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Torsten Linß

Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems

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Preface

This is a book on numerical methods for singular perturbation problems – in particular, stationary reaction-convection-diffusion problems exhibiting layer behaviour. More precisely, it is devoted to the construction and analysis of layer-adapted meshes underlying these numerical methods.

Numerical methods for singularly perturbed differential equations have been studied since the early 1970s and the research frontier has been constantly expanding since. A comprehensive exposition of the state of the art in the analysis of numerical methods for singular perturbation problems is [141] which was published in 2008. As that monograph covers a big variety of numerical methods, it only contains a rather short introduction to layer-adapted meshes, while the present book is exclusively dedicated to that subject.

An early important contribution towards the optimisation of numerical methods by means of special meshes was made by N.S. Bakhvalov [18] in 1969. His paper spawned a lively discussion in the literature with a number of further meshes being proposed and applied to various singular perturbation problems. However, in the mid 1980s, this development stalled, but was enlivened again by G.I. Shishkin's proposal of piecewise-equidistant meshes in the early 1990s [121, 150]. Because of their very simple structure, they are often much easier to analyse than other meshes, although they give numerical approximations that are inferior to solutions on competing meshes. Shishkin meshes for numerous problems and numerical methods have been studied since and they are still very much in vogue.

With this contribution we try to counter this development and lay the emphasis on more general meshes that – apart from performing better than piecewise-uniform meshes – provide a deeper insight in the course of their analysis.

In this monograph, a classification and a survey are given of layer-adapted meshes for reaction-convection-diffusion problems. The monograph aims at giving a structured and comprehensive account of current ideas in the numerical analysis for various methods on layer-adapted meshes. Both finite differences, finite elements and finite volumes will be covered.

While for finite difference schemes applied to one-dimensional problems, a rather complete convergence theory for arbitrary meshes is developed, the theory is more fragmentary for other methods and problems. They still require the restriction to certain classes of meshes.

The roots of this monograph are a survey lecture presented at the Oberwolfach seminar *Numerical Methods for Singular Perturbation Problems*, 8–14 April 2001 organised by Pieter W. Hemker, Hans-Görg Roos and Martin Stynes, and a review article [91] invited by Thomas J.R. Hughes. I am indebted to their invitations and their continued encouragement.

My thanks also go to a series of colleagues I had the pleasure of working with over the years and who consequently influenced this monograph: Sebastian Franz, Anja Fröhner, R. Bruce Kellogg, Natalia Kopteva, Niall Madden, Hans-Görg Roos, Martin Stynes and Relja Vulanović.

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Dresden and Prague *Torsten Linß*
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Notation

u	solution of boundary value problems
u^N	numerical approximations of u
ε, ε_d, ε_c	perturbation parameter(s)
$\mathcal{L}$, $\mathcal{L}^*$	differential operator, its adjoint
L, L^*	discrete operator (discretisation of $\mathcal{L}$), its adjoint
$\mathcal{G}$, G	continuous and discrete Green's functions
Ω	domain
$\partial\Omega = \Gamma$	boundary of Ω
n	outward pointing unit vector normal to $\partial\Omega$
N	number of mesh intervals (in each coordinate direction)
C	generic constant, independent of ε and N
$\bar{\omega}$, $\bar{\omega}_x \times \bar{\omega}_y$	sets of mesh points
h, h_i, $\hbar_i$, k, k_j, $\bar{k}_j$	mesh step sizes
v_x, $v_{\bar{x}}$, $v_{\breve{x}}$, $v_{\hat{x}}$, $v_{\tilde{x}}$, $v_{\check{x}}$	difference operators
u^I	nodal interpolant of u
$\lVert \cdot \rVert_\infty$	supremum norm
$\lVert \cdot \rVert_1$	L_1 norm
$\lVert \cdot \rVert_{-1,\infty}$	$W^{-1,\infty}$ norm
$\lvert\lVert \cdot \rVert\rvert_{\varepsilon,\infty}$	ε-weighted $W^{1,\infty}$ norm
$\lVert \cdot \rVert_{*,\omega}$	discrete version of the norm $\lVert \cdot \rVert_*$
V^ω	finite element space on the mesh ω
$a(\cdot,\cdot)$	bilinear form
$(\cdot,\cdot)$, $\lVert \cdot \rVert_0$	scalar product and norm in $L_2(\Omega)$
$\lvert \cdot \rvert_1$, $\lvert\lVert \cdot \rVert\rvert_\varepsilon$	semi norm and ε-weighted energy norm in $H^1(\Omega)$
$\lvert\lVert \cdot \rVert\rvert_{SD}$, $\lvert\lVert \cdot \rVert\rvert_\kappa$, $\lvert\lVert \cdot \rVert\rvert_\rho$	various method-dependent energy norms
$(\cdot,\cdot)_D$, $\lvert \cdot \rvert_{*,D}$, $\lVert \cdot \rVert_{*,D}$	scalar product and (semi) norm restricted to $D \subset \Omega$
C^l, $C^{l,\alpha}$	(Hölder) function spaces

Chapter 1
Introduction

Stationary linear reaction-convection-diffusion problems form the subject of this monograph:

$$-\varepsilon u'' - bu' + cu = f \;\text{ in }\; (0,1), \;\; u(0) = \gamma_0, \; u(1) = \gamma_1$$

and its two-dimensional analogue

$$-\varepsilon \Delta u - \boldsymbol{b} \cdot \nabla u + cu = f \;\text{ in }\; \Omega \subset I\!R^2, \;\; u|_{\partial\Omega} = g$$

with a small positive parameter ε.

Such problems arise in various models of fluid flow [52,53,73]; they appear in the (linearised) Navier-Stokes and in the Oseen equations, in the equations modelling oil extraction from underground reservoirs [32], flows in chemical reactors [3] and convective heat transport with large Péclet number [56]. Other applications include the simulation of semiconductor devices [130].

An Example

Consider the boundary-value problem of finding $u \in C^2(0,1) \cap C[0,1]$ such that

$$-\varepsilon u''(x) - u'(x) = 1 \quad \text{for } \; x \in (0,1), \quad u(0) = u(1) = 0 \tag{1.1}$$

with $0 < \varepsilon \ll 1$. Formally setting $\varepsilon = 0$, yields

$$-u'(x) = 1 \quad \text{for } \; x \in (0,1), \quad u(0) = u(1) = 0.$$

Unlike (1.1), this problem does not possess a solution in $C^2(0,1) \cap C[0,1]$. Consequently, when ε approaches zero, the solution of (1.1) is badly behaved in some way.

The solution of (1.1) is

$$u(x,\varepsilon) = \frac{\mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-x/\varepsilon}}{1 - \mathrm{e}^{-1/\varepsilon}} + 1 - x.$$

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Due to the presence of the exponential $\mathrm{e}^{-x/\varepsilon}$, the solution u and its derivatives change rapidly near $x = 0$ for small values of ε. Regions where this happens are referred to as **layers**. Singularly perturbed problems are typically characterised by the presence of such layers. The term **boundary layer** was introduced by Ludwig Prandtl at the Third International Congress of Mathematicians in Heidelberg in 1904.

The solution of (1.1) may be regarded as a function of two variables:

$$u : [0, 1] \times (0, 1] : (x, \varepsilon) \mapsto u(x, \varepsilon).$$

Taking limits of u for $(x, \varepsilon) \to (0, 0)$, we see that

$$\lim_{x\to 0} \lim_{\varepsilon\to 0} u(x, \varepsilon) = 1 \neq 0 = \lim_{\varepsilon\to 0} \lim_{x\to 0} . \tag{1.2}$$

Thus, u as a function of two variables possesses a *classical singularity* at the point $(0, 0)$ in the (x, ε)-plane. For this reason we may call (1.1) a singularly perturbed boundary-value problem.

What Is a Singularly Perturbed Problem?

Miller et al. [121] give the following characterisation:

> *The justification for the name 'singular perturbation' is that the nature of the differential equations changes completely in the limit case, when the singular perturbation parameter is equal to zero. For example, ... equations change from being nonlinear parabolic equations to nonlinear hyperbolic equations.*

This describes a phenomenon that *can* lead to the formation of boundary layers and *typically will*—if appropriate boundary conditions are imposed. Roos et al. [141] describe singularly perturbed problems as follows.

> *They are differential equations (ordinary or partial) that depend on a small positive parameter ε and whose solutions (or their derivatives) approach a discontinuous limit as ε approaches zero. Such problems are said to be* singularly perturbed*, where we regard ε as a perturbation parameter.*

Both sources avoid a formal definition:

In the present monograph we propose the following definition.

Definition 1.1. Let B be a function space with norm $\|\cdot\|_B$. Let $D \subset \mathbb{R}^d$ be a parameter domain. The continuous function $u : D \to B, \varepsilon \mapsto u(\varepsilon)$ is said to be *regular* for $\varepsilon \to \varepsilon^* \in \partial D$ if there exists a function $u^* \in B$ such that:

$$\lim_{\varepsilon\to\varepsilon^*} \|u_\varepsilon - u^*\|_B = 0,$$

otherwise u_ε is said to be **singular** for $\varepsilon \to \varepsilon^*$.

Let $(\mathcal{P}_\varepsilon)$ be a problem with solution $u(\varepsilon) \in B$ for all $\varepsilon \in D$. We say $(\mathcal{P}_\varepsilon)$ is **singularly perturbed** for $\varepsilon \to \varepsilon^* \in \partial D$ in the norm $\|\cdot\|_B$ if u is singular for $\varepsilon \to \varepsilon^*$. ♡

Remark 1.2. The definition is norm dependent. For example (1.1), is singularly perturbed in the C^0 norm and the L_∞ norm because of (1.2). However, it is not singularly perturbed in the L_2 norm. There exists a function $u^* : x \mapsto 1 - x$ with

$$\|u_\varepsilon - u^*\|_0 = \mathcal{O}\left(\varepsilon^{1/2}\right).$$

The L_2 norm fails to capture the boundary layer in u. ♣

Remark 1.3. Boundary conditions play an important role. Consider the boundary-value problem

$$-\varepsilon u''(x) - u'(x) = 1 \quad \text{for } x \in (0,1), \quad u'(0) = u(1) = 0.$$

This problem is singularly perturbed in the C^1 norm, but it is *not* perturbed in the C^0 norm. The Neumann boundary condition at $x = 0$ leads to the formation of a weak layer only. The first-order derivative remains bounded when $\varepsilon \to 0$. ♣

Uniform Convergence

Classical convergence results for numerical methods for boundary-value problems have the structure

$$\left\|u - u^h\right\| \le K h^k,$$

with the maximum mesh size h. The constant K depends on certain derivatives of u and typically tends to infinity as the perturbation parameter ε approaches zero. This means that the maximal step size h has to be chosen proportional to some positive power of ε which is impractical. Therefore, we are looking for so-called *uniform* or *robust* methods where the numerical costs are independent of the perturbation parameter ε. More precisely, we are looking for robust methods in the sense of the following definition:

Definition 1.4. Let u_ε be the solution of a singularly perturbed problem, and let u_ε^N be a numerical approximation of u_ε obtained by a numerical method with N degrees of freedom. The numerical method is said to be **uniformly convergent** or **robust** with respect to the perturbation parameter ε in the norm $\|\cdot\|$ if

$$\left\|u_\varepsilon - u_\varepsilon^N\right\| \le \vartheta(N) \text{ for } N \ge N_0$$

with a function ϑ satisfying

$$\lim_{N\to\infty} \vartheta(N) = 0 \text{ and } \partial_\varepsilon \vartheta \equiv 0,$$

and with some threshold value $N_0 > 0$ that is independent of ε. ♡

Scope of the Monograph

Well-developed techniques are available for the computation of solutions outside layers [123, 141], but the problem of resolving layers—which is of great practical importance—is still under investigation. This field has witnessed a stormy development. Layer-adapted meshes have first been proposed by Bakhvalov [18] in the context of reaction-diffusion problems. In the late 1970s and early 1980s, special meshes for convection-diffusion problems were investigated by Gartland [45], Liseikin [113, 114, 116], Vulanović [163–166] and others in order to achieve uniform convergence. The discussion has been livened up by the introduction of special piecewise-uniform meshes by Shishkin [150]. They will be described in more detail in Section 2.1.3. Because of their simple structure, they have attracted much attention and are now widely referred to as Shishkin meshes. A small survey of these meshes can be found in the monograph [141], while [109, 121] and [134] are devoted exclusively to them.

The performance of Shishkin meshes is however inferior to that of Bakhvalov meshes, which has prompted efforts to improve them while retaining some of their simplicity, in particular, the mesh uniformity outside the layers and the choice of mesh transition point where the mesh changes from fine to coarse. For instance, Vulanović [169] uses a piecewise-uniform mesh with more than one transition point. Linß [81, 82] combines the ideas of Bakhvalov and Shishkin, while Beckett and Mackenzie [20] combine an equidistribution idea [31] with a Shishkin-type transition point. With all these various mesh-construction ideas a natural question is:

> *Can a general theory be derived that allows one to immediately deduce the robust convergence of standard schemes on special meshes and a guaranteed rate of convergence?*

A first attempt towards this can be found in [137], where a first-order upwind scheme and a Galerkin FEM are studied on a class of so-called Shishkin-type meshes. A more general criterion was derived in [84, 85] for an upwind-difference scheme in one dimension.

The main purpose of this monograph is to give a survey of recent developments and present the state of the art in the analysis of layer-adapted meshes for a wide range of reaction-convection-diffusion problems.

Chapter 2
Layer-Adapted Meshes

Before surveying a few of the most important ideas from the literature for constructing layer-adapted meshes, we shall introduce some basic concepts for describing layer-adapted meshes.

Throughout $\bar{\omega} : 0 = x_0 < x_1 < \cdots < x_N = 1$ denotes a generic mesh with N subintervals on $[0, 1]$, while ω is the set of inner mesh nodes. Set $I_i := [x_{i-1}, x_i]$. The local mesh sizes are $h_i := x_i - x_{i-1}$, $i = 1, \ldots, N$, while the maximum step size is $h := \max\limits_{i=1,\ldots,N} h_i$.

Definition 2.1. A strictly monotone function $\varphi : [0, 1] \to [0, 1]$ that maps a uniform mesh $t_i = i/N$, $i = 0, \ldots, N$, onto a layer-adapted mesh by $x_i = \varphi(t_i)$, $i = 0, \ldots, N$, is called a **mesh generating function**. ♡

A related approach is that of **stretching functions** or **layer-damping transformation** [49,114,115], which are used to transform a problem with layers into a problem whose derivatives are bounded.

For a given mesh generating function $\varphi \in W^{1,1}(0, 1)$, the local mesh step sizes can be computed using the formula

$$h_i = \varphi(t_i) - \varphi(t_{i-1}) = \int_{t_{i-1}}^{t_i} \varphi'(t)\mathrm{d}t. \qquad (2.1)$$

Another important concept is that of **mesh equidistribution**.

Definition 2.2 (Equidistribution principle). Let $M : [0, 1] \to I\!R$ be a positive function a.e. A mesh $\bar{\omega}$ is said to equidistribute the **monitor function** M if

$$\int_{I_i} M(t)\mathrm{d}t = \frac{1}{N} \int_0^1 M(t)\mathrm{d}t \quad \text{for } i = 1, \ldots, N.$$

♡

Given a monitor function M the associated mesh generating function is implicitly defined by

$$\int_0^{\varphi(t)} M(s)\mathrm{d}s = \xi \int_0^1 M(s)\mathrm{d}s \quad \text{for } t \in [0, 1]$$

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and its derivative by

$$\varphi'(t) = \frac{1}{M(\varphi(t))} \int_0^1 M(s) \mathrm{d}s \quad \text{for } t \in [0,1].$$

2.1 Convection-Diffusion Problems

Consider the boundary-value problem

$$-\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \;\; u(0) = u(1) = 0, \tag{2.2}$$

where ε is a small positive parameter, $b \geq \beta > 0$ on $[0,1]$. The boundary value problem (2.2) has a unique solution that typically has an exponential boundary layer at $x = 0$ which behaves like $\mathrm{e}^{-\beta x/\varepsilon}$. Figure 2.1 gives a plot of a typical solution

A quantity that will appear frequently in the error estimates later and which characterises the convergence is

$$\vartheta_{cd}^{[p]}(\bar{\omega}) := \max_{i=1,\ldots,N} \int_{I_i} \left(1 + \varepsilon^{-1} \mathrm{e}^{-\beta s/p\varepsilon}\right) \mathrm{d}s. \tag{2.3}$$

For example, in Section 4.2 we shall establish for the maximum-norm error of a first-order upwind difference scheme that

$$\left\|u - u^N\right\|_\infty \leq C \vartheta_{cd}^{[1]}(\bar{\omega}) \tag{2.4}$$

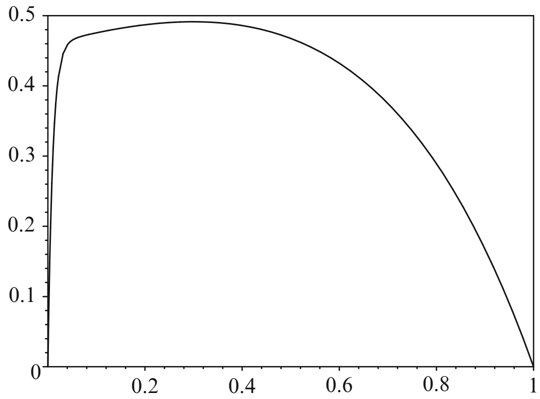


Fig. 2.1 Typical solution of (2.2)

on an arbitrary mesh $\bar{\omega}$. Noting that

$$\int_0^1 \left(1 + \varepsilon^{-1} \mathrm{e}^{-\beta s/p\varepsilon}\right) \mathrm{d}s \le C,$$

we see that an optimal mesh—optimal with respect to the order of convergence—equidistributes the monitor function $M : s \mapsto 1 + \varepsilon^{-1} \mathrm{e}^{-\beta s/p\varepsilon}$.

2.1.1 Bakhvalov Meshes

Bakhvalov's idea [18] is to use an equidistant t-grid near $x = 0$, then to map this grid back onto the x-axis by means of the (scaled) boundary layer function. That is, grid points x_i near $x = 0$ are defined by

$$q\left(1 - \mathrm{e}^{-\beta x_i/\sigma\varepsilon}\right) = t_i = \frac{i}{N} \quad \text{for } i = 0, 1, \ldots,$$

where the scaling parameters $q \in (0, 1)$ and $\sigma > 0$ are user chosen: q is roughly the portion of mesh points used to resolve the layer, while σ determines the grading of the mesh inside the layer. Away from the layer, a uniform mesh in x is used with the transition point τ such that, the resulting mesh generating function is $C^1[0, 1]$, i. e.,

$$\varphi(t) = \begin{cases} \chi(t) := -\dfrac{\sigma\varepsilon}{\beta} \ln \dfrac{q - t}{q} & \text{for } t \in [0, \tau], \\ \pi(t) := \chi(\tau) + \chi'(\tau)(t - \tau) & \text{otherwise,} \end{cases}$$

where the point τ satisfies

$$\chi'(\tau) = \frac{1 - \chi(\tau)}{1 - \tau}. \tag{2.5}$$

Geometrically this means that $(\tau, \chi(\tau))$ is the contact point of the tangent π to χ that passes through the point $(1, 1)$; see Fig. 2.2. When $\sigma\varepsilon \ge \rho q$, the equation (2.5) does not possess a solution. In this case the Bakhvalov mesh is uniform with mesh size N^{-1}.

The nonlinear equation (2.5) cannot be solved explicitly. However, the iteration

$$\tau_0 = 0, \qquad \chi'(\tau_{i+1}) = \frac{1 - \chi(\tau_i)}{1 - \tau_i}, \ \ i = 0, 1, 2 \ldots$$

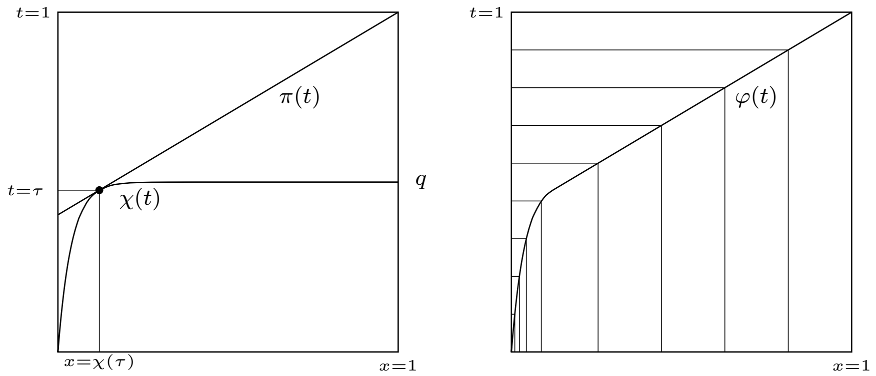


Fig. 2.2 Bakhvalov mesh: Construction of the mesh generating function (left) and the mesh generated (right)

converges fast. Moreover, the mesh obtained when the exact τ is replaced by the first iterate has very similar properties; see, e.g., [11,23]. In that case

$$\tau_1 = q - \frac{\sigma\varepsilon}{\beta} \qquad \text{and} \qquad \chi(\tau_1) = \frac{\sigma\varepsilon}{\beta} \ln \frac{\beta q}{\sigma\varepsilon},$$

are the mesh transition points in the t and x coordinates.

Alternatively, the Bakhvalov mesh can be generated by equidistributing the monitor function

$$M_{Ba}(s) := \max\left\{1, K\beta\varepsilon^{-1}\mathrm{e}^{-\beta s/\sigma\varepsilon}\right\}; \quad \text{see [85].}$$

Clearly, for $p \le \sigma$ and arbitrary $K > 0$ there exists a constant $C = C(\sigma, K)$ with

$$1 + \varepsilon^{-1}\mathrm{e}^{-\beta s/p\varepsilon} \le C \max\left\{1, K\beta\varepsilon^{-1}\mathrm{e}^{-\beta s/\sigma\varepsilon}\right\} = CM_{Ba}(s).$$

Thus,

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \le \frac{C}{N}\int_0^1 M_{Ba}(s)\mathrm{d}s \le \frac{C}{N}, \quad \text{if } \sigma \ge p, \tag{2.6}$$

for a Bakhvalov mesh since $\int_0^1 M_{Ba}(s)\mathrm{d}s \le C$.

Because (2.5) cannot be solved explicitly, Vulanović [163] proposed to replace the exponential in the above construction by its $(0,1)$-Padé approximation. Thus, in (2.5) we would take

$$\chi(t) = \frac{\sigma\varepsilon}{\beta}\frac{t}{t-q}.$$

Meshes that arise from an approximation of Bakhvalov's mesh generating function are called **meshes of Bakhvalov type** (B-type meshes). The following meshes belong to this class: the meshes proposed by Liseikin and Yanenko [116] (quadratic function outside layer) and meshes generated by equidistribution of monitor functions which, have been extensively studied by the group of Sloan and Mackenzie [20, 117, 132, 133], Gartland's graded mesh [45] and its modification by Roos and Skalický [140]. The estimate (2.6) holds for these meshes too.

These considerations and (2.4) give the typical convergence result for simple upwinding on B-type meshes:

$$\left\|u - u^N\right\|_\infty \le CN^{-1}.$$

I. e., uniform first-order convergence in the discrete maximum norm.

2.1.2 Shishkin Meshes

Another frequently-studied mesh is the so-called Shishkin mesh [121, 150]. This is because of its simplicity—it is piecewise uniform. We describe this mesh for problem (2.2). Let $q \in (0,1)$ and $\sigma > 0$ be two mesh parameters. We define a mesh transition point τ by

$$\tau = \min\left\{q, \frac{\sigma\varepsilon}{\beta}\ln N\right\}.$$

Then the intervals $[0,\tau]$ and $[\tau,1]$ are divided into qN and $(1-q)N$ equidistant subintervals (assuming that qN is an integer). This mesh (if $q \neq \tau$) may be regarded as generated by the mesh generating function

$$\varphi(t) = \begin{cases} \dfrac{\sigma\varepsilon}{\beta}\tilde{\varphi}(t) \quad \text{with} \quad \tilde{\varphi}(t) = \dfrac{t}{q}\ln N & \text{for } t \in [0,q], \\ 1 - \left(1 - \dfrac{\sigma\varepsilon}{\beta}\ln N\right)\dfrac{1-t}{1-q} & \text{otherwise} \end{cases} \tag{2.7}$$

see Fig. 2.3. Again the parameter q is the amount of mesh points used to resolve the layer. The mesh transition point τ has been chosen such that the layer term $\exp(-\beta x/\varepsilon)$ is smaller than $N^{-\sigma}$ on $[\tau,1]$. Typically σ will be chosen equal to the formal order of the method or sufficiently large to accommodate the error analysis.

Note that unlike the Bakhvalov mesh (and Vulanović's modification of it), the underlying mesh generating function is only piecewise $C^1[0,1]$ and depends on N, the number of mesh points. For simplicity we shall assume throughout that $q \geq \tau$ as otherwise N is exponentially large compared to $1/\varepsilon$, and a uniform mesh is sufficient to cope with the problem.

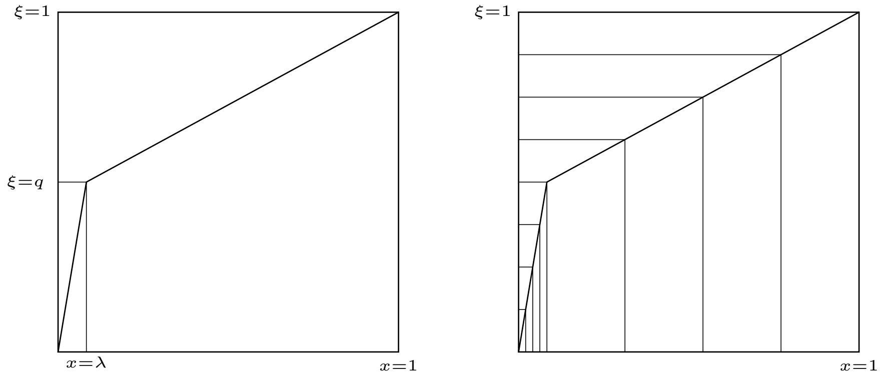


Fig. 2.3 Shishkin mesh: mesh generating function (left) and the mesh generated (right)

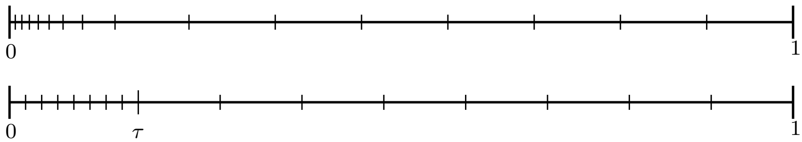


Fig. 2.4 Bakhvalov mesh (top) and Shishkin mesh (below) for a convection-diffusion equation

Although Shishkin meshes have a simple structure and many numerical methods are easier to analyse on a Shishkin mesh than on B-type meshes, they give numerical results that are inferior to those obtained by B-type meshes. For example,

$$\left\| u - u^N \right\|_\infty \le CN^{-1} \ln N,$$

for the aforementioned simple upwind scheme.

2.1.3 Shishkin-Type Meshes

The convergence on Shishkin meshes is spoilt by the logarithmic factor. This drawback prompted some work on improving Shishkin meshes. Roos and Linß [137] attempt at a general description of these improved meshes. They introduce the concept of a **Shishkin-type mesh** (S-type mesh) which is characterised by a "Shishkin"-transition point $\tau = \sigma\varepsilon\beta^{-1} \ln N$ and a uniform submesh on $[\tau, 1]$. Let the mesh be generated by (2.7) with a monotone function $\tilde{\varphi}$ satisfying

$$\tilde{\varphi}(0) = 0 \quad \text{and} \quad \tilde{\varphi}(q) = \ln N.$$

We introduce the **mesh characterising function** $\psi(t) = \exp(-\tilde{\varphi}(t))$ for $t \in [0, q]$. This function is monotonically decreasing with $\psi(0) = 1$ and $\psi(q) = N^{-1}$.

Lemma 2.3. *Let $\bar{\omega}$ be a Shishkin-type mesh with $\sigma \geq p > 0$. Assume there exists a constant κ such that*

$$\max_{t\in[0,q]} \tilde{\varphi}'(t) \leq \kappa N \tag{2.8}$$

Then

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \leq C\left(h + \max_{t\in[0,q]} |\psi'(t)|\, N^{-1}\right), \tag{2.9}$$

Proof. For $k = qN+1, \ldots, N$ we have

$$\int_{I_k} \left(1 + \varepsilon^{-1} \mathrm{e}^{-\beta s/p\varepsilon}\right) \mathrm{d}s = h_k - \frac{p}{\beta} \mathrm{e}^{-s\beta/p\varepsilon}\Big|_{x_{k-1}}^{x_k} \leq h + \frac{p}{\beta} N^{-1},$$

by the choice of the transition point τ and because $\sigma \geq p$.

The argument for $k = 1, \ldots, qN$ is slightly more laborious. First, (2.1) gives

$$h_k = \frac{\sigma\varepsilon}{\beta} \int_{t_{k-1}}^{t_k} \tilde{\varphi}'(t)\mathrm{d}t. \tag{2.10}$$

Therefore,

$$\frac{\beta h_k}{\sigma\varepsilon} \leq \kappa \quad \text{and} \quad \mathrm{e}^{\beta h_k/\sigma\varepsilon} \leq \mathrm{e}^{\kappa}, \tag{2.11}$$

by (2.8). Furthermore, the above integral representation gives

$$\frac{\beta h_k}{\sigma\varepsilon} \mathrm{e}^{-\beta x_k/\sigma\varepsilon} \leq N^{-1} \max|\psi'|, \tag{2.12}$$

because

$$\tilde{\varphi}' = -\frac{\psi'}{\psi} \quad \text{and} \quad \min_{t\in[t_{i-k},t_k]} \psi(t) = \psi(t_k) = \mathrm{e}^{-\beta x_k/\sigma\varepsilon}.$$

Next, use (2.11), (2.12) and $\sigma \geq p$ to obtain

$$\begin{aligned}\int_{I_k} \left(1 + \varepsilon^{-1} \mathrm{e}^{-\beta s/p\varepsilon}\right) \mathrm{d}s &\leq h_k + \frac{h_k}{\varepsilon} \mathrm{e}^{-x_{k-1}\beta/p\varepsilon} \\ &\leq h_k + \frac{\mathrm{e}^{\kappa} h_k}{\varepsilon} \mathrm{e}^{-x_k\beta/\sigma\varepsilon} \leq h + \frac{\sigma}{\beta} N^{-1} \max|\psi'|.\end{aligned}$$

Finally, note that $\max|\psi'| \geq \left(1 - N^{-1}\right)/q$. We are finished. □

We shall survey some of the S-type meshes proposed in the literature now. All meshes satisfy (2.8).

Shishkin mesh [121, 150]:

$$\psi(t) = \exp\left(-\frac{t\ln N}{q}\right) \quad \text{with} \quad \max|\psi'| = \frac{\ln N}{q} \quad \text{and} \quad h \le CN^{-1}.$$

Thus,

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \le CN^{-1}\ln N.$$

Bakhvalov-Shishkin mesh [81, 82]:

This mesh uses Bakhvalov's idea of inverting $\exp(-\beta x/\sigma\varepsilon)$ on $[0,\tau]$, while using a uniform mesh on $[\tau,1]$. Its mesh generating function is given by

$$\tilde{\varphi}(t) = -\ln\left(1 - \left(1 - \frac{1}{N}\right)\frac{t}{q}\right).$$

The corresponding mesh characterising function is

$$\psi(t) = 1 - \left(1 - \frac{1}{N}\right)\frac{t}{q}.$$

It satisfies

$$\max|\psi'| = \frac{1}{q}\left(1 - \frac{1}{N}\right) \le \frac{1}{q} \quad \text{and} \quad h \le C\left(\varepsilon + N^{-1}\right).$$

Thus,

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \le C\left(\varepsilon + N^{-1}\right).$$

In experiments it is observed that the convergence stalls when $\varepsilon \ge N^{-1}$, but in practise one typically has $\varepsilon \ll N^{-1}$.

A similar idea has been used by van Veldhuizen [161], but he chooses the transition point in a different manner: $\tau = 2\varepsilon\beta^{-1}\big(2 + k\ln N\big)$, where k is the formal order of the underlying scheme.

Vulanović-Shishkin mesh [137]:

Again Vulanović's idea of replacing the exponential by its $(0,1)$-Padé approximation can be used. Then,

$$\tilde{\varphi}(t) = \frac{t\ln N}{q + (q-t)\ln N}, \quad \psi(t) = \exp\left(-\frac{t\ln N}{q + (q-t)\ln N}\right)$$

and

$$\max|\psi'| \le \frac{4}{q}, \quad h \le C\left(1+\varepsilon\ln^2 N\right)N^{-1}.$$

For this mesh:

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \le C\left(1+\varepsilon\ln^2 N\right)N^{-1}.$$

Polynomial Shishkin mesh [137]:

Choose

$$\tilde{\varphi}(t) = \left(\frac{t}{q}\right)^m \ln N, \quad m \ge 1. \tag{2.13}$$

It can be shown that

$$\max|\psi'| \le C(\ln N)^{1/m} \quad \text{and} \quad h \le CN^{-1}.$$

Hence,

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \le CN^{-1}(\ln N)^{1/m}.$$

For $m=1$ we recover the original Shishkin mesh, while for $m>1$ the accuracy will be improved.

A mesh with a rational ψ [137].

Let $m>1$ and

$$\tilde{\varphi}(t) = \ln\bigl(1+(N-1)(t/q)^m\bigr), \quad \psi(t) = \frac{1}{1+(N-1)(t/q)^m}.$$

For this mesh $|\psi'| \le N^{1/m}$ on $[0,q]$. Thus,

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \le CN^{-1+1/m}.$$

Vulanović's improved Shishkin mesh [169]:

Introduce additional mesh transition points

$$\tau_1 = \frac{\sigma\varepsilon}{\beta}\ln N, \quad \tau_2 = \frac{\sigma\varepsilon}{\beta}\ln\ln N, \quad \dots, \quad \tau_\ell = \frac{\sigma\varepsilon}{\beta}\underbrace{\ln\ln\cdots\ln}_{\ell \text{ times}} N.$$

By formally setting $\tau_0 = 1$ and $\tau_{\ell+1} = 0$, we have $\tau_0 \geq \tau_1 \geq \tau_2 \geq \cdots \geq \tau_{\ell+1}$. Then each of the intervals $[\lambda_{i+1}, \lambda_i]$, $i = 0, \ldots, \ell$, is dissected uniformly.

This mesh can be characterised as a Shishkin-type mesh with

$$\max |\psi'| \leq C \underbrace{\ln \ln \cdots \ln}_{\ell \text{ times}} N \quad \text{and} \quad h \leq CN^{-1}.$$

Hence,

$$\vartheta_{cd}^{[p]}(\bar{\omega}) \leq CN^{-1} \underbrace{\ln \ln \cdots \ln}_{\ell \text{ times}} N.$$

Remark 2.4. All S-type meshes just mentioned also satisfy

$$\int_0^q \tilde{\varphi}'(t)^2 \mathrm{d}t \leq CN \tag{2.14}$$

with some constant C. Then by (2.10) and the Cauchy-Schwarz inequality

$$\sum_{k=1}^{qN} \left(\frac{\beta h_k}{\varepsilon} \right)^2 \leq \sigma^2 N^{-1} \int_0^q \tilde{\varphi}'(t)^2 \mathrm{d}t \leq C.$$

This property will be used later. ♣

Modified Shishkin meshes

Last but not least, two further modifications of the Shishkin mesh are worth mentioning. Both are due to Vulanović [168].

The first modification concerns the choice of transition point τ. It is chosen to be $\tau = \sigma \varepsilon \beta^{-1} L(N)$, where $L = L(N)$ solves $\mathrm{e}^{-L} = LN^{-1}$. This change is motivated by the appearance of two terms in the error analysis,

$$\mathrm{e}^{-\beta\tau/\sigma\varepsilon} \quad \text{and} \quad \left(\frac{\tau}{\varepsilon q N} \right)^{\sigma},$$

that should be balanced. For a standard Shishkin mesh they are N^{-1} and $N^{-1} \ln N$. Although $L(N)$ behaves asymptotically like $\ln N$, the modified transition point typically yields smaller error constants.

The second modification is a smoothing of the mesh generating function:

$$\varphi(t) = \begin{cases} \frac{\tau}{q} t & \text{for } t \in [0, q], \\ \tau + \frac{\tau}{q}(t - q) + k(t - q)^3 & \text{otherwise,} \end{cases}$$

where k is chosen such that $\varphi(1) = 1$. This construction ensures $\varphi \in C^2[0,1]$ with $\|\varphi''\|_\infty \le C$. A consequence is

$$|h_{i+1} - h_i| \le CN^{-2} \quad \text{for } i = 1, \dots, N-1,$$

a property that is not enjoyed by the Shishkin mesh. It helps to simplify the error analysis for certain difference schemes.

2.1.4 Turning-Point Boundary Layers

Turning-point layers are associated with zeros of the convection term. Consider the boundary-value problem of finding u such that

$$-\varepsilon x^{-\kappa} u''(x) - b(x)u'(x) + c(x)u(x) = f(x) \quad \text{for } x \in (0,1),$$
$$u(0) = u(1) = 0,$$

with $\kappa \ge 0$ and $b \ge \beta > 0$ on $[0,1]$. The solution will exhibit a boundary whose k^{th}-order derivatives behave like

$$\varepsilon^{-k/(\kappa+1)} \exp\left(-\frac{\beta x^{\kappa+1}}{\varepsilon(\kappa+1)}\right).$$

Bakhvalov meshes

Bakhvalov meshes for turning point layers equidistribute

$$\max\left\{1, K\varepsilon^{-1/(\kappa+1)} \exp\left(-\frac{\beta x^{\kappa+1}}{\sigma\varepsilon(\kappa+1)}\right)\right\}$$

with user chosen parameters $K > 0$ and $\sigma > 0$.

Shishkin meshes

Choose mesh parameters $q \in (0,1)$ and $\sigma > 0$. Determine a mesh transition point by solving

$$N^{-\sigma} = \exp\left(-\frac{\beta\tau^{\kappa+1}}{\varepsilon(\kappa+1)}\right), \text{ i.e.,} \quad \tau = \left(\frac{\sigma\varepsilon(\kappa+1)}{\beta}\ln N\right)^{1/(\kappa+1)}.$$

Then the interval $[0,\tau]$ is divided into qN equidistant subintervals (assuming that qN is an integer), while $[\tau,1]$ is divided into $(1-q)N$ subintervals.

2.1.5 Interior Layers

Interior layers may occur in convection-diffusion equations with a point source or when the convection coefficient is discontinuous. In order to achieve uniform convergence, these interior layers have to be resolved by the mesh.

Consider the following convection-diffusion problem:

$$-\varepsilon u'' - (bu)' + cu = f + \alpha\delta_d, \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{2.15}$$

where δ_d is the shifted Dirac-delta function. Also the convection coefficient b may have a discontinuity at $d \in (0,1)$. Suppose $b \geq \beta_1 > 0$ on $(0,d)$ and $b \geq \beta_2 > 0$ on $(d,1)$. The solution will exhibit two layers: a boundary layer $\mathrm{e}^{-\beta_1 x/\varepsilon}$ at $x = 0$ and an interior layer $\mathrm{e}^{-\beta_2(x-d)/\varepsilon}$ to the right of $x = d$. Figure 2.5 depicts a typical solution.

Uniform convergence for (2.15) will be established in terms of

$$\vartheta_{cd^i}^{[p]}(\bar{\omega}) := \max_{i=1,\dots,N} \int_{I_i} \left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta_1 s/p\varepsilon} + H_d(s)\varepsilon^{-1}\mathrm{e}^{-\beta_2(s-d)/p\varepsilon}\right) \mathrm{d}s,$$

where H_d is the shifted Heaviside function.

Bakhvalov meshes

Choosing mesh parameters $K_1, K_2 > 0$ and $\sigma_1, \sigma_2 > 0$, Bakhvalov meshes for (2.15) are constructed by equidistributing the monitor function

$$\max\left\{1, K_0\varepsilon^{-1}\mathrm{e}^{-\beta_1 s/\sigma_1\varepsilon}, H_d(s)K_1\varepsilon^{-1}\mathrm{e}^{-\beta_2(s-d)/\sigma_2\varepsilon}\right\};$$

A plot of the resulting mesh is found in Fig. 2.6. If $\sigma_1 \geq p$ and $\sigma_2 \geq p$ then $\vartheta_{cd^i}^{[p]}(\bar{\omega}) \leq CN^{-1}$.

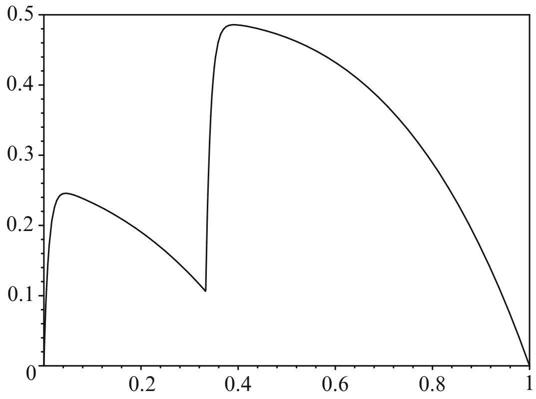


Fig. 2.5 Typical solution of (2.15). The point source is at $x = 1/3$

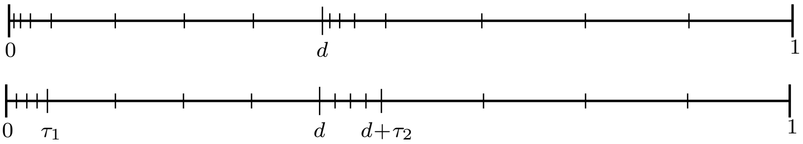


Fig. 2.6 Bakhvalov mesh (top) and Shishkin mesh (below) for a convection-diffusion problem with an internal layer right of $x = d$

Shishkin meshes

Let $q_i \in (0,1)$, $i = 1,\dots,4$ with $\sum q_i = 1$ and $\sigma_1, \sigma_2 > 0$ be mesh parameters. We set

$$\tau_1 = \min\left\{\frac{q_1 d}{q_1 + q_2}, \frac{\sigma_1 \varepsilon}{\beta_1} \ln N\right\} \quad \text{and} \quad \tau_2 = \min\left\{\frac{q_3(1-d)}{q_3+q_4}, \frac{\sigma_2\varepsilon}{\beta_2}\ln N\right\}.$$

Then the subintervals $J_1 = [0, \tau_1]$, $J_2 = [\tau_1, d]$, $J_3 = [d, d+\tau_2]$ and $J_4 = [d+\tau_2, 1]$ are divided into $q_i N$ equidistant subintervals (assuming that $q_i N$ are integers). The simplest choice is to take $q_i = 1/4$, $i = 1,\dots,4$, and N divisible by 4. Figure 2.6 depicts a Shishkin mesh with 16 mesh intervals for (2.15). If $\sigma_1 \geq p$ and $\sigma_2 \geq p$ then $\vartheta^{[p]}_{cd^i}(\bar{\omega}) \leq CN^{-1}\ln N$.

2.1.6 *Overlapping Layers*

Multiple layers can occur, for example, in systems of convection-diffusion equations:

$$-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' - (\boldsymbol{B}\boldsymbol{u})' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } (0,1), \ \ \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0},$$

where $\boldsymbol{u}$ and $\boldsymbol{f}$ are vectors with ℓ components and $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\ell \times \ell$ matrices, while the $\ell \times \ell$ matrix $\operatorname{diag}(\boldsymbol{\varepsilon})$ is diagonal with k^{th} entry ε_k.

We expect layers $\mathrm{e}^{-\beta_m x/\varepsilon_m}$ to form at $x = 0$ if $b_{mm} \geq \beta_m > 0$, and layers $\mathrm{e}^{-\beta_m(1-x)/\varepsilon_m}$ at $x = 1$, if $-b_{mm} \geq \beta_m > 0$; see Fig. 2.7.

Let J denote the set of indices $m \in \{1,\dots,\ell\}$ for which b_{mm} is negative and J^* its complement. Then set

$$\vartheta^{[p]}_{cd,\ell}(\bar{\omega}) := \max_{i=1,\dots,N} \int_{I_i} \left\{1 + \sum_{m\in I} \varepsilon_m^{-1}\mathrm{e}^{-\beta_m s/p\varepsilon_m} + \sum_{m\in I^*} \varepsilon_m^{-1} \mathrm{e}^{-\beta_m(1-s)/p\varepsilon_m}\right\} \mathrm{d}s.$$

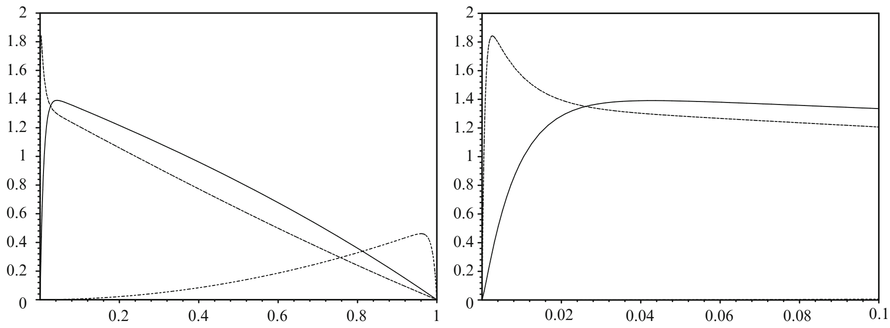


Fig. 2.7 Layers in a system of three convection-diffusion equations. The right plot zooms into the neighbourhood of $x = 0$ and reveals the presence of two layers there

Bakhvalov meshes

Bakhvalov meshes for multiple layers can be constructed by generalising the equidistribution principle for a single layer presented in Sect. 2.1.1. Define

$$M_{Ba}(s) := \max\Bigg\{1, \max_{m\in I} K_m\beta_m\varepsilon_m^{-1}\mathrm{e}^{-\beta_m s/\sigma_m\varepsilon_m}, \\ \max_{m\in I^*} K_m\beta_m\varepsilon_m^{-1}\mathrm{e}^{-\beta_m(1-s)/\sigma_m\varepsilon_m}\Bigg\} \quad \text{for } s\in[0,1]$$

with mesh parameters $\sigma_m > 0$ and $K_m > 0$. Choose the mesh points x_i such that the mesh equidistributes this monitor function. Clearly $\vartheta_{cd,\ell}^{[p]}(\bar{\omega}) \le CN^{-1}$ if $\sigma_m \ge p$ for all m.

Shishkin meshes

A Shishkin mesh for problem (4.81) is still piecewise equidistant, but now each layer in $\boldsymbol{u}$ requires its own fine mesh. The mesh is constructed as follows: Let N, the number of mesh intervals, be divisible by $\ell+1$. Let $\sigma > 0$ be arbitrary. Suppose the equations in the system are arranged such that b_k is positive for $k = 1,\dots,i$ and negative for $k = i+1,\dots,\ell$ and that

$$0 < \frac{\varepsilon_1}{\beta_1} \le \frac{\varepsilon_2}{\beta_2} \le \dots \le \frac{\varepsilon_i}{\beta_i} \quad \text{and} \quad 0 < \frac{\varepsilon_\ell}{\beta_\ell} \le \frac{\varepsilon_{\ell-1}}{\beta_{\ell-1}} \le \dots \le \frac{\varepsilon_{i+1}}{\beta_{i+1}}.$$

Fix the mesh transition points τ_k as follows. Set $\tau_0 = 0$ and $\tau_\ell = 1$. Then, if $i > 0$ set

$$\tau_i = \min\left\{\frac{i}{\ell+1}, \frac{\sigma\varepsilon_i}{\beta_i}\ln N\right\},$$
$$\tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \frac{\sigma\varepsilon_k}{\beta_k}\ln N\right\} \quad \text{for } k = i-1,\dots,1;$$

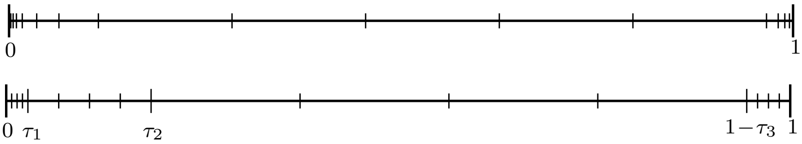


Fig. 2.8 Bakhvalov mesh (top) and Shishkin mesh (below) for a system of three convection-diffusion equations

and, if $i < \ell$,

$$\tau_{i+1} = 1 - \min\left\{\frac{\ell - i}{\ell + 1}, \frac{\sigma\varepsilon_{i+1}}{\beta_{i+1}} \ln N\right\},$$
$$\tau_{k+1} = 1 - \min\left\{\frac{(\ell - k)\tau_k}{\ell - k + 1}, \frac{\sigma\varepsilon_{k+1}}{\beta_{k+1}} \ln N\right\} \quad \text{for } k = i+1, \dots, \ell - 1.$$

Then the mesh is obtained by dividing each of the intervals $[\tau_k, \tau_{k+1}]$, for $k = 0, \dots, \ell$, into $N/(\ell + 1)$ subintervals of equal length. Figure 2.8 depicts a Shishkin mesh with 16 mesh intervals for a system of three equations.

For Shishkin meshes $\vartheta^{[p]}_{cd,\ell}(\bar{\omega}) \le CN^{-1} \ln N$ if $\sigma_m \ge 1$ for all m.

2.2 Reaction-Convection-Diffusion Problems

Now consider the boundary-value problem

$$-\varepsilon_d u'' - \varepsilon_c b u' + cu = f \quad \text{in } (0,1), \;\; u(0) = u(1) = 0, \tag{2.16}$$

with two small parameter ε_d and ε_c. Its solution typically has two exponential boundary layers – one at either end of the domain. These behave like $\mathrm{e}^{\mu_0 x}$ and $\mathrm{e}^{-\mu_1(1-x)}$, where the characteristic exponents $\mu_0 < 0$ and $\mu_1 > 0$ can be computed from the coefficients in the differential equation. See Fig. 2.9 for a plot of typical solution and Sect. 3.2 for details of the analysis.

The significant difference to convection-diffusion problems is the presence of two layers. The construction of adapted meshes has to provide for this.

For this class of problem convergence can be described in terms of

$$\vartheta^{[p]}_{rcd}(\bar{\omega}) := \max_{i=1,\dots,N} \int_{I_i} \left(1 + |\mu_0| \, \mathrm{e}^{\mu_0 s/p} + \mu_1 \mathrm{e}^{-\mu_1(1-s)/p}\right) \mathrm{d}s.$$

A special case of (2.16) is pure reaction-diffusion:

$$-\varepsilon^2 u'' + cu = f \quad \text{in } (0,1), \;\; u(0) = u(1) = 0, \tag{2.17}$$

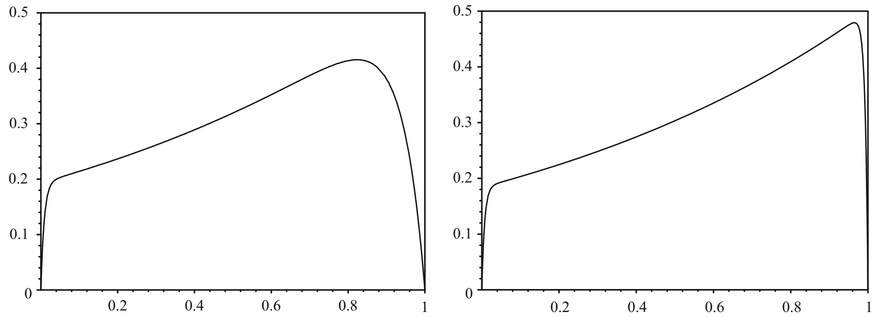


Fig. 2.9 Layers in reaction-convection-diffusion problems (left, different widths) and in reaction-diffusion problems (right, same widths)

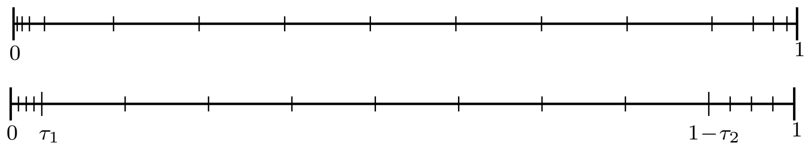


Fig. 2.10 Bakhvalov mesh (top) and Shishkin mesh (below) for a reaction-convection-diffusion equation. The layers at $x = 0$ and $x = 1$ have different widths

with $c \geq \rho^2$ on $[0,1]$, $\rho > 0$. Here $\mu_0 = -\mu_1 = -\rho/\varepsilon$ and the characteristic quantity is

$$\vartheta_{rd}^{[p]}(\bar{\omega}) := \max_{i=1,\dots,N} \int_{I_i} \left(1 + \varepsilon^{-1}\mathrm{e}^{-\rho s/p\varepsilon} + \varepsilon^{-1}\mathrm{e}^{-\rho(1-s)/p\varepsilon}\right) \mathrm{d}s.$$

Bakhvalov meshes

Choosing mesh parameters $K_0, K_1 > 0$ and $\sigma_0, \sigma_1 > 0$, Bakhvalov meshes for (2.16) can be generated by equidistributing the monitor function

$$\max\left\{1, K_0 \left|\mu_0\right| \mathrm{e}^{\mu_0 s/\sigma_0}, K_1 \mu_1 \mathrm{e}^{-\mu_1(1-s)/\sigma_1}\right\}.$$

In case of a reaction-diffusion equation (2.17) this becomes

$$\max\left\{1, K_0 \rho \varepsilon^{-1} \mathrm{e}^{-\rho s/\sigma_0 \varepsilon}, K_1 \rho \varepsilon^{-1} \mathrm{e}^{-\rho(1-s)/\sigma_1 \varepsilon}\right\}.$$

Figures 2.10 and 2.11 depict Bakhvalov meshes with 16 mesh intervals for (2.16) and (2.17).

We have

$$\vartheta_{rcd}^{[p]}(\bar{\omega}) \leq CN^{-1} \quad \text{and} \quad \vartheta_{rd}^{[p]}(\bar{\omega}) \leq CN^{-1} \quad \text{if } \sigma_0, \sigma_1 \geq p.$$

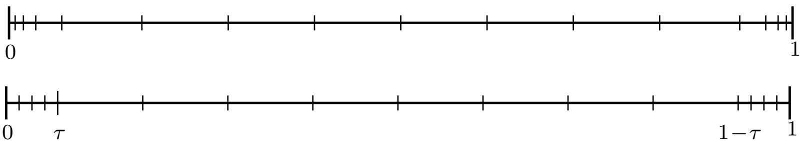


Fig. 2.11 Bakhvalov mesh (top) and Shishkin mesh (below, with $\tau_1 = \tau_2 = \tau$) for a reaction-diffusion problem. The layers at $x = 0$ and $x = 1$ have the same width

Shishkin meshes

Shishkin meshes are again piecewise uniform. Fixing mesh parameters $q_0, q_1 > 0$ and $\sigma_0, \sigma_1 > 0$ with $q_0 + q_1 < 1$, we define the mesh transition points

$$\tau_0 = \min\left\{q_0, \frac{\sigma_0}{|\mu_0|} \ln N\right\} \quad \text{and} \quad \tau_1 = \min\left\{q_1, \frac{\sigma_1}{\mu_1} \ln N\right\}.$$

Then the intervals $[0, \tau_0]$ and $[1-\tau_1, 1]$ are dissected into $q_0 N$ and $q_1 N$ subintervals, while $[\tau_0, 1 - \tau_1]$ is divided into $(1 - q_0 - q_1)N$ subintervals. Usually $\sigma_0 = \sigma_1$ and $q_0 = q_1 = 1/4$ are considered in the literature.

For pure reaction-diffusion the transition points are

$$\tau_0 = \min\left\{q_0, \frac{\sigma_0 \varepsilon}{\rho} \ln N\right\} \quad \text{and} \quad \tau_1 = \min\left\{q_1, \frac{\sigma_1 \varepsilon}{\rho} \ln N\right\}.$$

For these piecewise uniform meshes

$$\vartheta_{rcd}^{[p]}(\bar{\omega}) \le C N^{-1} \ln N \quad \text{and} \quad \vartheta_{rd}^{[p]}(\bar{\omega}) \le C N^{-1} \ln N \quad \text{if } \sigma_0, \sigma_1 \ge p.$$

Modifications of the above construction are conceivable by using ideas from Sect. 2.1.3 in order to improve the accuracy.

2.2.1 Interior Layers

Interior layers in reaction-diffusion problem occur when the reaction coefficient or the right-hand side have a discontinuity at a point d in the interior of the domain. Eq. (2.17) should be read as follows: Find $u \in C^2\big((0,d) \cup (d,1)\big) \cap C^1[0,1]$ such that

$$-\varepsilon^2 u'' + cu = f \quad \text{in } (0,d) \cup (d,1), \quad u(0) = u(1) = 0, \tag{2.18}$$

$c \ge \rho^2$ on $[0,1]$, $\rho > 0$. In addition to the two boundary layers $\mathrm{e}^{-\rho x/\varepsilon}$ and $\mathrm{e}^{-\rho(1-x)/\varepsilon}$, an internal layer $\mathrm{e}^{-\rho|x-d|/\varepsilon}$ will form because of the discontinuity in the data; see Fig. 2.12.

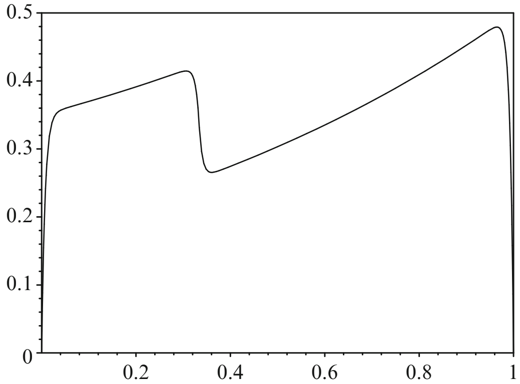


Fig. 2.12 Reaction-diffusion problems with an interior layer at $x = 1/3$

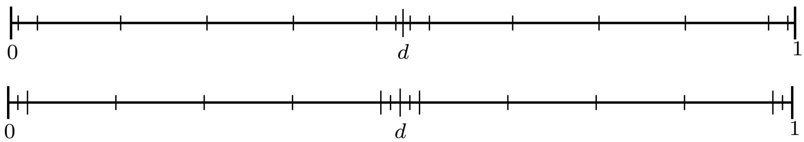


Fig. 2.13 Bakhvalov mesh (top) and Shishkin mesh (below) for a reaction-diffusion problem with an internal layer on both sides of $x = d$

Convergence of discretisations for (2.18) will be characterised by

$$\vartheta_{rd^i}^{[p]}(\bar{\omega}) := \max_{i=1,\ldots,N} \int_{I_i} \Big(1 + \varepsilon^{-1}\Big[\mathrm{e}^{-\rho s/p\varepsilon} + \mathrm{e}^{-\rho|s-d|/p\varepsilon} + \mathrm{e}^{-\rho(1-s)/p\varepsilon}\Big]\Big)\mathrm{d}s.$$

Bakhvalov meshes

Choosing mesh parameters $K > 0$ and $\sigma > 0$, we construct Bakhvalov meshes for (2.15) by equidistributing the monitor function

$$\max\left\{1, K\varepsilon^{-1}\mathrm{e}^{-\rho s/\sigma\varepsilon}, K\varepsilon^{-1}\mathrm{e}^{-\rho|s-d|/\sigma\varepsilon}, K\varepsilon^{-1}\mathrm{e}^{-\rho(1-s)/\sigma\varepsilon}\right\}.$$

A plot of the resulting mesh is found in Fig. 2.13. If $\sigma \geq p$ then $\vartheta_{cd^i}^{[p]}(\bar{\omega}) \leq CN^{-1}$. The construction can be modified by choosing different K and σ for different layers.

Shishkin meshes

A possible construction for a piecewise uniform mesh adapted to (2.18) is as follows: Let N be divisible by 8. Choose a mesh parameter $\sigma > 0$ and set

$$\tau_1 = \min\left\{\frac{d}{4}, \frac{\sigma\varepsilon}{\rho}\ln N\right\} \quad \text{and} \quad \tau_2 = \min\left\{\frac{1-d}{4}, \frac{\sigma\varepsilon}{\rho}\ln N\right\}.$$

Then the subintervals $[0, \tau_1]$, $[d - \tau_1, d]$, $[d, d + \tau_2]$ and $[1 - \tau_2, 1]$ are each divided into $N/8$ equidistant subintervals, while $[\tau_1, d - \tau_1]$ and $[d + \tau_2, 1 - \tau_2$ are divided into $N/4$ subintervals. Figure 2.13 contains of a plot a Shishkin mesh with 16 mesh intervals for (2.18). If $\sigma \geq p$ then $\vartheta^{[p]}_{rd^i}(\bar{\omega}) \leq CN^{-1} \ln N$.

2.2.2 Overlapping Layers

In systems of reaction-diffusion equations pairs of layers of different width occur that overlap. Consider the following system of ℓ equations: Find $\boldsymbol{u}$ such that

$$-\boldsymbol{E}^2 \boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in} \ (0, 1), \quad \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}. \tag{2.19}$$

where $\boldsymbol{E} = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_\ell)$ and the small positive parameters ε_m are distinct. Each give rise to a pair of boundary layers $\mathrm{e}^{-\kappa x/\varepsilon_m}$ and $\mathrm{e}^{-\kappa(1-x)/\varepsilon_m}$ at the two end points of the domain, where the constant $\kappa > 0$ depends on the coupling matrix $\boldsymbol{A}$ only. Consequently, layers from different perturbation parameters will overlap; see Fig. 2.14.

For the above system convergence is described in terms of

$$\vartheta^{[p]}_{rd,\ell}(\bar{\omega}) := \max_{i=1,\ldots,N} \int_{I_i} \left(1 + \sum_{m=1}^{\ell} \varepsilon_m^{-1} \left(\mathrm{e}^{-\kappa s/p\varepsilon_m} + \mathrm{e}^{-\kappa(1-s)/p\varepsilon_m} \right) \right) \mathrm{d}s$$

The presence of multiple layers in the solution $\boldsymbol{u}$ at each end of the interval $[0, 1]$, forces us to generalise the layer-adapted mesh constructions by refining the mesh separately for each layer. That is, when approaching an end-point of $[0, 1]$, one requires a fine mesh that undergoes a further refinement as one enters each new layer in $\boldsymbol{u}$.

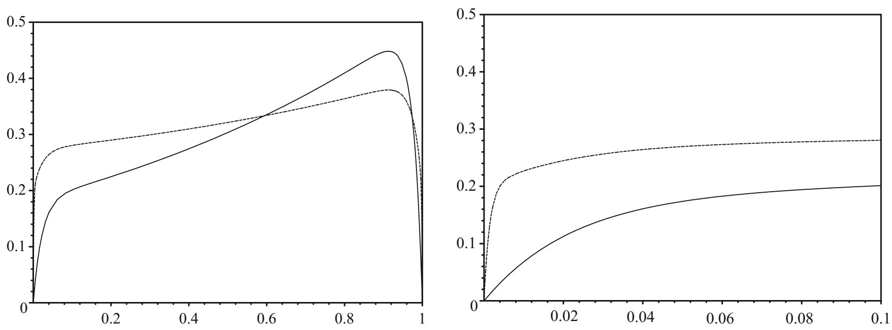


Fig. 2.14 Layers in a system of two reaction-diffusion equations. The right plot zooms into the neighbourhood of $x = 0$ and unveils a thinner sublayer

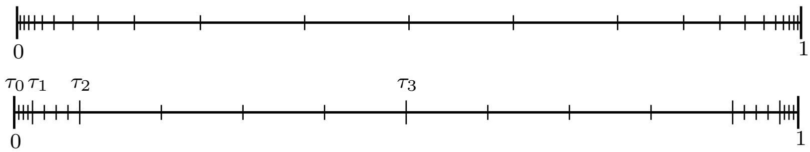


Fig. 2.15 Bakhvalov mesh (top) and Shishkin mesh (below) for a reaction-diffusion system with multiple layers

Bakhvalov meshes

Bakhvalov meshes for a system of reaction-diffusion equations can be constructed by equidistributing the monitor function

$$M_{Ba}(s) := \max\Big\{1, K_1\varepsilon_1^{-1}\mathrm{e}^{-\kappa s/\sigma_1\varepsilon_1}, K_1\varepsilon_1^{-1}\mathrm{e}^{-\kappa(1-s)/\sigma_1\varepsilon_1}, \\ \dots, K_\ell\varepsilon_\ell^{-1}\mathrm{e}^{-\kappa s/\sigma_\ell\varepsilon_\ell}, K_\ell\varepsilon_\ell^{-1}\mathrm{e}^{-\kappa(1-s)/\sigma_\ell\varepsilon_\ell}\Big\}$$

with positive user chosen constants σ_m and K_m.

For these meshes, if $\sigma_m \ge p$, $m = 1, \dots, \ell$, then $\vartheta^{[p]}_{rd,\ell}(\bar\omega) \le CN^{-1}$.

Shishkin meshes

Piecewise equidistant meshes for (2.19) are constructed as follows: Let N, the number of mesh intervals, be divisible by $2(\ell + 1)$. Let $\sigma > 0$ be arbitrary. Let the perturbation parameters be sorted by magnitude: $\varepsilon_1 \ge \varepsilon_2 \ge \cdots \ge \varepsilon_\ell$. Fix the mesh transition points τ_k by setting

$$\tau_{\ell+1} = 1/2, \quad \tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \frac{\sigma\varepsilon_{\ell+1-k}}{\kappa}\ln N\right\} \quad \text{for } k = \ell, \dots, 1,$$

and $\tau_0 = 0$. Then the mesh is obtained by dividing each of the intervals $[\tau_k, \tau_{k+1}]$ and $[1 - \tau_{k+1}, 1 - \tau_k]$, for $k = 0, \dots, \ell$, into $N/(2\ell + 2)$ subintervals of equal length. Figure 2.15 depicts a Shishkin mesh with 24 mesh intervals for a system of two equations.

If $\sigma \ge p$, then $\vartheta^{[p]}_{rd,\ell}(\bar\omega) \le CN^{-1}\ln N$ for a Shishkin mesh.

2.3 Two-Dimensional Problems

The second part of this book (Chap. 7–9) is concerned with two-dimensional reaction-convection-diffusion problems posed on the square $\Omega = (0, 1)^2$:

$$-\varepsilon\Delta u - \boldsymbol{b}^T\nabla u + cu = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g.$$

We shall only consider situations where the layers form along the four edges of the domain. Therefore, we can restrict ourselves to discretisations on tensor product meshes, i.e., meshes $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ with $\bar{\omega}_x : 0 = x_0 < x_1 < \cdots < x_N = 1$ and $\bar{\omega}_y : 0 = y_0 < y_1 < \cdots < y_M = 1$. For the moment, we will allow meshes with different numbers of mesh intervals in each coordinate direction. Later, we shall take $N = M$ for simplicity in the presentation.

2.3.1 Reaction-Diffusion Problems

We will start by considering the boundary-value problem

$$-\varepsilon^2 \Delta u + cu = f \quad \text{in } \Omega = (0,1)^2, \quad u|_{\partial\Omega} = g, \tag{2.20}$$

where $0 < \varepsilon \ll 1$ and $c > \rho^2$ on $\bar{\Omega}$ with a positive constant ρ. The convective field $\boldsymbol{b}$ vanishes identically. Its solution typically exhibits exponential boundary layers along all four edges of the domain. These layers behave like

$$\mathrm{e}^{-x\rho/\varepsilon}, \quad \mathrm{e}^{-(1-x)\rho/\varepsilon}, \quad \mathrm{e}^{-y\rho/\varepsilon} \quad \text{and} \quad \mathrm{e}^{-(1-y)\rho/\varepsilon}.$$

Figure 2.16 depicts a typical solution to (2.20).

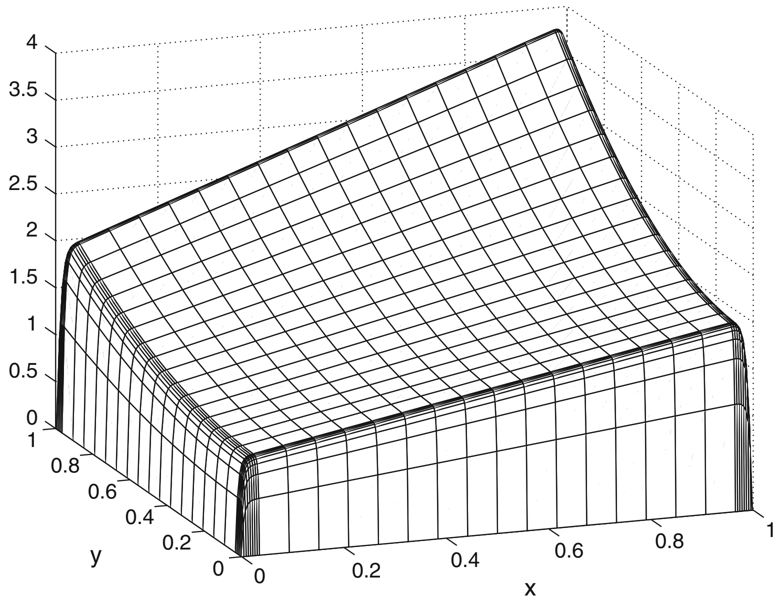


Fig. 2.16 Reaction-diffusion problem on the square. Layers of width $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$ at all four boundaries

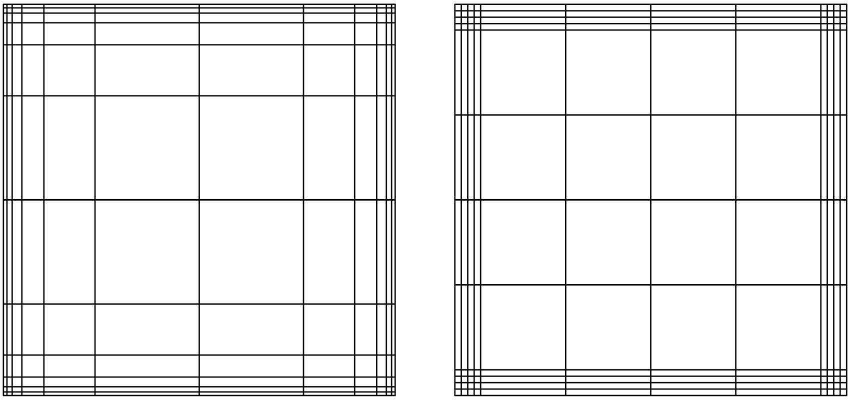


Fig. 2.17 Bakhvalov mesh (left) and Shishkin mesh (right) for a reaction-diffusion problem on the square

Appropriate meshes are constructed by taking tensor products of meshes for one-dimensional reaction-diffusion problems from Sect. 2.2. Figure 2.17 displays a tensor-product Bakhvalov mesh and a Shishkin mesh. One can also use tensor products of any of the other meshes proposed for one dimension. It is also possible to use different meshes in x- and y-direction.

2.3.2 Convection-Diffusion

The general convection-diffusion problem in two dimensions is

$$-\varepsilon \Delta u - \boldsymbol{b}^T \nabla u + cu = f \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u|_{\partial\Omega} = g, \tag{2.21}$$

where $0 < \varepsilon \ll 1$. Its solution may typically exhibit three different types of layers: interior layers, parabolic boundary layers and regular boundary layers. Let us assume that Ω is a domain with a regular boundary that has a uniquely defined outward normal n almost everywhere. Then the boundary can be divided into three parts:

$$\begin{aligned}
\Gamma^- &:= \left\{ \boldsymbol{x} \in \Gamma : \boldsymbol{b}^T n < 0 \right\} && \textit{inflow boundary,} \\
\Gamma^0 &:= \left\{ \boldsymbol{x} \in \Gamma : \boldsymbol{b}^T n = 0 \right\} && \textit{characteristic boundary} \text{ and} \\
\Gamma^+ &:= \left\{ \boldsymbol{x} \in \Gamma : \boldsymbol{b}^T n > 0 \right\} && \textit{outflow boundary.}
\end{aligned}$$

With this notation the layers can be classified as follows:

Regular Boundary Layers occur at the outflow boundary Γ^+ and have a width of $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$. They are often also called *exponential boundary layers*.

Parabolic Boundary Layers occur at characteristic boundaries Γ^0 where the boundary is parallel to the characteristics of the vector field $\boldsymbol{b}$. They are therefore also called *characteristic boundary layers*. In the non-degenerate case, i.e., when the convective field does not vanish at the boundary, their width is $\mathcal{O}\left(\sqrt{\varepsilon}\ln(1/\varepsilon)\right)$.

Interior Layers arise, e. g., from discontinuities in the boundary data at the inflow boundary Γ^- and are propagated across the domain along the characteristics of the vector field $\boldsymbol{b}$. They are similar in nature to parabolic boundary layers and therefore also called *characteristic or parabolic interior layers*. Their thickness is $\mathcal{O}\left(\sqrt{\varepsilon}\ln(1/\varepsilon)\right)$. They will not be the subject of the theoretical investigations in this monograph.

2.3.2.1 Regular Boundary Layers

Consider (2.21) on the square $\Omega = (0,1)^2$ with $b_1 \geq \beta_1$ and $b_2 \geq \beta_2$ on $\bar{\Omega}$ with positive constants β_1 and β_2. The outflow boundary consists of the two edges $x = 0$ and $y = 0$, where boundary layers form that behave like

$$\mathrm{e}^{-\beta_1 x/\varepsilon} \quad \text{and} \quad \mathrm{e}^{-\beta_2 y/\varepsilon}.$$

Figure 2.18 depicts a typical solution.

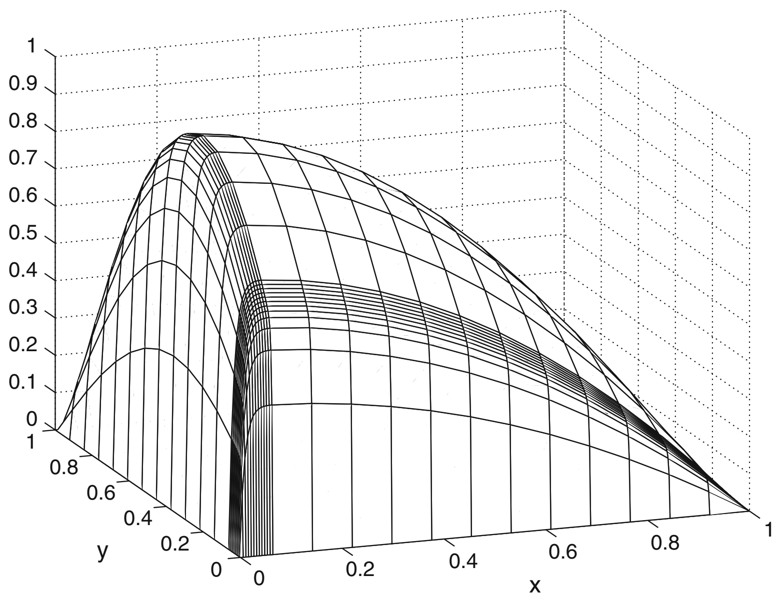


Fig. 2.18 Convection-diffusion problem on the square with two exponential layers at the outflow boundary

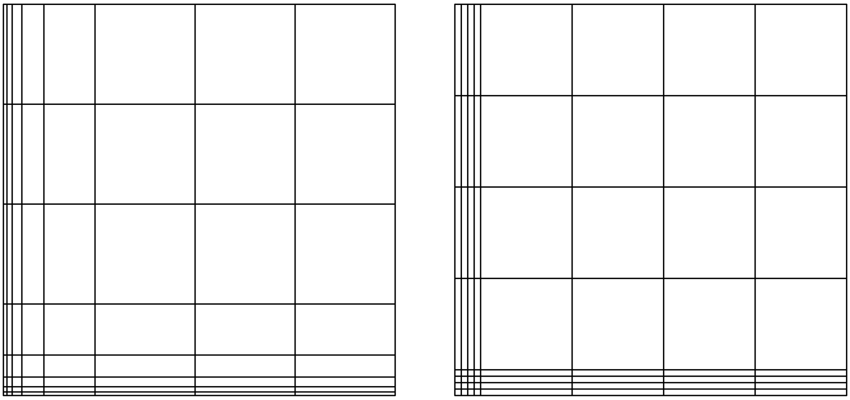


Fig. 2.19 Bakhvalov mesh (left) and Shishkin mesh (right) for a convection-diffusion problem with regular layers on the square

Appropriate meshes are constructed by taking tensor products of meshes for one-dimensional convection-diffusion problems from Sect. 2.1. For example, for a Shishkin mesh we take transition points

$$\tau_x = \min\left\{q, \frac{\sigma\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \tau_y = \min\left\{q, \frac{\sigma\varepsilon}{\beta_2}\ln N\right\}$$

for the meshes in x- and in y-direction, with mesh parameter $q \in (0,1)$ and $\sigma > 0$. Figure 2.19 displays both a Bakhvalov mesh and a Shishkin mesh.

2.3.2.2 Characteristic Boundary Layers

Again consider (2.21) on the square $\Omega = (0,1)^2$, but with $b_1 \geq \beta$ and $b_2 \equiv 0$ on $\bar{\Omega}$ with a positive constant β. The outflow boundary consists of the edge $x = 0$ only, but there are two characteristic boundaries at $y = 0$ and $y = 1$. Three boundary layers form behaving like

$$\mathrm{e}^{-\beta x/\varepsilon}, \quad \mathrm{e}^{-y^2/\varepsilon} \quad \text{and} \quad \mathrm{e}^{-(1-y)^2/\varepsilon}.$$

Figure 2.20 depicts a typical solution.

Note that for any $m > 0$ there exists a constant $C > 0$ such that

$$\mathrm{e}^{-y^2/\varepsilon} \leq \mathrm{e}^{-my/\sqrt{\varepsilon}} \quad \text{and} \quad \mathrm{e}^{-(1-y)^2/\varepsilon} \leq \mathrm{e}^{-m(1-y)/\sqrt{\varepsilon}} \quad \text{for all } \; y \in [0,1].$$

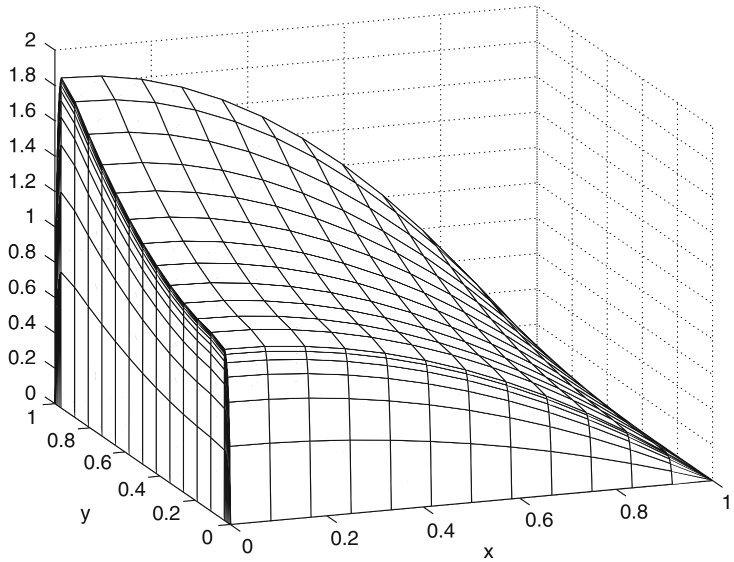


Fig. 2.20 Convection-diffusion problem on the square with an exponential layer (left) and two characteristic layers (front and back)

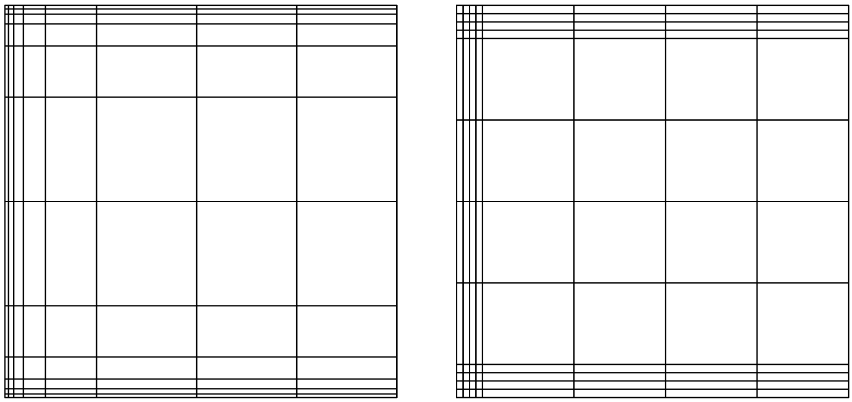


Fig. 2.21 Bakhvalov mesh (left) and Shishkin mesh (right) for a convection-diffusion problem with a regular layer and two characteristic layers

Therefore, an adapted mesh for a one-dimensional reaction-diffusion equation can be used in y-direction to resolve the characteristic layers. In x-direction a mesh for a one-dimensional convection-diffusion equation is used; see Fig 2.21.

Part I

One Dimensional Problems

Chapter 3
The Analytical Behaviour of Solutions

In this chapter, we gather a number of analytical properties for singularly perturbed boundary-value problems for second-order ordinary differential equations of the general type

$$-\varepsilon u'' - bu' + cu = f \text{ in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

with a small positive parameter ε and functions b, c, f : $[0,1] \to I\!R$, and of its vector-valued counterpart

$$-\boldsymbol{E}\boldsymbol{u}'' - \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{\gamma}_0, \;\; \boldsymbol{u}(1) = \boldsymbol{\gamma}_1,$$

with $\boldsymbol{E} = \operatorname{diag}(\boldsymbol{\varepsilon})$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_\ell)^T$ and small positive constants ε_i, $i = 1, \dots, \ell$, with matrix-valued functions $\boldsymbol{A}, \boldsymbol{B} : [0,1] \to I\!R^{\ell,\ell}$, and vector-valued functions $\boldsymbol{f}, \boldsymbol{u} : [0,1] \to I\!R^\ell$.

We shall study stability properties of the differential operators, their Green's functions and the behaviour of derivatives of the solutions of various boundary-value problems. Initial general considerations will be followed by specific results for various classes of singularly perturbed problems:

- reaction-convection-diffusion problems,
- reaction-diffusion problems,
- convection-diffusion problems with regular layers and
- convection-diffusion problems with turning-point layers.

These results form the basis for our analysis of numerical methods in later chapters.

Notation. *For functions $v, w : D \to I\!R$ we write "$v \le w$ in/on D" if $v(x) \le w(x)$ for all $x \in D$ and "$v \le 0$ in/on D" if $v(x) \le 0$ for all $x \in D$. Similarly, for vectors $\boldsymbol{v}, \boldsymbol{w} \in I\!R^n$ we write "$\boldsymbol{v} \le \boldsymbol{w}$" if $v_i \le w_i$ for $i = 1, \dots, n$. The notation $\boldsymbol{A} \le 0$ will be used if all components of the matrix $\boldsymbol{A}$ are non-positive. We shall also use "$v < w$" and $v \ge w$" etc. with the obvious meaning.*

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_3,

3.1 Preliminaries

In this section we consider basic stability properties of differential operators and of matrices that will be employed throughout this monograph.

3.1.1 Stability of Differential Operators

Maximum and comparison principles are important tools for studying the stability of differential operators. Consider the general second-order differential operator

$$\mathcal{L}u := -\left(au'\right)' + bu' + cu$$

with $a, b, c : [0,1] \to I\!R$ and $a > 0$ on $[0,1]$.

Definition 3.1. The differential operator $\mathcal{L}$ is called **inverse monotone** or **of positive type**, if it obeys a **comparison principle**. That is, for any two functions $v, w \in C^2(0,1) \cap C[0,1]$

$$\left.\begin{array}{r} \mathcal{L}v \le \mathcal{L}w \quad \text{in} \;\; (0,1), \\ v(0) \le w(0), \qquad\qquad \\ v(1) \le w(1) \qquad\qquad \end{array}\right\} \quad \Longrightarrow \quad v \le w \quad \text{on} \;\; [0,1].$$

♡

Remark 3.2. The inverse monotonicity can be equivalently characterised as follows: For any function $v \in C^2(0,1) \cap C[0,1]$

$$\left.\begin{array}{r} \mathcal{L}v \le 0 \quad \text{in} \;\; (0,1), \\ v(0) \le 0, \qquad\qquad \\ v(1) \le 0 \qquad\qquad \end{array}\right\} \quad \Longrightarrow \quad v \le 0 \quad \text{on} \;\; [0,1].$$

♣

Lemma 3.3. *Let there exist a function $\psi \in C^2(0,1) \cap C[0,1]$ with $\psi > 0$ on $[0,1]$ and $\mathcal{L}\psi > 0$ in $(0,1)$. Then the operator $\mathcal{L}$ is inverse monotone.*

Proof. The proof of the comparison principle is by contradiction; see, e.g. [131]. □

Corollary 3.4. *Let the assumptions of Lemma 3.3 be satisfied. Then for any two functions $v, w \in C^2(0,1) \cap C[0,1]$*

$$\left.\begin{array}{r} |\mathcal{L}v| \le \mathcal{L}w \quad \textit{in} \;\; (0,1), \\ |v(0)| \le w(0), \qquad\qquad \\ |v(1)| \le w(1) \qquad\qquad \end{array}\right\} \quad \Longrightarrow \quad |v| \le w \quad \textit{on} \;\; [0,1]$$

*and we call w a **barrier function** of v.*

Lemma 3.5. *Suppose there exists a function* $\psi \in C^2(0,1) \cap C[0,1]$ *with* $\psi > 0$ *on* $[0,1]$ *and* $\mathcal{L}\psi > 0$ *in* $(0,1)$. *Then for any function* $v \in C^2(0,1) \cap C[0,1]$ *with* $v(0) = v(1) = 0$

$$|v| \le \psi \left\| \frac{\mathcal{L}v}{\mathcal{L}\psi} \right\|_\infty \quad \textit{in } [0,1].$$

Proof. Set $\tilde{\mathcal{L}}v := \mathcal{L}v/\mathcal{L}\psi$. Clearly $\tilde{\mathcal{L}}\psi \equiv 1$. Therefore, $\tilde{\mathcal{L}}$ satisfies a comparison principle. Next,

$$\left|\left(\tilde{\mathcal{L}}v\right)(x)\right| = \left|\frac{(\mathcal{L}v)(x)}{(\mathcal{L}\psi)(x)}\right| \le \left\|\frac{\mathcal{L}v}{\mathcal{L}\psi}\right\|_\infty = \left\|\frac{\mathcal{L}v}{\mathcal{L}\psi}\right\|_\infty \left(\tilde{\mathcal{L}}\psi\right)(x) \quad \text{for } x \in (0,1).$$

Thus, $\psi \left\|\mathcal{L}v/\mathcal{L}\psi\right\|_\infty$ is a barrier function for v. The proposition of the lemma follows. □

Definition 3.6. Let A and B be two normed spaces. The operator $\Lambda : A \to B$ is said to be (A,B)**-stable** if there exists a constant K such that

$$\|v\|_A \le K\|\Lambda v\|_B \quad \text{for all } v \in A.$$

K is called the **stability constant**. ♡

Remark 3.7. In the case of discrete (difference) operators one has a family of spaces A^ω and B^ω and a family of operators Λ^ω. In this case, the stability inequality reads

$$\|v\|_{A^\omega} \le K\|\Lambda^\omega v\|_{B^\omega} \quad \text{for all } v \in A^\omega$$

and for all admissible meshes ω. ♣

Corollary 3.8. *Let the assumptions of Lemma 3.5 be satisfied. Then for any function* $v \in C^2(0,1) \cap C[0,1]$ *with* $v(0) = v(1) = 0$

$$\|v\|_\infty \le \|\psi\|_\infty \left\|\frac{\mathcal{L}v}{\mathcal{L}\psi}\right\|_\infty .$$

Hence, the operator $\mathcal{L}$ *is* (L_∞, L_∞)*-stable, or* ***maximum-norm*** *stable, i.e.,*

$$\|v\|_\infty \le K \|\mathcal{L}v\|_\infty$$

with the stability constant $K = \|\psi\|_\infty \Big/ \min_{x\in[0,1]} |(\mathcal{L}\psi)(x)|.$

3.1.2 Green's Functions

Green's functions form a very powerful tool in the analysis of differential equations. For example, given the Green's function $\mathcal{G}$ associated with $\mathcal{L}$ and Dirichlet boundary conditions, any function $v \in C^2(0,1) \cap C[0,1]$ with $v(0) = v(1) = 0$ can be represented as

$$v(x) = \int_0^1 \mathcal{G}(x,\xi)\left(\mathcal{L}v\right)(\xi)\mathrm{d}\xi \quad \text{for all } x \in (0,1). \tag{3.1}$$

Remark 3.9. By (3.1) the operator $\mathcal{L}$ is inverse monotone if and only if $\mathcal{G}(x,\xi) \geq 0$ for all $x, \xi \in (0,1)$. ♣

Lemma 3.10. *Suppose there exists a function $\psi \in C^2(0,1) \cap C[0,1]$ with $\psi > 0$ on $[0,1]$ and $\mathcal{L}\psi > 0$ in $(0,1)$. Then $\mathcal{G} \geq$ on $[0,1]$ and we have the following representation of the $(\mathcal{L}\psi)$-weighted L_1-norm of the Green's function:*

$$\int_0^1 \left(\mathcal{L}\psi\right)(\xi)\mathcal{G}(x,\xi)\mathrm{d}\xi = \psi(x) \quad \textit{for all } x \in (0,1).$$

If furthermore $\mathcal{L}\psi > 0$ on $[0,1]$ then

$$\|\mathcal{G}(x,\cdot)\|_1 \leq \frac{\|\psi\|_\infty}{\min_{\xi\in[0,1]}\left(\mathcal{L}\psi\right)(\xi)} \quad \textit{for all } x \in (0,1).$$

Proof. This is an immediate consequence of (3.1) and Lemma 3.3. □

Characterisation of the Green's function.

For fixed $\xi \in (0,1)$ it solves

$$\left(\mathcal{L}\mathcal{G}(\cdot,\xi)\right)(x) = \delta(x-\xi) \quad \text{for } x \in (0,1), \quad \mathcal{G}(0,\xi) = \mathcal{G}(1,\xi) = 0, \tag{3.2}$$

with the Dirac-δ distribution. For fixed $x \in (0,1)$ we have

$$\left(\mathcal{L}^*\mathcal{G}(x,\cdot)\right)(\xi) = \delta(\xi - x) \quad \text{for } \xi \in (0,1), \quad \mathcal{G}(x,0) = \mathcal{G}(x,1) = 0, \tag{3.3}$$

with the adjoint operator

$$\mathcal{L}^*v = -\left(av'\right)' + \left(bv\right)' + cv.$$

Eqs. (3.2) and (3.3) have to be read in the context of distributions. Alternatively, for fixed $\xi \in (0,1)$, one may seek $\mathcal{G}(\cdot,\xi) \in C^2\left((0,\xi)\cup(\xi,1)\right) \cap C[0,1]$ satisfying

$$\mathcal{L}\mathcal{G}(\cdot,\xi) = 0 \ \text{ in } \ (0,1)\setminus\{\xi\}, \ \ \mathcal{G}(0,\xi) = \mathcal{G}(1,\xi) = 0, \quad -\big[a\partial_x\mathcal{G}(\cdot,\xi)\big](\xi) = 1,$$

where $[v](\xi) := v(\xi+0) - v(\xi-0)$ denote the jump of v at $\xi \in (0,1)$. Similarly, we can interpret the adjoint problem as finding, for any fixed $x \in (0,1)$, a function $\mathcal{G}(x,\cdot) \in C^2\left((0,x)\cup(x,1)\right) \cap C[0,1]$ with

$$\mathcal{L}^*\mathcal{G}(x,\cdot) = 0 \ \text{ in } \ (0,1)\setminus\{x\}, \ \ \mathcal{G}(x,0) = \mathcal{G}(x,1) = 0, \quad -\left[a\partial_\xi \mathcal{G}(x,\cdot)\right](x) = 1.$$

When studying Green's functions for particular problems, we shall use the following comparison principle for functions with a derivative having a discontinuity at an interior point of the domain.

Lemma 3.11. *Suppose there exists a function $\psi \in C^2(0,1) \cap C[0,1]$ with $\psi > 0$ on $[0,1]$ and $\mathcal{L}\psi > 0$ in $(0,1)$. Let $d \in (0,1)$ be arbitrary, but fixed. Then for any two functions $v, w \in C[0,1] \cap C^2\left((0,\xi)\cup(\xi,1)\right)$*

$$\left.\begin{aligned} \mathcal{L}v \le \mathcal{L}w \quad &\textit{in} \ \ (0,1)\setminus\{\xi\} \\ v(0) \le w(0)& \\ v(1) \le w(1)& \\ -[av'](\xi) \le -[aw'](\xi)& \end{aligned}\right\} \implies \quad v \le w \quad \textit{on} \ \ [0,1].$$

Proof. The proof of the comparison principle is by contradiction, cf. [131]. □

3.1.3 *M*-matrices

M-matrices are the discrete counterparts of inverse-monotone differential operators. They are of utmost importance in the convergence analysis of finite difference methods and when studying systems of differential equations.

Definition 3.12. A matrix $\boldsymbol{A} \in \mathbb{R}^{n,n}$ is called

- an $\boldsymbol{L_0}$**-matrix** if $a_{ij} \le 0$ for all $i,j = 1,\ldots,n$ with $i \ne j$;
- **inverse monotone** if for any two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$

$$\boldsymbol{A v} \le \boldsymbol{A w} \quad \implies \quad \boldsymbol{v} \le \boldsymbol{w};$$

- an $\boldsymbol{M}$**-matrix** if it is an inverse monotone L_0-matrix. ♡

Remark 3.13. The inverse monotonicity of a matrix $\boldsymbol{A}$ can also be characterised by $\boldsymbol{A}^{-1} \ge 0$, where it is implicitly assumed that $\boldsymbol{A}^{-1}$ exists. This is the discrete counterpart of $\mathcal{G} \ge 0$ for the Green's function of a differential operator. ♣

Lemma 3.14 **($\boldsymbol{M}$-criterion).** *Let $\boldsymbol{A} \in \mathbb{R}^{n,n}$ be an L_0-matrix. Then $\boldsymbol{A}$ is an M-matrix if and only if there exists a vector $\boldsymbol{e} \in \mathbb{R}^n$ with $\boldsymbol{e} > 0$ and $\boldsymbol{A e} > 0$. Moreover,*

$$\left\|\boldsymbol{A}^{-1}\right\|_\infty \le \|\boldsymbol{e}\|_\infty \Big/ \min_{i=1,\ldots,n} \left|(\boldsymbol{A e})_i\right| =: K'.$$

Remark 3.15. The assumption that $\boldsymbol{A}$ is an L_0-matrix corresponds to the coefficient of the second-order derivative in the differential operator $\mathcal{L}$ being negative, while the existence of a particular vector $\boldsymbol{e}$ resembles the existence of a positive function ψ with $\mathcal{L}\psi > 0$. ♣

Remark 3.16. The estimate for $\boldsymbol{A}^{-1}$ implies

$$\|\boldsymbol{v}\|_\infty \le K' \|\boldsymbol{A}\boldsymbol{v}\|_\infty \quad \text{for all } \boldsymbol{v} \in I\!R^n,$$

i. e., $\boldsymbol{A}$ is $(\ell_\infty, \ell_i n f t y)$ stable, or **maximum-norm stable**. The stability constant K' is similar in nature to the constant K in Cor. 3.8. ♣

Lemma 3.17. *Let $\boldsymbol{A}$ be an L_0-matrix. Suppose there exists a vector $\boldsymbol{e} \in I\!R^n$ with $\boldsymbol{e} > 0$ and $\boldsymbol{A}\boldsymbol{e} > 0$. Then for any vector $v \in I\!R^n$*

$$|v_i| \le e_i \max_{j=1,\dots,n} \left| (\boldsymbol{A}v)_j \,/\, (\boldsymbol{A}\boldsymbol{e})_j \right|_\infty \quad \textit{for } i = 1, \dots, n.$$

Proof. Set $\tilde{\boldsymbol{A}} := \operatorname{diag}(\boldsymbol{A}\boldsymbol{e})^{-1} \boldsymbol{A}$. Clearly $\tilde{\boldsymbol{A}}$ is a L_0-matrix and $\tilde{\boldsymbol{A}}\boldsymbol{e} \equiv 1$. Therefore, it satisfies a comparison principle. Imitating the proof of Lemma 3.5, we obtain the proposition. □

3.2 Reaction-Convection-Diffusion Problems

In this section we consider the stationary linear reaction-convection-diffusion problem

$$\mathcal{L}u := -\varepsilon_d u'' - \varepsilon_c b u' + cu = f \text{ in } (0,1), \; u(0) = \gamma_0, \; u(1) = \gamma_1, \tag{3.4}$$

with two small parameters $0 < \varepsilon_d \ll 1$ and $0 \le \varepsilon_c \ll 1$. The functions b, c and f are assumed to be sufficiently smooth with $b(x) \ge 1$ and $c(x) \ge 1$ for $x \in [0,1]$.

The solution of (3.4) can be described by the two roots of the characteristic equation

$$-\varepsilon_d \lambda(x)^2 - \varepsilon_c b(x)\lambda(x) + c(x) = 0. \tag{3.5}$$

This quadratic equation defines two continuous functions $\lambda_i : [0,1] \to I\!R$ with

$$\lambda_0(x) = -\frac{\varepsilon_c b(x)}{2\varepsilon_d} - \sqrt{\left(\frac{\varepsilon_c b(x)}{2\varepsilon_d}\right)^2 + \frac{c(x)}{\varepsilon_d}} \le -\lambda_1(x),$$
$$\lambda_1(x) = -\frac{\varepsilon_c b(x)}{2\varepsilon_d} + \sqrt{\left(\frac{\varepsilon_c b(x)}{2\varepsilon_d}\right)^2 + \frac{c(x)}{\varepsilon_d}} > 0.$$

Set

$$\mu_0 := \max_{x\in[0,1]} \lambda_0(x) < \frac{\varepsilon_c}{\varepsilon_d} \le 0 \quad \text{and} \quad \mu_1 := \min_{x\in[0,1]} \lambda_1(x) > 0.$$

The decay of the boundary layers is determined by μ_0 and μ_1. At $x = 0$ there is a layer behaving like $\mathrm{e}^{\mu_0 x}$, while the layer at $x = 1$ is characterised by $\mathrm{e}^{-\mu_1(1-x)}$.

There are essentially three regimes:

		$\lvert\mu_0\rvert$	μ_1
convection-diffusion	$\varepsilon_d \ll \varepsilon_c = 1$	$1/\varepsilon_d$	1
reaction-convection-diffusion	$\varepsilon_d \ll \varepsilon_c^2 \ll 1$	$\varepsilon_c/\varepsilon_d$	$1/\varepsilon_c$
reaction-diffusion	$\varepsilon_c^2 \ll \varepsilon_d \ll 1$	$1/\sqrt{\varepsilon_d}$	$1/\sqrt{\varepsilon_d}$

Figure 2.9 depicts typical solutions of reaction-convection-diffusion problems. There are two layers of different width at the two end point of the interval.

Remark 3.18. The values of λ_0 and λ_1 do not vary significantly on $[0,1]$ because

$$\frac{\lambda_1(\xi)}{\lambda_1(\eta)} = \frac{c(\xi)\lambda_0(\eta)}{c(\eta)\lambda_0(\xi)} \ge \begin{cases} \|b\|_\infty^{-1}\|c\|_\infty^{-1} & \text{if } \varepsilon_c > 0, \\ \|c\|_\infty^{-1/2} & \text{if } \varepsilon_c = 0, \end{cases} \tag{3.6}$$

for all $\xi, \eta \in [0,1]$. ♣

3.2.1 Stability and Green's Function Estimates

Using the test function $\psi \equiv 1$, we see that for the operator $\mathcal{L}$ in (3.4) satisfies the assumptions of and Lemma 3.3 because $\mathcal{L}\psi = c \ge 1$. Therefore, Corollary 3.8 applies and yields

$$\|v\|_\infty \le \max\left\{ \left\| \frac{\mathcal{L}v}{c} \right\|_\infty, |v(0)|, |v(1)| \right\} \quad \text{for all } v \in C[0,1] \cap C^2(0,1). \tag{3.7}$$

Deeper insight into the stability properties of (3.4) is gained by studying the Green's function associated with $\mathcal{L}$. Note the adjoint operator is given by

$$\mathcal{L}^* v = -\varepsilon_d v'' + \varepsilon_c (bv)' + cv.$$

Figure 3.1 depicts a typical plot of $\mathcal{G}(x,\cdot)$. It is nonnegative, monotonically increasing for $\xi < x$, but decreasing for $\xi > x$, and has a maximum of order $\mathcal{O}(\mu_1)$ at $x = \xi$. These properties will be rigorously proved now.

