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FUNCTIONAL APPROACH TO NONLINEAR MODELS OF WATER FLOW IN SOILS

Gabriela Marinoschi

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by

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Contents

Part IV Appendix

Foreword

... a pure mathematician does
what he can do as well as he should,
whilst an applied mathematician does
what he should do as well as he can...

(Gr. C. Moisil
Romanian mathematician, 1906-1973)

Flows in porous media were initially the starting point for the study which has evolved into this book, because the acquirement and improving of knowledge about the analysis and control of water infiltration and solute spreading are challenging and demanding present issues in many domains, like soil sciences, hydrology, water management, water quality management, ecology. The mathematical modelling required by these processes revealed from the beginning interesting and difficult mathematical problems, so that the attention was redirected to the theoretical mathematical aspects involved. Then, the qualitative results found were used for the explanation of certain behaviours of the physical processes which had made the object of the initial study and for giving answers to the real problems that arise in the soil science practice. In this way the work evidences a perfect topic for an applied mathematical research.

This book was written in the framework of my research activity within the Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy. Some results were obtained within the project CNCSIS 33045/ 2004-2006, financed by the Romanian Ministry of Research and Education.

In a preliminary form, part of the results included here were lecture notes for master and Ph.D. students during the scientific stages (November-December 2003 and May-June 2004) of the author at the Center for Optimal Control and Discrete Mathematics belonging to the Central China Normal University in Wuhan.

The book addresses mathematicians, applied mathematicians and all researchers interested in mathematical problems susceptible to be solved within the semigroup and variational approaches, in particular applied to groundwater flows, and can be used as a basis for a graduate course in Applied Mathematics.

This work is a result of the suggestion made by Professor Viorel Barbu, to whom I also owe my initiation in this elegant domain of mathematics and the understanding of the perspective which the functional approach confers to applied problems. I take this opportunity to express my gratitude for the fruitful discussions and observations, as well as for the permanent encouragements provided during the elaboration of this work.

I am indebted to Professor Mimmo Iannelli for the helpful mathematical discussions I have had with him in the last years.

Also, I would like to thank Springer and in particular Prof. R. Lowen for the publication of my book, and Marlies Vlot, Marieke Mol and Werner Hermens for their kind assistance.

Gabriela Marinoschi January 2006, Bucharest

Introduction-motivation

This book is a work of applied mathematics focusing on the functional study of the nonlinear boundary value problems relating to the water flow in porous media and it was written with the belief that the abstract theory may be sometimes easier and richer in consequences for applications than standard classical approaches are. The volume deals with diffusion type models and emphasizes the mathematical treatment of their nonlinear aspects. An unifying approach to different boundary value problems modelling the water movement in porous media is presented, and the high degree of generality and abstraction, kept however within reasonable limits, is rewarded by the richness of the results obtained in this way.

Water infiltration, and transport and diffusion of solutes in porous media, are two underground flow processes whose study is of great importance due to the strong impact they have on the human life. Water supplied by rainfalls, irrigation or leakages from other water bodies crosses the soil carrying with it various soluble substances provided by surface or subsurface sources and can reach the phreatic aquifer from where the drinkable water is extracted. These processes evolve in time and in principle the problem is to detect the state of the system at any given time when knowing its initial state and the laws that govern the system changes with time. The mathematical model for such a system is an evolution equation, in the most cases a partial differential equation (PDE). Water infiltration and solute dispersion are not the only processes that develop in the underground, but we restricted the study especially to the first one because it is basic and the models describing it are fundamental in the theory of parabolic PDEs, being valid, with slight modifications, for solute dispersion, too.

The practical demands lie generally on the quantitative and numerical study of the system evolution, namely on its solution, but the mathematical point of view occurs before this and directs the interest to the proof of facts that allow the approach to make sense. These are the existence, uniqueness and the regularity properties of the solutions. The final intention is to apply these results to real physical systems, so it must be taken into account that the functions involved in the models may not be continuous or smooth, so that the success of the treatment is closely linked to the choice of suitable functional frameworks. These considerations persuade the researcher to adopt a more abstract approach and lead to the conviction that the functional approach is the most appropriate to describe the physical processes under discussion, with the hope of obtaining more information about what is expected during their evolution.

Since the monotonicity of the nonlinear term in the associated equations is a natural dissipativity assumption for these classes of problems and has an obvious physical meaning, the methods are related to the theory of nonlinear evolution equations with monotone operators in Hilbert spaces.

As far as revealed by the literature, a systematic study of these models within the above mentioned framework has not been done, but only some models have been occasionally dealt with.

Our main interest was centered upon the difficulties posed by the nonlinearity of the processes, so that specific nonlinear aspects were searched and modelled. Even if not all the models taken into account are accurate in describing the hydraulic behaviour of a soil and they may better characterize other phenomena related to the flow in a porous medium, they are discussed for the interesting mathematical aspects determined by their nonlinear particularities. Various boundary value problems are chosen to exemplify the characteristics induced by the degree of nonlinearity of the soil.

Using the theory of evolution equations with m-accretive operators new results of a higher degree of generality are obtained. This increases the possibility of a better approximation of the infiltration problems with direct applications to numerical schemes. An immediate implication is the approach to control and inverse problems which can answer a multitude of practical requirements (e.g., infiltration area control, long time behaviour control, pollution control).

This work has not the ambition of presenting an exhaustive study of the underground flow models, but rather to emphasize the modern methods of approaching these problems. Moreover, it intends to open new prospects relating to this topic, by confining it to some basic or specific models and by tackling certain particular aspects in this domain.

From the mathematical point of view the results obtained can be considered as general results in the theory of nonlinear parabolic equations. Nevertheless the mathematical models developed in this book do not confine only to the infiltration flow models. Such diffusion and transport models can be associated to other physical processes, too. Although the water flow in soils was the principal exemplification for the functional treatment, the techniques used within the book and the results obtained here turn out useful for studying other appropriate problems arising in general in the movement of fluids in porous media, in the heat theory, phase transitions, biology, chemistry or engineering.

Chapter 1 presents the necessary concepts, definitions and equations used in the theory of infiltration processes in porous media the next chapters will refer to. Water infiltration in unsaturated soils is formulated by the well known Richards' equation. The direct problem of solving it requires the knowledge of the hydraulic behaviour of the soil expressed by means of certain hydraulic functions that form the so-called hydraulic model. They are empirical models introduced by the soil scientists along the past century.

Chapter 2 is devoted to the presentation of the basic hydraulic models but it consists rather in a point of view than in an overview, while providing a mathematical explanation and deduction of various types of diffusion models based on the exploitation of the analytical properties of the hydraulic functions. The facts are interpreted from the mathematician point of view, having thus a certain degree of abstraction, and the hypotheses and properties of the hydraulic functions are synthesized in an analytical form. The purpose is to derive and explain a large variety of diffusion models and to search for the most nonlinear limit cases, all those being used further as examples in the theory application. A correspondence between general diffusion models and infiltration models is set up. The diffusive form provided by Richards' equation in the case of an unsaturated flow is extended, by the introduction of an appropriate multivalued operator, to a general equation which describes the simultaneous saturated-unsaturated flow. This turns into a realistic basic model studied in the next chapters.

Since the treatment of the models will be a functional one, related to the nonlinear evolution equations associated to monotone operators in Hilbert spaces, Chapter 3 brings together a presentation of the approach to the Cauchy problems in abstract spaces, the basic properties of m-accretive operators in Hilbert spaces, and the fundamental results within the semigroup and variational approaches.

Chapter 4 is concerned with the proofs of the existence results for some infiltration boundary value problems treated in the framework of a limit diffusive model here called quasi-unsaturated. The study of these models is also a good opportunity for presenting some fundamental results related to the properties of the nonlinear multivalued elliptic operators. A distinctive feature of these problems formulated as nonlinear infinite dimensional equations is that the state space is a distribution space which suits better the dissipative character of the flow process. As nonlinearities may occur as a combined action of the hydraulic functions, discussions on the proofs of the existence results put into evidence the modifications that arise due to the various assumptions considered in the hydraulic models.

Chapter 5 deals with the study of the complete process of water infiltration into an unsaturated soil and relies on the specific particularities involving the evolution of soil moisture up to saturation reaching and the advance of the interface between the saturated and unsaturated regions. The existence in both mathematical models implied by Richards' equation, i.e., the diffusive one and that in the pressure form, is investigated under the assumptions corresponding to strongly nonlinear hydraulic models. Special emphasis is laid on the main characteristics of a flow in a porous medium, namely on the free boundary problem occurred as a consequence of a simultaneously saturated-unsaturated flow. The existence of the free boundary and the formation of a unique connected saturated domain are studied for boundary conditions of flux type, corresponding to a rain-type inflow through the soil surface and an outflow through a semipermeable underground boundary.

In Chapter 6 some specific aspects relating to infiltration are investigated. First, degenerate problems occurring due to different causes are studied. To illustrate this, a situation often encountered, namely a water column ponding on the soil surface (or the contact of the lower boundary of the soil domain with the phreatic aquifer), was chosen to be modelled by the boundary conditions. Also, a case focusing on the particular behaviour of the hydraulic functions both at the saturation value and at the vanishing field capacity is dealt with. Up to now, the infiltration has been studied by disregarding a specific phenomenon associated, i.e., hysteresis. Hysteresis exhibited in underground processes is a well evidenced phenomenon that brings a serious complication into the models. It is not our intention to extend the discussion on this subject, the more that a whole work can be elaborated on it. The process is explained briefly and an example of an infiltration model with hysteresis is proposed.

As we have specified, this work refers with priority to the water flow in unsaturated, and saturated-unsaturated soils. The other aspect of the flow in completely saturated soils is not explicitly studied. At the first glance, diffusion and transport in fully saturated media may be a little apart from the infiltration problems. In particular, they are simpler from the mathematical point of view, because they generally involve linear PDEs, in which the coefficients can be functions of space variables and possibly time, only. However, nonlinear examples can be encountered, for instance in sorption processes. In most cases the same methods used to study the flow in unsaturated porous media can be also applied, that is why we chose not to dedicate them a separate part.

Chapter 7 approaches some optimal control problems issued from the theory of infiltration in unsaturated and saturated-unsaturated situations. It is the final achievement intended to exemplify the close relationship between the previously developed theory and the applications imposed by practical necessities and to sustain once again the motivation of such a study. All the mathematical issues converge now to respond to problems of major practical importance as parameter identification, optimization and control. A very sensible problem is that of the flow parameter reconstitution from recorded observations, especially when these are scarce. Methods of recovering the rain history which produces a certain infiltration in soil are presented for two cases with few and very few moisture observations, the final aim being to determine the optimality conditions. A regular geometry of the soil enhances a simpler method of control which is also presented.

The comments inserted at the end of each chapter emphasize the utility of the functional methods for the study of these models and especially that they lead, by choosing the appropriate functional framework, to better results as compared to those obtained by other techniques. For example, by a careful interpretation of the theoretical results, a correspondence with the correct physical sense of the solutions is established. Also, important properties useful for practical applications derived from a comparison theory may lead to a possible forecast of the time evolution of the flow. Nevertheless the results obtained in the proofs offer information and support for numerical approaches that must accompany an applied research. The estimates and constructive proofs of a solution enhance efficient procedures for the numerical computing. Of course the accuracy of the results obtained using these models depend on how well one of the proposed relationships fits the hydraulic properties of a real soil.

The mathematical background necessary in reading this book is restrained to the basic real analysis and functional analysis. In the Appendix some fundamental concepts, definitions and results of functional analysis, monotone operators and convex analysis are provided. The appendix is included with the precise aim to create a self-contained work, by exposing the significant results used in the proofs given in the previous chapters.

1

Brief overview of unsaturated flow concepts

The purpose of this chapter is to set up the background the mathematical approach developed in the next chapters will rely on. In this book we study nonlinear diffusion models in porous media and basically we envisage water infiltration in soils, comprising the particularities of both unsaturated and saturated-unsaturated flow, as well as some other specific aspects. In the first section we briefly review some proper concepts and notations encountered in the hydraulics of the unsaturated soils and the basic equations used in the theory of water infiltration. We provide only those concepts necessary in the formulation of the boundary value problems we deal with in the next chapters. The basic equations as they are encountered in soil hydraulics are given without proofs, this not being the purpose of this work, but appropriate references will be indicated in the bibliographical note.

1.1 Some basic definitions in the unsaturated flow

A soil is a porous medium consisting of a solid matrix and a void space (or pore space). The void space is filled with one or more miscible or immiscible fluids. We shall refer in the subsequent part usually to water and air. Water provided by rainfalls, irrigations, leakages from surface waters or underground sources may infiltrate into voids. The flow is said to be *unsaturated* as long as voids are still present. However, partially saturated zones may occur when all pores within them are filled with water. Then the interfaces between the saturated and the unsaturated regions of the soil become free boundaries. We call the water motion in this situation *saturated-unsaturated flow*. Hence, the term of infiltration is related in fact to water flow in an unsaturated or partially saturated soil. Sometimes the whole flow domain under study may become fully saturated and in this case the infiltration ceases and we face further a saturated flow. The soil may become again unsaturated by a drainage process of any type (natural evapo-transpiration, plant root uptake, pumping extraction, etc.).

Water volumetric content and capillary pressure

We shall consider water to be an incompressible fluid, i.e., with the density $\rho_w =$ constant, so that, as in the saturated flow, the concepts we introduce are associated to an *incompressible fluid* model.

Let us consider a reference elementary volume V_r, centered at the point $x = (x_1, x_2, x_3)$ belonging to the flow domain, let V_v be the volume of voids in V_r and V_w be the volume of water in volume V_r. The notions we shall use related to soil pores and fluid phase are: the *porosity* ϕ

$$\phi := \frac{V_v}{V_r}, \tag{1.1}$$

the *volumetric water content,* or simply the *soil moisture*, θ

$$\theta := \frac{V_w}{V_r} \tag{1.2}$$

and the *water saturation*

$$S_w := \frac{V_w}{V_v}.$$

It is obvious that

$$\theta = \phi S_w. \tag{1.3}$$

Similarly, the volumetric content and saturation of another phase can be defined. The porosity may be a function of x and t and $0 \leq \phi \leq 1$. The superior bound $\phi = 1$ is assigned to a fluid medium and $\phi = 0$ characterizes a totally impermeable medium. Generally, we shall consider that porosity is positive and constant. At some points it may be zero, revealing the existence of some totally impermeable intrusions in the soil.

We define (see [18], p. 203) the *residual moisture content* or *field capacity* θ_r, as being the water quantity that remains in soil after any drainage imposed by the gravitational forces has ceased, and the *saturation value* θ_s as the value reached by the moisture when all pores become filled with water, i.e., $\theta_s = \phi$, such that we deduce that

$$0 < \theta_r \leq \theta \leq \theta_s = \phi.$$

The most specific concept related to the unsaturated flow is that of the capillary force that holds water inside the pores against the gravitational force. It is determined by the attraction of water molecules for each other (cohesion) and the attraction of water molecules to the pore walls (adhesion). The level of this force is established in close relation with the size of the pores, being in fact inversely proportional to it. Therefore, in the unsaturated flow, the water-air system in soil is a two-phase system with partial pressures for water (p_w) and air (p_a), between whose values there is a discontinuity which entitles the introduction of a new relation,

$$p_c := p_a - p_w > 0. \tag{1.4}$$

This defines the *capillary pressure* which is a measure of the tendency of the partially saturated porous medium to suck in water or to repel air, (see [18], p. 194).

If we assume that the air in the void space is everywhere at atmospheric pressure, then the pressure in the water existent in the void space is $p_w < p_a$. Usually, the constant atmospheric pressure is taken as reference and is rescaled to zero. Therefore, in the water present in the void space in an unsaturated soil p_w is always negative ($p_w = -p_c$). Under such conditions, we introduce the definition of capillary pressure head, ψ also called *suction*, by

$$\psi := -\frac{p_w}{\rho_w g} > 0,$$

where g is the gravitational constant. Suction can be defined also for $p_a \neq 0$ by $\psi = \dfrac{p_c}{\rho_w g}$, (see [18]). Water inside pores in under suction and when suction is positive, the pressure in the water is negative. We are speaking here about the water pressure in the unsaturated soil, called *pressure head* and defined as the negative suction

$$h := \frac{p_w}{\rho_w g} < 0. \tag{1.5}$$

Some authors prefer to work with suction as a variable, rather than working with the negative water pressure, but in this work we shall use the pressure head as a basic function.

Obviously h and θ are functions of the spatial variables and time.

Moreover, we have to specify that in the unsaturated soil a basic relationship, called *constitutive law*, takes place between moisture and pressure, $\theta = \theta(h)$ and it describes a main aspect of the hydraulic behaviour of the soil.

Since the porosity ϕ does not depend on h, the constitutive law implies a relation between the water saturation and the pressure head, called the *retention curve*

$$S_w = S_w(h). \tag{1.6}$$

Hysteresis

Experimental evidence has shown that in reality a cycle of soil wetting-drying process exhibits hysteresis, roughly explained by the fact that the volumetric water content has different profiles with respect to the wetting and draining processes. A hysteretic behaviour means that at any point x belonging to the flow domain, the moisture $\theta(x,t)$ is influenced not only by the unsaturated pressure $h(x,t)$ at the time t, but also by the initial value of the moisture, $\theta_0(x)$ and by the previous time behaviour of the pressure at the point x. More precisely, $\theta(x,t)$ depends on its initial value and on the pressure history at the point x, $h(x,s)$, with $s \in (0,t)$. The essential aspect in the process history is how the monotonicity of the pressure function at the fixed point x has changed during the time interval $(0, t)$.

In soil sciences one assumes that if only one type of process develops, for example only infiltration, the flow is nonhysteretic and is represented by single-valued monotonically continuous hydraulic functions. In consequence, the inverse function $h = h(\theta)$ of the constitutive law can be introduced. During a process in which infiltration and drying are both occurring, the hysteretic character of the interdependence of these functions should be taken into account and the hydraulic functions do not preserve the same monotonicity on the branches of the hysteretic loop.

So, if a hysteretic behaviour is taken into account, the relationship between moisture and pressure at a point x is displayed in the form

$$\theta(t) = \mathcal{F}(h, \theta_0)(t).$$

Here $\mathcal{F}$ is a hysteretic function which allows θ to vary differently on different branches of the hysteretic loop, according to the nature of the current process. Moreover, it was observed that always infiltration takes place at lower moisture than drainage does. An example of hysteresis is represented in Fig. 1.1, where the infiltration ($\theta = \gamma_w(h)$) and drainage ($\theta = \gamma_d(h)$) curves are indicated by an upward ($\nearrow$) and a downward ($\swarrow$) arrow, respectively.

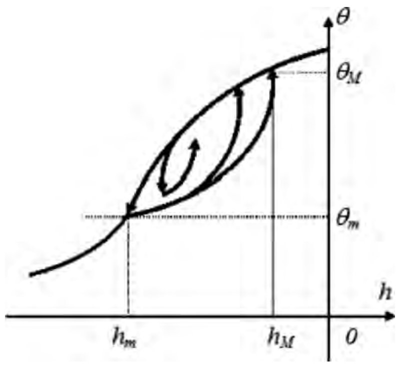

Fig. 1.1. Hysteresis in a wetting-drying process

1.2 Richards' equation

For simplicity we shall refer in this section to a nonhysteretic flow and, as a matter of fact, this case will be treated with priority in the book. The equations with hysteresis will be introduced and treated separately in a special section.

We consider a reference system in which the Ox_3 axis (the vertical axis) is downwards directed.

Richards' equation that describes the dynamics of flow in an unsaturated zone was deduced by combining *Darcy's law* for the specific discharge or *water flux vector* $q = (q_i)_{i\,=\,1,2,3}$

$$q := -\boldsymbol{k}(h) \cdot (\nabla h - i_3) \tag{2.1}$$

with the *equation of continuity* (or *mass conservation*)

$$\frac{\partial(\rho_w \theta)}{\partial t} + \nabla \cdot (\rho_w q) = f, \tag{2.2}$$

where i_3 is the unit vector along the Ox_3 axis and $\boldsymbol{k}$ represents the *hydraulic conductivity.* The hydraulic conductivity expresses the property of the soil of conducting water and, like the constitutive law, it is a function of material, i.e., it is specific to a given soil.

Here, $f(x,t)$ is some water source ($f > 0$), or sink ($f < 0$, characterizing, for example, plant root uptake or other drainage) existent in the flow domain.

We have to specify that if the medium is *anisotropic* (i.e., having different properties corresponding to different directions) the hydraulic conductivity is represented by a tensor, $\boldsymbol{k} = (k_{ij})_{i,j\,=\,1,2,3}$, while in *isotropic* media (whose properties do not depend on the direction) it is a scalar. It depends on the pressure, $k = k(h)$ and has the dimensions of a velocity.

The water incompressibility turns out in $\rho_w =$ constant, so we can write

$$\frac{\partial \theta}{\partial t} + \nabla \cdot q = f, \tag{2.3}$$

that will be used in this form only if ϕ is constant. Recall however that $\theta = \phi S_w$ and if ϕ depends on x or t then we have to take into account the contribution due to this relation in the equation

$$\frac{\partial(\phi S_w)}{\partial t} + \nabla \cdot q = f. \tag{2.4}$$

Homogeneous porous media

We assume that the medium is nondeformable in time (porosity is constant with respect to time) and *homogeneous,* i.e., it consists of a single type of texture.

Hence, Richards' equation representing the water infiltration equation into a three-dimensional anisotropic unsaturated soil is

$$\frac{\partial \theta}{\partial t} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{3} k_{ij}(h) \frac{\partial h}{\partial x_j} \right) + \sum_{i=1}^{3} \frac{\partial k_{i3}(h)}{\partial x_i} = f \tag{2.5}$$

and it is also called *the mixed form* of infiltration equation.

From here we have the equivalent *pressure form* given by

$$C(h)\frac{\partial h}{\partial t} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} k_{ij}(h)\frac{\partial h}{\partial x_j}\right) + \sum_{i=1}^{3} \frac{\partial k_{i3}(h)}{\partial x_i} = f \tag{2.6}$$

where

$$C(h) := \frac{d\theta}{dh} \tag{2.7}$$

is called the *specific water capacity.* As θ increases when pressure head increases, then $C(h) \geq 0$.

If h may be expressed as a function of θ, we shall denote by $\boldsymbol{K} = (K_{ij}(\theta))_{i,j\,=1,2,3}$ the conductivity tensor given in terms of θ. For $C(h) > 0$ we introduce now the notion of *water diffusivity,* $\boldsymbol{D} = (D_{ij})_{i,j=1,2,3}$ in the unsaturated flow

$$D_{ij}(\theta) := K_{ij}(\theta)\frac{dh}{d\theta} \tag{2.8}$$

and we can write (2.5) in the equivalent *diffusive form*

$$\frac{\partial \theta}{\partial t} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} D_{ij}(\theta)\frac{\partial \theta}{\partial x_j}\right) + \sum_{i=1}^{3} \frac{\partial K_{i3}(\theta)}{\partial x_i} = f. \tag{2.9}$$

Equations (2.5), (2.6) and (2.9) represent equivalent forms of Richards' equation for the case of an anisotropic and homogeneous medium.

For an anisotropic medium where we consider that the principal axes of the tensor $\boldsymbol{K}$ (or $\boldsymbol{D}$) are along the x_1, x_2 and x_3 directions ($\boldsymbol{K} = (K_j)_{j\,=1,2,3}$, $\boldsymbol{D} = (D_j)_{j\,=1,2,3}$)) equations (2.5) and (2.9) become

$$\frac{\partial \theta}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(k_j(h)\frac{\partial h}{\partial x_j}\right) + \frac{\partial k_3(h)}{\partial x_3} = f \tag{2.10}$$

and

$$\frac{\partial \theta}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(D_j(\theta)\frac{\partial \theta}{\partial x_j}\right) + \frac{\partial K_3(\theta)}{\partial x_3} = f. \tag{2.11}$$

Finally, if the medium is *isotropic*, the corresponding forms of Richards' equation are

$$\frac{\partial \theta}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(k(h)\frac{\partial h}{\partial x_j}\right) + \frac{\partial k(h)}{\partial x_3} = f, \tag{2.12}$$

or

$$\frac{\partial \theta}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(D(\theta)\frac{\partial \theta}{\partial x_j}\right) + \frac{\partial K(\theta)}{\partial x_3} = f, \tag{2.13}$$

where D and K are scalars depending nonlinearly on the unknown functions that may be either h or θ.

If the gravitational influence has no effect (in horizontal flows) we obtain the equation for the *horizontal infiltration*, called also sorption, having the form (e.g., in isotropic soils)

$$\frac{\partial\theta}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(D(\theta)\frac{\partial\theta}{\partial x_j}\right) = f. \tag{2.14}$$

All the above represent equations for the unsaturated flow, because when saturation occurs the water diffusivity can no longer be rigorously defined by (2.8).

We stress that we use the notation k for the hydraulic conductivity dependent on pressure, while the notation K represents the hydraulic conductivity written in terms of moisture.

In the unsaturated flow the functions θ, k, C depend in a specific nonlinear way on the pressure head h. Obviously, K and D depend nonlinearly on the soil moisture θ and all of them depend on the space variables through h or θ.

Nonhomogeneous (heterogeneous) porous media

We can encounter processes developing in *nonhomogeneous* media consisting of many types of soils structured more or less regularly. In this case each type of soil is characterized by its own porosity and hydraulic parameters and consequently ϕ becomes a function of position. For example, (2.12) becomes

$$\phi(x)\frac{\partial S_w}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(k(h)\frac{\partial h}{\partial x_j}\right) + \frac{\partial k(h)}{\partial x_3} = f. \tag{2.15}$$

However, often in reality a *heterogeneous* soil in which many homogeneous layers are disposed parallel to a given direction can be met. This type of soil is called *stratified* and in the literature it is considered a good enough approximation for certain heterogeneous media. If the stratification is parallel to the horizontal direction all the parameters characterizing this soil are functions of depth only and this model is easier to be studied from the mathematical point of view. However, in such structures at the interface between two layers some functions of interest may display a particular behaviour and we underline for instance that the water saturation (or water content) is no longer continuous, (see [18], p. 206). In return, the pressure head is required to be continuous at the separation surface between two different layers.

Initial conditions

We assume that the flow takes place in a bounded domain Ω, generally considered as three-dimensional, having a known geometry and a smooth boundary Γ, described by an equation written e.g., in an implicit form, $F_\Gamma(x_1, x_2, x_3) = 0$.

The initial conditions associated to one of the main forms of the equations we shall study, (2.5), (2.9) or (2.6) specify the value of θ or h at time $t = 0$, at every point of the flow domain Ω,

$$\theta(x, 0) = \theta_0(x) \text{ or } h(x, 0) = h_0(x),$$

θ_0 or h_0 being prescribed spatial functions in Ω.

Boundary conditions

The boundary conditions we shall choose in the models specify different physical situations occurring at the domain boundaries.

(a_1) The *Dirichlet boundary condition* is applied whenever the flow domain is adjacent to an open water body, like rivers or lakes, by knowing the value of θ or h at all points of the flow domain boundary Γ,

$$\theta(x, t) = \theta_\Gamma(x, t) \text{ or } h(x, t) = h_\Gamma(x, t) \text{ on the boundary } \Gamma, \text{ for } t > 0.$$

This condition written for h at the soil surface describes also an initial situation in which water is ponding, if $h_\Gamma > 0$.

(a_2) By the *Neumann boundary condition,* the normal derivative of the moisture or pressure is prescribed on the boundary,

$$\frac{\partial \theta}{\partial \nu}(x, t) = \theta_\nu(x, t) \text{ or } \frac{\partial h}{\partial \nu}(x, t) = h_\nu(x, t) \text{ on the boundary } \Gamma, \text{ for } t > 0.$$

The unit vector at the point x, in the direction of the outer normal to the boundary Γ is given by

$$\nu = \frac{\nabla F_\Gamma}{|\nabla F_\Gamma|}, \text{ where}$$

$$|\nabla F_\Gamma|^2 = \sum_{i=1}^{N} \left(\frac{\partial F_\Gamma}{\partial x_i} \right)^2, \quad N = 1, 2, 3$$

and the normal derivative is defined by

$$\frac{\partial \theta}{\partial \nu} = \nabla \theta \cdot \nu.$$

(a_3) By the *flux boundary condition,* the flux normal to the boundary surface Γ,

$$q \cdot \nu = q_\nu \text{ on } \Gamma, \text{ for } t > 0$$

is given. Here, $q_\nu(x, t)$ is the component of the specific discharge normal to the boundary. The function q_ν may depend also on the unknown functions

θ or h, directly or as a composed function and hence we get a *Robin boundary condition.* For example, such a condition may model a semipermeable boundary, case which will be intensively treated.

A special case is that of an impermeable boundary with $q_\nu(x,t) = 0$.

When the water supply is provided by a rainfall the boundary condition at the soil surface expresses the equality between the normal components of the water flux entering the soil, q and rainfall rate vector $R_u(x,t)$, i.e.,

$$q \cdot \nu = R_u \cdot \nu. \tag{2.16}$$

If the rainfall rate vector is along the direction of unit vector d, then it can be written $R_u(x,t) = u_R(x,t)d$, where the scalar $u_R(x,t) \geq 0$. We shall call u_R *rainfall rate* and we have $q \cdot \nu = u_R d \cdot \nu$. Obviously, the direction d can depend on time and since ν depends on x, then $d \cdot \nu$ which represents the cosine of the angle between the rain direction and the normal to the boundary may be variable in space and time.

In the particular case of a horizontal soil surface on which a vertical rain falls down, the direction of R_u is opposite to the outward normal to the soil surface, so we can write $R_u(x,t) = u_R(x,t) \cdot (-\nu)$ and therefore

$$q \cdot \nu = -u_R \leq 0. \tag{2.17}$$

Of course, mixed boundary conditions involving Dirichlet, Neumann and flux conditions on parts of the boundaries can be combined to describe an individual situation.

In practice, the directly measured function is the unsaturated pressure, by the means of a device called tensiometer, but this allows the determination of the moisture when one knows the constitutive law. Therefore, at least in the theoretical approach, initial and boundary conditions expressed function of moisture can be considered (as necessary in the diffusive model).

1.3 Presentation of the empirical hydraulic models

Finally, the infiltration models will be complete if information about the relevant physical coefficients (constitutive law, hydraulic conductivity) is provided. As we have specified, these functions depend on the soil structure and in particular on the pore dimensions. Thus, the suction is inversely proportional to the pore radius which explains the fact that soils with a larger size of the pores retain less water, at the same suction, than those with a smaller size of the pores. This specific feature is concentrated in the constitutive law, called also soil-water characteristic curve. The other basic property of the unsaturated soils of transmitting water is quantified by the hydraulic conductivity and it is directly proportional to the pore size. Together, these properties determine the hydraulic response of the soil, which may extend between large limits of nonlinearity, from a weakly nonlinear behaviour up to a strongly nonlinear one. Therefore, the knowledge of these two functions is essential for the direct problems where the goal is to determine the moisture profiles during infiltration, disposing of the known soil hydraulic properties.

In soil sciences these functions were introduced by empirical expressions but they have been used and are used nowadays with good results. We can cite the models established by Philip, Meyer and Warrick, Brooks and Corey, Gardner, Brutsaert, Haverkamp et al. Various hydraulic models are discussed, e.g., in [132]. To offer a general view, some empirical relationships reported in the literature, modelling soil hydraulic properties will be presented, for the isotropic case.

(GA) *Green-Ampt model* characterizes a strongly nonlinear soil and it is defined by the equations

$$D(\theta) := \text{constant} \cdot \delta(\theta - \theta_s) \text{ and } K(\theta) := \begin{cases} K_r = K(\theta_r) & \text{if } \theta < \theta_s, \\ K_s = K(\theta_s) & \text{if } \theta = \theta_s, \end{cases}$$

(where δ is the Dirac function).

(Bu) *Burgers model* corresponds to a weakly nonlinear hydraulic behaviour and is given by

$$D(\theta) := \text{constant and } K(\theta) := \theta^2.$$

(vG) *van Genuchten model* (see [118]) proposes the hydraulic functions defined for any $m \in (0,1)$ by

$$K(\Theta) := \begin{cases} K_s \Theta^{0.5}[1 - (1 - \Theta^{1/m})^m]^2 & \text{if } \Theta < 1, \\ K_s & \text{if } \Theta = 1, \end{cases}$$

$$\Theta(h) := \begin{cases} [1 + |\alpha h|^{1/(1-m)}]^{-m} & \text{if } h < 0, \\ \Theta_s & \text{if } h \geq 0, \end{cases}$$

where Θ is the dimensionless soil-water content

$$\Theta := \frac{\theta - \theta^r}{\theta_s - \theta^r},$$

with θ^r some reference value (usually taken equal to θ_r) and α a length scaling factor. Obviously, the dimensionless saturation value is equal to 1 if $\theta^r = \theta_r$.

The water capacity is then

$$C(h) := \frac{m}{1-m} \left\{ 1 + |\alpha h|^{\frac{1}{1-m}} \right\}^{-m-1} |\alpha h|^{\frac{m}{1-m}} \frac{|h|}{h}.$$

The various values of the parameter m correspond to more or less nonlinear behaviours of the soil, as illustrated in Figs. 1.2 and 1.3.

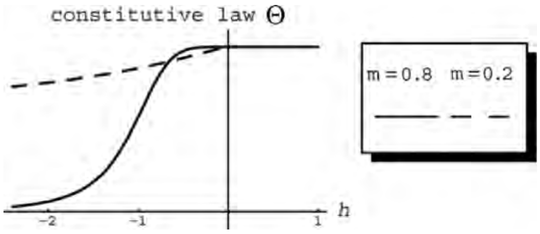

Fig. 1.2. Constitutive law in van Genuchten model

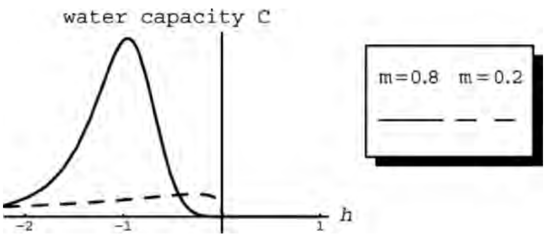

Fig. 1.3. Water capacity in van Genuchten model

Fig. 1.2 displays the graphics of the constitutive law in van Genuchten's model for $m = 0.2$ (dashed line) and $m = 0.8$ (continuous line) and Fig. 1.3 shows the graphic of the corresponding water capacity, for $\alpha = 1$.

Fig. 1.4 shows the graphics of the hydraulic conductivity in van Genuchten's model for $m = 0.2$ and $m = 0.8$.

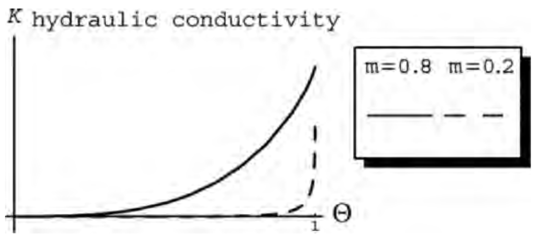

Fig. 1.4. Hydraulic conductivity in van Genuchten model

For example, we can notice that if m is close to 0, we have

$$\lim_{h \nearrow 0} C(h) = 0, \quad \lim_{\Theta \nearrow 1} K'(\Theta) = +\infty, \tag{3.1}$$

but the rate of variation of the constitutive law is very low. Moreover, the field capacity (the point at which the constitutive law becomes concave) is very close to the saturation point. The hydraulic conductivity evolves highly nonlinear.

If m is close to 1, we have that

$$\lim_{h \nearrow 0} C(h) = 0, \quad \lim_{\Theta \nearrow 1} K'(\Theta) < +\infty \tag{3.2}$$

and we can notice a nonlinear variation of the constitutive law, and a more linear behaviour of the hydraulic conductivity.

(BW) The *parametric model of Broadbridge and White*, (see [33]) is very suitable for analytical approaches due to its very simple form

$$D(\Theta) = \frac{c(c-1)}{(c-\Theta)^2}, \ K(\Theta) = \frac{(c-1)\Theta^2}{c-\Theta}, \tag{3.3}$$

with the same significance as before for Θ. Here, the hydraulic nonlinearity of the medium is characterized by the parameter c belonging to $(1, +\infty)$. If $c \to 1$ the medium is strongly nonlinear and if $c \to \infty$ the medium behaves weakly nonlinear. When c approaches these values, the limit models (GA) and (Bu) are recovered, for $c \to 1$ and $c \to \infty$, respectively. Moreover, we have

$$K'(\Theta) = \frac{(c-1)(2c-\Theta)\Theta}{(c-\Theta)^2} \tag{3.4}$$

and

$$\int_0^\Theta D(\xi) d\xi = \frac{(c-1)\Theta}{c-\Theta}. \tag{3.5}$$

Figs. 1.5 and 1.6 represent the water diffusivity and, respectively, the hydraulic conductivity obtained by (BW) model for two values of c, namely $c = 1.01$, corresponding to a highly nonlinear soil and $c = 1.2$ indicating a weakly nonlinear soil.

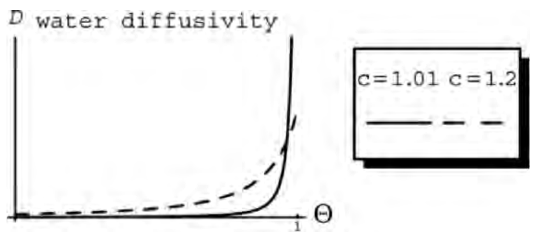

Fig. 1.5. Water diffusivity in Broadbridge model

In (3.3) and (3.4) we fix c and compute

$$D(1) = \frac{c}{c-1}, \ K(1) = 1, \ K'(1) = \frac{2c-1}{c-1}. \tag{3.6}$$

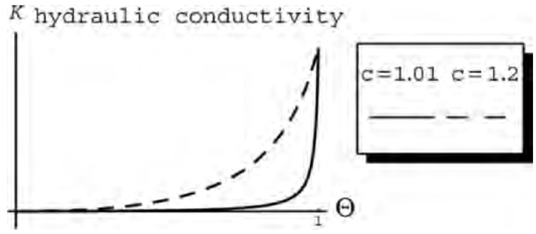

Fig. 1.6. Hydraulic conductivity in Broadbridge model

We observe that in the case of a weakly nonlinear behaviour, corresponding to c large, $c >> 1$, we have

$$D(1) < +\infty, \quad \int_0^1 D(\Theta)d\Theta = 1 < +\infty, \quad K'(1) < +\infty. \tag{3.7}$$

In [33] it is specified for example that for values of c greater than 1.02 the soil begins to behave weakly nonlinear.

When c is close to 1 ($c \searrow 1$) some limit values grow up to infinity

$$\lim_{\Theta \nearrow 1} D(\Theta) = +\infty, \quad \lim_{\Theta \nearrow 1} K'(\Theta) = +\infty, \tag{3.8}$$

which denotes a very strong nonlinearity. Anyway, we have

$$D(0) = \frac{c-1}{c} > 0 \text{ for } c > 1. \tag{3.9}$$

By these few examples we intended to show that the play of the parameters occurring in the empirical hydraulic models puts into evidence different properties of the hydraulic functions and reveals a large variety of soil responses, from a strongly nonlinear to a weakly nonlinear one.

1.4 Comments

This chapter was intended to familiarize the reader with the characters that populate the unsaturated flow world. The mathematical approach that follows will operate however with abstract concepts and so, the definitions introduced here will have no immediate importance. Their individuality will be regained when we shall interpret the qualitative results and particularize them to infiltration models.

We emphasize that, besides Richards' equation along with the initial data and boundary conditions which offer the individuality to a model, the nonlinear expressions of the hydraulic functions are crucial because they determine the embedding of the mathematical models in specific classes of diffusion problems. That is why the next chapter is entirely dedicated to a mathematical outlook of them and to a rigorous mathematical introduction of diffusion models in porous media.

Bibliographical note

Developments in the understanding of the physics of infiltration have been based on the statement of Richards' equation modelling the flow in unsaturated soils. In 1931 L.A. Richards derived in [108] the partial differential equation for the description of water flow in soils using Darcy's law and Buckingham's concept of capillary potential, (see [39]). The mathematical difficulties raised by this equation were overpassed when it was rewritten by E.C. Childs and N. Collis-George, in 1950, as a nonlinear diffusion equation with a water diffusivity dependent on the moisture content, (see [47]). Since then, a lot of approaches to the solution to this equation have been undertaken, but a particular advance in this direction was offered by the theoretical contributions of John Philip in his papers written within four decades beginning with 1954 (see [104], [105], [106], [107], [74], [75]) and more recently by those of M.T. van Genuchten (see [118]), P. Broadbridge and the co-authors (see [32], [33], [34], [133]). For a rigorous deduction of the (BW) hydraulic model we indicate the papers [33], [34]. In [33] and [65] analytical solutions for infiltration models with certain forms of hydraulic functions are given.

For background material on hydraulics of groundwater and details on unsaturated water flow in particular, we refer the reader to the monographs of J. Bear, [18], [19], the edited volumes [20], [21] and to the references given there.

2

Settlement of the mathematical models of nonhysteretic infiltration

In this chapter we shall introduce in a proper view the general models of saturated-unsaturated flow which are the basic ones for our study.

During the flow of a fluid in a porous medium its degree of saturation changes. Unsaturated parts may remain or the medium may become completely saturated. Due to the structure of the soil pores and to the particularities of the influencing factors, like the initial moisture distribution, the presence and the behaviour of the underground sources, the boundary permeability and the rate at which water is supplied, the possibly saturated zones develop in general in nonconnected domains. Obviously, if the whole flow domain becomes saturated the infiltration ceases. Therefore we can speak about infiltration in one of these two situations, namely when the soil is totally unsaturated, or partially saturated. In the simultaneous unsaturated-saturated flow, transition zones from between unsaturated and saturated parts exist and free boundary problems occur.

To deal with all these aspects the first step is to transpose the behaviour of the physical hydraulic functions in mathematical properties. The purpose is not to deduce again hydraulic models, i.e., to set new expressions for the constitutive law and hydraulic conductivity, but to explain the more linear or nonlinear character of them relying on general mathematical hypotheses made with respect to the hydraulic functions. While the unsaturated flow is governed by functions with stable mathematical properties, the mathematical modelling of saturation occurrence is more delicate and depends on the behaviour of the hydraulic functions around the saturation, where very fast variations may be encountered.

For example, some hydraulic models raise a difficult mathematical problem. When the pressure head in the unsaturated soil comes close to the saturation value, the water capacity vanishes and forces Richards' equation to degenerate. Correspondingly, the diffusion coefficient expressed as a function of moisture exhibits a blow-up development around saturation. The nonlinearity is described in one case by a Dirac pulse (see the Green-Ampt model in Sect. 1.3). In the most mathematical literature devoted to this subject this particularity was avoided, by considering a finite valued diffusivity, or studying the problem only in the pressure form.

Therefore the necessity of rigorous mathematical models to describe various types of infiltration and especially, an adequate mathematical model accounting for the simultaneous saturated-unsaturated flow with a blowing-up diffusivity is entitled.

There are not typical procedures to introduce a model meant here as a set of equations that describe a phenomenon in some limits of validity. Some physical processes, especially the ones for which experimental evidence is hardly collected, are very difficult, almost impossible to be trustily modelled, so that models cannot be an exact picture of reality. And even if this were done, the mathematics involved would be certainly extremely complicated so that numerous simplifications should be made. That is why it is normal to start with a simpler model, by letting apart at the beginning some nonessential aspects which do not change the sense of the phenomenon.

A model may be incomplete or may be not completely truthful but in any case must not be contradictory, in the sense that the equations that form it should not contradict one another. With the hope that we respect these considerations in our modelling, we shall investigate further some models considered fundamental for water infiltration in soils and closely related in general to other fluid flow in porous-type media. Even if some particular physical processes may lack in their description, we assert that the models reveal some features of the process studied and on the other hand, involve a certain mathematical interest.

The basic mathematical models of infiltration the next chapters deal with, will be set forth. The modelling developed here starts from the known properties of the hydraulic functions experimentally established by the soil scientists, transcripts them in analytical properties and combines the latter in order to put into evidence general infiltration mathematical models in the diffusive form. Also, particular diffusion types will be modelled, with a main accent on the aspects revealing the strong nonlinearities. Thus, the complete phenomenon of water infiltration into a (partially) unsaturated soil up to saturation occurrence and the evolution of the saturated-unsaturated flow will be set in a rigorous mathematical form and the particularities determined by more weakly or more strongly nonlinearities will be discussed. Some limit models, such as the very fast diffusion model, or degenerate cases, will be also presented.

2.1 Physical context and mathematical hypotheses

From the hydraulic point of view, the problems we shall study rely on the Darcian flow of an incompressible fluid in an isotropic, homogeneous nondeformable porous medium with a constant porosity, as we have seen.

Moreover, we shall assume that temperature variations are small enough to influence the process such that we shall not associate thermic laws to the infiltration model. We disregard some possible interactions that water may have with chemical substances from the soil particles, for the moment being interested only in the hydraulic process and not in a thermic, or a chemical one. Also, we consider here that the air movement does not influence the water flow. To fix the ideas, we state that we work under the following physical hypotheses:

(m_1) an isotropic, nondeformable and homogeneous porous medium with a constant porosity;

(m_2) an incompressible fluid with no physical or chemical reactions with the soil;

(m_3) a nonhysteretic flow.

The general boundary value problem

Assume that we have to study the water infiltration in a domain Ω, within the finite time interval $(0, T)$. The geometry of Ω having the boundary Γ is supposed to be known. More details about Ω and Γ will be given in the next chapters, for each model apart. The vector of space variables is denoted by $x = (x_1, x_2, x_3)$ and the time by t.

In our approach we consider as basic Richards' equation, for an isotropic and homogeneous medium (see (2.12) in Chap. 1), with initial data and various boundary conditions,

$$\frac{\partial \theta}{\partial t} - \nabla \cdot (k(h)\nabla h) + \frac{\partial k(h)}{\partial x_3} = f \text{ in } Q = \Omega \times (0, T), \tag{1.1}$$

$$h(x, 0) = h_0(x) \text{ in } \Omega, \tag{1.2}$$

$$\textit{boundary conditions in } h \text{ on } \Sigma = \Gamma \times (0, T). \tag{1.3}$$

We must specify the properties of the functions occurring in these equations, investigate actually the hydraulic models involved and analyze their levels of nonlinearity.

Description of the hydraulic models

We mention from the beginning that we intend to recover by this modelling the properties of the hydraulic models introduced for the various situations of nonlinearity that extend between the limit cases. In particular, we shall exemplify for the model of van Genuchten and for that of Broadbridge and White, both introduced in Sect. 1.3.

First, we present some assumptions generally valid for infiltration processes.

The hydraulic behaviour of a soil is determined by its microstructure that governs the way in which water enters the pores and imprints a more linear or a more nonlinear character of the porous medium response.

We reiterate the fact that the behaviour of an unsaturated soil, i.e., partially filled with water, is completely known from the hydraulic point of view if two functions are given: one is the constitutive relationship

$$\theta := C^*(h), \tag{1.4}$$

linking the volumetric water content, or moisture of the soil θ, to the pressure head, h, and the other is the hydraulic conductivity

$$k := k(h), \tag{1.5}$$

both depending nonlinearly on h. For the isotropic soil the latter is a scalar function, as we have already specified.

Since we study the nonhysteretic case, the constitutive law and the hydraulic conductivity are single-valued functions of pressure. We stress again that in the unsaturated flow we denote by h the negative value of capillary pressure, see (1.5) in Sect. 1.1.

Therefore, these functions are defined in the unsaturated flow for negative values of the unsaturated pressure between a minimum value, $h = h_r < 0$ and $h = 0$. They are relevant on this interval only because, practically, below h_r there is no flow. The value h_r corresponds to the residual moisture θ_r specified as the quantity of water resident in soil (see Sect. 1.1) and $h = 0$ is the pressure head value at which saturation is reached. Moreover, the value θ_r is related to the notion of field capacity which means, in other words, that infiltration may evolve from the field capacity up to the saturation value.

Correspondingly, the water capacity defined as the derivative of the moisture with respect to the pressure

$$C(h) := \frac{d\theta}{dh}, \tag{1.6}$$

has a unique maximum at h_r.

For the saturated flow, when h becomes zero and then positive, the previously defined functions take constant values all over $[0, \infty)$. Now, h represents the saturated hydraulic pressure that increases as the water column increases.

We intend to show how that the particular character of the hydraulic models is determined by the behaviour of the functions C^* and k around 0.

Mathematical hypotheses

For the unsaturated flow, where $h < 0$, we assume the following:

(m_4) $C^* : [h_r, 0) \to [\theta_r, \theta_s)$ is single-valued, positive, twice differentiable on $[h_r, 0)$, monotonically increasing and concave;

(m_5) $k : [h_r, 0) \to [K_r, K_s)$ is single-valued, positive, twice differentiable on $[h_r, 0)$, monotonically increasing, satisfying the property $k'(h_r) = 0$;

(m_6) $C : [h_r, 0) \to (C_0, C_r]$ is single-valued, non-negative, differentiable on $[h_r, 0)$ monotonically decreasing and satisfies $C'(h_r) = 0$;

(m_7) there exist

$$\theta_s := \lim_{h \nearrow 0} (C^*)(h) > 0, \tag{1.7}$$

$$C_0 := \lim_{h \nearrow 0} (C^*)'(h) \geq 0, \tag{1.8}$$

$$K_s := \lim_{h \nearrow 0} k(h) > 0. \tag{1.9}$$

The conductivity k is generally a convex function; in a certain case we shall assume that it becomes concave in a neighbourhood of the saturation point, $h = 0$. We denote

$$K_0' := \lim_{h \nearrow 0} k'(h), \;\; K_0' \in [0, \infty) \cup \{\infty\}\,. \tag{1.10}$$

In the saturated flow we have

(m_8) $C^*(h) = \theta_s$, $k(h) = K_s$ and $C(h) = 0$ for $h \geq 0$.

Therefore, we see that the unsaturated flow is characterized by $h < 0$ or by $\theta \in [\theta_r, \theta_s)$ while the saturated one by $h \geq 0$ or $\theta = \theta_s$.

The positive values θ_r, θ_s and their corresponding conductivities K_r, K_s are soil characteristics and are known. The properties $k'(h_r) = 0$ and $C'(h_r) = 0$ were put into evidence by experiments, (see [33]).

We notice that the functions C^* and k are continuous on $[h_r, \infty)$ and h_r is the maximum point for C and a saddle point for C^*. Also C is continuous on $[h_r, \infty)$, except possibly at the point 0.

We stress the fact that these properties are verified by the hydraulic models presented before (some properties like (1.8) and (1.10) are put into evidence for particular values of their parameters).

In fact, regarding from the perspective offered by the empirical hydraulic models presented in Sect. 1.3, we have gathered in the previous hypotheses their possible properties. We have noticed that the main role is played by the increase rate of the functions C^* and k around 0. The significant contribution is given by the behaviour of the constitutive law C^*, while the rate of k may determine a particular behaviour without equalizing however the main character imprinted by C^*. The specific particularities of the derivative of k at 0 will necessitate a special mathematical treatment. The next models are appropriate for the water infiltration in soils, and some of them reveal better the behaviours of other fluids infiltration in porous media in general.

2.2 Strongly nonlinear saturated-unsaturated diffusive model

Let us assume $(m_1) - (m_8)$ and

$$C_0 = 0$$

which is the main characteristic of this case. It follows then that C is continuous on $[h_r, \infty)$ and so we can write $C^* : [h_r, \infty) \to [\theta_r, \theta_s]$, as

$$C^*(h) = \begin{cases} \theta_r + \displaystyle\int_{h_r}^{h} C(\zeta) d\zeta, & h < 0, \\ \theta_s, & h \geq 0, \end{cases} \tag{2.1}$$

(see Fig. 2.1).

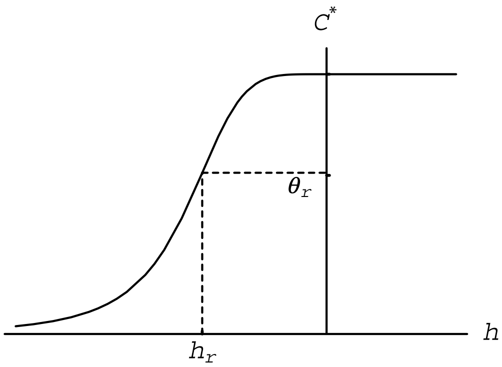

Fig. 2.1. Graphic of the constitutive law

Strongly nonlinear hydraulic conductivity

This situation corresponds to $K_0' \in \mathbf{R}_+ = (0, \infty)$ or $K_0' = +\infty$.

We define an antiderivative of K by

$$K^*(h) := \begin{cases} K_r^* + \displaystyle\int_{h_r}^{h} k(\zeta) d\zeta, & h < 0, \\ K_s^* + K_s h, & h \geq 0, \end{cases} \tag{2.2}$$

where $K^* : [h_r, \infty) \to [K_r^*, K_s^*]$ and

$$K_s^* := K^*(0) > 0. \tag{2.3}$$

(With no loss of generality K_r^* may be taken 0.)

The function K^* is differentiable, monotonically increasing on $[h_r, \infty)$ and with these notations Richards' equation (1.1) becomes

$$\frac{\partial \theta}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} = f \text{ in } Q. \tag{2.4}$$

We apply C^* to the initial condition (1.2) and obtain

$$\theta(x, 0) = \theta_0(x) \text{ in } \Omega, \text{ where } \theta_0 := C^*(h_0)$$

and corresponding replacements should be made in the boundary conditions.

Since it is more convenient to work with the variable θ, we introduce from (2.1) the inverse of C^*, $(C^*)^{-1} : [\theta_r, \theta_s] \to [h_r, +\infty)$, by

$$(C^*)^{-1}(\theta) := \begin{cases} (C^*)^{-1}(\theta), & \theta \in [\theta_r, \theta_s), \\ [0, +\infty), & \theta = \theta_s, \end{cases} \tag{2.5}$$

which is multivalued at $\theta = \theta_s$, but is continuous and monotonically increasing on $[\theta_r, \theta_s)$, see Fig. 2.2.

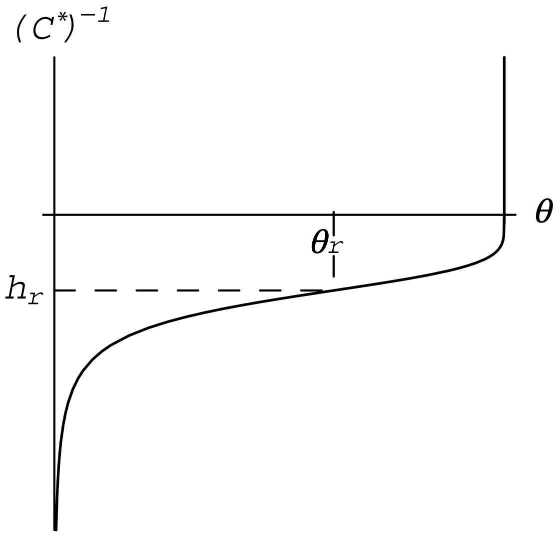

Fig. 2.2. Graphic of the inverse of the constitutive law

Then, we replace it all over in (1.1)-(1.3).

Thus, instead of the conductivity written in function of pressure, we obtain the conductivity expressed in terms of moisture

$$K : [\theta_r, \theta_s] \to [K_r, K_s], \; K(\theta) := (k \circ (C^*)^{-1})(\theta), \; \theta \in [\theta_r, \theta_s], \tag{2.6}$$

function which preserves some of the properties of k, i.e., it is positive, differentiable and monotonically increasing, since for any $\theta \in [\theta_r, \theta_s)$ we have that

$$K'(\theta) = k'((C^*)^{-1}(\theta)) \cdot ((C^*)^{-1})'(\theta)) = \frac{k'((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))} > 0. \tag{2.7}$$

We notice also that

$$K'(\theta_r) = 0 \tag{2.8}$$

and

$$\lim_{\theta \nearrow \theta_s} K'(\theta) = +\infty, \tag{2.9}$$

even if the limit of the derivative of k at $h = 0$, K_0' is either infinity or a finite value. We may assume in this case that k is convex on $[h_r, 0)$ and we see that the same property follows for K, too,

$$K'' = \frac{k''C - k'C'}{C^3} \geq 0, \text{ on } [\theta_r, \theta_s). \tag{2.10}$$

However, for $\theta \in [\theta_r, \theta_l]$ with $\theta_l < \theta_s$ the derivative of K is bounded, so that K follows to be Lipschitz on intervals strictly included in $[\theta_r, \theta_s)$,

$$\left|K(\theta) - K(\overline{\theta})\right| \leq M_l \left|\theta - \overline{\theta}\right|, \ \forall \theta, \overline{\theta} \in [\theta_r, \theta_l], \ \theta_l < \theta_s, \tag{2.11}$$

where

$$M_l = \max_{\theta \in [\theta_r, \theta_l]} \frac{k'((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))} < \infty. \tag{2.12}$$

Plugging (2.5) in (2.2) we get the function

$$\beta^*(\theta) := \begin{cases} (K^* \circ (C^*)^{-1})(\theta), & \theta \in [\theta_r, \theta_s), \\ [K_s^*, +\infty), & \theta = \theta_s \end{cases} \tag{2.13}$$

that turns out to be multivalued and notice immediately that

$$\lim_{\theta \nearrow \theta_s} \beta^*(\theta) = K_s^*, \tag{2.14}$$

(see Fig. 2.3).

For $\theta \in [\theta_r, \theta_s)$ the function $(C^*)^{-1}$ is monotonically increasing, so that we can calculate $\beta^*(\theta)$ by changing the variable in the integral (2.2), denoting $\zeta := (C^*)^{-1}(\xi)$. In this way we get

$$\beta^*(\theta) = K_r^* + \int_{\theta_r}^{\theta} \beta(\xi) d\xi, \text{ for } \theta \in [\theta_r, \theta_s),$$

where

$$\beta(\theta) := \frac{k((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))}, \text{ for } \theta \in [\theta_r, \theta_s), \tag{2.15}$$

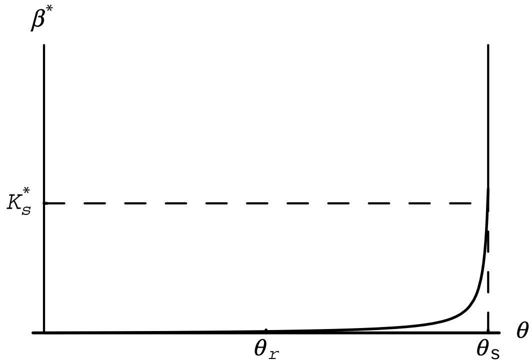

Fig. 2.3. Graphic of $\beta^*(\theta)$

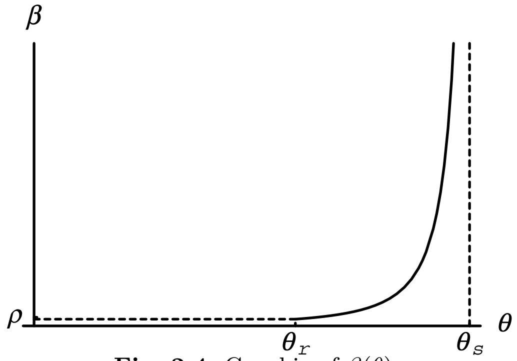

Fig. 2.4. Graphic of $\beta(\theta)$

(see Fig. 2.4).

In this way we have rigorously recovered the definition of the water diffusivity function which was denoted by D in Sect 1.2.

We notice that β has two important properties

$$\beta(\theta) \geq \rho := \beta(\theta_r) = \frac{K_r}{C_r} > 0, \ \forall \theta \in [\theta_r, \theta_s) \tag{2.16}$$

and

$$\lim_{\theta \nearrow \theta s} \beta(\theta) = +\infty. \tag{2.17}$$

Moreover, by the hypotheses made upon the functions C and k it follows that β is monotonically increasing and convex, i.e.,

$$\beta' = \frac{k'C - kC'}{C^3} \geq 0, \text{ on } [\theta_r, \theta_s), \tag{2.18}$$

$$\beta'(\theta_r) = 0, \tag{2.19}$$

$$\beta'' = \frac{(k''C - kC'')C - 3C'(k'C - kC')}{C^5} > 0, \text{ on } [\theta_r, \theta_s). \tag{2.20}$$

Hence, β^* is three times differentiable, monotonically increasing and convex on (θ_r, θ_s) and as a matter of fact we can write

$$\beta^*(\theta) = \begin{cases} K_r^* + \displaystyle\int_{\theta_r}^{\theta} \beta(\xi)d\xi & \text{for } \theta \in [\theta_r, \theta_s), \\ [K_s^*, +\infty) & \text{for } \theta = \theta_s. \end{cases} \tag{2.21}$$

Moreover, by (2.16) and (2.17) we deduce that the function β^* satisfies the inequality

$$(\beta^*(\theta) - \beta^*(\overline{\theta}))(\theta - \overline{\theta}) \geq \rho(\theta - \overline{\theta})^2, \forall \theta, \overline{\theta} \in [\theta_r, \theta_s]. \tag{2.22}$$

This can be very easily checked for $\theta, \overline{\theta} < \theta_s$, or $\theta = \overline{\theta} = \theta_s$.
If $\theta = \theta_s$ and $\overline{\theta} < \theta_s$, using (2.14) we have

$$\begin{aligned}(\beta^*(\theta_s) - \beta^*(\overline{\theta}))(\theta_s - \overline{\theta}) &\geq (K_s^* - \beta^*(\overline{\theta}))(\theta_s - \overline{\theta}) \\ &= \left(\lim_{\theta \nearrow \theta_s} K^*((C^*)^{-1}(\theta)) - \beta^*(\overline{\theta})\right)(\theta_s - \overline{\theta}) \\ &= \lim_{\theta \nearrow \theta_s} \left[\left(K^*((C^*)^{-1}(\theta)) - K^*((C^*)^{-1}(\overline{\theta}))\right)(\theta - \overline{\theta})\right] \\ &\geq \lim_{\theta \nearrow \theta_s} \left[(\beta^*(\theta) - \beta^*(\overline{\theta}))(\theta - \overline{\theta})\right] \geq \rho(\theta_s - \overline{\theta})^2,\end{aligned}$$

since $K^* \circ (C^*)^{-1}$ is a monotonically increasing function on $[\theta_r, \theta_s)$.
In conclusion we can set

Model 1.1. Let us assume (m_1)-(m_8), $C_0 = 0$ and $K_0' \in \mathbf{R}_+ \cup \{\infty\}$. Then, the diffusive model of the *strongly nonlinear saturated-unsaturated infiltration with a strongly nonlinear hydraulic conductivity* is given by

$$\frac{\partial \theta}{\partial t} - \Delta\beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \text{ in } Q, \tag{2.23}$$

$$\theta(x, 0) = \theta_0(x) \text{ in } \Omega, \tag{2.24}$$

$$\textit{boundary conditions in } \theta, \tag{2.25}$$

where β^* is the multivalued function defined by (2.21), β is given by (2.15) and K is the single-valued function (2.6). Moreover, β^* is strongly monotone, β satisfies (2.16)-(2.20) and K has the properties (2.8)-(2.12).

As a matter of fact, β^* is multivalued and the sign equal $(=)$ in (2.23) is not properly used. The appropriate symbol should be $\ni$, which will be used after the presentation of the functional interpretation of this type of equation. Also, we shall specify in a further chapter the exact meaning of the solutions to (2.23)-(2.25). It must be emphasized that equation (2.23) is multivalued. This must not be surprising if one takes into account that, roughly speaking, (2.23) models a free boundary problem, (see [11]). In fact, as we shall see later, at each time t the domain Ω can be decomposed into two regions: the saturated one, $\{x;\ \theta(x,t) = \theta_s\}$ and the unsaturated one $\{x;\ \theta(x,t) < \theta_s\}$, separated by a free boundary. The extension of a nonlinear function arising in such a problem to a multivalued one is common in the theory of nonlinear differential equations with discontinuous coefficients as well as in that modelling free boundary processes. The procedure of filling the jumps by using multivalued operators is necessary to enhance an existence theory.

In this way, the simultaneous saturated-unsaturated flow has been mathematically described by a unique multivalued function constructed with the aid of diffusivity.

Up to now the model is a general diffusion one in the sense that the functions have been not personalized, but we notice that the properties that are definitional for this model are (2.17) and (2.21). They are exemplified by functions of the form

$$\beta(\theta) = \frac{1}{(\theta_s - \theta)^{1-p}} \quad \text{with } 0 < p < 1,$$

which reveals the character of *fast diffusion* of this model.

In particular, we ascertain that this case is illustrated by the hydraulic functions stated in the (BW) model for the strongly nonlinear situation, corresponding to c approaching 1 (see (3.8) in Sect. 1.3) or in the (vG) model, e.g., for m close to 0, (see (3.1) in Sect. 1.3).

Weakly nonlinear hydraulic conductivity

A strongly nonlinear model, but with a weaker nonlinear behaviour of the conductivity may be obtained under conditions that lead to $\lim_{\theta \nearrow \theta_s} K'(\theta) < \infty$. To reach such a situation we have to impose just from the beginning a stronger condition for k, namely that there exists $M > 0$, such that

$$k'(h) \leq MC(h),\ \forall h \in [h_r, 0], \tag{2.26}$$

which implies that

$$K_0' = 0,\ \lim_{h \nearrow 0} \frac{k'(h)}{C(h)} = M. \tag{2.27}$$

This condition expresses the fact that k changes the convexity around $h = 0$, becoming concave in a neighbourhood of 0. In this way K turns out to be Lipschitz on $[\theta_r, \theta_s]$ with the constant M. We observe that the functions β and K remain monotonically increasing and we may assume, with no loss of generality, certain conditions to enhance their convexity.

This situation is put into evidence e.g., in the (vG) model for m close to 1, see (3.2) in Sect. 1.3. To conclude, this case can be resumed in

Model 1.2. Let us assume (m_1)-(m_8), $C_0 = 0$ and (2.26)-(2.27). Then, the diffusive model of *strongly nonlinear saturated-unsaturated infiltration with a weakly nonlinear hydraulic conductivity* is given by (2.23)-(2.25), where the functions β and β^* have the properties specified in Model 1.1 except for K which is given by (2.6), with

$$\lim_{\theta \nearrow \theta_s} K'(\theta) = M < \infty.$$

2.3 Weakly nonlinear saturated-unsaturated diffusive model

Having again a look at the hydraulic models, we see that for $c >> 1$ the diffusivity in the (BW) model is finite at $\theta = \theta_s$, (see (3.7) in Sect. 1.3). We intend to reveal which property of the functions C^* and k can provide such a value. Let us suppose that the constitutive law increases from the left to its maximum value with a nonzero rate at the left of zero,

$$> 0,$$

but very close to 0. In this case the function C^* is not differentiable at $h = 0$ and the water capacity

$$C : [h_r, \infty) \to [C_0, C_r] \cup \{0\}, \; C(h) = \begin{cases} \dfrac{d\theta}{dh}, & h < 0 \\ 0, & h \geq 0 \end{cases} \tag{3.1}$$

is no longer continuous at $h = 0$, having the jump $C_0 = -\lim\limits_{h \nearrow 0} \dfrac{d\theta}{dh}$.

The functions K and β^* and β will be defined in the same way as before, but in this case the value of β at $\theta = \theta_s$ exists and it is

$$\lim_{\theta \nearrow \theta_s} \beta(\theta) = \frac{K_s}{C_0} < \infty. \tag{3.2}$$

However, the function $\beta^*(\theta)$ follows still to be multivalued, even if the diffusivity is finite, this being the feature revealed by the free boundary problem that evolves in this case, too.

Weakly nonlinear hydraulic conductivity

Assume that the derivative of k at $h = 0$, has a finite value, $K_0' < \infty$. Hence, K is Lipschitz with the constant

$$M = \max_{\theta \in [\theta_r, \theta_s]} \frac{k'((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))} \leq \frac{K_0'}{C_0}, \tag{3.3}$$

so that we can settle

Model 1.3. Let us assume (m_1)-(m_8), $C_0 > 0$ and $K_0' < \infty$. Then, the diffusive model of *weakly saturated-unsaturated infiltration with a weakly nonlinear hydraulic conductivity* is given by (2.23)-(2.25), where β^* is the multivalued function defined by (2.21), β is given by (2.15) and K is the single-valued function (2.6) with $K'(\theta)$ finite on $[\theta_r, \theta_s]$. Moreover, β^* is strongly monotone, (2.22), β satisfies (2.16), (2.18)-(2.20) with

$$\lim_{\theta \nearrow \theta s} \beta(\theta) < +\infty \tag{3.4}$$

and K is Lipschitz on $[\theta_r, \theta_s]$,

$$\left|K(\theta) - K(\overline{\theta})\right| \leq M \left|\theta - \overline{\theta}\right|, \ \forall \theta, \overline{\theta} \in [\theta_r, \theta_s]. \tag{3.5}$$

It is obvious that this situation which is illustrated by nonsingular diffusivities including also power functions

$$\beta(\theta) = \theta^p, \text{ with } p > 1,$$

is related to a *slow diffusion* and to the *porous media equation.*

In this category is situated the hydraulic model (BW) with c taking large values, $c >> 1$.

Strongly nonlinear conductivity

A weakly nonlinear model but with a stronger nonlinear conductivity may be obtained for $K_0' = \infty$, which implies

$$\lim_{\theta \nearrow \theta s} K'(\theta) = +\infty. \tag{3.6}$$

Model 1.4. Let us assume (m_1)-(m_8), $C_0 > 0$ and $K_0' = +\infty$. Then, the diffusive model of a *weakly nonlinear saturated-unsaturated infiltration with a strongly nonlinear hydraulic conductivity* is given by (2.23)-(2.25) where the functions β^* and β have the properties specified in Model 1.3 and K is given by (2.6), with

$$\lim_{\theta \nearrow \theta_s} K'(\theta) = +\infty,$$

implying that K is Lipschitz on intervals strictly included in $[\theta_r, \theta_s)$ with M_l defined in (2.12).

2.4 Quasi-unsaturated model

Since we would like to study the most possible nonlinear models, we are interested in the investigation of the case when K_s^* may become very large. In fact we shall focus on the situation when $K_s^* \to \infty$, case illustrated by functions of the form

$$\beta(\theta) = \frac{1}{(\theta_s - \theta)^{1-p}}, \; p \leq 0, \tag{4.1}$$

where we have to separate two situations. The first corresponds to $p = 0$, when

$$\beta(\theta) = \frac{1}{\theta_s - \theta}, \; \beta^*(\theta) = \log \left| \frac{\theta_s - \theta_r}{\theta_s - \theta} \right| \tag{4.2}$$

which turns out in an analogy to the *very fast diffusion* characterized by an extremely fast diffusivity that acts in the sense of hindering the formation of saturation regions.

The other is for $p < 0$ which also describes a model with a very fast diffusivity, or a *superdiffusivity.*

We shall focus mainly on the case $p = 0$. Generally this behaviour is due to the intrinsic properties of the system (or material) but can also occur as a consequence of an external factor. Even if the soil which exhibits such a model of infiltration may be very particular, we shall study it as an example of a special diffusion in a porous medium.

As before, we try to explain how K_s^* can become infinity. From the mathematical point of view the definition of the function β satisfying (4.2) may be the consequence of two facts. One would be a blowing-up conductivity at $h = 0$, case which will be skipped from physical considerations. Another motive would be the asymptotical convergence of θ to θ_s as $h \to \infty$ which models a process in which the porous medium does not begin to saturate at $h = 0$, but for large values of the pressure. An explanation could be a high increase of the diffusivity in the neighbourhood of the saturation value, due to a material property or to an external cause (like a very high evaporation).

In this case $C^* : [h_r, +\infty) \to [\theta_r, \theta_s)$ is positive, twice differentiable, monotonically increasing all over in $[h_r, +\infty)$ and $k : [h_r, +\infty) \to [K_r, K_s)$ has the same properties, (see Fig. 2.5).

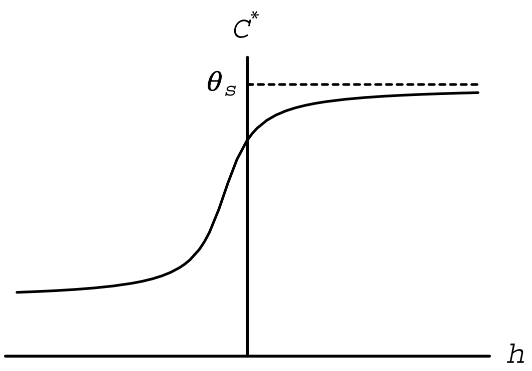

Fig. 2.5. Graphic of the constitutive law (quasi-unsaturated case)

We have

$$\lim_{h\nearrow\infty} C^*(h) = \theta_s, \quad \lim_{h\nearrow\infty} k(h) = K_s, \quad \lim_{h\nearrow\infty} k'(h) = 0. \tag{4.3}$$

The function $C : [h_r, +\infty) \to (0, C_r]$ is differentiable, monotonically decreasing and

$$\lim_{h\nearrow\infty} C(h) = 0. \tag{4.4}$$

For any $h \in [h_r, \infty)$ we define $(C^*)^{-1} : [\theta_r, \theta_s) \to [h_r, \infty)$ with

$$\lim_{\theta\nearrow\theta_s} (C^*)^{-1}(\theta) = +\infty, \tag{4.5}$$

(see Fig. 2.6) and $K^* : [h_r, \infty) \to [K_r^*, \infty)$ by

$$K^*(h) := K_r^* + \int_{h_r}^{h} k(\zeta)d\zeta, \text{ for } h \in [h_r, +\infty). \tag{4.6}$$

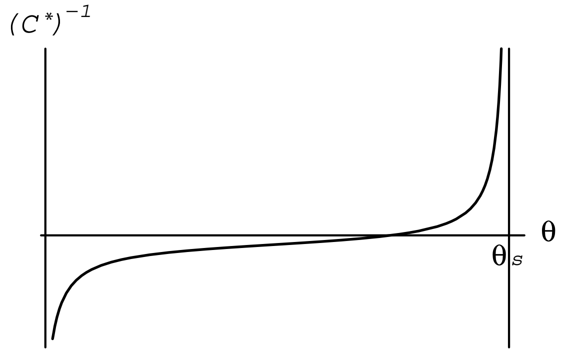

Fig. 2.6. Graphic of $(C^*)^{-1}$ (quasi-unsaturated case)

Then, we set

$$K(\theta) := (k \circ (C^*)^{-1})(\theta) \text{ for } \theta \in [\theta_r, \theta_s), \quad \lim_{\theta\nearrow\theta_s} K(\theta) = K_s, \tag{4.7}$$

with

$$K'_\infty := \lim_{\theta\nearrow\theta_s} K'(\theta) = \lim_{\theta\nearrow\theta_s} \frac{k'((C^*)^{-1}(\theta))}{C((C^*)^{-1}(\theta))} \in [0, \infty) \cup \{\infty\}. \tag{4.8}$$

Here we distinguish two cases, too. The first is when k' tends asymptotically to zero as fast as the function C (or faster than C) and then K'_∞ is finite (or zero). In the other case k' tends much slower than C to 0 and so K'_∞ is infinite. The function

$$\beta^*(\theta) = (K^* \circ (C^*)^{-1})(\theta), \ \theta \in [\theta_r, \theta_s) \tag{4.9}$$

is now single-valued, differentiable but blows up at θ_s, i.e.,

$$\lim_{\theta \nearrow \theta_s} \beta^*(\theta) = +\infty, \tag{4.10}$$

which is the main difference with respect to the previous cases.

This can be checked immediately since

$$\lim_{\theta \nearrow \theta_s} K^*((C^*)^{-1}(\theta)) = K^*(\lim_{\theta \nearrow \theta_s} (C^*)^{-1}(\theta)) = +\infty.$$

The diffusivity $\beta(\theta)$ is defined also by (2.15) and

$$\lim_{\theta \nearrow \theta_s} \beta(\theta) = +\infty.$$

As already mentioned, this model can been regarded as a limit model in infiltration and in this book we shall name it *quasi-unsaturated*, denomination justified also by the theoretical results we shall obtain with respect to it. Taking into account the possible behaviour of K at $\theta = \theta_s$, we identify two situations.

Model 1.5. Let us assume that $C^* : [h_r, +\infty) \to [\theta_r, \theta_s)$ and $k : [h_r, +\infty) \to [K_r, K_s)$ be twice differentiable, monotonically increasing and $\lim_{h \nearrow \infty} k'(h) = 0$. Let $K'_\infty \in [0, \infty)$. Then, the diffusive *quasi-unsaturated infiltration model with a weakly nonlinear conductivity* is given by (2.23)-(2.25), where K is the single-valued function (4.7), Lipschitz on the whole set $(-\infty, \theta_s]$,

$$\left|K(\theta) - K(\overline{\theta})\right| \le M \left|\theta - \overline{\theta}\right|, \ \forall \theta, \overline{\theta} \in [\theta_r, \theta_s] \tag{4.11}$$

and β^* is the single-valued function defined by (4.9), satisfying the properties

$$\lim_{\theta \nearrow \theta s} \beta^*(\theta) = +\infty, \tag{4.12}$$

$$(\beta^*(\theta) - \beta^*(\overline{\theta}))(\theta - \overline{\theta}) \ge \rho(\theta - \overline{\theta})^2, \forall \theta, \overline{\theta} \in [\theta_r, \theta_s).$$

Moreover,

$$\lim_{\theta \nearrow \theta s} \beta(\theta) = +\infty, \tag{4.13}$$

and β is monotonically increasing and positive, $\beta(\theta) \ge \rho > 0$ on $(-\infty, \theta_s)$.

Model 1.6. Assume that the hypotheses of Model 1.5 hold with $K'_\infty = +\infty$. Then, the *diffusive quasi-unsaturated infiltration model with a strongly nonlinear conductivity* is given by (2.23)-(2.25), where β and β^* have the properties specified in Model 1.5 and K is the single-valued function (4.7), Lipschitz on $(-\infty, \theta_l]$,

$$\left|K(\theta) - K(\overline{\theta})\right| \leq M_l \left|\theta - \overline{\theta}\right|, \ \forall \theta, \overline{\theta} \in [\theta_r, \theta_l], \ \theta_l < \theta_s. \qquad (4.14)$$

Note that in the quasi-unsaturated case the specific properties are the blow-up of both β and β^*.

2.5 Degenerate models

Richards' equation is by excellence an example of degenerate equation, due to the vanishing of the function $C(h)$. However, some other specific types of degeneracy in this equation, due to the vanishing of either the diffusivity or other coefficient in the time derivative term will be discussed further.

A diffusivity-degenerate limit case

We have also in view the situation when there exists a point θ at which $D(\theta) = 0$. We imagine, for example, an inferior limit model illustrating the case when infiltration continues below $h = h_r$, up to $h \to -\infty$, and the hydraulic functions decrease asymptotically to zero. In terms of θ this turns into $\beta(\theta) \to 0$ as $\theta \to 0$ which justifies the name of *diffusivity-degenerate*. This degenerate case can be associated with any model introduced before in which (2.16) is replaced by $\beta(\theta) > 0$ for $\theta \in (0, \theta_s)$ and $\beta(\theta) \to 0$, as $\theta \to 0$.

It will be studied because it offers an example of handling a degenerate equation and a comparison with the mathematical results obtained within the nondegenerate models.

A porosity-degenerate limit case

To introduce this model, we let apart the hypothesis (m_1) and consider a heterogeneous isotropic soil, nondeformable in time, in which the porosity $\phi(x) \geq 0$. This means that some completely nonporous intrusions can be found in the soil. We recall (2.15) in Sect. 1.2, and notice that since ϕ does not depend on h, the role of the function C^* can be played here by the function S_w. In fact, as function of h only, S_w has all the features of C^* and we can proceed in the same way as before by replacing $\theta = C^*(h)$ by $S_w = \widetilde{C}^*(h)$. Therefore, in this case, we obtain an analogous diffusion form

$$\frac{\partial}{\partial t}(\phi(x) S_w) - \Delta \widetilde{\beta}^*(S_w) + \frac{\partial \widetilde{K}(S_w)}{\partial x_3} = f \text{ in } Q, \qquad (5.1)$$

where $\widetilde{\beta}^*$ and $\widetilde{K}$ are functions obtained by a similar procedure as that developed for β^* and K in the precedent models and have the same mathematical properties. The blow-up happens when S_w comes close to 1. The initial condition can specify either the moisture

$$\phi(x)S_w(x,0) = \theta_0 \text{ in } \Omega,$$

or the saturation

$$S_w(x,0) = S_{w0} \text{ in } \Omega.$$

Obviously, the equation degenerates at the points where ϕ vanishes and certain degeneracy can occur also in the boundary conditions.

2.6 Extensions of the functions below the field capacity

Except for the diffusivity-degenerate case which is a very particular one, in all the other models the functions have been defined on the generally connived physical domain for the moisture, $\theta \in [\theta_r, \theta_s]$. To enable the application of the mathematical theory we envisage, we need to extend the functions to the left of θ_r, as is usually done by convenience in various cases and, for example, it suffices to mention the heat theory.

At this point, let us ignore for a while the physical meaning of the functions and let us think a little in a more abstract way. There is no fixed procedure to extend the functions but, in general, this is done by continuity. The most important thing is to succeed to prove, at the end, that the extension procedure did not introduce artificial solutions that may change the feature of the original problem.

In this work we shall extend the functions having in view two things: the first is to preserve some basic properties of the original functions, and the second to endow the extensions with the properties specified below. We introduce

$$\beta(\theta) = \rho,\ K(\theta) = K_r, \text{ for } \theta < \theta_r, \tag{6.1}$$

and as we can see, the properties of continuity and monotonicity of all functions β^* and K are still satisfied. Moreover, (2.22) is preserved for $\forall\theta, \overline{\theta} \in (-\infty, \theta_s]$ and we have

$$\lim_{\theta\to-\infty} \beta^*(\theta) = -\infty, \tag{6.2}$$

which is an important requirement in the application of the foreseen mathematical methods. We shall give more details in the next chapters.

Now, we would like to explain a little the motivation of this choice. Since the flow practically does not exist below the dimensional residual value θ_r, the most natural continuation of the water diffusivity and the hydraulic conductivity at the left of the field capacity would be by preserving the values $\rho = \beta(\theta_r)$ and $K_r = K(\theta_r)$ encountered at the level $\theta = \theta_r$.

On the other hand, the extensions made in this way ensure the continuity and other properties of β^*, especially the strong monotonicity, necessary for the application of the techniques related to monotone operators.

It is possible to find out other extensions which satisfy these properties as well. But as we have just said before, the most important thing is to show that the solution obtained in a way or another is the right one. In our case, the solution uniqueness which will be established for each model apart, will prove the fact that the solutions corresponding to certain extensions coincide, in particular, on the domain $[\theta_r, \theta_s]$. And last, but not least, it will be proved that under suitable conditions the moisture belongs exactly to the interval $[\theta_r, \theta_s]$, which shows that these extensions do not introduce unrealistic solutions.

We notice that in fact β^* is related to the Kirchhoff transform and soil scientists might consider that it is not worth working with a Kirchhoff variable that tends to $-\infty$ when $\theta \to -\infty$. If we look, for instance, at the procedure followed in [33] where the goal was to get an analytical approximate solution, we see that there the Kirchhoff variable is defined for a "negative" moisture, so that $\beta^*(\theta) \to 0$ as $\theta \to -\infty$. The setting of the Kirchhoff variable outside the physical domain is a convention without any physical signification, but with a necessary mathematical purpose. The extension we apply here is necessary only for proving the existence and the qualitative properties of the solutions, using a certain mathematical theory and not for calculating the solution. The solution which is unique will be calculated within the physical domain and the extension does not influence it, according to the above considerations.

This is why we will not extend the Kirchhoff variable to the left of θ_r, as done in [33], but we will work in our approach with a monotone one satisfying (6.2).

It is perspicuous that a special attention must be devoted to the setting of this variable for the diffusivity-degenerate model, because there $\beta(\theta) \to 0$ as $\theta \to 0$. According to the previous discussion, in this case it is not convenient to extend β at the left of the origin by its limit. We choose as extension a positive function β_{ext} vanishing at zero, that can confer β^* the properties previously specified.

The function K will be extended at the left of θ_r by K_r in all cases.

More details will be given in the appropriate sections.

2.7 Dimensionless form of the diffusive models

In physical models, the dimensionless form, indicated here by the decoration "~", has its important role, and we shall introduce it, by performing the generic variable transformations

$$\Theta = \frac{\theta - \theta_r}{\theta_s - \theta_r},\ \widetilde{t} = \frac{t}{t_d},\ \widetilde{x} = \frac{x}{\lambda_d},\ \widetilde{\beta}(\Theta) = \frac{\beta(\theta)}{\beta_d}, \tag{7.1}$$
$$\widetilde{K}(\Theta) = \frac{K(\theta) - K_r}{K_s - K_r},\ \widetilde{\beta^*}(\Theta) = \frac{\beta^*(\theta) - K_r^*}{K_s^* - K_r^*}.$$

As usually, λ_d, t_d and β_d are characteristic length, time and diffusivity for the problem. They are not independent but connected by relationships that allow to obtain the corresponding coefficients in the dimensionless diffusion equation equal to 1. For instance, the expression for λ_d given by Philip (see [132]) is related in our terms, to K_s^*,

$$\lambda_d = \frac{K_s^*}{K_s - K_r}.$$

A detailed dimensionless procedure is given in [33] for the parametric model. It is easy to check that, performing all these computations, we are lead to the dimensionless model that has the same form as (2.23)-(2.25)

$$\frac{\partial \Theta}{\partial \widetilde{t}} - \Delta \widetilde{\beta^*}(\Theta) + \frac{\partial \widetilde{K}(\Theta)}{\partial \widetilde{x_3}} = \widetilde{f} \text{ in } Q,$$

$$\Theta(\widetilde{x}, 0) = \Theta_0(\widetilde{x}) \text{ in } \Omega, \tag{7.2}$$

dimensionless boundary conditions in Θ.

The domain for the dimensionless moisture Θ is $(-\infty, \Theta_s)$ and in particular

$$\Theta_s = 1,\ \Theta_r = 0,\ \widetilde{K}(0) = 0,\ \widetilde{K_s} = \widetilde{K}(\Theta_s) = 1,$$

$$\widetilde{K^*}(\Theta_s) = 1,\ \widetilde{\beta}(0) = \widetilde{\rho} > 0.$$

However, for the sake of simplicity, we shall not indicate henceforth the decoration "˜", but we shall keep in mind that the models we shall further work with are written in the dimensionless form. At the same time, we shall keep for the dimensionless values the notations θ_s, K_s, K_s^*, θ_r, K_r (even these values are equal to 1, or, respectively, to 0), in order to put in evidence some properties of the functions at these points.

This dimensionless form is not essential in the theoretical approach of the problems, the qualitative results being the same both for the dimensional model and for the dimensionless one, but it is extremely useful in the numerical approaches. Also, some things are simplified due to the translations in (7.1) that rescales certain functions (θ, β^*, K) as zero at the residual moisture value.

2.8 Comments

In the mathematical literature the equation

$$u_t - \Delta \Phi(u) = 0$$

describes quite different phenomena according to the expression of the function $\Phi(u)$. The survey papers of D. G. Aronson (see [5]) and L. A. Peletier (see [102]) contains extensive references on this equation.

If Φ is a power function, $\Phi(u) = \dfrac{u^n}{n}$, then the following classification is done with respect to the values taken by the exponent n:

(a) the case $n > 1$ corresponds to a slow diffusion phenomenon and the equation is called the *porous media equation* ($\Phi'(u) = u^{n-1}$); typically it describes the infiltration of a gas in a porous medium;

(b) the case $n = 1$ represents the *heat equation* ($\Phi'(u) = 1$) and is associated to the classical heat theory, as known;

(c) the case $0 < n < 1$ is assigned to a *fast diffusion*

$$\Phi'(u) = \frac{1}{u^{1-n}}$$

and describes processes governed by high nonlinearities, like water infiltration in soils, for example;

(d) the case $\Phi(u) = \log |u|$ which we call a *very fast diffusion* $\left(\Phi'(u) = \dfrac{1}{u}\right)$ arises as a singular limit case of (a) in infiltration in porous media and models such as the diffusive limit for finite Boltzmann kinetic models (see [82]), diffusion in superconductors and polymers (see [111]); more recently it has been found to reveal significant diffusion features in population dynamics and biology flows;

(e) the case $n < 0$ reflects *superdiffusivity* phenomena and it was proposed by P. G. De Gennes as a model for the dynamics of thin liquid film subjected to long range Van der Waals interactions (see [52], [51], [57]); it also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see [46]); other physical applications are mentioned in [26].

As we have specified in Sect. 1.3, water infiltration in soils obeys laws involving more complex expressions for the function $\Phi(u)$.

Nonlinear models of previous types can be associated also to the solute diffusion in saturated porous media, in particular in groundwater and aquifers, when significant density differences caused by solute concentrations may influence the diffusion coefficient.

The functions accounting for $\Phi'(u)$, namely the diffusivities assigned to the empiric hydraulic models introduced in Sect. 1.3, have properties that situate them in one of the cases (a)-(d), according to the values taken by the parameters occurring in their expressions.

More specifically, the cases corresponding to (a)-(d) can be considered as particular types of infiltration, when the interaction between the hydraulic functions representing the soil properties turns out in a slower diffusivity (case (a)) or in a faster one (cases (c) and (d)). Case (b) corresponding to a constant diffusivity can be associated to the infiltration in a porous medium exhibiting an evident linear behaviour.

We notice that a common property is that $\Phi(u)$ is an increasing function mapping an interval included in $\mathbf{R}$ onto $\mathbf{R}$ with $\Phi(0) = 0$. The fact that the domain of Φ is not $\mathbf{R}$, but a subset of it, is a feature of the diffusion that develops in a porous medium which may become saturated when the fluid which diffuses fills all free pores. The corresponding physical function u ranges here between 0 and a finite value u_s.

The derivative of Φ, standing for the diffusivity of the process has some typical properties, as being non-negative and monotonically increasing. In some cases it may degenerate or become singular.

As the scope of this book is to extend the mathematical approach a little outside the framework of water infiltration models, to the classes of diffusion processes specified before, we have tried to reveal properties of the hydraulic models that include them in a diffusion category or another. Thus, the mathematical results which will be obtained in a general abstract framework for the various types of diffusion processes will apply in particular to the specific hydraulic models, but also to other models like those just enumerated.

Besides the dominant term $-\Delta\Phi(u)$, which confers the pure diffusive character, the equation may include also terms accounting for possible transport phenomena, like $\dfrac{\partial K(\theta)}{\partial x_3}$. The equation is then called of diffusion and transport. The transport term may have an important contribution in some cases and puts into evidence some features of the phenomenon. For example, in the infiltration models the flow is more accelerated towards saturation if $K'(\theta_s) = +\infty$.

From the mathematical point of view, the functional approach will be that adequate for the dominant term, but some particularities of the transport term may complicate the solution and require a special mathematical foresight.

The models have been considered for the isotropic case. The extension to the anisotropic case (at least for that with a diagonal form of the tensors) or the consideration of a transport with components along all three directions do not seem to induce particular mathematical difficulties, so that we restrict the study only to isotropic media.

Recalling the applications to water flow in soils, we conclude that the mathematical argumentation of this chapter has been materialized in some mathematical models for the general boundary value problems of saturated-unsaturated infiltration. Studying the behaviour of the constitutive law and hydraulic conductivity around $h = 0$, we have recovered some properties of K, β and β^* whose combinations determine certain classes of diffusion models, corresponding to more or less nonlinear diffusivities and hydraulic conductivities, covering a wide range between the known limit models. This enhances the possibility of situating various hydraulic models established by soil scientists in a class or another.

Generally, the diffusive dimensionless form of the models reads

$$\frac{\partial\theta}{\partial t} - \Delta\beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \text{ in } Q, \tag{8.1}$$

$$\theta(x, 0) = \theta_0(x) \text{ in } \Omega, \tag{8.2}$$

$$\textit{boundary conditions in } \theta, \tag{8.3}$$

where K, β and β^* still denote the extensions:

$$\beta(\theta) := \begin{cases} \rho, & \theta < 0, \\ \beta(\theta), & 0 \leq \theta < \theta_s, \end{cases} \qquad K(\theta) := \begin{cases} 0, & \theta < 0, \\ K(\theta), & 0 \leq \theta \leq \theta_s, \end{cases} \tag{8.4}$$

$$\beta(\theta) \geq \rho > 0, \; K(\theta) \geq 0, \; \forall \theta \in [0, \theta_s)$$

and

$$\beta^*(\theta) := \begin{cases} \rho\theta, & \theta < 0, \\ \displaystyle\int_0^\theta \beta(\xi) d\xi, & 0 \leq \theta < \theta_s, \\ [K_s^*, +\infty), & \theta = \theta_s. \end{cases} \tag{8.5}$$

We stress once again that the above extensions correspond to the dimensionless form which has no longer been marked by the symbol "~".

The properties that are common to all models are:

(i) $\quad (\beta^*(\theta) - \beta^*(\overline{\theta}))(\theta - \overline{\theta}) \geq \rho(\theta - \overline{\theta})^2, \forall \theta, \overline{\theta} \in (-\infty, \theta_s);$

(ii) $\quad \lim_{\theta \to -\infty} \beta^*(\theta) = -\infty;$

(i_K) $\quad \left| K(\theta) - K(\overline{\theta}) \right| \leq M \left| \theta - \overline{\theta} \right|, \forall \theta, \overline{\theta} \leq \theta_s,$

or

(ii_K) $\quad \left| K(\theta) - K(\overline{\theta}) \right| \leq M_l \left| \theta - \overline{\theta} \right|, \forall \theta, \overline{\theta} \leq \theta_l < \theta_s.$

We have established that the behaviour of the function β in the neighbourhood of $\theta = \theta_s$, correlated to that of $\beta^*(\theta)$, definitely determines the dominant type of the diffusion process.

Some other degrees of nonlinearity can be induced by the hydraulic function K, without changing the type of diffusion governed by the common action of β^* and β. A conductivity with an increase rate comparable to that of β^* will have an effect that will be better put into evidence by the mathematical analysis of the respective model.

Thus, the more or less nonlinear hydraulic behaviour for the saturated-unsaturated flow is expressed, besides (i)-(ii), by the following properties:

$$\lim_{\theta \nearrow \theta_s} \beta(\theta) = +\infty, \; \lim_{\theta \nearrow \theta_s} K'(\theta) = +\infty,$$

for the strongly nonlinear case and

$$\lim_{\theta \nearrow \theta_s} \beta(\theta) < \infty, \; \lim_{\theta \nearrow \theta_s} K'(\theta) < \infty,$$

for the weakly nonlinear situation. Some other choices accounting for possible combinations between the types of diffusion and conductivity nonlinearities have been specified.

Moreover, a fast diffusion process leading to a quasi-unsaturated model is characterized by $\lim_{\theta \nearrow \theta_s} \beta^*(\theta) = +\infty$.

We specify that in the diffusivity-degenerate limit case, when $\beta(0) = 0$, the function β is extended by a positive function β_{ext}, with $\lim_{\theta \to -\infty} \beta_{ext}(\theta) = \text{const.}$ and $\beta_{ext}(0) = 0$,

$$\beta(\theta) := \begin{cases} \beta_{ext}(\theta) & \text{if } \theta < 0, \\ \beta(\theta) & \text{if } 0 \le \theta < \theta_s, \end{cases} \quad \beta^*(\theta) := \begin{cases} \displaystyle\int_0^\theta \beta(\xi) d\xi & \text{if } \theta < \theta_s, \\ [K_s^*, +\infty) & \text{if } \theta = \theta_s \end{cases} \tag{8.6}$$

and (i) is verified with a zero right-hand side.

The most reliable model for water infiltration in soils is that of strongly saturated-unsaturated one, based on the assumption $C_0 = 0$. However, the weakly saturated-unsaturated case, corresponding to the hypothesis $C_0 > 0$ may be considered as an approximation of Model 1.1 with C_0 very close to 0 (illustrated with an acceptable approximation by m close to 0 in the (vG) model). This latter case was extensively considered in the mathematical literature related to the functional study of the infiltration models.

These models will be further studied under various boundary conditions motivated by real-life applications. A discussion concerning the modifications that must be done in the existence proofs when various boundary conditions are associated to each of the previous models, will be made.

Once the model has been formulated, the first step in the study of an applied mathematical problem is fulfilled and we can pass to the analysis of the problem, which represent the second and main part of this book.

3

Basic existence theorems for evolution equations with monotone operators in Hilbert spaces

The purpose of this chapter is to introduce the basic ideas of the functional state approach (or semigroup approach) that realizes the connection between the applied boundary value problem and the abstract theory of Cauchy problems and to present the fundamental results concerning the existence of solutions to nonlinear evolution equations associated with monotone operators in Hilbert spaces. Abstract Cauchy problem can be approached in two ways, from the semigroup perspective, and within the variational framework. Both approaches will be detailed. In order to fix the ideas, some definitions and the most important results related to m-accretive operators given in Appendix, Sect. 4, will be resumed in this chapter and enounced in Hilbert spaces.

3.1 The semigroup approach

In the first section, we intend to explain in an informal manner the philosophy of the semigroup approach as a very relevant mathematical tool for the qualitative study of the nonlinear boundary value problems evolving in time, in particular for the models presented in Chap. 2. The aim is also to justify in a heuristic way the appellation of semigroup theory. We shall exemplify this by the models of water infiltration that describe the time evolution of a physical system which initially is in a known state $\theta_0(x)$. At the time t we find the system in the state given by $\theta(x,t)$ which in our examples represents the moisture of the soil at the point x and time t. We stress that we consider that the state system (which is in fact described by a partial differential equation) is autonomous, i.e., it is time independent (its coefficients do not depend explicitly on time). Since for each t fixed θ is still a function of x, it follows that it is entitled to belong to a functional space X, which generally may be a Banach space. We introduce a writing convention, $\theta(t) := \theta(\cdot, t)$ to indicate the fact that a state in the Banach space X corresponds to $\theta(t)$ and we still write this as

$$\theta(t)(x) := \theta(x,t).$$

From the mathematical point of view this means that we have introduced a function

$$\theta : [0,T) \to X, \text{ where } T \leq \infty.$$

Roughly speaking we denote by θ both the original function depending on x and t and the function depending on t with values in X.

We recall that this evolution system has started from an initial state $\theta_0(x,0) := \theta_0(x)$ known at each point x belonging to the space domain. We assume that the evolution of the system develops continuously such that the relationship between the initial state θ_0 and the state at the current time t, $\theta(t)$ can be represented by an operator $S(t)$ that links these two states according to the law

$$\theta(t) = S(t)\theta_0, \ t \geq 0.$$

In this way a family of operators $\{S(t)\}_{t\geq 0}$ has been introduced and we shall describe their properties. We notice that they vary with time but are applied to θ_0, so that they have to be defined on a subset of a functional space, which is usually a Banach space X. Consequently, their domain should be the same for all t and it is included in X. They can behave in a linear or nonlinear way and we assume in the subsequent part that they are generally nonlinear operators.

Since at the initial moment of time the state of the system remains unchanged, it is obvious that $S(0) = I$, the identity of the space X.

Now, if the system evolution between the initial state and the state at time t is given by $S(t)\theta_0$, then the transition from this state to a state corresponding to time s would be normal to be realized by applying the operator $S(s)$ to the state $S(t)\theta_0$, i.e., to compute $S(s)\left(S(t)\theta_0\right)$. On the other hand, if we look at the state of the system when the time $t+s$ has elapsed since the initial moment, we can describe it as being $S(t+s)\theta_0$. In conclusion, we can write that $S(t+s)\theta_0 = S(s)S(t)\theta_0$ and we punctuate that this is allowed by the autonomous state system assumption.

If we consider that we deal with systems whose evolution is not submitted to an extremely discontinuous time behavior, heuristically thinking, we expect that as s gets *closer* to t, then $S(s)\theta_0$ should get *close* to $S(t)\theta_0$ in some way. Moreover, according to a similar argument, we state that two *near* initial states σ_0 and θ_0 lead to relative states at time t, namely $S(t)\sigma_0$ is *close* to $S(t)\theta_0$, in some sense. It remains now to translate into mathematical terms all these considerations and to give a mathematical meaning to the term *close*, while the both previous assumptions are turned out in continuity and boundedness conditions upon the family of operators $\{S(t)\}_{t\geq 0}$.

We introduce now the following definition:

Definition 1.1. Let X_C be a closed subset of X. A *semigroup of contractions* on X_C is a family $\{S(t)\}_{t\geq 0}$, $S : [0,T) \times X_C \to X_C$, that satisfy

(i) $S(s)S(t) = S(t+s)$ for all $s,t \geq 0$,

(ii) $S(0) = I$, the identity operator on X,

(iii) for each $u \in X_C$ fixed, we have $S(t)u \to u$ strongly in X, as $t \to 0^+$.

(iv) for every $u \in X_C$ the mapping $u \to S(t)u$ is a ω-contraction on X_C i.e., $\|S(t)u - S(t)v\|_X \le e^{\omega t} \|u - v\|_X$ for all u, $v \in X_C$ and $t \ge 0$, where $\omega \in \mathbf{R}$.

A C_0*-semigroup* (or *strongly continuous semigroup*) of operators on a Banach space X is a family satisfying only the first three properties.

Definition 1.2. The operator $G : D(G) \subset X_C \to X$

$$Gu = \lim_{h \to 0} \frac{S(h)u - u}{h}$$

is called the strong *generator* of S.

Thus, $\theta(t) = S(t)\theta_0$ may be viewed as the solution to the Cauchy problem

$$\frac{d\theta}{dt}(t) + A\theta(t) = 0, \ \forall t \ge 0,$$
$$\theta(0) = \theta_0,$$

where $A = -G$. The motivation for the name of semigroup is given by (i) and (ii). Condition (iii) ensures the fact that the evolution of the system does not exhibit a major break as time runs. Finally, the last condition (iv) expresses the fact that a slight perturbation of the initial data does not induce a pathological behavior in the evolution of the system.

We have now the background necessary to make the connection with the abstract Cauchy problem. Generally the state space is a Banach space, as we have seen, but in our approach we shall consider it a Hilbert space, H.

In most of the situations we shall encounter, the system state evolution is represented as a Cauchy problem in the form

$$\frac{d\theta}{dt}(t) + A\theta(t) = f(t), \ \text{a.e. } t \in (0,T) \tag{1.1}$$
$$\theta(0) = \theta_0,$$

for the function $\theta : [0,T] \to H$. Here, A is a nonlinear unbounded operator on H and θ_0 and f are given in appropriate functional spaces. The boundary conditions are not specified explicitly, they being included in the definition of the operator domain, $A : D(A) \subset H \to H$ (here H is identified with its own dual).

Certain assumptions and properties of A will imply that (1.1) has a unique solution. We shall see further that the main one is that A is (quasi) m-accretive.

If $f \equiv 0$ and A is quasi m-accretive, the solution satisfies the conditions (i)-(iv) above. We set $\theta(t) := S(t)\theta_0$ and, as mentioned above, the operator $G = -A$ generates a semigroup of contractions.

To conclude, the abstract Cauchy problem that replaced the original boundary value problem may have a solution given by a semigroup of contractions.

This method is known in the literature as the *semigroup approach*, or the *functional state approach*, (see [13]). We simply call it *functional approach.*

Thus, this technique presumes the definition of an operator (associated to the boundary value problem) in an appropriate functional space and the investigation of its properties that enhance to develop an existence theory for the Cauchy problem (1.1). Essentially, the property that can ensure the solution existence is the m-accretivity of the operator A.

That is the reason why we are going to resume in the first subsection of this chapter a few fundamental properties of operators in Hilbert spaces, related to this aspect. Then, the basic theorems that state the existence and uniqueness of the solution to (1.1) when A is an m-accretive operator will be presented.

Except for the fundamental theorems, the other results will be given without proofs, and for details and complete proofs the readers are referred to the monographs [9], [30].

3.2 Nonlinear m-accretive operators in Hilbert spaces

Let H be a Hilbert space with the scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and let $A : D(A) \subset H \to H$, where $D(A) := \{u \in H; Au \neq \emptyset\}$. We allow A to be a nonlinear multivalued operator and we particularize some definitions given in Appendix, Sect. 4, to Hilbert spaces.

Definition 2.1. The operator A is called *accretive* (monotone) if

$$(w_1 - w_2, u_1 - u_2) \geq 0,\ \forall [u_i, w_i] \in A,\ i = 1, 2. \tag{2.1}$$

If A is single valued this means that

$$(Au_1 - Au_2, u_1 - u_2) \geq 0,\ \forall\ u_i \in D(A),\ i = 1, 2. \tag{2.2}$$

Definition 2.2. The operator A is called *m-accretive* (maximal monotone) if it is accretive and

$$R(I + A) = H, \tag{2.3}$$

where I is the unit operator in H and $R(I + A)$ is the range of the operator $I + A$.

Definition 2.3. The operator A is said to be *quasi m-accretive* (or *ω-m-accretive*) if for every $[u_i, w_i] \in A, i = 1, 2$ and some $\omega \in \mathbf{R}_+$ we have

$$(w_1 - w_2, u_1 - u_2) \geq -\omega \|u_1 - u_2\|^2 \tag{2.4}$$

and

$$R(\lambda I + A) = H,\ \text{for}\ \lambda > \omega. \tag{2.5}$$

If A satisfies only (2.4) it is called *quasi-monotone* or *ω-quasi-accretive.*

Obviously, (2.4) with $\omega = 0$ means that the operator is monotone and for $\omega < 0$ defines a *strongly monotone* operator.

We recall that by Theorem 4.22 in Appendix, in Hilbert spaces the notions of accretive operators coincide with those of monotone operators.

Definition 2.4. The operator A is called *coercive* if there exists $u_0 \in H$ such that

$$\lim_{n\to\infty} \frac{(w_n, u_n - u_0)}{\|u_n\|} = +\infty \tag{2.6}$$

for any sequence $\{[u_n, w_n]\}_{n\geq 1} \subset A$ with $\lim\limits_{n\to\infty} \|u_n\| = +\infty$.

A sufficient condition for coercivity, which will be frequently used is that there exists $\alpha > 0$ such that

$$(w, u - u_0) \geq \alpha \|u\|^2, \ \forall [u, w] \in A, u_0 \in H. \tag{2.7}$$

The other definitions concerning the properties of continuity, boundedness and closure can be easily adapted from those given for Banach spaces in Appendix, Sect 4. However, since some results will be intensively used within the subsequent and the next chapters, they will be further enunciated in the framework of Hilbert spaces.

Proposition 2.5. *Let A be a linear m-accretive operator. Then $D(A)$ is dense in H and A is closed.*

Theorem 2.6. (Minty, [96]) *Let $A : H \to H$ be a single-valued, hemicontinuous and monotone operator. Then A is m-accretive* (see Theorem 4.17, in Appendix).

Theorem 2.7. (Minty, [96]) *If A is m-accretive and coercive, then A is surjective* (see Theorem 4.17, in Appendix).

Theorem 2.8. *Let $A : D(A) \subset H \to H$ be a monotone operator. Then A is m-accretive if and only if for every $\lambda > 0$ (equivalently for some $\lambda > 0$)*

$$R(I + \lambda A) = H.$$

Definition 2.9. Let A be m-accretive. Then

$$J_\lambda = (I + \lambda A)^{-1}, \ \forall \lambda > 0 \tag{2.8}$$

is the *resolvent* of A and

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \ \forall \lambda > 0 \tag{2.9}$$

is called the *Yosida approximation* of A.

According to Theorem 2.8, in Hilbert spaces, $D(J_\lambda) = D(A_\lambda) = H$. Moreover they are single-valued. We put together some immediate properties of A_λ, J_λ.

Proposition 2.10. *Let A be m-accretive. Then for $\lambda > 0$ we have*

(a) $A_\lambda u \in A(J_\lambda u)$, $\forall u \in H$,
(b) $A_\lambda u = J_\lambda(Au)$, $\forall u \in D(A)$,
(c) $\|A_\lambda u\| \leq \left\|A^0 u\right\|$, $\forall u \in H$; $\left\|A^0 u\right\| = \inf_{w \in Au}\{\|w\|\}$,
(d) $\lim_{\lambda \to 0} J_\lambda u = u$, $\forall u \in \overline{D(A)}$,
(e) $\lim_{\lambda \to 0} A_\lambda u = A^0 u$, $\forall u \in D(A)$,
(f) $(A_\lambda u - A_\lambda v, u - v) \geq 0$, $\forall u, v \in H$, $\lambda > 0$,
(g) $\|A_\lambda u - A_\lambda v\| \leq \frac{2}{\lambda}\|u - v\|$, $\forall u, v \in H$,
(h) $\|J_\lambda u - J_\lambda v\| \leq \|u - v\|$, $\forall u, v \in H$.

Here $A^0 : D(A) \to H$ is the minimal section of A. As we can see from these results, J_λ is a nonexpansive operator and $(A_\lambda)_{\lambda>0}$ is a family of Lipschitz operators that approximate A, as $\lambda \to 0$.

