$ into Y which maps the unit sphere $S(X^G \oplus \mathbb{R}^k)$ into D in such a way that $\sigma \mid S^G : S^G \simeq D^G$ is a homotopy equivalence. If $\mathbb{Z}/2$ acts on X by multiplication then $X^G = \{0\}$ and this is essentially the same invariant used in [25].

Analogously, if $G = \mathbb{S}^1$ we consider the representations $X^G \oplus \mathbb{C}^k$, where $\mathbb{S}^1 \subset \mathbb{C}$ acts trivially on X^G and by multiplication on $\mathbb{C}^k$. Then $\kappa_G(Y)$ is the greatest number $k \geq 0$ such that there exists an $\mathbb{S}^1$-map $\sigma : B(X^G \oplus \mathbb{C}^k) \to Y$ from the unit ball in $X^G \oplus \mathbb{C}^k$ into Y which maps the unit sphere $S(X^G \oplus \mathbb{C}^k)$ into D in such a way that $\sigma \mid S^G : S^G \simeq D^G$ is a homotopy equivalence.

Observe that if $\sigma \mid S^G : S^G \simeq D^G$ is a homotopy equivalence, then $\sigma^G : (B^G, S^G) \simeq (X^G, D^G)$ is a homotopy equivalence of pairs. So the notion of G-capacity given here is a special case of the one given in [11], cf. also [14].

The G-capacity satisfies the following monotonicity property.

Proposition 3.2. (Monotonicity) *If Y and Z are G-invariant subsets of X which contain $D \cup X^G$ and $\eta : (Y, D) \to (Z, D)$ is a G-map which induces a homotopy equivalence $\eta \mid D^G : D^G \simeq D^G$, then $\kappa_G(Y) \leq \kappa_G(Z)$.* □

Therefore we may use this invariant to obtain critical values of Φ via minimax. Let

$$\Gamma_k := \left\{ Y \subset X : Y \text{ is } G\text{-invariant, } Y \supset D \cup X^G \text{ and } \kappa_G(Y) \geq k \right\}.$$

We define

$$\begin{aligned} c_k &= \inf_{Y \in \Gamma_k} \sup_{u \in Y} \Phi(u) \\ &= \inf \{ c \geq 0 : \kappa_G(\Phi^c) \geq k \}. \end{aligned}$$

Observe that since $\dim X_n \to \infty$, there exists a G-map $\sigma : (B^G, S^G) * E_k G \to (X, D)$ as in Definition 3.1 for each $k \geq 0$ [11] Proposition 2.5. Hence $c_k \leq \max(\Phi\sigma) < \infty$ and, moreover,

$$0 = c_0 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdots < \infty.$$

By standard methods, using the G-invariant version [15] of the well-known Deformation Lemma [22], [24] and the monotonicity property of the G-capacity, one can easily show that

Proposition 3.3. *If $0 < c_k < \infty$ then c_k is a critical value of Φ.* □

Another invariant which provides critical values of Φ is defined, in a dual fashion, as follows.

Definition 3.4. Let (Y, Z) be a G-pair and let $\mathcal{A}$ be a family of G-spaces. The *$\mathcal{A}$-genus*, $\gamma_{\mathcal{A}}(Y, Z)$, is the smallest number j such that there exists a G-map

$$\tau : (Y, Z) \to (Z * A_1 * \cdots * A_j, Z)$$

with $A_i \in \mathcal{A}$ for all $i = 1, \ldots, j$, whose restriction $\tau \mid Z : Z \to Z$ is the identity.

If $Z = \emptyset$ this is just the $\mathcal{A}$-genus [4]. We shall be interested in the case when $\mathcal{A}$ is the set

$$\mathcal{G} = \{ G/H_1 \sqcup \cdots \sqcup G/H_m : H_i \subsetneq G \quad \text{a closed subgroup of} \quad i = 1, \ldots, m \}.$$

If G has the property that every finite set of proper subgroups is contained in a proper subgroup of G, then $\gamma_{\mathcal{G}}(Y, Z) = \gamma_{\mathcal{G}_0}(Y, Z)$ where

$$\mathcal{G}_0 = \{ G/H : H \text{ is a closed subgroup of } G, \ H \neq G \}.$$

This is true in particular for $G = \mathbb{Z}/2$ and for $G = \mathbb{S}^1$. Hence, if $G = \mathbb{Z}/2$, $\gamma_{\mathcal{G}}(Y, Z)$ is the smallest number j such that there exists an odd map

$$\tau : (Y, Z) \to (Z * \mathbb{S}^{k-1}, Z)$$

whose restriction $\tau \mid Z : Z \to Z$ is the identity. If $Z = \emptyset$, this is just the Krasnoselski genus. If $G = \mathbb{S}^1$, then $\gamma_{\mathcal{G}}(Y, Z)$ is the smallest number j such that there exist a fixed-point free $2j$-dimensional representation V of $\mathbb{S}^1$ and an $\mathbb{S}^1$-map

$$\tau : (Y, Z) \to (Z * SV, Z)$$

whose restriction $\tau \mid Z : Z \to Z$ is the identity.

This invariant is closely related to the equivariant Lusternik–Schnirelmann category [15]. It is easy to see that, if Z is closed in Y, then $\gamma_{\mathcal{A}}(Y, Z)$, is the smallest number j such that there exists an open covering $\{Y_0, Y_1, \ldots, Y_j\}$ of Y, a G-retraction $Y_0 \to Z$ and G-maps $Y_i \to A_i \in \mathcal{A}$, $i = 1, \ldots, j$, cf. [15], Proposition 2.4.

The $\mathcal{A}$-genus satisfies also a monotonicity property.

Proposition 3.5. (Monotonicity) *If (Y, Z) and (Y', Z) are G -pairs and $\eta : (Y, Z) \to (Y', Z)$ is a G-map whose restriction $\eta \mid Z : Z \to Z$ is the identity, then $\gamma_{\mathcal{A}}(Y, Z) \leq \gamma_{\mathcal{A}}(Y', Z)$.* □

So we may use this invariant to obtain critical values of Φ. Choose a regular value $0 < \vartheta < a$ of Φ, where a is as in $(\mathbf{MP}_2)$. Then the minimax values of Φ given by the $\mathcal{G}$ -genus

$$\widetilde{c}_j = \inf \left\{ c \geq \vartheta : \gamma_{\mathcal{G}}(\Phi^c, \Phi^\vartheta) \geq j \right\},$$

also give rise to critical values.

Proposition 3.6. *If $\vartheta < \widetilde{c}_j < \infty$, then $\widetilde{c}_j$ is a critical value of Φ.* □

Observe that, since ϑ is a regular value, there exists $\epsilon > 0$ such that $\gamma_{\mathcal{G}}(\Phi^{\vartheta+\epsilon}, \Phi^\vartheta) = 0$. Hence

$$\vartheta < \widetilde{c}_1 \leq \cdots \leq \widetilde{c}_j \leq \widetilde{c}_{j+1} \leq \cdots \leq \infty.$$

In fact, the $\mathcal{A}$-genus has also the usual properties of an index theory, which are useful for obtaining multiplicity results for symmetric functionals, cf. for example [1], [5], [6], [22], [15].

We wish to compare the minimax values provided by the G-capacity with those given by the $\mathcal{G}$-genus. We shall need the following property.

Definition 3.7. A compact Lie group G will be said to have the *Borsuk-Ulam property* **(BU)** if there exist b and j_0 such that, if there exists a G-map

$$\xi : \mathbb{S}^m * E_k G \to \mathbb{S}^m * A_1 * \cdots * A_j$$

with $A_i \in \mathcal{G}$ for $i = 1, \dots, j$, $j \ge j_0$, whose restriction $\xi^G : \mathbb{S}^m \simeq \mathbb{S}^m$ to the sphere of fixed points is a homotopy equivalence, then $k \le bj$.

Examples of groups satisfying **(BU)** have been given in [12] where the following result was proved.

Theorem 3.8. *a) If G is a torus $G = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ or a p-torus $G = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$, p prime > 1, then G has the Borsuk–Ulam property with $b = 1$.*

b) If $G = \mathbb{Z}/p^r$ is a cyclic p-group, p prime > 1, then G has the Borsuk-Ulam property with $b = p^{r-1}$. □

The Borsuk–Ulam property allows us to compare the minimax values given by the $\mathcal{G}$-genus with those provided by the G-capacity.

Proposition 3.9. *If G has the Borsuk–Ulam property* **(BU)** *and $\Phi : X \to \mathbb{R}$ is a G-invariant C^1-functional which satisfies* $(\mathbf{MP}_1)$ *and* $(\mathbf{MP}_2)$, *then there exists a k_0 such that, for all $k > k_0$,*

$$\vartheta < \tilde{c}_{j(k)} \le c_k < \infty$$

where $j(k)$ is the smallest integer $\ge \frac{k}{b} - \gamma_{\mathcal{G}}(BX_1, SX_1 \cup B^G)$.

Proof. Let Y be a G-subset of X such that $Y \supset \Phi^\vartheta \supset D \cup X^G$. Let $k = \kappa_G(Y)$, $j = \gamma_{\mathcal{G}}(Y, \Phi^\vartheta)$ and $i = \gamma_{\mathcal{G}}(BX_1, SX_1 \cup B^G)$. Consider the G-map

$$\begin{aligned}
\tilde{\xi} : (B^G, S^G) * E_k G &\xrightarrow{\sigma} (Y, D) \subset (Y, \Phi^\vartheta) \\
&\xrightarrow{\tau} (\Phi^\vartheta * A_1 * \cdots * A_j, \Phi^\vartheta) \\
&\xrightarrow{\rho * id} (S_R X \cup B_R X_1 * A_1 * \cdots * A_j, S_R X) \\
&\xrightarrow{\psi * id} ((S \cup B^G) * A'_1 * \cdots * A'_i * A_1 * \cdots * A_j, S)
\end{aligned}$$

where σ and τ are as in Definitions 3.1 and 3.4, $R = R_2$ is as in $(\mathbf{MP}_1)$,

$$\rho : \Phi^\vartheta \hookrightarrow X \setminus S_r X_1^\perp \to S_R X \cup B_R X_1$$

is the radial projection onto $S_R X$ on $X \setminus B_R X$ and the projection along straight lines onto $S_R X \cup B_R X_1$ on $B_R X \setminus S_r X_1^\perp$, and ψ is given by multiplying first by $\frac{1}{R}$ and then composing with the map given by $i = \gamma_{\mathcal{G}}(BX_1, SX_1 \cup B^G)$. Hence $\tilde{\xi}$ induces a G-map

$$B^G/S^G * E_k G \to B^G/S^G * A'_1 * \cdots * A'_i * A_1 * \cdots * A_j$$

which is homotopic to the identity on the fixed point sphere B^G/S^G obtained by identifying the boundary of the ball B^G with a point. Since G satisfies **(BU)**,

$$\kappa_G(Y) \le b(\max\{\gamma_{\mathcal{G}}(BX_1, SX_1 \cup B^G) + \gamma_{\mathcal{G}}(Y, \Phi^\vartheta), j_0\}).$$

In particular, $\kappa_G(\Phi^{\vartheta}) \leq b \max\{\gamma_{\mathcal{G}}(BX_1, SX_1 \cup B^G), j_0\} =: k_0$. Therefore

$$\{c \geq 0 : \kappa_G(\Phi^c) \geq k > k_0\} \subset \{c \geq \vartheta : \gamma_{\mathcal{G}}(\Phi^c, \Phi^{\vartheta}) \geq j(k) \geq 1\}$$

and the result follows. □

4 Morse indices at minimax orbits

The G-capacity, while being an invariant of G-invariant sets, has also a non-equivariant homotopy property called the rigidity property, which makes it very useful for obtaining critical point results for perturbed symmetric functionals [2], [21], [11], [13]. Neither the equivariant Lusternik–Schnirelmann category nor the $\mathcal{G}$-genus have this property. On the other hand, critical point results for perturbed symmetric functionals require good estimates on the growth of the critical values c_k. These are provided, for example, by the Morse index at some critical orbits [3], [25]. The critical values $\widetilde{c}_j$ given by the $\mathcal{G}$-genus have the advantage that one can easily estimate from below the Morse index at a corresponding critical orbit. We shall prove the following.

Theorem 4.1. *Let $\Phi : X \to \mathbb{R}$ be a G-invariant C^2-functional which satisfies* **(PS)**, $(\mathbf{MP}_1)$ *and* $(\mathbf{MP}_2)$ *and is such that $\Phi''(u)$ is a Fredholm operator for every $u \in X$. Assume that, for some $j \geq 1$, $\widetilde{c}_j < \infty$. Then there is a critical G-orbit Gu_j of Φ with critical value $\widetilde{c}_j$ which is either non-isolated or such that*

$$\mu(Gu_j) + \nu(Gu_j) \geq j$$

where μ is the Morse index and ν is the nullity of the G-orbit.

For $G = \mathbb{Z}/2$ a similar result was obtained by Tanaka [25] via a finite-dimensional approximation procedure. The relative notion of $\mathcal{G}$-genus allows us to obtain lower estimates for the Morse index of a mountain pass G-orbit directly, in a similar way as the Krasnoselki genus does for even fuctionals on a sphere [3], [10]. Theorem 4.1, together with Proposition 3.9 immediately gives the following.

Theorem 4.2. *Assume that G satisfies* **(BU)**. *Let $\Phi : X \to \mathbb{R}$ be a G-invariant C^2-functional which satisfies* **(PS)**, $(\mathbf{MP}_1)$ *and* $(\mathbf{MP}_2)$ *and is such that $\Phi''(u)$ is a Fredholm operator for every $u \in X$. Then there exists k_0 such that for every $k > k_0$ there is a critical G-orbit Gu_k of Φ with $\Phi(u_k) \leq c_k$ which is either non-isolated or such that*

$$\mu(Gu_k) + \nu(Gu_k) \geq \frac{k}{b} - \gamma_{\mathcal{G}}(BX_1, SX_1 \cup B^G).$$

□

The proof of Theorem 4.1 relies on the following topological lemma which is closely related to the cellular approximation theorem [23] 7.6.17, [16] II.2.1. For the notion of G-complex we refer to [16]. We denote by $Z^{(n)}$ the G-equivariant n-skeleton of the G-complex Z and by $\dim_G Z$ the dimension of the G-complex Z, that is, the ordinary dimension $\dim(Z/G)$ of its orbit space.

Lemma 4.3. *Let $Z, Y_0, \dots, Y_k$ be G-complexes such that $Y_i^H \neq \emptyset$ for every isotropy subgroup H of Z, $i = 0, \dots, k$, and let $\tau : Z \to Y_0 * \cdots * Y_k$ be a G-map. If $\dim_G Z \leq k-1$ then τ is G-homotopic to a G-map $\widetilde{\tau} : Z \to Y_0 * \cdots * Y_{k-1}$.*

Proof. First observe that any j-fold join of non-empty spaces $Z_0 * \cdots * Z_{j-1}$ is $(j-2)$-connected so, from the homotopy exact sequence of a pair [23] 7.2.3 it follows that $(Z_0 * \cdots * Z_m, Z_0 * \cdots * Z_{j-1})$ is $(j-1)$-connected for all $m \geq j$.

We will show, inductively, that there exist G-homotopies $\Theta_t^n : Z \to Y := Y_0 * \cdots * Y_k$ such that $\Theta_0^n = \tau$, $\Theta_1^{n-1} = \Theta_0^n$, $\Theta_1^n(Z^{(n)}) \subset Y_0 * \cdots * Y_n$ and Θ_t^n does not depend on t on $Z^{(n-1)}$. The desired homotopy is then given by glueing all of these together. For the induction step we assume that $\tau(Z^{(j)}) \subset Y_0 * \cdots * Y_j$ for $j < n$. Let $\chi : G/H \times (\mathbb{B}^n, \mathbb{S}^{n-1}) \to (Z^{(n)}, Z^{(n-1)})$ be a characteristic map for an n-cell. We wish to show that $\tau \circ \chi$ is G-homotopic rel. $G/H \times \mathbb{S}^{n-1}$ to a G-map $G/H \times \mathbb{B}^n \to Y_0 * \cdots * Y_n$. But such G-homotopies correspond to ordinary homotopies $(\mathbb{B}^n, \mathbb{S}^{n-1}) \times [0,1] \to (Y^H, Y_0^H * \cdots * Y_{n-1}^H)$ rel. $\mathbb{S}^{n-1}$ between $\tau \circ \chi \mid \mathbb{B}^n$ and a map $\mathbb{B}^n \to Y_0^H * \cdots * Y_n^H$. Since $(Y^H, Y_0^H * \cdots * Y_n^H)$ is n-connected such a homotopy does exist [23] 7.2.1. □

Proof of Theorem 4.1. Assume that the set of critical orbits with value $\widetilde{c}_j$ is finite and that the Morse index plus the nullity of each of them is $< j$. Choose $\epsilon > 0$ such that $\widetilde{c}_j - \epsilon > 0$ and such that $\widetilde{c}_j$ is the only critical value of Φ in $[\widetilde{c}_j - \epsilon, \widetilde{c}_j + \epsilon]$. By the G-equivariant version [26], [10] of the Marino–Prodi theorem [20] there is a G-invariant C^2-functional $\Psi : X \to \mathbb{R}$ which satisfies **(PS)** such that

(a) $\Psi = \Phi$ on $X \setminus \Phi^{-1}(\widetilde{c}_j - \frac{\epsilon}{2}, \widetilde{c}_j + \frac{\epsilon}{2})$,

(b) $|\Phi - \Psi|_{C^2} < \frac{\epsilon}{2}$, and

(c) the critical G-orbits of Ψ in $\Phi^{-1}[\widetilde{c}_j - \frac{\epsilon}{2}, \widetilde{c}_j + \frac{\epsilon}{2}]$ are nondegenerate and of Morse index $< j$.

Let $d_1 < \cdots < d_m$ be the critical values of the critical G -orbits of Ψ in $\Phi^{-1}[\widetilde{c}_j - \frac{\epsilon}{2}, \widetilde{c}_j + \frac{\epsilon}{2}]$. Then $\widetilde{c}_j - \epsilon < d_1$ and $d_m < \widetilde{c}_j + \epsilon$. Also, $\Psi^{-1}(\infty, \widetilde{c}_j - \epsilon] = \Phi^{-1}(\infty, \widetilde{c}_j - \epsilon]$ and $\Psi^{-1}[\widetilde{c}_j + \epsilon, \infty) = \Phi^{-1}[\widetilde{c}_j + \epsilon, \infty)$. We will show that for each d_i there is a $\delta > 0$ such that, if $\gamma_{\mathcal{G}}(\Psi^{d_i - \delta}, \Psi^0) < j$, then $\gamma_{\mathcal{G}}(\Psi^{d_i + \delta}, \Psi^0) < j$. Then, since $\gamma_{\mathcal{G}}(\Phi^{\widetilde{c}_j - \epsilon}, \Phi^0) = \gamma_{\mathcal{G}}(\Psi^{\widetilde{c}_j - \epsilon}, \Psi^0) < j$, it follows that $\gamma_{\mathcal{G}}(\Phi^{\widetilde{c}_j + \epsilon}, \Phi^0) = \gamma_{\mathcal{G}}(\Phi^{\widetilde{c}_j + \epsilon}, \Psi^0) < j$ which is a contradiction.

Fix $d = d_i$ and let $G/H_1, \dots, G/H_n$ be the critical orbits of Ψ with critical value d. Then there exists $\delta > 0$ such that, up to G -homotopy rel. Ψ^0, $\Psi^{d+\delta}$ is obtained from $\Psi^{d-\delta}$ by attaching the disk bundles of the negative normal bundles $p_k^- : N_k^- \to G/H_k$ of each one of these critical orbits along the sphere bundles [10], [18], [27]; that is,

$$\Psi^{d+\delta} \overset{G}{\simeq} \Psi^{d-\delta} \bigcup_{SN_1^- \sqcup \cdots \sqcup SN_n^-} (BN_1^- \sqcup \cdots \sqcup BN_n^-).$$

The bundle dimension of $p_k^- : N_k^- \to G/H_k$ is the Morse index of G/H_k, therefore $\dim_G SN_k^- \leq j-2$. Consider the composition of the attaching map ψ_k of SN_k^- with the G-map τ given by $\gamma_{\mathcal{G}}(\Psi^{d-\delta}, \Psi^0) < j$,

$$SN_k^- \xrightarrow{\psi_k} \Psi^{d-\delta} \xrightarrow{\tau} \Psi^0 * A_1 * \cdots * A_{j-1}.$$

By adding G/H to each A_i if necessary, we may assume that $A_i^H \neq \emptyset$ for each isotropy subgroup H of SN_k^-. So, by Lemma 4.3, $\tau \circ \psi_k$ is G-homotopic to a G-map $\widetilde{\tau} : SN_k^- \to \Psi^0 * A_1 * \cdots * A_{j-2}$. Let Θ be a G-homotopy with $\Theta_0 = \tau \circ \psi_k$ and $\Theta_1 = \widetilde{\tau}$. Then $\tau \circ \psi_k$ can be extended to a G-map

$$\tau_k : DN_k^- \to \Psi^0 * A_1 * \cdots * (A_{j-1} \sqcup G/H_k)$$

as follows: For $\zeta \in SN_k^-$ and $0 \leq t \leq 1$,

$$\tau_k(t\zeta) := \left\{ \begin{array}{ll} (\widetilde{\tau}(\zeta), 1-2t, p_k^-(\zeta)) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \Theta(\zeta, 2(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1 \end{array} \right\}.$$

Hence, τ can be extended to a G-map

$$\Psi^{d+\delta} \to \Psi^0 * A_1 * \cdots * (A_{j-1} \sqcup G/H_1 \sqcup \cdots \sqcup G/H_n),$$

and therefore $\gamma_{\mathcal{G}}(\Psi^{d+\delta}, \Psi^0) < j$. □

5 Further remarks and comments

For an arbitrary group G we may look at the minimum $\beta_G(k)$ of all numbers j such that there exists a G-map

$$\phi : \mathbb{S}^m * E_k G \to \mathbb{S}^m * A_1 * \cdots * A_j$$

with $A_i \in \mathcal{G}$, $i = 1, \dots, j$, whose restriction $\phi^G : \mathbb{S}^m \simeq \mathbb{S}^m$ to the sphere of fixed points is a homotopy equivalence. Obviously $\beta_G(k) \leq k$ and the function β_G is non-decreasing. If G satisfies (**BU**) then $\beta_G(k) \geq \frac{k}{b} - j_0$.

The same argument we gave to prove Proposition 3.9 shows that, for every G-invariant C^1-functional $\Phi : X \to \mathbb{R}$ satisfying ($\mathbf{MP}_1$) and ($\mathbf{MP}_2$),

$$\vartheta < \widetilde{c}_{\beta_G(k)-i} \leq c_k < \infty$$

for k large enough, where $i := \gamma_{\mathcal{G}}(BX_1, SX_1 \cup B^G)$. So, as a corollary of Theorem 4.1, we obtain the following.

Corollary 5.1. *Every G-invariant C^2-functional $\Phi : X \to \mathbb{R}$ satisfying* **(PS)**, $(\mathbf{MP}_1)$ *and* $(\mathbf{MP}_2)$ *and such that $\Phi''(u)$ is Fredholm for every $u \in X$, has a critical G-orbit Gu_k with $\Phi(u_k) \le c_k$ which is either non-isolated or such that*

$$\mu(Gu_k) + \nu(Gu_k) \ge \beta_G(k) - \gamma_G(BX_1, SX_1 \cup B^G). \qquad \square$$

Now, computing $\beta_G(k)$ or even obtaining lower bounds for it is, in general, not easy. It was shown in [12] that if G is a p-group, then $\beta_G(k) \to \infty$ as $k \to \infty$. For the applications to perturbed symmetric problems, however, this is not enough. One needs more information on the growth of $\beta_G(k)$, cf. for example [3], [8], [13], [25].

For arbitrary group actions, homological minimax values have been considered by Viterbo [26] who gave bounds for the Morse indices of the corresponding critical orbits under a strong cohomological assumption. This assumption is satisfied if G is $\mathbb{Z}/p$ or $\mathbb{S}^1$ but it is not satisfied for arbitrary actions of tori or p-tori of rank ≥ 2 or of cyclic p-groups of order p^r with $r \ge 2$, which are covered by Theorem 4.2.

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Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
04510 México, D.F., Mexico
email: mclapp@math.unam.mx

On Some Linear and Nonlinear Eigenvalue Problems in Relativistic Quantum Chemistry

Maria J. Esteban and Eric Séré

Dedicated to Antonio Marino

1 Introduction

In relativistic quantum mechanics [1], the bound states of an electron under the action of an external electrostatic potential V are represented by the wave functions $\phi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ which are solutions of the equation

$$(H_0 + V)\,\phi = \lambda\,\phi, \quad \lambda \in \mathbb{R} \tag{1.1}$$

$$\text{with } H_0 \;=\; -ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \;+\; mc^2\beta\,, \tag{1.2}$$

where c denotes the speed of light, $m > 0$, the mass of the electron, and $\hbar$ is Planck's constant. Moreover, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 complex matrices, whose standard form (in 2×2 blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \qquad (k = 1, 2, 3)\,,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can easily check the relations

$$\begin{cases} \alpha_k = \alpha_k^*\,, \quad \beta = \beta^*, \\ \alpha_k\alpha_\ell + \alpha_\ell\alpha_k \;=\; 2\delta_{k\ell}\,, \; \alpha_k\beta + \beta\alpha_k \;=\; 0\,, \end{cases} \tag{1.3}$$

which ensure that H_0 is a self-adjoint operator with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$, such that

$$H_0^2 \;=\; -c^2\Delta + m^2c^4. \tag{1.4}$$

Unless otherwise stated, in the rest of the paper we choose a system of units such that $m = c = \hbar = 1$.

Let us now fix some notation. The conjugate of $z \in \mathbb{C}$ will be denoted by z^*. For $X = \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ z_4 \end{pmatrix}$ a column vector in $\mathbb{C}^4$, we denote by X^* the row covector $(z_1^*, ..., z_4^*)$. Similarly, if $\mathcal{A} = (a_{ij})$ is a 4×4 complex matrix, we denote by $\mathcal{A}^*$ its adjoint, $(\mathcal{A}^*)_{ij} = a_{ji}^*$.

We denote by (X, X') the Hermitian product of two vectors X , X' in $\mathbb{C}^4$, and by $|X|$, the norm of X in $\mathbb{C}^4$, i.e., $|X|^2 = \sum_{i=1}^{4} X_i X_i^*$. The usual Hermitian product in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ is denoted

$$(\phi, \psi)_{L^2} = \int_{\mathbb{R}^3} \Big(\phi(x), \psi(x)\Big)\, d^3x. \tag{1.5}$$

The basic space in which we will work throughout this paper is $E := H^{1/2}(\mathbb{R}^3, \mathbb{C}^4)$, which is the form-domain of the first order operator H_0.

When the potential V is not too strong, the solutions of (1.1) can be found as critical points in E of the Rayleigh quotient

$$Q(\phi) := \frac{((H_0 + V)\phi, \phi)}{(\phi, \phi)}.$$

Since we are interested in particle states, we need to find critical values of $Q(\phi)$ at critical levels not belonging to the essential spectrum of the operator $H_0 + V$. The essential spectrum of H_0 is easily computable. It is the union of two unbounded intervals:

$$\sigma_{ess}(H_0) = (-\infty, -1] \cup [1, +\infty),$$

and when $V \not\equiv 0$, the essential spectrum of $H_0 + V$ is still the same set if V is not too singular. More precisely, we have the following:

Lemma 1.1. *Suppose that* $V = V_0 + V_1 + \cdots + V_k$, *with* $V_0 \in L^\infty(\mathbb{R}^3)$, $\lim_{|x| \to +\infty} V_0(x) = 0$, *and* $V_i(x)|x - x_i| < 1$ *for some* $x_i \in \mathbb{R}^3$, *for all* $i = 1, \ldots, k$. *Then* H *has a natural self-adjoint extension whose domain is a subspace of* E, *and whose essential spectrum is* $(-\infty, -1] \cup [1, +\infty)$.

This lemma is an easy extension of [31, 29, 37, 24].

A basic example of an eigenvalue problem arising in atomic and molecular models corresponds to the case of the Coulomb potential $-\mu|x|^{-1}$ created by a point-like nucleus at the origin of coordinates. The limitation $\mu < 1$ will have an impact on the assumptions made on the maximum nuclear charge under which our results will be valid. In our system of units, $\mu = \alpha Z$, where

$Z > 0$ is the number of protons in the nucleus, and α is a dimensionless scalar called the "fine structure constant", whose experimental value is slightly smaller than $1/137$. The constraint on the nucleus is thus $Z \leq 137$. This covers all known atoms. Note that, for a mathematician, it can be interesting to consider that Z is not necessarily an integer, and that α can be arbitrarily small (see Section 3).

In Section 2, we describe variational methods yielding critical points of $Q(\phi)$ with critical values in the gap $(-1, 1)$, for a class of potentials which contains the standard Coulomb case.

In Section 3, we treat more realistic systems corresponding to atoms or molecules with several electrons. The problem is to compute the stationary states of the electronic "cloud", given a fixed distribution of nuclear charges. The correct theory in this case should be Quantum Electrodynamics. However it seems extremely difficult to use this theory for the numerical computation of the electronic states of heavy atoms or molecules. This is why the Dirac–Fock approximation is used in quantum chemistry (see [32, 23, 18, 8, 27, 17]). In this model, the N electrons are represented by a family of N functions $\Phi = (\varphi_1, \ldots, \varphi_N)$, subject to the normalization constraints

$$\left(\varphi_\ell, \varphi_k\right)_{L^2} = \delta_{k\ell} \,. \tag{1.6}$$

The functions φ_k are solutions of a system of N coupled nonlinear eigenvalue problems. This corresponds to replacing the external potential V in (1.1) by a "mean-field" nonlocal operator, acting on functions $\psi \in E$, with values in its dual E':

$$\mathcal{V}_\Phi \psi = -\alpha Z \left(\mu * \frac{1}{|x|}\right) \psi + \alpha(\rho * \frac{1}{|x|})\psi - \alpha \int_{\mathbf{R}^3} \frac{R(x,y)\psi(y)}{|x-y|} \,.$$

Here, α is the already mentioned fine structure constant, Z is the total number of protons in the molecule, μ is a fixed probability measure on $\mathbf{R}^3$, and $Z\mu$ represents the density of protons in the molecule. A typical example is the case of a point-like nucleus containing Z protons, located at the origin: then μ is the Dirac mass δ_0 at the origin.

In the expression of $\mathcal{V}_\Phi$, the first term $-\alpha Z \left(\mu * \frac{1}{|x|}\right)$ represents the fixed nuclear potential acting on each electron. The other terms represent the mean field due to the N electrons acting on each electron. These terms depend on Φ in a nonlinear way: $\rho(x)$ is a scalar and $R(x,y)$ is a 4×4 complex matrix, given by

$$\rho(x) = \sum_{\ell=1}^{N} \left(\varphi_\ell(x), \varphi_\ell(x)\right) , \quad R(x,y) = \sum_{\ell=1}^{N} \varphi_\ell(x) \otimes \varphi_\ell^*(y) \,. \tag{1.7}$$

Physically, ρ is the electronic density, R is the "exchange matrix" which comes from Pauli's exclusion principle. Note that $R(y,x) = R(x,y)^*$, so that $\mathrm{tr}\Big(R(x,y)R(y,x)\Big) = \sum_{i,j} |R(x,y)_{ij}|^2$.

The Dirac–Fock system is

$$\overline{H}_\Phi\, \varphi_k = \sum_{\ell=1}^{N} \lambda_{k\ell}\, \varphi_\ell\ ,\ k = 1,\dots,N. \tag{1.8}$$

Here,

$$\overline{H}_\Phi \psi = H_0\psi + \mathcal{V}_\Phi \psi.$$

We can apply a unitary transformation $u \in \mathcal{U}(N)$ to the family Φ (i.e., replace φ_k by $\sum_{l=1}^N u_{kl}\varphi_l$) in order to diagonalize the above system and get

$$\overline{H}_\Phi\, \varphi_k = \lambda_k\, \varphi_k\ ,\ k = 1,\dots,N\ . \tag{1.9}$$

Moreover, once again we are interested in solutions of (1.9) such that all the λ_k lie in the interval $(-1,1)$. In fact, for physical reasons we are even more restrictive: we require $0 < \lambda_k < 1$, since negative energies would correspond to "positronic" states.

Some easy computations show that the solutions of (1.9) are critical points in E^N of the energy functional

$$\begin{aligned}\mathcal{E}_{DF}(\Phi) = &\sum_{\ell=1}^{N}\Big(\varphi_\ell, H_0\varphi_\ell\Big)_{L^2} - \alpha Z \sum_{\ell=1}^{N}\Big(\varphi_\ell, \Big(\mu * \frac{1}{|x|}\Big)\varphi_\ell\Big)_{L^2} \\ &+ \frac{\alpha}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} V(x-y)\Big[\rho(x)\rho(y) - \mathrm{tr}\Big(R(x,y)R(y,x)\Big)\Big]d^3x\,d^3y\,,\end{aligned} \tag{1.10}$$

under the orthonormality constraints (6), which can be summarized as

$$\mathrm{Gram}_{L^2}\Phi = \mathbb{1}.$$

Solutions of (1.9) can be found by variational methods, using the functional (10). The main difficulty, in the study of (1.9) as well as (1.1), lies in the fact that the energy functionals involved are all highly indefinite, since they are positive (resp. negative) definite in a space of infinite dimension.

2 The linear eigenvalue Dirac problem

In order to state the main results presented in this section, let us start with some technical lemmas which are necessary in the sequel of the paper. The first one is proved quite easily by using Fourier variables.

Lemma 2.1. *H_0 is a self-adjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$, with domain $\mathcal{D}(H_0) = H^1(\mathbb{R}^3, \mathbb{C}^4)$. Its spectrum is $(-\infty, -1] \cup [1, +\infty)$. There are two orthogonal projectors on $L^2(\mathbb{R}^3, \mathbb{C}^4)$, Λ_+ and $\Lambda_- = 1_{L^2} - \Lambda_+$, both with infinite rank, and such that*

$$\begin{cases} H_0\Lambda_+ &= \Lambda_+ H_0 = \sqrt{1-\Delta}\ \Lambda_+ = \Lambda_+ \sqrt{1-\Delta} \\ H_0\Lambda_- &= \Lambda_- H_0 = -\sqrt{1-\Delta}\ \Lambda_- = -\Lambda_- \sqrt{1-\Delta}\,. \end{cases} \tag{2.11}$$

The next lemma deals with the interplay between the Dirac potential and the potential V.

Lemma 2.2. *The Coulomb potential $\frac{1}{|x|}$ satisfies the following Hardy-type inequalities:*

$$\left(\phi, \left(\mu * \frac{1}{|x|}\right)\phi\right)_{L^2} \leq \frac{1}{2}\left(\frac{\pi}{2} + \frac{2}{\pi}\right)\left(\phi, |H_0|\phi\right)_{L^2}, \tag{2.12}$$

for all $\phi \in \Lambda_+(H^{1/2}) \cup \Lambda_-(H^{1/2})$ and for all probability measures μ on $\mathbb{R}^3$. Moreover,

$$\left(\phi, \left(\mu * \frac{1}{|x|}\right)\phi\right)_{L^2} \leq \frac{\pi}{2}\left(\phi, |H_0|\phi\right)_{L^2}, \ \forall \phi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^4), \tag{2.13}$$

$$\left\| \left(\mu * \frac{1}{|x|}\right)\phi\right\|_{L^2} \leq 2\|\nabla\phi\|_{L^2}, \quad \forall \phi \in H^1(\mathbb{R}^3, \mathbb{C}^4). \tag{2.14}$$

In the particular case where μ is equal to the Dirac mass at the origin δ_0, an inequality more precise than (2.12) was proved independently by Tix and Burenkov–Evans (see [35, 4, 36]). This inequality reads as follows

$$\left(\left(H_0 - \frac{\alpha Z}{|x|}\right)\phi, \phi\right) \geq ((1-\alpha Z)\phi, \phi)\ ,$$

for all $Z \leq Z_c := \frac{2}{(\frac{\pi}{2}+\frac{2}{\pi})\alpha}$, for all $\phi \in \Lambda_+(H^{1/2}(\mathbb{R}^3, \mathbb{C}^4))$. The technique used by Tix and Burenkov–Evans is based on ideas introduced by Evans, Perry and Siedentop in [15]. We refer to [21, 22] for inequality (2.12) in the case $\mu = \delta_0$. Thaller's book [34] gathers many results on the Dirac operator, including Lemma 2.1 and the standard Hardy inequalities (2.14) and (2.13) for $\mu = \delta_0$, with references. The extension of (2.12), (2.13) and (2.14) from $\mu = \delta_0$ to a general probability measure μ is immediate, since the projectors $\Lambda_\pm$, the gradient ∇ and the free Dirac operator H_0 commute with the space translations.

