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Global Propagation of Regular Nonlinear Hyperbolic Waves

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Preface

This book studies the global propagation of the regular nonlinear hyperbolic wave described by first-order quasilinear hyperbolic systems in the one-space-dimensioned case. Via the concept of weak linear degeneracy and the method of (generalized) normalized coordinates, a systematic theory is established on the global existence and the blow-up mechanism of the regular nonlinear hyperbolic wave with small amplitude not only for the Cauchy problem, but also for some other important problems such as the Cauchy problem on a semibounded initial data, the one-sided mixed initial-boundary value problem, the generalized Riemann problem, and the generalized nonlinear initial-boundary Riemann problem, etc, as well as not only for the direct problem, but also for inverse problems such as the inverse generalized Riemann problem and the inverse piston problem. Most of the material contained in this book is based on the results the authors obtained in recent years. Some material that was previously published has been revised and updated.

The whole approach in this book is based on the theory of the local regular solution and of the local piecewise regular solution for quasilinear hyperbolic systems. For more comprehensive information, the reader may refer to the book by Li Tatsien and Yu Wenci, *Boundary Value Problems for Quasilinear Hyperbolic Systems* (Duke University Mathematics Series V, 1985).

The first author would like to take this opportunity to give his warm thanks to Professor Gu Chaohao for having initiated and brought him into the fruitful area of quasilinear hyperbolic systems. The authors are very grateful to all the members on the Applied PDEs Seminar of Fudan University, organized by Qin Tiehu, Zhou Yi, and the first author, for their constant interest, discussion, and suggestion on the subject.

We would like to express our deep appreciation to the National Basic Research Program of China (973 Program)(2007CB814800) and, for the second author, also to the National Natural Science Foundation of China (10771038) for their support.

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Contents

Chapter I
Introduction

1.1 Cauchy Problem

In this monograph we shall consider the nonlinear hyperbolic waves described by the following first-order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth entries $a_{ij}(u)$ $(i, j = 1, \ldots, n)$.

By the definition of *hyperbolicity*, for any given u on the domain under consideration, the matrix $A(u)$ possesses n real eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$ and a complete set of left (resp. right) eigenvectors $l_1(u), \ldots, l_n(u)$ [resp. $r_1(u), \ldots, r_n(u)$]: For $i = 1, \ldots, n$,

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad [\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)]. \tag{1.1.2}$$

Without loss of generality, we assume that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i = 1, \ldots, n), \tag{1.1.3}$$

where δ_{ij} stands for Kronecker's delta.

In particular, if the matrix $A(u)$ possesses n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{1.1.4}$$

system (1.1.1) is said to be *strictly hyperbolic*.

We first consider the Cauchy problem for system (1.1.1) with the initial condition

$$t = 0: \ u = \phi(x), \tag{1.1.5}$$

where $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))^T$ is a C^1 vector function of x with bounded C^1 norm.

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If the matrix A is independent of u, we meet linear hyperbolic waves given by

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \tag{1.1.6}$$

with (1.1.5). The acoustic wave is a typical example of linear hyperbolic waves. In the scalar case, we have, for instance, the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0, \\ t = 0: \ u = \phi(x), \end{cases} \tag{1.1.7}$$

where $\phi(x)$ is a C^1 function of x with bounded C^1 norm. The wave speed is constant: $dx/dt = 1$ and the wave always keeps its shape in the course of propagation. In the general case, there are n linear waves given by (1.1.6) and (1.1.5) with constant speeds

$$\frac{dx}{dt} = \lambda_i \quad (i = 1, \dots, n), \tag{1.1.8}$$

respectively. Each wave keeps its shape in the propagation, and the interaction among waves is only a linear superposition. It is the reason that we can hear and distinguish many persons speaking at the same time. Otherwise, our life would be very complicated.

The situation for nonlinear hyperbolic waves is totally different. In the scalar case, let us consider, for instance, the Cauchy problem for Burger's equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0, \\ t = 0: \ u = \phi(x). \end{cases} \tag{1.1.9}$$

The wave speed depends on $u : dx/dt = u$ and then the wave cannot keep its shape in the course of propagation. Generically speaking, there will be a distortion of wave shape, such that the wave steepens and finally blows up in a finite amount of time. In the general case, there are n nonlinear hyperbolic waves given by (1.1.1) and (1.1.5) with speeds

$$\frac{dx}{dt} = \lambda_i(u) \quad (i = 1, \dots, n), \tag{1.1.10}$$

respectively, and there are nonlinear interactions among these waves, so that the situation is much more complicated.

As a conclusion, the Cauchy problem (1.1.1) and (1.1.5) always admits a unique C^1 solution $u = u(t, x)$ at least for a short time $0 \le t \le \delta$ (cf. [72] and the references therein); however, generically speaking, the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1.1) and (1.1.5) exists only locally in time and the singularity may occur in a finite time, i.e., there exists $t_0 > 0$ such that

$$\|u(t,.)\|_0 + \|u_x(t,.)\|_0 \quad \text{becomes unbounded as} \quad t \uparrow t_0, \tag{1.1.11}$$

no matter how smooth and how small the initial data are, where $||\cdot||_0$ stands for the C^0 norm (cf. [50] and the references therein).

Therefore, it is of great importance in both theory and applications to study the following two problems:

1. Under what conditions does the problem under consideration (the Cauchy problem, the mixed initial-boundary value problem, etc.) admit a unique global C^1 solution $u = u(t,x)$ in time?

2. Under what conditions does the C^1 solution to the problem under consideration blow up in a finite time? What is the sharp estimate on the life span of the C^1 solution, i.e., on the maximum length of existence of the t-interval? What is the mechanism of the formation of singularities and what is the character of singularity?

When $n = 1$ or 2, the answer to these two problems is relatively simple (cf. [50] and the references therein).

For the general hyperbolic system (1.1.1) of n equations, the first result in this direction was given by John [38]. Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, system (1.1.1) is strictly hyperbolic and *genuinely nonlinear* (GN) in the sense of Lax: For $i = 1, \ldots, n$,

$$\nabla \lambda_i(u) r_i(u) \neq 0. \tag{1.1.12}$$

Suppose furthermore that $\phi(x) \in C^2$ has a compact support:

$$\text{Supp } \phi \subseteqq [\alpha_0, \beta_0]. \tag{1.1.13}$$

John proved that if

$$\theta \overset{\text{def.}}{=} (\beta_0 - \alpha_0)^2 \sup_{x \in \mathbb{R}} |\phi''(x)| \tag{1.1.14}$$

is small enough, then the first-order derivative u_x of the C^2 solution $u = u(t,x)$ to the Cauchy problem (1.1.1) and (1.1.5) must blow up in a finite time.

Liu [77] generalized John's result to the case that in a neighbourhood of $u = 0$, a nonempty part of characteristics is GN, while the other part of characteristics is *linearly degenerate* (LD) in the sense of Lax: For the corresponding indices i,

$$\nabla \lambda_i(u) r_i(u) \equiv 0. \tag{1.1.15}$$

Under the additional hypothesis "linear waves do not generate nonlinear waves," he got the same result as in John [38] for a quite large class of initial data. His result can be applied to the system of one-dimensional gas dynamics with convexity.

Hörmander [32],[33] reproved John's result and, when $\phi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter, he got the following asymptotic behaviour of the life span $\tilde{T}(\varepsilon)$:

$$\lim_{\varepsilon \downarrow 0}\{\varepsilon\tilde{T}(\varepsilon)\} = M_0, \tag{1.1.16}$$

where M_0 is a positive constant independent of ε, defined by

$$M_0 = \Big(\max_{i=1,\dots,n} \sup_{x\in\mathbb{R}}\{-(\nabla\lambda_i(0)r_i(0))l_i(0)\psi'(x)\}\Big)^{-1}. \tag{1.1.17}$$

Thus, there exist two positive constants c and C independent of ε, such that the life span $\tilde{T}(\varepsilon)$ satisfies the following optimal estimate:

$$c\varepsilon^{-1} \le \tilde{T}(\varepsilon) \le C\varepsilon^{-1}, \tag{1.1.18}$$

denoted by

$$\tilde{T}(\varepsilon) \approx \varepsilon^{-1}. \tag{1.1.19}$$

On the other hand, Bressan [9] gave a result on the global existence of the classical solution as follows: Suppose that system (1.1.1) is strictly hyperbolic and LD in the sense of Lax: Equation (1.1.15) holds for $i = 1, \dots, n$. Suppose furthermore that the initial data ϕ have a compact support. If the total variation of ϕ is small enough,

$$\mathrm{TV}\{\phi\} << 1, \tag{1.1.20}$$

then the Cauchy problem (1.1.1) and (1.1.5) admits a unique global classical solution $u = u(t, x)$ for all $t \in \mathbb{R}$.

All the previous results are obtained under the following three hypotheses on system (1.1.1):

1. The system is strictly hyperbolic.
2. a. The system is GN, i.e., all the characteristics are GN; or
   b. A nonempty part of characteristics is GN, while the other part of the characteristics is LD.
   c. The system is LD, i.e., all the characteristics are LD.
3. In case 2b, "linear waves do not generate nonlinear waves."

In order to explain that these three hypotheses restrict the applications, we can see the examples given in Section 1.3.

Actually, many authors have pointed out the necessity of studying the quasilinear hyperbolic system with general characteristics. For instance, Majda proposed in [81] the open problem "investigate shock formation in non-GN systems for initial data of compact support." He also mentioned two especially interesting systems in nonlinear elasticity, namely, examples given in Sections 1.3.5 and 1.3.6.

One aim of this monograph is to establish a complete theory on both the global existence of and blow-up phenomenon of the C^1 solution to the Cauchy problem for general quasilinear hyperbolic system with small initial data with compact support or, more generally, with small and decaying initial data.

Remark 1.1.1 *In this monograph the weak entropy solution to quasilinear hyperbolic systems of conservation laws will be discussed in Chapters 7–10 only for the piecewise smooth solution to the generalized Riemann problem and to the piston problem in gas dynamics. The general theory on the weak entropy solution to the Cauchy problem in one dimension has been extensively studied by many authors, including Bianchini, Bressan, Chen, Glimm, LeFloch, Liu, Yang, etc.* (for instance, see [5], [6], [10]–[14], [16], [17], [27], [43], [44], [78], [79])*.*

Moreover, under some special hypotheses on the quasilinear hyperbolic system, the formation of singularities has also been studied for the Cauchy problem in [4], [8], [21], [23], [35]–[37], [82], [83], [99], [102].

1.2 Weak Linear Degeneracy

In what follows we first consider the strictly hyperbolic case.

In order to present a complete result, it is necessary to introduce a new concept—the weak linear degeneracy (see Chapter 2).

The ith characteristic $\lambda_i(u)$ is said to be **weakly linearly degenerate (WLD)** with respect to $u = u_0$ if, along the ith characteristic trajectory $u = u^{(i)}(s)$ passing through $u = u_0$ in the u-space, defined by

$$\begin{cases} \dfrac{du}{ds} = r_i(u), \\ s = 0: \ u = u_0, \end{cases} \tag{1.2.1}$$

we have

$$\nabla\lambda_i(u) r_i(u) \equiv 0, \quad \forall |u - u_0| \text{ small}, \tag{1.2.2}$$

namely,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(u_0), \quad \forall |s| \text{ small}. \tag{1.2.3}$$

Obviously, if $\lambda_i(u)$ is linearly degenerate (LD) in a neighbourhood of $u = u_0$, then $\lambda_i(u)$ is WLD with respect to $u = u_0$, whereas if $\lambda_i(u)$ is genuinely nonlinear (GN) in a neighbourhood of $u = u_0$, then $\lambda_i(u)$ is not WLD with respect to $u = u_0$.

By definition, if $\lambda_i(u)$ is not WLD, then $\lambda_i(u^{(i)}(s))$ is not identically equal to a constant for small $|s|$; therefore, either there exists an integer $\alpha_i \geq 0$

such that

$$\left.\frac{d^l\lambda_i(u^{(i)}(s))}{ds^l}\right|_{s=0}=0 \quad (l=1,\ldots,\alpha_i), \text{ but } \left.\frac{d^{\alpha_i+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha_i+1}}\right|_{s=0}\neq 0, \tag{1.2.4}$$

or

$$\left.\frac{d^l\lambda_i(u^{(i)}(s))}{ds^l}\right|_{s=0}=0 \quad (l=1,2,\ldots) \tag{1.2.5}$$

but (1.2.3) fails, denoted by $\alpha_i=+\infty$.

Thus, for each characteristic $\lambda_i(u)$, we have the following table:

<table>
<tr><td colspan="3">$\lambda_i(u)$</td></tr>
<tr><td colspan="2">non-WLD</td><td rowspan="2">WLD</td></tr>
<tr><td>$\alpha_i=\underbrace{\overset{(GN)}{0},1,2,\ldots}_{\text{finite}}$</td><td>$\alpha_i=+\infty$</td></tr>
</table>

If $\alpha_i=0$, then in a neighbourhood of $u=u_0$, $\lambda_i(u)$ is GN. Moreover, when α_i increases, $\lambda_i(u)$ is closer and closer to the WLD case.

System (1.1.1) is said to be WLD with respect to $u=u_0$ if all the characteristics $\lambda_1(u),\ldots,\lambda_n(u)$ are WLD with respect to $u=u_0$.

Hence, if system (1.1.1) is not WLD, then there exists a nonempty set of indices $J\subseteqq\{1,\ldots,n\}$ such that $\lambda_i(u)$ is not WLD if and only if $i\in J$.

Let

$$\alpha=\min_i\{\alpha_i \mid i\in J\}, \tag{1.2.6}$$

where α is an integer ≥ 0 or $+\infty$.

Thus, for any given quasilinear strictly hyperbolic system (1.1.1), all possible situations can be shown in the following table:

<table>
<tr><td colspan="2">non-WLD</td><td rowspan="2">WLD</td></tr>
<tr><td>$\alpha=\underbrace{0,1,2,\ldots}_{\text{finite}}$</td><td>$\alpha=+\infty$</td></tr>
</table>

It gives us a complete category.

Let

$$J_1=\{i \mid i\in J, \quad \alpha_i=\alpha\}. \tag{1.2.7}$$

When $\alpha = 0$, then for every $i \in J_1$, $\lambda_i(u)$ is GN in a neighbourhood of $u = u_0$. Furthermore, when α increases, system (1.1.1) is closer and closer to a WLD system.

1.3 Some Examples

In this section we give some examples of systems arising in mechanics, physics, or applications.

1.3.1 System of Nonlinear Elasticity

The *system of nonlinear elasticity* can be written as (see [50])

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial w}{\partial x} = 0, \\ \dfrac{\partial w}{\partial t} - \dfrac{\partial K(v)}{\partial x} = 0, \end{cases} \tag{1.3.1}$$

where $K(v)$ is a suitably smooth function of v such that

$$K'(v) > 0, \quad \forall v, \tag{1.3.2}$$

and, without loss of generality, we may assume that

$$K(0) = 0. \tag{1.3.3}$$

By (1.3.2), (1.3.1) is a strictly hyperbolic system with the following distinct real eigenvalues:

$$\lambda_1(U) = -\sqrt{K'(v)} < \lambda_2(U) = \sqrt{K'(v)}, \tag{1.3.4}$$

and the corresponding right eigenvectors can be taken as

$$r_1(U) = (1, \sqrt{K'(v)})^T, \quad r_2(U) = (1, -\sqrt{K'(v)})^T, \tag{1.3.5}$$

where $U = (v, w)^T$.

It is easy to see that $\lambda_1(U)$ and $\lambda_2(U)$ are GN if and only if

$$K''(v) \neq 0, \quad \forall v, \tag{1.3.6}$$

whereas, $\lambda_1(U)$ and $\lambda_2(U)$ are LD if and only if

$$K''(v) \equiv 0, \quad \forall v. \tag{1.3.7}$$

Now we consider system (1.3.1) in a neighbourhood of $U = U_0 \overset{\text{def.}}{=} (0, b)^T$, where b is an arbitrarily given constant.

If

$$K'(0) > 0, \tag{1.3.8}$$

(1.3.1) is a strictly hyperbolic system with the distinct real eigenvalues (1.3.4) in a neighbourhood of $U = U_0$. Suppose furthermore that

$$K''(0) \neq 0; \tag{1.3.9}$$

then (1.3.1) is GN in a neighbourhood of $U = U_0$. On the other hand, it is easy to see that $\lambda_1(U)$ and $\lambda_2(U)$ are WLD with respect to $U = U_0$ if and only if they are LD for small $|v|$, namely,

$$K''(v) \equiv 0, \quad \forall |v| \text{ small.} \tag{1.3.10}$$

Moreover, if

$$K''(0) = 0 \quad \text{but} \quad K''(v) \not\equiv 0, \quad \forall |v| \text{ small,} \tag{1.3.11}$$

then in a neighbourhood of $U = U_0$, (1.3.1) is neither GN nor LD (WLD). More precisely, if there exists an integer $\alpha \geq 0$ such that

$$K''(0) = K'''(0) = \cdots = K^{(\alpha+1)}(0) = 0, \text{ but } K^{(\alpha+2)}(0) \neq 0, \tag{1.3.12}$$

then in a neighbourhood of $U = U_0$, (1.3.1) is a non-WLD system with the index α. Obviously, when $\alpha = 0$, system (1.3.1) is GN in a neighbourhood of $U = U_0$.

1.3.2 System of Traffic Flow

The *traffic flow* can be described by the following system (see [3]):

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, \end{cases} \tag{1.3.13}$$

where $\rho > 0$ and v are the density and the velocity of cars at point x and time t, respectively, and $p(\rho)$ is a smooth increasing function of ρ.

Let

$$U = (\rho, v)^T. \tag{1.3.14}$$

It is easy to see that when $\rho > 0$, (1.3.13) is a strictly hyperbolic system with the following distinct real eigenvalues:

$$\lambda_1(U) = v - \rho p'(\rho) < \lambda_2(U) = v, \tag{1.3.15}$$

and the corresponding right eigenvectors can be taken as

$$r_1(U) = (1, -p'(\rho))^T, \quad r_2(U) = (1, 0)^T. \tag{1.3.16}$$

Obviously, $\lambda_2(U)$ is always LD.

Moreover, it is easy to see that $\lambda_1(U)$ is GN if and only if

$$\rho p''(\rho) + 2p'(\rho) \neq 0, \quad \forall \rho > 0. \tag{1.3.17}$$

In particular, under the assumption

$$p(\rho) = \rho^\gamma \quad (\gamma > 0 \text{ is a constant}), \tag{1.3.18}$$

$\lambda_1(U)$ is GN.

On the other hand, $\lambda_1(U)$ is LD if and only if

$$\rho p''(\rho) + 2p'(\rho) \equiv 0, \quad \forall \rho > 0, \tag{1.3.19}$$

namely,

$$p(\rho) = A - \frac{B}{\rho}, \quad \forall \rho > 0, \tag{1.3.20}$$

where A and $B > 0$ are real constants.

Now, we consider system (1.3.13) in a neighbourhood of $U = U_0 \stackrel{\text{def.}}{=} (\rho_0, v_0)^T$ with $\rho_0 > 0$.

$\lambda_1(U)$ is GN in a neighbourhood of $U = U_0$ if and only if

$$\rho_0 p''(\rho_0) + 2p'(\rho_0) \neq 0. \tag{1.3.21}$$

On the other hand, $\lambda_1(U)$ is WLD with respect to $U = U_0$ if and only if it is LD for small $|\rho - \rho_0|$, namely,

$$p(\rho) = A - \frac{B}{\rho}, \quad \forall |\rho - \rho_0| \text{ small}. \tag{1.3.22}$$

1.3.3 System of One-Dimensional Gas Dynamics

The *system of one-dimensional gas dynamics* can be written in Lagrangian representation as

$$\begin{cases} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(\tau, S)}{\partial x} = 0, \\ \dfrac{\partial S}{\partial t} = 0 \end{cases} \tag{1.3.23}$$

(see [22], [75]), where u is the velocity, S is the entropy, p is the pressure, and $\tau > 0$ is the specific volume. Moreover, $p = p(\tau, S)$ is the equation of state, satisfying

$$p_\tau < 0, \quad \forall \tau > 0, \tag{1.3.24}$$

which implies that (1.3.23) is a strictly hyperbolic system with three distinct eigenvalues:

$$\lambda_1(U) = -\sqrt{-p_\tau} < \lambda_2(U) = 0 < \lambda_3(U) = \sqrt{-p_\tau} \tag{1.3.25}$$

and the corresponding right eigenvectors can be taken as

$$r_1(U) = \left(1, \sqrt{-p_\tau}, 0\right)^T, \; r_2(U) = (p_S, 0, -p_\tau)^T, \; r_3(U) = \left(-1, \sqrt{-p_\tau}, 0\right)^T, \tag{1.3.26}$$

where

$$U = (\tau, u, S)^T. \tag{1.3.27}$$

Obviously, $\lambda_2(U)$ is always LD. On the other hand, $\lambda_1(U)$ and $\lambda_3(U)$ are LD if and only if

$$p_{\tau\tau}(\tau, S) \equiv 0, \quad \forall \tau > 0, \; \forall S, \tag{1.3.28}$$

whereas, $\lambda_1(U)$ and $\lambda_3(U)$ are GN if and only if $p = p(\tau, S)$ is a strictly convex or concave function with respect to τ.

We now consider system (1.3.23) in a neighbourhood of $U = U_0 \stackrel{\text{def.}}{=} (\tau_0, u_0, S_0)^T$ with $\tau_0 > 0$.

If

$$p_\tau(\tau_0, S_0) < 0, \tag{1.3.29}$$

then (1.3.23) is a strictly hyperbolic system with three distinct eigenvalues (1.3.25) in a neighbourhood of $U = U_0$. Moreover, it is easy to see that $\lambda_1(U)$ and $\lambda_3(U)$ are LD in a neighbourhood of $U = U_0$ if and only if

$$p_{\tau\tau}(\tau, S) \equiv 0, \quad \forall |\tau - \tau_0| \text{ and } |S - S_0| \text{ small}, \tag{1.3.30}$$

whereas, $\lambda_1(U)$ and $\lambda_3(U)$ are GN in a neighbourhood of $U = U_0$ if and only if

$$p_{\tau\tau}(\tau_0, S_0) \neq 0. \tag{1.3.31}$$

Noting that the first characteristic trajectory $U = U^{(1)}(s)$ passing through $U = U_0$ in the U-space is defined by

$$\begin{cases} \dfrac{dU}{ds} = r_1(U), \\ s = 0: \; U = U_0, \end{cases} \tag{1.3.32}$$

where $r_1(U)$ is given by (1.3.26), by the definition of WLD, we can see that $\lambda_1(U)$ is WLD with respect to $U = U_0$ if and only if

$$p_{\tau\tau}(\tau, S_0) \equiv 0, \quad \forall |\tau - \tau_0| \text{ small.} \tag{1.3.33}$$

Similarly, $\lambda_3(U)$ is WLD with respect to $U = U_0$ if and only if (1.3.33) holds. Obviously, if $\lambda_1(U)$ and $\lambda_3(U)$ are LD in a neighbourhood of $U = U_0$, then they are WLD with respect to $U = U_0$.

If

$$p_{\tau\tau}(\tau, S_0) \not\equiv 0, \quad \forall |\tau - \tau_0| \text{ small,} \tag{1.3.34}$$

then (1.3.23) is not WLD with respect to $U = U_0$. More precisely, if there exists an integer $\alpha \geq 0$ such that

$$p_{\tau\tau}(\tau_0, S_0) = \cdots = \frac{\partial^{\alpha+1} p}{\partial \tau^{\alpha+1}}(\tau_0, S_0) = 0, \quad \text{but} \quad \frac{\partial^{\alpha+2} p}{\partial \tau^{\alpha+2}}(\tau_0, S_0) \neq 0, \tag{1.3.35}$$

then (1.3.23) is a non-WLD system with the index α. Obviously, when $\alpha = 0$, system (1.3.23) is GN in a neighbourhood of $U = U_0$.

1.3.4 System of Compressible Elastic Fluids with Memory

The system of *1D compressible elastic fluids with memory* can be described by the following (cf. [45], [98]):

$$\begin{cases} \rho_t + v\rho_x + \rho v_x = 0, \\ \rho(v_t + vv_x) + p(\rho)_x = (\rho W'(F)F)_x, \\ F_t + vF_x - Fv_x = 0, \end{cases} \tag{1.3.36}$$

where $\rho > 0$ is the density, v is the velocity, p is the pressure, $W(F)$ is the strain energy function, and F corresponds to the deformation tensor.

Let

$$U = (\rho, v, F)^T. \tag{1.3.37}$$

When

$$p'(\rho) + W''(F)F^2 > 0, \quad \forall \rho > 0, \ \forall F, \tag{1.3.38}$$

(1.3.36) is a strictly hyperbolic system with the following distinct real eigenvalues:

$$\lambda_1 = v - \sqrt{p'(\rho) + W''(F)F^2} < \lambda_2 = v < \lambda_3 = v + \sqrt{p'(\rho) + W''(F)F^2}, \tag{1.3.39}$$

and the corresponding right eigenvectors can be taken as

$$\begin{aligned}
r_1(U) &= \left(-\frac{\rho}{\sqrt{p'(\rho)+w''(F)F^2}}, 1, \frac{F}{\sqrt{p'(\rho)+w''(F)F^2}}\right)^T,\\
r_2(U) &= \left(FW''(F)+W'(F), 0, \frac{p'(\rho)-FW'(F)}{\rho}\right)^T,\\
r_3(U) &= \left(\frac{\rho}{\sqrt{p'(\rho)+w''(F)F^2}}, 1, -\frac{F}{\sqrt{p'(\rho)+w''(F)F^2}}\right)^T.
\end{aligned} \tag{1.3.40}$$

Obviously, $\lambda_2(U)$ is always LD. On the other hand, $\lambda_1(U)$ and $\lambda_3(U)$ are LD if and only if

$$\rho p''(\rho)+2p'(\rho)-W'''(F)F^3 \equiv 0, \quad \forall \rho>0,\ \forall F, \tag{1.3.41}$$

whereas, $\lambda_1(U)$ and $\lambda_3(U)$ are GN if and only if

$$\rho p''(\rho)+2p'(\rho)-W'''(F)F^3 \neq 0, \quad \forall \rho>0,\ \forall F. \tag{1.3.42}$$

Now, we consider system (1.3.36) in a neighbourhood of $U = U_0 \overset{\text{def.}}{=} (\rho_0, v_0, 1)^T$ with $\rho_0 > 0$.

If

$$p'(\rho_0)+W''(1)>0, \tag{1.3.43}$$

(1.3.36) is a strictly hyperbolic system with three distinct eigenvalues (1.3.39) in a neighbourhood of $U = U_0$. Then, in a neighbourhood of $U = U_0$, $\lambda_1(U)$ and $\lambda_3(U)$ are LD if and only if

$$\rho p''(\rho)+2p'(\rho)-W'''(F)F^3 \equiv 0, \tag{1.3.44}$$

whereas, $\lambda_1(U)$ and $\lambda_3(U)$ are GN if and only if

$$\rho_0 p''(\rho_0)+2p'(\rho_0)-W'''(1) \neq 0. \tag{1.3.45}$$

The first characteristic trajectory $U = U^{(1)}(s)$ passing through $U = U_0$ in the U-space is defined by

$$\begin{cases} \dfrac{dU}{ds} = r_1(U),\\ s=0:\ U=U_0, \end{cases} \tag{1.3.46}$$

where $r_1(U)$ is given by (1.3.40). By the first equation and the third one in system (1.3.46), we immediately get

$$\frac{d(\rho F)}{ds} = 0. \tag{1.3.47}$$

Hence, $\rho F \equiv \rho_0$ always holds along the first characteristic trajectory passing through $U = U_0$. Thus, by the definition of WLD, $\lambda_1(U)$ and $\lambda_3(U)$ are WLD with respect to $U = U_0$ if and only if

$$\rho p''(\rho) + 2p'(\rho) - W'''(\rho_0\rho^{-1})(\rho_0\rho^{-1})^3 \equiv 0, \quad \forall |\rho - \rho_0| \text{ small.} \tag{1.3.48}$$

Obviously, if $\lambda_1(U)$ and $\lambda_3(U)$ are LD in a neighbourhood of $U = U_0$, then they are WLD with respect to $U = U_0$.

Let

$$Q(\rho) = \rho p''(\rho) + 2p'(\rho) - W'''(\rho_0\rho^{-1})(\rho_0\rho^{-1})^3. \tag{1.3.49}$$

If

$$Q(\rho) \not\equiv 0, \quad \forall |\rho - \rho_0| \text{ small,} \tag{1.3.50}$$

then (1.3.36) is not WLD with respect to $U = U_0$. More precisely, if there exists an integer $\alpha \geq 0$ such that

$$Q(\rho_0) = Q'(\rho_0) = \cdots = Q^{(\alpha-1)}(\rho_0) = 0, \text{ but } Q^{(\alpha)}(\rho_0) \neq 0, \tag{1.3.51}$$

then (1.3.36) is a non-WLD system with the index α. Obviously, when $\alpha = 0$, system (1.3.36) is GN in a neighbourhood of $U = U_0$.

1.3.5 System of the Motion of an Elastic String

The *dynamics of a nonlinear elastic string* can be expressed by the following system (cf. [15], [58], [59], [62], [76], [100], [101]):

$$\begin{cases} u_t - v_x = 0, \\ v_t - \left(\dfrac{T(r)}{r}u\right)_x = 0, \end{cases} \tag{1.3.52}$$

where $u = (u_1, \ldots, u_n)^T$, $v = (v_1, \ldots, v_n)^T$, $r = |u| = \sqrt{u_1^2 + \cdots + u_n^2}$ (in practice, $n = 2$ or 3), and $T = T(r)$ is a suitably smooth function of r such that

$$T'(r) > \frac{T(r)}{r} > 0, \quad \forall r > 1. \tag{1.3.53}$$

Under hypothesis (1.3.53), (1.3.52) is a hyperbolic system with $2n$ real eigenvalues:

$$\begin{aligned} \lambda_1(U) &\stackrel{\text{def.}}{=} -\sqrt{T'(r)} < \lambda_2(U) \equiv \cdots \equiv \lambda_n(U) \stackrel{\text{def.}}{=} -\sqrt{\frac{T(r)}{r}} \\ &< \lambda_{n+1}(U) \equiv \cdots \equiv \lambda_{2n-1}(U) \stackrel{\text{def.}}{=} \sqrt{\frac{T(r)}{r}} < \lambda_{2n}(U) \stackrel{\text{def.}}{=} \sqrt{T'(r)}, \end{aligned} \tag{1.3.54}$$

and the corresponding right eigenvectors can be taken as

$$
\begin{aligned}
r_1(U) &= (u^T, \sqrt{T'(r)}u^T)^T,\\
r_2(U) &= \left(-u_2, u_1, 0, \ldots, 0, -\sqrt{\frac{T(r)}{r}}u_2, \sqrt{\frac{T(r)}{r}}u_1, 0, \ldots, 0\right)^T,\\
&\cdots\\
r_n(U) &= \left(-u_n, 0, \ldots, 0, u_1, -\sqrt{\frac{T(r)}{r}}u_n, 0, \ldots, 0, \sqrt{\frac{T(r)}{r}}u_1\right)^T,\\
r_{n+1}(U) &= \left(u_2, -u_1, 0, \ldots, 0, -\sqrt{\frac{T(r)}{r}}u_2, , \sqrt{\frac{T(r)}{r}}u_1, 0, \ldots, 0\right)^T,\\
&\cdots\\
r_{2n-1}(U) &= \left(u_n, 0, \ldots, 0, -u_1, -\sqrt{\frac{T(r)}{r}}u_n, 0, \ldots, 0, \sqrt{\frac{T(r)}{r}}u_1\right)^T,\\
r_{2n}(U) &= (-u^T, \sqrt{T'(r)}u^T)^T,
\end{aligned}
\tag{1.3.55}
$$

in which

$$U = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.3.56}$$

When $n = 2$, system (1.3.52) is a strictly hyperbolic system, while, when $n > 2$, system (1.3.52) is a nonstrictly hyperbolic system with characteristics with constant multiplicity.

It is easy to see that $\lambda_2(U), \ldots, \lambda_{2n-1}(U)$ are always LD. Moreover, $\lambda_1(U)$ and $\lambda_{2n}(U)$ are GN if and only if

$$T''(r) \neq 0, \quad \forall r > 1, \tag{1.3.57}$$

whereas, $\lambda_1(U)$ and $\lambda_{2n}(U)$ are LD if and only if

$$T''(r) \equiv 0, \quad \forall r > 1. \tag{1.3.58}$$

Now, we consider system (1.3.52) in a neighbourhood of $U = U_0 \overset{\text{def.}}{=} \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$ with $|u_0| > 1$ and

$$T'(r_0) > \frac{T(r_0)}{r_0} > 0, \tag{1.3.59}$$

where $r_0 = |u_0|$ and u_0 is a constant vector.

$\lambda_1(U)$ and $\lambda_{2n}(U)$ are GN in a neighbourhood of $U = U_0$ if and only if

$$T''(r_0) \neq 0. \tag{1.3.60}$$

Moreover, it is easy to see that $\lambda_1(U)$ and $\lambda_{2n}(U)$ are WLD with respect to $U = U_0$ if and only if $\lambda_1(U)$ and $\lambda_{2n}(U)$ are LD in a neighbourhood of $U = U_0$, namely,

$$T''(r) \equiv 0, \quad \forall\ |r - r_0| \text{ small.} \tag{1.3.61}$$

Thus, if

$$T''(r) \not\equiv 0,\ \forall |r - r_0| \text{ small,} \tag{1.3.62}$$

then, in a neighbourhood of $U = U_0$, (1.3.52) is not LD (WLD). More precisely, if there exists an integer $\alpha \geq 0$ such that

$$T''(r_0) = T'''(r_0) = \cdots = T^{(\alpha+1)}(r_0) = 0,\ \text{but } T^{(\alpha+2)}(r_0) \neq 0, \tag{1.3.63}$$

then, in a neighbourhood of $U = U_0$, (1.3.52) is a non-WLD system with the index α. Obviously, when $\alpha = 0$, system (1.3.52) is GN in a neighbourhood of $U = U_0$.

1.3.6 System of Finite Amplitude Plane Elastic Waves for Hyperelastic Materials

The *system of finite amplitude plane elastic waves for hyperelastic materials* can be written as

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0 \tag{1.3.64}$$

(see [38]) with $u = (u_1, \ldots, u_6)^T$ and

$$A(u) = \begin{pmatrix} 0 & -I \\ -V'' & 0 \end{pmatrix}, \tag{1.3.65}$$

where $V'' = V''(u_1, u_2, u_3)$ is a 3×3 matrix determined by the material and I is the 3×3 unit matrix.

System (1.3.64) is a nonstrictly hyperbolic system.

For the material of Ciarlet-Geymonat (cf. [20]), it is easy to see that system (1.3.64) has two LD characteristics with multiplicity 2 and two simple GN characteristics (for details, see Section 7.4.3).

For the material of St.Venant-Kirchhoff (cf. [20]), system (1.3.64) has two simple GN characteristics; however, the other four characteristics coincide at $u = 0$ but have no constant multiplicity.

For the material of Odgen (cf. [20]), system (1.3.64) has two simple GN or non-GN characteristics; however, the other four characteristics coincide at $u = 0$ but have no constant multiplicity.

1.4 Main Results for the Cauchy Problem

We now consider the Cauchy problem for system (1.1.1) with small and decaying C^1 initial data (1.1.5) satisfying that there exists a number $\mu > 0$ such that

$$\theta \stackrel{\text{def.}}{=} \sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\phi(x)|+|\phi'(x)|)\} < +\infty \tag{1.4.1}$$

and θ is small enough.

For the global existence of the C^1 solution to the Cauchy problem, we will show in Chapter 3 that the following holds: Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, system (1.1.1) is strictly hyperbolic and WLD with respect to $u = 0$. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (1.1.1) and (1.1.5) admits a unique C^1 solution $u = u(t, x)$ with a small C^1 norm for all $t \in \mathbb{R}$.

Conversely, under the assumption that in a neighbourhood of $u = 0$, $A(u) \in C^1$ and system (1.1.1) is strictly hyperbolic, if the Cauchy problem (1.1.1) and (1.1.5) always admits a unique C^1 solution $u = u(t, x)$ on $t \geq 0$ for any given C^1 initial data $\phi(x)$ with small θ, then system (1.1.1) must be WLD with respect to $u = 0$.

Thus, for small θ, the weak linear degeneracy is equivalent to the global existence of the C^1 solution to the Cauchy problem (1.1.1) and (1.1.5); hence, if system (1.1.1) is not WLD, then we should meet the blow-up phenomenon.

For the blow-up phenomenon of the C^1 solution to the Cauchy problem, we will show in Chapter 4 that the following holds: Suppose that in a neighbourhood of $u = 0$, $A(u)$ is suitably smooth and system (1.1.1) is strictly hyperbolic. Suppose furthermore that system (1.1.1) is not WLD with respect to $u = 0$ and the corresponding index α defined by (1.2.6) is a finite nonnegative integer. Suppose finally that $\phi(x) = \varepsilon\psi(x)$, where $\varepsilon > 0$ is a small parameter and $\psi(x) \in C^1$ satisfies (1.4.1). Then, for a large class of initial data, precisely speaking, if there exists $i_0 \in J_1$ such that

$$l_{i_0}(0)\psi(x) \not\equiv 0, \tag{1.4.2}$$

then there exists $\varepsilon_0 > 0$ so small that for any given $\varepsilon \in (0, \varepsilon_0]$, the following conclusions hold:

a. The first-order partial derivative u_x of the C^1 solution $u = u(t, x)$ to the Cauchy problem (1.1.1) and (1.1.5) must blow up in a finite time, while the solution itself remains bounded and small. Moreover, the life span $\tilde{T}(\varepsilon)$ of the C^1 solution possesses the following asymptotic property:

$$\lim_{\varepsilon\downarrow 0}(\varepsilon^{\alpha+1}\tilde{T}(\varepsilon)) = M_0, \tag{1.4.3}$$

where M_0 is a positive constant independent of ε, given by

$$M_0 = \Big(\max_{i \in J_1} \sup_{x \in \mathbb{R}} \Big\{ -\frac{1}{\alpha!} \frac{d^{\alpha+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha+1}}\Big|_{s=0} (l_i(0)\psi(x))^\alpha l_i(0)\psi'(x) \Big\} \Big)^{-1}, \tag{1.4.4}$$

where $u = u^{(i)}(s)$ is defined by (1.2.1). Hence, there exist two positive constants c and C independent of ε such that

$$c\varepsilon^{-(\alpha+1)} \le \tilde{T}(\varepsilon) \le C\varepsilon^{-(\alpha+1)}, \tag{1.4.5}$$

denoted by

$$\tilde{T}(\varepsilon) \approx \varepsilon^{-(\alpha+1)}. \tag{1.4.6}$$

b. The singularity occurs at the beginning of the envelope of characteristics of the same family, i.e., at the point with the minimum t-value on the envelope.

c. For every $i \notin J_1$, the ith family of characteristics does not generate any envelope on the domain $0 \le t \le \tilde{T}(\varepsilon)$. In particular, every family of WLD characteristics and then every family of LD characteristics does not generate any envelope on the domain $0 \le t \le \tilde{T}(\varepsilon)$.

d. Let (t_0, x_0) $(t_0 \overset{\text{def.}}{=} \tilde{T}(\varepsilon))$ be a blow-up point. There exists $i_0 \in J_1$ such that along the i_0th characteristic passing through (t_0, x_0), the blow-up rate is given by

$$u_x(t,x) = O((t_0 - t)^{-1}), \quad \forall t < t_0, \tag{1.4.7}$$

which is independent of the index α.

e. On the line $t = \tilde{T}(\varepsilon)$, the set of blow-up points cannot possess a positive (even very small) measure.

These results imply all the previous ones given by John, Liu, and Hörmander and mentioned in Section 1.1.

As to the critical case that system (1.1.1) is not WLD, but the corresponding index α is equal to $+\infty$, we still have the blow-up phenomenon. However, it is impossible to get a unified sharp estimate on the life span in the critical case $\alpha = +\infty$. In fact, even in the scalar case

$$\begin{cases} u_t + \lambda(u)u_x = 0, \\ t = 0: \ u = \varepsilon\psi(x), \end{cases} \tag{1.4.8}$$

where $\lambda(u) \in C^\infty, \ \lambda'(u) \not\equiv 0$ with

$$\lambda^{(l)}(0) = 0 \quad (l = 1, 2, \ldots), \tag{1.4.9}$$

we may choose $\lambda(u)$ in different ways such that

$$\exp\{c\varepsilon^{-p}\} \le \tilde{T}(\varepsilon) \le \exp\{C\varepsilon^{-p}\}, \quad \forall p > 0, \tag{1.4.10}$$

or

$$\exp\{c(\ln\varepsilon)^2\} \le \tilde{T}(\varepsilon) \le \exp\{C(\ln\varepsilon)^2\}, \tag{1.4.11}$$

etc., where c and C are positive constants independent of ε.

The blow-up mechanism of the C^1 solution in the critical case is also presented in Chapter 4.

1.5 Normalized Coordinates

The basic idea of the proof of the results given in Section 1.4 is as follows: Since the initial data are small and decay as $|x| \to +\infty$, n waves should be essentially separated from each other in a finite amount of time and the interaction among n waves can be controlled to be relatively small. Thus, for every wave, the problem can be essentially reduced to the scalar case.

Noting that the definition of weak linear degeneracy depends on $u = u^{(i)}(s)(i = 1, \dots, n)$, the characteristic trajectories passing through, for instance, $u = 0$, the key point in the proof is to find new coordinates $\tilde{u} = \tilde{u}(u)[\tilde{u}(0) = 0]$ such that in the $\tilde{u}$-space the characteristic trajectories passing through $\tilde{u} = 0$ can be expressed in a simpler way.

In Chapter 2 we will prove that the following holds: Suppose that in a neighbourhood of $u = 0$, system (1.1.1) is strictly hyperbolic and $A(u) \in C^k$, where k is an integer ≥ 1. Then there exists a C^{k+1} diffeomorphism $u = u(\tilde{u})[u(0) = 0]$ such that in the $\tilde{u}$-space, for each $i = 1, \dots, n$, the ith characteristic trajectory passing through $\tilde{u} = 0$ coincides with the $\tilde{u}_i$-axis at least for small $|\tilde{u}_i|$, namely,

$$\tilde{r}_i(\tilde{u}_i e_i) // e_i, \quad \forall |\tilde{u}_i| \text{ small } \ (i = 1, \dots, n), \tag{1.5.1}$$

where $\tilde{r}_i(\tilde{u})$ denotes the corresponding ith right eigenvector in the $\tilde{u}$-space and $e_i = (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0)^T$.

We refer to the diffeomorphism given above as the *normalized transformation*, and the corresponding variables $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)^T$ are called the *normalized variables* or *normalized coordinates*.

In normalized coordinates $\tilde{u}$, the ith characteristic $\tilde{\lambda}_i(\tilde{u}) = \lambda_i(u(\tilde{u}))$ is WLD with respect to $u = 0$ if and only if

$$\tilde{\lambda}_i(\tilde{u}_i e_i) \equiv \tilde{\lambda}_i(0), \quad \forall |\tilde{u}_i| \text{ small}, \tag{1.5.2}$$

whereas, if $\lambda_i(\tilde{u})$ is not WLD with respect to $u = 0$, then either there exists an integer $\alpha_i \ge 0$ such that

$$\frac{d^l \tilde{\lambda}_i(\tilde{u}_i e_i)}{d\tilde{u}_i^l}\Big|_{\tilde{u}_i = 0} = 0 \quad (l = 1, \dots, \alpha_i), \quad \text{but} \quad \frac{d^{\alpha_i+1} \tilde{\lambda}_i(\tilde{u}_i e_i)}{d\tilde{u}_i^{\alpha_i+1}}\Big|_{\tilde{u}_i = 0} \ne 0, \tag{1.5.3}$$

or

$$\left.\frac{d^l\tilde{\lambda}_i(\tilde{u}_i e_i)}{d\tilde{u}_i^l}\right|_{\tilde{u}_i=0} = 0 \quad (l = 1, 2, \ldots) \tag{1.5.4}$$

but (1.5.2) fails, denoted by $\alpha_i = +\infty$.

The system in normalized coordinates can be regarded as a standard form of a strictly hyperbolic system. The proof of the results given in Section 1.4 are taken in normalized coordinates.

1.6 Weak Linear Degeneracy and Generalized Null Condition

The null condition was introduced in the study of nonlinear wave equations for getting the global existence of classical solutions with small initial data (cf. [19], [39]). For the first-order quasilinear strictly hyperbolic system (1.1.1), we can similarly introduce the following definition.

A strictly hyperbolic system (1.1.1) is said to satisfy the *null condition* if every small plane wave solution $u = u(s)$ $[u(0) = 0]$, where $s = ax+bt$, a and b being constants, to the corresponding linearalized system

$$u_t + A(0)u_x = 0 \tag{1.6.1}$$

is always a solution to the original quasilinear system (1.1.1), namely,

$$u_t + A(0)u_x = (A(0) - A(u))u_x \overset{\text{def.}}{=} B(u)u_x. \tag{1.6.2}$$

Without loss of generality, we may assume that

$$A(0) = \text{diag}\{\lambda_1(0), \ldots, \lambda_n(0)\}. \tag{1.6.3}$$

Thus, the general solution to system (1.6.1) is

$$u_i = u_i(x - \lambda_i(0)t) \quad (i = 1, \ldots, n), \tag{1.6.4}$$

where $u_i = u_i(s)$ $(i = 1, \ldots, n)$ are arbitrarily given smooth functions of s. Hence, noting the strict hyperbolicity, for each plane wave solution u to system (1.6.1), there exists an index $i \in \{1, \ldots, n\}$ such that

$$u = u_i(s)e_i, \tag{1.6.5}$$

where

$$s = x - \lambda_i(0)t \tag{1.6.6}$$

and

$$e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0)^T. \tag{1.6.7}$$

Therefore, we get

$$\text{Null condition for system (1.1.1)}$$
$$\Updownarrow$$
$$B(u_i(s)e_i)u_i'(s)e_i \equiv 0,\ \forall u_i(s)\ (u_i(0)=0) \text{ small} \quad (i=1,\dots,n)$$
$$\Updownarrow$$
$$B(u_ie_i)e_i \equiv 0,\ \forall |u_i| \text{ small} \quad (i=1,\dots,n)$$
$$\Updownarrow$$
$$A(u_ie_i)e_i \equiv \lambda_i(0)e_i,\ \forall |u_i| \text{ small} \quad (i=1,\dots,n)$$
$$\Updownarrow$$
$$\lambda_i(u_ie_i)e_i \equiv \lambda_i(0),\ \forall |u_i| \text{ small} \quad (i=1,\dots,n),$$
$$r_i(u_ie_i)//e_i,\ \forall |u_i| \text{ small} \quad (i=1,\dots,n)$$
$$\Updownarrow$$

$$\text{System (1.1.1) is WLD with respect to } u=0 \text{ and}$$
$$u=(u_1,\dots,u_n)^T \text{ are normalized coordinates.}$$

Moreover, since (1.1.1) is a system with n unknown variables, we may introduce the following definition.

System (1.1.1) is said to satisfy the *generalized null condition* if there exists a local C^2 diffeomorphism $\tilde{u}=\tilde{u}(u)$ $[\tilde{u}(0)=0]$ such that the corresponding system for $\tilde{u}$ satisfies the null condition.

In this definition, $\tilde{u}=(\tilde{u}_1,\dots,\tilde{u}_n)^T$ are normalized coordinates and $\tilde{u}=\tilde{u}(u)$ is nothing but a normalized transformation. Thus, saying system (1.1.1) satisfies the generalized null condition simply means that system (1.1.1) satisfies the null condition in normalized coordinates.

Noting that for every $i=1,\dots,n$, $\nabla\lambda_i(u)r_i(u)$ is an invariant under any given invertible C^2 transformation $\tilde{u}=\tilde{u}(u)$, we claim that system (1.1.1) is WLD with respect to $u=0$ if and only if system (1.1.1) satisfies the generalized null condition.

1.7 Nonstrictly Hyperbolic Case

Up to now, all the discussions have concentrated only on the strictly hyperbolic case. The nonstrictly hyperbolic system also has many applications (cf. Sections 1.3.5 and 1.3.6), but it is very complicated.

In the nonstrictly hyperbolic case, the following situation is the simplest one: Consider the quasilinear hyperbolic system of conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, \tag{1.7.1}$$

where $u = (u_1, \ldots, u_n)^T$ and $f(u) = (f_1(u), \ldots, f_n(u))^T$. Suppose that every eigenvalue of $A(u) = \nabla f(u)$ has a constant multiplicity. Without loss of generality, we may suppose that

$$\lambda(u) \stackrel{\text{def.}}{=} \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{1.7.2}$$

where $1 \leq p \leq n$. When $p = 1$, system (1.7.1) is strictly hyperbolic, whereas, when $p > 1$, (1.7.1) is a nonstrictly hyperbolic system of conservation laws with characteristics with constant multiplicity. According to the results given in [7] and [25], for the hyperbolic system of conservation laws, every characteristic with constant multiplicity $p > 1$ must be LD:

$$\nabla\lambda(u) r_i(u) \equiv 0 \qquad (i = 1, \ldots, p), \tag{1.7.3}$$

then WLD.

As a result, all previous results in the strictly hyperbolic case can be similarly extended to this situation (see Chapters 3 and 4). Moreover, several chapters in this monograph are also closely related to some more general nonstrictly hyperbolic cases.

In order to treat the nonstrictly hyperbolic system, the corresponding concepts such as the normalized transformation, the generalized normalized transformation, and weak linear degeneracy are introduced in Chapter 2.

1.8 Cauchy Problem on a Semibounded Initial Axis

For the Cauchy problem of system (1.1.1) on a semibounded initial axis:

$$t = 0: \; u = \phi(x), \quad x \geq 0, \tag{1.8.1}$$

where $\phi(x)$ is a C^1 vector function on $x \geq 0$, satisfying that there exists a number $\mu > 0$ such that

$$\theta \stackrel{\text{def.}}{=} \sup_{x \geq 0}\{(1+x)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < +\infty, \tag{1.8.2}$$

suppose that in a neighbourhood of $u = 0$,

$$\lambda_1(u), \ldots, \lambda_{n-1}(u) < \lambda_n(u). \tag{1.8.3}$$

In Chapter 5 we will show that the following holds: For θ small enough, the Cauchy problem (1.1.1) and (1.8.1) admit a unique global C^1 solution $u = u(t,x)$ on the maximum determinate domain $D = \{(t,x) |\; t \geq 0, x \geq x_n(t)\}$, where $x = x_n(t)$ is the nth characteristic curve passing through the origin $O(0,0)$:

$$\begin{cases} \dfrac{dx_n(t)}{dt} = \lambda_n(u(t, x_n(t))), \\ x_n(0) = 0 \end{cases} \tag{1.8.4}$$

if and only if $\lambda_n(u)$ is WLD with respect to $u = 0$.

Quite different from the corresponding result given in Section 1.4 for the Cauchy problem (1.1.1) and (1.1.5), in this situation, system (1.1.1) might be nonstrictly hyperbolic [see (1.8.3)]; moreover, only $\lambda_n(u)$ is asked to be WLD to get the global existence.

1.9 One-Sided Mixed Initial-Boundary Value Problem

In order to consider the interaction of nonlinear hyperbolic waves with the boundary, namely, the effect of boundary conditions on the global regularity of classical solution, under the hypothesis that in a neighbourhood of $u = 0$,

$$\lambda_1(u), \dots, \lambda_m(u) < 0 < \lambda_{m+1}(u) < \cdots < \lambda_n(u), \tag{1.9.1}$$

we consider the one-sided mixed initial-boundary value problem for system (1.1.1) with the initial condition (1.8.1) and the following boundary conditions:

$$x = 0: \ v_s = f_s(\alpha(t), v_1, \dots, v_m) + h_s(t) \quad (s = m+1, \dots, n), \tag{1.9.2}$$

where $f_s(\cdot)$, $\alpha(t) = (\alpha_1(t), \dots, \alpha_k(t))$, and $h_s(\cdot)$ $(s = m+1, \dots, n)$ are all C^1 functions,

$$v_i = l_i(u)u \quad (i = 1, \dots, n), \tag{1.9.3}$$

and, without loss of generality, we suppose that

$$f_s(\alpha(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{1.9.4}$$

Suppose that the conditions of C^1 compatibility are satisfied at the point $(t, x) = (0, 0)$. In Chapter 6 we will prove the following result: If all the positive characteristics $\lambda_s(u) (s = m+1, \dots, n)$ are WLD with respect to $u = 0$ and

$$\begin{aligned} \theta \overset{\text{def.}}{=} \max\{ & \sup_{x \geq 0} (1+x)^{1+\mu} (|\phi(x)| + |\phi'(x)|), \\ & \sup_{t \geq 0} (1+t)^{1+\mu} (|\alpha(t)| + |h(t)| + |\alpha'(t)| + |h'(t)|)\} \\ < & +\infty, \end{aligned} \tag{1.9.5}$$

in which $\mu > 0$ is a constant and

$$h(t) = (h_{m+1}(t), \ldots, h_n(t)), \tag{1.9.6}$$

then, for θ small enough, the one-sided mixed initial-boundary value problem (1.1.1), (1.8.1), and (1.9.2) admits a unique global C^1 solution $u = u(t,x)$ with a small C^1 norm on the domain $D = \{(t,x) |\ t \geq 0, x \geq 0\}$.

Comparing with the result on the Cauchy problem (see Section 1.4), this result shows that when there is only one boundary, the interaction of linear or nonlinear boundary conditions with nonlinear hyperbolic waves causes a positive effect on the global regularity of the solution: The weak linear degeneracy of all the negative characteristics $\lambda_r(u)$ $(r = 1, \ldots, m)$ is not necessary for the global existence of the C^1 solution; moreover, the system might be nonstrictly hyperbolic [see (1.9.1)].

1.10 Generalized Riemann Problem

For the quasilinear hyperbolic system of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1.10.1}$$

under the assumption that (1.10.1) is strictly hyperbolic:

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{1.10.2}$$

and for any given i $(i = 1, \ldots, n)$, $\lambda_i(u)$ is either GN or LD, we consider the generalized Riemann problem for system (1.10.1) with the following piecewise C^1 initial data:

$$t = 0 : \ u = \begin{cases} u_l(x), & x \leq 0, \\ u_r(x), & x \geq 0, \end{cases} \tag{1.10.3}$$

where $u_l(x)$ and $u_r(x)$ are C^1 vector functions on $x \leq 0$ and $x \geq 0$, respectively, and

$$\eta \overset{\text{def.}}{=} |u_r(0) - u_l(0)| > 0 \tag{1.10.4}$$

is suitably small.

Suppose that the corresponding Riemann problem for system (1.10.1) with the piecewise constant initial data

$$t = 0 : \ u = \begin{cases} u_l(0), & x \leq 0, \\ u_r(0), & x \geq 0, \end{cases} \tag{1.10.5}$$

admits a unique self-similar solution $u = U(x/t)$ composed of $n+1$ constant states and n small amplitude waves (nondegenerate shocks corresponding to GN characteristics and contact discontinuities corresponding to LD characteristics).

Suppose furthermore that there exists a constant $\mu > 0$ such that

$$\begin{aligned}\theta \stackrel{\text{def.}}{=} &\sup_{x\le 0}\{(1+|x|)^{1+\mu}(|u_l(x)-u_l(0)|+|u_l'(x)|)\}\\ &+\sup_{x\ge 0}\{(1+x)^{1+\mu}(|u_r(x)-u_r(0)|+|u_r'(x)|)\}\end{aligned} \tag{1.10.6}$$

is suitably small.

In Chapter 7 we will prove that the generalized Riemann problem (1.10.1) and (1.10.3) admits a unique global piecewise C^1 solution $u = u(t,x)$ on $t \ge 0$, which possesses a structure globally similar to that of the self-similar solution $u = U(x/t)$ to the corresponding Riemann problem (1.10.1) and (1.10.5). This result shows the global structural stability of the self-similar solution $u = U(x/t)$ to the Riemann problem.

1.11 Generalized Nonlinear Initial-Boundary Riemann Problem

Under assumption (1.9.1), we consider the following nonlinear initial-boundary Riemann problem with constant initial data:

$$\begin{cases}\dfrac{\partial u}{\partial t}+\dfrac{\partial f(u)}{\partial x}=0, \quad t>0,\ x>0, & (1.11.1)\\ t=0:\ u=u_+, \quad x\ge 0, & (1.11.2)\\ x=0:\ v_s=G_s(v_1,\dots,v_m) \quad (s=m+1,\dots,n), \quad t\ge 0, & (1.11.3)\end{cases}$$

where u_+ is a constant vector, $f(\cdot)\in C^3$, $G_s(\cdot)\in C^1$ $(s=m+1,\dots,n)$, and

$$v_i = l_i(u)u \quad (i=1,\dots,n). \tag{1.11.4}$$

However, the conditions of C^0 compatibility at the point $(t,x)=(0,0)$,

$$v_s^+ = G_s(v_1^+,\dots,v_m^+) \quad (s=m+1,\dots,n), \tag{1.11.5}$$

fail, where

$$v_i^+ = l_i(u_+)u_+ \quad (i=1,\dots,n). \tag{1.11.6}$$

Suppose that every positive characteristic $\lambda_s(u)$ $(s\in\{m+1,\dots,n\})$ is either GN or LD. If $|u_+|$ and $|v_s^+ - G_s(v_1^+,\dots,v_m^+)|$ $(s=m+1,\dots,n)$ are suitably small, in Chapter 8 we will prove that the nonlinear initial-boundary Riemann problem (1.11.1)–(1.11.3) admits a unique self-similar solution $u = U(x/t)$ composed of $n-m+1$ constant states and $n-m$ elementary waves with small amplitude (shocks or centered rarefaction waves corresponding to GN characteristics, contact discontinuities corresponding to LD characteristics).

Correspondingly, we consider the generalized nonlinear initial-boundary Riemann problem for system (1.11.1) with

$$t = 0 : u = u_r(x), \quad x \geq 0, \tag{1.11.7}$$

$$x = 0 : v_s = G_s(\alpha_s(t), v_1, \ldots, v_m) + h_s(t) \quad (s = m+1, \ldots, n), \quad t \geq 0, \tag{1.11.8}$$

where $u_r(\cdot)$, $\alpha_s(\cdot)$, $h_s(\cdot)$, and $G_s(\cdot) \in C^1$ $(s = m+1, \ldots, n)$,

$$G_s(\alpha_s(t), 0, \ldots, 0) \equiv 0 \quad (s = m+1, \ldots, n). \tag{1.11.9}$$

However, the conditions of C^0 compatibility at the point $(t, x) = (0, 0)$ fail.

Suppose that $\eta \stackrel{\text{def.}}{=} |u_r(0)| + |\alpha(0)| + |h(0)| > 0$ is suitably small, where

$$\alpha(t) = (\alpha_{m+1}(t), \ldots, \alpha_n(t)), \; h(t) = (h_{m+1}(t), \ldots, h_n(t)), \quad \forall t \geq 0, \tag{1.11.10}$$

and there is a constant $\mu > 0$ such that

$$\begin{aligned}\theta \stackrel{\text{def.}}{=} &\sup_{x \geq 0}\{(1+x)^{1+\mu}(|u_r(x) - u_r(0)| + |u_r'(x)|)\} \\ &+ \sup_{t \geq 0}\{(1+t)^{1+\mu}(|\alpha(t) - \alpha(0)| + |\alpha'(t)| + |h(t) - h(0)| + |h'(t)|)\}\end{aligned} \tag{1.11.11}$$

is also suitably small.

Suppose furthermore that the corresponding nonlinear initial-boundary Riemann problem for system (1.11.1) with

$$t = 0 : \; u = u_r(0), \quad x \geq 0, \tag{1.11.12}$$

$$\begin{aligned}x = 0 : \; v_s &= G_s(\alpha_s(0), v_1, \ldots, v_m) + h_s(0) \\ &\stackrel{\text{def.}}{=} \overline{G}_s(v_1, \ldots, v_m) \quad (s = m+1, \ldots, n), \quad t \geq 0,\end{aligned} \tag{1.11.13}$$

admits a unique self-similar solution $u = U(x/t)$ composed of $n-m+1$ constant states and $n-m$ small-amplitude elementary waves $x = \hat{\lambda}_k t$ $(k = m+1, \ldots, n)$ (nondegenerate shocks corresponding to GN characteristics and contact discontinuities corresponding to LD characteristics).

We will prove in Chapter 8 that under assumption (1.9.1), the generalized nonlinear initial-boundary Riemann problem (1.11.1) and (1.11.7)–(1.11.8) admits a unique global piecewise C^1 solution $u = u(x/t)$ with a structure globally similar to the self-similar solution $u = U(x/t)$ to the corresponding nonlinear initial-boundary Riemann problem (1.11.1) and (1.11.12)–(1.11.13). Thus, the self-similar solution $u = U(x/t)$ to the nonlinear initial-boundary Riemann problem possesses a global structural stability.

1.12 Inverse Generalized Riemann Problem

Suppose that all the characteristics $\lambda_i(u)$ $(i = 1, \ldots, n)$ are GN and the solution to the generalized Riemann problem (1.10.1) and (1.10.3) considered in Section 1.10 consists of n small-amplitude nondegenerate shocks $x = x_i(t)$ with $x_i(0) = 0$ $(i = 1, \ldots, n)$.

The inverse generalized Riemann problem asks us to solve the following problem: When the position of n small-amplitude nondegenerate shocks $x = x_i(t)$ with $x_i(0) = 0$ $(i = 1, \ldots, n)$ is given, to what degree can we determine the initial data (1.10.3)? In Chapter 9 we will prove that if one knows n small-amplitude nondegenerate shocks $x = x_i(t) \in C^2$ with $x_i(0) = 0$ $(i = 1, \ldots, n)$ satisfying

$$|x_i'(t) - x_i'(0)|,\ |x_i''(t)| \le \frac{\varepsilon}{1+t}, \quad \forall t \ge 0 \ \ (i = 1, \ldots, n), \tag{1.12.1}$$

and the initial data $u_l(x) \in C^1$ on $x \le 0$ satisfying

$$|u_l(x) - u_l(0)|,\ |u_l'(x)| \le \frac{\varepsilon}{1+|x|}, \quad \forall x \le 0, \tag{1.12.2}$$

where $\varepsilon > 0$ is suitably small, then we can uniquely determine the initial data $u_r(x) \in C^1$ on $x \ge 0$ satisfying

$$|u_r(x) - u_r(0)|,\ |u_r'(x)| \le \frac{K\varepsilon}{1+x}, \quad \forall x \ge 0, \tag{1.12.3}$$

where K is a positive constant independent of ε and x.

1.13 Inverse Piston Problem

Suppose that a piston originally located at the origin at $t = 0$ moves with the speed $v_p = \phi(t)$ $(t \ge 0)$ in a tube, whose length is assumed to be infinite and that the gas on the right side of the piston possesses an isentropic state or an adiabatic state with initial velocity $u = u_r(x)$ $(x \ge 0)$ at $t = 0$. When

$$\phi(0) > u_r(0), \tag{1.13.1}$$

the motion of the piston produces a forward shock $x = x(t)$ passing through the origin.

In Chapter 10 we will show that if the initial data of gas on the right side of the piston at $t = 0$ and the position of the forward shock $x = x(t)$ $[x(0) = 0]$ are all known, under suitable hypotheses similar to those in Section 1.12, we can uniquely determine the velocity $\phi(t)$ of the piston for all $t \geq 0$ in both the Lagrangian and Eulerian representations. This solves globally the inverse piston problem.

We must point out that the result given in Section 1.8 plays an important role for the resolution of the inverse piston problem in the case of adiabatic gas.

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Chapter II
Preliminaries

2.1 Definition of Quasilinear Hyperbolic System

In this book we consider the nonlinear hyperbolic wave described by the following first-order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{2.1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of (t, x), and $A(u)$ is a given $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \ldots, n)$.

Definition 2.1.1 *System* (2.1.1) *is said to be* **hyperbolic** *if, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$ and a complete set of left (resp. right) eigenvectors $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ [resp. $r_i(u) = (r_{1i}(u), \ldots, r_{ni}(u))^T$] $(i = 1, \ldots, n)$:*

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \tag{2.1.2}$$

and

$$A(u)r_i(u) = \lambda_i(u)r_i(u). \tag{2.1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \qquad [\textit{resp.}\ \det|r_{ij}(u)| \neq 0]. \tag{2.1.4}$$

Without loss of generality, we assume that on the domain under consideration,

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n), \tag{2.1.5}$$

where δ_{ij} stands for Kronecker's delta.

Let

$$L(u) = \begin{pmatrix} l_1(u) \\ \vdots \\ l_n(u) \end{pmatrix}, \tag{2.1.6}$$

$$R(u) = (r_1(u), \ldots, r_n(u)), \tag{2.1.7}$$

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and

$$\Lambda(u) = \mathrm{diag}\{\lambda_1(u), \dots, \lambda_n(u)\}. \tag{2.1.8}$$

By (2.1.2)–(2.1.3), we have

$$L(u)A(u) = \Lambda(u)L(u) \tag{2.1.9}$$

and

$$A(u)R(u) = R(u)\Lambda(u). \tag{2.1.10}$$

Moreover, by (2.1.5), we have

$$L(u)R(u) = L(u)R(u) = I, \tag{2.1.11}$$

where I is the $n \times n$ unit matrix.

Definition 2.1.2 *If, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues*

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u), \tag{2.1.12}$$

then the set of left (*resp. right*) *eigenvectors must be complete. This kind of hyperbolic system* (2.1.1) *is said to be* **strictly hyperbolic**.

Definition 2.1.3 *If, for any given u on the domain under consideration, each eigenvalue of $A(u)$ possesses a constant multiplicity and the corresponding set of left* (*resp. right*) *eigenvectors is still complete, then system* (2.1.1) *is said to be a* **hyperbolic system with characteristics with constant multiplicity**.

In this situation, without loss of generality, we assume that

$$\lambda(u) \overset{\text{def.}}{=} \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{2.1.13}$$

and the set of left (resp. right) eigenvectors corresponding to $\lambda(u)$ spans a p-dimensional space, where $1 \le p \le n$. When $p = 1$, system (2.1.1) is strictly hyperbolic, whereas, when $p > 1$, (2.1.1) is a nonstrictly hyperbolic system with characteristics with constant multiplicity.

It is easy to see that for any strictly hyperbolic system or for any nonstrictly hyperbolic system with characteristics with constant multiplicity, all $\lambda_i(u)$, $l_i(u)$, and $r_i(u)$ $(i = 1, \dots, n)$ have the same regularity as $A(u)$. However, it is not always the case for general hyperbolic systems.

Example 2.1.1 *Let*

$$A(u) = \begin{pmatrix} 0 & u \\ u^2 & 0 \end{pmatrix}. \tag{2.1.14}$$

Obviously, $A(u) \in C^\infty(\mathbb{R})$, but the eigenvalues $\lambda_{1,2}(u) = \pm u^{3/2} \overline{\in} C^\infty$ at $u = 0$.

Example 2.1.2 *Let*

$$A(u) = \begin{cases} e^{-\frac{1}{u^2}} \begin{pmatrix} \cos(\frac{2}{u}) & \sin(\frac{2}{u}) \\ \sin(\frac{2}{u}) & -\cos(\frac{2}{u}) \end{pmatrix}, & u \neq 0, \\ 0, & u = 0. \end{cases} \tag{2.1.15}$$

It can be shown that $A(u) \in C^\infty(\mathbb{R})$ *and the eigenvalues* $\lambda_{1,2}(u) = \pm e^{-1/u^2} \in C^\infty(\mathbb{R})$. *However, one cannot find a complete set of left (resp. right) eigenvectors* $\{l_1(u), l_2(u)\}$ [*resp.* $\{r_1(u), r_2(u)\}$] *depending continuously on* u *at* $u = 0$ (cf. [24]).

In what follows, we always suppose that all $\lambda_i(u)$, $l_i(u)$, and $r_i(u)$ $(i = 1, \ldots, n)$ have the same regularity as $A(u)$.

By means of the left eigenvectors, (2.1.1) can be rewritten in the following **characteristic form**:

$$l_i(u)\frac{du}{d_i t} = 0 \quad (i = 1, \ldots, n), \tag{2.1.16}$$

where

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.1.17}$$

denotes the directional derivative with respect to t along the ith characteristic

$$\frac{dx}{dt} = \lambda_i(u). \tag{2.1.18}$$

For each $i = 1, \ldots, n$, the ith equation of system (2.1.16) of characteristic form consists of only the directional derivatives of the unknown vector function $u = u(t, x)$ along the ith characteristic.

Remark 2.1.1 *For the inhomogeneous quasilinear hyperbolic system*

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u), \tag{2.1.19}$$

where $B(u) = (b_1(u), \ldots, b_n(u))^T$ *is a given vector function with suitably smooth elements, its hyperbolicity is defined in the same way as for the homogeneous system* (2.1.1).

2.2 Invariance Under a Smooth Invertible Transformation of Unknown Variables

The hyperbolicity of system (2.1.1) is invariant under any smooth invertible transformation of unknown variables.

In fact, we suppose that $u = u(\bar{u})$ is a suitably smooth invertible transformation, i.e.,

$$\det\left(\frac{\partial u}{\partial \bar{u}}\right) \neq 0. \tag{2.2.1}$$

Substituting $u = u(\bar{u})$ into (2.1.1), we get

$$\frac{\partial \bar{u}}{\partial t} + \bar{A}(\bar{u})\frac{\partial \bar{u}}{\partial x} = 0, \tag{2.2.2}$$

where

$$\bar{A}(\bar{u}) = \left(\frac{\partial u}{\partial \bar{u}}\right)^{-1} A(u(\bar{u}))\left(\frac{\partial u}{\partial \bar{u}}\right) \tag{2.2.3}$$

satisfies the same property as $A(u)$. More precisely, the eigenvalues $\bar{\lambda}_i(\bar{u})$ $(i = 1, \ldots, n)$ and the left (resp. right) eigenvectors $\bar{l}_i(\bar{u})$ [resp. $\bar{r}_i(\bar{u})$] $(i = 1, \ldots, n)$ of $\bar{A}(\bar{u})$ verify that

$$\bar{\lambda}_i(\bar{u}) = \lambda_i(u(\bar{u})) \quad (i = 1, \ldots, n), \tag{2.2.4}$$

$$\bar{l}_i(\bar{u})\left(\frac{\partial u}{\partial \bar{u}}\right)^{-1} // l_i(u(\bar{u})) \quad (i = 1, \ldots, n), \tag{2.2.5}$$

and

$$\left(\frac{\partial u}{\partial \bar{u}}\right)\bar{r}_i(\bar{u}) // r_i(u(\bar{u})) \quad (i = 1, \ldots, n). \tag{2.2.6}$$

Without loss of generality, we may take

$$\bar{l}_i(\bar{u}) = l_i(u(\bar{u}))\left(\frac{\partial u}{\partial \bar{u}}\right) \quad (i = 1, \ldots, n) \tag{2.2.7}$$

and

$$\bar{r}_i(\bar{u}) = \left(\frac{\partial u}{\partial \bar{u}}\right)^{-1} r_i(u(\bar{u})) \quad (i = 1, \ldots, n). \tag{2.2.8}$$

Then, noting (2.1.4) and (2.2.1), we get

$$\det\left|\bar{l}_{ij}(\bar{u})\right| \neq 0 \qquad [\text{resp. } \det|\bar{r}_{ij}(\bar{u})| \neq 0]. \tag{2.2.9}$$

Moreover, by (2.1.5), we have

$$\bar{l}_i(\bar{u})\bar{r}_j(\bar{u}) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n). \tag{2.2.10}$$

By Remark 2.1.1, the hyperbolicity of inhomogeneous quasilinear system (2.1.19) is also invariant under any invertible smooth transformation of unknown variables.

2.3 Genuine Nonlinearity and Linear Degeneracy

For system (2.1.1), we suppose that on the domain under consideration, $A(u) \in C^1$.

Definition 2.3.1 *For any given simple characteristic $\lambda_i(u)$ ($i \in \{1, \ldots, n\}$), if, on the domain under consideration, we have*

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0, \tag{2.3.1}$$

then $\lambda_i(u)$ is said to be **genuinely nonlinear** (**GN**) *in the sense of Lax. However, if, on the domain under consideration, we have*

$$\nabla\lambda_i(u) \cdot r_i(u) \equiv 0, \tag{2.3.2}$$

then $\lambda_i(u)$ is said to be **linearly degenerate** (**LD**) *in the sense of Lax* (cf. [42])*.*

Remark 2.3.1 *The simple characteristic $\lambda_i(u)$ is* **GN** *means that $\lambda_i(u)$ is strictly monotone along any given ith characteristic trajectory $u = u^{(i)}(s)$, defined by*

$$\frac{du}{ds} = r_i(u). \tag{2.3.3}$$

On the other hand, saying $\lambda_i(u)$ is **LD** *means that $\lambda_i(u)$ is a constant along any given ith characteristic trajectory $u = u^{(i)}(s)$.*

Definition 2.3.2 *For any characteristic $\lambda(u)$ with constant multiplicity p, if, on the domain under consideration, we have*

$$\nabla\lambda(u) \cdot r_i(u) \equiv 0 \quad (i = 1, \ldots, p), \tag{2.3.4}$$

*where $\{r_i(u)\ (i = 1, \ldots, p)\}$ is a complete set of linearly independent right eigenvectors corresponding to $\lambda(u)$, then $\lambda(u)$ is said to be **linearly degenerate** (**LD**).*

Definition 2.3.3 *For a strictly hyperbolic system* (2.1.1)*, if, on the domain under consideration, all the characteristics $\lambda_i(u)$ ($i = 1, \ldots, n$) are* **GN** [*resp.* ***LD***]*, then the system is said to be* **GN** [*resp.* ***LD***]*.*

Definition 2.3.4 *For a nonstrictly hyperbolic system* (2.1.1) *with characteristics with constant multiplicity, if, on the domain under consideration, all the simple characteristics and all the multiple characteristics with constant multiplicity are **LD**, then the system is said to be **LD**.*

Moreover, noting (2.2.4) and (2.2.8), it is easy to see that the genuine nonlinearity or linear degeneracy of a characteristic is invariant under any invertible smooth transformation of unknown variables.

2.4 Normalized Coordinates

By Section 2.2, the hyperbolicity of a system is invariant under any smooth invertible transformation of unknown variables. In this section, we want to find new coordinates $\widetilde{u} = \widetilde{u}(u)$ $[\widetilde{u}(0) = 0]$ such that in the $\widetilde{u}$-space the characteristic trajectories passing through $\widetilde{u} = 0$ can be expressed in a simpler way.

2.4.1 Normalized Coordinates for Strictly Hyperbolic Systems

Lemma 2.4.1 *Suppose that in a neighbourhood of $u = 0$, system* (2.1.1) *is strictly hyperbolic and $A(u) \in C^k$, where $k \geq 1$ is an integer. Then there exists an invertible C^{k+1} transformation $u = u(\widetilde{u})$ $[u(0) = 0]$ such that in the $\widetilde{u}$-space, for each $i = 1, \ldots, n$, the ith characteristic trajectory passing through $\widetilde{u} = 0$ coincides with the $\widetilde{u}_i$-axis at least for $|\widetilde{u}_i|$ small, namely,*

$$\widetilde{r}_i(\widetilde{u}_i e_i) // e_i, \quad \forall |\widetilde{u}_i| \text{ small} \quad (i = 1, \ldots, n), \tag{2.4.1}$$

where $\widetilde{r}_i(\widetilde{u})$ denotes the ith right eigenvector corresponding to $\lambda_i(u(\widetilde{u}))$ in the $\widetilde{u}$-space and $e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0)^T$ (see [50] *or* [75]*).*

Proof. Let $u^{(1)} = \hat{u}^{(1)}(\widetilde{u}_1)$ be the first characteristic trajectory passing through the origin $u = 0$, namely,

$$\begin{cases} \dfrac{du^{(1)}}{d\widetilde{u}_1} = r_1(u^{(1)}), \\ \widetilde{u}_1 = 0 : u^{(1)} = 0; \end{cases} \tag{2.4.2}$$

let $u^{(2)} = u^{(2)}(u^{(1)}, \widetilde{u}_2) = \hat{u}^{(2)}(\widetilde{u}_1, \widetilde{u}_2)$ be the second characteristic trajectory passing through $u = u^{(1)}$, namely,

$$\begin{cases} \dfrac{du^{(2)}}{d\widetilde{u}_2} = r_2(u^{(2)}), \\ \widetilde{u}_2 = 0 : u^{(2)} = u^{(1)}; \end{cases}$$

$$\vdots \tag{2.4.3}$$

let $u^{(n)} = u^{(n)}(u^{(n-1)}, \widetilde{u}_n) = \hat{u}^{(n)}(\widetilde{u}_1, \ldots, \widetilde{u}_{n-1}, \widetilde{u}_n)$ be the nth characteristic trajectory passing through $u = u^{(n-1)}$, namely,

$$\begin{cases} \dfrac{du^{(n)}}{d\widetilde{u}_n} = r_n(u^{(n)}), \\ \widetilde{u}_n = 0 : u^{(n)} = u^{(n-1)}. \end{cases} \tag{2.4.4}$$

Then $u = u(\widetilde{u}) = \hat{u}^{(n)}(\widetilde{u}_1, \ldots, \widetilde{u}_{n-1}, \widetilde{u}_n)$ must be a desired invertible C^{k+1} transformation.