Definition 2.11. Let $(0,T) \subset \mathbf{R}$ and let $A : D(A) \subset H \to H$, be an m-accretive operator. We set

$$D(\widetilde{A}) = \{u \in L^2(0,T;H);\ \exists\, v \in L^2(0,T;H) \text{ such that } v(t) \in Au(t) \text{ a.e. } t \in (0,T)\}.$$

The operator $\widetilde{A} : D(\widetilde{A}) \subset L^2(0,T;H) \to L^2(0,T;H)$, defined by

$$\widetilde{A}u = \{v \in L^2(0,T;H);\ v(t) \in Au(t) \text{ a.e. } t \in (0,T)\}$$

is called the *canonical extension* of A to $L^2(0,T;H)$ or the *realization* of A on $L^2(0,T;H)$.

If $(0,T)$ is finite, then $\widetilde{A}$ is m-accretive. Indeed, it is monotone and for every $g = g(t) \in L^2(0,T;H)$ we set $u(t) := (I + A)^{-1}g(t) \in L^2(0,T;H)$, which implies that $u + \widetilde{A}u \ni g$.

We also mention that the realizations of J_λ and A_λ are $(I + \lambda\widetilde{A})^{-1}$ and $\widetilde{A}_\lambda$, respectively.

Proposition 2.12. *A (quasi) m-accretive operator A is demiclosed, i.e., if $[x_n, y_n] \in A$, $x_n \to x$ strongly in H, $y_n \rightharpoonup y$ (weakly) in H, then $[x,y] \in A$.*

3.3 The Cauchy problem within the semigroup approach

Let H be a Hilbert space and let us consider the problem

$$\frac{du}{dt}(t) + Au(t) \ni f(t) \text{ a.e. } t \in (0,T), \qquad u(0) = u_0, \tag{3.1}$$

where $A : D(A) \subset H \to H$ is a nonlinear operator, time-independent and possibly multivalued. Let $f \in L^1(0,T;H)$ be given.

Definition 3.1. A function $u \in C([0,T];H)$ is said to be a *strong solution* to the Cauchy problem (3.1) if u is absolutely continuous on any compact subinterval of $(0,T)$, satisfies (3.1) a.e. $t \in (0,T)$, $u(0) = u_0$ and

$$u(t) \in D(A) \text{ a.e. } t \in (0,T). \tag{3.2}$$

We recall that the absolutely continuity on any compact subinterval of $(0,T)$ implies the a.e. differentiability on $(0,T)$, because H is a Hilbert space (generally this is true for a reflexive Banach space, see [9] and Theorem 3.9 in Appendix). Hence it is clear that a strong solution $u \in W^{1,1}(0,T;H)$.

Definition 3.2. A function $u \in C([0,T];H)$ is called a *weak solution* to (3.1) if there exist sequences $\{u_n\}_{n\geq 1} \subset W^{1,\infty}(0,T;H)$, $\{f_n\}_{n\geq 1} \subset L^1(0,T;H)$ and $\{u_n^0\}_{n\geq 1} \subset D(A)$ such that

$$\frac{du_n}{dt}(t) + Au_n(t) \ni f_n(t) \text{ a.e. } t \in [0,T),\ u_n(0) = u_n^0,\ n = 1,2,..., \tag{3.3}$$

$$u_n \longrightarrow u \text{ in } C([0,T];H), \tag{3.4}$$

$$u_n^0 \longrightarrow u_0 \text{ in } H \text{ and } f_n \longrightarrow f \text{ in } L^1(0,T;H). \tag{3.5}$$

Definition 3.3. The function $u : [0,T] \to H$ is called an *integral solution* to the initial value problem (3.1) if u is continuous on $[0,T]$, $u(0) = u_0$ and the inequality

$$\frac{1}{2}\|u(t) - x\|^2 \leq \frac{1}{2}\|u(s) - x\|^2 + \int_s^t (f(\tau) - y, u(\tau) - x)d\tau \tag{3.6}$$

holds for each $[x,y] \in A$ and $0 \leq s \leq t \leq T$.

Obviously, every strong solution is an integral solution.

Further we shall present two lemmas of Gronwall type (see [30]), both frequently used in the next proofs.

Lemma 3.4. (Gronwall-Bellman) *Let $m \in L^1(0,T;\mathbf{R})$ such that $m \geq 0$ a.e. on $(0,T)$ and let $a \geq 0$ be constant. Let ϕ be a continuous function from $[0,T]$ to $\mathbf{R}$ satisfying*

$$\phi(t) \leq a + \int_0^t m(s)\phi(s)ds,\ \forall t \in [0,T].$$

Then

$$\phi(t) \leq ae^{\int_0^t m(s)ds}.$$

Lemma 3.5. (Gronwall) *Let $m \in L^1(a,b;\mathbf{R})$, with $a,b \in \mathbf{R}$ and $m \geq 0$ a.e. on (a,b) and let c be a fixed real number. If $\phi \in C([a,b])$ verifies*

$$\frac{1}{2}\phi^2(t) \leq \frac{1}{2}c^2 + \int_a^t \phi(s)m(s)ds,\ \forall t \in [a,b]$$

then the following inequality

$$|\phi(t)| \leq |c| + \int_a^t m(s)ds,\ \forall t \in [a,b]$$

holds.

Next, we shall present a main result (see [9], [30]) concerning the existence and uniqueness of a strong solution to (3.1). Besides these, it states also the continuous (Lipschitz continuous) dependence of the solution on the data and shows that the class of m-accretive operators constitutes a perfect framework for studying Cauchy problems.

Theorem 3.6. *Assume that $A : D(A) \subset H \to H$ is an m-accretive operator and let $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$. Then problem* (3.1) *has a unique strong solution*

$$u \in L^\infty(0,T;D(A)) \cap W^{1,\infty}(0,T;H). \tag{3.7}$$

The function u is differentiable from the right at any point in $[0,T)$ and we have

$$\frac{d^+u}{dt}(t) = ((f(t) - Au(t))^0, \tag{3.8}$$

$$\left\| \frac{d^+u}{dt}(t) \right\| \leq \left\| (f(0) - Au_0)^0 \right\| + \int_0^t \left\| \frac{df}{ds}(s) \right\| ds \tag{3.9}$$

where $(f(t) - Au(t))^0$ denotes the element of minimum norm of the set $(f(t) - Au(t))$. Moreover, if u and v are the solutions corresponding to $(u_0, f), (v_0, g) \in D(A) \times W^{1,1}(0,T;H)$, then

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\| + \int_0^t \|f(s) - g(s)\| \, ds,\ 0 \leq t \leq T. \tag{3.10}$$

Proof. First we shall prove (3.10). Considering two solutions to problem (3.1), $(u_0, f), (v_0, g) \in D(A) \times W^{1,1}(0,T;H)$ we obtain, using the monotonicity of A, that

$$\frac{1}{2}\frac{d}{dt}\|u(t) - v(t)\|^2 \leq \|f(t) - g(t)\| \, \|u(t) - v(t)\| \text{ a.e. on } (0,T).$$

This inequality integrated over $(0,t)$ with $t \in [0,T]$, leads to

$$\frac{1}{2}\|u(t) - v(t)\|^2 \leq \frac{1}{2}\|u_0 - v_0\|^2 + \int_0^t \|f(s) - g(s)\| \, \|u(s) - v(s)\| \, ds.$$

Then, using Lemma 3.5 we obtain (3.10) as claimed. Obviously, from this relationship we obtain also the uniqueness of the solution.

In order to show the existence of the solution we notice first that for any $\lambda > 0$, the approximating Cauchy problem for the Yosida approximation of A

$$\frac{du_\lambda}{dt}(t) + A_\lambda u_\lambda(t) = f(t), \ 0 \le t \le T, \tag{3.11}$$

$$u_\lambda(0) = u_0 \tag{3.12}$$

has a unique solution $u_\lambda \in C^1([0,T];H)$. This result is a consequence of the fact that A_λ is Lipschitz on H. Because A_λ is monotone we deduce that $\forall t, t+h \in [0,T]$ we have

$$\frac{1}{2}\frac{d}{dt}\left\|u_\lambda(t+h) - u_\lambda(t)\right\|^2 \le \left\|f(t+h) - f(t)\right\| \left\|u_\lambda(t+h) - u_\lambda(t)\right\|$$

and by integration from 0 to t we obtain

$$\frac{1}{2}\left\|u_\lambda(t+h) - u_\lambda(t)\right\|^2 \le \frac{1}{2}\left\|u_\lambda(h) - u_0\right\|^2$$
$$+ \int_0^t \left\|f(s+h) - f(s)\right\| \left\|u_\lambda(s+h) - u_\lambda(s)\right\| ds, \ \forall t, t+h \in [0,T].$$

Then by Lemma 3.5 we get

$$\left\|u_\lambda(t+h) - u_\lambda(t)\right\| \le \left\|u_\lambda(h) - u_0\right\| + \int_0^t \left\|f(s+h) - f(s)\right\| ds. \tag{3.13}$$

From here, dividing by $h > 0$ and passing to limit as $h \to 0$ we obtain

$$\left\|\frac{du_\lambda}{dt}(t)\right\| \le \left\|f(0) - A_\lambda u_0\right\| + \int_0^t \left\|\frac{df}{ds}(s)\right\| ds \tag{3.14}$$
$$\le \left\|f(0)\right\| + \left\|A^0 u_0\right\| + \int_0^T \left\|\frac{df}{ds}(s)\right\| ds, \ 0 \le t \le T.$$

From (3.11) we get

$$\left\|A_\lambda u_\lambda(t)\right\| \le \left\|f(t)\right\| + \left\|f(0)\right\| + \left\|A^0 u_0\right\| \tag{3.15}$$
$$+ \int_0^T \left\|\frac{df}{ds}(s)\right\| ds \le \text{constant}, \ 0 \le t \le T.$$

Further we shall prove that $\{u_\lambda\}_{\lambda>0}$ converges in $C([0,T];H)$ as $\lambda \to 0$. We start from the equation

$$\frac{du_\lambda}{dt} - \frac{du_\mu}{dt} + A_\lambda u_\lambda - A_\mu u_\mu = 0, \ \lambda, \mu > 0$$

and we multiply it by $u_\lambda(t) - u_\mu(t)$. We have

$$\frac{1}{2}\frac{d}{dt}\left\|u_\lambda(t) - u_\mu(t)\right\|^2 = -(A_\lambda u_\lambda(t) - A_\mu u_\mu(t), u_\lambda(t) - u_\mu(t)).$$

Following some computations,

$$\begin{aligned}&-(A_\lambda u_\lambda(t)-A_\mu u_\mu(t), u_\lambda(t)-J_\lambda u_\lambda(t)+J_\lambda u_\lambda(t)-J_\mu u_\mu(t)+J_\mu u_\mu(t)-u_\mu(t))\\&=-(A_\lambda u_\lambda(t)-A_\mu u_\mu(t), \lambda A_\lambda u_\lambda(t)-\mu A_\mu u_\mu(t))\\&-(A_\lambda u_\lambda(t)-A_\mu u_\mu(t), J_\lambda u_\lambda(t)-J_\mu u_\mu(t)),\end{aligned}$$

we finally get

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda(t)-u_\mu(t)\|^2 \leq -(A_\lambda u_\lambda(t)-A_\mu u_\mu(t), \lambda A_\lambda u_\lambda(t)-\mu A_\mu u_\mu(t)). \quad (3.16)$$

Now from (3.15), (3.16) and from the relationship

$$A_\lambda u_\lambda(t) \in AJ_\lambda u_\lambda(t)$$

we obtain

$$\frac{d}{dt}\|u_\lambda(t)-u_\mu(t)\|^2 \leq C_0(\lambda+\mu), \ \forall\ \lambda,\mu>0 \text{ and } t\in[0,T].$$

It follows therefrom that

$$\|u_\lambda(t)-u_\mu(t)\| \leq \sqrt{C_0}\,\sqrt{t}\sqrt{\lambda+\mu}. \quad (3.17)$$

From (3.17) we deduce that there exists $u \in C([0,T];H)$ such that

$$u_\lambda \longrightarrow u \text{ as } \lambda \to 0 \text{ in } C([0,T];H). \quad (3.18)$$

Then (3.14) implies that $u \in W^{1,\infty}(0,T;H)$ and

$$\frac{du_\lambda}{dt} \longrightarrow \frac{du}{dt} \text{ weak-star in } L^\infty(0,T;H) \text{ as } \lambda \to 0, \quad (3.19)$$

hence

$$A_\lambda u_\lambda \longrightarrow f - \frac{du}{dt} \text{ weak-star in } L^\infty(0,T;H) \text{ as } \lambda \to 0. \quad (3.20)$$

Taking into account that $A_\lambda u_\lambda \in A(J_\lambda u_\lambda)$ and $J_\lambda u_\lambda - u = J_\lambda u_\lambda - u_\lambda + u_\lambda - u$ we obtain that

$$J_\lambda u_\lambda \longrightarrow u \text{ in } C([0,T];H). \quad (3.21)$$

Because the extension of A to $L^2(0,T;H)$ denoted by $\widetilde{A}$ is an m-accretive operator (see Definition 2.11) and then $\widetilde{A}$ is demiclosed, we obtain that

$$f - \frac{du}{dt} \in \widetilde{A}u,$$

meaning that u verifies a.e. (3.1). From (3.12) and (3.18) we deduce that $u(0)=u_0$. But (3.15) and (3.21) imply that $u(t)\in D(A)$, $\forall t \in (0,T]$, because A is demiclosed.

Then we have to prove (3.8). Let t_0 a point in $[0,T)$ and let multiply the equation

$$\frac{d}{dh}(u(t_0+h)-u(t_0)) \in f(t_0+h)-Au(t_0+h),\ \text{a.e. } h>0,\ t_0+h<T,$$

by $u(t_0+h)-u(t_0)$. Using again the fact that A is monotone, we obtain

$$\frac{1}{2}\frac{d}{dh}\|u(t_0+h)-u(t_0)\|^2$$
$$\leq \left(\left\|(f(t_0)-Au(t_0))^0\right\| + \|f(t_0+h)-f(t_0)\|\right)\|u(t_0+h)-u(t_0)\|.$$

We integrate this inequality over $(0,h)$ and apply Lemma 3.5 to obtain

$$\|u(t_0+h)-u(t_0)\| \leq h\left\|(f(t_0)-Au(t_0))^0\right\| \tag{3.22}$$
$$+\int_0^h \|f(t_0+s)-f(t_0)\|\,ds,\ \forall h>0 \text{ with } t_0+h<T.$$

This implies

$$\lim_{h\to 0^+}\sup\left(\frac{1}{h}\right)\|u(t_0+h)-u(t_0)\| \leq \left\|(f(t_0)-Au(t_0))^0\right\|. \tag{3.23}$$

But since we proved that u is a strong solution to (3.1), then u is an integral solution too, so we have for any $[x,y]\in A$

$$\frac{1}{2}\|u(t)-x\|^2 \leq \frac{1}{2}\|u(s)-x\|^2 + \int_s^t \|f(\tau)-y\|\,\|u(\tau)-x\|\,ds,$$

$\forall\ 0\leq s\leq t\leq T$.

Consequently

$$(u(t_0+h)-u(t_0),u(t_0)-x) \leq \frac{1}{2}\|u(t_0+h)-x\|^2 - \frac{1}{2}\|u(t_0)-x\|^2 \tag{3.24}$$
$$\leq \int_{t_0}^{t_0+h}(f(s)-y,u(s)-x)ds,$$

$\forall h>0$ such that $t_0+h<T$, $\forall [x,y]\in A$.

Then, by (3.23), there exists a sequence $h_n\to 0$, $h_n>0$ such that

$$\frac{u(t_0+h_n)-u(t_0)}{h_n} \longrightarrow l \text{ weakly in } H.$$

We take in (3.24) $h=h_n$, divide by h_n and pass to the limit with $h_n\to 0$. We obtain

$$(l-f(t_0)+y,x-u(t_0))\geq 0,\ \forall [x,y]\in A.$$

Using now (3.23) and the m-accretivity of A we get

$$l = (f(t_0) - Au(t_0))^0$$

meaning that l does not depend on the choice of h_n, so finally we have

$$\frac{u(t_0 + h) - u(t_0)}{h} \longrightarrow (f(t_0) - Au(t_0))^0 \text{ weakly in } H, \text{ as } h \to 0, \ h > 0.$$

But this relationship and (3.23) show that u is differentiable from the right at t_0 and we obtain

$$\frac{d^+u}{dt}(t_0) = (f(t_0) - Au(t_0))^0.$$

Also we can write, as before, that $\forall t, t + h \in [0, T]$ we have

$$\|u(t + h) - u(t)\| \leq \|u(h) - u_0\| + \int_0^t \|f(s + h) - f(s)\| \, ds, \tag{3.25}$$

and

$$\|u(h) - u_0\| \leq \int_0^h \left\|(f(s) - A(u_0))^0\right\| ds.$$

These two last relationships lead to

$$\|u(t + h) - u(t)\| \leq \int_0^h \left\|(f(s) - Au_0)^0\right\| ds + \int_0^t \|f(s + h) - f(s)\| \, ds$$

meaning that we can immediately derive (3.9). ∎

Corollary 3.7. *Assume that the operator $A : D(A) \subset H \to H$ is quasi m-accretive, let $u_0 \in D(A)$ and $f \in W^{1,1}(0, T; H)$. Then the conclusions of Theorem* 3.6 *remain true.*

Proof. The proof is exactly the same as that of Theorem 3.6, by replacing the approximating equation (3.11) by

$$\frac{du_\lambda}{dt} + (\omega I + A)_\lambda u_\lambda - \omega u_\lambda = f,$$
$$u_\lambda(0) = u_0,$$

but we do no longer give the details. However we mention that some slight modifications appear in the estimates (3.9) and (3.10). For instance (3.10) will be replaced by

$$\|u(t) - v(t)\| \leq e^{\omega t} \|u_0 - v_0\| + \int_0^t e^{\omega(t-s)} \|f(s) - g(s)\| \, ds, \ 0 \leq t \leq T.$$ ∎

Theorem 3.8. *Let $A : D(A) \subset H \to H$ be an m-accretive operator. Then for every $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; H)$ there exists a unique weak solution u, also called integral solution, to problem* (3.1) *satisfying*

$$\frac{1}{2}\|u(t)-x\|^2 \le \frac{1}{2}\|u(s)-x\|^2 + \int_s^t (f(\tau)-y, u(\tau)-x)d\tau, \tag{3.26}$$

for any $0 \le s \le t \le T$, $\forall [x,y] \in A$.

Moreover, if u *and* v *are the weak solutions corresponding to* (u_0, f) *and* $(v_0, g) \in \overline{D(A)} \times L^1(0,T;H)$, *then the following inequality*

$$\frac{1}{2}\|u(t)-v(t)\|^2 \le \frac{1}{2}\|u(s)-v(s)\|^2 + \int_s^t (f(\tau)-g(\tau), u(\tau)-v(\tau))d\tau \tag{3.27}$$

holds for any $0 \le s \le t \le T$.

Proof. Let $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;H)$ be taken arbitrarily, but fixed. Then, there exist some sequences $\{u_0^n\}_{n\ge 1} \subset D(A)$ and $\{f_n\}_{n\ge1} \subset W^{1,1}(0,T;H)$ such that

$$u_0^n \longrightarrow u_0 \text{ in } H$$

and

$$f_n \longrightarrow f \text{ in } L^1(0,T;H).$$

From Theorem 3.6 we have that for each $n \in \mathbf{N}^* = \{1,2,...\}$ there exists $u_n \in W^{1,\infty}(0,T;H)$, the strong solution to

$$\begin{aligned} \frac{du_n}{dt} + Au_n &\ni f_n \text{ a.e. on } (0,T), \\ u_n(0) &= u_0^n \end{aligned} \tag{3.28}$$

and we have the estimate

$$\|u_n(t)-u_m(t)\| \le \|u_0^n - u_0^m\| + \int_0^T \|f_n(s)-f_m(s)\|\,ds. \tag{3.29}$$

From here we deduce that $\{u_n\}_{n\ge1}$ is a Cauchy sequence in $C([0,T];H)$ and hence it converges strongly, $u_n \to u$ in $C([0,T];H)$. In particular $u(0) = u_0$. Evidently u is the weak solution to (3.1). The relations (3.26) and (3.27) are true for the strong solution to (3.28). They remain true by passing to limit as $n \to \infty$, since the function $w \to \|w\|$ is a proper, convex, l.s.c function (see Example 5.10 in Appendix). The uniqueness is an immediate consequence of (3.29). ∎

The previous result extends the general theorem of existence and uniqueness of the Cauchy problem to the situation when data are less regular. We shall give further a definition of the mild solution and we shall reformulate in these terms the existence result of Theorem 3.8. The reason of presenting this alternative approach is that it provides an approximate procedure (a discretizated scheme) for the numerical computation of the solution.

Definition 3.9. Let $f \in L^1(0,T;H)$ and $\varepsilon > 0$ be given. An *ε-discretization* on $[0,T]$ of the equation $\dfrac{du}{dt} + Au \ni f$ consists in a partition $0 = t_0 \le t_1 \le$

$t_2 \leq ... \leq t_L$ of the interval $[0, t_L]$ and a finite sequence $\{f_i\}_{i=1}^L \subset H$, such that

$$t_i - t_{i-1} < \varepsilon \text{ for } i = 1, ..., L, \; T - \varepsilon < t_L \leq T, \tag{3.30}$$
$$\sum_{i=1}^{L} \int_{t_{i-1}}^{t_i} \|f(s) - f_i\| \, ds < \varepsilon.$$

We shall denote by $D_A^\varepsilon(0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_L; f_1, ..., f_L)$ this ε- discretization.

Definition 3.10. A *solution to the ε-discretization* $D_A^\varepsilon(0 = t_0 \leq t_1 \leq t_2 \leq ... \leq t_L; f_1, ..., f_L)$ is a piecewise constant function $z : [0, t_L] \to H$ whose values z_i on $(t_{i-1}, t_i]$ satisfy the equation

$$\frac{z_i - z_{i-1}}{t_i - t_{i-1}} + Az_i \ni f_i, \; i = 1, ..., L. \tag{3.31}$$

Such a solution is called *ε-approximate solution* to the Cauchy problem (3.1) if it also satisfies

$$\|z(0) - u_0\| \leq \varepsilon.$$

Definition 3.11. A *mild solution* of the Cauchy problem (3.1) is a function $u \in C([0, T]; H)$ with the property that for each $\varepsilon > 0$ there is an ε-approximate solution z of $\frac{du}{dt} + Au \ni f$ on $[0, T]$ such that $\|u(t) - z(t)\| \leq \varepsilon$ for all $t \in [0, T]$ and $u(0) = u_0$.

Theorem 3.12. *Let A be quasi m-accretive (ω-accretive), $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; H)$. Then the Cauchy problem* (3.1) *has a unique mild solution u. Moreover, there is a continuous function $\delta = \delta(\varepsilon)$ such that $\delta(0) = 0$ and if z is an ε-approximate solution of* (3.1) *then*

$$\|u(t) - z(t)\| \leq \delta(\varepsilon) \text{ for } t \in [0, T - \varepsilon]. \tag{3.32}$$

Let $f, g \in L^1(0, T; H)$ and u, v be mild solutions to (3.1) *corresponding to f and g, respectively. Then*

$$\|u(t) - v(t)\| \leq e^{\omega(t-s)} \|u(s) - v(s)\| \tag{3.33}$$
$$+ \int_s^t e^{\omega(t-\tau)} (u(\tau) - v(\tau), f(\tau) - g(\tau)) \, ds,$$

for $0 \leq s < t \leq T$.

It turns out the mild solution u given by Theorem 3.12 is just the integral solution specified by Theorem 3.8. Our presentation will continue with an important existence result corresponding to the case when data are less regular. This shows that the irregularity of the initial data can be compensated by the regularizing action of the parabolic operator if it is a potential operator, i.e., the subdifferential of some convex lower semicontinuous function. For the approach of this situation the following differentiation lemma is useful.

Lemma 3.13. *Let $\varphi : H \to (-\infty, \infty]$ be a proper l.s.c. convex function and let $u \in W^{1,2}(0,T;H)$ be such that $u(t) \in D(\partial\varphi)$ a.e. on $(0,T)$. Assume also that there exists $g \in L^2(0,T;H)$ satisfying*

$$g(t) \in \partial\varphi(u)) \text{ a.e. } t \in (0,T).$$

Then the function $t \to \varphi(u(t))$ is absolutely continuous on $[0,T]$ and we have

$$\frac{d}{dt}\varphi(u(t)) = \left(h, \frac{du}{dt}(t)\right) \text{ a.e. on } (0,T),\ \forall h \in \partial\varphi(u(t)). \tag{3.34}$$

For the proof of this lemma we refer the reader to the monographs [9] and [30].

Theorem 3.14. (Brezis, [30]). *Let A be the subdifferential of a proper convex l.s.c. function $\varphi : H \longrightarrow (-\infty, \infty]$, i.e., $A = \partial\varphi$. Let $u_0 \in \overline{D(A)}$ and $f \in L^2(0,T;H)$. Then there exists a unique strong solution $u \in C([0,T];H)$ to* (3.1) *which satisfies*

$$u \in W^{1,2}(\delta, T; H) \text{ for every } 0 < \delta < T, \tag{3.35}$$

$$u(t) \in D(A) \text{ a.e. on } (0,T), \tag{3.36}$$

$$\sqrt{t}\frac{du}{dt} \in L^2(0,T;H),\ \varphi(u) \in L^1(0,T). \tag{3.37}$$

Moreover, if $u_0 \in D(\varphi)$ it follows that

$$u \in W^{1,2}(0,T;H),\ \varphi(u) \in L^\infty(0,T). \tag{3.38}$$

Proof. Let $[x_0, y_0] \in \partial\varphi$. We introduce the function

$$\widetilde{\varphi}(x) = \varphi(x) - \varphi(x_0) - (y_0, x - x_0).$$

Then (3.1) is equivalent to

$$\frac{du}{dt} + \partial\widetilde{\varphi}(u) \ni f(t) - y_0,\ 0 < t < T$$

and we assume, without any loss of generality that

$$\min_{u \in H} \varphi(u) = \varphi(x_0) = 0.$$

First we suppose that $u_0 \in D(\partial\varphi)$ and $f \in W^{1,2}(0,T;H)$. Then according to Theorem 3.6, problem (3.1) has a unique strong solution $u \in W^{1,\infty}(0,T;H)$. We multiply equation (3.1) by $t\dfrac{du}{dt}$ and using Lemma 3.13 we obtain

$$t\left\|\frac{du}{dt}(t)\right\|^2+t\frac{d}{dt}\varphi(u(t))=t(f(t),\frac{du}{dt}(t)),\ \text{a.e. } t\in(0,T).$$

Then, integrating this equation from 0 to T we get after some calculations that

$$\int_0^T t\left\|\frac{du}{dt}(t)\right\|^2 dt+T\varphi(u(T))=\int_0^T t\left(f(t),\frac{du}{dt}(t)\right)dt+\int_0^T\varphi(u(t))dt.$$

Due to the fact that $\varphi\geq 0$, we obtain

$$\int_0^T t\left\|\frac{du}{dt}(t)\right\|^2 dt\leq\int_0^T t\left\|f(t)\right\|^2 dt+2\int_0^T\varphi(u(t))dt. \tag{3.39}$$

But from

$$f(t)-\frac{du}{dt}(t)\in\partial\varphi(u(t))\ \text{a.e. } t\in(0,T)$$

and the definition of $\partial\varphi$ we obtain that

$$\varphi(u(t))\leq\left(f(t)-\frac{du}{dt}(t),u(t)-x_0\right)\ \text{ a.e. on } (0,T).$$

Therefore

$$\int_0^T\varphi(u(t))dt\leq\frac{1}{2}\left\|u_0-x_0\right\|^2+\int_0^T\left\|f(t)\right\|\left\|u(t)-x_0\right\|dt. \tag{3.40}$$

Now, multiplying (3.1) by $u(t)-x_0$ and integrating over $(0,t)$ we obtain, via Gronwall's lemma that

$$\left\|u(t)-x_0\right\|\leq\left\|u_0-x_0\right\|+\int_0^T\left\|f(t)\right\|dt. \tag{3.41}$$

From (3.40) and (3.41) we deduce that

$$\int_0^T\varphi(u(t))dt\leq\left(\left\|u_0-x_0\right\|+\int_0^T\left\|f(t)\right\|dt\right)^2, \tag{3.42}$$

hence $\varphi\in L^1(0,T)$. But (3.39) and (3.42) imply that

$$\int_0^T t\left\|\frac{du}{dt}(t)\right\|^2 dt\leq\int_0^T t\left\|f(t)\right\|^2 dt+2\left(\left\|u_0-x_0\right\|+\int_0^T\left\|f(t)\right\|dt\right)^2. \tag{3.43}$$

Further we assume that $u_0\in\overline{D(\varphi)}=\overline{D(\partial\varphi)}$ and $f\in L^2(0,T;H)$ and we consider the sequences $\{u_n^0\}_{n\geq 1}\subset D(\partial\varphi)$ and $\{f_n\}_{n\geq 1}\subset W^{1,2}(0,T;H)$, such that

$$u_n^0\longrightarrow u_0 \text{ strongly in } H \tag{3.44}$$

and

$$f_n \longrightarrow f \text{ strongly in } L^2(0,T;H). \tag{3.45}$$

We denote by u_n the strong solution to (3.1) corresponding to $u_0 = u_n^0$ and $f = f_n$ and replace in (3.10) $u = u_m$ and $v = u_m$. Hence we see that there exists some $u \in C([0,T];H)$ such that

$$u_n \longrightarrow u \text{ strongly in } C([0,T];H). \tag{3.46}$$

Also, the estimate (3.43) is true for u_n, u_n^0 and f_n, hence $\dfrac{du}{dt}$ exists a.e. on $(0,T)$, $\sqrt{t}\dfrac{du}{dt} \in L^2(0,T;H)$ and

$$\sqrt{t}\frac{du_n}{dt} \longrightarrow \sqrt{t}\frac{du}{dt} \text{ weakly in } L^2(0,T;H).$$

Particularly, for any $\varepsilon \in (0,T)$ we have

$$\frac{du_n}{dt} \longrightarrow \frac{du}{dt} \text{ weakly in } L^2(\varepsilon,T;H). \tag{3.47}$$

Now we denote by $\widetilde{A}$ the canonical extension of $A = \partial\varphi$ to $L^2(\varepsilon,T;H)$ and in virtue of (3.45), (3.46) and (3.47) we can pass to limit in

$$\frac{du_n}{dt} + \widetilde{A}u_n \ni f_n \text{ in } L^2(\varepsilon,T;H)$$

and obtain that u satisfies (3.1) a.e. on $(0,T)$. From (3.44) and (3.46) it follows also that $u(0) = u_0$.

Finally we assume that $u_0 \in D(\varphi)$. Because

$$\left\|\frac{du}{dt}(t)\right\|^2 + \frac{d}{dt}\varphi(u(t)) \le \|f(t)\| \left\|\frac{du}{dt}(t)\right\| \text{ a.e. on } (0,T)$$

we have

$$\frac{1}{2}\left\|\frac{du}{dt}(t)\right\|^2 + \frac{d}{dt}\varphi(u(t)) \le \frac{1}{2}\|f(t)\|^2 \text{ a.e. on } (0,T) \tag{3.48}$$

and hence the function

$$t \longrightarrow \varphi(u(t)) - \frac{1}{2}\int_0^t \|f(s)\|^2\, ds$$

is monotone nonincreasing on $[0,T]$. Since $u_0 \in D(\varphi)$ we deduce that

$$\varphi(u(t)) \le \varphi(u_0) + \frac{1}{2}\int_0^t \|f(s)\|^2\, ds. \tag{3.49}$$

Let $\varepsilon > 0$ be sufficiently small. We derive that

$$\frac{1}{2}\int_\varepsilon^T \left\|\frac{du}{dt}(t)\right\|^2 dt \le \varphi(u_0) + \frac{1}{2}\int_0^T \|f(s)\|^2\, ds.$$

Since ε is arbitrary, this means that $\dfrac{du}{dt} \in L^2(0,T;H)$ and from (3.49) we deduce that $\varphi(u) \in L^\infty(0,T)$. ∎

3.4 The Cauchy problem within the variational approach

In this section we shall present the variational approach to the Cauchy problem (3.1).

Let V be a reflexive Banach space and H a Hilbert space such that $V \subset H$ continuously and densely. Denote by V' the dual space of V, endowed with the dual norm

$$\|v\|_{V'} := \sup_{\|u\|_V \leq 1} \left| \langle u, v \rangle_{V',V} \right|$$

and identify H with its own dual,

$$V \subset H \equiv H' \subset V'. \tag{4.1}$$

Here, $\langle v, u \rangle_{V',V}$ is the pairing between V' and V, if $v \in V'$ and $u \in V$, which becomes the inner product in H, (v, u) if $v \in H$.

The space triplet V, H, V' is called in the literature *variational triplet.*

In the most situations we shall encounter in this book, V will be a Hilbert space.

We recall that in the semigroup approach the operator A acting in the Cauchy problem was allowed to be multivalued. In the variational approach the operator $A : V \to V'$, should be single-valued. In return, this approach allows the proof of existence results in the non-autonomous case, too.

Theorem 4.1. *Let $A : V \to V'$ be a single-valued, monotone, hemicontinuous and coercive operator. Let $f \in W^{1,1}(0, T; H)$ and let $u_0 \in V$ such that $Au_0 \in H$. Then, there exists a unique function $u : [0, T] \to V$ which satisfies* (3.1) *and*

$$u \in W^{1,\infty}(0, T; H), \ Au \in L^\infty(0, T; H). \tag{4.2}$$

Proof. Let A_H be the operator defined by

$$A_H u = Au \text{ for } u \in D(A_H) = \{u \in V; \ Au \in H\}.$$

By hypothesis, the operator $u \to u + Au$ is monotone, hemicontinuous and coercive from V to V', implying that it is surjective, i.e., $R(I + A) = V'$. It follows that $R(I + A_H) = H$ meaning that A_H is m-accretive in $H \times H$. Then we can apply Theorem 3.6. ∎

Theorem 4.2. (Lions, [83]) *Let V and H be given satisfying* (4.1) *and let $A : V \to V'$ be a monotone and hemicontinuous operator that satisfies*

$$\langle Au, u \rangle_{V',V} \geq \omega \|u\|_V^p + c_1, \ \omega > 0, \tag{4.3}$$

$$\|Au\|_{V'} \leq c_2 \left(\|u\|_V^{p-1} + 1 \right), \ \forall u \in V, \ p \geq 2. \tag{4.4}$$

Suppose that $u_0 \in H$ and $f \in L^{p'}(0, T; V')$, where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then, there exists a unique solution u to (3.1)*, that is V'-valued absolutely continuous on* $[0, T]$ *and satisfies*

$$u \in C([0,T];H) \cap L^p(0,T;V), \tag{4.5}$$

$$\frac{du}{dt} \in L^{p'}(0,T;V'). \tag{4.6}$$

Proof. Here we follow the proof presented in [9]. We assume first that $u_0 \in D(A_H)$ and $f \in W^{1,1}(0,T;H)$. According to Theorem 4.1, problem (3.1) has a solution $u \in W^{1,\infty}(0,T;H)$. From (4.3) and (4.4) there exists a constant c_1 such that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \omega\|u(t)\|_V^p \le \|f(t)\|_{V'}\|u(t)\|_V + c_1, \text{ a.e. on } (0,T).$$

We integrate this inequality from 0 to $t \in [0,T]$, apply Hölder's inequality (see Theorem 2.3 in Appendix)

$$\left|\int_0^t \phi\psi d\tau\right| \le \left(\int_0^t |\phi|^p\, d\tau\right)^{1/p}\left(\int_0^t |\psi|^{p'}\, d\tau\right)^{1/p'}, \ \frac{1}{p}+\frac{1}{p'}=1$$

for $\phi(\tau) = \|u(\tau)\|_V$ and $\psi(\tau) = \|f(\tau)\|_{V'}$ and then Young's inequality (see (2.2) in Appendix) in the form

$$\left(\omega\int_0^t |\phi|^p\, d\tau\right)^{1/p}\left(\omega^{1/p}\int_0^t |\psi|^{p'}\, d\tau\right)^{1/p'} \le \frac{\omega}{p}\int_0^t |\phi|^p\, d\tau + \frac{\omega^{p'/p}}{p'}\int_0^t |\psi|^{p'}\, d\tau.$$

We get

$$\frac{1}{2}\|u(t)\|^2 + \frac{\omega}{p}\int_0^t \|u(s)\|_V^p\, ds \le \frac{1}{2}\|u_0\|^2 + \frac{\omega^{p'/p}}{p'}\int_0^t \|f(s)\|_{V'}^{p'}\, ds + C_1.$$

Following some computations we obtain that

$$\|u(t)\|^2 + \int_0^t \|u(s)\|_V^p\, ds \le C\left(\|u_0\|^2 + \int_0^t \|f(s)\|_{V'}^{p'}\, ds + 1\right), \tag{4.7}$$

where C_1 and C are positive constants. Moreover, we obtain

$$\left\|\frac{du}{dt}(t)\right\|_{V'}^{p'} \le \|f(t)\|_{V'}^{p'} + c_3\|Au(t)\|_{V'}^{p'} \le \|f(t)\|_{V'}^{p'} + c_4\|u(t)\|_V^{(p-1)p'} + c_5.$$

Integrating with respect to t and using (4.7) and the fact that $(p-1)p' = p$ we get

$$\int_0^t \left\|\frac{du}{dt}(s)\right\|_{V'}^{p'} ds \le c_6\left(\|u_0\|^2 + \int_0^t \|f(s)\|_{V'}^{p'}\, ds + 1\right). \tag{4.8}$$

We prove now that $D(A_H)$ is a dense subset of H. For that we take u to be arbitrary in H and let $u_\varepsilon = (1+\varepsilon A_H)^{-1}u$. Equivalently, we have

$$u_\varepsilon + \varepsilon A u_\varepsilon = u$$

that implies, due to (4.3) and (4.4) that

$$\|u_\varepsilon\|^2 + \omega\varepsilon \|u_\varepsilon\|_V^p \le \|u_\varepsilon\| \|u\| + c_0\varepsilon, \ \varepsilon > 0$$

and

$$\|u_\varepsilon - u\|_{V'} \le \varepsilon \|Au_\varepsilon\|_{V'} \le c_2\varepsilon(\|u_\varepsilon\|_V^{p-1} + 1).$$

These last inequalities shows that $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in H and $u_\varepsilon \to u$ in V' as $\varepsilon \to 0$, proving that $D(A_H)$ is dense in H.

We consider now $u_0 \in H$ and $f \in L^{p'}(0,T;V')$. Due to the density of $W^{1,1}(0,T;H)$ in $L^{p'}(0,T;V')$, (which is a consequence of the density of H in V') we can choose $\{f_n\}_{n\ge1} \subset W^{1,1}(0,T;H)$ such that $f_n \to f$ in $L^{p'}(0,T;V')$. Similarly we take $\{u_n^0\}_{n\ge1} \subset D(A_H)$ such that $u_n^0 \to u_0$ in H, as $n \to \infty$.

Let $u_n \in W^{1,\infty}(0,T;H)$ be a solution to (3.1) corresponding to the initial value u_n^0 at the place of u_0 and with f_n instead of f. Because A is monotone, we have

$$\frac{1}{2}\frac{d}{dt}\|u_n(t) - u_m(t)\|^2 \le \langle f_n(t) - f_m(t), u_n(t) - u_m(t)\rangle_{V',V}, \text{ a.e. on } (0,T).$$

By integration with respect to t, we derive

$$\begin{aligned} &\|u_n(t) - u_m(t)\|^2 \le \left\|u_n^0 - u_m^0\right\|^2 \\ &+2\left(\int_0^t \|f_n(s) - f_m(s)\|_{V'}^{p'} ds\right)^{1/p'} \left(\int_0^t \|u_n(s) - u_m(s)\|_V^p ds\right)^{1/p}. \end{aligned} \tag{4.9}$$

By (4.7) and (4.8), it follows that u_n is bounded in $L^p(0,T;V)$ and $\dfrac{du_n}{dt}$ is bounded in $L^{p'}(0,T;V')$. Hence (4.9) implies that $u_n \to u$ in H, uniformly on $[0,T]$. So, we can extract a subsequence such that

$$u_n \longrightarrow u \text{ weakly in } L^p(0,T;V),$$

$$\frac{du_n}{dt} \longrightarrow \frac{du}{dt} \text{ weakly in } L^{p'}(0,T;V').$$

Hence $u \in W^{1,p'}(0,T;V')$ and due to Theorem 3.9 in Appendix, we deduce that $u : [0,T] \to V'$ is absolutely continuous, a.e. differentiable on $(0,T)$ and its derivative belongs to $L^{p'}(0,T;V')$, meaning in fact that $u \in A^{1,p}(0,T;V')$.

Let now $x \in V$ be fixed. Then

$$\frac{1}{2}\frac{d}{dt}\|u_n(t) - x\|^2 + \langle Au_n(t), u_n(t) - x\rangle_{V',V} = \langle f_n(t), u_n(t) - x\rangle_{V',V}$$

a.e. on $(0,T)$. By integration over (s,t) and on the basis of the monotonicity of A we obtain

$$\left\langle \frac{u_n(t) - u_n(s)}{t-s}, u_n(s) - x\right\rangle_{V',V} \le \frac{1}{t-s}\int_s^t \langle -Ax + f_n(\tau), u_n(\tau) - x\rangle_{V',V}\, d\tau.$$

By passing to limit as $n \to \infty$ we get

$$\left\langle \frac{u(t)-u(s)}{t-s}, u(s)-x \right\rangle_{V',V} \leq \frac{1}{t-s}\int_s^t \langle -Ax + f(\tau), u(\tau)-x\rangle_{V',V}\, d\tau.$$

It follows that

$$\left\langle \frac{du}{dt}(t_0) - f(t_0) + Ax, u(t_0)-x \right\rangle_{V',V} \leq 0,\ \forall x \in V \tag{4.10}$$

for every $t_0 \in (0,T)$, such that $\dfrac{du}{dt}(t_0)$ exists in V' and

$$\lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h} f(s)ds = f(t_0).$$

That means that (4.10) holds a.e. on $(0,T)$ for every x arbitrary in V. Because A is m-accretive, (3.1) follows to be satisfied a.e. on $(0,T)$. ∎

Remark 4.3. The conclusion of Theorem 4.2 remains valid if the assumption (4.3) is replaced by

$$(Au,u)_{V',V} \geq \omega \left\| u \right\|_V^2 - \gamma \left\| u \right\|_H^2 + c_1,\ \text{where } \omega, \gamma > 0, \tag{4.11}$$

see e.g., [9].

This type of approach may be used also in the case when the single-valued operator A depends measurably on $t \in (0,T)$ i.e., for the problem

$$\begin{aligned} \frac{du}{dt}(t) + A(t)u(t) &= f(t) \text{ a.e. } t \in (0,T), \\ u(0) &= u_0, \end{aligned} \tag{4.12}$$

situation in which we have the following result (see [9]):

Theorem 4.4. *Let V and H be given to satisfy* (4.1). *Let $A(t) : V \to V'$ be a family of nonlinear operators, defined on V a.e. $t \in (0,T)$ that satisfy*

(a) *$A(t)$ is monotone and hemicontinuous from V to V', a.e. $t \in (0,T)$,*
(b) *$t \to A(t)u$ is a measurable function from $[0,T]$ to V' for every $u \in L^p(0,T;V)$,*
(c) $\left\| A(t)u \right\|_{V'} \leq c_1 \left(\left\| u \right\|_V^{p-1} + 1 \right)$, $\forall u \in V$, $t \in (0,T)$, $c_1 > 0$,
(d) $\langle A(t)u,u\rangle_{V',V} \geq \omega \left\| u \right\|_V^p + c_2$, $\forall u \in V$, $t \in (0,T)$ *with* $\omega > 0$, *where* $2 \leq p < \infty$.

Suppose that $u_0 \in H$ and $f \in L^{p'}(0,T;V')$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then, there exists a unique solution u to (4.12) *which is V'-valued absolutely continuous on* $[0,T]$ *and satisfies*

$$u \in C([0,T];H) \cap L^p(0,T;V), \tag{4.13}$$

$$\frac{du}{dt} \in L^{p'}(0,T;V'). \tag{4.14}$$

Proof. We introduce the spaces

$$\mathcal{V} = L^p(0,T;V),\ \mathcal{H} = L^2(0,T;H),\ \mathcal{V}' = L^{p'}(0,T;V')$$

which obviously satisfy $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$, $\mathcal{V}$ and $\mathcal{V}'$ being dual pairs. Let u_0 be arbitrary but fixed in $\mathcal{H}$ and let $B : \mathcal{V} \to \mathcal{V}'$ be defined by

$$Bu = \frac{du}{dt} \text{ for } u \in D(B),$$

where $D(B) = \{u \in \mathcal{V} \text{ such that } \frac{du}{dt} \in \mathcal{V}' \text{ and } u(0) = u_0\}$. Here $\frac{du}{dt}$ is taken in the sense of vectorial distributions on $(0,T)$. We notice that $u(0)$ makes sense because $u \in D(B)$ implies that $u \in W^{1,p'}(0,T;V')$. We shall use a perturbation result for maximal monotone operators in $\mathcal{V}\times \mathcal{V}'$ (see Corollary 4.16 in Appendix). For this we have to prove that B is maximal monotone in $\mathcal{V} \times \mathcal{V}'$.

It can be immediately checked that B is monotone in $\mathcal{V}\times \mathcal{V}'$. To prove that it is maximal monotone we have to show that $R(B + \Phi) = \mathcal{V}'$, where Φ is defined by

$$\Phi(u)(t) = F(u(t))\,\|u(t)\|_V^{p-2},\ u \in \mathcal{V},\ t \in (0,T),$$

with $F : V \to V'$ the duality mapping on V. This follows from Theorem 4.15 in Appendix, noticing that $u \to \dfrac{\Phi(u)}{\|u(t)\|_V^{p-2}}$ is the duality mapping on $\mathcal{V}$.

Let f be an arbitrary element of $\mathcal{V}'$. The equation $Bu + \Phi(u) = f$ can be written as

$$\frac{du}{dt} + F(u)\,\|u\|^{p-2} = f,\ \text{a.e. } t \in (0,T), \tag{4.15}$$
$$u(0) = u_0.$$

Using the renorming theorem (see Theorem 1.24 in Appendix) we may assume that V and V' are strictly convex. Hence F is single-valued and demicontinuous and by Theorem 4.2, problem (4.15) has a unique solution which satisfies

$$u \in C([0,T];H) \cap L^p(0,T;V),\ \frac{du}{dt} \in L^{p'}(0,T;V').$$

This shows that B is maximal monotone and $D(B) \subset C([0,T];H)$.

If $u \in \mathcal{V}$ we define

$$(Au)(t) := A(t)u(t),\ \text{a.e. on } (0,T).$$

By hypotheses, A is monotone, everywhere defined, hemicontinuous and coercive. Since B is maximal monotone it follows that $A+B$ is maximal monotone and coercive, so it is surjective (see Corollary 4.16 in Appendix). ∎

Corollary 4.5. *Theorem* 4.2 *and* 4.4 *remain true if the hypothesis of monotony of the operator* A (*or* $A(t)$) *is replaced by that of quasi-monotony.*

Proof. Let assume, for instance, the hypotheses of Theorem 4.2 with A quasi-monotone. If A quasi-monotone, i.e., it satisfies $(Au - A\overline{u}, u - \overline{u}) \geq -\omega \left\| u - \overline{u} \right\|^2$, it follows that $\omega I + A$ is monotone. Then we make the function replacement

$$z = e^{-\omega t} u$$

and problem (3.1) becomes

$$\frac{dz}{dt} + \widehat{A}(t)z = f e^{-\omega t}, \;\; z(0) = u_0,$$

where $\widehat{A}(t)z = e^{-\omega t} A(e^{\omega t} z) + \omega z$.

It is easily seen that the operator $\widehat{A}(t) : V \to V'$ satisfies the conditions of Theorem 4.4 and then the previous problem has a unique solution $z \in C([0,T]; H) \cap L^p(0,T;V)$ with $\dfrac{dz}{dt} \in L^{p'}(0,T;V')$. Then $u = e^{\omega t} z$ is the unique solution to (3.1) belonging to the same spaces. ∎

3.5 Comments

The results presented in the last two sections have settled the basis for the analysis of the specific diffusion-type models that will be discussed in the next chapters.

The analysis of models is absolutely necessary even if the main and final interest in a real-life problem is the solution computation. Prior to the effective computation tentative, it should be established if the model is well-posed, i.e., if it has solution, if this is unique and if it depends continuously on data. The existence proof is basic, because if it fails the reliability of the model is doubted. Also, various pieces of information got in the proofs of the existence and uniqueness theorems and the specification of the solution qualitative properties are fundamental for the methods developed further for computing an approximate solution.

Existence and uniqueness proofs of the solutions to specific diffusion-type models will be based on the theorems presented in this chapter, after the boundary value problem is placed in an abstract framework, as explained in the first section.

The choice of the spaces $D(A)$ and H, or of the triplet V, H, V' and the introduction of the operator A is the first clue of these techniques. It is important to observe that it should be proved that the solution to the abstract

Cauchy problem (1.1) is a solution to the original boundary value problem in some generalized sense. Once established this, everything else turns out in the study of (1.1), more precisely in the investigation of the properties of A.

If the operator A does not have the necessary properties required by the basic theorems, then auxiliary procedures should be developed. Very often approximating problems solve the difficulties. They consist in replacing the original problem by a family of simpler (smoother) problems, for which existence and uniqueness results can be proved. The solution to the original problem can be obtained as a limit in some sense of the sequence of approximate solutions, via monotony and compacity techniques. Here, the concept of approximate solution is related to a theoretical procedure, rather than a numerical one.

The proof of the convergence expects the settlement of the so-called *a priori estimates* which are relationships that show that the approximating solutions are bounded in some spaces. Usually, the proof of the a priori estimates, especially of those necessary for the regularity properties is not trivial, and may involve a high level of technical aspects.

Finally, useful properties of the solutions can be established. Usually they include regularity, positiveness, boundedness and special asymptotic behaviors.

All these considerations serve as a justification for the material included in this chapter and as an introduction to the next chapters in which the detailed analysis of the models introduced in Chap. 2 will follow these steps.

Bibliographical note

As bibliographical material concerning this chapter, we indicate the monographs [9], [30], [83], [112], [127], [134]. For more details on semigroup of operators we refer also to [23], [101].

Theorem 3.6 is essentially due to Y. Komura (see [76]), but it has been extended in Banach spaces by M. G. Crandall and A. Pazy and H. Brezis (see [49], [30]). More details about the Definitions 3.9 - 3.11 and Theorem 3.12 can be found in [11]. Theorem 3.14 is due to H. Brezis ([30]).

Results similar to Theorem 4.2 and its extension, Theorem 4.4 were obtained by many authors, as J. L. Lions in [83] and F. E. Browder in [36]. More general results for the time-dependent case can be found in the work of M. G. Crandall and A. Pazy [49]. Here we presented the proofs indicated by V. Barbu in [9]. Generally, we refer to [9] for complete references and more general results of this type.

4

Functional approach to the quasi-unsaturated infiltration model

This chapter is concerned with the existence of the solutions to the boundary value problems treated under the quasi-unsaturated case and the proof of their properties. We recall that this is related in fact to a very fast diffusion being represented by Models 1.5 and 1.6 introduced in Sect. 2.4.

We have already specified that in general water diffusion in soils is not a very fast one such that it might not be completely realistic to discuss it in the framework of the quasi-unsaturated model. However, since this model is a limit case of the diffusion behaviour at saturation and involves a special mathematical treatment, we have considered it interesting to study it.

The decision of introducing this limit model at the beginning of the second part of the book is also due to the fact that it is an appropriate example for presenting some basic results in the theory of monotone operators we shall refer to many times in the next chapters.

Before resuming the quasi-unsaturated diffusive model, we shall introduce some notations that will be kept from now on in the book.

Some general notations

In the quasi-unsaturated model we shall denote, by convenience, the water diffusivity by D and its antiderivative by D^*. Thus, D and D^* stand here for β and β^* introduced by (8.4) and (8.5) in Sect. 2.8, because the latter group of notations will be assigned to the saturated-unsaturated models only.

Throughout this and the next chapters we shall use the following notations.

By Ω we designate an open, bounded subset of $\mathbf{R}^N$, $N = 1, 2, 3$, with the boundary $\Gamma = \partial\Omega$ sufficiently smooth, e.g., of class C^1. Sometimes it will be needed to assume more regularity for $\partial\Omega$ but this will be expressly specified.

Let $x \in \Omega$ represent the vector $x = (x_1, x_2, x_3)$. By $dx = dx_1 dx_2 dx_3$ and $d\sigma$ we denote the Lebesgue measure and the surface measure on Γ, respectively.

The time variable t runs within the interval $(0, T)$, where T has a finite value.

We shall also denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and, respectively, the norm in $L^2(\Omega)$. The scalar products and norms in other spaces than $L^2(\Omega)$ will be specified by corresponding subscripts.

Finally, if any confusion is avoided, sometimes we shall no longer indicate in the integrands those function arguments that represent the integration variables.

4.1 Basic hypotheses for the quasi-unsaturated model

We resume now the dimensionless model of the nonhysteretic infiltration of an incompressible fluid into an isotropic, homogeneous, unsaturated porous medium, with a constant porosity,

$$\begin{aligned} \frac{\partial \theta}{\partial t} - \Delta D^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} &= f \quad \text{in } Q = \Omega \times (0,T), \\ \theta(x,0) &= 0 \quad \text{in } \Omega, \\ \text{boundary conditions on } \Sigma = \Gamma \times (0,T), \end{aligned} \tag{1.1}$$

where D^* is the primitive of the diffusivity D that vanishes at 0,

$$D^*(\theta) = \int_0^\theta D(\xi)d\xi, \text{ for } \theta < \theta_s. \tag{1.2}$$

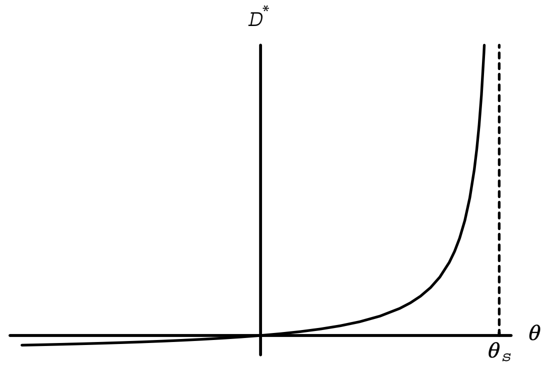

Fig. 4.1. Graphic of $D^*(\theta)$ in the quasi-unsaturated case

We recall that for simplicity we have denoted still by D and K (see Sect. 2.8) the following continuous extensions of the dimensionless diffusivity and hydraulic conductivity

$$D(\theta) := \begin{cases} \rho, & \text{if } \theta < 0 \\ D(\theta), & \text{if } 0 \le \theta < \theta_s, \end{cases} \tag{1.3}$$

$$K(\theta) := \begin{cases} 0, & \text{if } \theta < 0 \\ K(\theta), & \text{if } 0 \leq \theta \leq \theta_s. \end{cases} \tag{1.4}$$

In the quasi-unsaturated case we assumed that $D : (-\infty, \theta_s) \to [\rho, +\infty)$ is a continuous and monotonically increasing function that satisfies the following hypotheses:

(i$_\mathrm{D}$) $D(\theta) \geq \rho > 0$, $\forall \theta \in (-\infty, \theta_s)$;

(ii$_\mathrm{D}$) $\lim\limits_{\theta \nearrow \theta_s} D(\theta) = +\infty$;

(iii$_\mathrm{D}$) $\lim\limits_{\theta \nearrow \theta_s} \int_0^\theta D(\xi) d\xi = +\infty.$

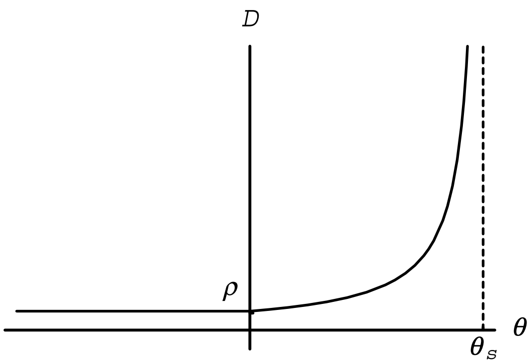

Fig. 4.2. Graphic of $D(\theta)$ in the quasi-unsaturated case

In what concerns $K : (-\infty, \theta_s] \to [0, K_s]$, we have established that it is a non-negative continuous and monotonically increasing function that may satisfy just one of the following properties:

$$\lim_{\theta \nearrow \theta_s} K'(\theta) < +\infty \tag{1.5}$$

or

$$\lim_{\theta \nearrow \theta_s} K'(\theta) = +\infty. \tag{1.6}$$

The first property corresponds to a weaker nonlinear contribution of the hydraulic conductivity around the saturation value (see (4.11) in Sect 2.4) and implies that K is Lipschitz continuous on $(-\infty, \theta_s]$, i.e.,

(i$_\mathrm{K}$) there exists $M > 0$ such that

$$|K(\theta_1) - K(\theta_2)| \leq M |\theta_1 - \theta_2|, \ \forall \theta_1, \theta_2 \in (-\infty, \theta_s].$$

The other property which characterizes a stronger nonlinear behaviour of the hydraulic conductivity around saturation endows K with a Lipschitz property only on an interval strictly included in $(-\infty, \theta_s)$, (see 4.14)),

($\mathrm{ii_K}$) there exist $\theta_l < \theta_s$ and $M_l > 0$ such that

$$|K(\theta_1) - K(\theta_2)| \le M_l \, |\theta_1 - \theta_2| \, , \ \forall \theta_1, \theta_2 \in (-\infty, \theta_l].$$

The model with a stronger nonlinear transport term, for which the derivative of K blows up at $\theta = \theta_s$, reduces to the case ($\mathrm{i_K}$) and we shall explain in a further section how it can be managed.

According to ($\mathrm{i_D}$)-($\mathrm{iii_D}$) the function D^* is differentiable and monotonically increasing on $(-\infty, \theta_s)$ and gets the basic properties held for the quasi-unsaturated model:

(i) $(D^*(\theta_1) - D^*(\theta_2))(\theta_1 - \theta_2) \ge \rho(\theta_1 - \theta_2)^2$, $\forall \theta_1, \theta_2 \in (-\infty, \theta_s)$;

(ii) $\lim\limits_{\theta \to -\infty} D^*(\theta) = -\infty$;

(iii) $\lim\limits_{\theta \to \theta_s} D^*(\theta) = +\infty$.

We shall see below that problem (1.1) can be written as an abstract Cauchy problem of the form

$$\frac{d\theta}{dt} + A\theta = \widetilde{f}, \ \text{a.e. } t \in (0, T),$$
$$\theta(0) = \theta_0$$

in an appropriate Hilbert space H, where A is a quasi m-accretive operator on H. More precisely, H will be either $H^{-1}(\Omega)$ or the dual $(H^1(\Omega))'$ of $H^1(\Omega)$.

The fact that the problem (1.1) is well posed not in $L^2(\Omega)$ but in a larger space (which in general might not be a distribution space on Ω), is a distinctive feature of the nonlinear parabolic equations of the form (1.1) and this is the price paid for the high nonlinearity arising there. However, under the above assumptions it will turn out that the solution obtained in such a way is smooth enough (for example $\theta \in L^2(0, T; H^1(\Omega))$) and this gives a physical meaning to the problem (1.1). More details about this aspect will be given in the following sections.

4.2 Preliminary results

We shall begin this chapter with a general result concerning the nonlinear multivalued elliptic operators. The result is due to Brezis (see [28]) and taking into account its importance for the treatment of nonlinear diffusion problems, we shall point out some steps of its proof.

Proposition 2.1. (Brezis) *Let j be a lower semicontinuous proper convex function from* $\mathbf{R}$ *into* $(-\infty, \infty]$ *and let* $\partial j = \eta$ *with*

$$\lim_{|r| \to \infty} \frac{j(r)}{|r|} = +\infty. \tag{2.1}$$

Let $\varphi : H^{-1}(\Omega) \longrightarrow (-\infty, \infty]$ *be defined by*

$$\varphi(u) = \begin{cases} \int_\Omega j(u(x))dx, & if\ u \in L^1_{loc}(\Omega),\ j(u) \in L^1(\Omega), \\ +\infty, & otherwise. \end{cases} \tag{2.2}$$

Consider the operator $A_\eta : D(A_\eta) \subset H^{-1}(\Omega) \to H^{-1}(\Omega)$ *defined by*

$$A_\eta u = \{-\Delta v;\ v \in H^1_0(\Omega) \text{ and } v(x) \in \eta(u(x)) \text{ a.e. } x \in \Omega\}$$

where

$$D(A_\eta) = \{u \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega);\ \exists v \in H^1_0(\Omega) \\ \text{such that } v(x) \in \eta(u(x)) \text{ a.e. } x \in \Omega\}.$$

Then, the function φ *is convex and lower semicontinuous on* $H^{-1}(\Omega)$ *and*

$$\partial\varphi(u) = A_\eta u,\ \forall u \in D(A_\eta). \tag{2.3}$$

Proof. We refer also to [9], pp. 67-71 for the complete proof, wherefrom we shall outline some steps. In order to prove that φ is lower semicontinuous on $H^{-1}(\Omega)$ we have to show that for a sequence $\{u_n\} \subset H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ convergent to $u \in H^{-1}(\Omega)$ and for which $\varphi(u_n) \le \lambda$, the inequality $\varphi(u) \le \lambda$ holds too. (In our case, since Ω is bounded we can replace $L^1_{loc}(\Omega)$ by $L^1(\Omega)$.) Because a convex and l.s.c. function on a Banach space is also weakly l.s.c. (see Proposition 5.5 in Appendix) and due to the fact that $u \to \int_\Omega j(u(x))dx$ is l.s.c. on $L^1(\Omega)$ (see Proposition 5.22, in Appendix) we have to show that the sequence $\{u_n\}_{n\ge1}$ is weakly compact in $L^1(\Omega)$. Then, by Dunford-Pettis criterion for weak compactness in L^1-spaces (see Theorem 2.13 in Appendix) it is enough to prove that the integrals $\int_\Omega |u_n(x)|\, dx$ are equi-absolutely continuous, meaning that for every $\varepsilon > 0$ there exists δ such that $\int_T |u_n(x)|\, dx < \varepsilon$, where meas$(T) < \delta$. Let $M_\varepsilon > \frac{2\lambda}{\varepsilon}$ and let R_M be such that $\frac{j(r)}{|r|} \ge M_\varepsilon$ for $|r| > R_M$, as a consequence of (2.1). If $\delta < \frac{\varepsilon}{2R_M}$ then

$$\int_T |u_n(x)|\, dx \le \int_{\{x\in T;\ u_n(x)\ge R_M\}} |u_n(x)|\, dx + \int_{\{x\in T;\ u_n(x)<R_M\}} |u_n(x)|\, dx \\ \le M_\varepsilon^{-1} \int_\Omega j(u_n(x))dx + R_M\delta < \varepsilon,$$

as needed.

In this proof we denote by $\langle \cdot\,, \cdot\rangle$ the pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

We recall here the following result (see e.g., [9], p. 68).

Let $f \in H^{-1}(\Omega) \cap L^1(\Omega)$, $v \in H_0^1(\Omega)$ and $g \in L^1(\Omega)$. Let h be measurable on Ω such that

$$f(x)v(x) \geq h(x) \geq g(x), \text{ a.e. } x \in \Omega. \tag{2.4}$$

Then it follows that $h \in L^1(\Omega)$ and $\langle f, v \rangle \geq \int_\Omega h(x)dx$.

The next step of the proof is to show the inclusion $A_\eta \subset \partial\varphi$. Let $f \in A_\eta u$. That means that $f = -\Delta v$ with $v \in H_0^1(\Omega)$ and $v(x) \in \eta(u(x))$ a.e. $x \in \Omega$.

If $w \in H^{-1}(\Omega) \cap L^1(\Omega)$ and $j(w) \in L^1(\Omega)$ (i.e., $w \in D(\varphi)$) then by $\eta = \partial j$ we have

$$j(u) - j(w) \leq v(u - w) \text{ a.e. on } \Omega$$

and we apply (2.4) with $f = u - w$, $h = j(u) - j(w)$ and $g = c_1 |u| - c_2 - j(w)$. Here c_1 and c_2 are constants such that $j(r) \geq c_1 |r| - c_2$ for each $r \in \mathbf{R}$. Hence $j(u) \in L^1(\Omega)$ and

$$\langle u - w, v \rangle \geq \int_\Omega (j(u) - j(w))dx$$

or

$$\int_\Omega j(w)dx - \int_\Omega j(u)dx \geq \langle w - u, v \rangle = \left\langle w - u, (-\Delta)^{-1} f \right\rangle.$$

This means that $f \in \partial\varphi(u)$ for each f, i.e., $A_\eta \subset \partial\varphi$. Further it will be shown that A_η is maximal monotone on $H^{-1}(\Omega)$. For f_0 given in $H^{-1}(\Omega)$ we must prove that there are $v \in H_0^1(\Omega)$ and $u \in H^{-1}(\Omega) \cap L^1(\Omega)$ such that

$$u - \Delta v = f_0 \ , \ v(x) \in \eta(u(x)) \text{ a.e. } x \in \Omega.$$

If we denote $\gamma = \eta^{-1}$, then the equivalent equation is

$$\gamma(v) - \Delta v = f_0. \tag{2.5}$$

We consider now the approximating equation

$$\gamma_\lambda(v_\lambda) - \Delta v_\lambda = f_0 \tag{2.6}$$

where $\gamma_\lambda = \lambda^{-1}\left(I - (I + \lambda\gamma)^{-1}\right)$ is the Yosida approximation of γ which is Lipschitz on $X = \mathbf{R}$. We multiply equation (2.6) by v_λ and integrate over Ω to obtain

$$\int_\Omega |\nabla v_\lambda|^2 dx + \int_\Omega \gamma_\lambda(v_\lambda)v_\lambda dx = \langle f_0, v_\lambda \rangle. \tag{2.7}$$

We may assume that $0 \in \gamma(0)$ and notice that from (2.1) it follows that $D(\gamma) = \mathbf{R}$. Then (2.7) implies that v_λ is bounded in $H_0^1(\Omega)$ as $\lambda \to 0$ and therefore we may extract a subsequence (denoted v_λ too) such that $v_\lambda \to v$ in $H_0^1(\Omega)$ and $v_\lambda \to v$ in $L^2(\Omega)$. Therefore we may assume that $v_\lambda(x) \to v(x)$ a.e. $x \in \Omega$ and

$$J_\lambda v_\lambda(x) \longrightarrow v(x) \text{ a.e. } x \in \Omega, \tag{2.8}$$

where $J_\lambda = (I + \lambda\gamma)^{-1}$ and $\lim_{\lambda \to 0} J_\lambda v = v$, $\forall v \in D(\gamma) = \mathbf{R}$. Then we denote $f_\lambda = \gamma_\lambda(v_\lambda)$ and $w_\lambda = J_\lambda v_\lambda$ with $f_\lambda(x) \in \gamma(w_\lambda(x))$ a.e. $x \in \Omega$ and $f_\lambda \in L^2(\Omega)$. Hence $f_\lambda v_\lambda \in L^1(\Omega)$ and from (2.7) it follows that

$$\int_\Omega f_\lambda v_\lambda dx \leq c, \ \forall \lambda > 0. \tag{2.9}$$

But, for $v_0 \in D(j)$ we have

$$j(f_\lambda(x)) \leq j(v_0) + \langle f_\lambda(x) - v_0, v \rangle, \ \forall v \in \eta(f_\lambda(x))$$

and by (2.8) and (2.9) we obtain that $\int_\Omega j(f_\lambda(x))dx \leq c$, with c some constant. Using Dunford-Pettis theorem we conclude that $\{f_\lambda\}_{\lambda > 0}$ is weakly compact in $L^1(\Omega)$ and in consequence $f_\lambda \to f$ in $L^1(\Omega)$ as $\lambda \to 0$. Passing to limit with $\lambda \to 0$ in (2.6) we obtain $f - \Delta v = f_0$. It remains to prove that $f(x) \in \gamma(v(x))$ a.e. $x \in \Omega$. It is sufficient to prove that for $\forall L$, $f(x) \in \gamma(v(x))$ a.e. $x \in \Omega_L = \{x \in \Omega; |v(x)| \leq L\}$. By the Egorov theorem (Theorem 2.12 in Appendix), for $\varepsilon > 0$ there exists $E \subset \Omega_L$ such that $\text{meas}(E) < \varepsilon$, $v_\lambda(x) \to v(x)$ uniformly on Ω and $v \in L^\infty(\Omega)$. Let $g : \mathbf{R} \to (-\infty, \infty]$ such that $\partial g = \gamma$. Then for every $\widetilde{v} \in L^\infty(\Omega)$ we have

$$\int_\Omega f_\lambda(x)(w_\lambda(x) - \widetilde{v}(x))dx \geq \int_\Omega g(v(x))dx - \int_\Omega g(\widetilde{v}(x))dx,$$

because f_λ converges weakly to $f \in L^1(\Omega)$. By Fatou's lemma (Lemma 2.11 in Appendix) we get

$$\int_\Omega f(x)(w(x) - \widetilde{v}(x))dx \geq \int_\Omega g(v(x))dx - \int_\Omega g(\widetilde{v}(x))dx,$$

and therefore

$$f(x)(v(x) - \widetilde{v}) \geq g(v(x)) - g(\widetilde{v}) \text{ a.e. } x \in \Omega, \ \forall \widetilde{v} \in \mathbf{R}.$$

Hence $f(x) \in \partial g(x) = \gamma(v(x))$ a.e. $x \in \Omega$ and the proof of Proposition 2.1 is complete. ∎

Using the previous proposition we shall prove a result which will be often used in the following existence proofs.

Let us consider the Hilbert space $V = H_0^1(\Omega)$ with its dual $V' = H^{-1}(\Omega)$ and introduce the operator

$$A_D\theta = -\Delta D^*(\theta), \ A_D : D(A_D) \subset H^{-1}(\Omega) \to H^{-1}(\Omega), \tag{2.10}$$

defined by

$$\langle A_D\theta, \psi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega \nabla D^*(\theta) \cdot \nabla\psi dx, \ \forall \psi \in V, \tag{2.11}$$

where

$$\begin{aligned} D(A_D) &= \{\theta \in H^{-1}(\Omega) \cap L^1(\Omega);\ D^*(\theta) \in H_0^1(\Omega)\} \\ &= \{\theta \in L^2(\Omega);\ D^*(\theta) \in H_0^1(\Omega)\}. \end{aligned} \tag{2.12}$$

It is obvious that the second set is included in the first one. Conversely, since $D^*(\theta) \in H_0^1(\Omega)$ and $D(\theta) \geq \rho$, we have

$$\frac{\partial \theta}{\partial x_i} = \frac{1}{D(\theta)} \frac{\partial D^*(\theta)}{\partial x_i} \in L^2(\Omega),\ i = 1, 2, 3. \tag{2.13}$$

Let us define $j : \mathbf{R} \to (-\infty, \infty]$ by

$$j(r) := \begin{cases} \int_0^r D^*(\xi)d\xi, & r < \theta_s, \\ +\infty, & r \geq \theta_s \end{cases} \tag{2.14}$$

and introduce $\varphi : H^{-1}(\Omega) \to (-\infty, \infty]$,

$$\varphi(\theta) := \begin{cases} \int_\Omega j(\theta)dx, & \text{if } \theta \in L^1(\Omega),\ j(\theta) \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.15}$$

Corollary 2.2. *Under the assumptions* (i)-(iii) *the functions j and φ are proper, convex, lower semicontinuous and we have*

$$\partial j(r) = \begin{cases} D^*(r), & r \in (-\infty, \theta_s), \\ \emptyset, & r \geq \theta_s \end{cases} \tag{2.16}$$

and

$$A_D\theta = \partial\varphi(\theta),\ \forall \theta \in D(A_D). \tag{2.17}$$

Moreover, A_D is m-accretive (maximal monotone) on $H^{-1}(\Omega)$.

Proof. First we prove the assertions for j. We recall that D is a positive continuous function on $(-\infty, \theta_s)$ and D^* is monotonically increasing on the same interval. Then it can be easily verified that j is well defined on $\mathbf{R}$ and convex. Indeed,

$$D^*(\xi) = \int_0^\xi D(\xi)d\xi \geq \rho \int_0^\xi d\xi = \rho\xi,\ \forall \xi < \theta_s$$

and j cannot take the value $-\infty$ and is not identically equal to $+\infty$, because

$$j(r) = \int_0^r D^*(\xi)d\xi \geq \int_0^r \rho\xi\, d\xi = \frac{\rho}{2}r^2,\ \forall r < \theta_s.$$

Moreover j is convex since its second derivative $D(r) \geq \rho > 0$, $\forall r < \theta_s$.

Then we prove that $\partial j(r) = D^*(r)$, where $D(\partial j) = (-\infty, \theta_s)$. To come to this end we show first the inclusion $D^* \subset \partial j$, i.e.,

$$j(r) - j(y) \leq D^*(r)(r - y), \ \forall y \in \mathbf{R} \text{ and } r \in D(D^*).$$

Indeed, for $r < \theta_s$ and $y < \theta_s$ we obtain the equality, $D^*(r)$ being exactly the derivative of $j(r)$, while for $y \geq \theta_s$, the inequality is also verified, because we get $-\infty < q$, where q is a negative number. Moreover, the function D^* is maximal monotone because it is obvious that $R(I+D^*) = \mathbf{R}$ i.e., the equation $r + D^*(r) = g \in \mathbf{R}$ has a (unique) solution in $(-\infty, \theta_s)$.

It remains to show that j is l.s.c. For $r < \theta_s$ the function j is continuous, so it suffices to show that for a sequence $\{r_n\}_{n\geq 1} \subset L^2(\Omega)$, $r_n < \theta_s$, such that $r_n \to \theta_s$ it follows $\liminf\limits_{n\to\infty} \int_0^{r_n} D^*(\xi)d\xi \geq j(\theta_s)$. We set

$$j(r_n) = \int_0^{r_n} D^*(\xi)d\xi = \int_0^{\theta_s} \chi_n(\xi)D^*(\xi)d\xi$$

where

$$\chi_n(\xi) = \begin{cases} 1 & \text{if } \ 0 \leq \xi \leq r_n, \\ 0 & \text{if } \ r_n < \xi \leq \theta_s. \end{cases}$$

We have $\chi_n(\xi)D^*(\xi) \geq 0$ and $\chi_n(\xi)D^*(\xi) \to D^*(\xi)$ a.e. on $(0, \theta_s)$, as $n \to \infty$. Consequently, using Fatou's lemma we have

$$\liminf_{n\to\infty} j(r_n) = \liminf_{n\to\infty} \int_0^{\theta_s} \chi_n(\xi)D^*(\xi)d\xi \geq \int_0^{\theta_s} D^*(\xi)d\xi = j(\theta_s).$$

According to hypothesis (iii), the range of D^* is $R(D^*) = (-\infty, \infty)$, so that using Proposition 5.15 in Appendix we conclude that $\lim\limits_{|r|\to\infty} \dfrac{j(r)}{|r|} = \infty.$

Because the hypotheses of Proposition 2.1 are satisfied, we are allowed to apply it by setting $\eta(\theta) = D^*(\theta)$ and $A_\eta\theta = A_D\theta$ which is a single-valued operator. It follows that $A_D = \partial\varphi$, where j is defined by (2.14) and φ by (2.15).

Since we proved that A_D is the subdifferential of a proper, convex, l.s.c function it follows by Rockafeller's theorem (see Theorem 5.13) that A_D is maximal monotone on $H^{-1}(\Omega)$ i.e., it is m-accretive. ∎

Denote now

$$M_{\theta_s} := \{\theta \in H^{-1}(\Omega); \ \theta \leq \theta_s \text{ a.e. in } \Omega\} \tag{2.18}$$

and

$$M_j := \{\theta \in L^2(\Omega); \ j(\theta) \in L^1(\Omega)\}. \tag{2.19}$$

In (2.18) the inequality $\theta \leq \theta_s$ is regarded in the sense of distributions, i.e.,

$$(\theta - \theta_s)(\psi) \leq 0,\ \forall \psi > 0,\ \psi \in \mathcal{D}(\Omega).$$

We note that

$$\begin{aligned} D(j) &= \{r \in \mathbf{R};\ j(\theta) < \infty\}, \\ D(\partial j) &= D(D^*) = (-\infty, \theta_s), \\ D(\varphi) &= \{\theta \in L^2(\Omega);\ j(\theta) \in L^1(\Omega)\}, \end{aligned}$$

and we notice that

$$D(A_D) = D(\partial\varphi) \subset D(\varphi) = M_j.$$

Corollary 2.3. *Assume* (2.10)-(2.12). *Then we have*

$$\overline{D(A_D)} = \overline{D(\partial\varphi)} = \overline{D(\varphi)} = \overline{M_j} = M_{\theta_s} \tag{2.20}$$

the closure being in $H^{-1}(\Omega)$.

Proof. The second and the third equalities from the left follow by Corollary 5.14 and Proposition 5.22 in Appendix.

In order to prove that $\overline{M_j} = M_{\theta_s}$ it suffices to show only the inclusion $M_{\theta_s} \subset \overline{M_j}$, the other one being obvious. We fix $\theta \in M_{\theta_s}$ and consider

$$\theta_\varepsilon = (1 + \varepsilon\partial j)^{-1}\theta.$$

Clearly $\theta_\varepsilon \in D(\partial j)$, i.e., $\theta_\varepsilon < \theta_s$ and $\theta_\varepsilon \to \theta$ a.e. as $\varepsilon \to 0$. Moreover,

$$|\theta_\varepsilon| \leq \left|(1+\varepsilon\partial j)^{-1}\theta - (1+\varepsilon\partial j)^{-1}0\right| + \left|(1+\varepsilon\partial j)^{-1}0\right| \leq |\theta| + C_0.$$

Hence, using Lebesgue's theorem (Theorem 2.10 in Appendix) we deduce that $\theta_\varepsilon \to \theta$ strongly in $L^2(\Omega)$ and therefore $\theta \in M_j$, as claimed. ∎

4.3 Weakly nonlinear conductivity. Homogeneous Dirichlet boundary conditions

Let us consider the model of infiltration with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial\theta}{\partial t} - \Delta D^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} &= f \quad \text{in } Q, \\ \theta(x,0) &= \theta_0 \quad \text{in } \Omega, \\ \theta(x,t) &= 0 \quad \text{on } \Sigma, \end{aligned} \tag{3.1}$$

where the water diffusivity function D satisfies ($\mathrm{i_D}$)-($\mathrm{iii_D}$), implying for D^* the properties (i)-(iii). For the beginning we shall study the model with a weaker nonlinear behaviour of the function K so that we consider the hypotheses (1.5) and ($\mathrm{i_K}$).

Functional framework

As often happens to highly nonlinear PDEs, problem (3.1) has not a classical solution. However, we shall see that it is well-posed in a class of generalized solutions defined below. Roughly speaking, a generalized solution is a function that satisfies the first equation in (3.1) in the sense of distributions on Q and the boundary conditions in a generalized sense, actually in the sense of the trace theory (see [84]). To come to this end we have to introduce a functional framework in which we define the solution.

We consider the space $V = H_0^1(\Omega)$ endowed with the usual Hilbertian norm

$$\|\psi\|_V = \left(\int_\Omega |\nabla\psi(x)|^2\, dx\right)^{1/2}, \tag{3.2}$$

and its dual $V^{'} = H^{-1}(\Omega)$, on which it is convenient to introduce the scalar product

$$\left\langle \theta, \overline{\theta}\right\rangle_{V'} = \theta(\psi),\ \forall \theta, \overline{\theta} \in V', \tag{3.3}$$

where $\psi \in V$ satisfies the boundary value problem

$$-\Delta\psi = \overline{\theta},\ \psi\,|_\Gamma = 0. \tag{3.4}$$

(This is enhanced by the fact that $-\Delta$ is the canonical isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$).

Here $\theta(\psi)$ represents the value of $\theta \in V^{'}$ at $\psi \in V$, or the pairing between $V^{'}$ and V and rigorously it should be denoted by $\langle \theta, \psi\rangle_{V',V}$. If any confusion is avoided sometimes we shall simply write it as $(\theta, \psi) = \int_\Omega \theta\psi dx$, since the pairing reduces to the scalar product in $L^2(\Omega)$ if $\theta \in L^2(\Omega)$.

As we specified before, we shall omit for convenience the function arguments in the integrands.

The norm of $\overline{\theta} \in V^{'}$ satisfies

$$\left\|\overline{\theta}\right\|_{V'} = \|\psi\|_V \tag{3.5}$$

that may be obtained by multiplying the first equation in (3.4) by $\overline{\theta}$ in V',

$$\begin{aligned}
\left\|\overline{\theta}\right\|_{V'}^2 &= \left\langle \overline{\theta}, \overline{\theta}\right\rangle_{V'} = \int_\Omega \overline{\theta}\psi\, dx = -\int_\Omega \psi\Delta\,\psi dx \\
&= -\int_\Gamma \frac{\partial\psi}{\partial\nu}\,\psi d\sigma + \int_\Omega |\nabla\psi|^2\, dx = \int_\Omega |\nabla\psi|^2\, dx = \|\psi\|_V^2\,.
\end{aligned}$$

Here we used Green's formula (see Theorem 6.1 in Appendix).

Further we introduce the operator

$$A : D(A) \subset V^{'} \to V^{'}$$

defined by

$$\langle A\theta, \psi \rangle_{V',V} = \int_\Omega \nabla D^*(\theta) \cdot \nabla \psi \, dx - \int_\Omega K(\theta) \frac{\partial \psi}{\partial x_3} \, dx, \ \forall \psi \in V, \tag{3.6}$$

where the domain is

$$D(A) = \{\theta \in L^2(\Omega); D^*(\theta) \in V\}.$$

Roughly speaking this means that

$$A = -\Delta D^* + \frac{\partial K}{\partial x_3}$$

in the sense of distributions.

Remark 3.1. By the property (i), the claim $D^*(\theta) \in V$ implies $\theta \in V$. Denote $\eta = D^*(\theta)$ a.e. on Ω. Since $D^* : (-\infty, \theta_s) \to \mathbf{R}$ is a continuous monotonically increasing function we can define its inverse $\eta \to (D^*)^{-1}(\eta)$ which is continuous and monotonically increasing. Writing (ii) in the form

$$(\eta - \overline{\eta}) \left((D^*)^{-1}(\eta) - (D^*)^{-1}(\overline{\eta}) \right) \geq \rho \left((D^*)^{-1}(\eta) - (D^*)^{-1}(\overline{\eta}) \right)^2 \tag{3.7}$$

it follows that $(D^*)^{-1}$ is Lipschitz

$$\left| (D^*)^{-1}(\eta) - (D^*)^{-1}(\overline{\eta}) \right| \leq \frac{1}{\rho} \left| \eta - \overline{\eta} \right|, \tag{3.8}$$

which means that $(D^*)^{-1}(\eta) = \theta \in H^1(\Omega)$. Next, the injectivity of D^* and $D^*(0) = 0$ imply that $\theta \in H_0^1(\Omega)$.

It is obvious also that ($\mathrm{i_K}$) implies $K(\theta) \in L^2(\Omega)$. Moreover, if $\theta \in V$ it follows by ($\mathrm{i_K}$) and (1.5) that $K(\theta) \in V$.

Definition 3.2. If

$$\theta_0 \in L^2(\Omega), \ \theta_0 < \theta_s, \ \text{a.e. } x \in \Omega \text{ and } f \in L^2(0, T; V')$$

we mean by *solution* to (3.1) a function $\theta \in C([0, T]; L^2(\Omega))$, such that $\frac{d\theta}{dt} \in L^2(0, T; V')$, $D^*(\theta) \in L^2(0, T; V)$, $\theta(x, 0) = \theta_0$ in Ω, and

$$\begin{aligned} &\left\langle \frac{d\theta}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega \left(\nabla D^*(\theta(t)) \cdot \nabla \psi - K(\theta(t)) \frac{\partial \psi}{\partial x_3} \right) dx \\ &= \langle f(t), \psi \rangle_{V',V}, \quad \text{a.e. } t \in (0, T), \ \forall \psi \in V. \end{aligned} \tag{3.9}$$

By $\frac{d\theta}{dt}$ we mean the strong derivative of $\theta(t)$ in V', i.e.,

$$\frac{d\theta}{dt}(t) = \lim_{\varepsilon \to 0} \frac{\theta(t + \varepsilon) - \theta(t)}{\varepsilon} \text{ in } V'.$$

Equivalently, this is the derivative in the sense of the V'-valued distributions on $(0,T)$ and very often we shall simply write it as $\dfrac{\partial\theta}{\partial t}$.

Taking into account the convention made before, we can write (3.9) in the form

$$\int_\Omega\left(\frac{\partial\theta}{\partial t}(t)\psi+\nabla D^*(\theta(t))\cdot\nabla\psi-K(\theta(t))\,\frac{\partial\psi}{\partial x_3}\right)dx=\int_\Omega f(t)\psi dx, \quad (3.10)$$
$$\text{a.e. } t\in(0,T),\ \forall\psi\in V,$$

or, equivalently

$$\int_Q\left(\frac{\partial\theta}{\partial t}\phi+\nabla D^*(\theta)\cdot\nabla\phi-K(\theta)\frac{\partial\phi}{\partial x_3}\right)dxdt=\int_Q f\phi dxdt, \quad (3.11)$$
$$\forall\phi\in L^2(0,T;V),$$

as it will be proved below in Lemma 3.4.

The latter, rigorously written, is in fact

$$\int_0^T\left\langle\frac{d\theta}{dt}(t),\phi(t)\right\rangle_{V',V}dt+\int_0^T\int_\Omega\left(\nabla D^*(\theta(t))\cdot\nabla\phi(t)-K(\theta(t))\frac{\partial\phi(t)}{\partial x_3}\right)dxdt$$
$$=\int_0^T\langle f(t),\phi(t)\rangle_{V',V}\,dt,\ \forall\phi\in L^2(0,T;V).$$

Moreover, let $\theta\in L^2(0,T,V)$ and $\phi\in L^2(0,T,V)$ with $\dfrac{d\theta}{dt}\in L^2(0,T,V')$ and $\dfrac{d\phi}{dt}\in L^2(0,T,V')$, which imply that $\theta\in C([0,T];L^2(\Omega))$ and $\phi\in C([0,T];L2(\Omega))$. We have that

$$\int_0^T\left\langle\frac{d\theta}{dt}(t),\phi(t)\right\rangle_{V',V}dt=\langle\theta(T),\phi(T)\rangle_{V',V}-\langle\theta(0),\phi(0)\rangle_{V',V}$$
$$-\int_0^T\left\langle\theta(t),\frac{d\phi}{dt}(t)\right\rangle_{V',V}dt,$$

and therefore we can write that

$$\begin{aligned}&\int_\Omega\theta(x,T)\phi(x,T)dx\\&-\int_Q\theta\frac{d\phi}{dt}\,dxdt+\int_Q\left(\nabla D^*(\theta)\cdot\nabla\phi-K(\theta)\frac{\partial\phi}{\partial x_3}\right)dxdt\\&=\int_\Omega\theta_0(x)\phi(x,0)dx+\int_Q f\phi dxdt,\end{aligned} \quad (3.12)$$
$$\forall\phi\in L^2(0,T;V)\text{ with }\frac{d\phi}{dt}\in L^2(0,T;V').$$

This relation is obvious if $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$ are in $L^2(Q)$ and follows by approximation if they are as before (see also [83]).

We have emphasized all these equivalent forms of the definition (3.9), because, when further required, the most appropriate one will be used.

Now, it is easy to check that a classical solution to (3.1), if it exists, is a generalized solution. Conversely, a generalized solution turns out to be a solution in the sense of distributions of the equation and satisfies the boundary conditions in the sense of trace. Let θ be a generalized solution in the sense of Definition 3.2. In (3.10) we take $\psi \in \mathcal{D}(\Omega)$ and we write

$$\begin{aligned}&\int_\Omega \frac{\partial\theta}{\partial t}(t)\psi dx - \int_\Omega (K(\theta(t))i_3 - \nabla D^*(\theta(t))) \cdot \nabla\psi dx \\ &= \int_\Omega \frac{\partial\theta}{\partial t}(t)\psi dx - \int_\Omega \nabla \cdot [\psi\, (K(\theta(t))i_3 - \nabla D^*(\theta(t)))]\, dx \\ &+ \int_\Omega \psi\nabla \cdot (K(\theta(t))i_3 - \nabla D^*(\theta(t)))\, dx,\ \text{a.e. } t \in (0,T).\end{aligned}$$

We use then the Gauss-Ostrogradsky formula (see Theorem 6.2 in Appendix)

$$\int_\Omega \nabla \cdot (\psi(K(\theta(t))i_3 - \nabla D^*(\theta(t))))\, dx = \int_{\partial\Omega} \psi\, [K(\theta(t))i_3 - \nabla D^*(\theta(t))] \cdot \nu d\sigma$$

and taking into account that ψ is with compact support in Ω, it follows that the previous integral vanishes and it remains

$$\int_\Omega \left(\frac{\partial\theta}{\partial t}(t) - \Delta D^*(\theta(t)) + \frac{\partial K(\theta(t))}{\partial x_3} - f(t)\right)\psi dx = 0,\ \text{a.e. } t \in (0,T),$$

for each $\psi \in \mathcal{D}(\Omega)$. This relation shows that

$$\frac{d\theta}{dt}(t) - \Delta D^*(\theta(t)) + \frac{\partial K(\theta(t))}{\partial x_3} - f(t) = 0 \text{ in } \mathcal{D}'(\Omega) \text{ a.e. } t \in (0,T), \quad (3.13)$$

expressing that (3.1) is satisfied in the sense of distributions. The boundary condition is satisfied in the sense of trace and this is observed immediately, due to the choice of the domain $D(A)$ and to the Remark 3.1.

Remark 3.3. It can be readily seen that Definition 3.2 implies that the solution to (3.1) necessarily satisfies $\theta \in V$ and

$$\theta(x,t) < \theta_s,\ \text{a.e. } (x,t) \in Q. \quad (3.14)$$

Indeed, $D^*(\theta) \in L^2(0,T;V)$ implies by Remark 3.1 that $\theta \in L^2(0,T;V)$. Then, $D^*(\theta) \in L^2(0,T;V)$ implies that $\int_\Omega |D^*(\theta(x,t)|^2\, dx < \infty$, a.e. $t \in (0,T)$. Since a summable function is almost everywhere finite, it follows that the Lebesgue measure of the set of points at which θ is exactly θ_s is zero,

$$\text{meas}\{(x,t) \in Q;\ \theta(x,t) = \theta_s\} = 0,$$

because $D^*(\theta_s) = +\infty$.

Now we prove an assertion made before, i.e.,

Lemma 3.4. *The equations* (3.10) *and* (3.11) *are equivalent.*

Proof. Assume (3.10), multiply by $\widetilde{\psi} \in L^2(0,T)$ and integrate over $(0,T)$. We obtain

$$\int_Q \left(\frac{\partial \theta}{\partial t}\, \widetilde{\phi} + \nabla D^*(\theta) \cdot \nabla \widetilde{\phi} - K(\theta) \frac{\partial \widetilde{\phi}}{\partial x_3} \right) dx\, dt = \int_Q f \widetilde{\phi}\, dx\, dt,\ \ \widetilde{\phi} \in L^2(0,T;V),$$

where $\widetilde{\phi}(x,t) = \psi(x)\widetilde{\psi}(t)$. It follows that the previous equality is true also for any $\widehat{\phi}$ which is an algebraic combination $\widehat{\phi}(x,t) = \sum_{i=1}^{n} \psi_i(x)\widetilde{\psi}_i(t)$, where $\psi_i \in V$ and $\widetilde{\psi}_i \in L^2(0,T)$. Next it is nothing else to do than noticing that the set $\left\{ \sum_{i=1}^{n} \psi_i(x)\widetilde{\psi}_i(t) \right\}_{n \geq 1}$ is dense in $L^2(0,T;V)$.

Indeed, for any $\phi \in L^2(0,T;H^1(\Omega))$ there exists a sequence $\{\phi_n\}_{n\geq 1}$, $\phi_n \in C(\overline{\Omega} \times [0,T])$ such that $\phi(x,t) = \lim_{n\to\infty} \phi_n(x,t)$. By the Weierstrass theorem, any continuous function can be approximated by polynomial functions

$$\phi_n(x,t) = \sum_{k=1}^{m} \sum_{l=1}^{p} \psi_{n,k}(x)\widetilde{\psi}_{n,l}(t),$$

where $\psi_{n,k}$ is a polynomial of k degree in x and $\widetilde{\psi}_{n,l}$ is a polynomial of l degree in t.

Assume now (3.11). If ϕ is arbitrary in $L^2(0,T;V)$, we take it of the form $\phi(x,t) = \psi(x)\widetilde{\psi}(t)$ with $\psi \in V$ and $\widetilde{\psi} \in L^2(0,T)$ arbitrary. We get

$$\int_0^T \widetilde{\psi}(t) \int_\Omega \left(\frac{\partial \theta}{\partial t}\, \psi + \nabla D^*(\theta) \cdot \nabla \psi - K(\theta) \frac{\partial \psi}{\partial x_3} \right) dx\, dt = \int_0^T \widetilde{\psi}(t) \int_\Omega f\psi\, dx\, dt.$$

Since $\widetilde{\psi}$ is arbitrary in $L^2(0,T)$ we obtain (3.10) as claimed. ∎

Eventually, we consider the Cauchy problem

$$\frac{d\theta}{dt} + A\theta = f,\ \text{ a.e. } t \in (0,T) \tag{3.15}$$

$$\theta(0) = \theta_0. \tag{3.16}$$

Since (3.15) is an equality in V' we can write it as

$$\left\langle \frac{d\theta}{dt}(t) + A\theta(t), \psi \right\rangle_{V',V} = \langle f(t), \psi \rangle_{V',V},\ \text{ a.e. } t \in (0,T),\ \forall \psi \in V$$

and it is obvious that the latter is in fact (3.9), thus we have

Lemma 3.5. *Let θ be a strong solution to* (3.15)-(3.16). *Then θ is a generalized solution to the boundary value problem* (3.1).

The assertion is obvious and so we replace the study of the existence in problem (3.1) by the investigation of the existence of a strong solution to (3.15)-(3.16).

Existence and uniqueness of the solution

To begin we shall prove

Proposition 3.6. *Assume that the hypotheses* (i)-(iii) *and* ($\mathrm{i_K}$) *are satisfied. Then the operator A is quasi m-accretive on $V^{'}$.*

Proof. We have to show first that A is quasi-monotone. Let $\lambda > 0$ and we calculate

$$\begin{aligned}&\left\langle (\lambda I + A)\theta - (\lambda I + A)\overline{\theta}, \theta - \overline{\theta}\right\rangle_{V'}\\ &\geq \lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \int_\Omega \left(\nabla(D^*(\theta) - D^*(\overline{\theta}))\cdot\nabla\psi - (K(\theta) - K(\overline{\theta}))\frac{\partial\psi}{\partial x_3}\right)dx\end{aligned}$$

where

$$-\Delta\psi = \theta - \overline{\theta}, \psi\,|_\Gamma\, = 0. \tag{3.17}$$

Due to Green's formula, (i), ($\mathrm{i_K}$) and Cauchy-Schwarz inequality (see Theorem 1.25 in Appendix) we have

$$\begin{aligned}&\left\langle (\lambda I + A)\theta - (\lambda I + A)\overline{\theta}, \theta - \overline{\theta}\right\rangle_{V'}\\ &\geq \lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \rho\left\|\theta - \overline{\theta}\right\|^2 - M\left\|\theta - \overline{\theta}\right\|\left\|\frac{\partial\psi}{\partial x_3}\right\|\\ &\geq \lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \rho\left\|\theta - \overline{\theta}\right\|^2 - M\left\|\theta - \overline{\theta}\right\|\left\|\theta - \overline{\theta}\right\|_{V'},\end{aligned}$$

so finally we get

$$\begin{aligned}&\left\langle (\lambda I + A)\theta - (\lambda I + A)\overline{\theta}, \theta - \overline{\theta}\right\rangle_{V'}\\ &\geq \left(\lambda - \frac{M^2}{2\rho}\right)\left\|\theta - \overline{\theta}\right\|_{V'}^2 + \frac{\rho}{2}\left\|\theta - \overline{\theta}\right\|^2 \geq 0,\end{aligned} \tag{3.18}$$

for λ large enough, $\lambda \geq \dfrac{M^2}{2\rho}$. Here we took into account that by (3.17) we have

$$\left\|\frac{\partial\psi}{\partial x_3}\right\| \leq \|\psi\|_V = \left\|\theta - \overline{\theta}\right\|_{V'}.$$

Next we must show that

$$R(\lambda I + A) = V^{'}, \tag{3.19}$$

meaning that for each $g \in V^{'}$ there exists $\theta \in D(A)$ such that

$$\lambda\theta + A\theta = g \tag{3.20}$$

or

$$(\lambda I + A_D)\theta = g - \frac{\partial K(\theta)}{\partial x_3}, \tag{3.21}$$

where $A_D\theta = -\Delta D^*(\theta)$, i.e.,

$$\langle A_D\theta, \psi\rangle_{V',V} = \int_\Omega \nabla D^*(\theta) \cdot \nabla\psi \, dx, \ \forall \psi \in V. \tag{3.22}$$

In fact A_D is the operator A without the contribution of K. The operator A_D is monotone

$$\left\langle A_D\theta - A_D\overline{\theta}, \theta - \overline{\theta}\right\rangle_{V'} \geq \rho \left\|\theta - \overline{\theta}\right\|^2 \geq \rho \left\|\theta - \overline{\theta}\right\|_{V'}^2$$

and actually by Corollary 2.2 it is m-accretive on V'.

Hence, $\lambda I + A_D$ is m-accretive on V', so that for each $\eta \in V'$ there exists a unique solution $\theta \in D(A_D)$ to

$$(\lambda I + A_D)\theta = \eta. \tag{3.23}$$

Setting $\eta = g - \dfrac{\partial K(\theta)}{\partial x_3} \in V'$ it follows that equation (3.21) has a solution $\theta \in D(A)$

$$\theta = (\lambda I + A_D)^{-1}\left(g - \frac{\partial K(\theta)}{\partial x_3}\right). \tag{3.24}$$

Now we denote

$$G(\theta) = (\lambda I + A_D)^{-1}\left(g - \frac{\partial K(\theta)}{\partial x_3}\right)$$

and we shall prove that equation (3.24) which may be rewritten as

$$G(\theta) = \theta, \tag{3.25}$$

has a solution, by checking that $G(\theta)$ is a contraction on $L^2(\Omega)$.

To this purpose we shall deduce first an estimate. We write two equations (3.23) for θ and $\overline{\theta}$, subtract them and multiply scalarly in $V^{'}$ the equation

$$\lambda(\theta - \overline{\theta}) - \Delta(D^*(\theta) - D^*(\overline{\theta})) = \eta - \overline{\eta}$$

by $\theta - \overline{\theta}$. We obtain

$$\lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \int_\Omega \nabla(D^*(\theta) - D^*(\overline{\theta})) \cdot \nabla\psi dx = \int_\Omega (\eta - \overline{\eta})\psi dx,$$

where ψ above is the solution to

$$-\Delta\psi = \theta - \overline{\theta}, \psi\,|_{\Gamma} = 0.$$

Further we have by (i) that

$$\begin{aligned}\lambda \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \rho \left\|\theta - \overline{\theta}\right\|^2 &\leq \|\eta - \overline{\eta}\|_{V'} \left\|\theta - \overline{\theta}\right\|_{V'} \\ &\leq \frac{\lambda}{2} \left\|\theta - \overline{\theta}\right\|_{V'}^2 + \frac{1}{2\lambda} \|\eta - \overline{\eta}\|_{V'}^2 .\end{aligned}$$

Hence we get

$$\left\|\theta - \overline{\theta}\right\| \leq \frac{1}{\sqrt{2\lambda\rho}} \|\eta - \overline{\eta}\|_{V'} , \tag{3.26}$$

which expresses that $I_\lambda = (\lambda I + A_D)^{-1}$ is Lipschitz from $V^{'}$ to $L^2(\Omega)$. Resuming now (3.21) and (3.24) with $\eta = g - \dfrac{\partial K(\theta)}{\partial x_3}$ and $\overline{\eta} = g - \dfrac{\partial K(\overline{\theta})}{\partial x_3}$ we get

$$\begin{aligned}&\left\|G(\theta) - G(\overline{\theta})\right\| = \left\|I_\lambda \left(g - \frac{\partial K(\theta)}{\partial x_3}\right) - I_\lambda \left(g - \frac{\partial K(\overline{\theta})}{\partial x_3}\right)\right\| \\ &\leq \frac{1}{\sqrt{2\lambda\rho}} \left\|\frac{\partial K(\theta)}{\partial x_3} - \frac{\partial K(\overline{\theta})}{\partial x_3}\right\|_{V'} \leq \frac{1}{\sqrt{2\lambda\rho}} \left\|K(\theta) - K(\overline{\theta})\right\| \leq \frac{M}{\sqrt{2\lambda\rho}} \left\|\theta - \overline{\theta}\right\|,\end{aligned}$$

that shows that for some $\lambda \left(\lambda > \dfrac{M^2}{2\rho}\right)$, $G(\theta)$ is a contraction on $L^2(\Omega)$. Hence G has a fixed point meaning that equation (3.24) has a solution $\theta \in L^2(\Omega)$. Correspondingly, equation (3.20) has a solution $\theta \in D(A)$ as claimed. ∎

Remark 3.7. In the above inequalities we have used the continuity of the linear operator $\dfrac{\partial}{\partial x_3}$ from $L^2(\Omega)$ to $V^{'}$. Indeed, for $\eta \in L^2(\Omega)$, we have

$$\left\|\frac{\partial \eta}{\partial x_3}\right\|_{V'} = \sup\left\{\left|\frac{\partial \eta}{\partial x_3}(\psi)\right|;\ \psi \in V,\ \ \|\psi\| \leq 1\right\}$$

and

$$\left|\int_\Omega \frac{\partial \eta}{\partial x_3}\,\psi\,dx\right| = \left|-\int_\Omega \eta \frac{\partial \psi}{\partial x_3}\,dx\right| \leq \|\eta\|\,\|\psi\|_V\,,\ \forall \psi \in V.$$

Finally we get

$$\left\|\frac{\partial \eta}{\partial x_3}\right\|_{V'} \leq \|\eta\| \tag{3.27}$$

that implies the stated assertion, which was particularized many times to $\eta = K(\theta)$.

Now we are ready to formulate the main existence result for the problem (3.15)-(3.16).

Theorem 3.8. *Assume* (i)-(iii) *and* ($\mathrm{i_K}$). *Let*

$$f \in W^{1,1}(0,T;V') \tag{3.28}$$

and

$$\theta_0 \in D(A). \tag{3.29}$$

Then there exists a unique strong solution $\theta \in C([0,T],V')$ *to problem* (3.15)-(3.16) *such that*

$$\theta \in W^{1,\infty}(0,T;V') \cap L^\infty(0,T;D(A)), \tag{3.30}$$

$$D^*(\theta) \in L^\infty(0,T;V), \tag{3.31}$$

$$\theta \in L^\infty(0,T;V). \tag{3.32}$$

The solution satisfies the estimate

$$\begin{aligned}\int_\Omega j(\theta(x,t))dx + \int_0^t \left\| \frac{d\theta}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|D^*(\theta(\tau))\|_V^2 \, d\tau \\ \leq \beta_0(t) \left(\int_\Omega j(\theta_0(x))dx + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \right),\end{aligned} \tag{3.33}$$

where $\beta_0(t) = 4\exp\left(\dfrac{2M^2 t}{\rho}\right)$, $\forall t \in [0,T]$, $T > 0$ *and* j *is defined by* (2.14).

Next, considering θ_λ *and* θ_μ *two solutions to problem* (3.15)-(3.16) *corresponding to the free terms and initial data* $f = f_\lambda$, $\theta_0 = \theta_\lambda^0$ *and, respectively,* $f = f_\mu$, $\theta_0 = \theta_\mu^0$, *we have the estimate*

$$\begin{aligned}&\|\theta_\lambda(t) - \theta_\mu(t)\|_{V'}^2 + \int_0^t \|\theta_\lambda(\tau) - \theta_\mu(\tau)\|^2 \, d\tau \\ &\leq \alpha_0(t) \left(\left\|\theta_\lambda^0 - \theta_\mu^0\right\|_{V'}^2 + \int_0^T \|f_\lambda(\tau) - f_\mu(\tau)\|_{V'}^2 \, d\tau \right),\end{aligned} \tag{3.34}$$

where $\alpha_0(t) = \dfrac{1}{\min\{1,\rho\}} \exp\left(\frac{M^2}{\rho} + 1\right) t$, $\forall t \in [0,T]$, $T > 0$.

Proof. Since A is quasi m-accretive in V', the proof of the existence of the solution with the properties (3.30)-(3.32) follows from Theorem 3.6 and Corollary 3.7 in Sect. 3.3. Moreover, this solution is upper bounded, because $\theta(t) \in D(A)$ implies, according to Remark 3.3, that

$$\theta < \theta_s \text{ a.e. } (x,t) \in Q. \tag{3.35}$$

In order to derive (3.34) we consider two solutions θ_λ and θ_μ to problem (3.15) with $f = f_\lambda$, $\theta_0 = \theta_\lambda^0$ and, respectively, $f = f_\mu$, $\theta_0 = \theta_\mu^0$. Then we multiply the equation

$$\frac{d}{dt}(\theta_\lambda - \theta_\mu) + A\theta_\lambda - A\theta_\mu = f_\lambda - f_\mu$$

by $\theta_\lambda - \theta_\mu$, scalarly in V' and integrate over $(0,t)$ with $t \in (0,T)$. We have

$$\begin{aligned} &\frac{1}{2}\int_0^t \frac{d}{d\tau}\|\theta_\lambda(\tau) - \theta_\mu(\tau)\|_{V'}^2\, d\tau \\ &+ \int_0^t \langle A\theta_\lambda(\tau) - A\theta_\mu(\tau), \theta_\lambda(\tau) - \theta_\mu(\tau)\rangle_{V'}\, d\tau \\ &= \int_0^t \langle f_\lambda(\tau) - f_\mu(\tau), \theta_\lambda(\tau) - \theta_\mu(\tau)\rangle_{V'}\, d\tau. \end{aligned} \tag{3.36}$$

We take into account that

$$\begin{aligned} &\langle A\theta_\lambda(\tau) - A\theta_\mu(\tau), \theta_\lambda(\tau) - \theta_\mu(\tau)\rangle_{V'} \\ &\geq \frac{\rho}{2}\|\theta_\lambda(\tau) - \theta_\mu(\tau)\|^2 - \frac{M^2}{2\rho}\|\theta_\lambda(\tau) - \theta_\mu(\tau)\|_{V'}^2 \end{aligned}$$

and obtain that

$$\begin{aligned} &\frac{1}{2}\|\theta_\lambda(t) - \theta_\mu(t)\|_{V'}^2 + \frac{\rho}{2}\int_0^t \|\theta_\lambda(\tau) - \theta_\mu(\tau)\|^2\, d\tau \\ &\leq \frac{1}{2}\left\|\theta_\lambda^0 - \theta_\mu^0\right\|_{V'}^2 + \frac{M^2}{2\rho}\int_0^t \|\theta_\lambda(\tau) - \theta_\mu(\tau)\|_{V'}^2\, d\tau \\ &+ \int_0^t \|f_\lambda(\tau) - f_\mu(\tau)\|_{V'}\|\theta_\lambda(\tau) - \theta_\mu(\tau)\|_{V'}\, d\tau \\ &\leq \frac{1}{2}\left\|\theta_\lambda^0 - \theta_\mu^0\right\|_{V'}^2 + \left(\frac{M^2}{2\rho} + \frac{1}{2}\right)\int_0^t \|\theta_\lambda(\tau) - \theta_\mu(\tau)\|_{V'}^2\, d\tau \\ &+ \frac{1}{2}\int_0^t \|f_\lambda(\tau) - f_\mu(\tau)\|_{V'}^2\, d\tau. \end{aligned}$$

Finally we get

$$\begin{aligned} &\|\theta_\lambda(t) - \theta_\mu(t)\|_{V'}^2 + \rho\int_0^t \|\theta_\lambda(\tau) - \theta_\mu(\tau)\|^2\, d\tau \\ &\leq \left\|\theta_\lambda^0 - \theta_\mu^0\right\|_{V'}^2 + \int_0^T \|f_\lambda(\tau) - f_\mu(\tau)\|_{V'}^2\, d\tau \\ &+ \left(\frac{M^2}{\rho} + 1\right)\int_0^t \|\theta_\lambda(\tau) - \theta_\mu(\tau)\|_{V'}^2\, d\tau. \end{aligned} \tag{3.37}$$

From here we deduce that

$$\begin{aligned} \|\theta_\lambda(t) - \theta_\mu(t)\|_{V'}^2 \leq &\left\|\theta_\lambda^0 - \theta_\mu^0\right\|_{V'}^2 + \int_0^T \|f_\lambda(\tau) - f_\mu(\tau)\|_{V'}^2\, d\tau \\ &+ \left(\frac{M^2}{\rho} + 1\right)\int_0^t \|\theta_\lambda(\tau) - \theta_\mu(\tau)\|_{V'}^2\, d\tau \end{aligned}$$

and we apply Gronwall's lemma for $g(t) = \|\theta_\lambda(t) - \theta_\mu(t)\|_{V'}^2$ (see Lemma 3.4 in Sect. 3.3). This yields

$$\|\theta_\lambda(t) - \theta_\mu(t)\|_{V'}^2 \leq \left(\left\|\theta_\lambda^0 - \theta_\mu^0\right\|_{V'}^2 + \int_0^T \|f_\lambda(\tau) - f_\mu(\tau)\|_{V'}^2 \, d\tau \right) e^{(M^2/\rho+1)t}.$$

Using this result in (3.37), we get after some simple computations (3.34) as claimed.

