Introduction to Fourier Optics
Third Edition
Problem Solutions

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Preface

Doing problems is an essential part of the learning process for any scientific or technical subject. This is particularly true for subjects that are highly mathematical, as is the subject of *Introduction to Fourier Optics*. However, there are many different types of problems that one could imagine. Some involve straightforward substitution into equations that have been established in the text; such problems are useful in so far as they relate an abstract mathematical result to a real situation, with physical numbers that might be encountered in practice. Other problems may ask students to apply methods similar to those used in the text, but to apply them to a problem that is different in some significant aspect from the one they have already encountered. By far the best problems are those that leave the student feeling that he or she has learned something new from the exercise.

With the above in mind, I would like to mention some of my favorite problems from this text, with some indication as to why they are especially valuable:

- Problem 2-4 introduces the student to the idea that a sequence of two Fourier transforms, perhaps with different scaling factors, results in an “image” with magnification or demagnification.
- Problem 2-8, which explores the conditions under which a cosinusoidal object results in a cosinusoidal image, is highly instructive.
- Problem 2-14 introduces the student to the Wigner distribution, a valuable concept which they will encounter nowhere else in the book.
- Problem 3-6 shows how the diffraction integrals for monochromatic light can be generalized to apply for non-monochromatic but narrowband light.
- Problem 4-4 is has a particularly simple and satisfying proof.
- Problem 4-11 asks the student to derive an important property of diffraction gratings.
- Problem 4-12 introduces a very important and simple method for calculating the diffraction efficiency of a grating.
- Problem 4-15 analyzes a particularly interesting and important type of grating.
- Problem 4-16 also introduces an important idea.
- Problem 4-18 is an excellent exercise that increases understanding of the self-imaging phenomenon.
- Problem 5-5 increases understanding of the vignetting problem.
- Problem 5-6 introduces the student to an astigmatic processor.
- Problem 5-9 increases understanding of the paraxial approximation.

- Problem 5-14 introduces the student to the idea of a Fresnel zone plate and its effects on a transmitted wave.
- Problem 6-2 introduces the concepts of line spread functions and step responses.
- Problem 6-3 illustrates the effects on the OTF of a central obscuration.
- Problem 6-7, which asks the student to derive the optimum size of the pinhole in a pinhole camera, is one of my personal favorites.
- Problem 6-8 is very instructive.
- Problem 6-15 provides further introduction to step responses.
- Problem 6-17, while very simple, carries an important message.
- Problem 7-5 provides an interesting exposure to Jones calculus in solving a problem somewhat different than those treated in the text itself.
- Problem 7-6 introduces a different method for use of the magneto-optic spatial light modulator.
- Problem 7-7 is an interesting and very practical problem.
- Problem 8-1 is a simple introduction to the importance of a coherent background in coherent imaging.
- Problem 8-5 provides the student with the real alignment requirements in a typical coherent optical processing system.
- Problem 8-10 provides a system that uses both coherent and incoherent light, and is an excellent exercise for the student.
- Problem 8-11 is a good exercise pertinent to the VanderLugt filter.
- Problem 8-15 requires some ingenuity and is instructive.
- Problem 8-16 is an excellent exercise related to inverse filtering, and happens to be another of my favorites.
- Problem 9-5 is a good exercise for the student to test his/her knowledge of image locations and magnifications in holography.
- Problem 9-6, concerning the problem of X-ray holography, is highly instructive.
- Problem 9-7, while rather lengthy and involved, provides some real numbers regarding wavelength and angle sensitivity of thick holograms.
- Problem 9-10 is highly recommended.
- Problem 9-12 is a worthwhile exercise in understanding the sampling requirements for computer-generated holograms.
- Problem 10-3 yields a simple and intuitively satisfying result about the allowable time separation of the reference and signal pulses in spectral holography.
- Problem 10-6 provides an excellent exercise to help the student understand the wavelength mapping properties of an arrayed waveguide grating.

In closing, I would thank Mr. Daisuke Teresawa, who served as my teaching Assistant in 1995 and who created LaTeX versions of the solutions to several of the problems.

I would be grateful if instructors would report to me any errors or possible simplifications of these solutions, so that changes can be introduced in future versions of this document.

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Chapter 2

2-1. (a) We can determine the meaning of this symbol by examining its behavior under integration. Let $g(x,y)$ be a function that is continuous at the origin, and let $x' = ax, y' = by$. Initially assume $a > 0, b > 0$. Then:

$$\iint_{-\infty}^{\infty} g(x,y)\,\delta(ax,by)\,dxdy = \iint_{-\infty}^{\infty} g\left(\frac{x'}{a},\frac{y'}{b}\right)\,\delta(x',y')\,d\frac{x'}{a}d\frac{y'}{b} = \frac{1}{ab}g(0,0).$$

If either a,b or both are negative, then by properly treating the changes of the limits of integration, the right-hand side of the equation can be represented in all cases by $\frac{1}{|ab|}g(0,0)$. We conclude that $\delta(ax,by) = \frac{1}{|ab|}\delta(x,y)$, since both yield the same result under integration.

(b) Using the result above, we have

$$\begin{aligned}\text{comb}(ax)\,\text{comb}(by) &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(ax-n)\,\delta(by-n) \\ &= \frac{1}{|ab|}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(x-\frac{n}{a}\right)\,\delta\left(y-\frac{m}{b}\right).\end{aligned}$$

2-2. (a) $\mathcal{F}\{\text{rect}\,x\ \text{rect}\,y\} = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\exp\left[-j2\pi\left(f_X x + f_Y y\right)\right]\,dxdy.$

The integral separates in rectangular coordinates. Examine one of the two separated integrals:

$$\begin{aligned}\int_{-1/2}^{1/2}\exp\left(-j2\pi f_X x\right)\,dx &= \frac{1}{-j2\pi f_X}\left[\exp(-j2\pi f_X x)\right]_{-1/2}^{1/2} \\ = \frac{1}{-j2\pi f_X}\left[-2j\sin\left(\pi f_X\right)\right] &= \frac{\sin\left(\pi f_X\right)}{\pi f_X} = \text{sinc}(f_X).\end{aligned}$$

The integral with respect to y is carried out the same way. The total result is the product of the two sinc functions:

$$\mathcal{F}\{\text{rect}\,x\ \text{rect}\,y\} = \text{sinc}\,f_X\ \text{sinc}\,f_Y.$$

(b) $\mathcal{F}\{\Lambda(x)\ \Lambda(y)\} = \int_{-1}^{1}\int_{-1}^{1}(1-|x|)\,(1-|y|)\ \exp\left[-j2\pi\left(f_X x + f_Y y\right)\right]\,dxdy.$

The integral again separates in rectangular coordinates. It suffices to concentrate on one of these separated integrals. Use the convolution relation

$$\Lambda(x) = \text{rect}(x)\otimes\text{rect}(x)$$

and the convolution theorem to write

$$\mathcal{F}\{\Lambda(x)\} = \mathcal{F}\{\text{rect}(x) \otimes \text{rect}(y)\} = \text{sinc}^2(f_X).$$

Thus

$$\mathcal{F}\{\Lambda(x)\,\Lambda(y)\} = \text{sinc}^2(f_X)\,\text{sinc}^2(f_Y).$$

(c) Clearly the function $g(x, y) = 1$ can be represented as the limit of a sequence of well behaved functions, e.g.

$$g(x, y) = 1 = \lim_{N\to\infty} \exp\left(-\pi\frac{x^2 + y^2}{N^2}\right).$$

Then according to the philosophy of transforms-in-the-limit, we would say

$$\mathcal{F}\left\{1\right\} = \lim_{N\to\infty} \mathcal{F}\left\{\exp\left(-\pi\frac{x^2+y^2}{N^2}\right)\right\} = \lim_{N\to\infty} N^2\, \exp\left[-\pi N^2\left(f_X^2 + f_Y^2\right)\right].$$

For every function in the sequence on the right, the area is unity. In addition, the width of these functions grows smaller and smaller with increasing N. Hence this sequence approaches $\delta(f_X, f_Y)$.

(d) By separability, we need only prove

$$\mathcal{F}\{\text{sgn}(x)\} = \frac{1}{j\pi f_X}.$$

The Fourier transform of this function doesn't exist. We have to find a generalized Fourier transform. To do so, use the following sequence definition:

$$\text{sgn}(x) = \lim_{N\to\infty} g_N(x)$$

where

$$g_N(x) = \begin{cases} \exp\left(-x/N\right) & x > 0 \\ -\exp\left(x/N\right) & x < 0 \\ 0 & x = 0 \end{cases}.$$

We Fourier transform each of the $g_N(x)$ to produce a sequence of transforms, $G_N(f_X)$, where

$$\begin{aligned} G_N(f_X) &= \int_0^\infty \exp(-x/N)\,\exp(-j2\pi f_X x)\,dx - \int_{-\infty}^0 \exp(x/N)\,\exp(-j2\pi f_X x)\,dx \\ &= \int_0^\infty \exp\left[-\left(\frac{1}{N} + j2\pi f_X\right)x\right]\,dx - \int_{-\infty}^0 \exp\left[-\left(-\frac{1}{N} + j2\pi f_X\right)x\right]\,dx \\ &= \frac{1}{\frac{1}{N} + j2\pi f_X} - \frac{1}{\frac{1}{N} - j2\pi f_X} = \frac{-j4\pi f_X}{\left(\frac{1}{N}\right)^2 + (2\pi f_X)^2}. \end{aligned}$$

Now if $N \to \infty$, the right hand side of this equation approaches

$$G(f_X) = \lim_{N\to\infty} G_N(f_X) = \frac{1}{j\pi f_X}$$

as was to be proved. Hence

$$\mathcal{F}\left\{\text{sgn}(x)\,\text{sgn}(y)\right\} = \left(\frac{1}{j\pi f_X}\right)\left(\frac{1}{j\pi f_Y}\right).$$

2-3. (a) $\mathcal{F}\mathcal{F}\left\{g(x,y)\right\} = \iint\limits_{-\infty}^{\infty} df_X df_Y \left\{ \iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta)\, e^{-j2\pi(\xi f_X + \eta f_Y)} \right\} e^{-j2\pi(f_X x + f_Y y)}.$

Interchange the orders of integration, yielding

$$\iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta) \iint\limits_{-\infty}^{\infty} df_X df_Y \exp\left\{-j2\pi\left[(\xi+x)f_X + (\eta+y)f_Y\right]\right\}.$$

But the right-hand double integral is identically the same as $\delta(\xi+x,\eta+y)$, and the sifting property can be applied to the remaining double integral,

$$\iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta)\, \delta(\xi+x,\eta+y)\, d\xi d\eta = g(-x,-y).$$

The result for $\mathcal{F}^{-1}\mathcal{F}^{-1}\{g(x,y)\}$ is derived the same way with a change of sign in both exponentials.

(b) The simplest method of proof is to show that

$$\mathcal{F}^{-1}\left\{G(f_X,f_Y) \otimes H(f_X,f_Y)\right\} = g(x,y)\, h(x,y).$$

Remembering that the $\mathcal{F}^{-1}$ operator operates on the variables (f_X, f_Y),

$$\begin{aligned} & \mathcal{F}^{-1}\left\{ \iint\limits_{-\infty}^{\infty} G(\xi,\eta) H(f_X-\xi, f_Y-\eta) d\xi d\eta \right\} \\ = & \iint\limits_{-\infty}^{\infty} G(\xi,\eta) \mathcal{F}^{-1}\left\{H(f_X-\xi,f_Y-\eta)\right\} d\xi d\eta \\ = & \iint\limits_{-\infty}^{\infty} G(\xi,\eta)\, \exp[j2\pi(\xi x+\eta y)]\, d\xi d\eta\; h(x,y) \\ = & \; g(x,y)\, h(x,y) \end{aligned}$$

where the shift theorem for inverse transforms has been used.

(c) $\mathcal{F}\left\{\nabla^2 g(x,y)\right\} = \mathcal{F}\left\{\frac{\partial^2}{\partial x^2} g(x,y) + \frac{\partial^2}{\partial y^2} g(x,y)\right\}$. Now

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^2}{\partial x^2} g(x,y)\right\} &= \frac{\partial^2}{\partial x^2} \iint\limits_{-\infty}^{\infty} G(f_X,f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y \\ &= \iint\limits_{-\infty}^{\infty} G(f_X,f_Y)\, \frac{\partial^2}{\partial x^2} e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y \\ &= \iint\limits_{-\infty}^{\infty} \left(-4\pi^2 f_X^2\right) G(f_X,f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y. \end{aligned}$$

Similarly,

$$\mathcal{F}\left\{\frac{\partial^2}{\partial y^2} g(x,y)\right\} = \iint\limits_{-\infty}^{\infty} \left(-4\pi^2 f_Y^2\right) G(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y.$$

We conclude that

$$\mathcal{F}\left\{\nabla^2 g(x,y)\right\} = -4\pi^2 (f_X^2 + f_Y^2)\, G(f_X, f_Y) = -4\pi^2 (f_X^2 + f_Y^2)\, \mathcal{F}\left\{g(x,y)\right\}.$$

2-4. (a) Apply the two operators one after the other,

$$\begin{aligned}\mathcal{F}_B\left\{\mathcal{F}_A\left\{g(x,y)\right\}\right\} &= \frac{1}{ab} \iint\limits_{-\infty}^{\infty} df_X df_Y\, e^{-j\frac{2\pi}{b}(x f_X + y f_Y)} \\ &\quad \times \iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta)\, e^{-j\frac{2\pi}{a}(f_X \xi + f_Y \eta)} \\ &= \frac{1}{ab} \iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta) \iint\limits_{-\infty}^{\infty} df_X df_Y\, e^{-j2\pi\left[f_X\left(\frac{\xi}{a}+\frac{x}{b}\right)+f_Y\left(\frac{\eta}{a}+\frac{y}{b}\right)\right]}.\end{aligned}$$

The last double integral is $\delta\left(\frac{\xi}{a}+\frac{x}{b}, \frac{\eta}{a}+\frac{y}{b}\right)$. But this delta function can be simplified as follows:

$$\delta\left(\frac{\xi}{a}+\frac{x}{b}, \frac{\eta}{a}+\frac{y}{b}\right) = a^2\, \delta\left(\xi + \frac{a}{b}x, \eta + \frac{a}{b}y\right).$$

Substituting this expression and using the sifting property of the delta function, we obtain

$$\mathcal{F}_B\left\{\mathcal{F}_A\left\{g(x,y)\right\}\right\} = \frac{a}{b}\, g\left(-\frac{a}{b}x, -\frac{a}{b}y\right).$$

(b) Interpretation:

Reversal of the signs of the arguments reflects the function $g(x,y)$ about the origin. We say that $g(x,y)$ has been "inverted".

The multiplier $\frac{a}{b}$ preceding the arguments x and y results in either a stretch ($a < b$) or a contraction ($a > b$) of $g(x,y)$. These two cases can be referred to as a "magnification" or a "demagnification" of $g(x,y)$.

2-5. Note that since $G(f_X, f_Y) = \iint\limits_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi(f_X x + f_Y y)}\, dxdy$, we see that

$$G(0,0) = \iint\limits_{-\infty}^{\infty} g(x,y)\, dxdy.$$

Similarly, since $g(x,y) = \iint\limits_{-\infty}^{\infty} G(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y$, we have

$$g(0,0) = \iint\limits_{-\infty}^{\infty} G(f_X, f_Y)\, df_X df_Y.$$

Thus

$$\Delta_{XY} = \left| \frac{\iint\limits_{-\infty}^{\infty} g(x,y)\,dxdy}{g(0,0)} \right| = \left| \frac{G(0,0)}{g(0,0)} \right| = \left| \frac{G(0,0)}{\iint\limits_{-\infty}^{\infty} G(f_X, f_Y)\,df_X df_Y} \right| = \frac{1}{\Delta_{f_X f_Y}}.$$

Hence $\Delta_{xy}\Delta_{f_X f_Y} = 1$.

2-6. (a) $\mathcal{B}\{\delta(r-r_0)\} = 2\pi\int_0^\infty r\,\delta(r-r_0)\,J_0(2\pi r\rho)\,dr = 2\pi r_0 J_0(2\pi r_0\rho)$.

(b) $\mathcal{B}\{g_R(r)\} = 2\pi\int_a^1 rJ_0(2\pi r\rho)\,dr$. Use the identity

$$\int xJ_0(x)\,dx = xJ_1(x).$$

Change variables to $x = 2\pi r\rho$, from which it follows that $dx = 2\pi\rho dr$. Then

$$\mathcal{B}\{g_R(r)\} = \frac{1}{2\pi\rho^2}\int_{2\pi a\rho}^{2\pi\rho} xJ_0(x)dx = \frac{1}{2\pi\rho^2}\left[xJ_1(x)\right]_{2\pi a\rho}^{2\pi\rho} = \frac{J_1(2\pi\rho) - aJ_1(2\pi a\rho)}{\rho}.$$

(c) $\mathcal{B}\{g_R(ar)\} = 2\pi\int_0^\infty r\,g_R(ar)\,J_0(2\pi r\rho)\,dr = 2\pi\int_0^\infty \frac{ar}{a}\,g_R(ar)J_0\left(2\pi ar\frac{\rho}{a}\right)\,d\frac{ar}{a}$. Define a new variable of integration $\sigma = ar$. The limits do not change. We have

$$\mathcal{B}\{g_R(ar)\} = \frac{2\pi}{a^2}\int_0^\infty \sigma\,g_R(\sigma)\,J_0\left(2\pi\sigma\frac{\rho}{a}\right)\,d\sigma = \frac{1}{a^2}G_0\left(\frac{\rho}{a}\right).$$

(d) The function $\exp(-\pi r^2)$ is separable in both polar coordinates and rectangular coordinates, since

$$\exp(-\pi r^2) = \exp[-\pi(x^2+y^2)].$$

From the table of two dimensional Fourier transforms, we know that

$$\mathcal{F}\left\{\exp[-\pi(x^2+y^2)]\right\} = \exp\left[-\pi\left(f_X^2 + f_Y^2\right)\right].$$

Hence

$$\mathcal{B}\left\{\exp(\pi r^2)\right\} = \exp(\pi\rho^2).$$

2-7. (a) Given $g(r,\theta) = g_R(r)e^{jm\theta}$,

$$\begin{aligned}
\mathcal{F}\{g(r,\theta)\} &= \mathcal{F}\left\{g_R(r)\,e^{jm\theta}\right\} = \int_0^{2\pi} d\theta e^{jm\theta}\int_0^\infty dr\,rg_R(r)\,e^{-j2\pi r\rho(\cos\theta\cos\phi+\sin\theta\sin\phi)} \\
&= \int_0^{2\pi} d\theta e^{jm\theta}\int_0^\infty dr\,rg_R(r)\,e^{-j2\pi r\rho\cos(\phi-\theta)} \\
&= \int_0^{2\pi} d\theta e^{jm\theta}\int_0^\infty dr\,rg_R(r)\,e^{-j2\pi r\rho\sin(\frac{\pi}{2}+\theta-\phi)}.
\end{aligned}$$

Using the hint, we obtain

$$\int_0^{2\pi} d\theta e^{jm\theta}\int_0^\infty dr\,rg_R(r)\sum_{k=-\infty}^{\infty} J_k(2\pi r\rho)\,e^{-jk\left(\frac{\pi}{2}+\theta-\phi\right)}.$$

Note $\exp(-jk\pi/2) = (-j)^k$. Invert the orders of the two integrations, yielding

$$\sum_{k=-\infty}^{\infty} (-j)^k e^{jk\phi} \int_0^\infty dr\, r g_R(r) J_k(2\pi r\rho) \int_0^{2\pi} d\theta e^{j(m-k)\theta}.$$

The last integral is zero except when $k = m$, in which case it is 2π. It follows that

$$\mathcal{F}\left\{g_R(r)\, e^{jm\theta}\right\} = (-j)^m e^{jm\phi} \mathcal{H}_m \left\{g_R(r)\right\},$$

where

$$\mathcal{H}_m \left\{g_R(r)\right\} = 2\pi \int_0^\infty r g_R(r)\, J_m(2\pi r\rho)\, dr.$$

(b) An arbitrary function separable in polar coordinates, $g_R(r)\, g_\Theta(\theta)$, is periodic in θ. Therefore $g_\Theta(\theta)$ can be expanded in a Fourier series, yielding

$$g_R(r)\, g_\Theta(\theta) = \sum_{m=-\infty}^{\infty} c_m e^{jm\theta}\, g_R(r)$$

where the Fourier coefficients c_m are given by

$$c_m = \frac{1}{2\pi} \int_0^{2\pi} g_\Theta(\theta)\, e^{-jm\theta}\, d\theta.$$

It follows that

$$\mathcal{F}\left\{g_R(r) g_\Theta(\theta)\right\} = \sum_{m=-\infty}^{\infty} c_m \mathcal{F}\left\{e^{jm\theta}\, g_R(r)\right\}$$

The results of part (a) can now be applied, demonstrating that

$$\mathcal{F}\left\{g_R(r)\, g_\Theta(\theta)\right\} = \sum_{m=-\infty}^{\infty} c_m (-j)^m \exp(jm\phi)\, \mathcal{H}_m \left\{g_R(r)\right\}.$$

2-8. To avoid confusion, let's call the frequencies of the applied cosinusoidal signal $(\bar{f}_X, \bar{f}_Y)$. Note that the input can be expanded into a sum of two complex exponentials,

$$g(x,y) = \cos[2\pi(\bar{f}_X x + \bar{f}_Y y)] = \frac{1}{2}\exp[j2\pi(\bar{f}_X x + \bar{f}_Y y)] + \frac{1}{2}\exp[-j2\pi(\bar{f}_X x + \bar{f}_Y y)].$$

Now to have any hope of producing a cosine at the output, we had better insist that the system be *invariant*, for only then can we expect the exponential nature of the two input components to be preserved. For an invariant system, each complex-exponential input produces a complex-exponential output of the same frequency, but with a possible change of amplitude and phase, as determined by the transfer function. Remembering that the complex exponentials are eigenfunctions of linear, invariant systems, we write the output $v(x,y)$ as

$$v(x,y) = \frac{1}{2}H(\bar{f}_X, \bar{f}_Y)\, \exp[j2\pi(\bar{f}_X x + \bar{f}_Y y)] + \frac{1}{2}H(-\bar{f}_X, -\bar{f}_Y)\, \exp[-j2\pi(\bar{f}_X x + \bar{f}_Y y)],$$

where $H(f_X, f_Y)$ is the transfer function of the system, given by the Fourier transform of the impulse response $h(x,y)$. The transfer function can be written as the product of an amplitude function and a phase function,

$$H(f_X, f_Y) = A(f_X, f_Y)\, e^{j\phi(f_X, f_Y)},$$

where $A(f_X, f_Y) \geq 0$. Thus the output can be written

$$\begin{aligned} v(x,y) &= \frac{1}{2}A(\bar{f}_X,\bar{f}_Y)\,\exp[j2\pi(\bar{f}_X x+\bar{f}_Y y)+\phi(\bar{f}_X,\bar{f}_Y)] \\ &+ \frac{1}{2}A(-\bar{f}_X,-\bar{f}_Y)\,\exp[-j2\pi(\bar{f}_X x+\bar{f}_Y y)+\phi(-\bar{f}_X,-\bar{f}_Y)] \end{aligned}$$

Now we ask under what conditions can the above two exponential terms be combined to form a cosinusoidal output of frequency $(\bar{f}_X, \bar{f}_Y)$? The answer is that the following two conditions must be satisfied:

$$\begin{aligned} A(-\bar{f}_X,-\bar{f}_Y) &= A(\bar{f}_X,\bar{f}_Y) \\ \phi(-\bar{f}_X,-\bar{f}_Y) &= -\phi(\bar{f}_X,\bar{f}_Y), \end{aligned}$$

i.e. the magnitude of the transfer function must be even and the phase must be odd. These symmetry relations will be satisfied if and only if the impulse response of the system, $h(x, y)$, is *real-valued*. Thus, to summarize, the required conditions are that the system be linear and invariant, and that its impulse response be real-valued.

2-9. Consider a linear, invariant system with a circularly-symmetric impulse response $h(r)$, and a corresponding circularly-symmetric transfer function $H(\rho)$. First take the Fourier-Bessel transform of the input $J_0(2\pi\rho_0 r)$, which from Prob. 2-6(a) is

$$\mathcal{B}\{J_0(2\pi\rho_0 r)\} = \frac{1}{2\pi\rho_0}\delta(\rho-\rho_0).$$

The output from the system is found by multiplying the spectrum of the input by the transfer function. Thus the spectrum of the output, $V(\rho)$, must be given by

$$V(\rho) = H(\rho)\,\frac{1}{2\pi\rho_0}\delta(\rho-\rho_0) = \frac{H(\rho_0)}{2\pi\rho_0}\,\delta(\rho-\rho_0).$$

An inverse Fourier-Bessel transform can now be applied, using the same transform pair listed above, yielding and output

$$v(r) = H(\rho_0)\,J_0(2\pi\rho_0 r).$$

Clearly the function $J_0(2\pi\rho_0 r)$ is an eigenfunction of the system. The corresponding eigenvalue is $H(\rho_0)$.

2-10. Consider the Fourier transform operator as a system.

(a) The system is linear by virtue of the linearity theorem of Fourier transforms.

(b) The system will have a transfer function only if it is shift-invariant. It will be shift-invariant only if a shift of the input produces a simple shift in the output. However, we know from the shift theorem of Fourier analysis that a shift of the function to be transformed produces a multiplicative phase factor in the transform domain, but no shift of the transform. Therefore the Fourier transform operator is not shift-invariant, and the system can have no transfer function.

2-11. (a) By the convolution theorem,

$$P(f_X, f_Y) = G(f_X, f_Y)\,XY\mathrm{comb}(Xf_X)\,\mathrm{comb}(Yf_Y),$$

where we have used the similarity theorem and the fact that the Fourier transform of a comb function is another comb function. Further simplification results from the following relation:

$$\begin{aligned} XY\,\text{comb}(Xf_X)\,\text{comb}(Yf_Y) &= XY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(Xf_X-n, Yf_Y-m) \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta(f_X-\frac{n}{X}, f_Y-\frac{m}{Y}), \end{aligned}$$

where we have used the fact that $\delta(ax, by) = \frac{1}{|a,b|}\delta(x, y)$. We have assumed in the above that $X \geq 0, Y \geq 0$.

(b) The Fourier transform of the given $g(x, y)$ is found as follows:

$$\mathcal{F}\{g(x,y)\} = \frac{XY}{4}\,\text{sinc}\left(\frac{X}{2}f_X\right)\,\text{sinc}\left(\frac{Y}{2}f_Y\right),$$

where the similarity theorem has been used. The figure below shows sketches of $g(x, 0)$ and $p(x, 0)$ in this case.

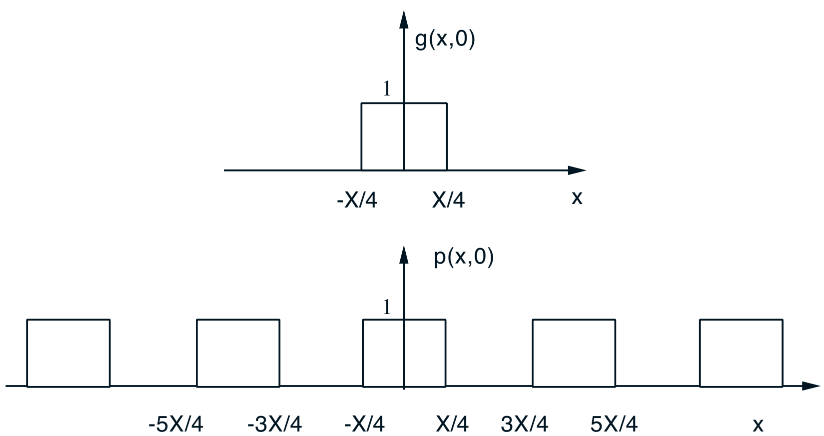

Figure 2-11:

2-12. For a function with no spectral components outside of a circle with radius B,

$$G(f_X, f_Y) = G_s(f_X, f_Y)\,\text{circ}\left(\frac{\rho}{B}\right),$$

where $G_s(f_X, f_Y)$ is the spectrum of the sampled function. By the convolution theorem, the equivalent expression in the space domain is

$$g(x,y) = \left[g(x,y)\,\text{comb}\left(\frac{x}{X}\right)\,\text{comb}\left(\frac{y}{Y}\right)\right] \otimes \mathcal{B}^{-1}\left\{\text{circ}\left(\frac{\rho}{B}\right)\right\}.$$

Recognizing that there is no difference between the forward and reverse transforms for circularly symmetric functions, and using the similarity theorem, we have

$$\mathcal{B}^{-1}\left\{\text{circ}\left(\frac{\rho}{B}\right)\right\} = B\frac{J_1(2\pi Br)}{r}.$$

Expanding the comb functions into sums of δ functions, we have

$$
\begin{aligned}
g(x,y) &= XY\sum_n\sum_m g(x,y)\,\delta(x-nX,y-mY)\otimes B\frac{J_1(2\pi Br)}{r}\\
&= XYB\sum_n\sum_m \iint_{-\infty}^{\infty} g(\xi,\eta)\,\delta(\xi-nX,\eta-nY)\,\frac{J_1\left(2\pi B\sqrt{(x-\xi)^2+(y-\eta)^2}\right)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\,d\xi d\eta\\
&= XYB\sum_n\sum_m g(nX,mY)\,\frac{J_1\left(2\pi B\sqrt{(x-nX)^2+(y-mY)^2}\right)}{\sqrt{(x-nX)^2+(y-mY)^2}}.
\end{aligned}
$$

By the same arguments used in the case of the rectangular band limitation, the maximum allowable sampling intervals without overlap of the spectral islands are $X=Y=\frac{1}{2B}$. With these values

$$
\begin{aligned}
g(x,y) &= \frac{1}{4B}\sum_n\sum_m g\left(\frac{n}{2B},\frac{m}{2B}\right)\,\frac{J_1\left(2\pi B\sqrt{(x-\frac{n}{2B})^2+(y-\frac{m}{2B})^2}\right)}{\sqrt{(x-\frac{n}{2B})^2+(y-\frac{m}{2B})^2}}\\
&= \sum_n\sum_m g\left(\frac{n}{2B},\frac{m}{2B}\right)\,\frac{\pi}{4}\left\{2\frac{J_1\left(2\pi B\sqrt{(x-\frac{n}{2B})^2+(y-\frac{m}{2B})^2}\right)}{2\pi B\sqrt{(x-\frac{n}{2B})^2+(y-\frac{m}{2B})^2}}\right\}.
\end{aligned}
$$

2-13. The object $U_o(x,y)$ has a band-unlimited spectrum, while the transfer function $H(f_X,f_Y)$ of the system is bandlimited to the region $|f_X|\le B_X, |f_Y|\le B_Y$. Because of the bandlimitation on H, it is possible to write

$$
H(f_X,f_Y)=H(f_X,f_Y)\,\mathrm{rect}\left(\frac{f_X}{2B_X}\right)\mathrm{rect}\left(\frac{f_Y}{2B_Y}\right).
$$

Since the imaging system is both linear and invariant, the image and object spectra, G_i and G_o, respectively, can be related by

$$
G_i(f_Y,f_Y)=H(f_X,f_Y)\,G_o(f_X,f_Y)=H(f_X,f_Y)\,\left[\mathrm{rect}\left(\frac{f_X}{2B_X}\right)\mathrm{rect}\left(\frac{f_Y}{2B_Y}\right)\,G_o(f_X,f_Y)\right].
$$

From this equation we can see directly that the output spectrum can be viewed as resulting from the application of a new fictitious object with spectrum

$$
G_o'(f_X,f_Y)=\mathrm{rect}\left(\frac{f_X}{2B_X}\right)\mathrm{rect}\left(\frac{f_Y}{2B_Y}\right)\,G_o(f_X,f_Y).
$$

In the space domain, the relation between the fictitious object and the actual object is

$$
\begin{aligned}
U_o'(x,y) &= U_o(x,y)\otimes 4B_XB_Y\,\mathrm{sinc}(2B_Xx)\mathrm{sinc}(2B_Yy)\\
&= 4B_XB_Y\iint_{-\infty}^{\infty} U_o(\xi,\eta)\,\mathrm{sinc}\left[2B_X(x-\xi)\right]\mathrm{sinc}\left[2B_Y(y-\eta)\right]d\xi d\eta.
\end{aligned}
$$

Since U_o' is bandlimited, it can be reconstructed from samples taken at the Nyquist rate, i.e. samples taken at coordinates $x_n=\frac{n}{2B_X}, y_m=\frac{m}{2B_Y}$. The sampled object which will yield U_o' after low pass filtering is given by

$$
\begin{aligned}
\hat{U}_o'(x,y) &= \mathrm{comb}(2B_Xx)\,\mathrm{comb}(2B_Yy)\,U_o'(x,y)\\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} U_o'\left(\frac{n}{2B_Xx},\frac{m}{2B_Yy}\right)\,\delta\left(x-\frac{n}{2B_X},y-\frac{m}{2B_Y}\right).
\end{aligned}
$$

Substituting the expression derived above for U_o',

$$\begin{aligned}\hat{U}_o'(x,y) &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\iint_{-\infty}^{\infty} U_o(x,y)\,\mathrm{sinc}(n-2B_X\xi)\,\mathrm{sinc}(m-2B_Y\eta)\,d\xi d\eta\right] \\ &\times\ \delta\left(x-\frac{n}{2B_X}, m-\frac{m}{2B_Y}\right).\end{aligned}$$

This array of point sources will yield the same image as the original object $U_o(x,y)$.