We are now ready to state our main results about the existence and multiplicity of the eigenvalues of $H_0 + V$ for potentials V satisfying the following assumptions:

$$V(x) \underset{|x|\to+\infty}{\longrightarrow} 0\,, \tag{2.15}$$

$$-\frac{\nu}{|x|} - c_1 \le V \le c_2 = \sup_{\mathbb{R}^3} V\,, \tag{2.16}$$

with $\nu \in (0,1)$, $c_1, c_2 \in \mathbb{R}$,

$$c_1, c_2 \ge 0,\ \ c_1 + c_2 - 1 < \sqrt{1-\nu^2}\,. \tag{2.17}$$

Theorem 2.3. **([10])** *Let V be a scalar potential satisfying (2.15), (2.16) and (2.17). Then, all the eigenvalues of $H_0 + V$ in the gap $(c_2 - 1, 1)$, counted with multiplicity, are given by the min-max values*

$$\lambda_k^V \;=\; \inf_{\substack{F \subset Y_+ \\ F \text{ vector space} \\ \dim F = k}} \;\; \sup_{\substack{\phi \in F \oplus Y_- \\ \phi \neq 0}} \;\; \frac{((H_0+V)\phi, \phi)}{(\phi,\phi)}\,, \tag{2.18}$$

where $Y_\pm := \Lambda_\pm(C_0^\infty(\mathbb{R}^3, \mathbb{C}^4))$.

Let us note that the above result does not correspond to a classical min-max situation: indeed, the classical theorems showing that min-max levels for some functional yield critical points need the condition that the sets in the min-max class are invariant under the action of the gradient flow related to the functional. This assumption is not satisfied here. The reason is that $\Lambda_\pm$ are spectral projectors corresponding to the free operator H_0, but not to $H_0 + V$. Therefore, no classical min-max method can prove Theorem 2.3 and *ad-hoc* arguments have to be used.

Under stronger assumptions on V, the result was first proved in [12]. A further improvement was contained in [9]. The proof of Theorem 2.3 in its final formulation, which seems to be optimal in the case of Coulomb potentials created by a point-like nucleus, is contained in [10]. Actually this theorem is a corollary of a general result about the point spectrum of operators with gaps (Theorem 1.1 in [10]). Let us state it in its full generality.

Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)$, and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ a self-adjoint operator. We denote by $\mathcal{F}(A)$ the form-domain of A. Let $\mathcal{H}_+, \mathcal{H}_-$ be two orthogonal Hilbert subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. We denote by P_+, P_- the projectors on $\mathcal{H}_+$, $\mathcal{H}_-$. We assume the existence of a core F (i.e., a subspace of $D(A)$ which is dense for the norm $\|\cdot\|_{D(A)}$), such that

(*i*) $F_+ = P_+F$ and $F_- = P_-F$ are two subspaces of $\mathcal{F}(A)$.

(*ii*) $a_- = \sup_{x_- \in F_- \setminus \{0\}} \frac{(x_-, Ax_-)}{\|x_-\|_{\mathcal{H}}^2} < +\infty$.

We consider the sequence of min-max levels:

$$\lambda_k = \inf_{\substack{V \text{ subspace of } F_+ \\ \dim V = k}} \;\; \sup_{x \in (V \oplus F_-) \setminus \{0\}} \frac{(x, Ax)}{\|x\|_{\mathcal{H}}^2}\,, \qquad k \ge 1. \tag{2.19}$$

Our last assumption is

(iii) $\lambda_1 > a_-$.

Now, let $b = \inf\,(\sigma_{\rm ess}(A) \cap (a_-, +\infty)) \in [a_-, +\infty]$. For $k \geq 1$, we denote by μ_k the $k^{\rm th}$ eigenvalue of A in the interval (a_-, b), counted with multiplicity, *if this eigenvalue exists.* If there is no $k^{\rm th}$ eigenvalue, we take $\mu_k = b$. Then,

Theorem 2.4. ([10]) *With the above notation, and under assumptions (i)-(iii),*

$$\lambda_k \;=\; \mu_k\ , \quad \textit{for all}\ \ k \geq 1.$$

As a consequence, $b = \lim\limits_{k\to\infty} \lambda_k = \sup\limits_k \lambda_k > a_-$.

The proof of this theorem is given in detail in [10]. What we will do here is sketch the proof of Theorem 2.3, i.e., explain why Theorem 2.4 implies Theorem 2.3.

Sketch of the proof of Theorem 2.3.

Here we choose $\mathcal{H}$ to be the space $L^2(\mathbb{R}^3, \mathbb{C}^4)$, $F = C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$, $A = H_0 + V$ and $P_\pm = \Lambda_\pm$. The self-adjointness of A follows from Lemma 2.1. From the explicit formulae defining $\Lambda_\pm$, it is easy to see that (i) and (ii) are satisfied with $a_- = c_2 - 1$. Also, here $b = 1$. So , only (iii) remains to be verified. Since λ_1^V is monotonic in V, it is enough to verify (iii) in the particular case $c_1 = c_2 = 0$. This is done in [10], by using a continuation argument and the fact that $\mu_1(H_0 - \frac{\nu}{|x|}) > 0 > -1$ is explicitly known: for all $\nu \in (0,1)$, $\mu_1(H_0 - \frac{\nu}{|x|}) = \sqrt{1-\nu^2} > 0 > -1$. □.

Theorem 2.3 is interesting from a theoretical point of view, since it can be used to obtain monotonicity results and some qualitative information about the eigenvalues of the operator $H_0 + V$ whenever V satisfies the assumptions of the theorem. However, the min-max defined in (2.3) is not excellent when one needs to compute (numerically) the eigenvalues.

In [10] we have proved that the min-max defining the first eigenvalue can be written as a minimum. The idea lying behind this is that the maximum contained in the min-max can be "explicitly" solved. But in order to do so in a very straightforward and easy way, we introduce another class of min-maxes related to different projectors $P_\pm$: for every $\phi = \binom{\varphi}{\chi}$, we define

$$P_+\phi = \binom{\varphi}{0}\ , \quad P_-\phi = \binom{0}{\chi}\ .$$

A min-max approach involving these projectors appears in [33] and [7]. On the other hand, this idea was first used in a rigorous proof by Griesemer and Siedentop in [20], but only for bounded potentials V.

The above projectors satisfy all the assumptions necessary to apply Theorem 2.4. With this definition, (i), (ii) and (iii) are again satisfied with the values $a_- = c_2 - 1$, $b = 1$. Hence, for all V satisfying the assumptions of Theorem 2.3, all the eigenvalues contained in the interval $(c_2 - 1, 1)$ are given by the min-max values defined in (2.19). But now it is not difficult to prove that for every $\varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \setminus \{0\}$,

$$\lambda(\varphi) := \sup \left\{ \frac{((H_0 + V)\phi, \phi)}{(\phi, \phi)} \; ; \; \phi \in span\{\binom{\varphi}{0}\} \oplus P_- \left({0 \atop C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)} \right) \right\}$$

is the unique number $\lambda \in (c_2 - 1, +\infty)$ such that

$$\lambda \int_{\mathbb{R}^3} |\varphi|^2 dx = \int_{\mathbb{R}^3} \left(\frac{|(\sigma \cdot \nabla)\varphi|^2}{1 - V + \lambda} + (1 + V)|\varphi|^2 \right) dx, \qquad (2.20)$$

where σ denotes the matrix vector with the Pauli matrices σ_i, $i = 1, 2, 3$, as entries. Moreover, the maximizer χ in the above supremum is the function

$$\chi(V, \varphi) := \frac{-i(\sigma \cdot \nabla)\varphi}{1 - V + \lambda(\varphi)}. \qquad (2.21)$$

Finally, it is easy to prove that under the assumptions of the theorem, the first eigenvalue of $H_0 + V$ in the interval $(c_2 - 1, 1)$ is given by

$$\inf \left\{ \lambda(\varphi) \, ; \; \varphi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \setminus \{0\} \right\}. \qquad (2.22)$$

In [11] we describe how to use (2.22) to write an algorithm which enables us to compute all the eigenvalues of $H_0 + V$ in a quite straightforward way. Tables containing numerical results are given in [11] for a family of Coulomb operators.

3 The nonlinear Dirac–Fock system

In [13] we proved, by a variational method, that under some assumptions on N and Z, there exists an infinite sequence of solutions of the Dirac–Fock equations. More precisely, we have the following:

Theorem 3.1. ([13]). *Let $N < Z + 1$. Then, for $\alpha \frac{\pi/2 + 2/\pi}{2} \max(Z, 3N - 1) < 1$, there exists a sequence of solutions of the Dirac–Fock equations, $\left\{ \Psi^i \right\}_{i \geq 1} \subset \left(H^{1/2}(\mathbb{R}^3) \right)^N$, such that*

(i) $0 < \mathcal{E}_{DF}(\Psi^i) < N$,

(ii) $\lim_{i \to +\infty} \mathcal{E}_{DF}(\Psi^i) = N$,

(iii) $0 < 1 - \frac{\mu_i}{\alpha} < \varepsilon_1^i \leq \ldots \leq \varepsilon_N^i < 1 - \frac{m_i}{\alpha}$, *with m_i, $\mu_i > 0$, independent of α.*

Some ideas in the proof of Theorem 3.1 are inspired, in particular, by the works [5, 3, 28], and we use a general result of Fang–Ghoussoub [16] on the Morse index of Palais–Smale sequences associated to a min-max.

With the physical value $\alpha = 1/137$ and Z an integer (the total number of protons in the molecule), our conditions become $N \le Z,\ N \le 41,\ Z \le 124$. The constraint $N \le 41$ is rather unnatural. In our proof, we need a certain concavity property of the Dirac–Fock functional, and the constraint on N ensures that this property is satisfied.

Our result was recently improved by Paturel [30], who relaxed the condition on N. Paturel obtains the same multiplicity result, assuming only that $N < Z+1$ and $\alpha\frac{\pi/2+2/\pi}{2}\max(Z,N) < 1$. This is an important improvement, since his result covers all existing neutral atoms. His proof is very technical and uses ideas coming from the variational theory of nonconvex Hamiltonian systems (in particular [6]).

As mentioned above, the physical value of α is approximately $1/137$. However, it is interesting to study the connection between the relativistic Dirac–Fock model and its nonrelativistic counterpart: the Hartree–Fock model. This comparison involves the so-called "nonrelativistic limit" $\alpha \to 0$. In [14], we obtain several results on this limit. The first one asserts that, under certain conditions and after rescaling, solutions of Dirac–Fock converge to solutions of Hartree–Fock as α goes to zero. More precisely, we fix N, Z with $N < Z+1$ and we take a fixed probability measure μ on $\mathbb{R}^3$. Let α_n be a sequence of positive numbers converging to zero. We define a sequence of probability measures μ_n by $\mu_n(E) = \mu(\alpha_n E)$, for all Borel sets E. We call (DF_n) the Dirac–Fock system associated to $\alpha_n,\ N,\ Z,\ \mu_n$. Take a sequence $\{\Psi^n\}_n$ of solutions of (DF_n) with eigenvalues $\varepsilon_{k,n}$ $(1 \le k \le N)$, for which

$$-\infty < \varliminf_{n\to+\infty} (\alpha_n)^{-2}(\varepsilon_{1,n}-1) \le \varlimsup_{n\to+\infty} (\alpha_n)^{-2}(\varepsilon_{N,n}-1) < 0\,. \tag{3.23}$$

Then $\{\alpha_n^{-3/2}\Psi^n(\cdot/\alpha_n)\}_n$ has a subsequence converging strongly in $H^1(\mathbb{R}^3,\mathbb{C}^4)$, towards $\overline{\Psi} = \begin{pmatrix}\overline{\Phi}\\ 0\end{pmatrix}$, where $\overline{\Phi} = (\varphi_1,\cdots,\varphi_N) : \mathbb{R}^3 \to (\mathbb{C}^2)^N$ is a solution of the Hartree–Fock equations:

$$\begin{cases} \widetilde{H}_{\Phi}\varphi_k = -\dfrac{\Delta\varphi_k}{2} - Z(\mu * V)\varphi_k + \Big(\rho_\Phi * \dfrac{1}{|x|}\Big)\varphi_k \\ \qquad\qquad -\displaystyle\sum_{j=1}^N \varphi_j(x)\int_{\mathbb{R}^3}\frac{(\varphi_k(y),\varphi_j(y))}{|x-y|}dy = \bar\lambda_k\varphi_k, \quad k=1,\dots N, \\ \displaystyle\int_{\mathbb{R}^3}\varphi_i\varphi_j^*\,dx = \delta_{ij}\ , \quad \bar\lambda_k = \lim_{n\to+\infty}(\alpha_n)^{-2}(\varepsilon_{k,n}-1)\,. \end{cases} \tag{HF}$$

Note that the Hartree–Fock equations are the Euler–Lagrange equations corresponding to critical points of the Hartree–Fock energy:

$$\mathcal{E}_{HF}(\Phi) = \sum_{i=1}^{N}\Big(\frac{1}{2}\|\nabla\varphi_i\|_2^2 - Z\int_{\mathbb{R}^3}(\mu * V)|\varphi_i|^2 dx\Big)$$
$$+\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_\Phi(x)\rho_\Phi(y) - |\rho_\Phi(x,y)|^2}{|x-y|}\,dxdy\,, \tag{3.24}$$

under the constraint $\int_{\mathbb{R}^3}\varphi_i\varphi_j^* = \delta_{ij}$, $i,j = 1,\ldots,N$, where

$$\rho_\Phi(x) = \sum_{i=1}^{N}(\varphi_i(x),\varphi_i(x)), \quad \rho_\Phi(x,y) = \sum_{i=1}^{N}(\varphi_i(x),\varphi_i(y)).$$

Particular solutions of the Hartree–Fock equations are the minimizers of

$$E_{1,HF} := \inf\Big\{\mathcal{E}_{HF}(\Phi)\,;\ \Phi\in (H^1(\mathbb{R}^3))^N,\ \mathrm{Gram}\,\Phi = \mathbf{1}\Big\}$$

The existence of such minimizers was proved by Lieb and Simon [26] under the assumption $N < Z+1$ (see also [28] for a multiplicity result on critical points that are not minimizers).

The second main result in [14] is that the first solution Ψ^1 of Dirac–Fock found in [13], whose energy level will be denoted $E_{1,DF}$, converges, after rescaling, to the ground state of Hartree–Fock.

Theorem 3.2. ([14]). *Fix $N < Z+1$ and a probability measure μ on $\mathbb{R}^3$. Take α very small, while the nuclear densities $Z\mu_\alpha$ are given by $\mu_\alpha(E) = \mu(\alpha E)$ for all Borel sets E. Then, with the above notation,*

$$E_{1,DF} \;=\; \mathcal{E}(\Psi^1) \;=\; N + \alpha^2 E_{1,HF} + o(\alpha^2)_{\alpha\to 0}. \tag{3.25}$$

If α_n goes to zero, then, after extraction of a subsequence,

$$(\alpha_n)^{-3/2}\Psi^1_{\alpha_n}(\cdot/\alpha_n) \underset{n\to+\infty}{\longrightarrow} \begin{pmatrix}\overline{\Phi}\\ 0\end{pmatrix} \text{ in } H^1(\mathbb{R}^3), \tag{3.26}$$

where $E_{1,HF}$ is the Hartree-Fock ground-state energy and $\overline{\Phi}$ is a minimizer for $E_{1,HF}$.

Moreover, for α small the eigenvalues corresponding to Ψ^1 in the Dirac–Fock system, $\varepsilon_1^1,\ldots,\varepsilon_N^1$ are the smallest positive eigenvalues of the linear (Dirac + mean-field) operator $\overline{H}_{\Psi^1}$ and the $(N+1)$-th positive eigenvalue of this operator is strictly larger than ε_N^1.

Finally, we are able to show that, in the neighborhood of the nonrelativistic limit, the function Ψ^1 can indeed be viewed as an electronic ground state for the Dirac–Fock equations, and this not only because it is close to the ground state of Hartree–Fock , but also it minimizes the Dirac–Fock energy among all possible "electronic configurations":

Theorem 3.3. ([14]). *Fix N, Z with $N < Z+1$ and take $\alpha > 0$ sufficiently small. Then Ψ^1 is a solution to the following minimization problem:*

$$\inf\{\mathcal{E}_{DF}(\Psi)\,;\ Gram\ \Psi = \mathbb{1}\,,\ \Lambda_\Psi^- \Psi = 0\ \} \tag{3.27}$$

where $\Lambda_\Psi^- = \chi_{(-\infty,0)}(\overline{H}_\Psi)$ is the negative spectral projector of the (Dirac + mean-field) operator $\overline{H}_\Psi$.

Part of the proof of Theorem 3.3 is inspired by the work of Buffoni and Jeanjean [2]. We also use an estimate on the negative energy projector Λ_Ψ^-, due to Griesemer, Lewis and Siedentop [19].

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CEREMADE - UMR C.N.R.S. 7534
Université Paris IX-Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16 - France
email: esteban@ceremade.dauphine.fr
sere@ceremade.dauphine.fr

Convexity at Infinity and Palais-Smale Conditions. Application to Hamiltonian Systems

A. Ioffe and E. Schwartzman

Dedicated to Antonio Marino

ABSTRACT We consider functionals $f(x) = (1/2)(Lx|x) + H(x)$ on Hilbert spaces and establish the relationship between convexity of H and conditions of Palais–Smale type. This is further extended to functionals with H exhibiting convex behavior "at infinity" and to Hamiltonian systems with Hamiltonians having such a property.

1 Introduction

We study the problem of existence of critical points for the function

$$f(x) = (1/2)(Lx|x) + H(x). \tag{1.1}$$

The starting point of the study was the dual action principle discovered by Clarke in the late 1970s which proved to be the main tool for existence proofs in problems with convex subquadratic H, especially for periodic solutions of Hamiltonian systems (e.g., [3, 6, 11, 12]).

At the same time, the natural question of whether classical minimax methods of calculus of variations (Ljusternik–Schnirelman theory, mountain pass theorem and its extensions, etc.) can be applied in such cases and produce similar results,remained open. As usual, this question reduces to that of the Palais–Smale condition: what kind of condition (if any) may be behind the existence results provided by the dual action principle?

A simple answer to this question, surprisingly unknown so far, is given here by Theorem 1. Namely, it turns out that, under very mild assumptions on L, f satisfies a slightly strengthened form of the Cerami version of the PS-condition (see [4] – we call this version the *weighted* Palais–Smale

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condition in this paper) if H is convex continuous and subquadratic at infinity.

The basic fact behind the theorem is that any collection of vectors x with bounded "measure of non-criticality" is norm bounded in this case (Proposition 3.3). This makes every weighted PS-sequence bounded and hence reduces the compactness test to verification of the PS-condition only for bounded PS-sequences.

The proof based on this scheme has one definite advantage over that based on the dual action principle: it requires "less" convexity as there is no need to pass to the dual problem and then to show that any solution of the latter gives a critical point of f. It turns out that all we need is that H behave like a convex function only "at infinity" when $\|x\| \to \infty$. This idea is realized in Theorems 3 and 4, the latter concerned with periodic solutions of Hamiltonian systems.

The deformation techniques used in the proofs of the mentioned results is very elementary — in fact, only finite dimensional deformations are needed. This, in turn, allows us to require very little of L. Practically the only assumption we impose on L is that zero does not belong to the essential spectrum of L. In fact, in this paper we assume for simplicity that the essential spectrum of L is empty, that is, that L has a purely discrete spectrum with every eigenvalue having finite multiplicity, but this assumption can be substantially weakened.

This is a brief description of the content of the paper. Elsewhere we shall consider the case of even H and corresponding multiplicity results: the simple finite dimensional deformation technique of this paper does not work in this case. Some of the results presented here were announced in [8]. Everywhere in what follows, we have:

X is a separable Hilbert space;
$(\cdot|\cdot)$ is the inner product in X;
B is the unit ball around the origin in X;
S is the unit sphere;
P_E is the orthogonal projection to the subspace $E \subset X$.

2 Preliminaries

We adopt the following hypotheses throughout the paper.

(A_1) X is a separable Hilbert space; L is a closed self-adjoint linear operator in X with dense domain dom L;

(A_2) $\sigma(L)$, the spectrum of L, is purely discrete and every eigenvalue has a finite multiplicity.

It follows from $(\mathbf{A}_2)$ that X admits an orthogonal decomposition

$$X = X^- \oplus X^0 \oplus X^+,$$

where $X^0 = \text{Ker } L$ is finite dimensional, X^+ and X^- are positive and negative subspaces of L and there are positive numbers ν^- and ν^+ such that

$$(Lx|x) \le -\nu^- \|x\|^2, \ \forall x \in X^- \cap \text{dom } L;$$
$$(Lx|x) \ge \nu^+ \|x\|^2, \ \forall x \in X^+ \cap \text{dom } L.$$

We set furthermore

$$Y = X^0 \oplus X^+,$$

and for any $x \in \text{dom } L$ we shall denote by x^-, x^0, x^+, y the corresponding components of x.

(A_3) there is a $\mu > 0$ such that $[-\mu, 0) \cap \sigma(L) = \emptyset$ and

$$\limsup_{\|x\| \to \infty} \|x\|^{-2} |H(x)| < \mu/2;$$

(A_4) $H(x) \to \infty$ if $\|x\| \to \infty$, $x \in \text{Ker } L$.

(A_5) H satisfies the Lipschitz condition on every ball;

By $\partial H(x)$ we denote Clarke's generalized gradient of H of x. Recall the definition (see, e.g., [3]):

$$\partial H(x) = \{u; \ (u|v) \le H^0(x, v), \ \forall v \in X\},$$

where

$$H^0(x, v) = \limsup_{x' \to x, t \to 0} t^{-1}(H(x' + tv) - H(x'))$$

is Clarke's directional derivative of H at x along v. Recall that $\partial H(x)$ coincides with the subdifferential of H at x in the sense of convex analysis if H is convex.

We say that x is a *critical point* of f if $x \in \text{dom } L$ and

$$0 \in Lx + \partial H(x). \tag{2.2}$$

Finally we shall denote

$$\rho(x) = \text{dist } (0, Lx + \partial H(x)).$$

This means that critical points are characterized by the relation $\rho(x) = 0$.

Furthermore, given a closed subspace $E \subset X$, we set

$$\rho_E(x) = \text{dist } (0, P_E(Lx + \partial H(x))).$$

3 Weighted Palais–Smale condition

Following tradition, it would be natural to call $\{x_n\}$ a Palais–Smale sequence if the corresponding sequence $\{f(x_n)\}$ is bounded and $\rho(x_n) \to 0$. But as was explained in the introduction we shall be interested in a narrower class of sequences. The following is the version of the PS-condition to be used.

Definition 3.1. (cf. [4, 1]). *A sequence* $\{x_n\}$ *is a weighted Palais–Smale sequence for* f *(at level* $c \in \mathbb{R}$*) if* $f(x_n) \to c$ *and* $\rho(x_n)(1 + \|x_n\|) \to 0$ *as* $n \to \infty$. *We say that* f *satisfies the weighted PS-condition at level* c *if any weighted PS-sequence for* f *at that level contains a convergent subsequence.*

More generally, we say that $\{x_n\}$ *is a weak (weighted) Palais–Smale sequence for* f *(at level* $c \in \mathbb{R}$*) if there is an increasing sequence of* L*-invariant subspaces* $E_n \subset X$ *such that* dom $L \subset \bigcup E_n$, $x_n \in E_n$, $f(x_n) \to c$ *and* $\rho_{E_n}(x_n) \to 0$ *(resp.* $\rho_{E_n}(x_n)(1+\|x_n\|) \to 0$*) as* $n \to \infty$. *We say that* f *satisfies the weak (weighted) PS-condition at level* c *if any weak (weighted) PS-sequence at that level contains a convergent subsequence.*

It is clear that every weighted PS-sequence is a Palais–Smale sequence in the usual sense (that is such that dist $(0, Lx_n + \partial H(x_n)) \to 0$ and a weak weighted PS-sequence (just take $E_n = X$).

Theorem 3.2. *We assume that* $(A_1)-(A_5)$ *hold and* H *is convex. Then* f *satisfies the weak weighted PS-condition at every level. Moreover, the limit of any convergent weak weighted PS-sequence is a critical point of* f.

The following result is crucial for the proof of the theorem.

Proposition 3.3. *Assume* $(\mathbf{A}_1)$*–*$(\mathbf{A}_5)$. *If in addition to these assumptions,* H *is convex, then for any* $-\infty < a \le b < \infty$ *and any* $\delta > 0$, *there is a* $K \ge 0$ *such that* $\|x\| \le K$ *for any* x *satisfying* $a \le f(x) \le b$, $x \in E$ *and* $\rho_E(x)(1+\|x\|) \le \delta$, *whenever* E *is a closed* L*-invariant subspace of* X.

Proof. Set $Q(x) = (1/2)(Lx|x)$ and define

$$r_E(t) = \sup\{\|x\| :\ x \in E^-,\ f(x) \ge -t\},$$
$$q_E(t) = \sup\{-Q(x) :\ x \in E^-,\ f(x) \ge -t\},$$

where we set $E^{\pm} = E \cap X^{\pm}$.

It follows from $(\mathbf{A}_3)$ that both $r_E(t)$ and $q_E(t)$ are finite for any t, nondecreasing and going to ∞ as $t \to \infty$, provided E^- contains nonzero elements. It is also clear that $r_E(t) \le r(t) = r_X(t)$ and $q_E(t) \le q(t) = q_X(t)$. Take an x satisfying the assumption. Then, as the function $\varphi(y) = f(x^- + y) - f(x^-)$ on $Y = X^0 \oplus X^+$ is convex, the inclusion $\emptyset \ne Ly + P_Y(\partial H(x^- + y)) \subset \partial\varphi$ holds for any $y \in Y \cap \text{dom } L$. Furthermore, if $z \in Lx + \partial H(x)$, then (as x^0, x^-, x^+ belong to E), setting $w = x - x^- = x^0 + x^+$, we get

$$\begin{aligned} f(x^-) - a &\ge (Lx|x - w) + H(x - w) - f(x) \\ &= (Lx| - w) + H(x - w) - H(x) \\ &\ge (z| - w) = (P_E z| - w) \end{aligned}$$

which implies that $f(x^-) - a \ge -\rho_E(x)\|x^+ + x^0\| \ge -\rho_E(x)\|x\|$, whence

$$-f(x^-) \le \rho_E(x)(1 + \|x\|) - a \le \delta - a.$$

It follows that

$$\|x^-\| \le r(\delta - a) = K_1, \qquad |Q(x^-)| \le q(\delta - a) = K_2.$$

We have therefore that

$$(\nu^+/2)\|x^+\|^2 + H(x) \le f(x) + K_2 \le b + K_2 = c. \tag{3.3}$$

By $(\mathbf{A}_4)$, H is bounded below on Ker L. It follows by standard rules of convex calculus that

$$H(x) = H_1(x) - (z|x),$$

where H_1 is bounded below by a certain α (which we may assume nonpositive) and z is orthogonal to Ker L. We therefore get from (3.3)

$$(\nu^+/2)\|x^+\|^2 \le c - \alpha + \|z\|(K_1 + \|x^+\|)$$

and, consequently,

$$\|x^+\| \le K_3 < \infty.$$

It remains to estimate the norm of $\|x^0\|$. Setting $\psi(y) = H_1(x^- + y) - H_1(x^-)$ we also get:

$$\begin{aligned}\psi(w) &= H_1(x) - H_1(x^-) = f(x) - Q(x) + (z|x) - H_1(x^-)\\ &\le f(x) - Q(x^-) + (z|x^- + x^+) - H_1(x^-)\\ &\le b + \|z\|\|x^-\| + \|z\|\|x^+\| - \alpha + K_2\\ &\le b + \|z\|(K_1 + K_3) - \alpha + K_2 = K_4. \end{aligned}\tag{3.4}$$

Set further

$$r_1 = \sup\{\|u\| :\ u \in \text{Ker } L,\ \psi(u) \le K_4\}.$$

By $(\mathbf{A}_4)$, $r_1 < \infty$. As $\psi(0) = 0$, it follows (due to convexity) that

$$u \in \text{Ker } L,\ \|u\| > (3/2)r_1 \ \Rightarrow\ \psi(u) > (3/2)K_4.$$

Let l_2 denote the Lipschitz constant of ψ on the ball of radius $R = \|w\|$ in $Y = X^0 \oplus X^+$. If $u \in Y$ is such that $\|u\| \le R$, $\|u^0\| > (3/2)r_1$ and $\|u^+\| \le \varepsilon_1 \le \min\{K_4/2l_2, K_3\}$, then

$$\psi(u) \ge \psi(u^0) - l_2\varepsilon_1 > K_4$$

Therefore if for a certain $u \in Y$ with $\|u\| \le R$ we have $\psi(u) \le K_4$, then either $\|u^0\| \le (3/2)r_1$ or $\|u^+\| > \varepsilon_1$.

If now $K_3/2l_2 \ge K_3 = \varepsilon_1$, then $\|x^+\| \le \varepsilon_1$ and we conclude from (3.4) that $\|w^0\| = \|x^0\| \le (3/2)r_1 = K_5$. Assume now that $\|x^0\| > (3/2)r_1$. In this case $K_4/2l_1 = \varepsilon_1 < \|x^+\| \le K_3$. Let $\lambda > 0$ be such that $\lambda\|x^0\| = (3/2)r_1$. Then $\|\lambda x^+\|$ is still not smaller than ε_1, $\lambda < 1$ and (as $\psi(0) = 0$) $\psi(\lambda w) \le \lambda\psi(w) < K_4$ by (3.4). Therefore

$$K_3 \ge \|x^+\| \ge \varepsilon_1/\lambda = K_4\|x^0\|/3r_1l_2,$$

that is $\|x^0\| \leq 3r_1 l_2 K_3/K_4 = K_7$. In either case, $\|x\| \leq K_1+K_3+K_5+K_6 = K$, which completes the proof.

As an immediate consequence of the proof of Proposition 3.3, we get

Corollary 3.4. *Under the assumptions of Proposition 3.3*

$$\lim_{\|x\|\to\infty,\ x\in Y} f(x) = \infty$$

and, consequently,

$$\inf_{x\in Y} f(x) > -\infty.$$

Proposition 3.5. *Assume (A_1),(A_2) and (A_5). Then any bounded sequence $\{x_k\}$ such that the sequence Lx_k is bounded is precompact. In particular, a bounded weak Palais–Smale sequence (at any level) is precompact and every its limit point is a critical point of $f(\cdot)$.*

*Proof.*The first statement is an obvious consequence of (A_2). Suppose now that $\{x_k\}$ is a bounded weak PS-sequence By the assumption there is an increasing sequence of L-invariant subspaces $E_k \subset X$ whose union is dense in X such that dist $(0, Lx_k + \partial P_{E_k}(H(x_k))) \to 0$. As $\{x_k\}$ is a bounded sequence, it follows from (A_5) that the sequence $\{Lx_k\}$ is also bounded, hence $\{x_k\}$is precompact.

Without loss of generality we may assume that $\{x_k\}$ converges to a certain x and $\{Lx_k\}$ weakly converges to a certain y. As L is a closed operator, it follows that $x \in \text{dom } L$ and $y = Lx$.

Furthermore, for any n and any $k \geq n$

$$\begin{aligned} \text{dist } (0, P_{E_n}(Lx_k) + P_{E_n}(\partial H(x_k))) &= \text{dist } (0, P_{E_n}(Lx_k + \partial H(x_k))) \\ &\leq \text{dist } (0, P_{E_k}(Lx_k + \partial H(x_k))) \\ &\leq \text{dist } (0, Lx_k + P_{E_k}(\partial H(x_k))) \end{aligned}$$

and we conclude (as the subdifferential mapping of a Lipschitz continuous function is upper norm-to-weak semicontinuous) that we have $0 \in P_{E_n}(Lx + \partial H(x))$ for any n. Since the union of E_n is dense in X, it follows that $0 \in Lx + \partial H(x)$, that is, x is a critical point of f.

Proof of Theorem 3.2: immediate from Propositions 3.3 and 3.5.

4 Existence theorem: the general case

Theorem 4.1. *Under the assumptions of Theorem 3.2 ($(A_1) - (A_5)$ and convexity of H), f has at least one critical point.*

Proof. If L is positive semi-definite, then $f(x)$ is a convex lower semi-continuous function which attains its minimum (as $f(x) \to \infty$ when $\|x\| \to$

∞, as follows from the corollary after Proposition 3.3). Thus in this case the theorem holds.

We therefore assume that the negative subspace X^- is nontrivial. Let E_n be finite dimensional L-invariant subspaces of dom L such that Ker $L \subset E_n \subset E_{n+1}$ and the union of E_n is dense in the domain of L. Denote by f_n the restriction of f to E_n. Then f_n is continuous and locally Lipschitz on E_n. Let further B_n^- and S_n^- be the intersections of E_n^- with the unit ball and the unit sphere of X, respectively. Let finally Y_n be the orthogonal complement of E_n^- in E_n which is the sum of E_n^+ and Ker $L = X^0$.

We shall show first, using the standard saddle point approach due to Rabinowitz [13], that for any n there is an $x_n \in E_n$ such that

$$0 \in P_n(Lx_n + \partial H(x_n)) \tag{4.5}$$

(which means that x_n is a critical point of f_n) and $f(x_n)$ converge to a certain finite number.

Set

$$\lambda_n = \inf_{x\in Y_n} f_n(x) = \inf_{x\in Y_n} f(x).$$

Then $\ldots \geq \lambda_n \geq \lambda_{n+1} \geq \ldots \geq \lambda = \inf_Y f(x)$ and $\lambda > -\infty$ by the corollary after Proposition 3.3.

Using $(\mathbf{A}_3)$, we can choose $r > 0$ so large that $\sup_{rS_n^-} f(x) < \lambda$. Consider the collection $\mathcal{P}_n$ of continuous mappings from rB_n^- into E_n which are the identity on rS_n^-. Every such mapping meets Y_n, as dim $E_n^- < \infty$. Therefore

$$c_n = \inf_{p\in\mathcal{P}_n} \max_{x\in rB_n^-} f(p(x)) \geq \lambda_n.$$

It follows that f_n has a critical point in the sense of the "nonsmooth critical point theory" of [5, 7, 10] at the level c_n. Let x_n be such a point. As f_n satisfies the Lipschitz condition, we get $0 \in \partial f_n(x_n)$ which is exactly (4.5).

We have $c_n \geq \lambda$ for all n. On the other hand, taking the identity mappings for any n meeting Y_n at zero, we conclude that $c_n \leq H(0) < \infty$. Therefore $c_n = f(x_n)$ are uniformly bounded. Applying Proposition 3.3, we conclude that $\{x_n\}$ is a bounded sequence. But it is obviously a weak PS-sequence, so by Proposition 3.5 it has a limit point which is a critical point of f.

In the above proof the convexity assumption was used only to show that the sequence $\{x_n\}$ is bounded. The rest of the proof is based exclusively on the assumptions $(\mathbf{A}_1) - -(\mathbf{A}_5)$. This allows one to weaken the convexity requirement in the following way.