To obtain (3.33) we multiply first equation (3.15) by $\psi = D^*(\theta)$ and integrate over $\Omega \times (0,t)$, or take into account (3.6). We have

$$\begin{aligned} &\int_\Omega \int_0^t \frac{d\theta}{d\tau} D^*(\theta) dx\, d\tau + \int_0^t \int_\Omega |\nabla D^*(\theta)|^2 \, dx\, d\tau \\ &= \int_0^t \int_\Omega K(\theta) \frac{\partial D^*(\theta)}{\partial x_3} \, dx\, d\tau + \int_0^t \int_\Omega f(x,\tau) D^*(\theta) dx\, d\tau. \end{aligned} \tag{3.38}$$

From the definition of j it follows that

$$\frac{\partial j(\theta)}{\partial t} = D^*(\theta) \frac{\partial \theta}{\partial t}$$

and we notice also that j is non-negative

$$j(\theta) \geq \frac{\rho}{2}\theta^2 \quad \text{and} \quad \int_\Omega j(\theta(t)) dx \geq \frac{\rho}{2} \|\theta(t)\|^2 .$$

Next we use the fact that K is Lipschitz and $K(0) = 0$,

$$|K(\theta)| \leq M\, |\theta|$$

and we apply the Schwarz and Hölder inequalities to obtain that

$$\begin{aligned} &\int_\Omega \int_0^t \frac{\partial j(\theta)}{\partial \tau} \, dx\, d\tau + \int_0^t \|D^*(\theta(\tau))\|_V^2 \, d\tau \\ &\leq \int_0^t M \|\theta(\tau)\| \, \|D^*(\theta(\tau))\|_V \, d\tau + \int_0^t \|f(\tau)\|_{V'} \|D^*(\theta(\tau))\|_V \, d\tau. \end{aligned}$$

This yields

$$\begin{aligned} &\int_\Omega j(\theta(t)) dx + \int_0^t \|D^*(\theta(\tau))\|_V^2 \, d\tau \\ &\leq \int_\Omega j(\theta_0) dx + \int_0^t \|f(\tau)\|_{V'}^2 \, d\tau + \frac{1}{4} \int_0^t \|D^*(\theta(\tau))\|_V^2 \, d\tau \\ &+ M^2 \int_0^t \|\theta(\tau)\|^2 \, d\tau + \frac{1}{4} \int_0^t \|D^*(\theta(\tau))\|_V^2 \, d\tau. \end{aligned}$$

It follows that

$$\frac{\rho}{2}\left\|\theta(t)\right\|^2 \leq \int_\Omega j(\theta(t))dx + \frac{1}{2}\int_0^t \left\|D^*(\theta(\tau))\right\|_V^2 d\tau \leq c_0 + M^2 \int_0^t \left\|\theta(\tau)\right\|^2 d\tau, \quad (3.39)$$

with

$$c_0 = \int_\Omega j(\theta_0)dx + \int_0^T \left\|f(\tau)\right\|_{V'}^2 d\tau < \infty.$$

Here we took into account that $\theta_0 \in D(A) = D(A_D) \subset D(j)$ and $f \in W^{1,1}(0,T;V') \subset L^2(0,T;V')$ (see Remark 3.10 in Appendix). Applying again Gronwall's lemma for $g(t) = \left\|\theta(t)\right\|^2$, we get

$$\left\|\theta(t)\right\|^2 \leq \frac{2c_0}{\rho}\, e^{2M^2 t/\rho} < \infty. \quad (3.40)$$

We denote $\gamma_0(t) = \exp\left(\dfrac{2M^2 t}{\rho}\right)$ and we obtain

$$\begin{aligned} &\int_\Omega j(\theta(x,t))dx + \frac{1}{2}\int_0^t \left\|D^*(\theta(\tau))\right\|_V^2 d\tau \quad (3.41)\\ &\leq \gamma_0(t)\left(\int_\Omega j(\theta_0(x))dx + \int_0^T \left\|f(\tau)\right\|_{V'}^2 d\tau\right). \end{aligned}$$

Then we multiply (3.15) by $\dfrac{d\theta}{dt}$ scalarly in V' and integrate over $(0,t)$. We have

$$\begin{aligned} &\int_0^t \left\|\frac{d\theta}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \int_0^t \int_\Omega \nabla D^*(\theta)\cdot\nabla\psi\, dx\, d\tau \\ &= \int_0^t \int_\Omega f(x,\tau)\psi\, dx\, d\tau + \int_0^t \int_\Omega K(\theta)\frac{\partial \psi}{\partial x_3}\, dx\, d\tau, \end{aligned}$$

where $\psi \in V$ is the solution to $-\Delta\psi = \dfrac{d\theta}{dt}(t)$, $\psi = 0$ on Γ.

But

$$\int_\Omega \nabla D^*(\theta)\cdot\nabla\psi dx = \int_\Omega D^*(\theta)(-\Delta\psi)dx = \int_\Omega D^*(\theta)\frac{d\theta}{dt}\, dx = \frac{d}{dt}\int_\Omega j(\theta)dx$$

and since $\left\|\psi\right\|_V = \left\|\dfrac{d\theta}{dt}(t)\right\|_{V'}$, by performing similar computations as before, we get

$$\begin{aligned} &\int_\Omega j(\theta(x,t))dx + \frac{1}{2}\int_0^t \left\|\frac{d\theta}{d\tau}(\tau)\right\|_{V'}^2 d\tau \\ &\leq \int_\Omega j(\theta_0(x))dx + \int_0^T \left\|f(\tau)\right\|_{V'}^2 d\tau + M^2 \int_0^t \left\|\theta(\tau)\right\|^2 d\tau. \end{aligned}$$

Via Gronwall's lemma we deduce that

$$\int_{\Omega} j(\theta(x,t))dx + \frac{1}{2}\int_0^t \left\| \frac{d\theta}{d\tau}(\tau) \right\|_{V'}^2 d\tau \tag{3.42}$$
$$\leq \gamma_0(t) \left(\int_{\Omega} j(\theta_0(x))dx + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \right).$$

Adding (3.41) and (3.42) we obtain

$$\int_{\Omega} j(\theta(x,t))dx + \int_0^t \left\| \frac{d\theta}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|D^*(\theta(\tau))\|_V^2 \, d\tau \tag{3.43}$$
$$\leq 4\gamma_0(t) \left(\int_{\Omega} j(\theta_0(x))dx + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \right)$$

and from here (3.33) as claimed. ∎

Regularity and comparison results

Remark 3.9. Recall Theorem 3.8 in Sect. 3.3. We see that if the initial datum θ_0 is not so regular, $\theta_0 \in \overline{D(A)}$ and $f \in L^1(0,T;V')$, problem (3.15)-(3.16) has still a unique mild solution $\theta \in C([0,T];V')$ that satisfies a relation of (3.26) type as in Sect. 3.3. However this solution will turn out to be a stronger solution, as we can see further.

It should be specified that in our case, due to the presence of the gravitational term, A is no longer a subdifferential of a l.s.c. function and hence Theorem 3.14 in Chap. 3 cannot be applied directly. However, we have

$$D(A) = D(A_D)$$

and we can use Corollary 2.3 to prove

Theorem 3.10. (a) *Let us assume that*

$$f \in L^2(0,T;V') \text{ and } \theta_0 \in M_j. \tag{3.44}$$

Then there exists a unique solution $\theta \in C([0,T];V')$ to problem (3.15)-(3.16) *that has the properties*

$$\theta \in W^{1,2}(0,T;V') \cap L^2(0,T;V), \tag{3.45}$$

$$D^*(\theta) \in L^2(0,T;V), \tag{3.46}$$

$$j(\theta) \in L^\infty(0,T;L^1(\Omega)). \tag{3.47}$$

(b) *Let*

$$f \in L^2(0,T;V') \text{ and } \theta_0 \in M_{\theta_s}. \tag{3.48}$$

Then there exists a unique solution θ to problem (3.15) *that satisfies*

$$\int_0^T t \left\| \frac{d\theta}{dt}(t) \right\|_{V'}^2 dt \leq \gamma_1 \|\theta_0\|_{V'}^2 + \gamma_2 \int_0^T t \|f(t)\|_{V'}^2 dt + \gamma_3$$

implying that

$$\theta \in W^{1,2}(\delta, T; V') \textit{ for every } 0 < \delta < T, \tag{3.49}$$

$$\theta(t) \in D(A) \textit{ a.e. } t \in (0,T), \tag{3.50}$$

$$\sqrt{t}\, \frac{d\theta}{dt} \in L^2(0,T;V') \textit{ and } \sqrt{t}\, D^*(\theta) \in L^2(0,T;V), \tag{3.51}$$

$$j(\theta) \in L^1(Q). \tag{3.52}$$

Proof. (a) Let $f \in L^2(0,T;V')$ and $j(\theta_0) \in L^1(\Omega)$. Hence, in virtue of density arguments (see Remark 3.10 in Appendix), there are sequences $f_n \in W^{1,1}(0,T;V')$ and $\theta_n^0 \in D(A)$ such that

$$\begin{aligned} f_n &\longrightarrow f \quad \text{strongly in } L^2(0,T;V'), \\ \theta_n^0 &\longrightarrow \theta_0 \ \text{ strongly in } V' \text{ as } n \to \infty. \end{aligned} \tag{3.53}$$

By Theorem 3.8, the Cauchy problem

$$\begin{aligned} \frac{d\theta_n}{dt} + A\theta_n = f_n,\ \text{a.e. } t \in (0,T), \\ \theta_n(0) = \theta_n^0 \end{aligned}$$

has, for each n, a unique strong solution $\theta_n \in C([0,T], V')$ such that

$$\theta_n \in W^{1,\infty}(0,T;V') \cap L^\infty(0,T;V)$$

and

$$D^*(\theta_n) \in L^\infty(0,T;V).$$

Also θ_n satisfies the estimates (3.34) and (3.33).

We write (3.34) for $\lambda = n$, $\mu = m$ and using the convergence of $\{f_n\}$ and $\{\theta_n^0\}$ we get by (3.34), for $n \geq N(\varepsilon)$ and $m \geq N(\varepsilon)$, that

$$\begin{aligned} \|\theta_n(t) - \theta_n(t)\|_{V'}^2 + \int_0^t \|\theta_n(\tau) - \theta_n(\tau)\|^2 d\tau \leq \alpha_0(t) \left\|\theta_n^0 - \theta_0 - (\theta_m^0 - \theta_0)\right\|_{V'}^2 \\ + \alpha_0(t) \int_0^T \|f_n(\tau) - f(\tau) - (f_m(\tau) - f(\tau))\|_{V'}^2 d\tau \leq 2\varepsilon\alpha_0(T)(1+T). \end{aligned}$$

From here we deduce that the sequence $\{\theta_n\}_{n\geq 1}$ is Cauchy in $C([0,T];V') \cap L^2(0,T;L^2(\Omega))$, hence

$$\theta_n \longrightarrow \theta \ \text{ strongly in } C([0,T];V'),\ \text{ as } n \to \infty, \tag{3.54}$$

$$\theta_n \longrightarrow \theta \ \text{ strongly in } L^2(0,T;L^2(\Omega)),\ \text{ as } n \to \infty. \tag{3.55}$$

From (3.33) we have on a subsequence that

$$\frac{d\theta_n}{dt} \longrightarrow \frac{d\theta}{dt} \text{ weakly in } L^2(0,T;V'), \text{ as } n \to \infty \tag{3.56}$$

and

$$D^*(\theta_n) \longrightarrow \eta \text{ weakly in } L^2(0,T;V), \text{ as } n \to \infty.$$

We are going now to prove that

$$\eta = D^*(\theta) \in L^2(0,T;V).$$

Since $\theta_n \to \theta$ in $L^2(0,T;L^2(\Omega))$, it follows that (on a subsequence again denoted by the subscript n), $\theta_n(x,t) \to \theta(x,t)$ a.e. on $\Omega \times (0,T)$. Due to Egorov's theorem, for each $\varepsilon > 0$, there exists a measurable subset $Q_\varepsilon \subset Q$ such that $\text{meas}(Q_\varepsilon) < \varepsilon$ and $\theta_n \to \theta$ uniformly on $Q \backslash Q_\varepsilon$. But the function D^* is continuous on $(-\infty, \theta_s)$ and we have $D^*(\theta_n(x,t)) \to D^*(\theta(x,t))$ on $Q \backslash Q_\varepsilon$. It follows that $D^*(\theta_n) \to D^*(\theta)$ strongly in $L^2(Q \backslash Q_\varepsilon)$, which means that $\eta = D^*(\theta)$ a.e. on Q and eventually that $D^*(\theta) \in L^2(0,T;V)$, since $\eta \in L^2(0,T;V)$.

We also observe that by (3.55) and (i_K)

$$\|K(\theta_n) - K(\theta)\| \leq M \|\theta_n - \theta\|$$

we can deduce that

$$K(\theta_n) \longrightarrow K(\theta) \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.57}$$

Since A is quasi m-accretive on V', its realization $\widetilde{A}$ on $L^2(0,T;V')$ is also quasi m-accretive (see Definition 2.11, Sect. 3.2), implying that $\widetilde{A}$ is demiclosed (see Proposition 2.12, Chap. 3), i.e.,

$$\widetilde{A}\theta_n \longrightarrow \widetilde{A}\theta \text{ weakly in } L^2(0,T;V'). \tag{3.58}$$

Now we recall the problem

$$\frac{d\theta_n}{dt} + \widetilde{A}\theta_n = f_n \text{ a.e. } t \in (0,T), \ \theta_n(0) = \theta_n^0$$

and pass to limit as $n \to \infty$. We take into account (3.56), (3.58) and (3.53) and obtain finally that

$$\frac{d\theta}{dt} + \widetilde{A}\theta = f \text{ a.e. } t \in (0,T), \ \theta(0) = \theta_0.$$

Moreover, $\theta \in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V')$ and $D^*(\theta(t)) \in V$ a.e. $t \in (0,T)$, so that we obtain the solution with the properties specified in part (a) of Theorem 3.10.

Since $\theta \in L^2(0,T;D(A))$ we have that

$$\theta < \theta_s \text{ a.e. } (x,t) \in Q.$$

From (3.33) we have that

$$\varphi(\theta_n) = \int_\Omega j(\theta_n)dx \leq \beta_0(T)c_0.$$

It follows that $\varphi(\theta_n) \in L^\infty(0,T)$ and by the lower semicontinuity of φ we get $\varphi(\theta) \leq \limsup_{n\to\infty} \varphi(\theta_n) \leq \beta_0(T)c_0$, i.e. $\varphi(\theta) \in L^\infty(0,T)$.

The uniqueness is enhanced by (3.34), because if we consider two different solutions θ and $\overline{\theta}$ corresponding to the same initial data, θ_0 and the free term, f, we obtain by the norm vanishing that $\theta(x,t) = \overline{\theta}(x,t)$ a.e. $(x,t) \in Q$.

(b) Let us consider first $\theta_0 \in D(A)$ and $f \in W^{1,1}(0,T;V')$. By Theorem 3.8 we know that there is a strong solution θ to ((3.15)-(3.16) that satisfies

$$\theta \in L^\infty(0,T;D(A)) \cap W^{1,\infty}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$$

and we are going to deduce some supplementary a priori estimates.

We multiply the equation (3.15) scalarly in V' by $t\dfrac{d\theta}{dt}$, and then we integrate over $(0,T)$. We obtain

$$\int_0^T t\left\|\frac{d\theta}{dt}(t)\right\|_{V'}^2 dt + T\varphi(\theta(T))$$
$$= \int_0^T \varphi(\theta(t))dt + \int_0^T \left\langle f(t), t\frac{d\theta}{dt}(t)\right\rangle_{V'} dt - \int_0^T \left\langle \frac{\partial K(\theta(t))}{\partial x_3}, t\frac{d\theta}{dt}(t)\right\rangle_{V'} dt.$$

But

$$\varphi(\theta(t)) \geq \frac{\rho}{2}\|\theta(t)\|^2 \geq 0, \ \varphi(0) = 0 \tag{3.59}$$

and after some calculations involving the Schwarz inequality we get

$$\int_0^T t\left\|\frac{d\theta}{dt}(t)\right\|_{V'}^2 dt$$
$$\leq \int_0^T \varphi(\theta(t))dt + \int_0^T \left(\sqrt{t}\,\|f(t)\|_{V'}\right)\left(\sqrt{t}\left\|\frac{d\theta}{dt}(t)\right\|_{V'}\right)dt$$
$$+ \int_0^t \left(\sqrt{t}M\,\|\theta\|\right)\left(\sqrt{t}\left\|\frac{d\theta}{dt}\right\|_{V'}\right)dt$$
$$\leq \int_0^T \varphi(\theta(t))dt + \int_0^T t\,\|f(t)\|_{V'}^2 dt + \frac{1}{4}\int_0^T t\left\|\frac{d\theta}{dt}(t)\right\|_{V'}^2 dt$$
$$+M^2\int_0^T t\,\|\theta(t)\|^2 dt + \frac{1}{4}\int_0^T t\left\|\frac{d\theta}{dt}(t)\right\|_{V'}^2 dt.$$

It follows that

$$\begin{aligned}&\int_0^T t\left\|\frac{d\theta}{dt}(t)\right\|_{V'}^2 dt\\&\leq 2\int_0^T \varphi(\theta(t))dt + 2\int_0^T t\,\|f(t)\|_{V'}^2\,dt + 2M^2\int_0^T t\,\|\theta(t)\|^2\,dt.\end{aligned} \tag{3.60}$$

Then we use the relations $\partial j(\theta) = D^*(\theta)$ with $j(0) = 0$ and again equation (3.15) and obtain

$$\begin{aligned}\int_\Omega j(\theta(t))dx \leq \int_\Omega D^*(\theta(t))\theta(t)dx &= \left\langle f - \frac{d\theta}{dt}(t) - \frac{\partial K(\theta(t))}{\partial x_3}, \theta(t)\right\rangle_{V'}\\&= -\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{V'}^2 + \left\langle f - \frac{\partial K(\theta(t))}{\partial x_3}, \theta(t)\right\rangle_{V'}.\end{aligned}$$

Integrating this relation over $(0,T)$ we deduce that

$$\begin{aligned}\int_0^T \varphi(\theta(t))dt \leq -\frac{1}{2}\|\theta(T)\|_{V'}^2 + \frac{1}{2}\|\theta_0\|_{V'}^2 + \int_0^T \|f(t)\|_{V'}\|\theta(t)\|_{V'}\,dt\\+\int_0^T \left\|\frac{\partial K(\theta(t))}{\partial x_3}\right\|_{V'}\|\theta(t)\|_{V'}\,dt.\end{aligned}$$

It follows by (3.59) that

$$\begin{aligned}\frac{\rho}{2}\int_0^T \|\theta(t)\|^2\,dt &\leq \int_0^T \varphi(\theta(t))dt\\&\leq \frac{1}{2}\|\theta_0\|_{V'}^2 + \frac{1}{2}\int_0^T \|\theta(t)\|_{V'}^2\,dt + \frac{1}{2}\int_0^T \|f(t)\|_{V'}^2\,dt\\&\quad+\frac{\rho}{4}\int_0^T \|\theta(t)\|^2\,dt + \frac{M^2}{\rho}\int_0^T \|\theta(t)\|_{V'}^2\,dt\end{aligned}$$

so finally we get

$$\begin{aligned}&\frac{\rho}{4}\int_0^T \|\theta(t)\|^2\,dt \leq \int_0^T \varphi(\theta(t))dt\\&\leq \frac{1}{2}\|\theta_0\|_{V'}^2 + \left(\frac{1}{2}+\frac{M^2}{\rho}\right)\int_0^T \|\theta(t)\|_{V'}^2\,dt + \frac{1}{2}\int_0^T \|f(t)\|_{V'}^2\,dt.\end{aligned} \tag{3.61}$$

We multiply (3.15) by $\theta(t)$ scalarly in V' and integrate it over $(0,t)$ with $t \in (0,T)$. The calculation is led exactly as for obtaining (3.34) and we get

$$\frac{1}{2}\|\theta(t)\|_{V'}^2 + \int_0^t \langle A\theta(\tau), \theta(\tau)\rangle_{V'}\,d\tau \leq \frac{1}{2}\|\theta_0\|_{V'}^2 + \int_0^t \|f(\tau)\|_{V'}\|\theta(\tau)\|_{V'}\,d\tau.$$

But (see (3.18))

$$\langle A\theta(t), \theta(t)\rangle_{V'} \geq \frac{\rho}{2}\|\theta(t)\|^2 - \frac{M^2}{2\rho}\|\theta(t)\|_{V'}^2,$$

so we obtain

$$\begin{aligned}&\frac{1}{4}\|\theta(t)\|_{V'}^2+\frac{\rho}{2}\int_0^t\|\theta(\tau)\|^2\,d\tau\\&\leq\frac{1}{2}\|\theta_0\|_{V'}^2+\int_0^t\|f(\tau)\|_{V'}^2\,d\tau+\frac{M^2}{2\rho}\int_0^t\|\theta(t)\|_{V'}^2\,d\tau.\end{aligned}$$

By Gronwall's lemma, in a similar manner as done for obtaining (3.34), we deduce now that

$$\|\theta(t)\|_{V'}^2\leq c_0\left(\|\theta_0\|_{V'}^2+\int_0^t\|f(\tau)\|_{V'}^2\,d\tau\right), \tag{3.62}$$

which introduced in (3.61) yields

$$\int_0^T\|\theta(t)\|^2\,dt\leq c_1\left(\|\theta_0\|_{V'}^2+\int_0^T\|\theta(\tau)\|_{V'}^2\,d\tau+1\right)<\infty, \tag{3.63}$$

where c_0 and c_1 are constants that depend on ρ, M and T. It follows also that

$$\int_0^T\varphi(\theta(t))dt<\infty. \tag{3.64}$$

Since

$$\int_0^T t\|\theta(t)\|^2\,dt\leq T\int_0^T\|\theta(t)\|^2\,dt$$

finally, using (3.60), (3.61)-(3.64), we obtain that

$$\int_0^T t\left\|\frac{d\theta}{dt}(t)\right\|_{V'}^2\,dt\leq c_2\left(\|\theta_0\|_{V'}^2+\int_0^T t\|f(t)\|_{V'}^2\,dt+1\right). \tag{3.65}$$

In a similar way as done for $\sqrt{t}\,\dfrac{d\theta}{dt}$ we can obtain an estimate for $\sqrt{t}\,D^*(\theta)$ (multiplying (3.15) by $tD^*(\theta)\in V$ and integrating over $\Omega\times(0,T)$),

$$\int_0^T t\|D^*(\theta(t))\|_{V'}^2\,dt\leq c_3\left(\|\theta_0\|_{V'}^2+\int_0^T t\|f(t)\|_{V'}^2\,dt+1\right). \tag{3.66}$$

Now we take $\theta_0\in M_{\theta_s}=\overline{D(A)}$ and $f\in L^2(0,T;V')$. In virtue of density arguments there exist $\{\theta_n^0\}_{n\geq1}\subset D(A)$ and $\{f_n\}_{n\geq1}\subset W^{1,2}(0,T;V')$ such that

$$f_n\to f\text{ strongly in }L^2(0,T;V')\text{ and }\theta_n^0\longrightarrow\theta_0\text{ strongly in }V',\ \text{as }n\to\infty,$$

and we proceed further like in the proof of (a). We obtain, for each $n\in\mathbf{N}^*$, a solution θ_n with the properties established both in part (a) and (b). By (a) it follows that

$$\theta_n \longrightarrow \theta \text{ strongly in } C([0,T];V') \cap L^2(0,T;L^2(\Omega)).$$

Next, we use (3.65) that implies that on a subsequence we have

$$\sqrt{t}\,\frac{d\theta_n}{dt} \longrightarrow \sqrt{t}\,\frac{d\theta}{dt} \text{ weakly in } L^2(0,T;V').$$

Since φ is lower semicontinuous

$$\liminf_{n\to\infty} \varphi(\theta_n) \geq \varphi(\theta),$$

which implies via Fatou's lemma that $\varphi(\theta) \in L^1(0,T)$, or equivalently $j(\theta) \in L^1(Q)$, as claimed.

Estimate (3.66) allows us to conclude (similarly as before) that

$$\sqrt{t}\,D^*(\theta_n) \longrightarrow \sqrt{t}\,D^*(\theta) \text{ weakly in } L^2(0,T;V).$$

In particular, for each $\delta \in (0,T)$ arbitrarily chosen we have

$$\frac{d\theta_n}{dt} \longrightarrow \frac{d\theta}{dt} \text{ weakly in } L^2(\delta,T;V')$$

and

$$D^*(\theta_n) \longrightarrow D^*(\theta) \text{ weakly in } L^2(\delta,T;V).$$

The proof of the last convergence is done like in (a). Passing then to the limit as $n \to \infty$, we can prove like in (a) that θ satisfies equation (3.15) a.e. on (δ,T) and $\theta(0) = \theta_0$. Since δ is arbitrary it follows that the equation

$$\frac{d\theta}{dt} + A\theta = f$$

is satisfied a.e. $t \in (0,T)$.

From (3.65) and the above equation we deduce

$$\int_0^T t\,\|A\theta(t)\|_{V'}^2\,dt \leq \text{ constant}$$

which implies that $\theta(t) \in D(A)$ a.e. $t \in (0,T)$.

The uniqueness follows from (3.34). ■

It is obvious that if in problem (3.1) we disregard the gravitational field influence we get the problem of horizontal infiltration for which the results remain true.

Up to now we have been interested in the proof of the existence and uniqueness of the solution to the unsaturated infiltration model in the domain $(-\infty, \theta_s)$, which represents in fact a necessary mathematical result. The connection with the physical model is made by analyzing whether this solution belongs to a certain reliable interval if the initial data correspond to some real situations.

The next result is intended to investigate whether the solution to the boundary value problem (3.1) belongs to the physical accepted domain.

Let θ_m be a non-negative constant, $\theta_m \geq 0$.

Theorem 3.11. *Let $f \in L^2(0,T;V')$, $\theta_0 \in M_j$ and assume that*

$$0 \leq \theta_0 \text{ a.e. in } \Omega, \tag{3.67}$$

$$0 \leq f \,. \tag{3.68}$$

Then the solution θ to (3.15)-(3.16) *satisfies*

$$0 \leq \theta(x,t) \text{ a.e. in } \Omega, \text{ for each } t \in [0,T]. \tag{3.69}$$

The conclusion is preserved even if $\theta_0 \in M_{\theta_s}$.

Proof. Assume $f \in L^2(0,T;V')$ and $\theta_0 \in M_j$. We specify that (3.68) is understood in the sense of distributions, i.e., $f(t)(\psi) \geq 0$, a.e. $t \in (0,T)$, for any $\psi \in V$, $\psi \geq 0$. Then by Theorem 3.10, part (a), there exists a unique solution θ to (3.15) that satisfies

$$\theta(x,t) < \theta_s \text{ a.e. } (x,t) \in Q.$$

We have to show that under the hypotheses (3.67)-(3.68) the negative part $(\theta(t))^- = 0$ a.e. on Ω, for each $t \in [0,T]$.

We notice that by Stampacchia's lemma (Theorem 2.14 in Appendix) we have that $(\theta(t))^- \in V$ a.e. t.

We multiply equation (3.15) by $(\theta(t))^-$ and we integrate it over $\Omega \times (0,t)$. We apply Green's formula and, after some calculations, we get

$$\begin{aligned}
&\int_0^t \int_\Omega \left\{ -\frac{1}{2}\frac{d}{d\tau}(\theta^-)^2 + \nabla D^*(\theta)\cdot\nabla\theta^- \right\} dx d\tau \\
&- \int_0^t \int_\Gamma \nabla D^*(\theta)\cdot \nu \theta^- d\sigma d\tau \\
&= \int_0^t \int_\Omega K(\theta)\frac{\partial \theta^-}{\partial x_3}\, dx\, d\tau + \int_0^t \int_\Omega f\theta^- dx\, d\tau,
\end{aligned} \tag{3.70}$$

where ν is the outward normal to Γ. Also, we have used Corollary 2.15 (in Appendix) to Stampacchia's lemma,

$$\nabla\theta^- = \begin{cases} -\nabla\theta, & \text{a.e. on } \theta < 0 \\ 0, & \text{a.e on } \theta \geq 0, \end{cases}$$

by which is follows also that the last term on the left-hand side in (3.70) is zero.

After integrating the first term on the left-hand side with respect to t, we get

$$-\frac{1}{2}\int_{\Omega}[\theta(t)^{-}]^{2}dx-\int_{0}^{t}\int_{\Omega}D(\theta)\left|\nabla\theta^{-}\right|^{2}dx\,d\tau-\int_{0}^{t}\int_{\Omega}K(\theta)\frac{\partial\theta^{-}}{\partial x_{3}}\,dx\,d\tau$$
$$=\frac{1}{2}\int_{\Omega}[(\theta_{0})^{-}]^{2}dx+\int_{0}^{t}\int_{\Omega}f\theta^{-}dx\,d\tau.$$

We took into account that $\theta\theta^{-}=-(\theta^{-})^{2}$. Further we have

$$\frac{1}{2}\int_{\Omega}[(\theta(t))^{-}]^{2}dx+\rho\int_{0}^{t}\left\|(\theta(t))^{-}\right\|_{V}^{2}d\tau$$
$$=-\int_{0}^{t}\int_{\Omega}K(\theta)\frac{\partial\theta^{-}}{\partial x_{3}}\,dx\,dt-\int_{0}^{t}\int_{\Omega}f\theta^{-}dx\,d\tau.$$

Using the hypotheses and the fact that $|K(\theta)|\leq M\,|\theta|$, we deduce that

$$\frac{1}{2}\int_{\Omega}[(\theta(t))^{-}]^{2}dx+\rho\int_{0}^{t}\left\|(\theta(\tau))^{-}\right\|_{V}^{2}d\tau$$
$$\leq\frac{M^{2}}{2\rho}\int_{0}^{t}\left\|(\theta(\tau))^{-}\right\|^{2}d\tau+\frac{\rho}{2}\int_{0}^{t}\left\|(\theta(\tau))^{-}\right\|_{V}^{2}d\tau.$$

This implies

$$\left\|(\theta(t))^{-}\right\|^{2}\leq\frac{M^{2}}{\rho}\int_{0}^{t}\left\|(\theta(\tau))^{-}\right\|^{2}d\tau,$$

wherefrom we deduce according to Gronwall's lemma that $\left\|(\theta(t))^{-}\right\|^{2}=0$, meaning that $\theta(x,t)\geq 0$ a.e. on Ω, for each $t\in[0,T]$.

If $\theta_0\in M_{\theta_s}$, the unique solution θ to (3.15) exists by Theorem 3.10, part (b), and we proceed further as before. ∎

Remark 3.12. The comparison result given by the previous theorem, combined with the upper boundedness of θ gives

$$0\leq\theta(x,t)<\theta_{s}\text{ a.e. }(x,t)\in Q,$$

and we conclude that the solution of the infiltration problem, namely the soil moisture is situated within the physical domain for moisture. This result sustains the assertions made in Sect. 2.6 that the extensions of the hydraulic functions performed at the left of the origin did not influence the correctness of the result.

Remark 3.13. Finally, we notice that, as expected when dealing with dissipative systems, the problem has a global time solution, i.e., the solution exists for T whatever large but finite, such that θ remains under θ_s. If T grows up to infinity, in certain cases it might be possible that the flow tends to a stationary regime, fact that opens the discussion upon the asymptotic behaviour of the solution, further treated.

Longtime behaviour of the solution

In problem (3.15), i.e.,

$$\frac{d\theta}{dt} + A\theta = f, \text{ a.e. } t \in (0,T), \quad (3.71)$$
$$\theta(0) = \theta_0,$$

we take now $T = +\infty$. The following result describes a type of behaviour of the solution for large time in a particular case with a slow varying conductivity.

Let f_∞ be a constant and c_Ω the constant in Poincaré's inequality (see Theorem 2.17 in Appendix).

Theorem 3.14. *Assume that* $\theta_0 \in \overline{D(A)}$, $f \in L^1_{loc}([0,+\infty); V')$ *and*

$$\operatorname*{ess\,sup}_{s\in(t,+\infty)} \|f(s) - f_\infty\|_{V'} \longrightarrow 0, \text{ as } t \to +\infty, \quad (3.72)$$

$$\rho > c_M := \max\{M, c_\Omega M\}. \quad (3.73)$$

Then there exists $\theta_\infty \in D(A)$, *such that*

$$\|\theta(t) - \theta_\infty\| \longrightarrow 0, \text{ as } t \to +\infty. \quad (3.74)$$

If $f \in W^{1,1}(0,\infty; V')$ *and* $\theta_0 \in D(A)$ *then*

$$\operatorname*{ess\,sup}_{s\in(t,+\infty)} \left\| \frac{d\theta}{dt}(s) \right\|_{V'} \longrightarrow 0, \text{ as } t \to +\infty. \quad (3.75)$$

Proof. Let $\rho > c_M$. We show that the operator A is strongly monotone (instead of quasi-monotone) and coercive. We have

$$\begin{aligned} \left\langle A\theta - A\overline{\theta}, \theta - \overline{\theta}\right\rangle_{V'} &\geq \rho \left\|\theta - \overline{\theta}\right\|^2 - M \left\|\theta - \overline{\theta}\right\| \left\| \frac{\partial \psi}{\partial x_3} \right\| \\ &\geq \rho \left\|\theta - \overline{\theta}\right\|^2 - M \left\|\theta - \overline{\theta}\right\| \left\|\theta - \overline{\theta}\right\|_{V'} \\ &\geq (\rho - M) \left\|\theta - \overline{\theta}\right\|^2 \geq (\rho - M) \left\|\theta - \overline{\theta}\right\|^2_{V'}, \end{aligned} \quad (3.76)$$

where we used the inequality $\left\|\theta - \overline{\theta}\right\|_{V'} \leq \left\|\theta - \overline{\theta}\right\|$.

From here we deduce in particular that A is coercive, i.e.,

$$\lim \frac{\langle A\theta, \theta\rangle_{V'}}{\|\theta\|_{V'}} = +\infty, \text{ as } \|\theta\|_{V'} \to +\infty. \quad (3.77)$$

It follows (see Theorem 2.7 in Sect. 3.2) that A is surjective, so that the equation

$$A\theta = f_\infty$$

has a unique solution denoted

$$\theta_\infty = A^{-1}(f_\infty) \in D(A). \tag{3.78}$$

In fact θ_∞ is the generalized solution to the stationary boundary value problem

$$\begin{aligned} -\Delta D^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} &= f_\infty \quad \text{in } \Omega, \\ \theta(x) &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.79}$$

Obviously, for the same consideration as that from Remark 3.1, it follows that $\theta_\infty(x) < \theta_s$, a.e. $x \in \Omega$.

On the other hand, since $f \in L^1_{loc}(0,\infty;V')$ and $\theta_0 \in \overline{D(A)}$, the problem (3.72) has a weak solution as established by Theorem 3.8, in Sect. 3.3.

Next we shall show that $\lim_{t\to\infty} \theta(t) = \theta_\infty$ a.e. in Ω. To come to this end we multiply the equation

$$\frac{\partial(\theta-\theta_\infty)}{\partial t} - \Delta\left(D^*(\theta) - D^*(\theta_\infty)\right) + \frac{\partial\left(K(\theta)-K(\theta_\infty)\right)}{\partial x_3} = f - f_\infty \tag{3.80}$$

by $(\theta-\theta_\infty)$ and integrate it over Ω. We have

$$\begin{aligned} &\int_\Omega \left\{ \frac{1}{2}\frac{\partial}{\partial \tau}(\theta-\theta_\infty)^2 + \nabla\left(D^*(\theta) - \nabla D^*(\theta_\infty)\right)\cdot\nabla(\theta-\theta_\infty)\right\} dx \\ &= \int_\Omega (K(\theta)-K(\theta_\infty))\frac{\partial(\theta-\theta_\infty)}{\partial x_3}\,dx + \int_\Omega (f-f_\infty)(\theta-\theta_\infty)dx. \end{aligned}$$

Using (i), ($\mathrm{i_K}$) and Poincaré's inequality (Theorem 2.17 in Appendix), we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\theta(t)-\theta_\infty\right\|^2 + \rho\left\|\theta(t)-\theta_\infty\right\|_V^2 \\ &\le Mc_\Omega\left\|\theta(t)-\theta_\infty\right\|_V^2 + \left\|f(t)-f_\infty\right\|_{V'}\left\|\theta(t)-\theta_\infty\right\|_V, \text{ a.e. } t>0. \end{aligned}$$

Since $\rho > c_M$ we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\theta(t)-\theta_\infty\right\|^2 + c_M\left\|\theta(t)-\theta_\infty\right\|_V^2 \\ &\le \frac{1}{2c_M}\left\|f(t)-f_\infty\right\|_{V'}^2 + \frac{c_M}{2}\left\|\theta(t)-\theta_\infty\right\|_V^2. \end{aligned}$$

This yields further

$$\frac{d}{dt}\left\|\theta(t)-\theta_\infty\right\|^2 + c_M\left\|\theta(t)-\theta_\infty\right\|^2 \le \frac{1}{c_M}\left\|f(t)-f_\infty\right\|_{V'}^2.$$

We multiply this inequality by $\exp(c_M t)$ and we obtain that

$$\frac{d}{dt}(R(t)) \le 0, \ \forall t \ge 0,$$

where

$$R(t) = e^{c_M t} \left|\left|\theta(t) - \theta_\infty\right|\right|^2 - \mathcal{P}(t)$$

and $\mathcal{P}(t)$ is an antiderivative of the function $\frac{1}{c_M} \left\| f(t) - f_\infty \right\|_{V'}^2 \exp(c_M t)$. It follows that R is monotonically decreasing and $R(t) \leq R(0)$. From here we obtain that

$$\left|\left|\theta(t) - \theta_\infty\right|\right|^2 \leq e^{-c_M t} \left|\left|\theta_0 - \theta_\infty\right|\right|^2 + \frac{1}{c_M} \int_0^t e^{-c_M (t-s)} \left\| f(s) - f_\infty \right\|_{V'}^2 ds. \quad (3.81)$$

Note that (3.72) is equivalent to the existence of a function $\widetilde{f} : (0, +\infty) \to V'$, such that $f = \widetilde{f}$ a.e. $t \in (0, +\infty)$ and $\widetilde{f}(t) \to f_\infty$ strongly in V' as $t \to +\infty$.

By (3.72) we have that for any ε there exists $T(\varepsilon)$ such that for any $s \geq T(\varepsilon)$ it follows that $\left\| f(s) - f_\infty \right\|_{V'} < \varepsilon$. Therefore, if t is large enough, $t \geq T(\varepsilon)$ we can write (3.81) in the following way

$$\begin{aligned} \left|\left|\theta(t) - \theta_\infty\right|\right|^2 &\leq e^{-c_M t} \left|\left|\theta_0 - \theta_\infty\right|\right|^2 + \frac{1}{c_M} \int_0^{T(\varepsilon)} e^{-c_M (t-s)} \left\| f(s) - f_\infty \right\|_{V'}^2 ds \\ &+ \frac{1}{c_M} \int_{T(\varepsilon)}^t e^{-c_M (t-s)} \left\| f(s) - f_\infty \right\|_{V'}^2 ds \end{aligned}$$

which implies that

$$\begin{aligned} \left|\left|\theta(t) - \theta_\infty\right|\right|^2 &\leq e^{-c_M t} \left|\left|\theta_0 - \theta_\infty\right|\right|^2 + \frac{1}{c_M^2} \left(1 - e^{-c_M (t - T(\varepsilon))}\right) \varepsilon^2 \\ &+ \frac{1}{a^2} e^{-c_M t} (e^{c_M N(\varepsilon)} - 1) \left(\left\| f(s) \right\|_{V'}^2 + \left| f_\infty \right|^2 \right). \end{aligned}$$

Passing now to limit we obtain that

$$\lim_{s \in (t, +\infty)} \left|\left|\theta(s) - \theta_\infty\right|\right|^2 = 0, \text{ as } t \to \infty,$$

as claimed.

If $\theta_0 \in D(A)$ and $f \in W^{1,1}(0, \infty; V')$, then there exists a solution θ pursuant to Theorem 3.8, this chapter. We multiply (3.80) by $\frac{\partial(\theta(t) - \theta_\infty)}{\partial t}$, scalarly in V' and we get

$$\begin{aligned} &\left\| \frac{d}{dt} (\theta(t) - \theta_\infty) \right\|_{V'}^2 d\tau + \int_\Omega \nabla \left(D^*(\theta(t)) - D^*(\theta_\infty) \right) \cdot \nabla \psi dx \\ &= \int_\Omega \left(f(t) - f_\infty \right) \psi dx + \int_\Omega \left(K(\theta(t)) - K(\theta_\infty) \right) \frac{\partial \psi}{\partial x_3} dx, \end{aligned}$$

where

$$-\Delta \psi = \frac{\partial \left(\theta(t) - \theta_\infty \right)}{\partial t}, \ \psi = 0 \text{ on } \Gamma.$$

Following the computations like in the proof of Theorem 3.8 we obtain

$$\left\| \frac{d}{dt} \left(\theta(t) - \theta_\infty \right) \right\|_{V'}^2 d\tau + \int_\Omega \frac{\partial j \left(\theta(x,t) - \theta_\infty \right)}{\partial t} dx$$
$$\leq \| f(t) - f_\infty \|_{V'} \| \theta(t) - \theta_\infty \|_V + M \| \theta(t) - \theta_\infty \| \| \theta(t) - \theta_\infty \|_V .$$

The function j is monotonically increasing, so its derivative is positive and we get

$$\left\| \frac{d}{dt} \left(\theta(t) - \theta_\infty \right) \right\|_{V'}^2 d\tau$$
$$\leq \| f(t) - f_\infty \|_{V'} \| \theta(t) - \theta_\infty \|_V + M \| \theta(t) - \theta_\infty \| \| \theta(t) - \theta_\infty \|_V .$$

Using now (3.74) we obtain (3.75) as claimed. ∎

Remark 3.15. In Theorem 3.14, the assumptions related to the initial data and the free term can be weakened by considering

$$\theta_0 \in \overline{D(A)} = M_{\theta_s} \text{ and } f \in L^2(0, +\infty; V'), \tag{3.82}$$

but the conclusion is still valid, by Theorem 3.10, (b).

Remark 3.16. If condition (3.73) does not hold, the longtime behaviour of the trajectory $\theta(t)$ might be more complex. However, since the trajectory $\{\theta(t);\ t \geq 0\}$ is bounded in $L^2(\Omega)$, (because $\theta(t) < \theta_s$ a.e. in Ω, for each $t \geq 0$), it is compact in V'. Then, the ω-limit set

$$\Gamma_\omega := \left\{ \lim_{t_n \to \infty} \theta(t_n) \text{ in } V' \right\}$$

which contains the stationary solutions to (3.71) is nonempty and the general theory of infinite dimensional attractors can be applied in order to investigate the structure of Γ_ω (see [115]).

4.4 Strongly nonlinear conductivity. Homogeneous Dirichlet boundary conditions

We resume now the case corresponding to a strongly nonlinear hydraulic conductivity where the property

$$\lim_{\theta \nearrow \theta_s} K'(\theta) = +\infty \tag{4.1}$$

holds. This implies that K is Lipschitz on any subset $(-\infty, \theta_l]$ of $(-\infty, \theta_s)$ i.e.,

(ii_K) there exists $M_l > 0$ such that

$$|K(\theta_1) - K(\theta_2)| \leq M_l \, |\theta_1 - \theta_2| \, , \; \forall \theta_1, \theta_2 \in (-\infty, \theta_l], \; \theta_l < \theta_s.$$

We shall prove that in this case the solution exists too, but it is obtained in a weaker sense than before.

Theorem 4.1. *Assume* (i)-(iii) *and* ($\mathrm{ii_K}$). *Let*

$$f \in W^{1,1}(0,T;V'), \; \theta_0 \in D(A). \tag{4.2}$$

Then there exists a weak solution $\theta \in C([0,T],V')$ *to problem* (3.15)- (3.16), *such that*

$$\theta \in W^{1,\infty}(0,T;V') \cap L^\infty(0,T;V), \tag{4.3}$$

$$D^*(\theta) \in L^\infty(0,T;V), \tag{4.4}$$

$$\theta(x,t) < \theta_s \text{ a.e. } (x,t) \in Q.$$

Also, the estimates

$$\begin{aligned} &\int_\Omega j(\theta(x,t))dx + \int_0^t \left\| \frac{d\theta}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|D^*(\theta(\tau))\|_V^2 \, d\tau \\ &\leq 4 \left(\int_\Omega j(\theta_0)dx + \int_0^t \|f(\tau)\|_{V'}^2 \, d\tau + K_s^2 t \right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\left\| \theta(t) - \overline{\theta}(t) \right\|_{V'}^2 + \int_0^t \left\| \theta(\tau) - \overline{\theta}(\tau) \right\|^2 d\tau \\ &\leq \alpha_1(t) \left(\left\| \theta_0 - \overline{\theta}_0 \right\|_{V'}^2 + \int_0^T \left\| f(\tau) - \overline{f}(\tau) \right\|_{V'}^2 d\tau + 2\overline{K}^2 T \right), \end{aligned} \tag{4.6}$$

hold, where θ_0, f *and* $\overline{\theta}_0$, $\overline{f}$ *are two pairs of initial data and free terms, and* $\alpha_1(t) = e^t$, $\overline{K} = K_s \, (\text{meas}(\Omega))^{1/2}$, $\forall t \in [0,T]$, $T > 0$.

Moreover, if $\theta_0 \geq 0$ *a.e.* $x \in \Omega$, *then* $\theta(x,t) \geq 0$ *a.e.* $x \in \Omega$, $\forall t \in [0,T]$.

Proof. We introduce an approximation of the operator A by replacing the function K by

$$K_\varepsilon(\theta) := \begin{cases} K(\theta), & \theta < \theta_s - \varepsilon \\ \dfrac{K_s - K(\theta_s - \varepsilon)}{\varepsilon} \, (\theta - (\theta_s - \varepsilon)) + K(\theta_s - \varepsilon), & \theta_s - \varepsilon \leq \theta \leq \theta_s. \end{cases} \tag{4.7}$$

It follows that K_ε is Lipschitz on $(-\infty, \theta_s]$ with the constant $M_\varepsilon = \dfrac{K_s - K(\theta_s - \varepsilon)}{\varepsilon}$.

Within the same functional framework specified in Sect. 4.3 we introduce the approximating operator $A_\varepsilon : D(A_\varepsilon) \subset V' \to V'$,

$$\langle A_\varepsilon \theta, \psi \rangle_{V',V} = \int_\Omega \nabla D^*(\theta) \cdot \nabla \psi \, dx - \int_\Omega K_\varepsilon(\theta) \frac{\partial \psi}{\partial x_3} \, dx, \; \forall \psi \in V, \tag{4.8}$$

whose domain is in fact that of A, i.e.,

$$D(A_\varepsilon) := \{\theta \in L^2(\Omega); D^*(\theta) \in V\}.$$

Consequently we consider the Cauchy problem

$$\begin{aligned} \frac{d\theta_\varepsilon}{dt} + A_\varepsilon \theta_\varepsilon &= f, \text{ a.e. } t \in (0,T), \\ \theta_\varepsilon(0) &= \theta_0. \end{aligned} \tag{4.9}$$

Since by regularization K becomes Lipschitz, all assumptions (i)-(iii) and (i_K) are fulfilled by D^* and K_ε, thence it follows that the operator A_ε is quasi m-accretive on V'. Then, by Theorem 3.8, the Cauchy problem (4.9) has, for each $\varepsilon > 0$, a unique strong solution

$$\theta_\varepsilon \in C([0,T]; V') \cap W^{1,\infty}(0,T;V') \cap L^\infty(0,T;V),$$

with $D^*(\theta_\varepsilon) \in L^2(0,T;V)$.

Moreover, $\theta_\varepsilon < \theta_s$ a.e. $x \in Q$ and θ_ε satisfies (3.33) with $\beta_0(t)$ depending on ε by M_ε. Note that we cannot use however this relation to pass to limit, so that we must seek for another estimate which is more appropriate for this case. Thus we prove (4.5) that follows in the same way as (3.33) except for the fact that we do no longer use the Lipschitz property in (3.38) but the boundedness of K_ε, $|K_\varepsilon(\theta_\varepsilon)| \le K_s$. We get the estimate

$$\begin{aligned} &\int_\Omega \int_0^t \frac{\partial j(\theta_\varepsilon)}{\partial \tau}\, dx\, d\tau + \int_0^t \|D^*(\theta_\varepsilon(\tau))\|_V^2\, d\tau \\ &\le \int_0^t K_s \|D^*(\theta_\varepsilon(\tau))\|_V\, d\tau + \int_0^t \|f(\tau)\|_{V'} \|D^*(\theta_\varepsilon(\tau))\|_V\, d\tau. \end{aligned}$$

This yields

$$\begin{aligned} &\int_\Omega j(\theta_\varepsilon(t))dx + \int_0^t \|D^*(\theta_\varepsilon(\tau))\|_V^2\, d\tau \\ &\le \int_\Omega j(\theta_0)dx + \int_0^t \|f(\tau)\|_{V'}^2\, d\tau + \frac{1}{4}\int_0^t \|D^*(\theta_\varepsilon(\tau))\|_V^2\, d\tau \\ &+ K_s^2 t + \frac{1}{4}\int_0^t \|D^*(\theta_\varepsilon(\tau))\|_V^2\, d\tau. \end{aligned}$$

It follows that

$$\frac{\rho}{2}\|\theta_\varepsilon(t)\|^2 \le \int_\Omega j(\theta_\varepsilon(t))dx + \int_0^t \|D^*(\theta_\varepsilon(\tau))\|_V^2\, d\tau \le c_1(t)$$

with

$$c_1(t) = 2\left(\int_\Omega j(\theta_0)dx + \int_0^t \|f(\tau)\|_{V'}^2\, d\tau + K_s^2 t\right) \le c_1(T) < \infty.$$

Performing the computations following the multiplication of (4.9) by $\frac{d\theta_\varepsilon}{dt}$ in V' we obtain finally (4.5) as claimed.

From these estimates it follows that $\{\theta_\varepsilon\}_{\varepsilon>0}$ lies in a bounded subset of $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ and $\left\{\frac{d\theta_\varepsilon}{dt}\right\}_{\varepsilon>0}$ lies in a bounded subset of $L^2(0,T;V)$. Therefore, extracting a subsequence we get that

$$\begin{aligned}\theta_\varepsilon \longrightarrow \theta \ \ &\text{weak-star in } L^\infty(0,T;V') \text{ and} \\ &\text{weakly in } L^2(0,T;V), \ \text{ as } \varepsilon \to 0,\end{aligned} \tag{4.10}$$

and

$$\frac{d\theta_\varepsilon}{dt} \longrightarrow \frac{d\theta}{dt} \text{ weakly in } L^2(0,T;V'), \ \text{ as } \varepsilon \to 0. \tag{4.11}$$

Since V is compact in $L^2(\Omega)$ it follows by the compacity result of Lions-Aubin (see Theorem 3.12 in Appendix) that

$$\theta_\varepsilon \longrightarrow \theta \text{ strongly in } L^2(0,T;L^2(\Omega)), \ \text{ as } \varepsilon \to 0. \tag{4.12}$$

By the previous estimate it still follows that

$$D^*(\theta_\varepsilon) \longrightarrow \eta \text{ weakly in } L^2(0,T;V), \ \text{ as } \varepsilon \to 0$$

and by a similar argument as in Theorem 3.10, part (a), involving Egorov's theorem, we can deduce that $\eta = D^*(\theta)$ a.e. on $L^2(0,T;L^2(\Omega))$.

By the strongly convergence of θ_ε to θ we get also the a.e. convergence which combined with the continuity of the function K implies that

$$K(\theta_\varepsilon) \longrightarrow K(\theta) \text{ a.e. } (x,t) \in \ \Omega \times (0,T), \ \text{ as } \varepsilon \to 0.$$

We use again the boundedness of K, i.e., $|K(\theta_\varepsilon)| \leq K_s$ a.e. on Q and applying Lebesgue's theorem (see Theorem 2.10 in Appendix) we conclude that

$$K(\theta_\varepsilon) \to K(\theta) \text{ strongly in } L^2(0,T;L^2(\Omega)), \ \text{ as } \varepsilon \to 0. \tag{4.13}$$

It is obvious that in this case $K(\theta)$ does no longer belong to $L^2(0,T;V)$.

Finally, it is nothing else to do now but passing to limit as $\varepsilon \to 0$ in (4.9), or in the equivalent equation

$$\int_Q \left(\frac{d\theta_\varepsilon}{dt}\phi + \nabla D^*(\theta_\varepsilon)\cdot\nabla\phi - K(\theta_\varepsilon)\frac{\partial\phi}{\partial x_3}\right) dx\,dt = \int_Q f\phi\,dx\,dt, \quad \forall \phi \in L^2(0,T;V) \tag{4.14}$$

and to obtain

$$\int_Q\left(\frac{d\theta}{dt}\phi + \nabla D^*(\theta)\cdot\nabla\phi - K(\theta)\frac{\partial\phi}{\partial x_3}\right) dx\,dt = \int_Q f\phi\,dx\,dt, \ \forall \phi \in L^2(0,T;V),$$

which, together with the initial condition $\theta(0) = \theta_0$, represent exactly the Cauchy problem (3.15). Since θ is in $C([0,T];V') \cap W^{1,\infty}(0,T;V')$ with $\theta(t) \in D(A)$, a.e. for $t \in (0,T)$ and it was obtained as a limit of strong solutions to an approximated problem, we can conclude that it is a weak solution in the sense of Definition 3.2 in Sect. 3.3.

Since $\theta \in L^2(0,T;D(A))$ we have that $\theta(x,t) < \theta_s$ a.e. $(x,t) \in Q$.

The proof of (4.6) is similar to that developed for proving (3.34) where instead $(\mathrm{i_K})$ we use

$$\left\|K(\theta) - K(\overline{\theta})\right\| \leq 2\overline{K}.$$

Then, we get, following the computations subsequent to (3.36) that

$$\begin{aligned}
&\left\|\theta(t) - \overline{\theta}(t)\right\|_{V'}^2 + \int_0^t \left\|\theta(\tau) - \overline{\theta}(t)\right\|^2 d\tau \\
&\leq \left\|\theta_0 - \overline{\theta}_0\right\|_{V'}^2 + \int_0^T \left\|f(\tau) - \overline{f}(\tau)\right\|_{V'}^2 d\tau + 2\overline{K}^2 T + \int_0^t \left\|\theta(\tau) - \overline{\theta}(\tau)\right\|_{V'}^2 d\tau
\end{aligned}$$

that implies (4.6). The solution is unique only if ess sup $\theta(x,t) < \theta_s$.

Concerning the boundedness, we notice that by Theorem 3.11 the solution $\theta_\varepsilon(x,t) \in [0,\theta_s)$ a.e. $(x,t) \in Q$. This property is preserved by θ due to the strong convergence of the sequence $\{\theta_\varepsilon\}_{\varepsilon>0}$ to θ. ∎

Remark 4.2. Generally speaking, the solution to the model considering the hypothesis $(\mathrm{ii_K})$ may have, under the corresponding hypotheses, the same properties as the solution to the model with the assumption $(\mathrm{i_K})$ has, because they are preserved by the strongly convergence. Thus, the solution may have a clear asymptotic behaviour, for example when $f_\infty = 0$. In this case we notice that a solution to the stationary equation (3.79) is $\theta_\infty = 0$ and hence the solution will vanish asymptotically in time.

However, the treatment of the asymptotic behaviour cannot be done exactly as before, for any constant f_∞. The fact that K is not Lipschitz implies that A is no longer coercive and the existence of the solution to (3.79) cannot be rigorously proved.

4.5 Weakly nonlinear conductivity. Nonhomogeneous Dirichlet boundary conditions

We hold forth the problem

$$\begin{aligned}
\frac{\partial\theta}{\partial t} - \Delta D^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} &= f && \text{in } Q, \\
\theta(x,0) &= \theta_0(x) && \text{in } \Omega, \\
\theta(x,t) &= g(x,t) < \theta_s && \text{on } \Sigma,
\end{aligned} \tag{5.1}$$

which describes the water infiltration into an isotropic, homogeneous, unsaturated porous medium with constant porosity whose moisture distribution on the boundary is time and space variable. We assume that saturation value of the moisture is not reached on the boundary during the flow. We confine ourselves to the quasi-unsaturated model with a weakly nonlinear hydraulic conductivity, so that we suppose that D^* and K satisfy the conditions (i)-(iii) and ($\mathrm{i_K}$), respectively.

Functional framework

The approach to this case requires that either Ω is of class C^2 or the boundary Γ is convex. We consider the space $V{=}H_0^1(\Omega)$ with the usual Hilbertian norm, its dual $V^{'}{=}H^{-1}(\Omega)$ with the norm derived from (3.3) and $H{=}L^2(\Omega)$. Identifying H with its dual we have

$$V \subset H \subset V^{'},$$

each space being densely embedded in the next one. Also V is compactly embedded in H (see Lemma 2.6 in Appendix).

We make now a hypothesis, requiring that there exists a function w such that

$$(H_w) \qquad \begin{cases} w \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q), \ w_t \in L^2(Q), \\ \|w\|_{L^\infty(Q)} < \theta_s, \ w = g \text{ on } \Sigma. \end{cases}$$

(Here $w_t = \dfrac{dw}{dt}$ is considered in the sense of distributions from $(0,T)$ to $L^2(\Omega)$.)

Because w satisfies (H_w) it follows that $\|D(w)\|_{L^\infty(Q)} < \infty$ and let us denote this norm by D_w. Hence

$$\rho \le D(w) \le D_w := \|D(w)\|_{L^\infty(Q)}, \text{ a.e. in } \Omega \times (0,T). \tag{5.2}$$

We introduce a new unknown function,

$$\phi = \theta - w \tag{5.3}$$

that vanishes on the boundary, $\phi\,|_\Sigma = 0$ and set

$$F^w(\phi) := D^*(\phi + w) - D^*(w), \ \forall \phi \in V, \tag{5.4}$$

$$f^B := f - (-\Delta D^*(w)). \tag{5.5}$$

Therefore, instead of problem (5.1), we can consider the problem for the unknown ϕ, with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \frac{\partial \phi}{\partial t} - \Delta F^w(\phi) + \frac{\partial K(\phi + w)}{\partial x_3} &= f^B - \frac{dw}{dt} && \text{in } Q, \\ \phi(x,0) &= \phi_0(x) && \text{in } \Omega, \\ \phi(x,t) &= 0 && \text{on } \Sigma, \end{aligned} \tag{5.6}$$

where

$$\phi_0(x) := \theta_0(x) - w_0(x).$$

Note also that $w_0 := w(x, 0)$ makes sense due to (H_w).

We denote by j the function defined in (2.14) and

$$M_j = \{\theta \in L^2(\Omega);\ j(\theta) \in L^1(\Omega)\}.$$

Definition 5.1. Let $\theta_0 \in M_j$ and $f \in L^2(0, T; V')$. We mean by *solution* to (5.6) a function $\phi \in C([0, T]; L^2(\Omega))$, such that $\dfrac{d\phi}{dt} \in L^2(0, T; V')$, $F^w(\phi) \in L^2(0, T; V)$ and

$$\begin{aligned}
&\left\langle \frac{d\phi}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega \left(\nabla F^w(\phi(t)) \cdot \nabla\psi - K(\phi(t) + w(t)) \frac{\partial\psi}{\partial x_3} \right) dx \\
&\qquad\qquad = \langle f(t), \psi \rangle_{V',V}\,, \ \text{a.e. } t \in (0, T),\ \forall \psi \in V, \\
&\phi(0) = \phi_0 \text{ in } \Omega.
\end{aligned}$$

The alternate form of the definition analogous to (3.12) holds too. It is obvious that if ϕ is the solution to (5.6), then, going back, it follows that $\theta = \phi + w$ is the solution to (5.1) belonging to the following spaces:

$$\begin{aligned}
&\theta \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
&\frac{d\theta}{dt} \in L^2(0, T; V'),\ D^*(\theta) \in L^2(0, T; H^1(\Omega)).
\end{aligned}$$

With all these considerations we shall study the problem (5.6).

We introduce the operator $B(t) : V \to V'$ (that depends on t by means of w), defined by

$$\langle B(t)\phi, \psi \rangle_{V',V} = \int_\Omega \nabla F^w(\phi) \cdot \nabla\psi dx - \int_\Omega K(\phi + w) \frac{\partial\psi}{\partial x_3} dx \tag{5.7}$$

and write the Cauchy problem

$$\begin{aligned}
\frac{d\phi}{dt} + B(t)\phi &= f^B - \frac{dw}{dt},\ \text{a.e. } t \in (0, T), \\
\phi(0) &= \phi_0.
\end{aligned} \tag{5.8}$$

On the basis of an argument analogous to that of Lemma 3.5, every solution to (5.8) is a solution in the sense of distributions to (5.6).

We shall prove that (5.8) has a unique solution which is obtained in a weaker form, as a limit of strong solutions, in the sense of Definition 3.2 in Sect. 3.3.

We notice that if $f \in L^2(0, T; V')$ then $f^B \in L^2(0, T; V')$. Indeed, we have

$$|-\Delta D^*(w)(\psi)| = \left| \int_\Omega \nabla D^*(w) \cdot \nabla\psi dx \right| \le \|D(w)\nabla w\| \, \|\nabla\psi\|\,,$$

whence

$$\|-\Delta D^*(w)\|_{V'} = \sup_{\|\psi\|_V \le 1} |-\Delta D^*(w)(\psi)| \le D_w \|w\|_{H^1(\Omega)} < \infty, \ \forall \psi \in V.$$

Then, since $w_t \in L^2(0,T;L^2(\Omega))$ we get that

$$f^B - \frac{dw}{dt} \in L^2(0,T;V'). \tag{5.9}$$

The next theorem states the main result for this problem (see also [85]).

Theorem 5.2. *Under the hypothesis* (H_w) *and the assumptions*

$$f \in L^2(0,T;V'), \tag{5.10}$$

$$\theta_0 \in L^2(\Omega), \tag{5.11}$$

$$j(\theta_0) \in L^1(\Omega), \tag{5.12}$$

problem (5.8) *has a unique solution* ϕ, *that satisfies*

$$\phi \in L^2(0,T;V), \ \frac{d\phi}{dt} \in L^2(0,T;V'). \tag{5.13}$$

Moreover, it follows that the generalized solution to (5.1) *satisfies*

$$\theta \in L^2(0,T;H^1(\Omega)), \ \frac{d\theta}{dt} \in L^2(0,T;V'), \tag{5.14}$$

$$D^*(\theta) \in L^2(0,T;H^1(\Omega)), \tag{5.15}$$

$$j(\theta) \in L^1(Q). \tag{5.16}$$

Although we have reduced the original problem to one with homogeneous Dirichlet boundary condition, the fact that the operator acting here is time dependent does not allow us to apply the theorems used in Sect. 3.3 (for example Theorem 3.6). The proof of this theorem is very technical, so that it will be split in a couple of steps containing some preliminary results.

The first step consists in the statement of an approximating problem obtained by replacing the blowing up function D^* by a smooth function, D^*_n, which will lead to an approximating problem involving D^*_n. The second step is the proof of the existence result for this subsidiary problem and finally the third one resides in a passing to limit technique and a compacity result that will provide the conclusion of Theorem 5.2.

The approximating problem

Since D as well as D^* are singular at $\theta = \theta_s$ we shall approximate problem (5.8) in the following way: for each $n \in \mathbf{N}^* = \{1, 2, ...\}$ we define the increasing sequence of functions

$$D_n(r) := \begin{cases} D(r), & r \le \theta_s - \dfrac{1}{n}, \\ D\left(\theta_s - \dfrac{1}{n}\right), & r > \theta_s - \dfrac{1}{n}, \end{cases} \tag{5.17}$$

as shown in Fig. 4.3.

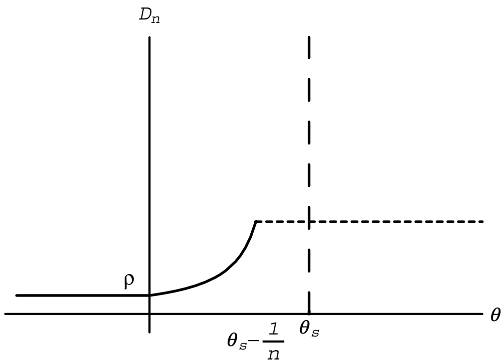

Fig. 4.3. Graphic of $D_n(\theta)$

The function D_n is bounded for each $n \ge 1$,

$$\rho \le D_n(r) \le \rho_n := D\left(\theta_s - \frac{1}{n}\right) < \infty \tag{5.18}$$

and because D is continuous monotonically increasing on $r \in \left(\theta_s - \dfrac{1}{n}, \theta_s\right)$, we have

$$D_n(r) \le D(r) \text{ and } \lim_{n\to\infty} D_n(r) = D(r), \text{ for } -\infty < r < \theta_s.$$

We also note that since $\|w\|_{L^\infty(Q)} < \theta_s$ then $D_n(w) = D(w) \le D_w := D(\|w\|_{L^\infty(Q)})$ for all n.

In the approximating problem we have also to extend the function K to the right of θ_s and we shall set for it the constant value K_s. Since it does not depend on n and the transport term $\dfrac{K(\theta)}{\partial x_3}$ has practically no contribution for $\theta \ge \theta_s$, we shall keep for this extended function the notation K.

With these notations we consider, for each $n \in \mathbf{N}^*$, the approximating problem

$$\frac{d\phi_n}{dt} + B_n(t)\phi_n = f^B - \frac{dw}{dt}, \text{ a.e. } t \in (0,T), \tag{5.19}$$
$$\phi_n(0) = \phi_0.$$

Here, the operator $B_n(t) : V \to V'$ is defined by

$$\langle B_n(t)\phi, \psi\rangle_{V',V} = \int_\Omega \nabla F_n^w(\phi)\cdot\nabla\psi dx - \int_\Omega K(\phi+w)\frac{\partial\psi}{\partial x_3}\,dx,\ \forall\psi\in V, \tag{5.20}$$

where

$$F_n^w(\phi) := D_n^*(\phi+w) - D_n^*(w). \tag{5.21}$$

The next step will be the proof of the existence of the solution to (5.19).

First we shall show that $F_n^w(\phi) \in H_0^1(\Omega)$, for $\phi \in H_0^1(\Omega)$. Indeed, from (5.21), (5.18) and (5.2) we have

$$\|F_n^w(\phi)\| \le \|D_n^*(\phi+w)\| + \|D^*(w)\| \le \rho_n \|\phi+w\| + D_w \|w\|.$$

Then using Poincaré's inequality (see Theorem 2.17 in Appendix), we have for each n fixed that

$$\|F_n^w(\phi)\| \le \sigma_n \|\phi\|_V + \sigma_n^1,\ \sigma_n = \rho_n c_\Omega \text{ and } \sigma_n^1 = (\rho_n + D_w)\|w\|. \tag{5.22}$$

We recall that $\|\cdot\|$ is the norm in $L^2(\Omega)$. Then

$$\left\|\frac{\partial F_n^w}{\partial x_i}\right\| \le \left\|\frac{\partial D_n^*(\phi+w)}{\partial x_i}\right\| + \left\|\frac{\partial D_n^*(w)}{\partial x_i}\right\| \le \rho_n\left\|\frac{\partial\phi}{\partial x_i}\right\| + (\rho_n + D_w)\left\|\frac{\partial w}{\partial x_i}\right\|,$$

wherefrom we obtain

$$\|F_n^w(\phi)\|_V \le \rho_n\|\phi\|_V + \rho_n^1 < \infty \tag{5.23}$$

with $\rho_n^1 = (\rho_n + D_w)\|w\|_{H^1(\Omega)}$.

Moreover, $F_n^w(\phi)|_\Gamma = 0$ and therefore $F_n^w(\phi) \in H_0^1(\Omega)$.

For a later use we are going to investigate the properties of the operator $B_n(t)$.

Properties of the operator $B_n(t)$

(a) The operator $B_n(t)$ is bounded. First we recall that the linear operator $\frac{\partial}{\partial x_3}$ is continuous from $L^2(\Omega)$ to V' (see Remark 3.7),

$$\left\|\frac{\partial K(\theta)}{\partial x_3}\right\|_{V'} \le \|K(\theta)\|.$$

Recall also that the linear operator $-\Delta$ is the isometric isomorphism from V to $V^{'}$. If $\Phi \in V$, then

$$\|-\Delta\Phi\|_{V'}^2 = \langle -\Delta\Phi, \psi\rangle_{V',V}$$

where $-\Delta\psi = -\Delta\Phi$, $\psi\,|_{\Gamma} = 0$. It follows that

$$\|-\Delta\Phi\|_{V'}^2 = \left\langle -\Delta\Phi, -\Delta^{-1}(-\Delta\Phi)\right\rangle_{V',V},$$

whence

$$\|-\Delta\Phi\|_{V'} = \|\Phi\|_V\,,\ \ \forall\Phi \in V. \tag{5.24}$$

In particular, applying (5.24) to $F_n^w(\phi) \in V$ and using a previous estimate for $F_n^w(\phi)$ we deduce that

$$\|-\Delta F_n^w(\phi)\|_{V'} = \|F_n^w(\phi)\|_V \leq \rho_n \|\phi\|_V + \rho_n^1. \tag{5.25}$$

Using again the continuity of $\dfrac{\partial}{\partial x_3}$ and a consequence of $(\mathrm{i_K})$ we get that

$$\left\|\frac{\partial K(\phi+w)}{\partial x_3}\right\|_{V'} \leq \|K(\phi+w)\| \leq M\,\|\phi+w\| \leq Mc_\Omega \|\phi\|_V + M\,\|w\| < \infty,$$

where c_Ω is the constant in Poincaré's inequality.

Now we have for each $\phi \in V$

$$\|B_n(t)\phi\|_{V'} = \left\|-\Delta F_n^w(\phi) + \frac{\partial K(\phi{+}w)}{\partial x_3}\right\|_{V'} \leq \|-\Delta F_n^w(\phi)\|_{V'} + \left\|\frac{\partial K(\phi{+}w)}{\partial x_3}\right\|_{V'}$$

and taking into account the previous estimates we can conclude that

$$\|B_n(t)\phi\|_{V'} \leq \varepsilon_n \|\phi\|_V + \varepsilon_n^1,\ \ \forall\phi \in V, \tag{5.26}$$

with $\varepsilon_n = \rho_n + Mc_\Omega$ and $\varepsilon_n^1 = \rho_n^1 + M\,\|w\|$.

(b) For each $\phi \in V$ we calculate now

$$\begin{aligned}
&\langle B_n(t)\phi, \phi\rangle_{V',V}\\
&= \int_\Omega \nabla\left(D_n^*(\phi+w) - D_n^*(w)\right)\cdot\nabla\phi dx - \int_\Omega K(\phi+w)\frac{\partial\phi}{\partial x_3}\,dx\\
&\geq \int_\Omega \left(D_n(\phi+w)\nabla(\phi+w) - D(w)\nabla w\right)\cdot\nabla\phi dx - M\,\|\phi+w\|\left\|\frac{\partial\phi}{\partial x_3}\right\|\\
&\geq \int_\Omega \left[\rho\,|\nabla\phi|^2 + \left(D_n(\phi+w) - D(w)\right)\nabla w\cdot\nabla\phi\right] dx - M\,\|\phi+w\|\,\|\phi\|_V\\
&\geq \rho\,\|\phi\|_V^2 - \frac{\rho}{4}\int_\Omega |\nabla\phi|^2\,dx - \frac{\delta_n^2}{\rho}\int_\Omega |\nabla w|^2\,dx\\
&-\left(\frac{\rho}{4}\,\|\phi\|_V^2 + \frac{2M^2}{\rho}\,\|\phi\|^2 + \frac{2M^2}{\rho}\,\|w\|^2\right),
\end{aligned}$$

where $\delta_n^2 = 2(\rho_n^2 + D_w^2)$. This yields

$$\langle B_n(t)\phi, \phi\rangle_{V',V} \geq \frac{\rho}{2} \|\phi\|_V^2 - \gamma_0 \|\phi\|^2 - \gamma_n, \tag{5.27}$$

with $\gamma_0 = \dfrac{2M^2}{\rho}$, $\gamma_n = \dfrac{1}{\rho}(\delta_n^2 + 2M^2) \|w\|_{H_1(\Omega)}$.

(c) Finally we show that the operator $B_n(t)$ is demicontinuous from V to V', i.e., it is strongly-weakly continuous from V to V'.

If we take a sequence $\{\phi_m\}_{m\geq 1} \in V$, strongly convergent to $\phi \in V$, as $m \to \infty$, then we have on a subsequence (still denoted ϕ_m) that $\phi_m \to \phi$ a.e. in Ω. Therefore, by the continuity of the function D_n^* we get $D_n^*(\phi_m) \to D_n^*(\phi)$ a.e. in Ω.

Since $\|\nabla D_n^*(\phi_m)\| \leq \rho_n \|\phi_m\|$ it follows that $D_n^*(\phi_m)$ is bounded in V (which is compact in $L^2(\Omega)$) and hence $\{D_n^*(\phi_m)\}_{m\geq 1}$ is compact in $L^2(\Omega)$. This implies, for each fixed $n \geq 1$, that

$$D_n^*(\phi_m) \to D_n^*(\phi) \text{ strongly in } L^2(\Omega), \text{ as } m \to \infty.$$

Next, for any $\psi \in L^2(\Omega)$ we have

$$\begin{aligned}
&\int_\Omega \left(\frac{\partial D_n^*(\phi_m)}{\partial x_i} - \frac{\partial D_n^*(\phi)}{\partial x_i}\right) \psi dx \\
&= \int_\Omega \left(D_n(\phi_m)\frac{\partial \phi_m}{\partial x_i} - D_n(\phi)\frac{\partial \phi}{\partial x_i}\right) \psi dx \\
&= \int_\Omega \left\{(D_n(\phi_m) - D_n(\phi))\frac{\partial \phi_m}{\partial x_i} + D_n(\phi)\left(\frac{\partial \phi_m}{\partial x_i} - \frac{\partial \phi}{\partial x_i}\right)\right\} \psi dx.
\end{aligned}$$

Since D_n is continuous we have $D_n(\phi_m) \to D_n(\phi)$ a.e. in Ω. On the other hand, since $|D_n(\phi_m)| \leq \rho_n$ a.e. in Ω, we infer by Lebesgue dominated convergence theorem that

$$D_n(\phi_m) \longrightarrow D_n(\phi) \text{ strongly in } L^2(\Omega), \text{ as } m \to \infty.$$

Moreover,

$$\frac{\partial \phi_m}{\partial x_i} \longrightarrow \frac{\partial \phi}{\partial x_i} \text{ strongly in } L^2(\Omega), \text{ as } m \to \infty$$

and hence

$$\int_\Omega \frac{\partial D_n^*(\phi_m)}{\partial x_i} \psi dx \longrightarrow \int_\Omega \frac{\partial D_n^*(\phi)}{\partial x_i} \psi dx, \text{ as } m \to \infty, \ \forall \psi \in L^2(\Omega),$$

i.e.,

$$\nabla D_n^*(\phi_m) \longrightarrow \nabla D_n^*(\phi) \text{ weakly in } L^2(\Omega), \text{ as } m \to \infty.$$

Actually we have got that

$$\nabla F_n^w(\phi_m) \longrightarrow \nabla F_n^w(\phi) \text{ weakly in } L^2(\Omega), \text{ as } m \to \infty.$$

Because K is Lipschitz we have

$$\left\| \frac{\partial \left(K(\phi_m + w) - K(\phi + w) \right)}{\partial x_3} \right\|_{V'} \leq \| K(\phi_m + w) - K(\phi + w) \| \leq \| \phi_m - \phi \| .$$

In conclusion, if $\phi_m \longrightarrow \phi$ strongly in V then

$$B_n(t)\phi_m \longrightarrow B_n(t)\phi \text{ weakly in } V', \text{ as } m \to \infty,$$

as claimed.

We resume now problem (5.19). However, we may observe that the operator $B_n(t)$ is not monotone from V to V' and this does not allow us to apply directly Lions' theorem (see Theorem 4.4 in Sect. 3.4). We have to resort to another way for proving the existence result for this Cauchy problem, that is to perform another approximation, this time for the operator $B_n(t)$ with n fixed. The existence result for the last approximating problem will be proved in Lemma 5.4 while in the following proposition we shall determine an a priori estimate for the approximating solution to (5.19). These two results form the second step of the procedure.

As we can see, some parts of these proofs are similar and can be easier watched in Proposition 5.3, where the notations are less loaded than those used in Lemma 5.4. That is why we have chosen to present them in this order, even if Lemma 5.4 should have been the first one.

Existence in the original problem

As we have just explained, we begin with a proposition intended to determine the a priori estimate necessary for the passing to limit procedure (Step 3 of the proof).

Proposition 5.3. *Let $f \in L^2(0,T;V')$ and $\theta_0 \in L^2(\Omega)$. Then the approximating problem (5.19) has, for each $n \in \mathbf{N}^*$, a unique solution*

$$\phi_n \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V), \tag{5.28}$$

$$\frac{d\phi_n}{dt} \in L^2(0,T;V'), \tag{5.29}$$

$$F_n^w(\phi_n) \in L^2(0,T;V). \tag{5.30}$$

Moreover ϕ_n satisfies the estimate

$$\begin{aligned} &\int_\Omega j_n((\phi_n + w)(t))\,dx + \int_0^t \left\| \frac{d\phi_n(\tau)}{d\tau} \right\|_{V'}^2 d\tau + \int_0^t \| F_n^w\,(\phi_n(\tau)) \|_V^2\,d\tau \\ &\leq \beta_0(t) \left(\int_\Omega j(\theta_0) dx + \|\phi_0 + w_0\|^2 + \|D^*(w_0)\|^2 \right. \\ &\left. + \int_0^T \left\| D\,(w(t))\,\frac{\partial w}{\partial t}\,(t) \right\|^2 dt + \int_0^T \left\| f^B(t) \right\|_{V'}^2 dt + D_w^2\,\theta_s^2 \operatorname{meas}(\Omega) \right) \end{aligned} \tag{5.31}$$

independently of n, *where*

$$j_n(r) = \int_0^r D_n^*(\xi)d\xi, \ \forall r \in \mathbf{R}. \tag{5.32}$$

Proof. Let $f \in L^2(0,T;V')$ and $\theta_0 \in L^2(\Omega)$. Hence $f^B - \dfrac{dw}{dt} \in L^2(0,T;V')$ and $\phi_0 = \theta_0 - w_0 \in L^2(\Omega)$, in virtue of (5.9) and (H_w). Then (5.28)-(5.30) follow from Lemma 5.4 below.

Using (5.3) and (H_w) we obtain also that

$$\theta_n \in L^2(0,T;H^1(\Omega)), \ \frac{d\theta_n}{dt} \in L^2(0,T;V'). \tag{5.33}$$

A priori estimate

In order to prove estimate (5.31) we multiply equation (5.19) by $F_n^w(\phi_n) \in H_0^1(\Omega)$ and integrate it over $\Omega \times (0,t)$ for $t \in (0,T)$. We obtain

$$\int_0^t \int_\Omega \frac{d(\phi_n + w)}{d\tau} F_n^w(\phi_n) dx\, d\tau + \int_0^t \int_\Omega |\nabla F_n^w(\phi_n)|^2 \, dx\, d\tau \tag{5.34}$$
$$= \int_0^t \int_\Omega f^B F_n^w(\phi_n) dx\, d\tau + \int_0^t \int_\Omega K(\phi_n + w) \frac{\partial F_n^w(\phi_n)}{\partial x_3}\, dx\, d\tau.$$

Note that j_n is exactly the differential of the function D_n^*, for $r \in \mathbf{R}$. We take into account the relations

$$\frac{\partial j_n(\phi_n)}{\partial t} = D_n^*(\phi_n)\frac{\partial \phi_n}{\partial t}, \ \text{a.e. } t \in (0,T) \tag{5.35}$$

and

$$j_n(r) = \int_0^r \int_0^\xi D_n(\sigma)d\sigma d\xi \geq \int_0^r \rho\xi\, d\xi = \frac{\rho}{2} r^2, \tag{5.36}$$

the last one being deduced from (5.32) and (5.18). Then we calculate

$$\int_0^t \int_\Omega \frac{d(\phi_n + w)}{d\tau}\, F_n^w(\phi_n) dx\, d\tau$$
$$= \int_\Omega \int_0^t \frac{d(\phi_n + w)}{d\tau}\, D_n^*(\phi_n + w) d\tau\, dx - \int_\Omega \int_0^t \frac{d(\phi_n + w)}{d\tau}\, D^*(w) d\tau\, dx$$
$$= \int_\Omega \int_0^t \frac{\partial j_n(\phi_n + w)}{\partial \tau}\, d\tau\, dx$$
$$- \int_\Omega \left[(\phi_n + w) D^*(w) \,\Big|_0^t - \int_0^t (\phi_n + w) D(w) \frac{\partial w}{\partial \tau}\, d\tau \right] dx$$
$$= \int_\Omega \left[j_n\left(\phi_n(t) + w(t)\right) - j_n(\theta_0) \right] dx - \int_\Omega \left(\phi_n(t) + w(t)\right) D^*\left(w(t)\right) dx$$
$$+ \int_\Omega (\phi_0 + w_0) D^*(w_0) dx + \int_\Omega \int_0^t (\phi_n + w) D(w) \frac{\partial w}{\partial \tau}\, d\tau\, dx.$$

By (5.36) $j_n(\phi_n + w) \geq \frac{\rho}{2}(\phi_n + w)^2$, and it follows from (5.34) that

$$\begin{aligned}
&\frac{\rho}{2}\left\|\phi_n(t)+w(t)\right\|^2 \qquad (5.37)\\
&\leq \int_\Omega j_n(\phi_n(t)+w(t))dx + \int_0^t\int_\Omega \left|\nabla F_n^w(\phi_n)\right|^2 dx\, d\tau\\
&= \int_\Omega j(\theta_0)dx + \int_0^t\int_\Omega K(\phi_n+w)\frac{\partial F_n^w(\phi_n)}{\partial x_3}\, d\tau\, dx\\
&- \int_\Omega (\phi_0+w_0)D^*(w_0)dx - \int_0^t\int_\Omega (\phi_n+w)D(w)\frac{\partial w}{\partial \tau}\, d\tau\, dx\\
&+ \int_\Omega (\phi_n(t)+w(t))D^*(w(t))dx + \int_0^t\int_\Omega f^B F_n^w(\phi_n)d\tau\, dx.
\end{aligned}$$

On the basis of the inequality

$$ab \leq \varepsilon^2 a^2 + \frac{b^2}{\varepsilon^2}$$

we have the following relations:

$$-(\phi_0+w_0)D^*(w_0) \leq \frac{1}{2}(\phi_0+w_0)^2 + \frac{1}{2}\left(D^*(w_0)\right)^2,$$

$$(\phi_n+w)D^*(w) \leq \frac{\rho}{4}(\phi_n+w)^2 + \frac{1}{\rho}\left(D^*(w)\right)^2,$$

$$-(\phi_n+w)D(w)\frac{\partial w}{\partial t} \leq \frac{1}{2}(\phi_n+w)^2 + \frac{1}{2}\left(D(w)\frac{\partial w}{\partial t}\right)^2,$$

$$\begin{aligned}
\left|\int_0^t\int_\Omega f^B F_n^w(\phi_n)d\tau dx\right| &\leq \int_0^t \left\|f^B(\tau)\right\|_{V'}\left\|F_n^w\left(\phi_n(\tau)\right)\right\|_V d\tau\\
&\leq \int_0^t \left\|f^B(\tau)\right\|_{V'}^2 + \frac{1}{4}\int_0^t \left\|F_n^w\left(\phi_n(\tau)\right)\right\|_V^2 d\tau,
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^t\int_\Omega K(\phi_n+w)\frac{\partial F_n^w(\phi_n)}{\partial x_3}d\tau dx\right| \leq M\int_0^t \left\|(\phi_n+w)(\tau)\right\|\left\|F_n^w\left(\phi_n(\tau)\right)\right\|_V d\tau\\
&\leq \frac{1}{4}\int_0^t \left\|F_n^w(\phi_n(\tau))\right\|_V^2 d\tau + M^2\int_0^t \left\|\phi_n(\tau)+w(\tau)\right\|^2 d\tau.
\end{aligned}$$

Plugging all these in (5.37) we obtain

$$\begin{aligned}
&\frac{\rho}{2}\left\|(\phi_n+w)(t)\right\|^2 \leq \int_\Omega j_n\left(\phi_n(t)+w(t)\right)dx + \int_0^t \left\|F_n^w\left(\phi_n(\tau)\right)\right\|_V^2 d\tau\\
&\leq \int_\Omega j_n(\theta_0)dx + \frac{1}{2}\left\|\phi_0+w_0\right\|^2 + \frac{1}{2}\left\|D^*(w_0)\right\|^2 + \frac{1}{\rho}\left\|D^*\left(w(t)\right)\right\|^2
\end{aligned}$$

$$+\frac{\rho}{4}\left\|(\phi_n+w)(t)\right\|^2+\frac{1}{2}\int_0^t\left\|D\left(w(\tau)\right)\frac{\partial w}{\partial\tau}(\tau)\right\|^2d\tau+\int_0^t\left\|f^B(\tau)\right\|_{V'}^2d\tau$$
$$+\frac{1}{2}\int_0^t\left\|F_n^w\left(\phi_n(\tau)\right)\right\|_V^2d\tau+\left(M^2+\frac{1}{2}\right)\int_0^t\left\|(\phi_n+w)(\tau)\right\|^2d\tau.$$

But from (5.32) and (5.18) we have

$$j_n(\theta_0)=\int_0^{\theta_0}D_n^*(\xi)d\xi=\int_0^{\theta_0}\int_0^{\xi}D_n(\sigma)d\sigma d\xi\le\int_0^{\theta_0}\int_0^{\xi}D(\sigma)d\sigma d\xi=j(\theta_0).$$

Hence

$$\int_\Omega j_n(\theta_0)dx\le\int_\Omega j(\theta_0)dx. \tag{5.38}$$

Finally, we get

$$\begin{aligned}&\frac{\rho}{4}\left\|(\phi_n+w)(t)\right\|^2 \\ &\le\int_\Omega j_n\left(\phi_n(t)+w(t)\right)dx+\frac{1}{2}\int_0^t\left\|F_n^w\left(\phi_n(\tau)\right)\right\|_V^2d\tau \\ &\le c_S+\left(M^2+\frac{1}{2}\right)\int_0^t\left\|(\phi_n+w)(\tau)\right\|^2d\tau,\end{aligned} \tag{5.39}$$

where

$$\begin{aligned}c_S=&\int_\Omega j(\theta_0)dx+\frac{1}{2}\left\|\phi_0+w_0\right\|^2+\frac{1}{2}\left\|D^*(w_0)\right\|^2 \\ &+\frac{1}{2}\int_0^T\left\|D\left(w(t)\right)\frac{\partial w}{\partial t}(t)\right\|^2dt+\int_0^T\left\|f^B(t)\right\|_{V'}^2dt+\frac{1}{\rho}D_w^2\theta_s^2\operatorname{meas}(\Omega)\Big)\end{aligned} \tag{5.40}$$

Note that by (H_w) we have

$$\left\|D(w(t))\frac{\partial w}{\partial t}(t)\right\|\le D_w\left\|\frac{\partial w}{\partial t}(t)\right\|<\infty.$$

Applying now Gronwall's lemma for $\left\|(\phi_n+w)(t)\right\|^2$ we obtain

$$\left\|(\phi_n+w)(t)\right\|^2\le c_S\gamma_0(t)<c_S\gamma_0(T)<\infty, \tag{5.41}$$

where

$$\gamma_0(t)=\frac{4}{\rho}\exp\left[\frac{4}{\rho}\left(M^2+\frac{1}{2}\right)t\right] \tag{5.42}$$

and consequently (5.39) yields

$$\begin{aligned}&\int_\Omega j_n\left(\phi_n(t)+w(t)\right)dx+\int_0^t\left\|F_n^w\left(\phi_n(\tau)\right)\right\|_V^2d\tau \\ &\le 2c_S\exp\left[\frac{4}{\rho}\left(M^2+\frac{1}{2}\right)t\right]<\infty.\end{aligned} \tag{5.43}$$

The constant c_S is independent of n. Then we multiply (5.19) scalarly in $V^{'}$ by $\dfrac{d(\phi_n + w)}{dt}$ and performing computations analogous to those before, we obtain that

$$\int_\Omega j_n\left(\phi_n(t) + w(t)\right) dx + \int_0^t \left\| \frac{d(\phi_n + w)(\tau)}{d\tau} \right\|_{V'}^2 d\tau \qquad (5.44)$$
$$\leq 2c_S \exp\left[\frac{4}{\rho}\left(M^2 + \frac{1}{2}\right) t\right].$$

Adding (5.43) and (5.44) we get (5.31) as claimed, with

$$\beta_0(t) = 4 \exp\left[\tfrac{4}{\rho}\left(M^2 + \tfrac{1}{2}\right) t\right].$$ ∎

This ends the second step which is the basis for approaching the passing to limit procedure.

***Proof of Theorem* 5.2.**

We recall that one of the assumptions in Theorem 5.2 was $\theta_0 \in M_j$. This implies immediately that c_S defined in (5.40) is finite. Then, from (5.41) and (5.31) we deduce the boundedness of some sequences of functions, namely: $\{\phi_n\}_{n\geq 1}$ lies in a bounded subset of $L^\infty(0,T;L^2(\Omega))$, $\left\{\dfrac{d\phi_n}{dt}\right\}_{n\geq 1}$ lies in a bounded subset of $L^2(0,T;V')$ and $\{F_n^w(\phi_n)\}_{n\geq 1}$ is included in a bounded subset of $L^2(0,T;V)$.

But the last assertion together with the remark that F_n^w satisfies the property (i) implies that the sequence $\{\phi_n\}_{n\geq 1}$ is bounded in $L^2(0,T;V)$, according to Remark 3.1.

From the boundedness of the sequences previously mentioned, we conclude that we can select a subsequence (that will be denoted by ϕ_n too) such that

$$\phi_n \longrightarrow \phi \quad \text{weak-star in } L^\infty(0,T;L^2(\Omega)), \text{ as } n \to \infty, \qquad (5.45)$$
$$\phi_n \longrightarrow \phi \quad \text{weakly in } L^2(0,T;V), \text{ as } n \to \infty, \qquad (5.46)$$
$$\frac{d\phi_n}{dt} \longrightarrow \frac{d\phi}{dt} \quad \text{weakly in } L^2(0,T;V'), \text{ as } n \to \infty, \qquad (5.47)$$

and

$$F_n^w(\phi_n) \longrightarrow \chi \text{ in } L^2(0,T;V), \text{ as } n \to \infty. \qquad (5.48)$$

The last implies that

$$D_n^*(\phi_n + w) \longrightarrow \eta \text{ weakly in } L^2(0,T;H^1(\Omega)), \text{ as } n \to \infty. \qquad (5.49)$$

Since $V = H_0^1(\Omega)$ is compact in $H = L^2(\Omega)$ we conclude, according to the Lions-Aubin theorem that $\{\phi_n\}_{n\geq 1}$ is compact in $L^2(0,T;L^2(\Omega))$, i.e., on a subsequence we have

$$\phi_n \longrightarrow \phi \text{ strongly in } L^2(0,T;L^2(\Omega)), \text{ as } n \to \infty, \tag{5.50}$$

implying that the corresponding sequence $\{\theta_n = \phi_n + w\}_{n\geq 1}$ converges to $\theta := \phi + w$ strongly in $L^2(Q)$. From (5.50) it follows that on a subsequence $\phi_n(x,t) \to \phi(x,t)$ a.e. on $\Omega \times (0,T)$.

We claim that $\eta = D^*(\phi + w)$ a.e. on Q.

We set

$$Q_s := \{(x,t) \in Q;\ \theta(x,t) = \theta_s\},\ Q_- := \{(x,t) \in Q;\ \theta(x,t) < \theta_s\}.$$

Then, if $(x,t) \in Q_-$ we have

$$\begin{aligned} D_n^*(\theta_n(x,t)) &= \int_0^{\theta_n(x,t)} D_n(r)dr = \int_0^{\theta_n(x,t)} D(r)dr \\ &\longrightarrow \int_0^{\theta(x,t)} D(r)dr = D^*(\theta(x,t)) \text{ a.e. on } Q_-, \text{ as } n \to \infty. \end{aligned}$$

If $(x,t) \in Q_s$, then two situations may arise:

(p$_1$) there is a sequence $n_k \to \infty$ such that $\theta_{n_k}(x,t) \geq \theta_s - \dfrac{1}{n_k}$ and

(p$_2$) for all n, except a finite number of them, we have $\theta_n(x,t) < \theta_s - \dfrac{1}{n}$.

In the second case the previous argument for $(x,t) \in Q_-$ applies and thus $D_n^*(\theta_n) \longrightarrow D^*(\theta)$ a.e. for $(x,t) \in Q_s$.

In the first case we have

$$\begin{aligned} D_{n_k}^*(\theta_{n_k}(x,t)) &= \int_0^{\theta_{n_k}(x,t)-\frac{1}{n_k}} D(r)dr + \int_{\theta_{n_k}(x,t)-\frac{1}{n_k}}^{\theta_{n_k}(x,t)} D\left(\theta_s - \frac{1}{n_k}\right) dr \\ &= \int_0^{\theta_{n_k}(x,t)-\frac{1}{n_k}} D(r)dr + \frac{1}{n_k} D\left(\theta_s - \frac{1}{n_k}\right) \longrightarrow +\infty = D^*(\theta_s), \\ &\text{as } \ n_k \to \infty, \end{aligned}$$

because $\displaystyle\int_0^{\theta_s} D(r)dr = +\infty$, pursuant to (iii). Hence, selecting a subsequence (denoted still by the subscript n), we have

$$D_n^*(\phi_n + w) \longrightarrow D^*(\phi + w) \text{ a.e. on } Q \text{ as } n \to \infty.$$

But $\{D_n^*(\theta_n)\}_{n\geq 1}$ is bounded, in particular, in $L^2(Q)$ and since it converges a.e. on Q, it follows by Theorem 2.23 in Appendix, that $D_n^*(\theta_n) \to D^*(\theta)$ strongly in $L^1(Q)$.

Relation (5.49) implies in particular that

$$D_n^*(\phi_n + w) \longrightarrow \eta \text{ weakly in } L^2(Q),$$

and consequently it converges weakly in $L^1(Q)$. By the uniqueness of the (weak) limit it follows that $\eta = D^*(\phi + w)$ a.e. $(x,t) \in Q$, as claimed.

By the weakly l.s.c. property it follows from (5.31) that $j(\theta) \in L^1(Q)$ and since $D^*(\theta) \in L^2(0,T;V)$ we infer once again that $\theta < \theta_s$ a.e. $(x,t) \in Q$.

Therefore

$$\chi = \lim_{n\to\infty} (D_n^*(\phi + w) - D_n^*(w)) = D^*(\phi + w) - D^*(w) = F^w(\phi)$$

weakly in $L^2(0,T;V)$.

Due to the continuity of the operator $\frac{\partial}{\partial x_3}$ we still have

$$\frac{\partial K(\phi_n + w)}{\partial x_3} \longrightarrow \frac{\partial K(\phi + w)}{\partial x_3} \text{ strongly in } L^2(0,T;V'), \text{ as } n \to \infty.$$

By the demicontinuity of $B_n(t)$ proved at point (c) in Step 1, we deduce finally that

$$B_n(t)\phi_n \longrightarrow B(t)\phi \text{ weakly in } L^2(0,T;V'), \text{ as } n \to \infty.$$

Now we can pass to limit as $n \to \infty$ in equation (5.19) and obtain

$$\frac{d(\phi + w)}{dt} + B(t)\phi = f^B \text{ a.e. } t \in (0,T)$$
$$(\phi + w)(0) = \phi_0 + w_0,$$

that proves that ϕ is the solution to (5.8). ∎

Existence in the approximating problem

It remains only to prove the existence result in the approximating problem, which would have been the first part of Step 2.

Lemma 5.4. *Let $f \in L^2(0,T;V')$ and $\theta_0 \in L^2(\Omega)$. Then the problem*

$$\frac{d\phi_n}{dt} + B_n(t)\phi_n = f^B - \frac{dw}{dt}, \tag{5.51}$$
$$\phi_n(0) = \phi_0$$

has, for each $n \in \mathbf{N}^*$, a unique solution

$$\phi_n \in L^2(0,T;V); \ \frac{d\phi_n}{dt} \in L^2(0,T;V').$$

Proof. Consider the operator $A_0 = -\Delta : D(A_0) \subset V \to V$, with the domain

$$D(A_0) = \{u \in H_0^1(\Omega) \cap H^2(\Omega); A_0 u \in V\}.$$

(We recall that $V = H_0^1(\Omega)$ with the usual Hilbertian norm). The operator A_0 is the restriction of $A_\Delta : V \to V'$, $A_\Delta = -\Delta$, to V and since A_Δ is monotone, continuous and coercive, we have by Theorem 2.7, in Sect. 3.2 that A_0 is m-accretive on V. Hence, we introduce its resolvent

$$J_\varepsilon : V \to V,\ J_\varepsilon = (I + \varepsilon A_0)^{-1}$$

and the Yosida approximation

$$(A_0)_\varepsilon : V \to V,\ (A_0)_\varepsilon = \frac{1}{\varepsilon}(I - J_\varepsilon),$$

with the properties specified in Proposition 2.10, in Sect. 3.2 and we easily notice that J_ε is an isomorphism between V' and V and also from $L^2(\Omega)$ to $H_0^1(\Omega) \cap H^2(\Omega)$. We recall that in this problem we considered either Ω of class C^2 or having a convex boundary, assumption necessary to obtain a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ to the equation $-\Delta u = f \in L^2(\Omega)$.

For each $\varepsilon > 0$, let us denote by $B_{n,\varepsilon}(t) : V \to V'$ the operator defined by

$$B_{n,\varepsilon}(t) = J_\varepsilon(B_n(t)) + \varepsilon A_0. \tag{5.52}$$

We can still write this as

$$B_{n,\varepsilon}(t)\phi = (I + \varepsilon A_0)^{-1} A_0 F_n^w(\phi) + \varepsilon A_0 \phi + (I + \varepsilon A_0)^{-1} \frac{\partial K(\phi + w)}{\partial x_3}$$

where we consider $-\Delta F_n^w(\phi)$ in the sense of distributions. We immediately remark (see also Sect. 3.2, Proposition 2.10, (b)) that

$$B_{n,\varepsilon}(t)\phi = (A_0)_\varepsilon F_n^w(\phi) + \varepsilon A_0 \phi + (I + \varepsilon A_0)^{-1} \frac{\partial K(\phi + w)}{\partial x_3},\ \forall \phi \in V. \tag{5.53}$$

Indeed, we can write

$$(I + \varepsilon A_0)^{-1} A_0 v = \frac{1}{\varepsilon}(I + \varepsilon A_0)^{-1}(I + \varepsilon A_0)v - \frac{1}{\varepsilon}(I + \varepsilon A_0)^{-1} v,\ \forall v \in V,$$

whence

$$(I + \varepsilon A_0)^{-1} A_0 v = \frac{1}{\varepsilon}(I - J_\varepsilon)v.$$

Next we introduce for each $n \in \mathbf{N}^*$ and $\varepsilon > 0$, the problem

$$\frac{d(\phi_n)_\varepsilon}{dt} + B_{n,\varepsilon}(t)(\phi_n)_\varepsilon = f^B - \frac{dw}{dt} \text{ a.e. } t \in (0, T), \tag{5.54}$$

$$(\phi_n)_\varepsilon(0) = \phi_0$$

and we prove that for $f^B \in L^2(0, T; V')$ and $\phi_0 \in L^2(\Omega)$ it has a unique solution

$$(\phi_n)_\varepsilon \in L^2(0, T; V),\ \frac{d(\phi_n)_\varepsilon}{dt} \in L^2(0, T; V') \tag{5.55}$$

that satisfies, for ε small enough, the estimate

$$\int_{\Omega} j_n\left((\phi_n)_\varepsilon + w)(t)\right) dx \tag{5.56}$$
$$+ \int_0^t \left\| \frac{d(\phi_n)_\varepsilon}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|F_n^w((\phi_n)_\varepsilon(\tau))\|_V^2 \, d\tau$$
$$\leq \beta_n \left(\int_{\Omega} j(\theta_0)dx + \|\phi_0 + w_0\|^2 + \|D^*(w_0)\|^2 + \int_0^T \left\| D(w(t)) \frac{\partial w}{\partial t}(t) \right\|^2 dt \right.$$
$$\left. + \int_0^T \left\| f^B(t) \right\|_{V'}^2 dt + D_w^2 \theta_s^2 \operatorname{meas}(\Omega) + D_w^2 \int_0^T \|w\|^2 \, dt + 1 \right)$$

with β_n independent of ε.

Moreover, we get also that $F_n^w((\phi_n)_\varepsilon) \in L^2(0,T;V)$.

For this we recall some properties established for $B_n(t)$ at points (a)-(c).

As seen earlier $B_n(t)$ is demicontinuous from V to V' (see (c)) so that it follows that $B_{n,\varepsilon}(t)$ is demicontinuous too from V to V' and it transforms a measurable function v from $[0,T]$ to V into a measurable function from $[0,T]$ to V' (see [83], p. 159).

The operator $B_{n,\varepsilon}(t)$ is quasi-monotone, i.e.,

$$\left\langle B_{n,\varepsilon}(t)\phi - B_{n,\varepsilon}(t)\overline{\phi}, \phi - \overline{\phi} \right\rangle_{V',V} \geq \varepsilon \left\| \phi - \overline{\phi} \right\|_V^2 - \mu_n \left\| \phi - \overline{\phi} \right\|^2, \tag{5.57}$$

for any $\phi, \overline{\phi} \in V$.

Indeed, using the properties of the resolvent and Yosida approximation and the fact that $D_n^*(\theta)$ is Lipschitz with the Lipschitz constant ρ_n, we have that

$$\left\langle B_{n,\varepsilon}(t)\phi - B_{n,\varepsilon}(t)\overline{\phi}, \phi - \overline{\phi} \right\rangle_{V',V}$$
$$= \left\langle (A_0)_\varepsilon F_n^w(\phi) - (A_0)_\varepsilon F_n^w(\overline{\phi}), \phi - \overline{\phi} \right\rangle_{V',V} + \varepsilon \left\langle A_0(\phi - \overline{\phi}), \phi - \overline{\phi} \right\rangle_{V',V}$$
$$+ \left\langle J_\varepsilon \left(\frac{\partial K(\phi + w)}{\partial x_3} - \frac{\partial K(\overline{\phi} + w)}{\partial x_3} \right), \phi - \overline{\phi} \right\rangle_{V',V}$$
$$= \frac{1}{\varepsilon} \int_{\Omega} \left((I - J_\varepsilon) F_n^w(\phi) - (I - J_\varepsilon) F_n^w(\overline{\phi}) \right) (\phi - \overline{\phi}) dx$$
$$+ \left\langle \frac{\partial K(\phi + w)}{\partial x_3} - \frac{\partial K(\overline{\phi} + w)}{\partial x_3}, J_\varepsilon(\phi - \overline{\phi}) \right\rangle_{V',V} + \varepsilon \left\| \phi - \overline{\phi} \right\|_V^2$$
$$\geq \frac{\rho}{\varepsilon} \left\| \phi - \overline{\phi} \right\|^2 + \varepsilon \left\| \phi - \overline{\phi} \right\|_V^2 - \frac{1}{\varepsilon} \left\| J_\varepsilon F_n^w(\phi) - J_\varepsilon F_n^w(\overline{\phi}) \right\| \left\| \phi - \overline{\phi} \right\|$$
$$- \left(K(\phi + w) - K(\overline{\phi} + w), \frac{\partial}{\partial x_3} \left(J_\varepsilon(\phi - \overline{\phi}) \right) \right)$$
$$\geq \frac{\rho}{\varepsilon} \left\| \phi - \overline{\phi} \right\|^2 + \varepsilon \left\| \phi - \overline{\phi} \right\|_V^2 - \frac{1}{\varepsilon} \left\| F_n^w(\phi) - F_n^w(\overline{\phi}) \right\| \left\| \phi - \overline{\phi} \right\|$$
$$- \left\| K(\phi + w) - K(\overline{\phi} + w) \right\| \left\| \frac{\partial}{\partial x_3} \left(J_\varepsilon(\phi - \overline{\phi}) \right) \right\|$$

$$\geq \frac{\rho}{\varepsilon}\left\|\phi-\overline{\phi}\right\|^2+\varepsilon\left\|\phi-\overline{\phi}\right\|_V^2-\frac{\rho_n}{\varepsilon}\left\|\phi-\overline{\phi}\right\|^2-M\left\|\phi-\overline{\phi}\right\|^2$$
$$=\varepsilon\left\|\phi-\overline{\phi}\right\|_V^2-\mu_n\left\|\phi-\overline{\phi}\right\|^2,\ \mu_n=\frac{\rho_n-\rho}{\varepsilon}+M>0.$$

Here we used the sequence of the following inequalities

$$\left\|\frac{\partial}{\partial x_3}\left(J_\varepsilon(\phi-\overline{\phi})\right)\right\|\leq\left\|J_\varepsilon(\phi-\overline{\phi})\right\|_V\leq\left\|\phi-\overline{\phi}\right\|_{V'}\leq\left\|\phi-\overline{\phi}\right\|.$$

If in (5.57) we take $\overline{\phi}=0$, we see that $B_{n,\varepsilon}(t)$ fulfils also the condition specified in Remark 4.3, in Sect. 3.4. Then, by (a) and (5.23) we have

$$\begin{aligned}\|B_{n,\varepsilon}(t)\phi\|_{V'} &\leq \|(A_0)_\varepsilon(F_n^w(\phi))\|_{V'}+\left\|J_\varepsilon\left(\frac{\partial K(\phi+w)}{\partial x_3}\right)\right\|_{V'}+\varepsilon\|A_0\phi\|_{V'}\\
&\leq \|A_0J_\varepsilon(F_n^w(\phi))\|_{V'}+\left\|J_\varepsilon\left(\frac{\partial K(\phi+w)}{\partial x_3}\right)\right\|_V+\varepsilon\|\phi\|_V\\
&\leq \|J_\varepsilon(F_n^w(\phi))\|_V+\left\|\frac{\partial K(\phi+w)}{\partial x_3}\right\|_{V'}+\varepsilon\|\phi\|_V\\
&\leq \|F_n^w(\phi)\|_{V'}+\|K(\phi+w)\|+\varepsilon\|\phi\|_V\\
&\leq \|F_n^w(\phi)\|+\|K(\phi+w)\|+\varepsilon\|\phi\|_V\\
&\leq ((\rho_n+M)c_\Omega+\varepsilon)\|\phi\|_V+\rho_n^1+(\rho_n+M)\|w\|.\end{aligned}$$

Hence, it follows that $B_{n,\varepsilon}(t)$ is bounded and for ε small enough we have

$$\|B_{n,\varepsilon}(t)\phi\|_{V'}\leq\nu_n(\|\phi\|_V+1),\tag{5.58}$$

with $\nu_n=\max\{\rho_n+Mc_\Omega,(\rho_n^1+(\rho_n+M))\|w\|_{H^1(\Omega)}\}$.

On the basis of all these results we can apply Corollary 4.5 in Sect. 3.4, to deduce that problem (5.54) has a unique solution with the properties specified by (5.55).

Since $F_n^w((\phi_n)_\varepsilon)\in L^2(0,T;V)$ and $(F_n^w)^{-1}$ is Lipschitz with the constant $\frac{1}{\rho}$ we obtain that

$$(\phi_n)_\varepsilon\in L^2(0,T;V).\tag{5.59}$$

To finish the proof of this lemma we have only to deduce estimate (5.56). To come to this end we need the following result.