2-14. (a) Substituting the infinite-length chirp function into the definition of the Wigner distribution function, we obtain

$$\begin{aligned}W(f,x) &= \int_{-\infty}^{\infty} e^{j\pi\beta(x+\xi/2)^2}\, e^{-j\pi\beta(x-\xi/2)^2}\, e^{-j2\pi f\xi}\, d\xi \\ &= \int_{-\infty}^{\infty} e^{-j2\pi(f-\beta x)\xi}\, d\xi = \delta(f-\beta x).\end{aligned}$$

(b) For the finite-length chirp function we obtain

$$\begin{aligned}W(f,x) &= \int_{\infty}^{\infty} e^{j\pi\beta(x+\xi/2)^2}\mathrm{rect}\left(\frac{x+\xi/2}{2L}\right) e^{-j\pi\beta(x-\xi/2)^2}\mathrm{rect}\left(\frac{x-\xi/2}{2L}\right) e^{-j2\pi f\xi} d\xi \\ &= \int_{\infty}^{\infty} e^{-j2\pi(f-\beta x)\xi}\,\mathrm{rect}\left(\frac{x+\xi/2}{2L}\right)\mathrm{rect}\left(\frac{x-\xi/2}{2L}\right) d\xi\end{aligned}$$

Now note that

$$\mathrm{rect}\left(\frac{x+\xi/2}{2L}\right)\mathrm{rect}\left(\frac{x-\xi/2}{2L}\right) = \mathrm{rect}\left(\frac{\xi}{4(L-|x|)}\right),$$

as can be verified by sketching the two rectangle functions and determining their region of overlap. Thus the Wigner distribution will be given by a Fourier transform of the rectangle on the right, evaluated at frequency $f-\beta x$. Using the similarity theorem, we have

$$W(f,x) = [4(L-|x|)]\,\mathrm{sinc}\left[(4(L-|x|))(f-\beta x)\right]$$

for $|x| \le L$ and 0 otherwise.

(c) The two requested figures are shown below.

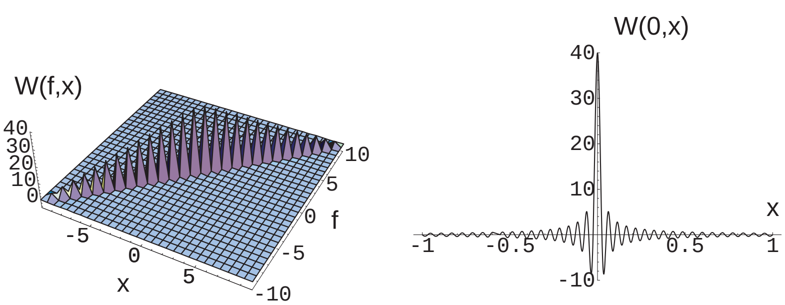

Figure 2-14:

Chapter 3

3-1. Begin with Maxwell's equations,

$$\begin{aligned} \nabla \times \vec{\mathcal{E}} &= -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t} \qquad & \nabla \times \vec{\mathcal{H}} = \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \\ \nabla \cdot \epsilon \vec{\mathcal{E}} &= 0 & \nabla \cdot \mu \vec{\mathcal{H}} = 0, \end{aligned}$$

where ϵ is dependent on spatial coordinates but independent of polarization, and μ is independent of both spatial coordinates and polarization. Eliminate $\vec{\mathcal{H}}$ on the left by taking the curl of the first equation for $\vec{\mathcal{E}}$, and using the vector identity $\nabla \times (\nabla \times \vec{\mathcal{E}}) = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \nabla^2 \vec{\mathcal{E}}$, giving

$$\nabla \times (\nabla \times \vec{\mathcal{E}}) = \nabla(\nabla \cdot \vec{\mathcal{E}}) - \nabla^2 \vec{\mathcal{E}} = -\mu \frac{\partial (\nabla \times \vec{\mathcal{H}})}{\partial t}.$$

Use the first equation for $\vec{\mathcal{H}}$ to reduce this equation to

$$\nabla^2 \vec{\mathcal{E}} - \nabla(\nabla \cdot \vec{\mathcal{E}}) - \mu\epsilon \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0.$$

The second Maxwell equation for $\vec{\mathcal{E}}$ can be expanded as follows:

$$\nabla \cdot (\epsilon \vec{\mathcal{E}}) = \epsilon(\nabla \cdot \vec{\mathcal{E}}) + \vec{\mathcal{E}} \cdot \nabla \epsilon = 0.$$

It follows that

$$\nabla \cdot \vec{\mathcal{E}} = -\vec{\mathcal{E}} \cdot \frac{\nabla \epsilon}{\epsilon} = -\vec{\mathcal{E}} \cdot \nabla \ln \epsilon.$$

Hence the wave equation becomes

$$\nabla^2 \vec{\mathcal{E}} + \nabla(\vec{\mathcal{E}} \cdot \nabla \ln \epsilon) - \mu\epsilon \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0.$$

Using the definitions given in Eqs. (3-5) and (3-6), this equation can be rewritten

$$\nabla^2 \vec{\mathcal{E}} + 2\nabla(\vec{\mathcal{E}} \cdot \nabla \ln n) - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0.$$

3-2. The Sommerfeld radiation condition is

$$\lim_{R \to \infty} R \left(\frac{\partial U}{\partial n} - jkU \right) = 0.$$

A diverging spherical wave can be written in the form

$$U = \frac{e^{jkr}}{r}.$$

For a diverging spherical wave ,

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial r} = jk\frac{e^{jkr}}{r} - \frac{e^{jkr}}{r^2}.$$

Letting $r \to R$ and inserting this expression in the left-hand side of the Sommerfeld radiation condition, we obtain

$$\lim_{R\to\infty} R\left(jk\frac{e^{jkR}}{R} - \frac{e^{jkR}}{R^2} - jk\frac{e^{jkR}}{R}\right) = \lim_{R\to\infty} \frac{e^{jkR}}{R}.$$

As $R \to \infty$, the result approaches zero, and thus the Sommerfeld radiation condition is satisfied.

3-3. We begin by stating Eq. (3-26),

$$U(P_0) = \frac{1}{4\pi} \iint_{\Sigma} \frac{\exp(jkr_{01})}{r_{01}} \left[\frac{\partial U}{\partial n} - jkU\cos(\vec{n}, \vec{r}_{01})\right] ds.$$

Assume that the aperture is illuminated with the diverging spherical wave

$$U(P_1) = \frac{A\exp(jkr_{21})}{r_{21}}.$$

Now at P_1,

$$\frac{\partial U}{\partial n} = A\cos(\hat{n}, \hat{r}_{21})\frac{e^{jkr_{21}}}{r_{21}}\left(jk - \frac{1}{r_{21}}\right).$$

If $jk \gg 1/r_{21}$, then

$$\frac{\partial U}{\partial n} \approx jkA\cos(\hat{n}, \hat{r}_{21})\frac{e^{jkr_{21}}}{r_{21}}.$$

Substitute this expression in Eq. (3-26),

$$\begin{aligned} U(P_0) &= \frac{1}{4\pi}\iint_{\Sigma} ds\frac{e^{jkr_{01}}}{r_{01}}\left[jkA\cos(\hat{n}, \hat{r}_{21})\frac{e^{jkr_{21}}}{r_{21}} - jkA\cos(\hat{n}, \hat{r}_{01})\frac{e^{jkr_{21}}}{r_{21}}\right] \\ &= \frac{jkA}{4\pi}\iint_{\Sigma} ds\frac{e^{jk(r_{21}+r_{01})}}{r_{21}r_{01}}[\cos(\hat{n}, \hat{r}_{21}) - \cos(\hat{n}, \hat{r}_{21})]. \end{aligned}$$

But $\frac{jkA}{4\pi} = -\frac{A}{2j\lambda}$, so

$$U(P_0) = \frac{A}{j\lambda}\iint_{\Sigma} \frac{\exp[jk(r_{21}+r_{01})]}{r_{21}r_{01}}\left[\frac{\cos(\vec{n}, \vec{r}_{01}) - \cos(\vec{n}, \vec{r}_{21})}{2}\right] ds,$$

which is Eq. (3-27).

3-4. We begin with an expression for G_+,

$$G_+(P_1) = \frac{e^{jkr_{01}}}{r_{01}} + \frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}}.$$

Taking the normal derivative of this function,

$$\frac{\partial G_+}{\partial n} = \cos(\hat{n}, \hat{r}_{01}) \left[jk - \frac{1}{r_{01}} \right] \frac{e^{jkr_{01}}}{r_{01}} + \cos(\hat{n}, \hat{\tilde{r}}_{01}) \left[jk - \frac{1}{\tilde{r}_{01}} \right] \frac{e^{jk\tilde{r}_{01}}}{\tilde{r}_{01}}.$$

Note that P_0 and $\tilde{P}_0$ are mirror images of each other, so $r_{01} = \tilde{r}_{01}$, and $\cos(\hat{n}, \hat{\tilde{r}}_{01}) = -\cos(\hat{n}, \hat{r}_{01})$. Substituting these facts into the above equation, we see that across the screen and aperture

$$\frac{\partial G_+}{\partial n} = 0.$$

3-5. Using Eq. (3-63) we have the following:

(a) For a circular aperture of diameter d:

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) = \mathcal{B}\left\{\text{circ}\left(\frac{2r}{d}\right)\right\}\Bigg|_{\substack{f_X=\alpha/\lambda \\ f_Y=\beta/\lambda}}.$$

Using the similarity theorem for Fourier-Bessel transforms (Eq. (2-34)) and the Fourier-Bessel transform pair of Eq. (2-35),

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) = \frac{d^2}{4} \frac{J_1\left(\frac{2\pi\rho d}{2}\right)}{\frac{d\rho}{2}} = \frac{d}{2} \frac{J_1(\pi\rho d)}{\rho}.$$

Finally, note that $\rho = \sqrt{f_X^2 + f_Y^2} = \sqrt{\left(\frac{\alpha}{\lambda}\right)^2 + \left(\frac{\beta}{\lambda}\right)^2}$ yielding

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) = \frac{d^2}{4} \frac{J_1\left(\frac{2\pi\rho d}{2}\right)}{\frac{d\rho}{2}} = \frac{d}{2} \frac{J_1\left(\pi\sqrt{\left(\frac{\alpha}{\lambda}\right)^2 + \left(\frac{\beta}{\lambda}\right)^2}\, d\right)}{\sqrt{\left(\frac{\alpha}{\lambda}\right)^2 + \left(\frac{\beta}{\lambda}\right)^2}}.$$

(b) A circular opaque disk of diameter d can be modeled by the following amplitude transmittance function:

$$t_A(x, y) = 1 - \text{circ}\left(\frac{2r}{d}\right).$$

From the linearity theorem of Fourier analysis it follows that the angular spectrum of this structure is

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) = \delta\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right) - \frac{d}{2} \frac{J_1\left(\pi\sqrt{\left(\frac{\alpha}{\lambda}\right)^2 + \left(\frac{\beta}{\lambda}\right)^2}\, d\right)}{\sqrt{\left(\frac{\alpha}{\lambda}\right)^2 + \left(\frac{\beta}{\lambda}\right)^2}}.$$

3-6. We start with the equation after Eq. (3-56),

$$u_-(P_0,t) = \iint\limits_\Sigma \frac{\cos(\vec{n},\vec{r}_{01})}{2\pi v r_{01}} \int_{-\infty}^{\infty} -j2\pi\nu' U(P_1,-\nu') e^{-j2\pi\nu'\left(t-\frac{r_{01}}{v}\right)}\, d\nu'\, ds.0$$

Since $u_-(P,t)$ has center frequency $-\bar{\nu}$ and bandwidth $\Delta\nu$, the inner integral is non-zero only for ν' in the range $(\bar{\nu}-\Delta\nu/2, \bar{\nu}+\Delta\nu/2)$. The first ν' in the equation varies by only a small fractional amount if $\Delta\nu \ll \bar{\nu}$; it can be replaced by $\bar{\nu}$ and taken out of the integral. Also, $\exp[j2\pi\nu' r_{01}/v] \rightarrow \exp[j2\pi\bar{\nu} r_{01}/v]$, provided $\Delta\nu \gg r_{01}/v$. Thus:

$$u_-(P_0,t) = \frac{1}{j\bar{\lambda}} \iint\limits_\Sigma \frac{\cos(\vec{n},\vec{r}_{01})}{r_{01}} \exp[j\bar{k}r_{01}] \int_{-\infty}^{\infty} U(P_1,-\nu') e^{-j2\pi\nu' t}\, d\nu'\, ds$$

Defining $u_-(P_1,t)=0$ for P_1 outside Σ,

$$u_-(P_0,t) = \frac{1}{j\bar{\lambda}} \iint\limits_{-\infty}^{\infty} u_-(P_1,t) \frac{\exp[j\bar{k}r_{01}]}{r_{01}} \cos(\vec{n},\vec{r}_{01})\, ds$$

(Note: $\bar{\lambda} = v/\bar{\nu}$, $\bar{k} = 2\pi/\bar{\lambda}$.)

3-7. (a) Substituting $U(x,y,z) \approx A(x,y,z)e^{jkz}$ into the Helmholtz equation $(\nabla^2 + k^2)U = 0$,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right] A(x,y,z)e^{jkz} = 0.$$

Then,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] Ae^{jkz} + \frac{\partial}{\partial z}\left[\frac{\partial A}{\partial z}e^{jkz} + jkAe^{jkz}\right] + k^2Ae^{jkz} = 0$$

$$\nabla_t^2 Ae^{jkz} + \frac{\partial^2 A}{\partial z^2}e^{jkz} + 2jk\frac{\partial A}{\partial z}e^{jkz} + (jk)^2Ae^{jkz} + k^2Ae^{jkz} = 0.$$

Dividing by e^{jkz} and simplifying,

$$\nabla_t^2 A + j2k\frac{\partial}{\partial z}A + \frac{\partial^2}{\partial z^2}A = 0.$$

The “slowly varying” approximation for A implies that:

$$\frac{\partial^2}{\partial z^2}A \ll j2k\frac{\partial A}{\partial z}$$

leaving,

$$\nabla_t^2 A + j2k\frac{\partial A}{\partial z} = 0.$$

(b) We first evaluate a number of different derivatives:

$$\begin{aligned}
A(x,y,z) &= \frac{A_1}{q} e^{jk\frac{x^2+y^2}{2q(z)}} \\
\frac{\partial}{\partial z} A(x,y,z) &= -\frac{A_1}{q^2}\frac{dq}{dz} e^{jk\frac{x^2+y^2}{2q}} - \frac{A_1}{q}\left(jk\frac{x^2+y^2}{2q^2}\right)\frac{dq}{dz} e^{jk\frac{x^2+y^2}{2q}} \\
&= -\left(\frac{1}{q} + jk\frac{x^2+y^2}{2q^2}\right)\frac{dq}{dz} A(x,y,z) \\
\frac{\partial}{\partial x} A(x,y,z) &= jk\frac{xA_1}{q^2} e^{jk\frac{x^2+y^2}{2q}} \\
\frac{\partial^2}{\partial x^2} A(x,y,z) &= jk\frac{A_1}{q^2} e^{jk\frac{x^2+y^2}{2q}} + \left(jk\frac{x}{q}\right)^2 \frac{A_1}{q} e^{jk\frac{x^2+y^2}{2q}} \\
&= \left(jk\frac{1}{q} - k^2\frac{x^2}{q^2}\right) A(x,y,z)
\end{aligned}$$

Similarly,

$$\frac{\partial^2}{\partial y^2} A(x,y,z) = \left(jk\frac{1}{q} - k^2\frac{y^2}{q^2}\right) A(x,y,z).$$

Now substitute the partial derivatives of A into the paraxial Helmholtz equation. Noting that dq/dz is equal to 1,

$$\begin{aligned}
\nabla_t^2 A + j2k\frac{\partial A}{\partial z} &= \left(2jk\frac{1}{q} - k^2\frac{x^2+y^2}{q^2} - 2jk\frac{1}{q}\frac{dq}{dz} + k^2\frac{x^2+y^2}{q^2}\frac{dq}{dz}\right) A \\
&= 0.
\end{aligned}$$

(c) Substituting the given expression into the result from part (b),

$$\begin{aligned}
A &= A_1\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\exp\left[\frac{jk}{2}(x^2+y^2)\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\right] \\
&= A_1\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\exp\left[-\frac{\rho^2}{W^2}\right]\exp\left[jk\frac{\rho^2}{2R}\right] \\
U &= A\exp[jkz] \\
&= A_1\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\exp\left[-\frac{\rho^2}{W^2}\right]\exp\left[jkz + jk\frac{\rho^2}{2R}\right] \\
&= A_0\frac{W_0}{W(z)}\exp\left[-\frac{\rho^2}{W^2}\right]\exp\left[jkz + jk\frac{\rho^2}{2R} + j\theta(z)\right]
\end{aligned}$$

where:

$$\begin{aligned}
\frac{A_0 W_0}{W} &= A_1\left[\left(\frac{1}{R}\right)^2 + \left(\frac{\lambda}{\pi W^2}\right)^2\right]^{1/2} \\
\theta(z) &= \tan^{-1}\left(\frac{\lambda R}{\pi W^2}\right).
\end{aligned}$$

To show that W_0 is independent of z, we differentiate W_0^2 with respect to z and show that it equals zero:

$$
\begin{aligned}
W_0^2 &= \left(\frac{A_1}{A_0}\right)^2 \left[\left(\frac{W}{R}\right)^2 + \left(\frac{\lambda}{\pi W}\right)^2\right] \\
\frac{d(W_0^2)}{dz} &= \left(\frac{A_1}{A_0}\right)^2 \left(\frac{2WR^2W' - 2RW^2R'}{R^4} - \frac{2\lambda^2 W'}{\pi^2 W^3}\right) \\
&= 2\left(\frac{A_1}{A_0}\right)^2 \frac{\pi^2 W^4 R W' - \pi^2 W^5 R' - \lambda^2 R^3 W'}{\pi^2 W^3 R^3},
\end{aligned}
$$

where the prime sign designates a derivative with respect to z. Now, using the condition $dq/dz = 1$, we can express R' and W' in terms of R and W:

$$
\begin{aligned}
\frac{d(1/q)}{dz} &= \frac{-1}{q^2}\frac{dq}{dz} = \frac{-1}{q^2} \\
\frac{d}{dz}\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right) &= -\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)^2 \\
\frac{-R'}{R^2} - 2j\frac{\lambda W'}{\pi W^3} &= \frac{-1}{R^2} + \frac{\lambda^2}{\pi^2 W^4} - 2j\frac{\lambda}{\pi R W^2}.
\end{aligned}
$$

Solving for both the real and the complex parts of the equation, we get

$$
\begin{aligned}
R' &= 1 - \frac{\lambda^2 R^2}{\pi^2 W^4} \\
W' &= \frac{W}{R}.
\end{aligned}
$$

Substituting,

$$
\begin{aligned}
\frac{d(W_0^2)}{dz} &= 2\left(\frac{A_1}{A_0}\right)^2 \frac{1}{\pi^2 W^3 R^3}\left[\pi^2 W^4 R\frac{W}{R} - \pi^2 W^5\left(1 - \frac{\lambda^2 R^2}{\pi^2 W^4}\right) - \lambda^2 R^3 \frac{W}{R}\right] \\
&= 2\left(\frac{A_1}{A_0}\right)^2 \frac{1}{\pi^2 W^3 R^3}\left(\pi^2 W^5 - \pi^2 W^5 + \lambda^2 R^2 W - \lambda^2 R^2 W\right) \\
&= 0.
\end{aligned}
$$

Chapter 4

4-1. (a) According to Eq. (4-21):

$$\mathcal{F}\left\{\frac{1}{j\lambda z}e^{j\frac{\pi}{\lambda z}(x^2+y^2)}\right\} \quad = \quad e^{-j\pi\lambda z\left(f_X^2+f_Y^2\right)}.$$

Using the area (or volume) property of the Fourier transform, the infinite integral of the function in the (x,y) domain is equal to the Fourier transform of the function evaluated at ($f_X = 0, f_Y = 0$). Thus,

$$\iint\limits_{-\infty}^{\infty} \frac{1}{j\lambda z}e^{j\frac{\pi}{\lambda z}(x^2+y^2)}\,dx\,dy \quad = \quad e^{-j\pi\lambda z\left(f_X^2+f_Y^2\right)}\Big|_{f_X=f_Y=0} = 1.$$

(b) Expanding the exponential,

$$\frac{1}{j\lambda z}e^{j\frac{\pi}{\lambda z}(x^2+y^2)} \quad = \quad \frac{1}{j\lambda z}\cos\left[j\frac{\pi}{\lambda z}(x^2+y^2)\right] + \frac{j}{j\lambda z}\sin\left[j\frac{\pi}{\lambda z}(x^2+y^2)\right]$$

The volume is one, so the volume under the imaginary part $\frac{-1}{\lambda z}\cos\left[\frac{\pi}{\lambda z}(x^2+y^2)\right]$ must be zero and all the volume must come from the real sine term.

4-2. Remember, if we start on the left and propagate a distance z to the right, there is a phase delay of kz radians incurred, which is represented by multiplication by $\exp(+jkz)$ since the phasors rotate counterclockwise.

(a) We first find the exact phase distribution:

$$\phi_{exact} \quad = \quad kz_0\sqrt{1+\frac{x^2+y^2}{z_0^2}}.$$

(b) Next we find the approximate phase distribution:

$$\phi_{approx} \quad = \quad kz_0 + \frac{k}{2z_0}(x^2+y^2).$$

(c) The phase error is

$$\begin{aligned}\Delta\phi &= \phi_{approx} - \phi_{exact} \\ &= kz_0 + \frac{k}{2z_0}(x^2+y^2) - kz_0\sqrt{1+\frac{x^2+y^2}{z_0^2}}.\end{aligned}$$

Now, we see that in binomial approximation, $(1+b)^{1/2} \leq 1 + b/2$ by noting that

$$LHS^2 = 1 + b \leq RHS^2 = 1 + b + b^2.$$

Hence, phase error ≥ 0, or $\phi_{approx} \geq \phi_{exact}$. Since ϕ_{exact} and ϕ_{approx} both correspond to phase *delay*, it follows that the approximate phase *lags* behind the exact phase. (In other words, since total phase $= -j\omega t + \phi$, more time is needed for the approximate phase to reach the same total phase as the exact phase.)

4-3. This time we can imagine propagating backwards (to the right) from the focus point to the sphere or parabolic surface, as if time were reversed. If we must go backwards distance z, then in effect the phase on the left leads the phase at the focal point by kz, as represented by a multiplicative factor $\exp(-jkz)$.

(a) Again we first find the exact phase distribution:

$$\phi_{exact} = -kz_0\sqrt{1 + \frac{x^2+y^2}{z_0^2}}.$$

(b) Next we find the approximate phase distribution:

$$\phi_{approx} = -kz_0 - \frac{k}{2z_0}(x^2+y^2).$$

(c) The phase error is

$$\begin{aligned} \Delta\phi &= \phi_{approx} - \phi_{exact} \\ &= -kz_0 - \frac{k}{2z_0}(x^2+y^2) + kz_0\sqrt{1 + \frac{x^2+y^2}{z_0^2}}. \end{aligned}$$

Now, we see that in binomial approximation, $(1+b)^{1/2} \leq 1 + b/2$ by noting that

$$LHS^2 = 1 + b \leq RHS^2 = 1 + b + b^2$$

Hence, phase error this time is ≤ 0, or $\phi_{approx} \leq \phi_{exact}$. Since ϕ_{exact} and ϕ_{approx} both correspond to phase *delay*, it follows that the approximate phase *leads* the exact phase. (In other words, since total phase $= -j\omega t + \phi$, less time is needed for the approximate phase to reach the same total phase as the exact phase.)

4-4. Over any distance z_k, Fresnel propagation can be described by the transfer function

$$H(f_X, f_Y; z_k) = e^{jkz_k}\, e^{-j\pi\lambda z_k\left(f_X^2+f_Y^2\right)}.$$

Propagation over several distances $z_1, z_2, \cdots, z_n$ can be represented by multiplication of the successive transfer functions,

$$H(f_X, f_Y; z_1 + z_2 + \cdots + z_n) = \prod_{k=1}^{n} H(f_X, f_Y; z_k)$$

Performing the product,

$$H(f_X, f_Y; z_1 + z_2 + \cdots + z_n) = e^{jk(z_1+z_2+\cdots+z_n)}\, e^{-j\pi\lambda(z_1+z_2+\cdots+z_n)\left(f_X^2+f_Y^2\right)}$$

Clearly, since $z = z_1 + z_2 + \cdots + z_n$, propagation over distance z is equivalent to propagation over the sum of the distances $z_1, z_2, \cdots, z_n$.

4-5. We have seen in Section 4.2.2 that the majority of the contribution to the convolution integral describing Fresnel diffraction comes from a square with sides of length $4\sqrt{\lambda z}$, centered on the point $(\xi = x, \eta = y)$. For a slit aperture (width $2w$), the boundary between the transition region and the dark region occurs when this square region lies entirely behind the opaque portion of the aperture. The figure below illustrates the geometries for both edges of the transition region.

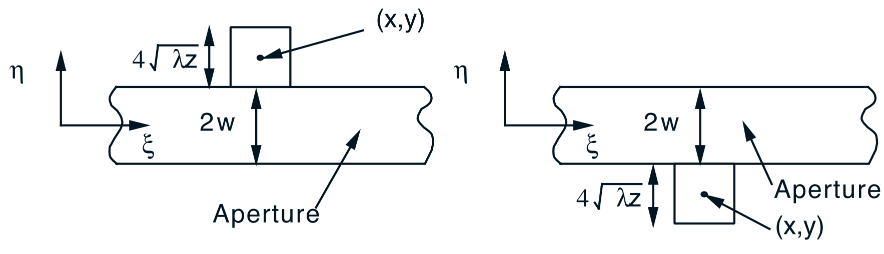

Figure 4-5:

The illustration on the left $(x > 0)$ defines an equation

$$(x - w) = 2\sqrt{\lambda z},$$

while that on the right $(x < 0)$ defines an equation

$$(x + w) = -2\sqrt{\lambda z}.$$

When these two equations are squared on the left and the right, we obtain the two parabolas of interest,

$$\begin{aligned}(w - x)^2 &= 4\lambda z \\ (w + x)^2 &= 4\lambda z.\end{aligned}$$

4-6. For converging illumination of the aperture, we see from Prob. 4-3 that:

$$\begin{aligned}\text{phase error} &= \phi_{approx} - \phi_{exact} \\ &= -kz_0 - \frac{k}{2z_0}(\xi^2 + \eta^2) + kz_0\sqrt{1 + \frac{\xi^2 + \eta^2}{z_0^2}} \\ &\leq 0,\end{aligned}$$

where z_0 is the distance from the point source to the aperture plane.

