Topics in Analysis and its Applications

Edited by

G.A. Barsegian and H.G.W. Begehr

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Topics in Analysis and its Applications

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Series II: Mathematics, Physics and Chemistry – Vol. 147

Topics in Analysis and its Applications

edited by

G.A. Barsegian
Institute of Mathematics of National Academy of Sciences of Armenia,
Yerevan, Armenia

and

H.G.W. Begehr
Mathematical Institute,
Freie Universität Berlin,
Berlin, Germany

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Preface

Subsequent to an International ISAAC Conference on "Complex Analysis, Differential Equations and Related Topics", the NATO ARW

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took place in Yerevan, Armenia with participants from Armenia, Austria, Belarus, Canada, Finland, Georgia, Germany, Italy, Marocco, Poland, Portugal, Romania, Russia, Spain, Sweden, Tajikistan, Turkey, Ukraine, USA. The present volume reflects the main contributions but neither the stimulating atmosphere nor the highly friendly relations among the participants. Thanks are due to many members of the family of Grigor Barsegian, the Armenian co-director of this ARW. They have helped a lot to make the four days pleasant and easy for all participants. Without the support from the Scientific and Environmental Affairs Devision of the NATO the event would not have taken place. All participants were very grateful to Dr. F. Pedrazzini, the Programme Director of Physical and Engineering Science and Technology and his staff for having arranged the support in time although it was very tough before the beginning.

Most topics in the workshop were devoted to complex analysis as well of one as of several complex variables. Several contributions came from elasticity theory.

One generalization of complex analysis is the theory of p-adic analysis, a field enjoying a lot of attention in recent years. Multiplicative semi-norms in an ultrametric normed algebra have many applications in the study of analytic elements in p-adic analysis. Here the existence of a Shilov boundary for a semi-multiplicative semi-norm defined on an algebra is found inside a set of multiplicative semi-norms.

In geometric function theory quasiconformal mappings are studied a lot since almost 80 years. This concept is called quasiregular in higher dimensions. More recent research involves mappings of bounded mean oscillations. This research is surveyed on as well for plane domains as for

Riemann and Klein surfaces. Classical subjects in the theory of Riemann surfaces are harmonic and analytic differentials. Spaces of differential forms are connected with the topological and the conformal structure of the surface. Here these connections are investigated for non-orientable i.e. for Klein surfaces.

A classical but still very modern subject is complex dynamics. This topic has started mainly with the works of P. Fatou and G. Julia some eighty years ago and later has lead to the very important areas of fractals and of dynamical systems. One contribution surveys on classical and recent results on complex dynamics related to inverse functions of rational or transcendental entire functions.

Boundary value problems for analytic functions were initiated already in the famous thesis of B. Riemann just 150 years ago. They have influenced not only mathematical analysis a lot but even algebraic geometry and topology. And still new insights are gotten. Here the nonlinear Riemann–Hilbert problem for analytic functions with nonsmooth target mainfolds are treated. The solutions can be attained as extremal functions in certain function classes. Disproving a conjecture an example is given of a nonsmooth topological target manifold for which the solution set of the Riemann-Hilbert problem is bigger than in the smooth case.

While complex function theory from the viewpoint of partial differential equations is nothing else but the theory of solutions to the Cauchy-Riemann equation generalized analytic functions are related in the same way to a more general first order elliptic system, the Carleman-Bers-Vekua system. Analytic and topological aspects of this system is studied and the Riemann-Hilbert monodromy problem solved for a special class of regular systems. Moreover, a holomorphic vector bundle on the Riemann sphere is constructed together with an L_p-connection on it using these systems. Another classical boundary value problem is the Riemann jump problem extensively studied in the former SU in particular in Belarus and in Georgia. Here this problem is reported on for analytic vectors in the cut plane.

Value distribution theory is the completion of the theory of analytic and meromorphic function theory. The logarithmic derivative is a main tool in this theory. It is here studied for meromorphic functions in angular domains, one important case of domains investigated in value distribution theory. Recently in Armenia an entirely new approach to value distribution theory is made. Instead of considering preimages of single points, preimages of more general sets e.g. lines are investigated leading to Γ-lines. This new generalization seems to open a wide field of applications of the methods of value distribution theory as the concept of level sets appears in many fields of mathematics, mathematical physics and natural sciences. The lengths of level sets of smooth functions are estimated, in particular those of Γ-lines, of monic polynomials for large classes of curves Γ. In case when Γ is closed this is related to the Erdös-Herzog-Piranian problem. Through Γ-lines also oscillations of solutions to ordinary differential equations including the Riccati, the Schrödinger and the Painlevé equations can be treated leading to applications in population dynamics and economics.

Also in real algebraic and real analytic geometry the question of zeros has a long history. Rather than localizing the roots of a real polynomial it is important just to effectively find their number e.g. in the left half plane. A general algorithm is described for computing the number of points of an arbitrary finite semi-algebraic subset and the complexity of the algorithm is estimated.

The behavior of polynomials of one variable at infinity is quite obvious. The situation is different for several variables. Necessary and sufficient conditions are given such that the polynomial tends to infinity when the variables do. This property is important for linear differential operators determining its typ.

One motivation to develop the theory of several complex variables was once to solve partial differential equations. But the theory became very abstract and far-reaching and not too many equations were solved explicitly. In recent years purely analytic methods were used to solve boundary value problems for mainly first and second order overdetermined systems either in the unit ball or in polydomains. Here first and second order systems in bounded domains degenerating at the boundary are solved explicitly. In order to explain the differences one and several variables are both treated.

A basic tool in treating partial differential equations in complex spaces are Pompeiu operators. For several variables they are known either for polydomains or for the unit ball. Using the weighted version of the Cauchy-Pompeiu representation explicit formulas for the derivatives of the solution to the inhomogeneous Cauchy-Riemann system in the unit ball having minimal norm in a particular L_2-space are given.

Through higher order Green functions modified higher order Pompeiu operators are used as well for one as for several variables to express all n-th order partial derivatives of a complex function by just one particular n-th order partial derivative via strongly singular integral operators of Calderon-Zygmund type. In case of several variables both polydomains and the unit ball are treated.

Complex methods to some extend are also available to problems in several real variables even in odd number. Quaternionic, octonionic and Clifford analysis are the proper frameworks. Many different kinds of higher order partial differential equations were recently treated in different properly adjusted Clifford algebras. A survey gives the results on solutions to initial and to boundary value problems via quadratures.

The most celebrated boundary value problem for elliptic systems is the Dirichlet problem. Nevertheless there are still new results on this classical problem. Different necessary and sufficient conditions for its unique solvability is given for some class of higher order elliptic systems. Another contribution focusses on the Dirichlet problem for second order elliptic equations in domains with non-smooth boundary. Also first order elliptic systems are reported on. Instead of generalizing the assumptions on the boundary the properties of the boundary functions can be weakend. The Dirichlet problem in the upper half plane is considered for harmonic functions under the assumption that the boundary functions belong to a certain weighted space with a weight function having finitely many singular points of finite order where it has a certain property called RO-varying.

The Carleman-Bers-Vekua equation was studied a lot in the former SU. This was done also intensively in Tajikistan where in particular equations with singular coefficients were investigated. Parallel to this school also ordinary differential equations with singularities were studied. Such equations with a singular or supersingular point inside the intervall are solved via a reduction to some Volterra integral equation of second kind.

Singular integral equations with generalized Cauchy kernel are related as well to complex analysis as to elasticity theory. Many problems like boundary value problems for partial differential equations or e.g. contact problems in elasticity theory are reducable to singular integral equations. Quadrature formulas of highest algebraic precision are obtained for singular integral equations and their effectiveness for applications shown.

Operator theory is an important tool in real and complex analysis whenever functional analytic methods are introduced to solve equations. Here the connection between localization operators, the Wigner transforms and paraproducts are discussed in the context of the Weyl-Heisenberg group, the affine group and SU(1.1).

Differential equations together with differential geometric methods can be very useful to solve concrete problems. This is perfectly demonstrated in describing the classes of curves along which the flight of an aircraft is possible under suitable choices of the initial velocity, the reactive forces and the fuel consumption. Depending on the data the optimal flight trajectory is determined.

The mathematical problem in tomography is to determine the inside construction of an object from its X-ray images in a finite number of directions. Here the plane problem is considered in a case when the directions are arbitrarily distributed. A solution of this inverse problem is given not minimizing the norm but being steady for certain distortions.

Using Laplace and Fourier transforms and the Canyard method analytic formulas are obtained for elastic displacements and stresses of the induced electromagnetic field for the Lamb magnetoelastic problem. An asymptotic method is developed for determining frequencies and forms of free vibrations of anisotropic strip-beams rigidly fastened along one of the facial surfaces. Two types of free vibrations arise unlike the orthotropic case not purely shear and longitudinal. Eigenfunctions are determined composing an orthonormal system. These results apply to thin bodies as e.g. plates and shells. An asymptotic method is used to solve boundary value problems of elasticity theory of thin bodies the dimensionless equations of which depend on a small geometric parameter. New classes of problems of statics and dynamics of thin bodies may be treated this way having applications in seismology and for seismosteady constructions. For the strain-stress state of an elastic plane with an elastic circular inlay of different elastic material and arc-type absolutely rigid inclusions one side of which is detached on joining lines the exact solution is constructed.

Quantum chaos understood as chaos in the wave function is described by stochastic quantum systems within the Langevin-Schrödinger type stochastic differential equation. Expressions for transition probabilities in a quantum subsystem are obtained. Some phase transitions of second kind may occur for microscopic quantum transitions depending on the coupling constants of the thermostat with the parametric quantum harmonic oscillator. The thermodynamic potentials of the quantum harmonic oscillator immersed into the thermostat is calculated by a method of stochastic density and its ground state energy level widening and shift are determined. Thus violation of the second law of thermodynamics due to quantum fluctuation becomes possible.

Generalizing the theory of periodic wavelets fast and stable algorithms are constructed for orthogonally decomposing a Hilbert space via classical orthogonal polynomials. Numerical results are attained by using the automatic integration package of MATHEMATICA.

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SHILOV BOUNDARY FOR NORMED ALGEBRAS

Alain Escassut
Laboratoire de Mathematiques Pures University Blaise Pascal (Clermont-Ferrand)
63177 AUBIERE CEDEX, FRANCE
escassut@math.univ-bpclermont.fr

Nicolas Mai Netti
LLAIC, IUT Département Informatique
Université d'Auvergne (Clermont-Ferrand)
63177 AUBIERE CEDEX, FRANCE
mainetti@llaic.u-clermont1.fr

Abstract Let E be a field provided with an absolute value for which it is complete and let A be a commutative E-algebra with unity provided with a semi-multiplicative (or power multiplicative) E-algebra semi-norm $\| \,.\, \|$. Let $Mult(A, \| \,.\, \|)$ be the set of multiplicative E-algebra semi-norms continuous with respect to $\| \,.\, \|$. We show the existence of a Shilov boundary for $\| \,.\, \|$, i.e. a closed subset F of $Mult(A, \| \,.\, \|)$, minimal for inclusion, such that for every $x \in A$, there exists $\varphi \in F$ such that $\varphi(x) = \|x\|$. In particular, if the field E is ultrametric, it applies to the spectral semi-norm of an ultrametric E-algebra.

Keywords: normed algebras, multiplicative semi-norm, Shilov boundary

Mathematics Subject Classification (2000): 46S10, 12J25

1. Introduction and Theorems

The idea of studying systematically multiplicative semi-norms in an ultrametric normed algebra is due to Bernard Guennebaud [10]. This theory found many applications in the study of analytic elements and actually gave the good explanation of several basic properties [6]. More generally, this is also true in ultrametric Banach algebras, where multiplicative semi-norms play a role somewhat similar to that of maximal ideals in complex Banach algebras [5, 7, 10]. Next, Vladimir Berkovich

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also used the set of multiplicative semi-norms in order to reconstruct Tate's Theory [1, 12] . In [4], Kamal Boussaf showed the existence of a Shilov boundary for the norm of an algebra of analytic elements and described it with help of circular filters that characterize multiplicative semi-norms. In [1], V. Berkovich showed the existence of a Shilov boundary for the norm of an affinoid algebra. The present paper is aimed at proving, in the general case, the existence of a Shilov boundary for the spectral semi-norm of an algebra provided with a semi-norm, over a field provided with an absolute value (for which it is complete), i.e. the existence of a closed subset F of the set of continuous multiplicative semi-norms, minimal for inclusion, such that for every $x \in A$, there exists $\varphi \in F$ satisfying $\varphi(x) = \|x\|$. Such a process was outlined by B. Guennebaud in his unpublished Thèse d'Etat, through an interesting strategy and our way widely takes from it. However, certain intermediate results were missing, or suffered a lack of rigour, while hypotheses and definitions were often confusing. Besides, there was a confusion between two different problems: on one hand, finding a "Shilov boundary" for a semi-multiplicative function defined on a semi-group, inside a set of multiplicative functions, and on the other hand, finding a "Shilov boundary" for a semi-multiplicative semi-norm defined on an algebra, inside a set of multiplicative semi-norms. Here, we have reconstructed the framework of the proof and only considered the problem of a Shilov boundary for a semi-multiplicative semi-norm on an algebra. Such a construction was made in [8] for an algebra A over an ultrametric field. Actually, all the elements of the proofs similarly apply to the case of an algebra on an Archimedean field.

Definitions and notation Throughout this paper, E will denote a field provided with a non trivial absolute value denoted by $|\ .\ |$ which makes it complete, hence is either $\mathbb{R}$ or $\mathbb{C}$, or a field L provided with a non trivial ultrametric absolute value for which it is complete. When there exists a risk of confusion, the Archimedean absolute value defined on $\mathbb{R}$ and $\mathbb{C}$ will be denoted by $|\ .\ |_\infty$. We will denote by B, B' semi-groups with unity, and by $A,\ A'$ commutative E-algebras with unity.

Given sets $D,\ G$, we will denote by $\mathcal{F}(D, G)$ the set of functions from D to G.

Let $\mu \in \mathcal{F}(B, \mathbb{R}_+)$. The function μ will be said to be *submultiplicative* (resp. *multiplicative*) if $\mu(xy) \leq \mu(x)\mu(y)\ \forall x, y \in B$ (resp. $\mu(xy) = \mu(x)\mu(y)\ \forall x, y \in B$). And μ will be said to be *semi-multiplicative* if $\mu(x^n) = \mu(x)^n\ \forall x \in B, \forall n \in \mathbb{N}^*$.

Let S be a subsemi-group of B. A submultiplicative function θ on B will be said to be *S-multiplicative* if $\theta(xy) = \theta(x)\theta(y)\ \forall x, y \in S$.

Let us recall that *a semi-norm* (resp. *a norm*) of E-algebra defined on A is a semi-norm (resp. a norm) of E-linear space φ that satisfies $\varphi(x.y) \leq \varphi(x)\ \varphi(y)\ \forall x,\ y \in A$.

Henceforth, $\|\ .\ \|$ will be a E-algebra semi-norm on A.

We will denote by $Mult(A, \|\ .\ \|)$ the set of multiplicative E-algebra semi-norms which are continuous for the semi-norm $\|\ .\ \|$ of A.

A subset F of $Mult(A, \|\ .\ \|)$ will be called a *boundary for* $(A, \|\ .\ \|)$ if for every $x \in A$, there exists $\psi \in F$ such that $\psi(x) = \|x\|$. A closed boundary (with respect to the topology of simple convergence on A) is called *Shilov boundary for* $(A, \|\ .\ \|)$ if it is the smallest of all closed boundaries for $(A, \|\ .\ \|)$ with respect to inclusion.

The paper is aimed at proving the existence of a Shilov boundary for a semi-multiplicative semi-norm on A. We must first recall two classical theorems on spectral semi-norms for ultrametric normed algebras.

Theorem A [6, 11] *Let $\|\ .\ \|$ be a semi-norm of E-algebra on A. Then for every $x \in A$ the sequence $\left(\|\ x^n\ \|^{1/n}\right)_{n\in\mathbb{N}}$ has a limit denoted by $\|\ x\ \|_{si}$, satisfying $\|x\|_{si} \leq \|x\|\ \forall x \in A$ and the mapping f defined in A as $f(x) = \|\ x\ \|_{si}$ belongs to $SM(A, \|\ .\ \|\ )$. Further, if E is ultrametric, so is $\|x\|_{si}$. Moreover $\|\ x\ \|_{si} < 1$ if and only if $\lim\limits_{n\to\infty} x^n = 0$.*

Theorem B [11] *Given $\varphi \in Mult(A)$, φ belongs to $Mult(A,\ \|\ .\ \|)$ if and only if it satisfies $\varphi(t) \leq \|t\|_{si}$ whenever $t \in A$ and we have $Mult(A, \|\ .\ \|) = Mult(A, \|\ .\ \|_{si})$. Further, for every $t \in A$ we have $\|t\|_{si} = \sup\{\varphi(t) \mid \varphi \in Mult(A, \|\ .\ \|)\}$ and there exists $\varphi \in Mult(A, \|\ .\ \|)$ such that $\varphi(t) = \|t\|_{si}$.*

In order to describe the Shilov boundary for a E-normed algebra, we must also recall a few basic definitions on ordered sets and introduce constructible sets.

Definitions and notation A set F provided with an order relation $\leq$ is said to be *well ordered* with respect to the inverse order $\geq$ if every subset of F admits a maximal element, and then the order $\geq$ is called *a good order*. Let $\geq$ be a good order on the set F and let $x \in F$. If there exists $y < x$, we call *follower of an element* x the element $\sup\{y \in F \mid y < x\}$. If there exists $z \in F$ such that x is the follower of z, then z is called *precedent of an element x*.

Let $\mu \in \mathcal{F}(B, \mathbb{R}_+)$ be submultiplicative and such that $\mu(s) \neq 0\ \forall s \in S$. We will denote by μ^S the function defined in B by $\mu^S(x) = \inf\{\mu(sx)/\mu(s)$ $|s \in S\}$.

Let N be a subset of $\mathcal{F}(B, \mathbb{R}_+)$. Then N will be said to be *constructible* if it satisfies the following three properties:

i) N is well ordered with respect to the order $\geq$.

ii) For all $\mu \in N$ having a follower μ' (with respect to the good order $\geq$), there exists a subsemi-group S of B such that μ is S-multiplicative and such that $\mu' = \mu^S$.

iii) For every $\mu \in N$ having no precedent, μ satisfies $\mu(x) = \inf\{\nu(x) \mid \nu \in N \mid \mu < \nu\}$.

Given two submultiplicative functions μ and ν, ν will be said to be *constructible from* μ if there exists a constructible set N admitting μ as its first element and ν as its last element, with respect to the good order on N.

Now, let θ be a function from D to $\mathbb{R}_+$ and let $\mathcal{B}(D, \theta)$ be the subset of $\mathcal{F}(D, \mathbb{R})$ consisting of the functions f from D to $\mathbb{R}_+$ such that $f(x) \leq \theta(x)\ \forall x \in D$.

Let $\theta \in \mathcal{F}(B, \mathbb{R}_+)$ be submultiplicative. We will denote by $Min(B, \theta)$ the set of multiplicative functions $\mu \in \mathcal{F}(B, \mathbb{R}_+)$ which are constructible from θ.

We are now able to state Theorem C:

Theorem C *$Min(A, \| \,.\, \|_{si})$ is included in $Mult(A, \| \,.\, \|)$ and there exists a Shilov boundary for $(A, \| \,.\, \|_{si})$ which is the closure of $Min(A, \| \,.\, \|_{si})$ in $Mult(A, \| \,.\, \|)$.*

Examples 1) Let D be compact set and let A be the $\mathbb{C}$-algebra of continuous functions from D to $\mathbb{C}$ provided with the norm $\| \,.\, \|_D$ of uniform convergence on D. For every $a \in D$, let φ_a be the multiplicative semi-norm defined as $\varphi_a(f) = |f(a)|$. In a $\mathbb{C}$-Banach algebra, we know that $Mult(A, \| \,.\, \|_D) = Mult_m(A, \| \,.\, \|_D)$, and since all maximal ideals have codimension 1, here it is easily seen that $\{\varphi_a \mid a \in D\} = Mult(A, \| \,.\, \|)$ is the Shilov boundary for $(A, \| \,.\, \|_D)$.

2) Let D be a bounded open set in $\mathbb{C}$, let $\overline{D}$ be its closure, let D^* be the boundary of D, and let A be the $\mathbb{C}$-algebra of the functions which are holomorphic in D and continuous in $\overline{D}$, provided with the norm $\| \,.\, \|_D$ of uniform convergence on D. Here, since any holomorphic function reaches its maximum on the boundary of D, it is easily seen that the Shilov boundary for $(A, \| \,.\, \|_D)$ is equal to $\{\varphi_a \mid a \in D^*\}$.

3) Let K be a complete ultrametric algebraically closed field, let D be a closed bounded subset of K and let $H(D)$ be the K-algebra of analytic elements in D, i.e. the completion of the K-algebra $R(D)$ of rational functions with no pole in D, with respect to the uniform convergence norm on D: $\| \,.\, \|_D$. Circular filters on K are defined in [9] and [6] and characterize the elements of $Mult(K[x])$ in the following way: for each circular filter $\mathcal{F}$ on K, for every polynomial P, then $|P(x)|$ has a limit $\varphi_{\mathcal{F}}(P)$ along $\mathcal{F}$, and the mapping from the set of circular filters to $Mult(K[x])$ which associates $\varphi_{\mathcal{F}}$ to $\mathcal{F}$ is a bijection. Then every element

$\varphi_{\mathcal{F}}$ of $Mult(K[x])$ has continuation to $Mult(H(D), \| \cdot \|_D)$ if and only if $\mathcal{F}$ is secant with D, such a continuation is unique, and every element of $Mult(H(D), \| \cdot \|_D)$ is of this form, so that $Mult(H(D), \| \cdot \|_D)$ is identified to a subset of $Mult(K[x])$. It is shown in [4] that the Shilov boundary for $(H(D), \| \cdot \|_D)$ is the set of $\varphi_{\mathcal{F}}$ such that $\mathcal{F}$ is secant with both D and $K \setminus D$. And in that case, (with some analogy to the previous case in $\mathbb{C}$) the Shilov boundary for $(H(D), \| \cdot \|_D)$ is the boundary (in the usual meaning) of $Mult(H(D), \| \cdot \|_D)$ in $Mult(K[x])$.
Remarks A boundary for $(A, \| \cdot \|)$ is not necessarily closed. Let K be a complete ultrametric algebraically closed field and let $D = \{x \in k \mid |x| \leq 1\}$. Then $Mult_m(H(D), \| \cdot \|_D)$ is known to be dense in $Mult(H(D), \| \cdot \|_D)$ but not closed [6], and is a boundary for $(A, \| \cdot \|_D)$. Actually, according to [4], the Shilov boundary for $(H(D), \| \cdot \|_D)$ has empty intesection with $Mult_m(H(D), \| \cdot \|_D)$, and therefore there exist no smallest boundary for $(H(D), \| \cdot \|_D)$.

2. The proofs

Definitions and notation Let D be a set and let F be a E-vector space of bounded functions from D to E. In the sequel, we will denote by $\| \cdot \|_D$ the norm of uniform convergence on D.

Let D be a set. $\mathcal{F}(D, \mathbb{R})$ is provided with the topology of simple convergence: the filter of neighborhoods of $\varphi \in \mathcal{F}(D, \mathbb{R})$ admits as a generating system the family of sets $\mathcal{V}(\varphi, a_1, ..., a_n, \epsilon) = \{\psi \in \mathcal{F}(D, \mathbb{R}) \mid |\psi(a_i) - \varphi(a_i)|_\infty \leq \epsilon\}$, with $a_1, ..., a_n \in D$ and $\epsilon > 0$.

By Tykhonov's Theorem, we have Lemma 1.

Lemma 1 $Mult(A, \| \cdot \|)$ *is compact with respect to the topology of simple convergence.*
Notation Let θ be a semi-multiplicative function from B to $\mathbb{R}_+$. We will denote by $Z(B, \theta)$ the set of the $\phi \in \mathcal{B}(B, \theta)$ which are constructible from θ. Given a subsemi-group S of B, we will denote by $Z(B, S, \theta)$ the set of $\phi \in Z(B, \theta)$ which are S-multiplicative.

Let X be a subset of $\mathcal{F}(B, \mathbb{R}_+)$. We put $\sigma_X(x) = \inf\{\nu(x) \mid \nu \in X\}$ $\forall x \in B$.

Lemma 2 is obvious, and comes from the definition of constructible sets.
Lemma 2 *Let X be a constructible subset of $F(B, \mathbb{R}R_+)$. Then $X \cup \{\sigma_X\}$ is a constructible subset of $F(B, \mathbb{R}_+)$. Let $\psi \in X$ and let Y be subset of X such that $\sigma_Y \leq \psi$. Then σY belongs to X.*
Lemma 3 *Let S be a subsemi-group of $(A, .)$ and let θ be a S-multiplicative E-algebra semi-norm on A . Then θ^S also is a S-multiplicative E-algebra semi-norm on A.*

Proof We have to show that θ is an E-algebra semi-norm. Let $x, y \in A$. First, it is obvious that $\theta^S(\lambda x) = |\lambda|\theta^S(x)\ \forall \lambda \in E$. Now, let ϵ be > 0. We can find $s, t \in S$ such that $\theta(sx)/\theta(s) < \theta^S(x) + \epsilon,\ \theta(ty)/\theta(t) < \theta^S(y) + \epsilon$. On the other hand, since θ is a E-algebra semi-norm and is S-multiplicative, we check that

$$\theta^S(x+y) \leq \frac{\theta(st(x+y))}{\theta(st)}$$

$$\leq \frac{\theta(stx) + \theta(sty)}{\theta(st)} \leq \frac{\theta(t)\theta(sx) + \theta(s)\theta(by)}{\theta(s)\theta(t)} = \frac{\theta(sx)}{\theta(s)} + \frac{\theta(ty)}{\theta(t)}.$$

Consequently, we have $\theta^S(x+y) \leq \theta^S(x) + \theta^S(y) + 2\epsilon$. Since ϵ is arbitrary, we have proven that $\theta^S(x+y) \leq \theta^S(x) + \theta^S(y)$. Next,

$$\theta^S(xy) \leq \frac{\theta(stxy)}{\theta(st)} \leq \Big(\frac{\theta(sx)}{\theta(s)}\Big)\Big(\frac{\theta(ty)}{\theta(t)}\Big) \leq (\theta^S(x) + \epsilon)(\theta^S(y) + \epsilon)$$

hence $\theta^S(xy) \leq \theta^S(x)\theta^S(y)$. Thus, we have proven that θ^S is an E-algebra semi-norm. Then, it is obviously seen that it is S-multiplicative in the same way as θ.

Lemma 4 is immediate

Lemma 4 *Let L be a totally ordered family of E-algebra semi-norms of A and let ϕ be the function defined on A as $\phi(x) = \inf\{\varphi(x) \mid \varphi \in L\}$, $\forall x \in A$. Then ϕ is a E-algebra semi-norm.*

Proposition 5 *Let $\| \,.\, \|$ be semi-multiplicative. Then every element of $Z(A, \| \,.\, \|)$ is a E-algebra semi-norm.*

Proof Suppose Proposition 5 is not true. Let $\phi \in Z(A, \| \,.\, \|)$ which is not a E-algebra semi-norm. Let $\mathcal{T}$ be a constructible set admitting $\| \,.\, \|$ as its first element and ϕ as its last element. Since $\mathcal{T}$ is well ordered, the subset $\mathcal{S}$ of the $\psi \in \mathcal{T}$ which are not E-algebra semi-norms admits a maximum element θ. If θ admits a precedent ξ, then there exists a subsemi-group S such that ξ is S-multiplicative and satisfies $\theta = \xi^S$. But by hypothesis ξ is a E-algebra semi-norm, and by Lemma 3 so is θ, a contradiction. Consequently, θ has no precedent, and then we have $\theta(x) = \inf\{\psi(x) \mid \psi \in \mathcal{T},\quad \theta < \psi\}\ \forall x \in A$ hence $\mathcal{T}$ is all ordered. Therefore, by Lemma 4 θ is a E-algebra semi-norm because so are all $\psi \in \mathcal{T}$ such that $\theta < \psi$.

Corollary 6 *Let $\| \,.\, \|$ be semi-multiplicative. Then $Min(A, \| \,.\, \|)$ is included in $Mult(A, \| \,.\, \|)$.*

Given a semi-norm of E-algebra φ on A, we will denote by $Ker(\varphi)$ the set of the $x \in A$ such that $\varphi(x) = 0$.

Lemma 7 *Let S be a subsemi-group of $(B,.)$ and let $\theta \in F(B,\mathbb{R}_+)$ be S-multiplicative. There exists $f \in Min(B,\theta)$ such that $f(x)=\theta(x)\ \forall x \in S$.*

Proof Let $\mathcal{H}$ be the set of constructible subsets of $Z(B,S,\theta)$. Clearly, $Z(B,S,\theta)$ is not empty, hence $\mathcal{H}$ isn't either. On $\mathcal{H}$ we denote by $\preceq$ the order relation defined as $N \preceq N'$ if N is a beginning section of N'. Then $\mathcal{H}$ is inductive for its order. Let N_0 be a maximal element of $\mathcal{H}$ and let $f_0(x) = \inf\{\sigma(x) \mid \sigma \in N_0\}$, $\forall x \in B$. Clearly f_0 lies in $Z(B,S,\theta)$. The family $\mathcal{G}$ of subsemi-groups $J \supset S$ such that f_0 is J-multiplicative and satisfies $f(s) \neq 0\ \forall s \in J$ is an inductive family with respect to inclusion. Let T_0 be a maximal element of $\mathcal{G}$. Suppose $T_0 \neq B \setminus Ker(f_0)$ and let $x \in B \setminus (T_0 \cup Ker(f_0)$. Let T_1 be the subsemi-group generated by x and T_0. Then f_0 is not T_1 -multiplicative. But since f_0 is submultiplicative and since T_0 is maximal, there exists $t \in T_0$ such that $f_0(tx) < f_0(t)f_0(x)$, hence $f_0^{T_0} \neq f_0$. On the other hand, since $f_0^{T_0}$ obviously is S-multiplicative, $f_0^{T_0}$ belongs to $Z(B,S,\theta)$. Consequently, we can check that $N_0 \cup \{f_0^{T_0}\}$ is a constructible subset of $Z(B,S,\theta)$, a contradiction to the hypothesis "N_0 is maximal". Consequently, $T_0 = B \setminus Ker(f_0)$. Thus, f_0 is T_0-multiplicatve. On the other hand, f_0 trivially satisfies $f(xy) \leq f(x)f(y) = 0\ \forall x \in Ker(f_0), y \in B$, hence $f(xy) = f(x)f(y) \forall x \in Ker(f_0), y \in B$. This finishes proving that f_0 is B-multiplicative, and therefore belongs to $Min(B,\theta)$. Finally, by definition, all elements of $Z(B,S,\theta)$ satisfies $f(x) = \theta(x)\ \forall x \in S$, so f_0 is the f we have looked for.

Theorem 8 *Let $\| \,.\, \|$ be semi-multiplicative. Then $Min(A,\| \,.\, \|)$ is not empty and is a boundary for $(A,\| \,.\, \|)$.*

Proof Let $x \in A$ and let S_x be the subsemi-group generated by x in A. Since $\| \,.\, \|$ is semi-multiplicative, $\| \,.\, \|$ is obviously S_x-multiplicative. Hence by Lemma 7 there exists $f \in Min(A,\| \,.\, \|)$ such that $f(x) = \|x\|$, and by Corollary 6 $Min(A,\| \,.\, \|)$ is included in $Mult(A,\| \,.\, \|)$, hence it is a boundary for $(A,\| \,.\, \|)$.

Notation Let γ be a semi-group homomorphism from B into B'. We will denote by $\overline{\gamma}$ the mapping from $\mathcal{F}(B',\mathbb{R})$ into $\mathcal{F}(B,\mathbb{R})$ defined by $\overline{\gamma}(\phi) = \phi \circ \gamma$, $\forall \phi \in \mathcal{F}(B',\mathbb{R})$.

Lemma 9 *Let γ be a semi-group homomorphism from B into B' and let θ (resp. θ') be semi-multiplicative functions such that $\theta = \theta' \circ \gamma$ it. The restriction of $\overline{\gamma}$ to $Z(B',\theta')$ is a surjection onto $Z(B,\theta)$. Moreover, $Min(B,\theta) \subset \overline{\gamma}(Min(B',\theta'))$.*

Proof Suppose Lemma 9 is not true. Then, there exists $\nu \in Z(B,\theta)$ such that $\nu \neq \phi \circ \gamma\ \forall \phi \in Z(B',\theta')$. Let $\mathcal{T}$ be a constructible ordered set admitting θ as its first element and ν as its last element. Let $\mathcal{S}$ be the set of $\phi \in \mathcal{T}$ such that $\phi \neq \nu' \circ \gamma\ \forall \nu' \in Z(B',\theta)$, let ψ be the maximal element of $\mathcal{S}$ and let $\mathcal{L}$ be the set of $\phi \in \mathcal{T}$ which are of the form $\phi' \circ \gamma$, with $\phi' \in Z(B', \theta')$ and satisfy $\phi \geq \psi$. Let $\mathcal{Q}$ be the family of constructible subsets of $Z(B', \theta')$ admitting θ' as a first element, consisting of ϕ' such that $\phi' \circ \gamma \in \mathcal{L}$. Then $\mathcal{Q}$ is inductive for inclusion. Let X be a maximal element of $\mathcal{Q}$ and let $\zeta = \inf\{\phi(x) \mid \phi \in X\}$, $\forall x \in B$. By Lemma 2 $X \cup \{\zeta\}$ is a constructible subset of $Z(B', \theta')$. In $\mathcal{T}$, we notice that $\zeta \circ \gamma(x) = \inf\{\xi \circ \gamma(x) \mid \xi \circ \gamma \in \mathcal{T}\ ,\ \xi \in X\}\ \forall x \in B$, hence of course $\psi \leq \zeta \circ \gamma$. Then, since $\psi \in \mathcal{T}$, by Lemma 2 $\zeta \circ \gamma$ belongs to $\mathcal{T}$, and therefore belongs to $\mathcal{L}$. But then, by Lemma 2 $X \cup \{\zeta\}$ belongs to Q, hence $\zeta \in X$ because X is maximal. Consequently, $\zeta \circ \gamma > \psi$, and hence, $\zeta \circ \gamma$ has follower ν such that $\psi \leq \nu$. Let S be a subsemi-group of B such that $\nu = (\zeta \circ \gamma)^S$ and let $\nu' = \zeta^{\gamma(S)}$. Then $\nu = \nu' \circ \gamma$. On the other hand, ν' is the follower of ζ in the set $X' = X \cup \{\nu'\}$, hence X' is a constructible set, so X' clearly belongs to $\mathcal{Q}$, a contradiction with the hypothesis "X maximal". This proves that for every $\nu \in Z(B, \theta)$ there exists $\nu' \in Z(B', \theta')$ such that $\nu = \nu' \circ \gamma$.

Now, we can show that $Min(B, \theta) \subset \overline{\gamma}(Min(B', \theta'))$. Let $\zeta \in Min(B, \theta)$. As we just showed, there exists $\xi \in Z(B', \theta')$ such that $\zeta = \xi \circ \gamma$. Let $Y = \gamma(B) \setminus \xi^{-1}(0)$. Then Y is a subsemi-group of $(B', .)$ and we can check that ξ is Y-multiplicative. By Lemma 17 there exists $\zeta' \in Min(B', \xi)$ such that $\zeta'(x) = \zeta(x)\ \forall x \in Y$, hence $\zeta = \zeta' \circ \gamma$. But of course $Min(B', \xi) \subset Min(B', \theta')$, hence f belongs to $\overline{\gamma}(Min(B', \theta'))$, which ends the proof of Lemma 9.

We must recall Urysohn's Theorem [2].

Theorem 10 (Urysohn) *D be a compact and let α, β be two different points of D. There exists a continuous mapping f from D into $\mathbb{R}_+$ such that $f(\alpha) = 0$ and $f(\beta) = 1$.*

Notation Given a topological space D we denote by $\mathcal{C}(D, \mathbb{R})$ the algebra of continuous functions from D into $\mathbb{R}$.

Lemma 11 *Let D be a compact, and let ϕ be the mapping from D into $Mult(C(D, \mathbb{R}), \| \ . \ \|D)$ defined by $\phi(\alpha)(f) = |f(\alpha)|$. Then ϕ is a bijection from D onto $Mult(C(D, \mathbb{R}), \| \ . \ \|_D)$ which is equal to $Min(C(D, \mathbb{R}), \| \ . \ \|_D)$.*

Proof Let $\mathcal{X}(\mathcal{C}(D, \mathbb{R}))$ be the set of $\mathbb{R}$-algebra homomorphisms from $\mathcal{C}(D, \mathbb{R})$ onto $\mathbb{R}$. According to classical results in Archimedean spectral theory [3], the Gelfand mapping ω from D into $\mathcal{X}(\mathcal{C}(D, \mathbb{R}))$ defined as $\omega(\alpha)(f) = f(\alpha)$ is a bijection. On the other hand, the injective mapping Ξ from $\mathcal{X}(\mathcal{C}(D, \mathbb{R}))$ into $Mult(\mathcal{C}(D, \mathbb{R}), \| \ . \ \|_D)$ defined as $\Xi(\chi)(f) = |\chi(f)|$ is a bijection from $\mathcal{X}(\mathcal{C}(D, \mathbb{R}))$ onto $Mult(\mathcal{C}(D, \mathbb{R}), \| \ . \ \|_D)$ [3]. Consequently, since $\phi = \Xi \circ \omega$, ϕ is a bijection from D onto $Mult(\mathcal{C}(D, \mathbb{R}), \| \ . \ \|_D)$. Now, by definition, $Min(\mathcal{C}(D, \mathbb{R}), \| \ . \ \|_D)$ is included in $Mult(\mathcal{C}(D, \mathbb{R}), \| \ . \ \|_D)$, so we just have to show that $Mult(\mathcal{C}(D, \mathbb{R}), \| \ . \ \|_D) \subset$

$Min(\mathcal{C}(D,\mathbb{R}),\|\ .\ \|_D)$. Let $\alpha \in D$, and let us show that $\phi(\alpha)$ belongs to $Min(\mathcal{C}(D,\mathbb{R}),\|\ .\ \|_D)$. Let $S = \{f \in \mathcal{C}(D,\mathbb{R}) \mid |f(\alpha)| = \|f\|_D = 1\}$. Then S is a subsemi-group of $\mathcal{C}(D,\mathbb{R})$ and the norm $\|\ .\ \|_D$ is S-multiplicative, hence by Lemma 7 there exists $\zeta \in Min(\ C(D,\mathbb{R}),\|\ .\ \|_D)$ such that $\zeta(f) = 1\ \forall f \in S$. But there exists $\beta \in D$ such that $\phi(\beta) = \zeta$, hence $|f(\beta)| = 1$. Consequently, for every $f \in \mathcal{C}(D,\mathbb{R})$ such that $|f(\alpha)| = 1,\ f$ must also satisfy $|f(\beta)| = 1$. Since D is compact, by Theorem 10 this implies $\alpha = \beta$ and therefore $\phi(\alpha)$ belongs to $Min(\mathcal{C}(D,\mathbb{R}),$ $\|\ .\ \|_D)$, which ends the proof.

Proof of Theorem C By Corollary 6 and Theorem 8, $Min(A,\|\ .\ \|)$ is not empty, is included in $Mult(A,\|\ .\ \|)$ and is a boundary for $(A,\|\ .\ \|)$. Now, let F be a closed boundary for $(A,\|\ .\ \|)$ and let H be the multiplicative semi-group $\mathcal{C}(F,\mathbb{R})$. Since F is a closed subset of $Mult(A,\|\ .\ \|)$, by Lemma 1 it is a compact. Let γ be the Gelfand transform defined on A, taking values in $C(F,\mathbb{R})$, as $\gamma(x)(\alpha) = \alpha(x),\ \alpha \in F$. Let $x \in A$. On one hand, we have $\|x\| \geq \sup_{\psi \in F} \psi(x)$. On the other hand, since F is a boundary for $(A,\|\ .\ \|)$ there exists $\varphi \in F$ such that $\|x\| = \phi(x)$, and consequently, $\|\gamma(x)\|_F = \|x\|$.

Now, since $\|\ .\ \|_F \circ \gamma = \|\ .\ \|$ and since γ is a semi-group homomorphism from $(A,.)$ into $(\mathcal{C}(F,\mathbb{R}),.)$, we can apply Lemma 9 and we have $Min(A,\|\ .\ \|) \subset \overline{\gamma}(Min(\mathcal{C}(F,\mathbb{R}),\|\ .\ \|_F))$. Let $\nu \in Min(A,\|\ .\ \|)$. So, there exists $\psi \in Min(\mathcal{C}(F,\mathbb{R}),\|\ .\ \|_F)$ such that $\nu = \psi \circ \gamma$. Now, let ϕ be the mapping from F into $Mult(\mathcal{C}(F,\mathbb{R}),\|\ .\ \|_F)$ defined as $\phi(\alpha)(f) = |f(\alpha)|,\ f \in \mathcal{C}(F,\mathbb{R})$. By Lemma 11, ϕ is a bijection from F onto $Min(\mathcal{C}(F,\mathbb{R}),\|\ .\ \|_F)$. Consequently, $\psi \circ \gamma$ is of the form $\phi(\alpha)$. And finally, for every $x \in A$ we have $\nu(x) = \overline{\gamma}(\phi(\alpha)(x) = (\phi(\alpha) \circ \gamma)(x) = |\gamma(x)(\alpha)| = |\alpha(x)| = \alpha(x)$, because by definition α lies in $Mult(A,\|\ .\ \|)$. Thus ν belongs to F. Therefore $Min(A,\|\ .\ \|)$ is included in F and then the closure of $Min(A,\|\ .\ \|)$ is the smallest of all closed boundaries for $(A,\|\ .\ \|)$, which ends the proof.

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BMO-MAPPINGS IN THE PLANE

Cabiria Andreian Cazacu
Faculty of Mathematics
University of Bucharest
Str. Academiei 14
Bucharest
701091 Romania
cabiria.andreian@imar.ro

Victoria Stanciu
Department of Mathematics
University "Politehnica"
Splaiul Independentei 313
Bucharest 6, c. 772061
vstanciu@math.pub.ro

Abstract This survey paper presents in the 2-dimensional case recent results which generalize plane quasiconformality and quasiregularity to the case of mappings whose distortion is dominated by a BMO-function. These are the so-called BMO-mappings. After a brief exposure on real BMO-functions in §2 classes of BMO-mappings are discussed in §3. §4 is devoted to $BMO - QC$ and $-QR$ mappings in the sense of Ryazanov, Srebro and Yakubov, and §4 to BMO–BD considered by Astala, Iwanieć, Koskela and Martin. BMO-mappings between Riemann and Klein surfaces are discussed in §4.5.

Keywords: mappings in the plane: quasiconformal (QC), quasiregular (QR), with finite distortion, $BMO - QR$, $BMO - QC$, BMO-bounded distortion, Riemann and Klein surfaces.

Mathematics Subject Classification (2000): 30C62, 30D45, 30D50, 46E35, 30F50

1. Introduction

Conformal mappings and analytic functions of one complex variable found, on the way opened by the outstanding work of H. Grötzsch [23],

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M.A. Lavrentiev [36], L.V. Ahlfors [2], O. Teichmüller [58] and many others, an important generalization to the spaces $\mathbb{R}^n$, $n \geq 2$, (and even more general), in the theory of quasiconformal (QC) and quasiregular (QR) mappings. Today this theory is presented in renowned monographs, e.g. by O. Lehto and K.I. Virtanen [37], L.V. Ahlfors [1] for $n = 2$, J. Väisälä [61], P. Caraman [15], Yu.G. Reshetnyak [41], M. Vuorinen [64], S. Rickman [42] for $n \geq 3$.

The specific feature of the quasiconformality (QCTY) and quasiregularity (QRTY) is that these mappings realize bounded deformations of the considered geometric objects, expressed by *bounded dilatations* or *distortions*. Gradually mappings with *unbounded dilatations* now usually called with *finite distortion* have also been studied by Teichmüller [58] p. 15, Volkovyskii [60], Andreian Cazacu [3], Lehto [35], Krushkal, Kühnau [34] and others, see also references in [34] and [46], p. 2. This research direction began to develop intensively especially after David's thesis [19], in connexion with PDEs and the nonlinear elasticity theory by J. Ball [12, 13] and others. The boundedness condition of the distortion by a constant was replaced by weaker conditions either controlling the measure of the set where the distortion is large or majorizing the distortion by a function Q with adequate properties.

An important case is obtained when Q is a function of bounded mean oscillation, denoted $Q \in BMO$, or a function locally (l.) of bounded mean oscillation, $Q \in BMO_{loc}$, and the aim of this survey is to present some essential aspects of the theory of these mappings in dimension 2. We shall limit us at this situation even if the exposed results have been established by their authors in general $n \geq 2$ dimensions.

Our paper is far from being exhaustive. We only desired to make an attractive introduction into the recent remarkable progress realized on the subject.

In the center of our exposure there are two closely related theories on BMO-mappings: the first one is due to V. Ryazanov, U. Srebro and E. Yakubov, which extended in an harmonious construction most of the classical function theory [44–47], while the other by K. Astala, T. Iwanieć, P. Koskela and G. Martin [26],[7] considered mappings in Orlicz-Sobolev classes. In §3.E we discuss BMO-mappings between Riemann and Klein surfaces.

References could not include all new significant papers. We wanted to emphasize the deep work about the Beltrami equation due to Iwanieć and his collaborators [25, 29].

2. BMO functions

Definition 1 Let G be a domain in $\mathbb{C}$. A function $u : G \to \mathbb{R}$ is called of *bounded mean oscillation* in G, $u \in BMO(G)$, if $u \in L^1_{loc}(G)$ and

$$\|u\|_* = \sup_B \frac{1}{|B|} \int_B |u(z) - u_B| dm(z) < \infty, \tag{1}$$

where the supremum is taken over all disks B in G,

$$u_B = \frac{1}{|B|} \int_B u(z) dm(z) \tag{2}$$

and $|A|$ denotes the Lebesgue measure of the measurable set $A \subset \mathbb{C}$.
Definition 2 The function $u \in BMO_{loc}(G)$ if $u|_U \in BMO(U)$ for every relatively compact subdomain U of G.

The definitions extend to $\widehat{\mathbb{C}}$ by using the spherical metric [43] p. 7; [46], p. 4 instead of the Euclidean one.

Obviously, $u \in BMO(G)$ and $c \in \mathbb{R}$ imply $u + c \in BMO(G)$ and $\|u\|_* = \|u+c\|_*$, such that the space of BMO functions modulo constants with the norm $\| \ \|_*$ (1.2) is a Banach space. In addition $BMO(G)$ is a lattice [43], p. 2. $L^\infty(G) \subset BMO(G)$.

The BMO functions have been introduced by F. John and L. Nirenberg in [33], where they proved the fundamental
Lemma 1 ([46], Lemma 2.3, p. 3) *If u is a non-constant function in $BMO(G)$, then*

$$|\{z \in B : |u(z) - u_B| > t\}| \le a e^{-bt/\|u\|_*} |B| \tag{3}$$

for every disk B in G and all $t > 0$, where a and b are positive constants, which do not depend on B and u. Conversely, if $u \in L^1_{loc}$, and for every disk B in G and all $t > 0$,

$$|\{z \in B : |u(z) - u_B| > t\}| < a e^{-bt} |B| \tag{4}$$

for some positive constants a and b, then $u \in BMO(G)$.

From Lemma 1 one obtains the following useful
Lemma 2 ([46], Lemma 2.6, p. 4) *If u is a non-constant function in $BMO(G)$, then*

$$|\{z \in B : |u(z)| > \tau\}| \le A e^{-\beta\tau} |B| \tag{5}$$

for every disk B in G and all $\tau > |u_B|$, where

$$\beta = b/\|u\|_* \ \text{ and } \ A = a e^{b|u_B|/\|u\|_*}, \tag{6}$$

with the constants a and b as in Lemma 1.

It follows immediately from Lemma 2 that $BMO(G) \subset L^p_{loc}(G)$ for $1 \leq p < \infty$.

C. Fefferman [20] characterized the BMO space as the dual of the Hardy space H^1. Other characterizations of BMO have been given by R.R. Coifman and R. Rochberg [18], by L. Carleson [16] and A. Uchiyama [59].

An important connexion between the BMO functions and the K-quasiconformal ($K-QC$) mappings was established by H.M. Reimann [40], p. 266. He proved that the logarithm of the Jacobian of a QC mapping is a BMO function and characterized the QCTY by means of BMO functions.

Theorem 1 ([40], Theorems 2 and 3, pp. 266-267) *Let $f : G \to G'$ be a sense-preserving (s.p.), absolutely continuous on lines (ACL),* [1], p. 23; [37] p. 8, p. 117, *homeomorphism of the domain G onto the domain G', G and G' in $\mathbb{C}$. The pull-back $f^\#$ realizes a bijective isomorphism between the spaces $BMO(G')$ and $BMO(G)$ associating to every $u' \in BMO(G')$ the function $u = u' \circ f \in BMO(G)$, and $\|u'\|_* \leq c\|u\|_*$ where c is a constant, if and only if f is QC.*

Remark 1 If f is $K - QC$, then c depends only on K. In this case, as f^{-1} is $K - QC$, one has $c^{-1}\|u\|_* \leq \|u'\|_*$ too.

These results are contained also in the H.M. Reimann and T. Rychener monograph [43], p. 100, p. 107.

K. Astala improved Theorem 1 by showing that the ACL property can be omitted through the localization of the problem.

Theorem 2 ([8], Theorem 3, pp. 209-210) *If $f : G \to G'$ is a s.p. homeomorphism and there is a constant $c > 0$ such that*

$$c^{-1}\|u\|_{*D'} \leq \|u \circ f\|_{*D} \leq c\|u\|_{*D'}$$

for any subdomain $D \subset G$ and any $u \in BMO(D')$, $D' = f(D)$, then f is QC.

Remark 2 In the proofs of the properties of the $BMO-QC$ and $-QR$ mappings in Section 2 it is useful the fact that the BMO functions can be extended over isolated singularities [43], p. 5; [46], Proposition 2.16, p. 6 or over a free boundary arc in ∂G, which is either a line segment or a circular arc, by symmetry [43], p. 8; [46], Lemma 2.20, p. 7, as well as over ∂G in the case that ∂G is a K-quasicircle [32], Theorem 2, p. 42; [46], Proposition 2.18, p. 6.

Remark 3 Important applications of BMO functions to quasidisks are contained in [21]. Much literature on BMO functions can be found in [22].

3. Classes of mappings

Let be G a domain in $\mathbb{C}$ and $f : G \to \mathbb{C}$ an ACL mapping. Then f has partial derivatives

$$\partial f = f_z = \frac{1}{2}(f_x - if_y) \text{ and } \bar{\partial} f = f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$$

a.e. in G. If in addition we suppose that f is open, it follows by a result of Gehring and Lehto [37], p. 128; [1], p. 24] that a.e. in G f is differentiable and has the Jacobian $J(z) = J_f(z) = |\partial f|^2 - |\bar{\partial} f|^2$.

The *complex dilatation* $\mu(z) = \mu_f(z)$ is defined a.e. in G by

$$\mu(z) = \begin{cases} \bar{\partial} f/\partial f & \text{if} \quad \partial f(z) \neq 0 \\ 0 & \text{if} \quad \partial f(z) = 0. \end{cases} \tag{7}$$

The local properties of f may also be expressed in terms of the (*quotient of*) *dilatation*

$$K(z) = K_f(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}. \tag{8}$$

If f is s.p., then a.e. in G: $J(z) \geq 0$, $|\mu(z)| \leq 1$ and $K(z) \geq 1$.

There exist several equivalent definitions for QC and QR mappings, based on different dilatations expressing the deviation from the conformal case, but their characteristic feature consists in the fact that the dilatations are bounded a.e. by a certain constant. This corresponds to the conditions: $\|\mu\|_\infty < 1$ or equivalently $K \in L^\infty(G)$. The theory is generalized by using instead of a constant, a measurable function $Q : G \to [1, \infty)$.

Throughout the paper mapping means continuous mapping, if nothing else is specified.

Definition 3 ([45], p. 2; [46], p. 10) A mapping $f : G \to \mathbb{C}$ is called $Q(z) - QR$ if f is either a constant or a s.p., open and discrete, ACL mapping with

$$K(z) \leq Q(z), \tag{9}$$

or equivalently

$$|\mu(z)| \leq (Q(z) - 1)/(Q(z) + 1) \tag{10}$$

a.e. in G. If f is an embedding (i.e. a homeomorphism into) then it is called $Q(z) - QC$.

Evidently, if $Q(z)$ is a constant $Q \geq 1$, then f is $Q - QR$ or $Q - QC$ respectively.

Ryazanov, Srebro and Yakubov define the $\mathbf{F}-QR$ and $\mathbf{F}-QCc$ mapping by the condition $Q(z) \in \mathbf{F}$, where $\mathbf{F}$ is some mapping class. We mention also the notations $\mathbf{F}-QC(G)$ and $\mathbf{F}-QR(G)$ for the class of $\mathbf{F}-QC$ and $\mathbf{F}-QR$ mappings respectively.

In the present paper $\mathbf{F}$ will usually be the class $BMO(G)$ or BMO_{loc} (G), but sometimes $\mathbf{F}$ will be $L^p_{loc}(G)$, $p \geq 1$.

From direct calculation, using Hölder's inequality, one obtains that $BMO_{loc}-QC \subset W^{1,s}_{loc}$ for all $s \in [1,2)$. However, there are $BMO-QC$ mappings that are not in $W^{1,2}_{loc}$ ([45], Proposition 2.2, p. 5).

Unlike the $QCTY$, there is a $BMO-QC$ mapping $f : G \to \mathbb{C}$ such that f^{-1} is not in $BMO-QC(f(G))$ and also there are $BMO-QC$ mappings $f : G \to \mathbb{C}$ and $g : f(G) \to \mathbb{C}$ such that $g \circ f$ is not in $BMO-QC(G)$ [45] Proposition 2.4, p. 6. $BMO-QC$ is invariant under pre- and post-composition with QC mappings [46] Corollary 4.14, p. 17.

If $f : G \to \mathbb{C}$ is a $Q(z)-QR$ or $Q(z)-QC$ mapping, then it verifies a.e. in G the *Beltrami equation*

$$\bar{\partial} f = \mu \partial f \tag{11}$$

with μ defined by (7) measurable and $\|\mu\|_\infty \leq 1$.

A homeomorphism $f : G \to f(G)$ which verifies (11) a.e. in G for a measurable $\mu : G \to \mathbb{C}$ with $\|\mu\|_\infty \leq 1$ is often called a *μ-homeomorphism* or even a *μ-conformal mapping*, under some additional conditions on μ and f in different papers.

Classes of μ-homeomorphisms have been introduced by O. Lehto [35], G. David [19], P. Tukia [57]. M.A. Brakalova and J.A. Jenkins [14].

Thus Lehto considered μ-homeomorphisms of the plane for a measurable μ with $|\mu(z)|$ bounded away from 1 in every compact subset of the complement of E, where E is a compact set of zero area, but μ may tend to 1 as z approaches E.

David's class $D(G)$ contains ACL μ-homeomorphisms with μ satisfying an exponential condition

$$|\{z \in G : |\mu(z)| > 1 - \varepsilon\}| \leq Ce^{-\alpha/\varepsilon} \tag{12}$$

for all $\varepsilon \in (0, \varepsilon_0]$, and $\varepsilon_0 \in (0,1]$, $C > 0$, $\alpha > 0$ – constants.

Tukia's class $T(G)$ replaces (12) by

$$\sigma\{z \in G : K(z) > t\} \leq Ce^{-\alpha t} \tag{13}$$

for all $t \geq t_0$, where σ designates the spherical area and $t_0 \geq 0$, $C > 0$, $\alpha > 0$ are constants.

Further by Brakalova and Jenkins' class $BJ(G)$, μ satisfies the conditions:

$$\int_B \exp\left[\frac{1}{1-|\mu|}/(1+\log\frac{1}{1-|\mu|})\right] dm < \Phi_B, \tag{14}$$

where B is an arbitrary bounded measurable set in G and Φ_B a constant depending on B, and

$$\int_{B(0,R)\cap G} \frac{1}{1-|\mu|} dm = O(R^2), \quad R \to \infty. \tag{15}$$

Locally the classes $D(G)$, $T(G)$ and $BMO_{loc}(G)$ coincide. Moreover $D(G) \subset T(G) \subset BMO-QC(G)$. If G is quasidisk, then $T(G) = BMO-QC(G)$. For some domains G, $T(G) \neq BMO-QC(G)$. For some domains G, $BMO-QC(G)\backslash BJ(G) \neq \emptyset$ ([45], Proposition 2.1, p. 4).

There are now many other papers concerning mappings with finite distortion (see Definition 4, below) related to the $Q(z)-QR$ mappings, especially to the $BMO-QR(G)$.

An example is given by the mappings of *BMO-bounded distortion* ($BMO-BD$) [7] and [26], whose class we shall denote by $BMO-BD(G)$. In spite of name similarity this class differs from $BMO-QR(G)$ by the regularity hypotheses about the considered mappings, which belong now to the Orlicz–Sobolev space $W^{1,P}_{loc}(G)$, and also by the fact that the topological properties: continuity, openness and discreteness are deduced and not contained in definition . The classes also differ by used methods, which are more geometric based on modulus estimates in the $BMO-QR$ case and more analytic for the $BMO-BD$.

For the sake of completeness we define some auxiliary notions, in our case $n = 2$ (see general case and results in [7] and [26].

A continuous function $P : [0,\infty] \to [0,\infty]$ is called an *Orlicz function* if it is strictly increasing with $P(0) = 0$ and $P(\infty) = \infty$, and *Young function* if P is also convex. The *Orlicz space* is $L^P(G) = \{h : G \to \mathbb{C}; h$ measurable and $\int_G P(\lambda^{-1}|h(z)|)dm(z) < \infty$ for some $\lambda = \lambda(h) > 0\}$. If

$$P(t) = t^p \log^\alpha(e+t), \tag{16}$$

then $L^P(G)$ is called a *Zygmund space*. The *Orlicz–Sobolev space* $W^{1,P}(G)$ consists of functions whose distributional gradient in $W^{1,1}_{loc}(G)$ belongs to $L^P(G)$ ([7], pp. 706–707).

Definition 4 ([7], p. 704) The mapping $f : G \to \mathbb{C}$ has *finite distortion* if

(i) $f \in W^{1,\Phi}_{loc}(G)$ with $\Phi(t) = t^2 \log^{-1}(e+t)$, $0 \le t < \infty$, and

(ii) there is a function $\mathcal{K}(z)$, $1 \leq \mathcal{K}(z) < \infty$ a.e. in G, such that

$$|Df(z)|^2 \leq \mathcal{K}(z)J(z) \text{ a.e. in } G, \tag{17}$$

$Df(z)$ is the differential of f at z and $|Df(z)| = \sup\{|Df(z)h| : |h| = 1\}$.

$\mathcal{K}$ is called a *distortion function* for f and K in (8) is one of the distortion functions verifying (17) with equality at a differentiability point of f.

Definition 5 ([7], p. 704) A mapping f has *BMO-bounded distortion* if

(i) $f \in W^{1,\Phi}_{loc}(G)$ and

(ii) there is a distortion function $M \in BMO(\mathbb{C})$.

One sees that M plays the same role as Q for $BMO - QR$.

The class $BMO - BD$ generalizes the $QRTY$ in both requirements. The QR mappings are obtained when $\mathcal{K}$ is bounded and M may be taken a constant, and when $f \in W^{1,2}_{loc}(G)$.

T. Iwanieć and C. Sbordone [30] established a necessary and sufficient condition that $\mathcal{K}$ could be majorized in G by a function $M \in BMO(\mathbb{C})$, namely

$$\int_G e^{\lambda\mathcal{K}(z)}(1 + |z|^3)^{-1} dm(z) < \infty \text{ for some } \lambda > 0.$$

Moreover then $\|M\|_* \leq C\lambda^{-1}$, with a constant C.

Important results are obtained in [7] if $\|M\|_*$ is sufficiently small or equivalently if λ is large enough (see Section 5 below).

$BMO - BD$ is invariant with respect to a QC–change of variables.

Under the generic name of mappings of BMO distortion, T. Iwanieć, P. Koskela and G. Martin studied in [26] mappings in Sobolev or Orlicz–Sobolev spaces, which admit a distortion function M in BMO, i.e. which satisfy condition (ii) in Definition 5 with different regularity assumptions instead of (i). By using the example of the radial stretchings, they prove that for each $\varepsilon > 0$, there is a mapping $f \in W^{1,1}_{loc}(G)$ with $\|K\|_* < \varepsilon$ and yet f^{-1} does not admit any BMO distortion function [26], Corollary 5.1, p. 19.

They work with the following definitions:

Definition 6 ([26], p. 17) The mapping $f : G \to \mathbb{C}$ has *finite distortion* if (i) $f \in W^{1,1}_{loc}(G)$ and (ii) as in Definition 4.

Definition 7 ([26], Definition 7.1, p. 22) The mapping $f : G \to \mathbb{C}$ is of $BMO - BD$, if (i) $f \in W^{1,P}_{loc}(G)$ with $P(t) = t\log^{-1}(e + t)$, and (ii) as in Definition 5.

Thus if $\|M\|_* < \varepsilon$ with certain constant ε, then $f \in W^{1,2}_{loc}(G)$ ([26], Theorem 7.1, p. 23).

Among the results obtained with Definition 7 let us quote that each $BMO-BD$ mapping f belongs to $W^{1,P}_{loc}(G)$ with $P(t) = t^2 \log\log^{\beta}(8+t)/\log(3+t)$, for all $\beta \in \mathbb{Z}^*$ ([26], Theorem 8.2, p. 40).

In his lecture at the 9th Romanian-Finnish Seminar (Braşov, August, 2001), U. Srebro presented, based on a joint work with O. Martio, V. Ryazanov and E. Yakubov, the more general class of *finite metric distortion* (FMD) *mappings* and classes based on the geometric definition of $QCTY$.

As in the metric definition of the $QCTY$, let us denote for a mapping $f : G \to \mathbb{C}$

$$L(z, f) = \limsup_{\zeta\to z} \frac{|f(\zeta) - f(z)|}{|\zeta - z|}, l(z, f) = \liminf_{\zeta\to z} \frac{|f(\zeta) - f(z)|}{|\zeta - z|}.$$

At points $z \in G$ where f is differentiable, $L(z, f) = |Df(z)|$.
Definition 8 f is an *FMD–mapping* in G if there is a measurable set $E \subset G$ with $|E| = |f(E)| = 0$ such that for every $z \in G\backslash E$

$$0 < l(z, f) \leq L(z, f) < \infty. \tag{18}$$

Then the interior transformations of S. Stoilow [55], 5, I, 6, p. 107, i.e. continuous, open and discrete, are called *coverings* and an *FDM*-mapping which is an s.p. covering (s.p. homeomorphism) is called an *FMD-covering* (*FMD-homeomorphism*).

Many properties are proved for the FMD–mappings and among them relations with $BMO - QR$.

4. $BMO - QC$ and - QR

Since the $BMO-QC$ and -QR mapping classes were thought by their authors as natural extension of the classic QC and QR, they had the successful idea to use as a main tool the good approximation.

A sequence of $Q(z) - QR$ mappings $f_n : G \to \widehat{\mathbb{C}}$ with complex dilatations μ_n, $n = 1, 2, \ldots$, is a *good approximation* of the $Q(z) - QR$ mapping $f : G \to \widehat{\mathbb{C}}$ with the complex dilatation μ if:

(i) $\|\mu_n\|_\infty < 1$ for all n,
(ii) $f_n \to f$ l. uniformly in G,
(iii) $\mu_n \to \mu$ a.e. in G ([37], IV, 5.4, p. 185; [45], 5.4, p. 14).

Thus Ryazanov, Srebro and Yakubov prove distortion and convergence results (see sections A and B from below) first for QC $Q(z) - QC$ mappings. Then they prove existence, uniqueness, representation for $Q(z)$–solutions of Beltrami equation and the essential fact: every $Q(z) - QR$ mapping f has a good approximation if $Q \in BMO_{loc}(G)$

8[45], Theorem 5.5, p. 14). Further by good approximation previous results (A, B) are obtained in general form for $Q(z)-QC$ or -QR mappings. In the following we present the results in A and B directly in all generality, for the sake of conciseness.

4.1 Distortion

V. Ryazanov, U. Srebro and E. Yakubov establish distortion lemmas based on path family modulus techniques and the modulus inequality [37], V, (6.6), p. 221).

Let be $f: G \to \widehat{\mathbb{C}}$ a $Q(z)$ a QC mapping for a given function $Q \in L^1_{loc}(G)$, Γ a path family in G and $\rho \in adm\Gamma$, the family of admissible functions for the modulus $M(\Gamma)$ of Γ. Then

$$M(f\Gamma) \leq \int_{\mathbb{C}} Q(z)\rho^2(z)dm(z). \tag{19}$$

The distortion lemmas deal with $Q(z)-QC$ mappings in the unit disk Δ.

Lemma 3 ([46], Lemma 5.17, p. 20) *Let $f: \Delta \to \widehat{\mathbb{C}}\backslash\{a,b\}$, $a,b \in \widehat{\mathbb{C}}$, $k(a,b)=\delta>0$, be a $Q(z)-QC$ mapping for some $Q \in BMO(\Delta)$. Then for $|z|<e^{-2}$,*

$$k(f(z), f(0)) \leq C(\log 1/|z|)^{-\alpha}, \tag{20}$$

where C and α are positive constants, depending only on δ, $\|Q\|_*$ and $Q_{B(0,e^{-1})}$, and k is the spherical distance.

The inequality (20) follows by using Lemma 2.21 in [46] p. 7, (3.1), Gehring's result [46], Lemma 5.13, p. 19 on the capacity of rings in $\widehat{\mathbb{C}}$ which separate pairs of points, the capacity of a Teichmüller ring, and the relation between capacity and modulus.

Lemma 4 ([46], Lemma 5.23, p. 21) *Let $f: \Delta \to \widehat{\mathbb{C}}\backslash\{a,b\}, a,b \in \widehat{\mathbb{C}}$, $k(a,b) \geq \delta > 0$, be a $Q(z)-QC$ mapping for some Q in $L^1(\Delta)$. If $z_1 \in \Delta\backslash\{0\}$ with $k(f(z_1), f(0)) \geq \delta$,*

$$k(f(z), f(0)) \geq \psi(|z|) \tag{21}$$

for every z in $|z|<r=\min(1-|z_1|, |z_1|/2)$, where ψ is a positive strictly increasing function which depends only on δ and Q_Δ.

4.2 Convergence

Starting with the classical theorems on normality and compactness of meromorphic functions, which have been extended to QC or QR mappings ([37], II, 5, p. 71), many authors generalized this topic on other classes of mappings, e.g. G. David [19], P. Tukia [57], V. Ryazanov [48], V. Ryazanov, U. Srebro and E. Yakubov [44–46], V. Stanciu [49].

Ryazanov, Srebro and Yakubov present several forms of the normality criteria by proving the equicontinuity of the considered mapping family. Thus, as a consequence of Lemma 3, they establish:

Corollary 1 ([45], Corollary 3.11, p. 10) *Given a function* Q *in* BMO_{loc} (G) *and distinct points* a *and* b *in* $\widehat{\mathbb{C}}$, *the family of all* $Q(z)-QC$ *mappings* $f : G \to \widehat{\mathbb{C}} \backslash \{a, b\}$ *is normal.*

Other normality criteria for $BMO_{loc}-QC$ follow directly from Tukia's Theorem 3B ([57], p. 52), since the classes $BMO_{loc}-QC$ and T coincide locally:

Theorem 3 *Let* $\mathcal{F}$ *be a family of* $Q(z)-QC$ *mappings* $f : G \to \widehat{\mathbb{C}}$ *with* $Q \in BMO_{loc}(G)$. *The family* $\mathcal{F}$ *is normal if there is* $\delta > 0$ *such that one of the following conditions is satisfied:*

(i) *Every* $f \in \mathcal{F}$ *omits two points* a_f *and* b_f *in* $\widehat{\mathbb{C}}$ *whose spherical distance is* $\geq \delta$.

(ii) *There are* $z_1, z_2 \in G$ *and* $a \in \widehat{\mathbb{C}} \backslash f(G)$ *such that* $k(f(z_j), a) > \delta$, $j = 1, 2$, *for every* $f \in \mathcal{F}$.

(iii) *There are* $z_1, z_2, z_3 \in G$ *such that* $k(f(z_j), f(z_l)) > \delta$, $j, l = 1, 2, 3$, $j \neq l$, *for all* $f \in \mathcal{F}$.

Another very useful result, based on Lemma 3.4 and various techniques in [37] and [41], is the following:

Theorem 4 ([46], Corollary 5.27, p. 22) *Let* $f_n : G \to \widehat{\mathbb{C}}$ *be a sequence of* $Q(z)-QC$ *mappings for* $Q(z) \in L^1_{loc}(G)$. *If* $f_n \to f$ *l. uniformly, then either* f *is a constant or* f *is* $Q(z)-QC$, *and* ∂f_n *and* $\bar{\partial} f_n$ *converge weakly in* L^1_{loc} *to* ∂f *and* $\bar{\partial} f$ *respectively. If in addition* $\mu_{f_n} \to \mu$ *a.e. in* G, *then* $\bar{\partial} f = \mu \partial f$ *a.e.*

4.3 Beltrami equation-Existence, uniqueness and representation of solutions

As the $BMO-QR$ and $-QC$ mappings are defined by means of a Beltrami equation, §3, (11), the existence and the uniqueness of its solution form one of the main themes.

Following again Ryazanov, Srebro and Yakubov's work ([45], 4, p.12; [46], 6, p. 23) a mapping $f : G \to \widehat{\mathbb{C}}$ is a solution of the Beltrami equation

$$\bar{\partial} w = \mu(z) \partial w \tag{22}$$

with $\mu : G \to \mathbb{C}$ measurable and $\|\mu\|_\infty \leq 1$, if it is ACL and verifies (22) a.e. in G. Moreover, μ or its corresponding K, (8), is submitted to different constraints.

First, homeomorphic solutions are looked for.

Theorem 5 ([46], Theorem 6.26, p. 28, Corollary 6.27, p. 29) *If* $|\mu(z)| < 1$ *a.e. in* G *and* $K(z) \leq Q(z)$ *a.e. in* G *for some function* $Q \in BMO_{loc}(G)$*, then*

(i) (22) *has a homeomorphic solution* $f^{\mu} : G \to \mathbb{C}$,

(ii) $(f^{\mu})^{-1} \in W^{1,2}_{loc}(f^{\mu}(G))$.

(iii) $(f^{\mu})^{-1}$ *is l. absolutely continuous and preserves null sets,*

(iv) f^{μ} *is a.e. regular, i.e. differentiable and* $J_{f^{\mu}}(z) > 0$.

The authors give a direct and beautiful proof of this theorem. E.g. for (i) they consider f_n the QC solution of the Beltrami equation $\bar{\partial} w = \mu_n(z)\partial w$, with $\mu_n(z) = \mu(z)$ if $|\mu_n(z)| \leq 1-1/n$ and $\mu_n(z) = 0$ otherwise, which fixes two points z_1 and z_2 in G (cf. [1], p.98). The sequence $\{f_n\}$ forms a normal family by Corollary 1 and a l. uniformly convergent subsequence has as a limit f^{μ} by Theorem 4.

Further, they effectively construct homeomorphic solutions for μ with $K \in L^1_{loc}$ in some special cases, e.g. when μ depends only: (i) on Im z, (ii) on $|z|$ or (iii) on $arg z$ – case treated by A. Schatz ([54], Theorem 2, p. 293). They provide thus interesting examples and one sees that conformal type is not always preserved. On the contrary, there exists μ with $K \in L^p(\Delta)$, for any $p \in [1, \infty)$, such that the Beltrami equation (22) has no homeomorphic solution in Δ ([45], Proposition 2.5, p. 7).

The uniqueness is based on David's Theorem 2 ([19], p. 55), namely

Theorem 6 ([45], Theorem 5.2, p. 14) *Let* f_i*,* $i = 1, 2$*, be* μ*–homeomorphisms in* G*, with* μ *satisfying David's condition* (12)*. Then* $f_2 \circ f_1^{-1}$ *is conformal.*

This theorem applies due to the local coincidence of the classes D and $BMO - QC$.

The $BMO - QR$ solutions of (22) are obtained by Stoilow's factorization theorem, which says that every open and discrete mapping $g : G \to \widehat{\mathbb{C}}$ admits a representation

$$g = h \circ \varphi, \tag{23}$$

where φ is a homeomorphism into and h a meromorphic function in $\varphi(G)$ ([55], V, 5, p. 120). Thus it follows:

(i) if g is $BMO - QR$ in G, then $g = h \circ \varphi$, where φ is $BMO - QC$ in G with $\mu_{\varphi} = \mu_g$ a.e. in G, and

(ii) if g is $BMO - QR$ and φ is $BMO - QC$ in G with $\mu_{\varphi} = \mu_g$ a.e., then $g = h \circ \varphi$,

h being a meromorphic function in $\varphi(G)$ in both cases ([45], Proposition 5.1 and Corollary 5.3, p. 14).

Another class of Beltrami equation solutions with the representation (23) (hence open and discrete) was considered by T. Iwanieć and V. Šverak, who proved:

Theorem 7 ([31], Theorem 1, p. 182) *Every $W^{1,2}_{loc}(G)$ solution $g : G \to \mathbb{C}$ of the Beltrami equation* (22) *with $J_g \geq 0$ and $K_g \in L^1_{loc}(G)$ has the representation $g = h \circ \varphi$, where h is holomorphic and φ a homeomorphism, $\varphi^{-1} \in W^{1,2}_{loc}(\varphi(G))$ and $h \in W^{1,2}_{loc}(\varphi(G))$.*

By using Theorem 5 (ii), a direct proof of Theorem 7 is given in [45], Theorem 4.3, p. 13 under the assumption that (22) has $|\mu(z)| < 1$ a.e. in G and $K(z) \leq Q(z)$ a.e. in G, with $Q \in BMO_{loc}(G)$: every solution $g \in W^{1,2}_{loc}(G)$ has a representation $g = h \circ f^\mu$ with f^μ as in Theorem 5 and h holomorphic in $f^\mu(G)$.

Hence if a homeomorphic solution belongs to $W^{1,2}_{loc}$ all homeomorphic solutions are in $W^{1,2}_{loc}$. Any homeomorphic solution generates by the representation (23) all open and discrete solutions and all $W^{1,2}_{loc}$ solutions ([44], Corollary 5.7, p. 21).

Iwanieć and Martin showed that there are ACL solutions outside $W^{1,2}_{loc}$, which are not open and discrete ([29], Theorem 9.2, p. 37).

4.4 Properties of the $BMO-QC$ and $-QR$ mappings

Ryazanov, Srebro and Yakubov extended for these mappings many classical theorems, especially by means of the representation (23), of other results in sections A–C and properties of BMO functions, e.g. extension of BMO functions, Remark 2. Thus they proved

- the reflection principle for $BMO-QC$ and $-QR$ ([45], Theorem 6.1, p. 15 and Theorem 6.6, p. 17),

- removability of isolated singularities: the $BMO-QC$ extension of the $BMO-QC$ mappings in $\Delta\backslash\{0\}$ on Δ and in $\mathbb{C}$ on $\widehat{\mathbb{C}}$ ([45], Theorem 6.3, p. 16), (however there is an L^p-QC mapping, $1 \leq p < \infty$ in $\Delta\backslash\{0\}$ with a nonremovable singularity at 0 ([45], Proposition 2.5, p. 7)),

- the Carathéodory theorem for $BMO-QC$ mappings between Jordan domains, which extend to homeomorphisms between the domain closures, and if the domains are quasidisks to $\widehat{\mathbb{C}}$ ([45], Corollary 6.2, p. 16),

- the conformal type invariance under $BMO-QC$ mappings ([45], Proposition 6.4, p. 16),

- Picard's theorems for $BMO-QR$ mappings ([45], Theorem 6.5, p. 17).

These results are not true for $BMO_{loc} - QC$ or $-QR$ mappings ([45], Remark 6.7, p. 18).

Let us also mention Sastry's paper [53], where necessary and sufficient conditions for a $BMO - QC$ extension from $\mathbb{R}$ to the upper half plane $\mathbb{H}_+$ as in Beurling-Ahlfors theory are given.

Jacobian's properties for $BMO_{loc} - QC$ mappings and applications in integral calculus: formulae for the change of variables, area of the image and pre-image of a measurable set under a $BMO_{loc} - QC$ mapping, generalized Green's formula for $BMO_{loc} - QR$ mappings are also treated in [46], 10, pp. 37-40.

The convergence Theorem 4 is completed by the weak convergence in L^1_{loc} of the Jacobian sequence J_{f_n} to the Jacobian of the limit mapping J_f, and the same for $J_{f_n^{-1}}$ and $J_{f^{-1}}$, as well as the weak convergence in L^s_{loc}, $s \in (1, 2)$ of the derivatives ([46], Theorem 10.10, p. 38 and Corollary 10.18, p. 40).

4.5 $BMO - QC$ and - QR mappings between Riemann and Klein surfaces

The classes $BMO - QC$ and -QR can be also defined and thoroughly studied for Riemann and Klein surfaces. In the sequel by a Klein surface we understand a non-orientable surface without border endowed with a dianalytic structure given by a dianalytic atlas ([6],1,§2), which is also sometimes called a non-orientable Riemann surface.

In [43], p. 9, Reimann and Rychener defined BMO-functions on Riemann surfaces by means of the universal covering and the definition applies to Klein surfaces too. In the same way we defined the $BMO - QC$ and $-QR$ mappings between such surfaces and in the following we shortly present our results concerning convergence and compactness properties of these mappings [4, 5, 49–52].

Let X and X' be Riemann or Klein homeomorphic surfaces, $(\hat{X}, \Pi, X)$ and $(\hat{X}', \Pi', X')$ their universal coverings, where $\Pi : \hat{X} \to X$ and $\Pi' : \hat{X}' \to X'$. Here $\hat{X}$ and $\hat{X}'$ are either $\mathbb{C}$, Δ or $\widehat{\mathbb{C}}$ with the corresponding metric: Euclidean, hyperbolic or spherical. The metrics on $\hat{X}$ and $\hat{X}'$ induce by Π and Π' the metrics of X and X'. The convergence will always be taken with respect to these metrics.

Definition 8 1. The function $Q : X \to \mathbb{R}$ belongs to $BMO(X)$ (or $BMO_{loc}(X)$) if its lifting $\widehat{Q} = Q \circ \Pi$ belongs to $BMO(\hat{X})$ (or $BMO_{loc}(\hat{X})$) [43] p. 9.

2. $f : X \to X'$ is a $Q(p) - QC$ or $-QR$ *mapping*, $p \in X$, if its lifting to the universal coverings $\hat{f} : \hat{X} \to \hat{X}'$ is a $\hat{Q}(\hat{p}) - QC$ respectively $-QR$ mapping, $\hat{p} \in \Pi^{-1}(p)$.

In the case of Klein surfaces X, X' we work with a sense-preserving lifting.

Let $\mathcal{F}$ be a family of $Q(p) - QC$ mappings $f : X \to X'$, $Q \in BMO_{loc}(X)$, $\mathcal{F}$ being normalized by a compacity condition

(24) $f(p_0) = p'_0$ for two arbitrary but fixed points, $p_0 \in X$ and $p'_0 \in X'$.

Theorem 9 1. *If X' is non conformally equivalent to $\mathbb{C}$ or $\widehat{\mathbb{C}}$, then $\mathcal{F}$ is normal* ([4], Theorem 2.0, p.411 and [5], Theorem 2.0, p.10).

2. *If X' is non conformally equivalent to $\mathbb{C}$ or Δ, and $\{f_n\}$ is a sequence of $\mathcal{F}$, which l. uniformly converges to f_0, then f_0 is a $Q(p){-}QC$ embedding* ([4], Theorem 3.0, p.413 and [5], Theorem 3.0, p.11).

3. *$\mathcal{F}$ is closed in the following cases:*

(i) *X' is compact (even without (3.15))* ([4], Theorem 4.0, p. 416 and [5], Theorem 4.0, p.12),

(ii) *$Q \in BMO(X)$ and X' is non conformally equivalent to $\mathbb{C}$* ([4], Theorem 5.0, p. 417 and [5], Theorem 5.0, p.12),

(iii) *If $X \neq \mathbb{C}$ and there exists a function $C \in BMO_{loc}(X')$ such that f^{-1} is a $C(p') - QC$ mapping for every $f \in \mathcal{F}$* ([4], Theorem 4.0*, Remark 4.1, pp. 416–417 and [5], Theorem 4.0*, p.12).

These results have been extended [49, 51] to the families $\mathcal{F}'$ and $\mathcal{F}''$ consisting of $Q(p) - QC$ homeomorphisms $f : X \to X'$ which map a given compact subset $M \subset X$ into respectively onto another given compact subset $M' \subset X'$.

By using [45], Corollary 5.6, p. 15, a normality criterion was proved for $BMO - QR$ between Riemann surfaces by Victoria Stanciu:

Theorem 10 ([52], Theorem C) *Let X and X' be two homeomorphic Riemann surfaces, X' non conformally equivalent to either $\widehat{\mathbb{C}}$ or $\mathbb{C}$, $p_j \in X$, $p'_j \in X'$, $j = 0, 1$, $p_0 \neq p_1$, $p'_0 \neq p'_1$ and $Q \in BMO_{loc}(X)$. If $\mathcal{F}$ is a family of $Q(p) - QR$ mappings $f : X \to X'$ such that $f^{-1}(p'_j) = p_j$ then $\mathcal{F}$ is normal.*

The proof method for non simply connected surfaces X and X' is based on the lifting of the family $\mathcal{F}$, $\mathcal{F}'$ or $\mathcal{F}''$ to the universal coverings. We use results of [45], §3; [46], §5; [57] and propositions which assure the preserving of the l. uniformly convergence by lifting or factorizing under certain conditions [4, 5].

5. Properties of $BMO - BD$

We complete our account with results concerning the $BMO - BD$ mappings and we begin with topological aspects. The problem consists in giving regularity or integrability conditions for mappings to assure openness and discreteness, or continuity, differentiability a.e. for mappings of finite distortion, in particular with $BMO - BD$.

The way has been opened by Yu.G. Reshetnyak [41], II, §6, 6.3, Theorem 6.3, p. 183, Theorem 6.4, p. 184, continued by S.K. Vodop'yanov and V. M. Gol'dstein [62], and research intensified together with the applications in non-linear elasticity and PDEs [12, 13, 56, 38]. In dimension 2 for continuous mappings the openness and discreteness is equivalent to Stoilow's factorization theorem and we already quoted [31], where the problem is solved for mappings f in $W^{1,2}_{loc}$ with $K_f \in L^1_{loc}$ (see Theorem 3.13 from above). Results were obtained in higher dimensions by J. Heinonen and P. Koskela [24] for mappings of finite distortion in Ball's class, and the research continued by many others among which [39], [27, 28], [63], [17], the last two quoted papers with emphasize on topological methods.

Concerning this problem for $BMO-BD$ Iwanieć, Koskela and Martin proved:

Theorem 11 ([26], Theorem 9.1, p. 40) *If $f : G \to \mathbb{C}$ is a $BMO - BD$ mapping (cf. Definition* 2.16*) with a distortion function M such that $\|M\|_* < \varepsilon$, then f is continuous (i.e. f is a.e. equal to a continuous mapping), differentiable a.e. and either constant or open and discrete (hence monotone) in G.*

They also proved the differentiability a.e. of an embedding $f : G \to \mathbb{C}$ of finite distortion (cf. Definition 2.15) with a distortion function $\mathcal{K} \in L^p_{loc}(G)$, $p > 1$ ([26], Theorem 9.3, p. 43). As a sequel of [26], other properties of $BMO - BD$ (Definition 2.14) are established in the paper by K. Astala, T. Iwanieć, P. Koskela and G. Martin [7], with the starting point in the fundamental cycle of works on $QCTY$ and $QRTY$ by K. Astala [8–11].

Thus they obtained estimates on the modulus of continuity of a monotone mapping in various Sobolev-Orlicz classes, among which the following theorem related to David's bounds ([19], Theorem 1, p. 27):

Theorem 12 ([7], Theorem 4.2, p. 712) *Let $f : \mathbb{C} \to \mathbb{C}$ be in $BMO-BD$ with $f(0) = 0$ and $f(1) = 1$. Then there are constants A and $b > 0$ such that for z, $\zeta \in B(0, 2)$*

$$|f(z) - f(\zeta)| \leq A \mid \log |z - \zeta| \mid^{-b/\|M\|_*} .$$

Other deep results on the continuity of monotone mappings are given in [27].

With their estimates for the modulus of continuity in [7] the authors succeed to measure the distortion of the Hausdorff dimension for sets in $\mathbb{C}$, by means of the Hausdorff measure $\mathcal{H}_h$ with weight function h besides the classical Hausdorff measure $\mathcal{H}^\beta$, under sufficiently small $\|M\|_*$. In particular, they deduce:

Corollary 2 ([7], Corollary 5.1 (respectively 5.2), p. 718) *There is an* $\varepsilon_1 > 0$, $(\varepsilon_2 > 0)$ *such that for each* $f \in BMO-BD(G)$ *with* $\|M\|_* < \varepsilon_1$, (ε_2), *if* $E \subset G$ *is compact with* $\mathcal{H}_h(E) = 0$ *for* $h(t) = \log^{-\beta(2+\beta)/2}(t)$, *(for* $\beta = 1$, $h(t) = \log^{-3/2}(t)$*) then* $\mathcal{H}^\beta(fE) = 0$ *(respectively* $\mathcal{H}^1(fE) = 0$*).*

Further bounds on the area distortion under an Orlicz-Sobolev, in particular a $BMO - BD$, mapping are given, e.g.

Corollary 3 ([7], Corollary 6.1, p. 719) *If* $f \in W^{1,\Phi}_{loc}(G)$ *with a BMO-bounded distortion function* M *of* $\|M\|_*$ *small enough, and if* $E \subset G$ *is compact, then*

$$|f(E)| \leq C \log^{-2}(|E|^{-1}) \int_G J(z)dm(z),$$

the constant C *depending on* G, M *and the distance of* E *to* ∂G.

As in Astala's work for $QRTY$ the previous results are applied to extend the Painlevé theorem of bounded analytic functions.

Theorem 13 ([7], Theorem 7.1, p. 720) *There is* $\varepsilon_3 > 0$ *with the property: If* $E \subset G$ *is a compact set with* $\mathcal{H}_h(E) = 0$ *for* $h(t) = \log^{-3/2}(t)$, *and* $f : G\backslash E \to \mathbb{C}$ *is a bounded mapping in* $BMO - BD(G\backslash E)$ *with* $\|M\|_* < \varepsilon_3$, *then* f *extends to a mapping in* $BMO - BD(G)$.

For the proof one extends by 0 the complex dilatation μ_f to G and one uses the existence of a homeomorphic solution for the corresponding Beltrami equation, Stoilow's factorization and the Painlevé theorem for analytic functions.

From here it follows that a compact set $E \subset G$ with the conformal capacity zero is also removable for a mapping f as in Theorem 13, since conformal capacity zero implies $\mathcal{H}_h(E) = 0$ for $h(t) = \log^{-3/2}(t)$ and Corollary 4.3 can be applied ([7], Corollary 7.1, p. 720).

Another remarkable result is Theorem 7.3, p.725 in [7], namely:

There are domains $G \subset \mathbb{C}$ which do not support any non constant bounded QR mappings, but admit non constant bounded mapping in $BMO-BD$ of arbitrarily small $\|M\|_*$. In particular, bounded $BMO-BD$ mappings cannot be uniformly approximated by QR mappings.

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HARMONIC FORMS ON NON-ORIENTABLE SURFACES

Angel Alonso
E.U.I.T. Agrícolas
Ciudad Universitaria 28040 Madrid
aalonso@agricolas.upm.es

Arturo Fernández, Javier Pérez
Facultad de Ciencias, U.N.E.D.
Ciudad Universitaria 28040 Madrid
afernan@mat.uned.es

Abstract A classical subject in the theory of Riemann surfaces is the study of harmonic and analytic differentials. In particular it is well-known that the dimensions of the spaces of harmonic and analytic differentials with respect to the corresponding fields are $2g$, where g is the topological genus of the surface. A further study of this classical setting about the connection between the topological and conformal structure with different spaces of differential forms is due to A. Pfluger, who obtained some important decomposition theorems in this connection derived from the Dirichlet principle. Here we shall consider some of these questions for Klein surfaces, that is, we allow the surface to be non-orientable or to have a boundary.

Keywords: Klein surface, harmonic differential, analytic differential

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1. Introduction

Let S be a Klein surface, that is, a surface with a dianalytic structure, that is, a surface which does not have to be orientable and can have a nonempty boundary ∂S, formed by a finite set of Jordan curves γ. The algebraic genus $\widetilde{g}$ of such a surface is given by $\widetilde{g} = \alpha g + k - 1$, where $\alpha = 2$ or 1, depending on S being orientable or non-orientable respectively, g is the topological genus and k is the number of boundary components.

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We shall consider differentials on S as they are presented in N.L. Alling and N. Greenleaf [2], in particular we shall be interested in harmonic and analytic differentials on Klein surfaces. In [7], the authors obtained some results on the dimensions of spaces of harmonic and analytic differentials. In this article this study will be continued paying attention to the work of A. Pfluger [8].

2. Definitions and basic facts

2.1. A real 1-form ω on a Klein surface S can be expressed in term of a local coordinate (U, z) in the form

$$\omega = u(z)\, dz + v(z)\, d\overline{z},$$

or in real terms

$$\omega = p(x, y)\, dx + q(x, y)\, dy.$$

We shall say that ω is closed if $d\omega = 0$ and exact or total when $\omega = df$ for f a continuously differentiable globally defined function on S.

We shall call ω harmonic if and only if we can express locally $\omega = df$, where f is a locally defined harmonic function, that is

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv 0.$$

If f is a globally defined differentiable functions on S, then df is called a total or exact harmonic form.

Given a proper Klein surface S, that is S is non-orientable or it has nonempty boundary, then we can consider the Schottky complex double S_c of S, which is a classical Riemann surface with a dianalytic projection

$$\pi : S_c \to S,$$

and there also exists an antianalytic involution σ on S such that $\pi \circ \sigma = \pi$, in such a way that $S \approx S_c / \langle \sigma \rangle$.

For a real 1-form ω on S_c, we can consider the associated form $\widetilde{\omega}$, given by

$$\widetilde{\omega} = \sigma^* \omega,$$

where σ^* is the pullback operator for σ.

A meromophic (analytic) complex 1-form φ on a Klein surface, in local coordinates (U, z), has an expression

$$\varphi = A(z)\, dz$$

where $A(z)$ is a meromorphic (analytic) function of the coordinate and in such a way that if we consider a new chart (U_1, z_1) with

$$T(z_1) = \begin{cases} \partial(z \circ z_1^{-1})/\partial z_1 & , \text{ when } z \circ z_1^{-1} \text{ is analytic,} \\ \partial(z \circ z_1^{-1})/\partial \overline{z_1} & , \text{ when } z \circ z_1^{-1} \text{ is antianalytic,} \end{cases}$$

then φ with respect to the new coordinates has the expression

$$\varphi = A_1(z_1)\, dz_1$$

where

$$\varphi_1(z_1) = \begin{cases} A\left(z \circ z_1^{-1}\right) \circ z_1 \circ T & , \text{ when } z \circ z_1^{-1} \text{ is analytic,} \\ \overline{A\left(z \circ z_1^{-1}\right) \circ z_1 \circ T} & , \text{ when } z \circ z_1^{-1} \text{ is antianalytic.} \end{cases}$$

In the complex case the associated form $\widetilde{\varphi}$ is defined by

$$\widetilde{\varphi} = \overline{\sigma^* \varphi}.$$

2.2. We shall denote by $\Lambda^1(S)$ the space of 1-forms with coefficients in C^1.
In $\Lambda^1(S)$ we define a product (ω_1, ω_2) in the following form. Let

$$\omega_1 = p_1 dx + q_1 dy$$

and

$$\omega_2 = p_2 dx + q_2 dy$$

with respect to a coordinate system $(U, z = x + iy)$, then we can consider the quantity

$$(\omega_1, \omega_2)_U = \int_U (p_1 p_2 + q_1 q_2)\, dxdy. \tag{1}$$

If we consider a new coordinate system $(U', z' = x' + iy')$ such that $U \cap U' \neq \emptyset$, and such that

$$\omega_1 = p_1' dx' + q_1' dy'$$

$$\omega_2 = p_2' dx' + q_2' dy'$$

we check that

$$\int_{U \cap U'} (p_1 p_2 + q_1 q_2)\, dxdy = \int_{U \cap U'} \left(p_1' p_2' + q_1' q_2'\right) dx' dy', \tag{2}$$

what follows from

$$p_1'p_2' + q_1'q_2' = |J|\left(p_1p_2 + q_1q_2\right), \tag{3}$$

where $|J|$ is the absolute value of the Jacobian of the transformation $z\left(z'\right)$ or in real coordinates

$$\begin{aligned} x &= x\left(x', y'\right) \\ y &= y\left(x', y'\right). \end{aligned}$$

In fact, we have

$$\begin{aligned} p_1' &= p_1\frac{\partial x}{\partial x'} + q_1\frac{\partial y}{\partial x'}\ ,\ p_2' = p_2\frac{\partial x}{\partial x'} + q_2\frac{\partial y}{\partial x'}, \\ q_1' &= p_1\frac{\partial x}{\partial y'} + q_1\frac{\partial y}{\partial y'}\ ,\ q_2' = p_2\frac{\partial x}{\partial y'} + q_2\frac{\partial y}{\partial y'}, \end{aligned}$$

so that

$$\begin{aligned} &p_1'p_2' + q_1'q_2' \\ &\quad = p_1p_2\left(\left(\frac{\partial x}{\partial x'}\right)^2 + \left(\frac{\partial x}{\partial y'}\right)^2\right) + q_1q_2\left(\left(\frac{\partial y}{\partial x'}\right)^2 + \left(\frac{\partial y}{\partial y'}\right)^2\right) \\ &\quad +p_1q_2\left(\left(\frac{\partial x}{\partial x'}\frac{\partial y}{\partial x'} + \frac{\partial x}{\partial y'}\frac{\partial y}{\partial y'}\right)\right) + p_2q_1\left(\left(\frac{\partial y}{\partial x'}\frac{\partial x}{\partial x'} + \frac{\partial x}{\partial y'}\frac{\partial y}{\partial y'}\right)\right). \end{aligned} \tag{4}$$

Now if T is orientable, i.e. T is analytic, the Cauchy-Riemann equations

$$\frac{\partial x}{\partial x'} = \frac{\partial y}{\partial y'}\ ,\ \frac{\partial x}{\partial y'} = -\frac{\partial y}{\partial x'}$$

hold and we obtain from (4)

$$\begin{aligned} p_1'p_2' + q_1'q_2' &= \left(p_1p_2 + q_1q_2\right)\left[\left(\frac{\partial x}{\partial x'}\right)^2 + \left(\frac{\partial x}{\partial y'}\right)^2\right] \\ &= J\left(p_1p_2 + q_1q_2\right). \end{aligned} \tag{5}$$

If T is non-orientable, i.e. T is antianalytic, the corresponding equations

$$\frac{\partial x}{\partial x'} = -\frac{\partial y}{\partial y'}\ ,\ \frac{\partial x}{\partial x'} = \frac{\partial y}{\partial x'}$$

hold and we obtain

$$\begin{aligned} p_1'p_2' + q_1'q_2' &= \left(p_1p_2 + q_1q_2\right)\left[\left(\frac{\partial x}{\partial x'}\right)^2 + \left(\frac{\partial x}{\partial y'}\right)^2\right] \\ &= -J\left(p_1p_2 + q_1q_2\right), \end{aligned} \tag{6}$$

so that (5) and (6) yield (3), and as a consequence we conclude (2).

On the other hand S is a locally compact and countable union of compact sets so that S is paracompact and therefore, for every open covering of S we can construct locally finite coverings. Also the existence of partitions of unity $\{\varphi_i\}_{i\in I}$ follows.

For an open set $U_i \subset S$ and $\varphi_i \in C^1$ with compact support in U_i, we can consider the integral

$$\int_{U_i} \varphi_i \left(p_1 p_2 + q_1 q_2\right) dx dy,$$

with respect to some coordinate $z = x + iy$ in U_i, and define

$$(\omega_1, \omega_2) = \sum_i \int_{U_i} \varphi_i \left(p_1 p_2 + q_1 q_2\right) dx_i dy_i, \tag{7}$$

where $\{(U_i, z_i = x_i + iy_i)\}_{i\in I}$ is a family with an open covering $\{U_i\}_{i\in I}$ of S and $z_i = x_i + iy_i$ a local coordinate defined in U_i.

Taking (2) into account, we can obtain the independence of (ω_1, ω_2) of $\{(U_i, z_i = x_i + iy_i)\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$.

For a form ω in $\Lambda^1(S)$ we can consider the associated number

$$\|\omega\|^2 = (\omega, \omega), \tag{8}$$

and we shall restrict our study to the subspace Γ^1 of Λ^1 of forms ω for which $\|\omega\| < \infty$.

In Γ^1, (7) defines a proper scalar product and (8) defines the associated norm, the so-called Dirichlet norm.

With respect to this norm Γ^1 is not complete. We can obtain a concrete completion of Γ^1 by considering the space Γ of all differentials

$$\omega = pdx + qdy,$$

whose coefficients in terms of local coordiantes are measurable functions which have finite norm.

We remark that the norm $\|\omega\|$ can be defined in the same way as in Γ^1. Two differentials in Γ are identified if their coefficients differ only on a set of measure zero.

Γ turns out to be a complete normed space and the completion of Γ^1 can be identified with its closure in Γ.

Now we shall introduce some spaces of harmonic forms on a Klein surface S, corresponding to those introduced by A. Pfluger [8] for Riemann surfaces. Those spaces will be subspaces of Γ, for which Ahlfors and Sario [1] use a different notation. We shall present and identify both notations though mainly we shall make use that by A. Pfluger.

$$\begin{aligned}
\Gamma_c^1 &= \{\omega \in \Gamma^1 \,|\omega \text{ closed}\} \text{ , A-S} \equiv \Omega^N \text{ , PF,}\\
\Gamma_e^1 &= \{\omega \in \Gamma^1 \,|\omega \text{ exact}\} \text{ , A-S} \equiv T^N \text{ , PF,}\\
\Gamma_h^1 &= \{\omega \in \Gamma^1 \,|\omega \text{ harmonic}\} \text{ , A-S} \equiv \Omega^H \text{ , PF.}
\end{aligned}$$

In the case of a surface with boundary, we shall also consider

$$\begin{aligned}
\Gamma_{c0}^1 &= \{\omega \in \Gamma_c^1 \,|\omega = 0 \text{ along the boundary}\} \text{ , A-S,}\\
\Gamma_{e0}^1 &= \{\omega \in \Gamma_e^1 \,|\omega = 0 \text{ along the boundary}\} \text{ , A-S,}\\
\Gamma_{h0}^1 &= \{\omega \in \Gamma_h^1 \,|\omega = 0 \text{ along the boundary}\} \text{ , A-S.}
\end{aligned}$$

Further A. Pfluger also considers the following subspaces of Γ ,

$$\begin{aligned}
\Omega_0 &= \{\omega \in \Omega^N \,|\omega \text{ has compact support}\},\\
\overline{\Omega_0} &= \text{closure of } \Omega_0 \text{ in } \Omega^N.
\end{aligned}$$

It is clear that $\Omega^N \equiv \Omega_0$ in case that S is compact,

$$\begin{aligned}
T_0 &= \Omega_0 \cap T^N\\
\overline{T_0} &= \text{closure of } T_0 \text{ in } \Omega^N,
\end{aligned}$$

and

$$T_{00} = \{\omega \in \Omega^N \,\big|\omega = df \text{ , } f \in C_0^1\},$$

i.e. the space of differentials of functions with compact support, and

$$\overline{T_{00}} = \text{closure of } T_{00} \text{ in } \Omega^N.$$

We have the following inclusions

$$\Gamma^1 \supset \Omega^N \supset T^N \supset \overline{T_0} \supset \overline{T_{00}} \text{ , } \Gamma^1 \supset \Omega^N \supset \overline{\Omega_0} \supset \overline{T_0}. \tag{9}$$

Now, following the notation of A. Pfluger for Riemann surfaces, we introduce for Klein surfaces the spaces

$$\Omega' = \Omega^H \cap \overline{\Omega_0} \text{ , } T_0^H = \Omega^H \cap \overline{T_0}.$$

In the case of orientable surfaces, the following relations follow from Stokes theorem

$$\begin{aligned}
\Gamma_{c0}^1 \text{ , A-S} &\equiv \overline{\Omega_0} \text{ , PF,}\\
\Gamma_{e0}^1 \text{ , A-S} &\equiv \overline{T_0} \text{ , PF,}\\
\Gamma_{h0}^1 \text{ , A-S} &\equiv \Omega' \text{ , PF.}
\end{aligned}$$

In the general case of Klein surfaces, in principle, we cannot ensure the above identities since we do not dispose of Stokes theorem.

2.3. Now we shall follow more closely the notation and exposition of A. Pfluger who takes a more traditional point of view in connection with the Dirichlet principle and its consequences in contrast with Ahlfors and Sario who derive the existence proofs for harmonic differentials from standard results in Hilbert space theory.

In Ω^N the scalar product allows to identify the quotient spaces

$$\frac{\Omega^N}{T^N}, \frac{\Omega^N}{\overline{T_0}}, \frac{\Omega^N}{\overline{T_{00}}},$$

with linear subspaces of Ω^N, namely the orthogonal spaces to T^N, $\overline{T_0}$, and $\overline{T}_{00}$. This follows from the

Dirichlet Principle *Given a closed subspace L of Ω^N, such that $T_{00} \subset L \subset T^N$, there exists for every $\omega \in \Omega^N$, a harmonic differential ω_0 such that*

$$\begin{array}{lrcl} \text{i)} & \omega - \omega_0 & \in & L \\ \text{ii)} & \omega_0 & \perp & L. \end{array}$$

As a consequence of ii) one concludes that

$$\|\omega_0\| = \min_{\omega' \in L} \left\|\omega - \omega'\right\|.$$

The Dirichlet principle for Klein surfaces can be derived from that for Riemann surfaces by the standard procedure of passing to the complex double. In fact, given a closed subspace L of $\Omega^N(S)$, we can consider the closed subspace L_c of $\Omega^N(S_c)$ given by

$$L_c = \{df_c \,|df \in L\},$$

where ω_c is the lifting of ω to S_c.

The correspondence

$$\begin{array}{rcl} Q & : & L \to L, \\ & & df \to df_c \end{array}$$

is a transformation R for which

$$\|Q(df_1) - Q(df_2)\| = \|df_{1c} - df_{2c}\| = 2\|df_1 - df_2\|$$

holds and therefore it is a homeomorphism. Since L is closed in $T^N(S)$, which is complete, L_c is also complete and therefore closed in $T(S_c)$.

Thus, we can apply the Dirichlet principle in S_c, to ω_c, the lifting of ω to S_c, and L_c, obtaining a harmonic form ω_{0c} in S_c, such that

$$\alpha_i)\ \omega_c - \omega_{0c} \in L_c,$$
$$\alpha_{ii})\ \|\omega_{0c}\| = \inf_{\omega' \in L_c} \left\|\omega_c - \omega'\right\|.$$

From α_i) we deduce that

$$\omega_{0c} = \omega_c + df_{0c},$$

and taking into account that ω_c and df_{0c} are symmetric, we conclude that ω_{0c} is symmetric in S_c. so that it gives rise to a form ω_0 in S, which is harmonic and such that, due to the fact that the above mentioned correspondence preserves the distances except for a multiplicative factor, satisfies i) and ii).

We can obtain similar conclusions to those from Dirichlet principle for the quotients Ω^N/L, for the quotients R/L, where R is a subspace of Ω^N, with $L \subset R \subset \Omega^N$.

Now we shall pay attention to the spaces introduced above

$$T^H = \Omega^H \cap T^N\ ,\ \Omega' = \Omega^H \cap \overline{\Omega_0}\ ,\ T_0^H = \Omega^H \cap \overline{T_0}.$$

We can consider the projections π, π_0, π_{00} from Ω^N onto the subspaces $T^N, \overline{T_0}, \overline{T_{00}}$ and also their restrictions to $\overline{\Omega_0}$ and Ω'.

We shall pay special attention to the projection π_0 restricted to the subspace Ω'

$$\pi_0 : \Omega' \to T_0^H \subset \overline{T_0},$$

so that we obtain the orthogonal decomposition

$$\Omega' = \Omega_0^H \dot{+} T_0^H$$

where $\Omega_0^H \subset \Omega^H$ is a subspace of harmonic differentials.

In the orientable case, when S is the interior of a bordered Riemann surface S_Γ, with border $\partial S_\Gamma = \Gamma$, formed by k boundary components which are Jordan curves $\gamma_1, .., \gamma_k$ we have the following description of T_0^H and Ω_0^H.

T_0^H consists of the differentials of harmonic functions on S and continuous on Γ, such that are constant on every boundary component.

If h_i is the harmonic measure of γ_i , $i = 1, .., k$, then the differentials $dh_1, .., dh_{k-1}$ form a basis of T_0^H.

Ω_0^H consists of those harmonic forms $\omega \in \Omega'$, such that

$$\int_{\gamma_i} \omega^* = 0\ ,\ i = 1, .., k$$

this follows from Stokes theorem.

The absence of Stokes theorem in the nonorientable case is the main differentiating feature in the theory of differential forms on Klein surfaces.

3. Dimensions of spaces of harmonic and meromorphic differentials on Klein surfaces

The following results, yielding the dimensions of spaces of harmonic and holomorphic differentials on a compact Klein surface S with no boundary, hold, see [7, 9, 10].

Theorem A *Let S be a compact unbordered Klein surface of topological genus g and algebraic genus $\widetilde{g}$, then*

$$\dim_{\mathbf{R}} h(S) = \dim_{\mathbf{R}} H_{\mathbf{C}}(S) = \widetilde{g}.$$

Here $h(S)$ denotes the space $\Gamma_h^1(S)$ of harmonic differentials considered as a real vector space and $H_{\mathbf{C}}(S)$ the space of holomorphic differentials, also considered as a real vector space.

In the orientable case, it is classical for bordered surfaces S, see Pfluger [9], p. 172, that for the space $\mathcal{S}(S)$ of Schottky differentials i.e. those differentials which can be obtained as restrictions to S of holomorphic differentials on the complex double S_c, it holds the following

Theorem B. *Let S be the interior of a bordered Riemann surface S_Γ with topological genus g, algebraic genus $\widetilde{g}$ and k boundary components, then it holds*

$$\dim_{\mathbf{R}} \mathcal{S}(S) = 2\widetilde{g} = 2(2g + k - 1).$$

The proof of Theorem B makes use again of Stokes theorem so that we cannot obtain similar conclusions for non-orientable surfaces.

4. Schottky differentials and harmonic forms vanishing at the boundary

The absence of Stokes theorem for non-orientable surfaces hinders again the proof of a decomposition theorem of Pfluger type for the space of harmonic differentials $\Omega^H(S)$ for non-orientable surfaces, i.e.

$$\Omega^H(S) = *\Omega_0^H \dot{+} *T_0^H \dot{+} T^H, \tag{10}$$

where $*$, in front position, denotes again conjugation, see Pfluger [8], p. 9.

However we can consider some questions, which were of interest in the context of this theorem, namely the dimensions of the involved spaces.

In the orientable case, the decomposition

$$\Omega^H = *\Omega_0^H \dot{+} T_1 \dot{+} *T_0^H \dot{+} T_0^H \dot{+} T_2, \tag{11}$$

can be derived from (10), where T_2 is defined by

$$T_2 = \mathrm{Re} T_A = \{\mathrm{Re}\omega \,|\omega \in T_A\},$$

where T_A is the subspace of total analytic differentials in Φ_A, with

$$\Phi_A = \left\{\varphi = \omega + i\omega^* = A(z)\,dz \,\middle|\, \|\varphi\|^2 = 2\int_S |A(z)|^2\,dxdy < \infty\right\}$$

and T_1 is the orthogonal complement to $T_0^H \dot{+} T_2$ in T^H.

A. Pfluger [8] proposes the problem of characterizing the classes of Riemann surfaces with previously given dimensions

$$\dim \Omega_0^H\ ,\ \dim T_0^H\ ,\ \dim T_1\ ,\ \dim T_2\ ,$$

in particular he gets interested in the class of surfaces for which $\dim T_1 = 0$, which we can denote by $\mathcal{A}$.

He finds a useful characterization for those

Theorem C *A surface S is in $\mathcal{A}$ if and only if*

$$\Omega_0^H(S) = *\Omega_0^H(S).$$

For Klein surfaces we do not have those decompositions theorems but we might consider the study of the dimensions of spaces corresponding to those considered in the orientable case.

As indicated in section (2.2), in the orientable case we obtain from Stokes theorem the identification

$$\begin{aligned}\Gamma^1_{c0}(S) &\equiv \overline{\Omega_0(S)},\\ \Gamma^1_{e0}(S) &\equiv \overline{T_0(S)},\\ \Gamma^1_{e0}(S) &\equiv \Omega'.\end{aligned}$$

In fact, let us check the first of them. We have for any $f \in C^\infty(S)$,

$$(\omega, df^*) = \int_S \omega \wedge df = -\int_S f d\omega + \int_{\partial S} f\omega, \tag{12}$$

so that if $\omega \in \Omega_0$, we know that $(\omega, df^*) = 0$, and since $d\omega = 0$, we also conclude

$$\int_S f\omega = 0,$$

and since f was arbitrary in $C^{\infty}(S)$ we can derive $\omega_{|\Gamma} \equiv 0$. For $\omega \in \overline{\Omega_0}$, we can approximate by a sequence $\{\omega_n\}_{n\in\mathbf{N}}$ in Ω_0 in the Dirichlet norm so that we also obtain by Cauchy-Schwarz inequality

$$(\omega, df^*) = \lim_{n\to\infty} (\omega_n, df^*) = 0$$

and obtain the same conclusion $\omega_{|\Gamma=0}$, i.e. $\omega \in \Gamma^1_{c0}$.

The argument is valid to show the last two identities.

In the non-orientable case we shall consider the corresponding decompositions

$$\Gamma^1_{h0} = \Gamma^1_{he0} \dot{+} \left(\Gamma^1_{hc0}\right)^{\perp}. \tag{13}$$

First of all, we remark that Γ^1_{he0} is closed in Γ^1_{h0}. To see this we can consider again the corresponding subspace $L \subset \Omega^H(S_c)$, where

$$L = \left\{\omega_c \in \Omega^H(S_c) \,\middle|\, \omega \in \Gamma^1_{h0}(S)\right\},$$

ω_c is the lifting to S_c of ω. Now L is closed in $\Omega^H(S_c)$, since it can be described as the subspace of symmetric forms which vanish at the closed set of fix points of σ, and recalling that the convergence in norm implies the locally uniform convergence, see Pfluger [9], p. 67, we conclude that the closure points of L should also be in L.

Now we obtain the following results concerning the dimensions of the subspaces involved in (13).

Theorem 1 *Let S be a compact bordered Klein surface of finite genus $g \geq 1$ and $k \geq 1$ components. Then it hold*

$$\dim \Gamma^1_{he0} = k - 1.$$

The argument to prove Theorem 1 is exactly the same as in the classical case. Γ_{he0} consists of the differentials of harmonic functions which are constants in every component γ_i of the boundary so that if h_i is the harmonic measure of γ_i , $i = 1, .., k-1$, then the differentials $dh_1, .., dh_{k-1}$ form a basis of Γ^1_{he0}. We recall that h_i is obtained as the solution of the Dirichlet problem for S with previously given boundary values

$$\begin{aligned} h_{i|\gamma_i} &\equiv 1, \\ h_{i|\gamma_j} &\equiv 0\ ,\ i \neq j, \end{aligned}$$

and the process is independent of the orientability of the surface

Theorem 2 *Let S be a compact bordered Klein surface of finite genus $g \geq 1$ and $k \geq 1$ boundary components. Then it holds*

$$\begin{aligned} \dim \left(\Gamma^1_{he0}\right)^{\perp} &= 2g\ ,\ \textit{if } S \textit{ is orientable}, \\ \dim \left(\Gamma^1_{he0}\right)^{\perp} &= g\ ,\ \textit{if } S \textit{ is non-orientable.} \end{aligned}$$

Proof of Theorem 2 The orientable case is known by Pfluger, however we shall indicate a way to prove it, in order to illustrate the non-orientable case.

Let $\mathcal{A}_r(S)$ be the space of analytic differentials on an orientable Klein surface S which are real on the boundary, then it is clear from the definitions that if

$$\varphi(z) = A_\varphi(z)\, dz = \omega_\varphi + i\omega_\varphi^* ,$$

then

$$\varphi \in \mathcal{A}_r \Longleftrightarrow \omega_{-i\varphi} \in \Omega' \equiv \Gamma_{h0}^1 \tag{14}$$

and equivalently

$$\varphi \in i\mathcal{A}_r \Longleftrightarrow \omega_\varphi \in \Omega' \equiv \Gamma_{h0}^1. \tag{15}$$

Then we can consider the complex double S_c as the union of two identical copies of S, say S and S^*, with the boundaries identified. This is not the case in the non-orientable case where further identifications are required.

Therefore in the orientable case, that is, the bordered Riemann surfaces as considered by A.Pfluger, we might consider the space $\mathcal{S}(S)$ of Schottky differentials, see [9], p. 172, of those analytic differentials in S, which can be obtained as restrictions of differentials in S_c.

Every Schottky differential can be written in the form

$$\varphi = \frac{\varphi_r + \varphi_i}{2}, \tag{16}$$

where $\varphi_r \in \mathcal{A}_r$ and $\varphi_i \in i\mathcal{A}_r$, and are given by

$$\begin{aligned} \varphi_r(p) &= \varphi(p) + \overline{\varphi(p^*)} \\ \varphi_i(p) &= \varphi(p) - \overline{\varphi(p^*)}, \end{aligned}$$

and $p^* = \sigma(p)$ is the conjugate point of p in S_c.

$\mathcal{A}_r$ and $i\mathcal{A}_r$ are subspaces of real dimension $2g+k-1$ of $\mathcal{A}(S)$, the full space of analytic differentials on S, so that the real dimension of $\mathcal{S}(S)$ is $2(2g+k-1)$.

Taking now into account (14) and (15) we get from (13) the following decompositions

$$\varphi_r = i(-i\varphi_r) = i\left[\omega^\circ_{-i\varphi_r} + df_{-i\varphi_r} + i\left(*\omega^\circ_{-i\varphi_r} + *df_{-i\varphi_r}\right)\right., \tag{17}$$

$$\varphi_i = \omega^\circ_{\varphi_i} + df_{\varphi_i} + i\left(*\omega^\circ_{\varphi_i} + *df_{\varphi_i}\right), \tag{18}$$

where $\omega^{\circ}_{-i\varphi_r}$, $\omega^{\circ}_{\varphi_i} \in \Omega^H_0 \equiv \Gamma^1_{he0}$ and $f_{-i\varphi_r}$, f_{φ_i} are harmonic functions on S and constant on every component of ∂S.

Putting together (16), (17) and (18) we conclude

$$\varphi = \frac{1}{2}\left[- * \omega^{\circ}_{-i\varphi_r} + \omega^{\circ}_{\varphi_i} - df^*_{-i\varphi_r} + df_{\varphi_i}\right.$$
$$\left. + \, i\left(\omega^{\circ}_{-i\varphi_r} + *\omega^{\circ}_{\varphi_i} + df_{-i\varphi_r} + df^*_{\varphi_i}\right)\right], \qquad (19)$$

and observe that the subspaces formed by the forms

$$-df^*_{-i\varphi_r} + idf_{-i\varphi_r}$$

and

$$df_{\varphi_i} + idf^*_{\varphi_i} \ ,$$

respectively have the same real dimension as $T^H_0 = \Gamma^1_{he0}$ i.e. $k-1$, and also the subspaces formed by the forms

$$- * \omega^{\circ}_{-i\varphi_r} + i\omega_{-i\varphi_r}$$

and

$$\omega^{\circ}_{\varphi_i} + i * \omega^{\circ}_{\varphi_i}$$

are subspaces of complex forms of real dimension equal to $\dim \Omega 0^H = \dim \Gamma^1_{he0}$.

We conclude from these remarks and Theorem B in Section 3

$$\dim \Omega^H_0 = \dim \Gamma^1_{he0} = 2g.$$

In the non-orientable case, we cannot define a Schottky form and the associated form φ_r and φ_i, since as it was pointed out above, we cannot consider S as a natural subspace of S_c, and as a consequence S^* is not well-defined either.

However we can consider the subspace $\mathcal{A}(S_c)^{\sigma_*}$ of $\mathcal{A}(S_c)$ which is invariant for the map

$$\sigma_* : \mathcal{A}(S_c) \to \mathcal{A}(S_c)$$

associated to the canonical involution

$$\sigma : S_c \to S_c.$$

For those $\varphi \in \mathcal{A}(S_c)^{\sigma_*}$, there is a well-defined form φ_S on S, which should be real at the boundary Γ of S, i.e. $\varphi \in \mathcal{A}(S)$.

Conversely every $\varphi_S \in \mathcal{A}(S)$ gives rise to a well-defined $\varphi \in \mathcal{A}(S_c)^{\sigma_*}$ by setting

$$\varphi(p) = \overline{\varphi(\sigma(p))} = \varphi_s(\pi(p)). \tag{20}$$

We remark that in the non-orientable case $i\mathcal{A}(S)$ is not well-defined since $\mathcal{A}(S)$ is not a complex vector space.

If we consider an analytic form $\varphi \in \mathcal{A}(S)$, and a local chart (U, z) at $p \in S$, then we can write

$$\varphi = A_\varphi(z)\, dz$$

in such a way that if we consider a new coordinate system (U_1, z_1), then the local representation of φ varies as described in section (2.1).

Here we can also decompose the local representation of φ in real and imaginary parts. In fact, if $z = x + iy$ and

$$\begin{aligned} \mathrm{Re}A_\varphi(z) &= \alpha(z) = \alpha(x,y), \\ \mathrm{Im}A_\varphi(z) &= \beta(z) = \beta(x,y), \end{aligned}$$

then we can write locally

$$\begin{aligned} \varphi &= (\alpha + i\beta)(dx + idy) \\ &= \alpha dx - \beta dy + i(\beta dx + \alpha dy), \end{aligned}$$

where

$$\omega_\varphi = \alpha dx - \beta dy$$

is again a proper harmonic form but the expression

$$\widehat{\omega_\varphi} = \beta dx + \alpha dy \tag{21}$$

is not a proper differential form in S for the required rules for non-orientable changes of coordinates do not follow.

In fact, with respect to the new coordinates (U_1, z_1), $z_1 = x_1 + iy_1$, $\widehat{\omega_\varphi}$ should have the form

$$\widehat{\omega_\varphi} = -\left[\left(\beta\frac{\partial x}{\partial x_1} + \alpha\frac{\partial y}{\partial x_1}\right)dx_1 + \left(\beta\frac{\partial x}{\partial y_1} + \alpha\frac{\partial x}{\partial x_1}\right)dy_1\right]. \tag{22}$$

Anyway, the entities which locally take on the form (21), in such a way that for orientable changes of coordinates, they follow the same rules as those of differential forms and with respect to non-orientable changes of coordinates, they change according to (22), form a real vector space.

In the particular case of the imaginary parts $\mathrm{Im}\varphi$ of analytic forms $\varphi \in \mathcal{A}(S)$, we shall obtain a real vector space which we shall denote by $\widehat{\Omega}(S)$.

We remark that those elements $\widehat{\omega} \in \widehat{\Omega}(S)$ satisfy

$$\widehat{\omega}_{|\Gamma} \equiv 0,$$

what is invariant for orientable and non-orientable changes of coordinates and we can establish a natural isomorphism

$$\begin{array}{rcl} \widehat{I} : \Gamma^1_{h0}(S) & \longrightarrow & \widehat{\Omega}(S), \\ \omega & \longmapsto & \widehat{\omega}, \end{array} \tag{23}$$

so that if ω locally admits an expression

$$\omega = \alpha(x,y)\,dx - \beta(x,y)\,dy$$

with respect to a parameter $z = x + iy$, then $\widehat{\omega}$ admits the expression

$$\widehat{\omega} = \beta(x,y)\,dx + \alpha(x,y)\,dy,$$

and we conclude

$$\dim_{\mathbf{R}} \widehat{\Omega}(S) = \dim_{\mathbf{R}} \Gamma^1_{h0}(S). \tag{24}$$

On the other hand the correspondence

$$\begin{array}{rcl} \mathcal{A}(S) & \to & \widehat{\Omega}(S), \\ \varphi & \to & \widehat{\omega_{\varphi}}, \end{array}$$

is also a real isomorphism so that we conclude from (13), (23) and (24)

$$\dim_{\mathbf{R}} \mathcal{A}(S) = \dim_{\mathbf{R}} \widehat{\Omega} = \dim_{\mathbf{R}} \Gamma^1_{h0}(S) = \dim_{\mathbf{R}} \Gamma^1_{he0} + \dim_{\mathbf{R}} \left(\Gamma^1_{he0}\right)^{\perp}. \tag{25}$$

The established correspondence through (20)

$$\begin{array}{rcl} \mathcal{A}(S_c)^* & \to & \mathcal{A}(S) \\ \varphi & \to & \varphi_S \end{array}$$

turns out to be a real isomorphism so that

$$\dim_{\mathbf{R}} \mathcal{A}(S_c)^{\sigma_*} = \dim_{\mathbf{R}} \mathcal{A}(S) \tag{26}$$

and on the other hand, as it is described in [7], we can decompose

$$\mathcal{A}(S_c) = \mathcal{A}(S_c)^{\sigma_*} \oplus *\mathcal{A}(S_c)^{\sigma_*} \tag{27}$$

through

$$\varphi = \frac{1}{2}(\varphi + \widetilde{\varphi}) + \frac{1}{2}(\varphi - \widetilde{\varphi}),$$

where

$$\widetilde{\varphi} = \overline{\sigma^* \varphi}.$$

From (26), (27) and the fact that the operator $*$ is a real isomorphism, we conclude

$$\dim_{\mathbf{R}} \mathcal{A}(S) = \frac{1}{2} \dim_{\mathbf{R}} \mathcal{A}(S_c) = g + k - 1 \tag{28}$$

and from (25), (28) and Theorem 1, we obtain finally

$$g + k - 1 = \dim \left(\Gamma^1_{he0}\right)^{\perp} + k - 1,$$

i.e.

$$\dim \left(\Gamma^1_{he0}\right)^{\perp} = g,$$

q.e.d.

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PERIODIC FATOU COMPONENTS AND SINGULARITIES OF THE INVERSE FUNCTION

Walter Bergweiler
Mathematisches Seminar
Christian–Albrechts–Universität zu Kiel
Ludewig–Meyn–Str. 4
D–24098 Kiel
Germany
bergweiler@math.uni-kiel.de

Abstract The periodic components of the Fatou set of a rational or entire function are closely connected to the singularities of the inverse function. This article is a survey of classical as well as more recent results concerning this relation.

Keywords: complex dynamics, rational or entire functions, periodic Fatou components, singularities of inverse functions

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1. Introduction

1.1 Basic definitions

Throughout this paper, let f be a rational function of degree at least two or a transcendental entire function. We denote by $D(f)$ the domain of definition of f so that $D(f) = \mathbb{C}$ if f is entire transcendental while $D(f) = \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ if f is rational. The basic objects studied in complex dynamics are the Fatou set $F(f)$ of f, which is defined to be the set of all points in $D(f)$ where the iterates of f form a normal family, and the Julia set $J(f)$, which is the complement of $F(f)$ with respect to $D(f)$. Thus $J(f) = \mathbb{C}\backslash F(f)$ for entire transcendental f and $J(f) = \widehat{\mathbb{C}}\backslash F(f)$ for rational f.

We denote by f^n the n–th iterate of f, with $f^0 = \mathrm{id}_{D(f)}$. We say that $z_0 \in D(f)$ is a *preperiodic point* of f if there exist $p > q \geq 0$ such

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that $f^p(z_0) = f^q(z_0)$. In the special case that $q = 0$ so that $f^p(z_0) = z_0$ for some $p \geq 1$ we say that z_0 is a *periodic point* of f. The smallest p with this property is called the *period* of z_0. For a periodic point $z_0 \in \mathbb{C}$ of period p we call $(f^p)'(z_0)$ the *multiplier* of z_0. If $z_0 = \infty$, which can happen only for rational function f of course, this has to be modified: in this case, the multiplier is defined to be $(g^p)'(0)$ where $g(z) := 1/f(1/z)$. A periodic point is called *attracting*, *indifferent*, or *repelling* depending on whether the modulus of its multiplier is less than, equal to, or greater than 1. Periodic points of multiplier 0 are called *superattracting*. The multiplier of an indifferent periodic point is of the form $e^{2\pi i\alpha}$ where $0 \leq \alpha < 1$. We say that z_0 is *rationally indifferent* if α is rational and *irrationally indifferent* otherwise. Finally, a periodic point of period 1 is called a *fixed point*.

A basic result in complex dynamics says that the Julia set is the closure of the set of repelling periodic points. Among other basic properties of the Fatou and Julia set we mention here only that both sets are completely invariant. Here, by definition, a subset S of $D(f)$ is called *completely invariant* if $f(z) \in S$ if and only if $z \in S$. For a an introduction to complex dynamics we refer to the textbooks [9, 21, 26, 43, 56] for rational functions. The case of transcendental entire functions (but also that of rational functions) is treated in [10, 29, 45].

1.2 The classification of periodic Fatou components

A maximal domain of normality of the iterates of f, that is, a connected component of $F(f)$, is called a *Fatou component*. If U is a Fatou component, then $f^p(U)$ is contained in a Fatou component which we denote by U_p. A Fatou component U is called *preperiodic* if there exist $p > q \geq 0$ such that $U_p = U_q$. In particular, if this is the case for $q = 0$ (where $U_0 = U$) and some $p \geq 1$, then U is called *periodic*, and $\{U, U_1, \ldots, U_{p-1}\}$ is called a *periodic cycle* of Fatou components. Again, the smallest p with this property is called the *period* of U. In the case $p = 1$, that is, if $f(U) \subset U$, the Fatou component U is called *invariant*. A Fatou component which is not preperiodic is called a *wandering component* (or *wandering domain*).

For rational functions we have $f^p(U) = U_p$, but for transcendental functions it is possible that $f^p(U) \neq U_p$. However, $U_p \backslash f^p(U)$ contains at most one point; see [18, 22, 23, 37]

The behavior of the iterates in periodic components is well understood. Let U be a periodic Fatou component of period p. Then we have one of the following possibilities:

- U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. If z_0 is superattracting, then U is called a *Böttcher domain*. Otherwise U is called a *Schröder domain*.

- ∂U contains a fixed point z_0 of f^p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. Then $(f^p)'(z_0) = 1$ if $z_0 \in \mathbb{C}$. (For $z_0 = \infty$ we have $(g^p)'(0) = 1$ where $g(z) := 1/f(1/z)$.) We call U a *Leau domain* at z_0.

- There exists an analytic homeomorphism $\phi : U \to \mathbb{D}$ where $\mathbb{D}$ is the unit disk such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In this case, U is called a *Siegel disk*.

- There exists an analytic homeomorphism $\phi : U \to A$ where A is an annulus, $A = \{z : 1 < |z| < r\}$, $r > 1$, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. In this case, U is called a *Herman ring*.

- $f^{np}(z) \to \infty \notin D(f)$ for $z \in U$ as $n \to \infty$. In this case, U is called a *Baker domain*.

Clearly, if f is rational, then Baker domains do not exist. Moreover, it is not difficult to see that entire functions do not have Herman rings.

The above classification of periodic Fatou components is due to Fatou and Cremer; see [10, p. 163] for a more detailed historic account with references. For a proof we refer to the textbooks mentioned in the introduction. Most of them deal only with the case that f is rational, but the case that f is entire transcendental requires only minor modifications. (This remark concerns also various other references made to these textbooks in this paper.)

If z_0 is an attracting periodic point, then z_0 is contained in a Schröder or Böttcher domain. Similarly, if z_0 is a periodic point of multiplier 1, then there is a Leau domain at z_0. More precisely, let $z_0 \in \mathbb{C}$ be a fixed point of multiplier 1; that is, $f(z_0) = z_0$ and $f'(z_0) = 1$. Then z_0 is a multiple pole of the function $h(z) := 1/(z - f(z))$, say of multiplicity $m + 1$ with $m \in \mathbb{N}$. (We also say that z_0 is a *multiple fixed point* of f of multiplicity $m + 1$.) We find that there are exactly m invariant Leau domains at z_0. There is of course an obvious modification of this result for the case that $z_0 = \infty$, as well as to the case that z_0 is not a fixed point of multiplier 1, but a rationally indifferent periodic point. The existence of Siegel disks and Herman rings was first shown by Siegel [54] and Herman [35]. The first example of an entire function with a Baker domain was already given by Fatou [32, Exemple I] who considered the function $f(z) = z + 1 + e^{-z}$ and noted that $\operatorname{Re} f^n(z) \to \infty$ as $n \to$

∞ for $\operatorname{Re} z > 0$ which implies that the right half-plane is contained in an invariant Baker domain. Examples of Baker domains of period greater than one were given in [44, 48]. The term "Baker domain" was introduced by Eremenko and Lyubich [29, 30].

1.3 Singularities of the inverse function

An important role in complex dynamics is played by the singularities of the inverse function. Let $a \in D(f)$ and let $\gamma : [0,1] \to D(f)$ be a curve in $D(f)$ with endpoint $\gamma(1) = a$. Let φ be a branch of f^{-1} defined in some neighborhood of $b := \gamma(0)$; that is, φ is meromorphic in some neighborhood V of b and $f(\varphi(z)) = z$ for all $z \in V$. Suppose that φ can be continued analytically along γ into the point $\gamma(t)$ for all $t \in [0,1)$, but not into the point $a = \gamma(1)$. Then (that is, if b, γ, φ as above exist) the point a is called a *singularity of the inverse function of* f.

One way this is possible is that $\varphi(\gamma(t)) \to z_0$ as $t \to 1$ for some $z_0 \in D(f)$ with the property that f is not locally univalent at z_0. Then $f(z_0) = a$, and we call z_0 a *critical point* and a a *critical value* of f. Note that if $z_0 \neq \infty$ and $f(z_0) \neq \infty$, then z_0 is a critical point if and only if $f'(z_0) = 0$. For rational functions f, critical values are the only singularities of f^{-1}. For transcendental entire f it is also possible that $\varphi(\gamma(t)) \to \infty \notin D(f)$ as $t \to 1$. Then a is called an *asymptotic value.*

We call the critical and asymptotic values of f also *singular values* and denote the set of all singular values of f by $\operatorname{sing}(f^{-1})$. We note that (see [1, Lemma 2])

$$\operatorname{sing}((f^p)^{-1}) = \bigcup_{n=0}^{p-1} f^n \left(\operatorname{sing}(f^{-1})\right). \tag{1}$$

The *postsingular set* $P(f)$ is defined by

$$P(f) := \overline{\bigcup_{n=0}^{\infty} f^n \left(\operatorname{sing}(f^{-1})\right)} = \overline{\bigcup_{n=0}^{\infty} \operatorname{sing}\left((f^n)^{-1}\right)}).$$

2. Relations between periodic Fatou components and singular values

By definition, a periodic cycle of Böttcher domains contains a superattracting fixed point and thus a critical point and a critical value of f. We shall discuss how the other types of Fatou components are related to singular values.

2.1 Schröder and Leau domains

The fundamental result in this case is the following theorem. A proof can be found in the textbooks mentioned in the introduction.

Theorem 1 *Let* $\{U_0, U_1, \ldots, U_{p-1}\}$ *be a periodic cycle of Schröder or Leau domains of* f*. Then* $U_j \cap \text{sing}(f^{-1}) \neq \emptyset$ *for some* $j \in \{0, 1, \ldots, p-1\}$*. More precisely, there exists* $j \in \{0, 1, \ldots, p-1\}$ *such that* $U_j \cap \text{sing}(f^{-1})$ *contains a point which is not preperiodic.*

This result can already be found in the memoirs by Fatou [31, 32] and Julia [39] that founded the theory. More recently, it has been shown that under a suitable additional hypothesis a periodic cycle of Leau domains contains at least two singular values. In order to formulate this result, let $z_0 \in \mathbb{C}$ be a fixed point of f. Then the function $h(z) := 1/(z - f(z))$ has a pole at z_0. The residue of h at z_0 is called the *residue fixed point index* and denoted by $\iota(f, z_0)$; see [43, §12] for a detailed discussion of this concept. The residue fixed point index is invariant under holomorphic changes of variables [43, Lemma 12.3]. This is used to define it for $z_0 = \infty$; that is, if ∞ is a fixed point of f and g is defined by $g(z) := 1/f(1/z)$, then $\iota(f, \infty) := \iota(g, 0)$. Of course, there is an obvious modification of the residue fixed point index to periodic points, but for simplicity we state the following result only for fixed points. It is easily transferred to the case of rationally indifferent periodic points using (1).