Lemma 5.5. *For any $F\in V$ and $F_0\in D(A_0)$ we have*

$$\left\|(I+\varepsilon A_0)^{-1}F-F\right\|_V\leq\left\|(I+\varepsilon A_0)^{-1}F_0-F_0\right\|_V+2\|F-F_0\|_V\tag{5.60}$$

and

$$\left\|(I+\varepsilon A_0)^{-1}F_0-F_0\right\|_V\leq\varepsilon\|A_0F_0\|_V.\tag{5.61}$$

Proof. To prove (5.60) we note that

$$\|J_\varepsilon F - F\|_V = \|J_\varepsilon(F - F_0) + J_\varepsilon F_0 - F_0 - (F - F_0)\|_V$$

and we use the properties of J_ε, while for getting (5.61) we compute

$$\begin{aligned}\|J_\varepsilon F_0 - F_0\|_V &= \left\|J_\varepsilon F_0 - J_\varepsilon J_\varepsilon^{-1} F_0\right\|_V \leq \left\|F_0 - J_\varepsilon^{-1} F_0\right\|_V \\ &= \|F_0 - (I + \varepsilon A_0)F_0\|_V = \varepsilon \|A_0 F_0\|_V .\end{aligned}$$

Eventually we obtain

$$\|J_\varepsilon F - F\|_V \leq \varepsilon \|A_0 F_0\|_V + 2\|F - F_0\|_V , \tag{5.62}$$

for any $F \in V$ and $F_0 \in D(A_0)$. ∎

Now we resume the proof of estimate (5.56) and we rewrite problem (5.54) in the following way

$$\begin{gathered}\frac{d(\phi_n)_\varepsilon}{dt} + B_n(t)(\phi_n)_\varepsilon = f^B - \frac{dw}{dt} + g_\varepsilon, \\ (\phi_n)_\varepsilon(0) = \phi_0\end{gathered} \tag{5.63}$$

where

$$\begin{aligned} g_\varepsilon = {} & (I - J_\varepsilon)A_0 F_n^w\left((\phi_n)_\varepsilon\right) \\ & +(I - J_\varepsilon)\left(\frac{\partial K\left((\phi_n)_\varepsilon + w\right)}{\partial x_3}\right) - \varepsilon A_0(\phi_n)_\varepsilon \end{aligned} \tag{5.64}$$

which belongs to V'. Indeed, for any $\phi \in V$ we have that

$$\begin{aligned}\left\|(I - J_\varepsilon)\left(\frac{\partial K(\phi)}{\partial x_3}\right)\right\|_{V'} &\leq \left\|\frac{\partial K(\phi)}{\partial x_3}\right\|_{V'} + \left\|J_\varepsilon\left(\frac{\partial K(\phi)}{\partial x_3}\right)\right\|_{V'} \\ &\leq \left\|\frac{\partial K(\phi)}{\partial x_3}\right\|_{V'} + \left\|J_\varepsilon\left(\frac{\partial K(\phi)}{\partial x_3}\right)\right\|_{V} \\ &\leq 2\left\|\frac{\partial K(\phi)}{\partial x_3}\right\|_{V'} \leq 2\|K(\phi)\| \leq 2M\|\phi\|\end{aligned}$$

and

$$\|(I - J_\varepsilon)A_0\phi\|_{V'} = \|-\Delta(I - J_\varepsilon)\phi\|_{V'} = \|(I - J_\varepsilon)\phi\|_V , \ \forall \phi \in V.$$

These lead to the estimate

$$\begin{aligned}\|g_\varepsilon(t)\|_{V'} \leq {} & \left\|(I + \varepsilon A_0)^{-1} F_n^w\left((\phi_n)_\varepsilon(t)\right) - F_n^w\left((\phi_n)_\varepsilon(t)\right)\right\|_V \\ & + 2M\left\|\left((\phi_n)_\varepsilon + w\right)(t)\right\| + \varepsilon\|(\phi_n)_\varepsilon(t)\|_V \\ \leq {} & \varepsilon\left\|A_0 F_n^w\left((\phi_n)_\varepsilon(t)\right)\right\|_V + 2M\left\|\left((\phi_n)_\varepsilon + w\right)(t)\right\| \\ & + \frac{\varepsilon}{\rho}\left\|F_n^w\left((\phi_n)_\varepsilon(t)\right)\right\|_V + \frac{\varepsilon}{\rho}\|w(t)\|_{H^1(\Omega)}(D_w + \rho_n).\end{aligned} \tag{5.65}$$

Now we multiply equation (5.63) by $F_n^w((\phi_n)_\varepsilon(t))$ and integrate over $\Omega\times(0,t)$. A straightforward computation following the operations performed to obtain estimate (5.31) provides the inequality

$$\begin{aligned}&\int_\Omega j_n((\phi_n)_\varepsilon(t)+w(t))\,dx+\frac{1}{2}\int_0^t \|F_n^w((\phi_n)_\varepsilon(\tau))\|_V^2\,d\tau \\ &\leq c_S+\left(M^2+\frac{1}{2}\right)\int_0^t \|((\phi_n)_\varepsilon+w))(\tau)\|^2\,d\tau \\ &+\int_0^t \|g_\varepsilon(\tau)\|_{V'}\|F_n^w((\phi_n)_\varepsilon(\tau))\|_V\,d\tau\end{aligned} \tag{5.66}$$

where c_S is defined by (5.40). We notice that the equation

$$(I+\varepsilon A_0)^{-1}F_n^w((\phi_n)_\varepsilon(\tau))=F_0$$

has, for any $F_0\in D(A_0)$, a unique solution which is involved in the equality

$$\varepsilon\|A_0F_0\|_V=\|F_n^w((\phi_n)_\varepsilon(\tau))-F_0\|_V.$$

Since F_0 is arbitrary in $D(A_0)$ we can choose it in an appropriate way and for example we select it to satisfy

$$\|F_n^w((\phi_n)_\varepsilon(\tau))-F_0\|_V\leq\frac{1}{8}\|F_n^w((\phi_n)_\varepsilon(\tau))\|_V. \tag{5.67}$$

Consequently we have

$$\begin{aligned}\|F_0\|_V&=\|F_n^w((\phi_n)_\varepsilon(\tau))-F_0\|_V+\|F_n^w((\phi_n)_\varepsilon(\tau))\|_V \\ &\leq\frac{9}{8}\|F_n^w((\phi_n)_\varepsilon(\tau))\|_V\end{aligned}$$

and

$$\varepsilon\|A_0F_0\|_V\leq\frac{1}{8}\|F_n^w((\phi_n)_\varepsilon(\tau))\|_V. \tag{5.68}$$

We shall denote by $C_n(w)$ several different positive constants depending on the H^1-norm of w, D_w, M, ρ and independent of ε.

By (5.65), (5.60), (5.61), (5.62) and (5.67) we obtain

$$\begin{aligned}&\int_0^t \|g_\varepsilon(\tau)\|_{V'}\|F_n^w((\phi_n)_\varepsilon(\tau))\|_V \\ &\leq\int_0^t \|(J_\varepsilon F_n^w((\phi_n)_\varepsilon(\tau))-F_n^w((\phi_n)_\varepsilon(\tau))\|_V\|F_n^w((\phi_n(\tau))\|_V\,d\tau \\ &+\int_0^t 2M\|((\phi_n)_\varepsilon+w)(t))\|\,\|F_n^w((\phi_n)_\varepsilon(\tau))\|_V\,d\tau \\ &+\frac{\varepsilon}{\rho}\int_0^t \|F_n^w((\phi_n)_\varepsilon(\tau))\|_V^2\,d\tau+\varepsilon C_n(w)\int_0^t \|F_n^w((\phi_n)_\varepsilon(\tau))\|_V\,d\tau\end{aligned}$$

$$\leq \frac{3}{8} \int_0^t \|F_n^w ((\phi_n)_\varepsilon(\tau))\|_V^2 \, d\tau + 8M^2 \int_0^t \|(\phi_n)_\varepsilon(\tau) + w(\tau)\|^2 \, d\tau$$
$$+ \frac{\varepsilon}{\rho} \int_0^t \|F_n^w ((\phi_n)_\varepsilon(\tau))\|_V^2 \, d\tau + \varepsilon^2 C_n(w).$$

This plugged into (5.66) yields

$$c_\rho \rho \, \|((\phi_n)_\varepsilon + w)(t)\|^2$$
$$\leq \int_\Omega j_n ((\phi_n)_\varepsilon(t) + w(t)) \, dx + \left(1 - \frac{\varepsilon}{\rho}\right) \int_0^t \|F_n^w ((\phi_n)_\varepsilon(\tau))\|_V^2 \, d\tau$$
$$\leq C_n(w) + \left(9M^2 + \frac{1}{2}\right) \int_0^t \|((\phi_n)_\varepsilon + w)(\tau)\|^2 \, d\tau,$$

where c_ρ is a constant. We apply then Gronwall's lemma and we obtain that

$$\|((\phi_n)_\varepsilon + w)(t)\|^2 \leq \frac{1}{c_\rho \rho} C_n(w) \exp\left[\frac{1}{c_\rho \rho}\left(9M^2 + \frac{1}{2}\right) T\right]. \tag{5.69}$$

Further, we proceed like in the proof of (5.31) and get (5.56).

These estimates allow us to conclude that the sequences $\{F_n^w ((\phi_n)_\varepsilon)\}_{\varepsilon>0}$ and $\{(\phi_n)_\varepsilon\}_{\varepsilon>0}$, with n fixed, are bounded in $L^2(0,T;V)$ by constants independent of ε.

Therefore it follows from (5.63) that

$$\left\{\frac{d(\phi_n)_\varepsilon}{dt}\right\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;V').$$

We continue like in the last part of Proposition 5.3 and conclude that for each $n \geq 1$, fixed, we have

$$\begin{aligned}
(\phi_n)_\varepsilon &\longrightarrow \phi_n && \text{strongly in } L^2(0,T;L^2(\Omega)) \text{ and} \\
& && \text{weakly in } L^2(0,T;V), \text{ as } \varepsilon \to 0, \\
\frac{d(\phi_n)_\varepsilon}{dt} &\longrightarrow \frac{d\phi_n}{dt} && \text{weakly in } L^2(0,T;V'), \text{ as } \varepsilon \to 0, \\
F_n^w ((\phi_n)_\varepsilon) &\longrightarrow F_n^w(\phi_n) && \text{weakly in } L^2(0,T;V), \text{ as } \varepsilon \to 0,
\end{aligned}$$

and

$$B_n(t)(\phi_n)_\varepsilon \longrightarrow B_n(t)\phi_n \text{ weakly in } L^2(0,T;V'), \text{ as } \varepsilon \to 0.$$

We stress that the convergences are with respect to ε for each n fixed.

Finally, we can pass to limit as $\varepsilon \to 0$ in (5.63), taking into account that

$$g_\varepsilon \longrightarrow 0 \text{ weakly in } L^2(0,T;V'), \text{ as } \varepsilon \to 0.$$

Indeed, for any $v \in V$ we have

$$\int_0^T \langle g_\varepsilon(\tau), v\rangle_{V',V}\, d\tau = \int_0^T \langle (I - J_\varepsilon)B_n(\tau)(\phi_n)_\varepsilon(\tau) - \varepsilon A_0(\phi_n)_\varepsilon(\tau), v\rangle_{V',V}\, d\tau$$

$$= \int_0^t \langle B_n(\tau)(\phi_n)_\varepsilon(\tau), v - J_\varepsilon v\rangle_{V',V}\, d\tau - \varepsilon \int_0^t \langle A_0(\phi_n)_\varepsilon(\tau), v\rangle_{V',V}\, d\tau \longrightarrow 0,$$

as $\varepsilon \to 0$.

The uniqueness can be proved via the same way as in Proposition 5.3. This ends the proof of Lemma 5.4. ∎

Properties of the solution

Remark 5.5. We notice that in the case of the nonhomogeneous Dirichlet boundary conditions depending on both space and time variables, the lack of monotonicity of the time dependent operator $B(t)$ did not allow the direct application of Theorem 3.6 in Sect. 3.3, with the initial datum in the domain of the operator. However, the existence could be obtained by choosing directly the initial datum in a larger space but the solution ϕ was obtained in a weak form.

Concerning the situation when the initial distribution of the moisture is less regular, i.e., $\theta_0 \in M_{\theta_s}$ defined by (2.18), we have the next result which states the regularization of the solution by the action of the parabolic operator.

Theorem 5.6. *Let $f \in L^2(0,T;V')$, $\theta_0 \in M_{\theta_s}$ and assume (H_w). Then the Cauchy problem* (5.8) *has a unique solution $\phi \in C([0,T],V')$ such that*

$$\phi \in W^{1,2}(\delta,T;V') \text{ for every } 0 < \delta < T,$$

$$j(\phi + w) \in L^1(Q),$$

$$\sqrt{t}\ \frac{d\phi}{dt} \in L^2(0,T;V'),\ \sqrt{t}\ D^*(\phi + w) \in L^2(0,T;V').$$

Proof. The proof of this result follows as the similar result of Theorem 3.10, or can be found in detail in [85]. ∎

A comparison result

Finally we prove that a comparison result still applies in the case of Dirichlet nonhomogeneous boundary conditions.

Theorem 5.7. *Let $f \in L^2(0,T;V')$, $\theta_0 \in M_j$ and assume that*

$$0 \le \theta_{0m} \le \theta_0,\ \text{a.e. in } \Omega, \tag{5.70}$$

$$f_m \le f, \tag{5.71}$$

$$\theta_m(t) \le g(x,t) < \theta_s,\ \text{a.e. on } \Sigma. \tag{5.72}$$

Then, the solution θ *to* (5.1) *satisfies*

$$\theta_m(t) \leq \theta(x,t) < \theta_s, \text{ a.e. in } \Omega, \text{ for each } t \in [0,T], \tag{5.73}$$

where

$$\theta_{0m} = \operatorname*{ess\,inf}_{x\in\Omega} \theta_0(x), \ f_m \in [0,+\infty), \ \theta_m(t) = \theta_{0m} + f_m t. \tag{5.74}$$

The result remains true if $\theta_0 \in M_{\theta_s}$.

Proof. Assume $f \in L^2(0,T;V')$ and $\theta_0 \in L^2(\Omega)$ such that $j(\theta_0) \in L^1(\Omega)$. Then by Theorem 5.2, there exists a unique solution ϕ to (5.8) and consequently a unique solution $\theta \in L^2(0,T;H^1(\Omega))$ to (5.1). We have to show that under the hypotheses (5.70)-(5.72) the negative part $(\theta(t)-\theta_m(t))^- = 0$ a.e. on Ω, for each $t \in [0,T]$.

We multiply directly the equation

$$\frac{\partial\theta}{\partial t} - \Delta D^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \text{ a.e. } t \in (0,T)$$

by $(\theta(t)-\theta_m(t))^- \in H^1(\Omega)$ and integrate it over $\Omega \times (0,t)$. We have

$$\begin{aligned}\int_0^t \int_\Omega \left\{ \frac{d\theta}{d\tau}(\theta-\theta_m)^- + \nabla D^*(\theta)\cdot\nabla(\theta-\theta_m)^- + \frac{\partial K(\theta)}{\partial x_3}(\theta-\theta_m)^- \right\} dx\,d\tau \\ = \int_0^t \int_\Omega (f-f_m)(\theta-\theta_m)^- dx\,d\tau \\ - \int_0^t \int_\Omega f_m(\theta-\theta_m)^- dx\,d\tau \\ + \int_0^t \int_\Gamma \nabla D^*(\theta)\cdot\nu(\theta-\theta_m)^- d\sigma\,d\tau.\end{aligned}$$

The last term on the right-hand side vanishes due to (5.72) and Stampacchia's lemma. Taking into account that θ_m does not depend on the space variables and $\theta_m(t) = \theta_{0m} + f_m t$, we can write

$$\begin{aligned}\int_0^t \int_\Omega \left\{ \frac{d(\theta - f_m\tau - \theta_{0m})}{d\tau}(\theta-\theta_m)^- - \nabla\left(D^*(\theta) - D^*(\theta_m)\right)\cdot\nabla(\theta-\theta_m)^- \right\} dx\,d\tau \\ = \int_0^t \int_\Omega (f-f_m)(\theta-\theta_m)^- dx\,d\tau \\ + \int_0^t \int_\Omega K'(\theta)\frac{\partial(\theta-\theta_m)^-}{\partial x_3}(\theta-\theta_m)^- dx\,d\tau.\end{aligned}$$

Some standard computations provide

$$\begin{aligned}\frac{1}{2}\int_{\Omega}\left[(\theta(t)-\theta_m(t))^-\right]^2 dx + \frac{\rho}{2}\int_0^t \left\|((\theta(\tau)-\theta_m(\tau))^-\right\|_V^2 d\tau \\ \leq \frac{M^2}{2\rho}\int_0^t \left\|(\theta(\tau)-\theta_m(\tau))^-\right\|^2 d\tau \\ -\int_0^t\int_{\Omega}(f-f_m)(\theta-\theta_m)^- dx\, d\tau \\ -\frac{1}{2}\int_{\Omega}\left[(\theta_0-\theta_m(0))^-\right]^2 dx.\end{aligned}$$

But $\theta_m(0)=\theta_{0m}$ and (5.70) imply that the last term on the right-hand side vanishes. Using (5.71) also we obtain that

$$\left\|(\theta(t)-\theta_m(t))^-\right\|^2 \leq \frac{M^2}{\rho}\int_0^t \left\|(\theta(\tau)-\theta_m(\tau))^-\right\|^2 d\tau.$$

According to Gronwall's lemma we deduce that $\left\|(\theta(t)-\theta_m(t))^-\right\|^2 = 0$, meaning that $\theta(x,t)\geq\theta_m(t)$ a.e. on Ω, for each $t\in[0,T]$.

Obviously, by density, the result remains true if $\theta_0 \in M_{\theta_s}$, but we let the details of the proof to the reader. ■

4.6 Comments

Using the semigroup method, nonlinear models involving various types of boundary value problems can be approached. In this chapter, devoted to a very fast diffusion in porous media, we studied problems with Dirichlet boundary conditions, but in the next chapters we shall apply the theory to models involving other types of boundary conditions.

The interpretation of abstract results obtained so far leads to some important conclusions concerning the behaviour of the physical process of diffusion in porous media, in particular to water infiltration. At the same time the proof requirements offer a perspective upon the motivation and reliability of the assumptions made in Chap. 2.

A. *Physical reliability of the mathematical assumptions*

A1. Recall the way in which we introduced the function extensions in Sect. 2.6.

We notice that in the proof of Proposition 2.1 and Corollary 2.2 an essential role is played by the fact that $R(D^*)=\mathbf{R}$. This property was envisaged when the extension of $D^*(\theta)$ was chosen at the left of $\theta=0$. Besides the monotonicity property, D^* had to map the definition domain $(-\infty,\theta_s)$ onto $\mathbf{R}$.

We can choose various extensions of D^* and as an example, one that fulfill these requirements might be represented by the maximal monotone operator

$$D^*(\theta) = \begin{cases} \mathbf{R}_-, & \text{if } \theta = 0 \\ \displaystyle\int_0^\theta D(\xi)d\xi, & \text{if } \theta \in (0, \theta_s). \end{cases}$$

But this is not differentiable at $\theta = 0$, so it should be regularized at this point too, fact that complicates the analysis just from the beginning.

On the other hand, the solution obtained by a certain extension should belong, under suitable conditions, to $[0, \theta_s)$. Since we have proved that the quasi m-accretivity of the operator enhances the uniqueness of the solution, it follows that the solutions corresponding to different extensions have to coincide on $[0, \theta_s)$. Hence we can work with the most suitable and convenient extension that ensures the m-accretivity of the operator and therefore the choice we have made for (1.3) and (1.4) turns out to be completely motivated.

A2. The results presented in Sect. 4.4 confirm the assertion that the assumption of the Lipschitz property of K is not necessary to obtain the existence and uniqueness of the solution. The difference between the results given in Sects. 4.3 and 4.4 consists in the direct proof of the quasi m-accretiveness of the operator A (in the first case), while in the second case A is merely the limit of a sequence of m-accretive operators.

A3. At a first glance the distribution space $H^{-1}(\Omega)$ might appear inappropriate and meaningless as a basic function space for the solution to (3.15). However we must emphasize that it is the unique energetic space in which the dissipative character of the equation is preserved.

Concerning the initial condition θ_0 or the free term f, in some situations they are more regular functions (belonging to $L^2(\Omega)$ or $L^2(Q)$) which has a clear physical meaning. In some other cases the hypotheses of the theorems require that f and θ_0 belong to $H^{-1}(\Omega)$ (e.g. in Theorem 3.10 where $\theta_0 \in \overline{D(A_D)}$). It means that they can be represented by measures (like Dirac type in the case $N = 1$). This situation is appropriate when we have to model punctual initial conditions or sources in such problems (e.g. $\theta(x, 0) = \delta(x)$ or $f(t, x) = f_0(t)\delta(x)$, with f_0 a continuous function of time).

In other cases the source f_0 may be distributed on a $(N-1)$-dimensional variety Γ_0 belonging to the flow domain. This separates the flow domain into two subdomains Ω_1 and Ω_0 between which the water flux has a jump, see Fig. 4.4.

In this case the model is described by

$$\begin{aligned} &\frac{\partial \theta}{\partial t} - \Delta D^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \quad \text{in } Q_1 = \Omega_1 \times (0, T), \\ &\frac{\partial \theta}{\partial t} - \Delta D^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \quad \text{in } Q_0 = \Omega_0 \times (0, T), \\ &[\nabla \beta^*(\theta)] \cdot \nu = f_0 \quad \text{on } \Gamma_0 \times (0, T), \\ &\theta(x, 0) = \theta_0 \qquad \text{in } \Omega. \end{aligned}$$

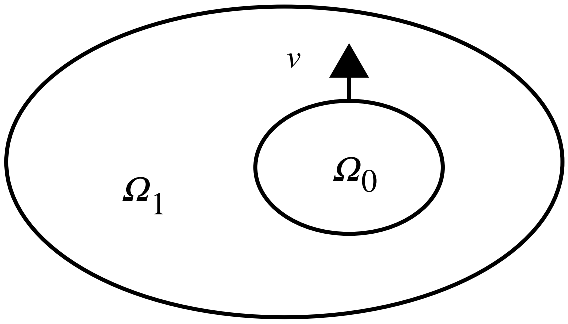

Fig. 4.4. Model with two domains and flux jump

The associated Cauchy problem will be then (3.15) where f is given by

$$f(\psi) = \int_{\Gamma_0} f_0(x)\psi(x)dx,\ \forall\psi \in C_0^\infty(\Omega),\ \text{where } f_0 \in L^1(\Gamma_0).$$

B. ***Consequences upon the solutions of the physical problems***

B1. A first aspect refers to the characterization of the moisture evolution in the unsaturated soil, given a certain initial distribution of it.

Let $N = 1$. By Theorem 3.8 we obtain that $D^*(\theta)(t) \in H_0^1(\Omega) \subset C(\overline{\Omega})$ a.e. $t \in (0,T)$, so it turns out to be continuous as function of x. Taking into account Remark 3.3 which proves that $\theta(t) < \theta_s(t)$ a.e. $x \in \Omega$, it follows that in the one-dimensional case, within the considered model, the values of the function θ remain under the saturation value θ_s.

Let $N = 2, 3$. Then from the conclusion of Theorem 3.8 and since $H^1(\Omega)$ is no longer embedded in $C(\overline{\Omega})$, it follows that it has to satisfy e.g., the relation

$$\int_\Omega |D^*(\theta(t))|^2\, dx < \infty,\ \text{a.e } t \in (0,T).$$

However, this does not exclude the possibility that some points $x \in \Omega$ may exist, such that $\theta(x,t) = \theta_s$. So, the water content may reach the saturation value at some points, or generally on $(N-1)$-dimensional manifolds (curves in $\mathbf{R}^2$ and surfaces in $\mathbf{R}^3$) even in the case when $\theta_0 < \theta_s$.

B2. Theorem 3.10, part (a) allows the study of the problem if the initial value is not so regular. Thus, θ_0 can be equal to θ_s on subsets of zero measure, i.e., such that $j(\theta_0) \in L^1(\Omega)$. From the physical point of view this corresponds to the case when the initial moisture distribution displays saturation only at points, curves or surfaces, in the 1-D, 2-D and 3-D cases, respectively. Moreover, Theorem 3.10, part (b) proves that if the initial data is irregular, the solution θ is finally regularized, and this is due to the parabolic operator action. This expresses the fact that a process which starts with moisture saturation in some subsets of Ω, can be still studied under this model, but the solution remains finally lesser than θ_s, almost everywhere.

B3. Other conclusion is that within the diffusive model with a blowing-up diffusivity, the abstract solution obtained by the existence theorems belongs to the physical accepted domain. Indeed, as specified at points **A1** or **A2,** the solution θ is less than θ_s or at most equal to θ_s on subsets of zero measure. Then, Theorem 3.11 for example, shows that the dimensionless θ is greater than 0, if the initial distribution is in the same way and f is non-negative. This last assumption means that if in the infiltration domain there are only sources and not sinks, then the moisture in the soil remains above the residual value. Thus, by the analysis of the situation under which an infiltration process begins to develop, one can forecast a qualitative result and also a quantitative one, according to the comparison results.

C. *Continuous media aspects*

Since from the physical point of view the quasi-unsaturated model might appear artificial, we would like to specify however that it does not contravene to the physics laws. In other words, we are going to show that the concept of solution we introduced here is compatible with the conservation laws of continuum mechanics.

Indeed, we have got that the boundary value problem (3.1) has, on every interval $[0, T]$, a solution θ in the following generalized sense

$$\int_Q \left(\frac{d\theta}{dt} \phi + \nabla D^*(\theta) \cdot \nabla \phi - K(\theta) \frac{\partial \phi}{\partial x_3} \right) dxdt = \int_Q f\phi \, dx \, dt, \tag{6.75}$$

for any $\phi \in L^2(0, T; V)$. If in addition, $\frac{d\phi}{dt} \in L^2(0, T; V')$ we can write

$$\begin{aligned} &\int_\Omega \theta(x,T)\phi(x,T)dx - \int_Q \theta \frac{\partial \phi}{\partial t} \, dx \, dt - \int_\Omega \theta_0(x)\phi(x,0)dx \\ &+ \int_0^T \int_\Omega D^*(\theta)(-\Delta\phi) dx \, dt = \int_Q K(\theta) \frac{\partial \phi}{\partial x_3} \, dx \, dt + \int_Q f\phi \, dx \, dt. \end{aligned} \tag{6.76}$$

For simplicity, let us take $\theta_0(x) = 0$, ϕ a time independent function, $K(\theta) = 0$ and $f =$ constant. In particular we may assume that ϕ is the solution to the boundary value problem

$$-\Delta\phi = c_0 > 0 \text{ in } \Omega, \ \phi = 0 \text{ on } \Gamma.$$

Then we have

$$\int_\Omega \theta(x,T)\phi(x)dx + c_0 \int_0^T \int_\Omega D^*(\theta) dx dt = fT\phi_\Omega, \tag{6.77}$$

where $\phi_\Omega = \displaystyle\int_\Omega \phi(x)dx < \infty$.

Now if T is finite the equality is satisfied for a whatever close θ to θ_s and the mass is conserved. From the mathematical point of view this corresponds to the fact that a dissipative system has always a global solution for T whatever large but finite, such that θ remains under θ_s.

If T grows up to infinity, the right-hand side term tends to infinity. However, this does not violate the conservation law since the second term in the left-hand side may become infinity (due to the blowing-up of $D^*(\theta)$ when θ approaches θ_s). Actually, (6.77) implies the ergodic equation

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_\Omega D^*(\theta) dx\, dt = f\phi_\Omega c_0$$

which expresses a conservation law reflecting that the time average of the inflow provided by the source f is transformed into the time average of the diffused water mass over Ω.

Bibliographical note

We mention that the problem $\frac{\partial \theta}{\partial t} - \nabla \cdot (D(\theta) \nabla \theta) = 0$ with initial and boundary conditions was approached in the form (3.1) by applying the transformation (1.2), for the first time by O.A. Oleinik & co-authors in [98]. Existence, uniqueness and other properties of the solution of the purely fast diffusion equation (without transport terms) were approached by many authors during the past decades by various methods different from these used here related to m-accretive operators. For the problems presented in this chapter we also indicate the papers [17], [85], [89], [86].

Many researchers paid much attention to various aspects raised by the singular diffusion equation $u_t - \Delta \log u = 0$, mainly for $x \in \mathbf{R}^N$. For $N = 1$ existence of multiple solutions in $\mathbf{R}^N$ was studied, among others, by J.R. Esteban, A. Rodriguez and J.L. Vázquez in [55], while for $N = 2$, the existence of solutions that extinct in time was approached by the same authors in [56], [57] and by K.M. Hui in [69].

Contributions to the study of the asymptotic behaviour of the solution in the case of the fast diffusion model have been given by P. Benilan and J. Bouillet (see [24]). Decay conditions for the solutions as $|x| \to \infty$ have been extensively investigated by K.M. Hui (see [70]) and S.Y. Hsu who studied the asymptotic behaviour of the solutions near the extinction time (see [71], [72]). Studies concerning solvability conditions have been done by J.L. Vázquez in [119], [120], and P. Daskalopoulos and M. Del Pino in [51]. Generally, all refer to the purely diffusion equation (without transport) in $\mathbf{R}^N$.

We also refer the reader to the bibliography indicated in these papers, specifying however that they are not directly related to infiltration in porous media, where the degree of saturation of the medium and the transport term induce specific mathematical aspects.

5

Functional approach to the saturated-unsaturated infiltration model

This chapter is concerned with the study of the saturated-unsaturated infiltration model, which describes the complete process of water infiltration into an unsaturated soil, including the evolution of soil moisture up to saturation and the advance of the interface between the saturated and unsaturated regions. The general hypotheses relating to the fluid incompressibility, homogeneity and isotropy of the porous medium with constant porosity, as well as to the nonhysteretic behaviour, are preserved.

Under certain conditions depending on the soil structure, the rate at which water is supplied on a part of the domain boundary, the initial moisture distribution in the soil, the presence of underground sources and the boundary permeability, saturation can occur within the flow domain. If certain conditions are fulfilled, saturation can be observed first at the soil surface, at the so-called saturation time. Consequently a waterfront starts to move downwards. A certain combination of the hydraulic factors can determine first the saturation of the soil basement and the advance of the free boundary from below up to the surface. Anyway, this represents the unknown interface between the saturated and unsaturated flow domains and practically consists in a mushy region (a very fine mixture between saturated and unsaturated zones).

It must be emphasized that the saturated-unsaturated process represented by a nonlinear parabolic equation involving a free boundary is better described as a nonlinear Cauchy problem with a multivalued nonlinearity. As settled in Chap. 2, the saturated-unsaturated infiltration is illustrated by the strongly saturated-unsaturated model (Model 1.1, Model 1.2), as well as by the weakly saturated-unsaturated model (with its variants Model 1.3 and Model 1.4).

We recall that the first two models account for a fast-type diffusion, while the latter two are related to a slow-type diffusion, represented in particular by the porous media equation.

This chapter is devoted to the saturated-unsaturated flow which is the most complete and reliable model of water infiltration in porous media. We intend to give answers to some basic problems arising in the saturated-unsaturated infiltration, namely the existence and properties of the solution, both for the model in the diffusive form and in the pressure form, and the existence of the free boundary.

The theory will be developed for the strongly nonlinear model with a weak nonlinear hydraulic conductivity (Model 1.2. in Sect. 2.2), because the study of Model 1.1, with a strongly hydraulic conductivity, can be reduced to the previous one. Also, we choose to illustrate this case by taking into account one of the most realistic and complete situations, in which water that infiltrates is supplied by a rain (or by an irrigation process) on the soil surface, Γ_u, and the boundary of the underground flow domain, Γ_α, is supposed to have a variable permeability. The chapter is organized in 7 sections including :

1. Basic hypotheses for the saturated-unsaturated model in the diffusive form and settlement of the functional framework;
2. Introduction of a certain approximating problem, existence, uniqueness, regularity and other properties of its solution;
3. Existence, uniqueness, regularity and properties of the solution to the original problem;
4. Existence of the solution in the pressure form;
5. Investigation of the conditions required for the separation of the flow domain in two well delimited parts and existence of the free boundary;
6. Uniqueness of the solution in the pressure form;
7. Comments on other saturated-unsaturated models (Models 1.1, 1.3 and 1.4 in Chap. 2).

5.1 Basic hypotheses for the saturated-unsaturated model

Assume that the flow domain Ω is an open bounded subset of $\mathbf{R}^N$ $(N = 1, 2, 3)$ with the boundary $\partial\Omega := \Gamma$ piecewise smooth, formed by the disjoint parts Γ_u and Γ_α, i.e.,

$$\Gamma = \Gamma_u \cup \overline{\Gamma_\alpha},\ \Gamma_u \cap \Gamma_\alpha = \emptyset.$$

The domain Ω extends from the soil surface, Γ_u up to an underground boundary, Γ_α, which is supposed to have a variable permeability, because it may have contact with types of soils different from that in which infiltration is monitored. The model reads

$$\frac{\partial\theta}{\partial t} - \Delta\beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} \ni f \quad \text{in} \quad Q = \Omega \times (0, T), \tag{1.1}$$

$$\theta(x, 0) = \theta_0(x) \quad \text{in} \quad \Omega, \tag{1.2}$$

$$(K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu \ni u \quad \text{on} \quad \Sigma_u = \Gamma_u \times (0, T), \tag{1.3}$$

$$(K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu - \alpha\beta^*(\theta) \ni f_0 \quad \text{on} \quad \Sigma_\alpha = \Gamma_\alpha \times (0, T). \tag{1.4}$$

Here ν is the outward normal to Γ, i_3 is the unit vector along Ox_3, downwards directed, f is some source in Q, f_0 and u are known on Σ_α and Σ_u, respectively, and K and β^* are defined below. The time runs within the finite interval $(0, T)$.

Equation (1.3) expresses the continuity of the normal component of the inflow flux and (1.4) describes the behaviour of the outflow. As a matter of fact it means that the flux through the underground boundary is directly proportional to the water diffusivity and to the permeability of the boundary. We made the convention of keeping for this case the notations β and β^* (which in the quasi-unsaturated case were denoted by D and D^*).

Basic assumptions for the saturated-unsaturated model

The functions β, β^* and K considered in the saturated-unsaturated model discussed here, are defined by

$$\beta^*(r) := \begin{cases} \rho r, & r \leq 0 \\ \int_0^r \beta(\xi)d\xi, & 0 < r < \theta_s \\ [K_s^*, +\infty), & r = \theta_s \end{cases}, \quad \beta(r) := \begin{cases} \rho, & r \leq 0 \\ \beta(r), & 0 < r < \theta_s \end{cases} \tag{1.5}$$

$$K(r) := \begin{cases} 0, & r \leq 0 \\ K(r), & 0 < r \leq \theta_s, \end{cases} \tag{1.6}$$

as presented in Sect. 2.2, (see also Figs. 5.1 and 5.2).

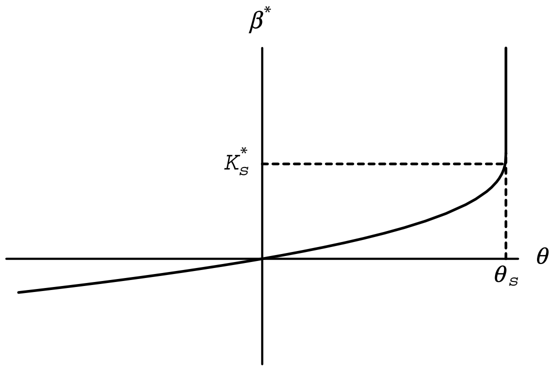

Fig. 5.1. Graphic of $\beta^*(\theta)$ (strongly nonlinear saturated-unsaturated case)

In this model $\beta : (-\infty, \theta_s) \to (\rho, +\infty)$ and $K : (-\infty, \theta_s] \to [0, K_s]$ are continuous and monotonic functions, twice differentiable on $[0, \theta_s)$. Moreover, it is assumed that β and K are convex (see the hypotheses made in Model 1.2. in Chap. 2) and satisfy the basic properties:

(i_β) $\beta(r) \geq \rho > 0$, $\forall r \in (-\infty, \theta_s)$;

(ii_β) $\lim\limits_{r \nearrow \theta_s} \beta(r) = +\infty$;

(iii_β) $K_s^* := \lim\limits_{r \nearrow \theta_s} \int_0^r \beta(\xi) d\xi$, $0 < K_s^* < \infty$.

They imply that β^* is differentiable, convex and satisfies:

(i) $(\beta^*(r) - \beta^*(\overline{r}))(r - \overline{r}) \geq \rho(r - \overline{r})^2$, $\forall r, \overline{r} \in (-\infty, \theta_s]$;

(ii) $\lim\limits_{r \to -\infty} \beta^*(r) = -\infty$;

(iii) $\lim\limits_{r \nearrow \theta_s} \beta^*(r) = K_s^*$.

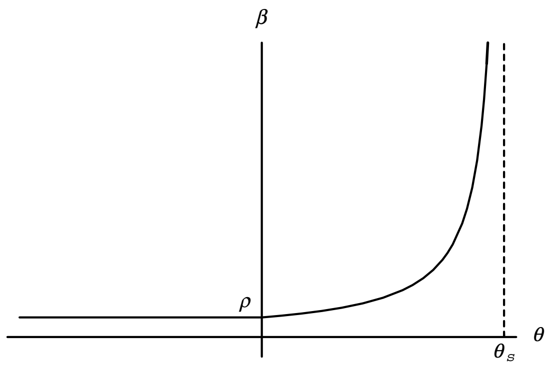

Fig. 5.2. Graphic of $\beta(\theta)$ (strongly nonlinear saturated-unsaturated case)

Since we study here the situation with a weakly nonlinear hydraulic conductivity, we assume that

$$M := \lim_{r \nearrow \theta_s} K'(r) < +\infty, \tag{1.7}$$

and K is Lipschitz, with the constant $M > 0$, i.e.,

(i_K) $|K(r) - K(\overline{r})| \leq M |r - \overline{r}|$, $\forall r, \overline{r} \leq \theta_s$.

Finally, we assume that $\alpha : \Gamma_\alpha \to [\alpha_m, \alpha_M]$ is positive and continuous

$$0 < \alpha_m \leq \alpha(x) \leq \alpha_M. \tag{1.8}$$

Functional framework

For the sake of simplicity we shall denote the scalar product and the norm in $L^2(\Omega)$ by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Also, we shall no longer write the function arguments which represent the integration variables.

The problem will be treated within the functional framework represented by $V = H^1(\Omega)$ with its dual $V' = (H^1(\Omega))'$. The norm on V is defined by

$$\|\psi\|_V = \left(\int_{\Omega} |\nabla\psi|^2 \, dx + \int_{\Gamma_\alpha} \alpha(x) \, |\psi|^2 \, d\sigma \right)^{1/2} \tag{1.9}$$

and it can be easily checked that it is equivalent with the standard Hilbertian norm on $H^1(\Omega)$. Indeed, we have

$$\|\psi\|_V^2 \leq \|\nabla\psi\|^2 + \alpha_M \|\psi\|_{L^2(\Gamma_\alpha)}^2 \leq \|\nabla\psi\|^2 + \alpha_M \|\psi\|_{L^2(\Gamma)}^2$$

and using the trace theorem (see Theorem 2.7 in Appendix) we obtain

$$\|\psi\|_V \leq c_{VH} \|\psi\|_{H^1(\Omega)}, \tag{1.10}$$

with $c_{VH}^2 := 1 + \alpha_M C^2$.

We still have from (1.9) that

$$\|\psi\|_{L^2(\Gamma_\alpha)} \leq c_{\Gamma_\alpha} \|\psi\|_V, \text{ with } c_{\Gamma_\alpha} := \frac{1}{\sqrt{\alpha_m}}. \tag{1.11}$$

Since $\Gamma_\alpha \subset \Gamma$ and $\text{meas}(\Gamma_\alpha) \neq 0$, we have from Poincaré inequality (see Theorem 2.18 in Appendix) that

$$\|\psi\|_{H^1(\Omega)}^2 \leq c_P (\|\psi\|_{L^2(\Gamma_\alpha)}^2 + \|\nabla\psi\|^2) \leq c_H^2 \|\psi\|_V^2,$$

where $c_H^2 := c_P \left(1 + \dfrac{1}{\alpha_m}\right)$. Hence

$$\|\psi\|_{H^1(\Omega)} \leq c_H \|\psi\|_V \tag{1.12}$$

which together with (1.10) implies that the two norms are equivalent.

Similarly, by the trace theorem we have

$$\|\psi\|_{L^2(\Gamma_u)} \leq \|\psi\|_{L^2(\Gamma)} \leq C \|\psi\|_{H^1(\Omega)} \leq c_{\Gamma_u} \|\psi\|_V, \tag{1.13}$$

with $c_{\Gamma_u} := c_H C$.

We have to underline that c_H^2, $c_{\Gamma_\alpha}^2$ and $c_{\Gamma_u}^2$ depend on $\dfrac{1}{\alpha_m}$ and we recall that we have assumed that $\alpha_m > 0$.

We endow the dual V' with the scalar product

$$\left\langle \theta, \overline{\theta} \right\rangle_{V'} := \theta(\psi), \ \forall \theta, \overline{\theta} \in V', \tag{1.14}$$

where $\psi \in V$ satisfies the boundary value problem

$$-\Delta\psi = \overline{\theta}, \ \frac{\partial\psi}{\partial\nu} + \alpha\psi = 0 \text{ on } \Gamma_\alpha, \ \frac{\partial\psi}{\partial\nu} = 0 \text{ on } \Gamma_u. \tag{1.15}$$

$\left(\dfrac{\partial}{\partial\nu} \text{ is the normal derivative.} \right)$

Here $\theta(\psi)$ represents the value of $\theta \in V'$ at $\psi \in V$, or the pairing between V' and V and by convention it is still written $\int_\Omega \theta(x)\psi(x)dx$, because it reduces exactly to the scalar product on $L^2(\Omega)$ when $\theta \in L^2(\Omega)$.

As we explained in Sect. 4.3, highly nonlinear partial differential equations do not have in general classical solutions, reason for which we have to introduce the definition of the solution to (1.1)-(1.4) in a generalized sense.

Definition 1.1. Let

$$\theta_0 \in L^2(\Omega),\ \theta_0 \leq \theta_s \text{ a.e. } x \in \Omega,$$

$$f \in L^2(0,T;V'),\ u \in L^2(0,T;L^2(\Gamma_u)),\ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)).$$

We mean by *solution* to (1.1)-(1.4) a function $\theta \in C([0,T];L^2(\Omega))$, such that

$$\frac{d\theta}{dt} \in L^2(0,T;V'), \tag{1.16}$$

$$\theta(x,t) \leq \theta_s \text{ a.e. } (x,t) \in Q, \tag{1.17}$$

$$\begin{aligned} &\left\langle \frac{d\theta}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega \left(\nabla\eta(t) \cdot \nabla\psi - K(\theta(t)) \frac{\partial \psi}{\partial x_3} \right) dx \\ &= \langle f(t), \psi \rangle_{V',V} - \int_{\Gamma_\alpha} (\alpha\eta(t) + f_0(t))\psi d\sigma - \int_{\Gamma_u} u(t)\psi d\sigma, \\ &\text{a.e. } t \in (0,T),\ \forall \psi \in V, \end{aligned} \tag{1.18}$$

where $\eta \in L^2(0,T;V)$ is such that $\eta(x,t) \in \beta^*(\theta(x,t))$ a.e. $(x,t) \in Q$,

and

$$\theta(x,0) = \theta_0 \text{ in } \Omega. \tag{1.19}$$

By $\frac{d\theta}{dt}$ we mean the strong derivative of $\theta(t)$ in V' (equivalently the derivative in the sense of the V'-valued distributions on $(0,T)$) and very often we shall simply write it as $\frac{\partial \theta}{\partial t}$. It is obvious that (1.18) can still be written

$$\begin{aligned} &\int_\Omega \left(\frac{\partial \theta}{\partial t}(t)\psi + \nabla\eta(t) \cdot \nabla\psi - K(\theta(t)) \frac{\partial \psi}{\partial x_3} \right) dx \\ &= \int_\Omega f(t)\psi dx - \int_{\Gamma_\alpha} (\alpha\eta(t) + f_0(t))\psi d\sigma - \int_{\Gamma_u} u(t)\psi d\sigma, \\ &\text{a.e. } t \in (0,T),\ \forall \psi \in V \text{ and } \eta \in \beta^*(\theta) \text{ a.e. on } Q, \end{aligned} \tag{1.20}$$

or still

$$\int_Q \left(\frac{\partial \theta}{\partial t}\phi + \nabla\eta \cdot \nabla\phi - K(\theta)\frac{\partial \phi}{\partial x_3} \right) dxdt \tag{1.21}$$
$$= \int_Q f\phi dxdt - \int_{\Sigma_\alpha} (\alpha\eta + f_0)\phi d\sigma dt - \int_{\Sigma_u} u\phi d\sigma dt,$$
$$\forall \phi \in L^2(0,T;V) \text{ and } \eta \in \beta^*(\theta) \text{ a.e. on } Q,$$

and the proof is similar to that of Lemma 3.4 in Chap. 4. Moreover, the latter is equivalent to

$$\int_\Omega \theta(x,T)\phi(x,T)dx - \int_Q \theta \frac{d\phi}{dt}dxdt + \int_Q \left(\nabla\eta \cdot \nabla\phi - K(\theta)\frac{\partial \phi}{\partial x_3} \right) dxdt \tag{1.22}$$
$$= \int_\Omega \theta_0(x)\phi(x,0)dx + \int_Q f\phi dxdt - \int_{\Sigma_\alpha} (\alpha\eta + f_0)\phi d\sigma dt - \int_{\Sigma_u} u\phi d\sigma dt,$$
$$\forall \phi \in L^2(0,T;V), \text{with } \frac{d\phi}{dt} \in L^2(0,T;L^2(\Omega)),\ \eta \in \beta^*(\theta) \text{ a.e. on } Q.$$

It is easily seen that a classical solution to (1.1)-(1.4), if it exists, is a solution in the sense of Definition 1.1.

Conversely, the solution introduced by the previous definition turns out to be a solution in the sense of distributions to (1.1) and satisfies the boundary conditions (1.3)-(1.4) in a generalized sense, i.e., in the sense of the trace theory (see [84]). Here is the argument. First we notice that by Green's and Gauss-Ostrogradski's formula, the left hand-side in (1.21) becomes

$$\int_Q \left(\frac{\partial \theta}{\partial t}\phi + \nabla\eta \cdot \nabla\phi - K(\theta)\frac{\partial \phi}{\partial x_3} \right) dxdt \tag{1.23}$$
$$= \int_Q \left(\frac{\partial \theta}{\partial t} - \Delta\eta + \frac{\partial K(\theta)}{\partial x_3} \right) \phi dxdt - \int_\Sigma \phi(K(\theta)i_3 - \nabla\eta) \cdot \nu d\sigma dt,$$

where $\eta \in \beta^*(\theta)$ and $\Sigma = \Gamma \times (0,T)$. Then, in (1.21) we take ϕ with compact support in Q, and we get

$$\int_Q \left(\frac{\partial \theta}{\partial t}\phi + \nabla\eta \cdot \nabla\phi - K(\theta)\frac{\partial \phi}{\partial x_3} \right) dxdt = \int_Q f\phi dxdt,$$

with ϕ arbitrary. This implies

$$\frac{\partial \theta}{\partial t} - \Delta\eta + \frac{\partial K(\theta)}{\partial x_3} = f \text{ in } \mathcal{D}'(Q), \text{ for } \eta \in \beta^*(\theta).$$

We define

$$\mathcal{D}(Q \cup \Sigma_\alpha) := \{\phi \in C^\infty(\overline{Q});\ \phi = 0 \text{ on } \Gamma_u \times [0,T]\}. \tag{1.24}$$

We multiply (1.1) by $\phi \in \mathcal{D}(Q \cup \Sigma_\alpha)$ and, after some integrations involving the Gauss-Ostrogradski formula, we obtain that

$$\int_Q \left(\frac{\partial \theta}{\partial t}\phi + \nabla\eta \cdot \nabla\phi - K(\theta)\frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$+ \int_{\Sigma_\alpha} \phi \left(K(\theta)i_3 - \nabla\eta \right) \cdot \nu d\sigma dt = \int_Q f\phi dxdt,$$

that compared with (1.21) yields

$$\int_{\Sigma_\alpha} \phi \left(K(\theta)i_3 - \nabla\eta \right) \cdot \nu d\sigma dt = \int_{\Sigma_\alpha} (f_0 + \alpha\eta)\, \phi d\sigma dt, \ \forall \phi \in \mathcal{D}(Q \cup \Sigma_\alpha).$$

Assume first that θ is smooth (such that $\eta \in L^2(0,T;H^1(\Omega))$, for instance). Therefore we get the boundary condition on Γ_α,

$$(K(\theta)i_3 - \nabla\eta) \cdot \nu = f_0 + \alpha(x)\eta \text{ a.e. } (x,t) \in \Sigma_\alpha.$$

If θ is not smooth in the sense specified above, the latter boundary condition is satisfied in the sense of the trace theory in the following way: there is a sequence of smooth functions $\theta_\varepsilon \longrightarrow \theta$ in $L^2(Q)$, such that $(K(\theta_\varepsilon)i_3 - \nabla\eta) \cdot \nu \longrightarrow \alpha\beta^*(\theta_\varepsilon) + f_0$ a.e. on Σ_α.

Multiplying now (1.1) by $\phi \in \mathcal{D}(Q \cup \Sigma_u)$ we obtain exactly in the same way the boundary condition on Σ_u.

Remark 1.2. By the property (i), the claim $\eta \in V$ where $\eta \in \beta^*(\theta)$ implies $\theta \in V$. The argument is the same as that given in Remark 3.1 in Chap. 4, based on the fact the inverse of β^* is a Lipschitz function with the constant $\frac{1}{\rho}$.

It is also obvious that $(\mathrm{i_K})$ implies $K(\theta) \in L^2(\Omega)$. Moreover, if $\theta \in V$, it follows by $(\mathrm{i_K})$ that $K(\theta) \in V$.

We introduce now

$$D(A) := \{\theta \in L^2(\Omega);\ \exists \eta \in V,\ \eta(x) \in \beta^*(\theta(x)),\ \text{a.e. } x \in \Omega\} \tag{1.25}$$

and we define the multivalued operator $A : D(A) \subset V' \to V'$ by

$$\langle A\theta, \psi \rangle_{V',V} := \int_\Omega \left(\nabla\eta \cdot \nabla\psi - K(\theta)\frac{\partial \psi}{\partial x_3} \right) dx + \int_{\Gamma_\alpha} \alpha\eta\psi d\sigma, \tag{1.26}$$
$$\forall \psi \in V,\ \text{for some } \eta \in \beta^*(\theta).$$

Moreover, we define the operator $B \in L(L^2(\Gamma_u); V')$ and the function $f_\Gamma \in L^2(0,T;V')$ by

$$Bu(\psi) := -\int_{\Gamma_u} u\psi d\sigma,\ \forall \psi \in V, \tag{1.27}$$

$$f_\Gamma(t)(\psi) := -\int_{\Gamma_\alpha} f_0\psi d\sigma,\ \forall \psi \in V. \tag{1.28}$$

With these notations we introduce the Cauchy problem

$$\frac{d\theta}{dt} + A\theta \ni f + Bu + f_\Gamma, \text{ a.e. } t \in (0,T), \tag{1.29}$$

$$\theta(0) = \theta_0. \tag{1.30}$$

Since (1.29) is an equality in V' we can write it as

$$\left\langle \frac{d\theta}{dt}(t) + A\theta(t), \psi \right\rangle_{V',V} = \langle f(t)+f_\Gamma(t)+Bu(t), \psi\rangle_{V',V}, \text{ a.e. } t\in(0,T),\ \forall\psi\in V$$

and it is obvious that the latter is in fact (1.18). Clearly, the strong solution to (1.29)-(1.30) is a solution in the generalized sense of (1.1)-(1.4), hence, we have to study the abstract Cauchy problem (1.29)-(1.30) that will be called the *original Cauchy problem*. Since the operator A is multivalued, we have to resort to an auxiliary problem by replacing β^* by a smooth function.

5.2 The approximating problem

In order to prove the existence results, we approximate the multivalued function β^* by the continuous function (see Fig. 5.3), defined for each $\varepsilon > 0$ by

$$\beta_\varepsilon^*(r) := \begin{cases} \beta^*(r), & r < \theta_s \\ K_s^* + \dfrac{r-\theta_s}{\varepsilon}, & r \geq \theta_s, \end{cases} \tag{2.1}$$

so that, besides the properties (i) (for $r, \overline{r} \in \mathbf{R}$) and (ii), $\beta_\varepsilon^*(r)$ satisfies

(iv) $\lim\limits_{r\to\infty} \beta_\varepsilon^*(r) = +\infty$.

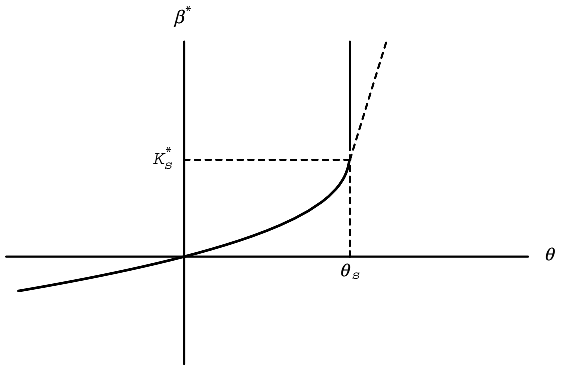

Fig. 5.3. Graphic of $\beta_\varepsilon^*(\theta)$ (given by (2.1))

This function is differentiable on $\mathbf{R}\backslash\{\theta_s\}$ only, because its left derivative at $\theta = \theta_s$ blows up. However, this approximation is good enough to prove the existence and uniqueness results.

In the approximating problem we extend K to the right of the saturation value by the constant value K_s, but for simplicity this function will be denoted still by K

$$K(r) := \begin{cases} 0, & r < 0 \\ K(r), & 0 \le r \le \theta_s \\ K_s, & r > \theta_s. \end{cases} \tag{2.2}$$

and it is Lipschitz on $\mathbf{R}$.

For a supplementary regularity that will be required in the proof of the free boundary existence, we need to work with smoother approximations of β^* and K than these used here. Obviously, all the results obtained using (2.1) and (2.2) will remain true for the smoother approximations.

Consequently, we introduce the associated approximating problem

$$\frac{d\theta_\varepsilon}{dt} + A_\varepsilon \theta_\varepsilon = f + Bu + f_\Gamma, \text{ a.e. } t \in (0,T), \tag{2.3}$$

$$\theta_\varepsilon(0) = \theta_0, \tag{2.4}$$

where $A_\varepsilon : D(A_\varepsilon) \subset V' \to V'$ is the single-valued operator defined by

$$\langle A_\varepsilon \theta, \psi \rangle_{V',V} \tag{2.5}$$
$$= \int_\Omega \left(\nabla \beta_\varepsilon^*(\theta) \cdot \nabla \psi - K(\theta) \frac{\partial \psi}{\partial x_3} \right) dx + \int_{\Gamma_\alpha} \alpha \beta_\varepsilon^*(\theta) \psi d\sigma, \ \forall \psi \in V,$$

with the domain

$$D(A_\varepsilon) := \{\theta \in L^2(\Omega);\ \beta_\varepsilon^*(\theta) \in V\}.$$

Obviously, the strong solution to (2.3)-(2.4) is the solution in the generalized sense (of Definition 1.1) to the boundary value problem

$$\frac{\partial \theta_\varepsilon}{\partial t} - \Delta \beta_\varepsilon^*(\theta_\varepsilon) + \frac{\partial K(\theta_\varepsilon)}{\partial x_3} = f \ \text{ in } \ Q, \tag{2.6}$$

$$\theta_\varepsilon(x,0) = \theta_0(x) \ \text{ in } \ \Omega, \tag{2.7}$$

$$(K(\theta_\varepsilon) i_3 - \nabla \beta_\varepsilon^*(\theta_\varepsilon)) \cdot \nu = u \ \text{ on } \ \Sigma_u, \tag{2.8}$$

$$(K(\theta_\varepsilon) i_3 - \nabla \beta_\varepsilon^*(\theta_\varepsilon)) \cdot \nu = \alpha \beta_\varepsilon^*(\theta_\varepsilon) + f_0 \ \text{ on } \ \Sigma_\alpha. \tag{2.9}$$

Existence and uniqueness in the approximating problem

We shall show the existence of the solution to the approximating problem using the property of quasi m-accretivity of the operator A_ε, proved below.

Proposition 2.1. *Let $\varepsilon > 0$ be fixed. Under the hypotheses* (i)-(ii), (iv) *and* ($\mathrm{i_K}$) *the operator A_ε is quasi m-accretive in V'.*

Proof. Let λ be a positive real number. We must prove that

$$\left\langle (\lambda I + A_\varepsilon)\theta - (\lambda I + A_\varepsilon)\overline{\theta}, \theta - \overline{\theta} \right\rangle_{V'} \ge 0 \tag{2.10}$$

and

$$R(\lambda I + A_\varepsilon) = V^{'}, \tag{2.11}$$

for λ large enough. We have

$$\begin{aligned}
&\left\langle (\lambda I + A_\varepsilon)\theta - (\lambda I + A_\varepsilon)\overline{\theta}, \theta - \overline{\theta} \right\rangle_{V'} \\
&= \lambda \left\| \theta - \overline{\theta} \right\|_{V'}^2 + \int_\Omega \nabla(\beta_\varepsilon^*(\theta) - \beta_\varepsilon^*(\overline{\theta})) \cdot \nabla \psi dx \\
&- \int_\Omega (K(\theta) - K(\overline{\theta})) \frac{\partial \psi}{\partial x_3} dx + \int_{\Gamma_\alpha} \alpha(\beta_\varepsilon^*(\theta) - \beta_\varepsilon^*(\overline{\theta}))\psi d\sigma,
\end{aligned}$$

where

$$-\Delta \psi = \theta - \overline{\theta},\ \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma_\alpha \text{ and } \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \Gamma_u.$$

Using Green's formula

$$\begin{aligned}
\int_\Omega \nabla \beta_\varepsilon^*(\theta) \cdot \nabla \psi dx &= -\int_\Omega \beta_\varepsilon^*(\theta) \Delta \psi dx + \int_{\partial\Omega} \beta_\varepsilon^*(\theta) \nabla \psi \cdot \nu d\sigma \\
&= -\int_\Omega \beta_\varepsilon^*(\theta) \Delta \psi dx - \int_{\Gamma_\alpha} \beta_\varepsilon^*(\theta) \alpha(x) \psi(x) d\sigma,
\end{aligned}$$

we have

$$\begin{aligned}
&\left\langle (\lambda I + A_\varepsilon)\theta - (\lambda I + A_\varepsilon)\overline{\theta}, \theta - \overline{\theta} \right\rangle_{V'} \\
&= \lambda \left\| \theta - \overline{\theta} \right\|_{V'}^2 + \int_\Omega (\beta_\varepsilon^*(\theta) - \beta_\varepsilon^*(\overline{\theta}))(\theta - \overline{\theta}) dx - \int_\Omega (K(\theta) - K(\overline{\theta})) \frac{\partial \psi}{\partial x_3} dx \\
&\geq \lambda \left\| \theta - \overline{\theta} \right\|_{V'}^2 + \rho \left\| \theta - \overline{\theta} \right\|^2 - M \left\| \theta - \overline{\theta} \right\| \left\| \theta - \overline{\theta} \right\|_{V'}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\langle (\lambda I + A_\varepsilon)\theta - (\lambda I + A_\varepsilon)\overline{\theta}, \theta - \overline{\theta} \right\rangle_{V'} \\
&\geq \left(\lambda - \frac{M^2}{2\rho} \right) \left\| \theta - \overline{\theta} \right\|_{V'}^2 + \frac{\rho}{2} \left\| \theta - \overline{\theta} \right\|^2 \geq 0,
\end{aligned} \tag{2.12}$$

for λ large enough, $\lambda \geq \dfrac{M^2}{2\rho}$.

Next we have to show that for every $g \in V'$ there exists $\theta \in D(A_\varepsilon)$ solution to

$$\lambda \theta + A_\varepsilon \theta = g. \tag{2.13}$$

If we denote $\zeta := \beta_\varepsilon^*(\theta) \in V$, due to the fact that β_ε^* is continuous and monotonically increasing on $(-\infty, \infty)$ and $R(\beta_\varepsilon^*) = (-\infty, \infty)$, it follows that its inverse

$$G(\zeta) := (\beta_\varepsilon^*)^{-1}(\zeta) \tag{2.14}$$

is Lipschitz, by (i), hence it is continuous from V to $L^2(\Omega)$, i.e.,

$$\left\|G(\zeta)-G(\overline{\zeta})\right\| \leq \frac{1}{\rho}\left\|\zeta-\overline{\zeta}\right\| \leq \frac{c_H}{\rho}\left\|\zeta-\overline{\zeta}\right\|_V. \tag{2.15}$$

Therefore, (2.13) can be rewritten as

$$\lambda G(\zeta)+A_G\zeta=g \tag{2.16}$$

with $A_G : V \to V^{'}$ defined by

$$\begin{aligned}&\langle A_G\zeta,\psi\rangle_{V',V} \\ &=\int_\Omega \nabla\zeta\cdot\nabla\psi dx-\int_\Omega K(G(\zeta))\frac{\partial\psi}{\partial x_3}dx+\int_{\Gamma_\alpha}\alpha\zeta\psi d\sigma,\ \forall\psi\in V.\end{aligned} \tag{2.17}$$

We intend to show that (2.16) has a solution and for that we are going to prove that $\lambda G + A_G$ is surjective. We have in virtue of assumptions (i_β), (i) and ($\mathrm{i_K}$) that

$$\begin{aligned}&\left\langle(\lambda G+A_G)\zeta-(\lambda G+A_G)\overline{\zeta},\zeta-\overline{\zeta}\right\rangle_{V',V} \\ &=\lambda\int_\Omega(G(\zeta)-G(\overline{\zeta}))(\zeta-\overline{\zeta})dx+\int_\Omega\left|\nabla(\zeta-\overline{\zeta})\right|^2dx \\ &-\int_\Omega\left(K(G(\zeta))-K(G(\overline{\zeta}))\right)\frac{\partial(\zeta-\overline{\zeta})}{\partial x_3}dx+\int_{\Gamma_\alpha}\alpha(\zeta-\overline{\zeta})^2d\sigma \\ &\geq\lambda\rho\int_\Omega(G(\zeta)-G(\overline{\zeta}))^2dx+\left\|\zeta-\overline{\zeta}\right\|_V^2-M\left\|G(\zeta)-G(\overline{\zeta})\right\|\left\|\zeta-\overline{\zeta}\right\|_V \\ &\geq\left(\lambda\rho-\frac{M^2}{2}\right)\left\|G(\zeta)-G(\overline{\zeta})\right\|^2+\frac{1}{2}\left\|\zeta-\overline{\zeta}\right\|_V^2\geq 0,\end{aligned}$$

for λ large enough $\left(\text{still for } \lambda \geq \dfrac{M^2}{2\rho}\right)$, so that $\lambda G+A_G$ is strongly monotone. This implies immediately that it is coercive, too.

By (2.15) it follows that the function $\zeta \to K(G(\zeta))$ is continuous on $L^2(\Omega)$, because

$$\left\|K(G(\zeta))-K(G(\overline{\zeta}))\right\| \leq M\left\|G(\zeta)-G(\overline{\zeta})\right\| \leq \frac{1}{\rho}\left\|\zeta-\overline{\zeta}\right\|. \tag{2.18}$$

Finally, we have deduced that the operator $\lambda G + A_G$ is continuous from V to $V^{'}$, monotone and coercive and, on the basis of Minty's theorem (Theorem 2.7 in Chap. 3) it is surjective, proving thus that (2.16) has a unique solution. This ends the proof of the quasi m-accretivity of A_ε. ∎

Denote

$$j_\varepsilon(r) := \int_0^r \beta_\varepsilon^*(\xi)d\xi,\ \forall r \in \mathbf{R}. \tag{2.19}$$

It follows that j_ε is a proper, convex and continuous function and

$$\partial j_\varepsilon(r) = \beta_\varepsilon^*(r),\ \forall r \in \mathbf{R}. \tag{2.20}$$

Theorem 2.2 (existence in the approximating problem) *Let*

$$f \in W^{1,1}(0,T;V'),\ f_0 \in W^{1,1}(0,T;L^2(\Gamma_\alpha)), \tag{2.21}$$

$$u \in W^{1,1}(0,T;L^2(\Gamma_u)),\ \theta_0 \in D(A_\varepsilon) \tag{2.22}$$

hold and let us assume hypotheses (i)-(ii), (iv) *and* (i_K). *Then, for each* $\varepsilon > 0$, *there exists a unique strong solution* $\theta_\varepsilon \in C([0,T];V')$ *to problem* (2.3)-(2.4) *such that*

$$\theta_\varepsilon \in W^{1,\infty}(0,T;V') \cap L^\infty(0,T;D(A_\varepsilon)) \cap L^\infty(0,T;V), \tag{2.23}$$

$$\beta_\varepsilon^*(\theta_\varepsilon) \in L^\infty(0,T;V), \tag{2.24}$$

$$j_\varepsilon(\theta_\varepsilon) \in L^\infty(0,T;L^1(\Omega)). \tag{2.25}$$

The solution satisfies the estimate

$$\begin{aligned}&\int_\Omega j_\varepsilon(\theta_\varepsilon(x,t))dx + \int_0^t \left\| \frac{d\theta_\varepsilon}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V^2\, d\tau \\ &\le \gamma_0(\alpha_m)\left(\int_\Omega j_\varepsilon(\theta_0(x))dx + \int_0^T \|f(\tau)\|_{V'}^2\, d\tau \right.\\ &\left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2\, d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2\, d\tau\right),\end{aligned} \tag{2.26}$$

where γ_0 *is independent of* ε.

Moreover, if $\theta_{\varepsilon,\lambda}$ *and* $\theta_{\varepsilon,\mu}$ *are two solutions to problem* (2.3)-(2.4) *corresponding to the data* $f = f_\lambda$, $\theta_0 = \theta_\lambda^0$, $f_0 = f_\lambda^0$, $f_\Gamma = f_\lambda^\Gamma$, $u = u_\lambda$ *and, respectively,* $f = f_\mu$, $\theta_0 = \theta_\mu^0$, $f_0 = f_\mu^0$, $f_\Gamma = f_\mu^\Gamma$, $u = u_\mu$, *the following estimate*

$$\begin{aligned}&\|\theta_{\varepsilon,\lambda}(t) - \theta_{\varepsilon,\mu}(t)\|_{V'}^2 + \int_0^t \|\theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\|^2\, d\tau \\ &\le \gamma_1(\alpha_m)\left(\left\|\theta_\lambda^0 - \theta_\mu^0\right\|_{V'}^2 + \int_0^T \|f_\lambda(\tau) - f_\mu(\tau)\|_{V'}^2\, d\tau \right.\\ &\left. + \int_0^T \|u_\lambda(\tau) - u_\mu(\tau)\|_{L^2(\Gamma_u)}^2\, d\tau + \int_0^T \left\|f_\lambda^0(\tau) - f_\mu^0(\tau)\right\|_{L^2(\Gamma_\alpha)}^2\, d\tau\right)\end{aligned} \tag{2.27}$$

holds for each $\varepsilon > 0$ *with* γ_1 *independent of* ε.

Proof. First we notice that by the trace theorem $Bu+f_\Gamma+f \in W^{1,1}(0,T;V')$. Indeed using (1.11) we have that

$$\|f_\Gamma(t)\|_{V'} = \sup_{\|\psi\|_V \leq 1} |f_\Gamma(t)(\psi)| \leq \frac{1}{\sqrt{\alpha_m}} \|f_0(t)\|_{L^2(\Gamma_\alpha)}.$$

From here we obtain that

$$\int_0^T \|f_\Gamma(t)\|_{V'} \, dt \leq \frac{1}{\sqrt{\alpha_m}} \int_0^T \|f_0(t)\|_{L^2(\Gamma_\alpha)} \, dt < \infty.$$

In the same way we deduce that

$$\int_0^T \left\| \frac{df_\Gamma}{dt}(t) \right\|_{V'} dt \leq \frac{1}{\sqrt{\alpha_m}} \int_0^T \left\| \frac{df_0}{dt}(t) \right\|_{L^2(\Gamma_\alpha)} dt < \infty,$$

meaning that $f_\Gamma \in W^{1,1}(0,T;V')$.

Similarly we can show via the trace theorem that $Bu \in W^{1,1}(0,T;V')$.

Since A_ε is quasi m-accretive, $\theta_0 \in D(A_\varepsilon)$ and $Bu+f_\Gamma+f \in W^{1,1}(0,T;V')$, the first part of the theorem follows from Theorem 3.6 and Corollary 3.7 in Chap. 3. From Remark 1.2 we get that $\theta_\varepsilon \in L^\infty(0,T;V)$.

Then, we multiply the equation

$$\frac{d}{dt}(\theta_{\varepsilon,\lambda} - \theta_{\varepsilon,\mu}) + A_\varepsilon \theta_{\varepsilon,\lambda} - A_\varepsilon \theta_{\varepsilon,\mu} = f_\lambda - f_\mu + f_\lambda^\Gamma - f_\mu^\Gamma + Bu_\lambda - Bu_\mu$$

by $\theta_{\varepsilon,\lambda} - \theta_{\varepsilon,\mu}$, scalarly in V', and integrate over $(0,t)$ with $t \in (0,T)$. We have

$$\begin{aligned}
&\frac{1}{2}\int_0^t \frac{d}{d\tau} \|\theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\|_{V'}^2 \, d\tau \\
&+ \int_0^t \langle A_\varepsilon\theta_{\varepsilon,\lambda}(\tau) - A_\varepsilon\theta_{\varepsilon,\mu}(\tau), \theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\rangle_{V'} \, d\tau \\
&= \int_0^t \langle f_\lambda(\tau) - f_\mu(\tau), \theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\rangle_{V'} \, d\tau \\
&+ \int_0^t \left\langle f_\lambda^\Gamma(\tau) - f_\mu^\Gamma(\tau) + Bu_\lambda(\tau) - Bu_\mu(\tau), \theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\right\rangle_{V'}.
\end{aligned}$$

But, by Proposition 2.1 we have that

$$\begin{aligned}
&\langle A_\varepsilon\theta_{\varepsilon,\lambda}(\tau) - A_\varepsilon\theta_{\varepsilon,\mu}(\tau), \theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\rangle_{V'} \\
&\geq -\frac{M^2}{2\rho} \|\theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\|_{V'}^2 + \frac{\rho}{2} \|\theta_{\varepsilon,\lambda}(\tau) - \theta_{\varepsilon,\mu}(\tau)\|^2,
\end{aligned}$$

and recalling (1.11)-(1.13) we obtain that

$$\frac{1}{2}\left\|\theta_{\varepsilon,\lambda}(t)-\theta_{\varepsilon,\mu}(t)\right\|_{V'}^{2}+\frac{\rho}{2}\int_{0}^{t}\left\|\theta_{\varepsilon,\lambda}(\tau)-\theta_{\varepsilon,\mu}(\tau)\right\|^{2}d\tau$$
$$\leq\frac{1}{2}\left\|\theta_{\lambda}^{0}-\theta_{\mu}^{0}\right\|_{V'}^{2}+\frac{1}{2}\left(1+c_{\Gamma_\alpha}^{2}+c_{\Gamma_u}^{2}+\frac{M^{2}}{\rho}\right)\int_{0}^{t}\left\|\theta_{\varepsilon,\lambda}(\tau)-\theta_{\varepsilon,\mu}(\tau)\right\|_{V'}^{2}d\tau$$
$$+\frac{1}{2}\int_{0}^{t}\left\|f_{\lambda}(\tau)-f_{\mu}(\tau)\right\|_{V'}^{2}d\tau+\frac{1}{2}\int_{0}^{t}\left\|f_{\lambda}^{0}(\tau)-f_{\mu}^{0}(\tau)\right\|_{L^{2}(\Gamma_\alpha)}^{2}d\tau$$
$$+\frac{1}{2}\int_{0}^{t}\left\|u_{\lambda}(\tau)-u_{\mu}(\tau)\right\|_{L^{2}(\Gamma_u)}^{2}d\tau.$$

Hence we apply Gronwall's lemma for $g(t)=\left\|\theta_{\varepsilon,\lambda}(t)-\theta_{\varepsilon,\mu}(t)\right\|_{V'}^{2}$ and we deduce that

$$\left\|\theta_{\varepsilon,\lambda}(t)-\theta_{\varepsilon,\mu}(t)\right\|_{V'}^{2}\leq c(\alpha_m,t)\left(\left\|\theta_{\lambda}^{0}-\theta_{\mu}^{0}\right\|_{V'}^{2}+\int_{0}^{T}\left\|f_{\lambda}(\tau)-f_{\mu}(\tau)\right\|_{V'}^{2}d\tau\right.$$
$$\left.+\int_{0}^{T}\left\|u_{\lambda}(\tau)-u_{\mu}(\tau)\right\|_{L^{2}(\Gamma_u)}^{2}d\tau+\int_{0}^{T}\left\|f_{\lambda}^{0}(\tau)-f_{\mu}^{0}(\tau)\right\|_{L^{2}(\Gamma_\alpha)}^{2}d\tau\right),$$

with

$$c(\alpha_m,t):=\exp\left[\left(1+c_{\Gamma_\alpha}^{2}+c_{\Gamma_u}^{2}+\frac{M^{2}}{\rho}\right)t\right]. \tag{2.28}$$

From here we get (2.27) as claimed, with $\gamma_1(\alpha_m):=c(\alpha_m,T)$.

For the estimate (2.26) we multiply equation (2.3) scalarly in V' by $\frac{d\theta_\varepsilon}{d\tau}$ and integrate over $(0,t)$. We obtain that

$$\int_{0}^{t}\left\|\frac{d\theta_\varepsilon}{d\tau}(\tau)\right\|_{V'}^{2}d\tau+\int_{0}^{t}\int_{\Omega}\nabla\beta_{\varepsilon}^{*}(\theta_\varepsilon)\cdot\nabla\psi dxd\tau+\int_{0}^{t}\int_{\Gamma_a}\alpha\beta_{\varepsilon}^{*}(\theta_\varepsilon)\psi d\sigma d\tau$$
$$=\int_{0}^{t}\int_{\Omega}f\psi dxd\tau-\int_{0}^{t}\int_{\Gamma_\alpha}f_0\psi dxd\tau-\int_{0}^{t}\int_{\Gamma_u}u\psi dxd\tau+\int_{0}^{t}\int_{\Omega}K(\theta_\varepsilon)\frac{\partial\psi}{\partial x_3}dxd\tau,$$

with ψ satisfying the boundary value problem

$$-\Delta\psi=\frac{d\theta_\varepsilon}{d\tau}(\tau),\ \frac{\partial\psi}{\partial\nu}=0\text{ on }\Gamma_u,\ \frac{\partial\psi}{\partial\nu}+\alpha\psi=0\text{ on }\Gamma_\alpha.$$

On the left side we obtain

$$\int_{\Omega}\nabla\beta_{\varepsilon}^{*}(\theta_\varepsilon)\cdot\nabla\psi dx+\int_{\Gamma_\alpha}\alpha\beta_{\varepsilon}^{*}(\theta_\varepsilon)\psi d\sigma$$
$$=\int_{\Gamma}\beta_{\varepsilon}^{*}(\theta_\varepsilon)\nabla\psi\cdot\nu d\sigma-\int_{\Omega}\beta_{\varepsilon}^{*}(\theta_\varepsilon)\Delta\psi dx+\int_{\Gamma_\alpha}\alpha\beta_{\varepsilon}^{*}(\theta_\varepsilon)\psi d\sigma$$
$$=-\int_{\Gamma_\alpha}\alpha\beta_{\varepsilon}^{*}(\theta_\varepsilon)\psi d\sigma+\int_{\Gamma_u}\beta_{\varepsilon}^{*}(\theta_\varepsilon)\nabla\psi\cdot\nu d\sigma$$

$$+\int_{\Omega}\beta_{\varepsilon}^{*}(\theta_{\varepsilon})\frac{d\theta_{\varepsilon}}{d\tau}(\tau)dx+\int_{\Gamma_{\alpha}}\alpha\beta_{\varepsilon}^{*}(\theta_{\varepsilon})\psi d\sigma=\int_{\Omega}\frac{dj_{\varepsilon}(\theta_{\varepsilon})}{d\tau}(\tau)dx$$
$$=\frac{d}{d\tau}\int_{\Omega}j_{\varepsilon}(\theta_{\varepsilon}(\tau))dx.$$

After the integration with respect to τ we get

$$\int_{\Omega}j_{\varepsilon}(\theta_{\varepsilon}(x,t))dx+\int_0^t\left\|\frac{d\theta_{\varepsilon}}{d\tau}(\tau)\right\|_{V'}^2 d\tau$$
$$\leq\int_{\Omega}j_{\varepsilon}(\theta_0)dx+\frac{1}{2}\int_0^t\left\|\frac{d\theta_{\varepsilon}}{d\tau}(\tau)\right\|_{V'}^2 d\tau+2\int_0^t\|f(\tau)\|_{V'}^2 d\tau$$
$$+2\int_0^t\left(c_{\Gamma_\alpha}^2\|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2+c_{\Gamma_u}^2\|u(\tau)\|_{L^2(\Gamma_u)}^2\right)d\tau+2M^2\int_0^t\|\theta_{\varepsilon}(\tau)\|^2 d\tau.$$

But we notice that

$$\frac{\rho}{2}\|\theta_{\varepsilon}(t)\|^2\leq\int_{\Omega}j_{\varepsilon}(\theta_{\varepsilon}(x,t))dx \tag{2.29}$$

and after some computations involving again Gronwall's lemma applied to $g(t)=\|\theta_{\varepsilon}(t)\|^2$we get

$$\|\theta_{\varepsilon}(t)\|^2\leq\frac{4}{\rho}\exp\left(\frac{4M^2}{\rho}t\right)S_{\varepsilon}\leq C_0S_{\varepsilon},\ \forall t\in[0,T], \tag{2.30}$$

where $C_0:=\frac{4}{\rho}\exp\left(\frac{4M^2}{\rho}T\right)$ and

$$S_{\varepsilon}:=\int_{\Omega}j_{\varepsilon}(\theta_0)dx \tag{2.31}$$
$$+\int_0^T\left(\|f(\tau)\|_{V'}^2+c_{\Gamma_u}^2\|u(\tau)\|_{L^2(\Gamma_u)}^2+c_{\Gamma_\alpha}^2\|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2\right)d\tau.$$

This implies that

$$\int_{\Omega}j_{\varepsilon}(\theta_{\varepsilon}(x,t))dx+\int_0^t\left\|\frac{d\theta_{\varepsilon}}{d\tau}(\tau)\right\|_{V'}^2 d\tau\leq 4S_{\varepsilon}\exp\left(\frac{4M^2}{\rho}t\right). \tag{2.32}$$

Then we apply (2.5) with $\phi=\beta_{\varepsilon}^{*}(\theta_{\varepsilon})$, integrate over $(0,t)$ and deduce by standard computations that

$$\int_{\Omega}j_{\varepsilon}(\theta_{\varepsilon}(x,t))dx+\int_0^t\|\beta_{\varepsilon}^{*}(\theta_{\varepsilon}(\tau))\|_V^2 d\tau\leq 4S_{\varepsilon}\exp\left(\frac{4M^2}{\rho}t\right). \tag{2.33}$$

Adding these two last inequalities we obtain (2.26) as claimed, with

$$\gamma_0(\alpha_m):=8\max\{1,c_{\Gamma_\alpha}^2,c_{\Gamma_u}^2\}\exp\left[\frac{4M^2T}{\rho}\right]. \tag{2.34}$$

Obviously

$$\gamma_0(\alpha_m) = O\left(\frac{1}{\alpha_m}\right) \text{ and } \gamma_1(\alpha_m) = O\left(\frac{1}{\alpha_m}\right) \text{ as } \alpha_m \to 0, \tag{2.35}$$

so that in the previous estimates we cannot consider the limit $\alpha_m \to 0$.

By (2.26) we get that $j_\varepsilon(\theta_\varepsilon) \in L^\infty(0,T;L^1(\Omega))$, for each $\varepsilon > 0$. ■

In Theorem 2.3 below, we shall see that the above existence result remains true under weaker regularity assumptions on f, u and f_0.

Theorem 2.3. *Let*

$$\begin{aligned} &f \in L^2(0,T;V'),\ \theta_0 \in L^2(\Omega), \\ &u \in L^2(0,T;L^2(\Gamma_u)),\ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)). \end{aligned} \tag{2.36}$$

Then, problem (2.3)-(2.4) *has, for each* $\varepsilon > 0$, *a unique solution*

$$\begin{aligned} \theta_\varepsilon &\in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V), \\ \beta^*_\varepsilon(\theta) &\in L^2(0,T;V), \end{aligned} \tag{2.37}$$

that satisfies estimates (2.26)-(2.27). *Moreover, if* $\theta_0 \le \theta_s$ *a.e. on* Ω, *then, the right-hand constants in* (2.26)-(2.27) *are independent of* ε.

Proof. Due to density arguments, let $\{f_n\}_{n\ge1}$, $\{u_n\}_{n\ge1}$ and $\{f_n^0\}_{n\ge1}$ be three sequences, such that

$$\begin{aligned} f_n &\in W^{1,1}(0,T;V'),\ f_n \longrightarrow f \text{ in } L^2(0,T;V') \\ u_n &\in W^{1,1}(0,T;L^2(\Gamma_u)),\ u_n \longrightarrow u \text{ in } L^2(0,T;L^2(\Gamma_u)) \\ f_n^0 &\in W^{1,1}(0,T;L^2(\Gamma_\alpha)),\ f_n^0 \longrightarrow f_0 \text{ in } L^2(0,T;L^2(\Gamma_\alpha)), \end{aligned}$$

and let $\theta_0 \in L^2(\Omega)$. Since $\theta \in D(A_\varepsilon)$ implies $\theta \in V$ which is dense in $L^2(\Omega)$, there exists $\{\theta_n^0\}_{n\ge1} \subset D(A_\varepsilon)$ such that $\theta_n^0 \to \theta_0$ strongly in $L^2(\Omega)$ and consequently strongly in V', too. Remember that ε is fixed. Then, for each $\varepsilon > 0$ there is a unique solution $\theta_{\varepsilon,n}$ to the approximating problem

$$\begin{aligned} \frac{d\theta_{\varepsilon,n}}{dt} + A_\varepsilon\theta_{\varepsilon,n} &= f_n + Bu_n + f_n^\Gamma,\ \text{a.e. } t \in (0,T), \\ \theta_{\varepsilon,n}(0) &= \theta_n^0, \end{aligned} \tag{2.38}$$

which satisfies the conclusions of Theorem 2.2, with the estimates

$$\begin{aligned} &\|\theta_{\varepsilon,n}(t) - \theta_{\varepsilon,m}(t)\|_{V'}^2 + \int_0^t \|\theta_{\varepsilon,n}(\tau) - \theta_{\varepsilon,m}(t)\|^2\, d\tau \\ &\le \gamma_1(\alpha_m)\left(\left\|\theta_n^0 - \theta_m^0\right\|_{V'}^2 + \int_0^T \|f_n(\tau) - f_m(\tau)\|_{V'}^2\, d\tau \right. \\ &\left. + \int_0^T \|u_n(\tau) - u_m(\tau)\|_{L^2(\Gamma_u)}^2\, d\tau + \int_0^T \left\|f_n^0(\tau) - f_m^0(\tau)\right\|_{L^2(\Gamma_\alpha)}^2 d\tau\right) \end{aligned} \tag{2.39}$$

and

$$\int_{\Omega} j_\varepsilon(\theta_{\varepsilon,n}(x,t))dx + \int_0^t \left\| \frac{d\theta_{\varepsilon,n}}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|\beta_\varepsilon^*(\theta_{\varepsilon,n}(\tau))\|_V^2 \, d\tau \quad (2.40)$$
$$\leq \gamma_0(\alpha_m) \left(\int_{\Omega} j_\varepsilon(\theta_n^0(x))dx + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \right.$$
$$\left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 \, d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 \, d\tau \right)$$

independent on n. Indeed, for example,

$$\int_0^t \|f_n(\tau)-f_m(\tau)\|_{V'}^2 \, d\tau \leq \int_0^t \|f_n(\tau)-f(\tau)\|_{V'}^2 \, d\tau + \int_0^t \|f(\tau)-f_m(\tau)\|_{V'}^2 \, d\tau < 2T\delta,$$
$$\text{for any } \delta > 0 \text{ and } n, m \geq n(\delta),$$

so that the right-hand side in (2.39) does no longer depend on n, if n is large enough. Also, for each $\varepsilon > 0$ we have

$$\int_{\Omega} j_\varepsilon(\theta_n^0(x))dx \leq \frac{1}{2\varepsilon} \left\|\theta_n^0\right\|^2$$

and if n is large enough, then $\left\|\theta_n^0\right\| \leq \|\theta_0\| + \delta$.

Then, from (2.39) we deduce then that the sequence $\{\theta_{\varepsilon,n}\}_{n\geq 1}$ is Cauchy in $L^\infty(0,T;V') \cap L^2(Q)$, so it is convergent

$$\theta_{\varepsilon,n} \longrightarrow \theta_\varepsilon \text{ strongly in } L^2(Q), \text{ as } n \to \infty. \quad (2.41)$$

Estimate (2.40) implies the boundedness of the sequences $\left\{ \dfrac{d\theta_{\varepsilon,n}}{dt} \right\}$ in $L^2(0,T;V')$, $\{\beta_\varepsilon^*(\theta_{\varepsilon,n})\}_{n\geq 1}$ in $L^2(0,T;V)$, the latter implying also the boundedness of $\{\theta_{\varepsilon,n}\}_{n\geq 1}$ in $L^2(0,T;V)$. Therefore, selecting a subsequence, if necessary, we obtain

$$\frac{d\theta_{\varepsilon,n}}{dt} \longrightarrow \frac{d\theta_\varepsilon}{dt} \quad \text{weakly in } L^2(0,T;V'), \text{ as } n \to \infty,$$
$$\beta_\varepsilon^*(\theta_{\varepsilon,n}) \longrightarrow \beta_\varepsilon^*(\theta_\varepsilon) \quad \text{weakly in } L^2(0,T;V), \text{ as } n \to \infty.$$

This convergence follows due to the continuity of β_ε^* in $\mathbf{R}$ and to the strongly convergence (2.41) that implies $\theta_{\varepsilon,n} \to \theta_\varepsilon$ a.e. in Q. We get also

$$\theta_{\varepsilon,n} \longrightarrow \theta_\varepsilon \text{ weakly in } L^2(0,T;V), \text{ as } n \to \infty.$$

Moreover, it follows that

$$K(\theta_{\varepsilon,n}) \longrightarrow K(\theta_\varepsilon) \text{ strongly in } L^2(Q) \text{ and weakly in } L^2(0,T;V),$$

because $K(\theta_{\varepsilon,n}) \in L^2(0,T;V)$. Since the operator A_ε is quasi m-accretive on V', it follows that its realization on $L^2(0,T;V')$ is quasi m-accretive too, hence it is demiclosed (see Definition 2.11 and Proposition 2.12, Chap. 3), meaning that

$$A_\varepsilon\theta_{\varepsilon,n} \longrightarrow A_\varepsilon\theta_\varepsilon \text{ weakly in } L^2(0,T;V), \text{ as } n \to \infty.$$

Eventually, we can pass to limit as $n \to \infty$ in (2.38) and obtain that

$$\begin{aligned} \frac{d\theta_\varepsilon}{dt} + A_\varepsilon\theta_\varepsilon &= f + Bu + f_\Gamma, \text{ a.e. } t \in (0,T), \\ \theta_\varepsilon(0) &= \theta_0, \end{aligned}$$

which shows that θ_ε is a solution to (2.3)-(2.4).

By passing at limit with $n \to \infty$ in (2.40) and by the lower-semicontinuity arguments it follows that θ_ε satisfies (2.26).

In a similar way we can get (2.27).

The uniqueness follows then from (2.27).

Now, if $\theta_0 \in L^2(\Omega)$ and $\theta_0 \le \theta_s$, a.e. $x \in \Omega$, we have

$$\begin{aligned} j_\varepsilon(\theta_0) &= \int_0^{\theta_0} \beta_\varepsilon^*(\xi)d\xi \le \int_0^{\theta_s} \beta_\varepsilon^*(\xi)d\xi \le \lim_{\theta \nearrow \theta_s} \int_0^{\theta} \beta_\varepsilon^*(\xi)d\xi \\ &= \lim_{\theta \nearrow \theta_s} \int_0^{\theta} \beta^*(\xi)d\xi \le \lim_{\theta \nearrow \theta_s} K_s^*\theta = K_s^*\theta_s < +\infty, \end{aligned} \tag{2.42}$$

hence we emphasize that the right-hand side in (2.26) becomes thus independent of ε, being equal to $\gamma_0(\alpha_m)S_0$, where

$$\begin{aligned} S_0 :&= K_s^*\theta_s \text{meas}(\Omega) + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \\ &+ \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 \, d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 \, d\tau. \end{aligned} \tag{2.43}$$

Finally, by (2.29) and (2.42) we obtain that

$$\|\theta_\varepsilon(t)\| \le C_S < +\infty, \ \forall t \in [0,T], \tag{2.44}$$

where

$$C_S^2 := \frac{2}{\rho} S_0 \gamma_0(\alpha_m) \tag{2.45}$$

does not depend on ε. ∎

Comparison results for the approximating solution

Consider two time dependent functions $\theta_M \in C^1[0,T]$ and $\theta_m \in C^1[0,T]$ such that

$$\theta_m(t) \le \theta_M(t) \text{ and } \theta_m'(t) \le \theta_M'(t),\ \forall t \in [0,T].$$

Assume also that $\theta_m(0)$ and $\theta_M(0)$ do not vanish simultaneously and the same property is true for $\theta_m'(0)$ and $\theta_M'(0)$. Then, let us denote

$$f_M(t) = \theta_M'(t),\ u_M(t) = -K(\theta_M(t)), \tag{2.46}$$

$$f_{0\varepsilon}^M(x,t) = K(\theta_M(t))i_3 \cdot \nu - \alpha(x)\beta_\varepsilon^*(\theta_M(t)) \tag{2.47}$$

and

$$f_m(t) = \theta_m'(t),\ u_m(t) = -K(\theta_m(t)),$$
$$f_{0\varepsilon}^m(x,t) = K(\theta_m(t))i_3 \cdot \nu - \alpha(x)\beta_\varepsilon^*(\theta_m(t)).$$

It is obvious that $\theta_M(t)$ is the classical solution to (1.1)-(1.4) in which f_M, u_M, $f_{0\varepsilon}^M$ stand for f, u, f_0, i.e.,

$$\begin{aligned} \frac{\partial \theta_M}{\partial t} - \Delta\beta_\varepsilon^*(\theta_M) + \frac{\partial K(\theta_M)}{\partial x_3} &= f_M(t) \ \text{ in } \ Q, \\ \theta_M(x,0) &= \theta_M(0) \ \text{ in } \ \Omega, \\ (K(\theta_M)i_3 - \nabla\beta_\varepsilon^*(\theta_M)) \cdot \nu &= u_M(t) \ \text{ on } \ \Sigma_u, \\ (K(\theta_M)i_3 - \nabla\beta_\varepsilon^*(\theta_M)) \cdot \nu &= \alpha\beta_\varepsilon^*(\theta_M) + f_{0\varepsilon}^M(x,t) \ \text{ on } \ \Sigma_\alpha. \end{aligned}$$

Analogously, $\theta_m(t)$ is the classical solution to (1.1)-(1.4) corresponding to f_m, u_m, $f_{0\varepsilon}^m$ instead of f, u, f_0.

Lemma 2.4. *Let*

$$f \in L^\infty(Q), \tag{2.48}$$

$$u \in L^\infty(\Sigma_u),\ f_0 \in L^\infty(\Sigma_\alpha), \tag{2.49}$$

$$\theta_0 \in L^2(\Omega) \tag{2.50}$$

hold and assume still that

$$\theta_m(0) \le \theta_0(x) \le \theta_M(0) \text{ a.e. in } \Omega, \tag{2.51}$$
$$\theta_m'(t) \le f(x,t) \le \theta_M'(t) \text{ a.e. in } Q, \tag{2.52}$$
$$u_M(t) \le u(x,t) \le u_m(t) \text{ a.e. on } \Sigma_u, \tag{2.53}$$
$$f_{0\varepsilon}^M(x,t) \le f_0(x,t) \le f_{0\varepsilon}^m(x,t) \text{ a.e. on } \Sigma_\alpha. \tag{2.54}$$

Then, for each $\varepsilon > 0$, *we have*

$$\theta_m(t) \le \theta_\varepsilon(x,t) \le \theta_M(t) \text{ a.e. in } \Omega,\ \text{for each } t \in [0,T]. \tag{2.55}$$

Proof. By Theorem 2.3, problem (2.3)-(2.4) has a unique solution

$$\theta_\varepsilon \in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V).$$

We multiply the equation

$$\frac{\partial(\theta_\varepsilon - \theta_M)}{\partial t} - \Delta(\beta_\varepsilon^*(\theta_\varepsilon) - \beta_\varepsilon^*(\theta_M)) + \frac{\partial K(\theta_\varepsilon)}{\partial x_3} - \frac{\partial K(\theta_M)}{\partial x_3} = f - f_M \quad (2.56)$$

by $(\theta_\varepsilon(x,t) - \theta_M(t))^+$ and then we integrate it over $\Omega \times (0,t)$. We get

$$\int_0^t \int_\Omega \left\{ \frac{1}{2} \frac{\partial}{\partial \tau} [(\theta_\varepsilon - \theta_M)^+]^2 + \nabla(\beta_\varepsilon^*(\theta_\varepsilon) - \beta_\varepsilon^*(\theta_M)) \cdot \nabla(\theta_\varepsilon - \theta_M)^+ \right\} dx d\tau$$
$$+ \int_0^t \int_{\Gamma_\alpha} \alpha(\beta_\varepsilon^*(\theta_\varepsilon) - \beta_\varepsilon^*(\theta_M))(\theta_\varepsilon - \theta_M)^+ d\sigma d\tau$$
$$= \int_0^t \int_\Omega (K(\theta_\varepsilon) - K(\theta_M)) \frac{\partial(\theta_\varepsilon - \theta_M)^+}{\partial x_3} dx d\tau + \int_0^t \int_\Omega (f - f_M)(\theta_\varepsilon - \theta_M)^+ dx d\tau$$
$$- \int_0^t \int_{\Gamma_\alpha} (f_0 - f_{0\varepsilon}^M)(\theta_\varepsilon - \theta_M)^+ d\sigma d\tau - \int_0^t \int_{\Gamma_u} (u - u_M)(\theta_\varepsilon - \theta_M)^+ d\sigma d\tau.$$

But

$$\alpha(\beta_\varepsilon^*(\theta_\varepsilon) - \beta_\varepsilon^*(\theta_M))(\theta_\varepsilon - \theta_M)^+ \geq \alpha\rho((\theta_\varepsilon - \theta_M)^+)^2$$

and by Stampacchia's lemma we have that

$$\nabla\beta_\varepsilon^*(\theta_\varepsilon) \cdot \nabla(\theta_\varepsilon - \theta_M)^+ = \beta_\varepsilon(\theta_\varepsilon)\nabla(\theta_\varepsilon - \theta_M) \cdot \nabla(\theta_\varepsilon - \theta_M)^+ \geq \rho \left|\nabla(\theta_\varepsilon - \theta_M)^+\right|^2.$$

It follows that

$$\frac{1}{2} \int_\Omega [(\theta_\varepsilon(t) - \theta_M(t))^+]^2 dx + \rho \int_0^t \left\|(\theta_\varepsilon(\tau) - \theta_M(\tau))^+\right\|_V^2 d\tau$$
$$\leq \frac{1}{2} \int_\Omega [(\theta_0 - \theta_M(0))^+]^2 dx + \int_0^t M \left\|\theta_\varepsilon(\tau) - \theta_M(\tau)\right\| \left\|(\theta_\varepsilon(\tau) - \theta_M(\tau))^+\right\|_V d\tau$$
$$+ \int_0^t \int_\Omega (f - f_M)(\theta_\varepsilon - \theta_M)^+ dx d\tau - \int_0^t \int_{\Gamma_\alpha} (f_0 - f_{0\varepsilon}^M)(\theta_\varepsilon - \theta_M)^+ d\sigma d\tau$$
$$- \int_0^t \int_{\Gamma_u} (u - u_M)(\theta_\varepsilon - \theta_M)^+ d\sigma d\tau.$$

By using the assumptions $\theta_0(x) \leq \theta_M(0)$ a.e. in Ω, $f \leq \theta_M'(t)$, $-u(x,t) \leq K(\theta_M(t))$ a.e. on Σ_u and $f_0(x,t) \geq f_{0\varepsilon}^M(x,t)$ a.e. on Σ_α we obtain that

$$\left\|(\theta_\varepsilon(t) - \theta_M(t))^+\right\|^2 + \rho \int_0^t \left\|(\theta_\varepsilon(\tau) - \theta_M(\tau))^+\right\|_V^2 d\tau$$
$$\leq \frac{M^2}{\rho} \int_0^t \left\|(\theta_\varepsilon(\tau) - \theta_M(\tau))^+\right\|^2 d\tau.$$

By Gronwall's lemma, we get that $\left\|(\theta_\varepsilon(t) - \theta_M(t))^+\right\|^2 = 0$, which implies that $\theta_\varepsilon(x,t) \leq \theta_M(t)$ a.e. on Ω, for each $t \in [0,T]$.

Similarly, if we multiply (2.56), with θ_m and f_m instead of θ_M and f_M, by $(\theta_\varepsilon(x,t) - \theta_M(t))^-$ and integrate it over $\Omega \times (0,t)$ we deduce that $\left\|(\theta_\varepsilon(t) - \theta_m(t))^-\right\|^2 = 0$, i.e., $\theta_\varepsilon(x,t) \geq \theta_m(t)$ a.e. on Ω, for each $t \in [0,T]$. ■

In the hypotheses of the previous lemma $f_{0\varepsilon}^M$ and $f_{0\varepsilon}^m$ depend on ε. However, for a particular choice of θ_m and θ_M, sufficient conditions that do not depend on ε may be found in the next result.

Corollary 2.5. *Let* $\theta_m, \theta_M \in C^1([0,T])$ *satisfy*

$$\theta_m(t) < \theta_s \leq \theta_M(t), \ \forall t \in [0,T], \ \textit{with } \theta_M(0) = \theta_s. \tag{2.57}$$

Assume (2.48)-(2.53) *and*

$$K_s - \alpha K_s^* \ \leq f_0(x,t) \leq K(\theta_m(t)) - \alpha\beta^*(\theta_m(t)), \ \textit{a.e. on } \Sigma_\alpha. \tag{2.58}$$

Then

$$\theta_m(t) \leq \theta_\varepsilon(x,t) \leq \theta_M(t), \ \textit{a.e. in } \Omega, \ \textit{for each } t \in [0,T]. \tag{2.59}$$

Proof. The hypothesis $\theta_m(t)<\theta_s$, $\forall t\in[0,T]$, implies that $\beta_\varepsilon^*(\theta_m)=\beta^*(\theta_m) <\beta^*(\theta_s)$, for any $\varepsilon < d(\theta_m(t),\theta_s)$, where $d(\theta_m(t),\theta_s) = \min\limits_{t\in[0,T]}(\theta_s - \theta_m(t))$. Hence, for ε small enough the term $K(\theta_m) - \alpha\beta_\varepsilon^*(\theta_m)$ may be replaced by $K(\theta_m) - \alpha\beta^*(\theta_m)$, so that $f_{0\varepsilon}^m$ turns out to be independent of ε. In particular θ_m can be chosen a constant lesser than θ_s.

Now, for $\theta_M(t) \geq \theta_s$ we have $\beta_\varepsilon^*(\theta_M) \geq K_s^*$, so that

$$K(\theta_M) - \alpha\beta_\varepsilon^*(\theta_M) \leq K_s - \alpha K_s^*.$$

In conclusion, using assumption (2.58) we can write that

$$K(\theta_M) - \alpha\beta_\varepsilon^*(\theta_M) \leq K_s - \alpha K_s^* \ \leq f_0(x,t) \leq K(\theta_m) - \alpha\beta^*(\theta_m), \text{ a.e. on } \Sigma_\alpha.$$

The latter, together with (2.51)-(2.53) implies the boundedness of θ_ε by $\theta_m(t)$ and $\theta_M(t)$. ∎

Additional regularity of the approximating solution

In this subsection we prove, for a later use, some regularity results for the approximating solution to (2.3)-(2.4). The proofs of these results require a function β_ε^* smoother (of class $C^3(\mathbf{R})$) than that given by (2.1). It can be obtained by regularizing the latter or by defining directly an approximation of $\beta^*$of $C^3(\mathbf{R})$.

Because β_ε^* previously defined in (2.1) is of class $C^3(\mathbf{R}\backslash\{0, \theta_s\})$ we have to smooth it only on the intervals $[\theta_{ext}, 0]$ and $[\theta_s - \varepsilon, \theta_s]$, where θ_{ext} is going to be specified. Thus, we consider the approximation

$$\beta_\varepsilon^*(r) = \begin{cases} \beta_m r, & r \leq \theta_{ext} \\ \beta_{ext}^*(r), & \theta_{ext} < r \leq 0 \\ \beta^*(r), & 0 < r \leq \theta_s - \varepsilon \\ \beta_{int}^*(r), & \theta_s - \varepsilon < r \leq \theta_s \\ \beta^*(\theta_s - \varepsilon) + \dfrac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}[r - (\theta_s - \varepsilon)], & r > \theta_s, \end{cases} \tag{2.60}$$

where θ_{ext} is chosen such that

$$(\beta^*_{ext})'(\theta_{ext}) := \beta_m > 0, \tag{2.61}$$

with β_m constant (independent of ε). The functions β^*_{ext} and β^*_{int} may read

$$\beta^*_{int}(r) = \sum_{j=1}^{5} a_j r^j, \; \beta^*_{ext}(r) = \sum_{j=1}^{5} b_j r^j$$

where a_j and b_j are determined such that $\beta^*_{int} \in C^3[\theta_s - \varepsilon, \theta_s]$ and $\beta^*_{ext} \in C^3[\theta_{ext}, 0]$. The derivatives of β^*_ε are:

$$\beta_\varepsilon(r) := (\beta^*_\varepsilon)'(r) = \begin{cases} \beta_m, & r \le \theta_{ext} \\ (\beta^*_{ext})'(r), & \theta_{ext} < r \le 0 \\ \beta(r), & 0 < r \le \theta_s - \varepsilon \\ (\beta^*_{int})'(r), & \theta_s - \varepsilon < r \le \theta_s \\ \dfrac{K^*_s - \beta^*(\theta_s - \varepsilon)}{\varepsilon}, & r > \theta_s, \end{cases} \tag{2.62}$$

$$\beta'_\varepsilon(r) = \begin{cases} 0, & r \le \theta_{ext} \\ (\beta^*_{ext})''(r), & \theta_{ext} < r \le 0 \\ \beta'(r), & 0 < r \le \theta_s - \varepsilon \\ (\beta^*_{int})''(r), & \theta_s - \varepsilon < r \le \theta_s \\ 0, & r > \theta_s, \end{cases} \tag{2.63}$$

$$\beta''_\varepsilon(r) = \begin{cases} 0, & r \le \theta_{ext} \\ (\beta^*_{ext})'''(r), & \theta_{ext} < r \le 0 \\ \beta''(r), & 0 < r \le \theta_s - \varepsilon \\ (\beta^*_{int})'''(r), & \theta_s - \varepsilon < r \le \theta_s \\ 0, & r > \theta_s. \end{cases} \tag{2.64}$$

(We recall that we considered β twice differentiable on $[0, \theta_s)$). Therefore β_ε, β'_ε and β''_ε are bounded on $\mathbf{R}$ and we have

$$0 < \beta_m \le \beta_\varepsilon(r), \; \forall r \in \mathbf{R}, \; \forall \varepsilon > 0,$$

so that relationship (i) is satisfied with the constant β_m instead of ρ.

For K we preserve here the same definition as in (2.2) and these functions are good enough to prove the regularity results we envisage. However, for the study of the free boundary existence it will be necessary to work with a smoother function $K \in C^2(\mathbf{R})$ which will be specified at that point.

Obviously all the results proved using the function (2.1) remain true for the smoother approximation introduced before.

Because the regularity proofs are very long and technical, they will be presented in two separate theorems.

Theorem 2.6. *Assume that*

$$f \in W^{1,2}(0,T;L^2(\Omega)), \tag{2.65}$$
$$u \in W^{1,2}(0,T;L^2(\Gamma_u)) \cap L^2(0,T;H^1(\Gamma_u)), \tag{2.66}$$
$$f_0 \in W^{1,2}(0,T;L^2(\Gamma_\alpha)) \cap L^2(0,T;H^1(\Gamma_\alpha)), \tag{2.67}$$
$$\theta_0 \in H^1(\Omega). \tag{2.68}$$

Then, the solution θ_ε to problem (2.3)-(2.4) *satisfies*

$$\theta_\varepsilon \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{2.69}$$
$$\beta_\varepsilon^*(\theta_\varepsilon) \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)). \tag{2.70}$$

Proof. By the hypotheses (2.65)-(2.68) it follows that the approximating problem has a unique solution satisfying the conclusions of Theorem 2.3.

Since we do not know a priori that $\frac{\partial \theta_\varepsilon}{\partial t}(t)$ and $\frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial t}(t)$ are in $L^2(\Omega)$ for $t \in (0,T)$, we should rigorously perform the next computations by replacing these derivatives by the corresponding finite differences

$$\frac{\theta_\varepsilon(t+\delta)-\theta_\varepsilon(t)}{\delta} \text{ and } \frac{\beta_\varepsilon^*(\theta_\varepsilon(t+\delta))-\beta_\varepsilon^*(\theta_\varepsilon(t))}{\delta}, \tag{2.71}$$

which belong to the same space as θ_ε does. However, for the writing simplicity, we denote, by convenience, these differences still by

$$\frac{\partial \theta_\varepsilon}{\partial t} \text{ and } \frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial t}, \text{ respectively,}$$

so that, in the proof below, the functions indicated by these notations are in fact the finite differences (2.71) and have the same regularity proved for θ_ε and $\beta_\varepsilon^*(\theta_\varepsilon)$ in Theorem 2.3.