For the Fresnel approximation, assuming an observation point (x, y) at distance z from the aperture,

$$\begin{aligned}\text{phase error} &= \phi_{approx} - \phi_{exact} \\ &= \left\{kz + \frac{k}{2z}\left[(x-\xi)^2 + (y-\eta)^2\right]\right\} - \left\{kz\sqrt{1 + \frac{(x-\xi)^2 + (y-\eta)^2}{z^2}}\right\} \\ &\geq 0,\end{aligned}$$

since $(1 + b)^{1/2} \leq 1 + b/2$. Hence, the phase error due to the quadratic approximation on the illumination and the phase error due to Fresnel approximation have opposite signs and at least partially cancel. Exact cancellation occurs when $z = z_0$ and $(x = 0, y = 0)$, i.e. for the particular point towards which the spherical illumination is converging.

4-7. (a) The amplitude transmittance of this aperture can be written

$$t_A(\xi,\eta) = \text{rect}\left(\frac{x}{X}\right)\left[\text{rect}\left(\frac{y-\Delta/2}{Y}\right) + \text{rect}\left(\frac{y+\Delta/2}{Y}\right)\right].$$

The Fourier transform of this function is

$$\mathcal{F}\{t_A(\xi,\eta)\} = 2XY\,\text{sinc}(Xf_X)\,\text{sinc}(Yf_Y)\,\cos(\pi\Delta f_Y).$$

The Fraunhofer diffraction pattern is therefore

$$I(x,y) = \left(\frac{4XY}{\lambda z}\right)^2 \text{sinc}^2\left(\frac{Xx}{\lambda z}\right)\,\text{sinc}^2\left(\frac{Yy}{\lambda z}\right)\,\cos^2\left(\frac{\pi\Delta y}{\lambda z}\right).$$

(b) The required sketch is shown below.

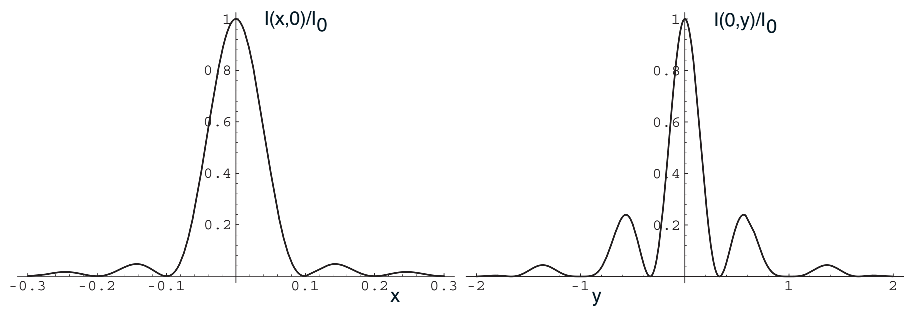

Figure 4-7:

4-8. (a) The amplitude transmittance function is separable and each factor can be considered separately; i.e. $t_A(\xi,\eta) = t_X(\xi)\,t_Y(\eta)$, where

$$\begin{aligned} t_X(\xi) &= \text{rect}\left(\frac{\xi}{X}\right)\otimes\delta(\xi) = \text{rect}\left(\frac{\xi}{X}\right) \\ t_Y(\eta) &= \left[\text{rect}\left(\frac{\eta}{Y}\right)\otimes\frac{1}{\Delta}\text{comb}\left(\frac{y}{\Delta}\right)\right]\text{rect}\left(\frac{\eta}{N\Delta}\right). \end{aligned}$$

The behavior of $t_X(\xi)$ is quite clear. The behavior of $t_Y(\eta)$ requires more thought. Since $\frac{1}{\Delta}\text{comb}\left(\frac{\eta}{\Delta}\right) = \sum_m \delta(\eta - m\Delta)$, we have

$$t_Y(\eta) = \left[\text{rect}\left(\frac{\eta}{Y}\right)\otimes\sum_m \delta(\eta - m\Delta)\right]\text{rect}\left(\frac{\eta}{N\Delta}\right).$$

Since $\Delta > Y$, the delta functions are more widely spaced that the width of the rectangles. The fact that N is odd means that the outer rectangle function subtends a symmetrical pattern of rectangle functions, i.e. with the same number of smaller rectangles above and below the small rectangle centered on the origin. The structure of $t_A(\xi,\eta)$ is illustrated in the figure above for $N = 5$.

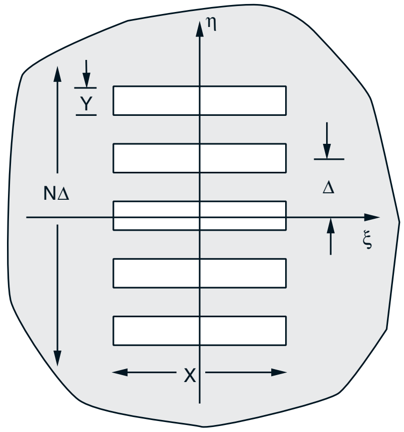

Figure 4-8:

(b) To find the Fraunhofer diffraction pattern, we must Fourier transform the amplitude transmittance and evaluate the scaled transform at properly scaled frequencies. Since the amplitude transmittance is separable, we can perform one-dimensional transforms on each of the factors:

$$\begin{aligned}
\mathcal{F}\{t_X(\xi)\} &= X\,\mathrm{sinc}(Xf_X) \\
\mathcal{F}\{t_Y(\eta)\} &= [Y\mathrm{sinc}(Yf_X)\times\Delta\mathrm{comb}(\Delta f_Y)]\otimes N\Delta\mathrm{sinc}(N\Delta f_Y) \\
&= \left[\sum_{m=-\infty}^{\infty} Y\mathrm{sinc}\left(\frac{mY}{\Delta}\right)\delta\left(f_Y-\frac{m}{\Delta}\right)\right]\otimes N\Delta\mathrm{sinc}(N\Delta f_Y) \\
&= N\Delta Y\sum_{m=-\infty}^{\infty}\mathrm{sinc}\left(\frac{mY}{\Delta}\right)\mathrm{sinc}\left[N\Delta\left(f_Y-\frac{m}{\Delta}\right)\right].
\end{aligned}$$

The full expression for the intensity distribution in the Fraunhofer diffraction pattern becomes:

$$I(x,y)=\left(\frac{N\Delta XY}{\lambda z}\right)^2\left\{\sum_{m=-\infty}^{\infty}\mathrm{sinc}\left(\frac{mY}{\Delta}\right)\mathrm{sinc}\left(\frac{Xx}{\lambda z}\right)\mathrm{sinc}\left[\frac{N\Delta}{\lambda z}\left(y-m\frac{\lambda z}{\Delta}\right)\right]\right\}^2.$$

(c) The sum appearing above can be viewed (considering only the y direction) as a sum of terms of the form sinc $\left(\frac{Xx}{\lambda z}\right)$ sinc $\left[\frac{N\Delta}{\lambda z}\left(y-m\frac{\lambda z}{\Delta}\right)\right]$, each with a weighting factor sinc $\left(\frac{mY}{\Delta}\right)$. We wish to find conditions under which the weighting factors of the terms for even values of m will be as small as possible. Since the sinc function has zero value at integer arguments, this requires

$$2kY/\Delta = \text{integer}$$

for each integer k. This will be the case if Y/Δ is any integer multiple of $1/2$. For example, if the slit spacing is twice the slit width, this will be the case. Note that the weighting factor for the $m=0$ term is independent of ratio of Y to Δ, so the strength of the "zero order" remains at its maximum possible value.

4-9. The amplitude transmittance of this aperture is given by

$$t_A(x,y)=\mathrm{rect}\left(\frac{x}{w_o}\right)\,\mathrm{rect}\left(\frac{y}{w_o}\right)-\mathrm{rect}\left(\frac{x}{w_i}\right)\,\mathrm{rect}\left(\frac{y}{w_i}\right).$$

The Fourier transform of this transmittance function is

$$\mathcal{F}\{t_A(x,y)\} = w_o^2 \,\mathrm{sinc}(w_o f_X)\,\mathrm{sinc}(w_o f_Y) - w_i^2 \,\mathrm{sinc}(w_i f_X)\,\mathrm{sinc}(w_i f_Y).$$

It follows that the Fraunhofer diffraction pattern of this aperture is

$$\begin{aligned} I(x,y) &= \left(\frac{4w_o^2}{\lambda z}\right)^2 \mathrm{sinc}^2\left(\frac{2w_o x}{\lambda z}\right)\,\mathrm{sinc}^2\left(\frac{2w_o y}{\lambda z}\right) \\ &- 2\left(\frac{4w_o w_i}{\lambda z}\right)^2 \mathrm{sinc}\left(\frac{2w_o x}{\lambda z}\right)\,\mathrm{sinc}\left(\frac{2w_o y}{\lambda z}\right)\,\mathrm{sinc}\left(\frac{2w_i x}{\lambda z}\right)\,\mathrm{sinc}\left(\frac{2w_i y}{\lambda z}\right) \\ &+ \left(\frac{4w_i^2}{\lambda z}\right)^2 \mathrm{sinc}^2\left(\frac{2w_i x}{\lambda z}\right)\,\mathrm{sinc}^2\left(\frac{2w_i y}{\lambda z}\right). \end{aligned}$$

4-10. The amplitude transmittance function for this aperture is

$$t_A(q) = \mathrm{circ}\left(\frac{q}{w_o}\right) - \mathrm{circ}\left(\frac{q}{w_i}\right).$$

The Fourier-Bessel transform of this function is

$$\mathcal{B}\{t_A(q)\} = w_o^2 \frac{J_1(2\pi w_o \rho)}{w_o \rho} - w_i^2 \frac{J_1(2\pi w_i \rho)}{w_i \rho}.$$

The Fraunhofer diffraction pattern is therefore

$$\begin{aligned} I(r) &= \left(\frac{w_o^2}{\lambda z}\right)^2 \left[\frac{J_1\left(\frac{2\pi w_o r}{\lambda z}\right)}{\frac{w_o r}{\lambda z}}\right]^2 \\ &- 2\left(\frac{w_o w_i}{\lambda z}\right)^2 \left[\frac{J_1\left(\frac{2\pi w_o r}{\lambda z}\right)}{\frac{w_o r}{\lambda z}}\right] \left[\frac{J_1\left(\frac{2\pi w_i r}{\lambda z}\right)}{\frac{w_i r}{\lambda z}}\right] \\ &+ \left(\frac{w_i^2}{\lambda z}\right)^2 \left[\frac{J_1\left(\frac{2\pi w_i r}{\lambda z}\right)}{\frac{w_i r}{\lambda z}}\right]^2. \end{aligned}$$

4-11. (a) From Eq. (4-41), for the case of a sinusoidal phase grating,

$$I(x,y) \approx \left(\frac{A}{\lambda z}\right)^2 \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right)\,\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x - q f_0 \lambda z)\right]\,\mathrm{sinc}^2\left(\frac{2wy}{\lambda z}\right).$$

The first zero for order q and wavelength λ_1 will occur at the value of x satisfying

$$\frac{2w}{\lambda_1 z}(x - q f_0 \lambda_1 z) = 1,$$

or

$$x = q f_0 \lambda_1 z + \frac{\lambda_1 z}{2w}.$$

Now consider the same order q but a different wavelength λ_2. This wavelength will produce a *maximum* of the order at

$$x = q f_0 \lambda_2 z.$$

Equating these two values of x yields the equation

$$qf_0 z(\lambda_2 - \lambda_1) = \frac{\lambda_1 z}{2w}.$$

Defining $\Delta\lambda = \lambda_2 - \lambda_1$, we obtain the resolving power of the grating in the qth diffraction order,

$$\frac{\lambda}{\Delta\lambda} = 2qwf_0,$$

where in the numerator $\lambda \approx \lambda_1 \approx \lambda_2$. Note that the number of grating periods M in the aperture is $2wf_0$, so

$$\frac{\lambda}{\Delta\lambda} = qM.$$

(b) The use of arbitrarily high diffraction orders is limited by (i) eventual decrease of diffraction efficiency in arbitrarily high orders, and (ii) the evanescent wave phenomenon, which prevents sufficiently high orders from propagating.

4-12. (a) The Fourier transform of the amplitude transmittance function is

$$\mathcal{F}\{t_A(\xi)\} = \sum_{n=-\infty}^{\infty} c_n \, \mathcal{F}\{e^{j\frac{2\pi n\xi}{L}}\} = \sum_{n=-\infty}^{\infty} c_n \delta\left(f_X - \frac{n}{L}\right).$$

Assuming unit amplitude, normally incident plane wave illumination, the intensity in any order will be proportional to the squared magnitude of the Fourier coefficient associated with that order. More generally, for arbitrary strength of illumination, the diffraction efficiency of a given order is the squared magnitude of the Fourier coefficient of the delta function corresponding to that order. Thus

$$\eta_k \quad = \quad |c_k|^2 .$$

(b) We must find the Fourier coefficients of the amplitude transmittance function

$$t_A(\xi) = \left|\cos\left(\frac{\pi\xi}{L}\right)\right|.$$

Do so as follows:

$$\begin{aligned} c_k &= \frac{1}{L}\int_{-L/2}^{L/2} \left|\cos\left(\frac{\pi\xi}{L}\right)\right| e^{-j\frac{2\pi k\xi}{L}}\, d\xi = \frac{1}{L}\mathcal{F}\left\{\text{rect}\left(\frac{\xi}{L}\right)\cos\left(\frac{\pi\xi}{L}\right)\right\}_{f_X=k/L} \\ &= \frac{1}{2L}\left(L\text{sinc}\left[L\left(f_X - \frac{1}{2L}\right)\right] + L\text{sinc}\left[L(f_X + \frac{1}{2L})\right]\right)_{f_X=k/L} \\ &= \frac{1}{2}\left[\text{sinc}\left(\frac{2k-1}{2}\right) + \text{sinc}\left(\frac{2k+1}{2}\right)\right]. \end{aligned}$$

The diffraction efficiency is seen to be

$$\eta_k = |c_k|^2 = \frac{1}{4}\left[\text{sinc}\left(\frac{2k-1}{2}\right) + \text{sinc}\left(\frac{2k+1}{2}\right)\right]^2.$$

For the particular case of the first diffraction order ($k = 1$),

$$|c_1|^2 = \frac{1}{4}\left[\text{sinc}\left(\frac{1}{2}\right) + \text{sinc}\left(\frac{3}{2}\right)\right]^2 = \frac{1}{4}\left[\frac{2}{\pi} - \frac{2}{3\pi}\right]^2 = \frac{4}{9\pi^2} = 4.5\%.$$

4-13. We first obtain a Fourier series expansion of the grating amplitude transmittance, and then use the results of the previous problem. First note that in the region $|\xi| \leq L/2$, the amplitude transmittance can be written

$$t_A(\xi) = \left(\frac{1}{2} - t_m\right) \text{rect}\left(\frac{\xi}{L}\right) + 2t_m \text{rect}\left(\frac{\xi}{L/2}\right).$$

Thus

$$\begin{aligned} c_n &= \frac{1}{L}\mathcal{F}\left\{\left(\frac{1}{2} - t_m\right) \text{rect}\left(\frac{\xi}{L}\right) + 2t_m \text{rect}\left(\frac{\xi}{L/2}\right)\right\}_{f_X=n/L} \\ &= \left(\frac{1}{2} - t_m\right) \text{sinc}(n) + t_m \,\text{sinc}\left(\frac{n}{2}\right). \end{aligned}$$

(a) The fraction of light absorbed by the grating is found by subtracting the spatial average (i.e. an average over one period) of $|t_X(\xi)|^2$ from unity,

$$\begin{aligned} \text{fraction absorbed} &= 1 - \frac{1}{L}\int_{-L/2}^{L/2} |t_A(\xi)|^2 \, d\xi \\ &= 1 - \frac{1}{L}\left[\frac{L}{2}\left(\frac{1}{2} - t_m\right)^2 + \frac{L}{2}\left(\frac{1}{2} + t_m\right)^2\right] = \frac{3}{4} - t_m^2. \end{aligned}$$

(b) The fraction of light transmitted by the grating is simply 1 minus the fraction absorbed. Therefore

$$\text{fraction transmitted} = \frac{1}{4} + t_m^2.$$

(c) The fraction of light appearing in a single first order will be given by $|c_1|^2$. We have

$$|c_1|^2 = \left(\frac{2t_m}{\pi}\right)^2 = \frac{4t_m^2}{\pi^2}.$$

4-14. We begin by writing an equation for the amplitude transmittance of the grating:

$$\begin{aligned} t_A(x) &= 1 - \left[(1 - e^{j\phi}) \times (\text{square wave})\right] \\ &= 1 - \left[(1 - e^{j\phi}) \times \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n x}{L}}\right] \end{aligned}$$

where,

$$c_n = \frac{1}{L}\int_{-\infty}^{\infty} \text{rect}\left(\frac{\xi}{L/2}\right) e^{-j\frac{2\pi n \xi}{L}} \, d\xi = \frac{1}{L}\mathcal{F}\left\{\text{rect}\left(\frac{x}{L/2}\right)\right\}_{f_X=n/L} = \frac{1}{2}\text{sinc}\left(\frac{n}{2}\right).$$

Continuing,

$$\begin{aligned} \mathcal{F}\left\{t_A(x)\right\} &= \delta(f_X) - (1 - e^{j\phi}) \sum_{n=-\infty}^{\infty} \frac{1}{2}\text{sinc}\left(\frac{n}{2}\right) \mathcal{F}\left\{e^{j\frac{2\pi n x}{L}}\right\} \\ &= \delta(f_x) - (1 - e^{j\phi}) \sum_{n=-\infty}^{\infty} \frac{1}{2}\text{sinc}\left(\frac{n}{2}\right) \delta\left(f_X - \frac{n}{L}\right). \end{aligned}$$

(a) Now finding the diffraction efficiency of the first order,

$$\begin{aligned}\eta_1 &= \eta_{-1} = \left|1 - e^{j\phi}\right|^2 \left[\frac{1}{2}\text{sinc}\left(\frac{1}{2}\right)\right]^2 = \frac{1}{4}\left(\frac{2}{\pi}\right)^2 (2 - 2\cos\phi) \\ &= \frac{2}{\pi^2}(1 - \cos\phi).\end{aligned}$$

(b) To maximize η_1 and η_{-1}, we require $\cos\phi = -1$, or $\phi = \pi$. In this case the diffraction efficiency becomes

$$\eta_1 = \eta_{-1} = \frac{4}{\pi^2} = 40.5\%.$$

4-15. (a) To find the diffraction efficiency, we find the Fourier coefficients as follows:

$$\begin{aligned}c_n &= \frac{1}{L}\int_0^L e^{j\frac{2\pi\xi}{L}}\, e^{j2\pi\xi\frac{n}{L}}\, d\xi = \frac{1}{L}\int_0^L e^{j\frac{2\pi}{L}(1+n)\xi}\, d\xi \\ &= \frac{1}{L}\frac{e^{\frac{j2\pi}{L}(1+n)\xi}}{j\frac{2\pi}{L}(1+n)}\Bigg]_0^L = e^{j\pi(1+n)}\, \text{sinc}(1+n).\end{aligned}$$

The diffraction efficiency is thus given by

$$\eta_n = |c_n|^2 = \text{sinc}^2(1+n).$$

Note that *all* of the light is transmitted into a first-order component $(n = -1)$.

(b) In this case,

$$\begin{aligned}c_n &= \frac{1}{L}\int_0^L e^{j\frac{\phi_o\xi}{L}}\, e^{j2\pi\xi\frac{n}{L}}\, d\xi = \frac{1}{L}\int_0^L e^{j\frac{2\pi\xi}{L}\left(n+\frac{\phi_o}{2\pi}\right)}\, d\xi = \frac{1}{L}\frac{e^{j\frac{2\pi}{L}(n+\frac{\phi_o}{2\pi})\xi}}{j\frac{2\pi}{L}\left(n+\frac{\phi_o}{2\pi}\right)}\Bigg]_0^L \\ &= e^{j\pi\left(n+\frac{\phi_o}{2\pi}\right)}\, \text{sinc}\left(n + \frac{\phi_o}{2\pi}\right).\end{aligned}$$

The diffraction efficiency becomes

$$\eta_n = \text{sinc}^2\left(n + \frac{\phi_o}{2\pi}\right).$$

4-16. (a) The wavefront in the aperture will be of the form $e^{-jkr_{01}}$, with a minus sign because the wave is converging, and with r_{01} being the distance between a point P_1 in the aperture (coordinates (ξ, η)) and the point P (coordinates $(0, Y)$) in the (x, y) plane. An exact expression for the distance r_{01} is

$$r_{01} = \sqrt{z^2 + \xi^2 + (\eta - Y)^2}.$$

Factoring the distance z outside the square root, and making the usual quadratic phase approximation, based on the assumption that z is much larger than the aperture and much larger than the distance of the point P off axis, we obtain

$$r_{01} = z\sqrt{1 + \frac{\xi^2 + (\eta - Y)^2}{z^2}} \approx z + \frac{\xi^2 + (\eta - Y)^2}{2z}.$$

(b) We must consider two quadratic phase factors, the one above representing the illumination, and a second one that occurs as part of the integrand in the Fresnel diffraction equation. Let the amplitude transmittance of the aperture be represented by $t_A(\xi, \eta)$, and suppose that the amplitude of the illuminating wave at the aperture is A. Then the full equation for amplitude of the Fresnel diffraction pattern in the (x, y) plane is

$$U(x,y) = \frac{Ae^{jkz}}{j\lambda z} \iint\limits_{-\infty}^{\infty} t_A(\xi,\eta) e^{-\frac{jk}{2z}\left[\xi^2+(\eta-Y)^2\right]} e^{\frac{jk}{2z}\left[(x-\xi)^2+(y-\eta)^2\right]} \, d\xi d\eta,$$

where the first quadratic phase factor represents the illumination, while the second is the normal exponential factor associated with the Fresnel diffraction integral. Expanding the quadratic phase factor and noting that the terms in $(\xi^2 + \eta^2)$ exactly cancel, we obtain

$$U(x,y) = \frac{Ae^{-\frac{jk}{2z}Y^2} e^{\frac{jk}{2z}\left(x^2+y^2\right)}}{j\lambda z} \iint\limits_{-\infty}^{\infty} t_A(\xi,\eta) e^{-j\frac{2\pi}{\lambda z}[\xi x + \eta(y-Y)]} \, d\xi d\eta.$$

Since all the phase factors in the integrand are linear in ξ and η, we see that the integral is a Fourier transform, and therefore the result is a *Fraunhofer* diffraction formula, with the Fraunhofer pattern centered on coordinates $(0, Y)$.

4-17. On the axis, $x = 0$ and $y = 0$, and therefore the Fresnel diffraction equation becomes

$$U(0,0) = \frac{e^{jkz}}{j\lambda z} \iint\limits_{-\infty}^{\infty} t_A(\xi,\eta) e^{j\frac{k}{2z}\left(\xi^2+\eta^2\right)} \, d\xi d\eta.$$

(a) For $t_A(\xi, \eta) = \mathrm{circ}\sqrt{\xi^2 + \eta^2}$, we can change to polar coordinates and write

$$U(0,0) = \frac{2\pi e^{jkz}}{j\lambda z} \int_0^1 q e^{\frac{jk}{2z}q^2} dq,$$

where $q = \sqrt{\xi^2 + \eta^2}$. Now change variables of integration, letting $\sigma = \frac{jk}{2z}q^2$, from which it follows that $d\sigma = \frac{jk}{z}qdq$. The lower limit remains 0 while the upper limit becomes $\frac{jk}{2z}$. Thus the integral becomes

$$\begin{aligned} U(0,0) &= \frac{2\pi e^{jkz}}{j\lambda z}\frac{z}{jk}\int_0^{\frac{jk}{2z}} \exp(\sigma)\, d\sigma = -e^{jkz}\exp(\sigma)\Big]_0^{\frac{jk}{2z}} \\ &= -e^{jkz}\left[\exp\left(\frac{jk}{2z}\right) - 1\right] = -2je^{jkz}e^{\frac{jk}{2z}}\sin\left(\frac{k}{2z}\right). \end{aligned}$$

The intensity is then given by

$$I(0,0) = |U(r)|^2 = 4\sin^2\left(\frac{k}{4z}\right) = 4\sin^2\left(\frac{\pi}{2\lambda z}\right).$$

Note that the argument of the sin is dimensionally correct, since we took the radius of the circular aperture to be unity. The more general argument for an aperture of radius r_1 would be $\frac{\pi r_1^2}{2\lambda z}$.

(b) For this aperture, the only change to the above equations comes from the fact that the lower limit of integration with respect to radius is now a and the upper limit is b. After the change to polar coordinates and the change of variable of integration, the field on axis is given by

$$U(0,0) = -e^{jkz} \int_{\frac{jka^2}{2z}}^{\frac{jkb^2}{2z}} \exp(\sigma) d\sigma = -e^{jkz} \left[e^{\frac{jkb^2}{2z}} - e^{\frac{jka^2}{2z}} \right].$$

Multiplying this expression by its complex conjugate to obtain the intensity, we find after some algebra that

$$I(0,0) = 2 \left\{ 1 - \cos \left[\frac{k}{2z}(b^2 - a^2) \right] \right\}.$$

4-18. Let the period of the periodic object be L, and assume that the variations run in the x direction. Then the spectrum of the object will have delta-function components at frequencies

$$f_X = \frac{m}{L} \qquad m = 0, \pm 1, \pm 2, \ldots.$$

In the paraxial or Fresnel approximation, the transfer function of the propagation phenomenon is

$$H(f_X, f_Y) = e^{jkz} \exp\left[-j\pi\lambda z \left(f_X^2 + f_Y^2 \right) \right].$$

For an image to appear, we require that

$$\exp\left[-j\pi\lambda z \left(\frac{m}{L} \right)^2 \right] = 1,$$

for all m, or

$$\pi\lambda z \left(\frac{m}{L} \right)^2 = k2\pi$$

for some integer k. Note that a different integer k can be chosen for each integer m. An infinite set of distances z, call them z_n, will satisfy this requirement. Solve for z to yield

$$z = \frac{2kL^2}{m^2\lambda}.$$

Clearly one solution would be, for each m choose $k = m^2$, yielding $z_1 = \frac{2L^2}{\lambda}$. Another solution would be to choose $k = 2m^2$, yielding $z_2 = \frac{4L^2}{\lambda}$. The general expression for the distances where self-imaging occurs is

$$z_n = \frac{2nL^2}{\lambda} \qquad n = 1, 2, \ldots.$$

4-19. Since the transfer function for propagation is, in the Fresnel approximation,

$$H(\rho) = e^{jkz} e^{-j\pi\lambda z \rho^2},$$

imaging of this object requires that

$$e^{-j\pi\lambda z(2ma)} = 1 \qquad m = 0, 1, 2, \ldots,$$

or equivalently

$$\pi\lambda z(2ma) = 2k\pi$$

for some integer k for each given m. If for each m we choose $k = m$, the we obtain a distance $z_1 = \frac{1}{\lambda z a}$. If we choose $k = 2m$, then the imaging distance is $z_2 = \frac{2}{\lambda a}$. More generally the self-imaging distances are

$$z_n = \frac{n}{\lambda a} \qquad n = 1, 2, \ldots.$$

4-20. Consider the Fourier transforms of each of the two components of the object:

$$\begin{aligned} \mathcal{F}\{2\pi J_0(2\pi r)\} &= \delta(\rho - 1) \\ \mathcal{F}\{4\pi J_0(4\pi r)\} &= \delta(\rho - 2). \end{aligned}$$

Since the transfer function for free-space propagation, under paraxial conditions, is

$$H(\rho) = e^{jkz} e^{-j\pi\lambda z \rho^2},$$

after propagation over distance z the field will be given by

$$U(r) = e^{jkz}\left[2\pi J_0(2\pi r) e^{-j\pi\lambda z} + 4\pi J_0(4\pi r) e^{-j4\pi\lambda z}\right],$$

or

$$U(r) = e^{jkz} e^{-j\pi\lambda z}\left[2\pi J_0(2\pi r) + 4\pi J_0(4\pi r) e^{-j3\pi\lambda z}\right].$$

Remembering that only intensity is important, for imaging to occur, we require that

$$3\pi\lambda z = 2k\pi$$

where k is any integer. Thus images will appear at distances given by

$$z_k = \frac{2k}{3\lambda} \qquad k = 0, 1, 2, \ldots.$$

4-21. Starting with the given wavefront,

$$U(y_1) = \exp\left[j\frac{\pi}{\lambda z}(y_1 - y_0)^2\right],$$

we calculate the local spatial frequency in the input plane,

$$f_{l y_1} = \frac{\theta_1}{\lambda} = \frac{1}{2\pi}\frac{\partial}{\partial y_1}\left[\frac{\pi}{\lambda z}(y_1 - y_0)^2\right]$$

yielding

$$\theta_1 = \frac{y_1 - y_0}{z}.$$

The above relationship between incident angle and incident position can also be derived geometrically by noting that the line source which gives rise to the wave is located at $(y_0, -z)$, where $z = 0$ corresponds to the plane where the wave has been specified.

Now,

$$\begin{aligned} \begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} &= \frac{1}{AD - BC}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix}. \end{aligned}$$

Substituting the expressions for y_1 and θ_1,

$$\frac{1}{AD-BC}(-Cy_2+A\theta_2) \quad = \quad \frac{1}{z}\left[\frac{Dy_2-B\theta_2}{AD-BC}-y_0\right]$$

which simplifies to

$$\theta_2 \quad = \quad \frac{y_2-y_{02}}{z_2}$$

where,

$$\begin{aligned} y_{02} &= \frac{AD-BC}{D+Cz}y_0 \\ z_2 &= \frac{Az+B}{D+Cz}. \end{aligned}$$

The phase angle ϕ_2 of the field at the output plane is given as

$$\begin{aligned} \phi_2 &= \int 2\pi f_{ly_2}\, dy_2 \\ &= \frac{2\pi}{\lambda}\int \theta_2\, dy_2 \\ &= \frac{2\pi}{\lambda}\int \frac{y_2-y_{02}}{z_2}\, dy_2 \\ &= \frac{\pi}{\lambda z_2}(y_2-y_{02})^2 \end{aligned}$$

where in the last equality, the integration constant was arbitrarily chosen so that the phase is zero at $y_2 = y_{02}$. Hence, the field at the output plane is

$$U_{out}(y_2) \quad = \quad \exp\left[j\frac{\pi}{\lambda z_2}(y_2-y_{02})^2\right].$$

Again, this result can also be obtained by geometrical considerations by noting that the emerging rays at the output plane represent a cylindrical wave with the source located at $(y_{02}, z_{out}-z_2)$ (or a cylindrical wave converging toward $(y_{02}, z_{out}-z_2)$ if z_2 is negative).