Theorem 2 *Let* z_0 *be a fixed point of* f *of multiplier* 1*. Let* $m + 1$ *be the multiplicity of* z_0*. If*

$$\text{Re}\, \iota(f, z_0) \geq \frac{7}{20} m + \frac{1}{2}, \tag{2}$$

then one of the m *Leau domains associated to* z_0 *contains at least two singular values.*

This theorem follows from results obtained by Bergweiler [14] and Buff and Epstein [25], which extended previous work by Shishikura [52]. In fact, for rational f this theorem follows immediately from [14, Theorem 2]. Moreover, it was shown in [14, Theorem 2] that the conclusion of Theorem 2.1 holds for functions f meromorphic in the plane if

$$\text{Re}\, \iota(f, z_0) > \frac{7}{20} m + \frac{1}{2}. \tag{3}$$

Here we refer to [10] for the basic definitions and results of complex dynamics in the setting of transcendental meromorphic functions. We note that we cannot replace (3) by (2) in this setting, as shown by the example $f(z) = \tan z$. Here $z_0 = 0$ is a fixed point of multiplicity 3, the two associated Leau domains being the upper and lower halfplane. As $\text{sing}(\tan^{-1}) = \{i, -i\}$, we see that each of them contains only one singular value. A simple computation shows that $\iota(\tan, 0) = \frac{6}{5}$ so that we have equality in (2) for $m = 2$. Similarly, $f(z) = \tan^2 \sqrt{z}$ provides an example of a multiple fixed point of multiplicity 2 having only one singular value in its Leau domain, yielding equality in (2) for $m = 1$.

These examples show that the conclusion of Theorem 2.1 need not hold for transcendental meromorphic f if we have equality in (2). On the other hand, the methods of [25] show that if f is a transcendental meromorphic functions for which we have equality in (2) and the conclusion of Theorem 2.1 does not hold, then the Julia set of f must be a straight line or a line segment. But this is impossible for transcendental entire functions by a result of Töpfer [58, §3].

We note that estimates sharper than (2) are available if all singular values are critical values and if the multiplicities of the critical points are bounded. For example, if all singular values are critical values and if the critical points are simple, then (2) can be replaced by $\operatorname{Re} \iota(f, z_0) > \frac{1}{4}m + \frac{1}{2}$; see [14, 25] for details.

2.2 Siegel disks and Herman rings

The classical result in this case is the following theorem, whose proof can be found in the textbooks mentioned in the introduction.

Theorem 3 *Let $\{U_0, U_1, \ldots, U_{p-1}\}$ be a periodic cycle of Siegel disks or Herman rings. Then $\partial U_j \subset P(f)$ for all $j \in \{0, 1, \ldots, p-1\}$.*

A more recent result by Mañé ([42], see also [53]) says that the boundary of a Siegel disc of a rational function is contained in the ω-limit set of a recurrent critical point. The question when the boundary of a Siegel disk or Herman ring actually contains a point of $\operatorname{sing}(f^{-1})$ is rather delicate; see [27, 33, 36, 46, 47, 50, 51, 59] for results in this direction.

2.3 Baker domains

As rational functions do not have Baker domains, f will always denote an entire transcendental function in this section. The following result is due to Eremenko and Lyubich [30, Theorem 1].

Theorem 4 *If $\operatorname{sing}(f^{-1})$ is bounded, then there is no Fatou component U such that $f^n(z) \to \infty$ for $z \in U$ as $n \to \infty$. In particular, f does not have Baker domains.*

The part concerning Baker domains was strengthened by Bargmann [7, Theorem 4] as follows.

Theorem 5 *If f has an invariant Baker domain, then there exist $c > 1$ and $r_0 > 0$ such that every annulus $\{z : r < |z| < cr\}$ with $r \geq r_0$ contains a singular value.*

In view of these results one might think that Baker domains always contain singular values. However, this is not the case. The first examples of entire functions f having a Baker domain U with $U \cap \operatorname{sing}(f^{-1}) = \emptyset$ were given by Herman [36, p. 609] and Eremenko and Lyubich [28, Example 3]. As a specific example with this property we mention the function $f(z) = 2 - \log 2 + 2z - e^z$ which even has a Baker domain U such that $\operatorname{dist}(P(f), U) > 0$, where $\operatorname{dist}(\cdot,\cdot)$ denotes the Euclidean distance in the plane; see [12, Theorem 1] for this example. A detailed study of invariant Baker domains U satisfying $U \cap \operatorname{sing}(f^{-1}) = \emptyset$ was given by Barański and Fagella [6]. Examples of periodic cycles of Baker domains of higher period which do not contain singular values were given by Rippon and Stallard [49, Theorem 4].

It was observed by Herman [36, p. 609] that Sullivan's method [57] to prove the non-existence of wandering domains for rational functions can also be applied to entire functions with Baker domains and leads to the following result.

Theorem 6 *If f has a periodic cycle of Baker domains which does not contain a singular value, then f has an infinite dimensional space of quasiconformal deformations.*

As specific examples where Theorem 2.3 applies we mention functions f of the form $f(z) = z + p(z)e^{q(z)}$ where p, q are polynomials, or functions f of the form $f(z) = z - g(z)/g'(z)$ where $g(z) = \int_0^z e^{q(t)}dt$ with a polynomial q. (Note that f arises from applying Newton's method for finding the zeros of g.) These functions have finite dimensional spaces of quasiconformal deformations, and thus every periodic cycle of Baker domains contains a singular value. We refer to [11, 55] for these examples, as well as to [20] for some other examples; see also [10, Theorem 14] for further discussion.

Another relation between Baker domains and singular values is given by the following result proved in [12, Theorem 3].

Theorem 7 *If U is an invariant Baker domain which does not contain a singular value, then there exists a sequence (p_n) such that $p_n \in P(f)$, $|p_n| \to \infty$, $|p_{n+1}/p_n| \to 1$, and $\operatorname{dist}(p_n, U) = o(|p_n|)$ as $n \to \infty$.*

We note that the example $f(z) = 2 - \log 2 + 2z - e^z$ already mentioned shows that $\operatorname{dist}(p_n, U) = o(|p_n|)$ and $|p_{n+1}/p_n| \to 1$ cannot be replaced by $\operatorname{dist}(p_n, U) = o(1)$ and $|p_{n+1} - p_n| = o(1)$. It is not clear whether the conditions that $\operatorname{dist}(p_n, U) = o(|p_n|)$ and $|p_{n+1}/p_n| \to 1$ can be improved.

If information about the asymptotics of f in a Baker domain U is given, then one can sometimes use this to prove that U contain at least one singular value. Hinkkanen [38, Theorem 2] obtained such a result for functions f satisfying $f(z) = z + az^{-m} + O(z^{-m-\delta})$ as $z \to \infty$ in a suitable sector, where $a \in \mathbb{C}\backslash\{0\}$ and $\delta > 0$. In some cases the asymptotic behavior in a Baker domain domain U even implies the existence of infinitely many singular values in U. The following result is due to Rippon and Stallard [49, Theorem 2]

Theorem 8 *Suppose that*

$$f(z) = az + bz^k e^{-z}(1 + o(1))$$

as $\operatorname{Re} z \to \infty$, *where* $a > 1, b > 0, k \in \mathbb{N}$. *Then there exists* $\rho > 0, R > 0$ *such that* $\{z : |z^k e^{-z}| < \rho, |z| > R\}$ *is contained in a Baker domain* U, *and* U *contains infinitely many singular values.*

Rippon and Stallard [49] have also given some other conditions implying that a Baker domain contains infinitely many singular values.

The following theorem [13] is a further result in this direction. In order to state it, we denote by λ_U the hyperbolic metric in a domain U and consider for a Baker domain U and $z \in U$ the sequence $(\rho_n(z))$ defined by $\rho_n(z) := \lambda_U(f^{n+1}(z), f^n(z))$. By Schwarz's lemma, $(\rho_n(z))$ is non-increasing so that $\rho(z) := \lim_{n\to\infty} \rho_n(z)$ exists. The sequences $(\rho_n(z))$ were also considered by Rippon and Stallard [49] in their proof of Theorem 2.3, and they have also been studied by Bargmann [8], Bonfert [24] and König [41].

Theorem 9 *If* U *is an invariant Baker domain such that* $U \cap \operatorname{sing}(f^{-1})$ *is bounded, then we have one of the following three cases:*

(i) $\inf_{z\in U} \rho(z) > 0$,

(ii) $\rho(z) > 0$ *for all* $z \in U$, *but* $\inf_{z\in U} \rho(z) = 0$, *and there exists* $a \geq 0$ *such that*

$$\rho_n(z) = \rho(z) + a \cdot \frac{1}{n^3} + O\left(\frac{1}{n^4}\right),$$

(iii) $\rho(z) = 0$ *for all* $z \in U$, *and there exists* $b \in \mathbb{R}$ *such that*

$$\rho_n(z) = \frac{1}{2n} + b \cdot \frac{\log n}{n^2} + O\left(\frac{1}{n^2}\right).$$

The main idea in the proof is to use that U is simply connected [2, Theorem 1] and that if $\varphi : \mathbb{D} \to U$ is simply-connected, then $g := \phi^{-1} \circ f \circ \phi$ is an inner function. This approach has been used in a number of papers [4, 5, 8, 40, 41]. It turns out that under the hypotheses of Theorem 2.3 the function g can be continued across some arc on $\partial\mathbb{D}$ and it has a fixed point ξ on this arc. If ξ is attracting, then we have case (i). Otherwise ξ is a multiple fixed point of multiplicity 2 or 3, and this leads to cases (iii) and (ii). The numbers a and b occurring in (ii) and (iii) can be expressed in terms of g and ξ. More precisely, we have

$$a = \frac{\iota(g,\xi) - 1}{3\tanh\left(\frac{\rho(z)}{2}\right)} \quad \text{and} \quad b = \frac{\iota(g,\xi)}{4} - \frac{3}{8}.$$

As an example where Theorem 2.3 applies we mention entire functions f which satisfy $f(z) = z + c + o(1)$ as $|z| \to \infty$ in some sector $\{z : |\arg z| \leq \eta\}$, where $c, \eta > 0$. It is easy to see that such a function f has an invariant Baker domain U containing $\{z : |\arg z| \leq \eta, \operatorname{Re} z > R\}$ for some $R > 0$. Using Theorem 2.3 one can show that if f has finite order, then $U \cap \operatorname{sing}(f^{-1})$ is unbounded; see [13, Theorem 3].

Finally we mention that for functions f satisfying $f(z) = z+c+o(1)$ as $|z| \to \infty$ in some sector $\{z : |\arg z| \leq \eta\}$, where $c, \eta > 0$, there is also an analogue of Theorem 2.1 for Baker domains; see also [15] besides [14, 25] for this result.

Theorem 10 *Suppose that* $f(z) = z + 1 + c/z + o(1/z)$ *as* $|z| \to \infty$ *in some sector* $\{z : |\arg z| \leq \eta\}$*, where* $c \in \mathbb{C}$ *and* $\eta > 0$*. Then* f *has a Baker domain* U *containing* $\{z : |\arg z| \leq \eta, \operatorname{Re} z > R\}$ *for some* $R > 0$*. If* $\operatorname{Re} c < \frac{3}{20}$*, then* U *contains at least two singular values.*

In order to see the analogy to Theorem 2.1 we note that if f is a rational function satisfying $f(z) = z + 1 + c/z + o(1/z)$ as $|z| \to \infty$, then f has a fixed point of multiplicity 2 at ∞, with $\iota(f, \infty) = 1 - c$. Thus the condition $\operatorname{Re} c < 3/20$ corresponds to (2) if $m = 1$.

Note that the hypothesis on c is satisfied in particular if $c = 0$.

2.4 Concluding remarks

Here we have concentrated on relations between periodic Fatou components and singular values. We mention that there are also relations between wandering domains and singular values. This concerns results about the non-existence of wandering domains [11, 19, 30, 34, 55], which extend the result of Sullivan [57] that rational functions do not have wandering domains to certain classes of transcendental entire functions, as well as results on the limit functions of the iterates in wandering domains [1, 16, 17].

There are many other results in complex dynamics where the singular values play an important role. For example, there are intimate relations between singular values and the geometry of the Julia set, its Hausdorff dimension, and various other properties. We refer the reader to the textbooks and articles mentioned in the introduction.

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ON THE NORMALITY OF TOPOLOGICAL TARGET MANIFOLDS FOR RIEMANN–HILBERT PROBLEMS

E. Wegert, G. Semmler
TU Bergakademie Freiberg
Institut Angewandte Mathematik I
D-09596 Freiberg, Germany
wegert@math.tu-freiberg.de

Abstract In recent years R. Belch proposed an approach for investigating nonlinear Riemann-Hilbert problems with non-smooth target manifold. His main result is a characterization of solutions to Riemann-Hilbert problems as extremal functions in certain function classes. However, a complete analogy to corresponding results for problems with smooth target manifold holds only for a subclass of the toplogical target manifolds introduced by Belch, which are called normal. The conjecture that this subclass coincides with the whole class of topological target manifolds was left unproved. In the present paper we give a (counter-)example of a topological target manifold for which the solution set of the Riemann-Hilbert problem is in some sense bigger than in the smooth case. The problem to characterize normal topological target manifolds in geometric terms arises now as a challenging question of ongoing research.

Keywords: Riemann-Hilbert problem, conformal mapping

Mathematics Subject Classification (2000): 30E25

The *non-linear Riemann-Hilbert problem* consists in finding all holomorphic functions w in the complex unit disc $\mathbb{D}$ which satisfy the boundary condition

$$w(t) \in M_t \quad \forall t \in \mathbb{T} \tag{1}$$

on the unit circle $\mathbb{T}$. Here $\{M_t\}$ is a given family of curves in the complex plane.

We refer to M_t as the *target curves* of the problem and define the *target manifold* as the set

$$M := \bigcup_{t\in\mathbb{T}} \{t\} \times M_t \subset \mathbb{T} \times \mathbb{C}.$$

G. A. Barsegian and H. G. W. Begehr (eds.), Topics in Analysis and its Applications, 61–68.

In the classical formulation it is assumed that w belongs to $H^\infty \cap C$, the space of holomorphic functions in $\mathbb{D}$ which have a continuous extension onto $\overline{\mathbb{D}}$.

Depending on the geometry of the target manifold, several types of Riemann-Hilbert problems can be distinguished (see [6] for Riemann-Hilbert problems on the disc, Efendiev and Wendland [4], for problems on multiply connected domains, and M. Černe [3] for problems on bordered Riemann surfaces). For the convenience of the reader we formulate one main result.

A *smooth compact target manifold* M is a connected totally real compact C^1-submanifold of $\mathbb{T}\times\mathbb{C}$. All target curves M_t of a smooth compact target manifold M are smooth simple closed curves in $\mathbb{C}$. The interior region of M_t is denoted by $\operatorname{int} M_t$.

A smooth compact target manifold M is said to be *regularly traceable* if there exists a function $w_0 \in H^\infty \cap C$ such that $w_0(t) \in \operatorname{int} M_t$ for all $t \in \mathbb{T}$. Any such function w_0 is said to be an *interior function* of M. For each solution $w \in H^\infty \cap C$ of (1) the function $w - w_0$ has a winding number about the origin giving the number of zeros of this difference in the open unit disc. We refer to this number as the *winding number* $\operatorname{wind}_M w$ *of* w *about* M.

The following statements about the solvability of the Riemann-Hilbert problem (1) can be traced back to A.I. Shnirel'man [5]. Here we quote from [6], Theorem 2.5.1.

Theorem 1 *Let M be a smooth compact regularly traceable target manifold.*

(i) *The set of solutions with winding number zero about M is homeomorphic to the unit circle.*

(ii) *Let w_0 be an interior function of M and $k \geq 0$. Fix k points $z_1, \ldots, z_k \in \mathbb{D}$, $t_0 \in \mathbb{T}$, and $x_0 \in M_{t_0}$. Then there is exactly one solution $w \in H^\infty \cap C$ with $\operatorname{wind}_M w = k$ satisfying the conditions $w(t_0) = x_0$ and*

$$w(z_j) = w_0(z_j), \quad j = 1, \ldots, k. \tag{2}$$

One main challenge in investigating Riemann-Hilbert problems is to reduce the smoothness assumptions on the target manifold M. In order to investigate non-smooth problems, R. Belch [1, 2] introduced a new concept of target manifolds.

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere. A *topological target manifold* M is a subset of $\mathbb{T} \times \widehat{\mathbb{C}}$, for which there exist open sets $A, B \subset \mathbb{T} \times \widehat{\mathbb{C}}$ with the following three properties.

(i) $\mathbb{T} \times \widehat{\mathbb{C}}$ is the disjoint union of A, B and M.

(ii) The fibers $A_t := \{w \in \widehat{\mathbb{C}} \colon\ (t, w) \in A\}$ and $B_t := \{w \in \widehat{\mathbb{C}} \colon\ (t, w) \in B\}$ are non-empty and simply connected for any $t \in \mathbb{T}$.

(iii) M is the common boundary of A and B, $\partial A = \partial B = M$.

Though the manifolds are called topological, the concept includes also Riemann-Hilbert problems with piecewise continuous boundary conditions. The fibers

$$M_t := \{w \in \widehat{\mathbb{C}} \colon\ (t, w) \in M\}$$

of a topological target manifold M need not even be curves.

In this general setting the meaning of the boundary condition is not immediately clear. Especially one cannot expect that solutions of problems with topological target manifolds are continuous up to the boundary of the disk. Assume we seek solutions in the Hardy space H^∞ of bounded holomorphic functions (which is a natural choice for bounded target manifolds $M \subset \mathbb{T} \times \mathbb{C}$). Functions in H^∞ have nontangential boundary values almost everywhere on $\mathbb{T}$ and we could formulate the boundary condition as

$$w(t) \in M_t \quad \text{a.e. on } \mathbb{T}. \tag{3}$$

In a moment we shall see that this does not give quite what we want, but before we have to mention another obstacle: The (geometric) winding number of a solution in H^∞ about M makes no sense. The natural substitute for the winding number of w about a target manifold M with an interior function w_0 is the number of zeros of $w - w_0$ in $\mathbb{D}$. Now consider, for example, the standard problem with $M = \mathbb{T} \times \mathbb{T}$. If the boundary condition is to be understood in the sense of (3), all inner functions are solutions of the Riemann-Hilbert problem. Among them are plenty of functions without zeros in $\mathbb{D}$: the unimodular constants (which are the "classical" solutions with winding number zero about M), but also all singular inner functions. So the class of solutions without zeros *is much wider* than the class of solutions with winding number zero.

In order to circumvent this difficulty, the boundary condition is interpreted in a different way. For this end, let $\operatorname{clus} w(t)$ denote the *cluster set* of a holomorphic (or meromorphic) function w at $t \in \mathbb{T}$,

$$\operatorname{clus} w(t) := \{w \in \widehat{\mathbb{C}} \colon\ \exists (z_n) \subset \mathbb{D},\ z_n \to t,\ w(z_n) \to w\}.$$

The boundary condition of a Riemann-Hilbert problem is then understood in the sense that

$$\operatorname{clus} w(t) \subset M_t \text{ for all } t \in \mathbb{T}.$$

If we define the cluster-graph of w as the set

$$\operatorname{cgr} w := \{(t,y) : \ t \in \mathbb{T}, \ y \in \operatorname{clus} w(t), \}$$

a short form of writing the boundary condition is

$$\operatorname{cgr} a \subset M. \tag{4}$$

Since Belch's target manifolds live in $\mathbb{T} \times \widehat{\mathbb{C}}$ it is natural that he looks for meromorphic solutions.

In this setting a topological target manifold $M \subset \mathbb{T} \times \widehat{\mathbb{C}}$ is called *regularly traceable* if there exist meromorphic functions a, b in $\mathbb{D}$ such that

$$\operatorname{cgr} a \subset A, \quad \operatorname{cgr} b \subset B, \quad \#\,[a = b] = 0.$$

Here the *intersection number* $\#\,[a = b]$ of the functions a and b is the number of zeros of $a - b$ as a mapping $\mathbb{D} \to \widehat{\mathbb{C}}$.

Since our main interest is in holomorphic solutions we assume for the sake of simplicity that A is bounded and $0 \in A_t$ for all $t \in \mathbb{T}$. We call A the interior and B the exterior of M and take $a \equiv 0$ and $b \equiv \infty$. Then for smooth target manifolds the meromorphic solutions with $\#\,[w = a] = k$ and $\#\,[w = b] = 0$ correspond to the holomorphic solutions with winding number k about M, which leads to the definition of the solution classes

$$\mathbb{W}_k := \{w \in \mathcal{O}(\mathbb{D}) : \ \#\,[w = a] = k\}.$$

In order to study the structure of the (most important) solution class $\mathbb{W}_0$, we follow Belch [1, 2] and consider families of holomorphic functions with restricted cluster sets

$$\begin{aligned}
\mathcal{A} &:= \{w \in \mathcal{O}(\mathbb{D}) : \ \operatorname{cgr} w \subset A, \ \#\,[w = b] = 0\} \\
\mathcal{B} &:= \{w \in \mathcal{O}(\mathbb{D}) : \ \operatorname{cgr} w \subset B, \ \#\,[w = a] = 0\} \\
\mathbb{A} &:= \{w \in \mathcal{O}(\mathbb{D}) : \ \operatorname{cgr} w \subset \overline{A}, \ \#\,[w = b] = 0\} \\
\mathbb{B} &:= \{w \in \mathcal{O}(\mathbb{D}) : \ \operatorname{cgr} w \subset \overline{B}, \ \#\,[w = a] = 0\}.
\end{aligned}$$

Let $\mathbb{X}$ be one of the sets $\mathcal{A}$, $\mathcal{B}$, $\mathbb{A}$, $\mathbb{B}$ or $\mathbb{W}_k$. The *range of values* of $\mathbb{X}$ at $z \in \mathbb{D}$ is defined by

$$\mathbb{X}(z) := \{w(z) : \ w \in \mathbb{X}\}, \quad z \in \mathbb{D}.$$

If M is a regularly traceable topological target manifold then, for each $z \in \mathbb{D}$,

(i) $\mathbb{A}(z)$ is homeomorphic to $\overline{\mathbb{D}}$ and $\mathbb{B}(z)$ is homeomorphic to $\mathbb{C} \setminus \mathbb{D}$,

(ii) $\mathcal{A}(z) = \mathbb{C} \setminus \mathbb{B}(z)$ and $\mathcal{B}(z) = \mathbb{C} \setminus \mathbb{A}(z)$,

(iii) $\mathbb{C}$ is the disjoint union of $\mathcal{A}(z)$, $\mathbb{W}_0(z)$, and $\mathcal{B}(z)$.

Belch's main result is the following existence theorem which generalizes an extremal principle for Riemann-Hilbert problems with smooth target manifold from [6].

Theorem 2 (R. Belch [1] Theorem 6.4.1; [2] Section D.4.6) *Let $z \in \mathbb{D}$. If the topological target manifold is regularly traceable, then for each point $y \in \partial\mathbb{A}(z) \cup \partial\mathbb{B}(z)$ there exists a unique solution $w \in \mathbb{W}_0$ with $w(z) = y$.*

A closer look reveals that this result is not quite what one expects. Recall that for smooth target manifolds the solution set $\mathbb{W}_0$ is homeomorphic to the unit circle $\mathbb{T}$. Now we have one solution for each $y \in \partial\mathbb{A}(z)$ and one solution for each $y \in \partial\mathbb{B}(z)$, which seems to be too many, compared with the case of smooth target manifolds.

Of course, everything would fit nicely with the former results if $\partial\mathbb{A}(z)$ and $\partial\mathbb{B}(z)$ coincide. Therefore a topological target manifold is said to be *normal*, if for one (and then for all) $z \in \mathbb{D}$

$$\overline{\mathcal{A}(z)} = \mathbb{A}(z), \tag{5}$$

which implies that $\partial\mathbb{A}(z) = \partial\mathbb{B}(z)$. Belch conjectured ([1] Section 1.2.1; [2] Section D4.7) that all topological target manifolds are normal.

Though all smooth regularly traceable target manifolds are normal, this conjecture was too optimistic. In the following we give a counterexample.

Theorem 3 *Not all topological target manifolds are normal.*

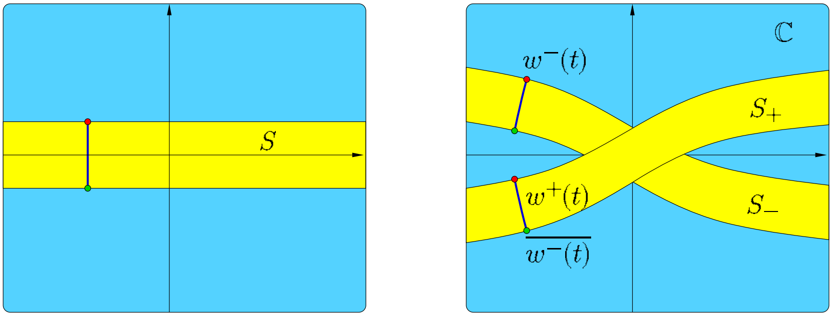
Proof We construct a topological target manifold for which $\mathbb{A}(0)$ and $\mathbb{B}(0)$ contain a common inner point.

Let $g\colon \mathbb{R} \to \mathbb{R}$ be a smooth strictly increasing bounded odd function. We define the strips S_- and S_+ by

$$S_\pm := \{z = x + \mathrm{i}y \in \mathbb{C}\colon\ |y \mp g(x)| < 1\}$$

and denote by $w^\pm = u^\pm + \mathrm{i}v^\pm$ the conformal maps of $\mathbb{D}$ onto $S_\pm$ normalized so that $\qquad w^\pm(0) = 0, \qquad u^\pm(1) = +\infty.$

It is convenient to consider these maps as composition of a map of $\mathbb{D}$ onto the horizontal strip $S := \{z \in \mathbb{C}\colon\ |\mathrm{Im}\, z| < 1\}$ and a map of S onto S_+ and S_-, respectively.



The conformal maps of S onto S+ and S-

The maps obey the symmetry relation $w^+(\overline{z}) = \overline{w^-(z)}$. Symmetry also implies that $u^\pm(-1) = -\infty$. If t is on the upper half circle $\mathbb{T}_+$ then $\operatorname{Re} w^-(t) > \operatorname{Re} w^+(t)$, if t is on the lower half circle $\mathbb{T}_-$ then $\operatorname{Re} w^-(t) < \operatorname{Re} w^+(t)$. The properties of w^+ and w^- guarantee the existence of a smooth bounded function $f\colon\ \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ with the following properties.

(i) For all $t \in \mathbb{T}$ the function $u \mapsto f(t,u)$ has compact support.

(ii) For $t = 1$ and $t = -1$ there holds $f(t,u) \equiv 0$.

(iii) For all $t \in \mathbb{T} \setminus \{-1, 1\}$ we have
$$v^-(t) < f(t, u^-(t)) - 1 \text{ and } v^+(t) > f(t, u^+(t)) + 1.$$

Now we choose a smooth real-valued function $\varphi\colon \mathbb{T} \to \mathbb{R}$ which vanishes outside a neighborhood of the point $t = \mathrm{i}$ and has integral mean 1 over $\mathbb{T}$. Let ψ denote the conjugate of φ, so that $w_1 = \varphi + \mathrm{i}\psi$ extends holomorphically into $\overline{\mathbb{D}}$. The extended function w_1 is smooth and satisfies $w_1(0) = 1$. For $\alpha, \beta \in \mathbb{R}$ we consider the perturbed functions

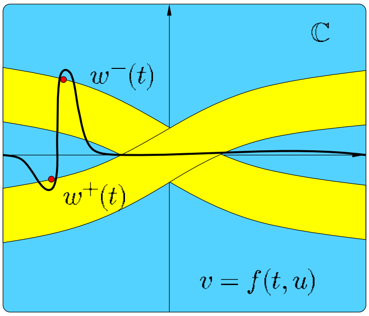
$$w^\pm_{\alpha,\beta} := w^\pm + \alpha w_1 + \beta\,\mathrm{i}.$$

Obviously, $w^-_{\alpha,\beta}(0) = w^+_{\alpha,\beta}(0) = \alpha + \mathrm{i}\,\beta$.

If α and β are sufficiently small, say $|\alpha + \mathrm{i}\,\beta| < \varepsilon$, then the perturbed functions $w^+_{\alpha,\beta}$ and $w^-_{\alpha,\beta}$ still satisfy

$$v^-_{\alpha,\beta}(t) < f\big(t, u^-_{\alpha,\beta}(t)\big), \quad v^+_{\alpha,\beta}(t) > f\big(t, u^+_{\alpha,\beta}(t)\big)$$

on $\mathbb{T} \setminus \{-1, 1\}$.



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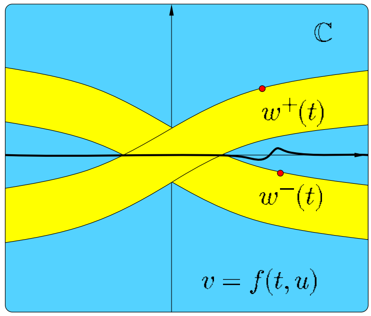

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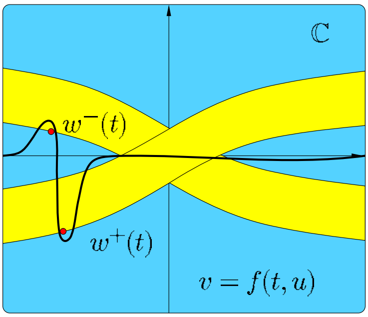

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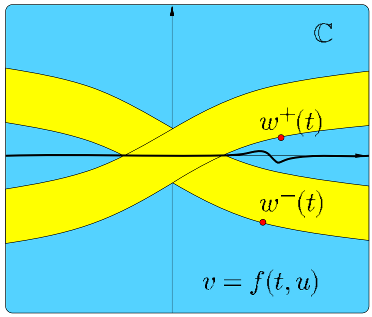

$\operatorname{Re} t > 0,\ \operatorname{Im} t < 0$

Further, there exists a real constant c which is an upper bound for the absolute values of f, $\operatorname{Re} w^-_{\alpha,\beta}$ and $\operatorname{Re} w^+_{\alpha,\beta}$ if only $|\alpha + \mathrm{i}\,\beta| < \varepsilon$.

Finally, let $T(z) := 1/(z - 2\mathrm{i}c)$ and define the curve $M^*_t \subset \mathbb{C}$ as the image of the graph of $u \mapsto f(t,u)$ under the transformation T. The (open smooth) curves M^*_t lie between two circles that touch each other at the origin. We complement the curves M^*_t by the origin and get closed Jordan curves M_t. Now it is not difficult to see that the set $M \subset \mathbb{T} \times \mathbb{C}$ defined by

$$M := \{(t,z) \colon\ t \in \mathbb{T}, z \in M_t\}$$

is a topological target manifold (it is the image of a torus under a fiber-preserving homeomorphism). If f is chosen so that $f(t,.) \equiv 0$ in a neighborhood of $t = 1$ it is even smooth with the exception of the single point $(-1,0)$.

In order to show that M is not normal, we remark that for all α and β with $|\alpha + \mathrm{i}\,\beta| < \varepsilon$ the functions $T \circ w^{\pm}_{\alpha,\beta}$ are holomorphic in $\mathbb{D}$ and continuous on the closed disk. They satisfy $\operatorname{cgr}\big(T \circ w^-_{\alpha,\beta}\big) \subset A$ and $\operatorname{cgr}\big(T \circ w^+_{\alpha,\beta}\big) \subset B$. The values $w^{\pm}_{\alpha,\beta}(0)$ with $|\alpha + \mathrm{i}\,\beta| < \varepsilon$ cover a neighborhood of the origin and consequently the point $T(0)$ is an interior point of $\mathbb{A}(0)$ and $\mathbb{B}(0)$.

Once we know that Belch's conjecture is not true, the question arises which topological target manifolds are normal.

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GEOMETRIC ASPECTS OF GENERALIZED ANALYTIC FUNCTIONS

Gia Giorgadze
Georgian Academy of Sciences
Razmadze Mathematical Institute
Tbilisi, Georgia
giorgadze@rmi.acnet.ge

Abstract The paper deals with analytic and topological aspects of elliptic Carleman-Bers-Vekua system. Using this system a holomorphic vector bundle on the Riemann sphere is constructed together with an L_p-connection on it. In this framework the Riemann-Hilbert monodromy problem is formulated and solved for a special class of regular systems.

Keywords: Elliptic Carleman-Bers-Vekua system, monodromy, Pfaff system, Riemann-Hilbert problem, holomorphic vector bundle, Chern class, differential form, L_p-connection

Mathematics Subject Classification (2000): 30E25, 30G35, 31A30

The global theory of generalized analytic functions [13] both in one-dimensional and multi-dimensional case [4], involves studying the space of horizontal sections of a holomorphic line bundle with connection on a complex manifold with singular divisor. In this context one needs to require that a connection is complex analytic. An interesting class of such connections is given by L_p-connections, and their moduli spaces have many applications. Such connections and their moduli spaces are the object of intensive study [12], [7].

We study holomorphic vector bundles with L_p-connections from the viewpoint of the theory of generalized analytic vectors [4]. To this end we consider a matricial elliptic system of the form

$$\partial_{\bar{z}}\Phi(z) = A(z)\Phi(z). \tag{1}$$

The system (1) is a particular case of the *Carleman-Bers-Vekua system* [13]

G. A. Barsegian and H. G. W. Begehr (eds.), Topics in Analysis and its Applications, 69–81.

$$\partial_{\bar{z}} f(z) = A(z)f(z) + B(z)\overline{f(z)}, \tag{2}$$

where $A(z)$, $B(z)$ are bounded matrix functions on a domain $U \subset \mathbf{C}$ and $f(z) = (f^1(z), ..., f^n(z))$ is an unknown vector function. The solutions of the system (2) are called *generalized analytic vectors*, by analogy with the one-dimensional case [13], [4].

Along with similarities between the one-dimensional and multi-dimensional cases, there also exist essential differences. One of them, as was noticed by B.Bojarski [4], is that there can exist solutions of system (1) for which there is no analogue of the Liouville theorem on the constancy of bounded entire functions.

At first we present some necessary fundamental results of the theory of generalized analytic functions [13],[2],[3],[4] in the form convenient for our purposes.

Let $f \in \mathrm{L}^p(U)$, where U is a domain in $\mathbf{C}$. We write $f \in \mathrm{W}_p(U)$, if there exist functions θ_1 and θ_2 of class $\mathrm{L}^p(U)$ such that the equalities

$$\iint_U f\frac{\partial\varphi}{\partial\bar{z}}dU = -\iint_U \theta_1\varphi dU, \ \iint_U f\frac{\partial\varphi}{\partial z}dU = -\iint_U \theta_2\varphi dU$$

hold for any function $\varphi \in \mathrm{C}^1(U)$.

Let us define two differential operators on $\mathrm{W}_p(U)$

$$\partial_{\bar{z}} : \mathrm{W}_p(U) \to \mathrm{L}_p(U), \quad \partial_z : \mathrm{W}_p(U) \to \mathrm{L}_p(U),$$

by setting $\partial_{\bar{z}} f = \theta_1$, $\partial_z f = \theta_2$. The functions θ_1 and θ_2 are called the generalized partial derivatives of f with respect to $\bar{z}$ and z respectively. Sometimes we will use a shorthand notation $f_{\bar{z}} = \theta_1$ and $f_z = \theta_2$. It is clear that ∂_z and $\partial_{\bar{z}}$ are linear operators satisfying the Leibniz equality.

Define the following singular integral operator on the Banach space $\mathrm{L}_p(U)$:

$$T : \mathrm{L}_p(U) \to W_p(U),$$

$$T(\omega) = -\frac{1}{\pi}\iint_U \frac{\omega(t)}{t-z}dU, \omega \in L_p(U). \tag{3}$$

The integral (3) makes sense for all $\omega \in \mathrm{L}_p(U)$, almost all $z \in U$, and all $z \notin \bar{U}$ and (3) determines a function $\varphi(z) = T(\omega)$ on the whole $\mathbf{C}$. For $\omega \in \mathrm{L}_p(U)$ with $p > 2$, the function φ is continuous.

Any element of $\mathrm{W}_p(U)$ can be represented by an integral (3). In particular, if $f_{\bar{z}} = \omega$, then $f(z)$ can be represented in the form

$$f(z) = h(z) - \frac{1}{\pi}\iint_U \frac{\omega(t)}{t-z}dU,$$

where $h(z)$ is holomorphic in U. The converse is also true, i.e., if $h(z)$ is holomorphic in U and $\omega \in \mathrm{L}_p(U)$, then $h(z) - (1/\pi) \iint_U \omega(t)/(t-z)dU$ determines an element $f(z)$ of $\mathrm{W}_p(U)$ satisfying the equality $f_{\bar{z}} = \omega$.

As we saw, the generalized derivative with respect to $\bar{z}$ of the integral (3) is ω. Similarly, there exists a generalized derivative of this integral with respect to z. It equals

$$-\frac{1}{\pi} \iint_U \frac{\omega(t)}{(t-z)^2} dU. \tag{4}$$

The integral (4) is understood in the sense of Cauchy principal value and by definition equals

$$S(\omega) = \lim_{\varepsilon \to 0} f_\varepsilon(z) \equiv -\frac{1}{\pi} \iint_{U_\varepsilon} \frac{\omega(t)}{(t-z)^2} dU, \tag{5}$$

where $U_\varepsilon = \mathbf{C} \setminus \overline{D_\varepsilon(z)}$, with $D_\varepsilon(z)$ being the disk of radius ε centered at z. In the equality (4), the limit converges to function $f(z)$ in L_p-metric, $p > 1$.

It is known [13] that in one dimensional case a solution of (1) can be represented as

$$\Phi(z) = F(z)\exp(\omega(z)), \tag{6}$$

where F is a holomorphic function in U, and $\omega = -(1/\pi) \iint_U A(z)/(\xi - z)dU$. In the multi-dimensional case an analogue of factorization (6) is given by the following theorem.

Theorem 1 [2]. E*ach solution of the matricial equation (1) in U can be represented as*

$$\Phi(z) = F(z)V(z), \tag{7}$$

where $F(z)$ is an invertible holomorphic matrix function in U, and $V(z)$ is a single-valued matrix function invertible outside $\overline{U}$.

We use the representation of the solution of system (1) in the form (7) for the construction of a holomorphic vector bundle on the Riemann sphere and for the solution of the Riemann-Hilbert monodromy problem [1] for the elliptic system inhe form (1).

We recall some properties of solutions of (1). The product of two solutions is again a solution. From Theorem 1 follows (see also [6]) that the solutions constitute an algebra and the invertible solutions are a subfield of this algebra.

Proposition 1 *Let $C(z)$ be a holomorphic matrix function, then* $[C(z), \partial_{\overline{z}}] = 0$.

Indeed,

$$[C(z), \partial_{\overline{z}}]\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - \partial_{\overline{z}}C(z)\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - C(z)\partial_{\overline{z}}\Phi(z) = 0.$$

Here we have used that $\partial_{\overline{z}}C(z) = 0$.

Definition Two systems $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ and $\partial_{\overline{z}}\Phi(z) = B(z)\Phi(z)$ called *gauge equivalent* if there exists a non-degenerate holomorphic matrix function $C(z)$, such that $B(z) = C(z)A(z)C(z)^{-1}$.

Proposition 2 *Let the matrix function $\Psi(z)$ be a solution of the system $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ and let $\Phi_1(z) = C(z)\Phi(z)$, where $C(z)$ is a nonsingular holomorphic matrix function. Then $\Phi(z)$ and $\Phi_1(z)$ are solutions of gauge equivalent systems.*

The converse is also true: if $\Phi(z)$ and $\Phi_1(z)$ satisfy systems of equations

$$\partial_{\overline{z}}\Phi(z) = A(z)\Phi_(z),$$

$$\partial_{\overline{z}}\Phi_1(z) = B(z)\Phi_1(z)$$

and $A(z) = C^{-1}(z)B(z)C(z)$, then $\Phi_1 = D(z)\Phi(z)$ for any holomorphic matrix function $D(z)$.

Proof By Proposition 1 we have $C(z)\partial_{\overline{z}}\Phi_1(z) = A(z)C(z)\Phi_1(z)$, and therefore $\Phi_1(z)$ satisfies the equation $\partial_{\overline{z}}\Phi_1(z) = C^{-1}(z)A(z)C(z)\Phi_1(z)$. To prove the converse let us substitute in $\partial_{\overline{z}}\Phi(z) = A(z)\Phi_(z)$, in place of $A(z)$ the expression of the form $C^{-1}B(z)C(z)$ and consider

$$\partial_{\overline{z}}\Phi_1(z) = C^{-1}B(z)C(z)\Phi(z) \Longrightarrow$$

$$C(z)\partial_{\overline{z}}\Phi(z) = B(z)C(z)\Phi(z)$$

but for the left-hand side of the last equation we have $C(z)\partial_{\overline{z}}\Phi(z) = \partial_{\overline{z}}C(z)\Phi(z)$, therefore

$$\partial_{\overline{z}}(C(z)\Phi(z)) = B(z)(C(z)\Phi(z)).$$

From this it follows that Φ and $C\Phi$ are the solutions of equivalent systems, which means that $\Phi_1 = D\Phi$.

The above arguments for solutions of (1) are of a local nature, so they are applicable for an arbitrary compact Riemann surface X, which enables us to construct a holomorphic vector bundle on X. Moreover, using the solutions of system (1) one can construct a matrix 1-form $\Omega = D_{\overline{z}}FF^{-1}$ on X which is analogous to holomorphic 1-forms on Riemann surfaces.

Let X be a Riemann surface. Denote by $L_p^{\alpha,\beta}(X)$ the space of L_p-forms of the type $(\alpha,\beta), \alpha,\beta = 0,1$, with the norm $\|\omega\|_{L_p^{\alpha,\beta}(X)} = \sum_j \|\omega\|_{L_p^{\alpha,\beta}(U_j)}$, where $\{U_j\}$ is an open covering of X and denote by $W_p(U) \subset L_p(U)$ the subspace of functions which have generalized derivatives.

We define the operators

$$D_z = \frac{\partial}{\partial z} : W_p(U) \to L_p^{1,0}(U), f \mapsto \omega_1 dz = \partial_z f dz,$$

$$D_{\bar z} = \frac{\partial}{\partial \bar z} : W_p(U) \to L_p^{0,1}(U), f \mapsto \omega_2 d\bar z = \partial_{\bar z} f d\bar z.$$

It is clear that $D_{\bar z}^2 = 0$ and hence the operator $D_{\bar z}$ can be used to construct the *de Rham cohomology.*

Let us denote by $\mathbf{C}L_p^1(X)$ the *complexification* of $L_p^1(X)$, i.e. $\mathbf{C}L_p^1(X) = L_p^1(X) \otimes \mathbf{C}$. Then we have the natural decomposition

$$\mathbf{C}L_p^1(X) = L_p^{1,0}(X) \oplus L_p^{0,1}(X) \tag{8}$$

according to the eigenspaces of the Hodge operator $* : L_p^1(X) \to L_p^1(X)$, $* = -\imath$ on $L_p^{1,0}(X)$ and $* = \imath$ on $L_p^{0,1}(X)$.

The decomposition (8) splits the operator $d : L_p^0(X) \to L_p^0(X)$ into the sum $d = D_z + D_{\bar z}$.

Next, let as above, $\mathcal{E} \to X$ be a C^∞-vector bundle on X, $L_p(X,\mathcal{E})$ be the sheaf of the L_p-sections of $\mathcal{E}$ and let $\Omega \in L_p^1(X,\mathcal{E}) \otimes \mathbf{GL}(n,\mathbf{C})$ be a matrix valued 1-form on X. If the above arguments are applied to the complex $L_p^*(X,\mathcal{E})$ with covariant derivative $\bigtriangledown_\Omega$, we obtain again the decompositions of the space $\mathbf{C}L_p^1(X,\mathcal{E})$ and the operator $\bigtriangledown_\Omega$:

$$\mathbf{C}L_p^1(X,\mathcal{E}) = L_p^{1,0}(X,\mathcal{E}) \oplus L_p^{0,1}(X,\mathcal{E}),$$

$$\bigtriangledown_\Omega = \bigtriangledown'_\Omega + \bigtriangledown''_\Omega.$$

Locally, on the domain U, we have $\bigtriangledown_\Omega^U = d_U + \Omega$, where $\Omega \in L_p^1(X,U) \otimes \mathbf{GL}(n,\mathbf{C})$ is a 1-form. Therefore $\bigtriangledown_\Omega^U = (D_z + \Omega_1) + (D_{\bar z} + \Omega_2)$, where Ω_1 and Ω_2 are, respectively, the holomorphic and anti-holomorphic parts of the matrix valued 1-form on U. We say that a W_p-section f of the bundle $\mathcal{E}$ with L_p-connection is holomorphic if it satisfies the system of equations

$$\partial_{\bar z} f(z) = A(z) f(z), \tag{9}$$

where $A(z)$ is an $n \times n$ matrix-function with entries in $L_p^0(X) \otimes \mathbf{GL}(n,\mathbf{C})$ and $f(z)$ is a vector function $f(z) = (f_1(z), f_2(z), ..., f_n(z))$, or in equivalent form if (9) may be written as

$$D_{\bar z} f = \Omega f,$$

where $\Omega \in L^1_p(X) \otimes \mathbf{GL}(n, \mathbf{C})$.

We now use the above arguments for constructing a holomorphic vector bundle over the Riemann sphere $\mathbf{C}P^1$ by system (1). Let $\{U_j\}$, j=1,2, be an open covering of the $\mathbf{C}P^1$. Then in any domain U_j, a solution $\Phi(z)$ can be represented as $\Phi(z) = V_j(z)F(z)$, where $V_j(z)$ is a holomorphic non-degenerate matrix function on $U_j^c - S_j$ with S_j being a finite set of points. Restrict $\Phi(z)$ on the $(U_1^{\mathbf{c}} \cap U_2^{\mathbf{c}}) - S = (U_1 \cup U_2)^{\mathbf{c}} - S$, $S = S_1 \cup S_2$ and consider the holomorphic matrix-function $\varphi_{12} = V_1(z)V_2(z)^{-1}$ on $(U_1 \cup U_2)^{\mathbf{c}} - S$. It is a cocycle and therefore defines a holomorphic vector bundle $\mathcal{E}'$ on $\mathbf{C}P^1 - S$. From the proposition 1 follows, that $\mathcal{E}' \rightarrow \mathbf{C}P^1 - S$ is independent of the choice of solutions in the same gauge equivalence class. The extension of this bundle to a holomorphic vector bundle $\mathcal{E} \rightarrow \mathbf{C}P^1$ can be done by a well-known construction (see [1]) and the obtained bundle is holomorphically nontrivial.

It is now possible to verify that the operator $\frac{\partial}{\partial \bar{z}} + \Omega(z, \bar{z})$ is an L_p-connection of this bundle. It turns out that its index coincides with the index of the Cauchy-Riemann operator on X. This follows since the index of Cauchy-Riemann operator is equal to the Euler characteristic of the sheaf of holomorphic sections of the holomorphic vector bundle $\mathcal{E}$.

Riemann-Hilbert boundary problem for generalized analytic vectors For a given loop $G : \Gamma \rightarrow GL_n(\mathbf{C})$, find the piecewise continuous generalized analytic vector $f(z)$ with a jump on the contour Γ such that on Γ it satisfies the conditions

$$a) f^+(t) = G(t)f^-(t), t \in \Gamma,$$

$$b) |f(t)| \leq c|z|^{-1}, |z| \rightarrow \infty.$$

It is known, that for G there exists a Birkhoff factorization, i.e.

$$G(t) = G_+(t)d_K(t)G_-(t).$$

Inserting this equality in a) we obtain the boundary problem

$$G_+^{-1}(t)f^+(t) = d_K(t)G_-(t)f^-(t).$$

Since $G_+^{-1}(t)f^+(t)$, $f^+(t)$ and $G_-(t)f^-(t)$, $f^-(t)$ are solutions of the gauge equivalent systems, the holomorphic type of corresponding vector bundles on the Riemann sphere is defined by $K = (k_1, ..., k_n)$.

Proposition 3 *The cohomology groups $H^i(\mathbf{C}\mathbb{P}^1, \mathcal{O}(\mathcal{E}))$ and $H^i(\mathbf{C}\mathbb{P}^1, \mathcal{G}(\mathcal{E}))$ are isomorphic for $i = 0, 1$, where $\mathcal{O}(\mathcal{E}))$ and $\mathcal{G}(\mathcal{E})$, respectively, are the sheaves of holomorphic and Q-holomorphic sections of $\mathcal{E}$.*

From this proposition follows that the number of linear independent solutions of the Riemann-Hilbert boundary problem is equal to $\sum_{k_j<0} k_j$. Its holomorphic type is determined by an integer vector. In terms of the cohomology groups $H^i(\mathbf{C}\mathbb{P}^1, \mathcal{O}(\mathcal{E}))$ and $H^i(\mathbf{C}\mathbb{P}^1, \mathcal{G}(\mathcal{E}))$ one can describe the number of solutions and stability of the Riemann-Hilbert problem [4]. The topological constructions related with the sheaf $\mathcal{O}(\mathcal{E})$ can be extended to the sheaf $\mathcal{G}(\mathcal{E})$ [8].

Denote by $\mathrm{L}_p^+(\gamma)$ the space of those holomorphic functions which are boundary values of functions from $\mathrm{L}^p(\gamma)$; similarly let $\mathrm{L}_p^-(\gamma)$ denote the space of those holomorphic functions on U_- whose extension to γ gives an element of $\mathrm{L}^p(\gamma)$.

Let $\mathrm{L}_\infty(\gamma)$ be the Banach space of Lebesgue measurable and essentially bounded functions. In this case it is possible to introduce the factorization of a matrix-function $G \in \mathrm{L}_\infty(\gamma)$ in the space $\mathrm{L}_p(\gamma)$. We say G is factorizable, if it is representable in the form

$$G(t) = G_+(t)\Lambda(t)G_-(t), \quad t \in \gamma, \tag{10}$$

where $\Lambda(t) = \mathrm{diag}(t^{k_1}, ..., t^{k_n})$, $k_i \in \mathbf{Z}$, $i = 1, ..., n$, G_+ is an invertible matrix-function with entries in $L_p^+(\gamma)$ and the elements of the G_+^{-1} are functions from $\mathrm{L}_q^+(\gamma)$, analogically, G_- is an invertible matrix-function with entries in $\mathrm{L}_q^-(\gamma)$, and the elements of the G_-^{-1} lie in $\mathrm{L}_p^-(\gamma)$, here $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\mathrm{PC}(\gamma)$ be the subspace of piecewise continuous matrix-functions. For elements of this space there exist the one-sided limits $G(t+0)$ and $G(t-0)$ for each $t \in \gamma$. Necessary and sufficient conditions for the existence of a Φ-factorization for such matrix-functions in the space $\mathrm{L}_p(\gamma)$ are the conditions

a) *the matrices $G(t+0)$ and $G(t-0)$ are invertible for each $t \in \gamma$;*

b) *for each $j = 1, ..., n$ and $t \in \gamma$ one has $1/2\pi \arg \lambda_j(t) + 1/p \notin \mathbf{Z}$, where $\lambda_1(t),...,\ \lambda_n(t)$ are eigenvalues of the matrix-function $G(t-0)G(t+0)^{-1}$.*

Suppose $G \in \mathrm{PC}(\gamma)$ is moreover a piecewise constant matrix function with singular points $s_1, ..., s_m \in \gamma$, occurring in this order on γ. Suppose G is factorizable in the space $\mathrm{L}_p(\gamma)$. Let us denote $M_k = G(s_k-0)G(s_k+0)^{-1}$, $k = 1, ..., m$. Thus G is constant on the arc (s_k, s_{k+1}), and clearly $M_1M_2 \cdots M_k = 1$.

Consider the Riemann-Hilbert boundary problem for generalized analytic vectors, where $G(t)$ is a lower triangular piecewise constant matrix function. Change $G(t)$ by a diagonal matrix function $\Lambda(t)$ from the representation of the factorization (10), then the spaces of solutions of these problems coincide.

Chen's iterated integral for L_p^1 For the investigation of the monodromy problem for the Pfaff system an important role is played by a representation of solutions of systems in exponential form, which in the one-dimensional case were studied by W. Magnus in [11]. We use iterated path integrals and the theory of formal connections (as a parallel transport operator) developed by K.-T.Chen [5].

Let $\omega_1, ..., \omega_r \in L_p^1(X)$ and $\gamma : [0,1] \to X$ be a piecewise continuous path. Let a_j be functions defined on $[0,1]$ and satisfying the identity $\gamma^*\omega_j = a_j(t)dt, j = 1, ..., r$.

Definition The r-iterated integral of 1-forms $\omega_1, ..., \omega_r$ is defined as the function on the space of piecewise continuous paths whose value on a path γ is the number $\gamma \to \int_\gamma \omega_1...\omega_r$, where $\int_\gamma \omega_1...\omega_r$ is

$$\int_\gamma \omega_1...\omega_r = \int_{\triangle_r} a_1(t_1)a_2(t_2)...a_r(t_r)dt_1dt_2...dt_r.$$

Here on the right-hand side is an ordinary Lebesque integral on the simplex

$$\triangle_r = \{(t_1, ..., t_r) : 0 \le t_1 \le t_2 \le ... \le t_r \le 1\}.$$

By passing to a multiple integral the value of an r-iterated integral can be expressed by the formula

$$\int_\gamma \omega_1...\omega_r = \int_0^1 a_r(t_r)... \int_0^{t_3} a_2(t_2)(\int_0^{t_2} a_1(t_1)dt_1)dt_2...dt_r.$$

If $r = 1$, then we obtain an ordinary path integral.

Let $P_{z_0}X$ be the space of piecewise continuous loops. It is known that it is a differentiable space [5] and the operator of exterior differentiation d is defined $P_{z_0}X$. Let $\mathrm{d}\int_\gamma \omega_1...\omega_r = 0$, then $\int_\gamma \omega_1...\omega_r = 0$ depends only on the homotopy class of γ and therefore we obtain a function on $\pi_1(X, z_0)$.

Let $\Omega_1, ..., \Omega_r$ be $m \times m$ matrix forms with entries of $L_p^1(X)$. The iterated integral of $\Omega_1, ..., \Omega_r$ is defined as follows: consider the form product of matrix forms $\Omega = \Omega_1...\Omega_r$ and define the iterated integrals of Ω elementwise.

Proposition 4 *The parallel transport corresponding to the elliptic system* (1) *has an exponential representation.*

Since the elliptic system (1) defines a connection the proof of the proposition follows from the general theory of formal connections.

From the identity $\partial_{\bar z}\Phi\Phi^{-1} = \Omega$ it follows that singular points of Ω are zeros of the matrix function Φ, in particular ∞. This means that it makes sense to speak of singular and apparent singular points of the system (1).

In case $n = 1$ from (6) it follows, that given an analytic function $F(z)$, one can define by (6) a generalized analytic function $\Phi(z)$ in a unique way. Besides that, if $z_1, ..., z_m$ are poles (or branching points) for $F(z)$, then they are poles (correspondingly, branching points) for $\Phi(z)$ too. For $n > 1$ this does not hold. Thus we wish to emphasize once more the difference between the one-dimensional and multi-dimensional theory of generalized analytic functions. In general the correspondence between holomorphic vectors and generalized analytic vectors is not one-to-one.

Despite of that, properties of the generalized analytic functions allow one to construct a generalized analytic function with given monodromy.

From the integrability of (1) it follows that for the iterated integral $\int \Omega\Omega...\Omega$ *we have* $d\int \Omega\Omega...\Omega = 0$ *and therefore we have a representation of the fundamental group* $\pi_1(X - S, z_0)$.

We can say that $z_i \in \{z_1, ..., z_m\}$ is a regular singular point of (1), if any element of $F(z)$ has at most polynomial growth as $z \to z_i$. If the solution $\Phi(z)$ at any singular point $z_i, i = 1, ..., m$ has a regular singularity, then we call the system (1) a regular system.

In case $n = 1$ the singular integral (6) is well studied. In particular, it is known that $\omega(z)$ is holomorphic in $\mathbf{C}_m \setminus \overline{U_{z_0}}$ and equal to zero at infinity. Here $\mathbf{C}_m = \mathbf{CP}^1 \setminus \{z_1, ..., z_m\}$.

Let $\tilde z \in U_{z_0}$ be any point and let $\gamma_1, \gamma_2, ..., \gamma_m$ be loops at $\tilde z$ such that γ_i goes around z_i without going around any $z_j \neq z_i$. Consider the holomorphic continuation of the function $F(z)$ around γ_i. Then we obtain an analytical element $\tilde F(z)$ of the holomorphic function $F(z)$, which are related by the equality $\tilde F_i(z) = m_i F(z)$, where $m_i \in \mathbf{C}^*$. It is independent on the choice of the homotopy type of loop γ_i. Therefore, we obtain a representation of the fundamental group $\pi_1(\mathbf{CP}^1 \setminus \{z_1, ..., z_m\}, \tilde z) \to \mathbf{C}^*$, which is defined by the correspondence $\gamma_i \to m_i$.

Let us sum up all what was said above.

Proposition 5 *Let the system (1) have regular singularities at points* $z_1, ..., z_m$. *Then it defines a monodromy representation of the fundamental group*

$$\rho : \pi_1(\mathbf{C} \setminus \{z_1, ..., z_m\}, \tilde z) \to GL_n(\mathbf{C}).$$

In this situation the monodromy matrices are given by Chen's iterated integrals

$$\rho(\gamma_j) = 1 + \int_{\gamma_j} \Omega + \int_{\gamma_j} \Omega\Omega + \int_{\gamma_j} \Omega\Omega\Omega + ... + ... \qquad (11)$$

The convergence properties of series (11) can be described as follows. Let a 1-form Ω be smooth except the points $s_1, ..., s_m \in X$. Let, as above, $S = \{s_1, s_2, ..., s_m\}$ and $X_m = X - S$. Thus, for every $\gamma \in PX_m$, there exists a constant $C > 0$ such that

$$|\int_{\gamma_j} \overbrace{\Omega...\Omega}^{r}| = 0(\frac{C^r}{r!})$$

and the series (11) converges absolutely [10].

In addition to system (1) consider the system of ordinary differential equations with regular singularities at the points $z_1, z_2, ..., z_m$,

$$\frac{df}{dz} = A(z)f(z). \tag{12}$$

Let

$$\rho : \pi_1(\mathbf{CP}^1 \backslash \{z_1, ..., z_m\}, \tilde{z}) \to GL_n(\mathbf{C}) \tag{13}$$

be the monodromy representation induced from (12). Denote by $F(z)$ the fundamental matrix of solutions of (12). Then $F(z)$ satisfies the system of equations

$$dF = \widehat{\Omega}F(z),$$

where $\widehat{\Omega} = A(z)dz$ denotes now the corresponding matrix valued $(1,0)$-form on the surface $\mathbf{CP}^1 \backslash \{z_1, ..., z_m\}$.

Let $\Omega(z) = *\widehat{\Omega}$, where $*$ is the Hodge star operator. Then $\Omega(z)$ is a (0,1)-form and it makes sense to consider a type (1) system

$$\partial_{\bar{z}}\Phi = \Phi\Omega(z), \tag{14}$$

which has regular singularities at the points $z_1, z_2, ..., z_m$.

Therefore we have proved the following theorem.

Theorem 2 *For a given representation (13), there exists an elliptic system (14) which has regular singularities at the given points $z_1, z_2, ..., z_m$ and its monodromy representation coincides with (13).*

Proof The proof of the theorem can be obtained by the following arguments. First we construct a holomorphic function Φ in the neighborhood $z_0 \in CP^1 - \{z_1, ..., z_m\}$, which is ramified at the points $z_1, z_2, ..., z_m$ and has the property that under continuation of Φ along small paths, when going around the singular points z_j, the matrix function $\Phi(z)$ transforms as $\Phi(z) = M_j\Phi(z)$. After that use the properties of generalized analytic vectors, which consist in the following: the set of ramification points coincides with the set of ramification points of the holomorphic matrix functions from (7).

Step 1. Construction of a holomorphic matrix function with given properties Let Γ be a contour, obtained by connecting the point z_1 with z_2, z_2 with z_3,..., z_m with z_1 by lines. Let on Γ be given a piecewise constant matrix function $G(t) = G_j...G_1$ for $t \in [z_j, z_{j+1}]$. It is clear, that $G(z_j + 0) = G(z_{j+1}), G(z_j - 0) = G(z_j + 0)^{-1} = M_j$ and $M_1...M_m = 1$. It is known, that there exist holomorphic vector functions $\varphi_1^{\pm}, ..., \varphi_n^{\pm}$, which are solutions of the Riemann-Hilbert boundary problem for the piecewise constant matrix function $G(t)$, and satisfy the conditions $\varphi_j^{\pm}(z - z_j)^{\varepsilon} \to 0$ $0 \leq \varepsilon < 1$, when $z \to z_j$. The determinant of the matrix function $\Psi = (\varphi_j^{\pm i})_{i,j=1}^n$ is nonzero on $CP^1 - \{z_1, ..., z_m\}$ and there exists such a diagonal matrix function $d_K = diag(z^{k_1}, ..., z^{k_m})$ that the matrix function $\widetilde{\Psi} = z^{d_K}\Psi(z)$ is holomorphically invertible at ∞. The matrix function has analytic continuation along any small path going around the point z_j and $\Phi \mapsto \Phi M_j^{-1}$.

Step 2. Construction of the generalized analytic vectors Let $F(z)$ be an invertible continuous matrix function on CP^1. Then the sought for analytic matrix function will be $\Phi = \widetilde{\Psi}F$.

Step 3. Construction of an elliptic system with given monodromy Consider the 1-form

$$D_{\bar{z}}(\widetilde{\Phi}F)(\Psi F)^{-1} = D_{\bar{z}}(\widetilde{\Phi}F)F^{-1}\Psi^{-1} = \widetilde{\Psi}(D_{\bar{z}}FF^{-1})\widetilde{\Psi}^{-1}$$

and denote by $\Omega = D_{\bar{z}}FF^{-1}$. Then $\Phi(z)$ is a solution of the elliptic system of form (1)

$$\partial_{\bar{z}}\Phi(z) = \Omega\Phi(z).$$

From the solution $\Phi(z)$ of this system it follows, that the 1-form Ω at the given points has a regular singularity and a prescribed monodromy.

For the regular system (12), the Poincaré theorem is valid, which gives that the fundamental matrix of solutions (12) has the form

$$F(z) = (z - z_i)^{E_i}Z(z), \tag{15}$$

where $E_i = \frac{1}{2\pi i}\ln M_i$ and M_i is the monodromy matrix corresponding to the singular point z_i. It follows that any solution of the system (14) in the neighborhood U_{z_i} has the form

$$\Phi(z) = (z - z_i)^{E_i}Z(z)V(z, \overline{z}). \tag{16}$$

All what was said above remains true for an arbitrary compact Riemann surface of genus g. It is clear, that in this case the singular integral (3) should contain the Cauchy kernel for the given Riemann surface.

As already noted, a system (14) without singularities induces a flat vector bundle $\mathcal{E} \to X$ on the Riemann surface X. Analogously, a system (14) with a regular singularity gives a flat vector bundle on the surface $X_m = X \setminus \{z_1, z_2, ..., z_m\}$ which we denote by $\mathcal{E}' \to X_m$. The representation of solutions in the form (15), (16) gives a possibility to extend $\mathcal{E}'$ to the whole of X, and if we choose the canonical extension then we obtain the uniquely defined (possibly topologically nontrivial) vector bundle $\mathcal{E} \to X$ which is induced from (14). The Chern number of the bundle $\mathcal{E} \to X$ can be calculated from monodromy matrices in the following way: $c_1(\mathcal{E}) = \sum_{i=1}^{m} tr(E_i)$, here the matrices E_j are chosen in such a way that the eigenvalues $\lambda_i, i = 1, ..., n$, satisfy the inequalities $0 \leq \text{ Re } \lambda_i^j < 1, j = 1, ..., n$.

The matrix-function Φ is a holomorphic section of the bundle $End\,\mathcal{E} \to X$. Assume that $\mathcal{E} \to X$ is stable in the sense of Mumford. Since stability implies $H^0(X, \mathcal{O}(End\,\mathcal{E})) \cong \mathbf{C}$, from the Riemann-Roch theorem for the bundle $End\,\mathcal{E} \to X$ we obtain

$$\dim H^1(X, \mathcal{O}(End\,\mathcal{E})) = n^2(g-1) + 1. \tag{17}$$

Since there exists a one-to-one correspondence between the gauge equivalent systems (14) and the holomorphic structures on the bundle $\mathcal{E} \to X$, we obtain that if the system (14) induces a stable bundle, then the dimension d of the gauge equivalent solutions of the system (14) is calculated by formula (17).

It seems interesting to distinguish further classes of L_p-forms of possibly simple shape (e.g., $\Omega = \sum_{j=1}^{m} A_j/(z - s_j)d\bar{z}$) with constant $n \times n$ matrices A_j for which it is still possible to construct a solution to the Riemann-Hilbert problem.

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THE RIEMANN-HILBERT BOUNDARY VALUE PROBLEM ON A CUT PLANE

N. Manjavidze
Georgian Technical University
Tbilisi, Georgia
ninomanjavidze@yahoo.com

Abstract The differential boundary value problem in case of several unknown functions is considered. The necessary and sufficient solvability conditions and the index formula are established.

Keywords: Boundary value, analytic function, cut plane, Hölder condition, singular integral, index formula

Mathematics Subject Classification (2000): 30E25

1°. Let D denote the plane of the complex variable $z = x + iy$, cut along some non-intersecting simple open Liapunov-smooth arcs $a_k b_k$, $k = 1, \ldots, m$. Denote $\Gamma_k = a_k b_k$ and $\Gamma = \bigcup_{k=1}^{m} \Gamma_k$.

Consider the function of the form

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)dt}{t - z} + P(z), \tag{1}$$

where $f(t) \in L_p(\Gamma, \rho)$, $p > 1$, the weight function

$$\rho(t) = \prod_{k=1}^{m} |t - c_k|^{\alpha_k},\ -1 < \alpha_k < p - 1,\ c_{2k-1} = a_k,\ c_{2k} = b_k,$$
$$(k = 1, \ldots, m), \tag{2}$$

$P(z)$ is an arbitrary polynomial. The class of the functions of the form (1) we denote by $E_p^{\pm}(\Gamma, \rho)$. The subclass of this class containing the functions of the form (1) in case $P(z) \equiv 0$ ($P(z) \equiv \text{const}$) denote by $E_{p,0}^{\pm}(\Gamma, \rho)$ ($E_{p,c}^{\pm}(\Gamma, \rho)$. Consider the following boundary value problem. Find a vector $\Phi(z) \in E_{P,0}^{\pm}(\Gamma, \rho)$ satisfying the boundary condition

$$\text{Re}[A_\pm(t)\Phi^\pm(t)] = f^\pm(t), \tag{3}$$

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almost everywhere on Γ; here $A_+(t),\ A_-(t)$ are given continuous non-singular quadratic matrices of order n on Γ, $f^+(t)=(f_1^+,\dots,f_n^+),\ f^-(t)=(f_1^-,\dots,f_n^-)$ are given real vectors on Γ belonging to the class $L_p(\Gamma,\rho)$, $\Phi^+(t),\ \Phi^-(t)$ denotes the boundary values of the vector $\Phi(z)$ from the left and from the right on Γ. Along with the problem (3) let us consider the following homogeneous problem:

$$\mathrm{Re}[A_\pm^*(t)\Psi^\pm(t)]=0,\ t\in\Gamma\ , \tag{4}$$

where $A_\pm^*(t)=t'(s)[A_\pm'(t)]^{-1}$, the solution of this problem $\Psi(z)$ will be found in the class $E_q^\pm(\Gamma,\rho^{1-q}),\ q=\dfrac{p}{p-1}$. We call the problem (4) the conjugate problem to the problem (3).

2°. Consider the following expression

$$B\equiv\mathrm{Im}\left\{\int_\Gamma[A_+^{-1}(t)f^+(t)\Psi^+(t)-A_-^{-1}(t)f^-(t)\Psi^-(t)]dt\right\}, \tag{5}$$

where $\Psi(z)$ is a solution of the problem (4) of the conjugate class $E_{q,0}^\pm(\Gamma,\rho^{1-q})$. Note, that

$$[A_\pm'(t)]^{-1}t'(s)\Psi^\pm(t)=iV^\pm(t),$$

where V^+, V^- are real vectors. Therefore

$$A_\pm^{-1}(t)f^\pm(t)\Psi^\pm(t)dt=f^\pm(t)[A_\pm'(t)]^{-1}\Psi^\pm(t)dt=if^\pm(t)V^\pm(t)dS$$

and the integral expression in (5) is purely imaginary valued and hence

$$B=-i\int_\Gamma[A_+^{-1}(t)f^+(t)\Psi^+(t)-A_-^{-1}(t)f^-(t)\Psi^-(t)]dt. \tag{6}$$

Let the problem (3) be solvable in the class $E_{p,0}^\pm(\Gamma,\rho)$ and $\varphi(z)$ be its arbitrary solution,

$$\mathrm{Re}[A_\pm(t)\varphi^\pm(t)]=f^\pm(t).$$

Then

$$B=\frac{1}{2}\mathrm{Im}\int_\Gamma\{A_+^{-1}(t)[A_+(t)\varphi^+(t)+\overline{A_+(t)}\ \overline{\varphi^+(t)}]\Psi^+(t)$$

$$-A_-^{-1}(t)[A_-(t)\varphi^-(t)+\overline{A_-(t)}\ \overline{\varphi^-(t)}]\Psi^-(t)\}dt$$

$$=\frac{1}{2}\mathrm{Im}\left\{\int_\Gamma[\varphi^+(t)\Psi^+(t)-\varphi^-(t)\Psi^-(t)]dt\right.$$

$$+\int_\Gamma \{\overline{\varphi^+(t)}\;\overline{A'_+(t)}[A'_+(t)]^{-1}\Psi^+(t)-\overline{\varphi^-(t)}\;\overline{A'_-(t)}[A'_-(t)]^{-1}\Psi^-(t)\}dt.$$

Since

$$\int_\Gamma [\varphi^+(t)\Psi^-(t)-\varphi^-(t)\Psi^-(t)]dt=0$$

and

$$[A'_\pm(t)]^{-1}t'(s)\Psi^\pm(t)=-\{\overline{[A'_\pm(t)]^{-1}t'(s)\Psi^\pm(t)}\},$$

we obtain $B=0$. Thus we have the following proposition:

Lemma 1 *For the problem* (3) *to be solvable in the class* $E^\pm_{p,0}(\Gamma,\rho)$ *the validity of the equality*

$$B(f,\psi)=0\;\left(\int_\Gamma [A_+^{-1}(t)f(t)\psi^+(t)-A_-^{-1}(t)f^-(t)\psi^-(t)]dt=0\right)\quad(7)$$

is necessary, where $\psi(z)$ *is an arbitrary solution of the problem* (4) *of the conjugate class* $E^\pm_{q,0}(\Gamma,\rho^{1-q})$.

Let $g_k^+(t)$, $g_k^-(t)$ be the vectors of the class $L_q(\Gamma,\rho^{1-q})$, $k=1,...,l$. For problem (3) to be solvable in the class $E^\pm_{p,o}(\Gamma,\rho)$ the fulfillment of the conditions

$$\mathrm{Im}\int_\Gamma [A_+^{-1}(t)f^+(t)g_k^+(t)-A_-^{-1}(t)f_-(t)g_k^-(t)]dt=0\,,\quad(k=1,...,l)\,,\quad(8)$$

is necessary and sufficient. It follows from (8) that

$$\mathrm{Im}\int_\Gamma \{A_+^{-1}(t)[A_+(t)\varphi^+(t)+\overline{A_+(t)}\;\overline{\varphi^+(t)}]g_k^+(t)-\quad(9)$$

$$-A_-^{-1}(t)[A_-(t)\varphi^-(t)+\overline{A_-(t)}\;\overline{\varphi^-(t)}]g_k^-(t)\}dt=0\,,$$

where $\varphi(z)=\dfrac{1}{2\pi i}\displaystyle\int_\Gamma\frac{h(t)}{t-z}$, $h(t)\in L_P(\Gamma,\rho)$, is an arbitrary vector from $E^\pm_{p,o}(\Gamma,\rho)$. We may rewrite (9) in the following way

$$\mathrm{Im}\int_\Gamma \{\varphi^+(t)[g_k^+(t)-\overline{t}'^2A'_+(t)[\overline{A'_+(t)}]^{-1}\overline{g_k^+(t)}]-$$

$$-\varphi^-(t)[g_k^-(t)-\overline{t}'^2A'_-(t)[\overline{A'_-(t)}]^{-1}\,\overline{g_k^-(t)}]\}dt=0\,.$$

Denoting

$$g_k^\pm(t)-\overline{t}'^2A'_\pm(t)[\overline{A'_\pm(t)}]^{-1}\overline{g_k^\pm(t)}=\omega_k^\pm(t)\,,\quad(10)$$

we get

$$\operatorname{Im} \int_\Gamma [\varphi^+(t)\omega_k^+(t) - \varphi^-(t)\omega_k^-(t)]dt = 0$$

or

$$\int_\Gamma [\varphi^+(t)\omega_k^+(t) - \varphi^-(t)\omega_k^-(t)]dt = 0\,. \tag{11}$$

It follows from (11), that $\omega_k^+(t)$, $\omega_k^-(t)$ are the boundary values of some vector $\omega_k(z)$ belonging to the class $E^{\pm}_{q,o}(\Gamma, \rho^{1-q})$. Note also that

$$2i \operatorname{Im} \int_\Gamma [A_+^{-1}(t)f^+(t)g_k^+(t) - A_-^{-1}(t)f^-(t)g_k^-(t)]dt$$

$$= \int_\Gamma \{A_+^{-1}(t)f^+(t)g_k^+(t) - \overline{A_-^{-1}(t)}f^+(t)\overline{g_k^+(t)}t'^2$$

$$-A_-^{-1}(t)f^-(t)g_k^-(t) + \overline{A_-^{-1}}(t)f^-(t)\overline{g_k^-(t)}\;\overline{t'^2}\}dt$$

$$= \int_\Gamma [A_+^{-1}(t)f^+(t)\omega_k^+(t) - A_-^{-1}(t)f^-(t)\omega_k^-(t)]dt.$$

Since the last integral is equal to zero, we get that if the necessary solvability conditions are fulfilled then the sufficient conditions are fulfilled also. Thus we have the following result.

Lemma 2 *If for the problem* (3) *to be solvable in the class* $E^{\pm}_{p,0}(\Gamma, \rho)$ *the necessary and sufficient conditions have the form*

$$\operatorname{Im} \int_\Gamma (A_+^{-1}f^+g_k^+ - A_-^{-1}f^-g_k^-)dt = 0,\;\; k = 1, \ldots, l, \tag{12}$$

then the conjugate problem (4) *has a finite number of linearly independent solutions* $\omega_k(z)$, $k = 1, \ldots, l' \le l$ *in the class* $E^{\pm}_{q,0}(\Gamma, \rho^{1-q})$ *and the necessary and sufficient solvability conditions for problem* (3) *in the class* $E^{\pm}_{p,0}(\Gamma, \rho)$ *have the form*

$$\operatorname{Im} \int_\Gamma [A_+^{-1}f^+\omega_k^+(t) - A_-^{-1}(t)f^-\omega_k^-(t)]dt = 0, \quad k = 1, \ldots, l'. \tag{13}$$

3°. We seek the solution of the problem (3) in the class $E^{\pm}_{p,0}(\Gamma, \rho)$. From the boundary condition we have

$$\Phi^{\pm}(t) = [A_{\pm}(t)]^{-1}[f^{\pm}(t) + i\mu_{\pm}(t)], \tag{14}$$

where $\mu^+(t),\ \mu^(t)$ are real vectors of the class $L_p(\Gamma,\rho)$ and

$$\Phi(z)=\frac{1}{2\pi}\int\limits_{\Gamma}\frac{A_+^{-1}(t)\mu^+(t)-A_-^{-1}(t)\mu^-(t)}{t-z}dt+F(z), \tag{15}$$

here

$$F(z)=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{A_+^{-1}(t)f^+(t)-A_-^{-1}(t)f^-(t)}{t-z}dt. \tag{16}$$

Substituting the formulas (14), (15) in the boundary condition (3) and introducing the vector with $2n$ components $\mu(t)=(\mu^+(t),\mu^-(t))$ we get the system of singular integral equations

$$a(t_0)\mu(t_0)+\frac{1}{\pi i}\int\limits_{\Gamma}\frac{K(t_0,t)\mu(t)}{t-t_0}dt=g(t_0)\,, \tag{17}$$

where

$$a=\mathrm{Im}\begin{pmatrix}A_-A_+^{-1} & 0\\ 0 & -A_+A_-^{-1}\end{pmatrix},\quad K(t_0,t)=\frac{i}{2}\begin{pmatrix}N_+^1 & N_-^1\\ N_+^2 & N_-^2\end{pmatrix}, \tag{18}$$

where the matrices $N_\pm^1$, $N_\pm^2$ are defined by

$$N_\pm^1(t_0,t)=A_-(t_0)A_\pm^{-1}(t)+\overline{A_-(t_0)}\ \overline{A_\pm^{-1}(t)}h(t_0,t),$$

$$N_\pm^2(t_0,t)=A_+(t_0)A_\pm^{-1}(t)+\overline{A_+(t_0)}\ \overline{A_\pm^{-1}(t)}h(t_0,t), \tag{19}$$

$$g=(g^+,g^-),\ g^\pm=2Ref^\pm-2Re[A_\pm F^\pm],\ h(t_0,t)=\frac{t-t_0}{\overline{t}-\overline{t}_0}\overline{t}'^2.$$

We have

$$a(t)+b(t)=i\begin{pmatrix}\overline{A}_-\ \overline{A_+^{-1}} & I\\ I & A_+A_-^{-1}\end{pmatrix},$$

$$a(t)-b(t)=-i\begin{pmatrix}A_-A_+^{-1} & I\\ I & \overline{A_+}\ \overline{A_-^{-1}}\end{pmatrix}.$$

Denote

$$Q(t)\equiv det(\overline{A_-}\ \overline{A_+^{-1}}A_+A_-^{-1}-I). \tag{20}$$

Lemma 3 *If* $Q(t)\neq 0,\ F(z)=0$ *and* $\Phi(z)=0$ *then* $\mu^+(t)=\mu^-(t)=0.$

Proof We have

$$\frac{1}{2\pi}\int\limits_{\Gamma}\frac{A_{+}^{-1}(t)\mu^{+}(t)-A_{-}^{-1}(t)\mu^{-}(t)}{t-z}dt=0$$

and from here

$$A_{+}^{-1}(t)\mu^{+}(t)-A_{-}^{-1}(t)\mu^{-}(t)=0,$$

$$A_{-}(t)A_{+}^{-1}(t)\mu^{+}(t)-\mu^{-}(t)=0,$$

$$\overline{A_{-}(t)}\;\overline{A_{+}^{-1}(t)}\mu^{+}(t)-\mu^{-}(t)=0,$$

$$[A_{-}(t)A_{+}^{-1}(t)-\overline{A_{-}(t)}\;\overline{A_{+}^{-1}(t)}]\mu^{+}(t)=0,$$

$$\mu^{+}(t)=0,\quad \mu^{-}(t)=0\,.$$

The conjugate homogeneous equation will have the form

$$-a'(t_0)\psi(t_0)+\frac{1}{\pi i}\int\limits_{\Gamma}\frac{K'(t,t_0)\psi(t)\,dt}{t-t_0}=0, \tag{21}$$

where

$$K'(t,t_0)=\frac{i\,\overline{t}_0'}{2}\begin{pmatrix}M_{+}^{1} & M_{+}^{2}\\ M_{-}^{1} & M_{-}^{2}\end{pmatrix}$$

and the block matrices $M_{\pm}^{1}, M_{\pm}^{2}$ are defined by

$$M_{\pm}^{1}=t_0'A_{\pm}^{'-1}(t_0)A_{-}'(t)+\overline{t_0'}\,\overline{A_{\pm}^{'-1}}(t_o)\overline{A_{-}'(t_o)}h_1(t_o,t)\,, \tag{22}$$

$$M_{\pm}^{2}=t_0'A_{\pm}'(t_0)A_{+}'(t_0)+\overline{t_0'}\,\overline{A_{\pm}^{'-1}}(t_0)\overline{A_{+}'(t)}h_1(t_0,t)\,,$$

$$h_1(t_0,t)=\frac{t-t_0}{\overline{t}-\overline{t_0}}\,.$$

If we compose the equation for the solution of the conjugate problem (4) in the class $E_{q,o}^{\pm}(\Gamma,\rho^{1-q})$ similarly as above (formula (13)) then we get the singular integral equation

$$a_1(t_0)\nu(t_0)+\frac{1}{\pi i}\int\limits_{\Gamma}\frac{K_1(t_0,t)\nu(t)}{t-t_0}dt=0, \tag{23}$$

where

$$a_1=I_m\begin{pmatrix}A_{-}^{'-1}A_{+}' & 0\\ 0 & -A_{+}^{'-1}A_{-}'\end{pmatrix},\; K_1(t_0,t)=\frac{i}{2}\overline{t'}\begin{pmatrix}M_{-}^{2} & M_{-}^{1}\\ M_{+}^{2} & M_{+}^{1}\end{pmatrix}\,. \tag{24}$$

It is easy to see that $a_1 = -\alpha a'\beta$ and $K_1(t_0, t) = \overline{t'} t'_0 \alpha K_1(t, t_0)\beta$, where α and β are some real non-singular matrices. If we make the substitution $\nu_1(t) = \overline{t'}\beta(t)\nu(t)$ in the equation (23) and multiply it from the left by the matrix α^{-1} then we obtain the equation

$$-a'(t_0)\nu_1(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{K'(t, t_0)\nu_1(t)\,dt}{t - t_0} = 0\,, \tag{25}$$

which coincides with equation (21). Thus the number of linearly independent solutions over the field of real numbers of system (23) in the class $L_q(\Gamma, \rho^{1-q})$ and the number of linearly independent solutions of problem (4) in the class $E^{\pm}_{q,o}(\Gamma, \rho^{1-q})$ are the same.

Theorem 1 *If $Q(t) \neq 0$ then the index of problem (3) in the class $E^{\pm}_{p,o}(\Gamma, \rho)$ is equal to the index of equation (17) of the class $L_p(\Gamma, \rho)$ (under the condition, that $1 + \alpha_k \neq p\,\mu_j^{(k)}$, where $\mu_j^{(k)} = \frac{arg\,\lambda_j^{(k)}}{2\pi}$, $0 \leq arg\,\lambda_j^{(k)} < 2\pi$, $\lambda_j^{(k)}$ are the roots of the equations $det[H(a_k) - \lambda I] = 0$ or $det[H^{-1}(b_k) - \lambda I] = 0$ for odd and even k correspondingly; $H(t) = [a(t) + b(t)]^{-1}[a(t) - b(t)]$; the necessary and sufficient solvability conditions for the problem (3) in the class $E_{p,o}(\Gamma, \rho)$ have the form (13).*

Remark 1 If $A_+(t) = A_-(t) = A(t)$ then $Q(t) \equiv 0$. In this case instead of the representation (15) we shall use the representation

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{A^{-1}(t)\mu(t)}{t - z}dt + F(z)\,, \text{ where } \mu = \mu^+ + \mu^-\,.$$

The equation (17) will have the form

$$\frac{1}{\pi i}\int_\Gamma \frac{K(t_0, t)\mu(t)}{t - t_0}dt = g(t_0)\,,$$

where

$$K(t_0, t) = \frac{i}{2}[A(t_0)A^{-1}(t) + \overline{A(t_0)}\,\overline{A^{-1}(t_0)}h(t_0, t)]\,.$$

Analogously we obtain the following result.

Theorem 2 *If $A_+(t) = A_-(t)$ then the index of problem (3) coincides with the index of the operator $\int_\Gamma \mu(t)/(t - t_o)dr$ of the class $L_p(\Gamma, \rho)$ under the condition that $2(1 + \alpha_k) \neq p$. In this case the necessary and sufficient solvability conditions have the form (13).*

Remark 2 If $Q(t) = 0$ in some points of Γ then introduce a new desired vector by the formula $\Phi(z) = \Lambda(z)\varphi(z)$, where $\Lambda = diag[e^{\omega_1(z)}, \ldots, e^{\omega_k(z)}]$,

$$\omega_k(z) = \frac{1}{2\pi i}\int_\Gamma \frac{h_k(t)dt}{t-z}, \quad h_k(t) \in H(\Gamma), \quad h_k(a_j) = h_k(b_j) = 0\,.$$

It is evident, that $\varphi(z) \in E^{\pm}_{p,0}(\Gamma, \rho)$. The matrices $A_\pm(t)$ are replaced by the matrices $A_\pm(t)\Lambda^\pm(t)$. Under the fulfillment of some conditions one may select the functions $h_k(t)$ such that the function $Q(t) \neq 0$ on Γ.

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ON THE LOGARITHMIC DERIVATIVE OF MEROMORPHIC FUNCTIONS

A.A. Mokhon'ko, A.Z. Mokhon'ko
Lviv Polytechnic National University
Carpatskaja str. 7/5, 79017, Lviv, Ukraine
maa@mail.univ.kiev.ua,mohon@yandex.ru

Abstract Estimations for the logarithmic derivative of functions, meromorphic in angular domains, are obtained.

Keywords: meromorphic functions in angular domains, logarithmic derivative, Nevanlinna characteristic

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We use notations of the theory of meromorphic functions [1]. Let $\{a_m\}$ be the set of zeros and $\{b_l\}$ the set of poles of a meromorphic function $f(z)$, $z \in D = \{z : \mathrm{Im} z \geqslant 0, |z| \geqslant r_0\}$. Let us denote by $\{c_q\}$ the union of the sequences $\{a_m\}$, $\{b_l\}$; $c_q = |c_q| \exp(i\theta_q)$. Throughout the article K denotes some distinct constant. Without loss of generality we can suppose $f(r_0 e^{i\theta}) \neq 0, \infty;\ 0 \leqslant \theta \leqslant \pi$, otherwise r_0 could be increased.

Theorem 1 *Let $f(z)$, $z \in D$ be a meromorphic function. If $z = r \exp i\varphi$, $r_0 < |z| < s$, $\mathrm{Im} z > 0$, then for all $n \in \mathbb{N}$*

$$\left|\frac{d^n \ln f(z)}{dz^n}\right| < \frac{K s^2 S(s,f)}{(s-r)^{n+1} \sin^{n+1}\varphi} \left(\frac{s}{r}\right)^n$$

$$+\frac{K}{\sin^n \varphi} \sum_{r_0<|c_q|<s} \left(\frac{\sin\theta_q}{(s-r)^n} + \frac{\sin\theta_q}{|z-c_q|^n}\right) + K,\ K = \mathrm{const} > 0, \quad (1)$$

where $S(r,f)$ is the Nevanlinna's characteristic of $f(z)$, $z \in D$, [1], *p.* 39. *If $f(z)$, $z \in D$ has finite order ρ, then* $(\forall \varepsilon > 0)$

$$\left|\frac{d^n \ln f(z)}{dz^n}\right| < \frac{K|z|^{(n+1)(\rho+1+\varepsilon)}}{\sin^{2n}\varphi},\ \left|\frac{f^{(n)}(z)}{f(z)}\right| < \frac{K|z|^{2n(\rho+1+\varepsilon)}}{\sin^{2n}\varphi},\ z \notin E, \quad (2)$$

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where E is a set of circles with finite sum of radii, $n \in \mathbb{N}$.

Remark 1 If function $f(z)$ is meromorphic in the angular domain $\{z : \alpha \leqslant \arg z \leqslant \beta, |z| > r_1\}$, then relations similar to (1), (2) can be obtained by using the function $f_1(z) = f(z^{1/k}e^{i\alpha})$, $k = \pi/(\beta - \alpha)$, that is meromorphic in the domain $\{z : \mathrm{Im} z \geqslant 0, |z| \geqslant r_0\}$, [1], p. 41. The estimate (2) depends on $\varphi = \arg z$. In the meantime, for a function f meromorphic in $\mathbb{C}$ of finite order of growth ρ the inequality of G. Valiron is known, [2], p. 87, $|f'(z)/f(z)| < K|z|^{2\rho+\varepsilon}$, $z \in \mathbb{C} \setminus E$, where E is a set of disks with finite sum of radii. In [3] there is an example, that shows, that Nevanlinna's lemma about logarithmic derivative, [1], p. 116, p. 137 for functions, meromorphic in a half-plane, is not valid. In particular, from this example it follows, that for such functions it is impossible to obtain an estimation for the absolute value of the logarithmic derivative, uniformly in $\arg z$.

Remark 2 In (2) for $n = 1$ both estimates are the same. If $n = 2, 3, 4, ...$, then the first and second inequalities of (2) estimate different expressions. In case of a meromorphic function $f(z), z \in \mathbb{C}$, from Valiron's inequality $|f'(z)/f(z)| < K|z|^{2\rho+\varepsilon}$, $z \in \mathbb{C} \setminus E$, the estimation for $|f^{(n)}(z)/f(z)|$ can be obtained, using the equality $\left|f^{(n)}/f\right| = \left|f^{(n)}/f^{(n-1)}\right| \left|f^{(n-1)}/f^{(n-2)}\right| \dots \ |f'/f|$, because Valiron's estimate can be applied for each factor on the right–hand side of the last equality. Such method implicitly uses the theorem, that the category of growth of the function f is not lower than the one for f', [1], p. 131, Th. 2.3. In the proof of this theorem the lemma about the logarithmic derivative was used. But as this lemma is not valid in the half plane, we should avoid a "fast" proof and initially obtain the estimate for $|d^n \ln f(z)/dz^n|$, which can have its own significance.

Let

$$w(z), z \in g_{\alpha\beta} = \{z = re^{i\theta} : \alpha \leqslant \theta \leqslant \beta, r_0 \leqslant r < +\infty\} \tag{3}$$

be a meromorphic function. Consider Nevanlinna's characteristic of the function $w(z)$, $z \in g_{\alpha\beta}$ [1, p. 40]. Denote $\ln^+ x = \max(\ln x, 0), x \geqslant 0; k = \pi/(\beta - \alpha)$. Let $b_l = |b_l| \exp(i\theta_l)$ be poles of the function $w(z)$, $z \in g_{\alpha\beta}$. We denote

$$A_{\alpha\beta}(r, w) = \frac{k}{\pi} \int_{r_0}^{r} \left(\frac{1}{t^{k+1}} - \frac{t^{k-1}}{r^{2k}} \right) [\ln^+ |w(te^{i\alpha})| + \ln^+ |w(te^{i\beta})|] dt,$$

$$B_{\alpha\beta}(r, w) = \frac{2k}{\pi r^k} \int_{\alpha}^{\beta} \ln^+ |w(re^{i\theta})| \sin k(\theta - \alpha) d\theta,$$

$$C_{\alpha\beta}(r, w) = 2k \int_{r_0}^{r} c_{\alpha\beta}(t, w) \left(\frac{1}{t^{k+1}} + \frac{t^{k-1}}{r^{2k}} \right) dt,$$

$$c_{\alpha\beta}(t,w) = c_{\alpha\beta}(t,\infty) = \sum_{r_0<|b_l|\leqslant t, \alpha\leqslant\theta_l\leqslant\beta} \sin k(\theta_l - \alpha), \tag{4}$$

$c_{\alpha\beta}(t,w)$ is the counting function of the poles; each pole is counted according to its multiplicity,

$$S_{\alpha\beta}(r,w) = A_{\alpha\beta}(r,w) + B_{\alpha\beta}(r,w) + C_{\alpha\beta}(r,w),\ r_0 \leqslant r < +\infty. \tag{5}$$

If $\alpha = 0$, $\beta = \pi$, then $k = \pi/(\beta-\alpha) = 1$ and we write $S_{0\pi}(r,w) = S(r,w)$, $A_{0\pi}(r,w) = A(r,w)$, $B_{0\pi}(r,w) = B(r,w)$, $C_{0\pi}(r,w) = C(r,w)$, $c_{0\pi}(r,w) = c(r,w)$. The number

$$\rho_{\alpha\beta} = \overline{\lim}_{r\to+\infty}, \ln^+ S_{\alpha\beta}(r,w)/\ln r, \tag{6}$$

is called the order of growth of the function $w(z)$, $z \in g_{\alpha\beta}$.

Similarly to Valiron's inequality [2] the estimates (2), can be used in the analytic theory of differential equations for investigating asymptotic properties of solutions in the neighborhood of a logarithmic singularity of solutions, meromorphic in an angular domain.

Let

$$F(z,\zeta) = \ln[(s^2 - z\overline{\zeta})(z-\overline{\zeta})(z-\zeta)^{-1}(s^2 - z\zeta)^{-1}], \tag{7}$$

$z,\zeta \in U = \{z : r_0 \leqslant |z| \leqslant s, \mathrm{Im} z \geqslant 0\}$, $z \neq \zeta$. As $\partial/\partial\eta$ we denote the differential operator along the inner normal to the boundary of U (on the arc $\{\zeta : \zeta = r_0 \exp(i\theta),\ 0 < \theta < \pi\}$) applied to the functions $\mathrm{Re}F(z,\zeta)$ and $\ln|f(\zeta)|$ of the variable ζ.

Lemma 1 *Let the function $f(z) \not\equiv 0$ be meromorphic in $U = \{z : r_0 \leqslant |z| \leqslant s,\ \mathrm{Im} z \geqslant 0\}$. Then the formula*

$$\ln f(z) = \frac{i}{2\pi} \int\limits_{[-s,-r_0]\cup[r_0,s]} \ln|f(t)| \left[\frac{t+z}{t-z} - \frac{s^2+tz}{s^2-tz}\right] \frac{dt}{t}$$

$$+\frac{1}{2\pi}\int\limits_0^\pi \ln|f(\zeta)| \left[\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right]_{\zeta=se^{i\theta}} d\theta \tag{8}$$

$$-\sum_{r_0<|a_m|<s} F(z,a_m) + \sum_{r_0<|b_l|<s} F(z,b_l) + Q(z,s),$$

$$Q(z,s) = \frac{r_0}{2\pi}\int\limits_0^\pi \left[\ln|f(\zeta)|\frac{\partial F(z,\zeta)}{\partial\eta} - F(z,\zeta)\frac{\partial \ln|f(\zeta)|}{\partial\eta}\right]_{\zeta=r_0e^{i\theta}} d\theta + iC, \tag{9}$$

holds, where the a_m *are the zeros, the* b_l *are the poles of the function* $f(z)$, $C = \text{const}$. *On the right-hand side of (8) summands, corresponding to multiple poles or zeros, are repeated according to their multiplicities.*

Proof If $t \in \mathbb{R}$ and $z = r\exp(i\varphi)$, then

$$\frac{1}{2t}\mathrm{Re}\left[i\left(\frac{t+z}{t-z} - \frac{s^2+tz}{s^2-tz}\right)\right] = \frac{r\sin\varphi}{|t-z|^2} - \frac{s^2 r\sin\varphi}{|s^2-tz|^2},$$

$$\mathrm{Re}\left[\frac{se^{i\theta}+z}{se^{i\theta}-z} - \frac{se^{-i\theta}+z}{se^{-i\theta}-z}\right] = \frac{s^2-r^2}{|se^{i\theta}-z|^2} - \frac{s^2-r^2}{|se^{-i\theta}-z|^2},$$

$$\mathrm{Re}F(z,a_m) = \ln\left|\frac{(s^2-z\overline{a}_m)(z-\overline{a}_m)}{(z-a_m)(s^2-za_m)}\right|. \tag{10}$$

Thus Nevanlinna's formula, [1], p. 15, Th. 2.1 and 2.3, can be represented as

$$\ln|f(z)| = \frac{1}{2\pi}\int\limits_{[-s,-r_0]\cup[r_0,s]} \ln|f(t)|\mathrm{Re}\left(i\left(\frac{t+z}{t-z} - \frac{s^2+tz}{s^2-tz}\right)\right)\frac{dt}{t}$$

$$+\frac{1}{2\pi}\int_0^\pi \ln|f(se^{i\theta})|\mathrm{Re}\left[\frac{se^{i\theta}+z}{se^{i\theta}-z} - \frac{se^{-i\theta}+z}{se^{-i\theta}-z}\right]d\theta$$

$$-\sum_{r_0<|a_m|<s}\mathrm{Re}F(z,a_m) + \sum_{r_0<|b_l|<s}\mathrm{Re}F(z,b_l)$$

$$+\frac{r_0}{2\pi}\int_0^\pi\left[\ln|f(\zeta)|\frac{\partial\mathrm{Re}F(z,\zeta)}{\partial\eta} - \mathrm{Re}F(z,\zeta)\frac{\partial\ln|f(\zeta)|}{\partial\eta}\right]_{\zeta=r_0e^{i\theta}}d\theta. \tag{11}$$

The right–hand and left–hand sides of (8) contain analytic functions of z. Due to (11), the real parts of these functions are the same. Thus from the Cauchy-Riemann conditions we obtain, that these functions are the same up to a constant. The proof is completed.

Suppose $f(z), z \in D = \{z : \mathrm{Im}z \geqslant 0,\ |z| \geqslant r_0\}$, is a meromorphic function, $c_q = |c_q|\exp(i\theta_q) \in \{c_q\}$. Denote (see (4)) the counting function of zeros and poles of f by $c(t,0,\infty)$. Then

$$c(t,0,\infty) = c(t,f) + c(t,1/f) = \sum_{r_0<|c_q|<t}\sin\theta_q. \tag{12}$$

Lemma 2 *Let $R > s > r_0$. Then*

$$c(s,0,\infty) \leqslant \frac{R^2 s C(R,0,\infty)}{2(R-s)(R+s)}. \tag{13}$$

Proof From the definition of the characteristic $C(r, f)$ it holds

$$C(R,0,\infty) = C(R,f) + C(R,f^{-1}) \geqslant 2\int\limits_s^R c(t,0,\infty)\left(\frac{1}{t^2}+\frac{1}{R^2}\right)dt$$

$$\geqslant 2c(s,0,\infty)\int\limits_s^R \left(\frac{1}{t^2}+\frac{1}{R^2}\right)dt = \frac{2(R-s)(R+s)}{R^2 s}c(s,0,\infty).$$

From here we obtain (13).

Proof of Theorem 1 Differentiating (8) n times with respect to z gives

$$\frac{d^n}{dz^n}\ln f(z) = \frac{i}{2\pi}\int\limits_{[-s,-r_0]\cup[r_0,s]} \ln|f(t)|\left[\frac{t+z}{t-z}-\frac{s^2+tz}{s^2-tz}\right]_z^{(n)}\frac{dt}{t}$$

$$+\frac{1}{2\pi}\int\limits_0^\pi \ln|f(\zeta)|\left(\left[\frac{\zeta+z}{\zeta-z}-\frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right]_{\zeta=se^{i\theta}}\right)_z^{(n)} d\theta$$

$$-\sum_{r_0<|a_m|<s} F_z^{(n)}(z,a_m) + \sum_{r_0<|b_l|<s} F_z^{(n)}(z,b_l) + Q_z^{(n)}(z,s). \tag{14}$$

The following equalities hold:

$$\left[\frac{t+z}{t-z}-\frac{s^2+tz}{s^2-tz}\right]_z^{(n)} = \frac{2tn!}{(t-z)^{n+1}} - \frac{2s^2t^n n!}{(s^2-zt)^{n+1}}, \tag{15}$$

$$\left(\frac{\zeta+z}{\zeta-z}-\frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right)_z^{(n)} = \frac{2\zeta n!}{(\zeta-z)^{n+1}} - \frac{2\overline{\zeta}n!}{(\overline{\zeta}-z)^{n+1}}, \tag{16}$$

$$(F(z,\zeta))_z^{(n)} = -\frac{(\overline{\zeta})^n(n-1)!}{(s^2-z\overline{\zeta})^n} - \frac{(n-1)!}{(\overline{\zeta}-z)^n} + \frac{(n-1)!}{(\zeta-z)^n} + \frac{\zeta^n(n-1)!}{(s^2-z\zeta)^n}. \tag{17}$$

On the arc $\{\zeta : \zeta = r_0 \exp(i\theta), 0 < \theta < \pi\}$ the derivative in direction of the inner normal with respect to the variable $\zeta = \rho e^{i\theta}$ has the form $\partial F(z,\zeta)/\partial\eta = \partial F(z, \rho\exp(i\theta))/\partial\rho$. Thus

$$\left.\frac{\partial F}{\partial \eta}\right|_{\zeta=r_0e^{i\theta}} = \frac{-z}{s^2e^{i\theta}-zr_0} + \frac{1}{ze^{-i\theta}-r_0} + \frac{z}{s^2e^{-i\theta}-zr_0} - \frac{1}{ze^{i\theta}-r_0}. \tag{18}$$

By differentiating (17) n times with respect to z, we obtain

$$n!\left(\frac{\partial F(z,r_0e^{i\theta})}{\partial\eta}\right)^{(n)}_z = -\frac{s^2e^{i\theta}r_0^{n-1}}{(s^2e^{i\theta}-zr_0)^{n+1}} - \frac{e^{-in\theta}}{(r_0-ze^{-i\theta})^{n+1}}$$
$$+\frac{s^2e^{-i\theta}r_0^{n-1}}{(s^2e^{-i\theta}-zr_0)^{n+1}} + \frac{e^{in\theta}}{(r_0-ze^{i\theta})^{n+1}}. \tag{19}$$

Let

$$r_0+1<|z|<s, \quad s>\max(2r_0, r_0+1), \tag{20}$$

then $|s^2e^{i\theta}-zr_0|$, $|s^2e^{-i\theta}-zr_0| > s^2-sr_0$; $|r_0-ze^{-i\theta}|>1$, $|r_0-ze^{i\theta}|>1$. That is why from (19) follows

$$\left|\left(\frac{\partial F(z,r_0e^{i\theta})}{\partial\eta}\right)^{(n)}_z\right| < 4n!\ . \tag{21}$$

Suppose (20) and $|\zeta| = r_0$ take place. Then $|z-\overline{\zeta}|>1$, $|z-\zeta|>1$; $|s^2-z\overline{\zeta}| > s^2-sr_0 = s(s-r_0) > sr_0$, $|s^2-z\zeta| > s^2-sr_0 > sr_0$, and (16) can be estimated as

$$\left.\left|(F(z,\zeta))^{(n)}_z\right|\right|_{|\zeta|=r_0} < 4(n-1)!\ . \tag{22}$$

Since $f(r_0e^{i\theta}) \neq 0, \infty$, then $|\ln|f(r_0e^{i\theta})|| < K$, $|\partial\ln|f(\zeta)|/\partial\eta|_{\zeta=r_0e^{i\theta}} < K$, $0 \leqslant \theta \leqslant \pi$, $K = \text{const}$, thus from (9), (21), (22), we obtain

$$|(Q(z,s))^{(n)}_z| < 4r_0Kn!, \quad K=\text{const}, \tag{23}$$

where K does not depend on z and s.

Using the binomial formula and simple transformations, (16) can be written as

$$\frac{1}{2n!}\left(\frac{\zeta+z}{\zeta-z} - \frac{\overline{\zeta}+z}{\overline{\zeta}-z}\right)^{(n)}_z = \frac{\sum_{j=0}^{n+1} C^j_{n+1}(-z)^{n+1-j}(\zeta\overline{\zeta}^j - \overline{\zeta}\zeta^j)}{(\zeta-z)^{n+1}(\overline{\zeta}-z)^{n+1}}. \tag{24}$$

Since

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}), \tag{25}$$

then numerator on the right-hand side of (24) can be represented as

$$(-z)^{n+1}(\zeta - \overline{\zeta}) + \sum_{j=2}^{n+1} C_{n+1}^j (-z)^{n+1-j} |\zeta|^2 (\overline{\zeta}^{j-1} - \zeta^{j-1})$$

$$= (\zeta - \overline{\zeta}) \left((-z)^{n+1} - \sum_{j=2}^{n+1} C_{n+1}^j (-z)^{n+1-j} |\zeta|^2 \sum_{p=0}^{j-2} \overline{\zeta}^p \zeta^{j-2-p} \right). \tag{26}$$

If $\zeta = se^{i\theta}$, then $\zeta - \overline{\zeta} = 2i|\zeta| \sin\theta$. Thus from (26) and formulae

$$\sum_{j=0}^{n} jC_n^j = n2^{n-1}, \quad \sum_{j=0}^{n} C_n^j = 2^n \tag{27}$$

we obtain an estimate for the numerator of the fraction on the right-hand side of (24) ($|z| < |\zeta| = s$),

$$\left| \sum_{j=0}^{n+1} C_{n+1}^j (-z)^{n+1-j} |\zeta|^2 (\overline{\zeta}^{j-1} - \zeta^{j-1}) \right|$$
$$< 2\sin\theta s^{n+2} (1 + \sum_{j=2}^{n+1} C_{n+1}^j (j-1)) < 2\sin\theta s^{n+2}(n2^n + 2).$$

From this and from (24), taking the inequalities $|\overline{\zeta} - z| > s\sin\varphi$, $|\zeta - z| > s - r$ into account we have

$$\left| \left(\frac{\zeta + z}{\zeta - z} - \frac{\overline{\zeta} + z}{\overline{\zeta} - z} \right)_z^{(n)} \right|_{\zeta = se^{i\theta}} < \frac{Ks\sin\theta}{(s-r)^{n+1}\sin^{n+1}\varphi}, \tag{28}$$

$K = 4n!(n2^n + 2)$ does not depend on s. In the same way we estimate (15), i.e.

$$\frac{1}{2n!} \left(\frac{t+z}{t-z} - \frac{s^2 + tz}{s^2 - tz} \right)_z^{(n)} = \frac{t(s^2 - zt)^{n+1} - s^2 t^n (t-z)^{n+1}}{(t-z)^{n+1}(s^2 - zt)^{n+1}}. \tag{29}$$

The numerator on the right-hand side of (29) can be written as

$$t\sum_{j=0}^{n+1} C_{n+1}^{j}(-zt)^{n+1-j}s^{2j} - s^2t^n\sum_{j=0}^{n+1} C_{n+1}^{j}(-z)^{n+1-j}t^j$$

$$= \sum_{j=0}^{n+1} C_{n+1}^{j}(-z)^{n+1-j}(t^{n+2-j}s^{2j} - s^2t^{n+j})$$

$$= ts^2\left(\sum_{j=2}^{n+1} C_{n+1}^{j}(-zt)^{n+1-j}\left((s^2)^{j-1} - (t^2)^{j-1}\right)\right) - (-z)^{n+1}t^n(s^2-t^2)$$

$$= (s^2-t^2)\left(ts^2\sum_{j=2}^{n+1} C_{n+1}^{j}(-zt)^{n+1-j}\sum_{k=0}^{j-2}(s^2)^{j-2-k}(t^2)^k - (-z)^{n+1}t^n\right). \tag{30}$$

Since $|z| \leqslant s, |t| \leqslant s$, using (29), (27), the numerator of the fraction on the right-hand side of (29) can be estimated as

$$|t(s^2-zt)^{n+1} - s^2t^n(t-z)^{n+1}| < (s^2-t^2)|t|s^{2n}2n!(n2^n+2). \tag{31}$$

As $z = re^{i\varphi}, t \in \mathbb{R}$, $|t| \leqslant s$, we obtain

$$|s^2 - zt| \geqslant s(s-r), \quad |z-t| \geqslant r\sin\varphi, \quad |z-t| \geqslant |t|\sin\varphi. \tag{32}$$

Thus from (29), (31), (32) it follows

$$\left|\left(\frac{t+z}{t-z} - \frac{s^2+tz}{s^2-tz}\right)_z^{(n)}\right| \leqslant \frac{|t|s^{n+2}2n!(n2^n+2)}{r^n(s-r)^{n+1}\sin^{n+1}\varphi}\left(\frac{1}{t^2} - \frac{1}{s^2}\right). \tag{33}$$

We write the relation (17) as

$$\frac{(F(z,\zeta))_z^{(n)}}{(n-1)!} = \left(\frac{\zeta^n}{(s^2-z\zeta)^n} - \frac{\overline{\zeta}^{\,n}}{(s^2-z\overline{\zeta}\,)^n}\right) + \left(\frac{1}{(\zeta-z)^n} - \frac{1}{(\overline{\zeta}-z)^n}\right). \tag{34}$$

The relations in the brackets on the right-hand side of (34) will be reduced to common denominators. In the considered relation $\zeta = |\zeta|\exp(i\theta)$; $z = r\exp(i\varphi)$; $|\zeta|, |z| < s$, thus

$$|s^2 - z\zeta|,\, |s^2 - z\overline{\zeta}| > s(s-r),\, |\overline{\zeta} - z| > r\sin\varphi. \tag{35}$$

One of the inequalities

$$|s^2 - z\zeta| > s^2 \sin\varphi,\ |s^2 - z\overline{\zeta}| > s^2 \sin\varphi,$$

also takes place. Since $0 < \varphi < \pi,\ 0 < \theta < \pi$, then the points $z\zeta = r|\zeta|e^{i(\varphi+\theta)}$, $z\overline{\zeta} = r|\zeta|e^{i(\varphi-\theta)}$ lie in distinct half-planes relative to the line $\Lambda = \{z : z = te^{i\varphi},\ \varphi = \text{const},\ -\infty < t < +\infty\}$. In one of these half-planes lies the number s^2. Let, for instance, the points s^2 and $z\overline{\zeta}$ lie on both sides of the line Λ. Then $|s^2 - z\overline{\zeta}|$ is greater than the distance from s^2 to the line Λ. Thus $|s^2 - z\overline{\zeta}| \geq s^2 \sin\varphi$. From these inequalities, from (25),(35) and the equality $\zeta - \overline{\zeta} = 2i|\zeta|\sin\theta$, we obtain the following estimation for (34)

$$\left|(F(z,\zeta))_z^{(n)}\right| < n!2^n \frac{\sin\theta}{\sin^n\varphi}\left(\frac{1}{(s-r)^n} + \frac{1}{|\zeta - z|^n}\right). \tag{36}$$

From (14),(23),(28),(33),(36), it follows

$$\begin{aligned}
&\frac{1}{K}\left|\frac{d^n \ln f(z)}{dz^n}\right| \\
&< \frac{s^{n+2}}{\pi r^n (s-r)^{n+1}\sin^{n+1}\varphi} \int\limits_{[-s,-r_0]\cup[r_0,s]} |\ln|f(t)||\left(\frac{1}{t^2} - \frac{1}{s^2}\right)dt \\
&+ \frac{s^2}{(s-r)^{n+1}\sin^{n+1}\varphi}\frac{1}{\pi s}\int\limits_0^{\pi} |\ln|f(se^{i\theta})||\sin\theta d\theta \\
&+ \frac{1}{\sin^n\varphi}\sum_{r_0<|c_q|<s}\left(\frac{\sin\theta_q}{(s-r)^n} + \frac{\sin\theta_q}{|z-c_q|^n}\right) + 1,\ K = \text{const}.
\end{aligned} \tag{37}$$

It is well-known, [1], p. 39–41, that

$$|\ln|f|| = \ln^+|f| + \ln^+|1/f|;\ \ \ln^+ x = \max(\ln x, 0),\ x \geqslant 0;$$

$$\begin{aligned}
&B(r,f) + B(r,\frac{1}{f}), A(r,f) + A(r,\frac{1}{f}), C(r,f) + C(r,\frac{1}{f}) \\
&\qquad < 2S(r,f) + \text{const}.
\end{aligned} \tag{38}$$

From the definition of Nevanlinna's characteristic (4) ($\alpha = 0, \beta = \pi, k = \pi/(\beta - \alpha) = 1$), and (37), (36) we obtain (1).

Taking (12), (13), (37) ($r < s < R$) into account

$$\sum_{r_0<|c_q|<s} \frac{\sin\theta_q}{(s-r)^n} = \frac{c(s,0,\infty)}{(s-r)^n}$$
$$< \frac{R^2 r C(R,0,\infty)}{(s-r)^n 2(R-s)(R+s)} < \frac{R^2 r(S(R,f)+K)}{(s-r)^n (R-s)(R+s)}.$$

Let $R = 2r,\ s = 3r/2$. Then we obtain

$$\sum_{r_0<|c_q|<s} \frac{\sin\theta_q}{(s-r)^n} < \frac{K\,S(2r,f)}{r^{n-1}}, \quad K = \text{const}.$$

There exists an increasing continuous function $\overset{\circ}{S}(r,f)$, such that $S(r,f) = \overset{\circ}{S}(r,f) + O(1),\ r \to +\infty$, [1], p. 43, Th. 5.4. Therefore from the previous and from (1) it follows ($s = 3r/2$)

$$\left|\frac{d^n \ln f(z)}{dz^n}\right| < \frac{K\,S(2r,f)}{r^{n-1}\sin^{n+1}\varphi} + \frac{K}{\sin^n\varphi} \sum_{r_0<|c_q|<2r} \frac{\sin\theta_q}{|z-c_q|^n} + K. \tag{39}$$

Suppose that f has finite order ρ. Then

$$S(r,f) < K r^{\rho+(\varepsilon/2)},\ r > r_0,\ \varepsilon > 0. \tag{40}$$

From (12), (13), (37), (40) we have

$$c(t,0,\infty) = \sum_{r_0<|c_q|<t} \sin\theta_q < K t^{\rho+1+(\varepsilon/2)}. \tag{41}$$

Consider the circles, centered at $c_q \in D$, with radii $|c_q|^{-\rho-1-\varepsilon}\sin\theta_q$. Denote by E the set of points inside all of these circles. Now it will be shown that the sum of radii of the circles E is finite. Using well-known properties of the Stieltjes integral and inequality (41), we have

$$\sum_{r_0<|c_q|<\infty} |c_q|^{-\rho-1-\varepsilon}\sin\theta_q = \int_{r_0}^{\infty} t^{-\rho-1-\varepsilon}\,dc(t,0,\infty)$$

$$= c(t,0,\infty)\,t^{-\rho-1-\varepsilon}\Big|_{r_0}^{\infty} + (\rho+1+\varepsilon)\int_{r_0}^{\infty} c(t,0,\infty)t^{-\rho-2-\varepsilon}dt$$

$$< (\rho+1+\varepsilon)K\int_{r_0}^{\infty} t^{\rho+(\varepsilon/2)-\rho-1-\varepsilon}dt < \text{const}. \tag{42}$$

Determine $\varphi_1 = \min(\varphi, \pi - \varphi)$, $z = re^{i\varphi}$, $0 < \varphi < \pi$. Then $0 < \varphi_1 \leqslant \frac{\pi}{2}$, $\sin\varphi = \sin\varphi_1 > \sin\frac{\varphi_1}{2} > \frac{\sin\varphi}{\pi}$. Denote $G = \{te^{i\theta} : r_0 < t < 2r, 0 < \theta < \pi\}$,

$$G_1 = \{te^{i\theta} : r_0 < t < 2r,\ \frac{\varphi_1}{2} < \theta < \pi - \frac{\varphi_1}{2}\},\ G_2 = G \setminus G_1. \tag{43}$$

If $c_q = |c_q|e^{i\theta_q} \in G_1$ and $z \in G \setminus E$, then $|z - c_q| > |c_q|^{-\rho-1-\varepsilon} \sin\theta_q \geqslant |c_q|^{-\rho-1-\varepsilon} \sin\varphi_1/2$. Thus

$$|z - c_q|^{-n} < (2r)^{n(\rho+1+\varepsilon)}/\sin^n(\varphi_1/2) < Kr^{n(\rho+1)}/\sin^n\varphi, K = \text{const}.$$

Therefore

$$\sum_{c_q \in G_1} \frac{\sin\theta_q}{|z - c_q|^n} < \frac{Kr^{n(\rho+1+\varepsilon)}}{\sin^n\varphi} \sum_{c_q \in G_1} \sin\theta_q. \tag{44}$$

If $c_q \in G_2$,then $|z - c_q| > r\sin(\varphi_1/2)$ and

$$\sum_{c_q \in G_2} \frac{\sin\theta_q}{|z - c_q|^n} < \frac{1}{r^n \sin^n(\varphi_1/2)} \sum_{c_q \in G_2} \sin\theta_q < \frac{K}{r^n \sin^n\varphi} \sum_{c_q \in G_2} \sin\theta_q.$$

From this and from (41), (44) we obtain

$$\sum_{c_q \in G} \frac{\sin\theta_q}{|z - c_q|^n} < \frac{Kr^{n(\rho+1+\varepsilon)}}{\sin^n\varphi} \sum_{c_q \in G} \sin\theta_q < \frac{Kr^{(n+1)(\rho+1+\varepsilon)}}{\sin^n\varphi}. \tag{45}$$

The first of the inequalities in (2) follows from (39), (40), (43), (45). To prove the second one we will use

Lemma 3 *Let $f(z)$ be meromorphic in the domain D. Then for $n = 1, 2, 3, \ldots$,*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{f'(z)}{f(z)}\right)^n + \sum_{\sum qi_q = n} B_{i_1 \ldots i_{n-1}} \prod_{q=1}^{n-1} \left(\frac{d^q \ln f(z)}{dz^q}\right)^{i_q} + \frac{d^n \ln f(z)}{dz^n} \tag{46}$$

holds, where the sum is performed for all $i_1, \ldots, i_{n-1}$, for which $0 \leqslant i_1, \ldots \ldots, i_{n-1} < n$, $\sum qi_q = 1i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = n$; $B_{i_1 \ldots i_{n-1}}$ are nonnegative numbers.

Proof The branches of the analytic function $\ln f(z)$, $z \in D$, are holomorphic functions in a small neighborhood of z, $z \neq c_q \in \{c_q\}$, $\{c_q\}$

is the set of zeros and poles of the function $f(z), z \in D$. Hence, there exists a neighborhood $\{\eta : |\eta| < \delta\}$, in which the series

$$\ln f(z+\eta) - \ln f(z) = \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j \ln f(z)}{dz^j} \eta^j \tag{47}$$

converges. Consider also the series

$$f(z+\eta) = f(z) + \sum_{j=1}^{\infty} \frac{1}{j!} f^{(j)}(z) \eta^j. \tag{48}$$

From (47) we have

$$f(z+\eta) = f(z) \exp\left\{ \sum_{j=1}^{\infty} \frac{\eta^j}{j!} \frac{d^j \ln f(z)}{dz^j} \right\}$$

$$= f(z) \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \sum_{j=1}^{\infty} \frac{\eta^j}{j!} \frac{d^j \ln f(z)}{dz^j} \right\}^k = f(z) \sum_{n=0}^{\infty} A_n \eta^n. \tag{49}$$

A direct calculation shows that $A_1 = f'(z)/f(z)$. In case $n = 2, 3, 4, \ldots$,

$$A_n = \frac{1}{n!} \left(\frac{f'(z)}{f(z)} \right)^n + \sum_{i_1,\ldots,i_{n-1}} B_{i_1,\ldots,i_{n-1}} \prod_{q=1}^{n-1} \left(\frac{d^q \ln f(z)}{dz^q} \right)^{i_q} + \frac{1}{n!} \frac{d^n \ln f(z)}{dz^n}, \tag{50}$$

where the sum is taken over all integers $i_1, i_2, \ldots, i_{n-1}$, for which $0 \leqslant i_1, i_2, \ldots,\ i_{n-1} < n,\ 1i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = n$; $B_{i_1 \ldots i_{n-1}}$ are non-negative numbers. From (48), (49) we obtain, that $A_n f(z) = \frac{1}{n!} f^{(n)}(z)$, thus from (50) (and after redefining the coefficients) we get (46).

From (2) and (46) we have ($\sum q i_q = n,\ |z| = r,\ z \notin E$),

$$\left| \prod_{q=1}^{n-1} \left(\frac{d^q \ln f(z)}{dz^q} \right)^{i_q} \right| < K \prod_{q=1}^{n-1} \frac{r^{(q+1)(\rho+1+\varepsilon)i_q}}{\sin^{2qi_q} \varphi} = K \frac{r^{\sum (q+1)(\rho+1+\varepsilon)i_q}}{(\sin\varphi)^{\sum 2qi_q}}. \tag{51}$$

Since $i_1 < n$, then $i_1 + i_2 + \cdots + i_{n-1} < i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = n$, and $\sum [(q+1)(\rho+1+\varepsilon)i_q] = (\rho+1+\varepsilon)(\sum q i_q + \sum i_q) \leqslant (\rho+1+\varepsilon)(2n-1)$. Thus from (51) we obtain

$$\left| \prod_{q=1}^{n-1} \left(\frac{d^q \ln f(z)}{dz^q} \right)^{i_q} \right| < K \frac{r^{(\rho+1+\varepsilon)(2n-1)}}{\sin^{2n} \varphi}. \tag{52}$$

From the first of the inequalities (2) we have

$$\left|\frac{f'(z)}{f(z)}\right|^n < K\frac{r^{2n(\rho+1+\varepsilon)}}{\sin^{2n}\varphi}, \ \left|\frac{d^n \ln f(z)}{dz^n}\right| < \frac{K|z|^{(n+1)(\rho+1+\varepsilon)}}{\sin^{2n}\varphi}, \ z \notin E. \quad (53)$$

The last of the inequalities in (2) is a consequence of (46), (52), (53).

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METHODS FOR STUDYING LEVEL SETS OF SMOOTH ENOUGH FUNCTIONS

G.A. Barsegian
Institute of Mathematics
of National Academy of Sciences of Armenia
24-b Marshal Bargramian ave., Yerevan 375019, Armenia
bars@netsys.am

G.A. Sukiasyan
Institute of Mathematics
of National Academy of Sciences of Armenia
24-b Marshal Bargramian ave., Yerevan 375019, Armenia
sugagik@mail.am

Abstract Many corn concepts in pure mathematics and applied science are in fact level sets of real functions. For particular cases of functions they were studied in different branches of mathematics. In the present paper we start investigations of the geometry of level sets for large classes of real functions. Some methods are established that permit to estimate the length of level sets of arbitrary "smooth enough" functions.

Keywords: level sets, Gamma-lines, value distribution

Mathematics Subject Classification (2000): 30D35

Introduction

Everywhere in the nature, in theoretical and applied investigations one can meet phenomena and problems, which are described or connected with level sets, that are sets of zeros of functions $u(x, y)$, or sets of solutions of equations $u(x, y) = 0$, where u is a real "smooth enough " function. In physics they are isoterms, isobars, potential lines, streaming lines, lines of given chemical, oil, radiation soil, etc. In pure mathematics level sets mean equilibrium sets, tangential sets (in ordinary differential equations), different type characteristic equations, boundary conditions, parabolic lines, where a given mixed type equation change its type

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(in partial differential equations). Level sets of polynomials are widely studied in algebraic geometry and topology. Hilbert's problem 16 deals with level sets. Almost any scientific calculation deals with level sets.

However, despite level sets are so often used in applied sciences and scientific calculations (during two centuries level sets were discussed in many sciences) they were little studied in pure mathematics for general classes of functions. Only recently (at the end of 70s) lengths of level sets were studied for the standard class of harmonic functions in arbitrary domains in the theory of Gamma-lines [1]-[4]. The obtained results were analogous to the main results in the classical Nevanlinna theory of a-points, [8]. Analogues of the deficiency relation for Gamma-lines were established in [1]-[4] describing particularly the distribution of the length of real or imaginary parts of meromorphic functions [1].

New methods developed in these papers have permitted to put forward a new program of investigations in analysis [5] embracing several subfields of mathematics connecting with investigations of level sets. A crucial role in this program play investigations of geometry of level sets of large classes of real functions.

In this paper we start the investigations of the geometry in fact for arbitrary "smooth enough" functions. Some methods are established that permit to study lengths and distributions of level sets of these functions. This seems gives a pertinent addition to investigations related to Hilbert's problem 16(a) that sounds qualitatively quite similar: to study the "topology" (number of connected components) of level sets for particular classes of real polynomials. This is one of the less investigated Hilbert's problems. On the other hand connections between the obtained inequalities and main results in classical Nevanlinna value distribution theory are established.

In this Section 1 of the paper we give two different methods for estimating the lengths of level sets of functions $v(x, y) - c$, where $v(x, y)$ is a good enough real function and c is a constant.

In Section 2 comparative properties of the *distribution* of solutions of $v(x, y) = c_k$, $k = 1, 2, ..., n$, will be studied similarly as the main theorems in the Nevanlinna theory desribe the solutions of $w(z) = c_k$, $k = 1, 2, ..., n$, for meromorphic functions $w(z)$ or the theory of Gamma-lines describes $\Gamma_1, \Gamma_2, ..., \Gamma_n$-lines of w. Particularly $v(x, y) = c_k$, $k = 1, 2, ..., n$, are investigated when $v(x, y)$ is a harmonic function.

In Section 3 we consider the case of several variables.

1. Length of level sets of real functions of two variables with “good” gradients.

1.1 Main result

We use the following notations: $D \subset \mathbb{C}$ is a bounded domain with a piecewise smooth boundary ∂D; $l(D)$ is the length of ∂D; $C^2(\overline{D})$ is the class of twice continuously differentiable functions $v(x, y)$ in the closure $\overline{D}$ of D with the condition $|\text{grad}\, v(x, y)| \neq 0$ in each $z \in \overline{D}$ except for probably isolated points in neighborhoods of which all integrals in the following theorem converge; $L(D, c, v)$ is the total length of level set $\{(x, y) \in D : v(x, y) - c = 0\}$ of the function $v(x, y)$, $c \in \mathbb{R}$. Through each point (x_0, y_0) passes a level curve $\{(x, y) \in D : v(x, y) - v(x_0, y_0) = 0\}$ whose curvature at the point (x_0, y_0) we denote by $K(x_0, y_0)$.

The following result gives upper bounds for the length $L(D, c, v)$ using some double integrals of the curvature $K(x, y)$ and the length $l(D)$.

Theorem 1 *For any function $v(x, y) \in C^2(\overline{D})$ and $c \in \mathbb{R}$*

$$L(D, c, v) \leq \frac{1}{2}\iint_D |K(x, y)|\, d\sigma + \frac{1}{2}\int_{\partial D} \frac{\left|v'_y(x, y)dx - v'_x(x, y)dy\right|}{|\text{grad}\, v(x, y)|}$$

$$\leq \frac{1}{2}\iint_D |K(x, y)|\, d\sigma + \frac{1}{2} l(D) \qquad (1)$$

is valid, where $d\sigma$ is the area element.

Hence we shall consider the magnitude

$$\begin{aligned} G(D) \quad &:= \quad \frac{1}{2}\iint_D |K(x, y)|\, d\sigma + \frac{1}{2}\int_{\partial D} \frac{\left|v'_y(x, y)dx - v'_x(x, y)dy\right|}{|\text{grad}\, v(x, y)|} \\ &=: \quad \frac{1}{2}\iint_D \left|\frac{\partial}{\partial x}\frac{v'_x(z)}{|\text{grad}\, v(z)|} + \frac{\partial}{\partial y}\frac{v'_y(z)}{|\text{grad}\, v(z)|}\right| d\sigma \\ &\quad + \frac{1}{2}\int_{\partial D} \frac{\left|v'_y(z)dx - v'_x(z)dy\right|}{|\text{grad}\, v(z)|} \end{aligned}$$

as a *characteristic for the length* and inequality (1) we shall refer to as the *length and curvature principle.*

Observe also that the second integral above is bounded by $l(D)$.

Proof of Theorem 1 The proof is based on Green’s formula. We consider the domain $D_1 = \{z \in D : v(z) > c\}$ and note that the boundary of this domain consists of curves $\{z \in D : v(z) = c\}$ and arcs of the boundary of D (see Figure 1).

Since the direction of the tangent to a curve $\{z \in D : v(z) = c\}$ at a point z under positive orientation with respect to a corresponding component $D_1 = \{z \in D : v(z) > c\}$ coincides with the direction of the vector $(v'_y(z); -v'_x(z))$ and

$$\cos\alpha(z) = \frac{v'_y(z)}{|\text{grad}\, v(z)|}, \quad \sin\alpha(z) = -\frac{v'_y(z)}{|\text{grad}\, v(z)|},$$

where $\alpha(z)$ is the angle between this tangent and the positive direction of the real axis, then (dl is the length element)

$$L(D,c,v) = \int_{\{z\in D: v(z)=c\}} dl = \int_{\{z\in D: v(z)=c\}} (\cos\alpha(z) + \sin\alpha(z))\, dl$$

$$= \int_{\{z\in D: v(z)=c\}} \frac{v'_y(z)}{|\text{grad}\, v(z)|} dx - \frac{v'_x(z)}{|\text{grad}\, v(z)|} dy.$$

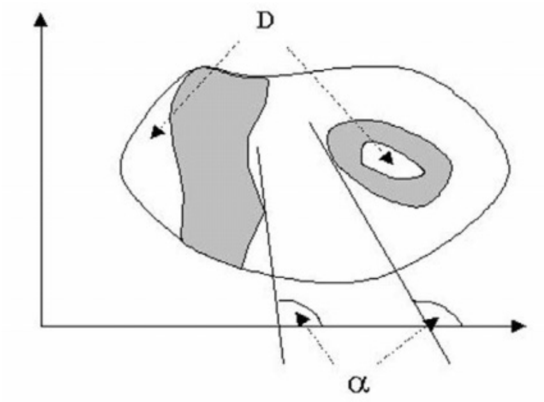

Figure 1. *Shaded part is the domain D_1.*

Adding the integrals along the lines $\partial D_1 \cap \partial D$ to both sides of the last relation and applying Green's formula for each component of D_1 separately, we get

$$L(D,c,v) + \int_{\partial D_1 \cap \partial D} \frac{v'_y(z)}{|\text{grad}\, v(z)|} dx - \frac{v'_x(z)}{|\text{grad}\, v(z)|} dy$$

$$= -\iint_{D_1} \left\{ \frac{\partial}{\partial x} \frac{v'_x(z)}{|\text{grad}\, v(z)|} + \frac{\partial}{\partial y} \frac{v'_y(z)}{|\text{grad}\, v(z)|} \right\} d\sigma. \quad (2)$$

Arguing in the same way for $D_2 = \{z \in D : v(z) < c\}$ we come to the relation

$$L(D, c, v) - \int_{\partial D_2 \cap \partial D} \frac{v'_y(z)}{|\text{grad } v(z)|} dx - \frac{v'_x(z)}{|\text{grad } v(z)|} dy$$

$$= \iint_{D_2} \left\{ \frac{\partial}{\partial x} \frac{v'_x(z)}{|\text{grad } v(z)|} + \frac{\partial}{\partial y} \frac{v'_y(z)}{|\text{grad } v(z)|} \right\} d\sigma. \qquad (3)$$

Inequality (1) follows from (2) and (3).

1.2 Connections with the Tangent variation principle for complex functions

Below we apply (1) to a widely studied class of real functions which are real or imaginary parts of complex analytic or meromorphic functions. Let $w(z)$ be a function meromorphic in D, $z = x + iy$, $v(z) := v(x, y) = \text{Im } w(z)$. Simple calculations yield

$$L(D, c, \text{Im } w) \leq \frac{1}{2} \iint_D \left| \frac{\partial}{\partial x} \sin(\arg w'(z)) + \frac{\partial}{\partial y} \cos(\arg w'(z)) \right| d\sigma$$

$$= \frac{1}{2} \iint_D \left| \text{Im } \frac{w''(z)}{w'(z)} e^{-i \arg w'(z)} \right| d\sigma + \frac{1}{2} \int_{\partial D} \left| \text{Re } (e^{i \arg w'(z)} dz) \right|. \qquad (4)$$

If we consider the class of real functions $v(z) := v(x, y) = |w(z)|$ which are modules of meromorphic functions then from (1) follows

$$L(D, c, |w|) \leq \frac{1}{2} \iint_D \left| \frac{\partial}{\partial x} \cos(\arg \frac{w'(z)}{w(z)}) - \frac{\partial}{\partial y} \sin(\arg \frac{w'(z)}{w(z)}) \right| d\sigma$$

$$+ \frac{1}{2} \int_{\partial D} \left| \sin(\arg \frac{w'(z)}{w(z)}) dx + \cos(\arg \frac{w'(z)}{w(z)}) dy \right|$$

$$= \frac{1}{2} \iint_D \left| \text{Re } \left(\frac{w''(z)}{w'(z)} - \frac{w'(z)}{w(z)} \right) e^{-i \arg w'(z)/w(z)} \right| d\sigma$$

$$+ \frac{1}{2} \int_{\partial D} \left| \text{Im } (e^{i \arg w'(z)/w(z)}) dz) \right|. \qquad (5)$$

Length of level sets of the imaginary parts and modules of meromorphic functions were first obtained as particular cases of so called Tangent

Variation Principle in the theory of Gamma-lines (see [1]-[3], especially [4]) that study Gamma-lines of functions w, that are preimages $w^{-1}(\Gamma)$ of curves Γ, similarly as complex analysis studies usual a-points, that are preimages $w^{-1}(a)$ of complex values a. According to this principle for any meromorphic function w and any smooth Jordan curve Γ with bounded variation of the angle between the tangent and the real axis is valid

$$L(D,\Gamma) \le K(\Gamma) \iint_D \left\{ \left| \frac{\partial}{\partial x} \arg w'(z) \right| + \left| \frac{\partial}{\partial y} \arg w'(z) \right| \right\} d\sigma$$

$$+K_1(D)l(D) \le 2K(\Gamma) \iint_D \left| \frac{w''(z)}{w'(z)} \right| d\sigma + K_1(D)l(D), \qquad (6)$$

where $L(D,\Gamma)$ is the total length of curves $w^{-1}(\Gamma) \cap D$, $l(D)$ is the length of ∂D, $K(\Gamma)$ and $K_1(\Gamma)$ are constants depending only on Γ.

Thus when Γ is the strait line $\{w | \operatorname{Im} w = c\}$ we have $L(D,\Gamma) = L(D,c,\operatorname{Im} w)$; when Γ is a circumference $\{w | \, |w| = c\}$ we have $L(D,\Gamma) = L(D,c,|w|)$. Now we see that the inequalities (4) and (5) are more precise than (6).

Besides from the inequality (1) by an appropriate choice of the function $v(z)$ one can deduce diverse particular estimates.

1.3 Sharpness

Below we illustrate the sharpeness of Theorem 1 (here $D = D(r) = \{z : |z| < r\}$).