In fact, according to (2.4.2)–(2.4.4), we have

$$\hat{u}^{(n)}(0) = \hat{u}^{(n-1)}(0) = \cdots = \hat{u}^{(1)}(0) = 0 \tag{2.4.5}$$

and

$$\frac{\partial u}{\partial \widetilde{u}_i}(0) = \frac{\partial \hat{u}^{(n)}}{\partial \widetilde{u}_i}(0) = \frac{\partial \hat{u}^{(n-1)}}{\partial \widetilde{u}_i}(0) = \cdots = \frac{\partial \hat{u}^{(i)}}{\partial \widetilde{u}_i}(0) = r_i(0) \quad (i = 1, \ldots, n). \tag{2.4.6}$$

Noting (2.1.4), we have

$$\det\left(\frac{\partial u}{\partial \widetilde{u}}(0)\right) \neq 0. \tag{2.4.7}$$

Then, by continuity, $u = u(\widetilde{u}) = \hat{u}^{(n)}(\widetilde{u}_1, \ldots, \widetilde{u}_{n-1}, \widetilde{u}_n)$ is invertible in a neighbourhood $\widetilde{u} = 0$.

Moreover, similarly to (2.4.6), for small $|\widetilde{u}_i|$ $(i = 1, \ldots, n)$, we have

$$\frac{\partial u}{\partial \widetilde{u}_i}(\widetilde{u}_i e_i) = \frac{\partial \hat{u}^{(n)}}{\partial \widetilde{u}_i}(\widetilde{u}_i e_i) = \cdots = \frac{\partial \hat{u}^{(i)}}{\partial \widetilde{u}_i}(\widetilde{u}_i e_i) = r_i(\widetilde{u}_i e_i) \quad (i = 1, \ldots, n). \tag{2.4.8}$$

Then

$$\left(\frac{\partial u}{\partial \widetilde{u}}(\widetilde{u}_i e_i)\right)^{-1} r_i(\widetilde{u}_i e_i) = e_i \quad (i = 1, \ldots, n). \tag{2.4.9}$$

Hence, noting (2.2.6), we get (2.4.1).

Definition 2.4.1 *The transformation given by Lemma 2.4.1 is said to be a* **normalized transformation**, *and the corresponding variables* $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$ *are called* **normalized variables** *or* **normalized coordinates** (cf. [50], [75]).

Remark 2.4.1 *For any strictly hyperbolic system* (2.1.1), *normalized coordinates are not unique. In fact, if* $u = (u_1, \ldots, u_n)$ *are normalized coordinates, then any smooth invertible transformation of the following form:*

$$\widetilde{u}_i = g_i(u_i) \quad (i = 1, \ldots, n) \tag{2.4.10}$$

with $g_i(0) = 0$ *and* $g_i'(0) \neq 0$ $(i = 1, \ldots, n)$ *gives other normalized coordinates* $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$.

According to Remark 2.4.1, we can choose suitable normalized coordinates $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$ such that

$$\widetilde{r}_i(\widetilde{u}_i e_i) \equiv e_i, \quad \forall |\widetilde{u}_i| \text{ small} \quad (i = 1, \ldots, n). \tag{2.4.11}$$

2.4.2 Normalized Coordinates for Nonstrictly Hyperbolic Systems with Characteristics with Constant Multiplicity

In this section we consider the nonstrictly hyperbolic system (2.1.1) with characteristics with constant multiplicity. Without loss of generality, we suppose that (2.1.13) holds.

First, we give the following (see [7], [25]).

Lemma 2.4.2 *For the quasilinear hyperbolic system of conservation laws with characteristics with constant multiplicity*

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{2.4.12}$$

where $f(u) = (f_1(u), \dots, f_n(u))^T \in C^2$*, the eigenvalue* $\lambda(u)$ *with constant multiplicity* p (> 1) *must be linearly degenerate, i.e.,* (2.3.4) *holds. Moreover, the right eigenvectors* $r_i(u)$ $(i = 1, \dots, p)$ $(p > 1)$ *corresponding to the multiple eigenvalue* $\lambda(u)$ *satisfy the following completely integrable condition:*

$$[r_i(u), r_j(u)] \in \mathrm{span}\{r_1(u), \dots, r_p(u)\}, \quad \forall i, j = 1, \dots, p, \tag{2.4.13}$$

where $[r_1, r_2] = (r_1 \cdot \nabla) r_2 - (r_2 \cdot \nabla) r_1$ *is the Poisson's bracket, and* $\mathrm{span}\{r_1(u), \dots, r_p(u)\}$ *stands for the linear space generated by* $r_1(u), \dots, r_p(u)$.

Similarly to Lemma 2.4.1, we have

Lemma 2.4.3 *Suppose that in a neighbourhood of* $u = 0$, (2.1.1) *is a nonstrictly hyperbolic system with characteristics with constant multiplicity. Also suppose that* (2.1.13) *holds, and* $A(u) \in C^k$*, where* $k \geq 1$ *is an integer. Suppose, furthermore, that the right eigenvectors* $r_i(u)$ $(i = 1, \dots, p)$ $(p > 1)$ *corresponding to the multiple eigenvalue* $\lambda(u)$ *satisfy the completely integrable condition* (2.4.13)*. Then there exists an invertible* C^{k+1} *transformation* $u = u(\widetilde{u})$ $[u(0) = 0]$ *such that in the* $\widetilde{u}$*-space*

$$\widetilde{r}_i\left(\sum_{h=1}^{p} \widetilde{u}_h e_h\right) // e_i \quad (i = 1, \dots, p), \quad \forall |\widetilde{u}_h| \ small \quad (h = 1, \dots, p), \tag{2.4.14}$$

and

$$\widetilde{r}_j(\widetilde{u}_j e_j) // e_j, \quad \forall |\widetilde{u}_j| \ small \quad (j = p+1, \dots, n) \tag{2.4.15}$$

(cf. [58])*.*

Thus, for the nonstrictly hyperbolic system with characteristics with constant multiplicity, we give the following.

Definition 2.4.2 *For the nonstrictly hyperbolic system with characteristics with constant multiplicity* (2.1.1)*, if there exists a smooth invertible transformation* $u = u(\widetilde{u})$ $[u(0) = 0]$ *such that in the* $\widetilde{u}$*-space*

$$\widetilde{r}_i \left(\sum_{h=1}^{p} \widetilde{u}_h e_h \right) // e_i \quad (i = 1, \ldots, p), \quad \forall |\widetilde{u}_h| \text{ small} \quad (h = 1, \ldots, p), \tag{2.4.16}$$

and

$$\widetilde{r}_j(\widetilde{u}_j e_j) // e_j, \quad \forall |\widetilde{u}_j| \text{ small} \quad (j = p+1, \ldots, n), \tag{2.4.17}$$

then the transformation is said to be a **normalized transformation***, and the corresponding variables* $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$ *are called* **normalized variables** *or* **normalized coordinates** (cf. [58])*.*

According to Lemmas 2.4.2 and 2.4.3, we have

Remark 2.4.2 *Any given quasilinear hyperbolic system of conservation laws with characteristics with constant multiplicity possesses normalized coordinates.*

Remark 2.4.3 *Similarly to Remark 2.4.1, the normalized coordinates are not unique. In fact, they are invariant under the following smooth invertible transformation:*

$$u_i = g_i(\widetilde{u}_1, \ldots, \widetilde{u}_p) \quad (i = 1, \ldots, p) \tag{2.4.18}$$

and

$$u_j = g_j(\widetilde{u}_j) \quad (j = p+1, \ldots, n), \tag{2.4.19}$$

with $g_i(0) = 0 (i = 1, \ldots, n)$ *and* $\partial(g_1, \ldots, g_n)/\partial(\widetilde{u}_1, \ldots, \widetilde{u}_n)(0) \neq 0$.

2.4.3 Generalized Normalized Coordinates for General Hyperbolic Systems

For the general hyperbolic system (2.1.1), similarly to Lemma 2.4.1, we have

Lemma 2.4.4 *Suppose that in a neighbourhood of* $u = 0$*, system* (2.1.1) *is hyperbolic and* $A(u) \in C^k$*, where* $k \geq 1$ *is an integer. Then for any given complete set of right eigenvectors* $r_i(u)$ $(i = 1, \ldots, n)$*, there exists an invertible* C^{k+1} *transformation* $u = u(\widetilde{u})$ $[u(0) = 0]$ *such that in the* $\widetilde{u}$*-space, for each* $i = 1, \ldots, n$*, the* i*th characteristic trajectory passing through* $\widetilde{u} = 0$ *coincides with the* $\widetilde{u}_i$*-axis at least for* $|\widetilde{u}_i|$ *small, namely,*

$$\widetilde{r}_i(\widetilde{u}_i e_i) // e_i, \quad \forall |\widetilde{u}_i| \text{ small} \quad (i = 1, \ldots, n), \tag{2.4.20}$$

where $\widetilde{r}_i(\widetilde{u})$, *given by* (2.2.8), *denotes the ith right eigenvector corresponding to* $\lambda_i(u(\widetilde{u}))$ *in the* $\widetilde{u}$*-space.*

The transformation given by Lemma 2.4.4 is called a **generalized normalized transformation**, and the corresponding variables $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_n)^T$ said to be **generalized normalized variables** or **generalized normalized coordinates** (cf. [90]).

Remark 2.4.4 *The generalized normalized coordinates depend on the special choice of right eigenvectors* $r_i(u)$ $(i = 1, \ldots, n)$.

Obviously, for any strictly hyperbolic system, generalized normalized coordinates must be normalized coordinates.

2.5 Weak Linear Degeneracy

The genuine nonlinearity and the linear degeneracy discussed in Section 2.3 are only two extreme cases. In applications, some characteristics may be neither GN nor LD (see Section 1.3). In such a case, it is necessary to introduce a new concept—the weak linear degeneracy (cf. [50], [75]).

2.5.1 Weak Linear Degeneracy for Strictly Hyperbolic Systems

Definition 2.5.1 *For the strictly hyperbolic system* (2.1.1), *the ith characteristic* $\lambda_i(u)$ *is* **weakly linearly degenerate** (**WLD**) *with respect to* $u = u_0$ *if, along the ith characteristic trajectory* $u = u^{(i)}(s)$ *passing through* $u = u_0$, *defined by*

$$\begin{cases} \dfrac{du}{ds} = r_i(u), \\ s = 0: \ \ u = u_0, \end{cases} \tag{2.5.1}$$

we have

$$\nabla\lambda_i(u) r_i(u) \equiv 0, \quad \forall |u - u_0| \ \textit{small}, \tag{2.5.2}$$

namely,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(u_0), \quad \forall |s| \ \textit{small}. \tag{2.5.3}$$

Obviously, any LD characteristic in a neighbourhood of $u = u_0$ must be WLD with respect to $u = u_0$ and any GN characteristic in a neighbourhood of $u = u_0$ is not WLD with respect to $u = u_0$.

For simplicity and without loss of generality, we may take $u_0 = 0$ and simply use WLD instead of "WLD with respect to $u = 0$" in what follows.

For any given $i = 1, \ldots, n$, the property that $\lambda_i(u)$ is WLD or not is invariant under any smooth invertible transformation $u = u(\bar{u})$ $[u(0) = 0]$.

According to Definition 2.5.1, for any $i = 1, \ldots, n$, if $\lambda_i(u)$ is not WLD, either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^k \lambda_i(u^{(i)}(s))}{ds^k}\right|_{s=0} = 0 \ (k = 1, \ldots, \alpha_i), \text{ but } \left.\frac{d^{\alpha_i+1} \lambda_i(u^{(i)}(s))}{ds^{\alpha_i+1}}\right|_{s=0} \neq 0 \tag{2.5.4}$$

or

$$\left.\frac{d^k \lambda_i(u^{(i)}(s))}{ds^k}\right|_{s=0} = 0 \ (k = 1, 2, \ldots), \text{ but } \lambda_i(u^{(i)}(s)) \not\equiv \lambda_i(0),$$
$$\text{denoted by } \alpha_i = +\infty, \tag{2.5.5}$$

where $u = u^{(i)}(s)$ is defined by (2.5.1). α_i is called the **non-WLD index of the characteristic** $\lambda_i(u)$, which is also invariant under any smooth invertible transformation $u = u(\bar{u})$ $[u(0) = 0]$.

In particular, if $\alpha_i = 0$, then in a neighbourhood of $u = 0$, $\lambda_i(u)$ is GN. Moreover, when α_i increases, $\lambda_i(u)$ is closer and closer to the WLD case.

Definition 2.5.2 *The strictly hyperbolic system* (2.1.1) *is said to be* **WLD** *if all characteristics* $\lambda_1(u), \ldots, \lambda_n(u)$ *are WLD.*

Therefore, if a strictly hyperbolic system (2.1.1) is not WLD, then there exists a nonempty set $J \subseteq \{1, \ldots, n\}$ such that $\lambda_i(u)$ is not WLD if and only if $i \in J$.

Let

$$\alpha = \min\{\alpha_i | i \in J\} \tag{2.5.6}$$

be the **non-WLD index of the strictly hyperbolic system**, where α_i is the non-WLD index of $\lambda_i(u)$ for each $i \in J$. α is either a nonnegative integer or $+\infty$.

Let

$$J_1 = \{i | i \in J, \alpha_i = \alpha\}. \tag{2.5.7}$$

When $\alpha = 0$, then for every $i \in J_1$, $\lambda_i(u)$ is GN in a neighbourhood of $u = 0$. Furthermore, when α increases, system (2.1.1) is closer and closer to a WLD system.

Obviously, in normalized coordinates, $\lambda_i(u)$ is WLD if and only if

$$\lambda_i(u_i e_i) \equiv \lambda_i(0), \quad \forall |u_i| \text{ small.} \tag{2.5.8}$$

Hence, if $\lambda_i(u)$ is not WLD, by (2.5.4)–(2.5.5), we have

$$\left.\frac{d^k\lambda_i(u_ie_i)}{du_i^k}\right|_{u_i=0} = 0 \ (k=1,\ldots,\alpha_i), \quad \text{but} \quad \left.\frac{d^{\alpha_i+1}\lambda_i(u_ie_i)}{du_i^{\alpha_i+1}}\right|_{u_i=0} \neq 0 \tag{2.5.9}$$

or

$$\left.\frac{d^k\lambda_i(u_ie_i)}{du_i^k}\right|_{u_i=0} = 0 \ (k=1,2,\ldots), \quad \text{but} \quad \lambda_i(u_ie_i) \not\equiv \lambda_i(0). \tag{2.5.10}$$

2.5.2 Weak Linear Degeneracy for Nonstrictly Hyperbolic Systems with Characteristics with Constant Multiplicity

Definition 2.5.3 *Suppose that there exist normalized coordinates for a nonstrictly hyperbolic system* (2.1.1) *with characteristics with constant multiplicity, say* (2.1.13) *holds. The ith characteristic $\lambda_i(u)$ is* **WLD** *if, in normalized coordinates, when $i \in \{1,\ldots,p\}$,*

$$\lambda_i\left(\sum_{h=1}^{p} u_he_h\right) \equiv \lambda(0), \quad \forall |u_h| \ \textit{small} \quad (h=1,\ldots,p), \tag{2.5.11}$$

whereas, when $i \in \{p+1,\ldots,n\}$,

$$\lambda_i(u_ie_i) \equiv \lambda_i(0), \quad \forall\ |u_i| \ \textit{small}. \tag{2.5.12}$$

If all characteristics $\lambda_i(u)$ $(i=1,\ldots,n)$ are WLD, system (2.1.1) *is said to be* **WLD**.

Remark 2.5.1 *By Lemma 2.4.2, for any hyperbolic system of conservation laws* (2.4.12) *with characteristics with constant multiplicity, the p-multiple characteristic $\lambda(u)$ $(p>1)$ must be LD, then WLD.*

2.5.3 Weak Linear Degeneracy for General Hyperbolic Systems

For a general hyperbolic system (2.1.1), if $\lambda_i(u)$ is a simple eigenvalue, we say that $\lambda_i(u)$ is WLD if, along the ith characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by (2.5.1), we have (2.5.2) or (2.5.3).

Obviously, in generalized normalized coordinates $u = (u_1, \ldots, u_n)$, a simple characteristic $\lambda_i(u)$ is WLD if and only if

$$\lambda_i(u_i e_i) \equiv \lambda_i(0), \quad \forall |u_i| \text{ small.} \tag{2.5.13}$$

If a simple characteristic $\lambda_i(u)$ is not WLD, then either there exists an integer $\alpha_i \geq 0$ such that

$$\left.\frac{d^k \lambda_i(u_i e_i)}{du_i^k}\right|_{u_i=0} = 0 \ (k = 1, \ldots, \alpha_i), \quad \text{but} \quad \left.\frac{d^{\alpha_i+1} \lambda_i(u_i e_i)}{du_i^{\alpha_i+1}}\right|_{u_i=0} \neq 0 \tag{2.5.14}$$

or

$$\left.\frac{d^k \lambda_i(u_i e_i)}{du_i^k}\right|_{u_i=0} = 0 \, (k = 1, 2, \ldots), \quad \text{but} \quad \lambda_i(u_i e_i) \not\equiv \lambda_i(0),$$
$$\text{denoted by } \alpha_i = +\infty. \tag{2.5.15}$$

2.6 Decomposition of Waves

In this section we assume that $A(u) \in C^2$.

2.6.1 Formulas on the Decomposition of Waves

Let

$$v_i = l_i(u)u \quad (i = 1, \ldots, n) \tag{2.6.1}$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \ldots, n). \tag{2.6.2}$$

By (2.1.5), we have

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.6.3}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.6.4}$$

For the general quasilinear hyperbolic system (2.1.1), using (2.1.16), we have (cf. [50], [75])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \ldots, n), \tag{2.6.5}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{2.6.6}$$

Obviously,

$$\beta_{iji}(u) \equiv 0, \quad \forall i, j. \tag{2.6.7}$$

Noting (2.6.4) and (2.6.5), we have

$$\begin{aligned} d[v_i(dx - \lambda_i(u)dt)] &= \left[\frac{\partial v_i}{\partial t} + \frac{\partial(\lambda_i(u)v_i)}{\partial x}\right] dt \wedge dx \\ &= \left[\frac{dv_i}{d_i t} + \nabla\lambda_i(u) u_x v_i\right] dt \wedge dx \\ &= \left[\sum_{j,k=1}^{n} B_{ijk}(u) v_j w_k\right] dt \wedge dx \quad (i = 1, \ldots, n), \end{aligned} \tag{2.6.8}$$

where

$$B_{ijk}(u) = \beta_{ijk}(u) + \nabla\lambda_i(u) r_k(u) \delta_{ij}. \tag{2.6.9}$$

By (2.6.7), it is easy to see that

$$B_{iji}(u) \equiv 0, \quad \forall j \neq i, \tag{2.6.10}$$

and

$$B_{iii}(u) = \nabla\lambda_i(u) r_i(u), \quad \forall i. \tag{2.6.11}$$

In the corresponding generalized normalized coordinates, noting (2.4.20), it follows from (2.6.6) and (2.6.9) that

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small} \quad (i, j = 1, \ldots, n), \tag{2.6.12}$$

$$B_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \quad \forall j \neq i. \tag{2.6.13}$$

On the other hand, we have (cf. [33], [38], [50], [75])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \ldots, n), \tag{2.6.14}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u) \delta_{ik} + (j|k)\}, \tag{2.6.15}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms,

$$\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i, \tag{2.6.16}$$

and

$$\gamma_{iii}(u) = -\nabla\lambda_i(u) r_i(u) \quad (i = 1, \ldots, n). \tag{2.6.17}$$

Noting (2.6.4) and (2.6.14), we have

$$
\begin{aligned}
d[w_i(dx - \lambda_i(u)dt)] &= \left[\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x}\right] dt \wedge dx \\
&= \left[\frac{dw_i}{d_i t} + \nabla\lambda_i(u)u_x w_i\right] dt \wedge dx \\
&= \left[\sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k\right] dt \wedge dx \quad (i = 1, \ldots, n),
\end{aligned}
\tag{2.6.18}
$$

where

$$
\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u))l_i(u)[\nabla r_k(u)r_j(u) - \nabla r_j(u)r_k(u)]. \tag{2.6.19}
$$

Obviously,

$$
\Gamma_{ijj}(u) \equiv 0 \quad (i, j = 1, \ldots, n). \tag{2.6.20}
$$

Using Stokes' formula, by (2.6.8) and (2.6.18), it is easy to prove the following (see [32], [33]).

Lemma 2.6.1 *Suppose that $u = u(t,x)$ is a C^1 solution to system* (2.1.1), *τ_1 and τ_2 are two C^1 arcs that are never tangent to the ith characteristic direction, and $\mathbb{D}$ is the domain bounded by τ_1, τ_2 and two ith characteristic curves L_i^- and L_i^+. Then*

$$
\int_{\tau_1} |v_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |v_i(dx - \lambda_i(u)dt)| + \iint\limits_{\mathbb{D}} \left|\sum_{j,k=1}^{n} B_{ijk}(u)v_j w_k\right| dtdx \tag{2.6.21}
$$

and

$$
\int_{\tau_1} |w_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |w_i(dx - \lambda_i(u)dt)| + \iint\limits_{\mathbb{D}} \left|\sum_{\substack{j,k=1\\ j\neq k}}^{n} \Gamma_{ijk}(u)w_j w_k\right| dtdx. \tag{2.6.22}
$$

Remark 2.6.1 *In order to get* (2.6.14), *we have formally used the second-order derivatives of u. However, it is easy to see that, by means of a difference technique, we can rigorously derive the integral equation corresponding to* (2.6.14). *Essentially, we only need this integral equation in what follows. For convenience, we still use the form of* (2.6.14).

Remark 2.6.2 *Noting* (2.2.7), *we see that* w_i $(i = 1, \ldots, n)$ *are invariant under any smooth invertible transformation* $u = u(\widetilde{u})$.

We now consider the special case where system (2.1.1) is strictly hyperbolic.

Obviously, if $\lambda_i(u)$ is LD, for any given u on the domain under consideration, from (2.6.11) and (2.6.17) we have

$$B_{iii}(u) \equiv 0 \tag{2.6.23}$$

and

$$\gamma_{iii}(u) \equiv 0. \tag{2.6.24}$$

In normalized coordinates, noting (2.4.1), it follows from (2.6.6) and (2.6.9) that

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small} \quad (i, j = 1, \ldots, n), \tag{2.6.25}$$

$$B_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small}, \quad \forall j \neq i. \tag{2.6.26}$$

Furthermore, if $\lambda_i(u)$ is WLD, it follows from (2.6.11) and (2.6.17) that

$$B_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small}, \tag{2.6.27}$$

and

$$\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small}. \tag{2.6.28}$$

We next consider the nonstrictly hyperbolic system with characteristics with constant multiplicity.

Without loss of generality, we still assume that the nonstrictly hyperbolic system (2.1.1) satisfies (2.1.13).

It follows from (2.6.6), (2.6.9), (2.6.15), and (2.6.19) that

$$\beta_{ijk}(u) \equiv 0, \quad \forall i, k \in \{1, \ldots, p\}, \quad \forall j, \tag{2.6.29}$$

$$B_{ijk}(u) \equiv 0, \quad \forall i, k \in \{1, \ldots, p\}, \quad \forall j \neq i, \tag{2.6.30}$$

$$\gamma_{ijk}(u) \equiv 0, \quad \forall i \in \{p+1, \ldots, n\}, \quad \forall j, k \in \{1, \ldots, p\}, \tag{2.6.31}$$

and

$$\varGamma_{ijk}(u) \equiv 0, \quad \forall j, k \in \{1, \ldots, p\}. \tag{2.6.32}$$

If the multiple characteristic $\lambda(u)$ is LD, we have

$$B_{ijk}(u) \equiv 0, \quad \forall i, k \in \{1, \ldots, p\}, \quad \forall j, \tag{2.6.33}$$

and

$$\gamma_{ijk}(u) \equiv 0, \quad \forall j, k \in \{1, \ldots, p\}, \quad \forall i, \tag{2.6.34}$$

whereas, if a simple characteristic $\lambda_i(u)$ $(i \in \{p+1, \ldots, n\})$ is LD, we still have (2.6.23)–(2.6.24).

In normalized coordinates, noting (2.4.16)–(2.4.17), it follows from (2.6.6) and (2.6.9) that

$$\beta_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \quad \forall j,k \in \{1,\dots,p\}, \quad \forall |u_h| \text{ small} \quad (h=1,\dots,p), \tag{2.6.35}$$

$$\beta_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small} \quad (j=p+1,\dots,n), \tag{2.6.36}$$

$$B_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \ \ \forall j,k \in \{1,\dots,p\} \text{ and } j \neq i, \ \ \forall |u_h| \text{ small } (h=1,\dots,p), \tag{2.6.37}$$

$$B_{ijj}(u_j e_j) \equiv 0, \quad \forall |u_j| \text{ small} \quad (j=p+1,\dots,n \quad \text{and} \quad j \neq i). \tag{2.6.38}$$

Furthermore, if $\lambda_i(u)$ is WLD, we have

$$B_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \ \ \forall j,k \in \{1,\dots,p\}, \forall |u_h| \text{ small } (h=1,\dots,p) \quad \text{for } i \in \{1,\dots,p\}, \tag{2.6.39}$$

$$B_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small for } i \in \{p+1,\dots,n\}, \tag{2.6.40}$$

$$\gamma_{ijk}\left(\sum_{h=1}^{p} u_h e_h\right) \equiv 0, \ \ \forall j,k \in \{1,\dots,p\}, \ \ \forall |u_h| \text{ small } (h=1,\dots,p) \quad \text{for } i \in \{1,\dots,p\}, \tag{2.6.41}$$

and

$$\gamma_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small} \quad \text{for } i \in \{p+1,\dots,n\}. \tag{2.6.42}$$

2.6.2 Formulas on the Decomposition of Waves (Continued)

In many cases, estimating u is more convenient than estimating v. For this purpose, in this section we derive the differential equations satisfied by u_i ($i=1,\dots,n$) instead of (2.6.5).

Noting (2.6.4), by (2.1.1), we have

$$\frac{du}{d_i t} = \sum_{k=1}^{n} (\lambda_i(u) - \lambda_k(u)) w_k r_k(u). \tag{2.6.43}$$

Suppose that $u = (u_1, \ldots, u_n)^T$ are generalized normalized coordinates corresponding to the complete set of right eigenvectors $\{r_1(u), \ldots, r_n(u)\}$. We have (cf. [64])

$$\begin{aligned} \frac{du_i}{d_i t} &= \sum_{k=1}^{n}(\lambda_i(u) - \lambda_k(u)) w_k r_k^T(u) e_i \\ &= \sum_{k=1}^{n}(\lambda_i(u) - \lambda_k(u)) w_k (r_k^T(u) - r_k^T(u_k e_k)) e_i \\ &= \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k, \end{aligned} \tag{2.6.44}$$

where

$$\rho_{ijj}(u) = 0 \quad (i, j = 1, \ldots, n) \tag{2.6.45}$$

and

$$\rho_{ijk}(u) = (\lambda_i(u) - \lambda_k(u)) \int_0^1 \frac{\partial r_{ik}}{\partial u_j}(u_k e_k + \sigma(u - u_k e_k)) d\sigma, \quad j \neq k, \tag{2.6.46}$$

with

$$\rho_{iji}(u) = 0 \quad (i, j = 1, \ldots, n). \tag{2.6.47}$$

Noting (2.6.4) and (2.6.44), we have

$$\begin{aligned} d[u_i(dx - \lambda_i(u)dt)] &= \left[\frac{du_i}{d_i t} + \nabla\lambda_i(u) u_x u_i\right] dt \wedge dx \\ &= \left[\sum_{j,k=1}^{n} F_{ijk}(u) u_j w_k\right] dt \wedge dx \quad (i = 1, \ldots, n), \end{aligned} \tag{2.6.48}$$

where

$$F_{ijk}(u) = \rho_{ijk}(u) + \nabla\lambda_i(u) r_k(u) \delta_{ij}. \tag{2.6.49}$$

Noting (2.6.45) and (2.6.47), we have

$$F_{iji}(u) = 0, \quad \forall j \neq i, \tag{2.6.50}$$

$$F_{ijj}(u) = 0, \quad \forall j \neq i, \tag{2.6.51}$$

and

$$F_{iii}(u) = \nabla\lambda_i(u) r_i(u) \quad (i = 1, \ldots, n). \tag{2.6.52}$$

On the other hand, (2.6.14)–(2.6.20) still hold.

Using Stokes' formula, similarly to Lemma 2.6.1, by (2.6.48) we have

Lemma 2.6.2 *Suppose that $u = u(t, x)$ is a C^1 solution to system* (2.1.1), *τ_1 and τ_2 are two C^1 arcs which are never tangent to the ith characteristic* *direction, and* $\mathbb{D}$ *is the domain bounded by* τ_1, τ_2 *and two* i*th characteristic curves* L_i^- *and* L_i^+. *Then*

$$\int_{\tau_1} |u_i(dx - \lambda_i(u)dt)| \le \int_{\tau_2} |u_i(dx - \lambda_i(u)dt)| + \iint_{\mathbb{D}} \left| \sum_{j,k=1}^{n} F_{ijk}(u)u_j w_k \right| dtdx. \tag{2.6.53}$$

For the strictly hyperbolic system (2.1.1), in normalized coordinates, if $\lambda_i(u)$ is WLD, it follows from (2.6.52) that

$$F_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \text{ small.} \tag{2.6.54}$$

For the nonstrictly hyperbolic system (2.1.1) with characteristics with constant multiplicity [say, (2.1.13) holds], we suppose that $u = (u_1, \ldots, u_n)$ are normalized coordinates, namely, (2.4.16)–(2.4.17) hold for u.

For $i = 1, \ldots, p$, noting (2.4.15), similarly to (2.6.44), we have

$$\begin{aligned}\frac{du_i}{d_i t} &= \sum_{k=1}^{n} (\lambda_i(u) - \lambda_k(u)) w_k r_k^T(u) e_i \\ &= \sum_{k=p+1}^{n} (\lambda_i(u) - \lambda_k(u)) w_k (r_k^T(u) - r_k^T(u_k e_k)) e_i \\ &= \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k, \end{aligned} \tag{2.6.55}$$

where

$$\rho_{ijk}(u) \equiv 0 \quad (k = 1, \ldots, p), \tag{2.6.56}$$

$$\rho_{ijj}(u) \equiv 0 \quad (j = p+1, \ldots, n), \tag{2.6.57}$$

and when $j \ne k$ $(k = p+1, \ldots, n)$, the $\rho_{ijk}(u)$ are defined by (2.6.46).

For $i = p+1, \ldots, n$, noting (2.4.16)–(2.4.17), similarly to (2.6.44), we have

$$\begin{aligned}\frac{du_i}{d_i t} &= \sum_{k=1}^{n} (\lambda_i(u) - \lambda_k(u)) w_k r_k^T(u) e_i \\ &= \sum_{k=1}^{p} (\lambda_i(u) - \lambda_k(u)) w_k \left(r_k^T(u) - r_k^T\left(\sum_{h=1}^{p} u_h e_h \right) \right) e_i \\ &\quad + \sum_{k=p+1}^{n} (\lambda_i(u) - \lambda_k(u)) w_k (r_k^T(u) - r_k^T(u_k e_k)) e_i \\ &= \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k, \end{aligned} \tag{2.6.58}$$

where

$$\rho_{ijk}(u) \equiv 0 \quad (j, k = 1, \ldots, p \quad \text{or} \quad k = i), \tag{2.6.59}$$

$$\rho_{ijj}(u) \equiv 0 \quad (j = p+1, \ldots, n), \tag{2.6.60}$$

and

$$\rho_{ijk}(u) = \begin{cases} (\lambda_i(u) - \lambda_k(u)) \displaystyle\int_0^1 \frac{\partial r_{ik}}{\partial u_j} \left(\sum_{h=1}^n u_h e_h + \sigma \left(u - \sum_{h=1}^n u_h e_h \right) \right) d\sigma \\ \qquad\qquad\qquad\qquad (k = 1, \ldots, p;\ j = p+1, \ldots, n), \\ (\lambda_i(u) - \lambda_k(u)) \left(\displaystyle\int_0^1 \frac{\partial r_{ik}}{\partial u_j} (u_k e_k + \sigma(u - u_k e_k)) \right) d\sigma \\ \qquad\qquad\qquad\qquad (k = p+1, \ldots, n,\ j \neq k). \end{cases} \tag{2.6.61}$$

Moreover, in the present situation, we still have (2.6.48)–(2.6.49) with

$$F_{ijk}(u) \equiv 0 \quad (j, k = 1, \ldots, p \quad \text{and} \quad j \neq i), \tag{2.6.62}$$

$$F_{ijj}(u) \equiv 0 \quad (j = p+1, \ldots, n \quad \text{and} \quad j \neq i), \tag{2.6.63}$$

and (2.6.52).

If $\lambda_i(u)$ is WLD, when $i = 1, \ldots, p$, we have

$$F_{ijk} \left(\sum_{h=1}^p u_h e_h \right) \equiv 0 \quad (j, k = 1, \ldots, p), \tag{2.6.64}$$

whereas, when $i = p+1, \ldots, n$, we have

$$F_{iii}(u_i e_i) \equiv 0. \tag{2.6.65}$$

Furthermore, if $\lambda_i(u)$ is LD, when $i = 1, \ldots, p$, we have

$$F_{ijk}(u) \equiv 0 \quad (j, k = 1, \ldots, p), \tag{2.6.66}$$

whereas, when $i = p+1, \ldots, n$, we have

$$F_{iii}(u) \equiv 0. \tag{2.6.67}$$

2.7 Two Lemmas on Ordinary Differential Equations

Lemma 2.7.1 *Suppose that $z = z(t)$ is a solution on the interval $[0, T]$ to the following ordinary differential equation:*

$$\frac{dz}{dt} = a_0(t) z^2 + a_1(t) z + a_2(t), \tag{2.7.1}$$

where $T > 0$ is a given number, $a_i(t) \in C[0,T]$ $(i = 0,1,2)$, and $a_0(t) \geq 0$ for $0 \leq t \leq T$. Let

$$K = \int_0^T |a_2(t)|dt \exp\left(\int_0^T |a_1(t)|dt\right). \tag{2.7.2}$$

If

$$z(0) > K, \tag{2.7.3}$$

then

$$\int_0^T a_0(t)dt \exp\left(-\int_0^T |a_1(t)|dt\right) < (z(0) - K)^{-1}. \tag{2.7.4}$$

Lemma 2.7.2 *For system (2.7.1), suppose that $a_i(t) \in C[0,T]$ $(i = 0,1,2)$ in which $T > 0$ is a given number. Let*

$$a_0^{\pm}(t) = \max\{\pm a_0(t), 0\}, \tag{2.7.5}$$

and let K still be defined by (2.7.2). If there exists z_0 satisfying

$$z_0 \geq 0, \tag{2.7.6}$$

$$\int_0^T a_0^+(t)dt \exp\left(\int_0^T |a_1(t)|dt\right) < (z_0 + K)^{-1}, \tag{2.7.7}$$

and

$$\int_0^T a_0^-(t)dt \exp\left(\int_0^T |a_1(t)|dt\right) < K^{-1}, \tag{2.7.8}$$

then (2.7.1) admits a unique solution $z = z(t)$ on $[0,T]$ with $z(0) = z_0$ and the following estimates hold:

$$(z(T))^{-1} \geq (z_0 + K)^{-1} - \int_0^T a_0^+(t)dt \exp\left(\int_0^T |a_1(t)|dt\right) \quad \text{if } z(T) > 0; \tag{2.7.9}$$

$$|z(T)|^{-1} \geq K^{-1} - \int_0^T a_0^-(t)dt \exp\left(\int_0^T |a_1(t)|dt\right) \quad \text{if } z(T) < 0. \tag{2.7.10}$$

The proofs of these lemmas can be found in [32] and [33].

Chapter III
The Cauchy Problem

The local existence and uniqueness of the classical solution to the Cauchy problem for quasilinear hyperbolic systems is well known (see [72]). In this chapter we consider the global existence and uniqueness of the classical solution to the Cauchy problem for quasilinear hyperbolic systems with small and smooth initial data satisfying certain decaying properties.

First, we study the strictly hyperbolic system.

3.1 Necessary Condition to Guarantee the Global Existence and Uniqueness of the C^1 Solution to the Cauchy Problem for the Strictly Hyperbolic System

Consider the **Cauchy problem** for the following quasilinear strictly hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0 \tag{3.1.1}$$

with the initial data

$$t = 0: \ u = \phi(x), \tag{3.1.2}$$

where $\phi(x)$ is a C^1 vector function. Let

$$\theta = \sup_{x \in \mathbb{R}} \{(1 + |x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\}, \tag{3.1.3}$$

in which $\mu > 0$ is a constant.

Let

$$u^{(1)} = (u_1, \ldots, u_m)^T, \quad u^{(2)} = (u_{m+1}, \ldots, u_n)^T \quad (1 \leq m < n). \tag{3.1.4}$$

Li Tatsien, Wang Libin, *Global Propagation of Regular Nonlinear Hyperbolic Waves*, DOI 10.1007/b78335_3,

The coefficient matrix $A(u) = (a_{ij})_{n\times n}$ can be correspondingly rewritten as

$$A(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix}, \tag{3.1.5}$$

where $A_{11}(u)$ is an $m \times m$ matrix, $A_{22}(u)$ is an $(n-m)\times(n-m)$ matrix, etc.

Lemma 3.1.1 *Suppose that in a neighbourhood of* $u = 0$, (3.1.1) *is a strictly hyperbolic system written in normalized coordinates. Then, in a neighbourhood of* $u^{(1)} = 0$, *the reduced system*

$$\frac{\partial u^{(1)}}{\partial t} + A_{11}(u^{(1)}, 0)\frac{\partial u^{(1)}}{\partial x} = 0 \tag{3.1.6}$$

is still strictly hyperbolic and $u^{(1)} = (u_1, \dots, u_m)^T$ *are the corresponding normalized coordinates.*

Proof. Noting that in normalized coordinates,

$$A(0) = \mathrm{diag}\{\lambda_1(0), \dots, \lambda_n(0)\}, \tag{3.1.7}$$

we have

$$A_{11}(0) = \mathrm{diag}\{\lambda_1(0), \dots, \lambda_m(0)\}. \tag{3.1.8}$$

Then, by continuity, it is easy to see that in a neighbourhood of $u^{(1)} = 0$, $A_{11}(u^{(1)}, 0)$ possesses m distinct real eigenvalues

$$\bar\lambda_1(u^{(1)}), \dots, \bar\lambda_m(u^{(1)}) \tag{3.1.9}$$

with

$$\bar\lambda_1(0) = \lambda_1(0), \dots, \bar\lambda_m(0) = \lambda_m(0). \tag{3.1.10}$$

Thus, (3.1.6) is a strictly hyperbolic system in a neighbourhood of $u^{(1)} = 0$. For $i = 1, \dots, m$, let $\bar r_i(u^{(1)})$ be a right eigenvector of $A_{11}(u^{(1)}, 0)$ corresponding to $\bar\lambda_i(u^{(1)})$.

For $i = 1, \dots, n$, the ith right eigenvector $r_i(u)$ of $A(u)$ can be written as

$$r_i(u) = \begin{pmatrix} r_i^{(1)}(u) \\ r_i^{(2)}(u) \end{pmatrix}, \tag{3.1.11}$$

where

$$r_i^{(1)}(u) = (r_{1i}(u), \dots, r_{mi}(u))^T, \quad r_i^{(2)}(u) = (r_{m+1,i}(u), \dots, r_{ni}(u))^T. \tag{3.1.12}$$

Noting (2.1.3) and that $u = (u_1, \dots, u_n)^T$ are normalized coordinates, we have

$$A(u_i e_i)e_i = \lambda_i(u_i e_i)e_i, \quad \forall |u_i| \text{ small} \quad (i = 1, \dots, n). \tag{3.1.13}$$

Then we get

$$A_{11}(u_i e_i)e_i^{(1)} = \lambda_i(u_i e_i)e_i^{(1)}, \quad \forall |u_i| \text{ small} \quad (i = 1, \dots, m), \tag{3.1.14}$$

namely,

$$A_{11}(u_i e_i^{(1)}, 0)e_i^{(1)} = \lambda_i(u_i e_i^{(1)}, 0)e_i^{(1)}, \quad \forall |u_i| \text{ small} \quad (i = 1, \dots, m). \tag{3.1.15}$$

In (3.1.14) and (3.1.15), $e_i^{(1)}$ is an m-dimensional unit vector. Hence, noting (3.1.9)–(3.1.10), we get

$$\bar{\lambda}_i(u_i e_i^{(1)}) = \lambda_i(u_i e_i^{(1)}, 0), \quad \forall |u_i| \text{ small} \quad (i = 1, \dots, m), \tag{3.1.16}$$

and

$$\bar{r}_i(u_i e_i^{(1)}) // e_i^{(1)}, \quad \forall |u_i| \text{ small} \quad (i = 1, \dots, m). \tag{3.1.17}$$

Thus, $u^{(1)} = (u_1, \dots, u_m)^T$ are the corresponding normalized coordinates for the reduced system (3.1.6).

Corollary 3.1.1 *Under the assumptions of Lemma 3.1.1, it follows from* (3.1.13) (*in which we take $i = 1$*) *that*

$$a_{11}(u_1 e_1) = \lambda_1(u_1 e_1), \quad \forall |u_1| \textit{ small}, \tag{3.1.18}$$

and

$$a_{k1}(u_1 e_1) \equiv 0 \quad (k = 2, \dots, n), \quad \forall |u_1| \textit{ small}. \tag{3.1.19}$$

From Corollary 3.1.1, it is easy to get the following.

Lemma 3.1.2 *Under the assumptions of Lemma 3.1.1, for the Cauchy problem of system* (3.1.1) *with the following initial data:*

$$t = 0: \ u_1 = \varphi(x), u_2 = \dots = u_n = 0, \tag{3.1.20}$$

where $\varphi(x)$ is an arbitrarily given C^1 function with small C^0 norm and bounded C^1 norm, the solution is given by

$$u_1 = u_1(t, x), \ u_2 = \dots = u_n = 0,$$

where $u_1 = u_1(t, x)$ is the C^1 solution to the following Cauchy problem for a single equation:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \lambda_1(u_1 e_1)\dfrac{\partial u_1}{\partial x} = 0, \\ t = 0: \ u_1 = \varphi(x). \end{cases} \tag{3.1.21}$$

Theorem 3.1.1 *Suppose that in a neighbourhood of* $u = 0$*,* $A(u) \in C^1$ *and system* (3.1.1) *is strictly hyperbolic. If there exists* $\theta_0 > 0$ *so small that for any given initial data* (3.1.2) *satisfying* $\theta \in [0, \theta_0]$*, where* θ *is defined by* (3.1.3)*, the Cauchy problem* (3.1.1)–(3.1.2) *always admits a unique global* C^1 *solution* $u = u(t, x)$ *on* $t \geq 0$*, then system* (3.1.1) *must be weakly linearly degenerate* (*WLD*) (cf. [51], [52])*.*

Proof. By Lemma 2.4.1, without loss of generality, we suppose that u are normalized coordinates. In particular, we take the initial data given by (3.1.20), where $\varphi(x)$ is an arbitrarily given C^1 function satisfying $\theta \in [0, \theta_0]$. Since the Cauchy problem (3.1.21) admits a unique global C^1 solution $u_1 = u_1(t, x)$ on $t \geq 0$ if and only if

$$\frac{d\lambda_1(\varphi(x)e_1)}{dx} \geq 0, \quad \forall x \in \mathbb{R}, \tag{3.1.22}$$

under the assumptions of Theorem 3.1.1, by Lemma 3.1.2 and noting (3.1.3), it is easy to get that

$$\lambda_1(u_1 e_1) \equiv \lambda_1(0), \quad \forall |u_1| \text{ small}, \tag{3.1.23}$$

namely, $\lambda_1(u)$ is WLD. Similarly, all other characteristics are also WLD. Hence, system (3.1.1) must be WLD.

3.2 Some Uniform a Priori Estimates Independent of Normalized Coordinates and Weak Linear Degeneracy for the Strictly Hyperbolic System

In this section we assume that the initial data (3.1.2) satisfy the following decaying property:

$$\theta \overset{\text{def.}}{=} \sup_{x \in \mathbb{R}} \{(1 + |x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < +\infty, \tag{3.2.1}$$

where $\mu > 0$ is a constant.

By the strict hyperbolicity, we have

$$\lambda_1(0) < \cdots < \lambda_n(0). \tag{3.2.2}$$

Without loss of generality, we may assume that

$$\lambda_1(0) > 0. \tag{3.2.3}$$

In fact, by the following invertible linear transformation of independent variables:

$$\begin{cases} t' = t, \\ x' = x + (1 - \lambda_1(0))t, \end{cases} \tag{3.2.4}$$

we can always realize (3.2.3).

Then, by continuity, there exist positive constants δ_0 $[< \lambda_1(0)]$ and δ so small that

$$\lambda_{i+1}(u) - \lambda_i(u') \geq 2\delta_0, \ \forall |u|, |u'| \leq \delta \quad (i = 1, \ldots, n-1), \tag{3.2.5}$$

and

$$|\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \ \forall |u|, |u'| \leq \delta \quad (i = 1, \ldots, n). \tag{3.2.6}$$

For any given $T \geq 0$, let

$$D_i^T = \begin{cases} \{(t,x) | 0 \leq t \leq T, \ x \leq (\lambda_1(0) + \delta_0)t\} \quad (i = 1), \\ \{(t,x) | 0 \leq t \leq T, \ (\lambda_i(0) - \delta_0)t \leq x \leq (\lambda_i(0) + \delta_0)t\} \ (i = 2, \ldots, n-1), \\ \{(t,x) | 0 \leq t \leq T, \ x \geq (\lambda_n(0) - \delta_0)t\} \quad (i = n). \end{cases} \tag{3.2.7}$$

Obviously,

$$\bigcup_{i=1}^{n} D_i^T \subset D(T) = \{(t,x) | 0 \leq t \leq T, \ -\infty < x < \infty\}. \tag{3.2.8}$$

On any given existence domain $D(T)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.1.1) and (3.1.2), let

$$W_\infty^c(T) = \max_{i=1,\ldots,n} \sup_{(t,x) \in D(T) \setminus D_i^T} \{(1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i(t,x)|\}, \tag{3.2.9}$$

$$\widetilde{W}_1(T) = \max_{i=1,\ldots,n} \max_{j \neq i} \sup_{c_j} \int_{c_j} |w_i(t,x)| dt, \tag{3.2.10}$$

where c_j denotes any given jth characteristic on $D(T)$,

$$W_1(T) = \max_{i=1,\ldots,n} \sup_{0 \leq t \leq T} \int_{-\infty}^{\infty} |w_i(t,x)| dx, \tag{3.2.11}$$

and

$$U_\infty(T) = ||u(t,x)||_{L^\infty(D(T))}. \tag{3.2.12}$$

For the time being, we assume that on any given existence domain $D(T)$,

$$|u(t,x)| \leq \delta. \tag{3.2.13}$$

At the end of the proof of Lemma 3.2.2, we will explain that this hypothesis is reasonable.

Lemma 3.2.1 *For each $i = 1, \ldots, n$ and any given point $(t, x) \,\overline{\in}\, D_i^T$, let $c_i : \xi = \xi_i(\tau)(0 \le \tau \le t)$ be the ith characteristic passing through (t, x) and intersecting the x-axis at $(0, x_{i0})$. Then there exist positive constants d_k $(k = 1, 2, 3)$ independent of (t, x) and i such that*

$$|x - \lambda_i(0)t| \ge \delta_0 t, \tag{3.2.14}$$

$$d_1|x| \le |x - \lambda_i(0)t| \le d_2|x_{i0}| \tag{3.2.15}$$

and if $(\tau, \xi_i(\tau)) \overline{\in} D_j^T$ for some j, then

$$|\xi_i(\tau) - \lambda_j(0)\tau| \ge d_3|x_{i0}| \tag{3.2.16}$$

(cf. [64]).

Proof. According to the definition of D_i^T $(i = 1, \ldots, n)$, we get (3.2.14) immediately.

When $i \in \{2, \ldots, n-1\}$, for any given point $(t, x) \,\overline{\in}\, D_i^T$, by the definition of D_i^T, we have

$$x \ge (\lambda_i(0) + \delta_0)t \quad \text{or} \quad x \le (\lambda_i(0) - \delta_0)t. \tag{3.2.17}$$

In what follows, we prove (3.2.15)–(3.2.16) for the case $x \ge (\lambda_i(0) + \delta_0)t$. When $x \le (\lambda_i(0) - \delta_0)t$, (3.2.15)–(3.2.16) can be proved similarly.

Noting (3.2.6), for $0 \le \tau \le t$, we easily get

$$\xi_i(\tau) \ge (\lambda_i(0) + \delta_0)\tau \tag{3.2.18}$$

and

$$\left(\lambda_i(0) - \frac{\delta_0}{2}\right)\tau \le \xi_i(\tau) - x_{i0} \le \left(\lambda_i(0) + \frac{\delta_0}{2}\right)\tau. \tag{3.2.19}$$

Then, noting that $\delta_0 < \lambda_i(0)$ $(i = 1, \ldots, n)$, we have

$$\xi_i(\tau) \le \frac{2(\lambda_i(0) + \delta_0)}{\delta_0} x_{i0}, \tag{3.2.20}$$

in particular,

$$x \le \frac{2(\lambda_i(0) + \delta_0)}{\delta_0} x_{i0}. \tag{3.2.21}$$

Thus, noting that $x \ge (\lambda_i(0) + \delta_0)t$, we get (3.2.15) immediately.

Since $(\tau, \xi_i(\tau)) \,\overline{\in}\, D_i^T$, in order to prove (3.2.16), we first consider the case $j = i$. In this case it is easy to see that

$$|\xi_i(\tau) - \lambda_i(0)\tau| \ge \frac{\delta_0}{\lambda_i(0) + \delta_0} x_{i0}. \tag{3.2.22}$$

Now we consider the case that there exists $j \neq i$ such that $(\tau, \xi_i(\tau)) \,\overline{\in}\, D_j^T$.

When $j < i$, we have

$$|\xi_i(\tau) - \lambda_j(0)\tau| \geq |\xi_i(\tau) - \lambda_i(0)\tau| \geq \frac{\delta_0}{\lambda_i(0) + \delta_0} x_{i0}. \tag{3.2.23}$$

When $j > i$, since $(\tau, \xi_i(\tau)) \,\overline{\in}\, D_j^T$, we have

$$\xi_i(\tau) \geq (\lambda_j(0) + \delta_0)\tau \quad \text{or} \quad \xi_i(\tau) \leq (\lambda_j(0) - \delta_0)\tau. \tag{3.2.24}$$

Hence, if $\xi_i(\tau) \geq (\lambda_j(0) + \delta_0)\tau$, we get

$$|\xi_i(\tau) - \lambda_j(0)\tau| \geq \frac{\delta_0}{\lambda_j(0) + \delta_0} x_{i0}, \tag{3.2.25}$$

whereas, if $\xi_i(\tau) \leq (\lambda_j(0) - \delta_0)\tau$, it is easy to get

$$|\xi_i(\tau) - \lambda_j(0)\tau| \geq \frac{\delta_0}{\lambda_j(0) - \delta_0} x_{i0}. \tag{3.2.26}$$

Then, (3.2.16) follows.

When $i = 1$ or n, noting the definition of D_1^T and D_n^T, we can get (3.2.15)–(3.2.16) in a similar way.

Lemma 3.2.2 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system* (3.1.1) *is strictly hyperbolic, i.e.,* (3.2.2) *holds. Suppose furthermore that the initial data* (3.1.2) *satisfy* (3.2.1). *Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem* (3.1.1)–(3.1.2), *we have the following uniform a priori estimates:*

$$W_\infty^c(T) \leq \kappa_1 \theta, \tag{3.2.27}$$

$$\widetilde{W}_1(T),\ W_1(T) \leq \kappa_2 \theta, \tag{3.2.28}$$

and

$$U_\infty(T) \leq \kappa_3 \theta, \tag{3.2.29}$$

where κ_1, κ_2, and κ_3 are positive constants independent of θ and T.

Proof. Noting (2.6.2) and (3.2.13), we have

$$(1 + |x|)^{1+\mu} |w_i(0, x)| \leq C\theta. \tag{3.2.30}$$

Here and henceforth, C denotes a (possibly different) positive constant independent of θ and T.

We first estimate $W_\infty^c(T)$.

For any given $i \in \{1, \ldots, n\}$, passing through any fixed point $(t, x) \in D(T) \backslash D_i^T$, we draw the ith characteristic c_i: $\xi = \xi_i(\tau)$ $(0 \leq \tau \leq t)$ that intersects the x-axis at a point $(0, x_{i0})$. Integrating (2.6.14) along c_i from 0 to t yields

$$w_i(t,x) = w_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k(\tau, \xi_i(\tau)) d\tau. \tag{3.2.31}$$

Noting (2.6.16) and (3.2.13) and using Lemma 3.2.1, we have

$$\begin{aligned}
&(1 + |x - \lambda_i(0)t|)^{1+\mu} |w_i(t,x)| \\
&\quad \le C(1 + |x_{i0}|)^{1+\mu} \{ |w_i(0,x_{i0})| \\
&\qquad + (W_\infty^c(T))^2 \sum_{j,k=1}^n \int_{\xi_i(\tau)\not\subseteq(D_j^T \cup D_k^T)} [(1 + |\xi_i(\tau) - \lambda_j(0)\tau|)(1 \\
&\qquad + |\xi_i(\tau) - \lambda_k(0)\tau|)]^{-(1+\mu)} d\tau \\
&\qquad + W_\infty^c(T) \sum_{j,k=1}^n \int_{\xi_i(\tau)\subseteq D_j^T} (1 + |\xi_i(\tau) - \lambda_k(0)\tau|)^{-(1+\mu)} |w_j(\tau, \xi_i(\tau))| d\tau \\
&\quad \le C\{ (1 + |x_{i0}|)^{1+\mu} |w_i(0,x_{i0})| + (W_\infty^c(T))^2 + W_\infty^c(T) \widetilde{W}_1(T) \}.
\end{aligned} \tag{3.2.32}$$

Then, noting (3.2.30), we find that

$$W_\infty^c(T) \le C\{ \theta + (W_\infty^c(T))^2 + W_\infty^c(T) \widetilde{W}_1(T) \}. \tag{3.2.33}$$

We next estimate $\widetilde{W}_1(T)$ and $W_1(T)$.

For $i \in \{1, \dots, n\}$, passing through the ends $A(t_A, x_A)$ and $B(t_B, x_B)$ of any given jth characteristic $c_j: \ \xi = \xi_j(\tau)$ $(t_A \le \tau \le t_B)$ $(j \ne i)$ on $D(T)$, we draw the ith characteristics which intersect the x-axis at the points $C(0, x_C)$ and $D(0, x_D)$ with $x_C \le x_D$, respectively. By (2.6.18), using Stokes' formula [see (2.6.22)] on the domain $ACDB$, we get

$$\begin{aligned}
&\int_{t_A}^{t_B} |w_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau \\
&\qquad \le \int_{x_C}^{x_D} |w_i(0,x)| dx + \iint_{ACDB} \Big| \sum_{j,k=1}^n \Gamma_{ijk}(u) w_j w_k(t,x) \Big| dt dx.
\end{aligned} \tag{3.2.34}$$

Then, noting (3.2.13), (3.2.30), and (2.6.20), we have

$$\begin{aligned}
&\int_{t_A}^{t_B} |w_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau \\
&\qquad \le C\Big\{ \theta \int_{x_C}^{x_D} (1 + |x|)^{-(1+\mu)} dx
\end{aligned}$$

$$+ (W_\infty^c(T))^2 \sum_{j,k=1}^{n} \iint_{(t,x)\overline{\in}(D_j^T \cup D_k^T)} [(1 + |x - \lambda_j(0)t|)(1 + |x - \lambda_k(0)t)]^{-(1+\mu)} dxdt \tag{3.2.35}$$

$$\left. + W_\infty^c(T) \sum_{j,k=1}^{n} \iint_{(t,x)\in D_j^T} (1 + |x - \lambda_k(0)t|)^{-(1+\mu)} |w_j(t,x)| dxdt \right\}. \tag{3.2.36}$$

Thus, noting (3.2.5) and using Lemma 3.2.1, we get

$$\int_{t_A}^{t_B} |w_i(\tau, \xi_j(\tau))| d\tau \le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T)\}, \tag{3.2.37}$$

namely,

$$\widetilde{W}_1(T) \le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T)\}. \tag{3.2.38}$$

For any given positive number r, similarly to (3.2.37), we have

$$\int_{-r}^{r} |w_i(t,x))| dx \le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T)\}, \tag{3.2.39}$$

where C is a positive constant independent of r. Then, taking $r \to +\infty$, we get

$$W_1(T) \le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T)\}. \tag{3.2.40}$$

By the definition of $W_\infty^c(T)$, $\widetilde{W}_1(T)$, and $W_1(T)$, it is easy to see that

$$W_\infty^c(0) = \widetilde{W}_1(0) = 0 \quad \text{and} \quad W_1(0) \le \frac{2}{\mu}\theta. \tag{3.2.41}$$

Then, by continuity, we see that there exists $\tau > 0$ so small that

$$W_\infty^c(\tau) \le \kappa_1 \theta \tag{3.2.42}$$

and

$$\widetilde{W}_1(\tau),\ W_1(\tau) \le \kappa_2 \theta, \tag{3.2.43}$$

where κ_1 and κ_2 are positive constants to be determined later. Hence, to show (3.2.27)–(3.2.28), we only need to prove that for any given T_0 with $0 < T_0 \le T$, if we have

$$W_\infty^c(T_0) \le 2\kappa_1 \theta \tag{3.2.44}$$

and

$$\widetilde{W}_1(T_0),\ W_1(T_0) \le 2\kappa_2 \theta, \tag{3.2.45}$$

then we have

$$W_\infty^c(T_0) \le \kappa_1 \theta \tag{3.2.46}$$

and

$$\widetilde{W}_1(T_0),\ W_1(T_0) \le \kappa_2\theta. \tag{3.2.47}$$

For this purpose, substituting (3.2.44)–(3.2.45) into the right-hand side of (3.2.33), (3.2.38), and (3.2.40) with $T = T_0$, we get

$$W^c_\infty(T_0) \le \bar{C}\{\theta + 4\kappa_1^2\theta^2 + 4\kappa_1\kappa_2\theta^2\} \tag{3.2.48}$$

and

$$\widetilde{W}_1(T_0),\ W_1(T_0) \le \bar{C}\{\theta + 4\kappa_1^2\theta^2 + 4\kappa_1\kappa_2\theta^2\}, \tag{3.2.49}$$

where $\bar{C}$ is a positive constant independent of θ. Hence, taking $\kappa_1,\ \kappa_2 \ge 2\bar{C}$ and noting that $\theta > 0$ is small, we get (3.2.46)–(3.2.47), and then we finish the proof of (3.2.27)–(3.2.28) (cf. [76]).

Finally, we estimate $U_\infty(T)$.

Passing through any given point $(t, x) \in D(T)$, we draw the ith characteristic $c_i:\ \xi = \xi_i(\tau)$ $(0 \le \tau \le t)$ which intersects the x-axis at a point $(0, x_{i0})$. Integrating (2.6.43) along c_i from 0 to t gives

$$u(t,x) = u(0,x_{i0}) + \int_0^t \sum_{k=1}^n (\lambda_i(u) - \lambda_k(u))w_k r_k(u)(\tau,\xi_i(\tau))d\tau. \tag{3.2.50}$$

Then, noting (3.2.1) and using Lemma 3.2.1 and (3.2.28), we get

$$|u(t,x)| \le C\{\theta + \widetilde{W}_1(T)\} \le C\theta, \tag{3.2.51}$$

from which (3.2.29) follows immediately. Noting that $\theta_0 > 0$ is small enough, (3.2.51) also implies that hypothesis (3.2.13) is reasonable.

3.3 Some Uniform a Priori Estimates Depending on Normalized Coordinates and Weak Linear Degeneracy for the Strictly Hyperbolic System

In this section we give some uniform a priori estimates depending on normalized coordinates and weak linear degeneracy.

Similarly to (3.2.9)–(3.2.12), let

$$U^c_\infty(T) = \max_{i=1,\dots,n} \sup_{(t,x)\in D(T)\setminus D_i^T} \{(1+|x-\lambda_i(0)t|)^{1+\mu}|u_i(t,x)|\}, \tag{3.3.1}$$

$$\widetilde{U}_1(T) = \max_{i=1,\dots,n} \max_{j\ne i} \sup_{c_j} \int_{c_j} |u_i(t,x)|dt, \tag{3.3.2}$$

where c_j denotes any given jth characteristic on $D(T)$,

$$U_1(T) = \max_{i=1,\dots,n} \sup_{0\le t\le T} \int_{-\infty}^{\infty} |u_i(t,x)|dx, \tag{3.3.3}$$

and

$$W_\infty(T) = ||w(t,x)||_{L^\infty(D(T))} \tag{3.3.4}$$

with $w(t,x) = (w_1(t,x), \dots, w_n(t,x))$.

Lemma 3.3.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system* (3.1.1) *is strictly hyperbolic, namely,* (2.1.12) *holds. Suppose furthermore that system* (3.1.1) *is WLD and the initial data* (3.1.2) *satisfy* (3.2.1). *Then in normalized coordinates there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem* (3.1.1)–(3.1.2), *we have the following uniform a priori estimates:*

$$U_\infty^c(T) \le \kappa_4\theta, \tag{3.3.5}$$

$$\widetilde{U}_1(T),\ U_1(T) \le \kappa_5\theta, \tag{3.3.6}$$

and

$$W_\infty(T) \le \kappa_6\theta, \tag{3.3.7}$$

where κ_4, κ_5, and κ_6 are positive constants independent of θ and T.

Proof. In what follows, we always assume that $u = (u_1, \dots, u_n)^T$ are normalized coordinates.

We now estimate $U_\infty^c(T)$.

Similarly to the estimate on $W_\infty^c(T)$, integrating (2.6.44) along c_i from 0 to t yields

$$u_i(t,x) = u_i(0, x_{i0}) + \int_0^t \sum_{j,k=1}^n \rho_{ijk}(u) u_j w_k(\tau, \xi_i(\tau)) d\tau. \tag{3.3.8}$$

Then, noting (3.2.1) and (2.6.45) and using Lemma 3.2.1, similarly to (3.2.32), we have

$$\begin{aligned}(1+&|x-\lambda_i(0)t|)^{1+\mu}|u_i(t,x)| \\ &\le C\{\theta + W_\infty^c(T)U_\infty^c(T) + U_\infty^c(T)\widetilde{W}_1(T) + \widetilde{U}_1(T)W_\infty^c(T)\}.\end{aligned} \tag{3.3.9}$$

Hence, by Lemma 3.2.2, we get

$$U_\infty^c(T) \le C\theta\{1 + U_\infty^c(T) + \widetilde{U}_1(T)\}. \tag{3.3.10}$$

We next estimate $\widetilde{U}_1(T)$ and $U_1(T)$.

Similarly to (3.2.34), it follows from (2.6.53) that

$$\int_{t_A}^{t_B} |u_i(\lambda_j(u)-\lambda_i(u))(\tau,\xi_j(\tau))|d\tau \\ \leq \int_{x_C}^{x_D} |u_i(0,x)|dx + \iint_{ACDB} \left|\sum_{j,k=1}^n F_{ijk}(u)u_j w_k(t,x)\right| dtdx. \tag{3.3.11}$$

Noting (2.6.51) and (2.6.54) and using Lemmas 3.2.1 and 3.2.2 and Hadamard's formula

$$F(u)-F(u_0) = \int_0^1 \nabla_u F(u_0+\tau(u-u_0))\cdot(u-u_0)d\tau, \tag{3.3.12}$$

we have

$$\begin{aligned}&\iint_{ACDB} \sum_{j,k=1}^n |F_{ijk}(u)u_j w_k(t,x)|dtdx \\ &\leq \iint_{ACDB} \left(\sum_{\substack{j,k=1\\ j\neq k}}^n |F_{ijk}(u)u_j w_k(t,x)| + |(F_{iii}(u)-F_{iii}(u_ie_i))u_i w_i(t,x)|\right)dtdx \\ &\leq C\{U_\infty^c(T)W_\infty^c(T)+U_1(T)W_\infty^c(T)+U_\infty^c(T)W_1(T)+U_\infty(T)U_\infty^c(T)W_1(T)\} \\ &\leq C\theta\{U_\infty^c(T)+U_1(T)\}.\end{aligned} \tag{3.3.13}$$

Then, if we note (3.2.1) and (3.2.5), it follows from (3.3.11) that

$$\int_{t_A}^{t_B} |u_i(\tau,\xi_j(\tau))|d\tau \leq C\theta\{1+U_\infty^c(T)+U_1(T)\}. \tag{3.3.14}$$

Hence,

$$\widetilde{U}_1(T) \leq C\theta\{1+U_\infty^c(T)+U_1(T)\}. \tag{3.3.15}$$

Similarly, we have [also see the proof of (3.2.40)]

$$U_1(T) \leq C\theta\{1+U_\infty^c(T)+U_1(T)\}. \tag{3.3.16}$$

Thus, the combination of (3.3.10) and (3.3.15)–(3.3.16) gives (3.3.5) and (3.3.6).

We finally estimate $W_\infty(T)$.

Similarly to (3.2.50), by (2.6.14), we have

$$w_i(t,x) = w_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u)w_j w_k(\tau,\xi_i(\tau))d\tau. \tag{3.3.17}$$

Noting (2.6.16) and (2.6.28) and using Lemmas 3.2.1 and 3.2.2 and Hadamard's formula, we have

$$\begin{aligned}
&\int_0^t \sum_{j,k=1}^n |\gamma_{ijk}(u) w_j w_k(\tau, \xi_i(\tau))| d\tau \\
&\quad \le \int_0^t \left(\sum_{\substack{j,k=1 \\ j\neq k}}^n |\gamma_{ijk}(u) w_j w_k(\tau, \xi_i(\tau))| + |\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)| w_i^2(\tau, \xi_i(\tau)) \right) d\tau \\
&\quad \le C\{(W_\infty^c(T))^2 + W_\infty^c(T) W_\infty(T) + U_\infty^c(T)(W_\infty(T))^2\} \\
&\quad \le C\{\theta^2 + \theta W_\infty(T) + U_\infty^c(T)(W_\infty(T))^2\}. \qquad (3.3.18)
\end{aligned}$$

Then, noting (3.2.30) and (3.3.5), it follows from (3.3.17) that

$$|w_i(t,x)| \le C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}. \qquad (3.3.19)$$

Hence,

$$W_\infty(T) \le C\theta\{1 + (W_\infty(T))^2\}. \qquad (3.3.20)$$

Then we get (3.3.7) using the method in the proof of Lemma 3.2.2.

3.4 Sufficient Condition to Guarantee the Global Existence and Uniqueness of the C^1 Solution to the Cauchy Problem for the Strictly Hyperbolic System

In this section we show that the necessary condition given in Section 3.1 is also a sufficient condition to guarantee that the Cauchy problem (3.1.1)–(3.1.2) admits a unique global C^1 solution, provided that the initial data satisfy (3.2.1).

Theorem 3.4.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system* (3.1.1) *is strictly hyperbolic, namely,* (2.1.12) *holds. Suppose furthermore that the initial data* (3.1.2) *satisfy* (3.2.1). *If system* (3.1.1) *is WLD, then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem* (3.1.1)–(3.1.2) *admits a unique global C^1 solution $u = u(t,x)$ with small C^1 norm for all $t \in \mathbb{R}$* (cf. [75], [76], a related result can be also found in [95]).

Proof. According to the existence and uniqueness of the local C^1 solution to the Cauchy problem (cf. [72]), there exists $\tau_0 > 0$ such that on $[0, \tau_0] \times \mathbb{R}$, Cauchy problem (3.1.1)–(3.1.2) has a unique C^1 solution $u = u(t,x)$.

Without loss of generality, we may consider the Cauchy problem in normalized coordinates. By Lemmas 3.2.2 and 3.3.1, we know that if $\theta_0 > 0$ is suitably small, then for any given $\theta \in [0, \theta_0]$, on any given existence domain

$[0,T]\times\mathbb{R}$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (3.1.1)–(3.1.2), we have the following uniform a priori estimate on the C^1 norm of $u = u(t,x)$:

$$\|u(t,\cdot)\|_{C^1} \stackrel{\text{def.}}{=} \|u(t,\cdot)\|_{C^0} + \|u_x(t,\cdot)\|_{C^0} \le C\theta, \tag{3.4.1}$$

where C is a positive constant independent of θ and T. Then the extension of the local solution immediately gives the global existence and uniqueness of the C^1 solution $u = u(t,x)$ to the Cauchy problem for all $t \ge 0$.

Taking $\bar{t} = -t$, we similarly get the same result for all $t \le 0$.

Remark 3.4.1 *The method used in this book to show Lemmas 3.2.2 and 3.3.1 is simpler than that in* [76], *where some intermediate estimates on* $v = (v_1,\dots,v_n)$ *with* $v_i = l_i(u)u$ $(i = 1,\dots,n)$ *are first needed and* $D(T)$ *is divided into more subdomains including* $D_\pm^T$ *and* D_0^T. *On the other hand, by means of Lemmas 3.2.2 and 3.3.1, we can easily obtain the corresponding estimates on* $D_\pm^T$ *and* D_0^T *given in* [57] *and* [76].

Remark 3.4.2 *In Theorem 3.4.1, the hypothesis* $\mu > 0$ *in* (3.2.1) *is necessary.*

To illustrate this fact, we consider (also see [40])

Example 3.4.1 *Consider the following Cauchy problem in a neighbourhood of* $(r,s) = (0,0)$ *for the system*

$$\begin{cases} r_t = 0, \\ s_t + (1+rs)s_x = 0 \end{cases} \tag{3.4.2}$$

with the initial data

$$t = 0:\ \ r = \varepsilon r_0(x),\ \ s = \varepsilon(1+x^2)^{-1}, \tag{3.4.3}$$

where $\varepsilon > 0$ *is a small parameter and* $r_0(x)$ *is a nonnegative fuction with*

$$\|r_0(x)\|_{C^1(\mathbb{R})} \le M \tag{3.4.4}$$

and

$$r_0(x) = (1+|x|)^{-(1+\mu)} \quad for\ |x| \ge 2, \tag{3.4.5}$$

in which $M > 0$ *and* $-1 < \mu \le 0$ *are constants.*

Obviously, in a neighbourhood of $(r,s) = (0,0)$, (3.4.2) is strictly hyperbolic and WLD.

It follows from the first equation of (3.4.2) that

$$r(t,x) = \varepsilon r_0(x), \quad \forall (t,x) \in \mathbb{R}^+\times\mathbb{R}. \tag{3.4.6}$$

Substituting it into the second equation of (3.4.2) gives

$$s_t + (1 + \varepsilon r_0(x)s)s_x = 0. \tag{3.4.7}$$

We now consider the Cauchy problem for Equation (3.4.7) with the initial data

$$t = 0: \ \ s = \varepsilon(1 + x^2)^{-1}. \tag{3.4.8}$$

On the existence domain of the C^1 solution $s = s(t, x)$ to this Cauchy problem, let $x = x(t, \beta)$ be the characteristic passing through any given point $(0, \beta)$ on the x-axis. Since along $x = x(t, \beta)$ we have

$$s = s(t, x(t, \beta)) = \varepsilon(1 + \beta^2)^{-1}, \tag{3.4.9}$$

$x = x(t, \beta)$ satisfies

$$\begin{cases} \dfrac{dx}{dt} = 1 + \varepsilon^2 r_0(x)(1 + \beta^2)^{-1}, \\ t = 0: \ \ x = \beta. \end{cases} \tag{3.4.10}$$

Then

$$x_\beta(t, \beta) = A(t, \beta) \exp(\triangle(t, \beta)), \tag{3.4.11}$$

where

$$A(t, \beta) = 1 - \frac{2\varepsilon^2\beta}{(1 + \beta^2)^2} \int_0^t r_0(x(\tau, \beta)) \exp(-\triangle(\tau, \beta))d\tau \tag{3.4.12}$$

and

$$\triangle(t, \beta) = \frac{\varepsilon^2}{(1 + \beta^2)} \int_0^t r_0'(x(\tau, \beta))d\tau. \tag{3.4.13}$$

Moreover, it follows from (3.4.9) that

$$s_x(t, x(t, \beta)) = \frac{-2\varepsilon\beta(1 + \beta^2)^{-2}}{x_\beta(t, \beta)}. \tag{3.4.14}$$

Noting (3.4.10) and that $r_0(x)$ is nonnegative, from (3.4.13) we have

$$\triangle(t, \beta) = \frac{\varepsilon^2}{(1 + \beta^2)} \int_\beta^{x(t,\beta)} \frac{r_0'(x)}{1 + \varepsilon^2 r_0(x)(1 + \beta^2)^{-1}} dx. \tag{3.4.15}$$

Then, noting (3.4.4)–(3.4.5), we have

$$\begin{aligned} |\triangle(t, \beta)| &\le C\varepsilon^2 \left\{ \int_{-2}^{2} |r_0'(x)|dx + \int_{-\infty}^{-2} \frac{1}{(1 - x)^{2+\mu}} dx + \int_2^\infty \frac{1}{(1 + x)^{2+\mu}} dx \right\} \\ &\le C\varepsilon^2, \quad \forall t \ge 0, \ \forall \beta \in \mathbb{R}. \end{aligned} \tag{3.4.16}$$

Here and henceforth, C denotes a positive constant. Thus, noting (3.4.4)–(3.4.5) and using (3.4.16), it follows from (3.4.12) that

$$\begin{aligned} A(t,\beta) &\le 1 - \frac{2\varepsilon^2\beta}{(1+\beta^2)^2}\exp(-C\varepsilon^2)\int_0^t r_0(x(\tau,\beta))d\tau \\ &\le 1 - \frac{2\varepsilon^2\beta}{(1+\beta^2)^2}\exp(-C\varepsilon^2)\int_\beta^{x(t,\beta)} \frac{r_0(x)}{1+\varepsilon^2 r_0(x)(1+\beta^2)^{-1}}dx \\ &\le 1 - \frac{c\varepsilon^2\beta}{(1+\beta^2)^2}\exp(-C\varepsilon^2)\int_\beta^{x(t,\beta)} r_0(x)dx, \quad \forall t\ge 0,\ \forall \beta\in\mathbb{R}, \end{aligned} \tag{3.4.17}$$

where c denotes a positive constant.

Taking $\beta = 2$ in (3.4.17) and noting (3.4.5), we get

$$A(t,2) \le 1 - c\varepsilon^2\exp(-C\varepsilon^2)\int_2^{x(t,2)} \frac{1}{(1+x)^{1+\mu}}dx. \tag{3.4.18}$$

Hence, noting that $x(t,\beta) \ge \beta + t$, we have that when $\mu = 0$,

$$A(t,2) \le 1 - c\varepsilon^2\exp(-C\varepsilon^2)[\log(3+t) - \log 3], \tag{3.4.19}$$

whereas, when $-1 < \mu < 0$,

$$A(t,2) \le 1 - c\varepsilon^2\exp(-C\varepsilon^2)[(3+t)^{-\mu} - 3^{-\mu}]. \tag{3.4.20}$$

Then, noting $A(0,\beta) = 1$ for any $\beta \in \mathbb{R}$, for any given $\varepsilon > 0$ small enough, there exists $t^* = t^*(\varepsilon) > 0$ such that

$$A(t^*, 2) = 0. \tag{3.4.21}$$

Thus, by (3.4.14) and (3.4.11), we get

$$s_x(t, x(t,2)) \to -\infty \quad \text{as} \quad t \uparrow t^*, \tag{3.4.22}$$

i.e., the C^1 solution to the Cauchy problem (3.4.2)–(3.4.3) must blow up in a finite time.

Remark 3.4.3 *By Theorems 3.1.1 and 3.4.1, for the Cauchy problem* (3.1.1)–(3.1.2) *with the initial data $\phi(x)$ satisfying* (3.2.1), *the weak linear degeneracy of the strictly hyperbolic system* (3.1.1) *is a necessary and sufficient condition to guarantee the global existence and uniqueness of the C^1 solution on $t \ge 0$ or for all $t \in \mathbb{R}$.*

Remark 3.4.4 *By means of the continuous Glimm functional, a global existence result for the Cauchy problem* (3.1.1)–(3.1.2) *in a slightly general class of initial data can be found in* [103] (some related results can be also found in [9], [56], and [97]).