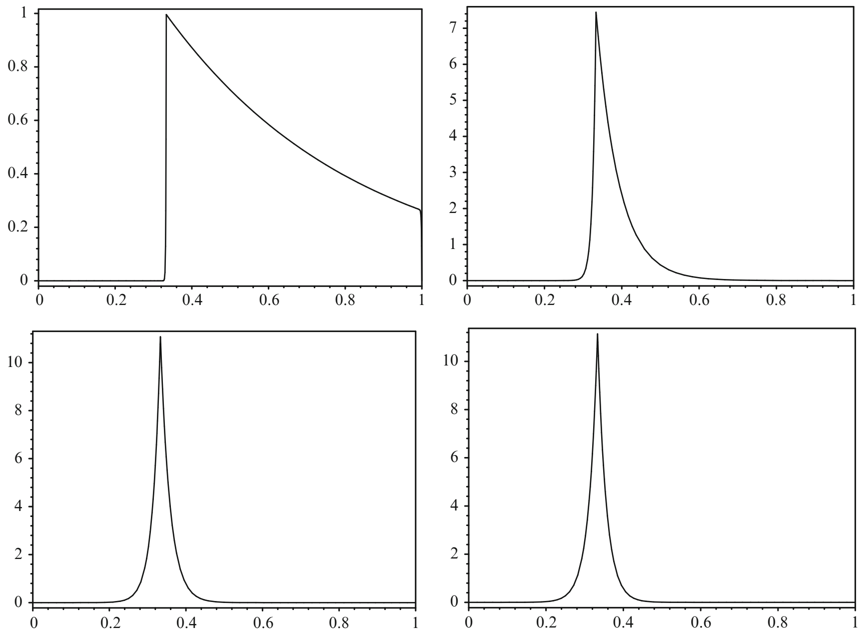


Fig. 3.1 Green's function $\mathcal{G}(x,\cdot)$ associated with $\mathcal{L}$ and $x = 1/4$; $\varepsilon_d = 10^{-3}$, $\varepsilon_c = 1, 10^{-1}, 10^{-2}, 0$ (left to right, top to bottom)

Proposition 3.19. *Let $p \in [0,1]$ be fixed. Then*

$$-\varepsilon_d p^2 \mu_i^2 - \varepsilon_c b(x) p \mu_i + c(x) \geq (1-p)c(x) \quad \textit{for} \;\; i = 0, 1 \;\; \textit{and all} \;\; x \in [0,1].$$

Proof. We consider $i = 0$ first. Let $x \in [0,1]$ be arbitrary. Then

$$-\varepsilon_d \left(p\mu_0\right)^2 - \varepsilon_c b(x) p \mu_0 \geq p\mu_0 \left(-\varepsilon_d \lambda_0(x) - \varepsilon_c b(x)\right)$$

because $p \leq 1$ and $0 > \mu_0 \geq \lambda_0(x)$ for all $x \in [0,1]$. The characteristic equation (3.5) yields

$$-\varepsilon_d \left(p\mu_0\right)^2 - \varepsilon_c b(x) p \mu_0 \geq -\frac{\mu_0}{\lambda_0(x)} p c(x)$$

Using $c(x) \geq 1$ and $\lambda_0(x) \leq \mu_0 < 0$ again, completes the proof for μ_0.

Now study $i = 1$.

$$-\varepsilon_d \left(p\mu_1\right)^2 - \varepsilon_c b(x) p \mu_1 \geq p\left(-\varepsilon_d \lambda_1(x)^2 - \varepsilon_c b(x) \lambda_1(x)\right)$$

because $p \leq 1$ and $0 < \mu_1 \leq \lambda_1(x)$ and $\mu_1^2 \leq \lambda_1(x)^2$ for all $x \in [0,1]$. Using (3.5), we are finished. □

Theorem 3.20. *The Green's function $\mathcal{G}$ associated with $\mathcal{L}$ satisfies*

$$0 \le \mathcal{G}(x,\xi) \le \bar{\mathcal{G}}(x,\xi) := \frac{1}{\beta^*} \begin{cases} \mathrm{e}^{\mu_1(x-\xi)} & \text{for } 0 \le x \le \xi \le 1, \\ \mathrm{e}^{\mu_0(x-\xi)} & \text{for } 0 \le \xi \le x \le 1, \end{cases}$$

where $\beta^* := \varepsilon_d\,(\mu_1 - \mu_0)$.

If furthermore

$$\varepsilon_c b' + c \ge 0 \quad \text{on } [0,1] \tag{3.8}$$

then $\mathcal{G}$ is piecewise monotone with

$$\begin{aligned} \partial_x \mathcal{G}(x,\xi) \le 0, \quad \partial_\xi \mathcal{G}(x,\xi) \ge 0 \quad \text{for } 0 \le \xi < x \le 1, \\ \partial_x \mathcal{G}(x,\xi) \ge 0, \quad \partial_\xi \mathcal{G}(x,\xi) \le 0 \quad \text{for } 0 \le x < \xi \le 1. \end{aligned}$$

and

$$\partial_x \partial_\xi \mathcal{G}(x,\xi) \le 0 \quad \text{for } x,\xi \in [0,1],\ x \ne \xi.$$

Proof. The inverse monotonicity of $\mathcal{L}$ implies $0 \le \mathcal{G}$ on $[0,1]^2$.

Proposition 3.19 gives $\mathcal{L}\bar{\mathcal{G}}(\cdot,\xi) \ge 0$ on $(0,1) \setminus \{\xi\}$. Clearly $\bar{\mathcal{G}}(0,\xi) > 0$ and $\bar{\mathcal{G}}(1,\xi) > 0$. The jump of $\partial_x \bar{\mathcal{G}}$ satisfies $-\varepsilon_d \big[\partial_x \bar{\mathcal{G}}(\cdot,\xi)\big](\xi) = 1$. Application of Lemma 3.11 establishes the upper bound on $\mathcal{G}$.

Next we prove the monotonicity of $\mathcal{G}$. Because $\mathcal{G}(x,0) = \mathcal{G}(x,1) = 0$ for $x \in [0,1]$ and $\mathcal{G} \ge 0$ on $[0,1]^2$ we have at the boundary of the domain

$$\partial_\xi \mathcal{G}(x,0) \ge 0 \quad \text{and} \quad \mathcal{G}_\xi(x,1) \le 0 \quad \text{for } x \in [0,1].$$

Integrating (3.3) over $[0,\xi]$, we get

$$-\varepsilon_d \Big(\partial_\xi \mathcal{G}(x,\xi) - \partial_\xi \mathcal{G}(x,0)\Big) + \varepsilon_c b(\xi)\mathcal{G}(x,\xi) = -\int_0^\xi c(s)\mathcal{G}(x,s)\mathrm{d}s \le 0$$

for $\xi < x$. Thus,

$$\varepsilon_d \partial_\xi \mathcal{G}(x,\xi) \ge \varepsilon_d \partial_\xi \mathcal{G}(x,0) + \varepsilon_c b(\xi)\mathcal{G}(x,\xi) \ge 0 \quad \text{for } \xi < x$$

because $\mathcal{G}(x,\xi) \ge 0$ and $\mathcal{G}_\xi(x,0) \ge 0$.

On the other hand, inspecting the differential equation (3.3), we see that $v = \partial_\xi \mathcal{G}(x,\cdot)$ satisfies

$$-\varepsilon_d v' + \varepsilon_c b v = -(\varepsilon_c b' + c)\mathcal{G} \le 0 \ \text{ in } \ (\xi,1) \ \text{ and } \ v(1) \le 0,$$

by (3.8). Application of a comparison principle for first-order operators yields $v \le 0$ on $[x,1]$.

Similarly, one proves that $\partial_x \mathcal{G}(x,\xi) \geq 0$ for $0 \leq x < \xi \leq 1$ and $\partial_x \mathcal{G}(x,\xi) \leq 0$ for $0 \leq \xi < x \leq 1$. Thus,

$$\partial_x \partial_\xi \mathcal{G}(x,0) \leq 0 \quad \text{and} \quad \partial_x \partial_\xi \mathcal{G}(x,1) \leq 0 \quad \text{for } \ x \in [0,1].$$

because $\partial_x \mathcal{G}(x,0) = \partial_x \mathcal{G}(x,1) = 0$ for $x \in [0,1]$. Differentiating (3.3) with respect to x and integrating with respect to ξ, we get

$$\begin{aligned} -\varepsilon_d \partial_x \partial_\xi \mathcal{G}(x,\xi) + \varepsilon_d \partial_x \partial_\xi \mathcal{G}(x,0) + \varepsilon_c b(\xi) \partial_x \mathcal{G}(x,\xi) - \varepsilon_c b(0) \partial_x \mathcal{G}(x,0) \\ = -\int_0^\xi c(s) \partial_x \mathcal{G}(x,s) \mathrm{d}s \quad \text{for } \ \xi < x. \end{aligned}$$

Hence,

$$\partial_x \partial_\xi \mathcal{G}(x,\xi) \leq 0 \quad \text{for } \ 0 \leq \xi < x \leq 1$$

because $\partial_x \mathcal{G}(x,\xi) \leq 0$, $\partial_x \partial_\xi \mathcal{G}(x,0) \leq 0$ and $\partial_x \mathcal{G}(x,0) = 0$. For $x < \xi$, differentiate (3.3) to see that $v = \partial_x \partial_\xi \mathcal{G}(x,\cdot)$ satisfies

$$-\varepsilon v' + bv = -(b' + c) \partial_x \mathcal{G} \leq 0 \quad \text{for } \ x \in (0,\xi), \ \ v(1) \leq 0,$$

by (3.8) and because $\partial_x \mathcal{G}(x,\xi) \geq 0$ for $x \leq \xi$. A comparison principle for first-order operators gives $\partial_x \partial_\xi \mathcal{G}(x,\cdot) \leq 0$ on $(x,1]$. □

Remark 3.21. The function $\bar{\mathcal{G}}$ attains its maximum for $x = \xi$. Because λ_0 is continuous there exists a $x^* \in [0,1]$ with $\lambda_0(x^*) = \mu_0$. Therefore,

$$\frac{1}{\beta^*} \leq -\frac{1}{\varepsilon_d \lambda_0(x^*)} = \frac{\lambda_1(x^*)}{c(x^*)} \leq C\mu_1, \quad \text{by (3.6)}.$$

If $\varepsilon_c = 0$ this estimate can be sharpened to

$$\frac{1}{\beta^*} \leq \frac{\mu_1}{2} \leq \frac{1}{2\sqrt{\varepsilon_d}}$$

since $\mu_1 = -\mu_0 \leq 1/\sqrt{\varepsilon_d}$. ♣

For our further investigations, let us introduce the L_∞ norm

$$\|v\|_\infty := \operatorname{ess} \sup_{x \in [0,1]} |v(x)|,$$

the L_1 norm

$$\|v\|_1 := \int_0^1 |v(x)| \mathrm{d}x$$

and the $W^{-1,\infty}$ norm $\|\cdot\|_{-1,\infty}$. Since $W^{-1,\infty} = \left(\mathring{W}^{1,1}\right)'$ the latter is defined by

$$\|v\|_{-1,\infty} := \sup_{u \in \mathring{W}^{1,1}: |u|_{1,1}=1} \langle u, v \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing. This norm can also be characterised by

$$\|v\|_{-1,\infty} = \min_{V: V'=v} \|V\|_\infty = \min_{c \in \mathbb{R}} \left\| \int_\cdot^1 v(s) \mathrm{d}s + c \right\|_\infty, \tag{3.9}$$

see [5] for a more detailed discussion.

Remark 3.22. By (3.9)

$$\|v\|_{-1,\infty} \le \left\| \int_\cdot^1 v(s) \mathrm{d}s - \frac{1}{2} \int_0^1 v(s) \mathrm{d}s \right\|_\infty .$$

Furthermore

$$\left| \int_x^1 v(s) \mathrm{d}s - \frac{1}{2} \int_0^1 v(s) \mathrm{d}s \right| = \frac{1}{2} \left| \int_x^1 v(s) \mathrm{d}s - \int_0^x v(s) \mathrm{d}s \right| .$$

Hence,

$$2 \|v\|_{-1,\infty} \le \|v\|_1 \le \|v\|_\infty .$$

♣

Theorem 3.23. *Let x and $\xi \in (0,1)$ be arbitrary. Then the Green's function $\mathcal{G}$ associated with $\mathcal{L}$ satisfies*

$$\|c\mathcal{G}(x,\cdot)\|_1 \le 1. \tag{3.10a}$$

If additionally (3.8) *holds then*

$$\|\partial_\xi \mathcal{G}(x,\cdot)\|_1 \le \frac{2}{\beta^*}, \quad \|\partial_x \mathcal{G}(\cdot,\xi)\|_1 \le \frac{2}{\beta^*}, \tag{3.10b}$$

$$\|\partial_x \partial_\xi \mathcal{G}(x,\cdot)\|_1 \le \frac{2}{\varepsilon_d} \tag{3.10c}$$

and

$$\varepsilon_d \left\| \partial_\xi^2 \mathcal{G}(x,\cdot) \right\|_1 \le \varepsilon_c \left\{ \frac{2\|b\|_\infty}{\beta^*} + \left\| \frac{b'}{c} \right\|_\infty \right\} + 2.$$

Proof. The first estimate, in the weighted L_1-norm, follows from Lemma 3.10.

Next, for fixed $x \in (0,1)$,

$$\|\partial_\xi \mathcal{G}(x,\cdot)\|_1 = \int_0^x \partial_\xi \mathcal{G}(x,\xi)\mathrm{d}\xi - \int_x^1 \partial_\xi \mathcal{G}(x,\xi)\mathrm{d}\xi = 2\mathcal{G}(x,x) \le \frac{2}{\beta^*},$$

by Theorem 3.20. Similarly, the bound for $\|\partial_x \mathcal{G}(\cdot,\xi)\|_1$ is obtained.

When calculating $\|\partial_x \partial_\xi \mathcal{G}(x,\cdot)\|_1$ note that

$$\int_0^1 \partial_x \partial_\xi \mathcal{G}(x,\xi)\mathrm{d}\xi = \partial_x \mathcal{G}(x,1) - \partial_x \mathcal{G}(x,0) = 0,$$

$\partial_x \partial_\xi \mathcal{G} \le 0$ for $x \neq \xi$ and $[\partial_x \mathcal{G}(x,\cdot)](x) = -\varepsilon_d^{-1}$.

Finally, to get the last inequality of the theorem, integrate (3.3) and apply the bounds for $\mathcal{G}$ and $\mathcal{G}_\xi$ just established. □

Remark 3.24. Because $c \ge 1$ on $[0,1]$, assuming (3.8) to hold true, provides an upper threshold value for ε_c, for which the analysis is valid in the case that b' attains negative values. If b' is non-negative everywhere then (3.8) is always satisfied. ♣

The bounds of Theorems 3.20 and 3.23 will be used to establish stability properties of the differential operator $\mathcal{L}$ now. To this end also introduce the weighted $W^{1,\infty}$ norm

$$|||v|||_\infty := \max\left\{\frac{\varepsilon_d}{2}\|v'\|_\infty, \frac{\beta^*}{2}\|v\|_\infty\right\}.$$

Theorem 3.25. *Set $\beta^* = \varepsilon_d(\mu_1 - \mu_0)$. The operator $\mathcal{L}$ satisfies*

$$\|v\|_\infty \le \|(\mathcal{L}v)/c\|_\infty \quad \textit{for all } v \in \mathring{W}^{1,\infty}(0,1) \cap W^{2,\infty}(0,1), \tag{3.11a}$$

$$\|v\|_\infty \le \frac{2}{\beta^*}\|\mathcal{L}v\|_1 \quad \textit{for all } v \in \mathring{W}^{1,1}(0,1) \cap W^{2,1}(0,1) \tag{3.11b}$$

$$\|v'\|_1 \le \frac{2}{\beta^*}\|\mathcal{L}v\|_1 \quad \textit{for all } v \in \mathring{W}^{1,1}(0,1) \cap W^{2,1}(0,1) \tag{3.11c}$$

and

$$|||v|||_\infty \le \|\mathcal{L}v\|_{-1,\infty} \quad \textit{for all } v \in \mathring{W}^{1,\infty}(0,1). \tag{3.11d}$$

Proof. Recall the representation (3.1). Then the Hölder inequality, Theorem 3.20 and (3.10a) give (3.11a) and (3.11b).

Next, let $V \in W^{0,\infty}(0,1)$ be an arbitrary function with $V' = \mathcal{L}v$. Integrating (3.1) by parts, we obtain

$$v(x) = -\int_0^1 \partial_\xi \mathcal{G}(x,\xi)V(\xi)\mathrm{d}\xi \quad \text{for } x \in (0,1)$$

and

$$v'(x) = -\int_0^1 \partial_x \partial_\xi \mathcal{G}(x,\xi) V(\xi) \mathrm{d}\xi \quad \text{for } x \in (0,1).$$

The Hölder inequality, (3.10b) and (3.10c) yield (3.11d).

Finally,

$$\int_0^1 |v'(x)| \le \int_0^1 \int_0^1 |\partial_x \mathcal{G}(x,\xi)| \, \left|(\mathcal{L}v)(\xi)\right| \mathrm{d}\xi \mathrm{d}x.$$

Changing the order of integration and using (3.10a), we obtain (3.11c). □

Remark 3.26. By (3.11a) the operator $\mathcal{L}$ is (L_∞, L_∞)-stable with a stability constant independent of ε_c and ε_d. It is also (L_∞, L_1)-stable and $(L_\infty, W^{-1,\infty})$-stable because of (3.11b) and (3.11d), but the stability constants depend on ε_d and ε_c. ♣

3.2.2 Derivative Bounds and Solution Decomposition

We derive bounds on the derivative of the solution of (3.4) now. The argument follows [105]. Define

$$w_{0,\alpha}(x) := \mathrm{e}^{\alpha\mu_0 x} \quad \text{and} \quad w_{1,\alpha}(x) := \mathrm{e}^{-\alpha\mu_1(1-x)}.$$

They will be used to describe the layers at $x = 0$ and $x = 1$, resp.

Proposition 3.27. *For any $p \in [0,1]$ and for $i = 0, 1$*

$$\mathcal{L} w_{i,p} \ge (1-p) w_{i,p} \quad \textit{in } (0,1).$$

Proof. The result follows readily from Proposition 3.19. □

Proposition 3.28. *For any $\vartheta \ge \|b\|_\infty$ and $i = 0, 1$*

$$-\mathcal{L} w_{i,\vartheta} \ge \left(\frac{\vartheta}{\|b\|_\infty} - \|c\|_\infty\right) w_{i,\vartheta} \quad \textit{in } (0,1).$$

Proof. We give the proof for $i = 0$ only because the case $i = 1$ is analogous. A direct calculation gives

$$-\left(\mathcal{L} w_{0,\vartheta}\right)(x) = \left\{\varepsilon_d \vartheta^2 \mu_0^2 - \varepsilon_c \vartheta b(x) \mu_0 - c(x)\right\} w_{0,\vartheta}(x).$$

The function λ_0 is continuous. Therefore, it attains its maximum μ_0 on $[0,1]$ and there exists a $\xi \in [0,1]$ such that $\lambda_0(\xi) = \mu_0$.

$$
\begin{aligned}
-\left(\mathcal{L}w_{0,\vartheta}\right)(x) &= \left\{\varepsilon_d \vartheta \lambda_0(\xi)^2 \left(\vartheta - \frac{b(x)}{b(\xi)}\right) + \frac{\vartheta c(\xi)}{b(\xi)} - c(x)\right\} w_{0,\vartheta}(x) \\
&\geq \left\{\varepsilon_d \vartheta \lambda_0(\xi)^2 \left(\vartheta - \|b\|_\infty\right) + \frac{\vartheta}{\|b\|_\infty} - \|c\|_\infty\right\} w_{0,\vartheta}(x),
\end{aligned}
$$

by (3.5) and because $b \geq 1$ and $c \geq 1$. The result follows. □

Theorem 3.29. *Let $b, c, f \in C^q[0,1]$ for some $q \in I\!N^+$. Let $p, \kappa \in (0,1)$ be arbitrary, but fixed. Assume*

$$
q\|b'\|_\infty \varepsilon_c \leq \kappa(1-p). \tag{3.12}
$$

Then

$$
\left|u^{(k)}(x)\right| \leq C\left\{1 + (-\mu_0)^k \mathrm{e}^{p\mu_0 x} + \mu_1^k \mathrm{e}^{-p\mu_1(1-x)}\right\} \quad \textit{for } x \in (0,1) \tag{3.13}
$$

and $k = 0, \ldots, q$.

Proof. The proof is by induction. For $k = 0$ the result follows from (3.7).

Now assume that (3.13) holds for $k = 0, \ldots, m < q$. Then differentiating (3.4) k-times, we get

$$
\mathcal{L}_k u^{(k)} := \mathcal{L}u^{(k)} - \varepsilon_c k b' u = g_k \quad \text{in } (0,1), \tag{3.14}
$$

where

$$
g_0 = f, \;\; c_0 = c, \;\; c_k = c - k\varepsilon_c b' \quad \text{and} \quad g_k = g'_{k-1} - c'_{k-1} u^{(k-1)}.
$$

We prove that (3.13) holds for $k = m+1$. Eqs. (3.13) and (3.14) give

$$
\left|\left(\mathcal{L}_{m+1} u^{(m+1)}\right)(x)\right| \leq C\left\{1 + (-\mu_0)^m w_{0,p}(x) + \mu_1^m w_{1,p}(x)\right\}, \;\; x \in (0,1).
$$

By Lemma 3.3 the operator $\mathcal{L}_{m+1}$ obeys a comparison principle, because for $\psi \equiv 1$

$$
\mathcal{L}_{m+1}\psi = c + \varepsilon_c(m+1)b' \geq 1 - \varepsilon_2 q\|b'\|_\infty \geq 1 - \kappa(1-p) > 0.
$$

Thus, if we had

$$
\left|u^{(m+1)}(0)\right| \leq C(-\mu_0)^{m+1} \quad \text{and} \quad \left|u^{(m+1)}(1)\right| \leq C\mu_1^{m+1}, \tag{3.15}
$$

then application of the comparison principle with the barrier function

$$w(x) = C_1\Big\{1 + (-\mu_0)^{m+1} w_{0,p}(x) + \mu_1^{m+1} w_{1,p}(x)\Big\}$$

with C_1 chosen sufficiently large, independently of ε_d and ε_c, would yield (3.13) for $k = m + 1$.

Therefore, we are left with proving (3.15). Introduce

$$u_m(x) := u^{(m)}(x) - u^{(m)}(0)(1 - x).$$

This function satisfies

$$\left|\left(\mathcal{L}_m u_m\right)(x)\right| \le C\left\{(-\mu_0)^k + (-\mu_0)^{k-1} w_{0,p}(x) + \mu_1^{k-1} w_{1,p}(x)\right\},$$

$x \in (0, 1)$, and

$$u_m(0) = 0, \quad |u_m(1)| \le C\mu_1^k.$$

Now consider

$$w(x) = C_2(-\mu_0)^k\big(1 + w_{0,p}(x) - 2w_{0,\vartheta}(x)\big) + C_3\mu_1^k\big(w_{1,p}(x) - w_{1,p}(0)\big)$$

***as a possible barrier function for u_m. Let $\vartheta \ge \|b\|_\infty$. Then Propositions 3.27 and 3.28 yield

$$\begin{aligned}\left(\mathcal{L}w\right)(x) \ge C_1(-\mu_0)^k\Bigg\{&1 - \varepsilon_c q\|b'\|_\infty + \left(1 - p - \varepsilon_c q\|b'\|_\infty\right) w_{0,p}(x)\\ &+ 2\left[\frac{\vartheta}{\|b\|_\infty} - \|c\|_\infty - \varepsilon_c q\|b'\|_\infty\right] w_{0,\vartheta}(x)\Bigg\}\\ + C_2\mu_1^k\Bigg\{&-\left(\|c\|_\infty + \varepsilon_c q\|b'\|_\infty\right) w_{1,p}(0)\\ &+ \left(1 - p - \varepsilon_c q\|b'\|_\infty\right) w_{1,p}(x)\Bigg\}.\end{aligned}$$

Fix $\vartheta \ge \|b\|_\infty(1 + \|c\|_\infty)$ and recall (3.12). Then there exist positive constants $C_4 = C_4(C_2, C_3)$, $C_5 = C_5(C_2)$ and $C_6 = C_6(C_3)$ with

$$\left(\mathcal{L}w\right)(x) \ge C_4\Big\{(-\mu_0)^k + (-\mu_0)^k w_{0,p}(x) + \mu_1^k w_{1,p}(x)\Big\} + C_5\mu_0^k - C_6,$$

because $\mu_1^k w_{1,p}(0) \le C$. Note $\mu_0 \ge 1\big/\big(\|b\|_\infty + \sqrt{\|c\|_\infty}\big)$. Thus, choosing C_2 and C_3 sufficiently large, independently of ε_d and ε_c, we see that w is a barrier function for u_m.

By the definition of the derivative we have

$$|u_m'(0)| \le \lim_{x\to+0} \frac{w(x)}{x} \le C_2(-\mu_0)^{m+1}(2\vartheta - p) + C_3\mu_1^{m+1} w_{1,p}(0).$$

Hence,

$$|u_m'(0)| \le C(-\mu_0)^{m+1}.$$

We get $|u^{(m+1)}(0)| \le C(-\mu_0)^{m+1}$, which is the first bound of (3.15). Analogously one estimates $u^{(m+1)}(1)$. □

3.3 Reaction-Diffusion Problems

This section is concerned with linear reaction-diffusion problems. First, we shall consider the scalar problem

$$\mathcal{L}u := -\varepsilon^2 u'' + cu = f \ \text{ in } \ (0,1), \ \ u(0) = \gamma_0, \ \ u(1) = \gamma_1,$$

while the second part of the section is devoted to vector-valued problems

$$\boldsymbol{\mathcal{L}u} := -\boldsymbol{E}^2\boldsymbol{u''} + \boldsymbol{Au} = \boldsymbol{f} \quad \text{in } (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{\gamma}_0, \ \ \boldsymbol{u}(1) = \boldsymbol{\gamma}_1,$$

where $\boldsymbol{E} = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_\ell)$ and the small parameters ε_k are in $(0,1]$.

3.3.1 Scalar Reaction-Diffusion Problems

Consider the boundary-value problem: Given $c, f \in C[0,1]$, find $u \in C^2(0,1) \cap C[0,1]$ such that

$$\mathcal{L}u := -\varepsilon^2 u'' + cu = f \quad \text{in } \ (0,1), \ \ u(0) = \gamma_0, \ \ u(1) = \gamma_1, \tag{3.16}$$

with the perturbation parameter $\varepsilon > 0$. Furthermore, we assume $c \ge 1$ on $[0,1]$.

We shall apply some of the results of Sect. 3.2. Note that $\varepsilon_d = \varepsilon^2$ and $\varepsilon_c = 0$. The characteristic roots satisfy

$$\lambda_1(x) = -\lambda_0(x) = \varepsilon^{-1}\sqrt{c(x)} \quad \text{for } \ x \in [0,1] \quad \text{and} \quad \mu_1 = -\mu_0 \ge \varepsilon^{-1}.$$

Therefore, two layers of equal widths will form at both ends of the domain.

Remark 3.30. The focus of our analysis is on linear problems. Using standard linearisation techniques, all results can be generalised to semilinear problems

$$\mathcal{T}u := -\varepsilon^2 u'' + c(u,\cdot) = 0 \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

with $\partial_u c \geq 1$ in $I\!R \times (0,1)$. ♣

3.3.1.1 Stability Properties

By (3.7) we have

$$\|v\|_\infty \leq \max\left\{ \left\| \frac{\mathcal{L}v}{c} \right\|_\infty, |v(0)|, |v(1)| \right\} \tag{3.17}$$

for all functions $v \in C[0,1] \cap C^2(0,1)$.

When deriving bounds on the Green's function we shall avail of Theorems 3.20 and 3.23. To this end, note that (3.8) is always satisfied and that $\beta^* \geq 2/\varepsilon$, see Remark 3.21. Also the Green's function is symmetric, i.e., $\mathcal{G}(x,\xi) = \mathcal{G}(\xi,x)$ for all $x, \xi \in [0,1]$, because the operator $\mathcal{L}$ is self-adjoint.

Theorem 3.31. *The Green's function $\mathcal{G}$ associated with (3.16) satisfies the pointwise bounds*

$$0 \leq \mathcal{G}(x,\xi) \leq \frac{\mathrm{e}^{-|x-\xi|/\varepsilon}}{2\varepsilon} \quad \textit{for } x, \xi \in [0,1],$$

and the (weighted) L_1-norm bounds

$$\|c\mathcal{G}(x,\cdot)\|_1 \leq 1, \quad \|\partial_\xi \mathcal{G}(x,\cdot)\|_1 \leq \varepsilon^{-1} \quad \textit{and} \quad \|\partial_{\xi\xi}\mathcal{G}(x,\cdot)\|_1 \leq 2\varepsilon^{-2}.$$

Remark 3.32. The bounds of the theorem are slight improvements over results first given by Kopteva [66]. ♣

Next stability inequalities for the operator will be derived. We shall make use of the following:

Lemma 3.33. *For any function $g \in W^{2,\infty}[a, a+\mu]$ with $\mu > 0$ there holds*

$$\|g'\|_{\infty,[a,a+\mu]} \leq \frac{2}{\mu} \|g\|_{\infty,[a,a+\mu]} + \frac{\mu}{2} \|g''\|_{\infty,[a,a+\mu]} \,.$$

Proof. This is Lemma 1 from [18]. □

Theorem 3.34. *For any function* $v \in \mathring{W}^{1,\infty}(0,1) \cap W^{2,\infty}(0,1)$ *the operator* $\mathcal{L}$ *satisfies*

$$\|v\|_\infty \le \|(\mathcal{L}v)/c\|_\infty, \quad \varepsilon \|v'\|_\infty \le \frac{5+\|c\|_\infty}{2} \|\mathcal{L}v\|_\infty$$

and

$$\varepsilon^2 \|v''\|_\infty \le (1+\|c\|_\infty) \|\mathcal{L}v\|_\infty .$$

Proof. The first inequality can be concluded from (3.17), Theorem 3.25 or Theorem 3.31.

Then, $\varepsilon^2 v'' = cv + \mathcal{L}v$, a triangle inequality, the bound on v and $c \ge 1$ yield the third estimate of the theorem.

Finally, Lemma 3.33 with $\mu = \varepsilon$ applied to $g = v$ gives the bound on v'. □

3.3.1.2 Bounds on Derivatives

Theorem 3.29 provides first bounds for the derivatives. Let $c, f \in C^q[0,1]$ for some $q \in I\!N^+$. Let $p \in (0,1)$ be arbitrary, but fixed. Then

$$\left|u^{(k)}(x)\right| \le C\left\{1 + \varepsilon^{-k}\mathrm{e}^{-px/\varepsilon} + \varepsilon^{-k}\mathrm{e}^{-p(1-x)/\varepsilon}\right\}, \quad \text{for } x \in (0,1) \tag{3.18}$$

and $k = 0, \dots, q$.

Note, in order to establish bounds for derivatives of order q, the data is assumed to lie in $C^q[0,1]$, while general theory for second-order differential equations guarantees the existence of a unique solution in $C^{q+2}[0,1]$. The following theory sharpens (3.18), gives bounds for higher-order derivatives and provides a solution decomposition, which is useful for the analysis of some numerical methods.

Theorem 3.35. *Suppose* $c, f \in C^q[0,1]$, $q \in I\!N$ *with* $c \ge 1$ *on* $[0,1]$. *Then* (3.16) *possesses a unique solution* $u \in C^{q+2}[0,1]$. *It can be decomposed as*

$$u = v + w_0 + w_1$$

with

$$\mathcal{L}v = f, \quad \mathcal{L}w_0 = 0 \quad \text{and} \quad \mathcal{L}w_1 = 0 \quad \text{in } (0,1).$$

The regular part v *satisfies*

$$\left\|v^{(m)}\right\|_\infty \le C(1+\varepsilon^{q-m}),$$

while for the layer pars w_0 and w_1 we have

$$\left|w_0^{(m)}(x)\right| \le C\varepsilon^{-m}\mathrm{e}^{-x/\varepsilon} \quad \textit{and} \quad \left|w_1^{(m)}(x)\right| \le C\varepsilon^{-m}\mathrm{e}^{-(1-x)/\varepsilon} \tag{3.19}$$

for $x \in [0,1]$ and $m = 0, 1, \ldots, q$.

Proof. The functions c and f can be extended to functions $\bar{c}, \bar{f} \in C^q[-1,2]$ with $\bar{c}|_{[0,1]} = c$, $\bar{f}\big|_{[0,1]} = f$ and $\bar{c} \ge 1/2$ on $[-1,2]$. Let $\bar{v}$ be the solution of

$$-\varepsilon^2\bar{v}'' + \bar{r}\bar{v} = \bar{f}, \text{ in } (-1,2), \;\; \bar{v}(-1) = \bar{v}(2) = 0 \text{ and set } v := \bar{v}|_{[0,1]}$$

An affine transformation and (3.18) yield $\left\|v^{(m)}\right\|_\infty \le C$ for $m = 0, \ldots, q$. Clearly $\mathcal{L}v = f$ on $(0,1)$. Therefore, $\left\|v^{(q+2)}\right\|_\infty \le C\varepsilon^{-2}$. Lemma 3.33 with $g = v^{(q)}$ and $\mu = \varepsilon$ gives $\left\|v^{(q+1)}\right\|_\infty \le C\varepsilon^{-1}$.

The layer components w_0 and w_1 solve

$$\mathcal{L}w_0 = 0 \quad \text{in } (0,1), \quad w_0(0) = u(0) - v(0), \quad w_0(1) = 0$$

and

$$\mathcal{L}w_1 = 0 \quad \text{in } (0,1), \quad w_1(0) = 0, \quad w_1(1) = u(1) - v(1).$$

Consider w_0. Application of Lemma 3.3 yields (3.19) for $m = 0$. From $\mathcal{L}w_0 = 0$ we get (3.19) for $m = 2$. Lemma 3.33 with $\mu = \varepsilon$ and $g = w_0$, gives (3.19) for $m = 1$. The bounds on the higher-order derivatives on w_0 follow upon differentiating $\mathcal{L}w_0 = 0$. The same argument is used for w_1. □

3.3.1.3 Discontinuous Data

The data, i.e. the reaction coefficient and the right-hand side, may have discontinuities at a number of points in the domain. For simplicity, let us consider a single point of discontinuity at $x = d \in (0,1)$. Then (3.16) takes the form: Given $c, f \in C\big((0,d) \cup (d,1)\big)$ find $u \in C^2\big((0,d) \cup (d,1)\big) \cap C^1[0,1]$ such that

$$\mathcal{L}u := -\varepsilon^2 u'' + cu = f \;\text{ in }\; (0,d) \cup (d,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1. \tag{3.20}$$

The consequence of the discontinuity in the data is the formation of an interior layer at $x = d$.

The modified operator obeys a comparison principle too: For any two functions $v, w \in C^2\big((0,d) \cup (d,1)\big) \cap C^1[0,1]$

$$\left.\begin{array}{l} \mathcal{L}v \le \mathcal{L}w \quad \text{in } (0,1), \\ v(0) \le w(0), \\ v(1) \le w(1) \end{array}\right\} \quad \Longrightarrow \quad v \le w \quad \text{on } [0,1].$$

By means of this comparison principle, stability of the operator $\mathcal{L}$ can be established. In particular, we have

$$\|v\|_\infty \le \max\left\{\left\|\frac{\mathcal{L}v}{c}\right\|_\infty, |v(0)|, |v(1)|\right\}$$

for any function $v \in C^2\big((0,d)\cup(d,1)\big)\cap C^1[0,1]$. This is a generalisation of (3.17).

The bounds on the Green's function associated with $\mathcal{L}$ remain unaffected by the discontinuity of the reaction coefficient c.

Estimates for the derivatives are obtained as follows. First note that the maximum-norm stability implies $|v(d)| \le C$. Then the solution of (3.20) is analysed on the two subdomains $(0,d)$ and $(d,1)$ separately. On each of these domains we have a problem with continuous data and the results of Sect. 3.3.1.2 apply. We get

$$\left|u^{(k)}(x)\right| \le C\left\{1+\varepsilon^{-k}\left(\mathrm{e}^{-x/\varepsilon}+\mathrm{e}^{-|d-x|/\varepsilon}+\mathrm{e}^{-(1-x)/\varepsilon}\right)\right\},$$

for $x \in (0,d)\cup(d,1)$.

3.3.2 *Systems of Reaction-Diffusion Equations*

We now leave the scalar equation (3.16) and move on to systems of equations of this type: Find $\boldsymbol{u} \in \big(C^2(0,1)\cap C[0,1]\big)^\ell$ such that

$$\boldsymbol{\mathcal{L}u} := -\boldsymbol{E}^2\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \tag{3.21}$$

where $\boldsymbol{E} = \operatorname{diag}(\varepsilon_1,\dots,\varepsilon_\ell)$ and the small parameter ε_k is in $(0,1]$ for $k = 1,\dots,\ell$. We set $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{f} = (f_i)$. Written out in full, (3.21) is

$$\begin{aligned}
-\varepsilon_1^2 u_1'' + a_{11}u_1 + a_{12}u_2 + \cdots + a_{1\ell}u_\ell &= f_1 \ \text{ in } (0,1), \quad u_1(0) = u_1(1) = 0,\\
-\varepsilon_2^2 u_2'' + a_{21}u_1 + a_{22}u_2 + \cdots + a_{2\ell}u_\ell &= f_2 \ \text{ in } (0,1), \quad u_2(0) = u_2(1) = 0,\\
&\ \vdots\\
-\varepsilon_\ell^2 u_\ell'' + a_{\ell 1}u_1 + a_{\ell 2}u_2 + \cdots + a_{\ell\ell}u_\ell &= f_\ell \ \text{ in } (0,1), \quad u_\ell(0) = u_\ell(1) = 0.
\end{aligned}$$

3.3.2.1 Stability

Assume that all entries a_{ij} of the coupling matrix $\boldsymbol{A}$ lie in $C[0,1]$ and that $\boldsymbol{A}$ has positive diagonal entries. Assume likewise that all f_i lie in $C[0,1]$. Our analysis follows that of [104] and is based on the stability properties of Sect. 3.3.1.1 for scalar operators.

For each k, the k^{th} equation of the system (3.21) can be written as

$$-\varepsilon_k^2 u_k'' + a_{kk} u_k = f_k - \sum_{\substack{m=1 \\ m \neq k}}^{\ell} a_{km} u_m.$$

The stability inequality (3.17) and a triangle inequality then yield

$$\|u_k\|_\infty - \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \left\| \frac{a_{km}}{a_{kk}} \right\|_\infty \|u_m\|_\infty \leq \left\| \frac{f_k}{a_{kk}} \right\|_\infty \tag{3.22}$$

Define the $\ell \times \ell$ constant matrix $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{A}) = (\gamma_{km})$ by

$$\gamma_{kk} = 1 \quad \text{and} \quad \gamma_{km} = -\left\| \frac{a_{km}}{a_{kk}} \right\|_\infty \quad \text{for } k \neq m.$$

Suppose that $\boldsymbol{\Gamma}$ is inverse-monotone, i.e., that $\boldsymbol{\Gamma}$ is invertible and

$$\boldsymbol{\Gamma}^{-1} \geq 0; \tag{3.23}$$

this can be verified by using Lemma 3.14. Then (3.22) immediately gives a bound on $\|\boldsymbol{u}\|_\infty$ in terms of the data $\boldsymbol{A}$ and $\boldsymbol{f}$. We obtain the following stability result for the operator $\boldsymbol{\mathcal{L}}$.

Theorem 3.36. *Assume the matrix $\boldsymbol{A}$ has positive diagonal entries. Suppose also that all entries of $\boldsymbol{A}$ lie in $C[0,1]$. Assume that $\boldsymbol{\Gamma}(\boldsymbol{A})$ is inverse-monotone. Then for $k = 1, \ldots, \ell$ one has*

$$\|v_k\|_\infty \leq \sum_{m=1}^{\ell} \left(\boldsymbol{\Gamma}^{-1}\right)_{km} \left\| \frac{(\boldsymbol{\mathcal{L}} \boldsymbol{v})_m}{a_{mm}} \right\|_\infty$$

for any function $\boldsymbol{v} = (v_1, \ldots, v_\ell)^T \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ with $\boldsymbol{v}(0) = \boldsymbol{v}(1) = \mathbf{0}$.

Corollary 3.37. *Under the hypotheses of Theorem 3.36, the boundary value problem (3.21) has a unique solution $\boldsymbol{u}$, and $\|\boldsymbol{u}\|_\infty \leq C \|\boldsymbol{f}\|_\infty$ for some constant C.*

Thus, the operator $\boldsymbol{\mathcal{L}}$ is (L_∞, L_∞) stable, or maximum-norm stable, although in general it is not inverse-monotone — the hypotheses of Theorem 3.36 do not in general imply that (3.21) obeys a comparison principle.

Remark 3.38. The obvious analogue of the stability inequality (3.17) is also valid for scalar reaction-diffusion problems posed in domains Ω lying in $\mathbb{R}^d$ for $d > 1$. Consequently, Theorem 3.36 holds true also for reaction-diffusion systems posed on $\Omega \subset \mathbb{R}^d$ with $d > 1$. ♣

Remark 3.39. The above argument is closely related to Ostrovski's concept of H-matrices [129]. He calls a matrix $A = (a_{ij}) \in C^{\ell,\ell}$ an H-matrix if its **companion matrix**

$$\begin{pmatrix} |a_{11}| & -|a_{12}| & \cdots & -|a_{1\ell}| \\ -|a_{21}| & |a_{22}| & \cdots & -|a_{2\ell}| \\ \vdots & \vdots & \ddots & \vdots \\ -|a_{\ell 1}| & |a_{\ell 2}| & \cdots & -|a_{\ell\ell}| \end{pmatrix}$$

is an M-matrix. ♣

In [18, 58] the coupling matrix $\boldsymbol{A}$ is assumed to be coercive, viz.,

$$\boldsymbol{v}^T \boldsymbol{A}(x)\boldsymbol{v} \geq \mu^2 \boldsymbol{v}^T \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in I\!R^\ell \text{ and } x \in [0,1], \tag{3.24}$$

where μ is some positive constant. The following result from [111], which slightly generalises [159], establishes a connection between (3.23) and (3.24).

Lemma 3.40. *Assume that* $\boldsymbol{A}$ *has positive diagonal entries and that* $\boldsymbol{\Gamma}$ *is inverse-monotone. Then there exists a constant diagonal matrix* $\boldsymbol{D}$ *and a constant* $\alpha > 0$ *such that*

$$\boldsymbol{v}^T \boldsymbol{D}\boldsymbol{A}(x)\boldsymbol{v} \geq \alpha \boldsymbol{v}^T \boldsymbol{v} \quad \textit{for all } \boldsymbol{v} \in I\!R^d,\ x \in [0,1],$$

i.e., the matrix $\boldsymbol{DA}$ *is coercive uniformly in* x.

Proof. As $\boldsymbol{\Gamma}^{-1}$ exists, one can define $\boldsymbol{y}, \boldsymbol{z} \in I\!R^\ell$ by $\boldsymbol{\Gamma y} = \boldsymbol{1}$ and $\boldsymbol{\Gamma}^T \boldsymbol{z} = \boldsymbol{1}$. Then $\boldsymbol{\Gamma}^{-1} \geq 0$ implies that $y_i > 0$ and $z_i > 0$ for $i = 1, \ldots, \ell$. Define the matrix-valued function $\boldsymbol{G} = (g_{ij})$ by $g_{ij}(x) = z_i a_{ij}(x) y_j$ for all i and j. Observe that both $\boldsymbol{G}$ and $\boldsymbol{G}^T$ are strictly diagonally dominant:

$$g_{ii}(x) - \sum_{j \neq i} |g_{ij}(x)| \geq a_{ii}(x) z_i \sum_j \gamma_{ij} y_j = a_{ii}(x) z_i > 0,$$
$$g_{ii}(x) - \sum_{j \neq i} |g_{ji}(x)| \geq a_{ii}(x) y_i \sum_j \gamma_{ji} z_j = a_{ii}(x) y_i > 0.$$

Thus $(\boldsymbol{G} + \boldsymbol{G}^T)/2$ is strictly diagonally dominant and symmetric. Hence, there exists a constant $\beta > 0$ such that

$$\boldsymbol{v}^T \boldsymbol{G}\boldsymbol{v} = \boldsymbol{v}^T \boldsymbol{G}^T \boldsymbol{v} = \boldsymbol{v}^T \frac{\boldsymbol{G} + \boldsymbol{G}^T}{2} \boldsymbol{v} \geq \beta \boldsymbol{v}^T \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in I\!R^\ell \quad \text{and } x \in [0,1].$$

Define the diagonal matrix $\boldsymbol{D} = (d_{ii})$ by $d_{ii} = z_i / y_i$ for all i. Then

$$\boldsymbol{v}^T \boldsymbol{D}\boldsymbol{A}(x)\boldsymbol{v} = \sum_{i,j} d_{ii} a_{ij} v_i v_j = \sum_{i,j} g_{ij} \frac{v_i}{y_i} \frac{v_j}{y_j} \geq \beta \sum_i \left(\frac{v_i}{y_i}\right)^2 \geq \alpha \sum_i {v_i}^2.$$

□

Remark 3.41. Lemma 3.40 remains valid for reaction-diffusion problems posed in $\Omega \subset \mathbb{R}^d$ with $d > 1$. ♣

Remark 3.42. As multiplication on the left by a positive diagonal matrix neither changes the structure of (3.21) nor alters $\boldsymbol{\Gamma}(\boldsymbol{A})$, Lemma 3.40 implies that, without loss of generality, if $\boldsymbol{A}$ has positive diagonal entries, then whenever (3.23) is satisfied, one can assume that (3.24) holds true also. Thus, the hypothesis that (3.24) alone holds true is more general than an assumption that (3.23) is valid, but the only analyses [18, 58] that are based solely on (3.24), are restricted to the special case $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_\ell$; see Sect. 7.2. ♣

3.3.2.2 Derivative Bounds

Let the coupling matrix $\boldsymbol{A}(x)$ be strictly diagonally dominant for all $x \in [0,1]$. Then $\boldsymbol{A}$ has positive diagonal entries and there exists a constant β such that

$$\sum_{\substack{m=1\\ m\neq k}}^{\ell} \left\| \frac{a_{km}}{a_{kk}} \right\|_\infty \le \beta < 1 \quad \text{for } k = 1, \dots, \ell. \tag{3.25}$$

An application of the M-criterion (Lemma 3.14) with a constant test vector $\boldsymbol{e}$ shows that $\boldsymbol{\Gamma}^{-1} \ge 0$. Define $\kappa = \kappa(\beta) > 0$ by

$$\kappa^2 := (1-\beta) \min_{k=1,\dots,\ell} \min_{x\in[0,1]} a_{kk}(x).$$

For arbitrary $\varepsilon \in (0,1]$ and $0 \le x \le 1$, set

$$\mathcal{B}_\varepsilon(x) := \mathrm{e}^{-\kappa x/\varepsilon} + \mathrm{e}^{-\kappa(1-x)/\varepsilon}.$$

For simplicity in our presentation we assume that

$$\varepsilon_1 \ge \varepsilon_2 \ge \cdots \ge \varepsilon_\ell \quad \text{and} \quad \varepsilon_1 \le \frac{\kappa}{4}\,;$$

the first chain of inequalities can always be achieved by renumbering the equations, while the last inequality provides a threshold value for the validity of our analysis.

The next result generalises Theorem 3.35.

Theorem 3.43. *Let $\boldsymbol{A}$ and $\boldsymbol{f}$ be twice continuously differentiable. Assume* (3.25) *holds true. Then the solution $\boldsymbol{u}$ of* (3.21) *can be decomposed as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ are defined by*

$$-\boldsymbol{E}^2\boldsymbol{v}'' + \boldsymbol{A}\boldsymbol{v} = \boldsymbol{f} \quad \text{in } (0,1), \quad \boldsymbol{v}(0) = \boldsymbol{A}(0)^{-1}\boldsymbol{f}(0), \;\; \boldsymbol{v}(1) = \boldsymbol{A}(1)^{-1}\boldsymbol{f}(1),$$

and

$$-\boldsymbol{E}^2\boldsymbol{w}'' + \boldsymbol{A}\boldsymbol{w} = \boldsymbol{0} \quad \textit{in } (0,1), \quad \boldsymbol{w}(0) = -\boldsymbol{v}(0), \;\; \boldsymbol{w}(1) = -\boldsymbol{v}(1).$$

For all $x \in [0,1]$ *and* $k = 1, \dots, \ell$, *the derivatives of* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *satisfy the bounds*

$$\left\|v_k^{(\nu)}\right\|_\infty \le C\big(1 + \varepsilon_k^{2-\nu}\big) \qquad \textit{for } \nu = 0, 1, \dots, 4,$$

$$\left|w_k^{(\nu)}(x)\right| \le C \sum_{m=1}^{k} \varepsilon_m^{-\nu} \mathcal{B}_{\varepsilon_m}(x) \qquad \textit{for } \nu = 0, 1, 2$$

and

$$\left|w_k^{(\nu)}(x)\right| \le C\varepsilon_k^{2-\nu} \sum_{m=1}^{\ell} \varepsilon_m^{-2} \mathcal{B}_{\varepsilon_m}(x) \quad \textit{for } \nu = 3, 4.$$

Proof. The proof in [104] is involved and full of technical details that will not be discussed here. For the layer components the argument proceeds via induction for the components w_1, w_2, …, w_ℓ. First, bounds for w_1 are established. Then w_1 is eliminated from the system and considered as an inhomogeneity for the system of the remaining components. Next, w_2 is bounded and subsequently eliminated; etc. □

The bounds of Theorem 3.43 say that each component u_k of the solution $\boldsymbol{u}$ can be written as a sum of a smooth part (whose low-order derivatives are bounded independently of the small parameters) and ℓ overlapping layers, though the full effect of these layers is manifested only in derivatives of order at least 3.

Figure 3.2 displays a typical solution in the case $\ell = 2$. The first plot shows the two components on the entire domain $[0,1]$; all that is apparent is that each component has layers at $x = 0$ and $x = 1$. The second plot is a blow-up of the layer

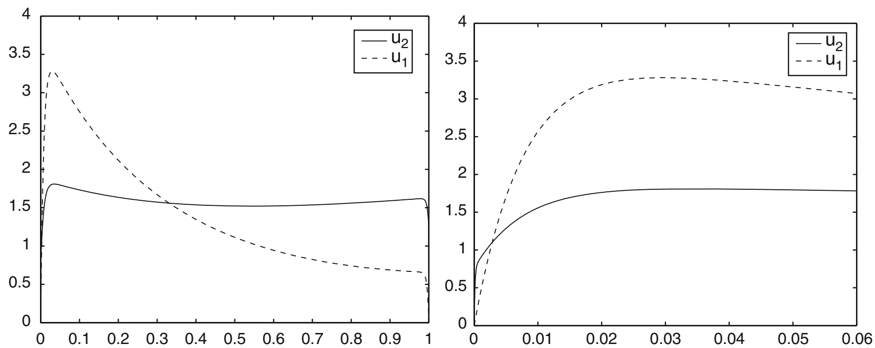


Fig. 3.2 Overlapping layers in a system of two reaction-diffusion equations

at $x = 0$ (the layer at $x = 1$ is similar) and we observe that while u_1 has a standard layer, in u_2 there are two separate layers—exactly as predicted by Theorem 3.43. This theorem also forecasts that u_1 has no such visible behaviour, since interactions between layers in u_1 appear only in the third-order and higher derivatives, and these are not easily noticed on graphs.

3.4 Convection-Diffusion Problems with Regular Layers

Now we consider the linear scalar convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

with $0 < \varepsilon \ll 1$, and its vector-valued counterpart

$$\boldsymbol{\mathcal{L}u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' - \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } (0,1), \;\; \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\ell)^T$ and the ε_k are small parameters.

3.4.1 Scalar Convection-Diffusion Problems

Given $b, c, f \in C[0,1]$, find $u \in C^2(0,1) \cap C[0,1]$ such that

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1, \tag{3.26}$$

where ε is a small positive parameter, $b \geq \beta > 0$ on $[0,1]$.

For the mere sake of simplicity we shall also assume that

$$c \geq 0 \quad \text{and} \quad b' + c \geq 0 \quad \text{on } [0,1]. \tag{3.27}$$

The results hold without these restrictions too, see [5], but the arguments become more complicated and the constants will be slightly different. Note that (3.27) can always be ensured for ε smaller than a certain threshold value ε_0 by a simple transformation $u = \hat{u}e^{\delta x}$ with δ chosen appropriately. This is because $b \geq \beta > 0$.

We shall apply some of the results of Sect. 3.2. This time $\varepsilon_d = \varepsilon$ and $\varepsilon_c = 1$. For the characteristic exponents we have

$$\mu_0 \leq -\frac{\beta}{\varepsilon} \quad \text{and} \quad \mu_1 = \mathcal{O}(1) \quad \text{for } \varepsilon \to 0.$$

Therefore, only one layer at $x = 0$ will be present in the solution.

3.4.1.1 Stability

Using the test function $\psi : x \mapsto \psi(x) = 1 - x$, we see that the differential operator $\mathcal{L}$ in (3.26) satisfies the hypotheses of Lemma 3.3, because $\mathcal{L}\psi \geq \beta > 0$. Consequently, it obeys a comparison principle and in association with Lemma 3.5 we get

$$|u(x)| \leq \max\{|\gamma_0|, |\gamma_1|\} + (1-x)\,\|f/b\|_\infty \quad \text{for } x \in [0,1].$$

This implies

$$\|v\|_\infty \leq \|\mathcal{L}v/b\|_\infty \quad \text{for all } v \in C^2[0,1] \text{ with } v(0) = v(1) = 0. \tag{3.28a}$$

Alternatively, if $c > 0$ on $[0,1]$, then Lemma 3.5 with $\psi \equiv 1$ yields

$$\|v\|_\infty \leq \|\mathcal{L}v/c\|_\infty \quad \text{for all } v \in C^2[0,1] \text{ with } v(0) = v(1) = 0, \tag{3.28b}$$

see also Theorem 3.25.

Remark 3.44. One can try to derive stability results for $\mathcal{L}$ that generalise both (3.28b) and (3.28a). This can be done, for example, by using as test function ψ a general linear or quadratic function φ. But the resulting stability equality will be more complicated and seem to be difficult to use. ♣

Green's function estimates

Theorem 3.20 applies to (3.26) because the general assumption $c \geq 1$ in Sect. 3.2 is not used in the argument leading to Theorem 3.20, but only $c \geq 0$. Note that $\beta^* \geq \beta$, $\mu_0 \leq -\beta/\varepsilon$ and $\mu_1 \geq 0$. We get

$$0 \leq \mathcal{G}(x,\xi) \leq \frac{1}{\beta}\begin{cases} 1 & \text{for } 0 \leq x \leq \xi \leq 1, \\ \mathrm{e}^{-\beta(x-\xi)/\varepsilon} & \text{for } 0 \leq \xi \leq x \leq 1. \end{cases}$$

These lower and upper bounds can also be verified by appealing to Lemma 3.3. The first plot in Fig. 3.1 depicts the typical behaviour of the Green's function for a convection-diffusion problem.

Clearly (3.27) implies that (3.8) is satisfied. Therefore, the derivative of $\mathcal{G}$ possess the sign pattern described in Theorem 3.20. Consequently, (3.10b,c) hold true and can be used to obtain stability estimates for $\mathcal{L}$.

We summarise the results.

Theorem 3.45. *Suppose* (3.27) *holds true. Then the operator* $\mathcal{L}$ *in* (3.26) *satisfies*

$$\|v\|_\infty \leq \min\left\{\left\|\frac{\mathcal{L}v}{b}\right\|_\infty, \left\|\frac{\mathcal{L}v}{c}\right\|_\infty\right\} \quad \textit{for all } v \in \mathring{W}^{1,\infty}(0,1) \cap W^{2,\infty}(0,1), \tag{3.29a}$$

$$\|v\|_\infty \le \frac{1}{\beta}\|\mathcal{L}v\|_1 \quad \textit{and} \quad \|v'\|_1 \le \frac{2}{\beta}\|\mathcal{L}v\|_1 \qquad \textit{for all} \;\; v \in \mathring{W}^{1,1}(0,1) \cap W^{2,1}(0,1) \tag{3.29b}$$

and

$$|||v|||_{\varepsilon,\infty} \le 2\|\mathcal{L}v\|_{-1,\infty} \quad \textit{for all} \;\; v \in \mathring{W}^{1,\infty}(0,1), \tag{3.29c}$$

with

$$|||v|||_{\varepsilon,\infty} := \max\{\varepsilon\|v'\|_\infty, \beta\|v\|_\infty\}.$$

Remark 3.46. Note that in view of Remark 3.22, the $(L_\infty, W^{-1,\infty})$-stability (3.29c) is the strongest of the three stability inequalities of Theorem 3.45. It was first given by Andreev and Kopteva [11] and later analysed in more detail by Andreev [6]. It implies that the operator $\mathcal{L}$ is also uniformly (L_∞, L_1)-stable and (L_∞, L_∞)-stable, i. e. with stability constants that are independent of ε. ♣

Remark 3.47. The same stability results hold true for the differential operator in conservative form, i. e., $\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu$. ♣

3.4.1.2 Derivative Bounds and Solution Decomposition

The boundary value problem (3.26) has a unique solution that typically has an exponential boundary layer at $x = 0$: u and its derivatives up to an arbitrary prescribed order q can be bounded by

$$\left|u^{(k)}(x)\right| \le C\left\{1 + \varepsilon^{-k}\mathrm{e}^{-\beta x/\varepsilon}\right\} \quad \text{for} \;\; k = 0, 1, \dots, q \quad \text{and} \quad x \in [0,1], \tag{3.30}$$

where the maximal order q depends on the smoothness of the data, see [61].

On a number of occasions, e. g. for the error analysis of a finite difference scheme in Sect. 4.2.6 or of the FEM in Sect. 5.2, we need more detailed information on u and its derivatives. In particular, a splitting of u into a regular solution component and a boundary layer component will be required. This decomposition will be derived now.

Defining $\mathcal{L}_0 v := -bv' + cv$, we follow [89] and construct the decomposition as follows. Let v and w be the solution of the boundary-value problems

$$\mathcal{L}v = f \;\; \text{in} \;\; (0,1), \;\; (\mathcal{L}_0 v)(0) = f(0), \;\; v(1) = \gamma_1 \tag{3.31a}$$

and

$$\mathcal{L}w = 0 \;\; \text{in} \;\; (0,1), \;\; w(0) = \gamma_0 - v(0), \;\; w(1) = 0. \tag{3.31b}$$

First we study the regular solution component v. The operator $\mathcal{L}$ equipped with mixed Robin and Dirichlet boundary conditions satisfies the following comparison principle [131]. For any two functions $z, \bar{z} \in C[0,1] \cap C^1[0,1) \cap C^2(0,1)$

$$\left.\begin{array}{r}\mathcal{L}z \le \mathcal{L}\bar{z} \quad \text{in} \;\; (0,1), \\ (\mathcal{L}_0 z)(0) \le (\mathcal{L}_0 \bar{z})(0), \\ z(1) \le \bar{z}(1)\end{array}\right\} \quad \Longrightarrow \quad z \le \bar{z} \quad \text{on} \;\; [0,1].$$

Using this comparison principle with the barrier functions $\bar{v}$ and $-\bar{v}$ defined by

$$\bar{v}(x) := \beta^{-1}\|f\|_\infty(1-x) + |\gamma_1| \quad \text{for} \;\; x \in [0,1],$$

we get

$$|v(x)| \le \bar{v} \le C \quad \text{for} \;\; x \in [0,1].$$

To derive bounds on the derivatives of v, set $h := f - cu$ and write v as

$$v(x) = \int_x^1 H_v(s)\mathrm{d}s + \frac{h(0)}{b(0)} \int_x^1 \mathrm{e}^{-B(s)}\mathrm{d}s + \gamma_1,$$

where

$$B(x) := \frac{1}{\varepsilon}\int_0^x b(s)\mathrm{d}s \;\; \text{and} \;\; H_v(x) := \frac{1}{\varepsilon}\int_0^x h(s)\mathrm{e}^{B(s)-B(x)}\mathrm{d}s.$$

Differentiating once, we get

$$v'(x) = -H_v(x) - \frac{h(0)}{b(0)}\mathrm{e}^{-B(x)}$$

which gives

$$|v'(x)| \le C \;\; \text{for} \;\; x \in [0,1]$$

because

$$H_v(x) \le \frac{C}{\varepsilon}\int_0^x \mathrm{e}^{\beta(s-x)/\varepsilon}\mathrm{d}s = \frac{C}{\beta}\left(1 - \mathrm{e}^{-\beta x/\varepsilon}\right) \le C. \tag{3.32}$$

Invoking the differential equation we get

$$|v''(x)| \le C\varepsilon^{-1} \;\; \text{for} \;\; x \in [0,1].$$

However, if $b, f \in C^1(0,1)$ then integration by parts and the boundary condition imposed on v at $x = 0$ yield

$$v''(x) = -\frac{b(x)}{\varepsilon} \int_0^x \left(\frac{h}{b}\right)'(s) \mathrm{e}^{B(s)-B(x)} \mathrm{d}s,$$

from which the sharper estimate

$$|v''(x)| \le C \quad \text{for} \quad x \in [0,1]$$

can be derived using (3.32). A bound for the third-order derivative is readily obtained from the differential equation and the bounds on v' and v'':

$$|v'''(x)| \le C\varepsilon^{-1} \quad \text{for} \quad x \in [0,1].$$

This completes our analysis of the regular component v of u.

Now consider the boundary-layer term w. Corollary 3.4 with the barrier function

$$\bar{w} : [0,1] \to \mathbb{R} : x \mapsto \bar{w}(x) := |\gamma_0 - v(0)| \, \mathrm{e}^{-\beta x/\varepsilon}$$

yields

$$|w(x)| \le C\mathrm{e}^{-\beta x/\varepsilon} \quad \text{for} \quad x \in [0,1]. \tag{3.33}$$

To bound the derivatives of w we use the fact that

$$w(x) = \int_x^1 H_w(s)\mathrm{d}s + \kappa \int_x^1 \mathrm{e}^{-B(s)}\mathrm{d}s$$

with

$$H_w(x) = -\frac{1}{\varepsilon} \int_0^x (bw)(s)\mathrm{e}^{B(s)-B(x)}.$$

Estimates for H_w are obtained using (3.33)

$$|H_w(x)| \le \frac{C}{\varepsilon} \int_0^x \mathrm{e}^{-\beta s/\varepsilon} \mathrm{e}^{B(s)-B(x)} \mathrm{d}s \le \frac{C}{\varepsilon} \exp(-\beta x/\varepsilon).$$

The coefficient κ is determined by the boundary condition for $w(0)$:

$$\kappa = \frac{1}{\alpha}\left(\gamma_0 - v(0) - \int_0^1 \vartheta_w(s)\mathrm{d}s\right),$$

where

$$\alpha = \int_0^1 \mathrm{e}^{-B(s)}\mathrm{d}s \geq \int_0^1 \mathrm{e}^{-\|b\|_\infty s/\varepsilon}\mathrm{d}s \geq \frac{\varepsilon}{\|b\|_\infty}.$$

Thus

$$|\kappa| \leq C\varepsilon^{-1}.$$

For w' we have

$$w'(x) = -H_w(x) - \kappa\mathrm{e}^{-B(x)}$$

and therefore

$$|w'(x)| \leq C\varepsilon^{-1}\mathrm{e}^{-\beta x/\varepsilon} \text{ for } x \in [0,1],$$

by the above bounds for κ and H_w.

Use the differential equation and the estimates for w and w', to get

$$|w''(x)| \leq C\varepsilon^{-2}\mathrm{e}^{-\beta x/\varepsilon} \text{ for } x \in [0,1].$$

If $b, c \in C^1(0,1)$ then differentiate (3.31b) once and apply the bounds for w, w' and w''. Thus

$$|w'''(x)| \leq C\varepsilon^{-3}\mathrm{e}^{-\beta x/\varepsilon} \text{ for } x \in [0,1].$$

We summarise the results.

Theorem 3.48. *Let $b, c, f \in C^k[0,1]$ with $k \in \{0,1\}$. Then $u \in C^{k+2}[0,1]$ can be decomposed as $u = v+w$, where the regular solution component v satisfies $\mathcal{L}v = f$ in $(0,1)$ and*

$$\left|v^{(i)}(x)\right| \leq C\big(1 + \varepsilon^{k+1-i}\big) \textit{ for } i = 0,1,\ldots,k+2, \ x \in [0,1], \tag{3.34a}$$

while the boundary layer component w satisfies $\mathcal{L}w = 0$ in $(0,1)$ and

$$\left|w^{(i)}(x)\right| \leq C\varepsilon^{-i}\mathrm{e}^{-\beta x/\varepsilon} \textit{ for } i = 0,1,\ldots,k+2, \ x \in [0,1]. \tag{3.34b}$$

Remark 3.49. A similar decomposition is given in [30,121]. However, the construction there requires more smoothness of the data of the problem because the regular solution component v is defined via solutions of first-order problems. ♣

Remark 3.50. Some applications, e. g., the analysis of higher-order schemes in Sect. 4.3.1.3 or [153] or of extrapolation schemes [124], require decompositions with bounds for higher-order derivatives. To derive those, note that the boundary condition $(-bv' + cv)(0) = f(0)$ imposed on v corresponds to $v''(0) = 0$. In order to prove Theorem 3.48 for $k = 2$, we would impose the boundary condition

$$\Big(-\left(b - \varepsilon(b' - c)\right)v' + \left(c - \varepsilon c'\right)v\Big)(0) = \left(f - \varepsilon f'\right)(0)$$

instead. This corresponds to setting $v'''(0) = 0$. The operator $\mathcal{L}$ with this boundary condition satisfies a comparison principle too, provided that ε is smaller than a certain threshold value ε_0. This principle implies the boundedness of v. Then proceed as above to get bounds for the derivatives. ♣

3.4.1.3 Discontinuous Coefficients and Point Sources

Consider the convection-diffusion problem in conservative form with a point source:

$$\begin{aligned} \mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f + \alpha\delta_d, \quad \text{in } (0,1), \\ u(0) = \gamma_0, \;\; u(1) = \gamma_1, \end{aligned} \tag{3.35}$$

where δ_d is the shifted Dirac-delta function $\delta_d(x) = \delta(x - d)$ with $d \in (0,1)$. The coefficient b may also have a discontinuity at $x = d$. Assume that $b \geq \beta_1 > 0$ in $(0,d)$ and $b \geq \beta_2 > 0$ in $(d,1)$. The argument follows [88].

Problem (3.35) has to be read in a distributional context. Alternatively, one may seek a solution $u \in C[0,1] \cap C^2((0,d) \cup (d,1))$ with

$$\mathcal{L}^c u = f \quad \text{in } (0,1) \setminus \{d\}, \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1, \;\; -\left(\varepsilon[u'] + [b]u\right)(d) = \alpha,$$

where $[v](d) := v(d+0) - v(d-0)$ is the jump of v at d.

The solution of (3.35) typically has an exponential boundary layer at the outflow boundary $x = 0$ and an internal layer at $x = d$ caused by the concentrated source or the discontinuity of the convective field. Figure 2.5 depicts a typical solution.

Using the negative-norm stability (3.29c), we obtain $\|u\|_\infty \leq C$. Then, u solves

$$\mathcal{L}u = f \quad \text{in } (0,d), \; u(0) = 0, \; u(d) = \rho,$$

and

$$\mathcal{L}u = f \quad \text{in } (d,1), \; u(d) = \rho, \; u(1) = 0$$

where ρ is uniformly bounded in ε, i.e., $|\rho| \leq C$.

Next apply the results from Sect. 3.4.1.2 separately on each of the two subdomains. We obtain

$$\left|u^{(k)}(x)\right| \leq C\left[1 + \varepsilon^{-k}\left\{\mathrm{e}^{-\beta_1 x/\varepsilon} + H_d(x)\mathrm{e}^{-\beta_2(x-d)/\varepsilon}\right\}\right] \quad \text{for } x \in (0,d) \cup (d,1) \tag{3.36}$$

where H_d denotes the shifted Heaviside function, i.e.,

$$H_d(x) = \begin{cases} 0 & \text{for } x < d, \\ 1 & \text{for } x > d. \end{cases}$$

Note that (3.36) holds for the one-sided derivatives at $x = d$ too.

3.4.2 Weakly Coupled Systems of Convection-Diffusion Equations

We now leave the scalar convection-diffusion equation and move on to systems of equations this type.

The system is said to be **weakly coupled** if the convective coupling matrix $\boldsymbol{B}$ is diagonal, so the system is coupled only through the lower-order reaction terms. We follow [96]. In one dimension such systems can be written as

$$\begin{aligned} \boldsymbol{\mathcal{L}u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' - \operatorname{diag}(\boldsymbol{b})\boldsymbol{u}' + \boldsymbol{Au} = \boldsymbol{f} \quad \text{on } (0,1), \\ \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \end{aligned} \tag{3.37}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\ell)^T$ and the small parameter ε_k is in $(0,1]$ for $k = 1, \ldots, \ell$. For $k = 1, \ldots, \ell$, the k^{th} equation of (3.37) is

$$-\varepsilon_k u_k'' - b_k u_k' + \sum_{j=1}^{\ell} a_{kj} u_j = f_k \quad \text{on } (0,1), \quad u_k(0) = u_k(1) = 0. \tag{3.38}$$

Assume that for each k one has $a_{kk} \geq 0$ and either $b_k \geq \beta_k$ or $b_k \leq -\beta_k$ on $[0,1]$ with positive constants β_k.

3.4.2.1 Stability

We follow the argument of [96, 111]. Rewrite (3.38) as

$$-\varepsilon_k u_k'' - b_k u_k' + a_{kk} u_k = -\sum_{\substack{m=1 \\ m \neq k}}^{\ell} a_{km} u_m + f_k. \tag{3.39}$$

Then (3.28) yields

$$\|u_k\|_\infty + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \tilde{\gamma}_{km} \|u_m\|_\infty \leq \min \left\{ \left\| \frac{f_k}{a_{kk}} \right\|_\infty, \left\| \frac{f_k}{b_k} \right\|_\infty \right\} \quad \text{for } i = 1, \ldots, \ell,$$

where the $\ell \times \ell$ constant matrix $\tilde{\boldsymbol{\Gamma}} = \tilde{\boldsymbol{\Gamma}}(\boldsymbol{A}, \boldsymbol{b}) = (\tilde{\gamma}_{km})$ is defined by

$$\tilde{\gamma}_{kk} = 1 \quad \text{and} \quad \tilde{\gamma}_{km} = -\min\left\{ \left\| \frac{a_{km}}{a_{kk}} \right\|_\infty, \left\| \frac{a_{km}}{b_k} \right\|_\infty \right\} \quad \text{for } k \neq m.$$

Repeating the analysis of Sect. 3.3.2.1, we reach the following stability result.

Theorem 3.51. *Assume that the matrix $\boldsymbol{A}$ has non-negative diagonal entries. Suppose also that all entries of $\boldsymbol{A}$ lie in $C[0,1]$. Assume that $\tilde{\boldsymbol{\Gamma}}(\boldsymbol{A}, \boldsymbol{b})$ is inverse-monotone. Then for $k = 1, \dots, \ell$ one has*

$$\|v_k\|_\infty \le \sum_{m=1}^{\ell} \left(\tilde{\boldsymbol{\Gamma}}^{-1}\right)_{km} \min\left\{ \left\| \frac{(\boldsymbol{\mathcal{L}} \boldsymbol{v})_m}{a_{mm}} \right\|_\infty, \left\| \frac{(\boldsymbol{\mathcal{L}} \boldsymbol{v})_m}{b_m} \right\|_\infty \right\}$$

for any function $\boldsymbol{v} = (v_1, \dots, v_\ell)^T \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ with $\boldsymbol{v}(0) = \boldsymbol{v}(1) = \mathbf{0}$. In this inequality the first term in $\min\{\dots\}$ should be omitted if it does not exist.

Corollary 3.52. *Under the hypotheses of Theorem 3.51 the boundary value problem (3.37) has a unique solution $\boldsymbol{u}$, and $\|\boldsymbol{u}\|_\infty \le C \|\boldsymbol{f}\|_\infty$ for some constant C.*

Remark 3.53. When bounding the source term, one can use (3.29c) to establish that

$$\|\boldsymbol{u}\|_\infty \le C \max_{m=1,\dots,\ell} \|f_m\|_{-1,\infty}\,.$$

This inequality allows the right-hand side to be a generalised function like the δ-distribution. ♣

3.4.2.2 Bounds on Derivatives

Applying the scalar-equation analysis of Sect. 3.4.1.2 to (3.39), it is shown in [96] that for $k = 1, \dots, \ell,\ x \in [0,1]$ and $\nu = 0, 1$ one has

$$\left| u_k^{(\nu)}(x) \right| \le C \begin{cases} 1 + \varepsilon_k^{-\nu} \mathrm{e}^{-\beta_k (1-x)/\varepsilon_k} & \text{if } b_k < 0, \\ 1 + \varepsilon_k^{-\nu} \mathrm{e}^{-\beta_k x/\varepsilon_k} & \text{if } b_k > 0. \end{cases}$$

Thus, there are boundary layers in the solution at $x = 0$ and/or $x = 1$, but no strong interaction between the layers in different components u_k is apparent when first-order derivatives are considered, which is quite unlike the reaction-diffusion case of Theorem 3.43. These bounds on the u_k' also reveal a lack of sharpness in the results of some papers published previously to [96] that gave derivative bounds for this problem with stronger interactions between different components.

3.4.3 Systems of Strongly Coupled Convection-Diffusion Equations

We now return to the general problem with **strong coupling**: this means that for each k one has $b_{km} \not\equiv 0$ for some $k \neq m$. The analysis follows [100]. The boundary-value problem under consideration is: Find $\boldsymbol{u} \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ such that

$$\begin{aligned} \boldsymbol{\mathcal{L}u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' - \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } (0,1), \\ \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \end{aligned} \tag{3.40}$$

where as before $\boldsymbol{f} : [0,1] \to I\!R^\ell$ is vector-valued, while $\boldsymbol{A}, \boldsymbol{B} : [0,1] \to I\!R^{\ell,\ell}$ are matrix-valued functions. The $\ell \times \ell$ matrix $\operatorname{diag}(\boldsymbol{\varepsilon})$ is diagonal with k^{th} entry $\varepsilon_k \in (0,1]$ for all k.

Assume for each k there exists a positive constant β_k such that

$$b_{kk} \geq \beta_k \quad \text{or} \quad b_{kk} \leq -\beta_k, \quad \text{on} \quad [0,1]. \tag{3.41a}$$

Furthermore let

$$a_{kk} \geq 0 \quad \text{and} \quad b'_{kk} + a_{kk} \geq 0 \quad \text{on} \quad [0,1]. \tag{3.41b}$$

Rewrite the k^{th} equation of the system (3.40) as

$$\begin{aligned} \mathcal{L}_k u_k &:= -\varepsilon_k u_k'' - b_{kk} u_k' + a_{kk} u_k \\ &= f_k + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \left[(b_{km} u_m)' - (b'_{km} + a_{km}) u_m \right] \end{aligned} \tag{3.42a}$$

with boundary conditions

$$u_k(0) = u_k(1) = 0. \tag{3.42b}$$

Apply the stability bound of (3.29c) and Remark 3.22 to obtain

$$|||u_k|||_{\varepsilon_k,\infty} \leq \sum_{\substack{m=1 \\ m \neq k}}^{\ell} \left\{ 2\left\| b_{km} \right\|_\infty + \left\| b'_{km} + a_{km} \right\|_1 \right\} \left\| u_m \right\|_\infty + 2\left\| f_k \right\|_{-1,\infty},$$

where

$$|||v|||_{\varepsilon_k,\infty} := \max\left\{ \varepsilon_k \|v'\|_\infty, \beta_k \|v\|_\infty \right\}$$

Then, after some minor manipulation, gather the u_j terms to the left-hand side in the manner of Sect. 3.3.2.1. Define the $\ell \times \ell$ matrix $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}(\boldsymbol{A}, \boldsymbol{B}) = (\gamma_{km})$ by

$$\gamma_{kk} = 1 \quad \text{and} \quad \gamma_{km} = -\frac{2\,\|b_{km}\|_\infty + \|b'_{km} + a_{km}\|_1}{\beta_k} \quad \text{for } k \neq m.$$

We get

$$|||u_k|||_{\varepsilon_k,\infty} + \sum_{\substack{m=1\\ m\neq k}}^{\ell} \gamma_{km}\, |||u_j|||_{\varepsilon_m,\infty} \leq 2\,\|f_k\|_{-1,\infty}\,.$$

Theorem 3.54. *Let $\boldsymbol{B}$ and $\boldsymbol{A}$ satisfy* (3.41)*. Suppose $\boldsymbol{\Upsilon}$ is inverse-monotone. Then the operator $\boldsymbol{\mathcal{L}}$ is $(L_\infty, W^{-1,\infty})$-stable with*

$$|||u_k|||_{\varepsilon_k,\infty} \leq 2 \sum_{m=1}^{\ell} \left(\boldsymbol{\Upsilon}^{-1}\right)_{km} \|(\boldsymbol{\mathcal{L}u})_m\|_{-1,\infty} \quad \text{for } k = 1, \dots, \ell.$$

Corollary 3.55. *Under the hypotheses of Theorem 3.54 the boundary value problem* (3.40) *possesses a unique solution $\mathbf{u}$, and*

$$|||\boldsymbol{u}|||_{\boldsymbol{\varepsilon},\infty} := \max_{k=1,\dots,\ell} |||u_k|||_{\varepsilon_k,\infty} \leq C \max_{k=1,\dots,\ell} \|f_k\|_{-1,\infty}$$

for some constant C. In view of Remark (3.22) *this implies the bounds*

$$\|\boldsymbol{u}\|_\infty \leq C\|\boldsymbol{f}\|_1 \quad \textit{and} \quad \|\boldsymbol{u}\|_\infty \leq C\|\boldsymbol{f}\|_\infty.$$

3.4.3.1 Derivative Bounds

Application of (3.29b) to (3.42) gives

$$\beta_k\, \|u'_k\|_1 \leq 2\left[\|f_k\|_1 + \sum_{\substack{m=1\\ k\neq k}}^{\ell} \left\{\|b_{km}\|_\infty\, \|u'_m\|_1 + \|a_{km}\|_1\, \|u_m\|_\infty\right\}\right].$$

Recalling the definition of the matrix $\boldsymbol{\Upsilon}$, we get

$$\begin{aligned}\beta_k\, \|u'_k\|_1 + \sum_{\substack{m=1\\ m\neq k}}^{\ell} \gamma_{km}\beta_m\, \|u'_m\|_1 &\leq 2\left[\|f_k\|_1 + \sum_{\substack{m=1\\ m\neq k}}^{\ell} \|a_{km}\|_1\, \|u_m\|_\infty\right] \\ &\leq 2\,\|f_k\|_1 + C\,\|\boldsymbol{u}\|_\infty\,,\end{aligned}$$

by Theorem 3.54. Thus, if $\boldsymbol{\Upsilon}$ is inverse monotone we get a uniform bound for the L_1 norm of $\boldsymbol{u}'$.

Theorem 3.56. *Let* $\boldsymbol{B}$ *and* $\boldsymbol{A}$ *satisfy* (3.41). *Suppose* $\boldsymbol{\Upsilon}$ *is inverse monotone. Then the solution* $\boldsymbol{u}$ *of* (3.40) *satisfies* $\|\boldsymbol{u}'\|_1 \le C\,\|\boldsymbol{f}\|_1$.

The L_1-norm bounds of Theorem 3.56 are sufficient to prove the existence of a mesh that yields robust uniform convergence for a first-order upwind difference scheme, see Sect. 4.4.2. However, in order to design layer-adapted meshes, a priori pointwise bounds are required. These are typically obtained by a splitting of the solution into layer parts and a regular part, that is independent of the perturbation parameter and captures the behaviour of the solution away from any layers; see e.g. Theorem 3.48.

To illustrate the difficulties in obtaining such a decomposition let us consider the following example of (3.40):

$$\begin{aligned} -\varepsilon_1 u_1'' + u_1' - u_2' &= 1, \quad u_1(0) = u_1(1) = 0, \\ -\varepsilon_2 u_2'' \qquad - 3u_2' &= 2, \quad u_2(0) = u_2(1) = 0. \end{aligned} \tag{3.43}$$

Figure 3.3 displays its solution for fixed $\varepsilon_2 = 10^{-4}$ and three different values of ε_1. Obviously there does not exist a regular solution component that is independent of the perturbation parameters, nor can there (in general) be anything like *the* solution of the reduced problem for strongly coupled convection-diffusion.

The special case where all diffusion coefficients ε_k are the same is studied in [127]. A lengthy analysis with some further assumptions on the data of the problem leads to the following result:

Theorem 3.57. *Suppose that* $\varepsilon_k = \varepsilon$ *for* $k = 1, \dots, \ell$. *Then the solution* $\boldsymbol{u}$ *of* (3.40) *can be decomposed as* $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ *where*

$$\left\|\boldsymbol{v}^{(\nu)}\right\|_\infty \le C\left(1 + \varepsilon^{2-\nu}\right) \quad \textit{for} \;\; \nu = 0, 1, 2, 3,$$

and for $x \in [0,1]$ *one has*

$$\left|w_k^{(\nu)}(x)\right| \le C\varepsilon^{-\nu} \mathrm{e}^{-\beta x/\varepsilon} \quad \textit{for} \;\; \nu = 0, 1, 2, 3 \quad \textit{and} \;\; k = 1, \dots, \ell.$$

with $\beta := \min\{\beta_1, \dots, \beta_\ell\}$.

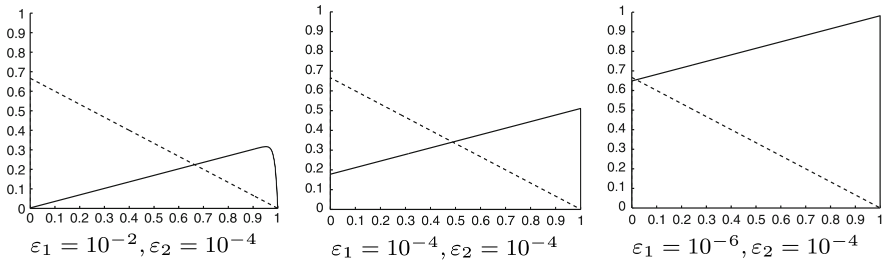


Fig. 3.3 Solution of (3.43) for different values of $\boldsymbol{\varepsilon}$ (u_2 dashed)

3.5 Convection-Diffusion Problems with Turning-Point Layers

Turning point layers are associated with zeros of the convection coefficient. Let us consider the boundary-value problem

$$-\varepsilon u'' - pbu' + c(\cdot, u) = 0 \text{ in } (q,1), \ u(q) = \gamma_q, \ u(1) = \gamma_1$$

with $q \in \{-1, 0\}$. We assume that $b(x) \geq \beta > 0$, $\partial_u c \geq 0$ and $\operatorname{sign} p(x) = \operatorname{sign} x$ for $x \in (q,1)$. The assumption on p implies that the point $x = 0$ is a turning point. If $q = 0$, then the turning point coincides with a boundary; we call this a boundary turning point problem. When $q = -1$ we have an interior layer.

Turning point problems with $\partial_u c(0,\cdot) > 0$ were considered in a number of papers in the 1980s. For interior turning point problems this additional assumption implies that the solution of the reduced problem is continuous and therefore, no strong layer is present. This means the problem is not singularly perturbed in the maximum norm. For boundary turning points, the situation is different since the solution of the reduced problem will in general not match the boundary condition prescribed at the outflow boundary. However, if $\partial_u c(0,\cdot) > 0$, then the dominating feature of the problem is the relation between the diffusion and reaction terms and the problem has the character of a reaction-diffusion problem. Here we restrict ourselves to the case $\partial_u c(0,\cdot) = 0$.

We shall study the linear convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - pbu' + pcu = pf \quad \text{in } (0,1), \quad u(0) = \gamma_0, \ u(1) = \gamma_1, \tag{3.44}$$

where $p, b, c, f \in C^1[0,1]$ with

$$p > 0, \quad p' > 0, \quad b \geq \beta > 0 \quad \text{and} \quad c \geq 0 \quad \text{in } (0,1). \tag{3.45}$$

Particular focus will be on the case $p(x) = x^\kappa$, $\kappa > 0$.

3.5.1 Stability

Suppose (3.45) holds. Then using the test function $\psi : x \mapsto \psi(x) = 1 - x$, we see that the differential operator $\mathcal{L}$ in (3.44) satisfies the hypotheses of Lemma 3.3, because $\mathcal{L}\psi \geq pb > 0$ in $(0,1)$. Therefore, $\mathcal{L}$ obeys a comparison principle and we have

$$\|v\|_\infty \leq \|\mathcal{L}v/pb\|_\infty \quad \text{for all } v \in C^2[0,1] \text{ with } v(0) = v(1) = 0,$$

by Lemma 3.5. If $c > 0$ on $[0,1]$, then Lemma 3.5 with $\psi \equiv 1$ yields also

$$\|v\|_\infty \le \|\mathcal{L}v/pc\|_\infty \quad \text{for all } v \in C^2[0,1] \text{ with } v(0) = v(1) = 0.$$

Thus, $\mathcal{L}$ is $(L_\infty, L_{\infty,1/p})$-stable with a $1/p$-weighted maximum norm.

Furthermore, we have for the solution of (3.44)

$$|u|_\infty \le \max\Big\{|\gamma_0|, |\gamma_1|\Big\} + \min\left\{\left\|\frac{f}{b}\right\|_\infty, \left\|\frac{f}{c}\right\|_\infty\right\}.$$

Green's function estimates

Lemma 3.58. *Assume* (3.45) *holds true. Then*

$$0 \le \mathcal{G}(x,\xi) \le \hat{\mathcal{G}}(x,\xi) := \frac{1}{p(\xi)\beta}\begin{cases} 1 & \textit{for } 0 \le x \le \xi \le 1, \\ \exp\left(-\frac{\beta}{\varepsilon}\int_\xi^x p(s)\mathrm{d}s\right) & \textit{for } 0 \le \xi \le x \le 1. \end{cases}$$

Proof. First note that the operator $\mathcal{L}$ satisfies the assumptions of Lemma 3.11 which we like to apply.

Let $\xi \in (0,1)$ be arbitrary, but fixed. Clearly $\hat{\mathcal{G}}(0,\xi) \ge 0$ and $\hat{\mathcal{G}}(1,\xi) \ge 0$. Furthermore,

$$\partial_x \hat{\mathcal{G}}(x,\xi) = \begin{cases} 0 & \text{for } 0 < x < \xi, \\ -\frac{\beta p(x)}{\varepsilon}\hat{\mathcal{G}}(x,\xi) & \text{for } \xi < x < 1. \end{cases}$$

and

$$\partial_x^2 \hat{\mathcal{G}}(x,\xi) = \begin{cases} 0 & \text{for } 0 \le x < \xi \\ -\frac{\beta}{\varepsilon}\left(\frac{\beta p(x)}{\varepsilon} - p'(x)\right)\hat{\mathcal{G}}(x,\xi) & \text{for } \xi < x \le 1. \end{cases}$$

Thus,

$$\mathcal{L}\hat{\mathcal{G}}(\cdot,\xi) \ge pc\hat{\mathcal{G}}(\cdot,\xi) \ge 0 \quad \text{in } (0,\xi) \cup (\xi,1),$$

because $p' \ge 0$ on $[0,1]$. The jump of $\partial_x \hat{\mathcal{G}}$ satisfies

$$-\varepsilon\left[\partial_x \hat{\mathcal{G}}(\cdot,\xi)\right](\xi) = 1.$$

Thus $\hat{\mathcal{G}}$ is a barrier function for $\mathcal{G}$. Applying Lemma 3.11, we are finished. □

Remark 3.59. By Lemma 3.58

$$\mathcal{G}(x,\xi) \le \frac{1}{\beta p(\xi)} \quad \text{for all } x, \xi \in (0,1).$$

However, numerical experiments indicate that when $p(x) = x^\kappa$, $\kappa \geq 0$, one has the sharper bound

$$\mathcal{G}(x,\xi) \leq C\left(\varepsilon^{1/(\kappa+1)} + \xi\right)^{-\kappa} \quad \text{for all } x,\xi \in (0,1).$$

But, we do not have a rigorous proof for this. ♣

Theorem 3.60. *Suppose* (3.45) *holds true. Then the operator* $\mathcal{L}$ *in* (3.44) *satisfies*

$$\|v\|_\infty \leq \beta^{-1}\,\|(\mathcal{L}v)/p\|_1 \quad \textit{for all } v \in \mathring{W}^{1,1}(0,1) \cap W^{2,1}(0,1).$$

Thus, the operator $\mathcal{L}$ *is* $(L_\infty, L_{1;1/p})$ *with a* $1/p$*-weighted* L_1 *norm.*

Proof. Use the representation (3.1) and Lemma 3.58 to obtain the assertion of the theorem. □

Corollary 3.61. *Suppose* (3.45) *holds true. Then the solution* u *of* (3.44) *satisfies*

$$\|v\|_\infty \leq \max\Big\{|\gamma_0|, |\gamma_1|\Big\} + \beta^{-1}\,\|f\|_1\,.$$

Remark 3.62. In view of the a posteriori error analysis for problems with regular layers (see Sect. 4.2.4) stronger negative-norm stability is desirable. Unfortunately, this is not available yet. ♣

3.5.2 Derivative Bounds and Solution Decomposition

We follow [92] to derive a decomposition of u into a regular solution component v and a layer part w for $p(x) = x^\kappa$ with $\kappa \geq 0$. In doing so, we generalise the analysis of Sect. 3.4.1.2 for problems with regular boundary layers.

Theorem 3.63. *Let* $b, c, f \in C^1[0,1]$ *and* $p(x) = x^\kappa$, $\kappa \geq 0$. *Assume* $b > \tilde{\beta} > 0$ *and* $c \geq 0$ *on* $[0,1]$. *Then* (3.44) *has a unique solution* $u \in C^3[0,1]$ *and this solution can be decomposed as* $u = v + w$*, where the regular solution component* v *satisfies* $\mathcal{L}v = f$ *in* $(0,1)$*,*

$$\|v\|_\infty + \|v'\|_\infty + \|v''\|_\infty \leq C \quad \textit{and} \quad \varepsilon\left|v'''(x)\right| \leq Cx^\kappa \quad \textit{for } x \in [0,1],$$

while the boundary layer component w *satisfies* $\mathcal{L}w = 0$ *in* $(0,1)$ *and*

$$\left|w^{(i)}(x)\right| \leq C\mu^{-i}\exp\left(-\frac{\tilde{\beta}x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \textit{for } i = 0,1,2,3, \;\; x \in (0,1),$$

where $\mu = \varepsilon^{1/(\kappa+1)}$.

Remark 3.64. For $\kappa = 0$, i.e. in the case of a regular boundary layer, this result is known from Theorem 3.48. ♣

Remark 3.65. In [92] the semilinear problem

$$-\varepsilon u'' - pbu' + pc(\cdot, u) = 0 \quad \text{in } (0,1), \quad u(0) = \gamma_0,\ u(1) = \gamma_1,$$

with $p(x) = x^\kappa$ and $\partial_u c \geq 0$ on $[0,1] \times I\!R$ is analysed. The same bounds on derivatives are obtained. ♣

Proof (of Theorem 3.63). The construction of the decomposition is similar to the one in Sect. 3.4.1.2. Set $\mathcal{B}z := -bz' + cz$, Let v and w be the solutions of the boundary-value problems

$$\mathcal{L}v = pf \quad \text{in } (0,1), \quad (\mathcal{B}v)(0) = f(0),\ v(1) = \gamma_1$$

and

$$\mathcal{L}w = 0 \quad \text{in } (0,1),\ w(0) = \gamma_0 - v(0),\ w(1) = 0.$$

Preliminaries

Before starting the main argument, let us provide some auxiliary results. Set

$$B(x) := \frac{1}{\varepsilon} \int_0^x s^p b(s) \mathrm{d}s$$

and let β^* with $b(x) \geq \beta^* > 0$ be arbitrary.

In the analysis bounds for certain integral expressions involving B are required. First,

$$B(s) - B(x) \leq \frac{\beta^*}{\varepsilon} \frac{s^{\kappa+1} - x^{\kappa+1}}{\kappa + 1} \quad \text{for } 0 \leq s \leq x \leq 1. \tag{3.46}$$

From this, for arbitrary $\nu \geq 0$ we get

$$\begin{aligned} &\frac{\beta^*}{\varepsilon} \int_0^x s^{(\kappa+\nu)} \exp(B(s) - B(x)) \mathrm{d}s \\ &\qquad \leq \frac{\beta^*}{\varepsilon} \int_0^x s^\kappa \exp\left(\frac{\beta^*}{\varepsilon} \frac{s^{\kappa+1} - x^{\kappa+1}}{\kappa+1}\right) \mathrm{d}s \leq 1. \end{aligned} \tag{3.47}$$

We shall also use

$$\begin{aligned} \int_0^1 \exp(-B(s)) \mathrm{d}s &\geq \int_0^1 \exp\left(-\frac{\|b\|_\infty s^{\kappa+1}}{(\kappa+1)\varepsilon}\right) \mathrm{d}s \\ &= \mu \int_0^{1/\mu} \exp\left(-\frac{\|b\|_\infty t^{\kappa+1}}{(\kappa+1)}\right) \mathrm{d}t \geq \mu \int_0^1 \exp\left(-\frac{\|a\|_\infty t^{\kappa+1}}{(\kappa+1)}\right) \mathrm{d}t = C\mu. \end{aligned} \tag{3.48}$$

Proposition 3.66. *For arbitrary $\nu > 0$ there exists a constant $C = C(\nu)$ such that*

$$\frac{x^\nu}{\varepsilon}\int_0^x \exp\left(\frac{\beta^*}{\varepsilon}\frac{s^{\nu+1}-x^{\nu+1}}{\nu+1}\right)\mathrm{d}s \le C \quad \textit{for all } \ x \ge 0,\ \varepsilon > 0.$$

Proof. Using the transformations

$$x = \left(\varepsilon(\nu+1)t/\beta^*\right)^{1/(\nu+1)} \quad \text{and} \quad s = \left(\varepsilon(\nu+1)\sigma/\beta^*\right)^{1/(\nu+1)},$$

we see that

$$\frac{\beta^* x^\nu}{\varepsilon}\int_0^x \exp\left(\frac{\beta^*}{\varepsilon}\frac{s^{\nu+1}-x^{\nu+1}}{\nu+1}\right)\mathrm{d}s$$
$$= \mathrm{e}^{-t}t^{\nu/(\nu+1)}\int_0^t \mathrm{e}^{\sigma}\sigma^{-\nu/(\nu+1)}\mathrm{d}\sigma =: F_\nu(t).$$

Clearly $F_\nu \in C^0[0,\infty)$ and $F_\nu(0) = 0$ for $\nu > 0$. Moreover, $\lim_{t\to\infty} F_\nu(t) = 1$. Thus, there exists a constant $C(\nu) > 0$ such that $\|F_\nu\|_\infty \le C(\nu)$. □

The regular solution component

The operator $\mathcal{L}$ satisfies the following comparison principle: For any two functions $z, \bar z \in C[0,1] \cap C^1[0,1) \cap C^2(0,1)$

$$\left.\begin{aligned} \mathcal{L}z &\le \mathcal{L}\bar z \ \text{ in } \ (0,1), \\ (\mathcal{B}z)(0) &\le (\mathcal{B}\bar z)(0), \\ z(1) &\le \bar z(1) \end{aligned}\right\} \quad \Longrightarrow \quad z \le \bar z \ \text{ on } \ [0,1].$$

Application of this principle with

$$v^\pm(x) := \pm\left(\beta^{-1}(1-x)\,\|f\|_\infty + \gamma_1\right)$$

yields

$$\|v\|_\infty \le C.$$

Now let us bound the derivatives of v. The function v can be written as

$$v(x) = \int_x^1 H_v(s)\mathrm{d}s + \frac{h_v(0)}{b(0)}\int_x^1 \exp(-B(s))\mathrm{d}s + \gamma_1,$$

where $h_v := f - cv$ and

$$H_v(x) = \frac{1}{\varepsilon}\int_0^x s^\kappa h_v(s)\exp\bigl(B(s) - B(x)\bigr)\mathrm{d}s \quad \text{for } \ x \in [0,1].$$

From this representation we immediately get

$$v'(x) = -\frac{1}{\varepsilon}\int_0^x s^\kappa h_v(s)\exp\bigl(B(s)-B(x)\bigr)\mathrm{d}s - \frac{h_v(0)}{b(0)}\exp\bigl(-B(x)\bigr). \tag{3.49}$$

Hence,

$$\|v'\| \le C,$$

by (3.47).

Differentiating (3.49) once and using integration by parts, we get

$$v''(x) = -\frac{x^\kappa b(x)}{\varepsilon}\int_0^x \left(\frac{h_v}{b}\right)'(s)\exp\bigl(B(s)-B(x)\bigr)\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\bigl|v''(x)\bigr| &\le C\frac{x^\kappa}{\varepsilon}\int_0^x \exp\bigl(B(s)-B(x)\bigr)\mathrm{d}s \\ &\le C\frac{x^\kappa}{\varepsilon}\int_0^x \exp\left(\frac{\beta}{\varepsilon}\frac{s^{\kappa+1}-x^{\kappa+1}}{\kappa+1}\right)\mathrm{d}s\end{aligned}$$

and

$$|v''| \le C.$$

by Prop. 3.66.

A bound for the third-order derivative is obtained from the differential equation and the bounds on v' and v'':

$$-\varepsilon v''' = x^\kappa\bigl(bv+h_v\bigr)' + \kappa x^{\kappa-1}\bigl(bv'+h_v\bigr).$$

Let $F(x) := bv' + h_v$. Eq. (3.31a) implies $F(0)=0$. On the other hand, we have

$$|F'(x)| = \bigl|\bigl(bv'+h_v\bigr)'(x)\bigr| \le C,$$

by our earlier bounds for v, v' and v''. Thus, $\bigl|F(x)\bigr| \le Cx$. We get

$$\varepsilon\bigl|v'''(x)\bigr| \le Cx^\kappa \quad \text{for } x\in(0,1).$$

This completes our analysis of the regular part of u.

The boundary layer component

Let β_i be arbitrary but fixed constants with

$$\min_{x\in[0,1]} b(x) = \beta_1 > \beta_2 > \beta_3 > \tilde{\beta} > 0.$$

The operator $\mathcal{L}$ with Dirichlet boundary conditions satisfies a comparison principle. With the barrier functions

$$w^{\pm}(x) := \pm|\gamma_0 - v(0)| \exp\left(-\frac{\beta_1}{\varepsilon}\frac{x^{p+1}}{p+1}\right),$$

it gives

$$\left|w(x)\right| \le C\exp\left(-\frac{\beta_1}{\varepsilon}\frac{x^{\kappa+1}}{\kappa+1}\right) \quad \text{for } x \in (0,1). \tag{3.50}$$

To bound the derivatives of w use the representation

$$w(x) = \int_x^1 H_w(s)\mathrm{d}s - \frac{v(0) - \gamma_0 + \int_0^1 H_w(s)\mathrm{d}s}{\int_0^1 \exp(-B(s))\mathrm{d}s}\int_x^1 \exp\bigl(-B(s)\bigr)\mathrm{d}s,$$

where

$$H_w(x) = -\frac{1}{\varepsilon}\int_0^x s^{\kappa}\bigl(cw\bigr)(s)\exp\bigl(B(s) - B(x)\bigr)\mathrm{d}s.$$

Thus,

$$w'(x) = -H_w(x) + \frac{v(0) - \gamma_0 + \int_0^1 H_w(s)\mathrm{d}s}{\int_0^1 \exp\bigl(-B(s)\bigr)\mathrm{d}s}\exp\bigl(-B(x)\bigr). \tag{3.51}$$

Using by (3.50) and then (3.46) with $\beta^* = \beta_1$, we obtain

$$\left|H_w(x)\right| \le C\frac{x^{\kappa+1}}{\varepsilon}\exp\left(-\frac{\beta_1}{\varepsilon}\frac{x^{\kappa+1}}{\kappa+1}\right) \le C\exp\left(-\frac{\beta_2}{\varepsilon}\frac{x^{\kappa+1}}{\kappa+1}\right). \tag{3.52}$$

From (3.47), (3.48), (3.51) and (3.52) we get

$$\left|w'(x)\right| \le C\mu^{-1}\exp\left(-\frac{\beta_2 x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \text{for } x \in (0,1).$$

Use the differential equation, the estimates for w and w' and Prop. 3.66 to get

$$\left|w''(x)\right| \leq C\mu^{-2} \exp\left(-\frac{\beta_3 x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \text{for } x \in (0,1).$$

Differentiate (3.31b), apply the bounds for w, w' and w'' and use Prop. 3.66 again in order to get

$$\left|w'''(x)\right| \leq C\mu^{-3} \exp\left(-\frac{\tilde{\beta} x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \text{for } x \in (0,1).$$

This completes the proof of Theorem 3.63. □

Chapter 4
Finite Difference Schemes for Convection-Diffusion Problems

This chapter is concerned with finite-difference discretisations of the stationary linear convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1, \tag{4.1}$$

with $b \geq \beta > 0$ on $[0,1]$. For the sake of simplicity we shall assume that

$$c \geq 0 \quad \text{and} \quad b' \geq 0 \quad \text{on } [0,1]. \tag{4.2}$$

Using (4.1) as a model problem, a general convergence theory for certain first- and second-order upwinded difference schemes on arbitrary and on layer-adapted meshes is derived. The close relationship between the differential operator and its upwinded discretisations is highlighted.