Theorem 4.2. *Assume in addition to* $(\mathbf{A}_1) - -(\mathbf{A}_5)$ *that* H *is "convex at infinity" in the following sense:* $H(x) = \tilde{H}(x) + \varphi(x)$*, where* $\tilde{H}$ *is convex continuous, the function* $\varphi(x)$ *is locally Lipschitz and bounded and* $\mathrm{Lip}\ \varphi(x) = O(\|x\|^{-1})$ *as* $\|x\|$ *goes to infinity (*$\mathrm{Lip}\ \varphi(x)$ *being the Lipschitz constant of* φ *at* x*). Then the conclusions of Theorems 3.2 and 4.1 remain valid.*

Proof. In view of Proposition 3.5 and the proof of Theorem 4.1, we have to show that under the assumptions every weak weighted Palais–Smale sequence is bounded. So assume the contrary: there is such a sequence $\{x_n\}$ with $\|x_n\| \to \infty$. Let $\{ E_n\}$ be a corresponding increasing sequence of L-invariant finite dimensional subspaces of X such that the domain of L the closure of belongs to the union of E_n and $(1+\|x_n\|)\rho_{E_n}(x_n) \to 0$.

Set $\tilde{f}(x) = (1/2)(Lx|x) + \tilde{H}(x)$ and $\tilde{\rho}_E(x) = \text{dist}\,(0, L(x) + P_E(\partial \tilde{H}(x)))$. By the assumptions, the sequence $\{\tilde{f}(x_n)\}$ is bounded, and $\partial \tilde{H}(x) \subset \partial H(x) - \partial\varphi(x)$ (see, e.g., [3] for the calculus of generalized gradients) so that

$$\text{dist}\,(0, L(x_n) + P_{E_n}(\partial \tilde{H}(x_n)) \le \text{dist}\,(0, L(x_n) + P_{E_n}(\partial H(x_n)) + \text{Lip}\,\varphi(x_n).$$

It follows that

$$(1+\|x_n\|))\tilde{\rho}_{E_n}(x_n) = O(1),$$

is a bounded sequence and, as $\tilde{H}$ is convex continuous and obviously satisfying $(\mathbf{A}_4)$–$(\mathbf{A}_5)$, Proposition 3.3 implies uniform boundedness of x_n.

5 Application to Hamiltonian systems

In this section we apply Theorem 3.2 to study the existence of T-periodic solutions of a non-autonomous Hamiltonian inclusion

$$J\dot{x} \in \partial H(t,x), \tag{5.6}$$

where $x \in \mathbb{R}^{2n}$ and $\partial H(t,x)$ stands for the Clarke subdifferential of H as a function of x. The corresponding functional is

$$f(x(\cdot)) = \int_0^T [(-1/2)(J\dot{x}|x) + H(t,x)]dt. \tag{5.7}$$

We shall consider the problem in the space $X = L_2[0,T]$. Then (5.7) has the form (1.1) with L being the operator that carries $x(t)$ into $-J\dot{x}(t)$ and $H(x(\cdot)) = \int_0^T H(t,x(t))dt$. The domain of L is of course the Hardy space H^1.

Then L satisfies $(\mathbf{A}_2)$, $(\mathbf{A}_3)$ (see, e.g., [6]). More precisely, the spectrum of L consists of all numbers $2k\pi/T$ with $k = 0, \pm1, \pm2, \dots$. The multiplicity of any eigenvalue is $2n$. In other words, when applying Proposition 3.3, we can take as E_m invariant subspaces of L, for instance, we can take E_m equal to the subspace spanned by all eigenvectors of L corresponding to eigenvalues $2k\pi/T$ with $|k| \le m$.

The following assumptions will be imposed on $H(t,x)$:

(H_1) $H(\cdot,x)$ is summable on $[0,t]$ for every x, $H(t,\cdot)$ is locally Lipschitz in x for almost any t and $\int_0^T H(t,x)dt \to \infty$ as $|x| \to \infty$;

(H_2) there is a positive $\alpha < \pi/T$ and a summable nonnegative function $r(t)$ such that

$$|H(t,x)| \le \alpha\|x\|^2 + r(t);$$

It follows from (H_1), (H_2) that $(\mathbf{A}_4)$ and $(\mathbf{A}_5)$ are satisfied as well.

Theorem 5.1. *Assume (H_1), (H_2) and (H_3) $H(t,x) = \tilde{H}(t,x) + \varphi(t,x)$, where $\tilde{H}$ is convex continuous in x and summable w.r.t. t for any x, $\varphi(t,0)$ is also summable, φ is locally Lipschitz as a function of x and* Lip $\varphi(t,x) \le l(t,\|x\|)$ *for all t, x, where $l(t,\xi)$ is a nonnegative Carathéodory function with the following properties:*

(i) $l(t,\xi)$ nonincreasing as a function of ξ;

(ii) $\int_0^T l^2(t,\xi)dt = O(1/\xi^2)$ as $\xi \to \infty$;

(iii) $\int_0^T \sup_{\xi} l(t,\xi)dt = C < \infty$.

Then there exists a solution of (5.6) *satisfying the periodic boundary condition $x(0) = x(T)$.*

Theorem 5.1 does not follow from Theorem 4.2 although the key element of the proof is similar: this time this is the demonstration that the weighted (PS)-condition is satisfied for $f(\cdot)$ in the L_2-metric.

We precede the proof by three technical lemmas. For any $x \in \mathbb{R}^n$ we denote by $m(x)$ the vector whose components are equal to absolute values of the corresponding components of x, that is, if $x = (\xi_1, ..., \xi_n)$, then $m(x) = (|\xi_1|, ..., |\xi_n|)$.

Lemma 5.2. *Let $\psi(x)$ be a convex function on $\mathbb{R}^n$ satisfying for some $\beta \ge 0$, $\gamma \ge 0$*

$$|\psi(x)| \le (\beta/2)\|x\|^2 + \gamma, \quad \forall x \in \mathbb{R}^n.$$

Then for any x and any $u \in \partial\psi(x)$

$$\frac{1}{n}(m(x)|m(u)) \le \frac{5}{2}\beta\|x\|^2 + 2\gamma.$$

Proof. Let $e \in \mathbb{R}^n$, $\|e\| = 1$. If $u \in \partial\psi(x)$, then

$$\psi(x \pm (x|e)e) - \psi(x) \ge \pm(x|e)(u|e),$$

so that

$$\begin{aligned} 2\beta\|x\|^2 + \gamma = \frac{\beta}{2}\|2x\|^2 + \gamma &\ge \frac{\beta}{2}\|x \pm (x|e)e\|^2 + \gamma \\ &\ge \psi(x \pm (x|e)e) \ge \pm(x|e)(u|e) + \psi(x) \\ &\ge \pm(x|e)(u|e) - \frac{\beta}{2}\|x\|^2 - \gamma. \end{aligned}$$

Thus, if $x = (\xi_1, ..., \xi_n)$, $u = (\eta_1, ...\eta_n)$, then taking $e = e_i = (0, ..., 1, ..., 0)$, we get

$$|\xi_i||\eta_i| \le \frac{5}{2}\beta\|x\|^2 + 2\gamma$$

and the result follows.

Lemma 5.3. *Assume (H_1)-(H_3). Let $\{x_k(\cdot)\}$ be a PS-sequence for* (5.6) *in L_2. Then there are numbers M_0 and M (not depending on k) such that*

$$\max_{0\le t\le T}\|x_k(t)\| \le M_0 + M\min_{0\le t\le T}\|x_k(t)\|$$

for all k.

Proof. In the proof, we denote $d(x,Q)$ the distance from $x\in\mathbb{R}^n$ to $Q\subset\mathbb{R}^n$. Then $\{x_k(\cdot)\}$ being the PS-sequence means (after selecting a suitable subsequence if necessary) that

$$f(x_k(\cdot))\to c;\quad \int_0^T d^2(0,-J\dot{x}_k(t)+\partial H(t,x_k(t))dt\to 0,$$

the latter being equivalent to the existence of measurable selections $u_k(\cdot)$ of set-valued mappings $t\mapsto\partial H(t,x(t))$ such that

$$\int_0^T |J\dot{x}_k(t)-u_k(t)|^2dt\to 0.$$

Set $\eta_k(t) = \|J\dot{x}_k(t)-u_k(t)\|$. By the standard rules of the calculus of generalized gradients of Clarke, $u_k(t)=w_k(t)+\xi_k(t)$, where $w_k(t)\in\partial\tilde{H}(t,x_k(t))$ and $\xi_k(t)\in\partial\varphi(t,x_k(t))$ (the possibility to choose these two functions measurable also follows from the standard measurable selection argument). By (H_3) the latter gives

$$\|\xi_k(t)\|\le l(t,\|x_k(t)\|)$$

As follows from Lemma 5.2 and (H_2),

$$\begin{aligned}(\dot{x}_k(t)|x_k(t)) &\le (m(\dot{x}_k(t))|m(x_k(t))) = (m(J\dot{x}_k(t))|m(x_k(t)))\\ &\le (m(w_k(t))|m(x_k(t))) + (\eta_k(t)+\|\xi_k(t)\|)\|x_k(t)\|\\ &\le 5n\alpha\|x_k(t)\|^2 + (\eta_k(t)+\|\xi_k(t)\|)\|x_k(t)\| + 2r(t)\\ &\le c_k(t)\|x_k\|^2 + r_k(t)),\end{aligned}$$

where, say $c_k(t)=5n\alpha+\eta_k(t)+\|\xi_k(t)\|$ and $r_k(t)=2r(t)+\eta_k(t)+\|\xi_k(t)\|$. It is clear that integrals of these functions are uniformly bounded by certain C_0 and R_0. Thus

$$\frac{d}{dt}\|x_k(t)\|^2\le c_k(t)\|x_k\|^2+r_k(t),$$

and applying Gronwall's lemma (e.g., [9]), we get that for any $\tau\in[0,T]$ and any k

$$\|x_k(t)\|^2\le\|x_k(\tau)\|^2+e^{\int_\tau^t c(s)ds}\int_\tau^t r_k(s)e^{-\int_\tau^s c(\sigma)}d\sigma\le\|x_k(\tau)\|^2+\text{const}$$

and this immediately implies the desired inequality.

Lemma 5.4. *Assume* (H_2), (H_3). *Then the function*

$$f(x(\cdot)) = \int_0^T \tilde{H}(t, x(t))dt$$

on L_2 *is convex continuous and satisfies*

$$\tilde{H}(x(\dot{})) \le \alpha \|x(\cdot)\|_2^2 + const.$$

Furthermore, $y(\cdot) \in \partial\tilde{H}(x(\cdot))$ *if and only if* $y(t) \in \partial\tilde{H}(t, x(t))$ *almost everywhere.*

Proof. The inequality is immediate from (H_2) and part (iii) of (H_3), whence continuity of f as it is clearly convex. The subdifferential inclusion is now a well-known fact of analysis of convex integral functionals.

Proof of the theorem. Let $\{x_k(\cdot)\}$ be a weighted PS-sequence for $f(\cdot)$ in L_2. This means that there are $u_k(\cdot)$ such that $u_k(t) \in \partial H(t, x_k(t))$ almost everywhere and

$$(1 + \|x_k(\cdot)\|_2)(\int_0^T \|u_k(t) - J\dot{x}_k(t)\|^2 dt)^{1/2} \to 0$$

as $k \to \infty$. Then of course $\int_0^T |u_k(t) - J\dot{x}_k(t)|^2 dt \to 0$ and we can apply Lemma 5.3. It follows from the lemma that either $\{x_k(\cdot)\}$ are uniformly bounded or $\min_t \|x_k(t)\| \to \infty$.

In the first case the sequence $\{x_k(\cdot)\}$ is obviously weakly precompact in H^1 and we get a critical point as the limit.

Assume now that $a_k = \min_t \|x_k(t)\| \to \infty$. We have $u_k(t) = y_k(t) + z_k(t)$, where $y_k(t) \in \partial\tilde{H}(t, x_k(t))$ and $z_k(t) \in \partial\varphi(t, x_k(t))$ almost everywhere. As $\|x_k(t)\| \ge a_k$, we have $\|z_k(t)\| \le l(t, a_k)$ by (i) which means that $\int_0^T \|z_k(t)\|^2 dt = O(a_k^{-2}) \to 0$ by (ii) and, on the other hand, $\|x(t)\| \le M_0 + M a_k$ for all t by Lemma 5.3 and therefore $\|x_k(\cdot)\|_2 \le O(a_k)$. It follows that the quantity

$$(1 + \|x_k(\cdot)\|_2)(\int_0^T \|y_k(t) - J\dot{x}_k(t)\|^2 dt)^{1/2}$$

is uniformly bounded for all k. But by Lemma 5.4 this quantity is precisely $(1 + \|x_k(\cdot)\|_2)\tilde{\rho}(x_k(\cdot))$, where $\tilde{\rho}(x(\cdot))$ is the distance from $Lx(\cdot) + \partial\tilde{H}(x(\cdot))$ to zero. Thus the sequence $(1 + \|x_k(\cdot)\|_2)\tilde{\rho}(x_k(\cdot))$ is bounded. This means that

$$\tilde{f}(x(\cdot)) = \int_0^T [(-1/2)(J\dot{x}|x) + \tilde{H}(t, x)]dt.$$

satisfies all the conditions of Proposition 3.3 (in L_2), and applying the proposition we find that the sequence $\{x_k\}$ is uniformly bounded, in contradiction with what has been assumed. This completes the proof of the theorem.

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Technion
Department of Mathematics
32000 Haifa, Israel
email: ioffe@math.technion.ac.il
fim@techunix.technion.ac.il

Periodic Solutions of Nonlinear Problems with Positive Oriented Periodic Coefficients

W. Marzantowicz

Dedicated to Antonio Marino

ABSTRACT We study nonlinear ODE problems in the complex Euclidean space, with the right hand side being a complex analytic function of the space variable z with nonconstant periodic coefficients in the time variable t. As the coefficients functions we admit only functions with vanishing Fourier coefficients for negative indices. This leads to an existence theorem which relates the number of solutions with the number of zeros of the averaged right hand side function, and finally gives a theorem of the existence of periodic solution which originates from infinity. The work generalizes and extends previous results of the author, joint with A. Borisovich, for the polynomial case.

1 Introduction

In this paper we study a non-autonomous system of ordinary differential equations in the plane of the form

$$\dot{u}(t) = f(u(t), t), \tag{1.1}$$

where $f(z, t)$ is a holomorphic function in the space variable $z \in \mathcal{U} \subset \mathbb{C}$ and T-periodic with respect to the time variable t. We note that if the coefficients of the Taylor expansion $\sum_{j=0}^{\infty} c_j(t)(z - z_0)^j$ of f, with respect to the variable z, belong to the trace of disc algebra on the boundary (i.e., are the restrictions to the sphere of these continuous functions on disc which are holomorphic in its interior) then the 0-th Fourier coefficient is a multiplicative functional.

The averaging of (1.1) leads to a complex analytic equation

$$0 = \mathbf{f}(\mathbf{u}), \tag{1.2}$$

where $\mathbf{f}(z) := \sum_{j=0}^{\infty} \mathbf{c}_j (z-z_0)^j$, with $\mathbf{c}_j := \frac{1}{T}\int_0^T c_j(t)dt$ and $\mathbf{u} := \frac{1}{T}\int_0^T u(t)dt$.

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First we observe that $u(t)$ is a periodic solution (in the discussed algebra) only if $\mathbf{u}$ is a root of $\mathbf{f}$, i.e., the T-periodic solutions of 1.1 are only in the fibres $\{F_l\}$, over (isolated !) zeros $\{z_l\}$, of $\mathbf{f}$ (4.1).

Next we give a sufficient condition for the existence of the T-periodic solution of (1.1) over a zero of $\mathbf{f}$. To do it we express f as

$$f(z, t) = (z - z_0)\, h(z, t) + r(t) \tag{1.3}$$

Theorem 3.7. The main result says that if the multiple of T by the sum of norms of h and r is smaller than 2, then there exists a T-periodic solution $u(t) \in C^0_+(T)$ of (1.1) with the mean value equal to z_0, which is nonconstant if $r \neq 0$ (Theorem 4.6). The proof of the main theorem is essentially the same as the corresponding theorem of [2], i.e., it is a kind of continuation method of Krasnosielski et al. and Mawhin et al. [13]. Indeed our hypothesis is equivalent to an assumption that equation (1.1) is either a small perturbation (but effectively given) of equation (1.2) or the frequency is large enough, which resembles previous approaches to the problem ([12], [11], [18]). As an application, we show an example of function $f(z, t)$ for which the equation (1.1) has infinitely many distinct T-periodic solutions (Corollary 4.12.)

Finally we prove the existence of a nontrivial periodic solution for a meromophic function f, which originates from infinity (Theorem 5.4).

To make the paper self-contained, in Section 1 we give a brief exposition on the Banach subalgebra of T-periodic C^k-functions of disc algebra type, called the positive oriented functions. Next in Section 2 we present properties of the Banach algebra of functions $f(z, t)$ which are holomorphic with respect to the first variable and T-periodic and positive oriented with respect to the second. The subject is presumably known but not unified in most of the available literature.

2 Banach algebras of positive oriented differentiable functions

Let $\mu >$ and $D_\mu := \{z \in \mathbb{C} : |z| \leq \mu\}$ be the disc of the radius μ. Let next $\mathcal{H}(D_\mu)$ be the class of all analytic functions in the interior $\overset{o}{D}_\mu = \{z \in \mathbb{C} : |z| < \mu\}$.

We use the notation $C^0(D_\mu)$, or for short $C(D_\mu)$, corresponding to $C^k(D_\mu)$ for the Banach space (algebra with the ordinary multiplication of functions) of continuous, or k-differentiable **in the real sense** respectively, complex-valued functions on D_μ, with the norm:

$$\|f\| := \sup_{z \in D_\mu} |f(z)| \quad \text{or}$$

$$\|f\|_k := \sum_{l=0}^{k} \sup_{z \in D_\mu} |f^{(l)}(z)| \qquad 0 \le l \le k \quad \text{respectively}.$$

Put $T = 2\pi\mu$. As above we have a natural linear map (of norm equal to 1): $\mathrm{res}_T : C^k(D_\mu) \to C^k(\partial D_\mu)$, assigning each $f \in C^k(D_\mu)$ its restriction $\tilde{f} := f_{|\partial D_\mu}$ to the boundary.

For an obvious reason the space $C^k(\partial D_\mu)$ is isomorphic to the space of all functions in $C^k(\mathbb{R}; \mathbb{C})$ which are T-periodic with all derivatives up to the k-th, which is denoted by $C^k(T)$. Set $\tilde{f}(t) := f(\mu \exp(\frac{2\pi}{T} \imath t))$. Since $S^1_\mu = \partial D_\mu = \mathbb{R}/2\pi\mu\mathbb{Z}$, we can identify the space $C^k(\partial D_\mu)$ with the space of k-th differentiable T-periodic functions with the usual norm

$$\|h\|_k := \sum_{l=0}^{k} \sup_{0 \le t \le T} |h^{(l)}(t)|.$$

Note that in this definition differentiability and differentials of $f : D \to \mathbb{C}$ are considered as corresponding notions of a map of $\mathbb{R}^2$. Consequently, for every $j \in \mathbb{Z}$ and $\tilde{f} \in C^k(T)$, the j-th Fourier coefficient of f is defined as

$$a_j(f) := \frac{1}{T} \int_0^T \tilde{f}(t) \exp(-\frac{2\pi}{T} \imath j\, t)\, dt.$$

Definition 2.1. *Let $\mathcal{A}^k(D_\mu)$ denote a subalgebra of $C^k(D_\mu)$ consisting of functions continuous on the boundary ∂D_μ and holomorphic in the open disc $\overset{o}{D}_\mu$, i.e.,*

$$\mathcal{A}^k(D_\mu) := \{f \in C^k(D_\mu) : f_{|\overset{o}{D}_\mu} \in \mathcal{H}(D_\mu)\}$$

with the ordinary multiplication of functions and the norm taken from $C(D_\mu)$. Let us also set

$$\tilde{\mathcal{A}}^k(D_\mu) := \mathrm{res}_T(\mathcal{A}^k(D_\mu)).$$

Finally put

$$C^k_+(T) := \{f \in C^k(T);\ a_j(f) = 0, \quad \text{for all} \quad j < 0\},$$

with the usual C^k-norm as in $C^k(T)$.

The norm in $\mathcal{A}^k(D_\mu)$ is defined as

$$\|f\|_k = \sum_{l=0}^{k} \sup_{|z| \le \mu} |f^{(l)}(z)| = \sum_{l=0}^{k} \sup_{|z| = \mu} |f(z)^{(l)}|.$$

We would like to emphasize that if $k = 0$, and $\mu = 1$, (i.e., $T = 2\pi$,) then $\mathcal{A}^0(D_1) := \mathcal{A}^0$ is called the disc algebra (cf. [7], [8], [9]).

To shorten this paper we present properties of the algebras $\mathcal{A}^k(D_\mu)$ without proofs. For the case $\mu = 1$, which is the most studied in the literature, they are presented in [7], [8], [9], [17]. The general case easily follows by Theorem 2.8.

First, for any derivative, up to the k-th, of a function $f \in \mathcal{A}^k(D_\mu)$ *the Cauchy integral formula and maximum modulus principle hold* (cf. [7]). From the maximum modulus principle we have

$$\|f\| = \sup_{|z|\le\mu} |f(z)| = \sup_{|z|=\mu} |f(z)|.$$

Theorem 2.2. *$\mathcal{A}^k$ and $\tilde{\mathcal{A}}^k(D_\mu)$ are closed Banach subalgebras of $C^k(D_\mu)$ and $C^k(\partial D_\mu)$, respectively. If $\tilde{f} = \mathrm{res}\, \mathrm{f} \in \mathcal{A}(\tilde{\mathrm{D}}_\mu)$, then for the j-th Fourier coefficient $a_j(f)$ we have*

$$a_j(\tilde{f}) = \frac{1}{Ti}\int_{|\zeta|=\mu} \frac{f(\zeta)}{\zeta^{j+1}} d\zeta .$$

For every $j < 0$, $\quad a_j(\tilde{f}) = 0$, and

$$a_j(\tilde{f}) = f^{(j)}(0)/j!,$$

for every $j \geq 0$. Furthermore the sequence of coefficients $a_j(\tilde{f}^{(l)})$, $j \geq 0$, $0 \leq l \leq k$, of a function $\tilde{f} \in \mathcal{A}^k(\tilde{D}_\mu)$ is determined by the sequence $\{f^{(j)}(0)\}$. □

Theorem 2.3. *The homomorphism* res : $\mathcal{A}^k(\mathrm{D}_\mu) \to \tilde{\mathcal{A}}^k(\mathrm{D}_\mu)$ *is an isometry of Banach algebras. We have $\tilde{\mathcal{A}}^k(D_\mu) = C^k_+(T)$, and consequently $C^k_+(T)$ is a Banach subalgebra of $C^k(T) = C^k(\partial D_\mu)$.* □

From now on we shall not distinguish between $\tilde{\mathcal{A}}^k(D_\mu)$ and $C^k_+(T)$ and use rather the second notion as the more natural one.

Definition 2.4. *For $n \in \mathbb{N} \cup \{0\}$ we define a functional*

$$\tilde{\varphi}_n : C^k_+(T) \to \mathbb{C},$$

given as the n-th Fourier coefficient:

$$\tilde{\varphi}_n(\tilde{f}) := a_n(\tilde{f}).$$

This functional corresponds to a functional $\varphi_n : \mathcal{A}^k(D_\mu) \to \mathbb{C}$, $\varphi_n(f) := \tilde{f}^{(n)}(0)/n!$, which means that $\tilde{\varphi}_n = \mathrm{res}\,\varphi$. It is called the averaging, or mean value functional. If $f \in \mathcal{A}^k(D_\mu)$, or $\tilde{f} \in C^k_+(T)$, then we denote by $\mathbf{f}$, *or* $\tilde{\mathbf{f}}$ *$\varphi(f)$, or $\tilde{\varphi}(\tilde{f})$ respectively.*

Corollary 2.5. *Functionals φ_0 and $\tilde{\varphi}_0$ are multiplicative on corresponding algebras, $\mathcal{A}^k(D_\mu)$ and $C_+^k(T)$, respectively.*

$$\varphi_0(fg) = \varphi_0(f)\varphi_0(g) \qquad \textit{for all } f, g \in \mathcal{A}^k(D_\mu)$$

$$\tilde{\varphi}_0(fg) = \tilde{\varphi}_0(f)\tilde{\varphi}_0(g) \qquad \textit{and equivalently}$$

$$\frac{1}{T}\int_0^T f(t)g(t)dt = \frac{1}{T}\int_0^T f(t)dt \cdot \frac{1}{T}\int_0^{2\pi} g(t)dt \quad \textit{for all } f, g \in C_+^k(T).$$

□

To shorten notation, from now on we shall drop the tilde in symbols of elements of $C_+^k(T) = \tilde{\mathcal{A}}^k(T)$, as well in the symbols of the Fourier functionals $\varphi_n : C_+^k(2\pi) \to \mathbb{C}$ if it does not lead to a disorientation. Also we write $\mathcal{A}^k$ for $\mathcal{A}^k(2\pi)$.

Corollary 2.6. *For given $k \geq 0$ the kernels*

$$\mathcal{I}_0^k := \ker \varphi_0 \subset \mathcal{A}^k(D_\mu) \qquad \textit{and} \qquad \tilde{\mathcal{I}}_0^k := \ker \tilde{\varphi}_0 \subset C_+^k(T)$$

are ideals in the corresponding Banach algebras. □

Definition 2.7. *Let $\rho : \mathcal{A}^k(D_\mu) \to \mathcal{A}^k$ be an operator defined as follows: for every $f \in \mathcal{A}^k(D_\mu)$ $\rho(f) = f_1$, where $f_1(z) = f(\mu z)$ for $|z| < 1$. For every $\tilde{f}$ in $\tilde{\mathcal{A}}^k(D_\mu)$ let $\rho^*(\tilde{f}) = \tilde{f}_1$ and $\tilde{f}_1(\sigma) = \tilde{f}(\mu\sigma)$, where $\sigma \in [0, 2\pi]$.*

Theorem 2.8. *Operators ρ and ρ^* are isometries and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}^k(D_\mu) & \xrightarrow{\ \rho\ } & \mathcal{A}^k \\ {\scriptstyle \mathrm{res}_T}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \tilde{\mathcal{A}}(D_\mu) = C_+^k(T) & \xrightarrow{\ \rho^*\ } & \tilde{\mathcal{A}} = C_+^k(2\pi)\,. \end{array}$$

□

Example 2.9. For every k any trigonometric polynomial

$$p(t) := \sum_0^n r_j \exp(\imath \frac{2\pi}{T} j\, t)$$

belongs to $C_+^k(T)$.

At the end of this section we define the dual notion of the "conjugated" disc algebra and negative oriented periodic functions.

Definition 2.10. *Let $\overline{\mathcal{A}}^k(D_\mu)$ denote a subalgebra of $C^k(D_\mu)$ consisting of C^k functions f on D_μ such that $\overline{f}$ is holomorphic in the open disc $\overset{\circ}{D}_\mu$.*
Analogously, for a given $k \in \mathbb{N} \cup \{0\}$ we define a closed subspace (subalgebra) of the T-periodic functions

$$C^k_-(T) := \{f \in C^k(T);\ a_j(f) = 0, \quad \text{for all} \quad j > 0\}.$$

We called $C^k_-(2\pi)$ the subspace of negative oriented periodic functions.

We end with a theorem whose proof is left to the reader.

Theorem 2.11. *The homomorphism $\mathrm{res}_\mathrm{T} : \overline{\mathcal{A}}^k(\mathrm{D}_\mu) \to \mathrm{C}^k_-(\mathrm{T})$ is an isometry of the complex Banach algebras. Moreover the mapping $f \mapsto \overline{f}$ defines an isomorphism of the Banach algebras (over $\mathbb{R}$!) $\mathcal{A}^k(D_\mu)$ and $\overline{\mathcal{A}}^k(D_\mu)$, and the mapping $u(t) \mapsto u(-t)$ defines an isomorphism of the real Banach algebras $C^k_+(T)$ and $C^k_-(T)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A}^k(D_\mu) & \xrightarrow{f \mapsto \overline{f}} & \overline{\mathcal{A}}^k(D_\mu) \\ {\scriptstyle \mathrm{res}_\mathrm{T}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_\mathrm{T}} \\ C^k_+(T) & \xrightarrow{u(t) \mapsto u(-t)} & C^k_-(T). \end{array}$$

Furthermore the hypotheses of all the statements of this section hold as well for the algebras $\overline{\mathcal{A}}^k(D_\mu)$ and $C^k_-(T)$ in appropriate formulations.

3 Algebras of analytic functions with periodic coefficients

In this section we introduce a special class of of functions of two variables (complex z and real-time t), which are analytic with respect to the first and periodic with respect to the second.

Our task is now to define an appropriate function space of functions on $\mathcal{U} \times [0, T]$ which we call the fine T-periodic functions.

Definition 3.1. *Let $\mathcal{U} \subset \mathbb{C}$ be an open set. We denote by $\mathcal{H}(\mathcal{U}, C^k_+(T))$ the set of all continuous functions $f : \mathcal{U} \times [0, T] \to \mathbb{C}$, which are T-periodic with respect to t and for a fixed $z \in \mathcal{U}$ belong to $C^k_+(T)$, and which are holomorphic with respect to z for a fixed t.*

Definition 3.2. *Let $\mathcal{U} \subset \mathbb{C}$ be an open set. We denote by $\mathcal{A}(\mathcal{U}, C^k_+(T))$ the set of all C^k- functions $f : \mathcal{U} \times [0, T] \to \mathbb{C}$, T-periodic with respect to t satisfying the following condition: For every $z^0 \in \mathcal{U}$, there exists $\rho > 0$ and* *a sequence* $\{c_j(t)\}$, $c_j(t) \in C^k_+(T)$, *with*

$$\limsup_{j\to\infty} \sqrt[j]{\|c_j\|_k} \le \frac{1}{\rho} \ , \tag{3.4}$$

such that

$$\forall_{\varepsilon} \ \exists_{n_0} \ \forall_{n \ge n+0} | \ \|f(z,t) - \sum_{j=0}^{j=n} c_j(t)(z-z_0)^j\|_k < \varepsilon \ , \tag{3.5}$$

uniformly with respect to z *in the disc* $D_\rho(z_0) = \{|z_1^0 - z| < \rho\}$. *We equip the space* $\mathcal{A}(\mathcal{U}, C^k_+(T))$ *in the topology of uniform convergence on every compact subset* $K \subset \mathcal{U}$. *More precisely, let* $\{K_n\}$, $K_n \subset K_{n+1}$, *be a sequence of compact subsets of* $\mathcal{U}$, *such that* $\bigcup_1^\infty K_n = \mathcal{U}$. *Every* K_n *defines a semi-norm*

$$p_{K_n}(f) := \sum_{l=0}^{k} \max_{z\in K_n,\, t\in[0,T]} |f^{(l)}(z,t)|.$$

The sequence of semi-norms p_{K_n} *defines a locally convex metrizable topology in* $\mathcal{A}(\mathcal{U}, C^k_+(T))$.

Remark 3.3. Since $|c| \le \|c(t)\|_0 \le \|c(t)\|_k$ for every $c(t) \in C^k(t)$, from (3.4) it follows that the series $\sum_j c_j(z-z_0)^j$ has the radius of convergence greater than ρ, and consequently it defines a holomorphic function in the disc $D_\rho(z_0)$.

We show that Definitions 3.2 and 3.1 describe the same space.

Theorem 3.4. *Let* $\mathcal{U} \subset \mathbb{C}$ *be an open set. Then*

$$\mathcal{A}(\mathcal{U}, C^k_+(T))) = \mathcal{H}(\mathcal{U}, C^k_+(T)).$$

Proof. Suppose that $f \in \mathcal{A}(\mathcal{U}, C^k_+(T))$. By Definition 3.2 and completeness of $C^k_+(T)$, the function $g(t) := f(z,t)$ belongs to $C^k_+(T)$. Furthermore, for a fixed $t \in [0,T]$ the function $h(z) := f(z,t)$ is defined by the power series $\sum_0^\infty c_j(t)(z-z_0)^j$ absolutely convergent in a disc $\{|z-z_0| < \rho\}$, and is thus holomorphic in this disc.

To show the inverse inclusion $\mathcal{H}(\mathcal{U}) \subset \mathcal{A}(\mathcal{U})$ we need the parameter Cauchy formula. Let $f \in \mathcal{H}(\mathcal{U}, C^k_+(T))$ and $z_0 \in \mathcal{U}$. For $j \in \mathbb{Z}$, define

$$c_j(\tilde{f})(t) = \frac{1}{Ti} \int_{|\zeta - z_0| = \rho_0} \frac{f(\zeta, t)}{(\zeta - z_0)^{j+1}} d\zeta,$$

where ρ_0 is so small that $|\zeta - z_0| \le \rho_0 \subset \mathcal{U}$. By the property of the integral of a function with parameter, each $c_j(\tilde{f})(t)$ is a continuous function and belongs to $C^k(T)$. For a given $t \in [0, T]$ from the Cauchy formula and the fundamental theorem of analytic functions, it follows that

$$f(z, t) = \sum_0^{\infty} c_j(t)(z - z_0)^j$$

and the series is convergent uniformly in every disc $|z - z_0| \leq \rho$, $\rho < \rho_0$. For $z = z_0$ we have $f(z_0, t) = c_0(t)$ which shows that $c_0(t) \in C^k_+(T)$. Set $f(z, t) = c_0(t) + (z - z_0)f_1(z, t)$. The function f_1, locally defined by the higher terms of the Taylor expansion, by definition belongs to $\mathcal{H}(D_\rho, C^k_+(T))$. Applying the above procedure to f_1 we get $c_1(t) \in C^k_+(T)$, and by the inductive argument $c_j(t) \in C^k_+(T)$ for every $0 \leq j < \infty$. Consequently $f \in \mathcal{A}(\mathcal{U}, C^k_+(T))$. □

(As a matter of fact f_1 has a unique extension to $\mathcal{H}(\mathcal{U}, C^k_+(T))$ by Lemma 3.5 and Theorem 3.7.)

Lemma 3.5. *Let $f \in \mathcal{H}(\mathcal{U}, C^k_+(T))$, $\mathcal{U} \in C$. Suppose that there exists $z^0 =\in \mathcal{U}$ such that $f(z_0, t) = 0$ for all $t \in [0, t]$. Then*

$$f(z, t) = (z - z_0)\, h(z, t),$$

where $h \in \mathcal{H}(\mathcal{U}, C^k_+(T))$ is uniquely defined.

Proof. Note that the function $z - z_0 \in \mathcal{H}(\mathcal{U}, C^k_+(T))$ and $\frac{f(z,t)}{(z-z_0)} \in \mathcal{H}(\mathcal{U} \setminus \{z_0\}, C^k_+(T))$. We show that $\frac{f(z,t)}{(z-z_0)} \in \mathcal{H}(\mathcal{U}, C^k_+(T))$.