We multiply equation (2.3) by $\frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial t} \in L^2(0,T;V)$ and integrate it over $\Omega \times (0,t)$. We get

$$\begin{aligned}
&\int_0^t \int_\Omega \beta_\varepsilon(\theta_\varepsilon)\left(\frac{\partial \theta_\varepsilon}{\partial \tau}\right)^2 d\tau dx + \frac{1}{2}\int_0^t \frac{d}{d\tau}\left\|\nabla \beta_\varepsilon^*(\theta_\varepsilon(\tau))\right\|^2 d\tau \\
&= \int_0^t\int_\Omega K(\theta_\varepsilon)\frac{\partial}{\partial x_3}\left(\frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial \tau}\right) d\tau dx + \int_0^t\int_\Omega f\frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial \tau} d\tau dx \\
&- \int_0^t \int_{\Gamma_\alpha} (\alpha\beta_\varepsilon^*(\theta_\varepsilon)+f_0)\frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial \tau} d\sigma d\tau - \int_0^t\int_{\Gamma_u} u\frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial \tau} d\sigma d\tau.
\end{aligned}$$

After the integration with respect to τ in the second term on the left-hand side, we obtain

$$\int_0^t \int_\Omega \beta_\varepsilon(\theta_\varepsilon) \left(\frac{\partial \theta_\varepsilon}{\partial \tau} \right)^2 d\tau dx + \frac{1}{2} \int_\Omega |\nabla \beta_\varepsilon^*(\theta_\varepsilon(t))|^2 \, dx - \frac{1}{2} \int_\Omega |\nabla \beta_\varepsilon^*(\theta_0)|^2 \, dx$$
$$+ \int_\Omega \left(K(\theta_\varepsilon(t)) \frac{\partial \beta_\varepsilon^*(\theta_\varepsilon(t))}{\partial x_3} - K(\theta_0) \frac{\partial \beta_\varepsilon^*(\theta_0)}{\partial x_3} \right) dx$$
$$- \int_0^t \int_\Omega \frac{\partial K(\theta_\varepsilon)}{\partial \tau} \frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial x_3} dx d\tau - \frac{1}{2} \int_{\Gamma_\alpha} \alpha \left((\beta_\varepsilon^*(\theta_\varepsilon(t)))^2 - (\beta_\varepsilon^*(\theta_0))^2 \right) d\sigma$$
$$+ \int_\Omega f(t) \beta_\varepsilon^*(\theta_\varepsilon(t)) dx - \int_\Omega f(0) \beta_\varepsilon^*(\theta_0) dx - \int_0^t \int_\Omega \frac{\partial f}{\partial \tau} \beta_\varepsilon^*(\theta_\varepsilon) dx d\tau$$
$$- \int_{\Gamma_\alpha} f_0(t) \beta_\varepsilon^*(\theta_\varepsilon(t)) d\sigma + \int_{\Gamma_\alpha} f_0(0) \beta_\varepsilon^*(\theta_0) d\sigma + \int_0^t \int_{\Gamma_\alpha} \frac{\partial f_0}{\partial \tau} \beta_\varepsilon^*(\theta_\varepsilon) d\sigma d\tau$$
$$- \int_{\Gamma_u} u(t) \beta_\varepsilon^*(\theta_\varepsilon(t)) d\sigma + \int_{\Gamma_u} u(0) \beta_\varepsilon^*(\theta_0) d\sigma + \int_0^t \int_{\Gamma_u} \frac{\partial u}{\partial \tau} \beta_\varepsilon^*(\theta_\varepsilon) d\sigma d\tau.$$

We use (1.10)-(1.13) which apply also for $\beta_\varepsilon^*(\theta_\varepsilon(t)) \in V$ and obtain

$$\int_0^t \int_\Omega \beta_\varepsilon(\theta_\varepsilon) \left(\frac{\partial \theta_\varepsilon}{\partial \tau} \right)^2 d\tau dx + \frac{1}{2} \int_\Omega |\nabla \beta_\varepsilon^*(\theta_\varepsilon(t))|^2 \, dx + \frac{1}{2} \int_{\Gamma_\alpha} \alpha (\beta_\varepsilon^*(\theta_\varepsilon(t)))^2 d\sigma$$
$$\leq C_0(\varepsilon) + \|K(\theta_\varepsilon(t))\| \, \|\beta_\varepsilon^*(\theta_\varepsilon(t))\|_V + M \int_0^t \left\| \frac{\partial \theta_\varepsilon}{\partial \tau}(\tau) \right\| \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V \, d\tau$$
$$+ c_H \|f(t)\| \, \|\beta_\varepsilon^*(\theta_\varepsilon(t))\|_V + c_H \int_0^t \left\| \frac{\partial f}{\partial \tau}(\tau) \right\| \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V \, d\tau$$
$$+ \|f_0(t)\|_{L^2(\Gamma_\alpha)} \|\beta_\varepsilon^*(\theta_\varepsilon(t))\|_{L^2(\Gamma_\alpha)} + \int_0^t \left\| \frac{\partial f_0}{\partial \tau}(\tau) \right\|_{L^2(\Gamma_\alpha)} \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_{L^2(\Gamma_\alpha)} \, d\tau$$
$$+ \|u(t)\|_{L^2(\Gamma_u)} \|\beta_\varepsilon^*(\theta_\varepsilon(t))\|_{L^2(\Gamma_u)} + \int_0^t \left\| \frac{\partial u}{\partial \tau}(\tau) \right\|_{L^2(\Gamma_u)} \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_{L^2(\Gamma_u)} \, d\tau,$$

where

$$C_0(\varepsilon) = \frac{1}{2} \int_\Omega |\nabla \beta_\varepsilon^*(\theta_0)|^2 dx - \int_\Omega K(\theta_0) \frac{\partial \beta_\varepsilon^*(\theta_0)}{\partial x_3} dx - \int_\Omega f(0) \beta_\varepsilon^*(\theta_0) dx \quad (2.72)$$
$$+ \frac{1}{2} \int_{\Gamma_\alpha} \alpha (\beta_\varepsilon^*(\theta_0))^2 d\sigma + \int_{\Gamma_\alpha} f_0(0) \beta_\varepsilon^*(\theta_0) d\sigma + \int_{\Gamma_u} u(0) \beta_\varepsilon^*(\theta_0) d\sigma.$$

Since $\beta_\varepsilon(\theta_\varepsilon) \geq \beta_m$ by (2.61), we have

$$\beta_m \int_0^t \int_\Omega \left(\frac{\partial \theta_\varepsilon}{\partial \tau} \right)^2 d\tau dx + \frac{1}{2} \|\beta_\varepsilon^*(\theta_\varepsilon)\|_V^2$$
$$\leq |C_0(\varepsilon)| + \frac{\beta_m}{2} \int_0^t \int_\Omega \left(\frac{\partial \theta_\varepsilon}{\partial \tau} \right)^2 d\tau dx + \frac{1}{2} \int_0^t \left(\frac{M^2}{\beta_m} + 3 \right) \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V^2 \, d\tau$$
$$+ \frac{1}{2} \int_0^t \left(c_H^2 \left\| \frac{\partial f}{\partial \tau}(\tau) \right\|^2 + c_{\Gamma_\alpha}^2 \left\| \frac{\partial f_0}{\partial \tau}(\tau) \right\|_{L^2(\Gamma_\alpha)}^2 + c_{\Gamma_u}^2 \left\| \frac{\partial u}{\partial \tau}(\tau) \right\|_{L^2(\Gamma_u)}^2 \right) d\tau$$

$$+\frac{1}{4}\left\|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\right\|_V^2+4c_H^2\left\|f(t)\right\|^2+4c_{\Gamma_\alpha}^2\left\|f_0(t)\right\|_{L^2(\Gamma_\alpha)}^2+4c_{\Gamma_u}^2\left\|u(t)\right\|_{L^2(\Gamma_u)}^2$$
$$+4M^2\left\|\theta_\varepsilon(t)\right\|^2.$$

Here we used (1.10)-(1.13) and (i_K). We notice that the assumptions (2.65)-(2.67) imply $f\in C([0,T];L^2(\Omega))\subset L^\infty(0,T;L^2(\Omega))$, $u\in L^\infty(0,T;L^2(\Gamma_u))$ and $f_0\in L^\infty(0,T;L^2(\Gamma_\alpha))$. We recall (2.30) and finally we get

$$\frac{\beta_m}{2}\int_0^t\int_\Omega\left(\frac{\partial\theta_\varepsilon}{\partial\tau}\right)^2 d\tau dx+\frac{1}{4}\left\|\beta_\varepsilon^*(\theta_\varepsilon(t))\right\|_V^2$$
$$\leq|C_0(\varepsilon)|+C_1(\varepsilon)+c_1+c_2\int_0^t\left\|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\right\|_V^2 d\tau,$$

where

$$c_1=\frac{1}{2}\left(c_H^2\left\|\frac{\partial f}{\partial t}\right\|_{L^2(Q)}^2+c_{\Gamma_\alpha}^2\left\|\frac{\partial f_0}{\partial t}\right\|_{L^2(\Sigma_\alpha)}^2+c_{\Gamma_u}^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(\Sigma_u)}^2\right),$$
$$C_1(\varepsilon)=4\left\{c_H^2\left\|f\right\|_{L^\infty(0,T;L^2(\Omega))}^2+c_{\Gamma_\alpha}^2\left\|f_0\right\|_{L^\infty(0,T;L^2(\Sigma_\alpha))}^2\right.$$
$$\left.+c_{\Gamma_u}^2\left\|u\right\|_{L^\infty(0,T;L^2(\Sigma_u))}^2+M^2a_0S_\varepsilon\right\},$$
$$c_2=\frac{1}{2}\left(\frac{M^2}{\beta_m}+3\right).$$

Using Gronwall's lemma we obtain that

$$\left\|\beta_\varepsilon^*(\theta_\varepsilon(t))\right\|_V^2\leq 4\left(|C_0(\varepsilon)|+C_1(\varepsilon)+c_1\right)e^{4c_2t},\ \forall t\in[0,T],$$

and therefore

$$\left\|\beta_\varepsilon^*(\theta_\varepsilon(t))\right\|_V^2\leq C_3(\varepsilon),\ \forall t\in[0,T], \tag{2.73}$$

i.e., $\beta_\varepsilon^*(\theta_\varepsilon)\in L^\infty(0,T;V)$ and

$$\left\|\frac{d\theta_\varepsilon}{dt}\right\|_{L^2(Q)}\leq C_4(\varepsilon)\ , \tag{2.74}$$

where

$$C_3(\varepsilon)=4\left(|C_0(\varepsilon)|+C_1(\varepsilon)+c_1\right)e^{4c_2T},$$
$$C_4(\varepsilon)=\frac{2}{\beta_m}\left(|C_0(\varepsilon)|+C_1(\varepsilon)+c_1+c_2C_3(\varepsilon)T\right).$$

We recall now that by $\frac{d\theta_\varepsilon}{dt}$ we have denoted the finite difference $\frac{\theta_\varepsilon(t+\delta)-\theta_\varepsilon(t)}{\delta}$, so that, as a matter of fact, relation (2.74) means that

$$\int_0^{T-\delta}\left\|\theta_\varepsilon(t+\delta)-\theta_\varepsilon(t)\right\|^2 dt\leq C_4^2(\varepsilon)T\delta^2,\ \forall\delta\in(0,T).$$

Then, by Theorem 3.11 in Appendix, it follows that $\theta_\varepsilon \in W^{1,2}(0,T;L^2(\Omega))$.

Condition (i) implies that the function $(\beta_\varepsilon^*)^{-1} : \mathbf{R} \to \mathbf{R}$ is Lipschitz, so that by (2.73) we also obtain that

$$\|\theta_\varepsilon(t)\|_V \leq C(\varepsilon),\ \forall t \in [0,T]. \tag{2.75}$$

From now on, everywhere within this proof, we shall denote by $C(\varepsilon)$ some constants that depend on ε, only by the means of $|C_0(\varepsilon)|$ and S_ε.

We also derive that $K(\theta_\varepsilon) \in L^\infty(0,T;H^1(\Omega))$, with

$$\|K(\theta_\varepsilon(t))\|_V \leq C(\varepsilon),\ \forall t \in [0,T]$$

and therefore we deduce that $K(\theta_\varepsilon)|_\Sigma \in L^\infty(0,T;H^{1/2}(\Gamma))$ (see e.g., [84]).

Now we resume (2.3) and get that

$$\Delta\beta_\varepsilon^*(\theta_\varepsilon) = \frac{\partial\theta_\varepsilon}{\partial t} - \frac{\partial K(\theta_\varepsilon)}{\partial x_3} - f \in L^2(0,T;L^2(\Omega)),$$

which yields

$$\|\Delta\beta_\varepsilon^*(\theta_\varepsilon)\|_{L^2(Q)} \leq C(\varepsilon). \tag{2.76}$$

By the boundary conditions (2.8), (2.9) and the hypotheses we deduce that

$$\begin{aligned} \nabla\beta_\varepsilon^*(\theta_\varepsilon)\cdot\nu|_{\Sigma_u} &= (K(\theta_\varepsilon)i_3\cdot\nu - u)|_{\Sigma_u} \in L^2(0,T;H^{1/2}(\Gamma_u)), \\ (\nabla\beta_\varepsilon^*(\theta_\varepsilon)\cdot\nu + \alpha\beta_\varepsilon^*(\theta_\varepsilon))|_{\Sigma_\alpha} &= (K(\theta_\varepsilon)i_3\cdot\nu - f_0)|_{\Sigma_\alpha} \in L^2(0,T;H^{1/2}(\Gamma_\alpha)), \end{aligned}$$

which together with (2.76) imply in virtue of the trace theorem (see [84]) that

$$\beta_\varepsilon^*(\theta_\varepsilon) \in L^2(0,T;H^2(\Omega)). \tag{2.77}$$

This implies that

$$\theta_\varepsilon \in L^2(0,T;H^2(\Omega)). \tag{2.78}$$

The last assertion is proved by noticing that by (2.77)

$$g := \frac{\partial\beta_\varepsilon^*(\theta_\varepsilon)}{\partial x_i} \in L^2(0,T;H^1(\Omega)), \tag{2.79}$$

$$\frac{\partial\theta_\varepsilon}{\partial x_i} = \frac{g}{\beta_\varepsilon(\theta_\varepsilon)} \tag{2.80}$$

and

$$\frac{\partial}{\partial x_j}\frac{\partial\theta_\varepsilon}{\partial x_i} = \frac{g_{x_j}}{\beta_\varepsilon(\theta_\varepsilon)} - g\frac{\beta_\varepsilon'(\theta_\varepsilon)}{\beta_\varepsilon^2(\theta_\varepsilon)}\frac{\partial\theta_\varepsilon}{\partial x_j}, \tag{2.81}$$

where $g_{x_j} \in L^2(0,T;L^2(\Omega))$, $\beta_\varepsilon(\theta_\varepsilon) \geq \beta_m > 0$ on Q and $\beta_\varepsilon'(\theta_\varepsilon)$ is bounded.

Now, by the Sobolev embedding theorems (see Theorem 2.21 in Appendix) we have for any $\eta \in H^1(\Omega) \subset L^6(\Omega)$ that

$$\int_\Omega \eta^4 dx \leq \left(\int_\Omega \eta^2 dx\right)^{1/2} \left(\int_\Omega \eta^6 dx\right)^{1/2} = \|\eta\| \, \|\eta\|^3_{L^6(\Omega)} \tag{2.82}$$

and therefore

$$\int_\Omega \eta^4 dx \leq C \|\eta\| \, \|\eta\|^3_{H^1(\Omega)}, \ \forall \eta \in H^1(\Omega). \tag{2.83}$$

Next,

$$\begin{aligned}
&\left\| g \frac{\partial \theta_\varepsilon}{\partial x_j} \right\|^2_{L^2(Q)} \\
&= \int_0^T \int_\Omega g^2 \left(\frac{\partial \theta_\varepsilon}{\partial x_j}\right)^2 dx dt \leq \int_0^T \left(\int_\Omega g^4 dx\right)^{1/2} \left(\int_\Omega \left(\frac{\partial \theta_\varepsilon}{\partial x_j}\right)^6 dx\right)^{1/2} dt \\
&\leq \int_0^T C^2 \|g(t)\|^{1/2} \|g(t)\|^{3/2}_{H^1(\Omega)} \left\| \frac{\partial \theta_\varepsilon}{\partial x_j}(t) \right\|^{1/2} \left\| \frac{\partial \theta_\varepsilon}{\partial x_j}(t) \right\|^{3/2}_{H^1(\Omega)} dt \\
&\leq \int_0^T C(\varepsilon) \left\| \frac{\partial \theta_\varepsilon}{\partial x_j}(t) \right\|^{1/2} \left\| \frac{\partial \theta_\varepsilon}{\partial x_j}(t) \right\|^{3/2}_{H^1(\Omega)} dt.
\end{aligned}$$

Recalling (2.80), (2.81) and Young's inequality we obtain that

$$\int_0^T \left\| \frac{\partial \theta_\varepsilon}{\partial x_j}(t) \right\|^2_{H^1(\Omega)} dt \leq C(\varepsilon) \left(1 + \int_0^T \left\| \frac{\partial \theta_\varepsilon}{\partial x_j}(t) \right\|^2 dt\right) + \frac{1}{2} \int_0^T \left\| \frac{\partial \theta_\varepsilon}{\partial x_j} \right\|^2_{H^1(\Omega)} dt,$$

implying finally that

$$\int_0^T \left\| \frac{\partial \theta_\varepsilon(t)}{\partial x_j} \right\|^2_{H^1(\Omega)} dt \leq C(\varepsilon). \tag{2.84}$$

By all these, together with the conclusions of Theorem 2.3 we get (2.69)-(2.70) as claimed. ∎

Remark 2.7. As we have seen c_1, c_2, are constants independent of ε and $C_0(\varepsilon), ..., C_4(\varepsilon)$ are constants depending on ε. Moreover, to avoid loading the notations we have denoted by $C(\varepsilon)$ other several ε-dependent constants, so the result is true only for each $\varepsilon > 0$ apart.

Therefore, we can prove a sharper regularity only for the approximating solution θ_ε, but generally, we cannot use this result in a passing to limit technique to get similar properties for θ. However, as we shall see later, under special assumptions, we may deduce a further regularity for θ too.

Theorem 2.8. *Assume that*

$$f \in W^{1,2}(0,T; L^2(\Omega)), \tag{2.85}$$

$$u \in W^{1,2}(0,T; L^2(\Gamma_u)) \cap L^2(0,T; H^1(\Gamma_u)), \tag{2.86}$$

$$f_0 \in W^{1,2}(0,T; L^2(\Gamma_\alpha)) \cap L^2(0,T; H^1(\Gamma_\alpha)), \tag{2.87}$$

$$\theta_0 \in H^2(\Omega). \tag{2.88}$$

Then, for each $\varepsilon > 0$, *problem* (2.3)-(2.4) *has a unique solution*

$$\theta_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \quad (2.89)$$
$$\beta_\varepsilon^*(\theta_\varepsilon) \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)). \quad (2.90)$$

Proof. By the hypotheses (2.85)-(2.88) it follows that the approximating problem has a unique solution satisfying the conclusions of Theorem 2.3. We shall show that $\beta_\varepsilon^*(\theta_\varepsilon) \in W^{1,\infty}(0,T;L^2(\Omega))$.

We denote

$$\eta := \beta_\varepsilon^*(\theta_\varepsilon),\ \theta_\varepsilon = (\beta_\varepsilon^*)^{-1}(\eta),\ \zeta(\eta) := K((\beta_\varepsilon^*)^{-1}(\eta)), \quad (2.91)$$
$$\omega(\eta) := \frac{1}{\beta_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta))} \quad (2.92)$$

and we notice that for each $\varepsilon > 0$ the functions $\beta_\varepsilon, \beta_\varepsilon', \beta_\varepsilon''$ are bounded on $\mathbf{R}$

$$\beta_m \le \beta_\varepsilon(r) \le \beta_M(\varepsilon),\ \beta_m' \le \beta_\varepsilon'(r) \le \beta_M'(\varepsilon),\ \beta_m'' \le \beta_\varepsilon''(r) \le \beta_M''(\varepsilon), \quad (2.93)$$

where

$$\beta_m = \min_{r\in\mathbf{R}} \beta_\varepsilon(r),\ \beta_m' = \min_{r\in\mathbf{R}} \beta_\varepsilon'(r),\ \beta_m'' = \min_{r\in\mathbf{R}} \beta_\varepsilon''(r),$$
$$\beta_M(\varepsilon) = \max_{r\in\mathbf{R}} \beta_\varepsilon(r),\ \beta_M'(\varepsilon) = \max_{r\in\mathbf{R}} \beta_\varepsilon'(r),\ \beta_M''(\varepsilon) = \max_{r\in\mathbf{R}} \beta_\varepsilon''(r).$$

We still mention that β_m, β_m' and $\beta_m^{''}$ do not depend on ε, but β_M, β_M', β_M'' depend and have the order of $\frac{1}{\varepsilon}$ as $\varepsilon \to 0$ (see their expressions in (2.62)-(2.64)).

Then, we replace $\beta_\varepsilon^*(\theta_\varepsilon)$ by η in (2.6)-(2.9) and equation (2.6) becomes

$$\omega(\eta)\eta_t - \Delta\eta + \frac{\partial\zeta(\eta)}{\partial x_3} = f \text{ in } Q,$$

where η_t is the derivative of η with respect to t. Correspondingly, we write the initial and boundary conditions in the new variable η. Then, we differentiate all these equations with respect to t and obtain

$$\omega'(\eta)(\eta_t)^2 + \omega(\eta)\eta_{tt} - \Delta\eta_t + \frac{\partial}{\partial x_3}(\zeta'(\eta)\eta_t) = f_t \quad \text{in} \quad Q, \quad (2.94)$$
$$(\zeta'(\eta)\eta_t i_3 - \nabla\eta_t)\cdot\nu = u_t \quad \text{on} \quad \Sigma_u, \quad (2.95)$$
$$(\zeta'(\eta)\eta_t i_3 - \nabla\eta_t)\cdot\nu = \alpha\eta_t + (f_0)_t \quad \text{on} \quad \Sigma_\alpha, \quad (2.96)$$
$$\eta_t(x,0) = \eta_{t0}(x) \quad \text{in} \quad \Omega, \quad (2.97)$$

where $\omega'(\eta)$ and $\zeta'(\eta)$ represent the derivatives of ω and ζ with respect to η, i.e.,

$$\omega'(\eta) = -\frac{1}{\beta_\varepsilon^3((\beta_\varepsilon^*)^{-1}(\eta))}, \qquad \zeta'(\eta) = \frac{K'((\beta_\varepsilon^*)^{-1}(\eta))}{\beta_\varepsilon((\beta_\varepsilon^*)^{-1}(\eta))} \le \frac{M}{\beta_m}. \quad (2.98)$$

Eq. (2.97) makes sense since $\theta_0 \in H^2(\Omega)$. In fact, for each $\varepsilon > 0$ we have

$$\|\eta_t(0)\| = \|\beta_\varepsilon(\theta_0)(\theta_\varepsilon)_t(0)\| \le \beta_M(\varepsilon)\,\|(\theta_\varepsilon)_t(0)\| < \infty, \tag{2.99}$$

since from (2.6) and the assumptions (2.85) and (2.88) we know that

$$\|(\theta_\varepsilon)_t(0)\| \le \|\Delta\beta^*_\varepsilon(\theta_0)\| + M\left\|\frac{\partial\theta_0}{\partial x_3}\right\| + \|f(0)\| < \infty.$$

Then we multiply equation (2.94) by η_t and integrate it over $\Omega \times (0,t)$. We have

$$\int_0^t\int_\Omega\left(\omega'(\eta)\eta_\tau^3 + \frac{1}{2}\omega(\eta)\frac{\partial}{\partial\tau}(\eta_\tau^2) + |\nabla\eta_\tau|^2 - \zeta'(\eta)\eta_\tau\frac{\partial\eta_\tau}{\partial x_3}\right)dxd\tau$$
$$+\int_0^t\int_{\Gamma_\alpha}(\alpha\eta_\tau^2 + (f_0)_\tau)\eta_\tau d\sigma d\tau = \int_0^t\int_\Omega f_\tau\eta_\tau dxd\tau - \int_0^t\int_{\Gamma_u} u_\tau\eta_\tau d\sigma d\tau.$$

We integrate with respect to τ the second term on the left-hand side and obtain

$$\frac{1}{2}\int_\Omega\omega(\eta)\eta_t^2(t)dx + \int_0^t\|\eta_\tau(\tau)\|_V^2\,d\tau \le \frac{1}{2}\int_0^t\int_\Omega\frac{1}{\beta_\varepsilon^3((\beta_\varepsilon^*)^{-1}(\eta))}\eta_\tau^3dxd\tau$$
$$+\frac{1}{2}\int_\Omega\omega(\eta_0)\eta_t^2(0)dx + \int_0^t\int_\Omega\zeta'(\eta)\eta_\tau\frac{\partial\eta_\tau}{\partial x_3}dxd\tau + \int_0^t\int_\Omega f_\tau\eta_\tau dxd\tau$$
$$-\int_0^t\int_{\Gamma_u}u_\tau\eta_\tau d\sigma d\tau - \int_0^t\int_{\Gamma_\alpha}(f_0)_\tau\eta_\tau d\sigma d\tau.$$

Taking into account (2.98) we have

$$\frac{1}{2\beta_M(\varepsilon)}\int_\Omega\eta_t^2(t)dx + \int_0^t\|\eta_\tau(\tau)\|_V^2\,d\tau \le \frac{1}{2\beta_m}\int_0^t\int_\Omega|\eta_\tau|^3\,dxd\tau$$
$$+\frac{1}{2}\int_\Omega\omega(\eta_0)\eta_t^2(0)dx + \frac{M}{\beta_m}\int_0^t\|\eta_\tau(\tau)\|\,\|\eta_\tau(\tau)\|_V\,d\tau + \int_0^t\|f_\tau(\tau)\|\,\|\eta_\tau(\tau)\|\,d\tau$$
$$+c_{\Gamma_u}\int_0^t\|u_\tau(\tau)\|_{L^2(\Gamma_u)}\|\eta_\tau(\tau)\|_V\,d\tau + c_{\Gamma_\alpha}\int_0^t\|(f_0)_\tau(\tau)\|_{L^2(\Gamma_\alpha)}\|\eta_\tau(\tau)\|_V\,d\tau.$$

Therefore

$$\frac{1}{2\beta_M(\varepsilon)}\int_\Omega\eta_t^2(t)dx + \int_0^t\|\eta_\tau(\tau)\|_V^2\,d\tau \le \frac{1}{2\beta_m}\int_0^t\int_\Omega|\eta_\tau|^3\,dxd\tau$$
$$+\frac{1}{2}\left(\frac{3M^2}{\beta_m}+1\right)\int_0^t\|\eta_\tau(\tau)\|^2\,d\tau + \frac{1}{2}\int_0^t\|\eta_\tau(\tau)\|_V^2\,d\tau + \frac{c_1^\varepsilon(t)}{2},$$

where

$$c_1^\varepsilon(t) = \int_\Omega \omega(\eta_0)\eta_t^2(0)dx + \int_0^t \|f_\tau(\tau)\|^2 \, d\tau \tag{2.100}$$
$$+ 3c_{\Gamma_u}^2 \int_0^t \|u_\tau(\tau)\|_{L^2(\Gamma_u)} \, d\tau + 3c_{\Gamma_\alpha}^2 \int_0^t \|(f_0)_\tau(\tau)\|_{L^2(\Gamma_\alpha)} \, d\tau$$
$$\leq \frac{1}{\beta_m} \|\eta_t(0)\|^2 + c_1,$$

and

$$c_1 = \int_0^T \|f_\tau(\tau)\|^2 \, d\tau + 3c_{\Gamma_u}^2 \int_0^T \|u_\tau(\tau)\|_{L^2(\Gamma_u)} \, d\tau + 3c_{\Gamma_\alpha}^2 \int_0^T \|(f_0)_\tau(\tau)\|_{L^2(\Gamma_\alpha)} \, d\tau.$$

After some computations we obtain

$$\begin{aligned} &\frac{1}{\beta_M(\varepsilon)} \|\eta_t(t)\|^2 + \int_0^t \|\eta_\tau(\tau)\|_V^2 \, d\tau \\ &\leq c_0^\varepsilon + c_2 \int_0^t \|\eta_\tau(\tau)\|^2 \, d\tau + c_3 \int_Q |\eta_\tau|^3 \, dx d\tau, \end{aligned} \tag{2.101}$$

where

$$c_0^\varepsilon = \frac{\|\eta_t(0)\|^2 + \beta_m c_1}{\beta_m}, \; c_2 = \frac{3M^2}{\beta_m^2} + 1, \; c_3 = \frac{1}{\beta_m}. \tag{2.102}$$

We are going to estimate now the term $\int_\Omega |\eta_\tau|^3 \, dx$, using Hölder's inequality.

$$\int_\Omega |\eta_\tau|^3 \, dx = \int_\Omega |\eta_\tau|^{3/2} |\eta_\tau|^{3/2} \, dx \leq \left(\int_\Omega |\eta_\tau|^2 \, dx\right)^{3/4} \left(\int_\Omega |\eta_\tau|^6 \, dx\right)^{1/4}.$$

Further we can write via Sobolev's embedding theorem that

$$\int_\Omega |\eta_\tau|^3 \, dx \leq \|\eta_\tau(\tau)\|^{3/2} \|\eta_\tau(\tau)\|_{L^6(\Omega)}^{3/2} \leq C \|\eta_\tau(\tau)\|^{3/2} \|\eta_\tau(\tau)\|_{H^1(\Omega)}^{3/2}. \tag{2.103}$$

We mention that by C and $C(\varepsilon)$, we shall further denote some constants independent of and dependent on ε, respectively. Therefore we have

$$\begin{aligned} \int_Q |\eta_\tau|^3 \, dx d\tau &\leq C \int_0^t \|\eta_\tau(\tau)\|^{3/2} \|\eta_\tau(\tau)\|_V^{3/2} \, d\tau \\ &\leq C \int_0^t \|\eta_\tau(\tau)\|^6 \, d\tau + \frac{3}{4} \int_0^t \|\eta_\tau(\tau)\|_V^2 \, d\tau. \end{aligned} \tag{2.104}$$

Here, we used Young's inequality (see (2.2) in Appendix) with $p = 4$ and $q = \frac{4}{3}$.

By plugging (2.104) in (2.101) we obtain that

$$\frac{1}{\beta_M(\varepsilon)} \|\eta_t(t)\|^2 + \frac{1}{4} \int_0^t \|\eta_\tau(\tau)\|_V^2 \, d\tau \leq c_0^\varepsilon + C \int_0^t (\|\eta_\tau(\tau)\|^2 + \|\eta_\tau(\tau)\|^6) d\tau. \tag{2.105}$$

If we denote

$$\varphi(t) = \|\eta_t(t)\|^2 \geq 0 \tag{2.106}$$

the inequality (2.105) becomes

$$0 \leq \frac{1}{\beta_M(\varepsilon)}\varphi(t) \leq c_0^\varepsilon + C\int_0^t (\varphi(\tau) + \varphi^3(\tau))d\tau \leq c_0^\varepsilon + 2C\int_0^t \varphi^3(\tau)d\tau,$$

if we assume that $\varphi(t) \geq 1$. In the other case ($\varphi(t) < 1$) the proof is finished, $\eta_t(t)$ being bounded in $L^2(\Omega)$. Thus we obtain

$$0 \leq \varphi(t) \leq c_1^\varepsilon + c_2^\varepsilon \int_0^t \varphi^3(\tau)d\tau, \tag{2.107}$$

where $c_1^\varepsilon = c_0^\varepsilon \beta_M(\varepsilon)$, $c_2^\varepsilon = 2C\beta_M(\varepsilon)$.

We shall prove that this implies the boundedness of $\varphi(t)$. We define

$$\Phi(t) = c_2^\varepsilon \int_0^t \varphi^3(\tau)d\tau,\ \Phi(0) = 0.$$

Therefore, $\Phi'(t) = c_2^\varepsilon \varphi^3(t)$ and we obtain the differential inequality

$$\frac{d\Phi}{(c_1^\varepsilon + \Phi)^3} \leq c_2^\varepsilon dt,\ \Phi(0) = 0,$$

whose solution is

$$c_1^\varepsilon + \Phi(t) \leq \frac{c_1^\varepsilon}{\sqrt{1 - 2(c_1^\varepsilon)^2 c_2^\varepsilon t}} \text{ for } 0 \leq t \leq \frac{1}{2(c_1^\varepsilon)^2 c_2^\varepsilon}. \tag{2.108}$$

In fact we have obtained

$$0 \leq \|\eta_t(t)\|^2 \leq \frac{c_1^\varepsilon}{\sqrt{1 - 2(c_1^\varepsilon)^2 c_2^\varepsilon t}} \text{ for } 0 \leq t < T_0, \tag{2.109}$$

where

$$T_0 = \frac{1}{2(c_1^\varepsilon)^2 c_2^\varepsilon}. \tag{2.110}$$

If $T_0 \geq T$ then we get from (2.105) that

$$\eta_t = \frac{\partial \beta_\varepsilon^*(\theta_\varepsilon)}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V). \tag{2.111}$$

If $T_0 < T$ we have the inequality (2.109) for any $t \in [0, t_0]$, where

$$t_0 = T_0 - \delta$$

with $\delta > 0$ arbitrary and sufficiently small.

Next, we repeat the procedure for the intervals $[t_0, t_1], ..., [t_{n-1}, t_n]$ until the whole time interval $[0, T]$ is covered by $\bigcup_{j=1,...,n} [t_{j-1}, t_j]$, where

$$t_j \in (T_j - \delta, T_j),\ T_j - t_{j-1} = \frac{1}{2(c^{\varepsilon}_{t_{j-1}})^2 c^{\varepsilon}_2},\ c^{\varepsilon}_{t_j} = \beta_M(\varepsilon)\frac{\|\eta_t(t_{j-1})\|^2 + \beta_m c_1}{\beta_m}.$$

Here, a problem arises due to the fact that the norm $\|\eta_t(t_{j-1})\|^2$ may increase, determining a high decrease of the time step $T_j - t_{j-1}$ and making thus impossible to reach the final time T.

However we can prove that this is not the case, by using a previous estimate for $\eta_t(t)$, namely (2.74) from where we deduce that

$$\int_0^T \|\eta_t(t)\|^2\, dt \le \beta_M^2(\varepsilon) \int_0^T \left\| \frac{d\theta_\varepsilon}{dt}(t) \right\|^2 dt \le \beta_M^2(\varepsilon) C_4(\varepsilon) = C^{\varepsilon}_M,$$

which is independent of the time step (here $C_4(\varepsilon)$ is that in formula (2.74)). Therefore the function $t \to \|\eta_t(t)\|^2$ belongs to $L^2(0, T)$. We apply Luzin's theorem, which works for measurable functions on bounded domains and says that for each $\delta > 0$ there exists E_δ with $\text{meas}(E_\delta) \le \dfrac{\delta}{2}$ such that $\|\eta_t(t)\|^2 \le C^{\varepsilon}_M$ for any $t \in [0, T]\backslash E_\delta$. In particular, the point t can be found in an interval of measure δ, so, for example, $t \in (T - \delta, T)$.

Applying this result to the interval (t_{j-1}, T_j), we can find a point $t_j \in [T_j - \delta, T_j)$ such that

$$\|\eta_t(t_j)\|^2 \le C^{\varepsilon}_M$$

and therefore

$$\begin{aligned} T_{j+1} - t_j &= \frac{1}{2(c^{\varepsilon}_{t_j})^2 c^{\varepsilon}_2} = \frac{1}{2c^{\varepsilon}_2} \frac{1}{\beta_M^2(\varepsilon)} \left(\frac{\beta_m}{\|\eta_t(t_j)\|^2 + \beta_m c_1} \right)^2 \\ &\ge \frac{1}{4C\beta_M^3(\varepsilon)} \left(\frac{\beta_m}{C^{\varepsilon}_M + \beta_m c_1} \right)^2, \end{aligned}$$

which is independent of the time step. The procedure stops when $[0, T] \subset \bigcup_{j=1,...,n} [t_{j-1}, t_j]$ and resuming (2.105) we obtain once again (2.111), i.e.,

$$\frac{\partial \beta^*_\varepsilon(\theta_\varepsilon)}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V).$$

Finally, taking into account (2.111) we obtain (taking into account that $\theta_\varepsilon \in L^2(0, T; V)$ and β_ε, β'_ε are bounded) that

$$\beta_\varepsilon(\theta_\varepsilon)\frac{\partial \theta_\varepsilon}{\partial t} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V), \tag{2.112}$$

$$\frac{\partial \theta_\varepsilon}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V). \tag{2.113}$$

From (2.6) we deduce that

$$\|\Delta\beta_\varepsilon^*(\theta_\varepsilon(t))\| \leq \|f(t)\| + \left\|\frac{\partial \theta_\varepsilon}{\partial t}(t)\right\| + \left\|\frac{\partial K(\theta_\varepsilon)}{\partial t}(t)\right\|,$$

i.e.,

$$\|\Delta\beta_\varepsilon^*(\theta_\varepsilon(t))\| \leq C(\varepsilon),\ \forall t \in [0,T]. \tag{2.114}$$

This implies successively that

$$\beta_\varepsilon^*(\theta_\varepsilon) \in L^\infty(0,T;H^2(\Omega)),$$

$$\nabla\beta_\varepsilon^*(\theta_\varepsilon) = \beta_\varepsilon(\theta_\varepsilon)\nabla\theta_\varepsilon \in L^\infty(0,T;H^1(\Omega)).$$

Completely similar as done before (see the relationships beginning with (2.79)) we get that

$$\left\|\frac{\partial \theta_\varepsilon}{\partial x_j}(t)\right\|_{H^1(\Omega)}^2 \leq C(\varepsilon) + \frac{1}{2}\left\|\frac{\partial \theta_\varepsilon}{\partial x_j}(t)\right\|_{H^1(\Omega)}^2,\ \forall t \in [0,T],$$

so finally we obtain

$$\theta_\varepsilon \in L^\infty(0,T;H^2(\Omega))$$

as claimed. ■

Remark 2.9. However, we have to mention that some of the constants occurring in the Theorem 2.8 before depend on ε by the means of $\beta_M(\varepsilon)$ which is unbounded if $\varepsilon \to 0$ (see especially C_M^ε). Consequently, the estimates are true only for each $\varepsilon > 0$ apart, and they cannot be used in order to obtain a similar regularity for the solution to the original problem.

5.3 The original problem

Let us define the function $j : \mathbf{R} \to (-\infty, \infty]$ by

$$j(r) = \begin{cases} \displaystyle\int_0^r \beta^*(\xi)d\xi, & \text{if } r \leq \theta_s \\ +\infty, & \text{if } r > \theta_s, \end{cases} \tag{3.1}$$

where $j(\theta_s)$ should be understood as

$$j(\theta_s) = \lim_{r \nearrow \theta_s} \int_0^r \beta^*(\xi)d\xi. \tag{3.2}$$

It follows that j is a proper, convex, lower semicontinuous function and

$$\partial j(r) = \begin{cases} \beta^*(r), & r < \theta_s \\ [K_s^*, +\infty), & r = \theta_s \\ \emptyset, & r > \theta_s \end{cases} \tag{3.3}$$

and the proof is similar to that of Corollary 2.2 in Sect. 4.2. Indeed, we see that

$$j(r) = \int_0^r \beta^*(\xi)d\xi \geq \frac{\rho}{2}r^2, \ \forall r \leq \theta_s,$$

and $j(r) \leq j(\theta_s) \leq K_s^*\theta_s$, for $r \leq \theta_s$, so j is proper. Then $j''(r) = \beta(r) > 0$, for $r < \theta_s$, so j and convex. We have to prove that

$$j(r) - j(y) \leq \beta^*(r)(r-y), \ \forall y \in \mathbf{R} \text{ and } r \leq \theta_s.$$

The inequality is obvious if $r < \theta_s$ and $y < \theta_s$ and if $r = y = \theta_s$.
Let $r = \theta_s$ and $y < \theta_s$. Then we have

$$j(\theta_s) - j(y) = \int_y^{\theta_s} \beta^*(\xi)d\xi \leq K_s^*(\theta_s - y) \leq \beta^*(\theta_s)(\theta_s - y).$$

If $r = \theta_s$ and $y > \theta_s$, then $j(y) = +\infty$ and the inequality is verified since $-\infty < a$, with $a < 0$.
Let $r < \theta_s$ and $y = \theta_s$. We have

$$j(r) - j(\theta_s) = -\int_r^{\theta_s} \beta^*(\xi)d\xi \leq -\beta^*(r)(\theta_s - r) = \beta^*(r)(r - \theta_s).$$

Existence and properties of the original solution

We are going now to prove the existence of the solution to the original problem, by passing to limit in the approximating problem corresponding to the function β_ε^* given by (2.1). We emphasize that the results provided by using this approximation of β^* are sufficient in the passing to limit procedure. Therefore, in Theorem 2.3, letting ε tend to 0 we obtain the following existence result:

Theorem 3.1. *Let f, u, f_0 and θ_0 satisfy*

$$f \in L^2(0,T;V'), \ u \in L^2(0,T;L^2(\Gamma_u)), \ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)), \tag{3.4}$$

$$\theta_0 \in L^2(\Omega), \ \theta_0 \leq \theta_s, \ a.e. \ x \in \Omega. \tag{3.5}$$

Then, there exists a unique solution θ to the original problem (1.29)-(1.30) *with the following properties:*

$$\theta \in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V), \tag{3.6}$$

$$\beta^*(\theta) \in L^2(0,T;V), \ K(\theta) \in L^2(0,T;V),$$

$$j(\theta) \in L^1(Q).$$

Moreover, the solution satisfies the estimates

$$\begin{aligned}
&\int_{\Omega} j(\theta(x,t))dx + \int_0^t \left\| \frac{d\theta}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|\beta^*(\theta(\tau))\|_V^2 \, d\tau \\
&\leq \gamma_0(\alpha_m) \left(\int_{\Omega} j(\theta_0(x))dx + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \right. \\
&\left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 \, d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 \, d\tau \right),
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\|\theta(t)\|_{V'}^2 + \int_0^t \|\theta(\tau)\|^2 \, d\tau \leq \gamma_1(\alpha_m) \bigg(\|\theta_0\|_{V'}^2 + \int_0^T \|f(\tau)\|_{V'}^2 \, d\tau \\
+ \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 \, d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 \, d\tau \bigg).
\end{aligned} \tag{3.8}$$

Proof. Assume that (3.4) and (3.5) hold. Then the approximating problem (2.3)-(2.4) has a strong solution θ_ε, satisfying conclusions of Theorem 2.3. Since here we have imposed $\theta_0 \leq \theta_s$ we have by (2.42) that

$$j_\varepsilon(\theta_0) \leq K_s^* \theta_s,$$

so the right-hand side term in (2.26) turns out to be bounded by $\gamma_0(\alpha_m)S_0$, independently of ε.

Hence from (2.44) and (2.26) we deduce that $\{\theta_\varepsilon\}_{\varepsilon>0}$ lies in a bounded subset of $L^\infty(0,T;L^2(\Omega))$, $\left\{\frac{d\theta_\varepsilon}{dt}\right\}_{\varepsilon>0}$ is included in a bounded subset of $L^2(0,T;V')$ and $\{\beta_\varepsilon^*(\theta_\varepsilon)\}_{\varepsilon>0}$ is in a bounded subset of $L^2(0,T;V)$. Using (i) we get that $(\beta_\varepsilon^*)^{-1}$ is uniformly Lipschitz and therefore $\{\theta_\varepsilon\}_{\varepsilon>0}$ is in a bounded subset of $L^2(0,T;V)$, too.

From the boundedness of the sequences previously mentioned, we conclude that there exists a subsequence (denoted by θ_ε, too) such that

$$\theta_\varepsilon \longrightarrow \theta \text{ weakly in } L^2(0,T;V) \text{ and weak-star in } L^\infty(0,T;L^2(\Omega)),$$

$$\frac{d\theta_\varepsilon}{dt} \longrightarrow \frac{d\theta}{dt} \text{ weakly in } L^2(0,T;V').$$

Since $V = H^1(\Omega)$ is compactly embedded in $H = L^2(\Omega)$, by Lions-Aubin compactness theorem (see Theorem 3.12 in Appendix) we conclude that $\{\theta_\varepsilon\}_{\varepsilon>0}$ is compact in $L^2(0,T;L^2(\Omega))$. Selecting a subsequence we have that

$$\theta_\varepsilon \longrightarrow \theta \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0\ . \tag{3.9}$$

Since $\theta \to K(\theta)$ is continuous from $L^2(0,T;V)$ to $L^2(0,T;L^2(\Omega))$ it follows that

$$K(\theta_\varepsilon) \longrightarrow K(\theta) \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0.$$

From (2.26) we also obtain that

$$\beta^*_\varepsilon(\theta_\varepsilon) \longrightarrow \eta \text{ weakly in } L^2(0,T;V). \tag{3.10}$$

Since by the trace theorem it follows that the trace operator is linear and continuous, we can write

$$\|\beta^*_\varepsilon(\theta_\varepsilon)\|_{L^2(0,T;L^2(\Gamma))} \leq C\,\|\beta^*_\varepsilon(\theta_\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \leq \text{ constant}$$

and we deduce therefore that the trace

$$\beta^*_\varepsilon(\theta_\varepsilon) \longrightarrow \eta \text{ weakly in } L^2(0,T;L^2(\Gamma)).$$

We shall prove now that

$$\eta \in \beta^*(\theta) \text{ a.e. on } Q. \tag{3.11}$$

We note that

$$j_\varepsilon(z) \longrightarrow j(z), \text{ as } \varepsilon \to 0, \ \forall z \in \mathbf{R}. \tag{3.12}$$

This assertion is clear for $z < \theta_s$, where $j_\varepsilon(z) \equiv j(z)$ and j is continuous. For $z = \theta_s$ we have

$$j_\varepsilon(\theta_s) = \int_0^{\theta_s} \beta^*_\varepsilon(\xi)d\xi = \lim_{r \nearrow \theta_s} \int_0^r \beta^*_\varepsilon(\xi)d\xi = \lim_{r \nearrow \theta_s} \int_0^r \beta^*(\xi)d\xi = j(\theta_s) \leq K^*_s\theta_s.$$

If $z > \theta_s$ we have

$$j_\varepsilon(z) = \int_0^{\theta_s} \beta^*_\varepsilon(\xi)d\xi + \int_{\theta_s}^z \left[K^*_s + \frac{\xi - \theta_s}{\varepsilon}\right] d\xi = j(\theta_s) + K^*_s(z-\theta_s) + \frac{1}{2\varepsilon}(z-\theta_s)^2$$

and so

$$\lim_{\varepsilon \to 0} j_\varepsilon(z) = +\infty = j(z) \text{ for } z > \theta_s.$$

Now, we are going to show that

$$\int_Q j(\theta)dxdt \leq \liminf_{\varepsilon \to 0} \int_Q j_\varepsilon(\theta_\varepsilon)dxdt. \tag{3.13}$$

To this end, we choose first a point (x,t) such that $\theta(x,t) < \theta_s$. Due to the strongly continuity of θ_ε to θ, it follows that for ε small $\theta_\varepsilon(x,t) < \theta_s$. Then we have

$$j_\varepsilon(\theta_\varepsilon(x,t)) = j(\theta_\varepsilon(x,t)) \text{ in } \{(x,t);\ \theta(x,t) < \theta_s\}$$

and since j is continuous we deduce that

$$\lim_{\varepsilon \to 0} j_\varepsilon(\theta_\varepsilon) = \lim_{\varepsilon \to 0} j(\theta_\varepsilon) = j(\theta) \text{ in } \{(x,t);\ \theta(x,t) < \theta_s\},$$

hence (3.13) is satisfied.

If the point (x,t) is such that $\theta(x,t) \geq \theta_s$, then $j(\theta)$ can be either bounded or become infinity and the following situations may occur:

(1°) There exists a subsequence $\{\varepsilon_j\}$, $\varepsilon_j \to 0$, such that $\theta_{\varepsilon_j}(x,t) > \theta_s$ (in other words we mean that this happens for an infinity of terms of $\{\theta_\varepsilon\}$). Then

$$j_\varepsilon(\theta_{\varepsilon_j}) = +\infty = j(\theta).$$

(2°) There exists a subsequence $\{\varepsilon_j\}$, $\varepsilon_j \to 0$, such that $\theta_{\varepsilon_j}(x,t) \leq \theta_s$. The case splits in two subcases: $\theta_{\varepsilon_j}(x,t) < \theta_s$ for an infinity of terms and this comes back to the first situation discussed, or, $\theta_{\varepsilon_j}(x,t) = \theta_s$ for an infinity of terms and then we have again $j(\theta_{\varepsilon_j}) = j(\theta_s)$.

(3°) There exist an infinite number of terms for which $\theta_\varepsilon(x,t) < \theta_s$ and an infinite number of terms for which $\theta_\varepsilon(x,t) \geq \theta_s$ and then we find again the results of either the point (1°) or the point (2°).

To resume, in whatever case, we can select a subsequence (if necessary) such that

$$\liminf_{\varepsilon\to 0} j_\varepsilon(\theta_\varepsilon) \geq j(\theta).$$

Since $j_\varepsilon(\theta_\varepsilon) > 0$ we have by Fatou's lemma that

$$\liminf_{\varepsilon\to 0} \int_Q j_\varepsilon(\theta_\varepsilon)dxdt \geq \int_Q \liminf_{\varepsilon\to 0} j_\varepsilon(\theta_\varepsilon)dxdt \geq \int_Q j(\theta)dxdt.$$

From here and (2.26) we see also that $j(\theta) \in L^1(Q)$, which implies, in particular, that $\theta \leq \theta_s$ a.e. $(x,t) \in Q$.

Also, since

$$j_\varepsilon(\theta_\varepsilon) \leq j_\varepsilon(z) + \beta_\varepsilon^*(\theta_\varepsilon)(\theta_\varepsilon - z),\ \forall\ z \in \mathbf{R},$$

we can write the inequality (in particular for $z : \Omega \times (0,T) \to \mathbf{R}$)

$$\int_Q j_\varepsilon(\theta_\varepsilon)dxdt \leq \int_Q j_\varepsilon(z)dxdt + \int_Q \beta_\varepsilon^*(\theta_\varepsilon)(\theta_\varepsilon - z)dxdt,\ \forall\ z \in L^2(Q). \quad (3.14)$$

Assume $z \leq \theta_s$ a.e. on Q. Then $j_\varepsilon(z) \leq K_s^*\theta_s$ and using (3.12), we deduce by the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon\to 0} \int_Q j_\varepsilon(z)dxdt = \int_Q j(z)dxdt.$$

We recall that $\beta_\varepsilon^*(\theta_\varepsilon) \to \eta$ weakly in $L^2(0,T;V)$ and $\theta_\varepsilon \to \theta$ strongly in $L^2(0,T;L^2(\Omega))$ and passing to limit as $\varepsilon \to 0$ in (3.14) and taking into account (3.13) we obtain that

$$\int_Q j(\theta)dxdt \leq \int_Q j(z)dxdt + \int_Q \eta(\theta - z)dxdt,\ \forall z \in L^2(Q),\ z \leq \theta_s \text{ a.e.} \quad (3.15)$$

Now, we fix $(x_0, t_0) \in Q$, choose v arbitrary in $\mathbf{R}$ and define

$$z(x,t) := \begin{cases} \theta(x,t), & (x,t) \notin B_r(x_0,t_0) \\ v, & (x,t) \in B_r(x_0,t_0), \end{cases}$$

where $B_r(x_0, t_0)$ is the ball of centre (x_0, t_0) and radius $r > 0$. We denote $\overline{B}_r(x_0,t_0) = Q \backslash B_r(x_0,t_0))$. Then, (3.15) yields

$$\int_{B_r(x_0,t_0)} j(\theta)dxdt + \int_{\overline{B}_r(x_0,t_0)} j(\theta)dxdt \leq \int_{B_r(x_0,t_0)} j(z)dxdt + \int_{\overline{B}_r(x_0,t_0)} j(z)dxdt$$
$$+ \int_{B_r(x_0,t_0)} \eta(\theta - z)dxdt + \int_{\overline{B}_r(x_0,t_0)} \eta(\theta - z)dxdt.$$

Taking into account the choice of $z(x,t)$ we have

$$\int_{B_r(x_0,t_0)} j(\theta)dxdt + \int_{\overline{B}_r(x_0,t_0)} j(\theta)dxdt \leq \int_{B_r(x_0,t_0)} j(v)dxdt + \int_{\overline{B}_r(x_0,t_0)} j(\theta)dxdt$$
$$+ \int_{B_r(x_0,t_0)} \eta(\theta - v)dxdt + \int_{\overline{B}_r(x_0,t_0)} \eta(\theta - \theta)dxdt$$

wherefrom it remains

$$\int_{B_r(x_0,t_0)} j(\theta)dxdt \leq \int_{B_r(x_0,t_0)} j(v)dxdt + \int_{B_r(x_0,t_0)} \eta(\theta - v)dxdt, \ \forall v \in L^2(Q),\ v \leq \theta_s.$$

Assume that (x_0, t_0) is a Lebesgue point for j. The point z_0 is called *a Lebesgue point* for a Lebesgue measurable function f if

$$\lim_{r\to 0} \frac{1}{\mathrm{meas}(B_r(z_0))} \int_{B_r(z_0)} f(x)dx = f(z_0).$$

Dividing by $\mathrm{meas}(B_r(x_0,t_0))$ and letting $r \to 0$ we get

$$j(\theta(x_0,t_0)) \leq j(v) + \eta(x_0,t_0)\left(\theta(x_0,t_0) - v\right), \ \forall v \in \mathbf{R},\ v \leq \theta_s.$$

This implies that $\partial j(\theta) = \eta$ and by (3.3) it follows that $\eta \in \beta^*(\theta)$ a.e. on Q.

Finally we show that θ is the solution to the original problem. Since θ_ε is a solution to (2.3)-(2.4) we have

$$\int_Q \left(\frac{\partial \theta_\varepsilon}{\partial t}\phi + \nabla \beta_\varepsilon^*(\theta_\varepsilon) \cdot \nabla \phi - K(\theta_\varepsilon)\frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$= \int_Q f\phi dxdt - \int_{\Sigma_\alpha} (\alpha \beta_\varepsilon^*(\theta_\varepsilon) + f_0)\phi d\sigma dt - \int_{\Sigma_u} u\phi d\sigma dt, \ \forall \phi \in L^2(0,T;V).$$

Passing to the limit as $\varepsilon \to 0$ we obtain that

$$\int_Q \left(\frac{\partial \theta}{\partial t}\phi + \nabla\eta \cdot \nabla\phi - K(\theta)\frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$= \int_Q f\phi dxdt - \int_{\Sigma_\alpha} (\alpha\eta + f_0)\phi d\sigma dt - \int_{\Sigma_u} u\phi d\sigma dt,$$
$$\forall \phi \in L^2(0,T;V), \text{ where } \eta(x,t) \in \beta^*(\theta(x,t)) \text{ a.e. } (x,t) \in Q,$$

which proves that θ is a solution to (1.29)-(1.30).

Next, we shall have a look at each term in the sum

$$\int_\Omega j_\varepsilon(\theta_\varepsilon(x,t))dx + \int_0^t \left\| \frac{d\theta_\varepsilon}{d\tau}(\tau) \right\|_{V'}^2 d\tau + \int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V^2 d\tau.$$

For the first term, we have shown (3.13). Next, the norm is convex (see Example 5.10) and continuous, so that all the sum is weakly l.s.c. and by passing to limit in (2.26) we obtain (3.7) as claimed.

Now, we consider two solutions to (1.29)-(1.30) θ and $\overline{\theta}$, corresponding to θ_0, f, f_0 and u, on the one hand and $\overline{\theta}_0$, $\overline{f}, \overline{f_0}$ and $\overline{u}$, on the other hand. We multiply the equation

$$\frac{d\theta}{dt} - \frac{d\overline{\theta}}{dt} + A\theta - A\overline{\theta} = f - \overline{f} + f_\Gamma - \overline{f_\Gamma} + Bu - B\overline{u}$$

by $\theta - \overline{\theta}$ scalarly in V' and integrate over $(0,t)$. Performing some standard computations we obtain that

$$\begin{aligned} &\left\|\theta(t) - \overline{\theta}(t)\right\|_{V'}^2 + \int_0^t \left\|\theta(\tau) - \overline{\theta}(\tau)\right\|^2 d\tau \\ &\leq \gamma_1(\alpha_m) \left(\left\|\theta_0 - \overline{\theta}_0\right\|_{V'}^2 + \int_0^T \left\|f(\tau) - \overline{f}(\tau)\right\|_{V'}^2 d\tau \right. \\ &\left. + \int_0^T \|u(\tau) - \overline{u}(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \left\|f_0(\tau) - \overline{f_0}(\tau)\right\|_{L^2(\Gamma_\alpha)}^2 d\tau \right). \end{aligned} \tag{3.16}$$

From here we immediately can derive the uniqueness of the solution. Also we obtain (3.8) if the data corresponding to the solution $\overline{\theta}$ are taken equal to 0.

However, we would like to include here, for a later use, a related result, i.e.,

$$\theta_\varepsilon(t) \longrightarrow \theta(t) \text{ strongly in } V' \text{ for each } t \in [0,T]. \tag{3.17}$$

This is obtained by the Ascoli-Arzelà theorem (see Theorem 3.14 in Appendix). We consider the family $\mathcal{M} = \{\theta_\varepsilon\}_\varepsilon \subset C([0,T];V')$ and we recall that

$$\|\theta_\varepsilon(t)\|_{V'} \leq \text{ constant},$$

(see (2.27)). Next, this family is equi-uniformly continuous. Indeed, let $\varepsilon > 0$ and consider that $\delta(\varepsilon)$ exists such that $|t-s| \leq \delta(\varepsilon)$, for $0 \leq s < t \leq T$. By Theorems 3.7 and 3.6 in Appendix we have

$$\|\theta_\varepsilon(t) - \theta_\varepsilon(s)\|_{V'} = \left\| \int_s^t \frac{d\theta_\varepsilon}{dt}(\tau)d\tau \right\|_{V'} \le \sqrt{t-s} \left\| \frac{d\theta_\varepsilon}{dt}(\tau)d\tau \right\|_{V'} \le \varepsilon,$$
$$\text{if } \delta(\varepsilon) \le \frac{\varepsilon}{\gamma_0(\alpha_m)S_0}, \forall \theta_\varepsilon \in \mathcal{M},$$

where $\gamma_0(\alpha_m)S_0$ is the constant right-hand side in (2.26) (see also (2.43)).

Finally, by (2.44) we have that $\|\theta_\varepsilon(t)\| \le C_S$, $\forall t \in [0,T]$ and since $L^2(\Omega)$ is compact in V' it follows that the sequence $\{\theta_\varepsilon(t)\}$ is compact in V', for each $t \in [0,T]$. Hence the set $\mathcal{M}$ is compact in $C([0,T];V')$, i.e., $\theta_\varepsilon(t) \to \theta(t)$ strongly in V', uniformly on $[0,T]$. ∎

As noticed earlier, since $j(\theta) \in L^1(Q)$, it follows by (3.1) that $\theta(x,t) \le \theta_s$ a.e. $(x,t) \in Q$. However, we shall give a direct proof of this property, via an auxiliary lemma which has also an intrinsic interest.

We set

$$Q^- = \{(x,t) \in Q;\, \theta(x,t) \le \theta_s\},\ Q^s = \{(x,t) \in Q;\, \theta(x,t) > \theta_s\}, \quad (3.18)$$
$$Q_\varepsilon^- = \{(x,t) \in Q;\, \theta_\varepsilon(x,t) \le \theta_s\}, Q_\varepsilon^+ = \{(x,t) \in Q;\, \theta_\varepsilon(x,t) > \theta_s\} \quad (3.19)$$

and denote by $\chi^+(x,t)$ and $\chi_\varepsilon^+(x,t)$ the characteristic functions of Q^s and Q_ε^+, respectively.

Lemma 3.2. *If Q^s and Q_ε^+ are defined by* (3.18) *and* (3.19) *we have the following inequality*

$$\liminf_{\varepsilon \to 0} \chi_\varepsilon^+(x,t) \ge \chi^+(x,t) \text{ a.e. on } Q. \quad (3.20)$$

Proof. We recall that $\theta_\varepsilon \to \theta$ strongly in $L^2(Q)$, which is the essential argument in this proof. It follows that $\theta_\varepsilon(x,t) \to \theta(x,t)$ a.e. $(x,t) \in Q$. We shall consider only those points $(x,t) \in Q$ where $\{\theta_\varepsilon\}_\varepsilon$ converges, because the set where the sequence does not converge is of zero measure and it will be let apart.

Let $(x,t) \in Q^-$ be a fixed point. Hence $\chi^+(x,t) = 0$ and the inequality is proved, since $\chi_\varepsilon^+(x,t)$ can be only either 1 or 0.

Let now $(x,t) \in Q^s$. Then $\theta(x,t) > \theta_s$ and $\chi^+(x,t) = 1$. Denote by $\delta > 0$ the difference $\theta(x,t) - \theta_s$. Then, there exists ε_δ that depends on δ such that

$$|\theta_\varepsilon(x,t) - \theta(x,t)| < \frac{\delta}{2},\ \forall \varepsilon \le \varepsilon_\delta.$$

This implies

$$\theta(x,t) - \frac{\delta}{2} < \theta_\varepsilon(x,t) < \theta(x,t) + \frac{\delta}{2}$$

so

$$\theta_\varepsilon(x,t) > \theta(x,t) - \frac{\delta}{2} = \theta_s + \frac{\delta}{2} > \theta_s.$$

This turns out in concluding that for $\varepsilon \le \varepsilon_\delta$ we have $\chi_\varepsilon^+(x,t) = 1$ that comes back to the fulfillment of (3.20). ∎

Corollary 3.3. *Let* f, u, f_0 *and* θ_0 *satisfy* (3.4)-(3.5). *Then, the solution* θ *to* (1.29)-(1.30) *has the property* $\theta(x,t) \leq \theta_s$, *a.e.* $(x,t) \in Q$.

Proof. By (2.26) we have that

$$\int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|^2 \, d\tau \leq \int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V^2 \, d\tau \leq \gamma_0(\alpha_m) S_0.$$

This can be written also as

$$\int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|^2 \, d\tau = \int_{Q_\varepsilon^-} (\beta_\varepsilon^*(\theta_\varepsilon))^2 dxdt + \int_{Q_\varepsilon^+} (\beta_\varepsilon^*(\theta_\varepsilon))^2 dxdt \leq c_0$$

from where, using that $\beta_\varepsilon^*(\theta_\varepsilon) \leq K_s^*$ on Q_ε^-, we get

$$\int_Q \chi_\varepsilon^+(x,t) \left(K_s^* + \frac{\theta_\varepsilon - \theta_s}{\varepsilon} \right)^2 dxdt \leq c_1.$$

This implies after some computations that

$$\int_Q \chi_\varepsilon^+(x,t)(\theta_\varepsilon - \theta_s)^2 dxdt \leq c_2 \varepsilon^2 \tag{3.21}$$

with c_0, c_1 and c_2 some constants.

By Lemma 3.2 we have that

$$\liminf_{\varepsilon\to 0} \left(\chi_\varepsilon^+(x,t)(\theta_\varepsilon - \theta_s)^2\right) = \left(\liminf_{\varepsilon\to 0} \chi_\varepsilon^+(x,t)\right) \lim_{\varepsilon\to 0} (\theta_\varepsilon - \theta_s)^2 \geq \chi^+(x,t)(\theta - \theta_s)^2,$$
$$\text{a.e. } (x,t) \in Q.$$

Applying Fatou's lemma we get from (3.21) that

$$\int_Q \chi^+(x,t)(\theta - \theta_s)^2 dxdt \leq \liminf_{\varepsilon\to 0} \int_Q \chi_\varepsilon^+(x,t)(\theta_\varepsilon - \theta_s)^2 dxdt = 0.$$

This yields that $\chi^+(x,t) = 0$, meaning that $\theta(x,t) \leq \theta_s$, a.e. $(x,t) \in Q$. ∎

Comparison results for the original solution

If we refer to the saturated-unsaturated infiltration model, we have to prove the fact that its solution is situated in the physical domain for moisture. To come to this end we provide a result concerning the comparison of the solution with two known smooth time dependent functions $\theta_m \in C^1[0,T]$ and $\theta_M \in C^1[0,T]$ which satisfy the conditions specified for the comparison results for the approximating solution.

Proposition 3.4. *Assume*

$$f \in L^\infty(Q), \; u \in L^\infty(\Sigma_u), \; f_0 \in L^\infty(\Sigma_\alpha), \; \theta_0 \in L^2(\Omega),$$

$$\theta_m(t) < \theta_s \le \theta_M(t),\ \forall t \in [0,T],\ with\ \theta_M(0) = \theta_s,$$
$$\theta_m(t) \le \theta_0(x) \le \theta_M(0)\ a.e.\ in\ \Omega,$$
$$\theta'_m(t) \le f(x,t) \le \theta'_M(t)\ a.e.\ in\ Q,$$
$$K(\theta_m(t)) \le -u(x,t) \le K(\theta_M(t))\ a.e.\ on\ \Sigma_u,$$
$$K_s - \alpha K_s^* \le f_0(x,t) \le K(\theta_m(t)) - \alpha\beta^*(\theta_m(t))\ a.e.\ on\ \Sigma_\alpha.$$

Then

$$\theta_m(t) \le \theta(x,t) \le \theta_s\ a.e.\ in\ \Omega,\ for\ each\ t \in [0,T]. \tag{3.22}$$

In particular, if $\theta_m(t) = 0$ *we have*

$$0 \le \theta(x,t) \le \theta_s\ a.e.\ in\ \Omega,\ for\ each\ t \in [0,T]. \tag{3.23}$$

Proof. The proof follows immediately from Theorem 3.1 and Corollary 2.5. Here is the argument. By the hypotheses of Theorem 3.1 we get an approximating solution θ_ε which tends strongly in $L^2(Q)$ to θ, the unique solution to the original problem. Next we have only to apply Corollary 2.5 and pass to limit in (2.59). Both inequalities are preserved by passing to limit (strongly) and we get (3.22), as claimed.

We notice that the smaller θ_m, the larger the interval of boundedness for θ_ε and θ. Finally, if $\theta_m(t) = 0$, we have (3.23). ∎

Remark 3.5. This last result ends the proof of the existence and uniqueness of the solution to the strongly nonlinear saturated-unsaturated model with a weakly nonlinear hydraulic conductivity. As in the quasi-unsaturated model, we obtained that under realistic assumptions, the solution is placed within the accepted physical domain $[0, \theta_s]$, confirming the fact that the extensions of the hydraulic functions to the left of 0 do not introduce inappropriate solutions (see also Comment A1 in Chap. 4).

It is obvious that if, in Corollary 2.5 and Proposition 3.4 we choose both functions $\theta_m(t)$ and $\theta_M(t)$ lesser than θ_s, the criterion of comparison may be applied only for solutions that remain under the saturation value θ_s all the time. That is why, in order to study the saturated-unsaturated flow the choice of $\theta_M(t) \ge \theta_s$ is essential.

Additional regularity of the original solution

To study some stronger regularity of the original solution we have to resort to the appropriate results proved for the approximating solution. We mentioned there that since the a priori estimates depend on ε, they cannot be used as a basis in a passing to limit procedure. However, under a particular assumption we may deduce a further regularity for θ too.

Let δ be a fixed positive number.

Theorem 3.6. *Let* (2.65)-(2.68) *and assume that*

$$\text{there exists } \delta > 0 \text{ such that } \operatorname*{ess\,sup}_{x\in\Omega} \theta_0 \leq \theta_s - \delta. \tag{3.24}$$

Then, the solution θ *to problem* (1.29)-(1.30) *satisfies in addition*

$$\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{3.25}$$
$$\beta^*(\theta) \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)). \tag{3.26}$$

Moreover, if $N = 1$, *then* θ *is continuous on* $\overline{Q}$.

Proof. Since $\theta_0 \in H^1(\Omega)$, $\operatorname*{ess\,sup}_{x\in\Omega} \theta_0 < \theta_s$, it follows that we may write

$$\theta_0 \leq \operatorname*{ess\,sup}_{x\in\Omega} \theta_0 \leq \theta_s - \delta < \theta_s.$$

Then, $\beta^*_\varepsilon(\theta_0) = \beta^*(\theta_0) \in H^1(\Omega)$, for any $\varepsilon < \delta$ (see (2.60)). By the hypotheses (2.65)-(2.67), Theorem 3.1 and Corollary 3.3 we obtain that problem (1.29)-(1.30) has a unique solution

$$\theta \in L^2(0,T;V) \cap W^{1,2}(0,T;V'),\ \beta^*(\theta) \in L^2(0,T;V),\ \theta \leq \theta_s \text{ a.e. in } Q.$$

At the same time we get by Theorem 2.8 that the approximating solution to (2.3)-(2.4) satisfies

$$\theta_\varepsilon \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)),$$
$$\beta^*_\varepsilon(\theta_\varepsilon) \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)).$$

But due to the hypothesis $\operatorname*{ess\,sup}_{x\in\Omega} \theta_0 \leq \theta_s - \delta < \theta_s$ we notice that the constant $|C_0(\varepsilon)|$ given by (2.72) becomes independent on ε, since we may replace $\beta^*_\varepsilon(\theta_0)$ by $\beta^*(\theta_0) \in H^1(\Omega)$ for all $\varepsilon < \delta$. Also S_ε is transformed into S_0 (see (2.43)).

In their turn, the right-hand constants in (2.73)-(2.76) and (2.84) do no longer depend on ε and we obtain the essential result

$$\|\theta_\varepsilon\|_{W^{1,2}(0,T;L^2(\Omega))} + \|\theta_\varepsilon\|_{L^\infty(0,T;V)} + \|\theta_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq \text{ constant}, \tag{3.27}$$

independently on ε. From here we may conclude that

$$\begin{aligned} \theta_\varepsilon &\longrightarrow \theta && \text{weakly in } L^2(0,T;H^2(\Omega)) \text{ and weak-star in } L^\infty(0,T;V), \\ \frac{d\theta_\varepsilon}{dt} &\longrightarrow \frac{d\theta}{dt} && \text{weakly in } L^2(0,T;L^2(\Omega)), \\ \beta^*_\varepsilon(\theta_\varepsilon) &\longrightarrow \beta^*(\theta) && \text{weakly in } L^2(0,T;H^2(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)) \\ & && \text{and weak-star in } L^\infty(0,T;V). \end{aligned}$$

Therefore, by the trace theorem, we still obtain that

$$\beta^*_\varepsilon(\theta_\varepsilon)|_{\Sigma_\alpha} \longrightarrow \beta^*(\theta)|_{\Sigma_\alpha} \text{ weakly in } L^2(0,T;H^{3/2}(\Gamma_\alpha)).$$

Finally, by Lions-Aubin theorem, we also get that

$$\theta_\varepsilon \longrightarrow \theta \text{ strongly in } L^2(0,T;V). \tag{3.28}$$

Hence we have proved that the solution to problem (1.29)-(1.30) belongs to the spaces indicated in (3.25).

Let $N = 1$.We apply Theorem 3.19 in Appendix for $p = 2$ and get that both θ_ε and θ belong to $W_2^{2,1}(Q)$ which is compact in $C(\overline{Q})$. Therefore, both functions θ_ε and θ are continuous on $\overline{Q}$ and the convergence is uniform, i.e.,

$$\theta_\varepsilon(x,t) \longrightarrow \theta(x,t) \text{ uniformly on } \overline{Q}. \ \blacksquare \tag{3.29}$$

5.4 The weak solution in the pressure form

In this section we introduce the definition of the weak solution to the model in the pressure form and prove its existence, on the basis of the results obtained for the solution to the model in the diffusive form.

The mathematical model in the pressure form is derived from Richards' equation (see (1.1), Sect. 2.1) with initial data and boundary conditions. Of course, we choose here the same boundary conditions as for the diffusive model previously studied.

We recall the notations and definitions of C^*, C, k and K^* introduced in Sect. 2.2 for the strongly nonlinear case with a weakly nonlinear hydraulic conductivity, i.e., Model 1.2 and consider, with no loss of generality, that C^* and k are differentiable functions on $\mathbf{R}$, strictly monotonically increasing for $h \in [h_r, 0]$, C is continuous on $\mathbf{R}$ and

$$\lim_{h \nearrow 0} (C^*)'(h) = C(0) = C_0 = 0. \tag{4.1}$$

Moreover, as functions of h, they are bounded, and generally $C^* : \mathbf{R} \to [0, \theta_s]$ and $k : \mathbf{R} \to [0, K_s]$. The model in the pressure form reads

$$\frac{\partial C^*(h)}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} = f \quad \text{in} \quad Q = \Omega \times (0,T), \tag{4.2}$$

$$h(x,0) = h_0(x) \quad \text{in} \quad \Omega, \tag{4.3}$$

$$q \cdot \nu = u(x,t) \quad \text{on} \quad \Sigma_u = \Gamma_u \times (0,T), \tag{4.4}$$

$$q \cdot \nu = \alpha(x) K^*(h) + f_0(x,t) \quad \text{on} \quad \Sigma_\alpha = \Gamma_\alpha \times (0,T). \tag{4.5}$$

Here, by q we denoted the flux defined by

$$q(x,t) = k(h) i_3 - \nabla K^*(h), \tag{4.6}$$

where ν is the outward normal to Γ and i_3 is the unit vector along Ox_3.

Existence of the weak solution in the pressure form

Let V be the space $H^1(\Omega)$ endowed with the usual Hilbertian norm. Let

$$h_0 \in L^2(\Omega),\ f \in L^2(0,T;V'),\ u \in L^2(0,T;L^2(\Gamma_u)),\ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)).$$

Definition 4.1. The function $h \in L^2(0,T;L^2(\Omega))$ is said to be a *weak solution* to problem (4.2)-(4.5) if $K^*(h) \in L^2(0,T;V)$, and

$$\begin{aligned}
&\int_Q \Big(-C^*(h)\phi_t(x,t)+\nabla K^*(h)\cdot\nabla\phi(x,t)-k(h)\frac{\partial\phi}{\partial x_3}(x,t)\Big)dxdt \\
&=\int_\Omega \phi(x,0)C^*(h_0(x))dx-\int_{\Sigma_\alpha}(\alpha(x)K^*(h)+f_0(x,t))\phi(x,t)d\sigma dt \qquad (4.7)\\
&-\int_{\Sigma_u} u(x,t)\phi(x,t)d\sigma dt+\int_Q f(x,t)\phi(x,t)dxdt,
\end{aligned}$$

for all $\phi \in L^2(0,T;V)$ with $\phi_t \in L^2(0,T;L^2(\Omega))$ and $\phi(x,T) \equiv 0$.

We specify, once again, that the notation $\int_Q f(x,t)\phi(x,t)dxdt$, where $f \in L^2(0,T;V')$ and $\phi \in L^2(0,T;V)$ means in fact $\int_0^T \langle f(t),\phi(t)\rangle_{V',V}\,dt$.

It is obvious (and we have already specified this in Sect. 2.7) that all theorems proved up to now apply for the dimensional form of the diffusion equations, as well as for the dimensionless one. In the next proof we assume that we work with the dimensional form, so that θ is the true dimensional solution. We do this in order to get directly the dimensional h, because in Sect. 2.7 we did not define a dimensionless transform for it.

Theorem 4.2. *Assume*

$$h_0 \in L^2(\Omega),\ f \in L^2(0,T;V'),\ u \in L^2(0,T;L^2(\Gamma_u)),\ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)).$$

Then, problem (4.2)-(4.5) *has a weak solution* $h \in L^2(0,T;V)$, *with* $C(h)\dfrac{dh}{dt} \in L^2(0,T;V')$.

Proof. Let $h_0 \in L^2(\Omega)$. We set $\theta_0 = C^*(h_0)$ and we note that since C^* is continuous and bounded (it belongs to $[0,\theta_s]$) we have that $\theta_0 \in L^2(\Omega)$ and $\theta_0 \le \theta_s$ a.e. $x \in \Omega$. Taking also into account the other hypotheses made upon f, u and f_0, we can apply Theorem 3.1 and obtain that the problem in the diffusive form (1.29)-(1.30) has a unique solution θ (which is in fact the generalized solution to (1.1)-(1.4)), such that $\theta \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V)$, $\dfrac{d\theta}{dt} \in L^2(0,T;V')$ and $\beta^*(\theta) \in L^2(0,T;V)$. Now, we define

$$h(x,t) := \begin{cases} (C^*)^{-1}(\theta(x,t)), & \text{if } \theta < \theta_s \\ (C^*)^{-1}(\theta_s) = [0,+\infty), & \text{if } \theta = \theta_s. \end{cases} \qquad (4.8)$$

The equality written on the second line in (4.8) means that h takes values in $[0,+\infty)$.

We shall show that the function defined in this way is a weak solution to (4.2)-(4.5). We apply K^* to (4.8) and obtain

$$K^*(h) = \begin{cases} K^*((C^*)^{-1}(\theta)), & \text{for } h < 0, \\ K^*([0,+\infty)) \in [K_s^* + \infty), & \text{for } h \geq 0, \end{cases} \tag{4.9}$$

and we note that, in fact, $K^*(h) = K^*((C^*)^{-1}(\theta)) = \beta^*(\theta) \in L^2(0,T;V)$.

From (4.8) we get $\theta = C^*(h)$ and since θ is a generalized solution to the diffusive form, it satisfies, for instance, (1.21). We have

$$\begin{aligned} &\int_Q \left(\frac{\partial C^*(h)}{\partial t}\phi + \nabla\eta \cdot \nabla\phi - K(C^*(h))\frac{\partial \phi}{\partial x_3} \right) dxdt \\ &= \int_Q f\phi dxdt - \int_{\Sigma_\alpha} (\alpha\eta + f_0)\phi d\sigma dt - \int_{\Sigma_u} u\phi d\sigma dt, \\ &\forall \phi \in L^2(0,T;V) \text{ and } \eta \in \beta^*(\theta). \end{aligned}$$

Here, $K(C^*(h)) = K(\theta) \in L^2(0,T;V)$. Since ϕ is arbitrary, we can apply the previous equality for those $\phi \in L^2(0,T;V)$, such that $\dfrac{d\phi}{dt} \in L^2(0,T;L^2(\Omega))$ and $\phi(x,T) = 0$. After integrating the first term on the left side with respect to t and replacing (4.9) we deduce that

$$\begin{aligned} &\int_Q \left(-C^*(h)\phi_t dxdt + \nabla K^*(h) \cdot \nabla\phi - k(h)\frac{\partial \phi}{\partial x_3} \right) dxdt \\ &= \int_\Omega \phi(x,0)C^*(h_0)dx - \int_{\Sigma_\alpha} (\alpha\eta + f_0)\phi d\sigma dt - \int_{\Sigma_\alpha} u\phi d\sigma dt + \int_Q f\phi dxdt, \\ &\forall \phi \in L^2(0,T;V),\ \frac{d\phi}{dt} \in L^2(0,T;L^2(\Omega)),\ \phi(x,T) = 0, \end{aligned}$$

meaning that h previously defined satisfies (4.7).

By (2.2) in Sect. 2, we remark that the dimensional $K^*((C^*)^{-1}(\theta))$ satisfies

$$\left(K^*(h) - K^*(\overline{h})\right)(h - \overline{h}) \geq K_r(h - \overline{h})^2, \quad \text{for any } h, \overline{h} \in [-h_r, +\infty).$$

From here we get that the inverse of K^* is Lipschitz, i.e., if we denote $\eta = K^*(h)$ it follows

$$\left|(K^*)^{-1}(\eta) - (K^*)^{-1}(\overline{\eta})\right| \leq \frac{1}{K_r}\left|\eta - \overline{\eta}\right|,$$

expressing the fact that $h = (C^*)^{-1}(\theta) \in L^2(0,T;V)$. From the equation (4.2) we get $C(h)\dfrac{dh}{dt} \in L^2(0,T;V')$. ∎

Remark 4.3. By the result above, we have proved that the weak solution in pressure exists on Ω, and it is smooth on the set $\{(x,t);\ h(x,t) \geq h_r\}$, whatever small h_r is such that $K_r = k(h_r) > 0$. This condition is essential in order to obtain that $h \in L^2(0,T;V)$ and as we shall show further, under standard conditions the pressure can not get under the value h_r. We also notice that, at this moment, we cannot say anything else about the time derivative $\frac{dh}{dt}$. A further regularity of h will be proved below.

Concerning the uniqueness of the solution we remark that the solution h is unique in the unsaturated domain, where $\theta(x,t) < \theta_s$ and this is due to the uniqueness of θ and to the strictly monotonicity of the function $\theta \to h = (C^*)^{-1}(\theta)$. However we cannot say the same thing about the solution h in all Q, where a mixture of saturated and unsaturated parts can evolve. More information that will allow the uniqueness proof will be obtained along with the study of the free boundary existence. As a matter of fact the solution uniqueness will be discussed separately for the saturated and unsaturated domains, after we prove that a clear separation in such domains is possible.

Let us now consider that $h_M \in C^1[0,T]$ is a strictly monotonically increasing function, and denote $\theta_M(t) := C^*(h_M(t))$. It is obvious that $\theta_M \in C^1[0,T]$, since C is continuous as function of h and $\theta_M'(t) = C(h_M(t))h_M'(t)$.

Corollary 4.4. *Assume the hypotheses of Theorem* 4.2 *and let*

$$\begin{aligned} h_r \le 0 \le h_M(t),\ \forall t \in [0,T],&\ \ \textit{with } h_M(0) > 0,\\ h_r \le h_0(x) \le h_M(0)&\ \ \textit{a.e. in } \Omega,\\ 0 \le f(x,t) \le C(h_M(t))h_M'(t)&\ \ \textit{a.e. in } Q,\\ K_r \le -u(x,t) \le k(h_M(t))&\ \ \textit{a.e. on } \Sigma_u,\\ K_s - \alpha K_s^* \le f_0(x,t) \le K_r - \alpha\beta^*(\theta_r)&\ \ \textit{a.e. on } \Sigma_\alpha. \end{aligned}$$

Then

$$h_r \le h(x,t) \ \textit{a.e. in } \Omega,\ \textit{for each } t \in [0,T]. \tag{4.10}$$

Proof. With the hypotheses and notations above we obtain by Theorem 4.2 a weak solution h. Moreover, we notice that the hypotheses turn out into the hypotheses of Proposition 3.4, that implies that the corresponding solution $\theta \in [\theta_r, \theta_s]$ a.e. $(x,t) \in Q$. Now we use again the strict monotonicity of the function $\theta \to (C^*)^{-1}(\theta)$ and get that $h \ge h_r$ a.e. $(x,t) \in Q$. Here, the solution $h \in L^2(0,T;V)$, as we have seen before. ∎

Corollary 4.5. *Let*

$$f \in W^{1,2}(0,T;L^2(\Omega)), \tag{4.11}$$

$$u \in W^{1,2}(0,T;L^2(\Gamma_u)) \cap L^2(0,T;H^1(\Gamma_u)), \tag{4.12}$$

$$f_0 \in W^{1,2}(0,T;L^2(\Gamma_\alpha)) \cap L^2(0,T;H^1(\Gamma_\alpha)), \tag{4.13}$$

$$h_0 \in H^1(\Omega),\ \textit{there exists } \delta > 0 \textit{ such that } \operatorname*{ess\,sup}_{x\in\Omega} h_0 \le -\delta \tag{4.14}$$

and assume the hypotheses of Corollary 4.4. *Then, the weak solution to* (4.2)-(4.5) *satisfies in addition*

$$h \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \quad (4.15)$$
$$K^*(h) \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)). \quad (4.16)$$

Proof. The proof is obvious, by applying Theorems 4.2 and 3.6 and Corollary 4.4, by which it follows that $h \geq h_r$ a.e. in Q. We have

$$K^*(h) = \beta^*(\theta) \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)),$$

with $(K^*)'(h) = k(h) \in [K_r, K_s]$. Then, if $K^*(h) \in W^{1,2}(0,T;L^2(\Omega))$ we have that

$$\frac{dh}{dt} = \frac{1}{k(h)} \frac{dK^*(h)}{dt} \in L^2(Q).$$

Then, since $K^*(h) = \beta^*(\theta) \in L^\infty(0,T;V)$, by the Lipschitz property of the inverse of K^* we deduce that $h \in L^\infty(0,T;V)$.

Finally, $K^*(h) = \beta^*(\theta) \in L^2(0,T;H^2(\Omega))$ with the boundary conditions

$$\nabla K^*(h) \cdot \nu|_{\Sigma_u} = (k(h)i_3 \cdot \nu - u)|_{\Sigma_u} \in L^2(0,T;H^{1/2}(\Gamma_u)),$$
$$(\nabla K^*(h) \cdot \nu + \alpha K^*(h))|_{\Sigma_\alpha} = (k(h)i_3 \cdot \nu - f_0)|_{\Sigma_\alpha} \in L^2(0,T;H^{1/2}(\Gamma_\alpha)),$$

provided by the hypotheses and the fact that $k(h) \in L^2(0,T;H^1(\Omega))$. Then, the computations that must be done to prove that all these imply that $h \in L^2(0,T;H^2(\Omega))$ follow exactly a procedure similar to that developed in Theorem 3.6 for θ_ε, beginning with the relation (2.77), in which $\beta^*_\varepsilon(\theta_\varepsilon)$ is replaced by $K^*(h)$. ∎

Problem with two well separated flow domains

The main purpose of this part is to study the mathematical aspects related to the formation and advance of the free boundary between the saturated and unsaturated domains. We will develop this analysis in the case when the soil saturation begins from the soil surface, advances to the bottom of the domain, determining the formation of two well delimited regions, one saturated, above, and the other unsaturated, below, situation presented in Fig. 5.4.

We specified that under certain conditions the simultaneous saturated-unsaturated flow can evolve such that the flow domain be separated into two well delimited parts, one saturated and the other unsaturated, separated by a free surface. The mathematical model of such a physical situation is composed of a set of equations in a domain whose boundary is free, namely unknown. To account for the lack of information about the equation of the free surface, some extra-conditions will be added at this interface. They couple the free boundary equation with the solution itself and the problem focuses on the determination of both the solution and free surface equation. We shall introduce the model with two separated flow domains and we will show that it is well posed in the sense of Definition 4.1.

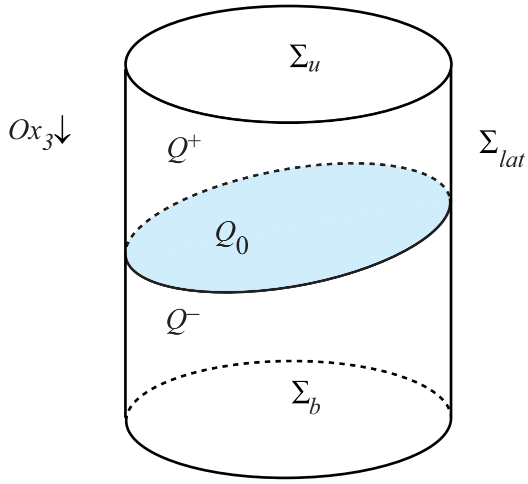

Fig. 5.4. Flow with separated saturated and unsaturated domains

We have assumed, for instance, that the saturated part is evolving from above. We shall study this problem within the model in pressure.

Denote again, but in terms of pressure

$$Q^- = \{(x,t);\ h(x,t) < 0\}$$

the unsaturated part,

$$Q^+ = \{(x,t);\ h(x,t) > 0\}$$

the saturated region and

$$Q_0 = \{(x,t);\ h(x,t) = 0\}$$

the free surface (boundary) separating the saturated part Q^+ from the unsaturated one.

We assume that

$$Q_0 \text{ is a smooth surface, i.e. of class } C^1 \text{ and } Q^+ \text{ and } Q^- \text{ are open.} \tag{4.17}$$

We specify that ν signifies in general the normal to a boundary, no matter which boundary is in discussion, but we should keep in mind that the respective normal is always directed to the exterior of the domain delimited by it. However, in order to avoid any confusion we shall mark by superscripts the normals to the interface, i.e., by ν^+ we mean the normal to Q_0 directed towards Q^- and by ν^- the normal to Q_0 directed to Q^+ and we notice that $\nu^+ = -\nu^-$. Moreover we denote

$$h^-(x_0,t_0) = \lim_{(x,t)\to(x_0,t_0)} h(x,t),$$
$$q^-(x_0,t_0) = \lim_{(x,t)\to(x_0,t_0)} (q\cdot\nu^-)(x,t), \text{ if } (x,t) \in Q^-.$$

Similarly by

$$\begin{aligned} h^+(x_0,t_0) &= \lim_{(x,t)\to(x_0,t_0)} h(x,t), \\ q^+(x_0,t_0) &= \lim_{(x,t)\to(x_0,t_0)} (q\cdot\nu^+)(x,t), \text{ if } (x,t)\in Q^+ \end{aligned}$$

we denote the corresponding Q^+-limits. We recall that Ox_3 is downwards directed and for simplicity we assume in this proof that Ω has a cylindrical form,

$$\Omega = \{x \in \mathbf{R}^3;\ s_u(x_1,x_2) < x_3 < s_L(x_1,x_2)\},$$

where $x_3 = s_u(x_1,x_2)$, $x_3 = s_L(x_1,x_2)$ are the equations of the soil surface and bottom of the flow domain, respectively.

The boundary value problem modelling a perfectly separated saturated-unsaturated situation is described by the system

$$C(h)\frac{\partial h}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} = f \quad \text{in} \quad Q^-, \tag{4.18}$$

$$-K_s\Delta h = f \quad \text{in} \quad Q^+, \tag{4.19}$$

$$h(x,0) = h_0(x) \quad \text{in} \quad \Omega, \tag{4.20}$$

$$q^+(x,t) = q^-(x,t) \quad \text{on} \quad Q_0, \tag{4.21}$$

$$h^+(x,t) = h^-(x,t) = 0 \quad \text{on} \quad Q_0, \tag{4.22}$$

$$q\cdot\nu = u(x,t) \quad \text{on} \quad \Sigma_u, \tag{4.23}$$

$$q\cdot\nu = \alpha(x)K^*(h) + f_0(x,t) \quad \text{on} \quad \Sigma_\alpha. \tag{4.24}$$

First we have to show that the model above is well-posed. To this end we prove

Proposition 4.6. *If h is a weak and smooth solution to* (4.2)-(4.5), *then h is the solution in a generalized sense to the model* (4.18)-(4.24) *describing the water infiltration into an saturated-unsaturated soil.*

Proof. To be more specific this means that we have to prove that h is a solution in the sense of distributions to (4.18)-(4.19) and satisfies the boundary conditions in the sense of the trace theory. If saturation occurs from above, we can represent Q^+, Q^- and Q_0 as

$$\begin{aligned} Q^+ &:= \{(x,t);\ s_u(x_1,x_2) < x_3 < s(t,x_1,x_2)\}, \\ Q^- &:= \{(x,t);\ s(t,x_1,x_2) < x_3 < s_L(x_1,x_2)\}, \\ Q_0 &:= \{(x,t);\ x_3 = s(t,x_1,x_2)\}, \end{aligned}$$

where $x_3 = s(t,x_1,x_2)$ was assumed to be smooth.

Let h be a solution to (4.2)-(4.5), like in Definition 4.1. Then, in (4.7), we take ϕ with compact support in Q^- and it follows that

$$\int_{Q^-}\left(\frac{\partial C^*(h)}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3}\right)\phi dxdt = \int_{Q^-} f\phi dxdt,\ \forall\phi\in\mathcal{D}(Q^-),$$

that implies that (4.18) is satisfied in the sense of distributions. Here, $\mathcal{D}(Q^-)$ is the space of indefinitely differentiable function with compact support in Q^-. We have also to remark that by Definition 4.1 $K^*(h) \in L^2(0,T;V)$ which implies that $h \in L^2(0,T;V)$, so that it has the necessary regularity to apply Green's formulae above.

Similarly, if we take ϕ with support compact in Q^+ we get (4.19).

Now we multiply (4.18) by ϕ, integrate it over Q^- and add with (4.19) multiplied by ϕ and integrated over Q^+. After some integrations by parts we obtain

$$\int_{Q^-}\left(-C^*(h)\phi_t + \nabla K^*(h)\cdot\nabla\phi - k(h)\frac{\partial\phi}{\partial x_3}\right)dxdt$$
$$+\int_{\Sigma_b} q\cdot\nu\phi d\sigma dt + \int_{\Sigma_{lat}^-} q\cdot\nu\phi d\sigma dt + \int_{Q_0} q\cdot\nu^-\phi d\sigma dt$$
$$+\int_{Q^+} K_s\nabla h\cdot\nabla\phi dxdt + \int_{\Sigma_u} q\cdot\nu\phi d\sigma dt + \int_{\Sigma_{lat}^+} q\cdot\nu\phi d\sigma dt + \int_{Q_0} q\cdot\nu^+\phi d\sigma dt$$
$$= \int_{Q^-} f\phi dxdt + \int_{Q^+} f\phi dxdt + \int_{\Omega^-}(C^*(h)\phi)(x,0)dx + \int_{\Omega^+}(C^*(h)\phi)(x,0)dx.$$

Here $\Omega^\pm$ are the spatial domains corresponding to $Q^\pm$, $\Sigma_{lat}^\pm$ are the lateral boundaries corresponding to $Q^\pm$, with $\Sigma_{lat}^+ \cup \overline{\Sigma_{lat}^-} = \Sigma_{lat}$, $\Sigma_{lat}^+ \cap \Sigma_{lat}^- = \emptyset$ and Σ_b corresponds to the bottom basis of Ω.

Taking into account (4.7) we get

$$\int_\Omega \phi(x,0)C^*(h_0)dx - \int_{\Sigma_\alpha}(\alpha K^*(h) + f_0)\phi d\sigma dt - \int_{\Sigma_u} u\phi d\sigma dt \tag{4.25}$$
$$+\int_{\Sigma_b} q\cdot\nu\phi d\sigma dt + \int_{\Sigma_{lat}^-} q\cdot\nu\phi d\sigma dt + \int_{Q_0} q^-\phi d\sigma dt$$
$$+\int_{\Sigma_u} q\cdot\nu\phi d\sigma dt + \int_{\Sigma_{lat}^+} q\cdot\nu\phi d\sigma dt + \int_{Q_0} q^+\phi d\sigma dt$$
$$= \int_{\Omega_-}(C^*(h)\phi)(x,0)dx + \int_{\Omega_+}(C^*(h)\phi)(x,0)dx,$$

for each ϕ with the properties specified in Definition 4.1. Since ϕ is arbitrary we obtain $q^+ = q^-$ on Q_0, $q\cdot\nu = \alpha K^*(h) + f_0$ on Σ_α, $q\cdot\nu = u$ on Σ_u and $C^*(h_0(x)) = (C^*(h))(x,0)$, by taking apart $\phi \in \mathcal{D}(Q_0)$, $\phi \in \mathcal{D}(Q \cup \Sigma_\alpha)$ and $\phi \in \mathcal{D}(Q \cup \Sigma_u)$, respectively (see also (1.24)). The condition related to the pressure continuity on Q_0 is implied by the assumption that h is smooth and by the definition of Q_0.

It must be emphasized that it should be not surprising that on the surface Q_0 there are two conditions, i.e., the flux and the pressure continuity (4.21)-(4.22), called also *transmission conditions.* Although they might appear superfluous, they both are necessary, as we mentioned before, because the free surface Q_0 is unknown, but can be determined by a supplementary condition. ∎

5.5 Existence of the free boundary

One might expect that in some conditions at least, the domain Q_0, separating the saturated region Q^+ from the unsaturated one, Q^-, is a surface. We shall see that indeed this is the case in some generalized sense.

We shall prove below that under certain conditions there exists a free boundary $s = s(t, x_1, x_2)$ that separates the saturated domain from the unsaturated one, with Q^+ above Q^-. To this end we shall prove that the function θ is monotonically decreasing with respect to x_3, i.e., $w = \dfrac{\partial \theta}{\partial x_3} \leq 0$. Consequently, the equation $\theta(x,t) = \theta_s$ can be solved with respect to x_3 and yields a unique solution $x_3 = s(t, x_1, x_2)$. It is not clear however if this is a smooth surface in the usual geometric sense and perhaps in general this is not true, but as we have seen, this assumption should have been considered in order to enhance some mathematical results.

Existence of the solution vertical derivative

This part is concerned with the proof of the existence of the vertical derivative of the approximating solution.

For simplicity we assume that Ω is a cylinder with horizontal bases.

All over this part we shall work with the smoother approximation β_ε^* given by (2.60). Moreover, here we shall use a regularization of K, namely the $C^2(\mathbf{R})$ approximation of K defined by

$$K_\varepsilon(r) := \begin{cases} K(r), & r < \theta_s - \varepsilon \\ K_{int}(r), & \theta_s - \varepsilon \leq r < \theta_s + \delta_{ext} \\ K_s, & r \geq \theta_s + \delta_{ext}, \end{cases} \tag{5.1}$$

where $\delta_{ext} > 0$, and K_{int} is determined such that $K_{int} \in C^2([\theta_s - \varepsilon, \theta_s + \delta_{ext}])$ and $K_{int}(r) \leq K(r)$.

Let θ_ε be the solution to the approximating problem (2.3)-(2.4) as given by Theorem 2.8. We introduce

$$w_\varepsilon := \frac{\partial \theta_\varepsilon}{\partial x_3} \tag{5.2}$$

and since $w_\varepsilon \in L^2(0,T;V)$ by Theorem 2.8, we can (formally) differentiate with respect to x_3 in (2.6)-(2.9) and obtain the equivalent model for the derivative w_ε

$$\frac{\partial w_\varepsilon}{\partial t} - \Delta(\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon) + \frac{\partial}{\partial x_3}(K_\varepsilon'(\theta_\varepsilon) w_\varepsilon) = f_{x_3} \quad \text{in} \quad Q, \tag{5.3}$$

$$K_\varepsilon(\theta_\varepsilon) - \beta_\varepsilon(\theta_\varepsilon) w_\varepsilon = -u \quad \text{on} \quad \Sigma_u, \tag{5.4}$$

$$K_\varepsilon(\theta_\varepsilon) - \beta_\varepsilon(\theta_\varepsilon) w_\varepsilon = \alpha \beta_\varepsilon^*(\theta_\varepsilon) + f_0 \quad \text{on} \quad \Sigma_b, \tag{5.5}$$

$$(K'_\varepsilon(\theta_\varepsilon)w_\varepsilon i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon)w_\varepsilon)) \cdot \nu = \alpha\beta_\varepsilon(\theta_\varepsilon)w_\varepsilon + \alpha_{x_3}\beta^*_\varepsilon(\theta_\varepsilon) + (f_0)_{x_3} \quad \text{on } \Sigma_{lat}, \tag{5.6}$$

to which we add the initial condition

$$w_\varepsilon(x,0) = w_0(x) \text{ in } \Omega. \tag{5.7}$$

This makes sense, because by (2.89), $\dfrac{\partial \theta_\varepsilon}{\partial x_3} \in W^{1,2}(0,T;L^2(\Omega))$ and implies $w_\varepsilon(x,0) \in L^2(\Omega)$.

In order to make this formal calculation rigorous, we shall prove next that if the initial and boundary data have a sufficient regularity, the problem (5.3)-(5.7) has a solution w_ε in some appropriate spaces.

Theorem 5.1. *Assume that* $\alpha \in C^1(\Gamma_\alpha)$, *and*

$$\theta_0 \in H^2(\Omega), \tag{5.8}$$

$$u \in W^{1,2}(0,T;L^2(\Gamma_u)) \cap L^2(0,T;H^1(\Gamma_u)) \cap L^\infty(\Sigma_u), \tag{5.9}$$

$$f_0 \in W^{1,2}(0,T;L^2(\Gamma_\alpha)) \cap L^2(0,T;H^1(\Gamma_\alpha)) \cap L^\infty(\Sigma_\alpha), \tag{5.10}$$

$$f \in W^{1,2}(0,T;L^2(\Omega)). \tag{5.11}$$

Then problem (5.3)-(5.7) *has a unique solution*

$$w_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V), \tag{5.12}$$

$$\frac{dw_\varepsilon}{dt} \in L^2(0,T;V'). \tag{5.13}$$

Proof. By Theorem 2.8 we know that problem (2.3)-(2.4) has a unique solution

$$\theta_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{5.14}$$

$$\beta^*_\varepsilon(\theta_\varepsilon) \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)). \tag{5.15}$$

We introduce the functions

$$w^u_\varepsilon = \left.\frac{K_\varepsilon(\theta_\varepsilon) + u}{\beta_\varepsilon(\theta_\varepsilon)}\right|_{\Sigma_u}, \quad w^b_\varepsilon = \left.\frac{K_\varepsilon(\theta_\varepsilon) - \alpha\beta^*_\varepsilon(\theta_\varepsilon) - f_0}{\beta_\varepsilon(\theta_\varepsilon)}\right|_{\Sigma_b}. \tag{5.16}$$

They are well defined on Σ_u and Σ_b, respectively, as we can see further. First, for any γ, $\eta \in H^1(\Omega)$ it follows that

$$\gamma\eta \in L^2(\Omega). \tag{5.17}$$

Indeed, by the Sobolev embedding theorems we have

$$\|\gamma\eta\|^2 = \int_\Omega \gamma^2\eta^2 dx \le \left(\int_\Omega \gamma^4 dx\right)^{1/2} \left(\int_\Omega \eta^4 dx\right)^{1/2}$$
$$\le C \|\gamma\|^{1/2} \|\gamma\|^{3/2}_{H^1(\Omega)} \|\eta\|^{1/2} \|\eta\|^{3/2}_{H^1(\Omega)} < \infty,$$

wherefrom the result. Since for each $\varepsilon > 0$, $\theta_\varepsilon \in L^\infty(0,T;H^2(\Omega))$ and $\beta_\varepsilon \in C^2(\mathbf{R})$, with bounded derivatives up to the second order, (see (2.62)-(2.64)) it follows that

$$\frac{\partial \beta_\varepsilon(\theta_\varepsilon)}{\partial x_i} = \beta_\varepsilon'(\theta_\varepsilon)\frac{\partial \theta_\varepsilon}{\partial x_i} \in L^\infty(0,T,L^2(\Omega)) \tag{5.18}$$

and

$$\frac{\partial^2 \beta_\varepsilon(\theta_\varepsilon)}{\partial x_i \partial x_j} = \beta_\varepsilon''(\theta_\varepsilon)\frac{\partial \theta_\varepsilon}{\partial x_j}\frac{\partial \theta_\varepsilon}{\partial x_i} + \beta_\varepsilon'(\theta_\varepsilon)\frac{\partial^2 \theta_\varepsilon}{\partial x_i \partial x_j} \in L^\infty(0,T;L^2(\Omega)), \tag{5.19}$$

the latter being implied by (5.17). We also used the fact that β_ε and its derivatives are bounded on $\mathbf{R}$. In conclusion we get that

$$\beta_\varepsilon(\theta_\varepsilon) \in L^\infty(0,T;H^2(\Omega)), \tag{5.20}$$

so its trace exists on Σ and

$$\beta_\varepsilon(\theta_\varepsilon)|_\Sigma \in L^\infty(0,T;H^{3/2}(\Gamma)). \tag{5.21}$$

Also we have that

$$\beta_\varepsilon'(\theta_\varepsilon) \in L^\infty(0,T;H^1(\Omega)), \tag{5.22}$$
$$\beta_\varepsilon'(\theta_\varepsilon)|_\Sigma \in L^\infty(0,T;H^{1/2}(\Gamma)). \tag{5.23}$$

Analogously, since $r \to K_\varepsilon(r) \in C^2(\mathbf{R})$, it follows that

$$K_\varepsilon(\theta_\varepsilon) \in L^\infty(0,T;H^2(\Omega)), \tag{5.24}$$

so its trace exists on Σ and $K_\varepsilon(\theta_\varepsilon)|_\Sigma \in L^\infty(0,T;H^{3/2}(\Gamma))$.

Hence w_ε^u is well defined on Σ_u. Then we calculate

$$\frac{\partial w_\varepsilon^u}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\left.\left(\frac{u}{\beta_\varepsilon(\theta_\varepsilon)} + \frac{K_\varepsilon(\theta_\varepsilon)}{\beta_\varepsilon(\theta_\varepsilon)}\right)\right|_{\Sigma_u}\right].$$

We shall detail the explanations only for the first term, the result being the same for the second, too.

Since the surface Γ_u, of equation $x_3 = s_u(x_1,x_2)$) (in our case $x_3 = 0$) is sufficiently smooth and $u \in L^\infty(\Sigma_u) \cap L^2(0,T;H^1(\Gamma_u))$, we have, e.g., for $i = 1,2,$ that

$$\frac{\partial}{\partial x_i}\left(\left.\frac{u}{\beta_\varepsilon(\theta_\varepsilon)}\right|_{\Sigma_u}\right)$$
$$= \left.\left(\frac{u_{x_i}}{\beta_\varepsilon(\theta_\varepsilon)} - \frac{1}{\beta_\varepsilon^2(\theta_\varepsilon)}\frac{\partial \beta_\varepsilon(\theta_\varepsilon)}{\partial x_i}u + \frac{\partial}{\partial s}\left(\frac{u}{\beta_\varepsilon(\theta_\varepsilon)}\right)\frac{\partial s_u}{\partial x_i}\right)\right|_{\Sigma_u} \in L^2(\Sigma_u),$$

so finally we get

$$w_\varepsilon^u \in L^2(0,T;H^1(\Gamma_u)). \tag{5.25}$$

In a similar way, using (5.14) and (5.15), we get that

$$\frac{\partial w_\varepsilon^u}{\partial t} = \frac{\partial}{\partial t}\left[\left(\frac{u}{\beta_\varepsilon(\theta_\varepsilon)} + \frac{K_\varepsilon(\theta_\varepsilon)}{\beta_\varepsilon(\theta_\varepsilon)}\right)\Big|_{\Sigma_u}\right] \in L^2(0,T;L^2(\Gamma_u)). \tag{5.26}$$

Analogously we deduce that

$$w_\varepsilon^b \in L^2(0,T;H^1(\Gamma_b)) \cap W^{1,2}(0,T;L^2(\Gamma_b)). \tag{5.27}$$

We shall prove that there exists a function

$$\widetilde{w}_\varepsilon \in L^2(0,T;H^{3/2}(\Omega)),$$

with

$$\frac{\partial \widetilde{w}_\varepsilon}{\partial t} \in L^2(0,T;L^2(\Omega)),$$

such that

$$\widetilde{w}_\varepsilon|_{\Sigma_u} = w_\varepsilon^u, \quad \widetilde{w}_\varepsilon|_{\Sigma_b} = w_\varepsilon^b \text{ and } \widetilde{w}_\varepsilon|_{\Sigma_{lat}} = w_\varepsilon^{lat}, \tag{5.28}$$

in the sense of traces, where $w_\varepsilon^{lat} \in L^2(0,T;H^1(\Gamma_{lat})) \cap W^{1,2}(0,T;L^2(\Gamma_{lat}))$ is fixed.

Indeed, by the surjectivity of the trace map, there exists $\widetilde{w}_\varepsilon \in L^2(0,T;H^{3/2}(\Omega))$, such that (5.28) holds.

Since the trace of $\widetilde{w}_\varepsilon$ on the boundary Σ is in $W^{1,2}(0,T;L^2(\Gamma))$, according to (5.27) and to the way in which we fixed w_ε^{lat} on Σ_{lat}, we can write that

$$\int_0^{T-\delta} \|\widetilde{w}_\varepsilon(t+\delta) - \widetilde{w}_\varepsilon(t)\|_{L^2(\Gamma)}^2 \, dt \le C\,|\delta|^2\,, \ \forall \delta \in (0,T),$$

(see Theorem 3.11 in Appendix).

This inequality remains true for $\widetilde{w}_\varepsilon$ in $H^{1/2}(\Omega)$ norm, by the continuity of the trace operator (see Dirichlet map, (2.8) in Appendix), that is,

$$\int_0^{T-\delta} \|\widetilde{w}_\varepsilon(t+\delta) - \widetilde{w}_\varepsilon(t)\|_{L^2(\Omega)}^2 \, dt \le \int_0^{T-\delta} \|\widetilde{w}_\varepsilon(t+\delta) - \widetilde{w}_\varepsilon(t)\|_{H^{1/2}(\Omega)}^2 \, dt$$
$$\le C \int_0^{T-\delta} \|\widetilde{w}_\varepsilon(t+\delta) - \widetilde{w}_\varepsilon(t)\|_{L^2(\Gamma)}^2 \, dt \le C\,|\delta|^2\,.$$

Hence, by the same Theorem 3.11 in Appendix, we deduce that

$$\widetilde{w}_\varepsilon \in W^{1,2}(0,T;L^2(\Omega)).$$

We define

$$\phi = w_\varepsilon - \widetilde{w}_\varepsilon \tag{5.29}$$

so that the problem (5.3)-(5.7) becomes

$$\frac{\partial \phi}{\partial t} - \Delta(\beta_\varepsilon(\theta_\varepsilon)\phi) + \frac{\partial}{\partial x_3}(K'_\varepsilon(\theta_\varepsilon)\phi) = f_\phi, \tag{5.30}$$

$$\phi = 0 \text{ on } \Sigma_u, \tag{5.31}$$

$$\phi = 0 \text{ on } \Sigma_b, \tag{5.32}$$

$$(K'_\varepsilon(\theta_\varepsilon)\phi i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon)\phi)) \cdot \nu = \alpha\beta_\varepsilon(\theta_\varepsilon)\phi + f_{0\phi} \text{ on } \Sigma_{lat}, \tag{5.33}$$

$$\phi(x,0) = \phi_0(x) \text{ in } \Omega, \tag{5.34}$$

where

$$f_\phi = f_{x_3} - \frac{\partial \widetilde{w}_\varepsilon}{\partial t} + \Delta\beta_\varepsilon(\theta_\varepsilon) - \frac{\partial}{\partial x_3}(K'_\varepsilon(\theta_\varepsilon)\widetilde{w}_\varepsilon), \tag{5.35}$$

$$f_{0\phi} = -(K'_\varepsilon(\theta_\varepsilon)\widetilde{w}_\varepsilon i_3 \cdot \nu - \nabla(\beta_\varepsilon(\theta_\varepsilon)\widetilde{w}_\varepsilon) \cdot \nu + \alpha_{x_3}\beta^*_\varepsilon(\theta_\varepsilon) + (f_0)_{x_3} \text{ on } \Sigma_{lat} \tag{5.36}$$

and $\phi_0(x) = w_\varepsilon(x,0) - \widetilde{w}_\varepsilon(x,0) \in L^2(\Omega)$.

