Dan Crisan
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Stochastic Analysis 2010

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Preface

In July 2009, Imperial College London hosted the 7th Congress of the International Society for Analysis, its Applications and Computations. One of the invited sections of the Congress was devoted to Stochastic Analysis and Applications. It proved to be one of the most popular sections with talks covering a variety of topics within Stochastic Analysis. Spearheaded by the success of the meeting, the participants agreed to contribute to a special volume dedicated to the subject. The book "Stochastic Analysis 2010" is the result of their joint efforts.

Springer Verlag has kindly agreed to publish the collected works of the participants and we owe a special thanks to the publisher and, in particular, to Dr Catriona Byrne, Springer's Editorial Director for Mathematics. I would also like to acknowledge the Mathematics editorial team, in particular, Dr Marina Reizakis Associate Editor for Mathematics and Annika Elting Editorial Assistant for Mathematics for the smooth and efficient handling of the book project.

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Introduction

Dan Crisan

Stochastic Analysis has emerged as one of the main branches of the latter half of twentieth century mathematics. Situated at the confluence between Analysis and Probability Theory, Stochastic Analysis impresses through its wide range of topics and applications. It aims to provide mathematical tools to describe and model random dynamical systems. Such tools arise in the study of stochastic differential equations and stochastic partial differential equations, infinite dimensional stochastic geometry, random media and interacting particle systems, super-processes, stochastic filtering, mathematical finance, etc. Its roots can be traced back to the work of Kiyosi Itô who has laid the foundations of the theory of stochastic differential equations and of stochastic integration in the 1940s.

Since the early 1950s, Stochastic Analysis has developed at an accelerated rate, gaining new perspectives through interactions with various branches of mathematics, including partial differential equations, potential theory, Lie groups and Lie algebras, differential geometry, and harmonic analysis. Stochastic Analysis embodies the main characteristics of twentieth century mathematical sciences – having elegance, mathematical depth and strong interactions with a wide range of areas. Its influence has gone far beyond the confines of mathematics by providing tools for analyzing random phenomena in fields as diverse as physics, biology, economics, and engineering. For example, the research on filtering theory with its many engineering applications could not have developed to its current stage without the use of stochastic analysis tools, including martingale theory, stochastic differential equations and stochastic partial differential equations. Similarly, in mathematical finance, stochastic differential equations and Itô calculus play crucial roles in pricing financial derivatives such as options (as embodied in the famous results of Black and Scholes), futures, interest rate derivatives, etc.

It is difficult, if not impossible, to summarize a subject that has experienced such rapid fundamental developments both in the vertical and in the horizontal direction.

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The special volume "Stochastic Analysis 2010" aims to provide only a sample of the current research in the different branches of the subject. It includes the collected works of the participants at the Stochastic Analysis section of the 7th Congress of the International Society for Analysis, its Applications and Computations organized at Imperial College London in July 2009. In the following, we give a brief description of the contributions comprising the volume:

The contribution of Bally and Clément considers a class of stochastic differential equations (SDEs) driven by Poisson point measures and aims to give sufficient conditions for proving the absolute continuity with respect to the Lebesgue measure of the law of their solution and for the smoothness of the corresponding density. The authors prove that, under certain non-degeneracy assumptions, the solution X_t of the stochastic differential equation will have a smooth density for sufficiently large t (explicit bounds are given). The results do not apply for small t: one has to wait for a while until the regularization effect takes place. The main tool of the analysis is a Malliavin type integration by parts formula for stochastic differential equations with the basic noise given by the jump times.

Ortiz–López and Sanz–Solé analyse in their contribution a family of stochastic wave equations in spatial dimension three, driven by a Gaussian noise, white in time and with a stationary spatial covariance. The family is constructed by perturbing the driving noise with a multiplicative parameter $\varepsilon \in]0,1]$. They show that this family satisfies a Laplace principle (as ε tends to 0) in the Hölder norm and identify the corresponding large deviation rate function. In the proof of main result, they use the weak convergence approach to large deviations developed by Dupuis and Ellis. An essential ingredient of this method is a variational representation for a reference Gaussian process, e.g., Brownian motion when studying diffusion processes, or various generalizations of the infinite-dimensional Wiener process when dealing with stochastic partial differential equations.

The paper of Li lies within the area of stochastic geometry. Let p be a differentiable map from a manifold N to another manifold M which intertwines a diffusion operator $\mathcal{B}$ on N with another (elliptic) diffusion operator $\mathcal{A}$ on M, that is $(\mathcal{A}f) \circ p = \mathcal{B}(f \circ p)$ for a given smooth function f from M to $\mathbb{R}$. The paper studies the geometry induced by the pair of diffusion operators $(\mathcal{A}, \mathcal{B})$. The geometry leads to an intrinsic point of view on the filtering framework. This is explained by examples, in local coordinates and in the metric setting. The article draws largely on recent work of Elworthy, LeJan and Li and aims to have a comprehensive account of the topic for a general audience.

The contribution of Gyurkó and Lyons explores high order numerical schemes for integrating linear parabolic partial differential equations with piece-wise smooth boundary data. The high order Monte-Carlo methods presented give accurate approximations with computation times comparable with much less accurate finite difference and basic Monte-Carlo schemes. A key feature of these algorithms is that the order of the approximation is tuned to the accuracy one requires. A considerable improvement in efficiency can be attained by using ultra high order cubature formulae. The methods presented belong to the Kusuoka–Lyons–Victoir (KLV) family of numerical methods for integrating solutions to partial differential equations that occur in mathematical finance and many other fields. Sofar, Lyons and Victoir have given a degree 5 approximation. Gyurkó and Lyons present here the cubature of degrees 9 and 11 in one-dimensional space-time and describe the advantages of using these new methods.

The paper of Kurtz studies the equivalence between stochastic equations and martingale problems. Stroock and Varadhan proved that the solution of a martingale problem for a diffusion process gives a weak solution of the corresponding Itô equation. The result is obtained by constructing the driving Brownian motion from the solution of the martingale problem. This constructive approach is much more challenging for more general Markov processes. A "soft" approach to this equivalence is presented here which begins with a joint martingale problem for the solution of the desired stochastic equation and the driving processes and applies a Markov mapping theorem to show that any solution of the original martingale problem corresponds to a solution of the joint martingale problem. These results coupled with earlier results on the equivalence of forward equations and martingale problems show that the three standard approaches to specifying Markov processes (stochastic equations, martingale problems, and forward equations) are, under very general conditions, equivalent in the sense that existence and/or uniqueness of one implies existence and/or uniqueness of the other two.

Gyöngy and Krylov present in their contribution a survey of numerical methods for some classes of deterministic and stochastic PDEs. The methods presented are distinguished from other methods as they all include *Richardson's extrapolation* method as an acceleration step. The first class of methods considered are monotone finite difference schemes for parabolic PDEs. The authors present theorems on power series expansions of finite difference approximations in terms of the mesh-size of the grid. These theorems imply that one can accelerate the convergence of finite difference approximations to any order by taking suitable mixtures of approximations corresponding to different mesh-sizes. The results are extended to degenerate elliptic equations in spaces with supremum norm. Similar methods and results are also presented for finite difference approximations of linear stochastic PDEs. Finally, the authors present results on power series expansion in the *parameters* of the approximations and introduce theorems on their acceleration for a large class of equations and various types of time discretizations.

The contribution of Papavasiliou studies the problem of estimating the parameters of the limiting equation of a multiscale diffusion in the case of averaging and homogenization. The parameters are estimated given data from the corresponding multiscale system. The author reviews recent results that make use of the maximum likelihood of the limiting equation. The results show that, in the averaging case, the maximum likelihood estimator will be asymptotically consistent in the limit. However, in the homogenization case, the maximum likelihood estimator will be asymptotically consistent only if one subsamples the data. The paper also contains a study of the problem of estimating the diffusion coefficient. Here, a novel approach is presented that makes use of the total p-variation. The author shows that in order to compute the diffusion coefficient, one should not use the quadratic variation commonly defined as a limit where the size of a partition goes to zero but rather as a supremum over all partitions. This definition is at the core of the theory of rough paths as it gives rise to a topology with respect to which the Itô map is continuous. The advantage of this approach is that it avoids the subsampling step. The method is applied to a multiscale Ornstein–Uhlenbeck process.

Stanciulescu and Tretyakov present in their contribution a class of numerical methods for solving the Dirichlet problem of linear parabolic stochastic partial differential equations. The methods are based on a probabilistic representation of the solution of the corresponding SPDE which involves averaging characteristics over an auxiliary Wiener process w independent of the Wiener process W that drives the SPDE. The numerical methods are constructed by approximating the stochastic characteristics for a fixed trajectory of W. The authors introduce first-order and order $1/2$ (in the mean-square sense) numerical methods. To make them fully implementable, the methods require an additional Monte-Carlo step. The corresponding convergence theorems, both in the mean-square sense and in the sense of almost sure convergence, are given. All results are proved under rather strong assumptions, in particular that the SPDE has a sufficiently smooth classical solution. This allows the authors to obtain convergence of the proposed methods in a strong norm and with optimal orders. However, the numerical algorithms of this paper can be used under broader conditions. The paper is concluded with results of some numerical experiments.

The contribution of Davie analyses the problem of uniqueness of solutions of stochastic differential equations. Consider the equation

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + b(t, x(t))\mathrm{d}W(t), \quad x(0) = x_0, \tag{1}$$

where W is a standard Brownian motion. Then, under suitable assumptions on the coefficients f and b, (1) has a unique strong solution. That is, there is a unique process x, adapted to the filtration of the Brownian motion, satisfying (1). Here, the author gives an answer to a different uniqueness problem: *Given* a particular Brownian path W, does (1) have a unique solution for that particular path? The first problem with this question is to interpret it, since the stochastic integral implied by the equation is not well-defined for individual paths. The author uses methods of rough path theory to give an interpretation of (1), under slightly stronger regularity conditions on b. The main result states that for almost all Brownian paths W there is a unique solution in this sense. The proof requires estimates for solutions of related equations which are given in the paper.

The paper by Kolokoltsov develops the theory of stochastic differential equations driven by nonlinear Lévy noise, aiming at applications to Markov processes. It is shown that a conditionally positive integro-differential operator (of the Lévy-Khintchine type) with variable coefficients (diffusion, drift and Lévy measure) depending Lipschitz continuously on its parameters generates a Markov semigroup. The analysis of SDEs driven by nonlinear Lévy noise was initiated by the author in previous publications. Here, the author suggests an alternative, more straightforward approach based on the path-wise interpretation of these integrals as nonhomogeneous Lévy processes. The paper uses a general result on the duality for propagators and develops stochastic integration with respect to a nonlinear Lévy noise. The well-posedness of SDEs driven by nonlinear Lévy noises is reformulated in terms of the uniqueness of an invariant measure for a certain probability kernel on the Skorohod space of cadlag paths. Uniqueness results are obtained for the case of Levy measures depending Lipschitz continuously on the position x in the Wasserstein–Kantorovich metric W_p. The paper is concluded with some basic coupling results of Lévy processes that form the cornerstone of the required W_p-estimates.

The contribution of Tunaru lies in the area of mathematical finance. Financial calculus is dominated by numerical approximations of integrals related to various moments of probability distributions used for modeling financial products. Here, the author develops a general technique that facilitates the numerical calculations of option prices for the difficult case of multi-assets, for the majority of European payoff contracts. The algorithms proposed here rely on known weak convergence results, making use of the gaussian probability kernel even when modeling with non-gaussian distributions. Even though the underpinning theory behind the approximations is probabilistic in nature, the methods described here are deterministic, thus they avoid common pitfalls related to Monte-Carlo simulation techniques. In addition, they can be adapted to a wide range of modeling situations. The techniques can also be employed for calculating greek parameters. The paper contains results that prove that the weak convergence characterizing condition can still be applied under some mild assumption on the payoff function of financial options.

Similar to the previous paper, Brody, Hughston and Macrina contribute with a paper in mathematical finance. The authors propose a model for the credit markets in which the random default times of bonds are assumed to be given as functions of one or more independent "market factors". Market participants are assumed to have partial information about each of the market factors, represented by the values of a set of market factor information processes. The market filtration is taken to be generated jointly by various information processes and by the default indicator processes of the various bonds. The value of a discount bond is obtained by taking the discounted expectation of the value of the default indicator function at the maturity of the bond, conditional on the information provided by the market filtration. Explicit expressions are derived for the bond price processes and the associated default hazard rates. The latter are not given *a priori* as part of the model but rather are deduced and shown to be functions of the values of the information processes. Thus, the "perceived" hazard rates, based on the available information, determine bond prices, and as perceptions change so do the prices. In conclusion, explicit expressions are derived for options on discount bonds, the values of which also fluctuate in line with the vicissitudes of market sentiment.

The contribution of Kelbert and Suhov addresses the continuity properties of mutual and conditional entropies between the input and output of a channel with additive noise. The authors study mainly a non-Gaussian situation for both large and small signal-to-noise ratio. This nontrivial aspect has not been discussed before at the level of generality adopted in this paper. The complex character of the continuity properties of various entropies was acknowledged as early as in the 1950s. An example can be found in Dobrushin's work where a number of important (and elegant) results about limiting behavior of various entropies have been proven. The paper studies the entropy power inequality (EPI), an important tool in the analysis of Gaussian channels of information transmission, proposed by Shannon. The authors analyse the continuity properties of the mutual entropy of the input and output signals in an additive memoryless channel and discuss assumptions under which the entropy-power inequality holds true.

Integration by Parts Formula with Respect to Jump Times for Stochastic Differential Equations

Vlad Bally and Emmanuelle Clément

Abstract We establish an integration by parts formula based on jump times in an abstract framework in order to study the regularity of the law for processes solution of stochastic differential equations with jumps.

Keywords Integration by parts formula · Poisson Point Measures · Stochastic Equations

MSC (2010): Primary: 60H07, Secondary: 60G55, 60G57

1 Introduction

We consider the one-dimensional equation

$$X_t = x + \int_0^t \int_E c(u, a, X_{u-}) \mathrm{d}N(u, a) + \int_0^t g(u, X_u) \mathrm{d}u \tag{1}$$

where N is a Poisson point measure of intensity measure μ on some abstract measurable space E. We assume that c and g are infinitely differentiable with respect to t and x, have bounded derivatives of any order, and have linear growth with respect to x. Moreover we assume that the derivatives of c are bounded by a function $\overline{c}$ such that $\int_E \overline{c}(a)\mathrm{d}\mu(a) < \infty$. Under these hypotheses, the equation has a unique solution and the stochastic integral with respect to the Poisson point measure is a Stieltjes integral.

Our aim is to give sufficient conditions in order to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. If $E = \mathbb{R}^m$ and if the measure μ admits a smooth density h, then one may develop a Malliavin calculus based on the amplitudes of the jumps in order

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to solve this problem. This has been done first in Bismut [4] and then in Bichteler, Gravereaux, and Jacod [3]. But if μ is a singular measure, this approach fails and one has to use the noise given by the jump times of the Poisson point measure in order to settle a differential calculus analogous to the Malliavin calculus. This is a much more delicate problem and several approaches have been proposed. A first step is to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure, without taking care of the regularity. A first result in this sense was obtained by Carlen and Pardoux [5] and was followed by a lot of other papers (see [1,7,11,13]). The second step is to obtain the regularity of the density. Recently two results of this type have been obtained by Ishikawa and Kunita [10] and by Kulik [12]. In both cases, one deals with an equation of the form

$$\mathrm{d}X_t = g(t, X_t)\mathrm{d}t + f(t, X_{t-})\mathrm{d}U_t \tag{2}$$

where U is a Lévy process. The above equation is multi-dimensional (let us mention that the method presented in our paper may be used in the multi-dimensional case as well, but then some technical problems related to the control of the Malliavin covariance matrix have to be solved – and for simplicity we preferred to leave out this kind of difficulties in this paper). Ishikawa and Kunita [10] used the finite difference approach given by Picard [14] in order to obtain sufficient conditions for the regularity of the density of the solution of an equation of type (1) (in a somehow more particular form, close to linear equations). The result in that paper produces a large class of examples in which we get a smooth density even for an intensity measure which is singular with respect to the Lebesgue measure. The second approach is due to Kulik [12]. He settled a Malliavin type calculus based on perturbations of the time structure in order to give sufficient conditions for the smoothness of the density. In his paper, the coefficient f is equal to one so the non-degeneracy comes from the drift term g only. As before, he obtains the regularity of the density even if the intensity measure μ is singular. He also proves that under some appropriate conditions, the density is not smooth for a small t so that one has to wait before the regularization effect of the noise produces a regular density.

The result proved in our paper is the following. We consider the function

$$\alpha(t,a,x) = g(x) - g(x + c(t,a,x)) + (g\partial_x c + \partial_t c)(t,a,x).$$

Except the regularity and boundedness conditions on g and c we consider the following non-degeneracy assumption. There exists a measurable function $\underline{\alpha}$ such that $|\alpha(t,a,x)| \geq \underline{\alpha}(a) > 0$ for every $(t,a,x) \in \mathbb{R}_+ \times E \times \mathbb{R}$. We assume that there exists a sequence of subsets $E_n \uparrow E$ such that $\mu(E_n) < \infty$ and

$$\underline{\lim}_{n\to\infty} \frac{1}{\mu(E_n)} \ln \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) \right) = \theta < \infty.$$

We need this hypothesis in order to control the error due to the fact that we localize our differential calculus on a non-degeneracy set. If $\theta = 0$, then for every $t > 0$, the law of X_t has a C^∞ density with respect to the Lebesgue measure. Suppose now that $\theta > 0$ and let $q \in \mathbb{N}$. Then, for $t > 16\theta(q+2)(q+1)^2$ the law of X_t has a density of class $\mathcal{C}^q$. Notice that for small t we are not able to prove that a density exists and we have to wait for a sufficiently large t in order to obtain a regularization effect.

In the paper of Kulik [12], one takes $c(t,a,x) = a$ so $\alpha(t,a,x) = g(x) - g(x + c(t,a,x))$. Then the non-degeneracy condition concerns just the drift coefficient g. And in the paper of Ishikawa and Kunita, the basic example (which corresponds to the geometric Lévy process) is $c(t,a,x) = xa(\mathrm{e}^a - 1)$ and g constant. So $\alpha(t,a,x) = a(\mathrm{e}^a - 1) \sim a^2$ as $a \to 0$. The drift coefficient does not contribute to the non-degeneracy condition (which is analogous to the uniform ellipticity condition).

The paper is organized as follows. In Sect. 2, we give an integration by parts formula of Malliavin type. This is analogous to the integration by parts formulas given in [2] and [1]. But there are two specific points: first of all the integration by parts formula take into account the border terms (in the above-mentioned papers the border terms cancel because one makes use of some weights which are null on the border; but in the paper of Kulik [12] such border terms appear as well). The second point is that we use here a "one shot" integration by parts formula: in the classical gaussian Malliavin calculus, one employs all the noise which is available – so one derives an infinite dimensional differential calculus based on "all the increments" of the Brownian motion. The analogous approach in the case of Poisson point measures is to use all the noise which comes from the random structure (jumps). And this is the point of view of almost all the papers on this topic. But in our paper, we use just "one jump time" which is chosen in a cleaver way (according to the non-degeneracy condition). In Sect. 3, we apply the general integration by parts formula to stochastic equations with jumps. The basic noise is given by the jump times.

2 Integration by Parts Formula

2.1 Notations-Derivative Operators

The abstract framework is quite similar to the one developed in Bally and Clément [2], but we introduce here some modifications in order to take into account the border terms appearing in the integration by parts formula. We consider a sequence of random variables $(V_i)_{i\in\mathbb{N}^*}$ on a probability space $(\Omega, \mathcal{F}, P)$, a sub σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and a random variable J, $\mathcal{G}$ measurable, with values in $\mathbb{N}$. Our aim is to establish a differential calculus based on the variables (V_i), conditionally on $\mathcal{G}$. In order to derive an integration by parts formula, we need some assumptions on the random variables (V_i). The main hypothesis is that conditionally on $\mathcal{G}$, the law of V_i admits a locally smooth density with respect to the Lebesgue measure.

H0. (a) Conditionally on $\mathcal{G}$, the random variables $(V_i)_{1\leq i\leq J}$ are independent and for each $i \in \{1,\ldots,J\}$ the law of V_i is absolutely continuous with respect to the Lebesgue measure. We note p_i the conditional density.

(b) For all $i \in \{1,\ldots,J\}$, there exist some $\mathcal{G}$ measurable random variables a_i and b_i such that $-\infty < a_i < b_i < +\infty$, $(a_i, b_i) \subset \{p_i > 0\}$. We also assume that p_i admits a continuous bounded derivative on (a_i, b_i) and that $\ln p_i$ is bounded on (a_i, b_i).

We define now the class of functions on which this differential calculus will apply. We consider in this paper functions $f : \Omega \times \mathbb{R}^{\mathbb{N}^*} \to \mathbb{R}$ which can be written as

$$f(\omega, v) = \sum_{m=1}^{\infty} f^m(\omega, v_1, \ldots, v_m) 1_{\{J(\omega)=m\}} \tag{3}$$

where $f^m : \Omega \times \mathbb{R}^m \to \mathbb{R}$ are $\mathcal{G} \times \mathcal{B}(\mathbb{R}^m)-$measurable functions.

In the following, we fix $L \in \mathbb{N}$ and we will perform integration by parts L times. But we will use another set of variables for each integration by parts. So for $1 \leq l \leq L$, we fix a set of indices $I_l \subset \{1,\ldots,J\}$ such that if $l \neq l'$, $I_l \cap I_{l'} = \emptyset$. In order to do l integration by parts, we will use successively the variables $V_i, i \in I_l$, then the variables $V_i, i \in I_{l-1}$ and end with $V_i, i \in I_1$. Moreover, given l we fix a partition $(\Lambda_{l,i})_{i\in I_l}$ of Ω such that the sets $\Lambda_{l,i} \in \mathcal{G}, i \in I_l$. If $\omega \in \Lambda_{l,i}$, we will use only the variable V_i in our integration by parts.

With these notations, we define our basic spaces. We consider in this paper random variables $F = f(\omega, V)$ where $V = (V_i)_i$ and f is given by (3). To simplify the notation we write $F = f^J(\omega, V_1, \ldots, V_J)$ so that conditionally on $\mathcal{G}$ we have $J = m$ and $F = f^m(\omega, V_1, \ldots, V_m)$. We denote by $\mathcal{S}^0$ the space of random variables $F = f^J(\omega, V_1, \ldots, V_J)$ where f^J is a continuous function on $O_J = \prod_{i=1}^J (a_i, b_i)$ such that there exists a $\mathcal{G}$ measurable random variable C satisfying

$$\sup_{v\in O_J} |f^J(\omega, v)| \leq C(\omega) < +\infty \quad \text{a.e.} \tag{4}$$

We also assume that f^J has left limits (respectively right limits) in a_i (respectively in b_i). Let us be more precise.

With the notations

$$V_{(i)} = (V_1, \ldots, V_{i-1}, V_{i+1}, \ldots, V_J), \quad (V_{(i)}, v_i) = (V_1, \ldots, V_{i-1}, v_i, V_{i+1}, \ldots, V_J),$$

for $v_i \in (a_i, b_i)$ our assumption is that the following limits exist and are finite:

$$\lim_{\varepsilon\to 0} f^J(\omega, V_{(i)}, a_i + \varepsilon) := F(a_i^+), \quad \lim_{\varepsilon\to 0} f^J(\omega, V_{(i)}, b_i - \varepsilon) := F(b_i^-). \tag{5}$$

Now for $k \geq 1$, $\mathcal{S}^k(I_l)$ denotes the space of random variables $F = f^J(\omega, V_1, \ldots, V_J) \in \mathcal{S}^0$, such that f^J admits partial derivatives up to order k with respect to the variables $v_i, i \in I_l$ and these partial derivatives belong to $\mathcal{S}^0$.

We are now able to define our differential operators.

- *The derivative operators.* We define $D_l : \mathcal{S}^1(I_l) \to \mathcal{S}^0(I_l)$: by

$$D_l F := 1_{O_J}(V) \sum_{i \in I_l} 1_{\Lambda_{l,i}}(\omega) \partial_{v_i} f(\omega, V),$$

where $O_J = \prod_{i=1}^{J}(a_i, b_i)$.
- *The divergence operators.* We note

$$p_{(l)} = \sum_{i \in I_l} 1_{\Lambda_{l,i}} p_i, \tag{6}$$

and we define $\delta_l : \mathcal{S}^1(I_l) \to \mathcal{S}^0(I_l)$ by

$$\delta_l(F) = D_l F + F D_l \ln p_{(l)} = 1_{O_J}(V) \sum_{i \in I_l} 1_{\Lambda_{l,i}} (\partial_{v_i} F + F \partial_{v_i} \ln p_i)$$

We can easily see that if $F, U \in \mathcal{S}^1(I_l)$ we have

$$\delta_l(FU) = F\delta_l(U) + U D_l F. \tag{7}$$

- *The border terms.* Let $U \in \mathcal{S}^0(I_l)$. We define (using the notation (5))

$$[U]_l = \sum_{i \in I_l} 1_{\Lambda_{l,i}} 1_{O_{J,i}}(V_{(i)})((Up_i)(b_i^-) - (Up_i)(a_i^+))$$

with $O_{J,i} = \prod_{1 \leq j \leq J, j \neq i}(a_j, b_j)$

2.2 Duality and Basic Integration by Parts Formula

In our framework, the duality between δ_l and D_l is given by the following proposition. In the sequel, we denote by $E_{\mathcal{G}}$ the conditional expectation with respect to the sigma-algebra $\mathcal{G}$.

Proposition 1. *Assuming H0 then* $\forall F, U \in \mathcal{S}^1(I_l)$ *we have*

$$E_{\mathcal{G}}(U D_l F) = -E_{\mathcal{G}}(F \delta_l(U)) + E_{\mathcal{G}}[FU]_l. \tag{8}$$

For simplicity, we assume in this proposition that the random variables F and U take values in $\mathbb{R}$ but such a result can easily be extended to $\mathbb{R}^d$ value random variables.

Proof. We have $E_{\mathcal{G}}(UD_l F) = \sum_{i \in I_l} 1_{\Lambda_{l,i}} E_{\mathcal{G}} 1_{O_J}(V)(\partial_{v_i} f^J(\omega, V) u^J(\omega, V))$. From H0 we obtain

$$E_{\mathcal{G}} 1_{O_J}(V)(\partial_{v_i} f^J(\omega, V) u^J(\omega, V)) = E_{\mathcal{G}} 1_{O_{J,i}}(V_{(i)}) \int_{a_i}^{b_i} \partial_{v_i}(f^J) u^J p_i(v_i) \mathrm{d}v_i.$$

By using the classical integration by parts formula, we have

$$\int_{a_i}^{b_i} \partial_{v_i}(f^J) u^J p_i(v_i) \mathrm{d}v_i = [f^J u^J p_i]_{a_i}^{b_i} - \int_{a_i}^{b_i} f^J \partial_{v_i}(u^J p_i) \mathrm{d}v_i.$$

Observing that $\partial_{v_i}(u^J p_i) = (\partial_{v_i}(u^J) + u^J \partial_{v_i}(\ln p_i)) p_i$, we have

$$\begin{aligned} E_{\mathcal{G}}(1_{O_J}(V) \partial_{v_i} f^J u^J) &= E_{\mathcal{G}} 1_{O_{J,i}}[(V_{(i)}) f^J u^J p_i]_{a_i}^{b_i} \\ &\quad - E_{\mathcal{G}} 1_{O_J}(V) F(\partial_{v_i}(U) + U \partial_{v_i}(\ln p_i)) \end{aligned}$$

and the proposition is proved. □

We can now state a first integration by parts formula.

Proposition 2. *Let H0 hold true and let* $F \in \mathcal{S}^2(I_l)$, $G \in \mathcal{S}^1(I_l)$ *and* $\Phi : \mathbb{R} \to \mathbb{R}$ *be a bounded function with bounded derivative. We assume that* $F = f^J(\omega, V)$ *satisfies the condition*

$$\min_{i \in I_l} \inf_{v \in O_J} |\partial_{v_i} f^J(\omega, v)| \geq \gamma(\omega), \tag{9}$$

where γ *is* $\mathcal{G}$ *measurable and we define on* $\{\gamma > 0\}$

$$(D_l F)^{-1} = 1_{O_J}(V) \sum_{i \in I_l} 1_{\Lambda_{l,i}} \frac{1}{\partial_{v_i} f(\omega, V)},$$

then

$$\begin{aligned} 1_{\{\gamma>0\}} E_{\mathcal{G}}(\Phi^{(1)}(F) G) &= -1_{\{\gamma>0\}} E_{\mathcal{G}}(\Phi(F) H_l(F, G)) \\ &\quad + 1_{\{\gamma>0\}} E_{\mathcal{G}}[\Phi(F) G (D_l F)^{-1}]_l \end{aligned} \tag{10}$$

with

$$H_l(F, G) = \delta_l(G(D_l F)^{-1}) = G \delta_l((D_l F)^{-1}) + D_l G (D_l F)^{-1}. \tag{11}$$

Proof. We observe that

$$D_l \Phi(F) = 1_{O_J}(V) \sum_{i \in I_l} 1_{\Lambda_{l,i}} \partial_{v_i} \Phi(F) = 1_{O_J}(V) \Phi^{(1)}(F) \sum_{i \in I_l} 1_{\Lambda_{l,i}} \partial_{v_i} F,$$

so that

$$D_l\Phi(F).D_lF = \Phi^{(1)}(F)(D_lF)^2,$$

and then $1_{\{\gamma>0\}}\Phi^{(1)}(F) = 1_{\{\gamma>0\}}D_l\Phi(F).(D_lF)^{-1}$. Now since $F \in \mathcal{S}^2(I_l)$, we deduce that $(D_lF)^{-1} \in \mathcal{S}^1(I_l)$ on $\{\gamma > 0\}$ and applying Proposition 1 with $U = G(D_lF)^{-1}$ we obtain the result. □

2.3 Iterations of the Integration by Parts Formula

We will iterate the integration by parts formula given in Proposition 2. We recall that if we iterate l times the integration by parts formula, we will integrate by parts successively with respect to the variables $(V_i)_{i\in I_k}$ for $1 \le k \le l$. In order to give some estimates of the weights appearing in these formulas, we introduce the following norm on $\mathcal{S}^l(\cup_{k=1}^l I_k)$, for $1 \le l \le L$.

$$|F|_l = |F|_\infty + \sum_{k=1}^{l}\sum_{1\le l_1<\ldots<l_k\le l}|D_{l_1}\ldots D_{l_k}F|_\infty, \tag{12}$$

where $|.|_\infty$ is defined on $\mathcal{S}^0$ by

$$|F|_\infty = \sup_{v\in O_J}|f^J(\omega, v)|.$$

For $l = 0$, we set $|F|_0 = |F|_\infty$. We remark that we have for $1 \le l_1 < \ldots < l_k \le l$

$$|D_{l_1}\ldots D_{l_k}F|_\infty = \sum_{i_1\in I_{l_1},\ldots,i_k\in I_{l_k}}\left(\prod_{j=1}^{k}1_{\Lambda_{l_j,i_j}}\right)|\partial_{v_{i_1}}\ldots\partial_{v_{i_k}}F|_\infty,$$

and since for each l $(\Lambda_{l,i})_{i\in I_l}$ is a partition of Ω, for ω fixed, the preceding sum has only one term not equal to zero. This family of norms satisfies for $F \in \mathcal{S}^{l+1}(\cup_{k=1}^{l+1}I_k)$:

$$|F|_{l+1} = |D_{l+1}F|_l + |F|_l \quad \text{so} \quad |D_{l+1}F|_l \le |F|_{l+1}. \tag{13}$$

Moreover, it is easy to check that if $F, G \in \mathcal{S}^l(\cup_{k=1}^l I_k)$

$$|FG|_l \le C_l|F|_l|G|_l, \tag{14}$$

where C_l is a constant depending on l. Finally for any function $\phi \in \mathcal{C}^l(\mathbb{R}, \mathbb{R})$ we have

$$|\phi(F)|_l \le C_l \sum_{k=0}^{l} |\phi^{(k)}(F)|_\infty |F|_l^k \le C_l \max_{0\le k\le l} |\phi^{(k)}(F)|_\infty (1+|F|_l^l). \tag{15}$$

With these notations, we can iterate the integration by parts formula.

Theorem 1. *Let H0 hold true and let $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be a bounded function with bounded derivatives up to order L. Let $F = f^J(w,V) \in \mathcal{S}^1(\cup_{l=1}^L I_l)$ such that*

$$\inf_{i\in\{1,\dots,J\}} \inf_{v\in O_J} |\partial_{v_i} f^J(\omega,v)| \ge \gamma(\omega), \quad \gamma \in [0,1] \quad \mathcal{G} \text{ measurable} \tag{16}$$

then we have for $l \in \{1,\dots,L\}$, $G \in \mathcal{S}^l(\cup_{k=1}^l I_k)$ and $F \in \mathcal{S}^{l+1}(\cup_{k=1}^l I_k)$

$$1_{\{\gamma>0\}}|E_{\mathcal{G}}\Phi^{(l)}(F)G| \le C_l ||\Phi||_\infty 1_{\{\gamma>0\}} E_{\mathcal{G}}\left(|G|_l(1+|p|_0)^l \Pi_l(F)\right) \tag{17}$$

where $||\Phi||_\infty = \sup_x |\Phi(x)|$, $|p|_0 = \max_{l=1,\dots,L}|p_{(l)}|_\infty$, C_l is a constant depending on l and $\Pi_l(F)$ is defined on $\{\gamma>0\}$ by

$$\Pi_l(F) = \prod_{k=1}^{l}(1+|(D_kF)^{-1}|_{k-1})(1+|\delta_k((D_kF)^{-1})|_{k-1}). \tag{18}$$

Moreover, we have the bound

$$\Pi_l(F) \le C_l \frac{(1+|\ln p|_1)^l}{\gamma^{l(l+2)}} \prod_{k=1}^{l}(1+|F|_k^{k-1}+|D_kF|_k^{k-1})^2, \tag{19}$$

where $|\ln p|_1 = \max_{i=1,\dots,J} |(\ln p_i)'|_\infty$.

Proof. We proceed by induction. For $l = 1$, we have from Proposition 2 since $G \in \mathcal{S}^1(I_1)$ and $F \in \mathcal{S}^2(I_1)$

$$\begin{aligned} 1_{\{\gamma>0\}}E_{\mathcal{G}}(\Phi^{(1)}(F)G) = &-1_{\{\gamma>0\}}E_{\mathcal{G}}(\Phi(F)H_1(F,G)) \\ &+1_{\{\gamma>0\}}E_{\mathcal{G}}[\Phi(F)G(D_1F)^{-1}]_1. \end{aligned} \tag{20}$$

We have on $\{\gamma>0\}$

$$\begin{aligned} |H_1(F,G)| &\le |G||\delta_1((D_1F)^{-1})| + |D_1G||(D_1F)^{-1}|, \\ &\le (|G|_\infty + |D_1G|_\infty)(1+|(D_1F)^{-1}|_\infty)(1+|\delta_1((D_1F)^{-1})|_\infty), \\ &= |G|_1(1+|(D_1F)^{-1}|_0)(1+|\delta_1((D_1F)^{-1})|_0). \end{aligned}$$

Turning to the border term $[\Phi(F)G(D_1F)^{-1}]_1$, we check that

$$|[\Phi(F)G(D_1F)^{-1}]_1| \leq 2||\Phi||_\infty |G|_\infty \sum_{i\in I_1} 1_{\Lambda_{1,i}} |\frac{1}{\partial_{v_i} F}|_\infty \sum_{i\in I_1} 1_{\Lambda_{1,i}} |p_i|_\infty,$$
$$\leq 2||\Phi||_\infty |G|_0 |(D_1F)^{-1}|_0 |p|_0.$$

This proves the result for $l = 1$.

Now assume that Theorem 1 is true for $l \geq 1$ and let us prove it for $l + 1$. By assumption, we have $G \in \mathcal{S}^{l+1}(\cup_{k=1}^{l+1} I_k) \subset \mathcal{S}^1(I_{l+1})$ and $F \in \mathcal{S}^{l+2}(\cup_{k=1}^{l+1} I_k) \subset \mathcal{S}^2(I_{l+1})$. Consequently, we can apply Proposition 2 on I_{l+1}. This gives

$$1_{\{\gamma>0\}} E_{\mathcal{G}}(\Phi^{(l+1)}(F)G) = -1_{\{\gamma>0\}} E_{\mathcal{G}} \left(\Phi^{(l)}(F) H_{l+1}(F,G)\right)$$
$$+1_{\{\gamma>0\}} E_{\mathcal{G}} [\Phi^{(l)}(F)G(D_{l+1}F)^{-1}]_{l+1}, \qquad (21)$$

with

$$H_{l+1}(F,G) = G\delta_{l+1}((D_{l+1}F)^{-1}) + D_{l+1}G(D_{l+1}F)^{-1},$$

$$[\Phi^{(l)}(F)G(D_{l+1}F)^{-1}]_{l+1} = \sum_{i\in I_{l+1}} 1_{\Lambda_{l+1,i}} 1_{O_{J,i}}(V_{(i)}) \left(\left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i} F} p_i\right)(b_i^-)\right.$$
$$\left. - \left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i} F} p_i\right)(a_i^+)\right).$$

We easily see that $H_{l+1}(F,G) \in \mathcal{S}^l(\cup_{k=1}^{l} I_k)$, and so using the induction hypothesis we obtain

$$1_{\{\gamma>0\}} |E_{\mathcal{G}} \Phi^{(l)}(F) H_{l+1}(F,G)|$$
$$\leq C_l ||\Phi||_\infty 1_{\{\gamma>0\}} E_{\mathcal{G}} |H_{l+1}(F,G)|_l (1 + |p|_0)^l \Pi_l(F),$$

and we just have to bound $|H_{l+1}(F,G)|_l$ on $\{\gamma > 0\}$. But using successively (14) and (13)

$$|H_{l+1}(F,G)|_l \leq C_l(|G|_l |\delta_{l+1}((D_{l+1}F)^{-1})|_l + |D_{l+1}G|_l |(D_{l+1}F)^{-1})|_l,$$
$$\leq C_l |G|_{l+1}(1 + |(D_{l+1}F)^{-1})|_l)(1 + |\delta_{l+1}((D_{l+1}F)^{-1})|_l).$$

This finally gives

$$|E_{\mathcal{G}} \Phi^{(l)}(F) H_{l+1}(F,G)| \leq C_l ||\Phi||_\infty E_{\mathcal{G}} |G|_{l+1} (1 + |p|_0)^l \Pi_{l+1}(F). \qquad (22)$$

So we just have to prove a similar inequality for $E_{\mathcal{G}}[\Phi^{(l)}(F)G(D_{l+1}F)^{-1}]_{l+1}$. This reduces to consider

$$\sum_{i\in I_{l+1}} 1_{\Lambda_{l+1,i}} p_i(b_i^-) E_{\mathcal{G}} 1_{O_{J,i}}(V_{(i)}) \left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i} F}\right)(b_i^-) \qquad (23)$$

since the other term can be treated similarly. Consequently, we just have to bound

$$|E_{\mathcal{G}}1_{O_{J,i}}(V_{(i)})\left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i}F}\right)(b_i^-)|.$$

Since all variables satisfy (4), we obtain from Lebesgue Theorem, using the notation (5)

$$\begin{aligned}&E_{\mathcal{G}}1_{O_{J,i}}(V_{(i)})\left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i}F}\right)(b_i^-)\\&\quad=\lim_{\varepsilon\to 0}E_{\mathcal{G}}1_{O_{J,i}}(V_{(i)})\Phi^{(l)}(f^J(V_{(i)},b_i-\varepsilon))\left(g^J\frac{1}{\partial_{v_i}f^J}\right)(V_{(i)},b_i-\varepsilon).\end{aligned}$$

To shorten the notation, we write simply $F(b_i-\varepsilon)=f^J(V_{(i)},b_i-\varepsilon)$.

Now one can prove that if $U\in\mathcal{S}^{l'}(\cup_{k=1}^{l+1}I_k)$ for $1\le l'\le l$ then $\forall i\in I_{l+1}$, $U(b_i-\varepsilon)\in\mathcal{S}^{l'}(\cup_{k=1}^{l}I_k)$ and $|U(b_i-\varepsilon)|_{l'}\le|U|_{l'}$. We deduce then that $\forall i\in I_{l+1}$ $F(b_i-\varepsilon)\in\mathcal{S}^{l+1}(\cup_{k=1}^{l}I_k)$ and that $(G\frac{1}{\partial_{v_i}F})(b_i-\varepsilon)\in\mathcal{S}^{l}(\cup_{k=1}^{l}I_k)$ and from induction hypothesis

$$\begin{aligned}&|E_{\mathcal{G}}\Phi^{(l)}(F(b_i-\varepsilon))1_{O_{J,i}}(G\frac{1}{\partial_{v_i}F})(b_i-\varepsilon)|\\&\quad\le C_l||\Phi||_\infty E_{\mathcal{G}}\{|G(b_i-\varepsilon)|_l|\frac{1}{\partial_{v_i}F(b_i-\varepsilon)}|_l(1+|p|_0)^l\Pi_l(F(b_i-\varepsilon))\},\\&\quad\le C_l||\Phi||_\infty E_{\mathcal{G}}|G|_l|\frac{1}{\partial_{v_i}F}|_l(1+|p|_0)^l\Pi_l(F).\end{aligned}$$

Putting this in (23) we obtain

$$\begin{aligned}&\left|E_{\mathcal{G}}\sum_{i\in I_{l+1}}1_{\Lambda_{l+1,i}}1_{O_{J,i}}\left(\Phi^{(l)}(F)G\frac{1}{\partial_{v_i}F}p_i\right)(b_i^-)\right|\\&\quad\le C_l||\Phi||_\infty E_{\mathcal{G}}\left\{|G|_l(1+|p|_0)^l\Pi_l(F)\sum_{i\in I_{l+1}}1_{\Lambda_{l+1,i}}|p_i|_\infty|\frac{1}{\partial_{v_i}F}|_l\right\},\\&\quad\le C_l||\Phi||_\infty E_{\mathcal{G}}\{|G|_l(1+|p|_0)^{l+1}\Pi_l(F)|(D_{l+1}F)^{-1}|_l\}.\end{aligned}\tag{24}$$

Finally plugging (22) and (24) in (21)

$$\begin{aligned}|E_{\mathcal{G}}(\Phi^{(l+1)}(F)G)|&\le C_l||\Phi||_\infty\Big(E_{\mathcal{G}}|G|_{l+1}(1+|p|_0)^l\Pi_{l+1}(F)\\&\qquad+E_{\mathcal{G}}|G|_l(1+|p|_0)^{l+1}\Pi_l(F)|(D_{l+1}F)^{-1}|_l\Big),\\&\le C_l||\Phi||_\infty E_{\mathcal{G}}|G|_{l+1}(1+|p|_0)^{l+1}\Pi_{l+1}(F),\end{aligned}$$

and inequality (17) is proved for $l+1$. This achieves the first part of the proof of Theorem 1.

It remains to prove (19). We assume that $\omega \in \{\gamma > 0\}$.

Let $1 \le k \le l$. We first notice that combining (13) and (14), we obtain

$$|\delta_k(F)|_{k-1} \le |F|_k \left(1 + \left|D_k \ln p_{(k)}\right|_\infty\right),$$

since $p_{(k)}$ only depends on the variables $V_i, i \in I_k$. So we deduce the bound

$$\left|\delta_k((D_k F)^{-1})\right|_{k-1} \le \left|(D_k F)^{-1}\right|_k (1 + |\ln p|_1). \tag{25}$$

Now we have

$$|(D_k F)^{-1}|_{k-1} = \sum_{i \in I_k} 1_{\Lambda_{k,i}} \left|\frac{1}{\partial_{v_i} F}\right|_{k-1}$$

From (15) with $\phi(x) = 1/x$

$$\left|\frac{1}{\partial_{v_i} F}\right|_{k-1} \le C_k \frac{(1 + |F|_k^{k-1})}{\gamma^k},$$

and consequently

$$|(D_k F)^{-1}|_{k-1} \le C_k \frac{(1 + |F|_k^{k-1})}{\gamma^k}. \tag{26}$$

Moreover, we have, using successively (13) and (26),

$$\begin{aligned} |(D_k F)^{-1}|_k &= |(D_k F)^{-1}|_{k-1} + |D_k (D_k F)^{-1}|_{k-1}, \\ &\le C_k \left(\frac{(1+|F|_k^{k-1})}{\gamma^k} + \frac{(1+|D_k F|_k^{k-1})}{\gamma^{k+1}}\right), \\ &\le C_k \frac{(1+|F|_k^{k-1} + |D_k F|_k^{k-1})}{\gamma^{k+1}}. \end{aligned}$$

Putting this in (25)

$$\left|\delta_k((D_k F)^{-1})\right|_{k-1} \le C_k \frac{(1 + |F|_k^{k-1} + |D_k F|_k^{k-1})}{\gamma^{k+1}} (1 + |\ln p|_1). \tag{27}$$

Finally from (26) and (27), we deduce

$$\Pi_l(F) \le C_l \frac{(1 + |\ln p|_1)^l}{\gamma^{l(l+2)}} \prod_{k=1}^{l} (1 + |F|_k^{k-1} + |D_k F|_k^{k-1})^2,$$

and Theorem 1 is proved. □

3 Stochastic Equations with Jumps

3.1 Notations and Hypotheses

We consider a Poisson point process p with measurable state space $(E, \mathcal{B}(E))$. We refer to Ikeda and Watanabe [9] for the notation. We denote by N the counting measure associated to p so $N_t(A) := N((0,t) \times A) = \#\{s < t; p_s \in A\}$. The intensity measure is $\mathrm{d}t \times \mathrm{d}\mu(a)$ where μ is a sigma-finite measure on $(E, \mathcal{B}(E))$ and we fix a non-decreasing sequence (E_n) of subsets of E such that $E = \cup_n E_n$, $\mu(E_n) < \infty$ and $\mu(E_{n+1}) \leq \mu(E_n) + K$ for all n and for a constant $K > 0$.

We consider the one-dimensional stochastic equation

$$X_t = x + \int_0^t \int_E c(s, a, X_{s}-)\mathrm{d}N(s, a) + \int_0^t g(s, X_s)\mathrm{d}s. \tag{28}$$

Our aim is to give sufficient conditions on the coefficients c and g in order to prove that the law of X_t is absolutely continuous with respect to the Lebesgue measure and has a smooth density. We make the following assumptions on the coefficients c and g.

H1. We assume that the functions c and g are infinitely differentiable with respect to the variables (t, x) and that there exist a bounded function $\overline{c}$ and a constant $\overline{g}$, such that

$$\forall (t, a, x) \quad |c(t, a, x)| \leq \overline{c}(a)(1 + |x|), \qquad \sup_{l+l' \geq 1} |\partial_t^{l'} \partial_x^l c(t, a, x)| \leq \overline{c}(a);$$
$$\forall (t, x) \quad |g(t, x)| \leq \overline{g}(1 + |x|), \qquad \sup_{l+l' \geq 1} |\partial_t^{l'} \partial_x^l g(t, x)| \leq \overline{g};$$

We assume moreover that $\int_E \overline{c}(a)\mathrm{d}\mu(a) < \infty$.

Under H1, (28) admits a unique solution.

H2. We assume that there exists a measurable function $\hat{c} : E \mapsto \mathbb{R}_+$ such that $\int_E \hat{c}(a)\mathrm{d}\mu(a) < \infty$ and

$$\forall (t, a, x) \quad |\partial_x c(t, a, x)(1 + \partial_x c(t, a, x))^{-1}| \leq \hat{c}(a).$$

To simplify the notation, we take $\hat{c} = \overline{c}$. Under H2, the tangent flow associated to (28) is invertible. At last we give a non-degeneracy condition which will imply (16). We denote by α the function

$$\alpha(t, a, x) = g(t, x) - g(t, x + c(t, a, x)) + (g\partial_x c + \partial_t c)(t, a, x). \tag{29}$$

H3. We assume that there exists a measurable function $\underline{\alpha} : E \mapsto \mathbb{R}_+$ such that

$$\forall (t, a, x) \quad |\alpha(t, a, x)| \geq \underline{\alpha}(a) > 0,$$

$$\forall n \int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) < \infty \quad \text{and} \quad \liminf_n \frac{1}{\mu(E_n)} \ln \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) \right) = \theta < \infty.$$

We give in the following some examples where $E = (0, 1]$ and $\underline{\alpha}(a) = a$.

3.2 Main Results and Examples

Following the methodology introduced in Bally and Clément [2], our aim is to bound the Fourier transform of X_t, $\hat{p}_{X_t}(\xi)$, in terms of $1/|\xi|$, recalling that if $\int_{\mathbb{R}} |\xi|^q |\hat{p}_{X_t}(\xi)| \mathrm{d}\xi < \infty$, for $q > 0$, then the law of X_t is absolutely continuous and its density is $\mathcal{C}^{[q]}$. This is done in the next proposition. The proof of this proposition relies on an approximation of X_t which will be given in the next section.

Proposition 3. *Assuming H1, H2 and H3 we have for all $n, L \in \mathbb{N}^*$*

$$|\hat{p}_{X_t}(\xi)| \le C_{t,L} \left(\mathrm{e}^{-\mu(E_n)t/(2L)} + \frac{1}{|\xi|^L} A_{n,L} \right),$$

with $A_{n,L} = \mu(E_n)^L (\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a))^{L(L+2)}$.

From this proposition, we deduce our main result.

Theorem 2. *We assume that H1, H2 and H3 hold. Let $q \in \mathbb{N}$, then for $t > 16\theta(q+2)(q+1)^2$, the law of X_t is absolutely continuous with respect to the Lebesgue measure and its density is of class $\mathcal{C}^q$. In particular if $\theta = 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure and its density is of class $\mathcal{C}^\infty$ for every $t > 0$.*

Proof. From Proposition 3, we have

$$|\hat{p}_{X_t}(\xi)| \le C_{t,L} \left(\mathrm{e}^{-\mu(E_n)t/2L} + \frac{1}{|\xi|^L} A_{n,L} \right).$$

Now $\forall k, k_0 > 0$, if $t/2L > k\theta$, we deduce from H3 that for $n \ge n_L$

$$t/2L > \frac{k}{\mu(E_n)} \ln \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) \right) + \frac{k \ln \mu(E_n)}{k_0 \mu(E_n)}$$

since the second term on the right-hand side tends to zero. This implies

$$\mathrm{e}^{\mu(E_n)t/2L} > \left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu(a) \right)^k \mu(E_n)^{k/k_0}.$$

Choosing $k = L(L+2)$ and $k/k_0 = L$, we obtain that for $n \ge n_L$ and $t/2L > L(L+2)\theta$

$$e^{\mu(E_n)t/2L} > A_{n,L}.$$

and then

$$|\hat{p}_{X_t}(\xi)| \leq C_{t,L}\left(e^{-\mu(E_n)t/2L} + \frac{1}{|\xi|^L} e^{\mu(E_n)t/2L}\right),$$
$$\leq C_{t,L}\left(\frac{1}{B_n(t)} + \frac{B_n(t)}{|\xi|^L}\right),$$

with $B_n(t) = e^{\mu(E_n)t/2L}$. Now recalling that $\mu(E_n) < \mu(E_{n+1}) \leq K + \mu(E_n)$, we have $B_n(t) < B_{n+1}(t) \leq K_t B_n(t)$. Moreover, since $B_n(t)$ goes to infinity with n we have

$$1_{\{|\xi|^{L/2} \geq B_{n_L}(t)\}} = \sum_{n \geq n_L} 1_{\{B_n(t) \leq |\xi|^{L/2} < B_{n+1}(t)\}}.$$

But if $B_n(t) \leq |\xi|^{L/2} < B_{n+1}(t)$, $|\hat{p}_{X_t}(\xi)| \leq C_{t,L}/|\xi|^{L/2}$ and so

$$\int |\xi|^q |\hat{p}_{X_t}(\xi)| d\xi = \int_{|\xi|^{L/2} < B_{n_L}(t)} |\xi|^q |\hat{p}_{X_t}(\xi)| d\xi + \int_{|\xi|^{L/2} \geq B_{n_L}(t)} |\xi|^q |\hat{p}_{X_t}(\xi)| d\xi,$$

$$\leq C_{t,L,n_L} + \int_{|\xi|^{L/2} \geq B_{n_L}(t)} |\xi|^{q-L/2} d\xi.$$

For $q \in \mathbb{N}$, choosing L such that $L/2 - q > 1$, we obtain $\int |\xi|^q |\hat{p}_{X_t}(\xi)| d\xi < \infty$ for $t/2L > L(L+2)\theta$ and consequently the law of X_t admits a density $\mathcal{C}^q$ for $t > 2L^2(L+2)\theta$ and $L > 2(q+1)$, that is $t > 16\theta(q+1)^2(q+2)$ and Theorem 2 is proved. □

We end this section with two examples

Example 1. We take $E = (0,1]$, $\mu_\lambda = \sum_{k \geq 1} \frac{1}{k^\lambda} \delta_{1/k}$ with $0 < \lambda < 1$ and $E_n = [1/n, 1]$. We have $\cup_n E_n = E$, $\mu(E_n) = \sum_{k=1}^n \frac{1}{k^\lambda}$ and $\mu_\lambda(E_{n+1}) \leq \mu_\lambda(E_n) + 1$. We consider the process (X_t) solution of (28) with $c(t,a,x) = a$ and $g(t,x) = g(x)$ assuming that the derivatives of g are bounded and that $|g'(x)| \geq \underline{g} > 0$. We have $\int_E a d\mu_\lambda(a) = \sum_{k \geq 1} \frac{1}{k^{\lambda+1}} < \infty$ so H1 and H2 hold. Moreover, $\alpha(t,a,x) = g(x) - g(x+a)$ so $\underline{\alpha}(a) = \underline{g}a$. Now $\int_{E_n} \frac{1}{a} d\mu_\lambda(a) = \sum_{k=1}^n k^{1-\lambda}$, which is equivalent as n, go to infinity to $n^{2-\lambda}/(2-\lambda)$. Now we have

$$\frac{1}{\mu_\lambda(E_n)} \ln\left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} d\mu_\lambda(a)\right) = \frac{\ln(\underline{g} \sum_{k=1}^n k^{1-\lambda})}{\sum_{k=1}^n \frac{1}{k^\lambda}} \sim_{n\to\infty} C \frac{\ln(n^{2-\lambda})}{n^{1-\lambda}} \to 0,$$

and then H3 is satisfied with $\theta = 0$. We conclude from Theorem 2 that $\forall t > 0$, X_t admits a density $\mathcal{C}^\infty$.

In the case $\lambda = 1$, we have $\mu_1(E_n) = \sum_{k=1}^{n} \frac{1}{k} \sim_{n\to\infty} \ln n$ then

$$\frac{1}{\mu_1(E_n)} \ln\left(\int_{E_n} \frac{1}{\underline{\alpha}(a)} \mathrm{d}\mu_1(a)\right) = \frac{\ln(\underline{g} \sum_{k=1}^{n} 1)}{\sum_{k=1}^{n} \frac{1}{k}} \sim_{n\to\infty} 1,$$

and this gives H3 with $\theta = 1$. So the density of X_t is regular as soon as t is large enough. In fact it is proved in Kulik [12] that under some appropriate conditions the density of X_t is not continuous for small t.

Example 2. We take the intensity measure μ_λ as in the previous example and we consider the process (X_t) solution of (28) with $g = 1$ and $c(t, a, x) = ax$. This gives $\overline{c}(a) = a$ and $\underline{\alpha}(a) = a$. So the conclusions are similar to example 1 in both cases $0 < \lambda < 1$ and $\lambda = 1$. But in this example we can compare our result to the one given by Ichikawa and Kunita [10]. They assume the condition

$$\liminf_{u\to 0} \frac{1}{u^h} \int_{|a|\le u} a^2 \mathrm{d}\mu(a) > 0, \quad (\star)$$

for some $h \in (0, 2)$. Here we have

$$\int_{|a|\le u} a^2 \mathrm{d}\mu(a) = \sum_{k\ge 1/u} \frac{1}{k^{2+\lambda}} \sim_{u\to 0} \frac{u^{1+\lambda}}{1+\lambda}.$$

So if $0 < \lambda < 1$, $(\star)$ holds and their results apply. In the case $\lambda = 1$, $(\star)$ fails and they do not conclude. However, in our approach we conclude that the density of X_t is $\mathcal{C}^q$ for $t > 16(q+2)(q+1)^2$.

The next section is devoted to the proof of Proposition 3.

3.3 Approximation of X_t and Integration by Parts Formula

In order to bound the Fourier transform of the process X_t solution of (28), we will apply the differential calculus developed in Sect. 2. The first step consists in an approximation of X_t by a random variable X_t^N which can be viewed as an element of our basic space $\mathcal{S}^0$. We assume that the process (X_t^N) is solution of the discrete version of (28)

$$X_t^N = x + \int_0^t \int_{E_N} c(s, a, X_{s-}^N)\mathrm{d}N(s, a) + \int_0^t g(s, X_s^N)\mathrm{d}s. \tag{30}$$

Since $\mu(E_N) < \infty$, the number of jumps of the process X^N on the interval $(0, t)$ is finite and consequently we may consider the random variable X_t^N as a function of these jump times and apply the methodology proposed in Sect. 2. We denote by

(J_t^N) the Poisson process defined by $J_t^N = N((0,t), E_N) = \#\{s < t; p_s \in E_N\}$ and we note $(T_k^N)_{k\geq 1}$ its jump times. We also introduce the notation $\Delta_k^N = p_{T_k^N}$. With these notations, the process solution of (30) can be written

$$X_t^N = x + \sum_{k=1}^{J_t^N} c(T_k^N, \Delta_k^N, X_{T_k^N -}^N) + \int_0^t g(s, X_s^N)\mathrm{d}s. \tag{31}$$

We will not work with all the variables $(T_k^N)_k$ but only with the jump times (T_k^n) of the Poisson process J_t^n, where $n < N$. In the following we will keep n fixed and we will make N go to infinity. We note $(T_k^{N,n})_k$ the jump times of the Poisson process $J_t^{N,n} = N((0,t), E_N \backslash E_n)$ and $\Delta_k^{n,N} = p_{T_k^{n,N}}$. Now we fix $L \in \mathbb{N}^*$, the number of integration by parts and we note $t_l = tl/L$, $0 \leq l \leq L$. Assuming that $J_{t_l}^n - J_{t_{l-1}}^n = m_l$ for $1 \leq l \leq L$, we denote by $(T_{l,i}^n)_{1\leq i \leq m_l}$ the jump times of J_t^n belonging to the time interval (t_{l-1}, t_l). In the following we assume that $m_l \geq 1$, $\forall l$. For $i = 0$ we set $T_{l,0}^n = t_{l-1}$ and for $i = m_l + 1$, $T_{l,m_l+1}^n = t_l$. With these definitions we choose our basic variables $(V_i, i \in I_l)$ as

$$(V_i, i \in I_l) = (T_{l,2i+1}^n, 0 \leq i \leq [(m_l - 1)/2]). \tag{32}$$

The σ-algebra which contains the noise which is not involved in our differential calculus is

$$\mathcal{G} = \sigma\{(J_{t_l}^n)_{1\leq l \leq L}; (T_{l,2i}^n)_{1\leq 2i \leq m_l, 1\leq l \leq L}; (T_k^{N,n})_k; (\Delta_k^N)_k\}. \tag{33}$$

Using some well-known results on Poisson processes, we easily see that conditionally on $\mathcal{G}$ the variables (V_i) are independent and for $i \in I_l$ the law of V_i conditionally on $\mathcal{G}$ is uniform on $(T_{l,2i}^n, T_{l,2i+2}^n)$ and we have

$$p_i(v) = \frac{1}{T_{l,2i+2}^n - T_{l,2i}^n} 1_{(T_{l,2i}^n, T_{l,2i+2}^n)}(v), \quad i \in I_l, \tag{34}$$

Consequently, taking $a_i = T_{l,2i}^n$ and $b_i = T_{l,2i+2}^n$ we check that hypothesis H0 holds. It remains to define the localizing sets $(\Lambda_{l,i})_{i\in I_l}$.

We denote

$$h_l^n = \frac{t_l - t_{l-1}}{2m_l} = \frac{t}{2Lm_l}$$

and $n_l = [(m_l - 1)/2]$. We will work on the $\mathcal{G}$ measurable set

$$\Lambda_l^n = \cup_{i=0}^{n_l} \{T_{l,2i+2}^n - T_{l,2i}^n \geq h_l^n\}, \tag{35}$$

and we consider the following partition of this set:

$$\Lambda_{l,0} = \{T^n_{l,2} - T^n_{l,0} \geq h^n_l\},$$
$$\Lambda_{l,i} = \cap_{k=1}^{i}\{T^n_{l,2k} - T^n_{l,2k-2} < h^n_l\} \cap \{T^n_{l,2i+2} - T^n_{l,2i} \geq h^n_l\}, \quad i = 1, \dots, n_l.$$

After $L - l$ iterations of the integration by parts we will work with the variables $V_i, i \in I_l$ so the corresponding derivative is

$$D_l F = \sum_{i \in I_l} 1_{\Lambda_{l,i}} \partial_{V_i} F = \sum_{i \in I_l} 1_{\Lambda_{l,i}} \partial_{T^n_{l,2i+1}} F.$$

If we are on Λ^n_l then we have at least one i such that $t_{l-1} \leq T^n_{l,2i} < T^n_{l,2i+1} < T^n_{l,2i+2} \leq t_l$ and $T^n_{l,2i+2} - T^n_{l,2i} \geq h^n_l$. Notice that in this case $1_{\Lambda_{l,i}} |p_i|_\infty \leq (h^n_l)^{-1}$ and roughly speaking this means that the variable $V_i = T^n_{l,2i+1}$ gives a sufficiently large quantity of noise. Moreover, in order to perform L integrations by parts we will work on

$$\Gamma^n_L = \cap_{l=1}^{L} \Lambda^n_l \tag{36}$$

and we will leave out the complementary of Γ^n_L. The following lemma says that on the set Γ^n_L we have enough noise and that the complementary of this set may be ignored.

Lemma 1. *Using the notation given in Theorem 1 one has*

(i) $|p|_0 := \max_{1 \leq l \leq L} \sum_{i \in I_l} 1_{\Lambda_{l,i}} |p_i|_\infty \leq \frac{2L}{t} J^n_t,$
(ii) $P((\Gamma^n_L)^c) \leq L \exp(-\mu(E_n)t/2L).$

Proof. As mentioned before $1_{\Lambda_{l,i}} |p_i|_\infty \leq (h^n_l)^{-1} = 2Lm_l/t \leq \frac{2L}{t} J^n_t$ and so we have (i). In order to prove (ii) we have to estimate $P((\Lambda^n_l)^c)$ for $1 \leq l \leq L$. We denote $s_l = \frac{1}{2}(t_l + t_{l-1})$ and we will prove that $\{J^n_{t_l} - J^n_{s_l} \geq 1\} \subset \Lambda^n_l$. Suppose first that $m_l = J^n_{t_l} - J^n_{t_{l-1}}$ is even. Then $2n_l + 2 = m_l$. If $T^n_{l,2i+2} - T^n_{l,2i} < h^n_l$ for every $i = 0, \dots, n_l$ then

$$T^n_{l,m_l} - t_{l-1} = \sum_{i=0}^{n_l} (T^n_{l,2i+2} - T^n_{l,2i}) \leq (n_l + 1) \times \frac{t}{2Lm_l} \leq \frac{t}{4L} \leq s_l - t_{l-1}$$

so there are no jumps in (s_l, t_l). Suppose now that m_l is odd so $2n_l + 2 = m_l + 1$ and $T^n_{l,2n_l+2} = t_l$. If we have $T^n_{l,2i+2} - T^n_{l,2i} < h^n_l$ for every $i = 0, \dots, n_l$, then we deduce

$$\sum_{i=0}^{n_l} (T^n_{l,2i+2} - T^n_{l,2i}) < (n_l + 1) \times \frac{t}{2Lm_l} < \frac{m_l + 1}{m_l} \frac{t}{4L} \leq \frac{t}{2L},$$

and there are no jumps in (s_l, t_l). So we have proved that $\{J^n_{t_l} - J^n_{s_l} \geq 1\} \subset \Lambda^n_l$ and since $P(J^n_{t_l} - J^n_{s_l} = 0) = \exp(-\mu(E_n)t/2L)$ the inequality (ii) follows. □

Now we will apply Theorem 1, with $F^N = X^N_t$, $G = 1$ and $\Phi_\xi(x) = e^{i\xi x}$. So we have to check that $F^N \in \mathcal{S}^{L+1}(\cup_{l=1}^{L} I_l)$ and that condition (16) holds.

Moreover, we have to bound $|F^N|_l^{l-1}$ and $|D_l F^N|_l^{l-1}$, for $1 \leq l \leq L$. This needs some preliminary lemma.

Lemma 2. *Let $v = (v_i)_{i\geq 0}$ be a positive non-increasing sequence with $v_0 = 0$ and $(a_i)_{i\geq 1}$ a sequence of E. We define $J_t(v)$ by $J_t(v) = v_i$ if $v_i \leq t < v_{i+1}$ and we consider the process solution of*

$$X_t = x + \sum_{k=1}^{J_t} c(v_k, a_k, X_{v_k-}) + \int_0^t g(s, X_s)\mathrm{d}s. \tag{37}$$

We assume that H1 holds. Then X_t admits some derivatives with respect to v_i and if we note $U_i(t) = \partial_{v_i} X_t$ and $W_i(t) = \partial^2_{v_i} X_t$, the processes $(U_i(t))_{t\geq v_i}$ and $(W_i(t))_{t\geq v_i}$ solve, respectively,

$$U_i(t) = \alpha(v_i, a_i, X_{v_i-}) + \sum_{k=i+1}^{J_t} \partial_x c(v_k, a_k, X_{v_k-})U_i(v_k-) + \int_{v_i}^t \partial_x g(s, X_s)U_i(s)\mathrm{d}s, \tag{38}$$

$$W_i(t) = \beta_i(t) + \sum_{k=i+1}^{J_t} \partial_x c(v_k, a_k, X_{v_k-})W_i(v_k-) + \int_{v_i}^t \partial_x g(s, X_s)W_i(s)\mathrm{d}s, \tag{39}$$

with

$$\begin{aligned}\alpha(t,a,x) &= g(t,x) - g(t, x + c(t,a,x)) + g(t,x)\partial_x c(t,a,x) + \partial_t c(t,a,x),\\ \beta_i(t) &= \partial_t\alpha(v_i, a_i, X_{v_i-}) + \partial_x\alpha(v_i, a_i, X_{v_i-})g(v_i, X_{v_i-}) - \partial_x g(v_i, X_{v_i})U_i(v_i)\\ &\quad + \sum_{k=i+1}^{J_t} \partial_x^2 c(v_k, a_k, X_{v_k-})(U_i(v_k-))^2 + \int_{v_i}^t \partial_x^2 g(s, X_s)(U_i(s))^2\mathrm{d}s.\end{aligned}$$

Proof. If $s < v_i$, we have $\partial_{v_i} X_s = 0$. Now we have

$$X_{v_i-} = x + \sum_{k=1}^{v_{i-1}} c(v_k, a_k, X_{v_k-}) + \int_0^{v_i} g(s, X_s)\mathrm{d}s,$$

and consequently

$$\partial_{v_i} X_{v_i-} = g(v_i, X_{v_i-}).$$

For $t > v_i$, we observe that

$$X_t = X_{v_i-} + \sum_{k=v_i}^{J_t} c(v_k, a_k, X_{v_k-}) + \int_{v_i}^t g(s, X_s)\mathrm{d}s,$$

this gives

$$\begin{aligned}\partial_{v_i} X_t &= g(v_i, X_{v_i-}) + g(v_i, X_{v_i-})\partial_x c(v_i, a_i, X_{v_i-}) + \partial_t c(v_i, a_i, X_{v_i-}) \\ &\quad - g(v_i, X_{v_i}) + \sum_{k=i+1}^{J_t} \partial_x c(v_k, a_k, X_{v_k-})\partial_{v_i} X_{v_k-} \\ &\quad + \int_{v_i}^t \partial_x g(s, X_s)\partial_{v_i} X_s \mathrm{d}s.\end{aligned}$$

Remarking that $X_{v_i} = X_{v_i-} + c(v_i, a_i, X_{v_i-})$, we obtain (38). The proof of (39) is similar and we omit it. □

We give next a bound for X_t and its derivatives with respect to the variables (v_i).

Lemma 3. *Let (X_t) be the process solution of (37). We assume that H1 holds and we note*

$$n_t(\overline{c}) = \sum_{k=1}^{J_t} \overline{c}(a_k).$$

Then we have:

$$\sup_{s \le t} |X_t| \le C_t(1 + n_t(\overline{c}))\mathrm{e}^{n_t(\overline{c})}.$$

Moreover $\forall l \ge 1$, there exist some constants $C_{t,l}$ and C_l such that $\forall (v_{k_i})_{i=1,\dots,l}$ with $t > v_{k_l}$, we have

$$\begin{aligned}&\sup_{v_{k_l} \le s \le t} |\partial_{v_{k_1}} \dots \partial_{v_{k_{l-1}}} U_{k_l}(s)| + \sup_{v_{k_l} \le s \le t} |\partial_{v_{k_1}} \dots \partial_{v_{k_{l-1}}} W_{k_l}(s)| \\ &\quad \le C_{t,l}(1 + n_t(\overline{c}))^{C_l} \mathrm{e}^{C_l n_t(\overline{c})}.\end{aligned}$$

We observe that the previous bound does not depend on the variables (v_i).

Proof. We just give a sketch of the proof. We first remark that the process (e_t) solution of

$$e_t = 1 + \sum_{k=1}^{J_t} \overline{c}(a_k) e_{v_k-} + \overline{g} \int_0^t e_s \mathrm{d}s,$$

is given by $e_t = \prod_{k=1}^{J_t}(1 + \overline{c}(a_k))\mathrm{e}^{\overline{g}t}$. Now from H1, we deduce for $s \le t$

$$\begin{aligned}|X_s| &\le |x| + \sum_{k=1}^{J_s} \overline{c}(a_k)(1 + |X_{v_k-}|) + \int_0^s \overline{g}(1 + |X_u|)\mathrm{d}u, \\ &\le |x| + \sum_{k=1}^{J_t} \overline{c}(a_k) + \overline{g}t + \sum_{k=1}^{J_s} \overline{c}(a_k)|X_{v_k-}| + \int_0^s \overline{g}|X_u|\mathrm{d}u, \\ &\le \left(|x| + \sum_{k=1}^{J_t} \overline{c}(a_k) + \overline{g}t \right) e_s\end{aligned}$$

where the last inequality follows from Gronwall lemma. Then using the previous remark

$$\sup_{s\leq t}|X_s| \leq C_t(1+n_t(\overline{c}))\prod_{k=1}^{J_t}(1+\overline{c}(a_k)) \leq C_t(1+n_t(\overline{c}))\mathrm{e}^{n_t(\overline{c})}. \tag{40}$$

We check easily that $|\alpha(t,a,x)| \leq C(1+|x|)\overline{c}(a)$, and we get successively from (38) and (40)

$$\sup_{v_{k_l}\leq s\leq t}|U_{k_l}(s)| \leq C_t(1+|X_{v_{k_l}-}|)\overline{c}(a_{k_l})(1+n_t(\overline{c}))\mathrm{e}^{n_t(\overline{c})} \leq C_t(1+n_t(\overline{c}))^2\mathrm{e}^{2n_t(\overline{c})}.$$

Putting this in (39), we obtain a similar bound for $\sup_{v_{k_l}\leq s\leq t}|W_{k_l}(s)|$ and we end the proof of Lemma 3 by induction since we can derive equations for the higher order derivatives of $U_{k_l}(s)$ and $W_{k_l}(s)$ analogous to (39). □

We come back to the process (X_t^N) solution of (30). We recall that $F^N = X_t^N$ and we will check that F^N satisfies the hypotheses of Theorem 1.

Lemma 4. *(i) We assume that H1 holds. Then* $\forall l \geq 1$, $\exists C_{t,l}, C_l$ *independent of* N *such that*

$$|F^N|_l + |D_l F^N|_l \leq C_{t,l}\left((1+N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_l},$$

with $N_t(\overline{c}) = \int_0^t\int_E \overline{c}(a)\mathrm{d}N(s,a)$.

(ii) Moreover, if we assume in addition that H2 and H3 hold and that $m_l = J_{t_l}^n - J_{t_{l-1}}^n \geq 1$, $\forall l \in \{1,\ldots,L\}$, *then we have* $\forall 1 \leq l \leq L$, $\forall i \in I_l$

$$|\partial_{V_i} F^N| \geq \left(\mathrm{e}^{2N_t(\overline{c})}N_t(1_{E_n}1/\underline{\alpha})\right)^{-1} := \gamma_n$$

and (16) holds.

We remark that on the non-degeneracy set Γ_L^n given by (36) we have at least one jump on (t_{l-1},t_l), that is $m_l \geq 1$, $\forall l \in \{1,\ldots,L\}$. Moreover, we have $\Gamma_L^n \subset \{\gamma_n > 0\}$.

Proof. The proof of (i) is a straightforward consequence of Lemma 3, replacing $n_t(\overline{c})$ by $\sum_{p=1}^{J_t^N}\overline{c}(\Delta_p^N)$ and observing that

$$\sum_{p=1}^{J_t^N}\overline{c}(\Delta_p^N) = \int_0^t\int_{E_N}\overline{c}(a)\mathrm{d}N(s,a) \leq \int_0^t\int_E \overline{c}(a)\mathrm{d}N(s,a) = N_t(\overline{c}).$$

Turning to (ii) we have from Lemma 2

$$\partial_{T_k^N} X_t^N = \alpha(T_k^N, \Delta_k^N, X_{T_k^N-}^N) + \sum_{p=k+1}^{J_t^N} \partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N) \partial_{T_k^N} X_{T_p^N-}^N + \int_{T_k^N}^t \partial_x g(s, X_s^N) \partial_{T_k^N} X_s \mathrm{d}s.$$

Assuming H2, we define $(Y_t^N)_t$ and $(Z_t^N)_t$ as the solutions of the equations

$$Y_t^N = 1 + \sum_{p=1}^{J_t^N} \partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N) Y_{T_k^N-} + \int_0^t \partial_x g(s, X_s^N) Y_s^N \mathrm{d}s,$$

$$Z_t^N = 1 - \sum_{p=1}^{J_t^N} \frac{\partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N)}{1 + \partial_x c(T_p^N, \Delta_p^N, X_{T_p^N-}^N)} Z_{T_k^N-} - \int_0^t \partial_x g(s, X_s^N) Z_s^N \mathrm{d}s.$$

We have $Y_t^N \times Z_t^N = 1$, $\forall t \geq 0$ and

$$|Y_t^N| \leq \mathrm{e}^{t\overline{g}} \mathrm{e}^{N_t(1_{E_N} \overline{c})} \leq \mathrm{e}^{N_t(\overline{c})}, \quad |Z_t^N| = \left| \frac{1}{Y_t^N} \right| \leq \mathrm{e}^{N_t(\overline{c})}.$$

Now one can easily check that

$$\partial_{T_k^N} X_t^N = \alpha(T_k^N, \Delta_k^N, X_{T_k^N-}^N) Y_t^N Z_{T_k^N}^N,$$

and using H3 and the preceding bound it yields

$$|\partial_{T_k^N} X_t^N| \geq \mathrm{e}^{-2N_t(\overline{c})} \underline{\alpha}(\Delta_k^N).$$

Recalling that we do not consider the derivatives with respect to all the variables (T_k^N) but only with respect to $(V_i) = (T_{l,2i+1}^n)_{l,i}$ with $n < N$ fixed, we have $\forall 1 \leq l \leq L$ and $\forall i \in I_l$

$$|\partial_{V_i} X_t^N| \geq \mathrm{e}^{-2N_t(\overline{c})} \left(\sum_{p=1}^{J_t^n} \frac{1}{\underline{\alpha}(\Delta_p^n)} \right)^{-1} = \left(\mathrm{e}^{2N_t(\overline{c})} N_t(1_{E_n} 1/\underline{\alpha}) \right)^{-1},$$

and Lemma 4 is proved. □

With this lemma we are at last able to prove Proposition 3.

Proof. From Theorem 1 we have since $\Gamma_L^n \subset \{\gamma_n > 0\}$

$$1_{\Gamma_L^n} |E_{\mathcal{G}} \Phi^{(L)}(F^N)| \leq C_L ||\Phi||_\infty 1_{\Gamma_L^n} E_{\mathcal{G}} (1 + |p_0|)^L \Pi_L(F^N).$$

Now from Lemma 1 (i) we have

$$|p_0| \le 2LJ_t^n/t$$

and moreover we can check that $|\ln p|_1 = 0$. So we deduce from Lemma 4

$$\begin{aligned} \Pi_L(F^N) &\le \frac{C_{t,L}}{\gamma_n^{L(L+2)}}\left((1+N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_L} \\ &\le C_{t,L}N_t(1_{E_n}1/\underline{\alpha})^{L(L+2)}\left((1+N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_L}. \end{aligned}$$

This finally gives

$$\begin{aligned} &|E1_{\Gamma_L^n}\Phi^{(L)}(F^N)| \\ &\quad \le ||\Phi||_\infty C_{t,L}E\left((J_t^N)^L N_t(1_{E_n}1/\underline{\alpha})^{L(L+2)}\left((1+N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^{C_L}\right). \end{aligned} \tag{41}$$

Now we know from a classical computation (see e.g., [2]) that the Laplace transform of $N_t(f)$ satisfies

$$E\mathrm{e}^{-sN_t(f)} = \mathrm{e}^{-t\alpha_f(s)}, \quad \alpha_f(s) = \int_E (1-\mathrm{e}^{-sf(a)})\mathrm{d}\mu(a). \tag{42}$$

From H1, we have $\int_E \overline{c}(a)\mathrm{d}\mu(a) < \infty$, so we deduce using (42) with $f = \overline{c}$ that, $\forall q > 0$

$$E\left((1+N_t(\overline{c}))\mathrm{e}^{N_t(\overline{c})}\right)^q \le C_{t,q} < \infty.$$

Since J_t^n is a Poisson process with intensity $t\mu(E_n)$, we have $\forall q > 0$

$$E(J_t^n)^q \le C_{t,q}\mu(E_n)^q.$$

Finally, using once again (42) with $f = 1_{E_n}1/\underline{\alpha}$, we see easily that $\forall q > 0$

$$EN_t(1_{E_n}1/\underline{\alpha})^q \le C_{t,q}\left(\int_{E_n}\frac{1}{\underline{\alpha}(a)}\mathrm{d}\mu(a)\right)^q.$$

Turning back to (41) and combining Cauchy–Schwarz inequality and the previous bounds, we deduce

$$\begin{aligned} |E1_{\Gamma_L^n}\Phi^{(L)}(F^N)| &\le ||\Phi||_\infty C_{t,L}\mu(E_n)^L\left(\int_{E_n}\frac{1}{\underline{\alpha}(a)}\mathrm{d}\mu(a)\right)^{L(L+2)} \\ &= ||\Phi||_\infty C_{t,L}A_{n,L}. \end{aligned} \tag{43}$$

We are now ready to give a bound for $\hat{p}_{X_t^N}(\xi)$. We have $\hat{p}_{X_t^N}(\xi) = E\Phi_\xi(F^N)$, with $\Phi_\xi(x) = \mathrm{e}^{i\xi x}$. Since $\Phi_\xi^{(L)}(x) = (i\xi)^L \Phi_\xi(x)$, we can write $|\hat{p}_{X_t^N}(\xi)| = |E\Phi_\xi^{(L)}(F^N)|/|\xi|^L$ and consequently we deduce from (43)

$$|\hat{p}_{X_t^N}(\xi)| \leq P((\Gamma_L^n)^c) + C_{t,L} A_{n,L}/|\xi|^L.$$

But from Lemma 1 (ii) we have

$$P((\Gamma_L^n)^c) \leq L\mathrm{e}^{-\mu(E_n)t/(2L)}$$

and finally

$$|\hat{p}_{X_t^N}(\xi)| \leq C_{L,t}\left(\mathrm{e}^{-\mu(E_n)t/(2L)} + A_{n,L}/|\xi|^L\right).$$

We achieve the proof of Proposition 3 by letting N go to infinity, keeping n fixed. □

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A Laplace Principle for a Stochastic Wave Equation in Spatial Dimension Three

Víctor Ortiz-López and Marta Sanz-Solé

Abstract We consider a stochastic wave equation in spatial dimension three, driven by a Gaussian noise, white in time and with a stationary spatial covariance. The free terms are non-linear with Lipschitz continuous coefficients. Under suitable conditions on the covariance measure, Dalang and Sanz-Solé [7] have proved the existence of a random field solution with Hölder continuous sample paths, jointly in both arguments, time and space. By perturbing the driving noise with a multiplicative parameter $\varepsilon \in]0, 1]$, a family of probability laws corresponding to the respective solutions to the equation is obtained. Using the weak convergence approach to large deviations developed in (A weak convergence approach to the theory of large deviations [10]), we prove that this family satisfies a Laplace principle in the Hölder norm.