Remark 3.4.5 *For the inhomogeneous quasilinear strictly hyperbolic system* (2.1.19)*, under certain quite strong restrictions on the inhomogeneous term* $B(u)$*, similar results as in Theorem 3.4.1 can be found* (cf. [46] and [96]).

Remark 3.4.6 *A continuous and piecewise C^1 vector function*

$$u = u(t,x) = \begin{cases} u_-(t,x), & x \le x_k(t), \\ u_+(t,x), & x \ge x_k(t), \end{cases} \tag{3.4.23}$$

is a **weakly discontinuous solution containing a kth weak discontinuity** $x = x_k(t)$ *for system* (2.1.1) *if $u = u(t,x)$ satisfies system* (2.1.1) *in the classical sense on both sides of $x = x_k(t)$,*

$$u_-(t, x_k(t)) = u_+(t, x_k(t)), \tag{3.4.24}$$

but the first-order derivatives of $u(t,x)$ have the first-kind discontinuity on $x = x_k(t)$. In this situation it is easy to see that $x = x_k(t)$ must be a characteristic curve, and, for the **kth weak discontinuity***, $x = x_k(t)$ is supposed to be the corresponding kth characteristic:*

$$\frac{dx_k(t)}{dt} = \lambda_k(u_-(t, x_k(t))) = \lambda_k(u_+(t, x_k(t))). \tag{3.4.25}$$

For the weakly discontinuous solution to the Cauchy problem of system (2.1.1) *with the following weakly discontinuous initial data:*

$$t = 0: \;\; u = \begin{cases} u_l(x), & x \le 0, \\ u_r(x), & x \ge 0, \end{cases} \tag{3.4.26}$$

where $u_l(x)$ and $u_r(x)$ are C^1 vector functions on $x \le 0$ and $x \ge 0$, respectively,

$$u_l(0) = u_r(0) \tag{3.4.27}$$

and

$$u_l'(0) \neq u_r'(0), \tag{3.4.28}$$

similar results as in Theorems 3.1.1 and 3.4.1 are still valid (cf. [64]).

3.5 Global C^1 Solution to the Cauchy Problem for the Hyperbolic System with Characteristics with Constant Multiplicity

In this section we consider the Cauchy problem for hyperbolic system (2.1.1) with characteristics with constant multiplicity. Without loss of generality, we assume that

$$\lambda(u) \stackrel{\text{def.}}{=} \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u), \tag{3.5.1}$$

where $p > 1$.

Suppose that system (2.1.1) possesses normalized coordinates.

In the present situation, similarly to Theorem 3.1.1, the weak linear degeneracy is still a necessary condition to guarantee the global existence and uniqueness of the C^1 solution to the Cauchy problem with small initial data (cf. [89]). In what follows, we will prove that this condition is also sufficient.

Noting (3.5.1), we have

$$\lambda(0) \stackrel{\text{def.}}{=} \lambda_1(0) = \cdots = \lambda_p(0) < \lambda_{p+1}(0) < \cdots < \lambda_n(0). \tag{3.5.2}$$

Similarly to (3.2.3), we assume that

$$\lambda(0) > 0. \tag{3.5.3}$$

Then, by continuity, there exist positive constants δ_0 $(< \lambda(0))$ and δ so small that

$$\lambda_{i+1}(u) - \lambda_i(u') \geq 2\delta_0, \quad \forall |u|, |u'| \leq \delta \quad (i = p, \ldots, n-1), \tag{3.5.4}$$

and

$$|\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \quad \forall |u|, |u'| \leq \delta \quad (i = 1, \ldots, n). \tag{3.5.5}$$

For any given $T \geq 0$, let

$$D_i^T = \begin{cases} \{(t,x) | 0 \leq t \leq T,\ x \leq (\lambda_1(0) + \delta_0)t\} \quad (i = 1, \ldots, p), \\ \{(t,x) | 0 \leq t \leq T,\ (\lambda_i(0) - \delta_0)t \leq x \leq (\lambda_i(0) + \delta_0)t\} \quad (i = p+1, \ldots, n-1), \\ \{(t,x) | 0 \leq t \leq T,\ x \geq (\lambda_n(0) - \delta_0)t\} \quad (i = n). \end{cases} \tag{3.5.6}$$

Obviously,

$$\bigcup_{i=1}^{n} D_i^T \subset D(T) = \{(t,x) | 0 \leq t \leq T,\ -\infty < x < \infty\}.$$

On any given existence domain $D(T)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (2.1.1) and (3.1.2), $W_\infty^c(T)$, $W_1(T)$, and $U_\infty(T)$ are still defined by (3.2.9), (3.2.11), and (3.2.12), respectively, and $U_\infty^c(T)$, $U_1(T)$, and $W_\infty(T)$ are still defined by (3.3.1), (3.3.3), and (3.3.4), respectively. However,

$\widetilde{W}_1(T)$ and $\widetilde{U}_1(T)$ are defined by

$$\widetilde{W}_1(T) = \max\left\{\max_{i=1,\dots,p}\max_{j=p+1,\dots,n}\sup_{c_j}\int_{c_j}|w_i(t,x)|dt,\right.$$
$$\left.\max_{i=p+1,\dots,n}\max_{j\neq i}\sup_{c_j}\int_{c_j}|w_i(t,x)|dt\right\} \tag{3.5.7}$$

and

$$\widetilde{U}_1(T)=\max\left\{\max_{i=1,\dots,p}\max_{j=p+1,\dots,n}\sup_{c_j}\int_{c_j}|u_i(t,x)|dt,\right.$$
$$\left.\max_{i=p+1,\dots,n}\max_{j\neq i}\sup_{c_j}\int_{c_j}|u_i(t,x)|dt\right\}, \tag{3.5.8}$$

where c_j denotes any given jth characteristic on $D(T)$.

In the present situation, Lemma 3.2.1 is still valid. Similarly to Lemmas 3.2.2 and 3.3.1, we have the following two lemmas.

Lemma 3.5.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and* (3.5.1) *holds. Suppose furthermore that the initial data* (3.1.2) *satisfy* (3.2.1). *Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem* (2.1.1) *and* (3.1.2), *we have the following uniform a priori estimates:*

$$W^c_\infty(T) \le \kappa_1\theta, \tag{3.5.9}$$

$$\widetilde{W}_1(T),\ W_1(T) \le \kappa_2\theta, \tag{3.5.10}$$

and

$$U_\infty(T) \le \kappa_3\theta, \tag{3.5.11}$$

where κ_1, κ_2, and κ_3 are positive constants independent of θ and T.

Proof. The proof of this lemma is similar to that of Lemma 3.2.2. In what follows, we only point out some essentially different points.

Noting (2.6.16) and (2.6.31), similarly to (3.2.33), we still have

$$W^c_\infty(T) \le C\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)\widetilde{W}_1(T)\}. \tag{3.5.12}$$

We next estimate $\widetilde{W}_1(T)$ and $W_1(T)$.

For $i = 1, \dots, p; j = p + 1, \dots, n$, noting (2.6.20) and (2.6.32), similarly to (3.2.37), we have

$$\int_{c_j}|w_i(t,x)|dt \le C\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)W_1(T)\}. \tag{3.5.13}$$

For $i = p+1,\dots,n; j \neq i$, noting (2.6.20) and (2.6.32), we still have (3.5.13). Hence, we get

$$\widetilde{W}_1(T) \le C\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)W_1(T)\}. \tag{3.5.14}$$

Similarly, we have

$$W_1(T) \le C\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T)W_1(T)\}. \tag{3.5.15}$$

Then, (3.5.9)–(3.5.10) follow from (3.5.12) and (3.5.14)–(3.5.15).

Finally, we estimate $U_\infty(T)$.

For any given $(t,x) \in D(T)$, (3.2.50) still holds. Noting (3.5.1) and using Lemma 3.2.1 and (3.5.9)–(3.5.10), similarly to (3.2.51), we have

$$|u(t,x)| \le C\{\theta + W^c_\infty(T) + \widetilde{W}_1(T)\} \le C\theta. \tag{3.5.16}$$

Then we get (3.5.11) immediately.

Lemma 3.5.2 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and* (3.5.1) *holds. Suppose furthermore that system* (2.1.1) *is WLD and the initial data* (3.1.2) *satisfy* (3.2.1)*. Then in normalized coordinates there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0,\theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem* (2.1.1) *and* (3.1.2)*, we have the following uniform a priori estimates:*

$$U^c_\infty(T) \le \kappa_4\theta, \tag{3.5.17}$$

$$\widetilde{U}_1(T),\ U_1(T) \le \kappa_5\theta, \tag{3.5.18}$$

and

$$W_\infty(T) \le \kappa_6\theta, \tag{3.5.19}$$

where κ_4, κ_5, and κ_6 are positive constants independent of θ and T.

Proof. We first estimate $U^c_\infty(T)$.

For $i = 1,\dots,p$, passing through any fixed point $(t,x) \in D(T)\backslash D^T_1$, we draw the ith characteristic c_i: $\xi = x_i(\tau)$ $(0 \le \tau \le t)$ which intersects the x-axis at a point $(0, x_{i0})$. Integrating (2.6.55) along c_i from 0 to t yields

$$u_i(t,x) = u_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^n \rho_{ijk}(u)u_j w_k(\tau, x_i(\tau))d\tau. \tag{3.5.20}$$

Then, noting (3.2.1) and (2.6.56)–(2.6.57) and using Lemmas 3.2.1 and 3.5.1, similarly to (3.3.9), we have

$$\begin{aligned}(1 + |x - \lambda_i(0)t|)^{1+\mu}|u_i(t,x)| &\le C\{\theta + U^c_\infty(T)W^c_\infty(T) + U^c_\infty(T)\widetilde{W}_1(T) + \widetilde{U}_1(T)W^c_\infty(T)\}\\ &\le C\theta\{1 + U^c_\infty(T) + \widetilde{U}_1(T)\}.\end{aligned} \tag{3.5.21}$$

For $i = p+1, \ldots, n$, noting (3.2.1) and (2.6.59)–(2.6.60), we still have (3.5.21). Hence, we get

$$U_\infty^c(T) \le C\theta\{1 + U_\infty^c(T) + \widetilde{U}_1(T)\}. \tag{3.5.22}$$

We next estimate $\widetilde{U}_1(T)$ and $U_1(T)$.

In the present situation, (3.3.11) still holds.

For $i = 1, \ldots, p$, since $\lambda(u)$ is WLD, noting (2.6.62)–(2.6.64) and using Lemmas 3.2.1 and 3.5.1, instead of (3.3.13), we have

$$\begin{aligned}
&\iint\limits_{ACDB} \sum_{j,k=1}^{n} |F_{ijk}(u)u_j w_k(t,x)| dtdx \\
&\le \iint\limits_{ACDB} \Bigg[\sum_{\substack{j \text{ or } k \in \{1,\ldots,p\} \\ j \ne k}} |F_{ijk}(u)u_j w_k(t,x)| \\
&\qquad + \sum_{j,k=1}^{p} \left| \left(F_{ijk}(u) - F_{ijk}\left(\sum_{h=1}^{p} u_h e_h \right) \right) u_j w_k(t,x) \right| \Bigg] dtdx \\
&\le C\{U_\infty^c(T)W_\infty^c(T) + U_1(T)W_\infty^c(T) + U_\infty^c(T)W_1(T) + U_\infty(T)U_\infty^c(T)W_1(T)\} \\
&\le C\theta\{U_\infty^c(T) + U_1(T)\}.
\end{aligned} \tag{3.5.23}$$

For $i = p+1, \ldots, n$, since $\lambda_i(u)$ is WLD, noting (2.6.62)–(2.6.63) and (2.6.65), similarly to (3.5.23), we have

$$\begin{aligned}
&\iint\limits_{ACDB} \sum_{j,k=1}^{n} |F_{ijk}(u)u_j w_k(t,x)| dtdx \\
&\le \iint\limits_{ACDB} \Bigg[\sum_{\substack{j \text{ or } k \in \{1,\ldots,p\} \\ j \ne k}} |F_{ijk}(u)u_j w_k(t,x)| + |(F_{iii}(u) - F_{iii}(u_i e_i))u_i w_i(t,x)| \Bigg] dtdx \\
&\le C\{U_\infty^c(T)W_\infty^c(T) + U_1(T)W_\infty^c(T) + U_\infty^c(T)W_1(T) + U_\infty(T)U_\infty^c(T)W_1(T)\} \\
&\le C\theta\{U_\infty^c(T) + U_1(T)\}.
\end{aligned} \tag{3.5.24}$$

Then, noting (3.2.1) and (3.5.4), similarly to (3.3.14), we get from (3.3.11) that

$$\int_{t_A}^{t_B} |u_i(\tau, x_j(\tau))| d\tau \le C\theta\{1 + U_\infty^c(T) + U_1(T)\}.$$

Hence,

$$\widetilde{U}_1(T) \le C\theta\{1 + U_\infty^c(T) + U_1(T)\}. \tag{3.5.25}$$

Similarly, we can get

$$U_1(T) \le C\theta\{1 + U_\infty^c(T) + U_1(T)\}. \tag{3.5.26}$$

Then the combination of (3.5.22) and (3.5.25)–(3.5.26) gives (3.5.17) and (3.5.18).

Finally, we estimate $W_\infty(T)$.

In the present situation, (3.3.17) still holds.

For $i = 1, \ldots, p$, and since $\lambda(u)$ is WLD, noting (2.6.16) and (2.6.41) and using Lemmas 3.2.1, 3.5.1 and (3.5.17), instead of (3.3.18), we have

$$\begin{aligned}
&\int_0^t \sum_{j,k=1}^n |\gamma_{ijk}(u) w_j w_k(\tau, x_i(\tau))| d\tau \\
&\quad \le \int_0^t \Bigg[\sum_{\substack{j \text{ or } k \overline{\in} \{1,\ldots,p\} \\ j \ne k}} |\gamma_{ijk}(u) w_j w_k(\tau, x_i(\tau))| \\
&\qquad + 2 \sum_{j=1}^p \left| \left(\gamma_{iji}(u) - \gamma_{iji}\left(\sum_{h=1}^p u_h e_h \right) \right) w_i w_j(\tau, x_i(\tau)) \right| \Bigg] d\tau \\
&\quad \le C\{(W_\infty^c(T))^2 + W_\infty^c(T) W_\infty(T) + U_\infty^c(T)(W_\infty(T))^2\} \\
&\quad \le C\theta\{\theta + W_\infty(T) + (W_\infty(T))^2\}.
\end{aligned} \tag{3.5.27}$$

For $i = p+1, \ldots, n$, since $\lambda_i(u)$ is WLD, noting (2.6.16), (2.6.31), and (2.6.42) and using Lemma 3.2.1, similarly to (3.5.27), we have

$$\begin{aligned}
&\int_0^t \sum_{j,k=1}^n |\gamma_{ijk}(u) w_j w_k(\tau, x_i(\tau))| d\tau \\
&\quad \le \int_0^t \Bigg[\sum_{\substack{j \text{ or } k \overline{\in} \{1,\ldots,p\} \\ j \ne k}} |\gamma_{ijk}(u) w_j w_k(\tau, x_i(\tau))| + |\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)| w_i^2(\tau, x_i(\tau)) \Bigg] d\tau \\
&\quad \le C\{(W_\infty^c(T))^2 + W_\infty^c(T) W_\infty(T) + U_\infty^c(T)(W_\infty(T))^2\} \\
&\quad \le C\theta\{\theta + W_\infty(T) + (W_\infty(T))^2\}.
\end{aligned} \tag{3.5.28}$$

Then, similarly to (3.3.19), it follows that

$$|w_i(t,x)| \le C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}.$$

Hence,

$$W_\infty(T) \le C\theta\{1 + (W_\infty(T))^2\}, \tag{3.5.29}$$

which implies (3.5.19).

By Lemmas 3.5.1 and 3.5.2, it is easy to get

Theorem 3.5.1 *Suppose that in a neighbourhood of* $u = 0$, $A(u) \in C^2$ *and* (3.5.1) *holds. Suppose furthermore that the initial data* (3.1.2) *satisfy* (3.2.1). *If system* (2.1.1) *is WLD, then there exists* $\theta_0 > 0$ *so small that for any given*

$\theta \in [0, \theta_0]$, the Cauchy problem (2.1.1) *and* (3.1.2) *admits a unique global C^1 solution $u = u(t,x)$ with a small C^1 norm for all $t \in \mathbb{R}$* (cf. [58]).

Remark 3.5.1 *When hyperbolic system* (2.1.1) *has a form of conservation laws, $\lambda(u)$ is LD, then WLD. In this situation Theorem 3.5.1 can be easily applied* (cf. [58]).

3.6 Applications

3.6.1 System of One-Dimensional Gas Dynamics

Consider the Cauchy problem for the **system of one-dimensional gas dynamics** in Lagrangian representation (cf. Section 1.3.3)

$$\begin{cases} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(\tau, S)}{\partial x} = 0, \\ \dfrac{\partial S}{\partial t} = 0 \end{cases} \tag{3.6.1}$$

with the initial data

$$t = 0: \quad \tau = \tau_0 + \tau_1(x),\ v = u_0 + u_1(x),\ S = S_0 + S_1(x), \tag{3.6.2}$$

where $\tau > 0$ is the specific volume, u is the velocity, S is the entropy, p is the pressure, and the equation of state is given by $p = p(\tau, S)$; moreover, $\tau_0\ (> 0), u_0$, and S_0 are constants, and $\tau_1(x), u_1(x)$, and $S_1(x) \in C^1$ satisfy the decaying property as shown in (3.2.1).

If

$$p_\tau(\tau_0, S_0) < 0, \tag{3.6.3}$$

then, in a neighbourhood of $U = U_0 \overset{\text{def.}}{=} (\tau_0, u_0, S_0)^T$, system (3.6.1) is a strictly hyperbolic system with three distinct eigenvalues,

$$\lambda_1(U) = -\sqrt{-p_\tau} < \lambda_2(U) = 0 < \lambda_3(U) = \sqrt{-p_\tau}, \tag{3.6.4}$$

and the corresponding right eigenvectors can be taken as

$$r_1(U) = (1, \sqrt{-p_\tau}, 0)^T,\ r_2(U) = (p_S, 0, -p_\tau)^T,\ r_3(U) = (-1, \sqrt{-p_\tau}, 0)^T, \tag{3.6.5}$$

where

$$U = (\tau, u, S)^T. \tag{3.6.6}$$

Obviously, $\lambda_2(U)$ is always LD in a neighbourhood of $U = U_0$. Moreover, it is easy to see that $\lambda_1(U)$ and $\lambda_3(U)$ are LD in a neighbourhood of $U = U_0$ if and only if

$$p_{\tau\tau}(\tau, S) \equiv 0, \quad \forall |\tau - \tau_0| \text{ and } |S - S_0| \text{ small.} \tag{3.6.7}$$

Noting that the first characteristic trajectory $U = U^{(1)}(s)$ passing through $U = U_0$ in the U-space is defined by

$$\begin{cases} \dfrac{dU}{ds} = r_1(U), \\ s = 0: \;\; U = U_0, \end{cases} \tag{3.6.8}$$

where $r_1(U)$ is given by (3.6.5), by the definition of WLD, we can see that $\lambda_1(U)$ is WLD with respect to $U = U_0$ if and only if

$$p_{\tau\tau}(\tau, S_0) \equiv 0, \quad \forall |\tau - \tau_0| \text{ small.} \tag{3.6.9}$$

Similarly, $\lambda_3(U)$ is WLD with respect to $U = U_0$ if and only if (3.6.9) holds. Obviously, if $\lambda_1(U)$ and $\lambda_3(U)$ are LD in a neighbourhood of $U = U_0$, then they are WLD with respect to $U = U_0$.

By Theorem 3.4.1, we have

Theorem 3.6.1 *Under assumption* (3.6.9), *there exists* $\theta_0 > 0$ *so small that for any fixed* $\theta \in [0, \theta_0]$, *the Cauchy problem* (3.6.1)–(3.6.2) *admits a unique global* C^1 *solution* $U = U(t, x)$ *for all* $t \in \mathbb{R}$.

3.6.2 System of Compressible Elastic Fluids with Memory

Consider the Cauchy problem for the **system of compressible elastic fluids with memory** (cf. Section 1.3.4)

$$\begin{cases} \rho_t + v\rho_x + \rho v_x = 0, \\ \rho(v_t + vv_x) + p(\rho)_x = (\rho W'(F)F)_x, \\ F_t + vF_x - Fv_x = 0 \end{cases} \tag{3.6.10}$$

with the initial data

$$t = 0: \quad \rho = \rho_0 + \rho_1(x), \;\; v = v_0 + v_1(x), \;\; F = 1 + F_1(x), \tag{3.6.11}$$

where $\rho > 0$ is the density, v is the velocity, p is the pressure, $W(F)$ is the strain energy function, and F corresponds to the deformation tensor. Moreover, $\rho_0(>0)$ and v_0 are constants, and $\rho_1(x), v_1(x)$, and $F_1(x) \in C^1$ satisfy the decaying property as shown in (3.2.1).

Let

$$U = (\rho, v, F)^T. \tag{3.6.12}$$

If

$$p'(\rho_0) + W''(1) > 0, \tag{3.6.13}$$

(3.6.10) is a strictly hyperbolic system in a neighbourhood of $U = U_0 \stackrel{\text{def.}}{=} (\rho_0, v_0, 1)^T$ with the following distinct real eigenvalues:

$$\lambda_1 = v - \sqrt{p'(\rho) + W''(F)F^2} < \lambda_2 = v < \lambda_3 = v + \sqrt{p'(\rho) + W''(F)F^2} \tag{3.6.14}$$

and the corresponding right eigenvectors can be taken as

$$\begin{aligned} r_1(U) &= \left(-\frac{\rho}{\sqrt{p'(\rho) + w''(F)F^2}}, 1, \frac{F}{\sqrt{p'(\rho) + w''(F)F^2}} \right)^T, \\ r_2(U) &= \left(FW''(F) + W'(F), 0, \frac{p'(\rho) - FW'(F)}{\rho} \right)^T, \\ r_3(U) &= \left(\frac{\rho}{\sqrt{p'(\rho) + w''(F)F^2}}, 1, -\frac{F}{\sqrt{p'(\rho) + w''(F)F^2}} \right)^T. \end{aligned} \tag{3.6.15}$$

Obviously, $\lambda_2(U)$ is always LD. On the other hand, $\lambda_1(U)$ and $\lambda_3(U)$ are LD in a neighbourhood of $U = U_0$ if and only if

$$\rho p''(\rho) + 2p'(\rho) - W'''(F)F^3 \equiv 0. \tag{3.6.16}$$

Noting that the first characteristic trajectory $U = U^{(1)}(s)$ passing through $U = U_0$ in the U-space is defined by

$$\begin{cases} \dfrac{dU}{ds} = r_1(U), \\ s = 0: \ U = U_0, \end{cases} \tag{3.6.17}$$

where $r_1(U)$ is given by (3.6.15), it is easy to see that $\rho F \equiv \rho_0$ always holds along the first characteristic trajectory passing through $U = U_0$. Hence, by the definition of WLD, $\lambda_1(U)$ and $\lambda_3(U)$ are WLD with respect to $U = U_0$ if and only if

$$\rho p''(\rho) + 2p'(\rho) - W'''(\rho_0\rho^{-1})(\rho_0\rho^{-1})^3 \equiv 0, \quad \forall |\rho - \rho_0| \text{ small}. \tag{3.6.18}$$

Obviously, if $\lambda_1(U)$ and $\lambda_3(U)$ are LD in a neighbourhood of $U = U_0$, then they are WLD with respect to $U = U_0$.

By Theorem 3.4.1, we have

Theorem 3.6.2 *Under assumption* (3.6.18), *there exists* $\theta_0 > 0$ *so small that for any fixed* $\theta \in [0, \theta_0]$, *the Cauchy problem* (3.6.10)–(3.6.11) *admits a unique global* C^1 *solution* $U = U(t,x)$ *for all* $t \in \mathbb{R}$.

3.6.3 System of the Motion of an Elastic String

Consider the following Cauchy problem for the **system of the motion of an elastic string** (cf. Section 1.3.5):

$$\begin{cases} u_t - v_x = 0, \\ v_t - \left(\dfrac{T(r)}{r}u\right)_x = 0 \end{cases} \tag{3.6.19}$$

with the initial data

$$t = 0: \ (u, v) = (\widetilde{u}^0 + u^0(x), \widetilde{v}^0 + v^0(x)), \tag{3.6.20}$$

where $u = (u_1, \ldots, u_n)^T$, $v = (v_1, \ldots, v_n)^T$, $r = |u| = \sqrt{u_1^2 + \cdots + u_n^2}$, and $T(r)$ is a suitably smooth function of $r > 1$ such that

$$T'(\widetilde{r}_0) > \frac{T(\widetilde{r}_0)}{\widetilde{r}_0} > 0, \tag{3.6.21}$$

in which $\widetilde{r}_0 = |\widetilde{u}^0| = \sqrt{(\widetilde{u}_1^0)^2 + \cdots + (\widetilde{u}_n^0)^2} > 1$. Moreover, $(\widetilde{u}^0, \widetilde{v}^0)$ is a constant vector and $(u^0(x), v^0(x)) \in C^1$ satisfies the decaying property as shown in (3.2.1).

Let

$$U = (u_1, \ldots, u_n, v_1, \ldots, v_n)^T = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.6.22}$$

By (3.6.21), in a neighbourhood of

$$U = U_0 \stackrel{\text{def.}}{=} \begin{pmatrix} \widetilde{u}^0 \\ \widetilde{v}^0 \end{pmatrix}, \tag{3.6.23}$$

(3.6.19) is a hyperbolic system with the following $2n$ real eigenvalues:

$$\begin{aligned} \lambda_1(U) &\stackrel{\text{def.}}{=} -\sqrt{T'(r)} < \lambda_2(U) \equiv \cdots \equiv \lambda_n(U) \stackrel{\text{def.}}{=} -\sqrt{\frac{T(r)}{r}} \\ &< \lambda_{n+1}(U) \equiv \cdots \equiv \lambda_{2n-1}(U) \stackrel{\text{def.}}{=} \sqrt{\frac{T(r)}{r}} < \lambda_{2n}(U) \stackrel{\text{def.}}{=} \sqrt{T'(r)}. \end{aligned} \tag{3.6.24}$$

When $n = 2$, system (3.6.19) is a strictly hyperbolic system in a neighbourhood of $U = U_0$, in which $\lambda_2(U)$ and $\lambda_3(U)$ are always LD, then WLD with respect to $U = U_0$. When $n > 2$, (3.6.19) is a quasilinear hyperbolic system of conservation laws with characteristics with constant multiplicity, then by Lemma 2.4.2, $\lambda_i(U)$ $(i = 2, \ldots, 2n-1)$ are always LD, then WLD with respect to $U = U_0$. Moreover, it is easy to see that $\lambda_1(U)$ and $\lambda_{2n}(U)$ are WLD with respect to $U = U_0$ if and only if $\lambda_1(U)$ and $\lambda_{2n}(U)$ are LD in a neighbourhood of $U = U_0$, namely,

$$T''(r) \equiv 0 \quad \text{for } |r - \widetilde{r}_0| \ \text{ small.} \tag{3.6.25}$$

By Theorems 3.4.1 and 3.5.1, we have

Theorem 3.6.3 *Under assumption* (3.6.25)*, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, the Cauchy problem* (3.6.19)–(3.6.20) *admits a unique global C^1 solution $U = U(t, x)$ for all $t \in \mathbb{R}$.*

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Chapter IV

The Cauchy Problem (Continued)

In this chapter, under the assumption that the quasilinear hyperbolic system is not weakly linearly degenerate (WLD), we consider the formation of singularities of the C^1 solution to the Cauchy problem and its blow-up mechanism.

4.1 Some Uniform a Priori Estimates Independent of Weak Linear Degeneracy

In this section we give some uniform a priori estimates independent of weak linear degeneracy for the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(u)\dfrac{\partial u}{\partial x} = 0, & (4.1.1) \\ t = 0: \;\; u = \phi(x), & (4.1.2) \end{cases}$$

where (4.1.1) is a strictly hyperbolic system and $\phi(x)$ satisfies (3.2.1). We already proved Lemmas 3.2.1 and 3.2.2 in Section 3.2.

In order to obtain a sharp estimate on the life span of the C^1 solution to the Cauchy problem, we assume that the initial data are of the form

$$\phi(x) = \varepsilon\psi(x), \tag{4.1.3}$$

where $\varepsilon > 0$ is a small parameter and $\psi(x) \in C^1$ satisfies

$$\sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\psi(x)| + |\psi'(x)|)\} < +\infty, \tag{4.1.4}$$

where $\mu > 0$ is a constant.

In this situation we rewrite Lemma 3.2.2 as

Lemma 4.1.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and system* (4.1.1) *is strictly hyperbolic, i.e.,* (2.1.12) *holds. Suppose furthermore that the initial data* (4.1.2) *satisfy* (4.1.3)–(4.1.4). *Then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $D(T)$*

Li Tatsien, Wang Libin, *Global Propagation of Regular Nonlinear Hyperbolic Waves*, DOI 10.1007/b78335_4,

of the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.1.1)–(4.1.2), *we have the following uniform a priori estimates:*

$$W^c_\infty(T) \le \kappa_1 \varepsilon, \tag{4.1.5}$$

$$\widetilde{W}_1(T),\ W_1(T) \le \kappa_2 \varepsilon, \tag{4.1.6}$$

and

$$U_\infty(T) \le \kappa_3 \varepsilon, \tag{4.1.7}$$

where κ_1, κ_2, and κ_3 are positive constants independent of ε and T.

4.2 Formation of Singularities of the C^1 Solution in the Noncritical Case $\alpha < +\infty$

In this section we discuss the formation of singularities of the C^1 solution to the Cauchy problem (4.1.1)–(4.1.2) in the noncritical case, namely, for the non-WLD hyperbolic system (4.1.1) with finite non-WLD index α (see Section 2.5).

4.2.1 Some Uniform a Priori Estimates Depending on Weak Linear Degeneracy

Lemma 4.2.1 *Suppose that in a neighbourhood of $u = 0$, $A(u)$ is suitably smooth and system* (4.1.1) *is strictly hyperbolic. Suppose furthermore that* (4.1.1) *is not WLD and its non-WLD index α is finite. Suppose finally that the initial data* (4.1.2) *satisfy* (3.2.1). *Then in normalized coordinates there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem* (4.1.1)–(4.1.2), *we have the following uniform a priori estimates:*

$$U^c_\infty(T) \le \kappa_4 \theta \{1 + \theta^{\alpha+2} T\}, \tag{4.2.1}$$

$$\widetilde{U}_1(T),\ U_1(T) \le \kappa_5 \theta \{1 + \theta^{\alpha+1} T\}. \tag{4.2.2}$$

When

$$T\theta^{\alpha+1} \le \kappa_6, \tag{4.2.3}$$

we have

$$W_\infty(T) \le \kappa_7 \theta, \tag{4.2.4}$$

in which $U^c_\infty(T)$, $\widetilde{U}_1(T)$, $U_1(T)$, and $W_\infty(T)$ are still defined by (3.3.1)–(3.3.4), *respectively, and $\kappa_4, \ldots, \kappa_7$ are positive constants independent of θ and T.*

Proof. Without loss of generality, we assume that $u = (u_1, \ldots, u_n)$ are normalized coordinates.

We first estimate $U_\infty^c(T)$.

For any given $i \in \{1, \ldots, n\}$, passing through any fixed point $(t, x) \in D(T) \backslash D_i^T$, we draw the ith characteristic c_i: $\xi = \xi_i(\tau)$ $(0 \le \tau \le t)$ which intersects the x-axis at a point $(0, x_{i0})$. Integrating (2.6.44) along c_i from 0 to t and noting (4.1.2) gives

$$u_i(t,x) = \phi_i(x_{i0}) + \int_0^t \sum_{j,k=1}^n \rho_{ijk}(u) u_j w_k(\tau, \xi_i(\tau)) d\tau. \tag{4.2.5}$$

Then, noting (3.1.3) and (2.6.45) and using Lemma 3.2.1, we have

$$\begin{aligned}(1+|x-\lambda_i(0)t|)^{1+\mu}|u_i(t,x)| \le & C\{\theta + U_\infty^c(T)W_\infty^c(T) + U_\infty^c(T)\widetilde{W}_1(T) \\ & + \widetilde{U}_1(T)W_\infty^c(T)\}. \end{aligned} \tag{4.2.6}$$

Hence, using Lemma 3.2.2, we get

$$U_\infty^c(T) \le C\theta\{1 + U_\infty^c(T) + \widetilde{U}_1(T)\}. \tag{4.2.7}$$

We next estimate $\widetilde{U}_1(T)$ and $U_1(T)$.

For $i \in \{1, \ldots, n\}$, passing through two endpoints $A(t_A, x_A)$ and $B(t_B, x_B)$ of any given jth characteristic c_j : $\xi = \xi_j(\tau)$ $(0 \le t_A \le \tau \le t_B)$ on $D(T)$ $(j \ne i)$, we draw the ith characteristic which intersects the x-axis at point $C(0, x_C)$ and point $D(0, x_D)$, respectively, with $x_C \le x_D$. By (2.6.48), using Stokes' formula [see (2.6.53)] on the domain $ACDB$ and noting (4.1.2), we have

$$\begin{aligned}&\int_{t_A}^{t_B} |u_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau \\ &\qquad \le \int_{x_C}^{x_D} |\phi_i(x)| dx + \iint_{ACDB} \left| \sum_{j,k=1}^n F_{ijk}(u) u_j w_k(t,x) \right| dt dx. \end{aligned} \tag{4.2.8}$$

Noting (2.6.52), we have

$$|F_{iii}(u_i e_i)| \le C|u_i|^\alpha. \tag{4.2.9}$$

Then, noting (3.1.3) and (2.6.51) and using Hadamard's formula, Lemmas 3.2.1, and 3.2.2, it follows from (4.2.8) that

$$\int_{t_A}^{t_B} |u_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau$$

$$
\begin{aligned}
&\le \int_{x_C}^{x_D} |\phi_i(x)|dx + \iint\limits_{ACDB} \Bigg[\sum_{\substack{j,k=1\\ j\neq k}}^{n} |F_{ijk}(u)u_j w_k(t,x)| + (|F_{iii}(u) - F_{iii}(u_i e_i)| \\
&\quad + |F_{iii}(u_i e_i)|)|u_i w_i(t,x)| \Bigg] dtdx \\
&\le C\{\theta + U_\infty^c(T)W_\infty^c(T) + U_\infty^c(T)W_1(T) + U_1(T)W_\infty^c(T) \\
&\quad + U_\infty^c(T)U_\infty(T)W_1(T) + (U_\infty(T))^{\alpha+1}W_1(T)T\} \\
&\le C\theta\{1 + U_\infty^c(T) + U_1(T) + \theta^{\alpha+1}T\}. \qquad (4.2.10)
\end{aligned}
$$

Then, noting (3.2.6), we get

$$
\widetilde{U}_1(T) \le C\theta\{1 + U_\infty^c(T) + U_1(T) + \theta^{\alpha+1}T\}. \qquad (4.2.11)
$$

Similarly, we have

$$
U_1(T) \le C\theta\{1 + U_\infty^c(T) + U_1(T) + \theta^{\alpha+1}T\}. \qquad (4.2.12)
$$

The combination of (4.2.7) and (4.2.11)–(4.2.12) gives (4.2.1)–(4.2.2).

We finally estimate $W_\infty(T)$.

Passing through any given point $(t,x) \in D(T)$, we draw the ith characteristic $c_i: \ \xi = \xi_i(\tau)$ $(0 \le \tau \le t)$ which intersects the x-axis at a point $(0, x_{i0})$. Integrating (2.6.14) along c_i from 0 to t, we have

$$
w_i(t,x) = w_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k(\tau, \xi_i(\tau)) d\tau. \qquad (4.2.13)
$$

Noting (2.6.17), we have

$$
|\gamma_{iii}(u_i e_i)| \le C|u_i|^\alpha. \qquad (4.2.14)
$$

Then, noting (3.2.1) and (2.6.16) and using Lemma 3.2.1 and Hadamard's formula, from (4.2.13) we have

$$
\begin{aligned}
&|w_i(t,x)| \\
&\le |w_i(0,x_{i0})| + \int_0^t \Bigg[\sum_{\substack{j,k=1\\ j\neq k}}^{n} |\gamma_{ijk}(u) w_j w_k(\tau,\xi_i(\tau))| + (|\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)| \\
&\quad + |\gamma_{iii}(u_i e_i)|) w_i^2(\tau, \xi_i(\tau)) \Bigg] d\tau \\
&\le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T)W_\infty(T) + U_\infty^c(T)(W_\infty(T))^2 \\
&\quad + (U_\infty(T))^\alpha T (W_\infty(T))^2\}. \qquad (4.2.15)
\end{aligned}
$$

Noting (4.2.1) and using Lemma 3.2.2, we get

$$W_\infty(T) \leq C\{\theta + (\theta + \theta^\alpha T)(W_\infty(T))^2\}. \tag{4.2.16}$$

Hence, noting (4.2.3), we can obtain (4.2.4).

For any given $i \overline{\in} J$, where J is defined in Section 2.5.1, $\lambda_i(u)$ is WLD. By (2.6.28) and (2.6.54), similarly to the proof of Lemma 4.2.1, it is easy to get

Lemma 4.2.2 *Under assumptions of Lemma 4.2.1, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem* (4.1.1)–(4.1.2)*, without restriction* (4.2.3) *we have the following uniform a priori estimate:*

$$|w_i(t, x)| \leq \kappa_8 \theta, \quad \forall i \overline{\in} J, \tag{4.2.17}$$

where κ_8 is a positive constant independent of θ and T.

Remark 4.2.1 *Under the assumptions of Lemma 4.2.1, suppose furthermore that the initial data* (4.1.2) *satisfy* (4.1.3) *and* (4.1.4)*. Then in normalized coordinates there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, on any given existence domain $D(T)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem* (4.1.1)–(4.1.2)*, we have the following uniform a priori estimates:*

$$U_\infty^c(T) \leq \kappa_4 \varepsilon\{1 + \varepsilon^{\alpha+2} T\}, \tag{4.2.18}$$

$$\widetilde{U}_1(T),\ U_1(T) \leq \kappa_5 \varepsilon\{1 + \varepsilon^{\alpha+1} T\}. \tag{4.2.19}$$

When

$$T\varepsilon^{\alpha+1} \leq \kappa_6, \tag{4.2.20}$$

we have

$$W_\infty(T) \leq \kappa_7 \varepsilon, \tag{4.2.21}$$

where $\kappa_4, \ldots, \kappa_7$ are positive constants independent of ε and T.

Remark 4.2.2 *According to Lemmas 3.2.1 and 4.2.1, under assumption* (3.2.1)*, there exists a unique C^1 solution $u = u(t, x)$ on $[0, \kappa_6 \theta^{\alpha+1}] \times \mathbb{R}$ to the Cauchy problem* (4.1.1)–(4.1.2)*. Thus, the life span $\widetilde{T}(\theta)$ of the C^1 solution $u = u(t, x)$ satisfies*

$$\widetilde{T}(\theta) > \kappa_6 \theta^{\alpha+1}. \tag{4.2.22}$$

Similarly, if the initial data (4.1.2) *satisfy* (4.1.3) *and* (4.1.4)*, by Lemma 4.1.1 and Remark 4.2.1, the life span $\widetilde{T}(\varepsilon)$ of the C^1 solution $u = u(t, x)$ satisfies*

$$\widetilde{T}(\varepsilon) > \kappa_6 \varepsilon^{\alpha+1}. \tag{4.2.23}$$

4.2.2 Sharp Estimate on the Life Span of the C^1 Solution

According to Theorem 3.1.1, when system (4.1.1) is not WLD, at least for certain small and decaying initial data, the C^1 solution to the Cauchy problem (4.1.1)–(4.1.2) must blow up in a finite time. The following theorem will further illustrate this fact. Moreover, an asymptotic behavior of the life span of the C^1 solution will be given (cf. [57], [75], [76]).

Theorem 4.2.1 *Suppose that in a neighbourhood of $u = 0$, $A(u)$ is suitably smooth and system* (4.1.1) *is strictly hyperbolic. Suppose furthermore that* (4.1.1) *is not WLD and its non-WLD index α is finite. Suppose finally that the initial data* (4.1.2) *satisfy* (4.1.3)–(4.1.4)*. If there exists $m_0 \in J_1$ such that*

$$l_{m_0}(0)\psi(x) \not\equiv 0, \tag{4.2.24}$$

where J_1 is defined by (2.5.7)*, then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the C^1 solution $u = u(t, x)$ to the Cauchy problem* (4.1.1)–(4.1.2) *must blow up in a finite time and the life span $\widetilde{T}(\varepsilon)$ of $u = u(t, x)$ satisfies*

$$\lim_{\varepsilon\downarrow 0}(\varepsilon^{\alpha+1}\widetilde{T}(\varepsilon)) = M_0, \tag{4.2.25}$$

where

$$M_0 = \left\{\max_{i\in J_1}\sup_{x\in\mathbb{R}}[-\frac{1}{\alpha!}\left.\frac{d^{\alpha+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha+1}}\right|_{s=0}(l_i(0)\psi(x))^{\alpha}l_i(0)\psi'(x)]\right\}^{-1}, \tag{4.2.26}$$

in which $u = u^{(i)}(s)$ is defined by (2.5.1)*.*

Proof. Let $u = u(\widetilde{u})$ [u(0) = 0] be the normalized transformation such that (2.4.11) holds. We have

$$\frac{\partial u}{\partial \widetilde{u}}(0) = R(0), \tag{4.2.27}$$

where $R(u)$ is the matrix composed of right eigenvectors $r_i(u)$ $(i = 1, \dots, n)$. Then, noting (2.1.5), we have

$$\frac{\partial \widetilde{u}}{\partial u}(0) = L(0), \tag{4.2.28}$$

where $L(u)$ is the matrix composed of left eigenvectors $l_i(u)$ $(i = 1, \dots, n)$. Thus, in normalized coordinates $\widetilde{u}$, (4.1.3) can be rewritten as

$$t = 0: \ \widetilde{u}_i = \widetilde{u}_i(\varepsilon\psi(x)) = \varepsilon l_i(0)\psi(x) + O(\varepsilon^2) \quad (i = 1, \dots, n). \tag{4.2.29}$$

Correspondingly, noting Remark 2.6.2, we have

$$\begin{aligned} t = 0: \ \widetilde{w}_i = w_i &= l_i(\varepsilon\psi(x))(\varepsilon\psi'(x)) \\ &= \varepsilon l_i(0)\psi'(x) + O(\varepsilon^2) \quad (i = 1, \dots, n). \end{aligned} \tag{4.2.30}$$

For simplicity of statement, in what follows, we will still denote normalized coordinates $\widetilde{u}$ by u. Then we rewrite (4.2.29)–(4.2.30) as

$$t = 0: \ u_i = \varepsilon l_i(0)\psi(x) + O(\varepsilon^2) \quad (i = 1, \dots, n), \tag{4.2.31}$$

$$t = 0: \ w_i = \varepsilon l_i(0)\psi'(x) + O(\varepsilon^2) \quad (i = 1, \dots, n). \tag{4.2.32}$$

In normalized coordinates, (4.2.26) is rewritten as

$$M_0 = \left\{ \max_{i \in J_1} \sup_{x \in \mathbb{R}} \left[-\frac{1}{\alpha!} \frac{\partial^{\alpha+1}\lambda_i}{\partial u_i^{\alpha+1}}(0)(l_i(0)\psi(x))^\alpha l_i(0)\psi'(x) \right] \right\}^{-1}. \tag{4.2.33}$$

Now we show (4.2.25), in which M_0 satisfies (4.2.33). For this purpose, it is sufficient to prove that

$$\overline{\lim_{\varepsilon \downarrow 0}} \{\varepsilon^{1+\alpha}\widetilde{T}(\varepsilon)\} \le M_0 \tag{4.2.34}$$

and

$$\underline{\lim_{\varepsilon \downarrow 0}} \{\varepsilon^{1+\alpha}\widetilde{T}(\varepsilon)\} \ge M_0. \tag{4.2.35}$$

We first prove (4.2.34).

For the time being, we suppose that

$$\widetilde{T}(\varepsilon) \le \varepsilon^{-(\alpha+2)}. \tag{4.2.36}$$

We will explain the validity of this hypothesis later.

Noting (4.1.4) and (4.2.24), it is easy to see that there exists $x_0 \in \mathbb{R}$ for certain $m \in J_1$ such that

$$M_0^{-1} = -\frac{1}{\alpha!} \frac{\partial^{\alpha+1}\lambda_m}{\partial u_m^{\alpha+1}}(0)(l_m(0)\psi(x_0))^\alpha l_m(0)\psi'(x_0) > 0, \tag{4.2.37}$$

namely,

$$M_0^{-1} = \frac{1}{\alpha!} \frac{\partial^\alpha \gamma_{mmm}}{\partial u_m^\alpha}(0)(l_m(0)\psi(x_0))^\alpha l_m(0)\psi'(x_0) > 0, \tag{4.2.38}$$

where γ_{mmm} is given by (2.6.17). Without loss of generality, we assume

$$a(l_m(0)\psi(x_0))^\alpha > 0 \quad \text{and} \quad l_m(0)\psi'(x_0) > 0, \tag{4.2.39}$$

in which

$$a = \frac{1}{\alpha!}\frac{\partial^\alpha \gamma_{mmm}}{\partial u_m^\alpha}(0). \tag{4.2.40}$$

Let $c_m:\ \xi = \xi_m(t;x_0)$ $\left[t \in [0,\widetilde{T}(\varepsilon))\right]$ be the mth characteristic passing through the point $(0,x_0)$. According to (2.6.14), along c_m we have

$$\begin{aligned}\frac{dw_m(t,\xi)}{dt} &= \sum_{j,k=1}^{n}\gamma_{mjk}(u)w_jw_k(t,\xi)\\ &= a_m^{(0)}(t)w_m^2(t,\xi)+a_m^{(1)}(t)w_m(t,\xi)+a_m^{(2)}(t), \quad \forall t\in[0,\widetilde{T}(\varepsilon)),\end{aligned} \tag{4.2.41}$$

where

$$a_m^{(0)}(t) = \gamma_{mmm}(u)(t,\xi), \tag{4.2.42}$$

$$a_m^{(1)}(t) = 2\sum_{j\neq m}\gamma_{mjm}(u)w_j(t,\xi), \tag{4.2.43}$$

and

$$a_m^{(2)}(t) = \sum_{j,k\neq m}\gamma_{mjk}(u)w_jw_k(t,\xi). \tag{4.2.44}$$

Integrating (2.6.44) (in which we take $i=m$) along c_m with respect to t, we have

$$u_m(t,\xi) = u_m(0,x_0)+\int_0^t\sum_{j,k=1}^{n}\rho_{mjk}(u)u_jw_k(\tau,\xi_m(\tau))d\tau, \quad \forall t\in[0,\widetilde{T}(\varepsilon)). \tag{4.2.45}$$

Then, noting (2.6.45), (2.6.47), and (4.2.31) and using Lemmas 3.2.1, 4.1.1, and 4.2.1, we get

$$\begin{aligned}|u_m(t,\xi)-\varepsilon l_m(0)\psi(x_0)| &\le C\{\varepsilon^2+U_\infty^c(T)W_\infty^c(T)+U_\infty^c(T)\widetilde{W}_1(T)\\ &\quad +U_\infty(T)W_\infty^c(T)\}\\ &\le C\varepsilon^2.\end{aligned} \tag{4.2.46}$$

Hence, noting (4.1.7), it is easy to see that

$$|(u_m(t,\xi))^\alpha-(\varepsilon l_m(0)\psi(x_0))^\alpha| \le C\varepsilon^{\alpha+1}. \tag{4.2.47}$$

By the definition of J_1 [see (2.5.7)] and noting (2.5.9), we have

$$\gamma_{mmm}(u_me_m)(t,\xi) = au_m^\alpha(t,\xi)+O(|u_m(t,\xi)|^{\alpha+1}), \tag{4.2.48}$$

in which a is defined by (4.2.40). Then, using (4.2.47) and noting (4.1.7), we get

$$\begin{aligned}\gamma_{mmm}(u_m e_m)(t,\xi) &= a(\varepsilon l_m(0)\psi(x_0))^\alpha + a[u_m^\alpha(t,\xi) - (\varepsilon l_m(0)\psi(x_0))^\alpha] \\ &\quad + O(|u_m(t,\xi)|^{\alpha+1}) \\ &= a(l_m(0)\psi(x_0))^\alpha \varepsilon^\alpha + O(\varepsilon^{\alpha+1}).\end{aligned} \tag{4.2.49}$$

On the other hand, taking $\varepsilon_0 > 0$ so small that when $t \in [\varepsilon^{-\alpha}, T]$ with $T \in [\varepsilon^{-\alpha}, \widetilde{T}(\varepsilon))$, c_m stays in D_m^T. Noting Lemma 3.2.1, we have

$$|a_m^{(0)}(t) - \gamma_{mmm}(u_m e_m)(t,\xi)| \leq C(1+t)^{-(1+\mu)} U_\infty^c(T), \quad \forall t \in [\varepsilon^{-\alpha}, T]. \tag{4.2.50}$$

Then, using Remark 4.2.1, we get

$$|a_m^{(0)}(t) - \gamma_{mmm}(u_m e_m)(t,\xi)| \leq C\varepsilon^{\alpha+1}, \quad \forall t \in [\varepsilon^{-\alpha}, T]. \tag{4.2.51}$$

The combination of (4.2.49) and (4.2.51) gives

$$a_m^{(0)}(t) = a(l_m(0)\psi(x_0))^\alpha \varepsilon^\alpha + O(\varepsilon^{\alpha+1}), \quad \forall t \in [\varepsilon^{-\alpha}, T]. \tag{4.2.52}$$

Hence, noting the first inequality in (4.2.39), we have

$$a_m^{(0)}(t) \geq \frac{1}{2} a(l_m(0)\psi(x_0))^\alpha \varepsilon^\alpha > 0, \quad \forall t \in [\varepsilon^{-\alpha}, T]. \tag{4.2.53}$$

Noting Remark 4.2.2, the integration of (4.2.41) from 0 to t ($t \leq T \leq \kappa_6 \varepsilon^{-(\alpha+1)}$) yields

$$w_m(t,\xi) = w_m(0,x_0) + \int_0^t [a_m^{(0)}(\tau) w_m^2(\tau,\xi) + a_m^{(1)}(\tau) w_m(\tau,\xi) + a_m^{(2)}(\tau)] d\tau. \tag{4.2.54}$$

Noting (2.6.16) and using Lemmas 3.2.1 and 4.1.1, we have

$$\int_0^T |a_m^{(1)}(t)| dt \leq c\{W_\infty^c(T) + \widetilde{W}_1(T)\} \leq c\varepsilon \tag{4.2.55}$$

and

$$\int_0^T |a_m^{(2)}(t)| dt \leq c\{(W_\infty^c(T))^2 + W_\infty^c(T)\widetilde{W}_1(T)\} \leq c\varepsilon^2. \tag{4.2.56}$$

By (4.2.42) and using Hadamard's formula, it is easy for us to get

$$a_m^{(0)}(t) = \gamma_{mmm}(u_m e_m)(t,\xi) + \sum_{j \neq m} \left[\int_0^1 \frac{\partial \gamma_{mmm}}{\partial u_j}(\tau u + (1-\tau) u_m e_m) d\tau \right] u_j(t,\xi). \tag{4.2.57}$$

Then, using Lemma 4.2.1 and noting (4.2.32) (in which $i = m$) and (4.2.49), we get

$$\begin{aligned}
&|w_m(t,\xi) - \varepsilon l_m(0)\psi'(x_0)| \\
&\le C\left\{\varepsilon^2 + (W_\infty(T))^2\left[\int_0^t |\gamma_{mmm}(u_m e_m)(\tau,\xi)|d\tau + \widetilde{U}_1(T)\right] + \varepsilon W_\infty(T)\right\} \\
&\le C\{\varepsilon^2 + \varepsilon^{\alpha+2}t\}. \qquad (4.2.58)
\end{aligned}$$

Hence,

$$w_m(\varepsilon^{-\alpha}, \xi_m(\varepsilon^{-\alpha}; x_0)) = \varepsilon l_m(0)\psi'(x_0) + O(\varepsilon^2). \qquad (4.2.59)$$

Then, noting the second inequality in (4.2.39), we get

$$w_m(\varepsilon^{-\alpha}, \xi_m(\varepsilon^{-\alpha}; x_0)) \ge \frac{1}{2}\varepsilon l_m(0)\psi'(x_0) > 0. \qquad (4.2.60)$$

Let

$$\bar{K} = \int_{\varepsilon^{-\alpha}}^{\widetilde{T}(\varepsilon)-\varepsilon} |a_m^{(2)}(t)|dt \exp\left(\int_{\varepsilon^{-\alpha}}^{\widetilde{T}(\varepsilon)-\varepsilon} |a_m^{(1)}(t)|dt\right). \qquad (4.2.61)$$

Noting (4.2.55)–(4.2.56), we have

$$\bar{K} \le C\varepsilon^2. \qquad (4.2.62)$$

The combination of (4.2.60) and (4.2.62) gives

$$w_m(\varepsilon^{-\alpha}, \xi_m(\varepsilon^{-\alpha}; x_0)) > \bar{K}. \qquad (4.2.63)$$

In what follows, on the interval $[\varepsilon^{-\alpha}, \widetilde{T}(\varepsilon) - \varepsilon]$ we consider the initial problem of Equation (4.2.41) with the initial data

$$t = \varepsilon^{-\alpha}: \ w_m = w_m(\varepsilon^{-\alpha}, \xi_m(\varepsilon^{-\alpha}; x_0)). \qquad (4.2.64)$$

By Lemma 2.7.1, we have

$$\begin{aligned}
&\int_{\varepsilon^{-\alpha}}^{\widetilde{T}(\varepsilon)-\varepsilon} |a_m^{(0)}(t)|dt \exp\left(-\int_{\varepsilon^{-\alpha}}^{\widetilde{T}(\varepsilon)-\varepsilon} |a_m^{(1)}(t)|dt\right) \\
&\qquad \le (w_m(\varepsilon^{-\alpha}, \xi_m(\varepsilon^{-\alpha}; x_0)) - \bar{K})^{-1}. \qquad (4.2.65)
\end{aligned}$$

Then, noting (4.2.55), we get

$$(w_m(\varepsilon^{-\alpha}, \xi_m(\varepsilon^{-\alpha}; x_0)) - \bar{K}) \int_{\varepsilon^{-\alpha}}^{\widetilde{T}(\varepsilon)-\varepsilon} a_m^{(0)}(t)dt < \exp(C\varepsilon). \qquad (4.2.66)$$

Moreover, noting (4.2.52), (4.2.59), and (4.2.62), we have

$$[\varepsilon^{\alpha+1}a(l_m(0)\psi(x_0))^\alpha l_m(0)\psi'(x_0)+O(\varepsilon^{\alpha+2})](\widetilde{T}(\varepsilon)-\varepsilon-\varepsilon^{-\alpha})<\exp(C\varepsilon). \tag{4.2.67}$$

Thus, we obtain (4.2.34) and the validity of hypothesis (4.2.36).

We next prove (4.2.35).

For any fixed $M\in(0,M_0)$, it suffices to prove

$$\widetilde{T}(\varepsilon)\ge M\varepsilon^{-(\alpha+1)}. \tag{4.2.68}$$

For this purpose, for each $i=1,\dots,n$ and any given $y\in\mathbb{R}$, we consider

$$\frac{dw_i(t,\xi)}{dt}=a_i^{(0)}(t)w_i^2(t,\xi)+a_i^{(1)}(t)w_i(t,\xi)+a_i^{(2)}(t), \tag{4.2.69}$$

where $\xi=\xi_i(t;y)$, and $a_i^{(0)}$, $a_i^{(1)}$, and $a_i^{(2)}$ are similarly defined as in (4.2.42)–(4.2.44).

(i) Suppose that

$$w_i(0,y)\ge 0. \tag{4.2.70}$$

Similarly to (4.2.55)–(4.2.57), we have

$$\int_0^T|a_i^{(1)}(t)|dt\le c\{W_\infty^c(T)+\widetilde{W}_1(T)\}\le C\varepsilon,$$
$$\int_0^T|a_i^{(2)}(t)|dt\le C\{(W_\infty^c(T))^2+W_\infty^c(T)\widetilde{W}_1(T)\}\le C\varepsilon^2,$$

and

$$a_i^{(0)}(t)=\gamma_{iii}(u_ie_i)(t,\xi)+\sum_{j\ne i}\left[\int_0^1\frac{\partial\gamma_{iii}}{\partial u_j}(\tau u+(1-\tau)u_ie_i)d\tau\right]u_j(t,\xi).$$

Then we have

$$K=\int_0^T|a_i^{(2)}(t)|dt\exp\left(\int_0^T|a_i^{(1)}(t)|dt\right)\le C\varepsilon^2,\quad\forall T\in[0,M\varepsilon^{-(\alpha+1)}], \tag{4.2.71}$$

$$\int_0^T(a_i^{(0)}(t))^+dt\le\int_0^T[\gamma_{iii}(u_ie_i)(t,\xi)]^+dt+\widetilde{U}_1(T),\quad\forall T\in[0,M\varepsilon^{-(\alpha+1)}], \tag{4.2.72}$$

and

$$\int_0^T |a_i^{(0)}(t)|dt \leq \int_0^T |\gamma_{iii}(u_i e_i)(t,\xi)|dt + \widetilde{U}_1(T), \quad \forall T \in [0, M\varepsilon^{-(\alpha+1)}]. \tag{4.2.73}$$

Similarly to (4.2.49), it is easy to see that

$$\gamma_{iii}(u_i e_i)(t,\xi) = \begin{cases} a(l_i(0)\psi(y))^\alpha + O(\varepsilon^{\alpha+1}), & i \in J_1, \\ O(\varepsilon^{\alpha+1}), & i \overline{\in} J_1, \end{cases} \tag{4.2.74}$$

in which

$$a = -\frac{1}{\alpha!}\frac{\partial^\alpha \gamma_{iii}}{\partial u_i^\alpha}(0). \tag{4.2.75}$$

Then, using Lemma 4.2.1, we get

$$\int_0^T (a_i^{(0)}(t))^+ dt \leq [a(l_i(0)\psi(y))^\alpha]^+ \varepsilon^{-1} M + C, \quad \forall T \in [0, M\varepsilon^{-(\alpha+1)}], \tag{4.2.76}$$

and

$$\int_0^T |a_i^{(0)}(t)|dt \leq |a(l_i(0)\psi(y))^\alpha)|\varepsilon^{-1} M + C, \quad \forall T \in [0, M\varepsilon^{-(\alpha+1)}]. \tag{4.2.77}$$

Moreover, we have

$$\begin{aligned} (w_i(0,y) + K)&\left[\int_0^T (a_i^{(0)}(t))^+ dt \exp\left(\int_0^T |a_i^{(1)}(t)|dt\right)\right] \\ &\leq (\varepsilon l_i(0)\psi'(y) + O(\varepsilon^2))\{[a(l_i(0)\psi(y))^\alpha]^+ \varepsilon^{-1} M + C\} \exp(C\varepsilon) \\ &\leq \left(\frac{M}{M_0} + C\varepsilon\right)\exp(C\varepsilon) < 1, \quad \forall T \in [0, M\varepsilon^{-(\alpha+1)}], \end{aligned} \tag{4.2.78}$$

and

$$K\left[\int_0^T (a_i^{(0)}(t))^- dt \exp\left(\int_0^T |a_i^{(1)}(t)|dt\right)\right] \leq C\varepsilon < 1, \quad \forall T \in [0, M\varepsilon^{-(\alpha+1)}]. \tag{4.2.79}$$

By Lemma 2.7.2, (4.2.69) with $w_i(0,\xi_i(0;y)) = w_i(0,y)$ admits a unique solution $w_i = w_i(t,\xi_i(t;y))$ on $[0,T]$. Furthermore, if $w_i(T,\xi_i(T;y)) > 0$, then

$$\frac{1}{w_i(T,\xi_i(T;y))} \geq \frac{1}{w_i(0,y) + K} - \int_0^T (a_i^{(0)}(t))^+ dt \exp\left(\int_0^T |a_i^{(1)}(t)|dt\right) \tag{4.2.80}$$

whereas, if $w_i(T, \xi_i(T; y)) < 0$, then

$$\frac{1}{|w_i(T, \xi_i(T; y))|} \geq K^{-1} - \int_0^T (a_i^{(0)}(t))^- dt \exp\left(\int_0^T |a_i^{(1)}(t)| dt\right). \quad (4.2.81)$$

Then, noting (4.2.78)–(4.2.79), we get

$$|w_i(T, \xi_i(T; y))| \leq C\varepsilon, \quad \forall T \in [0, M\varepsilon^{-(\alpha+1)}]. \quad (4.2.82)$$

(ii) Suppose that

$$w_i(0, y) < 0. \quad (4.2.83)$$

We can similarly obtain (4.2.82).

Since y is arbitrary, similarly to (4.2.82), we have

$$|w_i(t, x)| \leq C\varepsilon, \quad \forall (t, x) \in [0, M\varepsilon^{-(\alpha+1)}] \times \mathbb{R} \quad (i = 1, \ldots, n). \quad (4.2.84)$$

Hence, noting Lemma 4.1.1, we obtain the following uniform a priori estimate:

$$\|u(t, \cdot)\|_{C^1(\mathbb{R})} \leq C\varepsilon, \quad \forall t \in [0, M\varepsilon^{-(\alpha+1)}], \quad (4.2.85)$$

which implies (4.2.68).

Remark 4.2.3 *By Lemma 4.1.1, when the C^1 solution $u = u(t, x)$ to the Cauchy problem* (4.1.1)–(4.1.2) *with* (4.1.3)–(4.1.4) *blows up in a finite time, $u = u(t, x)$ itself is bounded and small on $[0, \widetilde{T}(\varepsilon))$, while the first-order partial derivative u_x tends to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$.*

Remark 4.2.4 *Under the assumptions of Theorem 4.2.1, the life span $\widetilde{T}(\varepsilon)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem* (4.1.1)–(4.1.2) *with* (4.1.3)–(4.1.4) *has the following sharp estimate:*

$$\bar{\kappa}\varepsilon^{-(\alpha+1)} \leq \widetilde{T}(\varepsilon) \leq \bar{\bar{\kappa}}\varepsilon^{-(\alpha+1)}, \quad (4.2.86)$$

where $\bar{\kappa}$ and $\bar{\bar{\kappa}}$ are positive constants independent of ε (see [76]).

Remark 4.2.5 *Using Lemma 2.7.2, similarly to the proof of* (4.2.35), *we can easily see that if*

$$a(l_i(0)\psi(y))^\alpha l_i(0)\psi'(y) \leq 0, \quad (4.2.87)$$

in which a is defined by (4.2.75), *then*

$$|w_i(t, x)| \leq C\varepsilon, \quad \forall (t, x) \in [0, \widetilde{T}(\varepsilon)) \times \mathbb{R}. \quad (4.2.88)$$

4.3 Blow-Up Mechanism of the C^1 Solution in the Noncritical Case $\alpha < +\infty$

4.3.1 Introduction and Main Results

In order to get a sharp estimate on the life span, the previous results about the blow-up phenomenon mainly focus on the special initial data (4.1.3)–(4.1.4). However, according to Theorem 3.1.1, if system (4.1.1) is not WLD, for any given $\theta_0 > 0$ suitably small, we can always find some initial data (4.1.2) with $\theta \in (0, \theta_0]$, where θ is defined by (3.2.1), such that the C^1 solution $u = u(t, x)$ to the Cauchy problem (4.1.1)–(4.1.2) blows up in a finite time. Thus, finding the blow-up mechanism is an interesting problem. In this section we study the blow-up phenomenon, particularly the geometric blow-up mechanism (cf. [1]) for the general initial data (4.1.2) satisfying (3.2.1) in the noncritical case $\alpha < +\infty$.

The main results are the following theorems (cf. [93]).

Theorem 4.3.1 *Suppose that in a neighbourhood of $u = 0$, $A(u)$ is suitably smooth and system* (4.1.1) *is strictly hyperbolic. Suppose furthermore that system* (4.1.1) *is not WLD and the non-WLD index*

$$\alpha < +\infty. \tag{4.3.1}$$

Suppose finally that the initial data (4.1.2) *satisfy* (3.2.1). *Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, the life span $\widetilde{T}(\theta)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem* (4.1.1)–(4.1.2) *satisfies*

$$\widetilde{T}(\theta) > \kappa \theta^{-(\alpha+1)}, \tag{4.3.2}$$

where κ is a positive constant independent of θ. Moreover, when $u = u(t, x)$ blows up in a finite time, $u = u(t, x)$ itself is bounded and small on the domain $[0, \widetilde{T}(\theta)) \times \mathbb{R}$, while the first-order partial derivative u_x of $u = u(t, x)$ tends to be unbounded as $t \nearrow \widetilde{T}(\theta)$.

Theorem 4.3.2 *Under the assumptions of Theorem 4.3.1, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, the C^1 solution $u = u(t, x)$ to the Cauchy problem* (4.1.1)–(4.1.2) *blows up in a finite time if and only if at least one family of characteristics forms an envelope in a finite time.*

Theorem 4.3.3 *Under the assumptions of Theorem 4.3.1, for each $i \,\overline{\in}\, J$, the family of the ith characteristics never forms any envelope on the domain $[0, \widetilde{T}(\theta)] \times \mathbb{R}$.*

Theorem 4.3.4 *Under the assumptions of Theorem 4.3.1, on the line $t = \widetilde{T}(\theta)$, the set of blow-up points cannot possess a positive measure.*

Theorem 4.3.5 *Under the assumptions of Theorem 4.3.1, if* (t^*, x^*)*, in which* $t^* = \widetilde{T}(\theta)$*, is a blow-up point of the* C^1 *solution* $u = u(t, x)$ *to the Cauchy problem* (4.1.1)–(4.1.2) *satisfying* (3.2.1)*, then there exists at least one mth characteristic* $x = x_m(t)$ $(m \in J)$ *passing through* (t^*, x^*) *with* $0 \leq t \leq t^*$ *such that along it, we have*

$$u_x = O((t^* - t)^{-1}) \quad as \quad t \uparrow t^*. \tag{4.3.3}$$

Otherwise, u_x *is a higher-order infinitely larger quantity than* $(t^* - t)^{-1}$ *as* $t \uparrow t^*$ (a related result can be found in [2])*.*

Remark 4.3.1 *Under the assumptions of Theorem 4.3.1, if* $\phi(x) = \varepsilon\psi(x)$*, where* $\psi(x)$ *satisfies* (4.1.4) *and the additional hypothesis* (4.2.24)*, for each* $i \,\overline{\in}\, J_1$*, the family of the ith characteristics never forms any envelope on the domain* $[0, \widetilde{T}(\varepsilon)] \times \mathbb{R}$ (see [57])*; moreover,* (4.3.3) *holds* (see [41])*.*

4.3.2 Proof of Main Results

Proof of Theorem 4.3.1. According to the existence and uniqueness of the local C^1 solution to the Cauchy problem (4.1.1)–(4.1.2) (see [72]) and noting Lemmas 3.2.2 and 4.2.1, it is easy to see that the life span $\widetilde{T}(\theta)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (4.1.1)–(4.1.2) satisfies (4.3.2). Moreover, by Lemma 3.2.2, when the C^1 solution $u = u(t, x)$ to the Cauchy problem (4.1.1)–(4.1.2) satisfying (3.2.1) blows up in a finite time, $u = u(t, x)$ itself must be bounded and small on $[0, \widetilde{T}(\theta))$. Hence, the first-order partial derivative u_x should tend to be unbounded as $t \nearrow \widetilde{T}(\theta)$. □

Proof of Theorem 4.3.2. Assume that (t^*, x^*) is a starting point of the blow-up of the C^1 solution $u = u(t, x)$ to the Cauchy problem (4.1.1)–(4.1.2). By Theorem 4.3.1, we have

$$t^* > \kappa\theta^{-(\alpha+1)}. \tag{4.3.4}$$

For each $i = 1, \ldots, n$, passing through any given point (t, x) with $0 \leq t < t^*$ and $x \in \mathbb{R}$, we draw the ith characteristic $c_i : \ \xi = x_i(\tau; y_i)$ in which $0 \leq \tau \leq t$ and y_i stands for the x-coordinate of the intersection point of this characteristic with the x-axis, i.e., we have

$$\frac{dx_i(\tau; y_i)}{d\tau} = \lambda_i(u(\tau, x_i(\tau; y_i))) \tag{4.3.5}$$

and

$$x_i(0; y_i) = y_i, \quad x_i(t; y_i) = x. \tag{4.3.6}$$

In what follows, we prove that

$$\left| w_i(t,x)\frac{\partial x_i(t;y_i)}{\partial y_i} \right| \le C_{t^*}\theta, \quad \forall (t,x)\in[0,t^*)\times\mathbb{R}. \tag{4.3.7}$$

Henceforth C_{t^*} denotes a positive constant possibly depending on t^*.

Noting (2.6.4), it follows from (4.3.5)–(4.3.6) that

$$\frac{d}{d\tau}\left(\frac{\partial x_i(\tau;y_i)}{\partial y_i}\right) = \sum_{k=1}^{n} \nabla\lambda_i(u)w_k r_k(u)(\tau, x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i} \tag{4.3.8}$$

and

$$\frac{\partial x_i(0;y_i)}{\partial y_i} = 1. \tag{4.3.9}$$

Hence,

$$\frac{\partial x_i(t;y_i)}{\partial y_i} = \exp\left(\int_0^t \sum_{k=1}^{n} \nabla\lambda_i(u)w_k r_k(u)(\tau, x_i(\tau;y_i))d\tau\right), \quad \forall t\in[0,t^*). \tag{4.3.10}$$

Noting (2.6.14) and (4.3.8), it is easy to derive

$$\frac{d}{d\tau}\left[w_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\right] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}, \tag{4.3.11}$$

where $\Gamma_{ijk}(u)$ is defined by (2.6.19). Then, noting (4.3.6), for any given $(t,x)\in[0,t^*)\times\mathbb{R}$, we have

$$w_i(t,x)\frac{\partial x_i(t;y_i)}{\partial y_i} = w_i(0,y_i) + \int_0^t \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}d\tau. \tag{4.3.12}$$

Let

$$I_0 = [0,\kappa_6\theta^{-(\alpha+1)}], \tag{4.3.13}$$

$$I_1 = [\kappa_6\theta^{-(\alpha+1)},t]\cap\{\tau|0\le\tau\le t, |w_i(\tau,x_i(\tau;y_i))|\le\kappa_1\theta\}, \tag{4.3.14}$$

and

$$I_2 = [\kappa_6\theta^{-(\alpha+1)},t]\cap\{\tau|0\le\tau\le t, |w_i(\tau,x_i(\tau;y_i))|>\kappa_1\theta\}, \tag{4.3.15}$$

where κ_1 is given in Lemma 3.2.2. Then, noting (2.6.20), (4.3.12) can be rewritten as

$$w_i(t,x)\frac{\partial x_i(t;y_i)}{\partial y_i} = w_i(0,y_i) + 2\int_0^t \sum_{\substack{j=1\\j\ne i}}^{n} \Gamma_{iji}(u)w_j w_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}d\tau$$

$$
\begin{aligned}
&+\left(\int_{I_0}+\int_{I_1}+\int_{I_2}\right)\sum_{\substack{j,k=1\\ j,k\neq i}}^{n}\Gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}d\tau\\
&= w_i(0,y_i)+E+E_0+E_1+E_2. \qquad (4.3.16)
\end{aligned}
$$

Now we estimate every term on the right-hand side of (4.3.16).
Obviously,

$$
|w_i(0,y_i)| \leq C\theta, \quad \forall y_i \in \mathbb{R}. \qquad (4.3.17)
$$

Let

$$
Q(t)=\sup_{(\tau,y_i)\in[0,t]\times\mathbb{R}}\left|w_i(\tau,x_i(\tau;y_i))\frac{\partial x_i(\tau;y_i)}{\partial y_i}\right|, \quad \forall t\in[0,t^*). \qquad (4.3.18)
$$

By Lemma 3.2.2, we get

$$
|E| \leq CQ(t)\widetilde{W}_1(t) \leq C\theta Q(t). \qquad (4.3.19)
$$

By Lemma 4.2.1, we have

$$
|w_i(\tau,x_i(\tau;y_i))| \leq C\theta, \quad \forall \tau \in I_0\cup I_1. \qquad (4.3.20)
$$

Then, noting (2.6.17) and using Lemmas 3.2.2 and 4.2.1, it follows from (4.3.10) that

$$
\begin{aligned}
\frac{\partial x_i(\tau;y_i)}{\partial y_i} &\leq \exp\left\{C\left[\int_0^\tau\left(\sum_{\substack{k=1\\ k\neq i}}^{n}|w_k|+|(\gamma_{iii}(u)-\gamma_{iii}(u_ie_i))w_i|+|\gamma_{iii}(u_ie_i)w_i|\right)\right.\right.\\
&\qquad \left.\left.\times(\sigma,x_i(\sigma;y_i))d\sigma\right]\right\}\\
&\leq \exp\{C[\widetilde{W}_1(\tau)+\theta(\widetilde{U}_1(\tau)+(U_\infty(\tau))^\alpha\tau)]\}\\
&\leq C_{t^*}, \quad \forall \tau\in I_0\cup I_1. \qquad (4.3.21)
\end{aligned}
$$

Hence, using Lemmas 3.2.2 and 4.2.1 and noting (2.6.20), we get

$$
\begin{aligned}
|E_0|+|E_1| &\leq C_{t^*}\left(\int_{I_0}+\int_{I_1}\right)\sum_{\substack{j,k=1\\ j,k\neq i}}^{n}|\Gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau;y_i))|d\tau\\
&\leq C_{t^*}\widetilde{W}_1(t)W_\infty^c(t) \leq C_{t^*}\theta^2. \qquad (4.3.22)
\end{aligned}
$$

We next estimate the last term on the right-hand side of (4.3.16).
According to Lemma 3.2.2, when $\tau\in I_2$, we have

$$
(\tau,x_i(\tau;y_i)) \in D_i^t. \qquad (4.3.23)
$$

Then, using Lemmas 3.2.1 and 3.2.2 and noting the definition of I_2, for any given $k \neq i$, when $\theta > 0$ is suitably small, we have

$$|w_k(\tau, x_i(\tau; y_i))| \leq CW_\infty^c(\tau)(1+\tau)^{-(1+\mu)} \leq C\theta^2 \leq |w_i(\tau, x_i(\tau; y_i))|, \quad \forall \tau \in I_2. \tag{4.3.24}$$

Hence, noting Lemma 3.2.2, we have

$$\begin{aligned} |E_2| &\leq C \int_{I_2} \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| w_j(\tau, x_i(\tau; y_i)) || w_i(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} \right| d\tau \\ &\leq C\widetilde{W}_1(t)Q(t) \leq C\theta Q(t). \end{aligned} \tag{4.3.25}$$

Noting (4.3.17), (4.3.19), (4.3.22), and (4.3.25), it follows from (4.3.16) that

$$\left| w_i(t, x) \frac{\partial x_i(t; y_i)}{\partial y_i} \right| \leq C_{t^*}\theta + C\theta Q(t). \tag{4.3.26}$$

Then, noting that $\theta > 0$ is suitably small, we immediately obtain (4.3.7).

By (4.3.7), if

$$w_i(t, x_i(t; y_i)) \to \infty \quad \text{as} \quad t \uparrow t^*, \tag{4.3.27}$$

then

$$\frac{\partial x_i(t; y_i)}{\partial y_i} \to 0 \quad \text{as} \quad t \uparrow t^*. \tag{4.3.28}$$

On the other hand, by (4.3.10) and noting Lemma 3.2.2, it is easy to see that (4.3.28) implies (4.3.27).

This proves Theorem 4.3.2. □

Proof of Theorem 4.3.3. For each $i \,\overline{\in}\, J$, by Lemma 4.2.2, we have (4.2.17). Hence, by the equivalence of (4.3.27) and (4.3.28), the family of the ith characteristics never forms any envelope on the domain $[0, \widetilde{T}(\theta)] \times \mathbb{R}$. □

We point out that, differently from the special situation mentioned in Remark 4.3.1, in the general case, the family of the characteristics which first forms an envelope may not correspond to J_1, namely, it is possible that the family of the ith characteristics ($i \in J \backslash J_1$) first forms an envelope.