1. $v(x,y) = \operatorname{Im} e^{ax+iby} = e^{ax} \sin by$, a and b are real number ($a, b \ne 0$).

$$L(D(r), 0, e^{ax} \sin by) = br^2 + O(r), \quad r \to \infty,$$

$$\frac{1}{2} \iint_{D(r)} \left| \frac{\partial}{\partial x} \frac{v'_x(z)}{|\text{grad } v(z)|} + \frac{\partial}{\partial y} \frac{v'_y(z)}{|\text{grad } v(z)|} \right| d\sigma$$

$$= \frac{1}{2} \iint_{D(r)} \left| \frac{\partial}{\partial y} \frac{b \cos by}{\sqrt{b^2 \cos^2 by + a^2 \sin^2 by}} \right| d\sigma = br^2 + O(r), \quad r \to \infty,$$

$$\frac{1}{2} \int_{\partial D(r)} \frac{|v'_y(z)dx - v'_x(z)dy|}{|\text{grad } v(z)|} \le \frac{1}{2} \int_{\partial D(r)} |dz| = \pi r.$$

2. $v(x,y) = \left|e^{ax+iby}\right| = e^{ax}$ $(a \neq 0)$.

$$L(D(r), 1, e^{ax}) = 2r,$$

$$\frac{1}{2}\iint_{D(r)} \left|\frac{\partial}{\partial x}\frac{v_x'(z)}{|\text{grad } v(z)|} + \frac{\partial}{\partial y}\frac{v_y'(z)}{|\text{grad } v(z)|}\right| d\sigma$$

$$= \frac{1}{2}\iint_{D(r)} \left|\frac{\partial}{\partial x}\frac{a}{|a|}\right| d\sigma = 0,$$

$$\frac{1}{2}\int_{\partial D(r)} \frac{\left|v_y'(z)dx - v_x'(z)dy\right|}{|\text{grad } v(z)|} = \frac{1}{2}\int_{\partial D(r)} |dy| = 2r.$$

The above examples show that inequality (1) is sharps. Also they show that each summand on the right- hand side of the inequality (1) may have a main contribution in the total sum.

1.4 Extremal curves of functions $v(x, y)$

Extremal curves τ_i of a given function $v(x,y)$ in D on which $|\text{grad } v(x,y)| = 0$ plays, clearly, a crucial role in the description of geometric behaviour of functions $v(x,y)$. The total length of all these extremal curves in D coincides with $L(D, 0, |\text{grad } v(x,y)|)$. Thus we have the following

Proposition 1 *Theorems 1 applied to the function* $|grad\, v(x,y)|$ *gives upper bounds for the total length of extremal curves provided that* $|grad\, v(x,y)| \in C^2(\overline{D})$.

1.5 Functions with level sets on which $|\text{grad } v(x, y)| = 0$

The above theorems hold only for those functions $v(x,y)$ given in D for which level sets of function $v(x,y) - c$ can not consist of curves, where $|\text{grad } v(x,y)| = 0$; according to the definition of the class $C^2(\overline{D})$ the function $|\text{grad } v(x,y)|$ can take zero only on isolated points. However, arbitrary smooth enough functions can have also some subsets of level sets consisting of such curves γ_i that at any point of γ_i $|\text{grad } v(x,y)| = 0$ is valid. Then we deal with those curves γ_i which consist only of extremal points of functions $v(x,y)$ lying on level sets of functions $v(x,y)$. Denoting by $L_E(D,c,v)$ the total length of all these curves in D we observe that $L_E(D,c,v) \leq L(D,0,|\text{grad } v(x,y)|)$. Therefore we have

Proposition 2 *Theorems 1 applied to the function* $|grad\, v(x,y)|$ *gives the upper bounds for* $L_E(D,c,v)$ *provided that* $|grad\, v(x,y)| \in C^2(\overline{D})$.

1.6 Average length

The following result gives the average length of level sets of real functions. The lemma has an independent interest. This result coincides in fact with the co-area formula [6] that usually is given for Lipschitz functions.

Theorem 2 *Under the conditions of Theorem 1 with an arbitrary non-negative continuous function* $f(c)$, $c \in (c_1, c_2)$,

$$\int_{c_1}^{c_2} L(D,c,v) f(c) dc = \iint_{\{z\in D: c_1<v(z)<c_2\}} |\text{grad } v(z)| \, f(v(z)) d\sigma \qquad (7)$$

is valid.

Proof We give a new short geometric proof, making use of notations clear from the Figures 2 and 3 below.

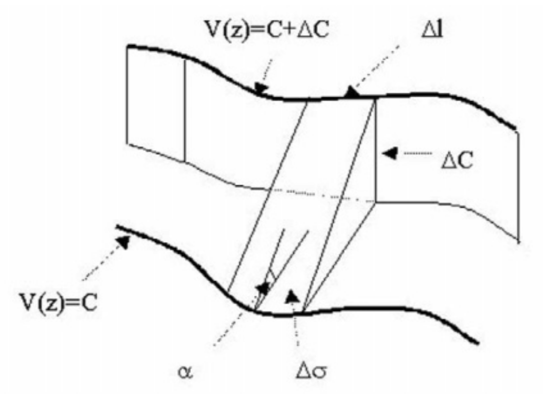

Figure 2.

For the angle $\beta(z)$ between the normal $(v'_x(z); v'_y(z); -1)$ to the surface $\{v(z) : c_1 < v(z) < c_2\}$ at a point z and the vector $(0;0;-1)$

$$\Delta l \Delta c \sim \tan\beta(z)\Delta\sigma, \quad \tan\beta(z) = |\text{grad } v(z)| \, .$$

holds. Therefore

$$\int_{c_1}^{c_2} L(D,c,v) f(c) dc = \int_{c_1}^{c_2} \int_{\{z\in D: v(z)=c\}} f(c) dl dc$$

$$= \iint_{\{z\in D: c_1<v(z)<c_2\}} |\text{grad } v(z)| \, f(v(z)) d\sigma.$$

The proof is complete.

From the identity (7) as a corollary for meromorphic functions the relations

$$\int_{c_1}^{c_2} L(D, c, |w|) f(c) dc = \iint_{\{z \in D: c_1 < v(z) < c_2\}} \left| w'(z) \right| f(|w(z)|) d\sigma,$$

$$\int_{c_1}^{c_2} L(D, c, \text{Im } w) f(c) dc = \iint_{\{z \in D: c_1 < v(z) < c_2\}} \left| w'(z) \right| f(\text{Im } w(z)) d\sigma,$$

follow, [4].

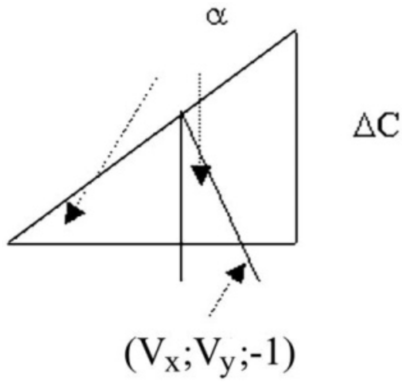

Figure 3.

Remark In Lemma 1 the function $f(z)$ can have finitely many isolated singularities, but it is necessary to require that the integrals in (7) converge in the neighborhoods of these points.

2. Length of level sets of real smooth functions

The next method can be applied to arbitrary functions $u(x, y)$ in $\overline{D}$ for which we suppose just that $u''_{xx}(x, y)$ and $u''_{yy}(x, y)$ are continuous in $\overline{D}$. Thus the function can be also indentical to zero in some sub domains D_0 of D. Respectively solutions of $u(x, y) = 0$ can consist of some curves γ_i as well as of some sub domains D_0. By total length $L(D, u)$ we mean total length of all γ_i plus total length of all boundaries ∂D_0 lying inside D.

Denote $J_x := \{z| \text{ Re } z = x\} \cap D$ and $J_y := \{z| \text{ Im } z = y\} \cap D$. The function $u(x, y)$ can be identically equal to zero on some intervals $(\dot{y}_\vartheta, \ddot{y}_\vartheta) \in J_x$. Denote by $N(J_x, u)$ the number of zeros of $u(x, y)$ which either are isolated zeros of u or they are those zeros $\dot{y}_\vartheta, \ddot{y}_\vartheta$ which lie inside J_x. Thus in our calculations we take into account only ends of the mentioned above intervals where $u \equiv 0$.

Similarly we define $N(J_y, u)$ substituting x by y. Using also the notations

$$\Theta_x(u,\varphi) := \begin{cases} \left|\frac{d}{dx} \arg \frac{d}{dx}\left[u(x,y) + i\varphi(x)u(x,y)\right]\right|, & \text{if } |u)| + |u'_x| > 0 \\ 0, & \text{for } |u| + |u'_x| = 0 \end{cases},$$

$$\Theta_y(u,\varphi) := \begin{cases} \left|\frac{d}{dy} \arg \frac{d}{dy}\left[u(x,y) + i\psi(y)u(x,y)\right]\right|, & \text{if } |u| + |u'_x| > 0 \\ 0, & \text{if } |u| + |u'_x| = 0 \end{cases},$$

we are now able to present our result.

Theorem 3 *Let $u(x,y)$ be an arbitrary smooth function in the closure of a given plane domain D, $\varphi(x) > 0$ respectively $\psi(y) > 0$ are arbitrary monotone functions defined for all x and y, with $z := x + iy \in D$. Then*

$$L(D,u) \le \sqrt{2}\int_{-\infty}^{\infty} N(J_x,u)dx + \sqrt{2}\int_{-\infty}^{\infty} N(J_y,u)dy$$

$$\le \sqrt{2}\int\int_D \left[\Theta_x(u,\varphi) + \Theta_y(u,\psi)\right] dydx + \sqrt{2}l(D), \tag{8}$$

where $l(D)$ is the length of the boundary ∂D of D.

Proof We use here a technique developed in [4]. First we give an evaluation method for the number of zeros of real functions [4], p. 157. Suppose $f(x)$ is a function twice continuously differentiable on a segment $a \le x \le b$, $-\infty < a < b < \infty$, and let $x_1, x_2, \ldots, x_n \in [a,b]$ be its zeros as was defined above: this means that $x_1, x_2, \ldots, x_n$ are isolated zeros of u or they are those zeros $\dot{x}_\vartheta, \ddot{x}_\vartheta$ (supposing that there are also some intervals $(\dot{x}_\vartheta, \ddot{x}_\vartheta)$ where $u \equiv 0$) which lie inside the interval (a,b).

Let us denote the union of all the segments $[\dot{x}_\vartheta, \ddot{x}_\vartheta]$, where $f(x)$ is identically equal to zero by X_0. Clearly at any point $x \in X_0 \cup \{x_1, x_2, \ldots, x_n\}$ we have $|f(x)| + |f'_x(x)| = 0$ and respectively for any $x \notin X_0$ is valid $|f(x)| + |f'_x(x)| > 0$.

We introduce a new, complex-valued function $F(x) = U(x) + iV(x)$ of the real variable $x \in [a,b]$, satisfying the condition $F(x) = 0$ at the points $x_1, x_2, \ldots, x_n$. As such a function one can take for example $F(x) = f(x) + i\varphi(x)f(x)$, where $\varphi(x)$ is such that $\varphi(x) \ne 0$ and $\varphi'(x) \ne 0$ for $x \in [a,b]$. Let us first suppose that for given i the segment $[x_i, x_{i+1}]$ does not belong to X_0. Obviously, the graph of $F(x)$, $x \in [x_i, x_{i+1}]$, is a curve closed at the origin (see Figure 4).

Therefore the variation of the angle $\arg(d/dx)F(x)$ between the tangent to the curve $F(x)$ and the real axis on any segment $[x_i, x_{i+1}]$ is not less than π:

$$\pi \leq \int_{x_i}^{x_{i+1}} \left| \frac{d}{dx} \arg \frac{d}{dx} F(x) \right| dx,$$

so that if $n \geq 2$ then for the number of zeros of $f(x)$ lying in $[a, b]$ we have

$$n - 1 \leq \frac{1}{\pi} \int_{[a,b]\setminus X_0} \left| \frac{d}{dx} \arg \frac{d}{dx} F(x) \right| dx.$$

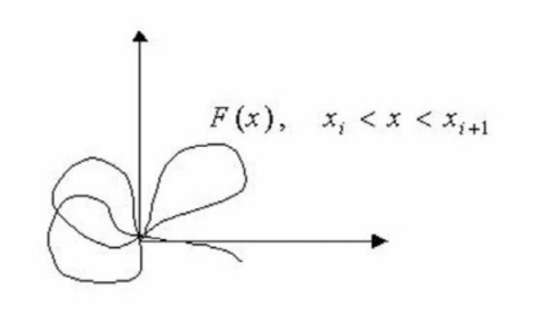

Figure 4

If $n = 0$ or $n = 1$ we can also write similarly

$$1 \leq \frac{1}{\pi} \int_{[a,b]\setminus X_0} \left| \frac{d}{dx} \arg \frac{d}{dx} F(x) \right| dx + 1,$$

so that from the last inequalities we get

Lemma 1 *For an arbitrary real twice continuously differentiable function $f(x)$ on a segment $a \leq x \leq b$*

$$n \leq \frac{1}{\pi} \int_{[a,b]\setminus X_0} \left| \frac{d}{dx} \arg \frac{d}{dx} F(x) \right| dx + 1 \tag{9}$$

is valid.

Consider the set $l(u)$ consisting of level sets γ_i plus all boundaries ∂D_0 lying inside D. Let us represent $l(u)$ as the sum $l_x(u) \cup l_y(u)$, where $l_x(u)$ is the totality of the arcs from $l(u) \cap D$, such that in any point belonging to $l_x(u)$ the smaller angle α_x between the tangent and x-axis is less than or equal to $\pi/4$; $l_y(u)$ is the totality of the arcs from $l(u) \cap D$, such that at any point belonging to $l_y(u)$ the smaller angle α_y between the tangent and the y-axis is less than $\pi/4$.

Let $L_x(D,u)$ and $L_y(D,u)$ be the total lengths of the arcs $l_x(u)$ and $l_y(u)$ respectively. Obviously $l(u) = l_x(u) \cup l_y(u)$ and therefore

$$L(D,u) = L_x(D,u) + L_y(D,u). \tag{10}$$

For a fixed x, we denote by J_x the set $D \cap \{z : \text{Re } z = x\}$, the number of points $z_{i,x}$ of the intersection of J_x with the union of arcs $l_x(u)$ by $\Lambda(J_x, u)$. Hence for the length element Δl of an arc from $l_x(u)$ the following relation is true: $\Delta l \sim \Delta x \cos \alpha_{i,x}$ at the point $z_{i,x}$. Consequently we have

$$L_x(D,u) \leq \sqrt{2} \int_{x_1}^{x_2} \Lambda(J_x, u) dx.$$

For a given x, let $\{m_x^{(p(x))}\}$ denote the collection of interval components of the set J_x, where $p(x)$ is the number of these components for given x. Suppose that the part of the domain D contained in the strip between the lines $\{z : \text{Re } z = x'\}$ and $\{z : \text{Re } z = x''\}$ satisfies the following conditions.
(A) The part is decomposed into n connected components each with boundaries having common points both with $J_{x'}$ and $J_{x''}$.
(B) Intersection of each of these components with J_x consisting of only one interval lying in the sets J_x.
Then

$$\int_{x_1}^{x_2} \Lambda(J_x, u) dx = \sum_{\{(x',x'')\}} \int_{x'}^{x''} \Lambda(J_x, u) dx,$$

where the union of all intervals (x', x'') is equal to (x_1, x_2).

Denoting the number of intersections of $l_x(u)$ with $m_x^{(p(x))}$ by $\Lambda(m_x^{(p(x))}, u)$ we get

$$\sum_{\{(x',x'')\}} \int_{x'}^{x''} \Lambda(J_x, u) dx = \sum_{\{(x',x'')\}} \int_{x'}^{x''} \sum_{\{p(x)\}} \Lambda(m_x^{(p(x))}, u) dx.$$

Also we have $\Lambda(m_x^{(p(x))}, u) \leq N(m_x^{(p(x))}, u)$. Therefore we get

$$\sum_{\{(x',x'')\}} \int_{x'}^{x''} \sum_{\{p(x)\}} \Lambda(m_x^{(p(x))}, u) dx$$

$$\leq \sum_{\{(x',x'')\}} \int_{x'}^{x''} \sum_{\{p(x)\}} N(m_x^{(p(x))}, u) dx := \int_{-\infty}^{\infty} N(J_x, u) dx.$$

Applying Lemma 1 to our function u we obtain

$$N(m_x^{(p(x))}, u) = \int_{m_x^{(p(x))}\setminus X_0} \left| \frac{d}{dy} \arg \frac{d}{dy} \left[u(x,y) + i\varphi(x)u(x,y) \right] \right| dy + 1$$

and henceforth

$$L_x(D,u) \le \sqrt{2} \sum_{\{(x',x'')\}}$$

$$\times \int_{x'}^{x''} \left(\int_{m_x^{(p(x))}\setminus X_0} \left| \frac{d}{dy} \arg \frac{d}{dy} \left[u(x,y) + i\varphi(x)u(x,y) \right] \right| dy + 1 \right) dx$$

$$\le \sqrt{2} \sum_{\{(x',x'')\}} \int_{x'}^{x''} \int_{m_x^{(p(x))}\setminus X_0} \left| \frac{d}{dy} \arg \frac{d}{dy} \left[u(x,y) + i\varphi(x)u(x,y) \right] \right| dydx$$

$$+ \sqrt{2} \sum_{\{(x',x'')\}} \int_{x'}^{x''} \sum_{\{m_x^{(p)}\}} 1dxdy$$

$$\le \sqrt{2} \int \int_D \Theta_x(u,\varphi)dydx + \sqrt{2} \sum_{\{(x',x'')\}} \int_{x'}^{x''} \sum_{\{m_x^{(p)}\}} 1dxdy \tag{11}$$

follows. For every $x \in (x', x'')$

$$\sum_{\{m_x^{(p)}\}} 1 = m,$$

where the constant m is independent of x and consequently

$$\int_{x'}^{x''} \sum_{\{m_x^{(p)}\}} 1dx = m(x'' - x').$$

But the quantity $m(x'' - x')$ is equal to the sum of projections on the x-axis of all components of ∂D lying in the strip between the lines $\{z : \text{Re } z = x'\}$ and $\{z : \text{Re } z = x''\}$. Therefore $2m(x'' - x')$ is less than the length of the part of ∂D contained in the mentioned strip. Hence splitting the interval (x_1, x_2) into appropriate parts we obtain

$$\sum_{\{(x',x'')\}} \int_{x'}^{x''} \sum_{\{m_x^{(p)}\}} 1dx \le \frac{l(D)}{2}. \tag{12}$$

Now from (11) and (12) we get

$$L_x(D,u) \leq \sqrt{2}\int\int_D \Theta_x(u,\varphi)dydx + \sqrt{2}l(D).$$

Similarly we get

$$L_y(D,u) \leq \sqrt{2}\int\int_D \Theta_y(u,\varphi)dydx + \sqrt{2}l(D).$$

Consequently from (10) follows inequality (8) of Theorem 3.

Notes

1. After [1], the length for real or imaginary parts of the particular case of univalent functions were studied first in paper [7] by Hayman and Wu and further by many other mathematicians: Bishop, C., Carleson, L., Garnet, J., Gehring, F., Fernandes, J., Heinonen, J., Jones, P., Martio, O., O'yma, K., Zinsmeister, M.; length of level sets of circumferentially mean p-valent functions have been studied by Sukiasyan, G. [9].

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GAMMA-LINES OF POLYNOMIALS AND A PROBLEM BY ERDÖS-HERZOG-PIRANIAN

G.A. Barsegian
Institute of Mathematics
of National Academy of Sciences of Armenia
24-b Marshal Bargramian ave., Yerevan 375019, Armenia
bars@netsys.am

Abstract Using a technique developed for meromorphic functions we give upper bounds for the length of Γ-lines of monic polynomials for large classes of curves Γ. When Γ is a circumference we deal with the Erdös-Herzog-Piranian problem.

Keywords: Gamma lines, polynomials, Erdös-Herzog-Piranian problem

Mathematics Subject Classification (2000): 30D35

We call the preimages $w^{-1}(\Gamma)$ Gamma-lines of a meromorphic function w, where Γ is a curve in the complex plane [4]. Using a technique of the Gamma-lines developed for meromorphic functions we give upper bounds for the length $L(\Gamma)$ of Γ-lines of monic polynomials $P := z^n + a_1 z^{n-1} + ... + a_n$ for a large classes of smooth Jordan curves Γ lying in the closure of the unit disk supposing only that the integrated absolute value of the curvature $v(\Gamma)$ of Γ is finite: we mean that $\int_\Gamma |k(s)|ds < \infty$, where $k(s)$ is the curvature of Γ at the point s.

Theorem *For any above type curve* $\Gamma \subset \{w : \ |w| \leq 1\}$ *and any monic polynomial* P *of degree* n

$$L(\Gamma) \leq K(\Gamma)n \tag{1}$$

where $K(\Gamma) = 15.54\pi(v(\Gamma) + 1)$.

Remark 1 A similar result for much lager classes of meromorphc functions has been established in [3], see also [4].

When Γ is a circumference we deal with the Erdös-Herzog-Piranian problem ([6], 1958) who have conjectured that the length L of the set $\{E : \ |P(z)| = 1\}$ for any monic polynomial $P := z^n + a_1 z^{n-1} + ... + a_n$

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is less than or equal to $(2\pi + o(n))n$. In light of Gamma-lines the set $\{E : \ |P(z)| = 1\}$ is the set of Γ-lines of $P(z)$ with a particular type of curve $\Gamma(1, 0) := \{w : \ |w| = 1\}$ so that due to this conjecture one should prove that $L(\Gamma(1, 0)) \leq (2\pi + o(n))n$.

The first essential progress has been made by Ch. Pommerenke in 1961 [10]), who showed that $L(\Gamma(1, 0)) \leq const \cdot n^2$; the next main step was done by D. Borwein in 1995 [[5]) who showed that

$$L(\Gamma(1, 0)) \leq const \cdot n,$$

and, thus, gave the correct rate of growth (dependence on degree n) in this long standing open problem. Then the best up to now constant has been obtained by A. Eremenko and W. Hayman in [7] who showed that the constant should be at most 9.173.

Remark 2 Our method applied more carefully in the particular case of $\Gamma(1, 0)$ gives a better constant that in [5] and worse than [7].

The proof follows almost immediately from the following inequality that has been proved in 1981 in [3], see also [4] p.20: for any meromorphic functions $w(z)$ in the closure of a domain D with piecewise smooth boundary ∂D and any curve Γ

$$L(D, \Gamma) \leq K_1(\Gamma) \int\int_D \left| \frac{w''}{w'} \right| d\sigma + K_1(\Gamma) l(D) \tag{2}$$

is valid, where $L(D, \Gamma)$ is the length of the Γ-lines of w in D, $K_1(\Gamma) = 3(v(\Gamma) + 1)$, $l(D)$ is the length of ∂D.

It is known that for a monic polynomial P of degree n the set $\{z : |P(z)| \leq M\}$ is contained in the union of some disks d_k, $k = 1, 2, ..., k^* \leq n$, the sum of whose radii r_k is $2eM^{1/n}$. This result is due to H. Cartan, see [8], p. 19; Ch. Pommerenke [9] improved the constant $2e$ to 2.59. Note that all our Gamma-lines lie in the union of similar disks taken for $M = 1$. With this M the union $\cup_k \left\{\overline{d_k}\right\}$, ($\overline{x}$ means the closure) consists, clearly, of some closed non overlapping domains D_j, $j = 1, 2, ...k^* \leq n$, with total length of the boundary $< 2\pi \sum_k r_k$. Applying (2) to P in these domains D_j we have

$$L(\Gamma) = L(D^*, P) \leq K_1(\Gamma) \int\int_{D^*} \sum_{t=1}^{n-1} \left| \frac{1}{z - z_t} \right| d\sigma + 5.18 K_1(\Gamma)\pi, \tag{3}$$

where $D_j^* := \cup_j D_j$ and z_t are zeros of the derivative P'. It is easy to see that if $|D^*|$ is the area of D^* then denoting $z - z_t = \rho e^{i\varphi}$ we have

$$\int\int_{D^*} \left| \frac{1}{z - z_t} \right| d\sigma = \int\int_{D^*} d\rho d\varphi \leq \int\int_{D_t^*} d\rho d\varphi. \tag{4}$$

where D_t^* is the disk with the center in z_t and with $|D_t^*| = |D^*|$. Indeed, for any two domain

$$d := \{z : \varphi_1 < \arg z < \varphi_2,\ \rho_1 < |z - z_t| < \rho_2\}$$

and

$$d' := \{z : \varphi_1' < \arg z < \varphi_2',\ \rho_1' < |z - z_t| < \rho_2'\}$$

with equal areas and $|\rho_2 - \rho_1| = |\rho_2' - \rho_1'|$ we have

$$\int\int_d d\rho d\varphi \leq \int\int_{d'} d\rho d\varphi$$

as soon as $\rho_1' < \rho_1$. Therefore dividing D^* into similar small domains d (we can take them as small as we please) and transferring them into domains d' with $\rho_1' < \rho_1$ we can compose from these disks d' the disk D_t^* satisfying $|D_t^*| = |D^*|$ for which, due to the above inequality, (4) is valid.

Since the diameter of D_t^* is less than or equal to $\sum d_k \leq 2.59$ we conclude that $\int\int_{D_t^*} d\rho d\varphi \leq 5.18\pi$ so that from (3) and (4) follows $L(\Gamma) \leq 5.18 K_1(\Gamma)\pi n$, i.e. inequality (1) of Theorem 1.

Concluding remark Usually we try to transfer to meromorphic functions what we know already about polynomials. Here we have in some sense an inverse situation: we have applied in the polynomial case what we knew already for meromorphic functions. It seems that there are other aspects in recent developments in the theory of meromorphic functions, say proximity property of a-points and other theorems about Gamma-lines and their applications whose analogues are not yet established for polynomials. I would propose to consider related problems.

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A METHOD FOR STUDYING OSCILLATIONS OF NONLINEAR DIFFERENTIAL EQUATIONS. APPLICATIONS TO SOME EQUATIONS IN BIOLOGY AND ECONOMICS

K. Barseghyan
Live Sciences Higher Educational School,
31 Avan-Acharyan str., 375006 Yerevan, Armenia
bars@web.am

G.A. Barsegian
Institute of Mathematics
of National Academy of Sciences of Armenia
24-b Marshal Bargramian ave., 374019 Yerevan, Armenia
bars@netsys.am

Abstract In this paper we give a new method for studying oscillations of large classes of nonlinear ordinary differential equations involving the Riccati equation, the Schrödinger and the Painlevé type equations. Some application are given as well.

Keywords: Gamma-lines, oscillations of ODE, population dynamics.

Mathematics Subject Classificatin (2000): 30C99, 34C05, 34C10, 92D25

1. Introduction and method

We study oscillations (distribution of zeros) of solutions of different type equations $F(t, y, y', y'') = 0$. Our approach permits to obtain upper bounds for the number $N(t_1, t_2, y)$ of zeros of solutions $y(t)$ in a given interval $t_1 < t < t_2$. There is enormously large number of investigations of zeros of solutions of some second order equations: the Internet shows around 30000 paper. The reason is somehow clear: in many applied problems namely zeros of the solutions are of primary interest. However, the majority of these investigations relate to the standard, linear

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Schrödinger type equations $y'' + A(t)y = 0$ or their immediate generalizations; they can be found in many textbooks and surveys, see for instance Ince [10], Sansone [15], Hartman [9], Kuzano and Nalto [14], Swanson [16], Wong [19], [20], Cheng [6]. What concerns "essentially" non linear equations we were not able to find any concrete references; moreover, we do not know about any pure mathematical investigation related to oscillations for first order non linear equations. However, zeros of solutions of many particular type equations play an important role in applications and we naturally suppose that they were likely considered in many applied papers related to those fields, where these particular type equations arise.

We are not experts in ODE and the above presented short survey is mainly due to our consultations with several experts in ODE. We should gratefully acknowledge valuable discussions and helpful information given by A. Agrachev, O. Çelebi, G. Csordas and A. Zafer, with special thanks to O. Çelebi, A. Zafer for telling us also a generalized oscillation problem considered in Section 5 below.

In this paper we just apply an approach used for the study of Gamma-lines of complex functions to ODE. The approach is based on the following evaluation method for the number of zeros of real functions, see [2], Chapter 5.

Lemma 1 *Suppose $f(x)$ is a function twice continuously differentiable on a segment $[a, b]$ and $x_1, x_2, \ldots, x_n \in [a, b]$ be those zeros of f for which $f' \neq 0$. Then*

$$n \leq \frac{1}{\pi} \int_a^b \left| \frac{d}{dx} \arctan \frac{f(x)}{f'(x)} \right| dx + 1. \tag{1}$$

To prove this estimate we note that in (x_i, x_{i+1}) there exists a point x^* such that $f'(x^*) = 0$ and $f(x^*) \neq 0$. Consequently the angles $\arctan f(x_i)/f'(x_i)$ and $\arctan f(x_{i+1})/f'(x_{i+1})$ are equal to zero, and the angle $\arctan f(x^*)/f'(x^*)$ is equal to $\pi/2$.

Hence

$$\frac{\pi}{2} \leq \int_{x_i}^{x^*} \left| \frac{d}{dx} \arctan \frac{f(x)}{f'(x)} \right| dx, \quad \frac{\pi}{2} \leq \int_{x^*}^{x_{i+1}} \left| \frac{d}{dx} \arctan \frac{f(x)}{f'(x)} \right| dx,$$

so that

$$\pi \leq \int_{x_i}^{x_{i+1}} \left| \frac{d}{dx} \arctan \frac{f(x)}{f'(x)} \right| dx \tag{2}$$

and summing up these estimates over all i we get the above lemma.

We can apply now (1) to a solution $y(t)$ of the equation $F(t, y, y', y'') = 0$ in $[t_1, t_2]$. Let $N(t_1, t_2, y)$ be the number of zeros t^*_ν of the solution

y in $[t_1, t_2]$. Suppose that for each zero there exists the non zero limit $\lim_{t \to t_\nu^*} f'(t)/f(t)$. Then it follows from (1)

$$N(t_1, t_2, y) \le \frac{1}{\pi} \int_{t_1}^{t_2} \left| \frac{d}{dt} \arctan \frac{y(t)}{y'(t)} \right| dt + 1$$

$$= \frac{1}{\pi} \int_{t_1}^{t_2} \left| \frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2} \right| dt + 1 \tag{3}$$

so that an upper bound for $N(t_1, t_2, y)$ immediately follows from an upper bound for the integral.

We show below the last integral can be easily estimated for solutions of some large classes of equations $F(...) = 0$.

2. Equations of type $y'(t) = P(t, y)$, where P is polynomial with respect to y

We start illustrating the method by considering the equation

$$\frac{dy}{dt} = P(t, y) := a_n(t)y^n + a_{n-1}(t)y^{n-1} + ... + a_0(t) \tag{4}$$

with an integer $n \ge 1$; for the coefficients we assume just that they are smooth in (t_1, t_2) and satisfy

$$M_1 := M_1(t_1, t_2) := \sup_{t_1 < t < t_2} \max_{0 \le j \le n} |a_j(t)| < \infty, \tag{5}$$

$$M_2 := M_2(t_1, t_2) := \sup_{t_1 < t < t_2} \max_{0 \le j \le n} \left|a_j'(t)\right| < \infty, \tag{5$'$}$$

$$m_1 := m_1(t_1, t_2) := \inf_{t_1 < t < t_2} |a_0(t)| > 0, \tag{6}$$

$$m_2 := m_2(t_1, t_2) := \inf_{t_1 < t < t_2} |a_n(t)| > 0. \tag{6$'$}$$

The class of these equations involve particularly the Riccati equation, that play an important role in many applied sciences particularly in population dynamics.

Theorem 1 *Let $y(t)$ is a solution of equation* (4) *in* (t_1, t_2) *with coefficients satisfying* $(5) - (6')$. *Then*

$$N(t_1, t_2, y) \le \frac{K}{\pi} |t_2 - t_1| + 1, \tag{7}$$

where K is a finite constant depending only on m_1, m_2, M_1, M_2, n.

Remark 1 Calculating (roughly) the constant K we obtain

$$K = \left(nM_2 + \frac{n^3 + 3n^2}{2} M_1^2 \right) \max \left(\frac{4}{m_1^2}; \frac{|2nM_1/m_2|^{2n}}{|m_1/(2nM_1)|^{2/n}}; \frac{2}{m_2^2} \right). \quad (8)$$

Remark 2 Equations (4) admit oscillatory solutions, see [3].

Proof Note that under the conditions of Theorem 1 the solutions $y(t)$ can have only zeros, say t_ν^*, $\nu = 1, 2, ..., N(t_1, t_2, y)$, that satisfy $y'(t_\nu^*) \neq 0$. Consequently by applying Lemma 1 and (3) to our solution $y(t)$ we obtain

$$N(t_1, t_2, y) \leq \frac{1}{\pi} \int_{t_1}^{t_2} \left| \frac{d}{dt} \arctan \frac{y(t)}{y'(t)} \right| dt + 1$$

$$= \frac{1}{\pi} \int_{t_1}^{t_2} \left| \frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2} \right| dt + 1, \quad (9)$$

where

$$y'(t) = \sum_{i=0}^{n} a_i(t) y^i(t). \quad (10)$$

We should obtain upper bounds for

$$\left| \frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2} \right|$$

$$= \frac{\left| \sum_{i=0}^{n} \left[a_i'(t) y^{i+1}(t) + i a_i(t) y^i(t) \sum_{i=0}^{n} a_i(t) y^i(t) \right] - \left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 \right|}{\left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + y^2(t)} \quad (11)$$

for arbitrary $y(t)$.

Consider the denominator of (11)

$$\left| \left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + y^2(t) \right|$$

$$= \left| a_0(t) + \sum_{i=1}^{n} a_i(t) y^i(t) \right|^2 + y^2(t). \quad (12)$$

For $|y| < |y^*|$, where $|y^*| := |m_1/(2nM_1)|^{1/n}$ we have

$$\left|\left(y'(t)\right)^2 + (y(t))^2\right| \geq \left(\frac{|a_0(t)|}{2}\right)^2 \geq \frac{m_1^2}{4}. \tag{13}$$

Indeed, for all those y for which

$$\left|a_i(t)y^i(t)\right| \leq \frac{|a_0(t)|}{2n},\ i = 1, 2, .., n,$$

we have

$$\left|\sum_{i=1}^{n} a_i(t)y^i(t)\right| \leq \frac{|a_0(t)|}{2},$$

so that from (12) follows (13).

In other word (13) is true when

$$|y(t)| \leq \left|\frac{a_0(t)}{2na_i(t)}\right|^{1/i},\ i = 1, 2, .., n,$$

and certainly this is also true for all those y satisfying $|y| < |y^*|$.

For those y that satisfy $|y| > y^*$ immediately follows

$$\left|\left(y'(t)\right)^2 + (y(t))^2\right| \geq (y^*)^2 \geq \left|\frac{m_1}{2nM_1}\right|^{2/n}, \tag{14}$$

where we take into account that $|m_1/(2nM_1)| < 1$.

To give upper bounds for (11) we should estimate $\left|(y'(t))^2 + (y(t))^2\right|$ more precisely when y is "large". Since

$$\left|\left(y'(t)\right)^2 + (y(t))^2\right| = \left|\left(\sum_{i=0}^{n} a_i(t)y^i(t)\right)^2 + (y(t))^2\right|$$

$$= \left|\left(a_n(t)y^n(t) + y^n(t)\sum_{i=0}^{n-1}\frac{a_i(t)}{y^{n-i}(t)}\right)^2 + (y(t))^2\right|$$

there exists a "large enough" value y^{**} such that

$$\left|\left(a_n(t)y^n(t) + y^n(t)\sum_{i=0}^{n-1}\frac{a_i(t)}{y^{n-i}(t)}\right)^2 + (y(t))^2\right|$$

$$\geq \frac{a_n^2(t)y^{2n}(t)}{2} \geq \frac{m_2^2 y^{2n}(t)}{2} \tag{15}$$

as soon as $|y| \geq |y^{**}|$.

To determine this value y^{**} we note now that for values y satisfying

$$\left|\frac{a_i}{y^{n-i}}\right| \leq \frac{|a_n|}{2n};\ i, = 0, 1, 2, \ldots, n-1,$$

we have

$$\left|y^n(t)\sum_{i=0}^{n-1}\frac{a_i(t)}{y^{n-i}(t)}\right| \leq \frac{|a_n(t)y^n(t)|}{2}$$

so that (15) is valid for any y satisfying

$$|y| \geq \left|\frac{2na_i}{a_n}\right|^{1/n-i},\ i = 0, 1, 2, \ldots, n-1.$$

Thus (15) is valid also for any y satisfying $|y| \geq |y^{**}|$, where

$$|y^{**}| := \left|\frac{2nM_1}{m_2}\right|.$$

Further we have

$$\left|y''(t)y(t) - \big(y'(t)\big)^2\right|$$

$$= \left|\sum_{i=0}^{n}\left[a_i'(t)y^{i+1}(t) + ia_i(t)y^i(t)y'(t)\right] - \left(\sum_{i=0}^{n}a_i(t)y^i(t)\right)^2\right| \tag{16}$$

$$= \left|\sum_{i=0}^{n}\left[a_i'(t)y^{i+1}(t) + ia_i(t)y^i(t)\sum_{i=0}^{n}a_i(t)y^i(t)\right] - \left(\sum_{i=0}^{n}a_i(t)y^i(t)\right)^2\right|$$

$$\leq \sum_{i=0}^{n}\left[\left|a_i'(t)y^{i+1}(t)\right| + \left|ia_i(t)y^i(t)\right|\sum_{i=0}^{n}\left|a_i(t)y^i(t)\right|\right] + \left(\sum_{i=0}^{n}\left|a_i(t)y^i(t)\right|\right)^2.$$

We will need also the following form of (16)

$$\left|y''(t)y(t) - \big(y'(t)\big)^2\right| \tag{17}$$

$$= \left|y^{2n}\right| \sum_{i=0}^{n} \left[\left| \frac{a_i'(t)}{y^{2n-(i+1)}(t)} \right| + \left| \frac{i a_i(t)}{y^{n-i}(t)} \right| \sum_{i=0}^{n} \left| \frac{a_i(t)}{y^{n-i}(t)} \right| \right] + \left(\sum_{i=0}^{n} \left| \frac{a_i(t)}{y^{n-i}(t)} \right| \right)^2 .$$

Now we give upper bounds for

$$\left| \frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2} \right| .$$

Due to (13) and (16) and the definition of y^* for y satisfying $|y| < |y^*|$ we have

$$\left| \frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2} \right|$$

$$\leq \frac{\sum_{i=0}^{n} \left[\left| a_i'(t) \left(y^*(t)\right)^{i+1} \right| + \left| i a_i(t) \left(y^*(t)\right)^{i} \right| \sum_{i=0}^{n} \left| a_i(t) \left(y^*(t)\right)^{i} \right| \right]}{\left(|a_0|/2\right)^2},$$

$$\frac{\left(\sum_{i=0}^{n} \left| a_i(t) \left(y^*(t)\right)^{i} \right| \right)^2}{\left(|a_0|/2\right)^2}$$

$$\leq \frac{\sum_{i=0}^{n} \left[|a_i'(t)| + |i a_i(t)| \sum_{i=0}^{n} |a_i(t)| \right] + \left(\sum_{i=0}^{n} |a_i(t)| \right)^2}{\left(|a_0|/2\right)^2}$$

$$\leq \frac{n M_2 + \left(n^3 + 3n^2\right) M_1^2/2}{m_1^2/4} = \frac{4n M_2 + 2\left(n^3 + 3n^2\right) M_1^2}{m_1^2}, \tag{18}$$

where we take into account that $|y^*| \leq 1$ and (5), (5′), (6).

Due to (14) and (17) and the definition of y^* and y^{**} for y such that $|y^*| < |y| < |y^{**}|$ we have

$$\left| \frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2} \right|$$

$$\leq \frac{\left(y^{**}\right)^{2n} \left[\sum_{i=0}^{n} \left[|a_i'(t)| + |i a_i(t)| \sum_{i=0}^{n} |a_i(t)| \right] + \left(\sum_{i=0}^{n} |a_i(t)| \right)^2 \right]}{\left(y^*\right)^2}$$

$$\leq \frac{(y^{**})^{2n}\left[nM_2+\left(n^3+3n^2\right)M_1^2/2\right]}{(y^*)^2}$$
$$= \frac{|2nM_1/m_2|^{2n}\left[nM_2+\left(n^3+3n^2\right)M_1^2/2\right]}{|m_1/(2nM_1)|^{2/n}}, \tag{19}$$

where we take into account that $|y^{**}| \geq 1$ and (4), $(5')$. Due to (15) and (17) for y such that $|y| > |y^{**}|$ we have

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|$$
$$\leq \frac{y^{2n}\left[\sum_{i=0}^n\left[|a_i'(t)|+|ia_i(t)|\sum_{i=0}^n|a_i(t)|\right]+\left(\sum_{i=0}^n|a_i(t)|\right)^2\right]}{(|a_ny^n|)^2/2}$$
$$= \frac{2\sum_{i=0}^n\left[|a_i'(t)|+|ia_i(t)|\sum_{i=0}^n|a_i(t)|\right]+2\left(\sum_{i=0}^n|a_i(t)|\right)^2}{|a_n|^2}$$
$$\leq \frac{2nM_2+\left(n^3+3n^2\right)M_1^2}{m_2^2}. \tag{20}$$

Combining (18)-(20) we obtain Theorem 1 and the constant K in Remark 1.

3. Equations of type $y'(t) = R(t, y)$, where R is rational with respect to y

Consider now the equation

$$\frac{dy}{dt} = R(t,y) := \frac{P_1(t,y)}{P_2(t,y)} := \frac{a_n(t)y^n+a_{n-1}(t)y^{n-1}+...+a_0(t)}{b_m(t)y^m+b_{m-1}(t)y^{m-1}+...+b_0(t)}, \tag{21}$$

with some integer $m \geq 1$ and $n \geq 1$, with coefficients $a_n, a_{n-1}, ..., a_0$ smooth in (t_1, t_2) satisfying (5)-$(6')$ and with coefficients $b_m, b_{m-1}, ..., b_0$ smooth in (t_1, t_2) satisfying respectively

$$M_3 := M_3(t_1,t_2) := \sup_{t_1<t<t_2}\max_{0\leq j\leq m}|b_j(t)| < \infty, \tag{22}$$

$$M_4 := M_4(t_1,t_2) := \sup_{t_1<t<t_2}\max_{0\leq j\leq m}\left|b_j'(t)\right| < \infty, \tag{22'}$$

$$m_3 := m_3(t_1,t_2) := \inf_{t_1<t<t_2} |b_0(t)| > 0, \tag{23}$$

$$m_4 := m_4(t_1,t_2) := \inf_{t_1<t<t_2} |b_m(t)| > 0. \tag{23'}$$

Theorem 2 *Let $y(t)$ be a solution of equation (21) in (t_1,t_2) with coefficients satisfying (5)-(6′) and (22)-(23′). Suppose $|P_2(t,y)| \geq \Psi(t)$ for $t \in (t_1,t_2)$, where $\Psi(t)$ is an arbitrary nonnegative, smooth function. Then*

$$N(t_1,t_2,y) \leq \frac{K}{\pi}|t_2-t_1|+1, \tag{24}$$

where K is a finite constant depending only on m_1, m_2, m_3, m_4, M_1, M_2, M_3, M_4, $\Psi(t)$, n, m.

Remark 3 Calculating (roughly) the constant K we obtain

$$K = C\max\left(\frac{4}{m_1^2};\frac{|4mM_3/m_4|^{1/2}}{\left(\Psi(t)\,|m_1/(2nM_1)|^{1/n}\right)^2};\frac{2}{m_4^2}\right) \quad \text{for} \quad m \geq n, \tag{25}$$

$$K = C\max\left(\frac{4}{m_1^2};\frac{|2nM_1/m_2|}{\left(\Psi(t)\,|m_1/(2nM_1)|^{1/n}\right)^2};\frac{2}{m_2^2}\right) \quad \text{for} \quad m < n, \tag{25'}$$

where C is

$$\frac{(M_3M_2+M_1M_4)\,2m\Psi(t)+(n+2)\,mM_3M_1^2+\left(m^2M_3+4\Psi(t)\right)M_1^2}{2\Psi(t)}n.$$

Remark 4 Theorem 1 is not a particular case of Theorem 2 since we suppose that $b_m > 0$.

Proof Idea of the proof is the same as in item 2; however there are several differences so that we have to give a separate complete proof.

Note that under the conditions of Theorem 2 the solutions $y(t)$ can have only zeros, say t_ν^*, $\nu = 1,2,...,N(t_1,t_2,y)$, that satisfy $y'(t_\nu^*) \neq 0$. Consequently by applying Lemma 1 and (3) to our solution $y(t)$ we obtain

$$\begin{aligned} N(t_1,t_2,y) &\leq \frac{1}{\pi}\int_{t_1}^{t_2}\left|\frac{d}{dt}\arctan\frac{y(t)}{y'(t)}\right|dt+1 \\ &= \frac{1}{\pi}\int_{t_1}^{t_2}\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|dt+1 \end{aligned} \tag{26}$$

where

$$y'(t) = \frac{\sum_{i=0}^{n} a_i(t) y^i(t)}{\sum_{j=0}^{m} b_j(t) y^j(t)}.$$

Thus we should obtain upper bounds for

$$\left| \frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2} \right|$$

$$= \frac{\sum\limits_{i=0}^{n} \left[\left| a_i'(t) y^{i+1}(t) \right| + \left| i a_i(t) y^i(t) \frac{\sum_{i=0}^{n} a_i(t) y^i(t)}{\sum_{j=0}^{m} b_j(t) y^j(t)} \right| \right] \sum\limits_{j=0}^{m} \left| b_j(t) y^j(t) \right|}{\left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + y^2(t) \left(\sum_{j=0}^{m} b_j(t) y^j(t) \right)^2}$$

$$+ \frac{\sum\limits_{i=0}^{n} \left| a_i(t) y^i(t) \right| \sum\limits_{j=0}^{m} \left[\left| b_j'(t) y^{j+1}(t) \right| + \left| j b_j(t) y^j(t) \frac{\sum_{i=0}^{n} a_i(t) y^i(t)}{\sum_{j=0}^{m} b_j(t) y^j(t)} \right| \right]}{\left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + y^2(t) \left(\sum_{j=0}^{m} b_j(t) y^j(t) \right)^2}$$

$$+ \frac{\left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2}{\left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + y^2(t) \left(\sum_{j=0}^{m} b_j(t) y^j(t) \right)^2} \tag{27}$$

for arbitrary $y(t)$.

Consider the denominator of (27)

$$\left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + y^2(t) \left(\sum_{j=0}^{m} b_j(t) y^j(t) \right)^2$$

$$= \left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + \left(\sum_{j=0}^{m} b_j(t) y^{j+1}(t) \right)^2$$

$$= \left(a_0(t) + \sum_{i=1}^{n} a_i(t) y^i(t) \right)^2 + \left(\sum_{j=0}^{m} b_j(t) y^{j+1}(t) \right)^2. \tag{28}$$

We show now that for $|y| < |y^*|$, where $|y^*| := |m_1/(2nM_1)|^{1/n}$ we have

$$\left| a_0(t) + \sum_{i=1}^{n} a_i(t) y^i(t) \right|^2 + \left(\sum_{j=0}^{m} b_j(t) y^{j+1}(t) \right)^2 \geq \left(\frac{|a_0(t)|}{2} \right)^2 \geq \frac{m_1^2}{4}. \tag{29}$$

Indeed, for all those y for which

$$\left| a_i(t) y^i(t) \right| \leq \frac{|a_0(t)|}{2n}, \; i = 1, 2, .., n,$$

we have

$$\left| \sum_{i=1}^{n} a_i(t) y^i(t) \right| \leq \frac{|a_0(t)|}{2}$$

and since $\left(\sum_{j=0}^{m} b_j(t) y^{j+1}(t) \right)^2 > 0$ from (28) and (6) follows (29).

In other words, (29) is true when

$$|y(t)| \leq \left| \frac{a_0(t)}{2n a_i(t)} \right|^{1/i}, \; i = 1, 2, \ldots, n,$$

and certainly this is also true for all those y that satisfy $|y| < |y^*|$.

For those y that satisfy $|y| > y^*$ immediately follows

$$\left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2 + y^2(t) \left(\sum_{j=0}^{m} b_j(t) y^j(t) \right)^2$$

$$\geq (y^*(t))^2 \left(\sum_{j=0}^{m} b_j(t) y^j(t) \right)^2 \geq \left(\left| \frac{m_1}{2nM_1} \right|^{1/n} \right)^2 \left(\sum_{j=0}^{m} b_j(t) y^j(t) \right)^2$$

$$\geq \left(\left| \frac{m_1}{2nM_1} \right|^{1/n} \Psi(t) \right)^2. \tag{30}$$

To give upper bounds for (27) we should estimate $(y'(t))^2 + (y(t))^2$ more precisely when y is “large”. Consider two cases.

Case 1: $m \geq n$. Since

$$\left(\sum_{i=0}^{n} a_i(t)y^i(t)\right)^2 + \left(\sum_{j=0}^{m} b_j(t)y^{j+1}(t)\right)^2$$
$$= \left(\sum_{i=0}^{n} a_i(t)y^i(t)\right)^2 + \left(b_m(t)y^{m+1}(t) + y^{m+1}(t)\left[\sum_{j=0}^{m-1} \frac{b_j(t)}{y^{m-j+1}(t)}\right]\right)^2$$

there exists a "large enough" value y^{**} such that

$$\left(\sum_{i=0}^{n} a_i(t)y^i(t)\right)^2 + \left(\sum_{j=0}^{m} b_j(t)y^{j+1}(t)\right)^2$$
$$\geq \frac{b_m^2(t)y^{2(m+1)}(t)}{2} \geq \frac{m_4^2 y^{2(m+1)}(t)}{2} \tag{31}$$

as soon as $|y| \geq |y^{**}|$.

To determine this value y^{**} we note now that for values y satisfying

$$\left|\frac{b_j}{y^{m-j+1}}\right| \leq \frac{|b_m|}{2m};\ j = 0, 1, 2, \ldots, m-1,$$

we have

$$\left|y^{m+1}(t) \sum_{j=0}^{m-1} \frac{b_j(t)}{y^{m-j+1}(t)}\right| \leq \frac{\left|b_m(t)y^{(m+1)}(t)\right|}{2},$$

so that (31) is valid for any y satisfying

$$|y| \geq \left|\frac{2mb_j}{b_m}\right|^{1/(m-j+1)},\ j = 0, 1, 2, \ldots, m-1.$$

Thus (31) is valid also for any y satisfying $|y| \geq |y^{**}|$, where

$$|y^{**}| := \left|\frac{2mM_3}{m_4}\right|^{1/2}.$$

Case 2: $m < n$. We can easily see that there exists a "large enough" value y^{**} such that

$$\left(a_n(t)y^n(t) + y^n(t)\sum_{i=0}^{n-1} \frac{a_i(t)}{y^{n-i}(t)}\right)^2 + \left(\sum_{j=0}^{n} b_j(t)y^{j+1}(t)\right)^2$$

$$\geq \frac{a_n^2(t) y^{2n}(t)}{2} \geq \frac{m_2^2 y^{2n}(t)}{2} \tag{31'}$$

as soon as $|y| \geq |y^{**}|$.

To determine this value y^{**} we note now that for values y satisfying

$$\left|\frac{a_i}{y^{n-i}}\right| \leq \frac{|a_n|}{2n};\ i = 0, 1, 2, \ldots, n-1,$$

we have

$$\left| y^{n-1}(t) \sum_{i=0}^{n-1} \frac{a_i(t)}{y^{n-i}(t)} \right| \leq \frac{|a_n(t) y^n(t)|}{2}$$

so that (31′) is valid for any y satisfying

$$|y| \geq \left|\frac{2na_i}{a_n}\right|^{1/n-i},\ i = 0, 1, 2, \ldots, n-1.$$

Thus (31′) is valid also for any y satisfying $|y| \geq |y^{**}|$, where

$$|y^{**}| := \left|\frac{2nM_1}{m_2}\right|.$$

Further for the nominator of (27) we have

$$\left| \sum_{i=0}^{n} \left[a_i'(t) y^{i+1}(t) + i a_i(t) y^i(t) y'(t) \right] \sum_{j=0}^{m} b_j(t) y^j(t) \right|$$

$$+ \left| \sum_{i=0}^{n} a_i(t) y^i(t) \sum_{j=0}^{m} \left[b_j'(t) y^{j+1}(t) + j b_j(t) y^j(t) y'(t) \right] \right|$$

$$+ \left(\sum_{i=0}^{n} a_i(t) y^i(t) \right)^2$$

$$\leq \sum_{i=0}^{n} \left[\left| a_i'(t) y^{i+1}(t) \right| + \left| i a_i(t) y^i(t) \frac{\sum_{i=0}^{n} a_i(t) y^i(t)}{\sum_{j=0}^{m} b_j(t) y^j(t)} \right| \right] \sum_{j=0}^{m} \left| b_j(t) y^j(t) \right|$$

$$+\sum_{i=0}^{n}\left|a_i(t)y^i(t)\right|\sum_{j=0}^{m}\left[\left|b_j'(t)y^{j+1}(t)\right|+\left|jb_j(t)y^j(t)\frac{\sum_{j=0}^{n}a_i(t)y^i(t)}{\sum_{j=0}^{m}b_j(t)y^j(t)}\right|\right]$$

$$+\left(\sum_{i=0}^{n}a_i(t)y^i(t)\right)^2$$

$$\leq\sum_{i=0}^{n}\left[M_2\left|y^{i+1}(t)\right|+i\frac{M_1^2}{\Psi(t)}\left|y^i(t)\right|\sum_{i=0}^{n}\left|y^j(t)\right|\right]M_3\sum_{j=0}^{m}\left|y^j(t)\right|$$

$$+M_1\sum_{i=0}^{n}\left|y^i(t)\right|\sum_{j=0}^{m}\left[M_4\left|y^{j+1}(t)\right|+j\frac{M_3M_1}{\Psi(t)}\left|y^j(t)\right|\sum_{i=0}^{n}\left|y^i(t)\right|\right]$$

$$+M_1^2\left(\sum_{i=0}^{n}y^i(t)\right)^2. \tag{32}$$

To estimate the last magnitude we consider two cases.
Case 1: $m\geq n$. Then

$$y^{2(m+1)}\left(\sum_{i=0}^{n}\left[\frac{M_2}{|y^{m-i}(t)|}+i\frac{M_1^2}{\Psi(t)}\sum_{i=0}^{n}\frac{1}{|y^{m-i}(t)|}\right]M_3\sum_{j=0}^{m}\frac{1}{|y^{m+1-j}(t)|}\right)$$

$$+y^{2(m+1)}\left(M_1\sum_{i=0}^{n}\frac{1}{|y^{m+1-i}(t)|}\sum_{j=0}^{m}\left[\frac{M_4}{|y^{m-i}(t)|}+j\frac{M_3M_1}{\Psi(t)}\sum_{i=0}^{n}\frac{1}{|y^{m-i}(t)|}\right]\right)$$

$$+y^{2(m+1)}M_1^2\left(\sum_{i=0}^{n}\frac{1}{y^{m+1-i}(t)}\right)^2. \tag{33}$$

Case 2: $m<n$. We write (32) as

$$y^{2n}\left(\sum_{i=0}^{n}\left[\frac{M_2}{|y^{n-i}(t)|}+i\frac{M_1^2}{\Psi(t)}\sum_{i=0}^{n}\frac{1}{|y^{n-i}(t)|}\right]M_3\sum_{j=0}^{m}\frac{1}{|y^{n-(j+1)}(t)|}\right)$$

$$+y^{2n}\left(M_1\sum_{i=0}^{n}\frac{1}{\left|y^{n-(i+1)}(t)\right|}\sum_{j=0}^{m}\left[\frac{M_4}{\left|y^{n-j}(t)\right|}+j\frac{M_3M_1}{\Psi\left(t\right)}\sum_{i=0}^{n}\frac{1}{\left|y^{n-i}(t)\right|}\right]\right)$$

$$+y^{2n}M_1^2\left(\sum_{i=0}^{n}\frac{1}{y^{n-i}(t)}\right)^2 \qquad (33')$$

Now we give upper bounds for

$$\left|\frac{y''(t)y(t)-\left(y'(t)\right)^2}{\left(y'(t)\right)^2+\left(y(t)\right)^2}\right|,$$

when $m\geq n$.

Due to (29) and (32) and the definition of y^* for y satisfying $|y|<|y^*|$ we have

$$\left|\frac{y''(t)y(t)-\left(y'(t)\right)^2}{\left(y'(t)\right)^2+\left(y(t)\right)^2}\right|$$

$$\leq\frac{\sum_{i=0}^{n}\left[M_2+i\frac{M_1^2}{\Psi(t)}\right]mM_3+nM_1\sum_{j=0}^{m}\left[M_4+j\frac{M_3M_1}{\Psi(t)}\right]+2nM_1^2}{m_1^2/4}$$

$$=\frac{(n/2)\frac{2mM_3M_2\Psi(t)+mM_3nM_1^2+2mM_3M_1^2+2M_1mM_4\Psi(t)+M_1^2m^2M_3+4M_1^2\Psi(t)}{\Psi(t)}}{m_1^2/4}$$

$$=\frac{2n}{m_1^2}\left(2mM_3M_2+2M_1mM_4+4M_1^2\right)$$

$$+\frac{2n}{\Psi\left(t\right)m_1^2}\left(mM_3nM_1^2+2mM_3M_1^2+M_1^2m^2M_3\right). \qquad (34)$$

Here we take into account that $|y^*|\leq 1$ and (5), (5′), (6) due to (30) and (33) and the definition of y^* and y^{**}. For y such that $|y^*|<|y|<|y^{**}|$ we have

$$\left|\frac{y''(t)y(t)-\left(y'(t)\right)^2}{\left(y'(t)\right)^2+\left(y(t)\right)^2}\right|$$

$$\leq \frac{\left|\frac{4mM_3}{m_4}\right|^{1/2}\left[\sum_{i=0}^{n}\left[M_2+i\frac{M_1^2}{\Psi(t)}\right]mM_3+nM_1\sum_{j=0}^{m}\left[M_4+j\frac{M_3M_1}{\Psi(t)}\right]+2nM_1^2\right]}{\left(\Psi(t)|m_1/(2nM_1)|^{1/n}\right)^2}$$

$$=\frac{n\left|4mM_3/m_4\right|^{1/2}}{2\left(\Psi(t)m_1/(2nM_1)|^{1/n}\right)^2}\ \frac{2mM_3M_2\Psi\left(t\right)+mM_3nM_1^2+2mM_3M_1^2}{\Psi\left(t\right)}$$

$$+\frac{n|4mM_3/m_4|^{1/2}}{2\left(\Psi(t)|m_1/(2nM_1)|^{1/n}\right)^2}\frac{2M_1mM_4\Psi\left(t\right)+M_1^2m^2M_3+4M_1^2\Psi\left(t\right)}{\Psi\left(t\right)}. \tag{35}$$

Here we take into account that $|y^{**}|\geq 1$ and (5), ($5'$) due to (31) and (33). For y such that $|y|>|y^{**}|$ we have

$$\left|\frac{y''(t)y(t)-\left(y'(t)\right)^2}{\left(y'(t)\right)^2+\left(y(t)\right)^2}\right|$$

$$\leq\frac{y^{2(m+1)}\left(t\right)\left[\sum_{i=0}^{n}\left[M_2+i\frac{M_1^2}{\Psi(t)}\right]mM_3+nM_1\sum_{j=0}^{m}\left[M_4+j\frac{M_3M_1}{\Psi(t)}\right]+2nM_1^2\right]}{m_4^2y^{2(m+1)}(t)/2}$$

$$=\frac{2\left[\sum_{i=0}^{n}\left[M_2+i\frac{M_1^2}{\Psi(t)}\right]mM_3+nM_1\sum_{j=0}^{m}\left[M_4+j\frac{M_3M_1}{\Psi(t)}\right]+2nM_1^2\right]}{m_4^2}$$

$$=\frac{n}{m_4^2}2mM_3M_2+2M_1mM_4+4M_1^2$$

$$+\frac{n}{m_4^2\Psi\left(t\right)}mM_3nM_1^2+2mM_3M_1^2+M_1^2m^2M_3. \tag{36}$$

Denote by

$$C:=n\left(mM_3M_2+M_1mM_4+2M_1^2\right)$$

$$+\frac{n}{2\Psi(t)}mM_3nM_1^2+2mM_3M_1^2+M_1^2m^2M_3. \tag{37}$$

Then due to (37) and (34) for y satisfying $|y|<|m_1/(2nM_1)|^{1/n}$ we have

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|\le\frac{4C}{m_1^2}. \tag{38}$$

Due to (37) and (35) for y such that $|m_1/(2nM_1)|^{1/n}<|y|<|4mM_3/m_4|^{1/2}$ we have

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|\le\frac{|4mM_3/m_4|^{1/2}C}{\left(\Psi(t)|m_1/(2nM_1)|^{1/n}\right)^2}. \tag{39}$$

Due to (37) and (36) for y such that $|y|>|4mM_3/m_4|^{1/2}$ we have

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|\le\frac{2C}{m_4^2}. \tag{40}$$

Now we give upper bounds for

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|,$$

when $m<n$. Due to (5)-(6′),(22)-(23′), (29),(32),(37) for y satisfying $|y|<|m_1/(2nM_1)|^{1/n}$ we have

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|\le\frac{4C}{m_1^2}. \tag{38′}$$

Due to (5)-(6′), (22)-(23′), (31′), (33′), (37) for y satisfying

$$\left|\frac{m_1}{2nM_1}\right|^{1/n}<|y|<\left|\frac{4nM_1}{m_2}\right|$$

we have

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|\le\frac{|4nM_1/m_2|C}{\left(\Psi(t)|m_1/(2nM_1)|^{1/n}\right)^2}. \tag{39′}$$

Due to (5)-(6′),(22)-(23′),(31′),(33′),(37) for y satisfying $|y|>|4nM_1/m_2|$ we have

$$\left|\frac{y''(t)y(t)-(y'(t))^2}{(y'(t))^2+(y(t))^2}\right|\le\frac{2C}{m_2^2}.$$

Combining (38)-(40, (38′)-(40′) and (37) we obtain Theorem 2 and the constant K in Remark 3.

4. Equations with non integer degrees of y

Consider the equation

$$\frac{dy}{dt} = a_n(t)y^{k_n} + a_{n-1}(t)y^{k_{n-1}} + ... + a_0(t)y^{k_0}, \tag{41}$$

where k_i, $i = 0, 1, \ldots, n$, are nonnegative numbers satisfying $k_n > k_{n-1} > \ldots > k_1 > k_0 = 0$ and the coefficients satisfying conditions of Theorem 1. Repeating almost word by word the proof in Section 2 we obtain

Theorem 3 *Let $y(t)$ is a solution of equation* (41) *in* (t_1, t_2). *Then*

$$N(t_1, t_2, y) \leq \frac{K}{\pi} |t_2 - t_1| + 1, \tag{42}$$

where K is a finite constant depending only on m_1, m_2, M_1, M_2, k_i, n.

Remark 5 Calculating (roughly) the constant K we obtain

$$K = \left(nM_2 + M_1^2 \sum_{i=0}^{n} k_i \right) \max \left(\frac{4}{m_1^2}; \frac{\left|4nM_1^2/m_2\right|^{k_n/(k_n-k_{n-1})}}{\left|m_1/(2nM_1)\right|^{2/k_n}}; \frac{2}{m_2^2} \right). \tag{43}$$

Consider the equation

$$\frac{dy}{dt} = R(t,y) = \frac{P_1(t,y)}{P_2(t,y)} := \frac{a_n(t)y^{k_n} + a_{n-1}(t)y^{k_{n-1}} + \ldots + a_0(t)y^{k_0}}{b_m(t)y^{p_m} + b_{m-1}(t)y^{p_{m-1}} + \ldots + b_0(t)y^{p_0}} \tag{44}$$

in a given interval $t_1 < t < t_2$, where k_i, $i = 0, 1, \ldots, n$, are nonnegative numbers satisfying

$$k_n > k_{n-1} > \ldots > k_1 > k_0 = 0 \tag{45}$$

and p_j, $j = 0, 1, \ldots, m$, are nonnegative, satisfying

$$p_m > p_{m-1} > \ldots > p_1 > p_0 = 0. \tag{46}$$

We also suppose that the coefficients $a_i, i = 0, 1, \ldots, n$, and $b_j, j = 0, 1, \ldots, m$, are satisfying the conditions of Theorem 2.

Repeating almost word by word the proof in Section 3 we obtain

Theorem 4 *Let $y(t)$ is a solution of equation* (44) *in* (t_1, t_2). *Suppose* $|P_2(t,y)| \geq \Psi(t)$ *for* $t \in (t_1, t_2)$, *where* $\Psi(t)$ *is an arbitrary nonnegative, smooth function. Then*

$$N(t_1, t_2, y) \leq \frac{K}{\pi} |t_2 - t_1|, \tag{47}$$

where K is a finite constant depending only on m_1, m_2, m_3, m_4, M_1, M_2, M_1, M_2, k_i, p_j, n, m.

Remark 6 Calculating (roughly) the constant K we obtain

$$K = C \max \left(\frac{4}{m_1^2}; \frac{\left| \frac{4mM_3}{m_4} \right|^{1/(p_m - p_{m-1}+1)}}{\left(\Psi\left(t\right) \left| \frac{m_1}{2nM_1} \right|^{1/k_n} \right)^2}; \frac{2}{m_4^2} \right) \quad \text{for} \quad p_m \geq k_n, \tag{48}$$

$$K = C \max \left(\frac{4}{m_1^2}; \frac{\left|2nM_1/m_2\right|^{1/(k_n - k_{n-1})}}{\left(\Psi\left(t\right) \left|m_1/(2nM_1)\right|^{1/k_n} \right)^2}; \frac{2}{m_2^2} \right) \quad \text{for} \quad p_m < k_n \tag{48'}$$

where

$$C = \left(nM_2 + \frac{M_1^2}{\Psi\left(t\right)} \sum_{i=0}^{n} k_i \right) mM_3 + nM_1 \left(mM_4 + \frac{M_3 M_1}{\Psi\left(t\right)} \sum_{j=0}^{m} p_j \right) + 2nM_1^2.$$

5. Oscillation of solutions around curves

The more interesting and certainly more applicable will be results related to oscillation of solutions around curves. This means that we should study the number $N(t_1, t_2, y-f)$ of zeros of $y(t) - f(t)$ in (t_1, t_2), where $y(t)$ is a solutions and $f(t)$ in a given function.

It turns out that our approach can be immediately applied for getting upper bounds of $N(t_1, t_2, y - f)$ for equations considered in the above sections. Indeed, let us again consider the equation

$$\frac{dy}{dt} = P(t, y) := a_n(t)y^n + a_{n-1}(t)y^{n-1} + ... + a_0(t). \tag{4}$$

Denoting $Y(t) := y(t) - f(t)$ we can rewrite it as

$$\frac{dy}{dt} = P(t, y) := A_n(t)Y^n + A_{n-1}(t)Y^{n-1} + ... + A_0(t), \tag{49}$$

where

$$A_n(t) = a_n(t), \quad A_0(t) = a_n(t)f^n + a_{n-1}(t)f^{n-1} + ... + a_0(t). \tag{50}$$

To get upper bounds for $N(t_1, t_2, y(t) - f(t))$ we need just to apply Theorem 1. To do that we need to get upper bounds for the magnitudes

$M_1(Y)$, $M_2(Y)$, $m_1(Y)$, $m_2(Y)$, for equation (49) (instead of M_1, M_2, m_1, m_2 considered for equation (4)). The magnitudes $M_1(Y)$ and M_2 (Y) for equation (49) can be easily obtained if we know M_1 and M_2 for equation (4), the magnitudes $m_2(Y) \equiv m_2$. To apply Theorem 1 we just need to suppose that

$$m_1(Y) := m_1(t_1, t_2) := \inf_{t_1 < t < t_2} |A_0(t)| > 0.$$

Then we will be able to calculate the magnitude $K(Y)$ in (8) with the magnitudes $M_1(Y)$, $M_2(Y)$, $m_1(Y)$, $m_2(Y)$ (instead of M_1, M_2, m_1, m_2).

Thus we immediately come to

Theorem 5 *Let $y(t)$ is a solution of equation (49) in (t_1, t_2). Suppose that inequalities (5) – (6′) are valid for $M_1(Y)$, $M_2(Y)$, $m_1(Y)$, $m_2(Y)$ (instead of M_1, M_2, m_1, m_2). Then*

$$N(t_1, t_2, y - f(t)) \leq \frac{K(Y)}{\pi} |t_2 - t_1| + 1. \tag{51}$$

Remark 7 Acting similarly we can get similar results on oscillation of solutions around curves for other equations considered in Sections 3 and 4.

6. Oscillations of some Painlevé and Schrödinger type equations

Due to the presence of $(y')^2$ equation

$$a(t)y'' + b(t) \left(y'\right)^2 f(y) + c(t)y'g(y) + d(t)h(y) = 0 \tag{52}$$

is, in a sense, of Painlevé type. The equation involves as particular cases many widely studied other equations: for instance standard Schrödinger type equations $ay'' + dy = 0$; Emden-Flower equations $ay'' + dy^\gamma = 0$. There are numerous investigations related to oscillations of solutions of these particular classes of equations: in those papers we met the coefficients $b(t)$ and $c(t)$ are identically equal to zero.

Our method permits to obtain immediately upper bounds for the number $N^*(t_1, t_2, y)$ of those zeros t_ν of solutions y of (52) in a given interval $t_1 < t < t_2$, for which there exists the non zero limit $\lim_{t \to t_\nu} y'(t)/y(t)$. By A we denote $\inf_{t_1<t<t_2} |a(t)|$; by B, C, D we denote $\sup_{t_1<t<t_2} |\psi(t)|$, where $\psi(t)$ is equal $b(t)$, $c(t)$, $d(t)$ respectively; by H_1, H_2, H_3 we denote $\sup_{t_1<t<t_2} |\varphi(t)|$, where $\varphi(t)$ is equal to $|f(y)y|$, $|g(y)/y|$, $|h(y)/y|$ respectively. With these notation the following result is valid.

Theorem 6 *Let $y(t)$ be a solution of equation (52) in (t_1, t_2) with smooth and bounded coefficients a, b, c, d in (t_1, t_2). Then*

$$N^*(t_1, t_2, y) \leq \frac{1}{\pi}\left(H_1\frac{B}{A} + H_2\frac{C}{A} + H_3\frac{D}{A}\right)|t_2 - t_1| + 1. \tag{53}$$

Proof Again by applying Lemma 1 to our solution $y(t)$ we have

$$N(t_1, t_2, y) \leq \frac{1}{\pi}\int_{t_1}^{t_2}\left|\frac{d}{dt}\arctan\frac{y(t)}{y'(t)}\right| dt + 1$$

$$= \frac{1}{\pi}\int_{t_1}^{t_2}\left|\frac{y''(t)y(t) - (y'(t))^2}{(y'(t))^2 + (y(t))^2}\right| dt + 1$$

so that we need just to obtain upper bounds for

$$\left|\frac{y''y - (y')^2}{(y')^2 + y^2}\right| \leq \left|\frac{y''y}{(y')^2 + y^2}\right| + 1.$$

For arbitrary $y(t)$ satisfying equation (52)

$$\left|\frac{y''y}{(y')^2 + y^2}\right| \leq \left|\frac{\left(b(t)\,(y')^2 f(y) + c(t)y'g(y) + d(t)h(y)\right)y}{a(t)\left((y')^2 + y^2\right)}\right|$$

$$\leq \left|f(y)y\frac{b(t)}{a(t)}\frac{(y')^2}{(y')^2 + y^2}\right| + \left|g(y)\frac{c(t)}{a(t)}\frac{y'y}{(y')^2 + y^2}\right| + \left|\frac{h(y)}{y}\frac{h(t)}{a(t)}\frac{y^2}{(y')^2 + y^2}\right|$$

$$\leq \left|f(y)y\frac{b(t)}{a(t)}\right| + \left|\frac{g(y)}{2}\frac{c(t)}{a(t)}\right| + \left|\frac{h(y)}{y}\frac{h(t)}{a(t)}\right|.$$

is valid from which Theorem 6 follows.

Particularly for the standard class of equations

$$a(t)y''(t) - d(t)y(t) = 0 \tag{54}$$

we have $H_1 \equiv 0$, $H_2 \equiv 0$,.$H_3 \equiv 1$ so that the next theorem immediately follows.

Theorem 7 *For any solution $y(t)$ of equation (54) in (t_1, t_2) is valid*

$$N^*(t_1, t_2, y) \leq \frac{D}{\pi A}|t_2 - t_1| + 1. \tag{55}$$

7. An application to some generalized mathematical models in biology and economics

Introduction and Problem *How long time is needed for restoring a given state (of a population, of an economical datum etc.).*
First we note that equations considered in the above Section 1 and 2 generalize many equations which play a key role in biomathematics and economical mathematics. We mention some of them.

Verhulst's differential equation in population dynamics

$$y' = (N - D)y + I - E, \tag{56}$$

where $y := y(t)$ is the number of individuals of a given population, Ny and Dy are respectively the velocities of births and deaths of the population, I is the number of immigrants entering the given area from outside and E is the number of emigrants departing from the area.

Differential equation of logistics

$$y' = ay^2 + by, \; a = const < 0, \; b = const > 0 \tag{57}$$

describing cell division.

Lotka-Volterra (or predator-pray) equation

$$\begin{cases} y' = ay + byx, \\ x' = cx + dyx, \end{cases} \tag{58}$$

where $y := y(t)$ and $x := x(t)$ are interpreted respectively as number of predators and prays, a, b, c and d are constants: a and c, are, respectively rates of birth and death of individual species (predators and prays), c and d, are, respectively decay and growth factors due to binary interactions. This system of equations is equivalent to the equation

$$y'_x = \frac{ay + byx}{cx + dyx}. \tag{58$'$}$$

We will also mention Zeeman's model describing muscle fibre

$$\begin{cases} ay' = -(y^3 + by + x), \\ x' = y - c, \end{cases} \tag{59}$$

where a, b, c are constants; this system is equivalent to the equation

$$ay'_x = \frac{-(y^3 + by + x)}{y - c}. \tag{59$'$}$$

These equations can be found in many textbooks and monographs, see, for instance, Lotka [13], Volterra [17], Akira [1], Frauenthal [7], Jones and Sleeman [11].

It has been mentioned in many books that these models provide oversimplified representations of real situations and in fact instead of constants there should be functions depending both on y and t. For instance Kolmogorov offered (see [12]) to consider the strengthened Lotka-Volterra model

$$\begin{cases} y' = yK_1(x, y), \\ x' = xK_2(x, y), \end{cases}$$

for the predator-pray problem with rather general K_1 and K_2; this system is equivalent to

$$y'_x = \frac{yK_1(x, y)}{xK_2(x, y)}.$$

Clearly more adequate models can be much more complicated if we try to take into account say age-structure of population, [8], [4], [5], concurrence inside individuals in the same species [18], immigration and emigration, seasonal changes etc., see [1], [7], [11]. In turn the more complicated coefficients can be approximated by polynomials; note that in fact we deal with reasonably simple coefficients and approximation should be considered in reasonable finite intervals. Thus we see that general models should be described mainly by equations of type (4) and (21) considered in the above Sections 1 and 2. By the way some of the above equations have interpretations also in economics. In some cases these two subjects (population dynamics and economics) are almost identical. For instance: payment of pensions to different categories of pensioners is mathematically identical to population dynamics; Verhulst's model is interpreted as growth of economics under given policy of investment etc. Thus some more adequate models in economics again will be reduced to equations of type (4) and (21).

However, there is no general theory for these equation and we can study only some of the aspects of solutions; this is even so for many simple models.

In this sections we note that the above methods permit to analyze the following practical question. Let us have a given policy of regulation of the given magnitude y (the magnitude can be the number of a population or an economical datum etc.); the policy is governed by coefficients of the related equations that in general can depend on time. Let the magnitude $Y(t)$ increase or descrease at the moment t^*. How long time is needed for restoring the given state of y? Mathematically this is the following

Problem 1 Let $Y(t)$, $t \in (t^*,\ t^{**})$, be a solutions of a differential equation $F(t, Y, Y', Y'') = 0$. Suppose $Y(t^*) = Q$ and t^{**} is the nearest next value $t = t^{**}$, $t^{**} > t^*$, where $Y(t^{**}) = Q$. Estimate $|t^{**} - t^*|$!

Theorem 8 *Suppose $Y(t)$, $t \in (t^*,\ t^{**})$, is a solution of a differential equation $F(t, Y, Y', Y'') = 0$ and $Y(t^*) = Y(t^{**}) = Q$, $Y(t) \neq 0$ for $t \in (t^*,\ t^{**})$. If the substitution $Y = y + Q$ reduces the equation $F(t, Y, Y', Y'') = 0$ to the equation* (4) *then*

$$|t^{**} - t^*| \geq \frac{\pi}{K}, \tag{60}$$

where K is defined in (8) *with substituting t_1 and t_2 by t^* and t^{**}.*

To obtain Theorem 8 we need just to repeat the proof of Theorem 1 making use of inequality (2) instead of inequality (3) of Lemma 1.

Taking into account (5)-(6′) and (8) we immediately obtain the following

Qualitative conclusion If the restoration happens very quickly, that is $|t^{**} - t^*|$ is "small" then either at least one of the coefficients $|a_j(t)|$, $\left|a_j'(t)\right|$ should be "large" or at least one of the coefficients $|a_0(t)|$, $|a_n(t)|$ should be "small".

Clearly similar results and conclusions will take place if we consider equations in the above Sections 3 to 6 instead of equation (4).

We can consider also the following more general question: how long will take the growth (or decay) of the magnitude Y from state $Y = Q^*$ till state $Y = Q^{**}$?

Problem 2 Let $Y(t)$, $t \in (t^*,\ t^{**})$, be a solutions of a differential equation $F(t, Y, Y', Y'') = 0$. Suppose $Y(t^*) = Q^*$ and t^{**} is the nearest value, $t^{**} > t^*$, where $Y(t^{**}) = Q^{**}$. Estimate $|t^{**} - t^*|$!

The case $Q^* = Q^{**}$ is subject of Theorem 8. For $Q^* \neq Q^{**}$ we have the following

Theorem 9 *Suppose $Y(t)$, $t \in (t^*,\ t^{**})$, is a solutions of a differential equation $F(t, Y, Y', Y'') = 0$; $Y(t^*) = Q^*$, $Y(t^{**}) = Q^{**}$. If the substitution $Y = y + Q$ reduces equation $F(t, Y, Y', Y'') = 0$ to the equation* (4) *then*

$$|t^{**} - t^*| \geq \frac{|Q^{**} - Q^*|}{M_1(t^*, t^{**})\,|\max\{Q^{**}, 1\}|^n}, \ \textit{when}\ Q^* < Q^{**}, \tag{61}$$

where $M_1(t^*, t^{**})$ is defined by (5), *that is*

$$M_1(t^*, t^{**}) : \sup_{t^* < t < t^{**}} \max_{0 \leq j \leq n} |a_j(t)| < \infty.$$

and

$$|t^{**} - t^*| \geq \frac{|Q^{**} - Q^*|}{M_1(t^*, t^{**})\,|\max\{Q^*, 1\}|^n} + \frac{N\pi}{K}, \ \textit{when}\ Q^* > Q^{**}, \tag{62}$$

where N is the number of points in $(t^*,\, t^{**})$, where Y attains the value Q^* and K is defined in (8) with substituting t_1 and t_2 by t^* and t^{**}.
Proof Consider two cases.
Case 1: $Q^* < Q^{**}$. Since y satisfies (4), we have

$$Q^{**} - Q^* = \int_{t^*}^{t^{**}} y' dt \leq \int_{t^*}^{t^{**}} \left|y'\right| dt = \int_{t^*}^{t^{**}} |P(t, y)|\, dt$$

$$\leq M_1(t^*, t^{**}) \left|\max\{Q^{**}, 1\}\right|^n \left|t^{**} - t^*\right|,$$

so that (61) follows.
Case 2: $Q^* > Q^{**}$. Suppose first that $Y(t) \leq Q^*$, for $t \in (t^*,\, t^{**})$. Then clearly

$$Q^{**} - Q^* = \int_{t^*}^{t^{**}} y' dt \leq \int_{t^*}^{t^{**}} \left|y'\right| dt = \int_{t^*}^{t^{**}} |P(t, y)|\, dt$$

$$\leq M_1(t^*, t^{**}) \left|\max\{Q^*, 1\}\right|^n \left|t^{**} - t^*\right|, \tag{63}$$

so that (62) follows. Now suppose that the inequality $Y(t) \leq Q^*$ is not valid for all t. Then it follows that there exist some values $t_1 = t^* < t_2 < ... < t_N$ such that $Y(t_i) = Q^*$ for $i = 1, 2, ..., N$ and $Y(t) \leq Q^*$ for $t \in (t_N,\, t^{**})$. According to inequality (63) we have

$$Q^{**} - Q^* \leq M_1(t_N, t^{**}) \left|\max\{Q^*, 1\}\right|^n \left|t^{**} - t_N\right|$$

$$\leq M_1(t^*, t^{**}) \left|\max\{Q^*, 1\}\right|^n \left|t^{**} - t_N\right|. \tag{64}$$

For any t_i and t_{i+1} according to Theorem 8

$$|t_{i+1} - t_i| \geq \frac{\pi}{K},$$

where K defined in (8) with substituting t_1 and t_2 by t_i and t_{i+1} so that

$$|t_N - t_1| \geq \frac{N\pi}{K} \tag{65}$$

follows. Theorem 9 follows now from (63) to (65).

Clearly similar results and conclusions will take place if we consider equations from the above Sections 3 to 6 instead of equation (4).

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COUNTING POINTS OF SEMI-ALGEBRAIC SUBSETS

T. Aliashvili
Georgian Technical University
Chair of Mathematics ♯99
77, Kostava str., Tbilisi, Georgia
aliashvili@hotmail.com

Abstract Effective computing of the number of points of a semi-algebraic subset given in explicit form is approached by a general algorithm.

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1. Introduction

The present paper is devoted to the problem of effective computing of the number of points of a semi-algebraic subset given in explicit form. This topic goes back to a problem of effective finding the number of roots of real polynomial in the left half-plane which was posed by J. Maxwell [12]. Under effective finding as usual is understood that one does not compute the roots themselves but indicates an algorithm which computes their number by means of a finite amount of algebraic and logical operations over the coefficients of given polynomials.

An analogous problem can be formulated for any polynomial endomorphism (vector field). The literature concerned with this topic is enormous. Some of the most fundamental results in this direction were described in [12]. Recently, new powerful methods were developed in the framework of the so-called signature formulae for topological invariants [10]. It turned out that the real roots number may be computed as the signature of a certain auxiliary quadratic form whose coefficients can be algebraically expressed through coefficients of given polynomials.

A natural generalization of Maxwell's problem is to find an algorithm for effective computing the number of points of an arbitrary finite semi-

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algebraic subset. The latter problem has not been investigated in full detail until today. The aim of the present paper is to describe a general algorithm for solving this problem and to estimate the computational complexity of the corresponding algorithm.

Our approach to this problem is based on the use of quadratic forms and multi-dimensional residues in the spirit of the aforementioned signature formulae for topological invariants. We first describe all necessary concepts and auxiliary results. We essentially use the same approach and concepts as in [1].

2. Counting the real roots rumber in the semi-algebraic subset on the plane

Let $\varphi = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathbb{C}$-proper polynomial endomorphism of type (n, m), that is $\deg f = n$, $\deg g = m$, and $C_\infty(f_\mathbb{C}) \cap C_\infty(g_\mathbb{C}) = \varnothing$. ($C_\infty(f_\mathbb{C})$ and $C_\infty(g_\mathbb{C})$ are tangent cones at infinity of complexifications of f and g) [1], [2].

From properness it follows that there exist exactly $N = n \cdot m$ common complex roots. Denote them by $p_i = (\alpha_j, \beta_j)$, $j = 0, \ldots, N-1$.

For the main construction we must be guaranteed that all these roots are simple and even have pair-wise different ordinates, say, $\beta_i \neq \beta_j$ for any $i \neq j$. Such mapping will be called *y-precise*. It was shown that it is possible to assume that everywhere without loosing generality (see Propositions 1.1 and 1.2 and Lemma 1.2 from [1]).

We also introduced the so-called "counting" quadratic form

$$Q^\varphi_\chi(\boldsymbol{\xi}) = \sum_{j=0}^{N-1} \chi(\alpha_j, \beta_j)(\xi_0 + \xi_1\beta_j + \xi_2\beta_j^2 + \ldots \xi_{N-1}\beta_j^{N-1})^2 \qquad (1)$$

defined on an auxiliary N-dimensional Euclidean space $\mathbb{R}^N$ and depending on arbitrary real rational function (or polynomial) $\chi \in \mathbb{R}_2$, where $\mathbb{R}_2 = \mathbb{R}[x, y]$ is the ring of real polynomials in two variables. If $\chi \equiv 1$ we write $Q^\varphi_1 \equiv Q^\varphi$ and call it the principal "counting" form [11].

An important statement in [1].

Theorem 1 *If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an y-precise, $\mathbb{C}$-proper polynomial endomorphism, then quadratic form Q^φ is nondegenerate and its signature $s(Q^\varphi)$ is equal to the real root number of endomorphism φ*

$$\#\, \varphi_\mathbb{R}^{-1}(0) = s(Q^\varphi). \qquad (2)$$

One can also formulated an effective criterion of y-preciseness.

Proposition 1 *If*

$$R\big(R_x(f, g)(y), R_x(f, g)'(y)\big) \neq 0, \qquad (3)$$

then map $\varphi = (f, g)$ is y-precise.

Here by $R_y(f, g) = R_y(f, g)(x)$ is denoted the resultant of two elements of ring $\mathbb{K}[x][y]$. This is polynomial of x.

Relying on the results [1], [2] we can investigate more complicated case, when we have inequalities apart from equations.

So, at the moment suppose we are given a polynomial in two variables $h \in \mathbb{R}$, such that the triple (f, g, h) has no common roots: $Z_{\mathbb{C}}(f, g, h) = \varnothing$.

Analyzing the proof of Theorem 1 it is easy to notice that introducing a coefficient $h(\alpha_j, \beta_j)$ enables us to take into account also the roots laying in the set $\{h > 0\}$. This enables us to get the main result giving a way to research zero-dimensional semi-algebraic subsets in $\mathbb{R}^2$.

Theorem 2 *If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is an y-precise, $\mathbb{C}$-proper polynomial endomorphism, $h \in \mathbb{R}_2$ and $Z_{\mathbb{C}}(f, g, h) = \varnothing$, then form Q^{φ}_h also non-degenerate and*

$$\#\, \varphi_{\mathbb{R}}^{-1}(0) \cap \{h > 0\} = \frac{1}{2}\big(s(Q^{\varphi}) + s(Q^{\varphi}_h)\big). \tag{4}$$

(when $h \equiv 1$, we get Theorem 1.

Lemma 1 *One has*

$$\begin{aligned} &(x+iy)(a+ib)^2 + (x-iy)(a-ib)^2 \\ &= \begin{cases} 2x\Big(a - \dfrac{yb}{x}\Big)^2 - 2\dfrac{y^2+x^2}{x}\, b^2, & \text{if} \quad x \neq 0, \\ -y(a+b)^2 + y(a-b)^2, & \text{if} \quad x = 0. \end{cases} \end{aligned} \tag{5}$$

Proof of Theorem 2 We write the quadratic form $Q^{\varphi}_h(\boldsymbol{\xi})$ in a standard form

$$Q^{\varphi}_h(\boldsymbol{\xi}) = \sum_{j=0}^{N-1} h(\alpha_j, \beta_j)(\xi_0 + \xi_1\beta_j + \xi_2\beta_j + \dots \xi_{N-1}\beta_j^{N-1})^2. \tag{6}$$

Since (α_j, β_j) are common real roots of f and g, and $Z_{\mathbb{C}}(f, g, h) = \varnothing$, then $h(\alpha_j, \beta_j) \neq 0$ for all $j = 0, \dots, N-1$. So some points of $\varphi_{\mathbb{R}}^{-1}(0)$ lie in the set $h(x, y) > 0$, and the rest there, where $h(x, y) < 0$.

For bringing to the normal form of (6), similarly as in the previous Theorem 4.2 [1] we have k real roots and l pair complex-conjugate root and we numerate them as in the mentioned theorem.

Now we will make the same change of variables and (6) will take the form

$$Q_h^{\varphi}(\eta) = \sum_{j=0}^{k-1} h(\alpha_j, \beta_j)\eta_j^2 + \sum_{j=1}^{l} \big[h(\alpha_{k+j-1}, \beta_{k+j-1})\eta_{k+j-1}^2$$
$$-h(\alpha_{k+l+j-1}, \beta_{k+l+j-1})\eta_{k+l+j-1}^2\big], \qquad (7)$$

where $\eta_j, h(\alpha_j, \beta_j) \in \mathbb{R}$, for $1 \le j \le l$. $\overline{\eta_{k+j-1}} = \beta_{k+k+j-1}$ and $h(\overline{\alpha_{k+j-1}}, \overline{\beta_{k+j-1}}) = h(\alpha_{k+l+j-1}, \beta_{k+l+j-1})$, for $1 \le j \le l$.

It is obvious, that this expression gives nondegenerate quadratic form, as $h(\alpha_j, \beta_j) \ne 0$ for all $0 \le j \le N-1$, hence the form is nondegenerate. Moreover, its determinant, in future Vandermonde determinant is easily calculated and equals to $\prod_{i \ne j}^{N} (\beta_i - \beta_j) \ne 0$.

So, Lemma 1 enables us to bring the second sum to the form, when the number of positive and negative squares are equal, i.e., they have no influence on signature of the form Q_h^{φ}. Hence

$$s(Q_h^{\varphi}) = \#\big[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h > 0\}\big] - \#\big[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h < 0\}\big].$$

On the other hand,

$$s(Q^{\varphi}) = \#\big[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h > 0\}\big] - \#\big[\varphi_{\mathbb{R}}^{-1}(0) \cap \{h < 0\}\big].$$

Summing these equalities, the formula (4) is obtained easily. The theorem is proved.

In the following theorem we estimate the real roots number of polynomial endomorphism in an arbitrary open semi-algebraic set, roots on the boundary excluded. It may be done with the help of formulae of the inclusion and exclusion, which we present in some detail for two and tree conditions of inequality type.

Theorem 3 *Let φ be of the same type as above and $h_1, h_2 \in \mathbb{R}_2$ is such that $Z_{\mathbb{C}}(f, g, h) = \varnothing$, $i = 1, 2$. Then*

$$\#\big(Z_{\mathbb{R}}(f, g) \cap \{h_1 > 0, h_2 > 0\}\big) = \frac{s(Q^{\varphi}) + s(Q_{h_1}^{\varphi}) + s(Q_{h_2}^{\varphi}) + s(Q_{h_1 h_2}^{\varphi})}{4}, \qquad (8)$$

where $s(Q^{\varphi})$, $s(Q_{h_1}^{\varphi})$ $s(Q_{h_2}^{\varphi})$ and $s(Q_{h_1 h_2}^{\varphi})$ are the signatures of the form Q^{φ}, $Q_{h_1}^{\varphi}$, $Q_{h_2}^{\varphi}$ and $Q_{h_1}^{\varphi}$ respectively. $\big(Z_{\mathbb{R}}(f, g)$ is the same as $\varphi_{\mathbb{R}}^{-1}(0)\big)$.

Proof Let us compose the following quadratic forms Q^{φ}, $Q_{h_1}^{\varphi}$, $Q_{h_2}^{\varphi}$ and $Q_{h_1}^{\varphi}$. Respectively, their signatures are s, s_1, s_2, s_{12} and m, n, p, q denote the number of real roots in the mentioned area. Using Theorem 1 and 2 the system

$$\begin{cases} m+n+p+q=s, \\ m+n=\dfrac{1}{2}(s+s_1), \\ n+q=\dfrac{1}{2}(s+s_2), \\ n+p=\dfrac{1}{2}(s+s_{12}), \end{cases}$$

can be easily deduced, from where

$$n=\frac{s+s_1+s_2+s_{12}}{4}$$

follows, which in fact gives formula (8). □

Remark 1 Particularly, if $h=J(f,g)$ is the Jacobian of φ then it is easily seen that $Z_{\mathbb{R}}(f,g,h)$ coincides with the set of multiple roots of $\varphi_{\mathbb{C}}$. Thus, our formulae enable us to calculate multiple root number for an arbitrary polynomial endomorphism.

Now we consider the case when there are three conditions of the inequality type $h_1,h_2,h_3\in\mathbb{R}_2$. We are interested in

$$\#\big(Z_{\mathbb{R}}(f,g)\cap\{h_1>0,\ h_2>0,\ h_3>0\}\big).$$

In the general case $\{h_1=0\}$, $\{h_2=0\}$ and $\{h_3=0\}$ curves cut the plane into eight pieces. Let $x_1,x_2,\dots,x_8$ be the real root number in the corresponding domains, i.e., x_1 is the number of the roots which are in the domain $\{h_1,h_2,h_3>0\}$, $(+++)$, x_2 - $\{h_1>0,\ h_2>0$ and $h_3<0\}$, $++-$, x_3 - $(+-+)$, x_4 - $(-++)$, x_5 - $(+--)$, x_6 - $(-+-)$, x_7 - $(--+)$, x_8 - $(---)$.

Let s,s_i,s_{ij} $(i,j=1,2,3;i\neq j)$ and s_{123} denote the signatures of the quadratic forms Q^{φ}, $Q^{\varphi}_{h_i}$, $Q^{\varphi}_{h_ih_j}$ and $Q^{\varphi}_{h_1h_2h_3}$ respectively.

Similarly, to the previous theorem we can construct the system of linear equalities. Let us construct all positive equalities so: we add the number of points from those domains by turn, where h_i or their multiplication are positive respectively. For example 6-th equation from following system is constructed like this, summing those numbers, for which on the first and the third position are "+" or product gives "+"

on same places

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & = c_1, \\ x_1 & +x_2 & +x_3 & & +x_5 & & & & = c_2, \\ x_1 & +x_2 & & +x_4 & & +x_6 & & & = c_3, \\ x_1 & & +x_3 & +x_4 & & & +x_7 & & = c_4, \\ x_1 & +x_2 & & & & & +x_7 & +x_8 & = c_5, \\ x_1 & & +x_3 & & & +x_6 & & +x_8 & = c_6, \\ x_1 & & & +x_4 & +x_5 & & & +x_8 & = c_7, \\ x_1 & & & & +x_5 & +x_6 & +x_7 & & = c_8, \end{cases} \tag{9}$$

where $c_1, c_2, \dots, c_8$ are the numbers, that can be calculated by signatures $c_1 = s$, $c_2 = \frac{1}{2}(s + s_1)$, $c_3 = \frac{1}{2}(s + s_2)$, $c_4 = \frac{1}{2}(s + s_3)$, $c_5 = \frac{1}{2}(s + s_{12})$, $c_6 = \frac{1}{2}(s + s_{13})$, $c_7 = \frac{1}{2}(s + s_{23})$, $c_8 = \frac{1}{2}(s + s_{123})$ (see Theorems 2 and 3).

The determinant of this system is $D = 32 \neq 0$, so the unknown values are

$$\begin{aligned} x_1 &= (-3c_1 + c_2 + c_3 + \cdots + c_8)/4, \\ x_2 &= (c_1 + c_2 + c_3 - c_4 + c_5 - c_6 - c_7 - c_8)/4, \\ x_3 &= (c_1 + c_2 - c_3 + c_4 - c_5 + c_6 - c_7 - c_8)/4, \\ x_4 &= (c_1 - c_2 + c_3 + c_4 - c_5 - c_6 + c_7 - c_8)/4, \\ x_5 &= (c_1 + c_2 - c_3 - c_4 - c_5 - c_6 + c_7 + c_8)/4, \\ x_6 &= (c_1 - c_2 + c_3 - c_4 - c_5 + c_6 - c_7 + c_8)/4, \\ x_7 &= (c_1 - c_2 - c_3 + c_4 + c_5 - c_6 - c_7 + c_8)/4, \\ x_8 &= (c_1 - c_2 - c_3 - c_4 + c_5 + c_6 + c_7 - c_8)/4. \end{aligned}$$

The value of x_1 can be expressed exactly by

$$\begin{gathered} \#\big(Z_{\mathbb{R}}(f, g) \cap \{h_1 > 0,\ h_2 > 0,\ h_3 > 0\}\big) \\ = x_1 = \frac{s + s_1 + s_2 + s_3 + s_{12} + s_{13} + s_{23} + s_{123}}{8}. \end{gathered}$$

Generally, the number of points can be calculated for arbitrary combinations h_1, h_2, h_3. For example

$$x_5 = \frac{s + s_1 - s_2 - s_3 - s_{12} - s_{13} + s_{23} + s_{123}}{8}.$$

On the other hand, as we see, this expression can be easily constructed from 5-th equation. It is obvious that we can do the same thing for any inequalities too.

Remark 2 a) Geometrically we can understand that $D \neq 0$ without calculation (we must use coordinate notation of matrix and consider product of vector-lines).

b) The matrix of the system has special symmetry, which is preserved in general case too and enables us to use this method for any numbers of conditions of inequality type. It gives opportunity to solve general problem on the point number of any semi-algebraic sets into $\mathbb{R}^n$.

In [1], [2] was shown the possibility of finding coefficients of form (1). There we proved an important statement into several stages.

Proposition 2 *The coefficients of the form* (1) *are polynomials in the coefficients of the given polynomials.*

Using transformation (1) we obtained an expression which depends on the so-called mixed Newton sums $\sum\limits_{j=0}^{N-1} \alpha_j^p \beta_j^q$. There were considered several cases, the most important being: a) the pure Newton sums ($p = 0$ or $q = 0$) and b) the simplest mixed sums ($p = 1$ or $q = 1$).

At last we have

Consequence Those two cases are sufficient for counting the number of real roots in the plane.

3. Counting forms and residues

The results obtained above and in [1] have interesting applications in the theory of polynomial endomorphism. In the same work we discussed $\mathbb{C}$-proper polynomial endomorphisms $f : (\mathbb{R}^n, \infty) \to (\mathbb{R}^n, \infty)$ and defined an affine ring $A(f)$ for this. It is then useful to consider the Jacobian J_f. It is known that J_f does not lie in the ideal (f), and its class was denoted by $d = \mathrm{cl}_{A(f)}\, J_f$. Then in two different ways bilinear forms $B(\widehat{\varphi}, \widehat{\psi}) = \mathrm{Res}_f(\varphi\psi)$ and $B_L^\lambda(\varphi, \psi) = \lambda(\pi_L(\varphi{\cdot}\psi))$ were introduced. Respectively, we get quadratic forms $Q^G(f)$ and Q_L^λ,

$$(Q^G(f)(\widehat{\varphi}) = \mathrm{Res}_f(\varphi^2), \quad \widehat{\varphi} \in A_{\mathbb{R}}(f), \quad Q_L^\lambda(\varphi) = \lambda(\pi_L(\varphi^2)).$$

As it was said, all such quadratic forms defined by residue and projection, have the same signature. This on its side enables us to calculate the signature of the Gorenstein form using $Q_L^\lambda(\varphi)$. Finally, topological degree of mappings was defined and its main properties were formulated.

We mentioned that there exist tight relationships between degree of the map and the Gorenstein form. The main result is

Theorem 4 *We have* $\deg f = \operatorname{sign} Q^G(f)$.

Proof Introduce compexification as usual: $f_{\mathbb{C}} : (\mathbb{C}^n, \infty) \to (\mathbb{C}^n, \infty)$ and shoots $y \in \mathrm{Reg}\, f_{\mathbb{C}} \cap \mathbb{R}^n$, then $\deg f$ can be calculated using formula

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \mathrm{sgn}\det J_f(x). \tag{10}$$

First assume that all roots of $f_{\mathbb{C}}$ are simple. On the other hand from the theory of coherent analytic sheaves (Oka-Cartan theorem [6]) follows that

$$\begin{aligned} & A(f_{\mathbb{C}}) \overset{def}{=} \mathbb{C}_n[z]/(f_{\mathbb{C}}) \\ & \cong \bigoplus_{w_j \in f_{\mathbb{C}}^{-1}(0)} \mathbb{C}_n[[z-w_j]]/((f_{\mathbb{C}}-w_j)) \cong \bigoplus_{j=1}^{N} (\mathbb{C})_j \cong \mathbb{C}^N, \end{aligned} \tag{11}$$

where $N = \# f_{\mathbb{C}}^{-1}(0)$, $f_{\mathbb{C}}^{-1}(0) = \{w_1, \ldots, w_N\}$.

As it is easily seen, always $A_{\mathbb{C}}(f) \cong A(f_{\mathbb{C}})$, and we get $A_{\mathbb{C}}(f) \cong \mathbb{C}^N$ and $A(f) \cong \mathbb{R}^N$. Hence

$$\dim_{\mathbb{C}} A_{\mathbb{C}}(f) = \dim_{\mathbb{R}} A(f).$$

The isomorphism of spaces $A(f)$ and $\mathbb{R}^N$ may be established as follows: if w_j points are

$$\begin{gathered} \{x_1, \ldots, x_p, z_1, \ldots, z_q, \overline{z}_1, \ldots, \overline{z}_q\} = f_{\mathbb{C}}^{-1}(0), \\ p + 2q = N, \quad x_i \in \mathbb{R}^n, \quad i = 1, \ldots, p, \\ z_k, \overline{z}_k \in \mathbb{C}^n, \quad k = 1, \ldots, q. \end{gathered}$$

Take $a = \operatorname{cl} \varphi \in A(f_{\mathbb{C}})$ and assign to it $\{\varphi(w_j)\}_{j=1}^N$. It is obvious, that this correspondence is well-defined and gives isomorphism of $A(f_{\mathbb{C}})$ on the $\mathbb{C}^N$ according to the (11).

In order to get an analogous isomorphism on subspace $A(f) \subset A(f_{\mathbb{C}})$, we need to take the real part of the introduced correspondence.

In other words: now take $\varphi \in \mathbb{R}_n(z)$, $a = \operatorname{cl} \varphi \in A(f)$ and consider it as an element from $A(f_{\mathbb{C}})$, then instead of complex-conjugate roots in pair, take real and imaginary parts, i.e.,

$$\operatorname{cl} \varphi \longmapsto \big\{\varphi(x_1), \ldots, \varphi(x_p), (\operatorname{Re} \varphi(z_k), \operatorname{Im}(z_k))_{k=1}^q\big\}.$$

It is obvious that we get an isomorphism of spaces $A(f)$ and $\mathbb{R}^N$. This circumstance suggests how to get convenient coordinates on the $A(f)$ and then find the expression of the Gorenstein form $Q^G(f)$ in these coordinates.

In order to give bilinear form $B(\widehat{\varphi}, \widehat{\psi})$, by one of the properties of residue we can write that

$$\operatorname{Res}_f(\gamma) = \sum_{j=1}^{N} \frac{\gamma(w_j)}{J(w_j)}.$$

Assuming that $\gamma = \varphi\psi$ we get

$$B(\varphi, \psi) = \sum_{j=1}^{N} \frac{\varphi(w_j)\psi(w_j)}{J(w_j)}.$$

Hence, for complexification on $A(f_{\mathbb{C}})$ by the corresponding Gorenstein form we have

$$Q^G_{\mathbb{C}}(f)(\varphi) = \sum_{j=1}^{N} \frac{\varphi^2(w_j)}{J(w_j)}, \quad \text{for} \quad \varphi \in A(f_{\mathbb{C}}). \tag{12}$$

Take $\varphi \in A(f)$ and research the restriction of the form in the introduced coordinates. At first rewrite (12) as

$$Q^G_{\mathbb{C}}(f)(\varphi) = \sum_{i=1}^{p} \frac{\varphi^2(x_i)}{J(x_i)} + \sum_{k=1}^{q} \Big[\frac{\varphi^2(z_k)}{J(z_k)} + \frac{\varphi^2(\overline{z}_k)}{J(\overline{z}_k)}\Big],$$

and let $\varphi(z_k) = x + iy$, $J(z_k) = a + ib$.

The expression in square brackets will be

$$\frac{(x+iy)^2}{a+ib} + \frac{(x-iy)^2}{a-ib} = \frac{2(ax^2 - ay^2 + 2bxy)}{a^2+b^2}.$$

According to Lemma 1, we have

$$\frac{2a(x+\frac{by}{a})^2 - 2\cdot\frac{a^2+b^2}{a}\cdot y^2}{a^2+b^2}, \quad \text{if} \quad a \neq 0$$

and

$$\frac{b(x+y)^2 - b(x-y)^2}{b^2}, \quad \text{if} \quad a = 0.$$

Then by an appropriate change of coordinates

$$\widehat{x} = x + \frac{b}{a}y, \;\; \widehat{y} = y, \;\; \text{or} \;\; \widetilde{x} = x + y, \;\; \widetilde{y} = x - y$$

we get the sum of squares.

Thus

$$Q^G(f)(\varphi) = \sum_{i=1}^{p} \frac{\varphi^2(x_i)}{J(x_i)}$$

$$+ \sum_{k=1}^{q} \Big[\frac{2\operatorname{Re} J(z_k)(\operatorname{Re}\varphi(z_k) + \frac{\operatorname{Im} J(z_k)\cdot\operatorname{Im}\varphi(z_k)}{\operatorname{Re} J(z_k)})^2}{[\operatorname{Re} J(z_k)]^2 + [\operatorname{Im} J(z_k)]^2} - \frac{2[\operatorname{Im}\varphi(z_k)]^2}{\operatorname{Re} J(z_k)}\Big],$$

if $\operatorname{Re} J(z_k) \neq 0$ and

$$Q^G(f)(\varphi) = \sum_{i=1}^{p} \frac{\varphi^2(x_i)}{J(x_i)} +$$

$$+ \sum_{k=1}^{q} \frac{\operatorname{Im} J(z_k)[\operatorname{Re}\varphi(z_k) + \operatorname{Im}\varphi(z_k)]^2 - \operatorname{Im} J(z_k)[\operatorname{Re}\varphi(z_k) - \operatorname{Im}\varphi(z_k)]^2}{[\operatorname{Im} J(z_k)]^2},$$

if $\operatorname{Re} J(z_k) = 0$.

It is obvious, that the second sum has no influence on signature, from what it immediately follows that

$$\operatorname{sig} Q^G(f) = \sum_{w_j \in f^{-1}(0)} \operatorname{sgn} J(w_j)$$

and in this way the theorem is proved when all the roots w_j, $1 \le j \le N$, are simple.

In case when there are multiple roots it is possible to use deformation f_t with simple roots. The degrees are the same for all t, $\deg f_1 = \deg f_0$, and signature $\operatorname{sig} Q^G(f)$ is constant according to the non-degeneracy from $Q^G(f_0)$. Thus taking a limit gives the required equality.

The way of practical calculating of the local degree by this method is described in full detail in [10]. In the global case an algorithm for finding a basis in the factor-algebra can be based on the results of A. Zich [15]. More exactly, here must be used the so-called Hefer decomposition of the components of the endomorphism f

$$f_j(\xi) - f_j(z) = \sum_{k=1}^{n} (\xi_k - z_k) p_{jk}(\xi, z), \quad j = 1, \ldots, n. \tag{13}$$

Here p_{jk} are polynomials which we get by expanding $f_j(\xi) - f_j(z)$ in powers of $(\xi_k - z_k)$, assuming z is constant. It is clear that it can be done by algebraic operators, as for that it is enough to know the coefficients of the Taylor's series of the function f_j at the point z.

Construct the determinant $H(\xi, z)$ of the matrix $\|p_{jk}(\xi, z)\|$ and divide the variables in every monomial, i.e., we write

$$H(\xi, z) = \sum_{k=1}^{L} \chi_k(\xi) \psi_k(z). \tag{14}$$

It turns out that the number L is uniquely defined and equals to the dimension of the factor-algebra $\dim A(f)$, and the functions $\{\chi_k\}$, $\{\psi_k\}$ give the basis in $A(f)$. All this follows from the results of Zich according to the preparation Malgrange theorem [3].

Proposition 3 *If polynomial map $f = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ has a discrete zero's set, then the polynomial h belongs to the ideal $I(f)$ if and only if $\operatorname{Res}_f(h, \psi_k) = 0$, for every k from 1 to L, where ψ_k are the polynomials from the decomposition* (14). *If the map $w = f(z)$ is proper, then the family of polynomials $\{\psi_k\}_{k=1}^{L}$ generates a ring $\mathbb{C}_n[z]$, considered as a module over the ring $\mathbb{C}_n[z]$, i.e., every polynomial $g \in$*

$\mathbb{C}_n[z]$ *admits the decomposition*

$$g(z) = c_1(w)\psi_1(z) + \cdots + c_L(w)\psi_L(z), \quad w = f(z). \tag{15}$$

We follow the argument from [15]. At first remark that by the Malgrange preparation theorem the decomposition (16) is equivalent to the basis $\mathbb{C}[z](f)$.

Let now $\{U_\alpha\}_{\alpha\in\mathbb{Z}}$ be disjunctive neighborhoods of the points $\alpha = z_j \in Z = f^{-1}(0)$ and r is so small that the polyhedra $\prod_\alpha = \prod_\alpha^r(f) = \{z : \|f(z) - f(\alpha)\| < r\}$ is relatively compact in U_α. Denote by $\prod^r = \bigcup\limits_{\alpha\in\mathbb{Z}} \prod_\alpha^r$ and let Γ^r be the skeleton of the polyhedron $\prod^r$ (i.e., union of skeletons of the polyhedra $\prod_\alpha^r$). Putting (13) in the usual Cauchy formula and according to the Weyl formula for each $g \in \mathbb{C}[z]$ we have

$$g(z) = (2\pi i)^{-1} \int_{\Gamma^r} \frac{g(\zeta)H(\zeta, z)}{f(\zeta) - f(z)} d\zeta, \quad z \in \prod\nolimits^r.$$

From this and (15) we obtain

$$g(z) = \sum_{k=1}^{L} c_k(f(z))\psi_k(z), \quad z \in \prod\nolimits^r, \tag{16}$$

where $c_k(w)$ are holomorphic functions in

$$B_r = \{w : |w_j| < r, \quad j = 1, \ldots, n\},$$

defined by integrals

$$g_k(w) = (2\pi i)^{-n} \int_{\Gamma^r} \frac{g(\zeta)\chi_k(\zeta)}{f(\zeta) - w} d\zeta, \quad w \in B_r. \tag{17}$$

Let $c_r(w) = \sum\limits_\alpha c_{k\alpha} w^\alpha$. As $f^\alpha \in I(f)$ for all $\alpha \neq 0$, from (16) properness of residues we have

$$\operatorname{Res}_f(hg) = \sum_{k=1}^{L} c_{k_0} \operatorname{Res}_f(h\psi_k).$$

The first part of our statement follows by the global duality theorem. Now, if the map $w = f(z)$ is proper then for any $w \in C_w^n$ the system $f - w$ has the same number of zeroes counted with multiplicities.

From the Rouché principle follows that for $w \in B_r$ all zeroes of this system lie into polyhedron

$$\Pi_{(0)}^r = \{z \in \mathbb{C}^n : |f_j(z) < r, \;\; j = 1, \ldots, n\},$$

where $r > 0$ is any real number.

Consequently, the integral (17) expresses the total sum of residues for the polynomial $g\chi_k$ with respect to map $f - w$. From this again by [15] we conclude, that the $c_k(w)$ are rational functions. Moreover, these functions are entire and the cycle Γ^r in (17) is homologous to any Γ^r, $R > r$, cycle in area, where the integrand is regular (from this follows holomorphic continuity of $c_k(w)$ from B_r to B_k). Thus, the polynomials c_k and foreseeing (16) we get that $\mathbb{C}[z]$ is a $\mathbb{C}[w]$-module with the system of generators $\{\psi_k(z)\}_{k=\overline{1,L}}$, q.e.d. □

Thus, the polynomials $\{\psi_k\}$ generate the ring $\mathbb{C}_n[w]$ introducing a module structure by f^*. So, the Malgrange theorem [3] shows that $e_i = \mathrm{cl}\,\psi_i \in A(f) = \mathbb{C}_n[z]/(f)$, $1 \le i \le L$, give a basis over the field $\mathbb{C}$. From this it follows that $L = \dim_{\mathbb{C}} A(\varphi) = N$ is the number of all roots, which in our case equals to $\prod_{i=1}^{N} \deg f_i)$ by the Bezout theorem .

So we described all necessary steps for counting the topological degree $\deg f$. It is obvious that this algorithm is easy to realize by computer.

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BEHAVIOUR AT INFINITY OF POLYNOMIALS OF TWO VARIABLES

H.G. Ghazaryan
Yerevan State University,
Alek Manukian 1 Yerevan 25, Armenia
Gaghakob@sun.ysu.am

V.N. Margaryan
Yerevan State University,
Alek Manukian 1 Yerevan 25, Armenia
Gaghakob@sun.ysu.am

Abstract In the present paper we study the behaviour at infinity of polynomials of two variables. Necessary and sufficient conditions are given for the polynomial tending to infinity when the variables tend to infinity.

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1. Formulation of the problem and preliminary facts

It is well known, that certain properties of general linear differential operators are determined corresponding to the behaviour of characteristic polynomials at infinity. Among this kind of properties hypoellipticity, almost hypoellipticity, simplicity by characteristic (see [1]), availability of coercive estimates (see [2]–[5]), comparison of strength or powers of operators (see [1] and [10]–[11]) may be mentioned.

The polynomials of one variable, as well as the elliptic and hypoelliptic polynomials of many variables increase to infinity together with increasing of arguments: $|P(\xi)| = |P(\xi_1, \dots, \xi_n)| \to \infty$ as $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2} \to \infty$. At the same time hyperbolic (by Petrovski or by Garding) polynomials and almost hypoelliptic polynomials (see [2] and [12]) fail to enjoy this property: they may remain bounded as their arguments increase. For these kind of polynomials lower order terms can play a decisive role

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for their behaviour at infinity (for example see [14]). Motivated from this in his work [6] V.P.Mikhailov introduced the set of complete non-degenerating polynomials, the behaviour of which at infinity does not depend on lower order terms. Then similar results, but in different form, have been obtained by many authors (see for example [7] and [13]).

Let N be the set of natural numbers and $N_0 = N \cup \{0\}$. Let R^n denote the n-dimensional real Euclidian space of points (vectors) $\xi = (\xi_1, \dots, \xi_n)$. We denote by N_0^n the set of n-dimensional multy-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in N_0$ $(j = 1, 2, \dots, n)$. For each $\xi \in R^n$, $\alpha \in N_0^n$ we write $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{;\alpha_n}$, $|\xi| = \sqrt{\xi_1^2 + \dots + \xi_n^2}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

If $P(\xi) = \sum \gamma_\alpha \xi^\alpha$ is a polynomial (that is a sum with a finite number of nonzero real coefficients $\{\gamma_\alpha\}$), then it can be represented in the form of the sum of homogeneous polynomials

$$P(\xi) = \sum_{j=0}^{M} P_j(\xi) = \sum_{j=0}^{M} \sum_{|\alpha|=d_j} \gamma_\alpha \xi^\alpha, \tag{1}$$

where $d_0 > d_1 > \dots > d_M \geqslant 0$.

Our objective is the description of the set $\{P\}$ of all polynomials with the property

$$|P(\xi)| \to \infty \quad \text{as} \quad |\xi| \to \infty. \tag{2}$$

In the note [14] criteria when (2) holds was obtained for the polynomials of two variables $(n = 2)$ in the *"three layered"* case, i.e. when $M = 3$ in the representation (1).

This paper, being sequel to the note [14], is devoted to obtaining such criteria for the polynomials of two variables in the *"multy layered"* case, i.e. when $M \in N$ is arbitrary.

It is obvious that if (2) holds (for the polynomials with real coefficients) then $P(\xi^s) \to +\infty$ or $P(\xi^s) \to -\infty$ for all sequences $\{\xi^s\}$ such that $|\xi^s| \to \infty$ as $s \to \infty$. Moreover, either $P(\xi) \geqslant 0$ or $P(\xi) \leqslant 0$ for all sufficiently large $|\xi|$. To proceed from this we denote by I_2 the set of polynomials $P(\xi) = P(\xi_1, \xi_2)$ of two variables with real coefficients such that $P(\xi) \geqslant 0$ for all $\xi \in R^2$ and

$$P(\xi) \to +\infty \quad \text{as} \quad |\xi| \to \infty. \tag{3}$$

Thus, in this paper we obtain criteria for $P \in I_2$ (or $(-P) \in I_2$). Our method does not work in the case $n > 2$.

Let $P(\xi) = P(\xi_1, \xi_2)$ be a polynomial represented in the form (1). We denote

$$\Sigma(P_0) = \big\{\eta;\ \eta \in R^2, |\eta| = 1, P_0(\eta) = 0\big\}$$

and

$$\Sigma(P_j) = \{\eta;\ \eta \in \Sigma(P_{j-1}), P_j(\eta) = 0\}\,,\ (j = 1, 2, \ldots, M).$$

For $\eta \in \Sigma(P_0)$ let $\ell_i = \ell_i(\eta)$ be the order of zero η of the polynomial P_i $(0 \leqslant i \leqslant M)$ and let

$$\chi(\eta, \delta) = \chi(P, \eta, \delta) = \max_{0 \leqslant i \leqslant M} \{d_i - \ell_i(\eta) \cdot \delta\}, \quad \delta \geqslant 0.$$

In addition we denote by $A(\eta) = A(P, \eta)$ the set of numbers $\delta \geqslant 0$ for which there exist indices $i,\ j$ $\ (i \neq j, 0 \leqslant i, j \leqslant M)$ such that

$$d_i - \ell_i(\eta)\delta = d_j - \ell_j(\eta)\delta = \chi(\eta, \delta).$$

Finally for a pair (η, δ): $(\eta \in \Sigma(P_0), \delta \in A(\eta))$ we write

$$J_0(\eta, \delta) = J_0(P, \eta, \delta) = \{i;\ 0 \leqslant i \leqslant M, d_i - \ell_i(\eta)\delta = \chi(P, \eta, \delta)\}\,,$$

$c(\eta, \delta) = c(P, \eta, \delta) = card J_0(\eta, \delta)$.

Remark 1 It is easy to see that

a) if $A(\eta) \neq \emptyset$ and $\delta \in A(\eta)$, then δ is a rational number,

b) for $P \in I_2$, $\delta \geqslant 0$, $\chi(P, \eta, \delta) > 0$ $\quad$ (see [14]),

c) if $J_0(\eta, \delta) = \{i_1, i_2, \ldots, i_c\}$ and $0 \leqslant i_1 < i_2 < \cdots < i_c \leqslant M$ then

$$\ell_{i_1}(\eta) > \ell_{i_2} > \cdots < \ell_{i_c}(\eta).$$

Remark 2 The stated problem was studied in (see [14]) in the case when $A(\eta) = \emptyset$ for all $\eta \in \Sigma(P_0)$. So we shall assume that $A(\eta^0) \neq \emptyset$, hence $c(\eta^0, \delta_0) \geqslant 2$ for a pair (η^0, δ_0): $\eta^0 \in \Sigma(P_0)$, $\delta_0 \in A(\eta^0)$.

Remark 3 Let $R(\xi) = R(\xi_1, \xi_2)$ be a homogeneous polynomial with real coefficients and with degree $d > 0$, $\eta \in R^2$, $|\eta| = 1$, $R(\eta) = 0$ and $\ell = \ell(\eta)$ be the order of η. Then we collate to η a unit vector $\varkappa = \varkappa(\eta) \in R^2$ such that $(\varkappa, \eta) = 0$. Then it is easy to see that

$$D_{\varkappa}^{\ell} R(\eta) = (\ell!) \sum_{|\alpha| = \ell} \varkappa^{\alpha} \frac{D^{\alpha} R(\eta)}{\alpha!} \neq 0, \tag{4}$$

where $D_{\varkappa}$ is the $\varkappa$-derivative.

Lemma 1 *Let R, η, d and $\ell = \ell(\eta)$ be as above. Then there exists a neighborhood $U(\eta)$ such that*

$$R(\xi) \geqslant 0 \quad , \quad \xi \in U(\eta), \tag{5}$$

if and only if ℓ is even and $D^\ell_\varkappa R(\eta) > 0$.

Remark 4 It is obvious that the condition (5) to hold for all $\eta \in \Sigma(R) = \left\{\xi \in R^2, |\xi| = 1, R(\xi) = 0\right\}$ is equivalent to the condition: d is even and $R(\xi) \geqslant 0$ for all $\xi \in R^2$. Therefore if $\ell(\eta) = d$ for a $\eta \in \Sigma(R)$ then the set $\Sigma(R)$ consists of two points (η_1, η_2) and $(-\eta_1, -\eta_2)$ and $\ell(\eta) = \ell(-\eta)$ is even.

Proof of Lemma 1 Let for $\eta \in \Sigma(R)$ there exists a neighborhood $U(\eta)$ satisfying (5) and $\ell(\eta) < d$. Let $x \in R^1$, $|x| = 1$, $s \in N$. Since $D^\ell_\varkappa R(\eta) \neq 0$ (see remark 0.3) and $\eta^s = \eta + \frac{x}{s}\varkappa \in U(\eta)$ for sufficiently large s, then from Taylor's formula we obtain (as $s \to \infty$)

$$0 \leqslant R(s\eta^s) = s^d R(\eta^s) = \sum_{j=\ell}^{d} \frac{x^j}{j!} D^j_\varkappa R(\eta) s^{d-j} = x^\ell s^{d-\ell} \left[\frac{D^\ell_\varkappa R(\eta)}{\ell!} + o(1)\right].$$

Taking $x = 1$ we obtain from this that $D^\ell_\varkappa R(\eta) > 0$. Then taking $x = -1$ we obtain that ℓ is even.

Conversely, let $\eta \in \Sigma(R)$, $D^\ell_\varkappa R(\eta) > 0$, $\ell = \ell(\eta)$ is even. We prove that there exists a number $\varepsilon > 0$ such that $P(\xi) \geqslant 0$ in the ε-neighborhood $U(\eta, \varepsilon)$ of η.

Let $\varepsilon > 0$, $|\xi - \eta| \leqslant \varepsilon$ and the vector $\xi - \eta$ be represented by the (orthonormal) basis $\{\eta, \varkappa\}$ in the form: $\xi - \eta = \theta_1\eta + \theta_2\varkappa$. Then $\xi = (1 + \theta_1)\eta + \theta_2\varkappa$ and $\theta_1^2 + \theta_2^2 \leqslant \varepsilon$. It is obvious that $1 + \theta_1 \to 1$, $\theta_2 \to 0$ as $\varepsilon \to 0$; hence we can choose $\varepsilon > 0$ such that $1 + \theta_1 > 0$, $|\theta_2| < 1 + \theta_1$. Then from Taylor's formula and from (4) we obtain

$$\begin{aligned} R(\xi) &= R\left[(1+\theta_1)\left(\eta + \frac{\theta_2}{1+\theta_1}\varkappa\right)\right] = (1+\theta_1)^d R\left(\eta + \frac{\theta_2}{1+\theta_1}\varkappa\right) \\ &= (1+\theta_1)^d \sum_{|\alpha| \geqslant \ell} \left(\frac{\theta_2}{1+\theta_1}\right)^{|\alpha|} \frac{\varkappa^\alpha}{\alpha!} D^\alpha R(\eta) \\ &= (1+\theta_1)^d \left[\left(\frac{\theta_2}{1+\theta_1}\right)^\ell D^\ell_\varkappa R(\eta) + \sum_{j=\ell+1}^{d} \left(\frac{\theta_2}{1+\theta_1}\right)^j D^j_\varkappa R(\eta)\right] \\ &= (1+\theta_1)^{d-\ell}\theta_2^\ell \left[D^\ell_\varkappa R(\eta) + \sum_{j=1}^{d-\ell} \left(\frac{\theta_2}{1+\theta_1}\right)^j D^{\ell+j}_\varkappa R(\eta)\right]. \end{aligned}$$

Let $\varepsilon > 0$ also be chosen such that for $\theta_1^2 + \theta_2^2 \leqslant \varepsilon$

$$\left|\sum_{j=1}^{d-\ell} \left(\frac{\theta_2}{1+\theta_1}\right)^j D^{\ell+j}_\varkappa R(\eta)\right| \leqslant \frac{1}{2} D^\ell_\varkappa R(\eta)$$

holds. Then since $D^{\ell}_{\varkappa}R(\eta) > 0$ and ℓ is even, we obtain from this

$$R(\xi) \geqslant \frac{1}{2}(1+\theta_1)^{d-\ell}\theta_2^{\ell}D^{\ell}_{\varkappa}R(\eta) \geqslant 0 \ , \quad \xi \in U(\eta, \varepsilon),$$

which proves the lemma.

Lemma 2 *Let $P \in I_2$. Then*

1) $P_0(\xi) \geqslant 0$, $\xi \in R^2$,

2) *for each $\eta \in \Sigma(P_0)$ there exists such a number $k = k(\eta)$ ($0 < k \leqslant M$) that $d_k > 0$ and $P_0(\eta) = P_1(\eta) = \cdots = P_{k-1}(\eta) = 0$, $P_k(\eta) > 0$,*

3) *let for $\eta \in \Sigma(P_0)$, $\delta \in A(\eta)$ $J_0(\eta,\delta) = \{i_1, i_2, \ldots, i_c\}$, $0 \leqslant i_1 < i_2 < \cdots < i_c \leqslant M$, then ℓ_{i_1}, ℓ_{i_c} are even and $D^{\ell_{i_1}}_{\varkappa}P_{i_1}(\eta) > 0$, $D^{\ell_{i_c}}_{\varkappa}P_{i_c}(\eta) > 0$.*

Proof Assertions 1) and 2) were proved in [12]. So we prove assertion 3). At first we prove it for P_{i_1}. Denote by

$$q^i_j = q^i_j(\eta) = \frac{1}{(\ell_j(\eta)+i)!}D^{\ell_j(\eta)+i}_{\varkappa}P_j(\eta) \ , \quad j \in J_0(\eta,\delta), \quad i = 0, 1, \ldots \ .$$

Assume that, on the contrary, one of the following assumptions is true

1) ℓ_{i_1} is odd, $q^0_{i_1} > 0$; 2) ℓ_{i_1} is odd, $q^0_{i_1} < 0$; 3) ℓ_{i_1} is even, $q^0_{i_1} < 0$. We remark that $q^0_{i_1} \neq 0$ by definition of the set $J_0(\eta,\delta)$ and by (4).

Let $x < 0$ in the case 1) and $x > 0$ in the cases 2)-3). Then $x^{\ell_{i_1}}q^0_{i_1} < 0$ in all cases 1)-3). Consider the polynomial P_{i_1} on the sequence $\xi^s = s\left(\eta + x\ s^{-\delta}\varkappa\right)$, $s \in N$. From Taylor's formula we have

$$P_{i_1}(\xi^s) = s^{d_{i_1}} \sum_{j=\ell_{i_1}}^{d_{i_1}} q^{\ell_{i_1}-j}_{i_1}(xts^{-\delta})^j$$

$$s^{\chi(\eta,\delta)}q^0_{i_1}x^{\ell_{i_1}}\left[1 + \sum_{j=\ell_{i_1}+1}^{d_{i_1}} \frac{q^{\ell_{i_1}-j}_{i_1}}{q^0_{i_1}}x^{j-l_{i_1}}s^{-\delta(j-\ell_{i_1})}\right].$$

This implies that for all three cases

$$P_{i_1}(\xi^s) = s^{\chi(\eta,\delta)}q^0_{i_1}x^{\ell_{i_1}}\left(1+o(1)\right) \to -\infty \quad \text{as} \quad s \to \infty. \tag{6}$$

For the polynomials P_{i_j} when $j > 1$ we have for s large enough

$$P_{i_j}(\xi^s) = s^{\chi(\eta,\delta)}q^0_{i_j}x^{\ell_{i_j}}\left(1+o(1)\right). \tag{7}$$

Since $\ell_{i_1} > \ell_{i_j}$ $(j = 2, \ldots, c)$, we can choose a number $x_0 \neq 0$ such that

$$\left| \sum_{j=2}^{c} q_{i_j}^0 x_0^{\ell_{i_j}} \right| \leqslant \frac{1}{2} \left| q_{i_1}^0 x^{\ell_{i_1}} \right|,$$

then from (6)-(7) when $x = x_0$ we have

$$\sum_{i \in J_0(\eta,\delta)} P_i(\xi^s) \leqslant s^{\chi(\eta,\delta)} \frac{q_{i_1}^0}{2} x^{\ell_{i_1}} (1 + o(1)) \to -\infty \quad \text{as} \quad s \to \infty. \tag{8}$$

For the polynomials P_i, when $i \notin J_0(\eta, \delta)$, $x = x_0$ and s is sufficiently large, we have

$$\left| \sum_{i \notin J_0(\eta,\delta)} P_i(\xi^s) \right| \leqslant C s^b, \tag{9}$$

where $C = C(x_0) \geqslant 0$ and

$$b = b(\eta, \delta) = \max_{i \notin J_0(\eta,\delta)} \left[d_i - \ell_i(\eta)\delta \right]. \tag{10}$$

From the definition of the set $J_0(\eta, \delta)$ it follows that $b < \chi(\eta, \delta)$. Then from (8)-(10) we deduce that $P(\xi^s) \to -\infty$ as $s \to \infty$. This contradicts the condition $P \in I_2$.

The same arguments bring us to a contradiction for the polynomial P_{i_c}, if we make use of representations (6), (8) (with replacing i_1 by i_c) and (9) taking $|x_0|$ sufficiently small.

Corollary 1 *Let $P \in I_2$, $\eta \in \Sigma(P_0)$ and $\delta \in A(\eta)$. Then it follows from Lemma 2 that there exists a (unique) number $k \geqslant 0$ such that $d_k > 0$, $d_k - \ell_k(\eta)\delta > d_j - \ell_j(\eta)\delta$ $(0 \leqslant j \leqslant M,\ j \neq k)$, $\ell_k = \ell_k(\eta)$ is even and* $D_{\varkappa}^{\ell_k} P_k(\eta) > 0$.

2. Reducing the problem to the study of functions of one variable

The following statement is basic in this section. It shows that the behaviour at infinity of the polynomial P is determined by its behaviour on the sequences of special type.

Theorem 1 *Let polynomial P satisfies the necessary conditions* $1) - 3)$ *of Lemma 2. Then $P \in I_2$ if and only if for each $\eta \in \Sigma(P_0)$, $\delta \in A(\eta)$,* $a \in (0, 1)$

$$\lim_{t \to \infty} \inf_{a \leqslant |x| \leqslant a^{-1}} P\left[t\left(\eta + t^{-\delta} x \varkappa \right) \right] = +\infty, \tag{11}$$

where the vector $\varkappa = \varkappa(\eta) \in R^2$ *is defined by* η *as above.*

Proof The *necessity* is obvious. To prove the sufficiency we suppose that while the conditions of the theorem are satisfied none the less $P \notin I_2$, i.e. there exist a sequence $\{\xi^s\}$ and a number $C \geqslant 0$ such that $|\xi^s| \to \infty$ as $s \to \infty$ and

$$|P(\xi^s)| \leqslant C; \qquad s \in N. \tag{12}$$

Let us set $\eta^s = \xi^s/|\xi^s|$ $(s \in N)$. By choosing a subsequence (this subsequence and all subsequences coming henceforth we denote by $\{\eta^s\}$) one may assume that $\eta^s \to \eta$ as $s \to \infty$. We prove that $\eta \in \Sigma(P_0)$. Suppose that, on the contrary, $\eta \notin \Sigma(P_0)$. Then for sufficiently large s we have

$$|P(\xi^s)| = \left| |\xi^s|^{d_0} P_0(\eta^s) + \sum_{j=1}^{M} |\xi^s|^{d_j} P_j(\eta^s) \right| \geqslant \frac{1}{2} |P_0(\eta)| |\xi^s|^{d_0}.$$

It means that $|P(\xi^s)| \to \infty$ as $s \to \infty$, which contradicts (12). Thus $\eta \in \Sigma(P_0)$.

We represent the vectors $\{\xi^s\}$ by (orthonormal) basis $\{\eta, \varkappa\}$: $\xi^s = \varphi_s \eta + \psi_s \varkappa$ $(s \in N)$. It is obvious that

$$\varphi_s \to \infty, \quad \frac{\psi_s}{\varphi_s} \to 0 \qquad \text{as} \quad s \to \infty. \tag{13}$$

Since one may take the vector $(-\varkappa)$ instead of $\varkappa = \varkappa(\eta)$, we can assume that $\psi_s \geqslant 0$, $\varphi_s > 1$ $(s \in N)$. If $\psi_s = 0$ for an infinite set of $\{s\}$ (assume that $\psi_s = 0$ for all $s \in N$), then $\xi^s = \varphi_s \eta$ and by the condition 2) of Lemma 2 there exists a number $k = k(\eta)$ such that $0 < k \leqslant M$, $d_k > 0$ and $P_0(\eta) = P_1(\eta) = \cdots = P_{k-1}(\eta) = 0$, $P_k(\eta) > 0$. Then

$$|P(\xi^s)| = |P(\varphi_s \eta)| = \left| \sum_{i \geqslant k} \varphi_s^{d_i} P_i(\eta) \right| \geqslant P_k(\eta) \varphi_s^{d_k} - \sum_{i > k} \varphi_s^{d_j} |P_i(\eta)|$$

$$= \varphi_s^{d_k} P_k(\eta) \, (1 + o(1)) \to \infty$$

as $s \to \infty$, which contradicts (12).