Keywords Large deviation principle · Stochastic partial differential equations · Wave equation

MSC (2010): 60H15, 60F10

1 Introduction

We consider the stochastic wave equation in spatial dimension $d = 3$

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = \sigma\big(u(t,x)\big)\dot{F}(t,x) + b\big(u(t,x)\big),\ t \in]0,T], \\ u(0,x) = v_0(x), \\ \frac{\partial}{\partial t} u(0,x) = \tilde{v}_0(x),\ x \in \mathbb{R}^3, \end{cases} \tag{1}$$

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where Δ denotes the Laplacian on $\mathbb{R}^3$. The coefficients σ and b are Lipschitz continuous functions and the process $\dot{F}$ is the *formal* derivative of a Gaussian random field, white in time and correlated in space. More precisely, for any $d \geq 1$, let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions and let Γ be a non-negative and non-negative definite tempered measure on $\mathbb{R}^d$. Then, on some probability space, there exists a Gaussian process $F = \big(F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\big)$ with mean zero and covariance functional

$$E\big(F(\varphi)F(\psi)\big) = \int_{\mathbb{R}_+} \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)(\varphi(s) * \tilde{\psi}(s))(x), \tag{2}$$

where $\tilde{\psi}(s)(x) = \psi(s)(-x)$ and the notation "$*$" means the convolution operator. As has been proved in [5], the process F can be extended to a martingale measure $M = \big(M_t(A), t \geq 0, A \in \mathcal{B}_b(\mathbb{R}^d)\big)$, where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the set of bounded Borel sets of $\mathbb{R}^d$.

For any $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, define the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^n} \Gamma(\mathrm{d}x)(\varphi * \tilde{\psi})(x)$$

and denote by $\mathcal{H}$ the Hilbert space obtained by the completion of $\mathcal{D}(\mathbb{R}^n)$ with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Using the theory of stochastic integration with respect to martingale measures (see for instance [16]), the stochastic integral $B_t(h) := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} h(y) M(\mathrm{d}s, \mathrm{d}y)$ is well defined, and for any $h \in \mathcal{H}$ with $\|h\|_{\mathcal{H}} = 1$, the process $(B_t(h), t \in [0,T])$ is a standard Wiener process. In addition, for any fixed $t \in [0,T]$, the mapping $h \to B_t(h)$ is linear. Thus, the process $(B_t, t \in [0,T])$ is a cylindrical Wiener process on $\mathcal{H}$ (see [9] for a definition of this notion). Let $(e_k, k \geq 1)$ be a complete orthonormal system of $\mathcal{H}$. Clearly, $B_k(t) := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} e_k(y) M(\mathrm{d}s, \mathrm{d}y)$, $k \geq 1$, defines a sequence of independent, standard Wiener processes, and we have the representation

$$B_t = \sum_{k \geq 1} B_k(t) e_k. \tag{3}$$

Let $\mathcal{F}_t$, $t \in [0,T]$, be the σ-field generated by the random variables $(B_k(s), s \in [0,t], k \geq 1)$. $(\mathcal{F}_t)$-predictable processes $\Phi \in L^2(\Omega \times [0,T]; \mathcal{H})$ can be integrated with respect to the cylindrical Wiener process $(B_t, t \in [0,T])$, and the stochastic integral $\int_0^t \Phi(s) \mathrm{d}B_t$ coincides with the Itô stochastic integral with respect to the infinite dimensional Brownian motion $(B_k(t), t \in [0,T], k \geq 1)$, $\sum_{k \geq 1} \int_0^t \langle \Phi(s), e_k \rangle_{\mathcal{H}} \mathrm{d}B_k(t)$.

We shall consider the mild formulation of (1),

$$u(t,x) = w(t,x) + \sum_{k \geq 1} \int_0^t \langle G(t-s, x-\cdot)\sigma(u(s,\cdot)), e_k(\cdot) \rangle_{\mathcal{H}} \mathrm{d}B_k(s) + \int_0^t [G(t-s) * b(u(s,\cdot))](x) \mathrm{d}s, \tag{4}$$

$t \in [0,T]$, $x \in \mathbb{R}^3$. Here

$$w(t,x) = \left(\frac{\mathrm{d}}{\mathrm{d}t} G(t) * v_0\right)(x) + (G(t) * \tilde{v}_0)(x),$$

and $G(t) = \frac{1}{4\pi t}\sigma_t$, where σ_t denotes the uniform surface measure (with total mass $4\pi t^2$) on the sphere of radius t.

Throughout the paper, we consider the following set of assumptions.

(H)

1. The coefficients σ, b are real Lipschitz continuous functions.
2. The spatial covariance measure Γ is absolutely continuous with respect to Lebesgue measure, and the density is $f(x) = \varphi(x)|x|^{-\beta}$, $x \in \mathbb{R}^3 \setminus \{0\}$. The function φ is bounded and positive, $\varphi \in \mathcal{C}^1(\mathbb{R}^3)$, $\nabla\varphi \in \mathcal{C}_b^{\delta}(\mathbb{R}^3)$ (the space of bounded and Hölder continuous functions with exponent $\delta \in]0,1]$) and $\beta \in]0,2[$.
3. The initial values v_0, $\tilde{v}_0$ are bounded and such that $v_0 \in \mathcal{C}^2(\mathbb{R}^3)$, ∇v_0 is bounded and Δv_0 and $\tilde{v}_0$ are Hölder continuous with degrees $\gamma_1, \gamma_2 \in]0,1]$, respectively.

We remark that the assumptions on Γ imply

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^3} |\mathcal{F}(G(t))(\xi)|^2 \mu(\mathrm{d}\xi) < \infty, \tag{5}$$

where $\mathcal{F}$ denotes the Fourier transform operator and $\mu = \mathcal{F}^{-1}\Gamma$. This is a relevant condition in connection with the definition of the stochastic integral with respect to the martingale measure M ([4]).

The set of hypotheses **(H)** are used in Chap. 4 of [7] to prove a theorem on existence and uniqueness of solution to (4) and the properties of the sample paths. More precisely, under a slightly weaker set of assumptions than **(H)** (not requiring boundedness of the functions v_0, $\tilde{v}_0$, ∇v_0), Theorem 4.11 in [7] states that for any $q \in [2,\infty[$, $\alpha \in]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}[$, there exists $C > 0$ such that for $(t,x), (\bar{t}, y) \in [0,T] \times D$,

$$E(|u(t,x) - u(\bar{t},y)|^q) \le C(|t-\bar{t}| + |x-y|)^{\alpha q}, \tag{6}$$

where D is a fixed bounded domain of $\mathbb{R}^3$. Consequently, a.s., the stochastic process $(u(t,x), (t,x) \in [0,T] \times D)$ solution of (4) has α-Hölder continuous sample paths, jointly in (t,x).

The reason for strengthening the assumptions of [7] is to ensure that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} |w(t,x)| < \infty \tag{7}$$

(see Hypothesis 4.1 and Lemma 4.2 in [8]), a condition that is needed in the proof of Theorem 2.3 below. This is in addition to (4.19) in [7], which provides an estimate of a fractional Sobolev norm of the function w.

We notice that in [7], the mild formulation of (1) is stated using the stochastic integral developed in [6]. Recent results by Dalang and Quer-Sardanyons (see [8], Proposition 2.11 and Proposition 2.6 (b)) show that this formulation is equivalent to (4).

In this paper, we consider the family of stochastic wave equations

$$u^\varepsilon(t,x) = w(t,x) + \sqrt{\varepsilon}\sum_{k\geq 1}\int_0^t \langle G(t-s,x-\cdot)\sigma(u(s,\cdot)), e_k(\cdot)\rangle_{\mathcal{H}} \mathrm{d}B_k(s)$$
$$+ \int_0^t [G(t-s) * b(u^\varepsilon(s,\cdot))](x)\mathrm{d}s, \tag{8}$$

$\varepsilon \in]0,1]$, and we establish a large deviation principle for the family $(u^\varepsilon, \varepsilon \in]0,1])$ in a Polish space closely related to $\mathcal{C}^\alpha([0,T]\times D)$, the space of functions defined on $[0,T]\times D$, Hölder continuous jointly in its two arguments, of degree $\alpha \in \mathcal{I}$, where

$$\mathcal{I} := \left]0, \gamma_1 \wedge \gamma_2 \wedge \frac{2-\beta}{2} \wedge \frac{1+\delta}{2}\right[.$$

To formulate the large deviation principle, we should consider a Polish space carrying the probability laws of the family $(u^\varepsilon, \varepsilon > 0)$. This cannot be $\mathcal{C}^\alpha([0,T]\times D)$, since this space is not separable. Instead, we consider the space $\mathcal{C}^{\alpha',0}([0,T]\times D)$ of Hölder continuous functions g of degree $\alpha' < \alpha$, with modulus of continuity

$$O_g(\delta) := \sup_{|t-s|+|x-y|<\delta} \frac{|g(t,s)-g(s,y)|}{(|t-s|+|x-y|)^{\alpha'}}$$

satisfying $\lim_{\delta\to 0^+} O_g(\delta) = 0$. This is a Banach space and $\mathcal{C}^\alpha([0,T]\times D) \subset \mathcal{C}^{\alpha',0}([0,T]\times D)$.

In the sequel, we shall denote by $(\mathcal{E}_\alpha, \|\cdot\|_\alpha)$ the Banach space $\mathcal{C}^{\alpha,0}([0,T]\times D)$ endowed with the Hölder norm of degree α, and consider values of $\alpha \in \mathcal{I}$.

Let $\mathcal{H}_T = L^2([0,T];\mathcal{H})$. For any $h \in \mathcal{H}_T$, we consider the deterministic evolution equation

$$V^h(t,x) = w(t,x) + \int_0^t \langle G(t-s,x-\cdot)\sigma(V^h(s,\cdot)), h(s,\cdot)\rangle_{\mathcal{H}} \mathrm{d}s$$
$$+ \int_0^t \left[G(t-s) * b(V^h(s)\right](x)\,\mathrm{d}s. \tag{9}$$

The second term on the right-hand side of this equation can be written as

$$\sum_{k\geq 1}\int_0^t \langle G(t-s,x-\cdot)\sigma(V^h(s,\cdot)), e_k(\cdot)\rangle_{\mathcal{H}} h_k(s)\,\mathrm{d}s,$$

with $h_k(t) = \langle h(t), e_k\rangle_{\mathcal{H}}$, $t\in[0,T]$, $k\geq 1$.

Existence and uniqueness of solution of (9) can be proved in a similar (but easier) way than for (4). This will be obtained in the next section as a by-product of Theorem 2.3, where it is also proved that $V^h \in \mathcal{E}_\alpha$. We will denote by $\mathcal{G}^0 : \mathcal{H}_T \longrightarrow \mathcal{E}_\alpha$ the mapping defined by $\mathcal{G}^0(h) = V^h$.

For any $f \in \mathcal{E}_\alpha$, define

$$I(f) = \inf_{h\in\mathcal{H}_T:\mathcal{G}^0(h)=f} \left\{ \frac{1}{2}\|h\|^2_{\mathcal{H}_T} \right\} \tag{10}$$

and for any $A \subset \mathcal{E}_\alpha$, $I(A) = \inf\{I(f), f \in A\}$.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that the set of hypotheses* **(H)** *are satisfied. Then, the family* $\{u^\varepsilon, \varepsilon \in]0,1]\}$ *given by (8) satisfies a large deviation principle on* $\mathcal{E}_\alpha$ *with rate function* I *given by* (10)*. That means, for any closed subset* $F \in \mathcal{E}_\alpha$ *and any open subset* $G \in \mathcal{E}_\alpha$*,*

$$\limsup_{\varepsilon\to 0+} \varepsilon \log P(u^\varepsilon \in F) \le -I(F),$$
$$\liminf_{\varepsilon\to 0+} \varepsilon \log P(u^\varepsilon \in G) \ge -I(G).$$

In the proof of this theorem, we will use the weak convergence approach to large deviations developed in [10]. An essential ingredient of this method is a variational representation for a reference Gaussian process (Brownian motion when studying diffusion processes, or different generalizations of infinite-dimensional Wiener process when dealing with stochastic partial differential equations). As it is shown in [2], a variational representation for an infinite-dimensional Brownian motion along with a transfer principle based on compactness and weak convergence allows to derive a large deviation principle for some functionals of this process. This method has been applied in [3] to establish a large deviation principle to reaction-diffusion systems considered in [12] and also in several subsequent papers, for instance in [11, 15, 17]. We next give the ingredients for the proof of Theorem 1.1 based on this method.

Variational Representation of Infinite Dimensional Brownian Motion

Let $B = (B_k(t), t \in [0,T], k \ge 1)$ be a sequence of independent standard Brownian motions. Denote by $\mathcal{P}(l^2)$ the set of predictable processes belonging to $L^2(\Omega \times [0,T]); l^2)$ and let g be a real-valued, bounded, Borel measurable function defined on $\mathcal{C}([0,T]; \mathbb{R}^\infty)$. Then,

$$-\log E(\exp[-g(B)]) = \inf_{u\in\mathcal{P}(l^2)} E\left(\frac{1}{2}\|u\|^2_{L^2([0,T];l^2)} + g\left(B + \int_0^\cdot u\right)\right) \tag{11}$$

(see Theorem 2 in [3]).

Weak Regularity

Denote by $\mathcal{P}_{\mathcal{H}}$ the set of predictable processes belonging to $L^2(\Omega\times[0,T]);\mathcal{H})$. For any $N>0$, we define

$$\mathcal{H}_T^N = \{h\in\mathcal{H}_T : \|h\|_{\mathcal{H}_T}\le N\},$$
$$\mathcal{P}_{\mathcal{H}}^N = \{v\in\mathcal{P}_{\mathcal{H}} : v\in\mathcal{H}_T^N, a.s.\},$$

and we consider $\mathcal{H}_T^N$ endowed with the weak topology of $\mathcal{H}_T$.

For any $v\in\mathcal{P}_{\mathcal{H}}^N$, $\varepsilon\in]0,1]$, let $u^{\varepsilon,v}$ be the solution to

$$\begin{aligned} u^{\varepsilon,v}(t,x) &= w(t,x) + \sqrt{\varepsilon}\sum_{k\ge1}\int_0^t \langle G(t-s,x-\cdot)\sigma(u^{\varepsilon,v}(s,\cdot)), e_k(\cdot)\rangle_{\mathcal{H}}\mathrm{d}B_k(s)\\ &\quad+\int_0^t \langle G(t-s,x-\cdot)\sigma(u^{\varepsilon,v}(s,\cdot)), v(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s\\ &\quad+\int_0^t [G(t-s)*b(u^{\varepsilon,v}(s,\cdot))](x)\mathrm{d}s. \end{aligned} \tag{12}$$

We will prove in Theorem 2.3 that this equation has a unique solution and that $u^{\varepsilon,v}\in\mathcal{E}_\alpha$ with $\alpha\in\mathcal{I}$.

Consider the following conditions:

(a) The set $\{V^h, h\in\mathcal{H}_T^N\}$ is a compact subset of $\mathcal{E}_\alpha$, where V^h is the solution of (9).
(b) For any family $(v^\varepsilon,\varepsilon>0)\subset\mathcal{P}_{\mathcal{H}}^N$ that converges in distribution as $\varepsilon\to0$ to $v\in\mathcal{P}_{\mathcal{H}}^N$, as $\mathcal{H}_T^N$-valued random variables, we have

$$\lim_{\varepsilon\to0} u^{\varepsilon,v^\varepsilon} = V^v,$$

in distribution, as $\mathcal{E}_\alpha$-valued random variables.

Here V^v stands for the solution of (9) corresponding to a $\mathcal{H}_T^N$-valued random variable v (instead of a deterministic function h). The solution is a stochastic process $\{V^h(t,x),(t,x)\in[0,T]\times\mathbb{R}^3\}$ defined path-wise by (9).

According to [3], Theorem 6 applied to the functional $\mathcal{G}:\mathcal{C}([0,T];\mathbb{R}^\infty)\to\mathcal{E}_\alpha$, $\mathcal{G}(\sqrt{\varepsilon}B):=u^\varepsilon$ (the solution of (8)), and $\mathcal{G}^0:\mathcal{H}_T\to\mathcal{E}_\alpha$, $\mathcal{G}^0(h):=V^h$ (the solution of (9)), conditions (a) and (b) above imply the validity of Theorem 1.1.

2 Laplace Principle for the Wave Equation

Following the discussion of the preceding section, the proof of Theorem 1.1 will consist of checking that conditions (a) and (b) above hold true. As we next show, both conditions will follow from a single continuity result. Indeed, the set $\mathcal{H}_T^N$ is a compact subset of $\mathcal{H}_T$ endowed with the weak topology (see [13], Chap. 12, Theorem 4). Thus, (a) can be obtained by proving that the mapping $h \in \mathcal{H}_T^N \mapsto V^h \in \mathcal{E}_\alpha$ is continuous with respect to the weak topology. For this, we consider a sequence $(h_n, n \ge 1) \subset \mathcal{H}_T^N$ and $h \in \mathcal{H}_T^N$ satisfying $\lim_{n\to\infty} \|h_n - h\|_w = 0$, which means that for any $g \in \mathcal{H}_T$, $\lim_{n\to\infty}\langle h_n - h, g\rangle_{\mathcal{H}_T} = 0$, and we will prove that

$$\lim_{n\to\infty} \|V^{h_n} - V^h\|_\alpha = 0. \tag{13}$$

As for (b), we invoke Skorohod Representation Theorem and rephrase this condition as follows. On some probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, consider a sequence of independent Brownian motions $\bar{B} = \{\bar{B}_k, k \ge 1\}$ along with the corresponding filtration $(\bar{\mathcal{F}}_t, t \in [0,T])$, where $\bar{\mathcal{F}}_t$ is the σ-field generated by the random variables $(\bar{B}_k(s), s \in [0,t], k \ge 1)$. Furthermore, consider a family of $(\bar{\mathcal{F}}_t)$-predictable processes $(\bar{v}^\varepsilon, \varepsilon > 0, \bar{v})$ belonging to $L^2(\bar{\Omega} \times [0,T]; \mathcal{H})$ taking values on $\mathcal{H}_T^N$, $\bar{P}$ a.s., such that the joint law of (v^ε, v, B) (under P) coincides with that of $(\bar{v}^\varepsilon, \bar{v}, \bar{B})$ (under $\bar{P}$) and such that,

$$\lim_{\varepsilon\to 0} \|\bar{v}^\varepsilon - \bar{v}\|_w = 0, \quad \bar{P} - a.s.$$

as $\mathcal{H}_T^N$-valued random variables. Let $\bar{u}^{\varepsilon,\bar{v}^\varepsilon}$ be the solution to a similar equation as (12) obtained by changing v into $\bar{v}^\varepsilon$ and B_k into $\bar{B}_k$. Then, we will prove that for any $q \in [0,\infty[$,

$$\lim_{\varepsilon\to 0} \bar{E}\left(\left\|\bar{u}^{\varepsilon,\bar{v}^\varepsilon} - V^{\bar{v}}\right\|_\alpha^q\right) = 0, \tag{14}$$

where $\bar{E}$ denotes the expectation operator on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. Notice that if in (12) we consider $\varepsilon = 0$ and $v := h \in \mathcal{P}_{\mathcal{H}}^N$ deterministic, we obtain the equation satisfied by V^h. Consequently, the convergence (13) can be obtained as a particular case of (14).

Therefore, we will focus our efforts on the proof of (14). In the sequel, we shall omit any reference to the bars in the notation, for the sake of simplicity.

According to Lemma A1 in [1], the proof of (14) can be carried out into two steps:

1. *Estimates on increments*

$$\sup_{\varepsilon\ge 1} E\left(\left|\left[u^{\varepsilon,v^\varepsilon}(t,x) - V^v(t,x)\right] - \left[u^{\varepsilon,v^\varepsilon}(r,z) - V^v(r,z)\right]\right|^q\right)$$
$$\le C\,[|t-r| + |x-z|]^{\alpha q}. \tag{15}$$

2. *Pointwise convergence*

$$\lim_{\varepsilon\to 0} E\left(|u^{\varepsilon,v^\varepsilon}(t,x) - V^v(t,x)|^q\right) = 0. \tag{16}$$

Here, $q \in [1,\infty[$, $(t,x), (r,z) \in [0,T] \times D$ and $\alpha \in \mathcal{I}$.

Before proving these facts, we will address the problem of giving a rigorous formulation of (12). As we have already mentioned, the stochastic integral with respect to $(B_k, k \geq 1)$ in (12) is equivalent to the stochastic integral $\int_0^t \int_{\mathbb{R}^3} G(t-s, x-y)\sigma(u^{\varepsilon,v^\varepsilon}(s,y))M(\mathrm{d}s,\mathrm{d}y)$ considered in the sense of [6]. We recall that such an integral is defined for stochastic processes $Z = (Z(s,\cdot), s \in [0,T])$ with values in $L^2(\mathbb{R}^3)$ a.s., adapted and mean-square continuous, and the integral

$$v_{G,Z}^t(\star) := \sum_{k\geq 1} \int_0^t \langle G(t-s, \star - \cdot)Z(s,\cdot), e_k(\cdot)\rangle_{\mathcal{H}} \mathrm{d}B_k(s) \tag{17}$$

satisfies

$$E\left(\|v_{G,Z}^t\|_{L^2(\mathbb{R}^3)}^2\right) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}\xi E(|\mathcal{F}Z(s)(\xi)|^2) \int_{\mathbb{R}^3} \mu(\mathrm{d}\eta)|\mathcal{F}G(t-s)(\xi-\eta)|^2. \tag{18}$$

(see [6], Theorem 6).

As a function of the argument x, for any $v \in \mathcal{P}_{\mathcal{H}}^N$, the path-wise integral

$$\int_0^t \langle G(t-s, x-\cdot)\sigma(u^{\varepsilon,v}(s,\cdot)), v(s,\cdot)\rangle_{\mathcal{H}} \,\mathrm{d}s,$$

is also a well-defined $L^2(\mathbb{R}^3)$-valued random variable. Indeed, let Z be a stochastic process satisfying the hypotheses described before. Set

$$\nu_{G,Z}^t(\star) := \int_0^t \langle G(t-s, \star - \cdot)Z(s,\cdot), v(s,\cdot)\rangle_{\mathcal{H}} \,\mathrm{d}s. \tag{19}$$

By Cauchy-Schwarz' inequality applied to the inner product on $\mathcal{H}_T$, we have

$$\begin{aligned}\|\nu_{G,Z}^t\|_{L^2(\mathbb{R}^3)}^2 &\leq N^2 \int_{\mathbb{R}^3} \mathrm{d}x \int_0^t \mathrm{d}s \|G(t-s, x-\cdot)Z(s,\cdot)\|_{\mathcal{H}}^2 \\ &= N^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}\xi |\mathcal{F}Z(s)(\xi)|^2 \int_{\mathbb{R}^3} \mu(\mathrm{d}\eta)|\mathcal{F}G(t-s)(\xi-\eta)|^2,\end{aligned}$$

where the last equality is derived following the arguments for the proof of Theorem 6 in [6]. We recall that this formula is firstly established for Z sufficiently smooth and by smoothing G by convolution with an approximation of the identity. The extension of the formula to the standing assumptions is done by a limit procedure.

From this, we clearly have

$$\begin{aligned}&E\left(\|\nu_{G,Z}^t\|_{L^2(\mathbb{R}^3)}^2\right) \\ &\quad\leq N^2 \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \mathrm{d}\xi E(|\mathcal{F}Z(s)(\xi)|^2) \int_{\mathbb{R}^3} \mu(\mathrm{d}\eta)|\mathcal{F}G(t-s)(\xi-\eta)|^2. \end{aligned} \tag{20}$$

Remark 2.1. Up to a positive constant, the $L^2(\Omega; L^2(\mathbb{R}^3))$-norm of the stochastic integral $v^t_{G,Z}$ and the path-wise integral $\nu^t_{G,Z}$ are bounded by the same expression.

Let $\mathcal{O}$ be a bounded or unbounded open subset of $\mathbb{R}^3$, $q \in [1,\infty[$, $\gamma \in]0,1[$. We denote by $W^{\gamma,q}(\mathcal{O})$ the fractional Sobolev Banach space consisting of functions $\varphi : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\|\varphi\|_{W^{\gamma,q}(\mathcal{O})} := \left(\|\varphi\|^q_{L^q(\mathcal{O})} + \|\varphi\|^q_{\gamma,q,\mathcal{O}}\right)^{\frac{1}{q}} < \infty,$$

where

$$\|\varphi\|_{\gamma,q,\mathcal{O}} = \left(\int_{\mathcal{O}} \mathrm{d}x \int_{\mathcal{O}} \mathrm{d}y \frac{|\varphi(x)-\varphi(y)|^q}{|x-y|^{3+\gamma q}}\right)^{\frac{1}{q}}.$$

For any $\varepsilon > 0$, we denote by $\mathcal{O}^\varepsilon$ the ε-enlargement of $\mathcal{O}$, that is,

$$\mathcal{O}^\varepsilon = \{x \in \mathbb{R}^3 : d(x,\mathcal{O}) < \varepsilon\}.$$

In the proof of Theorem 2.3 below, we will use a smoothed version of the fundamental solution G defined as follows. Consider a function $\psi \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}_+)$ with support included in the unit ball, such that $\int_{\mathbb{R}^3} \psi(x)\mathrm{d}x = 1$. For any $t \in]0,1]$ and $n \geq 1$, set

$$\psi_n(t,x) = \left(\frac{n}{t}\right)^3 \psi\left(\frac{n}{t}x\right),$$

and

$$G_n(t,x) = (\psi_n(t,\cdot) * G(t))\,(x). \tag{21}$$

Notice that for any $t \in [0,T]$, $\operatorname{supp} G_n(t,\cdot) \subset B_{t(1+\frac{1}{n})}(0)$.

Remark 2.2. Since $G_n(t)$ is smooth and has compact support, $v^t_{G_n,z}(x)$ is well-defined as a Walsh stochastic integral, and this integral defines a random field indexed by (t,x). By Burkholder's inequality, for any $q \in [2,\infty[$,

$$E\left(|v^t_{G_n,z}(x)|^q\right) \leq CE\left(\int_0^t \|G(t-s,x-\cdot)Z(s,\cdot)\|^2_{\mathcal{H}}\,\mathrm{d}s\right)^{\frac{q}{2}}.$$

As for the path-wise integral $\nu^t_{G_n,z}(x)$, by applying Cauchy-Schwarz' inequality to the inner product on $\mathcal{H}_T$, we have

$$E\left(|\nu^t_{G_n,z}(x)|^q\right) \leq N^q E\left(\int_0^t \|G(t-s,x-\cdot)Z(s,\cdot)\|^2_{\mathcal{H}}\,\mathrm{d}s\right)^{\frac{q}{2}}.$$

Hence, as in Remark 2.1, up to a constant, $L^q(\Omega)$-estimates for both type of integrals at fixed $(t,x) \in [0,T]\times\mathbb{R}^3$ coincide.

The following proposition is the analogue of Theorem 3.1 in [7] for the path-wise integral $\nu^t_{G,Z}$.

Proposition 2.1. *Fix $q \in]3, \infty[$ and a bounded domain $\mathcal{O} \subset \mathbb{R}^3$. Suppose that*

$$\tau_q(\beta, \delta) := \left(\frac{2-\beta}{2} \wedge \frac{1+\delta}{2}\right) - \frac{3}{q} > 0$$

and fix $\gamma \in]0,1[$, $\rho \in]0, \tau_q(\beta,\delta) \wedge \gamma[$. Let $\{Z_t, t \in [0,T]\}$ be a $L^2(\mathbb{R}^3)$-valued, $(\mathcal{F}_t)$-adapted, mean-square continuous stochastic process. Assume that for some fixed $t \in [0,T]$,

$$\int_0^t E\left(\|Z(s)\|^q_{W^{\gamma,q}(\mathcal{O}^{t-s})}\right) \mathrm{d}s < \infty.$$

We have the following estimates:

$$E\left(\|\nu^t_{G,Z}\|^q_{L^q(\mathcal{O})}\right) \le C \int_0^t E\left(\|Z(s)\|^q_{L^q(\mathcal{O}^{t-s})}\right) \mathrm{d}s, \tag{22}$$

$$E\left(\|\nu^t_{G_n,Z}\|^q_{\rho,q,\mathcal{O}}\right) \le C \int_0^t E\left(\|Z(s)\|^q_{W^{\rho,q}(\mathcal{O}^{(t-s)(1+\frac{1}{n})})}\right) \mathrm{d}s, \tag{23}$$

$$E\left(\|\nu^t_{G,Z}\|^q_{\rho,q,\mathcal{O}}\right) \le C \int_0^t E\left(\|Z(s)\|^q_{W^{\rho,q}(\mathcal{O}^{(t-s)})}\right) \mathrm{d}s. \tag{24}$$

Consequently,

$$E\left(\|\nu^t_{G,Z}\|^q_{W^{\rho,q}(\mathcal{O})}\right) \le C \int_0^t E\left(\|Z(s)\|^q_{W^{\rho,q}(\mathcal{O}^{(t-s)})}\right) \mathrm{d}s. \tag{25}$$

Proof. By virtue of Remark 2.2, we see that the estimate (22) follows from the same arguments used in [7], Proposition 3.4. We recall that this proposition is devoted to prove an analogue property for the stochastic integral $v^t_{G,Z}$. In the very same way, (23) is established using the arguments of the proof of Proposition 3.5 in [7]. Then, as in [7], (24) is obtained from (23) by applying Fatou's lemma. Finally, (25) is a consequence of (22), (24) and the definition of the fractional Sobolev norm $\|\cdot\|_{W^{\rho,q}(\mathcal{O})}$. □

Next, we present an analogue of Theorem 3.8 [7] for the path-wise integral $\nu^t_{G,Z}$, which gives the sample path properties in the argument t for this integral. As in Proposition 2.1, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^3$.

Proposition 2.2. *Consider a stochastic process $\{Z_t, t \in [0,T]\}$, $(\mathcal{F}_t)$-adapted, with values in $L^2(\mathbb{R}^3)$, mean-square continuous. Assume that for some fixed $q \in]3, \infty[$ and $\gamma \in]\frac{3}{q}, 1[$,*

$$\sup_{t\in[0,T]} E\left(\|Z(t)\|^q_{W^{\gamma,q}(\mathcal{O}^{T-t})}\right) < \infty.$$

Then, the stochastic process $\{\nu^t_{G,Z}(x), t \in [0,T]\}$, $x \in \mathcal{O}$, satisfies

$$\sup_{x\in\mathcal{O}} E\left(|\nu^t_{G,Z}(x) - \nu^{\bar{t}}_{G,Z}(x)|^{\bar{q}}\right) \le C|t-\bar{t}|^{\rho\bar{q}}, \tag{26}$$

for each $t, \bar{t} \in [0,T]$, any $\bar{q} \in]2, q[$, $\rho \in]0, (\gamma - \frac{3}{q}) \wedge (\frac{2-\beta}{2}) \wedge (\frac{1+\delta}{2})[$.

Proof. We follow the same scheme as in the proof of [7], Theorem 3.8. To start with, we should prove an analogue of (3.26) in [7], with $v^{\bar{t}}_{G_n,Z}$, $v^{t}_{G_n,Z}$ replaced by $\nu^{\bar{t}}_{G_n,Z}$, $\nu^{t}_{G_n,Z}$, respectively. Once again, we apply Remark 2.2, obtaining similar upper bounds for the $L^{\bar{q}}(\Omega)$-moments (up to a positive constant) as for the stochastic integrals considered in the above mentioned reference. More precisely, assume $0 \le t < \bar{t} \le T$; by applying Cauchy-Schwarz' inequality to the inner product on $\mathcal{H}_T$, we obtain

$$E\left(\left|\int_t^{\bar{t}} \langle G_n(\bar{t}-s, x-\cdot)Z(s,\cdot), v(s,\cdot)\rangle_{\mathcal{H}}\, \mathrm{d}s\right|^{\bar{q}}\right)$$

$$\le N^{\bar{q}} E\left(\int_0^{\bar{t}-t} \|G_n(s, x-\cdot)Z(\bar{t}-s,\cdot)\|^2_{\mathcal{H}}\, \mathrm{d}s\right)^{\frac{\bar{q}}{2}},$$

$$E\left(\left|\int_0^t \langle (G_n(\bar{t}-s, x-\cdot) - G_n(t-s, x-\cdot)Z(s,\cdot), v(s,\cdot)\rangle_{\mathcal{H}}\, \mathrm{d}s\right|^{\bar{q}}\right)$$

$$\le N^{\bar{q}} E\left(\int_0^t \|(G_n(\bar{t}-s, x-\cdot) - G_n(t-s, x-\cdot)Z(s,\cdot)\|^2_{\mathcal{H}}\, \mathrm{d}s\right)^{\frac{\bar{q}}{2}}.$$

These are, up to a positive constant, the same upper bounds obtained in [7] for the expressions termed $T_1^n(t,\bar{t},x)$ and $T_2^n(t,\bar{t},x)$, respectively. After this remark, the proof follows the same arguments as in [7]. □

For any $t \in [0,T]$, $a \ge 1$, let $K_a^D(t) = \{y \in \mathbb{R}^3 : d(y,D) \le a(T-t)\}$. For $a = 1$, we shall simply write $K^D(t)$; this is the light cone of $\{T\} \times D$.

In the next theorem, the statement on existence and uniqueness of solution, as well as (27), extend Theorem 4.3 in [8], while (28) and (29) are extensions of the inequality (4.24) of Theorem 4.6 and (4.41) of Theorem 4.11 in [7], respectively. Indeed in the cited references, the results apply to (4), while in the next theorem, they apply to (12).

Theorem 2.3. *Assuming* **(H)***, the following statements hold true:*

There exists a unique random field solution to (12), $\{u^{\varepsilon,v}(t,x), (t,x) \in [0,T] \times \mathbb{R}^3\}$, and this solution satisfies

$$\sup_{\varepsilon \in]0,1], v \in \mathcal{P}^N_{\mathcal{H}}} \; \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E\left(|u^{\varepsilon,v}(t,x)|^q\right) < \infty, \tag{27}$$

$$\sup_{\varepsilon \in]0,1], v \in \mathcal{P}^N_{\mathcal{H}}} \; \sup_{t\in[0,T]} E\left(\|u^{\varepsilon,v}(t)\|^q_{W^{\alpha,q}(K^D(t))}\right) < \infty, \tag{28}$$

for any $q \in [2,\infty[$, $\alpha \in \mathcal{I}$.

Moreover, for any $q \in [2,\infty[$ *and* $\alpha \in \mathcal{I}$*, there exists* $C > 0$ *such that for* $(t,x), (\bar{t}, y) \in [0,T] \times D$*,*

$$\sup_{\varepsilon \in]0,1], v \in \mathcal{P}_{\mathcal{H}}^N} E(|u^{\varepsilon,v}(t,x) - u^{\varepsilon,v}(\bar{t},y)|^q) \leq C(|t-\bar{t}| + |x-y|)^{\alpha q}. \tag{29}$$

Thus, a.s., $\{u^{\varepsilon,v}(t,x), (t,x) \in [0,T] \times D\}$ *has Hölder continuous sample paths of degree* $\alpha \in \mathcal{I}$*, jointly in* (t,x)*.*

Proof. For the sake of simplicity, we shall consider $\varepsilon = 1$ and write u^v instead of $u^{\varepsilon,v}$.

We start by proving existence and uniqueness along with (27). For this, we will follow the method of the proof of [8], Theorem 4.3 (borrowed from [14], Theorem 1.2 and [4], Theorem 13). It is based on the Picard iteration scheme:

$$\begin{aligned} u^{v,(0)}(t,x) &= w(t,x), \\ u^{v,(n+1)}(t,x) &= w(t,x) + \sum_{k\geq 1} \int_0^t \langle G(t-s, x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), e_k(\cdot)\rangle_{\mathcal{H}} \mathrm{d}B_k(s) \\ &\quad + \int_0^t \langle G(t-s, x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), v(s,\cdot)\rangle_{\mathcal{H}} \, \mathrm{d}s \\ &\quad + \int_0^t [G(t-s) * b(u^{v,(n)}(s,\cdot))](x) \, \mathrm{d}s, \quad n \geq 0. \end{aligned} \tag{30}$$

The steps of the proof are as follows. Firstly, we check that

$$\sup_{v \in \mathcal{P}_{\mathcal{H}}^N} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E\left(|u^{v,(n)}(t,x)|^q\right) < \infty, \tag{31}$$

and then

$$\sup_{n\geq 0} \sup_{v \in \mathcal{P}_{\mathcal{H}}^N} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E\left(|u^{v,(n)}(t,x)|^q\right) < \infty. \tag{32}$$

Secondly, by setting

$$M_n(t) := \sup_{(s,x)\in[0,t]\times\mathbb{R}^3} E\left(|u^{v,(n+1)}(s,x) - u^{v,(n)}(s,x)|^q\right), \ n \geq 0,$$

we prove

$$M_n(t) \leq C \int_0^t M_{n-1}(s) \left(1 + \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\mathcal{F}G(t-s)(\xi)|^2\right) \mathrm{d}s. \tag{33}$$

With these facts, we conclude that $(u^{v,(n)}(t,x), n \geq 0)$ converges uniformly in (t,x) in $L^q(\Omega)$ to a limit $u^v(t,x)$, which satisfies (12) with $\varepsilon = 1$.

In comparison with the proof of Theorem 4.3 in [8], establishing (31)–(33) requires additionally the analysis of the term given by the path-wise integral

$$\mathcal{I}^{v,(n+1)} := \int_0^t \langle G(t-s, x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), v(s,\cdot)\rangle_{\mathcal{H}} \, \mathrm{d}s, \; n \geq 0. \tag{34}$$

This is done as follows. We assume that (31) holds true for some $n \geq 0$. This is definitely the case for $n = 0$ (see (7)). By applying Cauchy-Schwarz inequality on the Hilbert space $\mathcal{H}_T$, and since $\|v\|_{\mathcal{H}_T} \leq N$ a.s., we have

$$E\left(\left|\int_0^t \langle G(t-s, x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), v(s,\cdot)\rangle_{\mathcal{H}} \, \mathrm{d}s\right|^q\right) \leq N^q E\left(\int_0^t \left\|G(t-s, x-\cdot)\sigma(u^{v,(n)}(s,\cdot))\right\|_{\mathcal{H}}^2 \, \mathrm{d}s\right)^{\frac{q}{2}}.$$

Notice that, by applying Burkholder's inequality to the stochastic integral term in (30), we obtain

$$E\left(\left|\sum_{k\geq 1}\int_0^t \langle G(t-s, x-\cdot)\sigma(u^{v,(n)}(s,\cdot)), e_k(\cdot)\rangle_{\mathcal{H}} \mathrm{d}B_k(s)\right|^q\right) \leq CE\left(\int_0^t \left\|G(t-s, x-\cdot)\sigma(u^{v,(n)}(s,\cdot))\right\|_{\mathcal{H}}^2 \, \mathrm{d}s\right)^{\frac{q}{2}}.$$

Thus, as has already been mentioned in Remark 2.2, up to a positive constant, $L^q(\Omega)$ estimates of the stochastic integral and of the path-wise integral $\mathcal{I}^{v,(n+1)}$ lead to the same upper bounds.

This simple but important remark yields the extension of properties (31)–(33), which are valid for (4), as is proved in Theorem 4.3 in [8], to (12) with $\varepsilon = 1$ and actually, for any $\varepsilon \in]0,1]$. In fact, those properties can be proved to hold uniformly in $\varepsilon \in]0,1]$.

Let us now argue on the validity of (28). We will follow the programme of Sect. 4.2 in [7], taking into account the new term

$$\int_0^t \langle G(t-s, x-\cdot)\sigma(u^v(s,\cdot)), v(s,\cdot)\rangle_{\mathcal{H}} \, \mathrm{d}s$$

of (12) (with $\varepsilon = 1$) that did not appear in [7]. This consists of the following steps.

Firstly, we need an extension of Proposition 4.3 in [7]. This refers to an approximation of the localized version of (12) on a light cone. In the approximating sequence, the fundamental solution G of the wave equation is replaced by a smoothed version G_n defined in (21). Going through the proof of that Proposition, we see that for the required extension, the term

$$M_n(t) := E\left(\|v^t_{G_n,Z} - v^t_{G,Z}\|^q_{L^q(K^D_a(t))}\right),$$

with $Z(s,y) = \sigma(u^v(s,y))1_{K^D_a(s)}(y)$, should be replaced by

$$\tilde{M}_n(t) := E\left(\|v^t_{G_n,Z} - v^t_{G,Z}\|^q_{L^q(K^D_a(t))}\right) + E\left(\|\nu^t_{G_n,Z} - \nu^t_{G,Z}\|^q_{L^q(K^D_a(t))}\right),$$

where we have used the notation introduced in (17), (19). Then, we should prove that $\lim_{n\to\infty} \tilde{M}_n(t) = 0$. This is carried out by considering first the case $q = 2$. By Remark 2.1, it suffices to have $\lim_{n\to\infty} M_n(t) = 0$ for $q = 2$, and this fact is proved in [7], Proposition 4.3.

To extend the convergence to any $q \in]2,\infty[$, we must establish that for some fixed $n_0 > 0$,

$$\sup_{n\ge n_0} E\left(\|\nu^t_{G_n,Z}\|^q_{L^q(K^D_a(t))}\right) < \infty \tag{35}$$

a result that holds true for $v^t_{G_n,Z}$. Once more, the first step in the proof of (35) consists in obtaining the upper bound

$$E\left(\|\nu^t_{G_n,Z}\|^q_{L^q(K^D_a(t))}\right) \le CE\int_0^t \mathrm{d}s \int_{K^D_a(t)} \mathrm{d}x E\left(\|G_n(t-s,x-\cdot)Z(s,\cdot)\|^q_{\mathcal{H}}\right). \tag{36}$$

This follows easily by applying first Cauchy-Schwarz' inequality to the inner product on $\mathcal{H}_T$ and then Hölder's inequality. Once we have (36), we can obtain (35) by following the steps of the proof of Proposition 3.4 in [7].

The last ingredient for the proof of (28) consist of the extension of Theorem 4.6 in [7]. This requires the following additional arguments. Firstly, using similar notations as in that reference, we set

$$R^{m,\gamma,D}_n(t) = E\left(\|u^{v,(m)}_n(t)\|^q_{W^{\gamma,q}(K^D_a(t))}\right),$$

where $u^{v,(m)}_n(t,x)$ stands for the mth Picard iteration of a similar equation as (12) with G replaced by the smoothed version G_n. In comparison with [7], in order to check that $\sup_{n,m\ge1} R^{m,\gamma,D}_n < \infty$, we have to study the additional term

$$T^{m,\gamma,D,3}_n(t) := E\left(\|\nu^t_{G_n,\sigma(u^{v,(m)}_n)1_{K^D_a}}\|^q_{W^{\gamma,q}(K^D_a(t))}\right)$$

and more specifically to check that

$$T^{m,\gamma,D,3}_n(t) \le C_1 + C_2\int_0^t \mathrm{d}s\, R^{m-1,\gamma,D}_n(s), \tag{37}$$

for some positive constants C_1, C_2.

This property holds true when $T_n^{m,\gamma,D,3}(t)$ is replaced by

$$E\left(\|v^t_{G_n,\sigma(u_n^{v,(m)})1_{K_a^D}}\|^q_{W^{\gamma,q}(K_a^D(t))}\right)$$

(see the arguments on page 42 of [7] based upon Proposition 3.5 of this reference). In a similar way, (37) follows from Proposition 2.1 and more precisely from (23).

This completes the proof of (28).

An important consequence of (28) is the following. For any $t > 0$, a.s., the sample paths of $(u^{\varepsilon,v}(t.x)1_{K^D(t)}(x), x \in \mathbb{R}^3)$ are α-Hölder continuous with $\alpha \in \mathcal{I}$. In addition, for any $q \in [2, \infty[$,

$$\sup_{\varepsilon \in]0,1], v \in \mathcal{P}_{\mathcal{H}}^N} \sup_{t \in [0,T]} E\left(|u^{\varepsilon,v}(t,x) - u^{\varepsilon,v}(t,y)|^q\right) \le C|x-y|^{\alpha q}, \tag{38}$$

for any $x, y \in K^D(t)$, $\alpha \in \mathcal{I}$. Hence, in order to prove (29), it remains to establish that for any $q \in [2, \infty[$ and $\alpha \in \mathcal{I}$, there exists $C > 0$ such that for every $t, \bar{t} \in [0, T]$,

$$\sup_{\varepsilon \in]0,1], v \in \mathcal{P}_{\mathcal{H}}^N} \sup_{x \in D} E\left(|u^{\varepsilon,v}(t,x) - u^{\varepsilon,v}(\bar{t},x)|^q\right) \le C|t-\bar{t}|^{\alpha q}, \tag{39}$$

For this, we will follow the steps of Sect. 4.3 in [7] devoted to the analysis of the time regularity of the solution to (4) and get an extension of Theorem 4.10.

As in the first part of the proof, we consider the case $\varepsilon = 1$. The additional required ingredient consists of showing that

$$\begin{aligned} E\Bigg(\Bigg|&\int_0^t \langle G(t-s, x-\cdot)\sigma(u^v(s,\cdot))1_{K^D(s)}(\cdot), v(s,\cdot)\rangle_{\mathcal{H}}\, ds \\ &- \int_0^{\bar{t}} \langle G(\bar{t}-s, x-\cdot)\sigma(u^v(s,\cdot))1_{K^D(s)}(\cdot), v(s,\cdot)\rangle_{\mathcal{H}}\, ds\Bigg|^q\Bigg) \\ \le\, & C|t-\bar{t}|^{\alpha q}, \end{aligned} \tag{40}$$

uniformly in $x \in D$.

Remark that the stochastic process

$$\{Z(s,y) := \sigma(u^v(s,y))1_{K^D(s)}(y),\ (s,y) \in [0,T] \times \mathbb{R}^3\},$$

satisfies the assumptions of Proposition 2.2 with $\mathcal{O} = D$ and arbitrarily large q. This fact is proved in Theorem 4.10 in [7]. Thus, (40) follows from that Proposition.

Going through the arguments, it is easy to realize that for $u^{v,\varepsilon}$, we can get uniform estimates in $\varepsilon \in]0, 1]$ and $v \in \mathcal{P}_{\mathcal{H}}^N$, and therefore (39) holds true. This ends the proof of (29) and of the Theorem. □

Remark 2.3. In connection with conclusion (4.8) of Theorem 4.1 in [7], we notice that property (27) implies

$$\sup_{\varepsilon\in]0,1],v\in\mathcal{P}_{\mathcal{H}}^N}\ \sup_{t\in[0,T]} E\left(\|u^{\varepsilon,v}(t)\|^q_{L^q(K^D(t))}\right)<\infty.$$

The estimates on increments described in (15) are a consequence of (29). Indeed, as has been already mentioned, for any $v\in\mathcal{P}_{\mathcal{H}}^N$, the stochastic process V^v is the solution to the particular (12) obtained by setting $\varepsilon=0$.

Proposition 2.4. *Assume* **(H)**. *Consider a family* $(v^\varepsilon,\varepsilon>0)\subset\mathcal{P}_{\mathcal{H}}^N$ *and* $v\in\mathcal{P}_{\mathcal{H}}^N$ *such that a.s.,*

$$\lim_{\varepsilon\to 0}\|v^\varepsilon-v\|_w=0.$$

Then, for any $(t,x)\in[0,T]\times D$ *and any* $q\in[2,\infty[$,

$$\lim_{\varepsilon\to 0}E\left(|u^{\varepsilon,v^\varepsilon}(t,x)-V^v(t,x)|^q\right)=0. \tag{41}$$

Proof. We write

$$u^{\varepsilon,v^\varepsilon}(t,x)-V^v(t,x)=\sum_{i=1}^4 T_i^\varepsilon(t,x),$$

with

$$T_1^\varepsilon(t,x)=\int_0^t\left[G(t-s)*\left(b(u^{\varepsilon,v^\varepsilon}(s,\cdot))-b(V^v(s,\cdot))\right)\right](x)\,\mathrm{d}s,$$

$$T_2^\varepsilon(t,x)=\int_0^t\left\langle G(t-s,x-\cdot)\left[\sigma(u^{\varepsilon,v^\varepsilon}(s,\cdot))-\sigma(V^v(s,\cdot))\right],v^\varepsilon(s,\cdot)\right\rangle_{\mathcal{H}}\mathrm{d}s,$$

$$T_3^\varepsilon(t,x)=\int_0^t\langle G(t-s,x-\cdot)\sigma(V^v(s,\cdot)),v^\varepsilon(s,\cdot)-v(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s,$$

$$T_4^\varepsilon(t,x)=\sqrt{\varepsilon}\sum_{k\geq 1}\int_0^t\langle G(t-s,x-\cdot)\sigma(u^{\varepsilon,v^\varepsilon}(s,\cdot)),e_k(\cdot)\rangle_{\mathcal{H}}\mathrm{d}B_k(s).$$

Fix $q\in[2,\infty[$. Hölder's inequality with respect to the measure on $[0,t]\times\mathbb{R}^3$ given by $G(t-s,\mathrm{d}y)\mathrm{d}s$, along with the Lipschitz continuity of b yield

$$\begin{aligned}E\left(|T_1^\varepsilon(t,x)|^q\right)&\leq\left(\int_0^t\mathrm{d}s\int_{\mathbb{R}^3}G(s,\mathrm{d}y)\right)^{q-1}\\&\quad\times\int_0^t\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}E\left(|u^{\varepsilon,v^\varepsilon}(r,z)-V^v(r,z)|^q\right)\left(\int_{\mathbb{R}^3}G(s,\mathrm{d}y)\right)\mathrm{d}s\\&\leq C\int_0^t\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}E\left(|u^{\varepsilon,v^\varepsilon}(r,z)-V^v(r,z)|^q\right)\,\mathrm{d}s\end{aligned}$$

To study $T_2^\varepsilon(t,x)$, we apply Cauchy-Schwarz' inequality to the inner product on $\mathcal{H}$ and then Hölder's inequality with respect to the measure on $[0,t]\times\mathbb{R}^3$ given by $|\mathcal{F}G(s)(\xi)|^2\mathrm{d}s\ \mu(\mathrm{d}\xi)$. Notice that this measure can also be written as $[G(s)*\tilde{G}(s)](x)\Gamma(\mathrm{d}x)\mathrm{d}s$. The Lipschitz continuity of σ along with (5) and the property $\sup_\varepsilon \|v^\varepsilon\|_{\mathcal{H}_T}\le N$ imply

$$\begin{aligned}
E\left(|T_2^\varepsilon(t,x)|^q\right) &\le E\left(\int_0^t \left\|G(t-s,x-\cdot)\left[\sigma(u^{\varepsilon,v^\varepsilon}(s,\cdot))-\sigma(V^v(s,\cdot))\right]\right\|_{\mathcal{H}}^2\mathrm{d}s\right)^{\frac{q}{2}}\\
&\quad\times\left(\int_0^t\|v^\varepsilon(s,\cdot)\|_{\mathcal{H}}^2\mathrm{d}s\right)^{\frac{q}{2}}\\
&\le CE\left(\int_0^t \left\|G(t-s,x-\cdot)\left[\sigma(u^{\varepsilon,v^\varepsilon}(s,\cdot))-\sigma(V^v(s,\cdot))\right]\right\|_{\mathcal{H}}^2\mathrm{d}s\right)^{\frac{q}{2}}\\
&\le C\left(\int_0^t\mathrm{d}s\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}G(t-s)(\xi)|^2\right)^{\frac{q}{2}-1}\\
&\quad\times\int_0^t\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}E\left(|u^{\varepsilon,v^\varepsilon}(r,z)-V^v(r,z)|^q\right)\\
&\quad\times\left(\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}G(s)(\xi)|^2\right)\mathrm{d}s\\
&\le C\int_0^t\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}E\left(|u^{\varepsilon,v^\varepsilon}(r,z)-V^v(r,z)|^q\right)\mathrm{d}s.
\end{aligned}$$

For any $(t,x)\in[0,T]\times\mathbb{R}^3$, the stochastic process

$$\{G(t-s,x-y)\sigma(V^v(s,y)),(s,y)\in[0,T]\times\mathbb{R}^3\}$$

satisfies the property

$$\sup_{v\in\mathcal{P}_{\mathcal{H}}^N}\sup_{s\in[0,T]}E\left(\|G(t-s,x-\cdot)\sigma(V^v(s,\cdot))\|_{\mathcal{H}}^q\right)<\infty. \tag{42}$$

Indeed, by applying (27) to the particular case $\varepsilon=0$, we get

$$\sup_{v\in\mathcal{P}_{\mathcal{H}}^N}\sup_{(t,x)\in[0,T]\times\mathbb{R}^3}E\left(|V^v(t,x)|^q\right)<\infty. \tag{43}$$

Then, we apply Hölder's inequality with respect to the measure on $\mathbb{R}^3$ given by $|\mathcal{F}G(t-s)(\xi)|^2\mu(\mathrm{d}\xi)$, along with the linear growth property of σ, and we obtain

$$E\left(\|G(t-s,x-\cdot)\sigma(V^v(s,\cdot))\|_{\mathcal{H}}^q\right) \le C\left(\int_{\mathbb{R}^3}|\mathcal{F}G(t-s)(\xi)|^2\mu(\mathrm{d}\xi)\right)^{\frac{q}{2}} \times\left(1+\sup_{(s,y)\in[0,T]\times\mathbb{R}^3}E\left(|V^v(s,y)|^q\right)\right).$$

With (5) and (43), we have (42).

From (42), it follows that $\{G(t-s,x-y)\sigma(V^v(s,y),(s,y)\in[0,T]\times\mathbb{R}^3\}$ takes its values in $\mathcal{H}_T$, a.s. Since $\lim_{\varepsilon\to 0}\|v^\varepsilon-v\|_w=0$, a.s.,

$$\lim_{\varepsilon\to 0}\left|\int_0^t\langle G(t-s,x-\cdot)\sigma(V^v(s,\cdot)),v^\varepsilon(s,\cdot)-v(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s\right|=0.$$

Applying (42) and bounded convergence, we see that the above convergence takes place in $L^q(\Omega)$ as well. Thus,

$$\lim_{\varepsilon\to 0}E\left(|T_3^\varepsilon(t,x)|^q\right)=0.$$

By the L^q estimates of the stochastic integral and (27), we have

$$\begin{aligned}
&E\left(\left|\sum_{k\ge 1}\int_0^t\langle G(t-s,x-\cdot)\sigma(u^{\varepsilon,v^\varepsilon}(s,\cdot)),e_k(\cdot)\rangle_{\mathcal{H}}\mathrm{d}B_k(s)\right|^q\right)\\
&=E\left(\int_0^t\|G(t-s,x-\cdot)\sigma(u^{\varepsilon,v^\varepsilon}(s,\cdot))\|_{\mathcal{H}_T}^2\,\mathrm{d}s\right)^{\frac{q}{2}}\\
&\le\left(\int_0^t\mathrm{d}s\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}G(t-s)(\xi)|^2\right)^{\frac{q}{2}-1}\\
&\quad\times\int_0^t\left(1+\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}E\left(|u^{\varepsilon,v^\varepsilon}(r,z)|^q\right)\right)\left(\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}G(s)(\xi)|^2\right)\mathrm{d}s\\
&\le C\int_0^t\left(1+\sup_{(r,z)\in[0,s]\times\mathbb{R}^3}E\left(|u^{\varepsilon,v^\varepsilon}(r,z)|^q\right)\right)\mathrm{d}s\\
&\le C.
\end{aligned}$$

This yields

$$\lim_{\varepsilon\to 0}E\left(|T_4^\varepsilon(t,x)|^q\right)=0.$$

We end the proof of the Proposition by applying the usual version of Gronwall's lemma.

Notice that we have actually proved the stronger statement

$$\lim_{\varepsilon\to 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E\left(|u^{\varepsilon,v^\varepsilon}(t,x) - V^v(t,x)|^p\right) = 0. \tag{44}$$

□

Proof of Theorem 1.1. As has been argued, it suffices to check the validity of (15) and (16). These statements follow from Theorem 2.3 and Proposition 2.4, respectively.

□

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Intertwinned Diffusions by Examples

Xue-Mei Li

Abstract We discuss the geometry induced by pairs of diffusion operators on two states spaces related by a map from one space to the other. This geometry led us to an intrinsic point of view on filtering. This will be explained plainly by examples, in local coordinates and in the metric setting. This article draws largely from the books On the geometry of diffusion operators and stochastic flows [11] and The Geometry of Filtering [13], and aims to have a comprehensive account of the geometry for a general audience.

Keywords Linear and equi-variant connections · stochastic differential equations · geometry of diffusion operators

MSC (2010): 60Hxx, 60Dxx

1 Introduction

Let p be a differentiable map from a manifold N to M which intertwines a diffusion operator $\mathcal{B}$ on N with another diffusion operator, $\mathcal{A}$ on M, that is $(\mathcal{A}f)\circ p = \mathcal{B}(f \circ p)$ for a given function f from M to $\mathbf{R}$. Suppose that $\mathcal{A}$ is elliptic and f is smooth. It is stated in [11] that this intertwining pair of operators determines a unique horizontal lifting map $\mathfrak{h}$ from TM to TN which is induced by the symbols of $\mathcal{A}$ and $\mathcal{B}$ and the image of the lifting map determines a subspace of the tangent space to N and is called the associated horizontal tangent space and denoted by H. The condition that $\mathcal{A}$ is elliptic can be replaced by cohesiveness, that is, the symbol $\sigma^{\mathcal{A}} : T_x^*M \to T_xM$ has constant non-zero rank and $\mathcal{A}$ is along the image of $\sigma^{\mathcal{A}}$. If $\mathcal{A}$ is of the form, $\mathcal{A} = \frac{1}{2}\sum_{i=1}^m L_{X^i}L_{X^i} + L_{X^0}$, it is cohesive if $\text{span}\{X^1(x), \dots, X^m(x)\}$ is of constant rank and contains $X^0(x)$. Throughout this paper, we assume that $\mathcal{A}$ is cohesive.

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For simplicity, we assume that the $\mathcal{B}$-diffusion does not explode. The pair of intertwining operators induces the splitting of TN in the case that $\mathcal{B}$ is elliptic or the splitting of $Tp^{-1}[Im(\sigma^{\mathcal{A}})] = \ker(T_u p) \oplus H_u$. Hence, a diffusion operator $\mathcal{A}$ in Hörmander form has a horizontal lift $\mathcal{A}^H$, operator on N, through the horizontal lift of the defining corresponding vector fields. For operators not in Hörmander form, an intrinsic definition of horizontal lift can also be defined by the lift of its symbols and another associated operator δ^A from the space of differential forms to the space of functions and such that $\delta^{\mathcal{A}}(\mathrm{d}f) = \mathcal{A}f$. In this case, the diffusion operator $\mathcal{B}$ splits and $\mathcal{B} = \mathcal{A}^H + B^V$ where B^V acts only on the vertical bundle, which leads to computation of the conditional distribution of the $\mathcal{B}$ diffusion given $\mathcal{A}$ diffusion. We describe this in a number of special cases.

This work was inspired by an observation for gradient stochastic flows. Let

$$\mathrm{d}x_t = X(x_t) \circ \mathrm{d}B_t + X_0(x_t)\mathrm{d}t$$

be a gradient stochastic differential equations (SDEs). As usual, (B_t) is an $\mathbf{R}^m$ valued Brownian motion. The bundle map $X : \mathbf{R}^m \times M \to TM$ is induced by an isometric embedding map $f : M \to \mathbf{R}^m$. Define $Y(x) := \mathrm{d}f(x) : T_x M \to \mathbf{R}^m$ and

$$\langle X(x)e, v\rangle := \langle e, Y(x)(v)\rangle.$$

Then, $\ker X(x)$ is the normal bundle νM and $[\ker X(x)]^{\perp}$ corresponds to the tangential bundle. It was observed by Itô that the solution is a Brownian motion, that is the infinitesimal generator of the solutions is $\frac{1}{2}\Delta$. It was further developed in [6] that if we choose an orthonormal basis $\{e_i\}$ of $\mathbf{R}^m$ and define the vector fields $X_i(x) = X(x)(e)$, then the SDE now written as

$$\mathrm{d}x_t = \sum_{i=1}^{m} X_i(x_t) \circ \mathrm{d}B_t^i + X_0(x_t)\mathrm{d}t \tag{1}$$

and the Itô correction term $\sum \nabla X^i(X^i)$ vanishes. In [16], this observation is used to prove a Bismut type formula for differential forms related to gradient Brownian flow, in [18] to obtain an effective criterion for strong 1-completeness, and in [17] to obtain moment estimates for the derivative flow $T\xi_t$ of the gradient SDEs. The key observation was that for each i either ∇X_i or X_i vanishes, and if $T\xi_t(v)$ is the derivative flow for the SDE, $T\xi_t(v)$ is in fact the derivative in probability of the solution $\xi_t(x)$ at x in the direction v satisfying

$$/\!/_t d(/\!/_t^{-1} v_t) = \sum_{i=1}^{m} \nabla X_i(v_t) \circ \mathrm{d}B_t^i + \nabla X_0(v_t)\mathrm{d}t.$$

where $/\!/_t(\sigma) : T_{\sigma_0}M \to T_{\sigma_t}M$ denotes the stochastic parallel translation corresponding to the Levi-Civita connection along a path σ which is defined almost surely for almost all continuous paths. Consider the Girsanov transform

$$B_t \to B_t + \int_0^t \sum_i \frac{\langle \nabla_{v_s} X_i, v_s\rangle_{x_s}}{|v_s|^2_{x_s}} e_i \mathrm{d}s$$

and $\int_0^t \sum_i \frac{\langle \nabla_{v_s} X_i, v_s\rangle_{x_s}}{|v_s|^2_{x_s}} e_i \mathrm{d}s = \int_0^t \frac{(\nabla_{v_s} X)^*(v_s)}{|v_s|^2_{x_s}} \mathrm{d}s$. Let $\tilde{x}_t$ and $\tilde{v}_t$ be the corresponding solutions to the above two SDEs. Then $\mathbf{E}|v_t|^p_{x_t} = \mathbf{E}|\tilde{v}_t|^p_{\tilde{x}_t} G_t$ where G_t is the Girsanov density. It transpires that the transformation does not change (1). Since $|v_t|^p = |v_0|^p G_t \mathrm{e}^{a_t^p(x_t)}$, where a_t is a term only depending on x_t not on v_t, see (18) in [18] $\mathbf{E}|v_t|^p = \mathbf{E}\mathrm{e}^{a_t^p(\tilde{x}_t)} = \mathbf{E}\mathrm{e}^{a_t^p(x_t)}$. In summary, the exponential martingale term in the formula for $|v_t|^p$ can be considered as the Radon–Nikodym derivative of a new measure given by a Cameron–Martin transformation on the path space and this Cameron–Martin transformation has no effect on the x-process.

Letting $\mathcal{F}_s^x = \sigma\{\xi_s(x) : 0 \le s \le t\}$, $\mathbf{E}\{/\!/_t^{-1} v_t | \mathcal{F}_t^x\}$ satisfies [7],

$$\frac{\mathrm{d}}{\mathrm{d}t} /\!/_t^{-1} W_t = -\frac{1}{2} /\!/_t^{-1}(\sigma)\mathrm{Ric}^\#(W_t)\mathrm{d}t$$

where $\mathrm{Ric}_x : T_xM \to T_xM$ is the linear map induced by the Ricci tensor. The process W_t is called damped stochastic parallel translation, and this observation allows us to give pointwise bounds on the conditional expectation of the derivative flow. Together with an intertwining formula

$$\mathrm{d}P_t(v) = \mathbf{E}\mathrm{d}f(T\xi_t(v)),$$

this gives an intrinsic probabilistic representation for $\mathrm{d}P_t f = \mathbf{E}\mathrm{d}f(W_t)$ and leads to

$$|\nabla P_t f|(x) \le |P_t(\nabla f)|_{L^p}(x)(\mathbf{E}|W_t|^q)^{\frac{1}{q}}(x)$$

and $\nabla|P_t f|(x) \le |\mathrm{d}f|_{L^\infty}\mathbf{E}|W_t|(x)$, which in the case of the Ricci curvature is bounded below by a positive constant that leads to:

$$|\nabla P_t|(x) \le \mathrm{e}^{-Ct}|P_t(\nabla f)|_{L^p}(x)$$

and

$$\nabla|P_t f|(x) \le |\mathrm{d}f|_{L^\infty}\mathrm{e}^{-Ct},$$

respectively.

If the Ricci curvature is bounded below by a function ρ, one has the following pointwise bound on the derivative of the heat semigroup:

$$|\nabla P_t|(x) \le |P_t(\nabla f)|_{L^p}(x)\left(\mathbf{E}\mathrm{e}^{-q/2\int_0^t \rho(x_s)\mathrm{d}s}\right)^{\frac{1}{q}}.$$

See [19] for an application, and [4, 5, 20, 22] for interesting work associated with differentiation of heat semi-groups.

It turns out that the discussion for the gradient SDE is not particular to the gradient system. Given a cohesive operator, the same consideration works provided that the linear connection, equivalently stochastic parallel transport or horizontal lifting map, we use is the correct one.

To put the gradient SDE into context we introduce a diffusion generator on GLM, the general linear frame bundle of M. Let η_t^i be the partial flow of X_i and let X_i^G

be the vector field corresponding to the flow $\{T\eta_t^i(u) : u \in GLM\}$. Let

$$\mathcal{B} = \frac{1}{2}\sum L_{X_i^G} L_{X_i^G} + L_{X_0^G}.$$

Then $\mathcal{B}$ is over $\mathcal{A}$, the generator of SDE (1). The symbol of $\mathcal{A}$ is $X^*(x)X(x)$ and likewise there is a similar formulation for $\sigma^{\mathcal{B}}$ and $h_u = X^{GL}(u)Y(\pi(u))$ where $Y(x)$ is the partial inverse of $X(x)$ and $X^G(e) = \sum X_i^G \langle e, e_i\rangle$.

Open Question. Let W_d be the Wasserstein distance on the space of probability measures on M associated to the Riemannian distance function, show that if

$$W_d(P_t^*\mu, P_t^*\nu) \le e^{ct} W_d(\mu,\nu),$$

the same inequality holds true for the Riemannian covering space of M. Note that if this inequality is obtained by an estimate through lower bound on the Ricci curvature, the same inequality holds on the universal covering space. It would be interesting to see a direct transfer of the inequality from one space to the other. On the other hand, if e^{cT} is replaced by $C e^{ct}$, we do not expect the same conclusion.

2 Horizontal Lift of Vectors and Operators

Let $p : N \to M$ be a smooth map and $\mathcal{B}$, $\mathcal{A}$ intertwining diffusions, that is

$$\mathcal{B}(f \circ p) = \mathcal{A} f \circ p$$

for all smooth function $f : M \to \mathbf{R}$, with semi-groups Q_t and P_t, respectively. Instead of intertwining, we also say that $\mathcal{B}$ is over $\mathcal{A}$.

Note that for some authors intertwining may refer to a more general concept for operators: $\mathcal{A}D = D(\mathcal{B} + k)$, where k is a constant and D an oprator. For example, if Δ^q is the Laplace–Beltrami operator on differential q-forms over a Riemannian manifold $d\Delta = \Delta^1 d$. The usefulness of such relation comes largely from the relation between their respective eigenfunctions. For h a smooth positive function, the following relation $(\Delta - 2L_{\nabla h})(e^h) = e^h(\Delta + V)$ relates to h-transform and links a diffusion operator $\Delta - 2L_{\nabla h}$ with $L + V$ for a suitable potential function V. See [1] for further discussion.

It follows that

$$\frac{\partial}{\partial t}(P_t f \circ p) = \frac{\partial}{\partial t}(P_t f)\circ p = \mathcal{A}(P_t f)\circ p = \mathcal{B}(P_t f \circ p).$$

Since $P_t f \circ p = f \circ p$ at $t = 0$ and $P_t f \circ p$ solves $\frac{\partial}{\partial t} = \mathcal{B}$, we have the intertwining relation of semi-groups:

$$P_t f \circ p = Q_t(f \circ p). \tag{2}$$

The $\mathcal{B}$ diffusion u_t is seen to satisfy Dynkin's criterion for $p(u_t)$ to be a Markov process. This intertwining of semi-groups has come up in other context. The relation $VP_t = Q_t V$, where V is a Markov kernel from N to M, is relates to this one when a choice of an inverse to p is made. For example, take M to be a smooth Riemannian manifold and N the orthonormal frame bundle. One would fix a frame for each point in M. Note that the law of the Horizontal Brownian motion has been shown to be the law of the Brownian motion and independent of the initial frame [6].

The symbol of an operator $\mathcal{L}$ on a manifold M is a map from $T^*M \times T^*M \to \mathbf{R}$ such that for $f, g : M \to \mathbf{R}$,

$$\sigma^{\mathcal{L}}(\mathrm{d}f, \mathrm{d}g) = \frac{1}{2}\left[\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f\right].$$

If $\mathcal{L} = \frac{1}{2}a_{ij}\frac{\partial^2}{\partial x_j \partial x_j} + b_k \frac{\partial}{\partial x_k}$ is an elliptic operator on $\mathbf{R}^n$, its symbol is (a_{ij}) which induces a Riemannian metric $(g_{ij}) = (a_{ij})^{-1}$ on $\mathbf{R}^n$.

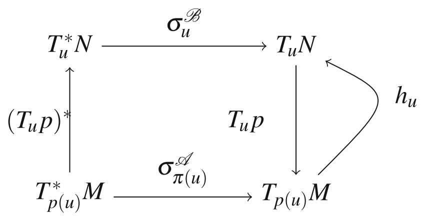
For the intertwining diffusions: $p^*\sigma^{\mathcal{B}} = \sigma^{A}$, or

$$T_u p \circ \sigma_u^{\mathcal{B}}((T_u p)^*) = \sigma_{p(u)}^{\mathcal{A}},$$

if the symbols are considered as linear maps from the cotangent to the tangent spaces. We stress again that throughout this article we assume that $\mathcal{A}$ is *cohesive*, i.e. $\sigma^{\mathcal{A}}$ has constant rank and $\mathcal{A}$ is along the distribution $E = Im[\sigma^{\mathcal{A}}]$.

There is a unique horizontal lifting map [11] such that now with the symbols considered as linear maps from the cotangent space to the tangent spaces

$$\mathfrak{h}_u \circ \sigma_{p(u)}^{\mathcal{A}} = \sigma_u^{\mathcal{B}}(T_u p)^*.$$



Let H_u to be the image of $\mathfrak{h}_u$, called the *horizontal distribution*. They consists of image of differential forms of the form $\phi(Tp-)$ for $\phi \in T^*M$ by $\sigma^{\mathcal{B}}$. Note that this cannot be reduced to the case of $\mathcal{A}$ being elliptic because E_x may not give rise to a submanifold of M.

If an operator $\mathcal{L}$ has the Hörmander form representation

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{m} L_{X^j} L_{X^j} + L_{X^0} \tag{3}$$

Define $X(x) : \mathbf{R}^m \to T_xM$ by $X(x) = \sum X^i(x)e_i$ for $\{e_i\}$ an orthonormal basis of $\mathbf{R}^m$. Then

$$\sigma_x^{\mathcal{L}} = \frac{1}{2}X(x)X(x)^* : T_x^*M \to T_xM.$$

In the elliptic case, $\sigma^{\mathcal{L}}$ induces a Riemannian metric and $X^*(x)\phi = Y(x)\phi^\#$.

An operator $\mathcal{L}$ is *along a distribution* $S := \{S_x : x \in M\}$, where each S_x is a subspace of T_xM, if $\mathcal{L}\psi = 0$ whenever $\psi_x(S_x) = \{0\}$. The horizontal lifts of tangent vectors induce a horizontal lift of the operator which is denoted as $\mathcal{A}^H$. To define a horizontal lift of a diffusion operator intrinsically, we introduced an operator $\delta^{\mathcal{L}}$. If M is endowed with a Riemannian metric let $\mathcal{L} = \Delta$ be the Laplace–Beltrami operator, this is d^*, the L^2 adjoint of d the differential operator d. Then, $d^*(f\phi) = fd^*\phi + \iota_{\nabla f}(\phi)$ for ϕ a differential 1-form and f a function, using the Riemannian metric to define the gradient operator, and $\Delta = d^*d$.

For a general diffusion operator, it was shown in [11] that there is a unique linear operator $\delta^{\mathcal{L}} : C^{r+1}T^*M \to C^r(M)$ determined by $\delta^{\mathcal{L}}(\mathrm{d}f) = \mathcal{L}f$ and $\delta^{\mathcal{L}}(f\phi) = f\delta^{\mathcal{L}}(\phi) + \mathrm{d}f\sigma^{\mathcal{L}}(\phi)$. If $\mathcal{L}$ has the representation (3),

$$\delta^{\mathcal{L}} = \frac{1}{2}\sum_{j=1}^m L_{X^j}\iota_{X^j} + \iota_{X^0}.$$

Here ι is the interior product, $\iota_v\phi := \phi(v)$. The symbol of the operator now plays the role of the Riemannian metric. For $\mathcal{B}$ over $\mathcal{A}$,

$$\delta^{\mathcal{B}}(p^*(\mathrm{d}f)) = p^*(\delta^{\mathcal{A}}\mathrm{d}f).$$

There are many operators over $\mathcal{A}$ and only one of which, $\mathcal{A}^H$, is horizontal. An operator $\mathcal{L}$ is *horizontal (respectively vertical)* if it is along the horizontal or the vertical distribution. An operator $\mathcal{B}$ is vertical if and only if $\mathcal{B}(f \circ p) = 0$ for all f and $\mathcal{B} - \mathcal{A}^H$ is a vertical operator.

The foundation of the noise decomposition theorem in [11] depends on the following decomposition of operator $\mathcal{B}$, when $\mathcal{A}$ is cohesive,

$$\mathcal{B} = \mathcal{A}^H + (\mathcal{B} - \mathcal{A}^H) \tag{4}$$

and it can be proven that $\mathcal{B} - \mathcal{A}^H$ is a vertical operator.

2.1 In Metric Form

Note that $\sigma^{\mathcal{A}}$ gives rise to a positive definite bilinear form on T^*M:

$$\langle \phi, \psi \rangle_x = \phi(x)(\sigma_x^{\mathcal{A}}(\psi(x)))$$

and this induces an inner product on E_x:

$$\langle u, v\rangle_x = (\sigma_x^{\mathcal{A}})^{-1}(u)(v).$$

For an orthonormal basis $\{e_i\}$ of E_x, let $e_i^* = (\sigma_x^{\mathcal{A}})^{-1}(e_i)$. Then, $e_j^* \sigma^{\mathcal{A}}(e_i^*) = (\sigma_x^{\mathcal{A}})^{-1}(e_j)(e_i) = \langle e_j, e_i\rangle$ and hence

$$\langle \phi, \psi\rangle_x = \sum_i \langle e_j, e_i\rangle \phi(e_i)\psi(e_j) = \sum_i \phi(e_i)\psi(e_i).$$

Likewise the symbol $\sigma^{\mathcal{A}^H}$ induces an inner product on T^*N with the property that $\langle \phi \circ Tp, \psi \circ Tp\rangle = \langle \phi, \psi\rangle$ and a metric on $H \subset TN$ which is the same as that induced by $\mathfrak{h}$ from TM. Note that $\sigma^{\mathcal{B}} = \sigma^{\mathcal{A}^H} + \sigma^{\mathcal{B}^V}$, where $\mathcal{B}^V$ is the vertical part of $\mathcal{B}$, and $Im[\sigma^{\mathcal{B}^V}] \cap H = \{0\}$. Let μ be an invariant measure for $\mathcal{A}^H$ and $\mu_M = p_*(\mu)$ the pushed forward measure which is an invariant measure for $\mathcal{A}$.

If $\mathcal{A}$ is symmetric,

$$\begin{aligned}\int_M \langle \mathrm{d}f, \mathrm{d}g\rangle\, \mu_M(\mathrm{d}x) &= \int \sigma^{\mathcal{A}}(\mathrm{d}f, \mathrm{d}g)\, \mu_M(\mathrm{d}x)\\ &= \frac{1}{2}\int [\mathcal{A}(fg) - f(\mathcal{A}g) - g(\mathcal{A}f)]\, \mu_M(\mathrm{d}x)\\ &= -\int_M f\mathcal{A}g\, \mathrm{d}\mu_M(x).\end{aligned}$$

Hence, $\mathcal{A} = -d^*d$ and

$$\delta^{\mathcal{A}} = -d^*$$

for d^* the L^2 adjoint. Similarly, we have an L^2 adjoint on N and $\mathcal{A}^H = -d^*d$. For a 1-form ϕ on M,

$$\begin{aligned}\int_N \langle \phi \circ Tp, d(g \circ p)\rangle \mathrm{d}\mu_N &= \int \langle d^*(\phi \circ Tp), g \circ p\rangle \mathrm{d}\mu_N\\ &= \int \langle \mathbf{E}\{d^*(\phi \circ Tp)|p\}, g \circ p\rangle \mathrm{d}\mu_N\end{aligned}$$

Hence, $\mathbf{E}\{d^*(\phi \circ Tp)|p\} = (d^*\phi) \circ p$. Since for $u + v \in H \oplus \ker[Tp]$, $h \circ Tp(u+v) = u$, every differential form ψ on N induces a form $\phi = \psi \circ \mathfrak{h}$ such that $\psi = \phi(T\pi)$ when restricted to H, hence $\mathbf{E}\{d^*\psi)|p\} = (d^*(\psi \circ h)) \circ p$.

2.2 On the Heisenberg Group

A Lie group is a group G with a manifold structure such that the group multiplication $G \times G \to G$ and taking inverse are smooth. Its tangent space at the identity

$\mathfrak{g}$ can be identified with left invariant vector fields on G, $X(a) = TL_a X(e)$ and we denote A^* the left invariant vector field with value A at the identity. The tangent space $T_a G$ at a can be identified with $\mathfrak{g}$ by the derivative TL_a of the left translation map. Let $\alpha_t = \exp(tA)$ be the solution flow to the left invariant vector field $TL_a A$ whose value at 0 is the identity then it is also the flow for the corresponding right invariant vector field: $\dot{\alpha}_s = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=s} \exp^{(t-s)A} \exp^{sA} = TR_{\alpha_s} A$. Then $u_t = a \exp(tA)$ is the solution flow through a.

Consider the Heisenberg group G whose elements are $(x, y, z) \in \mathbf{R}^3$ with group product

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1)).$$

The Lie bracket operation is $[(a, b, c), (a', b', c')] = (0, 0, ab' - a'b)$. Note that for $X, Y \in \mathfrak{g}$, $\mathrm{e}^X \mathrm{e}^Y = \mathrm{e}^{X+Y+\frac{1}{2}[X,Y]}$. If $A = (a, b, c)$, then $A^* = (a, b, c + \frac{1}{2}(xb - ya))$. Consider the projection $\pi : G \to \mathbf{R}^2$ where $\pi(x, y, z) = (x, y)$. Let

$$X_1(x, y, z) = \left(1, 0, -\frac{1}{2} y\right), \quad X_2(x, y, z) = \left(0, 1, \frac{1}{2} x\right),$$
$$X_3(x, y, z) = (0, 0, -1)$$

be the left invariant vector fields corresponding to the standard basis of $\mathfrak{g}$. The vector spaces $H_{(x,y,z)} = \mathrm{span}\{X_1, X_2\} = \{(a, b, \frac{1}{2}(xb - ya))\}$ are of rank 2. They are the horizontal tangent spaces associated to the Laplacian $\mathcal{A} = \frac{1}{2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ on $\mathbf{R}^2$ and the left invariant Laplacian $\mathcal{B} := \frac{1}{2} \sum_{i=1}^3 L_{X_i} L_{X_i}$ on G. The vertical tangent space is $\{(0, 0, c)\}$, and there is a a horizontal lifting map from $T_{(x,y)} \mathbf{R}^2$:

$$h_{(x,y,z)}(a, b) = \left(a, b, \frac{1}{2}(xb - ya)\right).$$

The horizontal lift of $\mathcal{A}$ is the hypo-elliptic diffusion operator $\mathcal{A}^H = \frac{1}{2} \sum_{i=1}^2 L_{X_i} L_{X_i}$ and the horizontal lift of a 2-dimensional Brownian motion, the horizontal Brownian motion, has its third component the Levy area. In fact for almost surely all continuous path $\sigma : [0, T] \to M$ with $\sigma(0) = 0$, we have the horizontal lift curve:

$$\tilde{\sigma}(t) = \left(\sigma^1(t), \sigma^2(t), \frac{1}{2} \int_0^t \left(\sigma^1(t) \circ \mathrm{d}\sigma^2(t) - \sigma^2(t) \circ \mathrm{d}\sigma^1(t)\right)\right).$$

The hypoelliptic semi-group Q_t in $\mathbf{R}^3$ and the heat semigroup P_t satisfy $Q_t(f \circ \pi) = \mathrm{e}^{\frac{1}{2} t \Delta} f \circ \pi$ and $d(\mathrm{e}^{\frac{1}{2} t \Delta} f) = Q_t(\mathrm{d} f \circ \pi) \circ \mathfrak{h}$.