4.1 Notation

Meshes and mesh functions

Throughout this chapter let $\bar{\omega} : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary partition of $[0,1]$ with mesh intervals $I_i := [x_{i-1}, x_i]$. The set of inner mesh points is denoted by ω. The midpoint of I_i is $x_{i-1/2} := (x_i - x_{i+1})/2$ and its length $h_i := x_i - x_{i-1}$. Let $h := \max_{i=1,\dots,N} h_i$ be the maximal mesh size.

We shall identify mesh functions $v : \bar{\omega} \to I\!R : x_i \mapsto v_i$ with vectors $v \in I\!R^{N+1}$ and with spline functions

$$v \in V^\omega := \mathcal{S}_1^0(\bar{\omega}) := \Big\{ w \in C^0[0,1] : w|_{I_i} \in \Pi_1 \;\text{ for }\; i = 1, \dots, N \Big\}.$$

Let $I\!R_0^{N+1} := \big\{ v \in I\!R^{N+1} : v_0 = v_N = 0 \big\}$ be the space of mesh functions that vanish at the boundary. Furthermore, $V_0^\omega := \mathcal{S}_1^0(\bar{\omega}) \cap H_0^1(0,1)$.

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_4,

Difference operators

In our notation of difference operators we follow Samarski's text book [146]. For any mesh function $v \in I\!R^{N+1}$ set

$$v_{x;i} := \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\bar{x};i} := v_{x;i-1} = \frac{v_i - v_{i-1}}{h_i}, \quad v_{\breve{x};i} := \frac{v_i - v_{i-1}}{h_{i+1}}$$

$$v_{\hat{x};i} := \frac{v_{i+1} - v_i}{\hbar_i}, \quad v_{\check{x};i} := \frac{v_i - v_{i-1}}{\hbar_i}, \quad v_{\mathring{x};i} := \frac{v_{i+1} - v_{i-1}}{2\hbar_i}$$

with the weighted mesh increment $\hbar$ defined by

$$\hbar_0 := \frac{h_1}{2}, \quad \hbar_i := \frac{h_i + h_{i+1}}{2}, \ \ i = 1, \dots, N-1, \quad \text{and} \quad \hbar_N := \frac{h_N}{2}.$$

To simplify the notation we set $g_i := g(x_i)$ for any $g \in C[0,1]$.

Further, less frequently used, difference operators will be introduced when needed.

Discrete norms and inner products

For any mesh function $v \in I\!R^{N+1}$ define the ℓ_∞ (semi-)norms

$$\|v\|_{\infty,\omega} := \max_{i=1,\dots,N-1} |v_i|, \quad \|v\|_{\infty,\bar{\omega}} := \max_{i=0,\dots,N} |v_i|,$$

$$\|[v\|_{\infty,\omega} := \max_{i=0,\dots,N-1} |v_i|, \quad |||v|||_{\varepsilon,\infty,\omega} := \max\Big\{\varepsilon \, \|[v_x\|_{\infty,\omega}, \beta\|v\|_{\infty,\bar{\omega}}\Big\},$$

the ℓ_1 norm

$$\|v\|_{1,\omega} := \sum_{j=0}^{N-1} h_{j+1}|v_j|$$

and the $w^{-1,\infty}$ norm

$$\|v\|_{-1,\infty,\omega} := \min_{V: V_x = v} \|V\|_{\infty,\bar{\omega}} = \min_{c \in I\!R} \Bigg\| \sum_{j=\cdot}^{N-1} h_{j+1} v_j + c \Bigg\|_{\infty,\bar{\omega}}.$$

We shall also use the following discrete inner products:

$$[w, v)_\omega := \sum_{i=0}^{N-1} h_{i+1} w_i v_i \quad \text{and} \quad (w, v)_\omega := \sum_{i=1}^{N-1} h_{i+1} w_i v_i.$$

4.2 A Simple Upwind Difference Scheme

In this section we study a first-order difference scheme for the discretisation of (4.1) on arbitrary meshes. Find $u^N \in I\!R^{N+1}$ such that

$$\left[Lu^N\right]_i = f_i \quad \text{for } i = 1, \dots, N-1, \; u_0^N = \gamma_0, \; u_N^N = \gamma_1 \tag{4.3}$$

with

$$[Lv]_i := -\varepsilon v_{\bar{x}x;i} - b_i v_{x;i} + c_i v_i \quad \text{for } v \in I\!R^{N+1}.$$

At first glance the discretisation of the second-order derivative is a bit non-standard, because on non-uniform meshes it is not consistent in the maximum norm, but it has advantages that become clearer in the course of our analysis. More frequently used is the central difference approximation $u_i'' \approx u_{\bar{x}\hat{x};i}$. An upwind scheme based on this discretisation of the second-order derivative will be studied in Sect. 4.2.6, because the technique used there becomes more important in 2D, see Sect. 9.1.

The difference scheme (4.3) can be generated by a finite-element approach. To this end consider (4.1) with homogeneous boundary conditions. Its weak formulation is: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) = f(v) \quad \text{for all } v \in H_0^1(0,1)$$

with

$$a(w,v) := \varepsilon\,(w',v') - (bw' - cw, v)\,, \quad f(v) := (f,v)$$

and the $L_2(0,1)$-scalar product $(w,v) := \int_0^1 (wv)(s)\mathrm{d}s$.

A standard FEM approximation is: Find $u^N \in V_0^\omega$ such that

$$a(u^N, v) = f(v) \quad \text{for all } v \in V_0^\omega.$$

The integrals in the bilinear form and the linear functional have to approximated. Use the left-sided rectangle rule $\int_{I_i} g(s)\mathrm{d}s \approx h_i g_{i-1}$, to arrive at: Find $u^N \in V_0^\omega$ such that

$$a_u\left(u^N, v\right) = f_u\left(v\right) \quad \text{for all } v \in V_0^\omega, \tag{4.4}$$

where

$$a_u\left(w,v\right) := \varepsilon\left[w_x, v_x\right)_\omega - \left(bw_x - cw, v\right)_\omega \quad \text{and} \quad f_u\left(v\right) := \left(f,v\right)_\omega.$$

Taking as test functions v the standard hat-function basis in V_0^ω, we see that (4.3) and (4.4) are equivalent. In particular,

$$a_u(w,v) = (Lw,v)_\omega = (w,L^*v)_\omega \quad \text{for all } w,v \in V_0^\omega,$$

with the adjoint operator L^* given by

$$[L^*v]_j = -\varepsilon v_{\bar\xi\xi;j} + (bv)_{\breve\xi;j} + c_j v_j.$$

This is verified using summation by parts; cf. [146].

4.2.1 Stability of the Discrete Operator

The matrix associated with the difference operator L is a L_0-matrix because all off-diagonal entries are non-positive. Application of the M-criterion (Lemma 3.14) with a test vector with components $e_i = 2 - x_i$, $i = 0, \ldots, N$ establishes the inverse monotonicity of L. Thus, L satisfies a comparison principle: For any mesh functions $v, w \in I\!R^{N+1}$

$$\left.\begin{aligned} Lv &\le Lw \quad \text{on } \omega, \\ v_0 &\le w_0, \\ v_N &\le w_N \end{aligned}\right\} \quad \Longrightarrow \quad v \le w \quad \text{on } \bar\omega. \tag{4.5}$$

This comparison principle and Lemma 3.17 give the stability inequality

$$\left|u_i^N\right| \le \max\{|\gamma_0|,|\gamma_1|\} + (1-x_i)\,\|f/b\|_{\infty,\omega} \quad \text{for } i = 0,\ldots,N.$$

Thus, the operator L is $(\ell_\infty,\ell_\infty)$-stable with

$$\|v\|_{\infty,\bar\omega} \le \|Lv/b\|_{\infty,\omega} \quad \text{for all } v \in I\!R_0^{N+1}.$$

Alternatively, if $c > 0$ on $[0,1]$, then Lemma 3.17 with $\boldsymbol{e} \equiv 1$ yields

$$\|v\|_{\infty,\bar\omega} \le \|Lv/c\|_{\infty,\omega} \quad \text{for all } v \in I\!R_0^{N+1}.$$

Note the analogy with (3.28).

Green's function estimates

Using the discrete Green's function $G : \bar\omega^2 \to I\!R : (x_i,\xi_j) \mapsto G_{i,j} = G(x_i,\xi_j)$ associated with L and Dirichlet boundary conditions, any mesh function $v \in I\!R_0^{N+1}$ can be represented as

$$v_i = a_u(v, G_{i,\cdot}) = (Lv, G_{i,\cdot})_\omega = (v, L^*G_{i,\cdot})_\omega \quad \text{for } i = 1,\ldots,N-1.$$

Taking for v the standard basis in V_0^ω, we see that for fixed $i = 1, \dots, N-1$

$$[L^* G_{i,\cdot}]_j = \delta_{i,j} \quad \text{for } j = 1, \dots, N-1, \; G_{i,0} = G_{i,N} = 0, \tag{4.6}$$

where

$$\delta_{i,j} := \begin{cases} h_{i+1}^{-1} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

is a discrete equivalent of the Dirac-δ distribution. As function of the first argument G solves, for fixed $j = 1, \dots, N-1$,

$$[LG_{\cdot,j}]_i = \delta_{i,j} \quad \text{for } i = 1, \dots, N-1, \; G_{0,j} = G_{N,j} = 0.$$

Theorem 4.1. *Suppose* (4.2) *holds true. Then the Green's function G associated with the discrete operator L and Dirichlet boundary conditions satisfies*

$$0 \le G_{i,j} \le \frac{1}{\beta} \begin{cases} 1 & \text{for } 0 \le i \le j \le N, \\ \prod_{k=j+1}^{i} \left(1 + \frac{\beta h_{k+1}}{\varepsilon}\right)^{-1} & \text{for } 0 \le j < i \le N, \end{cases}$$

$$G_{x;i,j} \le 0, \quad G_{\xi;i,j} \ge 0 \quad \text{for } 0 \le j < i < N,$$

$$G_{x;i,j} \ge 0, \quad G_{\xi;i,j} \le 0 \quad \text{for } 0 \le i \le j < N,$$

$$G_{x\xi;ij} \le 0 \quad \text{for } 0 \le i, j < N, \; i \ne j$$

and

$$0 \le G_{x\xi;ii} \le \frac{1}{\varepsilon h_{i+1}} \quad \text{for } i = 0, \dots, N-1.$$

Proof. The upper and lower bounds on G are verified using (4.5).

Since $G \ge 0$ on $\bar{\omega}^2$ and $G_{i,0} = 0$ for $i = 0, \dots, N$, we have $G_{\xi;i,0} \ge 0$ for $i = 0, \dots, N$. By multiplying (4.6) by h_{j+1} and summing over j, we get

$$-\varepsilon G_{\xi;i,j} + \varepsilon G_{\xi;i,0} + b_j G_{i,j} = -\sum_{k=1}^{j} h_{k+1} c_k G_{i,k} \quad \text{for } j = 1, \dots, i-1.$$

Hence,

$$\varepsilon G_{\xi;i,j} \ge \varepsilon G_{\xi;i,0} + b_j G_{i,j} \ge 0 \quad \text{for } j = 1, \dots, i-1,$$

since $G_{i,j} \geq 0$ and $G_{\xi;i,0} \geq 0$. On the other hand, $G_{\xi;i,N-1} \leq 0$ for $i = 0, \ldots, N$ because $G \geq 0$ on $\bar{\omega}^2$ and $G_{i,N} = 0$ for $i = 0, \ldots, N$. By inspecting the difference equation (4.6), we see that $v_j := G_{\xi;i,j}$ satisfies, for $i < j < N$,

$$-\frac{\varepsilon}{h_{j+1}}\left(v_j - v_{j-1}\right) + \frac{h_j b_{j-1}}{h_{j+1}} v_{j-1} = -\big(b_{\breve{\xi};j} + c_j\big) G_{i,j} \leq 0, \tag{4.7}$$

by (4.2). Since $v_{N-1} \leq 0$, induction for decreasing j yields $G_{\xi;i,j} = v_j \leq 0$ for $i \leq j < N$.

Similarly, one can prove that $G_{x;i,j} \geq 0$ for $0 \leq i < j-1$ and $G_{x;i,j} \leq 0$ for $j \leq j < N$. Thus,

$$G_{x\xi;i,0} \leq 0 \;\text{ and }\; G_{x\xi;i,N-1} \leq 0 \;\text{ for }\; i = 0, \ldots, N-1.$$

because $G_{x;i,0} = G_{x;i,N} = 0$ for $0 \leq i < N$. Taking differences of (4.6) with respect to i and summing over j, we get

$$-\varepsilon G_{x\xi;i,j} + \varepsilon G_{x\xi;i,0} + b_j G_{x;i,j} + \sum_{k=1}^{j} h_{k+1} c_k G_{x;i,k} = -\delta_{i,j} \quad \text{for } 0 < j \leq i.$$

Therefore,

$$G_{x\xi;i,j} \leq 0 \quad \text{for} \quad 0 \leq j < i < N \quad \text{and} \quad G_{x\xi;i,i} \leq \frac{1}{\varepsilon h_{i+1}} \quad \text{for} \quad 0 \leq i < N$$

because $G_{x;i,j} \geq 0$, $G_{x\xi;i,0} \leq 0$ and $G_{x;i,0} = 0$.

For $i < j$, take differences of (4.7) to see that $v_j = G_{x\xi;i,j}$ satisfies

$$-\frac{\varepsilon}{h_{j+1}}\left(v_j - v_{j-1}\right) + \frac{h_j b_{j-1}}{h_{j+1}} v_{j-1} = -\big(b_{\breve{\xi};j} + c_j\big) G_{x;i,j} \leq 0 \qquad \text{for } j = i+2, \ldots, N-1.$$

because $b', c \geq 0$ and $G_{x;i,j} \geq 0$ for $i < j$. We get $G_{x\xi;i,j} \leq 0$ for $0 \leq i < j < N$. Finally, for $i = j$, use

$$\sum_{j=0}^{N-1} h_{j+1} G_{x\xi;i,j} = G_{x;i,N} - G_{x;i,0} = 0$$

in order to obtain $h_{i+1} G_{x\xi;i,i} \geq 0$. □

Mimicking the arguments of Theorem 3.23 we obtain its discrete counterpart.

Theorem 4.2. *Suppose* (4.2) *holds true. Then the Green's function G associated with the discrete operator L satisfies*

$$\|G_{i,\cdot}\|_{1,\omega} \leq \frac{1}{\beta}, \quad \|G_{\xi;i,\cdot}\|_{1,\omega} \leq \frac{2}{\beta}, \quad \|G_{x;\cdot,j}\|_{1,\omega} \leq \frac{2}{\beta}$$

and

$$\|G_{x\xi;i,\cdot}\|_{1,\omega} \le \frac{2}{\varepsilon}.$$

for all $i, j = 1, \dots, N-1$.

The ℓ_1-norms bounds are used to establish stability properties for L that resemble those of Theorem 3.45 for the differential operator $\mathcal{L}$.

Theorem 4.3. *Suppose* (4.2) *holds true. Then the operator* L *satisfies*

$$\|v\|_{\infty,\omega} \le \min\left\{\|Lv/b\|_{\infty,\omega}, \|Lv/c\|_{\infty,\omega}\right\}, \tag{4.8a}$$

$$\|v\|_{\infty,\omega} \le \beta^{-1}\|Lv\|_{1,\omega}, \quad \|v_x\|_{1,\omega} \le 2\beta^{-1}\|Lv\|_{1,\omega} \tag{4.8b}$$

and

$$|||v|||_{\varepsilon,\infty,\omega} \le 2\|Lv\|_{-1,\infty,\omega} \tag{4.8c}$$

for all $v \in I\!R_0^{N+1}$.

Remark 4.4. Similar to Remark 3.22 we have

$$2\|v\|_{-1,\infty,\omega} \le \|v\|_{1,\omega} \le \|v\|_{\infty,\omega}.$$

Therefore, the $(\ell_\infty, w^{-1,\infty})$-stability (4.8c) is the strongest of the three stability inequalities of Theorem 4.3. It was first derived by Andreev and Kopteva [11], though their derivation is different. A systematic approach can be found in [5], where stability of both the continuous operator $\mathcal{L}$ and of its discrete counterpart L is investigated. So far the $(\ell_\infty, w^{-1,\infty})$-stability inequality gives the sharpest error bounds for one-dimensional problems. But unlike the (ℓ_∞, ℓ_1) stability, it is unclear whether it can be generalised to higher dimensions. ♣

Remark 4.5. The same stability results hold true if the convection-diffusion problem in conservative form

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in } (0,1), \; u(0) = \gamma_0, \; u(1) = \gamma_1, \tag{4.9}$$

is discretised by

$$\begin{gathered}\left[L^c u^N\right]_i := -\varepsilon u^N_{\bar{x}x;i} - \left(bu^N\right)_{x;i} + c_i u^N_i = f_i \quad \text{for } i = 1, \dots, N-1, \\ u^N_0 = \gamma_0, \; u^N_N = \gamma_1.\end{gathered} \tag{4.10}$$

♣

Remark 4.6. The (ℓ_∞, ℓ_1) stability (4.8b) was first given by Andreev and Savin [12] for a modification of Samarskii's scheme [144]. It has been used in a number of publications to establish uniform convergence on S-type and B-type meshes; see, e. g., [10, 12, 106, 154]. Details of a convergence analysis can be found in Sect. 4.2.5. This stability result can be generalised to study two-dimensional problems; see Sect. 9.3.2. ♣

Corollary 4.7. *By Theorem* (4.3) *there exists a unique solution of* (4.3) *and it satisfies*

$$\left\|u^N\right\|_{\infty,\bar\omega} \le \min\left\{\|f/b\|_{\infty,\omega}, \|f/c\|_{\infty,\omega}\right\}.$$

4.2.2 A Priori Error Bounds

Let us consider the approximation error of the simple upwind scheme (4.3) applied to the boundary value problem (4.1). We give a convergence analysis based on the negative-norm stability of Theorem 4.3.

Introduce the continuous and discrete operators and functions

$$\big(\mathcal{A}v\big)(x) := \varepsilon v'(x) + \big(bv\big)(x) + \int_x^1 \big((b'+c)v\big)(s)\mathrm{d}s, \quad \mathcal{F}(x) := \int_x^1 f(s)\mathrm{d}s$$

and

$$[Av]_i := \varepsilon v_{\bar x;i} + b_i v_i + \sum_{k=i}^{N-1} h_{k+1}\left(b_{x;k} v_{k+1} + c_k v_k\right), \quad F_i := \sum_{k=i}^{N-1} h_{k+1} f_k.$$

Note that $\mathcal{L}v = -(\mathcal{A}v)'$ and $f = -\mathcal{F}'$ on $(0,1)$, and $Lv = -(Av)_x$ and $f = -F_x$ on ω. Thus,

$$\mathcal{A}u - \mathcal{F} \equiv \alpha \text{ on } (0,1) \text{ and } Au^N - F \equiv a \text{ on } \omega \tag{4.11}$$

with constants α and a.

In view of the stability inequality (4.8c) we have

$$\left|\left|\left|u-u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le 2\left\|L\left(u-u^N\right)\right\|_{-1,\infty,\omega} = 2\min_{c\in I\!R}\left\|A\left(u-u^N\right)+c\right\|_{\infty,\omega}.$$

Taking $c = a - \alpha$, where a and α are the constants from (4.11), we get

$$\left|\left|\left|u-u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le 2\|Au - \mathcal{A}u - F + \mathcal{F}\|_{\infty,\omega}. \tag{4.12}$$

Furthermore,

$$\begin{aligned}
&(Au - \mathcal{A}u - F + \mathcal{F})_i \\
&\quad = \varepsilon\left(u_{\bar{x}} - u'\right)_i + \sum_{k=i}^{N-1} h_{k+1} b_{x;k} u_{k+1} - \int_{x_i}^{x_N} \left(b'u\right)(x)\mathrm{d}x \\
&\qquad + \sum_{k=i}^{N-1} h_{k+1}\left(c_k u_k - f_k\right) - \int_{x_i}^{x_N} \left(cu - f\right)(x)\mathrm{d}x.
\end{aligned} \tag{4.13}$$

Taylor expansions with the integral form of the remainder give

$$h_{k+1}\left(c_k u_k - f_k\right) - \int_{I_{k+1}} (cu - f)(x)\mathrm{d}x = \int_{I_{k+1}} \int_x^{x_k} \big(cu - f\big)'(s)\mathrm{d}s\mathrm{d}x,$$

$$h_{k+1} b_{x;k} u_{k+1} - \int_{I_{k+1}} \left(b'u\right)(x)\mathrm{d}x = \int_{I_{k+1}} b'(x) \int_x^{x_{k+1}} u'(s)\mathrm{d}s\mathrm{d}x$$

and

$$\varepsilon\left(u_{\bar{x}} - u'\right)_k = \frac{\varepsilon}{h_k} \int_{I_k} \int_x^{x_k} u''(s)\mathrm{d}s\mathrm{d}x = \frac{1}{h_k} \int_{I_k} \int_{x_k}^{x} \big(bu' - cu + f\big)(s)\mathrm{d}s\mathrm{d}x,$$

by (4.1). Combining these representations with (4.12) and (4.13) we get the following general convergence result.

Theorem 4.8. *Let u be the solution of* (4.1) *and u^N that of* (4.3). *Then*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le 2 \max_{k=1,\dots,N} \int_{I_k} \left(C_1\left|u'(x)\right| + C_2|u(x)| + C_3\right)\mathrm{d}x$$

with the constants

$$C_1 := \|c\|_\infty + \|b'\|_\infty + \|b\|_\infty, \quad C_2 := \|c\|_\infty + \|c'\|_\infty$$

and

$$C_3 := \|f\|_\infty + \|f'\|_\infty.$$

Remark 4.9. A similar result is given in [85] for the discretisation of the conservative form (4.9). When using the conservative form, the last two terms in (4.13) which involve b_x and b' disappear. ♣

Corollary 4.10. *Theorem 4.8 and the a priori bounds* (3.30) *yield*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le C\vartheta_{cd}^{[1]}(\bar{\omega}),$$

where the characteristic quantity $\vartheta_{cd}^{[p]}(\bar{\omega})$ *has been defined on p. 6:*

$$\vartheta_{cd}^{[p]}(\bar{\omega}) := \max_{i=1,\dots,N} \int_{I_i} \left(1 + \varepsilon^{-1} \mathrm{e}^{-\beta s/p\varepsilon}\right) \mathrm{d}s.$$

Remark 4.11. The mesh function u^N can be extended to a piecewise linear function on the mesh $\bar{\omega}$. For convenience we denote this extended function by u^N also. Then

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty} \le C\vartheta_{cd}^{[1]}(\bar{\omega})$$

follows from a triangle inequality and our bounds for the interpolation error. ♣

Remark 4.12. Corollary 4.10 allows to immediate deduce (almost) first-order uniform convergence for particular meshes. Suppose the mesh parameter σ in the definition of the meshes (see Sect. 2.1) satisfies $\sigma \ge 1$. Then

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty} \le \begin{cases} CN^{-1} & \text{for Bakhvalov meshes,} \\ C\left(h + N^{-1} \max\limits_{\xi\in[0,q]} |\psi'(\xi)|\right) & \text{for S-type meshes and} \\ CN^{-1}\ln N & \text{for Shishkin meshes,} \end{cases}$$

by (2.6) and (2.9). ♣

A numerical example

Table 4.1 displays numerical results for the upwind scheme (4.3) on a Bakhvalov mesh applied to the test problem

$$-\varepsilon u'' - u' + 2u = \mathrm{e}^{x-1}, \quad u(0) = u(1) = 0. \tag{4.14}$$

Table 4.1 Simple upwinding on a Bakhvalov mesh ($q = 1/2$)

	$\sigma = 0.2$		$\sigma = 0.4$		$\sigma = 0.6$		$\sigma = 0.8$		$\sigma = 1.0$	
N	error	rate	error	rate	error	rate	error	rate	error	rate
2^7	2.246e-2	0.23	1.173e-2	0.47	6.856e-3	0.69	4.658e-3	0.87	3.995e-3	0.97
2^8	1.913e-2	0.22	8.482e-3	0.45	4.258e-3	0.68	2.547e-3	0.88	2.036e-3	0.98
2^9	1.641e-2	0.21	6.201e-3	0.44	2.662e-3	0.67	1.388e-3	0.87	1.030e-3	0.99
2^{10}	1.416e-2	0.21	4.576e-3	0.43	1.675e-3	0.66	7.586e-4	0.87	5.193e-4	0.99
2^{11}	1.224e-2	0.20	3.403e-3	0.42	1.062e-3	0.65	4.155e-4	0.87	2.611e-4	0.99
2^{12}	1.063e-2	0.20	2.545e-3	0.41	6.784e-4	0.64	2.281e-4	0.86	1.310e-4	1.00
2^{13}	9.226e-3	0.20	1.911e-3	0.41	4.361e-4	0.63	1.256e-4	0.86	6.568e-5	1.00
2^{14}	8.030e-3	0.20	1.439e-3	0.41	2.819e-4	0.62	6.937e-5	0.85	3.290e-5	1.00
2^{15}	6.969e-3	0.21	1.086e-3	0.40	1.830e-4	0.62	3.846e-5	0.85	1.647e-5	1.00
2^{16}	6.026e-3	—	8.207e-4	—	1.193e-4	—	2.139e-5	—	8.245e-6	—

In our computations we have fixed the parameter q and varied σ to illustrate the sharpness of our theoretical results. The errors are measured in the discrete maximum norm $\|\cdot\|_{\infty,\bar\omega}$. Apparently, choosing $\sigma < 1$ adversely affects the order of convergence. Similar observations can be made for the Shishkin mesh and other meshes.

4.2.3 Error Expansion

In the previous section we have seen that the error of the simple upwind scheme (4.3) satisfies

$$\left|\!\left|\!\left| u - u^N \right|\!\right|\!\right|_{\varepsilon,\infty,\omega} \le C\vartheta_{cd}^{[1]}(\bar\omega).$$

Now an expansion of the error of this scheme is constructed. We shall show there exists a function ψ, the leading term of the error, such that

$$u - u^N = \psi + \text{second order terms}.$$

This result can be applied to analyse, e.g., derivative approximations, defect correction and Richardson extrapolation, see Sect. 4.2.9 and 4.3.3.

For the sake of simplicity, we will study the conservative form (4.9), i. e.,

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

and its discretisation by (4.10):

$$\begin{aligned} \left[L^c u^N\right]_i := -\varepsilon u^N_{\bar{x}x;i} - \left(bu^N\right)_{x;i} + c_i u^N_i = f_i \quad \text{for } i = 1,\dots,N-1,\\ u^N_0 = \gamma_0, \;\; u^N_N = \gamma_1. \end{aligned}$$

Corresponding to Sect. 4.2.2 we introduce

$$\left(\mathcal{A}^c v\right)(x) := \varepsilon v'(x) + \big(bv\big)(x) + \int_x^1 \big(cv\big)(s)\,\mathrm{d}s$$

and

$$[A^c v]_i := \varepsilon v_{\bar{x};i} + b_i v_i + \sum_{k=i}^{N-1} h_{k+1} c_k v_k.$$

Note that $\mathcal{L}^c v = -(\mathcal{A}^c v)'$ on $(0,1)$ and that $L^c v = -(A^c v)_x$ on ω.

4.2.3.1 Construction of the Error Expansion

We define the leading term of the error expansion as the solution of

$$\mathcal{L}^c\psi = \Psi' \quad \text{in } (0,1), \quad \psi(0) = \psi(1) = 0, \tag{4.15}$$

where

$$\Psi(x) = \varepsilon \frac{h(x)}{2} u''(x) - \int_x^1 \left(hg'\right)(s)\mathrm{d}s,$$

with

$$h(x) = x - x_{k-1} \quad \text{for } x \in (x_{k-1}, x_k) \quad \text{and} \quad g = f - cu.$$

Note that Ψ is discontinuous at the mesh nodes. Hence, $\mathcal{L}^c\psi$ is a generalised function. Therefore, (4.15) has to be interpreted in the context of distributions. Alternatively, one may seek a solution $\psi \in C^2((0,1) \setminus \omega) \cap C[0,1]$ such that

$$\mathcal{L}^c\psi = \Psi' \quad \text{in } (0,1) \setminus \omega, \;\; \psi(0) = \psi(1) = 0,$$

and

$$-\varepsilon[\psi'](x_i) = [\Psi](x_i) = -\varepsilon \frac{h_i}{2} u''(x_i) \quad \text{for } x_i \in \omega.$$

Since $\mathcal{A}^c\psi = -\Psi$ on $(0,1) \setminus \omega$, we have

$$\left[A^c\psi\right]_i = \varepsilon\left(\psi_{\bar{x};i} - \psi'_{i-0}\right) + \sum_{k=i}^{N-1} h_{k+1}c_k\psi_k - \int_{x_i}^1 (c\psi)(s)\mathrm{d}s + \Psi_{i-0}.$$

Thus,

$$\begin{aligned}\left[A^c\left(u - \psi - u^N\right)\right]_i = \varepsilon\left(u_{\bar{x};i} - u'_i + \frac{h_i}{2}u''_i\right) - \varepsilon\left(\psi_{\bar{x};i} - \psi'_{i-0}\right) \\ + \int_{x_i}^1 \left(g - hg'\right)(x)\mathrm{d}x - \sum_{k=i}^{N-1} h_{k+1}g_k \\ - \sum_{k=i}^{N-1} h_{k+1}c_k\psi_k + \int_{x_i}^1 (c\psi)(s)\mathrm{d}s.\end{aligned} \tag{4.16}$$

The function ψ has been designed such that the terms on the right-hand side that involve u are of second order. Those involving ψ are formally only first-order terms, but second order is gained since ψ itself is first order.

In order to bound the terms on the right-hand side, bounds for the derivatives of u up to order three are needed. These are provided by (3.30). The following theorem gives bounds for the leading term ψ of the error expansion and its derivatives up to order two, which are also required. Because of the number of technical details, its proof is deferred to the end of this section.

Lemma 4.13. *Let ψ be the solution of the boundary-value problem (4.15). Assume that $b, c \in C^2[0,1]$ and $f \in C^1[0,1]$. Then*

$$\left|\psi^{(k)}(x)\right| \leq C\vartheta_{cd}^{[2]}(\bar{\omega})\left(1+\varepsilon^{-k}\mathrm{e}^{-\beta x/2\varepsilon}\right) \quad \text{for } x \in (0,1)\setminus\omega, \quad k=0,1, \tag{4.17a}$$

and

$$\varepsilon\left|\psi''(x)\right| \leq C\vartheta_{cd}^{[2]}(\bar{\omega})\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right) \quad \text{for } x \in (0,1)\setminus\omega. \tag{4.17b}$$

Later we shall also show that (3.30), (4.16) and Lemma 4.13 yield

$$\left\|A^c(u-\psi-u^N)\right\|_{\infty,\omega} \leq C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2. \tag{4.18}$$

Then Theorem 4.3 yields our main result of this section.

Theorem 4.14. *Let u, u^N and ψ be the solutions of (4.9), (4.10) and (4.15), respectively. Assume that $b, c, f \in C^2[0,1]$. Then*

$$\left|\left|\left|u-\psi-u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \leq C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2.$$

4.2.3.2 Detailed Proofs

Proof of Lemma 4.13

Now we derive bounds for the derivatives of the leading term ψ in the error expansion. The following auxiliary result will be used several times in the subsequent analysis.

Proposition 4.15. *Let $x \in (x_{k-1}, x_k)$ and $\sigma > 0$ be arbitrary. Then*

$$h(x)\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta x/\sigma\varepsilon}\right) \leq \int_{x_{k-1}}^{x}\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta s/\sigma\varepsilon}\right)\mathrm{d}s$$

Proof. Let

$$F(x) := h(x)\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta x/\sigma\varepsilon}\right) \quad \text{and} \quad G(x) := \int_{x_{k-1}}^{x}\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta s/\sigma\varepsilon}\right)\mathrm{d}s.$$

Clearly $F(x_{k-1}) = G(x_{k-1}) = 0$ and

$$F'(x) = 1 + \varepsilon^{-1}\mathrm{e}^{-\beta x/\varepsilon} - \frac{h(x)\beta}{\sigma\varepsilon^2}\mathrm{e}^{-\beta x/\sigma\varepsilon} \le 1 + \varepsilon^{-1}\mathrm{e}^{-\beta x/\sigma\varepsilon} = G'(x)$$

for $x \in (x_{k-1}, x_k)$. The result follows. $\square$

First (3.30) implies

$$|\Psi(x)| \le C\varepsilon h(x)\left(1 + \varepsilon^{-2}\mathrm{e}^{-\beta x/\varepsilon}\right) + C\int_x^1 h(s)\left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta s/\varepsilon}\right)\mathrm{d}s.$$

This inequality, (3.29c) and Prop. 4.15 yield (4.17a) for $k = 0$.

Next, we derive bounds on ψ'. Set

$$B(x) := \frac{1}{\varepsilon}\int_0^x b(s)\mathrm{d}s, \quad a(x) := \Psi'(x) + (c - b')(x)\psi(x)$$

and

$$\chi(x) := \frac{1}{\varepsilon}\int_0^x a(s)\mathrm{e}^{B(s)-B(x)}\mathrm{d}s.$$

Then ψ can be written as

$$\psi(x) = \int_x^1 \chi(s)\mathrm{d}s + \kappa\int_x^1 \mathrm{e}^{-B(s)}\mathrm{d}s \quad \text{with} \quad \kappa = -\frac{\int_0^1 \chi(s)\mathrm{d}s}{\int_0^1 \mathrm{e}^{-B(s)}\mathrm{d}s}.$$

For ψ' we get

$$\psi'(x) = -\chi(x) - \kappa\mathrm{e}^{-B(x)}. \tag{4.19}$$

Apparently the critical point is to derive bounds on χ. Integration by parts and the definition of Ψ yield

$$2\chi(x) = (hu'')(x) - \zeta(x) \tag{4.20}$$

with

$$\zeta(x) := \frac{1}{\varepsilon}\int_0^x \big(hbu'' - 2h(f - cu)' - 2(c - b')\psi\big)(s)\mathrm{e}^{B(s)-B(x)}\mathrm{d}s.$$

For the first term on the right-hand side of (4.20) we have by (3.30) and Prop. 4.15

$$\left|(hu'')(x)\right| \le Ch(x)\left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right)^2 \le C\vartheta_{cd}^{[2]}(\bar\omega)\left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right). \tag{4.21}$$

To bound $\zeta(x)$, the second term in (4.20), we use (3.30), (4.17a) for $k = 0$ and (4.21):

$$|\zeta(x)| \le \frac{C}{\varepsilon}\int_0^x \left[h(s)\left(1+\varepsilon^{-2}\mathrm{e}^{-\beta s/\varepsilon}\right) + \vartheta_{cd}^{[2]}(\bar{\omega})\right]\mathrm{e}^{\beta(s-x)/\varepsilon}\mathrm{d}s$$
$$\le C\vartheta_{cd}^{[2]}(\bar{\omega})\int_0^x \left(1+\varepsilon^{-1}\mathrm{e}^{\beta s/2\varepsilon}\right)\mathrm{e}^{\beta(s-x)/\varepsilon}\mathrm{d}s$$
$$\le C\vartheta_{cd}^{[2]}(\bar{\omega})\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right).$$

This, eq. (4.20) and inequality (4.21) give

$$|\chi(x)| \le C\vartheta_{cd}^{[2]}(\bar{\omega})\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right). \tag{4.22}$$

Integrating (4.22), we obtain

$$|\kappa| \le C\varepsilon^{-1}\vartheta_{cd}^{[2]}(\bar{\omega}), \tag{4.23}$$

since $\int_0^1 \mathrm{e}^{-B(s)}\mathrm{d}s \ge \varepsilon/\|b\|_\infty$. Combining (4.19)-(4.23), we get (4.17a) for $k = 1$.

Finally, the bound (4.17b) for the second-order derivative of ψ follows from (4.15), (3.30), (4.17a) and Prop. 4.15.

Proof of (4.18)

We now bound the terms on the right-hand side of (4.16). For the first two terms a Taylor expansion with the integral form of the remainder yields

$$\varepsilon\left|u_{\bar{x};i} - u_i' + \frac{h_i}{2}u_i''\right| \le C\int_{I_i}(x-x_{i-1})\left(1+\varepsilon^{-2}\mathrm{e}^{-\beta x/\varepsilon}\right)\mathrm{d}x$$

by (3.30). To estimate the right-hand side we use the following result from [24].

Lemma 4.16. *Let g be a positive monotonically decreasing function on $[a,b]$. Let $p \in I\!N^+$. Then*

$$\int_a^b g(\xi)\,(\xi-a)^{p-1}\,\mathrm{d}\xi \le \frac{1}{p}\left\{\int_a^b g(\xi)^{1/p}\mathrm{d}\xi\right\}^p.$$

Proof. Consider the two integrals as functions of the upper integration limit. □

We get

$$\varepsilon\left|u_{\bar{x};i} - u_i' + \frac{h_i}{2}u_i''\right| \le C\left\{\int_{I_i}\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right)\mathrm{d}x\right\}^2 \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2. \tag{4.24}$$

Next we bound the third term in (4.16). Assuming $c, f \in C^2[0,1]$, we have

$$\int_{I_{k+1}} \big(g(x) - (x - x_k)g'(x)\big)\mathrm{d}x - h_{k+1}g_k = \int_{I_{k+1}} \int_{x_k}^{x} (s - x_k)g''(s)\mathrm{d}s.$$

Thus,

$$\left|\int_{I_{k+1}} (g(x) - (x - x_k)g')\,\mathrm{d}x - h_{k+1}g_k\right| \\ \le Ch_{k+1} \int_{I_{k+1}} (s - x_k)\left(1 + \varepsilon^{-2}\mathrm{e}^{-\beta s/\varepsilon}\right)\mathrm{d}s \le Ch_{k+1}\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,$$

by (3.30) and Lemma 4.16. Hence

$$\left|\int_{x_i}^{1} \big(g - hg'\big)(x)\mathrm{d}x - \sum_{k=i}^{N-1} h_{k+1}g_k\right| \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2. \tag{4.25}$$

To bound the remaining terms we use the bounds on ψ and its derivatives from Lemma 4.13. A Taylor expansion and (4.17b) yield

$$\varepsilon\left|\psi_{\bar{x};k} - \psi'_{k-0}\right| \le \varepsilon \int_{x_{k-1}}^{x_k} |\psi''(x)|\,\mathrm{d}x \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2. \tag{4.26}$$

Finally,

$$\left|\int_{x_k}^{x_{k+1}} (c\psi)(s)\mathrm{d}s - h_{k+1}(c\psi)_k\right| \\ \le h_{k+1}\int_{I_{k+1}} |(c\psi)'(s)|\,\mathrm{d}\xi\mathrm{d}s \le Ch_{k+1}\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,$$

by (4.17a). Therefore,

$$\left|\sum_{k=i}^{N-1} h_{k+1}c_k\psi_k - \int_{x_i}^{1} (c\psi)(s)\mathrm{d}s\right| \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2. \tag{4.27}$$

Applying (4.24)–(4.27) to (4.16) and taking the maximum over $i = 0, \ldots, N-1$, we get (4.18).

4.2.4 A Posteriori Error Estimation and Adaptivity

In Sect. 4.2.2 the stability of the *discrete operator* L was used to bound the error in the *discrete* maximum norm in terms of the derivative of the *exact solution*.

Now, in the first part of this section, roles are interchanged and the stability of the *continuous operator* $\mathcal{L}$ is used to bound the error in the *continuous* maximum norm in terms of finite differences of the *numerical solution*. We follow [64].

4.2.4.1 A Posteriori Error Bounds

Let u^N be the piecewise-linear function that solves (4.3). Then (3.29c) yields

$$|||u - u^N|||_{\varepsilon,\infty} \le 2 \left\| \mathcal{L}\left(u - u^N\right) \right\|_{-1,\infty} = 2 \min_{c \in I\!R} \left\| \mathcal{A}\left(u - u^N\right) + c \right\|_\infty .$$

Clearly

$$\min_{c \in I\!R} \left\| \mathcal{A}\left(u - u^N\right) + c \right\|_\infty \le \left\| \mathcal{A}\left(u - u^N\right) + a - \alpha \right\|_\infty , \tag{4.28}$$

where a and α are the constants from (4.11). Furthermore, for any $x \in (x_{i-1}, x_i)$,

$$\mathcal{A}\left(u - u^N\right) + a - \alpha = [Au^N]_i - \left(\mathcal{A}u^N\right)(x) - F_i + \mathcal{F}(x).$$

We bound the two terms on the right-hand side.

Since $\left(u^N\right)' = u^N_{\bar{x},i}$ for all $x \in (0,1) \setminus \omega$, we have

$$\begin{aligned} &[Au^N]_i - \left(\mathcal{A}u^N\right)(x) \\ &\quad = \sum_{k=i}^{N-1} h_{k+1} b_{x;k} u^N_{k+1} - \int_{x_k}^1 \left(b' u^N\right)(s) \mathrm{d}s + \int_x^{x_i} b(s) \left(u^N\right)'(s) \mathrm{d}s \\ &\quad\quad - \int_x^{x_i} \left(c u^N\right)(s) \mathrm{d}s - \int_{x_i}^1 \left(c u^N\right)(s) \mathrm{d}s + \sum_{k=i}^{N-1} h_{k+1} c_k u^N_k , \end{aligned}$$

by the definitions of A and $\mathcal{A}$ and by integration by parts. For the terms on the right-hand side, Taylor expansions give

$$\left| h_{k+1} b_{x;k} u^N_{k+1} - \int_{I_{k+1}} \left(b' u^N\right)(s) \mathrm{d}s \right| \le h_{k+1} \|b'\|_\infty \left| u^N_{k+1} - u^N_k \right| ,$$

$$\left| \int_x^{x_k} b(s) \left(u^N\right)'(s) \mathrm{d}s \right| \le \|b\|_\infty \left| u^N_k - u^N_{k-1} \right| ,$$

$$\begin{aligned} &\left| h_{k+1} c_k u^N_k - \int_{I_{k+1}} \left(c u^N\right)(s) \mathrm{d}s \right| \\ &\quad \le h^2_{k+1} \|c'\|_\infty \max\left\{ |u^N_{k+1}|, |u^N_k| \right\} + h_{k+1} \|c\|_\infty \left| u^N_{k+1} - u^N_k \right| \end{aligned}$$

and

$$\left| \int_x^{x_k} \left(cu^N\right)(s)\mathrm{d}s \right| \le h_k \|c\|_\infty \max\left\{ \left|u_k^N\right|, \left|u_{k-1}^N\right| \right\}.$$

Thus,

$$\left|[Au^N]_i - \left(\mathcal{A}u^N\right)(x)\right| \le C_1 \max_{k=0,\dots,N-1} |u_{k+1}^N - u_k^N| + C_2 h \|u^N\|_{\infty,\omega} \quad (4.29)$$

with the constants C_1 and C_2 form Theorem 4.8.

Next bound $F - \mathcal{F}$.

$$\left| \int_x^{x_k} f(s)\mathrm{d}s \right| \le h_k \|f\|_\infty$$

and

$$\left| h_{k+1} f_k - \int_{I_{k+1}} f(s)\mathrm{d}s \right| \le h_{k+1}^2 \|f'\|_\infty$$

yield

$$|F_k - \mathcal{F}(x)| \le C_3 h.$$

Combining this with (4.28) and (4.29), then taking the supremum over all $x \in (0,1) \setminus \omega$, we get

$$\left\|\mathcal{L}\left(u - u^N\right)\right\|_{-1,\infty} \le C_1 \max_{i=0,\dots,N-1} |u_{i+1}^N - u_i^N| + h\left(C_2 \|u^N\|_{\infty,\omega} + C_3\right)$$

with the constants C_1, C_2 and C_3 from Theorem 4.8.

Finally, use (3.29c) in order to obtain the main result of this section.

Theorem 4.17. *Let u be the solution of (4.1) and u^N that of (4.3). Then*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty} \le 2C_1 \max_{i=0,\dots,N-1} |u_{i+1}^N - u_i^N| + 2h\left(C_2 \|u^N\|_{\infty,\omega} + C_3\right).$$

Corollary 4.18. *Theorem 4.17 and Corollary 4.7 yield*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty} \le C \max_{i=0,\dots,N-1} h_{i+1}\left(1 + |u_{x;i}^N|\right).$$

Note the analogy of these results to Theorem 4.8 and to Corollary 4.10.

4.2.4.2 An Adaptive Method

From Theorem 4.8 it is easily concluded that the error of our upwind scheme satisfies

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le C \max_{i=1,\dots,N} \int_{I_i} \sqrt{1 + u'(x)^2}\mathrm{d}x.$$

On the other hand,

$$\int_0^1 \sqrt{1 + u'(x)^2}\mathrm{d}x \le C,$$

by (3.30). Thus, if the mesh is designed so that

$$\int_{I_{i-1}} \sqrt{1 + u'(x)^2}\mathrm{d}x = \int_{I_i} \sqrt{1 + u'(x)^2}\mathrm{d}x \tag{4.30}$$

for $i = 1, \dots, N-1$, i. e., if the mesh equidistributes the arc length of the exact solution, then

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le CN^{-1}. \tag{4.31}$$

However, u' is not available. An idea that leads to an adaptive method is to approximate the integrals in (4.30) by the mid-point quadrature rule, and $u'(x_{k-1/2})$ by a central difference quotient and finally to replace u by the numerical solution u^N. We get

$$\int_{I_i} \sqrt{1 + u'(x)^2}\mathrm{d}x \approx h_i \sqrt{1 + \left(u^N_{\bar{x};i}\right)^2}.$$

Thus setting

$$Q_i = Q_i\left(u^N, \omega\right) := \sqrt{1 + \left(u^N_{\bar{x};i}\right)^2},$$

we can replace (4.30) by

$$h_i Q_i = \frac{1}{N} \sum_{j=1}^{N} h_j Q_j \quad \text{for } i = 1, \dots, N-1. \tag{4.32}$$

Now solving the difference equation (4.3) and the discretised equidistribution principle (4.32) simultaneously, we get an adaptive method.

Kopteva and Stynes [69] proved that the nonlinear system of equations (4.3) and (4.32) possesses a solution and the error of the solution u^N obtained satisfies (4.31). An essential ingredient in the analysis is the a posteriori error bound of

Theorem 4.17. They proceed by considering a mesh movement algorithm, originally due to de Boor [25], which starts with a uniform mesh and aims to construct a mesh that solves the equidistribution problem (4.32).

In [69] it is shown that (4.32) does not need to be enforced strictly. The de Boor algorithm, which we are going to describe now, can be stopped when the relaxed equidistribution principle

$$Q_i h_i \le \frac{\gamma}{N} \sum_{j=1}^{N} h_j Q_j \quad \text{for } i = 1, \dots, N,$$

with a user-chosen constant $\gamma > 1$ is satisfied.

Algorithm:

1. Initialisation: Fix N and choose the constant $C_0 > 1$. The initial mesh $\omega^{[0]}$ is uniform with mesh size $1/N$.
2. For $k = 0, 1, \dots,$ given the mesh $\omega^{[k]}$, compute the discrete solution $u^{N,[k]}$ on this mesh. Set $h_i^{[k]} = x_i^{[k]} - x_{i-1}^{[k]}$ for each i. Let the piecewise-constant monitor function $\tilde{M}^{[k]}$ be defined by

$$\tilde{M}^{[k]}(x) := Q_i^{[k]} := Q_i\left(u^{N,[k]}, \omega^{[k]}\right) \quad \text{for} \quad x \in \left(x_{i-1}^k, x_i^k\right).$$

   Then the total integral of the monitor function $\tilde{M}^{[k]}$ is

$$I^{[k]} := \int_0^1 \tilde{M}^{[k]}(t)\mathrm{d}t = \sum_{j=1}^{N} h_j^{[k]} Q_j^{[k]}.$$

3. Test mesh: If

$$\max_{j=1,\dots,N} h_j^{[k]} Q_j^{[k]} \le \gamma \frac{I^{[k]}}{N},$$

   then go to Step 5. Otherwise, continue to Step 4.
4. Generate a new mesh by equidistributing the monitor function $\tilde{M}^{[k]}$ of the current computed solution: Choose the new mesh $\omega^{[k+1]}$ such that

$$\int_{x_{i-1}^{[k+1]}}^{x_i^{[k+1]}} \tilde{M}^{[k]}(t)\mathrm{d}t = \frac{I^{[k]}}{N}, \quad i = 0, \dots, N.$$

   (Since $\int_0^x M^{[k]}(t)\mathrm{d}t$ is increasing in x, the above relation clearly determines the mesh $\omega^{[k+1]}$ uniquely.) Return to Step 2.
5. Set $\omega^* = \omega_N^{[k]}$ and $u^{N,*} = u^{N,[k]}$ then stop.

In [69] it is shown that the stopping criterion is met after $\mathcal{O}\left(|\ln\varepsilon|\right)$ iterations and the error of the numerical solution obtained satisfies (4.31) with a constant $C = C(\gamma)$.

Remark 4.19. Beckett [19] notes that when γ is chosen close to 1 the algorithm becomes numerically unstable. The mesh starts to oscillate: Mesh points moved into the layer region in one iteration are moved back out of it in the next iteration. Thus, the parameter γ must not be chosen too small. Values used in various publications are 2 and 1.2, but may be problem dependent. ♣

To avoid these oscillations Linß [86] rewrites (4.31) as

$$(x_i - x_{i-1})^2 + (u_i^N - u_{i-1}^N)^2 \\ = (x_{i+1} - x_i)^2 + (u_{i+1}^N - u_i^N)^2 \quad \text{for } i = 1, \dots, N-1.$$

Then he treats the system of (4.3) and (4.31) as a map

$$(0,1] \to I\!\!R^{2(N+1)} : \varepsilon \mapsto \left(\bar{\omega}_\varepsilon, u_\varepsilon^N\right)$$

and applies a continuation method combining an explicit Euler method (predictor) with a Newton method (corrector). The iteration matrices in each Newton step are seven diagonal and in an example the numerical costs are approximately of order $N\,|\ln(N\varepsilon)|$. However, convergence of this method is not proved in [86].

4.2.5 An Alternative Convergence Proof

In this section we shall demonstrate how the (ℓ_∞, ℓ_1) stability (4.8b) can be exploited to study convergence of the scheme (4.3) on S-type meshes. The results are less general than those of Sect. 4.2.2, but can be generalised to two dimensions; cf. Sect. 9.3.2. In our presentation we follow [106].

By (4.8b), we have

$$\left\|u - u^N\right\|_{\infty,\bar{\omega}} \le \beta^{-1} \left\|Lu - f\right\|_{1,\omega}. \tag{4.33}$$

Thus, the maximal nodal error is bounded by a discrete ℓ_1 norm of the truncation error $\zeta := Lu - f$:

$$\|\zeta\|_{1,\omega} = \sum_{j=0}^{N-1} h_{j+1}|\zeta_j|.$$

Using the solution decomposition $u = v + w$ of Theorem 3.48 and a triangle inequality, we can bound the truncation error pointwise:

$$|\zeta_i| \le \left|\left[Lv\right]_i - f_i\right| + \left|\left[Lw\right]_i\right|.$$

Separate Taylor expansions for the two solution components and the derivative bounds of Theorem 3.48 yield

$$h_{i+1}\,|\zeta_i| \le C\left(h_{i+1} + h_i + \mathrm{e}^{-\beta x_{i-1}/\varepsilon}\right) \tag{4.34a}$$

and

$$\begin{aligned} h_{i+1}\,|\zeta_i| \le C\Big\{ &|h_{i+1} - h_i|\left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta x_{i-1}/\varepsilon}\right) \\ &+ \left(h_i^2 + h_{i+1}^2\right)\left(1 + \varepsilon^{-2}\mathrm{e}^{-\beta x_{i-1}/\varepsilon}\right)\Big\}. \end{aligned} \tag{4.34b}$$

In the analysis, assume the mesh generating function $\tilde{\varphi}$ of the S-type mesh satisfies (2.8) and that $\sigma \ge 2$. For the sake of simplicity suppose $\tilde{\varphi}'$ is nondecreasing. This leads to a mesh that does not condense on $[0, \tau]$ as we move away from the layer, i.e., $h_i \le h_{i+1}$ for $i = 1, \dots, qN-1$, which is reasonable for the given problem.

Now let us bound the ℓ_1 norm of the truncation error. Apply (4.34a) to bound $h_{i+1}\,|\zeta_i|$ for $i = qN, qN+1$ and (4.34b) otherwise. We get

$$\begin{aligned} \|\zeta\|_{1,\omega} \le C \sum_{i=1}^{qN-1} \Big\{ &(h_{i+1} - h_i)\left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta x_i/\varepsilon}\right) \\ &+ \left(h_i^2 + h_{i+1}^2\right)\left(1 + \varepsilon^{-2}\mathrm{e}^{-\beta x_{i-1}/\varepsilon}\right)\Big\} \\ + C&\left(h + \mathrm{e}^{-\beta x_{qN-1}/\varepsilon} + \mathrm{e}^{-\beta x_{qN}/\varepsilon}\right) \\ + C&\sum_{i=qN+2}^{N-1} N^{-2}\left(1 + \varepsilon^{-2}\mathrm{e}^{-\beta x_{i-1}/\varepsilon}\right). \end{aligned} \tag{4.35}$$

We bound the terms on the right-hand side separately in reverse order.

Let H denote the (constant) mesh size on $[\tau, 1]$. Then for $i = qN+2, \dots, N-1$

$$\varepsilon^{-2}\mathrm{e}^{-\beta x_{i-1}/\varepsilon} \le \varepsilon^{-2}\mathrm{e}^{-\beta H/\varepsilon}\mathrm{e}^{-\beta\tau/\varepsilon} \le C\big(H/\varepsilon\big)^2\mathrm{e}^{-\beta H/\varepsilon} \le C,$$

since $x_{i-1} \ge x_{N/2} + H = \tau + H$ and $\sigma \ge 2$. Thus,

$$\sum_{i=qN+2}^{N-1} N^{-2}\left(1 + \varepsilon^{-2}\mathrm{e}^{-\beta x_{i-1}/\varepsilon}\right) \le CN^{-1}. \tag{4.36}$$

Furthermore,

$$\begin{aligned} & h + \mathrm{e}^{-\beta x_{qN-1}/\varepsilon} + \mathrm{e}^{-\beta x_{qN}/\varepsilon} \\ & \quad \le h + \left(1 + \mathrm{e}^{\beta h_{qN}/\varepsilon}\right) \mathrm{e}^{-\beta x_{qN}/\varepsilon} \le h + CN^{-\sigma}, \end{aligned} \tag{4.37}$$

by (2.11).

Next we bound the first sum in (4.35). We have

$$\sum_{i=1}^{qN-1} \left(h_{i+1} - h_i + h_i^2 + h_{i+1}^2\right) \le 3h \tag{4.38}$$

and

$$\begin{aligned} & \sum_{i=1}^{qN-1} (h_{i+1} - h_i)\, \mathrm{e}^{-\beta x_i/\varepsilon} \\ & \quad = -h_1 \mathrm{e}^{-\beta x_1/\varepsilon} + \sum_{i=2}^{qN-1} h_i \left(\mathrm{e}^{-\beta x_{i-1}/\varepsilon} - \mathrm{e}^{-\beta x_i/\varepsilon}\right) + h_{qN} \mathrm{e}^{-\beta x_{qN-1}/\varepsilon}. \end{aligned}$$

The mean value theorem, (2.11) and (2.12) imply

$$\left|\mathrm{e}^{-\beta x_{i-1}/\varepsilon} - \mathrm{e}^{-\beta x_i/\varepsilon}\right| \le h_i \frac{\beta}{\varepsilon} \mathrm{e}^{-\beta x_{i-1}/\varepsilon} \le C\varepsilon N^{-1} \max|\psi'| \mathrm{e}^{-\beta x_{i-1}/(2\varepsilon)}.$$

Therefore, it follows that

$$\varepsilon^{-1} \sum_{i=1}^{qN-1} (h_{i+1} - h_i)\, \mathrm{e}^{-\beta x_i/\varepsilon} \le CN^{-1} \max|\psi'| \sum_{i=1}^{qN} \frac{h_i}{\varepsilon} \mathrm{e}^{-\beta x_{i-1}/(2\varepsilon)}.$$

Ineq. (2.11) also gives

$$\sum_{i=1}^{qN} \frac{h_i}{\varepsilon} \mathrm{e}^{-\beta x_{i-1}/(2\varepsilon)} \le C \int_0^{\tau} \varepsilon^{-1} \mathrm{e}^{-\beta x/(2\varepsilon)} \mathrm{d}x \le C.$$

Hence,

$$\varepsilon^{-1} \sum_{i=1}^{qN-1} (h_{i+1} - h_i)\, \mathrm{e}^{-\beta x_i/\varepsilon} \le CN^{-1} \max|\psi'|. \tag{4.39}$$

Similar calculations yield

$$\varepsilon^{-2} \left| \sum_{i=1}^{qN-1} \left(h_i^2 + h_{i+1}^2\right) \mathrm{e}^{-\beta x_{i-1}/\varepsilon} \right| \le CN^{-1} \max|\psi'|. \tag{4.40}$$

Substituting (4.36)–(4.40) into (4.35) and applying (4.33), we get the uniform error bound

$$\left\|u - u^N\right\|_{\infty,\bar\omega} \le C\left(h + N^{-1}\max|\psi'|\right).$$

In [106] the authors proceed—using more detailed bounds on the discrete Green's function—to prove the sharper bound

$$\left\|u - u^N\right\|_{\infty,\bar\omega\cap[\tau,1]} \le CN^{-1}$$

for the error outside of the layer region provided that (2.14) is satisfied by the mesh generating function.

4.2.6 The Truncation Error and Barrier Function Technique

We now consider the convection-diffusion problem

$$-\varepsilon u'' - bu' + cu = f \text{ in } (0,1), \quad u(0) = u(1) = 0 \tag{4.1}$$

discretised by

$$\begin{aligned}\left[\hat L u^N\right]_i := -\varepsilon u^N_{\bar x\hat x;i} - b_i u^N_{x;i} + c_i u^N_i = f_i \quad \text{for } i = 1,\dots,N-1,\\ u^N_0 = \gamma_0, \quad u^N_N = \gamma_1.\end{aligned} \tag{4.41}$$

In contrast to the scheme (4.3) this method is first-order consistent in the mesh nodes on arbitrary meshes.

The analysis of this section uses the truncation error and barrier function technique developed by Kellogg and Tsan [61]. This was adapted to the analysis of Shishkin meshes by Stynes and Roos [153] and later used for other meshes also [137]. This technique can be used for problems in two dimensions too; see Sect. 9.1 or [81, 107]. We demonstrate this technique by sketching the convergence analysis for S-type meshes. For more details the reader is referred to [137].

The matrix associated with $\hat L$ is an M-matrix. Similar to (4.5), we have the following comparison principle for two mesh functions $v, w \in I\!R^{N+1}$:

$$\left.\begin{aligned}\hat L v \le \hat L w \quad \text{on } \omega,\\ v_0 \le w_0,\\ v_N \le w_N\end{aligned}\right\} \implies v \le w \quad \text{on } \bar\omega. \tag{4.42}$$

Theorem 4.20. *Let ω be a S-type mesh with $\sigma \geq 2$; see Sect. 2.1.3. Assume that the function $\tilde{\varphi}$ is piecewise differentiable and satisfies (2.8) and (2.14). Then the error of the simple upwind scheme satisfies*

$$\left|u_i - u_i^N\right| \leq \begin{cases} C\left(h + N^{-1} \max|\psi'|\right) & \text{for } i = 0, \ldots, qN - 1, \\ C\left(h + N^{-1}\right) & \text{for } i = qN, \ldots, N. \end{cases}$$

Proof. The numerical solution u^N is split analogously to the splitting of $u = v + w$ of Theorem 3.48: $u^N = v^N + w^N$ with

$$[\hat{L}v^N]_i = f_i \quad \text{for } i = 1, \ldots, N-1, \;\; v_0^N = v(0), \; v_N^N = v(1) = \gamma_1$$

and

$$[\hat{L}w^N]_i = 0 \quad \text{for } i = 1, \ldots, N-1, \;\; w_0^N = w(0), \; w_N^N = w(1) = 0.$$

Then the error is $u - u^N = (v - v^N) + (w - w^N)$ and we can estimate the error in v and w separately. For the regular solution component v Taylor expansions and (3.34a) give

$$\left|\hat{L}(v - v^N)_i\right| = \left|[\hat{L}v]_i - (\mathcal{L}v)_i\right| \leq Ch \quad \text{for } i = 1, \ldots, N-1.$$

Furthermore, $(v - v^N)_0 = (v - v^N)_N = 0$. Then the comparison principle (4.42) yields

$$\left|v_i - v_i^N\right| \leq C(1 - x_i)h$$

with some constant C, which is independent of ε. Thus,

$$\left\|v - v^N\right\|_{\infty,\omega} \leq Ch. \tag{4.43}$$

Using (4.42), one can show that

$$\left|w_i^N\right| \leq \bar{w}_i^N := C \prod_{k=1}^{i} \left(1 + \frac{\beta h_k}{2\varepsilon}\right)^{-1} \quad \text{for } i = 0, \ldots, N. \tag{4.44}$$

For $\xi \geq 0$ we have $\ln(1 + \xi) \geq \xi - \xi^2/2$ which implies

$$\bar{w}_i^N \leq \bar{w}_{qN}^N \leq N^{-\sigma/2} \exp\left(\frac{1}{2} \sum_{k=1}^{qN} \left(\frac{\beta h_k}{2\varepsilon}\right)^2\right) \leq CN^{-1} \quad \text{for } i = qN, \ldots, N;$$

see Remark 2.4. Hence,

$$\left|w_i - w_i^N\right| \le |w_i| + \left|w_i^N\right| \le CN^{-1} \quad \text{for } i = qN, \dots, N, \tag{4.45}$$

where we have used (3.34b).

For the truncation error with respect to the layer part w, Taylor expansions and (3.34b) give

$$\begin{aligned}\left|[\hat{L}(w - w^N)]_i\right| = \left|[\hat{L}w]_i\right| &\le C\varepsilon^{-2}\,(h_i + h_{i+1})\,\mathrm{e}^{-\beta x_{i-1}/\varepsilon} \\ &\le C\varepsilon^{-1}\mathrm{e}^{-\beta x_i/(2\varepsilon)}N^{-1}\max|\psi'| \\ &\le C\varepsilon^{-1}\bar{w}_i^N N^{-1}\max|\psi'| \quad \text{for } i = 1, \dots, qN-1,\end{aligned}$$

by (2.11) and (2.12). Note that $w_0 - w_0^N = 0$ and $\left|w_{qN} - w_{qN}^N\right| \le CN^{-1}$. Therefore, (4.42) yields

$$\left|(w - w^N)_i\right| \le C\left\{N^{-1} + \bar{w}_i^N N^{-1}\max|\psi'|\right\}, \quad \text{for } i = 0, \dots, qN-1,$$

for C chosen sufficiently large. Thus,

$$\left|w_i - w_i^N\right| \le CN^{-1}\max|\psi'| \quad \text{for } i = 0, \dots, qN-1.$$

Combine (4.43) and (4.45) with the last inequality to complete the proof. □

Remark 4.21. We are not aware of any results for B-type meshes that make use of this truncation error and barrier function technique. Also note that this technique needs $\sigma \ge 2$, while in Sect. 4.2.2 only $\sigma \ge 1$ was assumed. ♣

Remark 4.22. The technique of Sect. 4.2.2 also provides error estimates for the approximation of the first-order differences:

$$\varepsilon\left\|[(u - u^N)_x\right\|_{\infty,\omega} \le C\vartheta_{cd}^{[1]}(\bar{\omega}).$$

In [33] the authors use the barrier function technique to establish that the upwind scheme (4.41) on standard Shishkin meshes satisfies

$$\varepsilon\left|\left(u^N - u\right)_{x;i}\right| \le \begin{cases} CN^{-1}\ln N & \text{for } i = 0, \dots, qN-1, \\ CN^{-1} & \text{for } i = qN, \dots, N-1. \end{cases}$$

However, the technique in [33] makes strong use of the piecewise uniformity of the mesh. ♣

4.2.7 Discontinuous Coefficients and Point Sources

Consider the convection-diffusion problem in conservative form with a point source or a discontinuity of the convection coefficient at $d \in (0,1)$:

$$\begin{aligned} \mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f + \alpha\delta_d, \quad \text{in } (0,1), \\ u(0) = \gamma_0, \ \ u(1) = \gamma_1, \end{aligned} \tag{4.46}$$

where δ_d is the shifted Dirac-delta function $\delta_d(x) = \delta(x-d)$ with $d \in (0,1)$. Assume that $b \geq \beta_1 > 0$ on $(0,d)$ and $b \geq \beta_2 > 0$ on $(d,1)$ and set $\beta = \min\{\beta_1, \beta_2\}$. For the sake of simplicity, also assume that $c \geq 0$ and $c - b' \geq 0$ on $[0,1]$. Properties of the exact solution are studied in Sect. 3.4.1.3.

Following [88], we consider the upwinded finite difference method: Find $u^N \in I\!R^{N+1}$ with

$$\begin{aligned} \left[L^c u^N\right]_i := -\varepsilon u^N_{\bar{x}x;i} - (b^- u^N)_{x;i} + c_i u^N_i = f_i + \Delta_{d,i} \quad \text{for } i = 1, \dots, N-1, \\ u^N_0 = \gamma_0, \ \ u^N_N = \gamma_1, \end{aligned} \tag{4.47}$$

where $v^-(x) := \lim_{s \to x-0} v(s)$, $v_i^- := v^-(x_i)$ and

$$\Delta_{d;i} := \begin{cases} h_{i+1}^{-1} & \text{if } d \in [x_i, x_{i+1}), \\ 0 & \text{otherwise} \end{cases}$$

is an approximation of the shifted *Dirac*-delta function.

The discrete operator L^c enjoys the stability property (4.8c). Therefore, it is sufficient to derive bounds for the truncation error $\left\|L^c(u-u^N)\right\|_{-1,\infty,\omega}$. Adapting the notation from Sect. 4.2.2, we set

$$\begin{aligned} \left(\mathcal{A}^c v\right)(x) &:= \varepsilon v'(x) + \left(b^- v\right)(x) + \int_x^1 \left(cv\right)(s)\mathrm{d}s, \\ \mathcal{F}(x) &:= \int_x^1 f(s)\mathrm{d}s + \begin{cases} \alpha & \text{if } x_i \leq d, \\ 0 & \text{otherwise,} \end{cases} \\ \left[A^c v\right]_i &:= \varepsilon v_{\bar{x};i} + \left(b^- v\right)_i + \sum_{k=i}^{N-1} h_{k+1} \left(cv\right)_k \end{aligned}$$

and

$$F_i := \sum_{k=i}^{N-1} h_{k+1} f_k + \begin{cases} \alpha & \text{if } x_i \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Inspecting (4.46) and (4.47), we see

$$\mathcal{A}^c u - \mathcal{F} \equiv \text{const} \quad \text{on} \;\; (0,1) \quad \text{and} \quad A^c u^N - F \equiv \text{const} \quad \text{on} \;\; \omega.$$

Then, analogously to (4.13), we obtain

$$\begin{aligned}&\left(A^c u - \mathcal{A}^c u - F + \mathcal{F}\right)_i \\ &\quad = \varepsilon \left(u_{\bar{x}} - u'\right)_i + \sum_{k=i}^{N-1} h_{k+1} \left(c_k u_k - f_k\right) - \int_{x_i}^{x_N} \left(cu - f\right)(x) \mathrm{d}x\end{aligned}$$

since the contributions from the δ functions and its discretisation cancel.

Proceeding along the lines of Sect. 4.2.2, we get.

Theorem 4.23. *Let u be the solution of* (4.46). *Then the error of the simple upwind scheme* (4.47) *satisfies*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le C \max_{i=1,\dots,N} \int_{I_i} \bigl(1 + |u'(x)|\bigr) \mathrm{d}x.$$

Corollary 4.24. *Theorem 4.23 and the derivative bounds* (3.36) *yield*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le C \vartheta_{cd^i}^{[1]}(\bar{\omega}),$$

where

$$\vartheta_{cd^i}^{[p]}(\bar{\omega}) := \max_{k=1,\dots,N} \int_{I_k} \left(1 + \varepsilon^{-1} \mathrm{e}^{-\beta_1 s / p\varepsilon} + H_d(s) \varepsilon^{-1} \mathrm{e}^{-\beta_2 (s-d)/p\varepsilon}\right) \mathrm{d}s,$$

and H_d is the shifted Heaviside function.

Remark 4.25. Layer-adapted meshes for (4.46) have been introduced in Sect. 2.1.5. We have the uniform error bounds

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le \begin{cases} CN^{-1} \ln N & \text{for the Shishkin mesh and} \\ CN^{-1} & \text{for the Bakhvalov mesh} \end{cases}$$

if $\sigma_1 \ge 1$ and $\sigma_2 \ge 1$; see Sect. 2.1.5 for the bounds on $\vartheta_{cd^i}^{[p]}(\bar{\omega})$. ♣

Numerical results

Let us verify experimentally the theoretical result of Theorem 4.23. Our test problem is

$$-\varepsilon u'' - u' = x + \delta_{1/2} \;\; \text{in} \;\; (0,1), \;\; u(0) = u(1) = 0. \tag{4.48}$$

Table 4.2 The upwind difference scheme for (4.48); errors in the discrete maximum norm

	Bakhvalov mesh		Shishkin mesh	
N	error	rate	error	rate
2^7	2.822e-2	0.95	3.898e-2	0.78
2^8	1.458e-2	0.97	2.277e-2	0.81
2^9	7.447e-3	0.98	1.299e-2	0.84
2^{10}	3.779e-3	0.99	7.280e-3	0.85
2^{11}	1.909e-3	0.99	4.027e-3	0.87
2^{12}	9.610e-4	0.99	2.204e-3	0.88
2^{13}	4.828e-4	1.00	1.197e-3	0.89
2^{14}	2.422e-4	1.00	6.454e-4	0.90
2^{15}	1.214e-4	1.00	3.462e-4	0.91
2^{16}	6.080e-5	—	1.848e-4	—

The results presented in Table 4.2 are in fair agreement with Theorem 4.23. Again the Bakhvalov mesh gives more accurate results than the Shishkin mesh.

Further remarks

The traditional truncation error and barrier function technique of Sect. 4.2.6 can also be applied to problems with interior layers. Farrell et al. [35] consider the problem of finding $u \in C^2((0,d) \cap (d,1)) \cup C^1[0,1]$ such that

$$-\varepsilon u'' - bu' = f \text{ in } (0,d) \cup (d,1), \quad u(0) = u(1) = 0,$$

where at the point $d \in (0,1)$ the convection coefficient changes sign:

$$b(x) > 0 \text{ for } x \in (0,d), \quad b(x) < 0 \text{ for } x \in (d,1) \text{ and } |b(x)| \geq \beta > 0.$$

The solution u and its derivatives satisfy

$$\left|u^{(k)}(x)\right| \leq C\left\{1 + \varepsilon^{-k} \mathrm{e}^{-\beta|x-d|/\varepsilon}\right\} \quad \text{for } k = 0, 1, \ldots, q \text{ and } x \in [0,1],$$

where the maximal order q depends on the smoothness of the data. Using the barrier function technique of Sect. 4.2.6, in [35] the authors establish the error bound

$$\left\|u - u^N\right\|_{\infty,\omega} \leq CN^{-1} \ln N$$

for the simple upwind scheme (4.41) on a Shishkin mesh.

4.2.8 Quasilinear Problems

We now extend the results of Sect. 3.4.1 and 4.2.1 to the class of quasilinear problems described by

$$\begin{aligned} \mathcal{T}^c u &:= -\varepsilon u'' - b(\cdot,u)' + c(\cdot,u) = 0 \quad \text{in } (0,1), \\ u(0) &= \gamma_0, \ \ u(1) = \gamma_1 \end{aligned} \tag{4.49}$$

with $0 < \varepsilon \ll 1$, $\partial_u b \geq \beta > 0$ and $\partial_u c \geq 0$ and its simple upwind discretisation

$$\begin{aligned} \left[T^c u^N\right]_i &:= -\varepsilon u^N_{\bar{x}x;i} - b(\cdot,u^N)_{x;i} + c(\cdot,u^N)_i = 0 \quad \text{for } i = 1,\dots,N-1, \\ u^N_0 &= \gamma_0, \ \ u^N_N = \gamma_1. \end{aligned}$$

First, for the solution u of (4.49) and its derivatives, the bounds (3.30) hold true too; see [166]:

$$\left|u^{(k)}(x)\right| \leq C\left\{1 + \varepsilon^{-k}\mathrm{e}^{-\beta x/\varepsilon}\right\} \quad \text{for } k = 0,1,\dots,q \ \text{ and } x \in [0,1],$$

where the maximal order q depends on the smoothness of the data.

Next we use a standard linearisation technique to study stability properties of $\mathcal{T}^c$. For any two functions $v, w \in W^{1,\infty}(0,1)$ define the linear operator

$$\mathcal{L}^c y = \mathcal{L}^c[v,w]y := -\varepsilon y'' - (py)' + qy,$$

with

$$p(x) = \int_0^1 \partial_u b\big(x, w(x) + s(v-w)(x)\big)\mathrm{d}s \geq \beta$$

and

$$q(x) = \int_0^1 \partial_u c\big(x, w(x) + s(v-w)(x)\big)\mathrm{d}s \geq 0.$$

The linearised operator $\mathcal{L}^c$ is constructed such that $\mathcal{L}^c(v-w) = \mathcal{T}^c v - \mathcal{T}^c w$ on $(0,1)$. The analysis of Sect. 3.4.1 can be applied to $\mathcal{L}^c$. We get

$$|||v-w|||_{\varepsilon,\infty} \leq \|\mathcal{T}^c v - \mathcal{T}^c w\|_{-1,\infty} \qquad \text{for all } v,w \in W^{1,\infty} \text{ with } v-w \in W^{0,\infty}_1.$$

Similarly, we linearise T^c. For arbitrary mesh functions $v, w \in I\!R^{N+1}$ set

$$[L^c y]_i = \left[L^c[v,w]y\right]_i := -\varepsilon y_{\bar{x}x;i} - (py)_{x;i} + q_i y_i$$

with p and q defined above. Again the linearised operator L^c is constructed such that $L^c(v-w) = T^c v - T^c w$ on ω. The technique from Sect. 4.2.1 can be used to obtain

$$|||v-w|||_{\varepsilon,\infty,\omega} \le \|Tv - Tw\|_{-1,\infty,\omega} \qquad \text{for all } \ v,w \in I\!R^{N+1} \ \text{ with } \ v-w \in I\!R_0^{N+1}.$$

In order to conduct an error analysis, take $v = u$ and $w = u^N$ and proceed as in Sect. 4.2.2 and 4.2.4 to get a priori and a posteriori error bounds.

Remark 4.26. Discretisations of quasilinear problems can also be analysed using the truncation error and barrier function technique of Sect. 4.2.6 or using the (ℓ_∞, ℓ_1) stability (4.8b); see [36, 149] and [106], respectively. ♣

4.2.9 Derivative Approximation

In a number of applications the user is more interested in the approximation of the gradient or of the flow than in the solution itself. In Sect. 4.2.2 the following error bound for the weighted derivative was established:

$$\varepsilon \left\| \left(u - u^N\right)' \right\|_\infty \le C \vartheta_{cd}^{[1]}(\bar{\omega}).$$

Note that $u'(0) \approx \varepsilon^{-1}$ by (3.30). Therefore, multiplying by ε in this estimate is the correct weighting. However, looking at the bounds (3.30) for the derivative of u, we see that the derivative is bounded uniformly away from the layer, where we therefore expect that a similar bound holds without the weighting by ε.

Theorem 4.27. *Let u be the solution of* (4.1) *and u^N that of* (4.3)*. Then*

$$\left|u_i' - u_{\bar{x};i}^N\right| \le C h_i^{-1} \left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2 \quad \textit{for } \ i = 1, \dots, N.$$

Proof. We work from the error expansion of Sect. 4.2.3:

$$\left(u - u^N\right)_{\bar{x};i} = \frac{u_i - \psi_i - u_i^N - \left(u_{i-1} - \psi_{i-1} - u_{i-1}^N\right)}{h_i} + \frac{\psi_i - \psi_{i-1}}{h_i}$$

Then

$$\left|\left(u - u^N\right)_{\bar{x};i}\right| \le C h_i^{-1} \left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2, \tag{4.50}$$

by Lemma 4.13 and Theorem 4.14. Furthermore,

$$\begin{aligned}|u_i' - u_{\bar{x};i}| &= \frac{1}{h_i}\left|\int_{I_i}(s-x_{i-1})u''(s)\mathrm{d}s\right| \\ &\le \frac{C}{h_i}\int_{I_i}(s-x_{i-1})\left(1+\varepsilon^{-2}\mathrm{e}^{-\beta s/\varepsilon}\right)\mathrm{d}s \le Ch_i^{-1}\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,\end{aligned}$$

by (3.30) and Lemma 4.16. Finally, a triangle inequality yields the assertion. □

Layer-adapted meshes.

Let us illustrate Theorem 4.27 by applying it to two standard layer-adapted meshes.

Bakhvalov meshes (Sect. 2.1.1) can be generated by equidistributing

$$M_{Ba}(\xi) = \max\left\{1, \frac{K\beta}{\varepsilon}\exp\left(-\frac{\beta\xi}{\sigma\varepsilon}\right)\right\} \quad \text{for } \xi \in [0,1].$$

Clearly M_{Ba} is continuous and monotonically decreasing. Therefore,

$$\frac{1}{N}\int_0^1 M_{Ba}(s)\mathrm{d}s = \int_{I_i} M_{Ba}(s)\mathrm{d}s \le h_i M_{Ba}(x_{i-1})$$

and

$$\frac{1}{h_i} \le CNM_{Ba}(x_{i-1}) = CN\max\left\{1, \frac{K\beta}{\varepsilon}\exp\left(-\frac{\beta x_{i-1}}{\sigma\varepsilon}\right)\right\}.$$

Now, (2.6) and Theorem 4.27 yield

$$\left|u_i' - u_{\bar{x};i}^N\right| \le CN^{-1}\max\left\{1, \frac{K\beta}{\varepsilon}\exp\left(-\frac{\beta x_{i-1}}{\sigma\varepsilon}\right)\right\} \quad \text{if } \sigma \ge 2.$$

A very similar result was established by Kopteva and Stynes [72] through a different technique.

Shishkin meshes (Sect. 2.1.3). For these meshes the local step sizes satisfy

$$h_i = \frac{\sigma\varepsilon}{q\beta}\frac{\ln N}{N} \quad \text{for } i = 1,\dots,qN \quad \text{and} \quad h_i \ge N^{-1} \quad \text{for } i = qN+1,\dots,N.$$

Hence, Theorem 4.27 gives

$$\left|u_i' - u_{\bar{x};i}^N\right| \le \begin{cases} C\varepsilon^{-1}N^{-1}\ln N & \text{for } i = 1,\dots,qN-1, \\ CN^{-1}\ln^2 N & \text{for } i = qN,\dots,N. \end{cases}$$

Outside the layer region this result is slightly suboptimal. Both in [43] and in [72] it was shown by means of barrier function techniques that the approximation is a factor of $\ln N$ better, i.e.,

$$\left|u'_{i+1} - u^N_{x;i}\right| \le CN^{-1} \ln N \quad \text{for } i = qN + 1, \dots, N.$$

4.3 Second-Order Difference Schemes

As simple upwinding yields only low accuracy, it is natural to look for higher-order alternatives. For one-dimensional problems inverse-monotone schemes exist that are second-order accurate. One will be studied in Sect. 4.3.1. However, the construction of inverse-monotone difference schemes in two or more dimensions is an open problem.

Sect. 4.3.2 summarises stability and convergence results for an unstabilised central difference scheme.

Possible other approaches to higher-order schemes include:

- the combination of two (or more) approximations by a first-order upwind scheme on nested meshes by means of the Richardson extrapolation technique.
- their combination of simple upwinding with higher-order unstabilised schemes using defect correction.

Both approaches have the advantage that linear problems involving only stabilised operators have to be solved. Sect. 4.3.3 is devoted to these techniques.

Finally, we like to mention the HODIE technique which was used by Clavero et al. [28] to construct and analyse second- and third-order compact schemes on Shishkin meshes.

4.3.1 Second-Order Upwind Schemes

Because of their stability properties, they can be analysed with the techniques similar to those of Sect. 4.2. Consider the convection-diffusion problem in conservative form:

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1. \tag{4.9}$$

Let ρ_i, $i = 1, \dots, N$ be arbitrary with $\rho_i \in [1/2, 1]$. Define the weighted step sizes

$$\chi_i = \rho_{i+1} h_{i+1} + (1 - \rho_i) h_i \quad \text{for } i = 1, \dots, N-1, \quad \chi_0 = \chi_N = 0.$$

Then following Andreev and Kopteva [11], our discretisation is: Find $u^N \in I\!R^{N+1}$ such that

$$\left[L^\rho u^N\right]_i = f_{\rho;i} \quad \text{for } i = 1,\ldots,N-1, \quad u_0^N = \gamma_0,\; u_N^N = \gamma_1, \tag{4.51}$$

where

$$[L^\rho v]_i := -\varepsilon v_{\bar{x}\acute{x};i} - (\rho b v + (1-\rho)(bv)_-)_{\acute{x};i} + (cv)_{\rho;i},$$
$$v_{\acute{x};i} = \frac{v_{i+1} - v_i}{\chi_i}, \quad v_{-;i} = v_{i-1}$$

and

$$v_{\rho;i} = \frac{\rho_{i+1} v_{i+1} + (1 - \rho_{i+1} + \rho_i) v_i + (1 - \rho_i) v_{i-1}}{2}.$$

The approximation of the first-order derivative is a weighted combination of upwinded and downwinded operators. At first glance the approximation of the lowest-order term and of the right-hand side seems to be very non-standard. It is chosen such that

$$\chi_i g_{\rho;i} \quad \text{is a second-order approximation of} \quad \int_{x_{\rho;i-1/2}}^{x_{\rho;i+1/2}} g(x)\mathrm{d}x$$

with $x_{\rho;i-1/2} = x_{i-1} + \rho_i h_i$. For $\rho \equiv 1/2$ we obtain a central difference scheme, while for $\rho \equiv 1$ the mid-point upwind scheme is recovered.

This second-order upwind scheme is very similar to the streamline-diffusion FEM, which is studied in Sect. 5.3.2.

4.3.1.1 Stability of the Discrete Operator

The stability analysis of the operator L^ρ is complicated by the positive contribution of the discretisation $(cu^N)_{\rho;i}$ of the lowest order term to the offdiagonal entries of the system matrix. It is difficult to ensure the correct sign pattern for the application of the M-matrix criterion (Lemma 3.14). Instead we follow [85] which adapts the technique from [11].

Set

$$[A^\rho v]_i := \varepsilon v_{\bar{x};i} + \rho(bv)_i + (1-\rho_i)(bv)_{i-1} - \sum_{j=1}^{i-1} \chi_j (cv)_{\rho;j}, \quad i = 1,\ldots,N.$$

This operator is related to L^ρ by $(A^\rho v)_{\acute{x}} = -L^\rho v$. Then any function $v \in I\!R_0^{n+1}$ can be represented as

$$v_i = \frac{W_N}{V_N} V_i - W_i,$$

where V and W are the solution of the difference equations

$$[A^\rho V]_i = 1, \quad i = 1, 2, \dots, N,\ V_0 = 0$$

and

$$[A^\rho W]_i = [A^\rho v]_i + c, \quad i = 1, 2, \dots, N,\ W_0 = 0$$

for any constant $c \in I\!R$.

Proposition 4.28. *Assume that*

$$1 \geq \rho_i \geq \max\left\{\frac{1}{2}, 1 - \frac{\varepsilon}{b_{i-1}h_i}\right\} \ \textit{for}\ \ i = 1, \dots, N, \tag{4.52a}$$

and

$$\|c\|_\infty h \leq \beta/4. \tag{4.52b}$$

Then the matrix associated with A^ρ is an M-matrix.

Proof. First (4.52a) ensures that the offdiagonal entries of A^ρ are nonpositive, while (4.52b) implies that the diagonal entries are positive.

For any monotonically increasing mesh function $z_i \geq 0$ we have

$$[A^\rho z]_i > \rho_i b_i z_i - \frac{\|c\|_\infty}{2} \sum_{j=1}^{i-1} \chi_j \left(z_{j+1} + z_j\right) \geq \frac{\beta}{4} z_i - \|c\|_\infty \sum_{j=1}^{i-2} \chi_j z_{j+1},$$

by (4.52).

Now let

$$z_0 = z_1 = z_2 = 1, \text{ and } z_i = \prod_{k=3}^{i} \left(1 + \frac{4\|c\|_\infty}{\beta} \chi_{k-2}\right) \ \text{ for } \ i = 3, \dots, N. \tag{4.53}$$

Clearly $z_i \leq \mathrm{e}^{4\|c\|_\infty/\beta}$ and

$$\frac{\beta}{4} z_i - \|c\|_\infty \chi_{i-2} z_{i-1}, \geq \frac{\beta}{4} z_{i-1}, \ \text{ by (4.52b)}.$$

Then induction for i yields

$$[A^\rho z]_i > \frac{\beta}{4} \ \text{ for } \ i = 1, \dots, N.$$

Finally, application of Lemma 3.14 with the test function $e_i = z_i$ completes the proof.

The M-matrix property of A^ρ and the function z from (4.53) can now be used to establish bounds on V and W:

$$0 < V_i \le \frac{4}{\beta} z_i \le \frac{4}{\beta} \mathrm{e}^{4\|c\|/\beta} \quad \text{and} \quad |W_i| \le V_i \, \|A^\rho v + c\|_{\infty,\omega}\,, \ \ i = 1, \ldots, N.$$

We get our main stability result.

Theorem 4.29. *Let ρ and h satisfy* (4.52). *Then the operator L^ρ is $(\ell_\infty, w^{-1,\infty})$-stable with*

$$\|v\|_{\infty,\omega} \le \frac{8}{\beta} \mathrm{e}^{4\|c\|_\infty/\beta} \min_{c \in \mathbb{R}} \|A^\rho v + c\|_{\infty,\omega} \quad \textit{for all} \ \ v \in \mathbb{R}_0^{N+1}.$$

Remark 4.30. The (ℓ_∞, ℓ_1) stability

$$\|v\|_{\infty,\omega} \le C \sum_{k=1}^{N-1} \chi_k \big| [L^\rho v]_k \big|$$

is an immediate consequence of the negative-norm stability.

Analyses of second-order upwind schemes based on this type of stability inequality were given by Andreev and Savin [12] for a modification of Samarskii's scheme on a Shishkin mesh [12], and on Bakhvalov meshes [4] and by Linß [87] for quasi-linear problems discretised on S-type meshes. ♣

4.3.1.2 Error Analysis

We now study the approximation error of the scheme (4.51). Following [11, 85], we base our analysis on the $(l_\infty, w^{-1,\infty})$ stability of Theorem 4.29.

Choose

$$\rho_i = \begin{cases} 1/2 & \text{if } \ h_i \le 2\varepsilon / b_{i-1}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.54}$$

This choice satisfies the assumptions of Theorem 4.29. Therefore,

$$\big\|u - u^N\big\|_{\infty,\omega} \le C \min_{c \in \mathbb{R}} \big\|A^\rho (u - u^N) + c\big\|_{\infty,\omega}\,. \tag{4.55}$$

Set

$$\left(\mathcal{A}^c v\right)(x) := \varepsilon v'(x) + \big(bv\big) + \int_x^{x_{\rho;1/2}} \big(cv\big)(s)\mathrm{d}s, \quad \mathcal{F} := \int_x^{x_{\rho;1/2}} f(s)\mathrm{d}s$$

and

$$F_i^\rho := -\sum_{k=1}^{i-1} \chi_k f_{\rho;k}$$

Inspecting (4.9) and (4.51), we see that

$$\mathcal{A}^c u - \mathcal{F} \equiv \alpha \ \text{ on } \ (0,1) \ \text{ and } \ A^\rho u^N - F^\rho \equiv a \ \text{ on } \ \omega$$

with constants α and a because $\mathcal{L}^c v = -(\mathcal{A}^c v)'$ and $f = -\mathcal{F}'$ on $(0,1)$, and $L^\rho v = (A^\rho v)_{\acute{x}}$ and $f = F^\rho_{\acute{x}}$ on ω. Take $c = a - \alpha$ in (4.55) in order to get

$$\left\|u - u^N\right\|_{\infty,\omega} \le C \max_{i=1,\dots,N} \left|[A^\rho u]_i - (\mathcal{A}^c u)(x_{\rho;i}) + \mathcal{F}(x_{\rho;i}) - F^\rho_i\right|. \tag{4.56}$$

Set $g := cu - f$,

$$[B^\rho u]_i := \varepsilon u_{\bar{x};i} + \rho_i b_i u_i + (1-\rho_i) b_{i-1} u_{i-1}$$

and

$$\mathcal{B}(x) := \varepsilon u'(x) + \big(bu\big)(x).$$

Then

$$\begin{aligned}
&[A^\rho u]_i - (\mathcal{A}^c u)(x_{\rho;i}) + \mathcal{F}(x_{\rho;i}) - F^\rho_i \\
&\qquad = [B^\rho u]_i - (\mathcal{B}^c u)(x_{\rho;i}) + \int_{x_{\rho;1/2}}^{x_{\rho;i-1/2}} g(s)\mathrm{d}s - \sum_{j=1}^{i-1} \chi_j g_{\rho;j}.
\end{aligned} \tag{4.57}$$

When bounding the first term on the right-hand side of (4.57), we have to distinguish two cases: $\sigma_i = 1$ and $\sigma_i = 1/2$.

For $\sigma_i = 1$ we have

$$\big[B^\rho u\big]_i - \big(\mathcal{B}u\big)(x_{\rho;i}) = \varepsilon \left\{ \frac{u_i - u_{i-1}}{h_i} - u_i' \right\} = \frac{\varepsilon}{h_i} \int_{I_i} u''(t)(t - x_{i-1})\mathrm{d}t.$$

Thus,

$$\left|\big[B^h u\big]_i - \big(Bu\big)(x_{\rho,i})\right| \le C \int_{I_i} \left(1 + \varepsilon^{-2} \mathrm{e}^{-\beta t/\varepsilon}\right)(t - x_{i-1})\mathrm{d}t, \tag{4.58}$$

by (3.30) and because $\varepsilon/h_i < \|b\|_\infty/2$ for $\rho_i = 1$.

Next, consider $\sigma_i = 1/2$. Then

$$\begin{aligned}
&\big[B^\rho u\big]_i - \big(\mathcal{B}u\big)(x_{\rho,i}) \\
&\qquad = \varepsilon \left\{ \frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2} \right\} + \frac{b_i u_i + b_{i-1} u_{i-1}}{2} - b_{i-1/2} u_{i-1/2},
\end{aligned}$$

where $u_{i-1/2} = u(x_{i-1/2})$. Taylor expansions for u and u' about x_i give

$$\varepsilon \left| \frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2} \right| \le \frac{3\varepsilon}{2} \int_{I_i} \left|u'''(t)\right|(t - x_{i-1})\mathrm{d}t$$

and

$$\left|\frac{b_i u_i + b_{i-1}u_{i-1}}{2} - b_{i-1/2}u_{i-1/2}\right| \le \frac{3}{2}\int_{I_i} \left|(bu)''(t)\right|(t-x_{i-1})\mathrm{d}t.$$

From this and (3.30) we see that (4.58) holds for $\sigma_i = 1/2$ too.

Finally, we bound the second term of the right-hand side of (4.57):

$$\int_{x_{\rho,j-1/2}}^{x_{\rho,j+1/2}} g(s)\mathrm{d}s - \chi_{\rho,j}g_{\rho,j} = \int_{x_{\rho,j-1/2}}^{x_{\rho,j+1/2}} \left(g(s) - g_{\rho,j}\right)\mathrm{d}s.$$

The representation

$$g(s) = g_{j+1} - g'_{j+1}(x_{j+1} - s) + \int_s^{x_{j+1}} g''(t)(t-s)\mathrm{d}t$$

yields

$$\left|g_{\rho,j} - g(s) - (x_{\rho,j} - s)g'_{j+1}\right| \le 2\int_{x_{j-1}}^{x_{j+1}} \left|g''(t)\right|(t-x_{j-1})\mathrm{d}t.$$

Next,

$$\begin{aligned}\left|\int_{x_{\rho,j-1/2}}^{x_{\rho,j+1/2}} g(s)\mathrm{d}s - \chi_{\rho,j}g_{\rho,j}\right| &\le 2(h_j + h_{j+1})\int_{x_{j-1}}^{x_{j+1}} \left|g''(t)\right|(t-x_{j-1})\mathrm{d}t \\ &\le C(h_j + h_{j+1})\int_{x_{j-1}}^{x_{j+1}} \left(1+\varepsilon^{-2}\mathrm{e}^{-\beta t/\varepsilon}\right)(t-x_{j-1})\mathrm{d}t,\end{aligned}$$

by (3.30) and because $g = cu - f$.

Combining this estimate with (4.56), (4.55) and (4.58), we get

$$\left\|u - u^N\right\|_{\infty,\omega} \le C \max_{i=1,\dots,N-1} \int_{x_{i-1}}^{x_{i+1}} \left(1+\varepsilon^{-2}\mathrm{e}^{-\beta t/\varepsilon}\right)(t-x_{i-1})\mathrm{d}t.$$

Finally, Lemma 4.16 gives the following convergence result.

Theorem 4.31. *Let u^N be the approximate solution to (4.9) obtained by the difference scheme (4.51) with ρ chosen according to (4.54). Assume $\|c\|_\infty h \le \beta/4$. Then*

$$\left\|u - u^N\right\|_{\infty,\omega} \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2.$$

Quasilinear problems

The conclusion of the Theorem also holds when (4.51) is adapted to discretise the quasilinear problem

$$-\varepsilon u'' - b(\cdot, u)' + c(\cdot, u) = 0 \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1$$

with $0 < \varepsilon \ll 1$, $\partial_u b \geq \beta > 0$ and $\partial_u c \geq 0$. The scheme reads: Find $u^N \in I\!R^{n+1}$ such that

$$-\varepsilon u^N_{\bar{x}\acute{x};i} - \left(\rho b(\cdot, u^N) + (1-\rho) b(\cdot, u^N)_-\right)_{\acute{x};i} + c(x_{\rho;i}, u^N_{\rho;i}) = 0 \quad \text{on } \omega,$$
$$u^N_0 = \gamma_0, \;\; u^N_N = \gamma_1$$

with the stabilisation parameter chosen to satisfy, e. g,

$$\rho_i = \begin{cases} 1/2 & \text{if } h_i \leq 2\varepsilon / \|b\|_\infty, \\ 1 & \text{otherwise.} \end{cases}$$

Discontinuous coefficients and point sources

Consider the convection-diffusion problem (4.46) with a point source:

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f + \alpha\delta_d, \;\; \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

with the shifted Dirac-delta function $\delta_d(x) = \delta(x - d)$. The coefficient b may also have a discontinuity at $x = d$. Assume that $b \geq \beta_1 > 0$ on $(0, d)$ and $b \geq \beta_2 > 0$ on $(d, 1)$.

Using (4.51) we seek an approximation $u^N \in I\!R^{n+1}$ with

$$\left[L^\rho u^N\right]_i = f_{\rho;i} + \Delta_{d,\rho;i} \quad \text{for } i = 1, \ldots, N-1, \;\; u^N_0 = \gamma_0, \;\; u^N_N = \gamma_1$$

with

$$\Delta_{d,\rho;i} := \begin{cases} \chi_i^{-1} & \text{if } d \in [x_{\rho;i-1/2}, x_{\rho;i+1/2}), \\ 0 & \text{otherwise.} \end{cases}$$

Then the above technique and the a priori bounds (3.36) for the derivatives of u yield the error estimate [88]

$$\left\|u - u^N\right\|_{\infty,\omega} \leq C \left(\vartheta^{[2]}_{cd^i}(\bar{\omega})\right)^2,$$

where $\vartheta^{[2]}_{cd^i}(\bar{\omega})$ has been defined in Sect. 2.1.5.

Remark 4.32. Roos and Zarin [143] study the difference scheme generated by the streamline diffusion FEM on Shishkin and on Bakhvalov-Shishkin meshes for the discretisation of a problem with a point source. They prove (almost) second-order convergence in the discrete maximum norm too. ♣

A posteriori error estimates

in the maximum norm for (4.9) discretised by (4.51) can be derived using the $(L_\infty, W^{-1,\infty})$-stability (3.29c) of the continuous operator $\mathcal{L}^c$. However, compared to Sect. 4.2.4 the analysis becomes more technical. Therefore, we refer the reader to the article by Kopteva [64]. A flavour of the technique is given in Sect. 4.3.3.1 where a defect-correction method is analysed.

4.3.1.3 The Barrier Function Technique

Stynes and Roos [153] study a hybrid difference scheme on a Shishkin mesh (with $q = 1/2$ and $\sigma > 4$). Their scheme uses central differencing on the fine part of the mesh and the mid-point upwind scheme on the coarse part.

Let us consider the convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1, \tag{4.1}$$

with $b \geq \beta > 0$ and $c \geq 0$ on $[0,1]$. This is discretised on a Shishkin mesh—see Sect. 2.1.3—using the difference scheme

$$\left[Lu^N\right]_i = \tilde{f}_i \quad \text{for } i = 1, \dots, N-1, \;\; u_0^N = \gamma_0, \;\; u_N^N = \gamma_1 \tag{4.59}$$

with

$$[Lv]_i := \begin{cases} -\varepsilon v_{\bar{x}\hat{x};i} - b_i v_{\mathring{x};i} + c_i v_i & \text{if } b_i h_i \leq 2\varepsilon, \\ -\varepsilon v_{\bar{x}\hat{x};i} - b_{i+1/2} v_{x;i} + (c_i v_i + c_{i+1} v_{i+1})/2 & \text{otherwise,} \end{cases}$$

and

$$\tilde{f}_i := \begin{cases} f_i & \text{if } b_i h_i \leq 2\varepsilon, \\ f_{i+1/2} & \text{otherwise.} \end{cases}$$

For N larger than a certain threshold value N_0, the matrix associated with L is an M-matrix and central differencing is used exclusively on the fine part of the mesh.

Theorem 4.33. *Let ω be a Shishkin mesh with $\sigma \geq 2$; see Sect. 2.1.3. Then the error of the upwinded scheme* (4.59) *applied to* (4.1) *satisfies*

$$\left|u_i - u_i^N\right| \leq \begin{cases} CN^{-2}\ln^2 N & \text{for } i = 0,\dots,qN-1, \\ CN^{-2} & \text{for } i = qN,\dots,N, \end{cases}$$

if N is larger than a certain threshold value.

Remark 4.34. A similar scheme generated by streamline-diffusion stabilisation was analysed by Stynes and Tobiska [154] with special emphasis on the choice of the mesh parameter σ. There it was first established that the mesh parameter σ should be chosen equal (or greater than) to the formal order of the scheme. ♣

Proof (of Theorem 4.33). Start with the truncation error. When $2\varepsilon < b_i h_i$ we have the bound

$$\left|[Lg]_i - (\mathcal{L}g)_{i+1/2}\right| \leq C\Bigg\{\varepsilon\int_{x_{i-1}}^{x_{i+1}} |g'''(s)|\,\mathrm{d}s + h_{i+1}\int_{x_i}^{x_{i+1}} \left[|g'''(s)| + |g''(s)|\right]\mathrm{d}s\Bigg\}, \tag{4.60a}$$

otherwise we use

$$\left|[Lg]_i - (\mathcal{L}g)_i\right| \leq C\int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon|g'''(s)| + |g''(s)|\right]\mathrm{d}s \tag{4.60b}$$

and, if $h_i = h_{i+1}$,

$$\left|[Lg]_i - (\mathcal{L}g)_i\right| \leq Ch_i\int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon|g^{(4)}(s)| + |g'''(s)|\right]\mathrm{d}s. \tag{4.60c}$$

For the analysis we split the numerical solution u^N analogously to the splitting $u = v + w$ of Theorem 3.48 and Remark 3.50: $u^N = v^N + w^N$ with

$$\left[Lv^N\right]_i = \tilde{f}_i \quad \text{for } i = 1,\dots,N-1,\ v_0^N = v(0),\ v_N^N = v(1)$$

and

$$\left[Lw^N\right]_i = 0 \quad \text{for } i = 1,\dots,N-1,\ w_0^N = w(0),\ w_N^N = w(1).$$

Then the error is $u - u^N = (v - v^N) + (w - w^N)$ and we estimate the error in v and w separately.