Thus, without loss of generality, we can assume that $\psi_s > 0$, $\varphi_s > 0$ $(s \in N)$. Let us set

$$\rho_s = 1 - \frac{\ln \psi_s}{\ln \varphi_s} \qquad (s \in N). \tag{14}$$

(13) implies that $\rho_s \geqslant 0$ for sufficiently large s, so we assume that $\rho_s \geqslant 0$ for all $s \in N$. We show that $\rho_s \leqslant \rho < \infty$ $(s \in N)$. Let, on the contrary

(for a subsequence) $\rho_s \to \infty$ as $s \to \infty$, and the number $k = k(\eta)$ be chosen according to Lemma 2. Then (see Remark 2) by Taylor's formula we have (for all i: $0 \leqslant i \leqslant k-1$, and for $s \in N$)

$$\begin{aligned} P_i(\xi^s) &= \varphi_s^{d_i} P_i\left(\eta + \frac{\psi_s}{\varphi_s}\varkappa\right) = \varphi_s^{d_i} \sum_{|\alpha| \geqslant \ell_i} \left[\frac{\varkappa^\alpha}{\alpha!} D^\alpha P_i(\eta)\right] \left(\frac{\psi_s}{\varphi_s}\right)^{|\alpha|} \\ &= \varphi_s^{d_i} \sum_{j=\ell_i}^{d_i} \frac{D_\varkappa^j P_i(\eta)}{j!} \left(\frac{\psi_s}{\varphi_s}\right)^j . \end{aligned}$$

This, together with (14), implies that for all $s \in N$, $i \in [0, k-1]$

$$P_i(\xi^s) = \varphi_s^{d_i - \ell_i \rho_s} \sum_{j=\ell_i}^{d_i} \frac{D_\varkappa^j P_i(\eta)}{j!} \varphi_s^{(\ell_i - j)\rho_s}. \tag{15}$$

Since $\rho_s \to \infty$, $\varphi_s \to \infty$ as $s \to \infty$ (see (13)) and $\ell_i = \ell_i(\eta) \geqslant 1$, $\ell_i - j \leqslant 0$ if $0 \leqslant i \leqslant k-1$, $j = \ell_i, \ell_i + 1, \ldots, d_i$ then it follows from (15) that

$$P_i(\xi^s) \to 0 \qquad (0 \leqslant i \leqslant k-1) \quad \text{as} \quad s \to \infty. \tag{16}$$

For the polynomial P_i $(k+1 \leqslant i \leqslant M)$ we have

$$P_i(\xi^s) = \varphi_s^{d_i} P_i(\eta)\,(1 + o(1)) \quad \text{as} \quad s \to \infty. \tag{17}$$

Therefore for a number $C \geqslant 0$ we have for sufficiently large s

$$|P_i(\xi^s)| \leqslant C\varphi_s^{d_i} = o\left(\varphi_s^{d_k}\right) \qquad (k+1 \leqslant i \leqslant M). \tag{18}$$

Then (16) to (18) implies that

$$|P(\xi^s)| \geqslant |P_k(\xi^s)| - \sum_{i \neq k} \left| P_i(\xi^) \right| \geqslant \frac{1}{2} P_k(\eta) \varphi_k^{d_k} \to \infty \quad \text{as} \quad s \to \infty,$$

which contradicts (12).

Thus (by choosing a subsequence) we can assume that $\rho_s \to \delta_0$ as $s \to \infty$ for a number δ_0, $0 \leqslant \delta_0 < \infty$.

Consider the cases 1) $\delta_0 \notin A(\eta)$ and 2) $\delta_0 \in A(\eta)$. In case 1), by the definition of the set $A(\eta)$, there exists a (unique) number $k = k(\eta)$ such that $d_k > 0$ and

$$d_i - \ell_i \delta_0 < d_k - \ell_k \delta_0 = \chi(\eta, \delta_0) \quad , \quad i = 0, 1, \ldots, M, \quad i \neq k.$$

By the continuity of the function χ in δ there exists a number $\varepsilon_0 > 0$ such that

$$d_k - \ell_k(\delta_0 + \varepsilon_0) > 0, \quad d_i - \ell_i(\delta_0 - \varepsilon_0) < d_k - \ell_k(\delta_0 + \varepsilon_0) \tag{19}$$

$$0 \leqslant i \leqslant M, \quad i \neq k.$$

On the other hand, because of $\rho_s \to \delta_0$ as $s \to \infty$ and $\psi_s = \varphi_s^{1-\rho_s}$ $(s \in N)$ (see (14)), there exists a number $s_0 \in N$ (assume that $s_0 = 1$) such that

$$\varphi_s^{-\delta_0-\varepsilon_0} \leqslant \frac{\psi_s}{\varphi_s} \leqslant \varphi_s^{-\delta_0+\varepsilon_0} \qquad (s \in N). \tag{20}$$

By Taylor's formula we have (for a constant $C > 0$ and for all $s \in N$)

$$|P(\xi^s)| \geqslant |P_k(\xi^s)| - \sum_{i \neq k} |P_i(\xi^s)|$$

$$= \left| \varphi_s^{d_k} \sum_{j=\ell_k}^{d_k} \frac{D_\varkappa^j P_k(\eta)}{j!} \left(\frac{\psi_s}{\varphi_s}\right)^j \right| - \left| \sum_{i \neq k} \varphi_s^{d_i} \sum_{j=\ell_i}^{d_i} \frac{D_\varkappa^j P_i(\eta)}{j!} \left(\frac{\psi_s}{\varphi_s}\right)^j \right|.$$

Since ℓ_k is even and $D_\varkappa^{\ell_k} P_k(\eta) > 0$, then this together with (20) implies that

$$|P(\xi^s)| \geqslant \frac{D_\varkappa^{\ell_k} P_k(\eta)}{\ell_k!} \varphi_s^{d_k - \ell_k(\delta_0+\varepsilon_0)} - \sum_{j=\ell_k+1}^{d_k} \varphi_s^{d_k - j(\delta_0-\varepsilon_0)} \frac{|D_\varkappa^j P_i(\eta)|}{j!}$$

$$- \sum_{i \neq k} \sum_{j=\ell_i}^{d_i} \frac{|D_\varkappa^j P_i(\eta)|}{j!} \varphi_s^{d_i - j(\delta_0-\varepsilon_0)} \qquad (s \in N).$$

We choose a number $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < \frac{j - \ell_k}{j + \ell_k} \delta_0 \qquad (j = \ell_k + 1, \ldots, d_k - \ell_k).$$

This, together with (19), implies that

$$|P(\eta^s)| \geqslant \frac{1}{2} \frac{|D_\varkappa^{\ell_k} P_k(\eta)|}{\ell_k!} \varphi_s^{d_k - \ell_k(\delta_0+\varepsilon_0)} \to \infty \quad \text{as} \quad s \to \infty,$$

which contradicts (12).

Let us consider the case 2) $\delta_0 \in A(\eta)$. It is obvious that in this case $\delta_0 > 0$. Proceeding as above, we get that for each $\varepsilon > 0$ there exist numbers $C = C(\varepsilon) \geqslant 0$ and $s_0 = s_0(\varepsilon)$ (we assume that $s_0 = 1$) such that

$$|P_i(\xi^s)| \leqslant C\varphi_s^{d_i-\ell_i(\delta_0-\varepsilon)}, \qquad i \notin J_0(\eta,\delta_0), \quad s \in N. \tag{21}$$

Writing $x_s = \psi_s/\varphi_s^{1-\delta_0}$ $(s \in N)$, for the polynomials P_i when $i \in J_0(\eta,\delta_0)$ we obtain

$$\sum_{i\in J_0(\eta,\delta_0)} P_i(\xi^s) = \sum_{i\in J_0(\eta,\delta_0)} \varphi_s^{d_i} \sum_{j=\ell_i}^{d_i} \frac{D_\varkappa^j P_i(\eta)}{j!} \left(\frac{\psi_s}{\varphi_s}\right)^j$$

$$= \varphi_s^{\chi(\eta,\delta_0)} \sum_{i\in J_0(\eta,\delta_0)} \sum_{j=\ell_i}^{d_i} \frac{D_\varkappa^j P_i(\eta)}{j!} \left(\frac{\psi_s}{\varphi_s}\right)^{j-\ell_i} x_s^{\ell_i}, \quad s \in N. \tag{22}$$

Consider the following subcases of the case 2): for a subsequence of the sequence $\{x_s\}$ (which we denote also by $\{x^s\}$) 2.1) $x_s \to \infty$, 2.2) $x_s \to 0$ as $s \to \infty$, 2.3) there exists a number $a \in (0,1)$ such that $a \leqslant x_s \leqslant a^{-1}$ $(s \in N)$. We notice that from the definition of the numbers $\{\rho_s\}$ and $\{x_s\}$ it follows that for arbitrary $\varepsilon > 0$ in all cases 2.1) to 2.3)

$$|x_s|\varphi_s^{\varepsilon} \to \infty, \quad |x_s|\varphi_s^{-\varepsilon} \to 0 \qquad \text{as} \quad s \to \infty. \tag{23}$$

Since $\psi_s/\varphi_s \to 0$ as $s \to \infty$ and (See Remark 1) $\ell_{i_1} > \ell_{i_2} > \cdots > \ell_{i_c}$, in the case 2.1 we have from (22)

$$\left|\sum_{i\in J_0(\eta,\delta_0)} P_i(\xi^s)\right| = \varphi_s^{\chi(\eta,\delta_0)} \frac{1}{\ell_{i_1}!} \left|D_\varkappa^{\ell_{i_1}} P_{i_1}(\eta)\right| x_s^{\ell_{i_1}} \,(1+o(1)) \tag{24}$$

for sufficiently large s.

We choose $\varepsilon_0 > 0$ such that

$$d_i - \ell_i(\delta_0 - \varepsilon_0) < \chi(\eta,\delta_0)\ ,\ i \notin J_0(\eta,\delta_0),$$

then from (21) and (24) we have

$$|P(\xi^s)| \geqslant \left|\sum_{i\in J_0(\eta,\delta_0)} P_i(\xi^s)\right| - \left|\sum_{i\notin J_0(\eta,\delta_0)} P_i(\xi^s)\right| \to \infty \quad \text{as} \quad s \to \infty,$$

which contradicts (12).

In the case 2.2) we choose $\varepsilon_1 > 0$ such that $\chi(\eta,\delta_0) - \varepsilon_1\ell_{i_c} > 0$. Then from (22) we obtain

$$\left|\sum_{i\in J_0(\eta,\delta_0)} P_i(\xi^s)\right| = \varphi_s^{\chi(\eta,\delta_0)} |x_s|^{\ell_{i_c}} \frac{1}{\ell_{i_c}!} D_\varkappa^{\ell_{i_c}} P_{i_c}(\eta)\,(1+o(1))$$

$$= \varphi_s^{\chi(\eta,\delta_0)-\varepsilon_1\ell_{i_c}} \left(\varphi_s^{\varepsilon_1}|x_s|\right)^{\ell_{i_c}} \frac{1}{\ell_{i_c}!} D_{\varkappa}^{\ell_{i_c}} P_{i_c}(\eta)\,(1+o(1))$$

for sufficiently large s (see Remark 1 and Lemma 2). Since $\varphi_s^{\varepsilon_1}\cdot|x_s| \to \infty$ as $s\to\infty$ (see (23)), then choosing a number $\varepsilon_2 \leqslant \varepsilon_1$ such that

$$\max_{i\notin J_0(\eta,\delta_0)} [\delta_i - \ell_i(\delta_0-\varepsilon_2)] < \chi(\eta,\delta_0) - \varepsilon_2\ell_{i_c},$$

we get from this and from (21) that $|P(\xi^s)| \to \infty$ as $s\to\infty$, which contradicts (12).

In case 2.3) $|P(\xi^s)| \to \infty$ as $s\to\infty$ by the theorem's condition (11) Theorem 1 is proved.

Theorem 2 *Let the polynomial P satisfy the necessary conditions 1)-3) of Lemma 2 and for each $\eta\in\Sigma(P_0)$, $\delta\in A(\eta)$, $c(\eta,\delta) = card J_0(P,\eta,\delta) = 2$. Then $P\in I_2$.*

Proof By Theorem 1 it is sufficient to prove the correlation (11) for each $\eta\in\Sigma(P_0)$, $\delta\in A(\eta)$, $a\in(0,1)$. From Taylor's formula we have

$$\begin{aligned}\sum_{i\notin J_0(\eta,\delta_0)} P_i[\xi(t)] &= \sum_{i\notin J_0(\eta,\delta_0)} P_i\left[t\left(\eta+t^{-\delta}x\varkappa\right)\right] \\ &= \sum_{i\notin J_0(\eta,\delta_0)} t^{d_i-\ell_i\delta}x^{\ell_i}\sum_{j=\ell_i}^{d_i}\frac{D_{\varkappa}^{j}P_i(\eta)}{j!}\left(t^{-\delta}x\right)^{j-\ell_i}.\end{aligned}$$

It follows from this that

$$\left|\sum_{i\notin J_0(\eta,\delta_0)} P_i[\xi(t)]\right| \leqslant C_1 a^{-d_0}t^b \tag{25}$$

for arbitrary $t\geqslant 1$, $|x|\leqslant a^{-1}$, where $C_1\geqslant 0$ is constant and the number b is defined by formula (10).

Since $C(\eta,\delta)=2$ then the set $J_0(\eta,\delta)$ consists of two indices i_1 and i_2. By Lemmas 1-2 the numbers ℓ_{i_1} and ℓ_{i_2} are even $\ell_{i_1} > \ell_{i_2}$ and $D_{\varkappa}^{\ell_{i_j}} P_{i_j}(\eta) > 0$ $(j=1,2)$. It follows from Taylor's formula that

$$\begin{aligned}\left|\sum_{i\in J_0(\eta,\delta_0)} P_i[\xi(t)]\right| &= \left|\sum_{j=1}^{2} P_{i_j}[\xi(t)]\right| \geqslant t^{\chi(\eta,\delta)}\left[\sum_{j=1}^{2}\frac{1}{\ell_{i_j}!}D_{\varkappa}^{\ell_{i_j}}P_{i_j}(\eta)x^{\ell_{i_j}}\right. \\ &\left. - \left|\sum_{j=1}^{2}\sum_{k=\ell_{i_j}+1}^{d_{i_j}}\frac{1}{k!}D_{\varkappa}^{k}P_{i_j}(\eta)x^k t^{-\delta(k-\ell_{i_j})}\right|\right] \geqslant C_2 a^{\ell_{i_1}} t^{\chi(\eta,\delta)}\end{aligned}$$

for sufficiently large t, with a constant $C_2 > 0$. From this and from (25), since $b < \chi(\eta, \delta)$ it follows that $|P[\xi(t)]| \to \infty$ as $t \to \infty$, which proves Theorem 2.

So the problem whether a polynomial P belongs to the set I_2 is completely solved when $c(\eta, \delta) = card J_0(P, \eta, \delta) = 2$ for all $\eta \in \sum(P_0)$, $\delta \in A(\eta)$. Therefore our further efforts will be devoted to obtaining conditions for $P \in I_2$ when $c(\eta, \delta) \geqslant 3$ for a pair (η, δ): $\eta \in \Sigma(P_0)$, $\delta \in A(\eta)$. We denote by $B = B(P)$ the set of these pairs.

Let $\eta \in \Sigma(P_0)$, $A(\eta) \neq \emptyset$ and $\delta \in A(\eta)$, then (see Remark 1) the number δ is rational. Let q be the smallest natural number such that $(q\delta)$ is natural. For the pair $(\eta, \delta) \in B$ and the number $k \in N_0$ we denote by

$$J_k(\eta, \delta) \equiv J_k(P, \eta, \delta) = \left\{ i : \ 0 \leqslant i \leqslant M, d_i - \ell_i(\eta)\delta = \chi(P, \eta, \delta) - \frac{k}{q} \right\}.$$

We notice that the set J_0 which we have just introduced is the same which we have introduced above. It is obvious that for arbitrary pair $(\eta, \delta) \in B$

$$i \in \bigcup_k J_k(\eta, \delta) \qquad i = 0, 1, \ldots, M. \tag{26}$$

So let $B(P) \neq \emptyset$. We shall show that in this case the investigation of the polynomial P at infinity reduces to the investigation of the finite set of functions of one variable. Having this in mind, we remark that, by Theorem 1, it is sufficient to consider the behavior of P on the set of points $\{\xi(t)\}$, where $\xi(t) = \xi(t, x) = t\left(\eta + t^{-\delta} x \varkappa\right) \equiv t\eta(t)$ as $t \to \infty$ and $(\eta, \delta) \in B(P)$, $a \leqslant |x| \leqslant a^{-1}$.

We write the following notations

$$q_i^j = q_i^j(\eta) = \frac{1}{(\ell_i + j)!} D_{\varkappa}^{\ell_i + j} P_i(\eta),$$

where $q_i^j(\eta) = 0$ if $\ell_i + j > d_i$ $(i = 0, 1, \ldots, M,\ j = 0, 1, \ldots)$, and consider the behaviour of $P[\xi(t, x)]$ as $t \to \infty$, $a \leqslant |x| \leqslant a^{-1}$. By Taylor's formula we obtain

$$\begin{aligned} P[\xi(t,x)] &= \sum_{i=0}^{M} t^{d_i} \sum_{j \geqslant 0} \frac{D_{\varkappa}^{\ell_i + j} P_i(\eta)}{(\ell_i + j)!} \left(t^{-\delta} x\right)^{\ell_i + j} \\ &= \sum_{i=0}^{M} t^{d_i - \ell_i \delta} \sum_{j \geqslant 0} \left[q_i^j x^{\ell_i} \right] \left(t^{-\delta} x\right)^j . \end{aligned}$$

From the definition of the sets $\{J_k\} = \{J_k(\eta,\delta)\}$ and from the (26) it follows that

$$\begin{aligned} P[\xi(t,x)] &= \sum_{k\geqslant 0} t^{\chi(\eta,\delta)-\frac{k}{q}} \sum_{i\in J_k}\sum_{j\geqslant 0} \left(t^{-\delta}x\right)^j q_i^j x^{\ell_i} \\ &= \sum_{k\geqslant 0} t^{\chi(\eta,\delta)-\frac{k}{q}} \sum_{j\geqslant 0} \left(t^{-\delta}x\right)^j \sum_{i\in J_k} q_i^j x^{\ell_i}. \end{aligned}$$

Writing for $k, j \in N_0$

$$Q_{k,j}(x) = Q_{k,j}(x,\eta,\delta) = \sum_{i\in J_k} q_i^j x^{\ell_i}, \tag{27}$$

we obtain from this that

$$\begin{aligned} P[\xi(t,x)] &= \sum_{k\geqslant 0} t^{\chi(\eta,\delta)-\frac{k}{q}} \sum_{j\geqslant 0} t^{-\frac{(\delta q)j}{q}} x^j Q_{k,j}(x) \\ &= \sum_{r\geqslant 0} t^{\chi(\eta,\delta)-\frac{r}{q}} \sum_{k+(\delta q)j=r} x^j Q_{k,j}(x). \end{aligned}$$

Let us set

$$Q_r(x) = \sum_{k+(\delta q)j=r} x^j Q_{k,j}(x)\ ,\ r = 0,1,\dots, \tag{28}$$

$$f(t,x) = f(t,x,\eta,\delta) = \sum_{r\geqslant 0} t^{\chi(\eta,\delta)-\frac{r}{q}} Q_r(x). \tag{29}$$

Then for arbitrary $(\eta,\delta) \in B$ and for $t \in (0,\infty)$, $a \leqslant |x| \leqslant a^{-1}$ we have $P[\xi(t,x)] = f(t,x)$ and the investigation of the polynomial P at infinity reduces to the investigation of the functions $f(t,x) = f(t,x,\eta,\delta)$ as $t \to \infty$ for each pair $(\eta,\delta) \in B(P)$.

3. Behaviour of functions depending on a parameter

Let $m > 0$, $q, M_0 \in N$ and

$$f(t,x) = \sum_{i=0}^{M_0} t^{m-\frac{i}{q}} Q_i(x), \tag{30}$$

where Q_i ($0 \leqslant i \leqslant M_0$) are polynomials with real coefficients. Our objective in this section is to establish conditions for (28) type polynomials

$\{Q_i\}$, under which

$$\lim_{t\to+\infty} \inf_{a\leqslant|x|\leqslant a^{-1}} f(t,x) = +\infty \tag{31}$$

for a given number $a \in (0,1)$.

As has already been noted, the problem of behaviour of the polynomial $P(\xi) = P(\xi_1, \xi_2)$ at infinity when $B(P) \neq \emptyset$ reduces to the study of a finite set of functions of (30) type.

The notation $f \in I_1(a)$ means that the function f satisfies the condition (31). We denote by

$$X_0 = X_0(a)\left\{x;\ a \leqslant |x| \leqslant a^{-1}, Q_0(x) = 0\right\},$$

$$X_i = X_i(a) = \{x;\ x \in X_{i-1}, Q_i(x) = 0\} \qquad i = 1, 2, \ldots, M_0.$$

It is obvious that we can choose a number $a \in (0,1)$ so that $Q_i(x) \neq 0$ for $|x| \notin (a,\ a^{-1})$ $(0 \leqslant i \leqslant M_0)$.

Let $a \leqslant |x| \leqslant a^{-1}$, $\delta \geqslant 0$. We denote by $\ell_i = \ell_i(x)$ the order of the root x of the polynomial Q_i $(i = 0, 1, \ldots, M_0)$ and put

$$\chi(x,\delta) = \chi(f,x,\delta) = \max_{0\leqslant i\leqslant M_0} \left\{m - \frac{i}{q} - \ell_i(x)\delta\right\}.$$

By $A(x) = A(f,x)$ we denote the set of numbers $\delta > 0$ for which there exist numbers i, j: $i \neq j$, $0 \leqslant i,\ j \leqslant M_0$ such that

$$m - \frac{i}{q} - \ell_i(x)\delta = m - \frac{j}{q} - \ell_j(x)\delta = \chi(x,\delta).$$

Finally for arbitrary $x \in X_0$, $\delta \in A(x)$ we denote by

$$J_0(x,\delta) = J_0(f,x,\delta) = \left\{i;\ 0 \leqslant i \leqslant M_0, m - \frac{i}{q} - \ell_i(x)\delta = \chi(x,\delta)\right\},$$

$$c = c(x,\delta) = card J_0(x,\delta).$$

We have introduced the notations $\{\ell_i\}$, χ, A, J_0, c both for the polynomials of two variables and points $\xi \in R^2$ and for the functions of one variable and points $x \in R^1$. We hope it will not make any misunderstanding.

The following elementary lemma comes from ([14] Lemma 3.2 and Lemma 3.4)

Lemma 3 *Let $f \in I_1(a)$ then*

1) $Q_0(x) \geqslant 0$, $a \leqslant |x| \leqslant a^{-1}$,

2) *for each point* $x_0 \in X_0$ *there exists a number* $k = k(x_0)$*:* $0 \leqslant k \leqslant M_0$ *such that* $m - k/q > 0$, $Q_0(x_0) = Q_1(x_0) = \cdots = Q_{k-1}(x_0) = 0$, $Q_k(x_0) > 0$.

Lemma 4 *Let* $Q_0(x) > 0$ *for all* x*:* $a \leqslant |x| \leqslant a^{-1}$*. Then* $f \in I_1(a)$*.*
Proof We notice that the set $[a, a^{-1}]$ is compact, so $Q_0(x) \geqslant C_0(a)$ and $|Q_j(x)| \leqslant C_1(a)$ for all x: $|x| \in [a, a^{-1}]$ and all $j = 1, \ldots, M_0$ with constants $C_0 = C_0(a) > 0$, $C_1 = C_1(a) \geqslant 0$. Hence for sufficiently large t

$$f(t,x) \geqslant C_0 t^m - C_1(M_0 - 1)t^{m-\frac{1}{q}}$$

and $f(t,x) \to \infty$ as $t \to \infty$. This proves the lemma.
Theorem 3 $f \in I_1(a)$ *if and only if the conditions 1)-2) of Lemma 3 are satisfied and 3) for every* $x \in X_0$, $\delta \in A(x)$, $b \in (0, a)$

$$\lim_{t\to\infty} \inf_{b\leqslant|y|\leqslant b^{-1}} f\left(t, x + t^{-\delta} \cdot y\right) = +\infty. \tag{32}$$

Proof *Necessity* of conditions 1) and 2) follows from Lemma 3. The necessity of condition 3) follows from the fact that $x_0 \in X$ is an inner point of the set $\{x\}$, hence the points $x_0 + t^{-\delta}y$ $|y| \in [b, b^{-1}]$ belong to this set for sufficiently large t and for any $b \in (0,1)$.

Sufficiency. Suppose that, on the contrary, while the conditions 1)–3) are satisfied none the less $f \notin I_1(a)$, i.e. there exist a number $C \geqslant 0$ and sequence $\{t_s, x_s\}$ such that $t_s \to \infty$ as $s \to \infty$, $a \leqslant |x_s| \leqslant a^{-1}$ and

$$|f(t_s, x_s)| \leqslant C \qquad (s \in N). \tag{33}$$

By extracting a suitable subsequence (this subsequence we also denote by $\{x_s\}$) of the (bounded) sequence $\{x_s\}$ one may assume that $x_s \to x_0$ as $s \to \infty$, $a \leqslant |x_0| \leqslant a^{-1}$. If $x_0 \notin X_0$ the contradiction is obvious. Let $x_0 \in X_0$. If $x_s = x_0$ for an infinite number of $s \in N$ then we get a contradiction to (33) as in proof of Theorem 1. Thus without loss of generality one can assume that $x_s \notin X_0$ for all $s \in N$. We put

$$\rho_s = -\frac{\ln|x_s - x_0|}{\ln t_s} \qquad (s \in N). \tag{34}$$

Then $\rho_s > 0$ for sufficiently large $s \in N$. Now for simplicity, let $sign(x_s - x) = 1$ $(s \in N)$, then from (34) it follows that

$$x_s = x_0 + t_s^{-\rho_s} sign(x_s - x) = x_0 + t_s^{-\rho_s} \qquad (s \in N).$$

At first we consider the case when (by choosing the subsequence) $\rho_s \to \infty$ as $s \to \infty$. Let the number $k = k(x_0)$ $(0 < k \leqslant M_0)$ be defined as in

Lemma 3. Since $\ell_i = \ell_i(x_0) > 0 \quad (i = 0, 1, \ldots, k-1)$ it follows from Taylor's formula

$$\left|\sum_{k=0}^{k-1} t_s^{m-\frac{i}{q}} Q_i(x_s)\right| = \left|\sum_{i=0}^{k-1} t_s^{m-\frac{i}{q}} \sum_{j\geqslant \ell_i} \frac{Q_i^{(j)}(x_0)}{j!} t_s^{-j\rho_s}\right| \to 0 \quad \text{as} \quad s\to\infty. \tag{35}$$

On the other hand, it is obvious that

$$\left|\sum_{i=k+1}^{M_0} t_s^{m-\frac{i}{q}} Q_i(x_s)\right| = o\left(t_s^{m-\frac{k}{q}}\right) \quad \text{as} \quad t_s\to\infty. \tag{36}$$

Since by the condition 2) $Q_k(x_0) > 0$, then for sufficiently large s

$$t_s^{m-\frac{k}{q}} Q_k(x_s) \geqslant \frac{Q_k(x_0)}{2} t_s^{m-\frac{k}{q}}. \tag{37}$$

Then (35)-(37) together imply that $|f(t_s, x_s)| \to \infty$ as $s\to\infty$, which contradicts (33).

Let us now assume that $0 \leqslant \rho_s \leqslant \rho < \infty \quad (s\in N)$. By choosing a subsequence we can assume that $\rho_s \to \delta_0$ as $s\to\infty$ for a number $\delta_0 \geqslant 0$. Since $x_0 \in X_0$ then $Q_0(x) = (x - x_0)^{\ell_0} r_0(x)$, where by condition 1) $r_0(x) > 0$. Then for sufficiently large s

$$t_s^m |Q_0(x_s)| \geqslant \frac{1}{2} r_0(x_0) |x_s - x_0|^{\ell_0} t_s^m = \frac{1}{2} r_0(x_0) t_s^{m-\frac{1}{2q}} \left[|x_s - x_0|\, t_s^{\frac{1}{2q\ell_0}}\right]^{\ell_0}.$$

We prove that $\delta_0 > 0$. Suppose $\delta_0 = 0$, i.e. $\rho_s \to 0$ as $s\to\infty$. Then we obtain

$$|x_s - x_0| t_s^{\frac{1}{2q\ell_0}} = t^{\frac{1}{2q\ell_0}-\rho_s} \to \infty \quad \text{as} \quad s\to\infty.$$

Therefore $st_s^m |Q_0(x_s)| \geqslant C_0 t^{m-\frac{1}{2q}}$, $C_0 > 0$, for sufficiently large s.

On the other hand, obviously we have

$$|f(t_s, x_s) - t_s^m Q_0(x_s)| \leqslant C_1 t_s^{m-\frac{1}{q}} \quad (s\in N),\ C_1 \geqslant 0.$$

From the last two relations it follows that $|f(t_s, x_s)| \to \infty$ as $s\to\infty$, which contradicts to (33) and proves that $\delta_0 > 0$.

First let $\delta_0 \notin A(x_0)$. Then there exists a number i_0: $0 \leqslant i_0 \leqslant M_0$ such that

$$m - \frac{i}{q} - \ell_i(x_0)\delta_0 < m - \frac{i_0}{q} - \ell_{i_0}(x_0)\delta_0 = \chi(x_0, \delta_0), \quad 0 \leqslant i \leqslant M_0,\ i \neq i_0.$$

Since $\chi(x_0, \delta_0) > 0$ (see condition 2) of Lemma 3) and the function $\chi(x_0, \delta)$ is continuous in δ, hence there exists a number $\varepsilon_0 \in (0, \delta_0)$ such that $\chi(x_0, \delta_0 + \varepsilon_0) > 0$ and

$$m - \frac{i}{q} - \ell_i(x_0)(\delta_0 - \varepsilon_0) < \chi(x_0, \delta_0 + \varepsilon_0), \quad i \neq i_0. \tag{38}$$

It is obvious (see Corollary 1) that $Q_{i_0}^{(\ell_0)}(x_0) > 0$. Then by Taylor's formula we obtain

$$t_s^{m-\frac{i_0}{q}} Q_{i_0}(x_s) = t_s^{m-\frac{i_0}{q}} \sum_{j \geqslant \ell_{i_0}} t_s^{-j\rho_s} \frac{Q_{i_0}^{(j)}(x_0)}{j!}$$

$$= \frac{1}{\ell_{i_0}!} Q_{i_0}^{(\ell_{i_0})}(x_0) t_s^{m-\frac{i_0}{q}-\ell_{i_0}\rho_s} + t_s^{m-\frac{i_0}{q}} \sum_{j \geqslant \ell_{i_0}+1} t_s^{-j\rho_s} \frac{Q_{i_0}^{(j)}(x_0)}{j!} \quad (s \in N).$$

Let ε_0 satisfy the condition (38). Since $\rho_s \to \delta_0$ as $s \to \infty$, there exists a number s_0 such that $\delta_0 - \varepsilon_0 \leqslant \rho_s \leqslant \delta_0 + \varepsilon$ for $s \geqslant s_0$. Let $0 < \varepsilon_0 < \delta_0 / (2\ell_{i_0} + 1)$, then from this

$$t_s^{m-\frac{i_0}{q}} Q_{i_0}(x_s) = \frac{1}{\ell_{i_0}!} Q_{i_0}^{(\ell_{i_0})}(x_0) t_s^{\chi(x_0,\delta_0+\varepsilon_0)} (1 + o(1)), \quad s \to \infty,$$

is obtained. Similarly for $i \neq i_0$ we have

$$\sum_{i \neq i_0} t^{m-\frac{i}{q}} |Q_i(x_s)| \leqslant C_2 t_s^{m-\frac{i}{q}-\ell_i(\delta_0-\varepsilon_0)}, \qquad s \geqslant s_0,\ C_2 \geqslant 0.$$

Then the last two relations together with (38) imply

$$|f(t_s, x_s)| \geqslant |Q_{i_0}(x_s)|\, t_s^{m-\frac{i_0}{q}} - \sum_{i \neq i_0} |Q_i(x_s)| t_s^{m-\frac{i}{q}} \to \infty \quad \text{as} \quad s \to \infty,$$

which contradicts (33).

It remains to consider the case $\delta_0 \in A(x_0)$. Let us write

$$\tau_s = t_s^{-\rho_s+\delta_0}, \quad \text{then} \quad x_s = x_0 + \tau_s t_s^{-\delta_0} \quad (s \in N). \tag{39}$$

We notice that because of $\rho_s \to \delta_0$ (as $s \to \infty$), then for any $\varepsilon > 0$ $t_s^{-\varepsilon}\tau_s \to 0,\ t_s^{\varepsilon}\tau_s \to \infty$ as $s \to \infty$.

We divide this case into the following subcases, when there exists subsequence of the sequence $\{\tau^s\}$ (which we write also by $\{\tau^s\}$) such that

1) $\tau_s \to \infty$, 2) $\tau_s \to 0$ as $s \to \infty$ and $b \leqslant |\tau_s| \leqslant b^{-1}$ for a number $b \in (0,1)$ ($s \in N$).

We let, as above, $J_0(x_0,\delta_0) = \{i_1, i_2, \ldots, i_c\}$, $i_1 < i_2 < \cdots < i_c$. In case 1) by Taylor's formula ($k = 1, 2, \ldots, c$)

$$t_s^{m-\frac{i_k}{q}} |Q_{i_k}(x_s)| = t_s^{m-\frac{i_k}{q}} \left| \sum_{j \geqslant \ell_{i_k}} \left(t_s^{-\delta_0}\tau_s\right)^j \frac{Q_{i_k}^{(j)}(x_s)}{j!} \right|$$

$$= t_s^{m-\frac{i_k}{q}-\ell_{i_k}\delta_0} \tau_s^{\ell_{i_k}} \left| Q_{i_k}^{\left(\ell_{i_k}\right)}(x_0) + \sum_{j\geqslant 1} \left(t_s^{-\delta_0}\tau_s\right)^{\ell_{i_k}+j} \frac{Q_{i_k}^{\left(\ell_{i_k}+j\right)}(x_0)}{(\ell_{i_k}+j)!} \right|.$$

Since $\tau_s \to \infty$ and $t_s^{-\delta_0}\tau_s \to 0$ as $s \to \infty$ (see also the definition of $J_0 = J_0(x_0,\delta_0)$) we have for sufficiently large s and $k = 1, 2, \ldots, c$

$$t_s^{m-\frac{i_k}{q}} |Q_{i_k}(x_s)| = \left| Q_{i_k}^{\left(\ell_{i_k}\right)}(x_0) \right| t_s^{\chi(x_0,\delta_0)} \tau_s^{\ell_{i_k}} \left(1 + o(1)\right).$$

Since $\ell_{i_1} > \ell_{i_k} (k=2, \ldots, c)$, we obtain from this that for sufficiently large s

$$\left| \sum_{i\in J_0} t_s^{m-\frac{i}{q}} Q_i(x_s) \right| = \left| Q_{i_1}^{\left(\ell_{i_1}\right)}(x_0) \right| t_s^{\chi(x_0,\delta_0)} \tau_s^{\ell_{i_1}} \left(1 + o(1)\right), \tag{40}$$

and

$$\left| \sum_{i\notin J_0} t_s^{m-\frac{i}{q}} Q_i(x_s) \right| \leqslant \sum_{i\notin J_0} t_s^{m-\frac{i}{q}} \sum_{j\geqslant \ell_i} \left(t_s^{-\delta_0}\tau_s\right)^j \frac{\left|Q_i^{(j)}(x_0)\right|}{j!}$$

$$= \sum_{i\notin J_0} t_s^{m-\frac{i}{q}-\ell_i\delta_0} \sum_{j\geqslant \ell_i} t_s^{-\delta_0(j-\ell_i)} \tau_s^j \frac{\left|Q_i^{(j)}(x_0)\right|}{j!}.$$

Let $\varepsilon > 0$ be arbitrary, then

$$\left| \sum_{i\in J_0} t_s^{m-\frac{i}{q}} Q_i(x_s) \right| \leqslant \sum_{i\notin J_0} t_s^{m-\frac{i}{q}-\ell_i\delta_0+\varepsilon} \sum_{j\geqslant \ell_i} \left[t_s^{-\frac{\delta_0(j-\ell_i)+\varepsilon}{j}} \tau_s \right]^j \frac{\left|Q_i^{(j)}(x_0)\right|}{j!}.$$

Since $\delta_0(j-\ell_i) + \varepsilon > 0$ for all $j \geqslant \ell_i$, then

$$t_s^{-\frac{1}{j}[\delta_0(j-\ell_i)+\varepsilon]} \tau_s \to 0 \quad \text{as} \quad s \to \infty,$$

hence for sufficiently large s we obtain from this that

$$\left|\sum_{i\notin J_0} t_s^{m-\frac{i}{q}} Q_i(x_s)\right| \leqslant C_3 t^{b_0+\varepsilon} \tag{41}$$

where $C_3 \geqslant 0$ is constant and

$$b_0 = \max_{i\notin J_0}\left\{m - \frac{i}{q} - \ell_i\delta_0\right\} < \chi(x_0,\delta_0). \tag{42}$$

If we choose a number $\varepsilon_0 > 0$ such that $b_0 + \varepsilon_0 < \chi(x_0,\delta_0)$, then from (40) and (42) we have $|f(t_s,x_s)| \to \infty$ as $s \to \infty$, which contradicts (33).

Similarly, in case 2) when $s \to \infty$ we obtain

$$\left|\sum_{i\in J_0} t_s^{m-\frac{i}{q}} Q_i(x_s)\right| = \left|Q_{i_c}^{(\ell_{i_c})}(x_0)\right| t_s^{\chi(x_0,\delta_0)} \tau_s^{\ell_{i_c}} \left(1 + o(1)\right), \tag{43}$$

and inequality (41) for $i \notin J_0$. We choose now a number $\varepsilon_1 > 0$ such that

$$b + \varepsilon_1 < \chi(x_0,\delta_0) - \varepsilon_1. \tag{44}$$

Then we obtain

$$t_s^{\chi(x_0,\delta_0)} \tau_s^{\ell_{i_c}} = t_s^{\chi(x_0,\delta_0)-\varepsilon_1} \left(t_s^{\frac{\varepsilon_1}{\ell_{i_c}}} \tau_s\right)^{\ell_{i_c}}.$$

On the other hand, since $t_s^{\varepsilon_1/\ell_{i_c}} \tau_s \to \infty$ as $s \to \infty$, then it follows from (43) that

$$\left|\sum_{i\in J_0} t_s^{m-\frac{i}{q}} Q_i(x_s)\right| \geqslant C_4 t_s^{\chi(x_0,\delta_0)-\varepsilon_1}, \qquad C_4 > 0,\ s \in N.$$

Clearly this inequality together with (41) (when $\varepsilon = \varepsilon_1$) and condition (44) imply that $|f(t_s,x_s)| \to \infty$ as $s \to \infty$ which contradicts (33).

In case 3) the contradiction comes from the condition 3) of the theorem. This completes the proof of Theorem 1.

Lemma 5 *Let $f \in I_1$, $x_0 \in X_0$, $A(x_0) \neq \emptyset$, $\delta_0 \in A(x_0)$ and $J_0 = J_0(x_0,\delta_0) = \{i_1, i_2, \dots, i_c\}$. Then the numbers $\ell_{i_1} = \ell_{i_1(x_0)}$, $\ell_{i_c} = \ell_{i_c(x_0)}$ are even and $r_{i_1} = r_{i_1}(x_0) > 0$, $r_{i_c} = r_{i_c}(x_0) > 0$, where $r_{i_j}(x) = Q_{i_j}(x)/(x-x_0)^{\ell_{i_j}}$ $(j = 1, 2, \dots, c)$.*

Proof It is sufficient to prove this for the index i_1; for the index i_c it can be proved analogously.

Let, on the contrary either 1) ℓ_{i_1} be even, $r_{i_1} < 0$ or 2) ℓ_{i_1} be odd, $r_{i_1} > 0$ or 3) ℓ_{i_1} be odd, $r_{i_1} < 0$.

Let $y > 0$ in the cases 1) and 3), $y < 0$ in the case 2). Then $y^{\ell_{i_1}} r_{i_1} < 0$ in all cases. Because of $\ell_{i_1} > \ell_{i_j}$ $(j = 2, \ldots, c)$ for a sufficiently large $|y|$ then

$$\mathcal{D}(y) = \sum_{i \in J_0} r_i y^{\ell_i} < 0.$$

Let $\mathcal{D}(y_0) < 0$, $t_s = s$, $x_s = x_0 + y_0 s^{-\delta_0}$ $(s \in N)$. Then by the definition of $J_0(x_0, \delta_0)$ and Taylor's formula we have

$$\sum_{i \in J_0} s^{m - \frac{i}{q}} Q_i(x_s) = s^{\chi(x_0, \delta_0)} \mathcal{D}(y_0) \left(1 + o(1)\right) \to -\infty \quad \text{as} \quad s \to \infty$$

and

$$\left| \sum_{i \notin J_0} s^{m - \frac{i}{q}} Q_i(x_s) \right| \leqslant C t^{b_0}, \qquad C > 0,\ (s \in N),$$

where the number $b_0 < \chi(x_0, \delta_0)$ is defined by the formula (42). It follows from the last two relations that $f(t_s, x_s) \to -\infty$ as $s \to \infty$, which contradicts to the condition $f \in I_2$ and proves the Lemma 5.

Theorem 4 *Let for each pair* (x, δ)*:* $x \in X_0$, $\delta \in A(x)$ $c(x, \delta) = card J_0(x, \delta) = 2$*. Then* $f \in I_1(a)$ *if and only if the conditions* 1)-2) *of Lemma 3 and the condition*

3) $\ell_i(x)$ *is even and* $r_i(x) > 0$ *for all* $x \in X_0$, $i \in \bigcup_{\delta \in A(x)} J_0(x, \delta)$,

are satisfied.

Proof The *necessity* immediately follows from Lemma 4.

Sufficiency. By the Theorem 1 it is sufficient to prove (32) for all $x \in X_0$, $\delta \in A(x)$ and $b \in (0, a)$. At first we notice that by the condition 3) there exists a number $C_1 = C_1(b) > 0$ such that for all $x \in X_0$ and y: $|y| \in \left(b, b^{-1}\right)$

$$\sum_{i \in J_0(x, \delta)} r_i(x) y^{\ell_i(x)} \geqslant C_1. \tag{45}$$

Then we have by Taylor's formula for sufficiently large s

$$\inf_{b \leqslant |y| \leqslant b^{-1}} f\left(t, x + t^{-\delta} y\right) = \inf_{b \leqslant |y| \leqslant b^{-1}} \left[t^{\chi(x, \delta)} \sum_{i \in J_0(x, \delta)} r_i(x) y^{\ell_i(x)} \left(1 + o(1)\right) \right.$$

$$+\sum_{i\notin J_0(x,\delta)} t^{m-\frac{i}{q}-\ell_i(x)\delta} r_i(x) y^{\ell_i(x)}\left(1+o(1)\right)\Bigg] .$$

By the definition of $J_0(x,\delta)$

$$m-\frac{i}{q}-\ell_i(x)\delta<\chi(x,\delta), \quad i\notin J_0(x,\delta),$$

therefore this together with (45) implies

$$\lim_{t\to\infty}\inf_{b\leqslant|y|\leqslant b^{-1}} f\left(t,x+t^{-\delta}y\right)\geqslant\frac{C_1}{2}\lim_{t\to\infty} t^{\chi(x,\delta)}=+\infty.$$

This completes the proof of Theorem 4.

Resuming the results of the first two sections, we see that the study of the behaviour of the polynomial P at infinity reduces to the study of the behaviour at infinity of the finite set of functions of type (30). While $cardJ_0(f,x,\delta)=2$ for all such functions and all admissible pairs (x,δ) then (see Theorem 4) the problem is solved. Therefore our further efforts will be devoted to the case when $cardJ_0(f,x,\delta)\geqslant 3$ for such a function f and for a pair (x,δ). This will be done in the next section.

4. The case $cardJ_0 \geqslant 3$

Let f be a function of type (30), $x\in X_0$, $\delta\in A(x)$. We denote by

$$\beta_i=\beta_i(f,x,\delta)=\chi(f,x,\delta)-\left[m-\frac{i}{q}-\ell_i(x)\delta\right], \quad i=0,1,\dots,M_0.$$

Then $\beta_i=0$ for $i\in J_0(x,\delta)$.

Let $q_1\in N$ be the smallest number for which $q_1(q\delta)\in N$ and $k_i=(q_1q)\beta_i$ $(i=0,1,\dots,M_0)$. It is obvious that $k_i\in N_0$ $(0\leqslant i\leqslant M_0)$. Thus the set $\{0,1,\dots,M_0\}$ may be divided into the subsets $(k\in N_0)$

$$J_k(x,\delta)=J_k(f,x,\delta)$$

$$=\left\{i;\ 0\leqslant i\leqslant M_0, m-\frac{i}{q}-\ell_i(x)\delta=\chi(f,x,\delta)-\frac{k}{q_1q}\right\}. \tag{46}$$

Notice that $J_k(x,\delta)$ can be empty for some $k\in N$.

Thus, by Theorem 1 the behaviour of functions of type (30) at infinity are defined by their behaviour on the sequences of special type: $\left(t,x+t^{-\delta}y\right)$ for $t\in(0,\infty)$, $y\in R^1$, $x\in X_0$, $\delta\in A(x)$.

Let $x_0 \in X_0,\ \ \delta \in A(x_0)$. We put

$$q_i^j = q_i^j(x_0) = \frac{1}{(\ell_i + j)!} Q_i^{(\ell_i + j)}(x_0) \quad (i = 0, 1, \ldots, M_0,\ j = 0, 1, \ldots), \tag{47}$$

where, as above, $\ell_i = \ell_i(x_0)$ denote the order of the root x_0 of Q_0 $(0 \leqslant i \leqslant M_0)$.

Consider the function (30) on the set $\left\{\left(t, x_0 + t^{-\delta_0} y\right)\right\}$. By Taylor's formula we have

$$f\left(t, x_0 + t^{-\delta_0} y\right) = \sum_{i=0}^{M_0} t^{m - \frac{i}{q}} Q_i\left(x_0 + t^{-\delta_0} y\right)$$

$$= \sum_{i=0}^{M_0} t^{m - \frac{i}{q}} \sum_{j \geqslant 0} q_i^j \left(t^{-\delta_0} y\right)^{\ell_i + j} = \sum_{i=0}^{M_0} t^{m - \frac{i}{q} - \ell_i \delta_0} \sum_{j \geqslant 0} q_i^j t^{-j\delta_0} y^{\ell_i + j}.$$

Hence (see also the definition of the sets $\{J_k(x, \delta)\}$)

$$\begin{aligned} f\left(t, x_0 + t^{-\delta_0} y\right) &= \sum_{k \geqslant 0} t^{\chi(x_0, \delta_0) - \frac{k}{q_1 q}} \sum_{j \geqslant 0} t^{-j\delta_0} \sum_{i \in J_k(x_0, \delta_0)} q_i^j y^{\ell_i + j} \\ &= \sum_{k \geqslant 0} \sum_{j \geqslant 0} t^{\chi(x_0, \delta_0) - \frac{k}{q_1 q} - j\delta_0} \end{aligned}$$

$$\times \sum_{j \in J_k(x_0, \delta_0)} q_i^j y^{\ell_i + j} = \sum_{r \geqslant 0} t^{\chi(x_0, \delta_0) - \frac{r}{q_1 q}} \sum_{k + q_1(q\delta_0) j = r} \ \sum_{i \in J_k(x_0, \delta_0)} q_i^j y^{\ell_i + j}. \tag{48}$$

We may put $m_1 = \chi(f, x_0, \delta_0)$,

$$Q_r^1(x) = Q_r^1(f, x_0, \delta_0, x) = \sum_{k + q_1(q\delta_0) j = r} \ \sum_{i \in J_k(x_0, \delta_0)} q_i^j x^{\ell_i + j} \quad (r = 0, 1, \ldots, M_1), \tag{49}$$

$$f_1(t, x) = f_1(f, x_0, \delta_0, t, x) = \sum_{i=0}^{M_1} t^{m_1 - \frac{i}{q q_1}} Q_i^1(x), \tag{50}$$

where the number $M_1 \in N_0$ is defined such that either $m_1 - \frac{k}{q_1 q} \leqslant 0$ or $Q_k^1(x) \equiv 0$ when $k > M_1$.

Now (48)-(50) imply that for arbitrary $x_0 \in X_0$, $\delta_0 \in A(x_0)$, x: $b \leqslant |x| \leqslant b^{-1}$ and $t \in (0, \infty)$

$$f\left(t, x_0 + t^{-\delta_0} x\right) = f_1(t, x) = f_1(f, x_0, \delta_0, t, x).$$

Thus, the study of the function f of *"order"* m (in t) leads to study a finite set of functions $\{f_1\}$ of *"order"* $m_1 < m$. For each function f_1 from this set we introduce

$$X_i^1 = X_i^1(f_1) = \left\{x;\ b \leqslant |x| \leqslant b^{-1}, Q_0^1(x) = \cdots = Q_i^1(x) = 0\right\},$$
$$i = 0, 1, \ldots, M_1.$$

Let $x_0 \in X_0^1$, $\ell_i^1 = \ell_i^1(f_1, x_0)$ be the order of the root x_0 of the polynomial Q_i^1 $(0 \leqslant i \leqslant M_1)$, $\delta_0 \geqslant 0$. We put

$$\chi_1(x_0, \delta_0) = \chi_1(f_1, x_0, \delta_0) = \max_{0 \leqslant i \leqslant M_1} \{m_1 - \frac{i}{qq_1} - \ell_i^1(x_0)\delta_0\},$$

$$\begin{aligned} A_1(x_0) &= A_1(f_1, x_0) \\ &= \Big\{\delta > 0, \ \text{ there exist numbers } \ i \neq j,\ 0 \leqslant i, j \leqslant M_1 \ \text{ such that} \\ &\quad m_1 - \frac{i}{qq_1} - \ell_i^1(x_0)\delta = m_1 - \frac{i}{qq_1} - \ell_j^1(x_0)\delta = \chi_1(x_0, \delta)\Big\}. \end{aligned}$$

Finally for arbitrary x: $b \leqslant x \leqslant b^{-1}$ and $\delta \in A_1(x)$ we put

$$\begin{aligned} J_0^1(x, \delta) &= J_0^1(f_1, x, \delta) \\ &= \left\{i;\ 0 \leqslant i \leqslant M_1, m_1 - \frac{i}{qq_1} - \ell_i^1(x)\delta = \chi_1(f_1, x, \delta)\right\}. \end{aligned}$$

Theorem 1 leads from the study of the polynomial P to the study of a finite set of functions $\{f\}$ of type (30). Theorem 2 leads this problem to study functions $\{f\}$ on special sets, what in turn leads to the problem to study a finite set of functions $\{f_1\}$ of type (50) etc.

If in some step s either $A_s = \bigcup_{x \in X_0^s} A_s(f_s, x) = \emptyset$ or $c_s(x, \delta) = 2$ for all (x, δ): $x \in X_0^s$, $\delta \in A_s(f_s, x)$ (when $A_s \neq \emptyset$) then the process comes to its end (see Theorems 1 and 2).

We prove that this process comes to its end if for some of such pairs (x, δ) $c_s(x, \delta) \geqslant 3$ $(s = 0, 1, \ldots)$ also. Namely we prove that after a definite number s_0 of steps we will come to one of the following cases: 1) $A_{s_0} = \emptyset$, 2) $A_{s_0} \neq \emptyset$ and for each pair (x, δ) $x \in X_0^{s_0}$, $\delta \in A_{s_0}(x)$ either $c_{s_0}(x, \delta) = 2$ or $m_{s_0} = m_{s_0}(x, \delta) \leqslant 0$ (and then $P \notin I_2$).

The next section is devoted to the consideration of this problem.

5. The Algorithm

In this section applying the basic results obtained above, an algorithm as an answer to the question: when does the polynomial $P(\xi) = P(\xi_1, \xi_2)$ belong to the set I_2, is offered.

If either $A(P) = \bigcup_{\eta \in \Sigma(P_0)} A(P, \eta) \neq \emptyset$ or $c_0(P, \eta, \delta) = card J_0(P, \eta, \delta) = 2$ for all pairs (η, δ): $\eta \in \Sigma(P_0)$, $\delta \in A(P, \eta)$ (when $A(P) \neq \emptyset$) then the problem is solved by Theorem 1 in [14] or Theorem 2 in this paper.

If $c_0(P, \eta^0, \delta_0) \geqslant 3$ for such a pair (η^0, δ_0) then this pair generates the numbers $q \in N$, $a \in (0, 1)$ and the function $f(t, x) = f(t, x, \eta^0, \delta_0)$ by formula (29). It is obvious that there are a finite number of pairs (η, δ) for which $c_0(\eta, \delta) \geqslant 3$. Assume that there are n_0 of such pairs. For simplicity of notation in the sequel we assume that $n_0 = 1$. Thus the problem leads to study of the function

$$f_0(t, x) = \sum_{i=0}^{M_0} t^{m_0 - \frac{i}{q_0}} Q_i^0(x), \tag{51}$$

on the set (t, x): $t \in (0, \infty)$, $a_0 \leqslant |x| \leqslant a_0^{-1}$. Here

$$m_0 = \chi(P, \eta^0, \delta_0) = \max_{0 \leqslant i \leqslant M} \left\{ d_i - \ell_i(\eta^0)\delta_0 \right\}, \tag{52}$$

$\ell_i = \ell_i(\eta^0)$ is the order of the zero η^0 of the polynomial P_i $(i = 0, 1, \ldots, M)$ (see (1)), the polynomials $Q_i^0 \equiv Q_i$ $(0 \leqslant i \leqslant M_0)$ are defined by the formulas (27)-(28) and $q_0 = q$ is the smallest natural number for which $q_0 \delta_0 \in N$.

For $0 \leqslant i \leqslant M_0$ we denote

$$X_i^0 = X_i^0(Q_i^0, a_0) = \left\{ x;\ a_0 \leqslant |x| \leqslant a_0^{-1}, Q_0^0(x) = Q_1^0(x) = \cdots = Q_i^0(x) = 0 \right\}.$$

For $x_0 \in X_0^0$ let $\ell_i^0 = \ell_i^0(x_0)$ be the order of the root x_0 of the polynomial Q_i^0 $(0 \leqslant i \leqslant M_0)$, $A_0(x_0) = A_0(f_0, x_0)$ be the set of numbers $\Delta > 0$ for which there exist indices $i \neq j$: $0 \leqslant i, j \leqslant M_0$ such that

$$\begin{aligned} m_0 - \frac{i}{q_0} - \ell_i^0 \Delta &= m_0 - \frac{j}{q_0} - \ell_j^0 \Delta = \chi_0(f_0, x_0, \Delta) \\ &= \max_{0 \leqslant k \leqslant M_0} \left\{ m_0 - \frac{k}{q_0} - \ell_k^0 \Delta \right\}, \end{aligned} \tag{53}$$

$$A_0 = A_0(f_0) = \bigcup_{x \in X_0^0} A_0(x).$$

Let $x \in X_0^0$, $\Delta \in A_0(x)$. We set

$$J_0^0(x,\Delta)=J_0^0(f_0,x,\Delta)=\left\{k;\ 0\leqslant k\leqslant M_0, m_0-\frac{k}{q_0}-\ell_k^0\Delta=\chi_0(f_0,x,\Delta)\right\},$$

$$c_0^0(x,\Delta) = card J_0^0(x,\Delta).$$

If either $A_0 = \emptyset$ or $c_0^0(x,\Delta) = 2$ for all pairs $\{(x,\Delta)\}$: $x \in X_0^0$, $\Delta \in A_0(x)$ (when $A_0 \neq \emptyset$) then the problem is solved by Theorem 2. If for a pair (x_0,Δ_0) $c_0^0(x_0,\Delta_0) \geqslant 3$ then this pair generates numbers $a_1 \in (0,1)$, and q_1 as the smallest natural number for which $q_1(q_0\Delta_0) \in N$ and the function $f_1(t,x) = f_1(x_0,\Delta_0,t,x)$ by formulas (49)-(50).

It is obvious that there is a finite number n_1 of such pairs $\{(x,\Delta)\}$ for which $c_0^0(x,\Delta) \geqslant 3$. We can assume for simplicity that $n_1 = 1$. In this case the problem leads to study the behaviour of the function

$$f_1(t,x) = \sum_{i=0}^{M_1} t^{m_1-\frac{i}{q_0q_1}} Q_i^1(x) \tag{54}$$

on the set (t,x): $t \in (0,\infty)$, $a_1 \leqslant |x| \leqslant a_1^{-1}$. Here $m_1 = \chi_0(f_0,x_0,\Delta_0)$.

We notice that in this case (while $c_0^0(x_0,\Delta_0 \geqslant 3)$

$$m_1 \leqslant m_0 - \frac{2}{q_0}. \tag{55}$$

Really, if $0 \in J_0^0 = J_0^0(x_0,\Delta_0)$ then by Lemma 5 $\ell_0^0(x_0) \geqslant 2$ and by definition of the number q_0, $q_0\Delta_0 \geqslant 1$, hence $m_1 = m_0 - \ell_0^0\Delta_0 \leqslant m_0 - 2/q_0$. If $0 \notin J_0^0$ then $i_0 \in J_0^0$ for a number $i_0 \geqslant 2$ and inequality (55) is obvious in this case.

We introduce sets $\{X_i^1\}$ for polynomials $\{Q_i^1\}$, numbers $\{\ell_i^1(x)\}$ for $x \in X_0^1$ $(i = 0,1,\dots,M_1)$, sets $A_1(f,x)$, $A_1(f_1)$, and $J_0^1(f_1,x,\delta)$ for $x \in X_0^1$, $\Delta \in A_1(x)$, and numbers $m_2 = m_2(f_1,x,\Delta)$, $c_0^1(x,\Delta) = card J_0^1(f_1,x,\Delta)$ for function f_1 as above.

Proceeding as above (see Theorem 2 and formula (55)) one may prove that

1) $m_2 \leqslant m_1 - \frac{2}{q_0q_1}$, when $f_1 \in I_1(a_1)$,

2) the problem will be solved if either $A_1(f_1) = \emptyset$ or $c_0^1(x,\Delta) = 2$ for all pairs (x,Δ): $x \in X_0^1$, $\Delta \in A_1(f_1,\ x)$.

Otherwise (when $c_0^1(x_1,\Delta_1) \geqslant 3$ for a (unique) pair (x_1,Δ_1): $x_1 \in X_0^1$, $\Delta_1 \in A_1(x_1)$) the numbers $q_2 \in N$, $a_2 \in (0,1)$ and the function f_2 arise and the problem leads to study this function behaviour at infinity etc.

To sum up, the polynomial P generates a tree. The branches of this tree are n_0 functions $\{f_0\}$ of type (51). The branches of these branches are $n_0 n_1$ functions $\{f_1\}$ of type (54) and so on. Assuming for simplicity of notation that $n_0 = n_1 = n_2 = \cdots = 1$ we get the set of functions (assuming that $c_0^k(x_s, \Delta_s) \geqslant 3,\ \ k = 0,\ 1,\ \ldots,\ s$)

$$f_s(t,x) = \sum_{i=0}^{M_s} t^{m_s - i/(q_0 q_1 \ldots q_{s-1})} Q_i^s(x), \quad s = 0, 1, \ldots, \tag{56}$$

for which

$$m_s \leqslant m_{s-1} - \frac{2}{q_0 q_1 \cdots q_{s-1}}, \quad s = 1, 2, \ldots\ . \tag{57}$$

Let σ_s be the number of nonzero coefficients of the polynomial Q_0^s $(s = 0, 1, \ldots)$. It is easy to see that

$$\sigma_{s+1} = c_0^s(x_s, \Delta_s) \equiv card J_0^s(f_s, x_s, \Delta_s) \qquad (s = 0, 1, \ldots). \tag{58}$$

We prove that after a finite number s_0 of steps we come either to the case when $m_{s_0} \leqslant 0$ or $c_0^{s_0}(x_{s_0}, \Delta_{s_0}) = 2$. In both cases the chain of functions $\{f_s\}$ will break down and consequently the problem will be solved completely.

Lemma 6 *Let $R(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ $(a_n \neq 0)$ be a polynomial with real coefficients, and σ the number of its nonzero coefficients. If $0 \neq x_0 \in R^1$, $R(x_0) = 0$ and $\ell = \ell(x_0)$ is the order of x_0, then $\ell \leqslant \sigma - 1$.*

Proof The result is obvious if $\sigma \leqslant 2$. Let $\sigma \geqslant 3$ and $a_{j_k} \neq 0$ $(k = 1, \ldots, \sigma)$, $j_1 < j_2 < \cdots < j_\sigma$. Let us consider the system of linear algebraic equations with respect to the unknowns $a_{j_1}, a_{j_2}, \ldots, a_{j_\sigma}$

$$R(x_0) = R'(x_0) = \cdots = R^{(\sigma-1)}(x_0) = 0. \tag{59}$$

If, on the contrary, $\ell \geqslant \sigma$ then it is easy to see that the determinant of this system

$$d = x_0^\alpha \prod_{s=2}^{\sigma-1} \prod_{i<s} (j_s - j_i) \neq 0, \qquad \alpha = \sum_{s=1}^{\sigma} \sum_{k=0}^{\min\{j_s, \sigma-1\}} (j_s - k).$$

Consequently the system (59) has only the trivial solution, which contradicts the conditions $a_{j_k} \neq 0$ $(1 \leqslant k \leqslant \sigma)$.

Lemma 7 *Let s be fixed, f_s be a function of type (56) and*

$$\sigma_{s+1} \geqslant \frac{\sigma_s}{2} + 2. \tag{60}$$

Then $(q_0 q_1 \dots q_s)\Delta_s \in N$, *i.e.* $q_{s+1} = 1$.

Proof First we prove that (60) implies the existence of a pair of indices $i, j \in J_0^s$ such that $\left|\ell_i^s - \ell_j^s\right| = 1$. Indeed, let $J_0^s = \{i_1, i_2, \dots, i_{c^s}\}$ where $i_1 < i_2 < \dots < i_{c_0^s}$. Then (see Remark 1) $\ell_{i_1}^s > \ell_{i_2}^s > \dots > \ell_{i_{c_0^s}}^s$. Let, on the contrary, $\left|\ell_{i_j}^s - \ell_{i_{j+1}}^s\right| \geqslant 2 \;\; (j = 1, 2, \dots, c_0^s - 1)$, then

$$\ell_{i_1}^s \geqslant \ell_{i_2}^s + 2 \geqslant \ell_{i_3}^s + 4 \geqslant \dots \geqslant \ell_{i_{c_0^s}}^s + 2\left(c_0^s - 1\right).$$

Since (see (58)) $c_0^s = \sigma_{s+1}$ then together with (60) it implies that

$$\ell_{i_1}^s \geqslant \ell_{i_{c_0^s}}^s + 2(\sigma_{s+1} - 1) \geqslant \ell_{i_{c_0^s}}^s + 2\left[\frac{\sigma_s}{2}\right] + 2 \geqslant \ell_{i_{c_0^s}}^s + \sigma_s + 1 \geqslant \sigma_s + 1.$$

On the other hand since $\ell_0^s \geqslant \ell_{i_1}^s$, then $\ell_0^s \geqslant \sigma_s + 1$, which contradicts to Lemma 6 and proves Lemma 7.

So, for each $s = 0, 1, \dots$ only one of the following cases is possible: either $q_{s+1} = 1$ or

$$\sigma_{s+1} \leqslant \frac{\sigma_s}{2} + 1. \tag{61}$$

Let $b_s \in N_0^n$ be the smallest number, for which

$$m_s > 0, \qquad m_s - \frac{2b_s}{q_0 q_1 \cdots q_{s-1}} \leqslant 0 \qquad (s = 0, 1, \dots). \tag{62}$$

If for a number $k \in N_0$ $q_{k+1} = q_{k+2} = \dots = q_{k+b_k} = 1$, then (57) and (62) imply that $m_{k+b_k} \leqslant 0$ and then $P \notin I_2$. If for each $s \in N_0$ there exists a number $k_s \in N_0$ such that $s \leqslant k_s \leqslant s + b_s$ and $q_{k_s} > 1$, then without loss of generality we can suppose that $q_s > 1$ for all $s \in N_0$. Then from Lemma 7 it follows that the numbers $\{\sigma_s\}$ satisfy inequality (61). Hence

$$\sigma_s \leqslant \left[\frac{\sigma_0}{2^s}\right] + 1 + \frac{1}{2} + \dots + \frac{1}{2^s} \leqslant \frac{\sigma_0}{2^s} + 2 \qquad (s \in N).$$

This implies that $\sigma_k = card J_0^{k-1} < 3$ for $k = [\log_2 \sigma_0] + 1$, i.e. $card J_0^{k-1} = 2$ and we come to the situation in Theorem 2.

Thus, we have finished the construction of the algorithm.

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ON SOME PROPERTIES OF DEGENERATE ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

Abduhamid Dzhuraev
Inst. of Math., Tajik Acad. Sci.
Pr. Rudaki 33, 734025 Dushanbe, Tajikistan
dzhuraev@khj.tajik.net

Abstract In this paper we study properties of some elliptic systems of partial differential equations in bounded domains of the plane degenerating on their boundaries and their generalizations to the unit ball of $\mathbb{C}^n$.

Keywords: elliptic systems degenerating at the boundary, Beltrami system in $\mathbb{C}$ and $\mathbb{C}^n$, invariant Laplace equation, invariant Cauchy-Riemann system, Fueter system

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1. Elliptic systems in the plane

1.1 FIRST ORDER ELLIPTIC SYSTEMS IN THE PLANE

Let G be a multiply connected domain in the complex plane $\mathbb{C}$ of the variable $z = x + iy$ which is bounded by a finite number of closed, non-intersecting smooth curves $\Gamma_k, 0 \le k \le m$. Any first order linear elliptic system in G with respect to two real valued functions $u_1(x, y), u_2(x, y)$ may be put into the following single complex equation

$$a(z)u_{\bar{z}} + b(z)\overline{u_{\bar{z}}} + c(z)u_z + d(z)\overline{u_z} + a_0(z)u + b_0(z)\bar{u} = f(z), \tag{1}$$

where all coefficients and the unknown function $u(z) = u_1 + iu_2$ are now complex valued. The ellipticy of (1) at the point $z \in G$ means the definiteness of the quadratic form

$$\chi(z,\zeta) \equiv |a(z)\zeta + c(z)\bar{\zeta}|^2 - |d(z)\zeta + b(z)\bar{\zeta}|^2 \equiv (|a(z)|^2$$

$$+|c(z)|^2 - |b(z)|^2 - |d(z)^2|)|\zeta|^2 + 2Re[(a(z)\overline{c(z)} - \overline{b(z)}d(z))\zeta^2] \tag{2}$$

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with respect to $\zeta = \xi_1 + i\xi_2$, i.e the negativeness of its discriminant $\sigma_0(z) = 4|\alpha(z)|^2 - (A(z) - B(z))^2 < 0$ at $z \in G$, where

$$A(z) = |a(z)|^2 - |b(z)|^2,\ B(z) = |d(z)|^2 - |c(z)|^2,$$
$$\alpha(z) = a(z)\overline{c(z)} - \overline{b(z)}d(z),\ \beta(z) = a(z)\overline{d(z)} - \overline{b(z)}c(z).$$

Since $|\beta(z)|^2 - |\alpha(z)|^2 = A(z)B(z)$, then it is easy to see that

$$\begin{aligned}\sigma_0(z)(|\alpha(z)| + |\beta(z)|)^2 \\ &= (|A(z)| + |\alpha(z)| + |\beta(z)|)(|B(z)| + |\alpha(z)| + |\beta(z)|) \\ &\times(|A(z)| - |\alpha(z)| - |\beta(z)|)(|B(z)| - |\alpha(z)| - |\beta(z)|)\end{aligned}$$

and therefore (1) is elliptic at $z \in G$ if and only if

i) $|A(z)| > |\alpha(z)| + |\beta(z)|, |B(z)| < |\alpha(z)| + |\beta(z)|$

or

ii) $|B(z)| > |\alpha(z)| + |\beta(z)|, |A(z)| < |\alpha(z)| + |\beta(z)|$

at $z \in G$.

In case i) eliminating $\overline{u_{\bar{z}}}$ equation (1.1) may be put into the form

$$u_{\bar{z}} - q_1(z)u_z - q_2(z)\overline{u_z} + A_0(z)u + B_0(z)\bar{u} = f_0(z) \tag{3}$$

with the condition

$$|q_1(z)| + |q_2(z)| < 1 \tag{4}$$

in G and in case ii) eliminating $\overline{u_z}$ it may be put into the form

$$u_z - q_1(z)u_{\bar{z}} - q_2(z)\overline{u_{\bar{z}}} + A_0(z)u + B_0(z)\bar{u} = f_0(z) \tag{5}$$

with condition (4) in G. Equation (1) is said to belong to $O(G)$ if i) holds in G and is said to belong to $\bar{O}(G)$ if ii) holds in G. Let us be given two elliptic equations

$$a_1(z)u_{\bar{z}} + b_1(z)\overline{u_{\bar{z}}} + c_1(z)u_z + d_1(z)\overline{u_z} + a_0(z)u + b_0(z)\bar{u} = 0 \tag{6}$$

and

$$a_2(z)v_{\bar{z}} + b_2(z)\overline{v_{\bar{z}}} + c_2(z)v_z + d_2(z)\overline{v_z} + a^0(z)v + b^0(z)\bar{v} = 0 \tag{7}$$

in G with coefficients in their principal parts being measurable bounded in G and with the lower coefficients belonging to $L_p(\bar{G}), p > 2$. We shall consider the following problem :

Problem I *Find a pair $(u(z), v(z))$ with $u(z), v(z)$ being solutions of equations (6) and (7) in G respectively, continuous up to $\Gamma = \Gamma_0 + \cdots + \Gamma_m$ such that*

$$u(\zeta) - v(\zeta) = \gamma(\zeta), \tag{8}$$

where $\gamma(\zeta)$ is a continuous function given on Γ .

Note that if all coefficients in (6),(7), except a_1, a_2 are equal zero, then problem I has infinitely many solutions $(\varphi(z), \varphi(z))$,where $\varphi(z)$ is an arbitrary holomorphic function in G. Moreover we have

Proposition 1 *Let both equations* $(6), (7)$ *coincide with* (1) *and* $f \equiv 0$ *in* G. *Then in case when i) holds Problem I is solvable if and only if for any function $\varphi(z)$ holomorphic in G and continuous in $\bar{G} = G + \Gamma$ the equality*

$$\int_G \rho(z)\varphi(z)d\bar{z} \wedge dz - \int_\Gamma \gamma(\zeta)\varphi(\zeta)d\zeta = 0$$

holds, where $\rho(z)$ is the solution of the following uniquely soluable singular integral equation

$$\begin{aligned} &a(z)\rho(z) + b(z)\bar{\rho}(z) - \frac{c(z)}{\pi}\int_G \frac{\rho(\zeta)d\zeta d\eta}{(\zeta - z)^2} - \frac{d(z)}{\pi}\int_G \frac{\bar{\rho}(\zeta)d\zeta d\eta}{(\bar{\zeta} - \bar{z})^2} \\ &-\frac{\alpha_0(z)}{\pi}\int_G \frac{\rho(\zeta)d\zeta d\eta}{\zeta - z} - \frac{b_0(z)}{\pi}\int_G \frac{\bar{\rho}(\zeta)d\zeta d\eta}{\bar{\zeta} - \bar{z}} \\ &= -\frac{c(z)}{2\pi i}\int_\Gamma \frac{\gamma(\zeta)d\zeta}{(\zeta - z)^2} + \frac{d(z)}{2\pi i}\int_\Gamma \frac{\bar{\gamma}(\zeta)d\bar{\zeta}}{(\bar{\zeta} - \bar{z})^2} - \frac{\alpha_0(z)}{2\pi i}\int_\Gamma \frac{\gamma(\zeta)d\zeta}{\zeta - z} \\ &+\frac{b_0(z)}{2\pi i}\int_\Gamma \frac{\bar{\gamma}(\zeta)d\bar{\zeta}}{\bar{\zeta} - \bar{z}}. \end{aligned} \tag{S1}$$

In case when ii) holds Problem I is soluble if and only if for any function $\psi(z)$, holomorphic in G and continuous in $\bar{G}$ the equality

$$\int_G \nu(z)\overline{\psi(z)}d\bar{z} \wedge dz + \int_\Gamma \gamma(\zeta)\overline{\psi(\zeta)}d\bar{\zeta} = 0$$

holds, where $\nu(z)$ is the solution of the following uniquely soluble singular integral equation

$$\begin{aligned} &d(z)\overline{\nu(z)} + c(z)\nu(z) - \frac{b(z)}{\pi}\int_G \frac{\overline{\nu(z)}d\zeta d\eta}{(\zeta - z)^2} - \frac{\alpha(z)}{\pi}\int_G \frac{\nu(\zeta)d\zeta d\eta}{(\bar{\zeta} - \bar{z})^2} \\ &-\frac{b_0(z)}{\pi}\int_G \frac{\overline{\nu(\zeta)}d\zeta d\eta}{\zeta - z} - \frac{\alpha_0(z)}{\pi}\int_G \frac{\nu(z)d\zeta d\eta}{\bar{\zeta} - \bar{z}} \\ &= \frac{b(z)}{2\pi i}\int_\Gamma \frac{\bar{\gamma}(\zeta)d\zeta}{(\zeta - z)^2} + \frac{\alpha(z)}{2\pi i}\int_\Gamma \frac{\gamma(\zeta d\bar{\zeta}}{\bar{\zeta} - \bar{z})^2} - \frac{b_0(z)}{2\pi i}\int_\Gamma \frac{\bar{\gamma}(\zeta)d\zeta}{\zeta - z} \\ &+\frac{\alpha_0(z)}{2\pi i}\int_\Gamma \frac{\gamma(\zeta d\bar{\zeta}}{\bar{\zeta} - \bar{z}}. \end{aligned} \tag{S2}$$

The homogeneous problem I $(\gamma \equiv 0)$ *has infinitely many solutions* $(u(z), u(z))$, *where $u(z)$ is an arbitrary generalized holomorphic function in G that is a solution of equation* (1) *with* $f \equiv 0$.

Proof As the difference $w(z) = u(z) - v(z)$ is a solution of (1) with $f \equiv 0$ in G and $w(\zeta) = \gamma(\zeta)$ on Γ, then the proposition follows from Theorem 1 (see [2], also [4], Theorem 2.1).

Theorem 1 *If equation (6) belongs to $O(G)$ and equation (7) belongs to $\overline{O}(G)$ (or (6) belongs to $\overline{O}(G)$ and (7) belongs to $O(G)$), then problem I is soluable for any right-hand side and the corresponding homogenous problem ($\gamma \equiv 0$) has exactly two linearly-independent solutions.*

Proof If (6) belongs to $O(G)$, then it is reduced to

$$u_{\bar{z}} - q_1^{(1)}(z)u_z - q_2^{(1)}(z)\overline{u_z} + A_1(z)u + B_1(z)\bar{u} \equiv 0 \tag{9}$$

with the condition

$$|q_1^{(1)}(z)| + |q_2^{(1)}(z)| < 1$$

in G and if (7) belongs to $\bar{O}(G)$, then it is reduced to

$$v_z - q_1^{(2)}(z)v_{\bar{z}} - q_2^2(z)(z)\overline{v_{\bar{z}}} + A_2(z)v + B_2(z)\bar{v} = 0 \tag{10}$$

with condition $|q_1^{(2)}(z)| + |q_2^{(2)}(z)| < 1$ in G.

Conjugating (10) we obtain the equation of type (9) for the function $w(z) = \bar{v}(z)$:

$$w_{\bar{z}} - \bar{q}_1^{(2)}(z)w_z - \bar{q}_2^2(z)\overline{w_z} + \bar{A}_2(z)w + B_2(z)\bar{w} = 0$$

in G and the boundary condition (18) is reduced to the Riemann-Hilbert condition for the vector function $v(z) = (u(z), w(z))$:

$$\mathrm{Re}[Gv(\zeta)] = H(\zeta)$$

with $H(\zeta) = (Re\gamma(\zeta), -Im\gamma(\zeta))$,

$$G = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

and the theorem follows as $\det G = -2i \neq 0$.

Theorem 2 *Let both equations (6), (7) belong to the same $O(G)$ (or $\bar{O}(G)$), but having different coefficients. If*

$$\left(1 - |q_1^{(1)}(\zeta)|^2 + |q_2^{(1)}(\zeta)|^2 + \sqrt{\sigma^{(1)}})q_2^{(2)}(\zeta)\right)$$
$$-\left(1 - |q_1^{(2)}(\zeta)|^2 + |q_2^{(2)}(\zeta)|^2 + \sqrt{\sigma^{(2)}})q_2^{(1)}(\zeta)\right) \neq 0$$

on Γ, where $\sigma^{(i)}(\zeta) = (1-|q_1^{(i)}(\zeta)|^2+|q_2^{(i)}(\zeta)|^2)^2-4|q_2^{(i)}(\zeta)|^2$, then Problem I is well posed and has the Noether property: it is normally solvable and has a finite index .

Proof If both equations (6),(7) belong to $O(G)$, then they are reduced to (9) with $q_1^{(i)}(z), q_2^{(i)}(z), A_i(z), B_i(z)$ in place of $q_1^{(1)}(z), q_2^{(1)}(z), A_1(z)$, $B_1(z)$ with the conditions

$$|q_1^{(i)}(z)| + |q_2^{(i)}(z)| < 1, i = 1, 2.$$

Introducing new unknown functions $\varphi(z)$and $\psi(z)$ by

$$\varphi(z) = \frac{1 - |q_1^{(1)}(z)|^2 + |q_2^{(1)}(z)|^2 + \sqrt{\sigma^{(1)}(z)}}{2(1 - |q_1^{(1)}(z)|^2)} u(z) - \frac{q_2^{(1)}(z)}{1 - |q_1^{(1)}(z)|^2} \overline{u(z)},$$

$$\psi(z) = \frac{1 - |q_1^{(2)}(z)|^2 + |q_2^{(2)}(z)|^2 + \sqrt{\sigma^{(2)}(z)}}{2(1 - |q_1^{(2)}(z)|^2)} v(z) - \frac{q_2^{(2)}(z)}{1 - |q_2^{(2)}(z)|^2} \overline{v(z)}$$

instead of (9) we obtain the following elliptic equations for φ and ψ:

$$\varphi_{\bar{z}} - q^{(1)}(z)\varphi_z + A^{(1)}(z)\varphi + B^{(1)}(z)\bar{\varphi} = 0, \tag{11}$$

$$\psi_{\bar{z}} - q^{(2)}(z)\psi_z + A^{(2)}(z)\psi + B^{(2)}(z)\bar{\psi} = 0, \tag{12}$$

in G and instead of (8) we obtain the following Riemann-Hilbert type boundary condition

$$\mathrm{Re}[G(\zeta)\mathcal{U}(\zeta)] = H(\zeta) \tag{13}$$

for the unknown vector-function $\mathcal{U}(z) = (\varphi(z), \psi(z))$, where

$$G(\zeta) = (G_{ij}(\zeta)), 1 \leq i, j \leq 2,$$

$$\begin{aligned}
G_{11} &= \frac{1 - |q_1^{(1)}|^2 + |q_2^{(1)}|^2 + \sqrt{\sigma^{(1)}}}{2\Delta_1^*(1 - |q_1^{(1)}|^2)} + \frac{\bar{q}_2^{(1)}}{\Delta_1^*(1 - |q_1^{(1)}|^2)}, \\
G_{12} &= -\frac{1 - |q_1^{(2)}|^2 + |q_2^{(2)}|^2 + \sqrt{\sigma^{(2)}}}{2\Delta_2^*(1 - |q_1^{(2)}|^2)} - \frac{\bar{q}_2^{(2)}}{\Delta_2^*(1 - |q_1^{(2)}|^2)}, \\
G_{21} &= i\Big(\frac{1 - |q_1^{(1)}|^2 + |q_2^{(1)}|^2 + \sqrt{\sigma^{(1)}}}{2\Delta_1^*(1 - |q_1^{(1)}|^2)} - \frac{\bar{q}_2^{(1)}}{\Delta_1^*(1 - |q_1^{(1)}|^2)}\Big), \\
G_{22} &= -i\Big(\frac{1 - |q_1^{(2)}|^2 + |q_2^{(2)}|^2 + \sqrt{\sigma^{(2)}}}{2\Delta_2^*(1 - |q_1^{(2)}|^2)} - \frac{\bar{q}_2^{(2)}}{\Delta_2^*(1 - |q_2^{(2)}|^2)}\Big),
\end{aligned}$$

$$\begin{aligned}
q^{(\nu)}(z) &= \frac{2\ \mathrm{Im}\ q_1^{(\nu)}(z) - i(\sqrt{\sigma^{(\nu)}(z)} - \delta_\nu(z))}{2\ \mathrm{Im}\ q_1^{(\nu)}(z) - i(\sqrt{\sigma^{(\nu)}(z)} + \delta_\nu(z))}, \\
\delta_\nu(z) &= |1 - q_1^\nu(z)|^2 - |q_2^\nu(z)|^2,
\end{aligned}$$

$$\Delta_\nu^*(z) = \frac{\sqrt{\sigma^{(\nu)}(z)}}{2(1-|q_1^{(\nu)}(z)|^2)}\Big(\sqrt{\sigma^{(\nu)}(z)} + 1 + |q_1^{(\nu)}(z)|^2 + |q_2^{(\nu)}(z)|^2\Big).$$

Now the theorem follows as

$$\det G(\zeta) = 2i(G_{11}G_{22} - G_{12}G_{21}) = \frac{1}{2\Delta_1^*\Delta_2^*(1-|q_1^{(1)}|^2)(1-|q_1^{(2)}|^2)}$$

$$\times\Big((1-|q_1^{(1)}|^2 + |q_2^{(1)}|^2 + \sqrt{\sigma^{(1)}})\bar{q}_2^{(2)}$$

$$-(1-|q_1^{(2)}|^2 + |q_2^{(2)}|^2 + \sqrt{\sigma^{(2)}})\bar{q}_2^{(1)})\Big) \neq 0$$

on Γ.

The problem considered in Theorem 1 as we have seen is reduced to the particular case of the following problem: Let us be given two equations

$$u_{\bar z} - q_1^{(1)}(z)u_z - q_2^{(1)}(z)\overline{u_z} + A_1(z)u + B_1(z)\bar u = f_1(z), \qquad (14)$$

and

$$v_{\bar z} - q_1^{(2)}(z)v_z - q_2^{(2)}(z)\overline{v_z} + A_2(z)v + B_2(z)\bar v = f_2(z), \qquad (15)$$

belonging to $O(G)$. Find solutions $u(z)$ and $v(z)$ of (14) and (15) in G, satisfying the condition

$$u(\zeta) - \lambda(\zeta)\overline{v(\zeta)} = 0, \qquad (16)$$

where $\lambda(\zeta)$ is a given function, satisfying the condition $\lambda(\zeta) \neq 0$ on Γ.

Let us describe the method how to solve this problem. For simplicity we assume for G to be the unit disc: $G = D = \{z \in \mathbb{C} : |z| < 1\}$. Let m be the winding number of $\lambda(\zeta)$:

$$m = \frac{1}{2\pi i}\{\log\lambda(\zeta)\}_\Gamma, \qquad (17)$$

and let $\lambda_0(\zeta) = \zeta^{-m}\lambda(\zeta)$, so $\{log\lambda_0(\zeta)\}_\Gamma = 0$. Then we can find a pair $(X(z), Y(z))$ of holomorphic functions in D such that

$$X(\zeta) - \lambda_0(\zeta)\overline{Y(\zeta)} = 0. \qquad (18)$$

The condition (16) then can be written as

$$\varphi(\zeta) - \zeta^m\overline{\psi(\zeta)} = 0, \qquad (19)$$

for the functions $\varphi(z) = u(z)/X(z)$ and $\psi(z) = v(z)/Y(z)$ satisfying the equations

$$\varphi_{\bar z} - q_1^{(1)}(z)\varphi_z(z) - \hat q_2^{(1)}(z)\overline{\varphi_z(z)} + \hat A_1(z)\varphi + \hat B_1(z)\bar\varphi = \hat f_1(z), \tag{20}$$

$$\psi_{\bar z} - q_1^{(2)}(z)\psi_z(z) + \hat q_2^{(2)}(z)\overline{\psi_z(z)} + \hat A_2(z)\psi + \hat B_2(z)\bar\psi = \hat f_2(z), \tag{21}$$

in D, where

$$\begin{aligned}
\hat q_2^{(1)}(z) &= q_2^{(1)}(z)\frac{\overline{X(z)}}{X(z)}, \hat A_1(z) = A_1(z) - q_1^{(1)}(z)(\log X(z))_z,\\
\hat B_1(z) &= B_1(z)\frac{\overline{X(z)}}{X(z)} - q_2^{(1)}(z)\frac{\overline{X_z(z)}}{X(z)}, \hat B_2(z) = B_1\frac{\overline{Y(z)}}{Y(z)} - q_2^{(2)}\frac{\overline{Y_z(z)}}{Y(z)},\\
\hat q_2^{(2)}(z) &= q_2^{(2)}(z)\frac{\overline{Y(z)}}{Y(z)}, \hat A_2(z) = A_2(z) - q_1^{(2)}(z)(\log Y(z))_z.
\end{aligned}$$

To solve the problem (19)-(21) put $\varphi_{\bar z} = \rho(z)$ and $\psi_{\bar z} = \nu(z)$ in D. Then

$$\varphi(z) = \Phi(z) - \frac{1}{\pi}\int_D \frac{\rho(\zeta)d\zeta d\eta}{\zeta - z}, \psi(z) = \Psi(z) - \frac{1}{\pi}\int_D \frac{\nu(z)d\zeta d\eta}{\zeta - z}, \tag{22}$$

where $\Phi(z), \Psi(z)$ are arbitrary holomorphic functions in D such that according (19) they satisfy the boundary condition

$$\Phi(\zeta) - \zeta^m\overline{\Psi(\zeta)} = \gamma(\zeta) \tag{23}$$

on the circle $|\zeta| = 1$, where

$$\gamma(\zeta) = \frac{\zeta^{m+1}}{\pi}\int_D \frac{\nu(\tau)dD_\tau}{1 - \bar\tau\zeta} - \frac{1}{\pi}\int_D \frac{\rho(\tau)dD_\tau}{\tau - \zeta}.$$

Solving (23) through the power series expansions of $\Phi(z), \Psi(\zeta)$ in the unit disc and substituting then (22) into (20) and (21) for the determination of $\rho(z)$ and $\nu(z)$ we obtain singular integral equations over the disc like the above equations $(S_1), (S_2)$ and derive the number of linearly independent solutions of the homogeneous problem (14)-(16) ($f_1 = f_2 \equiv 0$) in case $m \geq 0$ and the number of independent solvability conditions on $f_1(z), f_2(z)$ for the problem (14)-(16) to be solvable.

1.2 First order elliptic systems in the unit disk degenerating on the circle

Let $a^{(\gamma)}(x, y) = (a_{\gamma1}, a_{\gamma2})$ be a real vector field in G and let $a^{(1)}(x, y)$, $a^{(2)}(x, y)$ be linearly independent in G that is $\Delta \equiv a_{11}a_{22} - a_{12}a_{21} \neq 0$

in G. Consider the following first order system for two real functions $u_1(x,y), u_2(x,y)$:

$$a_{11}\frac{\partial u_1}{\partial x}+a_{12}\frac{\partial u_1}{\partial y}-a_{21}\frac{\partial u_2}{\partial x}-a_{22}\frac{\partial u_2}{\partial y}=f_1, \tag{24}$$

$$a_{21}\frac{\partial u_1}{\partial x}+a_{22}\frac{\partial u_1}{\partial y}+a_{11}\frac{\partial u_2}{\partial x}+a_{12}\frac{\partial u_2}{\partial y}=f_2.$$

This system is obviously elliptic in G. If $\Delta > 0$ in G, then this will be called a Beltrami system and an anti Beltrami system if $\Delta < 0$ in G. Multiplying the second equation by i and adding it to the first one we obtain the following single equation

$$a(z)w_{\bar{z}}+b(z)w_z=f(z), \tag{25}$$

for $w(z) = u_1 + iu_2$, where $a(z) = a_{11} + a_{12} + i(a_{21} - a_{12}), b(z) = a_{11} - a_{22} + i(a_{21} + a_{12})$, so that $|a(z)|^2 - |b(z)|^2 = 2\triangle(z)$.

Considering instead of (24) the system

$$\frac{\partial u_1}{\partial x}-a\left(a\frac{\partial u_1}{\partial x}+b\frac{\partial u_1}{\partial y}\right)-\frac{\partial u_2}{\partial y}=f_1,$$

$$\frac{\partial u_1}{\partial y}-b\left(a\frac{\partial u_1}{\partial x}+b\frac{\partial u_1}{\partial y}\right)+\frac{\partial u_2}{\partial x}=f_2, \tag{26}$$

with real $a(x,y), b(x,y)$ such that $a^2 + b^2 < 1$ in G it is elliptic in G because its principal symbol is equal to $\sigma = \xi_1(\xi_1 - a(a\xi_1 + b\xi_2) + \xi_2(\xi_2 - b(a\xi_1 + b\xi_2)) = \xi_1^2 + \xi_2^2 - (a\xi_1 + b\xi_2)^2 \geq (1 - a^2 - b^2)(\xi_1^2 + \xi_2^2)$. Multiplying the second equation by i and adding it to the first one we obtain the following single equation similar to (3) and belonging to $O(G)$:

$$w_{\bar{z}}-q_1(z)w_z-q_2\overline{w_z}=f(z) \tag{27}$$

for $w = u_1 + iu_2$ with

$$q_1(z)=\frac{a^2-b^2+2iab}{2(2-a^2-b^2)}, q_2(z)=\frac{a^2+b^2}{2(2-a^2-b^2)}$$

such that

$$|q_1|+|q_2|=\frac{a^2+b^2}{2-a^2-b^2}=\frac{1-(1-a^2-b^2)}{1+(1-a^2-b^2)}<1.$$

The transposed system to (26) is

$$\frac{\partial u_1}{\partial x} - a\left(a\frac{\partial u_1}{\partial x} + b\frac{\partial u_1}{\partial y}\right) + \frac{\partial u_2}{\partial y} - b\left(a\frac{\partial u_2}{\partial x} + b\frac{\partial u_2}{\partial y}\right) = f_1$$

$$-\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = f_2. \tag{28}$$

Multiplying the second equation by i and adding it to the first one we obtain the following single equation similar to (5) and belonging to $\overline{O(G)}$:

$$v_z - \frac{a^2 - b^2 - 2iab}{2(2 - a^2 - b^2)} v_{\bar{z}} - \frac{a^2 - b^2 + 2iab}{2(2 - a^2 - b^2)} \overline{v_{\bar{z}}} = f \tag{29}$$

for $v(z) = u_1 + iu_2$.

Considering system (24) in the unit disc D with

$$a_{11} = 1 - \frac{x^2}{1 + \sqrt{1 - x^2 - y^2}}, a_{12} = -\frac{xy}{1 + \sqrt{1 - x^2 - y^2}},$$

$$a_{21} = -\frac{xy}{1 + \sqrt{1 - x^2 - y^2}}, a_{22} = 1 - \frac{y^2}{1 + \sqrt{1 - x^2 - y^2}},$$

equation (25) becomes

$$B_1 w :\equiv w_{\bar{z}} - \frac{z^2}{(1 + \sqrt{1 - |z|^2})^2} w_z = f(z). \tag{30}$$

Considering systems (26) and (28) in D with $a = x, b = y$ equation (27) becomes

$$w_{\bar{z}} - \frac{z^2}{2(2 - |z|^2)} w_z - \frac{|z|^2}{2(2 - |z|^2)} \overline{w_z(z)} = f(z) \tag{31}$$

and equation (29) becomes

$$v_z - \frac{\bar{z}^2}{2(2 - |z|^2)} v_{\bar{z}} - \frac{z^2}{2(2 - |z|^2)} \overline{v_{\bar{z}}} = f(z). \tag{32}$$

Equations (30)-(32) being elliptic inside the disc D degenerate on its boundary $|z| = 1$. As

$$1 - \frac{|z|^2}{(1 + \sqrt{1 - |z|^2})^2} = \frac{2\sqrt{1 - |z|^2}}{1 + \sqrt{1 - |z|^2}},$$

then the degeneration on $|z| = 1$ of (30) is of order one half and moreover the circle $|z| = 1$ for (30) is the characteristic set. As $1 - \frac{|z|^2}{2-|z|^2} = \frac{2(1-|z|^2)}{2-|z|^2}$ then the degeneration on $|z| = 1$ of (31) and (32) is of order one and moreover the circle $|z| = 1$ is the characteristic set for (31) and for (32). The homogeneous equation corresponding to (30)

$$w_{\bar{z}} - \frac{z^2}{(1+\sqrt{1-|z|^2})^2} w_z = 0 \tag{33}$$

has the solution

$$\zeta = \frac{z}{1+\sqrt{1-|z|^2}} \tag{34}$$

which maps the unit disc $|z| < 1$ onto the unit disc $|\zeta| < 1$ homeomorphically, so that the general solution of (33) is a superposition of an arbitrary function $\phi(\zeta)$ holomorphic in the unit disc $|\zeta| < 1$ and the function (34)

$$w(z) = \phi\left(\frac{z}{1+\sqrt{1-|z|^2}}\right) = \phi \circ \zeta. \tag{35}$$

The inverse function to (34)

$$z = \frac{2\zeta}{1+|\zeta|^2} \tag{36}$$

is the homeomorphism of the unit disc $|\zeta| < 1$ into the unit disc $|z| < 1$, satisfying the following equation inverse to (33)

$$v_{\bar{\zeta}} + \zeta^2 v_\zeta = 0 \tag{33'}$$

the general solution of wich is a superposition of an arbitrary function $\psi(z)$ holomorphic in the unit disc $|z| < 1$ and the function (36)

$$v(z) = \psi \circ z = \psi\left(\frac{2\zeta}{1+|\zeta|^2}\right). \tag{35'}$$

Note that the circle is the set of degeneration of order one and is not a charachteristic set for (33′).
Let us return to system (26) with $a = x, b = y$

$$\frac{\partial u_1}{\partial x} - x\left(x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y}\right) - \frac{\partial u_2}{\partial y} = f_1,$$

$$\frac{\partial u_1}{\partial y} - y\left(x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y}\right) + \frac{\partial u_2}{\partial x} = f_2, \tag{26'}$$

in the unit disc D. Solving (26′) with respect to the derivatives $\partial u_1/\partial x$, $\partial u_1/\partial y$ gives

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= \frac{\partial u_2}{\partial y} + \frac{x^2\partial u_2/\partial y - xy\partial u_2/\partial x}{1-x^2-y^2} + \tilde{f}_1, \\ \frac{\partial u_1}{\partial y} &= -\frac{\partial u_2}{\partial x} + \frac{xy\partial u_2/\partial y - y^2\partial u_2/\partial x}{1-x^2-y^2} + \tilde{f}_2, \end{aligned} \tag{37}$$

because from (26′) it follows that

$$x\frac{\partial u_1}{\partial x} + y\frac{\partial u_1}{\partial y} = \frac{x\partial u_2/\partial y - y\partial u_2/\partial x}{1-x^2-y^2} + \tilde{f}.$$

Eliminating u_1 from (37) we obtain the following second order equation for u_2:

$$\Delta u_2 - x^2\frac{\partial^2 u_2}{\partial x^2} - 2xy\frac{\partial^2 u_2}{\partial x\partial y} - y^2\frac{\partial^2 u_2}{\partial y^2} - x\frac{\partial u_2}{\partial x} - y\frac{\partial u_2}{\partial y} = \tilde{f}$$

or

$$\triangle u_2 - E^2 u_2 = \tilde{f}, \tag{38}$$

where $E :\equiv x\partial/\partial x + y\partial/\partial y$. If we seek a solution of (38) with $\tilde{f} = 0$ in the form $u_2 = g(x^2+y^2)p(x,y)$, where $p(x,y)$ is a harmonic polynomial homogeneous of degree k : $(x\partial/\partial x + y\partial/\partial y)p = kp$, then we obtain $\triangle u_2 = 4(r^2g'' + (k+1)g')p, E^2u_2 = (4r^4g'' + 4(k+1)r^2g' + k^2g)p$, i.e. g is a solution of the hypergeometric equation

$$t(1-t)g''(t) + (\gamma - (\alpha+\beta+1)t)g'(t) - \alpha\beta g(t) = 0,$$

where $t = r^2 = x^2+y^2, \alpha = k/2, \beta = k/2, \gamma = k+1$, so that $Re(\gamma - \alpha - \beta) = 1$. Hence the solution of the Dirichlet problem, having the boundary value $p(x,y)$ is represented by

$$u_2(x,y) = \frac{F(k/2,k/2,k+1;|z|^2)}{F(k/2,k/2,k+1;1)} = \alpha_k(|z|^2)p(x,y)$$

with the Gauss hypergeometric function F and the solution of the Dirichlet problem with an arbitrary boundary function $f(\zeta)$ can be obtained through the expansion of $f(\zeta)$ into the series with respect to homogeneous harmonic polynomials by superposition of this special solution

$$u_2(x,y) = \sum_k \alpha_k(|z|^2)p_k(x,y).$$

Thus we can find $u_2(x,y)$ uniquely through the given Dirichlet boundary condition $u_2(\zeta) = f(\zeta)$ on $|\zeta| = 1$ and then the function $u_1(x,y)$ is determined from (37) up to a constant. But from (26′) with $f_1 = f_2 = 0$ it follows that $x\partial u_2/\partial y - y\partial u_2/\partial x = du_2/ds = 0$ on $| z |= 1$, so that $u_2 \equiv$ const and then $u_1 \equiv$ const again from (26). What we have solved for the elliptic system (26′) or (31) degenerating on the boundary is the Schwarz problem. The Schwarz problem for systems (33) and (33′) can be solved directly through the formulas (35), (35′) by means of the Schwarz integral for holomorphic functions. Considering system (24) in the unit disc D with coefficiets $a_{11} = 1-x^2, a_{12} = a_{21} = -xy, a_{22} = 1-y^2$ equation (25) becomes

$$w_{\bar{z}} - \frac{z^2}{2-|z|^2} w_z = f. \tag{39}$$

This equation being elliptic in the disc $|z| < 1$, degenerates of order one on the circle $|z| = 1$:

$$1 - \frac{|z|^2}{2-|z|^2} = \frac{2(1-|z|^2)}{2-|z|^2}$$

and moreover this circle is the characteristic set for (39). Because of the characteristic degeneration of order one this equation in contrast to (30) has no solution homeomorphically mapping the disc $|z| \le 1$ onto the disc $|\zeta| \le 1$. Besides of that this equation has the solution

$$\zeta = \frac{z}{\sqrt{1-|z|^2}}, \tag{40}$$

which maps the disc $|z| \le 1$ onto the whole complex plane $\mathbb{C}$ of the variable ζ, so that equation (31) is reduced to the inhomogeneous Cauchy-Riemann equation

$$W_{\bar{\zeta}} = \frac{(2+|\zeta|^2)^2}{2(1+|\zeta|^2)^{3/2}} f\left(\frac{\zeta}{\sqrt{1+|\zeta|^2)}}\right)$$

on the whole complex plane and hence is uniquely solvable without any boundary condition and the homogeneous equation (39), $f \equiv 0$ has no other solutions, continuous in the closed disc $|z| \le 1$ except constants. Note that the function

$$z = \frac{\zeta}{\sqrt{1+|\zeta|^2}} \tag{41}$$

inverse to (40) maps the whole plane $\mathbb{C}$ onto the unit disc $|z| < 1$ and is a solutions to the equation

$$v_{\bar{\zeta}} + \frac{\zeta^2}{2 + |\zeta|^2} v_\zeta = 0 \tag{42}$$

on $\mathbb{C}$ inverse to (39) wich is elliptic on the finite part of $\mathbb{C}$ and degenerates at infinity,

$$1 - \frac{|\zeta|^2}{2 + |\zeta|^2} \to 0$$

as $|\zeta| \to \infty$.

Another more simple equation, which possesses the same properties as (39) is

$$w_{\bar{z}} - z^2 w_z = f. \tag{43}$$

Instead of (40) this equation has the solution

$$\zeta = \frac{z}{1 - |z|^2} \tag{44}$$

which maps the disc $|z| < 1$ onto the whole complex plane $\mathbb{C}$, so that (43) is reduced to

$$w_{\bar{\zeta}} = \frac{1}{\sqrt{1 + 4|\zeta|^2}} f\left(\frac{2\zeta}{1 + \sqrt{1 + 4|\zeta|^2}}\right)$$

on the whole plane $\mathbb{C}$ and hence is uniquely solvable without any boundary condition and the homogeneous equation (43), $f \equiv 0$ has no other solutions, continous in $|z| \le 1$ except constants. The inverse function

$$z = \frac{2\zeta}{1 + \sqrt{1 + 4|\zeta|^2}} \tag{45}$$

maps the whole plane $\mathbb{C}$ onto the unit disc $|z| < 1$ and is a solutions to the inverse equation

$$v_{\bar{\zeta}} - \frac{4\zeta^2}{(1 + \sqrt{1 + 4|\zeta|^2})^2} v_\zeta = 0 \tag{46}$$

on $\mathbb{C}$, which is elliptic on the finite part of $\mathbb{C}$ and degenerates at infinity,

$$1 - \frac{4|\zeta|^2}{(1 + \sqrt{1 + 4|\zeta|^2})^2} \to 0$$

as $|\zeta| \to \infty$.

1.3 Second order elliptic systems in the unit disc degenerating on the circle

An operator adjoint to (30) is given by

$$B_1 v \equiv v_z - \left(\frac{\bar{z}^2}{1+\sqrt{1-|z|^2})^2} v \right)_{\bar{z}}$$

$$= v_{\bar{z}} - \frac{\bar{z}^2}{(1+\sqrt{1-|z|^2})^2} v_z - \frac{\bar{z}}{\sqrt{1-|z|^2}(1+\sqrt{1-|z|^2})} v. \qquad (47)$$

The composition of this operator with (30) produces the following second order equation

$$E_1 w :\equiv \left(1 + \frac{|z|^4}{(1+\sqrt{1-|z|^2|})^4}\right) w_{z\bar{z}}$$

$$-\frac{z^2}{(1+\sqrt{1-|z|^2}))^2} w_{z^2} - \frac{\bar{z}^2}{(1+\sqrt{1-|z|^2})^2} w_{\bar{z}^2}$$

$$-\frac{z}{\sqrt{1-|z|^2}(1+\sqrt{1-|z|^2})} \left(1 - \frac{2|z|^2}{(1+\sqrt{(1-|z|^2|})^3}\right) w_z$$

$$-\frac{\bar{z}}{\sqrt{1-|z|^2}(1+\sqrt{1-|z|^2})} w_{\bar{z}} = B^* f = f^* \qquad (48)$$

elliptic inside of the disk $|z| < 1$ and degenerating on its boundary $|z| = 1$, because its principal symbol is the quadratic form $4((1-x^2)\xi_1^2 - xy\xi_1\xi_2 + (1-y^2)\xi_2^2)$ having the discriminant $-16(1-|z|^2)$. Mapping the disk $|z| \le 1$ onto the disk $|\zeta| \le 1$ by (34) the equation (48) becomes

$$w_{\zeta\bar{\zeta}} - 2\bar{\zeta}|\zeta|^2(1-|\zeta|^4)^{-1} w_{\bar{\zeta}} = \frac{4}{(1+|\zeta|^2)^4} f^* \left(\frac{2\zeta}{1+|\zeta|^2}\right)$$

or

$$(1-|\zeta|^4) w_{\bar{\zeta}})_\zeta = \frac{4(1-|\zeta|^2)}{(1+|\zeta|^2)^3} f^* \left(\frac{2\zeta}{1+|\zeta|^2}\right) \qquad (49)$$

in $|\zeta| < 1$. Integrating (49) we obtain

$$(1-|\zeta|^4) w_{\bar{\zeta}} = -\frac{4}{\pi} \int_{|\tau|<1} \frac{(1-|\tau|^2) f^*(\frac{2\tau}{1+|\tau|^2})}{(1+|\tau|^2)^3(\bar{\tau}-\bar{\zeta})} dD_\tau.$$

As the left-hand side of this expression vanishes on the circle $|\zeta| = 1$ then the right-hand side must vanish as well

$$\int_{|\tau|<1} \frac{(1-|\tau|^2)f^*(\frac{2\tau}{1+|\tau|^2})dD_\tau}{(1+|\tau|^2)^3(1-\bar{\tau}\zeta)} = 0$$

or

$$\int_{|\tau|<1} f^*(\frac{2\tau}{1+|\tau|^2})\frac{\bar{\tau}^k(1-|\tau|^2)}{(1+|\tau|^2)^3}dD\tau = 0 \tag{50}$$

for $k = 0, 1, \ldots,$ i.e. in order to have a solution regular in $|z| \le 1$ the right-hand side of (48) must satisfy infinitely many conditions. In particular if $f \equiv 0$, then from (49) it follows that the function $(1-|\zeta|^4)w_{\bar{\zeta}}$ is antiholomorphic in $|\zeta| < 1$ and vanishes on the circle $|\zeta| = 1$, i.e. w is holomorphic in $|\zeta| < 1$ and therefore is given by (35).

Theorem 3 *The Dirichlet problem $w(z) = g(z)$ on $|z| = 1$ for the homogeneous equation* (48) *in $|z| < 1$ may have a solution if and only if for any function $\phi(z)$ holomophic in $|z| < 1$ the equality*

$$\int_{|z|=1} g(z)\phi(z)dz = 0 \tag{51}$$

holds.

Proof As we have seen in case $f \equiv 0$ any regular solution of (48) in $|z| \le 1$ is represented by (35), so that the Dirichlet condition means that $g(z)$ must be the boundary value of the function holomophic in $|z| < 1$ which is equivalent to (51).

Let us consider the Beltrami equation

$$Bw :\equiv w_{\bar{z}} + q(z)w_z = f(z) \tag{52}$$

in an arbitrary multiply connected domain G with a given coefficient $q(z)$ which is a measurable bounded function, satisfying the conditions

$$|q(z)| < 1 \tag{53}$$

inside of the domain G and

$$|q(z)| \equiv 1 \tag{54}$$

on the boundary Γ of G.

Theorem 4 *If $w(z)$ is a solution of the following second order equation*

$$\begin{aligned} Ew :\equiv (1+|q(z)|^2)w_{z\bar{z}} + q(z)w_{z^2} + \overline{q(z)}w_{\bar{z}^2} + (q_z + |q|^2_{\bar{z}})w_z + \overline{q_{\bar{z}}}w_{\bar{z}} \\ = f(z) \end{aligned} \tag{55}$$

in G, satisfying the boundary condition

$$Im(\theta(z)\frac{\partial}{\partial z})w(z) = 0 \tag{56}$$

on Γ, then $w(z)$ satisfies to the homogeneous Beltrami equation (52), $f \equiv 0$, where $\theta(z) = dz/ds - q(z)d\bar{z}/ds$, $z = z(s)$ is a parametric equation of the curves Γ.

Proof If $w \in C^2$ in G and $w \in C^1$ in $\bar{G} = G + \Gamma$, then using Green's indentity we obtain

$$\int_G |w_{\bar{z}} + q(z)w_z|^2 dG = \int_G (|w_{\bar{z}}|^2 + 2Req(z)w_z\bar{w}_z + |q(z)|^2|w_z|^2)dG$$

$$= \frac{1}{2i}\int_\Gamma \overline{w(z)}\left(\theta(z)\frac{\partial}{\partial z} - \overline{\theta(z)}\frac{\partial}{\partial \bar{z}}\right)w(z)ds - \int_G \bar{w}(z)EwdG. \tag{57}$$

Hence if w satisfies equation (55) in G and the condition (56) on Γ, then from (57) it follows that $w(z)$ satisfies (52) with $f \equiv 0$. Note that equation (55) is obtained by composition of the operator

$$v_z + (\overline{q(z)}v)_{\bar{z}}$$

with the Beltrami operator (52) and under conditions (53), (54) is elliptic inside of the domain G and degenerates on Γ, because its principial symbol is the quadratic form $(1 + |q|^2 + Req)\xi_1^2 + 4Imq\xi_1\xi_2 + (1 + |q|^2 - Req)\xi_2^2$ having the discriminant $-4(1 - |q|^2)^2$ which is negative inside of G and vanishes on Γ. Taking $q(z) = -z^2(1 + \sqrt{1 - |z|^2})^{-2}$ in the unit disk $|z| < 1$ we have $\theta(z) = 0$ on $|z| = 1$ and (57) becomes

$$\int_{|z|<1} |w_{\bar{z}} - \frac{z^2}{(1 + \sqrt{1 - |z|^2})^2}w_z|^2 dxdy = -\int_{|z|<1} \overline{w(z)}E_1 w dxdy,$$

where $E_1 w$ is given by (48), from which we obtain another proof of Theorem 3. Taking $q(z) \equiv z^2$ equation (55) becomes

$$E_1 w :\equiv (1 + |z|^4)w_{z\bar{z}} + z^2 w_{z^2} + \bar{z}^2 w_{\bar{z}} + 2z(1 + |z|^2)w_z + 2\bar{z}w_{\bar{z}} = 0. \tag{58}$$

Corollary 1 *The problem $w(z) = g(z), \partial w/\partial \nu = 0$ on $|z| = 1$ for the equation* (58) *in $|z| < 1$ may have a solution if and only if for any function $\phi(z)$ holomorphic in $|z| < 1$ and continous on $|z| \leq 1$ the equality* (51) *holds.*

Indeed in this case the indentity (57) becomes

$$\begin{aligned}\int_{|z|<1}|w_{\bar z}+z^2w_z|^2dv &= \int_{|z|=1}\overline{w(z)}\frac{\partial w}{\partial\nu}ds-\int_{|z|<1}\overline{w(z)}E_1wdv\\ &= -\int_{|z|<1}\overline{w(z)}E_1wdv,\end{aligned}$$

from which it follows that w is a solution of the equation $w_{\bar z}+z^2w_z=0$, satisfying the Dirichlet condition $w=g$. But as $w(z)=\phi(2z/(1+|z|^2))$, then this means that $g(z)$ must be the boundary value of a holomorphic function in $|z|<1$ that is equivalent to (51).

The corollary remains true if equation (58) is replaced by the equation

$$\begin{aligned}\hat E_1w :\equiv\ & \left(1+\frac{|z|^4}{(1+\sqrt{1-|z|^2})^2}\right)w_{z\bar z}\\ &+\frac{z^2}{(1+\sqrt{1-|z|^2})^2}w_{z^2}+\frac{\bar z^2}{(1+\sqrt{1-|z|^2})^2}w_{\bar z^2}\\ &+\frac{z}{\sqrt{1-|z|^2}(1+\sqrt{1-|z|^2})^2}\left(1+\frac{2|z|^2}{(1+\sqrt{1-|z|^2})^3}\right)w_z\\ &+\frac{\bar z}{\sqrt{1-|z|^2}(1+\sqrt{1-|z|^2})}w_{\bar z}=0\end{aligned}$$

and the equation $w_{\bar z}+z^2w_z=0$ is replaced by the equation

$$w_{\bar z}+\frac{z^2}{(1+\sqrt{1-|z|^2})^2}w_z=0.$$

Taking $q(z)=-z^2$ equation (55) becomes

$$E_2w:\equiv(1+|z|^4)w_{z\bar z}-z^2w_{z^2}-\bar z^2w_{\bar z^2}-2z(1-|z|^2)w_z-2\bar zw_{\bar z}=0 \tag{59}$$

and taking $q(z)=-z^2/2-|z|^2$ equation (55) becomes

$$\tilde E_2w:\equiv\left(w_{\bar z}-\frac{z^2}{2-|z|^2}w_z\right)_z-\left(\frac{\bar z^2}{2-|z|^2}w_{\bar z}-\frac{|z|^4w_z}{(2-|z|^2)^2}\right)_{\bar z}=0. \tag{60}$$

Corollary 2 *Equations* (59) *and* (60) *have no other solutions, continuous in* $|z|\le 1$ *except constants.*

Proof In both cases we have $\theta(z)\equiv 0$ on $|z|=1$, so that indentity (57) gives

$$\int_{|z|<1}|w_{\bar z}-z^2w_z|^2dv=-\int_{|z|<1}\bar wE_2wdv=0$$

and

$$\int_{|z|<1} |w_{\bar z} - \frac{z^2}{2-|z|^2} w_z|^2 d\nu = -\int_{|z|<1} \bar w \hat E_2 w d\nu = 0,$$

i.e. w satisfies the equations $w_{\bar z} - z^2 w_z = 0$ in $|z| < 1$ and $w_{\bar z} - z^2/(2 - |z|^2)w_z = 0$ in $|z| < 1$ respectively which as we have seen have no other solutions in $|z| < 1$, continuous in $|z| \le 1$, except constants.

2. Elliptic systems in the complex space

2.1 First order elliptic systems in the complex space

Let $B_n = \{z \in \mathbb{C}^n, |z| < 1\}$ be the unit ball of $\mathbb{C}^n$. The simplest first order elliptic system for a single complex valued function $u(z)$ in B_n is overdetermined

$$\frac{\partial u}{\partial \bar z_k} = f_k(z), 1 \le k \le n, \tag{61}$$

where $f_k(z)$ are given functions, satifying the conditions

$$\frac{\partial f_k}{\partial \bar z_l} - \frac{\partial f_l}{\partial \bar z_k} = 0, k \ne l,$$

in B_n. Let us be given two systems of equations: equations (61) and the equations

$$\frac{\partial v}{\partial \bar z_k} = g_k(z), 1 \le k \le n, \tag{62}$$

in B_n with given $g_k(z)$, satisfying the conditions

$$\frac{\partial g_k}{\partial \bar z_l} - \frac{\partial g_l}{\partial \bar z_k}, k \ne l,$$

in B_n and let $\lambda(\zeta) \ne 0, \zeta \in S_{2n-1} = \partial B_n$ be a function given on $S_{2n-1} = \{\zeta \in \mathbb{C}^n, |\zeta| = 1\}$. Consider the following problem:

Problem II *Find a pair of functions* $(u(z), v(z))$*, satisfying* (61) *and* (62) *respectively in* B_n *such that* u *is proportional to the complex conjugation of* v *with coefficient* λ

$$u(\zeta) = \lambda(\zeta)\overline{v(\zeta)} \tag{63}$$

on S_{2n-1}.

Here we give an explicit solution of this problem. As in case $n > 1$, $\log \lambda(\zeta)$ is a single-valued function, then the integrals

$$\Gamma_1(z) = \int_{S_{2n-1}} \log \lambda(\xi) \left(\frac{1}{(1-<z,\xi>)^n} - \frac{1}{2} \right) d\sigma(\xi),$$

$$\Gamma_2(z) = \int_{S_{2n-1}} \log \overline{\lambda(\xi)} \left(\frac{1}{(1- < z, \xi >)^n} - \frac{1}{2} \right) d\sigma(\xi)$$

are well defined holomorphic in B_n and by the analogue of the Plemelj formula

$$\Gamma_1(\zeta) = \frac{1}{2} \log \lambda(\zeta) + \int_{S_{2n-1}} \log \lambda(\eta) \left(\frac{1}{(1- < \zeta, \eta >)^n} - \frac{1}{2} \right) d\sigma(\eta)$$

where the integral on the right-hand side is understood as the Cauchy principal value.

Hence if $\log \lambda(\zeta) \perp H^{p,q}(S_{2n-1}), p, q \neq 0$, then (see [2]) we can represent $\lambda(\zeta)$ as factorized by $X(\zeta)$ and $1/\overline{Y(\zeta)}$,

$$\lambda(\zeta) = X(\zeta) \frac{1}{\overline{Y(\zeta)}},$$

where $X(z) = \exp \Gamma_1(z), Y(z) = \exp(-\Gamma_2(z))$. Inserting this into (63) we obtain the following condition

$$\varphi(\zeta) = \overline{\psi(\zeta)}, \tag{64}$$

on S_{2n-1} for solutions of the systems

$$\frac{\partial \varphi}{\partial \bar{z}_k} = f_k^0(z), \frac{\partial \psi}{\partial \bar{z}_k} = g_k^0(z), 1 \leq k \leq n, \tag{64'}$$

in B_n, where $f_k^0(z) = f_k(z)/X(z), g_k^0(z) = g_k(z)/Y(z)$. Integrating (64′) we obtain (see [7])

$$\varphi(z) = \Phi(z) + (Tf^0)(z), \psi(z) = \Psi(z) + (Tg^0)(z),$$

where Tf is the integral operator defined by

$$(Tf)(\zeta) = \frac{1}{n} \int_{B_n} \frac{< f(\eta), \zeta - \eta > d\nu(\eta)}{(1- < \zeta, \eta >)^n}$$

for $\zeta \in S_{2n-1}$ and by

$$(Tf)(z) = \int_{S_{2n-1}} \frac{1- < \zeta, z >}{|\zeta - z|^{2n}} (Tf)(\zeta) d\sigma(\zeta) - \frac{1}{n} \int_{B_n} \frac{< f(\eta), \eta - z >}{|\eta - z|^{2n}} d\nu(\eta)$$

for $z \in B_n$ and $\Phi(z), \Psi(z)$ are arbitrary functions, holomorphic in B_n, continuous in $\bar{B}_n = B_n + S_{2n-1}$, so that condition (64) is reduced to

$$\Phi(\zeta) - \overline{\Psi(\zeta)} = \gamma(\zeta),$$

on S_{2n-1}, where $\gamma(\zeta) = -(Tf^0)(\zeta) + (Tg^0)(\zeta), \zeta \in S_{2n-1}$. Splitting the real and the imaginary parts this is equivalent to two Schwarz conditions for the holomorphic functions $\Phi(z) + \Psi(z)$ and $\Phi(z) - \Psi(z)$, so that we obtain, see [7], for $\gamma(\zeta) \perp H^{p,q}(S_{2n-1}), pq \neq 0$,

$$\Phi(z) = \int_{S_{2n-1}} \gamma(\eta) \left(\frac{1}{(1- < z, \eta >)^n} - \frac{1}{2} \right) d\sigma(\eta) + \frac{C_2 + iC_1}{2},$$

$$\Psi(z) = -\int_{S_{2n}} \overline{\gamma(\eta)} \left(\frac{1}{(1- < z, \eta >)^n} - \frac{1}{2} \right) d\sigma(\eta) + \frac{C_2 - iC_1}{2},$$

where C_1, C_2 are real constants. As (see [2])

$$\int_{S_{2n-1}} \frac{(Tf^0)(\eta) d\sigma(\eta)}{(1- < z, \eta >)^n} = 0$$

then

$$\Phi(z) = \int_{S_{2n-1}} \overline{(Tg^0)}(\eta) \left(\frac{1}{(1- < z, \eta >)^n} - \frac{1}{2} \right) d\sigma(\eta) + \frac{C_2 + iC_1}{2},$$

$$\Psi(z) = \int_{S_{2n-1}} \overline{(Tf^0)(\eta)} \left(\frac{1}{(1- < z, \eta >)^n} - \frac{1}{2} \right) d\sigma(\eta) + \frac{C_2 - iC_1}{2}.$$

Hence if $\log \lambda(\zeta) \perp H^{p,q}(S_{2n-1}), (Tf^0)(\zeta) - \overline{(Tg^0)}(\zeta) \perp H^{p,q}(S_{2n-1}), pq \neq 0$, then the general solution of Problem II is given by

$$u(z) = X(z) \left((Tf^0)(z) + \int_{S_{2n-1}} \frac{\overline{(Tg^0)}(\eta) d\sigma\eta}{(1- < z, \eta >)^n} + \frac{C_2 + iC_1}{2} \right),$$

$$v(z) = Y(z) \left((Tg^0)(z) + \int_{S_{2n-1}} \frac{\overline{(Tf^0)}(\eta) d\sigma(\eta)}{(1- < z, \eta >)^n} + \frac{C_2 - iC_1}{2} \right).$$

Note that the conditions for the right-hand side of Problem II are also equivalent to the equalities

$$1) \quad \int_{S_{2n-1}} \log \lambda(\eta) \left(\frac{1}{1- < \zeta, \eta >)^n} + \frac{1}{1- < \eta, \zeta >)^n} - 1 \right) d\sigma(\eta) = 0,$$

$$2) \int_{S_{2n-1}} ((Tf^0)(\eta) - \overline{(Tg^0)}(\eta)) \times \left(\frac{1}{1- < \zeta, \eta >)^n} + \frac{1}{1- < \eta, \zeta >)^n} - 1 \right) d\sigma(\eta) = 0$$

for all $\zeta \in S_{2n-1}$. In the case $n = 1$ these conditions are authomatically fulfilled as

$$\frac{1}{1-\zeta\bar{\eta}} + \frac{1}{1-\eta\bar{\zeta}} - 1 = \frac{\eta}{\eta-\zeta} + \frac{\zeta}{\zeta-\eta} - 1 = 0.$$

The inhomogeneous Cauchy-Riemann system (61) is a particular case $q_{kl} \equiv 0$ of the following Beltrami system

$$B_k u :\equiv \frac{\partial u}{\partial \bar{z}_k} + \sum_{l=1}^{n} q_{kl}(z)\frac{\partial u}{\partial z_l} = f_k(z), 1 \le k \le n, \tag{65}$$

in $\Omega \subset \mathbb{C}^n$. From (65) it is obvious that in order for this system to have a solution in Ω it is necessary that its coefficients and right-hand sides satisfy the following compatibility conditions

$$B_l(a_{ki}) - B_k(a_{li}) = 0, B_l(f_k) - B_k(f_l) = 0, \tag{66}$$

in Ω for any $k \neq l, k = 1, 2, \ldots, n$.
Moreover we assume for (65) to be elliptic in Ω, which in terms of the coefficients means that

$$\det\,(I - q(z)\overline{q(z)}) \neq 0 \tag{67}$$

in Ω, where I is the indentity matrix and $q(z)$ is a square matrix with the elements $q_{kl}(z), 1 \le k, l \le n$. Let $(a_{\nu 1}(z), a_{\nu 2}(z)), \nu = 1, 2$, be complex vector fields in the domain $\Omega \subset \mathbb{C}^n$ and assume that they are linearly independent in Ω that is $\Delta(z) = a_{11}(z)a_{22}(z) - a_{12}(z)a_{21}(z) \neq 0$ in Ω. The following first order determined system of two equations for two unknown functions $u_1(z), u_2(z)$ is a counterpart of system (24) in $\mathbb{C}^2$:

$$\bar{\partial}_1 u_1 - \partial_2 u_2 = f_1, \bar{\partial}_2 u_1 + \partial_1 u_2 = f_2, \tag{68}$$

where

$$\partial_j :\equiv a_{j1}(z)\frac{\partial}{\partial z_1} + a_{j2}(z)\frac{\partial}{\partial z_2}, \overline{\partial_j} :\equiv \overline{a_{j1}(z)}\frac{\partial}{\partial \bar{z_1}} + \overline{a_{j2}(z)}\frac{\partial}{\partial \bar{z_2}}$$

and is elliptic in Ω, because its principal symbol is equal to

$$|\overline{a_{11}(z)}\zeta_1 + \overline{a_{12}(z)}\zeta_2|^2 + |\overline{a_{21}(z)}\zeta_1 + \overline{a_{22}(z)}\zeta_2|^2 \ .$$

When $a_{11} = a_{22} \equiv 1, a_{12} = a_{21}$ (68) turns to the well known Fueter system

$$\frac{\partial u_1}{\partial \overline{z_1}} - \frac{\partial u_2}{\partial z_2} = f_1, \frac{\partial u_1}{\partial \overline{z_2}} + \frac{\partial u_2}{\partial z_1} = f_2 \tag{69}$$

arising in quaternionic function theory as a counterpart to the Cauchy-Riemann equations in function theory of a complex variable.

Proposition 2 *System* (68) *remains invariant under biholomorphic maps of the domain* Ω.

Proof Let $\zeta_1 = \zeta_1(z), \zeta_2 = \zeta_2(z), z = (z_1, z_2)$ be a holomorphic map of $\mathbb{C}^2$ onto $\mathbb{C}^2$. Then we obtain the following expressions for the derivatives

$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}_1} &= \frac{\partial u}{\partial \bar{\zeta}_1}\overline{\frac{\partial \zeta_1}{\partial z_1}} + \frac{\partial u}{\partial \bar{\zeta}_2}\overline{\frac{\partial \zeta_2}{\partial z_1}},\\
\frac{\partial u}{\partial \bar{z}_2} &= \frac{\partial u}{\partial \bar{\zeta}_1}\overline{\frac{\partial \zeta_1}{\partial z_2}} + \frac{\partial u}{\partial \bar{\zeta}_2}\overline{\frac{\partial \zeta_2}{\partial z_2}},\\
\frac{\partial v}{\partial z_1} &= \frac{\partial v}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial z_1} + \frac{\partial v}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial z_1},\\
\frac{\partial v}{\partial z_2} &= \frac{\partial v}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial z_2} + \frac{\partial v}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial z_2}
\end{aligned}$$

and system (68) takes the form

$$\begin{aligned}
&\overline{a^0_{11}}(\zeta)\frac{\partial u_1}{\partial \bar{\zeta}_1} + \overline{a^0_{12}}(\zeta)\frac{\partial u_1}{\partial \bar{\zeta}_2} - a^0_{21}(\zeta)\frac{\partial u_2}{\partial \overline{\zeta_1}} - a^0_{22}(\zeta)\frac{\partial u_2}{\partial \overline{\zeta_2}} = f^0_1(\zeta),\\
&\overline{a^0_{21}}(\zeta)\frac{\partial u_1}{\partial \zeta_1} + \overline{a^0_{22}}(\zeta)\frac{\partial u_1}{\partial \zeta_2} + a^0_{11}(\zeta)\frac{\partial u_2}{\partial \zeta_1} + a^0_{21}(\zeta)\frac{\partial u_2}{\partial \zeta_2} = f^0_2(\zeta),
\end{aligned} \tag{70}$$

where

$$a^0_{j1}(\zeta) = a_{j1}\frac{\partial \zeta_1}{\partial z_1} + a_{j2}\frac{\partial \zeta_1}{\partial z_2}, a^0_{j1}(\zeta) = a_{j1}\frac{\partial \zeta_2}{\partial z_1} + a_{j2}\frac{\partial \zeta_2}{\partial z_2},$$

so that

$$\begin{aligned}
&\tilde{\Delta}(\zeta)\\
&= a^0_{11}(\zeta)a^0_{22}(\zeta) - a^0_{21}(\zeta)a^0_{22}(\zeta) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \frac{\partial \zeta_1}{\partial z_1} & \frac{\partial \zeta_2}{\partial z_1} \\ \frac{\partial \zeta_1}{\partial z_2} & \frac{\partial \zeta_2}{\partial z_2} \end{pmatrix} \neq 0
\end{aligned}$$

in Ω, because $a_{11}a_{22} - a_{12}a_{21} \neq 0$ and $\frac{\partial \zeta_1}{\partial z_1}\frac{\partial \zeta_2}{\partial z_2} - \frac{\partial \zeta_2}{\partial z_1}\frac{\partial \zeta_1}{\partial z_2} \neq 0$, i.e. system (70) is an elliptic system of the form (68).

Let us consider the following Fueter system perturbed with lower order terms

$$\begin{aligned}
&\frac{\partial u_1}{\partial \overline{z_1}} - \frac{\partial u_2}{\partial z_2} + A_{11}(z)u_1 + A_{21}(z)u_2 = 0,\\
&\frac{\partial u_2}{\partial \overline{z_2}} + \frac{\partial u_1}{\partial z_1} + A_{21}(z)u_1 + A_{22}(z)u_2 = 0,
\end{aligned} \tag{71}$$

in B_2 and let us assume its coefficients to satisfy the conditions

$$\frac{\partial A_{11}}{\partial \bar{z}_2} - \frac{\partial A_{21}}{\partial \bar{z}_1} = 0, \frac{\partial A_{12}}{\partial z_1} + \frac{\partial A_{22}}{\partial z_2} = 0 \tag{71'}$$

in B_2. Then there are functions $\omega(z)$ and $\nu(z)$ such that

$$\frac{\partial \omega}{\partial \bar{z}_1} = -A_{11}, \frac{\partial \omega}{\partial \bar{z}_2} = -A_{21}, \frac{\partial \nu}{\partial z_1} = -A_{22}, \frac{\partial \nu}{\partial z_2} = A_{12}$$

in B_2, so that (71) turns to the system

$$\frac{\partial u_1^0}{\partial \bar{z}_1} - \frac{\partial u_2^0}{\partial z_2} - \nu_{\bar{z}_1} u_1^0 + \omega_{z_2} u_2^0 = 0, \frac{\partial u_1^0}{\partial \bar{z}_2} + \frac{\partial u_2^0}{\partial z_1} - \nu_{\bar{z}_2} u_1^0 - \omega_{z_1} u_2^0 = 0 \tag{72}$$

in B_2 of the same type as (71), because

$$\frac{\partial}{\partial \bar{z}_2}\left(\frac{\partial \nu}{\partial \bar{z}_1}\right) - \frac{\partial}{\partial \bar{z}_1}\left(\frac{\partial \nu}{\partial \bar{z}_2}\right) \equiv 0, \frac{\partial}{\partial z_1}\left(-\frac{\partial \omega}{\partial z_2}\right) + \frac{\partial}{\partial z_2}\left(\frac{\partial \omega}{\partial z_1}\right) \equiv 0 \tag{72'}$$

Hence for every $u = (u_1, u_2)$ satisfying (71), (71′) there is a $u^0 = (u_1^0, u_2^0)$ satisfying (72), (72′) such that the factorization formula

$$U = U^0 e^{\omega + \nu}$$

holds in B_2. In particular, if A_{12}, A_{22} are holomorphic in B_2 and A_{11}, A_{21} are antiholomorphic in B_2 such that (71′) holds in B_2, then $\nu(z)$ is holomorphic and $\omega(z)$ is antiholomorphic, i.e. $p(z) = \nu(z) + \overline{\omega(z)}$ is pluriharmonic in B_2 and (72) turns to the homogeneous Fueter system

$$\frac{\partial u_1^0}{\partial \overline{z_1}} - \frac{\partial u_2^0}{\partial z_2} = 0, \frac{\partial u_1^0}{\partial \overline{z_2}} + \frac{\partial u_2^0}{\partial z_1} = 0. \tag{73}$$

That in this case means that for every solution $u = (u_1, u_2)$ of system (71) satisfying (71′) there is a solution u^0 of the homogeneous Fueter system in B_2 and a pluriharmonic function $p(z)$ such that

$$u = u^0 e^p \tag{74}$$

in B_2. If $u = (u_1, u_2)$ is a solution of the Fueter system (73) in $\Omega \subset \mathbb{C}^2$, then u_1 and u_2 are obviously harmonic in Ω and moreover

$$\begin{aligned} \rho_{z_1} \frac{\partial u_1}{\partial \bar{z}_1} + \rho_{z_2} \frac{\partial u_1}{\partial \bar{z}_2} &= -\left(\rho_{z_2} \frac{\partial u_2}{\partial z_1} - \rho_{z_1} \frac{\partial u_2}{\partial z_2}\right), \\ \rho_{\bar{z}_1} \frac{\partial u_2}{\partial z_1} + \rho_{\bar{z}_2} \frac{\partial u_2}{\partial z_2} &= -\left(\rho_{\bar{z}_1} \frac{\partial u_1}{\partial \bar{z}_2} - \rho_{\bar{z}_2} \frac{\partial u_1}{\partial \bar{z}_2}\right) \end{aligned} \tag{75}$$

on $\Gamma = \partial\Omega$, where $\rho(z)$ is the defining function of the domain Ω. Conversely if u_1 and u_2 are arbitrary functions, harmonic in Ω such that (75) holds on Γ, then $u = (u_1, u_2)$ satisfies the Fueter system (73) in Ω. Indeed for u_1, u_2 belonging to C^2 in Ω and to C^1 in $\bar{\Omega} = \Omega + \Gamma$

$$\int_\Omega \left\{ \left| \frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right|^2 + \left| \frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right|^2 \right\} d\Omega$$

$$= \int_\Omega \left\{ \frac{\partial}{\partial z_1} \bar{u}_1 \left(\frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right) + \frac{\partial}{\partial z_2} \bar{u}_1 \left(\frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right) \right.$$

$$\left. + \frac{\partial}{\partial \bar{z}_1} \bar{u}_2 \left(\frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right) - \frac{\partial}{\partial \bar{z}_2} \bar{u}_2 \left(\frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right) \right\} d\Omega$$

$$= 2 \int_\Gamma \left\{ \bar{u}_1 \left[\left(\frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right) \rho_{z_1} + \left(\frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right) \rho_{z_2} \right] \right.$$

$$\left. + \overline{u_2} \left[\left(\frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right) \rho_{\bar{z}_1} - \left(\frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right) \rho_{\bar{z}_2} \right] \right\} dS$$

$$- \int_\Omega \left\{ \bar{u_1} \left[\frac{\partial}{\partial z_1} \left(\frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right) + \frac{\partial}{\partial z_2} \left(\frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right) \right] \right.$$

$$\left. + \overline{u_2} \left[\frac{\partial}{\partial \bar{z}_1} \left(\frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right) - \frac{\partial}{\partial \bar{z}_2} \left(\frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right) \right] \right\} d\Omega$$

$$= 2 \int_\Gamma \left\{ \bar{u_1} \left(\rho_{z_1} \frac{\partial u_1}{\partial \bar{z}_1} + \rho_{z_2} \frac{\partial u_1}{\partial \bar{z}_2} + \rho_{z_2} \frac{\partial u_2}{\partial z_1} - \rho_{z_1} \frac{\partial u_2}{\partial z_2} \right) \right.$$

$$\left. + \bar{u_2} \left(\rho_{\bar{z}_1} \frac{\partial u_2}{\partial z_1} + \rho_{\bar{z}_2} \frac{\partial u_2}{\partial z_2} + \rho_{\bar{z}_1} \frac{\partial u_1}{\partial \bar{z}_2} - \rho_{\bar{z}_2} \frac{\partial u_1}{\partial \bar{z}_1} \right) \right\} dS$$

$$- \int_\Omega \left\{ \bar{u_1} \left(\frac{\partial^2 u_1}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2 u_1}{\partial z_2 \partial \bar{z}_2} \right) + \bar{u_2} \left(\frac{\partial^2 u_2}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2 u_2}{\partial z_2 \partial \bar{z}_2} \right) \right\} d\Omega,$$

so that if u_1, u_2 are harmonic in Ω, satisfying (75) on Γ, then

$$\int_\Omega \left(\left| \frac{\partial u_1}{\partial \bar{z}_1} - \frac{\partial u_2}{\partial z_2} \right|^2 + \left| \frac{\partial u_1}{\partial \bar{z}_2} + \frac{\partial u_2}{\partial z_1} \right|^2 \right) d\Omega = 0,$$

i.e. $u = (u_1, u_2)$ satisfies (73) in Ω.