Consider the function $h(z, t) := \frac{f(z,t)}{(z-z_0)}$, fix $z \in \mathcal{U}$, and take a small disc $D_\rho(z) \subset \mathcal{U}$. Obviously $h(z, t_0) \in \mathcal{H}(\mathcal{U})$ for a fixed t_0. Fixing t and using the Cauchy formula we have

$$\begin{aligned} h(z, t) = \frac{f(z, t)}{z - z_0} &= \frac{1}{2\pi i} \int_{|\zeta - z| = \rho} \frac{\frac{f(\zeta, t)}{\zeta - z_0}}{\zeta - z} d\zeta \\ &= \frac{1}{2\pi i} \int_{|\zeta - z| = \rho} \frac{f(\zeta, t)}{(\zeta - z_0)(\zeta - z)} d\zeta . \end{aligned}$$

Consequently $h(z, t)$ forms a continuous family of holomorphic functions, periodic and C^k in t thus gives a function in $\mathcal{H}(\mathcal{U}, C^k_+(T))$. The uniqueness is obvious. □

Definition 3.6. *We say that $z_0 \in \mathcal{U}$ is a zero of $f \in \mathcal{H}(\mathcal{U}, C^l_+(T))$, or a stationary zero of f, if $f(z_0, t) = 0$ for all $t \in [0, T]$ which is equivalent to*

$$f(z,t) = (z - z_0)\, \tilde{f}(z, t),$$

where $\tilde{f}(z, t) \in \mathcal{H}(\mathcal{U}, C^l_+(T))$. The set of all zeros of f is denoted by $\mathrm{Zer} f$

We say that $z_0 \in \mathcal{U}$ is a zero of $\mathbf{f} \in \mathcal{H}(\mathcal{U})$, or a nonstationary zero of f, if $\mathbf{f}(z_0) = 0$, which is equivalent to

$$\mathbf{f}(z) = (z - z_0)\,\tilde{\mathbf{f}}(z).$$

Observe that if z_0 is a zero of $f(z, t) \in \mathcal{H}(\mathcal{U}, C^l_+(T))$, then it is a zero of its averaging $\mathbf{f} \in \mathcal{H}(\mathcal{U})$ but not conversely.

Next, we need also a correspondent of the division Weierstrass theorem. (cf. [3] and references there).

Theorem 3.7. *Let $f \in \mathcal{H}(\mathcal{U}, C^k_+(T))$ and $z_0 \in \mathcal{U}$. Then there exist unique $h(z, t) \in \mathcal{H}(\mathcal{U}, C^k_+(T))$ and $r(t) \in C^k_+(T)$ such that*

$$f(z, t) \;=\; (z - z_0)\,h(z, t) \;+\; r(t)\,.$$

Moreover $r(t) = c_0(t)$ is the 0-coefficient of the local Taylor expansion of f at z_0.

Proof. Put $r(t) := c_0(t)$ the 0-coefficient of the local Taylor expansion of f at z_0. Then the function $g(z, t) := f(z, t) - r(t)$ belongs to $\mathcal{H}(\mathcal{U}, C^k_+(T))$ and has a stationary zero at z_0. The statement follows from Lemma 3.5. □

Properties of the function space $\mathcal{H}(\mathcal{U}, C^k_+(T)) = \mathcal{A}(\mathcal{U}, C^k_+(T))$ are given in the following statement.

Proposition 3.8. *We have $\mathcal{H}(\mathcal{U}, C^k_+(T)) \subset C(\mathcal{U} \times I) \subset L^1_C(\mathcal{U} \times I)$, where the last is the space of functions integrable on every compact subset of $\mathcal{U}$, and for every fixed $z \in \mathcal{U}$, the function $f_z(t) := f(z,t) \in C^k_+T$. $\mathcal{H}(\mathcal{U}, C^k_+(T))$ is a linear space and an algebra.*

For every $f \in \mathcal{H}(\mathcal{U}, C^k_+(T))$ the value $\varphi(f) := \frac{1}{T}\int_0^T f(z,t)dt$ is well defined and $\varphi(f) \in \mathcal{H}(\mathcal{U})$. Moreover φ is continuous and multiplicative, i.e.,

$$\frac{1}{T}\int_0^T f(z,t)g(z,t) = \frac{1}{T}\int_0^T f(z,t)dt \cdot \frac{1}{T}\int_0^T g(z,t)dt\,.$$

Furthermore for every $u \in C^l_+(T, \mathbb{C})$, such that $\forall t\ u(t) \in \mathcal{U}$ and $f \in \mathcal{H}(\mathcal{U}, C^k_+(T))$ the substitution function

$$t \to f(u(t), t) : [0, T] \to \mathbb{C}$$

belongs to $C^{\min(k,l)}_+(T)$.

Proof. These properties can be shown by the use of Theorems 3.4, 2.3. We left the proof for the reader. □

Now we wish to describe the set of all invertible elements of the algebra $\mathcal{H}(\mathcal{U}, C^k_+(T))$. For a given algebra $\mathbf{A}$, we denote by $(A)^{-1}$ the set of all invertible elements of $\mathbf{A}$. We begin with following theorem.

Theorem 3.9. *Let $k \geq 0$, $T = 2\pi\mu > 0$, $f \in (C^k_+(T)^{-1}$ iff $f = \mathrm{res}\,(\mathrm{u})$, where $u \in \mathcal{A}^k(D_\mu)^{-1}$. Moreover $u \in \mathcal{A}^k(D_\mu)^{-1}$ iff $u(z) \neq 0$ for $|z| \leq \mu$. Furthermore $f \in C^k_+(T)^{-1}$ iff its mean value* $\mathbf{f}$, *or equivalently the zero Fourier coefficient $a_0(f)$ is different from 0. Correspondingly, $u \in \mathcal{A}^k(D_\mu)^{-1}$ iff $u(0) \neq 0$.*

Proof. Suppose that $f = \mathrm{res}\,(\mathrm{u})$ and u^{-1} exists. Then $\mathrm{res}\,(\mathrm{u} \cdot \mathrm{u}^{-1}) = \mathrm{f} \cdot \mathrm{res}\,(\mathrm{u}^{-1}) = 1$ by the multiplicity property, which shows that $\mathrm{res}\,(\mathrm{u}^{-1}) = \mathrm{f}^{-1}$. Conversely, $f^{-1} = \mathrm{res}\,(\mathrm{v})$, $v \in \mathcal{A}^k(D_m u)$ by Theorem 2.3. This gives $1 = \mathrm{res}\,(\mathrm{u}) \cdot \mathrm{res}\,(\mathrm{v}) = \mathrm{res}\,(\mathrm{u} \cdot \mathrm{v})$ and consequently $u \cdot v = 1$, since res is an isomorphism. This shows the first part of the statement.

Of course, if u^{-1} exists, then for every $|z| \leq \mu$ $u(z) \neq 0$. Conversely, if $u(z) \neq 0$ in D_μ then u^{-1} is continuous in D_μ and holomorphic in $\overset{o}{D}_\mu$, which proves the statement. If $u(0) \neq 0$, then $u(z) \neq 0$ for all $|z| \leq \mu$, by the maximum modulus formula. The converse is obvious. Finally, if $f \in C^k_+(T)^{-1}$ then $a_0(f) = \varphi_0(f) \neq 0$, because φ_0 is a multiplicative functional (2.5). The inverse implication follows from the previous statement, since res is an isomorphism of Banach algebra and $a_0(f) = u(0)$ for $f = \mathrm{res}\,(\mathrm{u})$ (Theorem 2.2). □

Next we give a characterization of invertible elements in $\mathcal{H}(\mathcal{U}, C^k_+(T))$. We have the following theorem that is an analog of the characterization of invertible elements in the ring of formal power series.

Theorem 3.10.

$$f(z,t) \in \mathcal{H}(\mathcal{U}, C^k_+(T))^{-1}$$

iff for every $z_0 \in \mathcal{U}$ in the local representation 3.5 of f the zero coefficient $c_0(t) \in C^k_+(T)^{-1}$. Equivalently, $f(z,t) \in \mathcal{H}(\mathcal{U}, C^k_+(T))^{-1}$ iff

$$\varphi(f) \in \mathcal{H}(\mathcal{U})^{-1}.$$

Proof. For every $z \in D_\rho(z_0)$ we define $f^{-1}(z,t)$ by the formal power series

$$f^{-1}(z,t) := \sum_{i=0}^{\infty} b_i(t)(z - z_0)^i,$$

where $b_i(t) \in C^k_+(T)$ are defined by the recursion formula of the inverse in the ring $\mathbf{A}[[z - z_0]]$ of formal power series in the coefficients in ring $\mathbf{A} = C^k_+(T)$. It is known that this formula works if and only if the zero coefficient is in $\mathbf{A}^{-1}$. We have $b_0(t) = c_0^{-1}(t)$ and then $b_j(t)$ given by the above mentioned formula.

We are left with the task of proving that $f^{-1} \in \mathcal{H}(\mathcal{U}, C_+^k(T))$, i.e., the series is convergent in the sense of Definition 3.1.

If $f(z,t) \in \mathcal{H}(\mathcal{U}, C_+^k(T))^{-1}$, then $\varphi(f) \in \mathcal{H}(\mathcal{U})^{-1}$, since φ is a homomorphism of algebras. Conversely, if $\mathbf{f} := \varphi(f) \in \mathcal{H}(\mathcal{U})^{-1}$ then every $z_0 \in \mathbf{c}_0 = \varphi_0(f_{z_0}) \neq 0$ and $c_0(t) \in C_+^k(T)$ by the last part of statement of Theorem 3.9. This shows that $f(z,t) \in \mathcal{H}(\mathcal{U}, C_+^k(T))^{-1}$ with respect to the first part of the theorem. □

Example 3.11.

i) Every polynomial $p(z, t) := \sum_{j=0}^{n} c_j(t)z^j$, $c_j(t) \in C_+^k(T))$ belongs to $\mathcal{H}(\mathbb{C}, C_+^k(T))$.

ii) Let $f(z, t) := \frac{p(z,t)}{q(z,t)}$, where $z \in \mathbb{C}^N$, and $p(z, t)$, $q(z, t)$ are polynomials as above of degree n and m respectively. Then $f(z, t) \in \mathcal{H}(\mathcal{U}, C_+^k(T))$ for $\mathcal{U} = \mathbb{C} \setminus \operatorname{Zer} \mathbf{q}$, $\mathbf{q}(z) := \varphi(q(z, t))$, as follows from Theorem 3.10.

iii) Let $h(z, t)$, $g(z, t) \in \mathcal{H}(\mathbb{C}, C_+^k(T))$. Then

$$f(z, t) := \frac{h(z, t)}{g(z, t)} \in \mathcal{H}(\mathcal{U}, C_+^k(T)),$$

with $\mathcal{U} = \mathbb{C} \setminus \operatorname{Zer} \mathbf{g}(z)$, once more by 3.10.

Remark 3.12. There are dual notions of $\overline{\mathcal{A}}^k(D_\mu; \mathbb{C})$, $C_-^k(T; \mathbb{C})$ of the conjugated vector-valued disc algebras and vector-valued negative oriented periodic functions. Furthermore, let $\mathcal{U} \subset \mathbb{C}$ be an open set. We denote by $\overline{\mathcal{A}}^k(\mathcal{U}, C_-^k(T))$ the corresponding space of holomorphic functions of many variables with coefficients in the space of negative oriented vector-valued functions. All the stated above facts about $\mathcal{A}^k(\mathcal{U}, C_+^k(T))$ have their correspondents for the algebra $\overline{\mathcal{A}}^k(\mathcal{U}, C_-^k(T))$.

At the end of this section we would like to remind the reader of the Poincaré inequality (cf. [14]) in the form used in [2] for the scalar (complex) valued functions and the Wirtinger inequality ((cf. [14]).

Proposition 3.13. *For every function $u \in C^1(T; \mathbb{C}^N)$ we have the Poincaré inequality*

$$\|u\|_0 - |\mathbf{u}| \leq \|u - \mathbf{u}\|_0 \leq \frac{T}{2}\|\dot{u}\|_0,$$

and the Wirtinger inequality

$$\|u\|_{L^2} - |\mathbf{u}| \leq \|u - \mathbf{u}\|_{L^2} \leq \frac{T}{2\pi}\|\dot{u}\|_{L^2},$$

where $\mathbf{u} = \varphi(u)$ is the mean value of u and $\| \, \|_{L^2}$ is the L^2-norm. □

4 The existence of solutions for an equation with the right hand side being an analytic function with periodic coefficients

In this section we study the problem of existence of T-periodic solutions of the problem.

$$\frac{du}{dt} = f(u,t), \qquad \text{where}$$
$$f \in \mathcal{H}(\mathcal{U}, C^0_+(T)), \quad u(t) \in \mathcal{U}, \quad \mathcal{U} \subset \mathbb{C}. \tag{4.6}$$

More precisely, we will seek a positive oriented T-periodic of the equation 4.6. Our theorem extends the main result of [2] having a polynomial equation with positive oriented coefficients. Moreover, the use of the form of the right hand side of 4.6 given by the division theorem 3.7 gives a geometric interpretation for the suitable a priori bounds that are sufficient for the existence of solutions.

From now on we look for only the positive oriented periodic solutions of 4.6. This means that we pose our problem in the function spaces $C^k_+(T)$.

We begin with a necessary condition for the existence of T-periodic solutions of 4.6 that is a consequence of the multiplicity of the averaging functional on the algebra of positive oriented functions (2.5).

Theorem 4.1. *A function $u \in C^1_+(T)$ is a solution of the equation*

$$\dot{u}(t) = f(u,t), \qquad \textit{with} \qquad f \in \mathcal{H}(\mathcal{U}, C^0_+(T))$$

only if $\mathbf{u} = \varphi(u) \in Zer\mathbf{f}$, *i.e.* $\mathbf{f}(\mathbf{u}) = 0$.

Proof. Applying the averaging functional φ to both sides of 4.6 we split into a direct sum of two equations

$$\frac{d\tilde{u}}{dt} = \tilde{f}(\tilde{u}, t),$$
$$0 = \mathbf{f}(\mathbf{u}),$$

corresponding to the split $C^1_+(T) = \mathcal{I}^1 \oplus \mathbb{C}$, where $\mathcal{I}^1_0 = \ker\varphi$, $\mathbb{C} = \operatorname{im}\varphi$, and $\tilde{u} := u - \mathbf{u}$.

Note that for $D := \frac{d}{dt}$ we have $\ker D = \operatorname{coker} D = \operatorname{im}\varphi$, and $\operatorname{im} D = \ker\varphi$ since the derivative of constant and the mean value of the derivative of any C^1 periodic function is equal to 0. The statement follows from the fact that $f \in \mathcal{H}(\mathcal{U}, C^0_+(T))$, and thus $f(z,t) = f((\mathbf{u} + (u - \mathbf{u}), t) = \mathbf{f}(\mathbf{u}) + \tilde{f}(z - \mathbf{u}, t)$, where $\tilde{f}(z - \mathbf{u}, t) \in \mathcal{I}^1_0$, because $(z - \mathbf{u}) \in \mathcal{I}^1_0$ and the Taylor expansion at

u *has the form*

$$f(z,t) = c_0(t) + \sum_{j>0} c_j(t)\,(\mathbf{u} + (z-\mathbf{u}))^j$$
$$= \sum_{j=0} c_j(t)\mathbf{u}^j + \tau(z,t) = \sum_{j=0} \mathbf{c}_j\,\mathbf{u}^j + \mu(z,t) + \tau(z,t),$$

where $\mu, \tau \in \mathcal{I}_0^1$ *(cf. [2] for the polynomial case).* □

Remark 4.2. With respect to Theorem 4.1, we can say that the set of solutions of Equation 4.6 forms a fibration over the set Zer **f**.

Now we present a sufficient condition for the existence of periodic solutions (indeed positive oriented periodic solutions) at a given isolated zero of **f**. *To do this we need new notation. Let* $z^0 \in \mathcal{U}$ *be a zero of* **f** . *We define*

$$x := |z - z_0|.$$

Let next $\rho_0 > 0$ *be such that:*
1^0. $D_\rho(z_0) \subset \mathcal{U}$,
2^0. $D_\rho(z_0) \cap Zer(\mathbf{f}) = \{z_0\}$.

Let next $f(z,t) = (z - z_0)\,h(z,t) + r(t)$ *be the representation of* f *given by Theorem 3.7 . We define*

$$b := \|r(t)\|$$
$$a := \sup_{|z-z_0|\le\rho_0} \|h(z,t)\|, \tag{4.7}$$

with the convention that $\tilde{\rho} := \min\{1, \rho\}$. *We put*

$$d := \sum_{j>0} \|c_j\|\tilde{\rho}_0^{\,j-1} \tag{4.8}$$

with the same convention as above. Next we define

$$c = c(\|\ \|) := \begin{cases} 2 & for\ \ \|\ \|_0 \\ 2\pi & for\ \ \|\ \|_{L^2}\,. \end{cases} \tag{4.9}$$

To a given map $f(z,t) \in \mathcal{H}(\mathcal{U}, C^0_+(T))$ *we assign the following real functions of the variable* $x \in [0, \infty)$.

Definition 4.3. *Let* $f(z,t) \in \mathcal{H}(\mathcal{U}, C^0_+(T))$ *and let* a, b, c *be the con-*

stants defined above. Put

$$\lambda(x) = \tfrac{c}{T} x\,,$$

$$\theta(x) := a x + b\,,$$

$$\omega(x) := \sum_{j\geq 0} \|c_j\| x^j = \sum_{j>0} \|c_j\| x^j + b$$

$$\sigma(x) := d x + b\,.$$

Note that $\omega(x)$, and $\sigma(x)$ are well defined at least for $x \leq \rho_0$, because the Taylor series of the local expansion of f is absolutely convergent at least for $x \leq \rho_0$.

We begin with an obvious comparison of the above functions.

Proposition 4.4. *For every T-periodic solution $u(t)$, $\mathbf{u} = z_0$, $z_0 \in Zer\, \mathbf{f}$, $\|u - z_0\|_0 = x$, of 4.6 with $0 \leq x \leq \rho_0$ we have*

$$\lambda(x) \leq \theta(x) \leq \omega(x) \leq \sigma(x)\,.$$

Proof. Suppose that u is a solution of 4.6. First $u \in C^1(T)$. From the Wirtinger, or correspondingly Poincaré, inequality, it follows that

$$\lambda(x) \leq \|f(u(t), t)\| \leq \|r\| + \|h(u(t), t)\| = \|a_0(t) + \sum_{j>0} c_j(t) u(t)^j\|\,,$$

which gives directly the required inequalities in the case of the C^0-norm. For the case of the L^2 norm note that $\|u(t)\, v(t)\|_{L^2} \leq \|u(t)\|_{L^2}\, \|v(t)\|_0$ if $u \in L^2$, and $v \in C^0$. □

Remark 4.5. Let $z_0 \in a\, \mathrm{Zer}\, \mathbf{f}$. Then z_0 is a stationary zero of f iff and only $r(t) = 0$ in the local representation 3.7 of f. The last is equivalent to $b = 0$.

We are in a position to formulate our main result which gives a sufficient condition for the existence of a periodic solution near a given zero of the averaged right hand side (cf. 4.1).

Theorem 4.6. *Consider the nonautonomous, T-periodic ordinary differential equation 4.6 with positive oriented coefficients on the complex plane. Suppose the z_0 is a zero of the averaged function of the right hand side of 4.6 and let $f(z, t) = h(z, t) + r(t)$ be the local representation of f at z_0 given in 3.7 with the local Taylor expansion of the form 3.5 convergent over disc $|z - z_0| \leq \rho_0 < \rho$, where ρ is as in (3.4). Let next a, b, $c = 2$, d be real constants defined in 4.7, 4.9, 4.8 for the C^0-norm and $\tilde{\rho}_0 = \min\{\rho_0, 1\}$.*

If either $\quad T\,(a\tilde{\rho}_0 + b) < 2\,\tilde{\rho}_0 \quad$ *or* $\quad T\,(d\tilde{\rho}_0 + b) < 2\,\tilde{\rho}_0$

then the problem 4.6 has at least one T-periodic solution $u(t) \in C^0_+(T)$ with the mean value $\mathbf{u} = z_0$ and $\|u - z_0\|_0 \leq \tilde{\rho}_0$.

Proof. We pose our problem as a coincidence between maps of the function spaces $E = C^1_+(T)$ and $F = C^0_+(T)$. It could be transformed to a nonlinear equation of the form Id $- \Phi$, Φ a completely continuous map (see [2]).

Fix $z_0 \in \operatorname{Zer} \mathbf{f}$ and assume that $\rho_0 < \rho$ is such that $D_{\rho_0}(z_0) \cap \operatorname{Zer} \mathbf{f} = \emptyset$. Take $\mathcal{W} := D_{\rho_0}(z_0) \times \mathcal{I} \subset F$. Form a deformation of the problem 4.6

$$\dot{u} = f_\lambda(u, t) := \mathbf{f}(\mathbf{u}) + (1 - \lambda)\big(f(u, t) - \mathbf{f}(\mathbf{u})\big), \tag{4.10}$$

$\lambda \in [0, 1]$, which gives a deformation of the corresponding nonlinear problem.

Note first that if (u, λ) is a solution of (4.10), then $\mathbf{u} = z_0$, as follows from 4.1, i.e., every solution of the equation 4.10 in $\mathcal{W}$ is contained in $\{z_0\} \times \mathcal{I}$. Assume that for ρ one of the inequalities of the hypothesis of the theorem holds. We show that for the set $\mathcal{W}_\delta := D_{\rho_0}(z_0) \times B_\delta(\mathcal{I})$, $B_\delta(\mathcal{I}) := \{u \in \mathcal{I} : \|u\| \leq \delta\}$ there is no solutions of the deformation on the boundary $\partial\mathcal{W} = \partial D_{\rho_o}(z_0) \times B_\delta(\mathcal{I}) \cup D_{\rho_0}(z_0) \times \partial B_\delta(\mathcal{I})$ if $\delta = \rho_0$. Indeed if $u(t) \in C^0_+(T)$, $x = \|u - z_0\|$ is a solution of 4.6, then $\lambda(x) \leq \theta(x) \leq \omega(x) \leq \sigma(x)$ for every $0 \leq x \leq \rho_0$. In particular, $\lambda(\tilde{\rho}_0) \leq \theta(\tilde{\rho}_0) \leq \sigma(\tilde{\rho}_0)$ which contradicts the inequalities of the assumption. To complete the proof it is sufficient to use the continuation method (see [6], [13], because for $\lambda = 0$ $f_\lambda = f$. Thus to show that the Leray–Schauder degree of our problem is different from 0, it is enough to derive the Brouwer degree of the end of the deformation f at zero, i.e., the Brouwer degree of $\mathbf{f} : (D_{\rho_0}(z_0), D_{\rho_0}(z_0) \setminus \{z_0\})$. It is the local degree of a holomorphic map at its zero, which is always a positive integer. □

Note that if z_0 is a stationary zero of f, i.e., $r(t) \equiv 0$, then our method picks up the stationary solution $u = z_0$ of (4.6). On the other hand, from the local representation of Theorem (3.7), it follows that if z_0 is not stationary zero, then the solution u is not the constant solution.

Corollary 4.7. *If $z_0 \in \operatorname{Zer} \mathbf{f}$ is not a stationary zero, i.e., $f(z_0, t) \not\equiv 0$, then the solution $u(t)$ of (4.6) given by Theorem 4.6 is nonconstant.* □

Remark 4.8. Theorem 4.6 can be formulated and proved in the multidimensional case, i.e., if $f \in \mathcal{H}(\mathcal{U}, C^0_+(T); \mathbb{C}^N)$, $z \in \mathbb{C}^N$, $\mathcal{U} \subset \mathbb{C}^N$. We have to assume that $z_0 \in \mathcal{U}$ is an isolated zero of $\mathbf{f}$. A proof is analogous.

Remark 4.9. It seems that Theorem 4.6 has its correspondent for the L^2-norm. The main technical problem in a direct repetition of the argument is the fact that the L^2-norm is not multiplicative. Note that constant $c = 2\pi > 2$, and thus the supposed inequalities are weaker.

Remark 4.10. It is worth pointing out that the estimate of Theorem 4.6 is very restrictive and could be weakened for particular equations. On the other hand there are examples of equations with a simple right side as in (4.6), or in the complex Riccati equation, which do not have a T-periodic solution (cf. [15],[19], see also [5] for an example of real Riccati equation without periodic solutions and [4] for a complete discussion of solutions of the Riccati equation).

Remark 4.11. Theorems 4.1 and 4.6 have their correspondents for the equation 4.6 with $f \in \mathcal{H}(\mathcal{U}, C^k_-(T))$. The statements are the same for $C^k_+(T)$ replaced by $C^0_-(T)$.

Consider the equation

$$\dot{u} \;=\; a\,\sin(u) \,+\, r(t), \tag{4.11}$$

where $a > 0$, and $r(t) \in C^0_+(T)$ is a function with $\|r\|_0 = 0$, and $\mathbf{r} = 0$, e.g., $r(t) := b\,\exp(\frac{2\pi}{T} \imath\, t)$, $b \in \mathbb{R}$, $b > 0$.

Note that $a\,\sin(u) + r(t) \in \mathcal{H}(\mathbb{C}, C^0_+(2\pi))$. Moreover the local Taylor expansion has the form

$$r(t) \,+\, a\sum_{j=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

and is convergent over the whole plane $\mathbb{C}$, i.e., $\rho = \infty$. Furthermore $\mathbf{f}(z) = a\,\sin(z)$, and $\mathrm{Zer}\,\mathbf{f} = \{k\pi\}$, $k \in \mathbb{Z}$. Consequently at $z_0 = k\pi$, the local representation of f of Theorem 3.7 is of the form

$$r(t) \,+\, (z - z_0)\,h(z),$$

where $h(z) = \frac{a\,\sin(z)}{(z-z_0)}$, and $\rho_0 = 1$, $b = \|r(t)\|_0$. To apply Theorem 4.6 we have to estimate $|h(z)|$ for $|z - z_0| \leq 1$. Since $\sin(z)$ is a 2π- periodic function, it is enough to derive this estimate at $z_0 = 0$. From the maximum modulus principle we have

$$\begin{aligned}\max_{|z|\leq 1} |h(z)| &= \max_{|z|=1} |a\,(\sin(z)/z)| \\ &= \max_{0\leq t\leq 1} \frac{a\,|\exp(-2t) - \exp(2t)|}{2} \leq \frac{535.490\,a}{2} \leq 268\,a\,.\end{aligned}$$

This leads to the following fact.

Corollary 4.12. *For every perturbation $r(t) \in C^0_+(T)$, with $b = \|r\|_0 > 0$, $\mathbf{r} = 0$, the nonlinear equation (4.11) has infinitely many T-periodic non-constant solutions $\{u_k(t) \in C^0_+(T)\}$ provided*

$$T < \frac{2}{b + 268\,a}\,.$$

□

Remark 4.13. Using the same approach and the correspondent of the Theorem 4.6 for a system one can prove the existence of infinitely many non-constant periodic solutions for the perturbed complex pendulum equation

$$\ddot{u}(t) = a \sin(u(t)) + r(t), \qquad u(t) \in \mathbb{C}$$

where $r(t)$ is as in Corollary 4.12 (cf. [14] a discussion of the existence of periodic solutions of this equation in the real case).

5 Periodic solutions at infinity

In this section we show that the equation 4.6 with the right hand side being a positive oriented periodic meromorphic function could have a solution which originates at infinity.

Consider the problem

$$\dot{u}(t) = f(u(t), t), \qquad \text{with} \qquad f = \frac{p(z,t)}{q(z,t)}, \tag{5.12}$$

where $u \in \mathbb{C}$ and p, q are polynomials with positive oriented periodic coefficients of degree n and m, respectively. In this section study the problem of existence of T-periodic positive oriented solutions of the problem.

Note that the right hand side of the equation 5.12 belongs to $\mathbb{C}\backslash\{\mathrm{Zer}\,(\mathbf{q})\}$.

For every $u(t) \in C^k_+(T)$, $k \geq 0$, by $\bar{u}$ we denote its image in $C^k_+(T)$ given by $\bar{u}(t) := u(-t)$, and call it the conjugated function. The mapping $u \mapsto \bar{u}$ defines an isometry of the corresponding Banach algebras by Theorem 2.11.

Lemma 5.1. *$u \in C^k_+(T)$ is a solution of the equation 4.6 iff $\overline{u}$ is a solution of the equation*

$$\begin{aligned} \dot{u}(t) &= -\bar{f}(u,t), \qquad \textit{where} \\ \overline{f} \quad & \textit{has all coefficients formed of conjugated functions.} \end{aligned} \tag{5.13}$$

We can identify $C^k_\pm(T)$ with $\mathrm{res}\,\mathcal{A}^k_\mu(\mathrm{D}(\infty))$, i.e., with the space of all continuous functions $f : D_\mu(\infty) \subset S^2 = \mathbb{C} \cup \{\infty\}$, $D_\mu(\infty) := \{z \in \mathbb{C} : |z| \geq \frac{1}{\mu}\}$, $2\pi\mu = T$ which are holomorphic in $\overset{o}{D}_\mu(\infty) = \{z \in \mathbb{C} : |z| > \frac{1}{\mu}\} \subset S^2$.

Definition 5.2. *We say that $f(z,t) \in \mathcal{H}(\mathcal{U}, C^k_\pm(T))$, $\infty \in \mathcal{U} \subset S^2$ is a holomorphic at infinity if $f(1/z, t)$ expands in the Taylor series of Definition 3.1 at $z = 0$.*

Example 5.3. Every function f of the form

$$f(z,t) = \frac{p(z,t)}{q(z,t)}$$

where

$$p(z,\,t) \;=\; \sum_{j=0}^{n} p_j(t)\, z^{n-j}\,, \quad q(z,\,t) \;=\; \sum_{j=0}^{m} q_j(t)\, z^{n-j}\,,$$

$p_j(t),\ q_j(t) \in C^k_{\pm}(T))$, $k \geq 0$, and $n \leq m$, is holomorphic at ∞ provided $q_0(t) \in (C^k_{\pm}(T))^{-1}$ (cf. 3.9).

Now we show a theorem which gives a condition for the existence of positive oriented solutions that originated at infinity. To do it we need new notions. Let $R := \sup_{z\in Zerp(z)} |z|$. Note that $f(zt) \in \mathcal{H}(D_\mu(\infty), C^k_+(T))$ if $D_\mu(\infty) \subset \mathbb{C} \setminus D_R(0)$, i.e if $\mu \leq \frac{1}{R}$.

If $q(z,\,t) \;=\; \sum_{j=0}^{m} q_j(t)\, z^{n-j}$, is a polynomial of degree m, then there exists $\alpha > 0$ such that

$$\|q_0\| \;-\; x\Big(\sum_{j=1}^{m} \|q_j\|\, x^{j-1}\Big) \;\geq\; \frac{1}{2}\, \|q_0\|$$

for every $0 \leq x < \alpha$.

Let $\alpha_0 := \sup \alpha$, where α is as above. We put $\breve{\rho}_0 \;=\; \min\{1\,, \rho_0,\, \alpha_0)\}$. For a given polynomial $p(z,\,t) \;=\; \sum_{j=0}^{n} p_j(t)\, z^{n-j}$ we define a function $\varpi(x)$ of the real variable x by the formula

$$\varpi(x) \;=\; \sum_{j=1}^{n} \|p_j\|\, x^{j-1}.$$

Theorem 5.4. *Consider the equation 5.12*

$$\dot{u} \;=\; f(u,t), \qquad \textit{with} \qquad f(z,\,t) \;=\; \frac{p(z,\,t)}{q(z,\,t)}\,, \tag{5.14}$$

where $p(z,\,t)$, $q(z,\,t)$ are polynomials with positive oriented T-periodic coefficients. Assume that p and q are of the same degree, i.e., $\deg p = n = m = \deg q$. Suppose also that:

1^0 *The leading coefficient $p_0(t) \neq 0$ has zero mean value, i.e., $\mathbf{p}_0 = \varphi(p_0(t)) = 0$.*

2^0 *The leading coefficient $q_0(t)$ is invertible in $C^k_+(T)$ (see 3.9).*

Then the equation has a T-periodic nonconstant positive oriented solution $u(t)$ provided

$$\left(\frac{\|p_0\|}{\|q_0\|} + \frac{2\,\varpi(\breve{\rho}_0)\,\breve{\rho}_0)}{\|q_0\|}\right) T \;<\; 2\,,$$

where ϖ, $\breve{\rho}_0$ are defined above. Moreover this solution originates at infinity, i.e., $u(t) \in D_{\breve{\rho}}(\infty)$.

Proof. We show that the conjugated equation (5.13) has a negative oriented T-periodic solution that originates at infinity and which corresponds to a positive oriented T-periodic solution of (5.12) by Theorem 2.11. The conjugate equation has the form

$$-\frac{dz}{dt} = \bar{f}(z,t) = \frac{\sum_{j=0}^{n} \bar{p}_j(t)\, z^{n-j}}{\sum_{j=0}^{m} \bar{q}_j(t)\, z^{n-j}}.$$

Substituting $z = 1/z$ to $\bar{f}(z,t)$ we get

$$\bar{f}(t) = z^{n-m} \frac{\sum_{j=0}^{n} \bar{p}_j(t)\, z^{j}}{\sum_{j=0}^{m} \bar{q}_j(t)\, z^{j}} = \frac{\sum_{j=0}^{n} \bar{p}_j(t)\, z^{j}}{\sum_{j=0}^{n} \bar{q}_j(t)\, z^{j}} =: \frac{\hat{p}(z,t)}{\hat{q}(z,t)},$$

since $m = n$.

By assumption $\bar{f} \in \mathcal{H}(D_{\breve{\rho}_0}(\infty), C^k_-(T))$. To show the existence of a periodic solution we can use the results of Section 3, as follows from Remark 4.11.

Observe first that the necessary condition of Theorem (4.1) is satisfied, since $\bar{\mathbf{f}}(0) = \frac{\mathbf{p_0}}{\mathbf{q_0}} = 0$. Note that $\mathbf{q} \neq 0$ by Assumption 2^0 and Theorem 3.9.

Write $\hat{p}(z,t)$ as $\bar{p}_0(t) + z((\sum_{j=1}^{n} \bar{p}_j(t)\, z^{j-1}) =: \bar{p}_0(t) + z\tilde{p}(z\,t)$ and analogously $\hat{q}(z,t)$ as $\bar{q}_0(t) + z(\sum_{j=1}^{n} \bar{q}_j(t)\, z^{j-1}) =: \bar{q}_0(t) + z\tilde{q}(z,t)$.

By Theorem 3.10 $\hat{q}(z,t)$ is invertible in $\mathcal{H}(D_{\breve{\rho}_0}(\infty), C^0_-(T)$ and has the Taylor expansion at 0 with the zero coefficient equal to $\bar{q}_0^{-1}(t)$. By Theorem 3.7 we have

$$\hat{q}_0^{-1}(z,t) = \bar{q}_0^{-1}(t) + z\, g(z,t),$$

where $g(z,t) \in \mathcal{H}(D_{\breve{\rho}_0}(\infty), C^0_-(T))$. Multiplying $\hat{p}$ by $\hat{q}^{-1}$ we have

$$\bar{f}(z,t) = \bar{p}_0(t)\,\bar{q}_0^{-1}(t) + z\,h(z,t). \tag{5.15}$$

To apply Theorem 4.6 we have to estimate $\max\{\|h(z,t)\|\}$ for $|z| \le \breve{\rho}_0$. As previously put $x := |z|$. By the definition of $\varpi(x)$, α_0, $\breve{\rho}_0$, and $\|\bar{u}\| = \|u\|$, we have

$$\begin{aligned}\max_{|z|\le\breve{\rho}_0} \|h(z,t)\| &\le \max_{|z|\le\breve{\rho}_0} \|\tilde{p}(z,t)\| \max_{|z|\le\breve{\rho}_0} \|\hat{q}^{-1}(z,t)\| \\ &\le \varpi(\breve{\rho}_0) \frac{1}{\min_{|z|\le\breve{\rho}_0} \|\hat{q}(z,t)\|} \le \varpi(\breve{\rho}_0) \frac{1}{(1/2)\|q_0\|}.\end{aligned} \tag{5.16}$$

This shows that for the constants a, b of Definition 4.7, defining the function $\theta(x)$ (cf. Def. 4.3) we have $b = \frac{\|p_0\|}{\|q_0\|}$, and $a \le \frac{2\varpi(\breve{\rho}_0)}{\|q_0\|}$, which shows the existence of a T-periodic solution $u(t)$ by Theorem 4.6. $u(t) \neq \mathbf{u} = 0$, since 0 is not a stationary zero of $\bar{f}$. □

Remark 5.5. Theorem 5.4 can be used to study periodic solutions of systems with singularities. However we must say that the most important systems do not belong to the class studied here (cf [1]). This is because their singularities are of the form $\frac{1}{|z-z_0|^n}$. On the other hand the module $|z(t)|$ of a nonconstant function $z \in C^k_{\pm}(T)$ does not belong to $C^k_{\pm}(T)$.