2.3 The Local Coordinate Formulation

Let M be a smooth Riemannian manifold and $\pi : P \to M$ a principal bundle with group G acting on the right, of which we are mainly interested in the case when P is the general linear frame bundle of M or the orthonormal frame bundle with G the special general linear group or the special orthogonal group of $\mathbf{R}^n$. For $A \in \mathfrak{g}$, the Lie algebra of G, the action of the one parameter group $\exp(tA)$ on P induces the fundamental vector field A^* on P. Let VTP be the vertical tangent bundle consisting of tangent vectors in the kernel of the projection $T\pi$ so the fundamental vector fields are tangent to the fibres and $A \mapsto A^*(u)$ is a linear isomorphism from $\mathfrak{g}$ to VT_uP. At each point, a complementary space, called the horizontal space, can be assigned in a right invariant way: $HT_{ua}P = (R_a)_* HT_uP$.

For the general linear group $GL(n)$, its Lie algebra is the vector space of all n by n matrices and the value at a of the left invariant vector field A^* is aA. The Lie bracket is just the matrix commutator, $[A, B] = AB - BA$. Every finite dimensional Lie group is homomorphic to a matrix Lie group by the adjoint map. For $a \in G$, the tangent map to the conjugation $\phi : g \in G \mapsto aga^{-1} \in G$ induces the adjoint representation $\text{ad}(a) : G \to GL(\mathfrak{g}; \mathfrak{g})$. For $X \in \mathfrak{g}$, $\phi_* X^*(g) = TL_a TR_{a^{-1}}(X(a^{-1}ga)) = TR_{a^{-1}} X(ga) = (R_{a^{-1}})_* X(g)$ and is left invariant so $\text{ad}(a)(A) = TR_{a^{-1}} X^*(a)$. The Lie bracket of two left invariant vector fields $[X^*, Y^*] = \lim_{t\to 0} \frac{1}{t}(\exp(tY)_* X^* - X^*) = \lim_{t\to 0} \frac{1}{t}((R_{e^{tY}})_* X^* - X^*)$ is again a left invariant vector field and this defines a Lie bracket on $\mathfrak{g}$ by $[X, Y]^* = [X^*, Y^*]$. The Lie algebra homomorphism induced by $a \mapsto ad(a)$ is denoted by $\text{Ad} : \mathfrak{g} \to \mathfrak{g}l(n, \mathbf{R})$ is given by $\text{Ad}_X(Y) = [X, Y]$. A tangent vector at $a \in G$ can be represented in a number of different ways, notably by the curves of the form $a\exp(tA)$, $\exp(tB)a$. The Lie algebra elements are related by $B = aAa^{-1} = \text{ad}(a)A$ and $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} A\exp(TB)A^{-1} = \text{ad}(A)B$ so $A\exp(tB)A^{-1} = \exp(t\text{ad}(A)B)$. The left invariant vector fields provide a parallelism of TG, and there is a canonical left invariant 1-form on G, $\omega_g(TL_g(v)) = G_e(v)$, determined by $\theta(A^*) = A$.

The collection of left invariant vector fields on TP forms also an algebra and the map $A \to A^*$ is a Lie-algebra isomorphism. A horizontal subspace of the tangent space to the principal bundle TP is determined by the kernel of a connection 1-form ω, which is a $\mathfrak{g}$-value differential 1-form on P such that (i) $\omega(A^*) = A$, for all $A \in \mathfrak{g}$, and (2) $(R_a)^*\omega = ad(a^{-1})\omega(-)$. Here, A^* refers to the TP valued left invariant vector field. The first condition means that the connection 1-form restricts to an isomorphism from VTP to $\mathfrak{g}$ and the second is a compatibility condition following from that the fundamental vector field corresponding to $ad(a^{-1})A$ is $(R_a)_*A^*$. The kernel of ω is right invariant since $\omega_{ua}(TR_aV) = (R_a)^*\omega(V) = ad(a)\,\omega_u(V)$ for any $V \in T_uP$.

In a local chart $\pi^{-1}(U)$ with U an open set of M and $u \in \pi^{-1}(U) \to (\pi(u), \phi(u))$ the chart map where $\phi(ua) = \phi(u)a$, the connection map satisfies $\omega_{(x,a)}(0, B^*) = B$ for B^* the left invariant vector field of G corresponding to $B \in \mathfrak{g}$ and

$$\omega_{(x,a)}(v, B^*(a)) = ad(a^{-1})(M_x v) + B$$

where $M_x : T_x M \to \mathfrak{g}$ is a linear map varying smoothly with x. The trivial connection for a product manifold $M \times G$ would correspond to a choice of M_x with M_x identically zero and so the horizontal vectors are of the form $(v, 0)$. The horizontal tangent space at (x, a) is the linear space generated by

$$H_{(x,a)}TP = \{(v, -TR_a(M_x v)), \qquad v \in T_x U, a \in G\}.$$

Given a connection on P, for every differentiable path σ_t on M, through each frame u_0 over σ_0, there is a unique u_t which projects down to σ_t on M given by $\omega(\dot{u}_t) = 0$. In local coordinates $u_t = (\sigma_t, g_t)$, $\omega(\dot{u}_t) = \mathrm{ad}(g_t^{-1})\omega_{(\sigma_t,e)}(\dot{\sigma}_t, TR_{g_t}^{-1}\dot{g}_t)$ and $g_t{}^{-1}\dot{g}_t + ad(g_t^{-1})M_{\sigma_t}\dot{\sigma}_t = 0$. If u_t is a lift of x_t, then $u_t \circ g$ is the horizontal lift of x_t through $u_0 g$ so $u_t : \pi^{-1}(\sigma_0) \to \pi^{-1}(\sigma_t)$ is an isomorphism. This formulation works for continuous paths. Consider the path of continuous paths over M and a Brownian motion measure. For almost surely all continuous paths σ_t, a horizontal curve exists, as solution to the stochastic differential equation in Stratnovitch form:

$$\mathrm{d}g_t = -M_{\sigma_t}(e_i)(g_t) \circ \mathrm{d}\sigma_t^i.$$

Here, (e_i) is an orthonormal basis of $\mathbf{R}^n$, and the $M_{\cdot}(e_i)$s are matrices in $\mathfrak{g}$ and the solution u_t induces a transformation from the fibre at σ_0 to the fibre at σ_t.

2.4 The Orthonormal Frame Bundle

Let $N = OM$ be the orthonormal frame bundle with π the natural projection to a Riemannian manifold M and a right invariant Riemannian metric. Let $\mathcal{A} = \Delta$ be the Laplacian on M and $\mathcal{B}$ the Laplacian on N. We may choose the Laplacian $\mathcal{B}$ to be of the form $\frac{1}{2}L_{A_i^*}L_{A_i^*} + \frac{1}{2}L_{H_i}L_{H_i}$ where A_i are fundamental vector fields and $\{H_i\}$ the standard horizontal vector fields. The horizontal lifting map $\mathfrak{h}_u$ is: $v \in TM \mapsto (v, 0)$. We mention two Hörmander form representation for the horizontal lift. The first one consists of horizontal lifts of vector fields that defines $\mathcal{A}$. The second one is more canonical. Let $\{B(e), e \in \mathbf{R}^n\}$ be the standard horizontal vector fields on OM determined by $\theta(B(e)) = e$ where θ is the canonical form of OM, that is $T\pi[B(e)(u)] = u(e)$. Take an orthonormal basis of $\mathbf{R}^n$ and obtaining never vanishing vector fields $H_i =: B(e_i)$, then $\mathcal{A}^H = \sum L_{H_i}L_{H_i}$ and $\mathcal{A}^H$ is called the horizontal Laplacian. The two heat semigroups Q_t, upstairs, and P_t intertwine: $Q_t(f \circ \pi) = P_t f \circ \pi$. Let us observe that if Q_t^H is the semigroup corresponding to horizontal Laplacian $\mathcal{A}^H$, since $\mathrm{d}Q_t f \circ T\pi$ annihilates the vertical bundle, Q_t^H $(f \circ \pi) = Q_t(f \circ \pi)$ and Q_t^H restricts to a semigroup on the set of bounded measurable functions of the form $f \circ \pi$.

Denote by the semi-group corresponding to the Laplace–Beltrami operators by the same letters with the supscript one indicates the semi-group on 1-forms, then $\mathrm{d}P_t f = P_t^1 d$ and $\mathrm{d}Q_t = Q_t^1 d$, which follows from that the exterior differentiation d

and the Laplace–Beltrami operator commute. Now

$$d(P_t f \circ \pi) = d(P_t f) \circ T\pi = P_t^1(\mathrm{d}f) \circ T\pi.$$

Similarly, $d(Q_t(f \circ \pi)) = Q_t^1(\mathrm{d}f \circ T\pi)$. Now, we represent Q_t by the horizontal diffusion which does not satisfy the commutation relation: $\mathrm{d}\mathcal{A}^H \neq \mathcal{A}^H d$ in general. Let $\bar{W}_t$ be the solution to a differential equation involving the Weitzenbock curvature operator, see Proposition 3.4.5 in [11], $\bar{W}_t /\!/_t = W_t$ where $\frac{\mathrm{d}}{\mathrm{d}t}\bar{W}_t = -\frac{1}{2}u_t^{-1}\,\mathrm{Ric}^\#(u_t\bar{W}_t)$.

$$d(Q_t^H f \circ \pi)(\mathfrak{h}v) = d(P_t f)(v) = \mathbf{E}\mathrm{d}f(\bar{W}_t u_t \circ u_0^{-1}(v)),$$

the formula as we explained in the introduction, after conditioning the derivative flow. Note also that $d(Q_t^H(f \circ \pi)) = d(P_t f \circ \pi) = \mathrm{d}P_t f \circ T\pi$ and

$$d(P_t f)(-) = Q_t(\mathrm{d}f \circ T\pi)(\mathfrak{h}-).$$

3 Examples

3.1 Diffusions on the Euclidean Space

Take the example that $N = \mathbf{R}^2$ and $M = \mathbf{R}$. Any elliptic diffusion operators on M is of the form $a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and a diffusion operator on N is of the form $\mathcal{B} = a(x,y)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + d(x,y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + c(x,y)\frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}y}$ with $4ad > c^2$ and $a > 0$. Now, $\mathcal{B}$ is over $\mathcal{A}$ implies that $a(x,y) = a(x)$ for all y. If a, b, c are constants, a change of variable of the form $x = u$ and $y = (c/2\sqrt{a})u + v$ transforms $\mathcal{B}$ to $a^2\frac{\partial^2}{\partial u^2} + (d - c^2/4a)\frac{\partial^2}{\partial v^2}$. In this local coordinates, $\mathcal{B}$ and $\mathcal{A}$ have a trivial projective relation. In general, we may seek a diffeomorphism $\Phi : (x,y) \mapsto (u,v)$ so that Φ intertwines $\mathcal{B}$ and $\tilde{B}$ where $\tilde{B}$ is the sum of $a^2\frac{\partial^2}{\partial u^2}$ and an operator of the form $\frac{\partial^2}{\partial v^2}$. This calculation is quite messy. However, according to the theory in [11], the horizontal lifting map

$$v \mapsto \sigma^{\mathcal{B}}(Tp)^*(\sigma^{\mathcal{A}})^{-1}(v) = \sigma^{\mathcal{B}}\left(\frac{v}{a}, 0\right)^T = \begin{pmatrix} a & \frac{c}{2} \\ \frac{c}{2} & d \end{pmatrix}\begin{pmatrix} \frac{v}{a} \\ 0 \end{pmatrix} = \left(v, \frac{c}{2a}v\right).$$

where $p : (x,y) \to x$ and Tp is the derivative map and $(Tp)^*$ is the corresponding adjoint map. Hence, the lifting of $\mathcal{A}$, as the square of the lifting $\sqrt{a}\frac{\mathrm{d}}{\mathrm{d}x}$ gives $\sqrt{a}\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{c}{2a}\frac{\mathrm{d}}{\mathrm{d}y}\right)$ and resulting the completion of the square procedure and the splitting of $\mathcal{B}$:

$$\mathcal{B} = a\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{c}{2a}\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 + \left(d - \frac{c^2}{4a}\right)\frac{\mathrm{d}^2}{\mathrm{d}y^2}.$$

This procedure trivially generalises to multidimensional case $p : \mathbf{R}^{n+p} \to \mathbf{R}^n$ with $p(x, y) = x$. If $\pi : \mathbf{R}^N \to \mathbf{R}^m$ is a surjective smooth map not necessarily of the form $p(x, y) = x$, we may try to find two diffeomorphisms ψ on $\mathbf{R}^N$ and ϕ on $\mathbf{R}^m$ and so that $p = \phi \circ \pi\psi^{-1}$ if of simple form. The diffusion operators $\mathcal{B}$ and $\mathcal{A}$ induce two operators $\tilde{B}$ and $\tilde{\mathcal{A}}$. If $\mathcal{B}$ and $\mathcal{A}$ are intertwining, then so are $\tilde{B}$ and $\tilde{A}$. Indeed from

$$\begin{aligned}\tilde{\mathcal{B}}(gp)(y)) &= \mathcal{B}(gp \circ \psi)(\psi^{-1}(y)) = \mathcal{B}(g \circ \phi\pi)(\psi^{-1}(y)) \\ &= \mathcal{A}(g \circ \phi)(\pi\psi^{-1}(y)) = \mathcal{A}(g \circ \phi)(\phi^{-1}p(y)) = \tilde{\mathcal{A}}g(p(y)).\end{aligned}$$

This transformation is again not necessary because of the for-mentioned theorem.

In general, [11], if $p : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is the trivial projection and $\mathcal{B}$ is defined by

$$\mathcal{B}g(x, y) = \sigma^{ij}(x)\frac{\partial^2 g}{\partial x_i \partial x_j} + \sum b^k(x, y)\frac{\partial^2 g}{\partial y \partial x_k} + c(x, y)\frac{\partial^2 g}{\partial y^2}$$

with $a = (a_{ij})$ symmetric positive definite and of constant rank, $[b(x, y)]^T b(x, y) \le c(x, y)a(x)$, there is a horizontal lift induced by $\mathcal{B}$ and $\sigma^{ij}(x)\frac{\partial^2 g}{\partial x_i \partial x_j}$ given by

$$h_{(x,y)}(v) = (v, \langle a(x)^{-1}b, v\rangle).$$

Or even more generally, if $p : \mathbf{R}^{m+p+q} \to \mathbf{R}^{m+p}$ with $\mathcal{A}$ a $(m + p) \times (m + p)$ matrix and B a $(m+p)\times q$ matrix and C a $q\times q$ matrix with each column of $B(x, y)$ in the image of $\mathcal{A}$, the horizontal lift map is $h_{(x,y)}(v) = (v, B^T(x, y)A^{-1}v)$.

3.2 The SDE Example and the Associated Connection

Consider SDE (1). For each $y \in M$, define the linear map $X(y)(e) : \mathbf{R}^m \to T_y M$ by $X(y)(e) = \sum_{i=1}^m X_i(y)\langle e, e_i\rangle$. Let $Y(y) : T_y M \to [\ker X(y)]^\perp$ be the right inverse to $X(y)$. The symbol of the generator $\mathcal{A}$ is $\sigma_y^{\mathcal{A}} = \frac{1}{2}X(y)X(y)^*$, which induces a Riemannian metric on the manifold in the elliptic case, and a sub-Riemannian metric in the case of $\sigma^{\mathcal{A}}$ being of constant rank .

This map X also induces an affine connection $\breve{\nabla}$, which we called the LW connection, on the tangent bundle which is compatible with the Riemannian metric it induced as below. If $v \in T_{y_0}M$ is a tangent vector and $U \in \Gamma TM$ a vector field,

$$(\breve{\nabla}_v U)(y_0) = X(y_0)D(Y(y)U(y))(v).$$

At each point $y \in M$, the linear map

$$X(y) : \mathbf{R}^m = \ker X(y) \oplus [\ker X(y)]^\perp \to T_y M$$

induces a direct sum decomposition of $\mathbf{R}^m$. The connection defined above is a metric connection with the property that

$$\breve{\nabla}_v X(e) \equiv 0, \quad \forall e \in [\ker X(y_0)]^{\perp}, v \in T_{y_0}M.$$

This connection is the adjoint connection by the induced diffusion pair on the general linear frame bundle mentioned earlier. See [9] where it is stated any metric connection on M can be defined through an SDE, using Narasimhan and Ramanan's universal connection.

3.3 *The Sphere Example*

Consider the inclusion $i : S^n \to \mathbf{R}^{n+1}$. The tangent space to T_xS^n for $x \in S^n$ is of the form:

$$T_xS^n = \{v : \langle x, v\rangle = 0\}, \qquad \langle u, v\rangle_x = \langle u, v\rangle_{\mathbf{R}^{n+1}}.$$

Let P_x be the orthogonal projection of $\mathbf{R}^n$ to T_xS^n:

$$P_x : e \in \mathbf{R}^{n+1} \mapsto e - \langle e, x\rangle \frac{x}{|x|^2}.$$

This induces the vector fields $X_i(x) = P_x(e_i)$ and the gradient SDE

$$\mathrm{d}x_t = \sum_{i=1}^m P_{x_t}(e_i) \circ \mathrm{d}B_t^i.$$

For a vector field $U \in \Gamma TS^n$ on S^n and a tangent vector $v \in T_xS^n$, define the Levi–Civita connection as following:

$$\begin{aligned}\nabla_v U &:= P_x((DU)_x(v)) \\ &= (DU)_x(v) - \langle (DU)_x(v), x\rangle \frac{x}{|x|^2}.\end{aligned}$$

The term

$$\langle (DU)_x(v), x\rangle \frac{x}{|x|^2}$$

is actually tensorial since $\langle (DU)_x(v), x\rangle = \langle U, v\rangle$ and hence defines the Christoffel symbols Γ_{ij}^k, where

$$\nabla_{e_i} e_j = \Gamma_{ij}^k, \qquad (\nabla_v U)^k = D_v U^\nu + \Gamma_{ij}^k v_i u_j.$$

Solution to gradient SDE are BMs since $\nabla_{X_i} X_i = 0$ as observed by Itô. From tensorial property, get Gauss and Weingarten's formula,

$$(DU)_x(v) = \nabla_v U + \alpha_x(Z(x), v), \qquad v \in T_xM, U \in \Gamma TM$$
$$(D\xi)_x(v) - A(\xi(x), v) + [(D\xi)_x(v)]^\nu, \qquad \xi \in \nu M$$

For $e \in \mathbf{R}^m$, write $e = P_x(e) + \mathrm{e}^\nu(x)$ and obtain

$$D_v[P_x(e)] + D_v[\mathrm{e}^\nu] = 0.$$

Take the tangential part of all terms in the above equation to see that

$$\text{if } e \in [\ker X(x_0)]^\perp, \quad \nabla_v[P_x(e)] = A(v, \mathrm{e}^\nu(x_0)) = 0.$$

3.4 The Pairs of SDEs Example and Decomposition of Noise

In general, if we have $p : N \to M$ and the bundle maps $\tilde{X} : N \times \mathbf{R}^m \to TN$ and $X : M \times \mathbf{R}^m \to TM$ are p-related: $Tp\tilde{X}(u) = X(p(u))$, let $y_t = p(u_t)$ for u_t the solution to

$$\mathrm{d}u_t = \tilde{X}(u_t) \circ \mathrm{d}B_t + \tilde{X}_0(u_t)\mathrm{d}t.$$

Then, y_t satisfies

$$\mathrm{d}y_t = X(y_t) \circ \mathrm{d}B_t + X_0(y_t)\mathrm{d}t.$$

Consider the orthogonal projections at each $y \in M$,

$$K^\perp(y) : \mathbf{R}^m \to [\ker X(y)]^\perp, \qquad K^\perp(y) := Y(y)X(y)$$
$$K(y) : \mathbf{R}^m \to \ker[X(y)], \qquad K(y) := I - Y(y)X(y).$$

Then

$$\mathrm{d}y_t = X(y_t)K^\perp(y_t) \circ \mathrm{d}B_t + X_0(y_t)\mathrm{d}t \tag{5}$$

where the term $K^\perp(y_t) \circ \mathrm{d}B_t$ captures the noise in y_t.

To find the conditional law of y_t, we express the SDE for u_t use the term $K^\perp(y_t) \circ \mathrm{d}B_t$. For a suitable stochastic parallel translation [11] that preserves the splitting of $\mathbf{R}^m$ as the kernel and orthogonal kernel of $X(y)$, define two independent Brownian motions

$$B_t^\perp := \int_0^s /\!/_t^{-1} K^\perp(p(u_t))\mathrm{d}B_t$$
$$\beta_s := \int_0^s /\!/_t^{-1} K(p(u_t)) \circ \mathrm{d}B_t.$$

See also [10,12]. Assume now the parallel translation on $[\ker X(x)]^\perp$ is that given in Sect. 3.2. Since $\mathrm{d}x_t = X(x_t)K^\perp(x_t) \circ \mathrm{d}B_t + X_0(x_t)\mathrm{d}t$, the following filtrations are equal:

$$\sigma\{x_u : 0 \le u \le s\} = \sigma\{B_u^\perp : 0 \le u \le s\}.$$

The horizontal lifting map induced by the pair $(\mathcal{A}, \mathcal{B})$ is given as following:

$$\mathfrak{h}_u(v) = \tilde{X}(u)Y(\pi(u)v), \qquad u \in T_{p(u)}M.$$

From which we obtain the horizontal lift $X^H(u)$ of the bundle map X:

$$X^H(u) = \tilde{X}(u)K^{\perp}(p(u))$$

and it follows that

$$\begin{aligned} \mathrm{d}u_t &= \tilde{X}(u_t)K^{\perp}(p(u_t)) \circ \mathrm{d}B_t + \tilde{X}(u_t)K(p(u_t)) \circ \mathrm{d}B_t + \tilde{X}_0(u_t)\mathrm{d}t \\ &= X^H(u_t) \circ \mathrm{d}B_t + \tilde{X}(u_t)K(p(u_t)) \circ \mathrm{d}B_t + \tilde{X}_0(u_t)\mathrm{d}t \\ &= h_{u_t} \circ \mathrm{d}x_t + \tilde{X}(u_t)K(p(u_t)) \circ \mathrm{d}B_t + (\tilde{X}_0 - X_0^H)(u_t)\mathrm{d}t \end{aligned}$$

If this equation is linear in u_t, it is possible to compute the conditional expectation of u_t with respect to $\sigma\{x_u : 0 \le u \le s\}$ as in the derivative flow case (Sect. 2.8 below). This discussion is continued at the end of the article.

3.5 The Diffeomorphism Group Example

If M is a compact smooth manifold and X is smooth we may consider an equation on the space of smooth diffeomorphisms Diff(M). Define $\tilde{X}(f)(x) = X(f(x))$ and $\tilde{X}_0(f)(x) = X_0(f(x))$ and consider the SDE on Diff(M):

$$\mathrm{d}f_t = \tilde{X}(f_t) \circ \mathrm{d}B_t + \tilde{X}_0(f_t)\mathrm{d}t$$

with $f_0(x) = x$. Then, $f_t(x)$ is solution to $\mathrm{d}x_t = X(x_t) \circ \mathrm{d}B_t$ with initial point x.

Fix $x_0 \in M$, we have a map $\theta : \mathrm{Diff}(M) \to M$ given by $\theta(f) = f(x_0)$. Let $\mathcal{B} = \frac{1}{2}L_{\tilde{X}_i}L_{\tilde{X}_i}$ and $\mathcal{A} = \frac{1}{2}L_{X_i}L_{X_i}$. Then,

$$h_f(v)(x) = \tilde{X}(f)\,(Y(f(x_0))v)\,(x) = X(f(x))(Y(f(x_0))v).$$

3.6 The Twist Effect

Consider the polar coordinates in $\mathbf{R}^n$, with the origin removed. Consider the conditional expectation of a Brownian motion W_t on $\mathbf{R}^n$ on $|W_t|$ where $|W_t|$, and n-dimensional Bessel Process, $n > 1$, lives in $\mathbf{R}_+$. For $n = 2$, we are in the situation that $p : \mathbf{R}^2 \to \mathbf{R}$ given by $p : (r, \theta) \mapsto r$. The $\mathcal{B}$ and $\mathcal{A}$ diffusion are the Laplacians, $\mathcal{A}^H = \frac{\partial^2}{\partial r^2}$. The map $p(r, \theta) = r^2$ would result the lifting map $v\frac{\partial}{\partial x} \mapsto (\frac{v}{2r}, 0) = \frac{v}{2r}\frac{\partial}{\partial r}$.

At this stage, we note that if B_t is a one dimensional Brownian motion, l_t the local time at 0 of B_t and $Y_t = |B_t| + \ell_t$, a 3-dimensional Bessel process starting from 0. There is the following beautiful result of Pitman:

$$E\{f(|B_t|)|\sigma(Y_s : s \le t)\} = \int_0^1 f(xY_t)\mathrm{d}x = Vf(Y_t)$$

where V is the Markov kernel: $V(x, \mathrm{d}z) = \frac{\mathbf{1}_{0\le z\le x}}{x}\mathrm{d}z$ [2, 21].

A second example, [11], which demonstrates the twist effect is on the product space of the circle. Let $p : S^1 \times S^1 \to S^1$ be the projection on the first factor. For $0 < \alpha < \frac{\pi}{4}$, define the diffusion operator on $S^1 \times S^1$:

$$\mathcal{B} = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \tan\alpha\frac{\partial^2}{\partial x \partial y}.$$

and the diffusion operator $\mathcal{A} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$ on S^1. Then,

$$\begin{aligned}\mathcal{B}^V &= \frac{1}{2}(1 - (\tan\alpha)^2)\frac{\partial^2}{\partial y^2}\\ \mathcal{A}^H &= \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + (\tan\alpha)^2\frac{\partial^2}{\partial y^2}\right) + \tan\alpha\frac{\partial^2}{\partial x \partial y}.\end{aligned}$$

4 Applications

4.1 Parallel Translation

Let $P = GLM$, the space of linear frames on M with an assignment of metrics on the fibres. The connection on P is said to be metric if the parallel translation preserves the metric on the fibres. A connection on P reduces to a connection on the sub-bundle of oriented orthonormal frame bundles OM, i.e. the horizontal lifting belongs to OM if and only if it is metric. Let $F = P \times \mathbf{R}^n / \sim$ be the associated vector bundle determined by the equivalent relation $[u, e] \sim [ug, g^{-1}e]$ hence the vector bundle is $\{ue\}$ where $e \in \mathbf{R}^n, u \in P$. A section of F corresponds to a vector field over M. A parallel translation is induced on TM in the obvious way and given a connection on P let $H(e)$ be the standard horizontal vector field such that $H(e)_u$ is the horizontal lift through u of the vector $u(e)$. If $e \ne 0$, $H(e)$ are never vanishing vector fields such that $TR_a(H(e)) = H(a^{-1}e)$. The fundamental vector fields generated by a basis of $\mathfrak{gl}(n, \mathbf{R})$ and $H(e_i)$ for e_i a basis of $\mathbf{R}^n$ forms a basis of TP at any point and gives a global parallelism on TP.

If we have a curve σ_t with $\sigma_0 = x$ and $\dot\sigma_0 = v$,

$$\nabla_v Y = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[//_{\varepsilon}^{-1}Y(\sigma_\varepsilon) - Y(x)].$$

Alternatively, $\nabla_X Y(x) = u_0(\tilde{X} f)$ where $\tilde{X}$ is a horizontal lift of X, $f : P \to \mathbf{R}^n$ is defined by $f(u) = u^{-1}[Y(\pi(u))]$ and

$$\tilde{X} f(u_0) = \lim_{h\to 0} \frac{1}{h}(u_h^{-1} Y(\sigma_h) - u_0^{-1} Y(x))$$

for u_h a horizontal lift of x_t starting from u_0. Note that the linear maps $M_x(e)$ which define the connection form on TP are skew symmetric in the case of $P = OM$ and determine the Christoffel symbols. A vector field Y is horizontal along a curve σ_t if $\nabla_{\dot{\sigma}(t)} Y = \lim_{h\to 0} \frac{1}{h}[/\!/_h^{-1} Y(\sigma_h) - Y(x)] = 0$. Define the curvature form to be the 2-form $\Omega(-,-) := \mathrm{d}\omega(P_h-, P_h-)$ where P_h is the projection to the horizontal space. Then, the horizontal part of the Lie bracket of two horizontal vector fields X, Y is the horizontal lift of $[\pi(X), \pi(Y)]$ and its vertical part is determined by $\omega([X,Y]) = -2\Omega(X,Y)$.

The horizontal lift map u_t can also be thought of solutions to:

$$\mathrm{d}u_t = \sum H(e_i)(u_t) \circ \mathrm{d}\sigma_t.$$

In fact, if $\dot{v}_t$ is the horizontal lift of $\dot{\sigma}_t$, $\dot{v}_t = \sum_{i=1}^n \langle \dot{\sigma}_t, e_i\rangle H(e_i)(\tilde{\sigma}_t)$. Note that, $/\!/_t(\sigma)$ is not a solution to a Markovian equation, the pair $(/\!/_t(\sigma), u_t)$ is. In local coordinates for v_t^i the ith component of $/\!/_t(\sigma)(v)$, $v \in T_{\sigma_0}M$,

$$\mathrm{d}v_t^k = -\Gamma_{i,j}^k(\sigma_t) v_t^j \circ \mathrm{d}\sigma_t^i. \tag{6}$$

If σ_t is the solution of the SDE $\mathrm{d}x_t^k = X_i^k(x_t) \circ \mathrm{d}B_t^i + X_0^k(x_t)\mathrm{d}t$, then

$$\mathrm{d}v_t^k = -\Gamma_{i,j}^k(x_t) v_t^j X_i^k(x_t) \circ \mathrm{d}B_t^i - \Gamma_{i,j}^k(x_t) v_t^j X_0^k(x_t)\mathrm{d}t.$$

4.2 How Does the Choice of Connection Help in the Case of the Derivative Flow?

One may wonder why a choice of a linear connection removes a martingale term in a SDE? The answer is that it does not and what it does is the careful choice of a matrix which transforms the original objects of interest. Recall the differentiation formula:

$$d(P_t f)(v) = \mathbf{E}\mathrm{d}f(X_t^v)$$

where for each t, X_t^v is a vector field with $X(x) = v$. The choice of X_t^v is by no means unique. Both the derivative flows and the damped parallel translations are valid choices and the linear connection which is intrinsic to the SDE leads to the correct choice. To make this plain, let us now consider $\mathbf{R}^n$ as a trivial manifold with the non-trivial Riemannian metric and affine connection induced by X. In components, let U_i be functions on $\mathbf{R}^n$ and $U = (U_1, \dots, U_n)$ and $x_0, v \in \mathbf{R}^n$,

$$(\breve{\nabla}_v U)_k(x_0) = (DU_k)_{x_0}(v) + \sum_j \langle X(x_0) D(Y(x)(e_j), e_k\rangle(v) U_j e_k.$$

The last term determines the Christoffel symbols, c.f. [13].

Given a vector field along a continuous curve, there is the stochastic covariant differentiation defined for almost surely all paths, given by $\hat{D} V_t = \hat{/\!/}_t \frac{d}{dt} (\hat{/\!/}_t)^{-1} V_t$ where $\hat{/\!/}_t$ is the stochastic parallel translation using the connection $\hat{\nabla}$, the adjoint connection to $\breve{\nabla}$ to take into account of the torsion effect. Alternatively,

$$(\hat{D} V_t)^k = \frac{d}{dt} V_t^k + \Gamma_{ji}^k(\sigma_t) V_t^j \circ d\sigma_t^i.$$

The derivative flow $V_t = T\xi_t(v_0)$ satisfies the SDE:

$$\hat{D} V_t = \breve{\nabla} X_j(V_t) \circ dB_t^j + \breve{\nabla} X_0(V_t) dt.$$

Let $\bar{V}_t = \mathbf{E}\{V_t | \, x_s : 0 \le s \le T\}$. Then,

$$\hat{D} \bar{V}_t = -\frac{1}{2} (\breve{\text{Ric}})^{\#}(\bar{V}_t) dt + \nabla X_0(\bar{V}_t) dt.$$

In the setting of the Wiener space Ω and $\mathcal{I} = \xi_\cdot(x_0)$ the Itô map, let $V_t = T\mathcal{I}_t(h)$ for h a Cameron Martin vector then

$$\hat{D} V_t = \breve{\nabla} X_j(V_t) \circ dB_t^j + \breve{\nabla} X_0(V_t) dt + X(x_t)(\dot{h}_t) dt$$

and the corresponding conditional expectation of the vector field V_t satisfies

$$\hat{D} \bar{V}_t = -\frac{1}{2} (\breve{\text{Ric}})^{\#}(\bar{V}_t) dt + \breve{\nabla} X_0(\bar{V}_t) dt + X(x_t)(\dot{h}_t) dt.$$

This means, $\hat{/\!/}_t^{-1} \bar{V}_t$ is differentiable in t and hence a Cameron–Martin vector and $\bar{V}_t$ is the induced Bismut-tangent vector by parallel translation.

4.3 A Word About the Stochastic Filtering Problem

Consider the filtering problem for a one dimensional signal process $x(t)$ transmitted through a noise channel

$$\begin{aligned} dx_t &= \alpha(x_t) dt + \sigma \, dW_t \\ dy_t &= \beta(x_t) dt + \sqrt{a} dB_t \end{aligned}$$

where B_t and W_t are independent Brownian motions. The problem is to find the probability density of $x(t)$ conditioned on the observation process $y(t)$ which is closely associated to the following horizontal lifting problem.

Let $\mathcal{B}$ and $\mathcal{A}$ be intertwined diffusion operators. Consider the martingale problems on the path spaces, $C_{u_0}N$ and $C_{y_0}M$, on N and M, respectively. Let u_t and y_t be the canonical process on N and on M, assumed to exist for all time, so that for $f \in C_c^\infty(M)$ and $g \in C_c^\infty(N)$

$$M_t^{\mathrm{d}f,\mathcal{A}} := f(y_t) - f(y_0) - \int_0^t \mathcal{A}f(y_s)\mathrm{d}s$$
$$M_t^{\mathrm{d}g,\mathcal{B}} := g(u_t) - g(u_0) - \int_0^t \mathcal{B}g(u_s)\mathrm{d}s$$

are martingales. For a $\sigma\{y_s : 0 \le s \le t\}$-predictable T^*M-valued process ϕ_t which is along, y_t we could also define a local martingale $M_t^{\phi,\mathcal{A}}$ by

$$\langle M_t^{\phi,\mathcal{A}}, M_t^{\mathrm{d}f,\mathcal{A}}\rangle = 2\int_0^t \mathrm{d}f(\sigma^{\mathcal{A}}(\phi))(y_s)\mathrm{d}s.$$

It is also denoted by

$$M_t^{\phi,\mathcal{A}} \equiv \int_0^t \phi_s d\{y_s\}.$$

The conditional law of u_t given y_t is given by integration against function f from N to $\mathbf{R}$, define

$$\pi_t f(u_0)(\sigma) = \mathbf{E}\Big\{f(u_t)|p(u.) = \sigma\Big\}. \tag{7}$$

This conditional expectation is defined for $P^{\mathcal{A}}_{p(u_0)}$, the $\mathcal{A}$ diffusion measures, almost surely all σ and extends to $\phi_t \circ h_{u_t}$ for ϕ_t as before and h the horizontal lifting map. The following is from Theorem 4.5.1 in [11].

Theorem 1. *If f is C^2 with $\mathcal{B}f$ and $\sigma^{\mathcal{B}}(\mathrm{d}f, \mathrm{d}f) \circ \mathfrak{h}$ bounded, then*

$$\pi_t f(u_0) = f(u_0) + \int_0^t \pi_s(\mathcal{B}f)(u_0)\mathrm{d}s + \int_0^t \pi_s(\mathrm{d}f \circ h_{u.})(u_0)d\{y_s\}. \tag{8}$$

To see this holds, taking conditional expectation of the following equation:

$$f(u_t) = f(u_0) + \int_0^t \mathcal{B}f(u_s)\mathrm{d}s + M_t^{\mathrm{d}f,\mathcal{B}}$$

and use the following theorem, Proposition 4.3.5 in [11],

$$\mathbf{E}\{M_t^{\mathrm{d}f,\mathcal{B}}|p(u.) = x.\} = M_t^{\mathbf{E}\{\mathrm{d}f\circ h_{u_s}|p(u.)=x.\},\mathcal{A}}.$$

In the case that $p : M \times M_0 \to M$ is the trivial projection of the product manifold to M, let $\mathcal{A}$ be a cohesive diffusion operator on M, Ł the diffusion generator on M_0, and $u_t = (y_t, x_t)$ a $\mathcal{B}$ diffusion. If x_t is a Markov process with generator $\mathcal{L}$ and $\mathcal{B}$ a coupling of $\mathcal{L}$ and $\mathcal{A}$, by which we mean that $\mathcal{B}$ is intertwined with $\mathcal{L}$ and $\mathcal{A}$ by the projections p_i to the first or the second coordinates, there is a bilinear $\Gamma^{\mathcal{B}} : T^*M \times T^*M_0 \to \mathbf{R}$ such that

$$\mathcal{B}(g_1 \otimes g_2)(x, y) = (\mathcal{L}g_1)(x)g_2(y) + g_1(x)(\mathcal{A}g_2)(y) + \Gamma^{\mathcal{B}}((\mathrm{d}g_1)_x, (\mathrm{d}g_2)_y) \quad (9)$$

where $g_1 \otimes g_2 : M \times M_0 \to \mathbf{R}$ denotes the map $(x, y) \mapsto g_1(x)g_2(y)$ and g_1, g_2 are C^2. In fact, $\Gamma^{\mathcal{B}}((\mathrm{d}g_1)_x, (\mathrm{d}g_2)_y) = \sigma^{\mathcal{B}}_{(x,y)}(\mathrm{d}\tilde{g}_1, \mathrm{d}\tilde{g}_2)$ where $\tilde{g}_i = g(p_i)$. Then, $\sigma^{\mathcal{B}} : T^*M_1 \times T^*M_2 \to TM_1 \times TM_2$ is of the following form. For $\ell_1 \in T^*_x M_1$, $\ell_2 \in T^*_y M_2$

$$\sigma^{B}_{(x,y)}(\ell_1, \ell_2) = \begin{pmatrix} \sigma^L_x & \sigma^{1,2}_{(x,y)} \\ \sigma^{2,1}_{(x,y)} & \sigma^{\mathcal{A}}_x \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}.$$

The horizontal lifting map is given by

$$v \mapsto (v, \alpha \circ (\sigma^{\mathcal{A}})^{-1}(v))$$

where $\alpha : T_x M^* \to T_y M_0$ are defined by

$$\ell_2(\alpha(\ell_1)) = \frac{1}{2}\Gamma^{\mathcal{B}}(\ell_1, \ell_2).$$

In the theorem above, take $1 \otimes f$ to see that $\pi_s \mathcal{B}(1 \otimes f)$ reduces to $\mathcal{L}f$ and the filtering equation is:

$$\pi_t f(x_0) = f(x_0) + \int_0^t \pi_s(\mathcal{L}f)(x_0)\mathrm{d}s + \int_0^t \pi_s(\mathrm{d}f(\alpha \circ (\sigma^{\mathcal{A}})^{-1}))(x_0)d\{y_s\}.$$

The case of non-Markovian observation when the non-Makovian factor is introduced through the drift equation for the noise process y_t can be dealt with through a Girsanov transformation. See [11] for detail. Finally, we note that the field of stochastic filtering is vast and deep, and we did not and would not attempt to give historical references as they deserve. However, we would like to mention a recent development [3] which explore the geometry of the signal-observation system. See also [12, 14, 15] and recent work of T. Kurtz.

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Efficient and Practical Implementations of Cubature on Wiener Space

Lajos Gergely Gyurkó and Terry J. Lyons

Abstract This paper explores and implements high-order numerical schemes for integrating linear parabolic partial differential equations with piece-wise smooth boundary data. The high-order Monte-Carlo methods we present give extremely accurate approximations in computation times that we believe are comparable with much less accurate finite difference and basic Monte-Carlo schemes.

A key step in these algorithms seems to be that the order of the approximation is tuned to the accuracy one requires. A considerable improvement in efficiency can be attained by using ultra high-order cubature formulae. Lyons and Victoir (Cubature on Wiener Space [19]) give a degree 5 approximation of Brownian motion. We extend this cubature to degrees 9 and 11 in 1-dimensional space-time. The benefits are immediately apparent.

Keywords Stochastic differential equation · Numerical solution · Weak approximation · Cubature · Wiener space · Expected signature · High order

MSC (2010): 65C05, 65C30, 65M75, 91G60

1 Introduction

In this paper, we provide practical tools based on the Kusuoka–Lyons–Victoir (KLV) family of numerical methods for integrating solutions to partial differential equations that occur in mathematical finance and many other fields. These KLV methods, with the refinements of recombination, offer considerable potential for numerically solving partial differential equations. However, until now, these methods have only been implemented using at most a seventh order cubature formula, and more generally a fifth order cubature. A more detailed analysis shows that

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the maximum benefit of the method will only be achieved if one adapts genuinely high-order approximations and uses the expensive process of high-order recombination [13, 14] to keep the "Monte-Carlo" sample population size under control. This paper explores these numerical methods in the 1-dimensional space-time case and gives cubature formulae of order up to 11 for this setting.

The paper is structured as follows. In Sect. 2, a brief introduction to the KLV method is given with special focus on the Cubature on Wiener space framework that interprets the method as a quadrature-type numerical integration technique on the infinite dimensional Wiener space.

Section 3 focuses on the abstract algebraic background required for the construction of cubature formulae. The algebraic background consists of the representation of continuous paths in terms of elements in a certain Lie algebra and the derivation of the moment matching conditions in terms of simplified polynomial equations. The particular polynomial systems corresponding to degree 3 in high dimensions, degree 5 in dimensions 1 and 2, degree 7 in 1-dimension, degree 9 in 1-dimension and degree 11 cubature formula in 1-dimension are derived and solved in the appendix.

In Sect. 4, we consider piece-wise smooth terminal conditions allowing discontinuities. We combine high-order cubature formulae and iterative strategies and introduce a new extension that leads to efficient and highly accurate approximations. The new variant is referred to as *repeated cubature*. Furthermore, we outline a scheme that is expected to perform well if used for the approximation of solutions to Dirichlet boundary problems.

Some numerical results are presented in Sect. 5. We test different versions and extensions on parabolic partial differential equations with globally smooth and with piece-wise smooth terminal conditions.

2 Cubature on Wiener Space

The approximation of expectations of diffusion processes is required in many practical applications. Kusuoka [9, 10] constructed a high-order numerical method for this purpose, which we regard as the first version of (or Kusuoka's approach to) the KLV methods. Lyons and Victoir [18] described a framework that interprets the method as a generalisation of quadrature formulae on finite dimensional spaces to the infinite dimensional Wiener space. We refer to this latter approach as *Cubature on Wiener space*.

2.1 Background

Let the $\mathbb{R}^N$-valued process Y_t^y be the solution of the following Stratonovich stochastic differential equation

$$\mathrm{d}Y_t^y = V_0(Y_t^y)\mathrm{d}t + \sum_{i=1}^{d} V_i(Y_t^y) \circ \mathrm{d}B_t^i,\ Y_0^y = y \tag{1}$$

where $V_0, \ldots, V_d$ are elements of the space $C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ of $\mathbb{R}^N$-valued smooth functions defined in $\mathbb{R}^N$ with bounded derivatives of any order,[1] and $B = (B^1, \ldots, B^d)$ is a Brownian motion in $\mathbb{R}^d$. The underlying probability space is the Wiener space $(\Omega_T, \mathcal{F}, \mathbb{P})$, where $\Omega_T := C_0^0([0,T], \mathbb{R}^d)$ denotes the space of $\mathbb{R}^d$-valued continuous functions defined in $[0, T]$, which start at zero. For $\omega \in \Omega$, we make the convention $\omega^0(t) := t$ and $B_t^0(\omega) := t$.

In this section, we focus on the numerical approximation of the expectation

$$\mathbb{E}_{\mathbb{P}}\left[f(Y_T^y)\right] = \int_{\Omega_T} f(Y_T^y(\omega))\mathbb{P}(\mathrm{d}\omega)$$

where f is a smooth or Lipschitz continuous real valued function defined in $\mathbb{R}^N$.

In numerical analysis, integrals with respect to some measure μ on $\mathbb{R}^d$ are often approximated by integrals with respect to a finitely supported measure ν

$$\int_{\mathbb{R}^d} P(x)\mu(\mathrm{d}x) \approx \int_{\mathbb{R}^d} P(x)\nu(\mathrm{d}x) := \sum_{i=1}^{n} \nu_i P(x_i) \tag{2}$$

where $\nu = \sum_{i=1}^n \nu_i \delta_{x_i}$ and the points $x_1, \ldots, x_n$ are in the support of μ. By Tchakaloff's theorem [23], for any positive integer m, there exists a finitely supported measure $\nu = \nu_m$, such that the approximation (2) is exact for polynomials $P \in \mathbb{R}_m[X_1, \ldots, X_d]$; furthermore, $n \leq \dim \mathbb{R}_m[X_1, \ldots, X_d]$. For $d > 1$, the finitely supported approximating measure ν is referred to as cubature formulae, and for $d = 1$ we talk about quadrature formulae.

The approach by Lyons and Victoir generalises and adapts this idea to the infinite dimensional Wiener space by constructing a finitely supported measure $\mathbb{Q}$ determined by weights $\lambda_1, \ldots, \lambda_n$ and paths $\omega_1, \ldots, \omega_n \in \Omega_T$:

$$\mathbb{Q} = \sum_{i=1}^{n} \lambda_i \delta_{\omega_i},$$

such that

$$\int_{\Omega_T} f(Y_T^y(\omega))\mathbb{P}(\mathrm{d}\omega) \approx \sum_{i=1}^{n} \lambda_i f(Y_T^y(\omega_i)), \tag{3}$$

where

$$\mathrm{d}Y_t^y(\omega_i) = V_0(Y_t^y(\omega_i))\mathrm{d}t + \sum_{i=1}^{d} V_i(Y_t^y(\omega_i)) \circ \mathrm{d}\omega_t^i, \; Y_0^y(\omega_i) = y. \tag{4}$$

The nature of the approximation (3) is specified in Sect. 2.2.

[1] The condition can be relaxed; the boundedness of the derivatives is required up to certain order, which we will specify later.

The interpretation of (4) is not obvious; one might consider interpreting the $Y_T^y(\omega_i)$ as a solution to a differential equation driven by the rough path ω_i [16, 17]; however, the existence of $\mathbb{Q}$ supported by piece-wise linear paths is proven in [18], in which case, (4) is an ordinary differential equation. A general approach to constructing the measure $\mathbb{Q}$ supported by piece-wise linear paths is described in Sect. 3.

Finally, we recall the fact [6] that the function u defined by $u(t,x) = \mathbb{E}_{\mathbb{P}}\left[f(Y_{T-t}^x)\right]$ satisfies the partial differential equation (PDE)

$$\frac{\partial}{\partial t}u(t,x) + W_0 u(t,x) + \frac{1}{2}\sum_{i=1}^{d} W_i^2 u(t,x) = 0,\ u(T,x) = f(x) \tag{5}$$

where the operators $W_0, \ldots, W_d$ are defined by

$$W_i(x) := \sum_{j=1}^{N} V_i^j(x)\frac{\partial}{\partial x^j}.$$

Therefore, the KLV family is also useful for the approximation of certain PDEs.

2.2 The Scheme and Its Convergence

In this section, we specify the approximation (3). The cubature (quadrature) formula (2) on the finite dimensional space is chosen to be exact for polynomials up to certain degree. The cubature on Wiener space is constructed to be exact for Brownian iterated integrals up to certain degree. This idea is induced by the stochastic Taylor expansion.

For smooth functions $f : \mathbb{R}^N \to \mathbb{R}$, the stochastic Taylor expansion represents $f(Y_T^y)$ in terms of objects indexed by *multi-indices* of the form

$$I = (i_1, \ldots, i_k) \in \{0, 1, \ldots, d\}^k,\ k = 0, 1, 2, \ldots$$

Definition 1. The set of all finite multi-indices is denoted by $\mathcal{A}$. The empty multi-index is denoted by $\emptyset$. Given a multi-index $I = (i_1, \ldots, i_k)$, we define

(a) The *length* $|I|$ of the multi-index I by $|I| = |(i_1, \ldots, i_k)| := k$.
(b) The function $\|\cdot\| : \mathcal{A} \to \mathbb{N}$ – referred to as the *degree* of a multi-index – as

$$\|I\| = \|(i_1, \ldots, i_k)\| := |I| + \text{card}\{i_j = 0 \,|1 \le j \le k\}$$

(c) The *left decrement* as

$$_{-}I = {}_{-}(i_1, \ldots, i_k) := (i_2, \ldots, i_k)$$

Finally, for the multi-index $I = (i_1, \dots, i_k)$, we introduce the notation

$$W_I f(x) = W_{i_1} \cdots W_{i_k} f(x).$$

We define the norm $\|\cdot\|_\infty^W$ by

$$\|f\|_\infty^W = \max\{\|f\|_\infty, \|W_1 f\|_\infty, \dots, \|W_d f\|_\infty\}.$$

Lemma 1 (Stochastic Taylor expansion). *Let f be a real-valued smooth function defined in $\mathbb{R}^N$. Then,*

$$f(Y_t^y) = f(y) + \sum_{\|I\| \le m} W_I f(y) \int_{0<u_1<\cdots<u_k<t} \circ \mathrm{d}B_{u_1}^{i_1} \cdots \circ \mathrm{d}B_{u_k}^{i_k} + R_t^m. \quad (6)$$

Furthermore, the following inequality holds

$$\sqrt{\mathbb{E}_{\mathbb{P}}[(R_t^m)^2]} \le C \sum_{\substack{\|-I\| \le m \\ \|I\| = m+1}} \|W_I f\|_\infty^W t^{(m+1)/2} + C \sum_{\substack{\|-I\| \le m \\ \|I\| = m+2}} \|W_I f\|_\infty t^{(m+2)/2} \quad (7)$$

where C only depends on m and d.

The expansion (6) is based on the repeated use of Itô's lemma written in Stratonovich form:

$$f(Y_t^y) = f(y) + \sum_{i=0}^{d} \int_0^t W_i f(Y_s^y) \circ \mathrm{d}B_s^i.$$

The bound (7) on the remainder term is derived in [7] and also in [4].

Definition 2 (Cubature formula). Let m be a positive integer. A measure $\mathbb{Q}_T = \sum_{i=1}^n \lambda_i \delta_{\omega_i}$ supported by finitely many $\mathbb{R}^{d+1}$-valued paths $\omega_1, \dots, \omega_n$ defined in $[0, T]$ satisfying $\omega_i^0(t) = t$ is called a *degree m cubature formula on* $[0, T]$ if for all multi-indeces $I = (i_1, \dots, i_k)$, $k \ge 1$, $\|I\| \le m$ the following inequality holds

$$\mathbb{E}_{\mathbb{P}}\left[\int_{0<u_1<\cdots<u_k<T} \circ \mathrm{d}B_{u_1}^{i_1} \cdots \circ \mathrm{d}B_{u_k}^{i_k}\right] = \mathbb{E}_{\mathbb{Q}_T}\left[\int_{0<u_1<\cdots<u_k<T} \circ \mathrm{d}B_{u_1}^{i_1} \cdots \circ \mathrm{d}B_{u_k}^{i_k}\right]. \quad (8)$$

Note, that by the rescaling property of Brownian motion

$$\int_{0<u_1<\cdots<u_k<t} \circ \mathrm{d}B_{u_1}^{i_1} \cdots \circ \mathrm{d}B_{u_k}^{i_k} =^{\mathcal{L}} t^{\|(i_1,\dots,i_k)\|/2} \int_{0<u_1<\cdots<u_k<1} \circ \mathrm{d}B_{u_1}^{i_1} \cdots \circ \mathrm{d}B_{u_k}^{i_k}.$$

Therefore, the paths $\omega_1, \dots, \omega_n$ in the support of a cubature formula $\mathbb{Q}_1 = \sum_{i=1}^n \lambda_i \delta_{\omega_i}$ on $[0, 1]$ can be rescaled to a cubature formula $\mathbb{Q}_t = \sum_{i=1}^n \lambda_i \delta_{\langle t, \omega \rangle_i}$ on $[0, t]$ by

$$\langle t,\omega\rangle_i^j(s) := \begin{cases} t\omega_i^j(s/t) & \text{if } j=0 \\ \sqrt{t}\omega_i^j(s/t) & \text{if } 1\le j\le d \end{cases} \tag{9}$$

for $s \in [0,t]$ and $i = 1,\dots,n$.

In the rest of this section, we assume the existence of a degree m cubature formula $\mathbb{Q}_1 = \sum_{i=1}^n \lambda_i \delta_{\omega_i}$ on [0, 1] supported by piece-wise linear paths of finite length, and we assume that the cubature formula $\mathbb{Q}_t = \sum_{i=1}^n \lambda_i \delta_{\langle t,\omega\rangle_i}$ is constructed by rescaling (9) from $\mathbb{Q}_1$. This rescaling property implies the following bound on the *local cubature error*.

Lemma 2. *There exists a positive constant C depending on d and m and on the length of the paths in the support of the measure $\mathbb{Q}_1$, such that*

$$\left|(\mathbb{E}_{\mathbb{P}}-\mathbb{E}_{\mathbb{Q}_t})[f(Y_t^y)]\right| \le C \sum_{\substack{\|-I\|\le m \\ \|I\|=m+1}} \|W_I f\|_\infty^W t^{(m+1)/2} + C \sum_{\substack{\|-I\|\le m \\ \|I\|=m+2}} \|W_I f\|_\infty t^{(m+2)/2}.$$

The proof is based on the stochastic Taylor expansion of the SDE (1) and on the Taylor expansion of the ODE (4).

The lemma describes a high-order approximation on $[0,t]$ for small enough t. For a longer time interval $[0,T]$, we introduce the global cubature measure.

Definition 3. Let measure $\mathbb{Q}_1 = \sum_{i=1}^n \lambda_i \delta_{\omega_i}$ define a Cubature formula on [0, 1] and $\mathcal{D} = \{0 = t_0 < \cdots < t_k = T\}$ be a partition of $[0,T]$. The *global cubature measure* $\mathbb{Q}_{\mathcal{D}}$ is defined by

$$\mathbb{Q}_{\mathcal{D}} = \sum_{(i_1,\dots,i_k)\in\{1,\dots,n\}^k} \lambda_{i_1}\cdots\lambda_{i_k}\delta_{\langle t_1-t_0,\omega\rangle_{i_1}\circ\cdots\circ\langle t_k-t_{k-1},\omega\rangle_{i_k}}$$

where $\omega\circ\hat{\omega}$ denotes the concatenation of the paths ω and $\hat{\omega}$.

First, we establish a bound on the global error when the terminal condition function f is smooth with bounded derivatives up to order m.

Proposition 1. *Let $\mathcal{D} = \{0 = t_0 < \cdots < t_k = T\}$ be a partition of $[0,T]$ and f a real-valued smooth function with bounded derivatives up to order m. Then, there exists a constant C depending only on d, m and on the length of the paths in the support of $\mathbb{Q}_1$, such that*

$$\begin{aligned}\sup_{y\in\mathbb{R}^N}\left|(\mathbb{E}_{\mathbb{P}}-\mathbb{E}_{\mathbb{Q}_{\mathcal{D}}})[f(Y_T^y)]\right| \le C\sum_{i=2}^k\Bigg(&\sum_{\substack{\|-I\|\le m \\ \|I\|=m+1}} \|W_I P_{(T-t_{i-1})} f\|_\infty^W (t_i-t_{i-1})^{(m+1)/2} \\ &+ \sum_{\substack{\|-I\|\le m \\ \|I\|=m+2}} \|W_I P_{(T-t_{i-1})} f\|_\infty (t_i-t_{i-1})^{(m+2)/2}\Bigg),\end{aligned} \tag{10}$$

where $P_t f(x) := \mathbb{E}_{\mathbb{P}}\left[f(Y_t^x)\right]$.

Sketch of the proof: We introduce the operator Q_t by $Q_t f(x) := \mathbb{E}_{\mathbb{Q}_t}\left[f(Y_t^x)\right]$. Exploiting the semi-group property $P_t P_s = P_{t+s}$ of the operator P_t, we rewrite the left hand side of the inequality (10) in terms of the following sum.

$$\begin{aligned}&\left(P_T - Q_{(t_1-t_0)} \cdots \ Q_{(T-t_{k-1})}\right) f(x) \\ &\quad = \sum_{i=1}^{k} Q_{(t_1-t_0)} \cdots Q_{(t_{i-1}-t_{i-2})} \left(P_{(t_i-t_{i-1})} - Q_{(t_i-t_{i-1})}\right) P_{(T-t_i)} f(x). \quad (11)\end{aligned}$$

Then, the inequality (10) is implied by lemma 2 and by the fact that Q_t is a probability measure for each $t > 0$. □

In order to give a bound on the global error when the terminal condition f is not smooth but Lipschitz continuous, some stronger condition is required on the vector fields $W_0, \ldots, W_d$.

Definition 4 (UFG condition). The set of vector fields $\{W_0, \ldots, W_d\}$ satisfies the *UFG condition*, if there exists a positive integer l, such that for each multi-index $J = (j_1, \ldots, j_k) \neq (0)$, $k \geq 1$,

$$[W_{j_1}, [W_{j_2}, [\cdots, W_{j_k}]\cdots] = \sum_{\substack{I=(i_1,\ldots,i_r)\neq(0)\\ |I|\geq 1, \|I\|\leq l}} \phi_{I,J} [W_{i_1}, [W_{i_2}, [\cdots, W_{i_r}]\cdots]$$

for some smooth functions $\phi_{I,J} : \mathbb{R}^N \to \mathbb{R}$ with bounded derivatives, where $[W, V]$ denotes the Lie-bracket of the vector fields V and W.

Proposition 2. *Let $\mathcal{D} = \{0 = t_0 < \cdots < t_k = T\}$ be a partition of $[0, T]$ and f a real-valued Lipschitz continuous function defined in $\mathbb{R}^N$. Then, there exists a constant C depending only on d, m and on the length of the paths in the support of $\mathbb{Q}_1$, such that*

$$\begin{aligned}\sup_{y\in\mathbb{R}^N} \left|(\mathbb{E}_{\mathbb{P}} - \mathbb{E}_{\mathbb{Q}_{\mathcal{D}}})[f(Y_T^y)]\right| \leq C \|\nabla f\|_\infty \Bigg((T - t_{k-1})^{1/2} \\ + \sum_{i=1}^{k-1} \frac{(t_i - t_{i-1})^{(m+1)/2}}{(T - t_i)^{m/2}}\Bigg). \quad (12)\end{aligned}$$

Sketch of the proof: The decomposition (11) is used again. Since f is not smooth, the last term in the telescopic sum (11) is estimated as follows.

$$\begin{aligned}\left|(P_{(T-t_{k-1})} - Q_{(T-t_{k-1})}) f(x)\right| &\leq \left|(P_{(T-t_{k-1})} f(x) - f(x)\right| \\ &+ \left|(Q_{(T-t_{k-1})} f(x) - f(x)\right| \leq K_1 (T - t_{k-1})^{1/2} \|\nabla f\|_\infty\end{aligned}$$

for some constant K_1 depending on the vector fields $W_0, \ldots, W_d$.

Exploiting the fact that $P_t f$ is smooth for $t > 0$, the bound on the rest of the terms is derived from a result by Kusuoka and Stroock [8, 11] proving that under the UFG condition 4, for any multi-index I:

$$\|W_I P_t f\|_\infty \leq \frac{K_2 s^{1/2}}{s^{\|I\|/2}} \|\nabla f\|_\infty$$

for some constant K_2 depending on the vector fields $W_0, \ldots, W_d$. For detailed proof, the reader is referred to Kusuoka [10] and Litterer [13]. □

Remark 1. Note that the bound (12) is not optimal for partitions $\mathcal{D}$ of even steps. Kusuoka constructed the partition $\{0 = t_0 < t_1 < \cdots < t_k = T\}$

$$t_i = T\left(1 - \left(1 - \frac{i}{k}\right)^\gamma\right)$$

and proved that

$$\sup_{y \in \mathbb{R}^N} \left|(\mathbb{E}_\mathbb{P} - \mathbb{E}_{\mathbb{Q}_\mathcal{D}})[f(Y_T^y)]\right| \leq C\|\nabla f\|_\infty k^{-\gamma/2}$$

if $0 < \gamma < m - 1$,

$$\sup_{y \in \mathbb{R}^N} \left|(\mathbb{E}_\mathbb{P} - \mathbb{E}_{\mathbb{Q}_\mathcal{D}})[f(Y_T^y)]\right| \leq C\|\nabla f\|_\infty k^{-(m-1)/2} \log(k + 1)$$

if $\gamma = m - 1$ and

$$\sup_{y \in \mathbb{R}^N} \left|(\mathbb{E}_\mathbb{P} - \mathbb{E}_{\mathbb{Q}_\mathcal{D}})[f(Y_T^y)]\right| \leq C\|\nabla f\|_\infty k^{-(m-1)/2}$$

if $\gamma > m - 1$.

On Fig. 1, the structure of an equal step-size Cubature tree is demonstrated. On Fig. 2, a Cubature formula corresponding to Kusuoka's uneven partition is sketched. On both figures, the piece-wise linear paths are replace with straight arrows.

2.3 A Note on the Support Size of the Cubature Measure

The support of $\mathbb{Q}_\mathcal{D}$ grows exponentially with the number of time-steps in $\mathcal{D}$. If the support of $\mathbb{Q}_1$ is big and/or many time-steps are required to attain the desired accuracy, the method loses its tractability. This problem has been addressed in the literature.

Ninomiya and Kusuoka considered Monte-Carlo sampling from the global measure $\mathbb{Q}_\mathcal{D}$ as well as random re-sampling [12, 19]. They implemented the tree-based branching algorithm [3] and observed lower variance compared to the Monte-Carlo estimates. The variance reduction feature of the tree-based branching algorithm

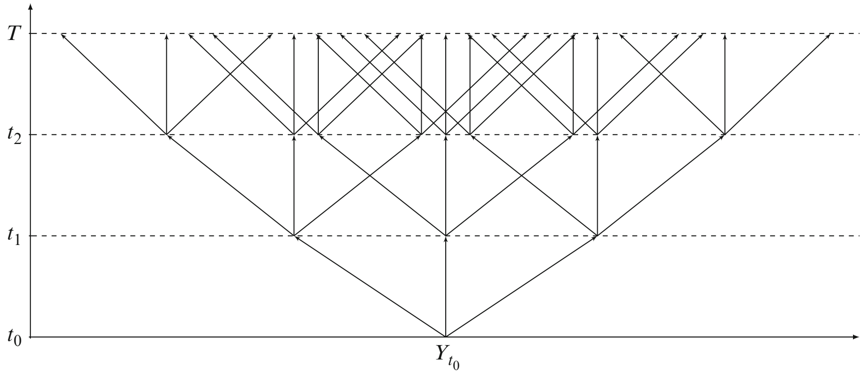

Fig. 1 Equal step-size cubature tree

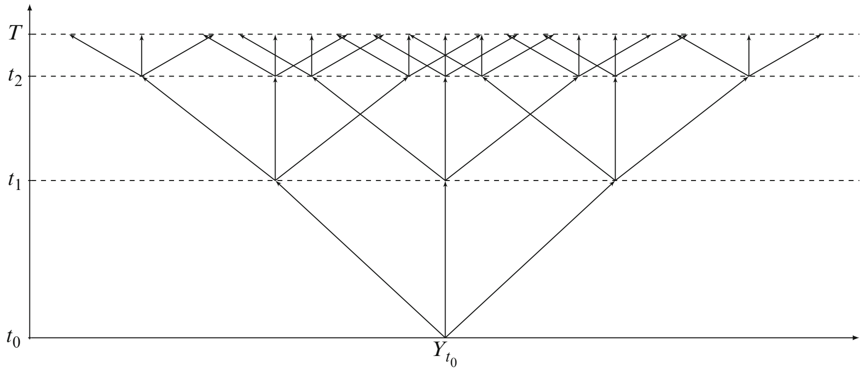

Fig. 2 Cubature tree with Kusuoka's partition

applied to the global cubature measure was further analysed by Gyurkó [5]. The drawback of these versions is that they introduce randomness and an approximation error component due to the sampling variance.

Litterer and Lyons [13, 14] considered the recombination of the global cubature measure. By recombination, they refer to an algorithm that constructs a measure $\hat{\mathbb{Q}}$ from a finitely supported measure $\mathbb{Q}$, such that the support of $\hat{\mathbb{Q}}$ is a subset of the support of $\mathbb{Q}$; furthermore, the expectations of certain test functions are the same under the two measures. Litterer and Lyons [13, 14] proved that the growth of the cubature measure can be reduced to a polynomial function of the number of time-steps using recombination. The cost of reducing a cloud of N points to a cloud of $d + 1$ points preserving d-moments is $C_1 3Nd + C_2 \log_2(N/d)d^4$ for some constants C_1 and C_2. In a typical implementation, the cloud of points reached by the cubature method at level k is split up into smaller subsets, such that the diameter of each subset is sufficiently small. The recombination is applied to each subset separately.

The recombination tool has proven to be essential for the extensions introduced in Sect. 4.

3 Cubature Formulae

This section is focused on the construction of cubature formulae satisfying the equality (8).

3.1 Signature of Paths

The Sect. 2.2 demonstrates the importance of the iterated integral

$$B^I_{s,t} := \int_{s<u_1<\dots<u_k<t} \circ\, \mathrm{d}B^{i_1}_{u_1} \cdots \circ\, \mathrm{d}B^{i_k}_{u_k}$$

where $I = (i_1, \dots, i_k)$ is a multi-index. Following some of the most crucial ideas in Rough Paths Theory [15–17], we introduce the following algebraic structure for handling a collection of iterated integrals simultaneously and for exploiting some algebraic properties of continuous paths.

Definition 5. Let $\mathcal{T}$ denote the associative and non-commutative tensor algebra with unit $\mathbf{1}$ generated by the letters $e_0, \dots, e_d$.

(a) For a multi-index $I = (i_1, \dots, i_k)$, we will use the notation

$$e_I := e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

(b) The exponential and logarithm functions on $\mathcal{T}$ are defined by power series:

$$\exp(\mathbf{a}) := \mathbf{1} + \sum_{i=1}^{\infty} \frac{\mathbf{a}^{\otimes i}}{i!}$$

$$\log(\mathbf{a}) := \log(a_\emptyset) + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \left(\frac{\mathbf{a}}{a_\emptyset} - \mathbf{1} \right)^{\otimes i}, \text{ assuming } a_\emptyset \neq 0$$

where $\mathbf{a} = \sum_{I \in \mathcal{A}} a_I e_I$

(c) The m-truncated tensor algebra $\mathcal{T}^{(m)}$ is spanned by the set $\{e_I \,|\, \|I\| \leq m\}$.

(d) The projection to $\mathcal{T}^{(m)}$ is denoted by π_m.

(e) The free Lie-algebra generated by the letters $e_0, \dots, e_d$, and the Lie-bracket $[\mathbf{a}, \mathbf{b}] := \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}$ is denoted by $\mathcal{L}$.

(f) The m-truncated Lie-algebra $\mathcal{L}^{(m)} := \pi_m \mathcal{L}$.

The key algebraic objects are the signature and log-signature of paths.

Definition 6. For $0 \leq s \leq t \leq T$, the *signature* of a continuous path $\omega : [0, T] \to \mathbb{R}^d$ of finite length is defined as

$$S_{s,t}(\omega) := \sum_{I \in \mathcal{A}} e_I \int_{s<u_1<\cdots<u_k<t} \mathrm{d}\omega_{u_1}^{i_1} \cdots \mathrm{d}\omega_{u_k}^{i_k}.$$

The *log-signature* of ω is defined by $\log(S_{s,t}(\omega))$.

The *Brownian signature* $S_{s,t}(B)$ is defined as

$$S_{s,t}(B) := \sum_{I \in \mathcal{A}} B_I e_I.$$

The *Brownian log-signature* is defined by $\log(S_{s,t}(B))$.

The (realization of) signatures of paths lie in $\mathcal{T}$; however, not all elements in $\mathcal{T}$ represent a signature. The following theorem by Chen has a crucial importance for the construction of cubature formulae.

Theorem 1 (Chen's theorem). *(a) The signature is multiplicative, that is for a continuous path $\omega : [0, T] \to \mathbb{R}^d$ of finite length and for $0 \leq s < t < u \leq T$, we have*

$$S_{s,t}(\omega) \otimes S_{t,u}(\omega) = S_{s,u}(\omega).$$

(b) The log-signature $\log(S_{s,t}(\omega))$ *of a continuous path $\omega : [0, T] \to \mathbb{R}^d$ of finite length lies in $\mathcal{L}$ for any* $0 \leq s \leq t \leq T$.

(c) The Brownian log-signature $\log(S_{s,t}(B))$ *a.s. lies in $\mathcal{L}$ for any* $0 \leq s \leq t$.

(d) Given an element L in $\mathcal{L}^{(m)}$, there exists a (not necessarily unique) continuous path $\omega : [s, t] \to \mathbb{R}^d$, such that $\pi_m \log(S_{s,t}(\omega)) = L$.

The reader is referred to [2, 15] for proof.

3.2 Construction of Cubature Measures in $\mathcal{L}^{(m)}$

The definition 2 can be rephrased in terms of the Brownian signature: a measure $\mathbb{Q}_1$ on Ω defines a degree m cubature formula if

$$\mathbb{E}_{\mathbb{P}}[\pi_m S_{0,1}(B)] = \mathbb{E}_{\mathbb{Q}_1}[\pi_m S_{0,1}(B)].$$

Assume that the Lie-elements $L_1, \ldots, L_n$ in $\mathcal{L}^{(m)}$ and the positive weights $\lambda_1, \ldots, \lambda_n$ satisfy

$$\mathbb{E}_{\mathbb{P}}[\pi_m S_{0,1}(B)] = \sum_{i=1}^{n} \lambda_i \pi_m \exp(L_i). \tag{13}$$

Then by part iv) of Chen's theorem, there exist paths $\omega_1, \ldots, \omega_n$, such that $\pi_m S_{0,1}(\omega_i) = \pi_m \exp(L_i)$ and $\mathbb{Q}_1 := \sum_{i=1}^n \lambda_i \delta_{\omega_i}$ defines a degree m cubature formula. In this section, the focus is on the solution of (13).

Lemma 3. *The expected value of the Brownian signature is*

$$\mathbb{E}_{\mathbb{P}}[S_{0,1}(B)] = \exp\left(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i \otimes e_i\right). \tag{14}$$

The lemma is proven in [18].

We have no general algorithm to construct the solution to (13); however, the equation can be simplified since the expected Brownian signature (14) is a sparse and symmetric element in $\mathcal{T}$. The following theorem has crucial importance.

Theorem 2 (Poincaré-Birkhoff-Witt). *Let $\mathcal{B}_{\mathcal{L}} = \{\ell_1, \ell_2, \ldots\}$ be a basis of $\mathcal{L}$ equipped with a total order $\preceq$. The set*

$$\bigcup_{k\geq 0}\{\ell_{i_1} \otimes \cdots \otimes \ell_{i_k} \,|\, \ell_{i_1} \preceq \cdots \preceq \ell_{i_k}\}$$

forms a basis of $\mathcal{T}$.

If $l_1, \ldots, l_k$ are elements in $\mathcal{T}$, we introduce their symmetrized product by

$$(l_1, \ldots, l_k) := \frac{1}{k!}\sum_{\sigma \in S_k} l_{\sigma(1)} \otimes \cdots \otimes l_{\sigma(k)},$$

where S_k denotes the set of all permutations of k elements.

Theorem 2 implies the following corollary.

Corollary 1. *Let $\mathcal{B}_{\mathcal{L}} = \{\ell_1, \ell_2, \ldots\}$ be a basis of $\mathcal{L}$ equipped with a total order $\preceq$. The set*

$$\bigcup_{k\geq 0}\{(\ell_{i_1}, \cdots, \ell_{i_k}) \,|\, \ell_{i_1} \preceq \cdots \preceq \ell_{i_k}\}$$

forms a basis of $\mathcal{T}$.

An approach to solving (13) could consist of exponentiating the Lie-elements $L_i = \sum_{\|\ell_j\| \leq m} \alpha_{i,j}\ell_j$ parametrized with some real valued coefficients $\alpha_{i,j}$, $i = 1, \ldots, n$ and $j = 1, \ldots, |\mathcal{B}_{\mathcal{L}}|$ and solving

$$\sum_{i=1}^{n} \lambda_i \pi_m \exp(L_i) = \pi_m \exp\left(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i \otimes e_i\right) \tag{15}$$

for λ_i and $\alpha_{i,j}$.

We propose two improvements to this strategy. First, we note that it is possible to narrow down the number of parameters. In this paper, we take the Philip Hall basis for $\mathcal{B}_{\mathcal{L}}$. Given the basis $\mathcal{B}_{\mathcal{L}}$ of $\mathcal{L}$, we construct the basis of the tensor algebra $\mathcal{T}$ determined by Corollary 1. We use the notation $\mathcal{B}_{\mathcal{T}}$ for this tensor basis. We denote by $\mathcal{S}_{\mathcal{T},m}$ the smallest subset of $\mathcal{B}_{\mathcal{T}}$ spanning the truncated expected signature $\pi_m \exp(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i \otimes e_i)$. Let $\mathcal{S}_{\mathcal{L},m}$ denote the smallest subset of $\mathcal{B}_{\mathcal{L}}$ that generates the set $\mathcal{S}_{\mathcal{T},m}$ via symmetrized products. In general, the set $\mathcal{S}_{\mathcal{L},m}$ is significantly smaller than the set $\{\ell \in \mathcal{B}_{\mathcal{L}}, \|\ell\| \le m\}$. This way we can reduce the number of parameters, and it is sufficient to look for L_i of the following form

$$L_i = \sum_{\ell_j \in \mathcal{S}_{\mathcal{L},m}} \alpha_{i,j} \ell_j .$$

The next step is to solve (15). Note the following lemma.

Lemma 4. $\pi_m \exp(L_i)$ *is a linear combination of symmetrized products of the Lie basis elements:*

$$\pi_m \exp(L_i) = \sum_{\substack{I=(i_1,\dots,i_k)\in\mathcal{A} \\ \ell_{i_j}\in\mathcal{S}_{\mathcal{L},m}, j=1,\dots,k \\ k\ge 0}} m_I(\alpha_{i,i_1},\dots,\alpha_{i,i_k})(\ell_{i_1},\dots,\ell_{i_k}) \tag{16}$$

where each m_I *is a monomial of its arguments.*

Sketch of the proof: The proof is based on the following two facts. On the one hand, by the definition of the exponential function, the product of the Lie basis elements $\ell_{i_1},\dots,\ell_{i_k} \in \mathcal{S}_{\mathcal{L},m}$, $k > 0$ will appear in all possible order in $\exp(L_i)$. Again by definition, the coefficient of the product $\ell_{i_1} \otimes \cdots \otimes \ell_{i_k}$ in $\exp(L_i)$ is $\frac{1}{k!}\alpha_{i,i_1}\cdots\alpha_{i,i_k}$, hence a monomial. The coefficient of the symmetrized product $(\ell_{i_1},\dots,\ell_{i_k})$ is a scalar multiple of this monomial.

On the other hand, due to the construction of the Philip Hall basis, each Lie basis element ℓ is a (possibly compound) bracket of the letters $e_0,\dots,e_d$, and hence it is a sum of tensor monomials of the same degree when expressed in the tensor basis $\{e_I, I \in \mathcal{A}\}$. Hence any product of Lie basis elements is a sum of tensor monomials of the same degree. Therefore, when applying π_m to a symmetrised product $(\ell_{i_1},\dots,\ell_{i_k})$, we either get **0** or $(\ell_{i_1},\dots,\ell_{i_k})$ itself. □

Equation (16) represents $\pi_m \exp(L_i)$ in the basis $\mathcal{B}_{\mathcal{T}}$. If we express the π_m-truncated expected signature in the basis $\mathcal{B}_{\mathcal{T}}$, then (15) specifies the values of the weighted monomials:

$$\sum_{i=1}^{n} \lambda_i m_I(\alpha_{i,i_1},\dots,\alpha_{i,i_k}) = c_I,\ I \in \{(i_1,\dots,i_k) | \ell_{i_j} \in \mathcal{S}_{\mathcal{L},m}, j=1,\dots,k, k \ge 0\} \tag{17}$$

where c_I is the coefficient of $(\ell_{i_1}, \ldots, \ell_{i_k})$ in the expected signature. Thus, the second improvement we propose is to derive the polynomial equations (17), resulting in a relatively simple system equivalent to (15).

We derived the set $\mathcal{S}_{\mathcal{L},m}$ and solved the corresponding simplified system of polynomial equations (17) for the following pairs of m and d: $(3,1)$, $(5,1)$, $(5,2)$, $(7,1)$, $(7,2)$, $(9,1)$ and $(11,1)$. We also derived general solutions to the degree 3 and degree 5 cases. See the appendix for details.

Remark 2 (Construction of Cubature measures on the Wiener space). Given the weights $\lambda_1, \ldots, \lambda_n$ and Lie elements $L_1, \ldots, L_n$ satisfying (15), we need to construct the piece-wise linear paths ω_i on $[0,1]$ satisfying $S_{0,1}(\omega) = \pi_m \exp(L_i)$. Chen's theorem guarantees the existence of such paths; however, it does not give a construction. We recall that the signature of a piece-wise linear path of l-many pieces is of the form

$$\exp\left(\phi_{0,1}e_0 + \sum_{j=1}^{d} \phi_{j,1}e_j\right) \otimes \cdots \otimes \exp\left(\phi_{0,l}e_0 + \sum_{j=1}^{d} \phi_{j,l}e_j\right)$$

where we make the additional assumptions $\sum_{i=1}^{n} \phi_{0,i} = 1$ and $\phi_{0,i} > 0$ for $i = 1 \ldots n$. The log-signature of the π_m-truncation of the above product can be worked out using the Campbell–Baker–Hausdorff formula. This determines a system of polynomial equations on the coefficients $\phi_{j,k}$ for $k = 1 \ldots l$ $j = 0 \ldots d$ for each L_i. The paths corresponding to the general degree 3 and degree 5 cubature formulae are derived in [18].