The function u is called biholomorphic in $\Omega \subset \mathbb{C}^n$ if in Ω

$$\frac{\partial^2 u}{\partial \bar{z}_k \partial \bar{z}_l} = 0, 1 \leq k, l \leq n.$$

The pluriharmonic functions are those which in Ω satisfy the equations

$$\frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} = 0, 1 \leq k, l \leq n.$$

If u is biholomophic in Ω, then it is obviously biharmonic and if u is pluriharmonic, then it is harmonic and so also biharmonic. Conversely if u is biharmonic in Ω and if

$$\sum_{i=1}^{n} \rho_{z_i} \frac{\partial}{\partial \bar{z}_i} \left(\frac{\partial u}{\partial z_k} \right) = 0, 1 \leq k \leq n, \sum_{i=1}^{n} \rho_{\bar{z}_i} \frac{\partial \Delta u}{\partial z_i} = 0,$$

on Γ, then u is a pluriharmonic in Ω and if

$$\sum_{i=1}^{n} \rho_{z_i} \frac{\partial}{\partial \bar{z}_i} \left(\frac{\partial u}{\partial \bar{z}_k} \right) = 0, 1 \leq k \leq n, \sum_{i=1}^{n} \rho_{z_i} \frac{\partial \Delta u}{\partial \bar{z}_i} = 0,$$

on Γ, then u is a biholomorphic in Ω, where $4\Delta = \sum_{i=1}^{n} \partial^2/(\partial z_i \partial \bar{z}_i)$. The first assertion follows from the indentity

$$\sum_{k,l=1}^{n} \int_\Omega |u_{\overline{z_k} z_l}|^2 d\Omega = \sum_{k,l=1}^{n} \int_\Omega \frac{\partial^2 \bar{u}}{\partial z_k \partial \bar{z}_l} \, \frac{\partial^2 u}{\partial \bar{z}_k \partial z_l} d\Omega$$

$$= \sum_{k,l=1}^{n} \int_\Omega \frac{\partial}{\partial z_k} \left[\overline{\left(\frac{\partial u}{\partial z_l} \right)} \frac{\partial^2 u}{\partial \bar{z}_k \partial z_l} \right] d\Omega - \sum_{k,l=1}^{n} \int_\Omega \left(\frac{\partial u}{\partial z_l} \right) \frac{\partial^3 u}{\partial z_k \partial \overline{z_k} \partial z_l} d\Omega$$

$$= \sum_{l=1}^{n} \int_\Gamma \left(\frac{\overline{\partial u}}{\partial z_l} \right) \sum_{i=1}^{n} \rho_{z_i} \frac{\partial}{\partial \bar{z}_i} \left(\frac{\partial u}{\partial z_l} \right) dS$$

$$- \sum_{k,l=1}^{n} \int_\Omega \frac{\partial}{\partial \overline{z_l}} \left[\bar{u} \frac{\partial^3 u}{\partial z_k \partial \bar{z}_k \partial z_l} \right] d\Omega + \sum_{k,l=1}^{n} \int_\Omega \bar{u} \frac{\partial^4 u}{\partial z_k \partial \bar{z}_k \partial z_l \partial \bar{z}_l} d\Omega$$

$$= \sum_{l=1}^{n} \int_\Gamma \left(\frac{\overline{\partial u}}{\partial z_l} \right) \sum_{i=1}^{n} \rho_{z_i} \frac{\partial}{\partial z_i} \left(\frac{\partial u}{\partial z_l} \right) dS - \frac{1}{4} \int_\Gamma \bar{u} \sum_{i=1}^{n} \rho_{\overline{z_i}} \frac{\partial \Delta u}{\partial z_i} dS = 0$$

and the second from the identity

$$\sum_{k,\ell=1}^{n} \int_\Omega |u_{\bar{z}_k, \bar{z}_\ell}|^2 \, d\Omega = \sum_{\ell=1}^{n} \int_\Gamma \frac{\partial \bar{u}}{\partial z_\ell} \sum_{i=1}^{n} \rho_{z_i} \frac{\partial}{\partial \bar{z}_i} \Big(\frac{\partial u}{\partial \bar{z}_\ell} \Big) ds$$

$$- \frac{1}{4} \int_\Gamma \bar{u} \sum_{i=1}^{n} \rho_{z_i} \frac{\partial \Delta u}{\partial \bar{z}_i} \, ds - \frac{1}{8} \int_\Omega \bar{u} \Delta \Delta u d\Omega = 0.$$

The homogeneous Fueter system (73) has the obvious solution $u = (\overline{\varphi(z)}, \overline{\psi(z)})$, where $\varphi(z)$ and $\psi(z)$ are arbitrary functions holomorphic in Ω. Such a solution is called trivial.

Proposition 3 *If $u = (u_1, u_2)$ is a solution of the Fueter system in the domain Ω and if $\sum_{i=1}^{2} \rho_{z_i}(z) \, \partial u_1/\partial \bar{z}_i = 0$ (or $\sum_{i=1}^{2} \rho_{\bar{z}_i}(z) \, \partial u_2/\partial z_i = 0$) on the boundary $\Gamma = \partial\Omega$ of Ω, then u is a trivial solution in Ω.*

Proof For an arbitrary functions u_1, u_2 belonging to C^2 in Ω and to C^1 in $\bar{\Omega}$ we have

$$\int_\Omega \left(\left| \frac{\partial u_1}{\partial \bar{z}_1} \right|^2 + \left| \frac{\partial u_1}{\partial \bar{z}_2} \right|^2 \right) d\Omega$$

$$= \int_\Omega \left\{ \frac{\partial}{\partial z_1} \left(\bar{u}_1 \frac{\partial u_1}{\partial \bar{z}_1} \right) + \frac{\partial}{\partial z_2} \left(\bar{u}_1 \frac{\partial u_1}{\partial \bar{z}_2} \right) - \bar{u}_1 \left(\frac{\partial^2 u_1}{\partial z_1 \partial \bar{z}_1} + \frac{\partial^2 u_1}{\partial z_2 \partial \bar{z}_2} \right) \right\} d\Omega$$

$$= \int_\Gamma \bar{u}_1 \left(\rho_{z_1} \frac{\partial}{\partial \bar{z}_1} + \rho_{z_2} \frac{\partial}{\partial \bar{z}_2} \right) u_1 dS - \frac{1}{4} \int_\Omega \bar{u}_1 \Delta u_1 d\Omega,$$

$$\int_\Omega \left(\left| \frac{\partial u_2}{\partial z_1} \right|^2 + \left| \frac{\partial u_2}{\partial z_2} \right|^2 \right) d\Omega$$

$$= \int_\Gamma \bar{u}_2 \left(\rho_{\bar{z}_1} \frac{\partial}{\partial z_1} + \rho_{\bar{z}_2} \frac{\partial}{\partial z_2} \right) u_2 dS - \frac{1}{4} \int_\Omega \bar{u}_2 \Delta u_2 d\Omega \tag{76}$$

As for every solution $u = (u_1, u_2)$ of (73) in Ω the functions u_1 and u_2 are harmonic in Ω, then from the first indentity (76) and the assumption $\rho_{z_1}\, \partial u_1/\partial \bar{z}_1 + \rho_{z_2}\, \partial u_1/\partial \bar{z}_2 = 0$ on Γ it follows that u_1 is holomorphic in Ω and from (73) it follows then that u_2 is antiholomorphic in Ω. From the second indentity (76) and the assumption $\rho_{\bar{z}_1}\, \partial u_2/\partial z_1 + \rho_{\bar{z}_2}\, \partial u_2/\partial z_2 = 0$ on Γ it follows that u_2 is antiholomorphic in Ω and from (76) it follows then that u_1 is holomorphic in Ω.

2.2 First order elliptic systems in the unit ball of $\mathbb{C}^n$ degenerating on its boundary

Let $\varphi_a(z)$ be the Moebius transformation of the ball $B_n = \{z \in \mathbb{C}^n, |z| < 1\}, \varphi_a(a) = 0, \varphi_\alpha(0) = a$:

$$\varphi_a(z) = \frac{a - sz - (1-s) < z, a > |a|^{-2} a}{1 - < z, a >}, \tag{77}$$

where $z = (z_1, \ldots, z_n) \in B_n, a = (a_1, \ldots, a_n) \in B_n, s = \sqrt{1 - |a|^2}$. The invariant Cauchy-Riemann system

$$-\frac{\partial}{\partial \bar{z}_k} (u_0 \varphi_z)(0) = \frac{\partial u}{\partial \bar{z}_k} - \frac{z_k}{1 + \sqrt{1 - |z|^2}} \overline{\mathcal{R}} u = f_k(z), 1 \le k \le n, \tag{78}$$

being elliptic inside the ball B_n degenerates on its boundary $S_{2n-1} = \{z \in \mathbb{C}^n, |z| = 1\}$, where $\mathcal{R}, \overline{\mathcal{R}}$ are the radial operators

$$\mathcal{R} :\equiv \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}, \overline{\mathcal{R}} :\equiv \sum_{i=1}^{n} \bar{z}_i \frac{\partial}{\partial \bar{z}_i}.$$

From (78) it follows that

$$\sqrt{1-|z|^2}\,\overline{\mathcal{R}}u = \sum_{i=1}^{n} \bar{z}_i f_i, 1 \le k \le n,$$

and hence (78) is equivalent to

$$\frac{\partial u}{\partial \bar{z}_k} = f_k(z) + \frac{z_k}{\sqrt{1-|z|^2}(1+\sqrt{1-|z|^2})} \sum_{i=1}^{n} \bar{z}_i f_i(z), 1 \le k \le n,$$

so that every solution of the homogeneous invariant Cauchy-Riemann system in B_n is a function holomorphic in B_n. The homogeneous system

$$\frac{\partial u}{\partial \bar{z}_k} - \frac{z_k}{(1+\sqrt{1-|z|^2})^2} \overline{\mathcal{R}}u = 0, 1 \le k \le n, \tag{79}$$

obtained from (65) when

$$q_{kl} \equiv \frac{z_k z_l}{(1+\sqrt{1-|z|^2})^2}$$

is also elliptic inside the ball B_n and degenerates on its boundary S_{2n-1}. The functions

$$\zeta_k = \frac{z_k}{1+\sqrt{1-|z|^2}}, 1 \le k \le n, \tag{80}$$

are solutions of (79) mapping the unit ball $|z| \le 1$ onto the unit ball $|\zeta| \le 1$ homeomorphically, so that equations (79) in the variables ζ_k have the forms

$$\frac{\partial u}{\partial \bar{\zeta}_k} + \zeta_k \overline{\mathcal{R}}_\zeta u = 0, 1 \le k \le n, \tag{81}$$

where

$$\overline{\mathcal{R}}_\zeta :\equiv \sum_{i=1}^{n} \overline{\zeta_i} \frac{\partial}{\partial \overline{\zeta_i}} .$$

From (81) it follows that $(1+|\zeta|^2)\overline{\mathcal{R}}_\zeta u = 0$ in B_n and hence $\partial u/\partial\bar\zeta_k = 0, 1 \le k \le n$, i.e. every solution of (79) is a superposition of a function ϕ holomorphic in the unit disc $|\zeta| < 1$ and the mapping (80):

$$u(z) = \phi(\frac{z}{1+\sqrt{1-|z|^2}}) = \phi \circ \zeta. \tag{82}$$

The inverse mapping to (80):

$$z_k = \frac{2\zeta_k}{1+|\zeta|^2}, 1 \le k \le n, \tag{83}$$

is a homeomorphism of the unit ball $|\zeta| < 1$ onto the unit ball $|z| < 1$, satifying to the following inverse system

$$\frac{\partial v}{\partial\bar\zeta_k} + \zeta_k \mathcal{R}_\zeta v = 0, 1 \le k \le n, \tag{84}$$

in $|\zeta| < 1$, where

$$\mathcal{R}_\zeta :\equiv \sum_{i=1}^n \zeta_i \frac{\partial}{\partial\zeta_i}.$$

The general solution of (84) is a superposition of an arbitrary function $\psi(z)$, holomorphic in the unit ball $|z| < 1$ and the mapping (83):

$$v(z) = \psi \circ z = \psi(\frac{2\zeta}{1+|\zeta|^2)}).$$

Note that the unit sphere S_{2n-1} is the set of degeneration of order one half for system (79) and of order one for system (84) being the charachteristic set for the first and not being the characteristic set for the second system. The system $(q_{kl} = z_k z_l/(2-|z|^2))$

$$\frac{\partial u}{\partial\bar z_k} - \frac{z_k}{2-|z|^2}\mathcal{R}u = 0, 1 \le k \le n, \tag{85}$$

is a counterpart in B_n to equation (39), being elliptic inside of B_n degenerating on S_{2n-1} of order one and S_{2n-1} is the charachteristic set for (85). So it has no solution mapping the ball $|z| \le 1$ onto the ball $|\zeta| \le 1$, but it has the solution

$$\zeta_k = \frac{z_k}{\sqrt{1-|z|^2}}, 1 \le k \le n, \tag{86}$$

which maps the ball $|z| < 1$ onto the whole space $\mathbb{C}^n$ of the variable $\zeta = (\zeta_1, \ldots, \zeta_n)$, so that (85) is reduced to

$$\frac{\partial u}{\partial\bar\zeta_k} + \frac{\zeta_k}{2+|\zeta|^2}\mathcal{R}_\zeta u = 0, 1 \le k \le n, \tag{87}$$

on $\mathbb{C}^n$, from which it follows that $u \equiv const$. Note that (87) is a Beltrami system elliptic everywhere in $\mathbb{C}^n$, except at infinity:

$$\lim_{|\zeta|\to\infty} \det\,(I - q(\zeta)\bar{q}(\zeta)) = 4 \lim_{\zeta\to\infty} \frac{1+|\zeta|^2}{(2+|\zeta|^2)^2} = 0.$$

Note also that the functions

$$z_\nu = \frac{\zeta_\nu}{\sqrt{1+|\zeta|^2}}\ , 1 \le \nu \le n, \tag{88}$$

are inverse to (86), satisfying the inverse system

$$\frac{\partial z_\nu}{\partial \bar{z}_k} + \frac{z_k}{2+|z|^2}\mathcal{R}(z_\nu) = 0, 1 \le k \le n,$$

and mapping the whole $\mathbb{C}^n$ to the unit ball,

$$|z| = \frac{|\zeta|}{\sqrt{1+|\zeta|^2}} < 1,$$

such that infinity is mapped to the unit sphere

$$\lim_{|\zeta|\to\infty} \frac{|\zeta|}{\sqrt{1+|\zeta|^2}} = 1,$$

The system

$$\frac{\partial u}{\partial \bar{z}_k} - z_k \mathcal{R} u = 0, 1 \le k \le n, \tag{89}$$

has the same property in B_n: instead of (86) it has the solution

$$\zeta_k = \frac{z_k}{1-|z|^2}, 1 \le k \le n, \tag{90}$$

mapping the ball $|z| \le 1$ on to the whole $\mathbb{C}^n$. The inverse mapping

$$z_k = \frac{2\zeta_k}{1+\sqrt{1+4|\zeta|^2}}, 1 \le k \le n,$$

maps $\mathbb{C}^n$ onto the unit ball $|z| < 1$ such that infinity is mapped to the sphere S_{2n-1}:

$$\lim_{|\zeta|\to\infty} \frac{2|\zeta|}{1+\sqrt{1+4|\zeta|^2}} = 1.$$

By means of (90) it can be shown that system (89) has no other solutions, continuous on the ball $|z| \leq 1$ except constants.The difference between (89) and the system

$$\frac{\partial u}{\partial \bar{z}_k} + z_k \mathcal{R} u = 0, 1 \leq k \leq n, \tag{81'}$$

is that the sphere S_{2n-1} is the charachteristic set for (89) and it is not for (81′) and the difference between (79) and (89) for both of which S_{2n-1} is the charachteristic set is that system (79) degenerates on S_{2n-1} of order one half,

$$\det(I - q(z)\overline{q(z)}) = \frac{4\sqrt{1-|z|^2}}{(1+\sqrt{1-|z|^2})^2},$$

and system (89) degenerates on S_{2n-1} of order one,

$$\det(I - q(z)\overline{q(z)}) = 1 - |z|^4.$$

2.3 Second order elliptic systems in the unit ball of $\mathbb{C}^n$ degenerating on the boundary

Among others the invariant Laplace equation

$$\sum_{k=1}^{n} \frac{\partial^2 (u_0 \varphi_z)(0)}{\partial z_k \partial \bar{z}_k} \equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - \mathcal{R}\overline{\mathcal{R}} u = 0 \tag{91}$$

in B_n is one of the most remarkable equations elliptic inside the ball B_n degenerating on its boundary S_{2n-1}. One of the important features of this equation is that the Dirichlet problem $u(\zeta) = \gamma(\zeta), \zeta \in S_{2n-1}$, for (91) is uniquely solvable in B_n and the solution is given by the Poisson-Szegö integral

$$u(z) = \frac{(n-1)!}{2\pi^n} \int_{S_{2n-1}} \gamma(\zeta) \frac{(1-|\zeta|^2)^n}{|1- < z, \zeta > |^{2n}} d\sigma(\zeta) \tag{92}$$

though in contrast to the usual Poisson integral giving the unique solution of the Dirichlet problem for the usual Laplace equation in B_n fails to be C^∞ in $\bar{B}_n = B_n + S_{2n-1}$ for any $\gamma \in C^\infty$ on S_{2n-1} (see [4]). If the complex numbers α, β are such that $\mathrm{Re}(n + \alpha + \beta) > 0$ and neither $n + \alpha$ or $n + \beta$ is zero or a negative integer, then the Dirichlet problem $u(\zeta) = \gamma(\zeta), \zeta \in S_{2n-1}$, for the equation

$$\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - \mathcal{R}\overline{\mathcal{R}} u + \alpha \mathcal{R} u + \beta \overline{\mathcal{R}} u - \alpha\beta u = 0 \tag{93}$$

in B_n also has a solution given by, see [1],

$$u(z) = \int_{S_{2n-1}} \gamma(\zeta) P_{\alpha\beta}(z,\zeta) d\sigma(\zeta),$$

where

$$P_{\alpha\beta}(z,\zeta) = \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\gamma_0 \Gamma(n+\alpha+\beta)} \frac{(1-|z|^2)^{n+\alpha+\beta}}{(1-<z,\zeta>)^{n+\alpha}(1-<\zeta,z>)^{n+\beta}}.$$

For the equation

$$Mu :\equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z_k}} - \mathcal{R}\overline{\mathcal{R}}u - n\overline{\mathcal{R}}u = 0 \tag{94}$$

with $\alpha = 0, \beta = -n$ we have $\mathrm{Re}(n+\alpha+\beta) \equiv 0, n+\beta \equiv 0$.
Theorem 5 *The Dirichlet problem $u(\zeta) = \gamma(\zeta), \zeta \in S_{2n-1}$ for equation (94) in B_n may have a solution if and only if $\gamma(\zeta)$ coincides with the boundary value on S_{2n-1} of a function holomorphic in B_n.*
Proof If u satisfies (94), then

$$\begin{aligned}
&\sum_{k=1}^{n} \int_{B_n} |\frac{\partial u}{\partial \bar{z}_k} - \frac{z_k}{1+\sqrt{1-|z|^2}} \overline{\mathcal{R}}u|^2 d\nu(z) \\
&= \sum_{k=1}^{n} \int_{B_n} |\frac{\partial u}{\partial \bar{z}_k}|^2 d\nu - \int_{B_n} |\overline{\mathcal{R}}u|^2 d\nu \\
&= \sum_{k=1}^{n} \int_{B_n} \frac{\partial}{\partial z_k}(\bar{u}\frac{\partial u}{\partial \bar{z}_k}) d\nu - \sum_{k=1}^{n} \int_{B_n} \bar{u} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} d\nu - \int_{B_n} \mathcal{R}\bar{u}\overline{\mathcal{R}}u d\nu \\
&= n \int_{S_{2n-1}} \bar{u}\overline{\mathcal{R}}u d\sigma - \int_{B_n} \bar{u} \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} d\nu + \int_{B_n} \bar{u}\mathcal{R}\overline{\mathcal{R}}u d\nu \\
&\quad + n \int_{B_n} \bar{u}\overline{\mathcal{R}}u d\nu - n \int_{S_{2n-1}} \bar{u}\overline{\mathcal{R}}u d\sigma = - \int_{B_n} \bar{u} M u d\nu = 0,
\end{aligned}$$

i.e u is holomorphic as a solution of the homogeneous invariant Cauchy-Riemann system (78), $f_k \equiv 0$. The Beltrami operator B_k defined by (65) acts from $\mathbb{C}$ to $\mathbb{C}^n$ and its adjoint defined by

$$B^* v :\equiv \sum_{k=1}^{n} \frac{\partial v_k}{\partial z_k} + \sum_{k,l=1}^{n} \frac{\partial}{\partial \bar{z}_l}(\bar{q}_{kl} v_k)$$

acts from $\mathbb{C}^n$ to $\mathbb{C}$ and the composition of B^* with the Beltrami operator produces the second order equation

$$Lu :\equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k}$$

$$+ \sum_{k,l=1}^{n} \left\{ \frac{\partial}{\partial z_k}(q_{kl}\frac{\partial u}{\partial z_l}) + \frac{\partial}{\partial \bar{z}_l}(\bar{q}_{kl}\frac{\partial u}{\partial \bar{z}_k}) + \frac{\partial}{\partial \bar{z}_l}\overline{q_{kl}(z)}\sum_{i=1}^{n} q_{ki}\frac{\partial u}{\partial z_i} \right\} = 0 \quad (95)$$

with the principial symbol equal to

$$\sum_k |\zeta_k + \sum_{l=1}^{n} q_{kl}\bar{\zeta}_l|^2 = |\zeta + q\bar{\zeta}|^2,$$

so that if we assume for the matrix $q = q_{kl}$ to be such that $\det(I - q\bar{q}) \neq 0$ in the domain $\Omega \subset \mathbb{C}^n$ and $\det(I - q\bar{q}) = 0$ on its boundary then the equation being strongly elliptic inside of Ω degenerates on its boundary Γ. Every solution of the homogeneous Beltrami system (65) obviously is a solution of the equation (95) by construction. Conversely, if u is a solution of the second order scalar equation (95) in Ω can we say that then u is a solution of the overdetermined homogeneous system (65) in Ω?

Theorem 6 *If $\Gamma = \partial\Omega$ is a characteristic set for* (65)*, then every solution of equation* (95) *is a solution of the homogeneous system* (65)*.*

Proof If u is a solution of (95) in Ω, then

$$\sum_{k=1}^{n} \int_\Omega |B_k u|^2 d\Omega$$

$$= \sum_{k=1}^{n} \int_\Omega \left(\frac{\partial \bar{u}}{\partial z_k} + \sum_{l=1}^{n} \overline{q_{kl}(z)} \frac{\partial \bar{u}}{\partial \bar{z}_l} \right) \left(\frac{\partial u}{\partial \bar{z}_k} + \sum_{l=1}^{n} q_{kl}(z) \frac{\partial u}{\partial z_l} \right) d\Omega$$

$$= \sum_{k=1}^{n} \int_\Omega \frac{\partial}{\partial z_k} \bar{u} B_k u d\Omega + \sum_{k,l=1}^{n} \int_\Omega \frac{\partial}{\partial \bar{z}_l} \bar{u}\overline{q_{kl}(z)} B_k u d\Omega - \sum_{k=1}^{n} \int_\Omega \bar{u} \frac{\partial}{\partial z_k} B_k u d\Omega$$

$$- \sum_{k,l=1}^{n} \int_\Omega \bar{u} \frac{\partial}{\partial \bar{z}_l} (\overline{q_{kl}(z)} B_k u) d\Omega = n \int_\Gamma \bar{u} \sum_{k=1}^{n} \overline{B_k \rho} B_k u dS - \int_\Omega \bar{u} L u d\Omega = 0, \quad (96)$$

because the set of degeneration Γ is the charachteristic set for (65) that is $B_k\rho = 0$ on Γ, where $\rho(z)$ is the defining function for the domain

$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}, grad\rho \neq 0$ on Γ. Hence $B_k u = 0, 1 \leq k \leq n$ in Ω.

Corollary 3 *The Dirichlet problem* $u(\zeta) = \gamma(\zeta), \zeta \in S_{2n-1}$ *for the equation* (95) *with* $q_{kl} = -z_k z_l (1 + \sqrt{1 - |z|^2})^{-2}, 1 \leq k, l \leq n$, *in* B_n *may have a solution if and only if* $\gamma(\zeta)$ *coincides with the boundary values of a holomorphic function in* B_n.

Proof Indeed, if $Lu = 0$ in B_n, then according to (96) u satisfies the system

$$\frac{\partial u}{\partial \bar{z}_k} - \frac{z_k}{(1 + \sqrt{1 - |z|^2})^2} \mathcal{R}u = 0, 1 \leq k \leq n, \tag{97}$$

because for $\rho(z) = |z|^2 - 1$ on S_{2n-1}

$$B_k \rho \equiv z_k - \frac{z_k}{(1 + \sqrt{1 - |z|^2})} \mathcal{R}\rho = z_k - z_k \frac{|z|^2}{(1 + \sqrt{1 - |z|^2})^2} = 0.$$

But as we know every solution of (97) is a superposition of a function holomorphic in the unit ball $|\zeta| \leq 1$ with the homeomorphism (80) of the ball $|z| \leq 1$ to the ball $|\zeta \leq 1$ satisfying (97) in B_n, i.e. $\gamma(\zeta)$ must be the boundary values of a holomorphic function in the ball B_n.

Corollary 4 *The Dirichlet problem* $u(\zeta) = \gamma(\zeta), \zeta \in S_{2n-1}$, *for the equation* (95) *in* B_n *with* $q_{kl}(z) \equiv z_k z_l, 1 \leq k \leq n$, *satisfying the homogeneous Neumann condition* $\partial u / \partial n = 0$ *on* S_{2n-1} *may have a solution if and only if* $\gamma(\zeta)$ *coincides with the boundary values of a holomorphic function in* B_n.

Proof In this case the set of degeneration S_{2n-1} is not a characteristic set:

$$B_k \rho = z_k (1 + |z|^2) = 2 z_k$$

on S_{2n-1}, so that the identity looks as

$$\sum_{k=1}^{n} \int_{B_n} \left| \frac{\partial u}{\partial \bar{z}_k} + z_k \mathcal{R}u \right|^2 dv = 2n \int_{S_{2n-1}} \bar{u} \sum_{k=1}^{n} \bar{z}_k B_k u d\sigma - \int_{B_n} \bar{u} L u dv$$
$$= 2n \int_{S_{2n-1}} \bar{u} (\mathcal{R}u + \bar{\mathcal{R}}\bar{u}) d\sigma - \int_{B_n} \bar{u} L u dv$$
$$= 2n \int_{S_{2n-1}} \bar{u} \frac{\partial u}{\partial n} d\sigma - \int_{S_{2n-1}} \bar{u} L u dv.$$

Hence if $Lu = 0$ in B_n and $\partial u / \partial n = 0$ on S_{2n-1}, then u satifies the system

$$\frac{\partial u}{\partial \bar{z}_k} + z_k \mathcal{R}u = 0, 1 \leq k \leq n,$$

every solution of which is a superposition of a function holomorphic in the unit ball $|\zeta| < 1$ with the homeomorphism $\zeta = 2z\,/(1+|z|^2)$ of the unit ball $|z| < 1$, satisfying this system.

Corollary 5 *Equation* (95) *with* $q_{kl}(z) \equiv -z_k z_l$ *or with* $q_{kl}(z) \equiv -\frac{z_k z_l}{2-|z|^2}$, $1 \le k \le n$, *has no solutions, continuous in* $\bar{B}_n$, *except constants.*

Proof In these cases S_{2n-1} is the characteristic set, so that from (96) it follows that u satifies the system of equations (89) and (85), which have no other solutions, continuous on $\bar{B_n}$, except constants.

The invariant pluriharmonic operator

$$\mathcal{P}_{kl}u :\equiv \frac{\partial^2}{\partial \bar{z}_k \partial z_l}(u_0\varphi_z)(0) = \frac{\partial^2 u}{\partial \bar{z}_k \partial z_l} + \frac{z_k \bar{z}_l}{(1+\sqrt{1-|z|^2})^2}\mathcal{R}\overline{\mathcal{R}}u$$

$$-\frac{z_k}{1+\sqrt{1-|z|^2}}\frac{\partial \mathcal{R}u}{\partial z_l} - \frac{\bar{z}_l}{1+\sqrt{1-|z|^2}}\frac{\partial \mathcal{R}u}{\partial \bar{z}_k} \tag{98}$$

acts from $\mathbb{C}$ to $\mathbb{C}^{n^2}$ and its adjoint

$$\mathcal{P}^*\varphi_{k,l} :\equiv \sum_{k,l=1}^{n}\left(\frac{\partial^2 \varphi_{kl}}{\partial z_k \partial \bar{z}_l} + \mathcal{R}\overline{\mathcal{R}}\left(\frac{z_k z_l}{(1+\sqrt{1-|z|^2})^2}\varphi_{kl}\right)\right.$$

$$-\mathcal{R}\left(\frac{\partial}{\partial \bar{z}_l}\left(\frac{\bar{z}_k}{1+\sqrt{1-|z|^2}}\varphi_{kl}\right)\right) - \overline{\mathcal{R}}\left(\frac{\partial}{\partial z_k}\left(\frac{z_l}{1+\sqrt{1-|z|^2}}\varphi_{kl}\right)\right)$$

$$-n\frac{\partial}{\partial z_k}\left(\frac{z_l}{1+\sqrt{1-|z|^2}}\varphi_{kl}\right) - n\frac{\partial}{\partial \bar{z}_l}\left(\frac{\bar{z}_k}{1+\sqrt{1-|z|^2}}\varphi_{kl}\right)$$

$$\left.+n\overline{\mathcal{R}}\left(\frac{\bar{z}_k z_l}{(1+\sqrt{1-|z|^2})^2}\varphi_{kl}\right) + \frac{n^2 \bar{z}_k z_l}{(1+\sqrt{1-|z|^2})^2}\varphi_{kl}\right)$$

acts from $\mathbb{C}^{n^2}$ to $\mathbb{C}$.

The invariant biharmonic operator is the composition of $\mathcal{P}^*$ with $\mathcal{P}_{kl}$: $B_h u :\equiv \mathcal{P}^*\mathcal{P}_{kl}u$ is an operator of fourth order with the principial part equal to

$$\sum_{k,l}\left(\frac{\partial^2}{\partial z_k \partial \bar{z}_k} - \frac{\bar{z}_k \mathcal{R}\overline{\mathcal{R}}}{1+\sqrt{1-|z|^2}}\right)\left(\frac{\partial^2}{\partial z_l \partial \bar{z}_l} - \frac{\bar{z}_l \mathcal{R}\overline{\mathcal{R}}}{1+\sqrt{1-|z|^2}}\right)u,$$

so that the principial symbol of B_h is equal to $(1/16)(|\zeta|^2 - | < z, \zeta > |^2)^2$.

Theorem 7 *The equation* $B_h u = 0$ *has no other solutions, continuous in* B_n, *except functions pluriharmonic in* B_n.

Proof The proof of this theorem is similar to that of Theorem 5 and uses the identity

$$\sum_{k,l=1}^{n} \int_{B_n} |\mathcal{P}_{k,l}u|^2 dv = \int_{B_n} \bar{u}\mathcal{P}^*\mathcal{P}_{kl}u dv = \int_{B_n} \bar{u}B_h u dv$$

from which it follows that u is a solution of the invariant pluriharmonic equations

$$\mathcal{P}_{k,l}u = 0. \tag{98'}$$

Multiplying both sides of (98′) by $\bar{z}_k z_l$ and summing up we obtain $(1 - |z|^2)\mathcal{R}\overline{\mathcal{R}}u = 0$, i.e. $\mathcal{R}\overline{\mathcal{R}}u = 0$ in B_n, so that (98′) turns to the system of equations

$$\frac{\partial^2 u}{\partial \bar{z}_k \partial z_l} - \frac{z_l}{1+\sqrt{1-|z|^2}}\frac{\partial}{\partial z_k}\overline{\mathcal{R}}u - \frac{\bar{z}_k}{1+\sqrt{1-|z|^2}}\frac{\partial}{\partial \bar{z}_l}\mathcal{R}u = 0, 1 \le k, l \le n. \tag{99}$$

Multiplying these equations by $\bar{z}_k$ and summing up we obtain $(\partial/\partial\bar{z}_l)\mathcal{R}u = 0$ in B_n and multiplying by z_l and summing up we obtain $(\partial/\partial z_\ell)\overline{\mathcal{R}}u = 0$ in B_n and then from (99) it follows that u is pluriharmonic in B_n.

The operator

$$\Delta_B^* v :\equiv \sum_{k=1}^{n} \frac{\partial^2 v}{\partial z_k \partial \bar{z}_k} - \mathcal{R}\overline{\mathcal{R}}v - n\mathcal{R}v - n\overline{\mathcal{R}}v - n^2 v \tag{100}$$

is an adjoint to the invariant Laplacian

$$\Delta_B u :\equiv \sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - \mathcal{R}\overline{\mathcal{R}}u$$

Theorem 8 *If u is a solution of $\Delta_B^*\Delta_B u = 0$ in B_n and if $\Delta_B u = 0$ on S_{2n-1}, then $\Delta_B u \equiv 0$ in B_n.*

Proof Using Green's indentity we have

$$\int_{B_n} |\sum_{k=1}^{n} \frac{\partial^2 u}{\partial z_k \partial \bar{z}_k} - \mathcal{R}\overline{\mathcal{R}}u|^2 dv = n \sum_{k=1}^{n} \int_{S_{2n-1}} \frac{\partial \bar{u}}{\partial \bar{z}_k} \bar{z}_k \Delta_B u d\sigma$$

$$-\sum_{k=1}^{n} \int_{B_n} \frac{\partial \bar{u}}{\partial \bar{z}_k} \frac{\partial \Delta_B u}{\partial z_k} d\ v - n \int_{S_{2n-1}} \mathcal{R}\bar{u}\Delta_B u d\sigma$$

$$+ \int_{B_n} \mathcal{R}\bar{u}\overline{\mathcal{R}}\Delta_B u dv + n \int_{B_n} \mathcal{R}\bar{u}\Delta_B u dv$$

$$= n^2 \int_{S_{2n-1}} \bar{u}\Delta_B u d\sigma + \int_{B_n} \bar{u}\Delta_B^* \Delta_B u dv$$

$$= \int_{B_n} \bar{u}\Delta_B^* \Delta_B u dv + n^2 \int_{S_{2n-1}} \bar{u}\Delta_B u d\sigma \tag{101}$$

because

$$\int_{B_n} \mathcal{R}uv dv = -\int_{B_n} u\mathcal{R}v dv - n \int_{B_n} uv dv + n \int_{S_{2n-1}} uv d\sigma$$

and

$$\int_{S_{2n-1}} \mathcal{R}\bar{u}\Delta_B u d\sigma = -\int_{S_{2n-1}} \bar{u}\mathcal{R}\Delta_B u d\sigma - n \int_{S_{2n-1}} \bar{u}\Delta_B u d\sigma,$$

so that if $\Delta_B^* \Delta_B u = 0$ in B_n and $\Delta_B u = 0$ on S_{2n-1}, then it follows from (101) that $\Delta_B u \equiv 0$ in B_n. Note that for the operator $\Delta_B^* v, \alpha = -n, \beta = -n, \text{Re}\ (n + \alpha + \beta) < 0, n + \alpha \equiv 0, n + \beta \equiv 0$, so that from $\Delta_B^* v = 0$ in B_n and $v = 0$ on S_{2n-1} it does not follow directly that $v \equiv 0$. From Theorem 5 it follows that the problem with the only condition $u(\zeta) = \gamma(\zeta), \zeta \in S_{2n-1}$, is uniquely solvable for the fourth order equation $\Delta_B^* \Delta_B u = 0$ in B_n, provided theat $\Delta_B u$ vanishes on the boundary S_{2n-1}.

However we expect that for the fourth order equation

$$\sum_{k=1}^{n} \left(\frac{\partial^2}{\partial z_k \partial \bar{z}_k} - \mathcal{R}\overline{\mathcal{R}} \right) \left(\sum_{l=1}^{n} \frac{\partial^2}{\partial z_l \partial \bar{z}_l} - \mathcal{R}\overline{\mathcal{R}} \right) u = 0$$

in B_n the Dirichlet problem

$$u(\zeta) = \gamma(\zeta), \frac{\partial u}{\partial n} = h(\zeta), \zeta \in S_{2n-1},$$

is uniquely solvable for any $\gamma(\zeta), h(\zeta) \in C^\infty$. But the solution will not be a C^∞ function in $\bar{B}_n$. We leave to the reader to prove whether this is true or not.

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FORMULAS FOR DERIVATIVES OF SOLUTIONS OF THE $\overline{\partial}$-EQUATION IN THE BALL

A.I. PETROSYAN
Yerevan State University
Aleck Manookian str. 1, Yerevan 375049, Armenia
albpet@yandex.ru and albpet@freenet.am

Abstract Let $f(z) = \sum_{k=1}^{n} f_k(z)d\overline{z}_k$ be a closed $(0,1)$ form in the unit ball $B \in \mathbb{C}^n$, and let u_α be the solution of the equation $\overline{\partial}u = f$, which has the minimal norm in the weighted space $L^2[(1-|z|^2)^\alpha dv]$. Some explicit integral formulas for the derivatives of u_α are obtained. These formulas are used for estimations of the derivatives of $u_\alpha(z)$ in C^m-norm. Similar formulas and estimates are obtained also for the derivatives of the "canonical" solution having the minimal L^2-norm on the unit sphere.

Keywords: $\bar{\partial}$-equation, minimal solution, representation formulas
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The Cauchy-Green formula

$$u(z) = P[u](z) + T[\overline{\partial}u](z) = \frac{1}{2\pi i}\int\limits_{\partial D} \frac{u(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int\limits_{D} \frac{\partial u(\zeta)}{\partial\overline{\zeta}}\frac{d\overline{\zeta}\wedge d\zeta}{\zeta - z} \quad (1)$$

and its weighted version are well known as one of the main tools in complex analysis. In the case of the unit disc D, the weighted formula is written in the form

$$u(z) = P_\alpha[u](z) + T_\alpha[\overline{\partial}u](z) = \frac{\alpha+1}{2\pi i}\int\limits_{D} u(\zeta)\frac{(1-|\zeta|^2)^\alpha}{(1-\overline{\zeta}z)^{\alpha+2}}d\overline{\zeta}\wedge d\zeta$$

$$-\frac{1}{2\pi i}\int\limits_{D} \frac{\partial u(\zeta)}{\partial\overline{\zeta}}\left(\frac{1-|\zeta|^2}{1-\overline{\zeta}z}\right)^{\alpha+1}\frac{d\overline{\zeta}\wedge d\zeta}{\zeta - z}. \quad (2)$$

The first summand of the right-hand side of this formula singles out the "analytic" part of $u(z)$. In the complex one-dimensional case, this fea-

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ture of the Cauchy-Green formula permits to find some explicit solutions of the $\overline{\partial}$-equation

$$\frac{\partial u(z)}{\partial \overline{z}} = f(z), \tag{3}$$

where f is a continuous function in $\overline{D}$. Namely, we mean the functions $u_0(z) = T[f(\zeta)d\overline{\zeta}](z)$ and $u_\alpha(z) = T_\alpha[f(\zeta)d\overline{\zeta}](z)$.
One has to note that in case of holomorphic functions formula (2) was first given by M.M. Djrbashian [2, 3].
In several complex variables the natural generalization of (3) is

$$\overline{\partial} u = f, \tag{4}$$

where f is a $\overline{\partial}$-closed $(0,1)$ form continuous in D. As in the one-dimensional case, the Cauchy-Green formula is used for finding integral representations of solutions of (4).
The integral representation method developed by G.M. Henkin (see, eg. [1]) and others permits to find explicit expressions for solutions of the $\overline{\partial}$-equation in different types of domains. However, we shall deal with (4) in the unit ball B of the space $\mathbb{C}^n$.
The well-known multidimensional counterparts of formulas (1) and (2) were found in [4]:

$$u(z) = P[u](z) + T[\overline{\partial} u](z) = \int\limits_{\partial B} u(\zeta)| \wedge C_0(\zeta, z) - \int\limits_{B} \overline{\partial} u(\zeta) \wedge C_0(\zeta, z), \tag{5}$$

$$u(z) = P_\alpha[u](z) + T_\alpha[\overline{\partial} u](z) =$$

$$= \frac{1}{nB(n,\alpha)} \int\limits_{B} u(\zeta) \frac{(1-|\zeta|^2)^\alpha}{(1-\langle z,\zeta\rangle)^{n+1+\alpha}} d\lambda(\zeta) - \int\limits_{B} \overline{\partial} u(\zeta) \wedge C_\alpha(\zeta, z) \tag{6}$$

where

$$C_0(\zeta, z) = c_n \frac{(1-\langle \zeta, z\rangle)^{n-1}}{D^n(\zeta, z)} \left[\sum_{i=1}^{n} (-1)^{i-1} (\overline{\zeta}_i - \overline{z}_i) \bigwedge_{j\neq i} d\overline{\zeta}_j \right] \bigwedge_{k=1}^{n} d\zeta_k,$$

$$D(\zeta, z) = |1 - \langle \zeta, z\rangle|^2 - (1-|\zeta|^2)(1-|z|^2), \qquad c_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi i)^n},$$

$d\lambda(\zeta)$ is the normed volume element of the ball,

$$C_\alpha(\zeta, z) = \Psi_\alpha(\zeta, z) C_0(\zeta, z)$$

and

$$\Psi_\alpha(\zeta, z) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)} \left(\frac{1-|\zeta|^2}{1-\langle z,\zeta\rangle}\right)^{\alpha+1} \times \left[\sum_{p=0}^{n-1} \binom{n-1}{p} \frac{(-1)^p}{\alpha+1+p} \left(\frac{(1-|\zeta|^2)(1-|z|^2)}{1-\langle z,\zeta\rangle}\right)^p\right].$$

One more remarkable property of formulas (5) and (6) is that the operators $P[u]$ and $P_\alpha[u]$ are the orthogonal projectors which map $L^2(d\sigma)$ and $L^2[(1-|z|^2)^\alpha d\lambda(z)]$ onto their subspaces of holomorphic functions. Therefore, the solutions of the $\overline{\partial}$-equation (4), which are given by $u = T[f]$ and $u_\alpha = T_\alpha[f]$, have minimal norms in $L^2(d\sigma)$ and $L^2[(1-|z|^2)^\alpha d\lambda(z)]$ correspondingly.

In applications, it is significant to have more than simply one solution of the $\overline{\partial}$-equation. It is necessary to have a solution which is estimated in some norm, and the minimal solutions are of this type as they are given by an explicit formula.

We are aimed to obtain some explicit formulas for derivatives. Using them we come to some estimates containing derivatives.

Below we assume that:

$C^m(\overline{B})$ is the space of all functions $u(z)$ which are m times continuously differentiable in $\overline{B}$;

$\|u\|_m$ is the norm in $C^m(\overline{B})$;

$C^m_{(0,1)}(\overline{B})$ is the space of all $(0,1)$ forms $f = \sum_{k=1}^n f_k(z) d\overline{z}_k$, the coefficients of which belong to $C^m(\overline{B})$;

$\|f\|_m = \sum_{k=1}^n \|f_k\|_m$.

Further, by $L_j^{s_j}$ we denote the differential operator of the order $r_1+\cdots+r_j-s_j-1$, $1 \le j \le n$, $0 \le s_j \le r_j - 1$:

$$L_j^{s_j} = \left(n + \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}\right) \cdots \left(n + r_1 + \cdots + r_j - s_j - 2 + \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}\right).$$

Here we assume that $L_j^{s_j}$ is the identical operator if for some j we have $s_j = r_1 + \cdots + r_j - 1$ (i.e. $r_1 = \cdots = r_{j-1} = 0$).

The main result of this work the following

Theorem 1 *Let $u \in C^m(\overline{B})$, $f \in C^m_{(0,1)}(\overline{B})$ and $r_1+\cdots+r_n \le m$. Then the following formula is true for derivatives of the minimal solution of the $\overline{\partial}$-equation $u_0 = T[f]$:*

$$D_1^{r_1}\cdots D_n^{r_n}T[f](z)$$
$$=\sum_{j=1}^{n}\sum_{s_j=0}^{r_j-1}L_j^{s_j}\left\{\overline{z}_1^{r_1}\cdots\overline{z}_{j-1}^{r_{j-1}}\overline{z}_j^{r_j-s_j}\left\langle D_j^{s_j}D_{j+1}^{r_{j+1}}\cdots D_n^{r_n}f(z),z\right\rangle\right\}$$
$$-\sum_{j=1}^{n}\sum_{s_j=0}^{r_j-1}L_j^{s_j}\left\{T\left[\overline{\partial}\left(\overline{\zeta}_1^{r_1}\cdots\overline{\zeta}_{j-1}^{r_{j-1}}\overline{\zeta}_j^{r_j-s_j}\right.\right.\right.$$
$$\left.\left.\left.\times\left\langle D_j^{s_j}D_{j+1}^{r_{j+1}}\cdots D_n^{r_n}f(\zeta),\zeta\right\rangle\right)\right](z)\right\}$$
$$+T\left[D_1^{r_1}\cdots D_n^{r_n}f(\zeta)\right](z). \tag{7}$$

Note that the multipliers of the form $\overline{z}_1^{r_1}\cdots\overline{z}_{j-1}^{r_{j-1}}\overline{z}_j^{r_j-s_j}$ arise in this formula by purely technical reasons and they are not significant in further evaluation.
A similar formula is true for the derivatives of the solutions $u_\alpha=T_\alpha[f]$. We omit this formula as it is more cumbersome.
One can observe that formula (7) expresses the derivatives of $T[f](z)$ of the order $r_1+\cdots+r_n$ by the derivatives of functions of the type $T[\overline{\partial}(\ldots)](z)$, which are of lower orders. This permits to use some inductive argument which leads to different estimates for $D_1^{r_1}\cdots D_n^{r_n}T[f](z)$. Particularly, the following theorem is true.
Theorem 2 *If $f\in C^m_{(0,1)}(\overline{B})$, then the following estimates are true for the minimal solutions of the $\overline{\partial}$-equation, i.e. for $u=T[f]$ and $u_\alpha=T_\alpha[f]$:*

$$\|T[f]\|_m\le\gamma\|f\|_m\,,\quad \|T_\alpha[f]\|_m\le\gamma_\alpha\|f\|_m.$$

Here γ and γ_α are some constants which are independent of f. In other words, the minimal solution operators $T\colon C^m_{(0,1)}(\overline{B})\to C^m(\overline{B})$ and $T_\alpha\colon C^m_{(0,1)}(\overline{B})\to C^m(\overline{B})$ are bounded.
One has to mention the works [5] and [6] containing some estimates for the derivatives of the solutions of the $\overline{\partial}$-equation in the polydisc.

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ON SOME COMPLEX DIFFERENTIAL AND SINGULAR INTEGRAL OPERATORS

H. Begehr
I. Math. Inst., FU Berlin
Arnimallee 3, D -14915 Berlin, Germany
begehr@math.fu-berlin.de

A. Dzhuraev
Inst. of Math., Tajik Acad. Sci.
Pr. Rudaki 33, 734025 Dushanbe, Tajikistan
dzhuraev@khj.tajik.net

Abstract The Pompeiu operator in complex analysis as the right inverse of the Cauchy-Riemann operator provides a particular solution to the inhomogeneous Cauchy-Riemann equation. In case of the entire plane $\mathbb{C}$ or the whole space $\mathbb{C}^n$ under proper decay conditions on the solution it gives even the unique solution. Taking the $z-$derivative then relates this derivative to the $\overline{z}-$derivative of the same function via the Ahlfors-Beurling operator. This area integral operator is singular of Calderon-Zygmund type. This situation is reflected to any higher order partial differential operator of fixed order. All $n-$th order derivatives are expressible by just one particular one through proper singular integral operators of Calderon-Zygmund type emerging from higher order Pompeiu operators within a hierarchy of integral operators through proper differentiation.

The situation is found true also in bounded domains. If the kernels of the higher order Pompeiu operators are altered by replacing them through proper derivatives of higher order Green functions then these operators turn out to be projections on L_2-subspaces orthogonal to the kernel of the related higher order partial differential operator. The unique solution to the related inhomogeneous partial differential equation is provided by this projective operator. All other derivatives of the same order of the solution are then expressed by the given one through singular integral operators. The situation is considered in particular for the unit disc in $\mathbb{C}$, the unit ball and the unit polydisc in $\mathbb{C}^n$. In $\mathbb{C}^2$ also the Fueter system is treated.

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Keywords: Ahlfors-Beurling transformation, Cauchy-Pompeiu representation, orthogonalized potentials, singular integral operators, higher order Green functions, Cauchy-Riemann system, Fueter system

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1. Ahlfors-Beurling transformation and its generalizations

In complex analysis of a single variable there are two basic first order elliptic differential operators, the Cauchy-Riemann operator and its adjoint

$$\partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big). \tag{1}$$

If w belongs to the Sobolev space $W^{1,p}(\mathbb{C})$ in the complex plane $\mathbb{C}$ for $1 < p < +\infty$ then the complex derivative $\partial_z w$ is expressable through $\partial_{\overline{z}} w$ in $\mathbb{C}$ by means of a singular integral operator S , at first introduced by Ahlfors and Beurling, see [1,6], as

$$(S\omega)(z) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{\omega(\zeta)}{(\zeta - z)^2}\ d\xi d\eta\ . \tag{2}$$

This operator and its adjoint $\overline{S}$ given by

$$(\overline{S}\omega)(z) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{\omega(\zeta)}{(\overline{\zeta - z})^2}\ d\xi d\eta \tag{3}$$

are Cauchy principle value integrals of Calderon-Zygmund type. They were substantially used in the theory of quasiconformal mappings and in the theory of elliptic first order systems. In case w vanishes at infinity and e.g. $w_{\overline{z}} \in L_{p,2}(\mathbb{C}), 2 < p < +\infty$, see [8], then

$$w(z) = -\frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z}\ d\xi d\eta\ .$$

Thus

$$\partial_z w(z) = (S\partial_{\overline{z}} w)(z)\ . \tag{4}$$

Similarly, when again $w(\infty) = 0$ and $w_z \in L_{p,2}(\mathbb{C}), 2 < p < +\infty$, then

$$\partial_{\overline{z}} w(z) = (\overline{S}\partial_z w)(z) \tag{5}$$

holds.

There are $(n+1)$ $n-$th order elliptic differential operators in the complex plane $\mathbb{C}$, namely $\partial_z^k \partial_{\overline{z}}^{n-k}, 0 \le k \le n$. In case w belongs to the Sobolev space $W^{n,p}(\mathbb{C}), 1 < p < +\infty$, and satisfies some decay conditions all these derivatives are expressable by just one fixed one again via singular integral operators.

Theorem *Let* $w \in W^{n,p}(\mathbb{C};\mathbb{C})$ *satisfy for some* $0 < \varepsilon$

$$\lim_{z\to\infty} |z|^{\rho+\sigma+\varepsilon}\ \partial_z^\rho \partial_{\overline{z}}^\sigma w(z) = 0 \text{ for } 0 \le \rho+\sigma \le n-1$$

then for $1 \le k \le n-1$

$$\begin{aligned} w(z) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{(z-\zeta)^{k-1}(\overline{z-\zeta})^{n-k-1}}{(k-1)!(n-k-1)!} \\ &\quad \times\Big[\log|\zeta-z|^2 - \sum_{\rho=1}^{k-1}\frac{1}{\rho} - \sum_{\sigma=1}^{n-k-1}\frac{1}{\sigma}\Big]\partial_\zeta^k \partial_{\overline{\zeta}}^{n-k} w(\zeta)d\xi d\eta \end{aligned}$$

and

$$w(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{(\overline{z-\zeta})^{n-1}}{(n-1)!(\zeta-z)}\ \partial_{\overline{\zeta}}^n w(\zeta)d\xi d\eta$$

and

$$w(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{(z-\zeta)^{n-1}}{(n-1)!(\overline{\zeta-z})}\ \partial_\zeta^n w(\zeta)d\xi d\eta.$$

For the last two formulas ε *may be replaced by 0 in the decay conditions, where then either* ρ *or* σ *are taken to be zero.*

Proof From the generalized Cauchy-Pompeiu formula, see [4], (3.6), for $|z| < R$

$$\begin{aligned} w(z) &= -\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{(z-\zeta)^\nu}{\nu_i(\overline{\zeta-z})}\ \partial_\zeta^\nu w(\zeta)d\overline{\zeta} \\ &\quad +\sum_{\nu=k}^{n-1}\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{(z-\zeta)^{k-1}(\overline{z-\zeta})^{\nu-k}}{(k-1)!(\nu-k)!} \\ &\quad \times\Big[\log|\zeta-z|^2 - \sum_{\rho=1}^{k-1}\frac{1}{\rho} - \sum_{\sigma=1}^{\nu-k}\frac{1}{\sigma}\Big]\partial_\zeta^k \partial_{\overline{\zeta}}^{\nu-k} w(\zeta)d\zeta \end{aligned}$$

$$+\frac{1}{\pi}\int\limits_{|\zeta|<R}\frac{(z-\zeta)^{k-1}(\overline{z-\zeta})^{n-k-1}}{(k-1)!(n-k-1)!}$$

$$\times\Big[\log|\zeta-z|^2-\sum_{\rho=1}^{k-1}\frac{1}{\rho}-\sum_{\sigma=1}^{\nu-k-1}\frac{1}{\sigma}\Big]\partial_\zeta^k\partial_{\overline{\zeta}}^{n-k}w(\zeta)d\xi d\eta\ ,$$

and

$$\begin{aligned} w(z) &= =\sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int\limits_{|\zeta|=R}\frac{(\overline{z-\zeta})^\nu}{\nu!(\zeta-z)}\ \partial_{\overline{\zeta}}^\nu w(\zeta)d\zeta \\ &\quad -\frac{1}{\pi}\int\limits_{|\zeta|<R}\frac{(\overline{z-\zeta})^{n-1}}{(n-1)!(\zeta-z)}\ \partial_{\overline{\zeta}}^n w(\zeta)d\xi d\eta, \end{aligned}$$

and

$$\begin{aligned} w(z) &= -\sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int\limits_{|\zeta|=R}\frac{(z-\zeta)^\nu}{\nu!(\overline{\zeta-z})}\ \partial_\zeta^\nu w(\zeta)d\overline{\zeta} \\ &\quad -\frac{1}{\pi}\int\limits_{|\zeta|<R}\frac{(z-\zeta)^{n-1}}{(n-1)!(\overline{\zeta-z})}\ \partial_\zeta^n w(\zeta)d\xi d\eta \end{aligned}$$

follow. From the growth condition the boundary integrals can be seen to tend to zero if R tends to ∞.
By differentiation

$$\partial_z^\mu\partial_{\overline{z}}^\nu w(z)=\frac{1}{\pi}\int\limits_{\mathbb{C}}\frac{(z-\zeta)^{k-\mu-1}(\overline{z-\zeta})^{n-k-\nu-1}}{(k-\mu-1)!(n-k-\nu-1)!}$$

$$\times\Big[\log|\zeta-z|^2-\sum_{\rho=1}^{k-\mu-1}\frac{1}{\rho}-\sum_{\sigma=1}^{k-\mu-1}\frac{1}{\sigma}\Big]\partial_\zeta^k\partial_{\overline{\zeta}}^{n-k}w(\zeta)d\xi d\eta$$

follows as long as $0\le\mu\le k-1$ and $0\le\nu\le n-k-1$. Let μ be fixed $0\le\mu\le k-1$. Then

$$\partial_z^{k-1-\mu}\partial_z^{n-k-1}w(z)=\frac{1}{\pi}\int\limits_{\mathbb{C}}\frac{(z-\zeta)^\mu}{\mu!}\Big[\log|\zeta-z|^2-\sum_{\rho=1}^{\mu}\frac{1}{\rho}\Big]\partial_\zeta^k\partial_{\overline{\zeta}}^{n-k}w(\zeta)d\xi d\eta$$

so that

$$\partial_z^{k-1-\mu}\partial_{\overline{z}}^{n-k+1+\mu}w(z)=-\frac{\mu+1}{\pi}\int\limits_{\mathbb{C}}\frac{(z-\zeta)^\mu}{(\overline{\zeta-z})^{\mu+2}}\ \partial_\zeta^k\partial_{\overline{\zeta}}^{n-k}w(\zeta)d\xi d\eta\ . \qquad (6)$$

Similarly fixing $\nu, 0 \leq \nu \leq n-k-1$, then

$$\partial_z^{k-1}\partial_{\overline{z}}^{n-k-1-\nu}w(z) = \frac{1}{\pi}\int\limits_{\mathbb{C}} \frac{(\overline{z-\zeta})^\nu}{\nu!}\Big[\log|\zeta - z|^2 - \sum_{\sigma=1}^{\nu}\frac{1}{\sigma}\Big]\partial_\zeta^k\partial_{\overline{\zeta}}^k w(\zeta)d\xi d\eta$$

so that

$$\partial_z^{k+1+\nu}\partial_{\overline{z}}^{n-k-1-\nu}w(z) = -\frac{\nu+1}{\pi}\int\limits_{\mathbb{C}} \frac{(\overline{z-\zeta})^\nu}{(\zeta - z)^{\nu+2}}\,\partial_\zeta^k\partial_{\overline{\zeta}}^{n-k}w(\zeta)d\xi d\eta\ . \quad (6')$$

Similarly for $1 \leq \nu \leq n$ as well

$$\partial_z^{\nu}\partial_{\overline{z}}^{n-\nu}w(z) = -\frac{\nu}{\pi}\int\limits_{\mathbb{C}} \frac{(\overline{z-\zeta})^{\nu-1}}{(\zeta - z)^{\nu+1}}\,\partial_{\overline{\zeta}}^{n}w(\zeta)d\xi d\eta \quad (7)$$

as

$$\partial_z^{n-\nu}\partial_{\overline{z}}^{\nu}w(z) = -\frac{\nu}{\pi}\int\limits_{\mathbb{C}} \frac{(z-\zeta)^{\nu-1}}{(\overline{\zeta - z})^{\nu+1}}\,\partial_{\zeta}^{n}w(\zeta)d\xi d\eta \quad (8)$$

follows.
The singular integral operators appearing in (6) to (8) are of the form

$$(S_\nu f)(z) = -\frac{\nu}{\pi}\int\limits_{\mathbb{C}} \Big(\frac{\overline{z-\zeta}}{\zeta - z}\Big)^{\nu-1}\frac{f(\zeta)}{(\zeta-z)^2}\ d\xi d\eta \quad (9)$$

and

$$(\overline{S}_\nu f)(z) = -\frac{\nu}{\pi}\int\limits_{\mathbb{C}} \Big(\frac{z-\zeta}{\overline{\zeta - z}}\Big)^{\nu-1}\frac{f(\zeta)}{(\overline{\zeta-z})^2}\ d\xi d\eta \quad (9')$$

for $1 \leq \nu$. $S_1 = S$ is the Ahlfors-Beurling, $\overline{S}_1 = \overline{S}$ its adjoint operator. The L_2-norm of S_ν is 1, $\| S_\nu \|_{L_2(\mathbb{C})} = 1$. This can be established for instance using Fourier transformation. It follows also from

$$\int\limits_{\mathbb{C}} |\ \partial_z^{k-1}\partial_{\overline{z}}^{n-k+1}\varphi(z)\ |^2\ dxdy = \int\limits_{\mathbb{C}} \partial_z^{k-1}\partial_{\overline{z}}^{n-k+1}\varphi(z)\partial_z^{n-k+1}\partial_{\overline{z}}^{k-1}\overline{\varphi(z)}dxdy$$

$$= \int\limits_{\mathbb{C}} \partial_z^{k}\partial_{\overline{z}}^{n-k}\varphi(z)\partial_z^{n-k}\partial_{\overline{z}}^{k}\overline{\varphi(z)}dxdy = \int\limits_{\mathbb{C}} |\ \partial_z^{k}\partial_{\overline{z}}^{n-k}\varphi(z)\ |\ dxdy,$$

$$\int_{\mathbb{C}} |\, \partial_z^{k+1} \partial_{\overline{z}}^{n-k-1} \varphi(z) \,|^2 \, dxdy = \int_{\mathbb{C}} \partial_z^{k+1} \partial_{\overline{z}}^{n-k-1} \varphi(z) \partial_z^{n-k-1} \partial_{\overline{z}}^{k+1} \overline{\varphi(z)} dxdy$$

$$= \int_{\mathbb{C}} \partial_z^{k} \partial_{\overline{z}}^{n-k} \varphi(z) \partial_z^{n-k} \partial_{\overline{z}}^{k} \overline{\varphi(z)} dxdy = \int_{\mathbb{C}} |\, \partial_z^{k} \partial_{\overline{z}}^{n-k} \varphi(z) \,|^2 \, dxdy$$

for compactly supported C^∞−functions φ obtained through integration by parts and observing that these functions are dense in $L_2(\mathbb{C})$.

2. Orthogonalized potentials and singular integral operators on bounded domains

In case of a bounded domain D of the complex plane $\mathbb{C}$ the situation is different. Let D be regular with respect to the Dirichlet problem and $G(z, \zeta)$ be twice its harmonic Green function satisfying

$$G(z, \zeta) = -\log |\, z - \zeta \,|^2 + h(z, \zeta)$$

with a function $h(\cdot, \zeta)$ harmonic in D. Then replacing the Cauchy kernel in the Cauchy-Pompeiu formula

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{w(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{\pi} \int_{D} \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} \, d\xi d\eta$$

for any $w \in W^{1,p}(D; \mathbb{C}), 1 < p < +\infty$, through

$$\frac{1}{\zeta - z} = \partial_z G(z, \zeta) - \partial_z h(z, \zeta)$$

proves

$$w(z) = -\frac{1}{\pi} \int_{D} w(\zeta) \partial_{\overline{\zeta}} \partial_z h(z, \zeta) d\xi d\eta - \frac{1}{\pi} \int_{D} \partial_z G(z, \zeta) w_{\overline{\zeta}}(\zeta) d\xi d\eta \,. \quad (10)$$

From the harmonicity of $h_{\overline{\zeta}}(\cdot, \zeta)$ the analyticity of the first term is seen.
By differentiating with respect to z gives

$$w_z(z) = -\frac{1}{\pi} \int_{D} w(\zeta) \partial_z^2 \partial_{\overline{\zeta}} h(z, \zeta) d\xi d\eta - \frac{1}{\pi} \int_{D} \partial_z^2 G(z, \zeta) w_{\overline{\zeta}}(\zeta) d\xi d\eta.$$

Hence, in general w_z cannot be expressed just through $w_{\overline{z}}$. But as the last term in (10) turns out to be orthogonal to the subspace of analytic functions $\mathcal{O}(D)$ in $L_2(D)$ this will be the case if one is restricting oneself to the subspace of the orthogonal complement $\mathcal{O}^{\perp}(D)$ of $\mathcal{O}(D)$ in $L_2(D)$. The orthogonality is seen from

$$(\partial_z G(z,\zeta),\phi(z)) = \int\limits_D \partial_z G(z,\zeta)\overline{\phi(z)}dxdy = (G(z,\zeta),\phi_{\overline{z}}(z))$$

which vanished if ϕ is analytic.

Let from now on D be the unit disc $\mathbb{D}$ of the complex plane. From

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{w(\zeta)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\zeta}w(\zeta)}{1-z\overline{\zeta}}d\zeta \\ &= \frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[\frac{w(\zeta)}{(1-z\overline{\zeta})^2}+\frac{\overline{\zeta}w_{\overline{\zeta}}(\zeta)}{1-z\overline{\zeta}}\Big]d\xi d\eta\end{aligned}$$

formula (10) becomes

$$w(z) = \frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{w(\zeta)}{(1-z\overline{\zeta})^2}d\xi d\eta - \frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta - z}-\frac{\overline{\zeta}}{1-z\overline{\zeta}}\Big)w_{\overline{\zeta}}(\zeta)d\xi d\eta. \tag{11}$$

Similarly

$$w(z) = \frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{w(\zeta)}{(1-\overline{z}\zeta)^2}\,d\xi d\eta - \frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\overline{\overline{\zeta - z}}}-\frac{\zeta}{1-\overline{z}\zeta}\Big)w_{\zeta}(\zeta)d\xi d\eta \tag{12}$$

follows for any $w \in W^{1,p}(\mathbb{D};\mathbb{C}), 1 < p < +\infty$. Here the first term is anti-analytic, i.e. in $\overline{\mathcal{O}}(\mathbb{D})$ while the second one is orthogonal to this set, hence belongs to $\overline{\mathcal{O}}^{\perp}(\mathbb{D})$. This is obvious as

$$G(z,\zeta) = \log \mid \frac{1-z\overline{\zeta}}{\zeta - z} \mid^2$$

is twice the Green function for $\mathbb{D}$.

Thus any $w \in W^{1,p}(\mathbb{D};\mathbb{C}), 1 < p < +\infty$, orthogonal to $\mathcal{O}(\mathbb{D})$ is representable as

$$\begin{aligned} w(z) &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta - z}-\frac{\overline{\zeta}}{1-z\overline{\zeta}}\Big)w_{\overline{\zeta}}(\zeta)d\xi d\eta \\ &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{1-\mid\zeta\mid^2}{1-z\overline{\zeta}}\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z}\,d\xi d\eta \end{aligned} \tag{13}$$

and those from $\overline{\mathcal{O}}^{\perp}(\mathbb{D})$ as

$$\begin{aligned} w(z) &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\overline{\zeta - z}} - \frac{\zeta}{1-\overline{z}\zeta}\Big) w_{\zeta}(\zeta)d\xi d\eta \\ &= -\frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{1-\mid\zeta\mid^2}{1-\overline{z}\zeta}\frac{w_{\zeta}(\zeta)}{\zeta - z}\; d\xi d\eta. \end{aligned} \tag{14}$$

From here

$$\begin{aligned} w_z(z) &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{(\zeta - z)^2} - \frac{\overline{\zeta}^2}{(1-z\overline{\zeta})^2}\Big) w_{\overline{\zeta}}(\zeta)d\xi d\eta \\ &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[2\frac{1-\mid\zeta\mid^2}{1-z\overline{\zeta}} - \Big(\frac{1-\mid\zeta\mid^2}{1-z\overline{\zeta}}\Big)^2\Big]\frac{w_{\overline{\zeta}}(\zeta)}{(\zeta - z)^2}\; d\xi d\eta, \end{aligned} \tag{15}$$

and

$$\begin{aligned} w_{\overline{z}}(z) &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{(\overline{\zeta - z})^2} - \frac{\zeta^2}{(1-\overline{z}\zeta)^2}\Big) w_{\zeta}(\zeta)d\xi d\eta \\ &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[2\frac{1-\mid\zeta\mid^2}{1-\overline{z}\zeta} - \Big(\frac{1-\mid\zeta\mid^2}{1-\overline{z}\zeta}\Big)^2\Big]\frac{w_{\overline{\zeta}}(\zeta)}{(\overline{\zeta - z})^2}\; d\xi d\eta \end{aligned} \tag{16}$$

follow.

(13) is called the orthogonalized Cauchy-Riemann potential, (14) the orthogonalized anti-Cauchy-Riemann potential.

Any function $w \in W^{2,p}(\mathbb{D};\mathbb{C}), 1 < p < +\infty$, can be represented, see [2], formula (50′), via

$$w(z) = \frac{1}{\pi}\int\limits_{|\zeta|<1} g_2(z,\zeta)w(\zeta)d\xi d\eta \tag{17}$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[\log\mid\frac{1-z\overline{\zeta}}{\zeta - z}\mid^2 - g_1(z,\zeta)(1-\mid\zeta\mid^2)\Big] w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta$$

where for $k \in \mathbb{N}$

$$g_k(z,\zeta) = \frac{1}{(1-z\overline{\zeta})^k} + \frac{1}{1-\overline{z}\zeta)^k} - 1$$

is a harmonic function in $|z|<1$ for $|\zeta|<1$.
While the first integral is a harmonic function in $|z|<1$ the last one differs by a harmonic function from

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2 w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta$$

which turns out to be orthogonal to the set of harmonic functions. To prove this let ϕ be harmonic, i.e. satisfy $\phi_{z\overline{z}}=0$. Then observing

$$\partial_{\overline{z}}\Big[-(\overline{\zeta-z})\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2+\frac{\overline{\zeta}(1-|\zeta|^2)(1-|z|^2)}{1-z\overline{\zeta}}\Big]$$

$$=\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2-g_1(z,\zeta)(1-|\zeta|^2)$$

and

$$\partial_z\Big[|\zeta-z|^2\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2-(1-|\zeta|^2)(1-|z|^2)\Big]$$

$$=-(\overline{\zeta-z})\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2+\frac{\overline{\zeta}(1-|\zeta|^2)(1-|z|^2)}{1-z\overline{\zeta}}$$

via the Gauss theorem

$$\int\limits_{|\zeta|<1}\Big[\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2-g_1(z,\zeta)(1-|\zeta|^2)\Big]\overline{\phi(z)}dxdy$$

$$=\int\limits_{|\zeta|<1}\Big[(\overline{\zeta-z})\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2-\frac{\overline{\zeta}(1-|\zeta|^2)(1-|z|^2)}{1-z\overline{\zeta}}\Big]\overline{\phi_z(z)}dxdy$$

$$=\int\limits_{|\zeta|<1}\Big[|\zeta-z|^2\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2-(1-|\zeta|^2)(1-|z|^2)\Big]\overline{\phi_{z\overline{z}}}(z)dxdy=0.$$

follows.
Any $w\in W^{2,p}(\mathbb{D};\mathbb{C})$ orthogonal to the space $\mathbb{H}(\mathbb{D})$ of harmonic functions in $\mathbb{D}$ hence can be represented as

$$w(z)=-\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[\log\left|\frac{1-z\overline{\zeta}}{\zeta-z}\right|^2-g_1(z,\zeta)(1-|\zeta|^2)\Big]w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta. \quad (18)$$

Differentiating leads to

$$\begin{aligned} w_z(z) &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[\frac{1}{\zeta - z} - \frac{\overline{\zeta}}{1-z\overline{\zeta}} - \frac{\overline{\zeta}(1-|\zeta|^2)}{(1-z\overline{\zeta})^2}\Big] w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta \\ &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^2 \frac{1}{\zeta - z}\, w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta\ , \\ w_{\overline{z}}(z) &= -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^2 \frac{1}{\overline{\zeta - z}}\, w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta\ , \end{aligned}$$

$$w_{zz}(z) = -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[3\Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^2 - 2\Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^3\Big]\frac{1}{(\zeta - z)^2} w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta, \tag{19}$$

$$w_{\overline{z}\overline{z}}(z) = -\frac{1}{\pi}\int\limits_{|\zeta|<1}\Big[3\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^2 - 2\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^3\Big]\frac{1}{(\overline{\zeta - z})^2}\, w_{\zeta\overline{\zeta}}(\zeta)d\xi d\eta. \tag{20}$$

Formula (17) can be generalized to functions from $W^{2n,p}(\mathbb{D};\mathbb{C})$ and used to treat functions from this space being orthogonal to the set of polyharmonic functions of order $n, \mathbb{H}^n(\mathbb{D})$, using [2], Theorem 3, Corollary 5. As the kernels become involved here only $n = 1$ is considered. From [2], Theorem 1, any $w \in W^{n,p}(\mathbb{D};\mathbb{C})$ is known to be expressible as

$$\begin{aligned} w(z) &= \sum_{\nu=0}^{n-1}\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{n}{\nu!}\Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^{n-1}\frac{(\overline{z-\zeta})^\nu}{(1-z\overline{\zeta})^2}\ \partial_{\overline{\zeta}}^{\nu} w(\zeta)d\xi d\eta \\ &\quad -\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{1}{(n-1)!}\Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^n\frac{(\overline{z-\zeta})^{n-1}}{\zeta - z}\partial_{\overline{\zeta}}^{n} w(\zeta)d\xi d\eta. \end{aligned} \tag{21}$$

The first n terms form a polyanalytic function of order n which also can be written as

$$\sum_{\nu=0}^{n-1}\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{n}{\nu!}\partial_{\overline{\zeta}}^{\nu}\Big[\frac{(1-|\zeta|^2)^{n-1}}{(1-z\overline{\zeta})^{n+1}}\ (\overline{\zeta - z})^\nu\Big] w(\zeta)d\xi d\eta$$

$$= \sum_{\nu=0}^{n-1} \frac{1}{\pi} \int\limits_{|\zeta|<1} (-1)^{n-1-\nu}(n-\nu)\binom{n}{\nu}^2 \tag{22}$$

$$\times \frac{|\zeta - z|^{2(n-1-\nu)} (1-|\zeta|^2)^\nu (1-|z|^2)^\nu}{(1-z\overline{\zeta})^{2n}} w(\zeta)d\xi d\eta.$$

Later on (21) combined with (22) will be written as

$$w(z) = \frac{1}{\pi} \int\limits_{|\zeta|<1} F_n(z,\zeta)w(\zeta)d\xi d\eta \tag{23}$$

$$-\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{1}{(n-1)!} \Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^n \frac{(\overline{z-\zeta})^{n-1}}{\zeta - z} \partial_{\overline{\zeta}}^n w(\zeta)d\xi d\eta.$$

Any $w \in W^{n,p}(\mathbb{D};\mathbb{C})$ orthogonal to the set of polyanalytic functions of order n in $\mathbb{D}$, $\mathcal{O}_n(\mathbb{D})$, can be represented as

$$w(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{1}{(n-1)!} \Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^n \frac{(\overline{z-\zeta})^{n-1}}{\zeta - z} \partial_{\overline{\zeta}}^n w(\zeta)d\xi d\eta, \tag{24}$$

see[2], Corollary 4.
Similarly any $w \in W^{n,p}(\mathbb{D};\mathbb{C})$ being orthogonal to the set $\overline{\mathcal{O}}_n(\mathbb{D})$ of polyantianalytic functions of order n in $\mathbb{D}$ are representable as

$$w(z) = -\frac{1}{\pi} \int\limits_{\mathbb{D}} \frac{1}{(n-1)!} \Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^n \frac{(z-\zeta)^{n-1}}{\overline{\zeta - z}} \partial_{\zeta}^n w(\zeta)d\xi d\eta. \tag{25}$$

Differentiation gives for $1 \le k \le n$

$$\partial_z^k \partial_{\overline{z}}^{n-k} w(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{1}{(k-1)!} \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \prod_{\mu=0,\mu\neq\nu}^{k} (n+\mu) \tag{26}$$

$$\times \Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^{n+\nu} \frac{(\overline{z-\zeta})^{k-1}}{(\zeta-z)^{k+1}} \partial_{\overline{\zeta}}^n w(\zeta)d\xi d\eta$$

and for $0 \le k \le n-1$

$$\partial_z^k \partial_{\overline{z}}^{n-k} w(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{1}{(n-k-1)!} \sum_{\nu=0}^{n-k} (-1)^\nu \binom{n-k}{\nu} \prod_{\mu=0,\mu\neq\nu}^{n-k} (n+\mu) \tag{27}$$

$$\times \Big(\frac{1-\mid \zeta \mid^2}{1-\overline{z}\zeta}\Big)^{n+\nu} \frac{(z-\zeta)^{n-k-1}}{(\overline{\zeta - z})^{n-k+1}}\ \partial_\zeta^n w(\zeta) d\xi d\eta.$$

These formulas follow from

$$\partial_z^k z^{n-1} = \frac{(n-1)!}{(n-1-k)!}\ z^{n-1-k}$$

and

$$\partial_z^k \Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big) \frac{1}{\zeta - z}$$

$$= \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \prod_{\mu=0,\mu\neq\nu}^{k} (n+\mu) \Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big)^{n+\nu} \frac{1}{(\zeta - z)^{k+1}}.$$

For $n = 2$ formulas (26) and (27) are

$$w_{zz}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big[12\Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big)^2 - 16\Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big)^3 \tag{26'}$$

$$+6\Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big)^4\Big] \frac{\overline{z-\zeta}}{(\zeta - z)^3}\ w_{\overline{\zeta\zeta}}(\zeta) d\xi d\eta\ ,$$

$$w_{z\overline{z}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big[3\Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big)^2 - 2\Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big)^3\Big] \frac{1}{(\zeta - z)^2} w_{\overline{\zeta\zeta}}(\zeta) d\xi d\eta \tag{26''}$$

and

$$w_{\overline{zz}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big[12\Big(\frac{1-\mid \zeta \mid^2}{1-\overline{z}\zeta}\Big)^2 - 16\Big(\frac{1-\mid \zeta \mid^2}{1-\overline{z}\zeta}\Big)^3 \tag{27'}$$

$$+6\Big(\frac{1-\mid \zeta \mid^2}{1-z\overline{\zeta}}\Big)^4\Big] \frac{z-\zeta}{(\overline{\zeta - z})^3}\ w_{\zeta\zeta}(\zeta) d\xi d\eta\ ,$$

$$w_{z\overline{z}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big[3\Big(\frac{1-\mid \zeta \mid^2}{1-\overline{z}\zeta}\Big)^2 - 2\Big(\frac{1-\mid \zeta \mid^3}{1-\overline{z}\zeta}\Big)^3\Big] \frac{1}{(\overline{\zeta - z})^2}\ w_{\zeta\zeta}(\zeta) d\xi d\eta\ . \tag{27''}$$

Formulas (26′), (26″) are valid for $w \in W^{2,p}(\mathbb{D};\mathbb{C})$ being orthogonal to the set $\overline{\mathcal{O}}_2(\mathbb{D})$ of biantianalytic functions.

If instead of this w and $w_{\overline{z}}$ are assumed to be orthogonal to the set $\mathcal{O}(\mathbb{D})$ of analytic functions then

$$w_{\overline{z}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{1}{\zeta - z} - \frac{\overline{\zeta}}{1 - z\overline{\zeta}} \Big) w_{\overline{\zeta}\overline{\zeta}}(\zeta) d\xi d\eta \ . \tag{28}$$

Similarly if w and w_z are orthogonal to the set $\overline{\mathcal{O}}(\mathbb{D})$ of antianalytic functions then

$$w_z(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{1}{\overline{\zeta - z}} - \frac{\zeta}{1 - \overline{z}\zeta} \Big) w_{\zeta\zeta}(\zeta) d\xi d\eta \ . \tag{29}$$

Put

$$\varphi(z) = w(z) - \overline{z} w_{\overline{z}}(z) \ , \ \psi(z) = w(z) - z w_z(z) \tag{30}$$

so that

$$\frac{\partial \varphi}{\partial \overline{z}} = -\overline{z} \frac{\partial^2 w}{\partial \overline{z}^2} \ , \ \frac{\partial \psi}{\partial z} = -z \frac{\partial^2 w}{\partial z^2}.$$

As φ is orthogonal to $\mathcal{O}(\mathbb{D})$ and ψ is orthogonal to $\overline{\mathcal{O}}(\mathbb{D})$, then

$$\varphi(z) = \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{1}{\zeta - z} - \frac{\overline{\zeta}}{1 - z\overline{\zeta}} \Big) \overline{\zeta} w_{\overline{\zeta}\overline{\zeta}}(\zeta) d\xi d\eta \ , \tag{31}$$

$$\psi(z) = \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{1}{\overline{\zeta - z}} - \frac{\zeta}{1 - \overline{z}\zeta} \Big) \zeta w_{\zeta\zeta}(\zeta) d\xi d\eta \ . \tag{32}$$

Hence from (30) by means of (28), (31), and (29), (32)

$$w(z) = \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{\overline{\zeta - z}}{\zeta - z} - \frac{\overline{\zeta}(\overline{\zeta - z})}{1 - z\overline{\zeta}} \Big) w_{\overline{\zeta}\overline{\zeta}} d\xi d\eta, \tag{33}$$

$$w(z) = \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{\zeta - z}{\overline{\zeta - z}} - \frac{\zeta(\zeta - z)}{1 - \overline{z}\zeta} \Big) w_{\zeta\zeta}(\zeta) d\xi d\eta. \tag{34}$$

From (33) we find the second order complex $z-$derivative

$$w_{zz}(z) = \frac{2}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{(\overline{\zeta - z})}{(\zeta - z)^3} - \frac{\overline{\zeta}^3 (\overline{\zeta - z})}{(1 - z\overline{\zeta})^3} \Big) w_{\overline{\zeta}\overline{\zeta}}(\zeta) d\xi d\eta \tag{35}$$

and the Laplacian

$$w_{z\overline{z}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{1}{(\zeta - z)^2} - \frac{\overline{\zeta}^2}{(1 - z\overline{\zeta})^2} \Big) w_{\overline{\zeta}\overline{\zeta}}(\zeta) d\xi d\eta \tag{36}$$

through its second order complex $\overline{z}-$derivative $w_{\overline{z}\overline{z}}$ for functions $w \in W^{2,p}(\mathbb{D};\mathbb{C}), 1 < p < +\infty$, such that $w_{\overline{z}}$ and $w(z) - \overline{z} w_{\overline{z}}$, i.e. w and $w_{\overline{z}}$, are orthogonal to every function holomorphic in $|\ z\ | < 1$. Also from (34) we find the second order complex $\overline{z}-$derivative $w_{\overline{z}\overline{z}}$

$$w_{\overline{z}\overline{z}}(z) = \frac{2}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{\zeta - z}{(\overline{\zeta - z})^3} - \frac{\zeta^3(\zeta - z)}{(1 - \overline{z}\zeta)^3} \Big) w_{\zeta\zeta}(\zeta) d\xi d\eta \tag{37}$$

and the Laplacian

$$w_{z\overline{z}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{1}{(\overline{\zeta - z})^2} - \frac{\zeta^2}{(1 - \overline{z}\zeta)^2} \Big) w_{\zeta\zeta}(\zeta) d\xi d\eta \tag{38}$$

through its second order complex $z-$derivative w_{zz} for functions $w \in W^{2,p}(\mathbb{D};\mathbb{C}), 1 < p < +\infty$, such that w_z and $w(z) - z w_z$, i.e. w and w_z , are orthogonal to every function antiholomorphic in $|\ z\ | < 1$.

As just another example consider for $w \in W^{3,p}(\mathbb{D};\mathbb{C})$ the combination of

$$w(z) = -\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Big[\frac{w(\zeta)}{\overline{\overline{\zeta - z}}} + \frac{z - \zeta}{\overline{\overline{\zeta - z}}}\ w_\zeta(\zeta) \Big] d\overline{\zeta} - \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{z - \zeta}{\overline{\overline{\zeta - z}}}\ w_{\zeta\zeta}(\zeta) d\xi d\eta$$

with

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{w(\zeta)}{\zeta - z}\ d\zeta - \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z}\ d\xi d\eta$$

leading to

$$\begin{aligned} w(z) \quad = \quad & -\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \Big[\frac{\zeta w(\zeta)}{1 - \overline{z}\zeta} - \frac{\zeta(\zeta - z)}{1 - \overline{z}\zeta}\ w_\zeta(\zeta) \Big] d\overline{\zeta} \\ & + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\zeta - z) \log \mid 1 - z\overline{\zeta} \mid^2 w_{\zeta\zeta}(\zeta) d\zeta \\ & - \frac{1}{\pi} \int\limits_{|\zeta|<1} (\zeta - z) \log \mid \zeta - z \mid^2 w_{\zeta\zeta\overline{\zeta}}(\zeta) d\xi d\eta \ . \end{aligned}$$

Introducing

$$G_3(z,\zeta) = \frac{1}{4} \mid \zeta - z \mid^4 \log \mid \frac{1-z\overline{\zeta}}{\zeta - z} \mid^2 - \frac{1}{4} \mid \zeta - z \mid^2 (1- \mid \zeta \mid^2)(1-|z|^2)$$
$$+\frac{1}{8}(1- \mid \zeta \mid^2)^2(1- \mid z \mid^2)^2$$

and applying the Gauss theorem repeatedly

$$\begin{aligned} w(z) &= \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big[\frac{1}{(1-z\overline{\zeta})^2} - 1 - 2z\overline{\zeta} \\ &\quad +2 \ \frac{2(1- \mid z \mid^2)(1- \mid \zeta \mid^2) - \mid \zeta - z \mid^2}{(1-\overline{z}\zeta)^4}\Big] w(\zeta) d\xi d\eta \\ &\quad -\frac{1}{\pi} \int\limits_{|\zeta|<1} G_{3z\overline{z}\overline{z}}(z,\zeta) w_{\zeta\zeta\overline{\zeta}}(\zeta) d\xi d\eta \end{aligned} \quad (39)$$

follows. Here the first term is a function $\varphi \in \overline{\mathcal{O}}\mathbb{H}(\mathbb{D})$ satisfying $\varphi_{zz\overline{z}} = 0$ while the last is obviously orthogonal to the set of all these functions. Therefore any $w \in W^{3,p}(\mathbb{D}), 1 < p < +\infty$, orthogonal to the set $\overline{\mathcal{O}}\mathbb{H}(\mathbb{D})$ is representable through

$$w(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} G_{3z\overline{z}\overline{z}}(z,\zeta) w_{\zeta\zeta\overline{\zeta}}(\zeta) d\xi d\eta \ . \quad (40)$$

Here

$$\begin{aligned} G_{3z\overline{z}\overline{z}}(z,\zeta) &= -(\zeta - z)\log \mid \frac{1-z\overline{\zeta}}{\zeta - z} \mid^2 + \frac{1}{2}(\zeta - z)(1- \mid \zeta \mid^2) \\ &\quad \times \Big[\frac{1}{1-z\overline{\zeta}} + \frac{2}{1-\overline{z}\zeta} - 1 + \frac{1- \mid \zeta \mid^2}{(1-\overline{z}\zeta)^2}\Big] \\ &\quad +\frac{1}{2} z(1- \mid \zeta \mid^2) + \frac{1}{2} z(1- \mid \zeta \mid^2)^2 \ . \end{aligned}$$

Then

$$w_{zzz}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} G_{3zzzz\overline{z}\overline{z}}(z,\zeta) w_{\zeta\zeta\overline{\zeta}}(\zeta) d\xi d\eta \ , \quad (41)$$

$$w_{\overline{z}\overline{z}\overline{z}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} G_{3z\overline{z}\overline{z}\overline{z}\overline{z}\overline{z}}(z,\zeta) w_{\zeta\zeta\overline{\zeta}}(\zeta) d\xi d\eta \ , \quad (42)$$

$$w_{z\overline{z}\overline{z}}(z) = -\frac{1}{\pi} \int\limits_{|\zeta|<1} G_{3zz\overline{z}\overline{z}\overline{z}\overline{z}}(z,\zeta) w_{\zeta\zeta\overline{\zeta}}(\zeta) d\xi d\eta \ , \tag{43}$$

where

$$\begin{aligned}
G_{3zzzz\overline{z}\overline{z}}(z,\zeta) &= \Big[4\Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^3 - 3\Big(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\Big)^4\Big]\frac{1}{(\zeta-z)^2},\\
G_{3zz\overline{z}\overline{z}\overline{z}\overline{z}}(z,\zeta) &= \Big[4\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^3 - 3\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^4\Big]\frac{1}{\overline{(\zeta-z)}^2},\\
G_{3z\overline{z}\overline{z}\overline{z}\overline{z}\overline{z}}(z,\zeta) &= \Big[-20\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^3 + 33\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^4\\
&\qquad -12\Big(\frac{1-|\zeta|^2}{1-\overline{z}\zeta}\Big)^5\Big]\frac{\zeta-z}{\overline{(\zeta-z)}^3}.
\end{aligned}$$

3. Orthogonalized potentials in polydiscs

Let $\mathbb{D}^2 = \{(z_1,z_2) \in \mathbb{C}^2 :| z_1 |< 1, | z_2 |< 1\}$ be the unit bidisc in $\mathbb{C}^2$ with the distinguished boundary $\partial_0\mathbb{D}^2 = \{(z_1,z_2) :| z_1 |= 1, | z_2 |= 1\}$. Iterating the modified Cauchy-Pompeiu formula (11) leads to

$$\begin{aligned}
w(z_1,z_2) &= \frac{1}{\pi}\int\limits_{|\zeta_1|<1} \frac{w(\zeta_1,z_1)}{(1-z_1\zeta_1)^2}\, d\xi_1 d\eta_1 + \frac{1}{\pi}\int\limits_{|\zeta_2|<1} \frac{w(z_1,\zeta_2)}{(1-z_2\overline{\zeta}_2)^2}\, d\xi_2 d\eta_2\\
&\quad -\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} \frac{w(\zeta_1,\zeta_2)}{(1-z_1\overline{\zeta}_1)^2(1-z_2\overline{\zeta}_2)^2}\, d\xi_1 d\eta_1 d\xi_2 d\eta_2\\
&\quad +\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} \Big(\frac{1}{\zeta_1-z_1} - \frac{\overline{\zeta}_1}{1-z_1\overline{\zeta}_1}\Big)\\
&\quad \times\Big(\frac{1}{\zeta_2-z_2} - \frac{\overline{\zeta}_2}{1-z_2\overline{\zeta}_2}\Big) w_{\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1,\zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2
\end{aligned}$$

for any $w \in W^{(1,1),p}(\mathbb{D}^2;\mathbb{C}), 1 < p < +\infty$. Here the first three terms are either analytic in z_1 or in z_2 i.e. belong to $\mathcal{O}(\mathbb{D}_1) \cup \mathcal{O}(\mathbb{D}_2)$ where $\mathbb{D}_k = \{z_k \in \mathbb{C} :| z_k |< 1\}, 1 \le k \le 2$. The last term is orthogonal to all solutions to $\phi_{\overline{z}_1\overline{z}_2} = 0$. This follows from

$$\int\limits_{|z_1|<1}\int\limits_{|z_2|<1} \partial_{z_1} \log \Big| \frac{1-z_1\overline{\zeta}_1}{\zeta_1-z_1} \Big|^2 \partial_{z_2} \log \Big| \frac{1-z_2\overline{\zeta}_2}{\zeta_2-z_2} \Big|^2 \overline{\phi(z_1,z_2)} d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

$$= \int\limits_{|z_1|<1}\int\limits_{|z_2|<1} \log \Big| \frac{1-z_1\overline{\zeta}_1}{\zeta_1-z_1} \Big|^2 \log \Big| \frac{1-z_2\overline{\zeta}_2}{\zeta_2-z_2} \Big|^2 \overline{\phi_{\overline{z},\overline{z}_2}(z,z_2)} d\xi_1 d\eta_1 d\xi_2 d\eta_2.$$

Thus if w is orthogonal to the space $\mathcal{O}_{1,1}(\mathbb{D}^2) = \{\phi \in W^{(1,1),p}(\mathbb{D}^2;\mathbb{C}) : \phi_{\overline{z}_1\overline{z}_2} = 0\}$ then

$$w(z_1,z_2) = \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1}\Big(\frac{1}{\zeta_1 - z_1} - \frac{\overline{\zeta}_1}{1 - z_1\overline{\zeta}_1}\Big)$$

$$\times\Big(\frac{1}{\zeta_2 - z_2} - \frac{\overline{\zeta}_2}{1 - z_2\overline{\zeta}_2}\Big) w_{\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

$$= \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} \frac{1-|\zeta_1|^2}{1 - z_1\overline{\zeta}_1}\frac{1}{\zeta_1 - z_1}\frac{1-|\zeta_2|^2}{1 - z_2\overline{\zeta}_2}\frac{1}{\zeta_2 - z_2}$$

$$\times w_{\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\ . \tag{44}$$

Similarly any function $w \in W^{(1,\dots,1),p}(\mathbb{D}^n;\mathbb{C})$ orthogonal to

$$\mathcal{O}_{1,\dots,1}(\mathbb{D}^n) = \{\phi \in W^{(1,\dots,1),p}(\mathbb{D}^n;\mathbb{C}) : \phi_{\overline{z}_1\cdots\overline{z}_n} = 0\}$$

can be represented as

$$w(z) = \frac{(-1)^n}{\pi^n}\int\limits_{\mathbb{D}^n} w_{\overline{\zeta}_1\cdots\overline{\zeta}_n}(\zeta)\prod_{\nu=1}^{n}\frac{1-|\zeta_\nu|^2}{1 - z_\nu\overline{\zeta}_\nu}\frac{d\xi_\nu d\eta_\nu}{\zeta_\nu - z_\nu}\ . \tag{45}$$

If $w \in W^{(1,\dots,1),p}(\mathbb{D}^n;\mathbb{C})$ is orthogonal to

$$\overline{\mathcal{O}}_{1,\dots,1}(\mathbb{D}^n) = \{\phi \in W^{(1,\dots,1),p}(\mathbb{D}^n;\mathbb{C}) : \phi_{z_1\cdots z_n} = 0\}$$

then similarly

$$w(z) = \frac{(-1)^n}{\pi^n}\int\limits_{\mathbb{D}^n} w_{\zeta_1\cdots\zeta_n}(\zeta)\prod_{\nu=1}^{n}\frac{1-|\zeta_\nu|^2}{1 - \overline{z}_\nu\zeta_\nu}\frac{\overline{d\xi_\nu d\eta_\nu}}{\overline{\zeta_\nu - z_\nu}}. \tag{46}$$

More generally any $w \in W^{(1,\dots,1),p}(\mathbb{D}^n;\mathbb{C})$ orthogonal to

$$\mathcal{O}^{\varrho_1,\dots,\varrho_\nu}_{1,\dots,1}\overline{\mathcal{O}}^{\varrho_{\nu+1},\dots,\varrho_n}_{1,\dots,1}(\mathbb{D}^n)$$

$$= \{\phi \in W^{(1,\dots,1),p}(\mathbb{D}^n;\mathbb{C}) : \phi_{\overline{z}_{\varrho_1}\cdots\overline{z}_{\varrho_\nu}z_{\varrho_{\nu+1}}\cdots z_{\varrho_n}} = 0\},$$

where $\{\varrho_1,\dots,\varrho_n\} = \{1,\dots,n\}$ and $1 \le \nu \le n$ is representable as

$$w(z) = \frac{(-1)^n}{\pi^n}\int\limits_{\mathbb{D}^n} w_{\overline{\zeta}_{\varrho_1}\cdots\overline{\zeta}_{\varrho_\nu}\zeta_{\varrho_{\nu+1}}\cdots\zeta_{\varrho_n}}(\zeta)\prod_{\tau=1}^{\nu}\frac{1-|\zeta_{\varrho_\tau}|^2}{1 - z_{\varrho_\tau}\overline{\zeta}_{\varrho_\tau}}\frac{d\xi_{\varrho_\tau}d\eta_{\varrho_\tau}}{\zeta_{\varrho_\tau} - z_{\varrho_\tau}} \tag{47}$$

$$\times \prod_{\tau=\nu+1}^{n} \frac{1-|\zeta_{\varrho_\tau}|^2}{1-\overline{z}_{\varrho_\tau}\zeta_{\varrho_\tau}} \frac{d\xi_{\varrho_\tau} d\eta_{\varrho_\tau}}{\overline{\zeta_{\varrho_\tau} - z_{\varrho_\tau}}} .$$

As mainly the case $n = 2$ will be considered (47) is repeated for this case

$$w(z_1, z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta_1}} \frac{1}{\zeta_1 - z_1} \frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2} \frac{1}{\overline{\zeta_2 - z_2}} \tag{48}$$

$$\times w_{\overline{\zeta}_1\zeta_2}(\zeta_1, \zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2.$$

If $w \in W^{(1,1),p}(\mathbb{D}^2;\mathbb{C})$ is orthogonal to the spaces $\mathcal{O}(\mathbb{D}_1), \mathcal{O}(\mathbb{D}_2)$ and $\mathcal{O}_{1,1}(\mathbb{D}^2)$ and $w_{\overline{z}_k}$ is orthogonal to $\mathcal{O}(\mathbb{D}_k), 1 \le k \le 2$, then prescribing e.g. the second order derivatives $w_{\overline{z}_1\overline{z}_1}, w_{\overline{z}_1\overline{z}_2}, w_{\overline{z}_2\overline{z}_2}$ all other second order derivatives can be expressed through these three. Using (26) for $\mathbb{D}_1$ and $\mathbb{D}_2$ expresses $w_{z_kz_k}$ and $w_{z_k\overline{z}_k}$ through $w_{\overline{z}_k\overline{z}_k}$. From (44)

$$w_{z_1z_2}(z_1, z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \Big[2\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta_1}} - \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta_1}}\Big)^2\Big] \frac{1}{(\zeta_1 - z_1)^2} \tag{49}$$

$$\times \Big[2\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta_2}} - \Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta_2}}\Big)^2\Big] \frac{1}{(\zeta_2 - z_2)^2} \; w_{\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1, \zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

and also

$$w_{\overline{z}_1z_2}(z_1, z_2) \tag{50}$$

$$= -\frac{1}{\pi} \int\limits_{|\zeta_2|<1} \Big[2\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta_2}} - \Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta_2}}\Big)^2\Big] \frac{1}{(\zeta_2 - z_2)^2} \; w_{\overline{z}_1\overline{\zeta}_2}(z_1, \zeta_2) d\xi_2 d\eta_2,$$

$$w_{z_1\overline{z}_2}(z_1, z_2) \tag{51}$$

$$= -\frac{1}{\pi} \int\limits_{|\zeta_1|<1} \Big[2\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta_1}} - \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta_1}}\Big)^2\Big] \frac{1}{(\zeta_1 - z_1)^2} \; w_{\overline{\zeta}_1\overline{z}_2}(\zeta_1, z_2) d\xi_1 d\eta_1$$

follows.
Assuming $w \in W^{(1,1),p}(\mathbb{D}^2;\mathbb{C})$ to be orthogonal to $\mathcal{O}_1\overline{\mathcal{O}}_1(\mathbb{D}^2)$ from (48)

$$w_{z_1\overline{z}_2}(z_1, z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \Big[2\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta_1}} - \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta_1}}\Big)^2\Big] \frac{1}{(\zeta_1 - z_1)^2}$$

$$\times\Big[2\frac{1-\mid\zeta_2\mid^2}{1-\overline{z}_2\zeta_2}-\Big(\frac{1-\mid\zeta_2\mid^2}{1-\overline{z}_2\zeta_2}\Big)^2\frac{1}{\overline{(\zeta_2-z_2)}^2}\ w_{\overline{\zeta}_1\zeta_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2,$$

$$w_{z_1z_2}(z_1,z_2)=-\frac{1}{\pi}\int\limits_{|\zeta_1|<1}\Big[2\frac{1-\mid\zeta_1\mid^2}{1-z_1\overline{\zeta}_1}-\Big(\frac{1-\mid\zeta_1\mid^2}{1-z_1\overline{\zeta}_1}\Big)^2\Big]w_{\overline{\zeta}_1z_2}(\zeta_1,z_2)d\xi_1 d\eta_1,$$

$$w_{\overline{z}_1\overline{z}_2}(z_1,z_2)=-\frac{1}{\pi}\int\limits_{|\zeta_2|<1}\Big[2\frac{1-\mid\zeta_2\mid^2}{1-\overline{z}_2\zeta_2}-\Big(\frac{1-\mid\zeta_2\mid^2}{1-\overline{z}_2\zeta_2}\Big)^2\Big]w_{\overline{z}_1\zeta_2}(z_1,\zeta_2)d\xi_2 d\eta_2.$$

Under proper assumptions formulas (19), (20) can be used to express $w_{z_kz_k}$ and $w_{\overline{z}_k\overline{z}_k}$ through $w_{z_k\overline{z}_k}$ for $1\le k\le 2$.

Iterating the representation (17) for a $w\in W^{(2,2),p}(\mathbb{D}^2;\mathbb{C})$ leads to

$$\begin{aligned}
w(z_1,z_2) \;=\; & \frac{1}{\pi}\int\limits_{|\zeta_1|<1} g_2(z_1,\zeta_1)w(\zeta_1,z_2)d\xi_1 d\eta_1\\
&+\frac{1}{\pi}\int\limits_{|\zeta_2|<1} g_2(z_2,\zeta_2)w(z_1,\zeta_2)d\xi_2 d\eta_2\\
&-\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} g_2(z_1,\zeta_1)g_2(z_2,\zeta_2)w(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\\
&+\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1}\Big[\log\mid\frac{1-z_1\overline{\zeta}_1}{\zeta_1-z_1}\mid^2-g_1(z_1,\zeta_1)(1-\mid\zeta_1\mid^2)\Big]\\
&\times\Big[\log\mid\frac{1-z_2\overline{\zeta}_2}{\zeta_2-z_1}\mid^2-g_1(z_2,\zeta_2)(1-\mid\zeta_2\mid^2)\Big]\\
&\times w_{\zeta_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\ .
\end{aligned}\tag{52}$$

The first three terms form a function satisfying $\partial_{z_1}\partial_{\overline{z}_1}\partial_{z_2}\partial_{\overline{z}_2}\omega=0$ while the last term is orthogonal to the kernel of the operator $\partial_{z_1}\partial_{\overline{z}_1}\partial_{z_2}\partial_{\overline{z}_2}$, denoted by $\mathbb{H}_1\mathbb{H}_1(\mathbb{D}^2)$. Hence any $w\in W^{(2,2),p}(\mathbb{D}^2;\mathbb{C})$ orthogonal to $\mathbb{H}_1\mathbb{H}_1(\mathbb{D}^2)$ may be written as

$$\begin{aligned}
w(z_1,z_2)=\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1}&\Big[\log\mid\frac{1-z_1\overline{\zeta}_1}{\zeta_1-z_1}\mid^2-g_1(z_1,\zeta_1)(1-\mid\zeta_1\mid^2)\Big]\\
&\times\Big[\log\frac{1-z_2\overline{\zeta}_2}{\zeta_2-z_2}\mid^2-g_1(z_2,\zeta_2)(1-\mid\zeta_2\mid^2)\Big]\\
&\times w_{\zeta_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\ .
\end{aligned}\tag{53}$$

From here

$$w_{z_1z_1z_2z_2}(z_1,z_2)=\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1}\Big[3\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2-2\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^3\Big]$$

$$\times\frac{1}{(\zeta_1-z_1)^2}\Big[3\Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^2-2\Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^3\Big]\frac{1}{(\zeta_2-z_2)^2}$$

$$\times w_{\zeta_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1d\eta_1d\xi_2d\eta_2\ , \tag{54}$$

$$w_{z_1z_1\overline{z}_2\overline{z}_2}(z_1,z_2)=\frac{1}{\pi}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1}\Big[3\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2-2\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^3\Big]$$

$$\times\frac{1}{(\zeta_1-z_1)^2}\Big[3\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^2-2\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^3\Big]\frac{1}{\overline{(\zeta_2-z_2)}^2}$$

$$\times w_{\zeta_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1d\eta_1d\xi_2d\eta_2\ , \tag{55}$$

$$w_{\overline{z}_1\overline{z}_1\overline{z}_2\overline{z}_2}(z_1,z_2)=\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1}\Big[3\Big(\frac{1-|\zeta_1|^2}{1-\overline{z}_1\zeta_1}\Big)^2-2\Big(\frac{1-|\zeta_1|^2}{1-\overline{z}_1\zeta_1}\Big)^3\Big]$$

$$\times\frac{1}{\overline{(\zeta_1-z_1)}^2}\Big[3\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^2-2\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^3\Big]\frac{1}{\overline{(\zeta_2-z_2)}^2}$$

$$\times\ w_{\zeta_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1d\eta_1d\xi_2d\eta_2\ , \tag{56}$$

$$w_{z_1\overline{z}_1z_2z_2}(z_1,z_2)=-\frac{1}{\pi}\int\limits_{|\zeta_2|<1}\Big[3\Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^2-2\Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^3\Big] \tag{57}$$

$$\times\frac{1}{(\zeta_2-z_2)^2}\ w_{z_1\overline{z}_1\zeta_2\overline{\zeta}_2}(z_1,\zeta_2)d\xi_2d\eta_2,$$

$$w_{z_1\overline{z}_1\overline{z}_2\overline{z}_2}(z_1,z_2)=-\frac{1}{\pi}\int\limits_{|\zeta_2|<1}\Big[3\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^2-2\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^3\Big] \tag{58}$$

$$\times\frac{1}{\overline{(\zeta_2-z_2)}^2}\ w_{z_1\overline{z}_1\zeta_2\overline{\zeta}_2}(z,\zeta_2)d\xi_2d\eta_2$$

can be seen.

Let now $w \in W^{(2,2),p}(\mathbb{D}^2;\mathbb{C})$. Iterating (23) for $n = 2$ and (17) gives

$$\begin{aligned}
&w(z_1,z_2)\\
&= \frac{1}{\pi}\int\limits_{|\zeta_1|<1} 2\frac{2(1-|\zeta_1|^2)(1-|z_1|^2)-|\zeta_1-z_1|^2}{(1-z_1\overline{\zeta}_1)^4}\, w(\zeta_1,z_2)d\xi_1 d\eta_1\\
&+\frac{1}{\pi}\int\limits_{|\zeta_2|<1} g_2(z_2,\zeta_2)w(z_1,\zeta_2)d\xi_2 d\eta_2\\
&-\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} 2\frac{2(1-|\zeta_1|^2)(1-|z_1|^2)-|\zeta_1-z_1|^2}{(1-z_1\overline{\zeta}_1)^4}\\
&\times g_2(z_2,\zeta_2)w(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\\
&+\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2 \frac{\overline{z_1-\zeta_1}}{\zeta_1-z_1}\Big[\log\Big|\frac{1-z_2\overline{\zeta}_2}{\zeta_2-z_2}\Big|^2\\
&-g_1(z_2,\zeta_2)(1-|\zeta_2|^2)\Big]w_{\overline{\zeta}_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\ .
\end{aligned}$$

Up to the last term the right-hand side is a function from $\mathcal{O}_2\mathbb{H}_1(\mathbb{D}^2)$ i.e. satisfies $\partial_{z_1}^2\partial_{z_2}\partial_{\overline{z}_2}w = 0$. Again it can be shown that any $w \in W^{(2,2),p}(\mathbb{D}^2;\mathbb{C})$ being orthogonal to $\mathcal{O}_2\mathbb{H}_1(\mathbb{D}^2)$ is given by

$$\begin{aligned}
w(z_1,z_2) = \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} &\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2 \frac{\overline{z_1-\zeta_1}}{\zeta_1-z_1}\Big[\log\Big|\frac{1-z_2\overline{\zeta}_2}{\zeta_2-z_2}\Big|^2 \qquad (59)\\
&-g_1(z_2,\zeta_2)(1-|\zeta_2|^2)\Big]w_{\overline{\zeta}_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\ .
\end{aligned}$$

From here

$$\begin{aligned}
w_{z_1z_1z_2z_2}(z_1,z_2) = \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} &\Big[12\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2 - 16\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^3 \qquad (60)\\
+6\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^4\Big]\frac{\overline{z_1-\zeta_1}}{(\zeta_1-z_1)^3}&\Big[3\Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^2-2\Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^3\Big]\frac{1}{(\zeta_2-z_2)^2}\\
&\times w_{\overline{\zeta}_1\overline{\zeta}_1\zeta_2\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2,
\end{aligned}$$

$$w_{z_1 z_1 \overline{z}_2 \overline{z}_2}(z_1, z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \Big[12\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2 - 16\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^3 \tag{61}$$

$$+6\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^4\Big] \frac{\overline{z_1-\zeta_1}}{(\zeta_1-z_1)^3} \Big[3\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^2 - 2\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^3\Big] \frac{1}{\overline{(\zeta_2-z_2)}^2}$$

$$\times\, w_{\overline{\zeta}_1 \overline{\zeta}_1 \zeta_2 \overline{\zeta}_2}(\zeta_1, \zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2,$$

$$w_{z_1 \overline{z}_1 z_2 z_2}(z_1, z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \Big[3\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2 - 2\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^3\Big] \tag{62}$$

$$\times \frac{1}{(\zeta_1-z_1)^2} \Big[3\Big(\frac{1-|\zeta_2|^2}{(1-z_2\overline{\zeta}_2}\Big)^2 - 2\Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^3\Big] \frac{1}{(\zeta_2-z_2)^2}$$

$$\times w_{\overline{\zeta}_1 \overline{\zeta}_1 \zeta_2 \overline{\zeta}_2}(\zeta_1, \zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2,$$

$$w_{z_1 \overline{z}_1 \overline{z}_2 \overline{z}_2}(z_1, z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \Big[3\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^2 - 2\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^3\Big] \tag{63}$$

$$\times \frac{1}{(\zeta_1-z_1)^2} \Big[3\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^2 - 2\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^3\Big] \frac{1}{\overline{(\zeta_2-z_2)}^2}$$

$$\times w_{\overline{\zeta}_1 \overline{\zeta}_1 \zeta_2 \overline{\zeta}_2}(\zeta_1, \zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

follow.

Let now $w \in W^{(m,n),p}(\mathbb{D}^2; \mathbb{C})$ for $m, n \in \mathbb{N}, 1 < p < +\infty$. Iterating (23) gives

$$\begin{aligned} &w(z_1, z_2) \\ &= \frac{1}{\pi} \int\limits_{|\zeta_1|<1} F_m(z_1, \zeta_1) w(\zeta_1, z_2) d\xi_1 d\eta_1 + \frac{1}{\pi} \int\limits_{|\zeta_2|<1} F_n(z_2, \zeta_2) w(z_1, \zeta_2) d\xi_2 d\eta_2 \\ &- \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} F_m(z_1, \zeta_1) F_n(z_2, \zeta_2) w(\zeta_1, \zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2 \\ &+ \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \frac{1}{(m-1)!} \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^m \frac{\overline{(z_1-\zeta_1)}^{m-1}}{\zeta_1-z_1} \\ &\times \frac{1}{(n-1)!} \Big(\frac{1-|\zeta_2|}{1-z_2\overline{\zeta}_2}\Big)^n \frac{\overline{(z_2-\zeta_2)}}{\zeta_2-z_2}\; \partial^m_{\overline{\zeta}_1} \partial^n_{\overline{\zeta}_2} w(\zeta_1, \zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2 \ . \end{aligned}$$

The first three terms compose a function which is a solution to $\partial_{\overline{z}_1}^m \partial_{\overline{z}_2}^n w = 0$. But the last term turns out to be orthogonal to the set of these functions. It will be denoted by $\mathcal{O}_m\mathcal{O}_n(\mathbb{D}^2)$. Any $w \in W^{(m,n),p}(\mathbb{D}^2;\mathbb{C})$ orthogonal to $\mathcal{O}_m\mathcal{O}_n(\mathbb{D}^2)$ is representable as

$$w(z_1,z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \frac{1}{(m-1)!} \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^m \frac{(\overline{z_1-\zeta_1})^{m-1}}{\zeta_1 - z_2} \tag{64}$$

$$\times \frac{1}{(n-1)!} \Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^n \frac{(\overline{z_2-\zeta_2})^{n-1}}{\zeta_2 - z_2} \; \partial_{\overline{\zeta}_1}^m \partial_{\overline{\zeta}_2}^n w(\zeta_1,\zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2.$$

By differentiation

$$\partial_{z_1}^\mu \partial_{\overline{z}_1}^{m-\mu} \partial_{z_2}^\nu \partial_{\overline{z}_2}^{n-\nu} w(z_1,z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \frac{1}{(\mu-1)!} \sum_{\kappa=0}^{\mu} (-1)^\kappa \binom{\mu}{\kappa}$$

$$\times \prod_{\tau=0,\tau\neq\kappa}^{\mu} (m+\tau) \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^{m+\kappa} \frac{(z_1-\zeta_1)^{\mu-1}}{(\zeta_1-z_1)^{\mu+1}} \frac{1}{(\nu-1)!} \sum_{\lambda=0}^{\nu} (-1)^\lambda \binom{\nu}{\lambda} \tag{65}$$

$$\times \prod_{\tau=0,\tau\neq\lambda}^{\nu} (n+\tau) \Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^{n+\lambda} \frac{(\overline{z_2-\zeta_2})^{\nu-1}}{(\zeta_2-z_2)^{\nu+2}} \; \partial_{\overline{\zeta}_1}^m \partial_{\overline{\zeta}_2}^n w(\zeta_1,\zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

follows for $1 \le \mu \le m, 1 \le \nu \le n$. As the variables z_1, z_2 in the polydomain $\mathbb{D}^2$ are independent from one another derivatives with respect to one variable cannot be expressed by those with respect to the other. E.g. for $\mu = 0$ (65) is not valid. Instead for $1 \le \nu \le n$

$$\partial_{\overline{z}_1}^m \partial_{z_2}^\nu \partial_{\overline{z}_2}^{n-\nu} w(z_1,z_2) = -\frac{1}{\pi} \int\limits_{|\zeta_2|<1} \frac{1}{(\nu-1)!} \sum_{\lambda=0}^{\nu} (-1)^\lambda \binom{\nu}{\lambda} \prod_{\tau=0,\tau\neq\lambda}^{\nu} (n+\tau)$$

$$\times \Big(\frac{1-|\zeta_2|^2}{1-z_2\overline{\zeta}_2}\Big)^{n+\lambda} \frac{(\overline{z_2-\zeta_2})^{\nu-1}}{(\zeta_2-z_2)^{\nu+1}} \partial_{\overline{z}_1}^m \partial_{\overline{\zeta}_2}^n w(z_1,\zeta_2) d\xi_2 d\eta_2.$$

In a similar way $w \in W^{(m,n),p}(\mathbb{D}^2;\mathbb{C})$ orthogonal to $\mathcal{O}_m\overline{\mathcal{O}}_n(\mathbb{D}^2)$ can be expressed as

$$w(z_1,z_2) = \frac{1}{\pi^2} \int\limits_{|\zeta_1|<1} \int\limits_{|\zeta_2|<1} \frac{1}{(m-1)!} \Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^m \frac{(\overline{z_1-\zeta_1})^{m-1}}{\zeta_1 - z_1} \tag{66}$$

$$\times \frac{1}{(n-1)!} \Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^n \frac{(z_2-\zeta_2)^{n-1}}{\overline{\zeta_2 - z_2}} \partial_{\overline{\zeta}_1}^m \partial_{\zeta_2}^n w(\zeta_1,\zeta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

for which

$$\partial_{z_1}^{\mu}\partial_{\overline{z}_1}^{m-\mu}\partial_{z_2}^{\nu}\partial_{\overline{z}_2}^{n-\nu}w(z_1,z_2)=\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1}\frac{1}{(\mu-1)!}\sum_{\kappa=0}^{\mu}(-1)^{\kappa}\binom{\mu}{\kappa}$$

$$\times\prod_{\tau=0,\tau\neq\kappa}^{\mu}(m+\tau)\Big(\frac{1-|\zeta_1|^2}{1-z_1\overline{\zeta}_1}\Big)^{m+\kappa}\frac{(\overline{z_1-\zeta_1})^{\mu-1}}{(\zeta_1-z_1)^{\mu+1}}$$

$$\times\frac{1}{(n-\nu-1)!}\sum_{\lambda=0}^{n-\nu}(-1)^{\lambda}\binom{n-\nu}{\lambda}\prod_{\tau=0,\tau\neq\lambda}^{n-\nu}(n+\tau)\Big(\frac{1-|\zeta_2|^2}{1-\overline{z}_2\zeta_2}\Big)^{n+\lambda}$$

$$\times\frac{(z_2-\zeta_2)^{n-\nu-1}}{(\overline{\zeta_2-z_2})^{n-\nu+1}}\ \partial_{\overline{\zeta}_1}^{m}\partial_{\zeta_2}^{n}w(\zeta_1,\zeta_2)d\xi_1d\eta_1d\xi_2d\eta_2$$

holds for $1\le\mu\le m$ and $1\le\nu\le n$.

For another kind of possible combination of differential operators consider now some $w\in W^{(3,1),p}(\mathbb{D}^2;\mathbb{C}), 1<p<+\infty$. Then rewriting (39) shortly as

$$\begin{aligned} w(z_1,z_2) &= \frac{1}{\pi}\int\limits_{|\zeta_1|<1}K_1(z_1,\zeta_1)w(\zeta_1,z_2)d\xi_1d\eta_1 \\ &\quad -\frac{1}{\pi}\int\limits_{|\zeta_1|<1}G_{3z_1\overline{z}_1\overline{z}_1}(z_1,\zeta_1)w_{\zeta_1\zeta_1\overline{\zeta}_1}(\zeta_1,z_2)d\xi_1d\eta_1\end{aligned}$$

with

$$\begin{aligned} K_1(z_1,\zeta_1) &= \frac{1}{(1-z_1\overline{\zeta}_1)^2}-1-2z_1\overline{\zeta}_1 \\ &\quad +2\frac{2(1-|z_1|^2)(1-|\zeta_1|^2)-|\zeta_1-z_1|^2}{(1-\overline{z}_1\zeta_1)^4}\end{aligned}$$

and combining it with

$$w(z_1,z_2)=\frac{1}{\pi}\int\limits_{\mathbb{D}_2}\frac{w(z_1,\zeta_2)}{(1-z_2\overline{\zeta}_2)^2}\,d\xi_2d\eta_2-\frac{1}{\pi}\int\limits_{\mathbb{D}_2}G_{1z_2}(z_2,\zeta_2)w_{\overline{\zeta}_2}(z_1,\zeta_2)d\xi_2d\eta_2,$$

see (11), where

$$G_1(z_2,\zeta_2)=\log\Big|\frac{1-z_2\overline{\zeta}_2}{\zeta_2-z_2}\Big|^2$$

and $G_3(z_1,\zeta_1)$ as given in connection with (39), shows

$$
\begin{aligned}
&w(z_1,z_2)\\
&= \frac{1}{\pi}\int\limits_{|\zeta_1|<1} K_1(z_1,\zeta_1)w(\zeta_1,z_2)d\xi_1 d\eta_1 + \frac{1}{\pi}\int\limits_{|\zeta_2|<1} \frac{w(z_1,\zeta_2)}{(1-z_2\overline{\zeta}_2)^2}\, d\xi_2 d\eta_2\\
&-\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} \frac{K_1(z_1,\zeta_1)}{(1-z_2\overline{\zeta}_2)^2}\, w(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\\
&+\frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} G_{3z_1\overline{z}_1\overline{z}_1}(z_1,\zeta_1)G_{1z_2}(z_2,\zeta_2)w_{\zeta_1\zeta_1\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2 .
\end{aligned}
$$

If now w is orthogonal to the space of functions ϕ satisfying $\phi_{z_1z_1\overline{z}_1\overline{z}_2} = 0$ then

$$
\begin{aligned}
&w(z_1,z_2)\\
&= \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} G_{3z_1\overline{z}_1\overline{z}_1}(z_1,\zeta_1)G_{1z_2}(z_2,\zeta_2)w_{\zeta_1\zeta_1\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2 .
\end{aligned}
$$

By differentiation

$$
\begin{aligned}
w_{z_1z_1z_1z_2}(z_1,z_2) = \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} & G_{3z_1z_1z_1z_1\overline{z}_1\overline{z}_1}(z_1,\zeta_1)G_{1z_2z_2}(z_2,\zeta_2)\\
&\times w_{\zeta_1\zeta_1\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2, \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
w_{\overline{z}_1\overline{z}_1\overline{z}_1z_2}(z_1,z_2) = \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} & G_{3z_1\overline{z}_1\overline{z}_1\overline{z}_1\overline{z}_1\overline{z}_1}(z_1,\zeta_1)G_{1z_2z_2}(z_2,\zeta_2)\\
&\times w_{\zeta_1\zeta_1\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2\ , \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
w_{z_1\overline{z}_1\overline{z}_1z_2}(z_1,z_2) = \frac{1}{\pi^2}\int\limits_{|\zeta_1|<1}\int\limits_{|\zeta_2|<1} & G_{3z_1z_1\overline{z}_1\overline{z}_1\overline{z}_1\overline{z}_1}(z_1,\zeta_1)G_{1z_2z_2}(z_2,\zeta_2)\\
&\times w_{\zeta_1\zeta_1\overline{\zeta}_1\overline{\zeta}_2}(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2 \qquad (69)
\end{aligned}
$$

follow.

4. Orthogonalized potentials in the whole space $\mathbb{C}^n$

An entire function in $\mathbb{C}^n$ vanishing at infinity is identically zero. Hence any function $w \in W^{1,p}(\mathbb{C}^n;\mathbb{C}), 1 < p < +\infty$, can be represented as

$$w(z) = \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{k=1}^{n} \frac{\overline{\zeta_k - z_k}}{\mid \zeta - z \mid^{2n}} \, w_{\overline{\zeta}_k}(\zeta) d\zeta \wedge d\overline{\zeta} \; . \tag{70}$$

This follows from the Bochner-Martinelli formula, see e.g. [3], (2.87). By differentiating

$$w_{z_k}(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{\ell=1}^{n} \frac{(\overline{\zeta_\ell - z_\ell})(\overline{\zeta_k - z_k})}{\mid \zeta - z \mid^{2(n+1)}} \, w_{\overline{\zeta}_\ell}(\zeta) d\zeta \wedge d\overline{\zeta} \tag{71}$$

follows for $1 \leq k \leq n$. The singular integral operator

$$(\Sigma_k \omega)(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}} \sum_{\ell=1}^{n} \frac{(\overline{\zeta_\ell - z_\ell})(\overline{\zeta_k - z_k})}{\mid \zeta - z \mid^{2(n+1)}} \, \omega_\ell(\zeta) d\zeta \wedge d\overline{\zeta}$$

converts all first order complex $\overline{z}-$derivatives of $w(z)$ into the first order complex $z-$derivatives. Using instead of (70) the representation

$$w(z) = \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}} \sum_{k=1}^{n} \frac{(\zeta_k - z_k)}{\mid \zeta - z \mid^{2n}} \, w_{\zeta_k}(\zeta) d\zeta \wedge d\overline{\zeta} \tag{70$'$}$$

and differentiating with respect to $\overline{z}_k$ gives

$$w_{\overline{z}_k}(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{\ell=1}^{n} \frac{(\zeta_\ell - z_\ell)(\zeta_k - z_k)}{\mid \zeta - z \mid^{2(n+1)}} w_{\zeta_\ell}(\zeta) d\zeta \wedge d\overline{\zeta}, 1 \leq k \leq n \; . \tag{72}$$

Thus the singular integral operator

$$(\overline{\Sigma}_k \omega)(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{\ell=1}^{n} \frac{(\zeta_\ell - z_\ell)(\zeta_k - z_k)}{\mid \zeta - z \mid^{2(n+1)}} \, \omega_\ell(\zeta) d\zeta \wedge d\overline{\zeta} \tag{73}$$

converts all first order complex $z-$derivatives of $w(z)$ into the first order complex $\overline{z}-$derivatives. From the equality

$$\sum_{k=1}^{n} \int\limits_{\mathbb{C}^n} \mid \frac{\partial \varphi}{\partial z_k} \mid^2 d\zeta \wedge d\overline{\zeta} = \sum_{k=1}^{n} \int\limits_{\mathbb{C}^n} \frac{\partial \varphi}{\partial z_k} \frac{\partial \overline{\varphi}}{\partial \overline{z}_k} \, d\zeta \wedge d\overline{\zeta}$$

$$= \sum_{k=1}^{n} \int_{\mathbb{C}^n} \frac{\partial \varphi}{\partial \overline{z}_k} \frac{\partial \overline{\varphi}}{\partial z_k} \, d\zeta \wedge d\overline{\zeta} = \sum_{k=1}^{n} \int_{\mathbb{C}^n} | \frac{\partial \varphi}{\partial \overline{z}_k} |^2 \, d\zeta \wedge d\overline{\zeta}$$

with compactly supported $C^{\infty}-$functions obtained by integration by parts and the density of such functions in $L_2(\mathbb{C}^n;\mathbb{C})$ it follows that the L_2-norms of the operators $(\Sigma_1, \Sigma_2, \dots, \Sigma_n)$ and $(\overline{\Sigma}_1, \overline{\Sigma}_2, \dots, \overline{\Sigma}_n)$ are equal to one.

If $w(z) \in W^{2,p}(\mathbb{C}^n;\mathbb{C}), 1 < p < +\infty$, then see [3],

$$w(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} \sum_{k,\ell=1}^{n} \frac{(\overline{\zeta_k - z_k})(\overline{\zeta_\ell - z_\ell})}{| \zeta - z |^{2n}} \, w_{\overline{\zeta}_k \overline{\zeta}_\ell}(\zeta) d\overline{\zeta} \wedge d\zeta \ , \tag{74}$$

$$w(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}} \sum_{k,\ell=1}^{n} \frac{(\zeta_k - z_k)(\zeta_\ell - z_\ell)}{| \zeta - z |^{2n}} \, w_{\zeta_k \zeta_\ell}(\zeta) d\overline{\zeta} \wedge d\zeta \ , \tag{75}$$

or

$$w(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} \sum_{k,\ell=1}^{n} I_{k,\ell}(\zeta, z) w_{\overline{\zeta}_k \zeta_\ell}(\zeta) d\zeta \wedge d\overline{\zeta} \tag{76}$$

with

$$I_{k,\ell}(\zeta, z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} \frac{| \overline{\zeta_k - \tau_k})(\tau_\ell - z_\ell)}{| \tau - \zeta |^{2n} | \tau - z |^{2n}} \, d\overline{\tau} \wedge d\tau.$$

Differentiating (74) gives

$$\begin{aligned} w_{\overline{z}_k}(z) &= \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} \sum_{\ell=1}^{n} \frac{(\overline{\zeta_1 - z_1})}{| \zeta - z |^{2n}} w_{\overline{\zeta}_k \overline{\zeta}_\ell}(\zeta) d\overline{\zeta} \wedge d\zeta, \\ w_{z_k}(z) &= \frac{n}{(2\pi i)^n} \int_{\mathbb{C}^n} \sum_{i,j=1}^{n} \frac{(\overline{\zeta_k - z_k})(\overline{\zeta_i - z_i})(\overline{\zeta_j - z_j})}{| \zeta - z |^{2(n+1)}} \, w_{\overline{\zeta}_i \overline{\zeta}_j}(\zeta) d\overline{\zeta} \wedge d\zeta. \end{aligned}$$

Differentiating these expressions with respect to z_ℓ all second order complex $z-$derivatives

$$w_{z_k z_\ell}(z) \tag{77}$$

$$= \frac{n(n+1)}{(2\pi i)^n} \int_{\mathbb{C}^n} \sum_{i,j=1}^{n} \frac{(\overline{\zeta_k - z_k})(\overline{\zeta_\ell - z_\ell})(\overline{\zeta_i - z_i})(\overline{\zeta_j - z_j})}{| \zeta - z |^{2(n+2)}} \, w_{\overline{\zeta}_i \overline{\zeta}_j}(\zeta) d\overline{\zeta} \wedge d\zeta$$

and all second order complex mixed derivatives

$$w_{\overline{z}_k z_\ell}(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}} \sum_{i=1}^{n} \frac{(\overline{\zeta_\ell - z_\ell})(\overline{\zeta_i - z_i})}{\mid \zeta - z \mid^{2(n+1)}} \; w_{\overline{\zeta}_k \overline{\zeta}_i}(\zeta) d\overline{\zeta} \wedge d\zeta \tag{78}$$

are expressed through the second order $\overline{z}$-derivatives.
Differentiating (75) for $1 \leq k \leq n$

$$\begin{aligned} w_{z_k}(z) &= \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{i=1}^{n} \frac{(\zeta_i - z_i)}{\mid \zeta - z \mid^{2n}} \; w_{\zeta_k \zeta_i}(\zeta) d\overline{\zeta} \wedge d\zeta, \\ w_{\overline{z}_k}(z) &= \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{i,j=1}^{n} \frac{(\zeta_k - z_k)(\zeta_i - z_i)(\zeta_j - z_j)}{\mid \zeta - z \mid^{2(n+1)}} \; w_{\zeta_i \zeta_j}(\zeta) d\overline{\zeta} \wedge d\zeta \end{aligned}$$

follow. Differentiating these expressions with respect to $\overline{z}_i$ all second order complex $\overline{z}-$derivatives

$$\begin{aligned} & w_{\overline{z}_k \overline{z}_\ell}(z) \\ &= \frac{n(n+1)}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{i,j=1}^{n} \frac{(\zeta_k - z_k)(\zeta_\ell - z_\ell)(\zeta_i - z_i)(\zeta_j - z_j)}{\mid \zeta - z \mid^{2(n+2)}} \; w_{\zeta_i \zeta_j}(\zeta) d\overline{\zeta} \wedge d\zeta \end{aligned}$$

and all second order complex mixed derivatives

$$w_{z_k \overline{z}_\ell}(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{i=1}^{n} \frac{(\zeta_\ell - z_\ell)(\zeta_i - z_i)}{\mid \zeta - z \mid^{2(n+1)}} \; w_{\zeta_k \zeta_i}(\zeta) d\overline{\zeta} \wedge d\zeta$$

in $\mathbb{C}^n$ are expressed through the second order $z-$derivatives.
Differentiating (76) the expressions

$$w_{\overline{z}_k}(z) = \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{i=1}^{n} \frac{(\zeta_i - z_i)}{\mid \zeta - z \mid^{2n}} \; w_{\overline{\zeta}_k \zeta_i}(\zeta) d\zeta \wedge d\overline{\zeta}, 1 \leq k \leq n,$$

$$w_{z_\ell}(z) = \frac{1}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{j=1}^{n} \frac{(\overline{\zeta_j - z_j})}{\mid \zeta - z \mid^{2n}} \; w_{\overline{\zeta}_j \zeta_\ell}(\zeta) d\zeta \wedge d\overline{\zeta}, 1 \leq \ell \leq n,$$

are obtained. Differentiating the first expressions with respect to $\overline{z}_\ell$ and the second with respect to z_k all second order complex $\overline{z}-$derivatives

$$w_{\overline{z}_k \overline{z}_\ell}(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{i=1}^{n} \frac{(\zeta_\ell - z_\ell)(\zeta_i - z_i)}{\mid \zeta - z \mid^{2(n+1)}} \; w_{\overline{\zeta}_k \zeta_i}(\zeta) d\zeta \wedge d\overline{\zeta} \tag{79}$$

and all second order complex $z-$derivatives

$$w_{z_k z_\ell}(z) = \frac{n}{(2\pi i)^n} \int\limits_{\mathbb{C}^n} \sum_{j=1}^{n} \frac{(\overline{\zeta_k - z_k})(\overline{\zeta_j - z_j})}{\mid \zeta - z \mid^{2(n+1)}} \; w_{\overline{\zeta}_j \zeta_\ell}(\zeta) d\zeta \wedge d\overline{\zeta} \tag{80}$$

in $\mathbb{C}^n$ are expressed through the second order mixed derivatives.