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Faculty of Mathematics and Computer Science
Adam Mickiewicz University of Poznań
ul. Matejki 47/48
60-769 Poznań, Poland
email: marzan@math.amu.edu.pl

The Conley Index and Rigorous Numerics for Attracting Periodic Orbits

M. Mrozek and P. Pilarczyk

Dedicated to Antonio Marino

1 Introduction

Despite the enormous number of papers devoted to the problem of the existence of periodic trajectories of differential equations, the theory is still far from satisfactory, especially when concrete differential equations are concerned, because the necessary conditions formulated in many theoretical criteria are difficult to verify in a concrete case. And even if some methods work for some concrete equations, it is usually difficult to carry them over to other problems. Thus quite often the only available method is to experiment numerically. Unfortunately, such an approach cannot be treated as reliable.

All this makes the problem a natural field of research in rigorous numerics. However, only recently some new techniques were developed, for which the amount of computations necessary is within the reach of present-day computers (see [4, 5, 15]). Especially powerful seem to be methods based on topological invariants like the Conley index [4] and the fixed point index [15].

In this paper we sketch an approach to the existence of periodic solutions of differential equations based on the discrete Conley index and rigorous numerics of dynamical systems. For details the reader is referred to [11, 12]. We briefly discuss the result of applying this method to two different periodic orbits in the Rössler equations and two periodic trajectories in the Lorenz equations.

2 Representable sets and maps

Let X, Y be locally compact metric spaces. For $\mathcal{G} \subset \mathcal{P}(X)$, $A \subset X$ and $\mathcal{C} \subset \mathcal{G}$ put

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$$\mathcal{G}(A) := \{ a \in \mathcal{G} \mid a \cap A \neq \emptyset \},$$
$$|\mathcal{G}| := \bigcup \mathcal{G},$$
$$\langle \mathcal{C} \rangle := \langle \mathcal{C} \rangle_{\mathcal{G}} := \{ x \in X \mid \mathcal{G}(x) = \mathcal{C} \}.$$

A family $\mathcal{G} \subset \mathcal{P}(X)$ will be called a *grid* in X if

(i) every element of $\mathcal{G}$ is a non-empty compact set,

(ii) for every compact $K \subset X$ we have $1 \leq \operatorname{card} \mathcal{G}(K) < \infty$,

(iii) for every $\mathcal{C} \subset \mathcal{G}$ we have $\operatorname{cl} \langle \mathcal{C} \rangle = \bigcap \mathcal{C}$.

A typical example of a grid in $\mathbb{R}^d$ is a set of d-dimensional hypercubes of the same size $\eta > 0$ which fill the space:

$$\mathcal{G}_\eta := \{ \prod_{i=1}^{d} [k_i \eta, (k_i + 1)\eta] \mid k_i \in \mathbb{Z},\ i = 1, \ldots, d \}.$$

Define the *diameter* of a grid $\mathcal{G}$ as

$$\operatorname{diam} \mathcal{G} := \sup \{ \operatorname{diam} a \mid a \in \mathcal{G} \}.$$

A set E is called an *elementary representable set* if $E = \langle \mathcal{C} \rangle$ for a finite subfamily $\mathcal{C} \subset \mathcal{G}$. A set A is called *representable* if it is a finite union of elementary representable sets. A set A is called *strongly representable* if it is a finite union of a subfamily of $\mathcal{G}$.

Theorem 2.1. (see [9]) *The family of representable sets is closed under the set-theoretical union, intersection, difference as well as topological closure and topological interior.*

The family of elementary representable sets over a grid $\mathcal{G}$ will be further denoted by $\mathrm{ER}(\mathcal{G})$, and the family of all representable sets by $\mathrm{R}(\mathcal{G})$.

A *multivalued map* $F\colon X \rightrightarrows Y$ is a map $F\colon X \to \mathcal{P}(Y)$. Its *domain* and *image* are defined as follows:

$$\operatorname{dom} F := \{ x \in X \mid F(x) \neq \emptyset \},$$
$$\operatorname{im} F := \bigcup_{x \in X} F(x).$$

The *image* and *preimage* of a set under a multivalued map is defined in the following way:

$$F(A) := \bigcup_{x \in A} F(x),$$
$$F^{-1}(B) := \{ x \in X \mid F(x) \cap B \neq \emptyset \}.$$

A multivalued map $F\colon X \rightrightarrows Y$ is called *representable* over grids $\mathcal{G}$, $\mathcal{H}$ in X, Y respectively if it satisfies the following conditions:

(i) $\operatorname{card} \mathcal{G}(\operatorname{dom} F) < \infty$,

(ii) for every $x \in X$ the set $F(x)$ is representable,

(iii) if E is an elementary representable set, then $F_{|E} = \text{const}$.

Theorem 2.2. (see [9]) *If $A \in \mathrm{R}(\mathcal{G})$, $B \in \mathrm{R}(\mathcal{H})$ and F is a representable multivalued map, then*

$$\operatorname{dom} F,\ F^{-1}(B) \in \mathrm{R}(\mathcal{G}),$$
$$F(A) \in \mathrm{R}(\mathcal{H}).$$

Let $N \subset X$ be a compact representable set. Define

$$F_N : X \ni x \to N \cap F(x) \subset X.$$

Proposition 2.3. *F_N is representable.*

We say that a multivalued map $F: X \rightrightarrows Y$ is *upper semicontinuous* if for every $x \in X$ the set $F(x)$ is compact and for every neighborhood U of $F(x)$ there exists a neighborhood V of x such that $F(V) \subset U$.

If a sequence of multivalued maps $\{F_n\}$ is given, then we say that this sequence converges to a multivalued map F, which we denote by $F_n \to F$, if the graphs of F_n converge to the graph of F as subsets of $X \times Y$ with respect to the Hausdorff metric.

A multivalued map f is called *single-valued* if $\operatorname{card} f(x) \leq 1$ for every $x \in X$ and may be identified with a map $X \multimap Y$ defined on a subset of X.

A single-valued map $f: X \multimap Y$ is called a *selector* of a multivalued map $F: X \rightrightarrows Y$ if $f(x) \in F(x)$ for every $x \in \operatorname{dom} f$ (in particular, $\operatorname{dom} f \subset \operatorname{dom} F$).

Assume X, Y are two locally compact metric spaces with given grids $\mathcal{G}$, $\mathcal{H}$. Let $f: X \multimap Y$ be a continuous map defined on a subset of X. We say that $F: X \rightrightarrows Y$ is a *representation* of f if F is representable and f is a selector of F.

Theorem 2.4. (see [8]) *Assume $\mathcal{G}_n$, $\mathcal{H}_n$ are sequences of grids in X, Y respectively, such that* $\operatorname{diam} \mathcal{G}_n \to 0$ *and* $\operatorname{diam} \mathcal{H}_n \to 0$. *Let $f: X \multimap Y$ be a Lipschitz function such that* $\operatorname{dom} f$ *is relatively compact. Then there exist sequences of multivalued maps $F_n, G_n : X \rightrightarrows Y$ such that*

(i) F_n, G_n are representations of f,

(ii) F_n is lower semicontinuous and G_n is upper semicontinuous,

(iii) $F_n \to f$, $G_n \to f$.

3 The Conley index

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a homeomorphism.

If $N \subset \mathbb{R}^d$, then the set

$$\operatorname{Inv} N := \operatorname{Inv}(N, f) := \{x \in N \mid \forall n \in \mathbb{Z}\ f^n(x) \in N\}$$

is called the *invariant part* of N.

A compact set $N \subset \mathbb{R}^d$ is called an *isolating neighborhood* if

$$\operatorname{Inv} N \subset \operatorname{int} N.$$

A set $S \subset \mathbb{R}^d$ is called an *isolated invariant set* if there exists an isolating neighborhood N such that $S = \operatorname{Inv} N$.

A pair $P = (P_1, P_2)$ of compact subsets of an isolating neighborhood N is called an *index pair* if $P_2 \subset P_1$ and

(i) $x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i,\quad i = 1, 2,$

(ii) $x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2,$

(iii) $\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2).$

Let H^* denote the Alexander–Spanier cohomology functor. Let i_P be the inclusion $(P_1, P_2) \to (P_1 \cup f(P_2), P_2 \cup f(P_2))$. Since f maps (P_1, P_2) to $(P_1 \cup f(P_2), P_2 \cup f(P_2))$ and i_P is an excision for the Alexander–Spanier cohomology, we can define the *index map* according to the formula

$$I_P := H^*(f_P) \circ H^*(i_P)^{-1}\colon H^*(P_1, P_2) \to H^*(P_1, P_2).$$

Define the *generalized kernel* of this map as

$$\operatorname{gker}(I_P) := \bigcup_{n \in \mathbb{N}} \ker I_P^n.$$

The Conley index is then defined as

$$CH^*(S, f) := (H^*(P_1, P_2)/\operatorname{gker}(I_P), [I_P]),$$

where $[I_P]$ stands for the automorphism induced by I_P on the quotient space $H^*(P_1, P_2)/\operatorname{gker}(I_P)$.

Consider now the multivalued case. Let $F\colon \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be an upper semicontinuous multivalued map and let $N \subset \mathbb{R}^d$ be a compact set. The invariant part of N is the set

$$\operatorname{Inv}(N, F) := \{x \in N \mid \exists \sigma\colon \mathbb{Z} \to N \text{ such that } \sigma(0) = x \text{ and } \sigma(n+1) \in F(\sigma(n))\}.$$

The set N is called an *isolating neighborhood* if

$$\operatorname{Inv} N \cup F(\operatorname{Inv} N) \subset \operatorname{int} N.$$

A pair $P = (P_1, P_2)$ of compact sets is an *index pair* in an isolating neighborhood N if $P_2 \subset P_1 \subset N$ and

(i) $F(P_i) \cap N \subset P_i, \ i = 1, 2,$

(ii) $F(P_1 \backslash P_2) \subset N,$

(iii) $\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2).$

The index map in this case is defined according to the formula

$$I_P := H^*(F_P) \circ H^*(i_P)^{-1} \colon H^*(P_1, P_2) \to H^*(P_1, P_2)$$

and the Conley index is defined as

$$CH^*(S, f) := (H^*(P_1, P_2)/\operatorname{gker}(I_P), [I_P]).$$

Assume $\mathcal{A}$ is a collection of multivalued maps. We recall that property φ of maps in $\mathcal{A}$ is *inheritable* if for every $F \in \mathcal{A}$ and every selector f of F

$$\varphi(F) \ \Rightarrow \ \varphi(f).$$

We say that φ is *strongly inheritable* if φ is inheritable and for any single-valued map $f \in \mathcal{A}$ such that $\varphi(f)$ and for any sequence $\{F_n\} \subset \mathcal{A}$ satisfying $F_n \to f$ we have $\varphi(F_n)$ for n sufficiently large. Finally, if $\alpha(F)$ is a term, then we say that α is *inheritable* (*strongly inheritable*) if for any x the property $\alpha(F) = x$ is inheritable (strongly inheritable).

Theorem 3.1. (see [8]) *Isolating neighborhood, index pair and Conley index are strongly inheritable terms.*

4 Existence of periodic orbits

Let $f \colon \mathbb{R}^d \to \mathbb{R}^d$ be a vector field on $\mathbb{R}^d$ of class C^1. Let $\varphi \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be the flow on $\mathbb{R}^d$ generated by the differential equation

$$x' = f(x). \tag{4.1}$$

A compact subset Ξ of a $(d-1)$-dimensional hyperplane Π is called a *local section* for φ if the vector field f is transverse to Π on Ξ. Such a set Ξ is called a *Poincaré section* for φ in an isolating neighbourhood N if $\Xi \cap N$ is closed and for every $x \in N$ there exists $t > 0$ such that $\varphi(x, t) \in \Xi$.

Given a $t \in \mathbb{R}$, define the *time-t map* by

$$\varphi_t \colon \mathbb{R}^d \ni x \mapsto \varphi(x, t) \in \mathbb{R}^d.$$

Fix $t > 0$ and $\eta > 0$. Assume we can construct a compact set N representable with respect to the grid $\mathcal{G}_\eta$ such that a certain representation F of φ_t on N satisfies the condition

$$F(N) \subset \operatorname{int} N. \tag{4.2}$$

In practice, we can expect that such a representation may be constructed if numerical simulations indicate the existence of an attracting periodic orbit, and the grid size η is chosen small enough.

As a consequence of (4.2), the set N is an isolating neighborhood for F and the pair $P = (N, \emptyset)$ is an index pair for F in N. If the cohomology of N is the cohomology of the circle and the index map I_P is an isomorphism, then the Conley index of N is an index of an attracting periodic orbit. Due to the inheritability property, this is also the Conley index of any selector of F, in particular of φ_t. Moreover, this is also the Conley index of N with respect to the flow φ, as proved in [7]. Therefore, it only remains to verify that N admits a Poincaré section in order to have checked all the assumptions of the following theorem, proved in a more general setting in [3]:

Theorem 4.1. *Assume N is an isolating neighborhood for the flow φ which admits a Poincaré section Ξ. If N has the cohomological Conley index of a hyperbolic periodic orbit, then* $\mathrm{Inv}(N, \varphi)$ *contains a periodic orbit.*

As a consequence, we obtain a computer assisted proof that the differential equation (4.1) admits a periodic orbit.

In practice, the verification of the assumptions of Theorem 4.1 involves a series of extensive, time-consuming computations. The algorithms which may be used for these computations are proposed in [11].

As a byproduct we obtain rigorous information concerning the location of the periodic orbit: it is contained in the interior of the isolating neighborhood N constructed in course of the computer assisted proof. Unfortunately, we do not prove anything about the period of this orbit. In particular, it is not ruled out that this orbit may make several turns along the neighborhood until it closes.

As an example consider the Rössler equations

$$\begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + by, \\ \dot{z} = b + z(x-a). \end{cases} \tag{4.3}$$

For $a = 5.7$ and $b = 0.2$ the existence of chaos in (4.3) was proved in [15]. The chaotic attractor observed there seems to emerge via a series of period-doubling bifurcations of stable periodic orbits as the parameter a is increased. The first orbit in this series was observed in numerical simulations for $a = 2.2$ in [2], but the existence of a periodic orbit close to the observed one was proved only recently [12] with the use of the method described in this section. This method also allows one to prove the existence of the second orbit, numerically best seen for $a = 3.1$ [13]. Summarizing, we can prove the following theorem.

Theorem 4.2. *Let $b = 0.2$. For $a = 2.2$ as well as for $a = 3.1$ the Rössler equations (4.3) admit a periodic orbit.*

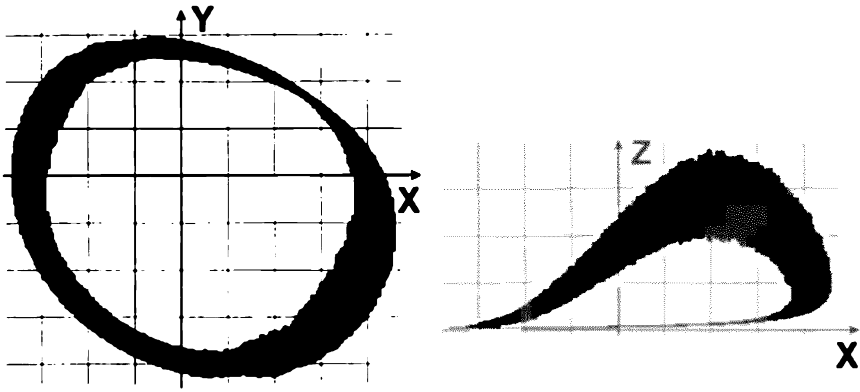

FIGURE 4.1. The isolating neighborhood constructed for the Rössler equations for the parameter value $a = 2.2$.

In Figure 4.1, projections to the XY and XZ planes of the neighborhood constructed for the Rössler equations (4.3) for the parameter value $a = 2.2$ are illustrated. The grid size used was $\eta = 1/32$. The time-step $t = 3$ was approximately a half of the period of the periodic trajectory. The thin lines in the picture indicate integer coordinates. Note that a much tighter neighborhood may be obtained if a finer grid is taken, but then the computations are more costly in terms of computer time and memory used.

Figure 4.2 shows projections to the XY and XZ planes of the neighborhood computed for the periodic trajectory which numerically is observed to appear after the first period-doubling bifurcation in the Rössler equations. This neighborhood was created with the grid size $\eta = 1/256$ and the time-step $t = 2$.

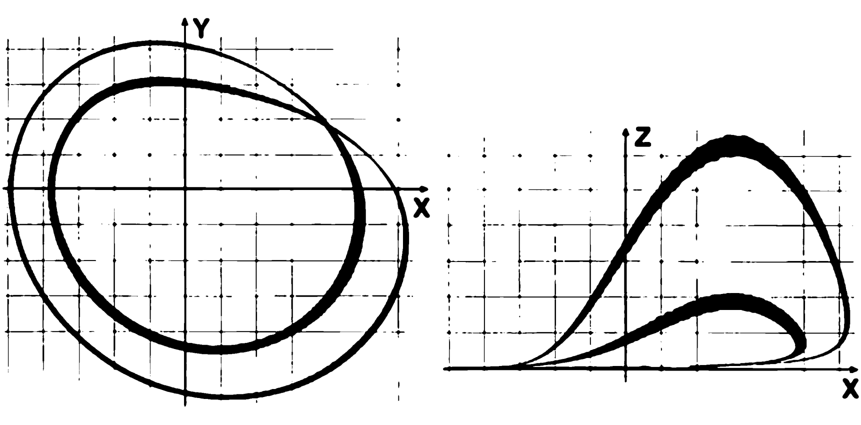

FIGURE 4.2. The neighborhood constructed for the Rössler equations with $a = 3.1$.

All the rigorous computations needed to complete the proof of the existence of the periodic orbit took about 2 hours ($a = 2.2$) and 3 days ($a = 3.1$) on an IBM compatible PC running a 450 MHz processor.

As our second example consider the Lorenz equations

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = Rx - y - xz, \\ \dot{z} = xy - bz. \end{cases} \tag{4.4}$$

For $R = 28$, $\sigma = 10$ and $b = 8/3$ the existence of chaos in these equations was proved in [1, 6]. However, when the parameter R is increased to $R = 260$ or to $R = 350$, attracting periodic orbits are observed in numerical simulations [14]: a symmetric one in the latter case and two mutually symmetric in the former case. These symmetries are due to the symmetry in the equations:

$$s : (x, y, z) \mapsto (-x, -y, z).$$

Our method allows us to prove that there exist periodic orbits close to the location of the numerically observed ones. The details are presented in [13].

Theorem 4.3. *Fix $\sigma = 10$ and $b = 8/3$. For $R = 260$ the Lorenz equations (4.4) admit two mutually symmetric periodic orbits, and for $R = 350$ the Lorenz equations (4.4) admit a periodic orbit.*

In Figure 4.3, projections to the XY and XZ planes of a neighborhood of one of the two mutually symmetric periodic trajectories for $R = 260$ are plotted. The grid marked in the picture is drawn every 10 units. The Z coordinate of the bottom of the right-hand picture is 180. The grid size used in rigorous computations was $\eta = 1/16$. The time step was taken to be $t = 1/16$ which is about $1/7$ of the approximate period of the trajectory observed in numerical simulations.

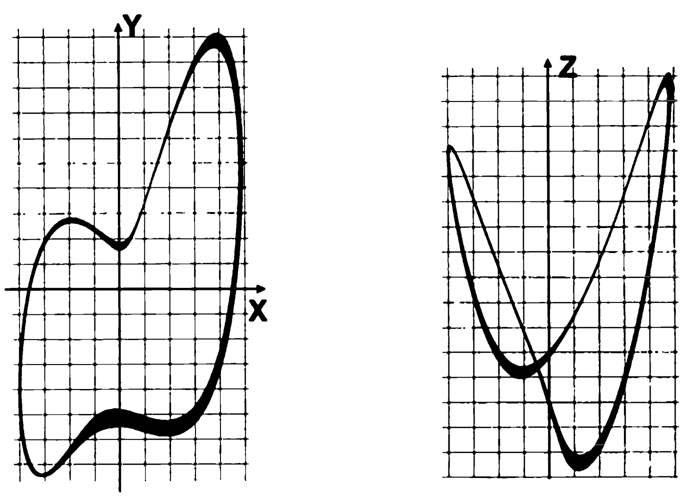

FIGURE 4.3. The isolating neighborhood constructed for the Lorenz equations for the parameter value $R = 260$.

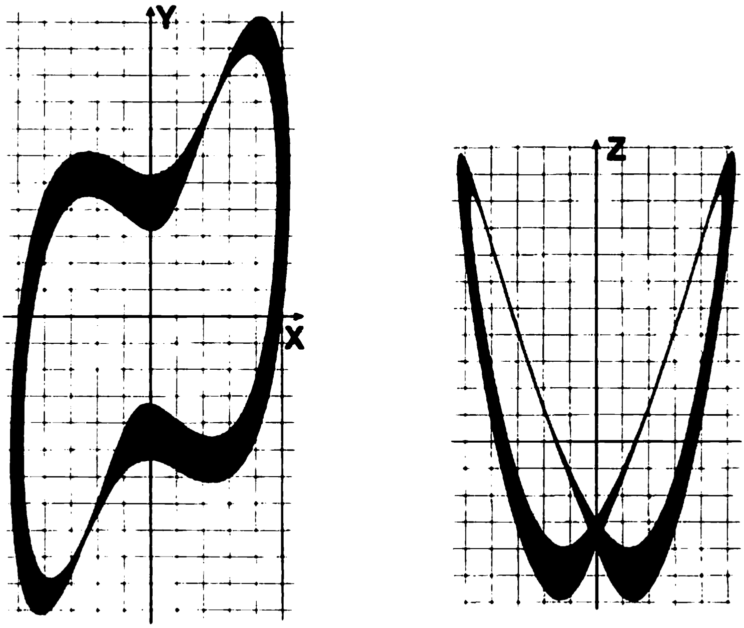

FIGURE 4.4. The neighborhood constructed for the Lorenz equations with $R = 350$.

In Figure 4.4, projections to the XY and XZ planes of the neighborhood found for $R = 350$ are visualized. Again, the grid marked in the picture is drawn every 10 units. The Z coordinate of the bottom of the right-hand picture is 280. The grid size used in rigorous computations was $\eta = 1/8$. The time step was chosen as $t = 1/16$, which is about $1/6$ of the approximate period of the observed trajectory.

The time of numerical computations on an IBM compatible PC running a 450 MHz processor amounted to about 4 days for the first orbit and 5 days for the other.

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Instytut Informatyki
Uniwersytet Jagielloński
Kraków, Poland
email: mrozek@ii.uj.edu.pl

Instytut Informatyki
Uniwersytet Jagielloński
Kraków, Poland
email: pilarczk@ii.uj.edu.pl

Dynamics of a Forced Oscillator having an Obstacle

R. Ortega

Dedicated to Antonio Marino

1 Introduction

Consider the scalar differential equation

$$\ddot{x} + g(x) = f(t) \tag{1.1}$$

where f is 2π-periodic, say $f \in C(\mathbb{T})$ with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and g satisfies

$$\lim_{x\to\pm\infty} g(x) = \pm\infty, \quad \limsup_{|x|\to\infty} \frac{g(x)}{x} < \infty. \tag{1.2}$$

The existence of 2π-periodic solutions has been analyzed by many authors using different variational and topological methods. For the linear case ($g(x) = \omega^2 x$) it is well known that the existence of a periodic solution is equivalent to the boundedness of all solutions, and one can ask whether such an equivalence still holds in nonlinear cases. In this paper we report on several results which give partial answers to this question. First we shall assume that g satisfies the assumptions of Lazer and Leach in [13] and we shall show that the condition for existence of a periodic solution obtained in that paper guarantees, in many cases, the boundedness of all solutions. For this class of nonlinearities the situation resembles the linear theory. Later we shall consider the asymmetric nonlinearities that were first discussed by Fucik [11] and Dancer [6, 5]. The situation now is more delicate because unbounded and periodic solutions can coexist. After this brief review of published results we shall analyze in detail the problem of boundedness for a forced linear oscillator which bounces elastically against a wall. This problem has not been considered previously and it will be employed to illustrate the techniques developed in [21] and [22]. We notice that the periodic problem for this bouncing oscillator was already studied by Lazer and McKenna in [15]. They interpreted the model as a limiting case of the asymmetric oscillator.

Moser's theorem on the existence of invariant curves will be crucial in the proofs. The use of this theorem in the study of boundedness for (1.1) is classical and one can refer to [20, 7, 16]. In all those papers the function

g was superlinear at infinity and did not satisfy (1.2). In many cases we shall be able to obtain additional information on the dynamics around infinity. For instance, when there is boundedness, the existence of large subharmonic and quasi-periodic solutions follows as a consequence of the method of proof and the theory of twist mappings.

2 Remarks on the Lazer–Leach condition

Let us now assume that the function g in (1.1) is of the type

$$g(x) = n^2 x + h(x)$$

where $n = 1, 2, \ldots$ and h is a continuous and bounded function having limits at infinity, $h(+\infty)$ and $h(-\infty)$. The main result in [13] implies that (1.1) has a 2π-periodic solution if the condition

$$|\hat{f}_n| < \frac{1}{\pi}|h(+\infty) - h(-\infty)|, \tag{2.3}$$

holds, where $\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{int}dt$.

We shall refer to (2.3) as to the Lazer–Leach condition. When h is not constant and satisfies

$$h(-\infty) \leq h(x) \leq h(+\infty) \quad \forall x \in \mathbb{R}, \tag{2.4}$$

this condition becomes necessary and sufficient for the solvability of the periodic problem. This is a remarkable consequence of [13].

Next we shall show that (2.3) also plays a role in the problem of boundedness. Let us first assume that h is the piecewise linear function

$$h_L(x) = \begin{cases} -L & \text{if } x \leq -1 \\ Lx & \text{if } |x| \leq 1 \\ L & \text{if } x \geq 1 \end{cases}$$

for some $L > 0$. This function satisfies (2.4) and the Lazer–Leach condition becomes

$$|\hat{f}_n| < \frac{2L}{\pi}. \tag{2.5}$$

It was proved in [22] that if $f \in C^5(\mathbb{R})$ and (2.5) holds, then all solutions of (1.1) with $g(x) = n^2x + h_L(x)$ are bounded. In this case (2.5) is sharp for the boundedness problem because all solutions are unbounded when it does not hold. This follows from [24] and [1]. More recently Liu has obtained similar results for a class of functions g, including the model nonlinearity $g(x) = \arctan x$ (see [18] and also [12]).

3 The asymmetric oscillator

Let us now consider another piecewise linear function. Namely, $g(x) = ax^+ - bx^-$ where $a, b > 0$ with $a \neq b$. The corresponding equation

$$\ddot{x} + ax^+ - bx^- = f(t) \tag{3.6}$$

can be thought of as a model of the motion of a particle subjected to an asymmetric restoring force. The periodic problem for (3.6) was analyzed by Fucik [11] and Dancer [6, 5] in the 1970s. In this context they realized the importance of the following set, lying in the plane of parameters (a, b),

$$\Sigma = \bigcup_{p=1}^{\infty} C_p, \quad C_p = \{(a,b) \in \mathbb{R}^2_+ : \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{p}\}.$$

It can be proved that if $(a, b) \notin \Sigma$, then (3.6) has a 2π-periodic solution for every $f \in C(\mathbf{T})$. On the contrary, when $(a, b) \in \Sigma$ the solvability of the periodic problem depends upon f (see [5, 14, 8]). The set Σ can be thought of as a sort of periodic spectrum and sometimes it is called the Fucik spectrum. In contrast to the Lazer–Leach situation, now there is no direct connection between the periodic problem and the boundedness of all solutions. In a joint paper with Alonso [2] we noticed that, given any (a, b) with

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \in \mathbb{Q},$$

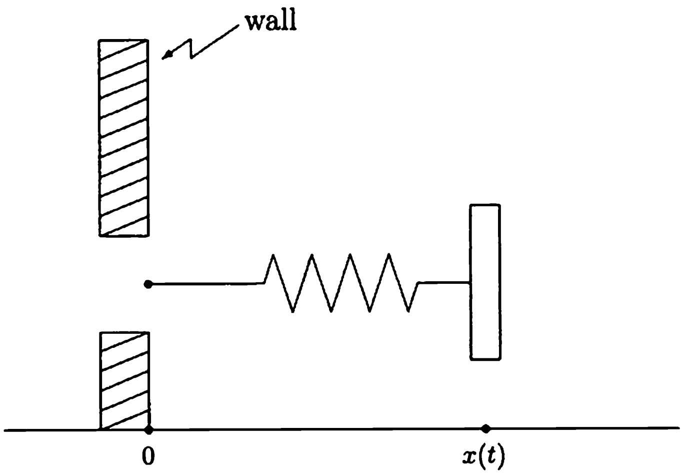
it is possible to construct many functions f for which (3.6) has unbounded solutions. Selecting the couple (a, b) so that it is not in Σ, one finds examples of coexistence of unbounded and periodic solutions. Sufficient conditions on f for the boundedness of all solutions have been obtained in [19] and also in [23].

The function $g(x) = ax^+ - bx^-$ is possibly the simplest example of a function satisfying $g'(+\infty) \neq g'(-\infty)$. Results for more general jumping nonlinearities can be seen in [9, 4, 26].

In the next sections we shall consider the following limit case of (3.6),

$$\ddot{x} + ax^+ - \infty x^- = f(t). \tag{3.7}$$

Of course this is not a well-defined differential equation but it will be interpreted in the sense proposed by Lazer and McKenna in [15]. The equation (3.7) can be thought of as the model of the motion of a particle which is attached to a spring $(-ax)$ that pushes the particle against a barrier situated at $x = 0$. At this barrier the particle bounces elastically. See the figure below.



Some discussions on the periodic problem for (3.7) can be found in [15]. We also mention [10] for some connections between jumping nonlinearities and bouncing problems.

4 A linear equation with obstacle

Given $a > 0$ and $f \in C(\mathbb{T})$, we consider the equation with obstacle

$$\begin{cases} \ddot{x} + ax = f(t), \\ x(t) \geq 0, \\ x(t_0) = 0 \quad \Rightarrow \quad \dot{x}(t_0+) = -\dot{x}(t_0-). \end{cases} \tag{4.8}$$

By a solution of (4.8) we understand a continuous function $x : I \to [0, \infty)$, defined on some closed interval $I \subset \mathbb{R}$, such that the conditions below hold:

(i) the set of zeros $Z = \{t \in I : \ x(t) = 0\}$ is discrete,

(ii) for any interval $J = [t_1, t_2]$ with $Z \cap (t_1, t_2) = \emptyset$, the function $x(t)$ belongs to $C^2(J)$ and satisfies

$$\ddot{x}(t) + ax(t) = f(t), \ \ t \in J,$$

(iii) given $t_0 \in Z \cap \text{int}\,(I)$, $\dot{x}(t_0+) = -\dot{x}(t_0-)$.
(Here $\dot{x}(t_0 \pm)$ denote the right and left derivatives of x at t_0. The condition (ii) guarantees that they are well defined).

In the previous definition the set Z can be empty. Then $x(t)$ is just a positive solution of the linear equation. When Z is non-empty we shall say that $x(t)$ is a bouncing solution. As an example consider the function

$x(t) = 1 + c \sin \sqrt{a} t$. It is a solution of (4.8) with $f \equiv a$ for any c with $|c| \leq 1$, but it is a bouncing solution if and only if $|c| = 1$.

Given $\tau \in \mathbb{R}$ and $(x_0, v_0) \in \mathbb{R}^2$ with $x_0 > 0$ or $x_0 = 0$ and $v_0 > 0$, we can always find a unique solution of (4.8) satisfying $x(\tau) = x_0$, $\dot{x}(\tau) = v_0$. Sometimes this solution cannot be defined in the whole line. For instance, assume that we can find a solution $x(t)$ of the linear equation $\ddot{x} + ax = f(t)$ satisfying (for some $t^* \in \mathbb{R}$): a) $x(t^*) = \dot{x}(t^*) = 0$, b) $x(t) > 0$ if $0 < t^* - t < \epsilon$, c) there exists a sequence $t_n \downarrow t^*$ such that $x(t_n) \leq 0$. Then we can construct a solution of (4.8) which coincides with $x(t)$ on the interval $(t^* - \epsilon, t^*]$. It is clear that this solution cannot be continued to the right of t^*. We also notice that all the solutions of (4.8) are well defined over $(-\infty, +\infty)$ if $f(t) \neq 0$ for every $t \in \mathbb{R}$.

The homogeneous equation ($f \equiv 0$) can be easily analyzed. Actually the solutions are

$$x(t) = A|\sin(\sqrt{a}t + \phi)|, \quad A > 0, \ \phi \in \mathbb{T}.$$

All of them are periodic with period

$$T = \frac{\pi}{\sqrt{a}}.$$

We shall distinguish the solution with initial conditions $x(0) = 0$, $\dot{x}(0) = 1$ and denote it by

$$\varphi_a(t) = \frac{1}{\sqrt{a}}|\sin(\sqrt{a}t)|.$$

The analysis of the non-homogeneous case ($f \not\equiv 0$) is more delicate and we shall distinguish two cases depending on whether the period T is commensurable with 2π or not. In the first case we can find positive integers p, q such that

$$\sqrt{a} = \frac{q}{2p}, \quad p \text{ and } q \text{ are relatively prime.} \tag{4.9}$$

Assuming that this condition holds, we define the function

$$\Phi(\tau) = \frac{1}{q}\int_0^{2\pi p} f(s+\tau)\varphi_a(s)ds.$$

This function is 2π-periodic and will play an important role in what follows. Sometimes it is more convenient to employ another expression of Φ, namely

$$\Phi(\tau) = \frac{1}{q}\sum_{h=0}^{q-1} \mu(\tau + hT)$$

where

$$\mu(\tau) = \int_0^T f(s+\tau)\varphi_a(s)ds = \frac{1}{\sqrt{a}}\int_\tau^{\tau+T} f(s)\sin(\sqrt{a}(s-\tau))ds. \tag{4.10}$$

These last expressions reveal that Φ is of class C^2.

Example 1. Computation of Φ.

Assume $a = 1$. Then (4.9) holds with $p = 1$, $q = 2$ and

$$\Phi(\tau) = \frac{1}{2}\int_0^{2\pi} f(s+\tau)|\sin s|ds.$$

Assuming that $f(t) = \alpha + \beta\sin t + \gamma\sin 2t$, with $\alpha, \beta, \gamma \in \mathbb{R}$, a computation shows that

$$\Phi(\tau) = 2\alpha - \frac{2}{3}\gamma\sin 2\tau.$$

We notice that the function Φ changes sign if and only if $3|\alpha| < |\gamma|$. Our next result will imply that in such a case there are unbounded solutions of (4.8).

Theorem 4.1. *Assume that $\sqrt{a} \in \mathbb{Q}$ and it satisfies (4.9). In addition Φ changes sign and all zeros of Φ are nondegenerate; that is,*

$$\Phi(\tau)^2 + \Phi'(\tau)^2 > 0 \quad \forall\tau \in \mathbb{R}.$$

Then there exists $R > 0$ such that any solution of (4.8) with

$$x(\tau) + |\dot{x}(\tau)| > R \quad (\text{for some } \tau \in \mathbb{R})$$

is unbounded.

Next we present a complementary result about the boundedness of solutions.

Theorem 4.2. *Assume that*

$$f \in C^4(\mathbb{T})$$

and one of the conditions below holds,

(i) $\sqrt{a} \in \mathbb{Q}$ *and it satisfies (4.9) with* $\Phi(\tau) \neq 0\ \forall\tau \in \mathbb{R}$,

(ii) $\sqrt{a} \notin \mathbb{Q}$ *and* $\int_0^{2\pi} f(t)dt \neq 0$.

Then there exists $R > 0$ such that every solution of (4.8) satisfying

$$x(\tau) + |\dot{x}(\tau)| > R \quad (\text{for some } \tau \in \mathbb{R})$$

is well defined in $(-\infty, +\infty)$ and bounded.

Remarks. 1. The proof of these theorems will give more insight into the dynamics of the equation as well as precise information about the oscillatory properties of solutions.