For the regular solution component Theorem 3.48, Remark 3.50 and (4.60) give

$$\left|\left[L(v-v^N)\right]_i\right| = \left|[Lv]_i - \tilde{f}_i\right| \le \begin{cases} CN^{-1} & \text{for } i = qN, \\ CN^{-2} & \text{otherwise.} \end{cases}$$

Note that $(v-v^N)_0 = (v-v^N)_N = 0$. Now set

$$\varphi_i = \begin{cases} 1 & \text{for } i = 0,\dots,qN, \\ \prod_{k=qN+1}^{i}\left(1+\dfrac{\beta h_k}{\varepsilon}\right)^{-1} & \text{for } i = qN+1,\dots,N. \end{cases}$$

Clearly $\varphi_0 \ge 0$ and $\varphi_N \ge 0$. Furthermore,

$$[L\varphi]_i \ge \begin{cases} 0 & \text{for } i \ne qN, \\ \dfrac{\beta}{2h_{qN+1}} \ge \dfrac{\beta(1-q)}{2}N & \text{for } i = qN. \end{cases}$$

Application of a comparison principle with the barrier function $CN^{-2}(1-x_i+\varphi_i)$ yields

$$\left\|v-v^N\right\|_{\infty,\omega} \le CN^{-2}, \tag{4.61}$$

since the matrix associated with L is inverse monotone as mentioned before.

Next, consider the layer component w. Let

$$\psi_i := \begin{cases} \prod_{k=1}^{i}\left(1+\dfrac{\beta h_k}{\varepsilon}\right)^{-1} + \prod_{k=1}^{qN}\left(1+\dfrac{\beta h_k}{\varepsilon}\right)^{-1} & \text{for } i = 1,\dots,qN, \\ 2\prod_{k=1}^{i}\left(1+\dfrac{\beta h_k}{\varepsilon}\right)^{-1} & \text{for } i = qN,\dots,N. \end{cases}$$

The inverse monotonicity of the discrete operator L yields

$$\left|w_i^N\right| \le |v(0)-\gamma_0|\,\psi_i \quad \text{for } i = 0,\dots,N,$$

because $L\psi \ge 0$. Furthermore, $|w_i| \le C\mathrm{e}^{-\beta x_i/\varepsilon} \le C\psi_i$. Thus,

$$\left|w_i - w_i^N\right| \le C\psi_i \quad \text{for } i = 0,\dots,N.$$

Now the argument that lead to (4.45) for the first-order scheme is imitated to establish

$$\left|w_i - w_i^N\right| \le CN^{-2} \;\text{ for } i = qN,\dots,N, \tag{4.62}$$

if $\sigma \ge 2$ in the construction of the Shishkin mesh (Sect. 2.1.3).

For $i = 1, \ldots, qN-1$ the truncation error with respect to w satisfies

$$\left|\left[L(w-w^N)\right]_i\right| \le CN^{-2}\ln^2 N\varepsilon^{-1}\mathrm{e}^{-\beta x_{i-1}/\varepsilon} \le CN^{-2}\ln^2 N\varepsilon^{-1}\tilde{\psi}_i,$$

by (4.60c), Theorem 3.48 and Remark 3.50, where

$$\tilde{\psi}_i := \prod_{k=1}^{i}\left(1+\frac{\beta h_k}{2\varepsilon}\right)^{-1}.$$

The inverse monotonicity of L gives

$$\left|(w-w^N)_i\right| \le CN^{-2}\ln^2 N\tilde{\psi}_i \quad \text{for } i = 1, \ldots, qN-1, \tag{4.63}$$

because

$$\left[L\tilde{\psi}\right]_i \ge C\varepsilon^{-1}\tilde{\psi}_i \quad \text{for } i = 1, \ldots, qN-1$$

and because both $\left|w_0 - w_0^N\right| \le CN^{-2}$ and $\left|w_{qN} - w_{qN}^N\right| \le CN^{-2}$.

Combining (4.61), (4.62) and (4.63), we are finished. □

4.3.2 Central Differencing

In numerical experiments [34, 50, 120] it was observed that central differencing on Shishkin meshes yields almost second-order accuracy.

A drawback of central difference approximations is their lack of stability. The discretisations are not maximum-norm stable. It will be seen in Sect. 4.3.2.1 that the use of layer-adapted meshes induce some additional stability. However, the discrete systems remain difficult to solve efficiently by means of iterative solvers. The system matrices have eigenvalues with large imaginary parts. This becomes a particularly important issue when solving higher-dimensional problems.

We shall consider the discretisation

$$\left[Lu^N\right]_i = f_i \quad \text{for } i = 1, \ldots, N-1, \;\; u_0^N = \gamma_0, \;\; u_N^N = \gamma_1 \tag{4.64}$$

of (4.9), where

$$[Lv]_i := -\varepsilon v_{\bar{x}\hat{x};i} - (bv)_{\mathring{x};i} + c_i v_i.$$

Similar to (4.4) this scheme is equivalent to a FEM with piecewise linear trial and test functions, but with the trapezium rule

$$\int_{I_i} g(s)\mathrm{d}s \approx h_i \frac{g_{i-1}+g_i}{2}$$

used to approximate the integrals on each subinterval I_i of the partition.

4.3.2.1 Stability

A first analysis was conducted by Andreev and Kopteva [10] who prove that central differencing on a Shishkin mesh is (l_∞, l_1) stable. This result was later generalised by Kopteva [70].

Theorem 4.35. *Assume that*

$$\left| \prod_{i=1}^{N} \left(\frac{\varepsilon}{h_i b_{i-1}} - \frac{1}{2} \right) \Big/ \left(\frac{\varepsilon}{h_i b_i} + \frac{1}{2} \right) \right| \le \frac{1}{4} \tag{4.65}$$

and that $h_i \le \mu h_j$ *for* $i \le j$ *with some constant* μ. *Then the central difference operator* L *is* (l_∞, l_1) *stable with*

$$\|v\|_{\infty,\omega} \le \frac{81}{4\beta} \sum_{i=1}^{N-1} \hbar_i \left| [Lv]_i \right| . \tag{4.66}$$

Furthermore, let m *be such that* $h_i \le 2\varepsilon/b_{i-1}$ *for* $i = 1, \ldots, m$ *and* $h_{m+1} > 2\varepsilon/b_m$. *Then the operator* L *is* $(l_\infty, w^{-1,\infty})$*-stable with*

$$\|v\|_{\infty,\omega} \le \frac{11}{2\beta} \max_{j=1,\ldots,N-1} \left| \sum_{k=j}^{N-1} \hbar_k \, [Lv]_k \right| .$$

for any mesh function v *with* $[Lv]_i = 0$ *for* $i > m$.

Proof. The argument is very technical and therefore not presented here. Instead the reader is referred to the original work by Kopteva [70]. □

4.3.2.2 A Priori Error Bounds

Based on Theorem 4.35 Kopteva [70] established convergence results for central differencing on two types of layer-adapted meshes:

$$\left\| u - u^N \right\|_{\infty,\omega} \le \begin{cases} CN^{-2} & \text{for Bakhvalov meshes with } \sigma > 2, \\ CN^{-2} \ln^2 N & \text{for Shishkin meshes with } \sigma > 2; \end{cases} \tag{4.67}$$

The (l_∞, l_1) stability (4.66) was used by Roos and Linß [138] to prove

$$\left\| u - u^N \right\|_{\infty,\omega} \le C \left(h + \max |\psi'| \, N^{-1} \right)^2 \tag{4.68}$$

on S-type meshes with $\sigma \ge 3$. A similar result was given by Kopteva and Linß [71] for certain quasilinear problems of type (4.49).

Another approach to study central differencing on Shishkin meshes is that of Lenferink [76, 77]. He eliminates every other unknown to get a scheme whose system matrix is an M-matrix.

4.3.2.3 Derivative Approximation

For the central-difference scheme (4.64) on S-type meshes with $\sigma \geq 3$ we have the second order bound

$$\varepsilon \left| u^N_{\bar{x};i} - u'_{i-1/2} \right| \leq C\big(h + N^{-1} \max |\psi'|\big)^2.$$

The proof in [138] uses the bound (4.68) for the discretisation error, then interprets the scheme as a finite element method with inexact integration and finally applies a finite element technique [173] to get the bound for the derivative approximation.

4.3.3 Convergence Acceleration Techniques

In the early 1980s Hemker [51] proposed the use of defect-correction methods when solving singularly perturbed problems. However, the first rigorous proof of uniform convergence of a defect-correction scheme was not published before 2001 (Fröhner et al. [43]). Various analyses by Nikolova and Axelsson [16, 126] are at least not rigorous with regard to the robustness, i. e. the ε-independence of the error constants, while the analysis by Fröhner and Roos [44] turned out to be technically unsound [42].

4.3.3.1 Defect Correction

Let us consider the defect correction method from [43] for our model convection-diffusion problem in conservative form:

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \ \text{ in } \ (0,1), \ \ u(0) = \gamma_0, \ \ u(1) = \gamma_1. \qquad (4.9)$$

It is based on the upwind scheme

$$\left[L^c u^N\right]_i := -\varepsilon u^N_{\bar{x}x;i} - \left(bu^N\right)_{x;i} + c_i u^N_i = f_i \qquad (4.10)$$

combined with the unstabilised second-order central difference scheme

$$\left[\widehat{L}^c u^N\right]_i := -\varepsilon u^N_{\bar{x}\hat{x};i} - \left(bu^N\right)_{\mathring{x};i} + c_i u^N_i = f_i.$$

With this notation we can formulate the defect correction method. This two-stage method is the following:

1. Compute an initial first-order approximation $\tilde{u}^N$ using simple upwinding:

$$\left[L^c\tilde{u}^N\right]_i = f_i \quad \text{for } i = 1, \dots, N-1, \;\; \tilde{u}_0^N = \gamma_0, \;\; \tilde{u}_N^N = \gamma_1. \tag{4.69a}$$

2. Estimate the defect ζ in the differential equation by means of the central difference scheme:

$$\zeta_i = [\widehat{L}^c\tilde{u}^N]_i - f_i \quad \text{for } i = 1, \dots, N-1. \tag{4.69b}$$

3. Compute the defect correction $\varDelta$ by solving

$$[L^c\varDelta]_i = \kappa_i\zeta_i, \qquad \text{for } i = 1, \dots, N-1, \;\; \varDelta_0 = \varDelta_N = 0. \tag{4.69c}$$

   with $\kappa_i = \hbar_i / h_{i+1}$.
4. Then the final computed solution is

$$u^N = \tilde{u}^N - \varDelta. \tag{4.69d}$$

Remark 4.36. At first glance both the upwind discretisation and the particular weighting of the residual in (4.69c) appear a bit non-standard. No justification for these choices is provided by [43, 93]. An argument that suggests this particular choice is presented in [101].

Furthermore, the weighting becomes the standard $\kappa_i = 1$ on uniform meshes; however, when used on non-uniform meshes, $\kappa_i = 1$ might reduce the order of convergence which is illustrated by numerical experiments in [101]. ♣

In the analysis of the method we use the following notation:

$$(\mathcal{A}^c v)(x) := \varepsilon v'(x) + \big(bv\big)(x) + \int_x^1 \big(cv\big)(s)\mathrm{d}s, \quad \mathcal{F}(x) := \int_x^1 f(s)\mathrm{d}s,$$

$$\left[A^c v\right]_i := \varepsilon v_{\bar{x};i} + (bv)_i + \sum_{k=i}^{N-1} h_{k+1}(cv)_k\,, \quad F_i := \sum_{k=i}^{N-1} h_{k+1} f_k$$

and

$$\left[\widehat{A}^c v\right]_i := \varepsilon v_{\bar{x};i} + \frac{(bv)_i + (bv)_{i-1}}{2} + \sum_{k=i}^{N} \hbar_k (cv)_k, \;\; \widehat{F}_i := \sum_{k=i}^{N} \hbar_k f_k.$$

The differential equation (4.9) yields

$$\mathcal{A}^c u - \mathcal{F} \equiv \alpha = \text{const}, \tag{4.70}$$

while (4.69b) and (4.69c) imply

$$A^c\varDelta - (\widehat{A}^c\tilde{u}^N - \widehat{F}) \equiv a = \text{const}\,. \tag{4.71}$$

A priori analysis

The negative norm stability (4.8c) of the operator L^c yields for the error of the defect-correction method

$$\begin{aligned} \left|\left|\left| u - u^N \right|\right|\right|_{\varepsilon,\infty,\omega} &\le 2\left\|(A^c - \widehat{A}^c)(u - \tilde{u}^N)\right\|_{\infty,\omega} \\ &\quad + 2\left\|\widehat{A}^c u - \widehat{F} + \alpha\right\|_{\infty,\omega}, \end{aligned} \tag{4.72}$$

by (4.70) and (4.71), where α is the constant from (4.70).

The second term in (4.72) is the truncation error of the central difference scheme. It is formally of second order. The first term is the so called **relative consistency error**. While the error $u - \tilde{u}^N$ of the simple upwind scheme is only of first order, the hope is that A^c and $\widehat{A}^c$ are sufficiently close to gain second order in this term too.

Consider the relative consistency error first. Let $\eta := u - \tilde{u}^N$ denote the error of the simple upwind scheme. A straight-forward calculation and summation by parts give

$$\left[(A^c - \widehat{A}^c)\eta\right]_i = \frac{(b\eta)_i - (b\eta)_{i-1}}{2} + \sum_{k=i+1}^{N-1} h_{k+1}\frac{(c\eta)_{k-1} - (c\eta)_k}{2} - \frac{h_i}{2}(c\eta)_i,$$

which can be bounded by

$$\begin{aligned} \left|\left[(A^c - \widehat{A}^c)\eta\right]_i\right| &\le \left(\|b\|_\infty + \frac{\|c\|_\infty}{2}\right)\max_{i=1,\dots,N}|\eta_i - \eta_{i-1}| \\ &\quad + h\left(\|b'\|_\infty + \frac{\|c'\|_\infty + \|c\|_\infty}{2}\right)\|\eta\|_{\infty,\omega}. \end{aligned}$$

Thus,

$$\left\|(A^c - \widehat{A}^c)\eta\right\|_{\infty,\omega} \le C\left(\max_{i=1,\dots,N}|\eta_i - \eta_{i-1}| + h\|\eta\|_{\infty,\omega}\right) \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2, \tag{4.73}$$

by (4.50) and because $h \le \vartheta_{cd}^{[1]}(\bar{\omega}) \le \vartheta_{cd}^{[2]}(\bar{\omega})$.

Remark 4.37. The first term, the maximum difference of the error of the upwind scheme in two adjacent mesh points, constituted the main difficulty in [43]. With the error expansion of Sect. 4.2.3 this has become a simple task. ♣

Next, let us bound the truncation error of the central difference scheme. By (4.70) we have $(\widehat{A}^c u - \widehat{F})_i - \alpha = (\widehat{A}^c u - \widehat{F})_i - (\mathcal{A}u - \mathcal{F})_{i-1/2}$. Hence,

$$\begin{aligned}\left|(\widehat{A}^c u - \widehat{F})_i - \alpha\right| \le \varepsilon \left|u_{\bar{x};i} - u'_{i-1/2}\right| + \left|\frac{(bu)_i + (bu)_{i-1}}{2} - (bu)_{i-1/2}\right| \\ + \left|\sum_{k=i}^{N} \hbar_k g_k - \int_{x_{i-1/2}}^{1} g(s)\mathrm{d}s\right|\end{aligned} \tag{4.74}$$

with $g = f - cu$. Using Taylor expansions for u, u' and $(bu)'$ about $x = x_i$, we obtain

$$\varepsilon \left|u_{\bar{x};i} - u'_{i-1/2}\right| \le \frac{3\varepsilon}{2} \int_{I_i} (s - x_{i-1})\left|u'''(s)\right|\mathrm{d}s \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2 \tag{4.75}$$

and

$$\left|\frac{(bu)_i + (bu)_{i-1}}{2} - (bu)_{i-1/2}\right| \le \frac{3}{2} \int_{I_i} (s - x_{i-1})\left|(bu)''(s)\right|\mathrm{d}s \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,$$

by (4.9), (3.30) and Lemma 4.16.

For the last term in (4.74) a Taylor expansion gives

$$\begin{aligned}\left|\frac{h_k}{2} g_k - \frac{h_k^2}{8} g'_{k-1/2} - \int_{x_{k-1/2}}^{x_k} g(s)\mathrm{d}s\right| &\le \frac{h_k^3}{8} \|g''\|_{\infty,(x_{k-1/2},x_k)} \\ &\le C h_k^3 \left(1 + \varepsilon^{-2} \mathrm{e}^{-\beta x_{k-1/2}/\varepsilon}\right) \le C h_k \left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,\end{aligned} \tag{4.76a}$$

where we have used (3.30) and Proposition 4.15 with $x = x_{k-1/2}$ and $\sigma = 2$. Furthermore, we have

$$\begin{aligned}&\left|\frac{h_{k+1}}{2} g_k + \frac{h_{k+1}^2}{8} g'_{k+1/2} - \int_{x_k}^{x_{k+1/2}} g(s)\mathrm{d}s\right| \\ &\qquad \le h_{k+1} \int_{x_k}^{x_{k+1/2}} (\sigma - x_k)\left|g''(\sigma)\right| \mathrm{d}\sigma \le C h_{k+1} \left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,\end{aligned} \tag{4.76b}$$

by (3.30) and Lemma 4.16. Combine these two estimates:

$$\begin{aligned}&\left|\sum_{k=i}^{N} \hbar_k g_k - \int_{x_{i-1/2}}^{1} g(s)\mathrm{d}s\right| \\ &\qquad \le C\left\{\vartheta_{cd}^{[2]}(\bar{\omega})^2 + h_i^2\left(1 + \varepsilon^{-2}\mathrm{e}^{-\beta x_{i-1/2}/\varepsilon}\right)\right\} \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,\end{aligned}$$

by Proposition 4.15.

Therefore,

$$\left\|\widehat{A}^c u - \widehat{F} - \alpha\right\|_{\infty,\omega} \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,$$

by (4.9), (3.30) and Lemma 4.16.

Collect (4.72), (4.73) and the last inequality to get the main result of this section.

Theorem 4.38. *Let* u *be the solution of* (4.9) *and* u^N *that of the defect correction method* (4.69). *Then*

$$|||u - u^N|||_{\varepsilon,\infty,\omega} \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2.$$

Derivative approximation

For $x \in I_i$ a triangle inequality gives

$$\begin{aligned}\varepsilon\left|u'(x) - u^N_{\bar{x};i}\right| &\le \varepsilon\left|u'(x) - u_{\bar{x};i}\right| + \varepsilon\left|(u^I - u^N)_{\bar{x};i}\right| \\ &\le \varepsilon\left|u'(x) - u_{\bar{x};i}\right| + C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,\end{aligned}$$

by Theorem 4.38. The representation

$$u'(x) - u_{\bar{x};i} = \frac{1}{h_i}\int_{I_i}\int_s^x u''(t)\mathrm{d}t\mathrm{d}s. \tag{4.77}$$

yields

$$\varepsilon\left|u'(x) - u_{\bar{x};i}\right| \le C\vartheta_{cd}^{[1]}(\bar{\omega}), \quad \text{by (3.30)}.$$

Thus, in general we only have a first-order approximation for the ε-weighted derivative:

$$\varepsilon\left|u'(x) - u_{\bar{x};i}\right| \le C\vartheta_{cd}^{[2]}(\bar{\omega}) \quad \text{for } x \in I_i.$$

This result is sharp.

For the midpoint $x_{i-1/2}$ of the mesh interval I_i we expand (4.77) to give

$$u'(x_{i-1/2}) - u_{\bar{x};i} = \frac{1}{h_i}\int_{I_i}\int_s^x\int_{x_{i-1/2}}^t u'''(\xi)\mathrm{d}\xi\mathrm{d}t\mathrm{d}s.$$

The right-hand side can be bounded using (3.30) and Lemma 4.16. We get

$$\varepsilon\left|u'(x_{i-1/2}) - u_{\bar{x};i}\right| \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2,$$

and finally

$$\varepsilon \left| u'(x_{i-1/2}) - u^N_{\bar{x};i} \right| \le C \left(\vartheta^{[2]}_{cd}(\bar{\omega}) \right)^2 .$$

This means that the midpoints of the mesh intervals are superconvergence points for the derivative and we can define a recovery operator R for the derivative. For a given mesh function $v \in I\!R^{N+1}$, let Rv be that function that is piecewise linear on the mesh $\hat{\omega} = \left\{0, x_{1+1/2}, x_{2+1/2}, \ldots, x_{N-1-1/2}, 1\right\}$ and satisfies

$$(Rv)_{i-1/2} = v_{\bar{x};i} \quad \text{for } \ i = 1, \ldots, N.$$

Then one can prove

$$\varepsilon \left\| u' - Ru^N \right\|_\infty \le C \left(\vartheta^{[2]}_{cd}(\bar{\omega}) \right)^2 .$$

A posteriori analysis

The next result is an extension of Lemma 2.2 in [64] which gave bounds in the mesh points only. It is an essential ingredient for the analysis of second-order approximations.

Theorem 4.39. *Let the hypothesis of Theorem 3.45 be satisfied. Let ψ be the solution of the boundary value problem*

$$\mathcal{L}^c \psi = -\Psi' \quad \textit{in } \ (0,1), \ \ u(0) = u(1) = 0$$

with

$$\Psi(x) = A_{i-1/2}(x - x_{i-1/2}) \quad \textit{for } \ x \in (x_{i-1}, x_i).$$

Then

$$\|\psi\|_\infty \le C^* \max_{i=1,\ldots,N} \left\{ \left|A_{i-1/2}\right| \min\left[\frac{h_i}{\|b\|_\infty}, \frac{h_i^2}{4\varepsilon} \right] \right\},$$

where

$$C^* = \frac{2\|b\|_\infty + \|c\|_\infty + \beta}{2\beta}.$$

Proof. Let $x \in (0,1)$ be arbitrary, but fixed. The Green's function representation gives

$$u(x) = \int_0^1 \partial_\xi \mathcal{G}(x,\xi) F(\xi) \mathrm{d}\xi = \sum_{i=1}^N A_{i-1/2} J_i$$

with

$$J_i := \int_{I_i} \partial_\xi \mathcal{G}(x,\xi)(\xi - x_{i-1/2})\mathrm{d}\xi.$$

A first bound for the J_i's is

$$|J_i| \le \left| \int_{I_i} \partial_\xi \mathcal{G}(x,\xi)(\xi - x_{i-1/2})\mathrm{d}\xi \right| \le \frac{h_i}{2} \int_{I_i} |\partial_\xi \mathcal{G}(x,\xi)| \,\mathrm{d}\xi, \tag{4.78}$$

while integration by parts yields

$$J_i = \int_{I_i} \partial_\xi^2 \mathcal{G}(x,\xi) \left[\frac{h_i^2}{8} - \frac{(\xi - x_{i-1/2})^2}{2} \right] \mathrm{d}\xi.$$

Hence

$$\begin{aligned} |J_i| &\le \frac{h_i^2}{8} \int_{I_i} \left| \partial_\xi^2 \mathcal{G}(x,\xi) \right| \mathrm{d}\xi \\ &\le \frac{h_i^2}{8\varepsilon} \int_{I_i} \left[\delta_x(\xi) + \|b\|_\infty \left| \partial_\xi \mathcal{G}(x,\xi) \right| + c(\xi)\mathcal{G}(x,\xi) \right] \mathrm{d}\xi. \end{aligned}$$

This estimate is combined with (4.78) to give

$$|J_i| \le \min \left[\frac{h_i^2}{8\varepsilon}, \frac{h_i}{2\|b\|_\infty} \right] \int_{I_i} \left[\delta_x(\xi) + \|b\|_\infty \left| \partial_\xi \mathcal{G}(x,\xi) \right| + \|c\|_\infty \mathcal{G}(x,\xi) \right] \mathrm{d}\xi.$$

Multiply by $\left|A_{i-1/2}\right|$, take sums for $i = 1, \dots, N$, use a discrete Hölder inequality and note that

$$\int_0^1 \left[\delta_x(\xi) + \|b\|_\infty \left| \partial_\xi \mathcal{G}(x,\xi) \right| + \|c\|_\infty \mathcal{G}(x,\xi) \right] \mathrm{d}\xi \le 1 + \frac{2\|b\|_\infty + \|c\|_\infty}{\beta},$$

by Theorem 3.20. This completes the proof. □

With these stability results at hand we can now derive our a posteriori error bounds. We shall identify any mesh function v with its piecewise linear nodal interpolant.

Theorem 4.40. *Let the hypothesis of Theorem 3.45 be satisfied. Set $g := f - cu^N$. Then the error of the defect-correction method satisfies*

$$\left\| u - u^N \right\|_\infty \le \eta := \eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5$$

with

$$\eta_1 := C^* \max_{i=1,\ldots,N} \min\left\{\frac{h_i}{\|b\|_\infty}, \frac{h_i^2}{\varepsilon}\right\} \left|g_{i-1/2} + \left(bu^N\right)_{\bar{x},i}\right|,$$

$$\eta_2 := \frac{1}{\beta} \max_{i=1,\ldots,N} h_i \left|(b\Delta)_{\bar{x},i}\right|, \quad \eta_3 := \frac{1}{\beta} \max_{i=1,\ldots,N-1} \left|\sum_{k=i}^{N-1} \frac{h_{k+1}-h_k}{2} c_k \Delta_k\right|,$$

$$\eta_4 := \frac{1}{6\beta} \sum_{i=1}^{N} h_i^3 \left\|g''\right\|_{\infty,I_i}$$

and

$$\eta_5 := \frac{3}{4\beta} \max_{i=1,\ldots,N} h_i^2 \left\{2\left\|g'\right\|_{\infty,I_i} + \left\|(bu^N)''\right\|_{\infty,I_i}\right\}.$$

Proof. By (4.70) and (4.71) we have, for $x \in (x_{i-1}, x_i)$,

$$\begin{aligned}&\mathcal{A}^c\left(u-u^N\right)(x)\\&\quad = \mathcal{F}(x) - \widehat{F}_i + \left[\widehat{A}^c u^N\right]_i - \left(\mathcal{A}^c u^N\right)(x) - \left[\left(A^c - \widehat{A}^c\right)\Delta\right]_i + \alpha - a.\end{aligned}$$

Recalling the definitions of $\mathcal{F}$, $\widehat{F}$, $\mathcal{A}^c$, A^c and $\widehat{A}^c$, we obtain the representation

$$\begin{aligned}&\mathcal{A}^c\left(u-u^N\right)(x)\\&\quad = \int_x^1 g(s)\mathrm{d}s - \sum_{k=i}^{N} \hbar_k g_k + \frac{\left(bu^N\right)_i + \left(bu^N\right)_{i-1}}{2} - \left(bu^N\right)(x)\\&\qquad - \frac{h_i}{2}(b\Delta)_{\bar{x},i} - \sum_{k=i}^{N-1} \frac{h_{k+1}-h_k}{2} c_k \Delta_k + \alpha - a.\end{aligned} \tag{4.79}$$

Taylor expansions yield

$$\int_x^1 g(s)\mathrm{d}s - \sum_{k=i}^{N} \hbar_k g_k = \int_{x_i}^1 (g-g^I)(s)\mathrm{d}s + (x_{i-1/2}-x) g_{i-1/2} + \mu_i(x),$$

where g^I is the piecewise linear interpolant of g, and

$$\frac{\left(bu^N\right)_i + \left(bu^N\right)_{i-1}}{2} - \left(bu^N\right)(x) = \left(x_{i-1/2}-x\right)\left(bu^N\right)_{\bar{x},i} + \tilde{\mu}_i(x)$$

with

$$\|\mu_i\|_{\infty,I_i} \le \frac{3h_i^2}{4}\,\|g'\|_{\infty,I_i} \quad \text{and} \quad \|\tilde{\mu}_i\|_{\infty,I_i} \le \frac{3h_i^2}{8}\,\left\|(bu^N)''\right\|_{\infty,I_i}$$

Substitute the above two equations into (4.79). We get

$$\begin{aligned}
&\mathcal{A}\left(u-u^N\right)(x)\\
&\quad = \int_{x_i}^1 (g-g^I)(s)\mathrm{d}s + \left(x_{i-1/2}-x\right)\left(g_{i-1/2} + \left(bu^N\right)_{\bar{x},i}\right)\\
&\qquad - \frac{h_i}{2}\left(b\Delta\right)_{\bar{x},i} - \sum_{k=i}^{N-1}\frac{h_{k+1}-h_k}{2}c_k\delta_k + \left(\mu_i+\tilde{\mu}_i\right)(x) + \alpha - a.
\end{aligned}$$

Furthermore,

$$\left|\int_{x_i}^1 (g-g^I)(s)\mathrm{d}s\right| \le \frac{1}{12}\sum_{i=1}^N h_i^3\,\|g''\|_{\infty,I_i}\,.$$

Finally, note that $(\mathcal{A}v)' = -\mathcal{L}v$. Use Theorems 3.45 and 4.39 to complete the proof. □

Remark 4.41. The error estimate of Theorem 4.40 contains terms, namely η_4 and η_5, that in general have to be approximated, for example

$$g' \approx \frac{g_i - g_{i-1}}{h_i}, \quad g'' \approx 4\frac{g_i - 2g_{i-1/2} + g_{i-1}}{h_i^2}$$

and

$$(bu^N)'' \approx 4\frac{(bu^N)_i - 2(bu^N)_{i-1/2} + (bu^N)_{i-1}}{h_i^2}.$$

The additional errors introduced this way are of third order and therefore decay rapidly when the mesh is refined. ♣

An adaptive mesh algorithm

Based on Theorem 4.40, the de Boor algorithm described in Sect. 4.2.4.2 can be adapted for the defect-correction method by choosing

$$Q_i = Q_i(u^N,\Delta,\omega) := \left\{\rho_0 + \sum_{k=1}^5 \rho_k\eta_{k;i}\right\}^{1/2} \tag{4.80}$$

with

$$\eta_{1;i} := \min\left\{\frac{h_i}{\|b\|_\infty}, \frac{h_i^2}{4\varepsilon}\right\}\left|g_{i-1/2} + \left(bu^N\right)_{\bar{x},i}\right|, \quad \eta_{2;i} := h_i\left|(b\Delta)_{\bar{x},i}\right|,$$

$$\eta_{3;i} := \left|\sum_{k=i}^{N-1}\frac{h_{k+1}-h_k}{2}c_k\Delta_k\right|, \quad \eta_{4;i} := \frac{2}{3\beta}\left|g_i - 2g_{i-1/2} + g_{i-1}\right|,$$

$$\eta_{5;i} := \frac{1}{2\beta}h_i\left|g_i - g_{i-1}\right| + \left|(bu^N)_i - 2(bu^N)_{i-1/2} + (bu^N)_{i-1}\right|$$

and non-negative weights ρ_ℓ.

Remark 4.42. The square root in (4.80) is necessary because the underlying method is formally of second order. ♣

Remark 4.43. The numerical experiments in [101] indicate that η_1 contains sufficient information to steer the mesh adaptation. Therefore η_k, $k = 2, \ldots, 5$, can be set to zero, however ρ_0 must not in order to avoid mesh starvation in regions where the solution does not vary much. This reduces the computational costs in the remeshing phase of the de Boor algorithm. ♣

4.3.3.2 Richardson Extrapolation

Richardson extrapolation on layer-adapted meshes was first analysed by Natividad and Stynes [124]. They study a simple first-order upwind scheme on a Shishkin mesh and prove that Richardson extrapolation improves the accuracy to almost second order, although the underlying scheme is only of first order. The analysis in [124] is based on comparison principles and barrier function techniques.

Here we shall pursue an alternative approach similar to the one in [93] that is based on the $(l_\infty, w^{-1,\infty})$ stability and on the error expansion of Sect. 4.2.3. Again consider the conservative form of our model problem:

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1. \qquad (4.9)$$

Given an arbitrary mesh $\bar{\omega}$, let $\bar{\omega}' : 0 = x_{1/2} < x_1 < x_{1+1/2} < \cdots < x_N = 1$ be the mesh obtained by uniformly bisecting $\bar{\omega}$. Let $\tilde{u}^N$ be the solution of the upwind scheme (4.10) on $\bar{\omega}$ and

$$u^{2N} = \left(u_0^{2N}, u_{1/2}^{2N}, u_1^{2N}, \ldots, u_{N-1/2}^{2N}, u_N^{2N}\right)$$

that of the difference scheme on $\tilde{\omega}$. Since (4.10) is a first-order scheme we combine $\tilde{u}^N$ and $\tilde{u}^{2N}$ by

$$u_i^N := 2\tilde{u}_i^{2N} - \tilde{u}_i^N \quad \text{for } i = 0, \ldots, N,$$

in order to get a second-order approximation defined on the coarser mesh $\bar{\omega}$.

In addition to the notation introduced on p. 122 set

$$\left[\tilde{A}^c v\right]_i := 2\varepsilon \frac{v_i - v_{i-1}}{h_i} + b_i v_i + \sum_{k=i}^{N-1} h_{k+1} \frac{c_k v_k + c_{k+1/2} v_{k+1/2}}{2}$$

and

$$\tilde{F}_i := \sum_{k=i}^{N-1} h_{k+1} \frac{f_k + f_{k+1/2}}{2}.$$

The differential equation (4.9) and the difference equation (4.10) yield

$$\mathcal{A}^c u - \mathcal{F} \equiv \alpha, \quad A^c \tilde{u}^N - F \equiv a \quad \text{and} \quad \tilde{A}^c \tilde{u}^{2N} - \tilde{F} \equiv \tilde{a}.$$

A direct calculation gives

$$\begin{aligned}
A^c(2\tilde{u}^{2N} - u^N - u)_i &+ (\mathcal{A}^c u - \mathcal{F})_{i-1/2} \\
&= -\varepsilon \left(\frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2} \right) \\
&\quad + \left\{ b_i(\tilde{u}_i^{2N} - u_i) - b_{i-1/2}(\tilde{u}_{i-1/2}^{2N} - u_{i-1/2}) \right\} \\
&\quad - \left\{ \frac{h_i}{2}(c\tilde{u}^{2N} - cu)_{i-1/2} \right. \\
&\qquad \left. + \sum_{k=i}^{N-1} h_{k+1} \left[(c\tilde{u}^{2N} - cu)_{k+1/2} - (c\tilde{u}^{2N} - cu)_k \right] \right\} \\
&\quad + \left\{ \int_{i-1/2}^{1} g(s)\mathrm{d}s - \frac{h_i}{2} g_{i-1/2} - \sum_{k=i}^{N-1} h_{k+1} g_{k+1/2} \right\}
\end{aligned}$$

with $g = cu - f$. The first term on the right-hand side is bounded by $C\vartheta_{cd}^{[2]}(\bar{\omega})^2$, see (4.75) The second and third term can be bounded by $C\vartheta_{cd}^{[2]}(\tilde{\omega})^2$ using the technique that gave (4.73). The last term is also bounded by $C\vartheta_{cd}^{[2]}(\bar{\omega})^2$, since similar to (4.76) we have

$$\left| \frac{h_k}{2} g_{k-1/2} - \frac{h_k^2}{8} g'_{k-1/2} - \int_{x_{k-1/2}}^{x_k} g(s)\mathrm{d}s \right| \leq C h_k \left(\vartheta_{cd}^{[2]}(\bar{\omega}) \right)^2,$$

and

$$\left| \frac{h_{k+1}}{2} g_{k+1/2} + \frac{h_{k+1}^2}{8} g'_{k+1/2} - \int_{x_k}^{x_{k+1/2}} g(s)\mathrm{d}s \right| \leq C h_{k+1} \left(\vartheta_{cd}^{[2]}(\bar{\omega}) \right)^2.$$

Finally, using the stability inequality (4.8c) we obtain the following convergence result.

Theorem 4.44. *Let u^N be the approximate solution to (4.9) obtained by the Richardson extrapolation technique applied to the simple upwind scheme (4.10). Then*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon,\infty,\omega} \le C \left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2 .$$

Corollary 4.45. *Theorem 4.44 and interpolation error bounds (see Sect. 5.1) give*

$$\left\|u - u^N\right\|_\infty + \varepsilon \left\|u' - Ru^N\right\|_\infty \le C \left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2 ,$$

where R is the recovery operator from Sect. 4.3.3.1.

4.3.4 A Numerical Example

The following tables contain the results of test computations for test problem (4.14):

$$-\varepsilon u'' - u' + 2u = \mathrm{e}^{x-1}, \quad u(0) = u(1) = 0.$$

We test the performance of:

- second-order upwinding
- central differencing
- defect correction and
- Richardson extrapolation.

In the experiments we have chosen $\varepsilon = 10^{-8}$. The meshes have been constructed with parameters $\sigma = 2$, $\beta = 1$ and $q = K = 1/2$.

The numerical results in Tables 4.3 and 4.4 are clear illustrations of the convergence estimates of Theorems 4.31, 4.38 and 4.44 and of (4.67). Furthermore, as the theory predicts, all four methods give higher accuracy on Bakhvalov meshes than on Shishkin meshes.

Finally, we consider a modified Shishkin mesh which is constructed as follows. Pick the transition point $\tau = 2\varepsilon\beta^{-1}\ln N$ as usual. Set $h = 2\tau/N$ and $H = 2(1-\tau)/N$. Then the mesh is defined by

$$h_i = \begin{cases} h & \text{if } i \le N/2, \\ 4H/3 & \text{if } i \text{ is odd and } i > N/2, \\ 2H/3 & \text{if } i \text{ is even and } i > N/2. \end{cases}$$

Thus, instead of a uniform mesh on each of the two subdomains $[0,\tau]$ and $[\tau,1]$ we use non-uniform, though very regular sub meshes. A similar mesh was considered in [26].

Table 4.3 Second-order schemes on Shishkin meshes

N	second order upwinding		central differencing		defect correction		Richardson extrapolation	
2^7	3.29e-04	1.61	1.37e-04	1.55	2.32e-04	1.64	2.33e-04	1.57
2^8	1.07e-04	1.66	4.66e-05	1.62	7.47e-05	1.68	7.86e-05	1.64
2^9	3.40e-05	1.70	1.52e-05	1.67	2.33e-05	1.72	2.53e-05	1.69
2^{10}	1.05e-05	1.73	4.79e-06	1.70	7.06e-06	1.75	7.86e-06	1.72
2^{11}	3.17e-06	1.75	1.47e-06	1.73	2.10e-06	1.78	2.38e-06	1.75
2^{12}	9.43e-07	1.77	4.43e-07	1.76	6.12e-07	1.79	7.07e-07	1.77
2^{13}	2.77e-07	1.79	1.31e-07	1.78	1.76e-07	1.81	2.07e-07	1.79
2^{14}	8.02e-08	1.80	3.83e-08	1.79	5.03e-08	1.82	5.97e-08	1.81
2^{15}	2.30e-08	1.81	1.11e-08	1.81	1.42e-08	1.83	1.71e-08	1.82
2^{16}	6.54e-09	1.83	3.16e-09	1.82	4.00e-09	1.84	4.84e-09	1.83
2^{17}	1.85e-09	1.84	8.95e-10	1.83	1.12e-09	1.85	1.36e-09	1.84
2^{18}	5.17e-10	—	2.52e-10	—	3.10e-10	—	3.81e-10	—

Table 4.4 Second-order schemes on Bakhvalov meshes

N	second order upwinding		central differencing		defect correction		Richardson extrapolation	
2^7	5.81e-04	2.36	9.67e-05	1.99	2.74e-04	1.94	1.16e-04	1.94
2^8	1.13e-04	2.11	2.43e-05	2.02	7.14e-05	1.97	3.01e-05	1.97
2^9	2.62e-05	2.05	5.98e-06	1.99	1.83e-05	1.98	7.67e-06	1.99
2^{10}	6.34e-06	2.27	1.51e-06	2.00	4.63e-06	1.99	1.93e-06	1.99
2^{11}	1.31e-06	2.36	3.77e-07	2.00	1.16e-06	2.00	4.86e-07	2.00
2^{12}	2.55e-07	2.11	9.45e-08	2.01	2.92e-07	2.00	1.22e-07	2.00
2^{13}	5.90e-08	2.05	2.34e-08	1.99	7.30e-08	2.00	3.05e-08	2.00
2^{14}	1.43e-08	2.05	5.89e-09	2.00	1.83e-08	2.00	7.63e-09	2.00
2^{15}	3.46e-09	2.00	1.47e-09	2.00	4.57e-09	2.00	1.91e-09	2.00
2^{16}	8.64e-10	2.00	3.68e-10	2.01	1.14e-09	2.00	4.77e-10	2.00
2^{17}	2.16e-10	2.00	9.16e-11	2.00	2.86e-10	2.00	1.19e-10	2.00
2^{18}	5.40e-11	—	2.30e-11	—	7.14e-11	—	2.98e-11	—

For this mesh $\vartheta_{cd}^{[2]}(\bar{\omega}) \le CN^{-1} \ln N$ and almost second order convergence is guaranteed for the upwind scheme, defect correction and Richardson extrapolation. This order of convergence is observed in our computational experiments; see Table 4.5. However, for central differencing the observed rate is only one. Thus, on this mesh the assumption (4.65) must be violated. In [26] it is shown that for $b = \text{const}$ and $c \equiv 0$ the stability constant in (4.66) blows up for $N \to \infty$.

Remark 4.46. This means that for central differencing a general result like

$$\left\| u - u^N \right\|_{\infty,\omega} \le C \left(\vartheta_{cd}^{[2]}(\bar{\omega}) \right)^2$$

cannot hold. ♣

Table 4.5 Second-order schemes on modified Shishkin meshes

N	second order upwinding		central differencing		defect correction		Richardson extrapolation	
2^7	1.30e-03	1.61	6.73e-03	0.98	5.94e-04	1.46	7.66e-04	1.46
2^8	4.27e-04	1.66	3.41e-03	0.99	2.16e-04	1.57	2.77e-04	1.58
2^9	1.35e-04	1.70	1.72e-03	0.99	7.25e-05	1.65	9.25e-05	1.65
2^{10}	4.17e-05	1.72	8.63e-04	1.00	2.31e-05	1.70	2.94e-05	1.70
2^{11}	1.26e-05	1.75	4.32e-04	1.00	7.11e-06	1.74	9.04e-06	1.74
2^{12}	3.76e-06	1.77	2.16e-04	1.00	2.13e-06	1.77	2.71e-06	1.76
2^{13}	1.10e-06	1.79	1.08e-04	1.00	6.24e-07	1.79	7.97e-07	1.79
2^{14}	3.20e-07	1.80	5.41e-05	1.00	1.81e-07	1.81	2.31e-07	1.80
2^{15}	9.18e-08	1.81	2.70e-05	1.00	5.16e-08	1.82	6.63e-08	1.82
2^{16}	2.61e-08	1.82	1.35e-05	1.00	1.46e-08	1.83	1.88e-08	1.83
2^{17}	7.37e-09	1.84	6.74e-06	1.02	4.11e-09	1.84	5.31e-09	1.84
2^{18}	2.07e-09	—	3.33e-06	—	1.15e-09	—	1.49e-09	—

This observation has far reaching consequences, in particular in higher dimensions, where it is difficult, if not impossible, to construct uniform or nearly uniform meshes. Therefore, stabilisation in regions where the mesh is coarse becomes essential.

4.4 Systems

We now leave the scalar convection-diffusion equation and move on to systems of equations of this type.

4.4.1 Weakly Coupled Systems in One Dimension

Consider the weakly coupled problem from Sect. 3.4.2:

$$\begin{aligned} \boldsymbol{\mathcal{L}u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' - \operatorname{diag}(\boldsymbol{b})\boldsymbol{u}' + \boldsymbol{Au} = \boldsymbol{f} \quad \text{on } (0,1), \\ \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \end{aligned} \tag{4.81}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_\ell)^T$ and the small parameter ε_k is in $(0,1]$ for $k = 1, \dots, \ell$. Assume that for each k one has $a_{kk} \geq 0$ and either $b_k \geq \beta_k$ or $b_k \leq -\beta_k$ on $[0,1]$ with positive constants b_k.

We follow [96] and discretise (4.81) by means of the simple upwind scheme that was studied in detail in Sect. 4.2: Find $\boldsymbol{u} \in \left(I\!R_0^{N+1}\right)^\ell$ such that

$$\left[\boldsymbol{L}\boldsymbol{u}^N\right]_i = \boldsymbol{f}_i \quad \text{for } k = 1, \dots, N-1, \tag{4.82}$$

where $\boldsymbol{f}_i = \boldsymbol{f}(x_i) = (f_{1;i}, f_{2;i}, \ldots, f_{\ell,i})^T$, $\boldsymbol{L}\boldsymbol{v} := \big((\boldsymbol{L}\boldsymbol{v})_1, (\boldsymbol{L}\boldsymbol{v})_2, \ldots, (\boldsymbol{L}\boldsymbol{v})_\ell\big)^T$,

$$(L\boldsymbol{v})_k := \Lambda_k v_k + \sum_{\substack{m=1\\ m\neq k}}^{\ell} a_{km} v_m$$

and

$$[\Lambda_k v]_i := \begin{cases} -\varepsilon_k v_{\bar{x}x;i} - b_{k;i} v_{x;i} + a_{kk;i} v_i & \text{if } b_k > 0,\\ -\varepsilon_k v_{x\bar{x};i} - b_{k;i} v_{\bar{x};i} + a_{kk;i} v_i & \text{if } b_k < 0.\end{cases}$$

4.4.1.1 Stability

The stability analysis for the discrete operator is conducted along the lines of the continuous analysis. By the definition of $\boldsymbol{L}$ and the Λ_k's we have, for any vector-valued mesh function $\boldsymbol{v} \in \left(I\!R_0^{N+1}\right)^{\ell}$,

$$\Lambda_k v_k = -\sum_{\substack{m=1\\ m\neq k}}^{\ell} a_{km} v_m + (\boldsymbol{L}\boldsymbol{v})_k \quad \text{on } \omega, \;\; k = 1, \ldots, \ell. \tag{4.83}$$

Then Theorem 4.3 yields

$$\|v_k\|_{\infty,\omega} + \sum_{\substack{m=1\\ m\neq k}}^{\ell} \tilde{\gamma}_{km} \|u_m\|_{\infty,\omega} \le \min\left\{ \left\| \frac{(\boldsymbol{L}\boldsymbol{v})_k}{a_{kk}} \right\|_{\infty,\omega}, \left\| \frac{(\boldsymbol{L}\boldsymbol{v})_k}{b_k} \right\|_{\infty,\omega} \right\}$$

for, $k = 1, \ldots, \ell$, where the $\ell \times \ell$ constant matrix $\tilde{\boldsymbol{\Gamma}} = \tilde{\boldsymbol{\Gamma}}(\boldsymbol{A}, \boldsymbol{b}) = (\tilde{\gamma}_{km})$ is—as in Sect. 3.4.2—defined by

$$\tilde{\gamma}_{kk} = 1 \quad \text{and} \quad \tilde{\gamma}_{km} = -\min\left\{ \left\| \frac{a_{km}}{a_{kk}} \right\|_{\infty,\omega}, \left\| \frac{a_{km}}{b_k} \right\|_{\infty,\omega} \right\} \quad \text{for } k \neq m.$$

We reach the following stability result.

Theorem 4.47. *Assume that the matrix $\boldsymbol{A}$ has non-negative diagonal entries. Assume that $\tilde{\boldsymbol{\Gamma}}(\boldsymbol{A})$ is inverse-monotone. Then for $i = 1, \ldots, \ell$ one has*

$$\|v_i\|_{\infty,\bar{\omega}} \le \sum_{k=1}^{\ell} \big(\tilde{\boldsymbol{\Gamma}}^{-1}\big)_{ik} \min\left\{ \left\| \frac{(\boldsymbol{L}\boldsymbol{v})_k}{a_{kk}} \right\|_{\infty,\omega}, \left\| \frac{(\boldsymbol{L}\boldsymbol{v})_k}{b_k} \right\|_{\infty,\omega} \right\}$$

for any mesh function $\boldsymbol{v} \in \left(I\!R_0^{N+1}\right)^{\ell}$.

Remark 4.48. Theorem 4.47 implies

$$\|\boldsymbol{v}\|_{\infty,\bar{\omega}} := \max_{k=1,\dots,\ell} \|v_k\|_{\infty,\bar{\omega}} \le C \|\boldsymbol{L}\boldsymbol{v}\|_{\infty,\omega} \quad \text{for all } \boldsymbol{v} \in \left(I\!\!R_0^{N+1}\right)^\ell,$$

i.e., the operator $\boldsymbol{L}$ is $(\ell_\infty, \ell_\infty)$-stable although it does not obey a comparison principle. ♣

Corollary 4.49. *Under the hypotheses of Theorem 4.47 the discrete problem* (4.81) *has a unique solution* $\boldsymbol{u}^N$*, and*

$$\left\|\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \le C \|\boldsymbol{f}\|_{\infty,\omega}$$

for some constant C*.*

Remark 4.50. One can also use (4.8c) when bounding $(\boldsymbol{L}\boldsymbol{v})_k$ in (4.83) to establish that

$$\|\boldsymbol{v}\|_{\infty,\bar{\omega}} \le C \max_{k=1,\dots,\ell} \|(\boldsymbol{L}\boldsymbol{v})_k\|_{-1,\infty} \quad \text{for all } \boldsymbol{v} \in \left(I\!\!R_0^{N+1}\right)^\ell.$$

This allows to analyse the difference scheme (4.82) when applied to problems whose source terms consist of generalised functions like the δ-distribution. ♣

4.4.1.2 A Priori Error Analysis

Following [96], we split the error $\boldsymbol{\eta} := \boldsymbol{u} - \boldsymbol{u}^N$ into two parts $\boldsymbol{\psi}, \boldsymbol{\varphi} \in (I\!\!R_0^{N+1})^\ell$ as $\boldsymbol{\eta} = \boldsymbol{\psi} + \boldsymbol{\varphi}$ with

$$\Lambda_k \psi_k = (\boldsymbol{L}\boldsymbol{\eta})_k \quad \text{and} \quad \Lambda_k \varphi_k = -\sum_{\substack{m=1\\m\neq k}}^{\ell} a_{km}\eta_m \ \text{ on } \omega, \quad k = 1, \dots, \ell.$$

A triangle inequality and Theorem 4.3 yield

$$\|\eta_k\|_{\infty,\omega} \le \|\psi_k\|_{\infty,\omega} - \sum_{\substack{m=1\\m\neq k}}^{\ell} \tilde{\gamma}_{km} \|\eta_m\|_{\infty,\omega}, \quad k = 1, \dots, \ell.$$

Assuming that $\tilde{\boldsymbol{\Gamma}}$ is an M-matrix, we obtain

$$\left\|\boldsymbol{u} - \boldsymbol{u}^N\right\|_{\infty,\omega} \le C \|\boldsymbol{\psi}\|_{\infty,\omega},$$

and we are left with bounding $\boldsymbol{\psi}$.

The components of $\boldsymbol{\psi}$ are the solutions of scalar problems to which the technique of Sect. 4.2.2 can be applied. The following general error bound is obtained.

Theorem 4.51. *Assume that the matrix $\boldsymbol{A}$ has non-negative diagonal entries. Assume that $\tilde{\boldsymbol{\Gamma}}(\boldsymbol{A})$ is inverse-monotone. Let $\boldsymbol{u}$ and $\boldsymbol{u}^N$ be the solutions of* (4.81) *and* (4.82)*. Then*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \le C \max_{i=1,\dots,N} \int_{I_i} \left[1+\sum_{m=1}^{\ell} \left|u_m'(s)\right|\right] \mathrm{d}s.$$

Corollary 4.52. *The a priori bounds on the u_m' in Sect. 3.4.2.2 can be used to derive more explicit error bounds:*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \le C\vartheta_{cd,\ell}^{[1]}(\bar{\omega}).$$

Note, the quantity $\vartheta_{cd,\ell}^{[p]}(\bar{\omega})$, $p>0$, has been defined in Sect. 2.1.6.

Remark 4.53. We immediately obtain, for example,

$$\left\|u-u^N\right\|_{\infty,\omega} \le \begin{cases} CN^{-1} & \text{for Bakhvalov meshes,} \\ CN^{-1}\ln N & \text{for Shishkin meshes,} \end{cases}$$

when the mesh parameters satisfy $\sigma_m \ge 1$. ♣

4.4.1.3 A Posteriori Error Analysis

Alternatively, one can appeal to the strong stability (3.29c) of the scalar continuous operators and combine the arguments of Sect. 4.2.4 and 4.4.1.1, in order to get the a posteriori bound

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty} \le C \max_{k=0,\dots,N-1} h_{k+1}\left[1+\sum_{m=1}^{\ell}\left|u_{m,x;k}^N\right|\right].$$

The constant(s) involved in this error bound can be specified explicitly; cf. Sect. 4.2.4.

4.4.2 Strongly Coupled Systems

We now consider strongly coupled systems of convection-diffusion type, i.e., for each k one has $b_{km} \not\equiv 0$ for some $k \ne m$. Strong coupling causes interactions between boundary layers that are not fully understood at present. The main papers on this problem are [1, 100, 127, 128].

The general strongly coupled two-point boundary-value problem in conservative form is: Find $\boldsymbol{u} \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ such that

$$\begin{aligned} \boldsymbol{\mathcal{L}^c u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' - (\boldsymbol{Bu})' + \boldsymbol{Au} = \boldsymbol{f} \quad \text{in } \Omega := (0,1), \\ \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \end{aligned} \tag{4.84}$$

where as before $\boldsymbol{f} = (f_1, \dots, f_\ell)^T$, while $\boldsymbol{A} = (a_{km})$ and $\boldsymbol{B} = (b_{km})$ are $\ell \times \ell$ matrices, and the $\ell \times \ell$ matrix $\operatorname{diag}(\boldsymbol{\varepsilon})$ is diagonal with k^{th} entry ε_k for all k. Furthermore, let $b_{kk} \geq \beta_k$ or $b_{kk} \leq -\beta_k$ on $[0,1]$ with positive constants β_k.

We follow [100] and discretise (4.84) using the upwind scheme of Sect. (4.2) for each equation of the system: Find $\boldsymbol{u}^N \in (\mathbb{R}_0^{N+1})^\ell$ such that

$$\left[\boldsymbol{L^c u}^N\right]_i = \boldsymbol{f}_i \quad \text{for } i = 1, \dots, N-1, \tag{4.85}$$

where $\boldsymbol{L^c v} := \left((\boldsymbol{L^c v})_1, (\boldsymbol{L^c v})_2, \dots, (\boldsymbol{L^c v})_\ell\right)^T$,

$$(\boldsymbol{L^c v})_k := \Lambda_k^c v_k - \sum_{\substack{m=1 \\ m \neq k}}^{\ell} (b_{km} v_m)_x + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} a_{km} v_m \quad \text{if } b_{kk} > 0,$$

$$(\boldsymbol{L^c v})_k := \Lambda_k^c v_k - \sum_{\substack{m=1 \\ m \neq k}}^{\ell} (b_{km} v_m)_{\bar{x}} + \sum_{\substack{m=1 \\ m \neq k}}^{\ell} a_{km} v_m \quad \text{if } b_{kk} < 0,$$

and

$$[\Lambda_k^c v]_i := \begin{cases} -\varepsilon_k v_{\bar{x}x;i} - (b_{kk} v)_{x;i} + a_{kk;i} v_i & \text{if } b_{kk} > 0, \\ -\varepsilon_k v_{x\bar{x};i} - (b_{kk} v)_{\bar{x};i} + a_{kk;i} v_i & \text{if } b_{kk} < 0. \end{cases}$$

4.4.2.1 Stability

The stability analysis for the difference operator $\boldsymbol{L}$ is analogous to that for the continuous operator $\boldsymbol{\mathcal{L}}$ in Sect. 3.4.3.

Define the $\ell \times \ell$ matrix $\boldsymbol{\Upsilon}_\omega = \boldsymbol{\Upsilon}_\omega(\boldsymbol{A}, \boldsymbol{B}) = (\gamma_{km})$ by

$$\gamma_{kk} = 1 \quad \text{and} \quad \gamma_{km} = -\frac{2\left\|b_{km}\right\|_{\infty,\omega} + \left\|a_{km}\right\|_{1,\omega}}{\beta_k} \quad \text{for } k \neq m.$$

Introduce the discrete maximum norms

$$|||v|||_{\varepsilon_k,\infty,\omega} := \varepsilon_k \left\|v_x\right\|_{\infty,\omega} + \beta_k \left\|v\right\|_{\infty,\bar{\omega}} \quad \text{for } v \in \mathbb{R}^{N+1}$$

and

$$|||\boldsymbol{v}|||_{\boldsymbol{\varepsilon},\infty,\omega} := \max_{k=1,\dots,\ell} |||v_k|||_{\varepsilon_k,\infty,\omega} \quad \text{for } \boldsymbol{v} \in \left(\mathbb{R}^{N+1}\right)^{\ell}.$$

Theorem 4.54. *Assume that for each* $k = 1, \dots, \ell$

$$b_{kk} \le -\beta_k \quad \textit{or} \quad b_{kk} \ge \beta_k \quad \textit{on} \ [0,1]$$

with positive constants β_k *and that*

$$a_{kk} \ge 0 \quad \textit{and} \quad b'_{kk} \ge 0 \quad \textit{on} \ [0,1].$$

Suppose $\boldsymbol{\Upsilon}_\omega(\boldsymbol{A}, \boldsymbol{B})$ *is inverse-monotone. Then the operator* $\boldsymbol{L^c}$ *is* $(\ell_\infty, w^{-1,\infty})$ *stable with*

$$|||v_k|||_{\varepsilon_k,\infty,\omega} \le \sum_{m=1}^{\ell} \left(\boldsymbol{\Upsilon}_\omega^{-1}\right)_{km} \|(\boldsymbol{L^c}\boldsymbol{v})_m\|_{-1,\infty,\omega} \quad \textit{for} \ k = 1, \dots, \ell,$$

and for all $\boldsymbol{v} \in (\mathbb{R}_0^{N+1})^\ell$.

Proof. For the sake of simplicity in the presentation, we restrict ourselves to the case when $b_{kk} > 0$ for all k.

Let $\boldsymbol{v} \in (\mathbb{R}_0^{N+1})^\ell$ be arbitrary. Then the definition of $\boldsymbol{L^c}$ and the $\varLambda_k^c$ yields

$$\varLambda_k^c v_k = (\boldsymbol{L^c}\boldsymbol{v})_k + \sum_{\substack{m=1\\ m\neq k}}^{\ell} (b_{km}v_m)_x - \sum_{\substack{m=1\\ m\neq k}}^{\ell} a_{km}v_m$$

Apply Theorem 4.3 and Remark 4.4 to get

$$|||v_k|||_{\varepsilon_k,\infty,\omega} \le \sum_{\substack{m=1\\ m\neq k}}^{\ell} \Big(2\,\|b_{km}\|_{\infty,\omega} + \|a_{km}\|_{1,\omega}\Big) \|v_m\|_{\infty,\omega} + \|(\boldsymbol{L^c}\boldsymbol{v})_k\|_{-1,\infty,\omega}\,.$$

Recall the definition of $\boldsymbol{\Upsilon}_\omega$ and rearrange the last inequality

$$\sum_{m=1}^{\ell} \gamma_{km}\, |||v_m|||_{\varepsilon_m,\infty,\omega} \le \|(\boldsymbol{L^c}\boldsymbol{v})_k\|_{-1,\infty,\omega} \quad \text{for } k = 1, \dots, \ell.$$

Using the inverse monotonicity of $\boldsymbol{\Upsilon}_\omega$, we are finished. □

Corollary 4.55. *Suppose the hypotheses of Theorem 4.54 are satisfied. Then the difference equation (4.85) possesses a unique solution $\boldsymbol{u}^N$, with*

$$|||\boldsymbol{u}^N|||_{\boldsymbol{\varepsilon},\infty,\omega} \le C \max_{m=1,\dots,\ell} \|f_m\|_{-1,\infty,\omega}$$

for some constant C that is independent of both ε and the mesh.

Remark 4.56. In general, the operator $\boldsymbol{L^c}$ does not obey a comparison principle. Nonetheless it is $(\ell_\infty, \ell_\infty)$-stable, i.e.,

$$\|\boldsymbol{v}\|_{\infty,\bar{\omega}} \le C\, \|\boldsymbol{L^c v}\|_{\infty,\omega} \quad \text{for all } \boldsymbol{v} \in \left(I\!R_0^{N+1}\right)^\ell,$$

by Theorem 4.54. ♣

4.4.2.2 A Priori Error Analysis

Adapt the argument of Sect. 4.4.1.2 as in [100] and split the error $\boldsymbol{\eta} := \boldsymbol{u}^N - \boldsymbol{u}$ into two parts $\boldsymbol{\psi}, \boldsymbol{\varphi} \in (I\!R_0^{N+1})^\ell$ as $\boldsymbol{\eta} = \boldsymbol{\psi} + \boldsymbol{\varphi}$ with

$$\Lambda_k^c \psi_k = (\boldsymbol{L\eta})_k \quad \text{and} \quad \Lambda_k^c \varphi_k = \sum_{\substack{m=1\\ m\neq k}}^{\ell} \left((b_{km}\eta_m)_x - a_{km}\eta_m\right) \quad \text{on } \omega.$$

Recalling the definition of $\boldsymbol{\Upsilon}_\omega = (\gamma_{km})$, we use a triangle inequality and Theorem 4.3 to obtain

$$\sum_{m=1}^{\ell} \gamma_{km}\, |||\eta_m|||_{\varepsilon_m \infty,\omega} \le |||\psi_k|||_{\varepsilon_k,\infty,\omega} \quad \text{for } k = 1,\dots,\ell.$$

Next, if $\boldsymbol{\Upsilon}$ is inverse monotone then

$$|||\boldsymbol{u} - \boldsymbol{u}^N|||_{\boldsymbol{\varepsilon},\infty,\omega} \le C\, |||\boldsymbol{\psi}|||_{\boldsymbol{\varepsilon},\infty,\omega},$$

and we are left with bounding $|||\psi_k|||_{\varepsilon_k,\infty,\omega}$, $k = 1,\dots,\ell$.

The components of $\boldsymbol{\psi}$ are the solutions of scalar problems. In [100] the technique of Sect. 4.2.2 is used to obtain the following general error bound.

Theorem 4.57. *Let the hypothesis of Theorem 4.54 be satisfied. Then the error of the upwind scheme (4.85) applied to (4.84) satisfies*

$$|||\boldsymbol{u} - \boldsymbol{u}^N|||_{\boldsymbol{\varepsilon},\infty,\omega} \le C \max_{i=1,\dots,N} \int_{I_i} \left[1 + \sum_{m=1}^{\ell} |u_m'(s)|\right] \mathrm{d}s.$$

Corollary 4.58. *By (3.56) we have* $\|u_k'\|_1 \le C$ *for* $k = 1, \dots, \ell$. *Therefore, there exists a mesh* ω^* *such that*

$$\int_{I_i} \left(1 + \sum_{m=1}^{\ell} |u_m'(x)|\right) \mathrm{d}x = \frac{1}{N} \int_0^1 \left(1 + \sum_{m=1}^{\ell} |u_m'(x)|\right) \mathrm{d}x \le CN^{-1}$$

and on this mesh one consequently has

$$|||\boldsymbol{u} - \boldsymbol{u}^N|||_{\boldsymbol{\varepsilon},\infty,\omega^*} \le CN^{-1}.$$

Remark 4.59. As satisfactory pointwise bounds on $|u_k'|$ are unavailable, this result does not give an immediate explicit convergence result on, e.g., a Bakhvalov or Shishkin mesh. ♣

Remark 4.60. When $\varepsilon_k = \varepsilon$ for $k = 1, \dots, \ell$, Theorem 3.57 yields

$$|||\boldsymbol{u} - \boldsymbol{u}^N|||_{\boldsymbol{\varepsilon},\infty,\omega} \le C \max_{i=1,\dots,N} \int_{I_i} \left(1 + \mathrm{e}^{-\beta x/\varepsilon}\right) \mathrm{d}x = C\vartheta_{cd}^{[1]}(\bar{\omega}).$$

The system behaves like the scalar equation of Section 4.2 and appropriately adapted meshes can be constructed as for scalar problems.

In [127] one also finds an error analysis for a system with a single parameter, but the analysis is limited to Shishkin meshes and uses a more traditional truncation error and barrier function argument. Furthermore, higher regularity of the solution is required. On the other hand, in certain situations the analysis of [127] is valid under less restrictive hypotheses on the entries of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ than the requirement that $\boldsymbol{\Upsilon}_\omega$ be inverse-monotone. ♣

4.4.2.3 A Posteriori Error Bounds

Using the strong stability results of Theorem 3.54, we can follow [100] to obtain the a posteriori error bound

$$|||\boldsymbol{u} - \boldsymbol{u}^N|||_{\boldsymbol{\varepsilon},\infty} \le C \max_{i=1,\dots,N} h_i \left(1 + \sum_{m=1}^{\ell} |u^N_{m;\bar{x};i}|\right).$$

Remark 4.61. The de Boor algorithm (see Sect. 4.2.4.2) can be used to adaptively generate meshes for (4.84) by choosing

$$Q_i = h_i \left(1 + \sum_{m=1}^{\ell} \left(u^N_{m;\bar{x};i}\right)^2\right)^{1/2}.$$

Numerical examples are presented in [100], but a complete analysis of the adaptive algorithm is not given. ♣

4.4.2.4 Numerical Results

We now present the results of two numerical experiments in order to illustrate the conclusions of Theorem 4.57.

First, we consider a test problem with two equations.

$$-\operatorname{diag}(\boldsymbol{\varepsilon})\,\boldsymbol{u}'' - \begin{pmatrix} 4+x & -1 \\ 1-2x & -3 \end{pmatrix} \boldsymbol{u}' + \begin{pmatrix} 0 & -x^2 \\ 1 & 1 \end{pmatrix} \boldsymbol{u} = \begin{pmatrix} \mathrm{e}^{1-2x} \\ \cos 2x \end{pmatrix} \quad \text{in } (0,1), \quad (4.86)$$

subject to homogeneous Dirichlet boundary conditions $\boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}$.

For this problem our convergence theory applies since

$$\boldsymbol{\Gamma} = \frac{1}{12}\begin{pmatrix} 12 & -7 \\ -12 & 12 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Gamma}^{-1} = \frac{1}{5}\begin{pmatrix} 12 & 7 \\ 12 & 12 \end{pmatrix} \geq 0.$$

We have $b_1 \geq 4$ and $-b_2 \geq 3$. Therefore, one expects that the solution exhibits two layers: one at $x = 0$ behaving like $\mathrm{e}^{-4x/\varepsilon_1}$ and the other at $x = 1$ that behaves like $\mathrm{e}^{-3(1-x)/\varepsilon_2}$. Also the first-order derivative of $\boldsymbol{u}$ is expected to satisfy

$$|u_i'(x)| \leq C\left\{1 + \varepsilon_1^{-1} e^{-4x/\varepsilon_1} + \varepsilon_2^{-1} e^{-3(1-x)/\varepsilon_2}\right\}, \quad i = 1, 2.$$

Note, we do not have proper proof for these derivative bounds. The difficulties in proving them have been explained in Section 3.4.3.1.

The exact solution of (4.86) is not available. Therefore, we compare the numerical solution with that obtained by Richardson extrapolation as before. We consider Bakhvalov and Shishkin meshes and the adaptive de Boor algorithm; see Remark 4.61. The construction of layer-adapted meshes for overlapping layers is explained in Section 2.1.6.

The results of our test computations are contained in Table 4.6. For both *a priori* adapted meshes the expected (almost) first order of uniform convergence is confirmed. For the adaptive algorithm first order is also observed although the numerical rates are “less stable”.

Table 4.6 Simple upwinding for a system of two convection-diffusion equations

N	Shishkin mesh		Bakhvalov mesh		adaptive algorithm	
	η_N	ρ^N	η_N	r^N	η_N	r^N
$3 \cdot 2^7$	3.131e-02	0.83	8.194e-03	0.99	7.600e-03	0.96
$3 \cdot 2^8$	1.843e-02	0.88	4.119e-03	0.99	3.903e-03	1.05
$3 \cdot 2^9$	1.051e-02	0.91	2.073e-03	0.99	1.880e-03	0.99
$3 \cdot 2^{10}$	5.870e-03	0.94	1.044e-03	0.99	9.477e-04	0.98
$3 \cdot 2^{11}$	3.229e-03	0.95	5.240e-04	1.00	4.821e-04	0.94
$3 \cdot 2^{12}$	1.758e-03	0.97	2.620e-04	1.00	2.505e-04	1.00
$3 \cdot 2^{13}$	9.491e-04	0.98	1.308e-04	1.00	1.254e-04	1.15
$3 \cdot 2^{14}$	5.092e-04	0.98	6.540e-05	1.00	5.643e-05	0.91
$3 \cdot 2^{15}$	2.717e-04	0.99	3.269e-05	1.00	3.006e-05	0.99
$3 \cdot 2^{16}$	1.444e-04	—	1.634e-05	—	1.516e-05	—

Table 4.7 Simple upwinding for a convection-diffusion problem with three equations

N	Shishkin mesh		Bakhvalov mesh		adaptive algorithm	
	η_N	ρ^N	η_N	r^N	η_N	r^N
$4 \cdot 2^3$	2.453e-01	0.49	1.780e-01	0.86	1.850e-01	0.64
$4 \cdot 2^4$	1.809e-01	0.63	9.777e-02	0.93	1.187e-01	0.75
$4 \cdot 2^5$	1.218e-01	0.75	5.124e-02	0.97	7.062e-02	0.87
$4 \cdot 2^6$	7.612e-02	0.84	2.623e-02	0.98	3.863e-02	0.96
$4 \cdot 2^7$	4.505e-02	0.90	1.327e-02	0.99	1.991e-02	0.92
$4 \cdot 2^8$	2.569e-02	0.94	6.673e-03	1.00	1.051e-02	1.02
$4 \cdot 2^9$	1.430e-02	0.97	3.346e-03	1.00	5.167e-03	0.97
$4 \cdot 2^{10}$	7.824e-03	0.98	1.676e-03	1.00	2.642e-03	1.06
$4 \cdot 2^{11}$	4.233e-03	1.00	8.384e-04	1.00	1.270e-03	1.05
$4 \cdot 2^{12}$	2.272e-03	—	4.193e-04	—	6.115e-04	—

The second test problem consists of three convection-diffusion equations.

$$-\operatorname{diag}(\boldsymbol{\varepsilon})\,\boldsymbol{u}'' - \begin{pmatrix} 3 & 1 & 0 \\ -x^2 & 5+x & -1 \\ 1-x & 0 & -5 \end{pmatrix} \boldsymbol{u}' = \begin{pmatrix} \mathrm{e}^x \\ \cos x \\ 1+x^2 \end{pmatrix} \quad \text{in } (0,1),$$

with boundary conditions $\boldsymbol{u}(0) = \boldsymbol{u}(1) = \mathbf{0}$. This time:

$$\boldsymbol{\Gamma} = \frac{1}{15} \begin{pmatrix} 15 & 10 & 0 \\ 7 & 15 & 6 \\ 9 & 0 & 15 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Gamma}^{-1} = \frac{1}{119} \begin{pmatrix} 225 & 150 & 60 \\ 159 & 225 & 90 \\ 135 & 90 & 155 \end{pmatrix} \geq 0.$$

We expect layers $\mathrm{e}^{-3x/\varepsilon_1}$, $\mathrm{e}^{-5x/\varepsilon_2}$ and $\mathrm{e}^{-5(1-x)/\varepsilon_3}$ to form and adapt the mesh accordingly.

Table 4.7 gives the numerical results for our second example. A comparison with Table 4.6 reveals that the behaviour of the method is similar to that for Example (4.86).

4.5 Problems with Turning Point Layers

This section considers linear convection-diffusion problems with a boundary turning point: Find $u \in C^2(0,1) \cap C[0,1]$ such that

$$\mathcal{L}u := -\varepsilon u'' - pbu' + pcu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \; u(1) = \gamma_1, \qquad (4.87)$$

where $p(x) = x^\kappa$, $\kappa > 0$, $b \geq \beta > 0$ and $c \geq 0$ in $(0,1)$.

We are aware of four publications analysing numerical methods for this problem with $\kappa = 1$. Liseikin [113] constructs a special transformation and solves the transformed problem on a uniform mesh. The method obtained is proved to be first-order uniformly convergent in the discrete maximum norm. Vulanović [167] studies an upwind-difference scheme on a layer-adapted Bakhvalov-type mesh and proves convergence in a discrete ℓ_1 norm. This result is generalised in [170] for quasilinear problems. However, this norm fails to capture the layers present in the solution. Therefore, the problem is not singularly perturbed in the sense of Def. 1.1. In [112] the authors establish almost first-order convergence for an upwind-difference scheme on a Shishkin mesh. Here we follow [92] and study (4.87) with arbitrary $\kappa > 0$.

4.5.1 A First-Order Upwind Scheme

The boundary-value problem (4.87) is discretised using simple upwinding: Find $u \in I\!R^{N+1}$ such that

$$\left[Lu^N\right]_i = p_i f_i \quad \text{for } i = 1, \dots, N-1, \;\; u_0^N = \gamma_0, \; u_N^N = \gamma_1 \tag{4.88}$$

with

$$[Lv]_i := -\varepsilon v_{\bar{x}x;i} - p_i b_i v_{x;i} + p_i c_i v_i.$$

4.5.1.1 Stability of the Discretisation

The matrix associated with it is an L_0-matrix. Lemma 3.14 with the test function $e_i = 1 - x_i$ verifies that it is an M-matrix. Therefore, the operator L satisfies a discrete comparison principle. That is, for any mesh functions $v, w \in I\!R^{N+1}$

$$\left.\begin{array}{r} Lv \le Lw \quad \text{on } \omega, \\ v_0 \le w_0, \\ v_N \le w_N \end{array}\right\} \quad \Longrightarrow \quad v \le w \quad \text{on } \bar{\omega}.$$

Lemma 3.17 with $e_i = 1 - x_i$ yields

$$\|v\|_{\infty,\bar{\omega}} \le \|Lv/pb\|_{\infty,\omega} \quad \text{for all } v \in I\!R_0^{N+1}.$$

Alternatively, if $c > 0$ on $[0, 1]$, then Lemma 3.17 with $\boldsymbol{e} \equiv 1$ gives

$$\|v\|_{\infty,\bar{\omega}} \le \|Lv/pc\|_{\infty,\omega} \quad \text{for all } v \in I\!R_0^{N+1}.$$

Thus, the operator L is $(\ell_\infty, \ell_{\infty,1/p})$ stable in the $1/p$-weighted maximum norm.

For the solution u^N of (4.88) this implies

$$\left\|u_i^N\right\|_{\infty,\bar{\omega}} \le \max\{|\gamma_0|,|\gamma_1|\} + \min\left\{\left\|\frac{f}{b}\right\|_{\infty,\omega}, \left\|\frac{f}{c}\right\|_{\infty,\omega}\right\}.$$

Green's function.

Lemma 4.62. *Assume that*

$$p > 0, \quad p' \ge 0, \quad b \ge \beta > 0 \quad \textit{and} \quad c \ge 0 \ \textit{on} \ (0,1). \tag{4.89}$$

Then

$$0 \le G_{i,j} \le \hat{G}_{i,j} := \frac{1}{p_j\beta}\begin{cases} 1 & \textit{for } i = 0,\dots,j, \\ \displaystyle\prod_{\nu=j+1}^{i}\left(1+\frac{\beta p_\nu h_{\nu+1}}{\varepsilon}\right)^{-1} & \textit{for } i = j+1,\dots,N. \end{cases}$$

Proof. Let j be arbitrary, but fixed. $G_{\cdot,j}$ solves

$$[LG_{\cdot,j}]_i = \delta_{i,j}, \quad i = 1,\dots,N-1, G_{0,j} = G_{N,j} = 0$$

with

$$\delta_{i,j} = \begin{cases} h_{i+1}^{-1} & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We shall show that $\hat{G}_{\cdot,j}$ is a barrier function for $G_{\cdot,j}$. Clearly $\hat{G}_{0,j} \ge 0$ and $\hat{G}_{N,j} \ge 0$.

Next, verify that

$$\hat{G}_{x;i,j} = \hat{G}_{i,j}\begin{cases} 0 & \text{for } i = 0,\dots,j-1, \\ -\dfrac{\beta p_i}{\varepsilon+\beta p_i h_{i+1}} & \text{for } i = j,\dots,N-1, \end{cases}$$

and

$$\hat{G}_{\bar{x};i,j} = \hat{G}_{i,j}\begin{cases} 0 & \text{for } i = 1,\dots,j, \\ -\dfrac{\beta p_{i-1}}{\varepsilon} & \text{for } i = j+1,\dots,N. \end{cases}$$

Hence,

$$\left[L\hat{G}_{\cdot,j}\right]_i \ge p_i c_i \hat{G}_{i,j} \ge 0 \quad \text{for } i = 1,\dots,j-1,$$

$$\left[L\hat{G}_{\cdot,j}\right]_j \geq \left(\frac{\beta pj}{h_{j+1}} + p_j c_j\right) \hat{G}_{j,j} \geq \frac{1}{h_{i+1}},$$

and

$$\left[L\hat{G}_{\cdot,j}\right]_i \geq \left(\beta \frac{p_i - p_{i-1}}{h_{i+1}} + p_i c_i\right) \hat{G}_{i,j} \geq 0 \quad \text{for } i = j+1, \ldots, N-1,$$

because $p' \geq 0$ on $[0,1]$. Thus, $\hat{G}$ is a barrier function for G. □

Remark 4.63. The proof of Lemma 4.62 simplifies the argument in [112] where a barrier function was constructed for the adjoint problem. ♣

Remark 4.64. Numerical results indicate that when $p(x) = x^\kappa$, $\kappa \geq 0$, one has the sharper bound

$$G_{i,j} \leq C \left(\varepsilon^{1/(\kappa+1)} + \xi_j\right)^{-\kappa} \quad \text{for all } i, j = 1, \ldots, N-1,$$

but, we do not have a rigorous proof for this.

Theorem 4.65. *Assume the data satisfies* (4.89)*. Then the discrete operator L is $(\ell_\infty, \ell_{1,1/p})$ stable*

$$\|v\|_{\infty,\omega} \leq \beta^{-1} \|Lv/p\|_{1,\omega}$$

for all $v \in I\!R_0^{N+1}$ with a $1/p$-weighted ℓ_1 norm.

Proof. For any function $v \in I\!R_0^{N+1}$ we have

$$v_i = \sum_{j=1}^{N} h_{j+1} G_{i,j} \left[Lv\right]_j, \quad i = 1, \ldots, N-1.$$

Then Lemma 4.62 yields the assertion of the theorem. □

Remark 4.66. An immediate consequence of Theorem 4.65 for the simple upwind scheme is

$$\left\|u - u^N\right\|_{\infty,\omega} \leq \beta^{-1} \left\|\left(Lu - pf\right)/p\right\|_{1,\omega}.$$

Thus, the error of the numerical solution in the maximum norm is bounded by an ℓ_1-type norm of the truncation error weighted with the inverse of the coefficient of the convection term. This was used in [112] to establish uniform almost first-order convergence on Shishkin meshes for $\kappa = 1$. ♣

4.5.2 Convergence on Shishkin Meshes

In [112] convergence of the upwind scheme (4.88) applied to (4.87) with $\kappa = 1$ was studied. Starting from the observation that for any fixed $m > 0$ there exists a constant $C = C(m)$ such that

$$\exp\left(-\frac{\tilde{\beta}x^2}{2\varepsilon}\right) \le C\exp\left(-m\frac{x}{\sqrt{\varepsilon}}\right)$$

a piecewise uniform mesh is constructed as follows: Fix the mesh transition point

$$\tau = \min\left\{q, \frac{2\sqrt{\varepsilon}}{m}\ln N\right\}.$$

Then divide $[0,\tau]$ uniformly into qN subintervals and $[\tau,1]$ into $(1-q)N$ subintervals.

Using the stability inequality of Theorem 4.65, it is then shown that

$$\left\|u - u^N\right\|_{\infty,\omega} \le CN^{-1}(\ln N)^2.$$

The details of the analysis are similar to the argument in Sect. 4.2.5.

The general case of an arbitrary $\kappa > 0$ has been considered in [92]. We will give a brief summary of that paper now. This time the transition point τ is chosen as follows:

$$\tau = \min\left\{q, \left(\sigma\frac{\varepsilon(\kappa+1)}{\tilde{\beta}}\ln N\right)^{1/(\kappa+1)}\right\}.$$

Here we shall consider $\tau < q$ which is the interesting case.

First Lemma 4.62 is sharpened to

$$G_{i,j} \le \begin{cases} C\tau\varepsilon^{-1} & \text{for } j = 1,\dots,qN-1, \\ \beta^{-1}\xi_j^{-\kappa} & \text{for } j = qN,\dots,N-1. \end{cases}$$

Note that $\tau\varepsilon^{-1} = C\mu^{-\kappa}(\ln N)^{1/(\kappa+1)}$.

Remark 4.67. It is argued in [92] based on numerical evidence that the logarithmic factor is superfluous and one has

$$G_{i,j} \le C(\mu + \xi_j)^{-\kappa} \quad \text{for } i,j = 1,\dots,N-1,$$

but no rigorous analysis is provided. ♣

The analysis in [92] proceeds along the lines of Sect. 4.2.5 using the solution decomposition in Theorem 3.63 to establish

$$\left\|u - u^N\right\|_{\infty,\omega} \le CN^{-1} (\ln N)^{2/(\kappa+1)} \tag{4.90}$$

if $\sigma \ge 2$.

Remark 4.68. If one had $G_{i,j} \le C(\mu + \xi_j)^{-\kappa}$, then (4.90) could be sharpened to

$$\left\|u - u^N\right\|_{\infty,\omega} \le CN^{-1} (\ln N)^{1/(\kappa+1)} .$$

Also we do not have any theory for arbitrary meshes. This is due to a lack of stronger negative-norm stability inequalities for both the continuous and the discrete operators. More work in this direction is required. ♣

4.5.3 A Numerical Example

We verify experimentally the convergence result of (4.90). Our test problem is the semilinear differential equation

$$-\varepsilon u''(x) - x^\kappa (2-x) u'(x) + x^\kappa \mathrm{e}^{u(x)} = 0 \quad \text{for } x \in (0,1),$$
$$u(0) = u(1) = 0.$$

The exact solution of this problem is not available. We therefore estimate the accuracy of the numerical solution by comparing it with the numerical solution on a higher order method: Richardson extrapolation. For our tests we take $\tilde{\beta} = 1$, $q = 1/2$ and $\varepsilon = 10^{-12}$.

Table 4.8 displays the results of the numerical test. For comparison reasons Table 4.9 contains the rates which can be expected if the error bound

Table 4.8 Simple upwinding on Shishkin meshes for turning point problems

	$\kappa = 1/2$		$\kappa = 1$		$\kappa = 2$		$\kappa = 3$	
N	error	rate	error	rate	error	rate	error	rate
2^7	6.171e-3	0.88	5.335e-3	0.92	4.675e-3	0.95	4.411e-3	0.96
2^8	3.358e-3	0.90	2.829e-3	0.93	2.426e-3	0.96	2.270e-3	0.97
2^9	1.803e-3	0.91	1.484e-3	0.94	1.249e-3	0.96	1.160e-3	0.97
2^{10}	9.592e-4	0.92	7.737e-4	0.95	6.401e-4	0.97	5.899e-4	0.98
2^{11}	5.069e-4	0.93	4.014e-4	0.95	3.269e-4	0.97	2.993e-4	0.98
2^{12}	2.666e-4	0.93	2.075e-4	0.96	1.666e-4	0.98	1.516e-4	0.98
2^{13}	1.396e-4	0.94	1.070e-4	0.96	8.473e-5	0.98	7.669e-5	0.98
2^{14}	7.292e-5	0.94	5.506e-5	0.96	4.305e-5	0.98	3.876e-5	0.99
2^{15}	3.798e-5	0.94	2.828e-5	0.96	2.185e-5	0.98	1.958e-5	0.99
2^{16}	1.973e-5	—	1.451e-5	—	1.108e-5	—	9.881e-6	—

Table 4.9 Expected "convergence rates" for $(\ln N)^{p/(\kappa+1)}\, N^{-1}$

	$\kappa = 1/2$		$\kappa = 1$		$\kappa = 2$		$\kappa = 3$	
N	$p = 2$	$p = 1$	$p = 2$	$p = 1$	$p = 2$	$p = 1$	$p = 2$	$p = 1$
2^7	0.74	0.87	0.81	0.90	0.87	0.94	0.90	0.95
2^8	0.77	0.89	0.83	0.92	0.89	0.94	0.92	0.96
2^9	0.80	0.90	0.85	0.92	0.90	0.95	0.92	0.96
2^{10}	0.82	0.91	0.86	0.93	0.91	0.95	0.93	0.97
2^{11}	0.83	0.92	0.87	0.94	0.92	0.96	0.94	0.97
2^{12}	0.85	0.92	0.88	0.94	0.92	0.96	0.94	0.97
2^{13}	0.86	0.93	0.89	0.95	0.93	0.96	0.95	0.97
2^{14}	0.87	0.93	0.90	0.95	0.93	0.97	0.95	0.98
2^{15}	0.88	0.94	0.91	0.95	0.94	0.97	0.95	0.98

is $(\ln N)^{p/(\kappa+1)}\, N^{-1}$ for $p = 2$ and $p = 1$. The rates observed are closer to those expected for $p = 1$. This supports the hypothesis of Remark 4.68.

Chapter 5
Finite Element and Finite Volume Methods

In this chapter we consider finite element and finite volume discretisations of

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } \ (0,1), \ \ u(0) = u(1) = 0, \tag{5.1}$$

with $b \geq \beta > 0$. Its associated variational formulation is: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) = f(v) \quad \text{for all } \ v \in H_0^1(0,1), \tag{5.2}$$

where

$$a(u,v) := \varepsilon\,(u',v') - (bu',v) + (cu,v)$$

and

$$f(v) := (f,v) := \int_0^1 \big(fv\big)(x)\mathrm{d}x. \tag{5.3}$$

Throughout assume that

$$c + b'/2 \geq \gamma > 0. \tag{5.4}$$

This condition guaranties the coercivity of the bilinear form in (5.2):

$$|||v|||_\varepsilon^2 := \varepsilon\,\|v'\|_0^2 + \gamma\,\|v\|_0^2 \leq a(v,v) \quad \text{for all } \ v \in H_0^1(0,1).$$

This is verified using standard arguments, see e.g. [141]. If $b \geq \beta > 0$ then (5.4) can always be ensured by a transformation $\bar{u}(x) = u(x)\mathrm{e}^{\delta x}$ with δ chosen appropriately. We assume this transformation has been carried out.

We start our investigations with interpolation-error estimates and a Galerkin discretisations of (5.1)—including aspects of convergence, superconvergence, and postprocessing of the derivatives. Then stabilised finite element methods are considered. We finish with an upwinded finite volume method.

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_5,

5.1 The Interpolation Error

In this section we study the error in linear interpolation. The argument follows [84]. Let $\bar{\omega}$ be an arbitrary mesh. Let w^I denote the piecewise-linear function that interpolates to w at the nodes of $\bar{\omega}$.

In this section let us assume the function $\psi \in C^2[0,1]$ admits the derivative bounds

$$|\psi''(x)| \le C\left\{1+\varepsilon^{-2}\mathrm{e}^{-\beta x/\varepsilon}\right\}. \tag{5.5}$$

For example, the solution u of the boundary-value problem (5.1) belongs to this class of functions, see Sect. 3.4.1.2.

Proposition 5.1. *Suppose ψ satisfies (5.5). Then*

$$\left\|\psi-\psi^I\right\|_{\infty,I_i} \le C\left[\int_{I_i}\left\{1+\varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right\}\mathrm{d}x\right]^2$$

for any mesh interval $I_i=[x_{i-1},x_i]$.

Proof. For the interpolation error on I_i we have the representation

$$\left(\psi^I-\psi\right)(x) = \frac{1}{h_i}\int_{I_i}\int_{x_{i-1}}^{x}\int_{\xi}^{s}\psi''(t)\mathrm{d}t\mathrm{d}\xi\mathrm{d}s.$$

The right-hand side can be estimated to give

$$\left|\left(\psi^I-\psi\right)(x)\right| \le \int_{I_i}\left(\xi-x_{i-1}\right)\left|\psi''(\xi)\right|\mathrm{d}\xi.$$

Using Lemma 4.16 and (5.5) to bound the right-hand side, we are finished. □

Theorem 5.2. *Suppose ψ satisfies (5.5). Then*

$$\left\|\psi^I-\psi\right\|_0 \le \left\|\psi^I-\psi\right\|_\infty \le C\left(\vartheta_{cd}^{[2]}(\bar{\omega})\right)^2$$

and

$$\left|\left|\left|\psi^I-\psi\right|\right|\right|_\varepsilon \le C\vartheta_{cd}^{[2]}(\bar{\omega}).$$

Remark 5.3. The quantity

$$\vartheta_{cd}^{[p]}(\bar{\omega}) := \max_{k=0,\dots,N-1}\int_{I_k}\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta s/p\varepsilon}\right)\mathrm{d}s,$$

was introduced in Sect. 2.1, where bounds on $\vartheta_{cd}^{[p]}(\bar{\omega})$ for various layer-adapted meshes are given too. ♣

Proof (of Theorem 5.2). The bound on the L_∞ error is an immediate consequence of Prop. 5.1 and the definition of $\vartheta_{cd}^{[2]}$.

For the error in the H^1 norm, use integration by parts to get

$$\left\|\left(\psi^I - \psi\right)'\right\|_0^2 = \int_0^1 \left(\left(\psi - \psi^I\right)'(x)\right)^2 \mathrm{d}x = -\int_0^1 \psi''(x)\left(\psi - \psi^I\right)(x)\mathrm{d}x.$$

Thus,

$$\left\|\left(\psi^I - \psi\right)'\right\|_0^2 \le \left\|\psi^I - \psi\right\|_\infty \int_0^1 |\psi''(x)|\,\mathrm{d}x \le C\varepsilon^{-1}\left\|\psi^I - \psi\right\|_\infty$$

by a Hölder inequality and (5.5). Finally, combine this with the bound for the L_2 norm of the interpolation error to obtain the energy-norm estimate. □

Remark 5.4. Proposition 5.1 can be used to give local estimates for the interpolation error too. For example on S-type meshes (see Sect. 2.1.3) one has

$$\left\|\psi^I - \psi\right\|_{0,[\tau,1]} \le \left\|\psi^I - \psi\right\|_{\infty,[\tau,1]} \le CN^{-2}, \quad \text{if } \sigma \ge 2,$$

for the interpolation error outside the layer region. This is in general a sharper bound than that implicitly given by Theorem 5.2. ♣

Remark 5.5. The maximum-norm interpolation error bound can be generalised to Lagrange interpolation with polynomial of arbitrary degree $p \ge 0$.

Fix $0 \le \xi_0 < \xi_1 < \cdots < \xi_p \le 1$. Define an interpolant $I_p\psi$ of ψ by

$$I_p\psi\big|_{I_i} \in \Pi_p$$

and

$$\left(I_p\psi\right)\left(x_{i-1} + \xi_k h_i\right) = \psi\left(x_{i-1} + \xi_k h_i\right) \quad \text{for } i = 1, \ldots, N, \;\; k = 0, \ldots, p.$$

If

$$\left|\psi^{(p+1)}(x)\right| \le C\left\{1 + \varepsilon^{-(p+1)}\mathrm{e}^{-\beta x/\varepsilon}\right\}$$

then

$$\left\|I_p\psi - \psi\right\|_\infty \le C\left(\vartheta_{cd}^{[p+1]}(\bar{\omega})\right)^{p+1}.$$

This result applies, for example, to the solution u of (5.1). ♣

5.2 Linear Galerkin FEM

We start from the weak formulation (5.2). Let $\bar{\omega}$ be an arbitrary mesh and let V^ω denote the space of continuous, piecewise linear functions on ω that vanish for $x = 0$ and $x = 1$. Then our discretisation is: Find $u^N \in V^\omega$ such that

$$a(u^N, v) = f(v) \quad \text{for all } \; v \in V^\omega. \tag{5.6}$$

The coercivity of $a(\cdot,\cdot)$ guarantees the existence of unique solutions of both (5.2) and of (5.6).

Notation. *Throughout this section we use* $\|\cdot\|_1$ *to denote the* L_1 *norm. This cannot be confused with the* H_1 *norm because we only use the weighted* H_1 *norm* $|||\cdot|||_\varepsilon$.

5.2.1 Convergence

Based on the interpolation error bounds of Sect. 5.1 we conduct an error analysis for the Galerkin FEM on S-type meshes (see Sect. 2.1.3). The technique we shall use was developed by Stynes and O'Riordan [152] for standard Shishkin meshes and later generalised for S-type meshes by Linß and Roos [82, 137]. The technique can be extended to discretisations of two-dimensional problems using triangular or rectangular elements on tensor-product S-type meshes; see Sect. 9.2.2.1.

Theorem 5.6. *Let* $\bar{\omega}$ *be an S-type mesh with* $\sigma \geq 2$ *whose mesh generating function* $\tilde{\varphi}$ *satisfies* (2.8) *and*

$$\max|\psi'| \ln^{1/2} N \leq CN. \tag{5.7}$$

Then

$$|||u - u^N|||_\varepsilon \leq C\left(h + N^{-1}\max|\psi'|\right)$$

for the error of the Galerkin FEM.

Remark 5.7. The additional assumption (5.7) does not constitute a major restriction. For example both the standard Shishkin mesh and the Bakhvalov-Shishkin mesh satisfy this condition. ♣

Proof (Proof of Theorem 5.6). Let $\eta = u^I - u$ and $\chi = u^I - u^N$. For the interpolation error η, we get from Sect. 5.1, the derivative bounds (3.30) and from (2.9) that

$$|||\eta|||_\varepsilon \leq C\left(h + N^{-1}\max|\psi'|\right). \tag{5.8}$$

To bound χ we start from the coercivity of $a(\cdot,\cdot)$ and the orthogonality of the Galerkin method:

$$|||\chi|||_\varepsilon^2 \le a(\chi,\chi) = a(\eta,\chi) = \varepsilon(\eta',\chi') + (b\eta,\chi') + ((c+b')\eta,\chi)\,.$$

Apply the Cauchy-Schwarz inequality to the diffusion and reaction terms and the Hölder inequality to the convection term to estimate

$$|||\chi|||_\varepsilon^2 \le C\,|||\eta|||_\varepsilon\,|||\chi|||_\varepsilon + C\left(\|\eta\|_{\infty,[0,\tau]}\,\|\chi'\|_{1,[0,\tau]} + \|\eta\|_{\infty,[\tau,1]}\,\|\chi'\|_{1,[\tau,1]}\right),$$

where τ is the mesh transition point in the S-type mesh. On $[0,\tau]$ we use

$$\|\chi'\|_{1,[0,\tau]} \le C\tau^{1/2}\,\|\chi'\|_{0,[0,\tau]} \le C\ln^{1/2} N\,|||\chi|||_\varepsilon\,,$$

while on $[\tau,1]$ we have by an inverse inequality

$$\|\chi'\|_{1,[\tau,1]} \le CN\,\|\chi\|_{1,[\tau,1]} \le CN\,\|\chi\|_{0,[\tau,1]} \le CN\,|||\chi|||_\varepsilon\,.$$

These two bounds and the interpolation results of Sect. 5.1 yield

$$\begin{aligned}|||\chi|||_\varepsilon \le C\Big\{&h + N^{-1}\max|\psi'| \\ &+ \left(h + N^{-1}\max|\psi'|\right)^2\ln^{1/2} N + N^{-1}\Big\}.\end{aligned}$$

Thus,

$$|||\chi|||_\varepsilon \le C\left(h + N^{-1}\max|\psi'|\right),$$

where we have used (5.7). Applying a triangle inequality and the bounds for $|||\eta|||_\varepsilon$ and $|||\chi|||_\varepsilon$, we complete the proof. □

Remark 5.8. Sun and Stynes [157] use a similar technique to study the Galerkin-FEM on standard Shishkin meshes for higher-order problems. ♣

Remark 5.9. We are not aware of a general convergence theory for the Galerkin FEM on arbitrary layer-adapted meshes.

Roos [136] proves the optimal uniform error estimate

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \le CN^{-1}$$

for the Galerkin FEM on a special B-type mesh under the assumption that $\varepsilon \le N^{-1}$. The key ingredient in his analysis is the use of a special quasi-interpolant with an improved stability property. However, he points out that this technique cannot be extended to higher dimensions. ♣

5.2.2 Supercloseness

In the preceding section we have seen that the Galerkin FEM is (almost) first-order convergent in the ε-weighted energy norm. Now we prove that $\left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon$ converges faster than $\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon$. This means the numerical approximation is *closer to the interpolant* of the exact solution than to the solution itself. This phenomenon is called **supercloseness**. Our analysis follows [83, 176] where two-dimensional problems are studied.

Theorem 5.10. *Let $\bar{\omega}$ be an S-type mesh with $\sigma \geq 5/2$ whose mesh generating function $\tilde{\varphi}$ satisfies* (2.8). *Then*

$$\left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon \leq C\left(h^2 \ln^{1/2} N + N^{-2} \max|\psi'|^2\right) \tag{5.9}$$

for the solution of the Galerkin FEM.

Proof. For the sake of simplicity we assume that b is constant. Let again $\eta = u^I - u$ and $\chi = u^I - u^N$. Then

$$a(\eta, \chi) = \varepsilon(\eta', \chi') - (b\eta', \chi) + (c\eta, \chi)$$

For the diffusion term, integration by parts gives

$$\int_{I_i} \eta'\chi' = \eta\chi'|_{x_{i-1}}^{x_i} - \int_{I_i} \eta\chi'' = 0,$$

because $\eta(x_{i-1}) = \eta(x_i) = 0$ and because χ is linear. Thus, $(\eta', \chi') = 0$. The reaction term is easily bounded using the Cauchy-Schwarz inequality:

$$|(c\eta, \chi)| \leq C\,\|\eta\|_0\,\|\chi\|_0 \leq C\left(h + N^{-1}\max|\psi'|\right)^2 \|\chi\|_0\,,$$

by Theorem 5.2.

We are left with the convection term. Recalling the decomposition (3.34), we split as follows:

$$\begin{aligned}(\eta', \chi) = -\int_0^\tau (w^I - w)\chi' - \int_0^\tau (v^I - v)\chi' \\ - \int_\tau^1 (w^I - w)\chi' + \int_\tau^1 (v^I - v)'\chi.\end{aligned} \tag{5.10}$$

The four terms on the right-hand side are bounded separately.