Chapter 5

5-1. We start with the expression for the focal length of an arbitrary lens in air,

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

or

$$f = \frac{1}{n-1}\,\frac{R_1 R_2}{R_2 - R_1}.$$

To determine whether a lens is positive or negative, we need only determine whether its focal length is positive or negative.

Double Convex: Since $R_1 > 0$ and $R_2 < 0$, $f > 0$.

Plano-convex: Since $R_1 = \infty$ and $R_2 < 0$, $f > 0$.

Positive Meniscus: Since $R_1 > 0$, $R_2 > 0$ and $R_1 < R_2$, $f > 0$.

Double Concave: Since $R_1 < 0$ and $R_2 > 0$, $f < 0$.

Plano-concave: Since $R_1 = \infty$ and $R_2 > 0$, $f < 0$.

Negative Meniscus: Since $R_1 < 0$ and $R_2 < 0$ and $|R_1| < |R_2|$, $f < 0$.

5-2. Consider the geometry shown in in the figure below, which is a top view of the cylindrical lens.

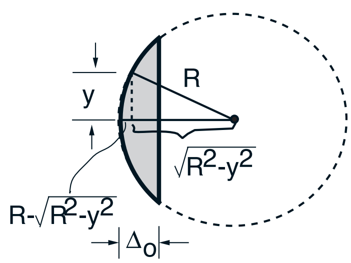

Figure 5-2:

(a) The thickness of the lens $\Delta(y)$ at an arbitrary vertical point y is seen to be

$$\begin{aligned}\Delta(y) &= \Delta_o - (R - \sqrt{R^2 - y^2}) = \Delta_o - R\left(1 - \sqrt{1 - \frac{y^2}{R^2}}\right) \\ &\approx \Delta_o - R\left[1 - \left(1 - \frac{1}{2}\frac{y^2}{R^2}\right)\right] = \Delta_o - \frac{y^2}{2R}.\end{aligned}$$

The lens amplitude transmittance function now becomes

$$
\begin{aligned}
t_l(x,y) &= e^{jk\Delta_o}\, e^{jk(n-1)\Delta(y)} = e^{jk\Delta_o}\, e^{-jk\Delta_o}\, e^{jkn\Delta_o}\, e^{-\frac{jk(n-1)}{2R}y^2} \\
&= e^{jkn\Delta_o} e^{-\frac{jk(n-1)}{R}\frac{y^2}{2}}.
\end{aligned}
$$

(b) Let $\frac{1}{f} = (n-1)\frac{1}{R}$. Then

$$
t_l(x,y) = e^{jkn\Delta_o}\, e^{-\frac{jk}{2f}y^2}.
$$

Such a lens will transform a normally incident plane wave into a converging cylindrical wave, bringing light to a line focus at distance $f = \frac{R}{n-1}$ behind the lens.

5-3. (a) Without loss of generality, assume that $b^2 > a^2$. The expression for the amplitude transmittance can be expanded as follows:

$$
\begin{aligned}
t_A(x,y) &= \exp\left\{-j\pi[a^2x^2 + (by+c)^2]\right\} \\
&= e^{-j\pi c^2}\, e^{-j2\pi bcy}\, e^{-j\pi a^2(x^2+y^2)}\, e^{-j\pi(b^2-a^2)y^2}.
\end{aligned}
$$

The first exponential term represents a constant phase shift, the second a prismatic wavefront tilt in the y direction, the third a positive spherical lens, and the fourth a positive cylindrical lens exerting focusing power in the y direction. By comparing these exponentials to the forms of the amplitude transmittances for a positive spherical lens, a positive cylindrical lens and a prism,

$$
\begin{aligned}
t_l(x,y) &= \exp\left[-j\frac{\pi}{\lambda f}(x^2+y^2)\right] \\
t_l(y) &= \exp\left[-j\frac{\pi}{\lambda f}y^2\right] \\
t_p(y) &= \exp\left[-j\frac{2\pi}{\lambda}\sin(\theta)y\right],
\end{aligned}
$$

respectively, we find the following parameter relations:

$$
\begin{aligned}
f_{\text{spherical}} &= \frac{1}{\lambda a^2} \\
f_{\text{cylindrical}} &= \frac{1}{\lambda(b^2-a^2)} \\
\theta &= \arcsin[\lambda bc].
\end{aligned}
$$

(b) Consider a positive cylindrical lens with focal length f, initially with power along the x axis. The corresponding amplitude transmittance is

$$
t_l(x,y) = \exp\left[-j\frac{\pi}{\lambda f}x^2\right].
$$

Now rotate this lens so that, instead of exerting power in the x direction, it exerts power along a direction at $+45$ degrees to the x-axis. The amplitude transmittance becomes

$$
t_1(x,y) = \exp\left[-j\frac{\pi}{\lambda f}\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)^2\right] = \exp\left[-j\frac{\pi}{2\lambda f}(x^2+2xy+y^2)\right].
$$

Now consider a negative cylindrical lens with focal length f, rotated to have power along a line at $+135$ degrees to the x-axis. The amplitude transmittance becomes

$$t_2(x,y) = \exp\left[+j\frac{\pi}{\lambda f}\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)^2\right] = \exp\left[+j\frac{\pi}{2\lambda f}(x^2-2xy+y^2)\right].$$

Now place the two lenses in contact. Their amplitude transmittances multiply, yielding

$$t_{\text{total}}(x,y) = e^{-j\frac{\pi}{2\lambda f}(x^2+2xy+y^2)}\, e^{+j\frac{\pi}{2\lambda f}(x^2-2xy+y^2)} = e^{-j\frac{2\pi}{\lambda f}xy}.$$

This has the form of the amplitude transmittance we sought, with

$$d = \frac{2}{\lambda f}.$$

5-4. (a) Following the logic of the problem dealing with a cylindrical lens, we can write the lens thickness function as

$$\Delta(x,y) = \Delta(y) - \frac{x^2}{2R(y)},$$

where $\Delta(y)$ is the thickness at y and $R(y)$ is the radius of curvature at y. Geometrical considerations yield

$$\begin{aligned} \Delta &= \Delta_o - \frac{y}{h}R \\ R(y) &= R\left(1-\frac{y}{h}\right). \end{aligned}$$

Substitution into the equation for $\Delta(y)$ gives

$$\Delta(x,y) = \Delta_o - \frac{y}{h}R - \frac{x^2}{R\left(1-\frac{y}{h}\right)}.$$

The lens amplitude transmittance function may now be written as

$$\begin{aligned} t_l(x,y) &= e^{jk\Delta_o}\exp\left[jk(n-1)\left(\Delta_o - \frac{y}{h}R - \frac{x^2}{R\left(1-\frac{y}{h}\right)}\right)\right] \\ &= \exp\left\{jk\left[n\Delta_o - \frac{(n-1)Ry}{h} - \frac{(n-1)x^2}{2R\left(1-\frac{y}{h}\right)}\right]\right\} \\ &= \exp\left\{jk\left[n\Delta_o - \frac{(n-1)Ry}{h} - \frac{x^2}{2f(y)}\right]\right\}, \end{aligned}$$

where

$$f(y) = \frac{R\left(1-\frac{y}{h}\right)}{n-1}.$$

(b) Start with the final result from part (a):

$$\begin{aligned} t_l(x,y) &= \exp\left\{jk\left[n\Delta_o - \frac{(n-1)Ry}{h} - \frac{x^2}{2f(y)}\right]\right\} \\ f(y) &= \frac{R\left(1-\frac{y}{h}\right)}{n-1}. \end{aligned}$$

The first term in the exponent is independent of coordinates and has no effect on the incoming light other than a constant phase delay. The second term acts like a prism in that we can identify $-\frac{(n-1)R}{h}$ with a direction cosine. Thus this term introduces a downward tilt to the wavefront, with the tilt angle σ with respect to the axis satisfying

$$\cos\left(\frac{\pi}{2}-\sigma\right)=\sin(\sigma)=-\frac{(n-1)R}{h}.$$

The third term represents a focusing of light along a line determined by the variable focal length. The axial distance of the top of this line focus away from the lens is $\frac{R}{n-1}\left(1-\frac{L}{h}\right)$ where L is the lens height, while the bottom of the line focus is at axial distance $\frac{R}{n-1}$ from the lens.

5-5. This is a vignetting problem. The two cases of interest in (a) and (c) below are shown in the figure.

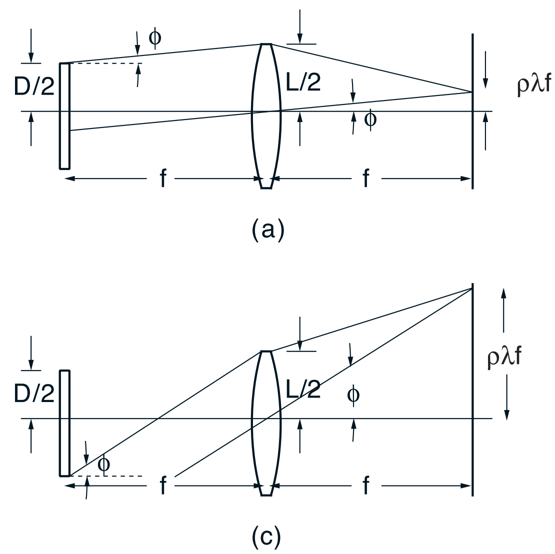

Figure 5-5:

(a) Vignetting occurs when the projection of the lens pupil onto the object plane does not cover the entire object. The center of the back-projected lens pupil is offset from the center of the object by $\rho\lambda f$, where $\rho=\sqrt{f_X^2+f_Y^2}$, so the object will start falling outside of the pupil when:

$$\begin{aligned}\frac{D}{2}+\rho\lambda f &= \frac{L}{2}\\ \rho &= \frac{1}{\lambda f}\left(\frac{L}{2}-\frac{D}{2}\right).\end{aligned}$$

(b)

$$\begin{aligned}\rho &= \frac{1}{(6\times10^{-7})0.5}\left(\frac{0.04}{2}-\frac{0.02}{2}\right)\\ &= \frac{1}{3\times10^{-7}}(0.01)\\ &= 0.33\times10^{5}\,\mathrm{m}^{-1}=33\,\text{cycles/mm}.\end{aligned}$$

(c) Here we are to find the ρ where the object falls completely outside of the projection of the lens pupil. This happens when:

$$\begin{aligned}\rho\lambda f &= \frac{L}{2}+\frac{D}{2}\\ \rho &= \frac{1}{\lambda f}\left(\frac{L}{2}+\frac{D}{2}\right).\end{aligned}$$

5-6. We wish to perform a 1-D Fourier transform in the ξ direction, and to *image* in the η direction. The imaging operation will preserve the array structure of the set of transforms (with an inversion); since only the intensity is of interest, we can ignore phase factors in ξ or η. There are a number of different possible solutions to this problem, of which we show only one for each part.

(a) Consider part (a) of the figure below:

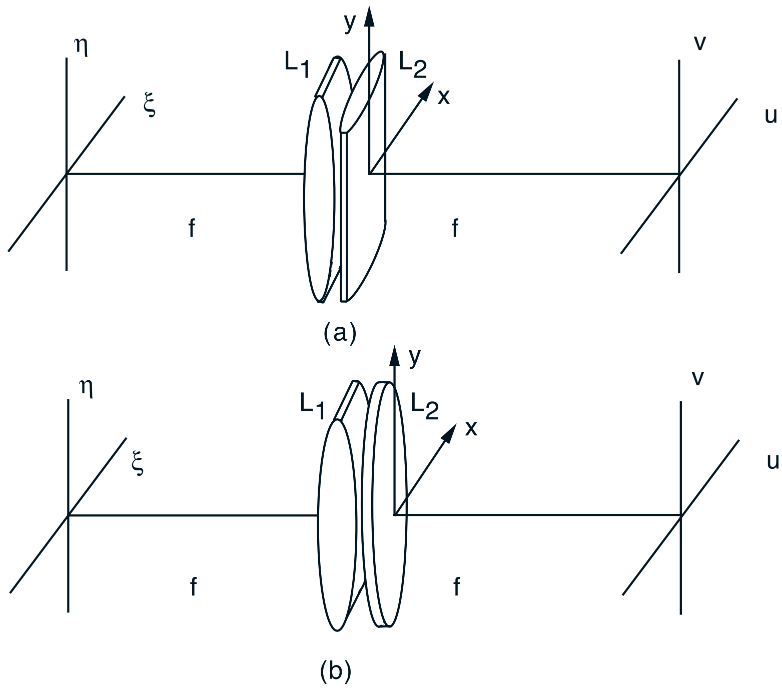

Figure 5-6:

Lens L_1 has power only in the y direction, and lens L_2 has power only in the x direction. The focal lengths of the two lenses are chosen to be

$$\begin{aligned}f_1 &= f/2\\ f_2 &= f.\end{aligned}$$

The cylindrical lenses are placed in contact at distances f from the input and output planes. This distance is two focal lengths with respect to power in the y direction, but only one focal length with respect to power in the x direction. Therefore the optical system *images* in the y direction and *Fourier transforms* in the x direction. Neglecting phase factors associated with the imaging operation, this yields the array of Fourier transforms that is desired.

(b) With reference to part (b) of the figure, in this case a spherical lens and a cylindrical lens are placed in contact. The focal lengths of both the spherical lens and the cylindrical lens are f. The cylindrical lens is oriented with power in the y direction, while the spherical lens has power in both directions. The input and output planes are placed at distances f in front of and behind the lens combination, respectively. In the x direction, only the spherical lens has power, and for this direction, the input and output planes are in the front and back focal planes of the lens combination. Hence the system performs a Fourier transform in the x direction. In the y direction, both lenses have power, and the equivalent focal length in this direction is only $f/2$, as can be seen by multiplying their two amplitude transmittances, considering only the y variation,

$$t_{tot}(y) = t_c(y)\, t_s(y) = \exp\left(-j\frac{\pi}{\lambda f}y^2\right)\exp\left(-j\frac{\pi}{\lambda f}y^2\right) = \exp\left(-j\frac{\pi}{\lambda f/2}y^2\right).$$

Thus the lens combination will *image* in the x direction.

5-7. Since the projected pupil function of the lens is considerably larger than the finite size of the object, we can neglect it. From Eq. (5-22), we then have the following expression for the field in the focal plane,

$$U_f(u,v) = \frac{A\exp\left[j\frac{k}{2d}(u^2+v^2)\right]}{j\lambda d}\frac{f}{d}\iint\limits_{-\infty}^{\infty} t_A(\xi,\eta)\exp\left[-j\frac{2\pi}{\lambda d}(u\xi+v\eta)\right]d\xi d\eta.$$

The problem is identical in form to that treated in section 4.4.3. Adapting the result of that analysis, Eq. (4-36), to the problem at hand, we see that

$$\begin{aligned} I(u,v) \quad \approx \quad & \left[\frac{Af}{2\lambda d^2}\right]^2 \text{sinc}^2\left(\frac{Lv}{\lambda d}\right)\left\{\text{sinc}^2\left(\frac{Lu}{\lambda d}\right)\right. \\ & + \left.\frac{1}{4}\text{sinc}^2\left[\frac{L}{\lambda d}(u+f_0\lambda d)\right]+\frac{1}{4}\text{sinc}^2\left[\frac{L}{\lambda d}(u-f_0\lambda d)\right]\right\}. \end{aligned}$$

For the particular parameter values given,

$$\begin{aligned} \frac{\lambda d}{L} &= \frac{0.633\times10^{-6}\times1}{10^{-2}} = 63.3\ \mu\text{m} \\ f_0\lambda d &= 10^4\times0.633\times10^{-6}\times1 = 0.633\times10^{-2}\text{m} = 6.33\ \text{mm} \end{aligned}$$

A plot of the (normalized) intensity pattern is shown below, with all distances expressed in meters.

5-8. (a) The Fourier plane is found in the plane where the source is imaged. Therefore the distance z_f of the Fourier plane to the right of the lens must satisfy

$$\frac{1}{z_1}+\frac{1}{z_f}=\frac{1}{f}$$

in which case z_f is given by

$$z_f = \frac{fz_1}{z_1-f}.$$

For the distance of the object to the left of the lens to equal the distance of the Fourier plane to the right of the lens, we require

$$d = z_f = \frac{fz_1}{z_1-f}.$$

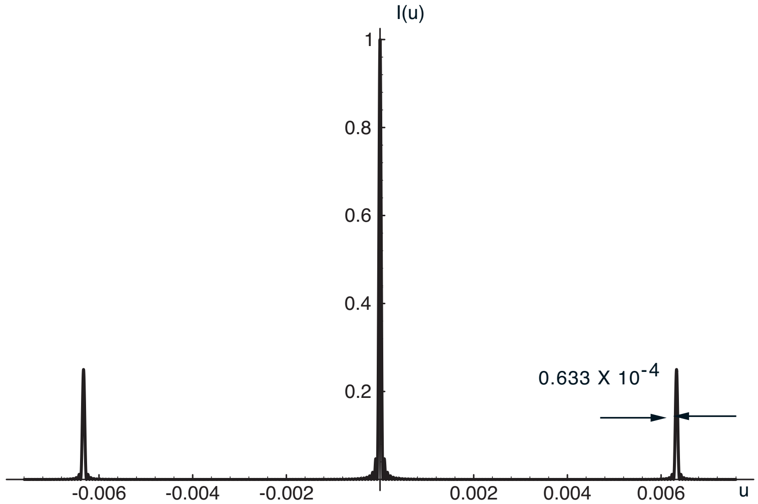

Figure 5-7:

(b) Let z_i represent the distance of the image from the lens. Then from the lens law,

$$\frac{1}{d} + \frac{1}{z_i} = \frac{1}{f}.$$

Substitute the expression for d obtained in part (a) into this equation and solve for z_i. The result is

$$z_i = z_1.$$

The magnification is given by

$$M = \left|\frac{z_i}{d}\right| = \left|\frac{z_1}{d}\right|.$$

5-9. The field in the plane at distance $f - \Delta$ from the lens is given by

$$U_{f-\Delta}(u, v) \quad = \quad \frac{1}{\lambda f} \quad \iint\limits_{-\infty}^{\infty} P(x, y) e^{-j\frac{k(x^2+y^2)}{2f}} e^{j\frac{k(x^2+y^2)}{2(f-\Delta)}} e^{-j\frac{2\pi(xu+yv)}{\lambda(f-\Delta)}} \, dxdy.$$

The first quadratic phase factor in the integrand represents the effect of the lens, while the second arises from the Fresnel diffraction kernel. In order for the diffraction pattern to be approximately Fraunhofer, we want the total quadratic-phase exponential factor to vanish. This requires

$$\exp\left[\frac{jk}{2}(x^2 + y^2)\left(\frac{1}{f-\Delta} - \frac{1}{f}\right)\right] \quad \approx \quad 1$$

or,

$$\left|\frac{2\pi}{2\lambda}(x^2 + y^2)\frac{\Delta}{f(f-\Delta)}\right|_{\text{max}} \quad \ll \quad 1 \text{ radian.}$$

In the worst case, $x^2 + y^2 \rightarrow (D/2)^2$. Also, assuming Δ is small compared to f, $f - \Delta \rightarrow f$ in the denominator. Thus,

$$\frac{\pi D^2 \Delta}{\lambda 4 f^2} \ll 1$$

or

$$\Delta \ll \frac{4 \lambda f^2}{\pi D^2}.$$

5-10. The distance d to the Fourier plane can be determined by finding where the source is imaged. The object transparency can be removed for this purpose. The normally incident plane wave at L_1 will be transformed into a spherical wave diverging from a point at distance f to the left of that lens, or equivalently at distance $2f$ to the left of lens L_2. That point source will be imaged at distance $2f$ to the right of lens L_2, so the Fourier plane will appear at distance $2f$ to the right of lens L_2.

As for the location of the image of the object, we can replace the object by a point-source in the object plane. According to the lens law, the negative lens produces an image of that point source at distance z_{i1} satisfying

$$\frac{1}{z_{i1}} + \frac{1}{2f} = -\frac{1}{f}$$

or

$$z_{i1} = -\frac{2}{3} f.$$

This initial image is thus $\frac{2}{3} f$ to the left of lens L_1. Now this point is imaged by lens L_2. The image distance z_{i2} from lens L_2 must now satisfy

$$\frac{1}{z_{i2}} + \frac{1}{\frac{5}{3} f} = \frac{1}{f}$$

or

$$z_{i2} = \frac{5f}{2}.$$

Thus the image appears $\frac{5f}{2}$ to the right of lens L_2.

5-11. Fourier planes will be found at the following locations:

- In the plane where the illumination beam comes to focus; i.e. distance f to the right of the object.
- In the plane where the above Fourier plane is imaged by the lens. According to the lens law, this will be at distance $2f$ to the right of the lens.

There will be only one image plane, namely the plane where the lens law is satisfied for an object $3f$ in front of the lens. We have

$$\frac{1}{z_i} + \frac{1}{3f} = \frac{1}{f}$$

from which it follows that

$$z_i = \frac{3f}{2}$$

to the right of the lens.

5-12. (a) Let $r^2 = \xi^2 + \eta^2$. Then we seek the radius r_0 for which

$$\frac{kr_0^2}{2z_1} = \pi.$$

Then

$$r_0^2 = \frac{2\pi z_1}{k} = \lambda z_1,$$

or

$$r_0 = \sqrt{\lambda z_1}.$$

(b) For an ideal image located at the origin, Eq. (5-33) predicts that

$$h(0,0;\xi,\eta) \approx \frac{1}{\lambda^2 z_1 z_2} \iint_{-\infty}^{\infty} P(x,y) e^{j\frac{2\pi M}{\lambda z_2}(\xi x + \eta y)}\, dx dy$$

The pupil function in this case is given by

$$P(x,y) = \text{circ}\left(\frac{\sqrt{x^2+y^2}}{R}\right),$$

yielding a (normalized) impulse response (from Eq. (4-31)) of the form

$$\tilde{h}(r) = 2\frac{J_1(kRr/z_1)}{kRr/z_1}$$

where this function is referred to the object space and we have used that fact that $M/z_2 = 1/z_1$. The radius to the first zero of this function will be

$$r_1 = 0.61\frac{\lambda z_1}{R}.$$

(c) We require that the radius r_1 of part (b) be smaller than the radius r_0 of part (a), in which case over the most important part of the impulse response the phase factor will not change appreciably. Thus we require

$$0.61\frac{\lambda z_1}{R} < \sqrt{\lambda z_1}$$

or

$$0.61\frac{\sqrt{\lambda z_1}}{R} < 1.$$

Consider a typical example: $R = 1\,\text{cm}$, $\lambda = 0.633\,\mu\text{m}$, $z_1 = 10\,\text{cm}$. The left-hand side of the above inequality is found to be 0.015, showing that the inequality is well satisfied.

5-13. (a) Expand the amplitude transmittance as follows:

$$t_A(r) = \left(\frac{1}{2} + \frac{1}{4}e^{j\gamma r^2} + \frac{1}{4}e^{-j\gamma r^2}\right),$$

and compare the second and third term with the amplitude transmittance of a lens with focal length f:

$$t_l(r) = e^{-j\frac{kr^2}{2f}}.$$

We see that the second and third terms of this transmittance function are of the same form as the transmittance function of a lens. Thus the structure behaves simultaneously as two different lenses, one positive and one negative, in addition to having a bias term that only attenuates the incident wavefront.

(b) If γ is positive, the first quadratic-phase term in t_A can be interpreted as a negative lens with focal length

$$f = -\frac{k}{2\gamma}$$

while the second quadratic-phase term can be interpreted as a positive lens with focal length

$$f = \frac{k}{2\gamma}.$$

(c) The focal lengths given by the above two equations are both functions of wavelength, since $k = 2\pi/\lambda$. Therefore if the object has any significant spectral spread, the image will experience severe degradation.

5-14. The circular bounding aperture will not affect the problem, so we ignore it. From the definition provided by Fig. P5.14, it is clear that the following is true:

$$\begin{aligned} t_A(r) &= \left[\frac{1}{2} + \frac{1}{2}\text{sgn}(\cos\gamma r^2)\right] \\ &= f(r^2) = \sum_{n=-\infty}^{\infty}\left[\frac{\sin(\pi n/2)}{\pi n}\right]\exp\left(jn\gamma r^2\right), \end{aligned}$$

where we have used the fact that the period X must be replaced by

$$X = \frac{2\pi}{\gamma}.$$

Noting that quadratic-phase structures can be interpreted as being equivalent to lenses, we see that the structure is equivalent to an infinite number of positive and negative lenses of different focal lengths, plus a bias term. Comparing these terms with the amplitude transmittance of a lens, $t_l(r) = e^{-j\frac{kr^2}{2f}}$, the focal length of the nth term in the series is seen to be

$$f_n = \pm\frac{k}{2n\gamma}.$$

where the positive sign is used for all terms having a negative quadratic-phase factor, the negative sign is used for those with a positive quadratic-phase factor, and $k = 2\pi/\lambda$. The relative amount of optical power contributing to the nth term is the squared magnitude of the corresponding Fourier coefficient in the expansion with respect to r^2. Thus for the nth term the fraction of power contributing is

$$\eta_n = \left[\frac{\sin(\pi n/2)}{\pi n}\right]^2.$$

5-15. Change variables of integration in Eq. (5-33) to $\hat{x} = x/\lambda z_2, \hat{y} = y/\lambda z_2$. Then the equation can be re-written

$$h(u, v; \xi, \eta) \approx M \iint_{-\infty}^{\infty} P(\lambda z_2\hat{x}, \lambda z_2\hat{y})\exp\left\{-j2\pi[(u - M\xi)\hat{x} + (v - M\eta)\hat{y}]\right\}\, d\hat{x}d\hat{y}.$$

Now consider the behavior of this Fourier transform as $\lambda \to 0$. Remembering that u, v, M and z_2 are to be considered fixed, we see that as λ shrinks the effect is to broaden the function $P(\lambda z_2\hat{x}, \lambda z_2\hat{y})$ in the $(\hat{x}, \hat{y})$ plane indefinitely. In the limit, the integral is the Fourier transform of a function that is unity everywhere, yielding

$$h(u,v;\xi,\eta) \approx M \iint_{-\infty}^{\infty} 1 \cdot \exp\{-j2\pi[(u - M\xi)\hat{x} + (v - M\eta)\hat{y}]\}\, d\hat{x}d\hat{y} = M\delta(u - M\xi, v - M\eta).$$

Equivalently we can write

$$h(u,v;\xi,\eta) \approx \frac{1}{|M|}\delta\left(\xi - \frac{u}{M}, \eta - \frac{v}{M}\right).$$

5-16. Referring to Eq. (5-57), we see the following:

(a) For $z_1 \to \infty$ and $d \to 0$, we have plane wave illumination and the object against the lens. Under such conditions, the distance z_2 is equal to the focal length f. The equation becomes

$$U_2(u) = \frac{\exp\left[j\frac{ku^2}{2f}\right]}{\sqrt{\lambda f}} \int_{-\infty}^{\infty} U_1(\xi) \exp\left[-j\frac{2\pi}{\lambda f}u\xi\right] d\xi.$$

(b) For $z_1 \to \infty$ and $d \to f$, we have the object illuminated by a normally incident plane wave and situated in the front focal plane. Again $z_2 \to f$. In this case we obtain

$$U_2(u) = \frac{1}{\sqrt{\lambda f}} \int_{-\infty}^{\infty} U_1(\xi) \exp\left[-j\frac{2\pi}{\lambda f}u\xi\right] d\xi.$$

(c) For $z_1 \to \infty$ and d an arbitrary distance, again we have $z_2 \to f$ and we find

$$U_2(u) = \frac{\exp\left[j\frac{k}{2f}\left(1 - \frac{d}{f}\right)u^2\right]}{\sqrt{\lambda f}} \int_{-\infty}^{\infty} U_1(\xi) \exp\left[-j\frac{2\pi}{\lambda f}u\xi\right] d\xi.$$

5-17. (a) Passage of light of wavelength λ from the front focal plane to the back focal plane of a positive lens with focal length f is described by the operator $\mathcal{V}\left[\frac{1}{\lambda f}\right]\mathcal{F}$. Thus the sequence of two Fourier transforms performed by this optical system can be represented by the operator $\mathcal{V}\left[\frac{1}{\lambda f_2}\right]\mathcal{F}\mathcal{V}\left[\frac{1}{\lambda f_1}\right]\mathcal{F}$.

(b) Equation (5-46) can be used to simplify these operations. We have

$$\mathcal{V}\left[\frac{1}{\lambda f_2}\right]\mathcal{F}\,\mathcal{V}\left[\frac{1}{\lambda f_1}\right]\mathcal{F} = \mathcal{V}\left[\frac{1}{\lambda f_2}\right]\mathcal{V}\left[\lambda f_1\right]\mathcal{F}\,\mathcal{F} = \mathcal{V}\left[\frac{f_1}{f_2}\right]\mathcal{V}\left[-1\right] = \mathcal{V}\left[-\frac{f_1}{f_2}\right]$$

where Eqs. (5-47) and (5-45) have both been used. Thus the image is both inverted and scaled by the magnification $M = f_2/f_1$.

Chapter 6

6-1. We can answer the questions posed in this problem if we find the intensity point-spread function. From Eqs. (6-4) and (6-5), we know that the intensity point-spread function of an incoherent system is the squared magnitude of the (properly scaled) Fourier transform of the exit pupil illumination. The amplitude transmittance of the exit pupil in this case can be written

$$t_A(x,y) = \text{circ}\left(\frac{2r}{d}\right) \otimes [\delta(x - s/2, y) + \delta(x + s/2, y)]$$

where $r = \sqrt{x^2 + y^2}$. The Fourier transform of this expression is

$$\mathcal{F}\left\{t_A(x,y)\right\} = \pi\left(\frac{d}{2}\right)^2 2\frac{J_1(\pi d\rho)}{\pi d\rho} \times 2\cos(\pi s f_X),$$

where $\rho = \sqrt{f_X^2 + f_Y^2}$. Taking the squared magnitude of this expression, using the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, and introducing the scaling parameters appropriate for the optical Fourier transform, we obtain the following expression for the intensity point-spread function (under the assumption that the intensity of the wave at the exit pupil is unity):

$$I(u,v) = |h(u,v)|^2 = \frac{\pi^2 d^4}{16\lambda^2 z_i^2}\left[2\frac{J_1\left(\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}\right)}{\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}}\right]^2 \left[1 + \cos\left(\frac{2\pi s u}{\lambda z_i}\right)\right].$$

We can now answer the specific questions of the problem:

(a) The spatial frequency of the fringe is clearly given by

$$f_0 = \frac{s}{\lambda z_i}.$$

Note that the fringe frequency increases as the separation between the two apertures increases.