Example 4.3.1 *We consider the following Cauchy problem:*

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x} = 0, \\ \dfrac{\partial u_2}{\partial t} + (1 + u_1 + u_2^2) \dfrac{\partial u_2}{\partial x} = 0, \\ t = 0: \;\; u_1 = \phi_1(x), u_2 = \phi_2(x), \end{cases} \tag{4.3.29}$$

where $\phi_i(x)$ $(i = 1, 2)$ *are* C^1 *functions with a bounded* C^1 *norm on* $\mathbb{R}$. *Obviously, the system is strictly hyperbolic. Moreover, both* $\lambda_1(u) = u_1$ *and* $\lambda_2(u) = 1 + u_1 + u_2^2$ *are not WLD and the corresponding non-WLD indexes are* $\alpha_1 = 0$ *and* $\alpha_2 = 1$, *respectively.*

Case 1. *Taking*

$$\phi_1(x) = \frac{\varepsilon}{1+x^2}, \quad \phi_2(x) \equiv 0, \tag{4.3.30}$$

for the C^1 *solution to the corresponding Cauchy problem for* (4.3.29), *we have*

$$u_2 \equiv 0.$$

Then the problem reduces to

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + u_1 \dfrac{\partial u_1}{\partial x} = 0, \\ t = 0: \ u_1 = \dfrac{\varepsilon}{1+x^2}. \end{cases}$$

Thus, the blow-up is formed by the envelope of the family of first characteristics $(\alpha = 1)$ *and the life span is*

$$\widetilde{T}(\varepsilon) \approx \varepsilon^{-1} \tag{4.3.31}$$

(see [50]).

Case 2. *Taking*

$$\phi_1(x) \equiv 0, \quad \phi_2(x) = \frac{\varepsilon}{1+x^2}, \tag{4.3.32}$$

for the C^1 *solution to the corresponding Cauchy problem for* (4.3.29), *we have*

$$u_1 \equiv 0.$$

Then the problem reduces to

$$\begin{cases} \dfrac{\partial u_2}{\partial t} + (1 + u_2^2) \dfrac{\partial u_2}{\partial x} = 0, \\ t = 0: \ u_2 = \dfrac{\varepsilon}{1+x^2}. \end{cases}$$

Thus, the blow-up is formed by the envelope of the family of second characteristics $(\alpha = 2)$ *and*

$$\widetilde{T}(\varepsilon) \approx \varepsilon^{-2} \tag{4.3.33}$$

(see [50]).

Theorem 4.3.4 can easily be obtained from the second inequality of (3.2.28) in Lemma 3.2.2.

Proof of Theorem 4.3.5. By Theorems 4.3.2 and 4.3.3, it is easy to see that there exist $m \in J$ such that along $c_m :\ \xi = x_m(t; y_m)\ (t \in [0, t^*])$ passing through (t^*, x^*), where y_m stands for the x-coordinate of the intersection point of c_m with the x-axis, we have

$$w_m(t, x_m(t; y_m)) \to \infty \quad \text{as} \quad t \uparrow t^*. \tag{4.3.34}$$

Then, noting (4.3.27)–(4.3.28), we get

$$\frac{\partial x_m(t, y_m)}{\partial y_m} \to 0 \quad \text{as} \quad t \uparrow t^*. \tag{4.3.35}$$

By (4.3.10), we have

$$\begin{aligned}\frac{\partial x_m(t; y_m)}{\partial y_m} &= \exp\left(\int_0^t \sum_{k=1}^n \nabla\lambda_m(u) w_k r_k(u)(\tau, x_m(\tau; y_m)) d\tau\right) \\ &= \exp\left\{\int_0^t \sum_{k \neq m} \nabla\lambda_m(u) w_k r_k(u)(\tau, x_m(\tau; y_m)) d\tau\right. \\ &\quad \left. + \int_0^t \nabla\lambda_m(u) w_m r_m(u)(\tau, x_m(\tau; y_m)) d\tau\right\}, \ \forall t \in [0, t^*).\end{aligned} \tag{4.3.36}$$

Then, noting Lemma 3.2.2 and (4.3.35), we obtain

$$\lim_{t \uparrow t^*} \int_0^t \nabla\lambda_m(u) w_m r_m(u)(\tau, x_m(\tau; y_m)) d\tau = -\infty. \tag{4.3.37}$$

Noting (2.6.43), we have

$$\begin{aligned}&u(t, x_m(t; y_m)) \\ &\quad = \phi(y_m) + \int_0^t \sum_{k=1}^n (\lambda_i(u) - \lambda_k(u)) w_k r_k(u)(\tau, x_m(\tau; y_m)) d\tau, \ \forall\ t \in [0, t^*).\end{aligned} \tag{4.3.38}$$

It is easy to see that the limit (denoted by u^*) of $u(t, x_m(t; y_m))$ exists as $t \uparrow t^*$. Then, noting that $\nabla\lambda_m(u) w_m r_m(u)(t, x_m(t; y_m))$ is continuous on $[0, t^*)$ and using (4.3.34), from (4.3.37), we have

$$\nabla\lambda_m(u) w_m r_m(u)(t, x_m(t; y_m)) \to -\infty \quad \text{as} \quad t \uparrow t^*. \tag{4.3.39}$$

Then, by (2.6.17), we get

$$\gamma_{mmm}(u) w_m(t, x_m(t; y_m)) \to +\infty \quad \text{as} \quad t \uparrow t^*. \tag{4.3.40}$$

By Lemma 3.2.2, it is easy to see that $(t^*, x^*) \in D_m^{t^*}$. Let t_0 be the t-coordinate of the intersection point of c_m with the boundary of $D_m^{t^*}$. We now consider

$$\begin{aligned}\frac{dw_m(t, x_m(t; y_m))}{ds} &= \gamma_{mmm}(u) w_m^2(t, x_m(t; y_m)) + 2 \sum_{j \neq m} \gamma_{mjm}(u) w_j w_m(t, x_m(t; y_m)) \\ &\quad + \sum_{j,k \neq m} \gamma_{mjk}(u) w_j w_k(t, x_m(t; y_m)) \end{aligned} \tag{4.3.41}$$

for $t \in [t_0, t^*)$. By Lemma 3.2.2, it is easy to see that

$$\begin{aligned}&\left| \sum_{j \neq m} \gamma_{mjm}(u) w_j(t, x_m(t; y_m)) \right|, \left| \sum_{j,k \neq m} \gamma_{mjk}(u) w_j w_k(t, x_m(t; y_m)) \right| \\ &\quad \leq C\theta, \forall t \in [t_0, t^*).\end{aligned} \tag{4.3.42}$$

Noting (4.3.40), there exists $t_1 \in (t_0, t^*)$ such that $\gamma_{mmm}(u)$ and $w_m(t, x_m(t; y_m))$ have the same sign on $[t_1, t^*)$. Without loss of generality, we assume that $\gamma_{mmm}(u)$ and $w_m(t, x_m(t; y_m))$ are positive on $[t_1, t^*)$. Then, noting (4.3.40) and (4.3.42), there exists $t_2 \in [t_1, t^*)$ such that along c_m,

$$\frac{1}{2}\gamma_{mmm}(u) w_m^2 + 2 \sum_{j \neq m} \gamma_{mjm}(u) w_j w_m + \sum_{j,k \neq m} \gamma_{mjk}(u) w_j w_k > 0, \quad \forall t \in [t_2, t^*), \tag{4.3.43}$$

and

$$\gamma_{mmm}(u) w_m^2 - 2 \sum_{j \neq m} \gamma_{mjm}(u) w_j w_m - \sum_{j,k \neq m} \gamma_{mjk}(u) w_j w_k > 0, \quad \forall t \in [t_2, t^*). \tag{4.3.44}$$

Then it follows from (4.3.41) that

$$\begin{aligned}\frac{1}{2}\gamma_{mmm}(u) w_m^2(t, x_m(t; y_m)) &\leq \frac{dw_m(t, x_m(t; y_m))}{dt} \\ &\leq 2\gamma_{mmm}(u) w_m^2(t, x_m(t; y_m)), \quad \forall t \in [t_2, t^*).\end{aligned} \tag{4.3.45}$$

Hence, noting (4.3.34), we have $w_m(t, x_m(t; y_m)) \to +\infty$ as $t \uparrow t^*$ and then

$$\begin{aligned}&\frac{1}{2}\int_t^{t^*} \gamma_{mmm}(u(\tau, x_m(\tau; y_m))) d\tau \\ &\quad \leq \frac{1}{w_m(t, x_m(t; y_m))} \\ &\quad \leq 2\int_t^{t^*} \gamma_{mmm}(u(\tau, x_m(\tau; y_m))) d\tau, \ \forall t \in [t_2, t^*).\end{aligned} \tag{4.3.46}$$

Thus, if $\gamma_{mmm}(u^*) > 0$, we immediately get

$$w_m(t, x_m(t; y_m)) = O((t^* - t)^{-1}) \quad \text{as} \quad t \uparrow t^*. \tag{4.3.47}$$

However, if $\gamma_{mmm}(u^*) = 0$, we have

$$\lim_{t \uparrow t^*} \frac{\int_t^{t^*} \gamma_{mmm}(u(\tau, x_m(\tau; y_m)))d\tau}{t^* - t} = 0. \tag{4.3.48}$$

It then follows from (4.3.46) that $w_m(t, x_m(t; y_m))$ is a higher-order infinitely larger quantity than $(t^* - t)^{-1}$ as $t \uparrow t^*$.

Thus, noting (2.6.4) and Lemma 3.2.2, we finish the proof of Theorem 4.3.5. □

4.4 Applications

In this section we give some applications of Theorems 4.2.1 and 4.3.2 to the Cauchy problems for some systems arising in mechanics, physics, or applications. Some other applications can be found in [76].

4.4.1 System of Traffic Flow

For the **system of traffic flow** (cf. Section 1.3.2)

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, \end{cases} \tag{4.4.1}$$

where $\rho > 0$ and v are the density and velocity of cars at point x and time t, respectively, and

$$p(\rho) = \rho^\gamma \quad (\gamma > 0 \text{ is a constant}), \tag{4.4.2}$$

we consider the Cauchy problem with the initial data

$$t = 0: \ (\rho, v) = (\widetilde{\rho}_0 + \varepsilon\rho_0(x), \widetilde{v}_0 + \varepsilon v_0(x)), \tag{4.4.3}$$

where $\widetilde{\rho}_0 > 0$ and $\widetilde{v}_0$ are constants, $\rho_0(x)$ and $v_0(x) \in C^1$, and

$$\sup_{x \in \mathbb{R}} \{(1 + |x|)^{1+\mu}(|\rho_0(x)| + |v_0(x)| + |\rho_0'(x)| + |v_0'(x)|)\} < +\infty, \tag{4.4.4}$$

where $\mu > 0$ is a constant.

Let

$$U = (\rho, v)^T. \tag{4.4.5}$$

It is easy to see that (4.4.1) is a strictly hyperbolic system with the following two distinct real eigenvalues:

$$\lambda_1(U) = v - \rho p'(\rho) < \lambda_2(U) = v, \tag{4.4.6}$$

and the corresponding left and right eigenvectors can be taken as

$$l_1(U) = (0, 1), \quad l_2(U) = (p'(\rho), 1), \tag{4.4.7}$$

and

$$r_1(U) = (1, -p'(\rho))^T, \quad r_2(U) = (1, 0)^T. \tag{4.4.8}$$

Obviously, $\lambda_1(U)$ is GN and $\lambda_2(U)$ is LD.

By Theorem 4.2.1, Theorem 4.3.2, and Remark 4.3.1, we have

Theorem 4.4.1 *Suppose that* (4.4.2) *holds and the initial data* (4.4.3) *satisfy* (4.4.4). *If*

$$v_0(x) \not\equiv 0, \tag{4.4.9}$$

then there exists $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$, *the* C^1 *solution* $U = U(t, x)$ *to the Cauchy problem* (4.4.1) *and* (4.4.3) *must blow up in a finite time and the life span* $\widetilde{T}(\varepsilon)$ *of* $U = U(t, x)$ *satisfies*

$$\lim_{\varepsilon \downarrow 0} (\varepsilon \widetilde{T}(\varepsilon)) = M_0, \tag{4.4.10}$$

where

$$M_0 = \left\{ \sup_{x \in \mathbb{R}} [\gamma(\gamma + 1)\rho_0^{\gamma-1} v_0'(x)] \right\}^{-1}. \tag{4.4.11}$$

Theorem 4.4.2 *Suppose that* (4.4.2) *holds and the initial data* (4.4.3) *satisfy* (4.4.4). *Then there exists* $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$, *the* C^1 *solution* $U = U(t, x)$ *to the Cauchy problem* (4.4.1) *and* (4.4.3) *blows up in a finite time if and only if the first family of characteristics forms an envelope in a finite time. If* (t^*, x^*), *in which* $t^* = \widetilde{T}(\varepsilon)$, *is a blow-up point of the* C^1 *solution* $U = U(t, x)$ *to the Cauchy problem* (4.4.1) *and* (4.4.3), *then there exists at least one first characteristic* $x = x_1(t)$ *passing through* (t^*, x^*) *with* $0 \le t \le t^*$ *such that along it, we have*

$$u_x = O((t^* - t)^{-1}) \quad \textit{as} \quad t \uparrow t^*. \tag{4.4.12}$$

4.4.2 System of One-Dimensional Gas Dynamics

Consider the Cauchy problem for the **system of one-dimensional gas dynamics** in Lagrangian representation (cf. Sections 1.3.3 and 3.6.1)

$$\begin{cases} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(\tau, S)}{\partial x} = 0, \\ \dfrac{\partial S}{\partial t} = 0 \end{cases} \tag{4.4.13}$$

with the initial data

$$t = 0: \quad \tau = \tau_0 + \varepsilon \tau_1(x), \; v = u_0 + \varepsilon u_1(x), \; S = S_0 + \varepsilon S_1(x), \tag{4.4.14}$$

where $\tau > 0$ is the specific volume, u is the velocity, S is the entropy, p is the pressure, and the equation of state is given by $p = p(\tau, S)$; moreover, τ_0 (> 0), u_0, and S_0 are constants, and $\tau_1(x), u_1(x)$, and $S_1(x) \in C^1$ satisfy the decaying property

$$\sup_{x \in \mathbb{R}} \{(1+|x|)^{1+\mu}(|\tau_1(x)|+|\tau_1'(x)|+|u_1(x)|+|u_1'(x)|+|S_1(x)|+|S_1'(x)|)\} < +\infty, \tag{4.4.15}$$

where $\mu > 0$ is a constant.

Let

$$U = (\tau, u, S)^T. \tag{4.4.16}$$

If

$$p_\tau(\tau_0, S_0) < 0, \tag{4.4.17}$$

then, in a neighbourhood of $U = U_0 \stackrel{\text{def.}}{=} (\tau_0, u_0, S_0)^T$, (4.4.13) is a strictly hyperbolic system with three distinct real eigenvalues,

$$\lambda_1(U) = -\sqrt{-p_\tau} < \lambda_2(U) = 0 < \lambda_3(U) = \sqrt{-p_\tau}, \tag{4.4.18}$$

and the corresponding left and right eigenvectors can be taken as

$$l_1(U) = (p_\tau, -\sqrt{-p_\tau}, p_S), \; l_2(U) = (0, 0, 1), \; l_3(U) = (p_\tau, \sqrt{-p_\tau}, p_S) \tag{4.4.19}$$

and

$$r_1(U) = (1, \sqrt{-p_\tau}, 0)^T, \; r_2(U) = (p_S, 0, -p_\tau)^T, \; r_3(U) = (-1, \sqrt{-p_\tau}, 0)^T. \tag{4.4.20}$$

Obviously, $\lambda_2(U)$ is always LD. On the other hand, $\lambda_1(U)$ and $\lambda_3(U)$ are GN in a neighbourhood of $U = U_0$ if and only if

$$p_{\tau\tau}(\tau_0, S_0) \neq 0. \tag{4.4.21}$$

Moreover, $\lambda_1(U)$ and $\lambda_3(U)$ are WLD with respect to $U = U_0$ if and only if

$$p_{\tau\tau}(\tau, S_0) \equiv 0, \quad \forall |\tau - \tau_0| \text{ small}. \tag{4.4.22}$$

Thus, if

$$p_{\tau\tau}(\tau, S_0) \not\equiv 0, \quad \forall |\tau - \tau_0| \text{ small}, \tag{4.4.23}$$

then (4.4.13) is not WLD with respect to $U = U_0$. More precisely, if there exists an integer $\alpha \geq 0$ such that

$$p_{\tau\tau}(\tau_0, S_0) = \cdots = \frac{\partial^{\alpha+1} p}{\partial \tau^{\alpha+1}}(\tau_0, S_0) = 0, \quad \text{but} \quad \frac{\partial^{\alpha+2} p}{\partial \tau^{\alpha+2}}(\tau_0, S_0) \neq 0, \tag{4.4.24}$$

then (4.4.13) is a non-WLD system with the index α. Obviously, when $\alpha = 0$, system (4.4.13) is GN in a neighbourhood of $U = U_0$.

By Theorem 4.2.1, Theorem 4.3.2, and Remark 4.3.1, we have

Theorem 4.4.3 *Suppose that* (4.4.24) *holds and the initial data* (4.4.14) *satisfy* (4.4.15). *If*

$$u_1(x) \not\equiv 0, \tag{4.4.25}$$

then there exists $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$, *the* C^1 *solution* $U = U(t, x)$ *to the Cauchy problem* (4.4.13) *and* (4.4.14) *must blow up in a finite time and the life span* $\widetilde{T}(\varepsilon)$ *of* $U = U(t, x)$ *satisfies*

$$\lim_{\varepsilon \downarrow 0} (\varepsilon^{\alpha+1} \widetilde{T}(\varepsilon)) = M_0, \tag{4.4.26}$$

where

$$M_0 = \left\{ \max_{i \in \{1,3\}} \sup_{x \in \mathbb{R}} \left[-\frac{1}{\alpha 2 \sqrt{-p_\tau(\tau_0, S_0)}} \frac{\partial^{\alpha+2} p}{\partial \tau^{\alpha+2}}(\tau_0, S_0) \right.\right.$$
$$\left.\left. (l_i(U_0)\psi(x))^\alpha l_i(U_0)\psi'(x) \right] \right\}^{-1}, \tag{4.4.27}$$

in which $\psi(x) = (\tau_1(x), u_1(x), S_1(x))^T$.

Theorem 4.4.4 *Suppose that* (4.4.24) *holds and the initial data* (4.4.14) *satisfy* (4.4.15). *Then there exists* $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$, *the* C^1 *solution* $U = U(t, x)$ *to the Cauchy problem* (4.4.13) *and* (4.4.14) *blows up in a finite time if and only if the first family* (*or the third family*) *of characteristics forms an envelope in a finite time. If* (t^*, x^*), *in which* $t^* = \widetilde{T}(\varepsilon)$, *is a blow-up point of the* C^1 *solution* $U = U(t, x)$ *to the Cauchy problem* (4.4.13) *and* (4.4.14), *then there exists at least one first* (*or third*) *characteristic* $x = x(t)$ *passing through* (t^*, x^*) *with* $0 \le t \le t^*$ *such that along it, we have*

$$u_x = O((t^* - t)^{-1}) \quad as \quad t \uparrow t^*. \tag{4.4.28}$$

4.4.3 System of Compressible Elastic Fluids with Memory

Consider the Cauchy problem for the **system of compressible elastic fluids with memory** (cf. Section 1.3.4)

$$\begin{cases} \rho_t + v\rho_x + \rho v_x = 0, \\ \rho(v_t + vv_x) + p(\rho)_x = (\rho W'(F)F)_x, \\ F_t + vF_x - Fv_x = 0, \end{cases} \tag{4.4.29}$$

with the initial data

$$t = 0: \quad \rho = \rho_0 + \varepsilon\rho_1(x), \ v = v_0 + \varepsilon v_1(x), \ F = 1 + \varepsilon F_1(x), \tag{4.4.30}$$

where $\rho > 0$ is the density, v is the velocity, p is the pressure, $W(F)$ is the strain energy function, and F corresponds to the deformation tensor. Moreover, ρ_0 (> 0) and v_0 are constants, and $\rho_1(x), v_1(x)$, and $F_1(x) \in C^1$ satisfy the decaying property

$$\sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\rho_1(x)|+|\rho_1'(x)|+|v_1(x)|+|v_1'(x)|+|F_1(x)|+|F_1'(x)|)\} < +\infty, \tag{4.4.31}$$

where $\mu > 0$ is a constant.

Let

$$U = (\rho, v, F)^T. \tag{4.4.32}$$

If

$$p'(\rho_0) + W''(1) > 0, \tag{4.4.33}$$

(4.4.29) is a strictly hyperbolic system in a neighbourhood of $U = U_0 \overset{\text{def.}}{=} (\rho_0, v_0, 1)^T$ with the following distinct real eigenvalues:

$$\lambda_1 = v - \sqrt{p'(\rho) + W''(F)F^2} < \lambda_2 = v < \lambda_3 = v + \sqrt{p'(\rho) + W''(F)F^2}, \tag{4.4.34}$$

and the corresponding left and right eigenvectors can be taken as

$$\begin{aligned} l_1(U) &= \left(\frac{FW'(F) - p'(\rho)}{\rho\sqrt{p'(\rho) + w''(F)F^2}}, 1, \frac{FW''(F) + W'(F)}{\sqrt{p'(\rho) + w''(F)F^2}} \right), \\ l_2(U) &= \left(\frac{F}{\rho}, 0, 1 \right), \end{aligned}$$

$$l_3(U) = \left(-\frac{FW'(F) - p'(\rho)}{\rho\sqrt{p'(\rho) + w''(F)F^2}}, 1, -\frac{FW''(F) + W'(F)}{\sqrt{p'(\rho) + w''(F)F^2}} \right), \qquad (4.4.35)$$

and

$$\begin{aligned} r_1(U) &= \left(-\frac{\rho}{\sqrt{p'(\rho) + w''(F)F^2}}, 1, \frac{F}{\sqrt{p'(\rho) + w''(F)F^2}} \right)^T, \\ r_2(U) &= \left(FW''(F) + W'(F), 0, \frac{p'(\rho) - FW'(F)}{\rho} \right)^T, \\ r_3(U) &= \left(\frac{\rho}{\sqrt{p'(\rho) + w''(F)F^2}}, 1, -\frac{F}{\sqrt{p'(\rho) + w''(F)F^2}} \right)^T. \end{aligned} \qquad (4.4.36)$$

Obviously, $\lambda_2(U)$ is always LD. $\lambda_1(U)$ and $\lambda_3(U)$ are GN in a neighbourhood of $U = U_0$ if and only if

$$\rho_0 p''(\rho_0) + 2p'(\rho_0) - W'''(1) \neq 0. \qquad (4.4.37)$$

On the other hand, $\lambda_1(U)$ and $\lambda_3(U)$ are WLD with respect to $U = U_0$ if and only if

$$\rho p''(\rho) + 2p'(\rho) - W'''(\rho_0\rho^{-1})(\rho_0\rho^{-1})^3 \equiv 0, \quad \forall |\rho - \rho_0| \text{ small}. \qquad (4.4.38)$$

Let

$$Q(\rho) = \rho p''(\rho) + 2p'(\rho) - W'''(\rho_0\rho^{-1})(\rho_0\rho^{-1})^3. \qquad (4.4.39)$$

If

$$Q(\rho) \not\equiv 0, \quad \forall |\rho - \rho_0| \text{ small}, \qquad (4.4.40)$$

then (4.4.29) is not WLD with respect to $U = U_0$. More precisely, if there exists an integer $\alpha \geq 0$ such that

$$Q(\rho_0) = Q'(\rho_0) = \cdots = Q^{(\alpha-1)}(\rho_0) = 0, \text{ but } Q^{(\alpha)}(\rho_0) \neq 0, \qquad (4.4.41)$$

then (4.4.29) is a non-WLD system with the index α. Obviously, when $\alpha = 0$, system (4.4.29) is GN in a neighbourhood of $U = U_0$.

By Theorem 4.2.1, Theorem 4.3.2, and Remark 4.3.1, we have

Theorem 4.4.5 *Suppose that* (4.4.41) *holds and the initial data* (4.4.30) *satisfy* (4.4.31). *If*

$$v_1(x) \not\equiv 0, \qquad (4.4.42)$$

then there exists $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$, *the* C^1 *solution* $U = U(t, x)$ *to the Cauchy problem* (4.4.29) *and* (4.4.30) *must blow up in a finite time and the life span* $\widetilde{T}(\varepsilon)$ *of* $U = U(t, x)$ *satisfies*

$$\lim_{\varepsilon \downarrow 0} (\varepsilon \widetilde{T}(\varepsilon)) = M_0, \qquad (4.4.43)$$

where

$$M_0 = \left\{ \max_{i\in\{1,3\}} \sup_{x\in\mathbb{R}} \left[-\frac{(\pm\rho_0)^\alpha Q^{(\alpha)}(\rho_0)}{2\alpha!(p'(\rho_0)+W''(1))^{\frac{\alpha+2}{2}}} (l_i(U_0)\psi(x))^\alpha l_i(U_0)\psi'(x) \right] \right\}^{-1}, \tag{4.4.44}$$

in which $\psi(x) = (\rho_1(x), v_1(x), F_1(x))^T$, *"–" corresponds to* $i = 1$, *and "+" corresponds to* $i = 3$.

Theorem 4.4.6 *Suppose that* (4.4.41) *holds and the initial data* (4.4.30) *satisfy* (4.4.31). *Then there exists* $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$, *the* C^1 *solution* $U = U(t,x)$ *to the Cauchy problem* (4.4.29) *and* (4.4.30) *blows up in a finite time if and only if the first family* (*or the third family*) *of characteristics forms an envelope in a finite time. If* (t^*, x^*), *in which* $t^* = \widetilde{T}(\varepsilon)$, *is a blow-up point of the* C^1 *solution* $U = U(t,x)$ *to the Cauchy problem* (4.4.29) *and* (4.4.30), *then there exists at least one first* (*or third*) *characteristic* $x = x(t)$ *passing through* (t^*, x^*) *with* $0 \le t \le t^*$ *such that along it, we have*

$$u_x = O((t^*-t)^{-1}) \quad as \quad t \uparrow t^*. \tag{4.4.45}$$

4.5 Blow-Up Mechanism of the C^1 Solution in the Critical Case $\alpha = +\infty$

4.5.1 Introduction and Main Results

The previous results mainly focus on the noncritical case that $\alpha < +\infty$. For the critical case $\alpha = +\infty$, however, only a few results are known (see [34], [76]). In this section, we study the blow-up phenomenon, particularly the geometric blow-up mechanism in the critical case.

Our main results are the following theorems, which show that although it is impossible to get a sharp estimate on the life span in the critical case, the blow-up mechanism in the critical case is almost the same as in the noncritical case. We point out that the method used in previous sections cannot be directly applied to the critical case; thus, some significant changes or improvements should be made in the proof (cf. [69]).

Theorem 4.5.1 *Suppose that in a neighbourhood of* $u = 0$, $A(u) \in C^\infty$ *and system* (4.1.1) *is strictly hyperbolic. Suppose furthermore that system* (4.1.1) *is not WLD and the non-WLD index*

$$\alpha = +\infty. \tag{4.5.1}$$

Suppose finally that the initial data (4.1.2) *satisfy* (3.2.1). *Then, for any given integer* $N \geq 1$, *there exists* $\theta_0 = \theta_0(N) > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$, *the life span* $\widetilde{T}(\theta)$ *of the* C^1 *solution* $u = u(t, x)$ *to the Cauchy problem* (4.1.1)–(4.1.2) *satisfies*

$$\widetilde{T}(\theta) > \theta^{-N}. \tag{4.5.2}$$

Moreover, when $u = u(t, x)$ *blows up in a finite time,* $u = u(t, x)$ *itself is bounded and small on the domain* $[0, \widetilde{T}(\theta)) \times \mathbb{R}$, *while the first-order partial derivative* u_x *of* $u = u(t, x)$ *tends to be unbounded as* $t \nearrow \widetilde{T}(\theta)$.

Theorem 4.5.2 *Under the assumptions of Theorem 4.5.1, for any given integer* $N \geq 1$, *there exists* $\theta_0 = \theta_0(N) > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$, *the* C^1 *solution* $u = u(t, x)$ *to the Cauchy problem* (4.1.1)–(4.1.2) *blows up in a finite time if and only if at least one family of characteristics forms an envelope in a finite time.*

Theorem 4.5.3 *Under the assumptions of Theorem 4.5.1, for each* $i \,\overline{\in}\, J$, *the family of the* i*th characteristics never forms any envelope on the domain* $[0, \widetilde{T}(\theta)] \times \mathbb{R}$.

Theorem 4.5.4 *Under the assumptions of Theorem 4.5.1, on the line* $t = \widetilde{T}(\theta)$, *the set of blow-up points cannot possess a positive measure.*

4.5.2 Some Uniform a Priori Estimates Depending on Weak Linear Degeneracy

Lemma 4.5.1 *Suppose that in a neighbourhood of* $u = 0$, $A(u) \in C^\infty$ *and system* (4.1.1) *is strictly hyperbolic. Suppose furthermore that system* (4.1.1) *is not WLD and* (4.5.1) *holds. Suppose finally that the initial data* (4.1.2) *satisfy* (3.2.1). *Then there exists* $\theta_0 > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$, *for any given positive integer* N, *on any given existence domain* $D(T)$ *of the* C^1 *solution* $u = u(t, x)$ *to the Cauchy problem* (4.1.1)–(4.1.2) *with*

$$T\theta^N \leq 1, \tag{4.5.3}$$

we have the following uniform a priori estimates:

$$U_\infty^c(T) \leq \kappa_4 \theta \tag{4.5.4}$$

and

$$\widetilde{U}_1(T),\ U_1(T) \leq \kappa_5 \theta. \tag{4.5.5}$$

Moreover, there exists $\theta_0 = \theta_0(N) > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$, *on any given existence domain* $D(T)$ *of the* C^1 *solution* $u = u(t, x)$ *to the Cauchy problem* (4.1.1)–(4.1.2), *where* T *still satisfies* (4.5.3), *we have*

$$W_\infty(T) \le \kappa_6 \theta, \tag{4.5.6}$$

where κ_4, κ_5, and κ_6 are positive constants independent of θ and T but possibly depending on N.

Proof. Without loss of generality, in order to prove Lemma 4.5.1, we assume that $u = (u_1, \ldots, u_n)$ are normalized coordinates.

We first estimate $U^c_\infty(T)$.

In the present situation, (4.2.7) still holds, i.e.,

$$U^c_\infty(T) \le C\theta\{1 + U^c_\infty(T) + \widetilde{U}_1(T)\}. \tag{4.5.7}$$

We next estimate $\widetilde{U}_1(T)$ and $U_1(T)$.

Similarly to (4.2.8), we have

$$\begin{aligned}
&\int_{t_A}^{t_B} |u_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau \\
&\le \int_{x_C}^{x_D} \left| u_i(0,x)|dx + \iint_{ACDB} \right| \sum_{j,k=1}^{n} F_{ijk}(u) u_j w_k(t,x)| dt dx.
\end{aligned} \tag{4.5.8}$$

Noting (4.5.1) and (2.6.52), for any given integer $N \ge 1$, we have

$$|F_{iii}(u_i e_i)| \le C_N |u_i|^N; \tag{4.5.9}$$

in what follows, C_N denotes a positive constant possibly depending on N. Then, noting (3.2.1) and (2.6.51) and using Hadamard's formula, Lemma 3.2.1, and Lemma 3.2.2, it follows from (4.5.8) that

$$\begin{aligned}
&\int_{t_A}^{t_B} |u_i(\lambda_j(u) - \lambda_i(u))(\tau, \xi_j(\tau))| d\tau \\
&\le \int_{x_C}^{x_D} |u_i(0,x)| dx + \iint_{ACDB} \Bigg[\sum_{\substack{j,k=1 \\ j \ne k}}^{n} |F_{ijk}(u) u_j w_k(t,x)| + (|F_{iii}(u) - F_{iii}(u_i e_i)| \\
&\quad + |F_{iii}(u_i e_i)|) |u_i w_i(t,x)| \Bigg] dt dx \\
&\le C\{\theta + U^c_\infty(T) W^c_\infty(T) + U^c_\infty(T) W_1(T) + U_1(T) W^c_\infty(T) \\
&\quad + U^c_\infty(T) U_\infty(T) W_1(T)\} + C_N (U_\infty(T))^{N+1} W_1(T) T \\
&\le C\theta\{1 + U^c_\infty(T) + U_1(T)\} + C_N \theta^{N+2} T.
\end{aligned} \tag{4.5.10}$$

Thus, noting (4.5.3), we get

$$\widetilde{U}_1(T) \le C\theta\{1 + U^c_\infty(T) + U_1(T)\} + C_N \theta^2. \tag{4.5.11}$$

Similarly, we have

$$U_1(T) \le C\theta\{1 + U^c_\infty(T) + U_1(T)\} + C_N\theta^2. \tag{4.5.12}$$

The combination of (4.5.7) and (4.5.11)–(4.5.12) gives (4.5.4)–(4.5.5).

We finally estimate $W_\infty(T)$.

Similarly to (4.2.13), we have

$$w_i(t,x) = w_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k(\tau, \xi_i(\tau)) d\tau. \tag{4.5.13}$$

Noting (4.5.1) and (2.6.17), for any given integer $N \ge 1$, we have

$$|\gamma_{iii}(u_i e_i)| \le C_N |u_i|^N. \tag{4.5.14}$$

Then, noting (2.6.16) and (4.5.4) and using Lemmas 3.2.1 and 3.2.2, from (4.5.13) we have

$$\begin{aligned}
&|w_i(t,x)| \\
&\le |w_i(0,x_{i0})| + \int_0^t \Bigg[\sum_{\substack{j,k=1\\ j\ne k}}^n |\gamma_{ijk}(u) w_j w_k(\tau,\xi_i(\tau))| + (|\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)| \\
&\quad + |\gamma_{iii}(u_i e_i)|) w_i^2(\tau, \xi_i(\tau)) \Bigg] d\tau \\
&\le C_N\{\theta + (W^c_\infty(T))^2 + W^c_\infty(T) W_\infty(T) + (U^c_\infty(T) \\
&\quad + (U_\infty(T))^N T)(W_\infty(T))^2\} \\
&\le C_N\{\theta(1 + W_\infty(T) + (W_\infty(T))^2) + \theta^N T (W_\infty(T))^2\}.
\end{aligned} \tag{4.5.15}$$

Hence, noting (4.5.3), we have

$$W_\infty(T) \le C_N\{\theta + (W_\infty(T))^2\}. \tag{4.5.16}$$

Thus, we can obtain (4.5.5) via the method in the proof of Lemma 3.2.2.

Remark 4.5.1 *For any given $i \,\overline{\in}\, J$, $\lambda_i(u)$ is WLD. By* (2.6.52) *and* (2.6.17)*, we have*

$$F_{iii}(u_i e_i) \equiv 0 \quad \textit{and} \quad \gamma_{iii}(u_i e_i) \equiv 0, \quad \forall |u_i| \ \textit{small}. \tag{4.5.17}$$

From the proof of Lemma 4.5.1, we know that there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $D(T)$ [*without restriction* (4.5.3)] *of the C^1 solution $u = u(t,x)$ to the Cauchy problem* (4.1.1)–(4.1.2) *satisfying* (3.2.1)*, we have the following uniform a priori* *estimate:*

$$|w_i(t,x)| \leq \kappa_7 \theta, \tag{4.5.18}$$

where κ_7 *is a positive constant independent of* θ *and* T.

4.5.3 Proof of Main Results

Proof of Theorem 4.5.1. According to the existence and uniqueness of the local C^1 solution to the Cauchy problem (4.1.1)–(4.1.2) (see [72]), there exists $\tau_0 > 0$ such that on $[0,\tau_0] \times \mathbb{R}$, the Cauchy problem (4.1.1)–(4.1.2) has a unique C^1 solution $u = u(t,x)$. By Lemmas 3.2.2 and 4.5.1, for any given integer $N \geq 1$, there exists $\theta_0 = \theta_0(N) > 0$ so small that for any given $\theta \in (0,\theta_0]$, on any given existence domain $[0,T] \times \mathbb{R}$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.1.1)–(4.1.2), where $0 < T \leq \theta^N$, we have the following uniform a priori estimate on the C^1 norm of $u = u(t,x)$:

$$\|u(t,\cdot)\|_{C^1} \triangleq \|u(t,\cdot)\|_{C^0} + \|u_x(t,\cdot)\|_{C^0} \leq C\theta, \quad \forall t \in [0,T], \tag{4.5.19}$$

where C is a positive constant independent of θ and T but possibly depending on N. By the C^1 extension, we immediately get the existence and uniqueness of the C^1 solution $u = u(t,x)$ on $[0,\theta^N] \times \mathbb{R}$. Hence, the life span $\widetilde{T}(\theta)$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.1.1)–(4.1.2) satisfies

$$\widetilde{T}(\theta) > \theta^{-N}. \tag{4.5.20}$$

Moreover, by Lemma 3.2.2, when the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.1.1)–(4.1.2) satisfying (3.2.1) blows up in a finite time, $u = u(t,x)$ itself must be bounded and small on $[0,\widetilde{T}(\theta))$. Hence, the first-order partial derivative u_x should tend to be unbounded as $t \nearrow \widetilde{T}(\theta)$. $\square$

Proof of Theorem 4.5.2. Assume that (t^*,x^*) is a starting point of the blow-up of the C^1 solution $u = u(t,x)$ to the Cauchy problem (4.1.1)–(4.1.2). Then, by Theorem 4.5.1, we have

$$t^* > \theta^{-N}. \tag{4.5.21}$$

On the other hand, we can find an integer $p > N$ such that

$$t^* < \theta^{-p}. \tag{4.5.22}$$

For each $i = 1,\ldots,n$, passing through any given point (t,x) with $0 \leq t < t^*$ and $x \in \mathbb{R}$, we draw the ith characteristic $c_i: \ \xi = x_i(\tau; y_i)$ in which $0 \leq \tau \leq t$ and y_i stands for the x-coordinate of the intersection point of this characteristic with the x-axis, i.e., we have

$$\frac{dx_i(\tau; y_i)}{d\tau} = \lambda_i(u(\tau, x_i(\tau; y_i))) \tag{4.5.23}$$

and

$$x_i(0; y_i) = y_i, \quad x_i(t; y_i) = x. \tag{4.5.24}$$

In what follows, we prove that

$$\left| w_i(t,x) \frac{\partial x_i(t; y_i)}{\partial y_i} \right| \le C_p \theta, \quad \forall (t,x) \in [0, t^*) \times \mathbb{R}. \tag{4.5.25}$$

Henceforth, C_p denotes a positive constant possibly depending on p.

In the present situation, (4.3.10) and (4.3.12) still hold, i.e., we have

$$\frac{\partial x_i(t; y_i)}{\partial y_i} = \exp\left(\int_0^t \sum_{k=1}^n \nabla\lambda_i(u) w_k r_k(\tau, x_i(\tau; y_i)) d\tau \right), \quad \forall t \in [0, t^*), \tag{4.5.26}$$

and

$$w_i(t,x) \frac{\partial x_i(t; y_i)}{\partial y_i} = w_i(0, y_i) + \int_0^t \sum_{j,k=1}^n \varGamma_{ijk}(u) w_j w_k(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} d\tau. \tag{4.5.27}$$

Let

$$I_0 = [0, \theta^{-N}], \tag{4.5.28}$$

$$I_1 = [\theta^{-N}, t] \cap \{\tau | 0 \le \tau \le t, |w_i(\tau, x_i(\tau; y_i))| \le \kappa_1 \theta\}, \tag{4.5.29}$$

and

$$I_2 = [\theta^{-N}, t] \cap \{\tau | 0 \le \tau \le t, |w_i(\tau, x_i(\tau; y_i))| > \kappa_1 \theta\}, \tag{4.5.30}$$

where κ_1 is given in Lemma 3.2.2. Then, similarly to (4.3.16), (4.5.27) can be rewritten as

$$\begin{aligned} w_i(t,x) \frac{\partial x_i(t; y_i)}{\partial y_i} &= w_i(0, y_i) + \int_0^t \sum_{\substack{j=1 \\ j \ne i}}^n \varGamma_{iji}(u) w_j w_i(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} d\tau \\ &\quad + \left(\int_{I_0} + \int_{I_1} + \int_{I_2} \right) \sum_{\substack{j,k=1 \\ j,k \ne i}}^n \varGamma_{ijk}(u) w_j w_k(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} d\tau \\ &= w_i(0, y_i) + E + E_0 + E_1 + E_2. \end{aligned} \tag{4.5.31}$$

Now we estimate every term on the right-hand side of (4.5.31).

Obviously,

$$|w_i(0, y_i)| \le C\theta, \quad \forall y_i \in \mathbb{R}. \tag{4.5.32}$$

Let

$$Q(t) = \sup_{(\tau,y_i)\in[0,t]\times\mathbb{R}} \left| w_i(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} \right|, \quad \forall t \in [0, t^*). \tag{4.5.33}$$

By Lemma 3.2.2, we get

$$|E| \le CQ(t)\widetilde{W}_1(t) \le C\theta Q(t). \tag{4.5.34}$$

By Lemma 4.5.1, we have

$$|w_i(\tau, x_i(\tau; y_i))| \le C\theta, \quad \forall \tau \in I_0 \cup I_1. \tag{4.5.35}$$

Then, noting (2.6.17) and (4.5.1) and using Lemmas 3.2.2 and 4.5.1, we see from (4.5.26) that

$$\begin{aligned}
&\left|\frac{\partial x_i(t; y_i)}{\partial y_i}\right| \\
&\le \exp\left\{ C \left[\int_0^t \left(\sum_{\substack{k=1\\k\ne i}}^n |w_k| + |(\gamma_{iii}(u) - \gamma_{iii}(u_i e_i)) w_i| \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + |\gamma_{iii}(u_i e_i) w_i| \right) (\tau, x_i(\tau; y_i)) d\tau \right] \right\} \\
&\le \exp\{C_p[\widetilde{W}_1(t) + \theta(\widetilde{U}_1(t) + (U_\infty(t))^p t)]\} \\
&\le C_p, \quad \forall t \in [0, t^*).
\end{aligned} \tag{4.5.36}$$

Hence, using Lemmas 3.2.1 and 3.2.2 and noting (2.6.20), we get

$$\begin{aligned}
|E_0| + |E_1| &\le C_p \left(\int_{I_0} + \int_{I_1} \right) \sum_{\substack{j,k=1\\j,k\ne i}}^n |\Gamma_{ijk}(u) w_j w_k(\tau, x_i(\tau; y_i))| d\tau \\
&\le C_p \widetilde{W}_1(t) W_\infty^c(t) \le C_p \theta^2.
\end{aligned} \tag{4.5.37}$$

We next estimate the last term.

According to Lemma 3.2.2, when $\tau \in I_2$, we have

$$(\tau, x_i(\tau; y_i)) \in D_i^t. \tag{4.5.38}$$

Then, using Lemmas 3.2.1 and 3.2.2 and noting the definition of I_2, for any given $k \ne i$, we have

$$|w_k(\tau, x_i(\tau; y_i))| \le CW_\infty^c(\tau)(1+\tau)^{-(1+\mu)} \le C\theta^2 \le |w_i(\tau, x_i(\tau; y_i))|, \quad \forall \tau \in I_2. \tag{4.5.39}$$

Hence, noting Lemma 3.2.2, we have

$$\begin{aligned}|E_2| &\le C \int_{I_2} \sum_{\substack{j=1\\ j\ne i}}^{n} |w_j(\tau, x_i(\tau; y_i))| \left| w_i(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} \right| d\tau \\ &\le C\widetilde{W}_1(t)Q(t) \le C\theta Q(t). \end{aligned} \tag{4.5.40}$$

Noting (4.5.32), (4.5.34), (4.5.37), and (4.5.40), from (4.5.31) we have

$$\left| w_i(t,x) \frac{\partial x_i(t; y_i)}{\partial y_i} \right| \le C_p\theta + C\theta Q(t). \tag{4.5.41}$$

Similarly, we have

$$\left| w_i(\tau, x_i(\tau; y_i)) \frac{\partial x_i(\tau; y_i)}{\partial y_i} \right| \le C_p\theta + C\theta Q(t), \quad \forall \tau \in [0,t]. \tag{4.5.42}$$

Hence, we have

$$Q(t) \le C_p\theta + C\theta Q(t), \tag{4.5.43}$$

which implies (4.5.25).

By (4.5.25), if

$$w_i(t, x_i(t; y_i)) \to \infty \quad \text{as} \quad t \uparrow t^*, \tag{4.5.44}$$

then

$$\frac{\partial x_i(t, y_i)}{\partial y_i} \to 0 \quad \text{as} \quad t \uparrow t^*. \tag{4.5.45}$$

On the other hand, by (4.5.26) and noting Lemma 3.2.2, we can easily see that (4.5.45) implies (4.5.44).

This proves Theorem 4.5.2. □

Proof of Theorem 4.5.3. For each $i \,\overline{\in}\, J$, by Remark 4.5.1, we have (4.5.18). Hence, by the equivalence of (4.5.44) and (4.5.45), the family of the ith characteristics never forms any envelope on the domain $[0, \widetilde{T}(\theta)] \times \mathbb{R}$. □

Theorem 4.5.4 can easily be obtained from the second inequality of (3.2.28) in Lemma 3.2.2.

4.6 Remarks

For the nonstrictly hyperbolic system (2.1.1) with characteristics with constant multiplicity, under the assumption that the characteristics with constant multiplicity are WLD, we can similarly obtain the corresponding results (see [58]). Moreover, some related results can be found in [91], [92]).

For the weakly discontinuous solution to the Cauchy problem of system (2.1.1) with the weakly discontinuous initial data (cf. Remark 3.4.6)

$$t = 0 : \ u = \begin{cases} u_l(x), & x \leq 0, \\ u_r(x), & x \geq 0, \end{cases} \tag{4.6.1}$$

we can similarly obtain the corresponding results also.

Chapter V
Cauchy Problem on a Semibounded Initial Axis

5.1 Introduction and Main Results

For the Cauchy problem of system (2.1.1) with the initial data

$$t = 0: \ u = \phi(x), \quad -\infty < x < +\infty, \tag{5.1.1}$$

where $\phi(x)$ is a C^1 vector function with bounded C^1 norm, in Chapter 3 we proved that if system (2.1.1) is a strictly hyperbolic system or a hyperbolic system with characteristics with constant multiplicity, then if the initial data $\phi(x)$ satisfy the following small and decaying property:

$$\theta \stackrel{\text{def.}}{=} \sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\phi(x)|+|\phi'(x)|)\} << 1, \tag{5.1.2}$$

where $\mu > 0$ is a constant, then the Cauchy problem (2.1.1) and (5.1.1) admits a unique global C^1 solution $u = u(t,x)$ with a small C^1 norm for all $t \in \mathbb{R}$ if and only if system (2.1.1) is weakly linearly degenerate (WLD), i.e., all the characteristics are WLD.

In the result mentioned above, in order to get the global existence of the C^1 solution with small and decaying initial data, all the characteristics should be WLD. However, in this chapter, under the assumption that all $\lambda_i(u)$, $l_{ij}(u)$, and $r_{ij}(u)$ $(i,j = 1,\dots,n)$ have the same regularity as $a_{ij}(u)$ $(i,j = 1,\dots,n)$, we will show that in order to get the global C^1 solution with small and decaying initial data to the **Cauchy problem on a semibounded initial axis**, it is only necessary to suppose that the leftmost or rightmost characteristic is WLD. Precisely speaking, assuming that

$$\lambda_1(0),\dots,\lambda_{n-1}(0) < \lambda_n(0), \tag{5.1.3}$$

i.e., in a neighbourhood of $u = 0$,

$$\lambda_1(u),\dots,\lambda_{n-1}(u) < \lambda_n(u), \tag{5.1.4}$$

Li Tatsien, Wang Libin, *Global Propagation of Regular Nonlinear Hyperbolic Waves*, DOI 10.1007/b78335_5,

for the Cauchy problem of system (2.1.1) with the initial data

$$t = 0: \ u = \phi(x), \quad x \geq 0, \tag{5.1.5}$$

we have the following:

Theorem 5.1.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, system* (2.1.1) *is hyperbolic and* (5.1.3) *holds. Suppose furthermore that $\lambda_n(u)$ is WLD, namely, along the nth characteristic trajectory $u = u^{(n)}(s)$ passing through $u = 0$, defined by*

$$\begin{cases} \dfrac{du}{ds} = r_n(u), \\ s = 0: \ u = 0, \end{cases} \tag{5.1.6}$$

we have

$$\nabla \lambda_n(u) r_n(u) \equiv 0, \quad \forall |u| \ small, \tag{5.1.7}$$

i.e.,

$$\lambda_n(u^{(n)}(s)) \equiv \lambda_n(0), \quad \forall |s| \ small. \tag{5.1.8}$$

Suppose finally that

$$\theta \overset{\text{def.}}{=} \sup_{x \geq 0} \{(1 + x)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} < +\infty, \tag{5.1.9}$$

where $\mu > 0$ is a constant. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (2.1.1) *and* (5.1.5) *admits a unique global C^1 solution $u = u(t, x)$ with a small C^1 norm on the domain $D = \{(t, x) | \ t \geq 0, x \geq x_n(t)\}$, where $x = x_n(t)$ is the nth characteristic passing through the origin $O(0, 0)$:*

$$\begin{cases} \dfrac{dx_n(t)}{dt} = \lambda_n(u(t, x_n(t))), \\ x_n(0) = 0. \end{cases} \tag{5.1.10}$$

On the other hand, under the assumption that in a neighbourhood of $u = 0$, $A(u) \in C^1$, system (2.1.1) *is hyperbolic and* (5.1.3) *holds, if for any given initial data $\phi(x)$ $(x \geq 0)$ such that*

$$\theta \overset{\text{def.}}{=} \sup_{x \geq 0} \{(1 + x)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} << 1, \tag{5.1.11}$$

the Cauchy problem (2.1.1) *and* (5.1.5) *always admits a unique global C^1 solution $u = u(t, x)$ on the domain $D = \{(t, x) | \ t \geq 0, x \geq x_n(t)\}$, then $\lambda_n(u)$ must be WLD* (cf. [63]).

Remark 5.1.1 *Suppose that in a neighbourhood of* $u = 0$,

$$\lambda_1(u), \ldots, \lambda_p(u) < \lambda_{p+1}(u) \equiv \cdots \equiv \lambda_n(u), \tag{5.1.12}$$

where $\lambda(u) \overset{\text{def.}}{=} \lambda_{p+1}(u) \equiv \cdots \equiv \lambda_n(u)$ *is a characteristic with constant multiplicity* $n - p$. *Suppose furthermore that* $\lambda_{p+1}(u), \ldots, \lambda_n(u)$ *are WLD* (see Definition 2.5.3). *Then the conclusion of Theorem 5.1.1 is still valid.*

Remark 5.1.2 *When*

$$\lambda_1(0) < \lambda_2(0), \ldots, \lambda_n(0) \tag{5.1.13}$$

or in a neighbourhood of $u = 0$,

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u), \ldots, \lambda_n(u), \tag{5.1.14}$$

for the initial data

$$t = 0: \;\; u = \phi(x), \quad x \leq 0, \tag{5.1.15}$$

such that

$$\theta \overset{\text{def.}}{=} \sup_{x \leq 0}\{(1 + |x|)^{1+\mu}(|\phi(x)| + |\phi'(x)|)\} << 1, \tag{5.1.16}$$

similar results hold as in Theorem 5.1.1 and Remark 5.1.1.

Remark 5.1.3 *For the inhomogeneous quasilinear hyperbolic system* (2.1.19)*, suppose that the inhomogeneous term* $B(u)$ *satisfies some conditions; then similar results hold* (see [30])*.*

Remark 5.1.4 *Some results about the formation of singularities of the* C^1 *solution to the Cauchy problem on a semibounded initial axis can be found in* [31]*.*

5.2 Proof of Theorem 5.1.1

We first prove the necessity—the second part of Theorem 5.1.1.

Since the weak linear degeneracy of $\lambda_n(u)$ is invariant under any smooth invertible transformation $u = u(\widetilde{u})$ $[u(0) = 0]$, without loss of generality, we may assume that system (2.1.1) is written in the corresponding generalized normalized coordinates. By the definition, $\lambda_n(u)$ is WLD if and only if

$$\lambda_n(u_n e_n) \equiv \lambda_n(0), \quad \forall |u_n| \text{ small}.$$

By the method in Section 3.1, we immediately get the necessity in Theorem 5.1.1.

Next, we prove the sufficiency—the first part of Theorem 5.1.1.

In what follows, we always assume that $\theta > 0$ is suitably small.

By (5.1.3), there exist positive constants δ_0 and δ so small that

$$\lambda_n(u) - \lambda_i(u') \geq 2\delta_0, \quad \forall |u|, |u'| \leq \delta \quad (i = 1, \ldots, n-1) \tag{5.2.1}$$

and

$$|\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \quad \forall |u|, |u'| \leq \delta \quad (i = 1, \ldots, n). \tag{5.2.2}$$

Without loss of generality, we suppose that

$$\lambda_i(0) \geq \delta_0 \quad (i = 1, \ldots, n). \tag{5.2.3}$$

For the time being, we assume that on any given existence domain of the C^1 solution $u = u(t,x)$ to the Cauchy problem (2.1.1) and (5.1.5), we have

$$|u(t,x)| \leq \delta. \tag{5.2.4}$$

At the end of the proof of Lemma 5.2.1, we will explain that this hypothesis is reasonable.

By (5.2.2) and (5.2.3), it is easy to get

$$x_n(t) \geq \left(\lambda_n(0) - \frac{\delta_0}{2}\right) t \geq \frac{\delta_0}{2} t. \tag{5.2.5}$$

In order to prove the sufficiency in Theorem 5.1.1, it is only necessary to establish a uniform a priori estimate on the C^1 norm of the C^1 solution $u = u(t,x)$ to the Cauchy problem (2.1.1) and (5.1.5) on any given existence domain (cf. [72]).

For any given $T > 0$, let

$$D^T = \{(t,x) |\ 0 \leq t \leq T, x \geq x_n(t)\}. \tag{5.2.6}$$

On each existence domain D^T of the C^1 solution $u = u(t,x)$, let

$$U_\infty^c(T) = \max_{i=1,\ldots,n-1} \sup_{(t,x)\in D^T} \{(1+x)^{1+\mu} |u_i(t,x)|\}, \tag{5.2.7}$$

$$W_\infty^c(T) = \max_{i=1,\ldots,n-1} \sup_{(t,x)\in D^T} \{(1+x)^{1+\mu} |w_i(t,x)|\}, \tag{5.2.8}$$

$$\widetilde{U}_1(T) = \max_{i=1,\ldots,n-1} \int_{c_i} |u_n(t,x)| dt, \tag{5.2.9}$$

$$\widetilde{W}_1(T) = \max_{i=1,\ldots,n-1} \int_{c_i} |w_n(t,x)| dt, \tag{5.2.10}$$

where c_i denotes any given ith characteristic on D^T,

$$U_1(T) = \sup_{0 \le t \le T} \int_{x_n(t)}^{+\infty} |u_n(t,x)| dx, \tag{5.2.11}$$

$$W_1(T) = \sup_{0 \le t \le T} \int_{x_n(t)}^{+\infty} |w_n(t,x)| dx, \tag{5.2.12}$$

$$U_\infty(T) = \max_{i=1,\dots,n} \sup_{(t,x)\in D^T} |u_i(t,x)|, \tag{5.2.13}$$

and

$$W_\infty(T) = \max_{i=1,\dots,n} \sup_{(t,x)\in D^T} |w_i(t,x)|. \tag{5.2.14}$$

Lemma 5.2.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$, system* (2.1.1) *is hyperbolic, and* (5.1.3) *holds. Suppose furthermore that $\phi(x)$ satisfies* (5.1.9)*. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, on any given existence domain $D^T = \{(t,x) | 0 \le t \le T, x \ge x_n(t)\}$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem* (2.1.1) *and* (5.1.5)*, we have the following uniform a priori estimates:*

$$\widetilde{W}_1(T),\ W_1(T) \le \kappa_1 \theta, \tag{5.2.15}$$

$$W_\infty^c(T) \le \kappa_2 \theta, \tag{5.2.16}$$

and

$$U_\infty(T) \le \kappa_3 \theta. \tag{5.2.17}$$

Here and henceforth, κ_i $(i = 1, 2, \dots)$ are positive constants independent of θ and T.

Proof. We first estimate $W_\infty^c(T)$.

Passing through any fixed point $(t,x) \in D^T$, we draw the ith characteristic c_i: $\xi = x_i(\tau)$ $(0 \le \tau \le t,\ i \ne n)$ that intersects the x-axis at a point $(0, x_{i0})$. Integrating (2.6.14) along c_i from 0 to t yields

$$w_i(t,x) = w_i(0, x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u) w_j w_k(\tau, x_i(\tau)) d\tau \quad (i \ne n). \tag{5.2.18}$$

By (5.2.2), it is easy to see that

$$\left(\lambda_i(0) - \frac{\delta_0}{2}\right) t \le x - x_{i0} \le \left(\lambda_i(0) + \frac{\delta_0}{2}\right) t \quad (i \ne n). \tag{5.2.19}$$

Moreover, by (5.2.5), we have

$$x \ge \left(\lambda_n(0) - \frac{\delta_0}{2}\right) t.$$

Then, noting (5.2.1) and (5.2.3), we see from (5.2.19) that

$$\frac{\delta_0}{\lambda_n(0)-\frac{\delta_0}{2}}x \le x_{i0} \le x \quad (i \neq n). \tag{5.2.20}$$

Similarly, we have

$$\frac{\delta_0}{\lambda_n(0)-\frac{\delta_0}{2}}x \le x_i(\tau) \le x, \quad \forall \tau \in [0,t] \quad (i \neq n). \tag{5.2.21}$$

Thus, noting (2.6.16), we see from (5.2.18) that

$$\begin{aligned}
&(1+x)^{1+\mu}|w_i(t,x)| \\
&\le (1+x)^{1+\mu}\left\{|w_i(0,x_{i0})| + \int_0^t \sum_{j,k=1}^n |\gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau))|d\tau\right\} \\
&\le C(1+x_{i0})^{1+\mu}|w_i(0,x_{i0})| + (1+x)^{1+\mu}\left\{\int_0^t\left(\sum_{j,k=1}^{n-1}|\gamma_{ijk}(u)w_jw_k(\tau,x_i(\tau))|\right.\right. \\
&\quad \left.\left. + 2\sum_{j=1}^{n-1}|\gamma_{ijn}(u)w_jw_n(\tau,x_i(\tau))|\right)d\tau\right\} \\
&\le C\left\{\theta + (W_\infty^c(T))^2\int_0^t(1+x_i(\tau))^{-(1+\mu)}d\tau + W_\infty^c(T)\widetilde{W}_1(T)\right\} \quad (i \neq n).
\end{aligned} \tag{5.2.22}$$

Then, noting (5.2.5), we get

$$W_\infty^c(T) \le C\left\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T)\widetilde{W}_1(T)\right\}. \tag{5.2.23}$$

We next estimate $\widetilde{W}_1(T)$.

Passing through any fixed point $A(t,x) \in D^T$, we draw the ith characteristic c_i: $\xi = x_i(\tau)$ $(0 \le \tau \le t,\ i \neq n)$, which intersects the x-axis at a point C. In the meantime, the nth characteristic passing through A intersects the x-axis at a point B. By Lemma 2.6.1, we get

$$\begin{aligned}
&\int_{c_i}|w_n(\lambda_i(u)-\lambda_n(u))(\tau,x_i(\tau))|d\tau \\
&\qquad \le \int_{BC}|w_n(0,x)|dx + \iint_{ABC}\sum_{j,k=1}^n|\Gamma_{njk}(u)w_jw_k(t,x)|dtdx.
\end{aligned} \tag{5.2.24}$$

Then, noting (2.6.20) and (5.2.1), we have

$$\begin{aligned}
&\int_{c_i} |w_n(\tau, x_i(\tau))| d\tau \\
&\quad \le C\left\{ \int_0^{+\infty} |w_n(0,x)|dx + \iint_{ABC} \left(\sum_{j,k=1}^{n-1} |\Gamma_{njk}(u) w_j w_k(t,x)| \right.\right.\\
&\qquad \left.\left. + \sum_{j=1}^{n-1} |\Gamma_{njn}(u) w_j w_n(t,x)| \right) dtdx \right\} \\
&\quad \le C\left\{ \theta + (W_\infty^c(T))^2 \iint_{ABC} (1+x)^{-2(1+\mu)} dtdx \right. \\
&\qquad \left. + W_\infty^c(T) \iint_{ABC} (1+x)^{-(1+\mu)} |w_n(t,x)| dtdx \right\}. \qquad (5.2.25)
\end{aligned}$$

Hence, noting (5.2.5), we get

$$\widetilde{W}_1(T) \le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T)\}. \qquad (5.2.26)$$

Now we estimate $W_1(T)$.

For any given t with $0 \le t \le T$, passing through $A(t,a)$ $(a > x_n(t))$, we draw the nth characteristic, which intersects the x-axis at a point C. Let B denote the point $(t, x_n(t))$. By Lemma 2.6.1, similarly to (5.2.24), we have

$$\int_{BA} |w_n(t,x)|dx \le \int_{OC} |w_n(0,x)|dx + \iint_{ABOC} \sum_{j,k=1}^{n} |\Gamma_{njk}(u) w_j w_k(t,x)| dtdx. \qquad (5.2.27)$$

Then, similarly to (4.2.8), it is easy to get

$$\int_{x_n(t)}^{a} |w_n(t,x)|dx \le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T)\}, \qquad (5.2.28)$$

where C is independent of a. Thus, letting $a \to +\infty$, we get

$$W_1(T) \le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_1(T)\}. \qquad (5.2.29)$$

The combination of (5.2.23), (5.2.26), and (5.2.29) gives (5.2.15)–(5.2.16).

Finally, we estimate $U_\infty(T)$.

Passing through any fixed point $(t,x) \in D^T$, we draw the nth characteristic c_n: $\xi = x_n(\tau)$ $(0 \le \tau \le t)$, which intersects the x-axis at a point $(0, x_0)$. Noting (2.6.43), we have

$$\frac{du}{d_n t} = \sum_{k=1}^{n-1} (\lambda_n(u) - \lambda_k(u)) w_k r_k. \qquad (5.2.30)$$

Integrating (5.2.30) along c_n from 0 to t yields

$$u(t,x) = \phi(x_0) + \int_0^t \sum_{k=1}^{n-1} (\lambda_n(u) - \lambda_k(u)) w_k r_k(\tau, x_n(\tau)) d\tau. \tag{5.2.31}$$

Then, noting (5.1.9) and (5.2.5) and using (5.2.16), we get

$$|u(t,x)| \leq C\{\theta + W_\infty^c(T)\} \leq C\theta. \tag{5.2.32}$$

Hence, (5.2.17) holds. Moreover, it turns out from (5.2.32) that hypothesis (5.2.4) is reasonable.

Lemma 5.2.2 *Under the assumptions of the first part of Theorem 5.1.1, in corresponding generalized normalized coordinates, there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, on any given existence domain $D^T = \{(t,x) | 0 \leq t \leq T, x \geq x_n(t)\}$ of the C^1 solution $u = u(t,x)$ to the Cauchy problem* (2.1.1) *and* (5.1.5)*, we have the following uniform a priori estimates:*

$$\widetilde{U}_1(T),\ U_1(T) \leq \kappa_4 \theta, \tag{5.2.33}$$
$$U_\infty^c(T) \leq \kappa_5 \theta, \tag{5.2.34}$$

and

$$W_\infty(T) \leq \kappa_6 \theta. \tag{5.2.35}$$

Proof. We first estimate $U_\infty^c(T)$.

Similarly to (5.2.18), integrating (2.6.44), we have

$$u_i(t,x) = u_i(0, x_{i0}) + \int_0^t \sum_{j,k=1}^{n} \rho_{ijk}(u) u_j w_k(\tau, x_i(\tau)) d\tau \quad (i \neq n). \tag{5.2.36}$$

Noting (2.6.45), we can rewrite (5.2.36) as

$$\begin{aligned} u_i(t,x) = u_i(0, x_{i0}) + \int_0^t \Bigg[\sum_{j,k=1}^{n-1} \rho_{ijk}(u) u_j w_k + \sum_{j=1}^{n-1} (\rho_{ijn}(u) u_j w_n \\ + \rho_{inj}(u) u_n w_j) \Bigg] (\tau, \xi(\tau)) d\tau \quad (i \neq n). \end{aligned} \tag{5.2.37}$$

Then, noting (5.2.15)–(5.2.17), similarly to (5.2.23), we have

$$\begin{aligned} U_\infty^c(T) &\leq C \left\{ \theta + U_\infty^c(T) W_\infty^c(T) + W_\infty^c(T) \widetilde{U}_1(T) + U_\infty^c(T) \widetilde{W}_1(T) \right\} \\ &\leq C\theta \{1 + U_\infty^c(T) + \widetilde{U}_1(T)\}. \end{aligned} \tag{5.2.38}$$

Hence, we have

$$U_\infty^c(T) \le C\theta\left\{1+\widetilde{U}_1(T)\right\}. \tag{5.2.39}$$

We next estimate $\widetilde{U}_1(T)$ and $U_1(T)$.

Since $\lambda_n(u)$ is WLD, by (2.6.52) we have

$$F_{nnn}(u_n e_n) \equiv 0. \tag{5.2.40}$$

Then, using (2.6.48) and noting (2.6.50), similarly to (5.2.25), we have

$$\begin{aligned}\int_{c_i} |u_n(\tau, x_i(\tau))| d\tau \le C\Bigg\{ & \int_0^{+\infty} |u_n(0,x)|dx \\ & + \iint_{ABC} \sum_{j,k=1}^{n} |F_{njk}(u)u_j w_k(t,x)| dtdx \Bigg\} \\ \le C\Bigg\{\theta + & \iint_{ABC} \Bigg[\sum_{j,k=1}^{n-1} |F_{njk}(u)u_j w_k(t,x)| \\ & + \sum_{j=1}^{n-1} |F_{nnj}(u)u_n w_j(t,x)| + |(F_{nnn}(u) \\ & - F_{nnn}(u_n e_n))u_n w_n(t,x)| \Bigg] dtdx \Bigg\} \quad (i \ne n).\end{aligned} \tag{5.2.41}$$

Hence, noting (5.2.5) and using Lemma 5.2.1, we get

$$\begin{aligned}\widetilde{U}_1(T) &\le C\{\theta + U_\infty^c(T)W_\infty^c(T) + W_\infty^c(T)U_1(T) + U_\infty(T)U_\infty^c(T)W_1(T)\} \\ &\le C\theta\{1+U_\infty^c(T)+U_1(T)\}.\end{aligned} \tag{5.2.42}$$

Moreover, similarly to (5.2.29), we have

$$U_1(T) \le C\theta\{1+U_\infty^c(T)+U_1(T)\},$$

which gives us

$$U_1(T) \le C\theta\{1+U_\infty^c(T)\}. \tag{5.2.43}$$

The combination of (5.2.39), (5.2.42), and (5.2.43) yields (5.2.33)–(5.2.34).

We finally estimate $W_\infty(T)$.

Since $\lambda_n(u)$ is WLD, by (2.6.17) we have

$$\gamma_{nnn}(u_n e_n) \equiv 0. \tag{5.2.44}$$

Similarly to (5.2.31), integrating (2.6.14) (in which we take $i = n$) along c_n from 0 to t gives

$$\begin{aligned} w_n(t,x) &= w_n(0,x_0) + \int_0^t \sum_{j,k=1}^{n} \gamma_{njk}(u) w_j w_k(\tau, x_n(\tau)) d\tau \\ &= w_n(0,x_0) + \int_0^t \left(\sum_{j,k=1}^{n-1} \gamma_{njk}(u) w_j w_k + 2\sum_{j=1}^{n-1} \gamma_{njn}(u) w_j w_n \right. \\ &\quad \left. + (\gamma_{nnn}(u) - \gamma_{nnn}(u_n e_n))(w_n)^2 \right)(\tau, x_n(\tau)) d\tau. \end{aligned} \tag{5.2.45}$$

Then, noting (5.1.9) and using (5.2.16) and (5.2.34), we get

$$\begin{aligned} |w_n(t,x)| &\le C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T) W_\infty(T) + (W_\infty(T))^2 U_\infty^c(T)\} \\ &\le C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}. \end{aligned} \tag{5.2.46}$$

Hence, we have

$$W_\infty(T) \le C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}, \tag{5.2.47}$$

which implies (5.2.35).

The sufficiency in Theorem 5.1.1 follows immediately from Lemmas 5.2.1 and 5.2.2.

5.3 Application

Consider the following Cauchy problem for the system of traffic flow (cf. Section 1.3.2):

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0 \end{cases} \tag{5.3.1}$$

with the initial data

$$t = 0: \ (\rho, v) = (\widetilde{\rho}_0 + \rho_0(x), \widetilde{v}_0 + v_0(x)) \quad (x \ge 0), \tag{5.3.2}$$

where $\rho(> 0)$ and v are the density and velocity of cars at point x and time t, respectively, $p(\cdot)$ is a suitably smooth and strictly increasing function, $\widetilde{\rho}_0 > 0$ and $\widetilde{v}_0$ are constants, and $(\rho_0(x), v_0(x)) \in C^1$ and satisfies the decaying property as shown in (5.1.9).

Let

$$U = (\rho, v)^T. \tag{5.3.3}$$

For $\rho > 0$, (5.3.1) is a strictly hyperbolic system with the following distinct real eigenvalues:

$$\lambda_1(U) = v - \rho p'(\rho) < \lambda_2(U) = v. \tag{5.3.4}$$

It is easy to see that $\lambda_2(U)$ is LD, then WLD with respect to $U = U_0 \stackrel{\text{def.}}{=} (\widetilde{\rho}_0, \widetilde{v}_0)^T$.

By Theorem 5.1.1, we have

Theorem 5.3.1 *There exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, the Cauchy problem* (5.3.1)–(5.3.2) *admits a unique global C^1 solution $U = U(t, x)$ on the domain $D = \{(t, x) | t \geq 0, x \geq x_2(t)\}$, where $x = x_2(t)$ is the second characteristic passing through the origin $O(0, 0)$.*

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Chapter VI
One-Sided Mixed Initial-Boundary Value Problem

In this chapter we consider the one-sided mixed initial-boundary value problem for quasilinear hyperbolic systems with nonlinear boundary conditions on the domain $\{(t,x)|\ t \geq 0, x \geq 0\}$.

6.1 Global Existence of the Classical Solution

6.1.1 Introduction and Main Results

Consider the following first-order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{6.1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of (t,x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i,j = 1,\ldots,n)$.

By the definition of hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u),\ldots,\lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i = 1,\ldots,n$, let $l_i(u) = (l_{i1}(u),\ldots,l_{in}(u))$ [resp. $r_i(u) = (r_{1i}(u),\ldots,r_{ni}(u))^T$] be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \tag{6.1.2}$$

and

$$A(u)r_i(u) = \lambda_i(u)r_i(u). \tag{6.1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \qquad [\text{resp.} \det|r_{ij}(u)| \neq 0]. \tag{6.1.4}$$

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Without loss of generality, we suppose that on the domain under consideration,

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n), \tag{6.1.5}$$

where δ_{ij} stands for Kronecker's delta.

We suppose that all $\lambda_i(u)$, $l_{ij}(u)$, and $r_{ij}(u)$ $(i, j = 1, \dots, n)$ have the same regularity as $a_{ij}(u)$ $(i, j = 1, \dots, n)$.

For the Cauchy problem of system (6.1.1) with the initial data

$$t = 0: \ u = \phi(x), \quad -\infty < x < +\infty, \tag{6.1.6}$$

where $\phi(x)$ is a C^1 vector function with a small C^1 norm and certain decaying properties, by means of the concept of weak linear degeneracy, the global existence and the blow-up phenomenon of C^1 solution have been studied in Chapters 3 and 4, respectively. Moreover, the global existence of the C^1 solution to the Cauchy problem on a semibounded initial axis has been studied in Chapter 5. In order to consider the effect of boundary conditions on the global regularity of classical solution, in this section we consider the one-sided mixed initial-boundary value problem for system (6.1.1) in a semibounded domain.

We suppose that the eigenvalues satisfy

$$\lambda_1(0), \dots, \lambda_m(0) < 0 < \lambda_{m+1}(0) < \cdots < \lambda_n(0). \tag{6.1.7}$$

On the domain

$$D = \{(t, x) |\ t \geq 0,\ x \geq 0\}, \tag{6.1.8}$$

we consider the following **one-sided mixed initial-boundary value problem** for system (6.1.1) with the initial condition

$$t = 0: \ u = \phi(x), \quad x \geq 0, \tag{6.1.9}$$

and the boundary condition

$$x = 0: \ v_s = f_s(\alpha(t), v_1, \dots, v_m) + h_s(t) \quad (s = m+1, \dots, n), \tag{6.1.10}$$

in which

$$v_i = l_i(u)u \quad (i = 1, \dots, n) \tag{6.1.11}$$

and

$$\alpha(t) = (\alpha_1(t), \dots, \alpha_k(t)). \tag{6.1.12}$$

Without loss of generality, we suppose that

$$f_s(\alpha(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{6.1.13}$$

Remark 6.1.1 *In a neighbourhood of $u = 0$, the boundary condition* (6.1.10) *takes a similar form under any possibly different choice of left eigenvectors.*

The main result of the section is

Theorem 6.1.1 *Suppose that in a neighbourhood of* $u = 0$, $A(u) \in C^2$ *and* (6.1.7) *holds. Suppose furthermore that* ϕ, f_s, α, *and* h_s $(s = m+1, \dots, n)$ *are all* C^1 *functions with respect to their arguments. Suppose finally that the conditions of* C^1 *compatibility are satisfied at the point* $(t, x) = (0, 0)$ *and* (6.1.13) *holds. If* $\lambda_s(u)$ $(s = m+1, \dots, n)$ *are weakly linearly degenerate* (*WLD*) *and*

$$\theta \overset{\text{def.}}{=} \max\left\{ \sup_{x\geq 0}(1+x)^{1+\mu}(|\phi(x)| + |\phi'(x)|), \right. \\ \left. \sup_{t\geq 0}(1+t)^{1+\mu}(|\alpha(t)| + |h(t)| + |\alpha'(t)| + |h'(t)|) \right\} \\ < +\infty, \tag{6.1.14}$$

where $\mu > 0$ *is a constant and*

$$h(t) = (h_{m+1}(t), \dots, h_n(t)), \tag{6.1.15}$$

then there exists $\theta_0 > 0$ *so small that for any given* $\theta \in [0, \theta_0]$, *the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10) *admits a unique global* C^1 *solution* $u = u(t, x)$ *with a small* C^1 *norm on the domain* $D = \{(t, x)|\ t \geq 0, x \geq 0\}$ (cf. [66]).

Remark 6.1.2 *Comparing with the result on the Cauchy problem in Chapter 3* (also see [75], [76]), *Theorem 6.1.1 shows that when there is only one boundary, the interaction of linear or nonlinear boundary conditions with nonlinear hyperbolic waves causes a positive effect on the global regularity of the solution: The weak linear degeneracy of negative characteristics* $\lambda_r(u)$ $(r = 1, \dots, m)$ *is not necessary for the global existence of the* C^1 *solution.*

Remark 6.1.3 *The global existence of the weakly discontinuous solution to the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10) *can be similarly discussed* (see [29]).

Remark 6.1.4 *For the one-sided mixed initial-boundary value problem of the inhomogeneous quasilinear hyperbolic system* (2.1.19) *with* (6.1.9)–(6.1.10), *suppose that the inhomogeneous term* $B(u)$ *satisfies some conditions. Then similar results hold* (see [18]).

Remark 6.1.5 *For the mixed initial-boundary value problem on a bounded domain* $\{(t, x)|t \geq 0, 0 \leq x \leq L\}$, *the results on the global regularity of the* C^1 *solution can be found in* [28], [50], [60], [61], *and* [85].

6.1.2 Proof of Theorem 6.1.1

In what follows, we always assume that $\theta > 0$ is suitably small.

By (6.1.7), there exist positive constants δ_0 and δ such that

$$\lambda_i(u) - \lambda_j(u') \geq 4\delta_0, \ \forall\ |u|, |u'| \leq \delta, \forall i \in \{m+1, \ldots, n\}, \\ \forall j \in \{1, \ldots, m\} \text{ or } j = i-1, \tag{6.1.16}$$

$$|\lambda_i(u) - \lambda_i(u')| \leq \frac{\delta_0}{2}, \quad \forall |u|, |u'| \leq \delta \quad (i = 1, \ldots, n), \tag{6.1.17}$$

and

$$|\lambda_i(0)| > \delta_0 \quad (i = 1, \ldots, n). \tag{6.1.18}$$

For the time being, we assume that on any given existence domain of the C^1 solution $u = u(t, x)$ to the one-sided mixed initial-boundary value problem (6.1.1) and (6.1.9)–(6.1.10), we have

$$|u(t, x)| \leq \delta. \tag{6.1.19}$$

At the end of the proof of Lemma 6.1.2, we will explain that this hypothesis is reasonable.

In order to prove Theorem 6.1.1, it is sufficient to establish a uniform a priori estimate on the C^1 norm of the C^1 solution $u = u(t, x)$ to the one-sided mixed initial-boundary value problem (6.1.1) and (6.1.9)–(6.1.10) (cf. [72]). By (2.6.3)–(2.6.4), it is easy to see that it is only necessary to establish a uniform a priori estimate on the C^0 norm of $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$.