2. Going back to the example before Theorem 4.1 one notices that Theorem 4.2 applies when $3|\alpha| > |\gamma|$. The case $3|\alpha| = |\gamma|$ and β arbitrary, is left open by the previous theorems.
3. In the second theorem we need some extra regularity for the forcing, $f \in C^4$. There are known examples where regularity plays a role in the problem of boundedness (see [17, 27]) and so this condition seems reasonable. We present an example in this direction, but first it is convenient to state explicitly an intuitive consequence of Theorem 4.2.

Corollary 4.3. *Assume that*

$$f \in C^4(\mathbb{T})$$

and

$$f(t) \geq 0 \ \ \forall t \in \mathbb{R}. \tag{4.11}$$

Then, for arbitrary $a > 0$, all solutions of (4.8) are bounded.

Example 2. A spring with impulses.

Let $\delta(t)$ denote the 2π-periodic extension of the Dirac mass concentrated at $t = 0$. More precisely, δ is the measure on $\mathbb{T}$ defined by

$$\delta \in C(\mathbb{T})^*, \ \ \langle \delta, \phi \rangle = \phi(0), \ \ \phi \in C(\mathbb{T}).$$

We shall consider the equation

$$\begin{cases} \ddot{x} + x = \delta(t), \\ x(t) \geq 0, \\ x(t_0) = 0 \ \ \Rightarrow \ \ \dot{x}(t_0+) = -\dot{x}(t_0-) \end{cases} \tag{4.12}$$

and we shall see that all solutions are unbounded. Since δ satisfies the condition (4.11) when it is interpreted in a liberal way, this example shows that the previous corollary is not valid when f is a measure.

A solution of (4.12) can be defined as a continuous function satisfying (4.8) with $f \equiv 0$ in each interval $[2\pi n, 2\pi(n+1)]$ and such that

$$\dot{x}(2\pi n+) = \dot{x}(2\pi n-) + 1, \qquad \text{if } x(2n\pi) > 0$$

and

$$\dot{x}(2\pi n+) = -\dot{x}(2\pi n-) + 1, \qquad \text{if } x(2n\pi) = 0.$$

Intuitively we can describe the situation as follows: in the absence of external force, our particle would bounce periodically with period π. Now we are adding an external force which is localized at times $t = 0, \pm 2\pi, \pm 4\pi, \ldots$ and has the effect of increasing the velocity of the particle in one unit.

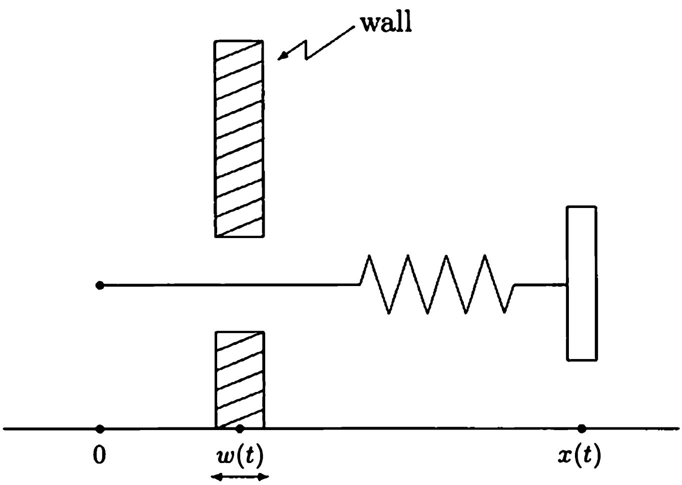
We consider the Poincaré mapping

$$P : (x(0), \dot{x}(0+)) \mapsto (x(2\pi), \dot{x}(2\pi+))$$

where $x(t)$ is a solution of (4.12). Since the period of δ is twice the period of the free oscillation, it is clear that P is just the translation of vector $(0,1)$, that is $x(2\pi) = x(0)$, $\dot{x}(2\pi+) = \dot{x}(0+) + 1$. In consequence all solutions are defined up to $+\infty$ and the corresponding energy goes to infinity .

Example 3. A spring with oscillating wall.

In principle one could consider more general oscillators by letting the barrier oscillate. More concretely, let us now assume that the wall is not fixed at $x = 0$ but it moves according to the known law $w = w(t)$. See the figure below.



We assume that w is smooth, positive and 2π-periodic. The particle follows the model

$$\begin{cases} \ddot{x} + ax = f(t) & \text{if } x(t) > w(t) \\ x(t) \geq w(t) & \\ x(t_0) = w(t_0) \quad \Rightarrow \quad \dot{x}(t_0+) = -\dot{x}(t_0-) + 2\dot{w}(t_0). & \end{cases}$$

The last condition reflects that the bouncing against the wall is elastic.

The change of reference system

$$x = y + w(t)$$

transforms the model into (4.8) where the new external force depends upon f, w and $\ddot{w}$. More discussions on this kind of oscillators as well as some connections with billiards can be seen in [3]. I thank R. Ramírez-Ros for informing me of this reference.

5 The successor map

Given $\tau \in \mathbb{R}$ and $v > 0$, let $x(t;\tau,v)$ be the solution of

$$\ddot{x} + ax = f(t), \quad x(\tau) = 0, \ \dot{x}(\tau) = v.$$

We denote by $\hat{\tau} > \tau$ the first zero of $x(t;\tau,v)$ to the right of τ. The corresponding velocity after bouncing will be denoted by

$$\hat{v} = -\dot{x}(\hat{\tau};\tau,v).$$

The properties of the map $S : (\tau,v) \mapsto (\hat{\tau},\hat{v})$ have been studied in [21] and [22] and we shall use the results in these papers. First of all we notice that S is well defined and one-to-one in the domain

$$\mathcal{R}_+ = \{(\tau,v) \in \mathbb{R}^2 : \ v > 0\}.$$

Moreover, it satisfies

$$S(\tau + 2\pi, v) = S(\tau,v) + (2\pi, 0).$$

In view of this property it is natural to identify τ with $\tau + 2\pi$ and we shall interpret τ and v as polar coordinates (τ =angle, v = radius). In this way the mapping S is defined on the cylinder,

$$S : \mathbb{T} \times (0,\infty) \to \mathbb{T} \times [0,\infty).$$

The iteration

$$(\tau_{n+1}, v_{n+1}) = S(\tau_n, v_n)$$

will reflect the dynamical properties of (4.8) as well as the oscillatory properties of solutions. Given an orbit of S, $\{(\tau_n, v_n)\}_{n\in\Lambda}$, $\Lambda \subset \mathbb{Z}$, such that $\{\tau_n \ : \ n \in \Lambda\}$ is a closed and discrete subset of $\mathbb{R}$, we can construct a bouncing solution of (4.8) defined as

$$x(t) = x(t;\tau_n, v_n) \ \text{ if } t \in [\tau_n, \tau_{n+1}].$$

Conversely, given a solution $x(t)$ of (4.8) we can label the set Z as a sequence $\{\tau_n\}$. If we define $v_n = \dot{x}(\tau_n+)$ and assume $v_n \neq 0$, then the sequence $\{(\tau_n, v_n)\}$ is an orbit of S. We notice that positive solutions of (4.8) do not correspond to any orbit of S. These solutions are always bounded. On the other hand a bouncing solution will be bounded if and only if $\sup_n v_n < \infty$ (see Lemma 4.3 in [21]).

We are now interested in the regularity of S. To this end we consider the singularity set

$$\Sigma = \{(\tau,v) : \ \hat{v} = 0\}.$$

Then S is of class C^1 on $\mathcal{R}_+ - \Sigma$. Another useful fact is that Σ is bounded in the cylinder, this means that there exists $\nu > 0$ such that

$$\Sigma \subset \mathbb{R} \times (0,\nu].$$

All these facts are proved in [21]. Now we present the expansion of S at infinity as obtained in Section 6 of [22]:

$$\begin{cases} \hat{\tau} = \tau + \frac{\pi}{\sqrt{a}} + \frac{1}{\sqrt{a}v}\sigma_0(\tau) + F(\tau, v) \\ \hat{v} = v + k_0(\tau) + G(\tau, v) \end{cases}$$

where

$$\sigma_0(\tau) = \int_\tau^{\tau+\frac{\pi}{\sqrt{a}}} f(t) \sin\sqrt{a}(t-\tau)dt, \quad k_0(\tau) = \int_\tau^{\tau+\frac{\pi}{\sqrt{a}}} f(t) \cos\sqrt{a}(t-\tau)dt.$$

The remainders F and G satisfy

$$F(\tau, v) = O(\frac{1}{v^2}), \quad G(\tau, v) = O(\frac{1}{v}) \text{ as } v \to +\infty,$$

uniformly in $\tau \in \mathbb{R}$. Moreover, if $f \in C^p(\mathbf{T})$, $p \geq 1$, then one can estimate the derivatives of the order $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 + \alpha_2 \leq p$, in the form

$$\partial^\alpha F(\tau, v) = O(\frac{1}{v^{2+\alpha_2}}), \quad \partial^\alpha G(\tau, v) = O(\frac{1}{v^{1+\alpha_2}}) \text{ as } v \to +\infty.$$

We are now in a position to prove the results of the previous section. To prove Theorem 4.1 we shall apply the results in Section 3 of [2]. When a satisfies (4.9) one can rewrite the expansion of S at infinity as

$$\begin{cases} \hat{\tau} = \tau + \frac{2\pi p}{q} + \frac{\mu(\tau)}{v} + F(\tau, v) \\ \hat{v} = v - \mu'(\tau) + G(\tau, v), \end{cases}$$

where $\mu(\tau)$ was defined by (4.10). At this point it is convenient to notice that $\sigma_0' = -\sqrt{a}k_0$.

The expansion for the q-iterate of S is

$$\begin{cases} \tau_q = \tau + 2\pi p + \frac{q\Phi(\tau)}{v} + \hat{F}(\tau, v) \\ v_q = v - q\Phi'(\tau) + \hat{G}(\tau, v) \end{cases}$$

where $\hat{F}$ and $\hat{G}$ are remainders satisfying the same conditions as F and G. We can now apply Proposition 3.1 in [2] to deduce that, when the conditions of Theorem 4.1 hold, there exists $R_1 > 0$ such that if $v_0 \geq R_1$, then $\{(\tau_n, v_n)\}$ is well defined in the future or in the past and satisfies $v_n \to +\infty$ as $n \to +\infty$ or $n \to -\infty$. Thus, any solution of (4.8) satisfying $x(\tau) = 0$, $\dot{x}(\tau) \geq R_1$ (for some τ) is unbounded. The proof of the theorem can be finished by an application of Lemma 4.3 in [21].

To prove Theorem 4.2 it is sufficient to find $R_1 > 0$ such that any orbit $\{(\tau_n, v_n)\}$ with $v_0 \geq R_1$ is well defined for $n \in \mathbb{Z}$ and $\sup_n v_n < \infty$. This will be achieved by means of the theory of invariant curves. First we assume that the condition (i) holds. We shall find a sequence of Jordan curves $\{\Gamma_n\}$

in $\mathbb{T} \times (0, \infty)$ which are homotopic to the circle $v = constant$ and such that $S^q(\Gamma_n) = \Gamma_n$. These curves are ordered and go to infinity as $n \to +\infty$. This means that Γ_{n+1} lies in the unbounded component of $[\mathbb{T} \times (0, \infty)] - \Gamma_n$ and $\min\{v : (\tau, v) \in \Gamma_n\} \to +\infty$ as $n \to \infty$. Since S is a topological mapping (for large v) and $\hat{v} \to +\infty$ as $v \to +\infty$, one deduces that Γ_n acts as a barrier for the orbits of S^q. Thus, if (τ_0, v_0) lies between Γ_N and Γ_{N+1}, then (τ_{kq}, v_{kq}) will also lie on this region for any $k \in \mathbb{Z}$. From here it is easy to prove that v_n is bounded.

To prove the existence of Γ_n one can proceed exactly in the same way as in the proof of Theorem 1.1 in [22]. First one notices that S^q has the intersection property and then, after the change of variables

$$\theta = \tau, \quad \delta r = \frac{1}{v}, \quad (\delta > 0 \text{ parameter}),$$

one can apply Theorem 3.1 of [22] to S^q. Notice that in that theorem one can replace C^5 by C^4. To realize this it is sufficient to employ a C^4 version of the Small Twist Theorem (see for instance the appendix in [23]).

To prove Theorem 4.2 when (ii) holds, one proves the existence of invariant curves of S. This is achieved by employing the main result in [23]. See also Example 2 in the same paper.

Final remarks.

1. In the assumptions of Theorem 4.1 it is possible to give an almost complete description of the dynamics of S around infinity. This can be achieved by combining the expansion of S^q previously obtained with the proof of Proposition 3.1 in [2].
2. In the assumptions of Theorem 4.2 we find the typical situation where KAM theory can be applied. In the annulus between two invariant curves we can apply the Poincaré–Birkhoff Theorem to deduce the existence of periodic points of S. This lead to subharmonic solutions of large amplitude. The solutions with initial conditions on an invariant curve will be quasi-periodic. All this is explained in the book [25].

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Departamento de Matemática Aplicada
Facultad de Ciencias, Universidad de Granada
18071 Granada, Spain
e-mail: rortega@ugr.es

Spike Patterns in the Super-Critical Bahri–Coron Problem

M. del Pino, P. Felmer, and M. Musso

Dedicated to Antonio Marino

1 Introduction

This paper deals with the construction of solutions of the problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}+\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $\varepsilon > 0$ is a small parameter.

It is well known that the problem

$$\begin{cases} -\Delta u = u^q & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has at least one solution when $1 < q < \frac{N+2}{N-2}$. However, when $q \geq \frac{N+2}{N-2}$, the existence of solutions to problem (1.2) depends strongly on the topology or geometry of Ω. A well-known result by Pohozaev [13], asserts that (1.2) has no solutions if $q \geq \frac{N+2}{N-2}$ and Ω is star-shaped. On the other hand Kazdan and Warner [10] showed that (1.2) has a radially symmetric solution for any $q > 1$ when Ω is a symmetric annulus. Coron in [5] considered the case $q = \frac{N+2}{N-2}$, and showed that (1.2) is solvable when Ω is a (non-symmetric) domain exhibiting a small hole, say $\Omega = \mathcal{D} \setminus \bar{B}(P_0, \mu)$, where $\mathcal{D}$ is a smooth bounded domain, $P_0 \in \mathcal{D}$ and μ is sufficiently small.

In [1], Bahri and Coron considerably generalize this result, proving that if $q = \frac{N+2}{N-2}$ and if some homology group of Ω with coefficients in $\mathbf{Z}_2$ is

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nontrivial, then problem (1.2) has a solution. While it may be expected that this solution *survives* a small supercritical perturbation of the exponent as in (1.1), the indirect variational arguments employed in [5] and [1] do not seem to give in principle a clue as to how to obtain this fact. Solvability when $q > \frac{N+2}{N-2}$ in domains "with topology" is not true in general as shown via counterexamples by Passaseo [11, 12], answering negatively the question posed by Brezis in [3]. In our recent work [6] we have considered problem (1.1) in Coron's situation of a domain with a small perforation, and proved solvability whenever ε is sufficiently small. The proof is constructive and, rather puzzlingly, the solutions found collapse as $\varepsilon \to 0$ in the form of a double spike: the solution tends to vanish everywhere except around two local maximum points which blow up at the rate $O(\varepsilon^{-\frac{1}{2}})$. This result generalizes to a domain exhibiting multiple holes, as we have recently established in [7]. In such a situation, multi-peak solutions exist, consisting of the gluing of double-spikes associated to each of the holes. More precisely, our setting in problem (1.1) is the following.

Let $\mathcal{D}$ be a bounded, smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $P_1, P_2, \dots, P_m$ points of $\mathcal{D}$. Let us consider the domain

$$\Omega = \mathcal{D} \setminus \bigcup_{i=1}^{m} B(P_i, \mu) \tag{1.3}$$

where $\mu > 0$ is a small number.

Theorem 1.1. *There exists a $\mu_0 > 0$, which depends on $\mathcal{D}$ and the points $P_1, \dots, P_m$ such that if $0 < \mu < \mu_0$ is fixed and Ω is the domain given by (1.3), then the following holds: Given an integer $1 \leq k \leq m$, there exists $\varepsilon_0 > 0$ and a family of solutions u_ε, $0 < \varepsilon < \varepsilon_0$ of (1.1), with the following property: u_ε has exactly k pairs of local maximum points $(\xi_{j1}^\varepsilon, \xi_{j2}^\varepsilon) \in \Omega^2$ $j = 1, \dots, k$ with $c\mu < |\xi_{ji}^\varepsilon - P_j| < C\mu$, for certain constants c, C independent of μ and such that for each small $\delta > 0$,*

$$\sup_{\{|x-\xi_{ij}^\varepsilon|>\delta \, \forall i,j\}} u_\varepsilon(x) \to 0$$

and

$$\sup_{|x-\xi_{ij}|<\delta} u_\varepsilon(x) \to +\infty, \quad \forall i, j$$

as $\varepsilon \to 0$.

The proof provides much finer information on the asymptotic profile of the blowup of these solutions, as $\varepsilon \to 0$: after scaling and translation one sees around each ξ_{ij}^ε a solution in entire $\mathbb{R}^N$ of the equation at the critical exponent. More precisely, we will find,

$$u_\varepsilon(x) = \sum_{i=1}^{k} \sum_{j=1}^{2} \left(\frac{\alpha_N \lambda_{ij} \varepsilon^{\frac{1}{N-2}}}{\varepsilon^{\frac{2}{N-2}} \lambda_{ij}^2 + |x - \xi_{ij}^\varepsilon|^2} \right)^{\frac{N-2}{2}} + \theta_\varepsilon(x), \tag{1.4}$$

where $\theta_\varepsilon(x) \to 0$ uniformly as $\varepsilon \to 0$, for certain positive constants α_N. The numbers λ and the points ξ will be further identified as critical points of certain functionals built upon the Green function of Ω. The role of Green's function in concentration phenomena associated to almost-critical problems on the subcritical side, $q = \frac{N+2}{N-2} - \varepsilon$, has already been considered in several works; see Brezis and Peletier [4], Rey [14], [15], [16], Han [9] and Bahri, Li and Rey [2].

In what follows we will denote by $G(x,y)$ the Green function of Ω, namely G satisfies

$$\Delta_x G(x,y) = \delta(x-y), \quad x \in \Omega,$$
$$G(x,y) = 0, \quad x \in \partial\Omega,$$

where $\delta(x)$ denotes the Dirac mass at the origin. We denote by $H(x,y)$ its regular part, namely

$$H(x,y) = \Gamma(x-y) - G(x,y)$$

where Γ denotes the fundamental solution of the Laplacian,

$$\Gamma(x) = b_N |x|^{2-N},$$

so that H satisfies

$$\Delta_x H(x,y) = 0, \quad x \in \Omega,$$
$$H(x,y) = \Gamma(x-y), \quad x \in \partial\Omega.$$

Its *diagonal* $H(x,x)$ is usually called the Robin function of the domain.

We shall concentrate next on the case of existence of a single two-spike solution, and state a general result derived in [6], which includes the case $k = 1$ in Theorem 1.1. In the two-spike concentration phenomenon, the following function will play a crucial role in our analysis:

$$\varphi(\xi_1,\xi_2) = H^{\frac{1}{2}}(\xi_1,\xi_1)H^{\frac{1}{2}}(\xi_2,\xi_2) - G(\xi_1,\xi_2). \tag{1.5}$$

We will construct solutions of (1.1) which as $\varepsilon \to 0$ develop a spike-shape, blowing up at exactly two distinct points ξ_1, ξ_2 while approaching zero elsewhere, provided that the set where $\varphi < 0$ is included in Ω^2 in a topologically nontrivial way. The pair (ξ_1,ξ_2) will be a critical point of φ with $\varphi(\xi_1,\xi_2) < 0$.

For a subspace B of Ω we will designate by $H^d(B)$ its d-th cohomology group with integral coefficients. We will consider the homomorphism $\iota^* : H^*(\Omega) \to H^*(B)$, induced by the inclusion $\iota : B \to \Omega$.

Theorem 1.2. *Assume $N \geq 3$ and let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, with the following property: There exists a compact manifold $\mathcal{M} \subset \Omega$ and an integer $d \geq 1$ such that, $\varphi < 0$ on $\mathcal{M} \times \mathcal{M}$, $\iota^* : H^d(\Omega) \to H^d(\mathcal{M})$ is nontrivial and either d is odd or $H^{2d}(\Omega) = 0$.*

Then there exists $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$, *problem (1.1) has at least one solution* u_ε. *Moreover, let* C *be the component of the set where* $\varphi < 0$ *which contains* $\mathcal{M} \times \mathcal{M}$. *Then, given any sequence* $\varepsilon = \varepsilon_n \to 0$, *there is a subsequence, which we denote in the same way, and a critical point* $(\xi_1, \xi_2) \in C$ *of the function* φ *such that* $u_\varepsilon(x) \to 0$ *on compact subsets of* $\Omega \setminus \{\xi_1, \xi_2\}$ *and such that for any* $\delta > 0$

$$\sup_{|x-\xi_i|<\delta} u_\varepsilon(x) \to +\infty, \quad i = 1, 2,$$

as $\varepsilon \to 0$.

The assumption of the above theorem does indeed hold true in the case of a small hole, as we explain next. Let us set

$$\Omega = \mathcal{D} \setminus \bar{B}(0, \mu). \tag{1.6}$$

Elementary properties of harmonic functions give the validity of the fact that

$$\lim_{\mu \to 0} H(x, y) = H_{\mathcal{D}}(x, y), \tag{1.7}$$

uniformly on x, y in compact subsets of $\bar{\mathcal{D}} \setminus \{0\}$, where $H_{\mathcal{D}}$ denotes the regular part of the Green function $G_{\mathcal{D}}$ on $\mathcal{D}$.

For any (fixed) sufficiently small number $\rho > 0$ there is a $\mu_0 > 0$ such that if $\mu < \mu_0$, and Ω is given by (1.6), then

$$\sup_{|\xi_1|=|\xi_2|=\rho} \varphi(\xi_1, \xi_2) < 0.$$

Hence, Theorem 1.2 applies to Ω given by (1.6), with

$$\mathcal{M} = \rho S^{N-1}.$$

This follows directly from (1.7) and the fact that $H_{\mathcal{D}}$ is smooth near $(0, 0)$ while $G_{\mathcal{D}}$ becomes unbounded as its arguments get close.

A second example is the following. Consider now a solid torus in $\mathbb{R}^3$ given by $T(l, r)$, where l is the radius of the axis circle, which we assume centered at 0, and r that of a cross-section. Assume now that there is an $r_0 > 0$ such that $T(l, r_0) \subset \mathcal{D}$. Consider now $\mathcal{D}_\delta$ defined as

$$\mathcal{D}_\delta = \mathcal{D} \setminus T(l, \delta).$$

Similarly, as in the previous example, the Green and Robin functions of $\mathcal{D}_\delta$ will approach that of $\mathcal{D}$. Then, fixing now a sufficiently small $\rho > 0$ and considering the boundary of a fixed section $S^1(\rho)$ of $T(l, \rho)$, we will have that if $\Omega = \mathcal{D}_\delta$ with δ sufficiently small, then

$$\sup_{\xi_1, \xi_2 \in S^1(\rho)} \varphi(\xi_1, \xi_2) < 0.$$

It follows that Theorem 1.2 applies now with

$$\mathcal{M} = S^1(\rho).$$

It is perhaps clear from the above argument that it suffices that for a torus not necessarily symmetric taken away, the same would be true, provided that it is "narrow" only in a certain region.

We explain next the main elements in the proofs of Theorems 1.1 and 1.2. One obvious difficulty to circumvent is the fact that Sobolev's embedding is no longer valid in our situation. We are able however to work out in "well-chosen" spaces a reduction to a finite dimensional problem, which we treat with a variational-topological approach. In the case of a single two-spike, the problem becomes basically reduced, as we will explain below, to that of finding a critical point of φ which persists under small C^1 perturbations. Such a critical point comes from a min-max quantity naturally defined from the assumptions of Theorem 1.2.

2 Recasting the problem: The finite-dimensional reduction

To find a multiple-spike solution, it is convenient to scale problem (1.1) into the expanding domain

$$\Omega_\varepsilon = \varepsilon^{-\frac{1}{N-2}}\Omega.$$

Let us consider the change of variables

$$v(y) = \varepsilon^{\overline{2+\varepsilon\frac{\frac{1}{N-2}}{2}}}u(\varepsilon^{\frac{1}{N-2}}y), \quad y \in \Omega_\varepsilon.$$

Then u solves (1.1) if and only if v satisfies

$$\begin{cases} \Delta v + v^{\frac{N+2}{N-2}+\varepsilon} = 0 & \text{in } \Omega_\varepsilon \\ v_\varepsilon > 0 & \text{in } \Omega_\varepsilon \\ v = 0 & \text{on } \partial\Omega_\varepsilon\,. \end{cases} \tag{2.8}$$

Since Ω_ε expands to the whole $\mathbb{R}^N$, and all positive solutions of

$$\Delta v + v^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N$$

are given by the functions

$$\bar{U}_{\lambda,\xi'}(y) = \lambda^{-\frac{N-2}{2}}\bar{U}\left(\frac{x-\xi'}{\lambda}\right)$$

with

$$\bar U(y) = \alpha_N \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}}$$

and $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$, $\xi' \in \mathbb{R}^N$, $\lambda > 0$, it is natural to seek solutions v of the form

$$v(y) \sim \sum_{i=1}^{h} \bar U_{\lambda_i, \xi_i'}(y) \tag{2.9}$$

for a certain set of h points $\xi_1, \dots, \xi_h$ in Ω and numbers $\lambda_1, \dots, \lambda_h > 0$, where now and in what follows we set, for $\xi \in \Omega$,

$$\xi' = \varepsilon^{-\frac{1}{N-2}} \xi \in \Omega_\varepsilon .$$

It turns out that this choice of scaling is precisely one at which we can find solutions satisfying (2.9), with the points ξ_i uniformly away from each other and from the boundary of Ω, and the positive scalars λ_i bounded above, and below away from zero. Such an approximation cannot be too good near the boundary, where v is supposed to vanish. A better approximation involves the orthogonal projections onto $H_0^1(\Omega_\varepsilon)$ of the functions $\bar U_{\lambda,\xi'}$. We denote by $V_{\lambda,\xi'}$ these projections, which are defined as the respective unique solutions of the equations

$$-\Delta V_{\lambda,\xi'} = \bar U_{\lambda,\xi'}^{\frac{N+2}{N-2}} \quad \text{in } \Omega_\varepsilon$$

$$V_{\lambda,\xi'} = 0 \quad \text{on } \partial\Omega_\varepsilon .$$

For a given set of points $\xi_1, \dots, \xi_h$ in Ω and numbers $\lambda_1, \dots, \lambda_h > 0$, we consider the functions

$$U_i = \bar U_{\lambda_i, \xi_i'}, \quad V_i = V_{\lambda_i, \xi_i'}, \quad i = 1, \dots, h. \tag{2.10}$$

Moreover, we write

$$U = \sum_{j=1}^{h} U_j, \qquad V = \sum_{j=1}^{h} V_j . \tag{2.11}$$

Consider further the functions

$$\bar Z_{ij} = \frac{\partial U_i}{\partial \xi_{ij}'}, \; j = 1, \dots, N, \quad \bar Z_{iN+1} = \frac{\partial U_i}{\partial \lambda_i} = (x - \xi_i') \cdot \nabla U_i + (N-2) U_i ,$$

and their respective $H_0^1(\Omega_\varepsilon)$-projections Z_{ij}, namely the unique solutions of

$$\Delta Z_{ij} = \Delta \bar Z_{ij} \quad \text{in } \Omega_\varepsilon$$

$$Z_{ij} = 0 \quad \text{on } \partial\Omega_\varepsilon.$$

We look for a solution v of problem (2.8) of the form

$$v = \sum_{i=1}^{h} V_i + \phi$$

where ϕ is some lower order term. In order to do so, we consider the following auxiliary problem: Find a (small) function ϕ such that for certain constants c_{ij}

$$\begin{cases} \Delta(V+\phi) + (V+\phi)_+^{p+\varepsilon} = \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \phi V_i^{p-1} Z_{ij} = 0 & \text{for all } i,j. \end{cases} \tag{2.12}$$

Here and in what follows we call $p = \frac{N+2}{N-2}$. Our task is then to solve (2.12) and find points ξ and scalars λ such that the associated c_{ij} are all zero, which determines a solution of (2.8).

The first equation in (2.12) can be rewritten in the following form:

$$\Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = -N_\varepsilon(\phi) - R^\varepsilon + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} \quad \text{in } \Omega_\varepsilon \tag{2.13}$$

where

$$N_\varepsilon(\phi) = (V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi, \tag{2.14}$$

and

$$R^\varepsilon = V^{p+\varepsilon} - \sum_{j=1}^{h} U_j^p. \tag{2.15}$$

It is then clear that we need to understand the following linear problem: given $h \in C^\alpha(\bar{\Omega}_\varepsilon)$, find a function ϕ such that

$$\begin{cases} \Delta\phi + (p+\varepsilon)V^{p+\varepsilon-1}\phi = h + \sum_{i,j} c_{ij} V_i^{p-1} Z_{ij} & \text{in } \Omega_\varepsilon \\ \phi = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij}\phi = 0 & \text{for all } i,j \end{cases} \tag{2.16}$$

for certain constants c_{ij}, $i = 1, \dots, h$, $j = 1, \dots, N+1$. In order to solve (boundedly) (2.16), it is convenient to work on functional spaces which depend on the chosen points ξ_i'. Let us consider the norms

$$\|\psi\|_* = \sup_{x\in\Omega_\varepsilon} \left| \left(\sum_{j=1}^{h} (1+|x-\xi_j'|^2)^{-\frac{N-2}{2}} \right)^{-\beta} \psi(x) \right|,$$

where $\beta = 1$ if $N = 3$, $\beta = \frac{2}{N-2}$ if $N > 3$, and

$$\|\psi\|_{**} = \sup_{x\in\Omega_\varepsilon} \Big| \left(\sum_{j=1}^{h} (1 + |x - \xi_j'|^2)^{-\frac{N-2}{2}} \right)^{-\frac{4}{N-2}} \psi(x) \Big|.$$

Let us fix a small number $\delta > 0$. From now on we will restrict ourselves to points $\xi_i' \in \Omega_\varepsilon$, and numbers $\lambda_i > 0$, $i = 1, \dots, h$, such that

$$|\xi_i' - \xi_j'| > \delta\varepsilon^{-\frac{1}{N-2}}, \quad \operatorname{dist}(\xi_i', \partial\Omega_\varepsilon) > \delta\varepsilon^{-\frac{1}{N-2}}, \quad \delta < \lambda_i < \delta^{-1}. \tag{2.17}$$

We have the validity of the following result.

Proposition 2.1. *There are numbers $\varepsilon_0 > 0$, $C > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, points (ξ', Λ) satisfying condition (2.17) and $h \in C^\alpha(\Omega_\varepsilon)$, we have that (2.16) has a unique solution $\phi = L_\varepsilon(h)$. Besides,*

$$\|L_\varepsilon(h)\|_* \le C\|h\|_{**} \tag{2.18}$$

for any $h \in C^\alpha(\Omega_\varepsilon)$.

Once this result is established, we see that Problem (2.12) is equivalent to the fixed point problem

$$\phi = -L_\varepsilon(N_\varepsilon(\phi) + R^\varepsilon).$$

We set even further

$$\psi_\varepsilon = -L_\varepsilon(R^\varepsilon), \quad \tilde\phi = \phi - \psi_\varepsilon \tag{2.19}$$

and rewrite the problem as

$$\tilde\phi = -L_\varepsilon(N_\varepsilon(\tilde\phi + \psi_\varepsilon)) \equiv T_\varepsilon(\tilde\phi).$$

It is not hard to check that $\|R_\varepsilon\|_{**} = O(\varepsilon)$, so that $\|\psi_\varepsilon\|_* = O(\varepsilon)$. From the fact that N_ε has a power behavior greater than one for small values of its argument, it can be shown that the operator T_ε defines a contraction mapping of a certain small ball in the $\| \ \|_*$ norm into itself. More precisely, we have that

$$\|N_\varepsilon(\phi)\|_{**} \le C\|\phi\|_*^{\min\{p\beta+1,2\}};$$

hence T_ε applies a ball with radius $O(\varepsilon^{\min\{p,2\}})$ into itself. Then the result follows from the Banach fixed point theorem applied in such a ball:

Proposition 2.2. *Assume the conditions of Proposition 2.1 are satisfied. Then there is a constant $C > 0$ such that, for all $\varepsilon > 0$ small enough, and all points ξ', λ satisfying (2.17) there exists a unique solution*

$$\phi = \phi(\xi', \lambda) = \tilde\phi + \psi_\varepsilon$$

to problem (2.12) with $\psi_\varepsilon = -L_\varepsilon(R_\varepsilon)$ such that

$$\|\tilde\phi\|_* \le C\varepsilon$$

It can be shown that the map $(\xi', \lambda) \to \tilde{\phi}(\xi', \lambda)$ is of class C^1 for the $\|\cdot\|_*$-norm and

$$\|\nabla_{(\xi',\lambda)}\tilde{\phi}\|_* \leq C\varepsilon. \tag{2.20}$$

We also have that $\|\nabla_{(\xi',\lambda)}\psi_\varepsilon\|_* \leq C\varepsilon$.

3 The energy approach

The functional associated to Problem (2.8) is given by

$$\mathcal{I}_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 - \frac{1}{p+1+\varepsilon}\int_{\Omega_\varepsilon} v^{p+1+\varepsilon}. \tag{3.21}$$

Regular critical points of it correspond exactly to the solutions of (2.8). Let us also observe that given points ξ and scalars λ, ϕ satisfies (2.12) if and only if

$$\mathcal{I}_\varepsilon'(V+\phi)[\eta] = 0 \tag{3.22}$$

for all η, which satisfies the orthogonality relations $\int_{\Omega_\varepsilon} V_i^{p-1} Z_{ij}\eta = 0$. On the other hand, it is readily checked that the scalars c_{ij} in (2.8) are all zero if and only if $\mathcal{I}_\varepsilon'(V+\phi)[Z_{ij}] = 0$ for all i, j. This last relation and (3.22) combined, plus the relationship up to lower order terms between the derivatives of V with respect to ξ and λ and the Z_{ij}'s, plus the smallness of these derivatives in ϕ, yield that the c_{ij}'s are zero in (2.8) if and only if

$$D_{\xi',\lambda}\mathcal{I}_\varepsilon(V+\phi) = 0.$$

Now we recall that we want to consider points

$$\xi_i' = \varepsilon^{-\frac{1}{N-2}}\xi_i, \tag{3.23}$$

with $\xi_i \in \Omega$. It will also be convenient, rather than working with the numbers λ_i, to do so with the Λ_i's given by

$$\lambda_i = (a_N\Lambda_i)^{\frac{1}{N-2}} \tag{3.24}$$

with

$$a_N = \frac{1}{p+1}\frac{\int_{\mathbb{R}^N}\bar{U}^{p+1}}{(\int_{\mathbb{R}^N}\bar{U}^p)^2}.$$

The role of this constant is to provide a simpler form for the expansion of the functional. Thus we search for critical points (ξ, Λ) of

$$I(\xi,\Lambda) \equiv \mathcal{I}_\varepsilon(V+\phi). \tag{3.25}$$

A crucial step is to give an asymptotic estimate for $I(\xi,\Lambda)$. Let us set

$$C_N = \frac{1}{2}\int_{\mathbb{R}^N} |D\bar{U}|^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |\bar{U}|^{p+1}.$$

Since the points ξ_i' are very far away from each other and from the boundary of the expanding domain Ω_ε, and the perturbation ϕ is a lower order term, then at first order

$$I(\xi,\Lambda) \sim \sum_{i=1}^{h} \mathcal{I}_\varepsilon(V_i) \sim hC_N.$$

A precise account of lower order terms in this expansion is given in the result below.