5. The Fueter system

In the space $\mathbb{C}^2$ these are the first order matrix elliptic operators

$$\partial \; :\equiv \begin{pmatrix} \partial_{z_1} & \partial_{z_2} \\ -\partial_{\overline{z}_2} & \partial_{\overline{z}_1} \end{pmatrix} \;\; , \;\; \overline{\partial} :\equiv \begin{pmatrix} \partial_{\overline{z}_1} & \partial_{\overline{z}_2} \\ -\partial_{z_2} & \partial_{z_1} \end{pmatrix} \; ,$$

$$\overline{\partial}' \; :\equiv \begin{pmatrix} \partial_{\overline{z}_1} & -\partial_{z_2} \\ \partial_{\overline{z}_2} & \partial_{z_1} \end{pmatrix} \;\; , \;\; \partial' :\equiv \begin{pmatrix} \partial_{z_1} & -\partial_{\overline{z}_2} \\ \partial_{z_2} & \partial_{\overline{z}_1} \end{pmatrix} .$$

The operator ∂ is called Fueter operator and arises in the function theory of a quaternionic variable [7].
If the vector-function $u(z) = (u_1, u_2)$ belongs to $W^{1,p}(\mathbb{C}^2; \mathbb{C}^2), 1 < p < +\infty$, then see [5],

$$u(z) = -\frac{1}{\pi^2} \int\limits_{\mathbb{C}^2} \frac{E_1(\zeta - z)}{\mid \zeta - z \mid^4} \; \partial_\zeta u(\zeta) d\mathbb{C}^2_\zeta \tag{81}$$

and

$$u(z) = -\frac{1}{\pi^2} \int\limits_{\mathbb{C}^2} \frac{E_2(\zeta - z)}{\mid \zeta - z \mid^4} \; \overline{\partial}'_\zeta u(\zeta) d\mathbb{C}^2_\zeta \; , \tag{82}$$

where

$$E_1(\zeta) = \begin{pmatrix} \zeta_1 & -\overline{\zeta}_2 \\ \zeta_2 & \overline{\zeta}_1 \end{pmatrix} \;\; , \;\; E_2(\zeta) = \begin{pmatrix} \overline{\zeta}_1 & \overline{\zeta}_2 \\ -\zeta_2 & \zeta_1 \end{pmatrix} .$$

Applying the operator $\overline{\partial}_z$ to both sides of (81)

$$\overline{\partial} u(z) = -\frac{2}{\pi^2} \int\limits_{\mathbb{C}^2} \frac{\sigma_1(\zeta - z)}{\mid \zeta - z \mid^6} \; \partial_\zeta u(\zeta) d\mathbb{C}^2_\zeta = \sigma_1(\partial u) \tag{83}$$

is obtained. Similarly acting with the operator ∂_z on (82) gives

$$\partial u(z) = -\frac{2}{\pi^2} \int\limits_{\mathbb{C}^2} \frac{\sigma_2(\zeta - z)}{\mid \zeta - z \mid^6} \; \overline{\partial}'_\zeta u(\zeta) d\mathbb{C}^2_\zeta = \sigma_2(\overline{\partial}' u) \tag{84}$$

i.e. the complex first order operator ∂u is converted to the complex first order operator $\overline{\partial} u$ by means of the singular matrix integral operator

$$(\sigma_1\omega)(z) = -\frac{2}{\pi^2}\int\limits_{\mathbb{C}^2} \frac{\sigma_1(\zeta - z)}{\mid \zeta - z \mid^6}\ \omega(\zeta)d\mathbb{C}^2_\zeta\ . \tag{85}$$

In the same way the complex first order operator $\overline{\partial}' u$ is converted to the complex first order operator ∂u by means of the singular matrix integral operator

$$(\sigma_2\omega)(z) = -\frac{2}{\pi^2}\int\limits_{\mathbb{C}^2} \frac{\sigma_2(\zeta - z)}{\mid \zeta - z \mid^6}\ \omega(\zeta)d\mathbb{C}^2_\zeta\ , \tag{86}$$

where

$$\sigma_1(\zeta) = \begin{pmatrix} \zeta_1^2 + \zeta_2^2 & \overline{\zeta}_1\zeta_2 - \zeta_1\overline{\zeta}_2 \\ \overline{\zeta}_1\zeta_2 - \zeta_1\overline{\zeta}_2 & \overline{\zeta}_1^2 + \overline{\zeta}_2^2 \end{pmatrix},$$

$$\sigma_2(\zeta) = \begin{pmatrix} \overline{\zeta}_1^2 + \zeta_2^2 & \overline{\zeta}_1\overline{\zeta}_2 - \zeta_1\zeta_2 \\ \overline{\zeta}_1\overline{\zeta}_2 - \zeta_1\zeta_2 & \zeta_1^2 + \overline{\zeta}_2^2 \end{pmatrix}. \tag{87}$$

If a vector-function $u(z) = (u_1, u_2)$ belongs to $W^{2,p}(\mathbb{C}^2; \mathbb{C}^2), 1 < p < +\infty$, then it can be represented as a Newtonian potential

$$u(z) = \frac{1}{\pi^2}\int\limits_{\mathbb{C}^2} \frac{1}{\mid \zeta - z \mid^2} \begin{pmatrix} \partial_{\zeta_1}\partial_{\overline{\zeta}_1} + \partial_{\zeta_2}\partial_{\overline{\zeta}_2} & 0 \\ 0 & \partial_{\zeta_1}\partial_{\overline{\zeta}_1} + \partial_{\zeta_2}\partial_{\overline{\zeta}_2} \end{pmatrix} u(\zeta)d\mathbb{C}^2_\zeta \tag{88}$$

in $\mathbb{C}^2$. Applying to both sides of (88) the second order elliptic matrix operator

$$\overline{D} = \begin{pmatrix} (\partial_{\overline{z}_1})^2 + (\partial_{\overline{z}_2})^2 & \partial_{z_1}\partial_{\overline{z}_2} - \partial_{\overline{z}_1}\partial_{z_2} \\ \partial_{z_1}\partial_{\overline{z}_2} - \partial_{\overline{z}_1}\partial_{z_2} & (\partial_{z_1})^2 - (\partial_{z_2})^2 \end{pmatrix}$$

gives

$$\overline{D}u(z) = \frac{1}{4\pi^2}\int\limits_{\mathbb{C}^2} \frac{\tau(\zeta - z)}{\mid \zeta - z \mid^6}\ \Delta u(\zeta)d\mathbb{C}^2_\zeta\ , \tag{89}$$

where Δ is the Laplacian in $\mathbb{C}^2$ and

$$\tau(\zeta) = \begin{pmatrix} \zeta_1^2 + \zeta_2^2 & \overline{\zeta}_2\zeta_2 - \zeta_1\overline{\zeta}_2 \\ \overline{\zeta}_1\zeta_2 - \zeta_1\overline{\zeta}_2 & \overline{\zeta}_1^2 + \overline{\zeta}_2^2 \end{pmatrix}.$$

Thus the second order complex elliptic differential operator applied to u is recovered through Δu by means of the following singular matrix integral operator

$$\frac{1}{4\pi^2}\int\limits_{\mathbb{C}^2}\frac{\tau(\zeta-z)}{|\,\zeta-z\,|^6}d\mathbb{C}^2_\zeta.$$

Applying any other elliptic matrix operator of second order to both sides of (89) expresses this operator application through Δu by means of a singular matrix integral operator. There are many such operators, among them, for instance

$$D_1 = \begin{pmatrix} (\partial_{\overline{z}_1})^2+(\partial_{z_2})^2 & \partial_{\overline{z}_1}\partial_{\overline{z}_2}-\partial_{z_1}\partial_{z_2} \\ \partial_{\overline{z}_1}\partial_{\overline{z}_2}-\partial_{z_1}\partial_{z_2} & (\partial_{z_1})^2+(\partial_{\overline{z}_2})^2 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} \partial_{z_1}\partial_{\overline{z}_1}-\partial_{z_2}\partial_{\overline{z}_2} & 2\partial_{\overline{z}_1}\partial_{z_2} \\ -2\partial_{z_1}\partial_{\overline{z}_2} & \partial_{z_1}\partial_{\overline{z}_1}-\partial_{z_2}\partial_{\overline{z}_2} \end{pmatrix}.$$

6. Orthogonalized potentials in the unit ball

Let $B^n = \{z \in \mathbb{C}^n :|\; z\; |< 1\}$ be the unit ball in $\mathbb{C}^n$ and $w \in W^{1,p}(B^n;\mathbb{C})$. From the Cauchy-Pompeiu formula ([3], (2.169))

$$\begin{aligned} w(z) &= \int\limits_{\partial B^n}\frac{w(\zeta)d\sigma(\zeta)}{(1-<z,\zeta>)^n} \\ &+\sum_{i=1}^n\int\limits_{B^n}\Big\{\frac{1}{n}\int\limits_{\partial B^n}\frac{1-<\eta,z>}{|\,\eta-z\,|^{2n}}\frac{(\overline{\eta_i-\zeta_i})d\sigma(\eta)}{(1-<\eta,\zeta>)^n(1-<\zeta,\eta>)} \\ &-\frac{\overline{\zeta_i-z_i}}{|\,\zeta-z\,|^{2n}}\Big\}w_{\overline{\zeta}_i}(\zeta)d\nu(\zeta) \end{aligned}$$

in the same way as formula (11) is attained w is representable as ([3], (2.170)–(2.172))

$$\begin{aligned} w(z) &= \int\limits_{B^n}\frac{w(\zeta)d\nu(\zeta)}{(1-<z,\zeta>)^{n+1}} \\ &+\sum_{i=1}^n\int\limits_{B^n}\Big\{\frac{1}{n}\int\limits_{\partial B^n}\frac{1-<\eta,z>}{|\,\eta-z\,|^{2n}}\frac{(\overline{\eta_i-\zeta_i})d\sigma(\eta)}{(1-<\eta,\zeta>)^n(1-<\zeta,\eta>)} \\ &+\frac{\overline{\zeta}_i}{n(1-<z,\zeta>)^n}-\frac{\overline{\zeta_i-z_i}}{|\,\zeta-z\,|^{2n}}\Big\}w_{\overline{\zeta}_i}(\zeta)d\nu(\zeta). \end{aligned}$$

The first term is an analytic function while the second is in the orthogonal complement of the set of analytic functions in B with respect to

$L_2(B^n;\mathbb{C})$.
Thus any $w \in W^{1,p}(B^n;\mathbb{C})$ orthogonal to the set $\mathcal{O}(B^n)$ of analytic functions in B^n is representable as

$$w(z) = \sum_{i=1}^{n} \int_{B^n} \Big\{ \frac{1}{n} \int_{\partial B^n} \frac{(1- <\eta, z>)(\overline{\eta_i - \zeta_i}) d\sigma(\eta)}{(1- <\eta,\zeta>)^n (1- <\zeta,\eta>) \mid \eta - z \mid^{2n}} \tag{90}$$

$$+ \frac{\overline{\zeta}_i}{n(1- <z,\zeta>)^n} - \frac{\overline{\zeta_i - z_i}}{\mid \zeta - z \mid^{2n}} \Big\} w_{\overline{\zeta}_i}(\zeta) d\nu(\zeta).$$

If the vector-function $f(z) = (f_1(z), \dots, f_n(z))$ belongs to $L^p(B^n;\mathbb{C}), 1 < p < +\infty$, and satisfies the conditions

$$\frac{\partial f_k}{\partial \overline{z}_\ell} - \frac{\partial f_\ell}{\partial \overline{z}_k} = 0, k \neq \ell,$$

then the potential

$$w(z) = \sum_{i=1}^{n} \int_{B^n} \Big\{ \frac{1}{n} \int_{\partial B^n} \frac{(1- <\eta, z>)(\overline{\eta_i - \zeta_i}) d\sigma(\eta)}{(1- <\eta,\zeta>)^n (1- <\zeta,\eta>) \mid \eta - z \mid^{2n}}$$

$$+ \frac{\overline{\zeta}_i}{n(1- <z,\zeta>)^n} - \frac{(\overline{\zeta}_i - \overline{z}_i)}{\mid \zeta - z \mid^{2n}} \Big\} f_i(\zeta) d\nu(\zeta)$$

as counterpart of the potential in (13) is the solution of the inhomogeneous Cauchy-Riemann system

$$w_{\overline{z}_k} = f_k, 1 \leq k \leq n \ ,$$

orthogonal to every function holomorphic in B^n.
From (90) we find all first order complex $z-$derivatives through the first order complex $\overline{z}-$derivatives

$$w_{z_k}(z) \quad = \quad \sum_{i=1}^{n} \int_{B^n} \left\{ \frac{1}{n} \int_{\partial B^n} \frac{(\overline{\eta_k - z_k})(1- <\eta, z>)(\overline{\eta_i - \zeta_i}) d\sigma(\eta)}{(1- <\eta,\zeta>)^n (1- <\zeta,\eta>) \mid \eta - z \mid^{2(n+1)}} \right. \tag{91}$$

$$\left. + \frac{\overline{\zeta}_i \overline{\zeta}_k}{(1- <z,\zeta>)^{n+1}} - \frac{(\overline{\zeta_i - z_i})(\overline{\zeta_k - z_k})}{\mid \zeta - z \mid^{2(n+1)}} \right\} w_{\overline{\zeta}_i}(\zeta) d\nu(\zeta).$$

Analogously if $w \in W^{1,p}(B^n;\mathbb{C}), 1 < p < +\infty$, and w is orthogonal to every function antiholomorphic in B^n , then all first order complex $\overline{z}-$derivatives can be expressed through the first order complex

$z-$derivatives by

$$w_{\overline{z}_k}(z) = \sum_{i=1}^{n} \int_{B^n} \left\{ \frac{1}{n} \left(\int_{\partial B^n} \frac{(\eta_k - z_k)(1- < z, \eta >)(\eta_i - \zeta_i) d\sigma(n)}{(1- < \zeta, \eta >)^n (1- < \eta, \zeta >) \mid \eta - z \mid^{2(n+1)}} \right. \right. \tag{92}$$

$$\left. + \frac{\zeta_i \zeta_k}{(1- < \zeta, z >)^{n+1}} - \frac{(\zeta_i - z_i)(\zeta_k - z_k)}{\mid \zeta - z \mid^{2(n+1)}} \right) w_{\zeta_i}(\zeta) d\nu(\zeta).$$

Let $w \in W^{2,p}(B^n;\mathbb{C})$. Choosing $w(z)$ from suitable subspaces of $W^{2,p}(B^n;\mathbb{C})$, $1 < p < +\infty$, all the second order complex derivatives $w_{z_k z_\ell}$ and $w_{\overline{z}_k \overline{z}_\ell}$ can be expressed through $w_{\overline{z}_k z_\ell}$ or the complex derivatives $w_{z_k z_\ell}$ and $w_{\overline{z}_k z_\ell}$ through $w_{\overline{z}_k \overline{z}_\ell}$ or the complex derivatives $w_{\overline{z}_k \overline{z}_\ell}$ and $w_{\overline{z}_k z_\ell}$ through $w_{z_k z_\ell}$. If for instance all the first order complex $\overline{z}-$derivatives $w_{\overline{z}_i}$ are assumed to be orthogonal to every function holomorphic in B^n , then

$$w_{\overline{z}_i}(z) = \sum_{j=1}^{n} \int_{B^n} T_j(\zeta, z) w_{\overline{\zeta}_i \overline{\zeta}_j}(\zeta) d\nu(\zeta),$$

where

$$T_j(\zeta, z) = \int_{\partial B^n} \frac{(1- < \eta, z >)(\overline{\eta_j - \zeta_j}) d\sigma(\eta)}{(1- < \eta, \zeta >)^n (1- < \zeta, \eta >) \mid \eta - z \mid^{2n}} + \frac{\overline{\zeta}_j}{n(1- < z, \zeta >)^n} - \frac{(\overline{\zeta_j - z_j})}{\mid \zeta - z \mid^{2n}}.$$

Introducing the function

$$\varphi(z) = w(z) - \sum_{i=1}^{n} \overline{z}_i w_{\overline{z}_i}(z),$$

satisfying the inhomogeneous Cauchy-Riemann system

$$\varphi_{\overline{z}_k} = - \sum_{i=1}^{n} \overline{z}_i w_{\overline{z}_k \overline{z}_i}$$

and assuming φ to be orthogonal to every function holomorphic in B^n , then

$$\varphi(z) = - \sum_{i,j=1}^{n} \int_{B^n} T_j(\zeta, z) \overline{\zeta}_i w_{\overline{\zeta}_i \overline{\zeta}_j}(\zeta) d\nu(\zeta).$$

Hence any element of $W^{2,p}(B^n;\mathbb{C}), 1 < p < +\infty$, such that $w_{\overline{z}_i}$ and $w(z) - \sum_{i=1}^{n} \overline{z}_i w_{\overline{z}_i}$ are orthogonal to every function holomorphic in B^n is representable as

$$w(z) = \sum_{i,j=1}^{n} \int_{B^n} T_{ij}(\zeta, z) w_{\overline{\zeta}_i \overline{\zeta}_j}(\zeta) d\nu(\zeta) \ , \tag{93}$$

where

$$\begin{aligned} T_{ij}(\zeta, z) &= -\frac{1}{n} \int_{B^n} \frac{(1- < \eta, z >)(\overline{\zeta_i - z_i})(\overline{\eta_j - \zeta_j}) d\sigma(\eta)}{(1- < \eta, \zeta >)^n (1- < \zeta, \eta >) \mid \eta - z \mid^{2n}} \\ & \quad - \frac{(\overline{\zeta_i - z_i})\overline{\zeta}_j}{n(1- < z, \zeta >)^n} + \frac{(\overline{\zeta_i - z_i})(\overline{\zeta_j - z_j})}{\mid \zeta - z \mid^{2n}}. \end{aligned}$$

From (93) all second order mixed complex derivatives can be found,

$$w_{\overline{z}_k z_\ell}(z) = \sum_{j=1}^{n} \int_{B^n} \frac{\partial}{\partial z_\ell} \ T_j(\zeta, z) w_{\overline{\zeta}_k \overline{\zeta}_j}(\zeta) d\nu(\zeta),$$

where

$$\begin{aligned} \frac{\partial T_j(\zeta, z)}{\partial z_\ell} &= \int_{\partial B^n} \frac{(1- < \eta, z >)(\overline{\eta_\ell - z_\ell})(\overline{\eta_j - \zeta_j}) d\sigma(\eta)}{(1- < \eta, \zeta >)^n (1- < \zeta, \eta >) \mid \eta - z \mid^{2(n+1)}} \\ & \quad + \frac{\overline{\zeta}_j \overline{\zeta}_\ell}{(1- < z, \zeta >)^{n+1}} - n \frac{(\overline{\zeta_j - z_j})(\overline{\zeta_\ell - z_\ell})}{\mid \zeta - z \mid^{2(n+1)}} \end{aligned}$$

and all second order complex $z-$derivatives

$$w_{z_k z_\ell}(z) = \sum_{i,j=1}^{n} \int_{B^n} \frac{\partial^2 T_{ij}(\zeta, z)}{\partial z_k \partial z_\ell} w_{\overline{\zeta}_i \overline{\zeta}_j}(\zeta) d\nu(\zeta),$$

where

$$\begin{aligned} & \frac{\partial^2 T_{ij}(\zeta, z)}{\partial z_k \partial z_\ell} \\ & = -(n+1) \Bigg[\int_{\partial B^n} \frac{(1- < \eta, z >)(\overline{\zeta_i - z_i})(\overline{\eta_j - \zeta_j})(\overline{\eta_k - z_k})(\overline{\eta_\ell - z_\ell}) d\sigma(\eta)}{(1- < \eta, \zeta >)^n (1- < \zeta, \eta >) \mid \eta - z \mid^{2(n+2)}} \\ & + \frac{(\overline{\zeta_i - z_i})\overline{\zeta}_j \overline{\zeta}_k \overline{\zeta}_\ell}{(1- < z, \zeta >)^{n+2}} - n \frac{(\overline{\zeta_i - z_i})(\overline{\zeta_j - z_j})(\overline{\zeta_k - z_k})(\overline{\zeta_1 - z_1})}{\mid \zeta - z \mid^{2(n+2)}} \Bigg] \end{aligned}$$

through the second order complex $\overline{z}-$derivatives $w_{\overline{z}_k \overline{z}_\ell}$.

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BOUNDARY AND INITIAL VALUE PROBLEMS FOR HIGHER ORDER PDES IN CLIFFORD ANALYSIS

E. Obolashvili
Mathematical Institute
Georgian Academy of Sciences
380093 Tbilisi, Georgia
helen@imath.acnet.ge

Abstract Some higher order PDE called elliptic-hyperbolic, elliptic-parabolic, hyperbolic-parabolic, elliptic-hyperbolic-parabolic equations are considered as they are constructed in [5]. They are related to harmonic-wave, harmonic-heat, wave-heat, harmonic-wave-heat equations. Moreover, parabolic type Beltrami and pluri Beltrami equations are considered. For all of them some boundary, initial value problems are solved in quadratures [5]. Here it is more strictly proved that the obtained solutions really satisfy all conditions of the problems.

Keywords: Clifford analysis, higher order pdes, initial boundary value problems

Mathematics Subject Classification (2000): 35G05, 30G30

1. Introduction

Let D be the Dirac operator in the Clifford algebras $R_{(n)}$, $R_{(n,n-1)}$, $R^0_{(n)}$, $n \geq 1$, [2, 6]

$$\overline{\partial} = \sum_0^n e_k \frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_0} e_0 + D, \quad \partial = \frac{\partial}{\partial x_0} e_0 - D, \tag{1}$$

where the vectors of the basis e_k satisfy the multiplication rules

$$\begin{gathered} e_0^2 = e_0, \quad e_j^2 = -e_0, \;\; j = 1, \ldots, n-1, \\ e_k e_i + e_i e_k = 0, \;\; k, i = 1, \ldots, n, \;\; k \neq i, \end{gathered} \tag{2}$$

G. A. Barsegian and H. G. W. Begehr (eds.), Topics in Analysis and its Applications, 271–286.

$$e_n^2 = \begin{cases} -e_0, & \text{in } R_{(n)}, \\ e_0, & \text{in } R_{(n,n-1)}, \\ 0, & \text{in } R^0_{(n)}, \end{cases} \tag{3}$$

e_0 is the identity element. These spaces are associative, 2^n-dimensional as real spaces, noncommutative for $n \geq 2$ with the basis $\{e_A\}$, $A = (\alpha_1, \alpha_2, \dots, \alpha_k)$, where α_k are integer numbers $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq n$. Thus any element of the space is represented as

$$u(x) = \sum_A u_A(x) e_A, \quad e_A = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}. \tag{4}$$

For u two conjugations are defined:

$$\bar{u} = \sum_A u_A(x) \bar{e}_A, \quad \widetilde{u}_A = \sum_A u_A \widetilde{e}_A, \tag{5}$$

where $\bar{e}_0 = \widetilde{e}_0 = e_0$, $\bar{e}_k = \widetilde{e}_k = -e_k$, $k = 1, \dots, n$,

$$\begin{gathered} \bar{e}_A = \bar{e}_{\alpha_k} \bar{e}_{\alpha_{k-1}} \cdots \bar{e}_{\alpha_1} = (-1)^{\frac{k(k+1)}{2}} e_A, \\ \widetilde{e}_A = \widetilde{e}_{\alpha_1} \cdots \widetilde{e}_{\alpha_k} = (-1)^k e_A. \end{gathered} \tag{6}$$

Thus one has

$$\partial \overline{\partial} = \overline{\partial} \partial = \begin{cases} \Delta_n & \text{in } R_{(n)}, \\ \Delta_{n-1} - \frac{\partial^2}{\partial x_n^2} & \text{in } R_{(n,n-1)}, \\ \Delta_{n-1} & \text{in } R^0_{(n)}, \end{cases} \tag{7}$$

where Δ_k $(k = n-1, n)$ is the Laplace operator with respect to the variables $x_0, x_1, \dots, x_k$. In [5] there are constructed
1) the elliptic-hyperbolic equation

$$\overline{\partial}\Big(\overline{\partial} + e_n \frac{\partial}{\partial x_n}\Big) u(x, x_n) = 0, \quad x(x_0, \dots, x_{n-1}), \tag{8}$$

where

$$\overline{\partial} = \sum_0^{n-1} e_k \frac{\partial}{\partial x_k}, \quad e_k^2 = -e_0, \quad k = 1, \dots, n-1, \quad e_n^2 = e_0; \tag{9}$$

2) the elliptic-parabolic equation

$$\overline{\partial}\Big(\overline{\partial} + e_n \frac{\partial}{\partial x_n} + P_n\Big) u(x, x_n) = 0, \quad e_n^2 = 0, \tag{10}$$

and $\overline{\partial}$ is defined as in (9);
3) the hyperbolic-parabolic equation

$$\Big(\overline{\partial} + e_{n-1} \frac{\partial}{\partial x_{n-1}}\Big)\Big(\overline{\partial} + e_n \frac{\partial}{\partial x_n} + P_n\Big) u(x, x_n) = 0, \tag{11}$$

where

$$\overline{\partial} = \sum_0^{n-2} e_k \frac{\partial}{\partial x_k}, \quad e_k^2 = -e_0, \ \ k = 1, \dots, n-2, \quad e_{n-1}^2 = e_0, \quad e_n^2 = 0; \tag{12}$$

4) the elliptic-hyperbolic-parabolic equation

$$\overline{\partial}\Big(\overline{\partial} + e_{n-1} \frac{\partial}{\partial x_{n-1}}\Big)\Big(\overline{\partial} + e_n \frac{\partial}{\partial x_n} + P_n\Big) u(x, x_n) = 0, \tag{13}$$

where $\overline{\partial}$ is defined as (12), the operator $P_n u$ is defined by the condition [6]

$$\partial P_n u = -\frac{\partial u}{\partial x_n} \,. \tag{14}$$

Representing u in the form

$$u = \sum_{A \neq n} u_A e_A + \sum_{A \neq n} u_{An} e_A e_n,$$

$A(\alpha_1, \dots, \alpha_k)$ does not contain n, then

$$P_n u = \sum_{A \neq n} (-1)^k u_{An} e_A. \tag{15}$$

By force of (7), (14), the solutions of equations (8), (10), (11), (13) are also the solutions of the equations respectively

$$\Delta\Big(\Delta - \frac{\partial^2}{\partial x_n^2}\Big)u(x, x_n) = 0, \quad x(x_0, \dots, x_{n-1}), \tag{16}$$

$$\Delta\Big(\Delta - \frac{\partial}{\partial x_n}\Big)u(x, x_n) = 0, \tag{17}$$

$$\Big(\Delta - \frac{\partial^2}{\partial x_{n-1}^2}\Big)\Big(\Delta - \frac{\partial}{\partial x_n}\Big)u(x, x_{n-1}, x_n) = 0, \quad x(x_0, \dots, x_{n-2}), \tag{18}$$

$$\Delta\Big(\Delta - \frac{\partial^2}{\partial x_{n-1}^2}\Big)\Big(\Delta - \frac{\partial}{\partial x_n}\Big)u(x, x_{n-1}, x_n) = 0. \tag{19}$$

They are called harmonic-wave, harmonic-heat, wave heat, harmonic-wave-heat equations respectively. For the above equations boundary initial value problems the solutions of which can be represented in quadratures are considered.

2. Dirichlet–Cauchy and Neumann–Cauchy problems for elliptic-hyperbolic equations

Let S be the domain $x_{n-1} > 0$, $x_n \equiv t > 0$, $(x_0, x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-1}$ and consider the equation (16).

Dirichlet–Cauchy Problem Find a regular solution of (16) in S vanishing at infinity and satisfying the conditions

$$\begin{aligned} u(x_0, x_1, \dots, x_{n-1}, 0) &= \varphi(x_0, \dots, x_{n-1}), \\ \frac{\partial u}{\partial t}\Big|_{t=0} &= \psi(x_0, \dots, x_{n-1}), \end{aligned} \tag{20}$$

$$u(x_0, \dots, x_{n-2}, 0, t) = f(x_0, \dots, x_{n-2}, t), \tag{21}$$

where these given functions satisfy the compatibility conditions

$$\begin{aligned} &\varphi(x_0, \dots, x_{n-2}, 0) = f(x_0, \dots, x_{n-2}, 0), \\ &\frac{\partial f}{\partial t}\Big|_{t=0} = \psi(x_0, \dots, x_{n-2}, 0). \end{aligned} \tag{22}$$

Solution The equation (16) can be written as

$$\Delta u = F(x, t), \tag{23}$$

$$\Delta F - \frac{\partial^2 F}{\partial t^2} = 0. \tag{24}$$

Then by force of (19), (24) the unknown function F satisfies the conditions

$$F(x,0) = \Delta\varphi(x), \quad \frac{\partial F}{\partial t}\Big|_{t=0} = \Delta\psi(x), \tag{25}$$

i.e., in order to define $F(x,t)$ one has Cauchy's IVP for the wave equation which is represented in quadratures [3, 6]. To define $u(x,t)$ one has the Dirichlet problem for the nonhomogeneous equation (23) with condition (21) and it can be written in quadratures too [6].

Now we will prove that this $u(x,t)$ is the solution of our problem. It is clear that it satisfies the equation (16) and the condition (21). We will show that it will satisfy (19) too. Really, by force of (23), (25) one has

$$\begin{aligned}&\Delta[u(x,0) - \varphi(x)] = 0,\\ &\Delta\Big[\frac{\partial u}{\partial t} - \psi(x)\Big] = 0, \quad t = 0, \quad x_{n-1} > 0.\end{aligned}$$

Thus $u(x,0) - \varphi(x)$ and $\partial u/\partial t - \psi(x)$ are harmonic functions in the half-space $x_{n-1} > 0$ and by force of (22) satisfy the homogeneous conditions

$$\begin{aligned}&u(x_0,\ldots,x_{n-2},0,0) - \varphi(x_0,\ldots,x_{n-2})\\ &= u(x_0,\ldots,x_{n-2},0,0) - f(x_0,\ldots,x_{n-2},0) = 0,\end{aligned} \tag{26}$$

$$\frac{\partial u}{\partial t}\Big|_{t=0} - \psi(x_0,\ldots,x_{n-2},0) = \frac{\partial u}{\partial t}\Big|_{t=0} - \frac{\partial f}{\partial t}\Big|_{t=0} = 0, \quad x_{n-1} = 0.$$

Harmonic functions in the half-space vanishing at infinity by force of homogeneous conditions as is well known are zero. Thus the conditions (19) are satisfied.

Neumann–Cauchy Problem Find the regular solution of (16) for $t > 0$, $x_{n-1} > 0$, $(x_0,\ldots,x_{n-2}) \in \mathbb{R}^{n-1}$ that vanishes at infinity and satisfies conditions (19) and

$$\frac{\partial u}{\partial x_{n-1}} = f(x_0,\ldots,x_{n-2},t), \quad x_{n-1} = 0, \quad t > 0. \tag{27}$$

The given functions satisfy the compatibility conditions

$$f(x_0,\dots,x_{n-2},0)=\frac{\partial\varphi(x_0,\dots,x_{n-1})}{\partial x_{n-1}},\quad x_{n-1}=0,$$
$$\frac{\partial\psi(x_0,\dots,x_{n-1})}{\partial x_{n-1}}=\frac{\partial f(x_0,\dots,x_{n-2},t)}{\partial t},\quad t=0,\quad x_{n-1}=0.$$

The solution can be constructed exactly in the above way in quadratures. For the harmonic-Klein-Gordon equation

$$\Delta\Big(\Delta-K^2-\frac{\partial^2}{\partial t^2}\Big)u(x,t)=0$$

the problems (19),(21) and (19),(27) can be solved in the same way. Consider the equation (8) in the domain S. One can be sure that the equation

$$\Big(\overline{\partial}+e_n\frac{\partial}{\partial t}\Big)\overline{\partial}u(x,t)=0 \tag{28}$$

differs from (8), i.e., these operators are not commutative. It must be noted that in [5] the possibility was stated to construct the solutions of some problems for the equation (8) in quadratures. It is found that this holds for (28) and does not hold for (8).

For the equations (8) and (28) boundary-initial value problems are formulated as follows.

Find a regular solution of (8) or of (28), in the domain $t>0$, $x_{n-1}>0$, vanishing at infinity and satitsfying the conditions

$$u(x_0,\dots,x_{n-1},0)=\varphi(x_0,\dots,x_{n-1}), \tag{29}$$
$$u_A(x_0,\dots,x_{n-2},0,t)=\psi_A(x_0,\dots,x_{n-2},t),\quad A=(\alpha_1,\dots,\alpha_k) \tag{30}$$

with the compatibility conditions

$$\varphi_A(x_0,\dots,x_{n-2},0)=\psi_A(x_0,\dots,x_{n-2},0),\quad 1\le\alpha_1<\dots<\alpha_k\le n-1.$$

This problem is correctly posed for both equations, but for (28) it can be solved in quadratures.

Solution (28) will be written as

$$\overline{\partial}u=F,\quad x_{n-1}>0,\quad t>0, \tag{31}$$
$$\overline{\partial}F+e_n\frac{\partial F}{\partial t}=0, \tag{32}$$

and by condition (29) we can obtain the condition for F

$$F(x_0,\ldots,x_{n-1},0)=\overline{\partial}\varphi.$$

Thus F will be defined and then u is defined by (30), (31) in quadratures [6]. Using the compatibility conditions one can prove that this $u(x,t)$ really is the solution of our problem. The equation (8) can be written as

$$\overline{\partial}u+e_n\,\frac{\partial u}{\partial t}=F,\quad \overline{\partial}F=0,$$

and by (30) one cannot define boundary conditions for F. It is interesting to consider the equation

$$\Big(\overline{\partial}+e_n\,\frac{\partial}{\partial t}\Big)\overline{\partial}^{\,m}u=0. \tag{33}$$

For simplicity consider the case $n=2$, i.e.,

$$u=u_0e_0-u_1e_1-u_2e_2-u_{12}e_1e_2,\quad e_1^2=-e_0,\quad e_2^2=e_0,$$
$$\overline{\partial}=\frac{\partial}{\partial x_0}\,e_0+e_1\,\frac{\partial}{\partial x_1}=\frac{d}{d\bar z}\,.$$

Then (32) has the form

$$\Big(\frac{d}{d\bar z}+e_2\,\frac{\partial}{\partial t}\Big)\frac{d^m u}{\partial\bar z^m}=0,\quad m\geq 2,\quad u(x,t),\quad t>0,\quad x_1>0. \tag{34}$$

Let this equation be written in the form

$$\frac{d^m u}{d\bar z^m}=F(z,t), \tag{35}$$
$$\frac{dF}{d\bar z}+e_2\,\frac{\partial F}{\partial t}=0 \tag{36}$$

and consider the boundary-initial conditions for (34)

$$u(z,0)=\varphi(z), \tag{37}$$

$$\begin{gathered}\operatorname{Re}\Big[\frac{d^k u}{d\bar z^k}\,e_j\Big]=\varphi_{kj}(x_0,0,t),\\ t>0,\ \ x_1=0,\ \ k=0,1,\ldots,m-1\end{gathered} \tag{38}$$

$$j=0,2\ \text{ or }\ j=1,2\ \text{ or }\ j=0,12\ \text{ or }\ j=1,12.$$

For $F(z,t)$ from the condition (37) one has $F(z,0)=d^m\varphi/\bar z^m$ and the solution of (36) will be defined in quadratures. Then $u(z,t)$ as the solution of equation (35) with the boundary conditions (38) can be defined [4, 5].

In the same way the system

$$\Big(\frac{d}{d\bar z}+e_2\,\frac{\partial}{\partial t}\Big)\mathbf{V}^m u=0,\quad m\ge 1,$$

can be considered where

$$\mathbf{V}u=\frac{du}{d\bar z}+B\overline{u}.$$

For this equation by conditions (37), (38) the solution is represented in quadratures when $B = const$ [5]. But if B is variable, then using results of [8, 1] one can study the solvability of the problems.

3. Boundary-initial value problems for elliptic-parabolic and hyperbolic-parabolic equations

Consider the elliptic-parabolic equation

$$\begin{gathered}\Big(\overline{\partial}+e_n\,\frac{\partial}{\partial t}+P_n\Big)\overline{\partial}u(x,t)=0,\\ t\equiv x_n>0,\quad x_{n-1}>0,\quad (x_0,\dots,x_{n-2})\in\mathbb{R}^{n-1},\end{gathered}\tag{39}$$

which differs from (10)because these two operators are not commutative. The solution of this equation is the solution of the equation

$$\Delta\Big(\Delta-\frac{\partial}{\partial t}\Big)u(x,t)=0,\quad x(x_0,\dots,x_{n-1}).\tag{40}$$

Dirichlet–Cauchy Problem for the equation (40) Find in the domain S the regular solution of (40) vanishing at infinity and satisfying the conditions

$$u(x,0)=\varphi(x),\tag{41}$$

$$u(x_0,\dots,x_{n-2},0,t)=\psi(x_0,\dots,x_{n-2},t),\quad t>0,\tag{42}$$

where the given functions satisfy the compatibility conditions

$$\varphi(x_0,\dots,x_{n-2},0)=\psi(x_0,\dots,x_{n-2},0).\tag{43}$$

Solution The equation (40) can be written as

$$\Delta u=F(x,t),\tag{44}$$

$$\Delta F-\frac{\partial F}{\partial t}=0.\tag{45}$$

Then by force of (41), (44) the unknown function F satisfies

$$F(x,0) = \Delta\varphi(x), \quad x \in \mathbb{R}^n. \tag{46}$$

Thus $F(x,t)$ will be defined as the solution of the Cauchy problem for the heat equation (45), (46), which is represented in quadratures. To define $u(x,t)$ one has the Dirichlet problem for the nonhomogeneous equation (44) with the condition (42). It can be represented in quadratures (e.g. [6]). $u(x,t)$ satisfies the condition (41) too. Really, by force of (44), (43) one has

$$\Delta[u(x,0) - \varphi(x)] = 0, \quad x_{n-1} > 0,$$

$$u(x_0,\ldots,x_{n-2},0,0) - \varphi(x_0,\ldots,x_{n-2},0) = u - \psi = 0, \quad t = 0, \quad x_{n-1} = 0.$$

Thus $u(x,0) - \varphi(x)$ is a harmonic function in the half-space $x_{n-1} > 0$, vanishes at infinity and on the boundary $x_{n-1} = 0$. By force of uniqueness it is zero and $u(x,t)$ will satisfy (41).

Neumann–Cauchy Problem Find the regular solution of the equation (40) vanishing at infinity by the conditions (41) and

$$\frac{\partial u}{\partial x_{n-1}} = \psi(x_0,\ldots,x_{n-2},t), \quad x_{n-1} = 0. \tag{47}$$

In this case the compatibility condition is

$$\psi(x_0,\ldots,x_{n-2},0) = \frac{\partial\varphi}{\partial x_{n-1}}, \quad x_{n-1} = 0.$$

The solution can be constructed in the above way too.

For (10) and (39) boundary-initial value problems can be formulated correspondingly, moreover, for equation (39) the solution by the above way can be represented in quadratures too.

Now consider equation (11) which can be written as

$$\overline{\partial}u + e_n\frac{\partial u}{\partial x_n} + P_n u = F, \tag{48}$$

$$\overline{\partial}F + e_{n-1}\frac{\partial F}{\partial x_{n-1}} = 0, \tag{49}$$

and the equation (18) can be written in the form

$$\Delta u - \frac{\partial u}{\partial x_n} = F(x, x_{n-1}, x_n), \tag{50}$$

$$\Delta F - \frac{\partial^2 F}{\partial x_{n-1}^2} = 0, \quad x_n \equiv t, \quad x_{n-1} \equiv \tau. \tag{51}$$

Cauchy's Problem Find the regular solution of (18) for $x_n > 0$, $x_{n-1} > 0$, $x(x_0, \ldots, x_{n-2}) \in \mathbb{R}^{n-1}$ satisfying the conditions

$$u(x,0,t) = \varphi_1(x,t), \quad \frac{\partial u}{\partial \tau} = \varphi_2(x,t), \quad \tau = 0, \tag{52}$$

$$u(x,\tau,0) = \psi(x,\tau) \tag{53}$$

with the compatibility condition

$$\varphi(x,0) = \psi(x,0). \tag{54}$$

Solution By force of (50) the unknown function $F(x,\tau,t)$ satisfies

$$\begin{aligned} F(x,0,t) &= \Delta\varphi_1(x,t) - \frac{\partial \varphi_1}{\partial t} \equiv f_1(x,t), \\ \frac{\partial F}{\partial \tau} &= \Delta\varphi_2(x,t) - \frac{\partial \varphi_2}{\partial t} \equiv f_2(x,t), \quad \tau = 0. \end{aligned} \tag{55}$$

Thus for the wave equation (51) one has Cauchy's problem the solution of which is represented in quadratures (e.g., [6]). Then for $u(x,\tau,t)$ we have Cauchy's problem for the inhomogeneous heat equation (50) with the condition (53), the solution is represented in quadratures [6] and as above it can be proved that it really satisfies all conditions.

To formulate the initial value problems for the equation (11), i.e., for (48), (49) all those conditions that are considered for the multiplier operators must be given [6]. Thus using the solutions of each problem, one can obtain the corresponding solution.

4. Parabolic type Beltrami and pluri Beltrami equations

Consider the space $R_{(n)}$, $x = \sum_0^n x_k e_k$, $e_k^2 = -e_0$, $k = 1, \ldots, n-1$, $e_n^2 = 0$. The classification of the Beltrami operator

$$\overline{\partial} u + q \partial u$$

with vectorial variable q in the spaces $R_{(n)}$, $R_{(n,n-1)}$ was given in [7]. In an analogue way one can obtain that in $R^0_{(n)}$ it is of parabolic type if $|q| \neq 1$. Moreover, the case $|q| > 1$ can be reduced to the case $|q| < 1$ where $|q|$ is defined correspondingly in $R^0_{(n)}$. The operator is parabolically degenerate if $|q| = 1$.

In the elliptic and hyperbolic cases the Beltrami equation

$$\overline{\partial} u + q \partial u = 0, \quad |q| < 1, \quad q = \sum_0^n q_k e_k,$$

with constant q can be reduced to the regular equation

$$\overline{\partial}_y u = 0$$

by the linear transformation [7]

$$y_k = \sum_0^n a_{kj} x_j, \quad k = 0, 1, \ldots, n. \tag{56}$$

For the equation

$$\overline{\partial} u + q \partial u + P_n u = 0 \tag{57}$$

in $R^0_{(n)}$, where q is a constant vectorial element of $R^0_{(n)}$ and P_n is defined by (14), one can prove the following result.

Theorem *The linear transformation* (56) *can be defined in such a way that the equation* (57) *can be reduced to the equation*

$$\overline{\partial}_y u + P_n u = 0. \tag{58}$$

Proof As

$$\frac{\partial}{\partial x_j} = \sum_{k=0}^n a_{kj} \frac{\partial}{\partial y_k}, \quad j = 0, 1, \ldots, n,$$

to obtain (58) it is sufficient that

$$\sum_{j=0}^n e_j a_{0j} + \sum_{i=0}^n q_i e_i \left(a_{00} - \sum_{j=1}^n e_j a_{0j} \right) = 1, \tag{59}$$

$$\sum_{j=0}^n e_j a_{kj} + \sum_{i=0}^n q_i e_i \left(a_{k0} - \sum_{j=1}^n e_j a_{kj} \right) = e_k, \quad k = 1, 2, \ldots, n. \tag{60}$$

This is a system for the unknown coefficients a_{kj}. For a_{0j}, $j = 0, \ldots, n$, we have

$$a_{00}(1 + q_0) + \sum_{j=1}^{n-1} q_j a_{oj} = 1, \tag{61}$$

$$a_{00} q_j + (1 - q_0) a_{0j} = 0, \quad j = 1, \ldots, n, \tag{62}$$

and the equations $q_j a_{0k} - q_k a_{oj} = 0$, $j, k = 1, \ldots, n$, which are not independent as they follow from (62).

The determinant of this system can be calculated by mathematical induction:

$$M(q) = \begin{vmatrix} 1 + q_0, q_1, \ldots, q_{n-1}, 0 \\ q_1, 1 - q_0, 0, \ldots, 0 \\ \vdots \\ q_n, 0, 0, \ldots, 1 - q_0 \end{vmatrix} = (1 - q_0)^{n-1}(1 - |q|^2) \neq 0. \tag{63}$$

The solutions are defined uniquely

$$a_{00} = \frac{1-q_0}{1-|q|^2}, \quad a_{0j} = \frac{-q_j}{1-|q|^2}, \quad j = 1, \dots, n,$$

for the other a_{kj} ($j = 0, \dots, n$) with each fixed $k \geq 1$ we have a system with the same determinant (63) and with the right-hand side of the kth equation equal to 1 and the right-hand side of all other equations equal to zero. Thus all a_{kj} are uniquely defined explicitly.

In particular, a_{nj} ($j = 0, \dots, n$) are defined as

$$a_{nj} = 0, \quad j = 0, \dots, n-1, \quad a_{nn} = \frac{1}{1-q_0}.$$

Thus by force of (56) $y_n = a_{nn}x_n$. The solution of equation (57) can be represented as

$$u = \varphi(y_0, \dots, y_n),$$

where φ is solution of equation (58).

All initial value problems which are solved for equation (58) with initial conditions on $y_n = 0$ in quadratures, can be solved for equation (57) with condition on $x_n = 0$ too.

Pluri Beltrami equations are defined as

$$B^m u = 0, \quad B \equiv \overline{\partial} + q\partial + P_n, \quad m \geq 2, \quad u(x). \tag{64}$$

It is obvious by the same linear transformation (56), where a_{kj} are defined by (59), (60), that these equations (64) can be reduced to the pluri parabolic equation

$$(\overline{\partial} + P_n)^m u = 0. \tag{65}$$

Thus all initial value problems which are solved for the equation (65) can be solved for (64) too.

The generalized Beltrami equations in $R^0_{(n)}$

$$\overline{\partial}u + q_1\partial u + q_2\overline{\partial}\widetilde{u} + P_n u = 0,$$

where q_1, q_2 are vectorial, can be considered. As in the cases $R_{(n)}$, $R_{(n,n-1)}$ it can be obtained that it is of parabolic type when $|q_1| + |q_2| < 1$.

5. Some more about pluri parabolic-hyperbolic equations

Let $R^0_{(n,n-1)}$ ($n \geq 2$) be the Clifford algebra for which the basis e_k ($k = 0, \dots, n$) satisfy the multiplication rules

$$e_k^2 = -e_0, \quad k = 1, \dots, n-2, \quad e_{n-1}^2 = e_0^2, \quad e_n^2 = 0. \tag{66}$$

Consider the higher order system of equations

$$\Big(\overline{\partial} + e_{n-1}\,\frac{\partial}{\partial x_{n-1}}\Big)^m (\overline{\partial} + P_n)^p u = 0, \qquad x(x_0,\dots,x_n),\quad m \geq 1,\quad p \geq 1, \tag{67}$$

where

$$\overline{\partial} = \sum_0^{n-2} e_k\,\frac{\partial}{\partial x_k} + e_n\,\frac{\partial}{\partial x_n}\,, \tag{68}$$

and the operator P_n is defined by the condition (12). We think it is interesting to consider also the system

$$\Big(\overline{\partial} + e_{n-1}\,\frac{\partial}{\partial x_{n-1}}\Big)^m \Big(\overline{\partial} + e_n\,\frac{\partial}{\partial x_n} + P_n\Big)^p u = 0, \tag{69}$$

where

$$\overline{\partial} = \sum_0^{n-2} e_k\,\frac{\partial}{\partial x_k}\,. \tag{70}$$

For both systems each component of u satisfies the equation

$$\Big(\Delta - \frac{\partial^2}{\partial t^2}\Big)^m \Big(\Delta - \frac{\partial}{\partial \tau}\Big)^p \phi = 0, \quad x_{n-1} \equiv t, \quad x_n = \tau, \tag{71}$$

where Δ is Laplace operator with respect to the variables $(x_0,\dots,x_{n-2})$. The system (67) is considered in [5] in the case $m = p = 1$. Now one can ask what is the difference between the systems (67) and (69). For visuality consider $m = p = 1$, $n = 2$,

$$u = u_0 e_0 + u_1 e_1 + u_2 e_2 + u_{12} e_1 e_2, \quad e_1^2 = e_0, \quad e_2^2 = 0,$$
$$P_2 u = u_2 e_0 - u_{12} e_1.$$

Let us consider (68) which can be written as

$$\overline{\partial} u + P_2 u = F, \tag{72}$$

$$\overline{\partial} F + e_1\,\frac{\partial F}{\partial x_1} = 0, \quad \overline{\partial} = \frac{\partial}{\partial x_0}\,e_0 + e_2\,\frac{\partial}{\partial x_2}, \tag{73}$$

i.e., for the components one has

$$\frac{\partial u_0}{\partial x_0} = -u_2 + F_0, \qquad \frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2} = F_2,$$
$$\frac{\partial u_1}{\partial x_0} = u_{12} + F_1, \qquad \frac{\partial u_{12}}{\partial x_0} - \frac{\partial u_1}{\partial x_2} = F_{12}, \tag{74}$$

where the components of F satisfy

$$\frac{\partial F_0}{\partial x_0}+\frac{\partial F_1}{\partial x_1}=0, \qquad \frac{\partial F_2}{\partial x_0}+\frac{\partial F_{12}}{\partial x_1}+\frac{\partial F_0}{\partial x_2}=0,$$
$$\frac{\partial F_0}{\partial x_1}+\frac{\partial F_1}{\partial x_0}=0, \qquad \frac{\partial F_2}{\partial x_1}+\frac{\partial F_{12}}{\partial x_0}+\frac{\partial F_1}{\partial x_2}=0. \tag{75}$$

In the case (70) all equations of (74) and the first two equations from (75) are the same but in place of the last two equations one will have

$$\frac{\partial F_2}{\partial x_0}+\frac{\partial F_{12}}{\partial x_1}=0,$$
$$\frac{\partial F_2}{\partial x_1}+\frac{\partial F_{12}}{\partial x_0}=0. \tag{76}$$

Thus in the case of (70) the equations for F_0, F_1 and for F_2, F_{12} are not connected while in the case of (68) they are connected (75).

First consider the initial value problems for the equation (71). Find the regular solution of (71) for $t>0$, $\tau>0$, $x(x_0,\dots,x_{n-2})\in\mathbb{R}^{n-1}$ vanishing at infinity and satisfying the conditions

$$\left.\frac{\partial^k\phi(x,t,\tau)}{\partial t^k}\right|_{t=0}=\varphi_k(x,\tau), \quad k=0,1,\dots,m-1, \tag{77}$$

$$\left.\frac{\partial^k\phi(x,t,\tau)}{\partial \tau^k}\right|_{\tau=0}=\psi_k(x,t), \quad k=0,1,\dots,p-1, \tag{78}$$

where the given functions must satisfy the compatibility conditions

$$\left.\frac{\partial^k\psi_j}{\partial t^k}\right|_{\substack{t=0\\ \tau=0}}-\varphi_k(x,0)=0, \quad k=0,\dots,2m-1, \quad j=0,\dots,p-1. \tag{79}$$

Solution The equation (71) can be written in the form

$$\left(\Delta-\frac{\partial}{\partial\tau}\right)^p\phi=v(x,t,\tau), \tag{80}$$

$$\left(\Delta-\frac{\partial^2}{\partial t^2}\right)^m v=0. \tag{81}$$

For the polywave equation (81) by force of conditions (77) from (80) one has

$$\left.\frac{\partial^k v(x,t,\tau)}{\partial t^k}\right|_{t=0}=\left(\Delta-\frac{\partial}{\partial\tau}\right)^p\varphi_k(x,\tau), \quad k=0,\dots,2m-1. \tag{82}$$

Thus we have Cauchy's problem for the polywave equation (81) the solution of which can be constructed in quadratures [5]. Then $u(x,t,\tau)$

as the solution of Cauchy's problem for nonhomogeneous polyheat equation (80), (78) can be constructed in quadratures too [5]. Now we will satisfy the conditions (77). Really, by force of (80) one has

$$\Big(\Delta - \frac{\partial}{\partial\tau}\Big)^p \frac{\partial^k \phi(x,t,\tau)}{\partial t^k}\Big|_{t=0} = \Big(\Delta - \frac{\partial}{\partial\tau}\Big)^p \varphi_k(x,\tau), \quad k = 0,\dots,2m-1,$$

which means that

$$\Big(\Delta - \frac{\partial}{\partial\tau}\Big)^p \left[\frac{\partial^k \phi(x,t,\tau)}{\partial t^k}\Big|_{t=0} - \Big(\Delta - \frac{\partial}{\partial\tau}\Big)^p \varphi_k(x,\tau)\right] = 0, \quad k = 0,\dots,2m-1.$$

Thus the functions

$$w^k(x,\tau) \equiv \frac{\partial^k \phi(x,t,\tau)}{\partial t^k}\Big|_{t=0} - \varphi_k(x,\tau)$$

are the solutions of the polyheat equation

$$\Big(\Delta - \frac{\partial}{\partial\tau}\Big)^p w^k(x,\tau) = 0, \quad k = 0,\dots,2m-1,$$

with the homogeneous conditions $w^k(x,0) = 0$, which follows from the compatibility conditions (79). By force of uniqueness of solutions $u(x,t,\tau)$ defined in the above way is the solution of our problem.

Now consider for equation (67) the initial value problem. Find the regular solution of (67) for $t > 0$, $\tau > 0$, $x(x_0,\dots,x_{n-2}) \in \mathbb{R}^{n-1}$ vanishing at infinity by the conditions

$$\frac{\partial^k u}{\partial t^k}\Big|_{t=0} = \varphi_k(x,\tau), \quad k = 0,\dots,m-1, \tag{83}$$

$$\left.\begin{array}{l} \dfrac{\partial^k u_A}{\partial\tau^k}\Big|_{\tau=0} = \psi_{kA}(x,t),\ k = 0,\dots,p-1, \\ \text{or} \\ \dfrac{\partial^k u_{An}}{\partial\tau^k}\Big|_{\tau=0} = \psi_{kA}(x,t),\ k = 0,\dots,p-1, \end{array}\right\} A = (\alpha_1,\dots,\alpha_k), A \neq n, \tag{84}$$

with corresponding compatibility conditions.

Solution Equation (67) can be written as

$$(\overline{\partial} + P_n)^p u(x,t,\tau) = F(x,t,\tau) \tag{85}$$

$$\Big(\overline{\partial} + \frac{\partial}{\partial t} e_1\Big)^m F = 0. \tag{86}$$

By condition (83) and equation (85) one can define

$$\frac{\partial^k F}{\partial t^k}\bigg|_{t=0} = (\overline{\partial} + P_n)^p \varphi_k(x,\tau), \quad k = 0, \dots, m-1. \tag{87}$$

Thus F will be defined as the solution of (86) with (87). Then $u(x,t,\tau)$ can be defined by the equation (85) with (84) [5]. Using the compatibility conditions $u(x,t,\tau)$ is the solution of the problem.

Analogously the problems for the equation

$$\Big(\overline{\partial} + \frac{\partial}{\partial x_{n-1}}\, e_{n-1}\Big)(\overline{\partial} + P_n)\overline{\partial} u(x, x_{n-1}, x_n) = 0$$

can be considered, where $\overline{\partial}$ is defined as (68).

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ON UNIQUE SOLVABILITY OF THE DIRICHLET PROBLEM FOR ONE CLASS OF PROPERLY ELLIPTIC EQUATIONS

A.O. Babayan
Armenian State Engineering University
Terian 105, Yerevan 375009, Armenia
hhairapet@seua.am

Abstract The paper studies the unique solvability of the Dirichlet problem for some class of higher order properly elliptic equations. The different forms of necessary and sufficient conditions rendering the corresponding problem in being uniquely solvable and some applications are found.

Keywords: higher order elliptic equations, Dirichlet problem
Mathematics Subject Classification (2000): 35J40

1. Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane with boundary $\Gamma = \{z : |z| = 1\}$. In D we consider the elliptic differential equation

$$\sum_{k=0}^{2n} A_k \frac{\partial^{2n} U}{\partial x^k \partial y^{2n-k}} = 0, \ \ (x, y) \in D, \tag{1}$$

where A_k are some complex constants, $A_0 \neq 0$. We are looking for a solution U in the class of functions 2n times continuously differentiable in D, which up to n-th order derivatives satisfy Hölder condition in $D \cup \Gamma$. On Γ we subject $U(x, y)$ to Dirichlet type boundary conditions

$$\frac{\partial^k u}{\partial r^k} \mid_\Gamma = f_k(x, y), \ \ (x, y) \in \Gamma, \ \ k = 0, 1, ..., n-1, \tag{2}$$

where $f_k(x, y) \in C^{(n-1-k,\alpha)}(\Gamma)$, and $\partial/\partial r, d/ds$ stand for the derivatives with respect to radius and arc length of Γ respectively. We suppose that

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the roots λ_k, $k = 1, ..., 2n$, of the characteristic equation

$$\sum_{k=0}^{2n} A_k \lambda^{2n-k} = 0, \tag{3}$$

are distinct, and satisfy

$$\text{Im } \lambda_i > 0,\ i = 1, \ldots, n,\ \text{Im } \lambda_i < 0,\ i = n+1, ..., 2n, \tag{4}$$

(the last condition means, that equation (1) is properly elliptic). The problem (1), (2) is Fredholmian, see [1]. In this paper we derive necessary and sufficient conditions, rendering the problem (1), (2) in being uniquely solvable. If $n = 1$ the unique solvability of the problem (1), (2) takes place for arbitrary roots λ_i (which satisfy (4)), see [2]. In the case $n = 2$ the necessary and sufficient conditions of the unique solvability were found in [3] and [4] (for simple and multiple roots).

We rewrite the equation (1) and boundary conditions (2) in complex form, setting

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The equation (1) is transformed to

$$\prod_{k=1}^{n}\left(\frac{\partial}{\partial \overline{z}} - \mu_k \frac{\partial}{\partial z}\right)\prod_{k=1}^{n}\left(\frac{\partial}{\partial z} - \nu_k \frac{\partial}{\partial \overline{z}}\right) U = 0, \tag{5}$$

where

$$\mu_k = -\frac{\lambda_k - i}{\lambda_k + i},\ \nu_k = -\frac{\lambda_{n+k} + i}{\lambda_{n+k} - i}, k = 1, 2, .., n.$$

Observe, that (4) implies $|\mu_k| < 1, |\nu_k| < 1, k = 1, 2, ..., n$. By using the equalities

$$z\frac{\partial}{\partial z} = \frac{1}{2}\left(r\frac{\partial}{\partial r} - i\frac{\partial}{\partial s}\right),\ \overline{z}\frac{\partial}{\partial\ \overline{z}} = \frac{1}{2}\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial s}\right),$$

which are valid for all $z = x + iy = r(\cos s + i \sin s)$, the boundary conditions (2) are transformed to

$$\frac{\partial^{n-1} U}{\partial z^k \partial \overline{z}^{n-1-k}} \,|_\Gamma = F_{n-1-k,k}(x, y) \equiv F_k(x, y),\ (x, y) \in D, \tag{6}$$

$$\frac{\partial^{i+k} U}{\partial z^i \partial \overline{z}^k}(1, 0) = F_{ik}(1, 0),\ 0 \le i + k \le n - 1. \tag{7}$$

The functions $F_{i,k}(x,y)$ are uniquely determined by the boundary functions f_k (for example $F_{10}(x,y) = ((x-iy)/z)(f_1 - idf_0/ds))$.

In the theorem that follows

$$A = \begin{pmatrix} \mu_1^{n-1} & \mu_2^{n-1} & ... & \mu_n^{n-1} \\ \mu_1^{n-2} & \mu_2^{n-2} & ... & \mu_n^{n-2} \\ ... & ... & ... & ...... \\ 1 & 1 & ... & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & ... & 1 \\ \nu_1 & \nu_2 & ... & \nu_n \\ ... & ... & ... & ... \\ \nu_1^{n-1} & \nu_2^{n-1} & ... & \nu_n^{n-1} \end{pmatrix},$$

$$M = diag(\mu_1, \ldots, \mu_n), \qquad N = diag(\nu_1, \ldots, \nu_n).$$

Theorem *The problem* (1) *and* (2) *is uniquely solvable if and only if the matrix*

$$\Omega_l = \begin{pmatrix} A & BN^l \\ AM^l & B \end{pmatrix} \tag{8}$$

is non-singular for $l = n+1, n+2, ...$, *that is*

$$\Delta_l = \det \Omega_l \neq 0, \;\; l = n+1, n+2, ... \tag{9}$$

If for any $k_0 \geq n+1 \quad \Delta_{k_0} = 0$, *then the homogeneous problem (1),(2), (when* $f_k \equiv 0$ *for all* $k = 0, 1, ..., n-1$*) has a nontrivial solution, which is a polynomial of degree* $n + k_0 - 1$.

Remark 1 According to our assumptions μ_k and ν_k are distinct numbers, which satisfy $|\mu_k| < 1, |\nu_k| < 1$. Hence, when $l \longrightarrow \infty$, we have $\Delta_l \longrightarrow \det A \det B \neq 0$, that is the condition (9) is satisfied for sufficiently large l.

The condition (9) may be reduced to another form.

Proposition *The condition* (9) *holds if and only if the matrix*

$$K_l = \begin{pmatrix} \dfrac{1-(\nu_1\mu_1)^l}{1-\nu_1\mu_1} & ... & \dfrac{1-(\nu_n\mu_1)^l}{1-\nu_n\mu_1} \\ ... & ... & ... \\ \dfrac{1-(\nu_1\mu_n)^l}{1-\nu_1\mu_n} & ... & \dfrac{1-(\nu_n\mu_n)^l}{1-\nu_n\mu_n} \end{pmatrix} \tag{10}$$

is non-singular for all $l = n+1, n+2, ...$.

Remark 2 Denoting $C_p^l = ((\mu_i\nu_j)^p), \; p = 0, 1, ..., l-1$, the matrix K_l may be represented in the form

$$K_l = \sum_{j=0}^{l-1} C_p^l = \sum_{j=0}^{l-1} \begin{pmatrix} \mu_1^j \\ \vdots \\ \mu_n^j \end{pmatrix} \left(\nu_1^j \nu_2^j ... \nu_n^j \right). \tag{11}$$

In some cases this representation helps to check non-singularity of the K_l. For example, if the coefficients of the equation (1) are real, we have $\overline{\mu}_k = \nu_k$ for $k = 1, ..., n$, and therefore for arbitrary nonzero vector $X = (x_1, ..., x_n)^T$

$$(K_l X, X) = \overline{X}^T K_l X = \sum_{j=0}^{l-1} (\overline{x}_1 ... \overline{x}_n) \begin{pmatrix} \mu_1^j \\ \vdots \\ \mu_n^j \end{pmatrix} \left(\nu_1^j \nu_2^j ... \nu_n^j\right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Hence,

$$(K_l X, X) = \sum_{j=0}^{l-1} \left| \mu_1^j \overline{x}_1 + \mu_2^j \overline{x}_2 + ... + \mu_n^j \overline{x}_n \right|^2 > 0$$

(the strict inequality is valid because $l \geq n+1$). Thus, $K_l X \neq 0$ for arbitrary nonzero vector X, that is the matrix K_l is non-singular. Therefore the problem (1), (2) for the properly elliptic equation (1) with real coefficients is uniquely solvable.

In the proof of Theorem we will use representations of analytic functions in a neighborhood of the unit circle. Let μ and ν be complex numbers $|\mu| < 1$ and $|\nu| < 1$. Denote by $D(\mu)$ and $D(\nu)$ the images of the unit disk D under the mappings $\varsigma = z + \mu\overline{z}$ and $\varsigma = \overline{z} + \nu z$ respectively. It is known (see [1]) that for $|z| = 1$ the functions $\varphi(z)$ and $\psi(z)$ admit the representations

$$\varphi(z + \mu\overline{z}) = \omega(z) + \omega(\mu\overline{z}), \quad \psi(\overline{z} + \nu z) = \rho(\overline{z}) + \rho(\nu z), \quad |z| = 1. \quad (12)$$

Here $|z| = 1$ and $\omega(z)$, $\rho(z)$ are analytic functions in the unit disk.

In D we have

$$\varphi(z + \mu\overline{z}) = \omega\left(\frac{z + \mu\overline{z} + \sqrt{(z + \mu\overline{z})^2 - 4\mu}}{2}\right)$$
$$+\omega\left(\frac{z + \mu\overline{z} - \sqrt{(z + \mu\overline{z})^2 - 4\mu}}{2}\right),$$
$$\psi(z + \mu\overline{z}) = \rho\left(\frac{1}{2}(\overline{z} + \nu z + \sqrt{(\overline{z} + \nu z)^2 - 4\mu})\right)$$
$$+\rho\left(\frac{1}{2}(\overline{z} + \nu z - \sqrt{(\overline{z} + \nu z)^2 - 4\mu})\right), \quad |z| < 1. \quad (13)$$

Here we take the branch of $\sqrt{\varsigma^2 - 4\mu}$, which is continuous outside the segment $\left[-2\sqrt{\mu}, 2\sqrt{\mu}\right]$ and satisfies the condition

$$\varsigma^{-1}\sqrt{\varsigma^2 - 4\mu} \longrightarrow 1, \ |\varsigma| \longrightarrow \infty.$$

In our proof we will use as well the following

Lemma *Let $P_l(z, \overline{z})$ be a polynomial of degree l. If the polynomial P_l on Γ satisfies the conditions*

$$\frac{\partial^k P_l}{\partial r^k} \mid_\Gamma = 0, \quad k = 0, 1, ... n - 1, \tag{14}$$

then this polynomial admits such a representation

$$P_l(z, z) = (z\overline{z} - 1)^n Q_{l-2n}(z, \overline{z}),$$

when $l \geq 2n$. In the case $l < 2n$ we have $P_l(z, \overline{z}) \equiv 0$.

Proof of Theorem The general solution of (1) can be represented in the form

$$u = \sum_{k=1}^{n} \Phi_k(z + \mu_k \overline{z}) + \sum_{k=1}^{n} \Psi_k(\overline{z} + \nu_k z), \tag{15}$$

where Φ_k and Ψ_k are analytic and continuous up to the boundaries in $D(\mu_k)$ and $D(\nu_k)$ respectively $(k = 1, ..., n)$. Substituting (15) into the boundary conditions (6) we get

$$\sum_{j=1}^{n} \mu_j^{n-k-1} \Phi_j^{(n-1)}(z + \mu_j \overline{z}) + \sum_{j=1}^{n} \nu_j^k \Psi_j^{(n-1)}(\overline{z} + \nu_j z) = F_k(x, y) \tag{16}$$

where $z = x + iy, (x, y) \in \Gamma, k = 0, 1, \ldots, n - 1$.

For $z \in \Gamma$, as in (12) the functions $\Phi_j^{(n-1)}$ and $\Psi_j^{(n-1)}$ admit the representations

$$\Phi_j^{(n-1)}(z + \mu_j \overline{z}) = \varphi_j(z) + \varphi_j(\mu_j \overline{z}), \quad \Psi_j^{(n-1)}(\overline{z} + \mu_j z) = \psi_j(\overline{z}) + \psi_j(\nu_j z), \tag{17}$$

where $\varphi_j(z)$ and $\psi_j(z)$ are analytic in D with Taylor expansions

$$\varphi_j(t) = \sum_{l=0}^{\infty} a_{jl} z^l, \quad \psi_j(t) = \sum_{l=0}^{\infty} b_{jl} z^l, \quad j = 1, \ldots, n, \tag{18}$$

a_{jl} and b_{jl} are complex constants to be determined. The functions $F_k(x, y)$ satisfy the Hölder condition on Γ, hence they posses uniformly convergent Fourier series [5]

$$F_k(x,y) = \sum_{l=0}^{\infty} d_{lk} z^l + \sum_{l=1}^{\infty} d_{-lk} \overline{z}^l, \quad k = 0,1,\dots,n-1. \tag{19}$$

Substituting (17), (18) and (19) into (16) and equating the coefficient of z and $\overline{z}$ we obtain the following linear system for a_{jl} and b_{jl}:

$$\begin{cases} \sum_{j=1}^{n} \mu_j^{n-k-1} a_{jl} + \sum_{j=1}^{n} \nu_j^{k+l} b_{jl} = d_{ek}, \\ \qquad\qquad\qquad\qquad k = 0,1,\dots,n-1,\ l = 0,1,\dots\ . \\ \sum_{j=1}^{n} \mu_j^{n-k-1+l} a_{jl} + \sum_{j=1}^{n} \nu_j^{k} b_{jl} = d_{-ek}, \end{cases} \tag{20}$$

Knowing the coefficients a_{jl}, b_{jl}, we can obtain the solution of the problem (1), (2), using (19), (13), (18) and (15). We consider the homogeneous problem (1), (2) when the conditions (9) are valid. The coefficient matrix of the system (20) is Ω_l, which is non-singular if $l \geq n$ (when $l = n$, Ω_n is a Vandermonde matrix with distinct terms, and when $l > n$, $det\Omega_l \neq 0$ in view of conditions (9)). Therefore all solutions of the homogeneous problem (1), (2) are polynomials of order less than $2n-1$. But from Lemma we have, that the order of the non-trivial polynomial, which satisfies the homogeneous condition (2), must be greater than $2n$. Hence the homogeneous problem (1), (2) has only the trivial solution. The problem (1), (2) is Fredholmian, that is the conditions (9) imply the unique solvability of (1), (2).

To prove necessity of these conditions, observe that if $\Delta_k = 0$ then the corresponding homogeneous system (20) has a nontrivial solution a_{jk}, b_{jk}, $\quad j = 1,...,n$, which generates the one linearly independent solution $u_k(z,\overline{z})$ of the homogeneous problem (1), (2). Observe that u_k is a polynomial of z and $\overline{z}$ of order $n+k-1$. Thus in this case problem (1), (2) is not uniquely solvable. This completes the proof of Theorem.

To prove Proposition, observe that the equality

$$\begin{pmatrix} I_n & 0 \\ -M^l A^{-1} & A^{-1} \end{pmatrix} \Omega_l = \begin{pmatrix} A & BN^l \\ 0 & A^{-1}B - M^l A^{-1} B N^l \end{pmatrix}$$

implies, that condition (9) is satisfied if and only if

$$det(A^{-1}B - M^l A^{-1} B N^l) \neq 0,\ l = n+1, n+2, \dots\ .$$

Using the result of problem (10) (part 7) from [6] we can explicitly calculate $A^{-1}B$ and after that verify, that $det(A^{-1}B - M^l A^{-1} B N^l)$ differs from $detK_l$ only by a nonzero factor, i.e. conditions (9) and (10) are equivalent.

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DIRICHLET PROBLEMS WITH NON-SMOOTH BOUNDARY

A. Okay Çelebi
Middle East Technical University,Department of Mathematics
06531 Ankara, Turkey
celebi@metu.edu.tr

Abstract In this paper, we give a survey on the solubility of the Dirichlet problems without imposing restrictions on the boundaries of the domains.

Keywords: elliptic equations, Dirichlet problem, nonsmooth boundary, balayage method, generalized solution in Wiener sense, generalized Cauchy-Riemann system

Mathematics Subject Classification (2000): 35J25, 35J70

1. Introduction

In this article we want to give a directed survey related the questions concerning the solubility of the Dirichlet problem without imposing restrictions on the boundary. Our starting point will be Wiener's works [33, 34]. But we do not dare to attempt to review all the publications by all their aspects while there exist very good and comprehensive surveys by Keldysh [11], Landis[17], Kondrat'ev and Landis [13] and Maly and Ziemer [21].

2. Dirichlet problem for Laplace equation

If $\Omega \subset \mathbb{R}^n$ is bounded and $f \in C(\partial\Omega)$, the Dirichlet problem

$$\triangle u = 0 \quad \text{in} \quad \Omega \tag{1}$$

$$u = f \quad \text{on} \quad \partial\Omega \tag{2}$$

has not always a solution in the classical sense for every domain Ω. One of the reasons the problem (1)-(2) fails to be soluble is that the harmonic function may not assume the prescribed boundary values at some points

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on $\partial\Omega$. However, we can construct a function u using the method of *balayage*, proposed by Poincaré (see for example [8], page 102). The statement of the theorem below, which exploits the method of balayage, is due to Wiener [33].

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded set, containing no points of its boundary. Let $\partial\Omega$ be its boundary. Let f be any function continuous on the set of points $\partial\Omega$. Then*

(i) *There is a function F continuous over $\overline{\Omega}$ and reducing to f on $\partial\Omega$;*

(ii) *There is a sequence $\{\partial\Omega_n\}$ of boundaries over the interior of which the Dirichlet problem (1)-(2) is soluble for any continuous boundary data, each contained in the interior of its successor, and together containing in their interiors every point of Ω;*

(iii) *The function u_n corresponding to the boundary conditions*

$$u_n(x) = F(x), \quad x \in \partial\Omega_n,$$

and harmonic in Ω_n, tends uniformly to a harmonic limit over any closed set of points interior to Ω as n increases without limit;

iv) *This limit, which we shall term $u_f(x)$, is independent of the choice of F and $\{\partial\Omega_n\}$.*

The limiting function u_f is called the *generalized solution in Wiener sense* of the boundary value problem (1)-(2). This must hold for every function f continuous on $\partial\Omega$.

The solubility of the Dirichlet problem in the classical sense for a given domain depends on the properties of the boundary, and may be reduced to the investigation of the "regularity" [34] of the individual points of the boundary.

Definition 1 *Let $x^0 \in \partial\Omega$ be a fixed point and u_f be the generalized solution of the boundary value problem (1)-(2) in Wiener sense. If*

$$\lim_{x \to x^0} u_f(x) = f(x^0), \quad x \in \Omega$$

is satisfied for every continuous function f, then x^0 is a regular point *of $\partial\Omega$.*

Many authors, among which we may mention Poincaré, Zaremba, Raynor, Lebesgue and Phillips, have given conditions for the regularity of a point on the boundary using different points of view. Wiener [34] has developed a complete necessary and sufficient characterization of regular points for the Dirichlet problem (1)-(2), using the concept of capacity. To define the *capacity*, let us start with a compact subset K of $\mathbb{R}^n$. For any nonnegative Borel measure μ on K, we introduce its potential by

$$u_\mu(x) = \int_K S(x-y)d\mu(y)$$

where $S(x)$ is a fundamental solution for the Laplacian in $\mathbb{R}^n$. We define the capacity of the compact set K by

$$Cap(K) = \sup\{\mu(K) : u_\mu(x) \le 1 \quad \text{for all} \quad x \in \mathbb{R}^n\}.$$

Capacity can be extended from compact sets to more general sets by the standard extension procedure.

Now we can state the *Wiener criterion.*

Theorem 2 [34] *Let $\Omega \subset \mathbb{R}^n, n \ge 3$, be an open set and $x^0 \in \partial\Omega$. Let λ be a positive quantity less than 1, and*

$$\gamma_m := Cap(\{x : x \in \mathbb{R}^n \setminus \Omega, \lambda^m \le |x - x^0| \le \lambda^{m-1}\}].$$

Then x^0 is regular or irregular according to

$$\sum_{m=1}^{\infty} \lambda^{-m(n-2)}\gamma_m$$

diverges or converges.

For domains in $\mathbb{R}^2$, a variant of the above result is valid.

Theorem 3 [34] *Let $\Omega \subset \mathbb{R}^2$, be an open set and $x^0 \in \partial\Omega$. Let λ be any positive quantity less than 1, and*

$$\gamma_m := Cap(\{x : x \in \mathbb{R}^2 \setminus \Omega, \lambda^{2^m} \le |x - x^0| \le \lambda^{2^{m-1}}\}].$$

Then x^0 is regular or irregular according to

$$\sum_{m=1}^{\infty} 2^m \gamma_m$$

diverges or converges.

The application of the Wiener's criterion shows that the point $O = (0, \cdots, 0) \in \partial\Omega$ is irregular if we take Ω whose complement has the form

$$\{x : 0 \le x_n \le e^{-(x_1^2 + \cdots + x_{n-1}^2)^{-1}}\}$$

near the origin.

Wiener's criterion gives a complete answer to the question of what condition have to be satisfied by a domain, in order that the Dirichlet problem (1)-(2) be soluble for arbitrary continuous boundary data. If the boundary of the domain contains irregular points, then we should determine the conditions on the boundary values so that the Dirichlet problem will be soluble. Thus a more detailed study of generalized solutions of the Dirichlet problems in the neighbourhood of an irregular point is necessary. The following theorem is important for the structure of the set of irregular points [8].

Theorem [Kellogg] *For any bounded region $\Omega \subset \mathbb{R}^n$, the set of irregular points of $\partial\Omega$ has capacity zero.*

Now we will extend the above results to second order elliptic equations and start with the Dirichlet problems for elliptic equations of non-divergence form.

3. Elliptic equations of nondivergence form

Let us consider the operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D_{x_i} D_{x_j} + \sum_{i=1}^{n} b_i(x) D_{x_i} + c(x) \tag{3}$$

in a domain $\Omega \subset \mathbb{R}^n$ where all the coefficients are measurable functions, the matrix (a_{ij}) is symmetric,

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \tag{4}$$

$$|b_i(x)| \leq m, \quad c(x) \leq 0. \tag{5}$$

A solution u of

$$Lu = 0 \tag{6}$$

in Ω will be assumed either in $C^2_{loc}(\Omega)$ or $W^{2,p}_{loc}(\Omega)$. We will utilize the following two definitions in the sequel.

Definition 2 *A function f is said to satisfy* Dini condition *in Ω if*

$$|f(x) - f(y)| \leq \varphi(|x-y|) \quad \text{where} \quad \int_{0^+} \frac{\varphi(r)}{r} dr < \infty, \quad \text{for all} \quad x, y \in \Omega.$$

Definition 3 *We say that $(a_{ij}(x))$ satisfies the* Cordes condition *if the spread of the characteristic roots is small.*

Let us assume that $f \in C(\Omega)$. If in a neigbourhood of any point $x \in \Omega$, the coefficients of L satisfies a Dini condition and there exists a solution of the Dirichlet problem in that neighbourhood, then the generalized Wiener solution u_f can be constructed. Generalized solution may also be obtained [5] if the matrix $(a_{ij}(x))$ satisfies the Cordes condition which is less restrictive than Dini condition.

In the following years, one of the aims of researchers in investigating the solubility of the Dirichlet problems involving regular boundary points in Wiener sense was to try to determine the minimal requirements on the coefficients of L such that the regularity condition for (6) resembles to, or coincides with, those for $\triangle u = 0$. The first essential result in this direction was obtained by Hervé [10] in 1962, which states that if the coefficients of (3) satisfy a Hölder condition, then the necessary and sufficient solubility condition to the Dirichlet problem defined for (6) is the same as the Wiener's criterion for the Dirichlet problem for harmonic functions. Five years later Krylov [16] has derived that the uniform fulfilment of the Dini condition is enough to get the same conclusion.

The Wiener type criterion for the regularity of boundary points with respect to the Dirichlet problem for second order elliptic linear equations has attracted many researchers; among them we may mention Littman-Stampacchia-Weinberger [20] (in which the remarkable result, stating that a boundary point is regular for a divergence type elliptic operator with bounded or measurable coefficients, if and only if it is regular for the Laplacian, is obtained in $H^1_{2loc}(\Omega)$), Keldysh [11], Oleinik [30] and Krylov [16]. In such investigations the authors impose some restrictions on the differential equations and their coefficients, to find that a regularity condition for the Dirichlet problem considered, coincides with or resembles to the one for Laplace's equation. We will consider two examples. The first of them is the article by Alkhutov [2]. He studies the Dirichlet problem for (4)-(6). The class of functions he has considered is such that the so-called "ellipticity function" satisfies the Dini condition. We should note that, to write down the ellipticity function, the inverse $(A_{ij}(x))$ of the matrix $(a_{ij}(x))$ is needed. The *elliptic distance* R is defined by

$$R^2 = \sum_{i,j=1}^{n} A_{ij}(y)(x_i - y_i)(x_j - y_j).$$

If

$$\gamma_i := (\sum_{j=1}^{n} A_{ij}(y)(x_j - y_j))/R$$

then

$$e_y(x) := \begin{cases} \dfrac{\sum_{i,j=1}^{n} a_{ij}(x)A_{ij}(y)}{\sum_{i,j=1}^{n} a_{ij}(x)\gamma_i\gamma_j} & \text{if} \quad x \neq y, \\ n & \text{if} \quad x = y, \end{cases}$$

is the *ellipticity function* for the differential operator (3). Thus if $e_y(x)$ satisfies the Dini condition, then he obtains that, the necessary and sufficient condition for the point $x^0 \in \partial\Omega$ being Wiener regular with respect to the Dirichlet problem for (4)-(6), is similar to the Wiener criterion for Laplace equation.

The second example we will take is due to Bauman [3]. In the operator L defined by (3), she assumes that the leading coefficients $a_{ij} = a_{ji}$ are bounded for $n = 2$ or continuous for $n \geq 3$ on $\overline{\Omega} \subset \mathbb{R}^n$. It is proved that a point $x^0 \in \partial\Omega$ is regular if and only if a criterion similar to Wiener is satisfied. The criterion obtained is reduced to the classical one if the leading coefficients are Hölder continuous.

4. Quasi-linear elliptic equations

The results obtained for linear elliptic equations have been extended to the quasi-linear equation

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x,u)u_{x_i x_j} + b(x,u,\bigtriangledown u) = 0 \tag{7}$$

in the domain Ω. Novruzov [24] considered the equation (7) with linear principal part. He proved that if the coefficients of the principal part satisfy a Dini condition, then the necessary and sufficient condition for the regularity of boundary points with respect to the Dirichlet problem defined for (7) are similar to the Wiener criterion for Laplace equation.

Afterwards, Novruzov [26] have considered the quasi-linear elliptic equation (7) with the coefficients satisfying

$$|a_{ij}(x,z_1) - a_{ij}(y,z_2)| \leq c_1[|x-y| + |z_1 - z_2|]$$

for all $z_1, z_2 \in [-\gamma, \gamma]$ and $x, y \in \Omega$

$$|b(x,z,\eta)| + |bx_i| + |bz| + ((1+|\eta||b\eta_i|) \leq c_2(1+|\eta|)^2$$

where $\eta = (\eta_1, \eta_2, \cdots, \eta_n), -\infty < \eta_i < \infty$ and γ, c_1, c_2 are positive constants. Then, he has proved that the point $x^0 \in \partial\Omega$ is regular for the Dirichlet problem

$$Lu = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = f,$$

if and only if the series

$$\sum_{m=1}^{\infty} 4^{m(n-2)}\gamma_m$$

diverges, where $f \in C(\partial\Omega)$, $|f| \leq \gamma$ and

$$\gamma_m = Cap(B(x^0, R) \setminus D).$$

5. Elliptic equations with discontinuous coefficients

In the case that the coefficients of the highest derivatives in the differential equation (6) are Dini continuous, then the conditions for the boundary points to be regular are the same as for the Laplace's equation [10, 30, 15]. But Landis [17] has given an example to demonstrate that the above statement is not true in the general case. To overcome such a difficulty, Landis defines the s-capacity.

Definition 4 [17] *Let s be a positive number, E be a B-set of n-dimensional Euclidean space. Let $\mathcal{M}$ be the set of measures μ defined on the subsets of E such that*

$$\int_E \frac{d\mu(y)}{|x-y|^s} \leq 1 \quad \text{for} \quad x \notin E.$$

Then the number

$$Cap_s(E) := \sup_{\mu \in \mathcal{M}} \mu(E)$$

is called the s-capacity *of the set E.*

The number $s > 0$ can be chosen so that $L(\mid x - x^0 \mid^{-s}) \geq 0$ where $x^0 \in \mathbb{R}^n$ is a fixed point, x is a point of the domain Ω and $\mid x - x^0 \mid < r_0$. Then, introducing a new definition for regularity of the points on the boundary $\partial\Omega$, Landis [17] proves that the point $x^0 \in \partial\Omega$ is regular if

$$\sum_{m=1}^{\infty} 2^{ms}\gamma_m = \infty,$$

where $\gamma_m = Cap_s(B(x^0, 2^{-m}) \cap H))$, and H denotes the complement of the domain Ω of the differential equation.

6. Degenerate elliptic equations

The Dirichlet problem for second order linear elliptic equations with degenerate principal part on the boundary of the domain has also been studied. The behavior of solutions near a boundary point of degenerate elliptic equations with continuous or discontinuous coefficients have been considered in detail by Novruzov [25, 28], Novruzov and Mamedov [29], Krupskaya [14] and Mamedov [22]. To give an example for the results obtained in this direction, we will take the article by Novruzov [27]. He has started with the Dirichlet problem

$$\varphi(r)\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = 0 \quad \text{in} \quad \Omega \tag{8}$$

and

$$u = f \quad \text{on} \quad \partial\Omega \tag{9}$$

where the coefficients of (8) satisfy the conditions

$$\alpha \mid \xi \mid^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \beta \mid \xi \mid^2, \tag{10}$$

$$\varphi(0) = 0, \quad \int_0 \frac{dt}{\varphi(t)} < \infty, \tag{11}$$

$$\mid b_i(x) \mid \leq b_0, -b_0 \leq c(x) \leq 0, \tag{12}$$

where the $a_{ij}(x)$ satisfy the Dini condition, $\varphi(t)$ is a positive increasing function of $t, 0 < t < t^0; \alpha, \beta, b_0$ are positive constants, $r = |x - x^0|$, $x^0 \in \partial\Omega$. The standard definition of the regularity of the point $x^0 \in \partial\Omega$, with respect to the Dirichlet problem (8)-(9) is assumed.Then it is proved that $x^0 \in \partial\Omega$ is a Wiener regular point if and only if the series

$$\sum_{m=1}^{\infty} 2^{m(n-2)}\gamma_m$$

is divergent where

$$\gamma_m = Cap(B(x^0, 2^{2-m}) \setminus D)$$

He also provides an example to show that the condition (11) is essential.

In the same period of time E. Fabes, D. Jerison and C. Kenig have published three consecutive articles. In one of them, [9], they have considered the degenerate elliptic equation

$$Lu = -\sum_{i,j=1}^{n} \partial_j(a_{ij}(x)\partial_j u(x)) = 0.$$

The coefficients a_{ij} are assumed to be real-valued, measurable, symmetric and to satisfy

$$c^{-1} \mid \xi \mid^2 w(x) \leq a_{ij}(x)\xi_i\xi_j \leq c \mid \xi \mid^2 w(x)$$

for all x and ξ in $\mathbb{R}^n$. The article contains two main results. The first of them is an approximate formula for the Green function of L. The second main result is a Wiener test, in which the capacity is related with the weight $w(x)$. As a consequence, they obtain that the set of regular points of $\partial\Omega$ depends only on w, not on the particular operator L.

7. Some recent results

Wiener-type tests for the Dirichlet problem for nonlinear elliptic differential equations are still attracting researchers. Now we want to present some of the recent results.

One of the essential questions was whether regular boundary points of Ω for the equation

$$div A(x, \bigtriangledown u) = 0$$

can be characterized by using Wiener test, where $u \in W^{1,p}_{loc}(\Omega)$. Naturally for nonlinear operators the classical Wiener test has to be modified so that the type p of the differential operator is involved. Maz'ya [23] established that the boundary point x^0 is regular if

$$W_p(\mathbb{R}^n \setminus \Omega, x^0) = +\infty$$

where

$$W_p(E, x^0) = \int_0^{\prime} \left(\frac{Cap_p(B(x^0,t) \cap E, B(x^0, 2t))}{Cap_p(B(x^0,t), B(x^0,2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t}$$

and $Cap_p(E, G)$ is the *p-capacity* [23]of the set E in G.

We assume that $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the following assumptions for some constants $0 < \alpha \leq \beta < \infty$:

(i) the function $x \mapsto A(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$,

(ii)the function $\xi \mapsto A(x, \xi)$ is continuous for a.e. $x \in \mathbb{R}^n$,

(iii) $A(x, \xi).\xi \geq \alpha \mid \xi \mid^p$ for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$,

(iv) $\mid A(x, \xi) \mid \leq \beta \mid \xi \mid^{p-1}$ for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$,

(v) $(A(x, \xi) - A(x, \zeta)).(\xi - \zeta) > 0$ whenever $\xi \neq \zeta$,

and

(vi) $A(x, \lambda\xi) = \lambda \mid \lambda \mid^{p-2} A(x, \xi)$ for all $\lambda \in \mathbb{R}, \lambda \neq 0$.

When $p \geq n$ for a.e. $x \in \Omega$, Lindqvist and Martio [19] have proved that, if $f \in W^{1,n}(\Omega)$ and u is an F- extremal which assumes the boundary values weakly (i.e. $u - f \in W_0^{1,n}(\Omega)$), then

$$\lim_{x \to x^0} u(x) = f(x^0), \quad x^0 \in \partial\Omega,$$

for all continuous functions f, if and only if the Wiener condition

$$\int_0 \varphi(t)^{\frac{1}{n-1}} t^{-1} dt = \infty$$

holds where

$$\varphi(t) = Cap(B(x^0, 2t), \mathbb{R}^n \setminus \Omega \cap \overline{B}(x^0, t)).$$

For this particular problem the characterization of the regular boundary points of Ω has been completed by Kilpeläinen and Malý [12]. They have proved that a finite boundary point $x^0 \in \partial\Omega$ is regular if and only if

$$W_p(\mathbb{R}^n \setminus \Omega, x^0) = \infty$$

where $p \in (1, n]$.

The problem

$$\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}\partial_{x_i}u) = f(x, u, Du), \quad (u - \psi) \in H_0^1(\Omega),$$

is considered by several authors, where $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$, the coefficients a_{ij} are measurable functions and $f(x, z, q)$ is a continuous function in (z, q) for each x, measurable in x for each fixed (z, q), with $z \in \mathbb{R}, q \in \mathbb{R}^n$. Assuming that $f(x, z, q) < B + C \mid q \mid^2$ a.e. in Ω, Biroli [4] showed that the Wiener criterion for the regularity of $x^0 \in \partial\Omega$ with respect to the Dirichlet problem for the Laplace equation holds also in this nonlinear case. This extension is derived by reduction of the equation into a linear one. For the same equation Adams and Heard [1] showed that the classical Wiener test is satisfied at each regular boundary point if f is at most quadratic in $\bigtriangledown u$.

The solubility of the boundary value problem with non-smooth boundary has also been extended by the author and K. Koca, to the class of generalized analytic functions in Vekua sense [31, 32]. In this extension we have investigated the conditions under which the solution of the boundary value problem

$$\begin{gathered} w_{\overline{z}} = Aw + B\overline{w} \quad \text{in} \quad \Omega, \\ Rew \mid_{\partial\Omega} = \varphi, \quad \varphi \in C(\partial\Omega), \\ Imw(z_0) = c, \quad z_0 \in \overline{\Omega}, \end{gathered} \tag{13}$$

defined on a Wiener-type domain $\Omega \subset \mathbb{C}$ exists in the class of generalized analytic functions [6]. The corresponding inhomogeneous Cauchy-Riemann system is

$$\begin{aligned} u_x - v_y &= a(x,y)u + b(x,y)v, \\ u_y + v_x &= c(x,y)u + d(x,y)v, \end{aligned} \tag{14}$$

where $w = u + iv$. Eliminating v from (14) we find the elliptic equation

$$Lu := \triangle u + p(z)u_x + q(z)u_y + k(z)u = 0.$$

We assume that the coefficients of this equation satisfy

$$\mid p(z) \mid, \mid q(z) \mid \quad \leq \quad \frac{c_1}{r\log(2\gamma/r)}$$

and

$$-\frac{c_2}{(r\log(2\gamma/r))^2} \leq k(z) \leq 0, \quad c_2 > 0,$$

where

$$r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}, \quad (x,y) \in \Omega, \quad (\xi,\eta) \notin \Omega,$$

and γ is a positive constant. Thus (13) gives us the Dirichlet problem

$$\begin{aligned} Lu(z) &= 0, \qquad z \in \Omega, \\ u(z) &= \varphi(z), \qquad z \in \partial\Omega, \end{aligned} \tag{15}$$

for the real part of $w(z)$. First, we have proved that every positive solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$Lu = 0, \quad \text{in} \;\; \Omega,$$

$$u = 0 \quad \text{on} \;\; \Gamma_1 \subset \partial\Omega$$

satisfies

$$\sup_{z\in\Omega} u(z) \geq [1 + C_3 Cap_{(L,s)}(E_R)] \sup_{z\in\Omega\cap B_R(z_0)} u(z),$$

where $Cap_{(L,s)}(E_R)$ is the logarithmic s-capacity of $E_R := B_R(z_0) \setminus \Omega \neq \phi$.

Secondly, we have proved that if

$$\sum_{m=1}^{\infty} Cap_{(L,s)}(E_{4^{-m}})$$

is divergent, then $z_0 \in \partial\Omega$ is ψ*-regular* [6], which means that the Dirichlet problem defined by (15) has a solution $u \in C^\alpha(\Omega)$ in a Wiener-type domain. Substituting u in (14) gives an exact differential equation for v. Hence v may be determined uniquely imposing the condition

$$Imw(z_0) = v(z_0) = c.$$

Thus the existence of the solution of the boundary value problem (13) is obtained as

$$w(x,y) = u(x,y) + iv(x,y).$$

It is well-know that the solution of the boundary value problem (13) has the representation

$$w(z) = \phi(z) + T_\Omega(Aw + B\overline{w})(z) \tag{16}$$

by use of the operator T_Ω for the domains Ω with smooth boundary. To extend this result to domains with non-smooth boundary, we employ the method of balayage. Thus we first define the boundary value problems

$$\frac{\partial w_m}{\partial \overline{z}} = Aw_m + B\overline{w}_m, \quad z \in \Omega_m,$$

$$\text{Re } w_m(z) = \phi_\Omega(z) \mid_{\partial\Omega_m} := \phi_{om}(z), z \in \partial\Omega_m,$$

$$\text{Im } w_m(z_0) = c_0, z_0 \in \Omega,$$

where $\phi_\Omega(z)$ is the Hölder-continuous extension of the boundary value φ into the domain Ω, $\phi_{om}(z) \in C^\alpha(\partial\Omega_m)$ and

$$\overline{\Omega}_m \subset \Omega_{m+1}, \quad \overline{\Omega}_m \subset \Omega, \quad \lim_{m\to\infty} \Omega_m = \Omega.$$

Then, by (16) the solutions have the representations

$$w_m(z) = \phi_m(z) + T_{\Omega_m}(Aw_m + B\overline{w}_m)(z), m = 1, 2, \cdots, \tag{17}$$

where ϕ_m are holomorphic functions satisfying proper boundary conditions. It is easy to show that $\{w_m(z)\}_1^\infty$ is a Cauchy sequence in $C^\alpha(\Omega)$. Since $C^\alpha(\overline{\Omega})$ is complete (17) has the limit

$$w(z) = \phi(z) + T_\Omega(Aw + B\overline{w})(z), \quad z \in \Omega.$$

Thus the integral representation (16) is also valid for the Wiener-type domains Ω.

We have also extended the above results, in $C^\alpha(\overline{\Omega})$ for the boundary value problems of type

$$w_{\overline{z}} = Aw + B\overline{w} + F, \qquad z \in \Omega,$$

$$\text{Re } w = \varphi(z), \qquad z \in \partial\Omega,$$

$$\text{Im } w(z_0) = c_0, \qquad z_0 \in \overline{\Omega},$$

to the domains with non-smooth boundary, where $A, B, F \in C^\alpha(\Omega)$ and $\varphi \in C^\alpha(\partial\Omega)$, see [7].

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DIRICHLET PROBLEM IN THE HALF-PLANE FOR RO–VARYING WEIGHT FUNCTIONS

H. Hairapetian
Armenian State Ingenering University
Terian 105, Yerevan 375009, Armenia
hhairapet@seua.am

Abstract The paper considers the Dirichlet problem in the upper half-plane, in the case where the boundary functions belong to the weighted space $L^1(\rho)$, where the weight function ρ has a finite number of singularities of finite order. Assuming that $\rho(x)$ is a function *RO*–varying at the singular points, we obtain a sufficient condition for normal solvability of the problem. We show that this condition is necessary provided the order of singularity differs from 1 and belongs to $(0, 2)$. Also, an explicit formula of the general solution of the corresponding homogeneous problem is obtained.

Keywords: harmonic functions, Dirichlet problem, *RO*-varying weight functions, normal solvability, half plane

Mathematics Subject Classification (2000): 35J25

1. Let B_1 be the class of harmonic functions $u(z)$ in the upper half-plane $G^+ = \{z : \mathrm{Im} z > 0\}$, that for any $y_0 > 0$ satisfy the inequality

$$|u(z)| < C \exp |z|^\gamma, \quad \gamma < 1, \quad \mathrm{Im} z > y_0 > 0,$$

where C is a constant which generally speacking depends on y_0. We consider the Dirichlet problem in the class B_1 in the following setting.
Problem *Find a real-valued harmonic function $u(x, y) \in B_1$, satisfying the boundary condition*

$$\lim_{y \to +0} \|u(x, y) - f(x)\|_{L^1(\rho)} = 0, \tag{1}$$

where $\rho(x)$ is a measurable nonnegative function defined on the real axis.

G. A. Barsegian and H. G. W. Begehr (eds.), Topics in Analysis and its Applications, 311–316.

The problem (1) is called normally solvable if the set of functions $f(x) \in L^1(\rho)$, for which there is a solution, is a closed subset in $L^1(\rho)$. The Dirichlet problem in the unit disc, for the case of weight function having a finite number of singularities of finite order, was studied in [1]. In the same work the weight functions were described, for which the problem is solvable for any $f \in L^p(\rho(x)dx)$. The Dirichlet problem in the unit disc for weight functions *RO*–varying (see below) at singular points was considered in [2]. The problem (1) in the half-plane for the set of power weight functions of the form $O(|x - x_k|^\alpha)$, where α is a nonnegative integer, was considered in [3].

The present paper considers the problem (1) for the set of weight functions *RO*–varying at singularity points.

2. A number $x \in (-\infty, +\infty)$ is called a singularity point of a function $\rho(x)$ if there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ either $\rho(x) \notin L^\infty(x - \delta, x + \delta)$ or $\rho^{-1} \notin L^\infty(x - \delta, x + \delta)$. Also infinity will be considered as a singularity point for $\rho(x)$. Let $x_1, x_2, \ldots, x_m$ be singularity points of $\rho(x)$. Further, let U_k $(k = 1, ..., m)$ be a neighborhood of x_k, such that $x_j \notin U_k$ for $k \neq j$, and let U_0 be a neighborhood of infinity. The numbers

$$\alpha_k = \sup\{\beta; \rho(x)\left|\frac{x - x_k}{1 + |x|}\right|^{-\beta} \in L^\infty(U_k)\}, \quad k = 1, 2...m,$$

$$\alpha_0 = \sup\{\beta; \rho(x)(1 + |x|)^\beta \in L^\infty(U_0)\}$$

are called the order of singularity of $\rho(x)$ at x_k and at infinity respectively. We shall assume that $0 \leq \alpha_k < \infty$, $k = 0, 1, ..., m$.

3. Following [4], we introduce some definitions.
A function $g(x)$, defined on (A_0, ∞), is said to be *RO*–varying at infinity from the left, if $g(x)$ is representable in the form

$$g(x) = \exp\Big(g_1(x) + \int_{A_1}^{x} \frac{g_2(t)}{t} dt\Big), \quad x \in (A_0, \infty), \tag{2}$$

where $A_1 > A_0$ and $g_1(x)$, $g_2(x)$ are measurable functions bounded in (A_0, ∞).

A function $g(x)$ is called *RO*–varying from the left at a finite singularity point x_k, if $g(x)$ is representable in the form

$$g(x) = exp\Big(g_1(x) + \int_{A_1}^{x} \frac{g_2(t)}{t - x_0} dt\Big), \quad x \in (x_0 - \delta, x_0), \tag{3}$$

where $A_1 \in (x_k - \delta, x_k)$ is some number and $g_1(x)$, $g_2(x)$ are measurable functions bounded in $(x_k - \delta, x_k)$.

The class of functions RO–varying from the right is defined similarly. A function $g(x)$ is called RO–varying at a given point, if it is RO–varying at the same point both from the left and from the right.

We say that a weight function $\rho(x)$ belongs to the class R, if the function

$$\rho_1(x) = (1+|x|)^{\alpha_0} \prod_{k=1}^{m} \Big(\frac{|x-x_k|}{1+|x|}\Big)^{-\alpha_k} \rho(x)$$

is RO–varying at each singularity point of $\rho(x)$ and the function $g_2(x)$ of (3) satisfies at all finite singular points the conditions

$$\limsup_{x\to x_k} \; g_2(x) < \{\alpha_k\}, \quad \liminf_{x\to x_k} \; g_2(x) > \{\alpha_k\} - 1,$$

if α_k is a non-integer, and

$$\limsup_{x\to x_k} \; g_2\,(x) < 1, \quad \liminf_{x\to x_k} \; g_2\,(x) \geq 0,$$

if α_k is an integer, while the function $g_2(x)$ from (2) at infinity satisfies the conditions

$$\limsup_{|x|\to\infty} \; g_2(x) < 1 - \{\alpha_k\}, \quad \liminf_{|x|\to\infty} \; g_2(x) > -\{\alpha_k\},$$

if α_0 is a non-integer, and

$$\limsup_{|x|\to\infty} \; g_2\,(x) \leq 0, \quad \liminf_{|x|\to\infty} \; g_2\,(x) > -1,$$

if α_0 is an integer.

4. We say that a weight function $\rho(x)$ satisfies the condition R_0 at a finite singularity point x_k $(k = 1, 2, ...m)$, if either α_k is a non-integer, or α_k is an integer and the following relation holds:

$$\lim_{h\to 0} \frac{1}{2h} \int_{x_k-h}^{x_k+h} \rho_1(x)dx = 0.$$

We say that a weight function $\rho(x)$ satisfies the condition R_0 at infinity if either α_0 is a non-integer or α_0 is an integer and

$$\int_{-\infty}^{\infty} \frac{\rho_1(x)dx}{1+|x|} < \infty.$$

A function $g(x)$, defined on (a, b), is called *almost monotone increasing* if there exists a constant $A > 0$ such that $g(x_1) < Ag(x_2)$ for any $x_1 < x_2$ from (a, b). Almost monotone decreasing functions are defined similarly.

Lemma 1 *Let $\rho(x) \in R$. Then the following assertions hold:*

(a) *If α_0 is not an integer, then there exist some $\delta_0 \in (0, 1 - \{\alpha_0\})$ and $\delta_1 \in (-\{\alpha_0\}, 0)$ such that for any $\delta > \delta_0$ the function $|x + i|^\delta \rho_1(x)$ is almost monotone increasing in (x'_m, ∞) and almost monotone decreasing in $(-\infty, x'_0)$, and at the same time for any $\delta < \delta_1$ the function $|x + i|^{-\delta} \rho_1(x)$ is almost monotone decreasing on (x'_m, ∞) and almost monotone increasing on $(-\infty, x'_0)$.*

(b) *If α_0 is an integer, then the function $|x + i|\rho_1(x)$ is almost monotone increasing in (x'_n, ∞) and almost monotone decreasing on $(-\infty, x'_0)$, while the function $|x+i|^\delta \rho_1(x)$ is almost monotone decreasing in (x'_m, ∞) and almost monotone increasing in $(-\infty, x'_0)$ for any $\delta < 0$.*

(c) *If α_k is a not an integer, then the function $|x - x_k|^{\{\alpha_k\}} \rho_1(x)$ is almost monotone increasing in (x_k, x'_k) and almost monotone decreasing in (x'_{k-1}, x_k) while the function $|x - x_k|^{-(1-\{\alpha_k\})} \rho_1(x)$ is almost monotone decreasing in (x_k, x'_k) and almost monotone increasing in (x'_{k-1}, x_k).*

(d) *If α_k is an integer, then there exists $\delta_0 \in (0, 1)$ such that the function $|x - x_k|^\delta \rho_1(x)$ is almost monotone increasing in (x_k, x'_k) and almost monotone decreasing in (x'_{k-1}, x_k) for any $\delta > \delta_0$, while the function $|x - x_k|^{-\delta} \rho_1(x)$ is almost monotone decreasing in (x_k, x'_k) and almost monotone increasing in (x'_{k-1}, x_k) for any $\delta > 0$.*

(e) *For any $\delta > 0$*

$$\int_{-\infty}^{\infty} \frac{\rho_1(x)dx}{(1 + |x|)^{1+\delta}} < \infty.$$

Lemma 1 was used for proving the following result.

Theorem 1 *Let $\rho(x) \in R$. Then the following assertions hold:*

(a) *If the weight function $\rho(x)$ satisfies the condition R_0 at the singularity points of $\rho(x)$, then the general solution of the homogeneous problem (1) can be represented in the form*

$$u_0(x, y) = \sum_{k=1}^{m} \operatorname{Re} \sum_{p=0}^{n_k} \frac{ia_{kp}}{(z - x_k)^p} + \operatorname{Re} \sum_{p=0}^{n_0} ia_{0p} z^p, \tag{4}$$

where a_{kp} $(k = 0, 1, 2, ..., m,\ p = 0, 1, ..., n_k)$ are arbitrary real numbers.

(b) *If the weight function $\rho(x)$ does not satisfy the condition R_0 at some singularity point x_k, then the general solution of the homogeneous problem (1) again can be represented in the form (4) with $a_{kn_k} = 0$ (in the case of infinity we set $a_{0n_0} = 0$).*

5. We put $x_0^{'} = x_1 - 1$, $x_1^{'} = 2^{-1}(x_1 + x_2)$, …, $x_k^{'} = 2^{-1}(x_k + x_{k+1})$, …, $x_m^{'} = x_m + 1$ and define the functions $\rho_k^{'}(x)$, $k = 0, 1, \ldots, m$, as

$$\rho_0^{'}(x) = \begin{cases} \rho_1(x), & \text{if } \ x \in (\infty, x_0^{'}) \cup (x_m^{'}, \infty), \\ 1, & \text{if } \ x \notin (\infty, x_0^{'}) \cup (x_m^{'}, \infty), \end{cases}$$

and for $k = 1, 2, \ldots, m$

$$\rho_k^{'}(x) = \begin{cases} \rho_1(x), & \text{if } \ x \in (x_{k-1}^{'}, x_k^{'}), \\ 1, & \text{if } \ x \notin (x_{k-1}^{'}, x_k^{'}). \end{cases}$$

For an arbitrary non-negative number λ we set

$$\lambda' = \begin{cases} 1 - \{\lambda\}, & \text{if } \ \lambda \ \text{ is non-integer}, \\ 0, & \text{if } \ \lambda \ \text{ is an integer}. \end{cases}$$

Theorem 2 *Let $\rho(x) \in R$ and*

$$M(|x - x_k|^{-\alpha_k^{'}} \rho_k^{'}(x)) < C|x - x_k|^{-\alpha_k^{'}} \rho_k^{'}(x), \ \ k = 1, 2, ...m, \tag{5}$$

where $M(f)$ is the Hardy–Littlewood maximal function. Then the problem (1) is solvable for any $f(x) \in L^1(\rho)$. The general solution can be represented as

$$u(x, y) = u_0(x, y) + u_1(x, y),$$

where $u_0(x, y)$ is the general solution of the homogeneous problem and

$$u_1(x, y) = \mathrm{Re}\left(\frac{1}{4\pi i r(z)} \int_{-\infty}^{\infty} \frac{f(t) r(t) dt}{t - z} + \frac{1}{4\pi i \overline{r(\bar{z})}} \int_{-\infty}^{\infty} \frac{f(t)\overline{r(t)} dt}{t - z} \right),$$

where

$$r(z) = (z + i)^{-n_0 + 1} \prod_{k=1}^{m} \frac{(z - x_k)^{n_k^{'}}}{(z + i)^{n_k^{'}}}$$

and $n_k^{'} = n_k$ if α_k is non-integer and $n_k^{'} = n_k - 1$ if α_k is an integer.

Theorem 3 *Let the weight function $\rho(t) \in R$ have no finite singularity points. Then problem (1) is normally solvable.*

The proofs for Theorems 2 and 3 are based on the following estimate (see [2]):

$$\int_{x_{k-1}^{'}}^{x_k^{'}} \frac{y|x - x_k|^{1 - \alpha_k^{'}} \rho_1(x)}{((x - x_k)^2 + y^2)|t - x - iy|} dx < C \frac{\rho_1(t)}{|t - x_k|^{\alpha_k^{'}}}$$

and also

Lemma 2 *Let $\rho(x) \in R$. Then*

$$\sup_{t\in(-\infty,x_0')\cup(x_m',\infty)} \frac{|t+i|^{\{\alpha_0\}}}{\rho_1(t)} \int_{-\infty}^{\infty} \frac{y\rho_1(x)dx}{((x-t)^2+y^2)|x+i|^{\{\alpha_0\}}} < \infty,$$

$$\sup_{t\in(-\infty,x_0')\cup(x_m',\infty)} \frac{|t+i|^{\{\alpha_0\}}}{\rho_1(t)} \int_{-\infty}^{\infty} \frac{y\rho_1(x)dx}{|t-x-iy||x+i|^{1+\{\alpha_0\}}} < \infty.$$

6. Lemma 3 *If (5) is not true for some k, $1 \leq k \leq m$, and α_k is not an integer, then for any natural p the family of functions*

$$A_y(t) = \frac{|t-x_k|^{\alpha_k'}}{\rho_k'(t)} \int_{x_{k-1}'}^{x_k'} \frac{(x-x_k)^{2p}y}{((x-x_k)^2+y^2)^p((x-t)^2+y^2)} \frac{\rho_k'(x)}{|x-x_k|^{\alpha_k'}} dx$$

is not uniformly bounded for $y \in (x_{k-1}', x_k')$.

Theorem 4 *Let $\rho(t) \in R$. If (5) is not true for some k, $k = 1, 2, ..., m$, and $\alpha_k \in (0,1) \cup (1,2)$, then the problem (1) is not normally solvable.*

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ABOUT ONE CLASS OF VOLTERRA TYPE LINEAR INTEGRAL EQUATIONS WITH AN INTERIOR FIXED SINGULAR OR SUPER-SINGULAR POINT

Nusrat Rajabov
Math. Inst. Tajik Acad. Sci.
Rudaki Av. 33
Dushanbe 734025, Tajikistan
nusrat@ac.tajik.net

Abstract The problems to find continuous solutions for the model and nonmodel linear ordinary differential equation of first and higher order with an interior singular or super-singular point, see [12], are reduced to the consideration of special cases of linear second kind Volterra type integral equation. In this paper the general solution to this integral equation is constructed depending on the power of the singularity and the signs of the constants $K(c+0) = K(c+0, c+0), K(c-0) = K(c-0, c-0), K(c) = K(c)$ (when $K(c+0) = K(c-0) = K(c)$). In the case, when $K(x,t) = A(t)$ the solution to this integral equation is found in an obvious form.

Keywords: Volterra integral equations, singular points, explicit solutions

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1. Assumptions

Let $\Gamma_0 = \{a < x < b\}$ be an open intervall on the real axis and $c \in \Gamma_0$. In $\Gamma = \Gamma_0 \setminus \{c\}$ we shall consider the integral equation

$$\varphi(x) + \int_c^x \frac{K(x,t)}{|t-c|^\alpha}\varphi(t)dt = f(x), \quad x \in \Gamma, \tag{1}$$

where $\varphi(x)$ is an unknown function, $f(x), K(x,t)$ are given functions in Γ and the rectangle $R = \{a < x < b, a < t < b\}, \alpha = const > 0$, respectively.

G. A. Barsegian and H. G. W. Begehr (eds.), Topics in Analysis and its Applications, 317–326.

Moreover, assume that $f(x) \in C(\Gamma)$ and it may have a first kind singularity at the point $x = c$, $K(x,t)$ is continuous everywhere in R besides the segments $\Gamma_1 = \{a < x < b, t = c\}, \Gamma_2 = \{x = c, a < t < b\}$, where $K(t,t)$ has a first kind singularity. When $K(c \pm 0, c \pm 0) \neq 0$ the solution to this integral equation will be seeked in the class of functions $\varphi(x) \in C(\bar{\Gamma}_0)$, satisfying in the neighborhood of the point $x = c$ the condition

$$\varphi(x) = o[|x-c|^{\gamma}], \gamma > \alpha - 1, \, as \, x \to c \pm 0.$$

2. A special kernel

In the first part in this work, we investigate the integral equation (1) in the case, when $K(x,t) = A(t)$, that is the integral equation

$$\varphi(x) + \int_c^x \frac{A(t)}{|t-c|^{\alpha}} \varphi(t) dt = f(x), \quad x, c \in \Gamma_0. \tag{2}$$

As in the theory of one-dimensional singular integral equations [11] the integral equation (2) will be called the characteristic integral equation corresponding to the integral equation (1). Without investigating the integral equation (2), it is not possible to develop the general theory of the integral equation (1). In this part of the work the solution of the integral equation (2) will be found in explicit form in depence of the constant $\alpha (\alpha < 1, \alpha = 1, \alpha > 1)$, the signs of the constants $A(c-0), A(c+0), A(c)$ (in the case, when $A(c-0) = A(c+0) = A(c)$) and of some conditions on the functions $A(x)$ and $f(x)$ in the neighborhood of the singular or super-singular point $x = c$. All possible cases will be investigated.

[1]-[10], [13] are dedicated to investigations of Volterra type integral equations and systems with left and right fixed singular and super-singular kernels.

In the case when $K(x,t) = A(t), \alpha = 1$, we have the following statement.

Theorem 1 *Let in equation* (2) $\alpha = 1, A(t) \in C(\Gamma)$, *the point* $x = c$ *be a first kind singularity for* A *and in the neighborhood of the point* $x = c$ *the conditions*

$$|A(x) - A(c \pm 0)| \leq H_1^{\pm} |x - c|^{\gamma_1}, \quad \gamma_1 > 0, \tag{3}$$

$A(c+0) < 0, A(c-0) > 0$ *be satisfied. Let the function* $f(x) \in C(\Gamma_0)$ *vanish at the point* $x = c$ *and its behaviour be determined by the asymptotic formulas*

$$f(x) = o[(c-x)^{\gamma_2}], \quad \gamma_2 > A(c-0), \quad as \quad x \to c - 0, \tag{4}$$

$$f(x) = o[(x-c)^{\gamma_3}], \quad \gamma_3 > |A(c+0)|, \quad as \quad x \to c+0. \tag{5}$$

Then the integral equation (2) *is always solvable and its general solution from the class* $C(\bar{\Gamma})$ *is given by*

$$\varphi(x) = \begin{cases} (c-x)^{A(c-0)} \exp[-W_{A,c}^{-,1}(x)]c_1 + K^-[f], \text{ when } a < x \leq c, \\ (x-c)^{-A(c+0)} \exp[W_{A,c}^{+,1}(x)]c_2 + K^+[f], \text{ when } c \leq x < b, \end{cases} \tag{6}$$

where c_1, c_2 *are arbitrary constants,*

$$W_{A,c}^{-,\alpha}(x) = \int_x^c \frac{A(c-0) - A(t)}{(c-t)^\alpha} dt, \; W_{A,c}^{+,\alpha}(x) = \int_c^x \frac{A(t) - A(c-0)}{(t-c)^\alpha} dt, \tag{7}$$

$$K^-[f] \equiv f(x) + \int_x^c \left(\frac{c-x}{c-t}\right)^{A(c-0)} \exp[W_{A,c}^{-,1}(t) - W_{A,c}^{-,1}(x)] \frac{A(t)}{c-t} f(t) dt, \tag{8}$$

$$K^+[f] \equiv f(x) - \int_c^x \left(\frac{t-c}{x-c}\right)^{A(c+0)} \exp[W_{A,c}^{+,1}(x) - W_{A,c}^{+,1}(t)] \frac{A(t)}{t-c} f(t) dt. \tag{9}$$

Remark 1 A solution of type (5) satisfying the conditions of Theorem 1, vanishes and its behaviour for $x \to c \pm 0$ is determined by the asymptotic formulas

$$\varphi(x) = o[(c-x)^{A(c-0)}], \quad when \quad x \to c-0, \tag{10}$$

and

$$\varphi(x) = o[(x-c)^{A(c+0)}], \quad when \quad x \to c+0. \tag{11}$$

In the case, when $A(c+0) > 0, A(c-0) < 0$ we have the following statement.

Theorem 2 *Let in the integral equation* (2) $A(t) \in C(\Gamma)$*, having at the point* $x = c$ *a first kind singularity and satisfying the condition* (3), $A(c+0) > 0, A(c-0) < 0$. *Let the function* $f(x) \in C(\Gamma)$*, having at the point* $x = c$ *a first kind singularity. Then the homogeneous integral equation* (2) *in the class* $C(\bar{\Gamma}_0)$ *has no solution except zero. The solution* *of the non homogeneous integral equation* (2) *has a unique solution, given by*

$$\varphi(x) = \begin{cases} K^-[f], & \textit{when} \quad a < x \le c, \\ K^+[f], & \textit{when} \quad c \le x < b. \end{cases} \tag{12}$$

From the integral representation (12) follows:

Remark 2 When the conditions of Theorem 2 hold, the solution (12) vanishes at $x = c$, and as $x \to c \pm 0$ it satisfies the equality

$$\varphi(c-0) = \varphi(c+0) = 0. \tag{13}$$

Remark 3 Similar assertions as in Theorems 1,2 may be obtained in the following cases:

1) $K(x,t) = A(t), \alpha = 1, A(c+0) < 0, A(c-0) < 0;$
2) $K(x,t) = A(t), \alpha = 1, A(c+0) > 0, A(c-0) > 0;$
3) $K(x,t) = A(t), \alpha = 1, A(c+0) = A(c-0) = A(c) < 0;$
4) $K(x,t) = A(t), \alpha = 1, A(c+0) = A(c-0) = A(c) > 0.$

In the case, when $K(x,t) = A(t), \alpha > 1$, we have the following statement.

Theorem 3 *Let in the integral equation* (2) $\alpha > 1, A(x) \in C(\Gamma)$, *having a first kind singularity at the point* $x = c$ *and satisfying*

$$|A(x) - A(c \pm 0)| \le H_2^{\pm}|x-c|^{\gamma_1},\ \gamma_1 > \alpha - 1. \tag{14}$$

Besides let $A(c+0) < 0, A(c-0) > 0$. *Let the function* $f(x) \in C(\bar{\Gamma}_0)$, *vanish at the point* $x = c$ *and its behaviour at* $x \to c \pm 0$ *be determined by the asymptotic formulas*

$$f(x) = o[\exp(-A(c-0)\omega_c^{\alpha}(x))(c-x)^{\gamma_2}], \quad \gamma_2 > \alpha - 1, \tag{15}$$

as $x \to c - 0$, and

$$f(x) = o[\exp(A(c+0)\omega_c^{\alpha}(x))(x-c)^{\gamma_3}], \quad \gamma_3 > \alpha - 1, \tag{16}$$

as $x \to c + 0$.

Then the integral equation (2) *is always solvable and its general solution from the class* $C(\bar{\Gamma}_0)$ *contains two arbitrary constants. The solution is given by the formula*

$$\varphi(x) = \begin{cases} \exp[-A(c-0)\omega_c^{\alpha}(x) - W_{A,c}^{-,\alpha}(x)]c_1 + K_{\alpha}^-[f], \textit{ when } a < x \le c, \\ \exp[A(c+0)\omega_c^{\alpha}(x) - W_{A,c}^{+,\alpha}(x)]c_2 + K_{\alpha}^+[f], \textit{ when } c \le x < b, \end{cases} \tag{17}$$

where $\omega_c^\alpha(x) = [(\alpha-1)|x-c|^{\alpha-1}]^{-1}$,

$$K_\alpha^-[f] \equiv f(x) + \int_x^c (c-t)^{-\alpha} A(t) \exp[A(c-0)(\omega_c^\alpha(t) - \omega_c^\alpha(x))$$

$$+W_{A,c}^{-;\alpha}(t) - W_{A,c}^{-;\alpha}(x)] f(t)dt, \qquad (18)$$

$$K_\alpha^+[f] \equiv f(x) - \int_c^x (t-c)^{-\alpha} A(t) \exp[A(c+0)(\omega_c^\alpha(x) - \omega_c^\alpha(t))$$

$$+W_{A,c}^{+}(x) - W_{A,c}^{+}(t)] f(t)dt, \qquad (19)$$

where c_1, c_2 *are arbitrary constants, and* $W_{A,c}^{-;\alpha}(x), W_{A,c}^{+;\alpha}(x)$ *are determined in formula* (7)*.*

From the integral representation (17) it follows

Remark 4 If the assumptions of Theorem 3 are satisfied, the solution (17) vanishes at $x = c$ and its behaviour for $x \to c \pm 0$ is determined by the asymptotic formulas

$$\varphi(x) = o[\exp(-A(c-0)\omega_c^\alpha(x))], \quad as \quad x \to c-0, \qquad (20)$$

and

$$\varphi(x) = o[\exp(A(c+0)\omega_c^\alpha(x))], \quad as \quad x \to c+0. \qquad (21)$$

In the case, when $A(c+0) > 0, A(c-0) < 0$, we have the following statement.

Theorem 4 *Let in the integral equation* (2) $\alpha > 1, A(x) \in C(\Gamma)$*, having in the point* $x = c$ *a first kind singularity and in the neighborhood of the point* $x = c$ *satisfying condition* (14)*. Besides let* $A(c+0) > 0, A(c-0) < 0$*. Let the function* $f(x) \in C(\bar{\Gamma}_0)$*, vanish at the point* $x = c$ *and its behaviour at* $x \to c \pm 0$ *be determined by the asymptotic formula*

$$f(x) = o[|x-c|^{\gamma_4}], \quad \gamma_4 > \alpha - 1. \qquad (22)$$

Then the integral equation (2) *has a unique solution, which is given by formula* (17) *with* $c_1 = c_2 = 0$*.*

We get from Theorem 4

Remark 5 If the conditions of Theorem 4 are satisfied, the solution (17) with $c_1 = c_2 = 0$ vanishes at $x = c$ and its behaviour for $x \to c \pm 0$ is determined by the asymptotic formula

$$\varphi(x) = o[|x-c|^{\gamma_4}), \quad \gamma_4 > \alpha - 1. \qquad (23)$$

Remark 6 A similar assertion as in Theorems 3,4 may be obtained in the following cases:
1) $K(x,t) = A(t), \alpha > 1, A(c+0) < 0, A(c-0) < 0;$
2) $K(x,t) = A(t), \alpha > 1, A(c+0) > 0, A(c-0) > 0;$
3) $K(x,t) = A(t), \alpha > 1, A(c+0) = A(c-0) = A(c) < 0;$
4) $K(x,t) = A(t), \alpha > 1, A(c+0) = A(c-0) = A(c) > 0.$

3. General kernel

Theorem 5 *Let in the integral equation* (1) $\alpha = 1, K(x,t) \in C(\bar{R} \backslash (\Gamma_1 \cup \Gamma_2))$. *In* Γ_1, Γ_2 *let* $K(t,t)$ *have a first kind singularity. Besides let* $K(x,t)$ *satisfy the following conditions*

$$|K(x,t) - K(t,t)| \leq H_3 (c-x)^{\delta_1} (c-t)^{\delta_2},\ a < x, t < c;$$

$$\delta_1 > K(c-0),\ \delta_2 > 0; \tag{K_1}$$

$$|K(x,t) - K(t,t)| \leq H_4 (x-c)^{\delta_3} (t-c)^{\delta_4},\ c < x, t < b;$$

$$\delta_3 > |K(c+0)|,\ \delta_4 > 0; \tag{K_2}$$

$$|K(t,t) - K(c \pm 0, c \pm 0)| \leq H_5 |t-c|^{\delta_5}, \quad \delta_5 > 0 \tag{K_3}$$

as $t \to c \pm 0$ *and* $K(c-0) > 0, K(c+0) < 0$. *Let the function* $f(x) \in C(\bar{\Gamma}_0)$ *and as* $x \to c \pm 0$ *vanish. Moreover, its asymptotic behaviour in the neighborhood of the point* $x = c$ *is assumed to be*

$$f(x) = o[(c-x)^{\delta_6}],\ \delta_6 > K(c-0), \quad at \quad x \to c-0; \tag{f_1}$$

and

$$f(x) = o[(x-c)^{\delta_7}],\ \delta_7 > |K(c+0)|, \quad at \quad x \to c+0. \tag{f_2}$$

Then the integral equation (1) *is always solvable and its solution from the class* $C(\bar{\Gamma})$ *is given by the formula*

$$\varphi(x) = \begin{cases} E_1^-[c_1, f(x)], & at \quad a < x \leq c, \\ E_1^+[c_2, f(x)], & at \quad c \leq x < b, \end{cases} \tag{24}$$

where c_1, c_2 *are arbitrary constants.*

In (24) *the integral operators* E_1^- *and* E_1^+ *are determined by the formulas*

$$E_1^-[c_1, f(x)] \equiv \Omega_1^-[c_1, f(x)] - \int_x^c \Gamma_1^-(x,t) \Omega_1^-[c_1, f(t)] dt,$$

$$E_1^+[c_2, f(x)] \equiv \Omega_1^+[c_2, f(x)] + \int_c^x \Gamma_1^+(x,t) \Omega_1^+[c_2, f(t)] dt,$$

$\Gamma_1^-(x,t),\ \Gamma_1^+(x,t)$ *are resolvents of the integral equations*

$$\varphi_1(x)+\int\limits_x^c \frac{K_1(x,t)}{c-t}\varphi_1(t)dt=\Omega_1^-[c_1,f(x)], \tag{25}$$

$$\varphi_2(x)+\int\limits_c^x \frac{K_2(x,t)}{t-c}\varphi_2(t)dt=\Omega_1^+[c_2,f(x)], \tag{26}$$

where

$$K_1(x,t)=(K(x,t)-K(t,t))$$

$$+\int\limits_x^t (\frac{c-x}{c-s})^{K(c-0)}\frac{K(s,s)}{c-s}(K(s,t)-K(t,t))\exp[W_{K,c}^{-,1}(s)-W_{K,c}^{-,1}(x)]ds,$$

$$K_2(x,t)=K(x,t)-K(t,t)$$

$$-\int\limits_t^x (\frac{s-c}{x-c})^{K(c+0)}\frac{K(s,s)}{s-c}(K(s,t)-K(t,t))\exp[W_{K,c}^{+,1}(x)-W_{K,c}^{+,1}(s)]ds,$$

$$\Omega_1^-[c_1,f(x)]=(c-x)^{K(c-0)}\exp(-W_{K,c}^{-,1}(x))c_1+K^-[f],$$

$$\Omega_1^+[c_2,f(x)]=(x-c)^{-K(c+0)}\exp[W_{K,c}^{+,1}(x)]c_2+K^+[f],$$

$$K^-[f]\equiv f(x)+\int\limits_x^c (\frac{c-x}{c-t})^{K(c-0)}\exp[W_{K,c}^{-,1}(t)$$

$$-W_{K,c}^{-,1}(x)]\frac{K(t,t)}{c-t}f(t)dt,\ a<x\leq c,$$

$$K^+[f]\equiv f(x)-\int\limits_c^x (\frac{t-c}{x-c})^{K(c+0)}\exp[W_{K,c}^{+,1}(x)$$

$$-W_{K,c}^{+,1}(t)]\frac{K(t,t)}{t-c}f(t)dt,\ c\leq x<b.$$

Remark 7 A similar assertion as in Theorem 5 may be obtained in the following cases:
1) $\alpha=1, K(c-0)=K(c-0,c-0)<0, K(c+0)=K(c+0,c+0)>0$;
2) $\alpha=1, K(c-0)<0, K(c+0)<0$;
3) $\alpha=1, K(c-0)>0, K(c+0)>0$;

4) $\alpha = 1, K(c+0) = K(c-0) = K(c) > 0$;
5) $\alpha = 1, K(c+0) = K(c-0) = K(c) < 0$.

In the case, when $\alpha > 1, K(x,t) \neq A(t)$ we have the following statement.

Theorem 6 *Let in equation* (1) $\alpha > 1, K(x,t) \in C[\bar{R}\backslash(\Gamma_1 \cup \Gamma_2)]$. *Let* $K(t,t)$ *have a first kind singularity in* Γ_1, Γ_2 *and in the neighborhood of the point* $x = t = c$ *let* $K(x,t)$ *satisfy the conditions*

$$K(x,t) - K(t,t) = o[\exp(-K(c-0)\omega_c^\alpha(x))(c-t)^{\beta_1}], \qquad (K_1^\alpha)$$

$$\beta_1 > \alpha - 1, \quad as \quad (x,t) \to (c-0, c-0),\ a < x, t < c;$$

$$K(x,t) - K(t,t) = o[\exp(K(c+0)\omega_c^\alpha(x))(t-c)^{\beta_2}], \qquad (K_2^\alpha)$$

$$\beta_2 > \alpha - 1, \quad as \quad (x,t) \to (c+0, c+0),\ c < x, t < b;$$

$$K(x,x) - K(c \pm 0, c \pm 0) = o(|x-c|^{\beta_3}),\ \beta_3 > \alpha - 1; \qquad (K_3^\alpha)$$

$K(c-0) > 0, K(c+0) < 0$, *where* $K(c+0) = K(c+0, c+0), K(c-0) = K(c-0, c-0)$. *Let the function* $f(x) \in C(\bar{\Gamma}_0)$ *and as* $x \to c \pm 0$ *satisfy the conditions*

$$f(x) = o[\exp(-K(c-0)\omega_c^\alpha(x))(c-x)^{\beta_4}], \beta_4 > \alpha - 1,\ as\ x \to c-0, \qquad (f_1^\alpha)$$

$$f(x) = o[\exp(K(c+0)\omega_c^\alpha(x))(x-c)^{\beta_5}], \beta_5 > \alpha - 1,\ as\ x \to c+0, \qquad (f_2^\alpha)$$

respectively. Then the integral equation (1) *is always solvable and its general solution from the class* $C(\bar{\Gamma}_0)$ *is given by the formula*

$$\varphi(x) = \begin{cases} E_\alpha^-[c_1, f(x)], & when \quad a < x \leq c \\ E_\alpha^+[c_2, f(x)], & when \quad c \leq x < b, \end{cases} \qquad (27)$$

where c_1, c_2 *are arbitrary constants.*

The integral operators E_α^- and E_α^+ in (27) are determined by the formulas

$$E_\alpha^-[c_1, f(x)] \equiv \Omega_\alpha^-[c_1, f(x)] + \int_x^c \Gamma_\alpha^-(x,t)\Omega_\alpha^-[c_1, f(t)]dt,$$

$$E_\alpha^+[c_2, f(x)] \equiv \Omega_\alpha^+[c_2, f(x)] + \int_c^x \Gamma_\alpha^+(x,t)\Omega_\alpha^+[c_2, f(t)]dt,$$

where $\Gamma_\alpha^-(x,t), \Gamma_\alpha^+(x,t)$ are the resolvents of the integral equations

$$\varphi_1(x) - \int_x^c (c-t)^{-\alpha} K_\alpha^-(x,t)\varphi_1(t)dt = \Omega_\alpha^-(c_1, f(x)), \qquad (28)$$

$$\varphi_2(x) + \int_c^x (t-c)^{-\alpha} K_\alpha^+(x,t)\varphi_2(t)dt = \Omega_\alpha^+(c_2, f(x)), \tag{29}$$

where

$$K_\alpha^-(x,t) = K(x,t) - K(t,t) - \int_x^t (c-s)^{-\alpha} K(s,s)(K(s,t) - K(t,t))$$

$$\times \exp[K(c-0)(\omega_c^\alpha(s) - \omega_c^\alpha(x)) + W_{K,c}^{-,\alpha}(s) - W_{K,c}^{-,\alpha}(x)]ds,$$

$$K_\alpha^+(x,t) = K(x,t) - K(t,t) - \int_t^x (s-c)^{-\alpha} K(s,s)(K(s,t) - K(t,t))$$

$$\times \exp[K(c+0)(\omega_c^\alpha(x) - \omega_c^\alpha(s)) + W_{K,c}^{+}(x) - W_{K,c}^{+}(s)]ds,$$

$$\Omega_\alpha^-[c_1, f(x)] \equiv \exp[-K(c-0)\omega_c^\alpha(x) - W_{K,c}^{-,\alpha}(x)]c_1 + K_\alpha^-[f],$$

$$\Omega_\alpha^+[c_2, f(x)] \equiv \exp[K(c+0)\omega_c^\alpha(x) - W_{K,c}^{+,\alpha}(x)]c_2 + K_\alpha^+[f].$$

Remark 8 A similar assertion as in Theorem 6 may be obtained in the following cases:
1) $\alpha > 1, K(c-0) < 0, K(c+0) > 0$;
2) $\alpha > 1, K(c-0) < 0, K(c+0) < 0$;
3) $\alpha > 1, K(c-0) > 0, K(c+0) > 0$;
4) $\alpha > 1, K(c+0) = K(c-0) = K(c) > 0$;
5) $\alpha > 1, K(c+0) = K(c-0) = K(c) < 0$.

In the proofs of the mentioned statements, we essentially use the relation of the characteristic integral equation (2) and the theory of ordinary differential equations with singular and super-singular coefficients [12].

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THE METHOD OF DISCRETE SINGULARITIES OF SOLUTIONS OF SINGULAR INTEGRAL EQUATIONS WITH UNMOVED SINGULARITY

AVETIK SAHAKYAN
Institute of Mechanics of NAS of Armenia,
Marshal Bagramian ave. 24b, 375019, Yerevan, Armenia
avsah@freenet.am

Abstract The mixed boundary value problems, particularly contact problems in elasticity theory are not infrequently reduced to the solution of singular integral equations with generalized Cauchy kernel, which are also called equations with unmoved singularity. Solving these equations is connected with certain difficulties.

The quadrature formulas of the highest algebraic precision for singular integral equations are obtained and the effectiveness of their application for solving similar equations is shown in the present work.

Keywords: singular integral equations with Cauchy kernel, unmoved singularity, quadrature formulae, compound elastic plane, cracks

Mathematics Subject Classification (2000): 45E05, 74B05

1. Introduction

The application of the Gauss type quadrature formulae for solving different types of singular integral equations is represented in work [1]. This work has an important significance, taking into account the wideness of the represented material and the successful selection of problems in order to represent the methods of investigating the behaviour of solutions at the singular points and the solution of different types of singular integral equations. However, unfortunately, there is no uniform method and the reduction of singular integral equations to systems of algebraic equations mainly has no justification, so far as Gauss type quadrature formulas, applied for ordinary integrals, are formally accepted to hold for singular integrals in the mentioned work. The validity of the last

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statement was shown for some weight functions and it was evidently the base for the generalization for wider classes of weight functions. The other method, the method of orthogonal polynomials is represented in the case of singular integral equations of the second kind, taking into account the impossibility of a formal approach.

The method of discrete singularities is the universal method for obtaining quadrature formulae of highest algebraic precision for singular integral equations with Jacobi weight function $(1-x)^{\alpha}(1+x)^{\beta}(Re[\alpha,\beta] > -1)$.

The application of the method of discrete singularities to solutions of singular integral equations of the second kind with real coefficient of the free term is shown in the work [2], and with complex coefficient in the work [3]. It is noticed that the system of algebraic equations obtained for the first kind singular integral equations completely coincides with the system in the work [1].

Taking the fundamental character of the work [1] into account, in the present work we will follow the thesis of this work and compare it with results, obtained there. Since in this case the comparison with approximate solution is considered, the solution of problem for the half-plane with crack on the border, which is reduced to the exactly solved similar equation [6], is also considered.

2. The description of the method of discrete singularities

Let consider the singular integral

$$I(z) = \int_{-1}^{1} \frac{f(x)}{x-z}(1-x)^{\alpha}(1+x)^{\beta}dx \quad z \in \mathbb{C}, z \neq \pm 1;\ \mathrm{Re}\ \ [\alpha,\beta] > -1, \tag{1}$$

where $f(x)$ is a function, satisfying a Hölder condition in the interval $(-1,1)$ and in the case of complex α and β has an analytical continuation into the complex plane.

For the calculation of the integral $I(z)$ let us substitute the function $f(x)$ by the following interpolation polynomial [4]

$$f(x) \approx f_n(x) = \frac{2}{n+\alpha+\beta+1}\sum_{i=1}^{n}\frac{f_i P_n^{(\alpha,\beta)}(x)}{(x-\xi_i)P_{n-1}^{(\alpha+1,\beta+1)}(\xi_i)}\ , \tag{2}$$

where $f_i = f(\xi_i)$ and $\{\xi_i\}_{i=1}^{n}$ are the roots of the polynomial $P_n^{(\alpha,\beta)}(x)$.