For any given $T > 0$, let

$$D_+^T = \{(t, x) |\ 0 \leq t \leq T, x \geq (\lambda_n(0) + \delta_0)t\}, \tag{6.1.20}$$

$$D_0^T = \{(t, x) |\ 0 \leq t \leq T, 0 \leq x \leq (\lambda_{m+1}(0) - \delta_0)t\}, \tag{6.1.21}$$

$$D^T = \{(t, x) |\ 0 \leq t \leq T, (\lambda_{m+1}(0) - \delta_0)t \leq x \leq (\lambda_n(0) + \delta_0)t\}, \tag{6.1.22}$$

and for $s = m+1, \ldots, n$,

$$D_s^T = \{(t, x) |\ 0 \leq t \leq T, |x - \lambda_s(0)t| \leq \delta_0 t\}. \tag{6.1.23}$$

It is easy to see that

$$D_+^T \cup D_0^T \cup D^T = \{(t, x) | 0 \leq t \leq T, x \geq 0\}$$

and

$$\bigcup_{s=m+1}^{n} D_s^T \subset D^T.$$

On each existence domain $\{(t,x)|0 \le t \le T, x \ge 0\}$ of the C^1 solution $u = u(t,x)$ to the one-sided mixed initial-boundary value problem (6.1.1) and (6.1.9)–(6.1.10), let

$$V(D_+^T) = \max_{i=1,\dots,n} \|(1+x)^{1+\mu} v_i(t,x)\|_{L^\infty(D_+^T)}, \tag{6.1.24}$$

$$W(D_+^T) = \max_{i=1,\dots,n} \|(1+x)^{1+\mu} w_i(t,x)\|_{L^\infty(D_+^T)}, \tag{6.1.25}$$

$$V(D_0^T) = \max_{i=1,\dots,n} \|(1+t)^{1+\mu} v_i(t,x)\|_{L^\infty(D_0^T)}, \tag{6.1.26}$$

$$W(D_0^T) = \max_{i=1,\dots,n} \|(1+t)^{1+\mu} w_i(t,x)\|_{L^\infty(D_0^T)}, \tag{6.1.27}$$

$$\begin{aligned} V_\infty^c(T) = \max \Bigg\{ & \max_{r=1,\dots,m} \sup_{(t,x)\in D^T} (1+t)^{1+\mu} |v_r(t,x)|, \\ & \max_{s=m+1,\dots,n} \sup_{(t,x)\in D^T \setminus D_s^T} (1+t)^{1+\mu} |v_s(t,x)| \Bigg\}, \end{aligned} \tag{6.1.28}$$

$$\begin{aligned} W_\infty^c(T) = \max \Bigg\{ & \max_{r=1,\dots,m} \sup_{(t,x)\in D^T} (1+t)^{1+\mu} |w_r(t,x)|, \\ & \max_{s=m+1,\dots,n} \sup_{(t,x)\in D^T \setminus D_s^T} (1+t)^{1+\mu} |w_s(t,x)| \Bigg\}, \end{aligned} \tag{6.1.29}$$

$$\widetilde{V}_1(T) = \max_{s=m+1,\dots,n} \max_{j\neq s} \sup_{\widetilde{c}_j} \int_{\widetilde{c}_j} |v_s(t,x)| dt, \tag{6.1.30}$$

$$\widetilde{W}_1(T) = \max_{s=m+1,\dots,n} \max_{j\neq s} \sup_{\widetilde{c}_j} \int_{\widetilde{c}_j} |w_s(t,x)| dt, \tag{6.1.31}$$

where $\widetilde{c}_j$ denotes any given jth characteristic on D_s^T $(j \neq s, s = m+1, \dots, n)$,

$$V_\infty(T) = \max_{i=1,\dots,n} \sup_{\substack{0\le t\le T \\ x \ge 0}} |v_i(t,x)|, \tag{6.1.32}$$

and

$$W_\infty(T) = \max_{i=1,\dots,n} \sup_{\substack{0\le t\le T \\ x \ge 0}} |w_i(t,x)|. \tag{6.1.33}$$

Lemma 6.1.1 *Suppose that in a neighbourhood of* $u = 0$, $A(u) \in C^2$, (6.1.1) *is hyperbolic and* (6.1.7) *holds. Suppose furthermore that* $\phi(x)$ *satisfies the requirement of* (6.1.14). *Then there exists* $\theta_0 > 0$ *so small that for any given* $\theta \in [0, \theta_0]$, *on any given existence domain* $\{(t,x)|0 \le t \le T, x \ge 0\}$ *of the* C^1 *solution* $u = u(t,x)$ *to the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10), *we have the following uniform a priori estimates:*

$$V(D_+^T),\ W(D_+^T) \le \kappa_1 \theta. \tag{6.1.34}$$

Henceforth, κ_i $(i=1,2,\dots)$ *are positive constants independent of* θ *and* T*.*

Proof. For each $i=1,\dots,n$, passing through any given point $(t,x)\in D_+^T$, we draw the ith characteristic $c_i:\ \xi=x_i(\tau;y_i)$ in which $0\le\tau\le t$ and y_i stands for the x-coordinate of the intersection point of this characteristic with the x-axis. Noting (6.1.7) and (6.1.16)–(6.1.18), it is easy to see that the whole characteristic c_i is included in D_+^T and there exist positive constants d_1 and d_2 such that

$$d_1(1+y_i)\le 1+x_i(\tau;y_i)\le d_2(1+y_i)\quad\text{for } 0\le\tau\le t. \tag{6.1.35}$$

Integrating (2.6.5) along c_i from 0 to t gives

$$v_i(t,x)=v_i(0,y_i)+\int_0^t\sum_{j,k=1}^n\beta_{ijk}(u)v_jw_k(\tau,x_i(\tau;y_i))d\tau. \tag{6.1.36}$$

Then, noting (6.1.14), (6.1.19), and (6.1.35), we get

$$\begin{aligned}
&(1+x)^{1+\mu}|v_i(t,x)|\\
&\quad\le C(1+x)^{1+\mu}\left\{v_i(0,y_i)+V(D_+^T)W(D_+^T)\int_0^t(1+x_i(\tau;y_i))^{-2(1+\mu)}d\tau\right\}\\
&\quad\le C\left\{\theta+V(D_+^T)W(D_+^T)\int_0^t(1+x_i(\tau;y_i))^{-(1+\mu)}d\tau\right\}\\
&\quad\le C\{\theta+V(D_+^T)W(D_+^T)\}.
\end{aligned}\tag{6.1.37}$$

Hence, we have

$$V(D_+^T)\le C\{\theta+V(D_+^T)W(D_+^T)\}. \tag{6.1.38}$$

Similarly, we have

$$W(D_+^T)\le C\{\theta+V(D_+^T)W(D_+^T)\}. \tag{6.1.39}$$

Then, we get (6.1.34) via the method in the proof of Lemma 3.2.2.

Lemma 6.1.2 *Suppose that in a neighbourhood of* $u=0$, $A(u)\in C^2$, (6.1.1) *is hyperbolic and* (6.1.7) *holds. Suppose furthermore that* (6.1.14) *holds. Then there exists* $\theta_0>0$ *so small that for any given* $\theta\in[0,\theta_0]$*, on any given existence domain* $\{(t,x)|0\le t\le T, x\ge 0\}$ *of the* C^1 *solution* $u=u(t,x)$ *to the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10)*, we have the following uniform a priori estimates:*

$$W(D_0^T)\le\kappa_2\theta, \tag{6.1.40}$$

$$\widetilde{W}_1(T)\le\kappa_3\theta, \tag{6.1.41}$$

$$W_\infty^c(T)\le\kappa_4\theta, \tag{6.1.42}$$

and

$$V_\infty(T) \le \kappa_5 \theta. \tag{6.1.43}$$

Proof. Differentiating the boundary condition (6.1.10) with respect to t gives

$$\begin{aligned} x = 0 : \frac{\partial v_s}{\partial t} =& \sum_{r=1}^{m} \frac{\partial f_s}{\partial v_r}(\alpha(t), v_1, \dots, v_m) \frac{\partial v_r}{\partial t} \\ &+ \left(\frac{\partial f_s}{\partial \alpha} \right) (\alpha(t), v_1, \dots, v_m)(\alpha'(t))^T + h_s'(t)(s = m+1, \dots, n). \end{aligned} \tag{6.1.44}$$

Noting (6.1.1) and (2.6.4), it is easy to see that

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial t}(l_i(u)u) = -\lambda_i(u) w_i + \sum_{k=1}^{n} a_{ik}(u) w_k \quad (i = 1, \dots, n), \tag{6.1.45}$$

where

$$a_{ik}(u) = -\lambda_k(u) r_k^T(u) \nabla l_i(u) u. \tag{6.1.46}$$

Then, noting (6.1.7) and (6.1.19), for $\delta > 0$ small enough, by (6.1.44)–(6.1.45) we have

$$\begin{aligned} x = 0 :\ w_s = \sum_{r=1}^{m} f_{sr}(t,u) w_r + \sum_{i=1}^{k} \overline{f}_{si}(t,u) \alpha_i'(t) + \sum_{\overline{s}=m+1}^{n} \widetilde{f}_{s\overline{s}}(t,u) h_{\overline{s}}'(t) \\ (s = m+1, \dots, n), \end{aligned} \tag{6.1.47}$$

where f_{sr}, $\overline{f}_{si}$, and $\widetilde{f}_{s\overline{s}}$ are continuous functions of t [via $\alpha(t)$] and u.

We first estimate $W(D_0^T)$.

i. For $r = 1, \dots, m$, passing through any fixed point $(t,x) \in D_0^T$, we draw the rth characteristic c_r, which intersects the straight lines $x = (\lambda_{m+1}(0) - \delta_0)t$ and $x = (\lambda_n(0) + \delta_0)t$ at points (t_1, x_1) and (t_2, x_2), respectively. Integrating (2.6.14) along c_r from t_2 to t yields

$$\begin{aligned} w_r(t,x) &= w_r(t_2, x_2) + \int_{t_2}^{t} \sum_{j,k=1}^{n} \gamma_{rjk}(u) w_j w_k d\tau \\ &= w_r(t_2, x_2) + \int_{t_2}^{t_1} \sum_{j,k=1}^{n} \gamma_{rjk}(u) w_j w_k d\tau + \int_{t_1}^{t} \sum_{j,k=1}^{n} \gamma_{rjk}(u) w_j w_k d\tau. \end{aligned} \tag{6.1.48}$$

By (6.1.17), we have

$$\lambda_r(0) - \frac{\delta_0}{2} \le \frac{x - x_2}{t - t_2} \le \lambda_r(0) + \frac{\delta_0}{2}. \tag{6.1.49}$$

Then, noting (6.1.18) and using the fact that the point (t_2, x_2) lies on the straight line $x = (\lambda_n(0) + \delta_0)t$, we get

$$t_2 \geq -\frac{\lambda_r(0) + \frac{\delta_0}{2}}{\lambda_n(0) - \lambda_r(0) + \frac{\delta_0}{2}} t \overset{\text{def.}}{=} a_r t \quad (a_r > 0). \tag{6.1.50}$$

Thus, noting Lemma 6.1.1, we see from (6.1.48) that

$$\begin{aligned}
&(1+t)^{1+\mu}|w_r(t,x)| \\
&\leq (1+t)^{1+\mu}\left\{|w_r(t_2,x_2)| + \int_{t_2}^{t_1} \sum_{j,k=1}^{n} |\gamma_{rjk}(u)w_j w_k| d\tau \right.\\
&\quad \left. + \int_{t_1}^{t} \sum_{j,k=1}^{n} |\gamma_{rjk}(u)w_j w_k| d\tau \right\} \\
&\leq C(1+t)^{1+\mu}\left\{(1+t_2)^{-(1+\mu)} W(D_+^T) + (W(D_0^T))^2 \int_{t_1}^{t} (1+\tau)^{-2(1+\mu)} d\tau \right.\\
&\quad \left. + \int_{t_2}^{t_1} \left(\sum_{j,k=1}^{m} + \sum_{\substack{j \in \{1,\cdots,m\} \\ k \overline{\in} \{1,\cdots,m\}}} + \sum_{\substack{k \in \{1,\cdots,m\} \\ j \overline{\in} \{1,\cdots,m\}}} + \sum_{\substack{j,k=m+1 \\ j \neq k}}^{n} \right) |\gamma_{rjk}(u) w_j w_k| d\tau \right\} \\
&\leq C\{W(D_+^T) + (W(D_0^T))^2 + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2\} \\
&\leq C\{\theta + (W(D_0^T))^2 + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2\}.
\end{aligned} \tag{6.1.51}$$

Henceforth, C denotes different positive constants independent of θ and T.

ii. For $s = m+1, \ldots, n$, passing through any fixed point $(t, x) \in D_0^T$, we draw the sth characteristic c_s, which intersects the t-axis at a point $(t_0, 0)$. Integrating (2.6.14) along c_s from t_0 to t yields

$$w_s(t,x) = w_s(t_0, 0) + \int_{t_0}^{t} \sum_{j,k=1}^{n} \gamma_{sjk}(u) w_j w_k d\tau. \tag{6.1.52}$$

By (6.1.47), we have

$$\begin{aligned}
w_s(t_0, 0) = &\sum_{r=1}^{m} f_{sr}(t_0, u) w_r(t_0, 0) + \sum_{i=1}^{k} \overline{f}_{si}(t_0, u)\alpha_i'(t_0) \\
&+ \sum_{\overline{s}=m+1}^{n} \widetilde{f}_{s\overline{s}}(t_0, u) h_{\overline{s}}'(t_0)(s = m+1, \ldots, n).
\end{aligned} \tag{6.1.53}$$

Then, passing through $(t_0, 0)$, we draw the rth characteristic c_r ($r \in \{1, \ldots, m\}$), which intersects the straight line $x = (\lambda_n(0) + \delta_0)t$ at a point (t_3, x_3).

By (6.1.51), we have

$$(1+t_0)^{1+\mu}|w_r(t_0,0)| \le C\{\theta + (W(D_0^T))^2 + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2\}. \tag{6.1.54}$$

Similarly to (6.1.50), we have

$$t_0 \ge \frac{\lambda_s(0) - \lambda_{m+1}(0) + \frac{\delta_0}{2}}{\lambda_s(0) - \frac{\delta_0}{2}} t \overset{\text{def.}}{=} b_s t \quad (b_s > 0). \tag{6.1.55}$$

Then, noting (6.1.14), (6.1.19), and (6.1.54)–(6.1.55), we see from (6.1.53) that

$$\begin{aligned}(1+t)^{1+\mu}|w_s(t_0,0)| &\le C(1+t_0)^{1+\mu}|w_s(t_0,0)| \\ &\le C\{\theta + (W(D_0^T))^2 + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2\}.\end{aligned} \tag{6.1.56}$$

Hence, it follows from (6.1.52) that

$$\begin{aligned}(1+t)^{1+\mu}|w_s(t,x)| \le\, &(1+t)^{1+\mu}|w_s(t_0,0)| \\ &+ (1+t)^{1+\mu}\int_{t_0}^{t}\sum_{j,k=1}^{n}|\gamma_{sjk}(u)w_jw_k|d\tau \\ \le\, &C\{\theta + (W(D_0^T))^2 + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2\}.\end{aligned} \tag{6.1.57}$$

Thus, by (6.1.51) and (6.1.57), we get

$$W(D_0^T) \le C\{\theta + (W(D_0^T))^2 + W_\infty^c(T)\widetilde{W}_1(T) + (W_\infty^c(T))^2\}. \tag{6.1.58}$$

We next estimate $\widetilde{W}_1(T)$.

Let

$$W_1(T) = \max_{s=m+1,\dots,n}\ \sup_{0\le t\le T}\int_{D_s^T(t)}|w_s(t,x)|dx, \tag{6.1.59}$$

where $D_s^T(t) = \{(\tau,x)|\ \tau = t, (\tau,x)\in D_s^T\}$.

We arbitrarily draw a jth characteristic $\widetilde{c}_j$ on D_s^T ($j \ne s, s = m+1,\dots,n$), which intersects the boundary of D_s^T at points P_1 and P_2. Then, we draw the sth characteristics passing through P_1 and P_2, respectively, which intersect, for instance, the straight lines $x = (\lambda_{m+1}(0)-\delta_0)t$ and $x = (\lambda_n(0)+\delta_0)t$ at points A and B, respectively [when these two sth characteristics intersect the straight line $x = (\lambda_{m+1}(0)-\delta_0)t$ or $x = (\lambda_n(0)+\delta_0)t$ twice, we have the same conclusion]. The sth characteristic passing through $O(0,0)$ intersects $\widetilde{c}_j$ at a point P_0. Applying Lemma 2.6.1 on the domain P_1AOP_0, we get

$$\int_{\widetilde{P_1P_0}} |w_s(dx - \lambda_s(u)dt)| \le \int_{OA} |w_s(dx - \lambda_s(u)dt)| + \iint\limits_{P_1AOP_0} \sum_{l,k=1}^{n} |\Gamma_{slk}(u)w_l w_k| dtdx; \tag{6.1.60}$$

then

$$\int_{\widetilde{P_1P_0}} |w_s(\lambda_j(u) - \lambda_s(u))|dt \le \int_{OA} |w_s(\lambda_{m+1}(0) - \delta_0 - \lambda_s(u))|dt + \iint\limits_{P_1AOP_0} \left(\sum_{l,k=1}^{m} + \sum_{\substack{l\in\{1,\cdots,m\}\\ k\bar{\in}\{1,\cdots,m\}}} + \sum_{\substack{k\in\{1,\cdots,m\}\\ l\bar{\in}\{1,\cdots,m\}}} + \sum_{\substack{l,k=m+1\\ l\neq k}}^{n} \right) |\Gamma_{slk}(u)w_l w_k| dtdx. \tag{6.1.61}$$

Thus, noting (6.1.16), we have

$$\int_{\widetilde{P_1P_0}} |w_s(t,x)|dt \le C\{W(D_0^T) + W_\infty^c(T)W_1(T) + (W_\infty^c(T))^2\}. \tag{6.1.62}$$

Similarly, we have

$$\int_{\widetilde{P_0P_2}} |w_s(t,x)|dt \le C\{W(D_+^T) + W_\infty^c(T)W_1(T) + (W_\infty^c(T))^2\}. \tag{6.1.63}$$

Then, if we note Lemma 6.1.1, it follows from (6.1.62)–(6.1.63) that

$$\widetilde{W}_1(T) \le C\{\theta + W(D_0^T) + W_\infty^c(T)W_1(T) + (W_\infty^c(T))^2\}. \tag{6.1.64}$$

Similarly, we have

$$W_1(T) \le C\{\theta + W(D_0^T) + W_\infty^c(T)W_1(T) + (W_\infty^c(T))^2\}. \tag{6.1.65}$$

Now we estimate $W_\infty^c(T)$.

i. For $r = 1, \ldots, m$, passing through any fixed point $(t,x) \in D^T$, we draw the rth characteristic c_r, which intersects the straight line $x = (\lambda_n(0) + \delta_0)t$ at a point $(\bar{t}, \bar{x})$. Integrating (2.6.14) along c_r from $\bar{t}$ to t yields

$$w_r(t,x) = w_r(\bar{t}, \bar{x}) + \int_{\bar{t}}^{t} \sum_{j,k=1}^{n} \gamma_{rjk}(u) w_j w_k d\tau. \tag{6.1.66}$$

Similarly to (6.1.50), it is easy to see that

$$\bar{t} \ge \frac{\lambda_{m+1}(0) - \lambda_r(0) - \frac{3}{2}\delta_0}{\lambda_n(0) - \lambda_r(0) + \frac{\delta_0}{2}} t \overset{\text{def.}}{=} d_r t \quad (d_r > 0). \tag{6.1.67}$$

Then, using Lemma 6.1.1, similarly to (6.1.51) we get

$$\begin{aligned}(1+t)^{1+\mu}|w_r(t,x)| &\le C\{W(D_+^T)+W_\infty^c(T)\widetilde{W}_1(T)+(W_\infty^c(T))^2\}\\ &\le C\{\theta+W_\infty^c(T)\widetilde{W}_1(T)+(W_\infty^c(T))^2\}. \qquad (6.1.68)\end{aligned}$$

ii. For $s=m+1,\dots,n$, passing through any fixed point $(t,x)\in D^T\backslash D_s^T$, we draw the sth characteristic c_s, which intersects the boundary of D^T at a point $(\bar{t},\bar{x})$. Similarly to (6.1.68), we have

$$\begin{aligned}(1+t)^{1+\mu}|w_s(t,x)| &\le C\{W(D_0^T)+W(D_+^T)+W_\infty^c(T)\widetilde{W}_1(T)+(W_\infty^c(T))^2\}\\ &\le C\{\theta+W(D_0^T)+W_\infty^c(T)\widetilde{W}_1(T)+(W_\infty^c(T))^2\}.\end{aligned} \qquad (6.1.69)$$

Thus, it follows from (6.1.68)–(6.1.69) that

$$W_\infty^c(T)\le C\{\theta+W(D_0^T)+W_\infty^c(T)\widetilde{W}_1(T)+(W_\infty^c(T))^2\}. \qquad (6.1.70)$$

Then, the combination of (6.1.58), (6.1.64)–(6.1.65), and (6.1.70) gives (6.1.40)–(6.1.42) and

$$W_1(T)\le \kappa_6\theta. \qquad (6.1.71)$$

Finally, we estimate $V_\infty(T)$.
By (6.1.11) and noting (6.1.19), we have

$$|v_i(t,x)|\le C|u(t,x)| \quad (i=1,\dots,n). \qquad (6.1.72)$$

For any fixed point (t,x) in the domain $\{(t,x)|0\le t\le T, x\ge 0\}$, we have

$$u(t,x)=-\int_x^{x_0}u_\xi(t,\xi)d\xi+u(t,x_0)=-\int_x^{x_0}\sum_{k=1}^n w_k r_k(t,\xi)d\xi+\sum_{k=1}^n v_k r_k(t,x_0), \qquad (6.1.73)$$

in which the point (t,x_0) lies on the straight line $x=(\lambda_n(0)+\delta_0)t$. Then, using Lemma 6.1.1 and noting (6.1.40), (6.1.42), and (6.1.71), we get

$$\begin{aligned}|u(t,x)| &\le C\{W(D_0^T)+W_\infty^c(T)+W_1(T)+W(D_+^T)+V(D_+^T)\}\\ &\le C\theta. \qquad (6.1.74)\end{aligned}$$

Thus, (6.1.43) follows immediately from (6.1.72) and (6.1.74). Moreover, by (6.1.74), we know that hypothesis (6.1.19) is reasonable.
The proof of Lemma 6.1.2 is finished.

Let

$$W_\infty^m(T)=\max_{r=1,\dots,m}\sup_{\substack{0\le t\le T\\ x\ge 0}}|w_r(t,x)|. \qquad (6.1.75)$$

By (6.1.34), (6.1.40), and (6.1.42), it is easy to get the following:

Lemma 6.1.3 *Under the assumptions of Lemma 6.1.2, there exists* $\theta_0 > 0$ *so small that for any given* $\theta \in [0, \theta_0]$, *on any given existence domain* $\{(t,x) | 0 \le t \le T, x \ge 0\}$ *of the* C^1 *solution* $u = u(t,x)$ *to the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10), *we have the following uniform a priori estimate:*

$$W_\infty^m(T) \le \kappa_7 \theta. \tag{6.1.76}$$

In order to consider the one-sided mixed initial-boundary value problem in generalized normalized coordinates, we prove

Lemma 6.1.4 *The boundary condition* (6.1.10) *keeps a similar form under any given smooth invertible transformation* $u = u(\widetilde{u})$ $[u(0) = 0]$.

Proof. Let $u = u(\widetilde{u})$ $[u(0) = 0]$ be a smooth invertible transformation. We have

$$v_i = l_i(u)u = l_i(0)\left(\frac{\partial u}{\partial \widetilde{u}}\right)(0)\widetilde{u} + o(|\widetilde{u}|). \tag{6.1.77}$$

Noting Remark 6.1.1, in the $\widetilde{u}$-space we may suppose

$$\widetilde{l}_i(0) = l_i(0)\left(\frac{\partial u}{\partial \widetilde{u}}\right)(0) \quad (i = 1, \dots, n). \tag{6.1.78}$$

Then

$$v_i = \widetilde{l}_i(0)\widetilde{u} + o(|\widetilde{u}|) = \widetilde{v}_i + o(|\widetilde{v}|) \quad (i = 1, \dots, n), \tag{6.1.79}$$

where

$$\widetilde{v}_i = \widetilde{l}_i(\widetilde{u})\widetilde{u} \quad (i = 1, \dots, n) \tag{6.1.80}$$

and $\widetilde{v} = (\widetilde{v}_1, \dots, \widetilde{v}_n)^T$.

Substituting (6.1.79) into (6.1.10), it is easy to see that in a neighbourhood of $\widetilde{u} = 0$, we have the following boundary condition similar to (6.1.10):

$$\begin{aligned} x = 0:\ \widetilde{v}_s &= g_s(\alpha(t), h_{m+1}(t), \dots, h_n(t), \widetilde{v}_1, \dots, \widetilde{v}_m) \\ &\stackrel{\text{def.}}{=} \widetilde{g}_s(\widetilde{\alpha}(t), \widetilde{v}_1, \dots, \widetilde{v}_m) + \widetilde{h}_s(t) \quad (s = m+1, \dots, n), \end{aligned} \tag{6.1.81}$$

where $g_s, \widetilde{g}_s \in C^1$ $(s = m+1, \dots, n)$,

$$\widetilde{\alpha}(t) = (\alpha(t), h_{m+1}(t), \dots, h_n(t)), \tag{6.1.82}$$

$$\widetilde{h}_s(t) = g_s(\alpha(t), h_{m+1}(t), \dots, h_n(t), 0, \cdots, 0) \quad (s = m+1, \dots, n); \tag{6.1.83}$$

then

$$\widetilde{g}_s(\widetilde{\alpha}(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{6.1.84}$$

Moreover, noting (6.1.13), it is easy to see that

$$g_s(\alpha(t), 0, \ldots, 0) \equiv 0 \quad (s = m+1, \ldots, n). \tag{6.1.85}$$

Hence, condition (6.1.14) is also invariant under any given smooth invertible transformation.

By Lemma 6.1.4, in what follows we still denote generalized normalized variables as $u = (u_1, \ldots, u_n)^T$.

Lemma 6.1.5 *Under the assumptions of Theorem 6.1.1, in generalized normalized coordinates, there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, on any given existence domain $\{(t,x) | 0 \le t \le T, x \ge 0\}$ of the C^1 solution $u = u(t,x)$ to the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10)*, we have the following uniform a priori estimates:*

$$V(D_0^T) \le \kappa_8 \theta, \tag{6.1.86}$$

$$\widetilde{V}_1(T) \le \kappa_9 \theta, \tag{6.1.87}$$

and

$$V_\infty^c(T) \le \kappa_{10} \theta. \tag{6.1.88}$$

Proof. Let

$$U_\infty^c(T) = \max\left\{ \max_{r=1,\ldots,m} \sup_{(t,x)\in D^T} (1+t)^{1+\mu} |u_r(t,x)|, \right.$$
$$\left. \max_{s=m+1,\ldots,n} \sup_{(t,x)\in D^T \setminus D_s^T} (1+t)^{1+\mu} |u_s(t,x)| \right\}. \tag{6.1.89}$$

By

$$u_i(t,x) = \sum_{k=1}^n v_k r_k^T(u) e_i = \sum_{k=1}^m v_k r_k^T(u) e_i + \sum_{k=m+1}^n v_k r_k^T(u) e_i \tag{6.1.90}$$

and noting (2.4.20), it is easy to see that

$$U_\infty^c(T) \le C V_\infty^c(T). \tag{6.1.91}$$

Now we estimate $V(D_0^T)$.

i. For $r = 1, \ldots, m$, for any fixed point $(t,x) \in D_0^T$, similarly to (6.1.48), we have

$$v_r(t,x) = v_r(t_2, x_2) + \int_{t_2}^t \sum_{j,k=1}^n \beta_{rjk}(u) v_j w_k d\tau$$
$$= v_r(t_2, x_2) + \int_{t_2}^{t_1} \sum_{j,k=1}^n \beta_{rjk}(u) v_j w_k d\tau + \int_{t_1}^t \sum_{j,k=1}^n \beta_{rjk}(u) v_j w_k d\tau. \tag{6.1.92}$$

Then, noting (2.6.12), by Hadamand's formula,

$$\begin{aligned}\beta_{rjj}(u) &= \beta_{rjj}(u) - \beta_{rjj}(u_j e_j)\\ &= \int_0^1 \sum_{\substack{l=1\\ l\neq j}}^n \frac{\partial \beta_{rjj}}{\partial u_l}(su_1, \ldots, su_{j-1}, u_j, su_{j+1}, \ldots, su_n)u_l ds, \end{aligned}\quad (6.1.93)$$

we have

$$\begin{aligned}&(1+t)^{1+\mu} v_r(t,x)\\ &= (1+t)^{1+\mu}\Bigg\{ v_r(t_2, x_2) + \int_{t_1}^t \sum_{j,k=1}^n \beta_{rjk}(u) v_j w_k d\tau\\ &\quad + \int_{t_2}^{t_1}\Bigg[\Bigg(\sum_{j,k=1}^m + \sum_{\substack{j\in\{1,\ldots,m\}\\ k\overline{\in}\{1,\ldots,m\}}} + \sum_{\substack{k\in\{1,\ldots,m\}\\ j\overline{\in}\{1,\ldots,m\}}} + \sum_{\substack{j,k=m+1\\ j\neq k}}^n\Bigg)\beta_{rjk}(u)v_j w_k\\ &\quad + \sum_{j=m+1}^n \Bigg(\int_0^1 \sum_{\substack{l=1\\ l\neq j}}^n \frac{\partial \beta_{rjj}}{\partial u_l}(su_1, \ldots, su_{j-1}, u_j, su_{j+1}, \ldots, su_n)u_l ds\Bigg) v_j w_j\Bigg] d\tau\Bigg\}.\end{aligned}\quad (6.1.94)$$

Hence, noting (6.1.50) and (6.1.91) and using Lemmas 6.1.1 and 6.1.2, we get

$$\begin{aligned}&(1+t)^{1+\mu}|v_r(t,x)|\\ &\leq C\{V(D_+^T) + V(D_0^T)W(D_0^T) + V_\infty^c(T)W_\infty^c(T) + V_\infty^c(T)\widetilde{W}_1(T)\\ &\quad + \widetilde{V}_1(T)W_\infty^c(T) + V_\infty^c(T)V_\infty(T)\widetilde{W}_1(T)\}\\ &\leq C\theta\{1 + V(D_0^T) + V_\infty^c(T) + \widetilde{V}_1(T)\}.\end{aligned}\quad (6.1.95)$$

ii. For $s = m+1, \ldots, n$, for any fixed point $(t,x) \in D_0^T$, similarly to (6.1.52), we have

$$v_s(t,x) = v_s(t_0, 0) + \int_{t_0}^t \sum_{j,k=1}^n \beta_{sjk}(u) v_j w_k d\tau. \quad (6.1.96)$$

Noting (6.1.13), by (6.1.10) it is easy to get

$$v_s(t_0, 0) = \sum_{r=1}^m g_{sr}(t_0) v_r(t_0, 0) + h_s(t_0), \quad (6.1.97)$$

in which

$$g_{sr}(t_0) = \int_0^1 \frac{\partial f_s}{\partial v_r}(\alpha(t_0), \tau v_1(t_0, 0), \ldots, \tau v_m(t_0, 0)) d\tau. \quad (6.1.98)$$

By (6.1.95), we have

$$(1+t_0)^{1+\mu}|v_r(t_0,0)| \le C\theta\{1+V(D_0^T)+V_\infty^c(T)+\widetilde{V}_1(T)\} \quad (r=1,\dots,m). \tag{6.1.99}$$

Then, noting (6.1.14) and (6.1.55), it easily follows from (6.1.96) that

$$(1+t)^{1+\mu}|v_s(t,x)| \le C\theta\{1+V(D_0^T)+V_\infty^c(T)+\widetilde{V}_1(T)\}. \tag{6.1.100}$$

Combining (6.1.95) and (6.1.100), we get

$$V(D_0^T) \le C\theta\{1+V_\infty^c(T)+\widetilde{V}_1(T)\}. \tag{6.1.101}$$

We next estimate $\widetilde{V}_1(T)$.

Let

$$V_1(T) = \max_{s=m+1,\dots,n} \sup_{0\le t\le T} \int_{D_s^T(t)} |v_s(t,x)|dx. \tag{6.1.102}$$

Similarly to (6.1.60), for $s=m+1,\dots,n$, we have

$$\int_{\widetilde{P_1P_0}} |v_s(dx-\lambda_s(u)dt)| \le \int_{OA} |v_s(dx-\lambda_s(u)dt)| + \iint\limits_{P_1AOP_0} \sum_{l,k=1}^n |B_{slk}(u)v_lw_k|dtdx. \tag{6.1.103}$$

Then we have

$$\begin{aligned}
&\int_{\widetilde{P_1P_0}} |v_s(\lambda_j(u)-\lambda_s(u))|dt \\
&\le \int_{OA} |v_s(\lambda_{m+1}(0)-\delta_0-\lambda_s(u))|dt \\
&\quad + \iint\limits_{P_1AOP_0} \Bigg[\Bigg(\sum_{l,k=1}^m + \sum_{\substack{l\in\{1,\dots,m\}\\ k\overline{\in}\{1,\dots,m\}}} + \sum_{\substack{k\in\{1,\dots,m\}\\ l\overline{\in}\{1,\dots,m\}}} + \sum_{\substack{l,k=m+1\\ l\ne k}}^n\Bigg)|B_{slk}(u)v_lw_k| \\
&\quad + \sum_{l=m+1}^n |B_{sll}(u)v_lw_l|\Bigg] dtdx.
\end{aligned} \tag{6.1.104}$$

Noting that $\lambda_s(u)$ $(s=m+1,\dots,n)$ are WLD, by (2.6.11) and (2.6.13), we have

$$B_{sll}(u_le_l) \equiv 0, \quad \forall l. \tag{6.1.105}$$

Then, using Lemma 6.1.2 and (6.1.91), we see from (6.1.104) that

$$\begin{aligned}
\int_{\widetilde{P_1P_0}} |v_s(t,x)|dt &\le C\{V(D_0^T)+V_1(T)W_\infty^c(T)+V_\infty^c(T)(W_1(T)+W_\infty^c(T)) \\
&\qquad + V_\infty(T)W_1(T)V_\infty^c(T)\} \\
&\le C\{V(D_0^T)+\theta V_1(T)+\theta V_\infty^c(T)\}.
\end{aligned} \tag{6.1.106}$$

Similarly, we have

$$\int_{\widetilde{P_0P_2}} |v_s(t,x)|dt \leq C\{V(D_+^T)+\theta V_1(T)+\theta V_\infty^c(T)\}. \tag{6.1.107}$$

Then, using Lemma 6.1.1, we get

$$\widetilde{V}_1(T) \leq C\{\theta+V(D_0^T)+\theta V_1(T)+\theta V_\infty^c(T)\}. \tag{6.1.108}$$

Similarly, we have

$$V_1(T) \leq C\{\theta+V(D_0^T)+\theta V_1(T)+\theta V_\infty^c(T)\}. \tag{6.1.109}$$

Noting that $\theta>0$ is suitably small, we see from (6.1.108)–(6.1.109) that

$$\widetilde{V}_1(T) \leq C\{\theta+V(D_0^T)+\theta V_\infty^c(T)\}. \tag{6.1.110}$$

We next estimate $V_\infty^c(T)$.

i. For $r=1,\dots,m$, similarly to (6.1.66), we have

$$\begin{aligned}
v_r(t,x) &= v_r(\bar{t},\bar{x}) + \int_{\bar{t}}^t \sum_{j,k=1}^n \beta_{rjk}(u)v_jw_k d\tau \\
&= v_r(\bar{t},\bar{x}) + \int_{\bar{t}}^t \Bigg[\Bigg(\sum_{j,k=1}^m + \sum_{\substack{j\in\{1,\dots,m\}\\ k\overline{\in}\{1,\dots,m\}}} + \sum_{\substack{k\in\{1,\dots,m\}\\ j\overline{\in}\{1,\dots,m\}}} + \sum_{\substack{j,k=m+1\\ j\neq k}}^n\Bigg)\beta_{rjk}(u)v_jw_k \\
&\quad + \sum_{j=m+1}^n (\beta_{rjj}(u)-\beta_{rjj}(u_je_j))v_jw_j\Bigg]d\tau.
\end{aligned} \tag{6.1.111}$$

Then, noting (6.1.67) and using Lemma 6.1.1, Lemma 6.1.2, and (6.1.91), similarly to (6.1.68), we get

$$\begin{aligned}
(1+t)^{1+\mu}|v_r(t,x)| &\leq C\{(V(D_+^T)+V_\infty^c(T)\widetilde{\widetilde{W}}_1(T)+W_\infty^c(T)(\widetilde{V}_1(T)+V_\infty^c(T)) \\
&\quad + V_\infty^c(T)V_\infty(T)\widetilde{\widetilde{W}}_1(T)\} \\
&\leq C\theta\{1+V_\infty^c(T)+\widetilde{V}_1(T)\}.
\end{aligned} \tag{6.1.112}$$

ii. For $s=m+1,\dots,n$, noting (6.1.91), similarly to (6.1.69), we have

$$\begin{aligned}
&(1+t)^{1+\mu}|v_s(t,x)| \\
&\leq C\Big\{V(D_0^T)+V(D_+^T)+V_\infty^c(T)\widetilde{\widetilde{W}}_1(T)+W_\infty^c(T)(\widetilde{V}_1(T)+V_\infty^c(T)) \\
&\quad + V_\infty^c(T)V_\infty(T)\widetilde{\widetilde{W}}_1(T)\Big\}.
\end{aligned} \tag{6.1.113}$$

Then, using Lemmas 6.1.1 and 6.1.2, we have

$$(1+t)^{1+\mu}|v_s(t,x)| \leq C\{V(D_0^T)+\theta(1+V_\infty^c(T)+\widetilde{V}_1(T))\}. \tag{6.1.114}$$

Noting that $\theta > 0$ is suitably small, we see from (6.1.112) and (6.1.114) that

$$V_\infty^c(T) \le C\{\theta + V(D_0^T) + \theta \widetilde{V}_1(T)\}. \tag{6.1.115}$$

From (6.1.101), (6.1.110), and (6.1.115), it is easy to get (6.1.86)–(6.1.88). The proof of Lemma 6.1.5 is finished.

Lemma 6.1.6 *Under the assumptions of Theorem 6.1.1, there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, on any given existence domain $\{(t,x) | 0 \le t \le T, x \ge 0\}$ of the C^1 solution $u = u(t,x)$ to the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10), *we have the following uniform a priori estimate:*

$$W_\infty(T) \le \kappa_{11}\theta. \tag{6.1.116}$$

Proof. By Lemmas 6.1.1–6.1.3, it is only necessary to estimate $|w_s(t,x)|$ for $(t,x) \in D_s^T$ $(s \in \{m+1, \ldots, n\})$. For this purpose, passing through any fixed point $(t,x) \in D_s^T$ $(s \in \{m+1, \ldots, n\})$, we draw the sth characteristic c_s, which intersects the boundary of D^T at a point $(\bar{t}_0, \bar{x}_0)$. Integrating (2.6.14) along c_s, from $\bar{t}_0$ to t yields

$$\begin{aligned} & w_s(t,x) \\ &= w_s(\bar{t}_0, \bar{x}_0) + \int_{\bar{t}_0}^t \sum_{j,k=1}^n \gamma_{sjk}(u) w_j w_k d\tau \\ &= w_s(\bar{t}_0, \bar{x}_0) + \int_{\bar{t}_0}^t \Bigg[\Bigg(\sum_{j,k=1}^m + \sum_{\substack{j \in \{1,\ldots,m\} \\ k \overline{\in} \{1,\ldots,m\}}} + \sum_{\substack{k \in \{1,\ldots,m\} \\ j \overline{\in} \{1,\ldots,m\}}} + \sum_{\substack{j,k=m+1 \\ j \ne k}}^n \Bigg) \gamma_{sjk}(u) w_j w_k \\ &\quad + \gamma_{sss}(u) w_s^2 \Bigg] d\tau. \end{aligned} \tag{6.1.117}$$

Noting that $\lambda_s(u)$ $(s = m+1, \ldots, n)$ are WLD, by (2.6.17), we have

$$\gamma_{sss}(u_s e_s) \equiv 0, \quad \forall |u_s| \text{ small} \quad (s = m+1, \ldots, n). \tag{6.1.118}$$

Then, using Lemmas 6.1.1, 6.1.2, and 6.1.5 and noting (6.1.91), we see from (6.1.117) that

$$\begin{aligned} |w_s(t,x)| &\le C\{W(D_0^T) + W(D_+^T) + (W_\infty^c(T))^2 + W_\infty^c(T) W_\infty(T) + V_\infty^c(T)(W_\infty(T))^2\} \\ &\le C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}. \end{aligned} \tag{6.1.119}$$

Hence, noting Lemmas 6.1.1–6.1.3, we get

$$W_\infty(T) \le C\theta\{1 + W_\infty(T) + (W_\infty(T))^2\}. \tag{6.1.120}$$

Equation (6.1.116) follows immediately from (6.1.120).

The proof of Lemma 6.1.6 is finished.

Proof of Theorem 6.1.1. By Lemmas 6.1.2 and 6.1.6, it is easy to see that on any given existence domain $\{(t,x)|0 \le t \le T, x \ge 0\}$ of the C^1 solution $u = u(t,x)$ to the one-sided mixed initial-boundary value problem (6.1.1) and (6.1.9)–(6.1.10), we have the following uniform a priori estimate:

$$\|u(t,\cdot)\|_{C^1} \stackrel{\text{def.}}{=} \|u(t,\cdot)\|_{C^0} + \|u_x(t,\cdot)\|_{C^0} \le C\theta. \tag{6.1.121}$$

Hence, the one-sided mixed initial-boundary value problem (6.1.1) and (6.1.9)–(6.1.10) admits a unique C^1 solution $u = u(t,x)$ with a small C^1 norm on the domain $D = \{(t,x)|t \ge 0, x \ge 0\}$.

This proves Theorem 6.1.1. □

6.2 Formation of Singularities of the C^1 Solution

For the one-sided mixed initial-boundary value problem (6.1.1) and (6.1.9)–(6.1.10), if the positive characteristics of system (6.1.1) are not all WLD, the following blow-up result can be obtained ([80]).

Theorem 6.2.1 *Suppose that in a neighbourhood of $u = 0$, $A(u)$ is suitably smooth and (6.1.7) holds. Suppose furthermore that $\phi(x) = \varepsilon\psi(x)$, $\alpha(t) = \varepsilon A(t)$, and $h(t) = \varepsilon H(t)$, where $\varepsilon > 0$ is a small parameter, $\psi(\cdot)$, $A(\cdot)$, and $H(\cdot)$ are all C^1 functions, and*

$$\max\left\{\sup_{x\ge 0}(1+x)^{1+\mu}(|\psi(x)| + |\psi'(x)|),\right.$$
$$\left.\sup_{t\ge 0}(1+t)^{1+\mu}(|A(t)| + |H(t)| + |A'(t)| + |H'(t)|)\right\}$$
$$< +\infty, \tag{6.2.1}$$

where $\mu > 0$ is a constant. Suppose finally that $f_s(\cdot)$ $(s = m+1, \dots, n)$ with (6.1.13) are C^1 functions with respect to their arguments and the conditions of C^1 compatibility are satisfied at the point $(t,x) = (0,0)$. If $\lambda_s(u)$ $(s = m+1, \dots, n)$ are not all WLD and the nonempty set of non-WLD indices is denoted by $J \subseteq \{m+1, \dots, n\}$, let

$$\alpha = \min\{\alpha_i | i \in J\} < +\infty, \tag{6.2.2}$$

where α_i is defined by (2.5.14)–(2.5.15) and

$$J_1 = \{i | i \in J, \alpha_i = \alpha\}. \tag{6.2.3}$$

If there exists $m_0 \in J_1$ *such that*

$$l_{m_0}(0)\psi(x) \not\equiv 0, \quad x \geq 0, \tag{6.2.4}$$

then there exists $\varepsilon_0 > 0$ *so small that for any fixed* $\varepsilon \in (0, \varepsilon_0]$, *the* C^1 *solution* $u = u(t,x)$ *to the one-sided mixed initial-boundary value problem* (6.1.1) *and* (6.1.9)–(6.1.10) *must blow up in a finite time and the life span* $\widetilde{T}(\varepsilon)$ *of* $u = u(t,x)$ *satisfies*

$$\lim_{\varepsilon \downarrow 0}(\varepsilon^{\alpha+1}\widetilde{T}(\varepsilon)) = M_0, \tag{6.2.5}$$

where

$$M_0 = \left\{ \max_{i \in J_1} \sup_{x \in \mathbb{R}} \left[-\frac{1}{\alpha!} \left. \frac{d^{\alpha+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha+1}} \right|_{s=0} (\varPhi_i(x))^\alpha \varPhi_i'(x) \right] \right\}^{-1}, \tag{6.2.6}$$

in which $u = u^{(i)}(s)$ *is defined by* (2.5.1) *and*

$$\varPhi_i(x) = \begin{cases} l_i(0)\psi(x), \quad x \geq 0, \\ \sum_{r=1}^{m} \dfrac{\partial f_i}{\partial v_r}(0, \ldots, 0) l_r(0)\psi\left(\dfrac{\lambda_r(0)}{\lambda_i(0)}x\right) + H_i\left(-\dfrac{x}{\lambda_i(0)}\right), \quad x < 0. \end{cases} \tag{6.2.7}$$

Remark 6.2.1 *Compared to the corresponding result on the Cauchy problem in Chapter 4* (also see [57] and [76]), *the negative characteristics* $\lambda_r(u)$ $(r = 1, \ldots, m)$ *have no effect on the blow-up of the* C^1 *solution.*

6.3 Applications

In this section we give some applications of Theorems 6.1.1 and 6.2.1 for the system of the planar motion of an elastic string.

6.3.1 Planar Motion of an Elastic String with a Fixed End

Consider the following one-sided mixed initial-boundary value problem for the **system of the planar motion of an elastic string** (cf. [15], [62], [76]):

$$\begin{cases} u_t - v_x = 0, \\ v_t - \left(\dfrac{T(r)}{r}u\right)_x = 0, \end{cases} \tag{6.3.1}$$

with the initial condition

$$t = 0: \ u = \widetilde{u}_0 + u_0(x), \ v = v_0(x), \quad x \geq 0, \tag{6.3.2}$$

and the boundary condition on the fixed end

$$x = 0: \ v = 0, \tag{6.3.3}$$

where $u = (u_1, u_2)^T$, $v = (v_1, v_2)^T$, $r = |u| = \sqrt{u_1^2 + u_2^2}$, $T(r)$ is a C^3 function of $r > 1$ such that

$$T'(\widetilde{r}_0) > \frac{T(\widetilde{r}_0)}{\widetilde{r}_0} > 0, \tag{6.3.4}$$

in which $\widetilde{r}_0 = |\widetilde{u}_0| > 1$, $\widetilde{u}_0$ is a constant vector, $(u_0^T(x), v_0^T(x)) \in C^1$, and there exists a constant $\mu > 0$ such that

$$\theta \stackrel{\text{def.}}{=} \sup_{x \geq 0}\{(1+x)^{1+\mu}(|u_0(x)| + |v_0(x)| + |u_0'(x)| + |v_0'(x)|)\} < +\infty. \tag{6.3.5}$$

Moreover, the conditions of C^1 compatibility are supposed to be satisfied at the point $(t, x) = (0, 0)$.

Let

$$U = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{6.3.6}$$

By (6.3.4), in a neighbourhood of $U_0 = \begin{pmatrix} \widetilde{u}_0 \\ 0 \end{pmatrix}$, (6.3.1) is a strictly hyperbolic system with the following distinct real eigenvalues:

$$\begin{aligned} \lambda_1(U) = -\sqrt{T'(r)} < \lambda_2(U) = -\sqrt{\frac{T(r)}{r}} < 0 < \lambda_3(U) \\ = \sqrt{\frac{T(r)}{r}} < \lambda_4(U) = \sqrt{T'(r)} \end{aligned} \tag{6.3.7}$$

and the corresponding left eigenvectors can be taken as

$$l_1(U) = (\sqrt{T'(r)}u^T, u^T), \quad l_2(U) = \left(\sqrt{\frac{T(r)}{r}}w, w\right), \tag{6.3.8}$$

$$l_3(U) = \left(\sqrt{\frac{T(r)}{r}}w, -w\right), \quad l_4(U) = (\sqrt{T'(r)}u^T, -u^T), \tag{6.3.9}$$

where $w = (-u_2, u_1)$.

$\lambda_2(U)$ and $\lambda_3(U)$ are LD, then WLD with respect to $U = U_0$. Moreover, $\lambda_1(U)$ and $\lambda_4(U)$ are WLD with respect to $U = U_0$, provided that

$$T''(r) \equiv 0, \quad \forall |r - r_0| \text{ small.} \tag{6.3.10}$$

Let

$$V_i = l_i(U)(U - U_0) \quad (i = 1, \ldots, 4). \tag{6.3.11}$$

The boundary condition (6.3.3) can be rewritten as

$$x = 0: \ V_3 = V_2, \ V_4 = V_1. \tag{6.3.12}$$

By Theorem 6.1.1, we have

Theorem 6.3.1 *Suppose that* (6.3.10) *holds. There exists $\theta_0 > 0$ so small that for any fixed $\theta \in [0, \theta_0]$, the one-sided mixed initial-boundary value problem* (6.3.1)–(6.3.3) *admits a unique global C^1 solution $U = U(t, x)$ on the domain $D = \{(t, x) | t \geq 0, x \geq 0\}$.*

However, if (6.3.10) fails, $\lambda_1(U)$ and $\lambda_4(U)$ are not WLD with respect to $U = U_0$. By Theorem 6.2.1, we have

Theorem 6.3.2 *Suppose that there exists an integer $\alpha \geq 0$ such that*

$$T''(\widetilde{r}_0) = \cdots = T^{(\alpha+1)}(\widetilde{r}_0) = 0, \quad \text{but } T^{(\alpha+2)}(\widetilde{r}_0) \neq 0. \tag{6.3.13}$$

Suppose furthermore that $u_0(x) = \varepsilon \bar{u}_0(x)$ and $v_0(x) = \varepsilon \bar{v}_0(x)$, where $\varepsilon > 0$ is a small parameter, $\bar{u}_0(\cdot)$ and $\bar{v}_0(\cdot)$ are C^1 functions, and

$$\sup_{x \geq 0} (1 + x)^{1+\mu} (|\bar{u}_0(x)| + |\bar{u}_0'(x)| + |\bar{v}_0(x)| + |\bar{v}_0'(x)|) < +\infty, \tag{6.3.14}$$

where $\mu > 0$ is a constant. If

$$\sqrt{T'(\widetilde{r}_0)} \widetilde{u}_0^T \bar{u}_0(x) \not\equiv \widetilde{u}_0^T \bar{v}_0(x), \quad x \geq 0, \tag{6.3.15}$$

then there exists $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the C^1 solution $U = U(t, x)$ to the one-sided mixed initial-boundary value problem (6.3.1)–(6.3.3) *must blow up in a finite time and the life span $\widetilde{T}(\varepsilon)$ of $U = U(t, x)$ satisfies*

$$\lim_{\varepsilon \downarrow 0} (\varepsilon^{\alpha+1} \widetilde{T}(\varepsilon)) = M_0, \tag{6.3.16}$$

where

$$M_0 = \left\{ \sup_{x \in \mathbb{R}} \left[\frac{(-1)^{\alpha+2} \widetilde{r}_0^{\alpha+1} T^{(\alpha+2)}(\widetilde{r}_0)}{2\alpha! \sqrt{T'(\widetilde{r}_0)}} (\Phi(x))^\alpha \Phi'(x) \right] \right\}^{-1}, \tag{6.3.17}$$

in which

$$\Phi(x) = \begin{cases} \sqrt{T'(\widetilde{r}_0)} \widetilde{u}_0^T \bar{u}_0(x) - \widetilde{u}_0^T \bar{v}_0(x), & x \geq 0, \\ \sqrt{T'(\widetilde{r}_0)} \widetilde{u}_0^T \bar{u}_0(-x) + \widetilde{u}_0^T \bar{v}_0(-x), & x < 0. \end{cases} \tag{6.3.18}$$

6.3.2 Planar Motion of an Elastic String with a Dissipative Boundary

If the initial condition (6.3.2) is replaced by

$$t = 0: \ u = \widetilde{u}_0 + u_0(x), \ v = \widetilde{v}_0 + v_0(x), \quad x \geq 0, \tag{6.3.19}$$

and the boundary condition (6.3.3) is replaced by the following dissipative boundary condition:

$$x = 0: \ \frac{T(r)}{r}u = \alpha v \quad (\alpha > 0 \text{ is a constant}), \tag{6.3.20}$$

where $\widetilde{u}_0$, $\widetilde{v}_0$ are constant vectors such that $\widetilde{r}_0 = |\widetilde{u}_0| > 1$ and

$$\frac{T(\widetilde{r}_0)}{\widetilde{r}_0}\widetilde{u}_0 = \alpha\widetilde{v}_0, \tag{6.3.21}$$

the conclusion of Theorem 6.3.1 is still valid. In fact, noting (6.3.11), in which $U_0 = \begin{pmatrix} \widetilde{u}_0 \\ \widetilde{v}_0 \end{pmatrix}$, it is easy to see that (6.3.20) can be rewritten as

$$x = 0: \ V_3 = f_3(V_1, V_2), \ V_4 = f_4(V_1, V_2), \tag{6.3.22}$$

where f_3 and f_4 are C^1 functions with respect to their arguments and

$$f_3(0,0) = f_4(0,0) = 0. \tag{6.3.23}$$

By Theorem 6.1.1, we have

Theorem 6.3.3 *Suppose that* (6.3.10) *holds. There exists* $\theta_0 > 0$ *so small that for any fixed* $\theta \in [0, \theta_0]$*, the one-sided mixed initial-boundary value problem* (6.3.1) *and* (6.3.19)–(6.3.20) *admits a unique global* C^1 *solution* $U = U(t,x)$ *on the domain* $D = \{(t,x) | t \geq 0, x \geq 0\}$.

If (6.3.10) fails, $\lambda_1(U)$ and $\lambda_4(U)$ are not WLD with respect to $U = U_0$. By Theorem 6.2.1, we can obtain a corresponding blow-up result.

Chapter VII
Generalized Riemann Problem

7.1 Introduction and Main Results

Consider the following quasilinear system of conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{7.1.1}$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector function of (t, x) and $f(u) = (f_1(u), \dots, f_n(u))^T$ is a given C^3 vector function of u.

Suppose that on the domain under consideration, system (7.1.1) is strictly hyperbolic, i.e., $\nabla f(u)$ possesses n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{7.1.2}$$

For $i = 1, \dots, n$, let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ and $r_i(u) = (r_{1i}(u), \dots, r_{ni}(u))^T$ be the left and right eigenvectors corresponding to $\lambda_i(u)$, respectively. Without loss of generality, we assume that on the domain under consideration

$$l_i(u) r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n), \tag{7.1.3}$$

where δ_{ij} stands for Kronecker's delta.

Consider the **generalized Riemann problem** for system (7.1.1) with the following piecewise C^1 initial data:

$$t = 0: \ u = \begin{cases} u_l(x), & x \le 0, \\ u_r(x), & x \ge 0, \end{cases} \tag{7.1.4}$$

where $u_l(x)$ and $u_r(x)$ are C^1 vector functions on $x \le 0$ and $x \ge 0$, respectively, and

$$\eta \overset{\text{def.}}{=} |u_r(0) - u_l(0)| > 0 \tag{7.1.5}$$

is suitably small.

Li Tatsien, Wang Libin, *Global Propagation of Regular Nonlinear Hyperbolic Waves*, DOI 10.1007/b78335_7,

Suppose that all the characteristics are genuinely nonlinear (GN) in the sense of Lax: Without loss of generality, for $i = 1, \dots, n$,

$$\nabla\lambda_i(u)r_i(u) \equiv 1. \tag{7.1.6}$$

Suppose furthermore that the corresponding **Riemann problem** for system (7.1.1) with the piecewise constant initial data

$$t = 0: \; u = \begin{cases} u_l(0), & x \le 0, \\ u_r(0), & x \ge 0, \end{cases} \tag{7.1.7}$$

admits a unique small amplitude self-similar solution $u = U(x/t)$ which contains n nondegenerate typical shocks $x = \hat{\lambda}_i t$ $(i = 1, \dots, n)$ (cf. Lax [42]). Suppose finally that $u_l(x)$ and $u_r(x)$ are C^1 vector functions with a small C^1 norm and certain decaying properties as $|x| \to +\infty$. Then the generalized Riemann problem (7.1.1) and (7.1.4) admits a unique piecewise C^1 solution $u = u(t, x)$ containing only n nondegenerate shocks, which possesses a similar structure to the self-similar solution $u = U(x/t)$ to the corresponding Riemann problem (see [50] and [73]).

On the other hand, suppose that all the characteristics are linearly degenerate (LD) in the sense of Lax:

$$\nabla\lambda_i(u)r_i(u) \equiv 0 \quad (i = 1, \dots, n). \tag{7.1.8}$$

By Lax [42], the corresponding Riemann problem (7.1.1) and (7.1.7) admits a unique small amplitude self-similar solution $u = U(x/t)$ that contains n typical contact discontinuities $x = \hat{\lambda}_i t$ $(i = 1, \dots, n)$ (some of them may degenerate). Under the same hypothesis on $u_l(x)$ and $u_r(x)$ as mentioned above, the generalized Riemann problem (7.1.1) and (7.1.4) admits a unique piecewise C^1 solution $u = u(t, x)$ containing only n contact discontinuities (some of them may degenerate to weak discontinuities), which possesses a similar structure to the self-similar solution $u = U(x/t)$ to the corresponding Riemann problem (see [55]).

Because that, for many practical quasilinear hyperbolic systems, have both GN and LD characteristics, in this chapter we generalize the previous results to the case that each characteristic is either GN or LD; correspondingly, the piecewise C^1 solution to the generalized Riemann problem (7.1.1) and (7.1.4) may contain both nondegenerate shocks and contact discontinuities. For this reason, the methods used in [73] and [55], which cannot be applied directly to this general situation, should be unified and improved.

We first give the following:

Definition 7.1.1 *A piecewise smooth vector function $u = u(t, x)$ is said to be a* **classical discontinuous solution containing a kth shock** $x = x_k(t)$ *for system* (7.1.1) *if $u = u(t, x)$ satisfies system* (7.1.1) *in the classical sense on both sides of $x = x_k(t)$ and verifies the* **Rankine–Hugoniot condition**

$$f(u^+) - f(u^-) = s(u^+ - u^-) \tag{7.1.9}$$

and the **entropy condition**

$$\begin{cases} \lambda_k(u^+) < s < \lambda_k(u^-), \\ \lambda_{k-1}(u^-) < s < \lambda_{k+1}(u^+) \end{cases} \tag{7.1.10}$$

on $x = x_k(t)$, *where* $u^\pm = u(t, x_k(t) \pm 0)$ *and* $s = dx_k(t)/dt$. *When* $[u] = u^+ - u^- \neq 0$, $x = x_k(t)$ *is a* **kth nondegenerate shock**.

Definition 7.1.2 *A piecewise smooth vector function* $u = u(t, x)$ *is said to be a* **classical discontinuous solution containing a kth contact discontinuity** $x = x_k(t)$ *for system* (7.1.1) *if* $u = u(t, x)$ *satisfies system* (7.1.1) *in the classical sense on both sides of* $x = x_k(t)$ *and verifies the Rankine–Hugoniot condition* (7.1.9) *and*

$$s = \lambda_k(u^+) = \lambda_k(u^-) \tag{7.1.11}$$

on $x = x_k(t)$, *where* $u^\pm = u(t, x_k(t) \pm 0)$ *and* $s = dx_k(t)/dt$. *When* $[u] = u^+ - u^- = 0$, *the kth contact discontinuity* $x = x_k(t)$ *is degenerate and becomes a kth weak discontinuity, where* $x = x_k(t)$ *is a kth characteristic.*

Remark 7.1.1 *If the kth characteristic* $\lambda_k(u)$ *is GN, then any given kth wave* $x = x_k(t)$, *on which u is discontinuous and the amplitude* $|u^+ - u^-|$ *is small, must be a kth shock. On the other hand, if* $\lambda_k(u)$ *is LD, then any given kth wave* $x = x_k(t)$, *on which u is discontinuous and the amplitude* $|u^+ - u^-|$ *is small, must be a kth contact discontinuity* (see [72]).

We now give the following hypotheses:

(H_1) For any given k $(k = 1, \ldots, n)$, $\lambda_k(u)$ is either GN or LD.

(H_2) There exists a constant $\mu > 0$ such that

$$\begin{aligned} \theta \stackrel{\text{def.}}{=} & \sup_{x \le 0}\{(1 + |x|)^{1+\mu}(|u_l(x) - u_l(0)| + |u_l'(x)|)\} \\ & + \sup_{x \ge 0}\{(1 + x)^{1+\mu}(|u_r(x) - u_r(0)| + |u_r'(x)|)\} < +\infty. \end{aligned} \tag{7.1.12}$$

(H_3) The corresponding Riemann problem (7.1.1) and (7.1.7) admits a unique self-similar solution $u = U(x/t)$ composed of $n + 1$ constant states $\hat{u}^{(0)} = u_l(0), \hat{u}^{(1)}, \ldots, \hat{u}^{(n-1)}, \hat{u}^{(n)} = u_r(0)$ and n small amplitude waves $x = \hat{\lambda}_k t$ $(k = 1, \ldots, n)$ (nondegenerate shocks corresponding to GN characteristics and contact discontinuities corresponding to LD characteristics):

$$u = U\left(\frac{x}{t}\right) = \begin{cases} \hat{u}^{(0)}, & x \le \hat{\lambda}_1 t, \\ \hat{u}^{(l)}, & \hat{\lambda}_l t \le x \le \hat{\lambda}_{l+1} t \quad (l = 1, \ldots, n - 1), \\ \hat{u}^{(n)}, & x \ge \hat{\lambda}_n t. \end{cases} \tag{7.1.13}$$

The main results of this chapter are the following theorems.

Theorem 7.1.1 *Suppose that system* (7.1.1) *is strictly hyperbolic and* $f(u) \in C^3$. *Suppose furthermore that* $u_l(x)$ *and* $u_r(x)$ *are* C^1 *vector functions on* $x \leq 0$ *and* $x \geq 0$, *respectively, and* $\eta = |u_r(0) - u_l(0)| > 0$ *is suitably small. Under assumptions* (H_1)*–*(H_3), *there exists* $\theta_0 > 0$ *so small that for any given* $\theta \in (0, \theta_0]$, *the generalized Riemann problem* (7.1.1) *and* (7.1.4) *admits a unique global piecewise* C^1 *solution* $u = u(t,x)$ *which contains* n *small amplitude waves* $x = x_k(t)$ *with* $x_k(0) = 0$ $(k = 1, \ldots, n)$ (*nondegenerate shocks corresponding to GN characteristics and contact discontinuities corresponding to LD characteristics*)*:*

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x), & (t,x) \in R_0, \\ u^{(l)}(t,x), & (t,x) \in R_l \ (l = 1, \ldots, n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n, \end{cases} \tag{7.1.14}$$

in which $u^{(l)}(t,x) \in C^1$ *satisfies system* (7.1.1) *in the classical sense on* R_l $(l = 0, 1, \ldots, n)$ *with*

$$R_0 = \{(t,x) | t \geq 0,\ x \leq x_1(t)\}, \tag{7.1.15}$$

$$R_l = \{(t,x) | t \geq 0,\ x_l(t) \leq x \leq x_{l+1}(t)\} \quad (l = 1, \ldots, n-1), \tag{7.1.16}$$

$$R_n = \{(t,x) | t \geq 0,\ x \geq x_n(t)\}. \tag{7.1.17}$$

Moreover, for $k = 1, \ldots, n$, $u^{(k-1)}(t,x)$ *and* $u^{(k)}(t,x)$ *are connected to each other by the* k*th wave* $x = x_k(t)$ (*the* k*th nondegenerate shock or the* k*th contact discontinuity*)*. This solution possesses a global structure similar to that of the self-similar solution* (7.1.13) *to Riemann problem* (7.1.1) *and* (7.1.7), *namely,*

$$u^{(l)}(0,0) = \hat{u}^{(l)} \quad (l = 0, 1, \ldots, n), \tag{7.1.18}$$

$$x_k'(0) = \hat{\lambda}_k \quad (k = 1, \ldots, n), \tag{7.1.19}$$

$$|u^{(l)}(t,x) - \hat{u}^{(l)}| \leq C\theta, \quad \forall (t,x) \in R_l \quad (l = 0, 1, \ldots, n), \tag{7.1.20}$$

$$\left| \frac{\partial u^{(l)}(t,x)}{\partial x} \right|, \ \left| \frac{\partial u^{(l)}(t,x)}{\partial t} \right| \leq C\theta, \quad \forall (t,x) \in R_l \quad (l = 0, 1, \ldots, n), \tag{7.1.21}$$

and

$$|x_k'(t) - \hat{\lambda}_k| \leq C\theta, \quad t \geq 0 \quad (k = 1, \ldots, n), \tag{7.1.22}$$

where C *is a positive constant independent of* t, x, *and* θ.

Remark 7.1.2 *The result and the proof in* [65] *have been improved in Theorem 7.1.1.*

Remark 7.1.3 *In (H_3), when some of contact discontinuities in the self-similar solution disappear, we still have Theorem 7.1.1, in which the corresponding contact discontinuities degenerate to weak discontinuities.*

Remark 7.1.4 *Suppose that* (7.1.1) *is a nonstrictly hyperbolic system with characteristics with constant multiplicity, say,*

$$\lambda_1(u) < \cdots < \lambda_k(u) < \lambda_{k+1}(u) \equiv \cdots \equiv \lambda_{k+p}(u) < \lambda_{k+p+1}(u) < \cdots < \lambda_n(u) \ (p > 1). \tag{7.1.23}$$

Let

$$\lambda(u) \stackrel{\text{def.}}{=} \lambda_{k+1}(u) \equiv \cdots \equiv \lambda_{k+p}(u). \tag{7.1.24}$$

Per Boillat [7] *and Freistühler* [25], $\lambda(u)$ *is always LD. Suppose that each simple eigenvalue* $\lambda_i(u)$ ($i = 1, \ldots, k, k+p+1, \ldots, n$) *is either GN or LD. Similar conclusions hold as in Theorem 7.1.1 and Remark 7.1.3* (cf. [49]).

Theorem 7.1.2 *Under the assumptions of Theorem 7.1.1, suppose furthermore that all the characteristics* $\lambda_k(u)$ ($k = 1, \ldots, n$) *are GN and* $\mu = 0$ *in* (H_2), *there exists* $\theta_0 > 0$ *so small that for any given* $\theta \in (0, \theta_0]$, *the generalized Riemann problem* (7.1.1) *and* (7.1.4) *admits a unique global piecewise* C^1 *solution* $u = u(t, x)$ *which contains only* n *small amplitude nondegenerate shocks* $x = x_k(t)$ *with* $x_k(0) = 0$ ($k = 1, \ldots, n$):

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x), & (t,x) \in R_0, \\ u^{(l)}(t,x), & (t,x) \in R_l \quad (l = 1, \ldots, n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n, \end{cases} \tag{7.1.25}$$

in which $u^{(l)}(t,x) \in C^1$ *satisfies system* (7.1.1) *in the classical sense on* R_l ($l = 0, 1, \ldots, n$) *given by* (7.1.15)–(7.1.17). *Moreover, for* $k = 1, \ldots, n$, $u^{(k-1)}(t,x)$ *and* $u^{(k)}(t,x)$ *are connected to each other by the kth nondegenerate shock* $x = x_k(t)$. *This solution possesses a global structure similar to that of the self-similar solution* (7.1.13) *to Riemann problem* (7.1.1) *and* (7.1.7), *namely,*

$$u^{(l)}(0,0) = \hat{u}^{(l)} \quad (l = 0, 1, \ldots, n), \tag{7.1.26}$$

$$x_k'(0) = \hat{\lambda}_k \quad (k = 1, \ldots, n), \tag{7.1.27}$$

$$|u^{(l)}(t,x) - \hat{u}^{(l)}| \le \frac{C\theta}{1+t}, \quad \forall (t,x) \in R_l \quad (l = 0, 1, \ldots, n), \tag{7.1.28}$$

$$\left|\frac{\partial u^{(l)}(t,x)}{\partial x}\right|, \ \left|\frac{\partial u^{(l)}(t,x)}{\partial t}\right| \le \frac{C\theta}{1+t}, \quad \forall (t,x) \in R_l \quad (l = 0, 1, \ldots, n), \tag{7.1.29}$$

and

$$|x'_k(t) - \hat{\lambda}_k| \leq \frac{C\theta}{1+t}, \quad t \geq 0 \quad (k = 1, \dots, n), \tag{7.1.30}$$

where C is a positive constant independent of t, x, and θ.

Remark 7.1.5 *The result of Theorem 7.1.2 can be found in* [73], *but in this book we prove it in a different way.*

In Section 7.2 we give some preliminaries. Then the main results are proved in Section 7.3. Finally, some applications are given in Section 7.4.

7.2 Preliminaries

7.2.1 Decomposition of Waves

Let

$$\hat{v}_i^{(l)} = l_i(u^{(l)})(u^{(l)} - \hat{u}^{(l)}) \quad (i = 1, \dots, n;\ l = 0, 1, \dots, n) \tag{7.2.1}$$

and

$$w_i^{(l)} = l_i(u^{(l)})u_x^{(l)} \quad (i = 1, \dots, n;\ l = 0, 1, \dots, n). \tag{7.2.2}$$

By (7.1.3), we have

$$u^{(l)} = \hat{u}^{(l)} + \sum_{k=1}^{n} \hat{v}_k^{(l)} r_k(u^{(l)}) \quad (l = 0, 1, \dots, n) \tag{7.2.3}$$

and

$$u_x^{(l)} = \sum_{k=1}^{n} w_k^{(l)} r_k(u^{(l)}) \quad (l = 0, 1, \dots, n). \tag{7.2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{7.2.5}$$

denote the directional derivative with respect to t along the ith characteristic. We have (see Chapter 2)

$$\frac{d\hat{v}_i^{(l)}}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u^{(l)})\hat{v}_j^{(l)} w_k^{(l)} \quad (i = 1, \dots, n;\ l = 0, 1, \dots, n), \tag{7.2.6}$$

where

$$\beta_{ijk}(u^{(l)}) = (\lambda_k(u^{(l)}) - \lambda_i(u^{(l)}))l_i(u^{(l)})\nabla r_j(u^{(l)})r_k(u^{(l)}). \tag{7.2.7}$$

Hence, we have

$$\beta_{iji}(u^{(l)}) \equiv 0, \quad \forall i, j. \tag{7.2.8}$$

On the other hand, we have (see Chapter 2)

$$\frac{dw_i^{(l)}}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u^{(l)}) w_j^{(l)} w_k^{(l)} \quad (i = 1, \ldots, n;\ l = 0, 1, \ldots, n), \tag{7.2.9}$$

where

$$\begin{aligned}\gamma_{ijk}(u^{(l)}) = \frac{1}{2}\{&(\lambda_j(u^{(l)}) - \lambda_k(u^{(l)}))l_i(u^{(l)})\nabla r_k(u^{(l)})r_j(u^{(l)}) \\ &- \nabla\lambda_k(u^{(l)})r_j(u^{(l)})\delta_{ik} + (j|k)\},\end{aligned} \tag{7.2.10}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. Hence,

$$\gamma_{ijj}(u^{(l)}) \equiv 0, \quad \forall j \neq i. \tag{7.2.11}$$

Moreover, if $\lambda_i(u)$ is LD, we have

$$\gamma_{iii}(u^{(l)}) \equiv 0. \tag{7.2.12}$$

Noting (7.2.4) and (7.2.9), we have (see Chapter 2)

$$\begin{aligned}d[w_i^{(l)}(dx - \lambda_i(u^{(l)})dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u^{(l)}) w_j^{(l)} w_k^{(l)} dt \wedge dx \\ (i = 1, \ldots, n;\ l = 0, 1, \ldots, n),\end{aligned} \tag{7.2.13}$$

where

$$\Gamma_{ijk}(u^{(l)}) = \frac{1}{2}(\lambda_j(u^{(l)}) - \lambda_k(u^{(l)}))l_i(u^{(l)})[\nabla r_k(u^{(l)})r_j(u^{(l)}) - \nabla r_j(u^{(l)})r_k(u^{(l)})]. \tag{7.2.14}$$

Obviously,

$$\Gamma_{ijj}(u^{(l)}) \equiv 0, \quad \forall i, j. \tag{7.2.15}$$

7.2.2 Rankine–Hugoniot Condition

Lemma 7.2.1 *For any fixed $k = 1, \ldots, n$, the Rankine–Hugoniot condition (7.1.9) on $x = x_k(t)$ (the kth small amplitude shock or the kth small amplitude contact discontinuity) can be rewritten in a neighbourhood of $(u^-, u^+) = (\hat{u}^{(k-1)}, \hat{u}^{(k)})$ as*

$$\hat{v}_i^- = G_i(\hat{v}_1^+, \ldots, \hat{v}_k^+, \hat{v}_k^-, \ldots, \hat{v}_n^-) \quad (i = 1, \ldots, k-1), \tag{7.2.16}$$

$$\hat{v}_i^+ = G_i(\hat{v}_1^+, \ldots, \hat{v}_k^+, \hat{v}_k^-, \ldots, \hat{v}_n^-) \quad (i = k+1, \ldots, n), \tag{7.2.17}$$

and

$$\frac{dx_k(t)}{dt} = \lambda_k(u^-, u^+), \tag{7.2.18}$$

where $u^\pm = u(t, x_k(t) \pm 0)$,

$$\hat{v}_i^- = l_i(u^-)(u^- - \hat{u}^{(k-1)}),\ \hat{v}_i^+ = l_i(u^+)(u^+ - \hat{u}^{(k)}) \quad (i = 1, \ldots, n), \tag{7.2.19}$$

$\lambda_k(u^-, u^+)$ *is the* k*th eigenvalue of the matrix*

$$\phi(u^-, u^+) = \int_0^1 \nabla f(u^- + \sigma(u^+ - u^-))d\sigma,$$

and $G_i(\cdot)$ $(i = 1, \ldots, k-1, k+1, \ldots, n)$ *and* $\lambda_k(\cdot, \cdot)$ *are all* C^2 *functions with respect to their arguments. Moreover, for* $i = 1, \ldots, k-1, k+1, \ldots, n$, *we have*

$$G_i(0, \ldots, 0) = 0, \tag{7.2.20}$$

$$\frac{\partial G_i}{\partial \hat{v}_j^+}(0, \ldots, 0) = \delta_{ij} + O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|) \quad (j = 1, \ldots, k), \tag{7.2.21}$$

and

$$\frac{\partial G_i}{\partial \hat{v}_j^-}(0, \ldots, 0) = \delta_{ij} + O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|) \quad (j = k, \ldots, n). \tag{7.2.22}$$

Proof. Noting that $|u^+ - \hat{u}^{(k)}|$ and $|u^- - \hat{u}^{(k-1)}|$ are sufficiently small, the proof is similar to that of the corresponding result in [72].

Lemma 7.2.2 *Suppose that* $|u^+ - \hat{u}^{(k)}|$ *and* $|u^- - \hat{u}^{(k-1)}|$ $[u^\pm = u(t, x_k(t) \pm 0)]$ *are sufficiently small. Then on* $x = x_k(t)$ (*the* k*th small amplitude shock or the* k*th small amplitude contact discontinuity*) *we have*

$$\hat{v}_i^- = \hat{v}_i^+ + \sum_{j=1}^{k} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)\hat{v}_j^+ + \sum_{j=k}^{n} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)\hat{v}_j^- + O(|\hat{v}^\pm|^2) \quad (i \neq k) \tag{7.2.23}$$

and

$$\begin{aligned} w_i^- = w_i^+ &+ \sum_{m=1}^{k} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)w_m^+ + \sum_{m=k}^{n} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)w_m^- \\ &+ \sum_{m=1}^{n} O(|\hat{v}^\pm|)w_m^+ + \sum_{m=1}^{n} O(|\hat{v}^\pm|)w_m^- \quad (i \neq k), \end{aligned} \tag{7.2.24}$$

where $\hat{v}^{\pm} = (\hat{v}_1^{\pm}, \ldots, \hat{v}_n^{\pm})^T$. *Moreover, if* $x = x_k(t)$ *is the kth contact discontinuity, the summation in* (7.2.24) *is only for* $m \neq k$.

Proof. By Lemma 7.2.1, it is easy to get (7.2.23).

We now prove (7.2.24).