3.3 Solving the ODEs

Once the cubature measure $\mathbb{Q}_1 = \sum_{i=1}^{n} \lambda_i \delta_{\omega_i}$ has been constructed, the method requires the solution of the ODE

$$\mathrm{d}Y_s^y(\hat{\omega}_i) = V_0(Y_s^y(\hat{\omega}_i))\mathrm{d}s + \sum_{i=1}^{d} V_i(Y_s^y(\hat{\omega}_i))\mathrm{d}\hat{\omega}_s^i,\ Y_0^y(\hat{\omega}_i) = y. \tag{18}$$

where $\hat{\omega}_i = \langle t, \omega_i \rangle$ for some $t \in [0,T]$. In general, the exact solution might not be known and numerical techniques are required. We note that the convergence results (propositions 1 and 2) are based on the inequality

$$\left|(P_t - Q_t)f(y)\right| \leq C \sum_{\substack{\|-I\| \leq m \\ \|I\| = m+1}} \|W_I f\|_\infty^W t^{(m+1)/2} + C \sum_{\substack{\|-I\| \leq m \\ \|I\| = m+2}} \|W_I f\|_\infty t^{(m+2)/2}.$$

The operator $Q_t f(y) = \sum_{i=1}^{n} \lambda_i f(Y_t^y(\langle t, \omega_i \rangle))$ can be replaced with any operator $\tilde{Q}_t$ satisfying

$$\left|(\tilde{Q}_t - Q_t) f(y)\right| \leq \tilde{C} \sum_{\substack{\|-I\| \leq m \\ \|I\| = m+1}} \|W_I f\|_\infty^W t^{(m+1)/2} + \tilde{C} \sum_{\substack{\|-I\| \leq m \\ \|I\| = m+2}} \|W_I f\|_\infty t^{(m+2)/2}.$$

for some constant $\tilde{C}$ not depending on t, and the order of the global convergence determined by propositions 1 and 2 remains the same. The choice of $\tilde{Q}_t$ considered in the literature [9, 10, 18, 20] etc. can be described by

$$\tilde{Q}_t f(y) = \sum_{i=1}^{n} \lambda_i f(\tilde{Y}_t^y(\langle t, \omega_i \rangle)),$$

where $\tilde{Y}_t^y(\langle t, \omega_i \rangle)$ is some numerical approximation of $Y_t^y(\langle t, \omega_i \rangle)$.

The numerical approximations in the literature can be split into two main categories. The first approach approximates the solution to the non-autonomous ODE (18) directly using high-order numerical ODE schemes. The reader is referred to [1] for particular high-order numerical ODE schemes.

The second approach is based on the results by [22] deriving an autonomous ODE approximating the solution of (18) on short time intervals. The solution of this autonomous ODE is then approximated by high-order ODE schemes. Let Γ denote the algebra homomorphism generated by $\Gamma(e_i) := W_i$, $i = 0 \ldots d$. Note that Γ assigns vector fields to the elements of $\mathcal{L}^{(m)}$. We introduce the rescaling operator $\langle t, \cdot \rangle$ on $\mathcal{T}$ by

$$\left\langle t, \sum_{I \in \mathcal{A}} a_I e_I \right\rangle := \sum_{I \in \mathcal{A}} t^{\|I\|/2} a_I e_I.$$

Then, the autonomous ODE derived in [22] can be represented as

$$\mathrm{d}Z_s^y(\langle t, L_i \rangle) = \Gamma(\langle t, L_i \rangle)(Z_s^y)\mathrm{d}s, \; Z_s^y(\langle t, L_i \rangle) = y. \tag{19}$$

Proposition 3. *There exists a constant C_1 depending on m, d, the length of ω_i and on L_i, such that*

$$|P_{(T-\tau)} f(Y_t^y(\langle t, \omega_i \rangle) - P_{(T-\tau)} f(Z_1^y(\langle t, L_i \rangle))| \leq C_1 \sum_{m < \|I\| \leq 2m} \|W_I P_{T-\tau} f\|_\infty t^{\|I\|/2}$$

Furthermore, if $W_0, \ldots, W_d$ satisfy the UFG condition 4 and $T - \tau \geq t$, there exists a constant C_2 depending on m, d, the length of ω_i and on L_i, such that

$$|P_{(T-\tau)} f(Y_t^y(\langle t, \omega_i \rangle) - P_{(T-\tau)} f(Z_1^y(\langle t, L_i \rangle))| \leq C_2 \frac{t^{(m+1)/2}}{(T-\tau)^m} \|\nabla f\|_\infty$$

for any Lipschitz continuous function f.

The proof is based on the termwise Taylor expansion of the ODEs (18) and (19). We refer the reader to [4, 9] and [10] for details.

Propositions 1 and 2 combined with 3 imply the following.

Corollary 2. *Let us define the operator* $\hat{Q}_t f(y) = \sum_{i=1}^{n} \lambda_i f(Z_1^y(\langle t, L_i \rangle))$ *and let* $0 = t_0 < \cdots < t_k = T$*. For a smooth function* f *with bounded derivatives up to order* $2m$*, we have*

$$\sup_y |(P_T - \hat{Q}_{(t_1-t_0)} \cdots \hat{Q}_{(T-t_{k-1})}) f(y)| \leq C_1 \Big(\sum_{i=1}^{k} \sum_{m<\|I\|\leq 2m} \|W_I P_{(T-t_{i-1})} f\|_\infty t^{\|I\|/2}$$
$$+ \sum_{\substack{\|-I\|\leq m \\ \|I\|=m+1}} \|W_I P_{(T-t_{i-1})} f\|_\infty^W (t_i - t_{i-1})^{(m+1)/2}$$
$$+ \sum_{\substack{\|-I\|\leq m \\ \|I\|=m+2}} \|W_I P_{(T-t_{i-1})} f\|_\infty (t_i - t_{i-1})^{(m+2)/2} \Big).$$

Furthermore, for Lipschitz continuous f

$$\sup_y |(P_T - \hat{Q}_{(t_1-t_0)} \cdots \hat{Q}_{(T-t_{k-1})}) f(y)|$$
$$\leq C_2 \|\nabla f\|_\infty \left((T - t_{k-1})^{1/2} + \sum_{i=1}^{k-1} \frac{(t_i - t_{i-1})^{(m+1)/2}}{(T - t_i)^{m/2}} \right).$$

4 Extensions

In this section, we outline the extension of the cubature on Wiener space method to problems with piece-wise smooth terminal conditions that are not required to be continuous and to certain problems with path dependent terminal conditions.

4.1 Piece-Wise Smooth Terminal Conditions

So far, the terminal condition f has been assumed to be smooth or Lipschitz continuous. In this section, we relax the conditions on the terminal condition function f and construct a strategy resulting in an accurate approximation of the expectation of f. We investigate the case when f is piece-wise smooth. The method sketched here works under two conditions. First, we require that $P_t f$ is smooth with bounded derivatives up to certain order. If this condition is satisfied, we can apply the cubature on Wiener space method on an accurate enough estimate of $P_t f$ as it is described in the previous section. The second condition guarantees the existence of accurate estimates of $P_t f$ for small enough t. In particular, we will assume that if the discontinuities are far enough from Y_{T-t}, then $P_t f(Y_{T-t}) \approx P_t \hat{f}(Y_{T-t})$ where $\hat{f}$ is a smooth function equal to f around Y_{T-t}.

In this section, we fix the initial value $Y_0 = y^*$. For a positive integer K, we define $\|\nabla^K f\|_1 := \mathbb{E}[\nabla^K f(Y_T)|Y_0 = y^*]$, where ∇ is understood in the weak sense. If $\|\nabla^K f\|_1 < \infty$, the boundedness of the derivatives of P_t is guaranteed under the following conditions.

Assumption 1 ((K,m)-assumption). For positive integers K and m and multi-indices I: $m+1 \le \|I\| \le m+2$, the semi-group P_t satisfies

$$\|W_I P_t f\|_\infty \le C_I t^{-(\|I\|+d-K)/2} \|\nabla^K f\|_1.$$

Lemma 5. *Let K be a positive integer and $\theta \in (0,1)$. Let us assume that $\|\nabla^K f\|_1$ is finite. Furthermore, let us assume that the semi-group P_t satisfies the (K,m)-assumption. Let the partition $\{0 = t_0 < t_1 < \cdots < t_k = T\}$ be defined by $t_j = T(1-(1-\theta)^j)$ for $j = 1,\ldots,k-1$. Then, there exists a positive constant C depending on C_I, $m+1 \le \|I\| \le m+2$ and possibly on T, such that*

$$\begin{aligned}&\left|\left(P_{t_{k-1}} - Q_{(t_1-t_0)}\cdots Q_{(t_{k-1}-t_{k-2})}\right) P_{(T-t_{k-1})} f(y^*)\right| \\ &\le C\|\nabla^K f\|_1 \left(\frac{\theta}{1-\theta}\right)^{\frac{m+1}{2}} \sum_{i=1}^{k-1} (T(1-\theta))^{i(K-d)/2}\end{aligned}$$

Proof. By definition, $T - t_i = T(1-\theta)^i$ and $t_i - t_{i-1} = T(1-\theta)^{i-1}\theta$ for $i = 1,\ldots,k-1$. Then, Lemma 2 and the assumptions imply

$$\begin{aligned}&\left|\left(P_{t_{k-1}} - Q_{(t_1-t_0)}\cdots Q_{(t_{k-1}-t_{k-2})}\right) P_{(T-t_{k-1})} f(y^*)\right| \\ &\le \sum_{i=2}^{k-1} \left|Q_{(t_1-t_0)}\cdots Q_{(t_{i-1}-t_{i-2})}\left(P_{(t_i-t_{i-1})} - Q_{(t_i-t_{i-1})}\right) P_{(T-t_i)} f(y^*)\right| \\ &\le C_2\|\nabla^K f\|_1 \left(\frac{\theta}{1-\theta}\right)^{\frac{m+1}{2}} \sum_{i=1}^{k-1} (T(1-\theta))^{i(K-d)/2}\end{aligned}$$

Remark 3. There is a trade-off when constructing numerical methods based on lemma 5. The closer θ is to 0, the higher the convergence order becomes. However, since $T - t_{k-1} = T(1-\theta)^{k-1}$, when θ is close to 0, more steps are required to get close enough to the boundary. There is another trade-off; the high value of m, that is a high-degree cubature measure improves the convergence; however, it increases the support of the global measure. In the remainder of this section, we highlight the importance of high-degree cubature measures.

To describe the accuracy of the global measure $\mathbb{Q}_{\mathcal{D}}$, we must combine the estimate of lemma 5 with an estimate on $|(P_t - Q_t) f(Y_{T-t})|$. For this purpose, we make the following assumption.

Assumption 2. $\mathbb{R}^N$ can be partitioned into finitely many disjoint connected domains $\mathbb{R}^N = \bigcup_{i=1}^h D_i$, such that $f = \sum_{i=1}^h f_i \mathbf{1}_{D_i}$ where $f_i : D_i \to \mathbb{R}$ is bounded and smooth with bounded derivatives for $i = 1 \ldots h$. Furthermore, each f_i can be extended to $\hat{f}_i : \mathbb{R}^N \to \mathbb{R}$, such that $\hat{f}_i$ is smooth satisfying $\|\hat{f}_i\|_\infty \le \|f\|_\infty$ and $\|W_I \hat{f}_i\|_\infty \le \|W_I f_i\|_\infty$ for $I\colon m+1 \le \|I\| \le m+2$.

Condition 1. Given $y \in \mathbb{R}^N$ and $\varepsilon > 0$, there exist an index $i\colon 1 \le i \le h$ such that $\mathbb{P}[Y_T \notin D_i | Y_{T-t} = y] \le \varepsilon$ and $y \in D_i$, $Y_t^y(\langle t, \omega_j \rangle) \in D_i$ for $j = 1 \ldots k$.

Under assumption 2, we cannot give a uniform bound on $|(P_t - Q_t) f(Y_{T-t})|$; however, we can make local estimates. We consider two cases. First, if condition 1 holds, then

$$|(P_t - Q_t) f(y)| \le |P_t f(y) - P_t \hat{f}(y)| + |(P_t - Q_t)\hat{f}(y)| + |Q_t f(y) - Q_t \hat{f}(y)| \\ \le 2\varepsilon \|f\|_\infty + C t^{(m+1)/2}. \tag{20}$$

Second, if $Y_t^y(\langle t, \omega_j \rangle) \in D_i$ does not hold for all $j = 1 \ldots n$, then

$$|(P_t - Q_t) f(y)| \le 2\|f\|_\infty. \tag{21}$$

If f is globally Lipschitz, the inequality (21) can be replaced with

$$|(P_t - Q_t) f(y)| \le \sqrt{t}\|\nabla f\|_\infty. \tag{22}$$

The Repeated Cubature Scheme

The inequality (20) makes the reduction of the computational cost possible. Let ε and τ be positive numbers chosen in advance. At a starting point y with time t to maturity, the repeated cubature estimate denoted by $Q_t^{\varepsilon,\tau} f(y)$ is computed as follows.

(a) At y, we check the condition 1.
(b) If the condition is met with a pre-defined ε, we jump straight to the boundary (type-(b) step), that is we estimate $P_t f(y)$ with $Q_t f(y)$.
(c) If the condition is not met, but $t \le \tau$, we jump straight to the boundary (type-(c) step). We estimate $P_t f(y)$ by $Q_t f(y)$.
(d) If the condition is not met and $t > \tau$, we solve the ODE along each rescaled cubature path $\langle \theta t, \omega_i \rangle$ for $i = 1 \ldots k$ generating the points $y_1, \ldots, y_k$ with time $t(1-\theta)$ to maturity (type-(d) step).
(e) We recursively repeat the algorithm starting at the points $y_1, \ldots, y_k$ generated by type-(c) steps resulting in an estimate denoted by $Q_{t(1-\theta)}^{\varepsilon,\tau} f(y_i)$. Then, we estimate $P_t f(y)$ by $\sum_{i=1}^k \lambda_i Q_{t(1-\theta)}^{\varepsilon,\tau} f(y_i)$.

Corollary 3. *Let ε and τ be positive real numbers. Let $\mathcal{N}_{\tau,\varepsilon}$ denote the set of starting points of the type-(c) steps using the algorithm above. Suppose that $\mathcal{N}_{\tau,\varepsilon}$ is non-empty and $\tau < T$. For $Y \in \mathcal{N}_{\tau,\varepsilon}$, λ_Y denotes the measure of the set of cubature paths along which we solved ODEs during the method and reached Y at time $t \le \tau$ to maturity. Under assumptions 1 and 2, the error of the repeated cubature scheme is bounded by*

$$\left|(P_T - Q_T^{\varepsilon,\tau}) f(y^*)\right| \le C_1 \left(\varepsilon + (T(1-\theta))^{(m+1)/2} + \sum_{Y \in \mathcal{N}_{\tau,\varepsilon}} \lambda_Y\right) + C_2 \|\nabla^K f\|_1 \left(\frac{\theta}{1-\theta}\right)^{\frac{m+1}{2}} \sum_{i=1}^{k-1} (T(1-\theta))^{i(K-d)/2} \tag{23}$$

where k satisfies $\tau \ge T(1-\theta)^{k-1}$, the constants C_1 and C_2 depend on m, d, $\|f\|_\infty$, and on $\|W_I f_i\|_\infty$ for $i = 1, \ldots, h$ and I: $m+1 \le \|I\| \le m+2$.

Sketch of the proof: The second term on the right-hand-side of the inequality (23) is the estimate of Lemma 5 on the difference

$$\left|\left(P_{t_{k-1}} - Q_{(t_1-t_0)} \cdots Q_{(t_{k-1}-t_{k-2})}\right) P_{(T-t_{k-1})} f(y^*)\right|$$

describing the approximation error when we approximate $P_{t_{k-1}} P_{(T-t_{k-1})} f(y^*)$ by $Q_{(t_1-t_0)} \cdots Q_{(t_{k-1}-t_{k-2})} P_{(T-t_{k-1})} f(y^*)$. When applying the repeated cubature, we work out a slightly different measure, because a few sub-trees are replaced with type-(b) steps (see Fig. 3). By the inequality (20), the total additional error due to type-(b) steps can be bounded by

$$C\left(\varepsilon + (T(1-\theta))^{(m+1)/2}\right),$$

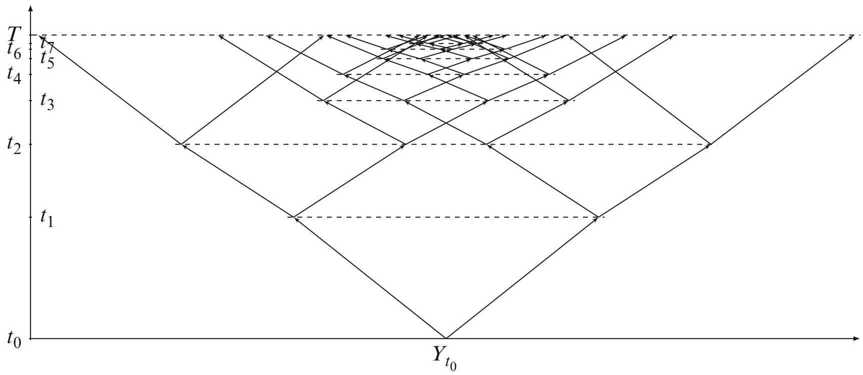

Fig. 3 Tree produced by the repeated cubature algorithm

where the constant C depends on m, d, $\|f\|_\infty$, and on $\|W_I f_i\|_\infty$ for $i = 1, \dots, h$ and $I\colon m+1 \le \|I\| \le m+2$.

There is another source of additional error due to type-(c) steps, which can be bounded by $2\|f\|_\infty \sum_{Y \in \mathcal{N}_{\tau,\varepsilon}} \lambda_Y$ (see inequality (21)). □

Remark 4. The method requires the estimation of the probability

$$\mathbb{P}[Y_T \notin D_i | Y_{T-t} = y].$$

If the expectation of some positive functional of $Y_T - Y_{T-t}$ (e.g. Laplace-transform) is known or can be estimated, one can use Chebyshev-type inequalities. If this is not the case, we suggest the comparison of $Q_t f(y)$ and $Q_{\theta t} Q_{(1-\theta)t} f(y)$. If the difference is of order $t^{(m+1)/2}$, then the condition 1 is likely to be met. Otherwise, the condition is unlikely to be satisfied. This latter decision rule works well, if the inner-diameter of the domains $D_1, \dots, D_h$ is big enough and the operator $Q_{\theta t} Q_{(1-\theta)t}$ is able to detect the non-smoothness of f.

4.2 Path Dependent Functions

In this section, we outline potential extensions to PDEs with Dirichlet boundary condition. Although no theoretical or empirical justification has been given yet, we believe these extensions should perform reasonably well.

In many applications, the focus is on the expectations of path dependent functionals. We consider path dependent functionals that can be approximated by functions of the signature of the underlying process. A particular example is the Asian option with payoff depending on $\int_0^T Y_t \mathrm{d}t$, which is one component in the signature of the process $t \mapsto (t, Y_t)$. Ninomiya [19] tested Kusuoka's version on Asian options.

As a second example, we consider adapting the cubature on Wiener space method to estimating solutions to boundary value problems. In particular, we aim to estimate the following expectation:

$$\mathbb{E}_{\mathbb{P}}\left[g(Y_T^y)\mathbf{1}_{\{\tau > T\}} + c\mathbf{1}_{\{\tau \le T\}}\right] \tag{24}$$

where D is a domain in $\mathbb{R}^N$, τ is the stopping time defined as $\tau = \min\{t | Y_t \in \partial D\}$ and c is some constant. We note that the expectation (24) is equal to the solution $u(y, 0)$ of the PDE (5) defined on D with boundary condition $u(x,t) = c$ for $x \in \partial D$.

To adapt the method, we extend the process Y_t as follows

$$\begin{aligned} \mathrm{d}Y_t^y &= V_0(Y_t^y)\mathrm{d}t + \sum_{i=1}^d V_i(Y_t^y) \circ \mathrm{d}B_t^i,\ Y_0^y = y, \\ \mathrm{d}A_t &= \phi(Y_t)\mathrm{d}t,\ A_0 = 0 \end{aligned}$$

where ϕ is zero in D and positive in $\mathbb{R}^N \setminus D$. Then, the terminal condition in the expectation (24) can be rewritten as a non-path dependent function

$$g(Y_T^y)\mathbf{1}_{\{\tau>T\}} + c\mathbf{1}_{\{\tau\leq T\}} = g(Y_T^y)\mathbf{1}_{\{A_T\leq 0\}} + c\mathbf{1}_{\{A_T>0\}}. \tag{25}$$

In the general case, the solution to the ODE (18) is numerically approximated, typically on a discrete time-scale. This approximation rarely gives us accurate information on the crossing of the boundary of the domain D. The role of the process A_t is to detect the boundary crossing of the process Y_t with high accuracy.

We claim that the repeated cubature algorithm can be adapted to approximate the expectation of (25).

One possible extension is based on the fact that a Wong-Zakai type strong approximation can be constructed from cubature paths on fine time-partitions;[2] that is cubature paths on very fine time scales are approaching Brownian paths. This application of the cubature paths leads to a low order approximation of the boundary value problems that we focus on. The fine scale global cubature measure is supported by a huge set of paths; however, fine scale paths are only required close to the boundary. The purpose of the repeated cubature is to detect a point far enough away from the boundary of the domain D so that a longer time-step can be taken.

The following second extension leads to a proper high-order scheme. We suggest refining the steps when the boundary of D is approached as follows. Let us assume that an accurate approximation $\hat{u}(y,t)$ of the solution $u(y,t)$ of the PDE is given exogenously in a neighbourhood of the boundary of D with radius δ.

(a) In order to compute an approximation of the solution to the PDE at a point Y_{t_1} with time $T - t_1$ to maturity, we attempt to take a step of length $T - t_1$. If the solution of an ODE (18) along one of the paths in the cubature formula started at Y_{t_1} with step length $T - t_1$ leaves the domain, we take a shorter step of length $\theta(T - t_1)$ from Y_{t_1} for some $\theta \in (0, 1)$.
(b) We keep refining this step until the paths of each of the solved ODEs stay inside D resulting in nodes $Y_{t_2}^1, \ldots, Y_{t_2}^k$ for $t_2 = t_1 + \theta^l(T - t_1)$ for some positive integer l.
(c) If any of these nodes, for example $Y_{t_2}^i$, are in the δ-radius neighbourhood of the boundary of D, we define the approximation to our PDE problem to be equal to the exogenously given approximation at the point $Y_{t_2}^i$ and time $T - t_2$ to maturity.
(d) From all other nodes, we repeat this procedure from point (a).

To ensure tractability, we put a lower bound ε on the length of the time-steps. If at any time t_i and point Y_{t_i}, the next step of length at least ε results in an ODE solution trajectory leaving the domain D, we use the exogenously given approximation at Y_{t_i} and time $T - t_i$ from maturity. Fig. 4 outlines the algorithm.

[2] We acknowledge Peter Friz who brought the Wong-Zakai approximation property of the cubature paths to our attention.

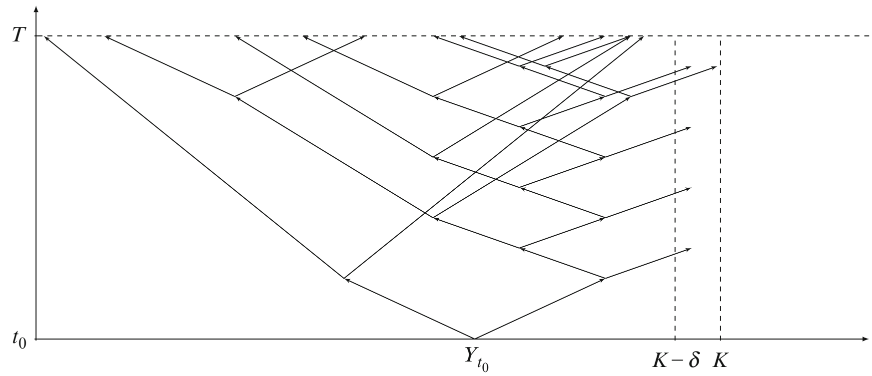

Fig. 4 Detection of boundary crossing

If the boundary condition is not constant, for example when one would like to approximate the expectation

$$\mathbb{E}_{\mathbb{P}}\left[g(Y_T^y)\mathbf{1}_{\{\tau>T\}} + h(Y_\tau^y)\mathbf{1}_{\{\tau\leq T\}}\right] \tag{26}$$

for some function h, the following extension of the state space might be useful:

$$\begin{aligned} \mathrm{d}Y_t^y &= V_0(Y_t^y)\mathrm{d}t + \sum_{i=1}^d V_i(Y_t^y)\circ \mathrm{d}B_t^i,\ Y_0^y = y, \\ \mathrm{d}A_t &= \phi(Y_t)\mathrm{d}t,\ A_0 = 0 \\ \mathrm{d}X_t &= \eta(A_t)\mathrm{d}Y_t^y,\ X_0 = y \end{aligned}$$

where η is a smooth approximation of the Dirac delta function assigned to 0. The process X_t approximates the stopped process $Y_{t\wedge\tau}$, and the payoff can be approximated by

$$g(Y_T^y)\mathbf{1}_{\{\tau>T\}} + h(Y_\tau^y)\mathbf{1}_{\{\tau\leq T\}} \approx g(Y_T^y)\mathbf{1}_{\{A_T\leq 0\}} + h(X_T)\mathbf{1}_{\{A_T>T\}}.$$

The accuracy of this method depends on the choice of η.

5 Numerical Results

In this section, we present some numerical results. We tested the classical version of the cubature on Wiener space method on PDEs with smooth terminal conditions; furthermore, we tested Kusuoka's uneven partition and the repeated cubature method on PDEs with piece-wise continuous terminal condition. The algorithm was implemented in C++ using double precision floating point numbers. We remark that the accuracy of high-order cubature formulae combined with the repeated cubature algorithm can be better exploited using higher precision numbers. Packages with high precision number are available for C++.

In all the test runs, we worked with autonomous ODEs defined by (19). One should use high-order numerical solvers to approximate the solution to these ODEs. However, we have chosen our test cases such that the exact solution to the arising ODEs is known, and there is no need for the numerical approximation of these ODEs. The impact of the high and the low accuracy numerical ODE solvers was numerically analysed in [5] and in [20].

5.1 Smooth Terminal Condition

The following SDE was chosen for the first set of test runs.

$$\begin{aligned} \mathrm{d}Y_t &= aY_t\mathrm{d}t + bY_t\mathrm{d}B_t \\ \mathrm{d}A_t &= Y_t\mathrm{d}t, \end{aligned}$$

that is $V_0(x, y) = (ax, x)$ and $V_1(x, y) = (bx, 0)$. This choice was motivated by the following two facts. Firstly, the solution to the above SDE and the explicit expectation of any polynomial of Y_T and A_T is known; the approximation error can directly be measured. Secondly, the Lie-bracket of the corresponding vector fields W_0 and W_1 is of the form

$$[W_0, W_1](x, y) = -bx\frac{\partial}{\partial y}.$$

This implies that for any Lie polynomial $L \in \mathcal{L}^{(m)}$, the vector field $\Gamma(\langle t, L\rangle)$ has the following form

$$\Gamma(\langle t, L\rangle) = c_1 x\frac{\partial}{\partial x} + c_2 x\frac{\partial}{\partial y}$$

where c_1 and c_2 are some constants depending on L. Therefore, the solution to the autonomous ODE driven by the vector field $\Gamma(\langle t, L\rangle)$ is explicitly known, and there is no need for any numerical approximation. This also implies that the set $\{W_0, W_1\}$ satisfies the UFG condition 4.

Furthermore, note that the system $\{W_0, W_1\}$ is not nilpotent, that is the algebra homomorphism Γ does not map any of the Lie-brackets appearing in the cubature formulae (written in terms of Lie polynomials, see the appendix) to the constant zero function. Therefore, the above SDE is also useful for testing the newly derived cubature formulae for typos and mistakes.

We implemented the degree 3, 5, 7, 9 and 11 cubature formulae corresponding to the 1-dimensional driving noise with respectively 2, 3, 6, 12 and 30 Lie elements

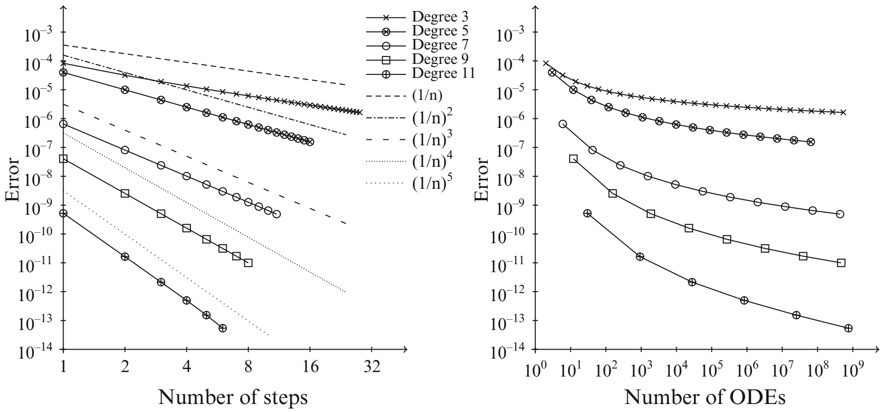

Fig. 5 Estimating $\mathbb{E}[A_T^3]$

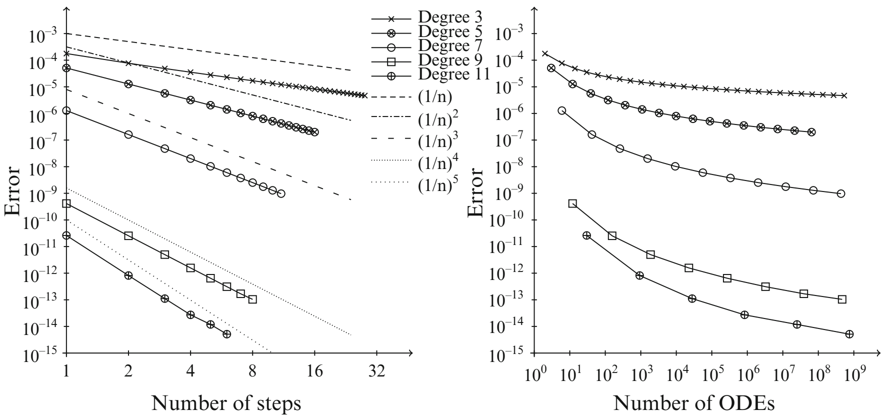

Fig. 6 Estimating $\mathbb{E}[A_T^2 Y_T]$

in their support (see the appendix for details). The method was applied to the estimation of the expectation of certain polynomials of Y_T and A_T with parameters $T = 0.25$, $a = 0.1$ and $b = 0.2$. Figures 5 and 6 show the results corresponding to $\mathbb{E}[A_T^3]$ and $\mathbb{E}[A_T^2 S_T]$, respectively. On the left-hand side chart of each figure the approximation error is plotted against the number of steps in the cubature tree. On the right-hand side graphs in each figure, we plotted the approximation error against the number of ODEs solved for the approximation. Since we were working with the exact solution of the arising ODEs and the evaluation of this solution had roughly equal computational expense for each of the different degree cubature formulae, the number of ODEs solved in each run results in a reasonable accurate measure of the computational expense.

We can observe that the empirical rate of convergence (also plotted) is very close to the theoretical one. The efficiency of high-degree cubature formulae in terms of accuracy at a given computational expense is conspicuous. We ran each formula up to a level with at most 10^9 ODEs. When estimating $\mathbb{E}[A_T^3]$, the degree 11 formula after 3 steps (corresponding to 27, 930 ODEs) outperformed each of the lower degree formulae.

The performance of the degree 11 formula becomes even more remarkable if we estimate the sample size required by a pure Monte-Carlo method (based on independent sampling without any variance reduction technique) to attain similar accuracy. The standard deviation of A_T^3 is approximately 0.00287597203, and the standard deviation of $A_T^2 S_T$ is approximately 0.01407034093. Recalling that the standard deviation of the Monte-Carlo estimate scales with the square root of the sample size, a sample of size at least 10^{20} is required of both A_T^3 and $A_T^2 S_T$ to result in an estimate with standard deviation of order of magnitude 10^{-12}. With the 4-step degree 11 formula (with less than 10^6 ODEs), we attained an error smaller than 10^{-12}.

Admittedly, in 1-dimension, the Monte-Carlo method is outperformed by finite difference and finite element methods. However, we believe that high-degree cubature methods (especially when combined with recombination) are significantly more efficient compared to finite difference or finite element methods on problems with smooth terminal conditions.

5.2 *Piece-Wise Smooth Terminal Condition*

In this section, we consider two piece-wise smooth terminal condition functions: $f(x) = (x_K)^+$ and $g(x) = \mathbf{1}_{\{x>K\}}$. The underlying SDE is $\mathrm{d}Y_t = \mathrm{d}B_t$, that is $V_0(x) = 0$, $V_1(x) = 1$. The parameters used are $T = 0.25$, $K = 0.6$. The function f is Lipschitz continuous; hence, Kusuoka's estimate based on uneven partitions applies. The function g is discontinuous and Kusuoka's results do not apply. Nevertheless, we experimented with Kusuoka's uneven partition. Partitions corresponding to different γ parameters were tested. Figure 7 illustrates the results corresponding to the least absolute error. We attained an error of order of magnitude 10^{-7} on the Lipschitz continuous terminal condition and 10^{-4} accuracy on the discontinuous terminal condition. Empirically, we observed convergence in both cases, although the order of convergence is not clear.

Since this system satisfies Assumption 1, we also applied the repeated cubature algorithm to the same SDE and terminal conditions. We used cubature formulae of degree[3] 9, 11 and 23, θ parameters between 0.2 and 0.4. Condition 1 was checked by comparing Q_t to $Q_{\theta t} Q_{(1-t)\theta}$ as described in Remark 4. Furthermore, the algorithm was using the recombination algorithm of Litterer and Lyons [13, 14]. Figure 8 shows the error of different approximations of $\mathbb{E}[(B_T - K)^+]$ and $\mathbb{E}[\mathbf{1}_{\{B_T<K\}}]$ respectively. Given a cubature formula of degree m and a $\theta \in (0, 1)$, the estimate

[3] This test case falls under the scope of the nilpotent problem; therefore, cubature formulae of degree greater than 11 are available. See Sect. A.11 for details.

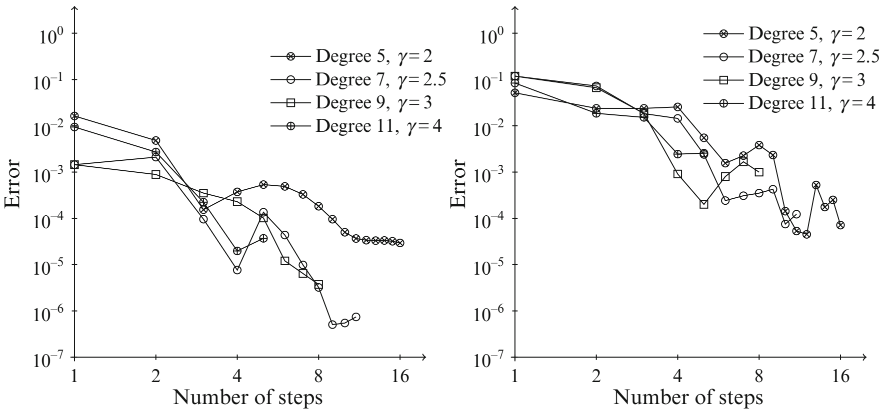

Fig. 7 Estimating $\mathbb{E}[(B_T - K)^+]$ (left-hand side graph) and $\mathbb{E}[\mathbf{1}_{\{B_T > K\}}]$ (right-hand side graph) using Kusuoka's uneven partition

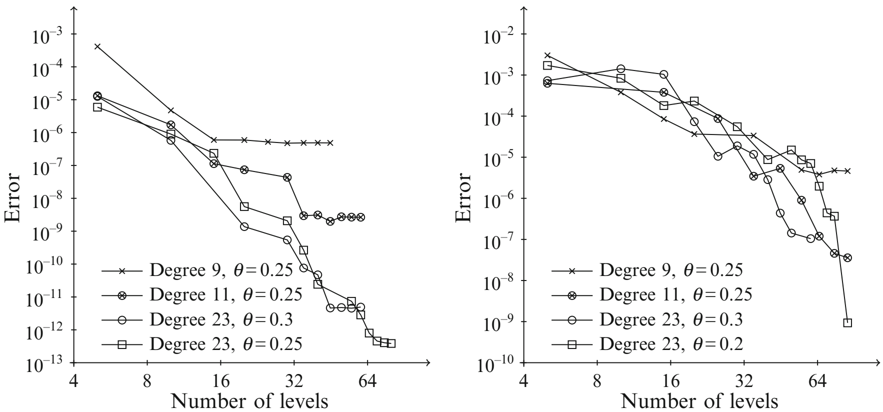

Fig. 8 Estimating $\mathbb{E}[(B_T - K)^+]$ (left-hand side graph) and $\mathbb{E}[\mathbf{1}_{\{B_T < K\}}]$ (right-hand side graph) with the repeated cubature algorithm

is not expected to be better than the accuracy of the degree m formula on a time step of length θT. Each curve on the graphs of Fig. 8 corresponds to a cubature formula and a fixed θ. In general, on each curve, we observe a downward trend up to certain number of levels (depending on the degree and θ), then the curve flattens. Corollary 3 implies this behaviour.

Running the repeated cubature method with many different sets of parameters, we observe that the smaller θ is, the higher accuracy is achievable, however the more levels we need to work out. The empirical results suggest that the repeated cubature is efficient with a combination of θ close to $1/2$ and a very high degree cubature formula. Furthermore, recombination is a crucial ingredient.

Appendix: Polynomial Equations and Solutions

The construction of the cubature formulae consists of two steps. First, the system (17) corresponding to the pair m and d is derived. We adapted and used the `libalgebra`[4] package for this purpose. This package contains an implementation of polynomial, lie algebra and tensor algebra objects and operations. This phase is completely done by software.

The second phase is to derive a solution to the system (17). We have no general algorithm solving the system; in each case, we have an ad-hoc approach. Typically, we use a degree m (or higher) Gaussian cubature formula in d-dimensions to determine the random coefficients of $e_1, \ldots, e_d$. Then, we split some cases into further sub-cases satisfying the rest of the conditions.

Note that in some special nilpotent cases, the elements in the cubature formulae introduced below can be simplified and the number of elements in the support of the formula can be reduced. For example, in the 3-nilpotent case (i.e. when all Lie-brackets formed by more than two vector fields in the set $\{W_0, \ldots, W_d\}$ vanish), the degree 3 cubature formula corresponding to the 1-dimensional case also satisfies the moment conditions of the degree 5 cubature formula. Coincidentally, in the 3-nilpotent case in 1-dimension, the degree 9 cubature formula also satisfies the moment conditions of the degree 11 cubature formula.

A.1 Degree 3, Dimension 1

For the degree 3 case in one dimension, (15) has a solution of the form

$$L_i = e_0 + \alpha_{i,1} e_1$$

where the corresponding system (17) is

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1} = 0 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2 = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^3 = 0$$

The vectors $(\lambda_1, \lambda_2) = (\frac{1}{2}, \frac{1}{2})$, $(\alpha_{1,1}, \alpha_{2,1}) = (-1, 1)$ solve the above system of polynomial equations.

[4] Implemented by the CoRoPa project, http://coropa.sourceforge.net/

A.2 Degree 3 in General

In general, for $d > 1$ and $m = 3$, (15) has a solution of the form

$$L_i = e_0 + \sum_{j=1}^{d} \alpha_{i,j} e_j$$

where the corresponding system (17) is

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,j}^2 = 1, \ j = 1 \dots d$$

and all coefficients of terms in $\sum_{i=1}^{n} \pi_m \exp(L_i)$ (when expressed in the basis given by the symmetrized products of Lie-basis elements described in Corollary 1) different from the ones above are all zero. There is a solution with $n = 2^d$ derived in [18]. Here we present a solution with $n = 2d$.

$$\alpha_{i,j} = \begin{cases} \frac{1}{\sqrt{2\lambda_i}} & \text{if } i = 2j - 1 \\ -\frac{1}{\sqrt{2\lambda_i}} & \text{if } i = 2j \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda_{2k} = \lambda_{2k-1} > 0$ for $k = 1 \dots d$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

A.3 Degree 5, Dimension 1

Equation (15) has a solution of the form

$$L_i = e_0 + \alpha_{i,1} e_1 + \alpha_{i,2} [e_1, [e_0, e_1]]$$

where the corresponding system (17) is

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2 = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^4 = 3 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1} \alpha_{i,2} = 0$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,1} = 0 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^3 = 0 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^5 = 0 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2} = -\frac{1}{12}$$

The vectors $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$, $(\alpha_{1,1}, \alpha_{2,1}, \alpha_{3,1}) = (-\sqrt{3}, 0, \sqrt{3})$ and

$$(\alpha_{1,2}, \alpha_{2,2}, \alpha_{3,2}) = \left(-\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}\right)$$

solve the above system of polynomial equations.

A.4 Degree 5, Dimension 2

The degree 5 cubature formula in 2 dimensions satisfies the following conditions.

$$\begin{aligned} L_i &= e_0 + \alpha_{i,1}e_1 + \alpha_{i,2}e_2 + \alpha_{i,3}[e_1, e_2] + \alpha_{i,4}[e_1, [e_0, e_1]] \\ &\quad + \alpha_{i,5}[e_1, [e_1, e_2]] + \alpha_{i,6}[e_2, [e_0, e_2]] + \alpha_{i,7}[e_2, [e_1, e_2]] \end{aligned}$$

where the corresponding system (17) is

$$\begin{array}{llll} \sum_{i=1}^n \lambda_i = 1 & \sum_{i=1}^n \lambda_i \alpha_{i,2}^2 = 1 & \sum_{i=1}^n \lambda_i \alpha_{i,4} = -\frac{1}{12} & \sum_{i=1}^n \lambda_i \alpha_{i,2}\alpha_{i,5} = \frac{1}{12} \\ \sum_{i=1}^n \lambda_i \alpha_{i,1}^2 = 1 & \sum_{i=1}^n \lambda_i \alpha_{i,2}^4 = 3 & \sum_{i=1}^n \lambda_i \alpha_{i,6} = -\frac{1}{12} & \sum_{i=1}^n \lambda_i \alpha_{i,1}^2\alpha_{i,2}^2 = 1 \\ \sum_{i=1}^n \lambda_i \alpha_{i,1}^4 = 3 & \sum_{i=1}^n \lambda_i \alpha_{i,3}^2 = \frac{1}{4} & \sum_{i=1}^n \lambda_i \alpha_{i,1}\alpha_{i,7} = -\frac{1}{12} & \end{array}$$

and all coefficients of terms in $\sum_{i=1}^n \lambda_i \pi_m \exp(L_i)$ (when expressed in the basis given by the symmetrized products of Lie-basis elements described in Corollary 1) different from the ones above are all zero. A solution to this system is derived in [18] with $n = 18$. Here we present an alternative solution with $n = 10$:

i	λ_i	$\alpha_{i,1}$	$\alpha_{i,2}$	$\alpha_{i,3}$
1	μ_1	$\sqrt{3}$	$\sqrt{3}$	0
2	μ_2	$\sqrt{3}$	0	0
3	μ_1	$\sqrt{3}$	$-\sqrt{3}$	0
4	μ_2	0	$\sqrt{3}$	0
5	μ_3	0	0	$1/(2\sqrt{2\mu_3})$
6	μ_3	0	0	$-1/(2\sqrt{2\mu_3})$
7	μ_2	0	$-\sqrt{3}$	0
8	μ_1	$-\sqrt{3}$	$\sqrt{3}$	0
9	μ_2	$-\sqrt{3}$	0	0
10	μ_1	$-\sqrt{3}$	$-\sqrt{3}$	0

furthermore, $\alpha_{i,4} = -1/12$, $\alpha_{i,5} = \alpha_{i,2}/12$, $\alpha_{i,6} = -1/12$ and $\alpha_{i,7} = \alpha_{i,1}/12$ for $i = 1 \ldots 10$ with $\mu_1 = 1/36$, $\mu_2 = 1/9$ and $\mu_3 = 2/9$.

Another solution, that is not permutation-invariant with $n = 8$:

i	λ_i	$\alpha_{i,1}$	$\alpha_{i,2}$	$\alpha_{i,3}$
1	ν_1	$\sqrt{3}$	1	0
2	ν_1	$\sqrt{3}$	-1	0
3	ν_1	0	2	0
4	ν_2	0	0	$1/(2\sqrt{2\nu_2})$
5	ν_2	0	0	$-1/(2\sqrt{2\nu_2})$
6	ν_1	0	-2	0
7	ν_1	$-\sqrt{3}$	1	0
8	ν_1	$-\sqrt{3}$	-1	0

furthermore, $\alpha_{i,4} = -1/12$, $\alpha_{i,5} = \alpha_{i,2}/12$, $\alpha_{i,6} = -1/12$ and $\alpha_{i,7} = \alpha_{i,1}/12$ for $i = 1\ldots 8$ with $\nu_1 = 1/12$, $\nu_2 = 1/4$.

A.5 Degree 5 in General

In [18], a general construction of degree 5 cubature formulae in d dimensions is derived that is based on the d-dimensional Gaussian cubature formula. If the underlying Gaussian cubature measure has k points in its support, the construction generates a cubature formula on the Wiener space with $2k$ points.

Using the key idea in Sect. A.2, it's possible to generate a cubature formula with $k + 2d - 1$ points.

A.6 Degree 7, Dimension 1

The degree 7 cubature formula in 1 dimension satisfies the following conditions.

$$L_i = e_0 + \alpha_{i,1}e_1 + \alpha_{i,2}[e_0, e_1] + \alpha_{i,3}[e_1, [e_0, e_1]] + \alpha_{i,4}[e_1, [e_1, [e_1, [e_0, e_1]]]]$$

where the corresponding system (17) is

$$\sum_{i=1}^{n}\lambda_i = 1 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^4 = 3 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,2}^2 = \frac{1}{12} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,4} = -\frac{1}{360}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2 = 1 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^6 = 15 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,3} = -\frac{1}{12} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,3} = -\frac{1}{12}$$

and all coefficients of terms in $\sum_{i=1}^{n}\lambda_i\pi_m\exp(L_i)$ (when expressed in the basis given by the symmetrized products of Lie-basis elements described in Corollary 1) different from the ones above are all zero.

A solution is derived in [13] with $n = 8$. Here we present a solution with $n = 6$.

Let the weights $\lambda_1, \ldots, \lambda_5$ and points $z_1, \ldots, z_5$ define a degree 5 Gaussian cubature formula, such that $z_3 = 0$. The solution is specified in the table below:

i	λ_i	$\alpha_{i,1}$	$\alpha_{i,2}$
1	μ_1	z_1	0
2	μ_2	z_2	0
3	$\mu_3/2$	0	$1/\sqrt{12\mu_3}$
4	$\mu_3/2$	0	$-1/\sqrt{12\mu_3}$
5	μ_4	z_4	0
6	μ_5	z_5	0

furthermore, $\alpha_{i,3} = -1/12$ and $\alpha_{i,4} = -1/360$ for $i = 1 \ldots 6$.

A.7 Degree 7, Dimension 2

The degree 7 cubature formula in 2 dimensions satisfies the following conditions. The parametrized lie element is

$$\begin{aligned} L_i = {} & e_0 + \alpha_{i,1}e_1 + \alpha_{i,2}e_2 + \alpha_{i,3}[e_0, e_1] + \alpha_{i,4}[e_0, e_2] + \alpha_{i,5}[e_1, e_2] + \alpha_{i,6}[e_1, [e_0, e_1]] \\ & + \alpha_{i,7}[e_1, [e_1, e_2]] + \alpha_{i,8}[e_2, [e_0, e_2]] + \alpha_{i,9}[e_2, [e_1, e_2]] + \alpha_{i,10}[e_1, [e_1, [e_1, e_2]]] \\ & + \alpha_{i,11}[e_2, [e_2, [e_1, e_2]]] + \alpha_{i,12}[e_1, [e_1, [e_1, [e_0, e_1]]]] + \alpha_{i,13}[e_1, [e_1, [e_1, [e_1, e_2]]]] \\ & + \alpha_{i,14}[e_2, [e_1, [e_1, [e_0, e_2]]]] + \alpha_{i,15}[e_2, [e_1, [e_1, [e_1, e_2]]]] \\ & + \alpha_{i,16}[e_2, [e_2, [e_1, [e_0, e_1]]]] + \alpha_{i,17}[e_2, [e_2, [e_1, [e_1, e_2]]]] \\ & + \alpha_{i,18}[e_2, [e_2, [e_2, [e_0, e_2]]]] + \alpha_{i,19}[e_2, [e_2, [e_2, [e_1, e_2]]]] \\ & + \alpha_{i,20}[[e_0, e_1], [e_2, [e_1, e_2]]] + \alpha_{i,21}[[e_0, e_2], [e_1, [e_1, e_2]]] \\ & + \alpha_{i,22}[[e_1, e_2], [e_1, [e_0, e_2]]] + \alpha_{i,23}[[e_1, e_2], [e_1, [e_1, e_2]]] \\ & + \alpha_{i,24}[[e_1, e_2], [e_2, [e_0, e_1]]] + \alpha_{i,25}[[e_1, e_2], [e_2, [e_1, e_2]]] \end{aligned}$$

where the corresponding system (17) is

$$\sum_{i=1}^{n} \lambda_i = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,16} = -\frac{1}{360} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^4 \alpha_{i,2}^2 = 3$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2 = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,18} = -\frac{1}{360} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2} \alpha_{i,7} = \frac{1}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,1}^4 = 3 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,20} = -\frac{1}{360} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2} \alpha_{i,13} = \frac{1}{360}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,1}^6 = 15 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,21} = \frac{1}{180} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2}\alpha_{i,17} = \frac{1}{360}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,2}^2 = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,22} = -\frac{1}{60} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2}\alpha_{i,25} = -\frac{1}{120}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,2}^4 = 3 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,24} = \frac{1}{90} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2}^2\alpha_{i,5}^2 = \frac{5}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,2}^6 = 15 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}\alpha_{i,9} = -\frac{1}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2}^2\alpha_{i,6} = -\frac{1}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,3}^2 = \frac{1}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}\alpha_{i,15} = -\frac{1}{360} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2}^2\alpha_{i,8} = -\frac{1}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,4}^2 = \frac{1}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}\alpha_{i,19} = -\frac{1}{360} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2}^3\alpha_{i,7} = \frac{1}{4}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,5}^2 = \frac{1}{4} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}\alpha_{i,23} = -\frac{1}{90} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,5}\alpha_{i,10} = \frac{1}{72}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,6} = -\frac{1}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2\alpha_{i,2}^2 = 1 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,5}\alpha_{i,11} = \frac{1}{72}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,7}^2 = \frac{5}{144} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2\alpha_{i,2}^4 = 3 \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}\alpha_{i,2}^2\alpha_{i,9} = -\frac{1}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,8} = -\frac{1}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2\alpha_{i,5}^2 = \frac{5}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}\alpha_{i,4}\alpha_{i,5} = \frac{1}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,9}^2 = \frac{5}{144} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2\alpha_{i,6} = -\frac{1}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2\alpha_{i,2}\alpha_{i,7} = \frac{1}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,12} = -\frac{1}{360} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^2\alpha_{i,8} = -\frac{1}{12} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,2}\alpha_{i,3}\alpha_{i,5} = -\frac{1}{12}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_{i,14} = -\frac{1}{360} \qquad \sum_{i=1}^{n} \lambda_i \alpha_{i,1}^3\alpha_{i,9} = -\frac{1}{4}$$

and all coefficients of terms in $\sum_{i=1}^{n} \lambda_i \pi_m \exp(L_i)$ (when expressed in the basis given by the symmetrized products of Lie-basis elements described in Corollary 1) different from the ones above are all zero. A solution to this system is derived in [13] with $n = 50$.

A.8 Degree 7, Dimension 3

The Lie-elements in the support of the 3-dimensional degree 7 cubature formula are spanned by 91 Lie-basis elements. There are over 150 conditions in the form of inhomogeneous polynomial equations satisfied by the parameters and a couple of more homogeneous polynomial equations.

A.9 Degree 9, Dimension 1

The degree 9 cubature formula in 1 dimension satisfies the following conditions.

$$\begin{aligned} L_i = e_0 &+ \alpha_{i,1}e_1 + \alpha_{i,2}[e_0, e_1] + \alpha_{i,3}[e_1, [e_0, e_1]] + \alpha_{i,4}[e_1, [e_1, [e_0, e_1]]] \\ &+ \alpha_{i,5}[e_1, [e_0, [e_0, [e_0, e_1]]]] + \alpha_{i,6}[e_1, [e_1, [e_0, [e_0, e_1]]]] + \alpha_{i,7}[e_1, [e_1, [e_1, [e_0, e_1]]]] \\ &+ \alpha_{i,8}[[e_0, e_1], [e_0, [e_0, e_1]]] + \alpha_{i,9}[[e_0, e_1], [e_1, [e_0, e_1]]] \\ &+ \alpha_{i,10}[e_1, [e_1, [e_1, [e_1, [e_1, [e_0, e_1]]]]]] \end{aligned}$$

where the corresponding system (17) is

$$\begin{array}{llll} \sum_{i=1}^n \lambda_i = 1 & \sum_{i=1}^n \lambda_i \alpha_{i,1}^8 = 105 & \sum_{i=1}^n \lambda_i \alpha_{i,7} = -\frac{1}{360} & \sum_{i=1}^n \lambda_i \alpha_{i,1}^2 \alpha_{i,2}^2 = \frac{1}{12} \\ \sum_{i=1}^n \lambda_i \alpha_{i,1}^2 = 1 & \sum_{i=1}^n \lambda_i \alpha_{i,2}^2 = \frac{1}{12} & \sum_{i=1}^n \lambda_i \alpha_{i,8} = -\frac{1}{720} & \sum_{i=1}^n \lambda_i \alpha_{i,1}^2 \alpha_{i,3} = -\frac{1}{12} \\ \sum_{i=1}^n \lambda_i \alpha_{i,1}^4 = 3 & \sum_{i=1}^n \lambda_i \alpha_{i,3} = -\frac{1}{12} & \sum_{i=1}^n \lambda_i \alpha_{i,10} = -\frac{1}{6720} & \sum_{i=1}^n \lambda_i \alpha_{i,1}^4, \alpha_{i,3} = -\frac{1}{4} \\ \sum_{i=1}^n \lambda_i \alpha_{i,1}^6 = 15 & \sum_{i=1}^n \lambda_i \alpha_{i,3}^2 = \frac{1}{72} & \sum_{i=1}^n \lambda_i \alpha_{i,1} \alpha_{i,6} = \frac{1}{360} & \sum_{i=1}^n \lambda_i \alpha_{i,2} \alpha_{i,4} = \frac{1}{144} \\ & \sum_{i=1}^n \lambda_i \alpha_{i,5} = \frac{1}{720} & \sum_{i=1}^n \lambda_i \alpha_{i,1} \alpha_{i,9} = -\frac{1}{720} & \end{array}$$

and all coefficients of terms in $\sum_{i=1}^n \lambda_i \pi_m \exp(L_i)$ (when expressed in the basis given by the symmetrized products of Lie-basis elements described in Corollary 1) different from the ones above are all zero.

A solution to this system with $n = 12$ is constructed as follows. Let the weights $\mu_1, \ldots, \mu_5$ and points z_1, z_2, 0, $-z_2$ and $-z_1$ define a degree 9 Gaussian quadrature formula in 1 dimension. The solution is specified in the table below:

i	λ_i	$\alpha_{i,1}$	$\alpha_{i,2}$	$\alpha_{i,3}$	$\alpha_{i,7}$
1	$\mu_1/2$	z_1	$1/\sqrt{12}$	$-1/12$	0
2	$\mu_1/2$	z_1	$-1/\sqrt{12}$	$-1/12$	0
3	$\mu_2/2$	z_2	$1/\sqrt{12}$	$-1/12$	0
4	$\mu_2/2$	z_2	$-1/\sqrt{12}$	$-1/12$	0
5	$\mu_3/4$	0	$1/\sqrt{12}$	$c_1 - c_2$	$-1/(360\mu_3)$
6	$\mu_3/4$	0	$-1/\sqrt{12}$	$c_1 - c_2$	$-1/(360\mu_3)$
7	$\mu_3/4$	0	$1/\sqrt{12}$	$c_1 + c_2$	$-1/(360\mu_3)$
8	$\mu_3/4$	0	$-1/\sqrt{12}$	$c_1 + c_2$	$-1/(360\mu_3)$
9	$\mu_4/2$	$-z_2$	$1/\sqrt{12}$	$-1/12$	0
10	$\mu_4/2$	$-z_2$	$-1/\sqrt{12}$	$-1/12$	0
11	$\mu_5/2$	$-z_1$	$1/\sqrt{12}$	$-1/12$	0
12	$\mu_5/2$	$-z_1$	-1/$\sqrt{12}$	$-1/12$	0

furthermore, $\alpha_{i,4} = \alpha_{i,2}/12$, $\alpha_{i,5} = 1/720$, $\alpha_{i,6} = \alpha_{i,1}/360$, $\alpha_{i,8} = -1/720$, $\alpha_{i,9} = -\alpha_{i,1}/720$ and $\alpha_{i,10} = -1/6720$ for $i = 1 \ldots 12$, where

$$c_1 = \left(\frac{1-\mu_3}{12} - \frac{1}{12}\right)\frac{1}{\mu_3}$$

$$c_2 = \sqrt{\left(\frac{1}{72} - \frac{1-\mu_3}{144} - c_1^2\mu_3\right)\frac{1}{\mu_3}}$$

A.10 Degree 11, Dimension 1

The degree 11 cubature formula in 1 dimensions satisfies the following conditions.

$$\begin{aligned}
L_i &= e_0 + \alpha_{i,1}e_1 + \alpha_{i,2}[e_0, e_1] + \alpha_{i,3}[e_0, [e_0, e_1]] + \alpha_{i,4}[e_1, [e_0, e_1]] \\
&\quad + \alpha_{i,5}[e_0, [e_0, [e_0, e_1]]] + \alpha_{i,6}[e_1, [e_0, [e_0, e_1]]] + \alpha_{i,7}[e_1, [e_1, [e_0, e_1]]] \\
&\quad + \alpha_{i,8}[e_1, [e_0, [e_0, [e_0, e_1]]]] + \alpha_{i,9}[e_1, [e_1, [e_0, [e_0, e_1]]]] \\
&\quad + \alpha_{i,10}[e_1, [e_1, [e_1, [e_0, e_1]]]] + \alpha_{i,11}[[e_0, e_1], [e_0, [e_0, e_1]]] \\
&\quad + \alpha_{i,12}[[e_0, e_1], [e_1, [e_0, e_1]]] + \alpha_{i,13}[e_1, [e_1, [e_1, [e_1, [e_0, e_1]]]]] \\
&\quad + \alpha_{i,14}[e_1, [e_1, [e_1, [e_0, [e_0, [e_0, e_1]]]]]] + \alpha_{i,15}[e_1, [e_1, [e_1, [e_1, [e_0, [e_0, e_1]]]]]] \\
&\quad + \alpha_{i,16}[e_1, [e_1, [e_1, [e_1, [e_1, [e_0, e_1]]]]]] + \alpha_{i,17}[[e_0, e_1], [e_1, [e_1, [e_0, [e_0, e_1]]]]] \\
&\quad + \alpha_{i,18}[[e_0, e_1], [e_1, [e_1, [e_1, [e_0, e_1]]]]] + \alpha_{i,19}[[e_0, e_1], [[e_0, e_1], [e_1, [e_0, e_1]]]] \\
&\quad + \alpha_{i,20}[[e_0, [e_0, e_1]], [e_1, [e_1, [e_0, e_1]]]] + \alpha_{i,21}[[e_1, [e_0, e_1]], [e_1, [e_0, [e_0, e_1]]]] \\
&\quad + \alpha_{i,22}[[e_1, [e_0, e_1]], [e_1, [e_1, [e_0, e_1]]]] \\
&\quad + \alpha_{i,23}[e_1, [e_1, [e_1, [e_1, [e_1, [e_1, [e_1, [e_0, e_1]]]]]]]]
\end{aligned}$$

where the corresponding system (17) is

$$\sum_{i=1}^{n}\lambda_i = 1 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,16} = -\frac{1}{6720} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,16} = -\frac{1}{2880}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2 = 1 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,17} = -\frac{1}{10080} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^3\alpha_{i,9} = \frac{1}{120}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,1}^4 = 3 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,19} = -\frac{1}{10080} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^3\alpha_{i,12} = -\frac{1}{240}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,1}^6 = 15 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,20} = \frac{1}{2520} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^4\alpha_{i,2}^2 = \frac{1}{4}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,1}^8 = 105 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,21} = -\frac{1}{1680} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^4\alpha_{i,4} = -\frac{1}{4}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,1}^{10} = 945 \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,23} = \frac{1}{201600} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^4\alpha_{i,10} = \frac{1}{120}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,2}^2 = \frac{1}{12} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}\alpha_{i,9} = \frac{1}{360} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^6\alpha_{i,4} = -\frac{5}{4}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,3}^2 = \frac{1}{720} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}\alpha_{i,12} = -\frac{1}{720} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,2}\alpha_{i,5} = -\frac{1}{720}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,4} = -\frac{1}{12} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}\alpha_{i,15} = -\frac{1}{5040} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,2}\alpha_{i,7} = \frac{1}{144}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,4}^2 = \frac{1}{72} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}\alpha_{i,18} = -\frac{1}{10080} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,2}\alpha_{i,13} = \frac{1}{2880}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,7}^2 = \frac{1}{960} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}\alpha_{i,22} = -\frac{1}{5040} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,2}^2\alpha_{i,4} = -\frac{11}{720}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,8} = \frac{1}{720} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,2}^2 = \frac{1}{12} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,4}\alpha_{i,10} = \frac{1}{1440}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,10} = -\frac{1}{360} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,4} = -\frac{1}{12} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}\alpha_{i,2}\alpha_{i,6} = -\frac{1}{360}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,11} = -\frac{1}{720} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,4}^2 = \frac{1}{60} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}\alpha_{i,3}\alpha_{i,4} = \frac{1}{720}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,14} = -\frac{1}{10080} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,8} = \frac{1}{720} \qquad \sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,2}\alpha_{i,7} = \frac{1}{240}$$

$$\sum_{i=1}^{n}\lambda_i\alpha_{i,1}^2\alpha_{i,11} = -\frac{1}{720}$$

and all coefficients of terms in $\sum_{i=1}^{n} \lambda_i \pi_m \exp(L_i)$ (when expressed in the basis given by the symmetrized products of Lie-basis elements described in Corollary 1) different from the ones above are all zero.

Here we present a solution to this system with $n = 30$. Let the weights $\mu_1, \ldots, \mu_7$ and the roots of the roots $z_1, \ldots, z_7$ of the 7th Hermite polynomial define a degree 13 Gaussian cubature formula, such that $\mu_i = \mu_{8-i}$, $z_i = -z_{8-i}$ for $i = 1, 2, 3$ and $z_4 = 0$, $\mu_4 > 0$. The solution is specified in the table below:

i	λ_i	$\alpha_{i,1}$	$\alpha_{i,2}$	$\alpha_{i,3}$	$\alpha_{i,4}$	$\alpha_{i,6}$	$\alpha_{i,7}$	$\alpha_{i,10}$
1	$\mu_1/4$	z_1	c_1	$c_3\alpha_{1,1}$	$c_5\alpha_{1,1} + c_6$	c_{10}	$\alpha_{1,2}/20$	$c_{13}\alpha_{1,1}^2 + c_{14}$
2	$\mu_1/4$	z_1	$-c_1$	$-c_3\alpha_{2,1}$	$-c_5\alpha_{2,1} + c_6$	$-c_{10}$		
3	$\mu_1/4$	z_1	c_1	$-c_3\alpha_{3,1}$	$-c_5\alpha_{3,1} + c_6$	c_{10}		
4	$\mu_1/4$	z_1	$-c_1$	$c_3\alpha_{4,1}$	$c_5\alpha_{4,1} + c_6$	$-c_{10}$		
5	$\mu_2/4$	z_2	c_1	$c_3\alpha_{5,1}$	$c_5\alpha_{5,1} + c_6$	c_{10}	·	·
6	$\mu_2/4$	z_2	$-c_1$	$-c_3\alpha_{6,1}$	$-c_5\alpha_{6,1} + c_6$	$-c_{10}$	·	·
7	$\mu_2/4$	z_2	c_1	$-c_3\alpha_{7,1}$	$-c_5\alpha_{7,1} + c_6$	c_{10}	·	·
8	$\mu_2/4$	z_2	$-c_1$	$c_3\alpha_{8,1}$	$c_5\alpha_{8,1} + c_6$	$-c_{10}$		
9	$\mu_3/4$	z_3	c_1	$c_3\alpha_{9,1}$	$c_5\alpha_{9,1} + c_6$	c_{10}		
10	$\mu_3/4$	z_3	$-c_1$	$-c_3\alpha_{10,1}$	$-c_5\alpha_{10,1} + c_6$	$-c_{10}$		
11	$\mu_3/4$	z_3	c_1	$-c_3\alpha_{11,1}$	$-c_5\alpha_{11,1} + c_6$	c_{10}		
12	$\mu_3/4$	z_3	$-c_1$	$c_3\alpha_{12,1}$	$c_5\alpha_{12,1} + c_6$	$-c_{10}$	$\alpha_{12,2}/20$	$c_{13}\alpha_{12,1}^2 + c_{14}$
13	$\mu_4 7/32$	0	0	c_4	$c_8 - c_9$	0	c_{12}	c_{16}
14	$\mu_4 7/32$	0	0	$-c_4$	$c_8 + c_9$	0	c_{12}	c_{16}
15	$\mu_4/16$	0	c_2	0	c_7	0	c_{11}	c_{15}
16	$\mu_4/16$	0	$-c_2$	0	c_7	0	$-c_{11}$	c_{15}
17	$\mu_4 7/32$	0	0	c_4	$c_8 + c_9$	0	$-c_{12}$	c_{16}
18	$\mu_4 7/32$	0	0	$-c_4$	$c_8 - c_9$	0	$-c_{12}$	c_{16}
19	$\mu_5/4$	z_5	c_1	$-c_3\alpha_{19,1}$	$c_5\alpha_{19,1} + c_6$	$-c_{10}$	$\alpha_{19,2}/20$	$c_{13}\alpha_{19,1}^2 + c_{14}$
20	$\mu_5/4$	z_5	$-c_1$	$c_3\alpha_{20,1}$	$-c_5\alpha_{20,1} + c_6$	c_{10}		
21	$\mu_5/4$	z_5	c_1	$c_3\alpha_{21,1}$	$-c_5\alpha_{21,1} + c_6$	$-c_{10}$		
22	$\mu_5/4$	z_5	$-c_1$	$-c_3\alpha_{22,1}$	$c_5\alpha_{22,1} + c_6$	c_{10}		
23	$\mu_6/4$	z_6	c_1	$-c_3\alpha_{23,1}$	$c_5\alpha_{23,1} + c_6$	$-c_{10}$		
24	$\mu_6/4$	z_6	$-c_1$	$c_3\alpha_{24,1}$	$-c_5\alpha_{24,1} + c_6$	c_{10}	·	·
25	$\mu_6/4$	z_6	c_1	$c_3\alpha_{25,1}$	$-c_5\alpha_{25,1} + c_6$	$-c_{10}$	·	·
26	$\mu_6/4$	z_6	$-c_1$	$-c_3\alpha_{26,1}$	$c_5\alpha_{26,1} + c_6$	c_{10}	·	·
27	$\mu_7/4$	z_7	c_1	$-c_3\alpha_{27,1}$	$c_5\alpha_{27,1} + c_6$	$-c_{10}$		
28	$\mu_7/4$	z_7	$-c_1$	$c_3\alpha_{28,1}$	$-c_5\alpha_{28,1} + c_6$	c_{10}		
29	$\mu_7/4$	z_7	c_1	$c_3\alpha_{29,1}$	$-c_5\alpha_{29,1} + c_6$	$-c_{10}$		
30	$\mu_7/4$	z_7	$-c_1$	$-c_3\alpha_{30,1}$	$c_5\alpha_{30,1} + c_6$	c_{10}	$\alpha_{30,2}/20$	$c_{13}\alpha_{30,1}^2 + c_{14}$

furthermore,

$$\alpha_{i,5} = \frac{-\alpha_{i,2}}{60} \qquad \alpha_{i,8} = \frac{1}{720}, \qquad \alpha_{i,9} = \frac{\alpha_{i,1}}{360}, \qquad \alpha_{i,11} = -\frac{1}{720},$$

$$\alpha_{i,12} = -\frac{\alpha_{i,1}}{720}, \quad \alpha_{i,13} = \frac{12\alpha_{i,2}}{2880} \quad \alpha_{i,14} = -\frac{1}{10080}, \quad \alpha_{i,15} = -\frac{\alpha_{i,1}}{5040},$$

$$\alpha_{i,16} = c_{17}\alpha_{i,1}^2 - c_{18}, \quad \alpha_{i,17} = -\frac{1}{10080}, \quad \alpha_{i,18} = -\frac{1}{10080}, \quad \alpha_{i,19} = -\frac{1}{10080},$$

$$\alpha_{i,20} = \frac{1}{2520}, \quad \alpha_{i,21} = -\frac{1}{1680}, \quad \alpha_{i,22} = -\frac{\alpha_{i,1}}{5040}, \quad \alpha_{i,23} = \frac{1}{201600}$$

for $i = 1 \ldots 30$, and

$$c_1 = \frac{1}{\sqrt{12}}, \qquad c_2 = \sqrt{\frac{2}{3}}, \qquad c_3 = \frac{1}{720 c_5 \sum_{i=1}^n \lambda_i |\alpha_{i,1}^3|},$$

$$c_4 = \sqrt{\frac{1/720 - c_3^2}{(1-1/8)\mu_4}}, \qquad c_5 = \sqrt{\frac{1/60 - c_6^2}{3}}, \qquad c_6 = -\frac{1}{12},$$

$$c_7 = 8\frac{-11/720 - c_1^2 c_6 (1-\mu_4)}{\mu_4 c_2^2}, \quad c_8 = \frac{-1/12 - (1-\mu_4)c_6 - c_7\mu_4/8}{(1-1/8)\mu_4},$$

$$c_9 = \sqrt{\frac{1/72 - c_5^2 - (1-\mu_4)c_6^2 - c_7^2\mu_4/8 - (1-1/8)\mu_4 c_8^2}{(1-1/8)\mu_4}},$$

$$c_{10} = \frac{-1/360}{c_1 \sum_{i=1}^n \lambda_i |\alpha_{i,1}|}, \qquad c_{11} = 8\frac{1/144 - (1-\mu_4)c_1^2/20}{\mu_4 c_2},$$

$$c_{12} = \sqrt{\frac{1/960 - (1-\mu_4)c_1^2/400 - \mu_4 c_{11}^2/8}{(1-1/8)\mu_4}}, \quad c_{13} = \frac{1}{720}, \quad c_{14} = -3c_{13},$$

$$\begin{pmatrix} c_{15} \\ c_{16} \end{pmatrix} = \begin{pmatrix} \frac{\mu_4}{8} & \left(1-\frac{1}{8}\right)\mu_4 \\ \frac{c_7\mu_4}{8} & \left(1-\frac{1}{8}\right)\mu_4 c_8 \end{pmatrix}^{-1} \begin{pmatrix} -\frac{1}{360} - c_{13} - (1-\mu_4)c_{14} \\ \frac{1}{1440} - c_6 c_{13} - (1-\mu_4)c_6 c_{14} \end{pmatrix}$$

$$c_{17} = \frac{1}{2}\left(\frac{1}{6720} - \frac{1}{2880}\right), \qquad c_{18} = \frac{-1}{6720} - c_{17}.$$

A.11 The 2-Nilpotent Case

Let the weights $\lambda_1, \dots, \lambda_k$ and points $z_1, \dots, z_k$ in $\mathbb{R}^d$ form a degree m Gaussian cubature formula in d-dimensions. Then, in the special case when $[W_i, W_j] = 0$ for all $0 \le i, j \le d$ the weights $\lambda_1, \dots, \lambda_k$ and Lie elements $L_1, \dots, L_k$ form a degree m cubature formula on the Wiener space, where

$$L_i = e_0 + \sum_{j=1}^{d} z_i^j e_j.$$

For example, when $N = 1$, that is when the functions $V_0, \dots, V_d$ are $\mathbb{R} \to \mathbb{R}$ functions and $V_i(x) = \alpha_i x^\beta$ for any real β and α_i, $i = 0, \dots, d$, the above nilpotent condition is satisfied. Furthermore, the path $t \mapsto (t, z_i^1 t, \dots, z_i^d t)$ has log-signature L_i on $[0, 1]$.

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Equivalence of Stochastic Equations and Martingale Problems

Thomas G. Kurtz

Abstract The fact that the solution of a martingale problem for a diffusion process gives a weak solution of the corresponding Itô equation is well-known since the original work of Stroock and Varadhan. The result is typically proved by constructing the driving Brownian motion from the solution of the martingale problem and perhaps an auxiliary Brownian motion. This constructive approach is much more challenging for more general Markov processes where one would be required to construct a Poisson random measure from the sample paths of the solution of the martingale problem. A "soft" approach to this equivalence is presented here, which begins with a joint martingale problem for the solution of the desired stochastic equation and the driving processes and applies a Markov mapping theorem to show that any solution of the original martingale problem corresponds to a solution of the joint martingale problem. These results coupled with earlier results on the equivalence of forward equations and martingale problems show that the three standard approaches to specifying Markov processes (stochastic equations, martingale problems, and forward equations) are, under very general conditions, equivalent in the sense that existence and/or uniqueness of one implies existence and/or uniqueness for the other two.

Keywords Existence · Forward equations · Markov mapping theorem · Markov processes · Martingale problems · Stochastic equations · Uniqueness

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1 Introduction

Let X be a solution of an Itô equation

$$X(t) = X(0) + \int_0^t \sigma(X(s))\mathrm{d}W(s) + \int_0^t b(X(s))\mathrm{d}s, \tag{1}$$

where X has values in $\mathbb{R}^d$, W is standard, m-dimensional Brownian motion, σ is a locally bounded $d \times m$-matrix-valued function, and b is a locally bounded $\mathbb{R}^d$-valued function. Let L be the corresponding differential generator

$$Lf(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_i b_i(x)\frac{\partial}{\partial x_i} f(x).$$

If we define $Af = Lf$ for $f \in \mathcal{D}(A) \equiv C_c^2(\mathbb{R}^d)$, the twice continuously differentiable functions with compact support in $\mathbb{R}^d$, then it follows from Itô's formula and the properties of the Itô integral that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))\mathrm{d}s = \int_0^t \nabla f(X(s))^T \sigma(X(s))\mathrm{d}W(s) \tag{2}$$

is a martingale and hence that X is a solution of the martingale problem for A (or more precisely, the $C_{\mathbb{R}^d}[0,\infty)$-martingale problem for A). That the converse to this observation is, in a useful sense, true is an important fact observed early in the study of martingale problems for diffusion processes (Stroock and Varachan [1]). To state precisely the sense in which the assertion is true, we say that a process X with sample paths in $C_{\mathbb{R}^d}[0,\infty)$ is a *weak solution* of (1) if and only if there exists a probability space $(\Omega, \mathcal{F}, P)$ and stochastic processes $\widetilde{X}$ and $\widetilde{W}$ adapted to a filtration $\{\mathcal{F}_t\}$ such that $\widetilde{X}$ has the same distribution as X, $\widetilde{W}$ is an $\{\mathcal{F}_t\}$-Brownian motion, and

$$\widetilde{X}(t) = \widetilde{X}(0) + \int_0^t \sigma(\widetilde{X}(s))\mathrm{d}\widetilde{W}(s) + \int_0^t b(\widetilde{X}(s))\mathrm{d}s. \tag{3}$$

We then have

Theorem 1.1. *X is a solution of the $C_{\mathbb{R}^d}[0,\infty)$-martingale problem for A if and only if X is a weak solution of (1).*

Taking expectations in (2), we obtain the identity

$$\nu_t f = \nu_0 f + \int_0^t \nu_s Af\,\mathrm{d}s, \quad f \in \mathcal{D}(A), \tag{4}$$

which is just the weak form of the forward equation for $\{\nu_t\}$, the one-dimensional distributions of X. The converse of the observation that every solution of the martingale problem gives a solution of the forward equation is also true, and we have the following theorem. (See the construction in [2], Theorem 4.9.19, or [3], Theorem 2.6.)

Theorem 1.2. *If X is a solution of the martingale problem for A, then $\{\nu_t\}$, the one-dimensional distributions of X, is a solution of the forward equation (4). If $\{\nu_t\}$ is a solution of (4), then there exists a solution X of the martingale problem for A such that $\{\nu_t\}$ are the one-dimensional distributions of X.*

Note that Theorem 1.1, as stated, applies to solutions of the $C_{\mathbb{R}^d}[0,\infty)$-martingale problem, that is, solutions whose sample paths are in $C_{\mathbb{R}^d}[0,\infty)$, while Theorem 1.2 does not have this restriction. In general, we cannot rule out the possibility that a solution of (1) hits infinity in finite time unless we add additional restrictions to the coefficients. One way around this issue is to allow X to take values in $\mathbb{R}^{d\Delta}$, the one-point compactification of $\mathbb{R}^d$, and to allow ν_t to be in $\mathcal{P}(\mathbb{R}^{d\Delta})$. To avoid problems with the definition of the stochastic integral in (1), we can replace (1) by the requirement that (2) hold for all $f \in C_c^2(\mathbb{R}^d)$, extending f to $\mathbb{R}^{d\Delta}$ by defining $f(\Delta) = 0$.

Given an initial distribution $\nu_0 \in \mathcal{P}(\mathbb{R}^{d\Delta})$, we say that uniqueness holds for the martingale problem (or $C_{\mathbb{R}^d}[0,\infty)$-martingale problem) for (A, ν_0) if any two solutions of the martingale problem (resp. $C_{\mathbb{R}^d}[0,\infty)$-martingale problem) for A with initial distribution ν_0 have the same finite dimensional distributions. Similarly, weak uniqueness holds for (2) (or (1)) with initial distribution ν_0 if any two weak solutions of (2) (resp. (1)) with initial distribution ν_0 have the same finite dimensional distributions, and uniqueness holds for the forward equation (4) if any two solutions with initial distribution ν_0 are the same.

Note that neither Theorem 1.1 nor Theorem 1.2 assumes uniqueness. Consequently, existence and uniqueness for the three problems are equivalent.

Corollary 1.3. *Let $\nu_0 \in \mathcal{P}(\mathbb{R}^d)$. The following are equivalent:*

(a) Uniqueness holds for the martingale problem for (A, ν_0).
(b) Weak uniqueness holds for (2) with initial distribution ν_0.
(c) Uniqueness holds for (4) with initial distribution ν_0.

The usual proof of Theorem 1.1 involves the construction of W in terms of the given solution X of the martingale problem. If $d = m$ and σ is nonsingular, this construction is simple. In particular, if we define

$$M(t) = X(t) - \int_0^t b(X(s))\mathrm{d}s,$$

then

$$W(t) = \int_0^t \sigma^{-1}(X(s))\mathrm{d}M(s),$$

where σ^{-1} denotes the inverse of σ. If σ is singular, the construction involves an auxiliary Brownian motion independent of X. (See, for example, [4], Theorems 4.5.1 and 4.5.2, or [2], Theorem 5.3.3.)

A possible alternative approach is to consider the process $Z = (X, Y) = (X, Y(0) + W)$. Of course,

$$\mathrm{d}Z(t) = \mathrm{d}\begin{pmatrix} X(t) \\ Y(0) + W(t) \end{pmatrix} = \begin{pmatrix} \sigma(X(t)) \\ I \end{pmatrix} \mathrm{d}W(t) + \begin{pmatrix} b(X(t) \\ 0 \end{pmatrix} \mathrm{d}t. \tag{5}$$

Note that each weak solution of (1) gives a weak solution of (5), and each weak solution of (5) gives a weak solution of (1). As before, using Itô's formula, it is simple to compute the generator $\widehat{A}$ corresponding to (5) (take the domain to be $C_c^2(\mathbb{R}^{d+m})$). Furthermore, since if one knows Z one knows W, it follows immediately that every solution of the martingale problem for $\widehat{A}$ is a weak solution of the stochastic differential equation. In particular, weak uniqueness for (5) implies uniqueness for the martingale problem for $\widehat{A}$. Note, however, that the assertion that every solution of the martingale problem for $\widehat{A}$ is a weak solution of (5) (and hence gives a weak solution of (1)) does not immediately imply that every solution of the martingale problem for A is a weak solution of (1) since we must obtain the driving Brownian motion. In particular, we cannot immediately conclude that uniqueness for (1) implies uniqueness for the martingale problem for A.