(b) The envelope of the fringe pattern is seen to be an Airy pattern of the form

$$E(u,v) = \left[2\frac{J_1\left(\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}\right)}{\frac{\pi d\sqrt{u^2+v^2}}{\lambda z_i}}\right]^2,$$

where the scaling factor preceding the Airy pattern has been neglected.

6-2. The physical quantities to follow are amplitudes in the case of a coherent system and intensities in the case of an incoherent system. $p(x,y)$ represents the (amplitude or intensity) point-spread function.

(a) A line excitation lying along the x axis would be represented by

$$o(x,y) = \delta(y).$$

The response to such an excitation would be

$$\begin{aligned} i_1(x,y) &= p(x,y) \otimes o(x,y) = p(x,y) \otimes \delta(y) \\ &= \iint_{-\infty}^{\infty} p(\xi,\eta)\,\delta(y-\eta)\,d\xi d\eta = \int_{-\infty}^{\infty} p(\xi,y)\,d\xi = l(y) \end{aligned}$$

(b) Consider a one-dimensional Fourier transform of the line-spread function:

$$\begin{aligned} \mathcal{F}\{l(y)\} &= \iint_{-\infty}^{\infty} p(\xi,y)\exp[-j2\pi f y]\,d\xi dy \\ &= \iint_{-\infty}^{\infty} p(\xi,y)\exp\left[-j2\pi((\xi f_X + y f_Y)\right]\,d\xi dy\ \Big|_{\substack{f_X=0\\ f_Y=f}} = P(0,f). \end{aligned}$$

(c) The unit step function will be represented by

$$s(x,y) = \begin{cases} 0 & y<0 \\ 1 & y>0 \end{cases}.$$

Therefore the response of the system will be

$$i_2(x,y) = p(x,y) \otimes s(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} p(\xi,\eta)\,d\xi d\eta = \int_{-\infty}^{y} l(\eta)d\eta$$

Thus

$$\text{step response} = \int_{-\infty}^{y} l(\eta)d\eta.$$

6-3. (a) The the f_X-axis and f_Y-axis sections of the OTF of a clear square pupil are already known to be identical triangle functions, dropping linearly to zero at frequency $2f_o = \frac{2w}{\lambda z_i}$ from value unity at the origin. Such a curve is included in part (a) of the figure. More interesting is the case with a central obscuration. We can calculate either the f_X section or the f_Y section, since they are identical. Note that the total area of the obscured pupil is $4w^2 - w^2 = 3w^2$, which must be used as a normalizing factor for the autocorrelation function. In calculating the autocorrelation function of the pupil, we shift one version of the pupil in the x direction with respect to the other version. As the shift takes place, the area of overlap drops from $3w^2$ with no shift, linearly to $3w^2/2$ at a shift of $f_0/2$. With further shift, the curve changes slope, dropping linearly to value w^2 at shift f_o. Continuing shift results in no change of overlap until the shift is $3f_o/2$, following which the curve falls linearly to zero at $2f_o$. Part (a) of the figure shows the properly normalized OTF that results.

(b) Suppose that the width of the stop is $2w - 2\epsilon$. The total clear area of the pupil become $4w^2 - (2w-2\epsilon)^2 = 8w\epsilon - 4\epsilon^2 \approx 8w\epsilon$. As the two pupils are shifted, the overlap area quickly drops to $2(2w-\epsilon)\epsilon \approx 4w\epsilon$ after a shift of ϵ. The overlap then continues to drop linearly, but with a shallower slope, reaching value $4\epsilon^2$ for a shift of $2w - 2\epsilon$. Continued shifting results in a rapid linear *rise* in the overlap to a value of $2w\epsilon$ when the displacement is $2w - \epsilon$, following which it falls linearly to zero at displacement $2w$. After proper normalization, the resulting OTF is as shown in part (b) of the figure.

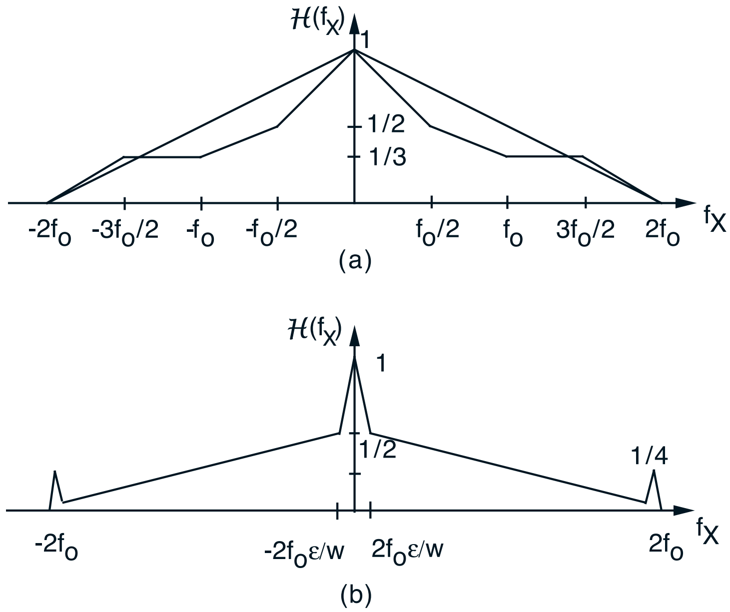

Figure 6-3:

6-4. For the calculation of the OTF section along the f_X axis, displacement in this direction does not change the *normalized* area of overlap with respect to the result for a full circle. Therefore,

$$\mathcal{H}(f_X, 0) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{f_X}{2f_o}\right) - \left(\frac{f_X}{2f_o}\right)\sqrt{1-\left(\frac{f_X}{2f_o}\right)^2}\right]$$

for $|f_X| \leq 2f_o$, and zero otherwise. Here $f_o = \frac{w}{\lambda z_i}$. The figure below shows the overlap of the half-circular pupils under displacement in the y direction.

Half of the area of overlap, represented by A in the figure, is found by taking the area of the circular sector defined by angle θ, and subtracting from it the triangle that composes the bottom portion of that sector. The resulting area of overlap is

$$2A = \frac{\theta}{2\pi}\pi w^2 - \frac{1}{2}(\Delta y)\sqrt{w^2 - (\Delta y)^2}.$$

The angle θ can be written

$$\theta = \cos^{-1}\left(\frac{\Delta y}{w}\right).$$

After normalization by the total area of the half circle , the OTF section becomes

$$\mathcal{H}(0, f_Y) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{f_Y}{f_o}\right) - \left(\frac{f_Y}{f_o}\right)\sqrt{1-\left(\frac{f_Y}{f_o}\right)^2}\right]$$

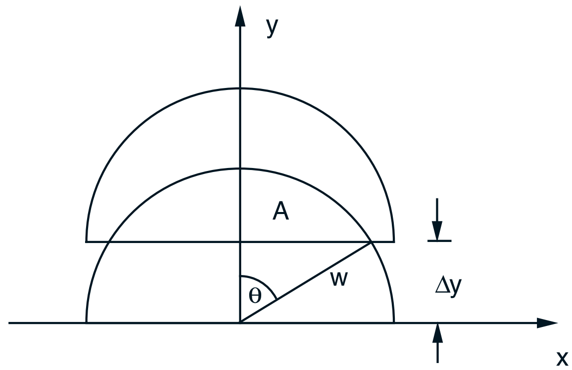

Figure 6-4:

for $|f_Y| \leq f_o$. Note that the OTF extends only half as far in the f_Y direction as it does in the f_X direction.

6-5. The figure below illustrates the overlap of two triangles when one is displaced along the x axis and also when one is displaced along the y axis.

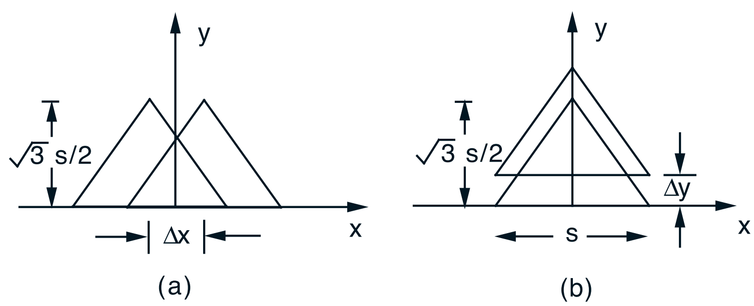

Figure 6-5:

Note that the area of the pupil is that of an equilateral triangle of side s, or

$$A = \frac{1}{2} s \times \frac{\sqrt{3}}{2} s = \frac{\sqrt{3} s^2}{4}.$$

When the shift between triangles is horizontal, as shown in part (a) of the figure, and equal to Δx, the region of overlap remains an equilateral triangle, but the length of a side is reduced to $s - \Delta x$. The area thus becomes $\frac{\sqrt{3}}{4}(s - \Delta x)^2$, which after normalization and proper scaling yields an OTF of the form

$$\mathcal{H}(f_X, 0) = \left(1 - \frac{f_X}{f_o}\right)^2,$$

where in this case $f_o = \frac{s}{\lambda z_i}$ and the OTF vanishes for $|f_X| > f_o$. If the displacement is vertical, as shown in part (b) of the figure, and equal to Δy, the region of overlap remains an equilateral triangle, but with a height $\frac{\sqrt{3}}{2}s - \Delta y$, and therefore with a side of length $\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}s - \Delta y\right)$. The area of this triangle is

$$A' = \frac{1}{2}\left(\frac{\sqrt{3}}{2}s - \Delta y\right) \times \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}s - \Delta y\right) = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}s - \Delta y\right)^2 .$$

Normalizing by the area of the pupil we obtain an OTF given by

$$\mathcal{H}(0, f_Y) = \left(1 - \frac{2}{\sqrt{3}}\frac{f_Y}{f_o}\right)^2 ,$$

where again $f_o = \frac{s}{\lambda z_i}$ and the OTF vanishes for $|f_Y| > \frac{\sqrt{3}}{2}f_o$.

6-6. In the f_X direction, shift of the pupil with respect to itself yields an overlap which, when normalized by the total area of the pupil, is indistinguishable from the autocorrelation of just one of the circular openings. Therefore

$$\mathcal{H}(f_X, 0) = Q(f_X) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{f_X}{2f_o}\right) - \left(\frac{f_X}{2f_o}\right)\sqrt{1 - \left(\frac{f_X}{2f_o}\right)^2}\right]$$

where $f_o = \frac{w}{\lambda z_i}$. When the displacement is in the y direction, the behavior of the autocorrelation is quite different. In this case the autocorrelation consists of a central island and two islands displaced to the left and the right of the origin, each with a strength that is half that of the central island. The shape of the islands is identical to the shape obtained in the case of an x-displacement. Thus we have

$$\mathcal{H}(0, f_Y) = Q(f_Y) + \frac{1}{2}Q(f_Y - \bar{f}) + \frac{1}{2}Q(f_Y + \bar{f}),$$

where the function Q is defined above and $\bar{f} = \frac{2d}{\lambda z_i}$. Plots of these functions are shown in the figure.

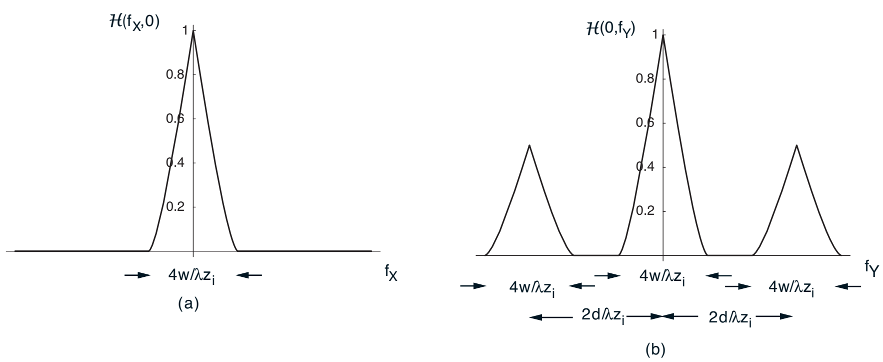

Figure 6-6:

6-7. To find the OTF of this system under various assumptions, we first find the intensity point-spread functions under those conditions. If the object is a point source, then under the assumption that z_o is very large, we can assume that the pinhole is illuminated by a normally-incident plane wave.

(a) Under the assumption that geometrical optics can be used when the pinhole is large, the point-spread function is in this case simply a projection of the pupil function onto the image plane. Since the incident wave has been approximated as plane, the diameter of the circular spread function is the same as the diameter of the circular pupil. Thus the point-spread function is given by

$$s(u,v) = A\,\mathrm{circ}\left(\frac{r}{w}\right)$$

where A is an arbitrary constant, and $r = \sqrt{u^2+v^2}$. The corresponding OTF is the normalized Fourier transform of $s(u,v)$, so

$$\mathcal{H}(\rho) = 2\frac{J_1(2\pi w\rho)}{2\pi w\rho},$$

where $\rho = \sqrt{f_X^2+f_Y^2}$. The first zero of this OTF occurs at

$$\rho_{o1} = \frac{0.61}{w}.$$

Note that the cutoff frequency *decreases* as the pinhole size increases.

(b) Now the pinhole is assumed to be so small that Fraunhofer diffraction occurs between the aperture and the image plane. The point-spread function of the system will now be the scaled optical Fourier transform of the circular aperture distribution, namely

$$s(u,v) = I_o\left[2\frac{J_1(2\pi wr/\lambda z_i)}{2\pi wr/\lambda z_i}\right]^2.$$

A scaled and normalized Fourier transform of this function yields the OTF

$$\mathcal{H}(\rho) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{\rho}{2w/\lambda z_i}\right) - \left(\frac{\rho}{2w/\lambda z_i}\right)\sqrt{1-\left(\frac{\rho}{2w/\lambda z_i}\right)^2}\right]$$

which vanishes at

$$\rho_{o2} = \frac{2w}{\lambda z_i}.$$

Note that this cutoff frequency *increases* as the diameter of the pinhole increases.

(c) If we start with a large pinhole, geometrical optics will hold, and the cutoff frequency will increase as we make the pinhole smaller. However, eventually the pinhole size will be so small that geometrical optics does not hold, and eventually the Fraunhofer approximation will be valid. In this case the cutoff frequency will decrease as we make the pinhole smaller. A good approximation to the optimum choice of pinhole diameter can be found by equating the two expressions for cutoff frequency,

$$0.61/w = 2w/\lambda z_i,$$

yielding a solution for the radius w given by

$$w_{\text{optimum}} = \sqrt{0.305}\sqrt{\lambda z_i}.$$

This solution has chosen the smallest pinhole size possible before diffraction spreads the point-spread function appreciably.

6-8. If the point-spread function is to be the convolution of the diffraction-limited spread function with the geometrical-optics spread function, the OTF must be the product of the two corresponding OTFs. We focus on the OTFs from this point on. We wish to compare the OTF of a misfocused system, given by Eq. (6-41) and repeated below,

$$\begin{aligned}\mathcal{H}(f_X, f_Y) &= \Lambda\left(\frac{f_X}{2f_o}\right)\Lambda\left(\frac{f_Y}{2f_o}\right)\\ &\times \ \text{sinc}\left[\frac{8W_m}{\lambda}\left(\frac{f_X}{2f_o}\right)\left(1-\frac{|f_X|}{2f_o}\right)\right]\text{sinc}\left[\frac{8W_m}{\lambda}\left(\frac{f_Y}{2f_o}\right)\left(1-\frac{|f_Y|}{2f_o}\right)\right],\end{aligned}$$

with the product of the diffraction-limited OTF, given by

$$\mathcal{H}(f_X, f_Y) = \Lambda\left(\frac{f_X}{2f_o}\right)\Lambda\left(\frac{f_X}{2f_o}\right),$$

and the geometrical-optics OTF (from Eq. (6-42))

$$\mathcal{H}(f_X, f_Y) = \text{sinc}\left[\frac{8W_m}{\lambda}\left(\frac{f_X}{2f_o}\right)\right]\text{sinc}\left[\frac{8W_m}{\lambda}\left(\frac{f_Y}{2f_o}\right)\right].$$

It is clear that the first equation is not the product of the second and third equations, due to the presence of the terms $\left(1-\frac{|f_X|}{2f_o}\right)$ and $\left(1-\frac{|f_Y|}{2f_o}\right)$ in the arguments of the sinc functions. Therefore the point-spread function can not be the convolution of the spread function due to diffraction and the spread function due to geometrical optics.

6-9. Note that the point-spread function, with or without aberrations, can be expressed (up to a constant multiplier) by

$$|h(u,v)|^2 = \iint\limits_{-\infty}^{\infty} \mathcal{H}(f_X, f_Y)\, e^{j2\pi(f_X u + f_Y v)}\, df_X df_Y.$$

Since the peak of the point-spread function is assumed to exist at the origin (i.e. on the optical axis), the relevant expression for that peak is

$$|h(0,0)|^2 = \iint\limits_{-\infty}^{\infty} \mathcal{H}(f_X, f_Y)\, df_X df_Y,$$

whether aberrations are present or not. Since the Strehl definition is the ratio of the peak intensities in the point-spread functions with and without aberrations, it follows that

$$\mathcal{D} = \frac{\iint\limits_{-\infty}^{\infty} \mathcal{H}(f_X, f_Y)_{\text{with}}\, df_X df_Y}{\iint\limits_{-\infty}^{\infty} \mathcal{H}(f_X, f_Y)_{\text{without}}\, df_X df_Y}.$$

6-10. The fundamental frequency of the square wave is $f_1 = 1/L = 100$ cycles/mm. Since the focal length is 10 cm and the object distance is 20 cm, the image distance will likewise be 20 cm.

(a) For coherent illumination we require that $f_1 \leq f_o$, where f_o is the coherent cutoff frequency,

$$f_o = \frac{w}{\lambda z_i}.$$

We are given the parameter values $\lambda = 10^{-3}$ mm, $z_i = 200$ mm, and we find the requirement that

$$f_1 \leq \frac{w}{\lambda z_i}$$

leads to the requirement that

$$w \geq \lambda z_i f_1 = 10^{-3} \times 200 \times 100 = 20\,\text{mm} = 2\,\text{cm},$$

which implies that the lens diameter $2w$ should be at least 4 cm.

(b) In the incoherent case the period of the square-wave object remains unchanged, but the cutoff frequency is now $2f_o$. It follows that the lens diameter can be half as big, or $2w \geq 2$ cm.

6-11. The intensity transmittance is given by

$$\tau(\xi,\eta) = \frac{1}{2}\left[1 + \cos\left(2\pi \tilde{f}\xi\right)\right]$$

and the object has uniform, constant phase shift. To find the amplitude transmittance, we note

$$\tau(\xi,\eta) = \cos^2\left[2\pi\left(\frac{\tilde{f}}{2}\right)\xi\right],$$

implying that

$$t_A(\xi,\eta) = \sqrt{\tau(\xi,\eta)} = \left|\cos\left[2\pi\left(\frac{\tilde{f}}{2}\right)\xi\right]\right| e^{j\phi}$$

where ϕ is a constant phase shift that will hereafter be dropped, and the absolute value signs are required due to the fact that the amplitude transmittance can not change sign if its phase is constant. It is clear that for incoherent illumination, the frequency of the variations of object intensity is $\tilde{f}$. For coherent illumination, we must use the fact that the fundamental frequency of the magnitude of a cosine is twice the frequency of the cosine without absolute signs. Therefore the fundamental frequency of amplitude variations is $\tilde{f}$. The coherent cutoff frequency is $f_o = \frac{w}{\lambda z_i} = \frac{w}{2\lambda f}$ while the cutoff frequency in the incoherent case is $2f_o = \frac{2w}{\lambda z_i} = \frac{w}{\lambda f}$. Thus in the coherent case we require

$$\bar{f} \leq \frac{w}{2\lambda f},$$

while in the incoherent case we require

$$\bar{f} \leq \frac{w}{\lambda f}.$$

Thus the frequency $\tilde{f}$ of the object can be twice as large in the incoherent case as it can be in the coherent case.

6-12. From the statement of the problem we can see that we are dealing with a coherent system. The object illumination can be represented by

$$U_o(\xi,\eta) = \exp\left(j2\pi f_i \xi\right)$$

where

$$f_i = \frac{\cos(\pi/2 - \theta)}{\lambda} = \frac{\sin\theta}{\lambda}.$$

(a) The light transmitted by the object will be the product of the illumination amplitude and the amplitude transmittance, or

$$U_o'(\xi,\eta) = \frac{1}{2}\left[1+\cos\left(2\pi\tilde{f}\xi\right)\right]\, e^{j2\pi f_i \xi}.$$

The Fourier spectrum of this object is

$$\mathcal{F}\{U_o'(\xi,\eta)\} = \left[\frac{1}{2}\delta(f_X - f_i) + \frac{1}{4}\delta(f_X - \tilde{f} - f_i) + \frac{1}{4}\delta(f_X + \tilde{f} - f_i)\right]\, \delta(f_Y).$$

(b) The figure illustrates the finite amplitude transfer function and the object frequency components present. Noting that $z_i = 2f$, the cutoff frequency of the amplitude transfer function is

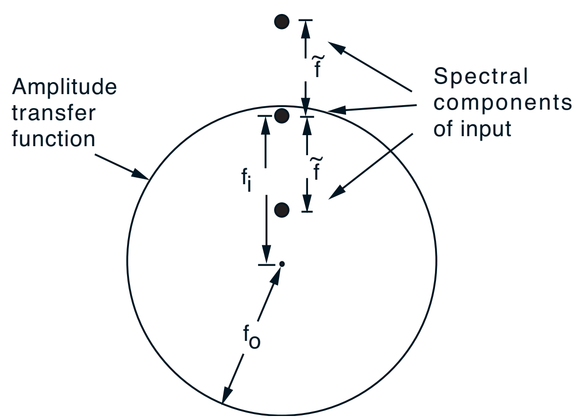

Figure 6-12:

$$f_o = w/2\lambda f.$$

To obtain any variations of intensity in the image, it is necessary that at least two spectral components of the object be passed by the amplitude transfer function. From the figure, this will be the case (assuming $\tilde{f} \le 2f_o$) provided

$$f_i \le f_o,$$

or equivalently provided

$$\sin\theta \le \frac{w}{2f}.$$

(c) Assuming two components of the spectrum of the object are passed by the amplitude transfer function, the intensity will be

$$\begin{aligned} I(u,v) &= \left|\frac{1}{2}\, e^{j2\pi f_i u} + \frac{1}{4} e^{j2\pi (f_i - \tilde{f})u}\right|^2 = \frac{1}{4}\left|1 + 2\, e^{-j2\pi\tilde{f}u}\right|^2 \\ &= \frac{1}{4}\left[5 + 4\,\cos\left(2\pi\tilde{f}u\right)\right]. \end{aligned}$$

The corresponding intensity when $\theta = 0$ is (assuming that $\tilde{f} \leq f_o$) is

$$\begin{aligned} I(u,v) &= \left|\frac{1}{2}\left[1+\cos\left(2\pi\tilde{f}u\right)\right]\right|^2 = \frac{1}{4}\left[1+2\cos\left(2\pi\tilde{f}u\right)+\cos^2\left(2\pi\tilde{f}u\right)\right] \\ &= \frac{1}{4}\left[\frac{3}{2}+2\cos\left(2\pi\tilde{f}u\right)+\frac{1}{2}\cos\left(4\pi\tilde{f}u\right)\right]. \end{aligned}$$

(d) When the maximum angle of illumination is used, the maximum value of $\tilde{f}$ that will yield intensity variations in the image is

$$\tilde{f}\text{max} = 2f_o,$$

which is twice the frequency that will yield intensity variations when the illumination is not oblique.

6-13. Recall that

$$f_o = \frac{w}{\lambda z_i} = \frac{w}{\lambda f},$$

where the last step holds because the object is at infinite distance from the lens, and the lens law implies that $z_i = f$. The F-number of the lens is

$$F\# = \frac{f}{2w}.$$

Solving for f in the equation above, and substituting that expression in the first equation yields

$$f_o = \frac{1}{2\lambda F\#}.$$

6-14. Let $s(u) = |h(u,0)|^2$. Then the Sparrow resolution distance (in the image space) will be the δ that satisfies the equation

$$\frac{d^2}{du^2}\left[s\left(u-\frac{\delta}{2}\right)+s\left(u+\frac{\delta}{2}\right)\right]_{u=0} = 0.$$

(a) By the symmetry of $s(u)$, $\frac{d^2}{du^2}s(u)$ is also symmetric in u, as proved by the following argument. Since $s(u)$ is real and even, $S(f_X)$ must also be real and even (from the symmetry properties of Fourier transforms). But

$$\mathcal{F}\{\frac{d^2}{du^2}s(u)\} = -(2\pi f_X)^2 S(f_X).$$

Since f_X^2 is real and even, we see that the entire transform of the second derivative is real and even, implying that its inverse transform (i.e. the second derivative) must be real and even. It now follows that

$$\frac{d^2}{du^2}s\left(u-\frac{\delta}{2}\right)+\frac{d^2}{du^2}s\left(u+\frac{\delta}{2}\right)\bigg|_{u=0} = 2\frac{d^2}{du^2}s(\delta/2) = 0$$

must be satisfied, as was to be proved.

(b) The intensity point-spread function for an incoherent system with a square aperture is known to have a u-dependence if the form $\text{sinc}^2\,\frac{2wu}{\lambda z_i}$. For simplicity, let $y = \frac{2wu}{\lambda z_i}$. Then we wish to find the value of y for which

$$\frac{d^2}{dy^2}\text{sinc}^2 y = 0.$$

Note that

$$\begin{aligned}\frac{d}{dy}\text{sinc}^2 y &= 2\,\text{sinc}\, y\ \frac{d}{dy}\text{sinc}\, y\\ \frac{d^2}{dy^2}\text{sinc}^2\, y &= 2\,\text{sinc}\, y\ \frac{d^2}{dy^2}\text{sinc}\, y + 2\left[\frac{d}{dy}\text{sinc}\, y\right]^2.\end{aligned}$$

Performing the required differentiations we find

$$\begin{aligned}\frac{d^2}{dy^2}\text{sinc}\, y &= \frac{1}{\pi}\frac{y(2-\pi^2y^2)\sin\pi y - 2\pi y^2\cos\pi y}{y^4}\\ \frac{d}{dy}\text{sinc}\, y &= \frac{1}{\pi}\frac{\sin\pi y - \pi y\cos\pi y}{y^2},\end{aligned}$$

so that

$$\frac{d^2}{dy^2}\text{sinc}^2\, y = \frac{2}{\pi^2}\frac{(3-\pi^2y^2)\sin^2\pi y - 4\pi y\ \cos\pi y\ \sin\pi y + \pi^2y^2\cos^2\pi y}{y^4}.$$

Finally, $\frac{d^2}{dy^2}\text{sinc}^2 y = 0$ implies

$$(3-\pi^2y^2)\sin^2\pi y - 4\pi y\ \cos\pi y\ \sin\pi y + \pi^2y^2\cos^2\pi y = 0,$$

which must be solved numerically. The result is $y = 0.415$. Since $y = \frac{2wu}{\lambda z_i}$, the solution is

$$u = 0.415\frac{\lambda z_i}{2w}.$$

The Sparrow separation is twice this distance, or

$$\delta = 0.83\frac{\lambda z_i}{2w}.$$

Note that this is a smaller separation than the Rayleigh resolution $1.22\frac{\lambda z_i}{2w}$.

6-15. Problem 6-2 is a great help in this problem. From Prob. 6-2(c) we know

$$\text{step response} = \int_{-\infty}^{v} l(\eta)\, d\eta$$

and from Prob. 6-2(b) we know that

$$\mathcal{F}\{l(v)\} = P(0, f)$$

where $P(f_X, f_Y)$ is the Fourier transform of the amplitude point-spread function in the coherent case, and the Fourier transform of the intensity point-spread function in the incoherent case.

(a) With coherent illumination, the Fourier transform of the amplitude point-spread function is a scaled version of the pupil. If the two pupils have the same width $2w$ in the y-direction, then $P(0, f)$ will be identical for the two systems. Therefore the line-spread functions will be the same and the step responses will be the same.

(b) With incoherent illumination, the Fourier transform of the intensity point-spread function is proportional to an autocorrelation function of the pupil. The autocorrelation functions of a circular pupil of diameter $2w$ and a square pupil of side $2w$, are different, even when evaluated only along the vertical axis. For the square pupil we have a (up to constants)

$$P(0,f) = \Lambda\left(\frac{f}{2f_o}\right),$$

while for the circular pupil we have

$$P(0,f) = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{f}{2f_o}\right) + \left(\frac{f}{2f_o}\right)\sqrt{1-\left(\frac{f}{2f_o}\right)^2}\right].$$

Therefore the line spread functions of the two systems are different and the step responses must likewise be different.

(c) The simplest approach to calculating step responses would be the following procedure:

- Since a unit step is the same function with coherent illumination as with incoherent illumination, we would first calculate the Fourier transform of a unit step using the Fast Fourier Transform algorithm. The calculation can be one-dimensional.
- We would then multiply this spectrum by the transfer function appropriate for the system of interest, whether it be a circular aperture or a square aperture, and whether the illumination be coherent or incoherent. The calculation would be one dimensional, using a slice of the appropriate transfer function.
- An inverse Fourier transform, again one dimensional and again performed using the Fast Fourier Transform, would yield the desired step response in each case.