Proposition 3.1. *Let us fix $\delta > 0$. Then there exist positive constants γ_N and w_N such that the following expansion holds.*

$$I(\xi,\Lambda) = hC_N + \varepsilon\big[\gamma_N + w_N\Psi(\xi,\Lambda) + o(1)\big], \tag{3.26}$$

where the quantity $o(1)$ tends to zero as $\varepsilon \to 0$ uniformly in the C^1-sense in the variables (ξ,Λ) for which ξ' given by (3.23) and λ given by (3.24). satisfy constraints (2.17). Here

$$\Psi(\xi,\Lambda) = \frac{1}{2}\{\sum_{j=1}^{h} H(\xi_j,\xi_j)\Lambda_j^2 - 2\sum_{i<j} G(\xi_i,\xi_j)\Lambda_i\Lambda_j\} + \log(\Lambda_1\cdots\Lambda_h), \tag{3.27}$$

The estimate given by the last proposition tells us that it is sufficient to find a critical point for Ψ which is *stable* under small C^1-perturbations. We construct such a critical point through a min-max characterization in the following section. We will sketch how to do so only for the case of a two-spike, under the assumption of Theorem 1.2. In that case the function Ψ becomes

$$\Psi(\xi,\Lambda) = \frac{1}{2}\{\sum_{j=1}^{2} H(\xi_j,\xi_j)\Lambda_j^2 - 2G(\xi_1,\xi_2)\Lambda_1\Lambda_2\} + \log(\Lambda_1\Lambda_2). \tag{3.28}$$

4 The min-max

In this section we set up a min-max scheme to find a critical point of the function Ψ given by (3.28). This scheme is then used to find a critical point for the reduced functional I (see (3.25), (3.26)). We recall that the function Ψ is well defined in $(\Omega \times \Omega \setminus \Delta) \times \mathbb{R}_+^2$, where Δ is the diagonal

$\Delta = \{(\xi_1, \xi_2) \in \Omega \times \Omega \,/\, \xi_1 = \xi_2\}$. In order to avoid the singularity of Ψ over Δ, we let $M > 0$ be a very large number, and we define

$$G_M(\xi) = \begin{cases} G(\xi) & \text{if } G(\xi) \le M \\ M & \text{if } G(\xi) > M, \end{cases} \tag{4.29}$$

and we consider $\Psi_{M,\rho} : \Omega_\rho \times \Omega_\rho \times \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$\Psi_{M,\rho}(\xi, \Lambda) = \Psi(\xi, \Lambda) - G_M(\xi)\Lambda_1\Lambda_2 + G(\xi)\Lambda_1\Lambda_2, \tag{4.30}$$

where $\rho > 0$ and $\Omega_\rho = \{\xi \in \Omega \,/\, \mathrm{dist}(\xi, \Omega) > \rho\}$. We will specify ρ later, and for notational convenience we will simply write $\Psi_{M,\rho} = \Psi$ and $D = \Omega_\rho \times \Omega_\rho \times \mathbb{R}^2_+$. We consider a further restriction $D_\varphi = \{(\xi, \Lambda) \in D \,/\, \varphi(\xi) < -\rho_0\}$, where $\rho_0 = \min\{\frac{1}{2}\exp(-2C_0 - 1), -\frac{1}{2}\max\{\varphi \,/\, \text{in } \mathcal{M}^2\}\}$, with

$$C_0 = \sup_{(\xi,\sigma)\in\mathcal{M}^2\times I_0} \Psi(\xi, \sigma).$$

With this choice certainly $\mathcal{M}^2 \times \mathbb{R}^2_+ \subset D_\varphi$.

Aiming to define the min-max class, for every $\xi \in \mathcal{M}^2$ we let $d(\xi) = (d_1(\xi), d_2(\xi)) \in S^1 \subset \mathbb{R}^2$ be the negative direction of the quadratic form defining Ψ. Such a direction exists since, by hypothesis of Theorem 1.2, the function φ is negative over $\mathcal{M}^2$. We easily see that there is a constant $c > 0$ such that $c < d_1(\xi)d_2(\xi) < c^{-1}$ for all $\xi \in \mathcal{M}^2$.

Next we let Γ be the class of continuous functions $\gamma : \mathcal{M}^2 \times I_0 \times [0,1] \to D_\varphi$, such that

1. $\gamma(\xi, \sigma_0, t) = (\xi, \sigma_0 d(\xi))$, and $\gamma(\xi, \sigma_0^{-1}, t) = (\xi, \sigma_0^{-1} d(\xi))$ for all $\xi \in \mathcal{M}^2$, $t \in [0,1]$, and
2. $\gamma(\xi, \sigma, 0) = (\xi, \sigma d(\xi))$ for all $(\xi, \sigma) \in \mathcal{M}^2 \times I_0$,

where $I_0 = [\sigma_0, \sigma_0^{-1}]$ with σ_0 is a small number to be chosen later. Then we define the min-max value

$$c(\Omega) = \inf_{\gamma\in\Gamma} \sup_{(\xi,\sigma)\in\mathcal{M}^2\times I_0} \Psi(\gamma(\xi, \sigma, 1)) \tag{4.31}$$

and we will prove in what follows that $c(\Omega)$ is a critical value of Ψ. For this purpose we will first prove an intersection lemma based on a topological continuation result of Fitzpatrick, Massabó and Pejsachowicz [8]. For every $(\xi, \sigma, t) \in \mathcal{M}^2 \times I_0 \times [0,1]$, we denote $\gamma(\xi, \sigma, t) = (\tilde{\xi}(\xi, \sigma, t), \tilde{\Lambda}(\xi, \sigma, t)) \in D_\varphi$, and we define $\mathcal{S} = \{(\xi, \sigma) \in \mathcal{M}^2 \times I_0 \,/\, \tilde{\Lambda}_1(\xi, \sigma, 1) \cdot \tilde{\Lambda}_2(\xi, \sigma, 1) = 1\}$,; then we have

Lemma 4.1. *For every open neighborhood V of $\mathcal{S}$ in $\mathcal{M}^2 \times I_0$, the map $g^* : H^*(\mathcal{M}^2) \longrightarrow H^*(V)$, induced by the projection $g : V \to \mathcal{M}^2$, is a monomorphism.*

As a consequence, we have

Proposition 4.2. *There is a constant K, independent of σ_0, so that*

$$\sup_{(\xi,\sigma)\in\mathcal{M}^2\times I_0} \Psi(\gamma(\xi,\sigma,1)) \geq -K \quad \textit{for all } \gamma\in\Gamma.$$

Proof. Since Ω is smooth, there is a $\delta_0 > 0$ such that if $\xi_1, \xi_2 \in \Omega_\rho$ and $|\xi_1 - \xi_2| < \delta_0$, then the line segment $[\xi_1, \xi_2] \subset \Omega$. Then we let $K > 0$ so that $G(\xi_1, \xi_2) \geq K$ implies $|\xi_1 - \xi_2| < \delta_0$.

Assume, for contradiction, that for certain $\gamma \in \Gamma$

$$\Psi(\gamma(\xi,\sigma,1)) \leq -K \qquad \text{for all} \quad (\xi,\sigma) \in \mathcal{M}^2 \times I_0.$$

This implies that, for a small neighborhood V of $\mathcal{S}$ in $\mathcal{M}^2 \times I_0$, we have

$$G(\tilde{\xi}(\xi,\sigma,1)) \geq K \quad \text{for all} \quad (\xi,\sigma) \in V. \tag{4.32}$$

Let $D_0 = \Omega \times \Omega \times \mathbb{R}^2_+$ and $\gamma_1 = \gamma(\cdot, 1)$. Consider the inclusion $i_2 : \gamma_1(V) \to D_0$ and the maps $p : \gamma_1(V) \to \Omega \times \mathbb{R}^2_+$ and $\delta : \Omega \times \mathbb{R}^2_+ \to D_0$ defined as $p(\xi_1, \xi_2, \Lambda) = (\xi_1, \Lambda)$ and $\delta(\xi_1, \Lambda) = (\xi_1, \xi_1, \Lambda)$. From (4.32) we find that the function $h : \gamma_1(V) \times [0,1] \to D_0$ defined as $h(\xi_1, \xi_2, \Lambda, t) = (\xi_1, \xi_2 + t(\xi_1 - \xi_2), \Lambda)$ is a homotopy between i_2 and $\delta \circ p$. Let d be the integer given in Theorem 1.2 and consider the following commutative diagram:

$$\begin{array}{ccc} H^{2d}(\mathcal{M}^2 \times I_0) & \xleftarrow{\gamma_1^*} & H^{2d}(D_0) \\ i_1^*\downarrow & & i_2^*\downarrow \\ H^{2d}(V) & \xleftarrow{\gamma_2^*} & H^{2d}(\gamma_1(V)), \end{array}$$

where i_1 is an inclusion map and $\gamma_2 = \gamma_1|_V$. From the hypothesis of Theorem 1.2 we find $u \in H^d(\mathcal{M})$ and $v \in H^d(\Omega)$ are nontrivial elements such that $\iota^*(v) = u$. If $\hat{v} \times \hat{v} \in H^{2d}(D_0)$ is the corresponding element, then by the homotopy axiom and Lemma 4.1 we have $i_1^* \circ \gamma_1^*(\hat{v} \times \hat{v}) \neq 0$. On the other hand we see that $\delta^*(\hat{v} \times \hat{v}) = \hat{v} \smile \hat{v} \in H^{2d}(\Omega \times \mathbb{R}^2_+)$ is zero, either because d is odd or because $H^{2d}(\Omega) = 0$. In both cases we have then $\gamma_2^* \circ i_2^*(\hat{v} \times \hat{v}) = 0$, providing a contradiction. □

In proving that $c(\Omega)$ is a critical value for Ψ, the next key step is to show that Ψ satisfies the Palais–Smale (P.S.) condition in D_φ. We do this now.

Proposition 4.3. *The function Ψ satisfies the P.S. condition in D_φ at level* $c(\Omega)$.

Proof. The following preliminary fact fixes the value of the parameter $\rho > 0$: Given $c \in \mathbb{R}$ there exists $\rho > 0$ sufficiently small so that if $(\bar{\xi}_1, \bar{\xi}_2) \in \partial(\Omega_\rho \times \Omega_\rho)$ is such that $\varphi(\bar{\xi}_1, \bar{\xi}_2) = c$, then there is a vector τ, tangent to $\partial(\Omega_\rho \times \Omega_\rho)$ at the point $(\bar{\xi}_1, \bar{\xi}_2)$, so that

$$\nabla\varphi(\bar{\xi}_1, \bar{\xi}_2) \cdot \tau \neq 0. \tag{4.33}$$

This choice of ρ allows us to prove a related property for Ψ. That is, given a sequence $\{(\xi_n, \Lambda_n)\} \subset D_\varphi$ such that $(\xi_n, \Lambda_n) \to (\bar{\xi}, \bar{\Lambda}) \in \partial D_\varphi$ and $\Psi(\xi_n, \Lambda_n) \to c(\Omega)$, there is a vector T, tangent to ∂D_φ at $(\bar{\xi}, \bar{\Lambda})$, such that

$$\nabla\Psi(\bar{\xi}, \bar{\Lambda}) \cdot T \neq 0. \tag{4.34}$$

In order to prove (4.34) we first observe that if $\Lambda_n \to \bar{\Lambda} \in \partial\mathbb{R}^2_+$ then $\Psi(\xi_n, \Lambda_n) \to -\infty$. Thus we can assume that $\bar{\Lambda} \in \mathbb{R}^2_+$, $\bar{\xi} \in \bar{\Omega}_\rho \times \bar{\Omega}_\rho$ and $\varphi(\bar{\xi}) \leq -\rho_0$. Two cases arise: if $\nabla_\Lambda\Psi(\bar{\xi}, \bar{\Lambda}) \neq 0$, then T can be chosen parallel to $\nabla_\Lambda\Psi(\bar{\xi}, \bar{\Lambda})$. Otherwise, when $\nabla_\Lambda\Psi(\bar{\xi}, \bar{\Lambda}) = 0$ we have that $\bar{\Lambda}$ satisfies

$$\bar{\Lambda}_1^2 = -\frac{H(\bar{\xi}_2, \bar{\xi}_2)^{1/2}}{H(\bar{\xi}_1, \bar{\xi}_1)^{1/2}\varphi(\bar{\xi}_1, \bar{\xi}_2)}, \quad \bar{\Lambda}_2^2 = -\frac{H(\bar{\xi}_1, \bar{\xi}_1)^{1/2}}{H(\bar{\xi}_2, \bar{\xi}_2)^{1/2}\varphi(\bar{\xi}_1, \bar{\xi}_2)},$$

and $\bar{\xi}$ satisfies $\varphi(\bar{\xi}) < 0$. Substituting back in Ψ, we get

$$\Psi(\bar{\xi}_1, \bar{\xi}_2, \bar{\Lambda}_1, \bar{\Lambda}_2) = -\frac{1}{2} + \frac{1}{2}\log\frac{1}{|\varphi(\bar{\xi}_1, \bar{\xi}_2)|}$$

and then $\varphi(\bar{\xi}) = -\exp(-2c(\Omega) - 1) \leq -2\rho_0 < -\rho_0$. Thus $\bar{\xi} \in \partial(\Omega_\rho \times \Omega_\rho)$ and the application of (4.33) completes the proof of (4.34). Now we can define an appropriate negative gradient flow that will remain in D_φ at level $c(\Omega)$.

To finish, we mention that the Palais–Smale condition indeed holds: if $\{(\xi_n, \Lambda_n)\} \subset D_\varphi$ satisfies $\Psi(\xi_n, \Lambda_n) \to c(\Omega)$ and $\nabla\Psi(\xi_n, \Lambda_n) \to 0$, then $\{(\xi_n, \Lambda_n)\}$ has a subsequence converging to some $(\bar{\xi}, \bar{\Lambda}) \in D$. In fact, it can be shown that the sequence Λ_n remains bounded. Finally we conclude using (4.34). □

In view of Proposition 4.1 and 4.2 we have that the number $c(\Omega)$ given in (4.31) is a critical value for Ψ in D. This min-max setting does survive a small C^1-perturbation of Ψ in the considered region, yielding a critical point of the functional I (see (3.25), (3.26)) as well, as required.

We finish this note by mentioning that extra care needs to be taken in the construction of multiple pairs of spikes as in Theorem 1.1. We need to work on a region for the reduced functional which indeed *isolates* pairs of spikes associated to distinct holes. This is possible provided that μ is chosen sufficiently small: in such a case, interactions of far away spikes become negligible, and the functional Ψ basically decouples into the sum of several functionals of the form φ.

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Departamento de Ingeniería Matemática
Centro de Modelamiento Matemático
Universidad de Chile, Casilla 170 Correo 3
Santiago, CHILE
email: delpino@dim.uchile.cl, pfelmer@dim.uchile.cl

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino, ITALY
email: musso@calvino.polito.it

A Semilinear Elliptic Equation on $\mathbf{R}^N$ with Unbounded Coefficients

P. Sintzoff and M. Willem

Dedicated to Antonio Marino

1 Introduction

We study the existence of localized solutions of the semilinear elliptic equation

$$-\Delta u + a(x)u = f(x,u)$$

on $\mathbb{R}^N$. Many papers deal with the case when a is "large" at infinity and f is subcritical : for some $c > 0$ and $p < \frac{2N}{(N-2)}$,

$$|f(x,u)| \leq c\left(1 + |u|^{p-1}\right),$$

see, e.g., [1], [2], [4], [5].

It was recently observed by Sirakov in his thesis [7] that the case when f is unbounded in x is rather delicate. Consider the model problem

$$\begin{cases} -\Delta u + |x|^a u = |x|^b u^{p-1}, \\ u > 0, u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $a \geq 0$, $b \geq 0$, $N \geq 3$. Sirakov proves the existence of a solution for

$$2 < p < p^{\#} = \frac{2N}{N-2} - \frac{4b}{a(N-2)}.$$

By the Derrick–Pohozaev identity there is no solution for

$$\tilde{p} = \frac{2N}{N-2} + \frac{2b}{N-2} < p.$$

Hence $2^* = \frac{2N}{N-2}$ lies in a gap between $p^{\#}$ and $\tilde{p}$. In [3], Ding and Ni obtain the existence of a solution of (1.1) with

$$a = 0, p < 2^*, 2b < (N-1)(p-2).$$

Our aim is to solve problem (1.1). Our approach is applicable in a more general setting, but, for simplicity, we consider only the model problem.

In section 2, we prove the existence of a radial least energy solution of (1.1) when

$$2 < p < \bar{p}, 2b - \left(1 + \frac{p}{2}\right) a < (N-1)(p-2). \tag{1.2}$$

In section 3, we consider the existence of nonradial solutions. Finally, in section 4, we consider necessary conditions for the existence of a solution of (1.1).

2 Radial solution

We denote by $H_r^1(\mathbb{R}^N)$ the space of radially symmetric functions in

$$H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}.$$

We denote by $H_{r,a}^1(\mathbb{R}^N)$ the space of radially symmetric functions in

$$H_a^1(\mathbb{R}^N) = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^a u^2 dx < \infty\right\}.$$

We denote by $\mathcal{D}_r^{1,2}(\mathbb{R}^N)$ the space of radially symmetric functions in

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\right\}.$$

The following radial lemma is an improvement of the Strauss radial lemma (see [8]).

Lemma 2.1. *If $N \geq 2$, there exists $A_N > 0$ such that, for every $u \in H_{r,a}^1(\mathbb{R}^N)$, $u \in C(\mathbb{R}^N \backslash \{0\})$ and*

$$|x|^{\frac{N-1}{2}+\frac{a}{4}} |u(x)| \leq A_N \left(\int_{\mathbb{R}^N} |x|^a |u|^2 dx\right)^{\frac{1}{4}} |\nabla u|_2^{\frac{1}{2}}.$$

Proof. By density, it suffices to consider $u \in H_{r,a}^1(\mathbb{R}^N) \cap \mathcal{D}(\mathbb{R}^N)$. Since

$$2u\frac{du}{dr} r^{\frac{a}{2}} r^{N-1} \leq \frac{d}{dr}\left(u^2 r^{\frac{a}{2}} r^{N-1}\right),$$

we obtain

$$\begin{aligned} r^{N-1} r^{\frac{a}{2}} u^2(r) &\leq 2\int_r^\infty |u| \left|\frac{du}{dr}\right| s^{N-1} s^{\frac{a}{2}} ds \\ &\leq A_N \left(\int s^a u^2\right)^{\frac{1}{2}} \left(\int |\nabla u|^2\right)^{\frac{1}{2}}. \end{aligned}$$

□

The following inequality is due to Rother [6].

Lemma 2.2. *If* $N \geq 3$, $1 \leq p < \infty$, $p = \frac{2N}{N-2} + \frac{2c}{N-2}$, *and there exists* $B_{N,c} > 0$ *such that, for every* $u \in \mathcal{D}^{1,2}_r(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N} |x|^c |u|^p dx\right)^{\frac{2}{p}} \leq B_{N,c} \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

We shall prove that, under some conditions,

$$m = m(a,b,p) = \inf_{\substack{u \in H^1_{r,a}(\mathbb{R}^N) \\ \int_{\mathbb{R}^N} |x|^b |u|^p dx = 1}} \int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a u^2 dx$$

is achieved. Then by the Lagrange multiplier rule, the symmetric criticality principle and the maximum principle, we obtain a solution of

$$\begin{cases} -\Delta v + |x|^a v = \lambda |x|^b v^{p-1}, \\ v \in H^1_{r,a}(\mathbb{R}^N), v > 0. \end{cases}$$

Hence $u = \lambda^{\frac{1}{p-2}} v$ is a solution of (1.1).

Theorem 2.3. *If* $a \geq 0$, $b \geq 0$, $N \geq 3$ *and*

$$2 < p < \bar{p} = \frac{2N}{N-2} + \frac{2b}{N-2}, 2b - \left(1 + \frac{p}{2}\right) a < (N-1)(p-2),$$

then $m(a,b,p)$ *is achieved and problem (1.1) has a radial solution.*

Proof. Let $(u_n) \subset H^1_{r,a}(\mathbb{R}^N)$ be a minimizing sequence for $m = m(a,b,p)$:

$$\int_{\mathbb{R}^N} |x|^b |u_n|^p dx = 1, \int_{\mathbb{R}^N} |\nabla u_n|^2 + |x|^a u_n^2 dx \to m.$$

By going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H^1_{r,a}(\mathbb{R}^N)$. Hence, by weak lower semicontinuity, we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a u^2 dx \leq m,$$

$$\int_{\mathbb{R}^N} |x|^b |u|^p dx \leq 1.$$

If c is defined by $p = \frac{2N}{N-2} + \frac{2c}{N-2}$, then $c < b$ and it follows from lemma 2.2 that

$$\begin{aligned} \int_{|x| \leq \epsilon} |x|^b |u_n|^p dx &\leq \epsilon^{b-c} \int_{\mathbb{R}^N} |x|^c |u_n|^p dx \\ &\leq c_1 \epsilon^{b-c}, \end{aligned}$$

since (u_n) is bounded in $H^1_{r,a}(\mathbb{R}^N)$. We deduce from Lemma 2.1 that

$$\begin{aligned}\int_{|x|\geq\frac{1}{\epsilon}} |x|^b|u_n|^p dx &= \int_{|x|\geq\frac{1}{\epsilon}} |x|^{b-a}|u_n|^{p-2}|x|^a u_n^2 dx \\ &\leq \left(\frac{1}{\epsilon}\right)^{b-a-(p-2)\left(\frac{(N-1)}{2}+\frac{a}{4}\right)} c_2 \int_{\mathbb{R}^N} |x|^a u_n^2 dx \\ &\leq c_3\epsilon^{a\left(\frac{1}{2}+\frac{p}{4}\right)-b+\frac{(p-2)(N-1)}{2}}.\end{aligned}$$

It follows from the two preceding inequalities that, for every $t < 1$, there exists $\epsilon > 0$ such that, for every n,

$$\int_{\epsilon\leq|x|\leq\frac{1}{\epsilon}} |x|^b|u_n|^p dx \geq t.$$

By the Rellich theorem and Lemma 2.1,

$$1 \geq \int_{\mathbb{R}^N} |x|^b|u|^p dx \geq \int_{\epsilon\leq|x|\leq\frac{1}{\epsilon}} |x|^b|u|^p dx \geq t.$$

Finally $\int_{\mathbb{R}^N} |x|^b|u|^p dx = 1$ and $m = m(a, b, p)$ is achieved at u. □

We consider now

$$M = M(a,b,p) = \inf_{\substack{u\in H^1_a(\mathbb{R}^N)\\ \int_{\mathbb{R}^N}|x|^b|u|^p dx=1}} \int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a u^2 dx.$$

It is clear that $M \leq m$. We shall prove that M is achieved under the condition used by Sirakov.

Theorem 2.4. *If $a > 0$, $b \geq 0$, $N \geq 3$ and*

$$2 < p < p^\# = \frac{2N}{N-2} - \frac{4b}{a(N-2)},$$

then $M(a,b,p)$ is achieved.

Proof. Let $(u_n) \subset H^1_a(\mathbb{R}^N)$ be a minimizing sequence for $M = M(a,b,p)$:

$$\int_{\mathbb{R}^N} |x|^b|u_n|^p dx = 1, \int_{\mathbb{R}^N} |\nabla u_n|^2 + |x|^a u_n^2 dx \to M.$$

By going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u$ in $H^1_a(\mathbb{R}^N)$. Hence, by weak lower semi-continuity we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a u^2 dx \leq M,$$

$$\int_{\mathbb{R}^N} |x|^b|u|^p dx \leq 1.$$

If c is defined by $p = \frac{2N}{N-2} - \frac{4c}{a(N-2)}$, then $c > b$ and

$$r = \frac{a}{c}, s = \frac{a2^*}{ap - 2c}$$

are conjugate. It follows from Hölder and Sobolev inequalities that

$$\begin{aligned}\int_{|x|\geq\frac{1}{\epsilon}} |x|^b|u_n|^p dx &\leq \left(\frac{1}{\epsilon}\right)^{b-c} \int_{\mathbf{R}^N} |x|^c|u_n|^p dx \\ &\leq \epsilon^{c-b}\left(\int_{\mathbf{R}^N} |x|^a u_n^2 dx\right)^{\frac{1}{r}} \left(\int_{\mathbf{R}^N} |u_n|^{2^*} dx\right)^{\frac{1}{s}} \\ &\leq c_4\epsilon^{c-b}.\end{aligned}$$

As in Theorem 2.3, for every $t < 1$, there exists $\epsilon > 0$ such that, for every n,

$$\int_{|x|\leq\frac{1}{\epsilon}} |x|^b|u_n|^p dx \geq t.$$

By the Rellich theorem, since $p < 2^*$,

$$1 \geq \int_{\mathbf{R}^N} |x|^b|u|^p dx \geq \int_{|x|\leq\frac{1}{\epsilon}} |x|^b|u|^p dx \geq t.$$

Hence $\int_{\mathbf{R}^N} |x|^b|u|^p dx = 1$ and $M = M(a, b, p)$ is achieved at u. □

3 Nonradial solutions

In this section, we use the preceding results in order to construct nonradial solutions of

$$\begin{cases} -\Delta u + |x|^a u = |x|^b u^{p-1}, \\ u > 0 \quad \text{in } B(0, R), \\ u = 0 \quad \text{on } \partial B(0, R). \end{cases} \tag{3.3}$$

Theorem 3.1. *If $a \geq 0$, $b \geq 0$, $N \geq 3$ and*

$$2 < p < \frac{2N}{N-2}, 2b - \left(1 + \frac{p}{2}\right) a < (N-1)(p-2), ap < 2b,$$

then, for every R large enough, problem (3.3) has a radial and a nonradial solution.

Proof. By Theorem 2.3, $m(a,b,p)$ is positive. Since $2b > ap$, it is easy to verify that $M(a,b,p) = 0$. Let us define

$$M(a,b,p,R) = \inf_{\substack{u \in H_0^1(B(0,R)) \\ \int_{B(0,R)} |x|^b |u|^p dx = 1}} \int_{B(0,R)} |\nabla u|^2 + |x|^a u^2 dx$$

$$m(a,b,p,R) = \inf_{\substack{u \in H_{0,rad}^1(B(0,R)) \\ \int_{B(0,R)} |x|^b |u|^p dx = 1}} \int_{B(0,R)} |\nabla u|^2 + |x|^a u^2 dx\,.$$

It is clear that, for every $R > 0$, $M(a,b,p,R)$ and $m(a,b,p,R)$ are achieved and

$$\lim_{R\to\infty} M(a,b,p,R) = M(a,b,p) = 0,$$
$$\lim_{R\to\infty} m(a,b,p,R) = m(a,b,p) > 0.$$

Using the Lagrange multiplier rule, the symmetric criticality principle and the maximum principle, we obtain, for every R large enough, a radial and a nonradial solution of (3.3). □

4 Necessary conditions

We derive from the Derrick–Pohozaev identity some necessary conditions for the existence of a solution of problem (1.1).

Theorem 4.1. *If*

$$\tilde{p} = \frac{2N}{N-2} + \frac{2b}{N-2} \le p$$

or

$$\frac{N+a}{2} \le \frac{N+b}{p},$$

then there is no solution for problem (1.1).

Proof. The Derrick–Pohozaev identity is verified by any solution u of (1.1) :

$$\frac{N-2}{2}\int_{\mathbf{R}^N} |\nabla u|^2 dx + \frac{N+a}{2}\int_{\mathbf{R}^N} |x|^a u^2 dx - \frac{N+b}{p}\int_{\mathbf{R}^N} |x|^b u^p dx = 0.$$

On the other hand, multiplying (1.1) by u and integrating, we see that

$$\int_{\mathbb{R}^N} |x|^b u^p dx = \int_{\mathbb{R}^N} |\nabla u|^2 + |x|^a u^2 dx.$$

Hence we obtain

$$\left(\frac{N-2}{2} - \frac{N+b}{p}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \left(\frac{N+a}{2} - \frac{N+b}{p}\right) \int_{\mathbb{R}^N} |x|^a u^2 dx = 0. \tag{4.4}$$

So, if u is a solution of problem (1.1), we must have

$$\frac{N-2}{2} - \frac{N+b}{p} < 0, \frac{N+a}{2} - \frac{N+b}{p} > 0.$$

□

Remarks

1. The first assumption of Theorem 2.3, $2 < p < \tilde{p}$, is optimal.
2. The second assumption of Theorem 2.3,

$$2b - \left(1 + \frac{p}{2}\right) a < (N-1)(p-2),$$

implies that

$$\frac{N+b}{p} < \frac{N+a}{2}.$$

3. The following example shows that $m(a, b, p) > 0$ implies

$$2b - ap \leq (N-1)(p-2).$$

This example is inspired by example 5.9 in [3].

Example 4.2. *We assume that*

$$2b - ap > (N-1)(p-2).$$

For each integer $n > 0$, we define $r = |x|$ and

$$u_n(r) = \begin{cases} \frac{1}{n^{\frac{N-1}{2} + \frac{a}{2}}} e^{\frac{r-n}{2}}, & 0 \leq r < n, \\ \frac{1}{r^{\frac{N-1}{2} + \frac{a}{2}}}, & n \leq r \leq n+1, \\ \frac{1}{(n+1)^{\frac{N-1}{2} + \frac{a}{2}}} e^{\frac{n+1-r}{2}}, & n+1 < r. \end{cases}$$

These (u_n) are in $H^1_{r,a}(\mathbb{R}^N)$. Moreover, we verify that

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^b |u_n|^p dx &\geq \int_n^{n+1} r^{N-1} r^b r^{-(\frac{N-1}{2}+\frac{a}{2})p} dr \\ &\geq n^{\frac{1}{2}(2b-ap-(N-1)(p-2))} \\ &\to +\infty \quad \text{as } n \to \infty.\end{aligned}$$

On the other hand, it is easy to verify that there exists a constant c_5 such that, for every n,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + |x|^a u_n^2 dx \leq c_5.$$

Hence

$$\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 + |x|^a u_n^2 dx}{\left(\int_{\mathbb{R}^N} |x|^b |u_n|^p dx\right)^{\frac{2}{p}}} \to 0 \text{ as } n \to \infty,$$

and thus $m(a,b,p) = 0$ when $2b - ap > (N-1)(p-2)$.

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Département de Mathématiques
Université Catholique
Louvain-La-Neuve
Belgium
email: willem@amm.ucl.ac.be

Traveling Waves in Natural Systems

R. E. L. Turner

Dedicated to Antonio Marino

1 Introduction

Waves that propagate along a real coordinate axis without change of form arise in a myriad of natural settings, from meteorology, oceanography, combustion theory, to name several areas in the physical sciences, and from biology, physiology, and population dynamics, to name several from the biological and health sciences. The phenomena usually involve some local "excitability" or change of state which can influence a neighboring region to pass to the altered state, thus provoking a chain type reaction. In many situations one can show that there is a set of conditions which permit the wave to move in one direction without change of form, and this is the type of wave we treat here. Our choice of examples is motivated by personal preference and experience with types in the realm of fluid waves and waves in physiology. We begin with a type which occurs in stratified fluids and then pass to a description of the classical study of Hodgkin and Huxley on propagating action potentials as a lead-in to our work on waves of contraction producing locomotion in the nematode *Ascaris*.

2 Fluid Waves

A common thread in many treatments of traveling waves is an underlying geometry which has the form of a domain $\Omega \times \mathbf{R}$ where Ω is a bounded domain in some Euclidean space and $\mathbf{R}$ is the real line. For a wide class of problems the solution of a differential equation, say $W(t, x, \omega)$, depending on time t, $x \in \mathbf{R}$, and $\omega \in \Omega$, can be sought in the form $U(x - ct, \omega)$, reducing the study to an equation for $U(\eta, \omega)$. This reduction is widely used in the study of water waves and we use this paradigm here. The resulting problem typically becomes an elliptic problem on the domain $\Omega \times \mathbf{R}$ where η lies in $\mathbf{R}$. Methods for studying the resulting elliptic problem include global bifurcation, variational methods, and, for solutions which are small in amplitude, the center-manifold approach introduced by Kirchgässner

([7]) and developed by him with his co-researchers. Our approach is in this latter line, but the machinery is different from that of Mielke ([11]) who worked closely with Kirchgässner. We describe the physical problem in the moving coordinates and then return to a further explanation of the point of view taken in the center-manifold approach.

The problem of a density stratified, non-diffusive fluid confined between horizontal boundaries can be formulated in Eulerian coordinates. One obtains a semilinear equation which reduces to the Laplace equation in any region of constant density. Substantial analytical difficulties arise, however, if this equation is used for the case of discontinuous density, for then interior free boundaries arise.

An alternative is to use semi-Lagrangian coordinates, a horizontal spatial coordinate and a streamline coordinate, as the two independent coordinates. The difficulty of the free boundary disappears, but a quasilinear equation is part of the trade. The streamline coordinate can be normalized to coincide with the height function as the horizontal coordinate x approaches $-\infty$ (we now use x in place of η). Thus all points on the streamline, which approaches height y at $-\infty$, have the same coordinate label y. The dependent coordinate is $w = w(x, y)$, the vertical deviation of the streamline with label y at position x, from its height in the underlying trivial flow. The flow region is taken to be $T \equiv \mathbf{R} \times I$ where $I = [-h, 1-h]$ and with our normalization T is also the domain of the new independent coordinates. We shall assume that the channel is occupied by two fluids, one of normalized density unity, corresponding to stream labels $-h < y < 0$, and one of density $\rho < 1$ in the coordinate region $0 < y < 1 - h$. In the new coordinates a flow connects to a parallel flow of speed c as $x \to -\infty$, provided $w \to 0$ as $x \to -\infty$. For any speed c the whole flow region can be the same trivial, parallel flow and satisfy the flow equations. In this case $w \equiv 0$. The problem is to find a speed c and a function $w \not\equiv 0$ corresponding to a nontrivial flow.

Some definitions are needed. Let I denote a bounded open set in $\mathbf{R}^2$ or a point and let $T = \mathbf{R} \times I$. When I is a point we merely identify T with $\mathbf{R}$. Let $C^j(T)$ denote the space of continuous real-valued functions on T with continuous derivatives through order j. For h in $C^0(T)$ and $0 < \alpha \leq 1$, the Hölder constant of h at x is measured by

$$(H_\alpha h)(x) = \sup_{\substack{|(x,y)-(x',y')|<1 \\ y \in I, (x',y') \in T}} \frac{|h(x,y) - h(x',y'|}{|(x,y) - (x',y')|^\alpha}, \tag{2.1}$$

where $|\quad|$ denotes the Euclidean distance in T. Using standard notation for partial derivatives of functions, for each real number μ we define

$$|h|_{j,\alpha,\mu} = \sum_{|\beta|\le j} \sup_{x\in\mathbf{R},\ y\in I} e^{-\mu|x|}|\partial^\beta h(x,y)| + \sum_{|\beta|=j} \sup_{x\in\mathbf{R}} e^{-\mu|x|}(H_\alpha \partial^\beta h)(x) \tag{2.2}$$

and let

$$C_\mu^{j,\alpha}(T) = \{h \in C^j(T) : |h|_{j,\alpha,\mu} < \infty\}. \tag{2.3}$$

The use of exponential weights in x appears to be crucial in making the center-manifold approach work.

The quasilinear elliptic problem for w contains an eigenvalue parameter $\lambda = g/c^2$ where g is the gravity constant. The lowest eigenvalue is

$$\lambda_d = \frac{1}{1-\rho}\left(\frac{1}{h} + \frac{\rho}{1-h}\right), \tag{2.4}$$

and one looks for a nontrivial solution in the neighborhood of $w \equiv 0$, $\lambda = \lambda_d$. Letting $p = \lambda_d - \lambda$ and expanding in linear and higher order terms in w, one can arrive at the system

$$\Delta w^\pm = div(g_1^\pm(\nabla w), g_2^\pm(\nabla w)) \quad \text{in } T^\pm, \tag{2.5}$$

$$\rho({w_y}^+ - \lambda_d w^+) - ({w_y}^- - \lambda_d w^-) = g_3(\nabla w, w, p) \quad \text{on } y = 0, \tag{2.6}$$

$$w^-(x,-h) = w^+(x, 1-h) = 0, \quad x \in \mathbf{R}. \tag{2.7}$$

Here $T^- = \mathbf{R} \times (-h, 0)$, $T^+ = \mathbf{R} \times (0, 1-h)$, and $\pm$ are used to denote values or limits taken within $T^\pm$. The first equation describes the fluid flow. The second gives pressure continuity at the internal fluid boundary, and the third merely expresses that no flow penetrates the walls. The terms $g_1^\pm, g_2^\pm$, and g_3 are higher order than linear w and $\nabla w = (w_x, w_y)$.