In fact, however, an argument along the lines described can be used to show that each solution of the martingale problem for A is a weak solution of (1). For simplicity, assume $d = m = 1$. Instead of augmenting the state by $Y(0) + W$, augment the state by

$$Y(t) = Y(0) + W(t) \bmod 2\pi.$$

We can still recover W from observations of the increments of Y. For example, if we set

$$\zeta(t) = \begin{pmatrix} \cos(Y(t)) + \int_0^t \frac{1}{2}\cos(Y(s))\mathrm{d}s \\ \sin(Y(t)) + \int_0^t \frac{1}{2}\sin(Y(s))\mathrm{d}s \end{pmatrix}, \tag{6}$$

and

$$W(t) = \int_0^t (-\sin(Y(s)), \cos(Y(s)))\mathrm{d}\zeta(s), \tag{7}$$

then W is a standard Brownian motion and ζ satisfies

$$\mathrm{d}\zeta(t) = \begin{pmatrix} -\sin(Y(t) \\ \cos(Y(t)) \end{pmatrix} \mathrm{d}W(t). \tag{8}$$

The introduction of Y may look strange, but the heart of our argument depends on being able to compute the conditional distribution of $Y(t)$ given $\mathcal{F}_t^X \equiv \sigma(X(s) : s \leq t)$. If $Y(0)$ is uniformly distributed on $[0, 2\pi]$ and is independent of W, then the conditional distribution of $Y(t)$ given $\mathcal{F}_t^X$ is uniform on $[0, 2\pi]$. In fact, that is the conditional distribution even if we condition on both X and W.

Let $\mathcal{D}(\widehat{A})$ be the collection of $f \in C_c^2(\mathbb{R} \times [0, 2\pi))$ such that $f(x, 0) = f(x, 2\pi-)$, $f_y(x, 0) = f_y(x, 2\pi-)$, and $f_{yy}(x, 0) = f_{yy}(x, 2\pi-)$. Applying Itô's formula, for $f \in \mathcal{D}(\widehat{A})$, we have

$$\widehat{A}f = \frac{1}{2}\sigma^2 f_{xx} + \sigma f_{xy} + \frac{1}{2} f_{yy} + b f_x.$$

Suppose $Z = (X, Y)$ is a solution of the martingale problem for $\widehat{A}$, and define ζ by (6) and W by (7). Applying Lemma A.2 with $f_1(x, y) = f(x)$, $f_2(x, y) = \cos(y)$, $f_3(x, y) = \sin(y)$, $g_1(x, y) = 1$, $g_2(x, y) = f'(x)\sigma(x)\sin(y)$, and $g_3(x, y) = -f'(x)\sigma(x)\cos(y)$ implies

$$\begin{aligned} M(t) = f(X(t)) - f(X(0)) &- \int_0^t Af(X(s))\mathrm{d}s \\ &+ \int_0^t f'(X(s))\sigma(X(s))\sin(Y(s))\mathrm{d}\zeta_1(s) \\ &- \int_0^t f'(X(s))\sigma(X(s))\cos(Y(s))\mathrm{d}\zeta_2(s) \end{aligned}$$

satisfies $\langle M \rangle \equiv 0$ so $M \equiv 0$ and hence

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\sigma(X(s))\mathrm{d}W(s) + \int_0^t Af(X(s))\mathrm{d}s. \tag{9}$$

It follows that any solution of the martingale problem for $\widehat{A}$ satisfying $\sup_{s \le t} |X(s)| < \infty$ a.s. for each t is a weak solution of (1).

Of course, this last observation does not prove Theorem 1.1. We still have the question of whether or not every solution of the martingale problem for A corresponds to a solution of the martingale problem for $\widehat{A}$. The following result from [3] provides the tools needed to answer this question affirmatively. Let (E, r) be a complete, separable metric space, $B(E)$, the bounded, measurable functions on E, and $\overline{C}(E)$, the bounded continuous functions on E. If E is locally compact, then $\widehat{C}(E)$ will denote the continuous functions vanishing at infinity. We say that an operator $B \subset B(E) \times B(E)$ is *separable* if there exists a countable subset $\{g_k\} \subset \mathcal{D}(B)$ such that B is contained in the bounded, pointwise closure of the linear span of $\{(g_k, Bg_k)\}$. B is a pregenerator if it is dissipative and there are sequences of functions $\mu_n : E \to \mathcal{P}(E)$ and $\lambda_n : E \to [0, \infty)$ such that for each $(f, g) \in B$

$$g(x) = \lim_{n \to \infty} \lambda_n(x) \int_{\mathbb{S}} (f(y) - f(x))\mu_n(x, \mathrm{d}y), \tag{10}$$

for each $x \in E$.

For a measurable, E_0-valued process U, $\widehat{\mathcal{F}}_t^U$ is the completion of

$$\sigma\left(\int_0^r h(U(s))\mathrm{d}s : r \le t, h \in B(E_0)\right) \vee \sigma(U(0)).$$

Let $\mathbf{T}^U = \{t : U(t) \text{ is } \widehat{\mathcal{F}}_t^U\text{-measurable}\}$. ($\mathbf{T}^U$ has full Lebesgue measure, and if U is cadlag with no fixed points of discontinuity, then $\mathbf{T}^U = [0,\infty)$. See Appendix A.2 of [5].) Let $M_{E_0}[0,\infty)$ be the space of measurable functions from $[0,\infty)$ to E_0 topologized by convergence in Lebesgue measure.

Theorem 1.4. *Suppose that $B \subset \overline{C}(E) \times \overline{C}(E)$ is separable and a pregenerator and that $\mathcal{D}(B)$ is closed under multiplication and separates points in E. Let (E_0, r_0) be a complete, separable metric space, $\gamma : E \to E_0$ be Borel measurable, and α be a transition function from E_0 into E ($y \in E_0 \to \alpha(y,\cdot) \in \mathcal{P}(E)$ is Borel measurable) satisfying $\alpha(y, \gamma^{-1}(y)) = 1$. Define*

$$C = \left\{\left(\int_E f(z)\alpha(\cdot,\mathrm{d}z), \int_E Bf(z)\alpha(\cdot,\mathrm{d}z)\right) : f \in \mathcal{D}(B)\right\}.$$

Let $\mu_0 \in \mathcal{P}(E_0)$, and define $\nu_0 = \int \alpha(y,\cdot)\mu_0(\mathrm{d}y)$. If $\widetilde{U}$ is a solution of the martingale problem for (C, μ_0), then there exists a solution V of the martingale problem for (B, ν_0) such that $\widetilde{U}$ has the same distribution on $M_{E_0}[0,\infty)$ as $U = \gamma \circ V$ and

$$P\{V(t) \in \Gamma | \widehat{\mathcal{F}}_t^U\} = \alpha(U(t), \Gamma), \quad \Gamma \in \mathcal{B}(E), t \in \mathbf{T}^U. \tag{11}$$

If $\widetilde{U}$ (and hence U) has a modification with sample paths in $D_E[0,\infty)$, then the modified $\widetilde{U}$ and U have the same distribution on $D_E[0,\infty)$.

Assume that σ and b in (1) are continuous. (This assumption can be removed with the application of more complicated technology. See Sect. 4.) Let B in the statement of Theorem 1.4 be $\widehat{A}$, $E = \mathbb{R} \times [0, 2\pi)$, $E_0 = \mathbb{R}$, $\gamma(x,y) = x$, and for $f \in B(\mathbb{R} \times [0,2\pi))$, define $\alpha f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,y)\mathrm{d}y$. For $f \in \mathcal{D}(\widehat{A})$, a straight forward calculation gives

$$\alpha\widehat{A}f(x) = A\alpha f(x),$$

so $A = C$. It follows that if X is a solution of the martingale problem for A, then there exists a solution $(\widetilde{X}, \widetilde{Y})$ of the martingale problem for $\widehat{A}$ such that X and $\widetilde{X}$ have the same distribution. Consequently, if X has sample paths in $C_{\mathbb{R}}[0,\infty)$, then X is a weak solution for (1), and Theorem 1.1 follows. Every solution of the martingale problem for A will have a modification with sample paths in $D_{\mathbb{R}^\Delta}[0,\infty)$, where $\mathbb{R}^\Delta$ denotes the one-point compactification of $\mathbb{R}$, and any solution with sample paths in $D_{\mathbb{R}}[0,\infty)$ will, in fact, have sample paths in $C_{\mathbb{R}}[0,\infty)$.

Invoking Theorem 1.4 is obviously a much less straight forward approach to Theorem 1.1 than the usual argument; however, the state augmentation approach extends easily to much more general settings in which the constructive argument becomes technically very complicated if not impossible.

2 Stochastic Differential Equations for Markov Processes

Typically, a Markov process X in $\mathbb{R}^d$ has a generator of the form

$$\begin{aligned} Af(x) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + \widehat{b}(x)\cdot \nabla f(x) \\ &\quad + \int_{\mathbb{R}^d} (f(x+y) - f(x) - \mathbf{1}_{B_1}(y) y \cdot \nabla f(x))\eta(x, \mathrm{d}y) \end{aligned}$$

where B_1 is the ball of radius 1 centered at the origin and η satisfies

$$\int 1 \wedge |y^2| \eta(x, \mathrm{d}y) < \infty \tag{12}$$

for each x. (See, for example, [6] and [7].) The three terms are, respectively, the diffusion term, the drift term, and the jump term. In particular, $\eta(x, \Gamma)$ gives the "rate" at which jumps satisfying $X(s) - X(s-) \in \Gamma$ occur. Note that B_1 can be replaced by any set C containing an open neighborhood of the origin provided that the drift term is replaced by

$$b_C(x)\cdot\nabla f(x) = \left(b(x) + \int_{\mathbb{R}^d} y(\mathbf{1}_C(y) - \mathbf{1}_{B_1}(y))\eta(x, \mathrm{d}y) \right)\cdot \nabla f(x).$$

Suppose that there exist $\lambda : \mathbb{R}^d \times S \to [0,1]$, $\gamma : \mathbb{R}^d \times S \to \mathbb{R}^d$, and a σ-finite measure ν on a measurable space $(S, \mathcal{S})$ such that

$$\eta(x, \Gamma) = \int_S \lambda(x,u)\mathbf{1}_\Gamma(\gamma(x,u))\nu(\mathrm{d}u).$$

This representation is always possible. In fact, there are many such representations. For example, we can rewrite

$$\begin{aligned} \eta(x,\Gamma) &= \int_S \lambda(x,u)\mathbf{1}_{[0,1]}(|\gamma(x,u|))\mathbf{1}_\Gamma(\gamma(x,u))\nu(\mathrm{d}u) \\ &\quad + \int_S \lambda(x,u)\mathbf{1}_{(1,\infty)}(|\gamma(x,u|))\mathbf{1}_\Gamma(\gamma(x,u))\nu(\mathrm{d}u) \\ &= \int_S \lambda_1(x,u)\mathbf{1}_\Gamma(\gamma(x,u))\nu(\mathrm{d}u) + \int_S \lambda_2(x,u)\mathbf{1}_\Gamma(\gamma(x,u))\nu(\mathrm{d}u) \\ &= \int_{S_1} \lambda(x,u)\mathbf{1}_\Gamma(\gamma(x,u))\nu(\mathrm{d}u) + \int_{S_2} \lambda(x,u)\mathbf{1}_\Gamma(\gamma(x,u))\nu(\mathrm{d}u), \end{aligned}$$

where S_1 and S_2 are copies of S, and λ on S_1 is given by λ_1 and λ on S_2 is given by λ_2. Noting that $\mathbf{1}_{S_1}(u) = \mathbf{1}_{B_1}(\gamma(x,u))$, we can replace S by $S_1 \cup S_2$, and assuming

$$\int_S \lambda(x,u)(\mathbf{1}_{S_1}(u)|\gamma(x,u)|^2 + \mathbf{1}_{S_2}(u))\nu(\mathrm{d}u) < \infty,$$

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x) + b(x)\cdot\nabla f(x) \tag{13}$$
$$+ \int_S \lambda(x,u)(f(x+\gamma(x,u)) - f(x) - \mathbf{1}_{S_1}(u)\gamma(x,u)\cdot\nabla f(x))\nu(\mathrm{d}u).$$

We will take $\mathcal{D}(A) = C_c^2(\mathbb{R}^d)$ and assume that for $f \in \mathcal{D}(A)$, $Af \in \overline{C}(\mathbb{R}^d)$. Removal of the continuity assumption is discussed in Sect. 4. The assumption that Af is bounded can also be relaxed, but that issue is not addressed here.

Let ξ be a Poisson random measure on $[0,1]\times S\times[0,\infty)$ with mean measure $m\times\nu\times m$, and let $\widetilde{\xi}(A) = \xi(A) - m\times\nu\times m(A)$. Let $(S_0, \mathcal{S}_0)$ be a measurable space, μ a σ-finite measure on $(S_0,\mathcal{S}_0)$, W a Gaussian white noise on $S_0\times[0,\infty)$ satisfying $E[W(A,s)W(B,t)] = s\wedge t\mu(A\cap B)$, and $\sigma : \mathbb{R}^d\times S_0 \to \mathbb{R}^d$ satisfying $\int_{S_0}|\sigma(x,u)|^2\mu(\mathrm{d}u) < \infty$ and

$$a(x) = \int_{S_0}\sigma(x,u)\sigma^T(x,u)\mu(\mathrm{d}u).$$

Again, there are many possible choices for μ and σ. The usual form for an Itô equation corresponds to taking μ to be counting measure on a finite set S_0.

Assume that for each compact $K\subset\mathbb{R}^d$

$$\sup_{x\in K}\Big(|b(x)| + \int_{S_0}|\sigma(x,u)|^2\mu(\mathrm{d}u) + \int_{S_1}\lambda(x,u)|\gamma(x,u)|^2\nu(\mathrm{d}u) \tag{14}$$
$$+ \int_{S_2}\lambda(x,u)|\gamma(x,u)|\wedge 1\nu(\mathrm{d}u)\Big) < \infty.$$

Then, X should satisfy a stochastic differential equation of the form

$$X(t) = X(0) + \int_{S_0\times[0,t]}\sigma(X(s),u)W(\mathrm{d}u\times\mathrm{d}s) + \int_0^t b(X(s))\mathrm{d}s \tag{15}$$
$$+ \int_{[0,1]\times S_1\times[0,t]}\mathbf{1}_{[0,\lambda(X(s-),u)]}(v)\gamma(X(s-),u)\widetilde{\xi}(\mathrm{d}v\times\mathrm{d}u\times\mathrm{d}s)$$
$$+ \int_{[0,1]\times S_2\times[0,t]}\mathbf{1}_{[0,\lambda(X(s-),u)]}(v)\gamma(X(s-),u)\xi(\mathrm{d}v\times\mathrm{d}u\times\mathrm{d}s),$$

for $t < \tau_\infty \equiv \lim_{k\to\infty}\inf\{t : |X(t-)| \text{ or } |X(t)| \geq k\}$. Stochastic equations of this form appeared first in [8].

An application of Itô's formula again shows that any solution of (15) gives a solution of the martingale problem for A. We will apply an extension of Theorem 1.4

to show that every solution of the $D_{\mathbb{R}^d}[0,\infty)$-martingale problem for A is a weak solution of (15), or more generally, we can replace (15) by the analog of (2) and drop the requirement that the solution have sample paths in $D_{\mathbb{R}^d}[0,\infty)$. (In any case, the solution will have a modification with sample paths in $D_{\mathbb{R}^{d\Delta}}[0,\infty)$.)

As in the introduction, we will need to represent the driving processes W and ξ in terms of processes whose conditional distributions given X are their stationary distributions. To avoid the danger of measure-theoretic or functional-analytic *faux-pas*, we will assume that S and S_0 are complete, separable metric spaces and that ν and μ are σ-finite Borel measures.

2.1 Representation of *W* by Stationary Processes

Let $\varphi_1, \varphi_2, \ldots$ be a complete, orthonormal basis for $L_2(\mu)$. Then, W is completely determined by

$$W(\varphi_i, t) = \int_{S_0\times[0,t]} \varphi_i(u) W(\mathrm{d}u \times \mathrm{d}s), \quad i = 1, 2, \ldots.$$

In particular, if H is an $\{\mathcal{F}_t\}$-adapted process with sample paths in $D_{L_2(\mu)}[0,\infty)$, then

$$\int_{S_0\times[0,t]} H(s-,u) W(\mathrm{d}u \times \mathrm{d}s) = \sum_{i=1}^{\infty} \int_0^t \langle H(s-,\cdot), \varphi_i\rangle \mathrm{d}W(\varphi_i, s).$$

In turn, if we define $Y_i(t) = Y_i(0) + W(\varphi_i, t) \bmod 2\pi$ and

$$\zeta_i(t) = \begin{pmatrix} \cos(Y_i(t)) + \int_0^t \frac{1}{2}\cos(Y_i(s))\mathrm{d}s \\ \sin(Y_i(t)) + \int_0^t \frac{1}{2}\sin(Y_i(s))\mathrm{d}s \end{pmatrix} = \begin{pmatrix} -\int_0^t \sin(Y_i(s))\mathrm{d}W(\varphi_i, s) \\ \int_0^t \cos(Y_i(s))\mathrm{d}W(\varphi_i, s) \end{pmatrix},$$

then

$$W(\varphi_i, t) = \int_0^t (-\sin(Y_i(s)),\ \cos(Y_i(s)))\,\mathrm{d}\zeta_i(s), \tag{16}$$

and hence,

$$\int_{S_0\times[0,t]} H(s,u) W(\mathrm{d}u\times\mathrm{d}s) = \sum_{i=1}^{\infty} \int_0^t \langle H(s,\cdot), \varphi_i\rangle(-\sin(Y_i(s)), \cos(Y_i(s)))\mathrm{d}\zeta_i(s).$$

Note that if $Y_i(0)$ is uniformly distributed on $[0, 2\pi)$ and independent of W, then Y_i is a stationary process and for each t, the $Y_i(t)$ are independent and independent of $\sigma(W(\varphi_j, s) : s \le t, j = 1, 2, \ldots)$. Identifying 2π with 0, $[0, 2\pi)$ is compact and $Y = \{Y_i\}$ is a Markov process with compact state space $[0, 2\pi)^\infty$.

2.2 Representation of ξ by Stationary Processes

Let $\{D_i\} \subset \mathcal{B}(S)$ be a partition of S satisfying $\nu(D_i) < \infty$, and define $\xi_i(C_1 \times C_2 \times [0,t]) = \xi(C_1 \times C_2 \cap D_i \times [0,t])$. Then, the ξ_i are independent Poisson random measures, and setting $N_i(t) = \xi([0,1] \times D_i \times [0,t])$, ξ_i can be written as

$$\xi_i(\cdot \times [0,t]) = \sum_{i=0}^{N_i(t)-1} \delta_{(V_{i,k},U_{ik})},$$

where $\{V_{i,k}, U_{i,k}, i \geq 1, k \geq 0\}$ are independent, $V_{i,k}$ is uniform-[0, 1], and $U_{i,k}$ is D_i-valued with distribution

$$\beta_i \equiv \frac{\nu(\cdot \cap D_i)}{\nu(D_i)}.$$

Define

$$Z_i(t) = (V_{i,N_i(t)}, U_{i,N_i(t)}).$$

Then, Z_i is a Markov process with stationary distribution $\ell \times \beta_i$, where ℓ is the uniform distribution on [0, 1], and $Z_i(t)$ is independent of $\sigma(\xi(\cdot \times [0,s]), s \leq t)$.

Since, with probability one, $V_{i,k} \neq V_{i,k+1}$, N_i can be recovered from Z_i, and since

$$\int_{[0,1]\times S\times[0,t]} H(v,u,s-)\xi(\mathrm{d}v \times \mathrm{d}u \times \mathrm{d}s) = \sum_i \int_0^t H(Z_i(s-), s-)\mathrm{d}N_i(s),$$

ξ can be recovered from $\{Z_i\}$.

2.3 Equivalence to Martingale Problem

To simplify notation, we will replace $\mathbf{1}_{[0,\lambda(x,u)]}(v)\gamma(x,u)$ by $\gamma(x,u)$. There is no loss of generality since S is arbitrary and we can replace $[0,1] \times S$ by S. Under the new notation, ξ is a Poisson random measure on $S \times [0,\infty)$ with mean measure $\nu \times m$. We will also assume that ν is nonatomic, so it is still the case that, with probability one, N_i can be recovered from observations of Z_i.

Let $\mathcal{D}_0 \subset C^2([0,2\pi))$ be the collection of functions satisfying $f(0) = f(2\pi-)$, $f'(0) = f'(2\pi-)$, and $f''(0) = f''(2\pi-)$, and let $\mathcal{D}_i = \overline{C}(D_i)$. Define

$$\mathcal{D}(\widehat{A}) = \left\{ f_0(x)\prod_{i=1}^{m_1} f_{1i}(y_i)\prod_{i=1}^{m_2} f_{2i}(z_i) : f_0 \in C_c^2(\mathbb{R}^d), f_{1i} \in \mathcal{D}_0, f_{2i} \in \mathcal{D}_i \right\},$$

and for $f \in \mathcal{D}(\widehat{A})$, derive $\widehat{A}f$ by applying Itô's formula to

$$f_0(X(t)) \prod_{i=1}^{m_1} f_{1i}(Y_i(t)) \prod_{i=1}^{m_2} f_{2i}(Z_i(t)).$$

Define L_x by

$$L_x f(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + b(x) \cdot \nabla f(x),$$

and L_y by

$$L_y f(y) = \frac{1}{2} \sum_k \frac{\partial^2}{\partial y_k^2} f(y).$$

Note that L_x would be the generator for X if γ were zero and L_y is the generator for $Y = \{Y_i\}$. The quadratic covariation of X_i and Y_k is

$$[X_i, Y_k] = \int_0^t c_{ik}(X(s)) \mathrm{d}s,$$

where $c_{ik}(x) = \int_{S_0} \sigma_i(x,u)\varphi_k(u)\mu(\mathrm{d}u)$, so define L_{xy} by

$$L_{xy} f(x,y) = \sum_{i,k} c_{ik}(x) \partial_{x_i} \partial_{y_k} f(x,y).$$

For $u \in S$ and $z \in \prod_i D_i$, let $\rho(z,u)$ be the element of $\prod_i D_i$ obtained by replacing z_i by u provided $u \in D_i$. Define

$$\begin{aligned} J_i f(x,y,z) &= \int_{D_i} \Big(f(x + \gamma(x, z_i), y, \rho(z,u)) - f(x,y,z) \\ &\qquad - \mathbf{1}_{S_1}(u)\gamma(x,u) \cdot \nabla_x f(x,y,z) \Big) \nu(\mathrm{d}u) \\ &= \int_{D_i} \Big(f(x + \gamma(x, z_i), y, \rho(z,u)) - f(x + \gamma(x,u), y, z) \Big) \nu(\mathrm{d}u) \\ &\qquad + \int \Big(f(x+\gamma(x,u), y, z) - f(x,y,z) - \mathbf{1}_{S_1}(u)\gamma(x,u) \cdot \nabla_x f(x,y,z) \Big). \end{aligned}$$

Then, at least formally, by Itô's formula,

$$f(X(t), Y(t), Z(t)) - f(X(0), Y(0), Z(0)) - \int_0^t \widehat{A} f(X(s), Y(s), Z(s)) \mathrm{d}s$$

is a martingale for

$$\begin{aligned}\widehat{A}f(x,y,z) &= L_x f(x,y,z) + L_y f(x,y,z) + L_{xy} f(x,y,z) + \sum_i J_i f(x,y,z)\\ &= \prod_{i=1}^{m_1} f_{1i}(y_i) \prod_{i=1}^{m_2} f_{2i}(z_i) A f_0(x) + L_y f(x,y,z) + L_{xy} f(x,y,z)\\ &\quad + \sum_i \int_{D_i} \Big(f(x+\gamma(x,z_i), y, \rho(z,u)) - f(x+\gamma(x,u), y, z) \Big) \nu(\mathrm{d}u).\end{aligned}$$

Unfortunately, in general, $\sum_i J_i$ may not converge. Consequently, the extension needs to be done one step at a time, so define $Z^n = (Z_1, \dots, Z_n)$ and observe that the generator for (X, Y, Z^n) is

$$\begin{aligned}\widehat{A}_n f(x,y,z) &= L_x f(x,y,z) + L_y f(x,y,z) + L_{xy} f(x,y,z) + \sum_{i=1}^{n} J_i f(x,y,z)\\ &= \prod_{i=1}^{m_1} f_{1i}(y_i) \prod_{i=1}^{m_2} f_{2i}(z_i) A f_0(x) + L_y f(x,y,z) + L_{xy} f(x,y,z)\\ &\quad + \sum_{i=1}^{n} \int_{D_i} \Big(f(x+\gamma(x,z_i), y, \rho(z,u)) - f(x+\gamma(x,u), y, z) \Big) \nu(\mathrm{d}u),\end{aligned}$$

where we take $\mathcal{D}(\widehat{A}_n) = \{f \in \mathcal{D}(\widehat{A}) : m_2 \le n\}$. Note that as long as $Af_0 \in B(\mathbb{R}^d)$, $\widehat{A}_n f \in B(\mathbb{R}^d \times [0, 2\pi)^\infty \times \prod_{i=1}^n D_i)$.

Instead of requiring (X, Y, Z) to be a solution of the martingale problem for $\widehat{A}$, for each n, we require (X, Y, Z^n) to be a solution of the martingale problem for $\widehat{A}_n$.

Lemma 2.1. *If for each n, (X, Y, Z^n) is a solution of the martingale problem for $\widehat{A}_n$ with sample paths in $D_{\mathbb{R}^{d\Delta} \times [0,2\pi)^\infty \times \prod_{i=1}^n D_i}[0, \infty)$, W is given by (16), and ξ is given by*

$$\int_{S\times[0,t]} g(u)\xi(\mathrm{d}u \times \mathrm{d}s) = \sum_{i=1}^{\infty} \int_0^t g(Z_i(s-))\mathrm{d}N_i(s),$$

then (X, W, ξ) satisfies (15) for $0 \le t < \tau_\infty$.

Remark 2.2. *Any process (X, Y, Z) such that for each n, (X, Y, Z^n) is a solution of the martingale problem for $\widehat{A}_n$ will have a modification with sample paths in $D_{\mathbb{R}^{d\Delta} \times [0,2\pi)^\infty \times \prod_{i=1}^\infty D_i}[0, \infty)$, and the modification will satisfy (17) for all $f \in C_c^2(\mathbb{R}^d)$, taking $f(\Delta) = 0$.*

Proof. As in the verification of (9), Lemma A.2 can again be used to show that (X, W, ξ) satisfies

$$f(X(t)) = f(X(0)) + \int_0^t Af(X(s))\mathrm{d}s + \int_0^t \nabla f(X(s))^T \sigma(X(s),u)W(\mathrm{d}u \times \mathrm{d}s)$$
$$+ \int_{S\times[0,t]} (f(X(s-) + \gamma(X(s-),u)) - f(X(s-))\widetilde{\xi}(\mathrm{d}u \times \mathrm{d}s), \qquad (17)$$

$f \in C_c^2(\mathbb{R}^d)$, $t \geq 0$, and it follows that X satisfies (15) for $0 \leq t < \tau_\infty$. □

Theorem 2.3. *Let A be given by (13), and assume that (14) is satisfied and that for $f \in C_c^2(\mathbb{R}^d)$, $Af \in B(\mathbb{R}^d)$. Then, any solution of the $D_{\mathbb{R}^d}[0,\infty)$-martingale problem for A is a weak solution of (15). More generally, any solution of the martingale problem for A has a modification with sample paths in $D_{\mathbb{R}^{d\Delta}}[0,\infty)$ and is a weak solution of (15) on the time interval $[0,\tau_\infty)$.*

Remark 2.4. *We need to relax the requirement in Theorem 1.4 that $\mathcal{R}(B) \subset \overline{C}(E)$. This extension is discussed in Sect. 4.*

Proof. Let $\beta_y \in \mathcal{P}([0,2\pi)^\infty)$ be the product of uniform distributions on $[0,2\pi)$ and $\beta_z^n \in \prod_{i=1}^n \beta_i$. For $x \in \mathbb{R}^d$, $\alpha_n(x,\cdot) = \delta_x \times \beta_y \times \beta_z^n \in \mathcal{P}(\mathbb{R}^d \times [0,2\pi)^\infty \times \prod_{i=1}^n D_i)$. ($\gamma_n$ is just the projection onto $\mathbb{R}^d$.) Computing $\alpha_n \widehat{A}_n f$, observe that $\alpha_n L_x f = L_x \alpha_n f$, that $\alpha_n L_y f = 0$ since β_y is the stationary distribution for L_y, and that $\alpha_n L_{xy} f = 0$ since $\int_0^{2\pi} \partial_{y_k} f(x,y,z)\mathrm{d}y_k = 0$. To see that $\alpha_n J_i f = J_i \alpha_n f$, note that

$$\int_{D_i}\int_{D_i} f(x+\gamma(x,z_i),y,\rho(z,u))\nu(\mathrm{d}u)\nu(\mathrm{d}z_i)$$
$$= \int_{D_i}\int_{D_i} f(x+\gamma(x,u),y,z)\nu(\mathrm{d}u)\nu(\mathrm{d}z_i).$$

Taking these observations together, we have $\alpha_n \widehat{A}_n f = A\alpha_n f$.

We apply Theorem 4.1. See Sect. 4. Note that $\mathcal{D}(\widehat{A}_n)$ is closed under multiplication. The separability condition follows from the separability of $\mathcal{D}(\widehat{A}_n)$ under the norm

$$\|f\|_* = \|f\| + \|\nabla_x f\| + \|\partial_x^2 f\|.$$

The pre-generator condition for $\mathbb{B}$ and $\mathbb{B}_n$ defined in Sect. 4 follows from existence of solutions of the martingale problem for $\mathbb{B}_n^v f \equiv \mathbb{B}_n f(\cdot,v)$. (See the discussion in Sect. 2 of [3].) Consequently, taking $C = A$ and $B = \widehat{A}_n$ in Theorem 4.1, any solution $\widetilde{X}$ of the martingale problem for A corresponds to a solution (X,Y,Z^n) of the martingale problem for $\widehat{A}_n$. But also note that $\beta_n \widehat{A}_{n+1} f = \widehat{A}_n \beta_n f$ for $f \in \mathcal{D}(\widehat{A}_{n+1})$. Consequently, any solution the martingale problem for $\widehat{A}_n$ extends to a solution of the martingale problem for $\widehat{A}_{n+1}$. By induction, we obtain the process (X,Y,Z) so the first part of the theorem follows by Lemma 2.1.

If X is a solution of the martingale problem for A, then by [2], Corollary 4.3.7, X has a modification with sample paths in $D_{\mathbb{R}^{d\Delta}}[0,\infty)$. For nonnegative $\kappa \in B(\mathbb{R}^d)$, let

$$\gamma(t) = \inf\left\{s : \int_0^s \kappa^{-1}(X(r))\mathrm{d}r \geq t\right\}.$$

Then, $\widetilde{X}(t) = X(\gamma(t))$ is a solution of the martingale problem for κA. If $\kappa(x) = 1$ for $|x| \leq k$ and $\kappa(x) = 0$ for $|x| \geq k+1$, then for $\tau_k \equiv \inf\{t : |X(t-)| \text{ or } |X(t)| \geq k\}$, $\widetilde{X}(t) = X(t)$ for $t < \tau_k$ and $\widetilde{X}$ has sample paths in $D_{\mathbb{R}^d}[0,\infty)$. It follows that $\widetilde{X}$ is a weak solution of (15) with σ replaced by $\sqrt{\kappa}\sigma$, b replaced by κb and λ replaced by $\kappa\lambda$, and hence X is a weak solution of the original equation (15) for $t \in [0, \tau_k)$. Since $\tau_\infty = \lim_{k\to\infty} \tau_k$, the theorem follows. □

Corollary 2.5. *Uniqueness holds for the $D_{\mathbb{R}^d}[0,\infty)$-martingale problem for (A, ν_0) if and only if weak uniqueness holds for (15) with initial distribution ν_0.*

3 Conditions for Uniqueness

In the orginal development by Itô [8] as well as in later presentations (for example, by Skorohod [9] and Ikeda and Watanabe [10]), L_2-estimates are used to prove uniqueness for (15). Graham [11] points out the possibility and desirability of using L_1-estimates. (In fact, for equations controlling jump rates with factors such as $\mathbf{1}_{[0,\lambda(X(t),u]}(v)$, L_1-estimates are essential.) Kurtz and Protter [12] develop methods that allow a mixing of L_1, L_2, and other estimates.

Theorem 3.1. *Suppose there exists a constant M such that*

$$|b(x)| + \int_{S_0} |\sigma(x,u)|^2\mu(\mathrm{d}u) + \int_{S_1} |\gamma(x,u)|^2\lambda(x,u)\nu(\mathrm{d}u) + \int_{S_2} \lambda(x,u)|\gamma(x,u)|\nu(\mathrm{d}u) < M, \tag{18}$$

and

$$\sqrt{\int_{S_0} |\sigma(x,u) - \sigma(y,u)|^2\mu(\mathrm{d}u)} \leq M|x-y| \tag{19}$$

$$|b(x) - b(y)| \leq M|x-y| \tag{20}$$

$$\int_{S_1} (\gamma(x,u) - \gamma(y,u))^2\lambda(x,u) \wedge \lambda(y,u)\nu(\mathrm{d}u) \leq M|x-y|^2 \tag{21}$$

$$\int_{S_1} |\lambda(x,u) - \lambda(y,u)||\gamma(x,u) - \gamma(y,u)|\nu(\mathrm{d}u) \leq M|x-y| \tag{22}$$

$$\int_{S_2} \lambda(x,u)||\gamma(x,u) - \gamma(y,u)|\nu(\mathrm{d}u) \leq M|x-y| \tag{23}$$

$$\int_{S} |\lambda(x,u) - \lambda(y,u)||\gamma(y,u)|\nu(\mathrm{d}u) \leq M|x-y|. \tag{24}$$

Then, there exists a unique solution of (15).

Proof. Suppose X and Y are solutions of (15). Then,

$$
\begin{aligned}
X(t) = X(0) &+ \int_{S_0\times[0,t]} \sigma(X(s),u)W(\mathrm{d}u\times\mathrm{d}s) + \int_0^t b(X(s))\mathrm{d}s \\
&+ \int_{[0,\infty)\times S_1\times[0,t]} \mathbf{1}_{[0,\lambda(X(s),u)\wedge\lambda(Y(s),u)]}(v)\gamma(X(s-),u)\widetilde{\xi}(\mathrm{d}v\times\mathrm{d}u\times\mathrm{d}s) \\
&+ \int_{[0,\infty)\times S_1\times[0,t]} \mathbf{1}_{(\lambda(Y(s-),u)\wedge\lambda(X(s-),u),\lambda(X(s-),u)]}(v) \\
&\qquad \gamma(X(s-),u)\widetilde{\xi}(\mathrm{d}v\times\mathrm{d}u\times\mathrm{d}s) \\
&+ \int_{[0,\infty)\times S_2\times[0,t]} \mathbf{1}_{[0,\lambda(X(s-),u)]}(v)\gamma(X(s-),u)\xi(\mathrm{d}v\times\mathrm{d}u\times\mathrm{d}s),
\end{aligned}
\tag{25}
$$

and similarly with the roles of X and Y interchanged. Then, (19) and (20) give the necessary Lipschitz conditions for the coefficient functions in the first two integrals on the right; (21) gives an L_2-Lipschitz condition for the third integral term; and (22), (23), and (24) give L_1-Lipschitz conditions for the fourth and fifth integral terms on the right. Theorem 7.1 of [12] gives uniqueness, and Corollary 7.7 of that paper gives existence. □

Corollary 3.2. *Suppose that there exists a function $M(r)$ defined for $r > 0$, such that (18)–(24) hold with M replaced by $M(|x| \vee |y|)$. Then, there exists a stopping time τ_∞ and a process $X(t)$ defined for $t \in [0,\tau_\infty)$ such that (15) is satisfied on $[0,\tau_\infty)$ and $\tau_\infty = \lim_{k\to\infty}\inf\{t : |X(t)| \text{ or } |X(t-)| \geq k\}$. If $(\widetilde{X},\widetilde{\tau})$ also has this property, then $\widetilde{\tau} = \tau_\infty$ and $\widetilde{X}(t) = X(t)$, $t < \tau_\infty$.*

Proof. The corollary follows by a standard localization argument. □

4 Equations with Measurable Coefficients

Let E and F be complete, separable metric spaces, and let $\mathbb{B} \subset \overline{C}(E)\times\overline{C}(E\times F)$. Then, Theorem 1.4 can be extended to generators of the form

$$
Bf(x) = \int_F \mathbb{B}f(x,v)\eta(x,\mathrm{d}v), \tag{26}
$$

where η is a transition function from E to F, that is, $x \in E \to \eta(x,\cdot) \in \mathcal{P}(F)$ is measurable. Note that $B \subset \overline{C}(E)\times B(E)$ but that B may not have range in $\overline{C}(E)$. (The boundedness assumption can also be relaxed with the addition of moment conditions.) Theorem 1.4 extends to operators of this form.

Theorem 4.1. *Suppose that B given by (26) is separable, that for each $v \in F$, $\mathbb{B}^v f \equiv \mathbb{B}f(\cdot,v)$ is a pregenerator, and that $\mathcal{D}(B)$ is closed under multiplication and separates points in E. Let (E_0,r_0) be a complete, separable metric space, γ :*

$E \to E_0$ be Borel measurable, and α be a transition function from E_0 into E ($y \in E_0 \to \alpha(y,\cdot) \in \mathcal{P}(E)$ is Borel measurable) satisfying $\alpha(y,\gamma^{-1}(y)) = 1$. Define

$$C = \left\{ \left(\int_E f(z)\alpha(\cdot, \mathrm{d}z), \int_E Bf(z)\alpha(\cdot,\mathrm{d}z) \right) : f \in \mathcal{D}(B) \right\}.$$

Let $\mu_0 \in \mathcal{P}(E_0)$, and define $\nu_0 = \int \alpha(y,\cdot)\mu_0(\mathrm{d}y)$. If $\widetilde{U}$ is a solution of the martingale problem for (C,μ_0), then there exists a solution V of the martingale problem for (B,ν_0) such that $\widetilde{U}$ has the same distribution on $M_{E_0}[0,\infty)$ as $U = \gamma \circ V$ and

$$P\{V(t) \in \Gamma | \widehat{\mathcal{F}}_t^U\} = \alpha(U(t),\Gamma), \quad \Gamma \in \mathcal{B}(E), t \in \mathbf{T}^U. \tag{27}$$

If $\widetilde{U}$ (and hence U) has a modification with sample paths in $D_E[0,\infty)$, then the modified $\widetilde{U}$ and U have the same distribution on $D_E[0,\infty)$.

Proof. See Corollary 3.5, Theorem 2.7, and Theorem 2.9d of [3]. □

To apply this result in the proof of Theorem 2.3, we must show that $\widehat{A}_n$ can be written in the form (26). Suppose that for each compact $K \subset \mathbb{R}^d$,

$$\sup_{x\in K} (|a(x)| + |b(x)| + \int_{S_1} |\gamma(x,u)|^2\nu(\mathrm{d}u) + \int_{S_2} |\gamma(x,u)| \wedge 1\nu(\mathrm{d}u) < \infty.$$

Let F_1 be the space of $d \times d$ nonnegative definite matrices with the usual matrix norm, $F_2 = \mathbb{R}^d$, and F_3 the space of $\mathbb{R}^d$-valued functions on S such that

$$\int_{S_1} |\gamma(u)|^2\nu(\mathrm{d}u) + \int_{S_2} |\gamma(u)| \wedge 1\nu(\mathrm{d}u)) < \infty.$$

We can define a metric on F_3 by

$$d_4(\gamma_1,\gamma_2) = \sqrt{\int_{S_1} |\gamma_1(u) - \gamma_2(u)|^2\nu(\mathrm{d}u)} + \int_{S_2} |\gamma_1(u) - \gamma_2(u)| \wedge 1\nu(\mathrm{d}u).$$

Then, $F = F_1 \times F_2 \times F_3$ is a complete, separable metric space, and for $v = (v^1, v^2, v^3) \in F$,

$$\begin{aligned} \mathbb{B}f(x,v) &= \frac{1}{2}\sum_{i,j=1}^d v_{ij}^1 \frac{\partial^2}{\partial x_i \partial x_j} f(x) + v^2 \cdot \nabla f(x) \\ &\quad + \int_S (f(x + v^3(u)) - f(x) - \mathbf{1}_{S_1}(u)v^3(u) \cdot \nabla f(x))\nu(\mathrm{d}u) \end{aligned} \tag{28}$$

is the generator of a Levy process in $\mathbb{R}^d$. Let

$$\eta(x,\cdot) = \delta_{(a(x),b(x),\gamma(x,\cdot))}.$$

Then,

$$Af(x) = \int \mathbb{B}f(x,v)\eta(x,\mathrm{d}v).$$

Similarly, we can define $\mathbb{B}_n$ to include Y and Z^n so that

$$\widehat{A}_n f(x,y,z) = \int \mathbb{B}_n f(x,y,z,v)\eta(x,\mathrm{d}v).$$

Appendix

Lemma A.2. *Let $A \subset B(E) \times B(E)$, and let X be a cadlag solution of the martingale problem for A. For each $f \in \mathcal{D}(A)$, define*

$$M_f(t) = f(X(t)) - \int_0^t Af(X(s))\mathrm{d}s.$$

Suppose $\mathcal{D}(A)$ is an algebra and that $f \circ X$ is cadlag for each $f \in \mathcal{D}(A)$. Let $f_1, \ldots, f_m \in \mathcal{D}(A)$ and $g_1, \ldots, g_m \in B(E)$. Then,

$$M(t) = \sum_{i=1}^{m} \int_0^t g_i(X(s-))\mathrm{d}M_{f_i}(s)$$

is a square integrable martingale with Meyer process

$$\begin{aligned}\langle M\rangle_t = \sum_{1\le i,j\le m} \int_0^t & g_i(X(s))g_j(X(s))(Af_i f_j(X(s)) - f_i(X(s))Af_j(X(s)) \\ & - f_j(X(s))Af_i(X(s)))\mathrm{d}s.\end{aligned}$$

Proof. The lemma follows by standard properties of stochastic integrals and the fact that

$$\langle M_{f_1}, M_{f_2}\rangle_t = \int_0^t (Af_1 f_2(X(s)) - f_1(X(s))Af_2(X(s)) - f_2(X(s))Af_1(X(s)))\mathrm{d}s.$$

This identity can be obtained by applying Itô's formula to $f_1(X(t))f_2(X(t))$ and the fact that $[f_1 \circ X, f_2 \circ X]_t = [M_{f_1}, M_{f_2}]_t$ to obtain

$$\begin{aligned}[M_{f_1}, M_{f_2}]_t = {} & f_1(X(t))f_2(X(t)) - f_1(X(0))f_2(X(0)) - \int_0^t Af_1 f_2(X(s))\mathrm{d}s \\ & - \int_0^t f_1(X(s-))\mathrm{d}M_{f_2}(s) - \int_0^t f_2(X(s-))\mathrm{d}M_{f_1}(s) \\ & + \int_0^t (Af_1 f_2(X(s)) - f_1(X(s))Af_2(X(s)) - f_2(X(s))Af_1(X(s)))\mathrm{d}s.\end{aligned}$$

Since the first five terms on the right give a martingale and the last term is predictable, the last term must be $\langle M_{f_1}, M_{f_2}\rangle_t$. □

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Accelerated Numerical Schemes for PDEs and SPDEs

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Abstract We give a survey of some results from Gyöngy and Krylov (An accelerated splitting-up method for parabolic equations [10], Cauchy problems with periodic controls [11], Expansion of solutions of arameterized equations and acceleration of numerical methods [12], Accelerated finite difference schemes for second order degenerate elliptic and parabolic problems in the whole space [13], Accelerated finite difference schemes for linear stochastic partial differential equations in the whole space [14]) on accelerated numerical schemes for some classes of deterministic and stochastic PDEs. First, we consider monotone finite difference schemes for parabolic (possibly degenerate) PDEs in the spatial variable. We present some theorems from Gyöngy and Krylov ("Accelerated finite difference schemes for second order degenerate elliptic and parabolic problems in the whole space, to appear in Mathematics of Computation") on power series expansions of finite difference approximations in terms of the mesh-size of the grid. These theorems imply that one can accelerate the convergence of finite difference approximations (in the spatial variables) to any order by taking suitable mixtures of approximations corresponding to different mesh-sizes. We extend these results to degenerate elliptic equations in spaces with supremum norm. Then, we establish power series expansions for finite difference approximations of linear stochastic PDEs, and hence we get, as before that the rate of convergence of these approximations can be accelerated to any order, provided the data and the coefficients of the equations are sufficiently smooth. Finally, for a large class of equations and various types of time discretizations for them, we present some results from Gyöngy and Krylov [10–12] on power series expansion in the parameters of the approximations and get theorems on their acceleration.

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1 Introduction

About a century ago, L.F. Richardson introduced an idea of accelerating the convergence of numerical solutions to deterministic equations. He implemented it in improving the accuracy of finite difference approximations for some parabolic PDE, which he wanted to solve as accurately as one could in the time when no computer existed. Richardson's idea is based on the assumption that finite difference approximations admit power series expansions in term of the mesh-size so that in appropriate mixtures of approximations corresponding to different mesh-sizes, the lower order terms vanish. Richardson showed that his idea works in some cases and demonstrated its usefulness in [28] and [29]. His method, often called *Richardson's method* or *extrapolation to the limit*, became an important method of modern numerical analysis, and his idea is applied to various types of approximations. The reader is referred to the survey papers [2] and [4] for a review on the history of the method and on the scope of its applicability and to the textbooks (for instance, [23] and [24]) concerning finite difference methods and their accelerations.

In the present paper, we give a survey of some of our recent results from [10]–[14] on accelerated numerical schemes for linear PDEs and stochastic PDEs. These results revolve around the possibility of power series expansions of approximations for the solutions in terms of the parameter of the discretizations of equations under consideration. First, in Sect. 2, we consider monotone finite difference schemes for parabolic PDEs in the spatial variable. We present some theorems from [13] on power series expansions of finite difference approximations in terms of the mesh-size of the grid. These theorems imply that if the coefficients and the data of the equations are sufficiently regular, then one can accelerate the convergence of finite difference approximations (in the spatial variables) to any order by taking suitable mixtures of approximations corresponding to different mesh-sizes. We also give conditions under which the constants in our estimates are independent of the time variable; it allows us to extend our theorems on expansions and accelerations to elliptic equations in Sect. 3. Our results seem to be the first ones to justify Richardson's method for degenerate elliptic and parabolic equations in spaces with supremum norm.

In Sect. 4, we consider stochastic linear PDEs of parabolic type and finite difference schemes for them in the spatial variables. The finite difference schemes are quite general; we do not require them to be monotone. We establish again a power series expansions in terms of the mesh-sizes and give an estimate of the second moment of suitable Sobolev norms of the remainders. Hence, as before, we get that if the coefficients and the free data are sufficiently regular, then by Richardson's method we can accelerate the rate of convergence of finite difference approximations to any order.

We note that there are some well-known applications of Richardson's method to stochastic differential equations (SDE) in the literature. In particular, in [33], and in its generalisations [19] and [16], *week convergence* of Euler's approximations of SDEs are accelerated. In contrast, here we consider acceleration of *strong convergence* of finite difference approximations in the space variable for stochastic parabolic equations. Our results have obvious applications in nonlinear filtering of partially observed diffusion processes. As far as we know, these are the first results in this direction.

In Sects. 2–4, we consider equations in the whole space having in mind treating equations in bounded smooth domains in a subsequent article. In connection with this, it may be worth noting that the results we present are applicable, for example, to the one dimensional ODE

$$(1-x^2)^2u''(x)-c(x)u(x)=f(x), \quad x\in(-1,1).$$

The point is that one need not prescribe any boundary value of u at the points ±1, and if one considers this equation on all of $\mathbb{R}$, the values of its coefficients and f outside $(-1,1)$ do not affect the values of $u(x)$ for $|x|<1$. Another rather standard example even of a uniformly nondegenerate equation with constant coefficients is the following. Consider the one-dimensional heat equation

$$u_t=u_{xx}, \quad t\geq 0, \quad u(0,x)=\cos x.$$

Then, the true solution is

$$u(t,x)=\mathrm{e}^{-t}\cos x, \quad u(1,0)=\mathrm{e}^{-1}=0.3678794,$$

and the obvious symmetric finite difference scheme in the space variable with mesh-size $h=1/2$ and $h=1/4$ give the approximations

$$v(1,0)=0.3755591, \quad \text{and} \quad w(1,0)=0.3697965,$$

respectively, for $u(1,0)$. These approximations are accurate up to the *second* digit only, whereas the Richardson's accelerated approximation

$$4w(1,0)/3-v(1,0)/3=0.3678756$$

gives the correct value up to the *fifth (out of seven)* digit. By the way, if one does not use the acceleration, then one gets such accuracy for $h=0.011$, which is almost 25 times smaller than $1/4$.

Finally, in Sect. 5, we consider a class of deterministic equations and various types of time discretizations for them. The class of equations includes linear parabolic PDEs, systems of parabolic PDEs, in particular symmetric hyperbolic systems of first order PDEs. The numerical approximations we consider include splitting-up approximations and finite difference schemes in time. We cite some results from [11] and [12] on power series expansion of numerical approximations in the parameters of the approximations. In applications, these parameters play the role of the mesh-size of time grids. Hence we get acceleration results for those numerical approximations that belong to the class of schemes presented here. We refer to [12], where more general equations and numerical schemes are considered, and more general theorems on power series expansions are proved.

Clearly, by combining the theorems of Sects. 2 and 5, one can get expansions and accelerations for various *fully discretized* parabolic PDEs and systems of parabolic PDEs. It is natural to look also for accelerated space and time discretized schemes of stochastic PDEs, say by using time discretization to solve the stochastic differential equations obtained after space discretizations. One knows, however, that if the values of the driving multidimensional Wiener process are given only on the grid, then, in general, one cannot have a numerical scheme for SDEs with (strong) rate of convergence better than $\sqrt{\tau}$, where τ is the mesh-size of the grid. On the other hand, in some particular cases, for example, when the Wiener process is one-dimenional or some special data, such as iterated stochastic integrals of the components of the Wiener processes are available, then one can have accelerated fully discretised numerical schemes for SPDEs. (See, e.g., [15] for high order strong approximations of stochastic differential equations when appropriate iterated stochastic integrals of the Wiener processes are used in the numerical schemes.)

2 Monotone Finite Difference Schemes for Parabolic Equations

We fix some numbers $h_0, T \in (0, \infty)$, and for each number $h \in (0, h_0]$, we consider the integral equation

$$u(t,x) = g(x) + \int_0^t \big(L_h u(s,x) + f(s,x)\big)\,\mathrm{d}s, \quad (t,x) \in H_T \tag{1}$$

for u, where $g(x)$ and $f(s,x)$ are given real-valued Borel functions of $x \in \mathbb{R}^d$ and $(s,x) \in H_T = [0,T] \times \mathbb{R}^d$, respectively, and L_h is a linear operator defined by

$$L_h\varphi(t,x) = L_h^0\varphi(t,x) - c(t,x)\varphi(x), \tag{2}$$

$$L_h^0\varphi(t,x) = \frac{1}{h}\sum_{\lambda\in\Lambda_1} q_\lambda(t,x)\delta_{h,\lambda}\varphi(x) + \sum_{\lambda\in\Lambda_1} p_\lambda(t,x)\delta_{h,\lambda}\varphi(x), \tag{3}$$

for functions φ on $\mathbb{R}^d$. Here Λ_1 is a finite subset of $\mathbb{R}^d$ such that $0 \notin \Lambda_1$,

$$\delta_{h,\lambda}\varphi(x) = \frac{1}{h}(\varphi(x+h\lambda) - \varphi(x)), \quad \lambda \in \Lambda_1,$$

$q_\lambda(t,x) \geq 0$, $p_\lambda(t,x)$, and $c(t,x)$ are given real-valued Borel functions of $(t,x) \in H_\infty = [0,\infty) \times \mathbb{R}^d$ for each $\lambda \in \Lambda_1$. Set $|\Lambda_1|^2 = \sum_{\lambda\in\Lambda_1} |\lambda|^2$.

As usual, we denote

$$D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}, \quad D_i = \frac{\partial}{\partial x_i}, \quad |\alpha| = \sum_i \alpha_i, \quad D_{ij} = D_i D_j$$

for multi-indices $\alpha = (\alpha_1, \dots \alpha_d)$, $\alpha_i \in \{0,1,\dots\}$. For smooth φ and integers $k \geq 0$, we introduce $D^k\varphi$ as the collection of partial derivatives of φ of order k, and define

$$|D^k\varphi|^2 = \sum_{|\alpha|=k} |D^\alpha\varphi|^2, \quad [\varphi]_k = \sup_{x\in\mathbb{R}^d} |D^k\varphi(x)|, \quad |\varphi|_k = \sum_{i\leq k} [\varphi]_i.$$

For functions ψ_h depending on $h \in (0,h_0]$, the notation $D_h^k\psi_h$ means the kth derivative of ψ in h. For Borel measurable bounded functions $\psi = \psi(t,x)$ on H_T, we write $\psi \in \mathfrak{B}^m = \mathfrak{B}_T^m$ if, for each $t \in [0,T]$, $\psi(t,x)$ is continuous in $\mathbb{R}^d$ and for all multi-indices α with $|\alpha| \leq m$ the generalized functions $D^\alpha\psi(t,x)$ are bounded on H_T. In this case, we use the notation

$$\|\psi\|_m^2 = \sup_{H_T} \sum_{|\alpha|\leq m} |D^\alpha\psi(t,x)|^2.$$

This notation will be also used for functions ψ independent of t.

Let $m \geq 0$ be a fixed integer. We make the following assumptions.

Assumption 2.1. *For any $\lambda \in \Lambda_1$, we have $p_\lambda, q_\lambda, c, f, g \in \mathfrak{B}^m$, and for $k = 0,\dots,m$ and some constants M_k, we have*

$$\sup_{H_T} \Big(\sum_{\lambda\in\Lambda_1} (|D^k q_\lambda|^2 + |D^k p_\lambda|^2) + |D^k c|^2 \Big) \leq M_k^2. \tag{4}$$

Remark 1. One can show, see Theorem 2.3 of [6], that under Assumption 2.1 for each $h \in (0,h_0]$, there exists a unique bounded solution u_h of (1); this solution is continuous in H_T, and all its derivatives in x up to order m are bounded. Actually, in Theorem 2.3 of [6], it is required that the derivatives of the data up to order m be continuous in H_T, but its proof can be easily adjusted to include our case (see Remark 5 below).

We view (1) as a finite difference scheme for the problem

$$\frac{\partial}{\partial t} u(t,x) = \mathcal{L}u(t,x) + f(t,x), \quad t \in (0,T],\ x \in \mathbb{R}^d, \tag{5}$$

$$u(0,x) = g(x), \quad x \in \mathbb{R}^d, \tag{6}$$

where

$$\mathcal{L} := \tfrac{1}{2} \sum_{\lambda \in \Lambda_1} \sum_{i,j=1}^{d} q_\lambda \lambda_i \lambda_j D_i D_j + \sum_{\lambda \in \Lambda_1} \sum_{i=1}^{d} p_\lambda \lambda_i D_i - c. \tag{7}$$

For this reason, we need the following assumption.

Assumption 2.2. *For all* $(t,x) \in H_T$

$$\sum_{\lambda \in \Lambda_1} \lambda q_\lambda(t,x) = 0.$$

Notice that under Assumption 2.2 for continuous functions φ on H_T, sufficiently smooth in $x \in \mathbb{R}^d$, we have

$$\lim_{h \to 0} L_h \varphi(t,x) = \mathcal{L}\varphi(t,x) \quad \text{for } (t,x) \in H_T,$$

which can be seen from the Taylor formulas

$$\sum_{\lambda \in \Lambda_1} p_\lambda \delta_{h,\lambda} \varphi = \sum_{\lambda \in \Lambda_1} p_\lambda \int_0^1 \partial_\lambda \varphi(t, x + h\theta\lambda)\,\mathrm{d}\theta,$$

$$\sum_{\lambda \in \Lambda_1} h^{-1} q_\lambda \delta_{h,\lambda} \varphi = \sum_{\lambda \in \Lambda_1} q_\lambda \int_0^1 (1-\theta) \partial_\lambda^2 \varphi(t, x + h\theta\lambda)\,\mathrm{d}\theta,$$

where $\partial_\lambda \varphi = \sum_{i=1}^d \lambda_i D_i \varphi$ denotes the derivative of φ in the direction of $\lambda = (\lambda_1, \dots, \lambda_d)$.

Remark 2. We often use also the following symmetry conditions:

(S) $\Lambda_1 = -\Lambda_1$ and $q_\lambda = q_{-\lambda}$ for all $\lambda \in \Lambda_1$.

(P) $\Lambda_1 = -\Lambda_1$ and $p_\lambda = -p_{-\lambda}$ for all $\lambda \in \Lambda_1$.

Clearly condition (S) implies Assumption 2.2. Moreover, if condition (S) holds, then

$$h^{-1} \sum_{\lambda \in \Lambda_1} q_\lambda(t,x) \delta_{h,\lambda} \varphi(x) = (1/2) \sum_{\lambda \in \Lambda_1} q_\lambda(t,x) \Delta_{h,\lambda} \varphi(x),$$

where

$$\Delta_{h,\lambda} \varphi(x) = h^{-2}(\varphi(x + h\lambda) - 2\varphi(x) + \varphi(x - h\lambda)).$$

By a solution of (5)–(6), we mean a bounded continuous function $u(t,x)$ on H_T, such that it belongs to $\mathfrak{B}^2$ and satisfies

$$u(t,x) = g(x) + \int_0^t [\mathcal{L}u(s,x) + f(s,x)]\,\mathrm{d}s \tag{8}$$

in H_T in the sense of generalized functions, that is, for any $t \in [0, T]$ and $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \phi(x)u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \phi(x)g(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} \phi(-cu+f)(s,x)\,\mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{R}^d} \phi \sum_{\lambda\in\Lambda_1} \Big(\tfrac{1}{2} \sum_{i,j=1}^d q_\lambda \lambda_i \lambda_j D_i D_j u$$
$$+ \sum_{i=1}^d p_\lambda \lambda_i D_i u\Big)(s,x)\,\mathrm{d}x\mathrm{d}s. \tag{9}$$

Observe that if $u \in \mathfrak{B}^2$, then (9) implies that (8) holds almost everywhere with respect to x, and if $u \in \mathfrak{B}^3$, then the second derivatives of u in x are continuous in x and (8) holds everywhere.

We remark that it is shown in [20] that in all practically interesting cases of parabolic equations such as (8), the operator $\mathcal{L}$ can be represented as in (7), which justifies considering operators L_h^0 in form (3).

The following theorem on existence and uniqueness of solutions is a classical result (see, for instance, [25–27]). In [13], we give an elementary proof by using finite-difference approximations.

Theorem 1. *Let Assumption 2.1 hold with $m \geq 2$. Then, (8) has a unique solution $u^{(0)} \in \mathfrak{B}^2 = \mathfrak{B}^2_T$. Moreover, $u^{(0)} \in \mathfrak{B}^m_T$ and*

$$\|u^{(0)}\|_m \leq N(\|f\|_m + \|g\|_m), \tag{10}$$

where N is a constant, depending only on d, m, $|\Lambda_1|$, $M_0, \dots, M_m$, and T.

Assuming now that Assumption 2.1 holds with $m \geq 2$, we want to know if u_h, the unique bounded solution of (1) (see Remark 1) admits an expansion

$$u_h(t,x) = u^{(0)}(t,x) + \sum_{1\leq j\leq k} \frac{h^j}{j!} u^{(j)}(t,x) + h^{k+1} r_h(t,x), \tag{11}$$

for all $(t,x) \in H_T$ and $h \in (0, h_0]$, where $u^{(0)}$ the unique solution of (8), $u^{(1)}, \dots, u^{(k)}$ are some functions on H_T, independent of h, and r_h is a function on H_T defined for each $h \in (0, h_0]$ such that

$$|r_h(t,x)| \leq N(\|f\|_m + \|g\|_m) \tag{12}$$

for all $(t,x) \in H_T$, $h \in (0, h_0]$ with some constant N independent of h.

To formulate our first result concerning the above expansion, we introduce

$$\chi_{h,\lambda} = q_\lambda + h p_\lambda,$$

and the following assumption.

Assumption 2.3. *For all* $(t, x) \in H_T$, $h \in (0, h_0]$, *and* $\lambda \in \Lambda_1$,

$$\chi_{h,\lambda}(t, x) \geq 0. \tag{13}$$

Theorem 2. *Let Assumption 2.1 with* $m \geq 3$ *and Assumption 2.3 hold. Let* $k \geq 0$ *be an integer. Then, expansion (11) holds with* r_h *satisfying (12), provided one of the following conditions is met:*

(i) $m \geq 3k + 3$ *and Assumption 2.2 holds*
(ii) $m \geq 2k + 3$ *and condition (S) holds*
(iii) k *is odd,* $m \geq 2k + 2$, *and conditions (S) and (P) are satisfied.*

In each of the cases (i)–(iii), the constant N *depends only on* d, m, $|\Lambda_1|$, $M_0, \ldots, M_m$, *and* T. *In case (iii), we have* $u^{(j)} = 0$ *for all odd* j *in expansion (11).*

The following corollary is one of the results of [3] proved there by using the theory of diffusion processes. It follows immediately from case (iii) of the above theorem with $k = 1$. Of course, the result is well known for uniformly nondegenerate equations, but we do not assume any nondegeneracy of $\mathcal{L}$, which becomes just a zero operator at those points where $q_\lambda = p_\lambda = c = 0$.

Corollary 1. *Let conditions (S) and (P) be satisfied. Let Assumption 2.1 with* $m = 4$ *and Assumption 2.3 hold. Then, we have* $|u_h - u_0| \leq N h^2$.

Actually, in [3], a full discretization in time and space is considered for parabolic equations, so that, formally, Corollary 1 does not yield the corresponding result of [3]. On the other hand, a similar corollary can be derived from Theorem 13 below, which treats elliptic equations, and it does imply the corresponding result of [3]. It also generalizes it because in [3], one of the assumptions, unavoidable for the methods used there, is that $q_\lambda = r_\lambda^2$ with functions r_λ that have four bounded derivatives in x, which may easily be not the case under the assumptions of Theorem 13.

To formulate a result implied by Theorem 2 on acceleration for parabolic equations, we fix an integer $k \geq 0$ and set

$$\bar{u}_h = \sum_{j=0}^{k} \rho_j u_{2^{-j}h}, \tag{14}$$

where, naturally, $u_{2^{-j}h}$ are the solutions to (1), with $2^{-j}h$ in place of h,

$$(\rho_0, \rho_1, \ldots, \rho_k) := (1, 0, 0, \ldots, 0)V^{-1} \tag{15}$$

and V^{-1} is the inverse of the Vandermonde matrix with entries

$$V^{ij} := 2^{-(i-1)(j-1)}, \quad i, j = 1, \ldots, k + 1.$$

The following result is a simple corollary of Theorem 2.

Theorem 3. *In each situation when Theorem 2 is applicable, we have that the estimate*

$$|\bar{u}_h(t,x) - u^{(0)}(t,x)| \leq N(\|f\|_m + \|g\|_m)h^{k+1} \tag{16}$$

holds for all $(t,x) \in H_T$, $h \in (0,h_0]$, *where* N *is a constant depending only on* d, m, $|\Lambda_1|$, $M_0, \ldots, M_m$, *and* T.

Proof. By Theorem 2

$$u_{2^{-j}h} = u^{(0)} + \sum_{i=1}^{k} \frac{h^i}{i!2^{ji}} u^{(i)} + \bar{r}_{2^{-j}h} h^{k+1}, \quad j = 0, 1, \ldots, k,$$

with $\bar{r}_{2^{-j}h} := 2^{-j(k+1)} r_{2^{-j}h}$, which gives

$$\bar{u}_h = \sum_{j=0}^{k} \rho_j u_{2^{-j}h} = (\sum_{j=0}^{k} \rho_j) u^{(0)} + \sum_{j=0}^{k} \sum_{i=1}^{k} \rho_j \frac{h^i}{i!2^{ij}} u^{(i)} + \sum_{j=0}^{k} \rho_j \bar{r}_{2^{-j}h} h^{k+1}$$

$$= u^{(0)} + \sum_{i=1}^{k} \frac{h^i}{i!} u^{(i)} \sum_{j=0}^{k} \frac{\rho_j}{2^{ij}} + \sum_{j=0}^{k} \rho_j \bar{r}_{2^{-j}h} = u^{(0)} + \sum_{j=0}^{k} \rho_j \bar{r}_{2^{-j}h} h^{k+1},$$

since

$$\sum_{j=0}^{k} \rho_j = 1, \quad \sum_{j=0}^{k} \rho_j 2^{-ij} = 0, \quad i = 1, 2, \ldots k$$

by the definition of $(\rho_0, \ldots, \rho_k)$. Hence,

$$\sup_{H_T} |\bar{u}_h - u^{(0)}| = \sup_{H_T} |\sum_{j=0}^{k} \rho_j \bar{r}_{2^{-j}h}| h^{k+1} \leq N(\|f\|_m + \|g\|_m) h^{k+1},$$

and the theorem is proved. □

Sometimes, it suffices to combine fewer terms $u_{2^{-j}h}$ to get accuracy of order $k+1$. To consider such a case for integers $k \geq 0$ define

$$\tilde{u}_h = \sum_{j=0}^{\tilde{k}} \tilde{\rho}_j u_{2^{-j}h},$$

where

$$(\tilde{\rho}_0, \tilde{\rho}_1, \ldots, \tilde{\rho}_{\tilde{k}}) := (1, 0, 0, \ldots, 0)\tilde{V}^{-1}, \quad \tilde{k} = [\tfrac{k}{2}], \tag{17}$$

and $\tilde{V}^{-1}$ is the inverse of the Vandermonde matrix with entries

$$\tilde{V}^{ij} := 4^{-(i-1)(j-1)}, \quad i, j = 1, \ldots, \tilde{k} + 1.$$

Theorem 4. *Suppose that the assumptions of Theorem 2 are satisfied and condition (iii) is met. Then, for $\tilde{u}_h$, we have*

$$\sup_{H_T} |u^{(0)} - \tilde{u}_h| \leq N(\|f\|_m + \|g\|_m)h^{k+1}$$

for all $h \in (0, h_0]$, where N depends only on d, m, $|\Lambda_1|$, $M_0, \ldots, M_m$, and T.

Proof. We obtain this result from Theorem 2 by a straightforward modification of the proof of the previous result, taking into account that for odd j, the terms with h^j vanish in expansion (11) when condition (iii) holds in Theorem 2. □

Example 1. Assume that in the situation of Theorem 4, we have $m = 8$. Then,

$$\tilde{u}_h := \tfrac{4}{3}u_{h/2} - \tfrac{1}{3}u_h$$

satisfies

$$\sup_{H_T} |u^{(0)} - \tilde{u}_h| \leq Nh^4$$

for all $h \in (0, h_0]$.

The above results show that if the data in (8) are sufficiently smooth, then the order of accuracy in approximating the solution $u^{(0)}$ can be as high as we wish if we use suitable mixtures of finite difference approximations calculated along nested grids with different mesh-sizes. Assume now that we need to approximate not only $u^{(0)}$ but its derivative $D^\alpha u^{(0)}$ for some multi-index α as well. What accuracy can we achieve? The answer is closely related to the question whether the expansion

$$D^\alpha u_h(t,x) = D^\alpha u^{(0)}(t,x) + \sum_{1 \leq j \leq k} \frac{h^j}{j!} D^\alpha u^{(j)}(t,x) + h^{k+1} D^\alpha r_h(t,x) \quad (18)$$

holds for all $(t,x) \in H_T$ and $h \in (0, h_0]$, such that

$$|D^\alpha r_h(t,x)| \leq N(\|f\|_m + \|g\|_m) \quad (19)$$

for all $(t,x) \in H_T$, $h \in (0, h_0]$. It is also natural to extend the above theorems from the parabolic to the elliptic case. The rather serious difficulty is that the constant in our estimates depends on T. To overcome this difficulty, we introduce some more notation and assumptions and investigate the smoothness of u_h in x. As a simple byproduct of this investigation, we also obtain smoothness of u_h with respect to h, which, by the way, cannot be derived from (11).

Let τ_λ be a function defined on Λ_1 taking values in $[0, \infty)$, and for $\lambda \in \Lambda_1$, introduce the operators

$$T_{h,\lambda}\varphi(x) = \varphi(x + h\lambda), \quad \bar{\delta}_{h,\lambda} = \tau_\lambda h^{-1}(T_{h,\lambda} - 1).$$

Set

$$\|\Lambda_1\|^2 = \sum_{\lambda \in \Lambda_1} |\tau_\lambda \lambda|^2.$$

For uniformity of notation, we also introduce Λ_2 as the set of fixed distinct vectors $\ell^1, \dots, \ell^d$, none of which is in Λ_1 and define

$$\bar{\delta}_{h,\ell^i} = \tau_0 D_i, \quad T_{h,\ell^i} = 1, \quad \Lambda = \Lambda_1 \cup \Lambda_2,$$

where $\tau_0 > 0$ is a fixed parameter. For $\lambda = (\lambda^1, \lambda^2) \in \Lambda^2$, introduce the operators

$$T_{h,\lambda} = T_{h,\lambda^1} T_{h,\lambda^2}, \quad \bar{\delta}_{h,\lambda} = \bar{\delta}_{h,\lambda^1} \bar{\delta}_{h,\lambda^2}.$$

For $k = 1, 2$, $\mu \in \Lambda^k$, we set

$$Q_{h,\mu}\varphi = h^{-1} \sum_{\lambda \in \Lambda_1} (\bar{\delta}_{h,\mu} q_\lambda) \delta_\lambda \varphi, \quad L^0_{h,\mu}\varphi = Q_{h,\mu}\varphi + \sum_{\lambda \in \Lambda_1} (\bar{\delta}_{h,\mu} p_\lambda) \delta_\lambda \varphi,$$

$$A_h(\varphi) = 2 \sum_{\lambda \in \Lambda} (\bar{\delta}_{h,\lambda}\varphi) L^0_{h,\lambda} T_{h,\lambda}\varphi, \quad \mathcal{Q}_h(\varphi) = \sum_{\lambda \in \Lambda_1} \chi_{h,\lambda} (\delta_{h,\lambda}\varphi)^2.$$

Below $B(\mathbb{R}^d)$ is the set of bounded Borel functions on $\mathbb{R}^d$, and $\mathfrak{K}$ is the set of bounded operators $\mathcal{K}_h = \mathcal{K}_h(t)$ mapping $B(\mathbb{R}^d)$ into itself preserving the cone of nonnegative functions and satisfying $\mathcal{K}_h 1 \le 1$.

Finally, we fix some constants $\delta \in (0, 1]$ and $K \in [1, \infty)$ and remark that below $\mathcal{K} = \mathcal{K}_h = \mathcal{K}_h(t)$ denotes generic elements from operators from $\mathfrak{K}$, which may be different from one appearance to another.

Assumption 2.4. *There exists a constant* $c_0 > 0$ *such that* $c \ge c_0$.

This assumption is almost irrelevant if we only consider (1) on a finite time interval. Indeed, if c is just bounded, say $|c| \le C = \text{const}$, by introducing a new function $v(t, x) = u(t, x)e^{-2Ct}$, we will have an equation for v similar to (1) with $L^0_h v - (c + 2C)v$ and $f e^{-2Ct}$ in place of $L_h u$ and f, respectively. Now for the new c, we have $c + 2C \ge C$.

Assumption 2.5. *We have* $m \ge 1$, *and for any* $h \in (0, h_0]$, *there exists an operator* $\mathcal{K}_h = \mathcal{K}_{h,m} \in \mathfrak{K}$, *such that*

$$mA_h(\varphi) \le (1 - \delta) \sum_{\lambda \in \Lambda} \mathcal{Q}_h(\bar{\delta}_{h,\lambda}\varphi) + K\mathcal{Q}_h(\varphi) + 2(1 - \delta)c\mathcal{K}_h\Big(\sum_{\lambda \in \Lambda} |\bar{\delta}_{h,\lambda}\varphi|^2\Big) \quad (20)$$

on H_T *for all smooth functions* φ.

Assumption 2.6. *We have* $m \ge 2$, *and for any* $h \in (0, h_0]$ *and* $n = 1, \dots, m$, *there exist operators* $\mathcal{K}_h = \mathcal{K}_{h,n} \in \mathfrak{K}$, *such that*

$$n\sum_{\nu\in\Lambda} A_h(\bar{\delta}_{h,\nu}\varphi) + n(n-1)\sum_{\lambda\in\Lambda^2}(\bar{\delta}_{h,\lambda}\varphi)Q_{h,\lambda}T_{h,\lambda}\varphi \le (1-\delta)\sum_{\lambda\in\Lambda^2}\mathcal{Q}_h(\bar{\delta}_{h,\lambda}\varphi)$$

$$+ K\sum_{\lambda\in\Lambda}\mathcal{Q}_h(\bar{\delta}_{h\lambda}\varphi) + 2(1-\delta)c\mathcal{K}_h\Big(\sum_{\lambda\in\Lambda^2}|\bar{\delta}_{h,\lambda}\varphi|^2\Big) + K\mathcal{K}_h\Big(\sum_{\lambda\in\Lambda}|\bar{\delta}_{h,\lambda}\varphi|^2\Big) \quad (21)$$

on H_T for all smooth functions φ.

Obviously, Assumptions 2.5 and 2.6 are satisfied if q_λ and p_λ are independent of x. In the general case, as it is discussed in [6], not only do the above assumptions impose analytical conditions, but they are related also to some structural conditions, which can somewhat easier be analized under the symmetry condition (S).

Assumption 2.7. *For all $t \in [0, T]$,*

$$\sum_{\lambda\in\Lambda_1}\lambda q_\lambda(t,x) \quad \textit{is independent of} \quad x. \quad (22)$$

In the main case of applications, we will require the last sum to be identically zero as in Assumption 2.2.

Remark 3. Assumptions 2.5 and 2.6 are discussed at length and in many details in [6] and [7], and sufficient conditions, without involving test functions φ are given for these assumptions to be satisfied. In particular, it is shown in [7] that if condition (S) holds, $m \ge 2$, $\tau_\lambda = 1$, Assumptions 2.1 and 2.3 are satisfied, and $q_\lambda \ge \kappa$ for a constant $\kappa > 0$, then both Assumptions 2.5 and 2.6 are satisfied for any $c_0 > 0$ and $\delta \in (0,1)$, if h_0 is sufficiently small and τ_0, K, and $\mathcal{K}$ are chosen appropriately. Moreover, the condition $\kappa > 0$ can be dropped, provided, additionally, that c_0 is large enough (this time we need not assume that h is small). Remember, that the condition that c_0 be large is, actually, harmless as long as we are concerned with equations on a finite time interval. Mixed situations, when c is large at those points where some of q_λ can vanish, are also considered in [7].

In [6], we have seen that Assumption 2.5 imposes certain nontrivial *structural* conditions on q_λ which cannot be guaranteed by the size of c_0 if q_λ is only once continuously differentiable. In contrast, even without condition (S), given that Assumptions 2.1, 2.5, and 2.7 are satisfied and $m \ge 2$, as is shown in [7], Assumption 2.6 is also satisfied if c_0 is large enough.

Theorem 5. *Let Assumption 2.1 through 2.6 hold with $m \ge 3$. Let $k \ge 0$ and $l \in [0, m]$ be integers. Then, for every multi-index α such that $|\alpha| \le l$, the function $D^\alpha u_h$ is a continuous function on H_T, and expansion (18) holds with $D^\alpha r_h$ satisfying (19), provided one of the following conditions is met:*

(i) $m \ge 3k + 3 + l$
(ii) $m \ge 2k + 3 + l$ and condition (S) holds
(iii) k is odd, $m \ge 2k+2+l$, and conditions (S) and (P) are satisfied. In each of the cases (i)–(iii), the constant N depends only on d, m, δ, K, τ_0, c_0, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$. In case (iii), we have $u^{(j)} = 0$ for all odd j in the expansion.

The following is an obvious consequence of Theorem 5. (See the proof of Theorem 3.)

Corollary 2. *Suppose that the assumptions of Theorem 5 are satisfied. Then,*

$$\sup_{H_T} |D^\alpha \bar{u}_h - D^\alpha u^{(0)}| \le N h^{k+1}(\|f\|_m + \|g\|_m),$$

and if condition (iii) is met, then

$$\sup_{H_T} |D^\alpha \tilde{u}_h - D^\alpha u^{(0)}| \le N h^{k+1}(\|f\|_m + \|g\|_m),$$

where N depends only on on d, m, δ, K, τ_0, c_0, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$.

Remark 4. Observe that for $k = 0$ Theorem 5 implies that

$$\sup_{H_T} |D^\alpha u_h - D^\alpha u^{(0)}| \le N h \tag{23}$$

if $m \ge 3 + |\alpha|$ and Assumption 2.1 through 2.6 hold. In addition, one can replace $D^\alpha u_h$ in (23) with δ_h^α, where

$$\delta_h^\alpha = \delta_{h,e_1}^{\alpha_1} \cdot \ldots \cdot \delta_{h,e_d}^{\alpha_d}$$

and e_i is the ith basis vector in $\mathbb{R}^d$. This follows easily from the mean value theorem and Theorem 6 below. The reader understands that similar assertion is true in case of Corollary 2 with the only difference that one needs larger m and better finite-difference approximations of D^α.

Next, we investigate the smoothness of u_h in x and h. Recall that for functions φ depending on h, we use the notation $D_h^r \varphi$ for the rth derivative of φ in h. As usual, $D_h^0 \varphi := \varphi$.

Remark 5. Suppose that Assumption 2.1 is satisfied. Take an $h_1 \in (0, h_0)$, consider (1) as an equation about a function $u_h(t,x)$ as function of $(h,t,x) \in [h_1, h_0] \times H_T$, and look for solutions in the space $\mathfrak{B}^m(h_1) = \mathfrak{B}_T^m(h_1)$, which is defined as the space of functions on $[h_1, h_0] \times H_T$ with finite norm

$$\sum_{|\alpha| + 3r \le m} \sup_{[h_1, h_0] \times H_T} |D^\alpha D_h^r u_h(t,x)|. \tag{24}$$

It is obvious that the integrand in (1) can be considered as the result of application of an operator, which is bounded in $\mathfrak{B}^m(h_1)$, to $u_h(s,x)$. Therefore, a standard abstract theorem on solvability of ODEs in Banach spaces shows that there exists a solution of (1) in $\mathfrak{B}^m(h_1)$. Since just bounded solutions are uniquely defined by (1), we conclude that our u_h belongs to $\mathfrak{B}^m(h_1)$ for any $h_1 \in (0, h_0)$. Obviously, if the derivatives of the data are continuous in x, the same will hold for u_h.

The above argument, actually, works if we replace $|\alpha|+3r \leq m$ with $|\alpha|+r \leq m$ in (24). We talk about (24) in the above form because we will show that under our future assumptions the quantity (24) is bounded independently of h_1.

Theorem 6. *Let $k \geq 0$ and $m \geq 2$ be integers and suppose that Assumptions 2.1 through 2.6 are satisfied. Then, for each integer $r \geq 0$ such that*

$$3k + r \leq m,$$

the generalized derivatives $D^r D_h^k u_h$ exist on $(0, h_0] \times H_T$, are bounded, and we have

$$|D^r D_h^k u_h| \leq N(\|f\|_m + \|g\|_m), \tag{25}$$

where N is a constant depending only on $m, \delta, c_0, \tau_0, K$, $M_0, \dots, M_m$, $|\Lambda_1|$, and $\|\Lambda_1\|$. In particular, $u_h \in \mathfrak{B}^m$ and

$$\|u_h\|_m \leq N(\|f\|_m + \|g\|_m).$$

From the above theorem, one can get, as simple corollary, the following theorem, which we use when we consider the elliptic case below.

Theorem 7. *Suppose that Assumptions 2.1 through 2.6 hold with $m \geq 2$. Then, the constant N in (10) depends only on $m, \delta, c_0, \tau_0, K$, $M_0, \dots, M_m$, $|\Lambda_1|$, and $\|\Lambda_1\|$ (thus, is independent of T). The same is true for the constants N in Theorems 2, 3, and 4.*

Additional information on the behaviour of $D^r D_h^k u_h$ for small h is provided by the following result.

Theorem 8. *Let $k \geq 1$ be an* odd *number and suppose that Assumptions 2.1 through 2.6 hold with $m \geq 3k + 1$. Assume that the symmetry conditions (S) and (P) are satisfied.*

Then, for any integer $r \geq 0$ such that

$$3k + r \leq m - 1,$$

we have

$$\sup_{H_T} |D^r D_h^k u_h| \leq N(\|f\|_m + \|g\|_m)h \tag{26}$$

for all $h \in (0, h_0]$, where N depends only on m, δ, c_0, τ_0, K, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$.