Differentiating (7.2.16) with respect to t gives

$$\frac{d\hat{v}_i^-}{dt} = \sum_{j=1}^{k} \frac{\partial G_i}{\partial \hat{v}_j^+} \frac{d\hat{v}_j^+}{dt} + \sum_{j=k}^{n} \frac{\partial G_i}{\partial \hat{v}_j^-} \frac{d\hat{v}_j^-}{dt} \quad (i = 1, \ldots, k-1). \tag{7.2.25}$$

Noting (7.2.4) and (7.2.18), it easily follows from (7.2.19) that on $x = x_k(t)$,

$$\begin{aligned}\frac{d\hat{v}_i^+}{dt} =& \frac{\partial \hat{v}_i^+}{\partial t} + s\frac{\partial \hat{v}_i^+}{\partial x} = (s - \lambda_i(u^+))w_i^+ \\ &+ \sum_{m=1}^{n} (s - \lambda_m(u^+)) r_m^T(u^+) \nabla l_i(u^+)(u^+ - \hat{u}^{(k)}) w_m^+, \quad \forall i,\end{aligned} \tag{7.2.26}$$

where

$$s = \lambda_k(u^-, u^+). \tag{7.2.27}$$

Similarly, we have

$$\begin{aligned}\frac{d\hat{v}_i^-}{dt} =& \frac{\partial \hat{v}_i^-}{\partial t} + s\frac{\partial \hat{v}_i^-}{\partial x} = (s - \lambda_i(u^-))w_i^- \\ &+ \sum_{m=1}^{n} (s - \lambda_m(u^-)) r_m^T(u^-) \nabla l_i(u^-)(u^- - \hat{u}^{(k-1)}) w_m^-, \quad \forall i.\end{aligned} \tag{7.2.28}$$

Then, noting (7.1.2) and (7.1.10)–(7.1.11), we get

$$\begin{aligned}w_i^- =& \sum_{j=1}^{k} \frac{\partial G_i}{\partial \hat{v}_j^+} \frac{s - \lambda_j(u^+)}{s - \lambda_i(u^-)} w_j^+ + \sum_{j=k}^{n} \frac{\partial G_i}{\partial \hat{v}_j^-} \frac{s - \lambda_j(u^-)}{s - \lambda_i(u^-)} w_j^- \\ &+ \sum_{m=1}^{n} \frac{s - \lambda_m(u^+)}{s - \lambda_i(u^-)} G_{im}^{(1)} (u^+ - \hat{u}^{(k)}) w_m^+ \\ &+ \sum_{m=1}^{n} \frac{s - \lambda_m(u^-)}{s - \lambda_i(u^-)} G_{im}^{(2)} (u^- - \hat{u}^{(k-1)}) w_m^- \quad (i = 1, \ldots, k-1),\end{aligned} \tag{7.2.29}$$

where

$$G_{im}^{(1)} = r_m^T(u^+) \sum_{j=1}^{k} \frac{\partial G_i}{\partial \hat{v}_j^+} \nabla l_j(u^+), \tag{7.2.30}$$

$$G_{im}^{(2)} = r_m^T(u^-)\left(\sum_{j=k}^{n} \frac{\partial G_i}{\partial \hat{v}_j^-}\nabla l_j(u^-) - \nabla l_i(u^-)\right). \tag{7.2.31}$$

Hence, noting (7.2.21) and (7.2.22), it is easy to get

$$\begin{aligned} w_i^- = w_i^+ &+ \sum_{m=1}^{k} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)w_m^+ + \sum_{m=k}^{n} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)w_m^- \\ &+ \sum_{m=1}^{n} O(|\hat{v}^{\pm}|)w_m^+ + \sum_{m=1}^{n} O(|\hat{v}^{\pm}|)w_m^- \quad (i = 1, \dots, k-1). \end{aligned} \tag{7.2.32}$$

Moreover, noting (7.1.11), if $x = x_k(t)$ is the kth contact discontinuity, the summation in (7.2.32) is only for $m \neq k$.

Similarly, we have

$$\begin{aligned} w_i^- = w_i^+ &+ \sum_{m=1}^{k} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)w_m^+ + \sum_{m=k}^{n} O(|\hat{u}^{(k)} - \hat{u}^{(k-1)}|)w_m^- \\ &+ \sum_{m=1}^{n} O(|\hat{v}^{\pm}|)w_m^+ + \sum_{m=1}^{n} O(|\hat{v}^{\pm}|)w_m^- \quad (i = k+1, \dots, n), \end{aligned} \tag{7.2.33}$$

and if $x = x_k(t)$ is the kth contact discontinuity, the summation in (7.2.33) is only for $m \neq k$.

The combination of (7.2.32) and (7.2.33) gives (7.2.24).

7.3 Proof of Main Results

In what follows, we always assume that $\theta > 0$ is suitably small.

By the existence and uniqueness of the local piecewise C^1 solution to the generalized Riemann problem (see [72]), there exists $T_0 > 0$ so small that the generalized Riemann problem (7.1.1) and (7.1.4) admits a unique piecewise C^1 solution $u = u(t,x)$ containing n small amplitude waves $x = x_k(t)$ $(k = 1, \dots, n)$ (shocks corresponding to GN characteristics and contact discontinuities corresponding to LD characteristics) on the domain $R(T_0) = \{(t,x) | 0 \le t \le T_0, -\infty < x < +\infty\}$:

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x), & (t,x) \in R_0(T_0), \\ u^{(l)}(t,x), & (t,x) \in R_l(T_0) \quad (l = 1, \dots, n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n(T_0), \end{cases} \tag{7.3.1}$$

where

$$R_0(T_0) = \{(t,x)|0 \le t \le T_0,\ x \le x_1(t)\},$$
$$R_l(T_0) = \{(t,x)|0 \le t \le T_0,\ x_l(t) \le x \le x_{l+1}(t)\} \quad (l = 1, \dots, n-1),$$

and

$$R_n(T_0) = \{(t,x)|0 \le t \le T_0,\ x \ge x_n(t)\}.$$

This solution possesses a structure similar to the self-similar solution $u = U(x/t)$ of the corresponding Riemann problem (7.1.1) and (7.1.7), namely,

$$u^{(l)}(0,0) = \hat{u}^{(l)} \quad (l = 0, 1, \dots, n) \tag{7.3.2}$$

and

$$x_k'(0) = \hat{\lambda}_k \quad (k = 1, \dots, n). \tag{7.3.3}$$

In order to prove Theorem 7.1.1, it suffices to establish a uniform a priori estimate on the piecewise C^1 norm of u on any given existence domain of the piecewise C^1 solution $u = u(t,x)$.

Without loss of generality, we suppose that

$$|u_l(0)|,\ |u_r(0)| \le \eta. \tag{7.3.4}$$

Noting (7.1.2), we have

$$\lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0). \tag{7.3.5}$$

Then there exist positive constants δ and δ_0 so small that

$$\lambda_{i+1}(u) - \lambda_i(u') \ge 4\delta_0, \quad \forall |u|, |u'| \le \delta \quad (i = 1, \dots, n-1), \tag{7.3.6}$$

and

$$|\lambda_i(u) - \lambda_i(u')| \le \frac{\delta_0}{2}, \quad \forall |u|, |u'| \le \delta \quad (i = 1, \dots, n). \tag{7.3.7}$$

Without loss of generality, we may suppose that

$$\lambda_i(0) > \delta_0 \quad (i = 1, \dots, n). \tag{7.3.8}$$

For the time being we suppose that on any given existence domain $R(T) = \{(t,x)|0 \le t \le T, -\infty < x < +\infty\}$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (7.1.1) and (7.1.4), we have

$$|u^{(l)}(t,x)| \le \delta, \quad \forall (t,x) \in R_l(T) \quad (l = 0, 1, \dots, n), \tag{7.3.9}$$

where

$$R_0(T) = \{(t,x)|0 \le t \le T,\ x \le x_1(t)\}, \tag{7.3.10}$$

$$R_l(T) = \{(t,x)|0 \le t \le T,\ x_l(t) \le x \le x_{l+1}(t)\} \quad (l = 1, \dots, n-1), \tag{7.3.11}$$

$$R_n(T) = \{(t,x)|0 \le t \le T,\ x \ge x_n(t)\}, \tag{7.3.12}$$

where $x = x_k(t)$ is the kth wave ($k = 1, \dots, n$). At the end of the proof of Lemma 7.3.3, we will explain that this hypothesis is reasonable.

Let I and J be the sets of indices such that $I \cup J = \{1, \dots, n\}$ and, when $i \in I$, $\lambda_i(u)$ is LD, whereas, when $i \in J$, $\lambda_i(u)$ is GN.

Let

$$D_+^T = \begin{cases} \{(t,x)|0 \le t \le T,\ x \ge x_n(t)\} \text{ if } n \in J, \\ \{(t,x)|0 \le t \le T,\ x \ge (\lambda_n(0) + \delta_0)t\} \text{ if } n \in I, \end{cases} \tag{7.3.13}$$

$$D_-^T = \{(t,x)|0 \le t \le T,\ x \le -t\}, \tag{7.3.14}$$

$$D_0^T = \begin{cases} \{(t,x)|0 \le t \le T,\ -t \le x \le x_1(t)\} \text{ if } 1 \in J, \\ \{(t,x)|0 \le t \le T,\ -t \le x \le (\lambda_1(0) - \delta_0)t\} \text{ if } 1 \in I, \end{cases} \tag{7.3.15}$$

$$D^T = \{(t,x)|0 \le t \le T, -\infty < x < +\infty\} \backslash (D_+^T \cup D_-^T \cup D_0^T), \tag{7.3.16}$$

and, for $k = 1, \dots, n$,

$$D_k^T = \begin{cases} \{(t,x)|0 \le t \le T,\ x = x_k(t)\} \text{ if } k \in J, \\ \{(t,x)|0 \le t \le T,\ (\lambda_k(0) - \delta_0)t \le x \le (\lambda_k(0) + \delta_0)t\} \text{ if } k \in I. \end{cases} \tag{7.3.17}$$

Obviously,

$$\bigcup_{k=1}^{n} D_k^T \subset D^T. \tag{7.3.18}$$

It is easy to see that

$$D_-^T \cup D_0^T \subseteq R_0(T) \tag{7.3.19}$$

and

$$D_-^T \cup D_0^T = R_0(T) \quad \text{iff} \quad 1 \in J, \tag{7.3.20}$$

$$D_+^T \subseteq R_n(T), \tag{7.3.21}$$

and

$$D_+^T = R_n(T) \quad \text{iff} \quad n \in J. \tag{7.3.22}$$

On any given existence domain

$$\{(t,x)|0 \le t \le T, -\infty < x < +\infty\} = D^T \cup D_+^T \cup D_-^T \cup D_0^T = \bigcup_{l=0}^{n} R_l(T) \tag{7.3.23}$$

of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (7.1.1) and (7.1.4), let

$$\hat{v}^{(l)} = (\hat{v}_1^{(l)}, \ldots, \hat{v}_n^{(l)}), \quad w^{(l)} = (w_1^{(l)}, \ldots, w_n^{(l)}) \quad (l = 0, 1, \ldots, n), \tag{7.3.24}$$

$$W(D_+^T) = \max_{i=1,\ldots,n} \sup_{(t,x)\in D_+^T} \{(1+x)^{1+\mu}|w_i^{(n)}(t,x)|\}, \tag{7.3.25}$$

$$W(D_-^T) = \max_{i=1,\ldots,n} \sup_{(t,x)\in D_-^T} \{(1+|x|)^{1+\mu}|w_i^{(0)}(t,x)|\}, \tag{7.3.26}$$

$$W(D_0^T) = \max_{i=1,\ldots,n} \sup_{(t,x)\in D_0^T} \{(1+t)^{1+\mu}|w_i^{(0)}(t,x)|\}, \tag{7.3.27}$$

$$W_\infty^c(T) = \max_{i=1,\ldots,n} \max_{l=0,1,\ldots,n} \sup_{(t,x)\in(D^T\setminus D_i^T)\cap R_l(T)} \{(1+t)^{1+\mu}|w_i^{(l)}(t,x)|\}, \tag{7.3.28}$$

$$\widetilde{W}_1(T) = \max_{i\in I} \max_{j\neq i} \left\{ \sup_{c_j} \int_{c_j} |w_i^{(i-1)}(t,x)|dt + \sup_{c_j} \int_{c_j} |w_i^{(i)}(t,x)|dt \right\}, \tag{7.3.29}$$

where c_j denotes any given jth characteristic on D_i^T,

$$W_1(T) = \max_{i\in I} \sup_{0\le t\le T} \left\{ \int_{(\lambda_i(0)-\delta_0)t}^{x_i(t)} |w_i^{(i-1)}(t,x)|dx + \int_{x_i(t)}^{(\lambda_i(0)+\delta_0)t} |w_i^{(i)}(t,x)|dx \right\}, \tag{7.3.30}$$

$$U_\infty(T) = \sum_{l=0}^{n} ||u^{(l)}(t,x) - \hat{u}^{(l)}||_{L^\infty(R_l(T))}, \tag{7.3.31}$$

$$V_\infty(T) = \sum_{l=0}^{n} ||\hat{v}^{(l)}(t,x)||_{L^\infty(R_l(T))}, \tag{7.3.32}$$

and

$$W_\infty(T) = \sum_{l=0}^{n} ||w^{(l)}(t,x)||_{L^\infty(R_l(T))}. \tag{7.3.33}$$

Lemma 7.3.1 *Let*

$$M = \{(t,x)|0 \le t \le T,\ g_1(t) \le x \le g_2(t)\} \tag{7.3.34}$$

be any given existence domain of the solution $u = u(t,x)$ (in the classical sense) to system (7.1.1)*, in which*

$$|u(t,x)| \le \delta, \quad \forall (t,x) \in M, \tag{7.3.35}$$

$g_i(0) = 0$ $(i = 1, 2)$, $g_1(t) < g_2(t)$, $\forall t \in (0, T]$, $g_1'(0) < g_2'(0)$, *and*

$$|g_i'(t) - g_i'(0)| \le \frac{\delta_0}{2}, \quad \forall t \in [0, T] \quad (i = 1, 2). \tag{7.3.36}$$

Suppose that

$$|g_i'(0) - \lambda_k(u(0,0))| \ge 2\delta_0 \quad (i = 1, 2), \tag{7.3.37}$$

which, noting (7.3.7) *and* (7.3.36), *implies that the kth characteristic passing through the origin $O(0,0)$ never enters the domain* (7.3.34). *Moreover, let c_k: $\xi = \xi_k(\tau)$ $(0 \le \tau \le t)$ be the kth characteristic passing through any given point $(t, x) \in M$ and (t_k, x_k) be the intersection point of c_k with the boundary of M. Then there exists a positive constant η_k independent of t and x such that*

$$t_k \ge \eta_k t. \tag{7.3.38}$$

Proof. Without loss of generality, we assume that

$$\lambda_k(u(0,0)) < g_1'(0) < g_2'(0). \tag{7.3.39}$$

Then, noting (7.3.7) and (7.3.36), we have

$$\begin{aligned}\frac{\left(g_1'(0) - \frac{\delta_0}{2}\right) t - \left(g_2'(0) + \frac{\delta_0}{2}\right) t_k}{t - t_k} &\le \frac{g_1(t) - g_2(t_k)}{t - t_k} \le \frac{x - x_k}{t - t_k} \\ &\le \lambda_k(u(0,0)) + \frac{\delta_0}{2};\end{aligned}$$

then

$$\left(g_1'(0) - \frac{\delta_0}{2}\right) t - \left(g_2'(0) + \frac{\delta_0}{2}\right) t_k \le \left(\lambda_k(u(0,0)) + \frac{\delta_0}{2}\right)(t - t_k).$$

Thus, noting (7.3.37) and (7.3.39), we get (7.3.38) immediately (cf. [73]).

In the present situation, similarly to Lemma 6.1.1 (also see the corresponding result in [57] and [76]), we have

Lemma 7.3.2 *Under the assumptions of Theorem 7.1.1, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $0 \le t \le T$ of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann problem* (7.1.1) *and* (7.1.4), *we have the following uniform a priori estimate:*

$$W(D_\pm^T), \ W(D_0^T) \le \kappa_1 \theta. \tag{7.3.40}$$

Here and henceforth, κ_i $(i = 1, 2, \ldots)$ denote positive constants independent of θ and T.

Lemma 7.3.3 *Under the assumptions of Theorem 7.1.1, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $0 \le t \le T$ of the piecewise C^1 solution $u = u(t, x)$ to the generalized Riemann* *problem* (7.1.1) *and* (7.1.4), *we have the following uniform a priori estimates:*

$$W_\infty^c(T) \le \kappa_2\theta, \tag{7.3.41}$$

$$\widetilde{W}_1(T),\ W_1(T) \le \kappa_3\theta, \tag{7.3.42}$$

and

$$V_\infty(T) \le \kappa_4\theta. \tag{7.3.43}$$

Proof. We first estimate $W_\infty^c(T)$.

For any given $i \in \{1,\dots,n\}$, passing through any fixed point $(t,x) \in (D^T \backslash D_i^T) \cap R_l(T)$ $(l = 0,1,\dots,n)$, we draw the ith characteristic c_i: $\xi = \xi_i(\tau)$ $(0 \le \tau \le t)$, which, noting (7.1.10) and (7.1.11), must intersect the boundary of D^T at a point (t_{i0}, x_{i0}). Without loss of generality, we assume that if $l \ge i$, then $i < n$ and (t_{i0}, x_{i0}) lies on the right boundary of D_n^T (if $n \in I$) or on $x = x_n(t)$ (if $n \in J$). Integrating (7.2.9) along c_i from t_{i0} to t yields

$$\begin{aligned} w_i^{(l)}(t,x) = {} & w_i^{(n-1)}(t_{i0}, x_{i0}) - \sum_{k=l+1}^{n-1} [w_i]_k + \int_{t_{i,l+1}}^{t} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(l)}) w_j^{(l)} w_m^{(l)}(\tau, \xi_i(\tau)) d\tau \\ & + \sum_{k=l+2}^{n} \int_{t_{ik}}^{t_{i,k-1}} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(k-1)}) w_j^{(k-1)} w_m^{(k-1)}(\tau, \xi_i(\tau)) d\tau \quad \text{if } n \in J \end{aligned} \tag{7.3.44}$$

and

$$\begin{aligned} w_i^{(l)}(t,x) = {} & w_i^{(n)}(t_{i0}, x_{i0}) - \sum_{k=l+1}^{n} [w_i]_k + \int_{t_{i,l+1}}^{t} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(l)}) w_j^{(l)} w_m^{(l)}(\tau, \xi_i(\tau)) d\tau \\ & + \sum_{k=l+2}^{n} \int_{t_{ik}}^{t_{i,k-1}} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(k-1)}) w_j^{(k-1)} w_m^{(k-1)}(\tau, \xi_i(\tau)) d\tau \\ & + \int_{t_{i0}}^{t_{in}} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(n)}) w_j^{(n)} w_m^{(n)}(\tau, \xi_i(\tau)) d\tau \quad \text{if } n \in I, \end{aligned} \tag{7.3.45}$$

where

$$[w_i]_k = w_i^{(k)}(t_{ik}, x_k(t_{ik})) - w_i^{(k-1)}(t_{ik}, x_k(t_{ik})), \tag{7.3.46}$$

in which $(t_{ik}, x_k(t_{ik}))$ stands for the intersection point of c_i with the kth wave $x = x_k(t)$ $(k = 1,\dots,n)$; moreover, when $n \in J$, $t_{i0} = t_{in}$.

When $n \in J$, noting (7.1.10), (7.3.7), and (7.3.9), we easily see that

$$\left(\lambda_n(0) - \frac{\delta_0}{2}\right) t_{i0} \le x_{i0} \le \left(\lambda_n(0) + \frac{\delta_0}{2}\right) t_{i0}. \tag{7.3.47}$$

By Lemma 7.2.2, we have

$$
\begin{aligned}
w_i^{(n-1)}(t_{i0}, x_{i0}) = w_i^{(n)}(t_{i0}, x_{i0}) &+ \sum_{m=1}^{n} O(|\hat{u}^{(n)} - \hat{u}^{(n-1)}|) w_m^{(n)}(t_{i0}, x_{i0}) \\
&+ O(|\hat{u}^{(n)} - \hat{u}^{(n-1)}|) w_n^{(n-1)}(t_{i0}, x_{i0}) \\
&+ \sum_{m=1}^{n} O(|\hat{v}^{\pm}|) w_m^{(n-1)}(t_{i0}, x_{i0}) + \sum_{m=1}^{n} O(|\hat{v}^{\pm}|) w_m^{(n)}(t_{i0}, x_{i0}).
\end{aligned} \tag{7.3.48}
$$

Then, noting (7.3.47) and using Lemma 7.3.2, we get

$$
\begin{aligned}
(1 + x_{i0})^{1+\mu} |w_i^{(n-1)}(t_{i0}, x_{i0})| &\le C\{W(D_+^T) + (\eta + V_\infty(T))(W_\infty^c(T) + W(D_+^T))\} \\
&\le C\{\theta(1 + V_\infty(T)) + (\eta + V_\infty(T)) W_\infty^c(T)\}.
\end{aligned} \tag{7.3.49}
$$

Here and henceforth, C denotes different positive constants independent of θ and T. Similarly, using Lemmas 7.2.2 and 7.3.1, noting (7.1.10), and noting that δ_0 is suitably small, it is easy to see that

$$
(1 + t)^{1+\mu} \left| \sum_{k=l+1}^{n-1} [w_i]_k \right| \le C(\eta + V_\infty(T)) W_\infty^c(T). \tag{7.3.50}
$$

Then, by Lemmas 7.3.1 and 7.3.2 and noting (7.2.11), it follows from (7.3.44) that

$$
\begin{aligned}
&(1 + t)^{1+\mu} |w_i^{(l)}(t, x)| \\
&\le C\{\theta(1 + V_\infty(T)) + (\eta + V_\infty(T)) W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)\widetilde{W}_1(T)\}.
\end{aligned} \tag{7.3.51}
$$

When $n \in I$, by the definition of D_n^T, instead of (7.3.47) we have

$$
x_{i0} = (\lambda_n(0) + \delta_0) t_{i0}. \tag{7.3.52}
$$

Then, in a completely similar manner, from (7.3.45) we still get (7.3.51).

Thus, noting that $\eta > 0$ is suitably small, we have

$$
W_\infty^c(T) \le C\{\theta(1 + V_\infty(T)) + V_\infty(T) W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)\widetilde{W}_1(T)\}. \tag{7.3.53}
$$

Next, we estimate $\widetilde{W}_1(T)$ and $W_1(T)$.

For $i \in I$, passing through any fixed point $A \in D_i^T \cap R_i(T)$, we draw the jth characteristic c_j: $\xi = \xi_j(\tau)$ $(j > i)$, which intersects $x = (\lambda_i(0) + \delta_0)t$ or the straight line $t = T$ at a point B and, in the meantime, intersects the ith contact discontinuity $x = x_i(t)$ at a point C.

When B lies on the straight line $x = (\lambda_i(0) + \delta_0)t$, by (7.2.13) and using Stokes' formula on the domain BCO, we have

$$\begin{aligned}
&\int_{c_j} |w_i^{(i)}(\lambda_j(u^{(i)}) - \lambda_i(u^{(i)}))(t, \xi_j(t))|dt \\
&\quad \leq \int_{OB} |w_i^{(i)}(\lambda_i(0) + \delta_0 - \lambda_i(u^{(i)}))(t, (\lambda_i(0) + \delta_0)t)|dt \\
&\qquad + \iint\limits_{BCO} \left| \sum_{j,m=1}^{n} \Gamma_{ijm}(u^{(i)}) w_j^{(i)} w_m^{(i)}(t, x) \right| dtdx. \qquad (7.3.54)
\end{aligned}$$

Then, noting (7.2.15) and (7.3.6), it is easy to get

$$\int_{c_j} |w_i^{(i)}(t, \xi_j(t))|dt \leq C\{W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T)\}. \quad (7.3.55)$$

On the other hand, when B lies on the straight line $t = T$, passing through B, we draw the ith characteristic, which intersects $x = (\lambda_i(0) + \delta_0)t$ at a point E. Using Stokes' formula on the domain $BCOE$, we still have (7.3.55). Moreover, (7.3.55) also holds for $j < i$. Similarly, we have

$$\int_{c_j} |w_i^{(i-1)}(t, \xi_j(t))|dt \leq C\{W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T)\}. \quad (7.3.56)$$

Then we get

$$\widetilde{W}_1(T) \leq C\{W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T)\}. \quad (7.3.57)$$

Similarly, we have

$$W_1(T) \leq C\{W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T)\}. \quad (7.3.58)$$

Finally, we estimate $V_\infty(T)$.

For $i = 1, \dots, n$, passing through any given point $(t, x) \in R_l(T)$ $(l = 0, 1, \dots, n)$, we draw the ith characteristic c_i: $\xi = \xi_i(\tau)$ $(0 \leq \tau \leq t)$, which intersects the x-axis at a point $(0, x_{i0})$. Integrating (7.2.6) along c_i from 0 to t gives

$$\begin{aligned}
\hat{v}_i^{(l)}(t, x) = \hat{v}_i^{(0)}(0, x_{i0}) &+ \sum_{k=1}^{l} [\hat{v}_i]_k + \int_{t_{il}}^{t} \sum_{j,m=1}^{n} \beta_{ijm}(u^{(l)}) \hat{v}_j^{(l)} w_m^{(l)}(\tau, \xi_i(\tau))d\tau \\
&+ \sum_{k=1}^{l-1} \int_{t_{ik}}^{t_{i,k+1}} \sum_{j,m=1}^{n} \beta_{ijm}(u^{(k)}) \hat{v}_j^{(k)} w_m^{(k)}(\tau, \xi_i(\tau))d\tau \\
&+ \int_{0}^{t_{i1}} \sum_{j,m=1}^{n} \beta_{ijm}(u^{(0)}) \hat{v}_j^{(0)} w_m^{(0)}(\tau, \xi_i(\tau))d\tau \quad \text{if } i > l \qquad (7.3.59)
\end{aligned}$$

and

$$\begin{aligned}\hat{v}_i^{(l)}(t,x) = \hat{v}_i^{(n)}(0,x_{i0}) - \sum_{k=l+1}^{n}[\hat{v}_i]_k + \int_{t_{i,l+1}}^{t}\sum_{j,m=1}^{n}\beta_{ijm}(u^{(l)})\hat{v}_j^{(l)}w_m^{(l)}(\tau,\xi_i(\tau))d\tau \\ + \sum_{k=l+2}^{n}\int_{t_{ik}}^{t_{i,k-1}}\sum_{j,m=1}^{n}\beta_{ijm}(u^{(k-1)})\hat{v}_j^{(k-1)}w_m^{(k-1)}(\tau,\xi_i(\tau))d\tau \\ + \int_0^{t_{in}}\sum_{j,m=1}^{n}\beta_{ijm}(u^{(n)})\hat{v}_j^{(n)}w_m^{(n)}(\tau,\xi_i(\tau))d\tau, \quad \text{if } i \le l \qquad (7.3.60)\end{aligned}$$

where

$$[\hat{v}_i]_k = \hat{v}_i^{(k)}(t_{ik},x_k(t_{ik})) - \hat{v}_i^{(k-1)}(t_{ik},x_k(t_{ik})), \qquad (7.3.61)$$

in which $(t_{ik},x_k(t_{ik}))$ stands for the intersection point of c_i with the kth wave $x = x_k(t)$ $(k=1,\dots,n)$.

Noting (7.1.12) and (7.3.9), by (7.2.1), we have

$$|\hat{v}_i^{(0)}(0,x_{i0})|, \quad |\hat{v}_i^{(n)}(0,x_{i0})| \le C\theta \quad (i=1,\dots,n). \qquad (7.3.62)$$

By Lemma 7.2.2, we have

$$|[\hat{v}_i]_k| \le C\{\eta V_\infty(T) + (V_\infty(T))^2\} \quad (i \ne k). \qquad (7.3.63)$$

Then, noting (7.2.8) and using Lemma 7.3.2, it follows from (7.3.59)–(7.3.60) that for $i = 1,\dots,n$ and for any given $(t,x) \in R_l(T)$ $(l = 0,1,\dots,n)$, we have

$$\begin{aligned}|\hat{v}_i^{(l)}(t,x)| &\le C\{\theta + V_\infty(T)(\eta + V_\infty(T) + W_\infty^c(T) + \widetilde{W}_1(T) \\ &\quad + W(D_\pm^T) + W(D_0^T))\} \\ &\le C\{\theta + V_\infty(T)(\eta + \theta + V_\infty(T) + W_\infty^c(T) + \widetilde{W}_1(T))\}. \qquad (7.3.64)\end{aligned}$$

Then we get

$$V_\infty(T) \le C\{\theta + V_\infty(T)(V_\infty(T) + W_\infty^c(T) + \widetilde{W}_1(T))\}. \qquad (7.3.65)$$

It is easy to see that the combination of (7.3.53), (7.3.57)–(7.3.58), and (7.3.65) gives (7.3.41)–(7.3.43). Then, noting (7.2.3), we have

$$U_\infty(T) \le C\theta, \qquad (7.3.66)$$

which means that hypothesis (7.3.9) is reasonable.

Lemma 7.3.4 *Under the assumptions of Theorem 7.1.1, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0,\theta_0]$, on any given existence domain $0 \le t \le T$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann* *problem* (7.1.1) *and* (7.1.4), *we have the following uniform a priori estimate:*

$$W_\infty(T) \le \kappa_5 \theta. \tag{7.3.67}$$

Proof. For $i = 1, \dots, n$ and for any given point $(t, x) \in R_l(T)$ $(l = 0, 1, \dots, n)$, we distinguish the following two cases:

i. $i \in J$.

By Lemmas 7.3.2 and 7.3.3, it is easy to see that

$$|w_i^{(l)}(t, x)| \le C\{W(D_\pm^T) + W(D_0^T) + W_\infty^c(T)\} \le C\theta \quad (l = 0, 1, \dots, n). \tag{7.3.68}$$

ii. $i \in I$.

When $(t, x) \in R_l(T)\backslash D_i^T$ $(l = 0, 1, \dots, n)$, similarly to (7.3.68), we have

$$|w_i^{(l)}(t, x)| \le C\{W(D_\pm^T) + W(D_0^T) + W_\infty^c(T)\} \le C\theta \quad (l = 0, 1, \dots, n), \tag{7.3.69}$$

whereas, when $(t, x) \in D_i^T$, passing through (t, x), we draw the ith characteristic c_i: $\xi = \xi_i(\tau)$ $(0 \le \tau \le t)$, which intersects the x-axis at a point $(0, x_{i0})$. Integrating (7.2.9) along c_i from 0 to t, we get

$$\begin{aligned} & w_i^{(i-1)}(t, x) \\ & = w_i^{(0)}(0, x_{i0}) + \sum_{k=1}^{i-1} [w_i]_k + \int_{t_{i,i-1}}^{t} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(i-1)}) w_j^{(i-1)} w_m^{(i-1)}(\tau, \xi_i(\tau)) d\tau \\ & \quad + \sum_{k=1}^{i-2} \int_{t_{ik}}^{t_{i,k+1}} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(k)}) w_j^{(k)} w_m^{(k)}(\tau, \xi_i(\tau)) d\tau \\ & \quad + \int_0^{t_{i1}} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(0)}) w_j^{(0)} w_m^{(0)}(\tau, \xi_i(\tau)) d\tau \ \text{if } (t, x) \in D_i^T \cap R_{i-1}(T) \end{aligned} \tag{7.3.70}$$

and

$$\begin{aligned} w_i^{(i)}(t, x) = {} & w_i^{(n)}(0, x_{i0}) - \sum_{k=i+1}^{n} [w_i]_k + \int_{t_{i,i+1}}^{t} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(i)}) w_j^{(i)} w_m^{(i)}(\tau, \xi_i(\tau)) d\tau \\ & + \sum_{k=i+2}^{n} \int_{t_{ik}}^{t_{i,k-1}} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(k-1)}) w_j^{(k-1)} w_m^{(k-1)}(\tau, \xi_i(\tau)) d\tau \\ & + \int_0^{t_{in}} \sum_{j,m=1}^{n} \gamma_{ijm}(u^{(n)}) w_j^{(n)} w_m^{(n)}(\tau, \xi_i(\tau)) d\tau \ \text{if } (t, x) \in D_i^T \cap R_i(T), \end{aligned} \tag{7.3.71}$$

where $[w_i]_k$ is given by (7.3.46) and t_{ik} is the same as in (7.3.46). Then, since $\lambda_i(u)$ $(i \in I)$ is LD, noting (7.1.12) and (7.2.11)–(7.2.12) and using Lemma 7.2.2, it is easy to get

$$|w_i^{(i-1)}(t,x)|,\ |w_i^{(i)}(t,x)| \le C\{\theta + V_\infty(T)(W_\infty^c(T) + W(D_+^T) + W(D_0^T)) + (W(D_\pm^T))^2 + (W(D_0^T))^2 + (W_\infty^c(T))^2 + W_\infty^c(T)W_\infty(T)\}, \quad \forall (t,x) \in D_i^T. \tag{7.3.72}$$

Hence, by Lemmas 7.3.2 and 7.3.3, we have

$$|w_i^{(i-1)}(t,x)|,\ |w_i^{(i)}(t,x)| \le C\theta\{1 + W_\infty(T)\}, \quad \forall (t,x) \in D_i^T. \tag{7.3.73}$$

The combination of (7.3.68)–(7.3.69) and (7.3.73) gives

$$W_\infty(T) \le C\theta\{1 + W_\infty(T)\}, \tag{7.3.74}$$

which implies (7.3.67).

From Lemmas 7.3.3 and 7.3.4 and noting (7.2.3)–(7.2.4), (7.2.18), system (7.1.1), and (7.3.2)–(7.3.3), we immediately get Theorem 7.1.1.

In order to prove Theorem 7.1.2, we take $\mu = 0$ in (7.3.25)–(7.3.28) and let

$$V(D_+^T) = \max_{i=1,\dots,n} \sup_{(t,x)\in D_+^T} \{(1+x)|\hat{v}_i^{(n)}(t,x)|\}, \tag{7.3.75}$$

$$V(D_-^T) = \max_{i=1,\dots,n} \sup_{(t,x)\in D_-^T} \{(1+|x|)|\hat{v}_i^{(0)}(t,x)|\}, \tag{7.3.76}$$

$$V(D_0^T) = \max_{i=1,\dots,n} \sup_{(t,x)\in D_0^T} \{(1+t)|\hat{v}_i^{(0)}(t,x)|\}, \tag{7.3.77}$$

$$V_\infty^c(T) = \max_{i=1,\dots,n} \max_{l=0,1,\dots,n} \sup_{(t,x)\in(D^T\setminus D_i^T)\cap R_l(T)} \{(1+t)|\hat{v}_i^{(l)}(t,x)|\}. \tag{7.3.78}$$

Similarly to Lemmas 7.3.2 and 7.3.3, we have

Lemma 7.3.5 *Under the assumptions of Theorem 7.1.2, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $0 \le t \le T$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem* (7.1.1) *and* (7.1.4), *we have the following uniform a priori estimate:*

$$V(D_\pm^T),\ V(D_0^T),\ W(D_\pm^T),\ W(D_0^T) \le \kappa_6\theta. \tag{7.3.79}$$

Lemma 7.3.6 *Under the assumptions of Theorem 7.1.2, there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $0 \le$*

$t \le T$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (7.1.1) *and* (7.1.4), *we have the following uniform a priori estimate:*

$$V_\infty^c(T), W_\infty^c(T) \le \kappa_7\theta. \tag{7.3.80}$$

Using Lemmas 7.3.5 and 7.3.6 and noting

$$|x| \ge Ct \quad \text{on } D_\pm^T, \tag{7.3.81}$$

we have

$$|\hat{v}_i^{(l)}(t,x)|,\ |w_i^{(l)}(t,x)| < \frac{C\theta}{1+t}, \quad \forall (t,x) \in R_l \quad (i = 1,\dots,n;\ l = 0,1,\dots,n). \tag{7.3.82}$$

Then, noting (7.2.3)–(7.2.4) and system (7.1.1), we get (7.1.28) and (7.1.29).

By (7.2.18) in Lemma 7.2.1, we have

$$\frac{dx_k(t)}{dt} = \lambda_k(u^{(k-1)}(t,x_k(t)), u^{(k)}(t,x_k(t))) \quad (k = 1,\dots,n). \tag{7.3.83}$$

Then, noting $\hat{\lambda}_k = \lambda_k(\hat{u}^{(k-1)}, \hat{u}^{(k)})$ $(k = 1,\dots,n)$ and (7.1.28), we obtain (7.1.30).

Thus, we finish the proof of Theorem 7.1.2.

7.4 Applications

7.4.1 System of Traffic Flow

For the system of traffic flow (cf. Section 1.3.2)

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, \end{cases} \tag{7.4.1}$$

where $\rho(> 0)$ and v are the density and velocity of cars at point x and time t, respectively, and

$$p(\rho) = \rho^\gamma \quad (\gamma > 0 \text{ is a constant}), \tag{7.4.2}$$

we consider the generalized Riemann problem with the initial data

$$t = 0:\ (\rho, v) = \begin{cases} (\widetilde{\rho}_0 + \rho_l(x), \widetilde{v}_0 + v_l(x)), & x \le 0, \\ (\widetilde{\rho}_0 + \rho_r(x), \widetilde{v}_0 + v_r(x)), & x \ge 0, \end{cases} \tag{7.4.3}$$

where $\widetilde{\rho}_0 > 0$ and $\widetilde{v}_0$ are constants, $\rho_l(x)$, $\rho_r(x)$, $v_l(x)$, and $v_r(x) \in C^1$,

$$|\rho_r(0) - \rho_l(0)| + |v_r(0) - v_l(0)| > 0 \tag{7.4.4}$$

is suitably small, and

$$\begin{aligned}\theta \overset{\text{def.}}{=} & \sup_{x \le 0}\{(1+|x|)^{1+\mu}(|\rho_l(x) - \rho_l(0)| + |v_l(x) - v_l(0)| + |\rho_l'(x)| + |v_l'(x)|)\} \\ & + \sup_{x \ge 0}\{(1+x)^{1+\mu}(|\rho_r(x) - \rho_r(0)| + |v_r(x) - v_r(0)| + |\rho_r'(x)| + |v_r'(x)|)\} \\ & < +\infty. \end{aligned} \tag{7.4.5}$$

Let

$$U = (\rho, v)^T. \tag{7.4.6}$$

It is easy to see that for $\rho > 0$, (7.4.1) is a strictly hyperbolic system with the following two distinct real eigenvalues:

$$\lambda_1(U) = v - \rho p'(\rho) < \lambda_2(U) = v, \tag{7.4.7}$$

in which $\lambda_1(U)$ is GN and $\lambda_2(U)$ is LD.

Let

$$\overline{U} = (u_1, u_2)^T \overset{\text{def.}}{=} (\rho, \rho(v + p(\rho)))^T. \tag{7.4.8}$$

System (7.4.1) can be rewritten in the form of (7.1.1) and the corresponding initial data satisfy the decaying property (7.1.12).

By Theorem 7.1.1 and Remark 7.1.3, we have

Theorem 7.4.1 *Suppose that the corresponding Riemann problem for system* (7.4.1) *with the initial data*

$$t = 0: \ (\rho, v) = \begin{cases} (\widetilde{\rho}_0 + \rho_l(0), \widetilde{v}_0 + v_l(0)), & x \le 0, \\ (\widetilde{\rho}_0 + \rho_r(0), \widetilde{v}_0 + v_r(0)), & x \ge 0, \end{cases} \tag{7.4.9}$$

admits a unique self-similar solution $U = U_0(x/t)$ containing one nondegenerate shock (*corresponding to the left characteristic*) *and one contact discontinuity* (*corresponding to the right characteristic*) *with small amplitude. Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, the generalized Riemann problem* (7.4.1) *and* (7.4.3) *admits a unique global piecewise C^1 solution $U = U(t,x)$ on $t \ge 0$, containing one nondegenerate shock* (*corresponding to the left characteristic*) *and one contact discontinuity* (*corresponding to the right characteristic*) *with small amplitude. This solution possesses a global structure similar to that of the self-similar solution $U = U_0(x/t)$. Moreover, the contact discontinuity degenerates into a weak discontinuity, if the contact discontinuity disappears in the self-similar solution $U = U_0(x/t)$.*

7.4.2 System of One-Dimensional Gas Dynamics

For the **system of one-dimensional gas dynamics in Lagrangian representation** (see [22], [72], [76], and [87])

$$\begin{cases} \partial_t \tau - \partial_x(\rho u) = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t(e + \frac{1}{2}u^2) + \partial_x(pu) = 0, \end{cases} \tag{7.4.10}$$

where $\tau(> 0)$ is the specific volume, u the velocity, p the pressure, $e(> 0)$ the internal energy, and

$$p = p(\tau, S) \tag{7.4.11}$$

with

$$p_\tau(\tau, S) < 0, \quad \forall \tau > 0, \tag{7.4.12}$$

in which S is the entropy:

$$S = S(\tau, e), \tag{7.4.13}$$

we consider the generalized Riemann problem with the initial data

$$t = 0: \ (\tau, u, e) = \begin{cases} (\widetilde{\tau}_0 + \tau_l(x), \widetilde{u}_0 + u_l(x), \widetilde{e}_0 + e_l(x)), & x \le 0, \\ (\widetilde{\tau}_0 + \tau_r(x), \widetilde{u}_0 + u_r(x), \widetilde{e}_0 + e_r(x)), & x \ge 0, \end{cases} \tag{7.4.14}$$

where $\widetilde{\tau}_0 > 0$, $\widetilde{e}_0 > 0$, and $\widetilde{u}_0$ are constants, $\tau_l(x)$, $\tau_r(x)$, $u_l(x)$, $u_r(x)$, $e_l(x)$, and $e_r(x) \in C^1$, satisfying the decaying property as shown in (7.1.12), and

$$|\tau_r(0) - \tau_l(0)| + |u_r(0) - u_l(0)| + |e_r(0) - e_l(0)| > 0 \tag{7.4.15}$$

is suitably small.

Let

$$U = (\tau, u, e)^T. \tag{7.4.16}$$

It is easy to see that under hypothesis (7.4.12), (7.4.10) is a strictly hyperbolic system with the following three distinct real eigenvalues:

$$\lambda_1(U) = -\sqrt{-p_\tau} < \lambda_2(U) = 0 < \lambda_3(U) = \sqrt{-p_\tau}. \tag{7.4.17}$$

$\lambda_2(U)$ is LD, and if

$$p_{\tau\tau}(\widetilde{\tau}_0, \widetilde{S}_0) \neq 0, \tag{7.4.18}$$

in which

$$\widetilde{S}_0 = S(\widetilde{\tau}_0, \widetilde{e}_0), \tag{7.4.19}$$

then $\lambda_1(U)$ and $\lambda_3(U)$ are GN in a neighbourhood of $U = \widetilde{U}_0 \overset{\text{def.}}{=} (\widetilde{\tau}_0, \widetilde{u}_0, \widetilde{e}_0)^T$.

By Theorem 7.1.1 and Remark 7.1.3, we have

Theorem 7.4.2 *Suppose that the corresponding Riemann problem for system* (7.4.10) *with the initial data*

$$t = 0: \ (\tau, u, e) = \begin{cases} (\widetilde{\tau}_0 + \tau_l(0), \widetilde{u}_0 + u_l(0), \widetilde{e}_0 + e_l(0)), & x \le 0, \\ (\widetilde{\tau}_0 + \tau_r(0), \widetilde{u}_0 + u_r(0), \widetilde{e}_0 + e_r(0)), & x \ge 0, \end{cases} \tag{7.4.20}$$

admits a unique self-similar solution $U = U_0(x/t)$ containing two nondegenerate
shocks (*corresponding to the first and third characteristics*) *and one contact discontinuity* (*corresponding to the second characteristic*) *with small amplitude. Then there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, the generalized Riemann problem* (7.4.10) *and* (7.4.14) *admits a unique global piecewise C^1 solution $U = U(t, x)$ on $t \ge 0$, which contains two nondegenerate shocks* (*corresponding to the first and the third characteristics*) *and one contact discontinuity* (*corresponding to the second characteristic*) *with small amplitude. This solution possesses a global structure similar to that of the self-similar solution $U = U_0(x/t)$. Moreover, the contact discontinuity degenerates into a weak discontinuity if the contact discontinuity disppears in the self-similar solution $U = U_0(x/t)$.*

7.4.3 System of Plane Elastic Waves for Hyperelastic Material

We now consider the time-dependent deformation of an elastic medium from the natural state, in which the position vector of a particle is denoted by $X = (X_1, X_2, X_3)^T$. At time t, the same particle has a position vector $Y = Y(t, X) = (Y_1, Y_2, Y_3)^T$. For homogeneous **hyperelastic materials**, there exists a stored energy function $W = W(p)$, where

$$p = (p_{ik}) = \left(\frac{\partial Y_i}{\partial X_k} \right)$$

is the strain tensor.

For Ciarlet–Geymonat material (see [20]), the stored energy function is given by

$$W(p) = a\|p\|^2 + b\|\mathrm{Cof}\ p\|^2 + \varGamma(\det p) + e, \tag{7.4.21}$$

where a, b are positive constants, e is a real number, and

$$\|p\| = (\mathrm{tr}(p^T p))^{\frac{1}{2}}, \ \mathrm{Cof}\ p = \det p \cdot (p^{-1})^T, \ \varGamma(\delta) = c\delta^2 - d \log \delta, \quad \forall \delta > 0, \tag{7.4.22}$$

in which c and d are two positive constants.

By [54] or [58] we know that for this kind of material, the **system of plane elastic waves** can be written as

$$\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = 0, \tag{7.4.23}$$

where $U = (u_1, \dots, u_6)^T$ and

$$\nabla f(U) = \begin{pmatrix} 0 & -I \\ -V''(u) & 0 \end{pmatrix}, \tag{7.4.24}$$

where I is the 3×3 unit matrix,

$$V''(u) = 2(a+b)I + [2(b+c) + d(1+\omega u)^{-2}]\omega^T \omega, \tag{7.4.25}$$

in which $u = (u_1, u_2, u_3)^T$, and ω is a three-dimensional constant vector with $|\omega| = 1$.

It is easy to see that (7.4.23) is a hyperbolic system with the following six real eigenvalues:

$$\lambda_1(U) = -\lambda,\ \lambda_2(U) \equiv \lambda_3(U) = -\lambda_0,\ \lambda_4(U) \equiv \lambda_5(U) = \lambda_0,\ \lambda_6(U) = \lambda, \tag{7.4.26}$$

where

$$\lambda = \sqrt{2(a+b) + 2(b+c) + d(1+\omega u)^{-2}}, \quad \lambda_0 = \sqrt{2(a+b)}. \tag{7.4.27}$$

Moreover, the simple characteristics $\lambda_1(U)$ and $\lambda_6(U)$ are GN, whereas, $\lambda_i(U)$ $(i = 2, \dots, 5)$ are LD characteristics with constant multiplicity 2.

We consider the generalized Riemann problem for system (7.4.23) with the following initial data:

$$t = 0: \ U = \begin{cases} U_l(x), & x \le 0, \\ U_r(x), & x \ge 0, \end{cases} \tag{7.4.28}$$

where $U_l(x)$ and $U_r(x)$ are two given C^1 vector functions satisfying the decay property as shown in (7.1.12) and

$$|U_r(0) - U_l(0)| > 0 \tag{7.4.29}$$

is suitably small.

By Theorem 7.1.1 and Remarks 7.1.3 and 7.1.4, we have

Theorem 7.4.3 *Suppose that the corresponding Riemann problem for system* (7.4.23) *with the initial data*

$$t = 0: \ U = \begin{cases} U_l(0), & x \le 0, \\ U_r(0), & x \ge 0, \end{cases} \tag{7.4.30}$$

admits a unique self-similar solution $U = U_0(x/t)$ *containing two nondegenerate shocks* (*corresponding to the leftmost and rightmost characteristics*) *and two contact discontinuities* [*corresponding to the second* (*or third*) *characteristic and the fourth* (*or fifth*) *characteristic*] *with small amplitude. Then there exists* $\theta_0 > 0$ *so small that for any fixed* $\theta \in (0, \theta_0]$, *the generalized Riemann problem* (7.4.23) *and* (7.4.28) *admits a unique global piecewise* C^1 *solution* $U = U(t, x)$ *on* $t \geq 0$, *which contains two nondegenerate shocks* (*corresponding to the leftmost and rightmost characteristics*) *and two contact discontinuities* [*corresponding to the second* (*or third*) *characteristic and the forth* (*or fifth*) *characteristic*] *with small amplitude. This solution possesses a global structure similar to that of the self-similar solution* $U = U_0(x/t)$. *Moreover, if a contact discontinuity disappears in the self-similar solution* $U = U_0(x/t)$, *then the corresponding contact discontinuity degenerates into a weak discontinuity.*

Chapter VIII
Generalized Nonlinear Initial-Boundary Riemann Problem

8.1 Introduction and Main Results

In hydrodynamics one often meets the mixed initial-boundary value problem for first-order quasilinear hyperbolic system of conservation laws. In certain situations, for example, the piston problem, the conditions of C^0 compatibility may fail at the corner where the initial axis meets the boundary (see [22], [26], [50], and [84]). Moreover, one also uses this kind of problem in numerical analysis (see [88] and [94]).

The aim of this chapter is to discuss this kind of problem in the general case with nonlinear boundary conditions.

First, we consider the following **nonlinear initial-boundary Riemann problem** with constant initial data:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, \quad t > 0,\ x > 0, \\ t = 0:\ u = u_+, \quad x \geq 0, \\ x = 0:\ v_s = G_s(v_1, \dots, v_m) \quad (s = m+1, \dots, n), \quad t \geq 0, \end{cases} \tag{8.1.1}$$

where $u = (u_1, \dots, u_n)^T$, $f(u) = (f_1(u), \dots, f_n(u))^T$ is a given C^3 vector function of u, such that $\nabla f(u)$ has n real eigenvalues:

$$\lambda_1(u), \dots, \lambda_m(u) < 0 < \lambda_{m+1}(u) < \cdots < \lambda_n(u) \tag{8.1.2}$$

and a complete set of left (resp. right) eigenvectors $l_1(u), \dots, l_n(u)$ [resp. $r_1(u), \dots, r_n(u)$], u_+ is a constant vector,

$$v_i = l_i(u)u \quad (i = 1, \dots, n), \tag{8.1.3}$$

and $G_s(\cdot) \in C^1$ $(s = m+1, \dots, n)$. However, the conditions of C^0 compatibility at the point $(t, x) = (0, 0)$,

$$v_s^+ = G_s(v_1^+, \dots, v_m^+) \quad (s = m+1, \dots, n), \tag{8.1.4}$$

Li Tatsien, Wang Libin, *Global Propagation of Regular Nonlinear Hyperbolic Waves*, DOI 10.1007/b78335_8,

fail, in which

$$v_i^+ = l_i(u_+)u_+ \quad (i = 1, \dots, n). \tag{8.1.5}$$

All $\lambda_i(u)$, $l_i(u)$, and $r_i(u)$ $(i = 1, \dots, n)$ are supposed to have the same regularity as $\nabla f(u)$, and, without loss of generality, we assume that on the domain under consideration,

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n), \tag{8.1.6}$$

where δ_{ij} stands for Kronecker's delta.

Suppose that on the domain under consideration, each positive eigenvalue $\lambda_s(u)$ $(s \in \{m+1, \dots, n\})$ is either genuinely nonlinear (GN): Without loss of generality,

$$\nabla\lambda_s(u)r_s(u) \equiv 1, \quad \forall u, \tag{8.1.7}$$

or linearly degenerate (LD):

$$\nabla\lambda_s(u)r_s(u) \equiv 0, \quad \forall u. \tag{8.1.8}$$

For the nonlinear initial-boundary Riemann problem (8.1.1), we have

Theorem 8.1.1 *Suppose that* (8.1.2) *holds and each positive eigenvalue is either GN or LD. If* $|u_+|$ *and* $|v_s^+ - G_s(v_1^+, \dots, v_m^+)|$ $(s = m+1, \dots, n)$ *are suitably small, then the nonlinear initial-boundary Riemann problem* (8.1.1) *admits a unique small amplitude self-similar solution* $u = U(x/t)$*. This solution is composed of* $n - m + 1$ *constant states* $\hat{u}^{(i)} (i = m, \dots, n)$*, with* $\hat{u}^{(n)} = u_+$ *and* $n - m$ *elementary waves with small amplitude* (*shocks or centered rarefaction waves corresponding to GN characteristics, contact discontinuities corresponding to LD characteristics*).

Remark 8.1.1 *The result of Theorem 8.1.1 can be found in* [67].

Remark 8.1.2 *If* u_+ *is suitably small and*

$$G_s(0, \dots, 0) = 0 \quad (s = m+1, \dots, n), \tag{8.1.9}$$

then $|v_s^+ - G_s(v_1^+, \dots, v_m^+)|$ $(s = m+1, \dots, n)$ *must be suitably small.*

Remark 8.1.3 *If there are positive characteristics with constant multiplicity* p (> 1)*, a similar result holds as in Theorem 8.1.1.*

Remark 8.1.4 *Theorem 8.1.1 generalizes Theorem 2.2 with the special boundary conditions in* [26].

We next investigate the following **generalized nonlinear initial-boundary Riemann problem**:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, \quad t > 0,\ x > 0, \\ t = 0:\ u = u_r(x), \quad x \geq 0, \\ x = 0:\ v_s = G_s(\alpha_s(t), v_1, \dots, v_m) + h_s(t) \quad (s = m+1, \dots, n), \quad t \geq 0, \end{cases} \tag{8.1.10}$$

where $f(\cdot) \in C^3$, $u_r(\cdot)$, $\alpha_s(\cdot)$, $h_s(\cdot)$, and $G_s(\cdot) \in C^1$ $(s = m+1, \dots, n)$. Moreover, without loss of generality, we assume that

$$G_s(\alpha_s(t), 0, \dots, 0) \equiv 0 \quad (s = m+1, \dots, n). \tag{8.1.11}$$

However, we assume that at the point $(t, x) = (0, 0)$, the conditions of C^0 compatibility fail.

In order to construct the globally defined piecewise C^1 solution to problem (8.1.10), we give the following hypotheses:

(H_1) $\eta \overset{\text{def.}}{=} |u_r(0)| + |\alpha(0)| + |h(0)| > 0$ is suitably small, where

$$\alpha(t) = (\alpha_{m+1}(t), \dots, \alpha_n(t)),\ h(t) = (h_{m+1}(t), \dots, h_n(t)), \quad \forall t \geq 0.$$

(H_2) There is a constant $\mu > 0$ such that

$$\begin{aligned} \theta \overset{\text{def.}}{=} & \sup_{x \geq 0}\{(1+x)^{1+\mu}(|u_r(x) - u_r(0)| + |u_r'(x)|)\} \\ & + \sup_{t \geq 0}\{(1+t)^{1+\mu}(|\alpha(t) - \alpha(0)| + |\alpha'(t)| + |h(t) - h(0)| + |h'(t)|)\} \\ < & + \infty. \end{aligned} \tag{8.1.12}$$

(H_3) The corresponding nonlinear initial-boundary Riemann problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0, \quad t > 0,\ x > 0, \\ t = 0:\ u = u_r(0), \quad x \geq 0, \\ x = 0:\ v_s = G_s(\alpha_s(0), v_1, \dots, v_m) + h_s(0) \\ \qquad\qquad \overset{\text{def.}}{=} \overline{G}_s(v_1, \dots, v_m) \quad (s = m+1, \dots, n), \quad t \geq 0, \end{cases} \tag{8.1.13}$$

admits a unique self-similar solution $u = U(x/t)$ composed of $n - m + 1$ constant states $\hat{u}^{(m)}, \dots, \hat{u}^{(n-1)}, \hat{u}^{(n)} = u_+(0)$, and $n - m$ small amplitude elementary waves $x = \hat{\lambda}_k t$ $(k = m+1, \dots, n)$ (nondegenerate shocks corresponding to GN characteristics and contact discontinuities corresponding to

LD characteristics):

$$u = U\left(\frac{x}{t}\right) = \begin{cases} \hat{u}^{(m)}, & 0 \le x \le \hat{\lambda}_{m+1}t, \\ \hat{u}^{(l)}, & \hat{\lambda}_l t \le x \le \hat{\lambda}_{l+1}t \quad (l = m+1, \dots, n-1), \\ \hat{u}^{(n)}, & x \ge \hat{\lambda}_n t. \end{cases} \tag{8.1.14}$$

For the generalized nonlinear initial-boundary Riemann problem (8.1.10), our main result is

Theorem 8.1.2 *Suppose that* (8.1.2) *holds and each positive characteristic is either GN or LD. Under assumptions* (H_1)–(H_3)*, there exists* $\theta_0 > 0$ *so small that for any given* $\theta \in (0, \theta_0]$*, the generalized nonlinear initial-boundary Riemann problem* (8.1.10) *admits a unique global piecewise* C^1 *solution*

$$u = u(t,x) = \begin{cases} u^{(m)}(t,x), & 0 \le x \le x_{m+1}(t), \\ u^{(l)}(t,x), & x_l(t) \le x \le x_{l+1}(t) \quad (l = m+1, \dots, n-1), \\ u^{(n)}(t,x), & x \ge x_n(t), \end{cases} \tag{8.1.15}$$

in which, for $l = m, \dots, n$, $u^{(l)}(t,x) \in C^1$ *satisfies the system in* (8.1.10) *in the classical sense on the corresponding angular domain. Moreover, for* $k = m+1, \dots, n$, $u^{(k-1)}(t,x)$ *and* $u^{(k)}(t,x)$ *are connected to each other by the* k*th small amplitude wave* $x = x_k(t)$ *with* $x_k(0) = 0$ [*the* k*th nondegenerate shock when* $\lambda_k(u)$ *is GN or the* k*th contact discontinuity when* $\lambda_k(u)$ *is LD*]*. This solution possesses a global structure similar to that of the self-similar solution* (8.1.14) *to the corresponding nonlinear initial-boundary Riemann problem* (8.1.13)*, namely,*

$$u^{(l)}(0,0) = \hat{u}^{(l)} \quad (l = m, \dots, n), \tag{8.1.16}$$

$$x_k'(0) = \hat{\lambda}_k \quad (k = m+1, \dots, n), \tag{8.1.17}$$

$$|u^{(l)}(t,x) - \hat{u}^{(l)}| \le C(\theta + \eta), \quad \forall (t,x) \in R_l \quad (l = m, \dots, n), \tag{8.1.18}$$

and

$$|x_k'(t) - \hat{\lambda}_k| \le C\theta, \quad t \ge 0 \quad (k = m+1, \dots, n), \tag{8.1.19}$$

where C *is a positive constant independent of* t, x, θ, *and* η.

Thus, we claim that the corresponding small amplitude self-similar solution $u = U(x/t)$ to the nonlinear initial-boundary Riemann problem (8.1.13) has global structural stability.

Remark 8.1.5 *The corresponding result and proof in* [67] *have been improved in Theorem 8.1.2.*

Remark 8.1.6 *Suppose that the system in* (8.1.10) *has some positive characteristics with constant multiplicity. Then similar results hold as in Theorem 8.1.2.*

Remark 8.1.7 *In* (H_3)*, when some contact discontinuities disappear in the self-similar solution* (8.1.14)*, we still have Theorem 8.1.2 in which the corresponding contact discontinuities degenerate to weak discontinuities.*

This chapter is organized as follows. In Section 8.2 we give some preliminaries. Then Theorems 8.1.1 and 8.1.2 proved in Sections 8.3 and 8.4, respectively.

8.2 Preliminaries

Let

$$\hat{v}_i^{(l)} = l_i(u^{(l)})(u^{(l)} - \hat{u}^{(l)}) \quad (i = 1, \ldots, n;\ l = m, \ldots, n) \tag{8.2.1}$$

and

$$w_i^{(l)} = l_i(u^{(l)})u_x^{(l)} \quad (i = 1, \ldots, n;\ l = m, \ldots, n). \tag{8.2.2}$$

By (8.1.6), we have

$$u^{(l)} = \hat{u}^{(l)} + \sum_{k=1}^{n} \hat{v}_k^{(l)} r_k(u^{(l)}) \quad (l = m, \ldots, n) \tag{8.2.3}$$

and

$$u_x^{(l)} = \sum_{k=1}^{n} w_k^{(l)} r_k(u^{(l)}) \quad (l = m, \ldots, n). \tag{8.2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{8.2.5}$$

denote the directional derivative with respect to t along the ith characteristic. We have (see Chapter 7)

$$\frac{d\hat{v}_i^{(l)}}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u^{(l)})\hat{v}_j^{(l)} w_k^{(l)} \quad (i = 1, \ldots, n;\ l = m, \ldots, n), \tag{8.2.6}$$

where

$$\beta_{ijk}(u^{(l)}) = (\lambda_k(u^{(l)}) - \lambda_i(u^{(l)}))l_i(u^{(l)})\nabla r_j(u^{(l)})r_k(u^{(l)}). \tag{8.2.7}$$

Hence, we have

$$\beta_{iji}(u^{(l)}) \equiv 0, \quad \forall i, j. \tag{8.2.8}$$

On the other hand, we have (see Chapter 7)

$$\frac{dw_i^{(l)}}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u^{(l)}) w_j^{(l)} w_k^{(l)} \quad (i = 1, \ldots, n;\ l = m, \ldots, n), \tag{8.2.9}$$

where

$$\begin{aligned}\gamma_{ijk}(u^{(l)}) = \frac{1}{2}\{&(\lambda_j(u^{(l)}) - \lambda_k(u^{(l)}))l_i(u^{(l)})\nabla r_k(u^{(l)})r_j(u^{(l)}) \\ &- \nabla\lambda_k(u^{(l)})r_j(u^{(l)})\delta_{ik} + (j|k)\},\end{aligned} \tag{8.2.10}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. Hence,

$$\gamma_{ijj}(u^{(l)}) \equiv 0, \quad \forall j \neq i. \tag{8.2.11}$$

Moreover, if $\lambda_i(u)$ is LD, we have

$$\gamma_{iii}(u^{(l)}) \equiv 0. \tag{8.2.12}$$

Noting (8.2.4) and (8.2.9), we have (see Chapter 7)

$$d[w_i^{(l)}(dx - \lambda_i(u^{(l)})dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u^{(l)}) w_j^{(l)} w_k^{(l)} dt \wedge dx (i=1, \ldots, n;\ l=m, \ldots, n), \tag{8.2.13}$$

where

$$\Gamma_{ijk}(u^{(l)}) = \frac{1}{2}(\lambda_j(u^{(l)}) - \lambda_k(u^{(l)}))l_i(u^{(l)})[\nabla r_k(u^{(l)})r_j(u^{(l)}) - \nabla r_j(u^{(l)})r_k(u^{(l)})]. \tag{8.2.14}$$

Obviously,

$$\Gamma_{ijj}(u^{(l)}) \equiv 0, \quad \forall i, j. \tag{8.2.15}$$

8.3 Proof of Theorem 8.1.1

First, we give the following lemma (see [42], [49] and [72]).

Lemma 8.3.1 *For any given GN or LD simple characteristic $\lambda_k(u)$ and any given left (resp. right) state $\underline{u}$, the right (resp. left) state $\overline{u}$, which can be connected with $\underline{u}$, by a kth elementary wave with small amplitude, forms a* *one-parameter family*

$$\overline{u} = W(\underline{u}; \varepsilon), \quad -\varepsilon_0 \le \varepsilon \le \varepsilon_0 \ (\varepsilon_0 > 0 \textit{ suitably small}), \tag{8.3.1}$$

where W is a C^1 function of ε, satisfying

$$W(\underline{u}; 0) = \underline{u} \tag{8.3.2}$$

and

$$\frac{dW}{d\varepsilon}(\underline{u}; 0) = r_k(\underline{u}). \tag{8.3.3}$$

Here, when $\lambda_k(u)$ is LD, the kth elementary wave with small amplitude is a contact discontinuity, whereas, when $\lambda_k(u)$ is GN, the kth wave with small amplitude is a shock or a centered rarefaction wave.

Proof of Theorem 8.1.1. By Lemma 8.3.1, there is a one-parameter family of states $u^{(n-1)} = u^{(n-1)}(u_+; \varepsilon_n)$ which can be connected with the right state u_+ by an nth elementary wave. Similarly, there is a one-parameter family of states $u^{(n-2)} = \widetilde{u}^{(n-2)}(u^{(n-1)}, \varepsilon_{n-1}) = u^{(n-2)}(u_+; \varepsilon_n, \varepsilon_{n-1})$ which can be connected with the right state $u^{(n-1)}$ by an $(n-1)$st elementary wave. Repeating this procedure, we obtain a one-parameter family of states $u^{(m)} = \widetilde{u}^{(m)}(u^{(m+1)}, \varepsilon_{m+1}) = u^{(m)}(u_+; \varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_{m+1})$, which can be connected with the right state $u^{(m+1)}$ by an $(m+1)$st elementary wave. Noting (8.3.2)–(8.3.3), for

$$u^{(m)} = u^{(m)}(u_+; \varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_{m+1}), \tag{8.3.4}$$

we have

$$u^{(m)}(u_+; 0, \dots, 0) = u_+, \tag{8.3.5}$$

$$\frac{\partial u^{(m)}}{\partial \varepsilon_i}(u_+; 0, \dots, 0) = r_i(u_+) \quad (i = m+1, \dots, n). \tag{8.3.6}$$

On the other hand, $u^{(m)}$ should satisfy the boundary condition in (8.1.1):

$$v_s^{(m)} = G_s(v_1^{(m)}, \dots, v_m^{(m)}) \quad (s = m+1, \dots, n), \tag{8.3.7}$$

where

$$v_i^{(m)} = l_i(u^{(m)})u^{(m)} \quad (i = 1, \dots, n). \tag{8.3.8}$$

Let

$$F_s(u_+; \varepsilon_{m+1}, \dots, \varepsilon_n; c_s) = v_s^{(m)} - G_s(v_1^{(m)}, \dots, v_m^{(m)}) - c_s \quad (s = m+1, \dots, n), \tag{8.3.9}$$

where c_s $(s = m+1, \dots, n)$ are independent variables. Noting (8.1.6) and (8.3.5)–(8.3.6), we can easily get

$$F_s(u_+; 0, \dots, 0; v_s^+ - G_s(v_1^+, \dots, v_m^+)) = 0 \quad (s = m+1, \dots, n), \tag{8.3.10}$$

and at $(\varepsilon_{m+1},\dots,\varepsilon_n)=(0,\dots,0)$ and $c_s=v_s^+-G_s(v_1^+,\dots,v_m^+)$ $(s=m+1,\dots,n)$, where v_i^+ $(i=1,\dots,n)$ are given by (8.1.5), we have

$$\frac{\partial(F_{m+1},\dots,F_n)}{\partial(\varepsilon_{m+1},\dots,\varepsilon_n)}\neq 0, \tag{8.3.11}$$

provided that $|u_+|$ is suitably small. Then when $|v_s^+-G_s(v_1^+,\dots,v_m^+)|$ $(s=m+1,\dots,n)$ are suitably small, by the implicit function theorem and taking $c_s=0$ $(s=m+1,\dots,n)$, we uniquely get $\hat\varepsilon_{m+1},\dots,\hat\varepsilon_n$ in a neighbourhood of $(\varepsilon_{m+1},\dots,\varepsilon_n)=(0,\dots,0)$ such that (8.3.4) with $(\varepsilon_{m+1},\dots,\varepsilon_n)=(\hat\varepsilon_{m+1},\dots,\hat\varepsilon_n)$ satisfies (8.3.7). Thus, the nonlinear initial-boundary Riemann problem (8.1.1) admits a unique self-similar solution with small amplitude, composed of the constant states $\hat u^{(m)}=u^{(m)}(u_+;\hat\varepsilon_n,\hat\varepsilon_{n-1},\dots,\hat\varepsilon_{m+1})$, $\hat u^{(m+1)}=u^{(m+1)}(u_+;\hat\varepsilon_n,\hat\varepsilon_{n-1},\dots,\hat\varepsilon_{m+2}),\dots,\hat u^{(n-1)}=u^{(n-1)}(u_+;\hat\varepsilon_n)$, and $\hat u^{(n)}=u_+$. By Lemma 8.3.1, for $k=m+1,\dots,n$, if $\lambda_k(u)$ is GN, $\hat u^{(k-1)}$ and $\hat u^{(k)}$ are connected by a shock or a centered rarefaction wave, whereas if $\lambda_k(u)$ is LD, $\hat u^{(k-1)}$ and $\hat u^{(k)}$ are connected by a contact discontinuity.

The proof of Theorem 8.1.1 is finished. □

8.4 Proof of Theorem 8.1.2

First, we show the existence and uniqueness of a local piecewise C^1 solution to the generalized nonlinear initial-boundary Riemann problem (8.1.10). Let

$$\widetilde v_i^{(k)}=l_i(\hat u^{(k)})u^{(k)}\quad (i=1,\dots,n;k=m,\dots,n), \tag{8.4.1}$$

where $\hat u^{(k)}$ $(k=m,\dots,n)$ stand for the constant states in the self-similar solution $u=U(x/t)$ to the nonlinear initial-boundary Riemann problem (8.1.13). It is easy to see that the boundary condition in (8.1.10) can be equivalently rewritten as

$$x=0:\ \widetilde v_s^{(m)}=\widetilde G_s(t,\widetilde v_1^{(m)},\dots,\widetilde v_m^{(m)})\quad (s=m+1,\dots,n). \tag{8.4.2}$$

Then, similarly to the proof of the existence and uniqueness of a local piecewise C^1 solution to the generalized Riemann problem (see [72] and [49]), we obtain that there exists $T_0>0$ such that the generalized nonlinear initial-boundary Riemann problem (8.1.10) admits a unique piecewise C^1 solution $u=u(t,x)$ on $R(T_0)$ containing $n-m$ small amplitude waves $x=x_k(t)$ with $x_k(0)=0$ $(k=m+1,\dots,n)$ (nondegenerate shocks corresponding to GN characteristics or contact discontinuities corresponding to

LD characteristics):

$$u = u(t,x) = \begin{cases} u^{(m)}(t,x), & (t,x) \in R_m(T_0), \\ u^{(l)}(t,x), & (t,x) \in R_l(T_0) \quad (l = m+1,\dots,n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n(T_0), \end{cases} \tag{8.4.3}$$

where

$$R_l(T_0) = \begin{cases} \{(t,x) | 0 \le t \le T_0,\ 0 \le x \le x_{m+1}(t)\} \quad (l = m), \\ \{(t,x) | 0 \le t \le T_0,\ x_l(t) \le x \le x_{l+1}(t)\} \quad (l = m+1,\dots,n-1), \\ \{(t,x) | 0 \le t \le T_0,\ x \ge x_n(t)\} \quad (l = n), \end{cases} \tag{8.4.4}$$

and

$$R(T_0) = \bigcup_{l=m}^{n} R_l(T_0). \tag{8.4.5}$$

Moreover, this solution possesses a structure similar to that of the self-similar solution $u = U(x/t)$ to the corresponding nonlinear initial-boundary Riemann problem (8.1.13), namely,

$$u^{(l)}(0,0) = \hat{u}^{(l)} \quad (l = m,\dots,n) \tag{8.4.6}$$

and

$$x_k'(0) = \hat{\lambda}_k \quad (k = m+1,\dots,n). \tag{8.4.7}$$

In order to obtain the global piecewise C^1 solution to the generalized nonlinear initial-boundary Riemann problem (8.1.10), we need to establish a uniform a priori estimate on the piecewise C^1 norm of the piecewise C^1 solution $u = u(t,x)$ on any given existence domain.

By (8.1.2), there exist small positive constants δ_0 and δ such that

$$\lambda_i(u) - \lambda_j(u') \ge 4\delta_0, \quad \forall |u|, |u'| \le \delta \ (i = m+1,\dots,n, \\ j = i-1 \text{ or } j = 1,\dots,m), \tag{8.4.8}$$

$$|\lambda_i(u) - \lambda_i(u')| \le \frac{\delta_0}{2}, \quad \forall |u|, |u'| \le \delta \quad (i = 1,\dots,n), \tag{8.4.9}$$

and

$$|\lambda_i(0)| \ge 3\delta_0 \quad (i = 1,\dots,n). \tag{8.4.10}$$

For the time being, we assume that on any given existence domain $R(T) = \{(t,x) | 0 \le t \le T, x \ge 0\}$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized nonlinear initial-boundary Riemann problem (8.1.10), we have

$$|u^{(l)}(t,x)| \le \delta, \ \ \forall (t,x) \in R_l(T) \quad (l = m,\dots,n), \tag{8.4.11}$$

where the piecewise C^1 solution

$$u = u(t,x) = \begin{cases} u^{(m)}(t,x), & (t,x) \in R_m(T), \\ u^{(l)}(t,x), & (t,x) \in R_l(T) \quad (l = m+1,\dots,n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n(T), \end{cases} \tag{8.4.12}$$

in which

$$R_l(T) = \begin{cases} \{(t,x) | 0 \le t \le T,\ 0 \le x \le x_{m+1}(t)\} \quad (l = m), \\ \{(t,x) | 0 \le t \le T,\ x_l(t) \le x \le x_{l+1}(t)\} \quad (l = m+1,\dots,n-1), \\ \{(t,x) | 0 \le t \le T,\ x \ge x_n(t)\} \quad (l = n), \end{cases} \tag{8.4.13}$$

where $x = x_k(t)$ with $x_k(0) = 0$ is the kth wave $(k = m+1,\dots,n)$. Later we will explain that hypothesis (8.4.11) is reasonable.

Let I and J be the sets of indices such that $I \cup J = \{m+1,\dots,n\}$. Also, when $i \in I$, $\lambda_i(u)$ is LD, whereas when $i \in J$, $\lambda_i(u)$ is GN. Let

$$D_i^T = \{(t,x) | 0 \le t \le T, x = x_i(t)\} \quad \text{for } i \in J, \tag{8.4.14}$$

and for $i \in I$, let

$$D_i^T = \begin{cases} \{(t,x) | 0 \le t \le T, (\lambda_i(0) - \delta_0)t \le x \le (\lambda_i(0) + \delta_0)t\} \quad (i \ne m+1, n), \\ \{(t,x) | 0 \le t \le T, 0 \le x \le (\lambda_{m+1}(0) + \delta_0)t\} \quad (i = m+1), \\ \{(t,x) | 0 \le t \le T, (\lambda_n(0) - \delta_0)t \le x\} \quad (i = n). \end{cases} \tag{8.4.15}$$

Obviously,

$$\bigcup_{i=m+1}^{n} D_i^T \subset R(T). \tag{8.4.16}$$

On any given existence domain $R(T)$ of the piecewise C^1 solution $u = u(t,x)$ to the generalized nonlinear initial-boundary Riemann problem (8.1.10), let

$$\hat{v}^{(l)} = (\hat{v}_1^{(l)},\dots,\hat{v}_n^{(l)}), \quad w^{(l)} = (w_1^{(l)},\dots,w_n^{(l)}) \quad (l = m,\dots,n), \tag{8.4.17}$$

in which

$$\hat{v}_i^{(l)} = l_i(u^{(l)})(u^{(l)} - \hat{u}^{(l)}), \quad w_i^{(l)} = l_i(u^{(l)})u_x^{(l)} \quad (i = 1,\dots,n), \tag{8.4.18}$$

$$W_\infty^c(T) = \max_{i=1,\dots,n} \left\{ \max_{l=m,\dots,n-1} \sup_{(t,x)\in R_l(T)\setminus D_i^T} (1+t)^{1+\mu} |w_i^{(l)}(t,x)|, \right.$$
$$\left. \sup_{(t,x)\in R_n(T)\setminus D_i^T} (1+x)^{1+\mu} |w_i^{(n)}(t,x)| \right\}, \tag{8.4.19}$$

in which $D_i^T = \emptyset$ for $i = 1, \ldots, m$,

$$\widetilde{W}_1(T) = \max_{i \in I} \max_{j \neq i} \sup_{c_j} \left\{ \int_{c_j} |w_i^{(i-1)}(t,x)| dt + \int_{c_j} |w_i^{(i)}(t,x)| dt \right\}, \quad (8.4.20)$$

where c_j denotes any given jth characteristic on D_i^T,

$$W_1(T) = \max_{i \in I} \sup_{0 \le t \le T} \left\{ \int_{a_i(t)}^{x_i(t)} |w_i^{(i-1)}(t,x)| dx + \int_{x_i(t)}^{b_i(t)} |w_i^{(i)}(t,x)| dx \right\}, \quad (8.4.21)$$

where

$$a_i(t) = \begin{cases} 0 \ \text{ for } i = m+1, \\ (\lambda_i(0) - \delta_0)t \ \text{ for } i \neq m+1, \end{cases} \quad \text{and}$$

$$b_i(t) = \begin{cases} +\infty \text{ for } i = n, \\ (\lambda_i(0) + \delta_0)t \text{ for } i \neq n, \end{cases} \quad (8.4.22)$$

$$V_\infty(T) = \sum_{l=m}^{n} ||\hat{v}^{(l)}(t,x)||_{L^\infty(R_l(T))}, \quad (8.4.23)$$

and

$$W_\infty(T) = \sum_{l=m}^{n} ||w^{(l)}(t,x)||_{L^\infty(R_l(T))}. \quad (8.4.24)$$

In what follows, we show that there exists $\theta_0 > 0$ so small that for any fixed $\theta \in (0, \theta_0]$, on any given existence domain $R(T)$ of the piecewise C^1 solution $u = u(t,x)$ [see (8.4.12)] to the generalized nonlinear initial-boundary Riemann problem (8.1.10), we have the following uniform a priori estimates:

$$W_\infty^c(T) \le \kappa_1 \theta, \quad (8.4.25)$$

$$\widetilde{W}_1(T), \ W_1(T) \le \kappa_2 \theta, \quad (8.4.26)$$

$$V_\infty(T) \le \kappa_3(\theta + \eta), \quad (8.4.27)$$

and

$$W_\infty(T) \le \kappa_4 \theta, \quad (8.4.28)$$

where κ_i $(i = 1, \ldots, 4)$ are positive constants independent of θ, η, and T.