We mention that under the specified assumptions

$$f_\phi \in L^2(0,T;L^2(\Omega)),\ f_{0\phi} \in L^2(0,T;L^2(\Gamma_{lat})).$$

We consider the spaces

$$V_0 = \{\psi \in H^1(\Omega);\ \psi = 0 \text{ on } \Gamma_u \text{ and } \psi = 0 \text{ on } \Gamma_b\ \},$$

with the norm $\|\psi\|_{V_0} = \left(\int_\Omega |\nabla\psi|\, dx\right)^{1/2}$ and its dual denoted V'_0 and we define the linear operator $A(t) : V_0 \to V'_0$ by

$$\begin{aligned}\langle A(t)\phi, \psi\rangle_{V'_0,V_0} =& \int_\Omega \left(\nabla(\beta_\varepsilon(\theta_\varepsilon(t))\phi) \cdot \nabla\psi - K'_\varepsilon(\theta_\varepsilon(t))\phi \frac{\partial \psi}{\partial x_3} \right) dx \\ &+ \int_{\Gamma_{lat}} \alpha\beta_\varepsilon(\theta_\varepsilon(t))\phi\psi d\sigma, \quad \text{for any } \psi \in V_0.\end{aligned} \tag{5.37}$$

The operator $A(t)$ is bounded and coercive. Indeed, we have

$$\begin{aligned}&\langle A(t)\phi, \phi\rangle_{V'_0,V_0} \\ &= \int_\Omega \left(\beta_\varepsilon(\theta_\varepsilon) |\nabla\phi|^2 + \beta'_\varepsilon(\theta_\varepsilon)\phi\nabla\theta_\varepsilon \cdot \nabla\phi - K'_\varepsilon(\theta_\varepsilon)\phi \frac{\partial \phi}{\partial x_3} \right) dx \\ &+ \int_{\Gamma_{lat}} \alpha\beta_\varepsilon(\theta_\varepsilon)\phi^2 d\sigma \\ &\geq \beta_m \|\phi\|^2_{V_0} + \alpha_m\beta_m \|\phi\|^2_{L^2(\Gamma_{lat})} - \beta'_M(\varepsilon) \|\phi\nabla\theta_\varepsilon\| \|\nabla\phi\| - M \|\phi\| \|\phi\|_{V_0} \\ &\geq \frac{\beta_m}{2} \|\phi\|^2_{V_0} + \alpha_m\beta_m \|\phi\|^2_{L^2(\Gamma_{lat})} - \frac{M^2}{\beta_m} \|\phi\|^2 - \frac{(\beta'_M(\varepsilon))^2}{\beta_m} \|\phi\nabla\theta_\varepsilon\|^2 .\end{aligned} \tag{5.38}$$

Using (2.83) we calculate

$$\|\phi\nabla\theta_\varepsilon\|^2 = \int_\Omega \phi^2 |\nabla\theta_\varepsilon|^2\, dx \le \left(\int_\Omega \phi^4 dx\right)^{1/2} \left(\int_\Omega |\nabla\theta_\varepsilon|^4\, dx\right)^{1/2}$$
$$\le C\,\|\phi(t)\|^{1/2}\,\|\phi(t)\|_{V_0}^{3/2}\,\|\nabla\theta_\varepsilon(t)\|^{1/2}\,\|\nabla\theta_\varepsilon(t)\|_{V_0}^{3/2}.$$

But θ_ε satisfies (5.14) and so

$$\|\phi\nabla\theta_\varepsilon\|^2 \le C(\varepsilon)\,\|\phi(t)\|^{1/2}\,\|\phi(t)\|_{V_0}^{3/2}. \tag{5.39}$$

Then we have

$$\frac{(\beta'_M(\varepsilon))^2}{\beta_m}\,\|\phi\nabla\theta_\varepsilon\|^2 \le C(\varepsilon)\,\|\phi(t)\|^{1/2}\,\|\phi(t)\|_{V_0}^{3/2} \le \frac{\beta_m}{4}\,\|\phi(t)\|_{V_0}^2 + C(\varepsilon)\,\|\phi(t)\|^2, \tag{5.40}$$

where we applied the Young inequality. Recalling (5.38) we obtain

$$\langle A(t)\phi,\phi\rangle_{V_0',V_0} \ge \frac{\beta_m}{4}\,\|\phi\|_{V_0}^2 + \alpha_m\beta_m\,\|\phi\|_{L^2(\Gamma_{lat})}^2 - \left(\frac{M^2}{\beta_m} + C(\varepsilon)\right)\|\phi\|^2. \tag{5.41}$$

By (5.37) we have

$$|A(t)\phi(\psi)| \le \beta_M(\varepsilon)\,\|\phi\|_{V_0}\,\|\psi\|_{V_0} + \beta'_M(\varepsilon)\,\|\phi\nabla\theta_\varepsilon\|\,\|\psi\|_{V_0}$$
$$+ M\,\|\phi\|\,\|\psi\|_{V_0} + \alpha_M\beta_M(\varepsilon)\,\|\phi\|_{L^2(\Gamma_{lat})}\,\|\psi\|_{L^2(\Gamma_{lat})}.$$

Using (5.39) and Poincaré's inequality we obtain

$$\|\phi\nabla\theta_\varepsilon\| \le C(\varepsilon)\,\|\phi(t)\|^{1/4}\,\|\phi(t)\|_{V_0}^{3/4} \le C(\varepsilon)\,\|\phi(t)\|_{V_0}$$

so that we finally can write (using also (1.10)-(1.13)) that

$$|A(t)\phi(\psi)| \le (\beta_M(\varepsilon) + C(\varepsilon))\,\|\phi\|_{V_0}\,\|\psi\|_{V_0}. \tag{5.42}$$

In conclusion we infer that

$$|A(t)\phi\|_{V_0'} \le C(\varepsilon)\,\|\phi\|_{V_0}, \tag{5.43}$$

so $A(t)$ is continuous. As previously, C and $C(\varepsilon)$ denote various constants independent of and dependent on ε, respectively. It follows that the operator $A(t)$ satisfies the hypotheses of Lions' theorem and since $f_\phi \in L^2(0,T;V_0')$ and $\phi_0 \in L^2(\Omega)$ we conclude that the system (5.30)-(5.34) has a unique solution

$$\phi \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V_0),\ \ \frac{d\phi}{dt} \in L^2(0,T;V_0'). \tag{5.44}$$

By (5.29) we obtain (5.12)-(5.13) as claimed. ∎

Vertical monotonicity of the solution

We can now pass to the proof of the vertical monotonicity of θ for the situation when $N \leq 3$, α is constant and f_0 is time dependent only. First, we shall prove this for the approximating solution θ_ε.

We use once again the approximates β_ε^* given by (2.60) and K_ε given by (5.1).

For the case of interest in our problem, meaning the study of the top saturation occurrence ($\theta = \theta_s$ on Σ_u) some supplementary conditions will be required and these include a monotonically vertical decreasing distribution of the initial data and source and some particular properties for the functions u and f_0. First we shall prove an intermediate result.

For each $\varepsilon > 0$, let us introduce the functions $F_\varepsilon : \mathbf{R} \to \mathbf{R}$,

$$F_\varepsilon(r) := K_\varepsilon(r) - \alpha\beta_\varepsilon^*(r) \tag{5.45}$$

and $F : (-\infty, \theta_s] \to \mathbf{R}$

$$F(r) := K(r) - \alpha \overset{\circ}{\beta}{}^*(r), \tag{5.46}$$

where $\overset{\circ}{\beta}{}^*$ is the minimal section of β^*. This means that $F(r) = K(r) - \alpha\beta^*(r)$ for $r < \theta_s$ and $F(\theta_s) = K_s - \alpha K_s^*$.

We notice that F_ε is differentiable on $\mathbf{R}$, while F is continuous (and differentiable on $(-\infty, \theta_s)$) and

$$F(r) = F_\varepsilon(r),\ \forall r \in (-\infty, \theta_s - \varepsilon). \tag{5.47}$$

We set

$$F_{\min} := \min_{r\in[0,\theta_s]} F(r),\ F_{\max} := \max_{r\in[0,\theta_s]} F(r).$$

Then $F : (-\infty, \theta_s] \to [F_{\min}, +\infty)$ and F is strictly monotonically decreasing on $(-\infty, 0]$ because $F'(r) = -\alpha\beta(r) < 0$ for $r \leq 0$.

Lemma 5.2. *Let α be a positive constant and $f_0 \in C^1[0,T]$. Then, if $f_0 \geq F_{\min}$, the equation*

$$F(r) = f_0(t) \tag{5.48}$$

has, for each t, at least one solution

$$r(t) = F^{-1}(f_0(t)). \tag{5.49}$$

This follows by the continuity of the function $F : (-\infty, \theta_s] \to [F_{\min}, +\infty)$.

In general, F^{-1} might be a multivalued function. Denote by θ_{left} the smallest solution to the equation $F(\theta) = F_{\max}$. We notice that if $f_0(t) \geq F_{\max}$, then the solution $F^{-1}(f_0(t))$ is unique and it is smaller than θ_{left}, see Fig. 5.5.

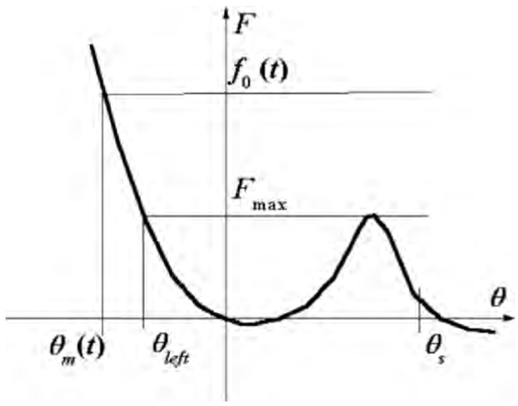

Fig. 5.5. Determination of $\theta_m(t)$

Lemma 5.3. *Assume the following conditions:*

$$f_0 \in C^1[0,T], \ f_0(t) \geq \max_{\theta \in [0,\theta_s]} F(\theta) = F_{\max}, \tag{5.50}$$

$$f \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(Q), \tag{5.51}$$

$$(F^{-1}(f_0))'(t) \leq f, \ a.e. \ in \ Q, \tag{5.52}$$

$$u \in W^{1,2}(0,T;L^2(\Gamma_u)) \cap L^\infty(0,T;H^1(\Gamma_u)) \cap L^\infty(\Sigma_u), \tag{5.53}$$

$$K_s \leq -u \ a.e. \ on \ \Sigma_u, \tag{5.54}$$

$$\theta_0 \in H^2(\Omega), \ 0 \leq \theta_0 \leq \theta_s. \tag{5.55}$$

Then, there exists $\theta_m(t)$ independent of ε, such that the approximating solution θ_ε satisfies

$$\theta_m(t) \leq \theta_\varepsilon(x,t), \ a.e. \ in \ \Omega, \ for \ any \ t \in [0,T] \ and \ \varepsilon > 0, \tag{5.56}$$

$$\frac{\partial \theta_\varepsilon}{\partial x_3}(x,t) \leq 0, \ a.e. \ on \ \Sigma_b, \tag{5.57}$$

$$\frac{\partial \theta_\varepsilon}{\partial x_3}(x,t) \leq 0, \ a.e. \ on \ \Sigma_u. \tag{5.58}$$

Proof. By Theorem 2.8 the approximating solution θ_ε exists and satisfies the boundary condition (2.9) which particularly written on the parts Σ_b and Σ_u becomes

$$\frac{\partial \theta_\varepsilon}{\partial x_3} = \frac{K_\varepsilon(\theta_\varepsilon) - \alpha\beta_\varepsilon^*(\theta_\varepsilon) - f_0}{\beta_\varepsilon(\theta_\varepsilon)}, \text{ a.e. on } \Sigma_b, \tag{5.59}$$

$$\frac{\partial \theta_\varepsilon}{\partial x_3} = \frac{K_\varepsilon(\theta_\varepsilon) + u}{\beta_\varepsilon(\theta_\varepsilon)}, \text{ a.e. on } \Sigma_u. \tag{5.60}$$

Condition (5.50) implies that

$$f_0(t) \geq F_{\max} \geq K(\theta_\varepsilon) - \alpha \overset{\circ}{\beta}{}^{*} (\theta_\varepsilon),\ \forall \theta_\varepsilon \in [0, \theta_s], \tag{5.61}$$

and in particular

$$f_0(t) \geq K_s - \alpha K_s^*. \tag{5.62}$$

We also mention here that, due to the convexity of the function β^*on $(0, \theta_s)$, on the interval $[\theta_s - \varepsilon, \theta_s)$ we have the inequality

$$\beta^*(\theta) \leq \beta^*(\theta_s - \varepsilon) + \frac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}[\theta - (\theta_s - \varepsilon)]\ ,$$

so that the function β^*_{int} inserted in (2.60) has the property

$$\beta^*(\theta) \leq \beta^*_{int}(\theta) \leq \beta^*(\theta_s - \varepsilon) + \frac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}[\theta - (\theta_s - \varepsilon)],$$

for $\theta \in [\theta_s - \varepsilon, \theta_s)$. Then it follows that

$$\beta^*_\varepsilon(\theta) \geq \beta^*(\theta) \text{ for } \theta \in [0, \theta_s). \tag{5.63}$$

We still have

$$\beta^*_\varepsilon(\theta) \geq K_s^* \text{ for } \theta \geq \theta_s. \tag{5.64}$$

Moreover, since

$$F(\theta_\varepsilon) = K(\theta_\varepsilon) - \alpha\beta^*(\theta_\varepsilon) \geq K(\theta_\varepsilon) - \alpha\beta^*_\varepsilon(\theta_\varepsilon),\ \forall \theta_\varepsilon \in [0, \theta_s),$$

(see (5.63)) and $F(\theta_s) = F_\varepsilon(\theta_s)$ we have that

$$f_0(t) \geq \max_{\theta_\varepsilon \in [0,\theta_s]} F(\theta_\varepsilon) \geq K(\theta_\varepsilon) - \alpha\beta^*_\varepsilon(\theta_\varepsilon) \text{ on } \Sigma_b,\ \forall \theta_\varepsilon \in [0, \theta_s]. \tag{5.65}$$

If $\theta_\varepsilon > \theta_s$ then $\beta^*_\varepsilon(\theta_\varepsilon) > K_s^*$ and by (5.62) we get

$$f_0(t) \geq K_s - \alpha K_s^* > K(\theta_\varepsilon) - \alpha\beta^*_\varepsilon(\theta_\varepsilon),\ \forall \theta_\varepsilon > \theta_s. \tag{5.66}$$

Now, for each t fixed the horizontal $y = f_0(t) \geq F_{\max}$ intersects the graphic of the function $y = F(\theta)$ yet at one point situated on the left decreasing branch of F (see Fig. 5.5).

Hence, for each t fixed, we define

$$\theta_m(t) := \min\{r_j(t);\ F(r_j(t)) = f_0(t)\} \tag{5.67}$$

and θ_m follows to be independent of ε.

By the decreasing monotonicity of F we obtain that if $f_0(t){=}F(\theta_m){\geq}F_{\max}$ then $\theta_m(t) \leq \theta_{left}$, where θ_{left} is the smallest solution to $F_{\max} = f_0(t)$.

Moreover, it follows that $t \to \theta_m(t)$ is differentiable,

$$\theta_m'(t) = \frac{1}{F'(F^{-1}(f_0(t)))}.$$

In particular, if $F_{\max} \geq 0$, then $\theta_m(t) \leq 0$, whence we obtain

$$f_0(t) \geq F_{\max} \geq F(\theta_\varepsilon) = F_\varepsilon(\theta_\varepsilon) \text{ for } \theta_\varepsilon \in [\theta_m(t), 0). \tag{5.68}$$

Finally we get from (5.65), (5.66) and (5.68) that assumption (5.50) turns in

$$f_0(t) \geq F(\theta_\varepsilon) \text{ for } \theta_m(t) \leq \theta_\varepsilon \text{ on } \Sigma_b \tag{5.69}$$

and for each t we have

$$f_0(t) = F(\theta_m(t)) = \sup_{\theta \geq \theta_m(t)} F(\theta). \tag{5.70}$$

Further, using (5.52) we can write

$$\theta_m'(t) \leq f \text{ on } Q. \tag{5.71}$$

Then, by (5.54) we have

$$K_\varepsilon(\theta_m(t)) \leq K(\theta_m(t)) \leq K_s \leq -u \text{ on } \Sigma_u, \tag{5.72}$$

because K is a monotonically increasing function for $\theta \leq \theta_s$ and $\theta_m(t) < \theta_s$.

In conclusion by (5.55), (5.66), (5.70), (5.71), (5.72) and Corollary 2.5 it follows that

$$\theta_m(t) \leq \theta_\varepsilon(x,t) \text{ a.e. on } \Omega,\ \forall t \in [0,T].$$

By (5.59), (5.65), (5.56) and the definition of K_ε (see (5.1)), we successively get

$$\begin{aligned}
\frac{\partial \theta_\varepsilon}{\partial x_3} &= \frac{K_\varepsilon(\theta_\varepsilon) - \alpha\beta_\varepsilon^*(\theta_\varepsilon) - f_0(t)}{\beta_\varepsilon(\theta_\varepsilon)} \leq \frac{K(\theta_\varepsilon) - \alpha\beta^*(\theta_\varepsilon) - f_0(t)}{\beta_\varepsilon(\theta_\varepsilon)} \\
&\leq \frac{\max\limits_{\theta_\varepsilon \geq \theta_m(t)} F(\theta_\varepsilon) - f_0(t)}{\beta_\varepsilon(\theta_\varepsilon)}, \text{ a.e. on } \Sigma_b, \\
\frac{\partial \theta_\varepsilon}{\partial x_3} &\leq \frac{F(\theta_m(t)) - f_0(t)}{\beta_\varepsilon(\theta_\varepsilon)} \leq 0, \text{ a.e. on } \Sigma_b
\end{aligned}$$

i.e., (5.57). Analogously we obtain

$$\frac{\partial \theta_\varepsilon}{\partial x_3} = \frac{K_\varepsilon(\theta_\varepsilon) + u}{\beta_\varepsilon(\theta_\varepsilon)} \leq \frac{K(\theta_\varepsilon) + u}{\beta_\varepsilon(\theta_\varepsilon)} \leq \frac{K_s + u}{\beta_\varepsilon(\theta_\varepsilon)} \leq 0, \text{ a.e. on } \Sigma_u,$$

as claimed. ■

Theorem 5.4. *Assume the hypotheses of Lemma* 5.3, *i.e.*,

$$\theta_0 \in H^2(\Omega), \tag{5.73}$$

$$0 \leq \theta_0 \leq \theta_s, \ a.e. \ in \ \Omega, \tag{5.74}$$

$$u \in W^{1,2}(0,T; L^2(\Gamma_u)) \cap L^\infty(0,T; H^1(\Gamma_u)) \cap L^\infty(\Sigma_u), \tag{5.75}$$

$$K_s \leq -u, \ a.e. \ on \ \Sigma_u, \tag{5.76}$$

$$f_0 \in C^1[0,T], \tag{5.77}$$

$$f_0(t) \geq \max_{\theta \in [0,\theta_s]} (K(\theta) - \alpha \overset{\circ}{\beta}{}^{*} (\theta)), \tag{5.78}$$

$$f \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(Q), \tag{5.79}$$

$$(F^{-1}(f_0))'(t) \leq f \ a.e. \ in \ Q. \tag{5.80}$$

In addition we suppose that α *is a positive constant and*

$$\frac{\partial \theta_0}{\partial x_3}(x,0) \leq 0 \ \ a.e. \ in \ \ \Omega, \tag{5.81}$$

$$f_{x_3}(x,t) \leq 0 \ \ a.e. \ in \ \ Q. \tag{5.82}$$

Then $w = \dfrac{\partial \theta}{\partial x_3} \leq 0$ *a.e. in* $\Omega \times (0,T)$ *and* $x_3 \to \theta(x_1,x_2,x_3,t)$ *is monotonically decreasing on* $[0,T]$ *for each* $t \in [0,T]$.

Proof. We recall system (5.3)-(5.7) which under the current assumptions has the following form:

$$\frac{\partial w_\varepsilon}{\partial t} - \Delta(\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon) + \frac{\partial}{\partial x_3}(K'_\varepsilon(\theta_\varepsilon) w_\varepsilon) = f_{x_3}, \tag{5.83}$$

$$w_\varepsilon(x,0) = w_0(x) \text{ in } \Omega, \tag{5.84}$$

$$K_\varepsilon(\theta_\varepsilon) - \beta_\varepsilon(\theta_\varepsilon) w_\varepsilon = -u \text{ on } \Sigma_u, \tag{5.85}$$

$$K_\varepsilon(\theta_\varepsilon) - \beta_\varepsilon(\theta_\varepsilon) w_\varepsilon = \alpha \beta^*_\varepsilon(\theta_\varepsilon) + f_0 \text{ on } \Sigma_b, \tag{5.86}$$

$$(K'_\varepsilon(\theta_\varepsilon) w_\varepsilon i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon)) \cdot \nu = \alpha \beta_\varepsilon(\theta_\varepsilon) w_\varepsilon \text{ on } \Sigma_{lat}. \tag{5.87}$$

Under the hypotheses (5.73)-(5.80) the approximating solution θ_ε has, by Theorem 2.8, the properties

$$\theta_\varepsilon \in W^{1,\infty}(0,T; L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T; H^2(\Omega)),$$
$$\beta^*_\varepsilon(\theta_\varepsilon) \in W^{1,\infty}(0,T; L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T; H^2(\Omega)).$$

Moreover, due to Lemma 5.3, there exists $\theta_m(t)$ such that $\theta_m(t) \leq \theta_\varepsilon$,

$$\frac{\partial \theta_\varepsilon}{\partial x_3} \leq 0 \text{ a.e. on } \Sigma_b, \ \frac{\partial \theta_\varepsilon}{\partial x_3} \leq 0 \text{ a.e. on } \Sigma_u,$$

implying that $w_\varepsilon^+ = 0$ a.e. on $\Sigma_b \cup \Sigma_u$.

With these considerations, we can multiply (5.83) by w_ε^+ and integrate it over $\Omega \times (0,t)$. We have

$$\begin{aligned}&\int_0^t \int_\Omega \frac{\partial w_\varepsilon}{\partial t} w_\varepsilon^+ dx d\tau + \int_0^t \int_\Omega \nabla(\beta_\varepsilon(\theta_\varepsilon) w_\varepsilon) \cdot \nabla w_\varepsilon^+ dx d\tau \\ &= \int_0^t \int_\Omega K_\varepsilon'(\theta_\varepsilon) w_\varepsilon \frac{\partial w_\varepsilon^+}{\partial x_3} dx d\tau \\ &- \int_0^t \int_{\Gamma_b} (\alpha \beta_\varepsilon^*(\theta_\varepsilon) + f_0) w_\varepsilon^+ d\sigma d\tau + \int_0^t \int_{\Gamma_u} u w_\varepsilon^+ d\sigma d\tau \\ &- \int_0^t \int_{\Gamma_{lat}} \alpha \beta_\varepsilon(\theta_\varepsilon) w_\varepsilon w_\varepsilon^+ d\sigma d\tau + \int_0^t \int_\Omega f_{x_3} w_\varepsilon^+ dx_3 d\tau.\end{aligned}$$

We have

$$\begin{aligned}&\frac{1}{2}\int_0^t \int_\Omega \frac{\partial (w_\varepsilon^+)^2}{\partial t} dx d\tau + \int_0^t \int_\Omega \beta_\varepsilon(\theta_\varepsilon) \left|\nabla w_\varepsilon^+\right|^2 dx d\tau \\ &+ \int_0^t \int_{\Gamma_{lat}} \alpha \beta_\varepsilon(\theta_\varepsilon)(w_\varepsilon^+)^2 d\sigma d\tau + \int_0^t \int_\Omega \beta_\varepsilon'(\theta_\varepsilon) w_\varepsilon \nabla\theta_\varepsilon \cdot \nabla w_\varepsilon^+ dx d\tau \\ &\leq M \int_0^t \left\|w_\varepsilon^+(\tau)\right\| \left\|\frac{\partial w_\varepsilon^+}{\partial x_3}(\tau)\right\| d\tau.\end{aligned} \tag{5.88}$$

The rest of the terms on the right-hand side vanishes, by hypotheses. We treat separately the last term on the left-hand side

$$\begin{aligned}&\int_0^t \int_\Omega \beta_\varepsilon'(\theta_\varepsilon) w_\varepsilon \nabla\theta_\varepsilon \cdot \nabla w_\varepsilon^+ dx d\tau \leq \int_0^t \left\|\beta_\varepsilon'(\theta_\varepsilon(\tau)) w_\varepsilon^+(\tau) \nabla\theta_\varepsilon(\tau)\right\| \left\|\nabla w_\varepsilon^+(\tau)\right\| d\tau \\ &\leq \frac{1}{2}\left(\int_0^t \frac{2(\beta_M'(\varepsilon))^2}{\beta_m} \left\|w_\varepsilon^+(\tau)\nabla\theta_\varepsilon(\tau)\right\|^2 d\tau + \frac{\beta_m}{2}\int_0^t \left\|\nabla w_\varepsilon^+(\tau)\right\|^2 d\tau\right).\end{aligned}$$

Noticing that by hypothesis $w_\varepsilon^+(0) = 0$, we deduce from (5.88) that

$$\begin{aligned}&\frac{1}{2}\left\|w_\varepsilon^+(t)\right\|^2 + \frac{\beta_m}{2}\int_0^t \left\|(\nabla w_\varepsilon(\tau))^+\right\|^2 d\tau + \int_0^t \int_{\Gamma_{lat}} \alpha \beta_\varepsilon(\theta_\varepsilon)(w_\varepsilon^+)^2 d\sigma d\tau \\ &\leq \frac{M^2}{\beta_m}\int_0^t \left\|w_\varepsilon^+(\tau)\right\|^2 d\tau + \frac{(\beta_M'(\varepsilon))^2}{\beta_m}\int_0^t \left\|w_\varepsilon^+(\tau)\nabla\theta_\varepsilon(\tau)\right\|^2 d\tau.\end{aligned} \tag{5.89}$$

Taking into account that $\frac{\partial \theta_\varepsilon}{\partial x_i} \in L^\infty(0,T;V)$ we have

$$\begin{aligned}&\int_\Omega (w_\varepsilon^+(\tau))^2 \left|\nabla\theta_\varepsilon(\tau)\right|^2 dx \leq \left(\int_\Omega (w_\varepsilon^+)^4(\tau) dx\right)^{1/2} \left(\int_\Omega \left|\nabla\theta_\varepsilon(\tau)\right|^4 dx\right)^{1/2} \\ &\leq C \left\|(w_\varepsilon^+)(\tau)\right\|^{1/2} \left\|(w_\varepsilon^+)(\tau)\right\|_{H^1(\Omega)}^{3/2} \left\|\nabla\theta_\varepsilon(\tau)\right\|^{1/2} \left\|\nabla\theta_\varepsilon(\tau)\right\|_{H^1(\Omega)}^{3/2} \\ &\leq \frac{\beta_m}{4}\left\|(\nabla w_\varepsilon(\tau))^+\right\|^2 + C(\varepsilon)\left\|w_\varepsilon^+(\tau)\right\|^2.\end{aligned}$$

From (5.89) we get finally that

$$\left\|w_\varepsilon^+(t)\right\|^2 \le C(\varepsilon)\int_0^t \left\|w_\varepsilon^+(\tau)\right\|^2 d\tau,\ \forall t\in[0,T],$$

which implies, by Gronwall's lemma that $w_\varepsilon^+(x,t) = 0$ a.e. on Ω, for each $t \in [0,T]$.

Consequently, for each $t\in[0,T]$, the function $x_3 \to \theta_\varepsilon(x,t)$ is monotonically decreasing, meaning that if $x_3, x_3' \in [s_u(x_1,x_2), s_L(x_1,x_2)]$ with $x_3 < x_3'$ we have

$$\theta_\varepsilon(x_1,x_2,x_3,t) \le \theta_\varepsilon(x_1,x_2,x_3',t).$$

This inequality is preserved by passing to limit strongly in $L^2(Q)$, as $\varepsilon \to 0$, so that we find that $x_3 \to \theta(x,t)$ is decreasing a.e. $t\in(0,T)$, on $(s_u(x_1,x_2), s_L(x_1,x_2))$. ∎

Remark 5.5. We have to mention that in the literature (see, for instance [33]) it was specified that experiments revealed that if the rain rate $|u|$ is greater than the conductivity at saturation K_s, then the saturation of the soil begins from the soil surface and extends to the basement. If the rain rate is lesser than K_s, then the reverse situation occurs, i.e., the saturation begins from the basement and the free boundary advances to the surface. That is why we have chosen to prove the result corresponding to the particular situation with the saturated domain above the unsaturated one, because we wanted to see if the same theoretical result could be obtained under the appropriate conditions. The answer was positive. However, we had to assume several hypotheses because our model is a more general one, including other boundary conditions than the 1-D model used in [33], where the basement was considered impermeable and the soil was initially completely dried ($\theta_0 = 0$). Under appropriate assumptions, the basement saturation occurrence can be proved, but we let this proof to the reader.

Corollary 5.6. *There is a graph $x_3 = s(x_1,x_2,t)$ that separates the saturated region Q^+ by the unsaturated region Q^-.*

Proof. By Theorem 5.4 we may conclude that under the specified conditions, either the flow remains all the time unsaturated, if the saturation does not occur first at the surface, or there exists only one saturated subset and only one unsaturated, separated by the free boundary s. Indeed, if $\theta(x_1,x_2,0,t) < \theta_s$, for all t, then $\theta(x_1,x_2,x_3,t) < \theta_s$ a.e. $x\in\Omega$.

If there exists $t_s > 0$ and (x_1^s, x_2^s) such that $\theta(x_1^s,x_2^s,0,t_s) = \theta_s$, then by the monotonicity of θ we have that $\theta(x_1^s,x_2^s,x_3,t_s) \le \theta_s$, $\forall x_3 \ge 0$. The equality may take place for $x_3 \in [s_u(x_1,x_2), x_3^s]$ while for $x_3 \in (x_3^s, s_L(x_1,x_2)]$ we have the strict inequality, where $x_3^s \in [s_u(x_1,x_2), s_L(x_1,x_2)]$.

Then, the proof of the corollary is immediate by defining

$$s(x_1,x_2,t) = \sup\{x_3; \theta(x_1,x_2,x_3,t) = \theta_s\} \tag{5.90}$$

or $s(x_1,x_2,t) = \inf\{x_3; \theta(x_1,x_2,x_3,t) < \theta_s\}$. ∎

5.6 Uniqueness of the weak solution

On the one hand, we have proved that the weak solution is the solution in some generalized sense of the model with two separated domains, if they exist. On the other hand we have shown that indeed, there exist such kind of situations which evolve under certain conditions. So, we are ready now to prove the uniqueness of the weak solution in the case of a well separation of the saturated and unsaturated flow domains. We have necessarily to work with a continuous weak solution h because this assumption implies that the separated domains are open and we recall that this can be obtained under the hypotheses of Corollary 4.5, for $N = 1$.

Theorem 6.1. *Let us assume the hypotheses of Theorem* 5.4. *Then, problem* (4.2)-(4.5) *has at the most one continuous weak solution.*

Proof. Under the previous hypotheses, problem (4.2)-(4.5) has a solution h as given by Theorem 4.2 and the saturated and unsaturated domains are separated according to Theorem 5.4. Moreover, if we assume that h is continuous, the sets Q^+ and Q^- are open. The weak and continuous solution h is the solution in the generalized sense to the problem (4.18)-(4.24) with two separated connected domains, as proved in Proposition 4.6. Thus, in the saturated domain Q^+ the model is described by

$$\begin{aligned} -K_s \Delta h &= f \quad \text{in} \quad Q^+, \\ h(x,0) &= h_0(x) \quad \text{in} \quad \Omega^+, \\ q \cdot \nu &= u(x,t) \quad \text{on} \quad \Sigma_u, \\ q \cdot \nu &= \alpha K^*(h) + f_0 \quad \text{on} \quad \Sigma_{lat}^+, \\ h(x,t) &= 0 \quad \text{on} \quad Q_0. \end{aligned}$$

This problem has a unique solution $h^+ \in L^2(0,T;H^1(\Omega^+))$. In the unsaturated domain Q^- the model consists in

$$\begin{aligned} C(h)\frac{\partial h}{\partial t} - \Delta K^*(h) + \frac{\partial k(h)}{\partial x_3} &= f \quad \text{in} \quad Q^-, \\ h(x,0) &= h_0(x) \quad \text{in} \quad \Omega^-, \\ (k(h)i_3 - \nabla K^*(h)) \cdot \nu &= \big(k(h^+)i_3 - \nabla K^*(h^+)\big) \cdot \nu \quad \text{on} \quad Q_0, \\ (k(h)i_3 - \nabla K^*(h)) \cdot \nu &= \alpha(x)K^*(h) + f_0(x,t) \quad \text{on} \quad \Sigma_{lat}^- \cup \Sigma_b, \end{aligned}$$

where $h^+(x,t)$ is the solution obtained in the saturated domain Q^+. As previously seen, this problem has, by Theorem 4.2, a unique solution $h^- \in L^2(0,T;H^1(\Omega^-))$, with $C(h^-)\dfrac{dh^-}{dt} \in L^2(0,T;V')$. Since h is continuous, in particular it is continuous on Q_0. Therefore, $h^- = h^+ = 0$ on Q_0 and so the weak solution is unique in Q. ∎

Remark 6.2. The continuity assumption for h may be considered a little forced, but as generally happens, the uniqueness in nonlinear problems can be proved under stronger conditions. However, this is not unrealistic, because, as we have seen, the global continuity of h with respect to the space and time variables can be proved, under certain hypotheses, in the 1-D case (see Corollary 4.5). Also, even in the 3-D case, when $h \in W_2^{2,1}(Q)$, it is continuous on each component apart, hence if we fix x_1 and x_2, then h follows to be continuous with respect to x_3 and t, meaning that it is continuous also when it crosses the boundary Q_0.

Example. The 1-D case

A better insight can be reached in one dimension, see Fig. 5.6. In this case we denote $z = x_3$, $\Omega = (0, L)$, $\Gamma_u = \{z;\ z = 0\}$, $\Gamma_\alpha = \{z;\ z = L\}$, $\alpha(L) = \alpha$, $f_0(x,t) = f_0(t)$.

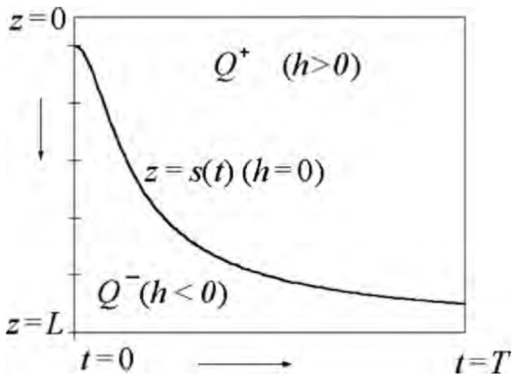

Fig. 5.6. Advance of the free boundary in the 1-D case

The system (4.2)-(4.5) reads

$$\begin{aligned} C(h)h_t - (k(h)h_z)_z + (k(h))_z &= f \text{ in } Q^- = \{z;\ s(t) < z < L\}, \\ -K_s h_{zz} &= f \text{ in } Q^+ = \{z;\ 0 < z < s(t)\}, \\ h(z,0) &= h_0(z) \text{ in } (0,L), \\ q^+(s(t),t) &= q^-(s(t),t),\ h^+(s(t),t) = h^-(s(t),t), \\ K_s - K_s h_z(0,t) &= -u(t) \ := u_R(t), \\ k(h(L,t)) - k(h(L,t))h_z(L,t) &= \alpha K^*(h(L,t)) + f_0(t). \end{aligned}$$

Solving this problem we determine the free boundary $z = s(t)$ from the equation $h(z,t) = 0$. The subscript "z" means the partial derivative with respect to z.

In the 1-D case the sets $\{(z,t);\ h(z,t)<0\}$ and $\{(z,t);\ h(z,t)>0\}$ are open and it follows that any solution h satisfies the equations

$$\begin{cases} h_{zz} = -\dfrac{f}{K_s}, & 0 < z < s(t), \\ -h_z(0,t) = \dfrac{u_R(t) - K_s}{K_s}, & 0 \le t \le T, \\ h(s(t),t) = 0. \end{cases}$$

Hence

$$h_z(z,t) = \frac{K_s - u_R(t)}{K_s} - \frac{1}{K_s}\int_0^z f(\xi,t)d\xi$$

and finally

$$h(z,t) = \frac{u_R(t) - K_s}{K_s}(s(t) - z) + \frac{1}{K_s}\int_z^{s(t)}\int_0^{\zeta} f(\xi,t)d\xi d\zeta,\ 0 < z < s(t).$$

Since $s(t)$ is defined by $\theta(s(t),t) = \theta_s$ and θ is unique, it follows that h is uniquely defined on $0 < z < s(t)$, i.e., in $\{(z,t);\ h(z,t) > 0\}$.

Separately we have

$$(C^*(h))_t - \Delta K^*(h) + (K(h))_z = f \text{ in } Q^-,$$

with flux boundary conditions on $\{z;\ z = L\} \cup \{z;\ z = s(t)\}$. Equivalently

$$\theta_t - \Delta\beta^*(h) + (K(h))_z = f \text{ in } \{\theta < \theta_s\}.$$

Since $C^*(h)$ is uniquely defined, so h is too.

5.7 Comments

Generally, the discussion upon the reliability of the mathematical assumptions made at the end of Chap. 4 applies for the saturated-unsaturated model, where the existence of a unique solution continuously dependent on data and situated within the physical accepted domain has been proved.

Mathematical and physical aspects

We have to remark that this model is in perfect agreement with the physical process from the point of view of saturation modelling. The introduction of the multivalued operator A, by the completion of the function β^* up to a maximal monotone operator was absolutely necessary to enhance a rigorous existence theory and permitted to obtain the solution within the interval $[0,\theta_s]$. This gave the possibility of surprising in the model the formation of the saturated domains.

We could not approach here the Cauchy problem directly, but using an auxiliary approximating problem and this is also necessary even though the boundary conditions are of other types, as for instance of homogeneous Dirichlet type. So, in comparison with the quasi-unsaturated case, we did not prove the m-accretiveness of the operator A, but that of the approximating operator A_ε, and obtained the solution as a limit of strong solutions to the approximating problems. The solution follows to be smooth, even if the initial data are less regular, which in terms of infiltration corresponds to the situation in which the porous medium contains initially some fully saturated domains.

Since this model reflects the saturation occurrence, a special part was devoted to the study of the separation of the flow into two well delimited connected domains, the saturated and the unsaturated ones. This may happen when the data satisfy certain conditions and in particular it was proved that their splitting according to a certain pattern correspond to the experimental evidence observed under the same conditions. In general, one might not expect to get a smooth free surface, especially this in not the case in practical situations. One may speculate that the region that separates the saturated and unsaturated zones has a more complex geometric shape, more appropriate to a fractal surface and the proof of the free boundary true shape remains an open problem.

Concerning the correlation with the mass conservation law, we specify that since the original model was deduced from it, its solution should preserve the same law. For example, let us resume (1.22)

$$\int_\Omega \theta(x,T)\phi(x,T)dx - \int_Q \theta \frac{d\phi}{dt} dxdt + \int_Q \left(\nabla\eta \cdot \nabla\phi - K(\theta)\frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$= \int_\Omega \theta_0(x)\phi(x,0)dx + \int_Q f\phi dxdt - \int_{\Sigma_\alpha} (\alpha\eta + f_0)\phi d\sigma dt - \int_{\Sigma_u} u\phi d\sigma dt,$$
$$\forall \phi \in L^2(0,T;V), \text{with } \frac{d\phi}{dt} \in L^2(0,T;L^2(\Omega)),\ \ \eta \in \beta^*(\theta) \text{ a.e. on } Q,$$

and write it for a time independent function ϕ that satisfies the problem

$$-\Delta\phi = c_0 > 0 \text{ in } \Omega, \quad \phi|_\Gamma = c_1 > 0.$$

Consider, for simplicity that $f_0 \equiv 0$, $f \equiv 0$ and $u = \text{constant} < 0$ (due to the downward direction of Ox_3). Hence $-u = u_R > 0$ (see (2.17) in Chap. 1). We have

$$\int_\Omega \theta(x,T)\phi(x,T)dx + \alpha c_1 \int_{\Sigma_\alpha} \eta d\sigma dt + \int_Q \left(\eta \cdot c_0 - K(\theta)\frac{\partial\phi}{\partial x_3} \right) dxdt$$
$$= \int_\Omega \theta_0(x)\phi(x,0)dx + c_1 u_R \text{meas}(\Sigma_u)T,\ \ \eta \in \beta^*(\theta) \text{ a.e. on } Q.$$

Assume now that the time T in which the soil receives the water supply grows indefinitely large. Here, $\eta \in \beta^*(\theta)$ is finite, this being the difference as against the quasi-unsaturated case. However, if the whole flow domain becomes saturated, $\theta = \theta_s$ remains still a solution and $\beta^*(\theta)$ is an arbitrary value greater than K_s^*. We can deduce that

$$\lim_{T\to\infty} \frac{1}{T}\left(c_0 \int_0^T \int_\Omega \beta^*(\theta)dxdt + \alpha c_1 \int_{\Sigma_\alpha} \beta^*(\theta)d\sigma dt \right) = c_1 u_R \text{meas}(\Sigma_u)$$

because the other terms vanish. This relation represents a conservation law for the average water inflow due to the rain u_R, that is transformed in the average diffused water mass over Ω and through the boundary Γ_α within the time T.

Model with a strongly nonlinear hydraulic conductivity

The strongly nonlinear saturated-unsaturated model with a strongly nonlinear hydraulic conductivity, i.e., Model 1.1. in Sect. 2.2, where the derivative $K'(\theta)$ tends to infinity at $\theta = \theta_s$ is treated in the same way as was done in Sect. 4.4, by replacing $K(\theta)$ with a smooth approximation, for example with (5.1). The difference is that $K(\theta)$ is no longer in $L^2(0,T;V)$, belonging only to $L^2(0,T;L^2(\Omega))$. All the results remain true, except for Theorem 3.6 and Corollary 4.5, in which $\theta \in L^2(0,T;H^2(\Omega))$ and $h \in L^2(0,T;H^2(\Omega))$ can not be obtained.

Weakly nonlinear saturated-unsaturated model

We refer now to the weakly saturated-unsaturated models (Model 1.3 and 1.4 in Sect. 2.3) for which $\beta(\theta_s)$ is finite being implied by the assumption $C(0) = C_0 > 0$, but the function $\beta^*(\theta)$ is multivalued at $\theta = \theta_s$. All the results concerning the existence, uniqueness, regularity and other properties of the problem in the diffusive form for θ remain in general true (with the appropriate exceptions for the case when $K'(\theta_s) = +\infty$).

A problem occurs however when studying the properties of the weak solution in the pressure form. Its existence follows as shown in Theorem 4.2, but the comparison result cannot be proved directly from that related to θ, because here the function $C(h)$ is not continuous at $h = 0$. As a matter of fact, we have to consider it as a multivalued function at this point,

$$C(h) = \begin{cases} (C^*)'(h), & h < 0 \\ C_0, & h = 0 \\ 0, & h > 0. \end{cases}$$

A comparison result should follow an approximating procedure, by replacing $C(h)$ by an approximating continuous function C_ε, having for each $\varepsilon > 0$ the expression

$$C_\varepsilon(h) := \begin{cases} C(h), & h < 0 \\ -\dfrac{C_0 - \varepsilon}{\varepsilon}(x - \varepsilon) + \varepsilon, & 0 \le h \le \varepsilon \\ \varepsilon, & h > \varepsilon. \end{cases}$$

This turns out in considering the approximating problem in θ for which the approximating solution θ_ε satisfies the conclusions of Theorem 3.1, including also the strongly convergence in $L^2(Q)$. Therefore, $\theta_\varepsilon \to \theta$ a.e. on Q and since $\theta \to (C^*)^{-1}(\theta)$ is a continuous function we obtain that the sequence $h_\varepsilon = (C^*)^{-1}(\theta_\varepsilon) \to (C^*)^{-1}(\theta) = h$ a.e. on Q. The strongly convergence cannot be proved, but for the unsaturated domain, where $(C^*)^{-1}(\theta_\varepsilon)$ follows to be bounded if $h_M(t)$ in Corollary 4.4 is taken ≤ 0. We let the reader to prove these details.

Bibliographical note

Most part of the results presented in this chapter have been announced in [87] and [88], where the saturated-unsaturated flow was modelled first time by a multivalued function.

Previous existence and uniqueness studies for solutions to the elliptic-parabolic equation

$$\frac{\partial(b(u))}{\partial t} + \nabla \cdot (a(\nabla u, b(u))) + f(b(u)) = 0 \text{ in } \Omega \times (0, T)$$

have been presented in the literature especially using a technique inspired by the method of entropy solutions introduced by S.N. Krushkov in [79]. Originally this method was devoted to prove L^1-contraction for entropy solutions for scalar conservation laws, i.e., generalized solutions in the sense of distributions satisfying admissibility conditions similar to those of entropy growth in gas dynamics (see also [25]). J. Carillo was probably the first to have applied Krushkov's method to second order equations (see [41], [42], [43], [44]). F. Otto (see [99]) proved a L^1-contraction principle and uniqueness of solutions for this type of equation by applying Krushkov's technique only to the time variable. H.W. Alt and S. Luckhaus showed in [1] that the natural solution space for this equation is given by all functions u of finite energy in the sense that

$$\sup_{t \in (0,T)} \int_\Omega \Psi(b(u(t)))dx + \int_Q |\nabla u|^r < \infty,$$

where Ψ is the Legendre transform of the primitive of b. In our terms this may be equivalent with (3.7). Some other aspects of the behaviour of diffusion problems with nonlinear terms are studied in [94], [114] and in the references given there.

More recently, in the paper [27] a model of the saturated-unsaturated flow lying on a special definition of the boundary conditions that changes during the phenomenon evolution, has been developed for a finite value of the diffusivity at saturation (which was implied by the assumption that $C(0) > 0$). Following the technique presented in [59] the model was reduced to systems in class of Stefan-like problems of high-order, see [58].

For other results on the modelling of infiltration problems we refer the reader to the works cited here in the chronological order: [8], [116], [58], [59], [67], [117], [2], [40], [66], [100], [64].

6

Specific problems in infiltration

Infiltration processes involve specific aspects which reveal problems of a certain mathematical interest. We have chosen some of them to present in this chapter. The first one is related to the possible degeneracy of the diffusion equation, which is also a specific feature of parabolic equations. In infiltration problems Richards' equation can become degenerate, this being a result of two different physical reasons related to flow in porous media, that may lead to the vanishing of either the diffusivity or the time derivative term.

The other specific particularity of infiltration problems which will be discussed in the second part of this chapter, is related to the hysteresis phenomenon that evolves during a combined infiltration-drainage process.

6.1 Analysis of the diffusivity-degenerate model

Up to now we have considered that the diffusivity function, denoted either by D in the quasi-unsaturated case, or by β in the saturated-unsaturated models is positive and greater than ρ. We are interested in what happens if this function vanishes at some points. Let us consider with no loss of generality that the diffusivity vanishes at only one point and let it be zero. In Sect. 2.5 we imagined a model in which such a situation can be met. If we consider that infiltration takes place along $h < 0$, the dimensional hydraulic functions are defined, in terms of θ, on the interval $(0, \theta_s)$ instead of (θ_r, θ_s), with $\lim_{\theta \searrow 0} D(\theta) = 0$ and $\lim_{\theta \searrow 0} K(\theta) = 0$, so that a limit situation appears. We have specified that this case of degeneracy can be associated with any model introduced before and we intend to study it here under the quasi-unsaturated one. Consequently, it will be a combination of two limit models, and this turns out in studying the problem by taking into account the particularities of both of them.

To illustrate this model we choose a problem with homogeneous Dirichlet boundary conditions and for simplicity we shall treat the horizontal infiltration case in which the gravitational term is absent. In this problem the water initially existent in the soil or supplied by a possible source infiltrates due to the action of diffusivity only, situation that can be met in a one or two-dimensional medium horizontally laid, or in a three-dimensional medium in which for some reasons the gravitational influence is not taken into account.

The boundary value problem we study is then

$$\begin{aligned} \frac{\partial \theta}{\partial t} - \Delta D^*(\theta) &= f \quad \text{in } Q, \\ \theta(x,0) &= \theta_0 \quad \text{in } \Omega, \\ D^*(\theta) &= 0 \quad \text{on } \Sigma, \end{aligned} \tag{1.1}$$

where we consider the boundary condition in a form a little different from those used up to now, even if it may be implied by the homogeneous Dirichlet condition $\theta = 0$ on Σ. For $f = 0$, this model characterizes the relaxation of the initial condition in a soil with a dry boundary. The solution to this problem follows almost like in Sect. 4.3, but the choice of presenting it is due to the fact that the mathematical results and their interpretation differ in some parts from that one.

Under the quasi-unsaturated model we have that D and D^* blow up at $\theta = \theta_s$, while the degeneracy particularity implies that $\lim\limits_{\theta \searrow 0} D(\theta) = 0$. This does no longer allow the extension of the function D by its value at 0, as we specified in Sect. 2.6, so D should be extended by a function of the form

$$D(r) := \begin{cases} D_{ext}(r), & r < 0 \\ D(r), & 0 \le r < \theta_s, \end{cases} \tag{1.2}$$

where $D_{ext} : (-\infty, 0] \to [0, \rho_0)$ is a non-negative continuous function, monotonically decreasing and

$$\lim_{r \to -\infty} D_{ext}(r) = \rho_0 > 0. \tag{1.3}$$

In fact we consider here D to be a positive function except for only one point $(r = 0)$ at which it vanishes, see Fig. 6.1.

The function D such defined is continuous and still satisfies:

($\mathrm{i_{qud}}$) $\lim\limits_{r \nearrow \theta_s} D(r) = +\infty$, $D(r) \ge 0$, $\forall r \in (-\infty, \theta_s)$ and $\int_0^{\theta_s} D(\xi) d\xi = +\infty$.

According to ($\mathrm{i_{qud}}$) the function

$$D^*(r) := \int_0^r D(\xi) d\xi, \; r \in (-\infty, \theta_s) \tag{1.4}$$

follows to be differentiable and monotonically increasing on $(-\infty, \theta_s)$ and satisfies

($\mathrm{i_0}$) $(D^*(r_1) - D^*(r_2))(r_1 - r_2) \ge 0$, $\forall r_1, r_2 \in (-\infty, \theta_s)$;
(ii) $\lim\limits_{r \to -\infty} D^*(r) = -\infty$;
(iii) $\lim\limits_{r \nearrow \theta_s} D^*(r) = +\infty$.

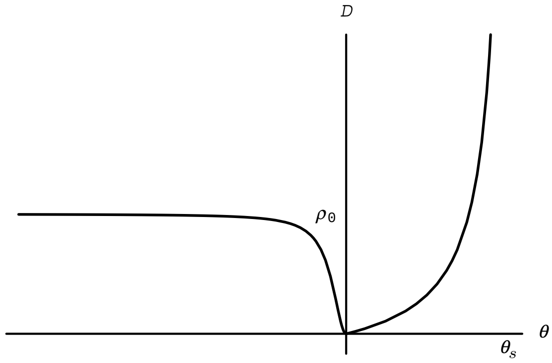

Fig. 6.1. Graphic of the extended function D in the diffusivity-degenerate case

Functional framework

We consider the space $V = H_0^1(\Omega)$ with the usual norm and its dual V' endowed with the norm (3.3)-(3.4) defined in Sect. 4.3. We introduce the operator $A_D : D(A_D) \subset V' \to V'$, defined by

$$\langle A_D\theta, \psi\rangle_{V',V} = \int_\Omega \nabla D^*(\theta) \cdot \nabla\psi dx, \ \forall \psi \in V, \tag{1.5}$$

with the domain

$$\begin{aligned} D(A_D) &= \{\theta \in H^{-1}(\Omega) \cap L^1(\Omega); \ D^*(\theta) \in H_0^1(\Omega)\} \\ &= \{\theta \in L^2(\Omega); \ D^*(\theta) \in H_0^1(\Omega)\}, \end{aligned}$$

(see (2.10)-(2.12) in Sect. 4.2).

Definition 1.1. If

$$\theta_0 \in L^2(\Omega), \ \theta_0 < \theta_s, \ \text{a.e. } x \in \Omega \text{ and } f \in L^2(0,T;V')$$

we mean by *solution* to (1.1) a function $\theta \in C([0,T];L^2(\Omega))$, such that

$$\frac{d\theta}{dt} \in L^2(0,T;V'), \ D^*(\theta) \in L^2(0,T;V), \ \theta(x,0) = \theta_0 \text{ in } \Omega \text{ and}$$

$$\left\langle \frac{d\theta}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega \nabla D^*(\theta(t)) \cdot \nabla\psi dx = \langle f(t), \psi\rangle_{V',V}, \quad \text{a.e. } t\in(0,T), \ \forall\psi\in V. \tag{1.6}$$

Within the stated functional framework we shall study the Cauchy problem

$$\begin{aligned} \frac{d\theta}{dt} + A_D\theta &= f, \ \text{a.e. } t \in (0,T), \\ \theta(0) &= \theta_0. \end{aligned} \tag{1.7}$$

In an equivalent form (1.7) can be rewritten as

$$\int_{\Omega} \left(\frac{\partial \theta}{\partial t}\psi + \nabla D^*(\theta) \cdot \nabla \psi \right) dx = \int_{\Omega} f \psi dx, \text{ a.e. } t \in (0,T), \ \forall \psi \in V, \quad (1.8)$$

with the initial datum $\theta(0) = \theta_0$.

It is readily seen that if θ is a strong solution to (1.7), or equivalently to (1.8), then it satisfies the diffusion equation in (1.1) in the sense of distributions. The boundary condition is verified in the sense of the trace, being implied by the choice of the space V.

We recall that this operator is m-accretive on V', as we proved in Corollary 2.2, in Sect. 4.2.

Theorem 1.2. *Assume* ($\mathrm{i_{qud}}$), ($\mathrm{i_0}$), (ii), (iii), *and let*

$$f \in W^{1,1}(0,T;V'), \quad (1.9)$$

$$\theta_0 \in D(A_D) = \{\theta \in H^{-1}(\Omega) \cap L^1(\Omega); \ D^*(\theta) \in V\}. \quad (1.10)$$

Then there exists a unique strong solution $\theta \in C([0,T],V')$ *to problem* (1.1) *such that*

$$\theta \in W^{1,\infty}(0,T;V') \cap L^{\infty}(0,T;D(A_D)), \quad (1.11)$$

$$D^*(\theta) \in L^{\infty}(0,T;V), \quad (1.12)$$

and $\theta < \theta_s$ *a.e.* $(x,t) \in Q$.

Proof. The conclusion (1.11) follows by applying directly Theorem 3.6 in Sect. 3.3), because A_D is an m-accretive operator on the space $V' = H^{-1}(\Omega)$. Equation (1.7) implies also that $A_D\theta = -\Delta D^*(\theta) \in L^{\infty}(0,T;V')$. Hence $D^*(\theta) \in L^{\infty}(0,T;V)$ because $-\Delta$ is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

However, in this case the operator is only monotone and not strongly monotone like in the non-degenerate diffusivity situation. Hereby we cannot prove that the solution belongs to $L^{\infty}(0,T;V)$, because the inverse of $D^*(\theta)$ is no longer Lipschitz. ■

Corollary 1.3. *Let* $\theta_0 \in \overline{D(A_D)}$ *and* $f \in L^1(0,T;V')$. *Then there exists* θ *a unique mild solution to problem* (1.7)

$$\theta(x,t) = \lim_{\varepsilon \to 0} \theta_{\varepsilon}(x,t) \text{ in } L^2(Q),$$

where $\theta_{\varepsilon}(x,t) = \theta_i(x) \in L^2(\Omega)$ *for* $t \in ((i-1)\varepsilon, i\varepsilon)$ *is the solution to the elliptic problem*

$$\frac{\theta_i - \theta_{i-1}}{\varepsilon} - \Delta D^*(\theta_i) = f_i(x), \ f_i(x) := \frac{1}{\varepsilon}\int_{(i-1)\varepsilon}^{i\varepsilon} f(x,t)dt, D^*(\theta_i) = 0 \textit{ on } \Gamma.$$

The proof follows from Theorem 3.12, in Sect. 3.3 and this can also be viewed as a convergence result for a difference scheme corresponding to problem (1.7).

But, as will be shown in the next result this mild solution is in fact a strong solution, because the operator A_D is a potential operator, $A_D\theta = \partial\varphi(\theta)$, for any $\theta \in D(A_D)$, with φ defined by (2.15) in Sect. 4.2.

Theorem 1.4. *Let*

$$\theta_0 \in \overline{D(A_D)},\ \ f \in L^2(0,T;V'). \tag{1.13}$$

Then there exists a unique strong solution $\theta \in C([0,T],V')$ *to problem* (1.7) *such that*

$$\theta \in W^{1,2}(\delta,T;V') \text{ for every } 0 < \delta < T, \tag{1.14}$$

$$\theta(t) \in D(A_D) \text{ a.e. } t \in (0,T), \tag{1.15}$$

$$\sqrt{t}\frac{d\theta}{dt} \in L^2(0,T;V') \text{ and } \sqrt{t}D^*(\theta) \in L^2(0,T;V). \tag{1.16}$$

If $\theta_0 \in D(\varphi)$ *then*

$$\theta \in W^{1,2}(0,T;V'), \tag{1.17}$$

$$D^*(\theta) \in L^2(0,T;V),\ \ A_D\theta \in L^2(0,T;V'). \tag{1.18}$$

All conclusions follow from Theorem 3.14, Sect. 3.3. Moreover, if $\theta_0 \in \overline{D(A_D)}$, we obtain from the same theorem that $\varphi(\theta) \in L^1(0,T)$ and if $\theta_0 \in D(\varphi)$ then $\varphi(\theta) \in L^\infty(0,T)$.

Remark 1.5. The previous results remain true for the complete model of infiltration, including the contribution due to the gravitational field, with the term $\frac{\partial K(\theta)}{\partial x_3}$, because we can apply Corollary 4.16 in Appendix. The abstract Cauchy problem can be reformulated as

$$\frac{d\theta}{dt} + A_D\theta + P\theta = f, \text{ a.e. } t \in (0,T),$$
$$\theta(0) = \theta_0,$$

where $P : V \to V'$ is the perturbation operator

$$\langle P\theta, \psi\rangle_{V',V} = -\int_\Omega K(\theta)\frac{\partial\psi}{\partial x_3}dx,\ \forall\psi \in V,$$

which is Lipschitz.

6.2 Analysis of the porosity-degenerate model

We shall discuss now the porosity-degenerate model, introduced in Sect. 2.5, by equation (5.1). This model describes the infiltration in a heterogeneous porous medium, in which porosity depends only on the position. In infiltration problems, we can often meet the situation in which water ponds at the soil surface, Γ_u. This happens when the rainfall rate is greater than the soil conductivity at saturation and the soil begins to saturate from the surface, or when the soil surface is in contact with an open water body, for example the bottom of a lake. Also, in some situations, the underground boundary, Γ_α, may be impermeable, meaning that the water flux across it is zero. We propose ourselves to study the equation with a variable porosity associated with boundary conditions corresponding to such a situation, under the strongly nonlinear saturated-unsaturated case with a weakly nonlinear conductivity.

In fact we intend to treat a little more general mathematical problem, in which we assume that the function $\widetilde{K}$ depends on the space variables, as well as on the solution, and we shall refer at the end to the connection that the theoretical results have with the real physical problem. Therefore the model is of the form

$$m(x)\frac{\partial S_w}{\partial t} - \Delta\widetilde{\beta}^*(S_w) + \frac{\partial \widetilde{K}(x,S_w)}{\partial x_3} = f \qquad \text{in } Q, \tag{2.1}$$

$$m(x)S_w(x,0) = \theta_0(x) \text{ in } \Omega, \tag{2.2}$$

$$S_w(x,t) = S_s \qquad \text{on } \Sigma_u, \tag{2.3}$$

$$\left(\widetilde{K}(x,S_w)i_3 - \nabla\widetilde{\beta}^*(S_w)\right)\cdot\nu = 0 \qquad \text{on } \Sigma_\alpha. \tag{2.4}$$

Here, m is the space-dependent porosity, S_w is the water saturation which is maintained at its value at saturation, denoted S_s, on the upper boundary (in dimensionless variables S_s is equal to 1). At the points where m vanishes, the equation degenerates. The function m is supposed to be essentially bounded, $m \in L^\infty(\Omega)$ and $0 \leq m(x) < 1$ a.e. $x \in \Omega$. However, we shall see that this assumption is not sufficient to get the solution existence in the standard spaces we work with, and a stronger hypothesis upon m is required. The functions $\widetilde{K}$ and $\widetilde{\beta}^*$ depend nonlinearly on S_w and have the properties corresponding to the strongly nonlinear saturated-unsaturated case with a weakly nonlinear conductivity, specified in Chap. 5, by (1.5), (1.6), (i_β)-(iii_β), (i)-(iii) and (i_K). Thus, $\widetilde{\beta}^*$ is multivalued at $r = S_s$, has the derivative $\widetilde{\beta}$, for $r < S_s$ and this blows up at $r = S_s$,

$$\widetilde{\beta}(r) \geq \widetilde{\rho},\ \forall r < S_s,\ \lim_{r\nearrow S_s}\widetilde{\beta}(r) = +\infty, \tag{2.5}$$

$$\widetilde{\beta}^*(r) := \begin{cases} \widetilde{\rho}r, & r < 0 \\ \displaystyle\int_0^r \widetilde{\beta}(\xi)d\xi, & 0 \leq r < S_s \\ [\widetilde{K}_s^*, +\infty), & r = S_s, \end{cases} \tag{2.6}$$

$$\widetilde{K}_s^* := \lim_{r\nearrow S_s}\widetilde{\beta}^*(r), \quad \lim_{r\to-\infty}\widetilde{\beta}^*(r) = -\infty.$$

In what concerns $\widetilde{K}$ we assume that it has the form

$$\widetilde{K}(x,r) := \begin{cases} \widetilde{K}_0(x) & \text{on } \{(x,t);\ m(x)=0\} \\ \widetilde{K}_m(r) & \text{on } \{(x,t);\ m(x)>0\}, \ \text{if } r\in(0,S_s] \\ 0 & \text{if } r\le 0, \end{cases} \tag{2.7}$$

where $\widetilde{K}\in L^\infty(\Omega)\times L^\infty((-\infty,S_s])$, $\widetilde{K}(x,r)\in[0,\widetilde{K}_s]$, a.e. $(x,r)\in\Omega\times(-\infty,S_s]$, $\widetilde{K}_s := \widetilde{K}(x,S_s)$ and $\widetilde{K}$ is Lipschitz with respect to r, uniformly with respect to x, i.e., there exists $M>0$, such that

(i$_K$) $\left|\widetilde{K}(x,r)-\widetilde{K}(x,\overline{r})\right| \le M\,|r-\overline{r}|\,,\ \forall r,\,\overline{r}\le S_s.$

Functional framework

We perform a function replacement by denoting

$$w := S_w - S_s, \tag{2.8}$$

so that we are led to the system

$$m(x)\frac{\partial w}{\partial t} - \Delta\widetilde{\beta}^*(w+S_s) + \frac{\partial \widetilde{K}(x,w+S_s)}{\partial x_3} = f \qquad \text{in } Q, \tag{2.9}$$

$$m(x)w(x,0) = v_0(x) \text{ in } \Omega, \tag{2.10}$$

$$w(x,t) = 0 \qquad \text{on } \Sigma_u, \tag{2.11}$$

$$\left(\widetilde{K}(x,w+S_s)i_3 - \nabla\widetilde{\beta}^*(w+S_s)\right)\cdot\nu = 0 \qquad \text{on } \Sigma_\alpha. \tag{2.12}$$

We denoted $v_0(x) := \theta_0(x) - m(x)S_s$. We shall indicate the value of w at saturation by w_s. Actually, by (2.8) it is equal to zero, but we shall keep the notation w_s in order to put into evidence the behaviour of the solution at this point).

We consider the spaces

$$V = \{w\in H^1(\Omega);\ w=0 \text{ on } \Gamma_u\}, \tag{2.13}$$

with the norm

$$\|\psi\|_V = \left(\int_\Omega |\nabla\psi|^2\,dx\right)^{1/2}, \tag{2.14}$$

and its dual V' on which we introduce the scalar product by

$$\langle w,\overline{w}\rangle_{V'} := \langle w,\psi\rangle_{V',V}\,,$$

where ψ is the solution to the boundary value problem

$$-\Delta\psi = \overline{w},\ \psi = 0 \text{ on } \Gamma_u,\ \nabla\psi\cdot\nu = 0 \text{ on } \Gamma_\alpha. \tag{2.15}$$

Definition 2.1. Let

$$m\in C^1(\overline{\Omega}),\ f\in L^2(0,T;V'),\ \frac{v_0}{m}\in L^2(\Omega),\ \frac{v_0}{m}\le w_s,\ \text{a.e. } x\in\Omega. \tag{2.16}$$

We call *solution* to (2.9)-(2.12) a function w that satisfies

$$\begin{aligned} w &\in L^2(0,T;V), \\ mw &\in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V'), \\ w &\le w_s, \text{ a.e. } (x,t) \in Q, \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\left\langle \frac{d(m(x)w)}{dt}(t), \psi \right\rangle_{V',V} + \int_\Omega \left(\nabla\widetilde{\zeta}(t)\cdot\nabla\psi - \widetilde{K}(x, w(t)+S_s)\frac{\partial\psi}{\partial x_3} \right) dx \\ &\qquad = \langle f(t), \psi\rangle_{V',V}, \text{ a.e. } t \in (0,T), \ \forall \psi \in V, \end{aligned} \tag{2.18}$$

where $\widetilde{\zeta}(x,t) \in \widetilde{\beta}^*(w(x,t)+S_s)$ a.e. on Q, $\widetilde{\zeta} \in L^2(0,T;H^1(\Omega))$, and

$$m(x)w(0) = v_0.$$

Equation (2.18) can be written also in the equivalent form

$$\begin{aligned} &\int_Q \frac{d(mw)}{dt}\phi dxdt + \int_Q \left(\nabla\widetilde{\zeta}\cdot\nabla\phi - \widetilde{K}(x, w+S_s)\frac{\partial\phi}{\partial x_3} \right) dxdt \\ &= \int_Q f\phi dxdt, \ \forall\phi \in L^2(0,T;V), \ \widetilde{\zeta}(x,t) \in \widetilde{\beta}^*(w(x,t)+S_s) \text{ a.e. on } Q. \end{aligned} \tag{2.19}$$

Using similar arguments to those presented in Sect. 5.1, we see that a solution (2.17)-(2.18) is a solution in the sense of distributions to (2.9)-(2.12), with the boundary conditions satisfied in the sense of the trace theory. We set

$$D(A) := \{ w \in L^2(\Omega); \ \exists \widetilde{\eta} \in H^1(\Omega), \ \widetilde{\eta}(x) \in \widetilde{\beta}^*(w(x)+S_s) \text{ a.e. } x \in \Omega \}$$

and we introduce the multivalued operator $A : D(A) \subset V' \to V'$ by

$$\begin{aligned} \langle Aw, \psi\rangle_{V',V} \ : &= \int_\Omega \left(\nabla\widetilde{\eta}\cdot\nabla\psi - \widetilde{K}(x, w+S_s)\frac{\partial\psi}{\partial x_3} \right) dx, \\ &\forall\psi \in V, \text{ for some } \widetilde{\eta} \in \widetilde{\beta}^*(w+S_s) \text{ a.e. } x \in \Omega. \end{aligned}$$

Thus, we can write the problem

$$\begin{aligned} m(x)\frac{dw}{dt} + Aw &\ni f, \text{ a.e. } t \in (0,T) \\ m(x)w(0) &= v_0, \end{aligned} \tag{2.20}$$

whose solution is a solution to (2.9)-(2.12), in the sense of Definition 2.1.

We consider now the multiplication operator

$$M : D(A) \to L^2(\Omega), \ Mw := mw, \tag{2.21}$$

whose inverse is multivalued and denoting

$$v(x,t) = m(x)w(x,t), \tag{2.22}$$

we can rewrite (2.20) in terms of v as,

$$\frac{dv}{dt} + A_M v \ni f, \text{ a.e. } t \in (0,T) \qquad v(0) = v_0, \tag{2.23}$$

where $A_M = AM^{-1}$ (formally indicated by $w = \dfrac{v}{m}$) and

$$D(A_M) := \left\{ v \in L^2(\Omega);\ \frac{v}{m} \in L^2(\Omega),\ \exists \widetilde{\eta} \in V,\ \widetilde{\eta} \in \widetilde{\beta}^* \left(\frac{v}{m} + S_s \right),\ \text{a.e. in } \Omega \right\}.$$

We see that $v \in D(A_M)$ implies $\dfrac{v}{m} \in D(A)$. Conversely, if $w = \dfrac{v}{m} \in D(A)$, then $v = mw \in D(A_M)$.

We define

$$\widetilde{j}(r) := \begin{cases} \displaystyle\int_0^r \widetilde{\beta}^*(\xi) d\xi, & r \le S_s \\ +\infty, & r > S_s, \end{cases}$$

where the left limit of $\widetilde{\beta}^*$ at S_s is specified in (2.6). As proved in Sect. 5.3, such a function is proper, convex, l.s.c., and

$$\partial \widetilde{j}(r) = \widetilde{\beta}^*(r),\ \forall r \in D(\widetilde{\beta}^*). \tag{2.24}$$

The approximating problem

Since the operator A_M is multivalued, in order to prove the existence in (2.20) we introduce an approximating problem replacing m by

$$m_\varepsilon(x) := m(x) + \varepsilon, \text{ for } \varepsilon > 0$$

and $\widetilde{\beta}^*$ by the continuous (single-valued) function

$$\widetilde{\beta}^*_\varepsilon(r) := \begin{cases} \widetilde{\beta}^*(r), & r < S_s \\ \widetilde{K}^*_s + \dfrac{r - S_s}{\varepsilon}, & r \ge S_s. \end{cases}$$

The function $\widetilde{K}$ will be extended to the right of S_s by its constant value $\widetilde{K}_s$. Then we denote

$$\beta^*_\varepsilon(r) := \widetilde{\beta}^*_\varepsilon(r + S_s) - \widetilde{K}^*_s \tag{2.25}$$

and define

$$A_\varepsilon : D(A_\varepsilon) \subset V' \to V',$$

$$\langle A_\varepsilon u, \psi \rangle_{V',V} := \int_\Omega \left(\nabla \beta^*_\varepsilon(u) \cdot \nabla \psi - \widetilde{K}(x, u + S_s) \frac{\partial \psi}{\partial x_3} \right) dx,\ \forall \psi \in V,$$

with

$$D(A_\varepsilon) := \{u \in L^2(\Omega);\ \beta^*_\varepsilon(u) \in V\}.$$

In this way we are led to the approximating Cauchy problem

$$\begin{aligned} m_\varepsilon(x)\frac{dw}{dt} + A_\varepsilon w = f,\ \text{a.e. } t \in (0,T), \\ m_\varepsilon(x)w(0) = v_{0\varepsilon}, \end{aligned} \tag{2.26}$$

where $v_{0\varepsilon} := m_\varepsilon \dfrac{v_0}{m}$.

We recall that $v_0 \in D(A_M)$ implies $\dfrac{v_0}{m} \in D(A)$, hence $\dfrac{v_{0\varepsilon}}{m_\varepsilon} \in D(A_\varepsilon)$, since

$$\frac{v_{0\varepsilon}}{m_\varepsilon} = \frac{v_0}{m}.$$

Definition 2.2. Let $\varepsilon > 0$ and

$$f \in L^2(0,T;V'),\ v_0 \in D(A_M).$$

We call *solution* to (2.26) a function $w_\varepsilon \in L^2(0,T;V)$ that satisfies

$$\begin{aligned} m_\varepsilon w_\varepsilon \in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V'), \\ \beta^*_\varepsilon(w_\varepsilon) \in L^2(0,T;V), \end{aligned}$$

$$\begin{aligned} \left\langle \frac{d(m_\varepsilon w_\varepsilon)}{dt}(t), \psi \right\rangle_{V',V} &+ \int_\Omega \left(\nabla \beta^*_\varepsilon(w_\varepsilon(t)) \cdot \nabla\psi - \widetilde{K}(x, w_\varepsilon(t) + S_s)\frac{\partial \psi}{\partial x_3}\right) dx \\ &= \langle f(t), \psi \rangle_{V',V},\ \text{a.e. } t \in (0,T),\ \forall \psi \in V \end{aligned} \tag{2.27}$$

and

$$m_\varepsilon w_\varepsilon(0) = v_{0\varepsilon}.$$

An equivalent form to (2.27) is

$$\begin{aligned} \int_Q \frac{d(m_\varepsilon w_\varepsilon)}{dt}\phi dxdt &+ \int_Q \left(\nabla \beta^*_\varepsilon(w_\varepsilon) \cdot \nabla \phi - \widetilde{K}(x, w_\varepsilon + S_s)\frac{\partial \phi}{\partial x_3}\right) dxdt \\ &= \int_Q f\phi dxdt,\ \forall \phi \in L^2(0,T;V). \end{aligned} \tag{2.28}$$

Denoting now

$$v_\varepsilon := m_\varepsilon w_\varepsilon$$

we can write the problem in the form

$$\begin{aligned} \frac{dv_\varepsilon}{dt} + B_\varepsilon v_\varepsilon = f,\ \text{a.e. } t \in (0,T), \\ v_\varepsilon(0) = v_{0\varepsilon}. \end{aligned} \tag{2.29}$$

The operator $B_\varepsilon : D(B_\varepsilon) \subset V' \to V'$ is single-valued, has the domain

$$D(B_\varepsilon) := \left\{ u \in L^2(\Omega);\ \beta_\varepsilon^*\left(\frac{u}{m_\varepsilon}\right) \in V \right\}$$

and is given by

$$\langle B_\varepsilon u, \psi\rangle_{V',V} := \int_\Omega \left(\nabla \beta_\varepsilon^*\left(\frac{u}{m_\varepsilon}\right) \cdot \nabla\psi - \widetilde{K}\left(x, \frac{u}{m_\varepsilon} + S_s\right) \frac{\partial \psi}{\partial x_3} \right) dx,\ \forall \psi \in V.$$

In fact $B_\varepsilon u = A_\varepsilon \left(\dfrac{u}{m_\varepsilon}\right)$.

First we shall prove that (2.29) has, for each $\varepsilon > 0$, a unique solution, v_ε in appropriate functional spaces. As we have done up to now, we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\Omega)$, respectively. Moreover, we define

$$\widetilde{j}_\varepsilon(r) := \int_0^r \widetilde{\beta}_\varepsilon^*(\xi) d\xi,\ \widetilde{j}_\varepsilon : \mathbf{R} \to \mathbf{R},$$

and notice that

$$\partial \widetilde{j}_\varepsilon(r) = \widetilde{\beta}_\varepsilon^*(r),\ \forall r \in \mathbf{R}. \tag{2.30}$$

Main results

Proposition 2.3. *Let*

$$m \in C^1(\overline{\Omega}),\ 0 \le m < 1,\ f \in L^2(0,T;V'),\ v_0 \in \overline{D(A_M)}.$$

Then, the Cauchy problem (2.29) *has, for each* $\varepsilon > 0$, *a unique solution*

$$v_\varepsilon \in C([0,T];L^2(\Omega)) \cap W^{1,2}(0,T;V') \cap L^2(0,T;V) \tag{2.31}$$

$$\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \in L^2(0,T;V), \tag{2.32}$$

$$\widetilde{j}_\varepsilon\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \in L^\infty(0,T;L^1(\Omega)), \tag{2.33}$$

that satisfies the estimates

$$\begin{aligned} &\int_\Omega m_\varepsilon \widetilde{j}_\varepsilon\left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s\right) dx + \int_0^t \left\| \frac{dv_\varepsilon}{d\tau}(\tau)\right\|_{V'}^2 d\tau \\ &+ \int_0^t \left\| \beta_\varepsilon^*\left(\frac{v_\varepsilon(\tau)}{m_\varepsilon}\right)\right\|_V^2 d\tau \\ &\le \beta_0 \left(\int_\Omega \widetilde{j}(S_s)dx + \int_0^T \|f(t)\|_{V'}^2 dt + 1\right), \end{aligned} \tag{2.34}$$

$$\left\| \sqrt{m_\varepsilon} \left(\frac{v_\varepsilon}{m_\varepsilon}(t) + S_s \right) \right\| \le c_0, \ \forall t \in [0,T], \tag{2.35}$$

$$\|v_\varepsilon(t)\| \le c_1, \ \forall t \in [0,T]. \tag{2.36}$$

Moreover, if v_ε *and* $\overline{v}_\varepsilon$ *are two solutions corresponding to the pairs of data* f, v_0 *and* $\overline{f}$, $\overline{v}_0$, *we have the estimate*

$$\begin{aligned} &\|v_\varepsilon(t) - \overline{v}_\varepsilon(t)\|_{V'}^2 + \int_0^t \|v_\varepsilon(\tau) - \overline{v}_\varepsilon(\tau)\|^2 \, d\tau \\ &\le \alpha_0 \left(\|v_0 - \overline{v}_0\|_{V'}^2 + \int_0^T \left\| f(t) - \overline{f}(t) \right\|_{V'}^2 dt \right), \end{aligned} \tag{2.37}$$

with α_0, β_0, c_0 *and* c_1 *independent of* ε.

Proof. The proof is based on the quasi m-accretivity of the operator B_ε, which follows by some standard computations, taking also into account the results given in the previous sections (see for example Sect. 5.2). We assume first that $f \in W^{1,1}(0,T;V')$ and $v_0 \in D(A_M)$ which is equivalent to $v_{0\varepsilon} \in D(B_\varepsilon)$.

Therefore, the existence of a unique solution to (2.29) follows from the general theorems for evolution equations with m-accretive operators and $v_\varepsilon \in W^{1,\infty}(0,T;V') \cap L^\infty(0,T;D(B_\varepsilon))$. Hence $\beta_\varepsilon^* \left(\frac{v_\varepsilon}{m_\varepsilon} \right) \in L^\infty(0,T;V)$, which implies by (2.25) that $\widetilde{\beta}_\varepsilon^* \left(\frac{v_\varepsilon}{m_\varepsilon} + S_s \right) \in L^\infty(0,T;H^1(\Omega))$. Since the inverse of β_ε^* is Lipschitz we deduce that $\frac{v_\varepsilon}{m_\varepsilon} \in L^\infty(0,T;V)$.

To prove the estimate (2.34) we multiply (2.29) by $\beta_\varepsilon^* \left(\frac{v_\varepsilon}{m_\varepsilon} \right)$ and integrate over $\Omega \times (0,t)$ and proceed, for instance, like in the proof of Proposition 5.3 in Sect. 4.5. Then we multiply (2.29) scalarly in V' by $\frac{dv_\varepsilon}{dt}(t)$ and integrate over $(0,t)$. We take into account that

$$\widetilde{\beta}_\varepsilon^* \left(\frac{v_\varepsilon}{m_\varepsilon}(t) + S_s \right) \frac{dv_\varepsilon}{dt}(t) = m_\varepsilon \frac{d}{dt} \left(\widetilde{j}_\varepsilon \left(\frac{v_\varepsilon}{m_\varepsilon}(t) + S_s \right) \right),$$

and subsequent to some standard computations we get (2.34) as claimed. Since $\widetilde{j}_\varepsilon(r) \ge \frac{\widetilde{\rho}}{2} r^2$, $\forall r \in \mathbf{R}$, we have

$$\int_\Omega m_\varepsilon(x) \widetilde{j}_\varepsilon \left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s \right) dx \ge \frac{\widetilde{\rho}}{2} \int_\Omega m_\varepsilon(x) \left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s \right)^2 dx,$$

so we deduce that

$$\frac{\widetilde{\rho}}{2} \left\| \sqrt{m_\varepsilon} \left(\frac{v_\varepsilon}{m_\varepsilon}(t) + S_s \right) \right\|^2 \le \int_\Omega m_\varepsilon(x) \widetilde{j}_\varepsilon \left(\frac{v_\varepsilon(x,t)}{m_\varepsilon} + S_s \right) dx, \ \forall t \in [0,T]. \tag{2.38}$$

Next, from the relation

$$v_\varepsilon(t) = \sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon}(t)\sqrt{m_\varepsilon} \tag{2.39}$$

using that $m_\varepsilon(x) \leq 1+\varepsilon$, for $x \in \Omega$, we get that

$$\|v_\varepsilon(t)\|^2 = \int_\Omega \left(\sqrt{m_\varepsilon(x)}\frac{v_\varepsilon(t)}{m_\varepsilon}\right)^2 m_\varepsilon(x)dx \leq 2\left\|\sqrt{m_\varepsilon}\left(\frac{v_\varepsilon}{m_\varepsilon}(t)+S_s\right)\right\|^2.$$

To show the estimate (2.37) we write two equations (2.29) corresponding to different pairs of data, subtract them, multiply the difference scalarly in V' by $v_\varepsilon - \overline{v}_\varepsilon$ and integrate over $(0,t)$. We get

$$\begin{aligned}
&\frac{1}{2}\|v_\varepsilon(t)-\overline{v}_\varepsilon(t)\|_{V'}^2 + \int_0^t \int_\Omega \frac{1}{m_\varepsilon}(v_\varepsilon(\tau)-\overline{v}_\varepsilon(\tau))^2 d\tau \\
&\leq \frac{1}{2}\|v_0-\overline{v}_0\|_{V'}^2 + \int_0^t \left\|f(\tau)-\overline{f}(\tau)\right\|_{V'}^2 \|v_\varepsilon(\tau)-\overline{v}_\varepsilon(\tau)\|_{V'} d\tau \\
&+M\int_0^t \|v_\varepsilon(\tau)-\overline{v}_\varepsilon(\tau)\| \|v_\varepsilon(\tau)-\overline{v}_\varepsilon(\tau)\|_{V'} d\tau.
\end{aligned}$$

Since $\dfrac{1}{m_\varepsilon} \geq \dfrac{1}{1+\varepsilon}$, we obtain for ε small enough the estimate (2.37), by similar computations to that of Theorem 3.8, in Sect. 4.3, via Gronwall's lemma.

Finally, assume that $f \in L^2(0,T;V')$ and $v_0 \in \overline{D(A_M)}$. Note that $\overline{D(A_M)} = \left\{v \in L^2(\Omega); \dfrac{v}{m} \leq w_s\right\}$ and $w_s = 0$, hence $\dfrac{v_0}{m} \leq w_s = 0$. We notice first that

$$j_\varepsilon\left(\frac{v_{0\varepsilon}}{m_\varepsilon}+S_s\right) = \int_0^{\frac{v_0}{m}+S_s} \widetilde{\beta}_\varepsilon^*(r)dr \leq \int_0^{S_s} \widetilde{\beta}_\varepsilon^*(r)dr \leq \int_0^{S_s} \widetilde{\beta}^*(r)dr \leq \widetilde{K}_s^* S_s,$$

which implies that the right-hand constant in the estimate (2.34) does not depend on ε. Since $W^{1,1}(0,T;V')$ is dense in $L^2(0,T;V')$, we can take $\{f_n\}_{n\geq 1} \subset W^{1,1}(0,T;V')$ and $\{v_0^n\}_{n\geq 1} \subset D(A_M)$, such that

$$f_n \longrightarrow f \text{ strongly in } L^2(0,T;V') \text{ and } v_0^n \longrightarrow v_0 \text{ strongly in } L^2(\Omega)$$

and continue like in Theorem 3.10, (a), in Sect. 4.3. We let the details of this proof to the reader. ∎

In the following we shall assume that the domains

$$\Omega_m := \{x \in \Omega;\ m(x) > 0\} \text{ and } \Omega_0 := \text{int}\{x \in \Omega;\ m(x) = 0\}$$

are connected and have the common C^1-boundary $\partial\Omega_0$. Here, the notation "int" represents the interior of the subset. To be more specific we shall assume that Ω_m and Ω_0 look like in Fig. 6.2. Denote $\beta^*(r) := \widetilde{\beta}^*(r) - \widetilde{K}_s^*$.

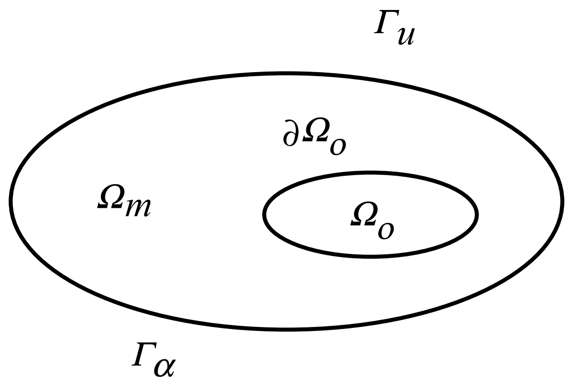

Fig. 6.2. Structure of the domain

Theorem 2.4. *Let*

$$m \in C^1(\overline{\Omega}),\ 0 \leq m < 1,\ f \in L^2(0,T;V'),\ \frac{v_0}{m} \in \overline{D(A)}.$$

Then, the Cauchy problem (2.20) *has a unique solution* w, *such that*

$$mw \in C([0,T]; L^2(\Omega)) \cap W^{1,2}(0,T;V'), \tag{2.40}$$

$$\beta^*(w) \in L^2(0,T;V), \tag{2.41}$$

$$w \in L^2(0,T;V),\ w \leq w_s,\ \text{a.e. } (x,t) \in Q. \tag{2.42}$$

Proof. Since $f \in L^2(0,T;V')$, $\frac{v_0}{m} \in \overline{D(A)}$ it follows that $v_0 \in \overline{D(A_M)}$ and the approximating problem (2.29) has, for each ε, a unique solution according to Proposition 2.3.

The estimates (2.34)-(2.36) which do not depend on ε imply then, that on subsequences we have

$$\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right) \longrightarrow \zeta \text{ weakly in } L^2(0,T;V), \tag{2.43}$$

$$\widetilde{\beta}_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon} + S_s\right) \longrightarrow \zeta + \widetilde{K}_s^* \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{2.44}$$

$$\frac{v_\varepsilon}{m_\varepsilon} \longrightarrow w \text{ weakly in } L^2(0,T;V), \tag{2.45}$$

$$\sqrt{m_\varepsilon}\left(\frac{v_\varepsilon}{m_\varepsilon} + S_s\right) \longrightarrow \overline{\omega} \text{ weak-star in } L^\infty(0,T;L^2(\Omega)). \tag{2.46}$$

For a later use we get that the trace of $\beta_\varepsilon^*\left(\frac{v_\varepsilon}{m_\varepsilon}\right)$ on Σ_u is well defined and $\zeta = 0$ on Σ_u. Now

$$v_\varepsilon = m_\varepsilon \frac{v_\varepsilon}{m_\varepsilon} \tag{2.47}$$

and since $m_\varepsilon \longrightarrow m$ uniformly on Ω and $m \in C(\overline{\Omega})$ it follows that

$$v_\varepsilon \longrightarrow v \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{2.48}$$

By (2.45) and (2.48) we get

$$v = mw \tag{2.49}$$

and obviously

$$v = 0, \text{ a.e. on } Q_0 := \Omega_0 \times (0,T). \tag{2.50}$$

Using (2.45), (2.46), (2.47) and (2.48) we still obtain that

$$\sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon} \longrightarrow \sqrt{m}v \quad \text{weak-star in } L^\infty(0,T;L^2(\Omega)),$$
$$v_\varepsilon = \sqrt{m_\varepsilon}\frac{v_\varepsilon}{m_\varepsilon}\sqrt{m_\varepsilon} \longrightarrow v \qquad \text{weak-star in } L^\infty(0,T;L^2(\Omega)).$$

Also it follows that $v(0) = v_0$. The sequence $\left\{\dfrac{dv_\varepsilon}{dt}\right\}_{\varepsilon>0}$ is bounded in $L^2(0,T;V')$ and therefore we have on a subsequence

$$\frac{dv_\varepsilon}{dt} \longrightarrow \frac{dv}{dt} \text{ weakly in } L^2(0,T;V'). \tag{2.51}$$

Again by (2.47) and $m \in C^1(\overline{\Omega})$ we deduce that

$$\|v_\varepsilon\|_{L^2(0,T;V)} \leq \text{ constant independent of } \varepsilon. \tag{2.52}$$

By Lions-Aubin compactness theorem we conclude then that $\{v_\varepsilon\}_\varepsilon$ is compact in $L^2(0,T;L^2(\Omega))$, i.e.,

$$v_\varepsilon \longrightarrow v \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ as } \varepsilon \to 0. \tag{2.53}$$

We set now

$$\Omega_\delta := \{x \in \Omega;\ m(x) > \delta\} \text{ for arbitrary } \delta > 0,$$
$$Q_\delta := \Omega_\delta \times (0,T),\ Q_m := \Omega_m \times (0,T),$$

and notice that Ω_δ and Ω_m are open because $m \in C^1(\overline{\Omega})$. We have

$$\frac{1}{m_\varepsilon} = \frac{1}{m+\varepsilon} < \frac{1}{m} < \frac{1}{\delta} \text{ on } \Omega_\delta$$

and by (2.53)

$$w_\varepsilon = \frac{1}{m_\varepsilon}v_\varepsilon \longrightarrow \frac{v}{m} = w \text{ strongly in } L^2(0,T;L^2(\Omega_\delta)),\ \forall \delta > 0.$$

Recall that $\beta_\varepsilon^*(r) = \widetilde{\beta}_\varepsilon^*(r + S_s) - \widetilde{K}_s^*$.

Let us fix $(x,t) \in Q_\delta$. Using the same argument like in the proof of Theorem 3.1, in Sect. 5.3 we obtain that $\widetilde{\beta}_\varepsilon^*(w_\varepsilon + S_s) \longrightarrow \widetilde{\zeta} \in \widetilde{\beta}^*(w + S_s)$ weakly in $L^2(0,T;H^1(\Omega_\delta))$.

By (2.25) and (2.44) we get that $\beta_\varepsilon^*(w_\varepsilon + S_s) \longrightarrow \widetilde{\beta}^*(w + S_s) - \widetilde{K}_s^*$ weakly in $L^2(0,T;H^1(\Omega_\delta))$. Since δ is arbitrary we obtain

$$\zeta(x,t) \in \widetilde{\beta}^*(w(x,t) + S_s) - \widetilde{K}_s^* \text{ a.e. } (x,t) \in Q_m = \bigcup_{\delta>0} Q_\delta. \tag{2.54}$$

Proving that the subset

$$Q_m^+ = \{(x,t) \in Q_m;\ w(x,t) > w_s\}$$

has zero measure, we deduce similarly to the proof of Corollary 3.3 in Sect. 5.3 that $w \le w_s$ a.e. $(x,t) \in Q_m$.

Finally, since $\left\{\widetilde{K}(x, w_\varepsilon + S_s)\right\}_\varepsilon$ is bounded in $L^2(Q)$, we have

$$\widetilde{K}(x, w_\varepsilon + S_s) \to \kappa \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{2.55}$$

and we assert that $\kappa(x,t) = \widetilde{K}(x, w(x,t))$, a.e. $(x,t) \in Q$. Indeed, $\left\{\widetilde{K}_m(w_\varepsilon + S_s)\right\}_\varepsilon$ is weakly convergent to κ, on Q_m, too. On the other hand, it is strongly convergent to $\widetilde{K}_m(w + S_s)$ on each Q_δ, because $\widetilde{K}_m$ is continuous. By the uniqueness of the limit the restriction of the weak limit function to Q_δ should coincide with $\widetilde{K}_m(w + S_s)$. This implies that

$$\kappa = \widetilde{K}(x, w + S_s), \text{ a.e. on } Q_m. \tag{2.56}$$

On the subset Q_0 the function K does not depend on w, so the limit is equal to $\widetilde{K}_0(x)$.

Now we can pass to limit as $\varepsilon \to 0$ in (2.28),

$$\int_Q \frac{dv_\varepsilon}{dt}\phi dxdt + \int_Q \left(\nabla\beta_\varepsilon^*(w_\varepsilon)\cdot\nabla\phi - \widetilde{K}(x, w_\varepsilon + S_s)\frac{\partial\phi}{\partial x_3}\right) dxdt \tag{2.57}$$
$$= \int_Q f\phi dxdt,\ \forall\phi \in L^2(0,T;V)$$

and we obtain

$$\int_Q \frac{d(mw)}{dt}\phi dxdt + \int_Q \left(\nabla\zeta\cdot\nabla\phi - \widetilde{K}(x, w + S_s)\frac{\partial\phi}{\partial x_3}\right) dxdt \tag{2.58}$$
$$= \int_Q f\phi dxdt,\ \forall\phi \in L^2(0,T;V),$$

where ζ is given by (2.43).

In (2.58) taking $\phi \in L^2(0,T;H_0^1(\Omega_m))$ we still deduce that

$$\int_{Q_m} \frac{d(mw)}{dt}\phi dxdt + \int_{Q_m}\left(\nabla\zeta\cdot\nabla\phi - \widetilde{K}_m(w+S_s)\frac{\partial\phi}{\partial x_3}\right)dxdt \tag{2.59}$$
$$= \int_{Q_m} f\phi dxdt,\ \forall\phi\in L^2(0,T;H_0^1(\Omega_m)),$$

where $\zeta(x,t)\in\widetilde{\beta}^*(w(x,t)+S_s)-\widetilde{K}_s^*$ a.e. on Q_m.

Taking now $\phi\in L^2(0,T;H_0^1(\Omega_0))$, we obtain the weak form of the equation on this subset

$$\int_{Q_0}\left(\nabla\zeta\cdot\nabla\phi-\widetilde{K}_0(x)\frac{\partial\phi}{\partial x_3}\right)dxdt = 0,\ \forall\phi\in L^2(0,T;H_0^1(\Omega_0)), \tag{2.60}$$

where ζ is given by (2.43).

On the other hand, (2.58)-(2.60) correspond to the following problems:

$$\begin{aligned}\frac{d(mw)}{dt}-\Delta\zeta+\frac{\partial\widetilde{K}(x,w+S_s)}{\partial x_3} &= f \text{ in } \Omega\times(0,T),\\ \zeta &= 0 \text{ on } \Sigma_u,\\ (\widetilde{K}(x,w+S_s)i_3-\nabla\zeta)\cdot\nu &= 0 \text{ on } \Sigma_\alpha,\end{aligned}$$

and, respectively,

$$\begin{aligned}\frac{d(mw)}{dt}-\Delta\zeta+\frac{\partial\widetilde{K}_m(w+S_s)}{\partial x_3} &= f \text{ in } \Omega_m\times(0,T),\\ -\Delta\zeta+\frac{\partial\widetilde{K}_0(x)}{\partial x_3} &= f \text{ in } \Omega_0\times(0,T),\\ \zeta &= 0 \text{ on } \Sigma_u,\\ (\widetilde{K}_m(w+S_s)i_3-\nabla\zeta)\cdot\nu &= 0 \text{ on } \Sigma_\alpha.\end{aligned} \tag{2.61}$$

Note that by assumption, the common boundary of the domains Ω_m and Ω_0 is regular. Since $\zeta\in L^2(0,T;V)$, we deduce that the trace of ζ on $\partial\Omega_m\times(0,T)=\partial\Omega_0\times(0,T)$ is well defined and continuous, due to the continuity of the trace operator across the boundary. Moreover, we take into account that $\zeta(x,t)\in\widetilde{\beta}^*(w(x,t)+S_s)-\widetilde{K}_s^*$ a.e. on Q_m, hence the trace of ζ satisfies

$$\zeta(x,t)\in\widetilde{\beta}^*(w(x,t)+S_s)-\widetilde{K}_s^*,\ \text{a.e. } (x,t)\in\partial\Omega_0\times(0,T). \tag{2.62}$$

Thus, ζ turns out to be the solution to the elliptic problem

$$\begin{aligned}-\Delta\zeta+\frac{\partial\widetilde{K}_0(x)}{\partial x_3} &= f \quad\text{in}\quad \Omega_0\times(0,T),\\ \zeta\in\widetilde{\beta}^*(w(t)+S_s)-\widetilde{K}_s^* &\quad\text{on}\quad \partial\Omega_0\times(0,T).\end{aligned} \tag{2.63}$$

We define the function

$$w^*(x,t) = \begin{cases} w(x,t), & \text{if } (x,t) \in Q_m \\ (\widetilde{\beta}^*)^{-1}(\zeta(x,t) + \widetilde{K}_s^*) - S_s, & \text{if } (x,t) \in Q_0 = \Omega_0 \times (0,T) \end{cases} \quad (2.64)$$

and we show that it is the solution to (2.20). Since $\zeta \in L^2(0,T;V)$ it follows that $\widetilde{\beta}^*(w^*(t)+S_s) \ni \zeta + \widetilde{K}_s^* \in V$, i.e., $(\widetilde{\beta}^*)^{-1}(\zeta(x,t)+\widetilde{K}_s^*) \in D(A)$, implying that $w^* \leq w_s$ a.e. on Q_0. Moreover, $mw^* = 0$ on Q_0, so w^* satisfies (2.17). We have to check that w^* verifies (2.19). Indeed, if we replace w^* in (2.19) we obtain

$$\int_0^T \int_{\Omega_m} \frac{d(mw)}{dt} \phi dxdt + \int_0^T \int_{\Omega_m} \left(\nabla \widetilde{\beta}^*(w^* + S_s) \cdot \nabla \phi - \widetilde{K}(x, w + S_s) \frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$+ \int_0^T \int_{\Omega_0} \left(\nabla \widetilde{\beta}^*(w^* + S_s) \cdot \nabla \phi - \widetilde{K}(x, w^* + S_s) \frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$= \int_0^T \int_{\Omega} \frac{d(mw)}{dt} \phi dxdt + \int_0^T \int_{\Omega_m} \left(\nabla(\zeta + \widetilde{K}_s^*) \cdot \nabla \phi - \widetilde{K}_m(w + S_s) \frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$+ \int_0^T \int_{\Omega_0} \left(\nabla(\zeta + \widetilde{K}_s^*) \cdot \nabla \phi - \widetilde{K}_0(x) \frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$= \int_Q \left(\frac{d(mw)}{dt} \phi + \nabla \zeta \cdot \nabla \phi - \widetilde{K}(x, w + S_s) \frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$= \int_0^T \int_{\Omega} f \phi dxdt, \ \forall \phi \in L^2(0,T;V).$$

We took into account the expressions assigned to w^* and $\widetilde{K}(x, w+S_s)$ on each subset, (2.54), (2.64) and (2.58). On the other hand, if we multiply (2.61) by $\phi \in L^2(0,T;V)$ and integrate the sum over Q we obtain

$$\int_0^T \int_{\Omega_m} \frac{d(mw)}{dt} \phi dxdt + \int_0^T \int_{\Omega_m} \left(\nabla \zeta \cdot \nabla \phi - \widetilde{K}_m(w + S_s) \frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$+ \int_0^T \int_{\partial \Omega_m} \left(\widetilde{K}_m(w + S_s) i_3 - \nabla \zeta \right) \cdot \nu^+ \phi d\sigma dt$$
$$+ \int_0^T \int_{\Omega_0} \left(\nabla \zeta \cdot \nabla \phi - \widetilde{K}_0(x) \frac{\partial \phi}{\partial x_3} \right) dxdt$$
$$+ \int_0^T \int_{\partial \Omega_0} \left(\widetilde{K}_0(x) i_3 - \nabla \zeta \right) \cdot \nu^- \phi d\sigma dt = \int_0^T \int_{\Omega} f \phi dxdt, \ \forall \phi \in L^2(0,T;V),$$

where ν^+ is the outer normal to $\partial \Omega_m$, ν^- is the outer normal to $\partial \Omega_0$ and $\zeta \in \widetilde{\beta}^*(w + S_s) - \widetilde{K}_s^*$ a.e. on Q_m. Taking into account (2.58) we obtain the flux continuity on the common boundary $\partial \Omega_0 \times (0,T)$

$$\left(\widetilde{K}_m(w + S_s) i_3 - \nabla \zeta \right) \cdot \nu^+ = \left(\widetilde{K}_0(x) i_3 - \nabla \zeta \right) \cdot \nu^+ \text{ on } \partial \Omega_0 \times (0,T). \quad (2.65)$$

The previous integrals on $\partial\Omega_m$ and $\partial\Omega_0$ are considered in the sense of distributions. By the trace theorem we see that, generally, the flux $(\widetilde{K}(x, w + S_s)i_3 - \nabla\zeta) \cdot \nu$ is well defined as an element of the space $L^2(0,T; H^{-1/2}(\partial\Omega_0))$.

Concerning the uniqueness we show that $v^* := mw^*$ is unique on Q. To this end we use an estimate obtained for the difference of two solutions w^* and $\overline{w}^*$ corresponding to different initial and free data (w_0, f) and $(\overline{w}_0, \overline{f})$. Actually we can pass to limit in (2.37) and use the weakly l.s.c. property. We deduce that

$$\|v^*(t) - \overline{v}^*(t))\|_{V'}^2 + \int_0^t \|v^*(\tau) - \overline{v}^*(\tau)\|^2 \, d\tau$$
$$\leq \alpha_0 \left(\|v_0 - \overline{v}_0)\|_{V'}^2 + \int_0^T \left\|f(t) - \overline{f}(t)\right\|_{V'}^2 dt \right),$$

that implies the uniqueness of v^* on Q. Then, on Q_m we have $\dfrac{v^*}{m} = \dfrac{\overline{v}^*}{m}$, implying that w^* is unique on Q_m.

Assume now that we have two different solutions w_1 and w_2 to the system composed of (2.61) and (2.63). We make the difference and denote $Z := \zeta_1 - \zeta_2$. We have, retaining only the equations we are interested in:

$$-\Delta Z = 0 \quad \text{in} \quad \Omega_0 \times (0,T),$$
$$Z \in \widetilde{\beta}^*(w_1 + S_s) - \widetilde{\beta}^*(w_2 + S_s) \quad \text{on} \quad \partial\Omega_0 \times (0,T).$$

We recall the fact that the solution is unique in Q_m, so that $Z = 0$ on the common boundary, implying that $Z = 0$ a.e. in Q_0. ∎

Remark 2.5. By the proof of the solution existence we also ascertain a consequence that can be inferred at an intuitive level, i.e., the boundary value problem is separated into two problems corresponding to the domains Q_m and Q_0, connected by the flux continuity.

We recall that the connection between moisture, water saturation and porosity is given by the relation $\theta(x,t) = m(x)S_w(x,t)$. This model describes the infiltration into a heterogeneous porous medium in which completely impermeable intrusions ($m = 0$), as well as parts with large voids (m close to 1) can be found. Now, we can particularize the results, assuming that inside the domain defined by $m(x) = 0$ there are no sources or sinks and the conductivity is zero, i.e., f and $\widetilde{K}_0$ are zero, on Q_0.

If we pass back to problem (2.1)-(2.4), we obtain that it has a unique solution $S_w \in L^2(0,T; H^1(\Omega))$, with $mS_w \in C([0,T]; L^2(\Omega)) \cap W^{1,2}(0,T; V')$ and $mS_w \equiv 0$ on Q_0. Moreover, if the common boundary is assumed impermeable, there is no water flux coming from Q_m through the boundary $\partial\Omega_0 \times (0,T)$,

$$\left(\widetilde{K}_m(w + S_s)i_3 - \nabla\zeta\right) \cdot \nu = 0 \text{ on } \partial\Omega_0 \times (0,T).$$

Hence (2.65) combined with the first equation (2.63), implies that the solution $\widetilde{\zeta}$ inside Q_0 is constant. It follows that $\widetilde{\beta}(w + S_s) = 0$, i.e., $S_w = 0$, which is a perfect agreement with the physical evidence.

In particular, if we consider the model in which $\partial\Omega_0 \cap \Gamma_u \neq \emptyset$, then since $\zeta = 0$ on Γ_u, we can conclude immediately that $\widetilde{\zeta} \equiv 0$ on Q_0, and $w + S_s = 0$, implying $S_w = 0$ on Q_0, too.

6.3 Analysis of an infiltration hysteretic model

Generally speaking, a hysteretic behaviour of a system whose state is described by two real valued functions ω (the dependent one) and u (the independent one) defined and continuous on a finite time interval $[0, T]$, is represented by a relationship of the type

$$\omega(t) = \mathcal{F}(u, \omega_0)(t).$$

It expresses the fact that at a certain spatial point x the function ω depends not only on $u(t)$ but also on the previous evolution of u in the interval $[0, t)$ and possibly on the initial state of the system. The functions u and ω are also called *input* and *output*, respectively. The dependence of ω on the history of u reflects the *memory effect* in the system evolution. A basic property requires that, at any instant t, $\omega(t)$ depend on the range of the values $u(\tau)$, for $\tau \in [0, t]$ and on the order in which they have been reached. So, there is no dependence on the derivatives of u, property which is called *rate-independence*. It is not our intention to enter into specific details of the hysteresis theory, for this referring the reader to the monograph [124]. The aim of this chapter is to analyze a hysteretic model (see [93]) represented by a system of equations involving a hysteretic operator (which is in fact a multivalued graph).

A hysteretic model for the wetting-drainage in a porous medium

We specified in Sect. 1.1 that the wetting-drying processes exhibit hysteresis and we explained how the wetting-drying cycle looks like. Here we shall consider that the behaviour of the moisture is of the form:

$$\frac{dh}{dt} \in -\partial I_{M(h)}(\theta),$$

where $\partial I_{M(h)}$ is the subdifferential of the indicator function of the set

$$M(h) = \{z;\ z \in [\gamma_w(h), \gamma_d(h)],\ h \in [h_m, h_M] \subset (h_r, 0)\}.$$

Here $z = \gamma_w(h)$ represents the infiltration curve and $z = \gamma_d(h)$ is the drainage curve, which is above the infiltration one, as put into evidence by experiments.

We shall consider further a model for a hysteretic wetting-drying process which turns out to be well posed and may reflect a feature of the physical process, by assuming that hysteresis occurs in the constitutive law, on an interval strictly included in $(h_r, 0)$. Heuristically speaking, since θ is a hysteretic function of h, it follows that the hysteretic behaviour will be involved in all functions depending on θ, namely in $\beta^*(\theta)$ and $K(\theta)$ (deduced from $k(h)$ and $K^*(h)$ defined in Sect. 2.2) which will become hysteretic functions of θ. For simplicity we shall develop this study disregarding the hysteretic component of K and taking into account that only β^* is a hysteretic function of θ, denoted $\mathcal{B}^*$. So, we introduce a model in which the state system is represented by the input θ and the output $\mathcal{B}^*$, or more exactly by its hysteretic component which will be denoted by w.

The structure of this function keeps inside the dependence on the history of the moisture evolution, e.g., $\mathcal{B}^*(t) = \mathcal{F}(\theta, \mathcal{B}_0^*)(t)$. The expression of $\mathcal{B}^*$ may be very complex, due to the fact that it inherits the specific time evolution that θ had on the wetting and drainage curves. In our model $\mathcal{B}^*$ is formed by superposing a hysteretic contribution over the values corresponding to the nonhysteretic case,

$$\mathcal{B}^*(\theta, h) = \beta^*(\theta) + w_t, \tag{3.1}$$

where w is a function that cumulates all the history which the hysteretic evolution of θ transmits to $\mathcal{B}^*$. More specifically, we shall choose a simple example,

$$\frac{dw}{dt} \in -H_{a,b}(w - \theta), \tag{3.2}$$

where $H_{a,b}$ is the multivalued function

$$H_{a,b}(r) := \begin{cases} R_+, & \text{if } r = -a \\ 0, & \text{if } r \in (-b, -a) \\ R_-, & \text{if } r = -b, \end{cases} \tag{3.3}$$

and $0 < a < b < \theta_s$, see Fig. 6.3.

Thus, the hysteretic equation (3.2) can be written as

$$\frac{dw}{dt} \in \begin{cases} R_-, & \text{if } w = \theta - a \\ 0, & \text{if } w \in (\theta - b, \theta - a) \\ R_+, & \text{if } w = \theta - b. \end{cases} \tag{3.4}$$

The form of this heuristic relationship may be mathematically argued by a procedure similar to that developed in Sect. 2.2 for constructing the function β^* in the nonhysteretic case. The idea is to reconstruct this function, starting from a hysteretic relationship between θ and h, e.g., of the form

$$h \in (C^*)^{-1}(\theta) + \left(\partial I_{M_1(\theta)}\right)^{-1}\left(\frac{dh}{dt}\right),$$

and working separately on each loop branch, where the functions are monotonic. To work with a simple example, the infiltration and drainage curves are

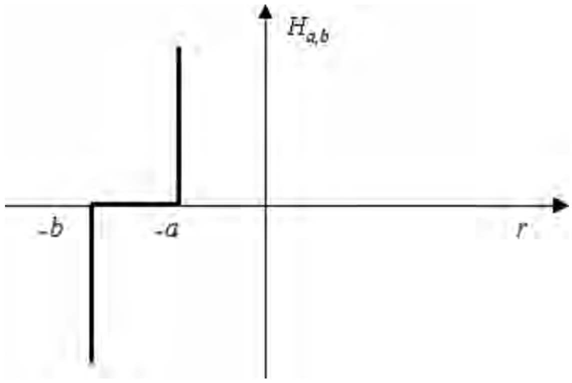

Fig. 6.3. The graphic of $H_{a,b}(r)$

considered straight lines, but in reality they are curves. However, this form, thought as a superposition of a hysteretic component over a nonhysteretic one, illustrates an enough reliable feature for an infiltration-drainage process.

Then, (3.4) means that w increases on the line $w = \theta - b$, where water infiltrates, decreases on the line $w = \theta - a$, where water is drained, and is constant between them, preserving thus the appropriate direction of variation of $\mathcal{B}^*$, by the term $\frac{dw}{dt}$.

As a matter of fact the evolution of w is more complex and it is explained below and illustrated in Fig. 6.4, where we can follow the behaviour of w at a fixed point x.