3 Acceleration for Elliptic Equations

Here we assume that p_λ, q_λ, c, and f are independent of t and turn now our attention to the equations

$$L_h v_h(x) + f(x) = 0 \quad x \in \mathbb{R}^d, \tag{27}$$

$$\mathcal{L}v(x) + f(x) = 0 \quad x \in \mathbb{R}^d, \tag{28}$$

where L_h and $\mathcal{L}_h$ are defined by (2)–(7) and (8).

Naturally by a solution of (28), we mean a function v on $\mathbb{R}^d$ such that it belongs to $\mathfrak{B}^2$, and (28) holds almost everywhere. Clearly, if a solution v belongs to $\mathfrak{B}^3$ and q_λ, p_λ, c, and f are continuous functions on $\mathbb{R}^d$, then (28) holds everywhere.

First, we prove the existence and uniqueness of the solutions of (27) and (28).

Theorem 9. *Suppose that Assumption 2.1 is satisfied with an $m \geq 0$ and let Assumptions 2.3 and 2.4 hold. Then, (27) has a unique bounded solution v_h. Moreover, v_h belongs to $\mathfrak{B}^m$.*

Proof. Observe that (27) is equivalent to

$$v_h(x) = h^2\xi(x)f(x) + \xi(x)\sum_{\lambda\in\Lambda_1}\chi_\lambda v_h(x+\lambda h),$$

where

$$\xi^{-1} = h^2 c + \sum_{\lambda\in\Lambda_1}\chi_\lambda.$$

Hence we can see that the existence and uniqueness of bounded solution of (27) follow by contraction principle. Using smooth successive iterations yields that $v_h \in \mathfrak{B}^m$. □

Theorem 10. *Let Assumptions 2.1 through 2.6 hold with an $m \geq 2$. Then, (28) has a unique solution v in the space $\mathfrak{B}^2$. Moreover, $v \in \mathfrak{B}^m$ and there is a constant N depending only on m, δ, c_0, τ_0, K, $M_0, \dots, M_m$, $|\Lambda_1|$, and $\|\Lambda_1\|$ such that*

$$\|v\|_m \leq N\|f\|_m. \tag{29}$$

Proof. First, we prove uniqueness. Let $v \in \mathfrak{B}^2$ satisfy (28) with $f = 0$. Take a constant $\nu > 0$, so small that $c - \nu \geq c_0/2$ and conditions (20) and (21) hold with $c - \nu$ and $\delta/2$ in place of c and δ, respectively. Then, for each $T > 0$, the function $u(t,x) := \mathrm{e}^{\nu t}v(x))$, $(t,x) \in H_T$, is a solution of class $\mathfrak{B}^2_T$ of the equation

$$\frac{\partial}{\partial t}u = (\mathcal{L} + \nu)u \quad \text{on } H_T \tag{30}$$

with initial condition $u(0,x) = v(x)$. Hence by virtue of Theorem 7 for every $T>0$,

$$e^{\nu T}|v(x)| = |u(T,x)| \le N\|v\|_2,$$

where N is independent of (T, x). Multiplying both sides of the above inequality by $e^{-\nu T}$ and letting $T \to \infty$, we get $v = 0$, which proves uniqueness.

To show the existence of a solution in $\mathfrak{B}^m$, let u be a function defined on H_∞ such that for each $T > 0$ its restriction onto H_T is the unique solution in $\mathfrak{B}^m_T$ of (30) with initial condition $u(0, x) = f(x)$ (see Theorem 1). By Theorem 7

$$\sup_{H_\infty} \sum_{r\le m} |D^r u| \le N\|f\|_m$$

with a constant N depending only on $m, \delta, c_0, \tau_0, K, M_0, \dots, M_m, |\Lambda_1|$, and $\|\Lambda_1\|$. Hence

$$v(x) := \int_0^\infty e^{-\nu t} u(t,x)\,dt, \quad x \in \mathbb{R}^d$$

is a well-defined function on $\mathbb{R}^d$, $v \in \mathfrak{B}^m$, and

$$\begin{aligned} \mathcal{L}v(x) &= \int_0^\infty e^{-\nu t}\mathcal{L}u(t,x)\,dt \\ &= \int_0^\infty e^{-\nu t}\left(\frac{\partial}{\partial t}u(t,x) - \nu u(t,x)\right)dt = -f(x), \end{aligned}$$

where the last equality is obtained by integration by parts. Consequently, v is a solution of (30) and it satisfies estimate (29). □

Theorem 11. *Let $k \ge 0$ and suppose that Assumptions 2.1 through 2.6 are satisfied with an $m \ge 3k$. Then, for any $h \in (0, h_0]$ and for each integer $r \ge 0$, such that*

$$3k + r \le m,$$

for the unique bounded solution v_h of (27), we have

$$\sup_{(0,h_0]\times\mathbb{R}^d} |D^r D_h^k v_h| \le N\|f\|_m, \tag{31}$$

where N is a constant depending only on $m, \delta, c_0, \tau_0, K, |\Lambda_1|, \|\Lambda_1\|, M_0, \dots, M_m$. In particular,

$$\|v_h\|_m \le N\|f\|_m.$$

Proof. To prove (31), take a constant $\nu > 0$ as in the proof of Theorem 10, define $u(t,x) := v_h(x)e^{\nu t}$, and observe that u is the unique bounded solution of

$$\frac{\partial}{\partial t}u = L_h^0 u - (c - \nu)u + e^{\nu t} f, \quad u(0,x) = v_h(x).$$

By Theorem 6 for any $T > 0$,

$$e^{\nu T}|D^r D_h^k v_h(x)| = |D^r D_h^k u(T,x)| \le N e^{\nu T}\|f\|_m + N\|v_h\|_m,$$

where N is a constant, depending only on $m, \delta, c_0, \tau_0, K, |\Lambda_1|, \|\Lambda_1\|, M_0, \dots, M_m$. By multiplying the extreme terms by $e^{-\nu T}$ and letting $T \to \infty$, we get the result. □

From estimate (26), we obtain the corresponding estimate for the derivatives of v_h.

Theorem 12. *Let the conditions of Theorem 8 hold. Then, for any integer $r \ge 0$ such that*

$$3k + r \le m - 1,$$

for the solution v_h of (27), we have

$$\sup_{\mathbb{R}^d} |D^r D_h^k v_h| \le N\|f\|_m h$$

for all $h \in (0, h_0]$, where N depends only on $m, \delta, c_0, \tau_0, K, |\Lambda_1|, \|\Lambda_1\|$ and $M_0, \dots, M_m$.

Proof. This theorem can be deduced from Theorem 8 in the same way as Theorem 11 is obtained from Theorem 6. □

Now we want to establish an expansion for v_h, that is, to show for an integer $k \ge 0$ the existence of some functions $v^{(0)}, \dots, v^{(k)}$ on $\mathbb{R}^d$, and a function R_h on $\mathbb{R}^d$ for each $h \in (0, h_0]$ such that for all $x \in \mathbb{R}^d$ and $h \in (0, h_0]$

$$v_h(x) = v^{(0)}(x) + \sum_{1 \le j \le k} \frac{h^j}{j!} v^{(j)}(x) + h^{k+1} R_h(x), \tag{32}$$

$$\sup_{h\in(0,h_0]} \sup_{\mathbb{R}^d} |R_h| \le N\|f\|_m \tag{33}$$

with a constant N.

Theorem 13. *Suppose that Assumptions 2.1 through 2.6 are satisfied with an $m \ge 3$. Let $k \ge 0$ be an integer. Then, expansion (32) holds with $v^{(0)}$ being the unique $\mathfrak{B}^m$ solution of (28) and R_h satisfying (33), provided one of the following conditions is met:*

(i) $m \ge 3k + 3$
(ii) $m \ge 2k + 3$ *and condition (S) holds*
(iii) *k is odd, $m \ge 2k + 2$, and conditions (S) and (P) are satisfied.*

In each of the cases (i)–(iii), the constant N in (33) depends only on d, m, $\delta, c_0, \tau_0, K, |\Lambda_1|, \|\Lambda_1\|, M_0, \dots, M_m$. Moreover, when (iii) holds, we have $v^{(j)} = 0$ for all odd j.

Proof. Take a small constant $\nu > 0$, as in the proof of Theorem 10; let u be a function defined on H_∞ such that for each $T > 0$, its restriction onto H_T is the unique solution in $\mathfrak{B}^m_T$ of

$$\frac{\partial}{\partial t}u_h = (L_h + \nu)u_h \quad (t,x) \in H_\infty$$

$$u_h(0,x) = f(x) \quad x \in \mathbb{R}^d,$$

(see Remark 1). As in the proof of Theorem 10, we get that

$$v_h(x) = \int_0^\infty e^{-\nu t} u_h(t,x)\, dt.$$

By Theorem 2 in each of the cases (i)–(iii), we have

$$u_h(t,x) = u^{(0)}(t,x) + \sum_{1 \le j \le k} \frac{h^j}{j!} u^{(j)}(t,x) + h^{k+1} r_h(t,x), \tag{34}$$

for all $(t,x) \in H_\infty$, $h \in (0,h_0]$, and by Theorem 7, we have

$$\sup_{h \in (0,h_0]} \sup_{H_\infty} \{|u_h| + \sum_{j=0}^{k} |u^{(j)}| + |r_h|\} \le N \|f\|_m \tag{35}$$

with a constant N depending only on d, m, δ, c_0, τ_0, K, $M_0, \dots, M_m$, $|\Lambda_1|$, and $\|\Lambda_1\|$. Multiplying both sides of (34) by $e^{-\nu t}$ and then integrating them over $[0,\infty)$ with respect to dt, we get expansion (32) with

$$R_h(x) := \int_0^\infty e^{-\nu t} r_h(t,x)\, dt,$$

$$v^{(j)}(x) := \int_0^\infty e^{-\nu t} u^{(j)}(t,x)\, dt, \quad \text{for } j = 0, \dots, k.$$

Clearly, (35) implies that (33) holds with N depending only on d, m, δ, c_0, τ_0, K, $M_0, \dots, M_m$, $|\Lambda_1|$, and $\|\Lambda_1\|$. As we know, the function $u^{(0)}$ in (34) is the $\mathfrak{B}^m$ solution of

$$\frac{\partial}{\partial t}u = (\mathcal{L} + \nu)u \quad (t,x) \in H_\infty,$$

$$u(0,x) = f(x) \quad x \in \mathbb{R}^d,$$

which as we have seen in the proof of Theorem 10 guarantees that $v^{(0)}$ is the unique $\mathfrak{B}^m$ solution of (28). □

The following result can be obtained easily from Theorem 5 by inspecting the proof of the previous theorem.

Theorem 14. *Let p_λ, q_λ, c, and f satisfy the conditions of Theorem 13, with $m-l$ in place of m in each of the conditions (i)–(iii) for an integer $l \in [0,m]$. Then, $D^\alpha v_h$ is a bounded continuous function on $\mathbb{R}^d$ for every multi-index α, $|\alpha| \le l$, and the expansion (32) is valid with $D^\alpha v_h$, $\{D^\alpha v^{(j)}\}_{j=0}^k$ and $D^\alpha R_h$ in place of v_h, $\{v^{(j)}\}_{j=0}^k$ and R_h, respectively. Furthermore, (33) holds with $D^\alpha R_h$ in place of R_h and a constant N depending only on d, m, δ, c_0, τ_0, K, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$. In case (iii), we have $v^{(j)} = 0$ for all odd j in the expansion.*

Set

$$\bar{v}_h = \sum_{j=0}^{k} \rho_j v_{2^{-j}h}, \quad \tilde{v}_h = \sum_{j=0}^{\tilde{k}} \tilde{\rho}_j v_{2^{-j}h},$$

where $(\rho_0, \rho_1, \dots, \rho_k)$ and $\tilde{k}$, $(\tilde{\rho}_0, \tilde{\rho}_1, \dots, \tilde{\rho}_{\tilde{k}})$ are defined by (15) and (17) in Sect. 2. Then, we have the following corollary.

Corollary 3. *Suppose that the assumptions of Theorem 14 are satisfied. Then, for every multi-index α with $|\alpha| \le l$,*

$$\sup_{\mathbb{R}^d} |D^\alpha \bar{v}_h - D^\alpha v^{(0)}| \le N \|f\|_m h^{k+1},$$

and if condition (iii) is met, then

$$\sup_{\mathbb{R}^d} |D^\alpha \tilde{v}_h - D^\alpha v^{(0)}| \le N \|f\|_m h^{k+1},$$

where N depends only on d, m, δ, K, τ_0, c_0, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$.

4 Acceleration for Stochastic Parabolic PDEs

We consider the equation

$$\mathrm{d}u(t) = (\mathcal{L}(t)u(t) + f(t))\,\mathrm{d}t + (\mathcal{M}^\rho(t)u(t) + g^\rho(t))\,\mathrm{d}w^\rho(t), \tag{36}$$

for $(t,x) \in [0,T] \times \mathbb{R}^d =: H_T$ with some initial condition, where

$$\mathcal{L}(t)\phi = a^{\alpha\beta}(t) D_\alpha D_\beta \phi, \quad \mathcal{M}^\rho(t)\phi = b^{\alpha\rho}(t) D_\alpha \phi,$$

and $\{w^\rho\}_{\rho=1}^\infty$ is a sequence of independent Wiener processes given on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F})_{t\ge 0}$ such that w_t^ρ is $\mathcal{F}_t$-measurable and $w_t^\rho - w_s^\rho$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t$ and integers $\rho \ge 1$. Here and below the summation with respect to α and β is performed over the set $\{0, 1, \dots, d\}$ and with respect to ρ in the range $\{1, 2, \dots\}$. Assume that, for $\alpha, \beta \in \{0, 1, \dots, d\}$, we have $a^{\alpha\beta} = a^{\beta\alpha}$, and $a^{\alpha\beta}(t) = a^{\alpha\beta}(t,x)$ are real-valued and $b_t^\alpha = (b^{\alpha\rho}(t,x))_{\rho=1}^\infty$ are l_2-valued $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable functions on $\Omega \times H_T$.

Let $m \geq 1$ be an integer and let W_2^m be the usual Hilbert-Sobolev space of functions on $\mathbb{R}^d$ with norm $\|\cdot\|_{W_2^m}$.

Assumption 4.1. *(i) For each (ω, t), the functions $a^{\alpha\beta}(t)$ are m times and the functions $b^\alpha(t)$ are $m+1$ times continuously differentiable in x. There exist constants K_l, $l = 0, \ldots, m+1$, such that for all values of indices and arguments, we have*

$$|D^l a_t^{\alpha\beta}| \leq K_l, \quad l \leq m, \quad |D^l b_t^\alpha|_{l_2} \leq K_l, \quad l \leq m+1.$$

(ii) There is a constant $\kappa > 0$ such that for all $(\omega, t, x) \in \Omega \times H_T$ and $z \in \mathbb{R}^d$

$$\sum_{i,j=1}^d (2a^{ij} - b^{i\rho} b^{j\rho}) z^i z^j \geq \kappa |z|^2.$$

Assumption 4.2. *We have $u_0 \in L_2(\Omega, \mathcal{F}_0, W_2^{m+1})$. The function f is W_2^m-valued, g^ρ, $\rho = 1, 2, \ldots$, are W_2^{m+1}-valued functions given on $\Omega \times [0, T]$ and they are predictable. Moreover, for $g := (g^\rho)_{\rho=1}^\infty$ and*

$$\|g\|_{W_2^l}^2 := \sum_{\rho=1}^\infty \|g^\rho\|_{W_2^l}^2$$

we have

$$E \int_0^T (\|f(t)\|_{W_2^m}^2 + \|g(t)\|_{W_2^{m+1}}^2) \, dt + E\|u_0\|_{W_2^{m+1}}^2 =: \mathcal{K}_m^2 < \infty.$$

Remark 6. If Assumption 4.2 holds with $m > d/2$, then by Sobolev's embedding of W_2^m into C_b, the space of bounded continuous functions, for almost all ω, we can find a continuous function of x that equals to u_0 almost everywhere. Furthermore, for each t and ω, we have continuous functions of x that coincide with f_t and g_t, for almost every $x \in \mathbb{R}^d$. Therefore, when Assumption 4.2 holds with $m > d/2$, we always assume that u_0, f_t and g_t are continuous in x for all t.

The solutions of (36) will be looked for in the Hilbert space

$$\mathbb{W}_2^{m+2}(T) = L_2(\Omega \times [0, T], \mathcal{P}, W_2^{m+2}).$$

One knows, see for example, [18] or [30], how to define stochastic integrals of Hilbert-space valued processes, and (36) is understood accordingly. Observe that since $u_0 \in L_2(\Omega, \mathcal{F}_0, W_2^m)$, the solutions of (36) automatically are continuous W_2^m-valued processes (a.s.).

We are going to use the following classical result (see, for instance, Theorem 5.1, Remark 5.6, and Theorem 7.1 of [17]).

Theorem 15. *Under the above assumptions, there exists a unique solution* $u \in \mathbb{W}_2^{m+2}(T)$ *of* (36) *with initial condition* u_0. *Furthermore, with probability one, the function* u_t *is a continuous* W_2^{m+1}*-valued function, and there exists a constant* N *depending only on* T, d, κ, m, *and* $K_l, l \le m+1$, *such that*

$$E \sup_{t \le T} \|u(t)\|^2_{W_2^{m+1}} + E \int_0^T \|u(t)\|^2_{W_2^{m+2}} \, \mathrm{d}t \le N \mathcal{K}_m^2.$$

Remark 7. In the future, we are going to assume that $m + 1 > d/2$. Then, by Sobolev embedding theorems, the solution $u_t(x)$ from Theorem 15 is a continuous function of (t, x) (a.s). More precisely, with probability one, for any t, one can find a continuous function of x that equals $u_t(x)$ for almost all x, and in addition, the so constructed modification is continuous with respect to the couple (t, x).

We are interested in approximating the solution by means of solving a semidiscretized version of (36) when partial derivatives are replaced with finite differences. For $\lambda = 0$, set $\delta_{h,\lambda}$ to be the unit operator and for the other values of $\lambda \in \mathbb{R}^d$, let

$$\delta_{h,\lambda} u(x) = \frac{u(x + h\lambda) - u(x)}{h} \quad \text{for } h \in \mathbb{R} \setminus \{0\}.$$

We draw the reader's attention to the fact that h can be of any sign. This will be important in the future.

To introduce difference equations, we take a finite set $\Lambda \subset \mathbb{R}^d$ containing the origin, and consider the equation

$$\mathrm{d}u^h(t) = (L^h(t)u^h(t) + f(t))\,\mathrm{d}t + (M^{h,\rho}(t)u^h(t) + g^\rho(t))\,\mathrm{d}w^\rho(t), \tag{37}$$

with

$$L^h(t)\phi = \mathfrak{a}^{\lambda\mu}(t)\delta_{h,\lambda}\delta_{-h,\mu}\phi, \quad M^{h,\rho}(t)\phi = \mathfrak{b}^{\lambda\rho}(t)\delta_{h,\lambda}\phi,$$

where the summation is performed over $\lambda, \mu \in \Lambda$ and in (37) also with respect to $\rho = 1, 2, \ldots$. Assume that, for $\lambda, \mu \in \Lambda$, $\mathfrak{a}^{\lambda\mu} = \mathfrak{a}^{\lambda\mu}(t, x)$ are real-valued and $\mathfrak{b}^\lambda = \mathfrak{b}^\lambda(t, x) = (\mathfrak{b}^{\lambda\rho}(t, x))_{\rho=1}^\infty$ are l_2-valued functions on $\Omega \times H_T$, measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$.

Set $\Lambda_0 := \Lambda \setminus \{0\}$. Let $\mathfrak{m} \ge 0$ be an integer. Set $\bar{\mathfrak{m}} = \max(\mathfrak{m}, 1)$, and let $A_0, A_1, \ldots, A_{\bar{\mathfrak{m}}}$ be some constants. The functions $\mathfrak{a}$ and $\mathfrak{b}$ are supposed to possess the following properties.

Assumption 4.3. *(i) For each* (ω, t) *and* $\lambda, \mu \in \Lambda_0$ *and* $\nu \in \Lambda$, $\mathfrak{a}_t^{\lambda\mu}$ *are* $\bar{\mathfrak{m}}$ *times continuously differentiable in* x, $\mathfrak{a}_t^{0\nu}$, $\mathfrak{a}_t^{\nu 0}$ *are* $\mathfrak{m}$ *times continuously differentiable in* x *and* $\mathfrak{b}_t^\nu$ *are* $\mathfrak{m}$ *times continuously differentiable in* x *as* l_2*-valued functions. For all values of arguments, we have*

$$|D^j \mathfrak{a}^{\lambda\mu}| \le A_j, \quad \lambda, \mu \in \Lambda_0, \quad j \le \bar{\mathfrak{m}},$$

$$|D^j \mathfrak{a}^{\lambda 0}| \le A_j, \quad |D^j \mathfrak{a}^{0\lambda}| \le A_j, \quad |D^j \mathfrak{b}^{\lambda}|_{l_2} \le A_j, \quad \lambda \in \Lambda, \quad j \le \mathfrak{m}.$$

(ii) For all $(\omega, t, x) \in \Omega \times H_T$ *and numbers* $z_\lambda, \lambda \in \Lambda_0$*, we have*

$$\sum_{\lambda,\mu \in \Lambda_0} (2\mathfrak{a}^{\lambda\mu} - \mathfrak{b}^{\lambda\rho}\mathfrak{b}^{\mu\rho}) z_\lambda z_\mu \ge \kappa \sum_{\lambda \in \Lambda_0} z_\lambda^2.$$

Introduce

$$\mathbb{G}_h = \{\lambda_1 h + \cdots + \lambda_n h : n = 1, 2, \ldots, \lambda_i \in \Lambda \cup (-\Lambda)\}$$

and let $l_2(\mathbb{G}_h)$ be the set of real-valued functions u on $\mathbb{G}_h$ such that

$$|u|^2_{l_2(\mathbb{G}_h)} := |h|^d \sum_{x \in \mathbb{G}_h} |u(x)|^2 < \infty.$$

The notation $l_2(\mathbb{G}_h)$ will also be used for l_2-valued functions like g.

Remark 8. Observe that, under Assumption 4.3 (i), (37) is an ordinary Itô equation with Lipschitz continuous coefficients for $l_2(\mathbb{G}_h)$-valued processes. Therefore, if, for instance, (a.s.)

$$\int_0^T (|f(t)|^2_{l_2(\mathbb{G}_h)} + |g(t)|^2_{l_2(\mathbb{G}_h)})\, \mathrm{d}t < \infty,$$

and Assumption 4.3 (i) holds, then (37) has a unique solution with continuous trajectories in $l_2(\mathbb{G}_h)$, provided that the initial data $u_0^h \in l_2(\mathbb{G}_h)$ (a.s.).

For (37) to be consistent with (36), we impose the following.

Assumption 4.4. *For all* $i, j = 1, \ldots, d$ *and* $\rho = 1, 2, \ldots$

$$\sum_{\lambda,\mu \in \Lambda_0} \mathfrak{a}^{\lambda\mu} \lambda^i \mu^j = a^{ij}, \quad \sum_{\lambda \in \Lambda_0} \mathfrak{a}^{\lambda 0} \lambda^i + \sum_{\mu \in \Lambda_0} \mathfrak{a}^{0\mu} \mu^i = a^{i0} + a^{0i}, \quad \mathfrak{a}^{00} = a^{00},$$

$$\sum_{\lambda \in \Lambda_0} \mathfrak{b}^{\lambda\rho} \lambda^i = b^{i\rho}, \quad \mathfrak{b}^{0\rho} = b^{0\rho}.$$

Remark 9. Clearly, if

$$a^{ij} = \sum_{\lambda,\mu \in \Lambda_0} \mathfrak{a}^{\lambda\mu} \lambda^i \mu^j, \quad i, j = 1, \ldots, d$$

is an invertible matrix for some ω, t, x, then Λ_0 spans the whole $\mathbb{R}^d$. On the other hand, if Λ_0 spans $\mathbb{R}^d$, then clearly a constant $\kappa' > 0$ exists such that

$$\sum_{\lambda \in \Lambda_0} |\sum_i z^i \lambda^i|^2 \ge \kappa' |z|^2, \quad \text{for all } z = (z^1, \ldots, z^d) \in \mathbb{R}^d,$$

and therefore Assumptions 4.3 (ii) and 4.4 imply Assumption 4.1 (ii). It is not hard to see that Assumptions 4.1 (ii) and 4.4 do not imply Assumption 4.3 (ii), in general, unless Λ_0 is a basis in $\mathbb{R}^d$.

There are several ways to construct appropriate $\mathfrak{a}$ and $\mathfrak{b}$.

Example 2. The most natural, albeit sometimes not optimal, way to choose $\mathfrak{a}$ and $\mathfrak{b}$ is to set $\Lambda = \{e_0, e_1, \dots, e_d\}$, where $e_0 = 0$ and e_i is the ith basis vector in $\mathbb{R}^d$ and let

$$\mathfrak{a}^{e_\alpha e_\beta} = a^{\alpha\beta}, \quad \mathfrak{b}^{e_\alpha \rho} = b^{\alpha\rho}, \quad \alpha, \beta = 0, 1, \dots, d.$$

Thus, in (37), the first order derivatives in (36) are approximated by usual finite differences and

$$\sum_{\lambda,\mu\in\Lambda_0} \mathfrak{a}^{\lambda\mu}\delta_{h,\lambda}\delta_{-h,\mu}u = -a^{ij}\delta_{h,e_i}\delta_{h,-e_j}u, \tag{38}$$

which is a standard finite-difference approximation of $a^{ij}D_iD_ju$. Also notice that

$$\sum_{\lambda,\mu\in\Lambda_0} \mathfrak{a}^{\lambda\mu}z_\lambda z_\mu = a^{ij}z_{e_i}z_{e_j}, \quad \sum_{\lambda\in\Lambda_0} \mathfrak{b}^{\lambda\rho}z_\lambda = b^{i\rho}z_{e_i}.$$

It follows that $\mathfrak{a}$ and $\mathfrak{b}$ satisfy the above assumptions as long as a and b do.

Example 3. The second choice is to use symmetric finite differences to approximate the first-order derivatives. Namely, we take $\Lambda_0 = \{\pm e_1, \dots, \pm e_d\}$ and

$$\mathfrak{a}^{0,\pm e_i} = \mathfrak{a}^{\pm e_i,0} = \pm(1/4)(a^{0i} + a^{i0}), \quad \mathfrak{b}^{\pm e_i,\rho} = \pm(1/2)b^{i,\rho},$$

$$\mathfrak{a}^{00} = a^{00}, \quad \mathfrak{b}^{0\rho} = b^{0\rho},$$

so that, for instance,

$$\sum_{\lambda\in\Lambda_0} \mathfrak{b}^{\lambda\rho}\delta_{h,\lambda}u(x) = \sum_{i=1}^{d} b^{i\rho}\frac{u(x+he_i) - u(x-he_i)}{2h}.$$

For $\lambda, \mu \in \Lambda_0$, we define $\mathfrak{a}_t^{\lambda\mu}$ by

$$\mathfrak{a}^{\pm e_i,\pm e_j} = \tfrac{1}{2}a^{ij}, \quad \mathfrak{a}^{\pm e_i,\mp e_j} = 0.$$

Then, Assumption 4.4 is satisfied, and formula (38) holds again ($a^{ij} = a^{ji}$). If Assumption 4.1 (ii) is satisfied, then for any numbers z_λ

$$\sum_{\lambda,\mu\in\Lambda_0}(2\mathfrak{a}^{\lambda\mu}-\mathfrak{b}^{\lambda\rho}\mathfrak{b}_t^{\mu\rho})z_\lambda z_\mu=\sum_{i,j=1}^d a^{ij}z_{e_i}z_{e_j}+\sum_{i,j=1}^d a^{ij}z_{-e_i}z_{-e_j}$$
$$-(1/4)\sum_\rho\Big|\sum_{i=1}^d b^{i\rho}z_{e_i}-\sum_{i=1}^d b^{i\rho}z_{-e_i}\Big|^2$$
$$\geq\sum_{i,j=1}^d a^{ij}z_{e_i}z_{e_j}-(1/2)\sum_\rho\Big|\sum_{i=1}^d b^{i\rho}z_{e_i}\Big|^2$$
$$+\sum_{i,j=1}^d a^{ij}z_{-e_i}z_{-e_j}-(1/2)\sum_\rho\Big|\sum_{i=1}^d b^{i\rho}z_{-e_i}\Big|^2$$
$$\geq\kappa\sum_{i=1}^d z_{e_i}^2+\kappa\sum_{i=1}^d z_{-e_i}^2=\kappa\sum_{\lambda\in\Lambda_0}z_\lambda^2,$$

so that Assumption 4.3 (ii) is also satisfied.

Our results revolve about the possibility to prove the existence of random processes $u^{(j)}(t,x)$, $t\in[0,T], x\in\mathbb{R}^d$, $j=0,\dots,k$, for some integer $k\geq 0$ such that they are independent of h; $u^{(0)}$ is the solution of (36) with initial value u_0, and almost surely, we have

$$u^h(t,x)=\sum_{j=0}^k\frac{h^j}{j!}u^{(j)}(t,x)+R^h(t,x)\tag{39}$$

for $h\neq 0$ and for all $t\in[0,T]$ and $x\in\mathbb{G}_h$, where u^h is the solution to (37) with initial data u_0 and R^h is a continuous $l_2(\mathbb{G}_h)$-valued adapted process, such that

$$E\sup_{t\in[0,T]}\sup_{x\in\mathbb{G}_h}|R^h(t,x)|^2\leq Nh^{2(k+1)}\mathcal{K}_m^2\tag{40}$$

with a constant N independent of h.

Theorem 16. *Let Assumptions 4.1, 4.2, 4.3, and 4.4 hold with*

$$\mathfrak{m}=m>k+1+d/2,$$

where $k\geq 0$ is an integer. Then, expansion (39) and estimate (40) hold with a constant N depending only on Λ, d, m, $K_0,\dots,K_{m+1}$, $A_0,\dots,A_m$, κ, and T.

Remark 10. Actually $u^h(t,x)$ is defined for all $x\in\mathbb{R}^d$ rather than only on $\mathbb{G}_h$, and as we will see from the proof of Theorem 16, one can replace $\mathbb{G}_h$ in (40) with $\mathbb{R}^d$.

Remark 11. Let Λ_0 be a basis in $\mathbb{R}^d$ such that Assumption 4.4 holds. Then, Assumption 4.1 (i) implies Assumption 4.3 (i), and Assumption 4.1 (ii) implies Assumption 4.3 (ii) with $\mathfrak{m}=m$. Thus, if Assumptions 4.1 and 4.2 hold with

$$m > k + 1 + d/2,$$

then the conditions of Theorem 16 are satisfied.

Equality (39) clearly yields

$$\delta_{h,\lambda} u^h(t,x) = \sum_{j=0}^{k} \frac{h^j}{j!} \delta_{h,\lambda} u^{(j)}(t,x) + \delta_{h,\lambda} R^h(t,x)$$

for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^n$ and integer $n \geq 0$, where $\Lambda^0 = \{0\}$ and

$$\delta_{h,\lambda} := \delta_{h,\lambda_1} \cdot \ldots \cdot \delta_{h,\lambda_n}.$$

Theorem 16 can be generalised as follows.

Theorem 17. *Let the conditions of Theorem 16 hold with*

$$\mathfrak{m} = m > k + n + 1 + d/2$$

for some integers $k \geq 0$ and $n \geq 0$. Then, expansion (39) holds and for $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^n$

$$E \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |\delta_{h,\lambda} R^h(t,x)|^2 + E \sup_{t\in[0,T]} \sum_{x\in\mathbb{G}_h} |\delta_{h,\lambda} R^h(t,x)|^2 |h|^d \leq N h^{2(k+1)} \mathcal{K}_m^2,$$

where N depends only on Λ, d, m, $K_0, \dots, K_{m+1}$, $A_0, \dots, A_m$, κ and T.

Hence in the already familiar way, we get the following acceleration result for

$$\bar{u}^h = \sum_{j=0}^{k} \rho_j u^{2^{-j}h},$$

where $k \geq 0$ is a fixed integer, $u^{2^{-j}h}$ are the solutions to (37), with $2^{-j}h$ in place of h, and the sequence of coefficients $\{\rho_j\}_{j=0}^k$ are defined by (15) in Sect. 2.

Theorem 18. *Under the assumptions of Theorem 16, we have*

$$E \sup_{t\leq T} \sup_{x\in\mathbb{G}_h} |\bar{u}^h(t,x) - u^{(0)}(t,x)|^2 \leq N|h|^{2(k+1)} \mathcal{K}_m^2, \tag{41}$$

where N depends only on Λ, d, m, $K_0, \dots, K_{m+1}$, κ, $A_0, \dots, A_m$, and T.

Remark 12. Let the conditions of Theorem 16 hold with

$$\mathfrak{m} = m > k + 1 + n + d/2,$$

where k and n are nonnegative integers. Then, (41) holds with $\delta_{h,\lambda}\bar{u}^h$ and $\delta_{h,\lambda}u^{(0)}$ in place of $\bar{u}^h$ and $u^{(0)}$, respectively, for $\lambda \in \Lambda^n$.

By the above remark, one can construct fast approximations for the derivatives of $u^{(0)}$ via suitable linear combinations of finite differences of $\bar{u}^h$.

Clearly it suffices to combine fewer terms $u^{2^{-j}h}$ to get accuracy of order $k+1$ if some of the terms in the expansion (39) vanish. For integers $k \geq 0$, define

$$\tilde{u}^h = \sum_{j=0}^{\tilde{k}} \tilde{\rho}_j u^{2^{-j}h},$$

where $\tilde{k} = [\frac{k}{2}]$ and $\tilde{\rho}_0, \tilde{\rho}_1, \ldots, \tilde{\rho}_{\tilde{k}}$ are defined by (17) in Sect. 2.

Theorem 19. *Let the conditions of Theorem 16 hold. Then, in the situation of Example 3, we have*

$$E \sup_{t \leq T} \sup_{x \in \mathbb{G}_h} |\tilde{u}^h(t,x) - u^{(0)}(t,x)|^2 \leq N|h|^{2(k+1)}\mathcal{K}_m^2, \tag{42}$$

where N is a constant depending only on d, m, κ, $K_0, \ldots, K_{m+1}$, $A_0, \ldots, A_m$, and T.

To prove this result, we need only repeat the proof of Theorem 18 taking into account that in (39), we have $u_t^{(j)} = 0$ for odd $j \leq k$ since $u_t^h = u_t^{-h}$ owing to the fact that in the case of Example 3, (37) does not change if we replace h with $-h$.

Remark 13. Notice that without acceleration, that is, when $k = 1$ in the above theorem, the mean square norm of the supremum in t and x of the error of the finite difference approximations in Example 3 is proportional to h^2. This is a sharp result see, for example, Remark 2.21 in [3] on finite difference approximations for deterministic parabolic PDEs.

Example 4. Assume that in the situation of Example 3, we have $d = 2$ and $m = 7$. Then,

$$\tilde{u}^h := \tfrac{4}{3}u^{h/2} - \tfrac{1}{3}u^h$$

satisfies

$$E \sup_{t \leq T} \sup_{x \in \mathbb{G}_h} |u_t^{(0)}(x) - \tilde{u}_t^h(x)| \leq Nh^4.$$

Example 5. Take $d = 1$ and consider the following SPDE:

$$\mathrm{d}u_t = 3D^2 u_t \,\mathrm{d}t + 2Du_t \,\mathrm{d}w_t$$

with initial data $u_0(x) = \cos x$, where w_t is a one-dimensional Wiener process. Then, a unique bounded solution is $u_t(x) = \mathrm{e}^{-t}\cos(x + 2w_t)$. Example 3 suggests the following version of (37):

$$\mathrm{d}u_t^h(x) = 3\frac{u_t^h(x+h) - 2u_t^h(x) + u_t^h(x-h)}{h^2}\,\mathrm{d}t + \frac{u_t^h(x+h) - u_t^h(x-h)}{h}\,\mathrm{d}w_t,$$

the unique bounded solution of which with initial condition $\cos x$ is given by

$$u_t^h(x) = \mathrm{e}^{-c_h t}\cos(x + 2\phi_h w_t), \quad h^2 c_h = 12\sin^2\frac{h}{2} - 2\sin^2 h, \quad \phi_h = \frac{\sin h}{h}.$$

For $t = 1$, $h = 0.1$, and $w_t = 0$, we have

$$u_1(0) \approx 0.3678794412, \quad u_1^h(0) \approx 0.366352748, \quad u_1^{h/2}(0) \approx 0.3674966179,$$

$$\tilde{u}_1^h(0) = \tfrac{4}{3}u_1^{h/2}(0) - \tfrac{1}{3}u_1^h(0) \approx 0.3678779079.$$

It is instructive to observe that such a level of accuracy is achieved for $u_1^{\tilde{h}}(0)$ with $\tilde{h} = 0.00316$, which is more than 15 times smaller than $h/2$.

Actually, this example does not quite fit into our scheme because u_0 is not square summable over $\mathbb{R}$. In connection with this, we note that our theorems can be extended to SPDEs with growing data by the help of weighted Sobolev spaces (see [5]), and then the above example can be included formally.

5 Accelerated Time Discretized Schemes

Instead of linear parabolic PDEs, we consider here a more general class of linear equations, and we approximate them by various types of time-discretized schemes whose convergence then we accelerate.

To formulate a general framework for our results, fix an integer $l \geq 1$ and assume that we have a sequence of Banach spaces

$$V_0, V_1, V_2, \ldots, V_l$$

such that V_i is continuously embedded into V_{i-1}, for every $i = 1, 2, \ldots, l$, and V_1 is dense in V_0. Consider

$$\mathrm{d}u(t) = (Lu(t) + f)\,\mathrm{d}t, \quad u(0) = u_0 \in V_l, \tag{43}$$

for $t \in [0, T]$, where f is an element from V_l and L is a bounded linear operator from V_1 to V_0 such that

$$\|Lv\|_i \leq K\|v\|_{i+1} \quad \text{for all } v \in V_{i+1}, i = 0, \ldots, l, \tag{44}$$

for a fixed constant K, where $\|\psi\|_i$ denotes the norm of ψ in V_i.

Definition 1. A V_1-valued weakly continuous function $u = u(t)$ on $[0, T]$ is called a solution of (43) if for all $t \in [0, T]$

$$u(t) = u_0 + \int_0^t (L(s)u(s) + f(s))\,\mathrm{d}s,$$

where the integral is understood in Bochner's sense.

Example 6. (Parabolic PDE) Consider the problem

$$D_t u(t,x) = Lu(t,x) + f(x), \qquad t \in (0,T], \ \ x \in \mathbb{R}^d, \tag{45}$$

$$u(0,x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{46}$$

where L is an operator of the form

$$L = a^{ij}(x)D_{ij} + a^i(x)D_i + a(x), \tag{47}$$

a^{ij}, a^i, a, f, and φ are real-valued functions on $\mathbb{R}^d$. Assume that the matrix (a^{ij}) is positive semidefinit, and that for some constant $K \geq 0$ and integer $\nu \geq 2$ the partial derivatives of a^{ij}, a^i and a up to order ν are bounded by a constant K, and $\varphi, f \in W_p^\nu$ such that

$$\|\varphi\|_{\nu,p} \leq K, \quad \|f\|_{\nu,p} \leq K,$$

for some $p > 1$, where

$$\|\psi\|_{\nu,p} = \Big(\sum_{|\gamma|\leq\nu} \int_{\mathbb{R}^d} |D^\gamma \varphi(x)|^p\,\mathrm{d}x\Big)^{1/p}$$

is the norm of a function ψ in the Sobolev spaces W_p^ν. Then, it is well known (see, for instance, Theorem 3.1 in [9] and recall that $\nu \geq 2$) that there is a unique W_p^ν-valued weakly continuous function $v(t)$, $t \geq 0$, such that

$$v(t) = \varphi + \int_0^t (Lv(s) + f)\,\mathrm{d}s, \tag{48}$$

where one understands the integral as Bochner's weak (= strong) integral, or equivalently, one understands the equation in the sense of integral identity obtained by multiplying by test functions and integrating with respect to x. Take $V_i = W_p^{\nu+2i}$ for some $\nu \geq 0$ and $i = 0, \ldots, l$. Notice that V_i is continuously and densely embedded into V_{i-1} for each $i = 1, \ldots, l$ and set

$$V_j := W_p^{\mu+2j}, \quad j = 0, 1, \ldots, l,$$

for an integer $\mu \geq 0$ satisfying $\mu + 2l \leq \nu$.

Example 7. (Systems of parabolic PDEs and hyperbolic PDEs)
As in the previous example, we consider the problem (45)–(46) with an operator L given by (47), but this time instead of unknown real-valued functions u, we consider

$\mathbb{R}^q$-valued ones, where q is a fixed number. Accordingly, we assume that a^i, a are $q\times q$-matrix valued functions with entries $a^{i,\alpha\beta}$, $a^{0\alpha\beta}$, respectively, and f and φ are $\mathbb{R}^q$-valued. Yet, a^{ij} is assumed to be real-valued as in the previous example. We set $p = 2$ and impose the same assumptions as in the previous example with obvious interpretation of the norms $\|\cdot\|_{\nu,2}$ for vector-valued functions. We also assume that for each $x, \lambda \in \mathbb{R}^d$ and $\alpha, \beta = 1, \ldots, q$, we have

$$\Big|\sum_{i=1}^{d}\bar{a}^{i,\alpha\beta}(x)\lambda^i\Big| \le K\Big(\sum_{i,j=1}^{d}a^{ij}(x)\lambda^i\lambda^j\Big)^{1/2}, \tag{49}$$

where $\bar{a}_r^{i,\alpha\beta} = a^{i,\alpha\beta} - a^{i,\beta\alpha}$. Observe that this assumption is obviously satisfied if

(a) The matrix (a^{ij}) is uniformly nondegenerate, so that the systems are uniformly parabolic, or
(b) $a^{ij} \equiv 0$ and the matrix a^i is symmetric, so that the systems are first-order symmetric hyperbolic. Then, under the above assumptions, there exists a unique $(W_2^\nu)^q$-valued continuous solution (see [12]), where $(W_2^\nu)^q$ denotes the q-fold product of W_2^ν. Take $V_j := (W_p^{\mu+2j})^q$ for $j = 0, 1, \ldots, l$ and integer $\mu \ge 0$ such that $\mu + 2l \le \nu$. Clearly, V_j is continuously and densely embedded into V_{j-1} for each $j = 1, \ldots, l$.

Together with the problem, (43) we consider for every $\tau \in (0, 1]$ the problem

$$\mathrm{d}w(t) = \sum_{r=1}^{m}(L_r\Theta_r w(t) + f_r)\,\mathrm{d}A_r(t), \quad t \in (0, T], \quad w(0) = u_0, \tag{50}$$

where $m \ge 1$ is a fixed integer, $u_0 \in V_1$, and for each $r = 1, \ldots, m$, we have that $f_r \in V_1$, L_r is a bounded linear operator from V_1 into V_0, $\Theta_r = \Theta_r^\tau$ is a bounded linear operator on the space of V_0-valued weakly cadlag functions on $[0, T]$, and $A_r = A_r^\tau(t)$, $t \in [0, T]$ is a right-continuos real function of finite variation over $[0, T]$.

We assume that

$$A_r(t) := \tau H_r(t/\tau), \quad t \ge 0, \quad r = 1, 2, \ldots, m$$

with some right-continuous functions $H_r = H_r(t)$ on $\mathbb{R}$, which have finite variation on every finite interval, such that $H_r(0) = 0$ and

$$H_r(t+1) - H_r(t) = H_r(1) = 1, \quad \forall t \in \mathbb{R}$$

for all $r = 1, 2, \ldots, m$. We assume, moreover, that

$$L = L_1 + L_2 + \ldots + L_m, \quad f = f_1 + f_2 + \ldots + f_m.$$

There are various types of time discretization schemes for problem (43), which can be represented by (50). Let us see some examples of such schemes defined on the grid

$$T_\tau := \{t_i = i\tau : i = 0, 1, 2, \dots\} \cap [0, T].$$

Example 8. (Finite differences in time.) Let $m = 1$, $L_1 = L$, $H_1(t) = [t]$, where $[t]$ denotes the integer part of t. Let $\Theta_1 = I$, the identity operator. Then, (50) represents the *implicit Euler method* with step size τ. If $\Theta_1 = \Theta$ is the operator defined by

$$(\Theta\varphi)(t) := \vartheta\varphi(t-) + (1-\vartheta)\varphi(t) \tag{51}$$

with some $\vartheta \in \mathbb{R}$, then (50) is the *Θ-method*, which is often called the *Crank-Nicolson* method when $\vartheta = 1/2$, and the *explicit Euler method* when $\vartheta = 1$.

Example 9. (Splitting up approximations.) Let $m > 1$. Consider the splitting-up approximations for (43) defined by

$$u(t_i) := \Big(P_m(\tau)\dots P_2(\tau)P_1(\tau)\Big)^i u_0, \tag{52}$$

at $t_i \in T_\tau$, where $P_k(t)\varphi$ denotes the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = L_k v(t) + f_k, \quad v(0) = \varphi, \quad k := 1, \dots, m \tag{53}$$

at time $t \in [0, T]$. Formula (52) means that to get the approximation at $t_{i+1} := t_i + \tau$ from the approximation at $t_i = i\tau$, we solve (53) with $k = 1$ on $[0, \tau]$ with initial value $u(t_i)$, then we solve (53) with $k = 2$ on $[0, \tau]$ again with the new initial value $P_1(\tau)u(t_i)$, and so on, and finally, taking the solution at τ of the preceding step as the initial value, we solve (53) with $k = m$ on $[0, \tau]$. We can easily represent this method as a problem of type (50) by using an idea from [8]. Namely, instead of going back and forth in time, when solving problems (53) one after another on $[0, \tau]$ with up-dated initial values, we 'stretch out the time' and rearrange solving these equations in forward time. This procedure can be described by the single equation

$$\mathrm{d}\bar{u}(t) = \sum_{k=1}^{m} (L_k\bar{u}(t) + f_k)\dot{h}_k(t/\tau)\,\mathrm{d}t,$$

where $\dot{h}_k := \frac{\mathrm{d}}{\mathrm{d}t}h_k$ is a function of period m, such that $\dot{h}_k(t) := 1_{[k-1,k)}(t)$ for $t \in [0, m]$, $k = 1, 2, \dots m$. Indeed, clearly $\bar{u}(mt) = u(t)$ for all $t \in T_\tau$. Hence $w(t) := \bar{u}(mt)$ satisfies

$$\mathrm{d}w(t) = \sum_{k=1}^{m} (L_k w(t) + f_k)\tau\mathrm{d}H_k(t/\tau), \quad w(0) = u_0, \tag{54}$$

with $H_k(t) := h_k(mt)$, which is the type of problem (50).

Example 10. (Fractional step) The splitting-up method described in the previous example for problem (43) can be generalised as follows. Let $p \geq m$ be an integer, $s_1, \ldots, s_p \in \mathbb{R}$ and $k_1, \ldots, k_p \in \{1, 2, \ldots, m\}$, such that $\sum_{j=1}^{p} s_j \delta_{rk_j} = 1$, for $r = 1, 2, \ldots, m$, where $\delta_{rk_j} = 1$ for $r = k_j$ and 0 otherwise. Then, we say that the product

$$S(\tau) := P_{k_p}(s_p \tau) \cdot \ldots \cdot P_{k_1}(s_1 \tau), \quad \tau \in (0, 1] \tag{55}$$

is a *fractional step method*, where $P_k(t)\varphi$ denotes the solution of (53). The *fractional step approximation*, $u(t) = u(\tau, t)$, $t \in T_\tau$ is defined as

$$u(\tau, t) = S^{t/\tau} u_0, \quad t \in T_\tau, \quad S^0 := I. \tag{56}$$

We characterise the fractional step method (55) by the absolutely continuous functions $\kappa_r = \kappa_r(t)$ on $\mathbb{R}$, for $r = 1, \ldots, m$, whose derivative $\dot{\kappa}_r(t)$ is periodic with period p, $\dot{\kappa}_r(t) := \sum_{j=1}^{p} s_j \delta_{rk_j} 1_{[j-1,j)}(t)$, for $t \in [0, p)$, and $\kappa_r(0) := 0$. We say that (55) is a symmetric product

$$\dot{\kappa}_r(p - t) = \dot{\kappa}_r(t) \quad \text{for } dt\text{-almost all } t \in (0, p), \quad r = 1, 2, \ldots, m.$$

The product

$$S(\tau) := P_1(\tau/2) \ldots P_{m-1}(\tau/2) P_m(\tau) P_{m-1}(\tau/2) \ldots P_1(\tau/2), \tag{57}$$

introduced and investigated in [31]–[32], is a simple example of a symmetric product. It is often called *Strang's splitting*. In the same way, as explained above, we can see that for $u(\tau, t)$ given by (56), we have

$$u(\tau, t) = w(\tau, t), \quad t \in T_\tau,$$

where $w(\tau, t)$ satisfies (54) with $H_r(t) := \kappa_r(pt)$.

Example 11. Let us now combine the fractional step method with the method of finite differences to solve (43) numerically. Namely, we solve (53) in each 'fractional' step by the Θ-method. Thus, we get the approximation $w(\tau, t_j)$, defined by (55)–(56) with

$$P_k(t)\varphi := (I - t\bar{\vartheta}_k L_k)^{-1}((I + t\vartheta_k L_k)\varphi + tf_k), \quad k \in \{1, 2, \ldots, m\},$$

provided $P_{k_j}(s_j \tau)$ is well-defined for for each j, where $\bar{\vartheta} := 1 - \vartheta$ and $\vartheta \in \mathbb{R}$ is the parameter of the Θ-method. Using the idea of rearranging in forward time from [9], we see that this approximation $u(\tau, t)$, $t \in T_\tau$ equals the solution of the equation

$$w(t) = u_0 + \int_0^t \sum_{k=1}^{m} (\Theta_k L_k w(s) + f_k)\, \tau dH_k(s/\tau),$$

at $t \in T_\tau$, where $H_k(t) := [\kappa_k(pt)]$, and Θ_k is defined by (51) with ϑ_k in place of ϑ.

Our aim is to present an expansion in powers of τ for the difference $w(\tau,t) - u(t)$ at the points t_i of the grid T_τ. To formulate our results, we use the notation $D_w([0,T] : V)$ and $C_w([0,T] : V)$ for the space of V-valued weakly right-continuos functions on $[0,T]$, having weak limits from the right, and for the space of V-valued weakly continuos functions on $[0,T]$, respectively. We equip these spaces with the supremum norm, and use the notation $W_j := D_w([0,T], V_j)$. To simplify the presentation of the results, assume that Θ_r in (50) is defined by

$$(\Theta_r\varphi)(t) := \vartheta_r\varphi(t-) + (1-\vartheta)\varphi(t)$$

for some $\vartheta_r \in \mathbb{R}$, $r = 1, 2\ldots, m$. Set $A_0(s) = s$ for $s \geq 0$.

Assumption 5.1. *(i) L_r is a bounded operator from V_{j+1} to V_j for $r = 1,\ldots,m$, and $\|L_r\varphi\|_j \leq K\|\varphi\|_{j+1}$,* for $\varphi \in V_{j+1}$, $j = 0,\ldots,l-1$;
(ii) $u_0 \in V_l$, $f_r \in V_l$, and $\|u_0\| \leq K$, $\|f_r\|_l \leq K$ for $r = 1,2,\ldots,m$.

Assumption 5.2. *For each $k = 0,1,\ldots,m$, there is a bounded linear operator $\mathcal{R}_k : W_0 \to W_0$, such that*

(i) $\sup_{t\in[0,T]} \|(\mathcal{R}_k g)(t)\|_j \leq K \sup_{t\in[0,T]} \|g(t)\|_j$, $g \in W_j$, $j = 0,1,\ldots,l$;
(ii) (existence) for any $g \in W_1$, the function $u = \mathcal{R}_k g$ satisfies

$$u(t) = \int_0^t Lu(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}A_k(s), \quad t \in [0,T] \tag{58}$$

(iii) (uniqueness) If $g_0,\ldots,g_m \in D_w([0,T] : V_0)$ and $u \in W_1$ satisfy

$$u(t) = \int_0^t Lu(s)\,\mathrm{d}s + \sum_{k=0}^m \int_0^t g_k(s)\,\mathrm{d}A_k(s), \quad t \in [0,T],$$

then $u = \sum_{k=0}^m \mathcal{R}_k g_k$.

Assumption 5.3. *For each $\tau(0,1]$, problems (43) and (50)) have unique solutions $u \in C_w([0,T] : V_l)$ and $w = w(\tau,t) \in D([0,T] : V_l)$, respectively, and*

$$\sup_{\tau\in(0,1]} \sup_{t\in[0,T]} \|w(\tau,t)\|_l < \infty.$$

Now we are in the position to formulate our first result on expansion in τ. To ease notation, we often suppress τ in some expressions.

Theorem 20. *Let $k \geq 0$ be an integer. Let Assumptions 5.1 5.2, and 5.3 hold with $l \geq 2(k+1)$. Then, there exist functions $v^{(1)},\ldots,v^{(k)} \in W_0$, independent from τ, and $R_k(\tau) \in W_0$, such that for all $\tau \in (0,1]$*

$$w(\tau,t) = u(t) + \tau v^{(1)}(t) + \tau^2 v^{(2)}(t) + \ldots + \tau^k v^{(k)}(t) + R_k(\tau,t) \tag{59}$$

for all $t \in T_\tau$, *and*

$$\sup_{t\in\mathbb{T}_\tau} \|R_k(\tau,t)\|_0 \le N\tau^{k+1}, \tag{60}$$

where N *is a constant, independent of* τ.

This theorem obviously implies that $w(\tau)$ approximates u on the grid T_τ, with accuracy proportional to τ. Moreover, for

$$u(\tau,t) := \sum_{j=0}^{k} \rho_j w(2^{-j}\tau,t),$$

where $(\rho_0,\ldots,\rho_k)$ is defined by (15) in Sect. 2, we get the following theorem.

Theorem 21. *Let the conditions of Theorem 20 hold. Then, for all* $\tau \in (0,1]$,

$$\max_{t\in T_\tau} \|u(\tau,t) - u(t)\|_0 \le N\tau^{k+1},$$

where N *is a constant independent of* τ.

It is worth emphasising that expansion (59) is valid only at the points t_i of the grid T_τ. This expansion follows from Theorem 22 below. To formulate it, we need more notation.

We call a sequence $\alpha = \alpha_1\alpha_2\ldots\alpha_i$ a multi-number of length $|\alpha| := i$, if $\alpha_j \in \{0,1,2,\ldots,m\}$. The set of all multi-numbers is denoted by $\mathcal{N}$. For every multi-number $\alpha = \alpha_1\ldots\alpha_j$, we define the operators $\Theta_\alpha := \Theta_{\alpha_j}$, $\bar{\Theta}_\alpha := \bar{\Theta}_{\alpha_j}$ on W_0, where $(\bar{\Theta}_{\alpha_j}\varphi)(t) := (1-\vartheta_j)\varphi(t-) + \vartheta_j\varphi(t)$ for $\alpha_j \neq 0$, and

$$\Theta_0\varphi := I\varphi = \varphi, \quad (\bar{\Theta}_0\varphi)(t) := \varphi(t-).$$

Moreover, for each $\gamma \in \mathcal{N}$, we define recursively an operator L_γ, $f_\gamma \in V_0$, a function b_γ and a number c_γ as follows

$$L_0 := 0, \quad f_0 := 0, \quad L_\gamma := L_r, \quad f_\gamma := f_r$$

$$b_0 := 0, \quad c_\gamma = c_r = c_0 = 0, \quad b_\gamma(t) := \tfrac{1}{\tau}(A_\gamma(t) - A_0(t)) \tag{61}$$

for $\gamma = r \in \{1,2,\ldots,m\}$, and

$$\begin{aligned} L_{\alpha 0} &:= LL_\alpha, \quad L_{\alpha r} := -L_\alpha L_r \\ f_{\alpha 0} &:= Lf_\alpha, \quad\ \ f_{\alpha r} := -L_\alpha f_r \end{aligned} \tag{62}$$

$$c_{\alpha\beta} := \frac{1}{\tau}\int_0^\tau \bar{\Theta}(\alpha) b_\alpha(s)\,\mathrm{d}A_\beta(s), \tag{63}$$

$$b_{\alpha\beta}(t) := \frac{1}{\tau}\left(\int_0^t \bar{\Theta}(\alpha) b_\alpha(s)\,\mathrm{d}A_\beta(s) - c_{\alpha\beta} A_0(t)\right) \tag{64}$$

for $\alpha \in \mathcal{N}$, $\beta \in \{0, 1, \ldots, m\}$, $r \in \{1, 2, \ldots, m\}$.

Let $\mathcal{M}$ denote the set of multi-numbers $\gamma_1\gamma_2\ldots\gamma_i$ with $\gamma_j \in \{1, 2, \ldots, m\}$, $j = 1, 2, \ldots, i$, and integers $i \geq 1$. Introduce the sequences $\sigma = (\beta_1, \beta_2, \ldots, \beta_i)$ of multi-numbers $\beta_j \in \mathcal{M}$, where $i \geq 1$ is any integer, and set

$$|\sigma| := |\beta_1| + |\beta_2| + \ldots + |\beta_i|.$$

Let $\mathcal{J}$ denote the set of all these sequences together with the 'empty sequence' e of length $|e| = 0$. For $\sigma = (\beta_1, \beta_2, \ldots, \beta_i)$, $i \geq 1$, we define

$$S_\sigma = \mathcal{R}L_{\beta_1} \cdot \ldots \cdot \mathcal{R}L_{\beta_i}, \quad \text{where} \quad \mathcal{R} := \mathcal{R}_0\Theta_0 = \mathcal{R}_0,$$

and for $\sigma = e$, we set $S_e := \mathcal{R}$. Notice that S_σ is well-defined as bounded linear operator from $W_{j+|\sigma|}$ to W_j if $j + |\sigma| \leq l$. If we have a collection of $g_\nu \in W_0$ indexed by a parameter ν taking values in a set A, then we use the notation $\sum^*_{\nu\in A} g_\nu$ for any linear combination of g_ν with coefficients depending only on c_α, A, and ν. For instance,

$$\sum_A^* S_\sigma w_\gamma = \sum_{(\sigma,\gamma)\in A}^* S_\sigma w_\gamma = \sum_{(\sigma,\gamma)\in A} c(\sigma, \gamma) S_\sigma w_\gamma,$$

where $c(\sigma, \gamma)$ are certain functions of $c_\alpha, \alpha \in \mathcal{N}$, and $(\sigma, \gamma) \in A$. These functions are allowed to change from one occurrence to another.

For $\mu = 0, \ldots, l, \kappa \geq 0$, any functions $u = u_\alpha(\tau, t)$ depending on the parameters $\alpha \in \mathcal{N}$ and τ, we write

$$u = O_\mu(\tau^\kappa) \quad \text{if} \quad \sup_{\tau\in(0,1]} \sup_{t\in[0,T]} \|u_\alpha(\tau, t)\|_\mu \tau^{-\kappa} < \infty.$$

Finally, set

$$A(i) = \{(\sigma, \beta) : \sigma \in \mathcal{J}, \beta \in \mathcal{M}, |\sigma| + |\beta| \leq i\},$$

$$B^*(i, j) = \{(\alpha, \beta) : \alpha \in \mathcal{N}, \beta \in \mathcal{M}, |\alpha| \leq i, |\beta| \leq j\}.$$

and $w_\beta(t) = w_\beta(\tau, t) := L_\beta w(\tau, t) + f_\beta, \quad u_\beta(t) := L_\beta u(t) + f_\beta.$

Theorem 22. *Under the assumptions of Theorem 20, we have*

$$w(t) = u(t) + \sum_{i=1}^k \tau^i \sum_{A(2i)}^* S_\sigma u_\beta(t) + \sum_{i=1}^k \tau^i \sum_{B^*(i,i+j)}^* b_\alpha(t) w_\beta(t) + O_0(\tau^{k+1}),$$

for all $t \in [0, T]$, where $b_\alpha(t) = b_\alpha(\tau, t)$ is defined by (61)–(64). Furthermore, if $k \geq 1$, then in the above equation,

$$\sum_{A(2)}^{*} S_\sigma u_\beta = \sum_{i,j=1}^{m} (c_{ij} - c_{j0})\mathcal{R}u_{ij},$$

so that it vanishes if $c_{ij} = c_{j0}$ *for all* $i, j = 1, \ldots, m$.

Theorem 22 immediately implies Theorem 20 by virtue of the following proposition, which can be proved by induction on the length of α and by change of variable $s := r\tau$ in the integrals in (63) and (64).

Proposition 1. *For every* $\alpha \in \mathcal{N}$, *the function* b_α *is* τ*-periodic, that is,* $b_\alpha(t+\tau) = b_\alpha(t)$ *for all* $t \geq 0$, *and* $b_\alpha(i\tau) = 0$ *for all integers* $i \geq 0$. *Moreover, the numbers* c_α, *the functions* $C_\alpha(t) := b_\alpha(\tau t)$, *and*

$$K_\alpha := \sup_{t \geq 0} |b_\alpha(t)| = \sup_{t \geq 0} |C_\alpha(t)| \tag{65}$$

are finite and do not depend on τ.

By Theorem 22, we can investigate the term $v^{(1)}$ in expansion (59). For example, in the case of fractional step methods, Example 10, we get the following specification of $v^{(1)}$.

Theorem 23. *Under the conditions of Theorem 20 in the case of Example 10, we have*

$$v^{(1)} = \frac{1}{2} \sum_{i,j=1}^{m} \int_0^p \big(\kappa_i(t)\, \mathrm{d}\kappa_i(t) - \kappa_j(t)\, \mathrm{d}\kappa_i(t)\big)\mathcal{R}u_{ij},$$

where $\mathcal{R}u_{ij}$ *is the solution of* (45) *with* $f = u_{ij} = L_i L_j u + L_i f_j$ *and 0 initial condition. Thus,* $v^{(1)}$ *vanishes if*

$$\int_0^p \big(\kappa_i(t)\, \mathrm{d}\kappa_j(t) - \kappa_j(t)\, \mathrm{d}\kappa_i(t)\big) = 0 \quad \text{for all } \ i, j = 1, 2, \ldots, m, \tag{66}$$

which is equivalent to

$$\int_0^p \kappa_i(t)\, \mathrm{d}\kappa_j(t) = \frac{1}{2} \quad \text{for all } \ 1 \leq i < j \leq m. \tag{67}$$

In particular, $v^{(1)} = 0$ *if* (55) *is a symmetric product, which is the case of, say Strang's splitting.*

Proof. By Theorem 22, expansion (59) holds with

$$v^{(1)} = \sum_{i,j=1}^{m} (c_{ij} - c_{j0})\mathcal{R}u_{ij},$$

so $v^{(1)} = 0$ if $c_{ij} - c_{j0} = 0$. Notice that for all $i, j = 0, 1, 2, \ldots, m$

$$c_{ij} = \int_0^p (\kappa_i(t) - \kappa_0(t))\,\mathrm{d}\kappa_j(t),$$

where $\kappa_0(t) := t/p$. Therefore,

$$2(c_{ij} - c_{j0}) = 2\int_0^p (\kappa_i(t) - \kappa_0(t))\,\mathrm{d}\kappa_j(t) - 2\int_0^p (\kappa_j(t) - \kappa_0(t))\,\mathrm{d}\kappa_0(t)$$

$$= 2\int_0^p \kappa_i(t)\,\mathrm{d}\kappa_j(t) - 2\kappa_0(p)\kappa_j(p) + \kappa_0^2(p) = 2\int_0^p \kappa_i(t)\,\mathrm{d}\kappa_j(t) - 1,$$

and we have,

$$2(c_{ij} - c_{j0}) = \int_0^p \kappa_i(t)\,\mathrm{d}\kappa_j(t) - \int_0^p \kappa_j(t)\,\mathrm{d}\kappa_i(t)$$

by taking into account

$$1 = \kappa_i(p)\kappa_j(p) = \int_0^p \kappa_i(t)\,\mathrm{d}\kappa_j(t) + \int_0^p \kappa_j(t)\,\mathrm{d}\kappa_i(t).$$

In particular, $c_{ij} - c_{j0} = -(c_{ji} - c_{i0})$, so $c_{ij} - c_{j0} = 0$ implies $c_{ji} - c_{i0} = 0$. Hence conditions (66), (67) and their equivalence follow immediately. If $\mathbb{S}(\tau)$ is a symmetric product, then obviously

$$\dot{\kappa}_i(p-t) = \dot{\kappa}_i(t) \quad \text{for all } t \in (0,p] \setminus \{1,\ldots,p\},$$

and $\kappa(t) + \kappa_i(p-t) = 1$ for all $t \in [0,p]$ and $i = 1,2,\ldots,p$. Hence

$$\begin{aligned}\int_0^p \kappa_i(t)\dot{\kappa}_j(t)\,\mathrm{d}t &= \int_0^p \kappa_i(p-s)\dot{\kappa}_j(p-s)\,\mathrm{d}s \\ &= \int_0^p (1-\kappa_i(s))\dot{\kappa}_j(s)\,\mathrm{d}s = 1 - \int_0^p \kappa_i(s)\dot{\kappa}_j(s)\,\mathrm{d}s,\end{aligned}$$

which immediately implies (67). The theorem is proved. □

Remark 14. Clearly, every symmetric product is a product of type (57) with respect to a new set of operators L_i' and free terms f_i', obtained from L_r and f_r by $L_1' := 2s_1 L_{k_1}$, $f_1' := 2s_1 f_{k_1}$,...

Remark 15. There are infinitely many nonsymmetric products, which still satisfy (67), and consequently define splitting-up approximations with accuracy of order τ^2. For example, when $m = 2$, every product of the form

$$\mathbb{P}(\tau) = \mathbb{P}^{(2)}_{(1-b)\tau}\mathbb{P}^{(1)}_{(1-a)\tau}\mathbb{P}^{(2)}_{b\tau}\mathbb{P}^{(1)}_{a\tau} \tag{68}$$

with $a \neq 1$, and $b = \frac{1}{2(1-a)}$, satisfies (67). If $a = \frac{1}{2}$, then (68) is Strang's product with $m = 2$. For $a \neq \frac{1}{2}$, these products are not symmetric.

Indeed, for κ_1, κ_2 characterising (68), we have

$$\dot{\kappa}_1(t) = a1_{[0,1)}(t) + (1-a)1_{[2,3)}(t), \quad \dot{\kappa}_2(t) = b1_{[1,2)}(t) + (1-b)1_{[3,4)}(t),$$

for $t \in (0, 4)$, and

$$\int_0^4 \kappa_1(t)\dot{\kappa}_2(t)\,\mathrm{d}t = ab + 1 - b = 1 - b(1-a) = \frac{1}{2},$$

that is, condition (67) holds. If $a \neq \frac{1}{2}$, then clearly (68) is not symmetric. If $a = \frac{1}{2}$, then $b = 1$, and (68) is Strang's symmetric product with $m = 2$.

For parabolic PDEs and systems of parabolic PDEs (see Examples 6 and 7), Assumptions 5.1 5.2 and 5.3 can be ensured by suitable smoothness and boundedness conditions on the coefficients and the free terms. Thus, we can apply the above results on accelerated schemes to the corresponding time discretizations of parabolic PDEs and systems of parabolic PDEs, in particular to symmetric hyperbolic system of first order PDEs. Hence one can also get results on accelerated numerical schemes for nonlinear ordinary differential equations by the method of characteristics (see [12]). For a more general framework of accelerated schemes, we refer to [12].

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Coarse-Grained Modeling of Multiscale Diffusions: The p-Variation Estimates

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Abstract We study the problem of estimating parameters of the limiting equation of a multiscale diffusion in the case of averaging and homogenization, given data from the corresponding multiscale system. First, we review some recent results that make use of the maximum likelihood of the limiting equation. In particular, it has been shown that in the averaging case, the MLE will be asymptotically consistent in the limit, while in the homogenization case, the MLE will be asymptotically consistent only if we subsample the data. Then, we focus on the problem of estimating the diffusion coefficient. We suggest a novel approach that makes use of the total p-variation, as defined in [15], and avoids the subsampling step. The method is applied to a multiscale OU process.

Keywords Diffusion estimation · multiscale Ornstein-Uhlenbeck process · p-variation

MSC (2010): 62M05, 60G17, 74Q99

1 Introduction

It is often the case that the most accurate models for physical systems are large in dimension and multiscale in nature. One of the main tasks for applied mathematicians is to find coarse-grained models of smaller dimension that can effectively describe the dynamics of the system and are efficient to use (see, for example [11, 12, 16, 17]). Once such a model is chosen, its free parameters are estimated by fitting the model to the existing data. Here, we study the challenges of this statistical estimation problem, in particular for the case where the coarse-grained model is a diffusion. Apart from the usual challenges of parameter estimation for diffusions, an additional problem that needs to be addressed in this setting is that of the mismatch

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between the full multiscale model that generated the data and the coarse-grained model that is fitted to the data. A first discussion of this issue, in the context of averaging and homogenization for multiscale diffusions, can be found in [1, 18, 19].

A similar statistical estimation problem arises in the context of "equation-free" modeling. In this case, coarse-grained equations exist only locally and are locally fitted to the data. The main idea of "equation-free" modeling is to use these locally fitted coarse-grained equations in combination with a global algorithm (for example, Newton-Raphson) in order to answer questions about the global dynamics of the coarse-grained model (for example, finding the roots of the drift). In this process, we go through the following steps: We simulate *short* paths of the system for given initial conditions. These are used to locally estimate the effective dynamics. Then, we carefully choose the initial conditions for the following simulations so that we reach an answer to whatever question we set on the global dynamics of the system, as quickly and efficiently as possible (see [14]). The statistical inference problem is similar to the one before: we have the data coming from the full model, we have a model for the effective local dynamics and we want to fit the data to this model. However, there is also an important difference: the available data is *short* paths of the full model. This issue has not been addressed in [18, 19] or [1], where it is assumed that the time horizon is either fixed or goes to infinity at a certain rate. We address this problem in Sect. 3, by letting the time horizon T be of order $\mathcal{O}(\varepsilon^{\alpha})$, where ε is the scale separation variable and $\alpha > 0$. Another important issue that we address here is that of estimating the scale separation variable ε.

We focus on a very simple Ornstein–Uhlenbeck model whose effective dynamics can be described by a scaled Brownian motion. This allows us to perform precise computations, reach definite conclusions and build our intuition about the behavior of more general diffusions. We only tackle the homogenization case and our goal be to estimate the diffusion coefficient of the effective dynamics. This problem has also been addressed in [1, 19]. In both these papers, the diffusion coefficient is constant. In fact, in [1], the authors also focus on the Ornstein–Uhlenbeck model. Our main contribution is to demonstrate that in order to compute the diffusion coefficient, one should not use the quadratic variation commonly defined as a limit where we let the size of a partition go to zero but rather as a supremum over all partitions. This definition is discussed in [15] and is at the core of the theory of rough paths, as it gives rise to a topology with respect to which the Itô map is continuous.

In Sect. 2, we review some of the core results for multiscale diffusions and their coarse-grained models. Then, we will review the results of [18, 19] and [1]. Finally, we give a more precise description of "equation-free" modeling.

In Sect. 3, we go on to define a new set of estimators for the diffusion parameter of the coarse-grained model, in the case of homogenization. We perform explicit computations of their L_2-error, which allows us to attest their performance. We conclude that they outperform the subsampled quadratic variance estimate studied in [1, 19]. Finally, we describe a heuristic way of estimating the scale separation parameter ε.

2 MLE for Multiscale Diffusions: A Review

In this section, we review some of the main concepts that come into play in multiscale modeling. First, we describe the limiting equations for multiscale stochastic differential equations. These allow us to reduce the dimension of the model. Then, we discuss the problem of the statistical estimation of parameters of the limiting equation given multiscale data and how this mismatch between model and data affects the result. Finally, we discuss a numerical algorithm that is applied when the limiting equations are completely unknown, which comes under the name of "equation-free" modeling.

2.1 Limiting Equations for Multiscale Diffusions

For reasons of consistency with the section that follows, the results of this section follow [18]. However, different versions of the same results – sometimes stronger – can be found in several places, such as [4, 6, 8, 9, 20].

There are two basic types of multiscale diffusions. The first is described by the following equations

$$\begin{aligned} dX_t &= f_1(X_t, Y_t)dt + \sigma_1(X_t, Y_t)dW_t \\ dY_t &= \frac{1}{\varepsilon^2} f_2(X_t, Y_t)dt + \frac{1}{\varepsilon}\sigma_2(X_t, Y_t)dV_t \end{aligned} \tag{1}$$

where $X_t \in \mathcal{X}$ and $Y_t \in \mathcal{Y}$ and $\mathcal{X}, \mathcal{Y}$ are finite dimensional Banach spaces. We call X the slow variable, Y the fast variable and ε the scale separation parameter. The main assumptions are the following:

Assumption 1. *(a) The solution of the system exists.*
(b) The equation

$$dY_t^x = \frac{1}{\varepsilon^2} f_2(x, Y_t^x)dt + \frac{1}{\varepsilon}\sigma_2(x, Y_t^x)dV_t$$

is ergodic with unique invariant measure μ_x, for every $x \in \mathcal{X}$.

We expect that by the time X takes a small step $\Delta \sim \mathcal{O}(1)$,

$$\frac{1}{\Delta}\int_t^{t+\Delta} f_1(X_s, Y_s)ds \approx \int f_1(X_t, y)\mu_{X_t}(dy)$$

as a result of the ergodicity of Y. Similarly,

$$\frac{1}{\Delta}\int_t^{t+\Delta} \sigma_1(X_s, Y_s)\sigma_1(X_s, Y_s)'ds \approx \int \sigma_1(X_t, y)\sigma_1(X_t, y)'\mu_{X_t}(dy).$$

where by $(\cdot)'$ we denote the transpose of a vector. We set

$$\bar{f}_1(x) = \int f_1(x,y)\mu_x(\mathrm{d}y), \ \ \bar{\sigma}_1(x) = \left(\int \sigma_1(x,y)\sigma_1(x,y)'\mu_x(\mathrm{d}y)\right)^{1/2}$$

and

$$\mathrm{d}\bar{X}_t = \bar{f}_1(\bar{X}_t)\mathrm{d}t + \bar{\sigma}_1(\bar{X}_t)\mathrm{d}W_t. \tag{2}$$

We call (2) the averaged limiting equation and $\bar{X}$ the averaged limit. We expect that $X_t \approx \bar{X}_t$, provided that they have the same initial conditions. Indeed, the following holds

Theorem 1 ([18]). *Let $\mathcal{X} = \mathbb{T}^{\ell}$ and $\mathcal{Y} = \mathbb{T}^{d-\ell}$, where $\mathbb{T}^k = [0,1]^k$ for any $k \geq 1$. We assume that all coefficients in (1) are smooth in both x and y and that the matrix $\Sigma_2(x,y) = \sigma_2(x,y)\sigma_2(x,y)'$ is positive definite, uniformly in x and y. Also, there exists a constant $C > 0$ such that*

$$\langle z, B(x,y)z\rangle \geq C|z|^2, \ \ \forall (x,y) \in \mathcal{X}\times\mathcal{Y} \text{ and } z \in \mathbb{R}^{d-\ell},$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product. Then, if $X_0 = \bar{X}_0$,

$$X \Rightarrow \bar{X} \text{ in } \mathcal{C}\left([0,T],\mathcal{X}\right).$$

The second basic type of a multiscale diffusion is described by the following equations

$$\begin{aligned} \mathrm{d}X_t &= \frac{1}{\varepsilon} f_1(X_t,Y_t)\mathrm{d}t \\ \mathrm{d}Y_t &= \frac{1}{\varepsilon^2} f_2(X_t,Y_t)\mathrm{d}t + \frac{1}{\varepsilon}\sigma_2(X_t,Y_t)\mathrm{d}V_t \end{aligned} \tag{3}$$

where $X_t \in \mathcal{X}$ and $Y_t \in \mathcal{Y}$ and $\mathcal{X},\mathcal{Y}$ are finite dimensional Banach spaces. As before, we call X the slow variable and Y the fast variable. In addition to assumption 1, we assume that

Assumption 2.

$$\int_{\mathcal{Y}} f_1(x,y)\mu_x(\mathrm{d}y) = 0, \ \ \forall x \in \mathcal{X}$$

where μ_x as defined in assumption 1.

Then, we expect that by the time X takes a small step $\Delta \sim \mathcal{O}(1)$,

$$\frac{1}{\Delta\varepsilon}\int_t^{t+\Delta} f_1(X_s,Y_s)\mathrm{d}s \approx \frac{1}{\Delta\varepsilon}\int_t^{t+\Delta} f_1(X_t,Y_s^{X_t})\mathrm{d}s$$

It follows from the Central Limit Theorem for ergodic Markov Processes (see [3]) that this will converge to a random number. More precisely, let us set

$$\bar{f}_1(x) = \int_{\mathcal{X}} \int_0^{\infty} f_1(x,y)\,(P_s \partial_x f_1(x,\cdot))\,(y)' \mu_x(\mathrm{d}y),$$

and

$$\bar{\tau}(x) = \left(2 \int_{\mathcal{X}} \int_0^{\infty} f_1(x,y)\,(P_s f_1(x,\cdot))\,(y)' \mu_x(\mathrm{d}y)\right)^{\frac{1}{2}},$$

where P_t are the transition kernels of the diffusion Y^x. Finally, we set

$$\mathrm{d}\bar{X}_t = \bar{f}_1(\bar{X}_t)\mathrm{d}t + \bar{\tau}(\bar{X}_t)\mathrm{d}W_t. \tag{4}$$

We call $\bar{X}$ the homogenized limiting equation. As before, we expect that $X_t \approx \bar{X}_t$, provided that they have the same initial conditions. Indeed, similar to the averaging case, we can prove the following:

Theorem 2 ([18]). *Let $\mathcal{X} = \mathbb{T}^{\ell}$ and $\mathcal{Y} = \mathbb{T}^{d-\ell}$. We assume that all coefficients in (3) are smooth in both x and y and that the matrix $\Sigma_2(x,y) = \sigma_2(x,y)\sigma_2(x,y)'$ is positive definite, uniformly in x and y. Also, there exists a constant $C > 0$ such that*

$$\langle z, B(x,y)z\rangle \geq C|z|^2, \quad \forall (x,y) \in \mathcal{X} \times \mathcal{Y} \quad \text{and} \quad z \in \mathbb{R}^{d-\ell},$$

where $\langle \cdot,\cdot \rangle$ denotes the Euclidean inner product. Then, if $X_0 = \bar{X}_0$ and assumption 2 holds, we get that

$$X \Rightarrow \bar{X} \quad \text{in} \quad \mathcal{C}([0,T],\mathcal{X}).$$

Theorems 1 and 2 allow us to replace the (X_t, Y_t) system by $\bar{X}_t$. If we are only interested in the slow dynamics of the system, this allows us to reduce the dimension of the problem. For example, using the limiting equations, we can simulate the slow dynamics of the process much faster, not only because of the dimension reduction but also because the dynamics of $\bar{X}$ do not depend on ε. Thus, the step of any numerical algorithm used to simulate the dynamics can be of order $\mathcal{O}(1)$ rather than $\mathcal{O}(\varepsilon^2)$, which would have been the case if we wanted to simulate the full multiscale system. However, in most cases, the drift and diffusion parameter of the limiting equation are not known in closed form and are approximated, with an additional computational cost. Thus, an efficient approximation is needed – which leads to the subject of the next section. Some results regarding the efficiency of the whole procedure (approximation and simulation of the limiting dynamics) can be found in [4].

2.2 Parameter Estimation for Multiscale Diffusions

The theory reviewed in Sect. 2.1 allows us to reduce the dimension of a multiscale system, approximating the slow dynamics by an diffusion of smaller dimension that does not have a multiscale structure anymore. In addition to multiscale diffusions, similar results hold for ordinary and partial differential equations (see [20]).

It is often the case that the dynamics of the full multiscale system – and consequently those of the limiting system – are not completely known. For example, in the case of multiscale diffusions, the drift and variance of the full system and thus the limiting system might depend on unknown parameters. This poses a statistical problem: how can we estimate these parameters give the multiscale data? In fact, it is even more realistic to ask to find the drift and diffusion coefficient of $\bar{X}$ given only X. This problem has been discussed in [1, 18, 19].