We first estimate $W_\infty^c(T)$.

For $r = 1, \ldots, m$, passing through any given point $(t,x) \in R(T)$, we draw the rth characteristic c_r: $\xi = \xi_r(\tau; t, x)$ $(0 \le \tau \le t)$, which, noting (8.1.2), must intersect the x-axis at a point $(0, x_{r0})$. If $(t,x) \in R_l(T)$ for some $l \in \{m, \ldots, n-1\}$, integrating the rth equation in (8.2.9) along c_r

from 0 to t yields

$$\begin{aligned} w_r^{(l)}(t,x)=w_r^{(n)}(0,x_{r0})-\sum_{k=l+1}^{n}[w_r]_k+\int_{t_{r,l+1}}^{t}\sum_{j,q=1}^{n}\gamma_{rjq}(u^{(l)})w_j^{(l)}w_q^{(l)}(\tau,\xi_r(\tau;t,x))d\tau \\ +\sum_{k=l+2}^{n}\int_{t_{rk}}^{t_{r,k-1}}\sum_{j,q=1}^{n}\gamma_{rjq}(u^{(k-1)})w_j^{(k-1)}w_q^{(k-1)}(\tau,\xi_r(\tau;t,x))d\tau \\ +\int_0^{t_{rn}}\sum_{j,q=1}^{n}\gamma_{rjq}(u^{(n)})w_j^{(n)}w_q^{(n)}(\tau,\xi_r(\tau;t,x))d\tau, \end{aligned} \tag{8.4.29}$$

where t_{rk} stands for the t-coordinate of the intersection point of c_r with the kth wave $x = x_k(t)$ and

$$[w_r]_k = w_r^{(k)}(t_{rk}, x_k(t_{rk})) - w_r^{(k-1)}(t_{rk}, x_k(t_{rk})). \tag{8.4.30}$$

Noting (7.1.10)–(7.1.11) and (8.4.8)–(8.4.11), by Lemma 7.3.1, we have

$$t_{rk} \geq Ct \quad (k = l+1, \ldots, n). \tag{8.4.31}$$

Here and henceforth, C denotes different positive constants independent of θ, η, and T. Moreover, noting (8.1.2) and (8.4.9)–(8.4.10), we have

$$x_{r0} \geq Ct_{rn} \tag{8.4.32}$$

and

$$\xi_r(\tau;t,x) \geq Cx_{r0}, \quad \forall(\tau, \xi_r(\tau;t,x)) \in R_n(T). \tag{8.4.33}$$

Then, noting (8.4.11), (8.2.11), (8.1.12), and Lemma 7.2.2, it follows from (8.4.29) that

$$(1+t)^{1+\mu}|w_r^{(l)}(t,x)| \leq C\{\theta+(\eta+V_\infty(T))W_\infty^c(T)+(W_\infty^c(T))^2+W_\infty^c(T)\widetilde{W}_1(T)\}. \tag{8.4.34}$$

If $(t,x) \in R_n(T)$, similarly to (8.4.34), we have

$$(1+x)^{1+\mu}|w_r^{(n)}(t,x)| \leq C\{\theta + (W_\infty^c(T))^2 + W_\infty^c(T)\widetilde{W}_1(T)\}. \tag{8.4.35}$$

For $s = m+1, \ldots, n$ and for any fixed point $(t,x) \in R(T) \setminus D_s^T$, if $(t,x) \in R_l(T)$ for some $l \geq s$, similarly we have (8.4.34)–(8.4.35), in which the subscript r should be replaced by s. If $(t,x) \in R_l(T)$ for some $l < s$, the s-th characteristic c_s passing through (t,x): $\xi = \xi_s(\tau;t,x)$ $(0 \leq \tau \leq t)$ intersects the t-axis at a point $(t_{s0}, 0)$. Integrating the sth equation in (8.2.9) along c_s from t_{s0} to t yields

$$w_s^{(l)}(t,x)=w_s^{(m)}(t_{s0},0)+\sum_{k=m+1}^{l}[w_s]_k+\int_{t_{sl}}^{t}\sum_{j,q=1}^{n}\gamma_{sjq}(u^{(l)})w_j^{(l)}w_q^{(l)}(\tau,\xi_s(\tau;t,x))d\tau$$

$$+\sum_{k=m+1}^{l-1}\int_{t_{sk}}^{t_{s,k+1}}\sum_{j,q=1}^{n}\gamma_{sjq}(u^{(k)})w_j^{(k)}w_q^{(k)}(\tau,\xi_s(\tau;t,x))d\tau$$
$$+\int_{t_{s0}}^{t_{s,m+1}}\sum_{j,q=1}^{n}\gamma_{sjq}(u^{(m)})w_j^{(m)}w_q^{(m)}(\tau,\xi_s(\tau;t,x))d\tau. \tag{8.4.36}$$

By the boundary condition in (8.1.10), noting (8.1.2) and (8.4.11), it is easy to get (cf. Chapter 6)

$$\begin{aligned} w_s^{(m)}(t_{s0},0) =& \sum_{r=1}^{m} f_{sr}(\alpha(t_{s0}),u^{(m)})w_r^{(m)}(t_{s0},0) \\ &+\sum_{\bar{s}=m+1}^{n}\overline{f}_{s\bar{s}}(\alpha(t_{s0}),u^{(m)})\alpha'_{\bar{s}}(t_{s0}) \\ &+\sum_{\bar{s}=m+1}^{n}\widetilde{f}_{s\bar{s}}(\alpha(t_{s0}),u^{(m)})h'_{\bar{s}}(t_{s0}) \quad (s=m+1,\dots,n), \end{aligned} \tag{8.4.37}$$

where $f_{sr}(\cdot)$, $\overline{f}_{s\bar{s}}(\cdot)$, and $\widetilde{f}_{s\bar{s}}(\cdot)$ are continuous functions of α and u. Similarly to (8.4.31), we have

$$t_{s0},\ t_{sk}\geq ct \quad (k=m+1,\dots,l;s=m+1,\dots,n). \tag{8.4.38}$$

Thus, noting (8.1.12), (8.4.11), (8.2.11), and Lemma 7.2.2, it follows from (8.4.36)–(8.4.37) that

$$\begin{aligned}(1+t)^{1+\mu}|w_s^{(l)}(t,x)| \leq C\Big\{&\theta+(1+t_{s0})^{(1+\mu)}\sum_{r=1}^{m}|w_r^{(m)}(t_{s0},0)| \\ &+(\eta+V_\infty(T))W_\infty^c(T)+(W_\infty^c(T))^2+W_\infty^c(T)\widetilde{W}_1(T)\Big\}.\end{aligned}$$

Then, by (8.4.34), we get

$$(1+t)^{1+\mu}|w_s^{(l)}(t,x)|\leq C\{\theta+(\eta+V_\infty(T))W_\infty^c(T)+(W_\infty^c(T))^2+W_\infty^c(T)\widetilde{W}_1(T)\}. \tag{8.4.39}$$

The combination of (8.4.34)–(8.4.35) and (8.4.39) gives

$$W_\infty^c(T)\leq C\{\theta+W_\infty^c(T)(V_\infty(T)+W_\infty^c(T)+\widetilde{W}_1(T))\}. \tag{8.4.40}$$

Next we estimate $V_\infty(T)$.

When $r=1,\dots,m$, for any given point $(t,x)\in R_l(T)$ with $l\in\{m,\dots,n-1\}$, similarly to (8.4.29), integrating the rth equation in (8.2.6) along c_r

from 0 to t gives

$$\begin{aligned}\hat{v}_r^{(l)}(t,x)=&\hat{v}_r^{(n)}(0,x_{r0})-\sum_{k=l+1}^{n}[\hat{v}_r]_k+\int_{t_{r,l+1}}^{t}\sum_{j,q=1}^{n}\beta_{rjq}(u^{(l)})\hat{v}_j^{(l)}w_q^{(l)}(\tau,\xi_r(\tau;t,x))d\tau\\&+\sum_{k=l+2}^{n}\int_{t_{rk}}^{t_{r,k-1}}\sum_{j,q=1}^{n}\beta_{rjq}(u^{(k-1)})\hat{v}_j^{(k-1)}w_q^{(k-1)}(\tau,\xi_r(\tau;t,x))d\tau\\&+\int_0^{t_{rn}}\sum_{j,q=1}^{n}\beta_{rjq}(u^{(n)})\hat{v}_j^{(n)}w_q^{(n)}(\tau,\xi_r(\tau;t,x))d\tau.\end{aligned}\tag{8.4.41}$$

Then, noting Lemma 7.2.2, (8.1.12), and (8.4.9)–(8.4.11) we get

$$|\hat{v}_r^{(l)}(t,x)|\le C\{\theta+\eta V_\infty(T)+(V_\infty(T))^2+V_\infty(T)W_\infty^c(T)+V_\infty(T)\widetilde{W}_1(T)\}.\tag{8.4.42}$$

Moreover, similarly to (8.4.35), for any given point $(t,x)\in R_n(T)$, we have

$$|\hat{v}_r^{(n)}(t,x)|\le C\{\theta+V_\infty(T)W_\infty^c(T)+V_\infty(T)\widetilde{W}_1(T)\}.\tag{8.4.43}$$

When $s=m+1,\dots,n$, if $l\ge s$, for $\hat{v}_s^{(l)}(t,x)$ we still have (8.4.42)–(8.4.43), whereas, if $l<s$, similarly to (8.4.36), we have

$$\begin{aligned}\hat{v}_s^{(l)}(t,x)=&\hat{v}_s^{(m)}(t_{s0},0)+\sum_{k=m+1}^{l}[\hat{v}_s]_k+\int_{t_{sl}}^{t}\sum_{j,q=1}^{n}\beta_{sjq}(u^{(l)})\hat{v}_j^{(l)}w_q^{(l)}(\tau,\xi_s(\tau;t,x))d\tau\\&+\sum_{k=m+1}^{l-1}\int_{t_{sk}}^{t_{s,k+1}}\sum_{j,q=1}^{n}\beta_{sjq}(u^{(k)})\hat{v}_j^{(k)}w_q^{(k)}(\tau,\xi_s(\tau;t,x))d\tau\\&+\int_{t_{s0}}^{t_{s,m+1}}\sum_{j,q=1}^{n}\beta_{sjq}(u^{(m)})\hat{v}_j^{(m)}w_q^{(m)}(\tau,\xi_s(\tau;t,x))d\tau.\end{aligned}\tag{8.4.44}$$

Noting (8.1.11)–(8.1.12) and (8.4.11), by the boundary condition in (8.1.10), it is easy to get

$$|\hat{v}_s^{(m)}(t_{s0},0)|\le C\left(\theta+\eta+\sum_{r=1}^{m}|\hat{v}_r^{(m)}(t_{s0},0)|\right).\tag{8.4.45}$$

Then, noting (8.4.42), it follows from (8.4.44) that

$$|\hat{v}_s^{(l)}(t,x)|\le C\{\theta+\eta+\eta V_\infty(T)+(V_\infty(T))^2+V_\infty(T)W_\infty^c(T)+V_\infty(T)\widetilde{W}_1(T)\}.\tag{8.4.46}$$

Thus, finally, we have

$$V_\infty(T)\le C\{\theta+\eta+V_\infty(T)(V_\infty(T)+W_\infty^c(T)+\widetilde{W}_1(T))\}.\tag{8.4.47}$$

Now we estimate $\widetilde{W}_1(T)$ and $W_1(T)$.

We first estimate $\int_{c_j} |w_i^{(i)}(t,x)|d\tau$. Suppose that for $i \in I$, $c_j: \ \xi = \xi_j(\tau)$ is a jth characteristic ($j \neq i$) on $R_i(T) \cap D_i^T$ whose ends are denoted by A and B. Passing through A and B, we draw the ith characteristics, respectively, which intersect the boundary of D_i^T at C and D, respectively, where if $i < n$, C and D lie on $x = (\lambda_i(0) + \delta_0)t$; however, if $i = n$, C and D lie on the x-axis. On the domain $ACDB$, by (8.2.13) and noting (8.4.9), it is easy to get

$$\begin{aligned} &\int_{c_j} |w_i^{(i)}(\lambda_j(u^{(i)}) - \lambda_i(u^{(i)}))(t,x)|dt \\ &\le \int_C^D |w_i^{(i)}(t,x)|(dx - \lambda_i(u^{(i)})dt) + \iint_{ACDB} \left| \sum_{j,q=1}^n \Gamma_{ijq}(u^{(i)}) w_j^{(i)} w_q^{(i)}(t,x) \right| dtdx. \end{aligned} \tag{8.4.48}$$

Then, noting (8.1.12), (8.4.8), (8.4.11), and (8.2.15), we have

$$\int_{c_j} |w_i^{(i)}(t,x)|dt \le C\{\theta + W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T)\}. \tag{8.4.49}$$

For $\int_{c_j} |w_i^{(i-1)}(t,x)|dt$, noting (8.1.12) and (8.4.37) (when $m+1 \in I$), we similarly have

$$\int_{c_j} |w_i^{(i-1)}(t,x)|dt \le C\{\theta + W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T)W_1(T))\}. \tag{8.4.50}$$

Thus, we get

$$\widetilde{W}_1(T) \le C\{\theta + W_\infty^c(T)(1 + W_\infty^c(T) + W_1(T))\}. \tag{8.4.51}$$

Similarly, we have

$$W_1(T) \le C\{\theta + W_\infty^c(T)(1 + W_\infty^c(T) + W_1(T))\}. \tag{8.4.52}$$

Then, by the method in Chapter 3 (see the proof of Lemma 3.2.2), the combination of (8.4.40), (8.4.47), and (8.4.51)–(8.4.52) gives (8.4.25)–(8.4.27).

Noting (8.2.3) and (8.4.27), we easily to see that hypothesis (8.4.11) is reasonable.

Finally, we estimate $W_\infty(T)$.

By the definition of $W_\infty^c(T)$, we need only to estimate $w_s^{(s)}(t,x)$ for $s \in I$ and $(t,x) \in R_s(T) \cap D_s^T$ and to estimate $w_s^{(s-1)}(t,x)$ for $s \in I$ and $(t,x) \in R_{s-1}(T) \cap D_s^T$. To this end, passing through any given point $(t,x) \in R_s(T) \cap D_s^T$ [resp. $(t,x) \in R_{s-1}(T) \cap D_s^T$], we draw the sth characteristic $c_s: \ \xi = \xi_s(\tau; t, x)$ $(0 \le \tau \le t)$, which intersects the boundary of D_s^T at $(t_0^{(s)}, x_0^{(s)})$ [resp. $(t_0^{(s-1)}, x_0^{(s-1)})$]. Then, integrating the sth equation in (8.2.9) along c_s

from $t_0^{(s-1)}$ to t and from $t_0^{(s)}$ to t, respectively, we get

$$\begin{aligned} w_s^{(s-1)}(t,x) = & \, w_s^{(s-1)}(t_0^{(s-1)}, x_0^{(s-1)}) \\ & + \int_{t_0^{(s-1)}}^{t} \sum_{j,q=1}^{n} \gamma_{sjq}(u^{(s-1)}) w_j^{(s-1)} w_q^{(s-1)}(\tau, \xi_s(\tau; t, x)) d\tau \end{aligned} \tag{8.4.53}$$

and

$$w_s^{(s)}(t,x) = w_s^{(s)}(t_0^{(s)}, x_0^{(s)}) + \int_{t_0^{(s)}}^{t} \sum_{j,q=1}^{n} \gamma_{sjq}(u^{(s)}) w_j^{(s)} w_q^{(s)}(\tau, \xi_s(\tau; t, x)) d\tau. \tag{8.4.54}$$

In (8.4.53), when $s = m+1$, $(t_0^{(s-1)}, x_0^{(s-1)})$ lies on the t-axis, whereas in (8.4.54), when $s = n$, $(t_0^{(s)}, x_0^{(s)})$ lies on the x-axis. Then, since $\lambda_s(u)$ $(s \in I)$ is LD, noting (8.2.12), (8.1.12), and (8.4.37), we easily get

$$|w_s^{(s-1)}(t,x)|, \; |w_s^{(s)}(t,x)| \le C\{\theta + W_\infty^c(T) + (W_\infty^c(T))^2 + W_\infty^c(T) W_\infty(T)\}. \tag{8.4.55}$$

Hence, noting (8.4.25), we have

$$W_\infty(T) \le C\theta\{1 + W_\infty(T)\}, \tag{8.4.56}$$

which implies (8.4.28).

Then, noting (8.2.3) and (7.2.18), we finish the proof of Theorem 8.1.2.

Chapter IX
Inverse Generalized Riemann Problem

9.1 Introduction and Main Results

Consider the following hyperbolic system of conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{9.1.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of t and x, and $f(u) = (f_1(u), \ldots, f_n(u))^T$ is a given C^2 vector function of u.

Suppose that on the domain under consideration,

(H_1) System (9.1.1) is strictly hyperbolic, i.e., the matrix $\nabla f(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{9.1.2}$$

For $i = 1, \ldots, n$, let $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ and $r_i(u) = (r_{1i}(u), \ldots, r_{ni}(u))^T$ be the left and right eigenvectors corresponding to $\lambda_i(u)$, respectively. Without loss of generality, we assume that on the domain under consideration,

$$l_i(u) r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n), \tag{9.1.3}$$

where δ_{ij} stands for Kronecker's delta.

(H_2) System (9.1.1) is genuinely nonlinear (GN) in the sense of Lax: Without loss of generality, for $i = 1, \ldots, n$,

$$\nabla \lambda_i(u) \cdot r_i(u) \equiv 1. \tag{9.1.4}$$

(H_3) For the Riemann problem of system (9.1.1) with the following piecewise constant initial data:

$$t = 0: \ u = \begin{cases} u_-, & x \le 0, \\ u_+, & x \ge 0 \end{cases} \tag{9.1.5}$$

Li Tatsien, Wang Libin, *Global Propagation of Regular Nonlinear Hyperbolic Waves*, DOI 10.1007/b78335_9,

with $|u_+ - u_-|$ small enough, suppose that the self-similar solution $u = U(x/t)$ is composed of $n+1$ constants states $\hat{u}^{(0)} = u_-, \hat{u}^{(1)}, \ldots, \hat{u}^{(n-1)}$ and $\hat{u}^{(n)} = u_+$ and n small amplitude nondegenerate typical shocks $x = s_i t$ $(i = 1, \ldots, n)$ with

$$s_1 < s_2 < \cdots < s_n, \tag{9.1.6}$$

on which we have the Rankine–Hugoniot condition

$$f(\hat{u}^{(i)}) - f(\hat{u}^{(i-1)}) = s_i(\hat{u}^{(i)} - \hat{u}^{(i-1)}) \tag{9.1.7}$$

and the entropy condition

$$\begin{cases} \lambda_i(\hat{u}^{(i)}) < s_i < \lambda_i(\hat{u}^{(i-1)}), \\ \lambda_{i-1}(\hat{u}^{(i-1)}) < s_i < \lambda_{i+1}(\hat{u}^{(i)}), \end{cases} \tag{9.1.8}$$

for $i = 1, \ldots, n$ (cf. [42]).

In order to consider the local or global structural stability of the self-similar solution $u = U(x/t)$ to Riemann problem (9.1.1) and (9.1.5), we consider the corresponding generalized Riemann problem of system (9.1.1) with the following piecewise smooth initial data:

$$t = 0: \ u = \begin{cases} u_l(x), & x \le 0, \\ u_r(x), & x \ge 0, \end{cases} \tag{9.1.9}$$

in which

$$u_l(0) = u_-, \quad u_r(0) = u_+. \tag{9.1.10}$$

In [72], Li and Yu got the local structural stability. On the other hand, in [50] and [73], Li and Zhao got the following (also see Theorem 7.1.2 in Chapter 7).

Proposition 9.1.1 *Under assumptions* (H_1)–(H_3), *suppose that* $u_l(x)$ *and* $u_r(x)$ *are* C^1 *functions on* $x \le 0$ *and* $x \ge 0$, *respectively, and that* $f(u)$ *is a* C^2 *vector function. Then there exists a positive constant* $\varepsilon_0 > 0$ *so small that for any given* ε *with* $0 < \varepsilon \le \varepsilon_0$, *if*

$$|u_l(x) - u_l(0)|, \ |u_l'(x)| \le \frac{\varepsilon}{1 + |x|}, \quad \forall x \le 0, \tag{9.1.11}$$

and

$$|u_r(x) - u_r(0)|, \ |u_r'(x)| \le \frac{\varepsilon}{1 + x}, \quad \forall x \ge 0, \tag{9.1.12}$$

then the generalized Riemann problem (9.1.1) *and* (9.1.9) *admits a unique global piecewise* C^1 *solution on* $t \ge 0$*:*

$$u = u(t,x) = \begin{cases} u^{(0)}(t,x), & (t,x) \in R_0, \\ u^{(l)}(t,x), & (t,x) \in R_l \quad (l = 1, \ldots, n-1), \\ u^{(n)}(t,x), & (t,x) \in R_n, \end{cases} \tag{9.1.13}$$

containing only n *small amplitude nondegenerate shocks* $x = x_i(t)$ $(i = 1, \ldots, n)$ *passing through the origin, in which* $u^{(l)}(t,x) \in C^1$ *with*

$$u^{(l)}(0,0) = \hat{u}^{(l)} \quad (l = 0, 1, \ldots, n) \tag{9.1.14}$$

satisfies system (9.1.1) *in the classical sense on the domain* $R_l (l = 0, 1, \ldots, n)$*, respectively, where*

$$\begin{cases} R_0 = \{(t,x) | t \geq 0, x \leq x_1(t)\}, \\ R_l = \{(t,x) | t \geq 0, x_l(t) \leq x \leq x_{l+1}(t)\} \quad (l = 1, \ldots, n-1), \\ R_n = \{(t,x) | t \geq 0, x \geq x_n(t)\}, \end{cases} \tag{9.1.15}$$

and $x_i(t) \in C^2$ *on* $t \geq 0$ *with*

$$x_i(0) = 0, \;\; x_i'(0) = s_i \quad (i = 1, \ldots, n). \tag{9.1.16}$$

For $i = 1, \ldots, n$*, on* $x = x_i(t)$ *we have the Rankine–Hugoniot condition*

$$f(u^{(i)}) - f(u^{(i-1)}) = x_i'(t)(u^{(i)} - u^{(i-1)}) \tag{9.1.17}$$

and the entropy condition

$$\begin{cases} \lambda_i(u^{(i)}) < x_i'(t) < \lambda_i(u^{(i-1)}), \\ \lambda_{i-1}(u^{(i-1)}) < x_i'(t) < \lambda_{i+1}(u^{(i)}), \end{cases} \tag{9.1.18}$$

where $u^{(i)} = u^{(i)}(t, x_i(t))$ *and* $u^{(i-1)} = u^{(i-1)}(t, x_i(t))$*. Moreover, we have*

$$|u^{(l)}(t,x) - \hat{u}^{(l)}| \leq \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in R_l \quad (l = 0, 1, \ldots, n), \tag{9.1.19}$$

$$\left| \frac{\partial u^{(l)}(t,x)}{\partial x} \right|, \; \left| \frac{\partial u^{(l)}(t,x)}{\partial t} \right| \leq \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in R_l \quad (l = 0, 1, \ldots, n), \tag{9.1.20}$$

and

$$|x_i'(t) - x_i'(0)| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0 \quad (i = 1, \ldots, n), \tag{9.1.21}$$

where K *is a positive constant independent of* ε*,* t*, and* x*.*

From Proposition 9.1.1, under perturbation (9.1.9) satisfying (9.1.10), we have the global structural stability of the self-similar solution $u = U(x/t)$ to Riemann problem (9.1.1) and (9.1.5). Moreover, under the assumptions of Proposition 9.1.1, noting (9.1.17), we have

$$x_i'(t) = \lambda_i(u^{(i-1)}, u^{(i)}), \tag{9.1.22}$$

where $\lambda_i(u^{(i-1)}, u^{(i)})$ is the ith eigenvalue of the matrix

$$\int_0^1 \nabla f(u^{(i-1)} + \sigma(u^{(i)} - u^{(i-1)}))d\sigma. \tag{9.1.23}$$

Differentiating (9.1.22) with respect to t gives

$$\begin{aligned} x_i''(t) = & \frac{\partial \lambda_i}{\partial u^{(i-1)}}(u^{(i-1)}, u^{(i)}) \left(\frac{\partial u^{(i-1)}}{\partial t} + \frac{\partial u^{(i-1)}}{\partial x} x_i'(t) \right) \\ & + \frac{\partial \lambda_i}{\partial u^{(i)}}(u^{(i-1)}, u^{(i)}) \left(\frac{\partial u^{(i)}}{\partial t} + \frac{\partial u^{(i)}}{\partial x} x_i'(t) \right). \end{aligned} \tag{9.1.24}$$

Then, noting (9.1.19)–(9.1.21), we easily to get

$$|x_i''(t)| \le \frac{K\varepsilon}{1+t}, \quad \forall t \ge 0, \tag{9.1.25}$$

where K is still a positive constant independent of ε and t.

Inversely, under assumptions (H_1)–(H_3), when the position of n small amplitude nondegenerate shocks $x = x_i(t) \in C^2$ $(i = 1, \dots, n)$ satisfying (9.1.16) is given, to what degree can we determine the initial data (9.1.9) satisfying (9.1.10), such that the corresponding generalized Riemann problem (9.1.1) and (9.1.9) admits a unique piecewise C^1 solution (9.1.13) in which n small amplitude nondegenerate shocks passing through the origin are just $x = x_i(t)$ $(i = 1, \dots, n)$? In [53], Li first considered the **inverse generalized Riemann problem** for system (9.1.1) and got a local result as follows: *Suppose that the position of n small amplitude nondegenerate shocks $x = x_i(t) \in C^2$ $(i = 1, \dots, n)$ satisfying* (9.1.16) *is prescribed. For any given $u_l(x) \in C^1$ with $u_l(0) = u_-$, in a neighbourhood of the origin, one can uniquely determine $u_r(x) \in C^1$ with $u_r(0) = u_+$ such that the corresponding generalized Riemann problem* (9.1.1) *and* (9.1.9) *admits a unique local piecewise C^1 solution* (9.1.13) *in which n small amplitude nondegenerate shocks passing through the origin are just $x = x_i(t)$ $(i = 1, \dots, n)$.*

In this chapter we generalize the previous local result to the global case and get

Theorem 9.1.1 *Under assumptions (H_1)–(H_3), there exists an $\varepsilon_0 > 0$ so small that for any given ε with $0 < \varepsilon \le \varepsilon_0$, if one knows the position of n*

small amplitude nondegenerate shocks $x = x_i(t) \in C^2$ $(i = 1, \ldots, n)$ *satisfying* (9.1.16) *and*

$$|x_i'(t) - x_i'(0)|, \; |x_i''(t)| \leq \frac{\varepsilon}{1+t}, \quad \forall t \geq 0 \quad (i = 1, \ldots, n), \tag{9.1.26}$$

then for any given $u_l(x) \in C^1$ $(x \leq 0)$ *satisfying* $u_l(0) = u_-$ *and*

$$|u_l(x) - u_l(0)|, \; |u_l'(x)| \leq \frac{\varepsilon}{1+|x|}, \quad \forall x \leq 0, \tag{9.1.27}$$

we can uniquely determine $u_r(x) \in C^1$ $(x \geq 0)$ *satisfying* $u_r(0) = u_+$ *and*

$$|u_r(x) - u_r(0)|, \; |u_r'(x)| \leq \frac{K\varepsilon}{1+x}, \quad \forall x \geq 0, \tag{9.1.28}$$

where K *is a positive constant independent of* ε *and* x *such that the corresponding generalized Riemann problem* (9.1.1) *and* (9.1.9) *admits a unique global piecewise* C^1 *solution* (9.1.13) *in which* n *small amplitude nondegenerate shocks passing through the origin are just* $x = x_i(t)$ $(i = 1, \ldots, n)$*, on which we have the Rankine–Hugoniot condition* (9.1.17) *and the entropy condition* (9.1.18)*.*

Remark 9.1.1 *In Theorem 9.1.1, if we arbitrarily give* $u_r(x) \in C^1$ $(x \geq 0)$ *satisfying* $u_r(0) = u_+$ *and*

$$|u_r(x) - u_r(0)|, \; |u_r'(x)| \leq \frac{\varepsilon}{1+x}, \quad \forall x \geq 0, \tag{9.1.29}$$

we can obtain a similar result.

Remark 9.1.2 *The results in this chapter can be found in* [68]*.*

This chapter is organized as follows. In Section 9.2 we first discuss the generalized Cauchy problem; then in Section 9.3 we prove Theorem 9.1.1.

9.2 Generalized Cauchy Problem

In this section we first consider the Cauchy problem for the quasilinear strictly hyperbolic system of the general form

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0 \tag{9.2.1}$$

with the initial data on the negative x-axis

$$t = 0: \; u = \phi(x), \quad x \leq 0, \tag{9.2.2}$$

where $A(u) \in C^1$. Suppose that the distinct real eigenvalues $\lambda_i(u)$ and the corresponding right eigenvectors $r_i(u)$ $(i = 1, \ldots, n)$ of $A(u)$ have the same regularity as $A(u)$.

Lemma 9.2.1 *There exists a positive constant* ε_0 *so small that for any given* ε *with* $0 < \varepsilon \leq \varepsilon_0$, *if* $\phi(x) \in C^1$ *and*

$$|\phi(x) - \phi(0)|, \ |\phi'(x)| \leq \frac{\varepsilon}{1+|x|}, \quad \forall x \leq 0, \tag{9.2.3}$$

then on the domain

$$\hat{D} = \{(t,x) | t \geq 0, x \leq \xi t\}, \tag{9.2.4}$$

where ξ *satisfies*

$$\xi < \min_{i=1,\ldots,n} \{\lambda_i(\phi(0))\}, \tag{9.2.5}$$

the Cauchy problem (9.2.1) *and* (9.2.2) *admits a unique global* C^1 *solution* $u = u(t,x)$ *with*

$$|u(t,x) - u(0,0)| \leq \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in \hat{D}, \tag{9.2.6}$$

$$\left|\frac{\partial u}{\partial x}(t,x)\right|, \ \left|\frac{\partial u}{\partial t}(t,x)\right| \leq \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in \hat{D}, \tag{9.2.7}$$

where K *is a positive constant independent of* ε, t, *and* x.

Proof. According to the existence and uniqueness of the local C^1 solution to the Cauchy problem (cf. [72]), there exists $\tau_0 > 0$ such that on the domain $\{(t,x) | 0 \leq t \leq \tau_0, \ x \leq \xi t\}$, the Cauchy problem (9.2.1)–(9.2.2) has a unique C^1 solution $u = u(t,x)$.

In order to obtain the unique global C^1 solution $u = u(t,x)$ on $\hat{D}$, it is only necessary to establish a uniform a priori estimate on the C^1 norm of the C^1 solution $u = u(t,x)$ to the Cauchy problem (9.2.1)–(9.2.2) on any given existence domain $\hat{D}(T) = \{(t,x) | 0 \leq t \leq T, \ x \leq \xi t\}$.

Let

$$V(\hat{D}(T)) = \max_{i=1,\ldots,n} \sup_{(t,x) \in \hat{D}(T)} \{(1 + |x - \lambda_i(\phi(0))t|) |\bar{v}_i(t,x)|\}, \tag{9.2.8}$$

$$W(\hat{D}(T)) = \max_{i=1,\ldots,n} \sup_{(t,x) \in \hat{D}(T)} \{(1 + |x - \lambda_i(\phi(0))t|) |w_i(t,x)|\}, \tag{9.2.9}$$

where

$$\bar{v}_i = l_i(u)(u - \phi(0)) \quad (i = 1, \ldots, n) \tag{9.2.10}$$

and

$$w_i = l_i(u) u_x \quad (i = 1, \ldots, n). \tag{9.2.11}$$

By (9.1.3), we have

$$u = \phi(0) + \sum_{k=1}^{n} \bar{v}_k r_k(u) \tag{9.2.12}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{9.2.13}$$

Similarly to (7.2.6) and (7.2.9) (also see Chapter 2), we have

$$\frac{d\bar{v}_i}{d_i t} \overset{\text{def.}}{=} \frac{\partial \bar{v}_i}{\partial t} + \lambda_i(u)\frac{\partial \bar{v}_i}{\partial x} = \sum_{j,k=1}^{n} \beta_{ijk}(u)\bar{v}_j w_k \quad (i = 1, \dots, n), \tag{9.2.14}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u))l_i(u)\nabla r_j(u) r_k(u), \tag{9.2.15}$$

and

$$\frac{dw_i}{d_i t} \overset{\text{def.}}{=} \frac{\partial w_i}{\partial t} + \lambda_i(u)\frac{\partial w_i}{\partial x} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \tag{9.2.16}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u) r_j(u) - \nabla\lambda_k(u) r_j(u)\delta_{ik} + (j|k)\}, \tag{9.2.17}$$

in which $(j|k)$ stands for all terms obtained by changing j and k in the previous terms.

By continuity, there exist, positive constants δ_0 $(< \min_{i=1,\dots,n}\{\lambda_i(\phi(0))\} - \xi)$ and δ so small that

$$|\lambda_i(u) - \lambda_i(\phi(0))| \le \frac{\delta_0}{2}, \quad \forall\ |u - \phi(0)| \le \delta \quad (i = 1, \dots, n). \tag{9.2.18}$$

For the time being, we assume that on any given existence domain $\hat{D}(T)$ of the C^1 solution $u = u(t, x)$ to the Cauchy problem (9.2.1)–(9.2.2), we have

$$|u(t, x) - \phi(0)| \le \delta. \tag{9.2.19}$$

At the end of the proof, we will explain why this hypothesis is reasonable.

For each $i = 1, \dots, n$ and any given point $(t, x) \in \hat{D}(T)$, let c_i : $\xi_i = \xi_i(\tau)$ $(0 \le \tau \le t)$ be the ith characteristic passing through (t, x), which intersects the x-axis at $(0, x_{i0})$. Noting $\delta_0 < \min_{i=1,\dots,n}\{\lambda_i(\phi(0))\} - \xi$ and (9.2.18), it is easy to see that c_i : $\xi_i = \xi_i(\tau)$ $(0 \le \tau \le t)$ always stays in $\hat{D}(T)$ and

$$|x - \lambda_i(\phi(0))t| \ge \delta_0 t, \tag{9.2.20}$$

$$\frac{2}{3}|x_{i0}| \le |\xi_i(\tau) - \lambda_i(\phi(0))\tau| \le 2|x_{i0}|. \tag{9.2.21}$$

Then, integrating the ith equation in (9.2.14) and (9.2.16) along c_i from 0 to t, respectively, we get

$$\bar{v}_i(t,x) = \bar{v}_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^n \beta_{ijk}(u)\bar{v}_j w_k(\tau,\xi_i(\tau))d\tau \tag{9.2.22}$$

and

$$w_i(t,x) = w_i(0,x_{i0}) + \int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u)w_j w_k(\tau,\xi_i(\tau))d\tau. \tag{9.2.23}$$

Noting (9.1.27) and (9.2.19)–(9.2.21), we get

$$\begin{aligned}
&(1+|x-\lambda_i(\phi(0))t|)|\bar{v}_i(t,x)| \\
&\le C(1+|x_{i0}|)\{|\bar{v}_i(0,x_{i0})|+V(\hat{D}(T))W(\hat{D}(T))\int_0^t(1+|\xi_i(\tau)-\lambda_i(\phi(0))\tau|)^{-2}d\tau\} \\
&\le C\{\varepsilon+V(\hat{D}(T))W(\hat{D}(T))\}.
\end{aligned} \tag{9.2.24}$$

Here and henceforth, C denotes different positive constants independent of ε and T. Hence, we have

$$V(\hat{D}(T)) \le C\{\varepsilon+V(\hat{D}(T))W(\hat{D}(T))\}. \tag{9.2.25}$$

Similarly, we have

$$W(\hat{D}(T)) \le C\{\varepsilon+(W(\hat{D}(T)))^2\}. \tag{9.2.26}$$

Then, similarly to the proof of Lemma 3.2.2, it follows from (9.2.25)–(9.2.26) that

$$V(\hat{D}(T)),\ W(\hat{D}(T)) \le C\varepsilon. \tag{9.2.27}$$

Thus, noting (9.2.20) and system (9.2.1), we get (9.2.6)–(9.2.7) immediately.

Moreover, noting (9.2.12), from (9.2.27) we see that hypothesis (9.2.19) is reasonable.

Remark 9.2.1 *The result of Lemma 9.2.1 can be found in Lemma 3.1 in Chapter 8 of* [50] *or Corollary 3.2 in* [73]. *Here we prove it in a different way.*

From Lemma 9.2.1, we have

Lemma 9.2.2 *Under the assumptions of Lemma 9.2.1, on the domain*

$$D = \{(t,x)|t \ge 0, x \le g(t)\}, \tag{9.2.28}$$

where $g(t) \in C^1$ satisfies $g(0) = 0$,

$$|g'(t) - g'(0)| \leq \varepsilon, \quad \forall t \geq 0, \tag{9.2.29}$$

and

$$g'(0) < \min_{i=1,\dots,n} \{\lambda_i(\phi(0))\}, \tag{9.2.30}$$

the Cauchy problem (9.2.1) *and* (9.2.2) *admits a unique global C^1 solution $u = u(t, x)$ with* (9.2.6)–(9.2.7) *on D.*

Proof. Let

$$\xi = \frac{1}{2} \min_{i=1,\dots,n} \{g'(0) + \lambda_i(\phi(0))\}. \tag{9.2.31}$$

Using (9.2.29)–(9.2.30) and noting that $\varepsilon_0 > 0$ is suitably small, we easily see that $D \subseteq \hat{D}$.

Next, we consider the **generalized Cauchy problem** for system (9.2.1) with the following generalized initial data:

$$x = g_1(t): \ u = \psi(t), \quad t \geq 0, \tag{9.2.32}$$

where $x = g_1(t)$ is a noncharacteristic curve.

In order to get the global C^1 solution to problem (9.2.1) and (9.2.32), we assume that $x = g_1(t) \in C^2$ and $x = g_2(t) \in C^2$ $(t \geq 0)$ satisfy

$$g_1(0) = g_2(0) = 0, \tag{9.2.33}$$

$$\lambda_r(\psi(0)) < g_1'(0) < g_2'(0) < \lambda_s(\psi(0)) \ \ (r = 1, \dots, m; s = m+1, \dots, n), \tag{9.2.34}$$

$$|g_i'(t) - g_i'(0)| \leq \frac{\varepsilon}{1+t}, \quad \forall t \geq 0 \ \ (i = 1, 2), \tag{9.2.35}$$

and

$$|g_1''(t)| \leq \frac{\varepsilon}{1+t}, \quad \forall t \geq 0, \tag{9.2.36}$$

where $0 < \varepsilon \leq \varepsilon_0$ with ε_0 suitably small. We have

Theorem 9.2.1 *Under assumptions* (9.2.33)–(9.2.36)*, if $\psi(t) \in C^1$ and*

$$|\psi(t) - \psi(0)|, \ |\psi'(t)| \leq \frac{\varepsilon}{1+t}, \quad \forall t \geq 0, \tag{9.2.37}$$

then on the domain

$$\widetilde{D} = \{(t, x) | t \geq 0, \ g_1(t) \leq x \leq g_2(t)\}, \tag{9.2.38}$$

the generalized Cauchy problem (9.2.1) *and* (9.2.32) *admits a unique global* C^1 *solution* $u = u(t,x)$ *with*

$$|u(t,x) - u(0,0)| \le \frac{K\varepsilon}{1+x-g_1(t)}, \quad \forall (t,x) \in \widetilde{D}, \tag{9.2.39}$$

$$\left|\frac{\partial u}{\partial x}(t,x)\right|, \ \left|\frac{\partial u}{\partial t}(t,x)\right| \le \frac{K\varepsilon}{1+x-g_1(t)}, \quad \forall (t,x) \in \widetilde{D}, \tag{9.2.40}$$

where K *is a positive constant independent of* ε, t, *and* x.

Proof. We first suppose that on the domain $\widetilde{D}$,

$$|u(t,x) - \psi(0)| \le \delta_0, \tag{9.2.41}$$

where $\delta_0 > 0$ is a suitably small constant. At the end of the proof, we will explain why this hypothesis is reasonable.

Taking the transformation of independent variables

$$\bar{x} = -t, \ \bar{t} = x - g_1(t) \tag{9.2.42}$$

and noting (9.2.33)–(9.2.35) and (9.2.41), we easily see that the original generalized Cauchy problem on $\widetilde{D}$ is reduced to the following Cauchy problem on $\bar{D} = \{(\bar{t},\bar{x}) | \bar{t} \ge 0, \ \bar{x} \le g(\bar{t})\}$:

$$\frac{\partial u}{\partial \bar{t}} - (A(u) - g_1'(-\bar{x})I)^{-1} \frac{\partial u}{\partial \bar{x}} = 0, \tag{9.2.43}$$

$$\bar{t} = 0: \ u = \psi(-\bar{x}), \quad \bar{x} \le 0, \tag{9.2.44}$$

where $\bar{x} = g(\bar{t})$ $(\le 0) \in C^2$ with $g(0) = 0$ is determined by

$$\bar{t} = g_2(-\bar{x}) - g_1(-\bar{x}). \tag{9.2.45}$$

Let

$$u_{n+1} = g_1'(-\bar{x}), \tag{9.2.46}$$

$$U = \begin{pmatrix} u \\ u_{n+1} \end{pmatrix}, \tag{9.2.47}$$

and

$$\Psi(\bar{x}) = \begin{pmatrix} \psi(-\bar{x}) \\ g_1'(-\bar{x}) \end{pmatrix}. \tag{9.2.48}$$

On the domain $\bar{D}$, (9.2.43)–(9.2.44) can be rewritten as

$$\frac{\partial U}{\partial \bar{t}} + \bar{A}(U) \frac{\partial U}{\partial \bar{x}} = 0, \tag{9.2.49}$$

$$\bar{t} = 0: \ U = \Psi(\bar{x}), \quad \bar{x} \le 0, \tag{9.2.50}$$

where

$$\bar{A}(U) = \begin{pmatrix} -(A(u) - u_{n+1}I)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{9.2.51}$$

The eigenvalues of $\bar{A}(U)$ are given by

$$\bar{\lambda}_1(U) = (u_{n+1} - \lambda_1(u))^{-1}, \ldots, \bar{\lambda}_n(U) = (u_{n+1} - \lambda_n(u))^{-1}, \bar{\lambda}_{n+1}(U) = 0, \tag{9.2.52}$$

and, noting (9.2.34)–(9.2.35) and (9.2.41), we see that (9.2.49) is still a strictly hyperbolic system with

$$\bar{\lambda}_r(U) > \bar{\lambda}_{n+1}(U) = 0 > \bar{\lambda}_s(U) \quad (r = 1, \ldots, m; s = m+1, \ldots, n). \tag{9.2.53}$$

It follows from (9.2.35)–(9.2.37) that

$$|\Psi(\bar{x}) - \Psi(0)|,\ |\Psi'(\bar{x})| \le \frac{\varepsilon}{1 + |\bar{x}|}, \quad \forall \bar{x} \le 0. \tag{9.2.54}$$

From (9.2.45), we have

$$g'(\bar{t}) = \frac{1}{g_1'(-g(\bar{t})) - g_2'(-g(\bar{t}))}, \quad \forall \bar{t} \ge 0. \tag{9.2.55}$$

then, noting (9.2.34)–(9.2.35) we easily get

$$|g'(\bar{t}) - g'(0)| \le K\varepsilon, \quad \forall \bar{t} \ge 0. \tag{9.2.56}$$

Here and henceforth, K denotes a positive constant independent of ε and $\bar{t}$. Moreover, noting (9.2.46), from (9.2.52)–(9.2.53), we have

$$\min_{i=1,\ldots,n+1} \bar{\lambda}_i(\Psi(0)) = \frac{1}{g_1'(0) - \min\limits_{s=m+1,\ldots,n} \lambda_s(\psi(0))}. \tag{9.2.57}$$

Then, noting (9.2.34) and $g'(0) = 1/[g_1'(0) - g_2'(0)]$, we get

$$g'(0) < \min_{i=1,\ldots,n+1} \bar{\lambda}_i(\Psi(0)). \tag{9.2.58}$$

Hence, by Lemma 9.2.2, the Cauchy problem (9.2.49)–(9.2.50) admits a unique global C^1 solution $U = U(\bar{t}, \bar{x})$ on $\bar{D}$ and

$$|U(\bar{t}, \bar{x}) - U(0,0)| \le \frac{K\varepsilon}{1 + \bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}, \tag{9.2.59}$$

$$\left|\frac{\partial U}{\partial \bar{x}}(\bar{t}, \bar{x})\right|,\ \left|\frac{\partial U}{\partial \bar{t}}(\bar{t}, \bar{x})\right| \le \frac{K\varepsilon}{1 + \bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}. \tag{9.2.60}$$

Then the Cauchy problem (9.2.43)–(9.2.44) admits a unique global C^1 solution $u = \bar{u}(\bar{t}, \bar{x})$ on $\bar{D}$ and

$$|\bar{u}(\bar{t}, \bar{x}) - \bar{u}(0,0)| \leq \frac{K\varepsilon}{1+\bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}, \tag{9.2.61}$$

$$\left|\frac{\partial \bar{u}}{\partial \bar{x}}(\bar{t}, \bar{x})\right|, \ \left|\frac{\partial \bar{u}}{\partial \bar{t}}(\bar{t}, \bar{x})\right| \leq \frac{K\varepsilon}{1+\bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}. \tag{9.2.62}$$

As a result, the generalized Cauchy problem (9.2.1) and (9.2.32) admits a unique global C^1 solution $u = u(t,x) = \bar{u}(x - g_1(t), -t)$ on $\widetilde{D}$. Moreover, since

$$\frac{\partial u}{\partial t}(t,x) = -g_1'(t)\frac{\partial \bar{u}}{\partial \bar{t}}(x - g_1(t), -t) - \frac{\partial \bar{u}}{\partial \bar{x}}(x - g_1(t), -t) \tag{9.2.63}$$

and

$$\frac{\partial u}{\partial x}(t,x) = \frac{\partial \bar{u}}{\partial \bar{t}}(x - g_1(t), -t), \tag{9.2.64}$$

noting (9.2.33)–(9.2.35) and (9.2.42), we see that (9.2.39)–(9.2.40) follow from (9.2.61)–(9.2.62).

Then, from (9.2.39), we have

$$|u(t,x) - u(0,0)| \leq K\varepsilon \leq K\varepsilon_0, \quad \forall (t,x) \in \widetilde{D}, \tag{9.2.65}$$

which implies the validity of hypothesis (9.2.41), provided that $\varepsilon_0 > 0$ is suitably small.

The proof of Theorem 9.2.1 is finished.

Remark 9.2.2 *In Theorem 9.2.1, when $x = g_2(t)$ is replaced by the positive x-axis and* (9.2.34) *is replaced by*

$$\lambda_i(\psi(0)) < g_1'(0) \quad (i = 1, \ldots, n), \tag{9.2.66}$$

Theorem 9.2.1 still holds.

9.3 Proof of Theorem 9.1.1

In what follows, we assume that all assumptions of Theorem 9.1.1 hold. First, we prove

Lemma 9.3.1 *For $i = 1, \ldots, n$, suppose that on the left side of the ith shock $x = x_i(t)$ satisfying $x_i(0) = 0$, $x_i'(0) = s_i$, and*

$$|x_i'(t) - x_i'(0)|, \ |x_i''(t)| \leq \frac{\varepsilon}{1+t}, \quad \forall t \geq 0, \tag{9.3.1}$$

the value of the solution $u_-^{(i)} = u_-^{(i)}(t) \in C^1$ *satisfies*

$$u_-^{(i)}(0) = \hat{u}^{(i-1)} \tag{9.3.2}$$

and

$$|u_-^{(i)}(t) - \hat{u}^{(i-1)}|, \ \left| \frac{du_-^{(i)}(t)}{dt} \right| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0. \tag{9.3.3}$$

Henceforth, K *stands for different positive constants independent of* ε*,* t*, and* x*. Then, on the right side of* $x = x_i(t)$*, we can uniquely determine the value of solution* $u_+^{(i)} = u_+^{(i)}(t)$ *such that*

$$u_+^{(i)}(0) = \hat{u}^{(i)} \tag{9.3.4}$$

and

$$|u_+^{(i)}(t) - \hat{u}^{(i)}|, \ |\frac{du_+^{(i)}(t)}{dt}| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0. \tag{9.3.5}$$

Proof. Similarly to [53], let

$$H(u_+^{(i)}, u_-^{(i)}, s^{(i)}) = f(u_+^{(i)}) - f(u_-^{(i)}) - s^{(i)}(u_+^{(i)} - u_-^{(i)}), \tag{9.3.6}$$

where $s^{(i)} = x_i'(t)$. Noting (9.1.7)–(9.1.8), we have

$$H(\hat{u}^{(i)}, \hat{u}^{(i-1)}, s_i) = 0 \tag{9.3.7}$$

and

$$\frac{\partial H}{\partial u_+^{(i)}}(u_+^{(i)}, u_-^{(i)}, s^{(i)}) = \nabla f(u_+^{(i)}) - s^{(i)} I \tag{9.3.8}$$

is nonsingular at $(u_+^{(i)}, u_-^{(i)}, s^{(i)}) = (\hat{u}^{(i)}, \hat{u}^{(i-1)}, s_i)$. Then, in a neighbourhood of $(\hat{u}^{(i)}, \hat{u}^{(i-1)}, s_i)$, the Rankine–Hugoniot condition

$$f(u_+^{(i)}) - f(u_-^{(i)}) = s^{(i)}(u_+^{(i)} - u_-^{(i)}) \tag{9.3.9}$$

can be rewritten as

$$u_+^{(i)} = G(u_-^{(i)}, s^{(i)}), \tag{9.3.10}$$

where $G(\cdot, \cdot) \in C^2$ and $\hat{u}^{(i)} = G(\hat{u}^{(i-1)}, s_i)$. Hence, as we have (9.3.1)–(9.3.3), the value of solution on the right side of $x = x_i(t)$ can be uniquely determined as $u_+^{(i)} = u_+^{(i)}(t) = G(u_-^{(i)}(t), x_i'(t))$, which satisfies (9.3.4).

From (9.3.10), we have

$$\begin{aligned} u_+^{(i)}(t) - \hat{u}^{(i)} &= G(u_-^{(i)}(t), s^{(i)}) - G(\hat{u}^{(i-1)}, s_i) \\ &= G(u_-^{(i)}(t), s^{(i)}) - G(u_-^{(i)}(t), s_i) + G(u_-^{(i)}(t), s_i) - G(\hat{u}^{(i-1)}, s_i) \end{aligned}$$

$$= \left[\int_0^1 \frac{\partial G}{\partial s^{(i)}}(u_-^{(i)}(t), s_i + \tau(s^{(i)} - s_i))d\tau\right](s^{(i)} - s_i)$$
$$+ \left[\int_0^1 \frac{\partial G}{\partial u_-^{(i)}}(\hat{u}^{(i-1)} + \tau(u_-^{(i)}(t) - \hat{u}^{(i-1)}), s_i)d\tau\right](u_-^{(i)}(t) - \hat{u}^{(i-1)}), \quad (9.3.11)$$

in which $s^{(i)} = x_i'(t)$. Noting (9.3.1) and (9.3.3), we have the first inequality of (9.3.5). On the other hand, differentiating $u_+^{(i)}(t) = G(u_-^{(i)}(t), x_i'(t))$ with respect to t yields

$$\frac{du_+^{(i)}(t)}{dt} = \frac{\partial G}{\partial u_-^{(i)}}(u_-^{(i)}(t), x_i'(t))\frac{du_-^{(i)}(t)}{dt} + \frac{\partial G}{\partial s^{(i)}}(u_-^{(i)}(t), x_i'(t))x_i''(t). \quad (9.3.12)$$

Then, noting (9.3.1) and (9.3.3), we get the second inequality of (9.3.5). The proof of Lemma 9.3.1 is finished.

Proof of Theorem. 9.1.1. First, on the domain R_0 [see (9.1.15)], we solve the Cauchy problem for system (9.1.1) with the initial data

$$t = 0: \; u = u_l(x), \quad x \le 0. \quad (9.3.13)$$

By the entropy condition (9.1.8) (in which we take $i = 1$) and noting (H_1), we have

$$x_1'(0) < \lambda_1(\hat{u}^{(0)}) < \cdots < \lambda_n(\hat{u}^{(0)}), \quad (9.3.14)$$

where

$$\hat{u}^{(0)} = u_l(0). \quad (9.3.15)$$

Then, noting (9.1.26) (in which we take $i = 1$), by Lemma 9.2.2, the Cauchy problem (9.1.1) and (9.3.13) admits a unique global C^1 solution $u = u^{(0)}(t, x)$ on the domain R_0, satisfying

$$u^{(0)}(0, 0) = \hat{u}^{(0)}, \quad (9.3.16)$$

$$|u^{(0)}(t, x) - \hat{u}^{(0)}| \le \frac{K\varepsilon}{1 + t}, \quad \forall (t, x) \in R_0, \quad (9.3.17)$$

and

$$\left|\frac{\partial u^{(0)}(t, x)}{\partial x}\right|, \left|\frac{\partial u^{(0)}(t, x)}{\partial t}\right| \le \frac{K\varepsilon}{1 + t}, \quad \forall (t, x) \in R_0. \quad (9.3.18)$$

Since $x = x_1(t)$ satisfies (9.1.16) and (9.1.26) and, according to (9.3.16)–(9.3.18), $u_-^{(1)}(t) = u^{(0)}(t, x_1(t))$ satisfies (9.3.2)–(9.3.3) (in which $i = 1$), we can use Lemma 9.3.1 to uniquely determine the value of the right side of $x = x_1(t)$ as $u_+^{(1)} = u_+^{(1)}(t)$ with

$$u_+^{(1)}(0) = \hat{u}^{(1)} \quad (9.3.19)$$

and

$$|u_+^{(1)}(t) - \hat{u}^{(1)}|, \quad \left|\frac{du_+^{(1)}(t)}{dt}\right| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0. \tag{9.3.20}$$

Then, on the domain R_1 [see (9.1.15)] we solve the generalized Cauchy problem for system (9.1.1) with the generalized initial data

$$x = x_1(t): \ u = u_+^{(1)}(t), \quad t \geq 0. \tag{9.3.21}$$

Noting (9.1.16), the entropy condition (9.1.8), and (H_1), we have

$$\lambda_1(\hat{u}^{(1)}) < x_1'(0) < x_2'(0) < \lambda_2(\hat{u}^{(1)}) < \cdots < \lambda_n(\hat{u}^{(1)}). \tag{9.3.22}$$

Noting (9.1.26) (in which we take $i = 1, 2$) and (9.3.19), by Theorem 9.2.1, the generalized Cauchy problem (9.1.1) and (9.3.21) admits a unique global C^1 solution $u = u^{(1)}(t,x)$ on the domain R_1, with

$$|u^{(1)}(t,x) - \hat{u}^{(1)}| \leq \frac{K\varepsilon}{1+x-x_1(t)}, \quad \forall (t,x) \in R_1, \tag{9.3.23}$$

and

$$\left|\frac{\partial u^{(1)}(t,x)}{\partial x}\right|, \ \left|\frac{\partial u^{(1)}(t,x)}{\partial t}\right| \leq \frac{K\varepsilon}{1+x-x_1(t)}, \quad \forall (t,x) \in R_1. \tag{9.3.24}$$

Noting (9.1.6), (9.1.16), and (9.1.26) (in which we take $i = 1, 2$), we have

$$x_2(t) - x_1(t) \geq Ct, \quad \forall t \geq 0,$$

where C is a positive constant independent of t. Then it follows from (9.3.23)–(9.3.24) that $u_-^{(2)}(t) = u^{(1)}(t, x_2(t))$ satisfies (9.3.2)–(9.3.3) (in which $i = 2$). Thus, by Lemma 9.3.1, the value of the right side of $x = x_2(t)$ can be uniquely determined as $u_+^{(2)} = u_+^{(2)}(t)$ with

$$u_+^{(2)}(0) = \hat{u}^{(2)} \tag{9.3.25}$$

and

$$|u_+^{(2)}(t) - \hat{u}^{(2)}|, \ \left|\frac{du_+^{(2)}(t)}{dt}\right| \leq \frac{K\varepsilon}{1+t}, \quad \forall t \geq 0. \tag{9.3.26}$$

Repeating the previous procedure, for any given $i = 2, \ldots, n-1$, we can uniquely determine the value of the right side of $x = x_i(t)$ as $u_+^{(i)} = u_+^{(i)}(t)$ with

$$u_+^{(i)}(0) = \hat{u}^{(i)} \tag{9.3.27}$$

and

$$|u_+^{(i)}(t) - \hat{u}^{(i)}|, \quad \left|\frac{du_+^{(i)}(t)}{dt}\right| \le \frac{K\varepsilon}{1+t}, \quad \forall t \ge 0. \tag{9.3.28}$$

Then for any given $i = 2, \ldots, n-1$, on the domain R_i [see (9.1.15)], we solve the generalized Cauchy problem for system (9.1.1) with the generalized initial data

$$x = x_i(t): \ u = u_+^{(i)}(t), \quad t \ge 0. \tag{9.3.29}$$

Noting (9.1.6) and (9.1.16), the entropy condition (9.1.8), and (H_1), we have

$$\lambda_1(\hat{u}^{(i)}) < \cdots < \lambda_i(\hat{u}^{(i)}) < x_i'(0) < x_{i+1}'(0) < \lambda_{i+1}(\hat{u}^{(i)}) < \cdots < \lambda_n(\hat{u}^{(i)}). \tag{9.3.30}$$

Noting (9.1.26) and (9.3.27), by Theorem 9.2.1, the generalized Cauchy problem (9.1.1) and (9.3.29) admits a unique global C^1 solution $u = u^{(i)}(t,x)$ on the domain R_i, with

$$|u^{(i)}(t,x) - \hat{u}^{(i)}| \le \frac{K\varepsilon}{1+x-x_i(t)}, \quad \forall (t,x) \in R_i, \tag{9.3.31}$$

and

$$\left|\frac{\partial u^{(i)}(t,x)}{\partial x}\right|, \quad \left|\frac{\partial u^{(i)}(t,x)}{\partial t}\right| \le \frac{K\varepsilon}{1+x-x_i(t)}, \quad \forall (t,x) \in R_i. \tag{9.3.32}$$

Finally, since $x = x_n(t)$ satisfies (9.1.16) and (9.1.26) (in which we take $i = n-1, n$), noting (9.1.6), we have

$$x_n(t) - x_{n-1}(t) \ge Ct, \quad \forall t \ge 0,$$

where C is a positive constant independent of t. Then according to (9.3.31)–(9.3.32) (in which $i = n-1$), $u_-^{(n)}(t) = u^{(n-1)}(t, x_n(t))$ satisfies (9.3.2)–(9.3.3) (in which $i = n$). By Lemma 9.3.1, on the right side of $x = x_n(t)$, we can uniquely determine the value of solution $u_+^{(n)} = u_+^{(n)}(t)$ satisfying

$$u_+^{(n)}(0) = \hat{u}^{(n)} \tag{9.3.33}$$

and

$$|u_+^{(n)}(t) - \hat{u}^{(n)}|, \quad \left|\frac{du_+^{(n)}(t)}{dt}\right| \le \frac{K\varepsilon}{1+t}, \quad \forall t \ge 0. \tag{9.3.34}$$

Then we solve the generalized Cauchy problem on the domain R_n [see (9.1.15)] for system (9.1.1) with the generalized initial data

$$x = x_n(t): \ u = u_+^{(n)}(t), \quad t \ge 0. \tag{9.3.35}$$

Noting (9.1.16), the entropy condition (9.1.8), and, (H_1), we have

$$x_n'(0) > \lambda_n(\hat{u}^{(n)}) > \cdots > \lambda_1(\hat{u}^{(n)}). \tag{9.3.36}$$

Using (9.1.26) (in which we take $i = n$) and (9.3.33), by Remark 9.2.2, the generalized Cauchy problem (9.1.1) and (9.3.35) admits a unique global C^1 solution $u = u^{(n)}(t, x)$ on the domain R_n, with

$$|u^{(n)}(t,x) - \hat{u}^{(n)}| \leq \frac{K\varepsilon}{1 + x - x_n(t)}, \quad \forall (t,x) \in R_n, \tag{9.3.37}$$

and

$$\left|\frac{\partial u^{(n)}(t,x)}{\partial x}\right|, \quad \left|\frac{\partial u^{(n)}(t,x)}{\partial t}\right| \leq \frac{K\varepsilon}{1 + x - x_n(t)}, \qquad \forall (t,x) \in R_n. \tag{9.3.38}$$

Thus, noting $x_n(0) = 0$, (9.3.33), and $\hat{u}^{(n)} = u_+$, on the positive x-axis, we see that the initial value $u_r(x) = u^{(n)}(0, x)$ satisfies

$$u_r(0) = u_+ \tag{9.3.39}$$

and

$$|u_r(x) - u_r(0)|, \quad |u_r'(x)| \leq \frac{K\varepsilon}{1 + x}, \quad \forall x \geq 0. \tag{9.3.40}$$

The proof of Theorem 9.1.1 is complete. $\square$

Chapter X
Inverse Piston Problem

10.1 Inverse Piston Problem for the System of One-Dimensional Isentropic Flow

10.1.1 Introduction and Main Results

Suppose that a piston originally located at the origin at $t = 0$ moves with the speed $v_p = \phi(t)$ $(t \geq 0)$ in a tube whose the length is assumed to be infinite and that the gas on the right side of the piston possesses an isentropic state. In order to determine the state of the gas on the right side of this piston, in **Lagrangian representation** this **piston problem** reduces to the following mixed initial-boundary value problem for the system

$$\begin{cases} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(\tau)}{\partial x} = 0, \end{cases} \tag{10.1.1}$$

with the initial data

$$t = 0: \ \tau = \tau_0^+(x) \ (> 0), \quad u = u_0^+(x), \quad x \geq 0, \tag{10.1.2}$$

and the boundary condition

$$x = 0: \ u = \phi(t), \quad t \geq 0, \tag{10.1.3}$$

where τ is the specific volume, u the velocity, and $p = p(\tau)$ the pressure. For polytropic gases,

$$p = p(\tau) = A\tau^{-\gamma}, \quad \forall \tau > 0, \tag{10.1.4}$$

where $\gamma > 1$ is the adiabatic exponent and A is a positive constant.

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Suppose that

$$\phi(0) > u_0^+(0). \tag{10.1.5}$$

The motion of the piston produces a forward shock $x = x_2(t)$ passing through the origin at least for a short time $T_0 > 0$ (see [50], [72], [74]) such that the corresponding piecewise C^1 solution on the domain

$$D(T_0) = \{(t,x)|0 \le t \le T_0, x \ge 0\} \tag{10.1.6}$$

is written as

$$(\tau, u) = \begin{cases} (\tau_0(t,x), u_0(t,x)), & 0 \le x \le x_2(t), \\ (\tau_+(t,x), u_+(t,x)), & x \ge x_2(t), \end{cases} \tag{10.1.7}$$

where $(\tau_0(t,x), u_0(t,x))$, $(\tau_+(t,x), u_+(t,x)) \in C^1$ satisfy system (10.1.1) in the classical sense on their domains, respectively, and verify the Rankine–Hugoniot condition

$$\begin{cases} [\tau]x_2'(t) + [u] = 0, \\ [u]x_2'(t) - [p(\tau)] = 0 \end{cases} \tag{10.1.8}$$

and the entropy condition

$$\begin{cases} \lambda_1(\tau_0(t, x_2(t))) < x_2'(t) < \lambda_2(\tau_0(t, x_2(t))), \\ x_2'(t) > \lambda_2(\tau_+(t, x_2(t))) \end{cases} \tag{10.1.9}$$

on $x = x_2(t)$, in which $[\tau] = \tau_+(t, x_2(t)) - \tau_0(t, x_2(t))$, etc. and

$$-\lambda_1(\tau) = \lambda_2(\tau) = \sqrt{-p'(\tau)}. \tag{10.1.10}$$

Introducing the Riemann invariants

$$\begin{cases} r = \frac{1}{2}(u - \int_\tau^\infty \sqrt{-p'(\eta)}d\eta) = \frac{1}{2}u - \frac{\sqrt{A\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}}, \\ s = \frac{1}{2}(u + \int_\tau^\infty \sqrt{-p'(\eta)}d\eta) = \frac{1}{2}u + \frac{\sqrt{A\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}} \end{cases} \tag{10.1.11}$$

as new unknown functions, we can reduce (10.1.1)–(10.1.3) to the following problem:

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r,s)\dfrac{\partial r}{\partial x} = 0 \\ \dfrac{\partial s}{\partial t} + \mu(r,s)\dfrac{\partial s}{\partial x} = 0, \end{cases} \tag{10.1.12}$$

$$t = 0: \ (r,s) = (r_0^+(x), s_0^+(x)), \quad x \ge 0, \tag{10.1.13}$$

and

$$x = 0: \; s = -r + \phi(t), \quad t \geq 0, \tag{10.1.14}$$

where

$$\begin{cases} r_0^+(x) = \frac{1}{2}u_0^+(x) - \frac{\sqrt{A\gamma}}{\gamma-1}(\tau_0^+(x))^{-\frac{\gamma-1}{2}}, \\ s_0^+(x) = \frac{1}{2}u_0^+(x) + \frac{\sqrt{A\gamma}}{\gamma-1}(\tau_0^+(x))^{-\frac{\gamma-1}{2}}, \end{cases} \tag{10.1.15}$$

with

$$s_0^+(x) - r_0^+(x) > 0, \quad \forall x \geq 0, \tag{10.1.16}$$

and

$$-\lambda(r,s) = \mu(r,s) = \sqrt{-p'(\tau(s-r))} = \frac{\left(\frac{\gamma-1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}}{(A\gamma)^{\frac{1}{\gamma-1}}}(s-r)^{\frac{\gamma+1}{\gamma-1}}. \tag{10.1.17}$$

In the special case that the piston moves with a constant speed u_p and the initial state is a constant state $(\bar{\tau}_+, \bar{u}_+)$ $(\bar{\tau}_+ > 0)$ with

$$u_p > \bar{u}_+, \tag{10.1.18}$$

(10.1.13) and (10.1.14) become, respectively,

$$t = 0: \; (r,s) = (\bar{r}_+, \bar{s}_+), \quad x \geq 0, \tag{10.1.19}$$

and

$$x = 0: \; s = -r + u_p, \quad t \geq 0, \tag{10.1.20}$$

where

$$\begin{cases} \bar{r}_+ = \frac{1}{2}\bar{u}_+ - \frac{\sqrt{A\gamma}}{\gamma-1}(\bar{\tau}_+)^{-\frac{\gamma-1}{2}}, \\ \bar{s}_+ = \frac{1}{2}\bar{u}_+ + \frac{\sqrt{A\gamma}}{\gamma-1}(\bar{\tau}_+)^{-\frac{\gamma-1}{2}}, \end{cases} \tag{10.1.21}$$

with

$$\bar{s}_+ - \bar{r}_+ > 0 \tag{10.1.22}$$

and

$$u_p > \bar{r}_+ + \bar{s}_+. \tag{10.1.23}$$

The solution to the previous problem is the typical forward shock

$$(r,s) = \begin{cases} (r_0, s_0), & 0 \leq x \leq Vt, \\ (\bar{r}_+, \bar{s}_+), & x \geq Vt, \end{cases} \tag{10.1.24}$$

where V is the speed of propagation of the typical forward shock:

$$V = G(\bar{r}_+, \bar{s}_+, r_0, s_0) \tag{10.1.25}$$

satisfying the entropy condition

$$\begin{cases} \lambda(r_0, s_0) < V < \mu(r_0, s_0), \\ V > \mu(\bar{r}_+, \bar{s}_+), \end{cases} \tag{10.1.26}$$

in which r_0, s_0, and (10.1.25) are uniquely determined by

$$r_0 + s_0 = u_p \tag{10.1.27}$$

and the Rankine–Hugoniot condition:

$$\begin{aligned} &(r_0 + s_0) - (\bar{r}_+ + \bar{s}_+) \\ &= \sqrt{-(p(\tau(s_0 - r_0)) - p(\tau(\bar{s}_+ - \bar{r}_+)))(\tau(s_0 - r_0) - \tau(\bar{s}_+ - \bar{r}_+))}, \end{aligned} \tag{10.1.28}$$

$$V = \sqrt{-\frac{p(\tau(s_0 - r_0)) - p(\tau(\bar{s}_+ - \bar{r}_+))}{\tau(s_0 - r_0) - \tau(\bar{s}_+ - \bar{r}_+)}} \tag{10.1.29}$$

(see [22] or [50]).

As a perturbation of the simplest piston problem mentioned above, in [50] and [74], the piston problem (10.1.1)–(10.1.3) is globally studied and we have the following:

Proposition 10.1.1 *Suppose that $\tau_0^+(x)$, $u_0^+(x)$, and $\phi(t) \in C^1$ and*

$$\tau_0^+(0) = \bar{\tau}_+, \quad u_0^+(0) = \bar{u}_+, \quad \phi(0) = u_p. \tag{10.1.30}$$

Suppose furthermore that

$$|\tau_0^+(x) - \bar{\tau}_+|,\ |u_0^+(x) - \bar{u}_+| \le \varepsilon, \quad \forall x \ge 0, \tag{10.1.31}$$

$$|\phi(t) - u_p| \le \varepsilon, \quad \forall t \ge 0, \tag{10.1.32}$$

$$|\tau_0^{+'}(x)|,\ |u_0^{+'}(x)| \le \frac{\eta}{1+x}, \quad \forall x \ge 0, \tag{10.1.33}$$

and

$$|\phi'(t)| \le \frac{\eta}{1+t}, \quad \forall t \ge 0, \tag{10.1.34}$$

where $\varepsilon > 0$ *and* $\eta > 0$ *are suitably small. Then the piston problem* (10.1.1)–(10.1.3) *admits a unique global piecewise* C^1 *solution*

$$(\tau(t,x), u(t,x)) = \begin{cases} (\tau_0(t,x), u_0(t,x)), & 0 \le x \le x_2(t), \\ (\tau_+(t,x), u_+(t,x)), & x \ge x_2(t), \end{cases} \tag{10.1.35}$$

on the domain

$$D = \{(t,x) | t \ge 0,\ x \ge 0\}. \tag{10.1.36}$$

This solution, containing only one forward shock $x = x_2(t)$ *passing through the origin with* $x_2'(0) = V$*, satisfies the following estimates: On the domain*

$$D_+ = \{(t,x) | t \ge 0,\ x \ge x_2(t)\}, \tag{10.1.37}$$

we have

$$|\tau_+(t,x) - \bar{\tau}_+|,\ |u_+(t,x) - \bar{u}_+| \le K_1\varepsilon, \tag{10.1.38}$$

$$\left|\frac{\partial \tau_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial \tau_+}{\partial t}(t,x)\right|,\ \left|\frac{\partial u_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial u_+}{\partial t}(t,x)\right| \le \frac{K_2\eta}{1+t}; \tag{10.1.39}$$

on the domain

$$D_- = \{(t,x) | t \ge 0,\ 0 \le x \le x_2(t)\}, \tag{10.1.40}$$

we have

$$|\tau_0(t,x) - \tau_0|,\ |u_0(t,x) - u_p| \le K_3\varepsilon, \tag{10.1.41}$$

$$\left|\frac{\partial \tau_0}{\partial x}(t,x)\right|,\ \left|\frac{\partial \tau_0}{\partial t}(t,x)\right|,\ \left|\frac{\partial u_0}{\partial x}(t,x)\right|,\ \left|\frac{\partial u_0}{\partial t}(t,x)\right| \le \frac{K_4\eta}{1+t}, \tag{10.1.42}$$

where τ_0 *is determined by* (10.1.11) (*in which* $r = r_0$, $s = s_0$, *and* $u = u_p$). *Additionally,*

$$|x_2'(t) - V| \le K_5\varepsilon, \quad \forall t \ge 0, \tag{10.1.43}$$

$$|x_2''(t)| \le \frac{K_6\eta}{1+t}, \quad \forall t \ge 0. \tag{10.1.44}$$

Here and henceforth, K_i $(i = 1, 2, \ldots)$ *are positive constants independent of* ε *and* η.

In this section we consider the corresponding **inverse piston problem**: Suppose that we know the original state $(\tau_0^+(x), u_0^+(x))$ of the gas on the right side of this piston and the position of the forward shock $x = x_2(t) \in C^2$ with

$$x_2(0) = 0 \tag{10.1.45}$$

and

$$x_2'(0) = V \tag{10.1.46}$$

Then can we determine the piston velocity $v_p = \phi(t)$?

For the supersonic plane flow past a curved wedge, the direct problem aims to determine the shock from the given curved wedge, while the target in the inverse problem is to determine the curved wedge from the given shock (see [47], [48], [72], [86]). The problem under consideration in this section can be regarded as an analogue in one-dimensional gas dynamics in which the moving piston corresponds to the curved wedge.

As in [53], this problem can be easily solved in the local sense. In this section we will give an affirmative answer to this problem in the global sense. We have

Theorem 10.1.1 *Suppose that the position of the forward shock $x = x_2(t) \in C^2$ $(t \geq 0)$ with* (10.1.45)–(10.1.46) *is prescribed and, for suitably small $\varepsilon > 0$ and $\eta > 0$, we have*

$$|x_2'(t) - V| \leq \varepsilon, \quad \forall t \geq 0, \tag{10.1.47}$$

and

$$|x_2''(t)| \leq \frac{\eta}{1+t}, \quad \forall t \geq 0, \tag{10.1.48}$$

where V satisfies (10.1.25)–(10.1.26)*. Then for any given $\tau_0^+(x)$ and $u_0^+(x) \in C^1$ $(x \geq 0)$ satisfying*

$$\tau_0^+(0) = \bar{\tau}_+, \ u_0^+(0) = \bar{u}_+, \tag{10.1.49}$$

$$|\tau_0^+(x) - \bar{\tau}_+|, \ |u_0^+(x) - \bar{u}_+| \leq \varepsilon, \quad \forall x \geq 0, \tag{10.1.50}$$

and

$$|{\tau_0^+}'(x)|, \ |{u_0^+}'(x)| \leq \frac{\eta}{1+x}, \quad \forall x \geq 0, \tag{10.1.51}$$

we can uniquely determine the piston velocity $v_p = \phi(t)$ $(t \geq 0)$ with

$$\phi(0) = u_p, \tag{10.1.52}$$

$$|\phi(t) - u_p| \leq K_7\varepsilon, \quad \forall t \geq 0, \tag{10.1.53}$$

and

$$|\phi'(t)| \leq \frac{K_8\eta}{1+t}, \quad \forall t \geq 0, \tag{10.1.54}$$

where u_p *is the same as in* (10.1.18)*, such that by Proposition 10.1.1, the corresponding direct piston problem* (10.1.1)–(10.1.3) *admits a unique global piecewise* C^1 *solution* $(\tau(t,x), u(t,x))$ *in which the forward shock passing through the origin is just* $x = x_2(t)$*.*

Remark 10.1.1 *The inverse piston problem under consideration can be regarded as a perturbation of the simplest inverse piston problem: to determine the constant piston velocity under the condition that the constant speed* V *of propagation of the forward typical shock and the constant initial state* $(\bar{\tau}_+, \bar{u}_+)$ *of the gas on the right side of the piston are given.*

Remark 10.1.2 *The results in this section can be found in* [71]*.*

Theorem 10.1.1 is proved in Section 10.1.2. Then the corresponding discussion in Eulerian representation is given in Section 10.1.3.

10.1.2 Proof of Theorem 10.1.1

By (10.1.15) and (10.1.21), it follows from (10.1.49)–(10.1.51) that

$$r_0^+(0) = \bar{r}_+, \ s_0^+(0) = \bar{s}_+, \tag{10.1.55}$$

$$|r_0^+(x) - \bar{r}_+|, \ |s_0^+(x) - \bar{s}_+| \le C_1\varepsilon, \quad \forall x \ge 0, \tag{10.1.56}$$

and

$$|r_0^{+'}(x)|, \ |s_0^{+'}(x)| \le \frac{C_2\eta}{1+x}, \quad \forall x \ge 0. \tag{10.1.57}$$

Here and henceforth, C_i $(i = 1, 2, \ldots)$ are positive constants independent of ε and η.