(i) By a standard interpolation error result

$$\left\|w^I - w\right\|_{0,I_i} \leq Ch_i^2\,\|w''\|_{0,I_i}\,.$$

This and the bounds for the derivatives of w give

$$\left\|w^I - w\right\|_{0,(0,\tau)}^2 \le C \sum_{i=1}^{qN} h_i^4 \int_{I_i} \varepsilon^{-4} \mathrm{e}^{-2\beta x/\varepsilon} \mathrm{d}x \le C \sum_{i=1}^{qN} \left(\frac{h_i}{\varepsilon}\right)^4 h_i \mathrm{e}^{-2\beta x_{i-1}/\varepsilon}$$
$$\le C \left(N^{-1} \max |\psi'|\right)^4 \sum_{i=1}^{qN} h_i \mathrm{e}^{(4/\sigma-2)\beta x_i/\varepsilon},$$

by (2.12). Next, from (2.11)

$$h_i \mathrm{e}^{(4/\sigma-2)\beta x_i/\varepsilon} \le C h_i \mathrm{e}^{(4/\sigma-2)\beta x_{i-1}/\varepsilon} \le C \int_{I_i} \mathrm{e}^{(4/\sigma-2)\beta x/\varepsilon} \mathrm{d}x.$$

Therefore,

$$\left\|w^I - w\right\|_{0,(0,\tau)}^2 \le C \left(N^{-1} \max |\psi'|\right)^4 \int_0^\tau \mathrm{e}^{(4/\sigma-2)\beta x/\varepsilon} \mathrm{d}x$$
$$\le C\varepsilon \left(N^{-1} \max |\psi'|\right)^4,$$

where we have used $\sigma > 2$. This result and the Cauchy-Schwarz inequality yield

$$\left| \int_0^\tau (w^I - w)\chi' \right| \le C \left(N^{-1} \max |\psi'|\right)^2 |||\chi|||_\varepsilon . \tag{5.11}$$

(ii) To bound the second term we proceed as follows:

$$\left\|v^I - v\right\|_{0,(0,\tau)}^2 \le C \sum_{i=1}^{qN} h_i^4 \left\|v''\right\|_{0,I_i}^2 \le Ch^4 \left\|v''\right\|_{0,(0,\tau)}^2 \le Ch^4 \varepsilon \ln N,$$

since $|v''| \le C$ on $[0,1]$. Hence,

$$\left| \int_0^\tau (v^I - v)\eta\chi' \right| \le Ch^2 \ln^{1/2} N \, |||\chi|||_\varepsilon . \tag{5.12}$$

(iii) Now we consider $\int_\tau^1 (w - w^I)\chi'$. The argument splits the integral once more, but first let us recall that the mesh on $(\tau, 1)$ is uniform with mesh diameter H: $N^{-1} \le H \le N^{-1}/(1-q)$. We have

$$\left\|w^I - w\right\|_{0,I_{qN}}^2 \le CN^{-1} \mathrm{e}^{-2\beta\tau/\varepsilon} \le CN^{-6}$$

since $\sigma \ge 5/2$. Thus,

$$\left| \int_\tau^{x_{qN+1}} (w - w^I)\chi' \right| \le CN^{-2} \|\chi\|_0,$$

by an inverse inequality. Next we have

$$\begin{aligned}\left\|w^I - w\right\|^2_{0,(x_{qN+1},1)} &\le 2 \sum_{i=qN+1}^{N-1} H \left\|w\right\|_{\infty,I_i} \\ &\le C \sum_{i=qN+1}^{N-1} H \mathrm{e}^{-2\beta x_i/\varepsilon} \le C \int_\tau^{x_{N-1}} \mathrm{e}^{-2\beta x/\varepsilon} \mathrm{d}x \le C\varepsilon N^{-5}.\end{aligned}$$

Thus,

$$\left|\int_\tau^1 (w - w^I)\chi'\right| \le CN^{-2} \left|\left|\left|\chi\right|\right|\right|_\varepsilon . \tag{5.13}$$

(iv) To bound the last term in (5.10) we use the integral identity

$$\begin{aligned}\int_{I_i} \left(v - v^I\right)' \chi = \frac{1}{6} \int_{I_i} v''' \left(E_i^2\right)' \chi' \\ - \frac{1}{3}\left(\frac{h_i}{2}\right)^2 \int_{I_i} v'''\chi + \frac{1}{3}\left(\frac{h_i}{2}\right)^2 v''\chi\Big|_{x_{i-1}}^{x_i}\end{aligned} \tag{5.14}$$

with

$$E_i(x) = \frac{1}{2}\left((x - x_{i-1/2})^2 - \left(\frac{h_i}{2}\right)^2\right) = \frac{1}{2}\left(x - x_{i-1}\right)\left(x - x_i\right).$$

This expansion formula holds true for arbitrary functions $v \in W^{3,\infty}(I_i)$ and linear functions χ; cf. [78]. We get

$$\int_\tau^1 \left(v - v^I\right)' \chi = \frac{1}{6} \int_\tau^1 v''' \left(E^2\right)' \chi' - \frac{H^2}{12}\left(v''\chi\right)(\tau) - \frac{H^2}{12} \int_\tau^1 v''\chi.$$

Assuming more regularity of the data, the decomposition (3.34) can be sharpened to give $|v'''| \le C$. This yields

$$\begin{aligned}\left|\int_\tau^1 \left(v - v^I\right)' \chi\right| &\le CH^3 \left\|\chi'\right\|_{1,(\tau,1)} + CH^2 \left|\chi(\tau)\right| + CH^2 \left\|\chi\right\|_{1,(\tau,1)} \\ &\le CH^2 \left(\left|\left|\left|\chi\right|\right|\right|_\varepsilon + \left|\chi(\tau)\right|\right),\end{aligned}$$

by an inverse inequality. Finally, we estimate

$$\left|\chi(\tau)\right| = \left|\int_0^\tau \chi'(s)\mathrm{d}s\right| \le \tau^{1/2} \left\|\chi'\right\|_{0,(0,\tau)} \le C \ln^{1/2} N \left|\left|\left|\chi\right|\right|\right|_\varepsilon .$$

Thus,

$$\left|\int_\tau^1 \left(v - v^I\right)' \chi\right| \le CH^2 \ln^{1/2} N \, |||\chi|||_\varepsilon \,. \tag{5.15}$$

Combine (5.10)-(5.15) to get for the convection term

$$|(\eta', \chi)| \le C\left(h^2 \ln^{1/2} N + N^{-2} \max |\psi'|^2\right) |||\chi|||_\varepsilon \,.$$

This inequality, the bounds for the diffusion and reaction terms and the coercivity of $a(\cdot,\cdot)$ yield the proposition of the theorem. □

Remark 5.11. Surprisingly, the major difficulty in the proof does not arise from the layer term, but from the regular solution component. To cope with this, the special integral expansion formula (5.14) by Lin had to be used. ♣

Remark 5.12. Another attempt at a supercloseness result for convection-diffusion problems is [175]. In that paper, finite elements with piecewise polynomials of degree $p \ge 1$ are used on a piecewise uniform mesh with transition point

$$\tau = \min\left\{\frac{1}{2}, \frac{\varepsilon(p+3/2)}{\beta} \ln(N+1)\right\}.$$

It was established that when the regular solution component v lies in the finite element space then

$$\left|\left|\left|Q_p u - u^N\right|\right|\right|_\varepsilon \le C\left(\frac{\ln(N+1)}{N}\right)^{p+1},$$

where $Q_p u$ is the $(p+1)$-point Gauss-Lobatto interpolant of u. This is a supercloseness result, because in general, one has for the interpolation error

$$|||Q_p u - u|||_\varepsilon \le C\left(\frac{\ln(N+1)}{N}\right)^{p}.$$

However, the assumption that the regular solution component lies in the finite element space is not very reasonable. If it were to hold for two different values of the mesh parameter N then $v \in \Pi_p$, because Shishkin meshes for different N are not nested.

This too illustrates the technical difficulties with the regular solution component just mentioned. Unfortunately—unlike (5.14) for linear elements—no expansion formulae for the convection term are available for quadratic or higher-order elements. ♣

5.2.3 Gradient Recovery and a Posteriori Error Estimation

Supercloseness results like Theorem 5.10 are basic ingredients for the superconvergent recovery of gradients, see for instance [2]. Furthermore, if a superconvergent recovery operator is available, then it is possible to define an a posteriori error estimator that is asymptotically exact. The presentation follows [138], where further details can be found.

First, we define for a given $v \in V^\omega$ a recovery operator for the derivative. Set

$$(Rv)(x) := \alpha_{i-1}\frac{x_i - x}{h_i} + \alpha_i \frac{x - x_{i-1}}{h_i} \quad \text{for } x \in I_i,\ i = 2, \ldots, N-1,$$

where α_i denotes the weighted average of the constant values of v' on the subintervals adjacent to x_i:

$$\alpha_i := \frac{h_{i+1} v'\big|_{I_i} + h_i v'\big|_{I_{i+1}}}{h_i + h_{i+1}}.$$

For the boundary intervals we simply extrapolate the well-defined linear function of the adjacent interval.

Our aim is to prove a superconvergence estimate for $\varepsilon^{1/2}\left\|u' - Ru^N\right\|_0$ that is superior to that of Theorem 5.6 for $\varepsilon^{1/2}\left\|u' - (u^N)'\right\|_0$. The key ingredients are the supercloseness property of the Galerkin solution, i.e. Theorem 5.10, and the consistency and stability of the recovery operator R.

Consistency: Let v be a quadratic function on $\tilde{I}_i$, the union of I_i and its adjacent mesh intervals. Then

$$R\left(v^I\right) = v' \quad \text{on } I_i. \tag{5.16a}$$

Stability:

$$\|Rv\|_{0,I_i} \le C\,\|v'\|_{0,\tilde{I}_i} \quad \text{for all } v \in V^\omega. \tag{5.16b}$$

We start our analysis from a triangle inequality:

$$\left\|u' - Ru^N\right\|_0 \le \left\|u' - R\left(u^I\right)\right\|_0 + \left\|R\left(u^I - u^N\right)\right\|_0.$$

The second term in this inequality can be bounded using Theorem 5.10 and (5.16b). Thus, we are left with the problem of estimating $\left\|u' - R\left(u^I\right)\right\|_0$. To take advantage of the consistency property (5.16a) we introduce a quadratic approximation of u on $\tilde{I}_i$: $Q_i u$. Using a triangle inequality, we obtain

$$\begin{aligned}\left\|u' - R\left(u^I\right)\right\|_{0,I_i} \le{}& \left\|u' - (Q_i u)'\right\|_{0,I_i} \\ &+ \left\|(Q_i u)' - R\left((Q_i u)^I\right)\right\|_{0,I_i} + \left\|R\left((Q_i u - u)^I\right)\right\|_{0,I_i}.\end{aligned}$$

The second term vanishes because of (5.16a). The last term can be bounded using (5.16b) and the stability of the linear interpolation in H^1, i. e., $|v^I|_1 \le C|v|_1$. We get

$$\left\| u' - R\left(u^I\right) \right\|_0^2 \le C \sum_{i=1}^{N} \left\| u' - (Q_i u)' \right\|_{0,\tilde{I}_i}^2 .$$

Note this H^1 stability of the interpolation operator holds true only in the one-dimensional case. In two dimensions the L_∞ stability of the interpolation operator has to be used instead, see Sect. 9.2.2.5.

Choosing $Q_i u$ to be, e. g., that bilinear function that coincides with u at the midpoint and both endpoints of $\tilde{I}_i$ and estimating the interpolation error carefully, see [138], we obtain

$$\varepsilon \sum_{i=1}^{N} \left\| u' - (Q_i u)' \right\|_{0,\tilde{I}_i}^2 \le C \left(h + N^{-1} \max |\psi'| \right)^4 \quad \text{if } \sigma \ge 2.$$

Combining these estimates, we get the following result.

Theorem 5.13. *Let $\bar{\omega}$ be a S-type mesh with $\sigma \ge 5/2$ whose mesh generating function $\tilde{\varphi}$ satisfies* (2.8)*. Then the error of the recovered gradient of the Galerkin FEM satisfies*

$$\varepsilon^{1/2} \left\| u' - Ru^N \right\|_0 \le C \left(h^2 \ln^{1/2} N + N^{-2} \max |\psi'|^2 \right) .$$

Remark 5.14. Using Ru^N instead of u', we get an asymptotically exact error estimator for the weighted H^1-seminorm of the finite element error $\varepsilon^{1/2} \left\| u' - U' \right\|_0$ on S-type meshes:

$$\varepsilon^{1/2} \left\| u' - (u^N)' \right\|_0 = \varepsilon^{1/2} \left\| Ru^N - (u^N)' \right\|_0 + \mathcal{O}\left(h^2 \ln^{1/2} N + N^{-2} \max |\psi'|^2 \right) .$$

In the generic case one has

$$\varepsilon^{1/2} \left\| u' - (u^N)' \right\|_0 = \mathcal{O}\left(h + N^{-1} \max |\psi'| \right) .$$

Thus, the above error estimator is asymptotically exact for $N \to \infty$. ♣

Remark 5.15. There are various other means of postprocessing to obtain superconvergent approximations for the derivatives. ♣

5.2.4 A Numerical Example

Let us briefly illustrate our theoretical results for the linear Galerkin FEM on S-type meshes when applied to the test problem

$$-\varepsilon u'' - u' + 2u = \mathrm{e}^{x-1} \quad \text{in } (0,1), \; u(0) = u(1) = 0.$$

For our tests we take $\varepsilon = 10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problem.

We consider three different S-type meshes: the original Shishkin mesh, the Bakhvalov-Shishkin mesh and a mesh with a rational mesh characterising function ψ. The results of our test computations are presented in Tables 5.1, 5.2 and 5.3. They are clear illustrations of the a priori error bounds given in Theorems 5.6, 5.10 and 5.13.

Table 5.1 Galerkin FEM on a S-type mesh with a rational ψ ($m = 2$)

	$\vert\vert\vert u - u^N \vert\vert\vert_\varepsilon$		$\vert\vert\vert u^I - u^N \vert\vert\vert_\varepsilon$		$\varepsilon^{1/2}\Vert u' - Ru^N\Vert_0$	
N	error	rate	error	rate	error	rate
2^8	1.158e-2	0.50	5.889e-4	0.99	3.289e-3	0.98
2^9	8.213e-3	0.50	2.959e-4	1.00	1.673e-3	0.99
2^{10}	5.817e-3	0.50	1.483e-4	1.00	8.435e-4	0.99
2^{11}	4.117e-3	0.50	7.424e-5	1.00	4.236e-4	1.00
2^{12}	2.912e-3	0.50	3.714e-5	1.00	2.123e-4	1.00
2^{13}	2.060e-3	0.50	1.858e-5	1.00	1.062e-4	1.00
2^{14}	1.456e-3	—	9.290e-6	—	5.315e-5	—

Table 5.2 Galerkin FEM on a standard Shishkin mesh

	$\vert\vert\vert u - u^N \vert\vert\vert_\varepsilon$		$\vert\vert\vert u^I - u^N \vert\vert\vert_\varepsilon$		$\varepsilon^{1/2}\Vert u' - Ru^N\Vert_0$	
N	error	rate	error	rate	error	rate
2^8	6.166e-3	0.83	1.624e-4	1.66	8.045e-4	1.64
2^9	3.470e-3	0.85	5.151e-5	1.69	2.585e-4	1.69
2^{10}	1.928e-3	0.86	1.592e-5	1.72	8.025e-5	1.72
2^{11}	1.060e-3	0.87	4.818e-6	1.75	2.432e-5	1.75
2^{12}	5.784e-4	0.88	1.434e-6	1.77	7.242e-6	1.77
2^{13}	3.133e-4	0.89	4.211e-7	1.79	2.126e-6	1.79
2^{14}	1.687e-4	—	1.221e-7	—	6.164e-7	—

Table 5.3 Galerkin FEM on a Bakhvalov-Shishkin mesh

	$\vert\vert\vert u - u^N \vert\vert\vert_\varepsilon$		$\vert\vert\vert u^I - u^N \vert\vert\vert_\varepsilon$		$\varepsilon^{1/2}\Vert u' - Ru^N\Vert_0$	
N	error	rate	error	rate	error	rate
2^8	1.357e-3	1.00	5.382e-6	1.99	4.173e-5	2.00
2^9	6.800e-4	1.00	1.353e-6	2.00	1.043e-5	2.00
2^{10}	3.403e-4	1.00	3.393e-7	2.00	2.610e-6	2.00
2^{11}	1.702e-4	1.00	8.497e-8	2.00	6.528e-7	2.00
2^{12}	8.514e-5	1.00	2.126e-8	2.00	1.632e-7	2.00
2^{13}	4.258e-5	1.00	5.317e-9	2.01	4.082e-8	2.00
2^{14}	2.129e-5	—	1.321e-9	—	1.020e-8	—

5.3 Stabilised FEM

We have seen that the Galerkin FEM on S-type meshes has good approximation properties. Unfortunately the linear systems generated are difficult to solve iteratively. Therefore, stabilisation is essential. We shall restrict ourselves to artificial viscosity stabilisation and to the streamline-diffusion FEM. Other stabilisation techniques, including:

- discontinuous Galerkin FEM (dGFEM),
- continuous interior penalties (CIP) and
- local projection stabilisation (LPFEM),

have been considered in the literature. However, with regard to the classification of layer-adapted meshes these contributions are negligible. Nonetheless, some of the results for these methods will be mentioned in Sect. 9.2.

5.3.1 Artificial Viscosity Stabilisation

The simplest way to stabilise discretisation methods for convection-diffusion problems consists of altering the diffusion coefficient a priori, the extra diffusion added being called **artificial viscosity**. Typically artificial viscosity proportional to the stepsize is used.

Let $\kappa > 0$ be an arbitrary constant. Then our stabilised FEM is: Find $u^N \in V^\omega$ such that

$$a_\kappa(u^N, v) = f(v) \quad \text{for all } \ v \in V^\omega,$$

where

$$a_\kappa(u, v) := \big((\varepsilon + \kappa\hbar)u', v'\big) - (bu' - cu, v)$$

and

$$\hbar(x) \equiv h_i \quad \text{for } \ x \in I_i.$$

The bilinear form $a_\kappa(\cdot,\cdot)$ is coercive with respect to the norm

$$|||v|||_\kappa := \Big\{\big((\varepsilon + \kappa\hbar)v', v'\big) + \gamma\|v\|_0^2\Big\}^{1/2},$$

which is stronger than the ε-weighted energy norm. This is the reason for the improved stability of the method.

Because of the added artificial viscosity, the method does not satisfy the orthogonality property which slightly complicates the convergence analysis. Assume a S-type mesh is used. Let $\eta = u^I - u$ and $\chi = u^I - u^N$ again. Then

$$\begin{aligned}|||\chi|||_\kappa^2 &\le a_\kappa(\chi,\chi) = a(\eta,\chi) + \left(\kappa\hbar(u^I)',\chi'\right) \\ &= a(\eta,\chi) + \kappa\left(\hbar\eta',\chi'\right) + \kappa\left(\hbar u',\chi'\right).\end{aligned}$$

Bounds for the first term have been derived in Sect. 5.2. The second term $(\hbar\eta',\chi')$ vanishes, while the last term, which is the inconsistency of the method, satisfies

$$\kappa\left|(\hbar u',\chi')\right| \le C\kappa\left(h\ln^{1/2}N + N^{-1}\max|\psi'|\right)|||\chi|||_\varepsilon .$$

The proof recycles some ideas from Sect. 5.2.1 and 5.2.2 and is therefore omitted. For more details see also [148]. We get

$$\begin{aligned}\left|\left|\left|u^I - u^N\right|\right|\right|_\kappa \le C\Big\{& h(h+\kappa)\ln^{1/2}N \\ &+ \left(\kappa + N^{-1}\max|\psi'|\right)N^{-1}\max|\psi'|\Big\}.\end{aligned} \tag{5.17}$$

If we choose $\kappa = \mathcal{O}(1)$, i. e., we add artificial viscosity proportional to the local mesh size, we get

$$\left|\left|\left|u^I - u^N\right|\right|\right|_\kappa + \left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \le C\left(h\ln^{1/2}N + N^{-1}\max|\psi'|\right),$$

by the interpolation error estimate (5.8).

Comparing (5.9) and (5.17), we see that the order of accuracy of the Galerkin FEM is not affected if we take $\kappa = \mathcal{O}\left(N^{-1}\right)$. This results in improved stability compared to the Galerkin method and the discrete systems—in particular for higher-dimensional problems—are slightly easier to solve by means of standard iterative methods. On the other hand, the method is not as stable as if $\kappa = \mathcal{O}(1)$ were chosen.

5.3.2 Streamline-Diffusion Stabilisation

The most popular and most frequently studied stabilised FEM is the streamline-diffusion finite element method (SDFEM) which is also referred to as the streamline-upwind Petrov-Galerkin method (SUPG). This kind of stabilisation was introduced by Hughes and Brooks [54]. Given a mesh ω and a finite element space V^ω, this method can be written as: Find $u^N \in V^\omega$ such that

$$a(u^N,v) + \sum_{i=1}^{N}\delta_i\int_{I_i}\left(f - \mathcal{L}u^N\right)bv' = (f,v) \quad \text{for all } v \in V^\omega, \tag{5.18}$$

where the stabilisation parameters δ_i are chosen according to the local mesh Peclét number:

$$\delta_i = \begin{cases} \kappa_0 h_i & \text{if } Pe_i > 1, \\ \kappa_1 h_i^2 \varepsilon^{-1} & \text{if } Pe_i \le 1, \end{cases} \quad \text{with} \quad Pe_i = \frac{\|b\|_{\infty,I_i} h_i}{2\varepsilon}$$

with user chosen positive constants κ_0 and κ_1. In contrast to the artificial-viscosity stabilisation, this method is consistent with (5.1) since u satisfies (5.18) for all $v \in H_0^1(0,1)$. Another advantage—though it becomes relevant only in higher dimensions—is the reduction of crosswind smear because artificial viscosity is added only in the streamline direction.

The second-order upwind schemes of Sect. 4.3.1 may be regarded as versions of the SDFEM with linear test and trial functions and inexact integration. While in the one-dimensional case it is always possible to choose the stabilisation parameters δ_i such that the resulting scheme is inverse monotone, this is in general impossible in higher dimensions. Therefore, alternative techniques have to be developed to study the SDFEM. Here we shall consider convergence in the streamline-diffusion norm $|||\cdot|||_{SD}$ naturally associated with the bilinear form of the method. This technique can be extended to two-dimensional problems; see Sect. 9.2.4 or [155].

5.3.2.1 Energy-Norm Error Estimates

We study the SDFEM on S-type meshes $\bar{\omega}$. For the sake of simplicity we consider (5.1) with constant b. Let $V_0^\omega \subset H_0^1(0,1)$ be the linear space of piecewise-affine functions on $\bar{\omega}$ that vanish at the boundary. We rewrite (5.18) as: Find $u^N \in V_0^\omega$ such that

$$a_{SD}(u^N, v) := a(u^N, v) + a_{stab}(u^N, v) = f(v) + f_{stab}(v) \quad \text{for all } v \in V_0^\omega$$

where $a(\cdot,\cdot)$ is the bilinear form of the Galerkin FEM,

$$a_{stab}(w, v) := -\delta \sum_{i=qN+1}^{N} \int_{I_i} (-\varepsilon w'' - bw' + cw) bv',$$

$$f_{stab}(v) := -\delta \sum_{i=qN+1}^{N} \int_{I_i} fbv'$$

and

$$\delta = \begin{cases} \kappa_0 H & \text{if } bH/2\varepsilon > 1, \\ \kappa_1 H^2/\varepsilon & \text{otherwise.} \end{cases}$$

Here H denotes again the mesh size on the coarse part of the mesh.

The streamline-diffusion norm naturally associated with $a_{SD}(\cdot,\cdot)$ is

$$|||v|||_{SD}^2 := \varepsilon \left\|v'\right\|_0^2 + \gamma \left\|v\right\|_0^2 + \left\|\delta^{1/2} b v'\right\|_{0,(\tau,1)}^2.$$

Provided the maximum step size h is smaller than some threshold value, the bilinear form $a_{SD}(\cdot,\cdot)$ is coercive with respect to the streamline-diffusion norm:

$$a_{SD}(v,v) \geq \tfrac{1}{2} |||v|||_{SD}^2 \quad \text{for all } \ v \in V_0^\omega;$$

see [141]. The bilinear form also satisfies the Galerkin-orthogonality property

$$a_{SD}(u - u^N, v) = 0 \quad \text{for all } \ v \in V_0^\omega.$$

This is the starting point of our error analysis. Let $\eta = u^I - u$ and $\chi = u^I - u^N$. Then

$$\tfrac{1}{2} |||\chi|||_{SD}^2 \leq a(\eta,\chi) + a_{stab}(\eta,\chi). \tag{5.19}$$

For the first term we have from the proof of Theorem 5.10

$$|a(\eta,\chi)| \leq C \left(h^2 \ln^{1/2} + N^{-2} \max |\psi'|^2 \right) |||\chi|||_\varepsilon \,.$$

It remains to bound $a_{stab}(\eta,\chi)$. We have

$$a_{stab}(\eta,\chi) = \delta \int_\tau^1 (\varepsilon u'' + b\eta' + c\eta)\, b\chi'.$$

Element-wise integration by parts yields $\int_\tau^1 \eta'\chi' = 0$. Furthermore,

$$\begin{aligned} \left| \delta \int_\lambda^1 c\eta b\chi' \right| &\leq C\delta^{1/2} \|\eta\|_{0,(\tau,1)} \|\delta^{1/2} b\chi'\|_0 \\ &\leq C\delta^{1/2} N^{-2} \|\delta^{1/2} b\chi'\|_0 \leq CN^{-2}\, |||\chi|||_{SD} \,, \end{aligned}$$

by the bounds for the interpolation error from Sect. 5.1.

To bound the remaining term $\int_\tau^1 u''\chi'$ we use the decomposition of u into a regular and a layer component; see Theorem 3.48. For the regular component v we have

$$\int_\tau^1 v''\chi' = -\int_\tau^1 v'''\chi - \int_0^\tau v''\chi'.$$

Hence,

$$\left| \int_\tau^1 v''\chi' \right| \leq C\|\chi\|_0 + C\left(\varepsilon \ln N\right)^{1/2} \|\chi'\|_0,$$

by the bounds for the derivatives of v. Thus,

$$\left|\varepsilon\delta \int_\tau^1 v''\chi'\right| \le CN^{-2}\ln^{1/2} N \,|||\chi|||_{SD}\,,$$

since the choice of δ implies $\varepsilon\delta \le CH^2 \le CN^{-2}$.

For the layer component w we estimate as follows:

$$\begin{aligned}\varepsilon\delta\left|\int_\tau^1 w''b\chi'\right| &\le \varepsilon\delta^{1/2}\|w''\|_{1,(\tau,1)}\|\delta^{1/2}b\chi'\|_{\infty,(\tau,1)}\\ &\le C\delta^{1/2}N^{-5/2}H^{-1/2}\|\delta^{1/2}b\chi'\|_{0,(\tau,1)},\end{aligned}$$

by an inverse inequality and because $\sigma \ge 5/2$. We get

$$\varepsilon\delta\left|\int_\tau^1 w''b\chi'\right| \le CN^{-2}\,|||\chi|||_{SD}\,.$$

Collecting these results, the second term in (5.19) is bounded by

$$|a_{stab}(\eta,\chi)| \le CN^{-2}\ln^{1/2} N\,|||\chi|||_\varepsilon\,.$$

We summarise our results.

Theorem 5.16. *Let $\bar\omega$ be an S-type mesh with $\sigma \ge 5/2$ whose mesh generating function $\tilde\varphi$ satisfies* (2.8)*. Then the error of the SDFEM satisfies*

$$\left|\left|\left|u^I - u^N\right|\right|\right|_{SD} \le C\left(h^2\ln^{1/2} N + N^{-2}\max|\psi'|^2\right).$$

Remark 5.17. This is a superconvergence result like Theorem 5.10. Similar to Sect. 5.2.3 it is possible to construct recovery operators to obtain a second-order approximations of the gradient of the exact solution. ♣

5.3.2.2 Maximum-Norm Error Estimates

Chen and Xu [26] consider a modification of (5.18). Find $\tilde u^N \in V_0^\omega$

$$a(\tilde u^N, v) + \sum_{i=1}^N \int_{I_i} \tilde\delta_i\left(f - \mathcal{L}\tilde u^N\right)bv' = (f,v) \quad \text{for all } \ v \in V_0^\omega,$$

where

$$\tilde\delta_i(x) = \min\left\{\frac{h_i^2}{2\varepsilon}, \frac{h_i}{b}\right\}(x_i - x)(x - x_{i-1}) \quad \text{for } \ x \in I_i,$$

i.e., $\tilde\delta_i$ is a bubble function on I_i instead of a constant.

For this modified SDFEM it is shown in [26] that if $b = \text{const.}$ and $c \equiv 0$ then

$$\left\| u - \tilde{u}^N \right\|_\infty \le C \min_{v \in V_0^\omega} \|u - v\|_\infty .$$

Thus, the method is quasi-optimal in the maximum norm.

Suppose $b = \text{const.}$ Then

$$\int_{I_i} \tilde{\delta}_i b w' b v' = \delta_i \int_{I_i} b w' b v' \quad \text{for all } \; w, v \in V_0^\omega$$

with

$$\delta_i = \frac{1}{6} \min \left\{ \frac{h_i^2}{2\varepsilon}, \frac{h_i}{b} \right\},$$

i. e., δ_i is the mean value of $\tilde{\delta}_i$ on I_i. Therefore, the modified and the original SDFEM generate the same difference stencil, but different discretisations of the right-hand side. It follows that

$$a_{SD}\left(u^N - \tilde{u}^N, v\right) = \sum_{i=1}^N \int_{I_i} \left(\tilde{\delta}_i - \delta_i\right)\left(f - f^I\right) b v'.$$

Next, the stability of the discretisation, which was established in [26], implies

$$\left\| u^N - \tilde{u}^N \right\|_\infty \le C \left\| f - f^I \right\|_\infty .$$

Finally, the triangle inequality yields for the solution of the original SDFEM

$$\left\| u - u^N \right\|_\infty \le C \left(\min_{v \in V_0^\omega} \|u - v\|_\infty + \left\| f - f^I \right\|_\infty \right).$$

5.4 An Upwind Finite Volume Method

Let us finish this chapter by considering finite volume discretisations of (5.1).

Although the construction of finite volume methods differs from finite difference and finite element methods, they are typically analysed as special finite difference methods or—more often—as nonconforming finite element methods. Here we like to highlight both approaches. In particular, this section is intended to prepare our later investigation of the FVM in two dimensions in Sect. 9.3. There a detailed construction of the method can be found too.

We shall assume

$$c \ge \gamma > 0, \quad c + b' \ge \gamma > 0 \qquad \text{when studying the FVM as a FDM} \qquad (5.20)$$

and

$$c + b'/2 \geq \gamma > 0 \qquad \text{in the FEM context.} \tag{5.21}$$

In the latter case the variational formulation (5.2) will be used.

Given an arbitrary mesh $\bar{\omega}$ our FVM reads: Find $u^N \in I\!R_0^{N+1}$ such that

$$\left[L_\rho u^N\right]_i = f_i \quad \text{for } i = 1, \ldots, N-1, \tag{5.22}$$

where

$$[L_\rho v]_i := -\varepsilon v_{\bar{x}\hat{x};i} - \rho\left(-\mu_{i+1/2}\right) b_{i+1/2} v_{\hat{x};i} - \rho\left(\mu_{i-1/2}\right) b_{i-1/2} v_{\check{x};i} + c_i u_i^N$$

with $\mu_{i+1/2} = b_{i+1/2} h_{i+1}/\varepsilon$ and $b_{i+1/2} = b\big(x_{i+1/2}\big)$.

The method can also be written in variational form: Find $u^N \in I\!R_0^{N+1}$ such that

$$a_\rho(u^N, v) := \left(L_\rho u^N, v\right)_\omega = (f, v)_\omega =: f_\rho(v) \quad \text{for all } v \in I\!R_0^{N+1},$$

where

$$(w, v)_\omega := \sum_{i=1}^{N-1} \hbar_i w_i v_i .$$

The crucial point is the choice of the controlling function $\rho : I\!R \to [0,1]$. It has to provide the correct weighting between the two one-sided difference approximations for the first-order derivative. Possible choices for ρ include:

$$\rho_I(t) = \begin{cases} \dfrac{1}{t}\left(1 - \dfrac{t}{\exp t - 1}\right) & \text{for } t \neq 0, \\ \dfrac{1}{2} & \text{for } t = 0, \end{cases}$$

$$\rho_S(t) = \begin{cases} 1/(2+t) & \text{for } t \geq 0, \\ (1-t)/(2-t) & \text{for } t < 0, \end{cases}$$

and, with $m \geq 0$,

$$\rho_{U,m}(t) = \begin{cases} 0 & \text{for } t > m, \\ \frac{1}{2} & \text{for } t \in [-m, m], \\ 1 & \text{for } t < -m; \end{cases}$$

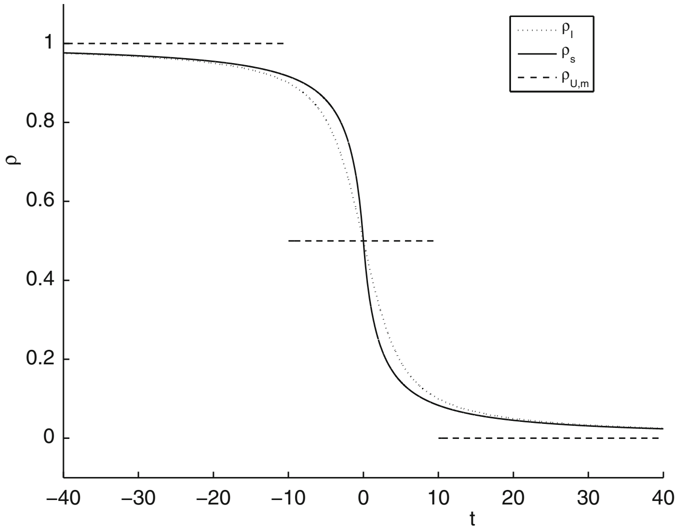


Fig. 5.1 The stabilising functions ρ_I, ρ_S and $\rho_{U,m}$

see Fig. 5.1. The full upwind stabilisation $\rho_{U,0}$ is due to Baba and Tabata [17], while $\rho_{U,m}$ with $m > 0$ was introduced by Angermann [13]. For ρ_I and ρ_S we get slight modifications of the schemes of Il'in [55] and of Samarski [144]. Further choices of ρ are mentioned in [13] and [62] where also a detailed derivation of the method in two dimensions can be found.

The constant choice $\rho \equiv \frac{1}{2}$ generates a central difference scheme, while the choice $\rho_{U,0}$ gives a scheme with upwinded one-sided difference approximation of the first-order derivative which is very similar to the upwind scheme analysed in Sect. 4.2. If a different ρ is used—in particular when ρ is Lipschitz continuous in a neighbourhood of 0—then the first-order derivatives are approximated by weighted combinations of upwinded and downwinded operators. This weighting provides an adaptive transition from an upwinded to a central difference approximation when the local mesh size is small enough. In this case higher accuracy is achieved while retaining the good stability of the scheme.

Important properties of ρ are

$$(\rho_0) \qquad t \mapsto t\rho(t) \text{ is Lipschitz continuous,}$$

$$(\rho_1) \qquad \big[\rho(t) + \rho(-t) - 1\big]t = 0 \quad \text{for all } t \in I\!R,$$

$$(\rho_2) \qquad \big[1/2 - \rho(t)\big]t \geq 0 \quad \text{for all } t \in I\!R,$$

$$(\rho_3) \qquad 1 - t\rho(t) \geq 0 \quad \text{for all } t \in I\!R.$$

Condition (ρ_1) ensures both the consistency of the scheme and the local conservation of the fluxes, while (ρ_2) guarantees the coercivity of the bilinear form $a_\rho(\cdot,\cdot)$ and (ρ_3) the inverse monotonicity of the scheme.

5.4.1 Stability of the FVM

Coercivity of the bilinear form $a_\rho(\cdot,\cdot)$

The consistency condition (ρ_1) and summation by parts yield

$$a_\rho(v,v) = \varepsilon \sum_{i=1}^{N} \frac{(v_i - v_{i-1})^2}{h_i} + \sum_{i=1}^{N} \left[\frac{1}{2} - \rho\left(\mu_{i-1/2}\right)\right] b_{i-1/2}(v_i - v_{i-1})^2 + \sum_{i=1}^{N-1} \left[\hbar_i c_i + \frac{1}{2}\left(b_{i+1/2} - b_{i-1/2}\right)\right] v_i^2. \tag{5.23}$$

Assume b' is Hölder continuous with coefficient $\alpha \in (0,1]$. Then

$$\left|b_{i+1/2} - b_{i-1/2} - \hbar_i b_i'\right| \le \hbar_i h^\alpha \|b\|_{C^{1,\alpha}[0,1]} . \tag{5.24}$$

Thus, if (5.21) is satisfied and if the maximum mesh size h is smaller than some threshold value h^* then

$$\sum_{i=1}^{N-1} \left[\hbar_i c_i + \frac{1}{2}\left(b_{i+1/2} - b_{i-1/2}\right)\right] v_i^2 \ge \frac{\gamma}{2} \sum_{i=1}^{N-1} \hbar_i v_i^2. \tag{5.25}$$

Let

$$|||v|||_\rho^2 := \varepsilon \, |v|_{1,\omega}^2 + |v|_{\rho,\omega}^2 + \frac{\gamma}{2} \|v\|_{0,\omega}^2$$

with

$$|v|_{1,\omega}^2 := \sum_{i=1}^{N} \frac{(v_i - v_{i-1})^2}{h_i}, \quad \|v\|_{0,\omega} := \sum_{i=1}^{N-1} \hbar_i v_i^2$$

and

$$|v|_{\rho,\omega}^2 := \sum_{i=1}^{N} \left[\frac{1}{2} - \rho\left(\mu_{i-1/2}\right)\right] b_{i-1/2}(v_i - v_{i-1})^2$$

Note, $|||\cdot|||_\rho$ is a well-defined norm when (ρ_2) is satisfied. The coercivity of the discrete bilinear form $a_\rho(\cdot,\cdot)$ follows from (5.23) and (5.25).

Theorem 5.18. *Assume conditions* (ρ_1), (ρ_2) *and* (5.21) *are satisfied. Let* $b \in C^{1,\alpha}[0,1]$ *with Hölder exponent* $\alpha \in (0,1]$. *Then the bilinear form* $a_\rho(\cdot,\cdot)$ *is coercive with respect to the FV-norm* $|||\cdot|||_\rho$, *i.e.,*

$$a_\rho(v,v) \geq |||v|||_\rho^2 \quad \textit{for all} \;\; v \in \mathbb{R}_0^{N+1},$$

provided the maximum mesh size h *is smaller than some threshold value which is independent of the perturbation parameter* ε.

Remark 5.19. Note when $\rho \equiv \frac{1}{2}$ the stabilisation is switched off. Nonetheless Theorem 5.18 states coercivity of the bilinear form with respect to the discrete ε-weighted energy norm $|||v|||_{\varepsilon,\omega}^2 := \varepsilon|v|_{1,\omega}^2 + \frac{\gamma}{2}\|v\|_{0,\omega}^2$. However, in the case $\rho \not\equiv \frac{1}{2}$ the scheme is coercive with respect to a stronger norm which results in enhanced stability of the method. ♣

Inverse monotonicity

Let $r^+, r^-, q \geq \gamma > 0$ and $\chi > 0$ be arbitrary mesh functions with

$$r_i^+ \geq \frac{\beta h_{i+1}}{\alpha\varepsilon} \quad \text{and} \quad 1 \geq r_i^- \geq 0 \quad \text{for } i = 1,\dots,N-1 \tag{5.26}$$

with a constant $\alpha > 0$. Consider the difference operator

$$[L_\chi v]_i := -\frac{\varepsilon}{\chi_i}\left[1 + r_i^+\right] v_{x;i} + \frac{\varepsilon}{\chi_i}\left[1 - r_i^-\right] v_{\bar{x};i} + q_i v_i. \tag{5.27}$$

We study this more general situation because it will also serve as an auxiliary result in Sect. 9.3.2 when two dimensional problems will be investigated. The FVM (5.22) belongs to this class of schemes provided that (ρ_3) holds.

Clearly, $1+r_i^+ \geq 1$ and $1-r_i^- \geq 0$. Hence, the system matrix associated with L_χ possesses nonnegative offdiagonal entries and is therefore a L_0-matrix. Application of Lemma 3.14 with the test vector $\boldsymbol{e} = \mathbf{1}$ yields the inverse monotonicity of L_χ. In particular, we get the stability inequality

$$\|v\|_{\infty,\bar{\omega}} \leq \|L_\chi v/q\|_{\infty,\omega} \quad \text{for all} \;\; v \in \mathbb{R}_0^{N+1}.$$

The inverse monotonicity can be used to study the Green's function associated with L_χ and derive stability inequalities similar to those of Sect. 4.2.1.

The (ℓ_∞, ℓ_1) *stability*

The Green's function $G_{\cdot,j}$ associated with the mesh node x_j satisfies

$$[L_\chi G_{\cdot,j}]_i = \delta_{i,j} \quad \text{for} \;\; i = 1,\dots,N-1, \;\; G_{0,j} = G_{N,j} = 0,$$

where

$$\delta_{i,j} := \begin{cases} \chi_i^{-1} & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Given G, we can represent any function $v \in I\!R_0^{N+1}$ as

$$v_i = (G_{i,\cdot}, L_\chi v)_\chi\,, \tag{5.28}$$

with the scalar product

$$(w, v)_\chi := \sum_{j=1}^{N-1} \chi_j w_j v_j.$$

Let

$$\hat{G}_{i,j} = \begin{cases} \dfrac{\alpha}{\beta} & \text{for } i = 0, \ldots, j, \\ \dfrac{\alpha}{\beta} \displaystyle\prod_{k=j+1}^{i} \left(1 + \frac{\beta h_k}{\alpha\varepsilon}\right)^{-1} & \text{for } i = j+1, \ldots, N. \end{cases}$$

Lemma 5.20. *Suppose* (5.26) *holds true. Then the Green's function G associated with L_χ satisfies*

$$0 \le G_{i,j} \le \hat{G}_{i,j} \le \alpha/\beta \quad \textit{for} \quad i, j = 0, \ldots, N.$$

Proof. If (5.26) holds then the operator L_χ is inverse monotone and therefore satisfies a discrete comparison principle. The lower bound on G is easily verified using the barrier function $v \equiv 0$.

In order to establish the upper bound, it is sufficient to show that for fixed j we have $G_{0,j} \le \hat{G}_{0,j}$, $G_{N,j} \le \hat{G}_{N,j}$ and

$$\left[L_\chi G_{\cdot,j}\right]_i \le \left[L_\chi \hat{G}_{\cdot,j}\right]_i \quad \text{for } i = 1, \ldots, N-1,$$

(i) First we check the boundary conditions. Clearly $\hat{G}_{i,j} > 0$ for all i, j. Thus,

$$0 = G_{0,j} \le \hat{G}_{0,j} \text{ and } 0 = G_{N,j} \le \hat{G}_{N,j} \text{ for } j = 1, \ldots, N-1.$$

(ii) Next, for $i < j$ we have $\left[L_\chi \hat{G}_{\cdot,j}\right]_i = q_i \hat{G}_{i,j} \ge 0$ since both q and $\hat{G}$ are positive.

(iii) For $i > j$ we have

$$\left[\hat{G}_{\cdot,j}\right]_{x;i} = \frac{\beta}{\alpha\varepsilon + \beta h_{i+1}} \hat{G}_{i,j} \quad \text{and} \quad \left[\hat{G}_{\cdot,j}\right]_{\bar{x};i} = -\frac{\beta}{\alpha\varepsilon} \hat{G}_{i,j}.$$

Thus,

$$\left[L_\chi \hat{G}_{\cdot,j}\right]_i \geq \frac{\varepsilon}{\chi_i}\left\{\frac{\left(1+r_i^+\right)\beta}{\alpha\varepsilon+\beta h_{i+1}} - \frac{\left(1-r_i^-\right)\beta}{\alpha\varepsilon}\right\}\hat{G}_{i,j} \geq 0,$$

by (5.26).

(iv) For $i = j$ a combination of the arguments from *(ii)* and *(iii)* yields

$$\left[L_\chi \hat{G}_{\cdot,j}\right]_j \geq \frac{\varepsilon}{\chi_j}\frac{\left(1+r_i^+\right)\beta}{\alpha\varepsilon+\beta h_{j+1}}\hat{G}_{j,j} \geq \frac{1}{\chi_j} = \left[L_\chi G_{\cdot,j}\right]_j,$$

by (5.26).

This Lemma and (5.28) give the (ℓ_∞, ℓ_1) stability of the method:

Theorem 5.21. *Suppose* (5.26) *holds. Then the operator* L_χ *defined in* (5.27) *satisfies the stability inequality*

$$\|v\|_{\infty,\omega} \leq \frac{\alpha}{\beta}\sum_{i=1}^{N-1}\chi_i\left|\left[L_\chi v\right]_i\right| \quad \textit{for all } v \in I\!R_0^{N+1}.$$

Remark 5.22. An error analysis of the upwind FVM using this (ℓ_∞, ℓ_1) stability can be conducted along the lines of Sect. 4.2.5; see also [90]. ♣

The $(\ell_\infty, w_{-1,\infty})$ *stability*

Now let us restrict our attention to difference operators of the type

$$\left[L_\kappa v\right]_i := -\frac{\varepsilon}{\chi_i}\left[1+\rho_i^+\right]v_{x;i} + \frac{\varepsilon}{\chi_i}\left[1-\rho_i^-\right]v_{\bar{x},i} + c_i v_i$$

with

$$\rho_i^+ := \rho\left(-\frac{b_{i+\kappa}h_{i+1}}{\varepsilon}\right)\frac{b_{i+\kappa}h_{i+1}}{\varepsilon}, \quad \rho_i^- := \rho\left(\frac{b_{i-1+\kappa}h_i}{\varepsilon}\right)\frac{b_{i-1+\kappa}h_i}{\varepsilon}$$

and

$$\kappa \in [0,1], \quad \chi_i = \kappa h_{i+1} + (1-\kappa)h_i \quad \text{and} \quad b_{i+\kappa} = b(x_i + \kappa h_{i+1}).$$

The FVM (5.22) is recovered for $\kappa = 1/2$, while the finite difference scheme of Sect. 4.2 is obtained when $\kappa = 1$ and $\rho = \rho_{U,0}$.

Remark 5.23. Condition (5.26) with $\alpha = \sup_{t<0} 1/\rho(t)$ follows from (ρ_3). ♣

Assuming that (ρ_1) holds, the Green's function G solves for fixed i

$$\left[L_\kappa^* G_{i,\cdot}\right]_j = \delta_{i,j} \qquad \text{for } j = 1,\ldots,N-1, \;\; G_{i,0} = G_{i,N} = 0$$

where L_κ^* is the adjoint operator to L_κ with respect to the scalar product $(\cdot,\cdot)_\chi$. Note (ρ_1) implies $\rho_i^+ + \rho_{i+1}^- = \varepsilon^{-1} b_{i+\kappa} h_{i+1}$. Then it is verified that

$$\left[L_\kappa^* v\right]_j = -\frac{\varepsilon}{\chi_j}\left\{\left[1-\rho_{j+1}^-\right] v_{x;j} - \left[1+\rho_{j-1}^+\right] v_{\hat{x};j}\right\} + \left(c_j + \frac{b_{j+\kappa} - b_{j-1+\kappa}}{\chi_j}\right) v_j .$$

Assume $c + b' \geq \gamma > 0$ and let b' be Hölder continuous with coefficient $\lambda \in (0,1]$. Then

$$c_j + \frac{b_{j+\kappa} - b_{j-1+\kappa}}{\chi_j} \geq 0$$

if the maximum step size h is sufficiently small, independent of ε; cf. (5.24). Proceeding as in Sect. 4.2.1, one can show

$$G_{i,j} \geq G_{i,j-1} \quad \text{for } j = 1,\dots,i \quad \text{and} \quad G_{i,j} \leq G_{i,j-1} \quad \text{for } j = i+1,\dots,N,$$

i.e., $G_{i,\cdot}$ is piecewise monotone.

Theorem 5.24. *Suppose (ρ_1), (ρ_3) and (5.20) hold true. Assume $b \in C^{1,\lambda}[0,1]$ with Hölder exponent $\lambda \in (0,1]$. Then the operator L_κ satisfies the stability inequality*

$$\|v\|_{\infty,\omega} \leq \frac{2\alpha}{\beta} \min_{C \in I\!R} \left\| \sum_{j=\cdot}^{N-1} \chi_j \left[L_\kappa v\right]_j + C \right\|_{\infty,\omega} \quad \textit{for all } v \in I\!R_0^{N+1},$$

with $\alpha = 1/\inf_{t<0} \rho(t) \leq 2$, provided the maximum step size h is smaller than some threshold value that is independent of the perturbation parameter ε.

5.4.2 Convergence in the Energy Norm

In this section we study the convergence in the energy norm $|||\cdot|||_\rho$ of the finite volume method on S-type meshes (see Sect. 2.1.3) with $\sigma \geq 2$. The controlling function ρ is assumed to satisfy conditions (ρ_0), (ρ_1) and (ρ_2).

Our analysis starts from coercivity of $a_\rho(\cdot,\cdot)$ (see Theorem 5.18) and follows the standard approach of the Strang Lemma. Let $\eta = u^I - u$ and $\chi = u^I - u^N$, where we use u^N for both the pointwise defined solution of (5.22) and its piecewise-linear extension on the mesh $\bar{\omega}$.

From (5.2) and (5.22) we get

$$a_\rho(\chi,\chi) = a(\eta,\chi) + a_\rho(u^I,\chi) - a(u^I,\chi) + f(\chi) - f_\rho(\chi).$$

Then Theorem 5.18 yields

$$
\begin{aligned}
\|\!|\chi|\!\|_\rho^2 \le |a(\eta,\chi)| + |f_\rho(\chi) - f(\chi)| \\
+ \left|r(u^I,\chi) - r_\rho(u^I,\chi)\right| + \left|c(u^I,\chi) - c_\rho(u^I,\chi)\right|
\end{aligned}
\tag{5.29}
$$

with

$$
r(u^I,\chi) = \int_0^1 c u^I \chi, \quad r_\rho(u^I,\chi) = \sum_{i=1}^{N-1} \hbar_i c_i u_i \chi_i, \quad c(u^I,\chi) = -\int_0^1 b(u^I)'\chi
$$

and

$$
\begin{aligned}
c_\rho(u^I,\chi) = -\sum_{i=1}^{N-1} \Big\{ \rho\left(\mu_{i+1/2}\right) b_{i+1/2}\left(u_{i+1} - u_i\right) \\
+ \rho\left(-\mu_{i-1/2}\right) b_{i-1/2}\left(u_i - u_{i-1}\right) \Big\} \chi_i.
\end{aligned}
$$

The four terms on the right-hand side of (5.29) will be bounded separately.

(i) The first term has been analysed in Sect. 5.2. We have under the assumptions of Theorem 5.6

$$
|a(\eta,\chi)| \le C\left(h + N^{-1}\max|\psi'|\right) \|\!|\chi|\!\|_{\varepsilon,\omega}\,, \tag{5.30}
$$

because the discrete and continuous energy norms are equivalent for functions from V^ω.

(ii) Next we bound the error arising from the discretisation of the right-hand side f. Denoting by φ_i the usual basis functions for linear finite elements, we have

$$
\begin{aligned}
&\left| \int_{I_i} (f\varphi_i)(x)\mathrm{d}x - \frac{h_i}{2} f_i \right| \\
&\quad = \left| \int_{I_i} \left\{ f_i + \int_{x_i}^x f'(s)\mathrm{d}s \right\} \varphi_i(x)\mathrm{d}x - \frac{h_i}{2} f_i \right| \le \frac{h_i^2}{2} \|f'\|_\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|f(\chi) - f_h(\chi)| &= \left| \sum_{i=1}^{N-1} \chi_i \left\{ \int_{x_{i-1}}^{x_{i+1}} (f\varphi_i)(x)\mathrm{d}x - \hbar_i f_i \right\} \right| \\
&\le \|f'\|_\infty h \sum_{i=1}^{N-1} \hbar_i |\chi_i| \le \|f'\|_\infty h \|\chi\|_{0,\omega}.
\end{aligned}
\tag{5.31}
$$

(iii) The next term in line is $r(u^I,\chi) - r_\rho(u^I,\chi)$. By the definition of $r_\rho(\cdot,\cdot)$ and $r(\cdot,\cdot)$, we have

$$r_\rho(u^I,\chi) - r(u^I,\chi) = \sum_{i=1}^{N-1} s_i\chi_i,$$

where

$$s_i := \int_{x_{i-1}}^{x_{i+1}} \left(cu^I\varphi_i\right)(x)\mathrm{d}x - \hbar_i c_i u_i = \int_{x_{i-1}}^{x_{i+1}} \left[\left(cu^I\right)(x) - c_iu_i\right]\varphi_i(x)\mathrm{d}x.$$

We have

$$\begin{aligned}&\left|\left(cu^I\right)(x) - c_iu_i\right| \\ &\qquad \le \left|\int_{x_i}^{x} \left(cu^I\right)' \mathrm{d}s\right| \le C\int_{x_{i-1}}^{x_{i+1}} \left\{1+\varepsilon^{-1}\mathrm{e}^{-\beta s/\varepsilon}\right\}\mathrm{d}s \le C\vartheta_{cd}^{[1]}(\bar\omega)\end{aligned}$$

for $x \in [x_{i-1},x_{i+1}]$. (The quantity $\vartheta_{cd}^{[p]}(\bar\omega)$ has been introduced in Sect. 2.1.) Hence, $|s_i| \le C\vartheta_{cd}^{[1]}(\bar\omega)\hbar_i$ and we obtain

$$\left|r_\rho(u^I,\chi)_i - r(u^I,\chi)\right| \le C\vartheta_{cd}^{[1]}(\bar\omega)\|\chi\|_{0,\omega}. \tag{5.32}$$

(iv) Finally consider the convection term. We have

$$\begin{aligned}&c_\rho(u^I,\chi) - c(u^I,\chi) \\ &\quad = \sum_{i=1}^{N}\Bigg\{\int_{I_i}\left(b(u^I)'\chi\right)(x)\mathrm{d}x \\ &\qquad\qquad - \left[\rho\left(-\mu_{i-1/2}\right)\chi_{i-1} + \rho\left(\mu_{i-1/2}\right)\chi_i\right] b_{i-1/2}\left(u_i-u_{i-1}\right)\Bigg\}\end{aligned}$$

and

$$\begin{aligned}\int_{I_i}\left(b(u^I)'\chi\right)(x)\mathrm{d}x &= b_{i-1/2}\left(u_i-u_{i-1}\right)\frac{\chi_i+\chi_{i-1}}{2} \\ &\qquad + \int_{I_i}\left\{\int_{x_{i-1/2}}^{x} b'(s)\mathrm{d}s\frac{u_i-u_{i-1}}{h_i}\chi(x)\right\}\mathrm{d}x.\end{aligned}$$

Combine these two equations and use (ρ_1) to get

$$\begin{aligned} c_\rho(u^I,\chi) - c(u^I,\chi) \\ &= \sum_{i=1}^{N}\left[\frac{1}{2} - \rho\left(\mu_{i-1/2}\right)\right]\left(\chi_i - \chi_{i-1}\right)\left(u_i - u_{i-1}\right)b_{i-1/2} \\ &\quad + \sum_{i=1}^{N}\int_{I_i}\left\{\int_{x_{i-1/2}}^{x} b'(s)\mathrm{d}s\,\frac{u_i - u_{i-1}}{h_i}\chi(x)\right\}\mathrm{d}x. \end{aligned} \tag{5.33}$$

Note that $|u_i - u_{i-1}| \le C\vartheta_{cd}^{[1]}(\bar\omega)$. Thus, for the second term in (5.33) one has

$$\left|\sum_{i=1}^{N}\int_{I_i}\left\{\int_{x_{i-1/2}}^{x} b'(s)\mathrm{d}s\,\frac{u_i - u_{i-1}}{h_i}\chi(x)\right\}\mathrm{d}x\right| \le C\vartheta_{cd}^{[1]}(\bar\omega)\,\|\chi\|_{0,\omega}\,. \tag{5.34}$$

Next we bound the first sum in (5.33). For $i \le qN$ use $|u_i - u_{i-1}| \le C\vartheta_{cd}^{[1]}(\bar\omega)$ again to obtain

$$\begin{aligned} &\left|\sum_{i=1}^{qN}\left[\frac{1}{2} - \rho\left(\mu_{i-1/2}\right)\right]\left(\chi_i - \chi_{i-1}\right)\left(u_i - u_{i-1}\right)b_{i-1/2}\right| \\ &\qquad \le C\vartheta_1(\omega)\sum_{i=1}^{qN}|\chi_i - \chi_{i-1}| \le C\vartheta_1(\omega)\varepsilon^{1/2}\ln^{1/2}N\,|\chi|_{1,\omega}\,, \end{aligned} \tag{5.35}$$

by a discrete Cauchy-Schwarz inequality.

For $i > qN$ we use the splitting $u = v + w$ of the exact solution according to Theorem 3.48. Starting with the layer term w, we have $w_i \le CN^{-2}$ for $i \ge qN$. Hence,

$$\begin{aligned} &\left|\sum_{i=qN+1}^{N}\left[\frac{1}{2} - \rho\left(\mu_{i-1/2}\right)\right]\left(\chi_i - \chi_{i-1}\right)\left(w_i - w_{i-1}\right)b_{i-1/2}\right| \\ &\qquad \le CN^{-2}\sum_{i=qN+1}^{N}\left(|\chi_i| + |\chi_{i-1}|\right) \le CN^{-1}\|\chi\|_{0,\omega}, \end{aligned} \tag{5.36}$$

by a discrete Cauchy-Schwarz inequality and an inverse inequality that can be used because $N^{-1} \le h_i$ for $i > qN$. Finally, consider the regular solution component v. To simplify the notation let

$$\gamma_{i-1/2} := b_{i-1/2}\left[\frac{1}{2} - \rho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right].$$

Summation by parts yields

$$\begin{aligned}\sum_{i=qN+1}^{N} \gamma_{i-1/2}\bigl(v_i - v_{i-1}\bigr)\bigl(\chi_i - \chi_{i-1}\bigr) \\ &= \gamma_{qN+1/2}\left(v_{qN} - v_{qN+1}\right)\chi_{qN} \\ &\qquad - \sum_{i=qN+1}^{N-1} \gamma_{i-1/2}\left(v_{i+1} - 2v_i + v_{i-1}\right)\chi_i \\ &\qquad + \sum_{i=qN+1}^{N-1} \bigl(\gamma_{i-1/2} - \gamma_{i+1/2}\bigr)\left(v_{i+1} - v_i\right)\chi_i .\end{aligned}$$

Taylor expansions give $|v_{i+1} - 2v_i + v_{i-1}| \le CN^{-2}$ and $|v_i - v_{i-1}| \le CN^{-1}$, while (ρ_0) implies $\left|\gamma_{i-1/2} - \gamma_{i+1/2}\right| \le CN^{-1}$. Thus,

$$\begin{aligned}\left|\sum_{i=qN+1}^{N} \gamma_{i-1/2}\left(v_i - v_{i-1}\right)\left(\chi_i - \chi_{i-1}\right)\right| \\ &\le CN^{-1}\Bigl(\|\chi\|_{0,\omega} + |\chi_{qN}|\Bigr) \le CN^{-1}\ln^{1/2} N\,|||\chi|||_{\rho},\end{aligned}$$

because

$$|\chi_{qN}| \le \sum_{i=1}^{qN} |\chi_i - \chi_{i-1}| \le C\ln^{1/2} N\varepsilon^{1/2}\,|\chi|_{1,\omega}.$$

Collecting (5.33)–(5.37), we get

$$\left|c_\rho(u^I,\chi) - c(u^I,\chi)\right| \le C\vartheta_1(\omega)\ln^{1/2} N\,|||\chi|||_{\rho}. \tag{5.37}$$

Now all terms on the right-hand side of (5.29) have been bounded; see (5.30), (5.32), (5.31), (5.34) and (5.37). Divide by $|||\chi|||_{\rho}$. Then recall the interpolation error bounds of Sect. 5.1 and note that $\vartheta_{cd}^{[1]}(\bar\omega) \le C\left(h + \max|\psi'|\right)$ for S-type meshes with $\sigma \ge 1$. We get the main result of this section.

Theorem 5.25. *Let $\bar\omega$ be an S-type mesh with $\sigma \ge 2$ whose mesh generating function $\tilde\varphi$ satisfies* (2.8) *and* $\max|\psi'|\ln^{1/2} N \le CN$. *Assume* (ρ_0), (ρ_1) *and* (ρ_2) *hold. Then*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon} + \left|\left|\left|u^I - u^N\right|\right|\right|_{\rho} \le C\left(h + N^{-1}\max|\psi'|\right)\ln^{1/2} N$$

for the error of the upwind FVM (5.22).

5.4.3 Convergence in the Maximum Norm

With the results of Sect. 5.4.1 at hand, the simplest maximum-norm analysis is based on the $(\ell_\infty, w_{-1,\infty})$ stability. Set

$$[A_\rho v]_i := \varepsilon\Big\{1 + \mu_{i-1/2}\big[\rho\left(-\mu_{i-1/2}\right) - \rho\left(\mu_{i-1/2}\right)\big]\Big\}\frac{v_i - v_{i-1}}{h_i} + b_{i-1/2}\frac{v_i + v_{i-1}}{2} + \sum_{j=i}^{N-1}\left(\hbar_j c_j + b_{j+1/2} - b_{j-1/2}\right) v_j.$$

If condition (ρ_1) is satisfied then $L_\rho v = -(A_\rho v)_{\hat{x}}$ and Theorem 5.24 yields

$$\|v\|_{\infty,\omega} \le \frac{4}{\beta}\min_{a\in\mathbb{R}}\|A_\rho v + a\|_{\infty,\omega} \quad \text{for all } v \in \mathbb{R}_0^{N+1}.$$

Integrate (5.1) to see that

$$\varepsilon u'_{i-1/2} + (bu)_{i-1/2} + \int_{x_{i-1/2}}^{x_{N-1/2}} \left((c+b')\,u - f\right)(s)\mathrm{d}s \equiv \alpha \quad \text{for all} \quad i = 1,\dots,N.$$

Thus,

$$\left\|u - u^N\right\|_{\infty,\omega} \le \frac{4}{\beta}\max_{i=1,\dots,N}|M_i|, \tag{5.38a}$$

where

$$\begin{aligned} M_i :=\ &\varepsilon\left(\frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2}\right) + b_{i-1/2}\left(\frac{u_i + u_{i-1}}{2} - u_{i-1/2}\right) \\ &+ \sum_{j=i}^{N-1}\left(\hbar_j c_j + b_{j+1/2} - b_{j-1/2}\right) u_j - \int_{x_{i-1/2}}^{x_{N-1/2}} (c+b')(s)u(s)\mathrm{d}s \\ &- \sum_{j=i}^{N-1}\hbar_j f_j + \int_{x_{i-1/2}}^{x_{N-1/2}} f(s)\mathrm{d}s \\ &+ b_{i-1/2}\left[\rho\left(-\frac{b_{i-1/2}h_i}{\varepsilon}\right) - \rho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right](u_i - u_{i-1}). \end{aligned} \tag{5.38b}$$

All terms except for the last one can be bounded by $\vartheta_{cd}^{[1]}(\bar{\omega})$ using the technique from Sect. 4.2.2. When bounding the last term note that $\rho(t) \in [0,1]$ for all $t \in \mathbb{R}$ and that

$$|u_i - u_{i-1}| \le \int_{I_i}|u'(s)|\,\mathrm{d}s \le C\vartheta_{cd}^{[1]}(\bar{\omega}).$$

Theorem 5.26. *Suppose* (ρ_1) *and* (ρ_3) *hold. Then the error of the upwind finite volume method* (5.22) *satisfies*

$$\left\|u - u^N\right\|_{\infty,\omega} \leq C\vartheta_{cd}^{[1]}(\bar{\omega}).$$

It was mentioned earlier that the accuracy of the scheme is improved when the function ρ is Lipschitz continuous in a neighbourhood of $t = 0$, say on an interval $[-m, m]$. We will briefly illustrate this using a standard Shishkin mesh with mesh parameter $\sigma \geq 2$.

We work from (5.38). The arguments from Sect. 4.3.3 are used to bound the first four terms by $\vartheta_{cd}^{[2]}(\bar{\omega})^2 \leq CN^{-2}\ln^2 N$.

We are left with the last term in (5.38). On a Shishkin mesh with $\sigma \geq 1$:

$$|u_i - u_{i-1}| \leq \begin{cases} CN^{-1}\ln N & \text{for } i = 1, \ldots, qN, \\ CN^{-1} & \text{for } i = qN+1, \ldots, N. \end{cases}$$

This can be verified using the decomposition of Theorem 3.48.

Furthermore, if $h_i \leq m\varepsilon\beta^{-1}$ then

$$\left|\rho\left(-\frac{b_{i-1/2}h_i}{\varepsilon}\right) - \rho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right| \leq C\frac{h_i}{\varepsilon},$$

because ρ is Lipschitz continuous on $[-m, m]$. Hence, in the layer region of the Shishkin mesh we have

$$\left|\rho\left(-\frac{b_{i-1/2}h_i}{\varepsilon}\right) - \rho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right| \leq CN^{-1}\ln N \quad \text{for } i = 1, \ldots, qN,$$

while on the coarse mesh region $\rho(t) \in [0, 1]$ for all $t \in I\!R$ is used. We obtain

$$\left|\rho\left(-\frac{b_{i-1/2}h_i}{\varepsilon}\right) - \rho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right| |u_i - u_{i-1}| \leq CN^{-1}.$$

All terms in (5.38) have been bounded by either CN^{-1} or by $CN^{-2}\ln^2 N$.

Theorem 5.27. *Assume* ρ *is Lipschitz continuous on* $[-m, m]$. *Let* (ρ_1) *and* (ρ_3) *hold. Then the error of the upwind FVM* (5.22) *on a standard Shishkin mesh satisfies*

$$\left\|u - u^N\right\|_{\infty,\omega} \leq CN^{-1}$$

if N *is larger than some threshold value which is independent of the perturbation parameter* ε.

Remark 5.28. On a standard Shishkin mesh the use of a Lipschitz continuous function ρ improves the accuracy from $\mathcal{O}\left(N^{-1}\ln N\right)$ to $\mathcal{O}\left(N^{-1}\right)$. ♣

Table 5.4 The upwind FVM on a standard Shishkin mesh

	$\rho_{U,0}$		$\rho_{U,10}$		ρ_S		ρ_I	
N	error	rate	error	rate	error	rate	error	rate
2^7	4.236e-3	0.84	3.855e-3	0.99	3.855e-3	0.99	3.855e-3	0.99
2^8	2.364e-3	0.86	1.942e-3	0.99	1.942e-3	0.99	1.942e-3	0.99
2^9	1.303e-3	0.87	9.745e-4	1.00	9.745e-4	1.00	9.745e-4	1.00
2^{10}	7.111e-4	0.89	4.882e-4	1.00	4.882e-4	1.00	4.882e-4	1.00
2^{11}	3.850e-4	0.89	2.443e-4	1.00	2.443e-4	1.00	2.443e-4	1.00
2^{12}	2.071e-4	0.90	1.222e-4	1.00	1.222e-4	1.00	1.222e-4	1.00
2^{13}	1.108e-4	0.91	6.115e-5	1.00	6.115e-5	1.00	6.115e-5	1.00
2^{14}	5.904e-5	0.91	3.059e-5	1.00	3.059e-5	1.00	3.059e-5	1.00
2^{15}	3.133e-5	0.92	1.531e-5	1.00	1.531e-5	1.00	1.531e-5	1.00
2^{16}	1.658e-5	—	7.672e-6	—	7.672e-6	—	7.672e-6	—

5.4.4 A Numerical Example

Table 5.4 displays the results of test computations using the upwind FVM with various stabilising functions ρ, when applied to the test problem (4.14) and contains the maximum nodal errors. For our tests we have chosen a standard Shishkin mesh with $\sigma = 1$ and $q = 1/2$. The results of the numerical tests are in agreement with Theorem 5.26 and 5.27. Comparing the numbers for $\rho_{U,0}$ with those for other ρs, we clearly see an improvement in the accuracy when ρ is Lipschitz continuous in a neighbourhood of $t = 0$. Also notice there is no (visible) difference in using either of those Lipschitz continuous ρs.

Chapter 6
Discretisations of Reaction-Convection-Diffusion Problems

This chapter is concerned with discretisations of the stationary linear reaction-convection-diffusion problem

$$-\varepsilon_d u'' - \varepsilon_c bu + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

with $b \geq 1$ and $c \geq 1$ on $[0,1]$.

In particular, we shall study the special case of scalar reaction-diffusion problems

$$-\varepsilon^2 u'' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1,$$

and its vector-valued counterpart

$$-\boldsymbol{E}^2 \boldsymbol{u''} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{\gamma}_0, \;\; \boldsymbol{u}(1) = \boldsymbol{\gamma}_1.$$

6.1 Reaction-Diffusion

This section is concerned with scalar reaction-diffusion problems

$$\mathcal{L}u := -\varepsilon^2 u'' + cu = f \quad \text{in } (0,1), \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1, \tag{6.1}$$

where $c \geq \rho^2$ on $[0,1]$ and $\rho > 0$ is a constant.

Analytical properties of (6.1) were studied in Sect. 3.3, while layer-adapted meshes for it have been introduced in Sect. 2.2. The crucial quantity for reaction-diffusion problems is

$$\vartheta_{rd}^{[p]}(\bar{\omega}) := \max_{k=1,\ldots,N} \int_{I_k} \left\{ 1 + \varepsilon^{-1} \left(\mathrm{e}^{-\rho x/p\varepsilon} + \mathrm{e}^{-\rho(1-x)/p\varepsilon} \right) \right\} \mathrm{d}x.$$

Using (6.1) as a model problem, a convergence analysis is conducted for variants of the linear FEM with convergence established in the energy norm.

Next a general convergence theory in the maximum norm is derived for central differencing on arbitrary meshes. The close relationship between the differential

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_6,

operator and its discretisation is highlighted. We then move on to a maximum-norm error analysis for the linear FEM and a special 4th-order scheme. Finally, central differencing for systems of reaction-diffusion equations is studied.

6.1.1 Linear Finite Elements

Consider (6.1) with homogeneous boundary conditions. Its weak formulation is: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) := \varepsilon^2 (u', v') + r(u,v) = f(v) \quad \text{for all } v \in H_0^1(0,1) \tag{6.2}$$

with $r(u,v) := (cu, v)$ and $f(v) := (f,v) := \int_0^1 \big(fv\big)(s)\mathrm{d}s$.

Given a mesh $\bar{\omega}$, let V_0^ω be the space of continuous functions that are piecewise linear on the mesh $\bar{\omega}$. Clearly $V_0^\omega \subset H_0^1(0,1)$. The standard Galerkin-FEM approximation is: Find $u^N \in V_0^\omega$ such that

$$a(u^N, v) = f(v) \quad \text{for all } v \in V_0^\omega.$$

Typically, the integrals in the bilinear form $r(\cdot,\cdot)$ and in the linear functional $f(\cdot)$ cannot be evaluated exactly. Therefore, they have to be approximated:

$$r(w,v) \approx \hat{r}(w,v) \quad \text{and} \quad f(v) \approx \hat{f}(v).$$

Different approximations yield different variants of the FEM. Later we shall consider four possible choices for $r(\cdot,\cdot)$ and $f(\cdot)$.

With this notation our FEM is: Find $u^N \in V_0^\omega$ such that

$$\hat{a}(u^N, v) := \varepsilon^2 \left((u^N)', v'\right) + \hat{r}\left(u^N, v\right) = \hat{f}(v) \quad \text{for all } v \in V_0^\omega. \tag{6.3}$$

The norm naturally associated with the weak formulation is the energy norm

$$|||v|||_{\varepsilon^2} := \left\{\varepsilon^2 \left\|v'\right\|_0^2 + \rho^2 \left\|v\right\|_0^2\right\}^{1/2}.$$

It is typically used in the convergence analysis of FEMs. Clearly $a(\cdot,\cdot)$ is coercive in the energy norm:

$$|||v|||_{\varepsilon^2}^2 \le a(v,v) \quad \text{for all } v \in H_0^1(0,1).$$

However, the coercivity of $\hat{a}(\cdot,\cdot)$ depends on the approximation used for the reaction term and has to be investigated separately. It is one ingredient in the error analysis for (6.3). The other ingredient is bounds for the interpolation error.

6.1.1.1 The Interpolation Error

Let again w^I denote the piecewise linear interpolant of w. Throughout this section let us assume the function $\psi \in C^2[0,1]$ admits the derivative bounds

$$|\psi''(x)| \le C\left\{1 + \varepsilon^{-2}\left(\mathrm{e}^{-\rho x/\varepsilon} + \mathrm{e}^{-\rho(1-x)/\varepsilon}\right)\right\}. \tag{6.4}$$

For example, the solution u of the boundary-value problem (6.1) belongs to this class of functions, see Sect. 3.3.1.2. And so does $cu - f = \varepsilon^2 u$ whose interpolation error will appear in the later analysis too.

Proposition 6.1. *Suppose ψ satisfies (6.4). Then*

$$\left\|\psi - \psi^I\right\|_{\infty,I_i} \le C\left[\int_{I_i}\left\{1 + \varepsilon^{-1}\left(\mathrm{e}^{-\rho x/2\varepsilon} + \mathrm{e}^{-\rho(1-x)/2\varepsilon}\right)\right\}\mathrm{d}x\right]^2$$

for all mesh intervals $I_i = [x_{i-1}, x_i]$.

Proof. For $x \in I_i$ and an arbitrary integrable functions χ set

$$(\mathcal{J}_i\chi)(x) := \frac{1}{h_i}\int_{I_i}\int_{x_{i-1}}^{x}\int_{\xi}^{s}\chi(t)\mathrm{d}t\mathrm{d}\xi\mathrm{d}s.$$

For triple integrals of this structure we have the two bounds

$$|(\mathcal{J}_i\chi)(x)| \le \int_{I_i}(\xi - x_{i-1})\,|\chi(\xi)|\,\mathrm{d}\xi \tag{6.5a}$$

and

$$|(\mathcal{J}_i\chi)(x)| \le \int_{I_i}(x_i - s)\,|\chi(s)|\,\mathrm{d}s. \tag{6.5b}$$

These integrals can be further bounded using Lemma 4.16. Let $\chi : I_j \to I\!R$ be any function with $\chi \ge 0$ on I_i. Then

$$(\mathcal{J}_i\chi)(x) \le \frac{1}{2}\left\{\int_{I_i}\chi(t)^{1/2}\mathrm{d}t\right\}^2 \quad \text{if } \chi \text{ is decreasing,} \tag{6.6a}$$

and

$$(\mathcal{J}_i\chi)(x) \le \frac{1}{2}\left\{\int_{I_i}\chi(t)^{1/2}\mathrm{d}t\right\}^2 \quad \text{if } \chi \text{ is increasing.} \tag{6.6b}$$

For the interpolation error on I_i we have the representation

$$\left(\psi - \psi^I\right)(x) = \big(\mathcal{J}_i(\psi'')\big)(x) \quad \text{for } x \in I_i. \tag{6.7}$$

Next we would like to apply (6.5) and (6.6). Therefore, we split ψ'' into two parts that can be bounded by monotone functions—one decreasing and the other increasing. Set

$$\bar{\psi}_I := \psi'' - \bar{\psi}_D \quad \text{and} \quad \bar{\psi}_D(x) := \begin{cases} \psi''(x) & \text{for } x \leq 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\left|\bar{\psi}_D(x)\right| \leq C\left\{1 + \varepsilon^{-2}\mathrm{e}^{-\rho x/\varepsilon}\right\} \quad \text{and} \quad \left|\bar{\psi}_I(x)\right| \leq C\left\{1 + \varepsilon^{-2}\mathrm{e}^{-\rho(1-x)/\varepsilon}\right\},$$

by (6.4). Hence, using (6.5) and (6.6), we obtain

$$\|\mathcal{J}_i(\psi'')\|_{\infty,I_i} \leq C\int_{I_i}\left(1 + \varepsilon^{-1}\mathrm{e}^{-\rho x/2\varepsilon} + \varepsilon^{-1}\mathrm{e}^{-\rho(1-x)/2\varepsilon}\right)\mathrm{d}x.$$

Recalling (6.7), we are finished. □

Theorem 6.2. *Suppose ψ satisfies* (6.4). *Then*

$$\left\|\psi - \psi^I\right\|_0 \leq \left\|\psi - \psi^I\right\|_\infty \leq C\left(\vartheta^{[2]}_{rd}(\bar{\omega})\right)^2$$

and

$$\left|\left|\left|\psi - \psi^I\right|\right|\right|_{\varepsilon^2} \leq C\left(\varepsilon^{1/2} + \vartheta^{[2]}_{rd}(\bar{\omega})\right)\vartheta^{[2]}_{rd}(\bar{\omega}).$$

Proof. The bound on the L_∞ error is an immediate consequence of Prop. 6.1 and the definition of $\vartheta^{[2]}_{rd}$.

For the error in the H^1 norm we proceed as follows using integration by parts

$$\left\|\left(\psi - \psi^I\right)'\right\|_0^2 = \int_0^1\left(\left(\psi - \psi^I\right)'(x)\right)^2\mathrm{d}x = -\int_0^1\psi''(x)\left(\psi - \psi^I\right)(x)\mathrm{d}x.$$

Next, a Hölder inequality gives

$$\left\|\left(\psi - \psi^I\right)'\right\|_0^2 \leq \left\|\psi - \psi^I\right\|_\infty\int_0^1|\psi''(x)|\,\mathrm{d}x \leq C\varepsilon^{-1}\left(\vartheta^{[2]}_{rd}(\bar{\omega})\right)^2,$$

by (6.4). Finally, combine this with the bound for the L_2 norm of the interpolation error to obtain the energy-norm estimate. □

Remark 6.3. The interpolation error in the energy norm is an order of magnitude better than for convection-diffusion problems. This is because this norm fails to capture the wider boundary layers in reaction-diffusion problems:

$$|||\mathrm{e}^{-\rho x/\varepsilon}|||_{\varepsilon^2} = \mathcal{O}\big(\varepsilon^{1/2}\big) \quad \text{for } \varepsilon \to 0.$$

Therefore, the reaction-diffusion problem is not singularly perturbed in the energy norm in the sense of Def. 1.1. ♣

Remark 6.4. For Lagrange interpolation with polynomials of arbitrary degree $p \geq 0$, cf. Remark 5.5, we have

$$\|I_p u - u\|_\infty \leq C\left(\vartheta_{rd}^{[p+1]}(\bar{\omega})\right)^{p+1}$$

for the solution u of (6.1). ♣

6.1.1.2 Convergence in the Energy Norm

With interpolation error bounds at hand, we can now return to the convergence analysis for the FEM (6.3).

Assume the bilinear form $\hat{a}(\cdot,\cdot)$ is V_0^ω-coercive with respect to the energy norm. That is, there exists a positive constant γ such that

$$\gamma\, |||v|||_{\varepsilon^2}^2 \leq \hat{a}(v,v) \quad \text{for all } v \in V_0^\omega. \tag{6.8}$$

Set $\eta := u^I - u$ and $\chi := u^I - u^N$. Then by a triangle inequality

$$|||u - u^N|||_{\varepsilon^2} \leq |||\eta|||_{\varepsilon^2} + |||\chi|||_{\varepsilon^2}.$$

Theorem 6.2 yields

$$|||\eta|||_{\varepsilon^2} \leq C\left(\varepsilon^{1/2} + \vartheta_{rd}^{[2]}(\bar{\omega})\right)\vartheta_{rd}^{[2]}(\bar{\omega}),$$

and we are left with bounding $|||\chi|||_{\varepsilon^2}$.

Starting from (6.8), we get

$$\gamma\, |||\chi|||_{\varepsilon^2}^2 \leq \hat{a}(\chi,\chi) = \hat{r}\left(u^I,\chi\right) - r(u,\chi) + f(\chi) - \hat{f}(\chi), \tag{6.9}$$

where we have used (6.2) and (6.3).

We shall consider four variants of the FEM characterised by different approximations of the reaction term and of the source term:

FEM-0: $\hat{r}(w,v) = r(w,v)$ and $\hat{f}(v) = f(v)$, i.e., no quadrature is used.
FEM-1: $\hat{r}(w,v) = (c^I w, v)$ and $\hat{f}(v) = (f^I, v)$,

FEM-2: $\hat{r}(w,v) = ((cw)^I, v)$ and $\hat{f}(v) = (f^I, v)$,
FEM-3: $\hat{r}(w,v) = (cw,v)_\omega$ and $\hat{f}(v) = (f,v)_\omega$ with the discrete ℓ_2-product

$$(w,v)_\omega := \sum_{i=1}^{N-1} \hbar_i w_i v_i.$$

This method is generated by applying the trapezium rule

$$\int_{I_i} g(s)\mathrm{d}s \approx \frac{h_{i+1}}{2}\left(g_i + g_{i+1}\right) = \int_{I_i} g^I(s)\mathrm{d}s$$

to (cw,v) and (f,v). It is equivalent to the standard central difference scheme which will be subject of Sect. 6.1.2.

Remark 6.5. A direct calculation shows that $(w,v)_\omega = (w,v)$ for all $w,v \in V_0^\omega$. ♣

Theorem 6.6. *Let u be the solution of (6.1) and u^N its approximation by FEM-0, FEM-1 or FEM-2. Then*

$$\left|\left|\left|u^I - u^N\right|\right|\right|_{\varepsilon^2} \le C\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2, \tag{6.10}$$

and

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon^2} \le C\left(\varepsilon^{1/2} + \vartheta_{rd}^{[2]}(\bar{\omega})\right)\vartheta_{rd}^{[2]}(\bar{\omega}).$$

Corollary 6.7. *For FEM-0, FEM-1 and FEM-2 we have the uniform first-order convergence result*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon^2} \le C\vartheta_{rd}^{[2]}(\bar{\omega}).$$

However, the worst case is not when ε is small, but when $\varepsilon = 1$. This is observed in numerical experiments [74].

We give short proofs of Theorem 6.6 for the various FEMs now.

FEM-0: $\hat{r}(w,v) = r(w,v)$, $\hat{f}(v) = f(v)$.

Clearly $\hat{a}(\cdot,\cdot) = a(\cdot,\cdot)$. Therefore, (6.8) holds with $\gamma = 1$. Inequality (6.9) yields

$$|||\chi|||_{\varepsilon^2}^2 \le r\left(u^I - u, \chi\right) = \left(c(u^I - u), \chi\right)$$

and

$$|||\chi|||_{\varepsilon^2} \le \|c\|_\infty \left\|u^I - u\right\|_0 \le C\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2,$$

by Theorem 6.2.

FEM-1: $\hat{r}(w,v) = (c^I w, v)$, $\hat{f}(v) = (f^I, v)$.

The coercivity of $\hat{a}(\cdot,\cdot)$ is verified upon noting that

$$\left(c^I w, w\right) \geq \left(\rho^2 w, w\right) = \rho^2 \|w\|_0^2 .$$

Thus, (6.8) is satisfied with $\gamma = 1$. Starting from (6.9) again, we get

$$|||\chi|||_{\varepsilon^2}^2 \leq \left((c^I - c)u^I, \chi\right) + \left(c(u^I - u), \chi\right) + \left(f - f^I, \chi\right) .$$

Appealing to Theorem 6.2 again, we get (6.10) for FEM-1.

FEM-2: $\hat{r}(w,v) = ((cw)^I, v)$, $\hat{f}(v) = (f^I, v)$.

This time establishing coercivity is slightly more involved. Let $w \in V_0^\omega$ be arbitrary. A direct calculation gives

$$\hat{r}(w,w) = \sum_{i=0}^{N-1} \frac{h_{i+1}}{3} \left(c_i w_i^2 + c_{i+1} w_{i+1}^2\right) + \sum_{i=0}^{N-1} \frac{h_{i+1}}{6} \left(c_i + c_{i+1}\right) w_i w_{i+1}.$$

We bound the second term from below:

$$\begin{aligned}\left(c_i + c_{i+1}\right) w_i w_{i+1} &\geq -\frac{c_i + c_{i+1}}{2} \left(w_i^2 + w_{i+1}^2\right) \\ &\geq -\left(c_i + \frac{h_{i+1}}{2} \|c'\|_\infty\right) w_i^2 - \left(c_{i+1} + \frac{h_{i+1}}{2} \|c'\|_\infty\right) w_{i+1}^2.\end{aligned}$$

Thus, if the maximum step size h is sufficiently small, dependent on κ, but independent of ε, then

$$\begin{aligned}\hat{r}(w,w) &\geq \sum_{i=0}^{N-1} \frac{h_{i+1}}{8} \left(c_i w_i^2 + c_{i+1} w_{i+1}^2\right) \\ &\geq \rho^2 \sum_{i=0}^{N-1} \frac{h_{i+1}}{8} \left(w_i^2 + w_{i+1}^2\right) = \frac{\rho^2}{4} \|w\|_0^2.\end{aligned}$$

Hence, $\hat{a}(\cdot,\cdot)$ is coercive and (6.8) holds true for $\gamma = 1/4$.

Next, (6.9) and the Cauchy-Schwarz inequality yield

$$\frac{1}{4} |||\chi|||_{\varepsilon^2} \leq \left\|q^I - q\right\|_0 , \quad \text{with } q := cu - f.$$

Theorem 6.2 applies to q and we obtain (6.10) for FEM-2.

FEM-3: $\hat{r}(w,v) = (cw,v)_\omega, \ \hat{f}(v) = (f,v)_\omega.$

In view of Remark 6.5, ineq. (6.8) holds true with $\gamma = 1$. Then by (6.9)

$$|||\chi|||_{\varepsilon^2}^2 \le (q,\chi)_\omega - (q,\chi) = \int_0^1 \left((q\chi)^I - q\chi\right)(x)\mathrm{d}x$$
$$= \left(q^I - q, \chi\right) + \sum_{i=0}^{N-1} \frac{h_{i+1}^2}{6}\left(q_{i+1} - q_i\right)\chi_{x;i},$$

where again $q = cu - f = \varepsilon^2 u''$. The first term on the right-hand has just been bounded when analysing FEM-2. Unfortunately, in view of the last term—in particular the presence of the discrete derivative χ_x—it seems impossible to obtain a convergence result as general as Theorem 6.6. On a S-type mesh, one might reason as in Sect. 5.2.1 by using an inverse inequality on the coarse-mesh region, but rely on the small mesh sizes inside the layer to gain the necessary powers of ε.

6.1.2 Central Differencing

Given an arbitrary mesh $\bar{\omega}$ the most frequently considered finite-difference approximation of (6.1) is: Find $u^N \in I\!R^{N+1}$ such that

$$\left[Lu^N\right]_i = f_i \quad \text{for } i = 1, \dots, N-1, \ \ u_0^N = \gamma_0, \ \ u_N^N = \gamma_1 \tag{6.11}$$

with

$$[Lv]_i := -\varepsilon^2 v_{\bar{x}\hat{x};i} + c_i v_i \quad \text{for } v \in I\!R^{N+1}.$$

The difference operators were introduced in Sect. 4.1. As mentioned before this difference scheme is equivalent to FEM-3 in the preceding section: Find $u^N \in V_0^\omega$ such that

$$a_c\left(u^N, v\right) = f_c\left(v\right) := (f,v)_\omega \quad \text{for all } v \in V_0^\omega, \tag{6.12}$$

where

$$a_c\left(w,v\right) := \varepsilon^2\left(w',v'\right) + (cw,v)_\omega = \varepsilon^2\left[w_x,v_x\right)_\omega + (cw,v)_\omega\,,$$

and

$$\left[w,v\right)_\omega := \sum_{i=0}^{N-1} h_{i+1} w_i v_i, \quad (w,v)_\omega := \sum_{i=1}^{N-1} \hbar_i w_i v_i.$$

Taking as test functions v the standard hat-function basis in V_0^ω, we see that (6.11) and (6.12) are equivalent. In particular, using summation by parts (cf. [146]) it is verified that

$$a_c(w, v) = (Lw, v)_\omega = (w, Lv)_\omega \quad \text{for all } w, v \in V_0^\omega.$$

Apparently the operator L is self-adjoint as is its continuous counterpart $\mathcal{L}$.

6.1.2.1 Stability

The matrix associated with the difference operator L is an L_0-matrix because all off-diagonal entries are non-positive. Application of the M-criterion (Lemma 3.14) with a test vector with components $e_i = 1$ establishes the inverse monotonicity of L. Thus, L satisfies a comparison principle: For any mesh functions $v, w\mathbb{R}^{N+1}$

$$\left.\begin{aligned} Lv &\le Lw \quad \text{on } \omega, \\ v_0 &\le w_0, \\ v_N &\le w_N \end{aligned}\right\} \quad \Longrightarrow \quad v \le w \quad \text{on } \bar\omega. \tag{6.13}$$

This comparison principle and Lemma 3.17 provide a priori bounds for the solution of (6.11):

$$\left\|u^N\right\|_{\bar\omega} \le \max\left\{|\gamma_0|, |\gamma_1|\right\} + \|f/c\|_{\infty,\omega} \quad \text{for } i = 0, \dots, N.$$

It also gives the stability inequality

$$\|v\|_{\infty,\bar\omega} \le \|Lv/c\|_{\infty,\omega} \quad \text{for all } v \in \mathbb{R}_0^{N+1}. \tag{6.14}$$

Green's function estimates

Using the discrete Green's function $G : \bar\omega^2 \to \mathbb{R} : (x_i, \xi_j) \mapsto G_{i,j} = G(x_i, \xi_j)$ associated with L and Dirichlet boundary conditions, any mesh function $v \in \mathbb{R}_0^{N+1}$ can be represented as

$$v_i = a_c(v, G_{i,\cdot}) = (Lv, G_{i,\cdot})_\omega = (v, LG_{i,\cdot})_\omega \quad \text{for } i = 1, \dots, N-1, \tag{6.15}$$

because the operator L is self-adjoint, i.e., $L = L^*$. This also implies $G_{i,j} = G_{j,i}$ for all $i, j = 0, \dots, N$.

Taking for v the standard basis in V_0^ω, we see that for fixed $i = 1, \dots, N-1$

$$[LG_{i,\cdot}]_j = \delta_{i,j} \quad \text{for } j = 1, \dots, N-1, \;\; G_{i,0} = G_{i,N} = 0, \tag{6.16}$$

where

$$\delta_{i,j} := \begin{cases} \hbar_i^{-1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

is the discrete equivalent of the Dirac-δ distribution.

Lemma 6.8. *For any fixed* $\mu > 0$

$$\|v\|_\infty^2 \le \mu \left[v_x, v_x\right)_\omega + \left(\mu^{-1} + 1\right)(v, v)_\omega \quad \textit{for all } v \in V_0^\omega.$$

Proof. This is Theorem 2 in chap. V, §4 of [145]. □

Theorem 6.9. *The Green's function G associated with the discrete operator L and Dirichlet boundary conditions satisfies*

$$0 \le G_{i,j} \le \frac{1 + \varepsilon\rho^{-1}}{\varepsilon\rho} \quad \textit{for } 0 \le i, j \le N,$$
$$G_{\xi;i,j} \ge 0 \quad \textit{for } 0 \le j < i < N,$$
$$G_{\xi;i,j} \le 0 \quad \textit{for } 0 \le i \le j < N.$$

Proof. The positivity of G follows from the inverse monotonicity of L.

The upper bound on G is proved using an argument from [147]. Let i be arbitrary, but fixed. Set $\Gamma := G_{i,\cdot}$. Then Lemma 6.8 with $\mu = \varepsilon/\rho$ yields

$$\begin{aligned} \|\Gamma\|_{\infty,\omega}^2 &\le \frac{1 + \varepsilon\rho^{-1}}{\rho} \left(\varepsilon \left[\Gamma_\xi, \Gamma_\xi\right)_\omega + \varepsilon^{-1}\gamma^2 (\Gamma, \Gamma)_\omega \right) \\ &\le \frac{1 + \varepsilon\rho^{-1}}{\rho} \left(\varepsilon \left[\Gamma_x, \Gamma_x\right)_\omega + \varepsilon^{-1} (c\Gamma, \Gamma)_\omega \right) \\ &= \frac{1 + \varepsilon\rho^{-1}}{\rho\varepsilon} \Gamma_i, \end{aligned}$$

by (6.15). The upper bound on G follows.

When proving the monotonicity, note that

$$\varepsilon^2 G_{\xi;i,j} = \varepsilon^2 G_{\xi;i,j-1} + \hbar_j c_j G_{i,j} \quad \text{for } 0 < j < i$$

and

$$\varepsilon^2 G_{\xi;i,j} = \varepsilon^2 G_{\xi;i,j+1} - \hbar_{j+1} c_{j+1} G_{i,j+1} \quad \text{for } i < j < N - 1.$$

The non-negativity of G implies $G_{\xi;i,0} \ge 0$ and $G_{\xi;i,N-1} \le 0$. Then the piecewise monotonicity of G follows by induction. □

Theorem 6.10. *The Green's function* G *associated with the discrete operator* L *satisfies*

$$\|cG_{i,\cdot}\|_{1,\omega} \leq 1, \quad \|G_{\xi;i,\cdot}\|_{1,{}^*\omega} \leq \frac{2\left(1+\varepsilon\rho^{-1}\right)^2}{\varepsilon\rho}, \quad \|G_{\xi\xi;i,\cdot}\|_{1,\omega} \leq \frac{2}{\varepsilon^2}$$

for all $i = 1, \ldots, N-1$.

Proof. First multiply (6.16) by $\hbar_j$ and sum for $j = 1, \ldots, N-1$.

$$\sum_{j=1}^{N-1} \hbar_j c_j G_{i,j} = 1 + \varepsilon^2 \left(G_{\bar{\xi};i,N} - G_{\xi;i,0}\right) \leq 1,$$

because $G_{\xi;i,0} \geq 0$ and $G_{\xi;i,N-1} \leq 0$. This is the bound on the c-weighted ℓ_1 norm.
Next

$$\|G_{\xi;i,\cdot}\|_{1,{}^*\omega} = \sum_{j=0}^{i-1} h_{j+1} G_{\xi;i,j} - \sum_{j=i}^{N-1} h_{j+1} G_{\xi;i,j} = 2G_{i,i} \leq \frac{2\left(1+\varepsilon\rho^{-1}\right)^2}{\varepsilon\rho}, \tag{6.17}$$

by Theorem 6.9.
Finally, a triangle inequality, (6.16) and (6.17) give the bound on $G_{\xi\xi}$. □

6.1.2.2 A Priori Error Bounds

The analysis follows [95, 103]. By (6.15) we have for the error $u - u^N$ in the mesh node x_i

$$\left(u - u^N\right)_i = a_c\left(u - u^N, G_{i,\cdot}\right) = a_c\left(u, G_{i,\cdot}\right) - f_c\left(G_{i,\cdot}\right).$$

For simplicity we set $\Gamma := G_i$. We identify the mesh function Γ with that function from V_0^ω that coincides with Γ at the mesh nodes. Using the weak form of the differential equation, we get

$$\begin{aligned}\left(u - u^N\right)_i &= a_c\left(u, \Gamma\right) - a\left(u, \Gamma\right) + f\left(\Gamma\right) - f_c\left(\Gamma\right) \\ &= \left(cu - f, \Gamma\right)_\omega - \left(cu - f, \Gamma\right).\end{aligned}$$

Note, if $v_0 = v_N = 0$ then

$$\left(w, v\right)_\omega = \int_0^1 \left(wv\right)^I(x)\mathrm{d}x, \tag{6.18}$$

where w^I denotes again the piecewise linear interpolant of w.

Setting $q := cu - f = \varepsilon^2 u''$, we obtain the error representation

$$\left(u - u^N\right)_i = \int_0^1 \left(\left(q\Gamma\right)^I - q\Gamma\right)(x)\mathrm{d}x. \tag{6.19}$$

We are left with bounding the interpolation error for $q\Gamma$. To this end we shall avail of the derivative bounds derived in Sect. 3.3.1.2 and repeat some of the arguments from Sect. 6.1.1.1.

By (6.7), for $x \in [x_j, x_{j+1}]$ we have

$$\left(q\Gamma\right)(x) - \left(q\Gamma\right)^I(x) = 2\Gamma_{x;j}\left(\mathcal{J}_j(q')\right)(x) + \left(\mathcal{J}_j(q''\Gamma)\right)(x). \tag{6.20}$$

Next, we wish to apply (6.5) and (6.6) to the right-hand side of (6.20). Therefore, we split q' into two parts that can be bounded by monotone functions—one decreasing and the other increasing. Set

$$q_I := q' - q_D \quad \text{and} \quad q_D(x) := \begin{cases} q'(x) & \text{for } x \le 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

From Sect. 3.3.1.2 we have

$$\varepsilon^{-1}\left|q_D(x)\right| \le C\left\{1 + \varepsilon^{-2}\mathrm{e}^{-\rho x/\varepsilon}\right\}$$

and

$$\varepsilon^{-1}\left|q_I(x)\right| \le C\left\{1 + \varepsilon^{-2}\mathrm{e}^{-\rho(1-x)/\varepsilon}\right\}.$$

Hence, using (6.5) and (6.6), we obtain

$$\left\|\mathcal{J}_j(q')\right\|_{\infty,I_j} \le C\varepsilon\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2. \tag{6.21}$$

The second integral in (6.20) is bounded in a similar manner. Set

$$\bar{q}_I := q'' - \bar{q}_D \quad \text{and} \quad \bar{q}_D(x) := \begin{cases} q''(x) & \text{for } x \le 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

From Sect. 3.3.1.2:

$$\left|\left(\Gamma\bar{q}_D\right)(x)\right| \le C\left(\Gamma_j + \Gamma_{j+1}\right)\left\{1 + \varepsilon^{-2}\mathrm{e}^{-\rho x/\varepsilon}\right\}$$

and

$$\left|\left(\Gamma\bar{q}_I(x)\right)\right| \le C\left(\Gamma_j + \Gamma_{j+1}\right)\left\{1 + \varepsilon^{-2}\mathrm{e}^{-\rho(1-x)/\varepsilon}\right\},$$

since Γ is piecewise linear and positive. Thus,

$$\|\mathcal{J}_j(q''\Gamma)\|_{\infty,I_j} \le C\left(\Gamma_j + \Gamma_{j+1}\right)\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2. \tag{6.22}$$

Applying (6.21) and (6.22) to (6.20), we obtain

$$\left|\left(\Gamma q - (\Gamma q)^I\right)(x)\right| \le C\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2 \left(\varepsilon \Gamma_{\bar{x};j} + \Gamma_{j+1} + \Gamma_j\right) \quad \text{for } x \in (x_j, x_{j+1}).$$

Integrate over $[0,1]$ to get

$$\left|\int_0^1 \left((\Gamma q) - (\Gamma q)^I\right)(x)\mathrm{d}x\right| \le C\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2 \left(\varepsilon \left\|\Gamma_x\right\|_{1,^*\omega} + \left\|\Gamma\right\|_{1,\omega}\right).$$

Finally, recall (6.19) and Theorem 6.10. We arrive at the main convergence result of this section.

Theorem 6.11. *Let u be the solution of the reaction-diffusion problem* (6.1) *and u^N its central difference approximation* (6.11)*. Suppose $c, f \in C^2[0,1]$. Then*

$$\left\|u - u^N\right\|_{\infty,\omega} \le C\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2.$$

Corollary 6.12. *Theorems 6.2 and 6.11 yield*

$$\left\|u - u^N\right\|_{\infty} \le C\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2,$$

by a triangle inequality and because $\left\|u^I - u^N\right\|_\infty = \left\|u^I - u^N\right\|_{\infty,\omega}$.

6.1.2.3 A Posteriori Error Analysis

The first a posteriori analysis for central differencing was conducted by Kopteva; see [66]. We slightly modify her argument. It is based on the bounds for the Green's function $\mathcal{G}$ associated with the continuous operator $\mathcal{L}$; see Theorem 3.31.

Let $x \in (0,1)$ be arbitrary, but fixed. Set $\Gamma = \mathcal{G}(x,\cdot)$. Then

$$\begin{aligned}\left(u - u^N\right)(x) &= a\left(u - u^N, \Gamma\right) = f(\Gamma) - a\left(u^N, \Gamma\right)\\ &= f(\Gamma) - f_c(\Gamma) + a_c(u^N, \Gamma) - a\left(u^N, \Gamma\right)\\ &= \left(cu^N - f, \Gamma\right)_\omega - \left(cu^N - f, \Gamma\right),\end{aligned}$$

by (6.12). Setting $\hat{q} := cu^N - f$, we have

$$\left(u - u^N\right)(x) = \int_0^1 \left(\left(\hat{q}\Gamma\right)^I - \hat{q}\Gamma\right)(s)\mathrm{d}s,$$

where (6.18) was used. Expand the integrand

$$\left(\hat{q}\Gamma\right)^I - \hat{q}\Gamma = \left(\hat{q}^I - \hat{q}\right)\Gamma + \hat{q}^I\left(\Gamma^I - \Gamma\right) + \left(\hat{q}\Gamma\right)^I - \hat{q}^I\Gamma^I$$

in order to obtain the error representation

$$\begin{aligned}\left(u - u^N\right)(x) &= \left(\hat{q}^I - \hat{q}, \Gamma\right) + \left(\hat{q}^I, \Gamma^I - \Gamma\right) \\ &\quad + \int_0^1 \left(\left(\hat{q}\Gamma\right)^I - \hat{q}^I\Gamma^I\right)(s)\mathrm{d}s.\end{aligned} \tag{6.23}$$

The terms on the right-hand side are estimated separately.

A Hölder inequality and Theorem 3.31 give

$$\left|\left(\hat{q} - \hat{q}^I, \Gamma\right)\right| \le \rho^{-2}\left\|\hat{q} - \hat{q}^I\right\|_\infty .$$

For the second term in (6.23), integration by parts yields

$$\int_{I_k} \hat{q}^I(s)\left(\Gamma^I - \Gamma\right)(s)\mathrm{d}s = \int_{I_k} (s - x_{k+1})(s - x_k)\, Q_k(s)\Gamma''(s)\mathrm{d}s,$$

where

$$Q_k(s) = \frac{\hat{q}_{k+1} + \hat{q}_k}{4} + \frac{\hat{q}_{k+1} - \hat{q}_k}{6h_{k+1}}\left(s - x_{k+1/2}\right), \quad x_{k+1/2} = \frac{x_k + x_{k+1}}{2}.$$

Thus

$$\left|\int_{I_k} \hat{q}^I(x)\left(\Gamma^I - \Gamma\right)(s)\mathrm{d}s\right| \le \frac{h_{k+1}^2}{8}\max\left\{\left|\hat{q}_k\right|, \left|\hat{q}_{k+1}\right|\right\}\int_{I_k} \left|\Gamma''(s)\right|\mathrm{d}s.$$

By Theorem 3.31, we have $\|\Gamma''\|_1 \le 2\varepsilon^{-2}$. Hence,

$$\left|\left(\hat{q}^I, \Gamma^I - \Gamma\right)\right| \le \max_{k=0,\dots,N-1} \frac{h_{k+1}^2}{4\varepsilon^2}\max\left\{\left|\hat{q}_k\right|, \left|\hat{q}_{k+1}\right|\right\}.$$

For the last term in (6.23) a direct calculation yields

$$\begin{aligned}\int_{I_k} \left(\left(\hat{q}\Gamma\right)^I - \hat{q}^I\Gamma^I\right)(s)\mathrm{d}s &= \frac{h_{k+1}}{6}\left(\hat{q}_{k+1} - \hat{q}_k\right)\left(\Gamma_{k+1} - \Gamma_k\right) \\ &= \frac{h_{k+1}}{6}\left(\hat{q}_{k+1} - \hat{q}_k\right)\int_{I_k} \Gamma'(s)\mathrm{d}s.\end{aligned}$$

Thus

$$\left| \int_0^1 \left((\hat{q}\Gamma)^I - \hat{q}^I \Gamma^I \right)(s)\mathrm{d}s \right| \le \max_{k=0,\dots,N-1} \frac{h_{k+1}}{6\varepsilon\rho} \left|\hat{q}_{k+1} - \hat{q}_k\right|,$$

because $\|\Gamma'\|_1 \le 1/(\varepsilon\rho)$, by Theorem 3.31.