6-16. The amplitude point-spread function for a coherent system with a square aperture of width $2w$ is given(up to a constant multiplier) by

$$h(u,v) = \text{sinc}\left(\frac{2wu}{\lambda z_i}\right)\ \text{sinc}\left(\frac{2wv}{\lambda z_i}\right).$$

The input to the system is a one-dimensional coherent step with the step taking place along the η axis. The response $\sigma(u,v)$ will be

$$\sigma(u,v) = h(u,v) \otimes s(u,v) = \iint\limits_{-\infty}^{\infty} h(\alpha,\beta)s(u-\alpha,\beta)\,d\alpha d\beta$$

where

$$s(u,v) = \begin{cases} 0 & u<0 \\ 1 & u>0 \end{cases}$$

is the unit step function. Since $s(\alpha,\beta)$ is independent of β, integration with respect to that variable yields a constant, which we drop. The result is

$$\sigma(u,v) = \int_{-\infty}^{u} \text{sinc}\left(\frac{2w\alpha}{\lambda z_i}\right)\,d\alpha = \int_{-\infty}^{u} \frac{\sin\left(\frac{2\pi w\alpha}{\lambda z_i}\right)}{\frac{2\pi w\alpha}{\lambda z_i}}\,d\alpha.$$

Changing the variable of integration to

$$t = \frac{2\pi w\alpha}{\lambda z_i}$$

we find

$$\sigma(u,v) = K \int_{-\infty}^{\frac{2\pi wu}{\lambda z_i}} \frac{\sin t}{t}\, dt = K \left[\frac{\pi}{2} + \text{Si}\left(\frac{2\pi wu}{\lambda z_i}\right)\right].$$

where K is a constant and

$$\text{Si}(z) = \int_0^z \frac{\sin t}{t}\, dt.$$

It follows that the intensity response to the step excitation is

$$I_i(u,v) = |\sigma(u,v)|^2 = K^2 \left|\frac{\pi}{2} + \text{Si}\left(\frac{2\pi wu}{\lambda z_i}\right)\right|^2.$$

6-17. The intensities in the two cases are as follows:

$$\begin{aligned} I &= |A+a|^2 = A^2 + 2Aa + a^2 && \text{coherent} \\ I &= A^2 + a^2 && \text{incoherent.} \end{aligned}$$

It follows that in the two cases

$$\begin{aligned} \frac{\Delta I}{|A|^2} &= \frac{2Aa + a^2}{A^2} && \text{coherent} \\ \frac{\Delta I}{|A|^2} &= \frac{a^2}{A^2} && \text{incoherent.} \end{aligned}$$

Since $A \gg a$, it is clear that the perturbation of the desired intensity is much greater in the case of coherent noise than in the case of incoherent noise.

6-18. Consider a coherent wavefield described by

$$U(x,y;t) = U(x,y)\, e^{-j2\pi\nu t}.$$

The mutual intensity of such a wavefield at points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{aligned} J(x_1,y_1;x_2,y_2) &= \langle U(x_1,y_1;t)\, U^*(x_2,y_2;t)\rangle = U(x_1,y_1)\, U^*(x_2,y_2) \left\langle e^{-j2\pi\nu t}\, e^{j2\pi\nu t}\right\rangle \\ &= U(x_1,y_1)\, U^*(x_2,y_2). \end{aligned}$$

From Eq. (6-11) we see that this wavefield is fully coherent.

Chapter 7

7-1. The intensity distribution exposing the film is known to be of the form

$$\mathcal{I}(x,y) = \mathcal{I}_o + \Delta\mathcal{I}(x,y),$$

with the further restriction that

$$\Delta\mathcal{I} \ll \mathcal{I}_0.$$

Using Eq. (7-2), we know that the intensity transmittance of the processed transparency is related to the exposing intensity by

$$\begin{aligned}\tau_n &= K_n\mathcal{I}^{-\gamma_n} = K_n(\mathcal{I}_o + \Delta\mathcal{I})^{-\gamma_n} \\ &= K_n\mathcal{I}_o^{-\gamma_n}(1 + \Delta\mathcal{I}/\mathcal{I}_o)^{-\gamma_n} \approx K_n\mathcal{I}_o^{-\gamma_n}(1 - \gamma_n\Delta\mathcal{I}/\mathcal{I}_o),\end{aligned}$$

where the first two terms of a binary expansion have been retained in the last step. Letting $\tau_n = \tau_{on} + \Delta\tau_n$, we have

$$\tau_{on} + \Delta\tau_n = K_n\mathcal{I}_o^{-\gamma_n} - \gamma_n K_n\mathcal{I}_o^{-\gamma_n}\frac{\Delta\mathcal{I}}{\mathcal{I}_o},$$

from which we conclude that

$$\begin{aligned}\tau_{on} &= K_n\mathcal{I}_o^{-\gamma_n} \\ \Delta\tau_n &= -\gamma_n K_n\mathcal{I}_o^{-\gamma_n}\frac{\Delta\mathcal{I}}{\mathcal{I}_o} \\ \frac{\Delta\tau_n}{\tau_{on}} &= -\gamma_n\frac{\Delta\mathcal{I}}{\mathcal{I}_o}.\end{aligned}$$

Thus the contrast of the variations of intensity transmittance is linearly related to the contrast of the exposing intensity distribution, regardless of the photographic gamma.

7-2. The intensity distribution in the interference pattern is given by

$$\mathcal{I} = |A\exp(j2\pi\beta_1 y) + B\exp(j2\pi\beta_2 y)|^2 = A^2 + B^2 + 2AB\cos(2\pi\Delta\beta y)$$

where $\Delta\beta = \beta_1 - \beta_2$. We then pass this intensity pattern through the frequency characteristic of the MTF yielding an effective exposing intensity $\mathcal{I}'$ as follows:

$$\mathcal{I}' = \mathcal{F}^{-1}\{\mathcal{F}\{\mathcal{I}\}M(f)\} = M(0)\,(A^2 + B^2) + 2M(\Delta\beta)\,AB\cos(2\pi\Delta\beta y).$$

A positive transparency with a gamma of -2 is made, yielding an intensity transmittance

$$\tau_p = K_p(\mathcal{I}')^2$$

and an amplitude transmittance

$$t_A = \sqrt{\tau_p} = \kappa \mathcal{I}' = \kappa \left[M(0)\,(A^2 + B^2) + 2M(\Delta\beta)\,AB\cos(2\pi\Delta\beta y) \right].$$

The field in the rear focal plane can now be written

$$\begin{aligned} U(x,y) &= \frac{1}{\lambda f}\mathcal{F}\left\{ t_A \,\text{rect}\frac{x}{L}\,\text{rect}\frac{y}{L} \right\}_{\substack{f_X = x/\lambda f \\ f_Y = y/\lambda f}} \\ &= L^2 \,\text{sinc}\frac{Lx}{\lambda f}\,\text{sinc}\frac{Ly}{\lambda f} \otimes \\ &\quad \left\{ \kappa M(0)\,(A^2+B^2)\,\delta(y) + \kappa M(\Delta\beta)\,AB\left[\delta(y - \lambda f\Delta\beta) + \delta(y + \lambda f \Delta\beta)\right] \right\}. \end{aligned}$$

Continuing, and noting that $M(0) = 1$,

$$\begin{aligned} U(x,y) &= L^2\,\kappa\,\text{sinc}\frac{Lx}{\lambda f}\left\{ (A^2+B^2)\,\text{sinc}\frac{y}{\lambda f} \right. \\ &\quad \left. + M(\Delta\beta)\,AB\left[\text{sinc}\left(\frac{L}{\lambda f}(y - \Delta\beta\lambda f)\right) + \text{sinc}\left(\frac{L}{\lambda f}(y + \Delta\beta\lambda f)\right)\right]\right\}. \end{aligned}$$

We plot the distribution of light intensity along the y-axis (It has been assumed that the cross-products between the three terms of the field can be ignored) :

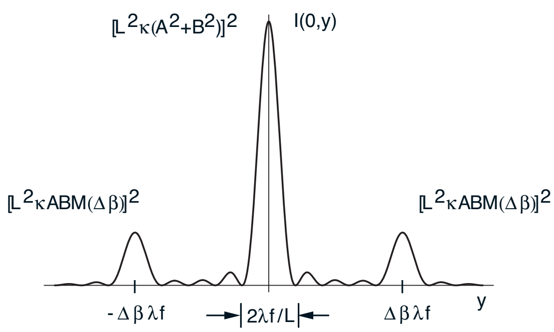

Figure 7-2:

7-3. The matrix manipulations required to prove the identities are outlined below:

$$\begin{aligned} \mathbf{L}_+ &= \begin{bmatrix} \cos\theta_t & -\sin\theta_t \\ \sin\theta_t & \cos\theta_t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\beta d} \end{bmatrix} \begin{bmatrix} \cos\theta_t & \sin\theta_t \\ -\sin\theta_t & \cos\theta_t \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_t & -\sin\theta_t \\ \sin\theta_t & \cos\theta_t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta_t & \sin\theta_t \\ -\sin\theta_t & \cos\theta_t \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_t & \sin\theta_t \\ \sin\theta_t & -\cos\theta_t \end{bmatrix} \begin{bmatrix} \cos\theta_t & \sin\theta_t \\ -\sin\theta_t & \cos\theta_t \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta_t - \sin^2\theta_t & 2\cos\theta_t\sin\theta_t \\ 2\cos\theta_t\sin\theta_t & \sin^2\theta_t - \cos^2\theta_t \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta_t & \sin 2\theta_t \\ \sin 2\theta_t & -\cos 2\theta_t \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\mathbf{L}_- &= \text{same as } \mathbf{L}_+ \text{ except } \theta_t \text{ is replaced by } -\theta_t \\
&= \begin{bmatrix} \cos 2\theta_t & -\sin 2\theta_t \\ -\sin 2\theta_t & -\cos 2\theta_t \end{bmatrix}.
\end{aligned}
$$

7-4. We will follow the path of the light incident on the cell and find the Jones matrix for each element (polarizer and FLC); by multiplying these matrices together, we can find the overall Jones matrix which relates the polarization vector of the light incident on the mirror with that of the light incident on the front of the device. We then multiply this matrix by its transpose to calculate the overall Jones matrix of the reflective device, in accord with Eq. (C-17) of Appendix C. Finally we apply the matrix $\mathbf{R}$ of Appendix C to return to a right-hand coordinate system.

First, we find the Jones matrices for the case with the long axis *not* parallel to the polarizer. We pick our coordinate system so that the direction of the polarizer coincides with the y axis.

- Polarizer: Using equation (C-16) with $\alpha = 90°$, we obtain:

$$
\mathbf{L}_{\text{polarizer}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
$$

- FLC: $\theta_t = 45°$:

$$
\begin{aligned}
\mathbf{L}_{\text{FLC}} &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\frac{\pi}{2}} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 1-j & 1+j \\ 1+j & 1-j \end{bmatrix}.
\end{aligned}
$$

- Single pass matrix:

$$
\mathbf{L}_{\text{single}} = \mathbf{L}_{\text{FLC}}\,\mathbf{L}_{\text{polarizer}} = \begin{bmatrix} 1-j & 1+j \\ 1+j & 1-j \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1+j \\ 0 & 1-j \end{bmatrix}.
$$

  For a double pass, we have

$$
\mathbf{L}_{\text{double}} = \mathbf{R}\,\mathbf{L}^t_{\text{single}}\,\mathbf{L}_{\text{single}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1+j & 1-j \end{bmatrix} \begin{bmatrix} 0 & 1+j \\ 0 & 1-j \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
$$

  Thus, we see that in the state where the molecular axis is oriented away from the polarizer direction, no light is transmitted by the cell.

When the molecular long axis is parallel to the polarizer, we have $\theta_t = 0°$ so that:

$$
\begin{aligned}
\mathbf{L}_{\text{FLC}} &= \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \\
\mathbf{L}_{\text{single}} &= \mathbf{L}_{\text{FLC}}\,\mathbf{L}_{\text{polarizer}} = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -j \end{bmatrix}.
\end{aligned}
$$

The double-pass Jones matrix becomes

$$
\mathbf{L}_{\text{double}} = \mathbf{R}\mathbf{L}^t_{\text{single}}\mathbf{L}_{\text{single}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -j \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.
$$

If the incident polarization vector is $\begin{bmatrix} U_X \\ U_Y \end{bmatrix}$, the emerging polarization vector is:

$$\vec{U}_{out} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} U_X \\ U_Y \end{bmatrix} = \begin{bmatrix} 0 \\ -U_Y \end{bmatrix}.$$

Thus, the two states of the FLC cell correspond to the output intensities 0 and $|U_Y|^2$; i.e. the cell can be used as a binary intensity modulator.

7-5. (a) Write the polarization vector of the given wave and express that vector as a weighted sum of left- and right-hand circularly polarized waves (coefficients C_R for right-hand and C_L for left-hand):

$$\begin{aligned} \vec{U}_\theta &= \mathbf{L}_R(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\ &= C_R \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -j \end{bmatrix} + C_L \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix}. \end{aligned}$$

This allows us to write two equations in the two unknowns C_R and C_L,

$$\begin{aligned} \frac{1}{\sqrt{2}}(C_R + C_L) &= \cos\theta \\ \frac{1}{\sqrt{2}} j(-C_R + C_L) &= \sin\theta. \end{aligned}$$

Solving for the unknowns, we find the coefficients to be

$$\begin{aligned} C_R &= \frac{1}{\sqrt{2}} e^{+j\theta} \\ C_L &= \frac{1}{\sqrt{2}} e^{-j\theta}. \end{aligned}$$

Thus we have demonstrated that such an expansion is possible and we have found the expansion coefficients.

(b) First make the following definitions:

$$\begin{aligned} \Delta_1 &= 2\pi n_1 d/\lambda_0, \\ \Delta_2 &= 2\pi n_2 d/\lambda_0, \\ \Delta &= 2\pi(n_1 - n_2)d/\lambda_0. \end{aligned}$$

Now, when the magnetic field points in the direction of wave propagation, we write the output polarization vector as

$$\begin{aligned} \vec{U}_{out} &= e^{j\Delta_1} C_L \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ j \end{bmatrix} + e^{j\Delta_2} C_R \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -j \end{bmatrix} \\ &= e^{j\frac{\Delta_1+\Delta_2}{2}} \frac{1}{2} \left(e^{j\frac{\Delta}{2}} e^{j\theta} \begin{bmatrix} 1 \\ j \end{bmatrix} + e^{-j\frac{\Delta}{2}} e^{+j\theta} \begin{bmatrix} 1 \\ -j \end{bmatrix} \right) \\ &= e^{j\frac{\Delta_1+\Delta_2}{2}} \begin{bmatrix} \cos(\Delta/2 + \theta) \\ \sin(\Delta/2 + \theta) \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= e^{j\frac{\Delta_1+\Delta_2}{2}} \begin{bmatrix} \cos\Delta/2\cos\theta - \sin\Delta/2\sin\theta \\ \sin\Delta/2\cos\theta + \cos\Delta/2\sin\theta \end{bmatrix} \\
&= e^{j\frac{\Delta_1+\Delta_2}{2}} \begin{bmatrix} \cos\Delta/2 & -\sin\Delta/2 \\ \sin\Delta/2 & \cos\Delta/2 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\
&= e^{j\frac{\Delta_1+\Delta_2}{2}} \mathbf{L}_+ \vec{U}_\theta.
\end{aligned}
$$

The constant phase factor in the front can be ignored, so $\mathbf{L}_+$ describes the given polarization transformation.

(c) Let the direction of the magnetic field be reversed. Switching the signs of n_2 and n_1 in the expression for Δ only changes its sign. Hence, we can obtain the expression for $\mathbf{L}_-$ by substituting $-\Delta$ for Δ in the expression for $\mathbf{L}_+$. Thus:

$$
\begin{aligned}
\mathbf{L}_- &= \begin{bmatrix} \cos(-\Delta)/2 & -\sin(-\Delta)/2 \\ \sin(-\Delta)/2 & +\cos(-\Delta)/2 \end{bmatrix} \\
&= \begin{bmatrix} \cos\Delta/2 & \sin\Delta/2 \\ -\sin\Delta/2 & +\cos\Delta/2 \end{bmatrix}.
\end{aligned}
$$

7-6. First write the Jones matrix of the polarization analyzer:

$$
\mathbf{L}_{\text{analyzer}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
$$

Now express the output polarization vector in terms of the input vector and the appropriate Jones matrices for the case of the magnetic field in the direction of wave propagation:

$$
\begin{aligned}
\vec{U}^+_{out} &= \mathbf{L}_{\text{analyzer}} \mathbf{L}_+ \begin{bmatrix} 0 \\ U_Y \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\Delta/2 & -\sin\Delta/2 \\ \sin\Delta/2 & \cos\Delta/2 \end{bmatrix} \begin{bmatrix} 0 \\ U_Y \end{bmatrix} \\
&= \begin{bmatrix} -U_Y \sin\Delta/2 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Now repeat the calculation with the magnetic field direction reversed:

$$
\begin{aligned}
\vec{U}^-_{out} &= \mathbf{L}_{\text{analyzer}} \mathbf{L}_- \begin{bmatrix} 0 \\ U_Y \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\Delta/2 & \sin\Delta/2 \\ -\sin\Delta/2 & \cos\Delta/2 \end{bmatrix} \begin{bmatrix} 0 \\ U_Y \end{bmatrix} \\
&= \begin{bmatrix} U_Y \sin\Delta/2 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Thus,

$$
|\vec{U}^+_{out}|^2 = |\vec{U}^-_{out}|^2 = |U_Y|^2 \sin^2\Delta/2
$$

and

$$
\vec{U}^+_{out} = -\vec{U}^-_{out}
$$

Thus the sign of the field has reversed, or a $180°$ phase shift has been introduced.

7-7. From equation (7-24),

$$\tau = \eta_P \, e^{-\alpha x} \, \sin^2(2\beta x).$$

Differentiating and setting the derivative equal to zero to find the maximum,

$$\begin{aligned}\frac{d\tau}{dx} &= \eta_P(-\alpha)e^{-\alpha x}\sin^2(2\beta x) + \eta_P e^{-\alpha x}(2)\sin(2\beta x)\cos(2\beta x)(2\beta) \\ &= \eta_P e^{-\alpha x}\sin(2\beta x)[-\alpha\sin(2\beta x) + 4\beta\cos(2\beta x)] \\ &= 0.\end{aligned}$$

The solution $\sin(2\beta x) = 0$ corresponds to $\tau = 0$, and therefore is an uninteresting solution. So set

$$-\alpha\sin(2\beta x) + 4\beta\cos(2\beta x) = 0.$$

Solving for the film thickness x,

$$\begin{aligned}x &= \frac{1}{2\beta}\tan^{-1}\left(\frac{4\beta}{\alpha}\right) \\ &= \frac{1}{2(1.46\times\pi/180)}\tan^{-1}\left(\frac{4\times 1.46\times\pi/180}{0.086}\right) \\ &= 17\,\mu\text{m}.\end{aligned}$$

7-8. (a) The amplitude transmittance of the unquantized grating within a single period can be represented by

$$t_A(x) = \begin{cases} e^{j2\pi(1+2x/L)} = e^{j4\pi x/L} & -\frac{L}{2} \le x < 0 \\ e^{j2\pi(1-2x/L)} = e^{-j4\pi x/L} & 0 \le x < \frac{L}{2} \end{cases}.$$

To find the diffraction efficiency of the grating, we must expand the amplitude transmittance in a Fourier series and find the Fourier coefficients. The required integrals can be expressed as

$$c_n = \frac{1}{L}\int_{-L/2}^{0} e^{-j2\pi(n-2)x/L}dx + \frac{1}{L}\int_{0}^{L/2} e^{-j2\pi(n+2)x/L}dx.$$

Skipping some of the steps in the evaluation, we have

$$\begin{aligned}c_n &= \left.\frac{e^{-j2\pi(n-2)x/L}}{-j2\pi(n-2)}\right|_{-L/2}^{0} + \left.\frac{e^{-j2\pi(n+2)x/L}}{-j2\pi(n+2)}\right|_{0}^{L/2} \\ &= \left[\frac{e^{j\pi(n-2)}-1}{j2\pi(n-2)}\right] + \left[\frac{1-e^{-j\pi(n+2)}}{j2\pi(n+2)}\right] \\ &= \left[\frac{e^{j\pi n}-1}{j2\pi(n-2)}\right] + \left[\frac{1-e^{-j\pi n}}{j2\pi(n+2)}\right].\end{aligned}$$

With some work the squared magnitude of the Fourier coefficients can be shown to be

$$|c_n|^2 = \left[\frac{n\,\sin\pi n}{\pi(n-2)(n+2)}\right]^2 + \left[\frac{2(1-\cos\pi n)}{\pi(n-2)(n+2)}\right]^2.$$

For n even and $n \neq \pm 2$, both terms vanish. These orders are missing. For n odd, both terms are non-zero. We focus only on the orders $n = -4, -3, -2, -1, 0, 1, 2, 3, 4$. The squared magnitudes are

$$\begin{aligned}|c_{-4}|^2 &= |c_4|^2 = 0\\ |c_{-3}|^2 &= |c_3|^2 = \frac{16}{25\pi^2} \approx 0.065\\ |c_{-2}|^2 &= |c_2|^2 = \frac{1}{4} = 0.25\\ |c_{-1}|^2 &= |c_1|^2 = \frac{16}{9\pi^2} \approx 0.18\\ |c_0|^2 &= 0.\end{aligned}$$

These are the diffraction efficiencies of the nine orders of interest. Note that the $\pm 1, \pm 2$, and ± 3 orders contain about 99% of the total power incident on the grating.

(b) Consider now the quantized grating. In this case we have an amplitude transmittance over one period of

$$t_A(x) = \begin{cases} e^{j0} = 1 & -L/2 \le x < -3L/8\\ e^{j\pi/2} = j & -3L/8 \le x < -L/4\\ e^{j\pi} = -1 & -L/4 \le x < -L/8\\ e^{j3\pi/2} = -j & -L/8 \le x < L/8\\ e^{j\pi} = -1 & L/8 \le x < L/4\\ e^{j\pi/2} = j & L/4 \le x < 3L/8\\ e^{j0} = 1 & 3L/8 \le x < L/2 \end{cases}.$$

Alternatively we can write $t_A(x)$ as

$$\begin{aligned} t_A(x) &= -j\left\{\text{rect}\left[\frac{8}{L}\left(x - \frac{L}{16}\right)\right] + \text{rect}\left[\frac{8}{L}\left(x + \frac{L}{16}\right)\right]\right\}\\ &- \left\{\text{rect}\left[\frac{8}{L}\left(x - \frac{3L}{16}\right)\right] + \text{rect}\left[\frac{8}{L}\left(x + \frac{3L}{16}\right)\right]\right\}\\ &+ j\left\{\text{rect}\left[\frac{8}{L}\left(x - \frac{5L}{16}\right)\right] + \text{rect}\left[\frac{8}{L}\left(x + \frac{5L}{16}\right)\right]\right\}\\ &+ \left\{\text{rect}\left[\frac{8}{L}\left(x - \frac{7L}{16}\right)\right] + \text{rect}\left[\frac{8}{L}\left(x + \frac{7L}{16}\right)\right]\right\}.\end{aligned}$$

The Fourier coefficients of this structure can now be evaluated as

$$\begin{aligned} c_n &= \frac{1}{L}\mathcal{F}\left\{t_A(x)\,\text{rect}\frac{x}{L}\right\}_{f_X=\frac{n}{L}}\\ &= \text{sinc}\frac{n}{8}\left[\frac{-j}{4}\cos\frac{\pi n}{8} + \frac{-1}{4}\cos\frac{3\pi n}{8} + \frac{j}{4}\cos\frac{5\pi n}{8} + \frac{1}{4}\cos\frac{7\pi n}{8}\right].\end{aligned}$$

It is now possible to evaluate $|c_n|^2$, either with a lengthy numerical calculation or with the help of a computer. The results are:

$$\begin{aligned}|c_0|^2 &= 0\\ |c_1|^2 &= |c_{-1}|^2 = 0.203\\ |c_2|^2 &= |c_{-2}|^2 = 0.203\\ |c_3|^2 &= |c_{-3}|^2 = 0.023\\ |c_4|^2 &= |c_{-4}|^2 = 0.\end{aligned}$$

Note that only about 86% of the light incident on this grating appears in this set of orders.

Chapter 8

8-1. The opaque stop on the optical axis blocks only the "DC" or constant Fourier component of the object, which is equivalent to subtracting the constant 4/5 from the given amplitude function. The intensity is found by taking the squared magnitude of this field. The resulting intensity of the filtered object is shown in the figure. Note the reversal of contrast in the image.

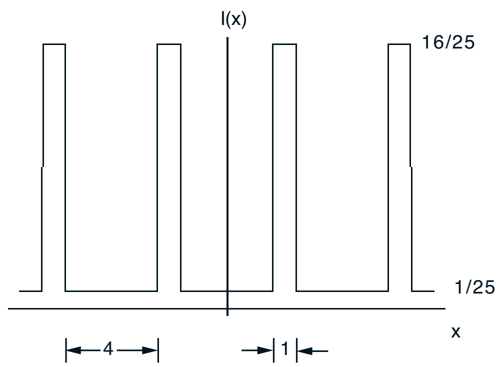

Figure 8-1:

8-2. Assume a unit magnification imaging system and neglect image inversion for simplicity. The phase object is represented by an amplitude transmittance

$$t_A(\xi,\eta) = e^{j\phi(\xi,\eta)} \approx 1 + j\phi(\xi,\eta),$$

where the approximation is allowable because $\phi \ll 2\pi$. Assuming that the spectrum of $\phi(\xi,\eta)$ is broad, the introduction of the small stop will have little effect on it, other than shifting its average value to zero. The shift of the phase by a constant phase factor is inconsequential, since we can always redefine the phase reference as we please. The stop will remove the constant 1, however. The intensity observed in the image plane will be

$$I(u,v) = |\phi(u,v)|^2.$$

8-3. (a) The Fourier transforms of the object and image amplitudes are related by

$$\mathcal{F}\{U_i(u,v)\} = \mathcal{F}\{U_o(x,y)\}\, t_A(\lambda f f_X, \lambda f f_Y).$$

Therefore the object and image amplitudes are related by

$$U_i(u,v) \quad = \quad U_o(u,v) \otimes \mathcal{F}^{-1}\{t_A(\lambda f f_X, \lambda f f_Y)\}$$

$$\begin{aligned}
&= U_o(u,v) \otimes \mathcal{F}^{-1}\{\frac{1}{2}(1+\text{sgn} f_X)\} \\
&= \frac{1}{2}U_o(u,v) \otimes \left[\delta(u) + \frac{j}{\pi u}\right] \\
&= \frac{1}{2}\left[U_o(u,v) + \frac{j}{\pi}\int_{-\infty}^{\infty}\frac{U_o(\xi,v)\,d\xi}{u-\xi}\right].
\end{aligned}$$

(b) We first approximate the object amplitude using the small phase approximation:

$$U_o(\xi,\eta) \quad = \quad e^{j\phi_o}\exp(j\Delta\phi) \approx e^{j\phi_o}(1+j\Delta\phi).$$

Using the result of part (a), the corresponding image intensity is

$$\begin{aligned}
I_i(u,v) &= |U_i(u,v)|^2 = \frac{1}{4}\left|U_o(u,v) + \frac{j}{\pi}\int_{-\infty}^{\infty}\frac{U_o(\xi,v)}{u-\xi}\,d\xi\right|^2 \\
&= \frac{1}{4}\left|e^{j\phi_o}\left(1+j\Delta\phi + \frac{j}{\pi}\int_{-\infty}^{\infty}\frac{1+j\Delta\phi(\xi,v)}{u-\xi}\,d\xi\right)\right|^2 \\
&= \frac{1}{4}\left[1-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Delta\phi(\xi,v)}{u-\xi}\,d\xi\right]^2 + \frac{1}{4}\left[\Delta\phi + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{u-\xi}\,d\xi\right]^2.
\end{aligned}$$

The second integral has value zero. In addition, we are assuming that $\Delta\phi$ is small, which allows us to make the approximation $(\Delta\phi)^2 \to 0$. Thus,

$$\begin{aligned}
I_i(u,v) &= \frac{1}{4}\left[1-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Delta\phi(\xi,v)}{u-\xi}\,d\xi\right]^2 \\
&\approx \frac{1}{4}\left[1-\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\Delta\phi(\xi,v)}{u-\xi}\,d\xi\right]
\end{aligned}$$

where we have used the binomial expansion in the last step, assuming that, due to the smallness of $\Delta\phi$,

$$\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Delta\phi(\xi,v)}{u-\xi}\,d\xi\right| \ll 1.$$

(c) The object is taken to have the phase distribution

$$\Delta\phi = \Phi\,\text{rect}\left(\frac{u}{U}\right).$$

The image intensity then takes the form

$$\begin{aligned}
I_i(u,v) &\approx \frac{1}{4}\left[1-\frac{2\Phi}{\pi}\int_{-U/2}^{U/2}\frac{d\xi}{u-\xi}\right] \\
&= \frac{1}{4}\left[1+\frac{2\Phi}{\pi}\ln\left|\frac{u-U/2}{u+U/2}\right|\right].
\end{aligned}$$

Note that the above expression is only valid when the assumption we have made in part (b) is satisfied. That is, it is only valid when the intensity is close to $1/4$. Thus the infinite discontinuities in the figure below are artifacts of the approximations. For the figure, the following values have been assumed: $U = 0.5$, $\Phi = 0.1$.