Corresponding to the eigenvalue λ_d there is an eigenfunction, $t(y)$ with no x dependence. Suppose we seek a solution w of the system (2.5)–(2.7) in the form

$$w = Q(x)t(y) + R(x,y); \tag{2.8}$$

that is, a "separation of variables" term plus a "remainder." Then the system (2.5)–(2.7) is a nonlinear problem with linear expressions in w on the left and nonlinear expressions in w on the right. As is common, to study the nonlinear problem we look first at a linear one in which the terms on the right side of (2.5)–(2.6), abbreviated as $G \equiv (g_1^+, g_1^-, g_2^+, g_2^-, g_3)$, are taken to be *given* functions of of x, y, and p. The resulting linear problem has a unique solution in the form (2.8) with (Q, Q', R) satisfying

$$\begin{aligned} Q(x) &= \xi_1 + \textstyle\int_0^x Q', \quad x \in \mathbf{R}, \\ Q'(x) &= \xi_2 + L_1 G|_0^x + \textstyle\int_0^x L_2 G, \quad x \in \mathbf{R}, \\ R(x,y) &= (L_3 G)(x,y), \quad (x,y) \in \overline{T} \end{aligned} \tag{2.9}$$

for suitable linear operators L_1, L_2, L_3 acting on weighted Hölder spaces defined on $T^{\pm}$ (cf. [1],[2]).

When G on the right of (2.5)–(2.6) is replaced by the original expressions $g_1^{\pm}(\nabla w^{\pm}, w)$, etc. from (2.5) and (2.6), one obtains a problem of fixed point type for $\Omega \equiv (Q_1, Q_2, R)$ with $Q_1 \equiv Q$ and $Q_2 \equiv Q'$. (In fact it is useful to put in a scale factor β so that $Q_2 \equiv \beta^{-1} Q'$.)

The result is a nonlinear map N from a product of weighted spaces to itself and the problem reduces to solving

$$\Omega = N(\xi, \Omega) \tag{2.10}$$

where $\xi = (\xi_1, \xi_2)$ represents the initial data in (2.9), and some of the parameters are suppressed. As with most center-manifold situations one must tailor the problem by putting in cutoff functions, resulting in an altered problem

$$\Omega = \tilde{N}(\xi, \Omega) \tag{2.11}$$

having the same "small" solutions as the original one. To show there is a solution of (2.11), we work in a closed subset of a weighted Hölder space, the closed subset having bounds on the derivatives of the admissible functions, but not on the functions themselves (cf. [2]). One must study the calculus of compositions in these spaces, an elementary though lengthy task. In this context one can show the existence of a unique solution $\Omega(\xi)(x)$.

A striking feature of the center-manifold equation (2.11) is that for a solution $\Omega = (Q_1, Q_2, R)$, the third component R is a pointwise function of the first two. That is, there is a function J defined on $\mathbf{R}^2$ and taking values in $C_\mu^{1,1/2}(I^+) \times C_\mu^{1,1/2}(I^-)$, $\mu > 0$, such that R as a function of x is

$$R = J(Q_1(x), Q_2(x)). \tag{2.12}$$

Thus, knowing the two components $(Q_1(x), Q_2(x))$ one can *lift* the orbit in $\mathbf{R}^2$ to the full infinite- dimensional solution. There is a problem, however, in that the map $\tilde{N}(\Omega)$ includes a truncation and only "small" solutions Ω correspond to solutions of the original partial differential equation. Returning to the equations (2.9) one now has the option of replacing the occurrence of R in the first two equations by the expression $R = J(Q_(x), Q_2(x))$. After some labor, one can obtain the lowest order terms in a pair of differential equations for $(Q_1(x), Q_2(x))$ alone. One can show that the function J is "smooth" (by restricting its domain) and that $J(0,0) = 0$. Hence a small solution of the equation for $(Q_1(x), Q_2(x))$ will lift to a small triple $\tilde{\Omega} = (Q_1, Q_2, \tilde{R})$ and hence will be a small fixed point of $\tilde{N}$. But a small fixed point of $\tilde{N}$ is also a fixed point of N and hence of the original elliptic problem. In this way one can succeed in characterizing all the "L^∞ small" solutions of the traveling wave problem (cf. [1], [2]). Among the resulting flows are solitary waves (homoclinic orbits), and surges connecting distinct parallel flows at the extreme ends of the channel (heteroclinic orbits). Each of these, of course, constitutes a traveling wave solution in the original coordinates.

3 Waves in Axons

In a classic paper in neurobiology Hodgkin and Huxley ([5]) gave a detailed model for the propagation of waves of depolarization in axons, the conduits that carry a variation in potential from the central part of a neuron to a distant synapse. The arrival of the voltage pulse at the synapse triggers the release of neurotransmitters enabling the passage of a signal to a neighboring neuron. The experimental basis for their work was the study of the giant axon of the squid Loligo, the culmination of studies by many researchers on cell electrophysiology. The mechanism can be understood by first looking at a situation in which there is no spatial dependence and then including the diffusive effect possible along the axon, a long cylindrical structure with a small diameter. In fact, Hodgkin and Huxley removed the spatial dependence by threading fine wires along the axon, maintaining constancy of voltage in space even while it changed in time.

The two main players in the passage of current across the axonal membrane are sodium and potassium ions. Being ions they move due to *diffusion* and to *electromotive force*. It is the combination of these two fluxes that affects the voltage excursions. The membrane has large numbers of tiny channels, some of which are selective for the passage of sodium and others for the passage of potassium. A crucial aspect of the dynamics is that each channel may be open or closed for the passage of ions and that its state, open or closed, is a random process. This is the simplest model for such a channel. The random aspect is fundamental, and without this stochastic behavior, the propagation would fail and the animal would "shut down" immediately.

The model of the kinetics of a potassium channel is somewhat simpler than that of a sodium channel, and so we start with the former, i.e., the K^+ channel. Modern usage takes the reference voltage to be that outside the cell and takes it to be $V = 0$ mV (millivolts). Hodgkin and Huxley took the reference $V = 0$ mV to be the the potential of the inside of the cell at rest. Moreover, the current standard is to use "depolarization" to mean an excursion of the interior of the cell (roughly -70 mV with respect to the exterior) toward more positive voltages. For Hodgkin and Huxley depolarization meant an excursion of the interior toward more negative voltages. We use the current standards.

There is a higher concentration of K^+ inside the cell than outside and this imbalance is maintained by ion pumps. If the voltage were 0 mV inside the cell and a K^+ channel were to open, diffusion would produce a net outward current of positive charges. If the inside were artificially maintained at -100 mV and a K^+ channel were to open, the outward diffusion would be countered by a flux of the charged ions due to the voltage gradient and a net inward flux of positive charge would result. If the interior were artificially held at about -90 mV, the two fluxes would cancel and produce a net zero current. The potential $V_K = -90$ mV, is called the *reversal potential* for potassium (for this particular cell).

The basis of the model can be understood in terms of a pure stochastic process for the state of a single K^+ channel. Let N denote a stochastic variable which can take the values $N = 0$ and $N = 1$, corresponding to closed and open, respectively. It is assumed that a single channel undergoes a random process independently of the other channels and remains in each state, 0 or 1, with an exponentially distributed waiting time (cf. [6]). The current through a single channel is taken to be $i_K = G_K * N * (V - V_K)$ where G_K is the conductance of a single open channel and $(V - V_K)$ is the net driving potential.

The stochastic variable N will have associated probability distributions. Suppose the infinitesimal transition probabilities are α for the transition $0 \rightharpoonup 1$ and β for the transition $0 \leftharpoondown 1$. By differentiating the Chapman–Kolmogorov equation ([6], p. 89) one can obtain the *forward equation* for the process, and the transition probability $P_{01}(t)$(that starting in state 0, one is in state 1 at time t) is found to satisfy the differential equation

$$\frac{d}{dt}P_{01}(t) = \alpha(1 - P_{01}(t)) - \beta P_{01}(t). \tag{3.13}$$

One sees that, given α and β, the equilibrium state for $P_{01}(t)$ is $\frac{\alpha}{\alpha+\beta}$, and it is approached by an exponential in time, with a time constant $\tau = \frac{1}{\alpha+\beta}$. For a large number of channels, starting closed, by the law of large numbers, we can expect the fraction n of channels in the open state at time t to be approximately $P_{01}(t)$. More generally, for any starting value, the fraction $n(t)$ will satisfy

$$\frac{dn}{dt} = \frac{n_\infty - n}{\tau_n}, \qquad n_\infty(V) = \frac{\alpha}{\alpha + \beta}, \tag{3.14}$$

$n_\infty(V)$ being the equilibrium state. The time constant is $\tau_n = \frac{1}{\alpha+\beta}$. Then took the reference $V = 0$ mV to be the the potential of the inside of the cell at rest. Moreover, the current standard is to use "depolarization" to mean an excursion of the interior of the cell (roughly -70 mV with respect to the exterior) toward more positive voltages. For Hodgkin and Huxley depolarization meant an excursion of the interior toward more negative voltages. We use the current standards.

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$n_\infty(V)$ being the equilibrium state. The time constant is $\tau_n = \frac{1}{\alpha+\beta}$. Then if there are N_K channels, the total potassium current will be

$$I_K = g_K \cdot n \cdot (V - V_K), \tag{3.15}$$

where $g_K = G_K \cdot N_K$ (in units of microsiemens).

In fact, Hodgkin and Huxley found a better fit to their measured data by using a form $I_K = g_K \cdot n^4 \cdot (V - V_K)$, arguing that this is consistent with a channel protein that has four independent "n gates" in the channel, all of which must be in the open state to allow passage of a K^+ ion.

Another wrinkle is that the kinetic parameters α and β change with the interior cellular potential V. In the case of potassium, the closed state is favored when V is substantially below zero and the open state is increasingly favored as V rises, with $n_\infty(V)$ approaching 1 for V well above zero. Among the data given in the article [5] are graphs of the dependence of $n_\infty(V)$ and $\tau_n(V)$ as V varies, based on extensive experiments, as well as explicit functions fit to this data.

The sodium current has a similar description in [5], but differs slightly. It has an m gate analogous to the n gate, favoring the open state as V becomes more depolarized (more positive), but also has an h gate (an inactivation gate) which favors being closed when V becomes more depolarized. Each gate has its own time scale and so the effect of an upward excursion of $V(t)$, starting at a substantially negative value where h is close to 1, is to first allow passage of calcium as V rises, and then to curtail the current as h decreases. The form is: $I_{Na} = g_{Na} \cdot m^3 h \cdot (V - V_{Na})$, with equations for the dynamics of m and h (see below). The net current also includes a "leak" current $I_L = g_L \cdot (V - V_L)$ which has no voltage dependence in its total conductivity.

The cell membrane acts as a capacitor of capacitance c_m (on the order of 1 microfarad per square centimeter) and the capacitative current acts in conjunction with the ionic currents, the latter often likened to batteries in a circuit. The total current equation plus the three kinetic equations for n, m, h are called the Hodgkin–Huxley equations and have the form

$$c_m \frac{dV}{dt} = g_{Na} m^3 \cdot h \cdot (V - V_{Na}) + g_K n^4 \cdot (V - V_K) + g_L \cdot (V - V_L) \tag{3.16}$$

$$\frac{dm}{dt} = \frac{m_\infty(V) - m}{\tau_m(V)} \tag{3.17}$$

$$\frac{dh}{dt} = \frac{h_\infty(V) - h}{\tau_h(V)} \tag{3.18}$$

$$\frac{dn}{dt} = \frac{n_\infty(V) - n}{\tau_n(V)}. \tag{3.19}$$

The set of four equations above has a stable equilibrium with V approximately −70 mV and with the kinetic variables at $m_\infty(-70)$, and so on.

The behavior of the system is sensitive to the injection of a current pulse. Let $I(t)$ be a function which is positive on the interval $0 \leq t \leq T$ where T is on the order of a millisecond. If one adds $A \cdot I(t)$ to the right side of (3.16) with A small, the voltage will undergo a small excursion and return to its equilibrium value. However, if A is larger than a critical value A_c, there will be a large excursion of 100 mV or so before the voltage returns to its equilibrium value. Moreover, the size of the excursion is essentially independent of A as long as $A > A_c$. This pulse is called a (non-propagating) *action potential* and is essentially equal to the plot labeled $V0$ in Figure 1. It differs from $V0$ in the fifth decimal place.

If one considers spatial dependence, taken to depend only on a single variable x measuring distance along a narrow axon, then one considers the various quantities to depend on x as well as on time t. One has dependent variables $V(x,t)$, $m(x,t)$, and so on. Now, of course, current can move axially along the axon as well as across its membrane, and the effect is to add a diffusive term

$$\frac{1}{r_i}\frac{\partial^2 V(x,t)}{\partial x^2} \tag{3.20}$$

on the right side of (3.16), where r_i is the interior resistance to current flow in units of ohms per unit length (the current equation now has units of current per units of length).

Without the active ion channels a current pulse inserted at a spatial position $x = 0$ would produce a spreading wave of voltage variation, much as an ordinary diffusion or heat equation. However, with active channels a pulse of current of super-critical amplitude inserted at $x = 0$ will produce a voltage excursion (a 'firing') and current influx near that point. It will be sufficient to bring a neighboring point to the critical "firing" level, and so on down the axon, so that the voltage excursion propagates along the axon and does so with essentially no change in form. Thus it arrives at the distant synaptic site at full strength. In Figure 1 we have plotted the voltage excursions at sites $x = 0$ and equally spaced increments $\Delta x > 0$, $2\Delta x$, $3\Delta x$. The computation is done by replacing the second partial derivative above by a second difference quotient and choosing $\Delta x = 1$ and $r_i = 1$.

4 Contraction waves in *Ascaris*

The simple nematode *Ascaris suum* has been studied for close to a century and there is a wealth of data regarding its anatomy, neural architecture, and electrophysiology. However, as regards locomotion, the data has never been brought to bear on an understanding of how the various components can produce movement. In a moving worm one sees body waves of sinusoidal shape progressing along the worm. At a fixed position along the worm,

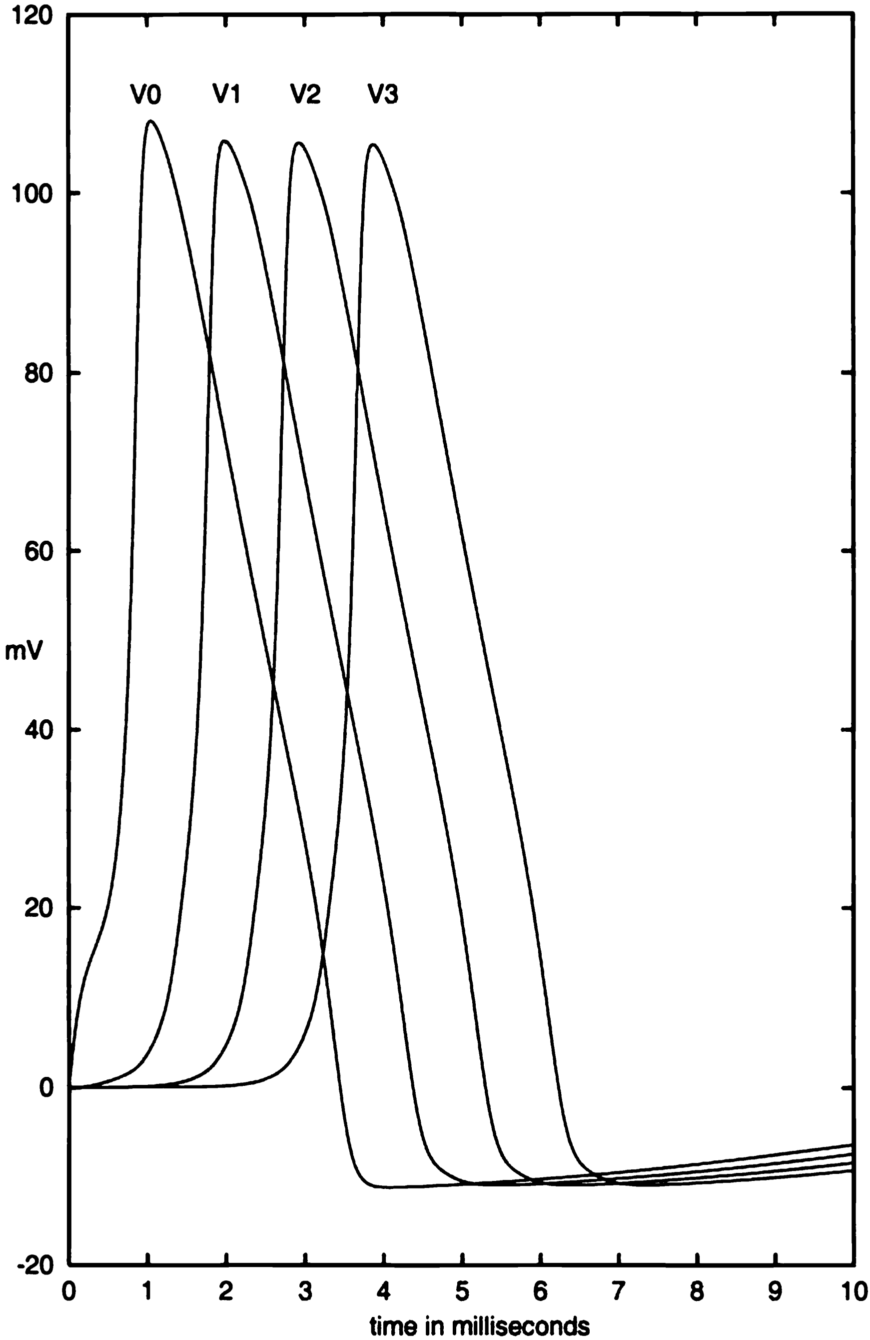

FIGURE 1. Plots of voltage excursions (from 0 mV inside the cell) for a model axon with a discrete second difference approximation to the diffusion term; plots of V0 at a site x=0, together with voltages Vk at sites $k \cdot \Delta x > 0$ for $k = 1, 2, 3$.

when the (ventral) underside is contracted, corresponding to positive voltage excursions, the (dorsal) upper side is relaxed; with the passage of time the dorsal muscles contract and the ventral muscles relax, this happening in alternation as the body waves progress. Here we present a model for an individual muscle cell and show how chains of cells can produce the progressing waves of activity that are seen in a moving worm. For the single cell and for the system we obtain a reasonable approximation to the types of recordings made by Weisblat and Russell ([17]) in dissected preparations and by Mead ([10]) in a semi-intact, behaving worm.

A longstanding question regards the origin of the contractile waves and signals that are observed in *Ascaris*. Do they arise in the muscle cells or in the neurons? We focus on the muscle cells in developing a model. A special characteristic of the muscle cells in *Ascaris* are that they appear to be capable of spontaneous oscillation. This is a feature that we emphasize in our model (cf. [15] for an earlier, minimal model).

Ascaris has approximately 50,000 muscle cells along its 30 cm. length and has 300 neurons, about 90 of which are involved in locomotion ([12],[13] and the references therein). One can view the muscle cells as being arrayed in two chains, a dorsal chain running along the top of the worm and a ventral chain running along the bottom. While each muscle cell has a complicated morphology, we consider it to be isopotential. The identified currents are listed below and each is modeled in the spirit of the Hodgkin–Huxley model. The data on the currents can be found in ([14], [8],[9]).

Calcium current

The calcium current across the cell membrane of a model muscle cell is taken to be

$$I_{Ca} = g_{Ca} \cdot e^2 \cdot f \cdot (V - V_{Ca}). \tag{4.21}$$

Here g_{Ca} is the maximum calcium conductance for the cell; e is a unitless activation parameter; f is an inactivation parameter; V is the cytoplasmic voltage in millivolts; and V_{Ca} is the calcium reversal potential. The dynamics of the gating variables e and f are analogous to those described in the previous section. However, in the present situation there is not sufficient data to do the fine type of fitting done by Hodgkin and Huxley in [5] and so we restrict ourselves here to having the infinitesimal transition probabilities $\alpha(V)$, $\beta(V)$ given by standard Boltzmann type rate functions (cf. [16]). The various free parameters entering the description of the kinetics are used to "fit" the calcium current in the model to the laboratory measurements in ([8]) done during voltage clamp experiments.

Potassium currents

A Hodgkin–Huxley type of "non-inactivating" current is given by

$$I_K = g_K \cdot n \cdot m \cdot (V - V_K), \tag{4.22}$$

the terms being similar to those in the calcium current. The activation parameter is n and we do, in fact, put in a slow inactivation m to capture a slow decline in I_K that one sees in Figure 6 of Martin et al. ([8]). The reversal potential for potassium is taken to be -50 mV, based on the discussion [8], pp. 81–82. Again, values for the controlling parameters are chosen to approximately capture the dynamic responses seen in ([8]), Figure 6. Here, however, we have replaced the Boltzmann relaxation times by constants: 5 milliseconds for n and 500 milliseconds for m.

A rapidly inactivating potassium current, I_A, is described by:

$$I_A = g_A \cdot p^4 \cdot q \cdot (V - V_K), \tag{4.23}$$

the activation parameter being p, and the inactivation, q (cf. ([4], page 118).

Leak current

This is taken to be

$$I_L = g_L \cdot (V - V_L) \tag{4.24}$$

with the parameters obtained from Martin et al. ([8]).

Calcium activated chloride current

This current is different from the type discussed in the previous section in that it is a two stage activation. First the calcium must enter the cell. Next the calcium acts at a site inside the cell to open chloride channels. The work of Thorn and Martin ([14]) was dedicated to studying a calcium dependent chloride channel in the muscle cell. The mechanics of activation of the chloride channels by calcium are not discussed in their paper and so we have incorporated a mechanism which allows for some accumulation of calcium and a subsequent opening of chloride channels when a threshold level of calcium is reached. We take into account the time lag (from 2 to 3 seconds, [14], p. 44) between the appearance of calcium and the opening of the chloride channels. We introduce a discrete version of a diffusion (a mathematical device). This allows for some time delay in the action of the entering calcium and has some smoothing effect on the calcium level.

We consider a segregated fraction r of the calcium to be that which accumulates and controls the gating of the chloride channel, through a parameter b satisfying $0 \leq b \leq 1$. The current has the form

$$I_{Cl} = g_{Cl} \cdot b \cdot (V - V_{Cl}). \tag{4.25}$$

The discrete diffusion takes place over hypothetical sites having calcium levels a_0 (segregated calcium), a_1, a_2, a_3. The amplitude of a_2 is used to trigger the value of the gating variable b through a sigmoidal function b_∞. We also include a buffering of calcium, with constant k (see below).

We used the anatomical data in Stretton ([12]), making a rough calculation of .002 cm^2 for the surface area of a muscle cell and multiplying by the commonly used value 1 microfarad per cm^2 to obtain the cell membrane capacitance c_m. Combining the components just described, we obtain the equations governing a model cell. Its state is described by the variables (V, e, f, n, m, q, b), and calcium concentrations a_i, $i = 0, 1, 2, 3$. The first is a current balance and the remainder regulate the kinetics. The form of these last four equations results from a spatially discrete approximation to the diffusion equation. See ([16]) for precise values of parameters.

$$c_m \frac{dV}{dt} = g_{Ca} e^2 \cdot f \cdot (V - V_{Ca}) + g_K n \cdot m \cdot (V - V_K) + g_L \cdot (V - V_L)$$
$$+ g_A \cdot p^4(V) \cdot q \cdot (V - V_K) + g_{Cl} \cdot b \cdot (V - V_{Cl}) \tag{4.26}$$

$$\frac{de}{dt} = \frac{e_\infty(V) - e}{\tau_e(V)} \tag{4.27}$$
$$\frac{df}{dt} = \frac{f_\infty(V) - f}{\tau_f(V)} \tag{4.28}$$
$$\frac{dn}{dt} = \frac{n_\infty(V) - n}{\tau_n(V)} \tag{4.29}$$
$$\frac{dm}{dt} = \frac{m_\infty(V) - m}{\tau_m(V)} \tag{4.30}$$
$$\frac{dq}{dt} = \frac{q_\infty(V) - q}{\tau_q(V)} \tag{4.31}$$
$$\frac{db}{dt} = \frac{b_\infty(a_2) - b}{\tau_b(V)} \tag{4.32}$$
$$\frac{da_0}{dt} = -rI_{Ca} - ka_0 \tag{4.33}$$
$$\frac{da_1}{dt} = \delta(a_2 - 2a_1 + a_0) - ka_1 \tag{4.34}$$
$$\frac{da_2}{dt} = \delta(a_3 - 2a_2 + a_1) - ka_2 \tag{4.35}$$
$$\frac{da_3}{dt} = \delta(-2a_3 + a_2) - ka_3 \tag{4.36}$$

The equations (4.26) to (4.36) are solved using Fortran and a differential equation solver "dvode" from Lawrence Livermore National Laboratory, Berkeley, California. In the simulations for a single cell the initial data are taken to be: $V = -30$ mV , $f = 1.0$, $q = 1.0$, $m = 1.0$. All of the remaining variables start at zero. A typical output of the model for a single cell is shown in Figure 2.

Figure 2 is a very good approximation to some of the recordings of Weisblat and Russell [17]. In their laboratory recordings there were trains of 3 to 8 spikes (Figure 2 shows an example with trains having 3 spikes) separated

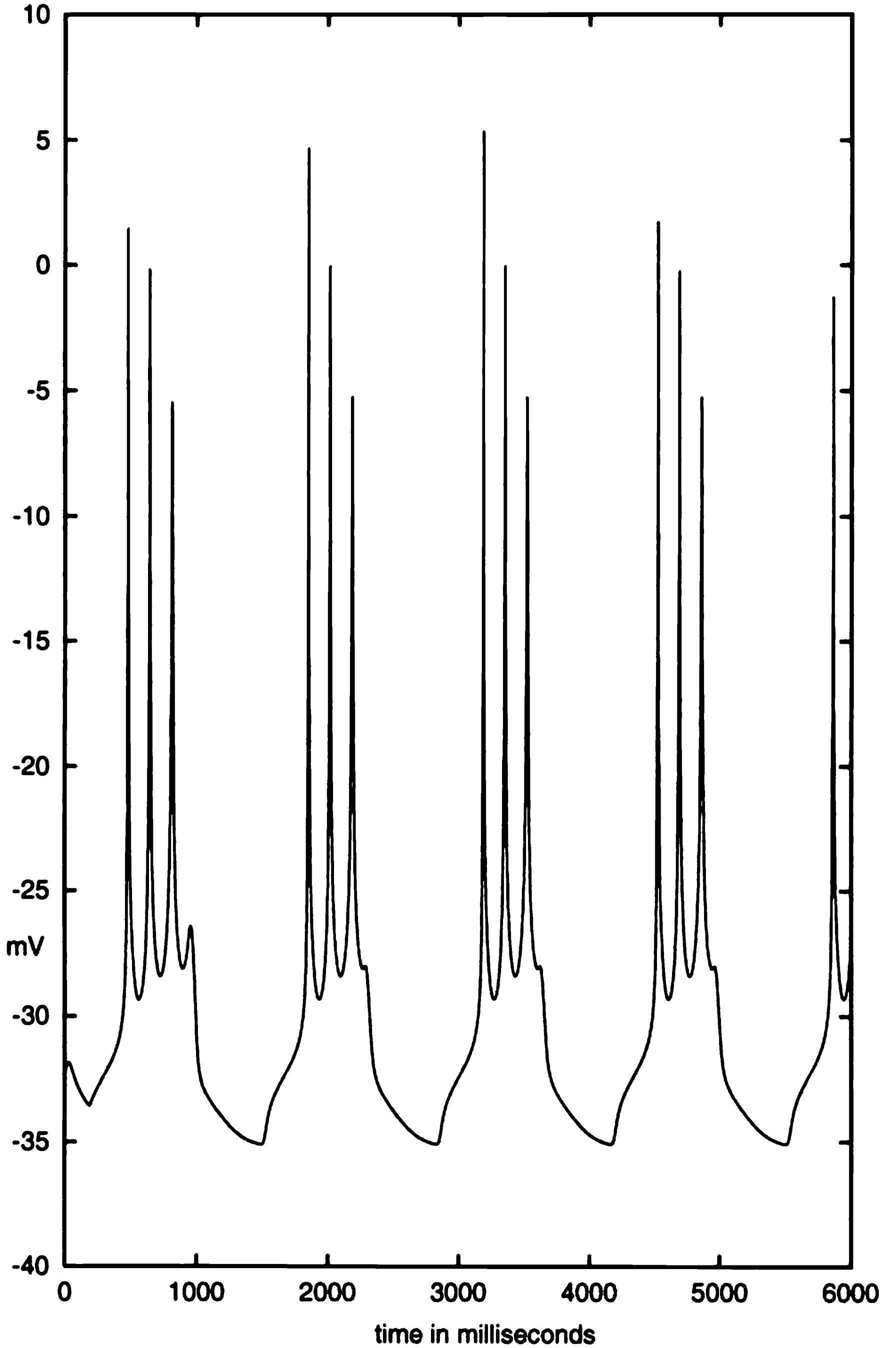

FIGURE 2. Plots of voltage excursions for a model muscle cell.

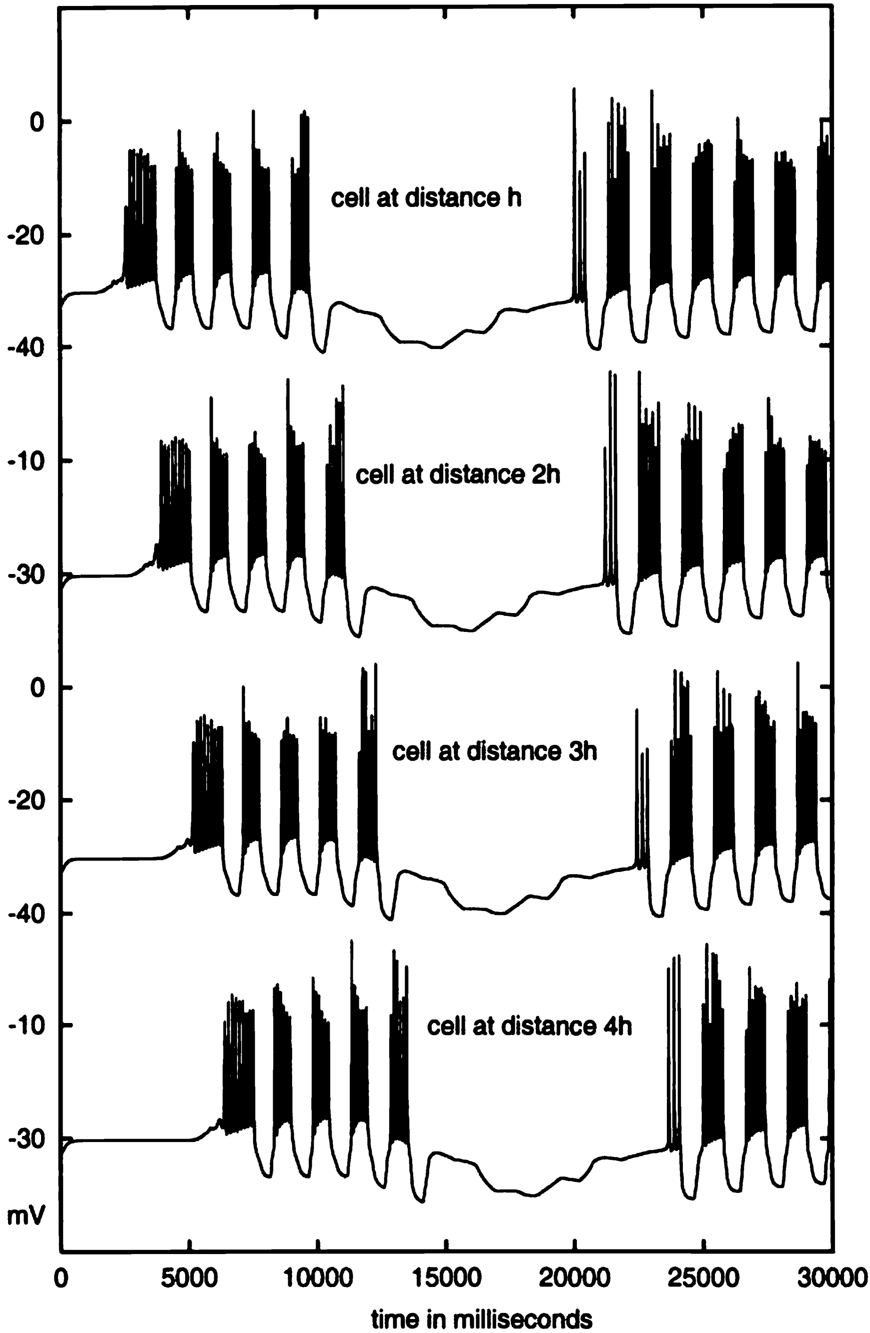

FIGURE 3. Plots of voltage excursions for a model axon with a discrete second difference approximation to the diffusion term; plots of V0 at a site x=0, together with voltages Vk at sites $k \cdot \Delta x > 0$ for $k = 1, 2, 3$. Scale markings $0, -20, -40$ are for first cell, etc.

by "quiet intervals" lasting for a few hundred milliseconds. The time scale for the individual spikes and the time scale for the "bursts" or groups of spikes is reasonably close to the laboratory recordings. It should be noted that for parameter values near those used in the simulation, the cell has an equilibrium solution with a voltage near −30 mV, an established value for a muscle cell at rest. For an individual muscle cell, Weisblat and Russell also saw a longer time scale, in that a string of bursts was interrupted every 7 to 20 seconds by a "quiet" period. We believe this is a system characteristic and to model it we extend the equations above with a term representing a "stretch" receptor. This aspect of *Ascaris* has not been carefully studied, but is firmly believed to be present (Davis, private communication).

For this element we propose a mechanism which measures the amount of bursting activity in a cell, effectively integrates it, and when it reaches a critical level, sends a hyperpolarizing signal to the cell. A further, well established effect is a reciprocal inhibition between cells in the dorsal chain and cells in the ventral chain ([13]). Each cell in the dorsal chain sends inhibitory signals to a group of cells in the ventral chain, the receiving group being roughly the same distance along the chain as the sending cell.

For our purposes we assume each dorsal cell is paired with a "twin" in the ventral chain. For the reciprocal inhibition we use a technique similar to that just described for stretch, but have activity in a dorsal(ventral) cell rapidly send a hyperpolarizing signal to its "twin" in the ventral(dorsal) chain. Likewise an active ventral cell inhibits its dorsal twin in the model. In the worm this effect is carried through the neural architecture, but for the purposes of our model at this stage, we have put in a *direct* reciprocal inhibition from a muscle cell to its twin. A last element is an influence of an active cell on the kinetic parameters of a neighboring cell, to rouse it from a quiescent mode and put it in an oscillatory mode. The results of the more extensive model (paper in preparation) can be seen in Figure 3 where we have used 15 dorsal cells "twinned" with 15 ventral cells, though with kinetics parameters differing from those yielding Figure 2. Shown are the voltage excursions at four successive (dorsal) cells in the chain, responding to a stimulus originally acting at one end of the chain and progressing along the chain.

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Department of Mathematics
University of Wisconsin at Madison
Madison, Wisconsin
email: turner@math.wisc.edu

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A. Ioffe
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M. Mrozek
M. Musso
R. Ortega
P. Pilarczyk
M. del Pino
E. Séré
E. Schwartzman
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