All terms in (6.23) have been bounded. We get

Theorem 6.13. *Let u be the solution of the reaction-diffusion problem (6.1) and u^N its central difference approximation (6.11). Let $\hat{q} := cu^N - f$. Then*

$$\left\|u - u^N\right\|_\infty \le \eta_1 + \eta_2 + \eta_3,$$

where

$$\eta_1 := \frac{1}{\rho^2}\left\|\hat{q} - \hat{q}^I\right\|_\infty, \quad \eta_2 := \max_{k=0,\dots,N-1} \frac{h_{k+1}^2}{4\varepsilon^2} \max\left\{\left|\hat{q}_k\right|, \left|\hat{q}_{k+1}\right|\right\}$$

and

$$\eta_3 := \max_{k=0,\dots,N-1} \frac{h_{k+1}}{6\varepsilon\rho} \left|\hat{q}_{k+1} - \hat{q}_k\right|.$$

Remark 6.14. The term $\left\|\hat{q} - \hat{q}^I\right\|_\infty$ in the a posteriori bound can be further expanded as follows

$$\hat{q} - \hat{q}^I = f - f^I - \left(c - c^I\right)u^N - c^I u^N + \left(cu^N\right)^I.$$

Thus

$$\begin{aligned}\left\|\hat{q} - \hat{q}^I\right\|_\infty \le \left\|f - f^I\right\|_\infty + \left\|c - c^I\right\|_\infty \left\|u^N\right\|_\infty \\ + \frac{1}{4} \max_{k=0,\dots,N-1} \left|u_{k+1}^N - u_k^N\right| \cdot \left|c_{k+1} - c_k\right|.\end{aligned}$$

The first two terms involve (continuous) norms of the data. These have to be approximated numerically with sufficient accuracy. At least $\mathcal{O}(h^2)$ is required. However, higher order is desirable to ensure all non explicitly computable terms in the error estimator are of higher order and decay rapidly as the mesh is refined. ♣

Remark 6.15. Invoking the difference equation (6.11), we see that η_3 implicitly contains third-order discrete derivatives of u^N.

$$\frac{h_{k+1}}{6\varepsilon\rho}\left|\hat{q}_{k+1} - \hat{q}_k\right| = \frac{\varepsilon h_{k+1}^2}{6\rho}\left|u_{\bar{x}\hat{x}x;k}^N\right|.$$

The a posteriori error bound in [66, Theorem 3.3] is given using these higher-order operators.

The argument in [66] proceeds to show that

$$\left\|u-u^N\right\|_\infty \le C \max_{i=0,\dots,N-1} \left\{h_{k+1}\left[1+\left|u_{\bar{x}\hat{x};k}\right|^{1/2}+\left|u_{\bar{x}\hat{x};k+1}\right|^{1/2}\right]\right\}^2 .$$

The constant C is independent of ε, but not specified in [66]. Therefore, this latest inequality cannot be used for reliable a posteriori error estimation. Nonetheless, it is useful for steering adaptive mesh generation. ♣

Adaptive mesh generation

Based on Theorem 6.13, the de Boor algorithm in Sect. 4.2.4.2 can be adapted for the problem and method under consideration. One only needs to redefine the Q_i:

$$Q_i = Q_i(u^N, \Delta, \omega) := \left\{\kappa_0 + \kappa_1\eta_{1;i}\kappa_2\eta_{2;i} + \kappa_3\eta_{3;i}\right\}^{1/2}$$

with

$$\eta_{1;i} := \left|\hat{q}_{i-1} - 2\hat{q}_{i-1/2} + \hat{q}_i\right|, \quad \eta_{2;i} := \frac{h_i^2}{\varepsilon^2}\max\left\{\left|\hat{q}_{i-1}\right|, \left|\hat{q}_i\right|\right\},$$
$$\eta_{3;i} := \frac{h_i}{\varepsilon}\left|\hat{q}_i - \hat{q}_{i-1}\right|$$

and non-negative weights κ_k. Note that $\max_i \eta_{1;i}$ is a second order approximation of η_1 in Theorem 6.13.

In view of Remark 6.15 one can also use the de Boor algorithm with

$$Q_i = 1 + \left|u_{\bar{x}\hat{x};i-1}\right|^{1/2} + \left|u_{\bar{x}\hat{x};i}\right|^{1/2} .$$

Numerical experiment for this variant of the algorithm are documented in [68].

6.1.2.4 An Alternative Convergence Proof

Traditional finite difference analysis aims at directly exploiting the maximum-norm stability or using barrier function techniques. In higher dimensions they are often the only tool available, because of a lack of stronger stability results.

We now present an error analysis for central differencing on a Bakhvalov mesh that solely uses (6.14). In the layer regions this mesh is not approximately equidistant. Consequently, the truncation error of the difference scheme is apparently only first order at points in the layers, but a more delicate analysis given in [18] shows that the truncation error at every mesh point is in fact $\mathcal{O}(N^{-2})$ uniformly in ε. We use the description of the mesh by a mesh generating functions, see Sect. 2.1.1.

For any $\psi \in C^4[0,1]$, Taylor expansions show that

$$\left|[\mathcal{L}\psi - L\psi]_i\right| \le C\varepsilon^2\left\|\psi''\right\|_{[x_{i-1},x_{i+1}]} \qquad (6.24a)$$

and

$$\left|[\mathcal{L}\psi - L\psi]_i\right| \le C\varepsilon^2 \left|h_i - h_{i+1}\right| \left|\psi_i'''\right| + (h_i + h_{i+1})^2 \left\|\psi^{(4)}\right\|_{[x_{i-1},x_{i+1}]}. \tag{6.24b}$$

When $\sigma\varepsilon \ge \rho q$ the mesh is uniform with mesh size N^{-1}. Furthermore, $\varepsilon^{-1} \le C$. Thus,

$$\|\mathcal{L}u - Lu\|_\omega \le CN^{-2},$$

by Theorem 3.35 and (6.24).

Now consider the case $\sigma\varepsilon < \rho q$. For simplicity we will consider only the layer at $x = 0$ and assume that $x_i = \varphi(t_i) \le 1/2$.

From the construction of φ one must have $\tau < q$. It follows that $1 < \chi'(\tau) < \bar{q}$, where we set $\bar{q} = 1/(1-2q)$. Define the auxiliary points τ_1 and τ_2 in $(0, q)$ by $\chi'(\tau_1) = \bar{q}$ and $\chi'(\tau_2) = 1$. Then $\tau_2 = q - \sigma\varepsilon/\rho < \tau < \tau_1 = q - \sigma\varepsilon(1-2q)/\rho$ because $\chi'' > 0$ on $[0, q)$.

(i) $\varphi'(t) \le \chi'(\tau) \le \tilde{q}$ for $t \in [0,1]$. Thus,

$$h_i = \int_{t_{i-1}}^{t_i} \varphi'(t)\mathrm{d}t \le \tilde{q}N^{-1} \quad \text{for } i = 1, \ldots, N. \tag{6.25}$$

(ii) For $t \le t_i < q$ we have $\varphi'(t) \le \chi'(t) = \sigma\varepsilon/\rho(q-t) \le \sigma\varepsilon/\rho(q-t_i)$. Hence, for $t_i \le q - N^{-1}$,

$$h_i = \int_{t_{i-1}}^{t_i} \varphi'(t)\mathrm{d}t \le N^{-1}\varphi'(t_i) \le \frac{\sigma\varepsilon}{\rho N(q-t_i)} \le \frac{2\sigma\varepsilon}{\rho N(q-t_{i-1})}\,. \tag{6.26}$$

(iii) $h_{i+1} - h_i = x_{i+1} - 2x_i + x_{i-1} = \varphi''(t_i^*)N^{-2}$ for some $t_i^* \in [t_{i-1}, t_{i+1}]$. Now

$$\varphi''(t) \le \chi''(\tau) = \frac{\sigma\varepsilon}{\rho(q-\tau)^2} \quad \text{and} \quad \frac{1}{q-\tau} \le \frac{1}{q-\tau_1} = \frac{\rho\bar{q}}{\sigma\varepsilon},$$

which gives

$$|h_{i+1} - h_i| \le \frac{\rho\bar{q}^2}{\sigma\varepsilon}N^{-2}. \tag{6.27}$$

Furthermore, we have the bound

$$\varphi''(t_i^*) \le \frac{\sigma\varepsilon}{\rho(q-t_{i+1})^2} \le \frac{4\sigma\varepsilon}{\rho(q-t_i)^2} \quad \text{for } t_i \le q - \frac{2}{N}\,,$$

which yields

$$|h_{i+1} - h_i| \le \frac{4\sigma\varepsilon}{N^2\rho(q-t_i)^2} \quad \text{for } t_i \le q - \frac{2}{N}\,. \tag{6.28}$$

(iv)

$$\mathrm{e}^{-\rho x_i/\varepsilon} = \left(\frac{q-t_i}{q}\right)^{\sigma} \quad \text{for } t_i \le \tau \tag{6.29}$$

and

$$\mathrm{e}^{-\rho x_i/\varepsilon} \le \left(\frac{\sigma\varepsilon}{\rho q}\right)^{\sigma} \quad \text{for } t_i \ge \tau_2. \tag{6.30}$$

Henceforth let $\sigma \ge 2$. Using (6.24), Theorem 3.35, (6.25) and (6.30), we get

$$\left|\left[\mathcal{L}u - Lu\right]_i\right| \le CN^{-2} \quad \text{for } \tau_2 \le t_{i-1},$$

which is the region outside the layer. For $t_i \le q - 2N^{-1}$ (the layer region), from (6.24) and Theorem 3.35 one arrives at

$$\begin{aligned}\left|\left[\mathcal{L}u - Lu\right]_i\right| \le C\left|h_i - h_{i+1}\right|\varepsilon^2 + C\left|h_i - h_{i+1}\right|\varepsilon^{-1}\mathrm{e}^{-\rho x_i/\varepsilon} \\ + C\left(h_i + h_{i+1}\right)^2\varepsilon^2 + C\left(h_i + h_{i+1}\right)^2\varepsilon^{-2}\mathrm{e}^{-\rho x_{i-1}/\varepsilon}.\end{aligned}$$

To bound the first term on the right-hand side use (6.27); for the second term, use (6.28) and (6.29); for the third term, use (6.25); and for the fourth, use (6.26), (6.29) and $q - t_{i-1} \le 3(q-t_i)/2$. This yields

$$\left|\left[\mathcal{L}u - Lu\right]_i\right| \le CN^{-2} \quad \text{for } t_i \le q - 2N^{-1}.$$

We are left with the transition region where $\tau_2 > t_{i-1}$ and $t_i > q - 2N^{-1}$. Thus,

$$q - \frac{2}{N} < t_i < \tau_2 + \frac{1}{N} = q - \frac{\sigma\varepsilon}{\rho} + \frac{1}{N} < q + \frac{1}{N}.$$

Notice that the first two inequalities here imply that $\varepsilon < 3\rho/(\sigma N)$. Use (6.24):

$$\left|\left[\mathcal{L}u - Lu\right]_i\right| \le C\left(\varepsilon^2 + \mathrm{e}^{-\rho x_{i-1}/\varepsilon}\right) \le CN^{-2},$$

by (6.30) and $\varepsilon \le CN^{-1}$.

Thus, on a Bakhvalov mesh the truncation error in the maximum norm is bounded uniformly by $\mathcal{O}\left(N^{-2}\right)$. Application of the stability inequality (6.14) gives the uniform error bound

$$\left\|u - u^N\right\|_{\infty,\omega} \le CN^{-2}, \quad \text{if } \sigma \ge 2.$$

We have recovered Theorem 6.11 for a Bakhvalov mesh by means of a different kind of analysis.

6.1.2.5 Discontinuous Data

Assume the reaction coefficient c or the source term f possess a discontinuity in a point $d \in (0,1)$. Then (6.1) reads: Find $u \in C^1[0,1] \cap C^2\big((0,1) \setminus \{d\}\big)$ such that

$$-\varepsilon^2 u'' + cu = f \quad \text{in } (0,1) \setminus \{d\}, \;\; u(0) = \gamma_0, \;\; u(1) = \gamma_1.$$

How should the central differencing scheme (6.11) *be generalised to deal with the discontinuous data?*

Assuming $d = x_\kappa \in \omega$, i.e., the discontinuity is in a mesh point, a naive finite difference approach would seek a mesh function u^N with

$$\left[Lu^N\right]_i = f_i \quad \text{for } x_i \in \omega \setminus \{x_\kappa\}, \quad u^N_{x;\kappa} = u^N_{\bar{x};\kappa}, \;\; u^N_0 = \gamma_0, \;\; u^N_N = \gamma_1.$$

This method on a Shishkin mesh was analysed in [37]. Only first order (up to a logarithmic factor) was established. The numerical experiments in [37] show that when ε is moderate at best first order can be achieved.

The continuity of the derivative in $x_\kappa = d$ is discretised by imposing

$$u^N_{x;\kappa} = u^N_{\bar{x};\kappa}.$$

However, the one-sided difference operators are first-order approximation only. This might explain the drop in accuracy.

Instead, we start from the variational formulation (6.12): Find $u^N \in V_0^\omega$ such that

$$\varepsilon^2 \left(\left(u^N\right)', v' \right) - \left(cu^N, v\right)_\omega = (f, v)_\omega \quad \text{for all } v \in V_0^\omega,$$

where

$$(w, v)_\omega := \sum_{i=1}^{N-1} \hbar_i w_i v_i = \int_0^1 (wv)^I (s) \mathrm{d}s.$$

For discontinuous functions the nodal interpolant does not exist. However, since the point d of discontinuity is a mesh point, we can define w^I element wise:

$$w^I(x) = \frac{x_{k+1} - x}{h_{k+1}} w(x_k + 0) + \frac{x - x_k}{h_{k+1}} w(x_{k+1} - 0) \quad \text{for } x \in I_k.$$

Clearly, when $w \in C[0,1]$, we recover the standard linear nodal interpolant.

Evaluating $\int_0^1 (wv)^I(s)\mathrm{d}s$, we get

$$(w,v)_\omega = \sum_{i=1}^{N-1} \hbar_i \tilde{w}_i v_i, \quad \text{with} \quad \tilde{w}_i = \frac{h_i w(x_i - 0) + h_{i+1} w(x_i + 0)}{2\hbar_i}.$$

Note that $\tilde{w}_i = w(x_i)$ for $i \neq \kappa$.

The resulting difference scheme is: Find $u^N \in I\!R^{N+1}$ such that

$$Lu^N := -\varepsilon^2 u^N_{\bar{x}\hat{x}} + \tilde{c}u^N = \tilde{f} \quad \text{on } \omega, \; u_0^N = \gamma_0, \; u_N^N = \gamma_1.$$

This scheme was analysed by Boglaev and Pack [22]. They establish uniform convergence of first order. Roos and Zarin [142] consider the scheme with $\tilde{w}_i = (w(x_i - 0) + w(x_i + 0))/2$, but in the critical point x_κ they have $h_\kappa = h_{\kappa+1}$. In that article uniform second order convergence is proved for Shishkin meshes and for Bakhvalov-Shishkin meshes.

Using the derivative bounds derived in Sect. 3.3.1.3, the analysis of Sect. 6.1.2.2 needs only minor modifications to get the pointwise error bound

$$\left\| u - u^N \right\|_\infty \leq C \left(\vartheta^{[2]}_{rd^i}(\bar{\omega}) \right)^2,$$

where $\vartheta^{[p]}_{rd^i}(\bar{\omega})$ has been defined in Sect. 2.2.1.

6.1.3 A Non-Monotone Scheme

In this section we shall present maximum-norm error bounds for a FEM applied to (6.1). We consider FEM-2. It generates the difference scheme

$$\begin{aligned} \left[Lu^N\right]_i &:= -\varepsilon^2 u^N_{\bar{x}\hat{x};i} + \widehat{cu^N}_i = \widehat{f}_i \qquad \text{for } i = 1, \dots, N-1, \\ u_0^N &= \gamma_0, \; u_N^N = \gamma_1, \end{aligned} \tag{6.31}$$

where

$$\widehat{w}_i := \frac{h_i}{\hbar_i}\frac{w_{i-1}}{6} + \frac{2w_i}{3} + \frac{h_{i+1}}{\hbar_i}\frac{w_{i+1}}{6}.$$

We see that the discretisation of the reaction term cu generates positive off-diagonal entries in the system matrix. This results in a scheme that is—unlike the central difference scheme studied before—not inverse monotone. Nonetheless, a maximum-norm error analysis can be conducted. We follow [97].

6.1.3.1 Stability

Although L is not inverse-monotone, it possesses a core that is:

$$[\Lambda v]_i := -\varepsilon^2 v_{\bar{x}\hat{x};i} + \frac{2c_i}{3} v_i.$$

This is the standard central finite difference approximation of $-\varepsilon^2 u'' + \frac{2}{3}cu$ and can be generated by means of the bilinear form

$$\varepsilon^2 (w', v') + \frac{2}{3}\int_0^1 (cwv)^I (s)\mathrm{d}s.$$

By (6.14) we have

$$\|v\|_{\infty,\bar{\omega}} \le \frac{3}{2}\left\|\frac{\Lambda v}{c}\right\|_{\infty,\omega} \quad \text{for all } v \in I\!R_0^{N+1}. \tag{6.32}$$

Theorem 6.16. *Suppose $c \in C^{0,\alpha}[0,1]$. Let $\kappa \in (0,1)$ be arbitrary, but fixed. Then*

$$\|v\|_{\infty,\bar{\omega}} \le \frac{3}{1-\kappa}\left\|\frac{Lv}{c}\right\|_{\infty,\omega} \quad \textit{for all } v \in I\!R_0^{N+1},$$

provided that the maximum step size h is smaller than some threshold value that is independent of ε.

Remark 6.17. Theorem 6.16 means the non-monotone scheme (6.31) is $(\ell_\infty, \ell_\infty)$-stable although the underlying operator L is not inverse monotone and does not satisfy a maximum principle. ♣

Proof. Let $v \in I\!R_0^{N+1}$ be an arbitrary mesh function. Then

$$[\Lambda v]_i = -\frac{h_i}{\hbar_i}\frac{c_{i-1}}{6}v_{i-1} - \frac{h_{i+1}}{\hbar_i}\frac{c_{i+1}}{6}v_{i+1} + [Lv]_i\,.$$

Thus,

$$|[\Lambda v]_i| \le \frac{h_i c_{i-1} + h_{i+1}c_{i+1}}{6\hbar_i}\|v\|_{\infty,\bar{\omega}} + |[Lv]_i| \quad \text{for } i = 1, \dots, N-1,$$

and the stability inequality (6.32) yields

$$\|v\|_{\infty,\bar{\omega}} \le \frac{3}{2}\max_{i=1,\dots,N-1}\frac{h_i c_{i-1} + h_{i+1}c_{i+1}}{6c_i\hbar_i}\|v\|_{\infty,\bar{\omega}} + \frac{3}{2}\left\|\frac{Lv}{c}\right\|_{\infty,\omega}. \tag{6.33}$$

Since $c \in C^{0,\alpha}[0,1]$, there exists a constant M with

$$|c(x) - c(\xi)| \le M\,|x-\xi|^\alpha \quad \text{for all } x, \xi \in [0,1]. \tag{6.34}$$

Therefore,

$$\frac{h_i c_{i-1} + h_{i+1} c_{i+1}}{6 c_i \hbar_i} \le \frac{1}{3} + \frac{M h^\alpha}{3\rho^2} \le \frac{1+\kappa}{3},$$

provided h is smaller than some threshold value that is independent of ε. Now, the proposition follows from (6.33). □

For our convergence analysis, we shall also require bounds on the discrete Green's function G_i associated with L and the mesh node x_i. With Theorem 6.10 we have

$$\int_0^1 (cG_i)^I(s)\mathrm{d}s = \sum_{j=1}^{N-1} \hbar_j c_j G_{i,j} \le \frac{3}{2} \quad \text{and} \quad \int_0^1 G_i(s)\mathrm{d}s \le \frac{3}{2\rho^2}\,. \tag{6.35}$$

6.1.3.2 A Priori Error Analysis

Theorem 6.18. *Let u be the solution of* (6.1) *and u^N that of* (6.31). *Then*

$$\left\|u - u^N\right\|_\infty \le C\left(\vartheta_{rd}^{[2]}(\bar\omega)\right)^2,$$

provided that h is smaller than some threshold value that is independent of ε.

Proof. By a triangle inequality

$$\left\|u - u^N\right\|_\infty \le \left\|u - u^I\right\|_\infty + \left\|u - u^N\right\|_{\infty,\bar\omega}.$$

The interpolation error was studied in Sect. 6.1.1.1.

Let $\eta = u^N - u$ and $q := f - cu$. Then the Green's-function representation and Eqs. (6.1) and (6.31) yield—after some calculations—

$$\eta_i = \left(q^I - q, G_i\right) - \left((c\eta)^I, G_i\right) + \frac{2}{3}\int_0^1 (ceG_i)^I \tag{6.36}$$

where we used $\left((u^I)', G_i'\right) = (u', G_i')$, because G_i' is piecewise constant. The first term on the right-hand side of (6.36) is bounded using a Hölder inequality, Theorem 6.2 and (6.35):

$$\left|\left(q^I - q, G_i\right)\right| \le C\left(\vartheta_{rd}^{[2]}(\bar\omega)\right)^2.$$

For the remaining terms in (6.36) a straight-forward calculation gives

$$\frac{2}{3}\int_0^1 (c\eta G_i)^I - \left((c\eta)^I, G_i\right) = -\frac{1}{3}\sum \frac{h_k}{2}\left(c_k\eta_k G_{i,k-1} + c_{k-1}\eta_{k-1}G_{i,k}\right).$$

Let $\kappa \in (0,1)$ be arbitrary, but fixed. Then, using (6.34) and (6.35), we can estimate as follows:

$$\left|\frac{2}{3}\int_0^1 (c\eta G_i)^I - \left((c\eta)^I, G_i\right)\right| \le \left(\frac{1}{2} + \frac{Mh^\alpha}{2\rho^2}\right)\|\eta\|_{\infty,\bar{\omega}} \le \frac{1+\kappa}{2}\|\eta\|_{\infty,\bar{\omega}} .$$

Thus,

$$|\eta_i| \le C\left(\vartheta_{rd}^{[2]}(\bar{\omega})\right)^2 + \frac{1+\kappa}{2}\|\eta\|_{\infty,\bar{\omega}} .$$

Taking the maximum over $i = 1, \ldots, N-1$, we get the general error bound of the theorem. □

Remark 6.19. In contrast to the analysis for central differencing, only bounds for the L_1 norm of the Green's function have been used, but no bounds on its derivative. Also no third-order derivative of u is required. Only the second-order derivative is used when Theorem 6.2 is invoked. ♣

Remark 6.20. The proof is easily adapted to deal with discontinuities in the right-hand side or in the reaction coefficient. ♣

6.1.3.3 A Posteriori Error Analysis

The analysis in [97] is along the lines of the analysis for central differencing in Sect. 6.1.2.3.

Set $\Gamma = \mathcal{G}(x,\cdot)$ and $\hat{q} := cu^N - f$. Then

$$\left(u - u^N\right)(x) = \left(\hat{q}^I - \hat{q}, \Gamma\right) + \left(\hat{q}^I, \Gamma^I - \Gamma\right) . \tag{6.37}$$

Note that compared with (6.23) the term

$$\int_0^1 \left((\hat{q}\Gamma)^I - \hat{q}^I\Gamma^I\right)(s)\mathrm{d}s$$

does not appear in (6.37).

Both terms on the right-hand side of (6.37) have been bounded in Sect. 6.1.2.3.

Theorem 6.21. *Let u be the solution of the reaction-diffusion problem* (6.1) *and u^N its approximation by* (6.31). *Let $\hat{q} := cu^N - f$. Then*

$$\left\|u - u^N\right\|_\infty \le \frac{1}{\rho^2}\left\|\hat{q} - \hat{q}^I\right\|_\infty + \max_{k=0,\ldots,N-1} \frac{h_{k+1}^2}{4\varepsilon^2}\max\{|\hat{q}_k|, |\hat{q}_{k+1}|\} .$$

Remark 6.22. Compared with central differencing, we see the term

$$\max_{k=0,\ldots,N-1} \frac{h_{k+1}}{6\varepsilon\rho} \left|\hat{q}_{k+1} - \hat{q}_k\right|$$

does not feature in the a posteriori estimate for the non-monotone scheme. By Remark 6.15 this term corresponds to a discrete third-order derivative of u^N.

Also note that in the analysis, no bounds on the derivative of the Green's function are required. ♣

6.1.4 A Compact Fourth-Order Scheme

In this section we consider a compact finite difference scheme of order four. Given an arbitrary mesh $\bar{\omega}$ we seek a mesh function $u^N \in \mathbb{R}^{N+1}$ satisfying

$$\begin{aligned}\left[Lu^N\right]_i &:= c_i^l u_{i-1}^N + c_i^c u_i^N + c_i^r u_{i+1}^N \\ &= q_i^l f_{i-1} + q_i^c f_i + q_i^r f_{i+1} =: \left[Qf\right]_i, \quad i = 1, \ldots, N-1.\end{aligned}$$

The coefficients c and q are determined so that the scheme is exact for polynomials up to degree four. That is $\left[Lp\right]_i = \left[Q(\mathcal{L}p)\right]_i$ for all $p \in \Pi_4$. This construction yields a difference scheme whose system matrix is not inverse monotone.

A similar approach was used in [27] where in order to ensure inverse monotonicity, the high-order approximation is used only when the local mesh size is small enough to give non-positive off-diagonal entries. In all other mesh points central differencing is used. This hybrid method is shown to be third-order convergent uniformly in ε on a Shishkin mesh; see [27, § 2]. An alternative approach to obtain a higher-order difference approximation while maintaining the M-matrix property can be found in [46]. However, the construction of that scheme requires explicit knowledge of the derivatives of the data (c and f) and subtle modifications in those points where the mesh is non-uniform.

We shall follow the analysis in [98] and see that inverse monotonicity is not a prerequisite for the maximum-norm error analysis of higher-order schemes.

6.1.4.1 Discretisation

The exactness of the scheme for polynomials up to degree four and the normalisation condition $q_{l;i} + q_{c;i} + q_{r;i} = 1$, $i = 1, \ldots, N-1$, yield the difference scheme: Find $u^N \in \mathbb{R}^{N+1}$ such that

$$\begin{aligned}\left[Lu^N\right]_i := -\varepsilon^2 u_{\bar{x}\hat{x},i}^N + \left[Q(cu^N)\right]_i = \left[Qf\right]_i \quad \text{for } i = 1, \ldots, N-1, \\ u_0^N = \gamma_0, \;\; u_N^N = \gamma_1,\end{aligned} \tag{6.38}$$

where

$$[Qv]_i := \frac{1-\mu_i^-}{6} v_{i-1} + \frac{4+\mu_i^- + \mu_i^+}{6} v_i + \frac{1-\mu_i^+}{6} v_{i+1}$$

with

$$\mu_i^- := \frac{h_{i+1}^2}{2h_i \hbar_i} \quad \text{and} \quad \mu_i^+ := \frac{h_i^2}{2h_{i+1}\hbar_i} \ .$$

6.1.4.2 Stability

For the stability analysis we consider an arbitrary mesh $\bar{\omega}$ with maximal step size h. Although L is not inverse-monotone, it possesses a core that is:

$$[\Lambda v]_i := -\varepsilon^2 \left[\delta_x^2 v\right]_i + \frac{\alpha_i^-}{6} r_i v_{i-1} + \frac{4+\mu_i^- + \mu_i^+}{6} r_i v_i + \frac{\alpha_i^+}{6} r_i v_{i+1}$$

where

$$\alpha_i^- = \min\left\{0, 1-\mu_i^-\right\}, \quad \alpha_i^+ = \min\left\{0, 1-\mu_i^+\right\}.$$

The matrix associated with Λ is an L_0 matrix with row sums $\beta_i/6$, where

$$\beta_i := 4 + \alpha_i^- + \alpha_i^+ + \mu_i^- + \mu_i^+.$$

Therefore, it is an M-matrix and we can conclude that

$$\|v\|_{\bar{\omega}} \le \max_{i=1,\dots,N-1} \left| \frac{6\,[\Lambda v]_i}{c_i \beta_i} \right| \quad \text{for all } v \in I\!R_0^{N+1}, \tag{6.39}$$

by Lemma 3.17.

Theorem 6.23. *Suppose $c \in C^1[0,1]$. Let $\kappa \in (0,1)$ be arbitrary, but fixed. Then*

$$\|v\|_{\bar{\omega}} \le \frac{3}{2-\kappa} \left\| \frac{Lv}{c} \right\|_{\omega} \quad \textit{for all } \ v \in I\!R_0^{N+1} \ ,$$

provided that h is smaller than some threshold value that depends on κ and c only.

Proof. First, note that

$$\mu_i^- + \mu_i^+ = \frac{h_{i+1}^3 + h_i^3}{h_i h_{i+1}(h_i + h_{i+1})} \ge 1, \tag{6.40}$$

by the relation between cubic, arithmetic and geometric means.

Furthermore,

$$h_{i+1} \geq h_i \implies \alpha_i^+ = 0, \qquad \text{and} \qquad h_{i+1} \leq h_i \implies \alpha_i^- = 0.$$

Therefore, at least one of α_i^- and α_i^+ is zero. Without loss of generality we assume $\alpha_i^- = 0$. This implies $0 \leq 1 - \mu_i^- \leq 1$.

We distinguish two cases $\alpha_i^+ = 0$ and $\alpha_i^+ = 1 - \mu_i^+$.

(i) Suppose $0 = \alpha_i^+$. Then $0 \leq 1 - \mu_i^+ \leq 1$ and

$$[\Lambda v]_i = [Lv]_i - \frac{1-\mu_i^-}{6} c_i v_{i-1} - \frac{1-\mu_i^+}{6} c_i v_{i+1}$$
$$- \frac{1-\mu_i^-}{6} \left(c_{i-1} - c_i\right) v_{i-1} - \frac{1-\mu_i^+}{6} \left(c_{i+1} - c_i\right) v_{i+1}.$$

We estimate as follows:

$$\left|\left(1-\mu_i^-\right)\left(c_{i-1} - c_i\right)\right| \leq h_i \|c'\|_\infty \leq \kappa c_i \ .$$

Similarly,

$$\left|\left(1-\mu_i^+\right)\left(c_{i+1} - c_i\right)\right| \leq h_{i+1} \|c'\|_\infty \leq \kappa c_i \ ,$$

provided h is sufficiently small—depending on κ and c only. We also have

$$\left|1-\mu_i^-\right| + \left|1-\mu_i^+\right| = 2 - \mu_i^- - \mu_i^+ \leq 1, \quad \text{by (6.40)}.$$

Therefore,

$$\left|[\Lambda v]_i\right| \leq \left|[Lv]_i\right| + \frac{1+2\kappa}{6} \|v\|_{\bar\omega} \, . \tag{6.41}$$

Note that $\beta_i = 4 + \mu_i^- + \mu_i^+ \geq 5$, by (6.40). Hence,

$$\left|\frac{6\,[\Lambda v]_i}{c_i \beta_i}\right| \leq \frac{6}{5} \left|\frac{[L_\varepsilon v]_i}{c_i}\right| + \frac{1+2\kappa}{5} \|v\|_{\bar\omega} \, . \tag{6.42}$$

(ii) If $\alpha_i^+ = 1 - \mu_i^+ \leq 0$ then

$$[\Lambda v]_i = [Lv]_i - \frac{1-\mu_i^-}{6} c_i v_{i-1} - \frac{1-\mu_i^-}{6} \left(c_{i-1} - c_i\right) v_{i-1}$$
$$- \frac{1-\mu_i^+}{6} \left(c_{i+1} - c_i\right) v_{i+1} \ .$$

The second and third term on the right-hand side are bounded as in case (i). For the last term note that $1-\mu_i^+ \le 0$ yields $\left|1-\mu_i^+\right| \le \mu_i^+$ and therefore,

$$\left|\left(1-\mu_i^+\right)(c_{i+1}-c_i)\right| \le \mu_i^+ h_{i+1}\|c'\|_\infty \le h_i\|c'\|_\infty \le \kappa c_i,$$

for sufficiently small h. Thus, (6.41) holds for this case too. Furthermore, for β_i we have $\beta_i = 5+\mu_i^- \ge 5$. Consequently, (6.42) holds for all $i=1,\ldots,N-1$.

Combining (6.39) with (6.42), we are finished. □

Remark 6.24. The discretisation (6.38) is $(\ell_\infty,\ell_\infty)$-stable although the underlying operator is in general not inverse monotone and therefore does not satisfy a maximum principle. ♣

Remark 6.25. The argument presented here sharpens Theorem 1 in [98] by giving a smaller stability constant. ♣

6.1.4.3 Nodal Error Analysis

We shall apply the difference scheme (6.38) on a Shishkin mesh with $\sigma \ge 4$. For this we have $h \le N^{-1}/(1-2q)$.

Let $\eta = u - u^N$ denote the error of the scheme. We start our analysis by decomposing the error of the scheme as $\eta = \psi + \varphi$, where the two parts $\psi, \varphi \in \mathbb{R}_0^{N+1}$ solve

$$[\Lambda\psi]_i = [L\eta]_i = \varepsilon^2\left[Q(u'') - u_{\bar{x}\hat{x}}\right]_i \quad \text{on } \omega,$$

and

$$\begin{aligned}[\Lambda\varphi]_i = &-\frac{c_{i-1}}{6}\eta_{i-1} - \frac{c_{i+1}}{6}\eta_{i+1} \\ &+ \frac{\mu_i^-}{6}(c_{i-1}-c_i)\,\eta_{i-1} + \frac{\mu_i^+}{6}(c_{i+1}-c_i)\,\eta_{i+1} \quad \text{on } \omega.\end{aligned}$$

Let $\kappa \in (0,1)$ be arbitrary, but fixed. Then using arguments from our stability analysis, we get

$$\left|\frac{6\,[\Lambda\varphi]_i}{\beta_i c_i}\right| \le \frac{1+2\kappa}{5}\,\|\eta\|_{\bar{\omega}} \quad \text{for } i=1,\ldots,N-1,$$

if N is greater than some threshold value that is independent of ε. The stability inequality (6.39) yields

$$\|\varphi\|_\omega \le \frac{1+2\kappa}{5}\,\|\eta\|_{\bar{\omega}}\,.$$

Application of a triangle inequality gives

$$\|\eta\|_{\bar\omega} \le \|\psi\|_{\bar\omega} + \|\varphi\|_{\bar\omega} \le \|\psi\|_{\bar\omega} + \frac{1+2\kappa}{5}\,\|\eta\|_{\bar\omega}\,.$$

Hence, the error can be bounded in terms of ψ:

$$\left\|u-u^N\right\|_{\bar\omega} \le \frac{5}{4-2\kappa}\,\|\psi\|_{\bar\omega}\,. \tag{6.43}$$

We are left with estimating ψ which will be done using a truncation error and barrier function technique.

Let $g \in C^6[x_{i-1},x_{i+1}]$. Then Taylor expansions give

$$\left|[Q(g'') - g_{\bar x\hat x}]_i\right| \le \begin{cases} C\,\|g''\|_{[x_{i-1},x_{i+1}]} & \text{if } h_i = h_{i+1},\\ C\,\|g''\|_{[x_i,x_{i+1}]} + C\mu_i^- h_i\,\|g'''\|_{[x_{i-1},x_i]} & \text{if } h_i \le h_{i+1},\\ C\,(h_i+h_{i+1})^3\,\left\|g^{(5)}\right\|_{[x_{i-1},x_{i+1}]} & \text{and}\\ Ch_i^4\,\left\|g^{(6)}\right\|_{[x_{i-1},x_{i+1}]} & \text{if } h_i = h_{i+1}. \end{cases} \tag{6.44}$$

We consider the two distinct cases for the mesh transition point: $\tau = q$ and $\tau < q$. In the first case, the mesh is uniform with $h_i = 1/N$ for $i = 1,\dots,N$. Moreover, $\varepsilon^{-1} \le C\ln N$. Thus, $\left\|u^{(6)}\right\| \le C\varepsilon^{-2}\ln^4 N$, by Theorem 3.35. Now the fourth bound of (6.44) yields

$$\left|[\Lambda\psi]_i\right| = \varepsilon^2\left|[Q(u'') - u_{\bar x\hat x}]_i\right| \le CN^{-4}\ln^4 N, \quad i = 1,\dots,N-1.$$

Hence,

$$\|\psi\|_{\bar\omega} \le CN^{-4}\ln^4 N \quad \text{if } \tau = q, \tag{6.45}$$

by (6.39).

The case when $\tau < q$ requires a more detailed argument employing the decomposition of u.

(i) For $x_i \in (0,\tau)\cup(1-\tau,1)$, use the fourth bound of (6.44), $\left\|u^{(6)}\right\| \le C\varepsilon^{-6}$ and $h_i = h_{i+1} \le C\varepsilon N^{-1}\ln N$ to get

$$\left|[\Lambda\psi]_i\right| = \varepsilon^2\left|[Q(u'') - u_{\bar x\hat x}]_i\right| \le CN^{-4}\ln^4 N \quad \text{for } x_i \in (0,\tau)\cup(1-\tau,1).$$

(ii) For $x_i \in (\tau, 1-\tau)$, split the truncation error according to the decomposition of u:

$$\varepsilon^2\left|[Q(u'') - u_{\bar x\hat x}]_i\right| = \varepsilon^2\left|[Q(v'') - v_{\bar x\hat x}]_i\right| + \varepsilon^2\left|[Q(w'') - w_{\bar x\hat x}]_i\right|.$$

The first term is bounded using the fourth estimate of (6.44), $\varepsilon^2 \left\| v^{(6)} \right\| \le C$ and $h_i = h_{i+1} \le CN^{-1}$. We obtain $\varepsilon^2 \left|[Q(v'') - v_{\bar{x}\hat{x}}]_i\right| \le CN^{-4}$. In order to bound the truncation error with respect to the layer part w, use the first estimate of (6.44) to get $\varepsilon^2 \left|[Q(w'') - w_{\bar{x}\hat{x}}]_i\right| \le CN^{-\sigma}$. Hence,

$$\left|[\Lambda\psi]_i\right| \le CN^{-4} \quad \text{for } \; x_i \in (\tau, 1-\tau).$$

(iii) For $x_i \in \{\tau, 1-\tau\}$, split the truncation error again. For the regular solution component v the third estimate of (6.44) gives

$$\varepsilon^2 \left|[Q(v'') - v_{\bar{x}\hat{x}}]_i\right| \le C\varepsilon N^{-3},$$

while for w we have by the second bound of (6.44)

$$\begin{aligned} \varepsilon^2 \left|[Q(w'') - w_{\bar{x}\hat{x}}]_{qN}\right| &\le C\left(\mathrm{e}^{-\rho x_{qN}/\varepsilon} + \mu_{qN}^- N^{-1} \ln N \mathrm{e}^{-\rho(1-x_{(1-q)N+1})/\varepsilon}\right) \\ &\le CN^{-\sigma} + CN^{\sigma/qN} N^{-\sigma-1} \mu_{qN}^- \ln N \\ &\le CN^{-\sigma} + CN^{-\sigma-1} \mu_{qN}^- \ln N, \end{aligned}$$

with an analogous estimate for $i = (1-q)N$. Collecting the various bounds for the truncation error, we get (for $\tau < q$)

$$\begin{aligned} &\varepsilon^2 \left|[Q(u'') - u_{\bar{x}\hat{x}}]_i\right| \\ &\quad \le CN^{-4}\ln^4 N + \begin{cases} C\varepsilon N^{-3} + CN^{-5}\mu_i^- \ln N & \text{if } i = qN, \\ C\varepsilon N^{-3} + CN^{-5}\mu_i^+ \ln N & \text{if } i = (1-q)N, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.46}$$

Finally, we use an idea from [122]. Define the mesh function $z \in I\!R_0^{N+1}$ by

$$z_i := \begin{cases} x_i/\tau & \text{for } \; i = 0, \ldots, qN, \\ 1 & \text{for } \; i = qN, \ldots, (1-q)N, \; \text{ and} \\ (1-x_i)/\tau & \text{for } \; i = (1-q)N, \ldots, N. \end{cases}$$

A direct calculation verifies

$$[\Lambda z]_i = \frac{2c_i}{3} z_i + \frac{1}{qN} \begin{cases} \frac{\varepsilon^2}{h_i \hbar_i} + \frac{\mu_i^- r_i}{6} & \text{if } \; i = qN, \\ \frac{\varepsilon^2}{h_{i+1}\hbar_i} + \frac{\mu_i^+ r_i}{6} & \text{if } \; i = (1-q)N \; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\frac{\varepsilon^2}{h_{qN}\hbar_{qN}}, \frac{\varepsilon^2}{h_{(1-q)N+1}\hbar_{(1-q)N}} \geq \frac{\varepsilon\rho(1-2q)N}{2\sigma\ln N}.$$

Therefore, recalling the truncation error bound (6.46), we can use the discrete comparison principle (6.13) with the barrier function $CN^{-4}\left(\ln^4 N + z_i \ln N\right)$ to see that (6.45) holds for $\tau < q$ too.

Finally, (6.43) yields the main convergence result of this section.

Theorem 6.26. *Let u be the solution of the boundary value problem* (6.1) *and u^N that of the finite difference scheme* (6.38) *on a Shishkin mesh with $\sigma \geq 4$. Then*

$$\left\|u - u^N\right\|_{\infty,\bar{\omega}} \leq CN^{-4}\ln^4 N,$$

provided that N is larger than some threshold value that depends on c only.

Corollary 6.27. *Clustering three adjacent and equidistant mesh intervals and fitting a cubic function through the numerical approximation on the four associated mesh points, we obtain a cubic C^0-spline $\mathcal{S}u^N$ that approximates u on the whole domain:*

$$\left\|u - \mathcal{S}u^N\right\|_{\infty} \leq CN^{-4}\ln^4 N.$$

Remark 6.28. Approximations of the derivatives can be obtained by differentiating $\mathcal{S}u^N$:

$$\varepsilon^k\left\|(u - \mathcal{S}u^N)^{(k)}\right\| \leq C\left(N^{-1}\ln N\right)^{(4-k)} \quad \text{for } k = 1, 2, 3.$$

Note that in those mesh points where two different cubic functions are concatenated to give $\mathcal{S}u^N$, we have different left- and right-sided derivatives, however for both, the above bound holds.

Better approximations of the second-order derivative are obtained by appealing to (6.1):

$$u''(x) \approx m(x) := \varepsilon^{-2}\left(c\mathcal{S}u^N - f\right)(x).$$

We have the error bound

$$\varepsilon^2\left\|u'' - m\right\| \leq CN^{-4}\ln^4 N.$$

This readily follows from Theorem 6.26. ♣

6.1.4.4 Numerical Results

To illustrate the theoretical results, we consider the problem

$$-\varepsilon^2 u'' + (1 + x^2 + \cos x)u = x^{9/2} + \sin x, \quad \text{in } (0,1), \tag{6.47}$$

with homogeneous Dirichlet boundary conditions.

The exact solution is not available. Therefore, we estimate the error for u^N and $\mathcal{S}u^N$ by comparing them to the numerical solution $\tilde{u}^N$ obtained on a mesh $\tilde{\omega}$ that is seven times as fine. More precisely $\tilde{\omega}$ is constructed from ω by uniformly dividing each of its mesh intervals into seven equidistant subintervals. We take the estimates

$$\left\|u - u^N\right\|_{\infty,\omega} \approx \eta^N := \left\|\tilde{u}^N - u^N\right\|_{\infty,\omega}$$

and

$$\left\|u - \mathcal{S}u^N\right\|_{\infty} \approx \tilde{\eta}^N := \left\|\tilde{u}^N - \mathcal{S}u^N\right\|_{\infty,\tilde{\omega}}.$$

Since we have an error bound of the form $C\left(N^{-1}\ln N\right)^p$, we also compute the "Shishkin" rates of convergence:

$$p^N = \frac{\ln \eta^N - \ln \eta^{2N}}{\ln(2\ln N) - \ln(\ln(2N))}.$$

We choose N divisible by three and define $\mathcal{S}u^N$ on macro intervals $[x_{3k}, x_{3(k+1)}]$, $k = 0, \ldots, N/3 - 1$.

The left half of Table 6.1 displays the results of our test computations for the Shishkin mesh. They are in good agreement with Theorem 6.26 and Corollary 6.27. The right half of the table contains results for a modified Bakhvalov mesh: First

Table 6.1 Compact fourth order scheme for (6.47), $\varepsilon = 10^{-4}$ (identical numbers for $\varepsilon = 10^{-4k}$, $k = 2, \ldots, 6$)

	Shishkin mesh				Bakhvalov mesh			
N	η^N	p^N	$\tilde{\eta}^N$	$\tilde{p}^N$	η^N	π^N	$\tilde{\eta}^N$	$\tilde{\pi}^N$
$3 \cdot 2^7$	1.151e-05	3.99	3.979e-04	3.66	2.644e-08	4.01	4.915e-07	3.99
$3 \cdot 2^8$	1.123e-06	4.00	4.701e-05	3.81	1.641e-09	4.00	3.090e-08	4.00
$3 \cdot 2^9$	1.045e-07	4.00	4.890e-06	3.89	1.024e-10	4.00	1.936e-09	4.00
$3 \cdot 2^{10}$	9.375e-09	4.00	4.672e-07	3.94	6.394e-12	4.00	1.212e-10	4.00
$3 \cdot 2^{11}$	8.160e-10	4.00	4.212e-08	3.97	3.996e-13	4.00	7.580e-12	4.00
$3 \cdot 2^{12}$	6.925e-11	4.00	3.644e-09	3.98	2.497e-14	4.00	4.739e-13	4.00
$3 \cdot 2^{13}$	5.750e-12	4.00	3.058e-10	3.99	1.561e-15	4.00	2.963e-14	4.00
$3 \cdot 2^{14}$	4.686e-13	4.00	2.506e-11	4.00	9.756e-17	4.00	1.852e-15	4.00
$3 \cdot 2^{15}$	3.756e-14	4.00	2.014e-12	4.00	6.097e-18	4.00	1.157e-16	4.00
$3 \cdot 2^{16}$	2.967e-15	—	1.594e-13	—	3.811e-19	—	7.234e-18	—

we construct a standard Bakhvalov mesh with $N/3$ mesh intervals, which are then subdivided into three subintervals of equal length. This gives our computational mesh ω. This modification is necessary because the stability constant of $\mathcal{S}$ depends on the local ratio of the mesh sizes, which on a Bakhvalov mesh depends on ε. For this mesh, we expect uniform convergence of order N^{-4}. This is clearly observed in the numerical experiments.

6.2 Systems of Reaction-Diffusion Type

We now leave the scalar problems and move on to systems of reaction-diffusion equations: Find $\boldsymbol{u} \in \left(C^2(0,1) \cap C[0,1]\right)^\ell$ such that

$$\boldsymbol{\mathcal{L}u} := -\boldsymbol{E}^2\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } (0,1), \quad \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}, \tag{6.48}$$

where $\boldsymbol{E} = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_\ell)$ and the small parameters ε_k are in $(0,1]$.

The analytical behaviour of the solution to (6.48) has been studied in Sect. 3.3.2. Again we assume the coupling matrix $\boldsymbol{A}$ has positive diagonal entries and is diagonally dominant with

$$\beta := \max_{k=1,\ldots,\ell} \sum_{\substack{m=1\\ m\neq k}}^{\ell} \left\| \frac{a_{km}}{a_{kk}} \right\|_\infty < 1. \tag{6.49}$$

Define $\kappa > 0$ by

$$\kappa^2 := (1-\beta) \min_{k=1,\ldots,\ell} \min_{x\in[0,1]} a_{kk}(x).$$

For simplicity in our presentation we assume that

$$\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_\ell \quad \text{and} \quad \varepsilon_1 \leq \frac{\kappa}{4}\,.$$

The first chain of inequalities can always be achieved by renumbering the equations, while the last inequality provides a threshold value for the validity of our analysis.

6.2.1 The Interpolation Error

We consider piecewise linear interpolation. Using the derivative bounds of Theorem 3.43 for the solution of (6.48), we can apply the technique in Section 6.1.1.1 to establish interpolation error bounds for the components of $\boldsymbol{u}$.

Theorem 6.29. *Suppose the assumptions of Theorem 3.43 hold true. Then*

$$\left\|u_k - u_k^I\right\|_0 \le \left\|u_k - u_k^I\right\|_\infty \le C\left(\vartheta_{rd,k}^{[2]}(\bar\omega)\right)^2, \quad k = 1,\dots,\ell,$$

and

$$\varepsilon_k \left|u_k - u_k^I\right|_1 \le C\varepsilon_k^{1/2}\vartheta_{rd,k}^{[2]}(\bar\omega), \quad k = 1,\dots,\ell,$$

where

$$\vartheta_{rd,k}^{[p]}(\bar\omega) := \max_{i=0,\dots,N-1}\int_{I_i}\left(1 + \sum_{m=1}^{k}\varepsilon_m^{-1}\left(\mathrm{e}^{-\kappa s/p\varepsilon_m} + \mathrm{e}^{-\kappa(1-s)/p\varepsilon_m}\right)\right)\mathrm{d}s$$

for $k = 1,\dots,\ell$.

6.2.2 Linear Finite Elements

In view of Lemma 3.40 we may assume without loss of generality that $\boldsymbol{A}$ is coercive, i.e., there exists a positive constant α such that

$$\boldsymbol{v}^T\boldsymbol{A}(x)\boldsymbol{v} \ge \alpha\boldsymbol{v}^T\boldsymbol{v} \quad \text{for all } x\in[0,1], \text{ and } \boldsymbol{v}\in I\!R^\ell. \tag{6.50}$$

As usual with finite element discretisations, we consider the weak formulation of (6.48): Find $\boldsymbol{u}\in H_0^1(0,1)^\ell$ such that

$$B(\boldsymbol{u},\boldsymbol{v}) = F(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v}\in H_0^1(0,1)^\ell,$$

with

$$B(\boldsymbol{u},\boldsymbol{v}) := \sum_{m=1}^{\ell}\varepsilon_m^2(u_m', v_m') + \sum_{m=1}^{\ell}\sum_{k=1}^{\ell}(a_{mk}u_k, v_m)$$

and

$$F(\boldsymbol{v}) := \sum_{m=1}^{\ell}(f_m, v_m), \quad (w,v) = \int_0^1 \big(wv\big)(s)\mathrm{d}s.$$

The natural norm on $H_0^1(0,1)^\ell$ associated with the bilinear form $B(\cdot,\cdot)$ is the energy norm $|||\cdot|||_{\boldsymbol{\varepsilon}^2}$ defined by

$$|||\boldsymbol{v}|||_{\boldsymbol{\varepsilon}^2}^2 := \sum_{m=1}^{\ell}\varepsilon_m^2\left|v_m\right|_1^2 + \alpha\left\|\boldsymbol{v}\right\|_0^2, \quad \left\|\boldsymbol{v}\right\|_0^2 := \sum_{m=1}^{\ell}\left\|v_m\right\|_0^2.$$

Because of (6.50) the bilinear form $B(\cdot,\cdot)$ is coercive with respect to the energy norm, i.e.,

$$|||\boldsymbol{v}|||_{\boldsymbol{\varepsilon}^2}^2 \le B(\boldsymbol{v},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H_0^1(0,1)^\ell .$$

If $\boldsymbol{f} \in L^2(0,1)^\ell$ then F is a linear continuous functional on $H_0^1(0,1)^\ell$. Therefore, the Lax-Milgram Lemma ensures the existence of a unique solution $\boldsymbol{u} \in H_0^1(0,1)^\ell$ of the variational formulation of (6.48).

Given a mesh $\bar{\omega}$, we seek an approximation $\boldsymbol{u}^N \in \left(V_0^\omega\right)^\ell$ such that

$$\hat{B}(\boldsymbol{u}^N,\boldsymbol{v}) = \hat{F}(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \left(V_0^\omega\right)^\ell , \tag{6.51}$$

where

$$\hat{B}(\boldsymbol{u},\boldsymbol{v}) := \sum_{m=1}^{\ell} \varepsilon_m^2 (u_m', v_m') + \sum_{m=1}^{\ell}\sum_{k=1}^{\ell} (a_{mk}^I u_k, v_m)$$

and

$$\hat{F}(\boldsymbol{v}) := \sum_{m=1}^{\ell} (f_m^I, v_m).$$

Thus, our FEM incorporates quadrature to approximate the integrals.

The coercivity of $\boldsymbol{A}$, i.e. (6.50), implies the coercivity of its piecewise linear interpolant $\boldsymbol{A}^I$:

$$\boldsymbol{v}^T \boldsymbol{A}^I(x)\boldsymbol{v} \ge \alpha \boldsymbol{v}^T\boldsymbol{v} \quad \text{for all } x \in [0,1], \text{ and } \boldsymbol{v} \in \mathbb{R}^\ell .$$

Consequently, the bilinear form $\hat{B}(\cdot,\cdot)$ is also coercive with

$$|||\boldsymbol{v}|||_{\boldsymbol{\varepsilon}^2}^2 \le \hat{B}(\boldsymbol{v},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in H_0^1(0,1)^\ell . \tag{6.52}$$

Set $\boldsymbol{\eta} := \boldsymbol{u}^I - \boldsymbol{u}$ and $\boldsymbol{\chi} := \boldsymbol{u}^I - \boldsymbol{u}^N$. Then by a triangle inequality

$$\left|\left|\left|\boldsymbol{u} - \boldsymbol{u}^N\right|\right|\right|_{\boldsymbol{\varepsilon}^2} \le |||\boldsymbol{\eta}|||_{\boldsymbol{\varepsilon}^2} + |||\boldsymbol{\chi}|||_{\boldsymbol{\varepsilon}^2} .$$

Theorem 6.29 yields

$$|||\boldsymbol{\eta}|||_{\boldsymbol{\varepsilon}^2} \le C\left(\varepsilon_1^{1/2} + \vartheta_{rd,\ell}^{[2]}(\omega)\right)\vartheta_{rd,\ell}^{[2]}(\omega),$$

and we are left with bounding $|||\boldsymbol{\chi}|||_{\boldsymbol{\varepsilon}^2}$.

Starting from (6.52), we get

$$|||\boldsymbol{\chi}|||_{\boldsymbol{\varepsilon}^2}^2 \le \hat{B}(\chi,\chi) = \sum_{m=1}^{\ell}\left(\sum_{k=1}^{\ell}\left(a_{mk}^I u_k^I - a_{mk}u_k\right) + f_m - f_m^I, \chi_m\right).$$

Use

$$a_{mk}^I u_k^I - a_{mk} u_k = \left(a_{mk}^I - a_{mk}\right) u_k^I + a_{mk}\left(u_k^I - u_k\right),$$

the Cauchy-Schwarz inequality and Theorem 6.29 to get

$$|||\boldsymbol{\chi}|||_{\boldsymbol{\varepsilon}^2}^2 \le C\left(\varepsilon_1^{1/2} + \vartheta_{rd,\ell}^{[2]}(\omega)\right)\vartheta_{rd,\ell}^{[2]}(\omega)\,\|\boldsymbol{\chi}\|_0 .$$

We obtain the following convergence results in the energy norm.

Theorem 6.30. *Let $\boldsymbol{u}$ be the solution of* (6.48) *and $\boldsymbol{u}^N$ its approximation by* (6.51). *Then*

$$\left|\left|\left|\boldsymbol{u}^I - \boldsymbol{u}^N\right|\right|\right|_{\boldsymbol{\varepsilon}^2} \le C\left(\vartheta_{rd,\ell}^{[2]}(\bar{\omega})\right)^2,$$

and

$$\left|\left|\left|\boldsymbol{u} - \boldsymbol{u}^N\right|\right|\right|_{\boldsymbol{\varepsilon}^2} \le C\left(\varepsilon_1^{1/2} + \vartheta_{rd,\ell}^{[2]}(\bar{\omega})\right)\vartheta_{rd,\ell}^{[2]}(\bar{\omega}).$$

Remark 6.31. A similar result is given in [99], but there the effect of numerical integration is not taken into account. ♣

Remark 6.32. Like in the scalar case, the energy norm $|||\cdot|||_{\boldsymbol{\varepsilon}^2}$ fails to capture the layers present in the solution. ♣

6.2.3 Central Differencing

We consider the discretisation of (6.48) by standard central differencing on meshes $\bar{\omega}$ that for the moment are arbitrary. That is, we seek $\boldsymbol{u}^N \in \left(I\!R_0^{N+1}\right)^\ell$ such that

$$\begin{aligned} \left[\boldsymbol{L}\boldsymbol{u}^N\right]_i &:= -\operatorname{diag}(\boldsymbol{E})^2 \boldsymbol{u}_{\bar{x}\hat{x};i}^N + \boldsymbol{A}_i \boldsymbol{u}_i^N = \boldsymbol{f}_i \quad \text{for } i = 1,\dots,N-1, \\ \boldsymbol{u}_0^N &= \boldsymbol{u}_N^N = \boldsymbol{0}. \end{aligned} \tag{6.53}$$

6.2.3.1 Stability

Our analysis follows that of [104] and is based on the stability properties of Section 6.1.2.1 for scalar operators.

Let $\boldsymbol{v} \in \left(I\!R_0^{N+1}\right)^\ell$ be arbitrary. Then

$$-\varepsilon_k^2 v_{k;\bar{x}\hat{x}} + a_{kk} v_k = (\boldsymbol{L}\boldsymbol{v})_k - \sum_{\substack{m=1 \\ m\neq k}}^{\ell} a_{km} v_m \quad \text{for } k = 1,\dots,\ell.$$

The stability inequality (6.14) and a triangle inequality then yield

$$\|v_k\|_{\infty,\bar{\omega}} - \sum_{\substack{m=1\\m\neq k}}^{\ell} \left\| \frac{a_{km}}{a_{kk}} \right\|_\infty \|u_m\|_{\infty,\omega} \leq \left\| \frac{(\boldsymbol{Lv})_k}{a_{kk}} \right\|_{\infty,\omega}.$$

Define the $\ell \times \ell$ constant matrix $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\boldsymbol{A}) = (\gamma_{km})$ by

$$\gamma_{kk} = 1 \quad \text{and} \quad \gamma_{km} = -\left\| \frac{a_{km}}{a_{kk}} \right\|_\infty \quad \text{for } k \neq m.$$

Suppose that $\boldsymbol{\Gamma}$ is inverse-monotone, i.e., that $\boldsymbol{\Gamma}$ is invertible and $\boldsymbol{\Gamma}^{-1} \geq 0$. Then we obtain the following stability result for the difference operator $\boldsymbol{L}$.

Theorem 6.33. *Assume the matrix $\boldsymbol{A}$ has positive diagonal entries. Assume that $\boldsymbol{\Gamma}(\boldsymbol{A})$ is inverse-monotone. Then for $k = 1, \ldots, \ell$ one has*

$$\|v_k\|_{\infty,\bar{\omega}} \leq \sum_{m=1}^{\ell} \left(\boldsymbol{\Gamma}^{-1}\right)_{km} \left\| \frac{(\boldsymbol{Lv})_m}{a_{mm}} \right\|_{\infty,\omega}$$

for any function mesh function $\boldsymbol{v} \in \left(\mathbb{R}_0^{N+1}\right)^\ell$.

Corollary 6.34. *Under the hypotheses of Theorem 6.33 the discrete problem (6.53) has a unique solution $\boldsymbol{u}^N$, and $\left\|\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \leq C \left\|\boldsymbol{f}\right\|_{\infty,\omega}$ for some constant C.*

Thus, the operator $\boldsymbol{L}$ is $(\ell_\infty, \ell_\infty)$ stable, or maximum-norm stable although in general it is not inverse-monotone.

6.2.3.2 A Priori Error Bounds

Let $\boldsymbol{\eta} := \boldsymbol{u} - \boldsymbol{u}^N$ denote the error of the discrete solution. We decompose the solution error as $\boldsymbol{\eta} = \boldsymbol{\varphi} + \boldsymbol{\psi}$, where the components φ_k and ψ_k of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ respectively are the solutions of

$$-\varepsilon_k^2 \varphi_{k;\bar{x}\hat{x}} + a_{kk}\varphi_k = -\varepsilon_k^2 \left(u_{k,\bar{x}\hat{x}} - u_k''\right) \quad \text{on } \omega, \quad \varphi_{k;0} = \varphi_{k;N} = 0$$

and

$$-\varepsilon_i^2 \psi_{k;\bar{x}\hat{x}} + a_{kk}\psi_k = -\sum_{\substack{m=1\\m\neq k}}^{\ell} a_{km}\eta_m \quad \text{on } \omega, \quad \psi_{k;0} = \psi_{k;N} = 0.$$

Assume that the matrix $\boldsymbol{\Gamma}(\boldsymbol{A})$ is inverse-monotone. Then for each k one has

$$\|\eta_i\|_{\infty,\bar{\omega}} \leq \|\varphi_i\|_{\infty,\bar{\omega}} + \|\psi_i\|_{\infty,\bar{\omega}} \leq \|\varphi_i\|_{\infty,\bar{\omega}} + \sum_{\substack{m=1\\m\neq k}}^{\ell} \left\| \frac{a_{km}}{a_{kk}} \right\|_\infty \|\eta_m\|_{\infty,\bar{\omega}},$$

by (6.14). Gathering together the η terms and invoking the inverse-monotonicity of $\boldsymbol{\Gamma}(\boldsymbol{A})$, we get

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \leq C\left\|\boldsymbol{\varphi}\right\|_{\infty,\bar{\omega}}.$$

Each component φ_i of $\boldsymbol{\varphi}$ is the solution of a scalar problem and can be analysed using the technique in Sect. 6.1.2.2 combined with the derivative bounds of Theorem 3.43. This was done in [104, § 3.2] to deduce the following result:

Theorem 6.35. *Let the matrix $\boldsymbol{A}$ and the source term $\boldsymbol{f}$ be twice continuously differentiable. Assume $\boldsymbol{A}$ possesses positive diagonal entries and satisfies* (6.49)*. Then the error in the solution of* (6.53) *satisfies*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \leq C\left(\vartheta^{[2]}_{rd,\ell}(\bar{\omega})\right)^2.$$

Corollary 6.36. *Identifying $\boldsymbol{u}^N$ with a piecewise linear function on $\bar{\omega}$, we have*

$$\left\|\boldsymbol{u}-\boldsymbol{u}^N\right\|_{\infty} \leq C\left(\vartheta^{[2]}_{rd,\ell}(\bar{\omega})\right)^2,$$

by Theorems 6.29 and 6.35.

Remark 6.37. An alternative analysis based on comparison principles with special barrier functions was used in [102, 118]. This technique has the more restrictive condition that $\boldsymbol{A}$ be an M-matrix and up to now has been applied successfully only to Shishkin meshes. ♣

6.2.3.3 Numerical Results

We now present the results of some numerical experiments in order to illustrate the conclusions of the preceding section, and to check if the error estimates of Theorem 6.35 are sharp.

The test problem is

$$\begin{aligned}
-\varepsilon_1^2 u_1'' + 3u_1 + (1-x)(u_2-u_3) &= \mathrm{e}^x, & u_1(0) = u_1(1) = 0,\\
-\varepsilon_2^2 u_2'' + 2u_1 + (4+x)u_2 - u_3 &= \cos x, & u_2(0) = u_2(1) = 0,\\
-\varepsilon_3^2 u_3'' + 2u_1 + 3u_3 &= 1 + x^2, & u_3(0) = u_3(1) = 0.
\end{aligned}$$

In the construction of the Bakhvalov and the Shishkin mesh (see Sect. 2.2.2) we take $\kappa/p = 0.8$.

The exact solutions to the test problems is not available, so we estimate the accuracy of the numerical solution by comparing it to the numerical solution of the Richardson extrapolation method, which is of higher order: Let $\boldsymbol{u}^N_{\boldsymbol{\varepsilon}}$ be the solution of the difference scheme on the original mesh and $\tilde{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}^{2N}$ that on the mesh obtained by uniformly bisecting the original mesh. This yields the estimated error

$$\eta_{\boldsymbol{\varepsilon}}^{N} := \frac{4}{3}\left\|\boldsymbol{u}_{\boldsymbol{\varepsilon}}^{N} - \tilde{\boldsymbol{u}}_{\boldsymbol{\varepsilon}}^{2N}\right\|_{\infty,\omega}.$$

The uniform errors for a fixed N are estimated by taking the maximum error over a wide range of $\boldsymbol{\varepsilon}$, namely

$$\eta^{N} := \max_{\varepsilon_1,\dots,\varepsilon_\ell = 10^{-0},\dots,10^{-24}} \eta_{\boldsymbol{\varepsilon}}^{N}.$$

For the Shishkin mesh we have an error bound of the form $C\left(N^{-1}\ln N\right)^{p}$. Therefore, we compute the “Shishkin” rates of convergence using the formula

$$p^{N} = \frac{\ln\eta^{N} - \ln\eta^{2N}}{\ln(2\ln N) - \ln(\ln(2N))},$$

while for the Bakhvalov mesh, the standard formula

$$\rho^{N} = \left(\ln\eta^{N} - \ln\eta^{2N}\right)/\ln 2$$

is used.

The results of our test computations displayed in Table 6.2 are in agreement with Theorem 6.35. The Bakhvalov mesh gives second order accuracy, while the results for the Shishkin mesh are spoiled by the typical logarithmic factor.

Table 6.2 Central differencing on layer-adapted meshes for systems of reaction-diffusion type

	Shishkin mesh		Bakhvalov mesh	
N	η^N	p^N	η^N	ρ^N
$8\cdot 2^3$	4.895e-02	1.22	4.910e-03	2.08
$8\cdot 2^4$	2.276e-02	1.50	1.157e-03	2.05
$8\cdot 2^5$	8.854e-03	1.67	2.790e-04	2.04
$8\cdot 2^6$	3.091e-03	1.77	6.771e-05	2.01
$8\cdot 2^7$	1.014e-03	1.83	1.676e-05	2.00
$8\cdot 2^8$	3.201e-04	1.88	4.179e-06	2.00
$8\cdot 2^9$	9.831e-05	1.91	1.044e-06	2.00
$8\cdot 2^{10}$	2.955e-05	1.94	2.610e-07	2.00
$8\cdot 2^{11}$	8.725e-06	1.96	6.527e-08	2.00
$8\cdot 2^{12}$	2.537e-06	—	1.632e-08	—

6.2.3.4 A Posteriori Error Bounds

By interpreting the components of $\boldsymbol{u}^N$ as piecewise linear functions, one can conduct an a posteriori error analysis that combines the technique in Sect. 6.1.2.3 with Theorem 3.36 to give

$$\left\|u_k - u_k^N\right\|_\infty \le \sum_{m=1}^{\ell} \left(\boldsymbol{\Gamma}^{-1}\right)_{km} \left(\eta_{1;m} + \eta_{2;m} + \eta_{3;m}\right), \quad k = 1, \ldots, \ell,$$

where

$$\eta_{1;m} := \frac{\left\|\hat{q}_m - \hat{q}_m^I\right\|_\infty}{\rho_m^2}, \quad \eta_{2;m} := \max_{i=0,\ldots,N-1} \frac{h_{i+1}^2}{4\varepsilon_k^2} \max\left\{\left|q_{k;i}\right|, \left|q_{k;i+1}\right|\right\}$$

$$\eta_{3;m} := \max_{i=0,\ldots,N-1} \frac{h_{i+1}}{6\varepsilon_m \rho_m} \left|\hat{q}_{m;i+1} - \hat{q}_{m;i}\right|$$

$$\hat{q}_m = f_m - \sum_{\nu=1}^{\ell} a_{m\nu} u_\nu^N \quad \text{and} \quad \rho_m = \min_{x\in[0,1]} a_{mm}(x)^{1/2}.$$

Remark 6.38. Since the a posteriori analysis uses the stability of the differential operator, a posteriori bounds can also be derived for non-monotone discretisations, like schemes generated by FEMs. If for example, FEM-2 (see p. 187) is used, then the η_3 terms in the above estimate will disappear; cf. Sect. 6.1.3.3. ♣

6.3 Reaction-Convection-Diffusion

Consider the scalar reaction-convection-diffusion problem

$$\mathcal{L}u := -\varepsilon_d u'' - \varepsilon_c b u' + cu = f \quad \text{in } (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1, \tag{6.54}$$

where $b \ge 1$ and $c \ge 1$ on $[0,1]$, $\varepsilon_d \in (0,1]$ and $\varepsilon_c \in [0,1]$.

Analytical properties of (6.54) were studied in Sect. 3.2, while layer-adapted meshes for it have been introduced in Sect. 2.2.

We briefly recall some of the notation. Let $\lambda_0 < 0$ and $\lambda_1 > 0$ be the solution of

$$-\varepsilon_d \lambda(x)^2 - \varepsilon_c b(x)\lambda(x) + c(x) = 0. \tag{6.55}$$

Set

$$\mu_0 := \max_{x\in[0,1]} \lambda_0(x) < 0 \quad \text{and} \quad \mu_1 := \min_{x\in[0,1]} \lambda_1(x) > 0.$$

The quantity used for presenting a priori error bounds is

$$\vartheta_{rd}^{[p]}(\bar{\omega}) := \max_{i=1,\dots,N} \int_{I_i} \left\{ 1 + |\mu_0| \, \mathrm{e}^{\mu_0 s/p\varepsilon} + \mu_1 \mathrm{e}^{-\mu_1(1-s)/p\varepsilon} \right\} \mathrm{d}s.$$

6.3.1 The Interpolation Error

Let w^I denote the piecewise linear interpolant of w again. Adapting the techniques from Sects. 5.1 and 6.1.1.1, and using the derivative bounds of Theorem 3.29 we obtain the following result.

Proposition 6.39. *Let u be the solution of (6.54). Let $p > 2$ be arbitrary, but fixed. Then*

$$\left\| u - u^I \right\|_{\infty, I_i} \le C \left\{ \int_{I_i} \left(1 + |\mu_0| \, \mathrm{e}^{\mu_0 s/p\varepsilon} + \mu_1 \mathrm{e}^{-\mu_1(1-s)/p\varepsilon} \right) \mathrm{d}s \right\}^2$$

for all mesh intervals $I_i = [x_{i-1}, x_i]$.

Define the ε_d-weighted energy norm by

$$|||v|||_{\varepsilon_d}^2 := \varepsilon_d \, |v|_1^2 + \|v\|_0^2 \, .$$

Theorem 6.40. *Let u be the solution of (6.54). Let $p > 2$ be arbitrary, but fixed. Then*

$$\left\| u - u^I \right\|_\infty \le C \left(\vartheta_{rcd}^{[p]}(\bar{\omega}) \right)^2$$

and

$$\left|\left|\left| u - u^I \right|\right|\right|_{\varepsilon_d} \le C \left(\mu_1^{-1/2} + \vartheta_{rcd}^{[p]}(\bar{\omega}) \right) \vartheta_{rcd}^{[p]}(\bar{\omega}).$$

for any $p > 2$.

Proof. The bound on the L_∞ error is an immediate consequence of Prop. 6.39 and the definition of $\vartheta_{rcd}^{[p]}$.

For the error in the H^1 semi-norm use integration by parts to get

$$\left| u - u^I \right|_1^2 = - \int_0^1 u''(x) \left(u - u^I \right)(x) \mathrm{d}x.$$

Note, that

$$\varepsilon_d \int_0^1 |u''(x)| \, \mathrm{d}x \le C \varepsilon_d \, |\mu_0| \le C \mu_1^{-1}.$$

The assertion of the theorem follows. □

Remark 6.41. The results of Theorem 6.40 hold for arbitrary $p > 2$, but not for $p = 2$. This is because of the derivative bounds provided by Theorem 3.29.

6.3.2 Simple Upwinding

This section studies a simple first-order upwind difference scheme for (6.54). The analysis essentially follows [95].

Our scheme is: Find $u^N \in I\!R^{N+1}$ such that

$$\begin{aligned} \left[Lu^N\right]_i := -\varepsilon_d u^N_{\bar{x}x;i} - \varepsilon_c b_i u^N_{x;i} + c_i u^N_i &= f_i \quad \text{for } i = 1, \dots, N-1, \\ u^N_0 = \gamma_0, \ \ u^N_N &= \gamma_1 . \end{aligned} \tag{6.56}$$

The difference operators were introduced in Sect. 4.1.

The variational formulation of (6.56) is: Find $u^N \in V^\omega$ with $u^N_0 = \gamma_0$ and $u^N_N = \gamma_1$ such that

$$a_u\big(u^N, v\big) = f_u\big(v\big) := (f, v)_\omega \quad \text{for all } v \in V^\omega_0 , \tag{6.57}$$

where

$$a_u\left(w, v\right) := \varepsilon_d \left[w_x, v_x\right)_\omega - \left(\varepsilon_c b u_x - c w, v\right)_\omega$$

and

$$\left[w, v\right)_\omega := \sum_{i=0}^{N-1} h_{i+1} w_i v_i, \quad \left(w, v\right)_\omega := \sum_{i=1}^{N-1} h_{i+1} w_i v_i.$$

Taking as test functions v the standard hat-function basis in V^ω_0, we see that (6.56) and (6.57) are equivalent. In particular, using summation by parts it is verified that

$$a_u\left(w, v\right) = \left(Lw, v\right)_\omega = \left(w, L^* v\right)_\omega \quad \text{for all } w, v \in V^\omega_0 ,$$

where

$$\left[L^* v\right]_i := -\varepsilon_d v_{\bar{x}x;i} + \varepsilon_c \left(bv\right)_{\breve{x};i} + c_i v_i \ .$$

is the adjoint operator to L with respect to the scalar product $(\cdot, \cdot)_\omega$.

6.3.2.1 Stability

The matrix associated with the difference operator L is an L_0-matrix. Application of Lemma 3.14 with a constant positive test function establishes the inverse monotonicity of L. Thus, L satisfies a comparison principle. This comparison principle gives the $(\ell_\infty, \ell_\infty)$-stability inequality

$$\|v\|_{\infty,\bar{\omega}} \le \|Lv/c\|_{\infty,\omega} \quad \text{for all } v \in I\!R_0^{N+1}$$

and provides a priori bounds for the solution of (6.56):

$$\left\|u^N\right\|_{\bar{\omega}} \le \max\left\{|\gamma_0|, |\gamma_1|\right\} + \|f/c\|_{\infty,\omega} \quad \text{for } i = 0, \dots, N.$$

Green's function estimates

Using the discrete Green's function $G : \bar{\omega}^2 \to I\!R : (x_i, \xi_j) \mapsto G_{i,j} = G(x_i, \xi_j)$ associated with L and Dirichlet boundary conditions, any mesh function $v \in I\!R_0^{N+1}$ can be represented as

$$v_i = a_u\left(v, G_{i,\cdot}\right) = \left(Lv, G_{i,\cdot}\right)_\omega = \left(v, L^* G_{i,\cdot}\right)_\omega \quad \text{for } i = 1, \dots, N-1.$$

Taking for v the standard hat basis in V_0^ω, we see that for fixed $i = 1, \dots, N-1$

$$\left[L^* G_{i,\cdot}\right]_j = \delta_{i,j} \quad \text{for } j = 1, \dots, N-1, \;\; G_{i,0} = G_{i,N} = 0, \tag{6.58}$$

where

$$\delta_{i,j} := \begin{cases} h_{i+1}^{-1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

is the discrete equivalent of the Dirac-δ distribution.

Theorem 6.42. *Assume that $b \ge 1$ and $c \ge 1$ on $[0,1]$, then there exists a positive constant C such that*

$$0 \le G_{i,j} \le K\mu_1 \quad \textit{for } i, j = 0, \dots, N,$$

where

$$K := \begin{cases} \|b\|_\infty \|c\|_\infty \max\left\{\|b\|_\infty, \|c\|_\infty\right\} & \textit{if } \varepsilon_c > 0, \\ \|c\|_\infty^{3/2} & \textit{if } \varepsilon_c = 0. \end{cases}$$

Furthermore, if $b' \ge 0$ on $[0,1]$ then

$$G_{\xi;i,j} \ge 0 \quad \textit{for } 0 \le j < i < N,$$
$$G_{\xi;i,j} \le 0 \quad \textit{for } 0 \le i \le j < N.$$

Proof. Let j be fixed. Set

$$\varGamma_i := \begin{cases} \prod_{k=i+1}^{j} (1+\mu_1 h_k)^{-1} & \text{for } 0 \le i < j \le N, \\ 1 & \text{for } 0 \le i = j \le N, \\ \prod_{k=j+1}^{i} (1-\mu_0 h_k)^{-1} & \text{for } 0 \le j < i \le N. \end{cases}$$

We shall use the discrete comparison principle to show that $K\mu_1\varGamma$ is an upper bound for $G_{\cdot,j}$.

Clearly $\varGamma_0 \ge 0$ and $\varGamma_N \ge 0$.

Next, a direct calculation gives

$$\varGamma_{x;i} = \varGamma_i \begin{cases} \mu_1 & \text{for } 0 \le i < j \le N, \\ \dfrac{\mu_0}{1-\mu_0 h_{i+1}} & \text{for } 0 \le j \le i < N \end{cases}$$

and

$$\varGamma_{\bar{x};i} = \varGamma_i \begin{cases} \dfrac{\mu_1}{1+\mu_1 h_{i+1}} & \text{for } 0 < i \le j \le N, \\ \mu_0 & \text{for } 0 \le j < i \le N. \end{cases}$$

For $i < j$ we have

$$\left[L\varGamma\right]_i = \varGamma_i \left(\frac{-\varepsilon_d \mu_1^2}{1+\mu_1 h_{i+1}} - \varepsilon_c b_i \mu_1 + c_i \right) \ge \varGamma_i \left(-\varepsilon_d \mu_1^2 - \varepsilon_c b_i \mu_1 + c_i \right) \ge 0,$$

by Proposition 3.19.

For $i > j$

$$\left[L\varGamma\right]_i = \varGamma_i \left(\frac{-\varepsilon_d \mu_0^2 - \varepsilon_c b_i \mu_0}{1-\mu_0 h_{i+1}} + c_i \right) \ge \varGamma_i \frac{-\varepsilon_d \mu_0^2 - \varepsilon_c b_i \mu_0 + c_i}{1-\mu_0 h_{i+1}} \ge 0,$$

where Proposition 3.19 was used again.

For $i = j$

$$\begin{aligned} \left[L\varGamma\right]_j &= -\frac{\varepsilon_d}{h_{j+1}} \left(\frac{\mu_0}{1-\mu_0 h_{j+1}} - \frac{\mu_1}{1+\mu_1 h_j} \right) - \varepsilon_c b_j \frac{\mu_0}{1-\mu_0 h_{j+1}} + c_j \\ &\ge \frac{-\varepsilon_d \mu_0 - (\varepsilon_c b_j \mu_0 - c_j)\, h_{j+1}}{h_{j+1}\,(1-\mu_0 h_{j+1})}. \end{aligned}$$

The function $x \mapsto \lambda_0(x)$ is continuous. Therefore there exists a $\xi \in [0,1]$ such that $\lambda_0(\xi) = \mu_0$. Furthermore, recall $b \geq 1$, $c \geq 1$ and $\mu_0 < 0$. Hence,

$$
\begin{aligned}
\left[L\Gamma\right]_j &\geq \frac{1}{\max\{b_j, c_j\}} \frac{-\varepsilon_d\mu_0 - \big(\varepsilon_c b(\xi)\mu_0 - c(\xi)\big)h_{j+1}}{h_{j+1}\left(1-\mu_0 h_{j+1}\right)} \\
&= \frac{-\varepsilon_d\mu_0}{h_{j+1}\max\{b_j, c_j\}} = \frac{c(\xi)}{h_{j+1}\max\{b_j, c_j\}\,\lambda_1(\xi)}, \quad \text{by (6.55)} \\
&\geq \frac{1}{h_{j+1}K\mu_1}, \quad \text{by (3.6)}.
\end{aligned}
$$

Thus, $K\mu_1\Gamma$ is a barrier function for $G_{\cdot,j}$, for all j.

In order to verify the monotonicity of $G_{i,\cdot}$ use the argument from the proof of Theorem 4.1. □

Theorem 6.43. *Assume that $b \geq 1$ and $c \geq 1$ on $[0,1]$, then there exists a positive constant C such that*

$$\|cG_{i,\cdot}\|_{1,\omega} \leq 1 \quad \textit{for} \;\; i = 0, \ldots, N,$$

Furthermore, if $b' \geq 0$ on $[0,1]$ then

$$\|G_{\xi;i,\cdot}\|_{1,\omega} \leq C\mu_1 \quad \textit{for} \;\; i = 0, \ldots, N.$$

Proof. To verify the bound on the ℓ_1-norm of G multiply (6.58) by h_{j+1} and sum for $j = 1, \ldots, N-1$. Then use the positivity of G.

For the bound on G_ξ use the piecewise monotonicity of G_ξ to get

$$\|G_{\xi;i,\cdot}\|_{1,\omega} \leq 2G_{i,i} \leq C\mu_1,$$

by Theorem 6.42. □

6.3.2.2 A Priori Error Analysis

Theorem 6.44. *Let the assumptions of Theorems 3.29 and 6.43 be satisfied. Let u be the solution of (6.54) and u^N its approximation by (6.56). Then*

$$\left\|u^N - u\right\|_\infty \leq C\vartheta_{rcd}^{[p]}(\bar{\omega})$$

for any $p > 1$.

Proof. Let $\varGamma := G_{i,\cdot} \in V_0^\omega$ be the Green's function associated with L and the mesh node x_i. Then

$$\begin{aligned} u_i^N - u_i &= a_u\left(u^N - u, \varGamma\right) = f_u(\varGamma) - a_u(u, \varGamma) \\ &= f_u(\varGamma) - f(\varGamma) + a(u, \varGamma) - a_u(u, \varGamma), \quad \text{by (6.54)} \\ &= \left(\varepsilon_c b u_x - cu + f, \varGamma\right)_\omega - \left(\varepsilon_c b u' - cu + f, \varGamma\right). \end{aligned}$$

We obtain the representation for the nodal error:

$$\begin{aligned} u_i^N - u_i &= \varepsilon_c \sum_{i=0}^{N-1} \int_{I_{i+1}} u'(s)\big[(b\varGamma)(s) - b_i \varGamma_i\big] \mathrm{d}s \\ &\quad + \sum_{i=0}^{N-1} \int_{I_{i+1}} \Big[(cu - f)_i \varGamma_i - (cu - f)(s)\varGamma(s)\Big] \mathrm{d}s \\ &=: T_1 + T_0 \ . \end{aligned} \tag{6.59}$$

To bound the first term we proceed as follows:

$$\left|(b\varGamma)(s) - b_i \varGamma_i\right| \le \|b\|_\infty \left|\varGamma_{i+1} - \varGamma_i\right| + \|b'\|_\infty h_{i+1} \varGamma_i \,,$$

because $\varGamma$ is piecewise linear. Thus,

$$|T_1| \le C \varepsilon_c \vartheta_{rcd}^{[p]}(\bar\omega) \Big\{ \|b\|_\infty \|\varGamma_x\|_{1,\omega} + \|b'\|_\infty \|\varGamma\|_{1,\omega} \Big\} \tag{6.60}$$

for any $p > 1$, by Theorem 3.29.

Next we estimate T_0.

$$\begin{aligned} T_0 = \sum_{i=0}^{N-1} \int_{I_{i+1}} &\big[(cu - f)_i - (cu - f)(s)\big] \varGamma_i \mathrm{d}s \\ &+ \sum_{i=0}^{N-1} \int_{I_{i+1}} (cu - f)(s)\big[\varGamma_i - \varGamma(s)\big] \mathrm{d}s. \end{aligned}$$

Hence,

$$\begin{aligned} |T_0| \le \|\varGamma\|_{1,\omega} &\max_{i=1,\dots,N} \int_{I_i} \big|(f - cu)'(s)\big| \mathrm{d}s \\ &+ \|\varGamma_x\|_{1,\omega} \max_{i=1,\dots,N} \int_{I_i} \big|\varepsilon_d u''(s) + \varepsilon_c b(s) u'(s)\big| \mathrm{d}s \,, \end{aligned}$$

where (6.54) was used. To bound u' and u'' apply Theorem 3.29. Then, for any $p > 1$,

$$|T_0| \le C\vartheta_{rcd}^{[p]}(\bar{\omega}) \left(\|\Gamma\|_{1,\omega} + \mu_1^{-1} \|\Gamma_x\|_{1,\omega} \right),$$

where we have used that $\varepsilon_d|\mu_0| + \varepsilon_d|\mu_1| \le C\mu_1^{-1}$ and $\varepsilon_c \le C\mu_1^{-1}$.

Finally, combining the last inequality with (6.59) and (6.60), we get

$$\left|u_i^N - u_i\right| \le C\vartheta_{rcd}^{[p]}(\bar{\omega}) \left(\|\Gamma\|_{1,\omega} + \mu_1^{-1} \|\Gamma_x\|_{1,\omega} \right).$$

Application of Theorem 6.43 completes the analysis. □

Remark 6.45. The truncation error and barrier function technique (see Sect. 4.2.6) was used in [80, 105] when studying the difference scheme

$$-\varepsilon_d u_{\bar{x}\hat{x};i}^N - \varepsilon_c b_i u_{x;i}^N + c_i u_i^N = f_i \tag{6.61}$$

on a Shishkin mesh. The authors establish the typical rate of $N^{-1} \ln N$ if the critical mesh parameter satisfies $\sigma > 2$. In [80, 105] hybrid difference schemes were also studied that raise the order of convergence to almost second order. ♣

Remark 6.46. Gracia et al. [47] combine (6.61) with the mid-point upwind scheme

$$-\varepsilon_d u_{\bar{x}\hat{x};i}^N - \varepsilon_c b_{i+1/2} u_{x;i}^N + c_{i+1/2} \left(u_i^N + u_{i+1}^N\right)/2 = f_{i+1/2}$$

and central differencing

$$-\varepsilon_d u_{\bar{x}\hat{x};i}^N - \varepsilon_c b_i u_{\mathring{x};i}^N + c_i u_i^N = f_i$$

in order to obtain an inverse monotone method that is second-order consistent in the maximum norm for all values of ε_d, ε_c and N.

In [47] that scheme is shown to be uniformly convergent on a Shishkin mesh with

$$\left\|u^N - u\right\| \omega \le CN^{-2} \ln^3 N.$$

The analysis in [47], using solution decompositions and distinguishing different parameter regimes, is very tedious. Results for general meshes are not available.

Surla et al. [158] design an inverse monotone spline difference scheme. They prove uniform convergence on a Shishkin mesh with

$$\left\|u^N - u\right\| \omega \le CN^{-2} \ln^2 N,$$

if $\varepsilon_d, \varepsilon_c \le CN^{-1}$. ♣

6.3.2.3 A Numerical Example

In this section a brief illustration for the a priori error bounds of Theorem 6.44 is shown. The test problem is:

$$-\varepsilon_d u'' + \varepsilon_c u' + u = \cos \pi x \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

Its solution is easily computed using, e. g., MAPLE.

Indicating by $u^N_{\varepsilon_d,\varepsilon_c}$ that the numerical approximation depends on N, ε_d and ε_c, and by $u_{\varepsilon_d,\varepsilon_c}$ that the exact solution depends on ε_d and ε_c, we estimate the uniform error by

$$\eta^N := \max_{\substack{\varepsilon_d=1,10^{-1},\dots,10^{-12}\\ \varepsilon_c=1,10^{-1},\dots,10^{-12},0}} \left\| u^N_{\varepsilon_d,\varepsilon_c} - u^I_{\varepsilon_d,\varepsilon_c} \right\|_\infty .$$

The rates of convergence are computed using the standard formula $r^N = \log_2 \left(\eta^N / \eta^{2N}\right)$. We also compute the constant in the error estimate, i. e., for a given theoretical error bound $\eta^N \le C\vartheta(N)$, we approximate the constant in the error estimate by $C^N = \eta^N / \vartheta(N)$.

Table 6.3 displays the results of our test computations. The numbers confirm the results of Theorem 6.44 for both the Shishkin and the Bakhvalov mesh.

6.3.2.4 A Posteriori Error Analysis

The analysis for all schemes starts from the Green's-function representation (3.1) and utilises Theorem 3.23, when bounds on the norms of the Green's function $\mathcal{G}$ are required.

Table 6.3 Uniform nodal errors of the upwind difference scheme applied to a model reaction-convection-diffusion problem

	Shishkin mesh			Bakhvalov mesh		
N	η^N	r^N	C^N	η^N	r^N	C^N
2^9	1.227e-2	0.83	1.007	3.404e-3	0.99	1.743
2^{10}	6.905e-3	0.85	1.020	1.713e-3	1.00	1.754
2^{11}	3.824e-3	0.87	1.027	8.592e-4	1.00	1.760
2^{12}	2.093e-3	0.88	1.031	4.302e-4	1.00	1.762
2^{13}	1.136e-3	0.89	1.032	2.153e-4	1.00	1.763
2^{14}	6.119e-4	0.90	1.033	1.077e-4	1.00	1.764
2^{15}	3.279e-4	0.91	1.033	5.384e-5	1.00	1.764
2^{16}	1.749e-4	0.91	1.033	2.692e-5	1.00	1.764
2^{17}	9.290e-5	0.92	1.033	1.346e-5	1.00	1.764
2^{18}	4.922e-5	—	1.034	6.731e-6	—	1.764

Let

$$q := f - cu^N + \varepsilon_c b \left(u^N\right)'.$$

Clearly q may have discontinuities at the mesh points because $u^N \in V^\omega$. Therefore, set

$$q_i^+ = \lim_{x \to x_i + 0} q(x) \quad \text{and} \quad q_i^- = \lim_{x \to x_i - 0} q(x)$$

for all mesh nodes x_i.

Fix $x \in (0,1)$ and set $\Gamma := \mathcal{G}(x,\cdot)$. Then (3.1) and (6.54) yield

$$\begin{aligned}\left(u - u^N\right)(x) &= a\left(u - u^N, \Gamma\right) = f(\Gamma) - a\left(u^N, \Gamma\right) \\ &= \left(f - f_u\right)(\Gamma) + \left(a_u - a\right)\left(u^N, \Gamma\right) \\ &= \sum_{i=1}^{N} \left[q_{i-1}^+ \Gamma_{i-1} h_i - \int_{I_i} \left(q\Gamma\right)(s) \mathrm{d}s \right].\end{aligned}$$

We see the approximation error is bounded by the error of the one-sided rectangle rule used for integrating $q\Gamma$. A Taylor expansion gives

$$\begin{aligned}&\left| h_i q_{i-1}^+ \Gamma_{i-1} - \int_{I_i} \left(b\Gamma\right)(\xi) \mathrm{d}\xi \right| \\ &\quad \le h_i \left\{ \int_{I_i} |q'(s)| \Gamma(s) \mathrm{d}s + \int_{I_i} |q(s)| \, |\Gamma'(s)| \mathrm{d}s \right\} \\ &\quad \le h_i \left\{ \|q'/c\|_{\infty, I_i} \int_{I_i} c(s)\Gamma(s) \mathrm{d}s + \|q\|_{\infty, I_i} \int_{I_i} |\Gamma'(s)| \mathrm{d}s \right\}.\end{aligned}$$

Using the bounds on the Green's function from Theorem 3.23, we arrive at the main result of this section.

Theorem 6.47. *Suppose* (3.8) *is satisfied, then the error of the first-order upwind scheme* (6.56) *satisfies*

$$\left\|u - u^N\right\|_\infty \le \eta_1^u + \eta_2^u,$$

where $\eta_k^u := \max_{i=1,\dots,N} \eta_{k,i}^u$ *and*

$$\eta_{1,i}^u := h_i \|q'/c\|_{\infty, I_i}, \qquad \eta_{2,i}^u := \frac{2h_i}{\varepsilon_d \left(\mu_1 - \mu_0\right)} \|q\|_{\infty, I_i}.$$

Remark 6.48. The error has been bounded in terms of the numerical solution u^N and of the data of the problem. The two parts of the error bound can be further expanded. For example:

$$h_i \left\| \frac{q'}{c} \right\|_{\infty,I_i} \leq \left\{ h_i \left\| \frac{f' - c' u^N}{c} \right\|_{\infty,I_i} + \left\| \frac{\varepsilon_c b'}{c} - 1 \right\|_{\infty,I_i} \left| u_{i+1}^N - u_i^N \right| \right\}. \quad (6.62)$$

Apparently, sampling of the data is inevitable. However, instead of sampling (6.62) it seems advisable to sample the η_i^u:

$$\eta_{1,i}^u \approx \tilde{\eta}_{1,i}^u := \left| \frac{q_i^- - q_{i-1}^+}{c_{i-1/2}} \right| \quad \text{and} \quad \eta_{2,i}^u \approx \tilde{\eta}_{2,i}^u := \frac{2h_i}{\varepsilon_d(\mu_1 - \mu_0)} \left| q_{i-1/2} \right|$$

This avoids the use of a triangle inequality and therefore gives in general sharper upper bounds for the error. ♣

Remark 6.49. The leading term in the estimate of Theorem 6.47 is a discrete first-order derivative. Therefore, the de Boor algorithm of Sect. 4.2.4.2 equidistributing the arc-length of the numerical solution can be used for adaptive mesh generation, when the simple upwind scheme (6.56) is applied to the boundary-value problem (6.54). ♣

Part II

Two Dimensional Problems

Chapter 7
The Analytical Behaviour of Solutions

In this chapter we gather a number of analytical properties for singularly perturbed elliptic boundary-value problems of the general type

$$-\varepsilon \Delta u - \boldsymbol{b}\nabla u + cu = f \ \text{ in } \ \Omega \subset \mathbb{R}^2, \ \ u|_{\partial\Omega} = g,$$

with a small positive parameter ε, the convection field $\boldsymbol{b} = (b_1, b_2)^T$ and functions b_1, b_2, c, $f : \bar{\Omega} \to \mathbb{R}$, and $g : \partial\Omega \to \mathbb{R}$.

7.1 Preliminaries

The function spaces required here are the spaces of Hölder continuous functions. Let $D \in \mathbb{R}^2$ be a convex domain and let $\alpha \in (0, 1]$. For nonnegative integer k, we use $C^k(D)$ to denote the space of functions whose derivatives up to order k are continuous on D; when $k = 0$ we write simply $C(D)$. We put the usual supremum (semi-)norms on $C^k(D)$:

$$|v|_{k,D} := \max_{i+j=k} \sup_{\boldsymbol{x}\in D} \left|\partial_x^i \partial_y^j v(\boldsymbol{x})\right| \quad \text{and} \quad \|v\|_{k,D} := \max_{0\le l\le k} |v|_{l,D}$$

For convenience we drop D from the notation when $D = \Omega$. We denote by $C^{0,\alpha}(D)$ the space of all functions that are Hölder continuous of degree α on D. Let

$$\lceil v \rceil_{0,\alpha,D} := \sup_{\substack{\boldsymbol{x}\neq\boldsymbol{x}' \\ \boldsymbol{x},\boldsymbol{x}'\in D}} \frac{|v(\boldsymbol{x}) - v(\boldsymbol{x}')|}{\|\boldsymbol{x} - \boldsymbol{x}'\|^\alpha},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. Then the norm in $C^{0,\alpha}(D)$ is

$$\|v\|_{0,\alpha,D} := |v|_{0,D} + \lceil v \rceil_{0,\alpha,D}.$$

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_7,

For each positive integer k, the space $C^{k,\alpha}(D)$ consists of all Hölder continuous functions whose derivatives up to order k are also Hölder continuous. For $v \in C^{k,\alpha}(D)$ we define the seminorms

$$\lceil v \rceil_{l,\alpha,D} := \max_{i+j=l} \lceil \partial_x^i \partial_y^j v \rceil_{0,\alpha,D} \quad \text{for } l = 1, \ldots, k$$

and

$$|v|_{l,\alpha,D} := |v|_{l,D} + \lceil v \rceil_{l,\alpha,D} \quad \text{for } l = 1, \ldots, k.$$

The norm in $C^{k,\alpha}(D)$ is

$$\|v\|_{k,\alpha,D} := \max_{l=0,\ldots,k} |v|_{l,\alpha,D} \, .$$

7.1.1 Stability

Consider the general elliptic second-order differential operator

$$\mathcal{L}u := -\nabla\left(a\nabla u\right) + \boldsymbol{b}\nabla u + cu$$

with $a, b_1, b_2, c : \Omega \to I\!R$ and $a > 0$ on $\bar{\Omega}$.

Definition 7.1. The differential operator $\mathcal{L}$ is called **inverse monotone** or **of positive type** if it obeys a **comparison principle**. That is, for any two functions $v, w \in C^2(0,1) \cap C[0,1]$

$$\left.\begin{array}{ll} \mathcal{L}v \le \mathcal{L}w & \text{in } \Omega, \\ v \le w & \text{on } \bar{\Omega} \end{array}\right\} \implies v \le w \quad \text{on } \bar{\Omega}.$$

♡

Lemma 7.2. *Let there exist a function $\psi \in C^2(\Omega) \cap C(\bar{\Omega})$ with $\psi > 0$ on $\bar{\Omega}$ and $\mathcal{L}\psi > 0$ in Ω. Then the operator $\mathcal{L}$ is inverse monotone.*

Proof. The proof is by contradiction; see, e.g. [131]. □

Similar to Lemma 3.5 we have:

Lemma 7.3. *Suppose there exists a function $\psi \in C^2(\Omega) \cap C(\bar{\Omega})$ with $\psi > 0$ on $\bar{\Omega}$ and $\mathcal{L}\psi > 0$ in Ω. Then for any function $v \in C^2(\Omega) \cap C(\bar{\Omega})$ with $v|_{\partial\Omega} = 0$*

$$|v| \le \psi \left\| \frac{\mathcal{L}v}{\mathcal{L}\psi} \right\|_\infty \quad \text{on } \bar{\Omega}.$$

Remark 7.4. Under the assumptions of Lemma 7.3 the operator $\mathcal{L}$ is (L_∞, L_∞) stable, or **maximum-norm** stable. That is

$$\|v\|_\infty \leq K \|\mathcal{L}v\|_\infty$$

with the stability constant $K = \|\psi\|_\infty \Big/ \min_{\boldsymbol{x}\in\bar{\Omega}} |(\mathcal{L}\psi)(\boldsymbol{x})|$. ♣

7.1.2 Regularity of Solutions

In this Section we state **compatibility conditions** from [48, 162] that guarantee smoothness of the solution of the boundary value problem

$$\begin{aligned} -\varepsilon\Delta u - \boldsymbol{b}^T\nabla u + cu = f \quad \text{in } \Omega = (0,1)^2, \\ u(\cdot,0) = g_s,\; u(\cdot,1) = g_n,\; u(0,\cdot) = g_w,\; u(1,\cdot) = g_e \;\text{ on } [0,1]. \end{aligned} \tag{7.1}$$

Let

$$F := f + \boldsymbol{b}^T\nabla u - cu.$$

We define linear functionals Λ_k at the corner $(0,0)$ of the domain by

$$\begin{aligned} \Lambda_k(f,\boldsymbol{b},c,g) := \varepsilon g_s^{(2k)}(0) + \varepsilon(-1)^{k+1} g_w^{(2k)}(0) \\ + \sum_{i=1}^{k}(-1)^{i-1}\partial_x^{2(k-i)}\partial_y^{2(i-1)}F(0,0). \end{aligned} \tag{7.2}$$

We also write $\Lambda_k^1 := \Lambda_k$. In a similar way linear functionals Λ_k^l, for $l = 2,3,4$, are defined at the other vertices of Ω. For Λ_k^l to be well defined, the data must possess sufficient smoothness. We set

$$\begin{aligned} X^{k,\alpha} := \Big\{(f,\boldsymbol{b},c,g) : f, b_1, b_2, c \in C^{k-2,\alpha}(\bar{\Omega}) \\ \text{and} \quad g_s, g_w, g_n, g_e \in C^{k,\alpha}[0,1]\Big\}. \end{aligned}$$

The following result is adapted from [48] which in turn is based on earlier work by Volkov [162].

Theorem 7.5. *Let $\alpha \in (0,1]$ be arbitrary, but fixed. Suppose that $(f,\boldsymbol{b},c,g) \in X^{2,\alpha}$ and that $\Lambda_0^l(f,\boldsymbol{b},c,g) = 0$ for $l = 1,2,3,4$. Then the solution u of (7.1) lies in $C^{1,\alpha}(\bar{\Omega})$.*

Suppose $(f,\boldsymbol{b},c,g) \in X^{2k,\alpha}$ and $u \in C^{2k-1,\alpha}(\bar{\Omega})$, then the linear functionals $\Lambda_j^l(f,\boldsymbol{b},c,g)$ are well defined for $j = 0,\dots,k$ and $\Lambda_j^l(f,\boldsymbol{b},c,g) = 0$ for

$j = 0, \ldots, k-1$. *We have* $\Lambda_k^l(f, \boldsymbol{b}, c, g) = 0$ *for* $l = 1, 2, 3, 4$ *if and only if* $u \in C^{2k,\alpha}(\bar{\Omega})$. *If in addition* $(f, \boldsymbol{b}, c, g) \in X^{2k+1,\alpha}$, *then* $u \in C^{2k+1,\alpha}(\bar{\Omega})$.