More precisely, in [19], the authors discuss the case where the drift of the limiting equation depends linearly on the unknown parameter while the diffusion parameter is constant. In [18], the authors extended the results of [19] for generic drift but did not discuss the problem of estimating the diffusion parameter. Finally, in [1], the authors extend the results in [19] by also proving the asymptotic normality of the estimators, but they limit their study to the Ornstein–Uhlenbeck system. The approach taken so far is the following:

(a) We pretend that the data come from the limiting equation, and we write down the corresponding maximum likelihood estimate (MLE) for the unknown parameters.
(b) We study whether the mismatch between model and data leads to errors, and if so, we try to find a way to correct them. It has been shown that in the limit as the scale separation parameter $\varepsilon \to 0$, the MLE corresponding to the averaged equation is consistent. However, this is not true in the case of homogenization. The method used so far to correct this problem has been that of subsampling the data by a parameter δ. Then, for $\delta \sim \mathcal{O}(\varepsilon^\alpha)$ and $\alpha \in [0, 2]$, it has been shown that the MLE that corresponds to the homogenized equation will be consistent in the limit $\varepsilon \to 0$. Also, an effort has been made to identify the optimal subsampling rate, i.e. the optimal α. However, since ε is usually an unknown, this is of little practical value.

Note that a separate issue is that of writing the maximum likelihood of the limiting diffusion, which in the general multi-dimensional case can still be challenging (see [2, 13]). We will not discuss this issue here, however.

We summarize the main results for the parameter estimation of the limiting equations of multiscale diffusions in the following theorems:

Theorem 3 (Drift estimation, averaging problem). *Suppose that $\bar{f}_1$ in (2) depends on unknown parameters θ, i.e. $\bar{f}_1(x) = \bar{f}_1(x;\theta)$. Let $\hat{\theta}(x;T)$ be the MLE of θ corresponding to (2). Suppose that we observe $\{X_t, t \in [0,T]\}$ of system (1) corresponding to $\theta = \theta_0$. Then, under appropriate assumptions described in [18] (Theorem 3.11), it is possible to show that*

$$\lim_{\varepsilon \to 0} dist\left(\hat{\theta}(X;T), \theta_\varepsilon\right) = 0, \quad \text{in probability}$$

where $dist(\cdot,\cdot)$ is the asymmetric Hausdorff semi-distance and θ_ε is a subset of the parameter space identified in the proof. Also

$$\lim_{\varepsilon \to 0} \mathrm{d}_H(\theta_\varepsilon, \theta_0) = 0, \quad \text{in probability}$$

where $\mathrm{d}_H(\cdot,\cdot)$ is the Hausdorff distance.

Theorem 4 (Drift estimation, homogenization problem). *Suppose that $\bar{f}_1$ in (4) depends on unknown parameters θ, i.e. $\bar{f}_1(x) = \bar{f}_1(x;\theta)$. Let $\hat{\theta}(x;N,\delta)$ be the maximizer of the discretized likelihood corresponding to (2) with step δ, where $T = N\delta$. Suppose that we observe $\{X_t, t \in [0,T]\}$ of system (3) corresponding to $\theta = \theta_0$. Then, under appropriate assumptions described in [18] (Theorem 4.5) and for $\delta = \varepsilon^\alpha$ with $\alpha \in (0,2)$ and $N = [\varepsilon^{-\gamma}]$ for $\gamma > \alpha$, it is possible to show that*

$$\lim_{\varepsilon \to 0} \hat{\theta}(X;N,\delta) = 0, \quad \text{in probability.}$$

The next two theorems deal with the estimation of the diffusion parameter of the limiting equation, given that this is constant. In that case, the MLE is the Quadratic Variation of the process. They assume that the dimension of the slow variable is 1.

Theorem 5 (Diffusion estimation, averaging problem). *Let X be the solution of (1) for $\bar{\sigma}_1 \equiv \theta$ a constant. Then, under appropriate conditions described in [19] (Theorem 3.4) and for every $\varepsilon > 0$, we have that*

$$\lim_{\delta \to 0} \frac{1}{N\delta} \sum_{n=0}^{N-1} |X_{(n+1)\delta} - X_{n\delta}|^2 = \theta^2 \ \text{ a.s.}$$

where $T = N\delta$ is fixed.

Theorem 6 (Diffusion estimation, homogenization problem). *Let X be the solution of (3), such that $\bar{\tau} \equiv \theta$ appearing in (4) is a constant. Then, under appropriate conditions described in [19] (Theorem 3.5) and for $\delta = \varepsilon^\alpha$ with $\alpha \in (0,1)$, we have that*

$$\lim_{\delta \to 0} \frac{1}{N\delta} \sum_{n=0}^{N-1} |X_{(n+1)\delta} - X_{n\delta}|^2 = \theta^2 \ \text{ a.s.}$$

where $T = N\delta$ is fixed.

It is conjectured that Theorem 6 should hold for any $\alpha \in (0,2)$ and that the optimal α, i.e. the one that minimizes the error, is $\alpha = \frac{4}{3}$. The reasoning is the following: there are two competing errors, one coming from the Monte-Carlo averaging, which should be of order $\frac{1}{\sqrt{N}} \propto \varepsilon^{\frac{\alpha}{2}}$, and the other one coming from homogenization, which we expect to be of order $\frac{\varepsilon^2}{\delta} \propto \varepsilon^{2-\alpha}$. To achieve optimal performance, these two errors should be balanced.

Clearly, the most interesting case is that of estimating the diffusion parameter of the homogenized system. This is the case that we study in detail in Sect. 3, assuming that the process is an Ornstein–Uhlenbeck process. Also, note that when estimating the diffusion parameter, the length of the time interval T is fixed. We will relax this condition later on, for reasons explained in the following section.

2.3 Equation-Free Modeling

In practical applications, it is often the case that the limiting equations (2) and (4) are completely unknown. More generally, let us say that we have good reasons to believe that a certain variable of a multiscale system that evolves slowly behaves like a diffusion at a certain scale but we have complete ignorance of its drift and diffusion coefficients. We would like to find a way to estimate these coefficients. In statistical terms, let us say that we are interested in the non-parametric estimation of the drift and diffusion coefficients of the limiting equation. Note that our data come "on demand" but for a certain cost, by simulating the multiscale model for given conditions.

A general algorithm for answering questions regarding the limiting dynamics of a quantity coming from a multiscale system that evolves slowly, when these are not explicitly known, comes under the name of "equation-free" algorithm (see [14] and also [5] for a similar approach). In our case, this would suggest pairing the problem of local estimation with an interpolation algorithm in order to estimate the drift and diffusion functions, denoted by $\bar{f}(x)$ and $\bar{\sigma}(x)$ respectively. We make this more concrete by describing the corresponding algorithm:

(0) Choose some initial condition x_0 and approximate $\bar{f}(x)$ and $\bar{\sigma}(x)$ by a local (polynomial) approximation around x_0. Simulate short paths of the multiscale system, so that the local approximation is acceptable. Note that the smaller the path, the better or simpler the local approximation.
(1) For $n \geq 1$, choose another starting point x_n using the knowledge of $\bar{f}(x_{n-1})$ and $\bar{\sigma}(x_{n-1})$ and possibly some of their derivatives on x_{n-1}, according to the rules of your interpolation algorithm.
(2) Repeat step 0, replacing x_0 by x_n.

As mentioned above, the size of the path T needs to be small and possibly comparable to ε. This is what led us to consider the estimation problem for $T = \varepsilon^{\alpha}$.

3 The p-Variation Estimate

In this section, we study the problem of estimating the diffusion parameter of the homogenization limit of a simple multiscale Ornstein–Uhlenbeck process. We hope that the detailed analysis will provide some intuition for the general problem.

Consider the following system:

$$\begin{aligned} dY_t^{1,\varepsilon} &= \frac{\sigma}{\varepsilon} Y_t^{2,\varepsilon} dt \\ dY_t^{2,\varepsilon} &= -\frac{1}{\varepsilon^2} Y_t^{2,\varepsilon} dt + \frac{1}{\varepsilon} dW_t \end{aligned} \tag{5}$$

with initial conditions $Y_0^{1,\varepsilon} = y_1$ and $Y_0^{2,\varepsilon} = y_2$. It is not hard to see that the homogenization limit as $\varepsilon \to 0$ is

$$Y_t^{1,\varepsilon} \to y_1 + \sigma W_t$$

and the convergence holds in a strong sense:

$$\sup_{t\in[0,T]} |Y_t^{1,\varepsilon} - y_1 - \sigma W_t| \xrightarrow{L_1} 0, \quad \text{as } \varepsilon \to 0. \tag{6}$$

Note that for this particular example, $Y_t^{1,\varepsilon}$ is exactly equal to

$$Y_t^{1,\varepsilon} = y_1 + \sigma W_t - \varepsilon\sigma \left(Y_t^{2,\varepsilon} - y_2\right),$$

and thus proving (6) is equivalent to proving that

$$\varepsilon \sup_{t\in[0,T]} |Y_t^{2,\varepsilon} - y_2| \xrightarrow{L_1} 0, \quad \text{as } \varepsilon \to 0.$$

This follows from [10].

We want to estimate the diffusion parameter σ given a path $\{Y_t^{1,\varepsilon}(\omega)\,; t \in [0, T]\}$, i.e. we assume we only observe the slow scale of the diffusion. If we were to follow the approach discussed in the previous section, we would use the maximum likelihood estimate that corresponds to the limiting equation. In this case, this would be the quadratic variation. However, as discussed earlier, this is not a good estimate since the quadratic variation for any fixed $\varepsilon > 0$ is zero. To correct this, we subsample the data, which leads to the following estimate:

$$\hat{\sigma}_\delta^2 = \frac{1}{N\delta} \sum_{i=1}^{N} \left(Y_{i\delta}^{1,\varepsilon} - Y_{(i-1)\delta}^{1,\varepsilon}\right)^2, \quad \text{for } N = \frac{T}{\delta} \tag{7}$$

The asymptotic behavior of this estimate has been studied in [1, 19]. In fact, taking advantage of the simplicity of the model, we can compute the L_2-error exactly, as a function of δ, ε and N. We find that

$$\begin{aligned}\frac{1}{\sigma^4}\mathbb{E}\left(\hat{\sigma}_\delta^2 - \sigma^2\right)^2 = {}& \frac{\varepsilon^4}{\delta^2}\left(1 - \mathrm{e}^{-\frac{\delta}{\varepsilon^2}}\right)^2 \\ & + \left(2 - 4\frac{\varepsilon^2}{\delta}\left(1 - \mathrm{e}^{-\frac{\delta}{\varepsilon^2}}\right) + \frac{\varepsilon^4}{\delta^2}\left(1 - \mathrm{e}^{-\frac{\delta}{\varepsilon^2}}\right)^2 \frac{3 + \mathrm{e}^{-\frac{\delta}{\varepsilon^2}}}{1 + \mathrm{e}^{-\frac{\delta}{\varepsilon^2}}}\right)\left(\frac{1}{N}\right) \\ & + \frac{\varepsilon^4}{\delta^2}\left(\frac{1 - \mathrm{e}^{-\frac{\delta}{\varepsilon^2}}}{1 + \mathrm{e}^{-\frac{\delta}{\varepsilon^2}}}\right)^2 \left(\frac{\mathrm{e}^{-\frac{2\delta N}{\varepsilon^2}} - 1}{N^2}\right)\end{aligned} \tag{8}$$

For reasons explained earlier, we are interested in the behavior of this error not only when T is fixed but also for $T \to 0$. Thus, we set $T = \varepsilon^{\alpha}$ and, as before, $\delta = \varepsilon^{\alpha+\beta}$, which lead to $N = \varepsilon^{-\beta}$. We are interested in the behavior of the error as $\varepsilon \to 0$. For these choices of T and δ, the square error will be

$$\mathbb{E}\left(\hat{\sigma}_{\delta}^{2} - \sigma^{2}\right)^{2} \sim \mathcal{O}\left(\varepsilon^{4-2(\alpha+\beta)} + \varepsilon^{2-\alpha} + \varepsilon^{\beta}\right) \tag{9}$$

For α fixed, we see that the error will be small if $0 < \beta < 2-\alpha$. In fact, the optimal choice for β is $\beta = \frac{4-2\alpha}{3}$, in which case the error becomes

$$\left(\mathbb{E}\left(\hat{\sigma}_{\delta}^{2} - \sigma^{2}\right)^{2}\right)^{\frac{1}{2}} \sim \mathcal{O}\left(\varepsilon^{\frac{2-\alpha}{3}}\right) \tag{10}$$

So, for $\alpha = 0$, we get that the optimal sub sampling rate is $\beta = \frac{4}{3}$, which results in an optimal error of order $\mathcal{O}\left(\varepsilon^{\frac{2}{3}}\right)$. However, if $\alpha > 0$, the error can increase significantly, especially for non-optimal choices of δ.

In the rest of this section, we are going to investigate the behavior of the p-variation norm as an estimator of σ. The intuition comes from the following observation: we know that at scale $\mathcal{O}(1)$, $\{Y_t^{1,\varepsilon}(\omega)\,; t \in [0,T]\}$ behaves like scaled Brownian motion, while at scale $\mathcal{O}(\varepsilon)$, it is a process of bounded variation (finite length). Could it be that at scale $\mathcal{O}(\varepsilon^{\alpha})$, the process behaves like a process of finite p-variation, for some p that depends on α? If so, would the p-variation norm be a better estimator of σ?

3.1 The Total p-Variation

We say that a real-valued continuous path $X : [0,T] \to \mathbb{R}$ has finite total p-variation if

$$D_p(X)_T := \sup_{\mathcal{D}([0,T])} \left(\sum_{t_\ell \in \mathcal{D}([0,T])} |X_{t_{\ell+1}} - X_{t_\ell}|^p \right)^{1/p} < +\infty, \tag{11}$$

where $\mathcal{D}([0,T])$ goes through the set of all finite partitions of the interval $[0,T]$ (see also [15]). It is clear by the definition that a process of bounded variation will always have finite total p-variation for any $p > 1$. Also, note that the total p-variation as defined above will only be zero if the process is constant. Thus, the total p-variation of a non-constant bounded variation process will always be a positive number.

For $\varepsilon > 0$ fixed, the process $Y^{1,\varepsilon} : [0,T] \to \mathbb{R}$ defined in (5) is clearly of bounded variation, but its total variation is of order $\mathcal{O}\left(\frac{T}{\varepsilon}\right)$. We will say that at scale

$\mathcal{O}(\varepsilon^{\alpha})$, the process $Y^{1,\varepsilon}$ behaves like a process of finite total p-variation *in the limit* if

$$\lim_{\varepsilon\to 0}\left(D_p\left(Y^{1,\varepsilon}\right)_{\varepsilon^{\alpha}}\right) < +\infty \text{ and } \forall q < p,\ \lim_{\varepsilon\to 0}\left(D_q\left(Y^{1,\varepsilon}\right)_{\varepsilon^{\alpha}}\right) = +\infty. \tag{12}$$

We will prove the following:

Theorem 7. *At scale $\mathcal{O}(\varepsilon^{\alpha})$ and $1 < \alpha < 2$, the process $Y^{1,\varepsilon} : [0,T] \to \mathbb{R}$ defined in (5) behaves like a process of finite total $(2-\alpha)$-variation in the limit.*

First, we prove the following lemma:

Lemma 1. *Let $X : [0,T] \to \mathbb{R}$ be a real-valued differentiable path of bounded variation. Then, its total p-variation is given by*

$$D_p(X)_T := \sup_{\mathcal{E}([0,T])}\left(\sum_{t_\ell\in\mathcal{E}([0,T])}|X_{t_{\ell+1}} - X_{t_\ell}|^p\right)^{1/p}, \tag{13}$$

where $\mathcal{E}([0,T])$ goes through all finite sets of extremals of X in the interval $[0,T]$.

Proof. Consider the function

$$f_{a,b}(t) = |X_t - X_a|^p + |X_b - X_t|^p, \quad a < t < b.$$

This is maximized for t an extremal point ($\dot{X}_t = 0$) or at $t = a$ or $t = b$. Thus, if $\mathcal{D} = \{0, t_1, \ldots, t_{n-1}, t_n = T\}$, there exists a set of extremals $\mathcal{E}$ with cardinality $|\mathcal{E}| \le n+1$, such that

$$\sum_{t_\ell\in\mathcal{D}}|X_{t_{\ell+1}} - X_{t_\ell}|^p \le \sum_{t_\ell\in\mathcal{E}}|X_{t_{\ell+1}} - X_{t_\ell}|^p.$$

The set $\mathcal{E}$ can be constructed by choosing τ_1 so that $f_{0,t_2}(t)$ is maximized and τ_k so that $f_{\tau_{k-1},t_{k+1}}(t)$ is maximized, for $k = 2,\ldots,n-1$. Thus,

$$\sup_{\mathcal{D}([0,T])}\left(\sum_{t_\ell\in\mathcal{D}([0,T])}|X_{t_{\ell+1}} - X_{t_\ell}|^p\right)^{1/p} \le \sup_{\mathcal{E}([0,T])}\left(\sum_{t_\ell\in\mathcal{E}([0,T])}|X_{t_{\ell+1}} - X_{t_\ell}|^p\right)^{1/p}.$$

The opposite inequality is obvious and completes the proof.

To prove the theorem, first we notice that

$$D_p(Y^{1,\varepsilon})_T = \varepsilon\sigma D_p(Z^1)_{\frac{T}{\varepsilon^2}}, \tag{14}$$

where (Z^1, Z^2) satisfy

$$\begin{aligned} dZ_t^1 &= Z_t^2 dt \\ dZ_t^2 &= -Z_t^2 dt + dW_t \end{aligned}$$

Now, Z^1 is clearly differentiable and thus, by the lemma

$$D_p(Z^1)_T = \sup_{\mathcal{E}([0,T])} \left(\sum_{t_\ell \in \mathcal{E}([0,T])} |Z^1_{t_{\ell+1}} - Z^1_{t_\ell}|^p \right)^{\frac{1}{p}}$$

The derivative of Z^1 is equal to Z^2, so all its extremal points correspond to zero-crossings of Z^2. So, for $s, t \in \mathcal{E}$,

$$Z_t^1 - Z_s^1 = (W_t - W_s) - \left(Z_t^2 - Z_s^2\right) = W_t - W_s$$

and $D_p(Z^1)_T$ becomes

$$\begin{aligned} D_p(Z^1)_T &= \sup_{\mathcal{E}([0,T])} \left(\sum_{t_\ell \in \mathcal{E}([0,T])} |W_{t_{\ell+1}} - W_{t_\ell}|^p \right)^{\frac{1}{p}} \\ &= \lim_{\delta \to 0} \left(\sum_{t_\ell \in \mathcal{E}_\delta([0,T])} |W_{t_{\ell+1}} - W_{t_\ell}|^p \right)^{\frac{1}{p}}, \end{aligned} \tag{15}$$

where

$$\mathcal{E}_\delta\left([0,T]\right) = \{0 = t_0, t_1, \ldots, t_{N_\delta(T)}, T\}$$

and $\{t_1, \ldots, t_{N_\delta(T)}\}$ is the set of all zero-crossings of Z^2 in $[0, T]$ that are at least distance δ apart from each other, i.e. if $t_k \in \mathcal{E}_\delta\left([0,T]\right)$ and $k < N_\delta(T)$, then t_{k+1} is the first time that Z^2 crosses zero after time $t_k + \delta$. Note that the set of zero-crossings of Z^2 in $[0, T]$ is an uncountable set that contains no intervals with probability 1. Equation (15) follows from the following two facts: (a) in general, adding any point to the partition will increase the L_p norm and thus the supremum is achieved for a countable set of zero-crossings and (b) any countable set that is dense in the set of all zero-crossings will give the same result.

If τ_δ is the stopping time of the first zero crossing of Z^2 after δ given $Z_0^2 = 0$, then the random variables $\{\tau_k^\delta = (t_k - t_{k-1}), t_k \in \mathcal{E}_\delta\left([0,T]\right), k \leq N_\delta(T)\}$ are i.i.d. with the same law as that of τ_δ. Thus, the sum $\sum_{t_\ell \in \mathcal{E}_\delta([0,T])} |W_{t_{\ell+1}} - W_{t_\ell}|^p$ is a sum of i.i.d. random variables of finite mean (to be computed in the following section), and as a consequence of the Law of Large Numbers, it grows like $N_\delta(T)$. From [7], we know that $N_\delta(T) \sim \mathcal{O}\left(\frac{T}{\varepsilon}\right)$. We conclude that

$$D_p(Z^1)_T \sim \mathcal{O}\left(T^{\frac{1}{p}}\right).$$

Finally, from (14), it is clear that

$$D_p(Y^{1,\varepsilon})_{\varepsilon^\alpha} \sim \mathcal{O}\left(\varepsilon\left(\frac{\varepsilon^\alpha}{\varepsilon^2}\right)^{\frac{1}{p}}\right) \sim \mathcal{O}\left(\varepsilon^{1+\frac{\alpha-2}{p}}\right),$$

which proves the theorem.

3.2 The p-Variation Estimates

Similar to the quadratic variation estimate $\hat{\sigma}^2$ defined in (7), we define the p-variation estimates as the properly normalized total p-variation of the process:

$$\hat{\sigma}^p := \frac{1}{C_p(T)}\left(D_p(Y^{1,\varepsilon})_T\right)^p. \tag{16}$$

We will study the L_2-error of this estimate in different scales. First, we need to define the constant $C_p(T)$. The natural choice would be to choose $C_p(T)$ so that $\mathbb{E}\left(\hat{\sigma}^p\right) = \sigma^p$. So,

$$C_p(T) = \frac{1}{\sigma^p}\mathbb{E}\left(\left(D_p(Y^{1,\varepsilon})_T\right)^p\right).$$

We need to compute $\mathbb{E}\left(\left(D_p(Y^{1,\varepsilon})_T\right)^p\right)$. From (14), we get that

$$\mathbb{E}\left(\left(D_p(Y^{1,\varepsilon})_T\right)^p\right) = \varepsilon^p\sigma^p\mathbb{E}\left(\left(D_p(Z^1)_{\frac{T}{\varepsilon^2}}\right)^p\right).$$

Using (15), we get that

$$\mathbb{E}\left(\left(D_p(Z^1)_T\right)^p\right) = \lim_{\delta\to 0}\mathbb{E}\left(\sum_{t_\ell\in\mathcal{E}_\delta([0,T])}|W_{t_{\ell+1}} - W_{t_\ell}|^p\right),$$

Note that for any $p > 1$, $D_p(Z^1)_T \leq D_1(Z^1)_T$, where $\mathbb{E}\left(D_1(Z^1)_T^p\right) < +\infty$. Thus, from the Dominated Convergence Theorem, the limit can come out of the expectation. To simplify our computations, from now on, we will assume that $Z_0^2 = Z_T^2 = 0$. We have already observed that the random variables $\{(W_{t_{\ell+1}} - W_{t_\ell}),\ t_\ell \in \mathcal{E}_\delta\left([0,T]\right), \ell < N_\delta(T)\}$ are independent and distributed like W_{τ_δ}, where τ_δ is the first time Z^2 crosses zero after $t = \delta$, given that $Z_0^2 = 0$. Thus,

$$\mathbb{E}\left(\sum_{t_\ell\in\mathcal{E}_\delta([0,T])}|W_{t_{\ell+1}} - W_{t_\ell}|^p\right) = \mathbb{E}N_\delta(T)\,\mathbb{E}|W_{\tau_\delta}|^p + \mathbb{E}|W_T - W_{t_{N_\delta(T)}}|^p,$$

where $N_\delta(T)$ is the number of zero-crossings of Z^2 in interval $[0, T]$ that are distance δ apart from each other. First, we notice that

$$\mathbb{E}|W_{\tau_\delta}|^p = \mathbb{E}\left(\mathbb{E}\left(|W_{\tau_\delta}|^p \middle| \tau_\delta\right)\right) = \frac{1}{\sqrt{\pi}} 2^{\frac{p}{2}} \Gamma\left(\frac{p+1}{2}\right) \mathbb{E}\left((\tau_\delta)^{\frac{p}{2}}\right).$$

To compute $\mathbb{E}\left((\tau_\delta)^p\right)$, we note that τ_δ can be written as $\tau_\delta = \delta + \tau(Z^2_\delta)$, where $\tau(z)$ is the first zero-crossing of the process Z^2 given that it starts at z. For Z^2, an Ornstein–Uhlenbeck process, the p.d.f. of $\tau(z)$ has been computed explicitly (see [21]) and is given by

$$f(t, z) = \frac{2}{\sqrt{\pi}} \frac{|z|\mathrm{e}^{-t}}{(1-\mathrm{e}^{-2t})^{\frac{3}{2}}} \exp\left(-\frac{z^2\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}}\right).$$

Since $Z^2_0 = 0$ by assumption, Z^2_δ is a Gaussian random variable with zero mean and variance $\frac{1}{2}(1-\mathrm{e}^{-2\delta})$. Let us denote its p.d.f. by $g_\delta(z)$. It follows that the p.d.f. of $\tau(Z^2_\delta)$ is given by

$$h_\delta(t) := \int_{-\infty}^{\infty} f(t, z) g_\delta(z)\mathrm{d}z = \frac{4\mathrm{e}^t \operatorname{csch}(\delta + t)\sinh(\delta)\sinh(t)}{\sqrt{(1-\mathrm{e}^{-2\delta})(1-\mathrm{e}^{-2t})}(-1+\mathrm{e}^{2t})\pi} \tag{17}$$

where $\operatorname{csch}(t) = \frac{1}{\sinh(t)}$ and $\sinh(t)$ the hyperbolic sine. We write

$$\begin{aligned}\lim_{\delta\to 0} \frac{1}{\sqrt{\delta}} \mathbb{E}\left((\tau_\delta)^p\right) &= \lim_{\delta\to 0} \frac{1}{\sqrt{\delta}} \int_0^\infty (\delta + t)^p h_\delta(t)\mathrm{d}t \\ &= \lim_{\delta\to 0} \int_0^\infty \frac{(\delta+t)^p h_\delta(t)}{\sqrt{\delta} t H(t)} t H(t)\mathrm{d}t.\end{aligned}$$

where

$$H(t) = \frac{4\mathrm{e}^{-t}\sqrt{\mathrm{e}^{-t}\sinh(t)}}{(1-\mathrm{e}^{-2t})^2\pi} \quad \text{and} \quad \int_0^\infty t H(t)\mathrm{d}t = \sqrt{2}.$$

The function $\frac{(\delta+t)^p h_\delta(t)}{\sqrt{\delta} t H(t)}$ is increasing to t^{-1+p} as $\delta \downarrow 0$, and thus by the dominated theorem, we find that

$$K_p := \int_0^\infty t^p H(t)\mathrm{d}t \tag{18}$$

Notice that for $t \to 0$, $H(t)$ behaves like $t^{-\frac{3}{2}}$ and thus the integral K_p is finite if and only if $p > \frac{1}{2}$. Also, for $p = 1, 2$ we find that $K_1 = \sqrt{2}$ and $K_2 = 2\sqrt{2}\log 2$.

Now, we need to compute the limit of $\sqrt{\delta}\mathbb{E}N_\delta(T)$ as $\delta \to 0$. We can use the results in [7] to get an upper and lower bound and show that $N_\delta(T)$ behaves like $\mathcal{O}\left(\frac{T}{\sqrt{\delta}}\right)$. However, we need to know the exact value of the limit. We proceed as follows: we write

$$\mathbb{E}N_\delta(T) = \sum_{n=1}^{\infty} \mathbb{P}\left(N_\delta(T) \geq n\right) = \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} \tau_i^\delta \leq T\right), \tag{19}$$

where $\tau_i^\delta = t_i - t_{i-1}$ for $t_i \in \mathcal{E}_\delta\left([0,T]\right)$ and $i \leq N_\delta(T)$. Using (17), we find that the Laplace transform of the distribution of τ_δ is

$$\hat{H}_\delta\left(\lambda\right) = \mathrm{e}^{-\lambda\delta}\hat{h}_\delta\left(\lambda\right) = \frac{2\mathrm{e}^{-d(\lambda+1)}\Gamma(\frac{\lambda+1}{2})\sinh(d)}{\sqrt{\pi(1-\mathrm{e}^{-2d})}}\bar{F}_1\left(1, \frac{\lambda+1}{2}, \frac{\lambda+2}{2}, \mathrm{e}^{-2d}\right), \tag{20}$$

where $\bar{F}_1\left(a,b,c,x\right)$ is the regularized hypergeometric function given by

$$\bar{F}_1\left(a,b,c,x\right) = \frac{1}{\Gamma(c)}\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \text{ and } (d)_n = \prod_{k=0}^{n-1}(d+k).$$

We find that for small $d > 0$, this behaves like

$$\hat{H}_\delta\left(\lambda\right) = 1 - \frac{2\sqrt{2}}{\sqrt{\pi}}\frac{\Gamma(\frac{\lambda+1}{2})}{\Gamma(\frac{\lambda}{2})}\sqrt{d} + \mathcal{O}\left(\delta\right). \tag{21}$$

Since the τ_i^δ's are i.i.d., the Laplace transform of the sum $\sum_{i=1}^{n}\tau_i^\delta$ will be $\hat{H}_\delta\left(\lambda\right)^n$ and thus we write

$$\mathbb{P}\left(\sum_{i=1}^{n}\tau_i^\delta \leq T\right) = \int_0^T \mathcal{L}^{-1}[\hat{H}_\delta\left(\lambda\right)^n](\mathrm{d}t),$$

where $\mathcal{L}^{-1}$ denotes the operator of the inverse Laplace transform. Substituting this back to (19), we get

$$\begin{aligned}\mathbb{E}N_\delta(T) &= \sum_{n=1}^{\infty}\int_0^T \mathcal{L}^{-1}[\hat{H}_\delta\left(\lambda\right)^n](\mathrm{d}t)\\ &= \int_0^T \mathcal{L}^{-1}\left[\sum_{n=1}^{\infty}\hat{H}_\delta\left(\lambda\right)^n\right](\mathrm{d}t)\\ &= \int_0^T \mathcal{L}^{-1}\left[\frac{\hat{H}_\delta\left(\lambda\right)}{1-\hat{H}_\delta\left(\lambda\right)}\right](\mathrm{d}t).\end{aligned} \tag{22}$$

Taking the limit inside the operator, we finally see that

$$\lim_{\delta\to\infty}\sqrt{\delta}\mathbb{E}\left(N_\delta(T)\right) = \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^T \mathcal{L}^{-1}\left[\frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}\right)}\right](\mathrm{d}t) = \frac{T}{\sqrt{2}}. \tag{23}$$

Finally, we note that since $Z^2_T = 0$ by assumption,

$$(T - t_{N_\delta(T)}) < \delta \Rightarrow \lim_{\delta \to 0} \mathbb{E}|W_T - W_{t_{N_\delta(T)}}|^p = 0.$$

For every $p > 1$, we set

$$a_p := \frac{1}{\sqrt{\pi}} 2^{\frac{p}{2}} \Gamma\left(\frac{p+1}{2}\right) \text{ and } c_p := \frac{a_p}{\sqrt{2}} K_{\frac{p}{2}}. \tag{24}$$

Putting everything together, we find that

$$\mathbb{E}\left(\left(D_p(Z^1)_T\right)^p\right) := c_p T \tag{25}$$

and consequently

$$\mathbb{E}\left(\left(D_p(Y^{1,\varepsilon})_T\right)^p\right) = \varepsilon^p \sigma^p c_p \frac{T}{\varepsilon^2} = \varepsilon^{p-2} \sigma^p c_p T.$$

Thus, we set

$$C_p(T) := \varepsilon^{p-2} c_p T. \tag{26}$$

By construction, the p-variation estimates $\hat{\sigma}^p$ defined in (16) are consistent, i.e. $\mathbb{E}\left(\hat{\sigma}^p\right) = \sigma^p$. We now compute its square L_2-error:

$$\begin{aligned}
\mathbb{E}\left(\hat{\sigma}^p - \sigma^p\right)^2 &= \mathbb{E}\left(\frac{\left(D_p(Y^{1,\varepsilon})_T\right)^p}{C_p(T)} - \sigma^p\right)^2 \\
&= \mathbb{E}\left(\frac{\left(D_p(Y^{1,\varepsilon})_T\right)^{2p}}{C_p(T)^2}\right) - \sigma^{2p} \\
&= \frac{1}{C_p(T)^2} \mathbb{E}\left(\left(D_p(Y^{1,\varepsilon})_T\right)^{2p}\right) - \sigma^{2p} \\
&= \frac{\varepsilon^{2p}\sigma^{2p}}{\varepsilon^{2p-4} c_p^2 T^2} \mathbb{E}\left(\left(D_p(Z^1)_{\frac{T}{\varepsilon^2}}\right)^{2p}\right) - \sigma^{2p} \\
&= \sigma^{2p}\left(\frac{\varepsilon^4}{c_p^2 T^2} \mathbb{E}\left(\left(D_p(Z^1)_{\frac{T}{\varepsilon^2}}\right)^{2p}\right) - 1\right)
\end{aligned} \tag{27}$$

To proceed, we need to compute the second moment of $\left(D_p(Z^1)_T\right)^p$. As with the computation of the first moment, we write:

$$\begin{aligned}\mathbb{E}\left(\left(D_p(Z^1)_T\right)^{2p}\right) &= \lim_{\delta\to 0}\mathbb{E}\left(\sum_{t_\ell\in\mathcal{E}_\delta([0,T])}|W_{t_{\ell+1}}-W_{t_\ell}|^p\right)^2\\ &= \lim_{\delta\to 0}\mathbb{E}\left(\sum_{n=1}^{N_\delta(T)}|W_{\tau_n^\delta}|^p+|W_T-W_{t_{N_\delta(T)}}|^p\right)^2\\ &= \lim_{\delta\to 0}\mathbb{E}\left(\sum_{n=1}^{N_\delta(T)}|W_{\tau_n^\delta}|^p\right)^2,\end{aligned}$$

where the last line comes from the fact that $\left(T-t_{N_\delta(T)}\right)<\delta$. To compute the above expectation, we write

$$\begin{aligned}\mathbb{E}\left(\sum_{n=1}^{N_\delta(T)}|W_{\tau_n^\delta}|^p\right)^2 &= \mathbb{E}\left(\sum_{m,n=1}^{N_\delta(T)}|W_{\tau_m^\delta}|^p|W_{\tau_n^\delta}|^p\right)^2\\ &= \mathbb{E}N_\delta(T)\mathbb{E}|W_{\tau_\delta}|^{2p}+\mathbb{E}\left(N_\delta(T)^2-N_\delta(T)\right)\left(\mathbb{E}|W_{\tau_\delta}|^p\right)^2\\ &= \mathbb{E}N_\delta(T)\mathbb{E}|W_{\tau_\delta}|^{2p}+\mathbb{E}N_\delta(T)^2\left(\mathbb{E}|W_{\tau_\delta}|^p\right)^2+\mathcal{O}\left(\sqrt{d}\right),\end{aligned}$$

where the last line follows from the fact that $N_\delta(T)\sim\mathcal{O}\left(\frac{T}{\sqrt{d}}\right)$ and $\mathbb{E}|W_{\tau_\delta}|^p\sim\mathcal{O}\left(\sqrt{\delta}\right)$. It remains to compute the limit of $\delta\mathbb{E}N_\delta(T)^2$. Following a similar approach to the one before, we write

$$\begin{aligned}\mathbb{E}N_\delta(T)^2 &= \sum_{n=1}^{\infty}(2n-1)\mathbb{P}\left(N_\delta(T)\geq n\right)=\sum_{n=1}^{\infty}(2n-1)\mathbb{P}\left(\sum_{i=1}^{n}\tau_i^\delta\leq T\right)\\ &= 2\sum_{n=1}^{\infty}n\mathbb{P}\left(\sum_{i=1}^{n}\tau_i^\delta\leq T\right)+\mathcal{O}\left(\frac{1}{\sqrt{\delta}}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}n\mathbb{P}\left(\sum_{i=1}^{n}\tau_i^\delta\leq T\right) &= \sum_{n=1}^{\infty}n\int_0^T\mathcal{L}^{-1}[\hat{H}_\delta(\lambda)^n](\mathrm{d}t)\\ &= \int_0^T\mathcal{L}^{-1}[\sum_{n=1}^{\infty}n\hat{H}_\delta(\lambda)^n](\mathrm{d}t)\\ &= \int_0^T\mathcal{L}^{-1}\left[\frac{\hat{H}_\delta(\lambda)}{\left(1-\hat{H}_\delta(\lambda)\right)^2}\right](\mathrm{d}t).\end{aligned}$$

Taking the limit as $\delta \to 0$, we get

$$\begin{aligned}
\lim_{\delta\to 0} \delta \mathbb{E} N_\delta(T)^2 &= \lim_{\delta\to 0} 2\int_0^T \mathcal{L}^{-1}\left[\frac{\delta \hat{H}_\delta(\lambda)}{\left(1-\hat{H}_\delta(\lambda)\right)^2}\right](\mathrm{d}t) \\
&= \frac{\pi}{4}\int_0^T \mathcal{L}^{-1}\left[\left(\frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}\right)}\right)^2\right](\mathrm{d}t) \\
&= \frac{T^2}{2} + (2\log 2)\, T
\end{aligned}$$

Putting everything together, we get

$$\begin{aligned}
\mathbb{E}\left(\left(D_p(Z^1)_T\right)^{2p}\right) &= \lim_{\delta\to 0}\left(\sqrt{\delta}\mathbb{E}N_\delta(T)\mathbb{E}\frac{|W_{\tau_\delta}|^{2p}}{\sqrt{\delta}} + \delta\mathbb{E}N_\delta(T)^2\left(\mathbb{E}\frac{|W_{\tau_\delta}|^{p}}{\sqrt{\delta}}\right)^2\right) \\
&= \frac{T}{\sqrt{2}}a_{2p}K_p + \left(\frac{T^2}{2} + (2\log 2)\,T\right)\left(a_p K_{\frac{p}{2}}\right)^2 \\
&= Tc_{2p} + \left(T^2 + (4\log 2)\,T\right)\left(c_p\right)^2 \\
&= T^2\left(c_p\right)^2 + T\left(c_{2p} + (4\log 2)\left(c_p\right)^2\right),
\end{aligned}$$

where a_p and c_p are defined in (24) and K_p is defined in (18). Finally, we get

$$\begin{aligned}
\mathbb{E}\left(\hat{\sigma}^p - \sigma^p\right)^2 &= \sigma^{2p}\left(\frac{\varepsilon^4}{T^2c_p^2}\left(\frac{T^2}{\varepsilon^4}\left(c_p\right)^2 + \frac{T}{\varepsilon^2}\left(c_{2p} + (4\log 2)\left(c_p\right)^2\right)\right) - 1\right) \\
&= \sigma^{2p}\frac{\varepsilon^2}{T}\left(\frac{c_{2p}}{c_p^2} + 4\log(2)\right) = \sigma^{2p}\frac{\varepsilon^2}{T}E(p),
\end{aligned} \tag{28}$$

where $E(p) = \frac{c_{2p}}{c_p^2} + 4\log(2)$. This is an increasing function for $p \in [1,2]$ and

$$4\log 2 =: E(1) \leq E(p) \leq E(2) := 10\log 2, \quad \forall p \in [1,2].$$

We summarize our conclusions in the following:

Theorem 8. *The L_2-error of the estimator $\hat{\sigma}^p$ defined in (16) is described by (28). At scale $\mathcal{O}(\varepsilon^\alpha)$, the error is of order $\mathcal{O}\left(\varepsilon^{\frac{2-\alpha}{2}}\right)$.*

We see that the performance of the estimators $\hat{\sigma}^p$ is the same for all $p > 1$ and they outperform the $\hat{\sigma}_\delta^2$ estimator defined in (7). In terms of the constant $E(p)$, the smaller the p, the smaller the error. However, there is a problem: except for scale $\mathcal{O}(1)$ ($\alpha = 0$), the normalizing constant C_p depends on ε, which will in general be unknown. We go on to define a new estimator that does not assume knowledge of ε.

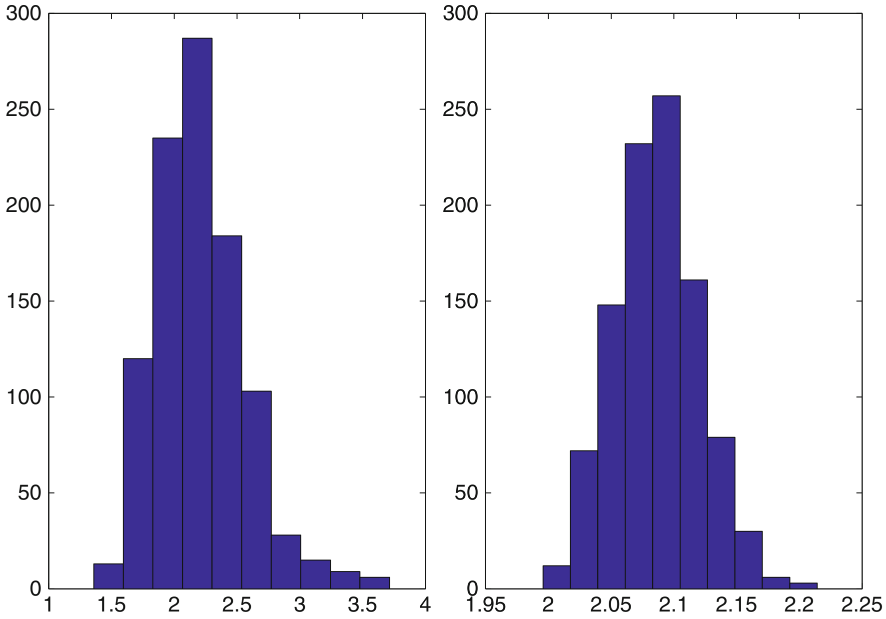

Fig. 1 The total 2-variation estimates for $\sigma = 2$, $\alpha = 0$, $\varepsilon = .1$ (*left*) and .01 (*right*) and $\delta = \varepsilon^3$

In Fig. 1, we plot the histogram of 1,000 realizations of the 2-variation estimate $\sqrt{\hat{\sigma}^2}$ for $\sigma = 2$ and $\alpha = 0$ or $T = 1$. We see that the estimator seems to be asymptotically normal, with a bias and asymptotic variance that decrease as ε gets smaller. For $\varepsilon = .1$, the mean is 2.2084 and the variance is 0.1256. For $\varepsilon = .01$, the mean is 2.0864 and the variance is 0.0012.

3.3 Estimating the Scale Separation Variable ε

Suppose that $T < 1$ and $T = \varepsilon^\alpha$ for some $\alpha > 0$. We define the new estimator $\tilde{\sigma}^p$ similar to $\hat{\sigma}^p$, and only use c_p rather than C_p as our normalization constant. Thus, we define

$$\tilde{\sigma}^p = \frac{1}{c_p}\left(D_p(Y^{1,\varepsilon})_T\right)^p, \tag{29}$$

where c_p is defined in (24). Then,

$$\begin{aligned}\mathbb{E}\,(\tilde{\sigma}^p - \sigma^p)^2 &= \mathbb{E}\left(\frac{\left(D_p(Y^{1,\varepsilon})_T\right)^p}{c_p} - \sigma^p\right)^2 \\ &= \frac{\varepsilon^{2p}\sigma^{2p}}{c_p^2}\mathbb{E}\left(\left(D_p(Z^1)_{\frac{T}{\varepsilon^2}}\right)^{2p}\right) - 2\sigma^p\frac{\varepsilon^p\sigma^p}{c_p}\mathbb{E}\left(\left(D_p(Z^1)_{\frac{T}{\varepsilon^2}}\right)^p\right) + \sigma^{2p}\end{aligned}$$

$$= \frac{\varepsilon^{2p}\sigma^{2p}}{c_p^2}\left(\frac{T^2}{\varepsilon^4}\left(c_p\right)^2 + \frac{T}{\varepsilon^2}\left(c_{2p} + (4\log 2)\left(c_p\right)^2\right)\right) - 2\sigma^p\frac{\varepsilon^p\sigma^p}{c_p}\left(c_p\frac{T}{\varepsilon^2}\right) + \sigma^{2p}$$

$$= \sigma^{2p}\left(\frac{T^2}{\varepsilon^{4-2p}} + \frac{T}{\varepsilon^{2-2p}}\left(\frac{c_{2p}}{\left(c_p\right)^2} + (4\log 2)\right) - 2\left(\frac{T}{\varepsilon^{2-p}}\right) + 1\right) \quad (30)$$

and by substituting T by ε^{α}, this becomes

$$\mathbb{E}\left(\hat{\sigma}^p - \sigma^p\right)^2 = \sigma^{2p}\left(\varepsilon^{2p+2a-4} + \varepsilon^{2p+a-2}E(p) - 2\varepsilon^{p+a-2} + 1\right). \quad (31)$$

Thus, we get the following behavior:

(a) For $p > 2-\alpha$, the error is of order $\mathcal{O}(1)$.
(b) For $p = 2-\alpha$, the error is well-behaved and of order $\mathcal{O}\left(\varepsilon^{\frac{2-\alpha}{2}}\right)$.
(c) For $p < 2-\alpha$ and $\alpha < 2$, the error explodes like $\mathcal{O}\left(\varepsilon^{2p+2a-4}\right)$.

We conclude that the optimal estimator is $\hat{\sigma}^2$, since it does not assume knowledge of ε and the estimators $\tilde{\sigma}^p$ do not outperform it even for $p = 2-\alpha$ (except that the constant $E(p)$ is smaller). However, the estimators $\tilde{\sigma}^p$ can be used to estimate the scale separation variable ε. We set

$$\hat{p} := \arg\min_{1<p<2} \left| \left(\tilde{\sigma}^p\right)^{\frac{1}{p}} - \left(\hat{\sigma}^2\right)^{\frac{1}{2}} \right|$$

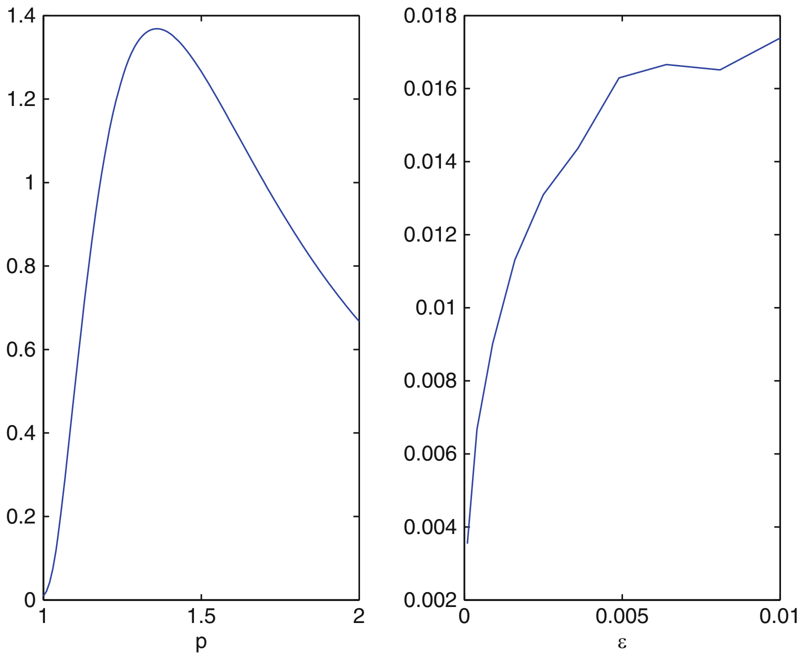

Fig. 2 *Left*: the estimator $\left(\tilde{\sigma}^p\right)^{\frac{1}{p}}$ for $\sigma = 2$, $\varepsilon = .01$ and $T = \varepsilon^{\frac{1}{2}}$. *Right*: the error $\hat{\varepsilon} - \varepsilon$ as a function of ε, for $\sigma = 2$ and $T = \varepsilon^{\frac{1}{2}}$

and

$$\hat{\alpha} := 2 - \hat{p}.$$

Then, we estimate ε by

$$\hat{\varepsilon} := T^{\frac{1}{\hat{\alpha}}}.$$

We demonstrate the method with an example. Let $\sigma = 2$. In Fig. 2 (left), we plot the estimator $(\tilde{\sigma}^p)^{\frac{1}{p}}$ for $\varepsilon = .01$, $T = \varepsilon^{\frac{1}{2}}$ and $\delta = \varepsilon^2/10$. For this realization, we find $(\hat{\sigma}^2)^{\frac{1}{2}} = 2.1098$, so the estimator $(\tilde{\sigma}^p)^{\frac{1}{p}}$ performs best around $p = 1.35$, and the corresponding $\hat{\varepsilon}$ is 0.0274. In Fig. 2 (right), we plot the error $\hat{\varepsilon} - \varepsilon$, for different values of ε varying from .0001 to .01, and for $\sigma = 2$, $T = \varepsilon^{\frac{1}{2}}$ and $\delta = \varepsilon^2/10$. Clearly, the error decreases with ε.

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Numerical Solution of the Dirichlet Problem for Linear Parabolic SPDEs Based on Averaging over Characteristics

Vasile N. Stanciulescu and Michael V. Tretyakov

Abstract Numerical methods for the Dirichlet problem for linear parabolic stochastic partial differential equations are constructed. The methods are based on the averaging-over-characteristic formula and the weak-sense numerical integration of ordinary stochastic differential equations in bounded domains. Their orders of convergence in the mean-square sense and in the sense of almost sure convergence are obtained. The Monte Carlo technique is used for practical realization of the methods. Results of some numerical experiments are presented.

Keywords Mean-square and almost sure convergence · Monte Carlo technique · Numerical integration of stochastic differential equations in bounded domains · Probabilistic representations of solutions of stochastic partial differential equations · The first boundary value problem

MSC (2010): 65C30, 60H15, 60H35, 60G35

1 Introduction

A great deal of attention has recently been paid to numerical methods for stochastic partial differential equations (SPDEs; a comprehensive list of references on this topic is available in, e.g., [27]). There are a number of approaches used to derive approximations for parabolic SPDEs. Of course, SPDEs themselves differ in their nature (see, e.g., [8, 14, 51]): linear and nonlinear; with various boundary conditions; different types of noise; and various interpretations of their solutions. This variety results in different regularity properties of the solutions, which (together with the aims of a particular application) usually affect the choice of an approximation technique used and norms for estimating errors of numerical methods. The most

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common approach to construct a numerical method for SPDEs is, to large extent, similar to the one used for approximating deterministic partial differential equations (PDEs). First, one discretizes an SPDE in space using spatial finite differences (see, e.g., [1, 24, 52, 54] and the references therein), finite element methods (see, e.g., [1, 21, 53] and the references therein) or spectral methods (see, e.g., [9, 23, 27]). The result of such a space discretization is a large (obviously growing with refining the space discretization) stiff system of ordinary stochastic differential equations (SDEs) which is then numerically solved by appropriate numerical schemes. In [5, 15], first an SPDE is discretized in time and then to this semi-discretization a finite-element or finite-difference method can be applied. The other numerical approaches include those making use of the Wiener chaos expansion (see, e.g., [26, 39]), splitting techniques (see, e.g., [2, 25, 32]), and quantization [22]. In [45], layer methods based on probabilistic representations are proposed for linear and semilinear parabolic SPDEs. Numerical methods of all these types are applicable to low dimensional SPDEs (say, of spatial dimension $d \le 3$).

For problems of mathematical physics associated with multi-dimensional (deterministic) linear PDEs, the Monte Carlo technique is the well-established numerical tool (see, e.g., [43] and references therein). In the case of the Cauchy problem for linear SPDEs, the method of characteristics (the averaging over the characteristic formula) and the weak-sense numerical integration of SDEs together with the Monte Carlo technique were used to propose numerical methods in [45] (see also [10, 31, 44, 46, 50]). The closely related approach is branching interacting particle systems methods (see, e.g., [11, 12]).

In this paper, we carry over the approach of [45] to numerically solve the *Dirichlet* problem for linear SPDEs (see (1)–(3) below) exploiting ideas of the simplest random walks for the deterministic Dirichlet problem for PDEs from [42] (see also [43, Chap. 6]). We note that using the weak-sense numerical integration of SDEs with reflection [41, 43] the approach of [45] can also be exploited for the Neumann problem for SPDEs.

Let G be a bounded domain in $\mathbf{R}^d$, $Q = [T_0, T) \times G$ be a cylinder in $\mathbf{R}^{d+1}$, and $\Gamma = \bar{Q} \backslash Q$ be the part of the cylinder's boundary consisting of the upper base and lateral surface. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\mathcal{F}_t,\ T_0 \le t \le T$, be a filtration satisfying the usual hypotheses, and $(w(t), \mathcal{F}_t) = ((w_1(t), \dots, w_r(t))^\top, \mathcal{F}_t)$ be an r-dimensional standard Wiener process. We consider the Dirichlet problem for the backward SPDE:

$$-\mathrm{d}v = [\mathcal{L}v + f(t,x)]\,\mathrm{d}t + \left[\beta^\top(t,x)v(t,x) + \gamma^\top(t,x)\right] * \mathrm{d}w(t), \quad (t,x) \in Q, \tag{1}$$

$$v\mid_\Gamma = \varphi(t,x), \tag{2}$$

where

$$\mathcal{L}v(t,x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j}v(t,x) + b^\top(t,x)\nabla v(t,x) + c(t,x)v(t,x), \tag{3}$$

$a(t,x) = \{a^{ij}(t,x)\}$ is a $d \times d$-matrix; $b(t,x)$ is a d-dimensional column-vector composed of the coefficients $b^i(t,x)$; $c(t,x)$ and $f(t,x)$ are scalar functions; $\beta(t,x)$ and $\gamma(t,x)$ are r-dimensional column-vectors composed of the coefficients $\beta^i(t,x)$ and $\gamma^i(t,x)$, respectively. The notation "$*dw$" in (1) means backward Ito integral [29, 47, 51] (see also Sect. 2).

The form of (1) is convenient for the probabilistic approach: the "initial" condition is prescribed at the final time moment $t = T$, and the equation is considered for $t < T$. As a result, stochastic characteristics for (1)–(2) are written in forward time (see Sect. 2). At the same time, we remark that by changing the time $s := T-(t-T_0)$ one can re-write the backward SPDE problem (1)–(2) in the one with forward direction of time [51] (see also Remark 2.1 in [45]), and for a given forward SPDE, one can write down the corresponding backward SPDE. Consequently, the methods for backward SPDEs considered in this paper can be used for solving forward SPDEs as well.

Linear SPDEs have a number of applications (see, e.g., [8, 51] and the references therein), the most popular among them is the nonlinear filtering problem (see, e.g., [30, 33, 47, 51]). The backward Zakai equation with Dirichlet boundary condition has the form of (1)–(2) with $c = 0$, $f = 0$, $\gamma = 0$. It arises in the nonlinear filtering problem when the unobservable signal is modelled by killed diffusions (see [37, 48] and also the related works [9, 35, 49]).

The rest of the paper is organized as follows. Section 2 contains auxiliary knowledge including the assumptions imposed on the problem, which are used in our proofs, and a probabilistic representation of the solution $v(t,x)$, which involves averaging characteristics over an auxiliary Wiener process $W(t)$ independent of $w(t)$. In Sect. 3, we construct numerical methods for (1)–(2) by approximating the stochastic characteristics for a fixed trajectory $w(t)$. We note that the stochastic characteristics belong to the bounded domain $\bar{Q}$, and hence the approximate characteristics should possess this property as well. To this end, we exploit ideas of the simplest random walks for the deterministic Dirichlet problem from [42] (see also [43, Chap. 6]) and propose first-order and order $1/2$ (in the mean-square sense) numerical methods for (1)–(2). To realize the proposed methods in practice, the Monte Carlo technique can be used. The corresponding convergence theorems (both in the mean-square sense and in the sense of almost sure convergence) are stated in Sect. 3 and proved in Sect. 4. The theorems are proved under rather strong assumptions, in particular, that the problem (1)–(2) has a sufficiently smooth in x^i classical solution, which allow us to obtain convergence of the proposed methods in a strong norm and with optimal orders. These assumptions are not necessary and the numerical algorithms of this paper can be used under broader conditions. Results of some numerical experiments are presented in Sect. 5.

2 Preliminaries

In this section, we state the assumptions imposed on the problem (1)–(2) and write a probabilistic representation for its solution $v(t,x)$.

Let us first recall the definition of a backward Ito integral [51]. Introduce the "backward" Wiener processes

$$\tilde{w}(t) := w(T) - w(T - (t - T_0)), \quad T_0 \le t \le T, \tag{4}$$

and a decreasing family of σ-subalgebras $\mathcal{F}_T^t$, $T_0 \le t \le T$ induced by the increments $w(T) - w(t')$, $t' \ge t$. A σ-algebra induced by $\tilde{w}(t')$, $t' \le t$ coincides with $\mathcal{F}_T^{T-(t-T_0)}$. Then, the backward Ito integral is defined as the Ito integral with respect to $\tilde{w}(s)$:

$$\int_t^{t'} \psi(t'') * \mathrm{d}w(t'') := \int_{T-(t'-T_0)}^{T-(t-T_0)} \psi(T - (t'' - T_0))\mathrm{d}\tilde{w}(t''), \quad T_0 \le t \le t' \le T,$$

where $\psi(T - (t - T_0))$, $t \le T$, is an $\mathcal{F}_T^{T-(t-T_0)}$-adapted square-integrable function. The solution $v(t,x)$ of the problem (1)–(2) is $\mathcal{F}_T^t$-adapted, it depends on $w(s) - w(t)$, $t \le s \le T$. The more precise notation for the solution of (1)–(2) is $v(t,x;\omega)$, $\omega \in \Omega$, but we use the shorter one $v(t,x)$.

We impose the following conditions on the problem (1)–(2).

Assumption 2.1. *(smoothness) We assume that the coefficients $a^{ij}(t,x)$, $b^i(t,x)$, $c(t,x)$, $f(t,x)$, $\beta^i(t,x)$, and $\gamma^i(t,x)$ in (1)–(3) are sufficiently smooth in $\bar{Q}$, $\varphi(t,x)$ is sufficiently smooth on Γ, and the domain G has a sufficiently smooth boundary ∂G.*

Assumption 2.2. *(ellipticity) We assume that $a = \{a^{ij}\}$ is symmetric and positive definite in $\bar{Q}$.*

Assumption 2.3. *(classical solution) We assume that the problem (1)–(2) has the classical solution $v(t,x)$, which has spatial derivatives up to a sufficiently high order for all $(t,x) \in \bar{Q}$; and the solution and its spatial derivatives are such that for some $p > 1$ they satisfy an inequality of the form*

$$E\left(\max_{(t,x)\in\bar{Q}} |v(t,x)|\right)^p \le K,$$

where $K > 0$ is a constant.

We note that Assumptions 2.1–2.2 together with some compatibility conditions ensure the existence of the classical solution with the properties described in Assumption 2.3 (see [3,4,17,18,48] and also [6,14,28,34,38,49,51]). Such *compatibility conditions* consist, e.g., in requiring that $\gamma(t,x)$ and its partial derivatives up to a sufficiently high order equal to zero for $(t,x) \in [T_0, T] \times \partial G$; the function $\beta(t,x)$ agrees with $\varphi(t,x)$ so that the function $\Phi(t,x)$ defined by

$$\Phi(t,x) := \varphi(t,x)\exp(-\eta(t;x)),$$
$$\eta(t;x) = \int_t^T \beta^\top(s,x) * \mathrm{d}w(s) = \int_t^T \beta^\top(s,x)\mathrm{d}w(s),$$

and its spatial derivatives up to a sufficiently high order are deterministic for $(t, x) \in [T_0, T] \times \partial G$ (e.g., $\Phi(t, x)$ is deterministic for $(t, x) \in [T_0, T] \times \partial G$, when $\varphi(t, x) = 0$ or $\beta(t, x) = 0$ for $(t, x) \in [T_0, T] \times \partial G$); and the function $f(t, x)$ agrees with $\varphi(t, x)$ so that

$$-\frac{\partial \varphi}{\partial t}(T, x) = \mathcal{L}\varphi(T, x) + f(T, x), \; x \in \partial G.$$

To demonstrate that Assumptions 2.1–2.2 together with these compatibility conditions imply the existence of a smooth classical solution, one can exploit the method of robust equation [7, 47, 51], i.e., by deriving the Dirichlet problem for the backward pathwise PDE with random coefficients corresponding to the SPDE (1)–(2). Indeed, the transform

$$\begin{aligned} v(t, x) &= \mathrm{e}^{\eta(t;x)} u(t, x) + \mathrm{e}^{\eta(t;x)} \varsigma(t; x), \qquad (5) \\ \varsigma(t; x) &= \int_t^T \gamma^\top(s, x) \mathrm{e}^{-\eta(s;x)} * \mathrm{d}w(s) \\ &= \mathrm{e}^{-\eta(t;x)} \int_t^T \gamma^\top(s, x) \exp\left(\int_t^s \beta^\top(s', x) \mathrm{d}w(s') \right) \mathrm{d}w(s) \\ &\quad + \int_t^T \gamma^\top(s, x) \beta(s, x) \mathrm{e}^{-\eta(s;x)} \mathrm{d}s, \end{aligned}$$

relates the solution $v(t, x)$ of the Dirichlet problem for the SPDE (1)–(2) with the solution $u(t, x)$ of the corresponding Dirichlet problem for the backward pathwise PDE. Existence of a smooth classical solution of this PDE problem follows from standard results on deterministic parabolic PDEs [20, 36], which together with the transform (5) implies the existence of a smooth classical solution of the SPDE problem (1)–(2).

Assumptions 2.1–2.3 are sufficient for the statements in this paper. At the same time, they are not necessary and the numerical methods presented here can be used under broader conditions. These rather strong assumptions allow us to prove convergence of the proposed methods in a strong norm and with optimal orders. Weakening the conditions (especially, substituting the compatibility assumptions by using weighted spaces [28, 34, 38]) requires further study.

The probabilistic representation (or in other words, the averaging-over-characteristic formula) for the solution of the problem (1)–(2) given below is analogous to the ones in [18] and also to the representations for solutions of the Cauchy problem for linear SPDEs from [29, 30, 45, 47, 51].

Let a $d \times d$-matrix $\sigma(t, x)$ be obtained from the equation

$$\sigma(t, x)\sigma^\top(t, x) = a(t, x).$$

The solution of the problem (1)–(2) has the following probabilistic representation:

$$v(t, x) = E^w \left[\varphi(\tau, X_{t,x}(\tau)) Y_{t,x,1}(\tau) + Z_{t,x,1,0}(\tau) \right], \;\; T_0 \le t \le T, \qquad (6)$$

where $X_{t,x}(s),\ Y_{t,x,y}(s),\ Z_{t,x,y,z}(s)$, $t \le s \le T$, $(t,x) \in Q$, is the solution of the SDEs

$$dX = b(s,X)ds + \sigma(s,X)dW(s), \quad X(t) = x, \tag{7}$$

$$dY = c(s,X)Y\,ds + \beta^{\top}(s,X)Y\,dw(s), \quad Y(t) = y, \tag{8}$$

$$dZ = f(s,X)Y\,ds + \gamma^{\top}(s,X)Y\,dw(s), \quad Z(t) = z, \tag{9}$$

$W(s) = (W_1(s),\dots,W_d(s))^{\top}$ is a d-dimensional standard Wiener process independent of $w(s)$ and $\tau = \tau_{t,x}$ is the first exit time of the trajectory $(s, X_{t,x}(s))$ to the boundary Γ. The expectation E^w in (6) is taken over the realizations of $W(s)$, $t \le s \le T$, for a fixed $w(s)$, $t \le s \le T$; in other words, $E^w(\cdot)$ means the conditional expectation $E\left(\cdot|w(s) - w(t), t \le s \le T\right)$. We note that the exit time $\tau_{t,x}$ does not depend on $w(\cdot)$.

To verify that (6)–(9) is a probabilistic representation for the solution $v(t,x)$ of (1)–(2), we can proceed, for instance, as follows. First, using the standard results of, e.g., [16, 19], we write down a probabilistic representation for the solution $u(t,x)$ of the discussed above pathwise PDE problem related to the SPDE problem (1)–(2). Then, using the relation (5), we transform this probabilistic representation for $u(t,x)$ into the one for $v(t,x)$ and arrive at (6)–(9).

3 Numerical Methods

To construct numerical methods for the Dirichlet problem for the SPDE (1)–(2), we use ideas of the simplest random walks for the deterministic Dirichlet problem from [42] (see also [43, Chap. 6]) together with the approach to solving SPDEs via averaging over characteristics developed in [45] (see also [44, 46]).

We propose three algorithms: two of them (Algorithms 1A and 1B) are of mean-square order one and Algorithm 2 is of mean-square order $1/2$.

Difficulties arising in the realization of the probabilistic representation for solving deterministic Dirichlet problems were discussed in [40, 42, 43]. They are inherited in the case of the Dirichlet problem for SPDEs. For instance, the difference $\tau - t$ in (6) can take arbitrary small values and, consequently, it is impossible to integrate numerically the system (7) with a fixed time step. In particular, we cannot use mean-square Euler approximations for simulating (7). Thanks to the fact that in (6) we average realizations of the solution $X(t)$ of (7), we can exploit simple weak approximations $\bar{X}$ of X imposing on them some restrictions related to nonexit of $\bar{X}$ from the domain $\bar{Q}$ as in the case of deterministic PDEs [40, 42, 43]. Namely, we will require that Markov chains approximating in the weak sense the solution of (7) remains in the domain $\bar{Q}$ with probability one.

In all the algorithms considered here, we apply the weak explicit Euler approximation with the simplest simulation of noise to (7):

$$X_{t,x}(s+h) \approx \bar{X} = x + hb(s,x) + h^{1/2}\sigma(s,x)\,\xi, \tag{10}$$

where $h > 0$ is a time step, $\xi = (\xi^1, \ldots, \xi^d)^\top$, ξ^i, $i = 1, \ldots, d$ are i.i.d. random variables taking the values ± 1 with probability $1/2$. As we will see, this approximation of X is used "inside" the space domain G.

Let us now introduce the boundary zone $S_{t,h} \subset \bar{G}$ for the time layer t while $\bar{G} \backslash S_{t,h}$ will become the corresponding "inside" part of G. Clearly, the random vector $\bar{X}$ in (10) takes 2^d different values. Introduce the set of points close to the boundary (a boundary zone) $S_{t,h} \subset \bar{G}$ on the time layer t : we say that $x \in S_{t,h}$ if at least one of the 2^d values of the vector $\bar{X}$ is outside $\bar{G}$. It is not difficult to see that due to compactness of $\bar{Q}$ there is a constant $\lambda > 0$ such that if the distance from $x \in G$ to the boundary ∂G is equal to or greater than $\lambda\sqrt{h}$, then x is outside the boundary zone and, therefore, for such x, all the realizations of the random variable $\bar{X}$ belong to $\bar{G}$.

Since restrictions connected with nonexit from the domain $\bar{G}$ should be imposed on an approximation of (7), the formula (10) can be used only for the points $x \in \bar{G} \backslash S_{t,h}$ on the layer t, and a special construction is required for points from the boundary zone. In Algorithms 1, we use a construction based on linear interpolation while in Algorithm 2 we just stop the Markov chain as soon as it reaches the boundary zone $S_{t,h}$. In the deterministic case, these constructions were exploited in [42, 43].

Below we also use the following notation. Let $x \in S_{t,h}$. Denote by $x^\pi \in \partial G$ the projection of the point x on the boundary of the domain G (the projection is unique because h is sufficiently small and ∂G is smooth) and by $n(x^\pi)$ the unit vector of internal normal to ∂G at x^π.

3.1 First-Order Methods

To define our approximation of $X(t)$ in the boundary zone, we introduce the random vector $X^\pi_{x,h}$ taking two values x^π and $x + h^{1/2}\lambda n(x^\pi)$, $x \in S_{t,h}$, with probabilities $p = p_{x,h}$ and $q = q_{x,h} = 1 - p_{x,h}$, respectively, where

$$p_{x,h} = \frac{h^{1/2}\lambda}{|x + h^{1/2}\lambda n(x^\pi) - x^\pi|}.$$

It is not difficult to check that if $v(x)$ is a twice continuously differentiable function with the domain of definition $\bar{G}$, then an approximation of $v(x)$ by the expectation $Ev(X^\pi_{x,h})$ corresponds to linear interpolation and

$$v(x) = Ev(X^\pi_{x,h}) + O(h) = pv(x^\pi) + qv(x + h^{1/2}\lambda n(x^\pi)) + O(h). \quad (11)$$

We emphasize that the second value $x + h^{1/2}\lambda n(x^\pi)$ does not belong to the boundary zone. We also note that p is always greater than $1/2$ (since the distance from x to ∂G is less than $h^{1/2}\lambda$) and that if $x \in \partial G$, then $p = 1$ (since in this case $x^\pi = x$).

Further, we apply the partly weak, partly mean-square explicit Euler approximation (cf. [45]) to the rest of the probabilistic representation, i.e., to (8)–(9), and obtain

$$
\begin{aligned}
Y_{t,x,y}(s+h) \approx \bar{Y} &= y + hc(s,x)\, y + \beta^{\top}(s,x) y \Delta w(s) \\
&\quad + \frac{1}{2} y \sum_{i=1}^{r} \left[\beta^{i}(s,x)\right]^2 \left([\Delta w_i(s)]^2 - h\right) \\
&\quad + y \sum_{i=1}^{r} \sum_{j=i+1}^{r} \beta^{i}(s,x)\beta^{j}(s,x)\Delta w_i(s)\Delta w_j(s), \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
Z_{t,x,y,z}(t+h) \approx \bar{Z} &= z + hf(s,x)\, y + \gamma^{\top}(s,x) y \Delta w(s) \\
&\quad + y \sum_{i,j=1}^{r} \gamma^{i}(s,x)\beta^{j}(s,x) I_{ij}(s), \qquad (13)
\end{aligned}
$$

where ξ is as in (10), $\Delta w(s) = w(s+h) - w(s)$, and $I_{ij}(s)$ are the Ito integrals

$$
I_{ij}(s) = \int_{s}^{s+h} \left(w_j(s') - w_j(s)\right) \mathrm{d}w_i(s'). \qquad (14)
$$

Let us observe that in the approximation (10), (12)–(13) we approximate the part of the system (7)–(9) related to the auxiliary Wiener process $W(t)$ in the weak sense and the part of the system related to the Wiener process $w(t)$, which drives our SPDE (1), in the mean-square sense. We also recall that X and w are independent.

Remark 1. We note that unless the commutativity condition $\beta^{i}\gamma^{j} = \beta^{j}\gamma^{i}$ holds we face the difficulty in simulating $\bar{Z}$ from (13) efficiently due to the presence of the integral $I_{ij}(s)$ (see various approaches to its approximation in, e.g., [43] and also [13]). As is well known, to realize (13) in the commutative noise situation, it suffices to simulate the Wiener increments $\Delta w_i(s)$ only. We also note that in the important case of $\gamma \equiv 0$ (e.g., it is so in the Zakai equation [33, 48, 51]) the term with $I_{ij}(s)$ in (13) is cancelled.

Now, we are ready to propose an algorithm for solving the SPDE problem (1)–(2). Let a point $(t_0, x_0) \in Q$. We would like to find the value $v(t_0, x_0)$. Introduce a discretization of the interval $[t_0, T]$, for definiteness the equidistant one:

$$
t_0 < t_1 < \cdots < t_N = T, \quad h := (T - t_0)/N.
$$

Let ξ_k, $k = 1, 2, \ldots$, be i.i.d. random variables with the same law as defined for the ξ in (10). We formulate the algorithm as follows.

Algorithm 1A

STEP 0. $X_0' = x_0,\ Y_0 = 1,\ Z_0 = 0,\ k = 0.$
STEP 1. If $X_k' \notin S_{t_k,h}$ then $X_k = X_k'$ and go to STEP 3.
If $X_k' \in S_{t_k,h}$ then either $X_k = X_k'^{\pi}$ with probability $p_{X_k',h}$
or $X_k = X_k' + h^{1/2}\lambda n(X_k'^{\pi})$ with probability $q_{X_k',h}$.
STEP 2. If $X_k = X_k'^{\pi}$ then STOP and $\varkappa = k,\ X_\varkappa = X_k'^{\pi},\ Y_\varkappa = Y_k, Z_\varkappa = Z_k.$
STEP 3. Simulate ξ_{k+1} and find $X_{k+1}',\ Y_{k+1},\ Z_{k+1}$ due to (10), (12)–(13) for $s = t_k,\ x = X_k, y = Y_k, z = Z_k,\ \xi = \xi_{k+1}, \Delta w(s) = \Delta w(t_k).$
STEP 4. If $k+1 = N$, STOP and $\varkappa = N,\ X_\varkappa = X_N',\ Y_\varkappa = Y_N,\ Z_\varkappa = Z_N$, otherwise $k := k+1$ and return to STEP 1.

Having obtained the end points of the chain $(t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$ with $(t_\varkappa, X_\varkappa) \in \Gamma$, we get the approximation of the solution to the SPDE problem (1)–(2):

$$v(t_0,x_0) \approx \bar{v}(t_0,x_0) = E^w\left[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa\right], \tag{15}$$

where the approximate equality corresponds to the numerical integration error (see Theorems 1 and 2 below). We note that in the case of the deterministic Dirichlet problem (i.e., when $\beta = 0$ and $\gamma = 0$) Algorithm 1A coincides with Algorithm 6.2.1 in [43, p. 355].

To realize the approximation (15) in practice, one can use the Monte Carlo technique:

$$\bar{v}(t_0,x_0) \approx \hat{v}(t_0,x_0) = \frac{1}{M}\sum_{m=1}^{M}\left[\varphi(t_\varkappa^{(m)}, X_\varkappa^{(m)})Y_\varkappa^{(m)} + Z_\varkappa^{(m)}\right], \tag{16}$$

where $(t_\varkappa^{(m)}, X_\varkappa^{(m)}, Y_\varkappa^{(m)}, Z_\varkappa^{(m)})$ are independent realizations of $(t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$, each obtained by Algorithm 1A. The approximate equality in (16) corresponds to the Monte Carlo (or in other words, statistical) error.

As it was noted, e.g., in [44, Remark 3.4], it can be computationally more efficient to approximate (8) as

$$Y_{t,x,y}(s+h) \approx \bar{Y} = y\exp\left(c(s,x)h - \beta^\top(s,x)\beta(s,x)h/2 + \beta^\top(s,x)\Delta w(s)\right) \tag{17}$$

rather than by (12) since, in particular, in comparison with (12) the approximation (17) preserves the property of positivity of Y.

By *Algorithm 1B*, we denote the algorithm which coincides with Algorithm 1A but uses (17) to approximate the component $Y(t)$ instead of (12). The approximation of the solution to the SPDE problem (1)–(2) in the case of Algorithm 1B has the same form (15) as for Algorithm 1A but with the new $Y_\varkappa$ and $Z_\varkappa$ (note that simulation of Z_k depends on simulation of Y_k).

The following two convergence theorems hold for both Algorithms 1A and 1B.

Theorem 1. *Algorithms 1A and 1B satisfy the inequality for* $p \geq 1$*:*

$$\left(E\left|\bar{v}(t_0, x_0) - v(t_0, x_0)\right|^{2p} \right)^{1/2p} \leq Kh, \tag{18}$$

where $K > 0$ *does not depend on the time step h, i.e., in particular, these algorithms are of mean-square order one.*

A proof of this theorem is given in Sect. 4. Theorem 1 together with the Markov inequality and Borel–Cantelli lemma (see the details of the corresponding recipe in, e.g., [44, 45]) implies the a.s. convergence as stated in the next theorem.

Theorem 2. *For almost every trajectory* $w(\cdot)$ *and any* $\varepsilon > 0$*, there exists* $C(\omega) > 0$ *such that*

$$|\bar{v}(t_0, x_0) - v(t_0, x_0)| \leq C(\omega) h^{1-\varepsilon}, \tag{19}$$

where the random variable C does not depend on the time step h, i.e., both Algorithm 1A and 1B converge with order $1 - \varepsilon$ *a.s.*

Remark 2. It might be useful to choose both h and λ adaptively, depending on the chain's state: h_k and λ_k. Then, in Theorems 1 and 2, one should put $h = \max_{0 \leq k < N} h_k$. In practice, one can take $\lambda_k = |\sigma(t_k, X_k)|$, possibly with small corrections.

3.2 Method of Order 1/2

The next algorithm (Algorithm 2) is obtained by a simplification of Algorithm 1A. In Algorithm 2, as soon as X_k gets into the boundary domain $S_{t_k,h}$, the random walk terminates, i.e., $\varkappa = k$, and $\bar{X}_\varkappa = X_k^\pi$, $Y_\varkappa = Y_k$, $Z_\varkappa = Z_k$ is taken as the final state of the Markov chain. Since this algorithm obviously cannot be of mean-square order higher than $1/2$ (it is already of the order $1/2$ in the case of deterministic PDEs [42, 43]), we also simplify the approximations (12)–(13). Namely, instead of (12)–(13), we use the approximations

$$Y_{t,x,y}(s+h) \approx \bar{Y} = y + hc(s,x)\, y + \beta^\top(s,x) y \Delta w(s), \tag{20}$$

$$Z_{t,x,y,z}(t+h) \approx \bar{Z} = z + hf(s,x)\, y + \gamma^\top(s,x) y \Delta w(s). \tag{21}$$

Let us write this algorithm formally.

Algorithm 2

STEP 0. $X_0 = x_0,\ Y_0 = 1,\ Z_0 = 0,\ k = 0.$
STEP 1. If $X_k \notin S_{t_k,h}$ then go to STEP 2.
If $X_k \in S_{t_k,h}$ then STOP and $\varkappa = k$, $\bar{X}_\varkappa = X_k^\pi$, $Y_\varkappa = Y_k$, $Z_\varkappa = Z_k$.
STEP 2. Simulate ξ_{k+1} and find $X_{k+1},\ Y_{k+1},\ Z_{k+1}$ due to (10), (20)–(21) for $s = t_k, x = X_k, y = Y_k,\ z = Z_k, \xi = \xi_{k+1}, \Delta w(s) = \Delta w(t_k)$.
STEP 3. If $k+1 = N$, STOP and $\varkappa = N$, $\bar{X}_\varkappa = X_N$, $Y_\varkappa = Y_N$, $Z_\varkappa = Z_N$, otherwise $k := k+1$ and return to STEP 1.

We form the approximation of the solution to the SPDE problem (1)–(2) as (cf. (15)):

$$v(t_0, x_0) \approx \tilde{v}(t_0, x_0) = E^w \left[\varphi(t_\varkappa, X_\varkappa) Y_\varkappa + Z_\varkappa\right] \tag{22}$$

with $(t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$ obtained by Algorithm 2. We note that in the deterministic case ($\beta = 0$ and $\gamma = 0$) Algorithm 2 coincides with Algorithm 6.2.6 in [43, p. 359]. We also remark that we can modify Algorithm 2 as we did with Algorithm 1A by using the approximation (17) of Y instead of (20). This modified algorithm will usually have better properties than Algorithm 2 using (20).

The following convergence theorem can be proved analogously to Theorems 1 and 2.

Theorem 3. *Algorithm 2 satisfies the inequality for* $p \geq 1$*:*

$$\left(E \left|\tilde{v}(t_0, x_0) - v(t_0, x_0)\right|^{2p}\right)^{1/2p} \leq K h^{1/2}, \tag{23}$$

where $K > 0$ *does not depend on the time step h, i.e., in particular, Algorithm 2 is of mean-square order* $1/2$.

For almost every trajectory $w(\cdot)$ *and any* $\varepsilon > 0$*, there exists* $C(\omega) > 0$ *such that*

$$|\tilde{v}(t_0, x_0) - v(t_0, x_0)| \leq C(\omega) h^{1/2-\varepsilon}, \tag{24}$$

where the random variable C *does not depend on the time step h, i.e., Algorithm 2 converges with order* $1/2 - \varepsilon$ *a.s.*

4 Proof of the Convergence Theorem

In this section, we prove Theorem 1 for Algorithm 1B, its proof for Algorithm 1A is analogous. The proof makes use of some ideas from [42] (see also [43, Sect. 6.2]) and [44]. Note that in this section we shall use the letter K to denote various constants which are independent of k and h.

We extend the definition of the chain corresponding to Algorithm 1B for all k by the rule: if $k > \varkappa$, then $(t_k, X_k, Y_k, Z_k) = (t_\varkappa, X_\varkappa, Y_\varkappa, Z_\varkappa)$. We denote by ν_{t_0,x_0} the number of those t_k at which X'_k gets into the set $S_{t_k,h}$.