Lemma 10.1.1 *Suppose that* (10.1.55)–(10.1.57) *hold for suitably small* $\varepsilon > 0$ *and* $\eta > 0$*. Then the Cauchy problem*

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r,s)\dfrac{\partial r}{\partial x} = 0, \\[2ex] \dfrac{\partial s}{\partial t} + \mu(r,s)\dfrac{\partial s}{\partial x} = 0, \\[2ex] t = 0: \ (r,s) = (r_0^+(x), s_0^+(x)), \quad x \ge 0, \end{cases} \tag{10.1.58}$$

admits a unique global C^1 solution $(r, s) = (\widetilde{r}_+(t, x), \widetilde{s}_+(t, x))$ on the domain

$$\hat{D}_+ = \{(t, x) | t \geq 0,\ x \geq \xi t\}, \tag{10.1.59}$$

where ξ is a constant satisfying

$$\xi > \mu(\bar{r}_+, \bar{s}_+). \tag{10.1.60}$$

Moreover, we have

$$\widetilde{s}_+(t, x) - \widetilde{r}_+(t, x) > 0, \quad \forall (t, x) \in \hat{D}_+, \tag{10.1.61}$$

$$|\widetilde{r}_+(t, x) - \bar{r}_+|,\ |\widetilde{s}_+(t, x) - \bar{s}_+| \leq K_9 \varepsilon, \quad \forall (t, x) \in \hat{D}_+, \tag{10.1.62}$$

and

$$\left|\frac{\partial \widetilde{r}_+}{\partial x}(t, x)\right|, \ \left|\frac{\partial \widetilde{r}_+}{\partial t}(t, x)\right|, \ \left|\frac{\partial \widetilde{s}_+}{\partial x}(t, x)\right|, \ \left|\frac{\partial \widetilde{s}_+}{\partial t}(t, x)\right| \leq \frac{K_{10}\eta}{1+t}, \quad \forall (t, x) \in \hat{D}_+. \tag{10.1.63}$$

Proof. Similarly to the proof of Lemma 9.2.1, it is easy to get (10.1.62)–(10.1.63). Then, noting that $\bar{s}_+ - \bar{r}_+ > 0$ and that $\varepsilon > 0$ is suitably small, we have (10.1.61).

Remark 10.1.3 *The result of Lemma 10.1.1 can be found in Chapter 6 of* [50].

The proof of Theorem 10.1.1 is as follows.

Step 1. We first solve the Cauchy problem (10.1.58) on the domain D_+ defined by (10.1.37).

Let

$$\xi = \frac{1}{2}(V + \mu(\bar{r}_+, \bar{s}_+)). \tag{10.1.64}$$

Noting (10.1.26), we have

$$V > \xi > \mu(\bar{r}_+, \bar{s}_+). \tag{10.1.65}$$

Hence, for suitably small $\varepsilon > 0$, by (10.1.45)–(10.1.47), we have

$$D_+ \subseteq \hat{D}_+.$$

Hence, by Lemma 10.1.1, the Cauchy problem (10.1.58) admits a unique global C^1 solution $(r, s) = (r_+(t, x), s_+(t, x))$ on the domain D_+ and we have

$$s_+(t, x) - r_+(t, x) > 0, \quad \forall (t, x) \in D_+, \tag{10.1.66}$$

$$|r_+(t, x) - \bar{r}_+|,\ |s_+(t, x) - \bar{s}_+| \leq C_3 \varepsilon, \quad \forall (t, x) \in D_+, \tag{10.1.67}$$

and

$$\left|\frac{\partial r_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial r_+}{\partial t}(t,x)\right|,\ \left|\frac{\partial s_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial s_+}{\partial t}(t,x)\right| \le \frac{C_4\eta}{1+t}, \quad \forall (t,x)\in D_+. \tag{10.1.68}$$

Then we obtain the value of (r,s) on the right side of $x = x_2(t)$:

$$(r,s) = (\widetilde{r}_+(t), \widetilde{s}_+(t)) = (r_+(t, x_2(t)), s_+(t, x_2(t))), \quad \forall t \ge 0, \tag{10.1.69}$$

and we have

$$\widetilde{r}_+(0) = \bar{r}_+, \quad \widetilde{s}_+(0) = \bar{s}_+, \tag{10.1.70}$$

$$\widetilde{s}_+(t) - \widetilde{r}_+(t) > 0, \quad \forall t \ge 0, \tag{10.1.71}$$

$$|\widetilde{r}_+(t) - \bar{r}_+|,\ |\widetilde{s}_+(t) - \bar{s}_+| \le C_3\varepsilon, \quad \forall t \ge 0. \tag{10.1.72}$$

In additions, noting (10.1.47), we also have

$$\left|\frac{d\widetilde{r}_+(t)}{dt}\right|,\ \left|\frac{d\widetilde{s}_+(t)}{dt}\right| \le \frac{C_5\eta}{1+t}, \quad \forall t \ge 0. \tag{10.1.73}$$

Step 2. By the Rankine–Hugoniot condition, we now find the value of (r,s) on the left side of $x = x_2(t)$.

On the forward shock $x = x_2(t)$, the Rankine–Hugoniot condition is written as

$$\begin{cases} [\tau(s-r)]x_2'(t) + [r+s] = 0, \\ [r+s]x_2'(t) - [p(\tau(s-r))] = 0, \end{cases} \tag{10.1.74}$$

where $[\tau] = \tau(t, x_2(t)+0) - \tau(t, x_2(t)-0)$, etc. Denoting the value of (r,s) on the left side of $x = x_2(t)$ as $(r,s) = (\widetilde{r}_-(t), \widetilde{s}_-(t))$ and $x_2'(t) = d$, we can rewrite (10.1.74) as

$$\begin{cases} (\tau(\widetilde{s}_- - \widetilde{r}_-) - \tau(\widetilde{s}_+ - \widetilde{r}_+))d + (\widetilde{r}_- + \widetilde{s}_- - \widetilde{r}_+ - \widetilde{s}_+) = 0, \\ (\widetilde{r}_- + \widetilde{s}_- - \widetilde{r}_+ - \widetilde{s}_+)d - (p(\tau(\widetilde{s}_- - \widetilde{r}_-)) - p(\tau(\widetilde{s}_+ - \widetilde{r}_+))) = 0. \end{cases} \tag{10.1.75}$$

Similarly to the proof of Lemma 9.3.1 (also see [53] and [68]), in a neighbourhood of $(\bar{r}_+, \bar{s}_+, r_0, s_0, V)$, (10.1.75) can be rewritten as

$$\begin{cases} \widetilde{r}_- = g(\widetilde{r}_+, \widetilde{s}_+, d), \\ \widetilde{s}_- = h(\widetilde{r}_+, \widetilde{s}_+, d), \end{cases} \tag{10.1.76}$$

where $g(\cdot), h(\cdot) \in C^2$ and

$$\begin{cases} r_0 = g(\bar{r}_+, \bar{s}_+, V), \\ s_0 = h(\bar{r}_+, \bar{s}_+, V). \end{cases} \tag{10.1.77}$$

Hence, the value of (r, s) on the left side of $x = x_2(t)$ can be uniquely determined as

$$\begin{cases} r = \widetilde{r}_-(t) = g(\widetilde{r}_+(t), \widetilde{s}_+(t), x_2'(t)), \\ s = \widetilde{s}_-(t) = h(\widetilde{r}_+(t), \widetilde{s}_+(t), x_2'(t)). \end{cases} \tag{10.1.78}$$

Moreover, noting (10.1.45)–(10.1.48), (10.1.70), (10.1.72), (10.1.73), and (10.1.77), we have

$$\widetilde{r}_-(0) = r_0, \quad \widetilde{s}_-(0) = s_0, \tag{10.1.79}$$

$$|\widetilde{r}_-(t) - r_0|, \ |\widetilde{s}_-(t) - s_0| \le C_6\varepsilon, \quad \forall t \ge 0, \tag{10.1.80}$$

and

$$|\frac{d\widetilde{r}_-(t)}{dt}|, \ |\frac{d\widetilde{s}_-(t)}{dt}| \le \frac{C_7\eta}{1+t}, \quad \forall t \ge 0. \tag{10.1.81}$$

Also, noting (10.1.26), (10.1.46), and (10.1.47), we have

$$\lambda(\widetilde{r}_-(t), \widetilde{s}_-(t)) < x_2'(t) < \mu(\widetilde{r}_-(t), \widetilde{s}_-(t)). \tag{10.1.82}$$

Step 3. We finally solve the generalized Cauchy problem

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r, s)\dfrac{\partial r}{\partial x} = 0, \\ \\ \dfrac{\partial s}{\partial t} + \mu(r, s)\dfrac{\partial s}{\partial x} = 0, \\ \\ x = x_2(t): \ (r, s) = (\widetilde{r}_-(t), \widetilde{s}_-(t)), \quad t \ge 0, \end{cases} \tag{10.1.83}$$

on the domain D_- defined by (10.1.40).

Noting (10.1.82), we see that the generalized Cauchy problem (10.1.83) always admits a unique local C^1 solution $(r, s) = (r_-(t, x), s_-(t, x))$ (see [72]). For the time being, we assume that on any existence domain of $(r, s) = (r_-(t, x), s_-(t, x))$, we have

$$|r_-(t, x) - r_0|, \ |s_-(t, x) - s_0| \le \delta, \tag{10.1.84}$$

where $\delta > 0$ is suitably small. At the end of the proof, we will why this hypothesis is reasonable.

Let

$$\begin{cases} \bar{t} = x_2(t) - x, \\ \bar{x} = t. \end{cases} \tag{10.1.85}$$

Then, noting (10.1.26), (10.1.45)–(10.1.47), and (10.1.84), we can reduce the generalized Cauchy problem (10.1.83) on the domain D_- to the following Cauchy problem:

$$\begin{cases} \dfrac{\partial \bar{r}}{\partial \bar{t}} + \bar{\lambda}(\bar{x}, \bar{r}, \bar{s}) \dfrac{\partial \bar{r}}{\partial \bar{x}} = 0, \\ \dfrac{\partial \bar{s}}{\partial \bar{t}} + \bar{\mu}(\bar{x}, \bar{r}, \bar{s}) \dfrac{\partial \bar{s}}{\partial \bar{x}} = 0, \\ \bar{t} = 0: \ (\bar{r}, \bar{s}) = (\widetilde{r}_-(\bar{x}), \widetilde{s}_-(\bar{x})), \quad \bar{x} \geq 0, \end{cases} \tag{10.1.86}$$

on the domain $\overline{D}_- = \{(\bar{t}, \bar{x}) | \bar{t} \geq 0, \ \bar{x} \geq \theta(\bar{t})\}$, where

$$(\bar{r}(\bar{t}, \bar{x}), \bar{s}(\bar{t}, \bar{x})) = (r_-(\bar{x}, x_2(\bar{x}) - \bar{t}), s_-(\bar{x}, x_2(\bar{x}) - \bar{t})), \tag{10.1.87}$$

$$\bar{\lambda}(\bar{x}, \bar{r}, \bar{s}) = \frac{1}{x_2'(\bar{x}) - \lambda(\bar{r}, \bar{s})}, \tag{10.1.88}$$

$$\bar{\mu}(\bar{x}, \bar{r}, \bar{s}) = \frac{1}{x_2'(\bar{x}) - \mu(\bar{r}, \bar{s})}, \tag{10.1.89}$$

and $\bar{x} = \theta(\bar{t}) \in C^2$ with $\theta(0) = 0$ is determined by

$$x_2(\bar{x}) = \bar{t}. \tag{10.1.90}$$

Additionally, by (10.1.26), (10.1.46), (10.1.47), and (10.1.84), we have

$$\frac{1}{x_2'(\bar{x})} > \bar{\lambda}(\bar{x}, \bar{r}, \bar{s}) > \bar{\mu}(\bar{x}, \bar{r}, \bar{s}), \tag{10.1.91}$$

and it follows from (10.1.80)–(10.1.81) that

$$|\widetilde{r}_-(\bar{x}) - r_0|, \ |\widetilde{s}_-(\bar{x}) - s_0| \leq C_8 \varepsilon, \quad \forall \bar{x} \geq 0, \tag{10.1.92}$$

and

$$|\widetilde{r}_-'(\bar{x})|, \ |\widetilde{s}_-'(\bar{x})| \leq \frac{C_9 \eta}{1 + \bar{x}}, \quad \forall \bar{x} \geq 0. \tag{10.1.93}$$

Obviously, problem (10.1.86) admits a unique local C^1 solution $(\bar{r}, \bar{s}) = (\bar{r}(\bar{t}, \bar{x}), \bar{s}(\bar{t}, \bar{x})) = (r_-(\bar{x}, x_2(\bar{x}) - \bar{t}), s_-(\bar{x}, x_2(\bar{x}) - \bar{t}))$ on the domain $\overline{D}_-(\delta_0) = \{(\bar{t}, \bar{x}) | 0 \leq \bar{t} \leq \delta_0, \ \bar{x} \geq \theta(\bar{t})\}$, where $\delta_0 > 0$ is a small number (see [72]). In order to get the global existence of the C^1 solution on $\overline{D}_-$, we need a uniform a priori estimate on the C^1 norm of the C^1 solution $(\bar{r}(\bar{t},\bar{x}),\bar{s}(\bar{t},\bar{x}))$ on any existence domain $\overline{D}_-(T)$.

Noting (10.1.80), we have

$$|\bar{r}(\bar{t},\bar{x})-r_0|,\ |\bar{s}(\bar{t},\bar{x})-s_0|\le C_6\varepsilon,\quad \forall(\bar{t},\bar{x})\in\overline{D}_-(T). \tag{10.1.94}$$

In what follows, we want to get a uniform a priori estimate on the C^0 norm of $\partial\bar{r}/\partial\bar{x}$, $\partial\bar{r}/\partial\bar{t}$, $\partial\bar{s}/\partial\bar{x}$ and $\partial\bar{s}/\partial\bar{t}$ on $\overline{D}_-(T)$. For this purpose, since the system in (10.1.86) depends explicitly on $\bar{x}$, differently from the usual Lax transformation, we introduce

$$w=e^{q(\bar{r},\bar{s})}\frac{\partial\bar{r}}{\partial\bar{t}}, \tag{10.1.95}$$

where $q(\bar{r},\bar{s})\in C^1$ satisfies

$$\frac{\partial q}{\partial\bar{s}}=\frac{1}{\lambda(\bar{r},\bar{s})-\mu(\bar{r},\bar{s})}\frac{\partial\lambda}{\partial\bar{s}}. \tag{10.1.96}$$

By (10.1.86)–(10.1.89), it is easy to get

$$\begin{cases}\dfrac{\partial w}{\partial\bar{t}}+\bar{\lambda}(\bar{x},\bar{r},\bar{s})\dfrac{\partial w}{\partial\bar{x}}=\dfrac{\partial\lambda(\bar{r},\bar{s})}{\partial\bar{r}}\bar{\lambda}(\bar{x},\bar{r},\bar{s})e^{-q(\bar{r},\bar{s})}w^2,\\ \bar{t}=0:\ w=-e^{q(\widetilde{r}_-(\bar{x}),\widetilde{s}_-(\bar{x}))}\bar{\lambda}(\bar{x},\widetilde{r}_-(\bar{x}),\widetilde{s}_-(\bar{x}))\widetilde{r}'_-(\bar{x}),\quad \bar{x}\ge 0.\end{cases} \tag{10.1.97}$$

By (10.1.91), each characteristic passing through any given point $(\bar{t},\bar{x})=(0,\beta)(\beta\ge 0)$ intersects the boundary $\bar{x}=\theta(\bar{t})$ $(\bar{t}\ge 0)$ of $\overline{D}_-$ in a finite time. Let $\bar{x}=\bar{x}_1(\bar{t},\beta)$ be the forward characteristic passing through a point $(0,\beta)$ and let $(\bar{T},\bar{x}_1(\bar{T},\beta))$ be the intersection point of $\bar{x}=\bar{x}_1(\bar{t},\beta)$ with $\bar{x}=\theta(\bar{t})$.

Noting (10.1.45), (10.1.47), (10.1.84), and (10.1.87), for suitably small $\varepsilon>0$ and $\delta>0$, we have

$$\begin{aligned}\frac{\bar{T}}{V-\frac14\lambda(r_0,s_0)}\le\bar{x}_1(\bar{T},\beta)=\beta+\int_0^{\bar{T}}\bar{\lambda}(\bar{x},\bar{r},\bar{s})(\tau,\bar{x}_1(\tau,\beta))\,d\tau\le\beta\\ +\frac{\bar{T}}{V-\frac12\lambda(r_0,s_0)}.\end{aligned} \tag{10.1.98}$$

Hence,

$$\bar{T}\le M_0\beta, \tag{10.1.99}$$

where

$$M_0=\frac{(4V-\lambda(r_0,s_0))(2V-\lambda(r_0,s_0))}{-2\lambda(r_0,s_0)}>0. \tag{10.1.100}$$

Noting (10.1.86), on $\bar{x} = \bar{x}_1(\bar{t}, \beta)$ we have

$$\bar{r}(\bar{t}, \bar{x}) = \bar{r}(\bar{t}, \bar{x}_1(\bar{t}, \beta)) = \widetilde{r}_-(\beta) \tag{10.1.101}$$

and

$$\bar{s}(\bar{t}, \bar{x}) = \bar{s}(\bar{t}, \bar{x}_1(\bar{t}, \beta)) = \widetilde{s}_-(\alpha(\bar{t}, \beta)), \tag{10.1.102}$$

where $\alpha(\bar{t}, \beta)$ is the $\bar{x}$-coordinate of the intersection point of the backward characteristic passing through $(\bar{t}, \bar{x}_1(\bar{t}, \beta))$ with the $\bar{x}$-axis. Then it follows from (10.1.97) that on $\bar{x} = \bar{x}_1(\bar{t}, \beta)$ we have

$$w(\bar{t}, \bar{x}) = w(\bar{t}, \bar{x}_1(\bar{t}, \beta)) = \frac{-e^{q(\widetilde{r}_-(\beta), \widetilde{s}_-(\beta))} \bar{\lambda}(\beta, \widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) \widetilde{r}'_-(\beta)}{1 + B}, \tag{10.1.103}$$

where

$$B = \int_0^{\bar{t}} \frac{\partial \lambda}{\partial \bar{r}} (\widetilde{r}_-(\beta), \bar{s}(\tau, \bar{x}_1(\tau, \beta))) \bar{\lambda}(\beta, \widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) \bar{\lambda}(\bar{x}_1(\tau, \beta), \widetilde{r}_-(\beta), \\ \bar{s}(\tau, \bar{x}_1(\tau, \beta))) \cdot \widetilde{r}'_-(\beta) e^{q(\widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) - q(\widetilde{r}_-(\beta), \bar{s}(\tau, \bar{x}_1(\tau, \beta)))} \, d\tau. \tag{10.1.104}$$

Hence, by (10.1.95), we get

$$\frac{\partial \bar{r}}{\partial \bar{t}} (\bar{t}, \bar{x}_1(\bar{t}, \beta)) = \frac{-e^{q(\widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) - q(\widetilde{r}_-(\beta), \bar{s}(\bar{t}, \bar{x}_1(\bar{t}, \beta)))} \bar{\lambda}(\beta, \widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) \widetilde{r}'_-(\beta)}{1 + B}. \tag{10.1.105}$$

By (10.1.47), (10.1.92), (10.1.101), and (10.1.102) and noting (10.1.26), on $\bar{x} = \bar{x}_1(\bar{t}, \beta)$ we have

$$\frac{1}{2}(V - \lambda(r_0, s_0)) < x'_2(\bar{x}) - \lambda(\bar{r}, \bar{s}) < 2(V - \lambda(r_0, s_0)). \tag{10.1.106}$$

Then, noting (10.1.92), (10.1.101), and (10.1.102), we get

$$e^{2|q|} |\bar{\lambda}| \le M_1, \qquad \left| \frac{\partial \lambda}{\partial \bar{r}} \right| \bar{\lambda}^2 e^{2|q|} \le M_2, \tag{10.1.107}$$

where M_1 and M_2 are two positive constants independent of ε and η.

We choose $\eta > 0$ so small that

$$C_9 M_0 M_2 \eta < \frac{1}{2}. \tag{10.1.108}$$

Then it follows from (10.1.105) that

$$\left| \frac{\partial \bar{r}}{\partial \bar{t}} (\bar{t}, \bar{x}_1(\bar{t}, \beta)) \right| \le M_1 \frac{C_9 \eta}{1 + \beta} \left(1 - M_2 \frac{C_9 \eta}{1 + \beta} \bar{T} \right)^{-1}. \tag{10.1.109}$$

Thus, noting (10.1.99) and (10.1.108), we get

$$\begin{aligned}\left|\frac{\partial \bar{r}}{\partial \bar{t}}(\bar{t}, \bar{x}_1(\bar{t}, \beta))\right| &\le C_9 M_1 \frac{\eta}{1+\beta}(1 - C_9 M_0 M_2 \eta)^{-1} \\ &\le 2C_9 M_1 \frac{\eta}{1+\beta} \\ &\le \frac{C_{10}\eta}{1+\bar{t}}, \quad 0 \le \bar{t} \le T. \end{aligned} \tag{10.1.110}$$

Hence, noting (10.1.88) and (10.1.106), we get

$$\left|\frac{\partial \bar{r}}{\partial \bar{x}}(\bar{t}, \bar{x}_1(\bar{t}, \beta))\right| \le \frac{C_{11}\eta}{1+\bar{t}}, \quad 0 \le \bar{t} \le T. \tag{10.1.111}$$

Finally, we obtain

$$\left|\frac{\partial \bar{r}}{\partial \bar{t}}(\bar{t}, \bar{x})\right|, \ \left|\frac{\partial \bar{r}}{\partial \bar{x}}(\bar{t}, \bar{x})\right| \le \frac{C_{12}\eta}{1+\bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \overline{D}_-(T). \tag{10.1.112}$$

Similarly, we have

$$\left|\frac{\partial \bar{s}}{\partial \bar{t}}(\bar{t}, \bar{x})\right|, \ \left|\frac{\partial \bar{s}}{\partial \bar{x}}(\bar{t}, \bar{x})\right| \le \frac{C_{13}\eta}{1+\bar{t}}, \quad \forall (\bar{t}, \bar{x}) \in \overline{D}_-(T). \tag{10.1.113}$$

Thus, we get a unique global C^1 solution $(\bar{r}, \bar{s}) = (\bar{r}(\bar{t}, \bar{x}), \bar{s}(\bar{t}, \bar{x}))$ to (10.1.86) on $\overline{D}_-$. Noting (10.1.85), for the generalized Cauchy problem (10.1.83), we obtain the unique global C^1 solution

$$(r, s) = (r_-(t, x), s_-(t, x)) = (\bar{r}(x_2(t) - x, t), \bar{s}(x_2(t) - x, t)) \tag{10.1.114}$$

on the domain D_-. Noting (10.1.94) and (10.1.114), we immediately obtain

$$|r_-(t, x) - r_0|, \ |s_-(t, x) - s_0| \le C_{14}\varepsilon, \quad \forall (t, x) \in D_-, \tag{10.1.115}$$

which also implies that hypothesis (10.1.84) is reasonable. In addition, noting (10.1.47), it follows from (10.1.112)–(10.1.113) that

$$\left|\frac{\partial r_-}{\partial t}(t, x)\right|, \ \left|\frac{\partial r_-}{\partial x}(t, x)\right|, \ \left|\frac{\partial s_-}{\partial t}(t, x)\right|, \ \left|\frac{\partial s_-}{\partial x}(t, x)\right| \le \frac{C_{15}\eta}{1+t}, \quad \forall (t, x) \in D_-. \tag{10.1.116}$$

Hence, we get the piston velocity

$$\phi(t) = r_-(t, 0) + s_-(t, 0), \quad t \ge 0. \tag{10.1.117}$$

Moreover, noting (10.1.79) and (10.1.27), we see that (10.1.52)–(10.1.54) hold.

Theorem 10.1.1 is then proved.

10.1.3 Related Problem in Eulerian Representation

In **Eulerian representation**, the system of one-dimensional isentropic flow is written as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho u)}{\partial x} = 0, \\ \dfrac{\partial (\rho u)}{\partial t} + \dfrac{\partial (\rho u^2 + p(\rho))}{\partial x} = 0, \end{cases} \tag{10.1.118}$$

where ρ is the density, u the velocity, and $p = p(\rho)$ the pressure. For polytropic gases,

$$p = p(\rho) = A\rho^{\gamma}, \quad \forall \rho > 0, \tag{10.1.119}$$

where $\gamma > 1$ is the adiabatic exponent and A is a positive constant. In this situation the corresponding piston problem asks us to solve the following mixed initial-boundary value problem for system (10.1.118) with the initial data

$$t = 0: \ \rho = \rho_0^+(x) \ (> 0), \quad u = u_0^+(x), \quad x \geq 0, \tag{10.1.120}$$

and the boundary condition

$$x = f(t): \ u = \varphi(t), \quad t \geq 0, \tag{10.1.121}$$

with

$$f(t) = \int_0^t \varphi(\xi) d\xi. \tag{10.1.122}$$

Suppose that

$$\varphi(0) > u_0^+(0). \tag{10.1.123}$$

Then the motion of the piston produces a forward shock $x = x_f(t)$ passing through the origin at least for a short time T_1 (see [72]) such that the corresponding piecewise C^1 solution on the domain

$$\varOmega(T_1) = \{(t,x) | 0 \leq t \leq T_1, x \geq f(t)\} \tag{10.1.124}$$

is written as

$$(\rho, u) = \begin{cases} (\rho_0(t,x), u_0(t,x)), & f(t) \leq x \leq x_f(t), \\ (\rho_+(t,x), u_+(t,x)), & x \geq x_f(t), \end{cases} \tag{10.1.125}$$

where $(\rho_0(t,x), u_0(t,x))$, $(\rho_+(t,x), u_+(t,x)) \in C^1$ satisfy system (10.1.118) in the classical sense on their domains, respectively, and verify the

Rankine–Hugoniot condition

$$\begin{cases} [\rho]x_f'(t) - [\rho u] = 0, \\ [\rho u]x_f'(t) - [\rho u^2 + p(\rho)] = 0 \end{cases} \tag{10.1.126}$$

and the entropy condition

$$\begin{cases} \lambda_1(\rho_0(t, x_f(t))) < x_f'(t) < \lambda_2(\rho_0(t, x_f(t))), \\ x_f'(t) > \lambda_2(\rho_+(t, x_f(t))) \end{cases} \tag{10.1.127}$$

on $x = x_f(t)$, in which $[\rho] = \rho_+(t, x_f(t)) - \rho_0(t, x_f(t))$, etc. and

$$-\lambda_1(\rho) = \lambda_2(\rho) = \sqrt{p'(\rho)}. \tag{10.1.128}$$

In the special case that the piston moves with a constant speed u_p and the initial state is a constant state $(\bar{\rho}_+, \bar{u}_+)$ $(\bar{\rho}_+ > 0)$ with $u_p > \bar{u}_+$, the solution to the previous problem is the typical forward shock (see [22])

$$(\rho, u) = \begin{cases} (\rho_0, u_p), & u_p t \le x \le Ut, \\ (\bar{\rho}_+, \bar{u}_+), & x \ge Ut, \end{cases} \tag{10.1.129}$$

where U, the speed of propagation of the typical forward shock, and ρ_0 are determined by the Rankine–Hugoniot condition:

$$\begin{cases} (\rho_0 u_p - \bar{\rho}_+ \bar{u}_+)^2 = (\rho_0 - \bar{\rho}_+)(\rho_0 u_p^2 + p(\rho_0) - \bar{\rho}_+ \bar{u}_+^2 - p(\bar{\rho}_+)), \\ U = \dfrac{\rho_0 u_p - \bar{\rho}_+ \bar{u}_+}{\rho_0 - \bar{\rho}_+}. \end{cases} \tag{10.1.130}$$

As a global perturbation of the simplest piston problem mentioned above, for the piston problem (10.1.118) and (10.1.120)–(10.1.121), we have the following:

Theorem 10.1.2 *Suppose that $\rho_0^+(x)$, $u_0^+(x) \in C^1$, and $f(t) \in C^2$ with*

$$\rho_0^+(0) = \bar{\rho}_+, \quad u_0^+(0) = \bar{u}_+, \quad \varphi(0) = u_p. \tag{10.1.131}$$

Suppose furthermore that

$$|\rho_0^+(x) - \bar{\rho}_+|, \ |u_0^+(x) - \bar{u}_+| \le \varepsilon, \quad \forall x \ge 0, \tag{10.1.132}$$

$$|\varphi(t) - u_p| \le \varepsilon, \quad \forall t \ge 0, \tag{10.1.133}$$

$$|\rho_0^{+'}(x)|, \ |u_0^{+'}(x)| \le \frac{\eta}{1+x}, \quad \forall x \ge 0, \tag{10.1.134}$$

and

$$|\varphi'(t)| \le \frac{\eta}{1+t}, \quad \forall t \ge 0, \tag{10.1.135}$$

where $\varepsilon > 0$ *and* $\eta > 0$ *are suitably small. Then the piston problem* (10.1.118) *and* (10.1.120)–(10.1.121) *admits a unique global piecewise* C^1 *solution*

$$(\rho(t,x), u(t,x)) = \begin{cases} (\rho_0(t,x), u_0(t,x)), & f(t) \le x \le x_f(t), \\ (\rho_+(t,x), u_+(t,x)), & x \ge x_f(t), \end{cases} \tag{10.1.136}$$

on the domain

$$\Omega = \{(t,x) | t \ge 0,\ x \ge f(t)\}. \tag{10.1.137}$$

This solution, containing only one forward shock $x = x_f(t)$ *passing through the origin with* $x_f'(0) = U$*, satisfies the following estimates: On the domain*

$$\Omega_+ = \{(t,x) | t \ge 0,\ x \ge x_f(t)\}, \tag{10.1.138}$$

we have

$$|\rho_+(t,x) - \bar{\rho}_+|,\ |u_+(t,x) - \bar{u}_+| \le K_{11}\varepsilon, \tag{10.1.139}$$

$$\left|\frac{\partial \rho_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial \rho_+}{\partial t}(t,x)\right|,\ \left|\frac{\partial u_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial u_+}{\partial t}(t,x)\right| \le \frac{K_{12}\eta}{1+t}; \tag{10.1.140}$$

on the domain

$$\Omega_- = \{(t,x) | t \ge 0,\ f(t) \le x \le x_f(t)\}, \tag{10.1.141}$$

we have

$$|\rho_0(t,x) - \rho_0|,\ |u_0(t,x) - u_p| \le K_{13}\varepsilon, \tag{10.1.142}$$

$$\left|\frac{\partial \rho_0}{\partial x}(t,x)\right|,\ \left|\frac{\partial \rho_0}{\partial t}(t,x)\right|,\ \left|\frac{\partial u_0}{\partial x}(t,x)\right|,\ \left|\frac{\partial u_0}{\partial t}(t,x)\right| \le \frac{K_{14}\eta}{1+t}. \tag{10.1.143}$$

Moreover,

$$|x_f'(t) - U| \le K_{15}\varepsilon, \quad \forall t \ge 0, \tag{10.1.144}$$

$$|x_f''(t)| \le \frac{K_{16}\eta}{1+t}, \quad \forall t \ge 0. \tag{10.1.145}$$

Proof. Taking the Lagrange coordinates $(\tilde{t}, m)$:

$$\begin{cases} m = \displaystyle\int_{(0,0)}^{(t,x)} \rho dx - \rho u dt, \\ \tilde{t} = t \end{cases} \tag{10.1.146}$$

as new variables, problem (10.1.118) and (10.1.120)–(10.1.121) reduces to (10.1.1)–(10.1.3) in which (t,x) is replaced by $(\tilde{t},m)$ and

$$\tau_0^+(m) = \frac{1}{\rho_0^+(x(m))}, \quad u_0^+(m) = u_0^+(x(m)), \tag{10.1.147}$$

$$\phi(\tilde{t}) = \varphi(\tilde{t}), \tag{10.1.148}$$

where $x = x(m)$ is determined by

$$m = \int_0^x \rho_0^+(\xi)\, d\xi. \tag{10.1.149}$$

By (10.1.131)–(10.1.135), it is easy to see that

$$\tau_0^+(0) = \frac{1}{\bar{\rho}_+} = \bar{\tau}_+, \quad u_0^+(0) = \bar{u}_+, \quad \phi(0) = u_p, \tag{10.1.150}$$

$$|\tau_0^+(m) - \bar{\tau}_+|,\ |u_0^+(m) - \bar{u}_+| \le C_{16}\varepsilon, \quad \forall m \ge 0, \tag{10.1.151}$$

$$|\phi(\tilde{t}) - \phi(0)| \le \varepsilon, \quad \forall \tilde{t} \ge 0, \tag{10.1.152}$$

$$|\tau_0^{+'}(m)|,\ |u_0^{+'}(m)| \le \frac{C_{17}\eta}{1+m}, \quad \forall m \ge 0, \tag{10.1.153}$$

and

$$|\phi'(\tilde{t})| \le \frac{\eta}{1+\tilde{t}}, \quad \forall \tilde{t} \ge 0. \tag{10.1.154}$$

By Proposition 10.1.1, we obtain that problem (10.1.1)–(10.1.3) corresponding to problem (10.1.118) and (10.1.120)–(10.1.121) admits a unique global piecewise C^1 solution

$$(\tau(\tilde{t},m), u(\tilde{t},m)) = \begin{cases} (\tau_0(\tilde{t},m), u_0(\tilde{t},m)), & 0 \le m \le m_2(\tilde{t}), \\ (\tau_+(\tilde{t},m), u_+(\tilde{t},m)), & m \ge m_2(\tilde{t}), \end{cases} \tag{10.1.155}$$

on the domain

$$\{(\tilde{t},m) | \tilde{t} \ge 0,\ m \ge 0\}. \tag{10.1.156}$$

This solution, containing only one forward shock $m = m_2(\tilde{t})$ passing through the origin with $m_2'(0) = V$, where

$$V = \bar{\rho}_+(U - \bar{u}_+), \tag{10.1.157}$$

satisfies the following estimates: On the domain

$$\{(\tilde{t},m) | \tilde{t} \ge 0,\ m \ge m_2(\tilde{t})\}, \tag{10.1.158}$$

we have

$$|\tau_+(\tilde{t},m)-\bar{\tau}_+|,\ |u_+(\tilde{t},m)-\bar{u}_+|\le C_{18}\varepsilon, \quad (10.1.159)$$

$$\left|\frac{\partial\tau_+}{\partial m}(\tilde{t},m)\right|,\ \left|\frac{\partial\tau_+}{\partial\tilde{t}}(\tilde{t},m)\right|,\ \left|\frac{\partial u_+}{\partial m}(\tilde{t},m)\right|,\ \left|\frac{\partial u_+}{\partial\tilde{t}}(\tilde{t},m)\right|\le\frac{C_{19}\eta}{1+\tilde{t}}; \quad (10.1.160)$$

on the domain

$$\{(\tilde{t},m)|\tilde{t}\ge 0,\ 0\le m\le m_2(\tilde{t})\}, \quad (10.1.161)$$

we have

$$|\tau_0(\tilde{t},m)-\tau_0|,\ |u_0(\tilde{t},m)-u_0|\le C_{20}\varepsilon, \quad (10.1.162)$$

$$\left|\frac{\partial\tau_0}{\partial m}(\tilde{t},m)\right|,\ \left|\frac{\partial\tau_0}{\partial\tilde{t}}(\tilde{t},m)\right|,\ \left|\frac{\partial u_0}{\partial m}(\tilde{t},m)\right|,\ \left|\frac{\partial u_0}{\partial\tilde{t}}(\tilde{t},m)\right|\le\frac{C_{21}\eta}{1+\tilde{t}}. \quad (10.1.163)$$

Moreover,

$$|m_2'(\tilde{t})-V|\le C_{22}\varepsilon,\quad \forall\tilde{t}\ge 0, \quad (10.1.164)$$

$$|m_2''(\tilde{t})|\le\frac{C_{23}\eta}{1+\tilde{t}},\quad \forall\tilde{t}\ge 0. \quad (10.1.165)$$

Using the inverse transformation of (10.1.146):

$$\begin{cases} x=\displaystyle\int_{(0,0)}^{(\tilde{t},m)}\tau dm+ud\tilde{t},\\ t=\tilde{t},\end{cases} \quad (10.1.166)$$

we get

$$m=m(t,x). \quad (10.1.167)$$

Then, by $(\rho(t,x),u(t,x))=(1/\tau(t,m(t,x)),u(t,m(t,x)))$, it is easy to see that the original piston problem (10.1.118) and (10.1.120)–(10.1.121) admits a unique global piecewise C^1 solution

$$(\rho(t,x),u(t,x))=\begin{cases}(\rho_0(t,x),u_0(t,x))=\left(\dfrac{1}{\tau_0(t,m(t,x))},u_0(t,m(t,x))\right),\\ \qquad f(t)\le x\le x_f(t),\\ (\rho_+(t,x),u_+(t,x))=\left(\dfrac{1}{\tau_+(t,m(t,x))},u_+(t,m(t,x))\right),\\ \qquad x\ge x_f(t),\end{cases} \quad (10.1.168)$$

on the domain (10.1.137), where

$$x_f(t) = \int_0^t (\tau(\sigma, m_2(\sigma))m_2'(\sigma) + u(\sigma, m_2(\sigma)))d\sigma; \tag{10.1.169}$$

moreover, (10.1.138)–(10.1.145) hold.

This proves Theorem 10.1.2.

For the global inverse piston problem, we have

Theorem 10.1.3 *Suppose that the position of the forward shock* $x = x_f(t) \in C^2$ $(t \geq 0)$ *with*

$$x_f(0) = 0 \tag{10.1.170}$$

and

$$x_f'(0) = U \tag{10.1.171}$$

is prescribed and for suitably small $\varepsilon > 0$ *and* $\eta > 0$*, we have*

$$|x_f'(t) - U| \leq \varepsilon, \quad \forall t \geq 0, \tag{10.1.172}$$

and

$$|x_f''(t)| \leq \frac{\eta}{1+t}, \quad \forall t \geq 0. \tag{10.1.173}$$

Then for any given $\rho_0^+(x)$ *and* $u_0^+(x) \in C^1$ $(x \geq 0)$ *satisfying*

$$\rho_0^+(0) = \bar{\rho}_+, \;\; u_0^+(0) = \bar{u}_+, \tag{10.1.174}$$

$$|\rho_0^+(x) - \bar{\rho}_+|, \;\; |u_0^+(x) - \bar{u}_+| \leq \varepsilon, \quad \forall x \geq 0, \tag{10.1.175}$$

and

$$|\rho_0^{+'}(x)|, \;\; |u_0^{+'}(x)| \leq \frac{\eta}{1+x}, \quad \forall x \geq 0, \tag{10.1.176}$$

we can uniquely determine the piston velocity $v_p = \varphi(t)$ $(t \geq 0)$ *with*

$$\varphi(0) = u_p, \tag{10.1.177}$$

$$|\varphi(t) - u_p| \leq K_{17}\varepsilon, \quad \forall t \geq 0, \tag{10.1.178}$$

and

$$|\varphi'(t)| \leq \frac{K_{18}\eta}{1+t}, \quad \forall t \geq 0, \tag{10.1.179}$$

where u_p *is the same as in* (10.1.129)*, such that by Theorem 10.1.2, the corresponding direct piston problem* (10.1.118) *and* (10.1.120)*–*(10.1.121) *admits* *a unique global piecewise* C^1 *solution* $(\rho(t,x), u(t,x))$ [see (10.1.136)] *in which the forward shock passing through the origin is just* $x = x_f(t)$*.*

Proof. First, we solve the Cauchy problem (10.1.118) and (10.1.120) on the domain Ω_+ defined by (10.1.138). Noting (10.1.174)–(10.1.176), just as we did in Lagrangian representation (see Lemma 10.1.1), the Cauchy problem (10.1.118) and (10.1.120) admits a unique global C^1 solution $(\rho, u) = (\rho_+(t,x), u_+(t,x))$ on the domain Ω_+ and we have

$$|\rho_+(t,x) - \bar{\rho}_+|,\ |u_+(t,x) - \bar{u}_+| \le C_{24}\varepsilon, \quad \forall (t,x) \in \Omega_+, \tag{10.1.180}$$

and

$$\left|\frac{\partial \rho_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial \rho_+}{\partial t}(t,x)\right|,\ \left|\frac{\partial u_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial u_+}{\partial t}(t,x)\right| \le \frac{C_{25}\eta}{1+t}, \quad \forall (t,x) \in \Omega_+. \tag{10.1.181}$$

Using the Lagrange transformation (10.1.146), the forward shock in Eulerian representation $x = x_f(t)$ reduces to the forward shock $m = m_2(\tilde{t})$ in Lagrangian representation with

$$m_2(\tilde{t}) = \int_0^{\tilde{t}} \rho_+(\sigma, x_f(\sigma))(x_f'(\sigma) - u_+(\sigma, x_f(\sigma)))\, d\sigma. \tag{10.1.182}$$

Noting (10.1.170)–(10.1.173) and (10.1.180)–(10.1.181), we have

$$m_2(0) = 0, \tag{10.1.183}$$

$$m_2'(0) = V, \tag{10.1.184}$$

$$|m_2'(\tilde{t}) - V| \le C_{26}\varepsilon, \quad \forall \tilde{t} \ge 0, \tag{10.1.185}$$

$$|m_2''(\tilde{t})| \le \frac{C_{27}\eta}{1+\tilde{t}}, \quad \forall \tilde{t} \ge 0, \tag{10.1.186}$$

where V is given by (10.1.157). Furthermore, $\tau_0^+(m)$ and $u_0^+(m)$ defined by (10.1.147) satisfy

$$\tau_0^+(0) = \bar{\tau}_+, \quad u_0^+(0) = \bar{u}_+, \tag{10.1.187}$$

$$|\tau_0^+(m) - \bar{\tau}_+|,\ |u_0^+(m) - \bar{u}_+| \le C_{28}\varepsilon, \quad \forall m \ge 0, \tag{10.1.188}$$

and

$$|\tau_0^{+'}(m)|,\ |u_0^{+'}(m)| \le \frac{C_{29}\eta}{1+m}, \quad \forall m \ge 0. \tag{10.1.189}$$

Thus, the inverse piston problem in Eulerian representation reduces to the corresponding one in Lagrangian representation. By Theorem 10.1.1, in Lagrangian representation we can uniquely determine the piston velocity $v_p = \phi(\tilde{t})$ $(\tilde{t} \ge 0)$ with (10.1.52)–(10.1.54), such that the corresponding direct piston problem (10.1.1)–(10.1.3), where (t, x) is replaced by $(\tilde{t}, m)$, admits a unique global piecewise C^1 solution $(\tau(\tilde{t}, m), u(\tilde{t}, m))$ [see (10.1.155)] in which the forward shock passing through the origin is just $m = m_2(\tilde{t})$.

Using the inverse transformation (10.1.166), in Eulerian representation we get the piston path

$$x = f(t) = \int_0^t \phi(\xi) d\xi. \tag{10.1.190}$$

Then, noting (10.1.52)–(10.1.54), the piston velocity $\varphi(t)$, which is nothing but $\phi(t)$, satisfies (10.1.177)–(10.1.179). Thus, by Theorem 10.1.2, the corresponding direct piston problem (10.1.118) and (10.1.120)–(10.1.121) admits a unique global piecewise C^1 solution $(\rho(t, x), u(t, x))$ [see (10.1.136)] in which the forward shock passing through the origin is just $x = x_f(t)$.

Theorem 10.1.3 is then proved.

10.2 Generalized Cauchy Problem with Cauchy Data Given on a Semibounded Noncharacteristic Curve

Consider the following first-order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} = 0, \tag{10.2.1}$$

where $u = (u_1, \ldots, u_n)^T$ is the unknown vector function of (t, x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \ldots, n)$.

By hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \ldots, \lambda_n(u)$ and a complete set of left (resp. right) eigenvectors. For $i = 1, \ldots, n$, let $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ [resp. $r_i(u) = (r_{1i}(u), \ldots, r_{ni}(u))^T$] be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u) A(u) = \lambda_i(u) l_i(u) \tag{10.2.2}$$

and

$$A(u) r_i(u) = \lambda_i(u) r_i(u). \tag{10.2.3}$$

Then we have

$$\det |l_{ij}(u)| \neq 0 \qquad [\text{resp.} \quad \det |r_{ij}(u)| \neq 0]. \tag{10.2.4}$$

Without loss of generality, we assume that on the domain under consideration,

$$l_i(u) r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n), \tag{10.2.5}$$

where δ_{ij} stands for Kronecker's delta.

We suppose that all $\lambda_i(u)$, $l_{ij}(u)$, and $r_{ij}(u)$ $(i,j=1,\ldots,n)$ have the same regularity as $a_{ij}(u)$ $(i,j=1,\ldots,n)$.

In Chapter 5 we showed that for the Cauchy problem of system (10.2.1) with small and decaying initial data given on a semibounded initial axis

$$t=0:\ \ u=\phi(x), \quad x\geq 0, \tag{10.2.6}$$

there exists a unique global C^1 solution on the corresponding maximum determinate domain if and only if the rightmost characteristic $\lambda_n(u)$ satisfying

$$\lambda_1(u),\ldots,\lambda_{n-1}(u)<\lambda_n(u) \tag{10.2.7}$$

is weakly linearly degenerate (WLD). Precisely speaking, we have the following:

Lemma 10.2.1 *Suppose that in a neighbourhood of $u=0$, $A(u)\in C^2$, system* (10.2.1) *is hyperbolic and* (10.2.7) *holds. Suppose furthermore that $\lambda_n(u)$ is WLD. Suppose finally that*

$$\theta \stackrel{\text{def.}}{=} \sup_{x\geq 0}\{(1+x)^{1+\mu}(|\phi(x)|+|\phi'(x)|)\}<+\infty,$$

where $\mu>0$ is a constant. Then there exists $\theta_0>0$ so small that for any given $\theta\in[0,\theta_0]$, the Cauchy problem (10.2.1) *and* (10.2.6) *admits a unique global C^1 solution $u=u(t,x)$ with*

$$|u(t,x)|,\ \left|\frac{\partial u}{\partial t}(t,x)\right|,\ \left|\frac{\partial u}{\partial x}(t,x)\right|\leq\kappa\theta$$

on the domain $D=\{(t,x)|\ t\geq 0, x\geq x_n(t)\}$, where κ is a positive constant independent of θ and $x=x_n(t)$ is the nth characteristic passing through the origin $O(0,0)$.

Recall that the ith characteristic $\lambda_i(u)$ is said to be WLD (see Chapter 2) if, along the ith characteristic trajectory $u=u^{(i)}(s)$ passing through $u=u_0$, defined by

$$\begin{cases}\dfrac{du}{ds}=r_i(u),\\ s=0:\ \ u=0,\end{cases} \tag{10.2.8}$$

we have

$$\nabla\lambda_i(u)r_i(u)\equiv 0, \quad \forall |u| \text{ small}, \tag{10.2.9}$$

i.e.,

$$\lambda_i(u^{(i)}(s))\equiv\lambda_i(0), \quad \forall |s| \text{ small}. \tag{10.2.10}$$

Obviously, if a characteristic $\lambda_i(u)$ is linearly degenerate (LD) in the sense of Lax:

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \tag{10.2.11}$$

then it is WLD.

In order to solve the inverse piston problem for the system of one-dimensional adiabatic flow, we should generalize the previous result to the generalized Cauchy problem with Cauchy data given on a semibounded non-characteristic curve. For this purpose, we consider the generalized Cauchy problem for system (10.2.1) with the following generalized Cauchy data:

$$x = g_2(t): \ u = \psi(t), \quad t \geq 0, \tag{10.2.12}$$

where the curve $x = g_2(t)$ is noncharacteristic everywhere.

Theorem 10.2.1 *Suppose that for any given u on the domain under consideration, the eigenvalues of $A(u)$ satisfy*

$$\lambda_1(u), \ldots, \lambda_{m-1}(u) < \lambda_m(u) < \lambda_{m+1}(u), \ldots, \lambda_n(u). \tag{10.2.13}$$

Suppose furthermore that $x = g_2(t) \in C^2$ with $g_2(0) = 0$ and $\psi(t) \in C^1$. Suppose finally that

$$\lambda_m(\psi(0)) < g_2'(0) < \lambda_s(\psi(0)) \quad (s = m+1, \ldots, n), \tag{10.2.14}$$

$$|g_2'(t) - g_2'(0)| \leq \frac{\varepsilon}{(1+t)^{1+\mu}}, \quad \forall t \geq 0, \tag{10.2.15}$$

$$|g_2''(t)| \leq \frac{\varepsilon}{(1+t)^{1+\mu}}, \quad \forall t \geq 0, \tag{10.2.16}$$

and

$$|\psi(t) - \psi(0)|, \ |\psi'(t)| \leq \frac{\varepsilon}{(1+t)^{1+\mu}}, \quad \forall t \geq 0, \tag{10.2.17}$$

where $0 \leq \varepsilon \leq \varepsilon_0$ with ε_0 suitably small and $\mu > 0$ is a constant. If $\lambda_m(u)$ is WLD, then the generalized Cauchy problem (10.2.1) *and* (10.2.12) *admits a unique global C^1 solution $u = u(t,x)$ on the maximum determinate domain $D = \{(t,x) | t \geq 0, \ g_1(t) \leq x \leq g_2(t)\}$, where $x = g_1(t)$ $(t \geq 0)$ is the mth characteristic passing through the origin $O(0,0)$:*

$$\begin{cases} \dfrac{dg_1(t)}{dt} = \lambda_m(u(t, g_1(t))), \\ g_1(0) = 0. \end{cases} \tag{10.2.18}$$

Moreover,

$$|u(t,x) - u(0,0)| \leq K\varepsilon, \quad \forall (t,x) \in D, \tag{10.2.19}$$

and

$$\left|\frac{\partial u}{\partial x}(t,x)\right|, \quad \left|\frac{\partial u}{\partial t}(t,x)\right| \le K\varepsilon, \quad \forall (t,x) \in D, \tag{10.2.20}$$

where K is a positive constant independent of ε, t, and x.

Remark 10.2.1 *If $x = g_1(t)$ and $x = g_2(t)$ are all noncharacteristic curves, the corresponding generalized Cauchy problem is as discussed in Chapter 9.*

Remark 10.2.2 *Suppose that on the domain under consideration,*

$$\lambda_1(u), \dots, \lambda_{m-1}(u) < \lambda_m(u) \equiv \cdots \equiv \lambda_{m+p}(u) < \lambda_{m+p+1}(u), \dots, \lambda_n(u), \tag{10.2.21}$$

where $\lambda(u) \stackrel{\text{def.}}{=} \lambda_m(u) \equiv \cdots \equiv \lambda_{m+p}(u)$ is a characteristic with constant multiplicity $p+1$, where $0 \le p \le n-m$. Suppose furthermore that $\lambda_m(u), \dots, \lambda_{m+p}(u)$ are WLD (see Chapter 2). Then the conclusion of Theorem 10.2.1 is still valid.

Remark 10.2.3 *The result in this section can be found in* [70].

Proof of Theorem 10.2.1. It is well known that the generalized Cauchy problem (10.2.1) and (10.2.12) always admits a local C^1 solution $u = u(t,x)$ (see [72]). For the time being, we assume that on any existence domain of $u = u(t,x)$, we have

$$|u(t,x) - \psi(0)| \le \delta_0, \tag{10.2.22}$$

where $\delta_0 > 0$ is a suitably small constant. At the end of the proof, we will, explain why this hypothesis is reasonable.

Taking the transformation of independent variables

$$\begin{cases} \bar{x} = t, \\ \bar{t} = g_2(t) - x \end{cases} \tag{10.2.23}$$

and noting $g_1(0) = g_2(0) = 0$, (10.2.14)–(10.2.15), (10.2.18), and (10.2.22), we can rewrite the original generalized Cauchy problem (10.2.1) and (10.2.12) on D as the following Cauchy problem on $\bar{D} = \{(\bar{t}, \bar{x}) | \bar{t} \ge 0,\ \bar{x} \ge \rho(\bar{t})\}$:

$$\frac{\partial \bar{u}}{\partial \bar{t}} + (g_2'(\bar{x})I - A(\bar{u}))^{-1} \frac{\partial \bar{u}}{\partial \bar{x}} = 0 \tag{10.2.24}$$

and

$$\bar{t} = 0: \ \bar{u} = \psi(\bar{x}), \quad \bar{x} \ge 0, \tag{10.2.25}$$

where

$$\bar{u}(\bar{t}, \bar{x}) = u(\bar{x}, g_2(\bar{x}) - \bar{t}), \tag{10.2.26}$$

and $\bar{x} = \rho(\bar{t}) (\ge 0) \in C^2$ with $\rho(0) = 0$ is determined by

$$\bar{t} = g_2(\bar{x}) - g_1(\bar{x}). \tag{10.2.27}$$

Let

$$\bar{u}_{n+1} = g_2'(\bar{x}) \tag{10.2.28}$$

and

$$\bar{U} = \begin{pmatrix} \bar{u} \\ \bar{u}_{n+1} \end{pmatrix}. \tag{10.2.29}$$

The Cauchy problem (10.2.24)–(10.2.25) on the domain $\bar{D}$ can be rewritten as

$$\frac{\partial \bar{U}}{\partial \bar{t}} + B(\bar{U})\frac{\partial \bar{U}}{\partial \bar{x}} = 0 \tag{10.2.30}$$

and

$$\bar{t} = 0: \ \bar{U} = \Psi(\bar{x}), \quad \bar{x} \geq 0, \tag{10.2.31}$$

where

$$B(\bar{U}) = \begin{pmatrix} (\bar{u}_{n+1}I - A(\bar{u}))^{-1} & 0 \\ 0 & 0 \end{pmatrix} \tag{10.2.32}$$

and

$$\Psi(\bar{x}) = \begin{pmatrix} \psi(\bar{x}) \\ g_2'(\bar{x}) \end{pmatrix}. \tag{10.2.33}$$

The eigenvalues of $B(\bar{U})$ are

$$\bar{\lambda}_i(\bar{U}) = (\bar{u}_{n+1} - \lambda_i(\bar{u}))^{-1} \quad (i = 1, \ldots, n), \quad \bar{\lambda}_{n+1}(\bar{U}) = 0. \tag{10.2.34}$$

Noting (10.2.13)–(10.2.15) and (10.2.22), it is easy to see that

$$\bar{\lambda}_1(\bar{U}), \ldots, \bar{\lambda}_{m-1}(\bar{U}), \bar{\lambda}_{m+1}(\bar{U}), \ldots, \bar{\lambda}_{n+1}(\bar{U}) < \bar{\lambda}_m(\bar{U}). \tag{10.2.35}$$

Moreover, the right eigenvectors corresponding to $\bar{\lambda}_i(\bar{U})$ $(i = 1, \ldots, n+1)$ can be taken as

$$\bar{r}_i(\bar{U}) = \begin{pmatrix} r_i(\bar{u}) \\ 0 \end{pmatrix} \quad (i = 1, \ldots, n) \tag{10.2.36}$$

and

$$\bar{r}_{n+1}(\bar{U}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10.2.37}$$

Then the weak linear degeneracy of $\lambda_m(u)$ implies the weak linear degeneracy of $\bar{\lambda}_m(\bar{U})$.

For the initial data $\Psi(\bar{x})$, noting (10.2.15)–(10.2.17), we have

$$|\Psi(\bar{x}) - \Psi(0)|, \ |\Psi'(\bar{x})| \leq \frac{\varepsilon}{(1+\bar{x})^{1+\mu}}, \quad \forall \bar{x} \geq 0. \tag{10.2.38}$$

Besides, noting (10.2.18), by (10.2.27), we have

$$\begin{aligned} \rho'(\bar{t}) &= \frac{1}{g_2'(\rho(\bar{t})) - g_1'(\rho(\bar{t}))} = \frac{1}{(\bar{u}_{n+1} - \lambda_m(\bar{u}))(\bar{t}, \rho(\bar{t}))} \\ &= \bar{\lambda}_m(\bar{U}(\bar{t}, \rho(\bar{t}))), \ \ \forall \bar{t} \geq 0. \end{aligned} \tag{10.2.39}$$

Hence, $\bar{x} = \rho(\bar{t})$ is the mth characteristic and the rightmost characteristic for system (10.2.30) passing through the origin $(\bar{t}, \bar{x}) = (0, 0)$.

By Lemma 10.2.1, it is easy to see that the Cauchy problem (10.2.30)–(10.2.31) admits a unique global C^1 solution $\bar{U} = \bar{U}(\bar{t}, \bar{x})$ on $\bar{D}$ and

$$|\bar{U}(\bar{t}, \bar{x}) - \Psi(0)| \leq K\varepsilon, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}, \tag{10.2.40}$$

$$\left| \frac{\partial \bar{U}}{\partial \bar{x}}(\bar{t}, \bar{x}) \right|, \ \left| \frac{\partial \bar{U}}{\partial \bar{t}}(\bar{t}, \bar{x}) \right| \leq K\varepsilon, \quad \forall (\bar{t}, \bar{x}) \in \bar{D}, \tag{10.2.41}$$

where K is a positive constant independent of ε. As a result, the generalized Cauchy problem (10.2.1) and (10.2.12) admits a unique global C^1 solution $u = u(t, x) = \bar{u}(g_2(t) - x, t)$ on D and (10.2.19)–(10.2.20) hold.

Finally, by (10.2.19), we have

$$|u(t, x) - \psi(0)| \leq K\varepsilon \leq K\varepsilon_0, \quad \forall (t, x) \in D, \tag{10.2.42}$$

which implies the validity of hypothesis (10.2.22), provided that $\varepsilon_0 > 0$ is suitably small.

The proof of Theorem 10.2.1 is finished.

10.3 Inverse Piston Problem for the System of One-Dimensional Adiabatic Flow

10.3.1 Introduction

Suppose that a piston originally located at the origin at $t = 0$ moves with the speed $v_p = \phi(t)$ in a straight tube, whose length is assumed to be infinite. In order to determine the state of the ideal gas on the right side of this piston, in Lagrangian representation this piston problem asks us to solve the following mixed initial-boundary value problem:

$$\begin{cases} \partial_t \tau - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t (e + \frac{1}{2}u^2) + \partial_x (pu) = 0, \end{cases} \quad t, \ x \geq 0, \tag{10.3.1}$$

$$t = 0: \ (\tau, u, e) = (\tau_0 + \tau_+(x), u_0 + u_+(x), e_0 + e_+(x)), \quad x \geq 0, \tag{10.3.2}$$

and

$$x = 0: \ u = \phi(t), \quad t \geq 0, \tag{10.3.3}$$

where τ is the specific volume, u the velocity, p the pressure, e the internal energy, and

$$p = p(\tau, S), \tag{10.3.4}$$

with

$$p_\tau(\tau, S) < 0, \quad \forall \tau > 0, \tag{10.3.5}$$

in which S is the entropy:

$$S = S(\tau, e). \tag{10.3.6}$$

Moreover, $\tau_0 > 0$, $e_0 > 0$, and u_0 are constants, and $\tau_+(x)$, $u_+(x)$, $e_+(x)$, and $\phi(t) \in C^1$.

Let

$$U = (\tau, u, e)^T. \tag{10.3.7}$$

It is easy to see that under hypothesis (10.3.5), (10.3.1) is a strictly hyperbolic system with three distinct real eigenvalues:

$$\lambda_1(U) = -\sqrt{-p_\tau} < \lambda_2(U) = 0 < \lambda_3(U) = \sqrt{-p_\tau} \tag{10.3.8}$$

Moreover, $\lambda_2(U)$ is linearly degenerate (LD), and if

$$p_{\tau\tau}(\tau_0, S_0) \neq 0, \tag{10.3.9}$$

where

$$S_0 = S(\tau_0, e_0), \tag{10.3.10}$$

then $\lambda_1(U)$ and $\lambda_3(U)$ are genuinely nonlinear (GN) in a neighbourhood of $U = U_0 \stackrel{\text{def.}}{=} (\tau_0, u_0, e_0)$.

If

$$\tau_+(x) = u_+(x) = e_+(x) = 0, \quad \forall x \geq 0, \tag{10.3.11}$$

and

$$\phi(t) = a, \quad \forall t \geq 0, \tag{10.3.12}$$

with

$$a > u_0, \tag{10.3.13}$$

then the solution to the piston problem is given by

$$(\tau, u, e) = \begin{cases} (\tau_-, a, e_-) \stackrel{\text{def.}}{=} \hat{U}_-, & 0 \leq x \leq Vt, \\ (\tau_0, u_0, e_0) \stackrel{\text{def.}}{=} \hat{U}_+, & x \geq Vt, \end{cases} \tag{10.3.14}$$

where V is the speed of propagation of the forward typical shock, satisfying

$$V = G(\hat{U}_+, \hat{U}_-) \tag{10.3.15}$$

and the entropy condition

$$\begin{cases} \lambda_3(\hat{U}_-) > V > \lambda_2(\hat{U}_-), \\ V > \lambda_3(\hat{U}_+), \end{cases} \tag{10.3.16}$$

in which τ_-, e_-, and (10.3.15) are uniquely determined by the Rankine–Hugoniot condition

$$\begin{cases} [\tau]V + [u] = 0, \\ [u]V - [p] = 0, \\ [e + \frac{1}{2}u^2]V - [pu] = 0, \end{cases} \tag{10.3.17}$$

where $[\tau] = \tau(t, Vt + 0) - \tau(t, Vt - 0) = \tau_0 - \tau_-$, etc. (see [22]).

As a perturbation of the simplest piston problem mentioned above, for the piston problem (10.3.1)–(10.3.3), suppose that

$$\tau_+(0) = u_+(0) = e_+(0) = 0 \tag{10.3.18}$$

and

$$\phi(0) = a \tag{10.3.19}$$

with (10.3.13). Then the motion of the piston should produce a forward shock $x = x_3(t)$ passing through the origin at least for a short time T_0. More precisely, the corresponding piecewise C^1 solution is written as (see [72])

$$U(t,x) = \begin{cases} (\tau^-(t,x), u^-(t,x), e^-(t,x)) \stackrel{\text{def.}}{=} U^-(t,x), & 0 \le x \le x_3(t), \\ (\tau^+(t,x), u^+(t,x), e^+(t,x)) \stackrel{\text{def.}}{=} U^+(t,x), & x \ge x_3(t), \end{cases} \tag{10.3.20}$$

where $U = U^-(t,x)$ and $U = U^+(t,x)$ satisfy system (10.3.1) in the classical sense on the domain $\{(t,x) | 0 \le x \le x_3(t)\}$ and $\{(t,x) | x \ge x_3(t)\}$, respectively, and verify the Rankine–Hugoniot condition

$$\begin{cases} [\tau]x_3'(t) + [u] = 0, \\ [u]x_3'(t) - [p] = 0, \\ [e + \frac{1}{2}u^2]x_3'(t) - [pu] = 0 \end{cases} \tag{10.3.21}$$

and the entropy condition

$$\begin{cases} \lambda_3(U^-) > x_3'(t) > \lambda_3(U^+), \\ x_3'(t) > \lambda_2(U^-) \end{cases} \tag{10.3.22}$$

on $x = x_3(t)$, where $U^\pm = U(t, x_3(t) \pm 0)$ and $[\tau] = \tau^+(t, x_3(t) + 0) - \tau^-(t, x_3(t) - 0)$, etc. Moreover, we have

$$U^+(0,0) = \hat{U}_+, \quad U^-(0,0) = \hat{U}_-, \tag{10.3.23}$$

and

$$x_3'(0) = V. \tag{10.3.24}$$

In this section we consider the corresponding inverse piston problem: Under the hypotheses that we know the original state (10.3.2) of the gas on the right side of the piston and the position of the forward shock $x = x_3(t)$, is it possible to determine the speed $v_p = \phi(t)$ of the piston?

In the case of one-dimensional isentropic flow, since the system consists of two equations and both the boundary $x = 0$ and the forward shock $x = x_3(t)$ are noncharacteristic, the inverse piston problem can be solved not only locally but also globally by means of the method given in Chapter 9 (see Section 9.1). In the case of the one-dimensional adiabatic flow considered in this section, the corresponding inverse piston problem is more difficult because the system consists of three equations and $x = 0$ is a characteristic curve. Even though this inverse piston problem can also be easily solved locally via the method in [53], however, to get a unique global solution to the inverse piston problem, we need a generalization of the global existence on the maximum determinate domain for the Cauchy problem with Cauchy data on a semibounded initial axis—this has been done in Section 10.2. Thus, in Sections 10.3.2 and 10.3.3 the inverse piston problem for one-dimensional adiabatic gas dynamics is discussed in Lagrangian representation and in Eulerian representation, respectively.

Remark 10.3.1 *The result in this section can be found in* [70].

10.3.2 Inverse Piston Problem in Lagrangian Representation

As mentioned in Section 10.3.1, the inverse problem can be easily solved in the local sense. We now discuss this inverse problem globally. We have

Theorem 10.3.1 *Suppose that the position of the forward shock $x = x_3(t) \in C^2$ is given and*

$$x_3(0) = 0, \quad x_3'(0) = V, \tag{10.3.25}$$

$$|x_3'(t) - V|,\ |x_3''(t)| \le \frac{\varepsilon}{(1+t)^{1+\mu}}, \quad \forall t \ge 0, \tag{10.3.26}$$

where $\varepsilon > 0$ is suitably small and $\mu > 0$ is a constant. Then for any given $(\tau_+(x), u_+(x), e_+(x)) \in C^1$ satisfying

$$\tau_+(0) = u_+(0) = e_+(0) = 0, \tag{10.3.27}$$

$$|\tau_+(x)|,\ |u_+(x)|,\ |e_+(x)| \le \frac{\varepsilon}{(1+x)^{1+\mu}}, \quad \forall x \ge 0, \tag{10.3.28}$$

and

$$|\tau_+'(x)|,\ |u_+'(x)|,\ |e_+'(x)| \le \frac{\varepsilon}{(1+x)^{1+\mu}}, \quad \forall x \ge 0, \tag{10.3.29}$$

there exists a unique speed $v_p = \phi(t)$ $(t \geq 0)$ *of the piston such that*

$$\phi(0) = a, \tag{10.3.30}$$

$$|\phi(t) - a| \leq \kappa\varepsilon, \quad \forall t \geq 0, \tag{10.3.31}$$

and

$$|\phi'(t)| \leq \kappa\varepsilon, \quad \forall t \geq 0, \tag{10.3.32}$$

where κ *is a positive constant independent of* ε*.*

Proof. Since $x = x_3(t)$ is a given forward shock, noting (10.3.27)–(10.3.29) and the entropy condition

$$\begin{cases} \lambda_3(U_-) > x_3'(t) > \lambda_2(U_-), \\ x_3'(t) > \lambda_3(U_+), \end{cases} \tag{10.3.33}$$

where $U_\pm$ denote the value of U on the right and left sides of $x = x_3(t)$, respectively, similarly to Lemma 9.2.2 (also see [50], [73] and [76]), we easily show that on the domain $D^+ = \{(t,x) | x \geq x_3(t),\ t \geq 0\}$, the Cauchy problem (10.3.1) and (10.3.2) admits a unique C^1 solution $U = U_+(t,x)$ satisfying

$$U_+(0,0) = \hat{U}_+, \tag{10.3.34}$$

$$|U_+(t,x) - \hat{U}_+|,\ \left|\frac{\partial U_+}{\partial t}(t,x)\right|,\ \left|\frac{\partial U_+}{\partial x}(t,x)\right| \leq \frac{K\varepsilon}{(1+t)^{1+\mu}}, \quad \forall (t,x) \in D^+. \tag{10.3.35}$$

Here and henceforth, K denotes a positive constant independent of ε. Hence, the value of U on the right side of $x = x_3(t)$ is given by

$$U_+(t) = U_+(t, x_3(t)) \tag{10.3.36}$$

with $U_+(0) = \hat{U}_+$. Then, by Lemma 9.3.1 in Chapter 9 (also see [53]), via the Rankine–Hugoniot condition, we can uniquely determine the value $U_-(t)$ of U on the left side of $x = x_3(t)$ such that

$$U_-(0) = \hat{U}_- \tag{10.3.37}$$

and

$$|U_-(t) - \hat{U}_-|,\ \left|\frac{dU_-(t)}{dt}\right| \leq \frac{K\varepsilon}{(1+t)^{1+\mu}}, \quad \forall t \geq 0. \tag{10.3.38}$$

Noting (10.3.16), (10.3.25)–(10.3.26), and that $\lambda_2(U)$ is LD, by Theorem 10.2.1, we get that on the domain $D^- = \{(t,x) | 0 \leq x \leq x_3(t), t \geq 0\}$, the generalized Cauchy problem (10.3.1) with the Cauchy data

$$x = x_3(t):\ U = U_-(t), \quad t \geq 0, \tag{10.3.39}$$

admits a unique C^1 solution $U = U_-(t,x) = (\tau_-(t,x), u_-(t,x), e_-(t,x))$ with

$$U_-(0,0) = \hat{U}_-, \tag{10.3.40}$$

and

$$|U_-(t,x) - \hat{U}_-|,\ \left|\frac{\partial U_-}{\partial t}(t,x)\right|,\ \left|\frac{\partial U_-}{\partial x}(t,x)\right| \le K\varepsilon, \quad \forall (t,x) \in D^-. \tag{10.3.41}$$

Thus, we obtain the speed of the piston:

$$\phi(t) = u_-(t,0). \tag{10.3.42}$$

Moreover, noting (10.3.40)–(10.3.41), we see that (10.3.30)–(10.3.32) hold.

10.3.3 Inverse Piston Problem in Eulerian Representation

Let us consider the system of one-dimensional gas dynamics in Eulerian representation

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0, \\ \dfrac{\partial(\rho u)}{\partial t} + \dfrac{\partial(\rho u^2 + p)}{\partial x} = 0, \\ \dfrac{\partial(\rho e + \frac{1}{2}\rho u^2)}{\partial t} + \dfrac{\partial((\rho e + \frac{1}{2}\rho u^2 + p)u)}{\partial x} = 0, \end{cases} \tag{10.3.43}$$

where ρ is the density, u the velocity, p the pressure, e the internal energy, and

$$p = p(\rho, S) \tag{10.3.44}$$

with

$$p_\rho(\rho, S) > 0, \quad \forall \rho > 0, \tag{10.3.45}$$

in which S is the entropy:

$$S = S(\rho, e). \tag{10.3.46}$$

The corresponding piston problem asks us to solve the mixed initial-boundary value problem for system (10.3.43) with the initial data

$$t = 0: \ (\rho, v, e) = (\rho_0 + \rho_+(x), u_0 + u_+(x), e_0 + e_+(x)), \quad x \ge 0, \tag{10.3.47}$$

and the boundary condition

$$x = f(t): \ u = \varphi(t), \quad t \geq 0, \tag{10.3.48}$$

where $\rho_0 > 0$, $e_0 > 0$, and u_0 are constants, and $\rho_+(x)$, $u_+(x)$, $e_+(x)$, and $\varphi(t) \in C^1$ with

$$\rho_+(0) = u_+(0) = e_+(0) = 0, \tag{10.3.49}$$

$$\varphi(0) = a, \tag{10.3.50}$$

and

$$f(t) = \int_0^t \varphi(\xi) d\xi. \tag{10.3.51}$$

Let

$$W = (\rho, u, e)^T. \tag{10.3.52}$$

It is easy to see that under hypothesis (10.3.45), (10.3.43) is a strictly hyperbolic system with the three distinct real eigenvalues

$$\lambda_1(W) = u - c, \quad \lambda_2(W) = u, \quad \lambda_3(W) = u + c, \tag{10.3.53}$$

where

$$c^2 = \frac{\partial p}{\partial \rho}. \tag{10.3.54}$$

Moreover, $\lambda_2(W)$ is LD, and if

$$\frac{\rho_0}{2} \frac{\partial^2 p}{\partial \rho^2}(\rho_0, S_0) + c^2(\rho_0, S_0) \neq 0, \tag{10.3.55}$$

where

$$S_0 = S(\rho_0, e_0), \tag{10.3.56}$$

then $\lambda_1(W)$ and $\lambda_3(W)$ are GN in a neighbourhood of $W = W_0 \overset{\text{def.}}{=} (\rho_0, u_0, e_0)^T$.

Suppose that

$$a > u_0. \tag{10.3.57}$$

Then the motion of the piston produces a forward shock $x = x_f(t)$ passing through the origin at least for a short time T_1 (see [72]). Furthermore, the corresponding piecewise C^1 solution is written as

$$W(t,x) = \begin{cases} (\rho^-(t,x), u^-(t,x), e^-(t,x)) \overset{\text{def.}}{=} W^-(t,x), & f(t) \leq x \leq x_f(t), \\ (\rho^+(t,x), u^+(t,x), e^+(t,x)) \overset{\text{def.}}{=} W^+(t,x), & x \geq x_f(t), \end{cases} \tag{10.3.58}$$

where $W = W^-(t,x)$ and $W = W^+(t,x)$ satisfy system (10.3.43) in the classical sense on the domain $\{(t,x) | f(t) \leq x \leq x_f(t)\}$ and $\{(t,x) | x \geq x_f(t)\}$, respectively, and verify the Rankine–Hugoniot condition

$$\begin{cases} [\rho]x_f'(t) - [\rho u] = 0, \\ [\rho u]x_f'(t) - [\rho u^2 + p] = 0, \\ [\rho e + \frac{1}{2}\rho u^2]x_f'(t) - [(\rho e + \frac{1}{2}\rho u^2 + p)u] = 0 \end{cases} \tag{10.3.59}$$

and the entropy condition

$$\begin{cases} \lambda_3(W^-) > x_f'(t) > \lambda_3(W^+), \\ x_f'(t) > \lambda_2(W^-) \end{cases} \tag{10.3.60}$$

on $x = x_f(t)$, where $W^\pm = W(t, x_f(t) \pm 0)$ and $[\rho] = \rho^+(t, x_f(t) + 0) - \rho^-(t, x_f(t) - 0)$ etc. Moreover,

$$\rho^-(0,0) = \rho_-, \quad u^-(0,0) = a, \quad e^-(0,0) = e_-, \tag{10.3.61}$$

$$\rho^+(0,0) = \rho_0, \quad u^+(0,0) = u_0, \quad e^+(0,0) = e_0, \tag{10.3.62}$$

and

$$x_f(0) = 0, \quad x_f'(0) = \widetilde{V}. \tag{10.3.63}$$

Here, ρ_-, e_-, and $\widetilde{V}$ are uniquely determined by the solution

$$(\rho, u, e) = \begin{cases} (\rho_-, a, e_-) \overset{\text{def.}}{=} \hat{W}_-, & at \le x \le \widetilde{V}t, \\ (\rho_0, u_0, e_0) \overset{\text{def.}}{=} \hat{W}_+, & x \ge \widetilde{V}t, \end{cases} \tag{10.3.64}$$

to the previous piston problem in the special case that the piston moves with a constant speed a and the initial state is a constant state (ρ_0, u_0, e_0) with $a > u_0$ (see [22]).

For the corresponding inverse piston problem, using Theorem 10.2.1 as in the proof of Theorem 10.3.1 or using the Lagrange transformation as in the proof of Theorem 10.1.3, we get

Theorem 10.3.2 *Suppose that the position of the forward shock* $x = x_f(t) \in C^2$ *with* (10.3.63) *is given and*

$$|x_f'(t) - \widetilde{V}|, \ |x_f''(t)| \le \frac{\varepsilon}{(1+t)^{1+\mu}}, \quad \forall t \ge 0, \tag{10.3.65}$$

where $\varepsilon > 0$ *is suitably small and* $\mu > 0$ *is a constant. Then for any given* $(\rho_+(x), u_+(x), e_+(x)) \in C^1$ *with* (10.3.49) *satisfying*

$$|\rho_+(x)|, \ |u_+(x)|, \ |e_+(x)| \le \frac{\varepsilon}{(1+x)^{1+\mu}}, \quad \forall x \ge 0, \tag{10.3.66}$$

and

$$|\rho'_+(x)|,\ |u'_+(x)|,\ |e'_+(x)| \leq \frac{\varepsilon}{(1+x)^{1+\mu}}, \quad \forall x \geq 0, \tag{10.3.67}$$

we can uniquely determine the speed $v_p = \varphi(t)$ $(t \geq 0)$ *of the piston with* (10.3.50) *and*

$$|\varphi(t) - a| \leq \bar{\kappa}\varepsilon, \quad \forall t \geq 0, \tag{10.3.68}$$

and

$$|\varphi'(t)| \leq \bar{\kappa}\varepsilon, \quad \forall t \geq 0, \tag{10.3.69}$$

where $\bar{\kappa}$ *is a positive constant independent of* ε.

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Index