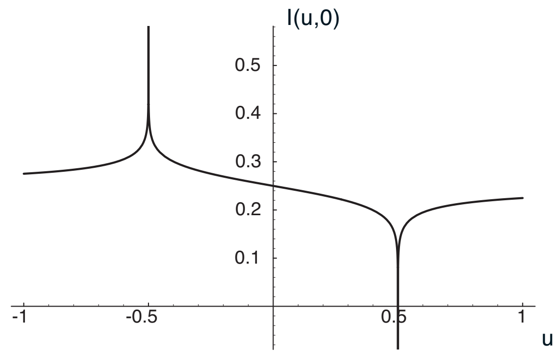

Figure 8-3:

8-4. Assume that the phase shifting dot retards the phase by $\pi/2$ radians. Represent the absorption of the dot by an amplitude transmittance $\sqrt{\alpha}$ applied only to the constant component. The intensity of the image (corresponding to Eq. (8-3)) becomes

$$I_i = |\sqrt{\alpha}\exp[j(\pi/2)] + j\Delta\phi|^2 = |j(\sqrt{\alpha} + \Delta\phi)|^2 \approx \alpha + 2\sqrt{\alpha}\,\Delta\phi.$$

Note that the contrast of the image variations,

$$C = \frac{2\sqrt{\alpha}\Delta\phi}{\alpha} = \frac{2\Delta\phi}{\sqrt{\alpha}},$$

is increased when $\alpha < 1$. A similar argument applies when the phase shift is $3\pi/2$, yielding

$$I_i \approx \alpha - 2\sqrt{\alpha}\,\Delta\phi.$$

8-5. Suppose we are trying to remove a delta function from the object spectrum (we choose the delta function because it gives the tightest alignment requirement). Treating the problem as one dimensional, with an input aperture function rect(ξ/D), the delta function appears in the Fourier plane as sinc$[D(x - x_o)/\lambda f]$, where $x_o/\lambda f$ is the spatial frequency corresponding to the delta function. Taking its width to be the width of the main lobe, we have:

$$W = 2 \times \frac{\lambda f}{D} = \frac{2(0.6328)(10)}{3} = 4.22\,\mu\text{m}.$$

Since the problem states that the mask has feature sizes comparable to the input spectrum, assume that the opaque spot has this width. To find the alignment requirement, we arbitrarily pick $\pm 1/10$ of this width to be the maximum we can be off and still block "most" of the sinc function. We then calculate the allowable misalignment as

$$\Delta W = \frac{W}{10} = 0.42\,\mu\text{m}.$$

That is, our tolerance is $\pm 0.42\,\mu$m. Notice that if the input aperture were infinite in extent, the sinc function would become a delta function and the alignment tolerance would become zero; that is, any misalignment would cause the opaque dot to miss the delta function completely.

8-6. Since the photographic gamma is -2, the amplitude transmittance of the input transparency is

$$t_A(\xi,\eta) = \kappa\,\mathcal{I}(\xi,\eta)\,\mathrm{rect}(\xi/L)\,\mathrm{rect}(\eta/L) = \frac{\kappa}{2}\left(1+\cos 2\pi f_o \xi\right)\,\mathrm{rect}(\xi/L)\,\mathrm{rect}(\eta/L).$$

This object, when optically Fourier transformed, will have sinc function spectral components centered at locations

$$\begin{aligned}(x,y) &= (0,0)\\ (x,y) &= (\lambda f f_o, 0)\\ (x,y) &= (-\lambda f f_o, 0).\end{aligned}$$

The widths of these sinc functions in the x and y directions, between first zeros, will be $\Delta x = \Delta y = 2\lambda f/L$.

(a) The locations of the absorbing spots should be at the places where the three sinc functions are centered, as listed above.

(b) The diameter of the absorbing spots should be at least the value of Δx above.

(c) At $(f_X = 0, f_Y = 0)$, we can not place a perfectly absorbing spot, for this would remove the constant component of the desired image amplitude, leading to strange effects on the ideal image intensity, such as contrast reversals. Rather, we need a partially absorbing spot there, with enough transmittance to allow the object variations to ride on a suitable bias, avoiding the creation of negative values of the image amplitude. Exactly how absorbing the spot should be depends on the structure of the desired object from which the noise is being removed.

8-7. The object amplitude transmittance is given by

$$t_A(x,y) = \frac{1}{2}[1+\cos 2\pi f_o x].$$

Since we are restricted to using a pure phase filter, we represent the amplitude transmittance of that filter in the frequency plane by $e^{j\phi(f_X)}$. Thus the image amplitude can be written

$$\begin{aligned}U_i &= \mathcal{F}^{-1}\left\{\mathcal{F}\{t_A\}e^{j\phi(f_X)}\right\}\\ &= \mathcal{F}^{-1}\{e^{j\phi(f_X)}\left[\frac{1}{2}\delta(f_X) + \frac{1}{4}\delta(f_X - f_o) + \frac{1}{4}\delta(f_X + f_o))\right]\}\\ &= \mathcal{F}^{-1}\{\frac{1}{2}e^{j\phi(0)}\delta(f_X) + \frac{1}{4}e^{j\phi(f_o)}\delta(f_X - f_o) + \frac{1}{4}e^{j\phi(-f_o)}\delta(f_X + f_o)\}\\ &= \frac{1}{2}e^{j\phi(0)} + \frac{1}{4}e^{j\phi(f_o)}e^{j2\pi f_o x} + \frac{1}{4}e^{j\phi(-f_o)}e^{-j2\pi f_o x}.\end{aligned}$$

The image intensity is given by

$$\begin{aligned}|U_i|^2 &= \frac{3}{8} + \frac{1}{4}\cos[\phi(f_o) - \phi(0) + 2\pi f_o x] + \frac{1}{4}\cos[\phi(-f_o) - \phi(0) - 2\pi f_o x]\\ &\quad + \frac{1}{8}\cos[\phi(f_o) - \phi(-f_o) + 4\pi f_o x].\end{aligned}$$

We wish to cancel out the first two cosine terms, since they are the only terms that have spatial frequency components corresponding to f_o. With some thought, we see that we can accomplish this by setting:

$$\begin{aligned}\phi(f_o) &= \pi/2,\\ \phi(-f_o) &= \pi/2,\\ \phi(0) &= 0.\end{aligned}$$

(Many other answers are possible.) Note that we only need to know the phase of the filter at three points, $0, -f_o$, and f_o, since the original object contains only these frequency components.

8-8. In the focal plane where the photographic transparency is recorded,

$$\mathcal{I} = |U_f|^2 = \frac{1}{(\lambda f)^2}\, T_A\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\, T_A^*\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right).$$

With $\gamma = -2$, the amplitude transmittance of the developed transparency is proportional to $\mathcal{I}$:

$$t'_A(x,y) \quad = \quad K\, T_A\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\, T_A^*\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right).$$

The intensity in the back focal plane when this transparency is placed against the lens is:

$$\begin{aligned} I_f(x,y) &= K' \left| \mathcal{F}\left\{ T_A\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\, T_A^*\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\right\}\right|^2_{\substack{f_X = x/\lambda f\\ f_Y = y/\lambda f}} \\ &= K'\,(\lambda f)^2 |t_A(-x,-y) \otimes t_A^*(x,y)|^2 \\ &= K\,|t_A(-x,-y) \star t_A(-x,-y)|^2. \end{aligned}$$

Thus, the intensity in the back focal plane during the second step is proportional to the squared magnitude of the autocorrelation of the amplitude transmittance of the original object, inverted or reflected about the x and y axes.

8-9. The image amplitude can be written

$$\begin{aligned} U_i(u,v) &= \frac{1}{\lambda f}\mathcal{F}^{-1}\{\frac{1}{\lambda f}\mathcal{F}\{t_A(x,y)\}\sqrt{\tau\,(\lambda f f_X, \lambda f f_Y)}\} \\ &= \frac{1}{(\lambda f)^2}\mathcal{F}^{-1}\{\mathcal{F}\{\exp[j\phi(x,y)]\}\sqrt{\alpha}(\lambda f)^2(f_X^2 + f_Y^2)\}. \end{aligned}$$

Using the Fourier transform property

$$\mathcal{F}\left\{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g(x,y)\right\} = -4\pi^2(f_X^2 + f_Y^2)\mathcal{F}\{g(x,y)\},$$

we obtain,

$$\begin{aligned} U_i(u,v) &= \frac{\sqrt{\alpha}}{-4\pi^2}\mathcal{F}^{-1}\{-4\pi^2(f_X^2 + f_Y^2)\mathcal{F}\{\exp[j\phi(x,y)]\}\} \\ &= \frac{\sqrt{\alpha}}{-4\pi^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\exp[j\phi(x,y)] \\ &= \frac{\sqrt{\alpha}}{-4\pi^2}\exp[j\phi(x,y)]\left\{ j\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) - \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]\right\} \end{aligned}$$

The image intensity is thus given by

$$I_i = |U_i|^2 = \frac{\alpha}{16\pi^4}\left\{\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi(x,y)\right]^2 + \left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]^2\right\}.$$

8-10. The first part of the system is purely coherent. Incident on the moving diffuser will be an amplitude distribution

$$U(x,y) = \frac{1}{\lambda f} S_1\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right),$$

where $S_1(f_X, f_Y) = \mathcal{F}\{s_1(\xi,\eta)\}$. The moving diffuser destroys the spatial coherence of the light and therefore the second part of the system must be modeled as an *incoherent* imaging system. For this system the object intensity is the squared magnitude of the field above,

$$I_o(x,y) = \frac{1}{(\lambda f)^2}\left|S_1\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)\right|^2.$$

Since the amplitude transmittance function in the pupil of the incoherent imaging system has the form

$$t_A(x,y) = s_2(x,y),$$

the intensity point-spread function of this system is

$$|h(u,v)|^2 = \frac{1}{(\lambda f)^2}\left|S_2\left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right)\right|^2.$$

The intensity distribution in the final image plane is therefore a convolution as follows

$$I_i(u,v) = |h(u,v)|^2 \otimes I_i(u,v) = \frac{1}{(\lambda f)^4}\left|S_2\left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right)\right|^2 \otimes \left|S_1\left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right)\right|^2.$$

This is the simplest form of the result.

8-11. As in the previous treatment of the VanderLugt filter, the reference wave is represented by

$$U_r(x_2,y_2) = r_o\ \exp(-j2\pi\alpha y_2).$$

In this case the wave from the object is the product of the Fourier transform of that object and a quadratic phase factor, because the object is against the lens. Thus

$$U_o(x_2,y_2) = \frac{\exp\left[\frac{\pi}{\lambda f}(x_2^2+y_2^2)\right]}{\lambda f} S\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right)$$

These two waves interfere at the film. After exposure and development, the amplitude transmittance of the film is given by

$$t_A(x_2,y_2) = k\left|r_o \exp(-j2\pi\alpha y_2) + \frac{\exp\left[j\frac{\pi}{\lambda f}(x_2^2+y_2^2)\right]}{\lambda f}\ S\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right)\right|^2$$

$$= \quad k\left\{ r_o^2 + \frac{1}{(\lambda f)^2}\left|S\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right)\right|^2 \right.$$
$$+ \frac{r_o \exp\left[j\frac{\pi}{\lambda f}(x_2^2+y_2^2) + j2\pi\alpha y_2\right]}{\lambda f} S\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right)$$
$$\left. + \frac{r_o \exp\left[-j\frac{\pi}{\lambda f}(x_2^2+y_2^2) - j2\pi\alpha y_2\right]}{\lambda f} S^*\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right) \right\}$$

Now by moving the input of the final processing system so that it is distance d in front of the first lens, rather than distance f, we are introducing a multiplicative quadratic phase factor associated with the spectrum of the input as it is incident on the filter. If $g(x_1, y_1)$ is the input, then (from Eq. (5-20)) incident on the filter in the Fourier plane we will have the amplitude distribution

$$U(x_2, y_2) = \frac{\exp\left[j\frac{k}{2f}\left(1-\frac{d}{f}\right)(x_2^2+y_2^2)\right]}{\lambda f} G\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right).$$

(a) To force cancellation of quadratic phase factors when the convolution term (impulse response $s(x_3, y_3)$) is to be used, we must introduce a quadratic phase factor of the form $\exp\left[-j\frac{\pi}{\lambda f}(x_2^2+y_2^2)\right]$. The previous equation shows this is achieved if $d = 2f$.

(b) To force cancellation of quadratic phase factors when the matched filter term is used (impulse response $s^*(-x_3, -y_3)$), we require a quadratic phase factor of the form $\exp\left[j\frac{\pi}{\lambda f}(x_2^2+y_2^2)\right]$. This is achieved if $d = 0$, i.e. the input is placed against the lens.

8-12. Suppose the input $g(x_1, y_1)$ is shifted by (x_0, y_0). Thus the input is $g(x_1 - x_0, y_1 - y_0)$. The effect of this space-domain shift will be, according to the shift theorem of Fourier analysis, the introduction of a linear phase shift in the frequency domain, changing the spectrum of the input as follows:

$$G\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right) \to G\left(\frac{x_2}{\lambda f}, \frac{y_2}{\lambda f}\right) \exp\left[-j\frac{2\pi}{\lambda f}(x_2 x_0 + y_2 y_0)\right].$$

This change must be made for every occurrence of G in the equation above Eq. (8-17). The inverse transform of that equation will then result simply in a shift of each output term involving $g(x_3, y_3)$, again by the shift theorem of Fourier analysis.

8-13. From Eqs. (8-22) through (8-24),

$$U_3(x_3, y_3) \quad = \quad \frac{1}{\lambda f}\Big[\underbrace{h(x_3,y_3)\otimes h^*(-x_3,-y_3)}_{2W_h} + \underbrace{g(x_3,y_3)\otimes g^*(-x_3,-y_3)}_{2W_g}$$
$$+ \underbrace{h(x_3,y_3)\otimes g^*(-x_3,-y_3)\otimes\delta(x_3,y_3-Y)}_{W_g+W_h}$$
$$+ \underbrace{h^*(-x_3,-y_3)\otimes g(x_3,y_3)\otimes\delta(x_3,y_3+Y)}_{W_g+W_h}\Big].$$

The width of each of the terms is indicated below it. The first two terms are centered at the origin, and the third and fourth terms are centered at $(0, Y)$ and $(0, -Y)$, respectively. Since the on-axis component (the first and second terms combined) is non zero between $-\max\{W_g, W_h\}$ and $\max\{W_g, W_h\}$, and the cross correlation components extend from $\pm[Y-(W_g+W_h)/2]$ to $\pm[Y+(W_g+W_h)/2]$, we need

$$Y - \frac{W_g + W_h}{2} > \max\{W_g, W_h\}$$

or,

$$Y > \max\{W_g, W_h\} + \frac{W_g + W_h}{2}$$

in order to ensure that the cross correlation terms are separated from the on-axis terms.

8-14. (a) By inspection, the point-spread function of this blurring process is

$$s(u,v) = \frac{K}{VT}\text{rect}\left(\frac{u - \frac{VT}{2}}{VT}\right)\delta(v),$$

where K is a constant.

(b) The Fourier transform of this impulse response is of the form

$$S(f_X, f_Y) = K\,\text{sinc}(VTf_X)\,e^{-j\pi VTf_X}.$$

The transfer function of an inverse filter would therefore be

$$H_{\text{inverse}}(f_X, f_Y) = \frac{1}{\text{sinc}(VTf_X)}\,e^{j\pi VTf_X}.$$

(c) Given a signal-to-noise ratio of 10 over all frequencies, the transfer function of a Wiener filter would be

$$H_{\text{Wiener}}(f_X, f_Y) = \frac{\text{sinc}(VTf_X)\,e^{j\pi VTf_X}}{\left[\text{sinc}^2(VTf_X) + 0.1\right]}.$$

A plot of the magnitude of this function for the special case $VT = 1$ is shown in part (a) of the figure. The sign of the central and even-numbered lobes is positive, that of odd-numbered lobes is negative.

(d) The impulse response of the filter is obtained by subjecting the transfer function to an inverse Fourier transform. Part (b) of the figure illustrates the impulse response obtained in this way. Note the the sign discontinuities of the impulse response of the Wiener filter occur with a separation equal to the width of the original rectangular blur. You may wish to contemplate the result of convolving the original blur function with this deblur impulse response.

8-15. (a) Since we wish to remove the periodic pattern, we need to remove the nearly impulsive components of its the spectrum while leaving the rest of the spectrum approximately intact. Thus we need to generate a mask with opaque spots at the locations of the impulsive components. One way to generate such a mask is to use a defect-free object to expose a film in the Fourier plane. If a defect-free object is not available, we can still generate the needed mask by developing the film in a non-linear region so as to make the film more sensitive to high incident intensities; this will allow the delta functions to get exposed while low intensity signals corresponding to defects get suppressed.

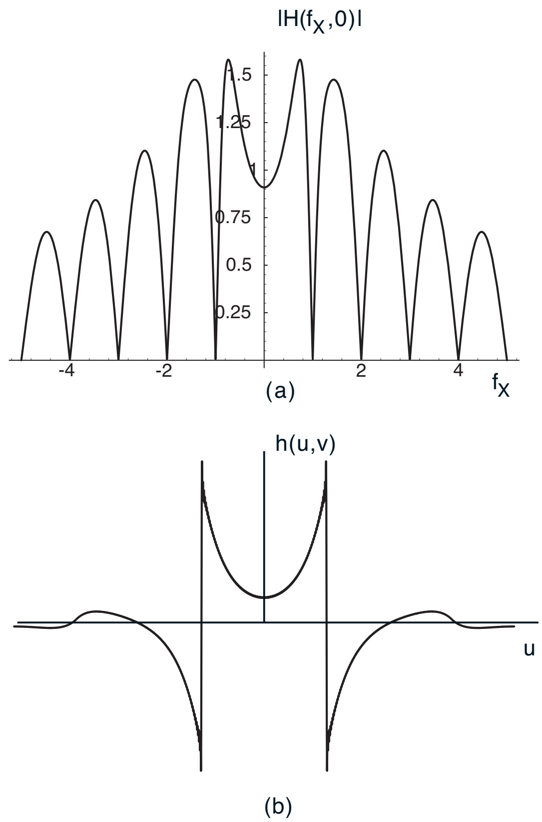

Figure 8-14:

(b) The object with the defect can be written as

$$U_o = p(x,y)\,[1 - d(x,y)],$$

where

$$d(x,y) = \begin{cases} 1 & \text{in the defect} \\ 0 & \text{outside the defect} \end{cases},$$

and $p(x,y)$ is the ideal periodic object. In the Fourier plane we have

$$\begin{aligned} U_f &= P(f_X,f_Y) \otimes [\delta(f_X,f_Y) - D(f_X,f_Y)] \\ &= P(f_X,f_Y) - P(f_X,f_Y) \otimes D(f_X,f_Y), \end{aligned}$$

where $P(f_X,f_Y)$ consists of a series of delta functions:

$$P(f_X,f_Y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm}\,\delta(f_X - n/L, f_Y - m/L).$$

Thus,

$$U_f = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{nm} \left[\delta(f_X - \frac{n}{L}, f_Y - \frac{m}{L}) - D(f_X - \frac{n}{L}, f_Y - \frac{m}{L})\right].$$

The filter removes the delta functions, and does not particularly affect the multiple replicas of $D(f_X, f_Y)$. Thus we have,

$$U'_f = -\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_{nm}\, D\left(f_X - \frac{n}{L}, f_Y - \frac{m}{L}\right).$$

Now, since we know that L is much larger than the size of the defect, in the Fourier domain the width of D must be much greater than the size of $1/L$; that is, all the replicas of D in the above equation are offset from one another by a very small distance compared to their width. Also, note that c_{nm} approaches zero as $|n|$ and $|m|$ are increased, so that the replicas which are offset significantly from the origin contribute very little to the summation. Thus, we have

$$U'_f \approx -D(f_X, f_Y)$$

and, taking the inverse Fourier transform,

$$\begin{aligned} U_i &\approx -d(u,v) \\ I_i &\approx d^2(u,v). \end{aligned}$$

8-16. The basic approach would be to construct a filter with a transfer function approximating

$$H(f_X, f_Y) = \frac{A(f_X, f_Y)}{F(f_X, f_Y)},$$

where $A(f_X, f_Y)$ is the Fourier transform of the letter “A” and $F(f_X, f_Y)$ is the Fourier transform of the letter “F”. We can do so by constructing a relative of the inverse filter. Construction of such a filter would be a two-step process, similar to what was described in Section 8.8.3. However, there are differences because what we are creating is not an inverse filter in the usual sense. The procedure would be as follows:

- Expose photographic film with the Fourier transform of the letter “F”, and process the film to create a negative transparency with a gamma of 2. The result will be a transparency with amplitude transmittance satisfying
$$t_{A1} \propto \frac{1}{|F(f_X, f_Y)|^2}.$$
- Now expose a second piece of film to the interference pattern between the Fourier transform of the letter “A” and the Fourier transform of the letter “F”. This can be done with an optical system such as is used to create the filter for a joint transform correlator. A transparency containing the letters “A” and “F”, side-by-side but separated from one another, is optically Fourier transformed and the resulting two spectra interfere on the film. This exposure is recorded in the linear region of the t_A vs. E curve. The result is a transparency with one component of amplitude transmittance satisfying
$$t_{A2} \propto F^*(f_X, f_Y)\, A(f_X, f_Y).$$
- Now place these two transparencies in contact and use them as the filter in a conventional “4f”coherent processing system. If the letter “F” is presented at the input to that system, then the field transmitted through the Fourier plane will be of the form

$$U_f \propto F(f_X, f_Y)\, \frac{F^*(f_X, f_Y)\, A(f_X, f_Y)}{|F(f_X, f_Y)|^2} = A(f_X, f_Y).$$

Inverse Fourier transformation of this field results in the letter "A" appearing at the output of the processor. If the letter "A" was placed above the letter "F" in the input plane when the second transparency was recorded, the output of interest will appear (in the inverted coordinate system of Fig. 8.16(b)), at location $(0, -Y)$, where Y is the separation of the centers of the letters "A" and "F".

8-17. (a) Since the illuminated patch on the ground is $\lambda_r r_1/D$ meters wide, the length of flight path over which the given scatterer on the ground would be illuminated is also $\lambda_r r_1/D$.

(b) The doppler shift encountered while approaching the target and receding from the target can be deduced from Eq. (8-65). Note from that equation that the phase of the radiation returned from a point scatterer at location x_n is

$$\phi_n(t) = -\frac{2\pi(v_a t - x_n)^2}{\lambda_r r_1}.$$

Without loss of generality we can consider the particular point scatterer at $x_n = 0$. The shift of instantaneous frequency, Δf, associated with this term is found from

$$\Delta f = \frac{1}{2\pi}\frac{d}{dt}\phi_n = -\frac{2v_a^2 t}{\lambda_r r_1}.$$

But we are interested in the frequency when the point scatterer just begins to enter the illumination beam and when it just leaves the illumination beam. Since the beam is $\lambda_r r_1/D$ meters wide, the times when the point scatterer enters and leaves the beam will be (respectively)

$$\begin{aligned} t_1 &= -\frac{\lambda_r r_1}{2v_a D} \\ t_2 &= \frac{\lambda_r r_1}{2v_a D}. \end{aligned}$$

Substituting these values into the expression for the frequency shift, we find in the two cases

$$\begin{aligned} \Delta f_1 &= \frac{2v_a^2}{\lambda_r r_1} \times \frac{\lambda_r r_1}{2v_a D} = \frac{v_a}{D} \\ \Delta f_2 &= -\frac{2v_a^2}{\lambda_r r_1} \times \frac{\lambda_r r_1}{2v_a D} = -\frac{v_a}{D}. \end{aligned}$$

(c) The signal arriving from the point scatterer chirps over a bandwidth

$$B = \Delta f_1 - \Delta f_2 = 2v_a/D.$$

This chirping signal can be compressed to a pulse of duration $T = 1/B$, and indeed such compression is done spatially by the coherent optical system. A compressed pulse of duration T corresponds to a dimension on the ground

$$\Delta x = v_a T = \frac{v_a}{B} = D/2.$$

This is the resolution achieved on the ground by a perfect processing system. The factor of $1/2$ arises because *both* the transmitter and the receiver are moving with respect to the scatterer.

8-18. Start with Eq. (7-34). While this has been derived for diffraction in the Raman-Nath regime, the only difference in the Bragg regime lies in the strengths of the various orders. In addition, this equation holds for only a single CW component driving the acousto-optic cell. However, it holds for a component of any frequency, and therefore by changing the frequency we can discover what happens to the many different frequency components of a broadband signal. In Eq. (7-34), the factor $e^{j2\pi y/\lambda}$ corresponds to a wavefront tilt, which results in this diffraction order being focused by the lens that follows to a diffraction-limited spot centered at coordinate

$$y_2 = \frac{\lambda f}{\Lambda} = \frac{\lambda f}{V}\, f_c.$$

Thus an RF frequency f_c is mapped to the spatial coordinate y_2 above. As is also evident from Eq. (7-34), the phasor representation of this field component has a time variation $e^{j2\pi f_c t}$, in addition to the rotation $e^{-j2\pi\nu t}$ at the optical frequency that has been suppressed. Hence the frequency of the light being focused to this position in the focal plane is $\nu - f_c$, which was to be proved. This equation holds for the $+1$ order. For the -1 order the frequency is $\nu + f_c$.

Chapter 9

9-1. A complicated but correct solution to the problem would write all the fields incident on the film, find the intensity, and find the fields transmitted by the hologram. A much simpler solution is based on Eq. (9-38) with $\lambda_2 = \lambda_1$. That equation states that

$$\frac{1}{z_i} = \frac{1}{z_p} \pm \frac{1}{z_r} \mp \frac{1}{z_o}.$$

This equation should now be compared with the lens law, which we must adapt to the sign convention used in the discussion of holographic image locations. Remembering that z_o is negative for an object to the left of the hologram or lens, the lens law can be written

$$\frac{1}{z_i} = \frac{1}{f} + \frac{1}{z_o}.$$

Equating these two expressions for $1/z_i$, we see immediately that

$$\frac{1}{f} = \frac{1}{z_p} \pm \frac{1}{z_r} \mp \frac{1}{z_o} - \frac{1}{z_o},$$

yielding two focal lengths

$$f = \left(\frac{1}{z_p} - \frac{1}{z_r}\right)^{-1} \qquad \text{and} \qquad f = \left(\frac{1}{z_p} + \frac{1}{z_r} - \frac{2}{z_o}\right)^{-1}.$$

Note that one of the two lenses has a focal length that depends on the location of the object.

9-2. Again we use Eq. (9-38) to find solutions. In both cases, $\lambda_1 = 0.488\,\mu$m, and $\lambda_2 = 0.6328\,\mu$m.

(a) Let $z_p = \infty,\ z_r = \infty,\ z_o = -10\,$cm. Then

$$\frac{1}{z_i} = 0 + 0 \pm \frac{0.6328}{0.488} \times \frac{1}{10},$$

or

$$z_i = \pm\frac{4.88}{0.6328} = \pm 7.71\,\text{cm}.$$

From Eqs. (9-40) and (9-41),

$$\begin{aligned} M_t &= 1 \\ M_a &= \frac{0.488}{0.6328} = 0.771. \end{aligned}$$

(b) In this case, $z_p = \infty,\ z_r = 2z_o,\ z_0 = -10\,\text{cm}$. Then

$$\frac{1}{z_i} = 0 \mp \frac{0.6328}{0.488} \times \frac{1}{20} \pm \frac{0.6328}{0.488} \times \frac{1}{10}$$

or

$$z_i = \pm 15.4\,\text{cm}.$$

As for the magnifications,

$$\begin{aligned} M_t &= \left|\frac{\lambda_2 z_i}{\lambda_1 z_o}\right| = \frac{0.6328 \times 15.4}{0.488 \times 10} = 2 \\ M_a &= \frac{\lambda_1}{\lambda_2} M_t^2 = \frac{0.6328}{0.488} \times 4 = 3.1. \end{aligned}$$

9-3. Take first the case of $z_p = z_r$. The image distance is

$$z_i = \left(\frac{1}{z_r} \pm \frac{1}{z_r} \mp \frac{1}{z_o}\right)^{-1}.$$

Thus the two solutions are (lower set of signs first)

$$z_i = z_o \qquad \text{and} \qquad z_i = \left(\frac{2}{z_r} - \frac{1}{z_o}\right)^{-1} = \frac{z_r z_o}{2z_o - z_r}.$$

Note that since $z_o < 0$ (i.e. the object lies to the left of the hologram), the first image also lies to the left of the hologram (a virtual image), while the second can lie to the left or to the right, depending on the particular values of z_o and z_r. The transverse magnifications in the two cases are

$$M_t = 1 \qquad \text{and} \qquad M_t = \left|\frac{z_i}{z_o}\right| = \left|\frac{z_r}{2z_o - z_r}\right|.$$

Thus one of the images is virtual and has unit transverse magnification.

Now consider the case for $z_p = -z_r$. The two solutions for image distance are now (again lower set of signs first)

$$z_i = \left(-\frac{2}{z_r} + \frac{1}{z_o}\right)^{-1} = \frac{-z_r z_o}{2z_o - z_r} \qquad \text{and} \qquad z_i = -z_o.$$

The transverse magnifications in the two cases are

$$M_t = \left|\frac{z_i}{z_o}\right| = \left|\frac{z_r}{2z_o - z_r}\right| \qquad \text{and} \qquad M_t = 1.$$

Note that, since z_o is negative, the second image is real (z_i positive) and has unit transverse magnification.

9-4. (a) The transverse magnification will be the ratio of the image distance to the object distance, since the wavelengths are the same during recording and reconstruction. The image appears in the rear focal plane of the Fourier-transforming lens, and therefore

$$M_t = \left|\frac{f}{z_o}\right|$$

(b) Let $U'(x_o, y_o)$ be defined to be

$$U'(x_o, y_o) = t_A(x_o, y_o)\, e^{j\frac{\pi}{\lambda z}(x_o^2+y_o^2)},$$

where z is taken to be a positive number (the object z-coordinate is $-z$), and the amplitude of the plane wave illuminating the object transparency has been taken to be unity. The reference point has been taken to be displaced from the center of the object plane by distance y_1 along the y_o axis. The intensity distribution incident on the hologram plane is

$$I(x, y) = \left| A e^{j\frac{2\pi}{\lambda z} y_1 y} + \frac{e^{j\frac{\pi}{\lambda z}(x^2+y^2)}}{\lambda z} \mathcal{F}\left\{U_o'\right\}_{\substack{f_X = x/\lambda z \\ f_Y = y/\lambda z}} \right|^2 .$$

For this problem, we are interested only in the on-axis term of the hologram transmittance, which in the linear region of the t_A vs. E curve yields

$$t_1(x, y) = A^2 + \frac{1}{(\lambda z)^2} \left| \mathcal{F}\{U_o'\}_{\substack{f_X = x/\lambda z \\ f_Y = y/\lambda z}} \right|^2 .$$

The reconstruction process subjects the film amplitude transmittance to a further Fourier transform, but with a slightly different scaling factor. The field in the focal plane of the reconstruction lens will be (assuming a unit-amplitude reconstruction plane wave)

$$\begin{aligned} U_f(u, v) &= \frac{1}{\lambda f} \mathcal{F}^{-1}\{t_1(x, y)\}_{\substack{f_X = u/\lambda f \\ f_Y = v/\lambda f}} = \lambda f\, \delta(u, v) \\ &+ \frac{1}{\lambda f} \iint_{-\infty}^{\infty} dx dy\, e^{-j\frac{2\pi}{\lambda f}(ux+vy)} \frac{1}{(\lambda z)^2} \iint_{-\infty}^{\infty} dx_o dy_o\, e^{-j\frac{2\pi}{\lambda z}(x_o x + y_o y)} U_o'(x_o, y_o) \\ &\times \iint_{-\infty}^{\infty} dx_o' dy_o'\, e^{j\frac{2\pi}{\lambda z}(x_o' x + y_o' y)} U_o'^*(x_o', y_o'). \end{aligned}$$

The integrals can be rearranged so that one of the double integrals reduces to

$$\begin{aligned} B &= \frac{1}{\lambda f} \iint_{-\infty}^{\infty} dx dy\, e^{-j2\pi\left[\left(\frac{u}{\lambda f} + \frac{x_o}{\lambda z} - \frac{x_o'}{\lambda z}\right)x + \left((\frac{v}{\lambda f} + \frac{y_o}{\lambda z} - \frac{y_o'}{\lambda z}\right)y\right]} \\ &= \frac{(\lambda z)^2}{\lambda f} \delta\left(x_o' - x_o - M_t u, y_o' - y_o - M_t v\right), \end{aligned}$$

where $M_t = f/z$. Substitution of this delta function into the remaining two double integrals causes one of those double integrals to vanish, leaving

$$U_f(u, v) = \frac{1}{\lambda f} \iint_{-\infty}^{\infty} U'(x_o, y_o)\, U_o'^*(x_o + u/M_t, y_o + v/M_t)\, dx_o dy_o.$$

(c) Since the object transparency has size $L \times L$, and since the autocorrelation of the object can have size at most $2L \times 2L$, the reference point source should be at least $3L/2$ above the object transparency.