Substituting (2) into (1), changing the order of integration and summing up and using the famous integral relation for the Jacobi polynomials [5]

$$\int_{-1}^{1} \frac{f(x)}{x-z}(1-x)^{\alpha}(1+x)^{\beta}dx \approx \sum_{i=1}^{n} \frac{w_i f(\xi_i)}{\xi_i - z}[1-q_i(z)], \quad z \neq \pm 1, \tag{3}$$

is obtained, where

$$w_i = \frac{2}{n+\alpha+\beta+1}\frac{Q_n^{(\alpha,\beta)}(\xi_i)}{P_n^{(\alpha+1,\beta+1)}(\xi_i)}, \qquad q_i(z) = \frac{Q_n^{(\alpha,\beta)}(z)}{Q_n^{(\alpha,\beta)}(\xi_i)}, \tag{4}$$

$$Q_n^{(\alpha,\beta)}(z) = \begin{cases} A(z)_2F_1(n+1, n+\alpha+1; 2n+\alpha+\beta+2; \frac{2}{1-z}) \ , \\ \frac{1}{2}[Q_n^{(\alpha,\beta)}(z+i0) + Q_n^{(\alpha,\beta)}(z-i0)] \quad (-1 < z < 1), \end{cases}$$

$$A(z) = (\frac{2}{z-1})^n 2^{\alpha+\beta}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \ .$$

It is noticed that formula (3) is true on the whole complex plane, which is cut along the interval $(-1, 1)$ and for multi-valued functions $Q_n^{(\alpha,\beta)}(z)$ the branch, having positive values for $x > 1$, is chosen.

By a similar way

$$\int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta}dx \approx \sum_{i=1}^{n} w_i f(\xi_i), \quad \mathrm{Re}(\alpha, \beta) > -1, \tag{5}$$

is obtained.

The formulae (3) and (5) are quadrature formulae of the highest algebraic precision. The formula (3) is precise, when function $f(x)$ is the polynomial of degree $(n-1)$ or less, and formula (5) is a polynomial of degree $(2n-1)$ or less.

3. Some examples

In order to show some applications and results of the method of discrete singularities on solution of the singular integral equation with unmoved singularity let us consider the solution of two problems, described by the equation

$$\int_{-1}^{1} \frac{F(t)}{t-x}dt + \int_{-1}^{1} K(x,t)F(t)dt = \pi f(x) \qquad (-1 < x < 1) \tag{6}$$

where

$$K(x,t) = [c_0 + c_1(1+x)\frac{d}{dx} + c_2(1+x)^2\frac{d^2}{dx^2}]\frac{1}{t+x+2} \ .$$

Problem 1 is the problem for a compound elastic plane, consisting of two semi-planes, in one of which there is a crack, perpendicular to the joining line. The approximate solution is given in work [1].

Problem 2 is the problem for an elastic semi-plane with crack on the boundary, at the banks of which the normal load is acting. The precise solution is given in [6].

For Problem 1 the following condition should be added to equation (6)

$$\int_{-1}^{1} F(t)dt = 0 \tag{7}$$

and c_0, c_1, c_2 are constants, depending on the ratio of the shear modules and the Poisson coefficients of the semi-planes. For Problem 2 there are equation (6) and $c_0 = -1, c_1 = -6, c_2 = -2$. The function $F(t)$ is represented in the form

$$F(t) = (1-t)^{\alpha}(1+t)^{\beta}g(t) \qquad (-1 < t < 1), \tag{8}$$

where $\alpha = -0.5$ and β is determined from the transcendental equation $2d_1 cos\pi(\beta+1) - d_2(\beta+1)^2 - d_3 = 0$, where d_1, d_2, d_3 are also dependent on the ratio of the shear modules and the Poisson coefficients of the semi-planes for Problem 1, and $\beta = 0$ for Problem 2.

Substituting (8) in equations (6) and (7), replacing integrals by their quadrature sums in accordance to formulae (3) and (5) and equating the left and right parts of equations in certain points, the following algebraic equations are obtained:

for Problem 1

$$\begin{aligned} &\sum_{i=1}^{n} w_i g(\xi_i)[\frac{[1-q_i(x_k)]}{\xi_i - x_k} + R(\xi_i,, x_k)] = \pi f(x_k), \\ &\sum_{i=1}^{n} w_i g(\xi_i) = 0, \end{aligned} \tag{9}$$

$$x_k = cos\frac{(2k-1)\pi}{2(n-1)} \qquad (k = 1, 2, ..., n-1),$$

for Problem 2

$$\sum_{i=1}^{n} w_i g(\xi_i)[\frac{[1-q_i(y_k)]}{\xi_i - y_k} + R(\xi_i,, y_k)] = \pi f(y_k), \tag{10}$$

$$y_k = cos\frac{(2k-1)\pi}{2n} \qquad (k = 1, 2, ..., n),$$

where

$$R(\xi_i, x) = [c_0 + c_1(1+x)\frac{d}{dx} + c_2(1+x)^2\frac{d^2}{dx^2}]\frac{1 - q_i(-x-2)}{\xi_i + x + 2}.$$

In work [1] the equations (6), (7) are reduced to the system of algebraic equations on the base of quadrature formulae, coinciding with (5), which is formally accepted as true for the singular integrals in equation (6).

The values of the coefficients of strains concentration at the ends of the cracks, calculating after solution of the system (9), are represented in the Table 1. In order to compare values the Table VIII of the work [1] is repeated here.

Table VIII

$p(x) = p_0$	$\mu_2/\mu_1 = 23.08$	
N	$\frac{k(b)}{p_0(a_0)^{0.5}}$	$\frac{k(a)}{p_0(a_0)^{0.5}}$
20	0.883063	2.630084
40	0.882811	2.624819
48	0.882759	2.624475
60	0.882716	2.624176
98	0.882650	2.623657

Table I

$p(x) = p_0$	$\mu_2/\mu_1 = 23.08$	
n	$\frac{k(b)}{p_0(a_0)^{0.5}}$	$\frac{k(a)}{p_0(a_0)^{\beta}}$
4	0.882438	2.871765
5	0.882529	2.828266
8	0.882541	2.800096
13	0.882542	2.799114
20	0.882542	2.800435

The comparison of the values N and n, represented in tables, is shown that using formula (5) for the singular integrals is beside the purpose. Besides, it is noticed that in the present work values $g(\pm 1)$ are determined by the interpolation formula (2), while in work [1] the authors assert that more stable results for the mentioned values were obtained by ignoring the $g(t_1)$ and $g(t_N)$ values and using formulae of quadratic extrapolation, based on the following or previous three values $g(t_k)$ ($k = 2, 3, 4$ and $k = N-3, N-2, N-1$).

The calculation results of system (10) are represented in Table 2.

Table 2

n	Calculated value	Precise value
4	1.121819	1.121522
6	1.121449	1.121522
9	1.121513	1.121522
12	1.121521	1.121522
16	1.121522	1.121522

There are values of interpolation degree in the first column, and calculated values of the coefficient of stress concentration at the inner end of the crack in the second column and precise values, calculated by the analytical formula, obtained in work [6] in the third column.

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LOCALIZATION OPERATORS, WIGNER TRANSFORMS AND PARAPRODUCTS

M.W. WONG
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3, Canada
mwwong@mathstat.yorku.ca

Abstract The connections between localization operators, Wigner transforms and paraproducts are elucidated in the context of the Weyl-Heisenberg group, the affine group and SU(1,1).

Keywords: localization operators, Wigner transform, paraproducts, Weyl-Heisenberg group, affine group, SU(1,1), wavelet

Mathematics Subject Classification (2000): 47G10, 47D03, 47B38

1. The Connections

Let G be a locally compact and Hausdorff group endowed with a left Haar measure μ. Let X be an infinite-dimensional, separable and complex Hilbert space in which the inner product and norm are denoted by $(\,,\,)$ and $\|\,\|$ respectively. A unitary representation π of G on X is said to be square-integrable if there exists a nonzero element φ in X such that

$$\int_G |(\varphi, \pi(g)\varphi)|^2 \, d\mu(g) < \infty.$$

This is known as the admissibility condition for the square-integrable representation π of G on X, and we define the constant c_φ by

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g).$$

We call c_φ the wavelet constant associated to the admissible wavelet φ.

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Theorem 1 *Let π be an irreducible and square-integrable representation of G on X. If φ is an admissible wavelet for π, then*

$$(x, y) = \frac{1}{c_\varphi} \int_G (x, \pi(g)\varphi)(\pi(g)\varphi, y)\, d\mu(g)$$

for all x and y in X.

Remark 1 *In order to understand the formula in Theorem 1 better, let us note that it tells us informally that*

$$I = \frac{1}{c_\varphi} \int_G (\cdot, \pi(g)\varphi)\pi(g)\varphi\, d\mu(g),$$

where I is the identity operator on X. In other words, the identity operator I on X can be resolved into a sum of rank-one operators $\frac{1}{c_\varphi}(\cdot, \pi(g)\varphi)\pi(g)\varphi$, which are parametrized by elements g in G. Thus, the formula in Theorem 1 is known as the resolution of the identity formula.

Theorem 1 is an abridged version of Theorem 7 in the paper [9] by Grossmann, Morlet and Paul, where the original contributions of Duflo and Moore in [8] are acknowledged. Chapter 14 of the book [7] by Dixmier is devoted to square-integrable representations. See also the paper [4] by Carey and the book [16] by Wong in this connection.

Let $F \in L^1(G) \cup L^\infty(G)$. Then for all $x \in X$, we define $L_{F,\varphi}x$ by

$$(L_{F,\varphi}x, y) = \frac{1}{c_\varphi} \int_G F(g)(x, \pi(g)\varphi)(\pi(g)\varphi, y)\, d\mu(g)$$

for all y in X. Then we have the following boundedness results, which are well-known and in fact very easy to prove.

Theorem 2 *Let $F \in L^1(G)$. Then $L_{F,\varphi} : X \to X$ is a bounded linear operator and*

$$\|L_{F,\varphi}\|_* \leq \frac{1}{c_\varphi} \|F\|_{L^1(G)},$$

where $\| \, \|_*$ is the norm in the C^*-algebra of all bounded linear operators from X into X.

Theorem 3 *Let $F \in L^\infty(G)$. Then $L_{F,\varphi} : X \to X$ is a bounded linear operator and*

$$\|L_{F,\varphi}\|_* \leq \|F\|_{L^\infty(G)}.$$

The bounded linear operators $L_{F,\varphi} : X \to X$ are dubbed localization operators in the paper [10] by He and Wong. The impetus for the terminology stems from the simple observation that if $F(g) = 1$ for all g in G, then the resolution of the identity forumla in Theorem 1 implies that the corresponding operator is simply the identity operator on X. Therefore the function or the symbol F is there to localize on G so as to produce a nontrivial bounded linear operator on X with various applications in the mathematical sciences.

Many deeper properties of localization operators than Theorems 2 and 3 can be found in the book [16] by Wong.

As a prime example, localization operators on the Weyl-Heisenberg group are analyzed. To recall, we let

$$(WH)^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}/2\pi\mathbb{Z}.$$

Then we define the binary operation $\cdot$ on $(WH)^n$ by

$$(q_1, p_1, t_1) \cdot (q_2, p_2, t_2) = (q_1 + q_2, p_1 + p_2, t_1 + t_2 + q_1 \cdot p_2)$$

for all (q_1, p_1, t_1) and (q_2, p_2, t_2) in $(WH)^n$, where $q_1 \cdot p_2$ is the Euclidean inner product of q_1 and p_2 in $\mathbb{R}^n$; t_1, t_2 and $t_1 + t_2 + q_1 \cdot p_2$ are cosets in the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ in which the group law is addition modulo 2π. It is easy to prove that with respect to the multiplication $\cdot$, $(WH)^n$ is a non-abelian group in which $(0, 0, 0)$ is the identity element and the inverse element of (q, p, t) is $(-q, -p, -t + q \cdot p)$ for all (q, p, t) in $(WH)^n$. In fact, $(WH)^n$, equipped with the multiplication $\cdot$, is a unimodular group on which the left (and right) Haar measure is the Lebesque measure on $\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty)$. It is known as the Weyl-Heisenberg group.

Let $U(L^2(\mathbb{R}^n))$ be the set of all unitary operators on $L^2(\mathbb{R}^n)$. Then it becomes a group with respect to the usual composition of mappings. We let $\pi : (WH)^n \to U(L^2(\mathbb{R}^n))$ be the mapping defined by

$$(\pi(q, p, t)f)(x) = e^{i(q \cdot x + \frac{1}{2} q \cdot p + t)} f(x + p)$$

for all x in $\mathbb{R}^n$, (q, p, t) in $(WH)^n$ and f in $L^2(\mathbb{R}^n)$. Then it can be shown that π is an irreducible and square-integrable representation of $(WH)^n$ on $L^2(\mathbb{R}^n)$, which is called the Schrödinger representation. It is a fact that every function φ in $L^2(\mathbb{R}^n)$ with $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$ is an admissible wavelet for π. The wavelet constant c_φ for every admissible wavelet φ is equal to $(2\pi)^{n+1}$.

The following result can be found in the book [15] by Wong.

Theorem 4 *Let F be a function in $L^1((WH)^n)$ such that*

$$F(q, p, t) = \sigma(q, p)$$

for all (q, p, t) in $(WH)^n$. Let φ be the function on $\mathbb{R}^n$ defined by

$$\varphi(x) = \pi^{-n/4} e^{-|x|^2/2}, \quad x \in \mathbb{R}^n.$$

Then the localization operator $L_{F,\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ associated to the symbol F and the admissible wavelet φ is the pseudo-differential operator associated to the symbol $\sigma * \Lambda$ given by

$$(L_{F,\varphi}u, v)_{L^2(\mathbb{R}^n)} = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\sigma * \Lambda)(x,\xi) W(u,v)(x,\xi)\, dx\, d\xi$$

for all u and v in $L^2(\mathbb{R}^n)$, where $\sigma * \Lambda$ is the convolution of σ and Λ,

$$\Lambda(x,\xi) = \pi^{-n} e^{-(|x|^2+|\xi|^2)}, \quad x, \xi \in \mathbb{R}^n,$$

and $W(u,v)$ is the Wigner transform of u and v defined by

$$W(u,v)(x,\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot p} u\left(x + \frac{p}{2}\right) \overline{v\left(x - \frac{p}{2}\right)}\, dp$$

for all x and ξ in $\mathbb{R}^n$.

In the case when $u = v$, $W(u,u)(x,\xi)$ is the substitute used by Wigner in [14] for the joint probability distribution of position x and momentum ξ in the state u in quantum mechanics.

That localization operators can be studied in the context of anti-Wick quantization is carried out in the book [1] by Boggiatto, Buzano and Rodino, the paper [2] by Boggiatto and Cordero and also the book [12] by Shubin.

Another canonical example is provided by localization operators on the affine group, which we now recall. Let U be the upper half plane given by

$$U = \{(b,a) : b \in \mathbb{R},\ a > 0\}.$$

Then we define the binary operation $\cdot$ on U by

$$(b_1, a_1) \cdot (b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2)$$

for all (b_1, a_1) and (b_2, a_2) in U. With respect to this multiplication, U is a non-abelian group in which $(0,1)$ is the identity element and the inverse element of (b,a) is $(-b/a, 1/a)$ for every (b,a) in U. The left and right Haar measures on U are given by

$$d\mu = \frac{db\, da}{a^2}$$

and

$$d\nu = \frac{db\, da}{a}$$

respectively. Therefore U is a non-unimodular group, which is called the affine group.

Let $H^2_+(\mathbb{R})$ be the subspace of $L^2(\mathbb{R})$ defined by

$$H^2_+(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq [0, \infty)\},$$

where $\operatorname{supp}(\hat{f})$ is the support of the Fourier transform $\hat{f}$ of f. The Fourier transform $\hat{f}$ of a function f in $L^2(\mathbb{R})$ is chosen to be the one defined by

$$\hat{f}(\xi) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-ix\xi} f(x)\, dx$$

for all ξ in $\mathbb{R}$, where the convergence is understood to take place in $L^2(\mathbb{R})$. We define $H^2_-(\mathbb{R})$ to be the subspace of $L^2(\mathbb{R})$ by

$$H^2_-(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq (-\infty, 0]\}.$$

$H^2_+(\mathbb{R})$ and $H^2_-(\mathbb{R})$ are known as the Hardy space and the conjugate Hardy space respectively. They are closed subspaces of $L^2(\mathbb{R})$. Only the Hardy space $H^2_+(\mathbb{R})$ is considered in this paper.

Let $U(H^2_+(\mathbb{R}))$ be the group of all unitary operators on $H^2_+(\mathbb{R})$. Let $\pi : U \to U(H^2_+(\mathbb{R}))$ be the mapping defined by

$$(\pi(b,a)f)(x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right)$$

for all x in $\mathbb{R}$, (b,a) in U and f in $H^2_+(\mathbb{R})$. Then it can be proved that π is an irreducible and square-integrable representation of U on $H^2_+(\mathbb{R})$. The following characterization of admissible wavelets for π is a well-known result in wavelet theory.

Theorem 5 *The set of admissible wavelets for π consists of all functions φ in $H^2_+(\mathbb{R})$ for which $\|\varphi\|_{L^2(\mathbb{R})} = 1$ and*

$$\int_0^\infty \frac{|\hat{\varphi}(\xi)|^2}{\xi} d\xi < \infty.$$

Moreover, the wavelet constant c_φ for every admissible wavelet φ is given by

$$c_\varphi = 2\pi \int_0^\infty \frac{|\hat{\varphi}(\xi)|^2}{\xi} d\xi.$$

Proofs of Theorem 5 can be found in the book by Daubechies [6] and the book by Wong [16].

Theorem 6 *Let F be a function in $L^\infty(U)$ such that*

$$F(b,a) = \beta(b)$$

for all (b,a) in U. Let φ be an admissible wavelet for π. Then the localization operator $L_{F,\varphi} : H^2_+(\mathbb{R}) \to H^2_+(\mathbb{R})$ associated to the symbol F and the admissible wavelet φ is given by

$$(L_{F,\varphi}u, v)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \beta(y) p_\psi(u,v)(y)\, dy$$

for all u and v in $H^2_+(\mathbb{R})$, where

$$\psi(x) = \overline{\varphi(-x)}, \quad x \in \mathbb{R},$$

ψ_a is the Friedrich mollifier of ψ defined by

$$\psi_a(x) = \frac{1}{a}\psi\left(\frac{x}{a}\right), \quad x \in \mathbb{R},$$

and $p_\psi(u,v)$ is the paraproduct of u and v with respect to the admissible wavelet ψ given by

$$p_\psi(u,v)(y) = \int_0^\infty (\psi_a * u)(y)\overline{(\psi_a * v)(y)}\frac{da}{a}$$

for all y in $(-\infty,\infty)$.

Theorem 6 is due to Rochberg [11] and is made precise in the paper [17] by Wong. It should be remarked that the notion of a paraproduct is rooted in Bony's work [3] on linearization of nonlinear partial differential equations. Several versions of paraproducts exist in the literature. We are using the one studied in the work [5] by Coifman and Meyer.

It is interesting to note that the formula for localization operators on the affine group is like the one for the Weyl-Heisenberg group. For the affine group, we have the inner product of β with the paraproduct $p_\psi(u,v)$. For the Weyl-Heisenberg case, we have the inner product of $\sigma * \Lambda$ with the Wigner transform. Thus, the paraproduct can be thought of as the Wigner transform on the affine group.

The aim of this paper is to consolidate this theme with the Lie group SU(1,1). We first give in Section 2 a recall of the Lie group SU(1,1) and its discrete series representations. The reason for using the discrete series representations of SU(1,1) is that they are infinite-dimensional, irreducible and square-integrable. That this is indeed the case is proved in Section 3. In Section 4, we show that localization operators associated to specific symbols on SU(1,1) and a specific admissible wavelet, *i.e.*, the vacuum state, for the discrete series representations, are given in terms of a paraproduct. This paraproduct can be used as the Wigner transform on SU(1,1).

2. Discrete series representations of SU(1,1)

Let SU(1,1) be the set of all 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$, where α and β are complex numbers such that $|\alpha|^2 - |\beta|^2 = 1$. With respect to matrix multiplication, it is a non-abelian group, which we call the pseudo-unitary group. According to the Cartan decomposition of SU(1,1), every element g in SU(1,1) is of the form

$$g = u_\phi a_t u_\psi, \quad 0 \le \phi < 4\pi,\, t \ge 0,\, 0 \le \psi < 2\pi,$$

where

$$u_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix},$$

$$u_\psi = \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}$$

and

$$a_t = \begin{pmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{pmatrix}.$$

In fact, for $0 \le \phi < 4\pi$, $t \ge 0$ and $0 \le \psi < 2\pi$, we have

$$\alpha = e^{i(\phi+\psi)/2}\cosh(t/2)$$

and

$$\beta = e^{i(\phi-\psi)/2}\sinh(t/2).$$

The Cartan decomposition of SU(1,1) as given is Proposition 5.2 in Chapter 5 of the book [13] by Sugiura.

In order to compute the Haar measure on SU(1,1), we recall the special linear group $\mathrm{SL}(2,\mathbb{R})$, which consists of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with real entries and such that $ad - bc = 1$. It is well-known that SU(1,1) acts on the unit disk with center at the origin, while $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane. In fact, they are related by the equation

$$\mathrm{SU}(1,1) = C\,\mathrm{SL}(2,\mathbb{R})C^{-1}, \tag{1}$$

where C is the unitary matrix given by

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

According to the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$, every element g in $\mathrm{SL}(2,\mathbb{R})$ is of the form

$$g = k_\theta \alpha_t n_\xi, \quad 0 \le \theta < 4\pi,\ t \in \mathbb{R},\ \xi \in \mathbb{R},$$

where

$$k_\theta = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

$$\alpha_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

and

$$n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

In fact, for $0 \le \theta < 4\pi$, $t \in \mathbb{R}$ and $\xi \in \mathbb{R}$, we have

$$e^{i\theta/2} = \frac{a - ic}{\sqrt{a^2 + c^2}},$$

$$e^t = a^2 + c^2$$

and

$$\xi = \frac{ab + cd}{a^2 + c^2}.$$

It follows from the Iwasawa decomposition that we can think of $\mathrm{SL}(2,\mathbb{R})$ as the set $[0, 4\pi) \times \mathbb{R} \times \mathbb{R}$, which becomes a non-abelian group when equipped with the multiplication induced by the matrix multiplication in $\mathrm{SL}(2,\mathbb{R})$. It is proved on page 266 of the book [13] by Sugiura that $\mathrm{SL}(2,\mathbb{R})$ is a unimodular group on which the left and right Haar measure ν is given by

$$\int_{\mathrm{SL}(2,\mathbb{R})} f(g)\, d\nu(g) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{4\pi} f(\theta, t, \xi) e^t \, d\theta\, dt\, d\xi,$$

for every continuous function f on $\mathrm{SL}(2,\mathbb{R})$ with compact support. Using this Haar measure on $\mathrm{SL}(2,\mathbb{R})$, the identification of SU(1,1) with

SL$(2,\mathbb{R})$ via (1), and the Cartan decomposition, it is proved on pages 267 and 268 that SU(1,1) is also a unimodular group and the left and right Haar measure μ on it is given by

$$\int_{\mathrm{SU}(1,1)} f(g)\, d\mu(g) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\infty} \int_0^{4\pi} f(\phi, t, \psi) \sinh t \, d\phi \, dt \, d\psi$$

for every continuous function f on SU(1,1) with compact support. For the sake of computations, it is helpful to think of SU(1,1) as the set $[0, 4\pi) \times [0, \infty) \times [0, 2\pi)$, which becomes a non-abelian group on which the multiplication is the one that comes from the matrix multiplication in SU(1,1) and the left and right Haar measure μ is given by

$$d\mu = \frac{1}{4\pi} \sinh t \, d\phi \, dt \, d\psi.$$

Thus, SU(1,1) is a unimodular, locally compact and Hausdorff group.

We can now introduce infinite-dimensional, irreducible and unitary representations of the pseudo-unitary group SU(1,1). To this end, we let D be the open unit disk with center at the origin in the complex plane. For $n \in \frac{1}{2}\mathbb{Z}$ with $|n| \geq 1$, we let X_n be the set of all analytic functions u on D such that

$$\frac{2n-1}{\pi} \int_D |u(z)|^2 (1 - |z|^2)^{2n-2} dz < \infty,$$

where dz is the Lebesque measure on $\mathbb{C}$. Thus, X_n is a Hilbert space in which the inner product $(\, , \,)_{X_n}$ is given by

$$(u, v)_{X_n} = \frac{2n-1}{\pi} \int_D u(z) \overline{v(z)} (1 - |z|^2)^{2n-2} dz$$

for all u and v in X_n. We now let $U(X_n)$ be the group of all unitary operators on X_n and let $\pi_n : \mathrm{SU}(1,1) \to U(X_n)$ be the mapping defined by

$$(\pi_n(g)u)(z) = (\overline{\beta} z + \overline{\alpha})^{-2n} u \left(\frac{\alpha z + \beta}{\overline{\beta} + \overline{\alpha}} \right)$$

for all g in SU(1,1) with

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

and all u in X_n. It is well-known that π_n is an irreducible and unitary representation of SU(1,1) on X_n. The representations π_n, $n \in \frac{1}{2}\mathbb{Z}, |n| \geq$

1, are known as the discrete series representations of SU(1,1) on X_n. We assume that $n \geq 1$ in this paper. The irreducible and unitary representations of SU(1,1) are computed in Chapter 5 of the book [13] by Sugiura.

3. Square-Integrability

Let $\mathbf{1}$ be the function on D defined by $\mathbf{1}(z) = 1$ for all z in D. Then we have the following results.

Theorem 7 *For $n \in \frac{1}{2}\mathbb{Z}$ with $n \geq 1$, we have $\|\mathbf{1}\|_{X_n} = 1$.*

Proof Using the inner product in X_n and polar coordinates, we get

$$\begin{aligned} \|\mathbf{1}\|_{X_n}^2 &= \frac{2n-1}{\pi}\int_D |\mathbf{1}(z)|^2(1-|z|^2)^{2n-2}dz \\ &= \frac{2n-1}{\pi}\int_0^{2\pi}\int_0^1 (1-r^2)^{2n-2}r\,dr\,d\theta \\ &= \frac{2n-1}{2\pi}\int_0^{2\pi}\int_0^1 r^{2n-2}dr\,d\theta = 1. \end{aligned}$$

□

Theorem 8 *For $n \in \frac{1}{2}\mathbb{Z}$ with $n \geq 1$, we have*

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\int_0^{4\pi} |(\mathbf{1}, \pi_n(\phi,t,\psi)\mathbf{1})_{X_n}|^2 \sinh t\, d\phi\, dt\, d\psi = \frac{4\pi}{2n-1}.$$

Proof For $0 \leq \phi < 4\pi$, $t \geq 0$ and $0 \leq \psi < 2\pi$, we have

$$\begin{aligned} &|(\pi_n(\phi,t,\psi)\mathbf{1}, \mathbf{1})_{X_n}| \\ &= \frac{2n-1}{\pi}\left|\int_D \left(-e^{-i\frac{\phi}{2}}\sinh\left(\frac{t}{2}\right)z + e^{i\frac{\phi}{2}}\cosh\left(\frac{t}{2}\right)\right)^{-2n}(1-|z|^2)^{2n-2}dz\right| \\ &= \frac{2n-1}{\pi}\left|\int_D \cosh^{-2n}\left(\frac{t}{2}\right)\left(1-e^{-i\phi}\tanh\left(\frac{t}{2}\right)z\right)^{-2n}(1-|z|^2)^{2n-2}dz\right| \\ &= \frac{2n-1}{\pi}\left|\int_D \cosh^{-2n}\left(\frac{t}{2}\right)\sum_{l=0}^{\infty}\binom{-2n}{l}e^{-il\phi}\tanh^l\left(\frac{t}{2}\right)\right. \\ &\qquad \left.\times z^l(1-|z|^2)^{2n-2}dz\right| = \frac{2n-1}{\pi}\left|\int_D \cosh^{-2n}\left(\frac{t}{2}\right)(1-|z|^2)^{2n-2}dz\right| \\ &= \frac{2n-1}{\pi}\cosh^{-2n}\left(\frac{t}{2}\right)\left|\int_0^{2\pi}\int_0^1 (1-r^2)^{2n-2}r\,dr\,d\theta\right| \\ &= \cosh^{-2n}\left(\frac{t}{2}\right), \end{aligned}$$

where

$$\binom{-2n}{l} = \frac{(-2n)(-2n-1)\cdots(-2n-l+1)}{l!}.$$

Thus, if we denote the integral in Theorem 8 by I, then

$$\begin{aligned} I &= 2\pi \int_0^\infty \cosh^{-4n} \sinh t\, dt \\ &= 4\pi \int_0^\infty \cosh^{-4n+1}\left(\frac{t}{2}\right) \sinh\left(\frac{t}{2}\right) dt \\ &= 8\pi \int_1^\infty t^{-4n+1} dt = \frac{4\pi}{2n-1} \end{aligned}$$

and the proof is complete. □

In light of Theorems 7 and 8, we see that for $n \in \frac{1}{2}\mathbb{Z}$ with $n \geq 1$, the discrete series representations $\pi_n : \mathrm{SU}(1,1) \to U(X_n)$ are square-integrable, the function $\mathbf{1}$ is an admissible wavelet for each $\pi_n : \mathrm{SU}(1,1) \to \mathrm{U(X_n)}$ and the wavelet constant for $\mathbf{1}$ is equal to $4\pi/(2n-1)$. The admissible wavelet $\mathbf{1}$ is the first element in the orthonormal basis $\{z^k : k = 0, 1, \dots\}$ for X_n. Hence we also call $\mathbf{1}$ the vacuum state associated to the discrete series representations of SU(1,1) on X_n.

4. Paraproducts

Let $F \in L^1(\mathrm{SU}(1,1))$. Then the localization operator $L_{F,\mathbf{1}} : X_n \to X_n$ associated to the symbol F and the admissible wavelet $\mathbf{1}$ is defined by

$$(L_{F,\mathbf{1}} u, v)_{X_n} = \frac{2n-1}{4\pi} \int_{\mathrm{SU}(1,1)} F(g)(u, \pi(g)\mathbf{1})_{X_n} (\pi(g)\mathbf{1}, v)_{X_n} d\mu(g)$$

for all functions u and v in X_n. We can give a formula for $L_{F,\mathbf{1}} : X_n \to X_n$ in terms of a paraproduct when the symbol F is a function of t only.

Theorem 9 *Let F be a function in $L^1(\mathrm{SU}(1,1))$ such that*

$$F(\phi, t, \psi) = \tau(\sinh^2 t), \quad (\phi, t, \psi) \in \mathrm{SU}(1,1).$$

Then the localization operator $L_{F,\mathbf{1}} : X_n \to X_n$ associated to the symbol F and the vacuum state $\mathbf{1}$ is given by

$$(L_{F,\mathbf{1}} u, v)_{X_n} = \int_0^\infty \tau(t) p_{\pi_n,\mathbf{1}}(u, v)(t)\, dt$$

for all u and v in X_n given by

$$u(z) = \sum_{k=0}^\infty a_k z^k, \quad z \in D,$$

and

$$v(z) = \sum_{k=0}^{\infty} b_k z^k, \quad z \in D,$$

in X_n, respectively, where the power series converge absolutely and uniformly on every compact subset of D, $p_{\pi_n,\mathbf{1}}(u,v)$ is the paraproduct of u and v associated to the vacuum state $\mathbf{1}$ of the representation π_n of $\mathrm{SU}(1,1)$ *on X_n and is given by*

$$p_{\pi_n,\mathbf{1}}(u,v)(t) = (2n-1)(1+t)^{-2n} \sum_{k=0}^{\infty} a_k \bar{b}_k \left(\frac{t}{1+t}\right)^k, \quad t > 0.$$

Proof For $k = 0, 1, 2, \ldots$ and $g = (\phi, t, \psi) \in \mathrm{SU}(1,1)$,

$$(\pi_n(g)\mathbf{1}, z^k)_{X_n} = (\pi_n(\phi,t,\psi)\mathbf{1}, z^k)_{X_n} = (2n-1)e^{-in\psi}$$
$$\times \cosh^{-2n}\left(\frac{t}{2}\right)\binom{-2n}{k} e^{-ik\phi} \tanh^k\left(\frac{t}{2}\right)\beta(k+1, 2n-1),$$

where β is the beta function, *i.e.*,

$$\beta(k+1, 2n-1) = \frac{\Gamma(k+1)\Gamma(2n-1)}{\Gamma(2n+k)}.$$

For $k = 0, 1, 2, \ldots,$

$$\begin{aligned}
&\binom{-2n}{k}\beta(k+1, 2n-1) \\
= \;& (-1)^k \frac{(2n)(2n+1)\cdots(2n+k-1)}{k!} \frac{\Gamma(k+1)\Gamma(2n-1)}{\Gamma(2n+k)} \\
= \;& (-1)^k \frac{(2n)(2n+1)\cdots(2n+k-1)\Gamma(2n-1)}{\Gamma(2n+k)} \\
= \;& (-1)^k \frac{\Gamma(2n-1)}{\Gamma(2n)} = \frac{(-1)^k}{2n-1}.
\end{aligned}$$

Therefore for $0 \le \phi < 4\pi$, $t \ge 0$ and $0 \le \psi < 2\pi$,

$$(\pi(\phi,t,\psi)\mathbf{1}, z^k)_{X_n} = e^{-in\psi}\cosh^{-2n}\left(\frac{t}{2}\right) e^{-ik\phi}\tanh^k\left(\frac{t}{2}\right)$$

for $k = 0, 1, 2, \ldots$, and hence

$$\begin{aligned}
& (L_{F,\mathbf{1}}u, v)_{X_n} \\
= \; & \frac{2n-1}{2} \int_0^\infty \tau(\sinh^2 t) \sum_{k=0}^\infty a_k \overline{b}_k \tanh^{2k}\left(\frac{t}{2}\right) \cosh^{-4n}\left(\frac{t}{2}\right) \sinh t \, dt \\
= \; & (2n-1) \int_0^\infty \tau(t) \sum_{k=0}^\infty a_k \overline{b}_k \left(\frac{t}{1+t}\right)^k (1+t)^{-2n} dt \\
= \; & \int_0^\infty \tau(t) p_{\pi_n,\mathbf{1}}(u,v)(t) \, dt,
\end{aligned}$$

as asserted. □

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THE FLIGHT OF AN AIRCRAFT ALONG A GIVEN TRAJECTORY AND OPTIMAL FLIGHT CONTROL

N. E. Tovmasyan
State Engineering University of Armenia,
105 Terian Str., Yerevan, 375009, Armenia
mamgin@instmath.sci.am

Abstract The paper describes classes of curves along which the flight of an aircraft is possible under suitable choice of the initial velocity, reactive force and fuel consumption. The optimal choice of the flight trajectory depending on the flight objectives also is detemined.

Keywords: flight along trajectories, optimal flight trajectory

Mathematics Subject Classification (2000): 34H05, 49K15

1. Introduction

On the actual stage of space technology the optimal choice of power setting and trajectory towards the target has great importance. These problems are also closely related to the construction of new types of aircrafts corresponding to predefined aims. They can be reduced to some nonlinear differential equations with parameters and to the optimal choice of these parameters.

The investigation of these problems leads to inverse problems of dynamics. Examples of such type of problems are Newton's problem of determination of the force, causing the rotation of planets around the sun and Bertrand's problem of determination of the force, causing the movement of a material point along a conic section [1] (Ch. XI, Sec. 222–233). Later such type of problems were considered by Zhukovskii [2], Meschersky [3], Chaplygin [4], Gorjachev [5], etc. A general approach to these problems is described in the monograph by Galiulin [6].

However, any special case requires a supplementary investigation in order to prove the possibility of the flight of an aircraft along a given

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trajectory and to determine the flight basic characteristics. Some results in this direction were obtained in our papers [7]–[10].

We suppose that the flight takes place near the earth's surface and the flight range is small compared with the earth's radius. So we assume that the earth's surface is a plane and the gravity acceleration $\mathbf{g}$ is constant.

The aim of this paper is to describe classes of curves along which the flight of an aircraft is possible under a suitable choice of initial velocity, reactive force, fuel consumption and to choose optimal flight trajectories.

2. Aircraft flight along a given trajectory

In this section we consider the question of the possibility of a flight of a wingless aircraft along a given trajectory, with simultaneous action of the gravity force $m(t)\mathbf{g}$, reactive force $\mathbf{F}_R$, friction force $\mathbf{F}_c$, where $m(t)$ is the aircraft mass at a given moment t and $\mathbf{g}$ is the gravity acceleration directed towards the earth surface, $g = 9.8m/sec^2$.

Let XOY be the coordinate plane, where the positive direction of the y-axis is directed upward to the earth surface and the x-axis is directed parallel to the surface. Assume that the aircraft starts from the origin and moves in the plane XOY, $x > 0$. It is known [4] that the value of reactive force is defined by

$$F_R = -k_0 \frac{dm(t)}{dt}, \tag{1}$$

where $-dm(t)/dt$ is the aircraft mass decreasing rate, due to fuel consumption, k_0 is a positive constant that depends on construction of reactive engine and fuel quality.

We will consider wingless aircrafts moving with velocities smaller than the acoustic velocity. We have

$$\mathbf{F}_c = -æ(x, y)\mathbf{V}, \tag{2}$$

where $æ(x, y)$ is a nonnegative continuous function of the point (x, y) and $\mathbf{V}$ is the velocity vector of the aircraft. Notice that the coefficient $æ(x, y)$ depends both on aircraft construction and weather conditions.

The flight of an aircraft with varying mass is defined by the well-known Meschersky's equation [4]

$$m(t)\frac{d\mathbf{V}}{dt} = m(t)\mathbf{g} + \mathbf{F}_R + \mathbf{F}_c. \tag{3}$$

We will assume that the direction of $\mathbf{F}_R$ coincides with that of $\mathbf{V}$. Then

$$\mathbf{F}_R = F_R \frac{\mathbf{V}}{V}, \tag{4}$$

where $\mathbf{V}$ is the magnitude of the velocity $\mathbf{V}$. Let V_1 and V_2 be the components of the vector $\mathbf{V}$ on the x– and y–axes. The equation (3) can be written as

$$m(t)\frac{dV_1}{dt} = F_R\frac{V_1}{V} - æ(x,y)V_1, \tag{5}$$

$$m(t)\frac{dV_2}{dt} = F_R\frac{V_2}{V} - æ(x,y)V_2 - gm(t). \tag{6}$$

As a new independent variable we take x. Then

$$\frac{dV_1}{dt} = \frac{dV_1}{dx}\cdot\frac{dx}{dt} = \frac{dV_1}{dx}V_1, \tag{7}$$

$$\frac{dV_2}{dt} = \frac{dV_2}{dx}, \qquad \frac{dm}{dt} = \frac{dm}{dx}V_1. \tag{8}$$

Suppose that the flight is carried out along the trajectory

$$y = f(x), \quad 0 \le x \le x_0, \tag{9}$$

where $f(x)$ is a three times differentiable function. Differentiating both sides of (9) by t we get

$$V_2 = f'(x)V_1, \quad 0 \le x \le x_0. \tag{10}$$

Using (1), (7) and (8) the equations (5) and (6) can be written in the form

$$F_R = -V_1\frac{dm}{dx}, \quad 0 \le x \le x_0, \tag{11}$$

$$mV_1\frac{dV_1}{dx} = F_R\frac{V_1}{V} - æ(x,y)V_1, \quad 0 \le x \le x_0, \tag{12}$$

$$mV_1\frac{dV_2}{dx} = F_R\frac{V_2}{V} - æ(x,y)V_2 - gm, \quad 0 \le x \le x_0. \tag{13}$$

Let m_1 be the mass of consumed combustible along the trajectory (9) and m_2 be the mass of the aircraft without combustible. We assume that m_2 is fixed, i.e.

$$m(x_0) = m_2. \tag{14}$$

Thus, the problem is reduced to the resolution of the system of nonlinear differential equations (10)–(13) with boundary condition (14) with respect to $V_1(x)$, $V_2(x)$, $F_R(x)$ and $m(x)$ in the segment $0 \le x \le x_0$. In problem (10)–(14) the terms $f(x)$, k_0, $æ(x,y)$ g and m_2 are known and $k_0 > 0$, $æ(x,y) \ge 0$ $g = 9.8$ and $m_2 > 0$. The physical considerations stipulate that $F_R(x)$ is a continuous function, while $m(x)$ $V_1(x)$and $V_2(x)$ are continously differentiable functions on the segment $[0, x_0]$ and

$$m'(x) \le 0,\ m(x) \ge m_2,\ F_R(x) \ge 0,\ V_1(0) > 0, \quad 0 \le x \le x_0. \tag{15}$$

Let V_1, V_2, F_R and m be the solution of problem (10)–(14). We have

$$m_1 = m(0) - m_2,$$

$$t = t(x) = \int_0^x \frac{dx}{V_1(x)}, \tag{16}$$

$$t_0 = \int_0^{x_0} \frac{dx}{V_1(x)}, \tag{17}$$

where $t(x)$ is the time to fly to the point (x,y) along the trajectory $y = f(x)$ and t_0 is the time to fly along the trajectory $y = f(x)$, $0 \le x \le x_0$. Let $x = x(t)$ be the inverse of the function $t = t(x)$. From (9) and (16) we have

$$x = x(t), \quad y = f(x(t)), \quad 0 \le t \le t_0. \tag{18}$$

Therefore, resolving the problem (10)–(14) and using (18) we can find all the parameters of the flight depending on the variable t.

In particular, it follows from the obtained equations that the flight along the trajectory (9) is possible if and only if the problem (10)–(14) is solvable.

Theorem 1 *For any function* $æ(x,y)$ *the problem* (10)–(14) *is solvable if and only if the function* $f(x)$ *satisfies the conditions:*

$$f''(x) < 0, \quad f'''(x) \ge 0, \quad 0 \le x \le x_0. \tag{19}$$

Proof Differentiating both sides of (10) by x we obtain

$$V_2'(x) = f''(x)(V_1'(x))^2 + f'(x)V_1'(x). \tag{20}$$

Using (9), (10) and (20) the equations (12) and (13) can be transformed as

$$m(x)V_1(x)V_1'(x) = \frac{F_R(x)}{\sqrt{1 + (f'(x))^2}} - æ(x, f(x))V_1(x), \quad 0 \le x \le x_0, \tag{21}$$

$$f''(x)V_1^2(x) = -g, \quad 0 \le x \le x_0. \tag{22}$$

By (22) we have

$$f''(x) < 0, \quad 0 \le x \le x_0. \tag{23}$$

Therefore, (23) is a necessary condition for solvability of problem (10)–(14). Assume that (23) is fulfilled. Since $V_1(x)$ is continuous and $V_1(0) > 0$ we obtain

$$V_1(x) = \frac{\sqrt{g}}{\sqrt{-f''(x)}}. \tag{24}$$

Substituting $V_1(x)$ from (24) into (21) we get

$$F_R(x) = (m(x)\psi_1(x) + \psi_2(x)) \frac{\sqrt{g}}{\sqrt{-f''(x)}}, \tag{25}$$

where

$$\psi_1(x) = \frac{1}{2} \frac{f'''(x)\sqrt{g}}{(-f''(x))^{3/2}} \sqrt{1 + (f'(x))^2}, \tag{26}$$
$$\psi_2(x) = æ(x, f(x))\sqrt{1 + (f'(x))^2}.$$

Assume that $æ(x, y) = 0$. Then from (15) and (25) we have

$$f'''(x) > 0, \quad 0 \le x \le x_0. \tag{27}$$

Therefore, (27) is also a necessary condition for the solvability of problem (10)–(14) for any function $æ(x, y) \ge 0$.

Now we prove that the condition (19) is sufficient for the solvability of problem (10)–(14). Substituting $F_R(x)$ from (11) into (25) we obtain

$$-k_0 \frac{dm(x)}{dx} = m(x)\psi_1(x) + \psi_2(x), \quad 0 \le x \le x_0. \tag{28}$$

So we have got the differential equation (28) with the boundary condition (14) to determine $m(x)$. We set

$$\psi_3(x) = \exp\left(\frac{1}{k_0} \int_0^x \psi_1(\tau)\, d\tau\right). \tag{29}$$

A solution of equation (28) with boundary condition (14) is given by

$$m(x) = \frac{1}{k_0\psi_3(x)} \int_x^{x_0} \psi_2(\tau)\psi_3(\tau)\, d\tau + \frac{m_2\psi_3(x_0)}{\psi_3(x)}, \quad 0 \le x \le x_0. \tag{30}$$

Since $\psi_1(x) \geq 0$ and $\psi_2(x) \geq 0$, the functions $F_R(x)$ and $m(x)$ defined by (25) and (30) respectively, satisfy (15). From (10) and (24) we have

$$V(x) = \frac{\sqrt{g}}{\sqrt{-f''(x)}}\sqrt{1 + (f'(x))^2}, \tag{31}$$

$$V_0^2(x) = \frac{-g(1 + (f'(0))^2)}{f''(0)}, \tag{32}$$

where V_0 is the aircraft initial velocity.

Next, substituting $V_1(x)$ from (24) into (16) and (17) we obtain

$$t = t(x) = \int_0^x \frac{\sqrt{-f''(x)}}{\sqrt{g}} dx, \tag{33}$$

$$t_0 = \int_0^{x_0} \frac{\sqrt{-f''(x)}}{\sqrt{g}}. \tag{34}$$

Since $f'''(x)$ is continuous and $f''(x) < 0$ the function $x = x(t)$ (the inverse of $t = t(x)$) is twice continuously differentiable on $[0, t_0]$. By (30)

$$m_1 = m(0) - m_2 = \frac{1}{k_0}\int_0^{x_0} \psi_2(\tau)\psi_3(\tau)\, d\tau + m_2(\psi_3(x_0) - 1). \tag{35}$$

This completes the proof of Theorem 1.

In the process of the proof of Theorem 1 we also have established the follwing result.

Theorem 2 *If the condition* (19) *is fulfilled, then the problem* (10)–(14) *has a unique solution determined by* (10), (24), (25) *and* (30).

Thus, if (19) is fulfilled, then to realize a flight along the trajectory $y = f(x)$ it is necessary to determine the reactive force $F_R(x)$, initial velocity V_0 and the mass of consumed combustible m_1 by formulae (25), (32) and (35) respectively, where m_2 is the mass of aircraft without combustible and $m(x)$ is defined by (30).

A special case. Let the flight trajectory be the parabola

$$y = f(x) = -bx^2 + kx, \quad 0 \leq x \leq x_0. \tag{36}$$

Clearly this curve satisfies (19) if and only if

$$b > 0. \tag{37}$$

Assume that the condition (37) is fulfilled. Then $F_R(x)$, V_0, m_1 and t_0 can be determined by

$$F_R(x) = \frac{\sqrt{g}}{\sqrt{2b}} æ(x, f(x))\sqrt{1+(k-2bx)^2}, \quad V_0 = \frac{\sqrt{g(1+k^2)}}{\sqrt{2b}}, \quad (38)$$

$$m_1 = \frac{1}{k_0}\int_0^{x_0} æ(x, f(x))\sqrt{1+(k-2bx)^2}\, dx, \quad t_0 = \frac{\sqrt{2b}x_0}{\sqrt{g}}, \quad (39)$$

where $f(x)$ is as in (36). It follows from (35)–(39) that in this special case F_R, V_0, m_1 and t_0 depend on the trajectory (36) and do not depend on the aircraft mass m_2. Cartainly, the mass m_2 is an essential parameter for the flight, since the initial velocity $V_0 = \sqrt{g(1+k^2)}/\sqrt{2b}$ should be transmitted to the aircraft with the mass $m(0) = m_1 + m_2$, where m_1 is defined by (39). In the general case the parameters F_R, V_0, m_1 and t_0 depend also on m_2.

Let now the initial velocity V_0 be also fixed. Then we have an additional boundary condition

$$V(0) = V_0. \quad (40)$$

In this case the above posed problem is reduced to the resolution of sytem (10)–(13) with boundary conditions (14) and (40). The following two theorems are similar in spirit to Theorem 1.

Theorem 3 *For any function* $æ(x, y) \geq 0$ *the problem* (10)–(14), (40) *is solvable if and only if the function* $f(x)$ *satisfies the conditions*

$$f''(x) < 0, \quad f'''(x) \geq 0, \quad 0 \leq x \leq x_0, \quad (41)$$

$$-\frac{(1+(f'(0))^2}{f''(0)} = \frac{V_0^2}{g}. \quad (42)$$

Theorem 4 *If the conditions* (41) *and* (42) *are fulfilled, the solution of problem* (10)–(14), (40) *coincides with that of* (10)–(14).

3. Integral representation of flight trajectory and fire area

Consider now the flight of an aircraft from the point $(0, 0)$ to the given point (x_0, y_0) $x_0 > 0$. According to Theorem 1 the flight trajectory $y = f(x)$ satisfies the conditions

$$f''(x) < 0, \quad f'''(x) \geq 0, \quad 0 \leq x \leq x_0, \quad (43)$$

$$f(0) = 0, \quad f(x_0) = y_0. \tag{44}$$

Consider the functional

$$\mathcal{Z} = \int_0^1 (1-x)^2 \omega(x)\,dx,$$

where the norm of ω is defined by

$$||\omega|| = \int_0^1 x(2-x)\omega(x)\,dx.$$

Theorem 5 *If $f(x)$ satisfies the conditions* (43), (44), *then it can be represented as*

$$f(x) = \frac{y_0}{x_0}x + kx_0 f_0\left(\frac{x}{x_0}\right), \quad 0 \le x \le x_0, \tag{45}$$

where

$$f_0(x) = x - bx^2 + \frac{\rho}{||\omega||}\int_0^x (x-t)^2\omega(t)\,dt, \quad 0 \le x \le 1, \tag{46}$$

$$b = 1 + \frac{\rho\mathcal{Z}(\omega)}{||\omega||}, \quad 0 \le \rho < 1, \quad k > 0 \tag{47}$$

and $\omega(x)$ is an arbitrary nonnegative function on $[0,1]$ $\omega(x) \neq 0$ (i.e. $\omega(x)$ does not vanish identically).

Proof It is easy to check that any function of the form (45) satisfies conditions (43) and (44). Now we prove the converse, that is any function $f(x)$ satisfying conditions (43) and (44) can be represented in the form (45). First we prove the assertion for $y_0 = 0$, that is,

$$f(0) = 0, \quad f(x_0) = 0. \tag{48}$$

It follows from (43) and (48) that $f'(0) > 0$. Indeed, assuming the opposite $f'(0) \le 0$, the condition implies $f'(x) < 0$ for $0 < x \le x_0$. Therefore, $f(x_0) < f(0)$, which is impossible.

Let $f(x)$ satisfy conditions (43) and (48). The function

$$f_0(x) = \frac{f(xx_0)}{x_0 f'(0)} \tag{49}$$

satisfies the conditions

$$f_0''(x) < 0, \quad f_0'''(x) \ge 0, \quad 0 \le x \le 1, \tag{50}$$

$$f_0(0) = 0, \quad f_0(1) = 0 \quad f_0'(0) = 1. \tag{51}$$

By (49)

$$f(x) = kx_0 f_0\left(\frac{x}{x_0}\right), \quad k = f'(0) > 0. \tag{52}$$

We have

$$f_0(x) = f_0(0) + f_0'(0)x + \frac{f_0''(0)x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 f'''(t)\,dt, \quad 0 \le x \le 1. \tag{53}$$

Let $f'''(x) \equiv 0$. By (51) and (53)

$$f_0(x) = x - x^2. \tag{54}$$

It follows from (52) and (54) that $f(x)$ admits the representation (45) for $y_0 = 0$ and $\rho = 0$. Let $f'''(x) \neq 0$. By (51) and (53) we obtain

$$f_0(x) = x - bx^2 + \frac{\rho}{||\omega||}\int_0^x (x-t)^2 \omega(t)\,dt, \tag{55}$$

where

$$\omega(x) = \frac{1}{2} f_0'''(x) \ge 0, \quad \rho = ||\omega|| > 0,$$

$$b = 1 + \frac{\rho \mathcal{Z}(\omega)}{||\omega||}. \tag{56}$$

Taking into acount that $f_0''(1) < 0$ from (55) and (56) we obtain $\rho < 1$. Therefore, $f(x)$ again admits the representation (45).

Let now $y_0 \neq 0$. consider the function

$$F(x) = f(x) - \frac{y_0}{x_0}x, \quad 0 \le x \le x_0. \tag{57}$$

It follows from (43) and (44) that $F(0) = 0$, $F(x_0) = 0$ and

$$F''(x) < 0, \quad F'''(x) \ge 0.$$

Therefore, $F(x)$ admits the representation (45) for $y_0 = 0$. Substituting this representation into (57), we get (45) for $y_0 \neq 0$. This completes the proof of Theorem 5.

Let the initial velocity V_0 be fixed and the flight be carried out from the point $(0, 0)$ to $(x_0, 0)$. Then, according to Theorem 3 the function $f(x)$ satisfies the conditions

$$f''(x) < 0, \quad f'''(x) \ge 0, \quad 0 \le x \le x_0, \tag{58}$$

$$-\frac{1+(f'(0))^2}{f''(0)} = \frac{V_0^2}{g}, \tag{59}$$

$$f(0) = 0, \quad f(x_0) = 0. \tag{60}$$

We have the following results.

Theorem 6 *Let $x_0 \le V_0^2/g$ and the conditions* (58)–(60) *be fulfilled. Then the function $f(x)$ can be represented in the form* (45) *where $y_0 = 0$ and*

$$k = b\gamma \pm \sqrt{b^2\gamma^2 - 1}, \quad \gamma = \frac{V_0^2}{gx_0}. \tag{61}$$

Proof Let $f(x)$ satisfy conditions (58) and (60). Then it can be represented in the form (45) with $y_0 = 0$. Substituting $f(x)$ from (45) into (59) we obtain

$$k^2 - 2bk\gamma + 1 = 0. \tag{62}$$

The condition $x_0 \le V_0^2/g$ implies $b\gamma \ge 1$. Hence both roots (61) of equation (62) are positive. Theorem 6 is proved.

Theorem 7 *Let $x_0 > V_0^2/g$ and the conditions* (58)–(60) *be fulfilled. Then the function $f(x)$ can be represented as*

$$f(x) = kx_0 f_0\left(\frac{x}{x_0}\right), \quad 0 \le x \le x_0, \tag{63}$$

where

$$f_0(x) = x - bx^2 + \frac{\rho}{||\omega||}\int_0^x (x-t)^2\omega(t)\,dt, \tag{64}$$

$$b = 1 + \frac{\rho\mathcal{Z}(\omega)}{||\omega||}, \tag{65}$$

$$k = b\gamma \pm \sqrt{b^2\gamma^2 - 1}, \quad \gamma = \frac{V_0^2}{gx_0}, \tag{66}$$

$$\frac{||\omega||}{\mathcal{Z}(\omega)}\left(\frac{1}{\gamma} - 1\right) \le \rho < 1, \tag{67}$$

and $\omega(x)$ is an arbitrary nonnegative continuous function on $[0,1]$ satisfying the condition

$$\mathcal{Z}(\omega) > \left(\frac{1}{\gamma} - 1\right)||\omega||. \tag{68}$$

Proof If $f(x)$ admits the representation (63), then clearly it satisfies the conditions (58)–(60). Now we prove the converse, that is any function $f(x)$ satisfying conditions (58)–(60) can be represented in the form (63). According to Theorem 5 this function can be represenred in the form (45) with $y_0 = 0$. Substituting this representation into (59) we get (62). Equation (62) has positive solutions if and only if

$$b\gamma \geq 1, \tag{69}$$

and these solutions are determined by (66). From (47) and (69) we obtain (67) and (68). Theorem 7 is proved.

Since $0 < \gamma < 1$ by (68)

$$0 < \frac{||\omega||}{\mathcal{Z}(\omega)} \left(\frac{1}{\gamma} - 1 \right) < 1. \tag{70}$$

This means that if $\omega \geq 0$, $\omega \neq 0$ and (68) is valid, then there always exist ρ and $k > 0$ satisfying (62) and (67).

Considering

$$\omega(x) = \omega_m(x) = (1 - x)^m,$$

where m is a nonnegative integer, we get

$$\frac{\mathcal{Z}(\omega_m)}{||\omega_m||} = \frac{m+1}{2}. \tag{71}$$

In view of (68) and (71) we have

Corollary 1 *For any x_0 and V_0 there exist an infinite number of polynomials satisfying* (58)–(60).

By the same way we can study the above problems where V_0 is given and the flight is carried out from the point $(0, 0)$ to the point (x_0, y_0).

Let now $f(x)$ be a polynomial of degree n ($n \geq 2$). We have $\omega(x) \equiv 0$ for $n = 2$ and $\omega(x)$ is a polynomial of degree $n - 3$ for $n \geq 3$. In this connection arises a question: if the order of the polynomial $f(x)$ and the initial velocity V_0 are fixed, is then the flight of the aircraft to an arbitrary point (x_0, y_0) possible? The theorem that follows answers this question. Let Q_m be the set of polynomials P_m of order m satisfying the conditions

$$P_m(x) \geq 0, \quad 0 \leq x \leq 1, \quad P_m(x) \neq 0.$$

Theorem 8 *The problem* (58)–(60) *is solvable in the class of polynomials of order n if and only if x_0 satisfies the inequalities*

$$x_0 \leq \frac{V_0^2}{g} \quad \text{for} \quad n = 2 \tag{72}$$

and

$$x_0 < (1 + ||\mathcal{Z}_{n-3}||)\, \frac{V_0^2}{g} \quad \text{for} \quad n > 2, \tag{73}$$

where $||\mathcal{Z}_m||$ *is the norm of the functional* $\mathcal{Z}$ *in the set* Q_m *defined by*

$$||\mathcal{Z}_m|| = \sup_{P_m \in Q_m} \frac{\mathcal{Z}(P_m)}{||P_m||}.$$

Proof Let $n > 2$ and $P_m(x)$ be a solution of problem (58)–(60) in the class of polynomials of order n. In the proof of Theorem 7 we have established (68) for $\omega = P_n'''(x)$. The inequality (68) can be written as

$$x_0 < \left(1 + \frac{\mathcal{Z}(\omega)}{||\omega||}\right) \frac{V_0^2}{g}. \tag{74}$$

From (74) follows (73). Hence (73) is a necessary condition for solvability of the considered problem in the class of polynomials of order n.

Let now (73) be fulfilled. There exists a polynomial $\omega = P_m \in Q_m$ satisfying (74). In view of Theorem 7 this implies that the problem (58)–(60) is solvable in the class Q_n for $n > 2$. The case $n = 2$ can be proved similarly. Theorem 8 is proved.

Thus, the fire range (surface-surface) with initial velocity V_0 along a trajectory $y = P_n(x)$ is determined by inequalities (72) and (73), where $P_n(x)$ is a polynomial of order n. It can be proved that $||\mathcal{Z}_m||$ is finite for any fixed m and monotonely tends to infinity as $m \to \infty$. It follows from (71) that

$$||\mathcal{Z}_m|| \geq \frac{m+1}{2}. \tag{75}$$

Theorem 8 and (75) imply

Corollary 2 *If* x_0 *satisfies the inequality*

$$x_0 < \frac{n}{2} \frac{V_0^2}{g},$$

then the flight of the aircraft from the point $(0, 0)$ *to the point* $(x_0, 0)$ *can be realized along the trajectory* $y = P_n(x)$*, where* $P_n(x)$ *is a polynomial of order* n.

We have calculated $||\mathcal{Z}_m||$ for $m = 0, 1, 2$. The corresponding values are $||\mathcal{Z}_0|| = 1/2$, $||\mathcal{Z}_1|| = 1$ and $||\mathcal{Z}_2|| = 1 + 4/\sqrt{19}$. It is of great interest to calculate $||\mathcal{Z}_m||$ for any $m > 2$.

4. Optimal control of flight trajectory

Let the aircraft be flying from point $(0,0)$ to the point $(x_0,0)$ along a trajectory of the form $y = f(x)$, where $f(x)$ is a polynomial of order n. Let x_0 satisfy the inequalities (72) and (73), where V_0 is the initial velocity of the aircraft. From Theorem 7 follows that the polynomial P_n contains $n-3$ arbitrary constants satisfying certain inequalities. Hence the fuel consumption and other basic characteristics of the flight depend on these parameters. The basic problem of optimal control is the optimal choice of these parameters, according to the flight objectives (e.g. minimal fuel consumption, maximal flight distance, minimal flight time, maximal flight altitude, etc).

In this section we consider a special case, when the flight is carried out along the trajectory

$$y = f(x) = ax + bx^2 \tag{76}$$

and the initial velocity V_0 satisfies the inequality

$$V_0 \leq \beta, \tag{77}$$

where β is a positive constant.

First the criterion of optimality will be the minimal fuel consumption along the trajectory joining the points $(0,0)$ and $(x_0,0)$ for fixed $x_0 > 0$. This trajectory is called optimal. It follows from (72) and (77) that

$$0 < x_0 \leq \frac{\beta^2}{g}. \tag{78}$$

On the other hand, if x_0 satisfies (78), then there exists $V_0 \leq \beta$ satisfying (72). Therefore (78) is a necessary and sufficient condition for the possibility of the flight from $(0,0)$ to the point $(x_0,0)$ along a trajectory of the form (76) ($V_0 \leq \beta$). So we will suppose that x_0 is a given number satisfying (78).

According to Theorem 7 and (77) the function $f(x)$ satisfies the conditions

$$f''(x) < 0, \quad f'''(x) \geq 0, \quad 0 \leq x \leq x_0, \tag{79}$$

$$f(0) = 0, \quad f(x_0) = 0, \tag{80}$$

$$-\frac{1 + (f''(0))^2}{f''(0)} \leq \frac{\beta^2}{g}. \tag{81}$$

It follows from (76), (79) and (80) that the flight trajectory is of the form

$$y = f(x) = kx\left(1 - \frac{x}{x_0}\right), \tag{82}$$

where k is a positive constant. Substituting $f(x)$ from (82) into (81) we get

$$\frac{1 + k^2}{2k} \leq \gamma, \tag{83}$$

where

$$\gamma = \frac{\beta^2}{g x_0}. \tag{84}$$

From (78) follows

$$\gamma \geq 1. \tag{85}$$

Consider the equation

$$\frac{1 + k^2}{2k} = \gamma. \tag{86}$$

Since $\gamma \geq 1$ both roots k_1 and k_2 of equation (86) are positive and

$$k_1 = \frac{1}{\gamma + \sqrt{\gamma^2 - 1}}, \quad k_2 = \gamma + \sqrt{\gamma^2 - 1}, \quad 0 < k_1 \leq 1, \quad k_2 \geq 1. \tag{87}$$

The inequality (83) is equivalent to

$$k_1 \leq k \leq k_2. \tag{88}$$

So the flight trajectory (82) contains one arbitrary parameter k that satisfies (88). Since $f'''(x) \equiv 0$ the formula (35) becomes

$$m_1 = \frac{1}{k_0} \int_0^{x_0} æ(x, f(x)) \sqrt{1 + (f'(x))^2}\, dx, \tag{89}$$

where m_1 is the mass of consumed fuel. Substituting $f(x)$ from (82) into (89) we get

$$m_1 = m_1(k), \tag{90}$$

where $m_1(k)$ is some positive function of k defined on the interval $[k_1, k_2]$. Hence

$$\min m_1 = m_1(k_3), \tag{91}$$

where k_3 is the minimum point of the function $m_1(k)$ on the segment $[k_1, k_2]$. So the optimal trajectory is defined by (82) with $k = k_3$. If $æ(x, y) = æ = \text{const} \geq 0$, then $k_3 = k_1$ and

$$\min m_1 = \frac{æx_0}{k_0} \int_0^1 \sqrt{1 + k_1^2 \tau^2}\, d\tau, \quad y = k_1 x \left(1 - \frac{x}{x_0}\right). \tag{92}$$

In particular if x_0 is the maximal admissible distance equal to β^2/g and $æ = \text{const}$, then

$$k_3 = k_1 = 1, \quad \min m_1 = \frac{æ\beta^2}{k_0 g} \int_0^1 \sqrt{1 + \tau^2}\, d\tau, \quad y = x - \frac{gx^2}{\beta^2}. \tag{93}$$

Let the initial velocity V_0 and the mass m_1 of consumed fuel satisfy

$$V_0 \leq \beta, \quad m_1 \leq m_0 \tag{94}$$

and let

$$\mathbf{F}_c = -æ\mathbf{V}, \quad F_R = -k_0 \frac{dm}{dt}, \tag{95}$$

where β, m_0, $æ$ and k_0 are given positive constants.

Let the flight be carried out along a trajectory of the form (76) from the point $(0, 0)$ to the point $(x_0, 0)$. Now the criterion of optimality is the maximal flight distance x_0. The problem is: choose a trajectory of the form (76) such that under the above restrictions the flight distance x_0 be maximal. Any such trajectory will be called optimal. To state the results we introduce the notations

$$\alpha = \frac{æ\beta^2}{gk_0} \int_0^1 \sqrt{1 + \tau^2}\, d\tau, \quad \nu = \frac{\beta^2}{g}, \tag{96}$$

$$\psi(\zeta) = \frac{æ\zeta}{k_0} \int_0^1 \sqrt{1 + k_1^2(\zeta)\tau^2}\, d\tau, \quad 0 \leq \zeta \leq \nu, \tag{97}$$

$$k_1(\zeta) = \frac{\zeta}{\nu + \sqrt{\nu^2 - \zeta^2}}, \quad 0 \leq \zeta \leq \nu. \tag{98}$$

It follows from (97) and (98) that $\psi(\zeta)$ and $k_1(\zeta)$ are increasing functions on $[0, \nu]$ and $\psi(0) = 0$, $\psi(\nu) = \alpha$. One has the following

Theorem 9 *If $m_0 \geq \alpha$, then the optimal flight trajectory is defined by*

$$y = x \left(1 - \frac{x}{\nu}\right), \quad 0 \leq x \leq \nu. \tag{99}$$

If $m_0 < \alpha$, then the optimal trajectory is defined by

$$y = k_1(\zeta)\left(1 - \frac{x}{\zeta}\right), \quad 0 \le x \le \zeta, \tag{100}$$

where ζ is the root of the equation

$$\psi(\zeta) = m_0, \quad 0 \le x \le \nu. \tag{101}$$

Proof Let $m_0 \ge \alpha$. Then the trajectory (99) joins the points $(0,0)$ and $(\nu, 0)$, and $V_0 = \beta$, $m_1 = \alpha \le m_0$ (see (32) and (89)). According to (78) the maximal flight distance along a trajectory of the form (76) is smaller than ν. Hence the trajectory (99) is optimal for $m_0 \ge \alpha$.

Let now $m_0 < \alpha$. Since $\psi(\zeta)$ is an increasing function on $[0, \nu]$ and $\psi(0) = 0$, $\psi(\nu) = \alpha > m_0$ the equation (101) has a unique root. We have shown above that if the flight is carried out from $(0,0)$ to the point $(x_0, 0)$ and $V_0 \le \beta$, then the flight trajectory of the form (76) is defined by equation (82), where x_0 and k satisfy conditions (78) and (88). Using (87) and (96) these conditions may be written as

$$0 < x_0 \le \nu, \tag{102}$$

$$k_1(x_0) \le k \le k_2(x_0), \tag{103}$$

where

$$k_1(x_0) = \frac{x_0}{\nu + \sqrt{\nu^2 - x_0^2}}, \quad k_2(x_0) = \frac{\nu + \sqrt{\nu^2 - x_0^2}}{x_0}. \tag{104}$$

By (102) and (104)

$$0 < k_1(x_0) < 1, \quad k_2(x_0) = \frac{1}{k_1(x_0)} > 1. \tag{105}$$

The fuel consumption m_1 along the trajectory (82) is defined by (89), which for $æ(x, y) = æ = \text{const}$ takes the form

$$m_1 = \frac{æ x_0}{k_0} \int_0^1 \sqrt{1 + k^2\tau^2}\, d\tau. \tag{106}$$

Hence condition $m_1 \le m_0$ may be written as

$$\frac{æ x_0}{k_0} \int_0^1 \sqrt{1 + k^2\tau^2}\, d\tau \le m_0. \tag{107}$$

Therefore, our problem is reduced to the resolution of system of inequalities (102), (103), (107) with respect to the pair (x_0, k) with the maximal value for x_0. We prove that such a solution is $(\zeta, k_1(\zeta))$, where ζ is the root of equation (101) and $k_1(\zeta)$ is defined by (98). Indeed, let ζ be a solution of (101). If $\psi(\zeta) = m_0$, $\psi(\nu) = \alpha$ and $m_0 < \alpha$ then $\zeta < \nu$, since $\psi(x)$ is an increasing function on $[0, \nu]$. It is clear that $x_0 = \zeta$ and $k = k_1(\zeta)$ satisfy (103) and (107). Hence $(\zeta, k_1(\zeta))$ is the solution of system (102), (103), (107). Let now (x_0, k) satisfy these inequalities. Then from (103) and (107) follows

$$\frac{æ x_0}{k_0} \int_0^1 \sqrt{1 + k_1^2(x_0)\tau^2}\, d\tau \le m_0. \tag{108}$$

It follows from (97), (101) and (108) that $\psi(x_0) \le \psi(\zeta)$. On the other hand, since $\psi(x)$ is an increasing function on $[0, \nu]$ and $x_0 \in [0, \nu]$, $\zeta \in [0, \nu]$ and $\psi(x_0) \le \psi(\zeta)$, we conclude that $x_0 \le \zeta$. Therefore $(\zeta, k_1(\zeta))$ is the solution of system (102), (103), (107) with the maximal value for x_0. Substituting $x_0 = \zeta$ and $k = k_1(\zeta)$ into (82) we get the optimal trajectory for $m_0 < \alpha$. Theorem 9 is proved.

Notice that we have also proved the following result.

Theorem 9 *If $V_0 \le \beta$ and $m_1 \le m_0$, then the maximal flight distance (surface-surface) along a trajectory of the form* (76) *is equal to β^2/g for $m_0 \ge \alpha$ and is ζ for $m_0 < \alpha$, where ζ is the root of equation* (101).

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ON MATHEMATICAL PROBLEMS OF TWO-DIMENSIONAL TOMOGRAPHY

A. Vagarshakyan
Institute of Matematics
of National Academy of Sciences of Armenia
24-b Marshal Bargramian ave., Yerevan 375019, Armenia
vagharshakyan@yahoo.com

Abstract In this paper the problem of restoring the inside construction of an object by its X-ray pictures is considered. We limit ourself on discussing the two dimensional case. Let us note that just this case is used in acting medical tomography devices. The problem is to find the formulas, which as good as possible restore the density, having on hand the X-ray pictures for a finite number of directions only and which are steady for $L_2(-1,1)$ distortions of the projections.

Keywords: Radon transformation, tomography, Legrende polynomials, Laplace representation

Mathematics Subject Classification (2000): 65R32, 44A12

1. Introduction

In the generally accepted mathematical model of the X-ray passage across an object, in tomography scattering processes are not taken into account. In the model, where the absorbing processes are only taken into account, lead to the Radon's transform, see [2].

The formulas, which restore the function $f(x,y)$ by its Radon's transform $\hat{f}(x,e^{i\vartheta})$ are well known, see [1], p. 131. However Radon's formula is of limited value for practical calculations.

In the real devices the function $\hat{f}(x,e^{i\vartheta})$, $-1 < x < 1, \quad 0 \le \vartheta < \pi$, is known only for a finite number $\vartheta_0, \vartheta_1, \ldots, \quad \vartheta_{n-1}$. So the problem of restoring $f(x,y)$ by the functions

$$f_j(x) = \hat{f}(x,e^{i\vartheta_j}), \quad -1 < x < 1, \quad j = 0,1,\ldots,n-1, \tag{1}$$

naturally arises.

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The data (1) are not sufficient for uniquely restoring the original function $f(x,y)$. So, there is a problem to choose one of them. The main complexity there is to find such a choice in which arbitrary permissible distortions of $f_j(x), -1 < x < 1, j = 0, 1, \ldots, n-1$, involve a small error on the result.

There are various interpolation methods, see [2], [3], which permit to approximate $\hat{f}(x, e^{i\vartheta}), -1 < x < 1, 0 < \vartheta < 2\pi$, by a finite number of functions (1). Further, one is trying to find the original function by Radon's inverse formula. In this step, characteristic for ill-posed problems, some effects arise.

There is another point of view, where one tries to find the solution of (1), which has the minimal $L_2(dxdy)$-norm. For the particular case of equally spaced directions, i.e. $\vartheta_j = \pi j/n, j = 0, 1, ..., n-1$, one can find the explicit formula for the solution in [5]. In this case the solution is steady for arbitrary distortions of the original function $f(x,y)$ in $L_2(dxdy)$.

Let us note, that in practical point of view it is desirable to find the solutions stable for $f_j(x), j = 0, 1, ..., n-1$, distortions, because only those functions are perceptible.

In this paper we suggest a new formula, which permits to present a solution of problem (6) for arbitrary directions $\vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1}$. Let us note that our solution does not minimize the norm, but it is steady for $f_j(x), j = 0, 1, ..., n-1$, distortions in $L_2(-1,1)$.

2. Representation of A. Legendre's polynomials

The polynomials

$$P_n(x) = \frac{1}{n2^n} \frac{d^n (x^2-1)^n}{dx^n}, \quad -1 < x < 1, \tag{2}$$

where $n = 0, 1, \ldots$, are known as A. Legendre's polynomials.

The family $P_n(x), n = 0, 1, \ldots$, is orthogonal and dense in $L_2(-1,1)$. For arbitrary $f(x) \in L_2(-1,1)$ the equality

$$\int_{-1}^{1} |f(x)|^2 dx = \sum_{n=0}^{\infty} (n+1/2) \left| \int_{-1}^{1} f(x) P_n(x) dz \right|^2$$

holds.

For A. Legendre's polynomials the recurrence formula

$$(n+1)P_{n+1}(x) + nP_{n-1}(l) = (2n+1)xP_n(x), \quad n = 1, 2, \ldots\ ,$$

is valid. This formula, with the conditions $P_0(x) = 1, \quad P_1(x) = x$, uniquely determine the family of all A. Legendre's polynomoals.

There are many formulas, which represent these polynomials. The following one is known as Ch. Laplace's formula

$$P_n(x) = \frac{1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2} \cos(t) \right)^n dt, \quad n = 0, 1, \ldots .$$

This formula is also possible to be rewritten in the form

$$P_n(x) = \frac{1}{\pi} \int_{l_x} \frac{z^n}{\sqrt{1-|z|^2}} dy, \quad -1 < x < 1,$$

where $z = x + iy$ and $l_x = \{x + iy; \quad |y| < \sqrt{1-x^2}\}$.

In this section we want to give a new representation for A. Legendre's polynomials. For that aim let us introduce a new family of polynomials.

Definition Let us denote by $Q_{n,k}(x), n = 0, 1, \ldots, k = 0, 1, \ldots, [n/2]$, the family of polynomials for which

1. $Q_{n,0}(x) = 1$, if $n = 0, 1, \ldots$;
2. $Q_{n,k}(x)$ are polynomials of the degree k;
3. for arbitrary $n = 0, 1, \ldots$ and $k = 0, 1, \ldots [n/2] - 1$ the equality

$$Q_{n,k+1}(x) = (1-x)Q_{n,k}(x) - (n-2k-1)\sqrt{x} \int_0^x \frac{Q_{n,k}(t)}{\sqrt{t}} dt$$

holds.

Lemma 1 *For arbitrary* $n = 2, 3, \ldots$ *and* $k = 0, 1, \ldots$ *the equality*

$$(k+1)I_{n,k}(x) = (k+1)I_{n-2,k}(x) - (1-x^2)(2n+k-1)I_{n-2,k+2}(x),$$

where

$$I_{n,k}(x) = \frac{1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2} \cos(t) \right)^n \sin^k(t) dt.$$

holds.

Proof For $n = 2, \ldots$ and $k = 0, 1, \ldots$ we have

$$i\sqrt{1-x^2}(n-1)I_{n-2,k+2}(x)$$

$$= -\frac{1}{\pi} \int_0^\pi \sin^{k+1}(t) d \left(x + i\sqrt{1-x^2} \cos(t) \right)^{n-1}$$

$$= \frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2} \cos(t) \right)^{n-1} \sin^k(t) \cos(t) dt.$$

Consequently,

$$-(n-1)(1-x^2)I_{n-2,k+2}(x)$$

$$= \frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2} \cos(t) \right)^{n-1} i\sqrt{1-x^2} \sin^k(t) \cos(t) dt$$

$$= \frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^n \sin^k(t)dt$$
$$-x\frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-1} \sin^k(t)dt$$
$$= (k+1)I_{n,k}(x) - x(k+1)I_{n-1,k}(x).$$

So, we have

$$x(k+1)I_{n-1,k}(x) = \frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2}$$
$$\times \sin^k(t) \left(x^2 + ix\sqrt{1-x^2}\cos(t)\right) dt$$
$$= \frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2} \sin^k(t)$$
$$\times \left(i\sqrt{1-x^2}\cos(t)\left(x + i\sqrt{1-x^2}\cos(t)\right) + x^2\right.$$
$$\left. +(1-x^2)\cos^2(t)\right) dt$$
$$= \frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2} \sin^k(t)$$
$$\times \left(\left(x + i\sqrt{1-x^2}\cos(t)\right) i\sqrt{1-x^2}\cos(t) + 1\right.$$
$$\left. -(1-x^2)\sin^2(t)\right) dt$$
$$= \frac{i\sqrt{1-x^2}}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-1} d\left(\sin^{k+1}(t)\right)$$
$$+\frac{k+1}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2} \sin^k(t)dt$$
$$-\frac{(k+1)(1-x^2)}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2} \sin^{k+2}(t)dt$$
$$= -\frac{(n-1)(1-x^2)}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2} \sin^{k+2}(t)dt$$
$$+\frac{(k+1)}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2} \sin^k(t)dt$$
$$-\frac{(k+1)(1-x^2)}{\pi} \int_0^\pi \left(x + i\sqrt{1-x^2}\cos(t)\right)^{n-2} \sin^{k+2}(t)dt.$$

Using the functions $I_{n,k}(x)$, we can rewrite this equality in the form

$$-(n-1)(1-x^2)I_{n-2,k+2}(x)$$

$$= (k+1)I_{n,k}(x) + (n-1)(1-x^2)I_{n-2,k+2}(x)$$
$$-(k+1)I_{n-2,k}(x) + (k+1)(1-x^2)I_{n-2,k+2}(x).$$

Theorem 1 *For arbitrary $n = 0, 1, \dots$ and $k = 0, 1, \dots, [n/2]$ the equality*

$$P_n(x) = \frac{1}{\pi}\int_{l_x} \frac{z^{n-2k}Q_{n,k}(1-|z|^2)}{\sqrt{1-|z|^2}}dy$$
$$= \frac{1}{\pi}\int_{l_x} \frac{\bar{z}^{n-2k}Q_{n,k}(1-|z|^2)}{\sqrt{1-|z|^2}}dy \qquad (3)$$

holds.

Proof Let us note that if z permits the presentation

$$z = x + i\sqrt{1-x^2}\cos(t),$$

then we have

$$1 - |z|^2 = (1-x^2)\sin^2(t).$$

So, for $n = 2, 3, \dots$ we have

$$P_n(x) = I_{n,0}(x) = I_{n-2,0}(x) - (2n-1)(1-x^2)I_{n-2,2}(x)$$
$$= \frac{1}{\pi}\int_{l_x} \frac{z^{n-2}\left(1-(2n-1)(1-|z|^2)\right)}{\sqrt{1-|z|^2}}dy = \frac{1}{\pi}\int_{l_x} \frac{z^{n-2}Q_{n,1}(1-|z|^2)}{\sqrt{1-|z|^2}}dy.$$

Let us denote

$$P_n(x) = a_{n,k}^{(0)}I_{n-2k,0}(x) + a_{2,k}^{(1)}I_{n-2k,2}(x)(1-x^2)$$
$$+\dots + a_{n,k}^{(k)}I_{n-2k,2k}(x)(1-x^2)^k,$$

where $n = 0, 1, \dots$ and $k = 0, 1, \dots, [n/2]$. By Lemma 1 we have

$$P_n(x) = a_{n,k}^{(0)}\left(I_{n-2k-2,0}(x) - (1-x^2)\left(1+2\frac{n-2k-1}{1}\right)I_{n-2k-2,2}(x)\right)$$
$$+(1-x^2)a_{n,k}^{(1)}\left(I_{n-2k-2,2}(x) - (1-x^2)\left(1+2\frac{n-2k-1}{3}\right)I_{n-2k-2,4}(x)\right)$$
$$+\dots + (1-x^2)^k a_{n,k}^{(k)}$$
$$\times\left(I_{n-2k-2,2k}(x) - (1-x^2)\left(1+2\frac{n-2k-1}{2k+1}\right)I_{n-2k-2,2k+2}(x)\right)$$
$$= a_{n,k}^{(0)}I_{n-2k-2,0}(x)$$

$$+\left(a_{n,k}^{(1)} - a_{n,k}^{(0)}\left(1 + 2\frac{n-2k-1}{1}\right)\right)(1-x^2)I_{n-2k-2,2}(x)$$
$$+\left(a_{n,k}^{(2)} - a_{n,k}^{(1)}\left(1 + 2\frac{n-2k-1}{3}\right)\right)(1-x^2)^2 I_{n-2k-2,4}(x) + \dots$$
$$+\left(a_{n,k}^{(k)} - a_{n,k}^{(k-1)}\left(1 + 2\frac{n-2k-1}{2k-1}\right)\right)(1-x^2)^k I_{n-2k-2,2k}(x)$$
$$-a_{n,k}^{(k)}\left(1 + 2\frac{n-2k-1}{2k+1}\right)(1-x^2)^{k+1} I_{n-2k-2,2k+2}(x).$$

Consequently,

$$
\begin{gathered}
a_{n,k+1}^{(0)} = a_{n,k}^{(0)}, \\
a_{n,k+1}^{(1)} = a_{n,k}^{(1)} - a_{n,k}^{(0)}\left(1 + 2\frac{n-2k-1}{1}\right), \\
\dots, \\
a_{n,k+1}^{(k)} = a_{n,k}^{(k)} - a_{n,k}^{(k-1)}\left(1 + 2\frac{n-2k-1}{2k-1}\right), \\
a_{n,k+1}^{(k+1)} = -a_{n,k}^{(k)}\left(1 + 2\frac{n-2k-1}{2k+1}\right).
\end{gathered}
\tag{4}
$$

For $n = 0, 1, \dots$ and $k = 0, 1, \dots, [n/2]$ let us denote

$$P_{n,k}(x) = a_{n,k}^{(0)} + a_{n,k}^{(1)}x + \dots + a_{n,k}^{(k)}x^k.$$

We want to prove that $P_{n,k}(x) = Q_{n,k}(x)$. It is sufficient to note that

$$P_{n,0}(x) = 1$$

and the polynomials $P_{n,k}(x)$ satisfy the same recurrence formula as $Q_{n,k}(x)$ does. Thanks to (4), we have

$$P_{n,k+1}(x) = P_{n,k}(x) - xP_{n,k}(x)$$
$$-2(n-2k-1)\left(a_{n,k}^{(0)}x + a_{n,k}^{(1)}\frac{x^2}{3} + \dots + a_{n,k}^{(k)}\frac{x^{k+1}}{2k+1}\right).$$

Taking into account that $P_{n,k}(0) = 1$, this formula can be rewritten in the form

$$P_{n,k}(x) = (1-x)P_{n,k}(x) - (n-2k-1)\sqrt{x}\int_0^x \frac{P_{n,k}(t)}{\sqrt{t}}dt.$$

Let us remark that for $k = 0$ the representation (3) coincides with Ch. Laplace's formula.

3. J. Radon's transform

J. Radon's transform, see [1], p. 117, is defined as follows.

Definition The Radon transform $\hat{f}(x, e^{i\varphi})$ of the function $f(z), |z| < 1$, is

$$\hat{f}(x, e^{i\varphi}) = \frac{1}{\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(xe^{i\varphi} + iye^{i\varphi}) dy,$$

where $-1 < x < 1, \quad 0 \leq \varphi < 2\pi$.

Lemma 2 *Let $f(z) \in L_p(D)$, where $D = \{|z| < 1\}, 4/3 < p < 2$. Then the integral*

$$\int_D \int_D \frac{f(x)\bar{f}(y)}{|x-y|} dxdy$$

converges absolutely.

Proof Let us denote

$$g(y) = \int_D \frac{|f(x)|}{|x-y|^{3/2}} dx, \quad y \in \mathbb{R}^2.$$

we determine the number $r, 1 < r < 4/3$ from the equation

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{r} - 1$$

For $|y| < 2$, the function $g(y)$ can be presented in the form

$$g(y) = \int_{\mathbb{R}^2} |f(x)| \frac{h(|y-x|)}{|y-x|^{3/2}} dx,$$

where we assume $f(x) = 0, x \notin D$, and $h(t) = 1$, for $0 \leq t \leq 3$, $h(t) = 0$, for $3 < t$.We have $f(x) \in L_p(\mathbb{R}^2)$, where $p > 4/3$ and $h(|x|)|x|^{-3/2} \in L_r(\mathbb{R}^2)$. So, the convolutions of these functions belongs to the space $L_2(\mathbb{R}^2)$. For $|y| > 2$ we have

$$|g(y)| \leq \frac{1}{(|y|-1)^{3/2}} \int_D |f(x)| dx.$$

So, $g(y) \in L_2(\mathbb{R}^2)$. Using the formula, see [4], p. 64,

$$\frac{1}{|x-y|} = \frac{1}{16\pi} \int_{\mathbb{R}^2} \frac{dz}{|x-z|^{3/2}|y-z|^{3/2}}$$

we have

$$\int_D \int_D \frac{f(x)\bar{f}(y)}{|x-y|} dxdy = \frac{1}{16\pi} \int_{R_2} \left| \int_D \frac{f(x)}{|z-x|^{3/2}} dx \right|^2 dz.$$

Definition Let us denote by H the space of the functions $f, g \in L_r(D), 4/3 < r$, with the scalar product

$$(f,g)_H = \int_D \int_D \frac{f(x)\bar{g}(y)}{|x-y|} dxdy.$$

Let us denote by **H** the completion of H.

Theorem 2 For an arbitrary function $f(z) \in L_p(D)$, where $4/3 < p$, the equality

$$\int_{-\pi}^{\pi} \int_{-1}^{1} \left| \hat{f}(x, e^{i\varphi}) \right|^2 dxd\varphi = \frac{1}{4\pi} \int_D \int_D \frac{f(x)\bar{f}(y)}{|x-y|} dxdy$$

holds.

Proof We have

$$\int_{-\pi}^{\pi} \int_{-1}^{1} |\hat{f}(x, e^{i\varphi})|^2 dxd\varphi$$

$$= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(xe^{i\varphi} + iye^{i\varphi}) \bar{f}(xe^{i\varphi} + ize^{i\varphi}) d\varphi dxdydz.$$

Let us introduce the new variables $t = (x+iy)e^{\varphi}, s = (x+iz)e^{i\varphi}$. We have

$$e^{i\varphi} = i\frac{s-t}{|s-t|}, \quad x = \frac{1}{2}\frac{s\bar{t} - \bar{s}t}{|s-t|}, \quad y = \frac{2|t|^2 - \bar{t}s - \bar{s}t}{2|s-t|}, \quad z = \frac{s\bar{t} + \bar{s}t - 2|s|^2}{2|s-t|}.$$

The Jacobian of this transform is

$$\frac{1}{4|s-t|},$$

which proves the theorem.

In this theorem we have norm equality. The different norm estimations for Radon's transform can be found in [2].

The family of the functions

$$\frac{z^{n-2k} Q_{n,k}(1-|z|^2)}{\sqrt{1-|z|^2}}, \quad \frac{\bar{z}^{n-2k} Q_{n,k}(1-|z|^2)}{\sqrt{1-|z|^2}}, \tag{5}$$

where $n = 0, 1, ...$ and $k = 0, 1, ..., [n/2]$ with the condition $n - 2k \neq 0$, is dense in $L_p(D)$. Indeed, the function of the form

$$\frac{P(z, \bar{z})}{\sqrt{1-|z|^2}},$$

where $P(z, \bar{z})$ is an arbitrary polynomial, can be presented as a linear combination of the functions (5).

Lemma 3 *Let*

$$f_j(x), \ -1 < x < 1, \quad j = 0, 1, \ldots, n-1,$$

be functions from $L_2(-1, 1)$ *and for some* $f(x, y) \in \mathbf{H}$

$$f_j(x) = \hat{f}(x, e^{i\vartheta_j}), \quad -1 < x < 1, \quad j = 0, 1, \ldots, n-1.$$

Let

$$c_{k,j} = \int_{-1}^{1} f_j(x) P_k(x) dx, \quad j = 0, \ldots, n-1, \quad k = 0, 1, \ldots .$$

Then, if $k, k+1 < n$, *is an even number, the rank of the matrix*

$$\begin{vmatrix} c_{k,0} & e^{ik\vartheta_0} & \ldots & e^{i2\vartheta_0} & 1 & e^{-i2\vartheta_0} & \ldots & e^{ik\vartheta_0} \\ c_{k,1} & e^{ik\vartheta_1} & \ldots & e^{i2\vartheta_1} & 1 & e^{-i2\vartheta_1} & \ldots & e^{ik\vartheta_1} \\ \\ c_{k,n-1} & e^{ik\vartheta_{n-1}} & \ldots & e^{i2\vartheta_{n-1}} & 1 & e^{-i2\vartheta_{n-1}} & \ldots & e^{ik\vartheta_{n-1}} \end{vmatrix} \tag{6}$$

is $k+1$. *If* $k, k+1 < n$, *is an odd number then the rank of the matrix*

$$\begin{vmatrix} c_{k,0} & e^{ik\vartheta_0} & \ldots & e^{i\vartheta_0} & e^{-i\vartheta_0} & \ldots & e^{ik\vartheta_0} \\ c_{k,1} & e^{ik\vartheta_1} & \ldots & e^{i\vartheta_1} & e^{-i\vartheta_1} & \ldots & e^{ik\vartheta_1} \\ \\ c_{k,n-1} & e^{ik\vartheta_{n-1}} & \ldots & e^{i\vartheta_{n-1}} & e^{-i\vartheta_{n-1}} & \ldots & e^{ik\vartheta_{n-1}} \end{vmatrix} \tag{7}$$

is $k+1$ *too.*

Proof Let us present the function $f(z) \in \mathbf{H}$ in the form

$$f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} \frac{Q_{n,k}(1-|z|^2)}{\sqrt{1-|z|^2}} (a_{n,k} z^{n-2k} + b_{n,k} \bar{z}^{n-2k}) \right)$$

where $b_{2n,n} = 0, n = 0, 1, \ldots$. Since for $n = 0, 1, \ldots$,

$$\left\| \frac{z^{n-2k} Q_{n,k}(1-|z|^2)}{\sqrt{1-|z|^2}} \right\|_H = \int_{-\pi}^{\pi} \int_{-1}^{1} |P_n(x)|^2 dx d\varphi = \frac{4\pi}{2n+1},$$

we have

$$\frac{1}{4\pi} \int_D \int_D \frac{f(x)\bar{f}(y)}{|x-y|} dx dy = \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=0}^{[n/2]} \left(|a_{n,k}|^2 + |b_{n,k}|^2 \right) < \infty.$$

By Theorem 1 we have

$$\hat{f}(x, e^{i\vartheta}) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} a_{n,k} e^{i(n-2k)\vartheta} + b_{n,k} e^{-i(n-2k)\vartheta} \right) P_n(x).$$

Since $\hat{f}(x, \vartheta_j) = f_j(x), j = 0, 1, \ldots, n-1$, so

$$c_{k,j} = \frac{2}{2k+1} \sum_{l=0}^{[k/2]} (a_{k,l} e^{i(n-2k)\theta_j} + b_{k,l} e^{-i(n-2k)\theta_j}), \tag{8}$$

where $j = 0, 1, \ldots, n-1, k = 0, 1, \ldots$.

For each fixed k, to determine the coefficients

$$a_{k,0}, \quad a_{k,1}, \quad \ldots, \quad a_{k,[k/2]},$$

$$b_{k,0}, \quad b_{k,1}, \quad \ldots, \quad b_{k,[k/2]},$$

we have $k+1$ unknowns and n equations. If $k+1 < n$ we have more equations and for the existence of a solution conditions on the numbers

$$c_{k,0}, \quad c_{k,1}, \quad \ldots, \quad c_{k,n-1}$$

are necessary. It is well known that those conditions are (6) and (7).

Theorem 3 *Let $f_j(x)$, $-1 < x < 1$, $j = 0, 1, \ldots, n-1$, be a family of functions from the space $L_2(-1,1)$ for which conditions* (6) *and* (7) *hold. Then, the function $f(z)$ defined by the formula*

$$f(z) = \sum_{k=0}^{n-1} \left(\sum_{l=0}^{[k/2]} \frac{Q_{k,l}(1-|z|^2)}{\sqrt{1-|z|^2}} (a_{k,l} z^{k-2l} + b_{k,l} \bar{z}^{k-2l}) \right)$$

$$+ \sum_{k=n}^{\infty} \left(\sum_{l=0}^{[n/2]} \frac{Q_{k,l}(1-|z|^2)}{\sqrt{1-|z|^2}} (a_{k,l} z^{k-2l} + b_{k,l} \bar{z}^{k-2l}) \right),$$

gives a solution $f(z) \in \mathbf{H}$ of the problem

$$f_j(x) = \hat{f}(x, e^{i\vartheta_j}), -1 < x < 1, j = 0, 1, \ldots, n-1.$$

Here the coefficients $a_{k,j}$ and $b_{k,j}$ are defined as the solutions of the equations

$$c_{k,j} = \frac{2}{2k+1} \sum_{l=0}^{[k/2]} (a_{k,l} e^{i(k-2l)\theta_j} + b_{k,l} e^{-i(k-2l)\theta_j}), \ j = 0, 1, \ldots, k, \tag{9}$$

for $k+1<n$ *and as the solutions of the equations*

$$c_{k,j} = \frac{2}{2k+1}\sum_{l=0}^{[n/2]}(a_{k,l}e^{i(k-2l)\theta_j} + b_{k,l}e^{-i(k-2l)\theta_j}), j = 0, 1, \dots, n-1, \quad (10)$$

for $k+1 \geq n$. *Moreover, the estimation*

$$\|f\|_{\mathbf{H}} \leq C\left(\sum_{j=0}^{n-1}\|f_j\|_{L_2}\right)$$

holds, where C *depends only upon the numbers* ϑ_j.

Proof Let us note that the equations (9) and (10) have Vandermonde matrices , which are not zero for arbitrary $0 \leq \vartheta_0 < \vartheta_1 < \cdots < \vartheta_{n-1} < \pi$. So, those equations have a unique solution. Moreover, there is a constant $C = C(\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1})$ such that the solution of the equations (9) and (10) permit the estimation

$$\sum_{j=0}^{[k/2]}(a_{k,j}^2 + b_{k,j}^2) \leq C\sum_{j=0}^{n-1}c_{k,j}^2$$

if $k+1<n$ and

$$\sum_{j=0}^{[n/2]}(a_{k,j}^2 + b_{k,j}^2) \leq C\sum_{j=0}^{n-1}c_{k,j}^2$$

if $k+1 \geq n$. These notes prove the theorem.

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ON A MIXED PROBLEM FOR A COMPOSITE PLANE WEAKENED BY ARC-TYPE CRACKS

V.N. Hakobyan, L.L. Dashtoyan
Institute of Mechanics
of National Academy of Sciences of Armenia,
Marshal Bagramian ave. 24b
375019 Yerevan, Armenia
mechins@sci.am

Abstract In the present work the precise solution is constructed for the problem on the strain-stress state of an elastic compound plane, obtained by junction of an elastic circular disk and an elastic plane with a circular hole of same radius, which are made of different materials. There are arc-type absolutely rigid inclusions, one side of which is detached from the matrix on joining lines of the circular disk and the plane.

Keywords: elastic compound plane, strain stress state, cracks, rigid inclusions, singular integral equations with Hilbert kernel

Mathematics Subject Classification (2000): 74A40, 74A45, 45E10

1. Introduction

Numerous works and monographs are dedicated to the research of strain-stress state of homogeneous and compound elastic massive bodies, containing concentrators such as thin inclusions and cracks. However, there are few works considering the interaction of different types of concentrators, acting on massive bodies and having common points simultaneously. This is explained by the mathematical difficulties, which are not permitting to build the effective solution for problems of the mentioned types. The problems for elastic bodies with inclusions are belonging to the mentioned ones, when one side of the inclusion¡ is detached from the matrix and mixed conditions take place on the banks of cuts or cracks, forming as a result detachments. The first in this direction, probably, is Sherman's work [1] in which the precise solution of

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the problem was built for an elastic homogeneous plane with a soldered thin rigid inclusion on one bank of the crack. Later N. Muskhelishvili [2] elaborated this solution and G. Cherepanov [3] built the solution for an analogous problem with any condition on the banks of the crack by reduction to Riemann's problem for two functions. The further development of this direction is considered in the works [4]-[5].

2. Statement of problem and deduction of determining equations

Let an elastic compound plane, obtained by junction of an elastic circular disk with radius R and Lamé coefficients μ_1, λ_1 and an elastic plane with a circular hole with the same radius and Lamé coefficients μ_2, λ_2 is strengthened by absolutely rigid inclusions on a joining line L, consisting of arcs $(\alpha_k, \beta_k)(k = 1, 2, \ldots, N)$. One side of the inclusion is detached from the matrix. It is assumed that the plane is deformed under the system of self-balanced forces, acting on the inclusions and the free bank of the line L. The contact strains, acting under the inclusions and in the zones of joining the circular disk with the plane, as well as the intensity coefficients at the end-points of the cracks are determined in this problem. After introduction of indices 1 and 2 for the characteristics of the circular disk and the plane with the circular hole correspondingly, the following auxiliary boundary problem for the compound plane is considered

$$(X_n^{(1)} + iY_n^{(1)})\Big|_{z=Re^{i\theta}} - (X_n^{(2)} + iY_n^{(2)})\Big|_{z=Re^{i\theta}} = \begin{cases} 0, & \theta \notin L, \\ \chi(\theta), & \theta \in L, \end{cases} \quad (1)$$

$$(U_1 + iV_1)\,|_{z=Re^{i\theta}} - (U_2 + iV_2)|_{z=Re^{i\theta}} = \begin{cases} 0, & \theta \notin L, \\ W(\theta), & \theta \in L, \end{cases}$$

where $X_n^{(j)} + iY_n^{(j)}$ and $U_1 + iV_1\,(j = 1, 2)$ are the complex combinations of strains components and the displacements of the disk and the plane with the circular hole correspondingly in the Cartesian coordinate system [2]. The functions $\chi(\theta)$ and $W(\theta)$ are the jumps of strains and displacements on the line L. It is assumed that there are no strains at infinity. The solution of the auxiliary boundary problem is built by Kolosov-Musckhelishvili's complex potentials $\varphi_j(z)$ and $\psi_j(z)\,(j = 1, 2)$. The complex combinations of strains and displacements of the circular disk and the plane with circular hole can be written in the form

$$f_1^{(j)} + if_2^{(j)} = \varphi_j(z) + z\overline{\varphi_j'}(z) + \overline{\psi_j}(z), \quad (2)$$
$$2\mu_j(U_j + iV_j) = H_j\varphi_j(z) - z\overline{\varphi_j'}(z) - \overline{\psi_j}(z), \quad j = 1, 2.$$

Here $H_j = 3 - 4\nu_j$, $j = 1, 2$, are Muskhelishvili's constants and

$$f_1^{(j)} + i f_2^{(j)} |_{z=Re^{i\theta}} = iR \int_{-\pi}^{\theta} (X_n^{(j)} + iY_n^{(j)}) d\theta.$$

The functions $\varphi_j(z)$ and $\psi_j(z)$ are represented in series form as

$$\begin{aligned} \varphi_1(z) &= \sum_{n=1}^{\infty} a_n z^n, \quad \psi_1(z) = \sum_{n=0}^{\infty} a'_n z^n, \quad |z| \leq R, \\ \varphi_2(z) &= \sum_{n=1}^{\infty} b_n z^{-n}, \quad \psi_2(z) = \sum_{n=1}^{\infty} b'_n z^{-n}, \quad |z| \geq R, \end{aligned} \tag{3}$$

and the functions $W(\theta)$ and $\chi(\theta)$ are represented as Fourier series

$$iR \int_{-\pi}^{\theta} \chi(\theta) d\theta = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}, \quad W(\theta) = \sum_{n=-\infty}^{\infty} B_n e^{in\theta}. \tag{4}$$

Then using the representations (2) and (4) and formulae (2) the conditions of boundary problem (1) are satisfied. As a result, equaling the coefficients of the same degrees of exponents the unknown coefficients a_n, a'_n, b_n and b'_n of the functions $\varphi_j(z)$ and $\psi_j(z)$ $(j = 1, 2)$ are expressed by Fourier's coefficients A_n and B_n.They are determined by the formulae

$$\begin{aligned} a_n &= \frac{\mu A_n + 2\mu_1 B_n}{R^n(\mu + H_1)}, \quad n \geq 2, \\ b_n &= -\frac{R^n(A_{-n} + 2\mu_1 B_{-n})}{1 + \mu H_2}, \quad n \geq 1, \\ a'_n &= -\frac{(n+2)a_{n+2}}{R^n} + \frac{\mu H_2 \overline{A}_{-n} - 2\mu_1 \overline{B}_{-n}}{R^n(1 + \mu H_2)}, \quad n \geq 1, \\ b'_n &= R^n \left[R^n \overline{a}_n + \frac{(n-2)b_{n-2}}{R^{n-2}} - \overline{A}_n \right], \quad n \geq 1, \\ a'_0 &= -2R^2 \overline{a}_2 - 2\mu_1 B_0, \\ a'_1 &= \frac{\mu[(\mu + H_1)A_1 - (\mu - 1)\overline{A}_1]}{R(H_1 + 1)(H_1 + 2\mu - 1)} + \frac{\mu_1[(\mu + H_1)B_1 - (\mu - 1)\overline{B}_1]}{R(H_1 + 1)(H_1 + 2\mu - 1)}, \\ b'_1 &= R\frac{\mu(A_1 + \overline{A}_1) + 2\mu_1(B_1 + \overline{B}_1)}{H_1 + 2\mu - 1}, \quad \mu = \mu_1/\mu_2, \\ A_n &= \frac{R}{2\pi n} \int_{-\pi}^{\pi} \chi(\theta) e^{-in\theta} d\theta, \quad B_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\theta) e^{-in\theta} d\theta. \end{aligned} \tag{5}$$

In this way Kolosov-Muskhelishvili's complex potentials are expressed by Fourier's coefficients of the jumps of strains and displacements of the functions $\chi(\theta)$ and $W(\theta)$. It is allowed to determine the strain-stress state of the compound plane by using these functions.

The consideration of the problem is described by the conditions

$$\begin{cases} X_n^{(2)} + iY_n^{(2)}|_{r=R} = \chi_1(\theta), \\ U_1 + iV_1|_{r=R} = W_0(\theta), \end{cases} \theta \in L, \tag{6}$$

$$(W_0(\theta) = \gamma_k + \delta_k e^{i\theta}, \quad \alpha_k < \theta < \beta_k, \quad k = 1, 2, ..., N),$$

where γ_k and δ_k $(k = 1, 2, ..., N)$ are unknown constants, which are determining the rigid displacements and turning angles of the rigid inclusions correspondingly, and $\chi_1(\theta)$ is a given function, determining the distribution of strains on the free bank of the line L. As a result for the determination of the functions $iR\chi(\theta) = \chi_*(\theta)$ and $W'(\theta)\theta_2^{(2)} = W'_*(\theta)$ the system of singular integral equations of second kind with Hilbert kernel

$$\begin{aligned} W'_*(\theta) - \frac{a_1}{2\pi i}\int\limits_L W'_*(\varphi) ctg\frac{\varphi-\theta}{2} d\varphi - \frac{a_2}{2\pi i}\int\limits_L ctg\frac{\varphi-\theta}{2}\chi_*(\varphi) d\varphi = F_1(\theta), \\ \chi_*(\theta) + \frac{b_1}{2\pi i}\int\limits_L W'_*(\varphi) ctg\frac{\varphi-\theta}{2} d\varphi - \frac{b_2}{2\pi i}\int\limits_L ctg\frac{\varphi-\theta}{2}\chi_*(\varphi) d\varphi = F_2(\theta) \end{aligned} \tag{7}$$

will be obtained. The system (2) is considered under the conditions

$$\int\limits_{\alpha_k}^{\beta_k} \chi(\theta) d\theta = 0, \int\limits_{\alpha_k}^{\beta_k} W'_*(\theta) d\theta = 0, \mathrm{Im}\int\limits_{\alpha_k}^{\beta_k} \chi(\theta) e^{i\theta} d\theta = 0, \ k = 1, 2, ..., N. \tag{8}$$

Here

$$F_1(\theta) = \frac{d_0\Delta^*}{l_1 d_0 + l_0^2}\left[\chi_1(\theta) - i\mathrm{Re}_2 e^{i\theta} - \frac{l_0}{d_0}(W'_0(\theta) - e_3 e^{i\theta})\right],$$

$$F_2(\theta) = \frac{l_0\Delta^*}{l_1 d_0 + l_0^2}\left[\chi_1(\theta) - i\mathrm{Re}_2 e^{i\theta} - \frac{l_1}{l_0}(W'_0(\theta) - e_3 e^{i\theta})\right],$$

$$a_1 = \frac{\theta_2^{(1)}\theta_1^{(2)}}{\theta}, \quad a_2 = \frac{\theta_2^{(1)}\theta_2^{(2)}}{2\theta},$$
$$b_1 = \frac{2\left((\theta_2^{(1)})^2 - (\theta_1^{(1)})^2\right)}{\theta}, \quad b_2 = \frac{\theta_1^{(1)}\theta_2^{(1)}}{\theta},$$
$$\theta = (\theta_2^{(1)})^2 - (\theta_1^{(1)})^2 + \theta_1^{(1)}\theta_1^{(2)}, \quad \theta_1^{(j)} = \frac{\mu_j}{2+\alpha_j},$$
$$\theta_2^{(j)} = \frac{(1+\alpha_j)\mu_j}{2+\alpha_j}, \quad \alpha_j = \frac{1}{1-2\nu_j}, j = 1, 2.$$

3. The solution of the determining equations

The system of the singular integral equations (6) is considered under condition (8). At first let us consider the case, when the equation

$$a_2\lambda^2 - \lambda(a_1 - b_2) + b_1 = 0 \tag{9}$$

has two different roots λ_1, λ_2. In this case the first equation of (6) is multiplied by λ_j $(j = 1, 2)$ and added to the second one. The two singular integral equations of the second kind with Hilbert kernel

$$\eta_j^*(\theta) + \frac{iq_j}{2\pi}\int\limits_L \eta_j^*(\varphi)ctg\frac{\varphi-\theta}{2}d\varphi = \overline{Q}_j^*(\theta), \quad \theta \in L, \ j = 1, 2,$$
$$\overline{Q}_j^*(\theta) = \lambda_j\theta_2^{(2)}F_1(\theta) + F_2(\theta), \tag{10}$$
$$\eta_j^*(\theta) = \chi_*(\theta) + \lambda_j W_*'(\theta),$$
$$q_j = \frac{a_1 + b_2 - (-1)^j\sqrt{(a_1 - b_2) - 4a_2b_1}}{2},$$

are obtained. The first two of conditions (8) can be rewritten in the form

$$\int\limits_{\alpha_k}^{\beta_k} \eta_j^*(\theta)d\theta = 0, \quad k = 1, 2, ..., N. \tag{11}$$

The solutions of equations (10) are given by the formulae, see [2],[6],

$$\eta_j^*(\theta) = \frac{1}{1-q_j}\left[\overline{Q}_j^*(\theta) + \frac{q_jX_j^+(tg(\theta/2))}{2\pi i\cos(\theta/2)}\int\limits_L \frac{\overline{Q}_j^*(\varphi)\cos(\varphi/2)d\varphi}{X_j^+(tg(\varphi/2))sin\frac{\varphi-\theta}{2}}\right], \tag{12}$$

for $j = 1, 2$, where for $\alpha_l < \theta < \beta_l$,

$$X_j^+(tg(\theta/2)) = i\sqrt{g_j}(-1)^{N-l+1}\prod_{k=1}^{N}\frac{|tg(\theta/2) - tg(\alpha_l/2)|^{-\gamma_j}}{|tg(\theta/2) - tg(\beta_l/2)|^{-\gamma_j+1}},$$

$$g_j = \frac{1+q_j}{1-q_j}, \quad \gamma_j = \frac{1}{2\pi i}ln|g_j| + \frac{\theta_j}{2\pi}, \quad 0 < \theta_j = arg(g_j) < 2\pi.$$

The expressions for the function $\overline{Q}_j^*$ include constants $\delta_k, e_2, e_3 (k = 1, 2, ..., N)$, which are determining from the last of the conditions (8) and the representations of the constants e_j $(j = 2, 3)$ by the unknown functions. Knowing the functions $\eta_j^*(\theta)$, the jump of stresses and displacements on the joining line are determined by the formulae

$$\chi_*(\theta) = \frac{\lambda_1\eta_2^*(\theta) - \lambda_2\eta_1^*(\theta)}{\lambda_1 - \lambda_2}, W_*'(\theta) = \frac{\eta_1^*(\theta) - \eta_2^*(\theta)}{\lambda_1 - \lambda_2}. \tag{13}$$

After that it is not difficult to determine the opening of cracks, the contact stresses, acting outside of the cracks on the joining line, and the intensity coefficients of these stresses at the end-points of the cracks.

Taking into account the awkwardness of the mentioned formulae they are omitted. It is noticed that in the case of real roots of equations (9) $\mathrm{Re}\gamma_j = 1/2$ and in the case of complex roots $1/2 < \mathrm{Re}\gamma_j < 1$. In the case of a homogeneous plane $\mathrm{Re}\gamma_j = 3/4$. In the case when the quadratic equation (9) has only one solution, in accordance with the work [4], the solution of system (6) is reduced to the sequential solution of two singular integral equations with Hilbert kernel and the closed solutions are obtained. We shall notice that these solutions can be obtained from the above mentioned solutions by using a formal limit letting λ_1 tend to λ_2.

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SOLUTION OF THE TWO-DIMENSIONAL MAGNETOELASTIC LAMB PROBLEM

G.Y. Baghdasaryan, Z.N. Danoyan, M.A. Mikilyan
Institute of Mechanics
of National Academy of Sciences of Armenia,
Marshal Baghramian ave. 24b
375019, Yerevan, Armenia
gevorgb@ysu.am

Abstract Solution of dynamical problems of the theory of magnetoelasticity is connected with certain mathematical difficulties as it is known from [1]-[4]. In the present work the analytical formulae for elastic displacements and stresses of the induced electromagnetic field for the Lamb magnetoelastic problem are obtained with the use of Laplace's and Fourier's transformation technique and consequently by applying the Canyard kernel. The noted problem is discussed in the works [5], [6] for certain conditions.

Keywords: dynamical problems of magnetoelasticity, Lamb magnetoelastic problem, Canyard kernel, Laplace transformation, Fourier transformation, Levi-Civita tensor

Mathematics Subject Classification (2000): 74F15, 74H05, 45A05

1. Introduction

One of most important problems when investigating the propagation of vibrations in elastic media is the displaying of singularities of non-stationary wave fields. In this meaning the problems for concentrated impulse sources of disturbances are of great significance. One of the problems the solutions of which can be derived in closed form is the Lamb problem for the half-space [8], [9]. This problem is solved with the integral transformation technique and with the use of other methods, for instance, with the Smirnov-Sobolev method [9]–[11]. The problem becomes complicated when investigating magnetoelastic bounded vibrations [3]–[5]. Solving these problems is connected with certain mathematical difficulties as is known from [1]–[4]. In the present work the

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analytical formulae for elastic displacements and stresses of the induced electromagnetic field for the Lamb magnetoelastic problem are obtained with the use of Laplace and Fourier transformation technique and consequently by applying the Canyard kernel. The noted problem is discussed in the works [5], [6] for certain conditions.

2. Formulation of the problem, basic equations and boundary conditions

Let the elastic isotropic perfectly conductive medium occupy the semi-infinite area $x_2 > 0$ (in the Cartesian rectangular coordinate system $0x_1x_2x_3$) and be under an external homogeneous constant magnetic field with the stress vector $\vec{H}_0(0;0;H_{03})$. It is assumed that there is a vacuum at the area $x_2 < 0$. Let later at the edge of the medium $x_2 = 0$ at the moment $t = 0$ the concentrated impulse pressure be applied, which is disturbed uniformly along the lines $x_1 = 0, x_2 = 0$ (Fig.1),

$$f = \{f_1; f_2; 0\} = \{f_1^0\delta(x_1)\delta(t); f_2^0\delta(x_1)\delta(t); 0\}, \tag{1}$$

where f_1^0 and f_2^0 are the coefficients of polarization of the source; $\delta(x_1)$, $\delta(t)$ are the Dirac delta functions.

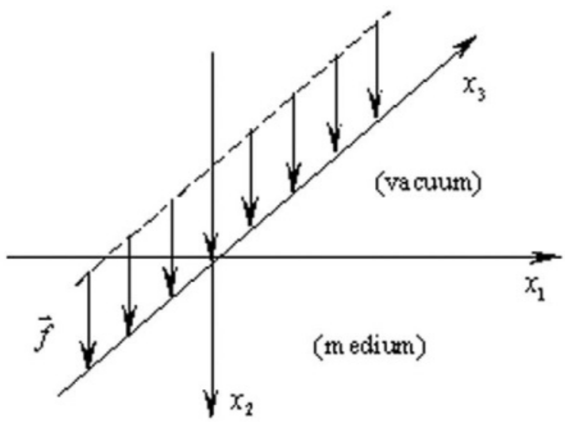

Figure 1

Elastic and electromagnetic disturbances are assumed to be small so the linearized equations of the theory of magnetoelasticity are used. The displacement current in elastic media is neglected. On the basis of these assumptions for the studied problem the following basic equations are obtained.

The equations of magnetoelasticity for the medium are

$$rot\vec{h} = \frac{4\pi}{c}\vec{j}\,, \quad rot\vec{e} = -\frac{\mu_e}{c}\frac{\partial \vec{h}}{\partial t}\,, \quad \vec{e} = -\frac{\mu}{c}\left(\frac{\partial \vec{u}}{\partial t} \times \vec{H}_0\right), \tag{2}$$
$$\mu\Delta\vec{u} + (\lambda+\mu) graddiv\vec{u} + \frac{\mu}{c}\vec{j} \times \vec{H}_0 = \rho_0 \frac{\partial^2 \vec{u}}{\partial t^2}\,,$$

where $\vec{h}$ and $\vec{e}$ are vectors of stresses of the induced electromagnetic field; $\vec{j}$ is the density of the induced current; $\vec{u}$ is the vector of elastic displacement; μ_e is the specific magnetic permeability; λ and μ are the Lamé coefficients; ρ_0 is the density of the medium; c is the velocity of light in the vacuum.

When obtaining equation (2) the relations

$$\sigma_{ik} = 2\mu\gamma_{ik} + \lambda div\vec{u}\delta_{ik}\,,$$
$$\gamma_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \tag{3}$$
$$\vec{B}_0 = \mu_e \vec{H}_0, \quad \vec{b} = \mu_e \vec{h}, \quad \vec{d} = \epsilon_e \vec{e},$$

are used, where σ_{ik} are the components of the tensor of stresses; γ_{ik} are the coefficients of the tensor of deformations; $\vec{B}_0$ is the vector of induction of external magnetic field; $\vec{d}$ and $\vec{b}$ are the vectors of induction of induced electromagnetic field.

After some transformations the system (2) can be rewritten in the form

$$\begin{cases} \vec{h} = rot\left(\vec{u} \times \vec{H}_0\right), \quad \vec{e} = -\dfrac{\mu_e}{c}\left(\dfrac{\partial \vec{u}}{\partial t} \times \vec{H}_0\right), \\ \mu\Delta\vec{u} + (\lambda+\mu)\, graddiv\vec{u} + \dfrac{\mu_e}{4\pi}\left[rotrot\left(\vec{u} + \vec{H}_0\right)\right] \times \vec{H}_0 = \rho_0 \dfrac{\partial^2 \vec{u}}{\partial t^2}\,. \end{cases} \tag{4}$$

The equations of electrodynamics for the vacuum are

$$\Delta \vec{e}^* = \frac{1}{c^2}\frac{\partial^2 \vec{e}^*}{\partial t^2}\,, \quad \Delta \vec{h}^* = \frac{1}{c^2}\frac{\partial^2 \vec{h}^*}{\partial t^2}\,, \tag{5}$$

where $\vec{e}^*$ and $\vec{h}^*$ are the vectors of stresses of the induced electromagnetic field in vacuum.

At the edge $x_2 = 0$ of the elastic medium and the vacuum the boundary conditions

$$\begin{cases} \left(\sigma_{ij} + t_{ij} - t^*_{ij}\right) n_j = f_i\,, \\ \left(b_j - b^*_j\right) n_j = 0\,, \\ \epsilon_{kij}\left(e_k - e^*_k\right) n_j = 0\,, \end{cases} \tag{6}$$

must be satisfied (here the summation convention with respect to repeated indices is applied) and the conditions

$$\vec{j}_s = -\frac{c}{4\pi}\vec{n} \times \left(\vec{h} - \vec{h}^*\right) , \quad \rho_s = -\frac{1}{4\pi}\vec{n}\left(\vec{d} - \vec{d}^*\right) , \tag{7}$$

defining the density of the surface current $\vec{j}_s$ and the charge ρ_s. Here $\vec{n} = n_j\vec{i}_j$ is the external normal to the edge of the elastic medium; t_{ij} and t^*_{ij} are the components of Maxwell tensor of stresses for the induced electromagnetic field in the medium and in vacuum,

$$\begin{aligned} t_{ij} &= \frac{1}{4\pi}\left(H_{0i}b_k + B_{0k}h_i\right) - \frac{\delta_{ik}}{8\pi}\left(\vec{H}_0\vec{b} + \vec{B}_0\vec{h}\right) , \\ t^*_{ij} &= \frac{1}{4\pi}\left(H^*_{0i}h_k + H^*_{0k}h_i\right) - \frac{\delta_{ik}}{4\pi}\vec{H}^*_0\vec{h}^* , \end{aligned} \tag{8}$$

ϵ_{kij} are the components of the Levi-Civita tensor.

It can be shown (analogously to the work [7]) that for the pressure (1) and the magnetic field $\vec{H}_0(0;0;H_{03})$ the equations of magnetoelasticity in the area $x_2 > 0$ (equations (4)), the equations of electrodynamics in the area $x_2 < 0$ (equations (5)) have solutions of the form

$$\begin{cases} u_1 = u_1\left(x_1, x_2, t\right) , \quad u_2 = u_2\left(x_1, x_2, t\right) , \quad u_3 \equiv 0 , \\ h^*_i = h^*_i\left(x_1, x_2, t\right) , \quad e^*_i = e^*_i\left(x_1, x_2, t\right) , i = 1, 2, 3. \end{cases} \tag{9}$$

For simplicity using $\mu_e = 1$ in the case of the two-dimensional problem (9) from (4), (5) and (6) the following mathematical problem will be obtained (Lamb problem of magnetoelasticity).

Find solutions of the equations of magnetoelasticity in the area $x_2 > 0$

$$\begin{cases} a_3\dfrac{\partial^2 u_1}{\partial x_1^2} + c_3\dfrac{\partial^2 u_2}{\partial x_1 \partial x_2} + d_3\dfrac{\partial^2 u_1}{\partial x_2^2} = \dfrac{\partial^2 u_1}{\partial t^2} , \\ d_3\dfrac{\partial^2 u_2}{\partial x_1^2} + c_3\dfrac{\partial^2 u_1}{\partial x_1 \partial x_2} + a_3\dfrac{\partial^2 u_2}{\partial x_2^2} = \dfrac{\partial^2 u_2}{\partial t^2} , \end{cases} \tag{10}$$

the equations of the induced magnetic field in the area $x_2 < 0$

$$\frac{\partial^2 h^*_3}{\partial x_1^2} + \frac{\partial^2 h^*_3}{\partial x_2^2} = \frac{1}{c^2}\frac{\partial^2 h^*_3}{\partial t^2} , \tag{11}$$

satisfying the homogeneous initial conditions and the boundary conditions for $x_2 = 0$

$$\begin{cases} d_3\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) = f^*_1 , \\ (c_3 - d_3)\dfrac{\partial u_1}{\partial x_1} + a_3\dfrac{\partial u_2}{\partial x_2} + \dfrac{H_{03}}{4\pi\rho_0}h^*_3 = f^*_2 , \\ \dfrac{\partial h^*_3}{\partial x_2} + \dfrac{H_{03}}{c^2}\dfrac{\partial^2 H_2}{\partial t^2} = 0 . \end{cases} \tag{12}$$

Here the notations

$$
\begin{aligned}
&f_i^* = \rho_0^{-1} f_i^0 \delta(x_1)\delta(t)\,, \quad f_i^0 = const \quad (i=1,2)\,,\\
&a_3 = a_0 + æ\,, \quad c_3 = c_0 + æ\,, \quad d_3 = d_0\,,\\
&a_0 = \frac{\lambda + 2\mu}{\rho_0}\,, d_0 = \frac{\mu}{\rho_0}\,, \quad c_0 = a_0 - d_0\,,\\
&æ = \frac{H_{03}^2}{4\pi\rho_0}\,,
\end{aligned}
$$

are used, where a_0, d_0 are the squares of velocities of the longitudinal and the transversal pure elastic waves in elastic medium.

The rest of characteristics of the main problem are defined by the above-mentioned corresponding expressions.

3. Solution of the problem

We will solve the addressed problem with Laplace and Fourier integral transformation technique. Moreover, the unknown functions on the Laplace's images are defined by the Canyard method [8].

Applying the Laplace transformation with respect to time t (with parameter s) to equations (10), (11) and boundary conditions (12) and the Fourier transformation with respect to the coordinate x_1 (with parameter α_1) after satisfying the initial conditions and the conditions on the dumping of disturbances at infinity for the Laplace's images of the unknown functions we will obtain

$$
\begin{cases}
\bar{u}_i(s,x_1,x_2) = \sum\limits_{k,j=1}^{2} \eta_{kj}^{(i)} f_j^0 \int\limits_{-i\infty}^{i\infty} f_{kj}^{(i)}(\theta) e^{-st_k(\theta)} d\theta\,,\\
\bar{h}_3(s,x_1,x_2) = \sum\limits_{k,j=1}^{2} \eta_{kj}^{(3)} f_j^0 s \int\limits_{-i\infty}^{i\infty} f_{kj}^{(3)}(\theta) e^{-st_3(\theta)} d\theta\,.
\end{cases}
\tag{13}
$$

Here

$$
\begin{cases}
f_{kj}^{(i)} = \Phi_{kj}^{(i)}(\theta)\Delta_3^{-1}(\theta)\,, \quad f_{kj}^{(3)} = \dfrac{\Phi_{kj}^{(2)}(\theta)}{\lambda_3(\theta)\Delta_3(\theta)}\,,\\
\Phi_{1j}^{(1)} = i\theta A_j(\theta)\,, \quad \Phi_{1j}^{(2)} = i\lambda_1(\theta)A_j(\theta)\,,\\
\Phi_{2j}^{(1)}(\theta) = i\lambda_2(\theta)B_j(\theta)\,, \quad \Phi_{2j}^{(2)}(\theta) = i\theta B_j(\theta)\,,
\end{cases}
\tag{14}
$$

$$
\begin{cases}
A_1(\theta) = \theta\left(2d_3c^2\lambda_1\lambda_3 + æ\right)\,,\\
A_2(\theta) = \lambda_3\left(\theta^2 - \lambda_2^2\right)\,,\\
B_1(\theta) = d_3c^2\lambda_3\left(\theta^2 - \lambda_2^2\right) - æ\lambda_1\,,\\
B_2(\theta) = \theta\lambda_1\lambda_3\,,
\end{cases}
\tag{15}
$$

$$
\Delta_3(\theta) = \left[\lambda_1(\theta) - \lambda_2(\theta)\right] R_3(\theta)\,,
\tag{16}
$$

$$\lambda_n(\theta) = \sqrt{V_n^{-2} - \theta^2}\left(Re\lambda_n(\theta) > 0\,, for|\theta| < V_n^{-1}\right)\,, \tag{17}$$

$$t_n(\theta) = \theta x_1 + \lambda_n(\theta)|x_2|\,, \tag{18}$$

$$V_1^2 = a_3\,, \quad V_3^2 = d_3\,, \quad V_3 = c\,, \tag{19}$$

$$\begin{cases} \eta_{11}^{(i)} = (2\pi\rho_0 d_3)^{-1} c_3\,, & \eta_{12}^{(i)} = (2\pi\rho_0)^{-1} c_3 c^2\,, \\ \eta_{21}^{(i)} = (-1)^{i+1}(2\pi\rho_0 d_3)^{-1} c_3\,, & \eta_{22}^{(i)} = (-1)^{i+1}(\pi\rho_0)^{-1} c_3 c^2\,, \\ \eta_{kj}^{(3)} = -H_{03}c^{-2}\eta_{kj}^{(2)}, & i,j,k = 1,2\,; n = 1,2,3\,, \end{cases} \tag{20}$$

where $R_3(\theta)$ is the Rayleigh function

$$R_3(\theta) = c^2\lambda_3\lambda_2\left(4c_3d_3\theta^2 - a_3\right) - a_3\left[c^2\lambda_3 + æ(\lambda_1 + \lambda_2)\right]\lambda_1\,. \tag{21}$$

When obtaining (13)-(21) the notation $\alpha_1 = is\theta$ is used assuming that the Laplace transformation parameter $s > 0$.

According to the work [4] $R_3(\theta)$ has two real roots $\pm\theta_R = \pm V_R^{-1}$, moreover,

$$V_R < \sqrt{d_3} = V_2\,. \tag{22}$$

In (22) V_R is the velocity of the Rayleigh magnetoelastic wave, V_2 is the velocity of the slow magnetoelastic wave.

From (14)-(17) it follows that the singularities of the integrated functions $f_{kj}^{(i)}(\theta)$ are the bifurcation points of the function $\lambda_n(\theta)$ and the simple pole

$$\begin{cases} \theta = \pm\theta_n^* = V_n^{-1}, & n = 1,2,3\,, \\ \theta = \pm\theta_R = \pm V_R^{-1}\,. \end{cases} \tag{23}$$

As the velocity V_1 of the longitudinal magnetoelastic waves is less than the velocity of light and greater that the velocity of the slow magnetoelastic waves V_2 and the condition (22) takes place then the singular points θ_n^* and θ_R at the real axis of the complex θ-plane are placed in the form

$$0 < \theta_3^* < \theta_1^* < \theta_2^* < \theta_R\,. \tag{24}$$

From (24) it follows that for the extraction of unique branches of the functions $f_{kj}^{(i)}(\theta)$ the sections $(-\infty; -\theta_3^*)$ and $(\theta_3^*; +\infty)$ can be constructed at the complex θ-plane.

At the complex θ-plane with the above-mentioned sections let us consider the line at which the function $t_n(\theta)$ from (18) has real positive values t, i.e.

$$\theta x_1 + \lambda_n(\theta)|x_2| = t > 0, \quad n = 1,2,3\,. \tag{25}$$

Solving the relation (25) with respect to θ we will obtain the parametric equation at the unknown line which is called Canyard line

$$\begin{aligned}&\theta = \theta_n^{\pm}|t| = \frac{x_1}{r^2}t \pm i\frac{x_2}{r^2}\sqrt{t^2-(t_n^*)^2}\,,\\ &t \geq t_n^*\,,\quad t_n^* = rV_n^{-1} = r\theta_n^*\,,\quad r^2 = x_1^2+x_2^2\,,\quad n=1,2,3\,.\end{aligned} \tag{26}$$

For each n equality (26) is the equation of the hyperbola's right branch Γ_n being obtained for $x_1>0$ and having the top at the point

$$\tilde{\theta}_n = \frac{x_1}{r^2}t_n^* = \frac{x_1}{r}\theta_n^* \quad (x_1>0)\,. \tag{27}$$

Moreover, if t varies in the interval $(t_n^*;+\infty)$ the function $\theta_n^+(t)$ characterizes the part Γ_n^+ of the curve Γ_n and the function $\theta_n^-(t)$ characterizes the part Γ_n^- of that curve (Fig.2). If $x_2<0$ the equality (26) characterizes the left branch of the hyperbola with the top being the point $-\tilde{\theta}_n$.

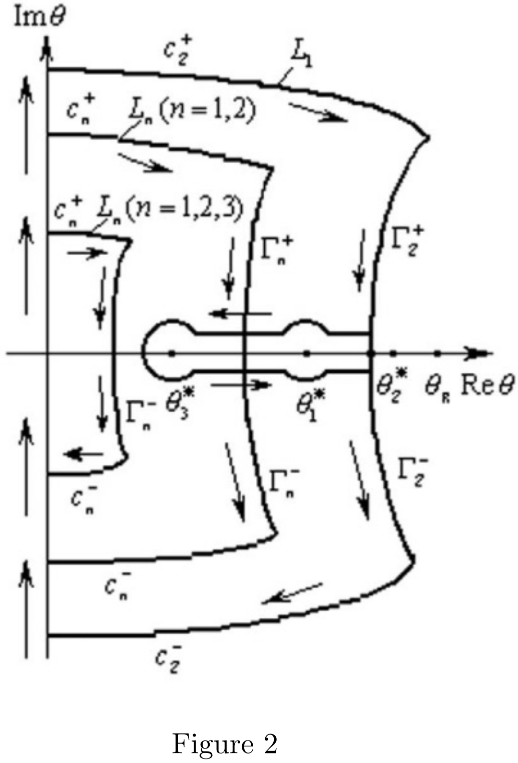

Figure 2

Further we will study only the area $x_1>0$ according to symmetry.

As the contour of integration contains the Canyard lines then it is necessary to explore the interlocation of sections and the Canyard lines in the complex θ-plane, i.e. the interlocation of the tops $\tilde{\theta}_n$ of the hyperbolas Γ_n and the singular points of the integrated functions θ_3^* θ_2^* θ_1^* and θ_R depending on the coordinates $x_1\,, x_2$ and on the external magnetic field $\vec{H}_0$.

On the basis of (23)-(27) we can conclude that depending on the coordinates of the points $M(x_1, x_2)$ of the wave area the following cases of location of tops of the above-mentioned hyperbolas $\tilde{\theta}_n$ with respect to the singular points θ_n^* and θ_R of the integrants $f_{kj}^{(i)}(\theta)$ are possible

$$\begin{cases} 1)\, 0 < \tilde{\theta}_3 < \theta_3^* < \tilde{\theta}_1 < \theta_1^* \le \tilde{\theta}_2 < \theta_2^* < \theta_R\,, & \text{for} \quad 0 < \varphi < \varphi_0\,, \\ 2)\, 0 < \tilde{\theta}_3 < \theta_3^* \le \tilde{\theta}_1 < \tilde{\theta}_2 < \theta_1^* < \theta_2^* < \theta_R\,, & \text{for} \quad \varphi_0 < \varphi \le \varphi_1\,, \\ 3)\, 0 < \tilde{\theta}_3 < \tilde{\theta}_1 < \theta_3^* \le \tilde{\theta}_2 < \theta_1^* < \theta_2^* < \theta_R\,, & \text{for} \quad \varphi_1 < \varphi \le \varphi_2\,, \\ 4)\, 0 \le \tilde{\theta}_3 \le \tilde{\theta}_1 \le \tilde{\theta}_2 < \theta_3^* < \theta_1^* < \theta_2^* < \theta_R\,, & \text{for} \quad \varphi_2 < \varphi \le \dfrac{\pi}{2}\,. \end{cases} \tag{28}$$

Here $\varphi = arctg|x_2/x_1|$ is the angle between the beam OM and the axis $0x_1$. The angels $\varphi_0\,, \varphi_1\,, \varphi_2$ are defined by the formulae

$$\varphi_0 = arctg\sqrt{V_1^2 V_2^{-1} - 1}\,,$$

$$\varphi_1 = arctg\sqrt{V_3^2 V_1^{-2} - 1}\,,$$

$$\varphi_2 = arctg\sqrt{V_3^2 V_2^{-1} - 1}\,.$$

Now having the interlocation (28) we can begin the inversion of integrals from the equation (13).

First of all let us discuss the inversion of the integrals having the form

$$\bar{I}_{1j}(s, x_1, x_2) = \int_{-i\infty}^{i\infty} f_{1j}^{(i)}(\theta) e^{-st_1(\theta)} d\theta\,, \tag{29}$$

which describes the role of the longitudinal magnetoelastic waves in the main solutions. Depending on the location of the points $\tilde{\theta}_1$ and $\tilde{\theta}_3$ we have the following cases of inversions and a certain area of the plane $0x_1x_2$ corresponds to each inversion:

a) $M(x_1, x_2) \in \Omega_1 = \{\varphi_1 < \varphi \le \pi/2\,; x_2 > 0\}\,.$

In this case according to (28) the line Γ_1 does not cut the branch $(\theta_3^*; +\infty)$. Therefore for the calculation of the integrals (29) the integration can be done along the closed contour L_1 generated by the line Γ_1, by the imaginary axis of the θ-plane and by the arcs $c_1^+\,, c_1^-$ of the circle with an infinitely large radius (Fig.2).

b) $M(x_1, x_2) \in \Omega_2 = \{0 < \varphi < \varphi_1\,; x_2 > 0\}\,.$

In this case according to (28) the line Γ_1 does not cut the branch $(\theta_3^*;+\infty)$. Therefore the contour of integration L_1 besides the above-mentioned elements also contains the upper γ_1^+ and the lower γ_1^- edges of the interval $\left[\theta_3^*;\tilde{\theta}_1\right]$ of the real axis of the θ-plane (Fig.2). Moreover, depending on the time t the parametric equation of the edges $\gamma_1^\pm$ has the form

$$\begin{cases} \theta = \theta_1^0(t) = \dfrac{x_1}{r^2}t - \dfrac{|x_2|}{r^2}\sqrt{(t_1^*)^2 - t^2}\,, \\ t_1^0 \le t \le t_1^*\,, \quad t_1^0 = \theta_3^* x_1 + |x_2|\sqrt{(\theta_1^*)^2 - (\theta_3^*)^2}\,. \end{cases} \tag{30}$$

Applying the Cauchy theorem and the Jordan lemma to the integrals (29) we will obtain

a) at the area Ω_1

$$\bar{I}_{1j}^{(i)} = 2\int_0^\infty \sigma_0(t-t_1^*) Re\left[f_{1j}^{(i)}(\theta_1^+(t))\frac{\partial\theta_1^+(t)}{\partial t}\right] e^{st}dt\,, \tag{31}$$

b) at the area Ω_2

$$\bar{I}_{1j}^{(i)} = 2\int_0^\infty \Big\{\sigma_0(t-t_1^0)\sigma_0(t_1^*-t) Re\left[f_{1j}^{(i)+}(\theta_1^0(t))\frac{\partial\theta_1^0(t)}{\partial t}\right]$$

$$+\sigma_0(t-t_1^*) Re\left[f_{1j}^{(i)}(\theta_1^+(t))\frac{\partial\theta_1^+(t)}{\partial t}\right]\Big\} e^{-st}dt\,, \tag{32}$$

where $\sigma_0(t)$ is the identity Heaviside function.