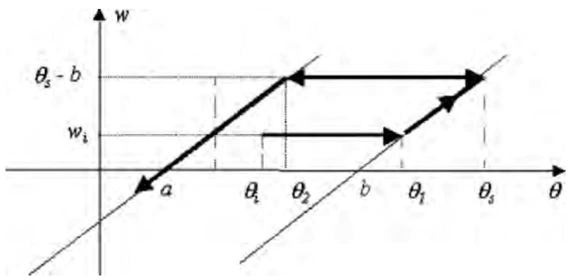

Fig. 6.4. Graphic of the hysteretic function w

Assume that at the initial time w starts from w_i, corresponding to $a < \theta_i < b$. When θ moves from θ_i towards θ_1, with $b \leq \theta_1 \leq \theta_s$, the function w remains constant at the level $w = w_i$. Between θ_1 and θ_s it increases along the line $w = \theta - b$ possibly up to the value $w = \theta_s - b$ when saturation occurs. Then, if moisture decreases from θ_s to θ_2, with $a \leq \theta_2 \leq b$, the function w

remains constant, equal to $\theta_s - b$. Next, consider that moisture increases again up to b. Consequently, w remains unchanged, equal to $\theta_s - b$, value still preserved if the moisture decreases again up to θ_2. Further, if moisture continues to decrease up to 0, w decreases up to $w = -a$ where the soil is completely dried. To conclude, w can move upwards on the line $w = \theta - b$, downwards on the line $w = \theta - a$ and in both directions on any horizontal between these two lines.

Consequently, we shall write the diffusion equation in the form

$$\frac{\partial \theta}{\partial t} - \Delta(\beta^*(\theta) + w_t) + \frac{\partial K(\theta)}{\partial x_3} \ni f_{hys}. \tag{3.5}$$

The model is completed with (3.2) and with initial and boundary conditions for θ and w.

We shall consider the problem under the strongly saturated-unsaturated nonlinear case with a weakly nonlinear conductivity and with flux type boundary conditions, i.e., the model described in Sect. 5.1. Hence the hysteretic model reads

$$\theta_t - \Delta\beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} + \Delta H_{a,b}(w - \theta) \ni f_{hys} \quad \text{in} \quad Q, \tag{3.6}$$

$$\theta(0) = \theta_0(x) \quad \text{in} \quad \Omega, \tag{3.7}$$

$$(K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu \ni u_{hys} \quad \text{on} \quad \Sigma_u, \tag{3.8}$$

$$(K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu - \alpha\beta^*(\theta) \ni f_0 \quad \text{on} \quad \Sigma_\alpha, \tag{3.9}$$

$$w_t + H_{a,b}(w - \theta) \ni 0, \tag{3.10}$$

$$w(0) = w_0, \tag{3.11}$$

$$-\nabla H_{a,b}(w - \theta) \cdot \nu \ni 0 \quad \text{on} \quad \Sigma_u, \tag{3.12}$$

$$-\nabla H_{a,b}(w - \theta) \cdot \nu + \alpha H_{a,b}(w - \theta) \ni 0 \quad \text{on} \quad \Sigma_\alpha, \tag{3.13}$$

where β^* and K are the functions defined in Sect. 5.1 by (1.5)-(1.6) and have the properties (i)-(iii) and ($\mathrm{i_K}$), respectively.

Functional framework

We shall work within the functional spaces, V and V' defined in Sect. 5.1, V being $H^1(\Omega)$ endowed with the norm

$$\|\psi\|_V = \left(\int_\Omega \psi^2 dx + \int_{\Gamma_\alpha} \alpha \psi^2 d\sigma \right)^{1/2}$$

and V' its dual, with the scalar product given by

$$\left\langle \theta, \overline{\theta} \right\rangle_{V'} = \theta(A_0^{-1}(\overline{\theta})),$$

where the operator $A_0 : V \to V'$ is defined by

$$\langle A_0\theta, \psi\rangle_{V',V} = \int_\Omega \nabla\theta \cdot \nabla\psi dx + \int_{\Gamma_\alpha} \alpha\theta\psi d\sigma, \ \forall\psi \in V. \tag{3.14}$$

We introduce the product spaces $W = V \times L^2(\Omega)$ and $W' = V' \times L^2(\Omega)$ and we endow W and W' with the scalar products

$$\langle y, z\rangle_W = \langle y_1, z_1\rangle_V + (y_2, z_2),$$

for $y = (y_1, y_2) \in W$ and $z = (z_1, z_2) \in W$ and, respectively

$$\left\langle \Theta, \overline{\Theta}\right\rangle_{W'} = \left\langle \theta, \overline{\theta}\right\rangle_{V'} + (w, \overline{w}), \tag{3.15}$$

where $\Theta = (\theta, w) \in W'$, $\overline{\Theta} = (\overline{\theta}, \overline{w}) \in W'$.

As usually, we denote the scalar product and the norm in $L^2(\Omega)$ by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively (considering that any confusion with the notation related to an element of the Cartesian product of two spaces is avoided).

We resume now the definition of the operator $A : D(A) \subset V' \to V'$, where

$$D(A) := \{\theta \in L^2(\Omega);\ \exists\eta \in V,\ \eta(x) \in \beta^*(\theta(x)) \text{ a.e. } x \in \Omega\},$$

$$\langle A\theta, \psi\rangle_{V',V} = \int_\Omega \nabla\eta \cdot \nabla\psi dx + \int_{\Gamma_\alpha} \alpha\eta\psi d\sigma - \int_\Omega K(\theta)\frac{\partial\psi}{\partial x_3}dx, \ \forall\psi \in V, \tag{3.16}$$

for $\eta \in \beta^*(\theta)$, (see Sect. 5.1).

If we assume that $f_0 \in L^2(0,T; L^2(\Gamma_\alpha))$ and $u \in L^2(0,T; L^2(\Gamma_u))$ we define $f_{\Gamma_\alpha} \in L^2(0,T;V')$ and $f_{\Gamma_u} \in L^2(0,T;V')$ by

$$f_{\Gamma_u}(t)\psi := -\int_{\Gamma_u} u\psi d\sigma,\ f_{\Gamma_\alpha}(t)\psi := -\int_{\Gamma_\alpha} f_0\psi d\sigma,\ \forall\psi \in V. \tag{3.17}$$

Further, setting the domain

$$\begin{aligned} D(\mathcal{A}) = \{(\theta, w) \in L^2(\Omega) \times L^2(\Omega);\ \exists\eta \in V,\ \eta(x) \in \beta^*(\theta(x)) \text{ a.e. } x \in \Omega, \\ \exists\xi \in V,\ \xi(x) \in H_{a,b}(w(x) - \theta(x)) \text{ a.e. } x \in \Omega\}, \end{aligned}$$

we introduce the operator

$$\mathcal{A} : D(\mathcal{A}) \subset W' \to W'$$

defined by

$$\begin{aligned} \langle \mathcal{A}\Theta, \Psi\rangle_{W',W} = \langle A\theta, \psi\rangle_{V',V} - \langle A_0\xi, \psi\rangle_{V',V} + (\xi, \phi), \forall\psi\in V, \forall\phi\in L^2(\Omega), \\ \text{for } \eta \in \beta^*(\theta) \text{ and } \xi \in H_{a,b}(w - \theta). \end{aligned} \tag{3.18}$$

In fact this can still be written

$$\begin{aligned} \langle \mathcal{A}\Theta, \Psi\rangle_{W',W} = &\int_\Omega \nabla\eta \cdot \nabla\psi dx + \int_{\Gamma_\alpha} \alpha\eta\psi d\sigma - \int_\Omega K(\theta)\frac{\partial\psi}{\partial x_3}dx \\ &- \int_\Omega \nabla\xi \cdot \nabla\psi dx - \int_{\Gamma_\alpha} \alpha\xi\psi d\sigma + \int_\Omega \xi\phi dx, \end{aligned}$$

$$\forall \Psi = (\psi, \phi) \in W, \text{ for } \eta \in \beta^*(\theta), \ \xi \in H_{a,b}(w-\theta) \text{ a.e. } x \in \Omega.$$

With these notations we can write the Cauchy problem

$$\frac{d\Theta}{dt} + \mathcal{A}\Theta = F_{hys} + F_{\Gamma_u}^{hys} + F_{\Gamma_\alpha}^{hys}, \text{ a.e. } t \in (0,T), \tag{3.19}$$

$$\Theta(0) = \Theta_0, \tag{3.20}$$

where $\Theta = (\theta, w)$, $\Theta_0 = (\theta_0, w_0)$, $F_{hys} := (f_{hys}, 0)$, $u := (u_{hys}, 0)$, $F_{\Gamma_u}^{hys} := (f_{\Gamma_u}, 0)$, $F_{\Gamma_\alpha}^{hys} := (f_{\Gamma_\alpha}, 0)$.

We mention that an equivalent form for the equation (3.19) is

$$\int_Q \frac{\partial\theta}{\partial t}\psi dxdt + \int_Q \nabla\eta\cdot\nabla\psi dxdt + \int_{\Sigma_\alpha} \alpha\eta\psi d\sigma dt - \int_Q K(\theta)\frac{\partial\psi}{\partial x_3} dxdt \tag{3.21}$$

$$- \int_Q \nabla\xi\cdot\nabla\psi dxdt - \int_{\Sigma_\alpha} \alpha\xi\psi d\sigma dt$$

$$= \int_Q f_{hys}\psi dxdt - \int_{\Sigma_u} u_{hys}\psi d\sigma dt - \int_{\Sigma_\alpha} f_0\psi d\sigma dt, \ \forall\psi \in L^2(0,T;V),$$

$$\int_Q \frac{\partial w}{\partial t}\phi dxdt + \int_Q \xi\phi dxdt = 0, \ \forall\phi \in L^2(0,T;L^2(\Omega)), \tag{3.22}$$

for some $\eta \in \beta^*(\theta)$, $\xi \in H_{a,b}(w-\theta)$. Notice that a strong solution to (3.19)-(3.20) is a solution in the sense of distributions to (3.6)-(3.7), (3.10)-(3.11) and satisfies the boundary conditions in the sense of the trace (the proof is similar to that of Sect. 5.1).

Main results

The existence of the solution to the previous problem will be based on the quasi m-accretivity of $\mathcal{A}$. This presumes some intermediate results that will be obtained in a couple of steps.

Proposition 3.1. *Under the assumptions* (i)-(iii), (i$_K$), $\mathcal{A}$ *is quasi m-accretive on* W'.

Proof. Let $\Theta = (\theta, w)$ and $\Theta_1 = (\theta_1, w_1)$ belong to W'. For proving the quasi-accretivity we compute

$$\langle(\lambda I + \mathcal{A})\Theta - (\lambda I + \mathcal{A})\Theta_1, \Theta - \Theta_1\rangle_{W'} \tag{3.23}$$

$$= \lambda \|\theta - \theta_1\|_{V'}^2 + \langle A\theta - A\theta_1, \theta - \theta_1\rangle_{V'}$$

$$- \langle A_0\xi - A_0\xi_1, \theta - \theta_1\rangle_{V'} + (\xi - \xi_1, w - w_1),$$

where $\xi \in H_{a,b}(w-\theta)$, $\xi_1 \in H_{a,b}(w_1 - \theta_1)$. Recalling now the definition of the scalar product in V' we have

$$\langle A\theta - A\theta_1, \theta - \theta_1\rangle_{V'} = \langle A\theta - A\theta_1, \psi\rangle_{V',V},$$

where $\psi = A_0^{-1}(\theta - \theta_1)$ with A_0 defined by (3.14).

Therefore, we can write

$$\langle A\theta - A\theta_1, \theta - \theta_1 \rangle_{V'} \geq \frac{\rho}{2} \|\theta - \theta_1\|^2 - \frac{M^2}{2\rho} \|\theta - \theta_1\|_{V'}^2 . \tag{3.24}$$

We took into account (i), ($\mathrm{i_K}$) and the obvious equality $\|\psi\|_V = \|\theta - \theta_1\|_{V'}$. Similarly,

$$\begin{aligned} \langle A_0\xi - A_0\xi_1, \theta - \theta_1 \rangle_{V'} &= \int_\Omega \nabla(\xi - \xi_1) \cdot \nabla\psi dx + \int_{\Gamma_\alpha} \alpha(\xi - \xi_1)\psi d\sigma \\ &= \int_\Omega (\xi - \xi_1)(\theta - \theta_1) dx, \end{aligned} \tag{3.25}$$

where $\psi = A_0^{-1}(\theta - \theta_1)$. Putting all together in (3.23) and using the monotonicity of $H_{a,b}$ we get

$$\begin{aligned} &\langle (\lambda I + \mathcal{A})\Theta - (\lambda I + \mathcal{A})\Theta_1, \Theta - \Theta_1 \rangle_{W'} \\ &\geq \left(\lambda - \frac{M^2}{2\rho}\right) \|\theta - \theta_1\|_{V'}^2 + \frac{\rho}{2} \|\theta - \theta_1\|^2 + (\xi - \xi_1, w - \theta - (w_1 - \theta_1)) \\ &\geq \left(\lambda - \frac{M^2}{2\rho}\right) \|\theta - \theta_1\|_{V'}^2 + \frac{\rho}{2} \|\theta - \theta_1\|^2 , \text{ for } \lambda > \lambda_0 = \frac{M^2}{2\rho}. \end{aligned} \tag{3.26}$$

By (3.26) the proof of the quasi-accretivity of $\mathcal{A}$ is ended.

To prove of the quasi m-accretivity we have to show that

$$R(\lambda I + \mathcal{A}) = W' \text{ for } \lambda \text{ large enough.}$$

This turns into proving that for any $(f, g) \in W'$ there exists $\Theta = (\theta, w) \in D(\mathcal{A})$ solution to the system

$$\lambda\theta + A\theta - A_0 H_{a,b}(w - \theta) \ni f, \tag{3.27}$$

$$\lambda w + H_{a,b}(w - \theta) \ni g. \tag{3.28}$$

Since $H_{a,b}$, or more exactly its realization in $L^2(\Omega)$ is m-accretive we can calculate $w - \theta$ from the second equation

$$w - \theta = (\lambda I + H_{a,b})^{-1}(g - \lambda\theta) \tag{3.29}$$

and replace it in the first resolvent equation

$$\lambda\theta + A\theta - A_0 H_{a,b}(\lambda I + H_{a,b})^{-1}(g - \lambda\theta) = f. \tag{3.30}$$

Eventually, the previous equation can be still written

$$\lambda\theta + A\theta - A_0 S_{\lambda,g}(\theta) = f, \tag{3.31}$$

where

$$S_{\lambda,g}(\theta) := S_\lambda(g - \lambda\theta),\ S_{\lambda,g} : L^2(\Omega) \to L^2(\Omega) \tag{3.32}$$

and

$$S_\lambda := I - \lambda(\lambda I + H_{a,b})^{-1}. \tag{3.33}$$

The operator acting in (3.31) is therefore

$$\widetilde{A} = \lambda I + A - A_0 S_{\lambda,g},\ \widetilde{A} : D(\widetilde{A}) \subset V' \to V',$$
$$D(\widetilde{A}) = \{\theta \in L^2(\Omega),\ \exists\eta \in V,\ \eta(x) = \beta^*(\theta(x)),\ S_{\lambda,g}(\theta) \in V \text{ a.e. } x \in \Omega\}.$$

It is obvious that $\theta \in D(\widetilde{A})$ implies $(\theta, w) \in D(\mathcal{A})$ and conversely. By (i) we deduce that the inverse of $\beta^*(\theta)$ is Lipschitz, so that if $\theta \in D(\widetilde{A})$ then $\beta^*(\theta) \in V$ which implies that $\theta \in V$.

Thus, instead of studying (3.27)-(3.28) we shall study the equivalent equation (3.31) and show the following result:

Lemma 3.2. *Let $f \in V'$ and $g \in L^2(\Omega)$ be arbitrary but fixed and let us fix $\lambda = \lambda_0 > \dfrac{M^2}{2\rho}$. Under the conditions* (i)-(iii), (i$_K$), (3.31) *has a unique solution $\theta \in D(\widetilde{A})$.*

Proof. Let $f \in V'$ and $g \in L^2(\Omega)$. Since A_0 defined by (3.14) is surjective, the equation $A_0 u = v$ has a unique solution for any $v \in V'$. Then, we define $A_0^{-1} : V' \to V$, by

$$\left\langle A_0^{-1}v, \phi\right\rangle_{V,V'} = \phi(A_0^{-1}v),\ \forall\phi \in V' \tag{3.34}$$

and notice that $A_0 \in L(V, V')$. If ϕ is in $L^2(\Omega)$, then the previous equality is exactly the scalar product in $L^2(\Omega)$.

Moreover, $\phi \in V'$ can be written as $\phi = A_0\psi$, with $\psi \in V$, so (3.34) becomes

$$\left\langle A_0^{-1}v, \phi\right\rangle_{V,V'} = \langle A_0\psi, u\rangle_{V',V,} \tag{3.35}$$

where u is the solution to the equation $A_0 u = v$ and ψ is the solution to the equation $A_0\psi = \phi$. Equivalently to (3.35) we can write

$$\left\langle A_0^{-1}v, \phi\right\rangle_{V,V'} = \int \nabla\psi \cdot \nabla u dx + \int_{\Gamma_\alpha} \alpha\psi u d\sigma. \tag{3.36}$$

Applying now the operator A_0^{-1} to equation (3.31) the proof of the existence of the solution to (3.31) reduces to the proof of the existence of the solution to the equation

$$\lambda A_0^{-1}\theta + A_0^{-1}A\theta - S_{\lambda,g}(\theta) = A_0^{-1}f \in V. \tag{3.37}$$

We shall prove that this equation has a solution on the basis of the surjectivity of the operator

$$\widehat{A} : D(\widehat{A}) \subset L^2(\Omega) \to L^2(\Omega),\ \widehat{A} = \lambda A_0^{-1} + A_0^{-1}A - S_{\lambda,g}, \tag{3.38}$$

where

$$D(\widehat{A}) := \{\theta \in L^2(\Omega);\ A\theta \in V'\} = D(\widetilde{A}).$$

Properties of the operator $\widehat{A}$.

The operator $\widehat{A}$ is a sum of three operators, A_0^{-1}, $-S_{\lambda,g}$ and $A_1 := A_0^{-1}A$. We shall specify the properties of each of them.

We shall denote still by A_0^{-1}, the restriction of the A_0^{-1} defined by (3.36) to $L^2(\Omega)$, i.e., the operator $A_0^{-1} : L^2(\Omega) \to L^2(\Omega)$ defined by

$$(A_0^{-1}\theta, \phi) = \langle A_0\psi, u\rangle_{V',V}\,, \tag{3.39}$$

where u is the solution to $A_0 u = \theta$ and ψ is the solution to $A_0\psi = \phi$. This operator is positive

$$(A_0^{-1}\theta, \theta) = \langle A_0 u, u\rangle_{V',V} = \|u\|_V^2 \geq 0 \tag{3.40}$$

and bounded. Indeed, if we multiply the equation $A_0 u = \theta \in L^2(\Omega)$ by $u \in V$ we get

$$\frac{1}{c_H^2}\|u\|^2 \leq \|u\|_V^2 \leq \int_\Omega \theta u dx \leq \|\theta\|\,\|u\|\,,$$

which implies

$$\left\|A_0^{-1}\theta\right\| \leq c_H^2\|\theta\|\,. \tag{3.41}$$

The constant c_H was defined by (1.12) in Sect. 5.1. Also it is obvious that

$$(A_0^{-1}\theta, \theta) = \|\theta\|_{V'}^2\,,\ \forall\theta \in L^2(\Omega). \tag{3.42}$$

Properties of the operator $-S_{\lambda,g}$.

The operator $-S_{\lambda,g} : L^2(\Omega) \to L^2(\Omega)$ is monotone, bounded and continuous.

For the continuity we show first an estimate, by multiplying the equation

$$(\lambda I + H_{a,b})\theta - (\lambda I + H_{a,b})\theta_1 = u - u_1$$

by $\theta - \theta_1$ and integrating over Ω. We obtain

$$\lambda\|\theta - \theta_1\|^2 + (H_{a,b}\theta - H_{a,b}\theta_1, \theta - \theta_1) \leq \|u - u_1\|\,\|\theta - \theta_1\|$$

and since $H_{a,b}$ is monotone we get

$$\|\theta - \theta_1\| = \left\|(\lambda I + H_{a,b})^{-1}u - (\lambda I + H_{a,b})^{-1}u_1\right\| \leq \frac{1}{\lambda}\|u - u_1\|\,. \tag{3.43}$$

Then it follows that

$$\begin{aligned} &\|S_{\lambda,g}\theta - S_{\lambda,g}\theta_1\| \\ &= \|S_\lambda(g - \lambda\theta) - S_\lambda(g - \lambda\theta_1)\| \\ &\leq \lambda\|\theta - \theta_1\| + \lambda\left\|(\lambda I + H_{a,b})^{-1}(g - \lambda\theta) - (\lambda I + H_{a,b})^{-1}(g - \lambda\theta_1)\right\| \\ &\leq \lambda\|\theta - \theta_1\| + \lambda\frac{1}{\lambda}\|\lambda\theta - \lambda\theta_1\| \leq 2\lambda\|\theta - \theta_1\|\,. \end{aligned} \tag{3.44}$$

Moreover, we have

$$\begin{aligned}
&(S_{\lambda,g}\theta - S_{\lambda,g}\theta_1, \theta - \theta_1)\\
&= \big((I - \lambda(\lambda I + H_{a,b})^{-1})(g - \lambda\theta) - (I - \lambda(\lambda I + H_{a,b})^{-1})(g - \lambda\theta_1), \theta - \theta_1\big)\\
&\geq -\lambda \left\|\theta - \theta_1\right\|^2 - \lambda \left\|(\lambda I + H_{a,b})^{-1}(g - \lambda\theta) - (\lambda I + H_{a,b})^{-1}(g - \lambda\theta_1)\right\| \left\|\theta - \theta_1\right\|\\
&\geq -\lambda \left\|\theta - \theta_1\right\|^2 - \frac{\lambda}{\lambda} \left\|g - \lambda\theta - g + \lambda\theta_1\right\| \left\|\theta - \theta_1\right\| = -2\lambda \left\|\theta - \theta_1\right\|^2.
\end{aligned}$$

Then,

$$-(S_{\lambda,g}\theta - S_{\lambda,g}\theta_1, \theta - \theta_1) \geq 0. \tag{3.45}$$

It is simple to show that in fact

$$-S_{\lambda,g}\theta = \begin{cases} \lambda\theta - g + \lambda b, & \theta \leq \dfrac{g}{\lambda} - b \\ \quad 0, & \theta \in \left(\dfrac{g}{\lambda} - b, \dfrac{g}{\lambda} - a\right] \\ \lambda\theta - g + \lambda a, & \theta > \dfrac{g}{\lambda} - a, \end{cases}$$

so that using (3.44) with $\theta_1 \in \left(\frac{g}{\lambda} - b, \frac{g}{\lambda} - a\right]$ we get that $S_{\lambda,g}$ is bounded

$$\left\|S_{\lambda,g}\theta\right\| \leq 2\lambda \left\|\theta - \left(\frac{g}{\lambda} - b\right)\right\| \leq 2\lambda \left\|\theta\right\| + 2 \left\|g\right\| + 2\lambda \left\|b\right\|,$$

where g is fixed in $L^2(\Omega)$ and λ is fixed.

By (3.45) it follows for $\theta_1 = 0$ that

$$-(S_{\lambda,g}\theta, \theta) = (-S_{\lambda,g}0, \theta) \geq -2\left\|g\right\| \left\|\theta\right\| - 2\lambda b \left\|\theta\right\|. \tag{3.46}$$

Properties of the operator A_1.

Finally, we shall show that the operator

$$\begin{aligned}
A_1 = A_0^{-1}A &: D(A_1) \subset L^2(\Omega) \to L^2(\Omega), \qquad (3.47)\\
(A_1\theta, \phi) &= \phi(A_1\theta), \ \forall \phi \in L^2(\Omega)\\
D(A_1) &= \{\theta \in L^2(\Omega);\ A\theta \in V'\} = D(\widetilde{A}),
\end{aligned}$$

is quasi m-accretive on $L^2(\Omega)$.
If $\phi \in L^2(\Omega)$, we may take $\phi = A_0\xi$, for $\xi \in V$. Let $u = A_0^{-1}v \in V$. We have

$$\begin{aligned}
(A_1\theta, \phi) &= \langle A_0\xi, u\rangle_{V',V}\\
&= \int_\Omega \nabla\xi \cdot \nabla A_0^{-1}v dx + \int_{\Gamma_\alpha} \alpha\xi A_0^{-1}v d\sigma = \int_\Omega \xi(A_0(A_0^{-1}v))dx\\
&= v(\xi) = A\theta(\xi) = \langle A\theta, \xi\rangle_{V',V}.
\end{aligned}$$

In fact this comes back to

$$(A_1\theta, \phi) = \langle A\theta, \xi \rangle_{V',V} = \left\langle A\theta, A_0^{-1}\phi \right\rangle_{V',V}, \ \forall \phi \in L^2(\Omega), \tag{3.48}$$

where $\xi \in V$ is the solution to the equation $A_0\xi = \phi \in L^2(\Omega)$.

The operator A_1 is quasi-monotone. Indeed, we have

$$(A_1\theta - A_1\theta_1, \theta-\theta_1) = \langle A\theta - A\theta_1, \psi \rangle_{V',V} \geq \frac{\rho}{2}\left\|\theta-\theta_1\right\|^2 - \frac{M^2}{2\rho}\left\|\theta-\theta_1\right\|_{V'}^2, \tag{3.49}$$

where ψ is the solution to the equation $A_0\psi = \theta - \theta_1$ (in the sense of (3.14) and $\|\psi\|_V = \|\theta - \theta_1\|_{V'}$). Hence, since $\|\theta\|_{V'} \leq \|\theta\|$, for $\theta \in L^2(\Omega)$, we get

$$((\mu I + A_1)\theta - (\mu I + A_1)\theta_1, \theta - \theta_1) \geq \left(\mu - \frac{M^2}{2\rho}\right)\left\|\theta-\theta_1\right\|^2 + \frac{\rho}{2}\left\|\theta-\theta_1\right\|^2 \tag{3.50}$$

for μ positive, large enough, $\mu > \mu_0 = \dfrac{M^2}{2\rho}$.

Quasi m-accretivity of A_1.

Further we shall prove that A_1 is quasi m-accretive, for $\mu > \mu_0$ large enough, i.e., for any $\widetilde{f} \in L^2(\Omega)$ there exists $\theta \in D(A_1)$ solution to

$$\mu\theta + A_1\theta = \widetilde{f}. \tag{3.51}$$

Case $\widetilde{f} \in V$.

We prove first for $\widetilde{f} \in V$, by the means of an approximating problem. We have

$$\mu\theta + A_0^{-1}A\theta = \widetilde{f} \in V \tag{3.52}$$

and since the domain of A_1 is exactly $D(\widetilde{A})$ it follows that $\theta \in D(A_1)$ implies $\theta \in V$. We apply A_0 to (3.52) and get

$$\mu A_0\theta + A\theta = A_0\widetilde{f} := G \in V'. \tag{3.53}$$

Approximating problem. We approximate β^* by $\beta_\varepsilon^* \in C^0(\mathbf{R})$ defined by

$$\beta_\varepsilon^*(r) = \begin{cases} \beta^*(r), & r < \theta_s - \varepsilon \\ \dfrac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}(r - \theta_s) + K_s^*, & r \geq \theta_s - \varepsilon, \end{cases} \tag{3.54}$$

which has a bounded derivative for each $\varepsilon > 0$, denoted by

$$\beta_\varepsilon(r) = \begin{cases} \beta(r), & r < \theta_s - \varepsilon \\ \dfrac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}, & r > \theta_s - \varepsilon \end{cases} \tag{3.55}$$

with $\beta_\varepsilon(\theta_\varepsilon) \geq \rho$ (since β^* and β are defined by (1.5) in Sect. 5.1).

We consider the equation

$$\mu A_0\theta + A_\varepsilon\theta = G \in V' \tag{3.56}$$

with the associated operator $\mu A_0 + A_\varepsilon : V \to V'$, defined by

$$\langle(\mu A_0 + A_\varepsilon)\theta, \phi\rangle_{V',V} = \langle\mu A_0\theta, \phi\rangle_{V',V} + \langle A_\varepsilon\theta, \phi\rangle_{V',V}\,, \tag{3.57}$$

where $A_\varepsilon : V \to V'$ is

$$\langle A_\varepsilon\theta, \psi\rangle_{V',V} = \int_\Omega \nabla\beta_\varepsilon^*(\theta)\cdot\nabla\psi dx + \int_{\Gamma_\alpha} \alpha\beta_\varepsilon^*(\theta)\psi d\sigma - \int_\Omega K(\theta)\frac{\partial\psi}{\partial x_3}dx \quad \forall\psi\in V.$$

We shall prove that the operator $\mu A_0 + A_\varepsilon$ defined from V to V' is monotone, bounded, demicontinuous and coercive, for each $\varepsilon > 0$. Indeed, A_0 is linear, continuous and coercive, so it remains to check the properties of A_ε. We have

$$\begin{aligned}\langle A_\varepsilon\theta - A_\varepsilon\theta_1, \theta - \theta_1\rangle_{V',V} &\geq -\frac{M^2}{2\rho}\|\theta - \theta_1\|_{V'}^2 + \frac{\rho}{2}\|\theta - \theta_1\|^2 \\ &\geq -\frac{M^2 c_H^2}{2\rho}\|\theta - \theta_1\|_V^2 + \frac{\rho}{2}\|\theta - \theta_1\|^2\,,\end{aligned}$$

since $\|\theta - \theta_1\|_{V'} \leq \|\theta - \theta_1\| \leq c_H\|\theta - \theta_1\|_V$. Moreover, we have

$$\|A_\varepsilon\theta\|_{V'} \leq c_\alpha(\varepsilon, \alpha_m)\|\theta\|_V + c_0(\varepsilon, \alpha_m),$$

where c_α and c_0 depend on $\dfrac{1}{\varepsilon}$ and $\dfrac{1}{\alpha_m}$. Indeed

$$\begin{aligned}|A_\varepsilon\theta(\phi)| &= \left|\int_\Omega\left(\nabla\beta_\varepsilon^*(\theta)\cdot\nabla\phi - K(\theta)\frac{\partial\phi}{\partial x_3}\right)dx + \int_{\Gamma_\alpha}\alpha\beta_\varepsilon^*(\theta)\phi d\sigma\right| \\ &\leq \frac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}\|\theta\|_V\|\phi\|_V + M\|\theta\|\|\phi\|_V \\ &\quad + \alpha_M\left(\frac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}\|\theta - \theta_s\|_{L^2(\Gamma_\alpha)} + K_s^*\right)\|\phi\|_{L^2(\Gamma_\alpha)} \\ &\leq \left[\frac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}\left(1 + \alpha_M c_{\Gamma_\alpha}^2\right) + Mc_H\right]\|\theta\|_V\|\phi\|_V + c_0(\varepsilon, \alpha_m),\end{aligned}$$

where $\alpha_M = \max_{x\in\Gamma_\alpha}\alpha(x)$ and $K_s^* - \beta^*(\theta_s - \varepsilon) > 0$ for $\varepsilon > 0$, (see (3.54)).

The operator is obviously demicontinuous because for a sequence $\{\theta_n\}_n \in V$, $\theta_n \to \theta$ strongly in V we obtain, using the definition of A_ε and the continuity of the functions $r \to \beta_\varepsilon^*(r)$ and $r \to K(r)$, that $A_\varepsilon\theta_n \to A_\varepsilon\theta$ weakly in V', as $n \to \infty$. Finally, the monotonicity

$$\langle(\mu A_0 + A_\varepsilon)\theta - (\mu A_0 + A_\varepsilon)\theta_1, \theta - \theta_1\rangle_{V',V} \geq \left(\mu - \frac{M^2 c_H^2}{2\rho}\right)\|\theta - \theta_1\|_V^2 + \frac{\rho}{2}\|\theta - \theta_1\|^2,$$

follows for μ large, $\mu > \mu_0 = \dfrac{M^2 c_H^2}{2\rho}$. This implies also the coercivity

$$\langle(\mu A_0 + A_\varepsilon)\theta, \theta\rangle_{V',V} \geq \left(\mu - \frac{M^2 c_H^2}{2\rho}\right) \|\theta\|_V^2 + \frac{\rho}{2}\|\theta\|^2$$

since $(\mu A_0 + A_\varepsilon)0 = 0$.

Thus, the operator $\mu A_0 + A_\varepsilon$ is surjective, so we conclude that equation (3.56) has a unique solution $\theta_\varepsilon \in V$.

A priori estimates. We are going now to obtain some estimates. Let us multiply eq. (3.56) by $\theta_\varepsilon \in V$ and integrate over Ω. By the monotonicity of the operator we obtain

$$\left(\mu - \frac{M^2 c_H^2}{2\rho}\right) \|\theta_\varepsilon\|_V^2 + \frac{\rho}{2}\|\theta_\varepsilon\|^2 \leq \|G\|_{V'} \|\theta_\varepsilon\|_V ,$$

hence

$$\|\theta_\varepsilon\|_V \leq \frac{1}{\mu - C_1(\alpha_m)} \|G\|_{V'} , \; C_1(\alpha_m) = \frac{M^2 c_H^2}{2\rho}. \tag{3.58}$$

Then we multiply eq. (3.56) by $\beta_\varepsilon^*(\theta_\varepsilon) \in V$ and we integrate it over Ω. We have

$$\mu\left(\int_\Omega \nabla\theta_\varepsilon \cdot \nabla\beta_\varepsilon^*(\theta_\varepsilon)dx + \int_{\Gamma_\alpha} \alpha\theta_\varepsilon\beta_\varepsilon^*(\theta_\varepsilon)d\sigma\right) + \int_\Omega (\nabla\beta_\varepsilon^*(\theta_\varepsilon))^2 dx$$
$$+ \int_{\Gamma_\alpha} \alpha(\beta_\varepsilon^*(\theta_\varepsilon))^2 d\sigma - \int_\Omega K(\theta_\varepsilon)\frac{\partial\beta_\varepsilon^*(\theta_\varepsilon)}{\partial x_3}dx = \int G\beta_\varepsilon^*(\theta_\varepsilon)dx.$$

Since $\nabla\beta_\varepsilon^*(\theta_\varepsilon) = \beta_\varepsilon(\theta_\varepsilon)\nabla\theta_\varepsilon$ and $\rho \leq \beta_\varepsilon(\theta_\varepsilon)$, we obtain

$$\mu\rho\|\theta_\varepsilon\|_V^2 + \|\beta_\varepsilon^*(\theta_\varepsilon)\|_V^2 \leq \|G\|_{V'}\|\beta_\varepsilon^*(\theta_\varepsilon)\|_V + Mc_H\|\theta_\varepsilon\|_V\|\beta_\varepsilon^*(\theta_\varepsilon)\|_V ,$$

and

$$\left(\mu\rho - M^2 c_H^2\right)\|\theta_\varepsilon\|_V^2 + \frac{1}{2}\|\beta_\varepsilon^*(\theta_\varepsilon)\|_V^2 \leq \|G\|_{V'}^2 ,$$

wherefrom we deduce that

$$\|\beta_\varepsilon^*(\theta_\varepsilon)\|_V^2 \leq C_2. \tag{3.59}$$

Passing to limit. Therefore we can extract a subsequence such that

$$\theta_\varepsilon \longrightarrow \theta \text{ weakly in } V, \text{ and} \tag{3.60}$$
$$\beta_\varepsilon^*(\theta_\varepsilon) \longrightarrow \eta \text{ weakly in } V. \tag{3.61}$$

Using $\|\theta_\varepsilon\|_V \leq C$ and the compacity of V in $L^2(\Omega)$ we get that

$$\theta_\varepsilon \longrightarrow \theta \text{ strongly in } L^2(\Omega). \tag{3.62}$$

Then, the inclusion $\eta \in \beta^*(\theta)$ a.e. on Ω follows in the same way as was done in Theorem 3.1, Sect. 5.3.

Next, we have nothing else to do than passing to limit in equation (3.56)

$$\mu\left(\int_\Omega \nabla\theta_\varepsilon\cdot\nabla\phi dx+\int_{\Gamma_\alpha}\alpha\theta_\varepsilon\phi d\sigma\right)+\int_\Omega\left(\nabla\beta_\varepsilon^*(\theta_\varepsilon)\cdot\nabla\phi-K(\theta_\varepsilon)\frac{\partial\phi}{\partial x_3}\right)dx$$
$$+\int_{\Gamma_\alpha}\alpha\beta_\varepsilon^*(\theta_\varepsilon)\phi d\sigma=\int_\Omega G\phi dx,\ \forall\phi\in V,$$

wherefrom we obtain that (3.53) has a solution $\theta\in V$, $\beta^*(\theta)\in V$, i.e., $\theta\in D(A)\cap D(A_0)$ and this solution is unique, by the monotonicity of μA_0+A.

Then, applying the inverse A_0^{-1} we get that the equation $\mu\theta+A_1\theta=\widetilde{f}$ has a unique solution $\theta\in D(A_1)$, where $\widetilde{f}\in V$.

Case $\widetilde{f}\in L^2(\Omega)$.

Now, we consider $\widetilde{f}\in L^2(\Omega)$. Then, there exists a sequence $\widetilde{f}_\varepsilon\in V$, with $\widetilde{f}=\lim\widetilde{f}_\varepsilon$ and we write the equation

$$\mu\theta_\varepsilon+A_1\theta_\varepsilon=\widetilde{f}_\varepsilon. \tag{3.63}$$

This equation has a solution $\theta_\varepsilon\in D(A_1)$, $\theta_\varepsilon\to\theta$ strongly in $L^2(\Omega)$ and weakly in V. But $A_1\theta$ is quasi m-accretive on $L^2(\Omega)$, so it is strongly-weakly closed and we have

$$(A_1\theta_\varepsilon,\phi)\to(A_1\theta,\phi),\ \forall\phi\in L^2(\Omega),\ \text{as }\varepsilon\to 0.$$

We pass to the limit as $\varepsilon\to 0$, in (3.63) in $L^2(\Omega)$ and we get that the equation $\mu\theta+A_1\theta=\widetilde{f}$ has a unique solution $\theta\in D(A_1)$. This ends the proof of the quasi m-accretivity of A_1.

We return now to the proof of Lemma 3.2, in fact to the assertion that $\widehat{A}$ given by (3.38) is surjective. We notice that the sum of λA_0^{-1} and $-S_{\lambda,g}$ is monotone, bounded and continuous and since A_1 is quasi m-accretive it follows that the operator $\widehat{A}=\lambda A_0^{-1}-S_{\lambda,g}+A_1$ is quasi m-accretive.

On the other hand, by (3.42), (3.46) and (3.49) we see that

$$\begin{aligned}&(\lambda A_0^{-1}\theta+A_0^{-1}A\theta-S_{\lambda,g}\theta,\theta)\\&\geq\lambda\|\theta\|_{V'}^2+\rho\|\theta\|^2-M\|\theta\|\|\theta\|_{V'}-2\|g\|\|\theta\|-2\lambda b\|\theta\|\\&\geq\frac{\rho}{4}\|\theta\|^2+C,\ \text{for }\lambda>\lambda_0.\end{aligned}$$

Hence $\widehat{A}=\lambda A_0^{-1}-S_{\lambda,g}+A_1$ is surjective as claimed, for $\lambda>\lambda_0$ and (3.37) has an unique solution $\theta\in D(\widehat{A})=D(\widetilde{A})$. With all proved up to now we have deduced in fact that (3.30) has a unique solution $\theta\in D(\widetilde{A})$ and this ends the proof of Lemma 3.2. ∎

Now, we resume the proof of Proposition 3.1 and recall that the previous result is equivalent to the existence of a unique solution to the second resolvent equation (3.29), i.e., we get

$$w = \theta + (\lambda I + H)^{-1}(g - \lambda\theta)$$

and we see that $w \in L^2(\Omega)$, since both terms belong there. Consequently, we have proved that the operator $\mathcal{A}$ is quasi m-accretive as claimed by Proposition 3.1. ∎

Theorem 3.3. *Let*

$$f_{hys} \in W^{1,1}(0,T;V'), \ \ (\theta_0, w_0) \in D(\mathcal{A}),$$
$$f_0 \in W^{1,1}(0,T;L^2(\Gamma_\alpha)), \ \ u_{hys} \in W^{1,1}(0,T;L^2(\Gamma_\alpha)).$$

Then the Cauchy problem (3.19)-(3.20) *has a unique strong solution*

$$(\theta, w) \in W^{1,\infty}(0,T;V' \times L^2(\Omega)) \cap L^\infty(0,T;D(\mathcal{A})), \tag{3.64}$$
$$\beta^*(\theta) \in L^\infty(0,T;V), \ \ H_{a,b}(w-\theta) \in L^\infty(0,T;V). \tag{3.65}$$

The proof is immediate, due to the quasi m-accretivity of $\mathcal{A}$. A priori estimates and a further existence result for data with a worse regularity can be obtained by a usual technique.

6.4 Comments

The models studied in the first two sections were intended to analyze how infiltration develops in situations that lead to degenerate equations and which is the specific behaviour of the solution. We can conclude that in both cases presented, the solution exists and it is unique, but it behaves in different ways, corresponding to the particular aspect that has led to the degeneracy of the equation. For the diffusivity degenerate-model we can notice that the solution loses the regularity it has in the situation when the equation does not degenerate (see Chap. 4, Sect. 4.3). Specifically it does no longer belong to $L^2(0,T;V)$ and this is explained by the fact that the parabolic operator that enhances this regularity in the models studied in Chap. 4, now degenerates.

In the porosity-degenerate model the regularity of the solution is better, because this type of degeneracy leads in fact to an elliptic equation on Q_0, so that the regularizing action of the operator is preserved and the solution w belongs to $L^2(0,T;H^1(\Omega))$. Moreover, the solution in the domain Q_0 had to be constructed with the aid of the solution obtained in Q_m, where the equation does not degenerate.

The study of the model with a variable porosity depending both on time and space can not be conducted by the same method, because the coefficients in the equation (5.1) in Sect. 2.5 become time-dependent. Under the weakly nonlinear saturated-unsaturated model, the approximating problem can be solved using Theorem 4.4 in Sect. 3.4, but under the strongly saturated-unsaturated case one should follow other types of arguments (for instance a result of Kato type).

In the third section, we briefly reviewed another typical aspect arising in infiltration, i.e., the hysteretic behaviour. The topic is very ample and cannot be investigated in detail in a single section but our aim was only to introduce a model with hysteresis, in a different way from those investigated up to now in the literature, and to prove that it is well posed. Even if the model is very simple from the hydraulic point of view, it involves some mathematical interest, because it requires many techniques of the theory of maximal monotone operators.

Bibliographical note

Concerning the degenerate problems an exhaustive overview dealing with linear degenerate operators is comprised in the monograph [61] of A. Favini and A. Yagi. We refer also to the papers [60], [62] and [63] (in which a nonlinear degenerate equation with homogeneous Dirichlet boundary conditions is studied).

Exhaustive presentations of hysteresis theory including also examples of hysteresis occurrence in various physical systems are given in the monographs of M. Brokate and J. Sprekels [35], M.A. Krasnosel'skiĭ and A.V. Pokrovskiĭ [77], P. Krejčí [78], I.D. Mayergoyz [95], A. Visintin [124] and [125]. We mention also the volumes [122] and [123], the latter referring to phase transitions in which hysteresis operators play a basic role. Studies on the infiltration cycle hysteretic behaviour have been published up to now and we mention here the papers [6], [7], [22], [80], [103], [113], [126] and the references given there. In [6], Richards' equation is coupled with a memory effect constitutive law consisting in a hysteresis operator of the *play* type and a rate-dependent component. A weak formulation of the problem is provided and an existence result is proved by time discretization. The result applies also for a *Preisach* hysteretic operator as is finally shown. In the paper [80], an infiltration problem is formulated in the form of a quasi-variational inequality and a time global solution is proved. In [103] the linear transport equation is supplemented with a hysteresis operator and the obtained local model of adsorption-desorption is studied when the hysteresis functional has a symmetric convex graph. The paper [113] describes some results regarding the well-posedness of initial boundary value problems for nonlinear parabolic PDE with memory effects and general boundary conditions. In [126] a forward-backward parabolic problem is obtained by coupling a linear parabolic equation with a nonmonotone relation. The latter is replaced by a relaxation dynamics which converges to a hysteresis operator. An approximate L^1-technique is used to obtain estimates and the existence of the solution is obtained by passing to limit as the relaxation parameter vanishes.

7

Identification of the boundary conditions from recorded observations

Often in natural sciences quantitative real-time information upon the causes which determine the evolution of a physical process may lack. Instead of the measurements of the causative factors, recorded observations upon their effect might be sometimes available. The factors we are speaking about may be especially natural phenomena. In other situations, when the determining factors are derived from a human activity, which is supposed to be controlled, one can act upon them in order to get the desired effect. These two categories of situations have entitled the approach of appropriate mathematical problems, namely the inverse and control problems, which have met a large development in the last decades. Groundwater hydrology, meteorology, agriculture or the management of water resources, and we are referring only to domains of interest for our topic, are by excellence domains which yield inverse and control problems. Specific needs in these domains require, for instance, that certain properties of physical variables be preserved for the considered flow, by controlling the action of certain parameters. The inverse problems which investigate the cause which determines a certain evolution in the state of a physical system are, in their turn, of major practical interest. From the mathematical point of view, both types of problems reduce to the same mathematical framework, namely, the inverse problems can be treated as optimal control problems, too. This chapter deals with some inverse and optimal control problems issued from the theory of water infiltration. They focus on the first order conditions of optimality, i.e., on the maximum principle. The approaches to specific problems which will be studied in this chapter are based on the previous results obtained in the book.

7.1 Basic concepts in the theory of optimal control

We begin by introducing briefly some notions used in the theory of optimal control. Consider two Banach spaces H and H_U. Very often in optimal control problems governed by partial differential equations, and, particulary in our case, H is a function space on a domain $\Omega \subset \mathbf{R}^N$ and f is a nonlinear differential operator on Ω, of the type encountered in the previous sections.

Definition 1.1. A *control system* is represented by a differential equation

$$\frac{dx}{dt}(x) = f(t, x(t), u(t)),\ t \in [t_0, T] \tag{1.1}$$

where $f : [t_0, T] \times H \times H_U \to H$, $x : [t_0, T] \to H$ and $u : [t_0, T] \to H_U$. The function $x(t)$ is the *state* and the measurable vectorial function $u(t)$ is called *control* function or *controller.* The Banach space H_U is called *controller space.*

Definition 1.2. A *solution to the control system* is an absolutely continuous function $x : [t_0, T] \to H$ which satisfies (1.1) a.e. $t \in [t_0, T]$.

Usually the controller is subject to certain constraints of the form $u(t) \in U(t)$ a.e. $t \in [t_0, T]$, where for each t, $U(t)$ is a closed subset of H_U. We denote by $AC([0, T]; H)$ the space of absolutely continuous functions from $[0, T]$ to H, $\mathcal{M}(0, T; H)$ is the space of all Lebesgue measurable functions from $[0, T]$ to H and

$$\mathcal{U} := \{u : [0, T] \to H_U;\ u(t) \in U\}.$$

Definition 1.3. The *cost functional* (or *payoff*) is a function

$$\Psi : C([0, T]; H) \times \mathcal{M}(0, T; H_U) \to (-\infty, \infty]$$

defined by

$$\Psi(x, u) = \int_0^T L(t, x(t), u(t))dt + l(x(0), x(T))$$

where $L : [0, T] \times H \times H_U \to \mathbf{R}$ and $l : H \times H \to (-\infty, \infty]$ are given functions satisfying specified conditions.

The general aim of the inverse problems theory is the following: given certain properties or objectives of the solution of a differential system, one requires to reconstitute the system, namely to find (some of) its parameters which are viewed as control variables. This reduces to looking for a pair (x^*, u^*) which minimizes a cost functional on the set of all admissible pair of functions $(x, u) \in AC([0, T]; H) \times \mathcal{U}$, satisfying the control system. That is why inverse problems are solved as control problems.

Definition 1.4. Let the pair (x^*, u^*) be a solution to the following optimal control problem: "minimize the function Ψ on the set of all $(x, u) \in AC([0, T]; H) \times \mathcal{U}$ satisfying system (1.1)"; u^* is called *optimal control* and x^* is called *optimal state.*

The solution to a control (or inverse problem) consists in accomplishing the following steps: problem statement (including the setting of the cost functional and admissible set for the controller), investigation of the state system (existence, uniqueness, regularity of the solution), proof of the existence of an optimal pair, determination of the necessary conditions of optimality (maximum principle). The last part refers to the determination of the expression of the optimal control, function of the solution to the dual system corresponding to the state one. It requires some auxiliary results including the introduction of the system in variations and the dual system and the proof of their well-posedness. All these results are going then to be used to elaborate a numerical algorithm.

In this last part of the book we shall treat, as examples, a few optimal control problems of interest in the theory of water infiltration in soils.

7.2 The identification problem settlement

Among other inverse problems in infiltration, the identification of the circumstances under which a process has evolved is absolutely necessary and this can be done using available observations of its effect recorded in time.

In this chapter we shall study an identification problem related to a specific mathematical model describing the rainfall type infiltration into a soil in which saturation can be partially or totally reached after some time. Actually, the problem of interest is whether the rain rate that produced a certain moisture of the soil, can be determined having available only scarce observations.

More specifically, we put the problem of retracing the rain history, on the basis of moisture observations recorded in the soil, in two situations. In the first case, the number of observations recorded in the flow domain Ω, at various times within the interval $(0,T)$, is enough to allow computing a reliable time average over $(0,T)$, denoted by

$$\theta^0(x) = \frac{1}{T}\int_0^T \theta^{observed}(x,t)dt. \tag{2.1}$$

From practical considerations, this function can not be considered continuous, but we assume it to be in $L^2(\Omega)$. The other situation is worse, in the sense that observations are very scarce in time, in fact only one observation in the domain, at the final time being available,

$$\theta^T(x) = \theta^{observed}(x,T). \tag{2.2}$$

As we will see later, in order to make possible the solution in this case, this observation should be more regular, $\theta^T \in H^1(\Omega)$.

The mathematical model we consider for these problems is that of strongly nonlinear saturated-unsaturated infiltration with weakly nonlinear hydraulic conductivity, i.e., the model discussed in Chap. 5,

$$\begin{cases} \dfrac{\partial\theta}{\partial t} - \Delta\beta^*(\theta) + \dfrac{\partial K(\theta)}{\partial x_3} \ni f & \text{in } Q, \\ \theta(x,0) = \theta_0(x) & \text{in } \Omega, \\ (K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu \ni u & \text{on } \Sigma_u, \\ (K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu - \alpha\beta^*(\theta) \ni f_0 & \text{on } \Sigma_\alpha, \end{cases} \tag{2.3}$$

which will be called here the *original state system*.

We consider a whatever space domain Ω, with the boundary Γ_u accounting for the soil surface, not necessarily horizontal. All notations and functions occurring in this model have the significance explained in Chap. 5, Sect. 5.1. Thus, β, β^* and K given by (1.5)-(1.6) in Sect. 5.1, are endowed with the properties (i$_\beta$)-(iii$_\beta$), (i)-(iii) and (i$_K$) respectively, and α is a positive, continuous function defined on Γ_α.

Generally, if we consider that the rainfall has an arbitrary direction along the vector d and the physical rate denoted by $u_R(x,t) \geq 0$, the boundary condition at the soil surface is

$$(K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu \ni u_R(x,t)d \cdot \nu := u(x,t).$$

Due to the downward orientation of the Ox_3 axis and to the fact that the flux is always oriented as the outward normal, in our model being opposite to i_3 on Γ_u, it follows that $\cos\iota = d \cdot \nu \leq 0$, and the flux normal component is non-positive, i.e.,

$$u(x,t) = u_R(x,t)d \cdot \nu \leq 0. \tag{2.4}$$

By ι we denoted the angle between the vertical and the outward normal at the soil surface at a certain point, so the cosine of ι may depend on x and t (if the rain changes its direction). For this study we shall consider that the rain direction d remains unchanged in time.

Moreover, a realistic assumption is that of essentially bounded rain rates $u_R(x,t)$ which implies bounded values for $u(x,t)$ too. By a language abuse we shall call u rain rate too, even if it is the normal component of the real rain rate.

We shall consider a more general case, of widely spread rains over large space domains, but taking into account the assumption of local rains, which is enough reliable from the physical point of view, as shown in Fig. 7.1.

Thus, a local rain rate corresponding to a subdomain Γ_i of the soil surface $\Gamma_u = \bigcup_{i=1}^{p} \overline{\Gamma_i}$ is constant with respect to the space variables over Γ_i, being only a function of time. Therefore we write that the global rain rate has the structure

$$u(x,t) = \sum_{i=1}^{p} \omega_i(t) g_i(x) \text{ on } \Sigma_u, \tag{2.5}$$

where g_i are constant non-negative bounded functions on Γ_i. If the surface Γ_i is horizontal, then g_i is the characteristic function of Γ_i. If Γ_i is non-horizontal,

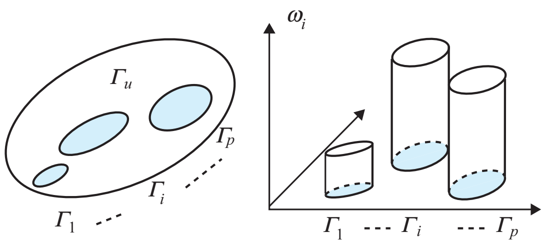

Fig. 7.1. Distribution of the global rain over Γ_u

but its geometry is such that cos ι is constant, then g_i are the characteristic functions of Γ_i, multiplied by $(-\cos\iota)$.

For our mathematical purposes we shall admit that $\cos\iota$ is constant on Γ_i and g_i are enough regular. In fact we shall choose g_i as being a smooth approximation of the characteristic function of Γ_i, that vanishes on the boundary $\partial\Gamma_i$. Specifically the hypotheses on g_i are:

$$g_i \in H^1(\Gamma_i) \cap L^\infty(\Gamma_i),\ 0 \leq g_i \leq g_i^M \text{ a.e. on } \Gamma_i, \tag{2.6}$$

as illustrated in Fig. 7.2. On Γ_j, with $j \neq i$ the function g_i is zero.

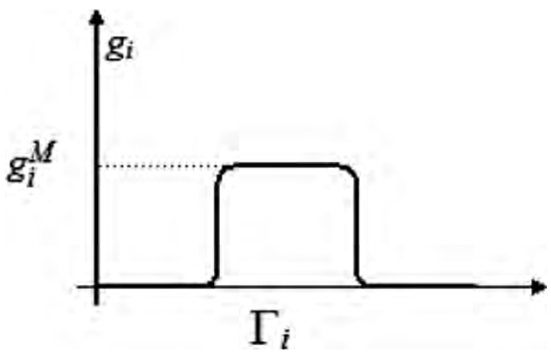

Fig. 7.2. Graphic of the function g_i

Since ω_i are the bounded local rain rates, assigned to Γ_i, we keep for them the convention made for u, i.e., $\omega_i \leq 0$, bounded by a constant R_i. Hence

$$\omega_i \in L^\infty(0,T),\ \ -R_i \leq \omega_i \leq 0,\ \text{a.e. } t \in (0,T),\ i = 1, ..., p. \tag{2.7}$$

Therefore we obtain from (2.5) that

$$u \in L^\infty(0,T; H^1(\Gamma_u)), \tag{2.8}$$

$$-R \leq u \leq 0 \text{ a.e. on } \Sigma_u, \; R = \max_{1\leq i\leq p}\{R_i g_i^M\}. \tag{2.9}$$

We set

$$M^0(\theta) = \frac{1}{T}\int_0^T \theta(x,t)dt. \tag{2.10}$$

With these notations, the problem of rain rate identification using the time average observations is expressed in mathematical terms as the cost functional minimization

$$\min_{\omega\in U}\int_\Omega \left(M^0(\theta) - \theta^0(x)\right)^2 dx \tag{P}$$

where θ is the solution to (2.3), and the admissible set U consists of

$$U = \{\omega = (\omega_1, ..., \omega_p); \; \omega_i \in L^\infty(0,T), \; -R_i \leq \omega_i \leq 0 \text{ a.e. } t \in (0,T)\}. \tag{2.11}$$

It is obvious that U is a closed convex set.

If the identification is made on the basis of the final time observation $\theta^T(x)$, the identification problem changes into

$$\min_{\omega\in U}\int_\Omega \left(\theta(x,T) - \theta^T(x)\right)^2 dx \tag{P_T}$$

where θ is the solution to (2.3).

Both problems will be treated as control problems, in which ω, and correspondingly u, is the controller and θ is the state.

7.3 Identification using time average observations

In this section we shall study the problem (P). The main steps are the following: investigation of the existence and uniqueness in the state system (2.3), proof of the existence of a solution to the problem (P) and determination of the conditions of optimality, which indicate the form of an optimal pair.

Existence in the state system

Since the state system corresponds to a model already studied, we can use all the results obtained in Chap. 5. We shall work within the same framework as that defined in Sect. 5.1, for studying the Cauchy problem (1.29)-(1.30), i.e.,

$$\begin{aligned} \frac{d\theta}{dt} + A\theta &= f + Bu + f_\Gamma, \text{ a.e. } t \in (0,T), \\ \theta(0) &= \theta_0, \end{aligned} \tag{3.1}$$

whose solution is the solution in a generalized sense to (2.3). We introduce the approximating problem (2.3)-(2.4) from Sect. 5.2, replacing β^* by a smooth function, i.e.,

$$\frac{d\theta_\varepsilon}{dt} + A_\varepsilon \theta_\varepsilon = f + Bu + f_\Gamma, \text{ a.e. } t \in (0,T), \qquad (3.2)$$
$$\theta_\varepsilon(0) = \theta_0.$$

Although the approximation β_ε^* defined by (2.1) in Sect. 5.2 is good enough to obtain the existence and uniqueness results for the state system, it will become insufficient for deriving a maximum principle type result. For that we need a better regularity of the approximating solution θ_ε which requires a smoother approximation $\beta_\varepsilon^* \in C^3(\mathbf{R})$. Consequently, we shall approximate from the beginning β^* by the smoother function $\beta_\varepsilon^* \in C^3(\mathbf{R})$, given by (2.60) in Sect. 5.2.

In what concerns K, we extend it to the right of θ_s by the constant value K_s and since this does not depend on ε, we denote the extension still by K.

It is obvious that all the main results given in Sect. 5.2, i.e., Theorem 2.3 (existence of the approximating solution), Theorem 2.6 and Theorem 2.8 (additional regularity of the approximating solution) are true. Moreover, all the results given in Sect. 5.3 for the original solution, i.e., Theorem 3.1 (existence of the solution to the original state system) and Theorem 3.6 (additional regularity of the original solution in the case $N = 1$) apply now, too.

Existence of the optimal control

The existence of a solution to problem (P) will be ensured by an appropriate result. Next, a family of approximating identification problems (P_ε) which approach problem (P) in some sense will be introduced. The approximating problems (P_ε) involve approximating differential equations (with smooth coefficients) instead of the multivalued problem (2.3). The existence of the solution to each problem (P_ε), is based on the result given for problem (P). Then, it will be proved that a sequence of solutions to (P_ε) converges to a solution to (P). These results are necessary for the next step referring to the optimality condition computation. The optimality conditions will be determined first for the approximating problems (P_ε). Then, a discussion about the possibility of deriving the optimality conditions for the original problem will be done.

Due to the form (2.5) of u, we have to keep in mind that a controller ω defines uniquely a rain rate u so that we shall consider corresponding pairs connected by (2.5) and denoted by the same subscript or decoration, e.g., (ω, u), $(\omega_\varepsilon, u_\varepsilon)$, (ω_n, u_n), $(\widetilde{\omega}, \widetilde{u})$, etc.

The following proofs take into account the details induced by the form of u we have considered. First we prove some auxiliary results.

Lemma 3.1. *Let*

$$\omega_n \longrightarrow \widetilde{\omega} \textit{ weak-star in } (L^\infty(0,T))^p.$$

Then

$$u_n \longrightarrow \widetilde{u} \textit{ weak-star in } L^\infty(0,T; H^1(\Gamma_u)).$$

Proof. Let us consider $\{\omega_n\} \subset U$, $\omega_n = (\omega_{n1}, ..., \omega_{np})$. Then, selecting a subsequence, we have

$$\omega_n \longrightarrow \widetilde{\omega} \text{ weak-star in } (L^\infty(0,T))^p \text{ and } \widetilde{\omega} \in U.$$

Consequently, (see (2.8), (2.9)) we have $u_n \in L^\infty(0,T;H^1(\Gamma_u))$ and $-R \leq u_n \leq 0$ a.e. on Σ_u, so that on a subsequence

$$u_n \longrightarrow \widetilde{u} \text{ weak-star in } L^\infty(0,T;H^1(\Gamma_u)).$$

We have to show that $\widetilde{u}$ is obtained from $\widetilde{\omega}$ by (2.5). Indeed, if $\omega_n \to \widetilde{\omega}$ weak-star in $(L^\infty(0,T))^p$ we have

$$\sum_{i=1}^p \int_0^T \omega_{ni}(t)\zeta_i(t)dt \longrightarrow \sum_{i=1}^p \int_0^T \widetilde{\omega}_i(t)\zeta_i(t)dt, \ \forall \zeta \in (L^1(0,T))^p.$$

Then, for $\phi \in L^1(\Sigma_u)$ we can write

$$\int_{\Sigma_u} u_n \phi d\sigma dt = \sum_{i=1}^p \int_{\Gamma_u} g_i(x) \left(\int_0^T \omega_{ni}(t)\phi(x,t)dt \right) d\sigma$$
$$\longrightarrow \sum_{i=1}^p \int_{\Gamma_u} g_i(x) \left(\int_0^T \widetilde{\omega}_i(t)\phi(x,t)dt \right) d\sigma = \int_{\Sigma_u} \widetilde{u}\phi d\sigma dt, \text{ as } n \to \infty.$$

Here we used the fact that

$$\int_0^T \omega_{ni}(t)\phi(x,t)dt \longrightarrow \int_0^T \widetilde{\omega}_i(t)\phi(x,t)dt \text{ strongly in } L^1(\Gamma_u). \qquad (3.3)$$

Indeed, for each $x \in \Omega$ we have $\displaystyle\int_0^T (\omega_{ni}(t) - \widetilde{\omega}_i(t))\phi(x,t)dt \longrightarrow 0$. Next,

$$\left| \int_0^T \omega_{ni}(t)\phi(x,t)dt \right| \leq \int_0^T |\omega_{ni}| \, |\phi(x,t)| \, dt \leq R_i \, \|\phi(t)\|_{L^1(\Gamma_u)}.$$

These two inequalities imply (3.3), by the Lebesgue dominated convergence theorem. The convergence of the derivatives with respect to x_i follows in the same way. ∎

The next result proves the existence of an optimal control in problem (P).

Theorem 3.2. *Let*

$$f \in L^2(0,T;V'), \ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)), \ \theta_0 \in L^2(\Omega), \ \theta_0 \leq \theta_s \ a.e. \ x \in \Omega.$$

Then, problem (P) *has at least one solution.*

Proof. Let $d = \min\limits_{\omega \in U} \left(\int_{\Omega} (M^0(\theta) - \theta^0(x))^2 dx \right)$ and let $\{\omega_n\}_{n \geq 1} \subset U$ be a minimizing sequence, i.e.,

$$d \leq \int_{\Omega} \left(M^0(\theta_n) - \theta^0(x)\right)^2 dx \leq d + \frac{1}{n}, \ n \geq 1, \tag{3.4}$$

where θ_n is the solution to the Cauchy problem (3.1), or the generalized solution to (2.3) with u replaced by $u_n(x,t) = \sum\limits_{i=1}^{p} \omega_{ni}(t) g_i(x)$. Here $\omega_n = (\omega_{n1}, ..., \omega_{np})$. Since $\{\omega_n\} \subset U$, selecting a subsequence, we have

$$\omega_n \longrightarrow \widetilde{\omega} \text{ weak-star in } (L^\infty(0,T))^p \text{ and } \widetilde{\omega} \in U.$$

Consequently, (see (2.8)) $u_n \in L^\infty(0,T;H^1(\Omega))$ and by Lemma 3.1 it follows that

$$u_n \longrightarrow \widetilde{u} \text{ weak-star in } L^\infty(0,T;H^1(\Gamma_u)).$$

Since θ_n is a solution to (3.1), it satisfies the estimates (3.7) and (3.8) of Theorem 3.1, in Sect. 5.3, with the constants $\gamma_0(\alpha_m)$ and $\gamma_1(\alpha_m)$ independent of n. It follows that there exists a subsequence of $\{\theta_n\}_{n \geq 1}$, still denoted in the same way, such that

$$\begin{aligned}
\theta_n &\longrightarrow \widetilde{\theta} && \text{weakly in } W^{1,2}(0, T; V') \cap L^2(0, T; V), \\
\frac{d\theta_n}{dt} &\longrightarrow \frac{d\widetilde{\theta}}{dt} && \text{weakly in } L^2(0,T;V'), \\
\theta_n &\longrightarrow \widetilde{\theta} && \text{strongly in } L^2(0, T; L^2(\Omega)), \\
K(\theta_n) &\longrightarrow K(\widetilde{\theta}) && \text{strongly in } L^2(0, T; L^2(\Omega)), \\
\eta_n &\longrightarrow \widetilde{\eta} && \text{weakly in } L^2(0, T; V),
\end{aligned}$$

where $\eta_n(x,t) \in \beta^*(\theta_n(x,t))$ and $\widetilde{\eta}(x,t) \in \beta^*(\widetilde{\theta}(x,t))$ a.e. $(x,t) \in Q$.
The proof of these convergences is similar to that developed in Theorem 3.1, Sect. 5.3.

In its turn, $\widetilde{\theta}$ is the solution to the problem (3.1), or the generalized solution to (2.3) with u replaced by $\widetilde{u}$. This assertion is obtained by passing to limit in the equality equivalent to (3.1) for $u = u_n$, i.e.,

$$\begin{aligned}
&\int_Q \left(\frac{\partial \theta_n}{\partial t} \phi + \nabla \eta_n \cdot \nabla \phi - K(\theta_n) \frac{\partial \phi}{\partial x_3} \right) dxdt \\
&= \int_Q f \phi dxdt - \int_{\Sigma_\alpha} (\alpha \eta_n + f_0) \phi d\sigma dt - \int_{\Sigma_u} u_n \phi d\sigma dt, \ \forall \phi \in L^2(0,T;V).
\end{aligned}$$

We obtain, using in addition Lemma 3.1, that

$$\begin{aligned}
&\int_Q \left(\frac{\partial \widetilde{\theta}}{\partial t} \phi + \nabla \widetilde{\eta} \cdot \nabla \phi - K(\widetilde{\theta}) \frac{\partial \phi}{\partial x_3} \right) dxdt \\
&= \int_Q f \phi dxdt - \int_{\Sigma_\alpha} (\alpha \widetilde{\eta} + f_0) \phi d\sigma dt - \int_{\Sigma_u} \widetilde{u} \phi d\sigma dt, \ \forall \phi \in L^2(0,T;V).
\end{aligned}$$

The latter equality shows that $\widetilde{\theta}$ is the solution to (3.1) with $u = \widetilde{u}$, or the generalized solution to the state system corresponding to $\widetilde{u}$, i.e., to $\widetilde{\omega}$. Moreover,

$$M^0(\theta_n) \longrightarrow M^0(\widetilde{\theta}) \text{ strongly in } L^2(\Omega), \text{ as } n \to \infty. \tag{3.5}$$

Indeed, because $\theta_n \longrightarrow \widetilde{\theta}$ strongly in $L^2(Q)$ and $\widetilde{\theta} \in C([0,T]; L^2(\Omega))$, we can write

$$\left\| M^0(\theta_n) - M^0(\widetilde{\theta}) \right\| \leq \frac{1}{T} \int_0^T \left\| \theta_n(t) - \widetilde{\theta}(t) \right\| dt \leq \frac{1}{\sqrt{T}} \left\| \theta_n - \widetilde{\theta} \right\|_{L^2(Q)},$$

getting thus (3.5).

Finally, we have to pass to limit in (3.4), taking into account (3.5) and the fact that the strongly lower semicontinuity implies the weakly lower semicontinuity if the function is convex. In our case this is $\theta \longrightarrow \int_\Omega \left(M^0(\theta) - \theta^0(x)\right)^2 dx$, so we obtain

$$d \leq \int_\Omega \left(M^0(\theta) - \theta^0(x)\right)^2 dx \leq d,$$

which shows that the pair $(\widetilde{\theta}, \widetilde{u})$ realizes the minimum of the cost functional and this ends the proof of an optimal pair existence in problem (P). ∎

The approximating control problem

Now we introduce the following approximating identification problem

$$\min_{\omega \in U} \int_\Omega \left(\frac{1}{T} \int_0^T \theta(x,t)dt - \theta^0(x) \right)^2 dx \tag{P_ε}$$

subjected to the system of approximating equations

$$\frac{\partial \theta}{\partial t} - \Delta \beta_\varepsilon^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \text{ in } Q, \tag{3.6}$$

$$\theta(x,0) = \theta_0(x) \text{ in } \Omega, \tag{3.7}$$

$$(K(\theta)i_3 - \nabla \beta_\varepsilon^*(\theta)) \cdot \nu = \nu_\varepsilon \text{ on } \Sigma_u, \tag{3.8}$$

$$(K(\theta)i_3 - \nabla \beta_\varepsilon^*(\theta)) \cdot \nu = \alpha \beta_\varepsilon^*(\theta) + f_0 \text{ on } \Sigma_\alpha, \tag{3.9}$$

where β_ε^* is given by (2.60) in Sect. 5.2. The functional form of this problem is (3.2) with u replaced by ν_ε. The function ν_ε is the regularization of u with respect to t using the mollifier ρ_ε (see Definition 3.16 in Appendix), i.e.,

$$\nu_\varepsilon(x,t) := u(x,t) * \rho_\varepsilon(t), \tag{3.10}$$

$$\nu_\varepsilon(x,t) = \sum_{i=1}^p g_i(x) \int_0^T \omega_i(\tau)\rho_\varepsilon(t-\tau)d\tau. \tag{3.11}$$

(We underline that no confusion with the constant ρ which represents the lower bound of $\beta(\theta)$ should be made).

Consequently we obtain a time regular function ν_ε (see Lemma 3.17 in Appendix, see also Theorem 1.1.5 in [13], pp. 4-6) and retain for our purposes only that

$$\nu_\varepsilon \in W^{1,\infty}(0,T;H^1(\Gamma_u)). \tag{3.12}$$

Corollary 3.3. *Let*

$$f \in L^2(0,T;V'),\ f_0 \in L^2(0,T;L^2(\Gamma_\alpha)),\ \theta_0 \in L^2(\Omega),\ \theta_0 \le \theta_s \text{ a.e. } x \in \Omega.$$

Then problem (P_ε) *has at least one solution.*

The proof is obvious, because the only difference as against the proof of Theorem 3.2 is that β^* and u are replaced by the smoother functions β^*_ε and ν_ε.

We shall show that under appropriate conditions, $\lim\limits_{\varepsilon\to 0}(P_\varepsilon) = (P)$, i.e., a sequence of solutions to (P_ε) tend in some sense to a solution to (P).

Due to the particular form of the approximation ν_ε introduced instead of u, we have to prove first a convergence result for the sequence $\{\nu_\varepsilon\}_{\varepsilon>0}$.

Consider $\omega_\varepsilon \in U$, $\omega_\varepsilon \to \omega \in U$, as $\varepsilon \to 0$, weak-star in $(L^\infty(0,T))^p$ and denote by u_ε and u the functions given by (2.5) corresponding to ω_ε and ω, respectively.

Lemma 3.4. *Consider*

$$\omega_\varepsilon \longrightarrow \omega \text{ weak-star in } (L^\infty(0,T))^p,\ \text{as } \varepsilon \to 0. \tag{3.13}$$

Then

$$\nu^\varepsilon := u_\varepsilon * \rho_\varepsilon \longrightarrow u \text{ weak-star in } L^\infty(\Sigma_u),\ \text{as } \varepsilon \to 0. \tag{3.14}$$

Proof. If $\omega_\varepsilon = (\omega_{\varepsilon 1}, ..., \omega_{\varepsilon p}) \in U$, $\omega_\varepsilon \to \omega$ weak-star in $(L^\infty(0,T))^p$, then we have immediately that the corresponding $u_\varepsilon \in L^\infty(0,T;H^1(\Gamma_u))$ and $u_\varepsilon \to u$ weak-star in $L^\infty(\Sigma_u)$. Consider ϕ arbitrary in $L^1(\Sigma_u)$. We have

$$\begin{aligned}
&\int_0^T \int_{\Gamma_u} \nu^\varepsilon(x,t)\phi(x,t)d\sigma dt \\
&= \sum_{i=1}^p \int_0^T \int_{\Gamma_u} \phi(x,t)g_i(x) \int_0^T \omega_{\varepsilon i}(\tau)\rho_\varepsilon(t-\tau)d\tau d\sigma dt \\
&= \sum_{i=1}^p \int_0^T \int_{\Gamma_u} g_i(x)\omega_{\varepsilon i}(\tau) \left(\int_0^T \phi(x,t)\rho_\varepsilon(t-\tau)dt\right) d\sigma d\tau \\
&= \int_0^T \int_{\Gamma_u} u_\varepsilon(x,\tau)\phi_\varepsilon(x,\tau)d\sigma d\tau,
\end{aligned}$$

where

$$\phi_\varepsilon(x,\tau) = \int_0^T \phi(x,t)\rho_\varepsilon(t-\tau)dt = \int_0^T \phi(x,t)\rho_\varepsilon(\tau-t)dt.$$

We know that the function $\phi_\varepsilon(x,\cdot)$ converges to $\phi(x,\cdot)$ strongly in $L^1(0,T)$, for each $x \in \Gamma_u$ and by the Lebesgue dominated convergence theorem we get that $\phi_\varepsilon \to \phi$ strongly in $L^1(\Sigma_u)$. Since $u_\varepsilon \to u$ weak-star in $L^\infty(\Sigma_u)$ it follows that

$$\int_0^T \int_{\Gamma_u} u_\varepsilon(x,\tau)\phi_\varepsilon(x,\tau)d\sigma dt \longrightarrow \int_0^T \int_{\Gamma_u} u(x,\tau)\phi(x,\tau)d\sigma dt. \tag{3.15}$$

This completes the proof. Obviously, $\nu^\varepsilon = u_\varepsilon * \rho_\varepsilon \to u$ weakly in $L^2(\Sigma_u)$. ∎

We give now the following result:

Lemma 3.5. *Let*

$$f \in L^2(0,T;V'),\ f_0 \in L^2(\Sigma_\alpha),\ \theta_0 \in L^2(\Omega),\ \theta_0 \le \theta_s \text{ a.e. on } \Omega.$$

Consider $\{\omega_\varepsilon\}_{\varepsilon>0} \subset U$, *a sequence such that* $\omega_\varepsilon \to \omega$ *weak-star in* $(L^\infty(0,T))^p$, *and let* θ_ε *be a solution to the approximating problem* (3.6)-(3.9) *corresponding to* ω_ε. *Then there exists a subsequence of* $\{\theta_\varepsilon\}_{\varepsilon>0}$, *such that*

$$\theta_\varepsilon \longrightarrow \theta \quad \text{weakly in } W^{1,2}(0,T;V') \cap L^2(0,T,V), \tag{3.16}$$
$$\text{and strongly in } L^2(Q),$$
$$M^0(\theta_\varepsilon) \longrightarrow M^0(\theta) \quad \text{strongly in } L^2(\Omega),\ \text{as } \varepsilon \to 0, \tag{3.17}$$

where θ *is the solution to the problem* (2.3) *corresponding to* u.

Proof. We recall that the solution to the Cauchy problem (3.2) is the solution in the generalized sense to (3.6)-(3.9).

Let us denote by u_ε the function defined by (2.5) corresponding to ω_ε and consider the approximating problem (3.6)-(3.9) corresponding to u_ε. Since $\nu^\varepsilon \in W^{1,\infty}(0,T;H^1(\Gamma_u))$, where $\nu^\varepsilon(x,t) = \int_0^T u_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)d\tau$, it follows that the approximating problem has a unique strong solution denoted θ_ε. Then, the proof of (3.16) follows exactly the steps of Theorem 3.1, in Sect. 5.3, while for (3.17) we revisit the last part of Theorem 3.2, in this section. We have only to take into account that by Lemma 3.4, $\nu^\varepsilon \to u$ weakly to u in $L^2(\Sigma_u)$. ∎

Theorem 3.6. *Assume that* f, f_0 *and* θ_0 *satisfy*

$$f \in L^2(0,T;V'),\ f_0 \in L^2(\Sigma_\alpha),\ \theta_0 \in L^2(\Omega),\ \theta_0 \le \theta_s \text{ a.e. on } \Omega,$$

and let $(\omega_\varepsilon, \theta_\varepsilon)$ *be a solution to the approximating problem* (P_ε). *Then,*

$$\begin{aligned}
\omega_\varepsilon &\longrightarrow \omega^* && \textit{weak-star in } (L^\infty(0,T))^p,\\
\theta_\varepsilon &\longrightarrow \theta^* && \textit{weakly in } L^2(0,T;V)\cap W^{1,2}(0,T,V')\\
& && \textit{and strongly in } L^2(Q),\\
M^0(\theta_\varepsilon) &\longrightarrow M^0(\theta^*) && \textit{strongly in } L^2(\Omega),
\end{aligned}$$

where $\omega^* \in U$ *and* θ^* *is the solution to the original problem* (2.3) *with* $u = u^*$. *Moreover,* ω^* *is a solution to* (P) *and* $\lim_{\varepsilon\to 0}(P_\varepsilon) = (P)$.

Proof. Let $\widetilde{\omega} \in U$ be a solution to the problem (P) and θ^ε be the solution to the approximating problem (3.2) where $\omega = \widetilde{\omega}$. By the optimality of $(\omega_\varepsilon, \theta_\varepsilon)$ in problem (P_ε) we have

$$\int_\Omega (M^0(\theta_\varepsilon) - \theta^0(x))^2 dx \le \int_\Omega (M^0(\theta^\varepsilon) - \theta^0(x))^2 dx.$$

By Lemma 3.5 it follows that there exists a subsequence of $\{\theta^\varepsilon\}$, such that $\theta^\varepsilon \to \widetilde{\theta}$ strongly in $L^2(Q)$, where $\widetilde{\theta}$ is the solution to (3.1) with $u = \widetilde{u}$ (deduced from $\widetilde{\omega}$). Hence, $(\widetilde{\theta}, \widetilde{\omega})$ follows to be optimal in problem (P). Alike to the last part of Lemma 3.5, it also follows that

$$M^0(\theta^\varepsilon) \longrightarrow M^0(\widetilde{\theta}) \text{ strongly in } L^2(\Omega). \tag{3.18}$$

From these relationships we get

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \int_\Omega (M^0(\theta_\varepsilon) - \theta^0(x))^2 dx &\le \limsup_{\varepsilon\to 0} \int_\Omega (M^0(\theta^\varepsilon) - \theta^0(x))^2 dx\\
&\le \int_\Omega (M^0(\widetilde{\theta}) - \theta^0(x))^2 dx = \min(P).
\end{aligned} \tag{3.19}$$

On the other hand, since $\{\omega_\varepsilon\}_{\varepsilon>0} \subset U$, there exists a subsequence of it, still denoted in the same way, such that $\omega_\varepsilon \to \omega^*$ weak-star in $(L^\infty(0,T))^p$. Consequently, there exists a subsequence of $\{\nu^\varepsilon\}$, such that $\nu^\varepsilon \to u^*$ weakly in $L^2(\Sigma_u)$, where u^* corresponds to ω^*. Then it follows from Lemma 3.5, that on a subsequence, we have that

$$\begin{aligned}
\theta_\varepsilon \longrightarrow \theta^* \text{ strongly in } L^2(Q) \text{ and weakly in } W^{1,2}(0,T;V') \cap L^2(0,T;V),\\
M^0(\theta_\varepsilon) \longrightarrow M^0(\theta^*) \text{ strongly in } L^2(\Omega),
\end{aligned}$$

where θ^* is the solution to (3.1) with $u = u^*$. This implies

$$\min(P) \le \int_\Omega (M^0(\theta^*) - \theta^0(x))^2 dx \le \liminf_{\varepsilon\to 0} \int_\Omega (M^0(\theta_\varepsilon) - \theta^0(x))^2 dx. \tag{3.20}$$

By (3.19) and (3.20) we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega (M^0(\theta_\varepsilon) - \theta^0(x))^2 dx = \int_\Omega (M^0(\theta^*) - \theta^0(x))^2 dx = \min(P).$$

This completes the proof. ∎

Necessary conditions of optimality for the approximating problem

The next step is to determine the necessary conditions of optimality for the problem (P_ε),

$$\min_{\omega\in U}\int_\Omega\left(\frac{1}{T}\int_0^T\theta(x,t)dt-\theta^0(x)\right)^2 dx$$

subjected to (3.6)-(3.9).

The justification for this intermediate step is related to two facts. Since we have proved that a sequence of the optimal pairs of the approximating problem converges to an optimal pair of the original problem, we expect to deduce the original conditions of optimality by passing to limit in the approximating ones. On the other hand, the latter are necessary in view of numerical computations, because in computer programs we may use the single valued function β_ε^* and not the multivalued one β^*.

To begin, we recall the regularity results for the solution to the approximating problem, Theorem 2.6 and Theorem 2.8 in Sect. 5.2, necessary for proving the existence in the systems in variations and in the dual one.

Assume that

$$u\in W^{1,2}(0,T;L^2(\Gamma_u))\cap L^2(0,T;H^1(\Gamma_u)), \tag{3.21}$$

$$f\in W^{1,2}(0,T;L^2(\Omega)), \tag{3.22}$$

$$f_0\in W^{1,2}(0,T;L^2(\Gamma_\alpha))\cap L^2(0,T;H^1(\Gamma_\alpha)), \tag{3.23}$$

$$\theta_0\in H^1(\Omega),\ \theta_0\le\theta_s \text{ a.e. on } \Omega. \tag{3.24}$$

Then, for each $\varepsilon>0$, *the solution to problem* (3.6)-(3.9) *satisfies in addition*

$$\theta_\varepsilon\in W^{1,2}(0,T;L^2(\Omega))\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega)), \tag{3.25}$$

$$\beta_\varepsilon^*(\theta_\varepsilon)\in W^{1,2}(0,T;L^2(\Omega))\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega)). \tag{3.26}$$

Assume (3.21)-(3.23) *and*

$$\theta_0\in H^2(\Omega),\ \theta_0\le\theta_s \text{ a.e. on } \Omega. \tag{3.27}$$

Then, for each $\varepsilon>0$, *the solution to problem* (3.6)-(3.9) *satisfies in addition*

$$\theta_\varepsilon\in W^{1,\infty}(0,T;L^2(\Omega))\cap W^{1,2}(0,T;V)\cap L^\infty(0,T;H^2(\Omega)), \tag{3.28}$$

$$\beta_\varepsilon^*(\theta_\varepsilon)\in W^{1,\infty}(0,T;L^2(\Omega))\cap W^{1,2}(0,T;V)\cap L^\infty(0,T;H^2(\Omega). \tag{3.29}$$

Obviously, under the previous hypotheses and since $\nu_\varepsilon\in W^{1,\infty}(0,T;H^1(\Gamma_u))$, it follows that the solution θ_ε to the approximating problem (3.6)-(3.9) satisfies the conclusions (3.28)-(3.29).

Let $\Sigma_i=\Gamma_i\times(0,T)$ and denote by p_ε the solution to the approximating dual system, which will be written a little farther.

Proposition 3.7. *Assume that* f, f_0 and θ_0 *satisfy the hypotheses* (3.22)-(3.23) *and* (3.27). *Let* $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ *be an optimal pair for the approximating problem* (P_ε). *Then*

$$\begin{cases} \omega_{\varepsilon i}^*(t) = -R_i & \text{on } \left[\int_{\Sigma_i} p_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)g_i(x)d\tau d\sigma > 0\right] \\ \omega_{\varepsilon i}^*(t) \in [-R_i, 0] & \text{on } \left[\int_{\Sigma_i} p_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)g_i(x)d\tau d\sigma = 0\right] \\ \omega_{\varepsilon i}^*(t) = 0 & \text{on } \left[\int_{\Sigma_i} p_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)g_i(x)d\tau d\sigma < 0\right]. \end{cases} \tag{3.30}$$

Proof. We specify e.g., that by $\left[\int_{\Sigma_i} p_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)g_i(x)d\tau d\sigma > 0\right]$ we denote the set

$$\left\{t \in (0,T); \int_{\Sigma_i} p_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)g_i(x)d\tau d\sigma > 0\right\}.$$

Assume that $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ is an optimal pair for problem (P_ε) as follows by Corollary 3.3. A function u_ε^* corresponds by (2.5) to ω_ε^*. We introduce the variation of $\omega_\varepsilon^*(t)$

$$\omega_\varepsilon^\lambda(t) = \omega_\varepsilon^*(t) + \lambda(v_\varepsilon(t) - \omega_\varepsilon^*(t)) \text{ for } v_\varepsilon \in U,\ \lambda > 0 \tag{3.31}$$

and the variation of θ_ε^*, denoted by

$$Y_\varepsilon = \lim_{\lambda \to 0} \frac{\theta_\varepsilon^{\omega_\varepsilon^* + \lambda w_\varepsilon} - \theta_\varepsilon^{\omega_\varepsilon^*}}{\lambda}, \text{ where } w_\varepsilon = v_\varepsilon - \omega_\varepsilon^*. \tag{3.32}$$

We recall that ω_ε^*, $\omega_\varepsilon^\lambda$, v_ε and w_ε are vectors with p components. The variation of u_ε is

$$u_\varepsilon^{var}(x,t) = \sum_{i=1}^p w_{\varepsilon i}(t) g_i(x) \tag{3.33}$$

and the system in variations reads

$$\frac{\partial Y_\varepsilon}{\partial t} - \Delta(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon) + \frac{\partial}{\partial x_3}(K'(\theta_\varepsilon^*)Y_\varepsilon) = 0 \text{ in } Q, \tag{3.34}$$

$$Y_\varepsilon(x,0) = 0 \text{ in } \Omega, \tag{3.35}$$

$$(K_\varepsilon'(\theta_\varepsilon^*)Y_\varepsilon i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon)) \cdot \nu = \nu_\varepsilon^{var} \text{ on } \Sigma_u, \tag{3.36}$$

$$(K'(\theta_\varepsilon^*)Y_\varepsilon i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon)) \cdot \nu = \alpha\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon \text{ on } \Sigma_\alpha, \tag{3.37}$$

where

$$\nu_\varepsilon^{var}(x,t) = \int_0^T u_\varepsilon^{var}(x,\tau)\rho_\varepsilon(t-\tau)d\tau. \tag{3.38}$$

Next, we shall give an existence and uniqueness result for the solution to the system in variations.

Proposition 3.8. *Assume that* f, f_0 and θ_0 *satisfy the hypotheses* (3.22)-(3.23) *and* (3.27). *Then, the system* (3.34)-(3.37) *has, for each* $\varepsilon > 0$, *a unique solution*

$$Y_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V),\ \frac{dY_\varepsilon}{dt} \in L^2 0,T;V'). \tag{3.39}$$

Proof. By Corollary 3.3 we know that the approximating problem (P_ε) has a solution $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ with θ_ε^* satisfying in addition

$$\theta_\varepsilon^* \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{3.40}$$
$$\beta_\varepsilon^*(\theta_\varepsilon^*) \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)). \tag{3.41}$$

First we notice that by the Sobolev embedding theorem we have for any $\eta \in H^1(\Omega) \subset L^6(\Omega)$ that

$$\int_\Omega \eta^4 dx \leq C_f \left\|\eta\right\| \left\|\eta\right\|^3_{H^1(\Omega)},\ \forall \eta \in H^1(\Omega), \tag{3.42}$$

with C_f a constant.

We recall that by the choice of β_ε^* and K it follows that $\beta_\varepsilon(r), \beta_\varepsilon'(r), \beta_\varepsilon''(r)$ are bounded on $\mathbf{R}$, for each ε, and $K'(r) \leq M$, on $\mathbf{R}$, so that by similar calculations like in Theorem 2.6, Sect. 5.2, (see (2.77)) we get

$$\beta_\varepsilon(\theta_\varepsilon^*) \in L^\infty(0,T;H^2(\Omega)). \tag{3.43}$$

Then, its trace exists on Σ_u and Σ_α and

$$\beta_\varepsilon(\theta_\varepsilon^*)|_{\Sigma_\alpha} \in L^\infty(0,T;H^{3/2}(\Gamma_\alpha)),\ \ \beta_\varepsilon(\theta_\varepsilon^*)|_{\Sigma_u} \in L^\infty(0,T;H^{3/2}(\Gamma_u)). \tag{3.44}$$

We introduce the linear operator $A_{Y,\varepsilon}(t) : V \to V'$, by

$$\begin{aligned} &\langle A_{Y,\varepsilon}(t)\phi, \psi\rangle_{V',V} \\ &= \int_\Omega \left(\nabla(\beta_\varepsilon(\theta_\varepsilon^*)\phi)\cdot\nabla\psi - K'(\theta_\varepsilon^*)\phi\frac{\partial\psi}{\partial x_3}\right) dx + \int_{\Gamma_\alpha} \alpha\beta_\varepsilon(\theta_\varepsilon^*)\phi\psi d\sigma,\ \forall\psi \in V, \end{aligned} \tag{3.45}$$

and we write the Cauchy problem

$$\frac{dY_\varepsilon}{dt} + A_{Y,\varepsilon}(t)Y_\varepsilon = B\nu_\varepsilon^{var} \text{ a.e. } t \in (0,T), \tag{3.46}$$
$$Y_\varepsilon(0) = 0. \tag{3.47}$$

Here B is the operator that acts in (3.2), defined in Sect. 5.2.

The operator $A_{Y,\varepsilon}(t)$ is well defined, since $\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon \in V$, as we can see immediately, as a consequence of (3.43) and of the previous properties of the function $\beta_\varepsilon(\theta_\varepsilon^*)$. We notice also that $\nu_\varepsilon^{var}(t) \in H^1(\Gamma_u)$, see (3.12), so

$B\nu_\varepsilon^{var} \in V'$. The function $t \to A_{Y,\varepsilon}(t)u$ is measurable and we have only to check that the operator $A_{Y,\varepsilon}(t)$ is bounded and coercive. Indeed, we have

$$\begin{aligned}
&\langle A_{Y,\varepsilon}(t)\phi, \phi\rangle_{V',V} \qquad (3.48)\\
&= \int_\Omega \left(\beta_\varepsilon(\theta_\varepsilon^*) \left|\nabla\phi\right|^2 + \beta_\varepsilon'(\theta_\varepsilon^*)\phi\nabla\theta_\varepsilon^* \cdot \nabla\phi - K'(\theta_\varepsilon^*)\phi\frac{\partial\phi}{\partial x_3}\right) dx + \int_{\Gamma_\alpha} \alpha\beta_\varepsilon(\theta_\varepsilon^*)\phi^2 d\sigma\\
&\geq \beta_m \left\|\phi\right\|_V^2 - \beta_M'(\varepsilon)\left\|\phi\nabla\theta_\varepsilon^*\right\| \left\|\nabla\phi\right\| - M\left\|\phi\right\| \left\|\phi\right\|_V\\
&\geq \frac{\beta_m}{2}\left\|\phi\right\|_V^2 - \frac{M^2}{\beta_m}\left\|\phi\right\|^2 - \frac{(\beta_M'(\varepsilon))^2}{\beta_m}\left\|\phi\nabla\theta_\varepsilon^*\right\|^2,
\end{aligned}$$

where $\beta_M'(\varepsilon) := \max_{r\in\mathbf{R}} \beta_\varepsilon'(r) < \infty$, for each ε (see the properties of β_ε^* in Sect. 5.2) and

$$\beta_\varepsilon(r) \geq \beta_m > 0, \ \forall r > 0.$$

Using (3.42) we calculate

$$\begin{aligned}
\left\|\phi\nabla\theta_\varepsilon^*\right\|^2 &= \int_\Omega \phi^2 \left|\nabla\theta_\varepsilon^*\right|^2 dx \leq \left(\int_\Omega \phi^4 dx\right)^{1/2} \left(\int_\Omega \left|\nabla\theta_\varepsilon^*\right|^4 dx\right)^{1/2}\\
&\leq C_f^2 \left\|\phi\right\|^{1/2} \left\|\phi\right\|_V^{3/2} \left\|\nabla\theta_\varepsilon^*\right\|^{1/2} \left\|\nabla\theta_\varepsilon^*\right\|_V^{3/2}.
\end{aligned}$$

But θ_ε^* satisfies (3.40) and so

$$\left\|\phi\nabla\theta_\varepsilon^*\right\|^2 \leq C(\varepsilon)\left\|\phi\right\|^{1/2} \left\|\phi\right\|_V^{3/2}. \qquad (3.49)$$

Then we have

$$\frac{(\beta_M'(\varepsilon))^2}{\beta_m}\left\|\phi\nabla\theta_\varepsilon^*\right\|^2 \leq C(\varepsilon)\left\|\phi\right\|^{1/2}\left\|\phi\right\|_V^{3/2} \leq \frac{\beta_m}{4}\left\|\phi\right\|_V^2 + C(\varepsilon)\left\|\phi\right\|^2. \qquad (3.50)$$

Recalling (3.48) we obtain

$$\langle A_{Y,\varepsilon}(t)\phi, \phi\rangle_{V',V} \geq \frac{\beta_m}{4}\left\|\phi\right\|_V^2 - \left(\frac{M^2}{\beta_m} + C(\varepsilon)\right)\left\|\phi\right\|^2. \qquad (3.51)$$

Further we calculate

$$\begin{aligned}
\left|A_{Y,\varepsilon}(t)\phi(\psi)\right| &\leq \beta_M(\varepsilon)\left\|\phi\right\|_V \left\|\psi\right\|_V + \beta_M'(\varepsilon)\left\|\phi\nabla\theta_\varepsilon^*\right\|_V \left\|\psi\right\|_V\\
&+ M\left\|\phi\right\| \left\|\psi\right\|_V + \alpha_M\beta_M(\varepsilon)\left\|\phi\right\|_{L^2(\Gamma_\alpha)} \left\|\psi\right\|_{L^2(\Gamma_\alpha)},
\end{aligned}$$

where $\beta_M(\varepsilon) := \max_{r\in\mathbf{R}} \beta_\varepsilon(r) < \infty$, for each ε. By (3.49) we obtain

$$\left\|\phi\nabla\theta_\varepsilon^*\right\| \leq C(\varepsilon)\left\|\phi\right\|^{1/4}\left\|\phi\right\|_V^{3/4} \leq C(\varepsilon)\left\|\phi\right\|_V$$

so that we finally can write that

$$\left|A_{Y,\varepsilon}(t)\phi(\psi)\right| \leq C(\varepsilon)\left\|\phi\right\|_V \left\|\psi\right\|_V \qquad (3.52)$$

and in conclusion we deduce that

$$\|A_{Y,\varepsilon}(t)\phi\|_{V'} \le C(\varepsilon)\,\|\phi\|_V\,. \tag{3.53}$$

By $C(\varepsilon)$ we have denoted several constants independent of and dependent on ε, respectively. It follows that the operator $A_{Y,\varepsilon}(t)$ satisfies the hypotheses of Lions' theorem, so we conclude that the system (3.34)-(3.37) has a unique solution (3.39), as claimed. This ends the proof of Proposition 3.8. ■

We continue now the proof of Proposition 3.7 by introducing the dual system

$$\frac{\partial p_\varepsilon}{\partial t} + \beta_\varepsilon(\theta_\varepsilon^*)\Delta p_\varepsilon + K'(\theta_\varepsilon^*)\frac{\partial p_\varepsilon}{\partial x_3} = F_\varepsilon(x) \ \text{ in } \ Q, \tag{3.54}$$

$$p_\varepsilon(x,T) = 0 \ \text{ in } \ \Omega, \tag{3.55}$$

$$\nabla p_\varepsilon \cdot \nu = 0 \ \text{ on } \ \Sigma_u, \tag{3.56}$$

$$\alpha p_\varepsilon + \nabla p_\varepsilon \cdot \nu = 0 \ \text{ on } \ \Sigma_\alpha, \tag{3.57}$$

where

$$F_\varepsilon(x) = \frac{1}{T}\int_0^T (\theta_\varepsilon^*(x,t) - \theta^0(x))dt. \tag{3.58}$$

We shall give the following result:

Proposition 3.9. *Assume that f, f_0 and θ_0 satisfy the hypotheses* (3.22)-(3.23) *and* (3.27). *Then, the system* (3.54)-(3.57) *has a unique solution*

$$p_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V), \ \frac{dp_\varepsilon}{dt} \in L^2(0,T;V'). \tag{3.59}$$

Proof. We change the variable t to $\widetilde{t} = T - t$ and the system becomes

$$\frac{\partial \widetilde{p}_\varepsilon}{\partial \widetilde{t}} - \beta_\varepsilon(\widetilde{\theta}_\varepsilon^*)\Delta \widetilde{p}_\varepsilon - K'(\widetilde{\theta}_\varepsilon^*)\frac{\partial \widetilde{p}_\varepsilon}{\partial x_3} = -F_\varepsilon(x) \ \text{ in } \ Q, \tag{3.60}$$

$$\widetilde{p}_\varepsilon(x,0) = 0 \ \text{ in } \ \Omega, \tag{3.61}$$

$$\nabla \widetilde{p}_\varepsilon \cdot \nu = 0 \ \text{ on } \ \Sigma_u, \tag{3.62}$$

$$\alpha \widetilde{p}_\varepsilon + \nabla \widetilde{p}_\varepsilon \cdot \nu = 0 \ \text{ on } \ \Sigma_\alpha, \tag{3.63}$$

where $\widetilde{p}_\varepsilon(x,\widetilde{t}) = p_\varepsilon(x, T - \widetilde{t})$ and $\widetilde{\theta}_\varepsilon^*(x,\widetilde{t}) = \theta_\varepsilon^*(x, T - \widetilde{t})$. We introduce the operator $A_{p,\varepsilon}(t) : V \to V'$ by

$$\left\langle A_{p,\varepsilon}(\widetilde{t})p, \psi \right\rangle_{V',V} \tag{3.64}$$

$$= \int_\Omega (\beta_\varepsilon(\widetilde{\theta}_\varepsilon^*)\nabla p \cdot \nabla\psi + \psi Z_\varepsilon^* \cdot \nabla p)dx + \int_{\Gamma_\alpha} \alpha\beta_\varepsilon(\widetilde{\theta}_\varepsilon^*)p\psi d\sigma, \ \forall \psi \in V,$$

where

$$Z_\varepsilon^* = \nabla\beta_\varepsilon(\widetilde{\theta_\varepsilon^*}) - K'(\widetilde{\theta_\varepsilon^*})i_3 \tag{3.65}$$

and we consider the Cauchy problem

$$\frac{d\,\widetilde{p}_\varepsilon}{d\,\widetilde{t}} + A_{p,\varepsilon}(\widetilde{t})\,\widetilde{p}_\varepsilon = -F_\varepsilon,\quad \widetilde{p}_\varepsilon\,(0) = 0. \tag{3.66}$$

We recall that since $\theta_0 \leq \theta_s$ a.e. $x \in \Omega$, we can use the relationship (2.44) in Theorem 3.2, Sect. 5.2,

$$\|\theta_\varepsilon^*\|_{L^2(Q)} \leq C_S, \tag{3.67}$$

which used in (3.58) implies $F_\varepsilon \in L^2(\Omega)$. Then applying Lions' theorem, it follows that problem (3.66) has a unique solution,

$$\widetilde{p}_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V),\ \frac{\partial\widetilde{p}_\varepsilon}{\partial\widetilde{t}} \in L^2(0,T;V'),$$

implying (3.59). This ends the proof of Proposition 3.9. ∎

Now, we come back again to the proof of Proposition 3.7. We multiply (3.34) by p_ε and integrate it over Q. We get

$$-\int_Q\left(\frac{\partial p_\varepsilon}{\partial t} + \beta_\varepsilon(\theta_\varepsilon^*)\Delta p_\varepsilon + K'(\theta_\varepsilon^*)\frac{\partial p_\varepsilon}{\partial x_3}\right)Y_\varepsilon dxdt + \int_\Omega p_\varepsilon Y_\varepsilon\big|_0^T dx \tag{3.68}$$
$$+\int_0^T\int_\Gamma p_\varepsilon q_\varepsilon^Y\cdot\nu d\sigma dt + \int_0^T\int_{\Gamma_\alpha}\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon\nabla p_\varepsilon\cdot\nu d\sigma dt = 0,$$

where we denoted q_ε^Y the variation of the flux vector, $q_\varepsilon^Y = K'(\theta_\varepsilon^*)Y_\varepsilon i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon)$.

Applying the initial and boundary conditions for Y_ε we get

$$-\int_Q\left(\frac{\partial p_\varepsilon}{\partial t} + \beta_\varepsilon(\theta_\varepsilon^*)\Delta p_\varepsilon + K'(\theta_\varepsilon^*)\frac{\partial p_\varepsilon}{\partial x_3}\right)Y_\varepsilon dxdt + \int_\Omega p_\varepsilon(x,T)Y_\varepsilon(x,T)dx$$
$$+\int_0^T\int_{\Gamma_u} p_\varepsilon\nu_\varepsilon^{var}d\sigma dt + \int_0^T\int_{\Gamma_\alpha}\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon(\alpha p_\varepsilon + \nabla p_\varepsilon\cdot\nu)d\sigma dt = 0$$

Using then (3.54)-(3.58) we obtain

$$\int_0^T\int_{\Gamma_u} p_\varepsilon\nu_\varepsilon^{var}d\sigma dt = \int_0^T\int_\Omega F_\varepsilon(x)Y_\varepsilon(x,t)dxdt. \tag{3.69}$$

By the assumption that $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ is optimal we can write

$$\int_\Omega\left(M^0(\theta_\varepsilon^\lambda) - \theta^0\right)^2 dx \geq \int_\Omega\left(M^0(\theta_\varepsilon^*) - \theta^0\right)^2 dx$$

and from here we deduce that

$$\int_\Omega M^0(Y_\varepsilon)M^0(\theta_\varepsilon^* - \theta^0)dx \geq 0,$$

or still

$$\int_0^T \int_\Omega Y_\varepsilon(x,t)F_\varepsilon(x)dxdt \geq 0. \tag{3.70}$$

By (3.69) and (3.70) we deduce the condition

$$\int_0^T \int_{\Gamma_u} p_\varepsilon \nu_\varepsilon^{var} d\sigma dt = \int_0^T \int_{\Gamma_u} p_\varepsilon u_\varepsilon^{var} * \rho_\varepsilon d\sigma dt \geq 0. \tag{3.71}$$

This can be still written by (3.33) and (3.38) as

$$\sum_{i=1}^p \int_0^T \int_{\Gamma_u} p_\varepsilon(x,t) \left(\int_0^T w_{\varepsilon i}(\tau) g_i(x) \rho_\varepsilon(t-\tau) d\tau \right) d\sigma dt \geq 0.$$

Inverting the order of the integrals we obtain that

$$\sum_{i=1}^p \int_0^T (\omega_{\varepsilon i}^*(\tau) - v_{\varepsilon i}(\tau)) \left(-\int_{\Gamma_u} g_i(x) \int_0^T p_\varepsilon(x,t)\rho_\varepsilon(t-\tau)dtd\sigma \right) d\tau \tag{3.72}$$

$$= \sum_{i=1}^p \int_0^T (\omega_{\varepsilon i}^*(\tau) - v_{\varepsilon i}(\tau)) \left(-\int_{\Sigma_u} g_i(x) p_\varepsilon(x,t)\rho_\varepsilon(\tau - t)dtd\sigma \right) d\tau \geq 0,$$

for any $v_\varepsilon = (v_{\varepsilon 1}, ..., v_{\varepsilon p}) \in U$. Inequality (3.72) means that

$$-\int_{\Sigma_u} p_\varepsilon(x,t)\rho_\varepsilon(\tau - t) g_i(x) dtd\sigma \in \partial I_U(\omega_\varepsilon^*), \tag{3.73}$$

where ∂I_U is the subdifferential of the indicator function of the set U, denoted I_U. But $\partial I_U(\omega_\varepsilon^*)$ is exactly the normal cone of the set U at the point ω^*, denoted $N_U(\omega_\varepsilon^*)$, (see Example 5.11 in Appendix). Since we notice that $U = \prod_{i=1}^p [-R_i, 0]$ it follows that $N_U(\omega_\varepsilon^*) = \left\{ N_{U_{R_i}}(\omega_{\varepsilon i}^*) \right\}_{1 \leq i \leq p}$, where $U_{R_i} = [-R_i, 0]$.

Therefore we obtain that

$$\begin{cases} -\int_{\Sigma_u} p_\varepsilon(x,t)\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma \leq 0 \text{ if } \omega_{\varepsilon i}^*(\tau) = -R_i \\ -\int_{\Sigma_u} p_\varepsilon(x,t)\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma = 0 \text{ if } \omega_{\varepsilon i}^*(\tau) \in (-R_i, 0) \\ -\int_{\Sigma_u} p_\varepsilon(x,t)\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma \geq 0 \text{ if } \omega_{\varepsilon i}^*(\tau) = 0. \end{cases}$$

It is obvious that

$$\int_{\Sigma_u} p_\varepsilon(x,t)\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma = \int_{\Sigma_i} p_\varepsilon(x,t)\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma.$$

In conclusion we get the final condition of optimality

$$\begin{cases} \omega_{\varepsilon i}^*(\tau) = -R_i & \text{on } \{\tau \in (0,T);\ \int_{\Sigma_i} (-p_\varepsilon(x,t))\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma < 0\} \\ \omega_{\varepsilon i}^*(\tau) \in [-R_i, 0] & \text{on } \{\tau \in (0,T); \int_{\Sigma_i} (-p_\varepsilon(x,t))\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma = 0\} \\ \omega_{\varepsilon i}^*(\tau) = 0 & \text{on } \{\tau \in (0,T);\ \int_{\Sigma_i} (-p_\varepsilon(x,t))\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma > 0\}, \end{cases}$$

which is in fact (3.30) and the proof of Proposition 3.7 is finished. ∎

The conditions of optimality for the original problem

We are ready now to study the possibility to determine the optimality conditions for the original problem (P)

$$\min_{\omega\in U}\int_{\Omega}\left(\frac{1}{T}\int_0^T\theta(x,t)dt-\theta^0(x)\right)^2dx$$

subject to (3.1).

Due to Theorem 3.6 which states that the sequence of solutions $(\omega_\varepsilon^*,\theta_\varepsilon^*)$ to problem (P_ε) tends to a solution (ω^*,θ^*) of problem (P), we shall try to determine the optimality conditions for (P) by passing to limit in (3.30).

But here we face with the fact that the in general inverse problems are not well-posed, specifically that their solutions are not unique. Therefore, nothing ensures us that the limit of $\{\omega_\varepsilon\}_{\varepsilon>0}$ tends exactly to the envisaged ω^*. Moreover, in (3.30) we deal only with weak convergences (for the controller and the normal cone). Since the monotone maximal operators are strongly-weakly closed, at least a strongly convergence is needed in order to apply this result by passing to limit. To ensure the strongly convergence of the sequence of controllers to a fixed considered optimal control ω^* we have to introduce further the adapted penalization procedure, see [10].

Before that, let us prove a further regularity for the solution p_ε to the dual system (3.54)-(3.57).

By C we shall denote several constants independent of ε.

Theorem 3.10. *Under the hypotheses* (3.22)-(3.23) *and* (3.27) *the solution* p_ε *to the dual problem* (3.54)-(3.57) *satisfies in addition*

$$\|p_\varepsilon\|_{L^2(0,T;H^2(\Omega))}\le C,\ \ \|p_\varepsilon\|_{L^\infty(0,T;V)}\le C, \tag{3.74}$$

where C *is independent of* ε.