9-5. (a) We know the magnification to be 2 from the object and image sizes given. We also know

$$M_t = \left| \frac{\lambda_2 z_i}{\lambda_1 z_o} \right| = 2.$$

Choosing the appropriate signs by noting that $z_i > 0$ and $z_o < 0$,

$$z_o = -\frac{\lambda_2 z_i}{\lambda_1 \times 2} = -\frac{(488\,\text{nm})(1\,\text{m})}{(632.8\,\text{nm})(2)} = -0.38\,\text{m}.$$

Using this result in the expression for z_i,

$$z_i = \left(\frac{1}{z_p} \pm \frac{\lambda_2}{\lambda_1 z_r} \mp \frac{\lambda_2}{\lambda_1 z_o} \right)^{-1} = \left(\frac{1}{z_p} \pm \frac{\lambda_2}{\lambda_1 z_r} \pm \frac{2}{z_i} \right)^{-1}.$$

Solving for z_r,

$$\begin{aligned} z_r &= \frac{\lambda_2}{\lambda_1} \left(-\frac{2}{z_i} \pm \frac{1}{z_i} \mp \frac{1}{z_p} \right)^{-1} \\ &= -\frac{\lambda_2}{\lambda_1} \left(\frac{-1}{z_i} - \frac{1}{z_p} \right)^{-1} \quad \text{or} \quad \frac{\lambda_2}{\lambda_1} \left(\frac{-3}{z_i} + \frac{1}{z_p} \right)^{-1} \\ &= 0.77\,\text{m} \quad \text{or} \quad -0.15\,\text{m}. \end{aligned}$$

Since the problem specifies that $z_r < 0$,

$$z_r = -0.15\,\text{m}.$$

(b) The same effect as flipping the hologram can be obtained by keeping the hologram unflipped but exposing the film from the other side. This has the effect of reversing the signs of z_o and z_r, making both of them positive. z_o must now be $+0.38$ m to assure $M_t = 2$. Carrying out the same calculations as part (a) with the new signs for z_o and z_r, we find

$$\begin{aligned} z_o &= +0.38\,\text{m} \\ z_r &= +0.15\,\text{m}. \end{aligned}$$

9-6. (a) We can find the maximum spatial frequency with the help of the figure.

The maximum spatial frequency will be the maximum distance from the reference point to any point on the object ($304\,\mu$m in this case), divided by λz_o,

$$f_{\text{max}} = \frac{l}{\lambda z_o} = \frac{304\,\mu\text{m}}{1 \times 10^{-4}\,\mu\text{m} \times 20\,\text{mm}} = 152000 \text{ cycles/mm}.$$

(b) The experiment will fail because the periods of all components of the holographic grating are much smaller than the wavelength of the reconstruction source. As a consequence, all diffraction orders will be evanescent, and there will be no way to form an image.

9-7. (a) Let λ and θ represent the wavelength and half-angle between beams outside the emulsion (i.e. in air where $n = 1$). In terms of these parameters the predicted fringe period is

$$\Lambda = \frac{\lambda}{2 \sin \theta}.$$

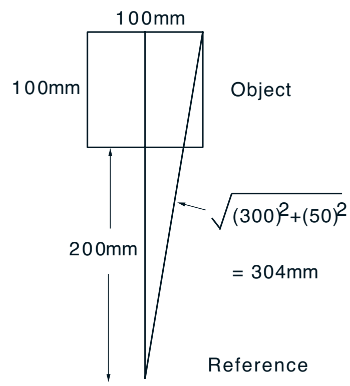

Figure 9-6:

For the values $\lambda = 0.488 \times 10^{-6}$m and $2\theta = 60°$, the period is given by $\Lambda = 0.488\mu$m. Inside the emulsion the wavelength is $\lambda' = \lambda/n = 488\text{nm}/1.52 = 321$nm and from Snell's law,

$$\sin\theta' = \frac{\sin\theta}{n} = 0.5/1.52 \text{ and } \theta = 19.2°.$$

Using the parameters inside the emulsion the predicted fringe period is

$$\Lambda = \frac{\lambda'}{2\sin\theta'} = \frac{\lambda}{2\sin\theta} = 0.488\mu\text{m}.$$

We conclude fringe period predicted from parameters outside the recording medium is exactly the same as that predicted from parameters inside the recording medium.

(b) From Eq. (9-70), under Bragg-matched conditions,

$$\eta_B = \sin^2\Phi,$$

where, from Eq. (9-68),

$$\Phi = \frac{\pi n_1 d}{\lambda'\cos\theta'},$$

and we are using the values of wavelength and angle inside the emulsion. To achieve 100% diffraction efficiency with the smallest possible refractive index modulation, we require $\Phi = \pi/2$. This implies that the peak refractive index modulation must be the n_1 that satisfies

$$\frac{\pi n_1 d}{\lambda'\cos\theta'} = \frac{\pi}{2}.$$

Remembering that θ is 1/2 the angle between the two beams ($30°$ in this case), the required value of n_1 is therefore

$$n_1 = \frac{\lambda'\cos\theta'}{2d} = \frac{\lambda'\sqrt{1-\sin^2\theta'}}{2d} = \frac{\frac{\lambda}{n}\sqrt{1-\frac{\sin^2\theta}{n^2}}}{2d} = \frac{0.321\,\mu\text{m}\times 0.9443}{2\times 15\,\mu\text{m}} = 0.0101.$$

(c) We assume that $\Phi = \pi/2$ (Bragg aligned), $\Delta\theta = 0$ but that $\Delta\lambda \neq 0$. From Eq. (9-69),

$$\eta = \frac{\sin^2\left(\frac{\pi}{2}\sqrt{1+\frac{\chi^2}{(\pi/2)^2}}\right)}{1+\frac{\chi^2}{(\pi/2)^2}},$$

where by Eq. (9-68)

$$\chi = \frac{Kd}{2\cos\theta_B'}\frac{\Delta\lambda'}{2\Lambda} = \frac{\pi d\Delta\lambda'}{2\Lambda^2\cos\theta_B'},$$

with θ_B' and $\Delta\lambda'$ being the Bragg angle and wavelength change within the emulsion. Again noting that $\Delta\lambda' = \Delta\lambda/n$, and $\cos\theta_B' = \sqrt{1-\frac{\sin^2\theta_B}{n^2}}$, as well as $\Lambda = \lambda/(2\sin\theta_B)$, we have

$$\chi = \frac{\pi d\Delta\lambda/n}{2(\frac{\lambda}{2\sin\theta_B})^2\sqrt{1-\frac{\sin^2\theta_B}{n^2}}} = \frac{2\pi d\Delta\lambda\sin^2\theta_B}{\lambda^2\sqrt{n^2-\sin^2\theta_B}}.$$

Next we must determine what value of χ causes the diffraction efficiency to drop to value 1/2. The desired value of χ satisfies the equation

$$\sin^2\left[\sqrt{\left(\frac{\pi}{2}\right)^2+\chi^2}\right] = \frac{1}{2}\left(1+\frac{\chi^2}{(\pi/2)^2}\right).$$

Equivalently, letting $X = \sqrt{(\pi/2)^2+\chi^2}$, we seek the value of X that satisfies

$$\sin^2 X = \frac{2}{\pi^2}X^2.$$

The solution is $X = 2.0$, or $(\pi/2)^2 + \chi^2 = 4.0$. Solving for χ, we obtain

$$\chi = 1.255.$$

Equating this value of χ to the general expression found above, and substituting $n = 1.52$, $\lambda = 0.488\,\mu$m, $d = 15\,\mu$m, and $\theta_B = 30°$, we obtain

$$\Delta\lambda = 2\times 1.255\times\frac{\lambda^2\sqrt{n^2-\sin^2\theta_B}}{2\pi d\sin^2\theta_B} = 18.2\,\text{nm}.$$

(d) In this case, according to Eq. (9-68) and noting that the grating is unslanted ($\psi = 0$),

$$\chi = \frac{Kd}{2\cos\theta_B'}\Delta\theta'\cos\theta_B' = \frac{\pi d}{\Lambda}\Delta\theta',$$

where θ_B' and $\Delta\theta'$ are both measured in the emulsion. In addition, Snell's law implies that $\sin(\theta_B - \Delta\theta) = \sin(\theta_B' - \Delta\theta')/n$ and for small $\Delta\theta'$, we have

$$\Delta\theta' = \Delta\theta\sqrt{\frac{1-\sin^2\theta_B}{n^2-\sin^2\theta_B}}.$$

Substituting $\Lambda = \lambda/(2\sin\theta_B)$ and noting that the same value of χ as found in part (c) is appropriate, we obtain for a change of angle external to the emulsion

$$\Delta\theta = 1.255\times\frac{\lambda}{2\pi d\sin\theta_B}\times\sqrt{\frac{n^2-\sin^2\theta_B}{1-\sin^2\theta_B}} = \frac{1.255\times 0.488\,\mu\text{m}}{2\pi\times 15\,\mu\text{m}\times\sin 30°}\times\sqrt{\frac{1.52^2-\sin^2 30°}{1-\sin^2 30°}}.$$

Thus

$$\Delta\theta = 1.23^\circ.$$

9-8. From Eq. (9-44),

$$Q = \frac{2\pi\lambda_0 d}{n\Lambda^2} = 2\pi.$$

Solving for Λ^2, we find

$$\Lambda^2 = \frac{\lambda_0 d}{n}.$$

In addition, we know that $\Lambda = \lambda_0/2\sin\theta$, where both λ_0 and θ are measured in air. Equating two expressions for λ^2 and solving for $\sin\theta$, we obtain

$$\sin\theta = \sqrt{\frac{n\lambda_0}{4d}},$$

from which it follows that

$$\theta = \sin^{-1}\sqrt{\frac{n\lambda_0}{4d}}.$$

Substituting the numerical values $n = 1.52$, $d = 15\,\mu$m and $\lambda_0 = 0.633\,\mu$m, we find

$$\theta = 0.127\,\text{radians} = 7.28^\circ.$$

The angle between the beams is $2\theta = 14.6^\circ$.

9-9. From Eq. (9-75) with $\alpha_1 = \alpha_0$, we have

$$\eta_B = e^{-\frac{2\alpha_0 d}{\cos\theta_B}}\ \sinh^2\left(\frac{\alpha_0 d}{2\cos\theta_B}\right).$$

As stated in the text, this quantity is maximized when

$$\Phi_a' = \frac{\alpha_0 d}{2\cos\theta_B} = 0.55,$$

from which we conclude

$$\alpha_0 = \alpha_1 = 1.10 \times \frac{\cos\theta_B}{d} = \frac{1.10 \times \cos 30^\circ}{d} = 0.953/d.$$

Now density D is related to intensity transmittance τ through $D = -\log_{10}\tau$, and the *local* value of τ is by definition (c.f. Eq. (9-72) and see Eq. (9-55) with $\alpha_1 = \alpha_0$)

$$\tau = \exp\left[-2\alpha_0 d - 2\alpha_0 d\cos\vec{K}\cdot\vec{r}\right].$$

It follows that the local value of density is

$$D = \frac{2\alpha_0 d}{\ln 10}\left[1 + \cos\vec{K}\cdot\vec{r}\right],$$

which when averaged over many cycles of the fringe pattern yields an average density given by

$$D_0 = \frac{2\alpha_0 d}{\ln 10}.$$

Now using the expression for α_0 in terms of d derived above, we find

$$D_0 = \frac{2 \times 0.953}{\ln 10} = 0.83.$$

9-10. From Eq. (9-66) with no wavelength mismatch,

$$\zeta = \Delta\theta\, K \cos(\theta_B - \psi).$$

Let α be the angular separation between the object and reference waves. Then from Eq. (9-48),

$$K = \frac{4\pi}{\lambda}\sin\left(\frac{\alpha}{2}\right).$$

Also, since the grating peaks run in the direction that bisects the object and reference wave directions,

$$\theta_B - \psi = \frac{\alpha}{2}.$$

Hence, the expression for the detuning parameter becomes

$$\begin{aligned}\zeta &= \Delta\theta\frac{4\pi}{\lambda}\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\\ &= \Delta\theta\frac{2\pi}{\lambda}\sin\alpha.\end{aligned}$$

To find the angular sensitivity, differentiate the detuning parameter with respect to $\Delta\theta$:

$$\frac{d\zeta}{d\Delta\theta} = \frac{2\pi}{\lambda}\sin\alpha.$$

The above quantity is easily seen to be maximized when $\alpha = 90°$.

9-11. First calculate the bandwidth of the object. Since the Fourier-transform hologram dimensions are (from the discussion of Section 9.9.1) $L_X \times L_Y$, the bandwidths of interest are

$$\begin{aligned}2b_X &= \frac{L_X}{\lambda f}\\ 2b_Y &= \frac{L_Y}{\lambda f}.\end{aligned}$$

Thus given an object of dimensions $L_\xi \times L_\eta$, if the object is sampled at the Nyquist rate, the number of samples will be

$$\begin{aligned}n_X &= L_\xi \times 2b_X = \frac{L_\xi L_X}{\lambda f}\\ n_Y &= L_\eta \times 2b_Y = \frac{L_\eta L_Y}{\lambda f}.\end{aligned}$$

This is precisely the same number of samples required in the Fourier plane, as evidenced by Eq. (9-90).

9-12. The geometry is such that we can apply Eq. (5-19) to relate the object and hologram fields,

$$\begin{aligned}U_h(x,y) &= \frac{\exp\left[j\frac{\pi}{\lambda f}\left(1-\frac{f+\Delta z}{f}\right)(x^2+y^2)\right]}{\lambda f}\\ &\times \iint_{-\infty}^{\infty} U_o(\xi,\eta)\exp\left[-j\frac{2\pi}{\lambda f}(\xi x+\eta y)\right]\, d\xi d\eta.\end{aligned}$$

We know from Eq. (9-88) that the bandwidths of the Fourier transform factor of this expression are

$$\begin{aligned} 2\tilde{B}_X &= \frac{L_\xi}{\lambda f} \\ 2\tilde{B}_Y &= \frac{L_\eta}{\lambda f}. \end{aligned}$$

In accord with the philosophy used in deriving Eq. (9-90), we add to this the bandwidth of the quadratic phase term. The local spatial frequencies of the factor

$$\exp\left[-j\frac{\pi \Delta z}{\lambda f^2}(x^2+y^2)\right]$$

are easily shown to be limited to

$$\begin{aligned} |f_{lX}| &\leq \frac{|\Delta z| L_X}{2\lambda f^2} \\ |f_{lY}| &\leq \frac{|\Delta z| L_Y}{2\lambda f^2}. \end{aligned}$$

The total bandwidths can now be approximated as

$$\begin{aligned} 2B_X &= 2\tilde{B}_X + 2|f_{lX}| = \frac{L_\xi + \frac{|\Delta z|}{f} L_X}{\lambda f} \\ 2B_Y &= 2\tilde{B}_Y + 2|f_{lY}| = \frac{L_\eta + \frac{|\Delta z|}{f} L_Y}{\lambda f}. \end{aligned}$$

It follows that the number of samples required in the hologram in this case becomes

$$\begin{aligned} N_X &= \frac{L_X\left(L_\xi + \frac{|\Delta z|}{f} L_X\right)}{\lambda f} \\ N_Y &= \frac{L_Y\left(L_\eta + \frac{|\Delta z|}{f} L_Y\right)}{\lambda f}. \end{aligned}$$

9-13. The figure illustrates the structure of the hologram for a spectrum that is constant.

(a) The coefficients of a two-dimensional Fourier series expansion are found by Fourier transforming the structure of a single cell, and substituting $f_X = n/L$, $f_Y = m/L$:

$$\begin{aligned} c_{n,m} &= \frac{1}{L^2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \text{rect}\frac{10x}{L}\,\text{rect}\frac{x}{aL} e^{-j2\pi(\frac{n}{L}x+\frac{m}{L}y)}\,dxdy \\ &= \frac{a}{10}\,\text{sinc}\left(\frac{n}{10}\right)\,\text{sinc}\,(am)\,. \end{aligned}$$

(b) The fraction of incident light that end up in the zero-order spot is given by

$$|c_{0,0}|^2 = \frac{a^2}{100}.$$

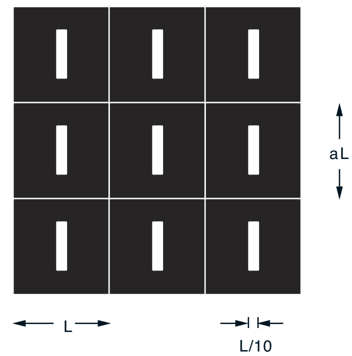

Figure 9-13:

(c) The fraction of incident light that is blocked by the opaque part of the hologram is simply the fractional area of a cell that is opaque. Simple geometry shows that this is $1 - a/10$.

(d) The diffraction efficiencies of the two first-order images are represented by $|c_{1,0}|^2$ and $|c_{-1,0}|^2$, since their direction of deflection from the origin is the x-direction. We have

$$|c_{1,0}|^2 = |c_{-1,0}|^2 = \frac{a^2}{100} \operatorname{sinc}^2\left(\frac{1}{10}\right) = 0.0097a^2.$$

9-14. For simplicity, assume that the reference point is on the optical axis. The spatial frequency associated with the fringe pattern generated by interference of this reference with a point-source object distance d away from the reference point will be

$$f_o = \frac{d}{\lambda z}.$$

Equating f_o to the cutoff frequency f_c of each type of film and solving for the resulting value of d, we obtain:

$$d = \lambda z f_c = \begin{cases} \text{Tri-X} & 1.6\,\text{mm} \\ \text{High-Contrast Copy} & 1.9\,\text{mm} \\ \text{SO-243} & 9.5\,\text{mm} \\ \text{Agepan FF} & 19.0\,\text{mm} \end{cases}.$$

9-15. The exposure to which the emulsion is subjected is

$$E = A^2 + a^2 + 2Aa\cos[2\pi\alpha x - \phi].$$

The variations of exposure about the bias contributed by the reference are

$$E_1 = a^2 + 2Aa\cos[2\pi\alpha x - \phi].$$

(a) Taking the cube of E_1 and expanding the square and the cube of the cosine obtained, we find

$$\begin{aligned} E_1^3 \quad = \quad & a^6 + 6a^4A^2 + (6a^5A + 6a^3A^3)\cos[2\pi\alpha x - \phi] \\ & +6a^4A^2\cos[4\pi\alpha x - 2\phi] + 2a^3A^3\cos[6\pi\alpha x - 3\phi]. \end{aligned}$$

The portion of the transmitted field that generates the first-order images is the term involving $\cos[2\pi\alpha x - \phi]$, or

$$U_1(x,y) = \left[6a^5(x,y)A + 6a^3(x,y)A^3\right]\,\cos\left[2\pi\alpha x - \phi(x,y)\right].$$

(b) If $A \gg a$, then

$$U_1(x,y) \approx 6a^3(x,y)A^3 \cos\left[2\pi\alpha x - \phi(x,y)\right].$$

(c) The phase modulation is correct, but the amplitude modulation is distorted from its ideal value of $2Aa(x,y)$.

Chapter 10

10-1. We start with the relation of Eq. 10-9,

$$\eta = \left[1 + \frac{1 - \frac{\chi^2}{\Phi^2}}{\sinh^2\left(\Phi\sqrt{1 - \frac{\chi^2}{\Phi^2}}\right)}\right]^{-1}.$$

First use the relation $1/\sinh^2 x = \cosh^2 x$ to write

$$\eta = \left[1 + \left(1 - \frac{\chi^2}{\Phi^2}\right)\text{csch}^2\left(\Phi\sqrt{1 - \frac{\chi^2}{\Phi^2}}\right)\right]^{-1}.$$

Then using Eq. 10-7,

$$\left(\frac{\chi}{\Phi}\right)^2 = \left(\frac{\pi\ell\,\Delta\tilde{\lambda}}{2\Lambda^2} \times \frac{\tilde{\lambda}}{\pi\,\delta n\,\ell}\right)^2 = \left(2\frac{\Delta\lambda}{\lambda_B\,\delta n}\right)^2 = \left(2\frac{x}{\delta n}\right)^2,$$

where we have used the fact that $\Delta\tilde{\lambda}/\tilde{\lambda} = \Delta\lambda/\lambda << 1$, allowing $\Delta\tilde{\lambda}\tilde{\lambda}/2\Lambda^2$ to be replaced by $2\Delta\lambda/\lambda_B = 2x$. In addition, using the definition $N = \ell/\Lambda$, we have

$$\Phi = \frac{\pi\,\delta n\,\ell}{\tilde{\lambda}} \approx \frac{\pi\,\delta n\,\ell}{\tilde{\lambda}_B} = \frac{\pi\,\delta n\,N}{2},$$

yielding the final result of Eq. 10-10,

$$\eta = \left[1 + \left(1 - \frac{4x^2}{\delta n^2}\right)\text{csch}^2\left(\frac{\pi\,\delta n\,N}{2}\sqrt{1 - \frac{4x^2}{\delta n^2}}\right)\right]^{-1}.$$

10-2. From Eq. 10-12, the effective length of the grating is given by

$$\ell_0 \approx \frac{6\Lambda}{\pi\,\delta n} = \frac{3\tilde{\lambda}_B}{\pi\,\delta n} = \frac{3\lambda_B}{\pi n\,\delta n_1}.$$

Now with $\lambda_B = 1550$ nm, $n_1 = 1.45$, and the three values of δn, we find

δn	ℓ_0
10^{-4}	1 cm
10^{-3}	1 mm
10^{-2}	100 μm

10-3. Choosing the time of the reference pulse as the time origin, the signal pulse is written $s(t-\tau_0)$, where τ_0 is the time difference between the reference pulse and the signal origin. Thus, from the shift theorem, the Fourier transform of the signal is given by

$$\mathcal{F}\left\{s(t-\tau_0)\right\} = S(\nu)\exp(-j2\pi\nu\tau_0),$$

where $S(\nu)$ is the Fourier transform of $s(t)$. Let $\Delta\nu$ represent the resolution of the grating in the spectral space. To avoid destructive cancellation of fringes arising from the various frequencies that lie within a single spectral resolution cell, we require that the phase shift above caused by the time delay τ_0 be less than 2π within any one resolution element of width $\Delta\nu$. Thus we require

$$2\pi\Delta\nu\tau_0 \leq 2\pi.$$

Now we call upon the result presented in Prob. 4-11, noting that $\Delta\lambda/\lambda = \Delta\nu/\nu$, with the result that

$$\frac{\Delta\nu}{\nu} = \frac{1}{N}$$

where N is the number of grating periods illuminated by the signal and reference pulses at the input grating, and a first diffraction order has been assumed. Solving this equation for $\Delta\nu$ and substituting this result in the equation above, we find that we require

$$\tau_0 \leq \frac{N}{\nu} = NT$$

to assure that the fringes do not cancel one another, where T is the period of the optical carrier frequency.

10-4. We begin the solution with a restatement of the grating equation (cf. Eq. 10-13 and Fig. 10.8),

$$\sin\theta_2 = \sin\theta_1 - \frac{\lambda}{\Lambda} = \sin\theta_1 - \frac{c}{\nu\Lambda},$$

where we have used the fact that $\lambda = c/\nu$, c being the velocity of light . We can solve this expression for ν, yielding

$$\nu = \frac{c/\Lambda}{\sin\theta_1 - \sin\theta_2}.$$

The ray traveling with angle θ_2 arrives at the focal plane at position

$$x = f\tan\theta_2 = \frac{f\sin\theta_2}{\sqrt{1-\sin^2\theta_2}}.$$

If we multiply through by the denominator of this equation for x, and square both sides of the equation, we can solve for $\sin^2\theta_2$ with the result

$$\sin^2\theta_2 = \frac{x^2}{f^2+x^2}.$$

Since θ_2 is positive and less than $90\,\text{deg}$, we must take the positive square root of both sides of this equation, yielding

$$\sin\theta_2 = \frac{x}{\sqrt{f^2+x^2}}$$

Substituting this expression for $\sin\theta_2$ in the previous expression for ν, we find

$$\nu = \frac{c/\Lambda}{\sin\theta_1 - \frac{x/f}{\sqrt{1-(x/f)^2}}}.$$

10-5. The location of wavelength $\lambda_m = \lambda_0 + m\,\delta\lambda$ at the output is found by first finding the location of the inverted image of the input port on which the wavelength appears, and then cyclic shifting (i.e. with wrapping) the output port downward by m locations. The result is

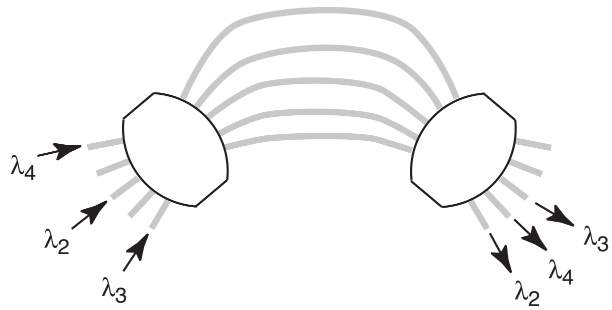

Figure 10-5:

10-6. A complicating factor in this case is that an odd number N of input ports yields an even number $2N$ of output ports. Thus some assumption must be adopted regarding how the input ports are mapped to output ports for the "design wavelength" λ_0. We assume the system has been designed such that for wavelength λ_0, the N input ports are mapped (with inversion) to the *top* N output ports. (Note for instructors: if you assign this problem to students, you may want to add this assumption in order to have uniform answers.) Other assumptions are equally valid, or course, and will yield different details in the answer to part (b).

(a) Since the grating section has $2N$ waveguides, we can use Eq. (10-35) for wavelength resolution, with N replaced by $2N$,

$$\delta\lambda = \frac{\lambda_0}{2Nm},$$

where m is the grating order used. Similarly, Eq. (10-36) can be used to yield the spatial resolution

$$\delta x = \frac{\lambda_0 f}{2n_s N\Lambda}.$$

(b) The problem does not specify how many wavelengths are present on each of the input ports. We assume that there are N wavelengths, as in the case shown in Fig. 10.23. Again, the wavelengths are assumed to be the same on all input ports and to consist of $\lambda_0 + p\,\delta\lambda$ for $p = 0, 1, \cdots, N-1$. We adopt the notation used earlier with two subscripts on each wavelength, the first being the label for the input port (0 at the bottom input port, $N-1$ at the top input port) and the second for the wavelength index. Thus $\lambda_{p,q}$ is the q^{th} wavelength on the p^{th} input port.

The geometry described is shown below for the particular case of $N = 5$. The numbering system for the output ports is also shown. Rather than attempting to squeeze the wavelength sequences for each output port into the figure above, instead we present a table, where the top row corre-

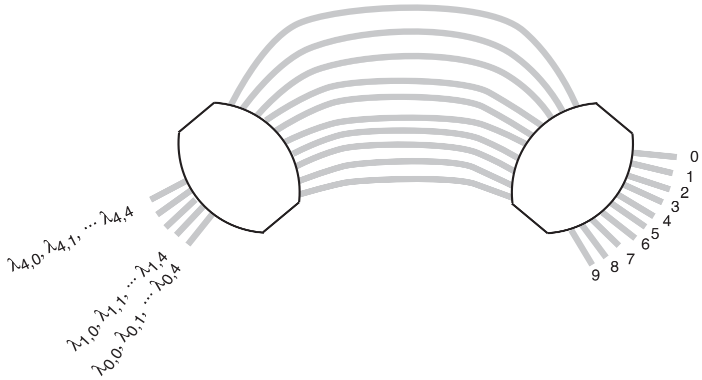

Figure 10-6:

sponds to the top output port, the next row the second output port from the top, etc.

$$\begin{bmatrix} \lambda_{0.0} & & & & \\ \lambda_{1.0} & \lambda_{0,1} & & & \\ \lambda_{2,0} & \lambda_{1,1} & \lambda_{0,2} & & \\ \lambda_{3,0} & \lambda_{2,1} & \lambda_{1,2} & \lambda_{0,3} & \\ \lambda_{4,0} & \lambda_{3,1} & \lambda_{2,2} & \lambda_{1,3} & \lambda_{0,4} \\ \lambda_{4,1} & \lambda_{3,2} & \lambda_{2,3} & \lambda_{1,4} & \\ \lambda_{4,2} & \lambda_{3,3} & \lambda_{2,4} & & \\ \lambda_{4,3} & \lambda_{3,4} & & & \\ \lambda_{4,4} & & & & \\ & & & & \end{bmatrix}$$

Note that only output ports 0 through 8 are occupied by wavelengths, and only output port 4 has a full complement of 5 wavelengths.