Thus using the Canyard lines the inverse Fourier transformation is turned into the Laplace transformations of the unknown functions. And this is the meaning of the Canyard method.

After the inversion of the Laplace transformation from (31) and (32) we will obtain

$$I_{1j}^{(i)}(t,x_1,x_2) = 2\Big\{\sigma_0(t-t_1^*) Re\left[f_{1j}^{(i)}(\theta_1^+(t))\frac{\partial\theta_1^+(t)}{\partial t}\right]$$

$$+sgn(m-1)\sigma_0(t-t_1^0)\sigma_0(t_1^*-t) Re\left[f_{1j}^{(i)+}(\theta_1^0(t))\frac{\partial\theta_1^0(t)}{\partial t}\right]\Big\}\,. \tag{33}$$

In (33) $m=1$ corresponds to the area Ω_1, $m=2$ to the area Ω_2, $f_{1j}^{(i)+}$ is the value of the function $f_{1j}^{(i)}$ at the edge γ_1^+.

The inversion of the integrals having the form $\bar{I}^{(i)}_{2j}(s, x_1, x_2)$ at the areas $\Omega^*_1 = \{\varphi_2 < \varphi \le \frac{\pi}{2}\,; x_2 > 0\}$, $\Omega^*_2 = \{0 < \varphi < \varphi_2\,; x_2 > 0\}$ and the integral $\bar{I}^{(3)}_{kj}(s, x_1, x_2)$ at the area $\Omega_3 = \{0 < \varphi \le \frac{\pi}{2}\,; x_2 > 0\}$ is handled also as in the previous case in accordance with (28).

Inverting the integrals from (13) for the displacements u_i and the induced magnetic field h^*_3 we will finally obtain

$$\begin{cases} u_i(t, x_1, x_2) = \sum\limits_{k,j=1}^{2} \eta^{(i)}_{kj} f^0_j I^{(i)}_{kj}(t, x_1, x_2)\,, \\ h^*_3(t, x_1, x_2) = \sum\limits_{k,j=1}^{2} \eta^{(3)}_{kj} f^0_j I^{(3)}_{kj}(t, x_1, x_2)\,, \end{cases} \tag{34}$$

where

$$\begin{cases} I^{(i)}_{kj}(t, x_1, x_2) = 2\left\{\sigma_0(t - t^*_k) Re\left[f^{(i)}_{kj}(\theta^+_k(t))\dfrac{\partial\theta^+_k(t)}{\partial t}\right]\right. \\ \qquad \left. +sgn(m-1)\sigma_0(t - t^0_k)\sigma_0(t^*_k - t) Re\left[f^{(i)+}_{kj}(\theta^0_k(t))\dfrac{\partial\theta^0_k(t)}{\partial t}\right]\right\}, \\ I^{(3)}_{kj}(t, x_1, x_2) = \dfrac{\partial}{\partial t}\left\{2\sigma_0(t - t^*_3) Re\left[f^{(3)}_{kj}(\theta^+_3(t))\dfrac{\partial\theta^+_3(t)}{\partial t}\right]\right\}, \end{cases} \tag{35}$$

$$\begin{cases} \theta^0_3(t) = \dfrac{x_1}{r^2} t - \dfrac{|x_2|}{r^2}\sqrt{(t^*_2)^2 - t^2}\,, \\ t^0_2 \le t \le t^*_2\,, \quad t^0_2 = \theta^*_3 x_1 + |x_2|\sqrt{(\theta^*_2)^2 - (\theta^*_3)^2}\,. \end{cases} \tag{36}$$

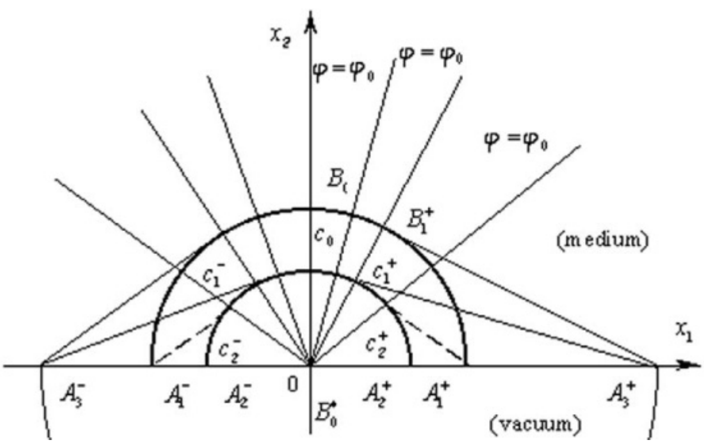

Figure 3

In Fig. 3 the wave fronts and wave fields of the magnetoelastic waves are shown. On the basis of the analysis of solutions (34) we have the following wave location. Just from the location of force two waves are propagated: the speed longitudinal magnetoelastic wave occupying the area constrained by the curve $A_1^+ B_0 A_1^- O A_1^+$ (Fig.3) and the transversal slow magnetoelastic wave occupying the area constrained by the curve $A_2^+ C_0 A_2^- O A_2^+$ and at the vacuum the electromagnetic wave is propagated occupying the area constrained by the curve $A_3^+ B_0^* A_3^- O A_3^+$. When propagating these waves generate additional disturbances having the form of magnetoelastic waves reflected from the plane of the half-space and propagating along the depth of the elastic medium and having the form of an electromagnetic wave propagating in the vacuum. As a result of the noted processes the speed magnetoelastic disturbances occupy the extended area constrained by the curve $A_3^+ B_0 A_3^- O A_3^+$; the slow magnetoelastic waves occupy the extended area constrained by the curve $A_3^+ C_0 A_3^- O A_3^+$; and the electromagnetic disturbances occupy the area constrained by the curve $A_3^+ B_0^* A_3^- O A_3^+$ as before.

Conclusions

In the presented work the two-dimensional magnetoelastic Lamb problem is solved with the use of integral transformations technique.

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ON AN EIGENVALUE PROBLEM FOR THE ANISOTROPIC STRIP

Mher Aghalovyan
Institute of Mechanics
of National Academy of Sciences of Armenia,
Marshal Bagramian ave. 24b
375019 Yerevan, Armenia
mechins@sci.am

Abstract Beginning with the equations of dynamics of elasticity theory, frequencies and forms of free vibrations of an anisotropic strip-beam rigidly fastened along one of the facial surfaces are determined by an asymptotic method. Characteristic equations for determining main values of frequencies are derived. It is proved that in the strip with general anisotropy two types of free vibrations arise, but unlike the orthotropic strip, they are not purely shear and longitudinal. Eigenfunctions which compose an orthonormalized system, are determined. The applied method of determining frequencies and forms of free vibrations may be used for the investigation of analogue questions for thin bodies, particularly, for plates and shells.

Keywords: elasticity of thin bodies, anisotropic strip-beam, dynamical equations of elasticity, singularly perturbed problems, asymtotic method, free vibrations, orthonormalized system of eigenfunctions

Mathematics Subject Classification (2000): 74K10

1. Introduction

A lot of papers, among them the papers on the basis of Bernoulli and Euler presuming the plane cross-section hypothesis, are devoted to the first boundary problem of elasticity theory for strip-beams (where on longitudinal borders the values of the stress tensor are given). There is considerably less work when on the longitudinal borders other conditions are given, for example, displacement values or mixed conditions. It is proved that for this class of problems the hypothesis of plane cross-sections is not applicable [1], [2]. For the solution of this

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kind of class of problems the asymptotic method for solving singularly perturbed equations [3], [4] appeared to be effective. For applying the fact that the dimensions of the cross-section of the bars and beams are much less than their length, passing to dimensionless coordinates and components of the displacement vector the system of elasticity theory equations becomes singularly perturbed by a small parameter. In this paper a boundary problem on eigenvalues of the anisotropic strip is solved by an asymptotic method. Problems like that are one of the basic problems of seismosteady building and seismology in fundamental constructions.

2. Frequencies and eigenfunctions

It is required to find null solutions of the dynamic equations of the homogeneous system of elasticity theory for the anisotropic strip $D = \{(x,y) : x \in [0;\ell],\ |y| \le h, h << \ell\}$ under the homogeneous boundary conditions

$$\sigma_{xy}(h) = \sigma_{yy}(h) = 0, \tag{1}$$
$$u(-h) = v(-h) = 0. \tag{2}$$

Finding the solution in the form

$$\begin{aligned} &\sigma_{\alpha\beta} = \sigma_{ik}(x,y)exp(i\omega t), \quad \alpha,\beta = x,y;\ i,k = 1,2, \\ &u,v = (u_1(x,y), v_1(x,y))exp(i\omega t), \end{aligned} \tag{3}$$

where $\sigma_{\alpha,\beta}, u, v$ are the components of the corresponding stress tensor and displacement vector, ω is the required frequency of free vibrations. Passing to dimensionless coordinates $\xi = x/\ell$, $\zeta = y/h$ and components $U = u_1/\ell$, $V = v_1/\ell$ we have a system singularly perturbed by a small parameter $\varepsilon = h/\ell$

$$\begin{aligned} &\frac{\partial\sigma_{11}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{12}}{\partial\zeta} + \omega_*^2\varepsilon^{-2}U = 0, \\ &\frac{\partial\sigma_{12}}{\partial\xi} + \varepsilon^{-1}\frac{\partial\sigma_{22}}{\partial\zeta} + \omega_*^2\varepsilon^{-2}V = 0, \quad \omega_*^2 = \rho h^2\omega^2, \\ &\frac{\partial U}{\partial\xi} = a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{16}\sigma_{12}, \\ &\varepsilon^{-1}\frac{\partial V}{\partial\zeta} = a_{12}\sigma_{11} + a_{22}\sigma_{22} + a_{26}\sigma_{12}, \\ &\varepsilon^{-1}\frac{\partial U}{\partial\zeta} + \frac{\partial V}{\partial\xi} = a_{16}\sigma_{11} + a_{26}\sigma_{22} + a_{66}\sigma_{12}. \end{aligned} \tag{4}$$

The solution of system (4) will be sought in the form of an asymptotic representation

$$\sigma_{ik} = \varepsilon^{-1+s}\sigma_{ik}^{(s)}, \quad U, V = \varepsilon^s (U^{(s)}, V^{(s)}), \qquad (5)$$
$$\omega_*^2 = \varepsilon^s \omega_{*s}^2, \ s = 0, \ldots, N.$$

Substituting (5) into (4) and comparing the coefficients of powers of ε in the equations we get the system

$$\begin{aligned}
&\frac{\partial \sigma_{11}^{(s-1)}}{\partial \xi} + \frac{\partial \sigma_{12}^{(s)}}{\partial \zeta} + \omega_{*k}^2 U^{(s-k)} = 0, \quad k = 0, \ldots, s,\\
&\frac{\partial \sigma_{12}^{(s-1)}}{\partial \xi} + \frac{\partial \sigma_{22}^{(s)}}{\partial \zeta} + \omega_{*k}^2 V^{(s-k)} = 0,\\
&\frac{\partial U^{(s-1)}}{\partial \xi} = a_{11}\sigma_{11}^{(s)} + a_{12}\sigma_{22}^{(s)} + a_{16}\sigma_{12}^{(s)},\\
&\frac{\partial V^{(s)}}{\partial \zeta} = a_{12}\sigma_{11}^{(s)} + a_{22}\sigma_{22}^{(s)} + a_{26}\sigma_{12}^{(s)},\\
&\frac{\partial U^{(s)}}{\partial \zeta} + \frac{\partial V^{(s-1)}}{\partial \xi} = a_{16}\sigma_{11}^{(s)} + a_{26}\sigma_{22}^{(s)} + a_{66}\sigma_{12}^{(s)},
\end{aligned} \qquad (6)$$

for determining the unknown coefficients $\sigma_{ik}^{(s)}, U^{(s)}, V^{(s)}, \omega_{*s}^2$ of the representation (5), where $k = 0, \ldots, s$ means that by the repeated index "k" a summation from null up to s takes place. The solution of system (6) is

$$\begin{aligned}
&\sigma_{11}^{(s)} = -\frac{1}{a_{11}}(a_{12}\sigma_{22}^{(s)} + a_{16}\sigma_{12}^{(s)}) + \frac{1}{a_{11}}\frac{\partial U^{(s-1)}}{\partial \xi},\\
&\sigma_{12}^{(s)} = \frac{1}{\Delta}\left(A_{11}\frac{\partial U^{(s)}}{\partial \zeta} - A_{16}\frac{\partial V^{(s)}}{\partial \zeta}\right) + \tilde{\sigma}_{12}^{(s-1)},\\
&\sigma_{22}^{(s)} = \frac{1}{\Delta}\left(A_{66}\frac{\partial V^{(s)}}{\partial \zeta} - A_{16}\frac{\partial U^{(s)}}{\partial \zeta}\right) + \tilde{\sigma}_{22}^{(s-1)},\\
&A_{11} = (a_{11}a_{22} - a_{12}^2)/a_{11}, \quad A_{66} = (a_{11}a_{66} - a_{16}^2)/a_{11},\\
&A_{16} = (a_{11}a_{26} - a_{12}a_{16})/a_{11}, \quad \Delta = A_{11}A_{66} - A_{16}^2.
\end{aligned} \qquad (7)$$

The functions $U^{(s)}, V^{(s)}$ are determined from the equations

$$\begin{aligned}
&A_{11}\frac{\partial^2 U^{(s)}}{\partial \zeta^2} + \Delta\omega_{*k}^2 U^{(s-k)} - A_{16}\frac{\partial^2 V^{(s)}}{\partial \zeta^2} = f_u^{(s-1)},\\
&A_{66}\frac{\partial^2 V^{(s)}}{\partial \zeta^2} + \Delta\omega_{*k}^2 V^{(s-k)} - A_{16}\frac{\partial^2 U^{(s)}}{\partial \zeta^2} = f_v^{(s-1)}, \ k = 0, \ldots, s,
\end{aligned} \qquad (8)$$

where the values $\tilde{\sigma}_{12}^{(s-1)}$, $\tilde{\sigma}_{22}^{(s-1)}$, $f_u^{(s-1)}$, $f_v^{(s-1)}$ are well-known if the previous approaches are built. It is not difficult to write down their expressions, $Q^{(m)} \equiv 0$ when $m < 0$. For the orthotropic layer $A_{16} = 0$ and system (7) separates into two independent equations, when $s = 0$. The main values of the free frequencies of the orthotropic strip are determined in [5]. When $s = 0$ system (7) becomes homogeneous and admitting

$$U^{(0)} = A_{16}\frac{\partial^2\Phi_0}{\partial\zeta^2}, \quad \Phi_0 = \varphi_0(\xi)\psi_0(\zeta), \tag{9}$$

$$V^{(0)} = \left(A_{11}\frac{\partial^2}{\partial\zeta^2} + \Delta\omega_{*0}^2\right)\Phi_0,$$

its solution is reduced to the solution of the equation

$$\frac{d^4\psi_0}{d\zeta^4} + (A_{11} + A_{66})\omega_{*0}^2\frac{d^2\psi_0}{d\zeta^2} + \Delta\omega_{*0}^4\psi_0 = 0. \tag{10}$$

Finding the solution of equation (10), calculating by formulae (7), (9) the components of the stress tensor and displacements vector, satisfying conditions (1), (2), we get the equations

$$\cos 2\beta_1\omega_{*0} = 0, \qquad \cos 2\beta_2\omega_{*0} = 0, \tag{11}$$

$$\beta_{1,2} = \frac{1}{\sqrt{2}}\sqrt{A_{11} + A_{66} \mp \sqrt{(A_{11} - A_{66})^2 + 4A_{16}^2}},$$

for the frequencies, from where the main values of the frequencies

$$\omega_{0n}^I = \frac{1}{h\sqrt{\rho}}\omega_{*0n}^I, \quad \omega_{*0n}^I = \frac{\pi}{4\beta_1}(2n+1),\ n \in N, \tag{12}$$

$$\omega_{0n}^{II} = \frac{1}{h\sqrt{\rho}}\omega_{*0n}^{II}, \quad \omega_{*0n}^{II} = \frac{\pi}{4\beta_2}(2n+1),\ n \in N, \tag{13}$$

and the eigenfunctions corresponding to them

$$\psi_{0n}^I = \sin\beta_1\omega_{*0n}^I\zeta + tg\beta_1\omega_{*0n}^I\cos\beta_1\omega_{*0n}^I\zeta, \tag{14}$$

$$\psi_{0n}^{II} = \sin\beta_2\omega_{*0n}^{II}\zeta + tg\beta_2\omega_{*0n}^{II}\cos\beta_2\omega_{*0n}^{II}\zeta, \tag{15}$$

follow. Each of the groups of eigenfunctions composes an orthogonal system. Unlike the orthotropic strip on the strength of (12), (13), the free vibrations of the strip with general anisotropy are not purely shear and purely longitudinal, which makes it easy to integrate physically.

When $s = 1$, system (7) is inhomogeneous and it is necessary to solve it under two independent cases a) $\omega_* = \omega_{*0n}^I$, b) $\omega_* = \omega_{*0n}^{II}$. In the first case using the orthogonality of the functions system $\{\psi_{0n}^{I}\}$ all the required functions are sought in the form of series in these functions. By the well–known procedure [3], [4] the coefficients of the decompositions are determined and therefore all the values for the approach $s = 1$, particularly, the correction ω_{*I}^{I} to the frequency.

In case b) the unknown functions decompose by the functions $\{\psi_{0n}^{II}\}$ and the same procedure is repeating. In the analogue way the approaches $s > 1$ are considered. It should be noted that in the problems of forced vibrations, it is practically enough to know the values of the frequencies $\omega_{*0n}^{I}, \omega_{*0n}^{II}$, as resonance arises under coincidence of the frequency of forced effect with the main parts of the values of the frequencies of the free vibrations.

In conclusion we note that the frequencies of free vibrations for plates may be found by the noted method.

3. Conclusions

The effectivity of the asymptotic method for solving singularly perturbed differential equations for determining frequencies and forms of free vibrations of strip-beams and plates type of thin bodies is shown.

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ON SINGULAR PERTURBED EQUATIONS OF THIN BODIES

Lenser Aghalovyan
Institute of Mechanics
of National Academy of Sciences of Armenia,
Marshal Bagramian ave. 24b
375019, Yerevan, Armenia
mechins@sci.am

Abstract The problem of solving systems of equations of static and dynamical boundary value problems of elasticity theory for thin bodies (beams, plates, shells) is considered. Taking into account the specific geometry of such bodies, it is shown that in dimensionless values the system of equations is singularly perturbed by a small geometrical parameter.
For solving such systems an asymptotic method is used. The solution is combined by the solutions of the inner problem and the boundary layers. Asymptotic orders of the required values are established and iteration processes for their determination are built. It is shown that asymptotics correctly react to the type of conditions, stated on the face surfaces. The connection of asymptotic solutions with the results on classical theories of Bernoulli-Coulomb-Euler beams, Kirchhoff-Love plates and shells is established. The connection of Saint-Venant's principle with the property of the solution for the boundary layer is revealed. A class of problems for which Saint-Venant's principle is mathematically exactly fulfilled, is selected. It is proved that the applied asymptotic method permits to solve new classes of problems of statics and dynamics of thin bodies.

Keywords: elasticity of thin bodies, singularly perturbed problems, asymptotic method, boundary layer, Saint-Venant's principle, dynamical equations of elasticity, seismology, seismosteady buildings

Mathematics Subject Classification (2000): 74K10, 74G50

1. Introduction

Thin bodies like bars, beams, plates and shells, which are composite elements of modern constructions, are characterised by the property that one of their geometrical dimensions substantially differs from the

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other two. For bars and beams their length is much more than the cross dimensions and for plates and shells the thickness is much less than the tangential dimensions. Taking into account this specific character, having written the equations of elasticity theory in dimensionless coordinates and dimensionless components of the displacement vector, it is easy to be convinced that the corresponding system of differential equations in individual derivatives is singularly perturbed relative to small (big) dimensionless geometrical parameter. And also the small (big) parameter is the coefficient not of the whole leading operator, as in classical problems on singular perturbation but of some of its parts. It leads, particularly, to the situation when the number of boundary functions becomes infinite, and besides, the methods of solving regularly perturbed differential equations are not applicable here. The system for orthotropic plates $D = \{x, y, z : x, y \in D_0, |z| \leq h, h << \ell\}$ in Cartesian coordinates is

$$\begin{aligned}
&\frac{\partial \sigma_{11}}{\partial \xi} + \frac{\partial \sigma_{12}}{\partial \eta} + \varepsilon^{-1}\frac{\partial \sigma_{13}}{\partial \zeta} + F_x(\xi, \eta, \zeta) = 0 \qquad (1, 2, 3; \xi, \eta, \zeta; x, y, z),\\
&\frac{\partial U}{\partial \xi} = a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{13}\sigma_{33},\\
&\frac{\partial V}{\partial \eta} = a_{12}\sigma_{11} + a_{22}\sigma_{22} + a_{23}\sigma_{33},\\
&\varepsilon^{-1}\frac{\partial W}{\partial \zeta} = a_{13}\sigma_{11} + a_{23}\sigma_{22} + a_{33}\sigma_{33},\\
&\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} = a_{66}\sigma_{12}, \qquad \frac{\partial W}{\partial \xi} + \varepsilon^{-1}\frac{\partial U}{\partial \zeta} = a_{55}\sigma_{13},\\
&\frac{\partial W}{\partial \eta} + \varepsilon^{-1}\frac{\partial V}{\partial \zeta} = a_{44}\sigma_{23},
\end{aligned} \tag{1}$$

where σ_{ik} are the tensor components of stresses, U, V, W are the dimensionless vector components of displacements: $U = u_x/\ell, V = u_y/\ell, W = u_z/\ell$, a_{ik} are the elasticity coefficients, $\xi = x/\ell, \eta = y/\ell, \zeta = z/h, \ell$ is the typical dimension of the middle surface D_0 of the plate, F_x, F_y, F_z are the components of the volume forces, $\varepsilon = h/\ell$ is the small parameter. In order to solve the singularly perturbed system (1) with parameter ε under boundary conditions on the facial surfaces $z = \pm h$ and the side surface ∂_D of the plate, it is natural to use the asymptotic method [1, 3]. The solution is combined from the solutions of an inner problem Q^{in} and the boundary layers R_b,

$$I = Q^{in} + R_b. \tag{2}$$

But for thin bodies in the inner problem the direct decomposition of required values by the small parameter does not lead to the desired result, for the asymptotic orders of the required values are not alike. Having the right determination of asymptotic orders we manage to get an iteration process for the determination of the coefficients of the asymptotic decomposition. In every concrete boundary value problem the right determination of asymptotic orders of the required values is much more difficult and a responsible moment of the asymptotic approach. This also has deeper physical roots, since formulating this or that physical law, this or other asymptotics are given.

2. The first boundary problem of elasticity theory for thin bodies

The solution of the system of equations (1) is required when on the facial surfaces $z = \pm h$ the values of the components of the stress tensor

$$\sigma_{13} = \sigma_{13}^{\pm}(\xi, \eta), \quad \sigma_{23} = \sigma_{23}^{\pm}(\xi, \eta), \quad \sigma_{33} = \sigma_{33}^{\pm}(\xi, \eta), \tag{3}$$

and on the side surface ∂_D some group of conditions of the boundary problem of elasticity theory are given. The solution of the inner problem Q^{in} is sought in the form of

$$Q^{in} = \varepsilon^{\chi_Q + s} Q^{(s)}, \quad s = 0, \ldots, N. \tag{4}$$

After substituting (4) into (1) and comparing the coefficients of the corresponding powers of ε we get a non contradictory system for determining the coefficients of the decomposition $Q^{(s)}$ only when

$$\begin{aligned} &\chi = -2 \quad for \quad \sigma_{11}, \sigma_{22}, U, V, \quad \chi = -1 \quad for \quad \sigma_{13}, \sigma_{23}, \\ &\chi = 0 \quad for \quad \sigma_{33}, \quad \chi = -3 \quad for \quad W. \end{aligned} \tag{5}$$

Substituting (4), (5) into (1) we have an iteration process of determining $Q^{(s)}$. In the space problem all the values for every s are expressed through three functions $U^{(s)}(\xi, \eta)$, $V^{(s)}(\xi, \eta), W^{(s)}(\xi, \eta)$. The equations for $U^{(s)}, V^{(s)}$ when $s = 0$ are corresponding to the equation of the classical theory of tension-pressure of plates and the equation for $W^{(s)}$ is corresponding to the equation of plates bending in the classical Kirchhoff-Love theory. When $s > 0$ the operators of the classical theory are preserved and only the right parts of the equations, i.e. the loading parts change.

In the plane problem all the values are expressed through two functions $U^{(s)}(\xi), W^{(s)}(\xi)$ which are determined from the equations

$$E_1 \frac{d^2 U^{(s)}}{d\xi^2} = q_x^{(s)}(\xi), \quad \frac{1}{3} E_1 \frac{d^4 W^{(s)}}{d\xi^4} = q^{(s)}(\xi), \tag{6}$$

where E_1 is the elasticity module in the longitudinal direction, $q_x^{(s)}, q^{(s)}$ are the loading elements. The first of the equations (6) when $s = 0$

corresponds to the tension-pressure equation of bars, and the second corresponds to the equation of beams bending in the classical Bernoulli-Coulomb-Euler theory. Being restricted to the solution of the inner problem the boundary conditions on the side surface may be satisfied only at certain points (lines) or integrally which indirectly satisfies the singular perturbation of the original edge problem. In order to remove the arising discrepancy, it is necessary to build also a new solution – a boundary layer. This solution has the character that it satisfies the system of equations (1) with homogeneous (trivial) boundary conditions when $y = \pm h(\zeta = \pm 1)$ and quickly fades when removing from the side surface into the inside region D. In order to build this solution near the side surface $\xi = 0$, in (1) the transformation of tension $t = \xi/\varepsilon$ is introduced and the solution of the once again obtained system is sought in the form of functions of boundary layer type [2],[3]

$$R_p = \varepsilon^{\chi_p + s} R_p^{(s)}(\eta, \zeta) exp(-\lambda t), \quad s = 0, \ldots, N. \tag{7}$$

From the consistency condition of the iteration process relative to $R_p^{(s)}$ follows, that $\chi_{\sigma_i} = \chi$, $\chi_u = \chi + 1$ where χ is an arbitrary still integer, the value of which is uniquely determined during the conjunction of the inner layer and boundary layer solutions, Re λ characterizes the velocity of diminution of values of the boundary layer by the degree of elimination from the side surface. In case of a plane problem the solution of the boundary layer is exact for arbitrary s and is expressed through a certain function F by the formulae

$$\begin{aligned}
&\sigma_{xp}^{(s)} = \frac{F''}{\lambda^2}, \qquad \sigma_{xyp}^{(s)} = \frac{F'}{\lambda} A^{(s)}, \qquad \sigma_{yp}^{(s)} = F A^{(s)}, \\
&U_p^{(s)} = -\left[a_{11} \frac{F''}{\lambda^3} + a_{12} \frac{F}{\lambda} \right] A^{(s)}, \\
&V_p^{(s)} = -\left[a_{11} \frac{F'''}{\lambda^4} + (a_{12} + a_{66}) \frac{F'}{\lambda^2} \right] A^{(s)},
\end{aligned} \tag{8}$$

where $A^{(s)}$ is an integration constant, and $F = F(\zeta)$ is the solution of the boundary problem

$$\begin{aligned}
&a_{11} F^{IV} + \lambda^2 (a_{66} + 2a_{12}) F'' + \lambda^4 a_{22} F = 0, \\
&F(\pm 1) = F'(\pm 1) = 0.
\end{aligned} \tag{9}$$

Depending on the values of the elasticity constants a_{ik}, three variants of the solution of problem (9) are possible. The corresponding values λ (their countable set, if we arrange them as an increasing sequence of values of material parts) are determined from the transcendental equations

$$\begin{aligned}
&a)\ \sin 2\lambda_n \pm 2\lambda_n = 0,\\
&b)\ \omega \sin z_n \pm \sin \omega z_n = 0, \quad z_n = (\beta_1 + \beta_2)\lambda_n,\\
&\quad \omega = \frac{\beta_2 - \beta_1}{\beta_1 + \beta_2}, \quad 0 < \omega < 1, \qquad (10)\\
&c)\ \omega \sin z_n + sh\omega z_n = 0, \quad z_n = 2\beta\lambda_n, \omega = \alpha/\beta, \quad 0 < \omega < 1,
\end{aligned}$$

see [3], where $\alpha, \beta, \beta_1, \beta_2$ are certain positive parameters.

The solution (8), (9) has quite an important property, namely

$$\int_{-1}^{+1} \sigma_{xp}^{(s)} d\zeta = 0, \qquad \int_{-1}^{+1} \zeta \sigma_{xp}^{(s)} d\zeta = 0, \qquad \int_{-1}^{+1} \sigma_{xyp}^{(s)} d\zeta = 0, \qquad (11)$$

which physically means self-balance of stresses of the boundary layer in the arbitrary cross-section $t = t_k$.

This solution is impossible to get by any hypothesis admitted in the theory of beams and plates. Therefore, admitting classical hypotheses of plane sections or non deformable normals the above mentioned exact solution gets lost. That is why those thin bodies theories, which do not take into account the boundary layers, cannot make the classical theory more precise.

In the space case two types of boundary layers exist – the antiplane and the plane ones, the values of which fade with different velocities: the antiplane layer fades as $exp\left(-\sqrt{G_{23}/G_{12}}\pi t/2\right)$, and the plane boundary layer fades as $exp(-Re\lambda_1 t)$, where λ_1 is the material part of the first root of the transcendental equations of type (10).

Having the solutions of the inner problem (4) and the boundary layer (7), using the general solution (2) we manage to satisfy the boundary conditions on the side surface of a thin body, i.e. to join the solutions of the inner problem and the boundary layer. In case of a plane problem the solution of the inner problem for stresses contains three arbitrary constants and if on the side surface the stress values are given, these constants are uniquely determined using the property of the boundary layer solution (11). In this case the same solution of the inner problem corresponds to the arbitrary static equivalent loading located on the side surface, and the boundary layer takes the self balanced part of the loading on itself. This purely mathematical result expresses validity of Saint-Venant's well known principle in case of the first boundary problem. In the rest of the cases Saint-Venant's principle is not valid and the conjunction of the solutions is fulfilled by other means by the method of Fourier's smallest squares and so on. In case of the space problem the original approach for the inner problem corresponds to the classical Kirchhoff-Love theory of plates and shells. Building the next approaches for the inner problem and boundary layers, a more precise definition of the results of the classical theory as well inside the thin body, as in the vicinity of its side surface becomes possible.

3. The second and third boundary value problems

Let on one of the facial surfaces of the thin body D the values of the displacement vector components

$$u_x(-h) = u_x^-, \quad u_y(-h) = u_y^-, \quad u_z(-h) = u_z^-, \tag{12}$$

and on the other one the values of displacement vector components (the second boundary problem)

$$u_x(h) = u_x^+, \quad u_y(h) = u_y^+, \quad u_z(h) = u_z^+, \tag{13}$$

or the stress tensor components (the third boundary problem)

$$\sigma_{13}(h) = \sigma_{13}^+, \quad \sigma_{23}(h) = \sigma_{23}^+, \quad \sigma_{33}(h) = \sigma_{33}^+, \tag{14}$$

be given. On the facial surface $z = h$ may be given and the mixed problems (the values of stress tensor and a part of displacement vector components).

The solution of the inner problem is again sought in the form of (4) but the asymptotics (5) are not admissible here.

We especially note that the asymptotics considerately reacts on the type of the boundary conditions on the facial surfaces. In the considered case the non contradictory values for χ are

$$\chi = -1 \quad \text{for stresses}, \quad \chi = 0 \quad \text{for displacements} \tag{15}$$

which principally differ from the asymptotics (5).

Substituting (4), (15) into (1) we determine $Q^{(s)}$. The solution of the inner problem is

$$\begin{aligned}
\sigma_{13}^{(s)} &= \sigma_{130}^{(s)}(\xi,\eta) + \sigma_{13*}^{(s)}(\xi,\eta,\zeta) \qquad (1,2,3),\\
\sigma_{11}^{(s)} &= A_{13}\sigma_{330}^{(s)} + A_{14}\sigma_{230}^{(s)} + A_{15}\sigma_{130}^{(s)} + \sigma_{11*}^{(s)}(\xi,\eta,\zeta), \quad (1,2),\\
\sigma_{12}^{(s)} &= A_{63}\sigma_{330}^{(s)} + A_{64}\sigma_{230}^{(s)} + A_{65}\sigma_{130}^{(s)} + \sigma_{12*}^{(s)}(\xi,\eta,\zeta),\\
U^{(s)} &= \zeta(A_{53}\sigma_{330}^{(s)} + A_{54}\sigma_{230}^{(s)} + A_{55}\sigma_{130}^{(s)}) + U_0^{(s)}(\xi,\eta) + U_*^{(s)}(\xi,\eta,\zeta),\\
V^{(s)} &= \zeta(A_{43}\sigma_{330}^{(s)} + A_{44}\sigma_{230}^{(s)} + A_{45}\sigma_{130}^{(s)}) + V_0^{(s)}(\xi,\eta) + V_*^{(s)}(\xi,\eta,\zeta),\\
W^{(s)} &= \zeta(A_{33}\sigma_{330}^{(s)} + A_{34}\sigma_{230}^{(s)} + A_{35}\sigma_{130}^{(s)}) + W_0^{(s)}(\xi,\eta) + W_*^{(s)}(\xi,\eta,\zeta),
\end{aligned} \tag{16}$$

where the coefficients A_{ik} are expressed through the constants of elasticity a_{ik}, and the values with an asterisk for every s are well-known, if the previous approaches are built.

The solution (16) contains six unknown functions $\sigma_{130}^{(s)}$, $\sigma_{230}^{(s)}$, $\sigma_{330}^{(s)}$, $U_0^{(s)}$, $V_0^{(s)}$, $W_0^{(s)}$, which are uniquely determined during the satisfaction of the boundary conditions (12), (13) or (12), (14). With this also this class of problems differs from the first boundary problem, where the part of the functions is determined from the conditions on the side surface. Therefore, in the given problems the boundary layer only removes the residual on the side surface, i.e. the fading solution of the boundary layer type corresponds to the arbitrary loading, applied on the side surface. If the conditions of the facial surfaces $z = \pm h$ are described by polynomials, the iteration process cuts off on a certain approach and the solution (4), (15), (16) becomes mathematically precise. The solution (16) holds also just for layered bodies. One should only ascribe the corresponding number of the layer to all the values. Besides the conditions on the facial surfaces, also the conditions of the contact between the layers satisfied, the unknown functions entering (16) are again uniquely determined. This, particularly, permits to find the solutions of all the problems of foundations and it is based on the model of a compressible layer.

We bring one of the solutions for a two-layered orthotropic strip-beam, corresponding to the case, when one of the facial surfaces ($z = -h_2$) of the beam is rigidly fastened and on the opposite surface ($z = h_1$) normal and tangential loadings of constant intensivity influence, $-\sigma^+$, τ^+, are given:

$$\begin{aligned}
&\sigma_{13}^I = \tau^+, \quad \sigma_{33}^I = -\sigma^+, \quad \sigma_{11}^I = \frac{a_{12}^I}{a_{11}^I}\sigma^+, \quad 0 \le z \le h_1,\\
&u^I = a_{66}^I \tau^+ z + a_{66}^{II}\tau^+ h_2, \quad W^I = -A_{11}^I \sigma^+ z - A_{11}^{II}\sigma^+ h_2,\\
&\sigma_{13}^{II} = \tau^+, \quad \sigma_{33}^{II} = -\sigma^+, \quad \sigma_{11}^{II} = \frac{a_{12}^{II}}{a_{11}^{II}}\sigma^+, \quad -h_2 \le y \le 0, \qquad (17)\\
&u^{II} = a_{66}^{II}\tau^+(z + h_2) \qquad W^{II} = -A_{11}^{II}\sigma^+(z + h_2),\\
&a_{66} = 1/G_{12}, \qquad A_{11} = (a_{11}a_{22} - a_{12}^2)/a_{11}.
\end{aligned}$$

From (17) follows that on the layer's contact line $z = 0$ we have

$$\sigma_{33}^c = \frac{1}{h_2 A_{11}^{II}} W^c \quad \text{or} \quad \sigma_{33}^{\mathrm{c}} = \mathrm{KW}^{\mathrm{c}}, \qquad (18)$$

where $K = 1/(A_{11}^{II} h_2)$ is the well known coefficient of bed. The value of this coefficient for a beam packet of an arbitrary number of layers may be obtained by elementary actions,

$$K = \frac{1}{\sum_{i=1}^{n} A_{11}^{(i)} h_i}, \tag{19}$$

at that time while obtaining (19) by other methods is a problem. Asymptotics of (4), (15) may be as well used when the parameters of elasticity are functions of the coordinates. It is not difficult to propagate the obtained results on layered thin plates and shells.

4. Dynamic boundary value problems

The asymptotic method appeared to be effective as well for the solutions of problems on free and forced vibrations of thin bodies. From this class of problems we consider some problems, having a significant value for seismology and seismosteady building. Consider free vibrations of plates (plate-like bases) on a rigid base (foundation). It is required to find null solutions of the dynamic equations of elasticity theory in the region of $D = \{(x, y, z) : x, y, \in D_0, |z| \leq h\}$ under the boundary conditions

$$u_x(-h) = 0 \quad u_y(-h) = 0 \quad u_z(-h) = 0, \tag{20}$$

$$u_x(h) = 0, \quad u_y(h) = 0 \quad u_z(h) = 0, \tag{21}$$

or

$$\sigma_{13}(h) = 0, \quad \sigma_{23}(h) = 0, \quad \sigma_{33}(h) = 0. \tag{22}$$

The solution is sought in the form of

$$R_{jk} = Q_{jk}(\xi, \eta, \zeta) exp(i\omega t), \tag{23}$$

where ω is the frequency of free vibrations. The corresponding equations with regard to (23) will differ from equations (1) because instead of F_x, F_y, F_z the iteration elements $\varepsilon^{-2}\omega_*^2 U$, $\varepsilon^{-2}\omega_*^2 V$, $\varepsilon^{-2}\omega_*^2 W$, where $\omega_*^2 = \rho h^2 \omega^2$ will appear. We describe the procedure of determining the required values of the coefficients Q_{jk} in the form of (4), and $\omega_*^2 = \varepsilon^s \omega_{*s}^2$, $s = 0, \ldots, N$, and substituting them in the transformed system (1) we have a noncontradictory system for determining $Q_{jk}^{(s)}, \omega_{*s}^2$. All the unknown functions may be expressed through the functions $U^s, V^{(s)}, W^{(s)}$, each of them in their turn is determined by a differential equation of second order. When $s = 0$, these equations are homogeneous and independent. Determining their solutions and satisfying the boundary conditions (20), (21) or (20), (22) we get an algebraic homogeneous system for the constants of integration. From the solvability of this system follows a transcendental equation for ω_{*0}, from which the main value of frequency of the free vibrations $\omega_0(\omega_0 = \omega_{*0}/(h\sqrt{\rho}))$, where ρ is the density of the plate, is determined. The next approaches make this value more precise, they are calculated by the well-known procedure of the theory of perturbations [1]. Shear free vibrations of orthotropic plates in the surfaces XZ, YZ with the main values of frequencies

$$\omega_{0n}^{xz} = \frac{\pi}{4h}\sqrt{\frac{G_{13}}{\rho}}(2n+1) = \frac{\pi}{4h}V_s^{xz}(2n+1), \quad n \in \mathbb{N}, \tag{24}$$
$$\omega_{0n}^{yz} = \frac{\pi}{4h}\sqrt{\frac{G_{23}}{\rho}}(2n+1) = \frac{\pi}{4h}V_s^{yz}(2n+1),$$

and longitudinal vibrations with frequency corresponding to the boundary conditions (20), (22)

$$\omega_{0n}^{p} = \frac{\pi}{4h}\sqrt{\frac{E_3}{\rho}\frac{1-\nu_{12}\nu_{21}}{1-\nu_{12}\nu_{21}-\nu_{31}\nu_{13}-\nu_{23}\nu_{32}-\nu_{31}\nu_{12}\nu_{23}-\nu_{32}\nu_{21}\nu_{13}}} \times(2n+1) = \frac{\pi}{4h}(2n+1)V_p, \tag{25}$$

where V_s^{xz}, V_s^{yz}, V_p in seismology and wave theory are well-known velocities of shear and longitudinal waves propagation. The main values of the frequencies are determined in the same way for other cases as well.

Now consider forced vibrations of plate-like foundations or bases of buildings. These vibrations arise if we perform to their feet displacements harmonically changing in time (seismic effects)

$$u_x(-h) = u_x^-(\xi,\eta)exp(i\Omega t) \quad (x,y,z), \tag{26}$$

where Ω is the frequency of the forcing effect. The solution of the dynamic equations of elasticity theory is sought in the form of

$$R_{jk} = Q_{jk}(\xi,\eta,\zeta)exp(i\Omega t). \tag{27}$$

By the procedure, described above for free vibrations the coefficients Q_{jk} are determined. If u_x^-, u_y^-, u_z^- are polynomials, the iteration process breaks down and a mathematically precise solution of the dynamic problem for a layer is obtained.

We describe the solution for orthotropic plates for u_x^-, u_y^-, $u_z^- = const$ when the surface of the plate $z = h$ is free $(\sigma_{13}(h) = \sigma_{23}(h) = \sigma_{33}(h) = 0)$

$$
\begin{aligned}
&u_x = \frac{u_x^-}{\cos 2\Omega_* \sqrt{a_{55}}} \cos \Omega_* \sqrt{a_{55}} (1-\zeta) exp(i\Omega t), \\
&u_y = \frac{u_y^-}{\cos 2\Omega_* \sqrt{a_{44}}} \cos \Omega_* \sqrt{a_{44}} (1-\zeta) exp(i\Omega t), \\
&u_z = \frac{u_z^-}{\cos \frac{2\Omega_*}{\sqrt{A_{11}}}} \cos \frac{\Omega_*}{\sqrt{A_{11}}} (1-\zeta) exp(i\Omega t), \qquad (28) \\
&\sigma_{13} = \frac{u_x^-}{h} \frac{\Omega_*}{\sqrt{a_{55}}} \frac{\sin \Omega_* \sqrt{a_{55}} (1-\zeta)}{\cos 2\Omega_* \sqrt{a_{55}} exp(i\Omega t)} \quad (1,2; x,y; a_{55}, a_{44}), \\
&\sigma_{12} = 0, \quad \sigma_{11} = -\frac{A_{23}}{h} \frac{\partial u_z}{\partial \zeta}, \quad \sigma_{22} = -\frac{A_{13}}{h} \frac{\partial u_z}{\partial \zeta}, \quad \sigma_{33} = \frac{A_{11}}{h} \frac{\partial u_z}{\partial \zeta}, \\
&\Omega_*^2 = \rho h^2 \Omega^2, \quad A_{11} = (a_{11}a_{22} - a_{12}^2)/\Delta, \\
&A_{13} = (a_{11}a_{23} - a_{12}a_{13})/\Delta, \; A_{23} = (a_{22}a_{13} - a_{12}a_{23})/\Delta, \\
&\Delta = a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2.
\end{aligned}
$$

System (28) is solvable, if

$$
\cos 2\Omega_* \sqrt{a_{55}} \neq 0, \quad \cos 2\Omega_* \sqrt{a_{44}} \neq 0, \quad \cos \frac{2\Omega_*}{\sqrt{A_{11}}} \neq 0. \qquad (29)
$$

Those values Ω for which the conditions (29) are not fulfilled, coincide with the main values of frequencies of free vibrations. Under such values of the frequency Ω of a forced effect a resonance will take place. But the parameters of the plates (foundation, base) can be chosen so that under seismic and other dynamic effects conditions (29) were fulfilled, i.e. resonance does not arise.

5. Conclusions

Using specificity of geometrical dimensions of thin bodies, it is proved that for them the differential equations of elasticity theory are singularly perturbed. For the solution of such kind of systems an asymptotic method is effective. It is shown that the asymptotics on the solution essentially depends on the type of boundary conditions on facial surfaces of the thin body. The method allows us to find the solutions of wide classes of static and dynamic problems having significant value for applications.

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EXACTLY SOLVABLE MODELS OF STOCHASTIC QUANTUM MECHANICS WITHIN THE FRAMEWORK OF LANGEVIN-SCHROEDINGER TYPE EQUATIONS

A.S. Gevorkyan
Institute for Informatics and Automation Problems
National Academy of Sciences of Armenia,
Ind. 375014, Parujr Sevak 1, Yerevan, Armenia
gashot@sci.am

Abstract A closed, uncountably infinite dimensional system of "quantum object + thermostat" is mathematically stated in terms of a complex probabilistic process for the wave function, i.e. the solution of Langevin-Schrödinger (L-Sch)-type stochastic differential equation (SDE). The L-Sch SDE has been studied for nonstationary 1D random potential with quadratic space form. It was proved that when the random processes in potential determined by δ-shaped correlators, then closed, exactly solvable nonrelativistic quantum mechanics may be constructed. In this case all physically measured parameters of the system are built as multiple integrals of the fundamental solution of some second-order partial differential equations (etalon equations). In the present work we obtain expressions for transition probabilities in a quantum subsystem. It is shown that depending on the coupling constant of the thermostat with the parametric quantum harmonic oscillator (PQHO), some phase transitions of second kind may occur in the expressions for microscopic quantum transitions. A method of stochastic density matrix is developed for calculation of thermodynamic potentials of the quantum harmonic oscillator (QHO) immersed into a thermostat. The analytic expressions for the ground state energy level widening and shift of QHO are obtained. In other words the possibility of violating the second law of thermodynamics due to quantum fluctuations, (i.e., spontaneous transitions in QHO from vacuum to the excited states) is shown.

Keywords: Langevin-Schrödinger stochastic differential equation, random potential, thermodynamic potentials, nonrelativistic quantum mechanics, etalon equations, quantum harmonic oscillator, phase transition, second law of thermodynamics, quantum fluctuation

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1. Introduction

There are three primary sources of quantum chaos or of chaos in the main object of quantum mechanics [1–3], namely, in the wave function. The first one is related to measurements made over the quantum system [4, 5, 7]. The second one is due to the openess of a quantum system under study. It arises as a result of interaction of the system with environment [8]. The openness may be caused by a more fundamental reason, namely, the immersion of a quantum system in physical vacuum (see e.g. [9]). And, the third, the quantum chaos may be also generated in dynamical systems, in chemical reactions [10–12], at the scattering of particles on singular potentials [13], multichannel scattering in a many-body system [14, 15], mesoscopic systems [16] and in many other fields of physics [17–20]. Besides that, as was recently shown by the authors in a number of works [21–23], the chaos in the wave function may arise also in relatively simple problems such as the collinear three-body collision system. It is sufficient for the chaos to be developed enough in the corresponding classical problem, i.e., the chaotic region in phase space be greater than the volume of an elemental quantum cell $\hbar^N$, where N is the system dimensionality (see e.g. [24]). In other words, the hypothesis of organic relation between the classical nonintegrability and the quantum chaos is thus proved [25]. It should be pointed out that in all mentioned cases one is faced somehow or the other with the necessity to give the statistical description of a quantum system. The above problems require different research approaches including the Gutzwiller track formulas for description of quantum scattering of the system having a chaotic classical analogue [3], the theory of Mehta random matrix for calculation of statistical properties of cross section fluctuations for a series of processes (see e.g. [26]), the representation of classical Marcus-Miller S-matrix for calculation of a chemical reaction, the properties of which are in nature nearer to the behavior of a classical statistical system rather than to a quantum one and other approximate quasi-classical methods (see e.g. [27]). Nevertheless, many quantum-mechanical problems of fundamental importance (the Lamb shift of energy levels, spontaneous transitions between atomic levels, the Zeno quantum effect [28], the issues of chaos and self-organization in quantum systems especially with phase transition type phenomena etc.) may be rigorously described quantitatively and qualitatively only in the framework of a nonperturbative approach. Note that the Lindblad representation [29, 30] for the density matrix of the “quantum object + thermostat” system gives an a priory description of the most possible general situation. However, at the study of properties of a quantum subsystem one has to pass to a reduced density matrix on a semigroup [31], that is a rather ambiguous procedure and besides, its technical realization is feasible only within the framework of this or that perturbation scheme. In the present article, the works [32] on development of a novel scheme for investigation of stochastic quantum systems within the framework of L-Sch-type SDE for the wave function are continued. Note, that since the potential energy of the system is assumed to be a random function, the wave function of the system "quantum object + thermostat" is described by SDE. The basic point of the method consists in the fact that for small time periods the L-Sch SDE may be replaced by the ordinary Schrödinger equation. Here the dynamics of the system "PQHO + thermostat" is investigated within the framework of a 1D model with nonstationary random potential, the space part of which has a quadratic form. It was shown that when the random terms in the potential are determined by δ-shaped correlators (the white noise), the problem admits an exact solution in the statistical sense. In other words, all the sought physical parameters of the quantum subsystem are calculated in closed way and are given as finite multiplicity integrals of fundamental solutions of some reference tasks (second order partial differential equations). In particular, the average values of transition probabilities in PQHO have been calculated with due regard for the nonperturbative effect exerted by the thermostat on an elemental process. Exact expressions for energy level widening and shift (an analogue of the Lamb shift) of QHO immersed in the thermostat were obtained. Expressions for thermodynamic potentials of the problem were constructed, the entropy of the vacuum (ground) state of a quantum oscillator was investigated in detail.

2. Formulation of the problem

We shall consider the closed system "quantum object + thermostat" within the framework of L-Sch type SDE

$$i\partial_t\Psi_{stc} = \hat{H}\Psi_{stc}, \tag{1}$$

where the 1D evolution operator $\hat{H}$ is assumed to be quadratic over the space variable,

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\Omega^2(t)x^2 - F(t)x, \tag{2}$$

and the functions $\Omega^2(t)$ and $F(t)$ are stochastic functions of time. Let them have the forms

$$\begin{aligned} \Omega^2(t) &= \Omega_0^2(t) + \sqrt{2\epsilon_1 p_1} f_1(t)\Theta(t-t_1), \\ F(t) &= F_0(t) + \sqrt{2\epsilon_2 p_2} f_2(t)\Theta(t-t_2), \end{aligned} \tag{3}$$

where $\Omega_0^2(t)$ and $F_0(t)$ are regular functions and $f_1(t), f_2(t)$ are independent Gaussian stochastic processes with zero mean and $\delta-$shaped correlators

$$< f_i(t) f_j(t') >= \delta_{ij}\delta(t-t'), \quad i,j=1,2. \tag{4}$$

The constants ϵ_i control the power of forces $f_i(t)$, the functions $p_i(t)$ are assumed to be nonnegative: $p_1, p_2 \geq 0$. The step-function $\Theta(x)$ is defined by

$$\Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0. \end{cases} \tag{5}$$

Let us assume that the asymptotic conditions

$$\Omega_0(t) \underset{t\to\pm\infty}{\longrightarrow} \Omega_{\left(\substack{in \\ out}\right)}, \quad F_0(t) \underset{t\to\pm\infty}{\longrightarrow} 0, \quad p_i(t) \underset{t\to\pm\infty}{\longrightarrow} 0, \tag{6}$$

hold, which guarantee that the autonomous states $\phi_n^{in}(x,t)$ exist as $t \to -\infty$,

$$\phi_n^{in}(x,t) = e^{-i(n+1/2)\Omega_{in}t}\phi_n^{in}(x),$$
$$\phi_n^{in}(x) = \left(\frac{1}{2^n n!}\sqrt{\frac{\Omega_{in}}{\pi}}\right)^{1/2} e^{-\Omega_{in}x^2/2} H_n(\sqrt{\Omega_{in}}x), \tag{7}$$

where $\phi_n^{in}(x)$ is the wave function of a stationary oscillator and $H_n(x)$ is the Hermitian polynomial. It also follows from (6) that the autonomous states $\phi_n^{out}(x,t)$, which are obtained from (7) by replacing Ω_{in} by Ω_{out}, exist in the limit $t \to +\infty$ as well. The Θ-functions in (3) reflect the fact that the random processes $f_1(t)$ and $f_2(t)$ are activated at the moments t_1 and t_2, respectively. If necessary, the functions p_1 and p_2 may be chosen having the form which prevents the jumps of Ω and F when the noise is activated. The moments t_1 and t_2 are assumed to be finite to make the following inference correct. The aim of the paper is to find the average probabilities W_{nm} of transitions from the initial $\phi_n^{in}(x,t)$ to the final $\phi_m^{out}(x,t)$ autonomous states when the evolution is governed by stochastic operators $\hat{H}$, see (2). This is the same way as to construct a thermodynamic potential of a problem.

3. Expressions for the wave functions: Complex probabilistic processes and transition amplitudes

Proposition 1 *The formal solution of the problem (1)-(2) may be written down explicitly for arbitrary* $\Omega^2(t)$ *and* $F(t)$. *It has the form*

$$\Psi_{stc}(x,t|\{\vec{\zeta}\}) = \frac{1}{\sqrt{r}} exp\left\{ i \left[\dot{\eta}(x-\eta) + \frac{\dot{r}}{2r}(x-\eta)^2 + \sigma \right] \right\} \chi\left(\frac{x-\eta}{r}, \tau \right), \tag{8}$$

where the function $\chi(y,\tau)$ *satisfies the Schrödinger equation for a harmonic oscillator with the constant frequency* Ω_{in}

$$i\frac{\partial\chi}{\partial\tau} = -\frac{1}{2}\frac{\partial^2\chi}{\partial^2 y} + \frac{\Omega_{in}^2 y^2}{2}\chi, \qquad y = \frac{x-\eta}{r}, \tag{9}$$

the function $\eta(t)$ *is a solution of the classical equation of motion for the oscillator with the frequency* $\Omega(t)$, *subjected to the external force* $F(t)$,

$$\ddot{\eta} + \Omega^2(t)\eta = F(t), \quad \eta(-\infty) = \dot{\eta}(-\infty) = 0, \tag{10}$$

$\sigma(t)$ *is a classical action, corresponding to the solution* $\eta(t)$,

$$\sigma(t) = \int_{-\infty}^{t} \left[\frac{1}{2}\dot{\eta}^2 - \frac{1}{2}\Omega^2\eta^2 + F\eta \right] dt', \tag{11}$$

and $r(t)$ *and* $\tau(t)$ *are expressed in terms of the solution* $\xi(t)$ *of the homogeneous equation, corresponding to (10)*

$$\ddot{\xi} + \Omega^2(t)\xi = 0, \quad \xi(t) \underset{t\to-\infty}{\sim} e^{i\Omega_{in}t}, \tag{12}$$

as

$$\xi(t) = r(t)e^{\gamma(t)}, \quad r(t) = |\xi(t)|, \quad \tau = \gamma(t)/\Omega_{in}.$$

A special notation $\vec{\zeta} = (\xi,\eta)$ for the set of functions $\xi(t)$ and $\eta(t)$ is introduced in (8).

The proof is based on the substitutions first used in [34]. As it was shown in [32] it works also for a certain type of random processes.

The set of solutions of type (8) which is important for the following considerations in this paper is obtained from (8) after replacing $\chi(y,\tau)$ by $\phi_n^{in}(y,\tau)$ from (7). It is thus defined as

$$\Psi_{stc}^{(n)}(x,t|\{\vec{\zeta}\}) = \frac{1}{\sqrt{r}} exp\left\{ i \left[\dot{\eta}(x-\eta) + \frac{\dot{r}}{2r}(x-\eta)^2 + \sigma \right] \right\} \phi_n^{in}(\frac{x-\eta}{r}, \tau), \quad n = 1, 2, \ldots \, . \tag{13}$$

The main properties of the set of functionals (13), which are important in what follows are

1 For any n the functional $\Psi^{(n)}_{stc}(x,t|\{\vec{\zeta}\})$ reduces to the autonomous state $\phi^{in}_n(x,t)$ in the limit $t \to -\infty$.

2 For any fixed $\vec{\zeta}$ the elements of the set (13) are mutually orthogonal in the sense of $L_2(R^1)$, the space of square-integrable functions,

$$\int_{-\infty}^{\infty} \Psi^{(n)}_{stc}(x,t|\{\vec{\zeta}\})\overline{\Psi^{(m)}_{stc}(x,t|\{\vec{\zeta}\})}\, dx = \delta_{nm}, \tag{14}$$

where a bar denotes the complex conjugation procedure and $\delta_{nm} = 1$, for $n = m$ and $\delta_{nm} = 0$ for $n \neq m$.

Definition 1 *The average probabilities W_{nm} of the transitions from the states $\Psi^{(n)}_{stc}(x,t|\{\vec{\zeta}\})$ to the autonomous ones $\phi^{out}_m(x,t)$ in the limit $t \to +\infty$ are defined by*

$$\Psi^{(n)}_{stc}(x,t|\{\vec{\zeta}\}) = \sum_{m=0}^{\infty} c_{nm}(t|\{\vec{\zeta}\})\phi^{out}_m(x,t), \tag{15}$$

$$W_{nm} = \lim_{t\to+\infty} \left\langle |c_{nm}|^2 \right\rangle, \tag{16}$$

where the symbol $< \cdot >$ denotes the procedure of averaging with respect to f_1 and f_2.

Definition 2 *The generating function $I_{stc}(z_1,z_2,t|\{\vec{\zeta}\})$ for the coefficients c_{nm} is defined by the expression*

$$I_{stc}(z_1,z_2,t|\{\vec{\zeta}\}) = \int_{-\infty}^{+\infty} dx \overline{\Psi_{out}(\bar{z}_1,x,t)}\Psi_{stc}(z_2,x,t|\{\vec{\zeta}\}), \tag{17}$$

where

$$\Psi_{stc}(z,x,t|\{\vec{\zeta}\}) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\Psi^{(n)}_{stc}(x,t|\{\vec{\zeta}\}),$$

$$\Psi_{out}(z,x,t) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\phi^{(out)}_n(x,t),$$

so that

$$c_{nm}(t|f_1,f_2) = \left.\frac{\partial^{n+m} I_{stc}}{\partial z_1^n \partial z_2^m}\right|_{z_1=z_2=0}. \tag{18}$$

Proposition 2 *The explicit expression for the generating function* (17) *is given by*

$$I_{stc}(z_1, z_2, t|\{\vec{\zeta}\}) = \left(\frac{2\sqrt{\Omega_{in}\Omega_{out}}}{K\xi}\right)^{1/2} \exp\{Az_1^2 + Bz_2^2 + Cz_1z_2 + Dz_1 + Lz_2 + M\}, \quad (19)$$

where the notations

$$A = \frac{1}{2}e^{2i\Omega_{out}t}\left(\frac{2\Omega_{out}}{K} - 1\right), \quad B = \frac{1}{2}\left(\frac{2\Omega_{in}}{K\xi^2} - e^{-2i\gamma}\right),$$

$$C = \frac{2\sqrt{\Omega_{in}\Omega_{out}}}{K\xi}e^{i\Omega_{out}t}, \quad L = -\frac{\sqrt{2\Omega_{in}}}{K\xi}(\Omega_{out}\eta - i\dot{\eta}),$$

$$D = \sqrt{2\Omega_{out}}e^{i\Omega_{out}t}\left[\left(1 - \frac{\Omega_{out}}{K}\right)\eta + \frac{i\dot{\eta}}{K}\right],$$

$$M = \frac{\Omega_{out}}{2}\left(\frac{\Omega_{out}}{K} - 1\right)\eta^2 - \frac{1}{2K}\dot{\eta}^2 - \frac{i\Omega_{out}}{K}\eta\dot{\eta} + i\left(\frac{1}{2}\Omega_{out}t + \sigma\right),$$

$$K = -i\frac{\dot{\xi}}{\xi} + \Omega_{out},$$

are introduced.

The proof is carried out by the direct summation of the series over Hermitian polynomials followed by the calculation of the Gaussian integral.

Expanding the expression (19) in powers of z_1 and z_2, we obtain the coefficients c_{nm}. The first ones are

$$c_{00} = \left(\tfrac{2\sqrt{\Omega_{in}\Omega_{out}}}{K\xi}\right)^{1/2} e^M, \; c_{01} = Dc_{00}, \; c_{10} = Lc_{00}, \quad (20)$$
$$c_{11} = (C + DL)c_{00}, \; c_{20} = \sqrt{2}\left(B + \tfrac{L^2}{2}\right)c_{00}, \; c_{02} = \sqrt{2}\left(A + \tfrac{D^2}{2}\right)c_{00}.$$

Given expressions for the objects to be averaged it is necessary to reduce the averaging procedure to a form convenient for the subsequent analytical or numerical treatment. The following sections are devoted to the solution of the problem (1)-(6) in different situations.

4. Transition probabilities in case of $\epsilon_1 = 0$

If $\epsilon_1 = 0$ the $\xi(t)$ from (12) is a regular function. We denote it as $\xi_0(t)$, and get

$$\ddot{\xi}_0 + \Omega_0^2(t)\xi_0 = 0, \quad \xi_0(t) \underset{t\to-\infty}{\sim} e^{i\Omega_{in}t}, \quad (21)$$

$$\xi_0(t) = r_0(t)e^{\gamma_0(t)} = \xi_{01}(t) + i\xi_{02}(t).$$

We also introduce the notation $\eta_0(t)$ for a function satisfying the equation

$$\ddot{\eta}_0 + \Omega_0^2(t)\eta_0 = F_0(t), \quad \eta_0(-\infty) = \dot{\eta}_0(-\infty) = 0. \tag{22}$$

Theorem 1 *For any quantity $G(\eta(t), \dot{\eta}(t))$ locally with respect to $\eta(t)$ and $\dot{\eta}(t)$ (such are the coefficients c_{nm} in case of $\epsilon_1 = 0$) the averaging formula has the form*

$$\langle G(\eta(t), \dot{\eta}(t))\rangle = \int\int_{-\infty}^{+\infty} dx_1 dx_2 \; G(x_1, x_2) P_1(x_1, x_2, t|\eta_0(t_2), \dot{\eta}_0(t_2), t_2),$$
$$t > t_2, \tag{23}$$

where $x_1 = \eta$ and $x_2 = \dot{\eta}$. In addition

$$P_1(x_1, x_2, t|x_{10}, x_{20}, t_0) = \frac{(4b_1b_3 - b_2^2)^{-1/2}}{2\pi\Omega_{in}} \exp\left\{-\frac{b_3y_1^2 + b_1y_2^2 - b_2y_1y_2}{4b_1b_3 - b_2^2}\right\}, \tag{24}$$

and

$$b_1(t) = \frac{\epsilon_2}{\Omega_{in}^2}\int_{t_2}^{t} p_2(t')\xi_{01}^2(t')dt', \quad b_3(t) = \frac{\epsilon_2}{\Omega_{in}^2}\int_{t_2}^{t} p_2(t')\left(\dot{\xi}_{01}(t')\right)^2 dt',$$

$$b_2(t) = \frac{2\epsilon_2}{\Omega_{in}^2}\int_{t_2}^{t} p_2(t')\xi_{01}(t')\dot{\xi}_{01}(t')dt',$$

$$\begin{cases} y_1 = -\left[\dot{\xi}_{01}(t)\left(x_1 - \eta_0(t)\right) - \xi_{01}(t)\left(x_2 - \dot{\eta}_0(t)\right)\right]/\Omega_{in}, \\ y_2 = \left[\dot{\xi}_{02}(t)\left(x_1 - \eta_0(t)\right) - \xi_{02}(t)\left(x_2 - \dot{\eta}_0(t)\right)\right]/\Omega_{in}. \end{cases}$$

It is obvious that the function $\eta(t)$ is non-stochastic in the time interval $t < t_2 : \eta(t) = \eta_0(t)$, while in the interval $t > t_2$ it represents a random process with the evolution governed by the equation

$$\ddot{\eta} + \Omega_0^2(t)\eta = F_0(t) + \sqrt{2\epsilon_2 p_2} f_2(t). \tag{25}$$

The initial condition for (25) is defined by the requirement for the trajectory and its first derivative that they be continuous at the moment t_2, i.e. $\eta(t_2) = \eta_0(t_2)$, $\dot{\eta}(t_2) = \dot{\eta}_0(t_2)$. To make the analysis of equation

(25), containing random processes, correct it is convenient to rewrite it as the set of two first order differential equations

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = F_0 - \Omega_0^2 x_1 + \sqrt{2\epsilon_2 p_2} f_2, \end{cases} \qquad \begin{cases} x_1(t_2) = \eta_0(t_2), \\ x_2(t_2) = \dot{\eta}_0(t_2). \end{cases} \tag{26}$$

Equations (26) are naturally interpreted as SDE for the stochastic processes (thermostat or vacuum fields) $x_1(t)$ and $x_2(t)$. Proceeding from them it is not difficult to write down the Fokker-Planck equation for the conditional probability density

$$P_1(x_1, x_2, t|x_{10}, x_{20}, t_0) = \Big\langle \delta\big(x_1(t) - x_1\big)\delta\big(x_2(t) - x_2\big)\Big\rangle\Big|_{\substack{x_1(t_0) = x_{10}, \\ x_2(t_0) = x_{20},}}$$

describing the probability that the trajectory (thermostat or vacuum fields) $\big(x_1(t), x_2(t)\big)$ finds itself in the vicinity of the point (thermostat or vacuum coordinates) $\big(x_1, x_2\big)$ at the moment t, having started from the point $\big(x_{10}, x_{20}\big)$ at the moment t_0. It can be shown that the equation for P_1 has the form (see [35] or [36])

$$\frac{\partial P_1}{\partial t} = -x_2 \frac{\partial P_1}{\partial x_1} - (F_0 - \Omega_0^2 x_1) \frac{\partial P_1}{\partial x_2} + \epsilon_2 p_2 \frac{\partial^2 P_1}{\partial x_2^2}. \tag{27}$$

The solution of (27) must be integrable and satisfy the obvious initial condition

$$P_1\Big|_{t=t_0} = \delta(x_1 - x_{10})\delta(x_2 - x_{20}). \tag{28}$$

As the position of the trajectory at the moment t_2 is known, it is natural to set $t_0 = t_2$, $x_{10} = \eta_0(t_2)$, $x_{20} = \dot{\eta}_0(t_2)$. The integrable solution of (27), satisfying (28), may be expressed in terms of the functions $\xi_{01}(t)$, $\xi_{02}(t)$, $\eta_0(t)$. It is easy to test that it has the form (24) by explicit verification. The theorem is proved.

To find probabilities W_{nm} in this case we calculate the Gaussian integrals and then compute their limiting values at $t \to +\infty$.

5. Transition probabilities in case of $\epsilon_2 = 0$

In the case of $\epsilon_2 = 0$ the time axis is broken into two parts. At $t < t_1$ the functions $\eta(t)$ and $\xi(t)$ are non-stochastic: $\eta(t) = \eta_0(t)$, $\xi(t) = \xi_0(t)$. For $t > t_1$ both $\eta(t)$ and $\xi(t)$ trajectories become random.

Theorem 2 *In the case of* $\epsilon_2 = 0$, *for* $t > t_1$ *the set of equations (10), (12) gives rise to the set of SDE, describing the evolution of four random* *processes*

$$\vec{u}(t) \equiv \big(u_1(t), u_2(t), u_3(t_1), u_4(t)\big) \equiv \left(\eta(t), \dot{\eta}(t), \mathrm{Re}\left(\frac{\dot{\xi}(t)}{\xi(t)}\right), \mathrm{Im}\left(\frac{\dot{\xi}(t)}{\xi(t)}\right)\right),$$

with the joint probability distribution function $P_2(\vec{u}, t|\vec{u}_0, t_1),\ t > t_1$, *satisfying the Fokker-Planck equation*

$$\frac{\partial P_2}{\partial t} = \hat{L}_2 P_2, \tag{29}$$

$$\hat{L}_2\left(\vec{u}\right) \equiv -\sum_{i=1}^{4} K_i \frac{\partial}{\partial u_i} + \epsilon_1 p_1 u_1^2 \frac{\partial^2}{\partial u_2^2} + \epsilon_1 p_1 \frac{\partial^2}{\partial u_3^2} + 2\epsilon_1 p_1 u_1 \frac{\partial^2}{\partial u_2 \partial u_3} + 4u_3,$$

$$K_1 = u_2, \quad K_2 = F_0 - \Omega_0^2 u_1, \quad K_3 = u_4^2 - u_3^2 - \Omega_0^2, \quad K_4 = -2u_3 u_4,$$

and the initial condition

$$P_2\big|_{t=t_1} = \delta\left(\vec{u} - \vec{u}_0\right) \equiv \prod_{i=1}^{4} \delta(u_i - u_{0i}).$$

In the case under consideration the equation (10) is transformed to a set of SDE in the same way as it has been done in the previous section, namely, by introducing the quantities $u_1 = \eta,\ u_2 = \dot{\eta}$ which reduce (10) to (26) with the only distinction in the initial condition: $u_1(t_1) = \eta_0(t_1),\ u_2(t_1) = \dot{\eta}_0(t_1)$.

The equation (12) may be reduced to a nonlinear first order differential equation by the substitution

$$\xi(t) = \begin{cases} \xi_0(t), & t < t_1, \\ \xi_0(t_1) \exp\left\{ \int\limits_{t_1}^{t} \Phi(t')dt' \right\}, & t > t_1, \end{cases} \tag{30}$$

which gives upon being applied to (12) the following SDE for $\Phi(t)$ in the interval $t_1 < t < \infty$

$$\dot{\Phi}(t) + \Phi^2(t) + \Omega_0^2(t) + \sqrt{2\epsilon_1 p_1} f_1 = 0, \quad \Phi(t_1) = \dot{\xi}_0(t_1)/\xi_0(t_1), \tag{31}$$

where the second equation expresses a condition which guarantees continuity of the function $\xi(t)$ and its first derivative at $t = t_1$. The function $\Phi(t)$ is a complex-valued random process due to the initial condition. As a result the SDE (31) is equivalent to a set of two SDE for real-valued stochastic processes. Namely, introducing real and imaginary parts of $\Phi(t)$

$$\Phi(t) = u_3(t) + iu_4(t),$$

we finally obtain the following set of SDE for the components of the vector field $\vec{u}$

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = F_0 - \Omega_0^2 u_1 - \sqrt{2\epsilon_1 p_1} u_1 f_1, \\ \dot{u}_3 = -u_3^2 + u_4^2 - \Omega_0^2(t) - \sqrt{2\epsilon_1 p_1} f_1(t), \\ \dot{u}_4 = -2u_3 u_4, \end{cases}$$

$$\begin{cases} u_1(t_1) = \eta_0(t_1), \\ u_2(t_1) = \dot{\eta}_0(t_1), \\ u_3(t_1) = Re\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right), \\ u_4(t_1) = Im\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right). \end{cases} \tag{32}$$

The pairs of fields (u_1, u_2) and (u_3, u_4) are not independent, because their evolution is influenced by the common stochastic force $f_1(t)$. This means that the joint probability distribution

$$P_2\big(\vec{u}, t|\vec{u}_0, t_1\big) = \left\langle \prod_{i=1}^{4} \delta\big(u_i(t) - u_i\big) \right\rangle \Bigg|_{\vec{u}(t_1) = \vec{u}_0},$$

$$\vec{u}_0 = \Big(\eta_0(t_1), \dot{\eta}_0(t_1), \mathrm{Re}\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right), \mathrm{Im}\left(\dot{\xi}_0(t_1)/\xi_0(t_1)\right)\Big),$$

is a non-factorable function. Proceeding from the known evolution equations (32), we obtain by the standard method the Fokker-Planck equation for P_2 (see e.g. [35] or [36]), which has the form (29). The theorem is proved.

Given P_2, one can average any quantity $G(\vec{u}(t))$ which is local with respect to $\vec{u}(t)$:

$$\langle G(\vec{u}(t))\rangle = \int d\vec{u} P_2(\vec{u}, t|\vec{u}_0, t_1)\, G(\vec{u}), \quad d\vec{u} = du_1 du_2 du_3 du_4. \tag{33}$$

But this formula fails to give a result if it is used for averaging the objects containing the coefficients c_{nm} from (18), which are nonlocal with respect to $\vec{u}$. There does not exist a general approach to calculating the average value of any quantity nonlocal with respect to a random process. But it is known that for some types of such objects the averaging procedure may be reduced to finding a fundamental solution of some parabolic partial differential equation and its subsequent weighted integration. The description of the simplest case of this kind is given in [37]. It is not difficult to generalize the result obtained in [37], and therefore the formulas (A.4)–(A.5) can be derived (see Appendix). Using (A.4)–(A.5), we have the following proposition.

Proposition 3 *If the components of the random vector process $\vec{u}$ satisfy the set of SDE (32), then the averaging procedure can be represented as*

$$\left\langle \exp\left\{-\int_{t_1}^{t} V_1\left(\vec{u}(\tau),\vec{u}(t),t\right)d\tau - V_2\left(\vec{u}(t)\right)\right\}\right\rangle = \int d\vec{u} e^{-V_2(\vec{u})} Q\left(\vec{u},\vec{u},t\right), \tag{34}$$

where the function $Q\left(\vec{u},\vec{u}',t\right)$ is a solution of the problem

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \left(\hat{L}_2\left(\vec{u}\right) - V_1\left(\vec{u},\vec{u}',t\right)\right)Q, \\ Q\left(\vec{u},\vec{u}',t\right) &\underset{t\to t_1}{\longrightarrow} \delta\left(\vec{u}-\vec{u}_0\right), \quad Q\left(\vec{u},\vec{u}',t\right) \underset{||\vec{u}||\to\infty}{\longrightarrow} 0, \end{aligned} \tag{35}$$

where $||\cdot||$ is a norm in R^4. It is not difficult to show that when $V_1 = 0$, the formula (34) transforms into (33) with the substitution $G = \exp\{-V_2\}$, because in this case the equation (35) for Q transforms into the Fokker-Planck equation (29) for P_2.

Using the Proposition 3, we obtain the representation for the average values $\left\langle |c_{nm}|^2\right\rangle$, which are equal to the probabilities W_{nm} in the limit $t \to +\infty$. The explicit expressions for the first four of them are presented here $(n = 0,1;\ m = 0,1)$:

$$\left\langle |c_{nm}|^2\right\rangle = \int d\vec{u} H_{nm}\left(\vec{u}\right) Q_{nm}\left(\vec{u},t\right), \tag{36}$$

where

$$\begin{aligned} H_{00}\left(\vec{u}\right) &= \frac{2\sqrt{\Omega_{in}\Omega_{out}}}{|\xi_0(t_1)K|}\exp\left\{\frac{\Omega_{out}}{2}\left(\frac{2\Omega_{out}(u_4+\Omega_{out})}{|K|^2}-2\right)u_1^2\right. \\ &\left. -\frac{u_4+\Omega_{out}}{|K|^2}u_2^2 + \frac{2\Omega_{out}u_3}{|K|^2}u_1u_2\right\}, \\ H_{01}\left(\vec{u}\right) &= \frac{2\Omega_{out}}{|K|^2}\left((u_2-u_1u_3)^2+u_1^2u_4^2\right)H_{00}\left(\vec{u}\right), \\ H_{10}\left(\vec{u}\right) &= \frac{2\Omega_{in}}{|\xi_0(t_1)K|^2}\left(\Omega_{out}^2u_1^2+u_2^2\right)H_{00}\left(\vec{u}\right), \\ H_{11}\left(\vec{u}\right) &= \frac{4\Omega_{in}\Omega_{out}}{|\xi_0(t_1)K^2|^2}\left|K-(\Omega_{out}u_1-iu_2)(u_1u_4+i(u_2-u_1u_3))\right|^2 \\ &\times H_{00}\left(\vec{u}\right), \end{aligned} \tag{37}$$

$$K = \Omega_{out} + u_4 - iu_3,$$

and functions $Q_{nm}(\vec{u},t)$ satisfy the equations

$$\begin{aligned} &\frac{\partial Q_{nm}}{\partial t} = \hat{L}_2 Q_{nm} - V_{nm} Q_{nm}, \\ &Q_{nm}(\vec{u},t) \underset{t\to t_1}{\longrightarrow} \delta(\vec{u}-\vec{u}_0), \quad Q_{nm}(\vec{u},t) \underset{||\vec{u}||\to\infty}{\longrightarrow} 0, \end{aligned} \tag{38}$$

where

$$V_{nm} = p_{nm} u_3, \quad p_{00} = p_{01} = 1, \quad p_{10} = p_{11} = 3.$$

To obtain W_{nm} it is necessary to proceed in (36) to the limit $t \to +\infty$. The representation (36)-(38) is exact and free from any simplifying assumptions. Given a specific realization of $p_1(t)$, (36)-(38) is used as a basis for numerical calculations of the probabilities W_{nm}.

6. Transition probabilities in case of $\epsilon_1, \epsilon_2 \neq 0$

If influence of both random forces is taken into account, i.e. $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$, the evolution of the system depends on the correlation of the moments t_1 and t_2. If $t < t_> := max(t_1, t_2)$ equations (10) and (12) for the trajectories $\eta(t)$ and $\xi(t)$ are different, depending on whether $t_> = t_1$ or $t_> = t_2$. The two possibilities correspond to the probability distributions of random variables $\eta(t_>)$ and $\xi(t_>)$, which are the initial values for the corresponding trajectories on the interval $t > t_>$. Denote $z_1 = \eta,\ z_2 = \dot{\eta},\ z_3 = \mathrm{Re}(\dot{\xi}/\xi),\ z_4 = \mathrm{Im}(\dot{\xi}/\xi)$. For the probability density function $R(\vec{z},t)$, which is defined as a probability for the trajectory $\vec{z}(t)$ to be found in the interval $[\vec{z},\ \vec{z}+d\vec{z}]$ at the moment $t > t_< := \min(t_1,t_2)$, we can write down the following expression

$$R(\vec{z},t_>) = \begin{cases} P_1(z_1,z_2,t_>|z_{01},z_{02},t_<)\delta(z_3-z_{03})\delta(z_4-z_{04}), & t_> = t_1, \\ P_2(\vec{z},t_>|\vec{z}_0,t_<), & t_> = t_2, \end{cases} \tag{39}$$

where

$$\vec{z}_0 = (z_{01},z_{02},z_{03},z_{04}) = \left(\eta_0(t_<),\ \dot{\eta}_0(t_<), \mathrm{Re}\left(\frac{\dot{\xi}_0(t_<)}{\xi_0(t_<)}\right), \mathrm{Im}\left(\frac{\dot{\xi}_0(t_<)}{\xi_0(t_<)}\right)\right),$$

and the functions P_1 and P_2 are the solutions of (27) and (29), respectively. It is obvious that the normalization condition $\int R(\vec{z},t_>)d\vec{z} = 1$ holds. At $t > t_>$ we have the same set of SDE for the components of the stochastic vector process $\vec{z}(t)$ both at $t_> = t_1$ and $t_> = t_2$. Its inference literally reproduces the derivation of the set (32) and results in

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = F_0 - \Omega_0^2 z_1 - \sqrt{2\epsilon_1 p_1} z_1 f_1 + \sqrt{2\epsilon_2 p_2} f_2, \\ \dot{z}_4 = -2 z_3 z_4, \\ \dot{z}_3 = z_4^2 - z_3^2 - \Omega_0^2(t) - \sqrt{2\epsilon_1 p_1} f_1(t), \qquad \vec{z}(t_>) = \vec{z}_>, \end{cases} \tag{40}$$

where the distribution $R(\vec{z}_>, t_>)$ of the components of the random vector $\vec{z}_>$ is given by formula (39). The representation of the joint probability density

$$P_3(\vec{z}, t | \vec{z}_>, t_>) = \left\langle \delta(\vec{z}(t) - \vec{z}) \right\rangle \Big|_{\vec{z}(t_>) = \vec{z}_>},$$

derived by the standard method from (40), is given by the Fokker-Planck equation. Thus we arrive at an analogue of Theorem 2.

Theorem 3 *In the case of* $\epsilon_1 \neq 0$ *and* $\epsilon_2 \neq 0$ *the set of equations* (10), (12) *generates for* $t > t_>$ *the set of SDE describing the evolution of the vector field (four stochastic processes)*

$$\vec{z}(t) \equiv \big(z_1(t), z_2(t), z_3(t), z_4(t)\big) \equiv \left(\eta(t), \dot{\eta}(t), \mathrm{Re}\left(\frac{\dot{\xi}(t)}{\xi(t)}\right), \mathrm{Im}\left(\frac{\dot{\xi}(t)}{\xi(t)}\right)\right)$$

with the joint probability distribution $P_3(\vec{z}, t | \vec{z}_>, t_>)$*, satisfying the Fokker-Planck equation*

$$\frac{\partial P_3}{\partial t} = \hat{L}_3 P_3, \tag{41}$$

$$\hat{L}_3(\vec{z}) \equiv -\sum_{i=1}^{4} K_i \frac{\partial}{\partial z_i} + (\epsilon_2 p_2 + \epsilon_1 p_1 z_1^2) \frac{\partial^2}{\partial z_2^2} + \epsilon_1 p_1 \frac{\partial^2}{\partial z_3^2} + 2\epsilon_1 p_1 z_1 \frac{\partial^2}{\partial z_2 \partial z_3} + 4 z_3,$$

$$K_1 = z_2, \quad K_2 = F_0 - \Omega_0^2 z_1, \quad K_3 = z_4^2 - z_3^2 - \Omega_0^2, \quad K_4 = -2 z_3 z_4,$$

and the initial condition

$$P_3 \Big|_{t = t_>} = \delta(\vec{z} - \vec{z}_>),$$

with the probability distribution of components of the vector field $\vec{z}_>$ *given by formula* (39).

It is not difficult to show that if $\epsilon_1 = 0$ equation (41) transforms into (27) by the substitution

$$P_3(\vec{z}, t | \vec{z}_>, t_>) = \delta\left(z_3 - \mathrm{Re}\frac{\dot{\xi}_0(t)}{\xi_0(t)}\right) \delta\left(z_4 - \mathrm{Im}\frac{\dot{\xi}_0(t)}{\xi_0(t)}\right) P(z_1, z_2, t).$$

When $\epsilon_2 = 0$ equation (41) transforms directly into (29). Thus the relation of the general case with the particular situations, considered in the previous sections, is established. Using again the formulas (A.4)–(A.5), the representation (36) of $\langle |c_{nm}|^2 \rangle$ may be generalized to the case under consideration

$$\langle |c_{nm}|^2 \rangle = \int d\vec{z}_> R(\vec{z}_>, t_>) \int d\vec{z} H_{nm}(\vec{z}) Q_{nm}(\vec{z}, t). \tag{42}$$

As compared with (36), an additional integration with the weighting function $R(\vec{z}_>, t_>)$ allowing for the dispersion of initial values of the trajectory, was included into (42). The functions $H_{nm}(\vec{z})$ are defined in (37) and the functions $Q_{nm}(\vec{z}, t)$, depending on $z_>$ as a parameter, are solutions of the following problem

$$\frac{\partial Q_{nm}}{\partial t} = \hat{L}_3 Q_{nm} - V_{nm} Q_{nm},$$

$$Q_{nm}(\vec{z}, t) \underset{t \to t_>}{\longrightarrow} \delta(\vec{z} - \vec{z}_>), \quad Q_{nm}(\vec{z}, t) \underset{||\vec{z}|| \to \infty}{\longrightarrow} 0, \tag{43}$$

where

$$V_{nm} = p_{nm} z_3, \quad p_{00} = p_{01} = 1, \quad p_{10} = p_{11} = 3.$$

7. Thermodynamics within the framework of representation by a stochastic density matrix

It is well known (see e.g. [38]) that the key object of interest in quantum mechanics is the density matrix.

Definition 3 *The stochastic density matrix is defined by the expression*

$$\rho_{stc}\left(x, t; \{\vec{\zeta}\} | x', t'; \{\vec{\zeta}'\}\right) = \sum_{m=0}^{\infty} w_0^{(m)} \rho_{stc}^{(m)}\left(x, t; \{\vec{\zeta}\} | x', t'; \{\vec{\zeta}'\}\right), \tag{44}$$

$$\rho_{stc}^{(m)}\left(x, t; \{\vec{\zeta}\} | x', t'; \{\vec{\zeta}'\}\right) = \sqrt{\frac{\pi}{\Omega_{in}}} \Psi_{stc}^{(m)}\left(x, t | \{\vec{\zeta}\}\right) \overline{\Psi_{stc}^{(m)}\left(x', t' | \{\vec{\zeta}'\}\right)}, \tag{45}$$

where $w_0^{(m)}$ has the meaning of the initial distribution over quantum states with energies $E_m = \left(\frac{1}{2} + m\right)\Omega_{in}$, until the moment when the generator of random excitations is activated.

Definition 4 *The expected value of the operator $\hat{A}\left(x, t | \{\vec{\zeta}\}\right)$ in quantum state with the index m is*

$$A_m = \lim_{t \to +\infty} \left\{ Sp_x \left[Sp_{\{\vec{\zeta}\}} \hat{A} \rho_{stc}^{(m)} \right] \Big/ Sp_x \left[Sp_{\{\vec{\zeta}\}} \rho_{stc}^{(m)} \right] \right\}. \tag{46}$$

The mean value of the operator $\hat{A}\left(x,t|\{\vec{\zeta}\}\right)$ *over the whole ensemble of states will respectively be given by*

$$A = \lim_{t\to+\infty}\left\{ Sp_x\left[Sp_{\{\vec{\zeta}\}}\hat{A}\rho_{stc}\right]\Big/ Sp_x\left[Sp_{\{\vec{\zeta}\}}\rho_{stc}\right]\right\}. \tag{47}$$

The operation Sp_x in (46) and (47) is defined by

$$Sp_x\{K(x,x')\} = \sqrt{\frac{\Omega_{in}}{\pi}}\int dx K(x,x), \tag{48}$$

for any function $K(x,x')$.

Using (45) and the properties of the functionals $\Psi^{(m)}_{stc}\left(x,t|\{\vec{\zeta}\}\right)$ we easily obtain the expression for the total nonstationary distribution function

$$w_0 = Sp_x Sp_{\{\vec{\zeta}\}}\left\{\rho_{stc}\left(x,t;\{\vec{\zeta}\}|x',t';\{\vec{\zeta}'\}\right)\right\} = \sum_{m=0}^{\infty} w_0^{(m)}. \tag{49}$$

If the initial weighting functions $w_0^{(m)}$ are given by the canonical distribution $w_0^{(m)} = \exp\left(-E_m/\mathrm{k}T\right)$, the expression (49) takes the form of the Planck distribution (see [38])

$$w_0\left(\beta\right) = \frac{e^{\beta/2}}{e^{\beta}-1}, \qquad \beta = \frac{\Omega_{in}}{\mathrm{k}T}. \tag{50}$$

Substituting the expansion (15) of the wave functional in *out*-states into (44), (45) we have the representation

$$\rho_{stc}\left(x,t;\{\vec{\zeta}\}|x',t';\{\vec{\zeta}'\}\right)$$

$$= \sum_{m,k,l=0}^{\infty} w_0^{(m)} c_{mk}\left(t|\{\vec{\zeta}\}\right)\overline{c_{ml}\left(t|\{\vec{\zeta}\}\right)}\phi_k^{out}(x,t)\overline{\phi_l^{out}(x',t)}. \tag{51}$$

Definition 5 *The nonequilibrium partial distribution function is defined by*

$$w^{(m)}\left(\epsilon_1,\epsilon_2,t\right) = Sp_{\{\vec{\zeta}\}}\left\{\sum_{k=0}^{\infty}\left[w_0^{(k)}\left|c_{km}\left(t|\{\vec{\zeta}\}\right)\right|^2 - w_0^{(m)}\left|c_{mk}\left(t|\{\vec{\zeta}\}\right)\right|^2\right]\right\}$$

$$+w_0^{(m)} = \sum_{k=0}^{\infty}\left\{\left[w_0^{(k)}\Delta_{km}\left(t\right) - w_0^{(m)}\Delta_{mk}\left(t\right)\right]\right\} + w_0^{(m)}, \tag{52}$$

where

$$\Delta_{km}(t) = Sp_{\{\vec{\zeta}\}} \left| c_{km}\left(t|\{\vec{\zeta}\}\right)\right|^2 = \left\langle \left| c_{mm}\left(t|\{\vec{\zeta}\}\right)\right|^2 \right\rangle .$$

In this case the total distribution function is equal to the sum

$$w_0 = \sum_{m=0}^{\infty} w^{(m)}(\epsilon_1, \epsilon_2, t) . \tag{53}$$

In the case under consideration one can introduce different definitions for such a thermodynamical quantity as an entropy. Despite formal similar definitions given may or not provide the connection of the defined quantity with the irreversibility of the system of evolution. For example one can define the total and the partial entropy in the following way.

Definition 6 *The total entropy of nonequilibrium state in the extended space-time $R^1 \otimes R^0 \otimes R_{\{\vec{\xi}\}}$ is defined as*

$$S(\epsilon_1, \epsilon_2, t) = -Sp_{\{\vec{\zeta}\}} Sp_x \{\rho_{stc} \ln \rho_{stc}\} , \tag{54}$$

where R^0 characterizes the time.

Definition 7 *The partial nonequilibrium entropy in the extended space-time $R^1 \otimes R^0 \otimes R_{\{\vec{\xi}\}}$ is defined as*

$$S^{(m)}(\epsilon_1, \epsilon_2, t) = -Sp_{\{\vec{\zeta}\}} Sp_x \left\{\rho^{(m)}_{stc} \ln \rho^{(m)}_{stc}\right\} . \tag{55}$$

It is not difficult to show that the formal partial entropy does not depend on time and has no relation to thermodynamical irreversibility.

Proposition 4 *For any m the formal partial entropy $S^{(m)}(\epsilon_1, \epsilon_2, t)$ is equal to zero.*

Let us consider the N-dimensional square matrix $\hat{A}$ with elements $\hat{A}_{ik} = a_i a_k$, where a_i, $i = 1, \dots, N$, are the elements of an N-dimensional vector. It is possible to find all eigenvalues λ_i and to find out the structure of eigen-subspaces for the matrix A. Namely, one can show that $\lambda_1 = a_1^2 + a_2^2 + \cdots + a_N^2$, $\lambda_2 = \lambda_3 = \cdots = \lambda_N = 0$. At that eigenvector $\boldsymbol{e_1}$ coincides with $\boldsymbol{a}$, and the eigen-subspace corresponding to zero eigenvalues is orthogonal to $\boldsymbol{a}$.

Generalizing this result to the case of an infinitely dimensional matrix $\rho^{(m)}_{stc}$, one obtains: there is one eigenvector $(\pi/\Omega_{in})^{1/4}\, \Psi^{(m)}_{stc}\left(x, t|\{\vec{\zeta}\}\right)$, corresponding to nonzero eigenvalue $\lambda_1 = Sp_{\{\vec{\zeta}\}} Sp_x \left\{\rho^{(m)}_{stc}\right\}$, and there is an infinitely dimensional eigen-subspace, corresponding to the zero eigenvalue, which is orthogonal to this vector. Supplementing the vector $(\pi/\Omega_{in})^{1/4}\,\Psi_{stc}^{(m)}$ $\left(x,t|\{\vec{\zeta}\}\right)$ with any orthonormal set of vectors lying in the subspace mentioned above, one obtains the basis of the whole space which brings the matrix $\rho_{stc}^{(m)}$ in diagonal form. Understanding the uncertainty $0\ln 0$ as a limit

$$0\ln 0 = \lim_{s\to 0} s\ln s = 0,$$

one obtains for the formal partial entropy

$$S_f^{(m)}(\epsilon_1,\epsilon_2,t) = Sp_{\{\vec{\zeta}\}}Sp_x\left\{\rho_{stc}^{(m)}\right\}Sp_{\{\vec{\zeta}\}}Sp_x\left\{\ln\rho_{stc}^{(m)}\right\} = 0,$$

which makes the proof complete.

If one wishes to have the quantity describing the irreversible behavior of the system, it is necessary to change the definition of entropy.

Definition 8 *The total and partial entropies of the nonequilibrium state in space-time $R^1\otimes R^0$ are defined as*

$$S(\epsilon_1,\epsilon_2,t) = -Sp_x\left\{\rho_{av}\ln\rho_{av}\right\}. \tag{56}$$

and

$$S^{(m)}(\epsilon_1,\epsilon_2,t) = -Sp_x\left\{\rho_{av}^{(m)}\ln\rho_{av}^{(m)}\right\}, \tag{57}$$

correspondingly, where

$$\rho_{av} = Sp_{\{\vec{\zeta}\}}\left\{\rho_{stc}\right\},\quad \rho_{av}^{(m)} = Sp_{\{\vec{\zeta}\}}\left\{\rho_{stc}^{(m)}\right\}.$$

Unfortunately, at the moment we have no simple enough analytical representation for the quantities defined in such a way. To illustrate the definitions given above we calculate the average energy of the oscillator in the ground, vacuum, state (i.e. at $m=0$) assuming that both regular and stochastic parts of the external force are absent. In this case the density matrix has the form

$$\rho_{stc}^{(0)}\left(x,t;\{\vec{\zeta}\}\mid x',t';\{\vec{\zeta}'\}\right) = \exp\left\{-\frac{\Omega_{in}}{2}(x^2+x'^2)\right.$$
$$\left.-\frac{1}{2}\int_{t_1}^{t}\Phi(\tau)\,d\tau - \frac{1}{2}\int_{t_1}^{t'}\Phi^*(\tau)\,d\tau - \frac{i}{2}\left[\Phi(t)\,x^2 - \Phi^*(t')\,x'^2\right]\right\}. \tag{58}$$

Proposition 5 *Let* $\Omega(t) \equiv \Omega_{in}$, $F_0(t) \equiv 0$, $\epsilon_2 = 0$, $p_1(t) \equiv 1$. *Then the average energy*

$$E_{osc}^{(0)}(\lambda) = Sp_x Sp_{\{\vec{\zeta}\}}\left(\hat{H}\tilde{\rho}_{stc}^{(0)}\right)$$

is represented by

$$E_{osc}^{(0)}\left(\lambda\right) = \frac{1}{2}\Omega_{in}\left\{1 - \frac{1}{\sqrt{\lambda}}\int\limits_{-\infty}^{+\infty} d\bar{u}_1 K_1\left(\lambda, \bar{u}_1\right) + \frac{i}{\sqrt{\lambda}}\int\limits_{-\infty}^{+\infty} d\bar{u}_1 K_2\left(\lambda, \bar{u}_1\right)\right\}, \tag{59}$$

with the notations

$$K_1\left(\lambda, \bar{u}_1\right) = C_0 \bar{u}_1 \bar{q}^{st}\left(\lambda, \bar{u}_1\right)$$

$$\times\left\{-\sqrt{\frac{A_{00}-1}{2A_{00}^2}} + \frac{\bar{u}_1}{\sqrt{\lambda}A_{00}^2}\left[\sqrt{\frac{A_{00}+1}{2A_{00}^2}} + \frac{\bar{u}_1}{\sqrt{\lambda}}\sqrt{\frac{A_{00}-1}{2A_{00}^2}}\right]\right\},$$

$$K_2\left(\lambda, \bar{u}_1\right) = C_0 \bar{u}_1 \bar{q}^{st}\left(\lambda, \bar{u}_1\right)$$

$$\times\left\{-\sqrt{\frac{A_{00}-1}{2A_{00}^2}} + \frac{\bar{u}_1}{\sqrt{\lambda}A_{00}^2}\left[-\sqrt{\frac{A_{00}-1}{2A_{00}^2}} + \frac{\bar{u}_1}{\sqrt{\lambda}}\sqrt{\frac{A_{00}-1}{2A_{00}^2}}\right]\right\},$$

$$A_{00}\left(\lambda, \bar{u}_1\right) = \sqrt{1 + \frac{\bar{u}_1^2}{\lambda}}. \tag{60}$$

The function $q^{st}(u_1)$ *is an arbitrary solution of the equation*

$$\epsilon_1 \frac{d^2 q^{st}}{du_1^2} + (u_1^2 + \Omega_{in}^2)\frac{dq^{st}}{du_1} + u_1 q^{st} = 0, \tag{61}$$

decreasing as $|u_1| \to \infty$, *and the constant* C_0 *given by*

$$C_0 = \frac{Y^{st}(u_{0;1}, u_{0;2})}{\int du_1\ Y^{st}(u_1, 0) q^{st}(u_1)}, \tag{62}$$

where $u_{0;1}$ *and* $u_{0;2}$ *define the initial coordinates. Here the function* $Y^{st}(u_1, u_2)$ *is an arbitrary solution of the equation*

$$\epsilon_1 \frac{d^2 Y^{st}}{du_1^2} - (u_1^2 - u_2^2 + \Omega_{in}^2)\frac{dY^{st}}{du_1} - 2u_1 u_2 \frac{dY^{st}}{du_2} - u_1 Y^{st} = 0, \tag{63}$$

decreasing as $u_1^2 + u_2^2 \to \infty$.

In fact the proof copies the manipulations performed repeatedly in this paper and thus may be omitted.

The second term inside the figure brackets in (59) is a level shift which is well known from quantum electrodynamics as the Lamb shift, the third term determines the magnitude of the ground state energy broadening. Note that the lifetime at this level is proportional to the inverse of the broadening

$$\Delta t^{(0)} \sim 2\frac{\sqrt{\lambda}}{\Omega_{in}} \left\{ \int_{-\infty}^{+\infty} d\bar{u}_1 K_2\left(\lambda, \bar{u}_1\right) \right\}^{-1} . \tag{64}$$

The average energy of a randomly wandering (QHO) for any quantum level is calculated quite similarly.

Conclusion

A fundamentally novel scheme for construction of quantum mechanics for the "quantum object + thermostat" closed system was proposed in [32, 33]. The problem was mathematically formulated in terms of a complex stochastic process $\Psi_{stc}(x,t|\{\vec{\zeta}\}$, the solution of L-Sch-type SDE on the extended space $R^n \otimes R_{\{\xi\}}$, where R^n denotes the n-dimensional Eucledian space and $R_{\{\xi\}}$ the functional space respectively. Different models of stochastic quantum processes on $R^n \otimes R_{\{\xi\}}$ with a quadratic space form for the random potential have been considered. It was proved that using a reference nonlinear Langevin-type SDE (determined on the $R_{\{\xi\}}$ space), one can reduce the initial L-Sch SDE to an autonomous form of the Schrödinger equation with a random space-time continuum $\{y,\tau\}$ {see (9)}. In other words, the separation of variables in the L-Sch SDE is made conditionally with the help of a nonlinear transformation that is a solution of the nonlinear Langevin SDE. It was shown that if the random force generators are given as δ-shaped correlators {see (4)}, then by using the Langevin SDE one can obtain the Fokker-Planck equation for the density of the conditional probability in $R_{\{\xi\}}$ space. Using the solution of the Fokker-Planck equation for an infinitesimal time interval, a measure in the $R_{\{\xi\}}$ space is constructed, with respect to which the instantaneous values of elemental quantum transition amplitudes are averaged in the form of functional integrals. Then, it proved possible in the most general case to reduce the functional integral to a multiple integral of the basic solution of some parabolic equation by using the generalized Feynman-Kac theorem {see Appendix}. The qualitative analysis of the parabolic equation shows that it may have discontinuous solutions [39], and, this, in its turn, may lead to phase transitions in the expression for microscopic transitions depending on the coupling constant of the quantum subsystem with the thermostat. A representation of the stochastic density matrix within the framework of harmonic models has been developed to permit a construction of closed scheme of nonequilibrium quantum statistical mechanics. In particular, representations in the form of multiple integrals for the breadth, shift (an analogue of Lamb shift) of energy levels of the oscillator, as well as for the entropy of an arbitrary quantum state characterized by an integer index n were obtained. The ground state of energy of the oscillator has been analyzed in detail and it was shown that a spontaneous decay of the vacuum level took place. This fact is important for substantiation of the violation of the second law of thermodynamics due to quantum fluctuations [40]. A further development of the proposed formalism based on the example of other exactly solvable models [1], especially multidimensional ones, may essentially deepen our understanding of the quantum space and lead to new nontrivial discoveries. The investigation of exactly solvable models of stochastic quantum processes may be highly important for creation of quantum computers and for quantum cryptography. Besides, these investigations will serve in the future as samples for direct quantum numerical simulations, because, as is well known, the quantum chaos may be investigated exclusively with quantum computers [41].

At the end we pay attention to one of the most important properties of developed quantum representation. On the one hand, it provides the execution of the vacuum average conservation law for the operators of energy, momentum, etc. on the extended space-time $R^1 \otimes R^0 \otimes R_{\{\vec{\xi}\}}$. On the other hand, for the same operators, which are local field operators $\{\vec{\eta}\}$ on the space-time $R^1 \otimes R^0$, provides non-zero values of vacuum averages (in standard quantum representation vacuum averages of the mentioned operators in whole $R^1 \otimes R^0$ are equal to zero). The last fact, particularly, indicates the spontaneous symmetry breakdown and the degeneration of physical vacuum, and also violation of the second law of thermodynamics [40]. Nevertheless, in context of this approach such important problems as properties and structure of free and interacting with "quantum object" vacuum remain unexplored. Note, that the static properties of vacuum (thermostat) are described by second-order partial differential equations for the conditional probability of vacuum coordinates distribution. At that, the equations which describe free vacuum, are generated by etalons SDE (see (32) and (40)). The equations, which describe vacuum interacting with "quantum object", are obtained as a result of reduction of continual integral representation for the amplitude of transition probability to multiple integral representations (see Appendix). Analysis of zeros of those equations, in time slice, will provide valuable information about self-organization of vacuum, about process of it's degeneration following particularly by excitation of Goldstone bosom [42]. Analysis of geometrical characteristics and topological features of sets of zeros in the configuration space of vacuum coordinates, i.e. in the limit of stationary processes, is an open mathematical problem. In one-dimensional task the general number of that zeros and distances between them is enough to represent fully the types of excitation of vacuum. More complicated but interesting and informative for analysis of vacuum, is a problem of zeros in multi-dimensional task. Here it is necessary to analyse the geometry of set of zeros, their topological features, for classifying the types of excitation of vacuum. In the first case we intend to find the common number of sets of zeros, developing closed hyper-surfaces (in case 2D closed curve), calculation of their squares (lengths) and enclosed with them cubatures (squares).

In that case, it is presumed, that the attitude of hyper-square and hyper-cubature should be minimal. Besides, hyper-cubature of isolated topology should be discrete and invariant regarding external interactions. The above-mentioned problems are not simply solved, if one takes into account that the describing of level lines for even so simple functions as polynomial is far from being solved (the problem 16a of Hilbert). Note, that till recently there were no general methods for the analysis of zeros of real one-variable functions, level lines of two-variable functions (level surfaces in case of multi-variables). Such methods have been appeared in a recently developed theory of Gamma-lines [43], [44]. Therefore one of our immediate tasks is testing of Gamma-line theory for researching zeros of solutions of the mentioned equations and their following physical interpretation.

Appendix

Let the set of random processes $(\xi_1, \xi_2, .., \xi_n) \equiv \vec{\xi}$ satisfy the set of SDE

$$\dot{\xi}_i(t) = a_i\left(\vec{\xi}, t\right) + \sum_{j=1}^{n} b_{ij}\left(\vec{\xi}, t\right) f_j(t), \quad i = 1, ..., n,$$

$$\left\langle f_i(t) f_j(t')\right\rangle = \delta_{ij}\delta(t - t'),$$

so that the Fokker-Planck equation for the conditional transition probability density

$$P^{(2)}\left(\vec{\xi}_2, t_2|\vec{\xi}_1, t_1\right) = \left\langle \delta\left(\vec{\xi}(t_2) - \vec{\xi}_2\right)\right\rangle\Big|_{\vec{\xi}(t_1)=\vec{\xi}_1} \quad t_2 > t_1 \qquad \text{(A.1)}$$

is given by

$$\frac{\partial P^{(2)}}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} \left(a_i P^{(2)} \right) + \sum_{i,j,l,n} \frac{\partial}{\partial \xi_i} \left(b_{li} \frac{\partial}{\partial \xi_n} \left(b_{nj} P^{(2)} \right) \right) \equiv \hat{L}^{(n)}(\vec{\xi}) P^{(2)}. \tag{A.2}$$

The processes ξ_i are assumed to be markovian and satisfy the condition $\vec{\xi}(t_0) = \vec{\xi}_0$. At the same time the function (A.1) gives their exhaustive description

$$P^{(n)} \left(\vec{\xi}_n, t_n; ...\vec{\xi}_1, t_1; \vec{\xi}_0, t_0 \right) = P^{(2)} \left(\vec{\xi}_n, t_n | \vec{\xi}_{n-1}, t_{n-1} \right) ...P^{(2)} \left(\vec{\xi}_1, t_1 | \vec{\xi}_0, t_0 \right) \tag{A.3}$$

where $P^{(n)}$ is a density of the probability that the trajectory $\vec{\xi}(t)$ would pass through the sequence of intervals $\left[\vec{\xi}_1, \vec{\xi}_1 + d\vec{\xi}_1 \right], \ldots, \left[\vec{\xi}_n, \vec{\xi}_n + d\vec{\xi}_n \right]$ at the subsequent moments of time $t_1 < \ldots < t_n$, respectively.

Under these assumptions we can obtain the following representation for an averaging procedure

$$\left\langle \exp \left\{ - \int_{t_0}^{t} V_1 \left(\vec{\xi}(\tau), \vec{\xi}(t) \right) d\tau - V_2 \left(\vec{\xi}(t) \right) \right\} \right\rangle = \int d\vec{\xi} e^{-V_2(\vec{\xi},t)} Q \left(\vec{\xi}, \vec{\xi}, t \right), \tag{A.4}$$

where $d\vec{\xi} = d\xi_1 ... d\xi_n$, and the function $Q \left(\vec{\xi}, \vec{\xi}', t \right)$ is a solution of the problem

$$\frac{\partial Q}{\partial t} = \left[\hat{L}^{(n)}(\vec{\xi}) - V_1(\vec{\xi}, \vec{\xi}') \right] Q, \tag{A.5}$$

$$Q \left(\vec{\xi}, \vec{\xi}', t \right) \underset{t \to t_0}{\longrightarrow} \delta \left(\vec{\xi} - \vec{\xi}_0 \right), \quad Q \left(\vec{\xi}, \vec{\xi}', t \right) \underset{||\vec{\xi}|| \to \infty}{\longrightarrow} 0,$$

where $|| \cdot ||$ is a norm in R^n. The proof is performed formally under the assumption that all the manipulations are legal. Denote the left-hand side of the equality (A.4) by I and expand the averaging quantity into the Taylor series

$$I = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n(t), \tag{A.6}$$

where

$$\mu_n(t) = \left\langle \left\{ \int_{t_0}^{t} V_1(\tau) d\tau + V_2(t) \right\}^n \right\rangle$$

$$= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \left\langle V_2^{n-m}(t) \left\{ \int_{t_0}^{t} V_1(\tau) d\tau \right\}^m \right\rangle \tag{A.7}$$

$$= \sum_{m=0}^{n} \frac{n!}{(n-m)!} \left\langle V_2^{n-m}(t) \int_{t_0}^{t} d\tau_m \int_{t_0}^{\tau_m} d\tau_{m-1} \ldots \int_{t_0}^{\tau_2} d\tau_1 V_1(\tau_m) \ldots V_1(\tau_1) \right\rangle .$$

For the sake of brevity in (A.7) the notation $V_1(\tau) \equiv V_1(\vec{\xi}(\tau), \vec{\xi}(t)),\ V_2(t) \equiv V_2\left(\vec{\xi}(t)\right)$ is introduced. Using the Fubini theorem, we can represent the averaging procedure in (19) as an integration with the weight $P^{(n)}$ from (A.3)

$$\left\langle V_2^{n-m}(t) \int_{t_0}^{t} d\tau_m \int_{t_0}^{\tau_m} d\tau_{m-1} \ldots \int_{t_0}^{\tau_2} d\tau_1 V_1(\tau_m) \ldots V_1(\tau_1) \right\rangle$$

$$= \int d\vec{\xi} \int d\vec{\xi}_m \ldots \int d\vec{\xi}_1 \int_{t_0}^{t} d\tau_m \ldots$$

$$\times \int_{t_0}^{\tau_2} d\tau_1 P^{(2)} \left(\vec{\xi}, t \middle| \vec{\xi}_m, t_m \right) P^{(2)} \left(\vec{\xi}_m, t_m \middle| \vec{\xi}_{m-1}, t_{m-1} \right) \ldots$$

$$\times\ P^{(2)} \left(\vec{\xi}_2, t_2 \middle| \vec{\xi}_1, t_1 \right) V_2^{n-m} \left(\vec{\xi} \right) V_1 \left(\vec{\xi}_m, \vec{\xi} \right) \ldots V_1 \left(\vec{\xi}_1, \vec{\xi} \right) .$$

Changing, where necessary, the order of integration we can obtain the following representation for the n-th moment $\mu_n(t)$

$$\mu_n(t) = \sum_{m=0}^{n} \frac{n!}{(n-m)!} \int d\vec{\xi} V_2^{n-m} \left(\vec{\xi} \right) Q_m \left(\vec{\xi}, \vec{\xi}', t \right) , \tag{A.8}$$

where the countable set of functions $Q_m \left(\vec{\xi}, \vec{\xi}', t \right)$ is determined from the recurrence relations

$$Q_m \left(\vec{\xi}, \vec{\xi}', t \right)$$

$$= \int_{t_0}^{t} d\tau \int d\vec{\eta}\ V_2^{n-m} \left(\vec{\xi} \right) P^{(2)} \left(\vec{\xi}', t \middle| \vec{\eta}, \tau \right) V_1 \left(\vec{\eta}, \vec{\xi}' \right) Q_{m-1} \left(\vec{\eta}, \vec{\xi}', \tau \right) , \tag{A.9}$$

for $m = 0, 1, 2, ...$, where

$$Q_0\left(\vec{\xi}, \vec{\xi}', t\right) = P^{(2)}\left(\vec{\xi}, t \middle| \vec{\xi}_0, t_0\right) \tag{A.10}$$

i.e. in fact the function Q_0 is independent of $\vec{\xi}'$. Upon the substitution of (A.8) into (A.6) we insert the summation procedure under the integration sign and then, changing the order of double summation, get the expression

$$I = \int d\vec{\xi}\, e^{-V_2(\vec{\xi},t)} Q\left(\vec{\xi}, \vec{\xi}, t\right), \tag{A.11}$$

where

$$Q\left(\vec{\xi}, \vec{\xi}', t\right) = \sum_{n=0}^{\infty} (-1)^n Q_n\left(\vec{\xi}, \vec{\xi}', t\right). \tag{A.12}$$

The representation (A.4) is thus obtained.

It remains to prove that the function Q from (A.11) is a solution of the problem (A.5). Using (A.12) and (A.9) we can easily show that Q satisfies the integral equation

$$Q\left(\vec{\xi}, \vec{\xi}', t\right) + \int_{t_0}^{t} d\tau \int d\vec{\eta}\, P^{(2)}\left(\vec{\xi}, t \middle| \vec{\eta}, \tau\right) V_1\left(\vec{\eta}, \vec{\xi}'\right) Q\left(\vec{\eta}, \vec{\xi}', \tau\right) = Q_0\left(\vec{\xi}, t\right). \tag{A.13}$$

Taking into account that Q_0 satisfies (A.2) and the initial condition $Q_0\left(\vec{\xi}, t_0\right) = \delta\left(\vec{\xi} - \vec{\xi}_0\right)$ and is an integrable function, it can be deduced from (A.13) that Q coincides with the solution of the problem (A.5). The representation (A.4), (A.5) is thus obtained.

Notes

1. It is worthwhile to mention that under the exactly solvable models of stochastic quantum processes, only the models are implied, in the frameworks of which one can construct closed representations in the form of multiple integrals of basic solutions of some reference parabolic type equations, for physically measured parameters of the problem.

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GENERATING FUNCTIONS AND WAVE-LET-LIKE DECOMPOSITIONS

Anry Nersessian
Institute of Mathematics of
National Academy of Sciences of Armenia
Baghramian ave. 24 B, Yerevan 375019, Armenia
nerses@instmath.sci.am

Abstract A generalization of periodic wavelet theory is presented. The obtained results are applicable for constructing fast and stable algorithms for decompositions by classical orthogonal polynomials.

Keywords: periodic wavelet theory, periodic decomposition of orthogonal bases of Hilbert space, generating functions, stable algorithms, Gegenbauer polynomials, discrete Fourier transform

Mathematics Subject Classification (2000): 42C40, 46C07, 33C45, 65T50

1. Introduction

In [1] a general scheme of construction of orthogonal systems by periodic decomposition of orthogonal bases in the Hilbert space was suggested. In the case of the classical Fourier basis this approach allows to generalize the scheme of constructing periodic (including interpolating) wavelets (see also [2]).

In [3] it was shown that for some special functions the corresponding decompositions can be realized on the basis of the known generating functions. We describe this method sometimes leading to the construction of fast and stable algorithms. Particularly, this scheme allows to avoid integrals of fast oscillating functions while calculating coefficients of expansions by orthogonal polynomials. Here as an example a generating function for Gegenbauer polynomials is considered.

The given numerical results are obtained by means of MATHEMATICA 4.1 system ([4]).

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2. Generating functions and decompositions of bases

2.1 Periodic decomposition

Consider a basis $\{\alpha_n\}_{n=a}^{\infty}$ ($||\alpha_n|| = 1, n \geq a$) in an infinite-dimensional Hilbert space H. We are mainly interested in two generally accepted cases of this basic numeration

- Case $A: \ a = -\infty$;
- Case $B: \ a = 0$.

Let $\Omega = \{n\}_{n=a}^{n=\infty}$, N be an integer and

$$\Omega(N) = \begin{cases} \{n\},\ n = -[\frac{N}{2}], -[\frac{N}{2}] + 1 \ldots, -[\frac{N}{2}] + N - 1, & \text{in case } A, \\ \{n\},\ n = 0, 1 \ldots, N-1, & \text{in case } B. \end{cases}$$

We use the following notations of numeration suitable for both cases.

$$\sum_n \cdot = \sum_{n\in\Omega} \cdot\ ; \qquad \sum_{n,N} \cdot = \sum_{n\in\Omega(N)} \cdot\ ; \qquad \sum_{N,n} \cdot = \sum_{n\in\Omega\setminus\Omega(N)} \cdot\ . \qquad (1)$$

Let now $\{c_n\}$, $c_n \in \mathbb{C}$, $n \in \Omega$ be a sequence such that $\sum_n (1+\varepsilon)^n |c_n| < \infty$, $\varepsilon = \text{const} > 0$. Then the function

$$F(z) = \sum_n c_n z^n \alpha_n \qquad (2)$$

with values in H, is analytic in a ring neighborhood of the unit circle $|z| = 1$ (in the case B it is analytic in the disc $|z| < 1 + \varepsilon$). It is easy to see, that even for $\varepsilon = 0$, $|z| = 1$ the series (2) converges in norm, i.e. $F(z) \in H$.

If $c_n \neq 0$, $n \in \Omega$, it is assumed (for instance, in the theory of special functions) to call the function $F(z)$ as a *generating function* of the system $\{\alpha_n\}$.

The further constructions of this paper are based on the following easily proved formula ($N \geq 1$ is an integer)

$$N^{-1} \sum_{k,N} e^{-2i\pi kn/N} F(e^{2i\pi k/N}) = \sum_s c_{n+Ns} \alpha_{n+Ns}, \quad n \in \Omega(N). \qquad (3)$$

2.2 Decomposition of orthogonal systems

Consider the case of complete orthonormal basis $\{\alpha_n\}_{n=a}^{\infty}$ ($||\alpha_n|| = 1$, $\overline{\{\alpha_n\}} = H$). It is easy to see from (3) that the finite system $\{\varphi_n\}$, $n \in \Omega(N)$,

$$\varphi_n = \varphi_n(N) = \frac{1}{\omega_n} \sum_{k,N} e^{-2i\pi kn/N} F(e^{2i\pi k/N}), \quad n \in \Omega(N), \qquad (4)$$

where

$$\omega_n^2 = \sum_s |c_{n+Ns}|^2 = N^{-1} \sum_{k,N} e^{-2i\pi kn/N} \sum_s |c_s|^2 e^{2i\pi ks/N} \neq 0, \quad (5)$$

is also orthonormal.

Using the properties of discrete sine-cosine Fourier transforms we also obtain the two orthogonal systems $\{\psi_n^\pm\}$,

$$\psi_n^+ = \sum_{k=0}^{N-1} \cos \frac{2\pi kn}{N} F_k^+, \quad \psi_n^- = \sum_{k=0}^{N-1} \sin \frac{2\pi kn}{N} F_k^-, \quad n = 0, 1, \ldots, N-1, \quad (6)$$

where

$$F_k^+ = F(e^{2i\pi k/N}) + F(e^{-2i\pi k/N}) = 2 \sum_n c_n \cos \frac{2\pi kn}{N}\ \alpha_n,$$

$$F_k^- = F(e^{2i\pi k/N}) - F(e^{-2i\pi k/N}) = 2i \sum_n c_n \sin \frac{2\pi kn}{N}\ \alpha_n. \quad (7)$$

Sometimes (6)-(7) are more convenient than (4)-(5) because $F_k^\pm$ are real decompositions of $\{\alpha_n\}$ if $\{c_n\}$ is real.

Consider now the projection of an element $f \in H$ on the N-dimensional subspace $H_N = \mathrm{Span}\{\varphi_n\}$

$$S_N(f) = \sum_{n,N} (f, \varphi_n)\varphi_n. \quad (8)$$

We have ($\mu_{n,s} = \mu_{n,s}(N) = c_{n+Ns}/\omega_n$, $n \in \Omega(N)$, $s \in \Omega$)

$$S_N(\alpha_{n+Ns}) = \overline{\mu}_{n,s}\varphi_n, \quad \|\alpha_{n+Ns} - S_N(\alpha_{n+Ns})\|^2 = 1 - |\mu_{n,s}|^2. \quad (9)$$

It is not difficult to verify that the following estimates ($f_n = (f, \alpha_n)$)

$$||S_N(f)|| = \sqrt{\sum_{n,N} \left| \sum_s f_{n+Ns}\overline{\mu}_{n,s} \right|^2} \leq ||f||, \quad (10)$$

$$||f - S_N(f)|| \leq 2\sqrt{\sum_{N,n} |f_n|^2} + \sqrt{\sum_{n,N} |f_n|^2(1 - |\mu_{n,0}|^2)} \quad (11)$$

hold. From here it follows immediately

Theorem 1 [1] *The condition*

$$\lim |\mu_{n,0}(N)|^2 \to 1, \quad N \to \infty,\ n = \text{const}\ , \quad (12)$$

is necessary and sufficient for the convergence $S_N(f) \to f\,(N \to \infty)$ *with respect to the norm of* H *for any* $f \in H$.

Thus, if the condition of Theorem 1 holds then the orthogonal system $\{\varphi_n\}$, $n \in \Omega(N)$, has a decomposition by the system $\{\alpha_{n+Ns}\}$, $n \in \Omega(N)$, $s \in \Omega$, and from (3), (4) it follows that $\{\varphi_n\} \simeq \{\alpha_n\}$ if $N \gg 1$.

For evaluation of the rate of convergence $S_N(f) \to f$ it is convenient to use the following criterion based on (11) and similar to the Strang-Fix condition, known in theory of periodic interpolation wavelets [5].

Theorem 2 [1] *Let* $\varepsilon_n \downarrow 0\,(n \to \infty)$ *(in case A:* $\varepsilon_n = \varepsilon_{-n}$*). If* $f \in H$, $f_n = (f, \alpha_n)$, $\sum_n |f_n|^2/\varepsilon_n^2 < +\infty$ *(in case A:* $\sum_n |f_n|^2/\varepsilon_{2n}^2 < \infty$*) and*

$$1 - |\mu_{n,0}|^2 \leq \begin{cases} (\varepsilon_N/\varepsilon_{2n})^2, & \text{in case } A, \\ (\varepsilon_N/\varepsilon_n)^2, & \text{in case } B, \end{cases} \tag{13}$$

then $\|f - S_N(f)\| \leq C\varepsilon_N$, *where* C *does not depend on* N.

2.3 Biorthogonal expansions

Form (4) and (8) we can represent $S_N(f)$ by the biorthogonal system $\{F(e^{2i\pi k/N}), \phi_k\}$, $k \in \Omega(N)$ when

$$\phi_k = \sum_{n,N} \frac{1}{\omega_n^2} \sum_{k,N} e^{2i\pi n(p-k)/N} F(e^{2i\pi k/N}). \tag{14}$$

On the other hand, the scheme of this section holds also in the more general case of a biorthogonal basis $\{\alpha_n, \beta_n\}$, $n \in \Omega$, $\alpha_n \in B$, $\beta_n \in B^*$, where B is a Banach space. In this case the system $\{\varphi_n, \psi_n\}$, $n \in \Omega(N)$,

$$\varphi_n = \sum_s u_{n+Ns}\alpha_{n+Ns}, \quad \psi_n = \sum_s v_{n+Ns}\beta_{n+Ns}, \tag{15}$$

where $\sum_s |u_n|^2$, $\sum_s |v_n|^2 < \infty$, $n \in \Omega(N)$, is biorthogonal under the natural condition

$$\sum_s u_{n+Ns}\overline{v}_{n+Ns} \neq 0, \quad n \in \Omega(N). \tag{16}$$

2.4 Transition to an infinite orthonormal system

If the condition of Theorem 1 holds, $M > 0$ is an integer, $N = M^s$ $(s = 1, 2, \ldots)$ and $\mathrm{Span}\{\varphi_n(M^s)\} \subset \mathrm{Span}\{\varphi_n(M^{s+1})\}$ one can construct an infinite orthogonal system by the scheme $H_{M^{s+1}} = H_{M^s} \oplus W_s$, $H = H_M \oplus \cup_{s=1}^{\infty} \oplus W_s$. The general conditions for this construction are described in [1] (Theorem 3). It is clear that this scheme is quite similar to the method of construction of periodic wavelets on a finite segment (see [6]).

Unfortunately, even when this system is complete in H it may be not convenient for stable calculation and in practice it may be better to stay with a fixed N (see below, Sections 4.2 and 4.3).

2.5 Numeration of the bases

Cases A and B in the beginning of Section 2.1 have been picked out only for convenience in applications. Actually the simple renumeration reduces one case to the other. For example, in case B we can bring the basis $\{\alpha_n\}_{n=0}^{\infty}$ to the basis $\{\beta_n\}_{n=-\infty}^{\infty}$ by putting $\beta_n = \alpha_{2n}$ for $n \geq 0$ and $\beta_n = \alpha_{-2n-1}$ for $n < 0$. On the contrary, case A can be reduced to case B.

Let now n be an integer multi-index $n = (n_1, n_2, \ldots, n_m) \quad (m \geq 2)$, where $n_p \in \Omega$, $(p = 1, 2, \ldots, m)$ and $N = (N_1, ..., N_m)$. For every p the set $\Omega = \Omega_p$ may correspond to one of cases A or B). Then all the results of the previous sections are valid if we give the corresponding sense to summation (2) and use in (3) the multidimensional Fourier transform. This approach can be applied in multidimensional functional spaces H (see [3]).

3. Analysis of expansions

3.1 Expansion algorithms

Let us now consider an expansion of an element $f \in H$ through the basis $\{\alpha_n\}, n \in \Omega$. Note that in practice only the approximate expansion by the truncated basis is possible, i.e. for $n \in \Omega(N)$ in the N-dimensional subspace $\text{Span}\{\alpha_n\} = H_N \subset H$.

When calculating the coefficients (f, α_n) in classical weighted functional spaces H, one of the main difficulties is fast oscillations of α_n for $n \gg 1$, leading to a high waste of time of automatic integration of corresponding inner products and essential round-off errors (see Section 4.3 below).

The suggested decomposition allows to overcome these obstacles in some important cases. The algorithm of evaluating the expansion $S_N(f)$ by $\{F(e^{2i\pi k/N})\}$(see (16)) based on a generating function should be realized by the following general scheme

a) calculation of inner products $u_k = (f, F(e^{2i\pi k/N})), \quad k \in \Omega(N)$,

b) calculation of the Discrete Fourier Transform (DFT) $v_n = \sum_{k,N} e^{-2i\pi kn/N} u_k, \quad n \in \Omega(N)$,

c) calculation of the DFT $w_p = \sum_{n,N} e^{-2i\pi pn/N} (v_n/\omega_n^2), \quad p \in \Omega(N)$,

d) output of the expansion

$$S_N(f) = \sum_{p,N} w_p F\left(e^{2i\pi p/N}\right). \tag{17}$$

Note that the evaluation of the representation (8) will be completed already on the stages a)+b). Besides (see (5)), sometimes it is convenient to calculate also the normalizing factor ω_n by means of the DFT. As it is well-known, calculation of DFT by Fast Fourier Transform (FFT) requires $O(N \log N)$ $(N \gg 1)$ arithmetic operations. As we see from this scheme, stages b) and c) (and also the mentioned normalization) can be realized numerically fast.

Inner products on stage a) can be realized stable, for example, if H is a space of functions of several variables and the functions $\{F(e^{2i\pi k/N})\}$ are not fast oscillating.

4. Examples and numerical results

4.1 Classical wavelets

The periodical wavelet theory corresponds to the case $h = L_2(-1,1)$, $\{\alpha_n\} = \{e^{i\pi nx}\}$, $n = 0, \pm 1, ...$, when the generating function (3) is actually the Laurent series of a function $F(z)$. The system $\{F(e^{2i\pi k/N})\}$, $k \in \Omega(N)$, now consists of translations of a given function $g(x) = F(e^{i\pi x})$

$$F(e^{2i\pi k/N}) = g(x + \frac{2k}{N}), k \in \Omega(N).$$

In this case it is preferable to use such a function $g(x)$ that is both non-oscillatory and has local (or almost local with respect to the required precision) support. On this basis many fast and stable algorithms are created for different problems (see, for instance, [1], [7] - [10]).

Below we consider a completely other situation.

4.2 Gegenbauer polynomials

In [3] the systems $\{\varphi_n\}$ were constructed for all classical orthogonal polynomials, including two-dimensional and discrete ones (see also Section 2.4 above). Here our consideration is connected with the family of Gegenbauer polynomials $\{C_n^\lambda(x)\}$, $\lambda > -1/2$, $n = 0, 1, \dots$, which are orthogonal on the interval $(-1, 1)$ with the weight $(1 - x^2)^{\lambda - 1/2}$.

Consider the known generating function (see [11], vol. II, item 10.9)

$$\sum_{n=0}^{\infty} C_n^\lambda(x) r^n z^n = (1 - 2xrz + r^2 z^2)^{-\lambda}, \quad |z| = 1, |r| < 1. \tag{18}$$

Using the orthonormal shape $\{g_n^\lambda\} = \left\{\sqrt{\frac{\Gamma(2\lambda)\Gamma(\lambda)n!(n+\lambda)}{\sqrt{\pi}\Gamma(2\lambda+n)\Gamma(1/2+\lambda)}}C_n^\lambda\right\}$ of the system $\{C_n^\lambda(x)\}$ we have from (3)

$$F(z) = \left(\frac{\sqrt{\pi}\Gamma(1/2+\lambda)}{\Gamma(2\lambda)\Gamma(\lambda)}\right)^{1/2} \sum_{n=0}^{\infty} \left(\frac{\Gamma(2\lambda+n)}{n!(n+\lambda)}\right)^{1/2} z^n g_n^\lambda. \qquad (19)$$

Thus the system (see (3), (4) and (12))

$$\varphi_n(x) = \frac{1}{\omega_n} \sum_{k=0}^{N-1} e^{-2i\pi kn/N}(1 - 2xre^{2i\pi k/N} + r^2 e^{4i\pi k/N})^{-\lambda} \qquad (20)$$

can be considered as an approximation of the system $\{g_n^\lambda\}$.

Note that if $\lambda \neq 0$ then ω_n^2 is represented by the hypergeometric function ${}_pF_q((\lambda, 2\lambda), (1+\lambda), z^2)$ (see [11], vol. I, item 4.1). Thus we can use FFT algorithm for calculating the ω_n (see (5)). Note if 2λ is integer this function is represented by elementary functions.

The results of Sections 1.1 and 1.2 sufficiently characterize the error $\|f - S_N(f)\|$, $f \in H$, of the expansion $S_N(f)$ (see (8)) in the general case. The criterion of Theorem 2 is especially convenient here. As we see, the closer the expansion $S_N(f)$ to the expansion of f by the basis $\{\alpha_n\}$, $n \in \Omega(N)$, the faster $|c_n/\omega_n|$ tends to 1 as $N \to \infty$.

Practical realization of the approximate Gegenbauer expansion in this way is connected with essential difficulties. First of all, the problem of choosing r (for a given λ and N) is rather complex. If the value r^{-2N} falls outside of computer capacity accumulation of errors take place. Besides, the function $(1 - 2xre^{2i\pi k/N} + r^2e^{4i\pi k/N})^{-\lambda}$ can have up to $(\lambda+1)$ extrema on the segment $[-1, 1]$. For $\lambda > 10$ this oscillation really influences on automatic integration. Nevertheless if we use system (19) instead of $\{g_n^\lambda\}$ $(n = 0, 1, ..., N-1)$ we avoid the sharp oscillation of g_n^λ if $n > 30$. Besides, elementary functions on the right-hand side of (17) are evaluated much faster and more stable than the Gegenbauer polynomials (for example, $C_{50}^{3.9}(x) = -2653.77 + ... + 1.26202 \times 10^{21}x^{46} - 2.32265 \times 10^{20}x^{48} + 2.00601 \times 10^{19}x^{50}$).

4.3 Numerical results

In our algorithm an empirical choice of r (depending on λ and N) was developed. In Table below some results of calculations of Gegenbauer coefficients for the rather smooth function $f(x) = e^{iax}$, $a = 5.231$, are presented. Exact values of these coefficients are known (see [11], vol. II, item 10.20). We have calculated them using the automatic integration package of MATHEMATICA 4.1 on the computer Pentium 3, 500 MHz, 128 RAM. The coefficients $u_n = (f, g_n^\lambda)$ $(n = 0, 1, ..., N-1)$ are calculated directly (with l_2-error denoted in Table by $DirErr$) and with the help of our scheme (with l_2-error denoted in Table by $NewErr$) (see above 3.1, scheme a)+b)). Numerical integration options "AccuracyGoal$- > 6$, PrecisionGoal$- > 5$" were used. The results (see Table) show that our method is rather stable in contrast to the traditional one. In addition note that for $\lambda = 1.9$ even for $N = 300$ the error of our method is a little bit smaller than requested: $DirErr = 4e + 95$, $NewErr = 2e - 4$. Application of any other options of MATHEMATICA code for automatic integration do not decrease $DirErr$ for $N > 40$ while the evaluation time increases sharply.

From this and some other experiments we conclude that our algorithm is rather efficient for $30 < N < 150$ and $-0.3 \leq \lambda \leq 9$. One can use the corresponding package from the MATHEMATICA site

http//library.wolfram.com/ databases/MathSource/4748/ .

Note that in the case of Chebyshev polynomials ($\lambda = 0, 1$) the classical wavelet theory is used and this package works superfast and stable. In this software we use only non-oscillation of the right-hand side of (17) though it is possible to use the other properties of this function. For instance, if $N \gg 1$ one can choose $r \simeq 1$ and use the functions $F(e^{2i\pi k/N})$ having local (with respect to the required precision) small supports. Because of some technical difficulties this property was not realized yet.

	$N- >$	25	35	45	55	100	200
$\lambda = -0.1$	DirErr (time)	1.3e-7 (4 sec)	7e-5 (5.5 sec)	6.5e-1 (7 sec)	5e+4 (9 sec)	2e+22 (21 sec)	7e+60 (63 sec)
	NewErr (time)	3e-8 (2 sec)	8e-8 (3.7 sec)	7.5e-8 (4.6 sec)	2e-7 (6 sec)	4e-7 (7.5 sec)	2.5e-6 (15 sec)
$\lambda = 1.9$	DirErr (time)	6e-10 (2 sec)	8.7e-7 (3.2 sec)	4e-3 (4 sec)	1.4e+1 (5 sec)	5e+19 (8.5 sec)	3e+57 (30 sec)
	NewErr (time)	1e-8 (0.5 sec)	7.8e-9 (0.8 sec)	3e-8 (1.1 sec)	5e-9 (1.3 sec)	7.6e-9 (2.5 sec)	1.3e-5 (5 sec)
$\lambda = 3.9$	DirErr (time)	1e-10 (2 sec)	4.5e-8 (3.5 sec)	1e-4 (4.4 sec)	1e+0 (5.2 sec)	1.5e+18 (9 sec)	2e+55 (24 sec)
	NewErr (time)	4e-10 (2.8 sec)	6.6e-10 (4.6 sec)	5.3e-10 (6.1 sec)	5.7e-10 (8 sec)	1.1e-6 (21 sec)	2e-3 (47 sec)

Table *The l_2-errors of calculation of Gegenbauer coefficients $\{(f, g_n^\lambda)\}$, $n = 0, 1, \ldots, N-1$, for the function $f(x) = e^{i\pi ax}$, $a = 5.231$.*

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