Proof. Since we do not know a priori that $p_\varepsilon\in L^2(0,T;H^2(\Omega))$, a rigorous calculus should be done by approximating equation (3.54) by a finite difference equation which is elliptic. To shorten the computations we multiply (3.54) by Δp_ε and integrate over $\Omega\times(t,T)$, for any $t\in(0,T)$. We have

$$\begin{aligned}&\int_t^T\int_\Omega\frac{\partial p_\varepsilon}{\partial\tau}\Delta p_\varepsilon dxd\tau+\int_t^T\int_\Omega\beta_\varepsilon(\theta_\varepsilon^*)(\Delta p_\varepsilon)^2dxd\tau\\&+\int_t^T\int_\Omega K'(\theta_\varepsilon^*)\frac{\partial p_\varepsilon}{\partial x_3}\Delta p_\varepsilon dxd\tau=\int_t^T\int_\Omega F_\varepsilon\Delta p_\varepsilon dxd\tau.\end{aligned}\tag{3.75}$$

Therefore, recalling that $|K'(\theta_\varepsilon^*)|\le M$ and $F_\varepsilon\in L^2(\Omega)$ we have, using the Gauss-Ostrogradsky formula, that

$$\begin{aligned}&\int_t^T\int_\Gamma\frac{\partial p_\varepsilon}{\partial\tau}\nabla p_\varepsilon\cdot\nu d\sigma d\tau-\int_t^T\int_\Omega\nabla\left(\frac{\partial p_\varepsilon}{\partial\tau}\right)\cdot\nabla p_\varepsilon dxd\tau+\int_t^T\int_\Omega\beta_\varepsilon(\theta_\varepsilon^*)(\Delta p_\varepsilon)^2dxd\tau\\&\le M\int_t^T\left\|\frac{\partial p_\varepsilon}{\partial x_3}(\tau)\right\|\|\Delta p_\varepsilon(\tau)\|\,d\tau+\int_t^T\|F_\varepsilon\|\,\|\Delta p_\varepsilon(\tau)\|\,d\tau.\end{aligned}$$

After some calculations in which we use the conditions (3.55)-(3.57) we obtain for the sum of the two first terms on the left-hand side that

$$\begin{aligned}
&\int_t^T \int_\Gamma \frac{\partial p_\varepsilon}{\partial \tau} \nabla p_\varepsilon \cdot \nu d\sigma d\tau - \int_t^T \int_\Omega \nabla \left(\frac{\partial p_\varepsilon}{\partial \tau} \right) \cdot \nabla p_\varepsilon dx d\tau \\
&= \int_t^T \int_{\Gamma_\alpha} \frac{\partial p_\varepsilon}{\partial \tau} (-\alpha p_\varepsilon) d\sigma d\tau - \frac{1}{2} \int_t^T \frac{\partial}{\partial \tau} \int_\Omega |\nabla p_\varepsilon|^2 \, dx d\tau \\
&= -\frac{1}{2} \int_t^T \frac{\partial}{\partial \tau} \int_{\Gamma_\alpha} \alpha p_\varepsilon^2 d\sigma d\tau + \frac{1}{2} \int_\Omega |\nabla p_\varepsilon(t)|^2 \, dx \\
&= \frac{1}{2} \int_{\Gamma_\alpha} \alpha p_\varepsilon^2(t) d\sigma + \frac{1}{2} \int_\Omega |\nabla p_\varepsilon(t)|^2 \, dx = \frac{1}{2} \|p_\varepsilon(t)\|_V^2 .
\end{aligned}$$

Hence, since $\beta_\varepsilon(\theta_\varepsilon^*) \geq \beta_m > 0$, (see (2.60) in Sect. 5.2), we deduce from (3.75) that

$$\begin{aligned}
&\frac{1}{2} \|p_\varepsilon(t)\|_V^2 + \beta_m \int_t^T \int_\Omega (\Delta p_\varepsilon)^2 dx d\tau \leq \frac{\beta_m}{4} \int_t^T \int_\Omega (\Delta p_\varepsilon)^2 dx d\tau \\
&+ \frac{M^2}{\beta_m} \int_t^T \|p_\varepsilon(\tau)\|_V^2 \, d\tau + \frac{\beta_m}{4} \int_t^T \int_\Omega (\Delta p_\varepsilon)^2 dx d\tau + \frac{1}{\beta_m} \int_t^T \|F_\varepsilon\|^2 \, d\tau.
\end{aligned}$$

Finally we obtain

$$\|p_\varepsilon(t)\|_V^2 + \beta_m \int_t^T \int_\Omega (\Delta p_\varepsilon)^2 dx d\tau \leq \frac{2M^2}{\beta_m} \int_t^T \|p_\varepsilon(\tau)\|_V^2 \, d\tau + \frac{2}{\beta_m} \int_t^T \|F_\varepsilon\|^2 \, d\tau.$$

Now, we use once again (3.67) and $\theta^0 \in L^2(\Omega)$, and get that the last term in the right-hand side in the previous inequality is independent on ε, because

$$\|F_\varepsilon\| \leq \frac{1}{T} \int_0^T \left\| \theta_\varepsilon^*(t) - \theta^0 \right\| dt \leq \frac{1}{\sqrt{T}} \left\| \theta_\varepsilon^* - \theta^0 \right\|_{L^2(Q)} \leq C.$$

Using Gronwall's lemma we obtain that

$$\|p_\varepsilon(t)\|_V \leq C, \ \forall t \in [0, T], \tag{3.76}$$

independently of ε. Hence

$$\{p_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^\infty(0, T; H^1(\Omega)), \tag{3.77}$$

and

$$\{\Delta p_\varepsilon\}_{\varepsilon > 0} \text{ is bounded in } L^2(0, T; L^2(\Omega)), \tag{3.78}$$

which implies that

$$\{p_\varepsilon\} \text{ is bounded in } L^2(0, T; H^2(\Omega)), \tag{3.79}$$

as claimed. The latter assertion is based on the fact that we have

$$\nabla p_\varepsilon \cdot \nu = 0 \ \in L^2(0,T;L^2(\Gamma_u)), \ \nabla p_\varepsilon \cdot \nu = -\alpha p_\varepsilon \in L^2(0,T;L^2(\Gamma_\alpha)).$$

According to a known result (see [84]) these last two together with (3.78) yield (3.79). ∎

Corollary 3.11. *Under the hypotheses* (3.22)-(3.23) *and* (3.27) *the solution* p_ε *to the dual problem* (3.54)-(3.57) *converges on a subsequence as follows:*

$$p_\varepsilon \longrightarrow p \text{ weakly in } L^2(0,T;H^2(\Omega)), \tag{3.80}$$

$$p_\varepsilon \longrightarrow p \text{ weak-star in } L^\infty(0,T;H^1(\Omega)), \tag{3.81}$$

$$\begin{aligned} &\int_0^T \int_{\Gamma_i} p_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)g_i(x)d\sigma d\tau \\ &\longrightarrow \int_{\Gamma_i} p(x,t)g_i(x)d\sigma \text{ weakly in } L^2(0,T). \end{aligned} \tag{3.82}$$

Proof. The first two conclusions are obvious from the boundedness of $\{p_\varepsilon\}_{\varepsilon>0}$ in the spaces $L^2(0,T;H^2(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$, while the last one is derived by a similar proof to Lemma 3.4. ∎

The penalized problem

To prove the strongly convergence of ω_ε to ω^* we apply the adapted penalization method introduced in [10]. For our problem we define the penalized minimization problem $(\widetilde{P_\varepsilon})$ in the following way:

$$\begin{aligned} &\min_{\omega\in U} \Psi(\omega) \\ &= \min_{\omega\in U} \left\{ \int_\Omega \left(\frac{1}{T}\int_0^T \theta(x,t)dt - \theta^0(x) \right)^2 dx + \frac{1}{2}\sum_{i=1}^p \int_0^T (\omega_i(t)-\omega_i^*(t))^2 dt \right\} \end{aligned} \tag{3.83}$$

subjected to the approximating problem (3.6)-(3.9) where ω^* is optimal in problem (P).

Concerning the problem $(\widetilde{P_\varepsilon})$ we can notice without any difficulty that, under the appropriate hypotheses, it has at least a solution and for the proof we can follow the same steps like in Theorem 3.2. In the next theorem we shall prove the existence of a minimum of the penalized functional and the most important result stating that the sequence of solutions to $(\widetilde{P_\varepsilon})$ tends strongly exactly to that ω^* fixed in (3.83).

Theorem 3.12. *Assume the hypotheses* (3.22)-(3.23) *and* (3.27) *upon* f, f_0 *and* θ_0, *and let* $(\widetilde{\omega}_\varepsilon, \widetilde{\theta}_\varepsilon)$ *be optimal in problem* $(\widetilde{P_\varepsilon})$. *Then,*

$$\widetilde{\omega}_\varepsilon \longrightarrow \omega^* \text{ strongly in } (L^2(0,T))^p, \ \widetilde{\theta}_\varepsilon \longrightarrow \theta^* \text{ strongly in } L^2(Q), \tag{3.84}$$

where (ω^*, θ^*) *is optimal in problem* (P).

Proof. Assume now that $(\widetilde{\omega}_\varepsilon, \widetilde{\theta}_\varepsilon)$ is optimal in problem $(\widetilde{P_\varepsilon})$ and denote by θ_ε^* the solution to the approximating problem (3.6)-(3.9) corresponding to ω^* (which was considered to be optimal in problem (P)). Therefore we can write

$$\int_\Omega \left(\frac{1}{T} \int_0^T \widetilde{\theta}_\varepsilon(x,t)dt - \theta^0(x) \right)^2 dx + \frac{1}{2} \sum_{i=1}^p \int_0^T (\widetilde{\omega}_{\varepsilon i}(t) - \omega_i^*(t))^2 dt \tag{3.85}$$
$$\leq \int_\Omega \left(\frac{1}{T} \int_0^T \theta_\varepsilon^*(x,t)dt - \theta^0(x) \right)^2 dx + \frac{1}{2} \sum_{i=1}^p \int_0^T (\omega_i^*(t) - \omega_i^*(t))^2 dt.$$

Letting ε tend to 0, we get like in Theorem 3.6 that

$$\widetilde{\omega}_\varepsilon \longrightarrow \widetilde{\omega} \text{ weak-star in } (L^\infty(0,T))^p,$$
$$\widetilde{\nu}^\varepsilon = \widetilde{u}_\varepsilon * \rho_\varepsilon \longrightarrow \widetilde{u} \text{ weak-star in } L^\infty(\Sigma_u),$$

$$\widetilde{\theta}_\varepsilon \longrightarrow \widetilde{\theta} \text{ weakly in } L^2(0,T;\, V) \cap W^{1,2}(0,T,\, V') \text{ and strongly in } L^2(Q),$$
$$M^0(\widetilde{\theta}_\varepsilon) \longrightarrow M^0(\widetilde{\theta}) \text{ strongly in } L^2(\Omega),$$
$$\theta_\varepsilon^* \longrightarrow \theta^* \text{ weakly in } L^2(0,T;\, V) \cap W^{1,2}(0,T,\, V') \text{ and strongly in } L^2(Q),$$
$$M^0(\theta_\varepsilon^*) \longrightarrow M^0(\theta^*) \text{ strongly in } L^2(\Omega),$$

where (ω^*, θ^*) is optimal in problem (P), by Theorem 3.6. Moreover, it follows by the same theorem that the pair $(\widetilde{\omega}, \widetilde{\theta})$ is optimal in problem (P). Hence

$$\int_\Omega \left(\frac{1}{T} \int_0^T \widetilde{\theta}(x,t)dt - \theta^0(x) \right)^2 dx + \frac{1}{2} \limsup_{\varepsilon \to 0} \sum_{i=1}^p \int_0^T (\widetilde{\omega}_{\varepsilon i}(t) - \omega_i^*(t))^2 dt$$
$$\leq \int_\Omega \left(\frac{1}{T} \int_0^T \theta^*(x,t)dt - \theta^0(x) \right)^2 dx.$$

Taking into account that both pairs are optimal we can write that

$$\int_\Omega \left(\frac{1}{T} \int_0^T \theta^*(x,t)dt - \theta^0(x) \right)^2 dx \leq \int_\Omega \left(\frac{1}{T} \int_0^T \widetilde{\theta}(x,t)dt - \theta^0(x) \right)^2 dx$$

which implies

$$\sum_{i=1}^p \int_0^T (\widetilde{\omega}_i(t) - \omega_i^*(t))^2 dt = 0.$$

This yields

$$\widetilde{\omega}(x,t) = \omega^*(x,t) \text{ a.e. in } (L^2(0,T))^p \tag{3.86}$$

and consequently we get $\widetilde{\theta}(x,t) = \theta^*(x,t)$ a.e. on Q.

By passing to limit in (3.85) we obtain after some calculations that

$$\omega_i^* = \lim_{\varepsilon\to 0} \omega_{\varepsilon i}^* \text{ strongly in } L^2(0,T),$$

as claimed. ∎

We are going now to determine the conditions of optimality for the approximating problem $(\widetilde{P_\varepsilon})$.

Proposition 3.13. *Assume that* f, f_0 and θ_0 *satisfy the hypotheses* (3.22)-(3.23) *and* (3.27). *Let* $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ *be an optimal pair for the approximating problem* $(\widetilde{P_\varepsilon})$. *Then*

$$-\int_{\Sigma_i} p_\varepsilon(x,\tau)\rho_\varepsilon(t-\tau)g_i(x)d\tau d\sigma + (\omega_i^*(t) - \omega_{\varepsilon i}^*(t)) \in N_{R_i}(\omega_{\varepsilon i}^*). \tag{3.87}$$

Proof. Assume that $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ is an optimal pair for problem $(\widetilde{P_\varepsilon})$. The proof is completely similar to that of Proposition 3.7. We specify that the dual system is the same with (3.54)-(3.57), hence p_ε has the properties proved in Theorem 3.10. From the assumption that $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ is optimal we get that

$$\begin{aligned} &\int_\Omega \left(M^0(\theta_\varepsilon^\lambda) - \theta^0\right)^2 dx + \frac{1}{2}\sum_{i=1}^p \int_0^T (\omega_{\varepsilon i}^\lambda - \omega_i^*)^2 dt \\ &\geq \int_\Omega \left(M^0(\theta_\varepsilon^*) - \theta^0\right)^2 dx + \frac{1}{2}\sum_{i=1}^p \int_0^T (\omega_{\varepsilon i}^* - \omega_i^*)^2 dt, \end{aligned}$$

where ω^* is optimal in problem (P) and this yields

$$\int_0^T \int_\Omega Y_\varepsilon(x,t)F_\varepsilon(x)dxdt + \sum_{i=1}^p \int_0^T (\omega_{\varepsilon i}^* - \omega_i^*)w_{\varepsilon i}dt \geq 0. \tag{3.88}$$

By similar calculations to those performed in the previous section, Proposition 3.7, we deduce once again the relationship (3.69)

$$\int_Q F_\varepsilon Y_\varepsilon dxdt = \int_0^T \int_{\Gamma_u} p_\varepsilon \nu_\varepsilon^{var} d\sigma dt.$$

Combining this equality with (3.88) we get finally that

$$\begin{aligned} &\sum_{i=1}^p \int_0^T (\omega_{\varepsilon i}^*(\tau) - v_{\varepsilon i}(\tau)) \left(- \int_{\Sigma_i} p_\varepsilon(x,t)\rho_\varepsilon(\tau - t)g_i(x)dtd\sigma\right) d\tau \quad (3.89) \\ &+\sum_{i=1}^p \int_0^T (\omega_{\varepsilon i}^*(\tau) - v_{\varepsilon i}(\tau))(\omega_i^*(\tau) - \omega_{\varepsilon i}^*(\tau))d\tau \geq 0, \end{aligned}$$

for any $v_\varepsilon = (v_{\varepsilon 1}, ..., v_{\varepsilon p}) \in U$. By the same argument as in Proposition 3.7 this implies that

$$\int_{\Sigma_i} (-p_\varepsilon(x,t))\rho_\varepsilon(t-\tau)g_i(x)dtd\sigma + (\omega_i^*(\tau) - \omega_{\varepsilon i}^*(\tau)) \in N_{R_i}(\omega_{\varepsilon i}^*),$$

as claimed. ∎

We have the following maximum principle type result for problem (P).

Proposition 3.14. *Assume the hypotheses* (3.22)-(3.23), (3.27) *and let* ω^* *be optimal in problem* (P). *Then* ω^* *has the following form*

$$\begin{cases} \omega_i^*(t) = -R_i & \text{on } \left\{t \in (0,T);\ \int_{\Gamma_i} p(x,t)g_i(x)d\sigma > 0\right\} \\ \omega_i^*(t) \in [-R_i, 0] & \text{on } \left\{t \in (0,T);\ \int_{\Gamma_i} p(x,t)g_i(x)d\sigma = 0\right\} \\ \omega_i^*(t) = 0 & \text{on } \left\{t \in (0,T);\ \int_{\Gamma_i} p(x,t)g_i(x)d\sigma < 0\right\}, \end{cases} \tag{3.90}$$

where the dual state $p \in L^2(0,T;H^2(\Omega))$ *is given by*

$$p = \lim_{\varepsilon\to 0} p_\varepsilon \text{ weakly in } L^2(0,T;H^2(\Omega)) \tag{3.91}$$

and p_ε *is the solution to* (3.54)-(3.57).

Proof. For the proof there is nothing else to do than passing to limit as $\varepsilon \to 0$ in (3.87), applying (3.82) from Corollary 3.11 and then Theorem 3.12 that states that $\widetilde{\omega}_{\varepsilon i}^* \to \omega_i^*$ strongly in $L^2(0,T)$. We get

$$-\int_{\Gamma_i} p(x,t)g_i(x)d\sigma \in N_{R_i}(\omega_i^*)$$

that comes back to (3.90). ∎

Apparently p should be the solution to the dual system corresponding to the original problem and we expect to write for it a system of equations by passing to limit in (3.54)-(3.57). Unfortunately, due to the particularities of the direct model, involving a blowing up function $\beta(\theta)$ at $\theta = \theta_s$, we did not find a possibility for passing to limit in the approximating dual equation (3.54), in the 3-D case.

However a further study presents some conclusions for the one-dimensional case (corresponding to the domain $\Omega = (0, L)$ along the vertical axis Ox_3) for a particular regular initial situation.

We recall first the result of regularity given in Theorem 3.6, in Sect. 5.3, for the solution to the original problem, (3.1) i.e.,

Let $0 < d < \theta_s$ *be fixed. Under the hypotheses* (3.21)-(3.23) *and*

$$\theta_0 \in H^1(\Omega), \operatorname*{ess\,sup}_{x\in\Omega} \theta_0 \le \theta_s - d < \theta_s, \tag{3.92}$$

the solution θ *to problem* (3.1) *satisfies in addition*

$$\begin{aligned}\theta &\in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)),\\ \beta^*(\theta) &\in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)).\end{aligned} \tag{3.93}$$

Moreover, if θ_ε is the solution to the approximating problem (3.2) *we have*

$$\theta_\varepsilon \longrightarrow \theta \text{ uniformly with respect to } (x,t) \in \overline{Q}. \tag{3.94}$$

Denote

$$Q^- = \{(x,t) \in Q;\ \theta^*(x,t) < \theta_s\},\ Q^+ = \{(x,t) \in Q;\ \theta^*(x,t) = \theta_s\}. \tag{3.95}$$

Corollary 3.15. *Let $N = 1$. Assume*

$$f \in W^{1,2}(0,T;L^2(\Omega)),$$

$$f_0 \in W^{1,2}(0,T;L^2(\Gamma_\alpha)) \cap L^2(0,T;H^1(\Gamma_\alpha)),$$

$$\theta_0 \in H^1(\Omega), \operatorname*{ess\,sup}_{x\in\Omega} \theta_0 \le \theta_s - d < \theta_s,$$

and let (ω^*, θ^*) be optimal in problem (P). Then, the dual state p satisfies the following equations

$$\frac{\partial p}{\partial t} + \beta(\theta^*)\Delta p + K'(\theta^*)\frac{\partial p}{\partial x} = F(x) \quad \text{in} \quad Q^-, \tag{3.96}$$

$$p(x,T) = 0 \quad \text{for } (x,T) \in Q^-, \tag{3.97}$$

$$\nabla p \cdot \nu = 0 \quad \text{on} \quad \Sigma_u, \tag{3.98}$$

$$\alpha p + \nabla p \cdot \nu = 0 \quad \text{on} \quad \Sigma_\alpha, \tag{3.99}$$

where

$$F(x) = \frac{1}{T}\int_0^T (\theta^*(x,t) - \theta^0(x))dt. \tag{3.100}$$

Proof. By hypotheses it follows that the approximating penalized problem $(\widetilde{P_\varepsilon})$ has a solution $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ and this converges to the optimal pair (ω^*, θ^*) to (P) as specified by Theorem 3.12. Using the regularity result previously mentioned it follows that in addition the sequence $\{\theta_\varepsilon^*\}_{\varepsilon>0}$ converges uniformly to θ^* in $C(\overline{Q})$.

Since θ^* is continuous it follows that the set Q^- is open. In fact Q^- represents the unsaturated subdomain of Q which is separated from the saturated domain Q^+ by the free boundary $x = s(t)$.

Assume that $Q^+ \neq \emptyset$. Let δ be an arbitrary fixed positive number and denote

$$\begin{aligned}Q_-^\delta &= \{(x,t) \in Q;\ \theta^*(x,t) < \theta_s - \delta\},\\ Q_+^\delta &= \{(x,t) \in Q;\ \theta^*(x,t) \ge \theta_s - \delta\}.\end{aligned}$$

We denote by p_ε the solution to the approximating dual system (3.54)-(3.57).

Let $(x,t) \in Q_-^\delta$. Since θ_ε^* converges uniformly with respect to (x,t) to θ^* we have for ε small, $\varepsilon < \dfrac{\delta}{2}$ that

$$|\theta_\varepsilon^*(x,t) - \theta^*(x,t)| < \frac{\delta}{2},\ \forall (x,t) \in \overline{Q},$$

which implies $\theta_\varepsilon^* < \theta^* + \dfrac{\delta}{2} < \theta_s - \delta + \dfrac{\delta}{2} < \theta_s - \varepsilon$, i.e., $\theta_\varepsilon^* \in Q_-^\varepsilon$.

Then, by the definition of β_ε we have that $\beta_\varepsilon(\theta_\varepsilon^*) = \beta(\theta_\varepsilon^*) < \infty$ on Q_-^δ, (because $\theta_\varepsilon^* < \theta_s - \varepsilon$) so that (3.54) becomes

$$\frac{\partial p_\varepsilon}{\partial t} + \beta(\theta_\varepsilon^*)\Delta p_\varepsilon + K'(\theta_\varepsilon^*)\frac{\partial p_\varepsilon}{\partial x_3} = F_\varepsilon(x) \text{ in } Q_-^\delta. \tag{3.101}$$

Recalling now the conclusions of Theorem 3.10, true also for $(x,t) \in Q_-^\delta$ we deduce from (3.54) that

$$\left\| \frac{\partial p_\varepsilon}{\partial t} \right\|_{L^2(Q_-^\delta)} \le \|\beta(\theta_\varepsilon^*)\Delta p_\varepsilon\|_{L^2(Q_-^\delta)} + M \left\| \frac{\partial p_\varepsilon}{\partial x} \right\|_{L^2(Q_-^\delta)} \le \text{ constant},$$

so that on a subsequence we have that

$$\frac{\partial p_\varepsilon}{\partial t} \longrightarrow \frac{\partial p}{\partial t} \text{ weakly in } L^2(Q_-^\delta),\ \forall \delta > 0.$$

Next we pass to limit as $\varepsilon \to 0$ in (3.101) and obtain that

$$\frac{\partial p}{\partial t} + \beta(\theta^*)\Delta p + K'(\theta^*)\frac{\partial p}{\partial x_3} = F(x) \text{ in } Q_-^\delta,$$

where p is given by (3.91). But δ is arbitrary and noticing that $Q^- = \bigcup\limits_{\delta>0} Q_-^\delta$, we deduce that $\beta_\varepsilon(\theta_\varepsilon^*) = \beta(\theta_\varepsilon^*) < \infty$ on Q^-, so that the previous equation is true on Q^-. Moreover, we have that

$$\left\| \frac{\partial p_\varepsilon}{\partial t} \right\|_{L^2(Q^-)} \le C \tag{3.102}$$

and on a subsequence

$$\frac{\partial p_\varepsilon}{\partial t} \to \frac{\partial p}{\partial t} \text{ weakly in } L^2(Q^-). \tag{3.103}$$

The previous estimate shows that $\{p_\varepsilon\}$ is bounded in $W^{1,2}(Q^-)$ and recalling (3.74) we deduce that it is compact in $C(\overline{Q^-})$,so it tends uniformly to p on Q^- (see again Theorem 3.19 in Appendix). In particular by passing to limit in (3.55) we get $p(x,T) = 0$ for $(x,T) \in Q^-$.

Still by Theorem 3.10 we deduce that $\nabla p_\varepsilon \in L^2(0,T; H^1(\Omega))$, so its trace exists on the boundary $\Gamma \times (0,T)$. Moreover, we have that

$$\nabla p_\varepsilon \cdot \nu|_{\Gamma\times(0,T)} \in L^2(0,T;L^2(\Gamma)).$$

In conclusion, we can pass weakly to limit in (3.56) and (3.57) and obtain

$$\nabla p \cdot \nu = 0 \text{ on } \Sigma_u, \ \alpha p + \nabla p \cdot \nu = 0 \text{ on } \Sigma_\alpha.$$

This ends the proof. ∎

Remark 3.16. We notice that (3.98) and (3.99) remain true in the $3-D$ case too. For the moment we cannot say anything about the other two equations in the 3-D case. The regularity proved for θ_ε is not sufficient to deduce for example that Q^- is open in the 3-D case. However, if $Q^+ = \emptyset$, i.e., when the situation is restrained to the unsaturated case ($\theta < \theta_s$), the dual system for the original problem can be obtained by passing directly to limit in (3.54)-(3.57), because $\beta_\varepsilon(\theta_\varepsilon^*) = \beta(\theta_\varepsilon^*) < \infty$ and we get the system (3.96)-(3.99) even in the 3-D case. The determination of p can be done by a numerical technique developed on the basis of the approximate results presented before for the approximating optimality conditions.

7.4 Case of a plane soil surface

In the particular case when Γ_u is a plane surface (a subdomain of $\mathbf{R}^2$), the problem may be solved in a simpler way by regularizing u with respect to both time and space variables. For example let us consider that Γ_u is a horizontal surface

$$\Gamma_u := \{x \in \Gamma;\ x_3 = 0\}$$

and u a global rain spread over the whole Γ_u. Here it is no longer necessary to assume local smooth rains because we can directly regularize u with both time and space variables and replace (3.11) by

$$\nu_\varepsilon(x,t) := \int_0^T \int_{\Gamma_u} u(\xi,\tau)\rho_\varepsilon(x-\xi,t-\tau)d\xi d\tau. \tag{4.1}$$

We have that $\nu_\varepsilon \in W^{1,\infty}(0,T;H^1(\Gamma_u))$ and

$$\nu_\varepsilon \longrightarrow u \text{ strongly in } L^2(0,T;H^1(\Gamma_u)).$$

Similarly to Lemma 3.4 we have now that if $u_\varepsilon \to u$ weak-star in $L^\infty(\Sigma_u)$, then $\nu^\varepsilon \to u$ weak-star in $L^\infty(\Sigma_u)$. Therefore the control problems may be treated by considering the admissible set U_R,

$$U_R = \{u \in L^\infty(\Sigma_u); -R \le u \le 0 \text{ a.e. on } \Sigma_u\}. \tag{4.2}$$

The control problems (P), (P_ε) and $(\widetilde{P_\varepsilon})$ are

$$(P) \qquad \min_{u \in U_R} \int_{\Omega} \left(\frac{1}{T} \int_0^T \theta(x,t)dt - \theta^0(x) \right)^2 dx, \tag{4.3}$$

subjected to (2.3),

$$(P_\varepsilon) \qquad \min_{u \in U_R} \int_{\Omega} \left(\frac{1}{T} \int_0^T \theta(x,t)dt - \theta^0(x) \right)^2 dx, \tag{4.4}$$

subjected to (3.6)-(3.9), with ν_ε given by (4.1) and

$$\min_{u \in U_R} \int_{\Omega} \left(\frac{1}{T} \int_0^T \theta(x,t)dt - \theta^0(x) \right)^2 dx + \frac{1}{2} \int_0^T (u(x,t) - u^*(x,t))^2 dt, \tag{4.5}$$

subjected to (3.6)-(3.9), with ν_ε given by (4.1).

All the results concerning the existence of the optimal pairs in problems (P), (P_ε) and $(\widetilde{P_\varepsilon})$ as well as the convergence results remain true, as proved in the previous sections of this paper. Proposition 3.7 becomes

Proposition 4.1. *Assume that* f, f_0 and θ_0 *satisfy the hypotheses* (3.22)-(3.23) *and* (3.27). *Let* $(u_\varepsilon^*, \theta_\varepsilon^*)$ *be an optimal pair for the approximating problem* (P_ε). *Then*

$$\begin{cases} u_\varepsilon^*(x,t) = -R & \text{on } \left[\int_{\Sigma_u} p_\varepsilon(\xi,\tau)\rho_\varepsilon(x-\xi,t-\tau)d\tau d\sigma > 0\right] \\ u_\varepsilon^*(x,t) \in [-R,0] & \text{on } \left[\int_{\Sigma_u} p_\varepsilon(\xi,\tau)\rho_\varepsilon(x-\xi,t-\tau)d\tau d\sigma = 0\right] \\ u_\varepsilon^*(x,t) = 0 & \text{on } \left[\int_{\Sigma_u} p_\varepsilon(\xi,\tau)\rho_\varepsilon(x-\xi,t-\tau)d\tau d\sigma < 0\right]. \end{cases} \tag{4.6}$$

Proof. In the proof of Proposition 3.7 we replace $w_\varepsilon = v_\varepsilon - u_\varepsilon^*$, $v_\varepsilon \in U_R$ and $\nu_\varepsilon^{var}(x,t) = w_\varepsilon(x,t) * \rho_\varepsilon(x,t)$. By the same calculations we get

$$\int_0^T \int_{\Gamma_u} (u_\varepsilon^* - v_\varepsilon) \left(- \int_0^T \int_{\Gamma_u} p_\varepsilon(x,t)\rho_\varepsilon(\xi - x, \tau - t) d\sigma_x dt \right) d\sigma_\xi d\tau \geq 0,$$

from where we obtain the conclusion. ∎

Moreover, as seen before, similarly to Corollary 3.11, we have

$$\int_0^T \int_{\Gamma_u} p_\varepsilon(x,t)\rho_\varepsilon(\xi - x, \tau - t) dx dt \longrightarrow p \text{ weakly in } L^2(\Sigma_u). \tag{4.7}$$

Concerning the new adapted penalization problem (4.5) where u^* is optimal in (P), we have by Theorem 3.12 that $u^* \to u$ strongly in $L^2(\Sigma_u)$ and we can enounce

Proposition 4.2. *Assume that* f, f_0 and θ_0 *satisfy the hypotheses* (3.22)-(3.23) *and* (3.27). *Let* $(\omega_\varepsilon^*, \theta_\varepsilon^*)$ *be an optimal pair for the approximating problem* $(\widetilde{P}_\varepsilon)$. *Then*

$$\int_{\Sigma_u} (-p_\varepsilon(\xi,\tau))\, \rho_\varepsilon(x-\xi, t-\tau) d\tau d\xi + (u^*(x,t) - u_\varepsilon^*(x,t)) \in N_R(u_\varepsilon^*).$$

Proposition 4.3. *Assume the hypotheses* (3.22)-(3.23), (3.27) *and let* u^* *be optimal in problem* (P). *Then* u^* *has the following form*

$$\begin{cases} u^*(x,t) = -R & \text{on } \{(x,t) \in \Sigma_u;\ p(x,t) > 0\} \\ u^*(x,t) \in [-R, 0] & \text{on } \{(x,t) \in \Sigma_u;\ p(x,t) = 0\} \\ u^*(x,t) = 0 & \text{on } \{(x,t) \in \Sigma_u;\ p(x,t) < 0\}, \end{cases} \tag{4.8}$$

where $p \in L^2(0,T; H^2(\Omega))$ is given by

$$p = \lim_{\varepsilon \to 0} p_\varepsilon \text{ weakly in } L^2(0,T; H^2(\Omega)).$$

The result proved for $N = 1$ in Corollary 3.15 remains true.

7.5 Identification problem using final time observations

We shall study now problem (P_T),

$$\min_{\omega \in U} \int_\Omega \left(\theta(x,T) - \theta^T(x)\right)^2 dx \tag{P_T}$$

where θ is the solution to (2.3) and

$$U = \{\omega = (\omega_1, ..., \omega_p);\ \omega_i \in L^\infty(0,T),\ \ -R_i \le \omega_i \le 0 \text{ a.e. } t \in (0,T)\}.$$

The solution to this problem follows the same steps as for the problem (P). However, some differences will occur and they will be emphasized. The part related to the existence in the state system is skipped, all results mentioned before being the same.

Existence of the optimal control

Obviously, Lemma 3.1 remains true.

Theorem 5.1. *Let*

$$f \in L^2(0,T;V'),\ f_0 \in L^2(0,T; L^2(\Gamma_\alpha)),\ \theta_0 \in L^2(\Omega),\ \theta_0 \le \theta_s \text{ a.e. } x \in \Omega.$$

Then, problem (P_T) *has at least one solution.*

Proof. Let $d = \min_{\omega \in U} \left(\int_\Omega (\theta(x,T) - \theta^T(x))^2 dx \right)$ and let $\{\omega_n\}_{n \geq 1} \subset U$ be a minimizing sequence, i.e.,

$$d \leq \int_\Omega \left(\theta_n(x,T) - \theta^T(x) \right)^2 dx \leq d + \frac{1}{n}\,,\ n \geq 1, \tag{5.1}$$

where θ_n is the solution to (2.3) with u replaced by $u_n(x,t) = \sum_{i=1}^{p} \omega_{ni}(t) g_i(x)$. This means that θ_n is a solution to the Cauchy problem (3.1) with $u = u_n$. The sequence $\{\theta_n\}_{n \geq 1}$ tends as specified in Theorem 3.2 to the solution $\widetilde{\theta}$ to (2.3) with u replaced by $\widetilde{u}$, the limit of the sequence $\{u_n\}$. The proof is identical to that of Theorem 3.2. The only difference is in the proof of the convergence of the cost functional. In this case, we need to prove that

$$\theta_n(T) \longrightarrow \widetilde{\theta}(T) \text{ weakly in } L^2(\Omega). \tag{5.2}$$

For that we shall show that for $y \in L^2(\Omega)$, the set $\{(\theta_n(t), y)\}_{n \geq 1}$, is compact in $\mathbf{R}$ for any $t \in [0,T]$. We have $(\theta_n, y) \in C([0,T]; \mathbf{R})$ and we use again (3.67), $\|\theta_n(t)\| \leq C_S$. Therefore if $y \in L^2(\Omega)$, it follows that

$$\left| \int_\Omega \theta_n(t) y dx \right| \leq \|\theta_n(t)\| \, \|y\| \leq \text{ constant.}$$

The set $\{(\theta_n(t), y)\}$ is equi-uniformly continuous.

First we show this for $y \in V$, $y \neq 0$. Let $\epsilon > 0$. We have

$$\begin{aligned} \left| \int_\Omega (\theta_n(t) - \theta_n(s)) y dx \right| &= \left| \int_\Omega y \int_s^t \theta_n'(\xi) d\xi dx \right| \leq \int_s^t \|\theta_n'(\xi)\|_{V'} \|y\|_V \, d\xi \\ &\leq (t-s)^{1/2} \|\theta_n'\|_{L^2(0,T;V')} \|y\|_V \leq \frac{\epsilon}{2}, \\ \text{for } |t-s| \leq \delta_\epsilon &:= \left(\frac{\epsilon}{2 \|\theta_n'\|_{L^2(0,T;V')} \|y\|_V} \right)^2. \end{aligned}$$

Let now $y \in L^2(\Omega)$. Since V is compact in $L^2(\Omega)$ we have that for any $\epsilon > 0$, there exists $y_V \in V$ such that $\|y - y_V\| \leq \dfrac{\epsilon}{2C_S}$, where $\|\theta_n(t)\| \leq C_S$. Then

$$\begin{aligned} &\left| \int_\Omega (\theta_n(t) - \theta_n(s)) y dx \right| \\ &= \left| \int_\Omega (\theta_n(t) - \theta_n(s))(y - y_V) dx \right| + \left| \int_\Omega (\theta_n(t) - \theta_n(s)) y_V dx \right| \\ &\leq \|\theta_n(t) - \theta_n(s)\| \, \|y - y_V\| + (t-s)^{1/2} \|\theta_n'\|_{L^2(0,T;V')} \|y_V\|_V \\ &\leq \frac{\epsilon}{2} + (t-s)^{1/2} \|\theta_n'\|_{L^2(0,T;V')} \|y_V\|_V \leq \epsilon, \text{ for } |t-s| \leq \delta_\epsilon. \end{aligned}$$

Then, by Arzelà's theorem it follows that the set $\{(\theta_n, y)\}$ is compact in $C([0,T]; \mathbf{R})$, for each $t \in [0,T]$. This implies that on a subsequence $(\theta_n, y) \to (\widetilde{\theta}, y)$ strongly in $\mathbf{R}$, implying that $\theta_n(t) \to \widetilde{\theta}(t)$ weakly in $L^2(\Omega)$, $\forall t \in [0,T]$. In particular we get (5.2). Therefore, by weakly lower semicontinuity, we have that

$$d \leq \int_\Omega (\widetilde{\theta}(x,T) - \theta^T(x))^2 dx \leq \liminf_{n\to\infty} \int_\Omega (\theta_n(x,T) - \theta^T(x))^2 dx \leq d$$

showing that $\widetilde{u}$ is a solution to problem (P_T). ∎

The approximating control problem

We introduce the following approximating identification problem

$$\min_{\omega \in U} \int_\Omega \left(\theta(x,T) - \theta^T(x)\right)^2 dx \qquad (P_{T\varepsilon})$$

subjected to (3.6)-(3.9).

Lemma 3.4 and Lemma 3.5 remain valid. Theorem 3.6 will suffer a modification, imposed by the requirement of the strongly convergence of the cost functional.

Theorem 5.2. *Assume that* θ_0, f *and* f_0 *satisfy*

$$f \in L^2(0,T;V'),\ f_0 \in L^2(\Sigma_\alpha),$$
$$\theta_0 \in H^1(\Omega), \operatorname*{ess\,sup}_{x\in\Omega} \theta_0 \leq \theta_s - d < \theta_s,\ d > 0,$$

and let $(\omega_\varepsilon, \theta_\varepsilon)$ *be a solution to the approximating problem* $(P_{T\varepsilon})$. *Then,*

$$\omega_\varepsilon \longrightarrow \omega^* \text{ weak-star in } (L^\infty(0,T))^p,$$
$$\theta_\varepsilon \longrightarrow \theta^* \text{ weakly in } L^2(0,T;\, V) \cap W^{1,2}(0,T, V') \text{ and strongly in } L^2(Q),$$
$$\theta_\varepsilon(T) \longrightarrow \theta^*(T) \text{ strongly in } L^2(\Omega),$$

where $\omega^* \in U$ *and* θ^* *is the solution to the original problem* (2.3) *with* $u = u^*$. *Moreover,* ω^* *is a solution to* (P_T) *and* $\lim_{\varepsilon\to 0}(P_{T\varepsilon}) = (P_T)$.

Proof. The proof is similar to that of Theorem 3.6 but instead of the convergences of the type (3.18) we have to show the strong convergence

$$\theta_\varepsilon(T) \longrightarrow \theta^*(T) \text{ strongly in } L^2(\Omega).$$

Here we use the regularity result given in Theorem 3.6, Sect. 5.3. Since $\theta_0 \in V$, by the proof of this theorem (see 3.27) in Sect. 5.3) we have that $\|\theta_\varepsilon(t)\|_V \leq C$ independently of ε. It follows that $\{\theta_\varepsilon(t)\}$ is compact in $L^2(\Omega)$ for each $t \in [0,T]$. Then

$$\theta_\varepsilon(t) \longrightarrow \theta(t) \text{ strongly in } L^2(\Omega),\ \forall t \in [0,T].$$

Therefore we obtain (3.17) as claimed. ∎

Necessary conditions of optimality

All the results given in the corresponding previous subsections remain true. The only modification is the dual system which now has the following form

$$\begin{aligned} \frac{\partial p_\varepsilon}{\partial t} + \beta_\varepsilon(\theta_\varepsilon^*)\Delta p_\varepsilon + K'(\theta_\varepsilon^*)\frac{\partial p_\varepsilon}{\partial x_3} &= 0 && \text{in } Q, \\ p_\varepsilon(x,T) &= -F_\varepsilon^T(x) && \text{in } \Omega, \\ \nabla p_\varepsilon \cdot \nu &= 0 && \text{on } \Sigma_u, \\ \alpha p_\varepsilon + \nabla p_\varepsilon \cdot \nu &= 0 && \text{on } \Sigma_\alpha, \end{aligned} \tag{5.3}$$

where

$$F_\varepsilon^T(x) = \theta_\varepsilon^*(x,T) - \theta^T(x), \tag{5.4}$$

but the optimal pair has the same form as that given by Proposition 3.7 for an approximating optimal pair. Also, the form of the optimal pair of the original problem found in Proposition 3.14 does not change.

Some modifications will occur in the proof of Theorem 3.10, due to the change of the dual system. Also, in this proof a better regularity of the observed data is required, i.e., $\theta^T \in H^1(\Omega)$. We shall not enter into the details of the proofs, they being essentially similar to that developed in the previous case.

Corollary 3.15 remains true, with the appropriate modification in the original dual system implied by the form (5.3), with $F^T(x) = \theta^*(x,T) - \theta^T(x)$, where θ^* is the original optimal state.

7.6 Comments

By presenting these two inverse problems we have intended also to put into evidence the complications which turn up in the mathematical approach when available data are less regular. Comparing the problems (P) and (P_T) we can conclude that the lack of time measurements in the second problem should be compensated by an increase in the space regularity of the measured data and in the regularity of the control. Thus, the existence of the optimal control can be proved in both cases on the basis of the same hypotheses. From this point on, the differences between them are noticeable.

Following carefully the proof of the convergence of the sequence of approximating optimal controls in (P_ε) to an optimal control in (P), (see Theorem 3.6) we notice that it would not have been necessary the regularization of u. The supplementary regularity of θ_ε is necessary in (P) only for the application of the maximum principle. In return, the proof of the convergence result in problem (P_T) requires this regularization, because this implies the supplementary regularity of the approximating solution θ_ε absolutely necessary to enable the strongly convergence of the cost functional (see Theorem 5.2). We emphasize that to this end also a more regular initial condition is necessary in addition.

Next, in the precursory results to the determination of the optimality conditions for the original problem (P_T) we need a higher regularity of the measured data, i.e, $\theta^T \in H^1(\Omega)$, while this result is obtained in the case (P) with the data in $L^2(\Omega)$. Finally, the difficulty of the direct problem, which is a free boundary problem with a blowing-up coefficient, forecloses the possibility of writing the form of the 3-D dual system in both cases, this result could be obtained in 1-D, assuming a pure unsaturated moisture distribution in the soil. Thus, we can see that the lack of data or the difficulties of the real physical problem costs more, in the sense of involving a greater theoretical and numerical effort, and the result can be a little far from the real one due to some compulsive assumptions. Anyway, we have to resort to numerical algorithms for computing the optimal pair in the approximating problems.

Eventually, the regularity assumption made initially for u (using the local smooth ω_i) can be skipped when the geometry of the soil is smoother.

This chapter was intended to justify the utility of the theoretical approach to infiltration problems and to familiarize the reader with some types of possible problems arising in the infiltration control. Besides the identification of specific practical problems that require the theoretical approach developed in the book, we would like to emphasize the mathematical interest that specific inverse and control problems in infiltration theory may have. The examples we have discussed can be viewed as control problems or inverse problems. Applications are envisaged in agriculture, for controlling the irrigation activity, when a certain moisture evolution is aimed, or in hydro-meteorology, when retracing the rain history from available observations. Many other practical applications can be imagined and the theoretical approach will be accordingly adapted and improved.

Bibliographical note

Generally, the literature covering various control problems is very rich. For results concerning the general theory of optimal control we refer the reader to the monographs [12], [14], [83], [97] and to the works [10], [15], [73], [128], [129], [130], [131]. For numerical techniques in optimal control problems we cite the books [3] and [45]. Specifically, for infiltration problems we indicate the papers [16], [91] and [92]. In the last two papers the problem of identification of the rain rate from time average observations was studied. Results for the case regarding the rain history reconstitution on the basis of a final time observation were obtained in [90].

A

Background tools

The appendix is included in this work with the precise aim of exposing the fundamental concepts, definitions and the significant results used in the book. However, some mathematical background in real analysis and vector spaces is required. Here we list some basic references the reader can consult and specify that we shall recall the results without proofs:

Section 1: [9], [13], [23], [31], [50], [68], [134];
Section 2: [9], [13], [31], [50], [53], [54], [68], [109], [110], [134];
Section 3: [9], [13], [31], [53], [54], [68], [109], [110], [134];
Section 4: [9], [30], [68], [134];
Section 5: [9], [13], [14], [30], [134];
Section 6: [13].

A.1 Some definitions and results in Banach spaces

Let $(X, \|\cdot\|_X)$ be a real normed vector space, where $\|\cdot\|_X$ represents the norm on X. The dual of X is the space of all linear and continuous real valued functionals defined on X and is denoted by X'. It is known that X' is always a Banach space (complete normed vector space), even if X is not complete.

An element f of X' maps X into $\mathbf{R}$, namely $f(x) \in \mathbf{R}$ for any $x \in X$. Moreover,

$$|f(x)| \leq M_f \|x\|_X , \tag{1.1}$$

where M_f is a number depending in general on f. The norm in the dual space is defined by

$$\|f\|_{X'} := \sup_{\|x\|_X \leq 1} |f(x)| . \tag{1.2}$$

By $|\cdot|$ we denote the norm on $\mathbf{R}$.

The value of $f \in X'$ at $x \in X$ is still indicated by the notation $\langle f, x\rangle_{X',X}$ which is the scalar product for the duality X', X, also called the *pairing* between X and X'. If there is no danger of confusion, we shall drop sometimes the subscripts.

Let $(X, \|\cdot\|_X)$ be a real normed vector space.

Definition 1.1. A subset G of X is called *open* if, for each $x \in G$ there exists $\varepsilon = \varepsilon(x) > 0$ such that the set

$$B(x, \varepsilon) := \{y;\ y \in X,\ \ \|x - y\|_X < \varepsilon\}$$

lies entirely inside G. The set $B(x, \varepsilon)$ is called the *open ball* with centre x and radius ε.

A subset M of X is *closed* if its complement $X \setminus M$ is an open set.

Definition 1.2. Let M be a subset of X. An element $x \in X$ is called a *limit point* of M if there exists a sequence $\{x_n\}_{n\geq 1} \subset M$ such that $x_n \to x$ as $n \to \infty$. The set of all limit points of M is called the *closure* of M and it is denoted $\overline{M}$.

It is obvious that $M \subseteq \overline{M}$.

Definition 1.3. A subset $M \subset X$ is said to be *dense* in X if one of the following equivalent assertions holds:

(i) $\overline{M} = X$;

(ii) for each $x \in X$ and each $\varepsilon > 0$ there exists $m \in M$, such that $\|m - x\|_X < \varepsilon$.

Proposition 1.4. *Let Y be a vector subspace of X, such that $\overline{Y} \neq X$. Then, there exists $f \in X'$, $f \neq 0$, such that $\langle f, y\rangle_{X',X} = 0$, $\forall y \in Y$.*

(This result is often applied to prove that a subspace is dense.)

Definition 1.5. The space X is said to be *separable* if it contains a countable subset D which is dense in X.

Definition 1.6. The subset K of X is called (*sequentially*) *compact* if every sequence $\{x_n\}_{n\geq 1}$ of K contains a subsequence strongly convergent (i.e., convergent in the norm of X) to $x \in K$.

Definition 1.7. The subset K of X is called *relatively compact* if and only if its closure is compact.

Definition 1.8. A subset $C \subset X$ is called *convex* if for all $x, y \in C$ and $\lambda \in [0, 1]$ it follows that $\lambda x + (1 - \lambda)y \in C$.

Definition 1.9. The space X is called *strictly convex* if $\|tx + (1 - t)y\|_X < 1$, for any $t \in (0, 1)$ and $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$, $x \neq y$.

Definition 1.10. The space X is said *uniformly convex* if $\forall\varepsilon > 0$, $0 < \varepsilon \leq 2$, $\exists\ \delta(\varepsilon) > 0$ such that $x, y \in X$, $\|x\|_X \leq 1$, $\|y\|_X \leq 1$, $\|x - y\|_X \geq \varepsilon$ implies $\left\|\dfrac{x + y}{2}\right\|_X \leq 1 - \delta(\varepsilon)$.

Definition 1.11. Let X be a real vector space of real-valued functions on a fixed domain $\Omega \subseteq \mathbf{R}$. The set

$$X_+ := \{f;\ f \in X,\ f(x) \geq 0 \text{ for all } x \in \Omega\}$$

is called the (closed) *positive cone* of X.

Proposition 1.12. *Let X be a normed vector space. Then for each $x_0 \in X$ there exists $f_0 \in X'$ such that $\|f_0\|_{X'} = \|x_0\|_X$ and $\langle f_0, x_0\rangle_{X',X} = \|x_0\|_X^2$.*

The element f_0 is not generally unique. But if X' is strictly convex (a Hilbert space for example) then f_0 is unique.

Definition 1.13. Let X be a normed vector space and X' its dual. For each $x_0 \in X$ we denote

$$F(x_0) := \left\{ f_0 \in X';\ \|f_0\|_{X'} = \|x_0\|_X, \langle f_0, x_0\rangle = \|x_0\|_X^2 \right\}.$$

The multivalued application $x_0 \to F(x_0)$ is called the *duality mapping* from X to X'.

Now, let X be a Banach space and X' its dual.

Definition 1.14. The sequence $\{x_n\}_{n\geq 1} \subset X$ converges *strongly* to $x \in X$ (and we also write it $x_n \to x$) if $\|x_n - x\|_X \to 0$.

Definition 1.15. The sequence $\{x_n\}_{n\geq 1} \subset X$ converges *weakly* to $x \in X$ (and we also write it $x_n \rightharpoonup x$) if

$$\langle f, x_n\rangle_{X',X} \longrightarrow \langle f, x\rangle_{X',X},\ \forall f \in X'. \tag{1.3}$$

Proposition 1.16. *Let $\{x_n\}_{n\geq 1}$ be a sequence in X. Then*

(i) *if $x_n \to x$, then $x_n \rightharpoonup x$,*

(ii) *if $x_n \rightharpoonup x$, then $\|x_n\|_X$ is bounded and $\|x\|_X \leq \liminf_{n\to\infty} \|x_n\|_X$,*

(iii) *if $x_n \rightharpoonup x$ and $f_n \to f$ in X', then $\langle f_n, x_n\rangle_{X',X} \to \langle f, x\rangle_{X',X}$.*

Definition 1.17. The sequence $\{f_n\}_{n\geq 1} \subset X'$ converges *weak-star* to $f \in X'$, if

$$\langle f_n, x\rangle_{X',X} \longrightarrow \langle f, x\rangle_{X',X},\ \forall x \in X. \tag{1.4}$$

Proposition 1.18. *Let $\{f_n\}_{n\geq 1}$ be a sequence in X'. Then*

(i) *if $f_n \to f$, then $f_n \rightharpoonup f$ (i.e., $\langle \xi, f_n\rangle_{X'',X'} \to \langle \xi, f\rangle_{X'',X'}, \forall \xi \in X''$),*

(ii) *if $f_n \rightharpoonup f$, then $f_n \to f$ weak-star ($\langle f_n, x\rangle_{X',X} \to \langle f, x\rangle_{X',X}, \forall x \in X$),*

(iii) *if $f_n \rightharpoonup f$, then $\|f_n\|_{X'}$ is bounded and $\|f\|_{X'} \leq \liminf_{n\to\infty} \|f_n\|_{X'}$,*

(iv) *if $f_n \to f$ weak-star and $x_n \to x$ in X, then $\langle f_n, x_n\rangle_{X',X} \to \langle f, x\rangle_{X',X}$.*

Definition 1.19. A mapping between two Banach spaces is said *compact* if and only if it maps bounded sets onto relatively strongly compacts sets.

Let us now define a canonical injection $J : X \to X''$ in the following way

$$\langle Jx, f\rangle_{X'',X'} := \langle f, x\rangle_{X',X}\ ,\ \forall x \in X,\ \forall f \in X'.$$

This application is linear and isometric, i.e., $\|Jx\|_{X''} = \|x\|_X$, where the norm in X'' is given by

$$\|g\|_{X''} = \sup_{\|\phi\|_{X'} \le 1} |g(\phi)|\,.$$

Definition 1.20. A Banach space X is called *reflexive* if $J(X) = X''$.

We note that if X is reflexive the weak-star convergence is equivalent with the weakly convergence in X'.

Proposition 1.21. *Let X be a separable Banach space and let $\{f_n\}_{n\ge1}$ be a bounded set in X'. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f \in X'$, weak-star.*

Proposition 1.22. (Eberlein-Smulyan) *A Banach space X is reflexive if and only if it is locally sequentially compact, i.e., if every strongly bounded sequence $\{x_n\}_{n\ge1}$ in X contains a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x \in X$.*

Proposition 1.23. *Let X be a uniformly convex Banach space and let $\{x_n\}_{n\ge1}$ a sequence such that $x_n \rightharpoonup x$ and $\limsup_{n\to\infty} \|x_n\|_X \le \|x\|_X$.* Then $x_n \to x$.

Theorem 1.24. (renorming theorem) *Let X be a reflexive Banach space with the norm $\|\cdot\|_X$. Then there exists an equivalent norm $\|\cdot\|_0$ on X such that X is strictly convex under this norm and X' is strictly convex under the dual norm $\|\cdot\|_0'$.*

Theorem 1.25. (Cauchy-Schwarz inequality) *In a Hilbert space $(H, (\cdot,\cdot))$ the scalar product satisfies*

$$|(x, y)_H| \le \|x\|_H \|y\|_H \text{ for all } x, y \in H.$$

A.2 L^p spaces and Sobolev spaces

We recall some concepts concerning L^p spaces and Sobolev spaces whose detailed presentation can be found, for instance, in [53]. Let Ω be an open subset of $\mathbf{R}^N$ ($N \in \mathbf{N}^* = \{1, 2, ...\}$), endowed with the Lebesgue measure dx. We assume that $\partial\Omega$ is of Lipschitz class. (For basic elements of Lebesgue measure we refer the reader to [13].)

By $C^k(\Omega)$, $0 \le k \le \infty$, we denote the set of real-valued functions defined in Ω which have continuous partial derivatives of an order up to and including k. If $k = 0$ we shall denote also $C(\Omega)$ instead of $C^0(\Omega)$.

The space denoted $C_0^k(\Omega)$ represents the functions belonging to $C^k(\Omega)$ with compact support included in Ω. We recall that

$$\text{supp } u := \overline{\{x \in \Omega;\ u(x) \neq 0\}}.$$

Denote by $\mathcal{D}(\Omega)$ the space of real-valued infinitely differentiable functions defined in Ω, with compact support in Ω, equipped with the inductive limit topology (see [109]).

Denote by $\mathcal{D}'(\Omega)$ the dual space of $\mathcal{D}(\Omega)$, i.e., the space of all linear functionals defined on $\mathcal{D}(\Omega)$ with values in $\mathbf{R}$ that are continuous with respect to the inductive limit topology of $\mathcal{D}(\Omega)$. The elements of $\mathcal{D}'(\Omega)$ are called *scalar distributions* defined on Ω.

A partial derivative of a distribution $u \in \mathcal{D}'(\Omega)$ with respect to x_j is given by

$$\frac{\partial u}{\partial x_j}(\phi) := -u\left(\frac{\partial \phi}{\partial x_j}\right) \text{for all } \phi \in \mathcal{D}(\Omega).$$

If $D_i = \dfrac{\partial}{\partial x_i}$, $i = 1, 2, ..., N$, then the distribution $D^\alpha u$, called the *derivative of order* α of $u \in \mathcal{D}'(\Omega)$ is defined by

$$D^\alpha u(\varphi) := (-1)^{|\alpha|} u(D^\alpha \varphi),\ \forall \varphi \in \mathcal{D}(\Omega)$$

where $D^\alpha u(x) = D_1^{\alpha_1} ... D_N^{\alpha_N} u(x)$, $x \in \Omega$, $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbf{N} = \mathbf{N}^* \cup \{0\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

We consider the equivalence classes of real-valued functions that coincide almost everywhere (a.e.) on the Lebesgue measurable subset $\Omega \subset \mathbf{R}^N$. We denote by $L^1(\Omega)$ the space of Lebesgue integrable functions defined on Ω with values in $\mathbf{R}$ and let

$$\|f\|_{L^1(\Omega)} := \int_\Omega |f(x)|\, dx.$$

If $p \in \mathbf{R}$, $1 < p < \infty$ we define

$$L^p(\Omega) := \{f : \Omega \to \mathbf{R};\ f \text{ is Lebesgue measurable and } |f(x)|^p \in L^1(\Omega)\}$$

and denote

$$\|f\|_{L^p(\Omega)} := \left[\int_\Omega |f(x)|^p\, dx\right]^{1/p}.$$

We also define

$$L^\infty(\Omega) := \{f : \Omega \to \mathbf{R};\ f \text{ is measurable and } \exists C \text{ such that } |f(x)| \le C \text{ a.e. on } \Omega\}$$

and put

$$\|f\|_{L^\infty(\Omega)} := \inf\{C; |f(x)| \le C \text{ a.e. on } \Omega\} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| .$$

If D is a open bounded subset of Ω, with $\overline{D} \subset \Omega$ we define

$$L^p_{loc}(\Omega) := \{f : \Omega \to \mathbf{R};\ f \text{ is measurable, } f\,|_D \in L^p(D),\ \forall p \in [1, +\infty)\}.$$

Theorem 2.1. (Fischer-Riesz) *$L^p(\Omega)$ is a Banach space for $1 \le p \le \infty$.*

Theorem 2.2. *For $1 < p < \infty$, $L^p(\Omega)$ is reflexive and separable and $(L^p(\Omega))' = L^{p'}(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.*

The space L^1 is separable but it is not reflexive and $(L^1(\Omega))' = L^\infty(\Omega)$.
The space $L^\infty(\Omega)$ is nor reflexive or separable and $(L^\infty(\Omega))' \supset L^1(\Omega)$.

Theorem 2.3. (Hölder's inequality) *Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \le p \le \infty$. Then $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_{L^p} \|g\|_{L^{p'}} . \tag{2.1}$$

We also recall the Young inequality

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \text{ for } a \ge 0,\ b \ge 0 \tag{2.2}$$

and give two consequences of Hölder's inequality.

Let $f_i \in L^{p_i}(\Omega)$, $i = 1, 2, ..., k$, with $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2} + \cdots + \dfrac{1}{p_k} \le 1$.

Then, $f = f_1 f_2 \cdots f_k \in L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} \le \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)} \cdots \|f_p\|_{L^{p_k}(\Omega)}. \tag{2.3}$$

In particular, if $f \in L^p(\Omega) \cap L^q(\Omega)$, with $1 \le p \le q \le \infty$, then $f \in L^r(\Omega)$ for any $p \le r \le q$ and we have

$$\|f\|_{L^r(\Omega)} \le \|f\|^\alpha_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)} , \text{ where } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q},\ 0 \le \alpha \le 1. \tag{2.4}$$

Theorem 2.4. *Let $\{f_n\}_{n\ge1}$ be a strongly convergent sequence to f in $L^p(\Omega)$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ a.e. $x \in \Omega$.*

For $1 \le p \le \infty$ and $k \ge 1$, we introduce the Sobolev spaces

$$W^{k,p}(\Omega) := \{f : \Omega \to \mathbf{R};\ f \text{ is measurable and } D^\alpha f \in L^p(\Omega), |\alpha| \le k\}$$

endowed with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \le k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{if } 1 \le p < \infty \tag{2.5}$$

and

$$\|f\|_{W^{k,\infty}(\Omega)} := \max_{|\alpha| \le k} \|D^\alpha f\|_{L^\infty(\Omega)}, \ \ |\alpha| \le k. \tag{2.6}$$

Here $D^\alpha f$ is considered in the sense of distributions.

The space $W^{k,p}(\Omega)$ is a Banach space under the norms (2.5) or (2.6) defined before.

The space $W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \le p < \infty$.

We denote by $W_0^{k,p}(\Omega)$ the completion of $C_0^k(\Omega)$ in the norm of $W^{k,p}(\Omega)$.

Finally, the space $W^{-k,p'}(\Omega)$, $1 \le p' \le \infty$ is the set of all distributions u belonging to $\mathcal{D}'(\Omega)$ that can be represented as the sums

$$u = \sum_{|\alpha| \le k} D^\alpha f_\alpha, \ \ f_\alpha \in L^{p'}(\Omega).$$

Theorem 2.5. *The dual of space of* $W_0^{k,p}(\Omega)$ *coincides with the space* $W^{-k,p'}(\Omega)$, $1 \le p \le \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

For proofs we refer to [54].

For simplicity one denotes $H^k(\Omega) := W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega) := H_0^k(\Omega)$. These are both Hilbert spaces with respect to the scalar product

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

The dual of $H_0^k(\Omega)$ will be denoted by $H^{-k}(\Omega)$.

Lemma 2.6. *Let* Ω *be an open and bounded subset of* $\mathbf{R}^N$ *of class* C^1. *Then the injection of the space* $H^1(\Omega)$ *into* $L^2(\Omega)$ *is compact.*

If Ω is an open subset of $\mathbf{R}^N$ of class C^1, with the boundary $\partial\Omega$, then each $u \in C(\overline{\Omega})$ is well defined on $\partial\Omega$. We shall call the restriction of u to $\partial\Omega$ the *trace* of u to $\partial\Omega$ and it will be denoted by $\gamma_0(u)$.

We denote $\mathbf{R}^N_+ := \{(x_1, ..., x_N); \, x_N > 0\}$.

Theorem 2.7. (trace theorem) *Let* Ω *be an open subset of* $\mathbf{R}^N$, *of class* C^1 *with compact boundary* $\partial\Omega$ *or* $\Omega = \mathbf{R}^N_+$. *Then, there is* $C > 0$ *such that*

$$\|\gamma_0(u)\|_{L^2(\partial\Omega)} \le C \, \|u\|_{H^1(\Omega)}, \ \forall u \in C_0^\infty(\mathbf{R}^N).$$

Consider now that Ω is a bounded subset of $\mathbf{R}^N$ with a sufficiently smooth boundary $\partial\Omega = \Gamma$. Then

$$H_0^1(\Omega) = \{u \in H^1(\Omega);\ \text{the trace of } u \text{ on } \Gamma \text{ vanishes}\}.$$

Let $\mathcal{D}(\overline{\Omega})$ be the space of all infinitely differentiable functions on $\mathbf{R}^N$ with compact support in $\overline{\Omega}$.

For any $u \in \mathcal{D}(\overline{\Omega})$ we define the outward normal derivative of order j to Γ, $\dfrac{\partial^j u}{\partial \nu^j}$ and the mapping $u \longrightarrow \left\{ u|_\Gamma , \left.\dfrac{\partial u}{\partial \nu}\right|_\Gamma , ..., \left.\dfrac{\partial^{k-1} u}{\partial \nu^{k-1}}\right|_\Gamma \right\}$ can be extended by continuity to all $u \in H^k(\Omega)$.

For every $s \geq 0$ define

$$H^s(\mathbf{R}^N) := \{u \in L^2(\mathbf{R}^N);\ \left(1 + |\xi|^2\right)^{s/2} \hat{u}(\xi) \in L^2(\mathbf{R}^N)\} \tag{2.7}$$

where $\hat{u}$ denotes the Fourier transform of u.

Theorem 2.8. *The mapping* $u \longrightarrow \left\{\dfrac{\partial^j u}{\partial \nu^j};\ j = 0, 1, ..., \mu\right\}$ *from* $\mathcal{D}(\overline{\Omega})$ *to* $(\mathcal{D}(\Gamma))^{\mu+1}$ *extends to a linear continuous operator*

$$u \longrightarrow \left\{\frac{\partial^j u}{\partial \nu^j};\ j = 0, 1, ..., \mu\right\}$$

from $H^k(\Omega)$ *onto* $\prod_{j=0}^{\mu} H^{k-j-1/2}(\Gamma)$ *where* μ *is the largest integer such that* $\mu < k - \dfrac{1}{2}$.

The proof of this theorem can be found in [84].

Moreover, we emphasize that the inverse of the trace operator is continuous from

$$L^2(\Gamma) \longrightarrow H^{1/2}(\Omega), \tag{2.8}$$

or more generally, the mapping

$$D \text{ is continuous} : H^s(\Gamma) \longrightarrow H^{s+1/2}(\Omega),\ \ s \in \mathbf{R}, \tag{2.9}$$

(see also Dirichlet map, D in [84], p.187, or [81], p.181).

Theorem 2.9. (monotonically convergence theorem of Beppo-Levi) *Let* $\{f_n\}_{n\geq 1} \in L^1(\Omega)$ *be a monotonically increasing sequence with*

$$\sup_{n\to\infty} \int_\Omega f_n(x)dx < \infty.$$

Then $f_n(x) \to f(x)$ *a.e. on* Ω, $f \in L^1(\Omega)$ *and* $\|f_n - f\|_{L^1(\Omega)} \to 0$.

Theorem 2.10. (dominated convergence theorem of Lebesgue) *Let* $\{f_n\}_{n\geq 1} \in L^1(\Omega)$. *Suppose that*

(i) $f_n(x) \to f(x)$ *a.e.* $x \in \Omega$
(ii) *there exists* $g \in L^1(\Omega)$ *such that for* $\forall n,\ |f_n(x)| \leq g(x)$ *a.e. in* Ω.

Then $f \in L^1(\Omega)$ *and* $\|f_n - f\|_{L^1(\Omega)} \to 0$.

Lemma 2.11. (Fatou) *Let* $\{f_n\}_{n\geq 1}$ *be a sequence of nonnegative measurable functions. Then*

$$\int_\Omega \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_\Omega f_n(x)dx.$$

Theorem 2.12. (Egorov) *Suppose that* $\text{meas}(\Omega) < \infty$. *Let* $\{f_n\}_{n\geq 1}$ *be a measurable function sequence from* Ω *to* $\mathbf{R}$ *such that* $f_n(x) \to f(x)$ *a.e. on* Ω *and* $|f(x)| < \infty$ *a.e. Then for* $\forall \varepsilon > 0$, $\exists\ A \subset \Omega$ *measurable such that* $meas(\Omega \setminus A) < \varepsilon$ *and* $f_n \to f$ *uniformly on* A.

Theorem 2.13. (Dunford-Pettis) *Let* Ω *be an open bounded subset of* $\mathbf{R}^N$ *and let* F *be a bounded subset of* $L^1(\Omega)$. *Then* F *is weakly compact in* $L^1(\Omega)$ *if and only if for any* $\varepsilon>0$ *there exists* $\delta>0$ *such that* $\int_A |f(x)|\,dx < \varepsilon$, $\forall A \subset \Omega$ *with* $\text{meas}(A) < \delta$, $\forall f \in F$.

We enounce now some important results in Sobolev spaces.

Let $\xi \in \mathbf{R}$. We call the *positive part* ξ^+ and the *negative part* ξ^- of ξ the non-negative numbers defined as

$$\xi^+ := \max\{\xi, 0\} = \frac{\xi + |\xi|}{2}, \tag{2.10}$$

$$\xi^- := -\min\{\xi, 0\} = \frac{|\xi| - \xi}{2}, \tag{2.11}$$

Theorem 2.14. (Stampacchia's lemma) *Let* $u \in H^1(\Omega)$. *Then* $u^+ \in H^1(\Omega)$ *and*

$$\left(\frac{\partial}{\partial x_i} u^+\right)(x) = \begin{cases} \dfrac{\partial u}{\partial x_i}(x) & a.e.\ in\ \{x \in \Omega;\ u(x) > 0\}, \\ 0 & a.e.\ in\ \{x \in \Omega;\ u(x) \leq 0\}, \end{cases} \tag{2.12}$$

for all $i = 1, 2, ..., N$. *If* $\gamma_0(u) \leq 0$ *a.e. in* $\partial\Omega$, *then* $u^+ \in H_0^1(\Omega)$.

Corollary 2.15. *Let* $u \in H^1(\Omega)$. *Then* $u^- \in H^1(\Omega)$ *and*

$$\left(\frac{\partial}{\partial x_i} u^-\right)(x) = \begin{cases} -\dfrac{\partial u}{\partial x_i}(x) & a.e.\ in\ \{x \in \Omega;\ u(x) < 0\}, \\ 0 & a.e.\ in\ \{x \in \Omega;\ u(x) \geq 0\}, \end{cases} \tag{2.13}$$

for all $i = 1, 2, ..., N$. *If* $\gamma_0(u) \geq 0$ *a.e. in* $\partial\Omega$, *then* $u^- \in H_0^1(\Omega)$.

Corollary 2.16. *Let* $u \in H^1(\Omega)$ *and let* k *be a real number. Then the function* $\widetilde{u} = \max\{u, k\}$ *belongs to* $H^1(\Omega)$ *and*

$$\left(\frac{\partial}{\partial x_i} \widetilde{u}\right)(x) = \begin{cases} \dfrac{\partial \widetilde{u}}{\partial x_i}(x) & a.e.\ in\ \{x \in \Omega;\ u(x) > k\}, \\ 0 & a.e.\ in\ \{x \in \Omega;\ u(x) = k\}, \end{cases} \tag{2.14}$$

for all $i = 1, 2, ..., N$.

For the proofs of these three last results we refer the readers to the monograph [13].

Theorem 2.17. (Poincaré inequality) *Let Ω be a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary. For each $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega |u(x)|^2 \, dx \leq c_\Omega \int_\Omega |\nabla u(x)|^2 \, dx \tag{2.15}$$

with c_Ω depending only on Ω and on the dimension N.

Theorem 2.18. (Poincaré inequality) *Let Ω be a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary* (*Lipschitz*) *and let $\Gamma_0 \subset \partial\Omega$ such that* $\text{meas}(\Gamma_0) \neq 0$. *For each $u \in H^1(\Omega)$, we have*

$$\|u\|^2_{H^1(\Omega)} \leq c_P \left(\int_{\Gamma_0} |u(x)|^2 \, d\sigma + \int_\Omega |\nabla u(x)|^2 \, dx \right) \tag{2.16}$$

with c_P depending only on Ω and on the dimension N.

These results remain valid if the domain is bounded in one direction only.

Now we shall recall the Sobolev inequalities, meaning the following inclusions in the topological and algebraic sense:

Theorem 2.19. *Let m be an integer ≥ 1 and $p \in [1, +\infty)$. Then*

$$\begin{aligned}
&\textit{if } \frac{1}{p} - \frac{m}{N} > 0,\ W^{m,p}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N),\ \textit{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{N},\\
&\textit{if } \frac{1}{p} - \frac{m}{N} = 0,\ W^{m,p}(\mathbf{R}^N) \subset L^q(\mathbf{R}^N),\ \forall q \in [p, +\infty),\\
&\textit{if } \frac{1}{p} - \frac{m}{N} < 0,\ W^{m,p}(\mathbf{R}^N) \subset L^\infty(\mathbf{R}^N)
\end{aligned}$$

with continuous injections.

We have for any $u \in W^{m,p}(\mathbf{R}^N)$

$$\|D^\alpha u\|_{L^\infty(\mathbf{R}^N)} \leq C \, \|u\|_{W^{m,p}(\mathbf{R}^N)}\,,\ \forall\, |\alpha| \leq k = \left[m - \frac{N}{p} \right].$$

In particular, $W^{m,p}(\mathbf{R}^N) \subset C^k(\mathbf{R}^N)$.

Theorem 2.20. *Let $\Omega \subset \mathbf{R}^N$ and we assume that either Ω is an open subset of C^1 class with bounded boundary $\partial\Omega$, or $\Omega = \mathbf{R}^N_+$. Let $1 \leq p \leq \infty$. Then*

$$\begin{aligned}
&\textit{if } 1 \leq p < N,\ W^{1,p}(\Omega) \subset L^{p'}(\Omega),\ \textit{where } \frac{1}{p'} = \frac{1}{p} - \frac{1}{N},\\
&\textit{if } p = N,\ W^{1,p}(\Omega) \subset L^q(\Omega),\quad \forall q \in [p, +\infty),\\
&\textit{if } p > N,\ W^{1,p}(\Omega) \subset L^\infty(\Omega),
\end{aligned}$$

with continuous injections.

Moreover, if $p > N$, *we have for any* $u \in W^{1,p}(\Omega)$

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,p}(\Omega)} |x-y|^{\alpha} \ \text{a.e. } x, y \in \Omega, \ \alpha = 1 - \frac{N}{p}.$$

In particular $W^{1,p}(\Omega) \subset C(\overline{\Omega})$.

Theorem 2.21. (Rellich-Kondrachov) *Let* Ω *be an open subset of* $\mathbf{R}^N$ *of class* C^1 *with compact boundary* $\partial\Omega$. *Then*

$$\begin{aligned} &\text{if } p < N,\ W^{1,p}(\Omega) \subset L^q(\Omega),\ \forall q \in [1, p'), \frac{1}{p'} = \frac{1}{p} - \frac{1}{N}, \\ &\text{if } p = N,\ W^{1,p}(\Omega) \subset L^q(\Omega),\ \forall q \in [1, +\infty), \\ &\text{if } p > N,\ W^{1,p}(\Omega) \subset C(\overline{\Omega}), \end{aligned}$$

with compact injections.

Theorem 2.22. (Vitali) *Let* $\{f_n\}_{n\geq 1} \subset L^1(\Omega)$. *If* $\{f_n\}_{n\geq 1}$ *converges weakly to* f *in* $L^1(\Omega)$ *and* $f_n(x) \to f(x)$ *a.e.* $x \in \Omega$, *then*

$$f_n \longrightarrow f \text{ strongly in } L^1(\Omega).$$

Generally, let $f_n(x) \to f(x)$ *a.e.* $x \in \Omega$ *and* $f \in L^p(\Omega)$, $1 \leq p < \infty$. *Then,* $f_n \to f$ *strongly in* $L^p(\Omega)$ *if and only if* $\left\{g_n := |f_n|^p_{n\geq 1}\right\}$ *is bounded in* $L^1(\Omega)$ *and* $g_n \to g$ *weakly in* $L^1(\Omega)$, *where* $g(x) := |f(x)|^p$.

Theorem 2.23. *Assume that* $\text{meas}(\Omega) < \infty$. *Let* $\{f_n\}_{n\geq 1}$ *be bounded in* $L^2(\Omega)$ *and* $f_n(x) \to f(x)$ *a.e.* $x \in \Omega$. *Then*

$$\begin{aligned} &f_n \longrightarrow f \text{ weakly in } L^2(\Omega), \\ &f_n \longrightarrow f \text{ strongly in } L^1(\Omega). \end{aligned}$$

Generally, if $\{f_n\}_{n\geq 1}$ *is bounded in* $L^p(\Omega)$ *and* $f_n(x) \to f(x)$ *a.e.* $x \in \Omega$, *then*

$$f_n \longrightarrow f \text{ strongly in } L^q(\Omega),$$

where $1 \leq q < p < \infty$.

A.3 Vectorial distributions and $W^{k,p}$ spaces

Consider now $[0, T]$ a fixed real interval and let X be a Banach space. We denote by $\mathcal{D}'(0, T; X)$ the space of all linear and continuous operators from $\mathcal{D}(0, T)$ to X. An element of $\mathcal{D}'(0, T; X)$ is called an X-valued, or *vectorial distribution* on $(0, T)$. For a detailed presentation see [110]. We denote by

$$\begin{aligned} L^p(0, T; X) := \big\{ &f : (0, T) \to X \text{ a.e. } t;\ f \text{ measurable}, \\ &\text{and } \|f(t)\|_X^p \text{ is Lebesgue integrable over } (0, T) \big\}. \end{aligned}$$

This is a Banach space with the norm

$$\|f\|_{L^p(0,T;X)} := \left(\int_0^T \|f(t)\|_X^p \, dt \right)^{1/p}, \text{ if } p \in [1, \infty)$$

and

$$\|f\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|f(t)\|_X, \text{ if } p = \infty.$$

For k be a positive integer, $W^{k,p}(0,T;X)$ is the space of all vectorial distributions $u \in \mathcal{D}'(0,T;X)$ with the property that

$$W^{k,p}(0,T;X) = \left\{ u \in \mathcal{D}'(0,T;X);\ \frac{d^j u}{dt^j} \in L^p(0,T;X),\ j = 0, 1, ..., k \right\}$$

with $\frac{d^j u}{dt^j}$ (denoted also $(u^{(j)})$ the derivative in the sense of distributions

$$\frac{d^j u}{dt^j}(\varphi) = (-1)^j u \left(\frac{d^j \varphi}{dt^j} \right), \ \forall \varphi \in \mathcal{D}(0,T).$$

The space $W^{k,p}(0,T;X)$ is a Banach space with the norm

$$\|u\|_{W^{k,p}(0,T;X)} := \left\{ \sum_{j=0}^{k} \left\| \frac{d^j u}{dt^j} \right\|_{L^p(0,T;X)}^p \right\}^{1/p}, \text{ if } 1 \le p < \infty$$

and

$$\|u\|_{W^{k,\infty}(0,T;X)} := \max_{0 \le j \le k} \left\| \frac{d^j u}{dt^j} \right\|_{L^\infty(0,T;X)}, \text{ if } p = \infty.$$

Definition 3.1. Let $[a,b] \subset \mathbf{R}$, let X be a real normed vector space and let $u : [a,b] \to X$ be such that $u(t) \in X$, $\forall t \in [a,b]$. The function u is *strongly continuous* at $c \in [a,b]$ if, for each $\varepsilon > 0$, a positive δ can be found, such that

$$\|u(t) - u(c)\|_X < \varepsilon, \text{ whenever } t \in [a,b] \text{ and } |t - c| < \delta.$$

A function strongly continuous at every $c \in [a,b]$ is called strongly continuous on $[a,b]$. The space of all strongly continuous functions on $[a,b]$ will be denoted by $C([a,b];X)$.

Definition 3.2. The function u is (strongly) *uniformly continuous* on $[a,b]$ if for each $\varepsilon > 0$, a $\delta = \delta(\varepsilon)$ can be found, such that

$$\|u(t) - u(t')\|_X < \varepsilon, \text{ whenever } t, t' \in [a,b] \text{ and } |t - t'| < \delta. \tag{3.1}$$

Definition 3.3. The function u satisfies a *Lipschitz* condition on $[a,b]$ if there exists a positive constant M such that

$$\|u(t) - u(t')\|_X < M\,|t - t'|, \text{ for all } t, t' \in [a,b].$$

Theorem 3.4. *Any function $u \in C([a,b];X)$ is uniformly continuous on $[a,b]$.*

Theorem 3.5. *If X is a Banach space, then $C([a,b];X)$ is a Banach space with respect to the norm*

$$\|u\|_\infty = \sup_{t\in[a,b]} \{\|u(t)\|\}.$$

Theorem 3.6. *Let X be a Banach space and let $u : [a,b] \to X$ be continuous. Then there exists the Riemann integral $\int_a^b u(t)dt \in X$ and we have*

$$\left\| \int_a^b u(t)dt \right\|_X \leq \int_a^b \|u(t)\|_X \, dt.$$

Theorem 3.7. *Let X be a Banach space and let $u : [a,b] \to X$ be continuous. Then, for each $t \in [a,b]$ the Riemann integral $\int_a^t u(\tau)d\tau$ exists in X and*

$$\frac{d}{dt}\left(\int_a^t u(\tau)d\tau\right) = u(t).$$

Definition 3.8. An X-valued function $x(t)$ defined on $[0,T]$ is said to be *absolutely continuous* on $[0,T]$ if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\sum_n \|x(\beta_n) - x(\alpha_n)\|_X \leq \varepsilon$, whenever $\sum_n |\beta_n - \alpha_n| \leq \delta(\varepsilon)$ and $(\alpha_n, \beta_n) \cap (\alpha_m, \beta_m) = \emptyset$ for $m \neq n$.

Now we introduce $A^{k,p}(0,T;X)$, the space of all *absolutely continuous functions* $u : [0,T] \to X$ whose derivatives $\frac{d^j u}{dt^j}$ (defined almost everywhere) are absolutely continuous for $j = 1, 2, ..., k-1$ and belong to $L^p(0,T;X)$. In particular $A^{1,p}(0,T;X)$ consists of all absolutely continuous functions $u : [0,T] \to X$ with the property that the function $t \to \frac{du}{dt}$ exists a.e. on $(0,T)$ and belongs to $L^p(0,T;X)$. Here, $\frac{du}{dt}$ is the *strongly derivative* of u defined as

$$\frac{du}{dt}(t) = \lim_{h\to 0} \frac{u(t+h) - u(t)}{h} \text{ strongly in } X.$$

Theorem 3.9. *Let X be a Banach space and $u \in L^p(0,T;X)$, $1 \leq p \leq \infty$. Then the following conditions are equivalent:*

(i) $u \in W^{k,p}(0,T;X)$,

(ii) *there is $u_1 \in A^{k,p}(0,T;X)$ such that $u(t) = u_1(t)$ a.e. on $(0,T)$.*

Remark 3.10. Let X be a Banach space. Then $W^{1,1}(0,T;X)$ is densely and continuously embedded in $L^2(0,T;X)$. Indeed, since $u \in W^{1,1}(0,T;X)$ is absolutely continuous we have $W^{1,1}(0,T;X) \subset C([0,T];X) \subset L^2(0,T;X)$.

Theorem 3.11. *Let X be reflexive and let $f \in L^p(0,T;X)$, $1 < p \leq \infty$. Then the following conditions are equivalent:*

(i) *there exists $f_1 \in W^{1,p}(0,T;X)$ such that $f(t) = f_1(t)$ a.e. on $(0,T)$,*

(ii) $\displaystyle\int_0^{T-h} \|f(t+h) - f(t)\|_X^p \, dt \leq Ch^p, \forall h \in (0,T)$.

Here, C is a constant. For the proofs see [9].

Theorem 3.12. (Lions-Aubin, [4], [83]) *Let X_1, X_2, X_3 be three Banach spaces, X_1 and X_3 reflexive, $X_1 \subset X_2 \subset X_3$ with dense and continuous inclusions and the inclusion $X_1 \subset X_2$ is compact. Let $\{u_n\}_{n\geq 1}$ be a bounded sequence in $L^{p_1}(0,T;X_1)$ such that $\left\{\dfrac{du_n}{dt}\right\}_{n\geq 1}$ is bounded in $L^{p_3}(0,T;X_3)$. Then $\{u_n\}_{n\geq 1}$ is compact in $L^{p_2}(0,T;X_2)$, where $1 \leq p_1$, p_2, $p_3 < \infty$.*

Theorem 3.13. (Arzelà) *A subset X_0 of $C([a,b])$ is compact if and only if it is bounded and equicontinuous, i.e., if and only if*

(i) *there exists a constant M such that $\|f\|_{L^\infty(a,b)} \leq M$ for all $f \in X_0$ and*

(ii) *for each $\varepsilon > 0$ a $\delta > 0$ can be found such that $|f(x) - f(y)| < \varepsilon$ for all $f \in X_0$ and for all $x, y \in [a,b]$ such that $|x - y| < \delta$ (where δ may depend on ε but it is independent of f and of x, y).*

If X is a Banach space this result is extended to $\mathcal{M} \subset C([a,b];X)$ provided that $\{u(x);\, u \in \mathcal{M}\}$ is relatively strongly compact in X for any $x \in [a,b]$.

Theorem 3.14. (Ascoli-Arzelà) *Let X be a Banach space and let $\mathcal{M} \subset C([0,T];X)$ be a family of functions such that*

(i) $\|u(t)\|_X \leq M$, $\forall t \in [0,T]$, $u \in \mathcal{M}$,

(ii) *$\mathcal{M}$ is equi-uniformly continuous i.e., $\forall \varepsilon$, $\exists \delta(\varepsilon)$ such that*

$$\|u(t) - u(s)\|_X \leq \varepsilon \text{ if } |t - s| \leq \delta(\varepsilon), \ \forall u \in \mathcal{M},$$

(iii) *For each $t \in [0,T]$ the set $\{u(t); u \in \mathcal{M}\}$ is compact in X.*

Then, $\mathcal{M}$ is compact in $C([0,T];X)$.

Finally, we define the convolution between two functions f and g, by stating

Theorem 3.15. *Let $f \in L^1(\mathbf{R}^N)$ and $g \in L^p(\mathbf{R}^N)$ with $1 \leq p \leq \infty$. Then the function $y \to f(x-y)g(y)$ is integrable a.e. $x \in \mathbf{R}^N$ and we define*

$$(f * g)(x) = \int_{\mathbf{R}^N} f(x-y)g(y)dy.$$

Moreover, $f * g \in L^1(\mathbf{R}^N)$ and

$$\|f * g\|_{L^p(\mathbf{R}^N)} \leq \|f\|_{L^1(\mathbf{R}^N)} \|g\|_{L^p(\mathbf{R}^N)} .$$

Consider now the function $\rho : \mathbf{R}^N \to \mathbf{R}$,

$$\rho(x) := \begin{cases} \mu_\rho \exp[(|x|-1)^{-1}], & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \tag{3.2}$$

where $|x|$ is the norm in $\mathbf{R}^N$ and $\mu_\rho = \int_{\mathbf{R}^N} \rho(x)dx$. Then supp $\rho \subset \overline{B(0,1)}$ and

$$\rho(x) \geq 0,\ \rho(x) = \rho(-x),\ \rho \in C^\infty(\mathbf{R}^N),\ \int_{\mathbf{R}^N} \rho(x)dx = 1. \tag{3.3}$$

Let $\varepsilon > 0$, $f \in L^1_{\text{loc}}(\mathbf{R}^N)$ and define

$$\rho_\varepsilon(x) = \varepsilon^{-N} \rho\left(\frac{x}{\varepsilon}\right),\ \forall x \in \mathbf{R}^N, \tag{3.4}$$

$$f_\varepsilon(x) = \int_{\mathbf{R}^N} f(y)\rho_\varepsilon(x-y)dy = \int_{\mathbf{R}^N} f\left(x-\varepsilon\right)\rho(x)dy,\ \forall x \in \mathbf{R}^N.$$

Definition 3.16. The sequence $\{\rho_\varepsilon\}_{\varepsilon>0}$ is called *mollifier* and the function f_ε is called the *regularization* of f.

Lemma 3.17. (see [13], p. 5) Let $f \in L^p(\mathbf{R}^N)$, $1 \leq p < \infty$. Then, $f_\varepsilon \in C^\infty(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$ and $\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{L^p(\mathbf{R}^N)} = 0$.

Theorem 3.18. *Let Ω be an open subset of $\mathbf{R}^N$. Then the space $C^\infty(\overline{\Omega})$ is dense in $L^p(\Omega)$, $1 \leq p < \infty$.*

Let $\Omega \subset \mathbf{R}^N$ and $Q = \Omega \times (0,T)$. Denote

$$W_p^{2,1}(Q) = \{y \in L^p(Q);\ D_t^r D_x^\alpha y \in L^p(Q),\ 2r + |\alpha| \leq 2\}.$$

In particular $W_2^{2,1}(Q) = W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

Theorem 3.19. (see [82]) *If $p > \dfrac{N+2}{2}$ and $p \neq N+2$, the space $W_p^{2,1}(Q)$ is compactly embedded in $C(\overline{Q})$.*

A.4 Operators in Banach spaces

In the main part of the book the problems we deal with involve mainly nonlinear operators. That is why in this section, definitions and results related in general to nonlinear operators there are presented. Most of them apply also for linear operators (some with slight modifications).

We consider X and Y two normed vector spaces and $X \times Y$ their Cartesian product space. An element of $X \times Y$ will be denoted $[x,y]$ for $x \in X$ and $y \in Y$.

Definition 4.1. A *multivalued operator* A from X to Y is a subset of $X \times Y$.

If $A \subset X \times Y$ we define

$$\begin{aligned} Ax &:= \{y \in Y;\ [x,y] \in A\} \\ D(A) &:= \{x \in X;\ Ax \neq \emptyset\} \text{ is the domain of } A, \\ R(A) &:= \bigcup_{x \in D(A)} Ax \text{ is the range of } A, \\ A^{-1} &:= \{[y,x];\ [x,y] \in A\} \text{ is the inverse of } A, \\ G(A) &:= \bigcup_{x \in D(A)} [x, Ax] \subset X \times Y \text{ is the graph of } A. \end{aligned} \tag{4.1}$$

In fact the operators from X to Y will not be distinguished from their graphs in $X \times Y$ and generally A^{-1} may be a multivalued operator.

If $A, B \subset X \times Y$ and $\lambda \in \mathbf{R}$, one sets

$$\begin{aligned} \lambda A &:= \{[x, \lambda y];\ [x,y] \in A\}, \\ A + B &:= \{[x, y+z];\ [x,y] \in A,\ [x,z] \in B\}, \\ AB &:= \{[x,z];\ [x,y] \in B,\ [y,z] \in A \text{ for some } y \in Y\}. \end{aligned}$$

If A is single-valued, Ax will denote either the value of A at x or the set defined by (4.1). Generally $X \supset D(A) \to R(A) \subset Y$ and it is said that A maps $D(A)$ *into* Y. If $R(A) \equiv Y$, then it is said that A maps $D(A)$ *onto* Y.

Definition 4.2. An *unbounded* operator from X to Y is an application defined on the subspace $D(A)$ of X, $A : D(A) \subset X \to Y$.

Definition 4.3. Let X and X' be a Banach space and, respectively, its dual. An operator A defined from X to X' is called *monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle_{X',X} \geq 0,\ \forall [x_i, y_i] \in A,\ i = 1, 2. \tag{4.2}$$

Remark 4.4. If A is single-valued, then $y_i \in Ax_i$ is replaced in all definitions and results by Ax_i for $x_i \in D(A)$.

Definition 4.5. The operator A is called *maximal monotone* if it is not properly contained in any other monotone subset of $X \times X'$.

Definition 4.6. The operator $A : X \to X'$ is said to be *strongly monotone* if

$$\langle y_1 - y_2, x_1 - x_2 \rangle_{X',X} \geq \rho \left\| x_1 - x_2 \right\|_X^2, \tag{4.3}$$

for any $[x_i, y_i] \in A$, $i = 1, 2$ with $\rho > 0$ fixed.

Definition 4.7. The operator $A : X \to X'$ is said to be *locally bounded* at $x_0 \in X$ if there exists a neighbourhood V_0 of x_0 such that $A(V_0) = \{Ax;\ x \in V_0 \cap D(A)\}$ is a bounded subset of X'. The operator A is *bounded* if it maps every bounded subset of X into a bounded subset of X'.

Definition 4.8. The operator A is said to be *injective* if

$$Ax \cap Ay \neq \emptyset \text{ implies } x = y. \tag{4.4}$$

Definition 4.9. The operator A is called *closed* if

$$[x_n, y_n] \in A \text{ and } x_n \longrightarrow x,\ y_n \longrightarrow y \text{ imply } [x, y] \in A. \tag{4.5}$$

Definition 4.10. A subset A of $X \times X'$ is called *demiclosed* if it is strongly-weakly closed in $X \times X'$, i.e.,

$$x_n \longrightarrow x,\ y_n \rightharpoonup y \text{ where } [x_n, y_n] \in A \text{ imply } [x, y] \in A. \tag{4.6}$$

Definition 4.11. The single-valued operator A of $X \times X'$ is called *continuous* at $x_0 \in X$ if

$$\begin{aligned}&\forall \varepsilon > 0, \exists \delta(\varepsilon, x_0) > 0 \text{ such that}\\ &\text{for any } x \in X \text{ with } \|x - x_0\|_X < \delta \text{ it follows } \|Ax - Ax_0\|_{X'} \leq \varepsilon.\end{aligned}$$

It is said that the single-valued operator A has the *Lipschitz* property at $x_0 \in X$ if there exists a positive constant M such that

$$\|Ax - Ax_0\|_{X'} \leq M \|x - x_0\|_X .$$

If the operator A is linear, then the boundedness is equivalent to the continuity.

Definition 4.12. The single-valued operator $A : D(A) = X \to X'$ is said to be *hemicontinuous* if

$$A(x + \lambda y) \rightharpoonup Ax, \text{ as } \lambda \longrightarrow 0,\ \forall x, y \in X. \tag{4.7}$$

Definition 4.13. The single-valued operator $A : X \to X'$ is called *demicontinuous* if it is strongly-weakly continuous from X to X', i.e.,

$$Ax_n \rightharpoonup Ax \text{ for any } x_n \longrightarrow x \text{ in } X. \tag{4.8}$$

Definition 4.14. The (multivalued) operator $A : X \to X'$ is *coercive* if there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} \frac{\langle y_n, x_n - x_0 \rangle_{X',X}}{\|x_n\|_X} = +\infty \tag{4.9}$$

for all $[x_n, y_n] \in A$ such that $\lim\limits_{n\to\infty} \|x_n\|_X = +\infty$.

In the most applications it is enough to prove that there exists $\alpha > 0$ such that $\langle y, x - x_0 \rangle_{X',X} \geq \alpha \|x\|_X^2$, $\forall [x,y] \in A$.

Theorem 4.15. *Let X and X' be reflexive and strictly convex and let $F : X \to X'$ be the duality mapping of X. Let A be a monotone operator of $X \times X'$. Then A is maximal monotone in $X \times X'$ if and only if, for any $\lambda > 0$ (equivalently for some $\lambda > 0$) $R(A + \lambda F) = X'$.*

We also mention a result stating that if X' is strictly convex, then the duality mapping $F : X \to X'$ is single valued and demicontinuous (see [9]).

Corollary 4.16. *Let X be reflexive and let B be a monotone and hemicontinuous operator from X to X'. If A is maximal monotone from X to X', then $A + B$ is maximal monotone.*

Theorem 4.17. *Let X be reflexive and let A be monotone, everywhere defined and hemicontinuous from X to X'. Then A is maximal monotone. If A is coercive, then A is surjective, i.e.,*

$$R(A) = X'.$$

Theorems 4.15 and 4.17 were given by Minty, [96] in the case of Hilbert spaces. They were extended to the general case by Browder, [37], [38].

Definition 4.18. The multivalued operator $A : X \to X$ is called *accretive* if for any $[x_i, y_i] \in A$, $i = 1, 2$, there exists $f \in F(x_1 - x_2)$ such that

$$\langle y_1 - y_2, f \rangle \geq 0. \tag{4.10}$$

Definition 4.19. An accretive operator is said to be *maximal accretive* if it is not properly contained in any accretive subset of $X \times X$.

Definition 4.20. An accretive operator is called *m-accretive* if $R(I+A) = X$. The operator $A \subset X \times X$ is said to be *quasi m-accretive* if $\lambda I + A$ is accretive for $\lambda > \omega$, $\lambda > 0$.

Definition 4.21. An operator A is called *dissipative* (maximal dissipative, m-dissipative) is $(-A)$ is accretive (maximal accretive, m-accretive).

Theorem 4.22. *Any m-accretive operator in $X \times X$ is maximal accretive. If $X = X' = H$ is a Hilbert space, then the notions of maximal accretive operators coincide with those of m-accretive operators.*

The last part of this theorem is owed to Minty, [96].

Let K be a closed subset of X. we denote

$$|K| := \inf \ \{\|x\|_X ; \ x \in K\}.$$

Definition 4.23. Let $A \subset X \times X$ be a multivalued operator. The operator A^0 defined by

$$A^0x := \{y \in Ax;\ \|y\|_X = |Ax|\}$$

is called the *minimal section* of A. If A is single valued, then $A^0 = A$.

We end this section recalling the Banach fixed point theorem.

Definition 4.24. Let X be a vector normed space and $A : D(A) \subset X \to R(A) \subset X$. A is a *contraction* if there exists a constant $\alpha \in [0,1]$ such that

$$\|Au_1 - Au_2\|_X \le \alpha \|u_1 - u_2\|_X\,,\ \text{for all } u_1,\ u_2 \in D(A). \tag{4.11}$$

A is called a *strict contraction* if (4.11) holds with $0 \le \alpha < 1$.

Theorem 4.25. (Banach fixed point theorem) *Let X be a Banach space and let $A : X \to X$ be a strict contraction. Then the equation $Au = u$ has a unique solution in X, i.e., A has a unique fixed point u.*

A.5 Convex functions and subdifferential mappings

In convex analysis the extended real line $\widetilde{\mathbf{R}} = (-\infty, \infty) \cup \{-\infty, +\infty\}$ is considered.

Definition 5.1. Let X be a real Banach space and X' its dual. The function $\varphi : X \to \widetilde{\mathbf{R}}$ is called *convex* if

$$\varphi\left(\lambda x + (1-\lambda)y\right) \le \lambda\varphi(x) + (1-\lambda)\varphi(y), \tag{5.1}$$

for $\lambda \in [0,1]$ and $x, y \in X$. A convex function $\varphi : X \to \widetilde{\mathbf{R}}$ is said to be *proper* if $(-\infty) \notin \varphi(X)$ and $\varphi(X) \ne \{+\infty\}$. Here it is assumed that

$$(+\infty) - (-\infty) = (-\infty) + (+\infty) = +\infty. \tag{5.2}$$

Definition 5.2. The set

$$D(\varphi) := \{x \in X;\ \varphi(x) < +\infty\}$$

is called the *effective domain* and the set

$$K(\varphi) = \{[x, \lambda] \in X \times \mathbf{R};\, \varphi(x) \le \lambda\} \tag{5.3}$$

is called the *epigraph* of φ and is denoted epi φ.

Definition 5.3. A function $\varphi : X \to \widetilde{\mathbf{R}}$ is called *strongly* (*weakly*, resp.) *lower semicontinuous* (l.s.c.) if for any $a \in \mathbf{R}$ the set $\{v \in X;\, \varphi(v) \le a\}$ is strongly (weakly, resp.) closed.

It is obvious that $K(\varphi)$ is a convex set. If φ is l.s.c, then epi φ is closed in $X \times \mathbf{R}$ and reciprocally. Note also that a proper, convex, l.s.c. function φ is continuous on the interior of $D(\varphi)$.

Proposition 5.4. *The function $\varphi : X \to \widetilde{\mathbf{R}}$ is strongly lower semicontinuous on X if it is strongly sequentially lower semicontinuous, i.e., for any sequence $\{x_n\}_{n\geq 1}$ which converges strongly to x, we have*

$$\liminf_{x_n \to x} \varphi(x_n) \geq \varphi(x), \ \forall x \in X. \tag{5.4}$$

Proposition 5.5. *Any convex function $\varphi : X \to \widetilde{\mathbf{R}}$ is strongly lower semicontinuous if and only if it is weakly lower semicontinuous.*

Remark 5.6. Any strongly sequentially convex l.s.c. function is *weakly sequentially l.s.c.,* the latter meaning that,

$$\liminf_{x_n \to x} \varphi(x_n) \geq \varphi(x_0), \text{ if } x_n \rightharpoonup x_0 \in X. \tag{5.5}$$

If the function is not convex, the above assertion does not function in both senses, namely the weakly lower semicontinuity is a stronger property and implies the strongly lower semicontinuity, but the reverse is not true.

Proposition 5.7. *Any proper l.s.c. convex function on X is bounded below by an affine function, i.e., there exist $x^* \in X'$ and $\mu \in \mathbf{R}$ such that*

$$\varphi(x) \geq \langle x^*, x \rangle_{X',X} + \mu, \ \forall x \in X. \tag{5.6}$$

We shall review below without proofs some facts of convex analysis in Banach spaces. We refer to H. Brezis [30] and V. Barbu [9] for complete proofs.

Definition 5.8. Let φ be a proper convex function on X and $x \in X$. Then the set

$$\partial\varphi(x) := \{x^* \in X'; \ \varphi(x) - \varphi(y) \leq \langle x^*, x - y \rangle_{X',X}, \forall y \in X\} \tag{5.7}$$

is called the *subdifferential* of φ at x and its elements are called *subgradients* of φ at x.

Example 5.9. Let φ be Gâteaux differentiable at x, meaning that the function $\varphi : X \to \mathbf{R}$,

$$\lim_{\lambda \to 0} \frac{\varphi(x + \lambda y) - \varphi(x)}{\lambda} = \eta_y$$

exists for all $y \in X$ and $y \to \eta_y$ is a linear continuous functional on X. Then $\partial\varphi(x)$ consists of a single element, namely the Gâteaux differential of φ at x.

Example 5.10. Let $\varphi : X \to (-\infty, \infty]$, $\varphi(x) = \frac{1}{2} \|x\|_X^2$. It is easy to see that φ is a proper, convex, l.s.c. function (in fact it is exactly continuous). For the convexity we have to check that

$$\begin{aligned}\|\lambda u + (1-\lambda)v\|_X^2 &\leq \lambda^2 \|u\|_X^2 + 2\lambda(1-\lambda)\|u\|_X \|v\|_X + (1-\lambda)^2 \|v\|_X^2 \\ &\leq \lambda^2 \|u\|_X^2 + \lambda(1-\lambda)(\|u\|_X^2 + \|v\|_X^2) + (1-\lambda)^2 \|v\|_X^2 \\ &\leq \lambda \|u\|_X^2 + (1-\lambda)\|v\|_X^2 .\end{aligned}$$

Then, $\partial\varphi$ coincides with the duality mapping $F : X \to X'$.

Example 5.11. Let K be a closed convex subset of X and let $I_K : X \to (-\infty, \infty]$ be the indicator function defined by

$$I_K(x) := \begin{cases} 0, & \text{if } x \in K \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.8}$$

Then, I_K is convex and l.s.c. on X, $D(\partial I_K) = D(I_K) = K$ and $\partial I_K(x) = \{0\}$ for $x \in \overset{\circ}{K}$ (interior of K). If $x \in \partial K$, then $\partial I_K(x)$ coincides with the cone of normals to K at point x. We mean by the normal cone to K at x the set $N_K(x) \subset X'$, defined by

$$N_K(x) := \{x^* \in X';\ \ \langle x^*, x - y\rangle_{X',X} \geq 0,\ \forall y \in K\}.$$

Proposition 5.12. *Any set $K \subset X$ is convex (respectively closed) if and only if I_K is convex (respectively l.s.c.).*

Theorem 5.13. (Rockafeller) *If $\varphi : X \to (-\infty, \infty]$ is a proper convex function, then $\partial\varphi$ is a monotone operator from X to X'. If φ is still l.s.c. then $\partial\varphi$ is maximal monotone.*

Corollary 5.14. *Let φ be a l.s.c. proper convex function on X. Then $D(\partial\varphi)$ is a dense subset of $D(\varphi)$.*

Proposition 5.15. *Let X be reflexive and $A = \partial\varphi$, $\varphi : X \to (-\infty, \infty]$ being a proper convex l.s.c. function on X. Then, the following conditions are equivalent*

(i) $\dfrac{\varphi(x)}{\|x\|_X} \longrightarrow +\infty$ *as* $\|x\|_X \to \infty$ *with* $x \in D(\varphi)$,

(ii) $R(A) = X'$ *and* A^{-1} *is bounded.*

Definition 5.16. A multivalued operator $A : X \to X'$ is said to be *cyclically monotone* if

$$\begin{aligned}&\langle x_0^*, x_0 - x_1\rangle_{X',X} + \cdots + \left\langle x_{n-1}^*, x_{n-1} - x_n\right\rangle_{X',X} + \langle x_n^*, x_n - x_0\rangle_{X',X} \geq 0, \\ &\forall x_i^* \in Ax_i,\ i = 0, 1, ..., n.\end{aligned} \tag{5.9}$$

Definition 5.17. The cyclically monotone operator A is said to be *maximal cyclically monotone* if it has no cyclically monotone extensions in $X \times X'$.

Remark 5.18. Obviously, every cyclically monotone operator is also monotone. If φ is a proper convex function on X, then $\partial\varphi$ is cyclically monotone.

Theorem 5.19. *Let X be a real Banach space and A an operator $A{:}X{\to}X'$. The necessary and sufficient condition to exist φ, a l.s.c. proper convex function on X, such that $A = \partial\varphi$, is that A is a maximal cyclically monotone operator. Moreover, in this case A uniquely determines φ up to an additive constant.*

Example 5.20. (maximal monotone graphs in $\mathbf{R}^2$) Every maximal monotone graph in $\mathbf{R}^2$ (every maximal monotone operator from $\mathbf{R}$ to $\mathbf{R}$) is cyclically monotone.

Example 5.21. (convex integrands) Let $j : \mathbf{R} \to (-\infty, \infty]$ be a l.s.c. proper convex function on $\mathbf{R}$ and let $\beta{=}\partial j$. Let $\varphi : L^2(\Omega){\to}(-\infty, \infty]$ be defined by

$$\varphi(u) = \begin{cases} \displaystyle\int_\Omega j(u(x))dx, & \text{if } j(u) \in L^1(\Omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$.

Proposition 5.22. *The function φ is l.s.c. and convex on $L^2(\Omega)$. Moreover, $w \in \partial\varphi(u)$ if and only if $w(x) \in \beta(u(x))$ a.e. on Ω and*

$$\overline{D(\varphi)} = \{u \in L^2(\Omega);\ u(x) \in \overline{D(j)} \text{ a.e. on } \Omega\}.$$

Detailed proofs of the results given in examples 5.20-5.22 can be found in [9].

Example 5.23. Let Ω be a bounded and open domain of $\mathbf{R}^N$ with a smooth boundary $\Gamma = \partial\Omega$ (e.g., of class C^2). Let j be a l.s.c. proper convex function from $\mathbf{R}$ to $(-\infty, \infty]$ and $\beta = \partial j$. For every $u \in L^2(\Omega)$ let us define

$$\varphi(u) = \begin{cases} \displaystyle\frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx + \int_\Gamma j(u)d\sigma, & \text{if } u \in H^1(\Omega);\ j(u) \in L^1(\Gamma) \\ +\infty, & \text{otherwise.} \end{cases}$$

The function $\varphi : L^2(\Omega) \to (-\infty, \infty]$ is proper, convex and l.s.c.

Proposition 5.24. (Brezis, [29]) *We have $\partial\varphi = -\Delta$ with*

$$D(\partial\varphi) = \left\{u \in H^2(\Omega); -\frac{\partial u}{\partial \nu} \in \beta(u) \text{ a.e. on } \Gamma\right\}$$

where $\dfrac{\partial u}{\partial \nu} = \nabla u \cdot \nu$, ν is the outward normal to Γ. Moreover, there exist c_1 and c_2 such that

$$\|u\|_{H^2(\Omega)} \le c_1 \|u - \Delta u\|_{L^2(\Omega)} + c_2,\ \forall u \in D(\partial\varphi).$$

A.6 Various formulas

Let Ω be an open bounded subset of $\mathbf{R}^N$. Given a function $u \in C^1(\Omega)$ we denote by ∇ the *gradient* operator,

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x)\right).$$

If $u \in C^2(\Omega)$ the *Laplacian* operator of u is defined by

$$\Delta u(x) = \frac{\partial^2 u}{\partial x_1^2}(x) + \cdots + \frac{\partial^2 u}{\partial x_N^2}(x), \;\; \Delta : C^2(\Omega) \longrightarrow C(\Omega).$$

If w is a vector $w = (w_1, ..., w_N)$, with $w_i \in C^1(\Omega)$, then the *divergence* of w is defined as

$$(\nabla \cdot w)(x) = \frac{\partial u}{\partial x_1}(x) + \cdots + \frac{\partial u}{\partial x_N}(x).$$

If Ω is of class C^1 and u is differentiable on $\overline{\Omega}$ we may define the *outward normal derivative* of u on $\partial\Omega$ by

$$\frac{\partial u}{\partial \nu}(x) = \lim_{\lambda \searrow 0} \frac{u(x) - u(x - \lambda\nu(x))}{\lambda} = \nabla u(x) \cdot \nu(x), \;\; \forall x \in \partial\Omega.$$

Here $\nu = (\nu_1, ..., \nu_N)$ is the unit *outward normal* to $\partial\Omega$.

This definition can be extended to functions $u \in C^1(\Omega)$. We set

$$\frac{\partial u}{\partial \nu}(x) = \lim_{\lambda \searrow 0} \nabla u(x - \lambda\nu(x)) \cdot \nu(x), \;\; \forall x \in \partial\Omega$$

and if the limit exists it is equal to the directional derivative

$$\lim_{\lambda \searrow 0} \frac{u(x) - u(x - \lambda\nu(x))}{\lambda}.$$

Theorem 6.1. (Green formula) *Let Ω be an open, relatively compact subset of $\mathbf{R}^N$ with the boundary $\partial\Omega$ of class C^1.*

(i) *If $u, v : \overline{\Omega} \to \mathbf{R}^N$, $u, v \in C^1(\overline{\Omega})$* and $\Delta v \in C(\overline{\Omega})$, *then the following relation*

$$\int_\Omega u\Delta v dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} d\sigma - \int_\Omega \nabla u \cdot \nabla v dx \tag{6.1}$$

takes place, where ν is the outward normal to $\partial\Omega$ and $\frac{\partial v}{\partial \nu}$ is the normal derivative.

(ii) *If $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and Δu, $\Delta v \in C(\overline{\Omega})$, then*

$$\int_\Omega (u\Delta v - v\Delta u) dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) d\sigma. \tag{6.2}$$

Here, dx is the Lebesgue measure on Ω and $d\sigma$ is the surface measure on $\partial\Omega$.

Theorem 6.2. (Gauss-Ostrogradski formula) *Let* Ω *be an open and bounded subset of* $\mathbf{R}^N$ *with the boundary* $\partial\Omega$ *of class* C^1 *and* $v : \overline{\Omega} \to \mathbf{R}^N$, $v \in C^1(\overline{\Omega})$. *Then the following relation*

$$\int_{\Omega} \nabla \cdot v dx = \int_{\partial\Omega} v \cdot \nu d\sigma \tag{6.3}$$

holds.

We note that (6.1) extends by density for all u, $v \in H^1(\Omega)$ such that $\Delta v \in (H^1(\Omega))'$. Also (6.3) extends by density for $v \in H^1(\Omega)$.

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