This theorem is used to derive compatibility conditions in terms of the data of the problem (7.1), i. e., in terms of the functions b_1, b_2, c, f and the gs. For $k = 0$ the linear functional defined in (7.2) becomes

$$\Lambda_0(f, \boldsymbol{b}, c, g) = \varepsilon g_s(0) - \varepsilon g_w(0) = 0.$$

Similarly $\Lambda_0^l(f, \boldsymbol{b}, c, g) = 0$ for $l = 2, 3, 4$, so $u \in C^{1,\alpha}(\bar{\Omega})$. Consequently, we have $u(0,0) = g_s(0)$, $\partial_x u(0,0) = g_s'(0)$ and $\partial_y u(0,0) = g_w'(0)$. Thus, the linear functional $\Lambda_1(f, \boldsymbol{b}, c, g)$ can be reformulated as a local functional of the data:

$$\begin{aligned}\Lambda_1(f, \boldsymbol{b}, c, g) = \varepsilon g_s''(0) + \varepsilon g_w''(0) + f(0,0) \\ + b_1(0,0) g_s'(0) + b_2(0,0) g_w'(0) - c(0,0) g_s(0).\end{aligned}$$

Remark 7.6. Han and Kellogg [48] point out that in general the functionals Λ_k cannot be reformulated as local functionals for $k \geq 2$. For example, $\Lambda_2(f, \boldsymbol{b}, c, g)$ contains the term $\partial_y b_1(0,0) \partial_x \partial_y u(0,0)$ that cannot be expressed in terms of the data of the problem by using (7.1) and continuity arguments. However, for reaction-diffusion equations, i.e. vanishing convective field $\boldsymbol{b}$, we have

$$\begin{aligned}\Lambda_2(f, \boldsymbol{0}, c, g) = \varepsilon g_s^{(4)}(0) + \Big(f(\cdot,0) - c(\cdot,0) g_s\Big)''(0) \\ - \varepsilon g_w^{(4)}(0) - \Big(f(0,\cdot) - c(0,\cdot) g_w\Big)''(0)\ .\end{aligned}$$

♣

7.2 Reaction-Diffusion

We consider a singularly perturbed system of $\ell \geq 2$ of reaction-diffusion equations on the square. We seek the solution $\boldsymbol{u} = (u_1, u_2, \ldots, u_\ell)^T$ such that

$$\boldsymbol{\mathcal{L}u} := -\varepsilon^2 \Delta \boldsymbol{u} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega = (0,1)^2, \ \ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}, \tag{7.3}$$

where the parameter ε satisfies $0 < \varepsilon \ll 1$.

Assume that a diagonal constant matrix $\boldsymbol{D} = \operatorname{diag}(d_{11}, \ldots, d_{\ell\ell})$ exists with $d_{kk} > 0$ for all k and a constant $\mu > 0$ such that

$$\boldsymbol{v}^T \boldsymbol{D}\boldsymbol{A}\boldsymbol{v} \geq \mu^2 \boldsymbol{v}^T \boldsymbol{v} \quad \text{in } \Omega \quad \text{for all } \boldsymbol{v} \in I\!R^\ell. \tag{7.4}$$

Under this hypothesis it is not true in general that the system (7.3) obeys a comparison principle.

Remark 7.7. In [57], which generalises the one-dimensional analysis of [18], the authors analyse the case $\boldsymbol{D} = \boldsymbol{I}$. In [58], the coupling matrix $\boldsymbol{A}$ is assumed to be a strictly diagonally dominant M-matrix, so Lemma 3.40 ensures that (7.4) is satisfied.

Here we present an idea from [111] which extends the analysis of [18, 57] by showing that it remains valid under the more general hypothesis (7.4). ♣

For vector-valued functions $\boldsymbol{v} \in C(G)^\ell$, we introduce the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{D},G} := \sup_{\boldsymbol{x}\in G}\Big\{\boldsymbol{v}(\boldsymbol{x})^T\boldsymbol{D}\boldsymbol{v}(\boldsymbol{x})\Big\}^{1/2}.$$

Furthermore, set $\delta^2 := \max_{k=1,\ldots,\ell} d_{kk}$.

7.2.1 Stability

The following stability result is a slight modification of an argument from [18, 57]. Note how neatly it side-steps the absence of a comparison principle for $\boldsymbol{\mathcal{L}}$.

Theorem 7.8. *Let $\boldsymbol{w} \in C^2(\Omega)^\ell \cap C(\bar\Omega)^\ell$. Then*

$$\|\boldsymbol{w}\|_{\boldsymbol{D},\bar\Omega} \le \max\Big\{\delta^2\mu^{-2}\,\|\boldsymbol{\mathcal{L}}\boldsymbol{w}\|_{\boldsymbol{D},\Omega}\,,\|\boldsymbol{w}\|_{\boldsymbol{D},\partial\Omega}\Big\}.$$

Proof. Set $\varphi = \boldsymbol{w}^T\boldsymbol{D}\boldsymbol{w}/2$. Observe that $2\varphi \le \delta^2\boldsymbol{w}^T\boldsymbol{w}$ on $\bar\Omega$ and

$$-\boldsymbol{w}^T\boldsymbol{D}\Delta\boldsymbol{w} = -\Delta\varphi + \partial_x\boldsymbol{w}^T\boldsymbol{D}\partial_x\boldsymbol{w} + \partial_y\boldsymbol{w}^T\boldsymbol{D}\partial_y\boldsymbol{w} \ge -\Delta\varphi.$$

Taking the scalar product of $\boldsymbol{D}\boldsymbol{w}$ with $-\varepsilon^2\Delta\boldsymbol{w} + \boldsymbol{A}\boldsymbol{w} = \boldsymbol{\mathcal{L}}\boldsymbol{w}$ and using the coercivity of $\boldsymbol{D}\boldsymbol{A}$, we get

$$-\varepsilon^2\Delta\varphi + \frac{2\mu^2}{\delta^2}\varphi \le \boldsymbol{w}^T\boldsymbol{D}\boldsymbol{\mathcal{L}}\boldsymbol{w} \quad \text{in } \Omega.$$

Hence, by the comparison principle for scalar problems,

$$\|\varphi\|_{\infty,\bar\Omega} \le \max\left\{\frac{\delta^2}{2\mu^2}\left\|\boldsymbol{w}^T\boldsymbol{D}\boldsymbol{\mathcal{L}}\boldsymbol{w}\right\|_{\infty,\Omega}, \frac{1}{2}\|\boldsymbol{w}\|^2_{\boldsymbol{D},\partial\Omega}\right\}.$$

It follows from the Cauchy-Schwarz inequality that

$$\|\boldsymbol{w}\|^2_{\boldsymbol{D},\bar\Omega} = 2\,\|\varphi\|_{\infty,\bar\Omega} \le \|\boldsymbol{w}\|_{\boldsymbol{D},\bar\Omega}\max\Big\{\delta^2\mu^{-2}\,\|\boldsymbol{\mathcal{L}}\boldsymbol{w}\|_{\boldsymbol{D},\Omega}\,,\|\boldsymbol{w}\|_{\boldsymbol{D},\partial\Omega}\Big\},$$

and we are done. □

Remark 7.9. Theorem 7.8 implies that $\boldsymbol{\mathcal{L}}$ is (L_∞, L_∞) stable. ♣

Corollary 7.10. *The solution* $\boldsymbol{u}$ *of* (7.3) *satisfies* $\|\boldsymbol{u}\|_{\infty,\bar{\Omega}} \le C$*, where the constant* C *depends only on* $\boldsymbol{A}$*,* $\boldsymbol{f}$ *and* $\boldsymbol{g}$*.*

7.2.2 Derivative Bounds

Let $q \in \{0,1,2\}$ and $\alpha \in (0,1]$. Assume that we have $\boldsymbol{A} \in C^{2(q+1),\alpha}(\bar{\Omega})^{\ell\times\ell}$ and $\boldsymbol{f} \in C^{2(q+1),\alpha}(\bar{\Omega})^{\ell}$, and that the restriction of $\boldsymbol{g}$ to each edge of $\bar{\Omega}$ yields a function lying in $C^{2(q+1),\alpha}[0,1]^{\ell}$, and that $\boldsymbol{g}$ is continuous at each of the corners; this implies that $\boldsymbol{g} \in C^{0,\alpha}(\partial\Omega)^{\ell}$. Finally, suppose that at the south-west corner of the domain we have the compatibility conditions

$$\boldsymbol{g}_s(0) - \boldsymbol{g}_w(0) = 0. \tag{7.5a}$$

If $q \ge 1$ also assume

$$-\varepsilon^2\boldsymbol{g}_s''(0) - \varepsilon^2\boldsymbol{g}_w''(0) + \boldsymbol{A}(0,0)\boldsymbol{g}_s(0) = \boldsymbol{f}(0,0) \tag{7.5b}$$

and, if $q = 2$,

$$\begin{aligned}\varepsilon^2\boldsymbol{g}_s^{(4)}(0) &+ \Big(\boldsymbol{f}(\cdot,0) - \boldsymbol{A}(\cdot,0)\boldsymbol{g}_s\Big)''(0)\\ &- \varepsilon^2\boldsymbol{g}_w^{(4)}(0) - \Big(\boldsymbol{f}(0,\cdot) - \boldsymbol{A}(0,\cdot)\boldsymbol{g}_w\Big)''(0) = 0\end{aligned} \tag{7.5c}$$

with corresponding conditions at the other three corners of the domain $\bar{\Omega}$.

These assumptions imply that the boundary-value problem (7.3) has a unique solution $\boldsymbol{u}$ in $\in C^{2q+2,\alpha}(\Omega)^{\ell} \cap C^{2q+1,\alpha}(\bar{\Omega})^{\ell}$; see [75].

Corollary 7.10 enables the invocation of Schauder-type estimates; see [75]. Consequently, we have the following result:

Lemma 7.11. *Let* $q \in \{1,2,3\}$ *be fixed. Suppose the compatibility conditions* (7.5) *are satisfied at all corners of the domain* $\bar{\Omega}$*. Then the solution* $\boldsymbol{u}$ *of* (7.3) *satisfies the bounds*

$$\left\|\partial_x^m \partial_y^{k-m}\boldsymbol{u}\right\|_{\infty,\bar{\Omega}} \le C\varepsilon^{-k} \quad \textit{for}\ \ m = 0,\dots,k,\ \ k = 0,\dots,2q-1; \tag{7.6a}$$

furthermore,

$$\left\|\partial_x^{2m} \partial_y^{2(q-m)}\boldsymbol{u}\right\|_{\infty,\bar{\Omega}} \le C\varepsilon^{-2q} \quad \textit{for}\ \ m = 0,\dots,q. \tag{7.6b}$$

Remark 7.12. The inequality (7.6b), which stems from a result of Volkov [162], is somewhat surprising since in general, one does not have $\boldsymbol{u} \in C^{2q,\alpha}(\bar{\Omega})^{\ell}$; note that (7.6b) is valid only for certain derivatives of order $2q$. ♣

We now derive sharper bounds on the pure derivatives of $\boldsymbol{u}$, that show that the large values seen in Lemma 7.11 do in fact decay rapidly as one moves away from $\partial\Omega$.

Theorem 7.13. *Let $\boldsymbol{u}$ be the solution of (7.3). Let $\rho \in (0, \mu/\delta)$ be arbitrary but fixed. Suppose (7.5) hold true for some $q \in \{0, 1, 2\}$, then there exists a constant C, which is independent of ε, such that*

$$\left|\partial_x^k \boldsymbol{u}(x,y)\right| \le C\left[1 + \varepsilon^{-k}\left(\mathrm{e}^{-\rho x/\varepsilon} + \mathrm{e}^{-\rho(1-x)/\varepsilon}\right)\right] \tag{7.7a}$$

and

$$\left|\partial_y^k \boldsymbol{u}(x,y)\right| \le C\left[1 + \varepsilon^{-k}\left(\mathrm{e}^{-\rho y/\varepsilon} + \mathrm{e}^{-\rho(1-y)/\varepsilon}\right)\right], \tag{7.7b}$$

for all $(x, y) \in \bar{\Omega}$ and $k = 1, \ldots, 2(q+2)$.

Proof. Fix $\rho \in (0, \mu/\delta)$ and set

$$B_k(x) := 1 + \varepsilon^{-k}\left(\mathrm{e}^{-\rho x/\varepsilon} + \mathrm{e}^{-\rho(1-x)/\varepsilon}\right).$$

(i) First let us consider $k \le 2q+1$. We use induction on k. For $k = 0$ the theorem follows from Corollary 7.10. For $k = 1, \ldots, 2q+1$, differentiating (7.3) k times with respect to x gives

$$-\varepsilon^2 \Delta \partial_x^k \boldsymbol{u} + \boldsymbol{A}\partial_x^k \boldsymbol{u} = \partial_x^k \boldsymbol{f} - \sum_{m=0}^{k-1} \binom{k}{m} \left(\partial_x^{k-m} \boldsymbol{A}\right) \partial_x^m \boldsymbol{u} =: \boldsymbol{\varphi}_k$$

with

$$|\boldsymbol{\varphi}_k(x,y)| \le C B_{k-1}(x),$$

where the bound on $\boldsymbol{\varphi}_k$ is a consequence of the inductive hypothesis. Define $\tilde{\boldsymbol{u}}$ by $\partial_x^k \boldsymbol{u} = B_k \tilde{\boldsymbol{u}}$. Then

$$-\varepsilon^2 \Delta \tilde{\boldsymbol{u}} - 2\varepsilon^2 \frac{B_k'}{B_k} \partial_x \tilde{\boldsymbol{u}} + \left(\boldsymbol{A} - \varepsilon^2 \frac{B_k''}{B_k}\right) \tilde{\boldsymbol{u}} = \frac{\varphi_k}{B_k}.$$

Taking the inner product with $\boldsymbol{D}\tilde{\boldsymbol{u}}$ and setting $v_k = \frac{1}{2}\tilde{\boldsymbol{u}}^T\boldsymbol{D}\tilde{\boldsymbol{u}}$ we obtain, as in the proof of Theorem 7.8,

$$-\varepsilon^2 \Delta v_k - 2\varepsilon^2 \frac{B_k'}{B_k}\partial_x v_k + 2(\mu^2\delta^{-2} - \rho^2)v_k \le C\|\tilde{\boldsymbol{u}}\|,$$

by (7.4) and because $B_k''(x) \le \varepsilon^{-2}\rho^2 B_k(x)$ and $\boldsymbol{\varphi}_k(x,y) \le CB_k(x)$. Boundary conditions for v_k follow from Lemma 7.11: $|v_k| \le C\|\tilde{\boldsymbol{u}}\|$ on $\partial\Omega$. Application of a comparison principle for scalar equations yields

$$\|\tilde{\boldsymbol{u}}\|^2_{\infty,\bar{\Omega}} = 2\,\|v_k\|_{\infty,\bar{\Omega}} \le C\,\|\tilde{\boldsymbol{u}}\|_{\infty,\bar{\Omega}}\,.$$

Dividing by $\|\tilde{\boldsymbol{u}}\|_{\infty,\bar{\Omega}}$ gives $\|\tilde{\boldsymbol{u}}\|_{\infty,\bar{\Omega}} \le C$; then, recalling the definition of $\tilde{\boldsymbol{u}}$, we obtain the first inequality of the theorem. The second inequality is proved in the same way.

(ii) Now consider the pure derivatives of order $(2q+2)$. The proof is different since these derivatives are discontinuous at the corners of the domain and consequently do not lie in $C(\bar{\Omega})$.

For each $(x,y) \in \Omega$, let $\Gamma(x,y)$ denote the distance from (x,y) to the nearest of the four corners of Ω. Set $\Omega_\varepsilon = \{(x,y) \in \Omega : \Gamma(x,y) < \varepsilon\}$. From (7.6b) we see immediately that (7.7a) holds true for $k = 2q+2$ on $\bar{\Omega}_\varepsilon$, because the sum of exponentials in (7.7a) is then bounded below by a positive constant.

On the other hand, $\boldsymbol{u} \in C^{2q+2,\alpha}(\overline{\Omega \setminus \Omega_\varepsilon})^\ell$ since the corners of $\bar{\Omega}$ are excluded from $\overline{\Omega \setminus \Omega_\varepsilon}$; as regards boundary data for $\partial_x^{2q+2}\boldsymbol{u}$ on $\partial(\Omega \setminus \Omega_\varepsilon)$, first (7.6b) implies that (7.7a) is valid on $\partial\Omega_\varepsilon$ and on $\{(x,y) \in \partial(\Omega \setminus \Omega_\varepsilon) : x = 0 \text{ or } 1\}$, then (7.3) gives $\left\|\partial_x^{(2q+2)}\boldsymbol{u}\right\| \le C$ on $\{(x,y) \in \partial(\Omega \setminus \Omega_\varepsilon) : y = 0 \text{ or } 1\}$. The maximum principle argument of *(i)* can now be invoked to prove (7.7a) on $\Omega \setminus \Omega_\varepsilon$ for $k = 2q+2$.

The inequality (7.7b) follows similarly. □

Remark 7.14. The bounds of Theorem 7.13 were obtained without constructing any decomposition of $\boldsymbol{u}$. ♣

Remark 7.15. Consider the case of a single equation, i.e., $\ell = 1$. Then one can apply a maximum principle argument directly to (7.3), i.e., most of the argument of Theorem 7.8 can be discarded, and following our subsequent analysis, one again obtains the bounds of Theorems 7.13. These bounds are sharper than the bounds obtained in [29] via a lengthy decomposition of $\boldsymbol{u}$. ♣

Remark 7.16. When the above arguments are applied to a reaction-diffusion system posed on a one-dimensional domain, this yields a slight improvement of the a priori bounds of [18]. ♣

7.3 Convection-Diffusion

We now consider the two-dimensional convection-diffusion problem

$$-\varepsilon\Delta u - \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in } \Omega, \;\; u = g \;\text{ on }\; \Gamma = \partial\Omega. \tag{7.8}$$

Its solution may typically exhibit three different types of layers: interior layers, parabolic boundary layers and regular boundary layers. Let us assume that Ω is a domain with a regular boundary that has a uniquely defined outward normal n almost everywhere. Then the boundary can be divided into three parts:

$$\begin{aligned}
\Gamma^- &:= \left\{\boldsymbol{x}\in\Gamma : \boldsymbol{b}^T n < 0\right\} && \textit{inflow boundary,}\\
\Gamma^0 &:= \left\{\boldsymbol{x}\in\Gamma : \boldsymbol{b}^T n = 0\right\} && \textit{characteristic boundary} \text{ and}\\
\Gamma^+ &:= \left\{\boldsymbol{x}\in\Gamma : \boldsymbol{b}^T n > 0\right\} && \textit{outflow boundary.}
\end{aligned}$$

With this notation the layers can be classified as follows:

Regular Boundary Layers occur at the outflow boundary Γ^+ and have a width of $\mathcal{O}\left(\varepsilon\ln(1/\varepsilon)\right)$. They are often also called *exponential boundary layers*.

Parabolic Boundary Layers occur at characteristic boundaries Γ^0 where the boundary is parallel to the characteristics of the vector field $\boldsymbol{b}$. They are therefore also called *characteristic boundary layers*. In the non-degenerate case, their width is $\mathcal{O}\left(\sqrt{\varepsilon}\ln(1/\varepsilon)\right)$.

Interior Layers arise, e. g., from discontinuities in the boundary data at the inflow boundary Γ^- and are propagated across the domain along the characteristics of the vector field $\boldsymbol{b}$. They are similar in nature to parabolic boundary layers and therefore also called *characteristic or parabolic interior layers*. Their thickness is $\mathcal{O}\left(\sqrt{\varepsilon}\ln(1/\varepsilon)\right)$.

7.3.1 Regular Layers

Consider the convection-diffusion problem

$$\mathcal{L}u := -\varepsilon\Delta u - \boldsymbol{b}\cdot\nabla u + cu = f \;\text{ in }\; \Omega = (0,1)^2, \;\; u = 0 \;\text{ on }\; \Gamma = \partial\Omega, \tag{7.9}$$

i. e., (7.8) on the unit square with homogeneous Dirichlet boundary conditions. The regularity of its solution was studied in Sect. 7.1.2.

We assume that $(b_1,b_2) > (\beta_1,\beta_2) > 0$ on $\bar{\Omega}$ with constants β_1 and β_2. These assumptions on $\boldsymbol{b}$ imply that the solution has exponential or regular layers along the sides $x = 0$ and $y = 0$.

A standard method to gain insight into the layer structure of the solution is the method of matched asymptotic expansions. In [108] this approach is complemented with a careful analysis of the remainder term of the expansion to obtain a solution decomposition.

Theorem 7.17. *Let* $f \in C^{4,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$. *Let* $n \geq 2$ *be an integer. Suppose that* f *satisfies the compatibility conditions*

$$f(0,0) = f(0,1) = f(1,1) = f(1,0) = 0,$$

that

$$\partial_y \left(\frac{f}{b_1}\right)(1,1) = \partial_x \left(\frac{f}{b_2}\right)(1,1),$$

$$\partial_y \left(\partial_x \left(\frac{f}{b_1}\right) - \mathcal{D}_0 \left(\frac{f}{b_1}\right) \right)(1,1) = \partial_x^2 \left(\frac{f}{b_2}\right)(1,1),$$

$$\partial_y \left(\partial_x^2 \left(\frac{f}{b_1}\right) - \mathcal{D}_0 \left(\partial_x \left(\frac{f}{b_1}\right) - \mathcal{D}_0 \left(\frac{f}{b_1}\right) \right) - 2\mathcal{D}_1 \left(\frac{f}{b_1}\right) \right)(1,1) = \partial_x^3 \left(\frac{f}{b_2}\right)(1,1)$$

and that

$$\left(b_2 \partial_x^2 \left(\frac{f}{b_2}\right) \right)(1,1) = \left(b_1 \partial_y^2 \left(\frac{f}{b_1}\right) \right)(1,1),$$

where $\mathcal{D}_0 v := -\partial_y v b_2/b_1 + vc/b_1$ *and* $\mathcal{D}_1 v := \partial_y v \partial_x (b_2/b_1) - v\partial_x(c/b_1)$. *If* $n \geq 4$, *we assume in addition that*

$$\partial_x b_2(0,0) = \partial_y b_1(0,0).$$

Then the boundary value problem (7.9) *has a solution* $u \in C^{3,\alpha}(\bar{\Omega})$, *and this solution can be decomposed as* $u = v + w_1 + w_2 + w_{12}$, *where*

$$\|v\|_{C^2(\bar{\Omega})} + \varepsilon^{\alpha} |v|_{C^{2,\alpha}(\bar{\Omega})} \leq C \ ,$$

while for all $x, y \in [0,1]$ *we have*

$$\left\| \partial_x^i w_1(x,\cdot) \right\|_{C^{\nu,\alpha}(\{x\}\times[0,1])} \leq C\varepsilon^{-i} \mathrm{e}^{-\beta_1 x/\varepsilon},$$

$$\left\| \partial_y^j w_2(\cdot,y) \right\|_{C^{\mu,\alpha}([0,1]\times\{y\})} \leq C\varepsilon^{-j} \mathrm{e}^{-\beta_2 y/\varepsilon}$$

and

$$\left|\partial_x^i \partial_y j w_{12}(x,y)\right| \le C\varepsilon^{-(i+j)} \mathrm{e}^{-(\beta_1 x+\beta_2 y)/\varepsilon}$$

for $0 \le \mu, \nu \le 2$ *and* $0 \le i, j \le n$*. Moreover, for all* $(x,y) \in \Omega$ *we have*

$$|\mathcal{L}w_1(x,y)| \le C\varepsilon \mathrm{e}^{-\beta_1 x/\varepsilon}, \ \ |\mathcal{L}w_2(x,y)| \le C\varepsilon \mathrm{e}^{-\beta_2 y/\varepsilon}$$

and

$$|\mathcal{L}w_{12}(x,y)| \le C\varepsilon \mathrm{e}^{-(\beta_1 x+\beta_2 y)/\varepsilon}.$$

Proof. The argument is very technical an therefore omitted. The interested reader is referred to the original publication [108]. □

The regular solution component is defined via solutions of hyperbolic problems. Unlike elliptic operators, these first-order operators do not possess smoothing properties. Because of this we have to assume high regularity of f and a large number of compatibility conditions in Theorem 7.17, but we expect such a decomposition to exist under less restrictive assumptions. Similar ideas have been pursued in [30] and [121], but compatibility issues are either not considered or dealt with incorrectly; see Remarks 5.1, 5.2 and 5.5 in [108].

Remark 7.18. If for the analysis of a scheme less regularity of the various components of the decomposition is required, then some of the compatibility conditions can be discarded, see [108, Remark 5.3]. ♣

Remark 7.19. A different approach is used by Roos [135] who defines the regular solution component as the solution of an elliptic problem on an extended domain. Therefore, the construction requires less regularity and compatibility of the data, but only bounds for the first order derivatives of the components of u are obtained in [135]. ♣

7.3.2 Characteristic Layers

Consider (7.8) with convective field $\boldsymbol{b} = (b, 0)$, i.e.,

$$-\varepsilon\Delta u - bu_x + cu = f \ \text{ in } \Omega = (0,1)^2, \ \ u = 0 \ \text{ on } \partial\Omega \tag{7.10}$$

with $b \ge \beta > 0$ and $c \ge 0$ on $\bar{\Omega}$.

Near the outflow boundary at $x = 0$ a regular layer $\mathrm{e}^{-\beta x/\varepsilon}$ will form, while along the characteristic boundaries $y = 0$ and $y = 1$ characteristic (or parabolic) layers appear which behave like $\mathrm{e}^{-y/\sqrt{\varepsilon}}$ and $\mathrm{e}^{-(1-y)/\sqrt{\varepsilon}}$, resp.

A detailed study of (7.10) has been conducted by Kellogg and Stynes [59, 60] for the case of constant b and c. Their analysis also takes into account the formation of corner singularities when the data does not satisfy certain compatibility condition.

Theorem 7.20. *Suppose $b > 0$ and c are constant. Let $f \in C^8(\bar{\Omega})$ satisfy*

$$f(0,0) = f(0,1) = f(1,1) = f(1,0) = 0.$$

Then the solution u of (7.10) can be decomposed as

$$u = v + w_1 + w_2 + w_{12},$$

where for all $x, y \in [0,1]$ and $0 \le i + j \le 2$, we have the pointwise estimates

$$\left|\partial_x^i \partial_y^j v(x,y)\right| \le C, \quad \left|\partial_x^i \partial_y^j w_1(x,y)\right| \le C\varepsilon^{-i} \mathrm{e}^{-\beta x/\varepsilon},$$
$$\left|\partial_x^i \partial_y^j w_2(x,y)\right| \le C\varepsilon^{-j/2} \left(\mathrm{e}^{-y/\sqrt{\varepsilon}} + \mathrm{e}^{-(1-y)/\sqrt{\varepsilon}}\right),$$
$$\left|\partial_x^i \partial_y^j w_{12}(x,y)\right| \le C\varepsilon^{-(i+j/2)} \mathrm{e}^{-\beta x/\varepsilon} \left(\mathrm{e}^{-y/\sqrt{\varepsilon}} + \mathrm{e}^{-(1-y)/\sqrt{\varepsilon}}\right)$$

and for $0 \le i + j \le 3$ the L_2 bounds

$$\left\|\partial_x^i \partial_y^j v\right\|_{0,\Omega} \le C, \quad \left\|\partial_x^i \partial_y^j w_1\right\|_{0,\Omega} \le C\varepsilon^{-i+1/2},$$
$$\left\|\partial_x^i \partial_y^j w_2\right\|_{0,\Omega} \le C\varepsilon^{-j/2+1/4}, \quad \left\|\partial_x^i \partial_y^j w_{12}\right\|_{0,\Omega} \le C\varepsilon^{-i-j/2+3/4},$$

except for $\partial_x^2 \partial_y w_2$ for which

$$\left\|\partial_x^2 \partial_y w_2\right\|_{0,\Omega} \le C\varepsilon^{-1/2}.$$

Proof. See [59, 60]. □

Chapter 8
Reaction-Diffusion Problems

In this chapter numerical methods for singularly perturbed reaction-diffusion equations on the square are studied. Find $u \in C^2(\Omega) \cup C(\bar{\Omega})$ such that

$$\mathcal{L}u := -\varepsilon^2 \Delta u + cu = f \quad \text{in} \;\; \Omega = (0,1)^2, \;\; \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}, \tag{8.1}$$

where the parameter ε satisfies $0 < \varepsilon \ll 1$ and $c \geq \gamma > 0$ on Ω.

8.1 Central Differencing

We consider central differencing for a system of $\ell \geq 2$ singularly perturbed reaction-diffusion equations on the square. Thus, instead of (8.1) we consider the more general problem of finding a vector-valued function $\boldsymbol{u} = (u_1, u_2, \ldots, u_\ell)^T$ such that

$$\boldsymbol{\mathcal{L}u} := -\varepsilon^2 \Delta \boldsymbol{u} + \boldsymbol{Au} = \boldsymbol{f} \quad \text{in} \;\; \Omega = (0,1)^2, \;\; \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{g}, \tag{8.2}$$

where the parameter ε satisfies $0 < \varepsilon \ll 1$.

Assume that a diagonal constant matrix $\boldsymbol{D} = \operatorname{diag}(d_{11}, \ldots, d_{\ell\ell})$ exists with $d_{kk} > 0$ for all k and a constant $\mu > 0$ such that

$$\boldsymbol{v}^T \boldsymbol{DAv} \geq \mu^2 \boldsymbol{v}^T \boldsymbol{v} \quad \text{in} \; \Omega \quad \text{for all} \;\; \boldsymbol{v} \in I\!R^\ell.$$

Let $\delta^2 = \max\limits_{k=1,\ldots,\ell} d_{kk}$. The analytical behaviour of the solution to (8.2) was studied in Sect. 7.2.

Consider an arbitrary tensor-product mesh $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ on $\bar{\Omega}$, with

$$\bar{\omega}_x : 0 = x_0 < x_1 < \cdots < x_N = 1$$

and

$$\bar{\omega}_y : 0 = y_0 < y_1 < \cdots < y_N = 1.$$

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_8,

Let $\omega = \bar{\omega} \cap \Omega$ and $\partial\omega = \bar{\omega} \cap \partial\Omega$. Set $h_i := x_i - x_{i-1}$ and $k_j := y_j - y_{j-1}$. Given a mesh function $\{v_{i,j}\}_{i,j=0}^N$, introduce the standard difference operators

$$v_{\bar{x};ij} = \frac{v_{ij} - v_{i-1,j}}{h_i} \quad \text{and} \quad v_{\hat{x};ij} = \frac{v_{i+1,j} - v_{ij}}{\hbar_i} \quad \text{with} \quad \hbar_i = \frac{h_i + h_{i+1}}{2}.$$

Analogously we define $v_{\bar{y};ij}$, $v_{\hat{y};ij}$ and $\bar{k}_j$.

Our discretisation is: Find $\boldsymbol{u}^N$ such that

$$\left[\boldsymbol{L}\boldsymbol{u}^N\right]_{i,j} := -\varepsilon^2 \left(\boldsymbol{u}^N_{\bar{x}\hat{x}} + \boldsymbol{u}^N_{\bar{y}\hat{y}}\right)_{i,j} + \boldsymbol{A}(x_i, y_j)\boldsymbol{u}^N_{i,j} = \boldsymbol{f}_{i,j} \quad \text{in } \omega, \tag{8.3a}$$

$$\boldsymbol{u}^N_{i,j} = \boldsymbol{g}_{i,j} \quad \text{on } \partial\omega, \tag{8.3b}$$

where $\boldsymbol{v}_{\bar{x}\hat{x}} = \left(v_{1;\bar{x}\hat{x}}, v_{2;\bar{x}\hat{x}}, \ldots, v_{\ell;\bar{x}\hat{x}}\right)^T$ for vector-valued mesh functions.

8.1.1 Stability

For vector-valued mesh functions $\boldsymbol{v}$, we set

$$\|\boldsymbol{v}\|_{\boldsymbol{D},G} := \max_{\boldsymbol{x}\in G} \left\{\boldsymbol{v}(\boldsymbol{x})^T \boldsymbol{D}\boldsymbol{v}(\boldsymbol{x})\right\}^{1/2}, \quad \text{for any } G \subset \bar{\omega}.$$

Theorem 8.1. *The discrete operator* $\boldsymbol{L}$ *is* $(\ell_\infty, \ell_\infty)$*-stable and satisfies the stability inequality*

$$\|\boldsymbol{w}\|_{\boldsymbol{D},\bar{\omega}} \le \delta^2 \mu^{-2} \max\left\{\|\boldsymbol{L}\boldsymbol{w}\|_{\boldsymbol{D},\omega}, \|\boldsymbol{w}\|_{\boldsymbol{D},\partial\omega}\right\}.$$

for arbitrary vector-valued functions $\boldsymbol{w}$ *defined on* $\bar{\omega}$.

Proof. This proof is a discrete analogue of the argument for Theorem 7.8. Let $\varphi = \frac{1}{2}\boldsymbol{w}^T \boldsymbol{D}\boldsymbol{w}$. Note that $2\varphi \le \delta^2 \boldsymbol{w}^T\boldsymbol{w}$ on $\bar{\omega}$ and

$$\varphi_{\bar{x}\hat{x};i,j} = \boldsymbol{w}^T_{i,j}\boldsymbol{D}\boldsymbol{w}_{\bar{x}\hat{x};i,j} + \sum_{k=1}^{\ell} \frac{d_{kk}}{2\hbar_i}\left(h_{i+1} w^2_{k;x;i,j} + h_i w^2_{k;\bar{x};i,j}\right)$$

with a similar identity for $\varphi_{\bar{y}\hat{y}}$. Thus,

$$\boldsymbol{w}^T_{i,j}\boldsymbol{D}\left(\boldsymbol{w}_{\bar{x}\hat{x};i,j} + \boldsymbol{w}_{\bar{y}\hat{y};i,j}\right) \le \varphi_{\bar{x}\hat{x};i,j} + \varphi_{\bar{y}\hat{y};i,j}.$$

Taking the scalar product of $\boldsymbol{D}\boldsymbol{w}$ with $-\varepsilon^2\left(\boldsymbol{w}_{\bar{x}\hat{x}} + \boldsymbol{w}_{\bar{y}\hat{y}}\right) + \boldsymbol{A}\boldsymbol{w} = \boldsymbol{L}\boldsymbol{w}$ and using the coercivity of $\boldsymbol{D}\boldsymbol{A}$, we get

$$-\varepsilon^2\left(\varphi_{\bar{x}\hat{x}} + \varphi_{\bar{y}\hat{y}}\right) + 2\mu^2\delta^{-2}\varphi \le \boldsymbol{w}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{w} \quad \text{on } \omega\,.$$

Furthermore, $|\varphi| \le \frac{1}{2}\|\boldsymbol{w}\|^2_{\boldsymbol{D},\partial\omega}$ on $\partial\Omega$. A standard discrete maximum principle for scalar problems, with a constant barrier function, yields

$$\|\varphi\|_{\infty,\bar{\omega}} \le \max\left\{\frac{\delta^2}{2\mu^2}\left\|\boldsymbol{w}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{w}\right\|_{\infty,\omega}, \frac{1}{2}\|\boldsymbol{w}\|^2_{\boldsymbol{D},\partial\omega}\right\}.$$

It follows from the Cauchy-Schwarz inequality that

$$\|\boldsymbol{w}\|^2_{\boldsymbol{D},\bar{\omega}} = 2\|\varphi\|_{\infty,\bar{\omega}} \le \|\boldsymbol{w}\|_{\boldsymbol{D},\bar{\omega}} \max\Big\{\delta^2\mu^{-2}\|\boldsymbol{L}\boldsymbol{w}\|_{\boldsymbol{D},\omega}, \|\boldsymbol{w}\|_{\boldsymbol{D},\partial\omega}\Big\},$$

and we are finished. □

Remark 8.2. Theorem 8.1 implies that the linear system (8.3) possesses a unique solution $\boldsymbol{u}^N$. ♣

8.1.2 Convergence on Layer-Adapted Meshes

In the layer regions the Bakhvalov mesh is not approximately equidistant. Therefore, the truncation error of the difference scheme is only first order at points in the layers. Nonetheless, it is straightforward to modify the analysis from Sect. 6.1.2.4 for a two-point boundary value problem to the two-dimensional problem (8.1), and to show that the truncation error is of order N^{-2} uniformly in ε.

The analysis of central differencing for reaction-diffusion problems on Shishkin meshes on the domain Ω has been carried out by Clavero et al. [29] in the case of a single equation, and by Kellogg et al. [58] for a system of equations, but the error analysis in both papers relies on the lengthy construction of a decomposition of the solution. Here we shall present the much simpler analysis from [57].

Theorem 8.3. *Suppose that the compatibility conditions (7.5a,b) are satisfied. Let $\rho \in (0, \mu/\delta)$ be arbitrary, but fixed. Assume that the mesh parameter σ satisfies $\sigma \ge 2$. Then*

$$\left\|\boldsymbol{u} - \boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \le \begin{cases} CN^{-2} & \text{for Bakhvalov meshes,} \\ CN^{-2}\ln^2 N & \text{for Shishkin meshes.} \end{cases}$$

Proof. Theorem 8.1 yields

$$\left\|\boldsymbol{u} - \boldsymbol{u}^N\right\|_{\infty,\bar{\omega}} \le C\left\|\boldsymbol{L}(\boldsymbol{u} - \boldsymbol{u}^N)\right\|_{\infty,\omega} = C\left\|\boldsymbol{L}\boldsymbol{u} - \boldsymbol{\mathcal{L}}\boldsymbol{u}\right\|_{\infty,\omega} \tag{8.4}$$

because $\boldsymbol{u} = \boldsymbol{u}^N$ on $\partial\omega$. For the m^{th} component of the truncation error we have

$$(\boldsymbol{L}\boldsymbol{u} - \boldsymbol{\mathcal{L}}\boldsymbol{u})_m = \varepsilon^2\left(\partial_x^2 u_m - u_{m;\bar{x}\hat{x}}\right) + \varepsilon^2\left(\partial_y^2 u_m - u_{m;\bar{y}\hat{y}}\right) \quad \text{for } m = 1, \ldots, \ell.$$

For any $\psi \in C^4(\Omega)$ with $\partial_x^k \psi$ bounded on $\bar{\Omega}$ for $k = 0, \ldots, 4$, Taylor expansions show that

$$\left| \left[\partial_x^2 \psi - \psi_{\bar{x}\hat{x}} \right]_{i,j} \right| \le C \left\| \partial_x^2 \psi \left(\cdot, y_j \right) \right\|_{\infty,[x_{i-1},x_{i+1}]}, \tag{8.5a}$$

$$\left| \left[\partial_x^2 \psi - \psi_{\bar{x}\hat{x}} \right]_{i,j} \right| \le C \left(h_i + h_{i+1} \right) \left\| \partial_x^3 \psi \left(\cdot, y_j \right) \right\|_{\infty,[x_{i-1},x_{i+1}]} \tag{8.5b}$$

and

$$\left| \left[\partial_x^2 \psi - \psi_{\bar{x}\hat{x}} \right]_{i,j} \right| \le C \left| h_i - h_{i+1} \right| \left| \partial_x^3 \psi(x_i, y_j) \right| + C(h_i + h_{i+1})^2 \left\| \partial_x^4 \psi(\cdot, y_j) \right\|_{\infty,[x_{i-1},x_{i+1}]}. \tag{8.5c}$$

There are analogous bounds for $\partial_y^2 \psi - \psi_{\bar{y}\hat{y}}$.

Bakhvalov meshes

On choosing $\sigma \ge 2$, an application of the technique in Sect. 6.1.2.4, combined with the bound (7.7a) on $\partial_x^k u_m$ for $k = 2, 3, 4$, yields

$$\varepsilon^2 \left| \left(\partial_x^2 u_m - u_{m;\bar{x}\hat{x}} \right)_{i,j} \right| \le CN^{-2} \quad \text{for } i, j = 1, \ldots, N-1 \text{ and } m = 1, \ldots, \ell.$$

There is a corresponding bound for $\partial_y^2 u_m - u_{m,\bar{y}\hat{y}}$. Thus, the truncation error is uniformly bounded by CN^{-2}. Invoking (8.4), the proof of Theorem 8.3 for the Bakhvalov mesh is complete.

Shishkin meshes

First consider the case where $\tau = q \le \sigma\varepsilon\rho^{-1} \ln N$. Then the mesh is uniform with mesh size N^{-1}. Furthermore, $\varepsilon^{-1} \le C \ln N$. Hence, Theorem 7.13 and (8.5c) give $\left\| \boldsymbol{Lu} - \boldsymbol{\mathcal{L}u} \right\|_{\infty,\omega} \le CN^{-2} \ln^2 N$. Invoking (8.4), Theorem 8.3 follows in this case.

Now suppose that $\tau = \sigma\varepsilon\rho^{-1} \ln N < q$. Let $x^* = 2\varepsilon\rho^{-1} \ln(1/\varepsilon)$. For each $m \in \{1, \ldots, \ell\}$ and $(x, y) \in \bar{\Omega}$, set

$$v_m(x,y) = \begin{cases} \displaystyle\sum_{\nu=0}^{4} \frac{(x - x^*)^\nu}{\nu!} \partial_x^\nu u_m(x^*, y) & \text{for } 0 \le x \le x^*, \\ u_m(x,y) & \text{for } x^* \le x \le 1 - x^*, \\ \displaystyle\sum_{\nu=0}^{4} \frac{(x - x^*)^\nu}{\nu!} \partial_x^\nu u_m(1 - x^*, y) & \text{for } 1 - x^* \le x \le 1, \end{cases}$$

and $w_m(x, y) = u_m(x, y) - v(x, y)$. Then Theorem 7.13, and the choice of x^*, imply that

$$\left| \partial_x^k v_m(x,y) \right| \le C \left(1 + \varepsilon^{2-k} \right)$$

and

$$\left|\partial_x^k w_m(x,y)\right| \le C\varepsilon^{-k}\left(\mathrm{e}^{-\rho x/\varepsilon} + \mathrm{e}^{-\rho(1-x)/\varepsilon}\right) \quad \text{for } 0 \le k \le 4.$$

That is, $\boldsymbol{u} = \boldsymbol{v}+\boldsymbol{w}$, i.e., we have decomposed $\boldsymbol{u}$ into a sum of a regular component $\boldsymbol{v}$ and a layer component $\boldsymbol{w} = (w_1, \ldots, w_\ell)^T$. We remark that our decomposition does not in general satisfy $\boldsymbol{\mathcal{L}}\boldsymbol{v} = \boldsymbol{f}$ and $\boldsymbol{\mathcal{L}}\boldsymbol{w} = \boldsymbol{0}$; these additional properties, which are not needed here, have been obtained for various singular perturbation problems via more complicated analyses.

Split the truncation error by writing

$$\left[\partial_x^2 u_m - u_{m;\bar{x}\hat{x}}\right]_{i,j} = \left[\partial_x^2 v_m - v_{m;\bar{x}\hat{x}}\right]_{i,j} + \left[\partial_x^2 w_m - w_{m;\bar{x}\hat{x}}\right]_{i,j}.$$

When bounding the truncation error in $\boldsymbol{v}$, use (8.5b) for $i = qN$ or $i = (1-q)N$, i.e., at the mesh transition points, and at other points use (8.5c). For the layer term $\boldsymbol{w}$, invoke (8.5a) for $i = qN, \ldots, (1-q)N$ and (8.5c) otherwise. This yields

$$\varepsilon^2\left|\left[\partial_x^2 u_m - \delta_x^2 u_m\right]_{i,j}\right| \le CN^{-2}\ln^2 N + \begin{cases} C\varepsilon N^{-1} & \text{for } i \in \{qN, (1-q)N\}, \\ 0 & \text{otherwise.} \end{cases}$$

We cannot invoke (8.4) to get the desired error bound. Instead we proceed as follows. Write $\boldsymbol{u} - \boldsymbol{u}^N = \boldsymbol{\psi} + \boldsymbol{\varphi}$, where $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ are the solutions of

$$\begin{aligned} \boldsymbol{L}\boldsymbol{\psi} &= \varepsilon^2\left(\partial_x^2 \boldsymbol{u} - \boldsymbol{u}_{\bar{x}\hat{x}}\right) \quad \text{in } \omega, \quad \boldsymbol{\psi} = \boldsymbol{0} \quad \text{on } \partial\omega, \\ \boldsymbol{L}\boldsymbol{\varphi} &= \varepsilon^2\left(\partial_y^2 \boldsymbol{u} - \boldsymbol{u}_{\bar{y}\hat{y}}\right) \quad \text{in } \omega, \quad \boldsymbol{\varphi} = \boldsymbol{0} \quad \text{on } \partial\omega. \end{aligned}$$

In order to bound $\boldsymbol{\psi}$ we use the technique of Theorem 8.1. Set $\Psi = \frac{1}{2}\boldsymbol{\psi}^T\boldsymbol{\psi}$. Then

$$-\varepsilon^2\left(\Psi_{\bar{x}\hat{x}} + \Psi_{\bar{y}\hat{y}}\right) + 2\mu^2\delta^{-2}\Psi \le C\left\|\boldsymbol{\psi}\right\|_{\infty,\omega}\left|\boldsymbol{L}\boldsymbol{\psi}\right| \quad \text{in } \omega, \quad \Psi = 0 \quad \text{on } \partial\omega.$$

One can apply a discrete comparison principle for scalar equations using the barrier function from [122]:

$$\bar{\Psi}_{i,j} = C_1\left\|\boldsymbol{\psi}\right\|_{\infty,\omega} N^{-2}\left(\ln^2 N + \tau\varepsilon^{-1}\chi_i\right),$$

where

$$\chi_i := \begin{cases} x_i\tau^{-1} & \text{for } i = 0, \ldots, qN, \\ 1 & \text{for } i = qN, \ldots, (1-q)N, \\ (1-x_i)\tau^{-1} & \text{for } i = (1-q)N, \ldots, N. \end{cases}$$

Then with C_1 chosen sufficiently large, independently of ε, one obtains $\|\Psi\|_{\infty,\bar{\omega}} \leq C\|\bar{\Psi}\|_{\infty,\bar{\omega}}$. Hence,

$$\|\boldsymbol{\psi}\|_{\infty,\bar{\omega}} \leq CN^{-2}\ln^2 N.$$

An identical bound is obtained for $\boldsymbol{\varphi}$. The assertion of the theorem follows for the Shishkin mesh. □

Remark 8.4. Our convergence analysis is based on derivative bounds whose derivation requires that $\boldsymbol{A} \in C^{4,\alpha}(\bar{\Omega})^{\ell\times\ell}$ and $\boldsymbol{f} \in C^{4,\alpha}(\bar{\Omega})^{\ell}$, and that the compatibility conditions (7.5a) and (7.5b) are satisfied. Therefore, $\boldsymbol{u} \in C^{3,\alpha}(\bar{\Omega}) \cap C^6(\Omega)$.

Andreev [8] analyses central differencing for a scalar reaction-diffusion equation assuming that only the lowest order compatibility condition (7.5a) is satisfied. In this case one has $u \in C^{1,\alpha}(\bar{\Omega}) \cap C^{6,\alpha}(\Omega)$ only. Nonetheless, it is shown in [8] that on a Shishkin mesh

$$\left\|u - u^N\right\|_{\infty,\bar{\omega}} \leq CN^{-2}\ln^4 N.$$

The key ingredient is a careful analysis of the corner singularity arising from the violation of the compatibility condition (7.5b). ♣

Remark 8.5. The situation becomes more involved when the reaction-diffusion equation is considered on an L-shaped domain:

$$-\varepsilon^2\Delta u + cu = f \quad \text{in } \Omega = (-1,1) \setminus [0,1]^2, \quad u|_{\partial\Omega} = g,$$

$\varepsilon \in (0,1]$, $c \geq \rho^2$ on $\bar{\Omega}$ with a positive constant ρ.

In addition to boundary layers along all six sides of the domain, there is also a singularity at the re-entrant corner $(0,0)$. Because of this, the solution only lies in the Hölder space $C^{2/3}(\bar{\Omega})$. Andreev and Kopteva [9] consider central differencing on a Shishkin mesh with additional refinement at the corner $(0,0)$.

Fix the transition parameter $\tau = \min\left\{1/3, 2\varepsilon\rho^{-1}\ln N, \frac{1}{3}\right\}$. The mesh $\bar{\omega}_x$ in x-direction is then constructed by dividing each of the four intervals $[-1,\tau-1]$, $[\tau-1,-\tau]$, $[\tau,1-\tau]$ and $[1-\tau,1]$ into $M = N/6$ equidistant subintervals. On $[-\tau,\tau]$ a polynomial Shishkin mesh with $m = 3$ is used; see (2.13), i.e., the mesh points are fixed by $x_i = \tau\,(i/M)^3$, $i = -M,\dots,M$.

The final mesh on $\bar{\Omega}$ is $\bar{\omega} = \bar{\omega}_x^2 \setminus (0,1)^2$; see Fig. 8.1. For central differencing on this mesh the pointwise error bound

$$\left\|u - u^N\right\|_{\infty,\bar{\omega}} \leq CN^{-2}\ln^2 N$$

is established in [9]. ♣

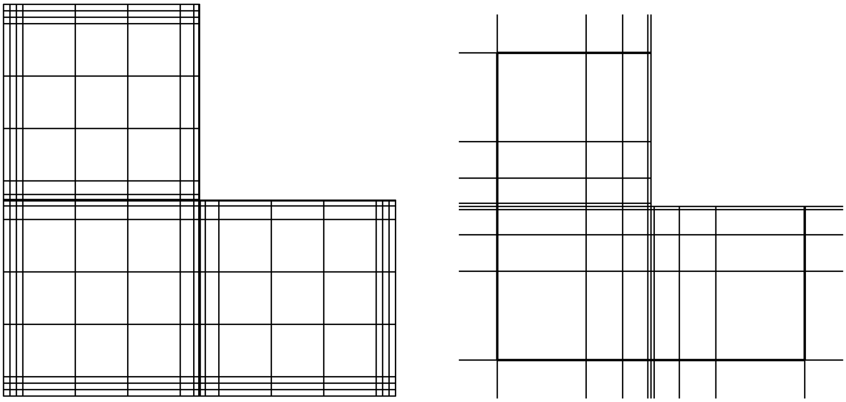


Fig. 8.1 Shishkin mesh and geometric refinement for a reaction-diffusion problem with boundary layers and a corner singularity. The right plot zooms into the region $[-\tau, \tau]^2$ to better visualise the polynomial refinement

8.1.3 Numerical Results

We now present the results of some numerical experiments in order to illustrate the conclusions of Theorem 4.8, and to check if they are sharp.

Example 1.

$$\begin{aligned} -\varepsilon^2 \Delta u_1 + 2u_1 + xu_2 &= \sin\bigl(\pi(x+y)\bigr) \quad \text{in } \Omega,\ u_1|_{\partial\Omega} = 0 \\ -\varepsilon^2 \Delta u_2 + (1+y^2)u_1 + (3+x)u_2 &= 3x(1-x) \qquad\quad \text{in } \Omega,\ u_1|_{\partial\Omega} = 0. \end{aligned}$$

For this problem $\boldsymbol{D} = \boldsymbol{I}$ and $\mu^2 \approx 1.12$.

Example 2.

$$\begin{aligned} -\varepsilon^2 \Delta u_1 + (3+x)u_1 + yu_2 + x^2 u_3 &= 1 \quad \text{in } \Omega,\ u_1|_{\partial\Omega} = 0 \\ -\varepsilon^2 \Delta u_2 + yu_1 + (4-y)u_2 + xyu_3 &= 0 \quad \text{in } \Omega,\ u_2|_{\partial\Omega} = 0 \\ -\varepsilon^2 \Delta u_3 + x^2 u_1 + xyu_2 + (3-x^2)u_3 &= 1 \quad \text{in } \Omega,\ u_3|_{\partial\Omega} = 0 \end{aligned}$$

with $\gamma^2 \approx 1.32$.

For both problems we take $\rho = 1$ and $\sigma = 2$ in the construction of the meshes. The perturbation parameter ε is chosen to be 10^{-8}.

The exact solutions to both test problems is not available, so we estimate the accuracy of the numerical solution by comparing it with the numerical solution obtained from Richardson extrapolation over two meshes, which has a higher order of accuracy. Thus, let $\boldsymbol{u}_\varepsilon^N$ be the solution of the difference scheme on the original Shishkin mesh and $\tilde{\boldsymbol{u}}_\varepsilon^{2N}$ that on the mesh obtained by uniformly bisecting the original mesh. Then the extrapolated solution is

$$\boldsymbol{u}_\varepsilon^{R,N} := \frac{4\tilde{\boldsymbol{u}}_\varepsilon^{2N} - \boldsymbol{u}_\varepsilon^N}{3}.$$

We then compute the error

$$\eta_\varepsilon^N := \left\| \boldsymbol{u}_\varepsilon^N - \boldsymbol{u}_\varepsilon^{R,N} \right\|_{\infty,\omega} = \frac{4}{3} \left\| \boldsymbol{u}_\varepsilon^N - \tilde{\boldsymbol{u}}_\varepsilon^{2N} \right\|_{\infty,\omega}.$$

The results of our test computations are displayed in the following table.

	Example 1						Example 2					
	Shishkin mesh			Bakhvalov mesh			Shishkin mesh			Bakhvalov mesh		
N	η^N	r^N	C^N	η^N	r^N	C^N	η^N	r^N	C^N	η^N	r^N	C^N
16	1.45e-2	0.93	0.48	5.17e-3	1.81	1.32	1.18e-2	0.80	0.39	4.53e-3	1.82	1.16
32	7.57e-3	1.41	0.65	1.37e-3	1.94	1.40	6.47e-3	1.20	0.55	1.19e-3	1.95	1.22
64	2.84e-3	1.47	0.67	3.52e-4	1.99	1.44	2.82e-3	1.46	0.67	3.03e-4	1.98	1.24
128	1.02e-3	1.58	0.71	8.86e-5	2.00	1.45	1.02e-3	1.58	0.71	7.61e-5	2.00	1.25
256	3.43e-4	1.65	0.73	2.22e-5	2.00	1.45	3.41e-4	1.65	0.73	1.90e-5	2.00	1.25
512	1.09e-4	—	0.73	5.55e-6	—	1.45	1.09e-4	—	0.73	4.76e-6	—	1.25

As well as the uniform errors η^N, we also give experimental rates of convergence r^N which are computed by means of the formula

$$r^N = \log_2 \left(\eta^N / \eta^{2N} \right).$$

We also estimate the constants in the error estimate, i. e., if we have the theoretical error bound $\eta^N \le C\vartheta(N)$, then we compute the quantity $C^N = \eta^N / \vartheta(N)$.

The numerical results are in accordance with Theorem 8.3 and illustrate its sharpness. The 2nd example deliberately does not satisfy the compatibility condition $\boldsymbol{f} = \boldsymbol{0}$ in the corners. Nonetheless, we observe (almost) second order convergence. This illustrates Andreev's findings.

8.2 Arbitrary Bounded Domains

So far we have considered only rectangular domains or domains that can be assembled from two rectangles. Kopteva [67] considers the semilinear problem

$$-\varepsilon^2 \Delta u + b(\cdot, u) = 0, \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega, \tag{8.6}$$

with an arbitrary bounded domain $\Omega \subset \mathbb{R}^2$. Depending on the precise assumptions on the data b and g, the problem may possess multiple solutions with a boundary layer of width $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$ along all of $\partial\Omega$; see Fig. 8.2.

The domain is discretised as in Figure 8.3, using layer-adapted tensor-product meshes of Bakhvalov and Shishkin types along the boundary with the typical refinement perpendicular to it. On this part of the mesh Kopteva's method uses a central finite difference approximation adapted to the curvilinearity of the mesh. In the interior region, lumped mass linear finite elements on a quasiuniform Delaunay triangulation is used. The number of mesh nodes is $\mathcal{O}\left(h^{-2}\right)$, where h is the maximum side length of mesh elements. The key feature of the method is that it uses an M-matrix discretisation of the Laplace operator.

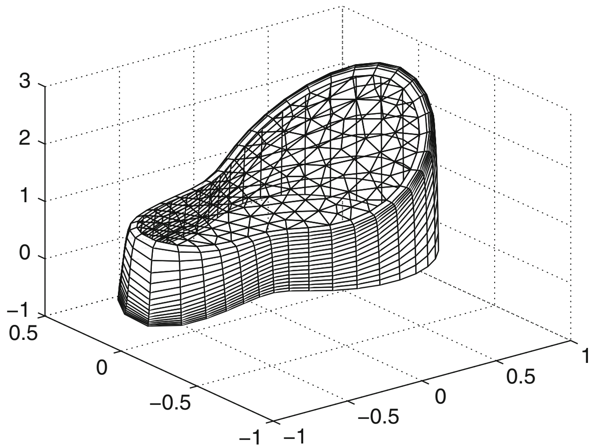


Fig. 8.2 Typical solution of (8.6)

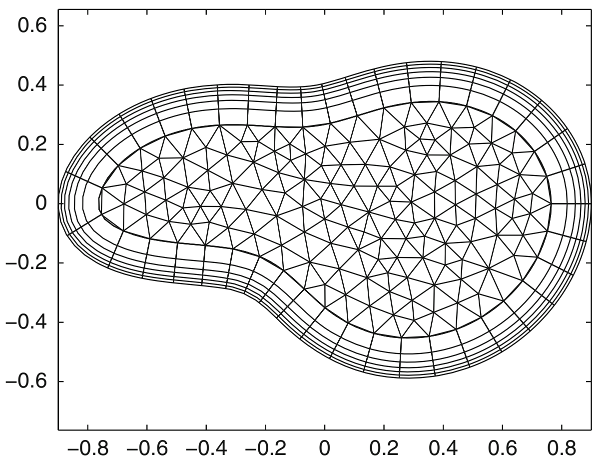


Fig. 8.3 A Bakhvalov-type mesh for (8.6)

The resulting method is shown in [67] to be uniformly convergent if $\varepsilon \le Ch$ in the following sense. For any solution u of (8.6), there exists a discrete solution u^N such that the nodal error satisfies

$$\left\| u - u^N \right\|_{\infty,\bar{\omega}} \le \begin{cases} Ch^2 |\ln h|^2 & \text{for a Shishkin mesh,} \\ Ch^2 & \text{for a Bakhvalov mesh.} \end{cases}$$

Figs. 8.2 and 8.3 were kindly provided by N. Kopteva.

Chapter 9
Convection-Diffusion Problems

This chapter is devoted to numerical methods for the convection-diffusion problem

$$-\varepsilon \Delta u - \boldsymbol{b}\nabla u + cu = f \ \text{ in } \ \Omega = (0,1)^2, \ \ u|_{\partial\Omega} = 0, \tag{9.1}$$

with $b_1 \geq \beta_1 > 0$, $b_2 \geq \beta_2 > 0$ on $[0,1]^2$, i.e., problems with regular boundary layers at the outflow boundary $x = 0$ and $y = 0$. The analytical behaviour of the solution of (9.1) was studied in Sect. 7.3.1.

Results for problems with characteristic layers will only be mentioned briefly.

9.1 Upwind Difference Schemes

We shall consider discretisations of (9.1) on a tensor product mesh $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ with N mesh intervals in both coordinate directions.

The simple upwind scheme for (9.1) is: Find $u^N \in I\!R_0^{(N+1)^2}$ such that

$$\left[Lu^N\right]_{ij} = f_{ij} \quad \text{for } \ i,j = 1,\ldots,N-1 \tag{9.2}$$

with

$$\left[Lu^N\right]_{ij} := -\varepsilon\left(u^N_{\bar{x}\hat{x};ij} + u^N_{\bar{y}\hat{y};ij}\right) - b_{1;ij}u^N_{x;ij} - b_{2;ij}u^N_{y;ij} + c_{ij}u^N_{ij},$$

and

$$v_{x;ij} = \frac{v_{i+1,j} - v_i}{h_{i+1}}, \quad v_{\bar{x};ij} = \frac{v_{ij} - v_{i-1,j}}{h_i} \quad \text{and} \quad v_{\hat{x};ij} = \frac{v_{i+1,j} - v_{ij}}{\hbar_i},$$

$\hbar_i = (h_i + h_{i+1})/2$ and analogous definitions for $v_{y;ij}$, $v_{\bar{y};ij}$, $v_{\hat{y};ij}$ and $\bar{k}_j$.

This scheme on layer-adapted meshes was first studied by Shishkin who established the maximum-norm error estimate

$$\|u - u^N\|_{\infty,\omega} \leq CN^{-1}\ln^2 N$$

T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Mathematics 1985, DOI 10.1007/978-3-642-05134-0_9,

on Shishkin meshes; see [121]. He also proved [151, §3, Theorem 2.3]

$$\|u - u^N\|_{\infty,\omega} \le C\big(N^{-1}\ln^2 N\big)^p$$

with $p = 1/4$ and $p = 1/8$ (depending on the precise assumptions on the data) if the solution is less smooth.

Here we shall present the technique from [107] which gives a sharper error estimate. This technique is an extension of the truncation error and barrier function technique from Sect. 4.2.6 to two dimensions.

9.1.1 Stability

The matrix associated with L is an L_0-matrix. Application of the M-criterion (Lemma 3.14) with the test function $v_{ij} = (1 - x_i)/\beta_1$ establishes the inverse monotonicity of L. As a consequence, we have the $(\ell_\infty, \ell_\infty)$-stability inequality

$$\|v\|_{\infty,\bar\omega} \le \min\left\{\left\|\frac{Lv}{b_1}\right\|_{\infty,\omega}, \left\|\frac{Lv}{b_2}\right\|_{\infty,\omega}, \left\|\frac{Lv}{c}\right\|_{\infty,\omega}\right\} \quad \text{for all } v \in \left(I\!R_0^{N+1}\right)^2,$$

by Lemma 3.17.

Because of the inverse monotonicity we also have the following discrete comparison principle. For any two mesh functions $v, w \in \left(I\!R^{N+1}\right)^2$

$$\left.\begin{array}{ll} Lv \le Lw & \text{in } \omega \text{ and} \\ v \le w & \text{on } \partial\omega \end{array}\right\} \quad \Longrightarrow \quad v \le w \quad \text{on } \bar\omega.$$

It will be used repeatedly in the convergence analysis.

9.1.2 Pointwise Error Bounds

Consider the upwind scheme (9.2) on a tensor-product Shishkin-type mesh $\bar\omega = \bar\omega_x \times \bar\omega_y$ where the two one-dimensional meshes in x- and y-direction are constructed as described in Sect. 2.1.3. Give mesh parameters $\sigma > 0$ and $q > 0$ the mesh uses transition points in the S-type mesh are

$$\tau_x := \min\left\{q, \frac{\sigma\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \tau_y := \min\left\{q, \frac{\sigma\varepsilon}{\beta_2}\ln N\right\}.$$

Fig. 2.19 displays a plot of the resulting mesh.

Theorem 9.1. *Assume the solution u of (9.1) can be decomposed as in Theorem 7.17 with $\alpha = 1$ and $n = 3$. Let the mesh be a tensor-product S-type mesh with $\sigma \geq 2$. Suppose the mesh generating function $\tilde{\varphi}$ satisfies (2.8) and (2.14). Then the error of the simple upwind scheme (9.2) satisfies*

$$\left|u_{ij} - u_{ij}^N\right| \leq \begin{cases} C\left(h + N^{-1}\right) & \textit{for } i, j = qN, \dots, N, \\ C\left(h + N^{-1} \max\left|\psi'\right|\right) & \textit{otherwise.} \end{cases}$$

Proof. We adapt the truncation error and barrier function technique of Sect. 4.2.6 to two space dimensions.

Recalling the decomposition of Theorem 7.17, we split the numerical solution in a similar manner:

$$u^N = v^N + w_1^N + w_2^N + w_{12}^N,$$

where

$$Lv^N = \mathcal{L}v, \quad Lw_1^N = \mathcal{L}w_1, \quad Lw_2^N = \mathcal{L}w_2, \quad Lw_{12}^N = \mathcal{L}w_{12} \quad \text{on } \omega,$$

and

$$v^N = v, \quad w_1^N = w_1, \quad w_2^N = w_2, \quad w_{12}^N = w_{12} \quad \text{on } \partial\omega.$$

For the regular solution component a Taylor expansion, the derivative bounds of Theorem 7.17 and the inverse monotonicity of L give

$$\left\|v - v^N\right\|_{\infty,\omega} \leq Ch.$$

For the term representing the layer at $x = 0$, we have, similarly to (4.44),

$$0 \leq w_{1;ij}^N \leq \bar{w}_{1;i}^N := C \prod_{k=1}^{i} \left(1 + \frac{\beta_1 h_k}{2\varepsilon}\right)^{-1} \quad \text{for } i, j = 0, \dots, N.$$

Thus

$$\left|w_{1;ij} - w_{1;ij}^N\right| \leq CN^{-1} \quad \text{for } i = qN, \dots, N, \ j = 0, \dots, N;$$

see the argument that led to (4.45). Now let $i < qN$. Taylor expansions and Theorem 7.17 give

$$\begin{aligned} \left|L(w_1 - w_1^N)_{ij}\right| &\leq C\left(h + \varepsilon^{-2}\left(h_i + h_{i+1}\right) \mathrm{e}^{-\beta_1 x_{i-1}/\varepsilon}\right) \\ &\leq C\left(h + \varepsilon^{-1} \bar{w}_{1;i}^N N^{-1} \max\left|\psi'\right|\right). \end{aligned}$$

Application of the comparison principle with the barrier function

$$C\left(N^{-1}+h+\bar{w}_{1;i}^{N}N^{-1}\max|\psi'|\right)$$

with C sufficiently large yields

$$\left|w_{1;ij}-w_{1;ij}^{N}\right|\le C\left(h+N^{-1}\max|\psi'|\right)\quad\text{for } i=0,\dots,qN-1,\ j=0,\dots,N.$$

For the boundary layer at $y=0$, the same type of argument is used in order to obtain

$$\left|w_{2;ij}-w_{2;ij}^{N}\right|\le CN^{-1}\quad\text{for } i=0,\dots,N,\ j=qN,\dots,N$$

and

$$\left|w_{2;ij}-w_{2;ij}^{N}\right|\le C\left(h+N^{-1}\max|\psi'|\right)\quad\text{for } i=0,\dots,N,\ j=0,\dots,qN-1.$$

Finally, for the corner layer term one first shows

$$\left|w_{12;ij}-w_{12;ij}^{N}\right|\le\bar{w}_{12;ij}^{N}:=C\prod_{k=1}^{i}\left(1+\frac{\beta_1h_k}{2\varepsilon}\right)^{-1}\prod_{l=1}^{j}\left(1+\frac{\beta_2k_l}{2\varepsilon}\right)^{-1}\quad\text{for } i,j=0,\dots,N,$$

which implies

$$\left|w_{12;ij}-w_{12;ij}^{N}\right|\le CN^{-1}\quad\text{if } i\ge qN \text{ or } j\ge qN.$$

In a second step the truncation error is estimated using Taylor expansions:

$$\left|L(w_{12}-w_{12}^{N})_{ij}\right|\le C\varepsilon^{-1}\bar{w}_{12;ij}^{N}N^{-1}\max|\psi'|.$$

And the discrete comparison principle yields

$$\left|w_{12;ij}-w_{12;ij}^{N}\right|\le CN^{-1}\max|\psi'|\quad\text{for } i,j=0,\dots,qN-1.$$

Collecting the bounds for the various components, we are finished. □

Remark 9.2. We are not aware of any a priori error estimates for arbitrary meshes similar to those of Sect. 4.2.2 for one-dimensional problems. This seems to be due to a lack of strong negative-norms stability inequalities. ♣

Remark 9.3. In [107] a modified, hybrid scheme on a standard Shishkin mesh is considered. It is based on simple upwinding, but employs central differencing whenever the mesh allows one to do this without losing stability. For this scheme the above technique gives the maximum-norm error bound

$$\left\|u - u^N\right\|_{\infty,\omega} \le CN^{-1}.$$

The improved bound is because central differencing improves the error terms of order $N^{-1} \max |\psi'|$ in the above proof to order $N^{-2} \max |\psi'|^2$. ♣

A numerical example

We briefly illustrate our theoretical findings for the simple upwind difference scheme on S-type meshes and for the hybrid scheme when applied to the test problem

$$-\varepsilon\Delta u - (2+x)u_x - (3+y^2)u_y + u = f \quad \text{in} \quad \Omega = (0,1)^2, \tag{9.3a}$$

$$u = 0 \quad \text{on} \quad \Gamma = \partial\Omega, \tag{9.3b}$$

where the right-hand side is chosen such that

$$u(x,y) = \cos\frac{\pi x}{2}\left(1 - \mathrm{e}^{-2x/\varepsilon}\right)(1-y)^3\left(1 - \mathrm{e}^{-3y/\varepsilon}\right) \tag{9.3c}$$

is the exact solution. This function exhibits typical boundary layer behaviour. For our tests we take $\varepsilon = 10^{-8}$, which is a sufficiently small choice to bring out the singularly perturbed nature of the problem. Table 9.1 displays the results of our text computations. They are in agreement with the theoretical findings.

Table 9.1 Upwind and hybrid difference scheme on S-type meshes

	simple upwinding						hybrid scheme	
	standard Shishkin mesh		Shishkin mesh with 2 transition points		Bakhvalov-Shishkin mesh		standard Shishkin mesh	
N	error	rate	error	rate	error	rate	error	rate
16	9.6379e-2	0.50	9.0430e-2	0.73	9.3261e-2	0.74	1.1072e-1	0.88
32	6.8194e-2	0.59	5.4533e-2	0.76	5.5803e-2	0.90	5.9962e-2	0.94
64	4.5364e-2	0.66	3.2138e-2	0.79	2.9916e-2	0.93	3.1328e-2	0.97
128	2.8636e-2	0.72	1.8606e-2	0.84	1.5665e-2	0.97	1.6031e-2	0.98
256	1.7360e-2	0.77	1.0416e-2	0.87	8.0140e-3	0.98	8.1081e-3	0.99
512	1.0182e-2	0.80	5.6941e-3	0.90	4.0529e-3	0.99	4.0768e-3	1.00
1024	5.8286e-3	0.83	3.0602e-3	0.91	2.0379e-3	1.00	2.0440e-3	1.00
2048	3.2776e-3	—	1.6247e-3	—	1.0219e-3	—	1.0234e-3	—

9.1.3 Error Expansion

Kopteva [65] derives an error expansion for the simple upwind scheme (9.2) on standard Shishkin meshes. Let h_c and h_f denote the coarse and fine mesh sizes in the Shishkin mesh. Provided that $\varepsilon \le CN^{-1}$, she proves that the error can be expanded as

$$u_{ij}^N - u_{ij} = h_c \Phi_{ij} + \frac{h_f}{\varepsilon} \Psi_{ij} + R_{ij}$$

with

$$\Phi(x,y) = \varphi(x,y) + \varphi(0,0) \exp\left(-\frac{b_1(0,0)x + b_2(0,0)y}{\varepsilon}\right) - \varphi(0,y) \exp\left(-\frac{b_1(0,y)x}{\varepsilon}\right) - \varphi(x,0) \exp\left(-\frac{b_2(x,0)y}{\varepsilon}\right)$$

and

$$\Psi(x,y) = \frac{x}{\varepsilon} \frac{b_1^2(0,y)\tilde{w}_1 + b_1^2(0,0)\tilde{w}_{12}}{2} + \frac{y}{\varepsilon} \frac{b_2^2(x,0)\tilde{w}_2 + b_2^2(0,0)\tilde{w}_{12}}{2},$$

where the $\tilde{w}$'s satisfy bounds similar to those of Theorem 7.17 and $\|\varphi\|_{C^{1,1}} \le C$, while for the remainder we have

$$R_{ij} \le \begin{cases} CN^{-2} & \text{for } i,j = qN, \dots, N, \\ CN^{-2} \ln^2 N & \text{otherwise.} \end{cases}$$

This expansion is used in [65] to derive error bounds for Richardson extrapolation and for the approximation of derivatives.

Richardson extrapolation

Let $\tilde{u}^N$ be the upwind difference solution on the mesh obtained by uniformly bisecting the original mesh $\bar{\omega}$ and let $\Pi \tilde{u}^N$ be the obvious restriction of $\tilde{u}^N$ to $\bar{\omega}$. Then

$$\left| \left([2\Pi \tilde{u}^N - u^N] - u \right)_{ij} \right| \le C \begin{cases} N^{-2} & \text{for } i,j = qN, \dots, N-1, \\ N^{-2} \ln^2 N & \text{otherwise [65].} \end{cases}$$

These results are neatly illustrated by the numbers in Table 9.2 which display the results of Richardson extrapolation applied to our test problem (9.3).

Table 9.2 Richardson extrapolation on a Shishkin mesh

	fine mesh region		coarse mesh region	
N	error	rate	error	rate
16	1.3869e-2	1.08	3.7171e-3	1.44
32	6.5448e-3	1.23	1.3733e-3	1.74
64	2.7918e-3	1.38	4.1086e-4	1.87
128	1.0703e-3	1.49	1.1271e-4	1.93
256	3.8049e-4	1.58	2.9616e-5	1.96
512	1.2701e-4	1.64	7.5975e-6	1.98
1024	4.0623e-5	—	1.9234e-6	—

Derivative approximation

In [65] the bounds

$$\left|(u^N - u)_{x;ij}\right| \le C \begin{cases} N^{-1} & \text{for } i,j = qN,\dots,N-1, \\ N^{-1}\ln^2 N & \text{for } i = qN,\dots,N-1,\ j = 0,\dots,qN-1, \\ \varepsilon^{-1}N^{-1}\ln N & \text{otherwise} \end{cases}$$

are given with analogous results for $(u^N - u)_y$.

9.2 Finite Element Methods

This section is concerned with finite element discretisations of (9.1).

The variational formulation of (9.1) is as follows. Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v) \quad \text{for all } v \in H_0^1(\Omega), \tag{9.4}$$

where

$$a(u,v) = \varepsilon(\nabla u, \nabla v) - (\boldsymbol{b}\cdot\nabla u, v) + (cu, v) \quad \text{and} \quad f(v) = (f, v)$$

with

$$(u,v) := \int_\Omega u(x,y)v(x,y)\,\mathrm{d}x\mathrm{d}y.$$

If

$$c + \tfrac{1}{2}\operatorname{div}\boldsymbol{b} \ge \gamma > 0 \tag{9.5}$$

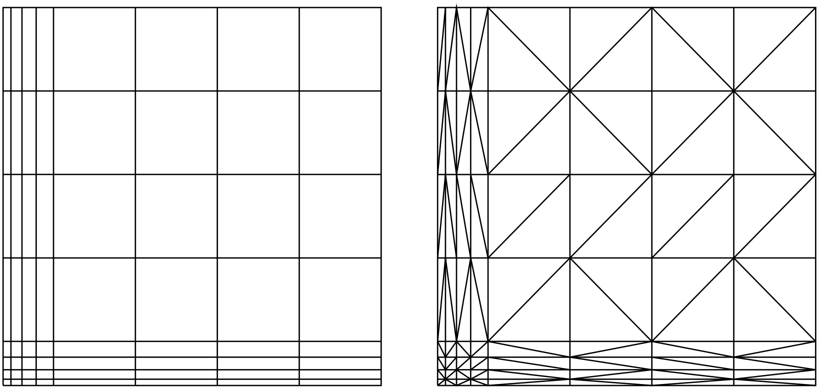


Fig. 9.1 Triangulations into rectangles and triangles on tensor-product layer-adapted meshes

then the bilinear form $a(\cdot,\cdot)$ is coercive, i.e.,

$$a(v,v) \geq |||v|||_\varepsilon^2 := \varepsilon\left(\|\partial_x v\|_0^2 + \|\partial_y v\|_0^2\right) + \gamma\|v\|_0^2 \quad \text{for all } v \in H_0^1(\Omega),$$

and (9.4) possesses a unique solution $u \in H_0^1(\Omega)$.

We shall restrict ourselves to tensor-product meshes $\bar{\omega} := \bar{\omega}_x \times \bar{\omega}_y$ as in Sect. 9.1. Set $I_i := [x_{i-1}, x_i]$, $J_j := [y_{j-1}, y_j]$ and $T_{ij} := I_i \times J_j$. We shall consider both bilinear elements on rectangles and linear elements on triangles with the triangulation obtained by drawing either diagonal in each of the mesh rectangles; see Fig. 9.1.

9.2.1 The Interpolation Error

The first important results are bounds for the interpolation error. We denote by ψ^I the piecewise bilinear function that interpolates to ψ at the nodes of the mesh $\bar{\omega}$. The meshes we consider are characterised by high aspect ratios of the mesh elements. Because of this anisotropy, standard interpolation theory cannot be applied. There have been a number of contributions to extend the theory to anisotropic elements, e.g., [15, 171, 172]. The first uniform interpolation error estimates for layer-adapted meshes, namely Shishkin meshes, were derived by Stynes and O'Riordan [152] and Dobrowolski and Roos [30]. Here we shall give the more general results from [84].

Set

$$\Theta_{cd}^{[p]}(T_{ij}) := \int_{I_i}\left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta_1 x/(p\varepsilon)}\right)\mathrm{d}x + \int_{J_j}\left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta_2 y/(2\varepsilon)}\right)\mathrm{d}y$$

and

$$\vartheta_{cd}^{[p]}(\bar{\omega}) := \max_{i,j=1,\ldots,N} \Theta_{cd}^{[p]}(T_{ij}).$$

Theorem 9.4. *Suppose the assumptions of Theorem 7.17 are satisfied. Then the maximum-norm error of bilinear interpolation on a tensor-product mesh satisfies*

$$\left\|u^I - u\right\|_{\infty,T_{ij}} \le C\left(\varTheta_{cd}^{[2]}(T_{ij})\right)^2, \tag{9.6}$$

$$\varepsilon\left\|\nabla(u^I - u)\right\|_{\infty,T_{ij}} \le C\varTheta_{cd}^{[1]}(T_{ij}) \tag{9.7}$$

and for the ε*-weighted energy norm*

$$\left|\left|\left|u^I - u\right|\right|\right|_\varepsilon \le C\vartheta_{cd}^{[2]}(\bar\omega)\,.$$

Proof. First Theorem 7.17 implies

$$\left|\partial_x^i\partial_y^j u(x,y)\right| \le C\Big(1+\varepsilon^{-i}\mathrm{e}^{-\beta_1 x/\varepsilon}\Big)\times\Big(1+\varepsilon^{-j}\mathrm{e}^{-\beta_2 y/\varepsilon}\Big) \tag{9.8}$$

for $i+j\le 2$.

(i) Let $(x,y)\in T_{ij}$. Then

$$\begin{aligned}\left(u-u^I\right)(x,y) = {} & \frac{1}{h_i}\int_{I_i}\int_{x_{i-1}}^{x}\int_\sigma^\xi \partial_x^2 u(\tau,y_{j-1})\mathrm{d}\tau\mathrm{d}\xi\mathrm{d}\sigma \\ & + \frac{1}{k_j}\int_{J_j}\int_{y_{j-1}}^{y}\int_\sigma^\xi \partial_y^2 u(x_{i-1},\tau)\mathrm{d}\tau\mathrm{d}\xi\mathrm{d}\sigma \\ & + \int_{x_{i-1}}^{x}\int_{y_{j-1}}^{y}\partial_x\partial_y u(\xi,\tau)\mathrm{d}\tau\mathrm{d}\xi \\ & - \frac{x-x_{i-1}}{h_i}\frac{y-y_{j-1}}{k_j}\int_{I_i}\int_{J_j}\partial_x\partial_y u(\xi,\tau)\mathrm{d}\tau\mathrm{d}\xi.\end{aligned}$$

Applying the technique from Prop. 5.1, we immediately see that the first two terms are bounded by $\varTheta_{cd}^{[2]}(T_{ij})^2$. The third and fourth term are clearly bounded by $\varTheta_{cd}^{[1]}(T_{ij})^2$. Ineq. (9.6) follows.

(ii) Next, we have

$$\begin{aligned}\partial_x\left(u-u^I\right)(x,y) = {} & \frac{1}{h_i}\int_{I_i}\int_\sigma^x \partial_x^2 u(\tau,y_{j-1})\mathrm{d}\tau\mathrm{d}\sigma + \int_{y_{j-1}}^{y}\partial_x\partial_y u(x,\tau)\mathrm{d}\tau \\ & - \frac{1}{h_i}\frac{y-y_{j-1}}{k_j}\int_{I_i}\int_{J_j}\partial_x\partial_y u(\xi,\tau)\mathrm{d}\tau\mathrm{d}\xi.\end{aligned}$$

Thus,

$$\left|\partial_x\left(u-u^I\right)(x,y)\right| \le \int_{I_i}\left|\partial_x^2 u(\tau,y_{j-1})\right|\mathrm{d}\tau + 2\int_{J_j}\left|\partial_x\partial_y u(x,\tau)\right|\mathrm{d}\tau$$

and (9.7) follows from (9.8).

(iii) In order to bound the interpolation error in the H^1 seminorm, use integration by parts:

$$\begin{aligned}\left\|\partial_x(u^I - u)\right\|_0^2 = &\int_\Omega \partial_x^2 u(x,y)\big(u^I - u\big)(x,y)\mathrm{d}x\mathrm{d}y \\ &+ \sum_{i=1}^{N-1}\int_0^1 \big(u^I - u\big)(x_i,y)K_i(y)\mathrm{d}y\,,\end{aligned} \tag{9.9}$$

where

$$K_i(y) := \partial_x u^I(x_i - 0, y) - \partial_x u^I(x_i + 0, y)\,.$$

For $y \in J_j$ we have

$$K_i(y) = \frac{y - y_{j-1}}{k_j}\left(u_{\bar{x};ij} - u_{x;i+1,j}\right) + \frac{y_j - y}{k_j}\left(u_{\bar{x};i,j-1} - u_{x;i+1,j-1}\right).$$

By the mean-value theorem there exists a $\xi_{i,j} \in I_i$, such that $u_{\bar{x};i,j} = \partial_x u(\xi_{i,j}, y_j)$. Therefore,

$$\left|u_{\bar{x};ij} - u_{x;i+1,j}\right| = \left|\partial_x u(\xi_{i,j}, y_j) - \partial_x u(\xi_{i+1,j}, y_j)\right| \le \int_{x_{i-1}}^{x_{i+1}} \left|\partial_x^2 u(\xi, y_j)\right| \mathrm{d}\xi\,.$$

We get

$$|K_i(y)| \le \max_{y\in[0,1]}\int_{x_{i-1}}^{x_{i+1}} \left|\partial_x^2 u(\xi, y)\right| \mathrm{d}\xi\,.$$

This and a Hölder inequality applied to (9.9) yield

$$\begin{aligned}&\left\|(u^I - u)_x\right\|_0^2 \\ &\quad\le \left\|u^I - u\right\|_\infty \left\{\int_\Omega \left|\partial_x^2 u(x,y)\right| \mathrm{d}x\mathrm{d}y + 2\max_{y\in[0,1]}\int_0^1 \left|\partial_x^2 u(x,y)\right| \mathrm{d}x\right\} \\ &\quad\le C\varepsilon^{-1}\left\|u^I - u\right\|_\infty,\end{aligned}$$

by (9.8). The interpolation error in the L_2 norm is bounded by its L_∞ norm. We get the second bound of the theorem. □

Remark 9.5. The second part of the proof when the H^1 seminorm is considered works for bilinear elements, but not for linear ones. Nonetheless, for S-type meshes and linear elements, the conclusions of the theorem hold too; see [82, 137]. ♣

Remark 9.6. Error bounds for particular layer-adapted meshes can be immediately concluded using the results from Sections 2.1.1 and 2.1.3. ♣

9.2.2 Galerkin FEM

Let $V_0^\omega \subset H_0^1(\Omega)$ be the space of piecewise linear/bilinear functions on the given triangulation that vanish on the boundary of Ω. Then our discretisation is as follows: Find $u^N \in V_0^\omega$ such that

$$a\left(u^N, v\right) = f(v) \;\text{ for all }\; v \in V_0^\omega \,.$$

The coercivity of $a(\cdot,\cdot)$ guarantees the existence of a unique solution $u^N \in V_0^\omega$.

9.2.2.1 Convergence

Convergence of the (bi)linear Galerkin FEM on standard Shishkin meshes was first studied by Stynes and O'Riordan [152]. Their technique was later adapted by Linß and Roos to the analysis of more general S-type meshes [82, 137].

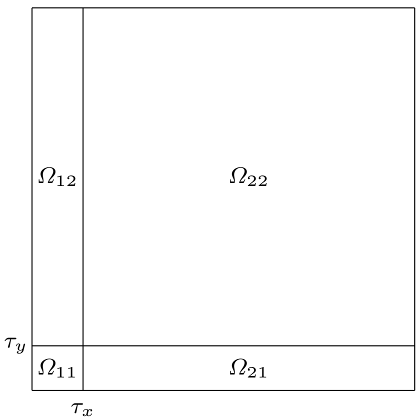
The mesh transition points in the S-type mesh are

$$\tau_x := \min\left\{q, \frac{\sigma\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \tau_y := \min\left\{q, \frac{\sigma\varepsilon}{\beta_2}\ln N\right\}$$

with mesh parameters $\sigma > 0$ and $q \in (0,1)$ arbitrary, but fixed and with $qN \in \mathbb{N}$. Divide the domain Ω as in Fig. 9.2: $\bar{\Omega} = \Omega_{11} \cup \Omega_{21} \cup \Omega_{12} \cup \Omega_{22}$.

Corollary 9.7. *Let $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ be a S-type mesh with $\sigma \geq 2$. Then Theorem 9.4 implies*

$$\left\|u - u^I\right\|_{\infty,\Omega\setminus\Omega_{22}} \leq C\left(h + N^{-1}\max|\psi'|\right)^2, \quad \left\|u - u^I\right\|_{\infty,\Omega_{22}} \leq CN^{-2},$$



$$\Omega_{11} = [0, \tau_x] \times [0, \tau_y],$$
$$\Omega_{21} = [\tau_x, 1] \times [0, \tau_y],$$
$$\Omega_{12} = [0, \tau_x] \times [\tau_y, 1],$$
$$\Omega_{22} = [\tau_x, 1] \times [\tau_y, 1].$$

Fig. 9.2 Dissection of Ω for tensor-product S-type meshes

and

$$\left|\left|\left|u - u^I\right|\right|\right|_\varepsilon \le C\left(h + N^{-1}\max|\psi'|\right) .$$

Furthermore, by the Cauchy-Schwarz inequality,

$$\left\|u - u^I\right\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}\ln^{1/2} N\left(h + N^{-1}\max|\psi'|\right)^2$$

and

$$\left\|u - u^I\right\|_{0,\Omega_{22}} \le CN^{-2} .$$

Where we have used the results of Sect. 2.1.3 too.

Theorem 9.8. *Let $\bar\omega = \bar\omega_x \times \bar\omega_y$ be a tensor-product S-type mesh with $\sigma \ge 2$, whose mesh generating function $\tilde\varphi$ satisfies (2.8) and*

$$\left(h + N^{-1}\max|\psi'|\right)\ln^{1/2} N \le C. \tag{9.10}$$

Then

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \le C\big(h + N^{-1}\max|\psi'|\big)$$

for both linear elements on triangles and bilinear elements on rectangles.

Proof. The proof is along the lines of Sect. 5.2.1 exploiting the tensor-product structure of the mesh and the solution decomposition of Theorem 7.17; see also [82]. Let $\eta = u^I - u$ and $\chi = u^I - u^N$. Bounds for the interpolation error η are provided by Corollary 9.7.

Bounding χ, we start from the coercivity of $a(\cdot,\cdot)$ and the orthogonality of the Galerkin method, i. e.,

$$\begin{aligned}|||\chi|||_\varepsilon^2 &\le a(\chi,\chi) = a(\eta,\chi) = \varepsilon(\nabla\eta,\nabla\chi) + (\eta, \boldsymbol{b}^T\nabla\chi) + \big((\operatorname{div} b + c)\eta,\chi\big)\\ &\le C\,|||\eta|||_\varepsilon\,|||\chi|||_\varepsilon + C\Big(\|\eta\|_{0,\Omega_{22}}\|\nabla\chi\|_{0,\Omega_{22}}\\ &\qquad\qquad + \|\eta\|_{L_\infty(\Omega\setminus\Omega_{22})}\|\nabla\chi\|_{L_1(\Omega\setminus\Omega_{22})}\Big).\end{aligned}$$

On $\Omega\setminus\Omega_{22}$ the Cauchy-Schwarz inequality yields

$$\|\nabla\chi\|_{L_1(\Omega\setminus\Omega_{22})} \le C\sqrt{\tau_x + \tau_y}\,\|\nabla\chi\|_0 \le C\ln^{1/2} N\,|||\chi|||_\varepsilon ,$$

while on Ω_{22} an inverse inequality yields

$$\|\nabla\chi\|_{0,\Omega_{22}} \le CN\,\|\chi\|_{0,\Omega_{22}} \le CN\,|||\chi|||_\varepsilon .$$

because $h_i \geq N^{-1}$ and $k_j \geq N^{-1}$ for $T_{ij} \in \Omega_{22}$. These two bounds and Corollary 9.7 give

$$|||\chi|||_\varepsilon \leq C\left\{h + N^{-1}\max|\psi'| + \left(h + N^{-1}\max|\psi'|\right)^2 \ln^{1/2} N + N^{-1}\right\}.$$

Thus,

$$|||\chi|||_\varepsilon \leq C\left(h + N^{-1}\max|\psi'|\right),$$

where we have used (9.10). A triangle inequality and the bounds for $|||\eta|||_\varepsilon$ and $|||\chi|||_\varepsilon$ complete the proof. □

9.2.2.2 Supercloseness

Similar to the one dimensional case, the Galerkin FEM using bilinear elements on rectangular S-type meshes enjoys a superconvergence property; see [83, 176]. Note that this superconvergence result generally does not hold for linear elements on triangles as numerical experiments confirm [109].

In contrast to the one-dimensional case where we have $((u^I - u)', \chi') = 0$ for arbitrary $\chi \in V^\omega$, we do *not* have $(\nabla(u^I - u^N), \nabla\chi) = 0$ here because $u^I - u$ vanishes in the mesh points only, but not at the inter-element boundaries. This complicates the analysis and requires higher regularity of the solution. In particular, we shall assume that the solution u can be decomposed as $u = v + w_1 + w_2 + w_{12}$, where

$$\left|\partial_x^i \partial_y^j v(x,y)\right| \leq C, \tag{9.11a}$$

$$\left|\partial_x^i \partial_y^j w_1(x,y)\right| \leq C\varepsilon^{-i}\mathrm{e}^{-\beta_1 x/\varepsilon}, \tag{9.11b}$$

$$\left|\partial_x^i \partial_y^j w_2(x,y)\right| \leq C\varepsilon^{-j}\mathrm{e}^{-\beta_2 y/\varepsilon} \tag{9.11c}$$

and

$$\left|\partial_x^i \partial_y^j w_{12}(x,y)\right| \leq C\varepsilon^{-(i+j)}\mathrm{e}^{-(\beta_1 x + \beta_2 y)/\varepsilon} \tag{9.11d}$$

for $i + j \leq 3$ and $x, y \in [0,1]$.

Theorem 9.9. *Let $\bar\omega = \bar\omega_x \times \bar\omega_y$ be a tensor-product S-type mesh with $\sigma \geq 5/2$ that satisfies (2.8). Then the error of the Galerkin FEM u^N satisfies*

$$\left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon \leq C\left(h^2 \ln^{1/2} N + N^{-2}\max|\psi'|^2\right).$$

Proof. The coercivity and Galerkin orthogonality of $a(\cdot,\cdot)$ give

$$\begin{aligned}\left|\left|\left|u^I-u^N\right|\right|\right|_\varepsilon^2 &\le \left|a\big(u-u^I,u^I-u^N\big)\right| \\ &\le \varepsilon\left|\big(\nabla(u-u^I),\nabla(u^I-u^N)\big)\right| \\ &\qquad + \left|\Big(\boldsymbol{b}^T\nabla(u-u^I)-c(u-u^I),u^I-u^N\Big)\right|.\end{aligned}$$

In the Sect. 9.2.2.3 we shall show that for all $\chi \in V^\omega$

$$\varepsilon\left|\big(\nabla(u-u^I),\nabla\chi\big)\right| \le C\left(h^2+N^{-2}\max|\psi'|^2\right)|||\chi|||_\varepsilon \tag{9.12}$$

and

$$\begin{aligned}&\left|\Big(\boldsymbol{b}^T\nabla(u-u^I)-c(u-u^I),\chi\Big)\right| \\ &\quad\le C\left(h^2\ln^{1/2}N+N^{-2}\max|\psi'|^2\right)|||\chi|||_\varepsilon\,.\end{aligned} \tag{9.13}$$

Thus,

$$\left|\left|\left|u^I-u^N\right|\right|\right|_\varepsilon^2 \le C\left(h^2\ln^{1/2}N+N^{-2}\max|\psi'|^2\right)\left|\left|\left|u^I-u^N\right|\right|\right|_\varepsilon\,.$$

Divide by $\left|\left|\left|u^I-u^N\right|\right|\right|_\varepsilon$ to complete the proof.

Theorem 9.9 yields

$$\left|\left|\left|u^I-u^N\right|\right|\right|_\varepsilon \le \begin{cases} CN^{-2}\ln^2 N & \text{for the standard Shishkin mesh and} \\ C(\varepsilon^2+N^{-2})\ln^{1/2}N & \text{for the Bakhvalov-Shishkin mesh;}\end{cases}$$

see Sect. 2.1.3 for the bounds on h and $\max|\psi'|$.

Remark 9.10. Another superconvergence result was established by Zhang [176], who studied convergence of the Galerkin FEM on Shishkin meshes in a discrete version of the energy norm, where $\nabla(u-u^N)$ is replaced by a piecewise-constant approximation based on the midpoints of the rectangles of the triangulation. ♣

9.2.2.3 Detailed Analysis, Proofs of (9.12) and (9.13)

In the analysis we require error estimates for interpolation on anisotropic elements, which were derived by Apel and Dobrowolski [15]. Furthermore, a sharp bound for the L_2-norm error of the interpolation error for the layer terms is needed. We shall also use special error expansion formulae derived by Lin [78].

Preliminaries

Let $T_{ij} = I_i \times J_j$ be an element of the triangulation. Set

$$F_i(x) = \frac{(x - x_{i-1/2})^2}{2} - \frac{h_i^2}{8} \qquad \text{and} \qquad G_j(y) = \frac{(y - y_{j-1/2})^2}{2} - \frac{k_j^2}{8},$$

where $(x_{i-1/2}, y_{j-1/2})$ is the midpoint of the mesh rectangle T_{ij}. Denote the east, north, west and south edges of T_{ij} by $l_{k;ij}$ for $k = 1, \ldots, 4$, respectively.

Lemma 9.11 (Lin Identities [78]). *For any function $g \in C^3(\overline{T}_{ij})$ and any $\chi \in V^\omega$ we have the identities*

$$\int_{T_{ij}} \partial_x (g - g^I) \chi_x = \int_{T_{ij}} \Big[G_j \partial_x \chi - \frac{1}{3} (G_j^2)' \partial_x \partial_y \chi \Big] \partial_x^2 \partial_y g \,, \tag{9.14a}$$

$$\begin{aligned} \int_{T_{ij}} \partial_x (g - g^I) \chi_y = \Big(\int_{l_{4;ij}} - \int_{l_{2;ij}} \Big) F_j \partial_x^2 g \chi_x \\ + \int_{T_{ij}} \Big[G_j \partial_x \partial_y^2 g \left(\partial_y \chi - F_i' \partial_x \partial_y \chi \right) + F_j \partial_x^2 \partial_y g \partial_x \chi \Big] \end{aligned} \tag{9.14b}$$

and

$$\begin{aligned} &\int_{T_{ij}} \partial_x (g - g^I) \chi \\ &\quad = \int_{T_{ij}} \Big[G_j \left(\chi - F_i' \partial_x \chi \right) - \frac{1}{3} \left(G_j^2 \right)' \left(\partial_y \chi - F_i' \partial_x \partial_y \chi \right) \Big] \partial_x \partial_y^2 g \\ &\qquad + \Big(\int_{l_{1;ij}} - \int_{l_{2;ij}} \Big) \frac{h_i^2}{12} \chi \partial_x^2 g + \int_{T_{ij}} \Big[\frac{1}{6} \left(F_i^2 \right)' \partial_x \chi - \frac{h_i^2}{12} \chi \Big] \partial_x^3 g \,. \end{aligned} \tag{9.14c}$$

An immediate consequence of (9.14a) is

$$\Big| \partial_x \left((g - g^I), \partial_x \chi \right)_{T_{ij}} \Big| \le \frac{k_j^2}{8} \int_{T_{ij}} |\partial_x \chi| \, \big|\partial_x \partial_y^2 g\big| + \frac{k_j^3}{24} \int_{T_{ij}} |\partial_x \partial_y \chi| \, \big|\partial_x \partial_y^2 g\big| \,.$$

with the Cauchy-Schwarz and an inverse inequality giving

$$\Big| \partial_x \left((g - g^I), \partial_x \chi \right)_{T_{ij}} \Big| \le C k_j^2 \, \|\partial_x \chi\|_{0,T_{ij}} \, \big\|\partial_x \partial_y^2 g\big\|_{0,T_{ij}} \,. \tag{9.15}$$

Lemma 9.12 ([15, Theorem 3]). *Let $T_{ij} \in \Omega^N$ and $p \in [1, \infty]$. Assume that g lies in $W_p^2(T_{ij})$. Let g^I denote the bilinear function that interpolates to g at the vertices of T_{ij}. Then*

$$\big\|\partial_x (g - g^I)\big\|_{L_p(T_{ij})} \le C \Big\{ h_i \|\partial_x^2 g\|_{L_p(T_{ij})} + k_j \|\partial_x \partial_y g\|_{L_p(T_{ij})} \Big\}.$$

Proposition 9.13. *Let $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ be a tensor-product S-type mesh that satisfies* (2.8)*. Then for $w = w_1 + w_2 + w_{12}$*

$$\left\|w - w^I\right\|_{0,\Omega_{22}} \le C\left(\varepsilon^{1/2}N^{-\sigma} + N^{-\sigma-1/2}\right) \tag{9.16a}$$

and

$$\left\|w - w^I\right\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2 \quad \textit{if } \sigma > 2. \tag{9.16b}$$

Proof. (i) When proving (9.16a), we bound $\|w\|_{0,\Omega_{22}}$ and $\left\|w^I\right\|_{0,\Omega_{22}}$ separately and apply a triangle inequality. Clearly,

$$\|w\|_{0,\Omega_{22}} \le C\varepsilon^{1/2}N^{-\sigma}, \tag{9.17}$$

by (9.11) and a direct calculation.
In order to bound the L_2 norm of w^I we split Ω_{22} into two subdomains

$$S := [x_{qN+1}, 1] \times [y_{qN+1}, 1] \quad \text{and} \quad \Omega_{22} \setminus S.$$

Note that $\Omega_{22} \setminus S$ consists of only one ply of mesh rectangles along the interface between the coarse and the fine mesh regions. We have

$$\left\|w^I\right\|_{0,\Omega_{22}\setminus S}^2 \le \big(2(1-q)N - 1\big)h_{qN+1}k_{qN+1}\left\|w^I\right\|_{\infty,\Omega_{22}}^2.$$

Hence,

$$\left\|w^I\right\|_{0,\Omega_{22}\setminus S} \le CN^{-\sigma-1/2}. \tag{9.18}$$

For $T_{ij} \subset S$ we estimate as follows:

$$\begin{aligned}\left\|w^I\right\|_{0,T_{ij}}^2 &\le h_i k_j \left\|w^I\right\|_{\infty,T_{ij}}^2 \\ &\le C\int_{I_{i-1}}\int_{J_{j-1}}\left(\mathrm{e}^{-2\beta_1 x/\varepsilon} + \mathrm{e}^{-2\beta_2 y/\varepsilon} + \mathrm{e}^{-2(\beta_1 x + \beta_2 y)/\varepsilon}\right),\end{aligned}$$

by (9.11) and since the mesh on Ω_{22} is uniform. We get

$$\left\|w^I\right\|_{0,S}^2 \le C\int_{\tau_x}^1\int_{\tau_y}^1\left(\mathrm{e}^{-2\beta_1 x/\varepsilon} + \mathrm{e}^{-2\beta_2 y/\varepsilon} + \mathrm{e}^{-2(\beta_1 x + \beta_2 y)/\varepsilon}\right)$$

and

$$\left\|w^I\right\|_{0,S} \le C\varepsilon^{1/2}N^{-\sigma}. \tag{9.19}$$

Collecting (9.17), (9.18) and (9.19), we obtain (9.16a).

(ii) Before starting the proof of (9.16b) note that by (2.11) and (2.12)

$$\frac{\beta h_i}{\sigma\varepsilon} \le N^{-1}\max|\psi'|\mathrm{e}^{\beta_1 x_i/\sigma\varepsilon} \le CN^{-1}\max|\psi'|\,\mathrm{e}^{\beta_1 x/\sigma\varepsilon} \quad \text{for } x\in I_i.$$

(α) First let us study $w_1 - w_1^I$. For $T_{ij}\subset \Omega_{12}\cup\Omega_{11}$ we have by Lemma 9.12 and (9.11)

$$\begin{aligned}\left\|w_1 - w_1^I\right\|_{0,T_{ij}}^2 &\le C\Big\{h_i^4 k_j\int_{I_i}\varepsilon^{-4}\mathrm{e}^{-2\beta_1 x/\varepsilon}\mathrm{d}x \\ &\qquad + h_i^2k_j^3\int_{I_i}\varepsilon^{-2}\mathrm{e}^{-2\beta_1 x/\varepsilon}\mathrm{d}x + k_j^5\int_{I_i}\mathrm{e}^{-2\beta_1 x/\varepsilon}\mathrm{d}x\Big\} \\ &\le C\left(N^{-1}\max|\psi'| + h\right)^4 k_j\int_{I_i}\mathrm{e}^{-(2-4/\sigma)\beta_1 x/\varepsilon}\mathrm{d}x.\end{aligned}$$

Summing over all elements in $\Omega_{11}\cup\Omega_{12}$, we get

$$\left\|w_1 - w_1^I\right\|_{0,\Omega_{11}\cup\Omega_{12}}^2 \le C\left(N^{-1}\max|\psi'| + h\right)^4\int_0^{\tau_x}\mathrm{e}^{-(2-4/\sigma)\beta_1 x/\varepsilon}\mathrm{d}x.$$

Thus,

$$\left\|w_1 - w_1^I\right\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\varepsilon^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2$$

because $\sigma > 2$ is assumed.
On Ω_{21} we estimate as follows

$$\begin{aligned}\left\|w_1 - w_1^I\right\|_{0,\Omega_{21}} &\le \sqrt{\text{meas }\Omega_{21}}\left\|w_1 - w_1^I\right\|_{\infty,\Omega_{21}} \le \sqrt{\text{meas }\Omega_{21}}\left\|w_1\right\|_{\infty,\Omega_{21}} \\ &\le C\varepsilon^{1/2}\ln^{1/2}N\mathrm{e}^{-\beta_1\tau_x/\varepsilon} \le C\varepsilon^{1/2}N^{-2}\end{aligned}$$

because $\sigma > 2$.

Therefore,

$$\left\|w_1 - w_1^I\right\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2 \tag{9.20}$$

since $\max|\psi'| \ge 1$.

(β) Clearly a similar argument yields

$$\left\|w_2 - w_2^I\right\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2. \tag{9.21}$$

(γ) Finally, let us bound $w_{12} - w_{12}^I$. For $T_{ij} \subset \Omega_{11}$ we have by Lemma 9.12 and (9.11)

$$\left\|w_{12} - w_{12}^I\right\|_{0,T_{ij}}^2 \le C\left(h_i^2 + k_j^2\right)^2 \int_{I_i}\int_{J_j} \varepsilon^{-4}\mathrm{e}^{-2(\beta_1 x+\beta_2 y)/\varepsilon}\mathrm{d}y\mathrm{d}x$$
$$\le C\left(h + N^{-1}\max|\psi'|\right)^4 \int_{I_i}\int_{J_j} \mathrm{e}^{-(2-4/\sigma)(\beta_1 x+\beta_2 y)/\varepsilon}\mathrm{d}y\mathrm{d}x\,.$$

Summing over all elements in Ω_{11}, we get

$$\left\|w_{12} - w_{12}^I\right\|_{0,\Omega_{11}}^2$$
$$\le C\left(N^{-1}\max|\psi'| + h\right)^4 \int_0^{\tau_x}\int_0^{\tau_y} \mathrm{e}^{-(2-4/\sigma)(\beta_1 x+\beta_2 y)/\varepsilon}\mathrm{d}y\mathrm{d}x\,.$$

Hence,

$$\left\|w_{12} - w_{12}^I\right\|_{0,\Omega_{11}} \le C\varepsilon\left(N^{-1}\max|\psi'| + h\right)^2. \tag{9.22}$$

On $\Omega_{12} \cup \Omega_{21}$ we estimate as follows:

$$\left\|w_{12} - w_{12}^I\right\|_{0,\Omega_{12}\cup\Omega_{21}} \le \sqrt{\text{meas}\,\Omega_{12}\cup\Omega_{21}}\,\left\|w_{12} - w_{12}^I\right\|_{\infty,\Omega_{12}\cup\Omega_{21}}$$
$$\le \varepsilon^{1/2}N^{-\sigma}\ln^{1/2}N\,\|w_{12}\|_{\infty,\Omega_{12}\cup\Omega_{21}}$$
$$\le C\varepsilon^{1/2}N^{-\sigma}\ln^{1/2}N \le C\varepsilon^{1/2}N^{-2}\,, \tag{9.23}$$

because $\sigma > 2$.

Collect (9.20)–(9.23) to complete the proof. □

Proof of (9.12)

(i) Using (9.15), we obtain for $T_{ij} \subset \Omega_{11} \cup \Omega_{21}$

$$\left|\left(\partial_x(u-u^I), \partial_x\chi\right)_{T_{ij}}\right|$$
$$\le Ck_j^2\left\|\left(1+\varepsilon^{-1}\mathrm{e}^{-\beta_1 x/\varepsilon}\right)\left(1+\varepsilon^{-2}\mathrm{e}^{-\beta_2 y/\varepsilon}\right)\right\|_{0,T_{ij}}\|\partial_x\chi\|_{0,T_{ij}}$$
$$\le Ck_j^2\left(1+\varepsilon^{-2}\mathrm{e}^{-\beta_2 y_{j-1}/\varepsilon}\right)\left\|1+\varepsilon^{-1}\mathrm{e}^{-\beta_1 x/\varepsilon}\right\|_{0,T_{ij}}\|\partial_x\chi\|_{0,T_{ij}}$$
$$\le C\left(h + N^{-1}\max|\psi'|\right)^2\left\|1+\varepsilon^{-1}\mathrm{e}^{-\beta_1 x/\varepsilon}\right\|_{0,T_{ij}}\|\partial_x\chi\|_{0,T_{ij}}\,,$$

by (2.12) and since $e^{\beta_2 k_j/\varepsilon} \le C$ because of (2.8). Application of the Cauchy-Schwarz inequality yields

$$\varepsilon \left| \left(\partial_x(u-u^I), \partial_x\chi\right)_{\Omega_{11}\cup\Omega_{21}} \right| \\ \le C\varepsilon \left(h + N^{-1}\max|\psi'|\right)^2 \left\| 1+\varepsilon^{-1}e^{-\beta_1 x/\varepsilon} \right\|_{0,\Omega_{11}\cup\Omega_{21}} \|\partial_x\chi\|_{0,\Omega_{11}\cup\Omega_{21}}.$$

Hence,

$$\varepsilon \left| \left(\partial_x(u-u^I), \partial_x\chi\right)_{\Omega_{11}\cup\Omega_{21}} \right| \le C\left(h + N^{-1}\max|\psi'|\right)^2 |||\chi|||_\varepsilon \,. \tag{9.24}$$

(ii) An argument similar to *(i)* gives

$$\varepsilon \left| \left(\partial_x((v+w_1)-(v+w_1)^I), \partial_x\chi\right)_{\Omega_{12}\cup\Omega_{22}} \right| \\ \le C\varepsilon h^2 \left\| 1+\varepsilon^{-1}e^{-\beta_1 x/\varepsilon} \right\|_{0,\Omega_{12}\cup\Omega_{22}} \|\partial_x\chi\|_{0,\Omega_{12}\cup\Omega_{22}} \\ \le Ch^2 |||\chi|||_\varepsilon \,. \tag{9.25}$$

(iii) Next we consider $w := w_2 + w_{12}$ for $T_{ij} \subset \Omega_{12}$. The stability of the interpolation operator and our bounds on the derivatives of w_2 and w_{12} yield

$$\left\|\partial_x(w-w^I)\right\|_{\infty,T_{ij}} \\ \le \|\partial_x w\|_{\infty,T_{ij}} + \left\|\partial_x w^I\right\|_{\infty,T_{ij}} \le C\|\nabla w\|_{\infty,T_{ij}} \le C\varepsilon^{-1}N^{-\sigma}.$$

Thus,

$$\varepsilon \left| \left(\partial_x(w-w^I), \partial_x\chi\right)_{\Omega_{12}} \right| \le CN^{-\sigma}\|\partial_x\chi\|_{1,\Omega_{12}} \le CN^{-\sigma}\varepsilon^{1/2}\ln^{1/2}N\, |||\chi|||_\varepsilon \,,$$

since $\operatorname{meas}\Omega_{12} = \mathcal{O}(\varepsilon \ln N)$. Therefore,

$$\varepsilon \left| \partial_x\left((w_2+w_{12})-(w_2+w_{12})^I\right), \partial_x\chi\right)_{\Omega_{12}} \right| \le CN^{-2}|||\chi|||_\varepsilon \,, \tag{9.26}$$

because $\sigma > 2$.

(iv) Finally, let us bound the terms involving w_2 and w_{12} on Ω_{22}. Using Lemma 9.12 and (9.11) we get

$$\left\|\partial_x(w_2-w_2^I)\right\|_{0,\Omega_{22}} \le C\varepsilon^{-1/2}N^{-\sigma}$$

and

$$\left\|\partial_x(w_{12}-w_{12}^I)\right\|_{0,\Omega_{22}} \le C\varepsilon^{-1}N^{-2\sigma} \,.$$

Thus,

$$\varepsilon \left| \left(\partial_x (w_2 - w_2^I), \partial_x \chi\right)_{\Omega_{22}} \right| \le C N^{-2} \, |||\chi|||_\varepsilon \tag{9.27}$$

and

$$\varepsilon \left| \left(\partial_x (w_{12} - w_{12}^I), \partial_x \chi\right)_{\Omega_{22}} \right| \le C N^{-2\sigma} \, \|\partial_x \chi\|_{0,\Omega_{22}} \le C N^{-2\sigma+1} \, \|\chi\|_{0,\Omega_{22}} \,, \tag{9.28}$$

by an inverse inequality.

Collect (9.24)-(9.28) to obtain

$$\varepsilon \left| \left(\partial_x (u - u^I), \partial_x \chi\right) \right| \le C \left(h + N^{-1} \max |\psi'|\right)^2 |||\chi|||_\varepsilon \quad \text{for all } \chi \in V^\omega .$$

Obviously, we have an identical bound for $\left|\left(\partial_y (u - u^I), \partial_y \chi\right)\right|$ which completes the proof of (9.12).

Proof of (9.13)

Recalling the decomposition (9.11), we set $w = w_1 + w_2 + w_{12}$. Then integration by parts yields

$$\begin{aligned} &\left| - \left(\boldsymbol{b}^T \nabla (u - u^I), \chi\right) + \left(c(u - u^I), \chi\right) \right| \\ &\quad \le \left| \left(\boldsymbol{b}^T \nabla (v - v^I), \chi\right) \right| + \left| \left(w - w^I, \boldsymbol{b}^T \nabla \chi\right) \right| \\ &\qquad\qquad + \left| \left(c(v - v^I), \chi\right) + \left((c + \operatorname{div} \boldsymbol{b})(w - w^I), \chi\right) \right| \end{aligned} \tag{9.29}$$

The terms on the right-hand side are analysed separately.

First

$$\begin{aligned} &\left| \left(c(v - v^I), \chi\right) + \left((c + \operatorname{div} \boldsymbol{b})(w - w^I), \chi\right) \right| \\ &\qquad\qquad \le C \left(\|v - v^I\|_0 + \|w - w^I\|_0 \right) \|\chi\|_0 \,. \end{aligned}$$

Adapting the technique from Sect. 9.2.1 it is shown that

$$\|v - v^I\|_0 + \|w - w^I\|_0 \le C \left(h + N^{-1} \max |\psi'|\right)^2 ,$$

since v and w satisfy derivative bounds similar to those of u. Thus,

$$\left| \left(c(v - v^I), \chi\right) + \left((c + \operatorname{div} \boldsymbol{b})(w - w^I), \chi\right) \right| \le C \left(h + N^{-1} \max |\psi'|\right)^2 |||\chi|||_\varepsilon \,. \tag{9.30}$$

Next let us bound the second and third term in (9.29). The Cauchy-Schwarz inequality and Proposition 9.13 yield

$$\begin{aligned}
&\left|\left(w - w^I, \boldsymbol{b}^T \nabla\chi\right)\right| \\
&\quad \le C\|w - w^I\|_{0,\Omega_{22}}\|\nabla\chi\|_{0,\Omega_{22}} + C\|w - w^I\|_{0,\Omega\setminus\Omega_{22}}\|\nabla\chi\|_{0,\Omega\setminus\Omega_{22}} \\
&\quad \le C\left(\varepsilon^{1/2}N^{-5/2} + N^{-3}\right)\|\nabla\chi\|_{0,\Omega_{22}} \\
&\qquad\qquad + C\varepsilon^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2\|\nabla\chi\|_{0,\Omega\setminus\Omega_{22}} \\
&\quad \le C\left(h + N^{-1}\max|\psi'|\right)^2 |||\chi|||_{\varepsilon}\,,
\end{aligned} \tag{9.31}$$

where we have used an inverse inequality and that on Ω_{22} the mesh is uniform with mesh size $\mathcal{O}\left(N^{-1}\right)$.

Finally, we study the term $\left(\boldsymbol{b}^T\nabla(v - v^I), \chi\right)$. Let $b_{1;ij} = b_1(x_i, y_j)$ for all i, j. Using the second identity of Lemma 9.11, we get

$$\begin{aligned}
&\left(b_1\partial_x(v - v^I), \chi\right) \\
&\quad = \sum_{T_{ij}\in\Omega^N}\left\{\left(b_{1;ij}\partial_x(v - v^I), \chi\right)_{T_{ij}} + \left((b_1 - b_{1;ij})\partial_x(v - v^I), \chi\right)_{T_{ij}}\right\} \\
&\quad = \sum_{T_{ij}\in\Omega^N} b_{1;ij}\int_{T_{ij}}\left\{\left[\frac{1}{6}\left(F_i^2\right)'\partial_x\chi - \frac{1}{12}h_i^2\chi\right]\partial_x^3 v\right. \\
&\qquad\qquad \left. + \left[G_j\left(\chi - F_i'\partial_x\chi\right) - \frac{1}{3}\left(G_j^2\right)'\left(\partial_y\chi - F_{ij}'\partial_x\partial_y\chi\right)\right]\partial_x\partial_y^2 v\right\} \\
&\qquad + \frac{1}{12}\sum_{i=1}^{N-1}\sum_{j=1}^{N}\left(b_{1;i+1,j}h_{i+1}^2 - b_{1;ij}h_i^2\right)\int_{J_j}\left(\chi\partial_x^2 v\right)(x_i, y)\mathrm{d}y \\
&\qquad + \sum_{T_{ij}\in\Omega^N}\left((b_1 - b_{1;ij})\partial_x(v - v^I), \chi\right)_{T_{ij}} \\
&\quad =: I_1 + I_2 + I_3\,.
\end{aligned} \tag{9.32}$$

Use (9.11) to obtain

$$\begin{aligned}
|I_1| \le C\sum_{T_{ij}\in\Omega^N}&\left\{\left(h_i^2 + k_j^2\right)\left(\|\chi\|_{1,T_{ij}} + h_i\|\partial_x\chi\|_{1,T_{ij}}\right)\right. \\
&\qquad\left. + k_j^2\left(k_j\|\partial_y\chi\|_{1,T_{ij}} + h_i k_j\|\partial_x\partial_y\chi\|_{1,T_{ij}}\right)\right\}.
\end{aligned}$$

Thus,

$$|I_1| \le C\sum_{T_{ij}\in\Omega^N}\left(h_i^2 + k_j^2\right)\|\chi\|_{1,T_{ij}} \le Ch^2\|\chi\|_{L_1(\Omega)} \le Ch^2\|\chi\|_0\,. \tag{9.33}$$

For I_2 we proceed as follows: First

$$\int_{J_j} \big(\chi v_{xx}\big)(x_i, y)\mathrm{d}y = \sum_{k=1}^{i} \int_{J_j} \int_{I_k} \big(\partial_x \chi \partial_x^2 v + \chi \partial_x^3 v\big)(x, y)\mathrm{d}x\mathrm{d}y$$

yields

$$\begin{aligned}
&\sum_{i=1}^{qN} \big(b_{1;i+1,j} h_{i+1}^2 - b_{1;ij} h_i^2\big) \int_{J_j} \big(\chi \partial_x^2 v\big)(x_i, y)\mathrm{d}y \\
&\quad = \sum_{i=1}^{qN} \big(b_{1;qN+1,j} h_{qN+1}^2 - b_{1;ij} h_i^2\big) \int_{J_j} \int_{I_i} \big(\partial_x \chi \partial_x^2 v + \chi \partial_x^3 v\big)(x, y)\mathrm{d}x\mathrm{d}y\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Big|\sum_{i=1}^{qN} \sum_{j=1}^{N} \big(b_{1;i+1,j} h_{i+1}^2 - b_{1;ij} h_i^2\big) \int_{J_j} \big(\chi \partial_x^2 v\big)(x_i, y)\mathrm{d}y\Big| \\
&\qquad \le Ch^2 \big\|\partial_x \chi \partial_x^2 v + \partial_x^3 \chi v\big\|_{1,\Omega_{11}\cup\Omega_{12}} \\
&\qquad \le Ch^2 \varepsilon^{1/2} \ln^{1/2} N \big(\|\partial_x \chi\|_0 + \|\chi\|_0\big) \\
&\qquad \le Ch^2 \ln^{1/2} N \, |||\chi|||_\varepsilon\,,
\end{aligned} \tag{9.34}$$

by (9.11). Furthermore, for $i = qN+1, \dots, N$, we have

$$\Big|\big(b_{1;i+1,j} h_{i+1}^2 - b_{1;ij} h_i^2\big) \int_{J_j} \big(\chi \partial_x^2 v\big)\,(x_i, y)\mathrm{d}y\Big| \le Ch_i^2 \|\chi\|_{1,T_{ij}},$$

because $\big|b_{1;i+1,j} - b_{1;ij}\big| \le Ch_{i+1}$, $h_i = h_{i+1} \le h$ and

$$\int_{J_j} \big(\chi \partial_x^2 v\big)(x_i, y)\mathrm{d}y \le Ch_i^{-1} \|\chi\|_{1,T_{ij}},$$

by an inverse inequality. We get

$$\left|\sum_{i=qN+1}^{N-1} \sum_{j=1}^{N} \big(b_{1;i+1,j} h_{i+1}^2 - b_{1;ij} h_i^2\big) \int_{J_j} \big(\chi \partial_x^2 v\big)(x_i, y)\mathrm{d}y\right| \le Ch^2 \|\chi\|_0. \tag{9.35}$$

For I_3 we have the following bound:

$$\begin{aligned}
|I_3| &\le \sum_{T_{ij}\in\Omega_N} \big\|b_1 - b_{1;ij}\big\|_{\infty,T_{ij}} \Big(h_i \big\|\partial_x^2 v\big\|_{\infty,T_{ij}} + k_j \big\|\partial_x \partial_y v_{xy}\big\|_{\infty,T_{ij}}\Big) \big\|\chi\big\|_{1,T_{ij}} \\
&\le Ch^2 \|\chi\|_0,
\end{aligned} \tag{9.36}$$

by Lemma 9.12 and (9.11).

Collect (9.32)–(9.36) to obtain

$$\left|\left(b_1\partial_x(v-v^I),\chi\right)\right| \le Ch^2\ln^{1/2}N\,|||\chi|||_\varepsilon \tag{9.37}$$

with the analogously bound

$$\left|\left(b_2\partial_y(v-v^I),\chi\right)\right| \le Ch^2\ln^{1/2}N\,|||\chi|||_\varepsilon\,. \tag{9.38}$$

Substituting (9.30), (9.31), (9.37) and (9.38) into (9.29), we are finished.

9.2.2.4 Maximum-Norm Error Bounds

In this section we use Theorem 9.9 and the interpolation error bounds from Sect. 9.2.1 to obtain bounds for the error of the Galerkin method in the maximum norm.

Start with the region Ω_{22}, where the mesh is quasi-uniform with mesh size $\mathcal{O}\left(N^{-1}\right)$:

$$\begin{aligned}\left\|u^I-u^N\right\|_{\infty,\Omega_{22}} &\le CN\left\|u^I-u^N\right\|_{0,\Omega_{22}}\\ &\le C\left(Nh^2\ln^{1/2}N+N^{-1}\max|\psi'|^2\right).\end{aligned}$$

Thus, on a standard Shishkin mesh, where $h=\mathcal{O}\left(N^{-1}\right)$, one gets

$$\left\|u-u^N\right\|_{\infty,\Omega_{22}} \le CN^{-1}\ln^2 N,$$

while for the Bakhvalov-Shishkin mesh we have

$$\left\|u-u^N\right\|_{\infty,\Omega_{22}} \le CN^{-1}\ln^{1/2} \quad \text{if } \varepsilon\le CN^{-1},$$

because for this mesh $h=\mathcal{O}\left(\max\{N^{-1},\varepsilon\}\right)$.

Now let (x_i,y_j) be any mesh node in Ω_{21}. Then following [152, pp. 11,12] we obtain

$$\begin{aligned}&\left|(u^I-u^N)(x_i,y_j)\right|\\ &\quad=\left|\int_0^{x_i}(u^I-u^N)(x,y_j)\mathrm{d}x\right| \le CN\int_0^{\tau_x}\int_{J_j}\left|\partial_x(u^I-u^N)\right|\\ &\quad\le CN\left(\varepsilon N^{-1}\ln N\right)^{1/2}\left\|\nabla(u^I-u^N)\right\|_{0,[0,\tau_x]\times J_j}\\ &\quad\le CN^{1/2}\ln^{1/2}N\left|\left|\left|u^I-u^N\right|\right|\right|_\varepsilon\,.\end{aligned}$$

Thus,

$$\|u - u^N\|_{\infty,\Omega_{21}} \le CN^{1/2} \ln^{1/2} N \left(h^2 \ln^{1/2} N + N^{-2} \max |\psi'|^2\right),$$

by Theorems 9.4 and Theorem 9.9. Clearly identical bounds hold on Ω_{12}.

Apply this result to get bounds for particular S-type meshes:

$$\|u - u^N\|_{\infty,\Omega_{12}\cup\Omega_{21}} \le \begin{cases} CN^{-3/2} \ln^{5/2} N & \text{for standard Shishkin meshes,} \\ CN^{-3/2} \ln N & \text{for Bakhvalov-Shishkin meshes} \\ & \text{with } \varepsilon \le CN^{-1}. \end{cases}$$

9.2.2.5 Gradient Recovery

Similar to Sect. 5.2.3 a gradient recovery operator can be defined for the bilinear Galerkin FEM, that gives approximations of the gradient which are superior to those of Theorem 9.8. We follow [139].

Notation. *In this section let* $\|\cdot\|_{1,D}$ *denote the standard norm in* $H^1(D)$.

Let T be a rectangle of Ω^N and let $\widetilde{T}$ be the patch associated with T, consisting of all rectangles that have a common corner with T (see Fig. 9.3). We define for $v \in V^\omega$ the recovered gradient $\boldsymbol{R}v$ as follows: First we compute the gradient of v at the midpoints of the mesh rectangles $\left(\gamma_{ij} := \nabla v(x_{i-1/2}, y_{j-1/2})\right)$. Then these values are bilinearly interpolated to give the values of Rv at the mesh points of the triangulation, viz.,

$$\left(\boldsymbol{R}v\right)_{ij} = \boldsymbol{\alpha}_{ij} := \sum_{m,n=0}^{1} \frac{h_{i+1-m}}{h_i + h_{i+1}} \frac{k_{j+1-n}}{k_j + k_{j+1}} \gamma_{i+m,j+n} \,. \tag{9.39}$$

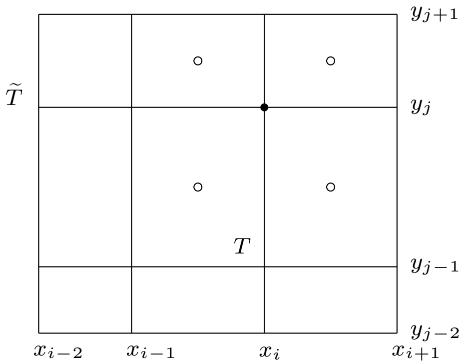


Fig. 9.3 Mesh rectangle T and associated patch $\widetilde{T}$

Bilinear interpolation is again used to extend the recovered gradient from the mesh nodes to the whole of Ω:

$$\begin{aligned}\big(\boldsymbol{R}w^N\big)(x,y) := \boldsymbol{\alpha}_{i-1,j-1}\frac{x_i-x}{h_i}\frac{y_j-y}{k_j} + \boldsymbol{\alpha}_{i,j-1}\frac{x-x_{i-1}}{h_i}\frac{y_j-y}{k_j}\\ + \boldsymbol{\alpha}_{i-1,j}\frac{x_i-x}{h_i}\frac{y-y_{j-1}}{k_j} + \boldsymbol{\alpha}_{i,j}\frac{x-x_{i-1}}{h_i}\frac{y-y_{j-1}}{k_j}\\ \text{for } (x,y)\in T_{ij},\quad i,j=2,\dots,N-1.\end{aligned}$$

For the boundary rectangles, we simply extrapolate the well-defined bilinear function of the adjacent rectangles.

Lemma 9.14. $\boldsymbol{R} : V^\omega \to V^\omega \times V^\omega$ *is a linear operator with the following properties:*

(locality) $\boldsymbol{R}v$ *on* T *depends only on values of* v *on the patch* $\widetilde{T}$,

(stability)
$$\big\|\boldsymbol{R}v\big\|_{\infty,T} \le C\big\|v\big\|_{1,\infty,\widetilde{T}} \quad \text{for all } v\in V^\omega, \tag{9.40a}$$
$$\big\|\boldsymbol{R}v\big\|_{0,T} \le C\big\|v\big\|_{1,\widetilde{T}} \quad \text{for all } v\in V^\omega, \tag{9.40b}$$

(consistency)
$$\boldsymbol{R}v^I = \nabla v \quad \text{on } T \quad \text{for all } v \text{ that are quadratic on } \widetilde{T}. \tag{9.40c}$$

Proof. The first three properties are immediate consequences of the definition of $\boldsymbol{R}$, while (9.40c) is verified by a Taylor expansion of v.

Now, given any continuous function v on $\widetilde{T}$, we denote by Qv that quadratic function on $\widetilde{T}$ with

$$\big(Qv\big)(P_k) = v(P_k) \quad \text{for } k=1,\dots,6 \text{ (see Fig. 9.4).}$$

This set of degrees of freedom is unisolvent and thus our Lagrange interpolant Qv is well defined.

The decomposition (9.11) and a careful analysis yield the following bounds for quadratic interpolation.

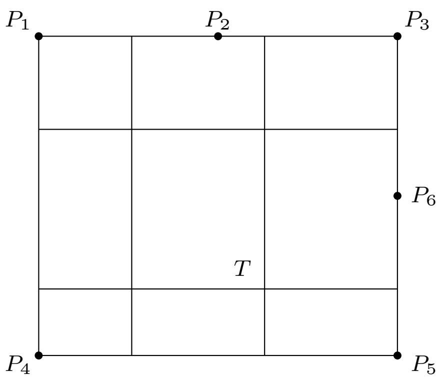


Fig. 9.4 Definition of the quadratic interpolant on the patch $\widetilde{T}$

Lemma 9.15. *Let $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ be a tensor-product S-type mesh with $\sigma \geq 3$ that satisfies* (2.8)*. Assume that the solution u of* (9.1) *can be decomposed as in* (9.11)*. Then*

$$\varepsilon \big|u - Qu\big|_{1,\infty,\tilde{T}} \leq \begin{cases} C\big(h + N^{-1} \max|\psi'|\big)^2 & \text{for } T \subset \Omega \setminus \Omega_{22}, \\ CN^{-2} & \text{for } T \subset \Omega_{22}, \end{cases}$$

$$\big\|u - Qu\big\|_{\infty,\tilde{T}} \leq \begin{cases} C\big(h + N^{-1} \max|\psi'|\big)^3 & \text{for } T \subset \Omega \setminus \Omega_{22}, \\ CN^{-3} & \text{for } T \subset \Omega_{22} \end{cases}$$

and

$$\varepsilon^{1/2} \big\|u - Qu\big\|_{1,\tilde{T}} \leq \begin{cases} C(\operatorname{meas} \tilde{T})^{1/2}\big(h + N^{-1} \max|\psi'|\big)^2 & \text{for } T \subset \Omega \setminus \Omega_{22}, \\ CN^{-3} & \text{for } T \subset \Omega_{22}. \end{cases}$$

We would like to estimate the difference between the gradient and the recovered gradient in the ε-weighted H^1 seminorm. We start from

$$\varepsilon^{1/2} \big\|\nabla u - \boldsymbol{R} u^N\big\|_0 \leq \varepsilon^{1/2} \big\|\nabla u - \boldsymbol{R} u^I\big\|_0 + \varepsilon^{1/2} \big\|\boldsymbol{R}(u^I - u^N)\big\|_0, \tag{9.41}$$

by a triangle inequality. For the second term in (9.41), the stability property (9.40b) of the recovery operator and the superconvergence result of Theorem 9.9 yield

$$\varepsilon^{1/2} \big\|\boldsymbol{R}(u^I - u^N)\big\|_0 \leq C\left(h + N^{-1} \max|\psi'|\right)^2 \ln^{1/2} N. \tag{9.42}$$

In the next result we estimate the first term in (9.41).

Proposition 9.16. *Let $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ be a tensor-product S-type mesh with $\sigma \geq 3$ that satisfies* (2.8)*. Assume that the solution u of* (9.1) *can be decomposed as in* (9.11) *and that*

$$\min\{h_{qN}, k_{qN}\} \geq C\varepsilon N^{-1}. \tag{9.43}$$

Then

$$\varepsilon^{1/2} \big\|\nabla u - \boldsymbol{R} u^I\big\|_0 \leq C\big(h + N^{-1} \max|\psi'|\big)^2 \ln^{1/2} N.$$

Proof. For any $T \in \Omega^N$, the consistency property (9.40c) of the recovery operator yields

$$\big\|\nabla u - \boldsymbol{R} u^I\big\|_{0,T} \leq \big\|\nabla(u - Qu)\big\|_{0,T} + \big\|\boldsymbol{R}(u - Qu)^I\big\|_{0,T}, \tag{9.44}$$

since $\boldsymbol{R}(Qu)^I = \nabla Qu$. For the interpolation operator we can use the stability estimates

$$\left\|v^I\right\|_{\infty,T} \le C\|v\|_{\infty,T} \quad \text{and} \quad \left\|v^I\right\|_{1,\infty,T} \le C\left\|v\right\|_{1,\infty,T}.$$

To estimate the second term in (9.44), we bound the L_2 norm by the L_∞ norm and apply the stability property (9.40a) of the recovery operator:

$$\begin{aligned}\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} &\le (\operatorname{meas} T)^{1/2}\left\|\boldsymbol{R}(u-Qu)^I\right\|_{\infty,T} \\ &\le C(\operatorname{meas} T)^{1/2}\left\|(u-Qu)^I\right\|_{1,\infty,\widetilde{T}}. \end{aligned} \tag{9.45}$$

Thus, for $T \notin \Omega_{22}$ we have

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \le C\varepsilon^{-1}(\operatorname{meas} T)^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2, \tag{9.46}$$

by Lemma 9.15.

Next we consider $T \in \Omega_{22}$. We apply an inverse inequality and the L_∞ stability of bilinear interpolation to (9.45) to get

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \le CN^{-1}\left(\min_{\widetilde{T}} h\right)^{-1}\left\|u-u^*\right\|_{\infty,\widetilde{T}}.$$

If $\min_{\widetilde{T}} h = \mathcal{O}\left(N^{-1}\right)$ then

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \le C\left\|u-Qu\right\|_{\infty,\widetilde{T}} \le CN^{-3}, \tag{9.47}$$

by Lemma 9.15. Otherwise — for the elements T along the transition from the fine to the coarse mesh — we have to estimate more carefully:

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \le \left\|(u-Qu)^I\right\|_{1,\widetilde{T}} \le \sum_{T\in\widetilde{T}} \frac{(\operatorname{meas} T)^{1/2}}{\min_T h}\left\|u-Qu\right\|_{\infty,T}.$$

From (9.43), we have

$$\varepsilon^{1/2}\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \le C\left\|u-Qu\right\|_{\infty,\widetilde{T}} \le CN^{-3}, \tag{9.48}$$

by Lemma 9.15. Combining (9.46), (9.47) and (9.48), we have

$$\begin{aligned}&\varepsilon^{1/2}\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \\ &\qquad \le \begin{cases} C\varepsilon^{-1/2}(\operatorname{meas} T)^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2 & \text{for } T \subset \Omega\setminus\Omega_{22}, \\ CN^{-3} & \text{for } T \subset \Omega_{22}. \end{cases}\end{aligned}$$

We use the last estimate of Lemma 9.15 and (9.44) to obtain

$$\varepsilon^{1/2}\left\|\nabla u - \boldsymbol{R}u^I\right\|_{0,T} \le \begin{cases} C\varepsilon^{-1/2}(\operatorname{meas} T)^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2 & \text{for } T \subset \Omega \setminus \Omega_{22}, \\ CN^{-3} & \text{for } T \subset \Omega_{22}\,. \end{cases}$$

Recalling that

$$\left\|\nabla u - \boldsymbol{R}u^I\right\|_0^2 = \sum_{T\in\Omega^N} \left\|\nabla u - \boldsymbol{R}u^I\right\|_{0,T}^2$$

and $\operatorname{meas}\left(\Omega \setminus \Omega_{22}\right) = \mathcal{O}\left(\varepsilon \ln N\right)$, the proof is complete. □

Remark 9.17. The condition (9.43) is satisfied if for example, $\tilde{\varphi}'$ in Sect. 2.1.3 is bounded from below by a positive constant independently of ε and N. Both the original Shishkin mesh and the Bakhvalov-Shishkin mesh satisfy this condition. ♣

As a consequence of (9.41), (9.42) and Proposition 9.16 we have the following result:

Theorem 9.18. *Let $\bar{\omega} = \bar{\omega}_x \times \bar{\omega}_y$ be a tensor-product S-type mesh with $\sigma \ge 5/2$ that satisfies* (2.8) *and* (9.43)*. Assume that the solution u of* (9.1) *can be decomposed as in* (9.11)*. Then*

$$\varepsilon^{1/2}\left\|\nabla u - \boldsymbol{R}u^N\right\|_0 \le C\left(h + N^{-1}\max|\psi'|\right)^2 \ln^{1/2} N.$$

Remark 9.19. Similar to the one-dimensional case, using $\boldsymbol{R}u^N$ instead of ∇u^N, we get an asymptotically exact error estimator for the weighted H^1-seminorm of the finite element error $\varepsilon^{1/2}\left\|\nabla(u - u^N)\right\|_0$ on S-type meshes. ♣

9.2.2.6 Numerical Tests

Let us verify our theoretical results for the Galerkin FEM using bilinear trial and test functions on S-type meshes when applied to the test problem (9.3). In our computations we have chosen $\varepsilon = 10^{-8}$ and $\sigma = 3$ for the meshes. In the tables we compare both the error in the ε-weighted energy norm $|||u - u^N|||_\varepsilon$ with the error in the discrete energy norm $|||u^I - u^N|||_\varepsilon$, and the accuracy of the gradient approximation ∇u^N with that of the recovered gradient approximation $\boldsymbol{R}u^N$. The errors are estimated using a 4^{th}-order Gauß-Legendre formula on each mesh rectangle. The rates of convergence are computed in the usual way. Tables 9.3 and 9.4 are clear illustrations of Theorems 9.8, 9.9 and 9.18.

Table 9.3 Shishkin mesh

	$\|\|\|u - u^N\|\|\|_\varepsilon$		$\|\|\|u^I - u^N\|\|\|_\varepsilon$		$\varepsilon^{1/2}\|\nabla u - \nabla u^N\|_0$		$\varepsilon^{1/2}\|\nabla u - \boldsymbol{R}u^N\|_0$	
N	error	rate	error	rate	error	rate	error	rate
16	2.6900e-1	0.63	5.2110e-2	1.25	2.6898e-1	0.63	9.5425e-1	2.86
32	1.7359e-1	0.72	2.1896e-2	1.43	1.7359e-1	0.72	1.3141e-1	1.81
64	1.0556e-1	0.77	8.1467e-3	1.53	1.0556e-1	0.77	3.7507e-2	1.48
128	6.1881e-2	0.80	2.8137e-3	1.60	6.1881e-2	0.80	1.3479e-2	1.56
256	3.5421e-2	0.83	9.2543e-4	1.65	3.5421e-2	0.83	4.5685e-3	1.64
512	1.9936e-2	0.85	2.9398e-4	1.69	1.9936e-2	0.85	1.4687e-3	1.69
1024	1.1078e-2	—	9.0961e-5	—	1.1078e-2	—	4.5612e-4	—

Table 9.4 Bakhvalov-Shishkin mesh

	$\|\|\|u - u^N\|\|\|_\varepsilon$		$\|\|\|u^I - u^N\|\|\|_\varepsilon$		$\varepsilon^{1/2}\|\nabla u - \nabla u^N\|_0$		$\varepsilon^{1/2}\|\nabla u - \boldsymbol{R}u^N\|_0$	
N	error	rate	error	rate	error	rate	error	rate
16	1.2475e-1	1.00	7.9084e-3	2.00	1.2471e-1	1.00	5.0012e-1	3.43
32	6.2574e-2	1.00	1.9800e-3	2.00	6.2569e-2	1.00	4.6315e-2	3.09
64	3.1312e-2	1.00	4.9620e-4	2.00	3.1311e-2	1.00	5.4227e-3	2.43
128	1.5659e-2	1.00	1.2425e-4	2.00	1.5659e-2	1.00	1.0044e-3	2.08
256	7.8298e-3	1.00	3.1096e-5	2.00	7.8298e-3	1.00	2.3690e-4	2.01
512	3.9149e-3	1.00	7.7789e-6	2.00	3.9149e-3	1.00	5.8638e-5	2.00
1024	1.9575e-3	—	1.9460e-6	—	1.9575e-3	—	1.4624e-5	—

9.2.3 Artificial Viscosity Stabilisation

In Sect. 5.3 we studied a FEM with artificial viscosity stabilisation in one dimension. It can be generalised to two dimensions as follows: Set

$$\boldsymbol{\hbar} := \operatorname{diag}(\hbar, \bar{k}) \quad \text{with} \quad \hbar := h_i \quad \text{in } I_i \times (0,1) \text{ and } \bar{k} := k_j \quad \text{in } (0,1) \times J_j$$

and let $\kappa \geq 0$ be an arbitrary constant. Then we add artificial viscosity of order $\kappa\hbar$ in x-direction and of order $\kappa\bar{k}$ in y-direction, i. e., we consider the following discretisation. Find $u^N \in V_0^\omega$ such that

$$a_\kappa(u^N, v) := a(u^N, v) + \kappa\left(\boldsymbol{\hbar}\nabla u^N, \nabla v\right) = (f, v) \quad \text{for all } v \in V_0^\omega.$$

The norm naturally associated with $a_\kappa(\cdot,\cdot)$ is

$$|||v|||_\kappa := \left[|||v|||_\varepsilon^2 + \kappa\left(\boldsymbol{\hbar}\nabla v, \nabla v\right)\right]^{1/2} \geq |||v|||_\varepsilon, \quad \text{for all } v \in H^1(\Omega).$$

The bilinear form $a_\kappa(\cdot,\cdot)$ is coercive with respect to this norm, because

$$a_\kappa(v, v) \geq |||v|||_\kappa^2 \quad \text{for all } v \in H_0^1(\Omega). \tag{9.49}$$

In our analysis we follow Schneider et al. [148], but refine it by explicitly monitoring the dependence on κ. Let again $\eta = u^I - u$ denote the interpolation error and

$\chi = u^I - u^N$ the difference between interpolated and exact solution. Because of the added artificial viscosity the discretisation does not satisfy the Galerkin orthogonality condition, but we have

$$a_\kappa(\chi, \chi) = a(\eta, \chi) + \kappa(\boldsymbol{\hbar}\nabla\eta, \nabla\chi) + \kappa(\boldsymbol{\hbar}\nabla u, \nabla\chi) . \tag{9.50}$$

(i) For the first term we have two bounds from Sections 9.2.2.1 and 9.2.2.2:

$$|a(\eta, \chi)| \\ \le C\,|||\chi|||_\varepsilon \begin{cases} h + N^{-1}\max|\psi'| & \text{for general linear and,} \\ & \text{bilinear elements,} \\ h^2 \ln^{1/2} N + N^{-2}\max|\psi'|^2 & \text{for bilinear elements.} \end{cases} \tag{9.51}$$

(ii) Next we bound $\kappa\big(\boldsymbol{\hbar}\nabla\eta, \nabla\chi\big)$. Let T_{ij} be arbitrary. Then

$$\big(\hbar\eta_x, \chi_x\big)_{T_{ij}} = h_i \int_{J_j}\int_{I_i} \partial_x\eta\partial_x\chi = \int_{J_j}\int_{I_i}\partial_x\eta\int_{I_i}\partial_x\chi.$$

Thus,

$$\Big|\big(\hbar\partial_x\eta, \partial_x\chi\big)_{T_{ij}}\Big| \le 2\big\|\eta\big\|_{\infty,T_{ij}}\big\|\partial_x\chi\big\|_{1,T_{ij}}.$$

Consequently, we have

$$\big|\big(\hbar\partial_x\eta, \partial_x\chi\big)\big| \\ \le C\Big\{N^{-2}\big\|\partial_x\chi\big\|_{0,\Omega_{22}} + \big(N^{-1}\max|\psi'|\big)^2\big(\varepsilon\ln N\big)^{1/2}\big\|\partial_x\chi\big\|_{\Omega\setminus\Omega_{22}}\Big\} \\ \le CN^{-1}\max|\psi'|\ln^{1/2}N\,|||\chi|||_\varepsilon\,,$$

by an inverse inequality and (9.5). An analogous estimate holds for $\big|\big(\hbar\partial_y\eta, \partial_y\chi\big)\big|$. Hence,

$$\kappa\,\big|\big(\boldsymbol{\hbar}\nabla\eta, \nabla\chi\big)\big| \le C\kappa N^{-1}\max|\psi'|\ln^{1/2}N\,|||\chi|||_\varepsilon\,. \tag{9.52}$$

(iii) Finally, $\big(\boldsymbol{\hbar}\nabla u, \nabla\chi\big)$ has to be considered. We restrict ourselves to bounding $\big(\hbar\partial_x u, \partial_x\chi\big)$ since the term $\big(\hbar\partial_y u, \partial_y\chi\big)$ can be treated analogously. Using the decomposition of Theorem 7.17, we get

$$\big(\hbar\partial_x u, \partial_x\chi_x\big) \\ = \big(\hbar\partial_x(v + w_2), \partial_x\chi\big)_{\Omega_{11}\cup\Omega_{12}} + \big(\hbar\partial_x(v + w_2), \partial_x\chi\big)_{\Omega_{21}\cup\Omega_{22}} \\ + \big(\hbar\partial_x(w_1 + w_{12}), \partial_x\chi\big)_{\Omega_{11}\cup\Omega_{12}} + \big(\hbar\partial_x(w_1 + w_{12}), \partial_x\chi\big)_{\Omega_{21}\cup\Omega_{22}}\,. \tag{9.53}$$

The Cauchy-Schwarz inequality and Theorem 7.17 yield

$$\begin{aligned}&\left|\left(\hbar\partial_x(v+w_2),\partial_x\chi\right)_{\Omega_{11}\cup\Omega_{12}}\right|\\&\qquad\le Ch\left(\varepsilon\ln N\right)^{1/2}\left\|\partial_x\chi\right\|_{\Omega_{11}\cup\Omega_{12}}\le Ch\ln^{1/2}N\,|||\chi|||_\varepsilon\,.\end{aligned}\tag{9.54}$$

On $\Omega_{21}\cup\Omega_{22}$ we have

$$\begin{aligned}&\left(\hbar\partial_x(v+w_2),\partial_x\chi\right)_{\Omega_{21}\cup\Omega_{22}}\\&\qquad=H\int_0^1\int_{\tau_x}^1\partial_x(v+w_2)\partial_x\chi\mathrm{d}x\mathrm{d}y\\&\qquad=-H\int_0^1\left\{\left(\partial_x(v+w_2)\chi\right)(\tau_x,y)+\int_{\tau_x}^1\partial_x^2(v+w_2)\chi\mathrm{d}x\right\}\mathrm{d}y.\end{aligned}$$

Thus,

$$\left|\left(\hbar\partial_x(v+w_2),\partial_x\chi\right)_{\Omega_{21}\cup\Omega_{22}}\right|\le CN^{-1}\left\{\|\chi\|_0+\int_0^1|\chi(\tau_x,y)|\,\mathrm{d}y\right\}.\tag{9.55}$$

Note that

$$\int_0^1|\chi(\tau_x,y)|\,\mathrm{d}y=\int_0^1\left|\int_0^{\tau_x}\partial_x\chi\mathrm{d}x\right|\mathrm{d}y\le\|\partial_x\chi\|_{1,\Omega_{11}\cup\Omega_{12}}\le C\ln^{1/2}N\,|||\chi|||_\varepsilon\,.$$

We apply this inequality to (9.55) to obtain

$$\left|(\hbar\partial_x(v+w_2),\partial_x\chi)_{\Omega_{21}\cup\Omega_{22}}\right|\le CN^{-1}\ln^{1/2}N\,|||\chi|||_\varepsilon\,.\tag{9.56}$$

Now we bound the last two terms in (9.53). Using Theorem 7.17 we get, for any $T_{ij}\in\Omega^N$,

$$\begin{aligned}&\left|\left(\hbar\partial_x(w_1+w_{12}),\partial_x\chi\right)_{T_{ij}}\right|\\&\qquad\le C\int_{J_j}\left\{\int_{I_i}\varepsilon^{-1}\mathrm{e}^{-\beta_1x/\varepsilon}\mathrm{d}x\int_{I_i}\left|\partial_x\chi\right|\mathrm{d}x\right\}\mathrm{d}y.\end{aligned}$$

This implies that

$$\begin{aligned}&\left|\left(\hbar\partial_x(w_1+w_{12}),\partial_x\chi\right)_{T_{ij}}\right|\\&\qquad\le\begin{cases}CN^{-1}\max|\psi'|\left\|\partial_x\chi\right\|_{1,T_{ij}}&\text{for }T_{ij}\subset\Omega_{11}^N\cup\Omega_{12}^N,\\CN^{-2}\left\|\partial_x\chi\right\|_{1,T_{ij}}&\text{for }T_{ij}\subset\Omega_{21}^N\cup\Omega_{22}^N.\end{cases}\end{aligned}$$

Therefore,

$$\left|\left(\hbar\partial_x(w_1+w_{12}),\partial_x\chi\right)_{\Omega_{11}\cup\Omega_{12}}\right|$$
$$\le CN^{-1}\max|\psi'|\left\|\chi_x\right\|_{1,\Omega_{11}\cup\Omega_{12}} \le CN^{-1}\max|\psi'|\ln^{1/2}N\,|||\chi|||_\varepsilon$$

and

$$\left|\left(\hbar\partial_x(w_1+w_{12}),\partial_x\chi\right)_{\Omega_{21}\cup\Omega_{22}}\right| \le CN^{-1}\left\|\chi\right\|_0,$$

by an inverse inequality.

Combine the last two bounds with (9.53), (9.54) and (9.56) to get

$$\left|\left(\hbar\partial_x u,\partial_x\chi\right)\right| \le CN^{-1}\max|\psi'|\ln^{1/2}N\,|||\chi|||_\varepsilon.$$

With an analogous estimate for $\left(\bar{k}\partial_y u,\partial_y\chi\right)$ we have

$$\kappa\left|(\boldsymbol{\hbar}\nabla u,\nabla\chi)\right| \le C\kappa N^{-1}\max|\psi'|\ln^{1/2}N\,|||\chi|||_\varepsilon. \tag{9.57}$$

Finally, combine (9.49)–(9.52) and (9.57) in order to obtain the main result of this section.

Theorem 9.20. *Let $\bar\omega := \bar\omega_x\times\bar\omega_y$ be a tensor-product S-type mesh with $\sigma\ge 2$ that satisfies* (2.8)*. Then the upwind-FEM solution u^N satisfies*

$$|||u^I-u^N|||_\kappa \le C\left(1+\kappa\ln^{1/2}N\right)N^{-1}\max|\psi'|$$

and, for bilinear elements and $\sigma\ge 5/2$,

$$|||u^I-u^N|||_\kappa \le C\left\{\kappa N^{-1}\max|\psi'|\ln^{1/2}N + h^2\ln^{1/2}N + N^{-2}\max|\psi'|^2\right\}.$$

A consequence of Theorem 9.20 and Sect. 9.2.1 is the following bound of the error in the ε-weighted energy norm:

$$|||u-u^N|||_\varepsilon \le C\left(h+N^{-1}\max|\psi'|\ln^{1/2}N\right).$$

Remark 9.21. The supercloseness property of the Galerkin FEM with bilinear elements is not affected if we take $\kappa=\mathcal{O}\left(N^{-1}\right)$. However, for the efficient treatment of the discrete systems, the choice $\kappa=\mathcal{O}\left(1\right)$ is more appropriate which then results in a loss of the supercloseness property. ♣

Remark 9.22. The $|||\cdot|||_\kappa$ bounds imply that the method gives uniform convergent approximations of the gradient on the coarse mesh region Ω_{22}. For example, for a Shishkin mesh, where $\max|\psi'|\le C\ln N$ and $h\le 2N^{-1}$, we have

$$\kappa^{1/2}N^{-1/2}\left\|\nabla\left(u^I-u^N\right)\right\|_{0,\Omega_{22}} \le C\left\{\kappa N^{-1}\ln^{3/2}+N^{-2}\ln^2 N\right\}.$$

Thus,

$$\left\|\nabla\left(u^I-u^N\right)\right\|_{0,\Omega_{22}} \le \begin{cases} CN^{-1/2}\ln^{3/2} & \text{if } \kappa=\mathcal{O}(1)\,,\\ CN^{-1}\ln^2 & \text{if } \kappa=\mathcal{O}\left(N^{-1}\right). \end{cases}$$

Note that in contrast to the streamline-diffusion FEM, we have full control of the gradient, while for SDFEM one has uniform bounds for the streamline derivative $\|\boldsymbol{b}\cdot\nabla(u^I-u^N)\|_{0,\Omega_{22}}$ only; see Sect. 9.2.4. ♣

Remark 9.23. Suboptimal maximum-norm error bounds on Ω_{22} can be obtained by application of the discrete Sobolev inequality

$$\|\chi\|_{\infty,\Omega_{22}} \le C\ln^{1/2}N\,\|\nabla\chi\|_{0,\Omega_{22}}\,, \tag{9.58}$$

that holds true for piecewise-polynomial functions χ that vanish on a part of the boundary of finite length, see [160, Lemma 5.4] or [63]. We get

$$\left\|u-u^N\right\|_{\infty,\Omega_{22}} \le \begin{cases} CN^{-1/2}\ln^2 N & \text{if } \kappa=\mathcal{O}(1)\,,\\ CN^{-1}\ln^{5/2}N & \text{if } \kappa=\mathcal{O}\left(N^{-1}\right). \end{cases}$$

Bounds for the maximum-norm error on $\Omega_{21}\cup\Omega_{12}$ can be obtained using the technique from Sect. 9.2.2.4. ♣

9.2.4 Streamline-Diffusion FEM

Introduced by Hughes and Brooks [54], this method is the most commonly used stabilised FEM for the discretisation of convection-diffusion and related problems. Starting from the weak formulation (9.4), we add weighted residuals in order to stabilise the method. Then the SDFEM reads: Find $u^N\in V_0^\omega$ such that

$$a_{SD}(u^N,v)=a(u^N,v)+a_{stab}(u^N,v)=f_{SD}(v)\quad\text{for all } v\in V_0^\omega$$

with

$$a_{stab}(u^N,v):=\sum_{T\in\Omega^N}\delta_T(\mathcal{L}u^N,-\boldsymbol{b}\cdot\nabla v)_T$$

and

$$f_{SD}(v):=f(v)+\sum_{T\in\Omega^N}\delta_T(f,-\boldsymbol{b}\cdot\nabla v)_T$$

and user chosen stabilisation parameters $\delta_T \geq 0$. We clearly have the Galerkin orthogonality property

$$a_{SD}(u - u^N, v) = 0 \quad \text{for all } \ v \in V_0^\omega . \tag{9.59}$$

Let V_0^ω be our finite element space consisting of piecewise (bi)linear functions that vanish on $\partial\Omega$. It is shown in, e.g., [141, §III.3.2.1], that if

$$0 \leq \delta_T \leq \gamma \|c\|_{\infty,T}^{-2} \quad \text{for all } \ T \in \Omega^N , \tag{9.60}$$

then

$$a_{SD}(v, v) \geq \frac{1}{2} \, |||v|||_{SD}^2 \quad \text{for all } \ v \in V_0^\omega , \tag{9.61}$$

with the streamline-diffusion norm

$$|||v|||_{SD}^2 := |||v|||_\varepsilon^2 + \sum_{T \in \Omega^N} \delta_T (\boldsymbol{b} \cdot \nabla v, \boldsymbol{b} \cdot \nabla v)_T \ .$$

9.2.4.1 Convergence in the Streamline-Diffusion Norm

Stynes and Tobiska [155] analyse the SDFEM using piecewise bilinear finite elements on standard Shishkin meshes for problems with regular layers. Here we shall extend the technique from [155] to our more general class of S-type meshes, but still consider piecewise bilinear test and trial functions.

Partition the domain $\bar{\Omega}$ as in Fig. 9.2. We follow standard recommendations [141, p. 307] and set

$$\delta_T := \begin{cases} \delta & \text{if } T \subset \Omega_{22}, \\ 0 & \text{otherwise}, \end{cases} \tag{9.62a}$$

and

$$\delta := \begin{cases} \delta_0 N^{-1} & \text{if } \varepsilon \leq N^{-1}, \\ \delta_1 \varepsilon^{-1} N^{-2} & \text{otherwise}. \end{cases} \tag{9.62b}$$

with positive constants δ_0 and δ_1. Clearly $\delta \leq \max\{\delta_0, \delta_1\} N^{-1}$ and therefore (9.60) is satisfied for N sufficiently large, independent of ε.

Note that in the layer regions $\Omega \setminus \Omega_{22}$, the stabilisation is switched off because there the streamline-diffusion stabilisation would be negligible compared to the natural stability induced by the discretisation of the diffusion term.

Our error analysis again starts from the coercivity (9.61) and the Galerkin orthogonality (9.59). Let again $\eta = u^I - u$ and $\chi = u^I - u^N$. Then

$$\frac{1}{2} \, |||\chi|||_{SD}^2 \leq a(\eta, \chi) + a_{stab}(\eta, \chi).$$

For the first term we have

$$|a(\eta,\chi)| \le C\left(h^2 \ln^{1/2} N + N^{-2}\max|\psi'|^2\right) |||\chi|||_\varepsilon\,,$$

see Sect. 9.2.2.3, while the stabilisation term

$$a_{stab}(\eta,\chi) = \delta \sum_{T\subset\Omega_{22}} (\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla\eta - c\eta, \boldsymbol{b}\cdot\nabla\chi)_T$$

still has to be analysed. This was done in [155]. Using (9.14b) as a crucial ingredient, Stynes and Tobiska derive the bound

$$|a_{stab}(\eta,\chi)| \le CN^{-2}\ln^{1/2} N\, |||\chi|||_{SD}\,.$$

Eventually we get the following convergence results.

Theorem 9.24. *Let $\omega_x \times \omega_y$ be a tensor-product S-type mesh with $\sigma \ge 5/2$ that satisfies* (2.8)*. Then the SDFEM solution u^N satisfies*

$$\left|\left|\left|u^I - u^N\right|\right|\right|_{SD} \le C\left(h^2\ln^{1/2} N + N^{-2}\max|\psi'|^2\right)$$

Remark 9.25. Theorems 9.4 and 9.24 give

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \le C\left(h + N^{-1}\max|\psi'|\right).$$

Thus, Theorem 9.24 is a supercloseness result for the SDFEM.
Furthermore,

$$\varepsilon\left\|\nabla u - \boldsymbol{R}u^N\right\|_0 \le C\left(h^2 + N^{-2}\max|\psi'|^2\right)\ln^{1/2} N,$$

where $\boldsymbol{R}$ is the recovery operator from Sect. 9.2.2.5. ♣

9.2.4.2 Maximum-Norm Error Bounds

Clearly the technique for the Galerkin FEM from Sect. 9.2.2.4 can be applied to give pointwise error bounds for the SDFEM with bilinear test and trial functions within the layer regions Ω_{12} and Ω_{21}, while on the coarse mesh region Ω_{22}, we can employ (9.58). We get

$$\|u - u^N\|_{\infty,\Omega\setminus\Omega_{11}} \le \begin{cases} CN^{-3/2}\ln^{5/2} N & \text{for standard Shishkin meshes} \\ CN^{-3/2}\ln N & \text{for Bakhvalov-Shishkin meshes} \\ & \text{with } \varepsilon \le CN^{-1}. \end{cases} \tag{9.63}$$

Adapting Niijima's technique [125], Linß & Stynes [110] study the SDFEM with piecewise linear test and trial functions on Shishkin meshes. For technical reasons a modified version of the SDFEM with artificial crosswind diffusion added on Ω_{22} is studied. Furthermore, it is assumed that the convective field $\boldsymbol{b}$ is constant. The method reads as follows. Find $u^N \in V_0^\omega$ such that

$$a_{SD}(u^N, v) + (\varepsilon^* \boldsymbol{b}^\perp \cdot \nabla u^N, \boldsymbol{b}^\perp \cdot \nabla v) = f_{SD}(v) \quad \text{for all } v \in V_0^\omega$$

with

$$\boldsymbol{b}^\perp := \frac{1}{\|\boldsymbol{b}\|} \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} \quad \text{and} \quad \varepsilon^* := \begin{cases} \max\left\{0, N^{-3/2} - \varepsilon\right\} & \text{on } \Omega_{22}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\varepsilon \le N^{-3/2}$, then for any point $(x, y) \in \Omega$ the analysis in [110] yields

$$\left|(u - u^N)(x, y)\right| \le \begin{cases} CN^{-1/2} \ln^{3/2} N & \text{if } (x, y) \in \Omega_{22}, \\ CN^{-3/4} \ln^{3/2} N & \text{if } (x, y) \in \Omega \setminus \Omega_{22}, \\ CN^{-11/8} \ln^{1/2} N & \text{if } (x, y) \in (\lambda^*, 1)^2, \end{cases}$$

where $\lambda^* = \mathcal{O}\left(N^{-3/4} \ln N\right)$. The analysis in [110] includes more detailed results and also deals with the case $\varepsilon \ge N^{-3/2}$. Numerical experiments in [109] show convergence of almost second order on the coarse part of the mesh, while inside the boundary layers, the rates are smaller than one. For bilinear elements, almost second-order convergence in the maximum norm is observed globally, but no rigorous analysis is yet available.

9.2.4.3 A Numerical Example

Let us verify the theoretical results when the SDFEM is applied to our test problem (9.3). In the computations we have chosen $\varepsilon = 10^{-8}$ and $\sigma = 3$.

The tables display the error in the ε-weighted energy norm $\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon$, in the discrete SD-norm $\left|\left|\left|u^I - u^N\right|\right|\right|_{SD}$ and in the maximum-norm. Tables 9.5 and 9.6 clearly illustrate Theorem 9.24, while for the maximum-norm errors (9.63) appears to be suboptimal: instead of convergence of order (almost) $3/2$ we observe (almost) 2nd order.

9.2.4.4 Higher-Order Elements

In [156] Stynes and Tobiska study the SDFEM with Q_p elements, $p > 1$ on tensor-product Shishkin meshes.

Table 9.5 The SDFEM on a Shishkin mesh

	$\|\|\|u-u^N\|\|\|_\varepsilon$		$\|\|\|u^I-u^N\|\|\|_{SD}$		$\|u-u^N\|_\infty$	
N	error	rate	error	rate	error	rate
16	3.3542e-1	0.75	2.0654e-1	1.04	1.7673e-1	1.14
32	1.9932e-1	0.82	1.0021e-1	1.33	8.0261e-2	1.41
64	1.1259e-1	0.83	3.9957e-2	1.50	3.0251e-2	1.51
128	6.3418e-2	0.83	1.4151e-2	1.59	1.0635e-2	1.61
256	3.5718e-2	0.84	4.6849e-3	1.65	3.4956e-3	1.66
512	1.9989e-2	0.85	1.4886e-3	1.69	1.1063e-3	1.70
1024	1.1087e-2	—	4.5993e-4	—	3.4131e-4	—

Table 9.6 The SDFEM on a Bakhvalov-Shishkin mesh

	$\|\|\|u-u^N\|\|\|_\varepsilon$		$\|\|\|u^I-u^N\|\|\|_{SD}$		$\|u-u^N\|_\infty$	
N	error	rate	error	rate	error	rate
16	1.3415e-1	1.07	4.9909e-2	1.92	5.1204e-2	1.89
32	6.3934e-2	1.02	1.3161e-2	1.98	1.3793e-2	1.96
64	3.1488e-2	1.01	3.3354e-3	2.00	3.5346e-3	1.99
128	1.5681e-2	1.00	8.3621e-4	2.00	8.8983e-4	2.00
256	7.8326e-3	1.00	2.0910e-4	2.00	2.2291e-4	2.00
512	3.9153e-3	1.00	5.2263e-5	2.00	5.5756e-5	2.00
1024	1.9575e-3	—	1.3063e-5	—	1.3940e-5	—

The transition points in the Shishkin mesh are

$$\tau_x := \min\left\{q, \frac{(p+1)\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \tau_y := \min\left\{q, \frac{(p+1)\varepsilon}{\beta_2}\ln N\right\},$$

otherwise the construction of the mesh is unchanged. The stabilisation parameters δ_T are chosen as in (9.62).

We introduce a special vertices-edges-element interpolant [79] as follows. Let v be a given function. On each element $T \in \Omega^N$ the interpolant Iv is defined by

$$(Iv)(\boldsymbol{x}_i) = v(\boldsymbol{x}_i) \quad \text{for } i = 1, \dots, 4,$$
$$\int_{\ell_i} (Iv)\varphi = \int_{\ell_i} v\varphi \quad \text{for all } \varphi \in P_{p-2}(\ell_i) \text{ and for } i = 1, \dots, 4,$$

and

$$\int_T (Iv)\psi = \int_T v\psi \quad \text{for all } \psi \in Q_{p-2}(T),$$

where $\boldsymbol{x}_i$ are the four vertices of T and ℓ_i are the four edges of T. $P_k(\ell_i)$ is the space of polynomials of degree at most k in the single variable whose axis is parallel to the edge ℓ_i.

In [156] the authors proceed by deriving interpolation error bounds for Iu on anisotropic meshes. For example, the pointwise interpolation error is shown to satisfy

$$|(u - Iu)(\boldsymbol{x})| \leq \begin{cases} CN^{-(p+1)} & \text{if } \boldsymbol{x} \in \bar{\Omega}_{22}, \\ C\left(N^{-1}\ln N\right)^{p+1} & \text{otherwise.} \end{cases}$$

However, the main result in [156] is

$$\left|\left|\left|u^N - Iu\right|\right|\right|_{SD} \leq CN^{-(p+1/2)}.$$

This is a supercloseness result, because in general for Q_p elements, one can expect at best

$$\left|\left|\left|u^N - u\right|\right|\right|_{\varepsilon} \leq C\left(N^{-1}\ln N\right)^p.$$

Postprocessing can be used to obtain an approximation of u of order $p + 1/2$.

Remark 9.26. Matthies [119] considers a different approach to stabilised FEM: local projection stabilisation on Shishkin-type meshes. Using Q_p elements inside the layers and enriched Q_p elements in the region where the mesh is coarse, he proves convergence and supercloseness results that resemble those by Stynes and Tobiska for SDFEM. ♣

9.2.5 Characteristic Layers

We now consider (9.1) with parabolic layers. This is, we assume the convective field is $\boldsymbol{b} = (b, 0)$ and seek a solution u to

$$-\varepsilon\Delta u - bu_x + cu = f \text{ in } \Omega = (0,1)^2, \quad u = 0 \text{ on } \partial\Omega \tag{9.64}$$

with $b \geq \beta > 0$ and $c \geq 0$ on $\bar{\Omega}$. The main contributions here are by Franz et al. [38–41].

Analytical properties of this problem have been studied in Sect. 7.3.2. There will be an exponential layer at $x = 0$ and parabolic layers along the boundaries $y = 0$ and $y = 1$.

When discretising (9.64), we use tensor-product Shishkin-type meshes with N mesh intervals in each coordinate direction. The mesh in x-direction is a mesh for one-dimensional convection-diffusion equations (Sect. 2.1.3), while the mesh in y-direction is a mesh for a reaction-diffusion problem (Sect. 2.2). The transition parameters for these meshes are

$$\lambda_x := \min\left\{q, \frac{\sigma\varepsilon}{\beta}\ln N\right\} \quad \text{and} \quad \lambda_y := \min\left\{\frac{q}{2}, \sigma\sqrt{\varepsilon}\ln N\right\}$$

with the mesh parameters $\sigma > 0$ and $q \in (0, 1/2)$. A plot of the resulting mesh is displayed in Fig. 2.21 on p. 29. The mesh characterising function is again denoted by ψ.

Interpolation error

Adapting the technique from Sect. 9.2.1, we obtain for all mesh rectangles T_{ij}

$$\left\|u - u^I\right\|_{\infty,T_{ij}} \le C\left\{\int_{I_i} \left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right) \mathrm{d}x + \int_{J_j} \left(1 + \varepsilon^{-1/2}\mathrm{e}^{-y/2\sqrt{\varepsilon}}\right) \mathrm{d}y\right\}^2$$

and

$$\left\|u - u^I\right\|_{\infty} \le C\left\{\max_{i=1,\dots,N} \int_{I_i} \left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta x/2\varepsilon}\right) \mathrm{d}x + \max_{j=1,\dots,N} \int_{J_j} \left(1 + \varepsilon^{-1/2}\mathrm{e}^{-y/2\sqrt{\varepsilon}}\right) \mathrm{d}y\right\}^2 .$$

Bounds for particular meshes (Bakhvalov or Shishkin meshes) can immediately be concluded as has been done before.

Energy-norm bounds. The L_2 part of the norm is easily bounded by the maximum norm. Therefore, we consider the H_1 part only. Reasoning as in Sect. 9.2.1 and using Theorem 7.20, we get

$$\varepsilon^{1/2} \left\|\partial_x \left(u - u^I\right)\right\|_0 \le C \left\|u - u^I\right\|_{\infty}^{1/2}$$

and

$$\varepsilon^{1/2} \left\|\partial_y \left(u - u^I\right)\right\|_0 \le C\varepsilon^{1/4} \left\|u - u^I\right\|_{\infty}^{1/2} .$$

The last estimate highlights a problem of the energy norm applied to problems with characteristic layers: It fails to capture these layers. Nonetheless, it is the natural norm associated with the weak formulation of (9.64).

Galerkin FEM

In [38, 39] for finite elements with piecewise bilinear test and trial functions the following bounds in the ε-weighted energy norm are given.

Theorem 9.27. *Let u^N be the piecewise bilinear Galerkin approximation on a S-type mesh with $\sigma \geq 5/2$ then*

$$|||u^I - u^N|||_\varepsilon \leq C\left(\left(h + N^{-1}\right)\ln^{1/4} N + k + N^{-1}\max|\psi'|\right)^2$$

and

$$|||u - u^N|||_\varepsilon \leq C\left(\varepsilon^{1/4}h + k + N^{-1}\max|\psi'|\right),$$

where h and k are the maximum step sizes in x- and y-direction, resp.

Remark 9.28. The first bound is a supercloseness result and allows for postprocessing that gives higher-order accurate approximations of the gradient. ♣

Streamline-diffusion FEM

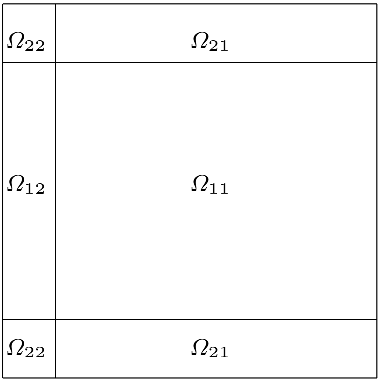
The choice of S-type meshes naturally divides the domain Ω into four (six) subregions, see Fig. 9.5. Ω_{12} covers the exponential layer, Ω_{21} the parabolic layer and Ω_{22} the corner layer and Ω_{11} the remaining region which does not have layers. On each of the four subdomains we allow the streamline-diffusion parameter δ to take different values: δ_{ij} on Ω_{ij}.

Theorem 9.29. *Let u^N be the piecewise bilinear streamline-diffusion approximation on a S-type mesh with $\sigma \geq 5/2$. Suppose the stabilisation parameters satisfy*

$$\delta_{12} \leq \delta_0\varepsilon\left(h + N^{-1}\max|\psi'|\right)^2, \quad \delta_{21} \leq \delta_0\varepsilon^{-1/4}N^{-2}, \quad \delta_{22} \leq \delta_0\varepsilon^{3/4}N^{-2}$$

and

$$\delta_{11} \leq \delta_0 \min\left\{N^{-1}, \varepsilon^{-1}N^{-2}\right\}$$



$$\begin{aligned}
\Omega_{11} &:= [\lambda_x, 1] \times [\lambda_y, 1-\lambda_y] \\
\Omega_{12} &:= [0, \lambda_x] \times [\lambda_y, 1-\lambda_y] \\
\Omega_{21} &:= [\lambda_x, 1] \times ([0, \lambda_y] \cup [1-\lambda_y, 1]) \\
\Omega_{22} &:= [0, \lambda_x] \times ([0, \lambda_y] \cup [1-\lambda_y, 1])
\end{aligned}$$

Fig. 9.5 Dissection of Ω

with a constant δ_0 that is independent of ε. Then

$$|||u^I - u^N|||_{SD} \le C\left(\left(h + N^{-1}\right)\ln^{1/4} N + k + N^{-1}\max|\psi'|\right)^2$$

and

$$|||u - u^N|||_{\varepsilon} \le C\left(\varepsilon^{1/4}h + k + N^{-1}\max|\psi'|\right),$$

where h and k are the maximum step sizes in x- and y-direction, resp.

Proof. See [38, 40]. □

Remark 9.30. The amount of stabilisation inside the exponential layer Ω_{12} and in the corner Ω_{22} is negligible and can be switched off. On the coarse-mesh region Ω_{11} the stabilisation is standard. However, inside the characteristic layer, i.e. on Ω_{21} the negative power of ε in the upper bound for δ_{21} is surprising.

This is essentially due to the aforementioned weakness of the SD-norm and the ε-weighted energy norm, which fail to capture the parabolic layer. For the term w_2 in the decomposition of Theorem 7.20 one has $|||w_2|||_{\varepsilon} = |||w_2|||_{SD} = \mathcal{O}\left(\varepsilon^{1/4}\right)$.

An alternative, though heuristic approach, in [94] using residual free bubbles suggests the choice $\delta_{22} = \mathcal{O}\left(N^{-2}\right)$. ♣

Local projection stabilisation

LPFEM for problems with characteristic layers is studied by Franz and Matthies [41]. The results are similar to those for streamline-diffusion stabilisation.

9.3 Finite Volume Methods

In this section we consider an inverse-monotone finite volume discretisation for problem (7.8). This scheme was introduced by Baba and Tabata [17] and later generalised by Angermann [13, 14]. For a detailed derivation of the method, the reader is referred to [13, 62].

When working on arbitrary partitions we follow Angermann [13]. Further stability results for tensor-product meshes and uniform convergence of the method in both a discrete energy norm and in the maximum norm are due to other authors. References will be given when appropriate.

For the moment let $\Omega \subset I\!R^2$ be an arbitrary domain with polygonal boundary. Consider the problem

$$-\varepsilon\Delta u - \boldsymbol{b}\nabla u + cu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma = \partial\Omega \tag{9.65}$$

with $0 < \varepsilon \ll 1$ and

$$c + \frac{1}{2}\operatorname{div} \boldsymbol{b} \geq \gamma > 0 \quad \text{on} \quad \Omega. \tag{9.66}$$

Let $\bar{\omega} = \{\boldsymbol{x}_i\} \subset \bar{\Omega}$ be a set of mesh points. Let Λ and $\partial\Lambda$ be the sets of indices of interior and boundary mesh points, i.e.,

$$\Lambda := \{i : \boldsymbol{x}_i \in \Omega\} \quad \text{and} \quad \partial\Lambda := \{i : \boldsymbol{x}_i \in \partial\Omega\}.$$

Set $\bar{\Lambda} := \Lambda \cup \partial\Lambda$. Partition the domain Ω into subdomains

$$\Omega_i := \left\{\boldsymbol{x} \in \Omega : \|\boldsymbol{x} - \boldsymbol{x}_i\| < \|\boldsymbol{x} - \boldsymbol{x}_j\| \text{ for all } j \in \bar{\Lambda} \text{ with } i \neq j\right\} \text{ for } i \in \bar{\Lambda},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. We define $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and we say that two mesh nodes $\boldsymbol{x}_i \neq \boldsymbol{x}_j$ are adjacent if and only if $m_{ij} := \operatorname{meas}_{1D} \Gamma_{ij} \neq 0$. By Λ_i we denote the set of indices of all mesh nodes that are adjacent to $\boldsymbol{x}_i$. Furthermore, set $d_{ij} := \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$ and $m_i = \operatorname{meas}_{2D} \Omega_i$. We denote by n_{ij} the outward normal on the boundary part Γ_{ij} of Ω_i. Let h, the mesh size, be the maximal distance between two adjacent mesh nodes. Set $N_{ij} := -n_{ij} \cdot \boldsymbol{b}\,((\boldsymbol{x}_i + \boldsymbol{x}_j)/2)$; see Fig. 9.6.

For a reasonable discretisation of the boundary conditions, we shall assume that $\Gamma \subset \bigcup_{i \in \partial\Lambda} \bar{\Omega}_i$.

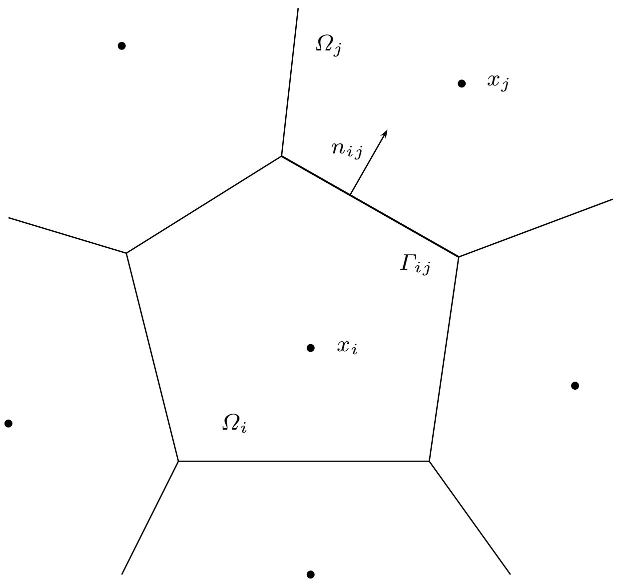


Fig. 9.6 Mesh cell of the FVM

Then our discretisation of (9.65) is as follows. Find $u^N : \bar{\Lambda} \to \mathbb{R}$ such that

$$[L_\rho u^N]_i = f_i m_i \quad \text{for } i \in \Lambda, \quad u_i^N = 0 \quad \text{for } i \in \partial\Lambda, \tag{9.67a}$$

with u_i^N being the numerical approximation of $u(\boldsymbol{x}_i)$,

$$[L_\rho v]_i := \sum_{j \in \Lambda_i} m_{ij} \left(\frac{\varepsilon}{d_{ij}} - N_{ij}\rho_{ij} \right) (v_i - v_j) + c_i m_i v_i, \tag{9.67b}$$

$\rho_{ij} = \rho(N_{ij} d_{ij}/\varepsilon)$ and a function $\rho : \mathbb{R} \to [0,1]$. Possible choices for ρ are given in Sect. 5.4 which studies the one-dimensional version of the FVM. Again we shall assume that ρ satisfies

$$
\begin{aligned}
&(\rho_0) && t \mapsto t\rho(t) \text{ is Lipschitz continuous,}\\
&(\rho_1) && \big[\rho(t) + \rho(-t) - 1\big]t = 0 \quad \text{for all } t \in \mathbb{R},\\
&(\rho_2) && \big[1/2 - \rho(t)\big]t \geq 0 \quad \text{for all } t \in \mathbb{R},\\
&(\rho_3) && 1 - t\rho(t) \geq 0 \quad \text{for all } t \in \mathbb{R}.
\end{aligned}
$$

Note that the constant choice $\rho \equiv \frac{1}{2}$, which generates a generalised central difference scheme, satisfies conditions (ρ_1) and (ρ_2), but not (ρ_3). Conditions (ρ_1) and (ρ_2) guarantee the coercivity of the weak formulation associated with (9.67), while (ρ_3) ensures the inverse monotonicity of the scheme when the coefficient c is strictly positive.

9.3.1 Coercivity of the Method

The FVM can be written in variational form: Find

$$u^N \in V_0^\omega := \left\{ v \in \mathbb{R}^{\operatorname{card}\bar{\Lambda}} : v_k = 0 \text{ for } k \in \partial\Lambda \right\}$$

such that

$$a_\rho(u^N, v) = f_\rho(v) \quad \text{for all } v \in V_0^\omega,$$

with

$$a_\rho(w,v) := \sum_{i \in \Lambda} [L_\rho w]_i v_i \quad \text{and} \quad f_\rho(v) := \sum_{i \in \bar{\Lambda}} f_i m_i v_i.$$

When studying the coercivity of the scheme we split the bilinear form into three parts representing the diffusion, convection and reaction terms:

$$a_\rho(w,v) = \varepsilon d_\rho(w,v) + c_\rho(w,v) + r_\rho(w,v)$$

with

$$d_\rho(w,v) = \sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} \frac{m_{ij}}{d_{ij}}(w_i - w_j)v_i,$$
$$c_\rho(w,v) = -\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\rho_{ij}\,(w_i - w_j)\,v_i$$

and

$$r_\rho(w,v) = \sum_{i\in\bar{\Lambda}} c_i m_i w_i v_i.$$

These three terms will be studied separately.

Changing the order of summation and renaming the indices yields

$$\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} \frac{m_{ij}}{d_{ij}}(v_i - v_j)v_i = -\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} \frac{m_{ij}}{d_{ij}}(v_i - v_j)v_j$$

Therefore,

$$d_\rho(v,v) = \frac{1}{2}\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} \frac{m_{ij}}{d_{ij}}(v_i - v_j)^2 =: |v|^2_{1,\omega} \tag{9.68}$$

which is a positive definite term.

Remark 9.31. Given a mesh function $v \in V_0^\omega$ define a function $\tilde{v} \in H_0^1(\Omega)$ that coincides with v in the mesh points, and that is piecewise linear on a Delaunay triangulation associated with the set of mesh points $\bar{\omega}$. Then

$$|v|^2_{1,\omega} = (\nabla\tilde{v}, \nabla\tilde{v}) = |\tilde{v}|_1^2;$$

see [13]. ♣

Next consider the convection term. By definition we have $m_{ij} = m_{ji}$, $d_{ij} = d_{ji}$ and $N_{ij} = -N_{ji}$. Furthermore, (ρ_1) implies $N_{ji}\rho_{ji} = N_{ij}(\rho_{ij} - 1)$. Hence,

$$\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\rho_{ij}(v_i - v_j)v_i = -\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\,(\rho_{ij} - 1)\,(v_i - v_j)v_j$$

and

$$c_\rho(v,v) = \frac{1}{2}\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\left(\frac{1}{2}-\rho_{ij}\right)(v_i-v_j)^2 - \frac{1}{4}\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\left(v_i^2-v_j^2\right).$$

Introducing

$$|v|^2_{\rho,\omega} := \frac{1}{2}\sum_{i\in\bar{\Lambda}}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\left(\frac{1}{2}-\rho_{ij}\right)\left(v_i-v_j\right)^2,$$

which is a well-defined semi-norm when (ρ_2) is satisfied, we have

$$c_\rho(v,v) = |v|^2_{\rho,\omega} - \frac{1}{2}\sum_{i\in\bar{\Lambda}} v_i^2 \sum_{j\in\Lambda_i} m_{ij}N_{ij}.$$

This and (9.68) yield

$$a_\rho(v,v) = \varepsilon|v|^2_{1,\omega} + |v|^2_{\rho,\omega} + \sum_{i\in\bar{\Lambda}} m_i v_i^2\left(c_i - \frac{1}{2}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\right).$$

Note that

$$m_i \operatorname{div} \boldsymbol{b}_i + \sum_{j\in\Lambda_i} m_{ij}N_{ij} = \mathcal{O}(h).$$

This implies

$$a_\rho(v,v) \ge \varepsilon|v|^2_{1,\omega} + |v|^2_{\rho,\omega} + \frac{\gamma}{2}\|v\|^2_{0,\omega} =: |||v|||^2_\rho \quad \text{with} \quad \|v\|^2_{0,\omega} := \sum_{i\in\bar{\Lambda}} m_i v_i^2,$$

provided h is sufficiently small, independent of the perturbation parameter ε.

We summarise the result of our stability analysis.

Theorem 9.32. *Assume the discretisation* (9.67) *satisfies conditions* (ρ_1) *and* (ρ_2). *Suppose* (9.66) *holds true. Then the bilinear form* $a_\rho(\cdot,\cdot)$ *is coercive with respect to the norm* $|||\cdot|||_\rho$*, i.e.,*

$$a_\rho(v,v) \ge |||v|||^2_\rho \quad \textit{for all} \;\; v \in V_0^\omega$$

if h *is sufficiently small, independent of the perturbation parameter* ε.

Remark 9.33. When $\rho \equiv \frac{1}{2}$, i.e. when the stabilisation is switched off, the bilinear form is coercive with respect to the discrete ε-weighted energy norm

$$|||v|||_{\varepsilon,\omega}^2 := \varepsilon|v|_{1,\omega}^2 + \frac{\gamma}{2}\|v\|_{0,\omega}^2.$$

However, when $\rho \not\equiv \frac{1}{2}$ then we have coercivity of the scheme in a stronger norm, which results in enhanced stability properties of the FVM. ♣

9.3.2 Inverse Monotonicity

Let the function ρ, which describes the FVM method, satisfy (ρ_1) and (ρ_3). Furthermore, assume that $c > 0$ on $\bar{\Omega}$. Then recalling the definition (9.67), we have

$$\frac{\varepsilon}{d_{ij}} - N_{ij}\rho_{ij} \geq 0.$$

Hence, the diagonal entries of the matrix associated with L_ρ are positive while the off-diagonal ones are non-positive. Thus, the system matrix is an L_0 matrix. Next note that for $v \equiv 1$ we have $[L_\rho v]_i = c_i m_i > 0$. Therefore, application of the M-criterion (Lemma 3.14) verifies the inverse monotonicity of L_ρ, while Lemma 3.17 gives the $(\ell_\infty, \ell_\infty)$-stability inequality

$$\|v\|_{\infty,\omega} \leq \|f/c\|_{\infty,\omega}.$$

Since L_ρ is inverse monotone, it enjoys a comparison principle. That is if two mesh functions v and w satisfy

$$\left.\begin{array}{ll} [L_\rho v]_i \leq [L_\rho w]_i & \text{for all } i \in \Lambda, \\ v_i \leq w_i & \text{for all } i \in \partial\Lambda \end{array}\right\} \implies v_i \leq w_i \quad \text{for all } i \in \bar{\Lambda}.$$

Remark 9.34. These results hold true with no restrictions imposed on the convective field $\boldsymbol{b}$ and with (ρ_2) possibly violated. ♣

The Green's function on a tensor-product mesh

Using the inverse monotonicity of L_ρ, we now study the Green's functions associated with L_ρ and derive an anisotropic stability inequality on a general tensor-product mesh $\bar{\omega} := \bar{\omega}_x \times \bar{\omega}_y$, with N mesh intervals in each coordinate direction. A stability result of this kind was first established by Andreev for a simple upwind difference scheme; see [7].

Setting $\hbar_i := (h_{i+1} + h_i)/2$,

$$\rho^+_{1;ij} := \rho\left(-\frac{b_{1;i+1/2,j}h_{i+1}}{\varepsilon}\right)b_{1;i+1/2,j}, \quad \rho^-_{1;ij} := \rho\left(\frac{b_{1;i-1/2,j}h_i}{\varepsilon}\right)b_{1;i-1/2,j},$$

$$v_{\bar{x};ij} := \frac{v_{ij} - v_{i-1,j}}{h_i}, \quad v_{\hat{x};ij} = \frac{v_{i+1,j} - v_{ij}}{\hbar_i} \quad \text{and} \quad v_{\check{x};ij} = \frac{v_{i,j} - v_{i-1,j}}{\hbar_i}$$

with analogous definitions for ρ^+_2, ρ^-_2, $v_{\bar{y}}$, $v_{\hat{y}}$, $v_{\check{y}}$ and $\hbar$, we can rewrite (9.67) as: Find $u^N \in \left(I\!R_0^{N+1}\right)^2$ such that

$$\begin{aligned}\left[L_\rho u^N\right]_{ij} := -\varepsilon\left(u^N_{\bar{x}\hat{x};ij} + u^N_{\bar{y}\hat{y};ij}\right) - \rho^+_{1,ij}u^N_{\hat{x},ij} - \rho^-_{1,ij}u^N_{\check{x},ij} \\ - \rho^+_{2,ij}u^N_{\hat{y},ij} - \rho^-_{2,ij}u^N_{\check{y},ij} + c_{ij}u^N_{ij} = f_{ij}\end{aligned}$$

for $i, j = 1, \ldots, N-1$.

Any mesh function v that vanishes on the boundary can be represented using the Green's function:

$$v_{ij} = (v, G_{ij,\cdot\cdot})_\rho := \sum_{k,l=1}^{N-1} \hbar_k \hbar_l G_{ij,kl}\,[Lv]_{kl}\,, \tag{9.69}$$

where $G_{ij,kl} = G(x_i, y_j, \xi_k, \eta_l)$ solves for fixed k and l

$$[L_\rho G_{\cdot\cdot,kl}]_{ij} = \delta_{x;ik}\,\delta_{y;jl} \quad \text{on } \omega, \quad G_{ij,kl} = 0 \quad \text{on } \partial\omega$$

with

$$\delta_{x;ik} = \begin{cases} \hbar_i^{-1} & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \delta_{y;jl} = \begin{cases} \hbar_j^{-1} & \text{if } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

The adjoint operator to L_ρ is

$$\begin{aligned}\left[L^*_\rho v\right]_{kl} = -\varepsilon\left(v_{\bar{\xi}\hat{\xi};kl} + v_{\bar{\eta}\hat{\eta};kl}\right) + \left(\rho^+_1 v\right)_{\check{\xi};kl} + \left(\rho^-_1 v\right)_{\hat{\xi};kl} \\ + \left(\rho^+_2 v\right)_{\check{\eta};kl} + \left(\rho^-_2 v\right)_{\hat{\eta};kl} + c_{kl}v_{kl}\end{aligned}$$

and the Green's function solves, for fixed i and j,

$$[L^* G_{ij,\cdot\cdot}]_{kl} = \delta_{x;ik}\,\delta_{y;jl} \quad \text{on } \omega, \quad G_{ij,kl} = 0 \quad \text{on } \partial\omega. \tag{9.70}$$

In our subsequent analysis the following mean value theorem is used.

Lemma 9.35. *Let* $\varphi, g \in I\!R^{N+1}$ *with* $g_j \geq 0$ *and* $m \leq \varphi_j \leq M$ *for* $j = 1, \ldots, N-1$. *Then there exists a constant* $\tilde{\varphi} \in [m, M]$ *with*

$$\sum_{j=1}^{N-1} \hbar_j \varphi_j g_j = \tilde{\varphi} \sum_{j=1}^{N-1} \hbar_j g_j .$$

Let i and j be fixed. First, the inverse monotonicity of L yields $G_{ij,kl} \geq 0$. Next, multiplying (9.70) by $\hbar_l$ and summing for $l = 1, \ldots, N-1$, we obtain the one-dimensional equation

$$- \varepsilon \left(\sum_{l=1}^{N-1} \hbar_l G_{ij,\cdot l} \right)_{\bar{\xi}\check{\xi},k} + \left(\sum_{l=1}^{N-1} \hbar_l \rho^+_{1,\cdot l} G_{ij,\cdot l} \right)_{\check{\xi},k}$$

$$+ \left(\sum_{l=1}^{N-1} \hbar_l \rho^-_{1,\cdot l} G_{ij,\cdot l} \right)_{\hat{\xi},k} + \sum_{l=1}^{N-1} \hbar_l c_{kl} G_{ij,kl} = \delta_{x;ik} - F_k \ ,$$

where

$$F_k = -\varepsilon \left[1 + \frac{\rho^+_{2;k,N-1} h_N}{\varepsilon} \right] G_{\bar{\eta};ij,kN} + \varepsilon \left[1 - \frac{\rho^+_{2;k,0} h_1}{\varepsilon} \right] G_{\bar{\eta};ij,k1} \geq 0,$$

by (ρ_3) and since $G \geq 0$.

Defining

$$\tilde{G}_k := \sum_{l=1}^{N-1} \hbar_l G_{ij,kl} = \| G_{ij,\cdot l} \|_{1,\omega} \, , \quad \text{for } k = 0, \ldots, N,$$

we see that according to Lemma 9.35 there exist mesh functions $\tilde{\rho}^+, \tilde{\rho}^-, \tilde{c}$ with $\tilde{\rho}^+ \geq \beta_1$, $\tilde{\rho}^- \geq \beta_1$ and $\tilde{c} \geq \gamma$ such that

$$-\varepsilon \tilde{G}_{\bar{\xi}\check{\xi},k} + \left(\tilde{\rho}^+ \tilde{G} \right)_{\check{\xi},k} + \left(\tilde{\rho}^- \tilde{G} \right)_{\hat{\xi},k} + \tilde{c}_k \tilde{G}_k = \delta_{x;ik} - F_k .$$

Let $\Gamma = \Gamma_{m,k}$ be the Green's function of the operator

$$[Lv]_k = -\varepsilon v_{\bar{\xi}\check{\xi},k} - \tilde{\rho}^+_k v_{\check{\xi},k} - \tilde{\rho}^-_k v_{\hat{\xi},k} + \tilde{c}_k v_k .$$

Then $\tilde{G}$ can be written as

$$\tilde{G}_k = \Gamma_{i,k} - \sum_{m=1}^{N-1} \hbar_m \Gamma_{m,k} F_m \ ,$$

The nonnegativity of Γ and F gives

$$\tilde{G}_k \leq \Gamma_{i,k} \leq \frac{1}{\beta_1 \inf_{t<0} \rho(t)},$$

by Lemma 5.20 and Remark 5.23. We get the first inequality of the following theorem. The other one is proved analogously.

Theorem 9.36. *Suppose the control function ρ enjoys properties (ρ_0) and (ρ_3). Then the Green's function associated with L satisfies*

$$\max_{i,j,k=1,\dots,N-1} \sum_{l=1}^{N-1} \hbar_l G_{ij,kl} \leq \frac{\alpha}{\beta_1} \quad \textit{and} \quad \max_{i,j,l=1,\dots,N-1} \sum_{k=1}^{N-1} \hbar_k G_{ij,kl} \leq \frac{\alpha}{\beta_2}$$

with $\alpha = 1/\inf_{t<0} \rho(t) \leq 2$.

Finally, we use these bounds on the Green's function to derive stability estimates for the operator L_ρ. For any mesh function $v : \bar{\omega} \to I\!R$ that vanish on $\partial\omega$, introduce the norm

$$\|v\|_A := \sum_{k=1}^{N-1} \hbar_k \max_{l=1,\dots,N-1} |v_{kl}| \, .$$

Its dual norm with respect to the discrete scalar product $(\cdot,\cdot)_\rho$ is

$$\|v\|_{A^*} = \max_{k=1,\dots,N-1} \sum_{l=1}^{N-1} \hbar_k |v_{kl}| \, ,$$

cf. [21, Theorem 2]. The representation (9.69) gives

$$|v_{ij}| \leq \|G_{ij,\cdot\cdot}\|_{A^*} \|v\|_A \, .$$

Application of Theorem 9.36 yields our final stability result which is an extension of the (ℓ_∞, ℓ_1) stability of Sect. 4.2.5 and a generalisation of [7].

Theorem 9.37. *Suppose the control function ρ enjoys properties (ρ_0) and (ρ_3). Then the operator L_ρ is $(\ell_\infty, \ell_1 \otimes \ell_\infty)$ stable with*

$$\|v\|_{\infty,\omega} \leq \frac{\alpha}{\beta_1} \|L_\rho v\|_{\ell_1 \otimes \ell_\infty} \quad \textit{and} \quad \|v\|_{\infty,\omega} \leq \frac{\alpha}{\beta_2} \|L_\rho v\|_{\ell_\infty \otimes \ell_1}$$

with $\alpha = 1/\inf_{t<0} \rho(t) \leq 2$,

$$\|w\|_{\ell_1 \otimes \ell_\infty} := \sum_{k=1}^{N-1} \hbar_k \max_{l=1,\dots,N-1} |w_{kl}|$$

and

$$\|w\|_{\ell_\infty \otimes \ell_1} := \sum_{l=1}^{N-1} \hbar_l \max_{k=1,\dots,N-1} |w_{kl}|.$$

9.3.3 Convergence

Energy norm

Starting from the coercivity of the bilinear form $a_\rho(\cdot,\cdot)$, see Theorem 9.32, the analysis proceeds along the lines of Sect. 5.4.2 resembling many of the details also used for the Galerkin FEM in two dimensions, see Sect. 9.2.2. Eventually one gets

$$|||u - u^N|||_\rho + |||u - u^N|||_\varepsilon \le CN^{-1} \max |\psi'| \ln^{1/2} N$$

for tensor-product meshes of Shishkin-type with $\sigma \ge 2$; see also [174].

Maximum norm

The pointwise errors can be bounded using the hybrid stability inequalities from Theorem 9.37, see [90]. We give a very brief outline of the argument.

The truncation error is split according to the decomposition of Theorem 7.17. Then either of the two bounds from Theorem 9.37 is applied. Section 4.2.5 gives a flavour of the technical details. For a S-type mesh with $\sigma \ge 2$ we obtain

$$\|u - u^N\|_{\infty,\omega} \le CN^{-1} \max |\psi'|.$$

If ρ is Lipschitz continuous in $(-m, m)$ with $m > 0$, then there exists an $N_m > 0$ independent of the perturbation parameter ε such that on a standard Shishkin mesh with $\sigma \ge 2$

$$\|u - u^N\|_{\infty,\omega} \le CN^{-1} \quad \text{for} \ \ N \ge N_m.$$

In the latter case the stabilisation is reduced when the local mesh size is small enough, thus giving higher accuracy inside the layers.

9.3.3.1 Numerical Tests

We verify our theoretical results for the upwind FEM on Shishkin meshes when applied to the test problem (9.3). For our tests we take $\varepsilon = 10^{-8}$, which is a sufficiently small choice to bring out the singularly perturbed nature of the problem.

We test the method for three different choices of the controlling function ρ. The errors are measured in the discrete energy and maximum norm and in the FVM-norm.

For $\rho_{U,0}$ (see Table 9.7) we observe convergence of almost first order, namely $N^{-1}\ln N$, in all three norms, while for both $\rho_{U,2}$ and ρ_I—which are Lipschitz continuous—the errors behave like $\mathcal{O}(N^{-1})$; see Tables 9.8 and 9.9. Note, this is covered by our analysis for the maximum norm only.

Table 9.7 FVM on Shishkin meshes, $\rho = \rho_{U,0}$

	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\rho$		$\lVert u-u^N\rVert_{\varepsilon,\omega}$		$\lVert u-u^N\rVert_{\infty,\omega}$	
N	error	rate	error	rate	error	rate
16	2.7575e-1	0.68	2.0623e-1	0.55	1.8112e-1	0.62
32	1.7198e-1	0.75	1.4052e-1	0.66	1.1770e-1	0.71
64	1.0230e-1	0.79	8.9046e-2	0.73	7.1880e-2	0.76
128	5.8999e-2	0.83	5.3575e-2	0.79	4.2537e-2	0.80
256	3.3292e-2	0.85	3.1081e-2	0.82	2.4483e-2	0.83
512	1.8493e-2	0.87	1.7579e-2	0.85	1.3786e-2	0.85
1024	1.0153e-2	0.88	9.7672e-3	0.87	7.6456e-3	0.87
2048	5.5247e-3	—	5.3576e-3	—	4.1908e-3	—

Table 9.8 FVM on Shishkin meshes, $\rho = \rho_I$

	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\rho$		$\lVert u-u^N\rVert_{\varepsilon,\omega}$		$\lVert u-u^N\rVert_{\infty,\omega}$	
N	error	rate	error	rate	error	rate
16	1.5894e-1	0.83	8.9598e-2	0.80	7.5370e-2	0.70
32	8.9627e-2	0.92	5.1417e-2	0.90	4.6297e-2	0.84
64	4.7445e-2	0.96	2.7514e-2	0.95	2.5790e-2	0.92
128	2.4388e-2	0.98	1.4222e-2	0.98	1.3610e-2	0.96
256	1.2360e-2	0.99	7.2279e-3	0.99	6.9899e-3	0.98
512	6.2219e-3	1.00	3.6430e-3	0.99	3.5418e-3	0.99
1024	3.1214e-3	1.00	1.8288e-3	1.00	1.7827e-3	1.00
2048	1.5633e-3	—	9.1618e-4	—	8.9431e-4	—

Table 9.9 FVM on Shishkin meshes, $\rho = \rho_{U,2}$

	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\rho$		$\lVert u-u^N\rVert_{\varepsilon,\omega}$		$\lVert u-u^N\rVert_{\infty,\omega}$	
N	error	rate	error	rate	error	rate
16	1.5359e-1	0.81	8.2430e-2	0.77	7.6384e-2	0.72
32	8.7574e-2	0.91	4.8263e-2	0.88	4.6337e-2	0.85
64	4.6686e-2	0.95	2.6272e-2	0.93	2.5790e-2	0.92
128	2.4120e-2	0.98	1.3773e-2	0.96	1.3610e-2	0.96
256	1.2270e-2	0.99	7.0752e-3	0.98	6.9899e-3	0.98
512	6.1928e-3	0.99	3.5935e-3	0.99	3.5418e-3	0.99
1024	3.1122e-3	1.00	1.8132e-3	0.99	1.7827e-3	1.00
2048	1.5605e-3	—	9.1144e-4	—	8.9431e-4	—

Conclusions and Outlook

This monograph has attempted to provide a general framework for the convergence analysis of a variety of numerical methods using layer-adapted meshes for the solution of certain classes of singularly perturbed problems. While for some problems satisfactory answers have been presented, there is still a large number of open issues. We would like to summarise the results here and to point out some directions for future research.

Difference schemes in one dimension have been covered successfully to a large extent, although there are a few minor open questions. For example, the analysis for the fourth-order scheme for reaction-diffusion problems in Sect. 6.1.4 and the analysis for turning-point problems are restricted to Shishkin meshes so far. Similarly, for the two-parameter problem in Sect. 6.3, no analysis for arbitrary meshes and a second-order scheme is available.

For finite element methods—in both one and two dimensions—the situation is different. Here results on arbitrary meshes are restricted to the interpolation error. Convergence and superconvergence estimates are known for special meshes, namely Shishkin-type meshes, only. A general framework for this very important class of methods is still missing.

For systems of strongly coupled convection-diffusion problems, we only have a limited grasp of the situation. Even for one-dimensional problems there are still basic difficulties: when different diffusion parameters are present, can sharp pointwise bounds on derivatives be proved?

For two-dimensional convection-diffusion problems, further work on stability bounds is needed to improve our understanding of these problems. In particular, sharp estimates for the Green's functions and negative-norm stability inequalities are required. In one dimensions these turned out to be the key ingredient for the convergence theory for arbitrary meshes, for the a posteriori error analysis and for dealing with strongly coupled convection-diffusion equations.

Two parameter problems of reaction-convection-diffusion type in two dimensions were considered in a small number of publications, but the presentations are typically very technical, mainly because of complicated solution decompositions involving a large number of different terms. Also the techniques used are very similar to that known from convection-diffusion problems with a single parameter. Further research is required to derive a general comprehensive theory.

Time-dependent problems have not been included in this book because only a few results are available that are insufficient for the development of a general theory. The vast majority of results use first-order backward Euler for discretisation in time. Higher-order time discretisations by A-stable Runge-Kutta methods have been considered in the literature, but their analysis is incomplete because off a lack of resolvent estimates for non-uniform meshes.

This account of open issues in the field is naturally incomplete. Studying some of the material presented in the book, the reader will certainly discover further mathematical problems worthy of investigating.

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Edited by J.-M. Morel, F. Takens, B. Teissier, P.K. Maini

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