Let us first provide some intuitive guidance to the proof. "Inside" the domain, i.e., when $X_k \notin S_{t_k,h}$, the Markov chain (t_k, X_k, Y_k, Z_k) is analogous to the one used in the case of the Cauchy problem for the linear SPDE (1), for which the first mean-square order of convergence was proved in [44] (see also [45, 46]). Then, it is reasonable to expect that the approximation "inside" the domain contributes $O(h)$ to the global error of Algorithm 1B. Near the boundary, i.e., when $X_k \in S_{t_k,h}$, the local error is of order $O(h)$ as it follows from the interpolation relation (11) while the number of steps ν_{t_0,x_0} on which we should count contributions of this local error is such that its any moment is uniformly (with respect to the time step) bounded (see

(25) below), i.e., roughly speaking, the local error $O(h)$ is counted finite number of times to the global error. Consequently, we can expect that the total error "inside" the domain and the total error near the boundary should sum up to $O(h)$.

We now proceed to the formal proof. We note that since we use here the probabilistic representation (6)–(9) in which the Wiener process $w(t)$ driving the SPDE (1) does not enter the X component, the Markov chain X_k coincides with the corresponding Markov chain used in [43, p. 355] for the deterministic Dirichlet problem. As a result, we can exploit here some of the properties of this chain X_k established in [42, 43]. The following lemma is proved in [43, p. 356].

Lemma 1. *The inequalities*

$$P\{\nu_{t_0,x_0} = n\} \leq \frac{1}{2^{n-1}}, \quad P\{\nu_{t_0,x_0} > n\} \leq \frac{1}{2^n}, \; n \in \mathbf{N},$$

hold.

This lemma implies that for any p

$$E\nu_{t_0,x_0}^p \leq K, \tag{25}$$

where K does not depend on the time step h.

We will also need the following uniform estimate for Y_k.

Lemma 2. *For any $p > 1$, the following inequality for Y_k from Algorithm 1B holds:*

$$E\left(\max_{0\leq k\leq N} Y_k\right)^p \leq K,$$

where K does not depend on h.

Proof. (cf. [43, p. 358]) We have for $k \leq \varkappa$

$$Y_k = M_k \exp\left(h\sum_{i=0}^{k-1} c(t_i, X_i)\right) \leq M_k \exp(\bar{c}(t_k - t_0)),$$

where M_k is the corresponding positive martingale and $\bar{c} = \max_{(s,x)\in\overline{Q}} c(s,x)$. Note that $M_k = M_\varkappa$ for $k > \varkappa$ and the moments EM_N^p are finite. Then, the above inequality together with Doob's martingale inequality implies

$$E\left(\max_{0\leq k\leq N} Y_k\right)^p \leq \exp(\bar{c}(T-t_0)p)E\left(\max_{0\leq k\leq N} M_k\right)^p$$
$$\leq \exp(\bar{c}(T-t_0)p)\left[\frac{p}{1-p}\right]^p EM_N^p \leq K.$$

□

Now, we prove one-step error lemmas for Algorithm 1B. Introduce

$$
\begin{aligned}
r_k &= v(t_k, X_k)\, Y_k + Z_k - v(t_k, X_k')\, Y_k - Z_k,\\
R_k &= v(t_{k+1}, X_{k+1}')\, Y_{k+1} + Z_{k+1} - v(t_k, X_k)\, Y_k - Z_k,\\
k &= 0, \dots, N-1.
\end{aligned}
$$

We recall that X_k' belongs to the layer $t = t_k$; the variable r_k can be nonzero in the case of $X_k' \in S_{t_k,h}$ only, i.e., r_k has the meaning of the local error in the boundary zone. If $\varkappa > k$, then $X_k \notin S_{t_k,h}$ and all the 2^d realizations of the random variable X_{k+1}' belong to $\bar{G}$; if $\varkappa \le k$, then $t_k = t_{k+1} = t_\varkappa$, $X_{k+1}' = X_k = X_\varkappa$, $Y_{k+1} = Y_k = Y_\varkappa$, $Z_{k+1} = Z_k = Z_\varkappa$ and, consequently, $R_k = 0$. The variable R_k has the meaning of the local error "inside" the domain.

Lemma 3. *The following relation for the local error in the boundary zone holds:*

$$
|E^w(r_k \mid X_k', Y_k, Z_k)| = \rho(t_k, X_k', Y_k) h I_{S_{t_k,h}}(X_k')\, \chi_{\varkappa \ge k}, \tag{26}
$$

where $I_{S_{t_k,h}}(x)$ and $\chi_{\varkappa \ge k}$ are the indicator functions and the random variable ρ is such that for $p > 1$:

$$
E\left[\max_{0 \le k \le N-1} \rho(t_k, X_k', Y_k)\right]^{2p} \le K \tag{27}
$$

with K being independent of h and k.

Proof. We have

$$
E^w(r_k \mid X_k', Y_k, Z_k) = Y_k I_{S_{t_k,h}}(X_k')\, \chi_{\varkappa \ge k} \left[E^w(v(t_k, X_k) \mid X_k') - v(t_k, X_k')\right]
$$

and for $X_k' \in S_{t_k,h}$ and $\varkappa \ge k$

$$
\rho(t_k, X_k', Y_k) = Y_k \left|E^w(v(t_k, X_k) \mid X_k') - v(t_k, X_k')\right|.
$$

Due to the interpolation relation (11), we have

$$
\begin{aligned}
&E^w(v(t_k, X_k) \mid X_k') - v(t_k, X_k')\\
&\quad= \frac{1}{2} p_{X_k',h} \left(X_k'^{\pi} - X_k'\right)^\top H(v)(t_k, X_k' + \theta_1(X_k'^{\pi} - X_k'))\left(X_k'^{\pi} - X_k'\right)\\
&\qquad+ \frac{h\lambda}{2} q_{X_k',h} n^\top(X_k'^{\pi}) H(v)(t_k, X_k' + \theta_2 h^{1/2}\lambda n(X_k'^{\pi})) n(X_k'^{\pi}),
\end{aligned}
$$

where $H(v)$ is the Hessian of v and θ_1 and θ_2 are some values in $(0, 1)$. Using the assumed properties of the solution v and its spatial derivatives and recalling that $|X_k'^{\pi} - X_k'| \le \lambda h^{1/2}$, we get

$$E\left[\max_{0\le k\le N-1}|E^w(v(t_k,X_k)\mid X'_k)-v(t_k,X'_k)|\right]^{2p}\le K$$

with K being independent of h and k. Then, the Cauchy–Bunyakovskii inequality and Lemma 2 imply (26). □

Lemma 4. *The following relation for the local error "inside" the domain holds:*

$$\begin{aligned}E^w(R_k\mid X_k,Y_k,Z_k)=\chi_{\varkappa>k}Y_k\Bigg\{&\sum_{i=1}^{r}[h\Delta w_i(t_k)\alpha_i(t_{k+1},X_k)\\&+I_{0i}(t_k)\alpha_{0i}(t_{k+1},X_k)]\\&+\sum_{i,j,l=1}^{r}I^*_{ijl}(t_k)\alpha^*_{ijl}(t_{k+1},X_k)\Bigg\}+\chi_{\varkappa>k}O(h^2),\end{aligned}\tag{28}$$

where

$$I_{0i}(t_k)=\int_{t_k}^{t_{k+1}}(w_i(s)-w_i(t_k))\mathrm{d}s,$$

$$I^*_{ijl}(t_k)=\int_{t_k}^{t_{k+1}}\int_{t}^{t_{k+1}}[w_l(t_{k+1})-w_l(s)]*\mathrm{d}w_j(s)*\mathrm{d}w_i(t),$$

$\alpha_i(t_{k+1},X_k)$, $\alpha_{0i}(t_{k+1},X_k)$, *and* $\alpha^*_{ijl}(t_{k+1},X_k)$ *are combinations of the coefficients of (1) and their partial derivatives at* (t_k,X_k) *and the solution* $v(t_{k+1},X_k)$ *and its spatial first and second derivatives; and* $O(h^2)$ *is a random variable such that for* $p>1$

$$E|O(h^2)|^{2p}\le Kh^{4p}\tag{29}$$

with K *being independent of* h *and* k.

Proof. The error $R_k=0$ for $\varkappa\le k$. Let us analyze it in the case $\varkappa>k$. Introduce the notation $b_k=b(t_k,X_k)$, $\sigma_k=\sigma(t_k,X_k)$, and so on. We have

$$\begin{aligned}E^w(R_k\mid X_k,Y_k,Z_k)&=E^w(v(t_{k+1},X'_{k+1})\,Y_{k+1}-v(t_k,X_k)\,Y_k\\&\quad+Z_{k+1}-Z_k\mid X_k,Y_k,Z_k)\\&=Y_k\{E^w[v(t_{k+1},X'_{k+1})\mid X_k]\times\exp\left(c_kh-\beta_k^{\top}\beta_kh/2+\beta_k^{\top}\Delta w(t_k)\right)\\&\quad+hf_k+\gamma_k^{\top}\Delta w(t_k)+\sum_{i,j=1}^{r}\gamma_k^{i}\beta_k^{j}I_{ij}(t_k)-v(t_k,X_k)\}.\end{aligned}\tag{30}$$

Using the Taylor formula, we expand $v(t_{k+1},X'_{k+1})$ at (t_{k+1},X_k) and obtain

$$E^w[v(t_{k+1}, X'_{k+1}) \mid X_k] = E^w(v(t_{k+1}, X_k + hb_k + h^{1/2}\sigma_k \xi_{k+1}) \mid X_k)$$
$$= v(t_{k+1}, X_k) + hb_k^\top \nabla v(t_{k+1}, X_k)$$
$$+ \frac{h}{2} \sum_{i,j=1}^{d} a_k^{ij} \frac{\partial^2}{\partial x^i \partial x^j} v(t_{k+1}, X_k) + O(h^2), \quad (31)$$

where $O(h^2)$ is as in (29). By the Taylor expansion, we also get

$$\exp\left(c_k h - \beta_k^\top \beta_k h/2 + \beta_k^\top \Delta w(t_k)\right) = 1 + \beta_k^\top \Delta w(t_k) + c_k h$$
$$+\frac{1}{2}\sum_{i=1}^{r}\left[\beta_k^i\right]^2\left([\Delta w_i(t_k)]^2 - h\right) + \sum_{i=1}^{r}\sum_{j=i+1}^{r}\beta_k^i\beta_k^j \Delta w_i(t_k)\Delta w_j(t_k) \quad (32)$$
$$+h(c_k - \beta_k^\top \beta_k/2)\beta_k^\top \Delta w(t_k) + \frac{1}{6}\left[\beta_k^\top \Delta w(t_k)\right]^3 + O(h^2).$$

Now, we consider the last term in (30), $v(t_k, X_k)$, which we expand around (t_{k+1}, X_k) by iterative application of (1). Due to (1), we have

$$v(t_k, X_k) = v(t_{k+1}, X_k) + \int_{t_k}^{t_{k+1}} [(\mathcal{L}v)(t, X_k) + f(t, X_k)]\,\mathrm{d}t \quad (33)$$
$$+ \int_{t_k}^{t_{k+1}} \left[\beta^\top(t, X_k)v(t, X_k) + \gamma^\top(t, X_k)\right] * \mathrm{d}w(t).$$

Clearly,

$$\int_{t_k}^{t_{k+1}} f(t, X_k)\mathrm{d}t = hf_k + O(h^2), \quad (34)$$
$$\int_{t_k}^{t_{k+1}} \gamma^\top(t, X_k) * \mathrm{d}w(t) = \gamma_k^\top \Delta w(t_k) + \frac{\partial}{\partial t}\gamma_k^\top \int_{t_k}^{t_{k+1}} (t - t_k)\mathrm{d}w(t) + O(h^2).$$

By using the expression for $v(t, X_k)$ from (1) twice, we get

$$\int_{t_k}^{t_{k+1}} (\mathcal{L}v)(t, X_k)\mathrm{d}t = h\mathcal{L}v(t_{k+1}, X_k) \quad (35)$$
$$+ \int_{t_k}^{t_{k+1}} \int_{t}^{t_{k+1}} \mathcal{L}\left[\beta^\top(s, X_k)v(s, X_k)\right.$$
$$\left.+\gamma^\top(s, X_k)\right] * \mathrm{d}w(s)\mathrm{d}t + O(h^2)$$
$$= h\mathcal{L}v(t_{k+1}, X_k) + \mathcal{L}\left[\beta_k^\top v(t_{k+1}, X_k) + \gamma_k^\top\right]$$
$$\times \int_{t_k}^{t_{k+1}} (w(t_{k+1}) - w(t))\mathrm{d}t + O(h^2),$$

where the coefficients of $\mathcal{L}$ in the right-hand side are evaluated at (t_k, X_k). Analogously, we obtain

$$\int_{t_k}^{t_{k+1}} \beta^\top(t, X_k) v(t, X_k) * \mathrm{d}w(t) = \beta_k^\top \Delta w(t_k) v(t_{k+1}, X_k) + v(t_{k+1}, X_k)$$
$$\times \sum_{i,j=1}^{r} \beta_k^i \beta_k^j \int_{t_k}^{t_{k+1}} \left[w_j(t_{k+1}) - w_j(t)\right] * \mathrm{d}w_i(t)$$
$$+ \sum_{i,j=1}^{r} \beta_k^i \gamma_k^j \int_{t_k}^{t_{k+1}} \left[w_j(t_{k+1}) - w_j(t)\right] * \mathrm{d}w_i(t)$$
$$+ v(t_{k+1}, X_k) \sum_{i,j,l=1}^{r} \beta_k^i \beta_k^j \left(\beta_k^l + \gamma_k^l\right)$$
$$\times \int_{t_k}^{t_{k+1}} \int_{t}^{t_{k+1}} [w_l(t_{k+1}) - w_l(s)] * \mathrm{d}w_j(s) * \mathrm{d}w_i(t)$$
$$+ (\beta_k^\top \mathcal{L} v(t_{k+1}, X_k) + f_k) \int_{t_k}^{t_{k+1}} (t_{k+1} - t)\mathrm{d}w(t) + O(h^2). \quad (36)$$

It is not difficult to show that

$$\int_{t_k}^{t_{k+1}} \left[w_j(t_{k+1}) - w_j(t)\right] * \mathrm{d}w_i(t) = \int_{t_k}^{t_{k+1}} [w_i(t) - w_i(t_k)]\,\mathrm{d}w_j(t) = I_{ji}(t_k),\ i \neq j; \quad (37)$$

$$\int_{t_k}^{t_{k+1}} \left[w_i(t_{k+1}) - w_i(t)\right] * \mathrm{d}w_i(t) = \frac{[\Delta w_i(t_k)]^2}{2} - \frac{h}{2};$$
$$\sum_{i,j,l=1}^{r} \beta_k^i \beta_k^j \beta_k^l \int_{t_k}^{t_{k+1}} \int_{t}^{t_{k+1}} \left[w_l(t_{k+1}) - w_l(s)\right] * \mathrm{d}w_j(s) * \mathrm{d}w_i(t)$$
$$= \frac{1}{6}\left[\beta_k^\top \Delta w(t_k)\right]^3 - \frac{h}{2}\beta_k^\top \beta_k \beta_k^\top \Delta w(t_k).$$

Also, recall that

$$I_{ij}(t_k) = \Delta w_i(t_k)\Delta w_j(t_k) - I_{ji}(t_k),\ i \neq j; \quad (38)$$
$$\int_{t_k}^{t_{k+1}} (t - t_k)\mathrm{d}w_i(t) = h\Delta w_i(t_k) - I_{0i}(t_k);$$
$$\int_{t_k}^{t_{k+1}} (w_i(t_{k+1}) - w_i(t))\mathrm{d}t = h\Delta w_i(t_k) - I_{0i}(t_k); \int_{t_k}^{t_{k+1}} (t_{k+1} - t)\mathrm{d}w_i(t) = I_{0i}(t_k).$$

Finally, substituting (31)–(38) in (30), we arrive at (28). □

Now, we prove the convergence theorem for Algorithm 1B.

Proof of Theorem 1. By the standard technique (see [42, 43]) of re-writing the global error as a sum of local errors, we obtain

$$E^w\left[\varphi(t_\varkappa, X_\varkappa)Y_\varkappa + Z_\varkappa\right] - v(t_0, x_0) = E^w\left[v(t_\varkappa, X_\varkappa)\, Y_\varkappa + Z_\varkappa\right] - v(t_0, x_0) \quad (39)$$
$$= E^w \sum_{k=0}^{\varkappa-1} \left[v(t_{k+1}, X_{k+1})Y_{k+1} + Z_{k+1} - v(t_k, X_k)Y_k - Z_k\right]$$
$$= \sum_{k=0}^{N-1} E^w(r_k) + \sum_{k=0}^{N-1} E^w(R_k),$$

where the first sum is related to the total error of Algorithm 1B in the boundary zone while the second one is the total error "inside" the domain.

Using (26) and the conditional Jensen inequality, we obtain for the first sum in (39):

$$E\left[\sum_{k=0}^{N-1} E^w(r_k)\right]^{2p} = E\left[\sum_{k=0}^{N-1} E^w E^w(r_k \mid X_k', Y_k, Z_k)\right]^{2p} \quad (40)$$
$$\le Kh^{2p}\, E\left[E^w \sum_{k=0}^{N-1} \rho(t_k, X_k', Y_k) I_{S_{t_k,h}}(X_k')\, \chi_{\varkappa \ge k}\right]^{2p}$$
$$\le Kh^{2p}\, E\left[\sum_{k=0}^{N-1} \rho(t_k, X_k', Y_k) I_{S_{t_k,h}}(X_k')\, \chi_{\varkappa \ge k}\right]^{2p}$$
$$\le Kh^{2p}\, E\left(\max_{0\le k\le N-1} \rho(t_k, X_k', Y_k) \times \sum_{k=0}^{N-1} I_{S_{t_k,h}}(X_k') \chi_{\varkappa \ge k}\right)^{2p}$$
$$= Kh^{2p}\, E\left(\max_{0\le k\le N-1} \rho(t_k, X_k', Y_k) \times \nu_{t_0,x_0}\right)^{2p}.$$

Applying the Cauchy–Bunyakovskii inequality, (25), and (27) to the expression in the last line of (40), we get

$$E\left[\sum_{k=0}^{N-1} E^w(r_k)\right]^{2p} \le Kh^{2p}\left[E\left(\max_{0\le k\le N} \rho(t_k, X_k', Y_k)\right)^{4p}\right]^{1/2} \left[E\nu_{t_0,x_0}^{4p}\right]^{1/2} \le Kh^{2p}. \quad (41)$$

Consider the second sum in (39). Let

$$\tilde{R}_k := \chi_{\varkappa > k} Y_k \left\{ \sum_{r=1}^{q} \left[h\Delta w_r(t_k)\alpha_r(t_{k+1}, X_k) + I_{0r}(t_k)\alpha_{0r}(t_{k+1}, X_k)\right]\right.$$
$$\left. + \sum_{r,n,l=1}^{q} \left[I_{rnl}^*(t_k)\alpha_{rnl}^*(t_{k+1}, X_k) + \Delta w_r(t_k)\Delta w_n(t_k)\Delta w_l(t_k)\alpha_{rnl}(t_{k+1}, X_k)\right]\right\}.$$

Due to Lemma 4, we have

$$E\left[\sum_{k=0}^{N-1} E^w(R_k)\right]^{2p} = E\left[\sum_{k=0}^{N-1} E^w E^w(R_k \mid X_k, Y_k, Z_k)\right]^{2p}$$
$$= E\left[\sum_{k=0}^{N-1} \left(E^w\left(\tilde{R}_k\right) + \chi_{\varkappa>k} O(h^2)\right)\right]^{2p}$$
$$\leq E\left[\sum_{k=0}^{N-1} E^w\left(\tilde{R}_k\right)\right]^{2p} + O(h^{2p}). \quad (42)$$

According to Lemma 4, we have $E\left(\tilde{R}_k\right)^{2p} = O(h^{3p})$ and $E\left(\tilde{R}_k\right) = 0$. Then (using only the former estimate for $\tilde{R}_k$), the sum in the right-hand side of (42) can immediately, but roughly, be estimated as $O(h^p)$. If the coefficients $\alpha_i(t_{k+1}, X_k)$, $\alpha_{0i}(t_{k+1}, X_k)$, $\alpha^*_{ijl}(t_{k+1}, X_k)$ in (28) were $\mathcal{F}_{t_k}$-measurable random variables (remember that Y_k is $\mathcal{F}_{t_k}$-measurable; it depends on the increments $w(s) - w(t_0)$, $t_0 \leq s \leq t_k$ for $\varkappa > k$), then one could estimate this sum by $O(h^{2p})$ as it is done in the case of deterministic PDE problems [43]. However, here, these coefficients include the SPDE solution and its derivatives at time t_{k+1} making them $\mathcal{F}_T$-measurable; they depend on $w(T) - w(s)$, $t_{k+1} \leq s \leq T$. This fact makes accurate estimation of this sum difficult. In [44], such a difficulty was overcome in the case of the Cauchy problem for SPDEs. Since the structure of $\tilde{R}_k$ here and of the one-step error in the Cauchy problem case (cf. the result of Step 1 in the proof of Theorem 3.3 in [44]) are analogous, the approach developed in [44, Steps 2–5 in the proof of Theorem 3.3] can be used to obtain

$$E\left[\sum_{k=0}^{N-1} E^w\left(\tilde{R}_k\right)\right]^{2p} \leq K h^{2p}. \quad (43)$$

We omit here the corresponding lengthy details of the proof of (43) but they can be restored from [44].

The inequalities (41), (42), and (43) imply (18). Theorem 1 is proved. □

5 Numerical Experiments

For our numerical tests, we take the model which solution can be simulated exactly. We consider the following Dirichlet problem

$$-\,\mathrm{d}v = \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}\mathrm{d}t + \beta v * \mathrm{d}w(t), \quad (t,x) \in [T_0, T) \times (-1, 1), \quad (44)$$
$$v(t, \pm 1) = 0, \quad (t,x) \in [T_0, T), \quad (45)$$
$$v(T, x) = \varphi(x), \quad x \in (-1, 1), \quad (46)$$

where $w(t)$ is a standard scalar Wiener process, and σ and β are constants. The solution of this problem has the probabilistic representation (see (6)–(9)):

$$v(t,x) = E^w\left[\mathbf{1}_{\{\tau \geq T\}}\varphi(X_{t,x}(T))Y_{t,x,1}(T)\right], \tag{47}$$

$$\mathrm{d}X = \sigma \mathrm{d}W(s), \tag{48}$$

$$\mathrm{d}Y = \beta Y \mathrm{d}w(s), \tag{49}$$

where $W(s)$ is a standard Wiener process independent of $w(s)$ and $\tau = \tau_{t,x}$ is the first exit time of the trajectory $X_{t,x}(s)$, $s \geq t$, from the interval $(-1, 1)$.

For the experiments, we choose

$$\varphi(x) = A \cdot (x^2 - 1)^3 \tag{50}$$

with some constant A. We note that the coefficients of (44)–(46), (50) satisfy the assumptions made in Sect. 2.

The problem (44)–(46) with (50) has the explicit solution, which can be written in the form

$$v(t,x) = \frac{18\,432}{\pi^7} A \exp\left(-\frac{\beta^2}{2}(T-t) + \beta(w(T) - w(t))\right) \tag{51}$$

$$\times \sum_{k=0}^{\infty} \frac{(-1)^k(\pi^2\,(2k+1)^2 - 10)}{(2k+1)^7} \cos\frac{\pi(2k+1)x}{2} \exp\left(-\frac{\sigma^2\pi^2(2k+1)^2(T-t)}{8}\right).$$

We use the three algorithms from Sect. 3 to solve (44)–(46), (50). In the tests, we fix a trajectory $w(t)$, $0 \leq t \leq T$, which is obtained with a small time step equal to 0.0001. In Table 1, we present the errors for Algorithms 1A and 1B from our tests. In the table, the "±" reflects the Monte Carlo error only, it gives the confidence interval for the corresponding value with probability 0.95 while the values before "±" gives the difference $\hat{v}(0,0) - v(0,0)$ (see $\hat{v}$ in (16)). In Table 1, one can observe convergence of both algorithms with order one that is in good agreement with our

Table 1 Errors of Algorithms 1A and 1B. Evaluation of $v(0,0)$ from (44)–(46), (50) with various time steps h. Here, $\sigma = 0.5$, $\beta = 0.5$, $A = 10$, and $T = 5$. The expectations are computed by the Monte Carlo technique simulating M independent realizations. The "±"reflects the Monte Carlo error only. All simulations are done along the same sample path $w(t)$. The boundary zone parameter λ is taken equal to σ. The corresponding reference value is -2.074421, which is found due to (51)

h	M	Algorithm 1A	Algorithm 1B
0.1	10^7	$4.823 \cdot 10^{-2} \pm 0.247 \cdot 10^{-2}$	$3.604 \cdot 10^{-2} \pm 0.249 \cdot 10^{-2}$
0.05	10^8	$2.591 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$	$2.219 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$
0.025	10^8	$1.616 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$	$0.815 \cdot 10^{-2} \pm 0.079 \cdot 10^{-2}$
0.01	$2.5 \cdot 10^8$	$0.545 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$	$0.338 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$
00.005	$2.5 \cdot 10^8$	$0.210 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$	$0.135 \cdot 10^{-2} \pm 0.050 \cdot 10^{-2}$

Table 2 *Errors of Algorithm 2.* Evaluation of $v(0,0)$ from (44)–(46), (50) with various time steps h. The parameters here are the same as in Table 1

h	M	Algorithm 2
0.0004	10^7	$0.886 \cdot 10^{-1} \pm 0.025 \cdot 10^{-1}$
0.0003	10^7	$0.380 \cdot 10^{-1} \pm 0.025 \cdot 10^{-1}$
0.0002	10^7	$0.404 \cdot 10^{-1} \pm 0.025 \cdot 10^{-1}$
0.0001	10^7	$0.306 \cdot 10^{-1} \pm 0.025 \cdot 10^{-1}$

theoretical results. We note that in this example the trajectory of $Y(t)$ is simulated exactly by Algorithm 1B.

Algorithm 2 produced less accurate results than Algorithms 1 and it was possible to observe its convergence only after choosing very small time steps. Some results are presented in Table 2. They demonstrate an evidence of convergence of Algorithm 2 with order $1/2$ and even smaller time steps are required for further confirmation. We can conclude that, with the computational costs of Algorithms 1 and 2 being essentially the same, Algorithm 1B is the more efficient method than the others.

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Individual Path Uniqueness of Solutions of Stochastic Differential Equations

Alexander M. Davie

Abstract We consider the stochastic differential equation $\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + b(t, x(t))\mathrm{d}W(t)$, $x(0) = x_0$ for $t \geq 0$, where $x(t) \in \mathbb{R}^d$, W is a standard d-dimensional Brownian motion, f is a bounded Borel function from $[0, \infty) \times \mathbb{R}^d$ to $\mathbb{R}^d$, and b is an invertible matrix-valued function satisfying some regularity conditions. We show that, for almost all Brownian paths $W(t)$, there is a unique $x(t)$ satisfying this equation, interpreted in a "rough path" sense.

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1 Introduction

In this paper, we consider the stochastic differential equation

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + b(t, x(t))\mathrm{d}W(t), \qquad x(0) = x_0 \tag{1}$$

for $t \geq 0$, where $x(t) \in \mathbb{R}^d$, W is a standard d-dimensional Brownian motion, f is a bounded Borel function from $[0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ to $\mathbb{R}^d$, and b is an invertible $d \times d$ matrix valued function on $[0, \infty) \times \mathbb{R}^d$ satisfying a suitable regularity condition. If b satisfies a Lipschitz condition in x, then it follows from a theorem of Veretennikov [4] that (2) has a unique strong solution, i.e. there is a unique process $x(t)$, adapted to the filtration of the Brownian motion, satisfying (1).

Here we consider a different question, posed by N. V. Krylov (Gyöngy, Personal communication): we choose a Brownian path W and ask whether (1) has a unique solution for that particular path. The first problem with this question is to interpret it, since the stochastic integral implied by the equation is not well defined for individual paths. One case for which there is a simple interpretation is when $b(t, x)$ is the identity matrix for all t, x, since in that case we can write the equation as

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$$x(t) = W(t) + x_0 + \int_0^t f(s, x(s))\mathrm{d}s, \qquad t \geq 0 \tag{2}$$

and the existence of a unique solution to (1), for almost every path W, was proved in [2].

In this paper, we use methods of rough path theory to give an interpretation of (1), under slightly stronger regularity conditions on b. Our main result is then that for almost all Brownian paths W there is a unique solution in this sense. The proof is similar to that in [2] but requires estimates for solutions of a related equation ((3) below) similar to those for Brownian motion in Sect. 2 of [2], and different arguments including a suitable $T(1)$ theorem are needed for this. The rough path interpretation and formulation of the main theorem are described in Sect. 2, the estimates for solutions of (3) are given in Sect. 3 and the proof of the theorem in Sect. 4.

2 Interpretation of Individual Path Solution

We now describe our interpretations of (1) precisely. Let U be a domain in $\mathbb{R}^d$ containing (x_0). We assume that $f(t,x)$ and $b_{ij}(t,x)$ are defined on $[0,\infty) \times U$ and that f is bounded and b_{ij} is differentiable with respect to x and that b_{ij} and $\partial b_{ij}/\partial x_k$ satisfy locally a Hölder condition of order α for some $\alpha > 0$ in (t,x). We write (1) in components as

$$\mathrm{d}x_i(t) = f_i(t, x(t))\mathrm{d}t + \sum_{j=1}^{d} b_{ij}(t, x(t))\mathrm{d}W_j(t)$$

for $i = 1, \cdots, n$. For $s < t$ we write $A_{rj}(s,t) = \int_s^t (W_r(\tau) - W_r(s))\mathrm{d}W_j(\tau)$ and $\Psi_i(s,t,x) = \sum_{j=1}^d \int_s^t b_{ij}(\tau, x)\mathrm{d}W_j(\tau)$ for $x \in \mathbb{R}^d$. On a suitable set of probability 1, these quantities can be defined simultaneously for all $0 \leq s < t$ and $x \in \mathbb{R}^d$, depending continuously on (s,t,x), and we assume that such definitions have been fixed.

We say that a continuous $x : [0,T) \to U$ is a solution *in rough path sense* of (1) on an interval $0 \leq t < T$ where $0 < T \leq \infty$ if $x(0) = x_0$ and, for any T' with $0 < T' < T$ we can find $\delta > 0$ and $C > 0$ such that

$$\left| x_i(t) - x_i(s) - \int_s^t f_i(\tau, x(\tau))\mathrm{d}\tau - \Psi_i(s,t,x(s)) - \sum_{j,r} g_{ijr}(x(s))A_{rj}(s,t) \right| \leq C(t-s)^{1+\delta}$$

where $g_{ijr} = \sum_{k=1}^n \frac{\partial b_{ij}}{\partial x_k} b_{kr}$.

It is not hard to show that the strong solution of (1) is, for almost all Brownian paths, a solution in this sense.

We can now formulate the main result.

Theorem 1. *For almost every Brownian path W, there is a unique T with $0 < T \leq \infty$ and a unique continuous $x : [0, T) \to U$, satisfying (1) in rough path sense, and such that in case $T < \infty$ we have that $|x(t)|$ 'escapes from U as $t \to T$, in the sense that for any compact subset K of U there is $t \in [0, T)$ with $x(t) \notin K$.*

This theorem is essentially local in character, and this enables us to reduce it to the case when the coefficients b_{ij} are globally well behaved. We use a standard localisation argument to deduce Theorem 1 from the following:

Proposition 1. *Suppose f and b_{ij} are defined on $[0, \infty) \times \mathbb{R}^d$ and satisfy the above regularity conditions, and also that there is $K > 0$ such that $b_{ij}(t, x) = \delta_{ij}$ if $|x| > K$. Then (1) has a solution x on $[0, \infty)$ in rough path sense, and is unique in the sense that if y is a solution on $[0, t)$ for some $T > 0$ then $x(t) = y(t)$ on $[0, T)$.*

To deduce Theorem 1 from Proposition 1 we let $U_1, U_2, \cdots$ be relatively compact open subsets of U with union U and $\overline{U}_n \subseteq U_{n+1}$. Then for each n, we can find functions $f^{(n)}$ and $b_{ij}^{(n)}$ satisfying the requirement of the proposition and agreeing, respectively, with f and b_{ij} on U_n. The proposition then gives a solution $x^{(n)}$ on $[0, \infty)$ to (1) with $f^{(n)}, b_{ij}^{(n)}$ replacing f, b_{ij}. Define $T_n \in (0, \infty]$ by $T_n = \sup\{t : x^{(n)}([0, t)) \in U_n\}$. Then by the uniqueness part of the proposition $x^{(n+1)} = x^{(n)}$ on $[0, T_n)$ and $T_{n+1} \geq T_n$. Hence T_n converges to a limit T and we can define $x(t)$ on $[0, T)$ by $x(t) = x^{(n)}(t)$ for $t < T_n$. Then x satisfies (1).

If T is finite and K is a compact subset of U, then for some n we have $K \subseteq U_n$. Then $T_n < \infty$ and by continuity $x^{(n+1)}(T_n) \in \overline{U}_n \subseteq U_{n+1}$ so $T_n < T_{n+1}$ and hence there is t so that $x(t) \notin U_n$, so $x(t) \notin K$.

The proof of Proposition 1 follows the same lines as [2]. First it suffices to prove the uniqueness for $0 \leq t \leq 1$. Then we need an estimate analogous to Proposition 2.1 of [2], where instead of $W(t)$ we have a solution $y(t)$ of the equation

$$\mathrm{d}y(t) = a(t, y(t))\mathrm{d}t + b(t, y(t))\mathrm{d}W(t) \tag{3}$$

where the vector $a(t, x)$ is defined by $a_i = \sum_{k,j} \frac{\partial b_{ij}}{\partial x_k} b_{kj}$.

This estimate is proved in the next section, and the deduction of Proposition 1 is described in Sect. 4.

3 The Basic Estimate

In this section, we establish the following estimate for (3). We assume throughout that b is fixed and satisfies the conditions of Proposition 1.

Proposition 2. *Let g be a Borel function on $[0, 1] \times \mathbb{R}^d$ with $|g(s, z)| \leq 1$ everywhere, let $I \subseteq [0, 1]$ be an interval, and let y satisfy the SDE (3). Then for any even positive integer p and $x \in \mathbb{R}^d$, we have*

$$\mathbb{E}\left(\int_I \{g(t, y(t)+x) - g(t, y(t))\}\mathrm{d}t\right)^p \le C^p (p/2)!|x|^p|I|^{p/2}$$

where C is an absolute constant and $|x|$ denotes the usual Euclidean norm.

The proof of this is similar to that of Proposition 2.1 of [2] with some differences. One is that the reduction to the one-dimensional case no longer works, so the entire argument has to be carried out in $\mathbb{R}^d$ – this does not in fact change much. A more substantial difference is that we have to replace the Gaussian transition densities $E(t-s, w-z)$ by transition densities $E(s, t, z, w)$ for the diffusion defined by (3). Then the proof of Lemma 2.3 of [2], which uses translation invariance, no longer works. Instead we apply a $T(1)$ theorem to prove L^2 boundedness of an integral operator with kernel $E(s, t, z, w)$.

Proposition 2 will be deduced from the following variant for smooth g. We denote by g' the derivative w.r.t. the first component of the vector x, i.e. $g'(t, x) = \frac{\partial g}{\partial x_1}(t, x)$.

Proposition 3. *There is a constant C such that, if g is a compactly supported smooth function on* $[0, 1] \times \mathbb{R}^d$ *with* $|g(s, z)| \le 1$ *everywhere and* g' *bounded, and* $0 \le t_0 < T \le 1$*, then for any even positive integer p we have, conditional on* $y(t_0) = y_0$*,*

$$\mathbb{E}\left(\int_{t_0}^T g'(t, y(t))\mathrm{d}t\right)^p \le (C|I|)^{p/2}(p/2)!$$

This is proved in a manner similar to that of Proposition 2.2 of [2]. If $I = [t_0, T]$ where $0 \le t_0 < T \le 1$ we can write the LHS as

$$p! \int_{t_0 < t_1 < \dots < t_p < T} \mathbb{E} \prod_{j=1}^p g'(t_j, y(t_j))\mathrm{d}t_1 \cdots \mathrm{d}t_p$$

and using the joint distribution of $y(t_1), \dots, y(t_p)$ this can be expressed as

$$p! \int_{t_0 < t_1 < \dots < t_p < T} \int_{\mathbb{R}^{dp}} \prod_{j=1}^p \{g'(t_j, z_j)E(t_{j-1}, t_j, z_{j-1}, z_j)\}\mathrm{d}z_1 \cdots \mathrm{d}z_p\mathrm{d}t_1 \cdots \mathrm{d}t_p$$

where $z_0 = y_0$.

We introduce the notation

$$J_k(t_0, z_0) = \int_{t_0 < t_1 < \dots < t_k < T} \int_{\mathbb{R}^{kd}} \prod_{j=1}^k \{g'(t_j, z_j)E(t_{j-1}, t_j, z_{j-1}, z_j)\} \mathrm{d}z_1 \cdots \mathrm{d}z_k\mathrm{d}t_1 \cdots \mathrm{d}t_k$$

and we shall show that $J_p(t_0, y_0) \le C^p(T - t_0)^{p/2}/\Gamma(\frac{p}{2} + 1)$; Proposition 3 will then follow since $p! \le 2^p((p/2)!)^2$.

As in [2] we use integration by parts to shift the derivatives to the exponential terms. We introduce some notation to handle the resulting terms – we define $B_1(s,t,z,w) = -\frac{\partial}{\partial z_1}E(s,t,w,z)$, $B_2(s,t,z,w) = -\frac{\partial}{\partial w_1}E(s,t,z,w)$ and $D(s,t,z,w) = \frac{\partial^2}{\partial z_1 \partial w_1}E(s,t,z,w)$.

If $S = S_1 \cdots S_k$ is a word in the alphabet $\{E, B_1, B_2, D\}$ then we define

$$I_S(t_0, z_0) = \int_{t_0<t_1<\cdots<t_k<1} \int_{\mathbb{R}^{kd}} \prod_{j=1}^{k} \{g(t_j, z_j) S_j(t_{j-1}, t_j, z_{j-1}, z_j)\} \, \mathrm{d}z_1 \cdots \mathrm{d}z_k \mathrm{d}t_1 \cdots \mathrm{d}t_k$$

We say a word is *allowed* if it is of the form $B_2^{m_0} E B_1^{n_1} D B_2^{m_1} E \cdots E B_1^{n_r} D B_2^{m_r}$ for some non-negative integers $r, m_0 \cdots m_r, n_1 \cdots, n_r$. This is equivalent to the definition in[2] except that there B_1 and B_2 are not distinguished. There are 2^{k-1} allowed words of length k. For example, the allowed words of length 3 are $B_2B_2B_2$, EDB_2, B_2ED and EB_1D.

Just as in [2], we show by induction on k that

$$J_k(t_0, z_0) = \sum_{j=1}^{2^{k-1}} I_{S^{(j)}}(t_0, z_0) \tag{4}$$

where each $S^{(j)}$ is an allowed word of length k. (in fact each allowed word of length k appears exactly once in this sum, but we do not need this fact). The proof will then be completed by obtaining a bound for I_S.

We prove (4) by induction on k. So, assuming (4) for J_k, we have

$$\begin{aligned} J_{k+1}(t_0, z_0) &= \int_{t_0}^{1} \mathrm{d}t_1 \int g'(t_1, z_1) E(t_1 - t_0, z_1 - z_0) J_k(t_1, z_1) \mathrm{d}z_1 \\ &= -\int_{t_0}^{1} \mathrm{d}t_1 \int g(t_1, z_1) B(t_1 - t_0, z_1 - z_0) J_k(t_1, z_1) \mathrm{d}z_1 \\ &\quad - \int_{t_0}^{1} \int g(t_1, z_1) E(t_1 - t_0, z_1 - z_0) J_k'(t_1, z_1) \mathrm{d}z_1 \end{aligned}$$

Now we observe that, if S is an allowed string, then $I_S' = -I_{\tilde{S}}$ where $\tilde{S}$ is defined as BS^* if $S = ES^*$ and as DS^* if $S = BS^*$ (note that $\tilde{S}$ is not an allowed string). Applying this to (4) we find $J_k'(t_0, z_0) = \sum_{j=1}^{2^{k-1}-1} \mp I_{\tilde{S}^j}(t_0, z_0)$ and then we obtain

$$J_{k+1}(t_0, z_0) = \mp \sum_{j=1}^{2^{k-1}-1} I_{BS^j}(t_0, z_0) \pm \sum_{j=1}^{2^{k-1}-1} I_{E\tilde{S}^j}(t_0, z_0)$$

Noting that, if S is an allowed string, BS and $E\tilde{S}$ are also allowed, this completes the inductive proof of (4).

We now proceed to the estimation of $I_S(t_0, z_0)$, when S is an allowed string. We start with some standard properties on the transition density $E(s, t, z, w)$.

E satisfies the Kolmogorov equations

$$\frac{\partial E}{\partial s} = -\sum_i a_i(s, x)\frac{\partial E}{\partial z_i} - \frac{1}{2}\sigma_{ij}\frac{\partial^2 E}{\partial z_i \partial z_j}$$
$$\frac{\partial E}{\partial t} = -\sum_i a_i(t, w)\frac{\partial E}{\partial w_i} + \frac{1}{2}\sigma_{ij}\frac{\partial^2 E}{\partial w_i \partial w_j}$$

recalling that $a_i = \sum_{k,j} \frac{\partial b_{ij}}{\partial x_k} b_{kj}$; σ is defined by $\sigma_{ik} = \sum_j b_{ij} b_{kj}$.

Then we can combine standard decay bounds for E with Schauder interior estimates and obtain the following bounds: there exist $c, C > 0$ such that for $0 \le s < t \le 1$ and $y, z \in \mathbb{R}^d$ we have $|E(s, t, z, w)| \le C(t-s)^{-d/2}\mathrm{e}^{-c|z-w|^2/(t-s)}$ and

$$\left|\frac{\partial^2 E(s, t, z, w)}{\partial z_i \partial z_j}\right| \le C(t-s)^{-1-d/2}\mathrm{e}^{-c|z-w|^2/(t-s)}$$

with a similar bound for the 2nd partial derivatives w.r.t. w. We can combine these to get $|D(s, t, z, w)| \le C(t-s)^{-1-d/2}\mathrm{e}^{-c|z-w|^2/(t-s)}$. As a consequence we have a bound $|D(s, t, z, w)| \le Cd(s, z; t, w)^{-d-2}$ using the parabolic metric $d(s, y; t, z) = |z-w| + |s-t|^{1/2}$. Moreover, the Schauder estimates give Hölder estimates from which it follows that for some $\delta > 0$ we have a bound

$$|D(s, t, z, w) - D(s, t', z, w')| \le C\frac{d(t, w; t', w')^{\delta}}{d(s, z; t, w)^{d+2+\delta}}$$

whenever $\frac{d(t,w;t',w')}{d(s,z;t,w)} < \frac{1}{2}$.

We now assert:

Lemma 1. *The operator T defined by*

$$Th(s, y) = \int_s^1 \int_{\mathbb{R}^d} D(s, t, y, z)h(t, z)\mathrm{d}z\mathrm{d}t$$

is bounded on $L^2([0,1] \times \mathbb{R}^d)$.

This is proved by applying a '$T(1)$ theorem on spaces of homogeneous type' for an operator T of Calderon–Zygmund type, which, by virtue of the above bounds on D, our T is (on $[0,1] \times \mathbb{R}^d$, with Lebesgue measure and the parabolic metric, which is a space of homogeneous type). The $T(1)$ theorem asserts that T is bounded on L^2 provided $T(1)$ and $T^*(1)$ are in BMO. See e.g. [1]. Our operators T_{ij} are of the required type, provided we equip $\mathbb{R} \times \mathbb{R}^d$ with the parabolic metric, and we have in fact $T_{ij}(1) = 0$, and $T^*_{ij}(1) = 0$ (this follows from the fact that $\int_{\mathbb{R}^d} D(s, t, y, z)\mathrm{d}z = 0$ and $\int_{\mathbb{R}^d} D(s, t, y, z)\mathrm{d}y = 0$).

We also note the bound $|B_i(s,t,z,w)| \le C(t-s)^{-(d+1)/2}\mathrm{e}^{-c|z-w|^2/(t-s)}$

We now apply Lemma 1 to show

Lemma 2. *Given* $\alpha > 0$ *we can find a constant* C *such that, if* ϕ *and* h *are real-valued Borel functions on* $[0,1]\times\mathbb{R}$ *with* $|\phi(t,y)| \le \mathrm{e}^{-\alpha y^2/t}$ *and* $|h(t,y)| \le 1$ *everywhere, and* $0 \le t_0 < T \le 1$*, then*

$$\left|\int_{(t_0+T)/2}^{T}\mathrm{d}t\int_{(t_0+t)/2}^{t}\mathrm{d}s\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(s,z)h(t,y)D(t-s,y-z)\mathrm{d}y\mathrm{d}z\right| \le C(T-t_0)$$

Proof. We follow the proof of Lemma 2.3 of [2], using Lemma 1 in place for the Fourier transform calculation. Instead the covering of $\mathbb{R}$ by intervals of length 1 in [2], we use a covering of $\mathbb{R}^d$ by cubes of side $(T-t_0)^{1/2}$. Then with notation as there we have $I_{lm} = \langle Th_m, \phi_l\rangle$ and $\|h_l\|_2 \le C_1(T-t_0)^{1/2}$, $\sum_l \|\phi_l\|_2 \le C_2(T-t_0)^{1/2}$. Also the sum of $|I_{lm}|$ over pairs of cubes separated by at least $(T-t_0)^{1/2}$ is bounded by $C_3(T-t_0)$. For other l,m pairs we use $|I_{lm}| \le \|T\|\|\phi_l\|_2\|h_m\|_2$ to deduce $\sum_{l,m}|I_{lm}| \le C(T-t_0)$.

Corollary 1. *There is a constant* C *such that if* g *and* h *are Borel functions on* $[0,1]\times\mathbb{R}$ *bounded by 1 everywhere,* $z_0\in\mathbb{R}^d$ *and* $0 \le t_0 < T \le 1$*, then*

$$\left|\int_{(t_0+T)/2}^{T}\mathrm{d}t\int_{(t_0+t)/2}^{t}\mathrm{d}s\int_{\mathbb{R}^{2d}} g(s,z)E(t_0,s,z_0,z)h(t,w)D(s,t,z,w)\mathrm{d}z\mathrm{d}w\right| \le C(T-t_0)$$

and

$$\left|\int_{(t_0+T)/2}^{T}\mathrm{d}t\int_{(t_0+t)/2}^{t}\mathrm{d}s\int_{\mathbb{R}^{2d}} g(s,z)B_1(t_0,s,z_0,z)h(t,w)D(s,t,z,w)\mathrm{d}z\mathrm{d}w\right| \le C(T-t_0)^{1/2}$$

Proof. These follow easily from Lemma (2).

Lemma 3. *There is a constant* C *such that if* g *and* h *are Borel functions on* $[0,1]\times\mathbb{R}$ *bounded by 1 everywhere, and* $r\ge 0$ *then*

$$\left|\int_{t_0}^{T}\mathrm{d}t\int_{t_0}^{t}\mathrm{d}s\int_{\mathbb{R}^{2d}} g(s,z)E(t_0,s,z_0,z)h(t,w)D(s,t,z,w)\mathrm{d}z\mathrm{d}w(1-t)^r\right| \le C(1+r)^{-1}(T-t_0)^{r+1}$$

and

$$\left| \int_{t_0}^{T} \mathrm{d}t \int_{t_0}^{t} \mathrm{d}s \int_{\mathbb{R}^{2d}} g(s,z) B_1(t_0,s,z_0,z) h(t,w) D(s,t,z,w) \mathrm{d}z\mathrm{d}w (1-t)^r \right|$$
$$\leq C(1+r)^{-1/2}(T-t_0)^{r+\frac{1}{2}}$$

This is proved in the same way as Lemma 2.5 of [2].

As in [2] we can now complete the proof of Proposition 3 by proving by induction on k, for suitable choice of M, that

$$|I_S(t_0,z_0)| \leq \frac{M^k}{\Gamma(\frac{k}{2}+1)}(1-t_0)^{k/2} \tag{5}$$

for any allowed string S of length k. The proof is split into 3 cases as in [2]. The role of B in [2] is played by B_1 in case (1) and B_1 in case (3). With this adjustment the proof is the same as in [2].

We can then deduce Proposition 2 from Proposition 3 in the same way that Proposition 2.1 in [2] is deduced (but omitting the reduction to $d=1$).

The following corollary is what we mainly use. For $s \geq 0$ let $\mathcal{F}_s$ be the σ-field generated by $\{W(\tau) : 0 < \tau < s\}$.

Corollary 2. *There is a constant $c > 0$ such that if g is a Borel function on $[0,1] \times \mathbb{R}^d$ with $|g| \leq 1$ everywhere and $0 \leq s \leq a < b \leq 1$, then for $x \in \mathbb{R}^d$ and $\lambda > 0$ we have*

$$\mathbb{P}\left(\left|\int_a^b \{g(z(t)+x) - g(z(t))\}\mathrm{d}t\right| \geq \lambda(b-a)^{1/2}|x| \,|\mathcal{F}_s\right) \leq 2\mathrm{e}^{-c\lambda^2}$$

This is deduced from Proposition 2 in the same way as Corollary 2.6 in [2].

We also need the following technical lemma.

Lemma 4. *There exists $\delta > 0$ such that, given $K > 0$ we can find $C > 0$ so that, if $0 \leq s_0 < s < t$, and A is a matrix with $\|A\| \leq K$ and $\|(I+A)^{-1}\| \leq K$, then the following holds, where we define the random vector X by $X = z(t) + A(z(s) - z(s_0))$:*

For any Borel function h on $\mathbb{R}^d$ with $|h| \leq 1$ everywhere, we have

$$\mathbb{E}|(h(X)|\mathcal{F}_{s_0}) - \mathbb{E}h(V)| < C\{(t-s_0)^{\varepsilon}\lambda^{-r} + \lambda\}$$

where $\lambda = \frac{t-s}{t-s_0}$ and V is a random vector, normally distributed with mean $z(s_0)$ and covariance $(I+A)BB^t(I+A)^t)$, where $B = b(z(s_0))$.

Proof. Let f_X be the density of X, conditional on $\mathcal{F}_{s_0}$. Then we can write $F(x) = \mathbb{E}(E(s,t,(I+A)Z(s),x)|\mathcal{F}_{s_0})$ and using our regularity results for E we deduce $|\nabla F(x)| \leq C_1(t-s)^{-\frac{d+1}{2}}$ for all $x \in \mathbb{R}^d$.

Now define

$$Y = z(s_0) + B(W(t) - W(s_0)) + AB(W(x) - W(s_0))$$

Then (conditional on $\mathcal{F}_{s_0}$) Y has a normal distribution with mean $z(s_0)$ and covariance matrix $(s - s_0)(I + A)BB^t(I + A^t) + (t - s)BB^t$. Now straightforward estimates give $\mathbb{E}(|X - Y||\mathcal{F}_{s_0}) \le C_2(t - s_0)^{\frac{1}{2}+\eta}$ for some constant $\eta > 0$. We need to convert this to a bound for the difference of densities $f_X - f_Y$. Fix $x \in \mathbb{R}^d$, let $\delta > 0$ (to be specified later) and let ϕ be a nonnegative smooth function on $\mathbb{R}^d$, supported on the δ-neighbourhood of x, such that $\int \phi = 1$ and $|\nabla\phi| \le C_3\delta^{-d-1}$. Using the derivative bound for f_X we have $|f_X(x) - \int \phi f_X| \le C_4\delta|t - s|^{-\frac{d+1}{2}}$. And

$$\left|\int \phi f_X - \int \phi f_Y| \le \mathbb{E}|\phi(X) - \phi(Y)\right| \le C_3\delta^{-d-1}\mathbb{E}|X - Y| \le C_5\delta^{-d-1}|t - s_0^{\frac{1}{2}+\eta}$$

Putting these bounds together, and choosing $\delta = (t - s_0)^{\frac{1}{2}+\frac{\eta}{d+2}}\lambda^{\frac{d+1}{2(d+2)}}$ we obtain

$$|f_X(x) - f_Y(x)| \le C_6(t - s_0)^{\frac{\eta}{d+2}-\frac{d}{2}}\lambda^{-r}$$

and since $f_X(x)$ decays like $\mathrm{e}^{-c|x|^2/(t-s_0)}$ we deduce that if $0 < \varepsilon < \frac{\eta}{d+2}$ then $\int |f_X(x) - f_Y(x)|\mathrm{d}x \le C_7(t - s_0)^{\varepsilon}\lambda^{-r}$. Finally Y and V both have normal distributions with the same mean and with covariance matrices having a relative difference $O(\lambda)$ so $\int |f_Y(x) - f_V(x)|\mathrm{d}x \le C_8\lambda$ and the result follows.

4 Proof of Theorem

As we have seen, it suffices to prove Proposition 1, so we assume the conditions of that proposition hold. We let $z(t)$ denote the strong solution of (1), so we have to show that for almost all paths W, z is the only solution of (1) for $0 \le t \le 1$ in our 'rough path' sense. We write $x(t) = z(t) + u(t)$ and then we have to show that the only solution u of

$$u(t) = \int_0^t \{f(s, z(s) + u(s)) - f(s, z(s))\}\mathrm{d}s + \int_0^t \{b(s, z(s) + u(s)) - b(s, z(s))\}\mathrm{d}W(s) \tag{6}$$

satisfying $u(0) = 0$ is $u(t) = 0$, where the $\int \mathrm{d}W$ is interpreted in rough path sense.

The basic idea is to approximate u by a sequence of step functions u_n, such that u_n is constant on each interval $I_{nk} = [k2^{-n}, (k + 1)2^{-n}]$, $k = 0, 1, 2, \ldots, 2^n - 1$ and then use

$$\begin{aligned}
&\int_{I_{nk}} \{f(t, z(t) + u(t)) - f(t, z(t))\}\mathrm{d}t \\
&\quad = \lim_{l\to\infty} \int_{I_{nk}} \{f(t, z(t) + u_l(t)) - f(t, z(t))\}\mathrm{d}t \\
&\quad = \int_{I_{nk}} \{f(t, z(t) + u_n(t)) - f(t, z(t))\}\mathrm{d}t \\
&\qquad + \sum_{l=n}^{\infty} \int_{I_{nk}} \{f(t, z(t) + u_{l+1}(t)) - f(t, z(t) + u_l(t))\}\mathrm{d}t
\end{aligned} \tag{7}$$

with a similar expansion for $\int_{I_{nk}} \{b(t, z(t) + u(t)) - b(t, z(t))\}\mathrm{d}W(t)$.

We introduce the notation

$$\theta_{nk}(x) = \int_{I_{nk}} \{f(t, z(t) + x) - f(t, z(t))\}\mathrm{d}t$$
$$\phi_{nk} = \int_{I_{nk}} \{b(t, z(t) + x) - b(t, z(t))\}\mathrm{d}W(t)$$

and then $\sigma_{nk}(x) = \theta_{nk}(x) + \phi_{nk}(x)$, $\mu_{nk}(x, y) = \theta_{nk}(x) - \theta_{nk}(y)$, $\nu_{nk}(x, y) = \phi_{nk}(x) - \phi_{nk}(y)$ and finally

$$\rho_{nk}(x, y) = \mu_{nk}(x, y) + \nu_{nk}(x, y) = \sigma_{nk}(x) - \sigma_{nk}(y)$$

Using this notation, from (7), and the similar expansion for the dW, (6) implies

$$u((k+1)2^{-n}) - u(k2^{-n}) = \sigma_{nk}(u(k2^{-n})) + \sum_{l=n}^{\infty} \sum_{r=k2^{l-n}}^{(k+1)2^{l-n}-1} \rho_{l+1,2r+1} \\ (u(2^{-l-1}(2r+1)), u(2^{-l}r)) \tag{8}$$

As in [2], the first stage of the proof is to show that a number of estimates for these quantities hold with probability 1. In doing this, we need to use two distinct probability measures on our sample space, the original measure which we denote by $\mathbb{P}$ and the equivalent measure, given by the Girsanov theorem, w.r.t. which the process z had the same law as y in Sect. 3. We denote this measure by $\tilde{\mathbb{P}}$. We use $\mathbb{E}$ and $\tilde{\mathbb{E}}$ for the corresponding expectations. As the measures are mutually absolutely continuous, statements of the form 'with probability 1' can be made without specifying the measure. The first estimate we need is

Lemma 5. *With probability 1, for every cube $Q \subseteq \mathbb{R}^d$ there is a constant $C > 0$ such that the following bounds hold:*

$$|\rho_{nk}(x, y)| \le C \left\{ n^{1/2} + \left(\log^+ \frac{1}{|x - y|} \right)^{1/2} \right\} 2^{-n/2}|x - y|$$

for all dyadic $x, y \in Q$ *and all choices of integers* n, k *with* $n > 0$ *and* $0 \leq k \leq 2^n - 1$.

This is proved by proving separately the same bound for μ and for ν. The bound for μ is identical to that of Lemma 3.1 in [2], using and working with $\tilde{\mathbb{P}}$. That for ν is proved using and working with $\mathbb{P}$.

Next we have

Lemma 6. *With probability 1, for every cube* $Q \subseteq \mathbb{R}^d$ *there is a constant* $C > 0$ *such that for all* $n \in \mathbb{N}$, $k \in \{0, 1, \cdots, 2^n - 1\}$ *and dyadic* $x \in Q$ *we have*

$$|\sigma_{nk}(x)| \leq C n^{1/2} 2^{-n/2} (|x| + 2^{-2^n})$$

Again this is proved separately for θ and ϕ, using the argument of Lemma 3.2 of [2] and the same bounds as in the previous lemma.

We also need an analogue of Lemma 3.6 of [2], for sums of ρ_{nk} terms. This requires martingale arguments as used in the proof of Lemma 3.5 of [2], but they are complicated by the fact these arguments involve both probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$. We manage to arrange the proof to keep them separate. First we prove the following bound for sums of σ_{nk} terms.

Lemma 7. *With probability 1, for any cube* Q *we can find* $C > 0$ *such that, for any choice of* $n, r \in \mathbb{N}$ *with* $r \leq 2^{n/4}$, $k \in \{0, 1, \cdots, 2^n - r$ *and* $x_0 \in Q$*, if we define* $x_1, \cdots, x_r$ *by the recurrence relation* $x_{q+1} = x_q + \sigma_{n,k+q}(x_q)$*, then*

$$\sum_{q=0}^{r-1} |\sigma_{n,k+q}(x_q)| \leq C(2^{-n/4}|x_0| + 2^{-2^{-n/2}})$$

Proof. Again we prove the estimate separately for θ and ϕ, starting with θ. For $K > 0$ and dyadic t define an event χ_t^K: for all $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, 2^n - 1\}$ with $(k+1)2^{-n} \leq t$ and dyadic $x \in Q$ we have

$$|\sigma_{nk}(x)| \leq K n^{1/2} 2^{-n/2} (|x| + 2^{-2^n})$$

Then χ_t^K increases with K and Lemma 6 says that the union has full measure. One easily checks that for given K there is $C(K)$ such that in χ_t^K one has, in the situation of the statement of this lemma, $|x_q| \leq C(K)|x_0|$ provided $(k+q)2^{-n} \leq t$. We abuse notation and write χ_q for χ_t when $t = (k+q)2^{-n}$.

We fix K and define $Y_q = |\theta_{n,k+q-1}(x_{q-1})|\chi_{q-1}^K$, $Z_q = \tilde{\mathbb{E}}(Y_q|\mathcal{F}_{q-1}$ and $X_q = Y_q - Z_q$. Then we have $Z_q \leq C_1 2^{-n/2} C(K)|x_0|$ for each q. Also, by Burkholder's inequality, for any $p \geq 2$,

$$\tilde{\mathbb{E}}|\sum_{q=1}^{r} X_q|^p \leq C_p r^{p/2-1} \sum \tilde{\mathbb{E}}(Y_q^p) \leq C_2 C(K)^p C(p) r^{p/2-1} 2^{-np/2} |x_0|^p$$

Using this, we can use the methods of Lemmas 3.5 and 3.6 of [2] to obtain the required bound.

The estimate for ϕ is similar, using $\mathbb{P}$, but simpler as the analogue of Z_q vanishes.

Now we apply Lemma 7 to obtain the required bound for sums of ρ_{nk}.

Lemma 8. *With probability 1, for any cube Q we can find $C > 0$ such that for any $n, r \in \mathbb{N}$ with $r \leq 2^{n/4}$, and $k \in \{0, 1, \cdots, 2^n - r\}$ and any $y_0, \cdots, y_r \in Q$ we have*

$$\sum_{q=1}^{r} |\rho_{n,k+q}(y_{q-1}, y_q)| \leq C \left(2^{-3n/4}|y_0| + 2^{-n/4} \sum_{q=0}^{r-1} |\gamma_q| + 2^{-2^{n/2}} \right)$$

where $\gamma_q = y_{q+1} - y_q - \sigma_{n,k+q}(y_q)$.

Proof. Again we obtain bounds for μ and ν separately. We give the proof for μ.

We proceed in a manner similar to the previous lemma. Fix Q, then define χ_t^K to be the event that the conclusion of Lemma 7 holds with $C = K$ whenever $(k + r + 1)2^{-n} \leq t$. Then define $Y_q = |\mu_{n,k+q}(x_{q-1}, x_q)|\chi_q^K$, $Z_q = \tilde{\mathbb{E}}(Y_q|\mathcal{F}_q$ and $X_q = Y_q - Z_q$. We get $\sum Z_q \leq CK2^{-3n/4}(|x_0| + 2^{-2^{n/2}})$ and

$$\tilde{\mathbb{E}}|\sum_{q=1}^{r} X_q|^p \leq C_p K^p r^{p/2} n^{p/2} 2^{-np}(|x_0| + 2^{-2^n})^p$$

which suffices for the method of proof of Lemma 3.6 in [2].

Lemma 9. *There exists $\delta > 0$ such that, with probability 1, for any cube Q we can find $C > 0$ such that for any $n, r \in \mathbb{N}$ with $r \leq 2^{n/4}$, and $k \in \{0, 1, \ldots, 2^n - 2r\}$ and any $y_0, \ldots, y_r \in Q$ we have*

$$\left| \sum_{q=1}^{r} \rho_{n,k+2q}(y_{2q-1}, y_{2q}) \right| \leq C \left(2^{-(\frac{3}{4}+\delta)n} + 2^{-n/4} \sum_{q=0}^{r-1} |\gamma_q| \right)$$

where $\gamma_q = y_{q+1} - y_q - \sigma_{n,k+q}(y_q)$.

Proof. Again we consider μ and ν separately, and we only discuss μ which is harder. We simply outline the idea, which is to use a martingale argument similar to those in the previous two lemmas, but using a better bound for the expectation term. To do this we consider a short-term approximation to the solutions $z(t)$ and $x(t) = z(t) + u(t)$. Given a time s_0, for $t > s_0$ but close to s_0 we expect $z(t) \approx z(s_0) + B(W(t) - W(t_0))$ and $x(t) \approx x(s_0) + C(W(s) - W(s_0))$ where $B = b(z(s_0))$ and $C = b(x(s_0))$. Then we have $u(t) \approx Az(s) + w$ where $A = CB^{-1} - I$ and $w = u(s_0) - Az(s_0)$.

Now consider $\tilde{\mathbb{E}}(\mu_{n,k+2q}(x, y)|\mathcal{F}_{s_0})$ where $s_0 = j2^{-n}$ and $k + 2q - j >> 1$. We can write this as

$$\int_{(k+2q)2^{-n}}^{(k+2q+1)2^{-n}} \{\tilde{\mathbb{E}}(g(t, x+z(t))|\mathcal{F}_{s_0}) - \tilde{\mathbb{E}}(g(t, y+z(t))|\mathcal{F}_{s_0})\}\mathrm{d}t$$

where we have suppressed the conditioning. We want to estimate these expectations when x, y are y_{2q-1}, y_{2q}. The idea is that y_j is an approximation to $u(2^{-n}(k+j))$ which can in turn be approximated by $Az(2^{-n}(k+j))+w$. We can use this approximation and Lemma 4 to approximate each of the two terms in the integral, and the resulting V's are the same in the two cases, so the approximations cancel. We are left with the various error terms which give a bound for $\tilde{\mathbb{E}}(\mu_{n,k+2q}(y_{2q-1}, y_{2q})|\mathcal{F}_{s_0})$ which can then be used in a martingale estimate which gives the stated result.

Note that, in contrast with Lemma 8, the bound in Lemma 9 does not involve y_0. Lemma 9 is applied for very large n and it is the dependence on n which is important, the extra δ in the exponent being crucial as it ensures that the bound is small relative to the total length of the intervals involved.

The next step is to use all the above results to obtain the following:

Lemma 10. *With probability 1, for any cube Q there are positive constants K and m_0 such that, for all integers $m > m_0$, if u is a solution of (1) with $u(t) \in Q$ for all $t \in [0, 1]$ and for some $j \in \{0, 1, \ldots, 2^m - 1\}$ and some β with $2^{-2^{3m/4}} \leq \beta \leq 2^{-2^{2m/3}}$ we have $|u(j2^{-m})| \leq \beta$, then*

$$|u((j+1)2^{-m})| \leq \beta\{1 + K2^{-m}\log(1/\beta)\}$$

The proof of this Lemma uses (8) and the above bounds and mostly follows exactly the proof of Lemma 3.7 of [2], so we merely mention the points that are different. The estimate for $\sum_{l=N+1}^{\infty}\Omega_l$ in (33) in [2] does not work here because u does not satisfy a Lipschitz condition, but Lemma 9 is an adequate substitute. The other point of difference is the technical justification of the limiting process as $l \to \infty$ involved in (7). In [2], this is handled using Lemma 3.4 of that paper. The proof of that lemma uses the fact that u is Lipschitz, so it does not work in the present context, but we can modify the proof using Lemma 9 as a substitute again, and this deals with the liming question for the dt integral. For the dW integral, it can be handled by a straightforward "rough path" argument.

Finally, Proposition 1, and hence Theorem 1, then follows from Lemma 10 as in [2].

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Stochastic Integrals and SDE Driven by Nonlinear Lévy Noise

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Abstract We develop the theory of SDE driven by nonlinear Lévy noise, aiming at applications to Markov processes. It is shown that a conditionally positive integro-differential operator (of the Lévy–Khintchine type) with variable coefficients (diffusion, drift and Lévy measure) depending Lipschitz continuously on its parameters generates a Markov semigroup, where the measures are metricized by the Wasserstein–Kantorovich metrics W_p. The analysis of SDE driven by nonlinear Lévy noise was initiated by the author in Probability Theory Related Fields, [12] 2009 (inspired partially by Carmona and Nualart, Nonlinear Stochastic Integrators, Equations and Flows [4]; see also Kolokoltsov, Nonlinear Markov Processes and Kinetic Equations [11]). Here, we suggest an alternative (seemingly more straightforward) approach based on the path-wise interpretation of these integrals as nonhomogeneous Lévy processes. Moreover, we are working with more general W_p-distances rather than with W_2.

Keywords SDE driven by Lévy noise · Nonlinear integrators · Wasserstein-Kantorovich metric · Pseudo-differential operators · Markov processes

MSC (2010): 60J25, 60H05

1 Introduction

It is well known that the generator L of a conservative (i.e., preserving constants) Feller semigroup in $\mathbf{R}^d$, with a domain containing the space $C_c^2(\mathbf{R}^d)$, has the following Lévy–Khintchine form with variable coefficients:

$$Lf(x) = \frac{1}{2}(G(x)\nabla, \nabla) f(x) + (b(x), \nabla f(x)) + \int (f(x+y) - f(x) \\ -(\nabla f(x), y)\mathbf{1}_{B_1}(y))\nu(x, \mathrm{d}y), \tag{1}$$

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where $G(x)$ is a symmetric non-negative matrix and $\nu(x,.)$ a Borel measure on $\mathbf{R}^d$ (called Lévy measure) such that

$$\int_{\mathbf{R}^n} \min(1,|y|^2)\nu(x;\mathrm{d}y) < \infty, \quad \nu(\{0\}) = 0. \tag{2}$$

The inverse question on whether a given operator of this form (or better to say its closure) actually generates a Feller semigroup is nontrivial and attracted lots of attention. One can distinguish analytic and probabilistic approaches to this problem. The existence results obtained by analytic techniques require certain non-degeneracy condition on ν, e.g., a lower bound for the symbol of pseudo-differential operator L (see, e.g., [2, 6, 9] and references therein), and for the construction of the processes via usual stochastic calculus (integration with respect to the Wiener process combined with a Poisson measure), one needs to have a family of transformations F_x of $\mathbf{R}^d$ preserving the origin, regularly depending on x and pushing a certain Lévy measure ν to the Lévy measures $\nu(x,.)$, i.e., $\nu(x,.) = \nu^{F_x}$ (see, e.g., [1]). Yet more nontrivial is the problem of constructing the so called nonlinear Markov semigroups solving the weak equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(f,\mu_t) = (L_{\mu_t} f, \mu_t), \quad \mu_t \in \mathcal{P}(\mathbf{R}^d), \quad \mu_0 = \mu, \tag{3}$$

that should hold, say, for all $f \in C_c^2(\mathbf{R}^d)$, where L_μ has form (1), but with all coefficients additionally depending on μ, i.e.,

$$\begin{aligned} L_\mu f(x) = {} & \frac{1}{2}(G(x,\mu)\nabla,\nabla)f(x) + (b(x,\mu),\nabla f(x)) \\ & + \int (f(x+y) - f(x) - (\nabla f(x), y)\mathbf{1}_{B_1}(y))\nu(x,\mu,\mathrm{d}y). \end{aligned} \tag{4}$$

Equations of type (3) play indispensable role in the theory of interacting particles (mean field approximation) and exhaust all positivity preserving evolutions on measures subject to certain mild regularity assumptions (see, e.g., [10, 17]).

Our aim is to extend stochastic calculus in the way that would allow the treatment of general Markov processes generated by (1) or their nonlinear counterpart generated by (3) as solutions to certain SDEs. We will concentrate only on the linear problem specified by (1) (nonlinear extension can be done in the same way as in [12]). But unlike [12], here, we follow a more straightforward approach based on the path-wise interpretation of these integrals as nonhomogeneous Lévy or additive processes.

Next section is devoted to a result on the duality for propagators, on the level of generality required. In Sects. 3 and 4, a general approach to the stochastic integration with respect to a nonlinear Lévy noise is developed. In particular, the well-posedness of the corresponding SDEs is reformulated in terms of the uniqueness of an invariant measure for a certain probability kernel on the Skorohod space of cadlag paths.[1] In Sect. 4, this uniqueness is obtained for the case of Levy measures $\nu(x, dy)$ depending Lipschitz continuous on the position x in the Wasserstein–Kantorovich metric W_p. Appendix describes the basic coupling of Lévy process forming the cornerstone of our W_p-estimates.

2 Analytic Preliminaries

Suppose $\{U^{t,r}\}$, $t \le r$, is a strongly continuous backward propagator (see, e.g., Chap. 5 of [14]) of bounded linear operators on a Banach space B with a common invariant domain D, which is itself a Banach space with the norm $\| \|_D \ge \| \|_B$. Let $\{A_t\}$, $t \ge 0$, be a family of bounded linear operators $D \to B$ depending strongly measurably on t (i.e., $A_t f$ is a measurable function $t \mapsto B$ for each $f \in D$). Let us say that the family $\{A_t\}$ *generates* $\{U^{t,r}\}$ on the invariant domain D if the equations

$$\frac{d}{ds} U^{t,s} f = U^{t,s} A_s f, \quad \frac{d}{ds} U^{s,r} f = -A_s U^{s,r} f, \quad t \le s \le r, \tag{5}$$

hold a.s. in s for any $f \in D$, that is there exists a set of zero-measure S in $\mathbf{R}$ such that for all $t < r$ and all $f \in D$ (5) hold for all s outside S, where the derivatives exist in the Banach topology of B. In particular, if the operators A_t depend strongly continuously on t, this implies that (5) hold for all s and $f \in D$, where for $s = t$ (resp. $s = r$) it is assumed to be only a right (resp. left) derivative.

For a Banach space B or a linear operator A, we shall denote, as usual, its Banach dual by $B^\star$ or $A^\star$, respectively.

Theorem 1. *Let $U^{t,r}$ be a strongly continuous backward propagator of bounded linear operators in a Banach space B with a common invariant domain D, which is itself a Banach space with the norm $\| \|_D \ge \| \|_B$, and let the family $\{A_t\}$ of bounded linear operators $D \to B$ generates $U^{t,r}$ on D. Then, the following holds.*

(i) The family of dual operators $V^{s,t} = (U^{t,s})^\star$, $t \le s$, forms a weakly continuous in s, t propagator of bounded linear operators in $B^\star$ such that

$$\frac{d}{ds} V^{s,t} \xi = A_s^\star V^{s,t} \xi, \quad t \le s \le r, \tag{6}$$

weakly in $D^\star$, that is

$$\frac{d}{ds} (f, V^{s,t} \xi) = (A_s f, V^{s,t} \xi), \quad t \le s \le r, \quad f \in D, \tag{7}$$

for s outside a zero-measure subset of $s \in \mathbf{R}$.

[1] recall that a probability or stochastic kernel on a Borel space is a measurable mapping from this space to the set of its probability measures

(ii) *$V^{s,t}\xi$ is the unique solution to the Cauchy problem of (7), i.e., if $\xi_t = \xi$ for a given $\xi \in B^\star$ and ξ_s, $s \in [t, r]$, is a weakly continuous family in $B^\star$ satisfying*

$$\frac{d}{ds}(f, \xi_s) = (A_s f, \xi_s), \quad t \le s \le r, \quad f \in D, \tag{8}$$

for s outside a zero-measure subset of $\mathbf{R}$, then $\xi_s = V^{s,t}\xi$ for all $s \in [t, r]$.

(iii) *$U^{s,r} f$ is the unique solution to the inverse Cauchy problem of the second equation in (5), i.e., if $f_r = f$, $f_s \in D$ for $s \in [t, r]$ and satisfies the equation*

$$\frac{d}{ds} f_s = -A_s f_s, \quad t \le s \le r, \tag{9}$$

for s outside a zero-measure subset of $\mathbf{R}$ (with the derivative existing in the norm topology of B), then $f_s = U^{s,r} f$ for all $s \in [t, r]$.

Proof. Statement (i) is a direct consequence of duality and (5). (ii) For a given r, let $g(s) = (U^{s,r} f, \xi_s)$ for a given $f \in D$. Writing

$$(U^{s+\delta,r} f, \xi_{s+\delta}) - (U^{s,r} f, \xi_s) = (U^{s+\delta,r} f - U^{s,r} f, \xi_s) + (U^{s,r} f, \xi_{s+\delta} - \xi_s) \\ +(U^{s+\delta,r} f - U^{s,r} f, \xi_{s+\delta} - \xi_s)$$

and using (5), (8) and the invariance of D, allows one to conclude that

$$\frac{d}{ds} g(s) = -(A_s U^{s,r} f, \xi_s) + (U^{s,r} f, A_s^\star \xi_s) = 0,$$

because a.s. in s

$$\left(\frac{U^{s+\delta,r} f - U^{s,r} f}{\delta}, \xi_{s+\delta} - \xi_s\right) \to 0,$$

as $\delta \to 0$ (since the family $\delta^{-1}(U^{s+\delta,r} f - U^{s,r} f)$ is relatively compact, being convergent, and ξ_s is weakly continuous). Hence, $g(r) = (f, \xi_r) = g(t) = (U^{t,r} f, \xi_t)$ showing that ξ_r is uniquely defined. (iii) Similar to (ii) it follows from the observation that

$$\frac{d}{ds}(f_s, V^{s,t}\xi) = 0.$$

Theorem 2. *Suppose we are given a sequence of propagators $\{U_n^{t,r}\}$, $n = 1, 2, \ldots$, generated by the families $\{A_t^n\}$ and a propagator $\{U^{t,r}\}$ generated by the family $\{A_t\}$. Suppose all these propagators satisfy the same conditions as $U^{t,r}$ and A_t from Theorem 1 with the same D, B. Suppose also that all $U^{t,r}$ are uniformly bounded as operators in D.*

(i) *Let the families A_t^n and A_t depend cadlag on t in the Banach topology of the space $\mathcal{L}(D, B)$ of bounded operators $D \to B$ and A_t^n converge to A_t as $n \to \infty$ in the corresponding Skorohod topology of the space $D([0, T], \mathcal{L}(D, B))$. Then, $U_n^{t,r}$ converge to $U^{t,r}$ strongly in B as $n \to \infty$.*

(ii) Let the families A_t^n and A_t depend cadlag on t in the strong topology of the space $\mathcal{L}(D, B)$, $A_t^n f$ converge to $A_t f$, as $n \to \infty$, in the Skorohod topology of the space $D([0, T], B)$ for any f and uniformly for f from any compact subset of D. Moreover, let the family $U^{t,r}$ is strongly continuous as a family of operators in D. Then again, $U_n^{t,r}$ converge to $U^{t,r}$ strongly in B.

Proof. (i) By the density argument (taking into account that $U_n^{t,r} g$ are uniformly bounded), in order to prove the strong convergence of $U_n^{t,r}$ to $U^{t,r}$, it is sufficient to prove that $U_n^{t,r} g$ converge to $U^{t,r} g$ for any $g \in D$. If $g \in D$,

$$(U_n^{t,r} - U^{t,r})g = U_n^{t,s} U^{s,r} g \mid_{s=t}^{r} = \int_t^r U_n^{t,s}(A_s^n - A_s)U^{s,r} g \, \mathrm{d}s. \tag{10}$$

In order to prove that expression (10) converges to zero as $n \to \infty$, it is sufficient to show that

$$\int_0^r \|A_s^n - A_s\|_{D \to B} \, \mathrm{d}s \to 0$$

as $n \to \infty$. Since A_t^n converge to A_t in the Skorohod topology, there exists a sequence λ_n of the monotone bijections of $[0, r]$ such that

$$\sup_s |\lambda_n(s) - s| \to 0, \quad \sup_s \|A_s^n - A_{\lambda_n(s)}\|_{D \to B} \to 0,$$

as $n \to \infty$. Hence, writing

$$\int_0^r \|A_s^n - A_s\|_{D \to B} \, \mathrm{d}s = \int_0^r \|A_s^n - A_{\lambda_n(s)}\|_{D \to B} \, \mathrm{d}s + \int_0^r \|A_s - A_{\lambda_n(s)}\|_{D \to B} \, \mathrm{d}s,$$

we see that both terms tend to zero (the second one by the dominated convergence, as the limiting function vanishes a.s.).

(ii) This is proved similar to (i).

3 Additive Processes as Stochastic Integrals Driven by Nonhomogeneous Noise

Here, we are interested in the processes generated by a time dependent family of Lévy–Khintchine operators

$$\begin{aligned} L_t f(x) = & \frac{1}{2}(G_t \nabla, \nabla) f(x) + (b_t, \nabla f)(x) \\ & + \int [f(x+y) - f(x) - (y, \nabla f(x)) \mathbf{1}_{B_1}(y)] \nu_t(\mathrm{d}y), \end{aligned} \tag{11}$$

where for any t, G_t is a non-negative symmetric $d \times d$-matrix, $b_t \in \mathbf{R}^d$ and ν_t is a Lévy measure. The set of Lévy measures is equipped with the weak topology, where the continuous dependence of the family ν_t on t means that $\int f(y)\nu_t(\mathrm{d}y)$ depends continuously on t for any continuous f on $\mathbf{R}^d$ with $|f(y)| \le c \min(|y|^2, 1)$.

We shall denote by $C_\infty(\mathbf{R}^d)$ the space of continuous functions on $\mathbf{R}^d$ vanishing at infinity, and by $C^k_\infty(\mathbf{R}^d)$ its subspace consisting of functions, whose derivatives up to and including order k also belong to $C_\infty(\mathbf{R}^d)$.

Proposition 3.1. *For a given family $\{L_t\}$ of form (11) with bounded coefficients G_t, b_t, ν_t, i.e.,*

$$\sup_t(\|G_t\| + \|b_t\| + \int (1 \wedge y^2)\nu_t(\mathrm{d}y)) < \infty,$$

that depend continuously on t a.s., i.e., outside a fixed zero-measure subset $S \subset \mathbf{R}$, there exists a unique family $\{\Phi^{s,t}\}$ of positive linear contractions in $C_\infty(\mathbf{R}^d)$ depending strongly continuously on $s \le t$ such that for any $f \in C^2_\infty(\mathbf{R}^d)$ the functions $f_s = \Phi^{s,t} f$ belong to $C^2_\infty(\mathbf{R}^d)$ and solve a.s. (i.e., for s outside a zero-measure set) the inverse-time Cauchy problem

$$\dot{f}_s = -L_s f_s, \quad s \le t, \quad f_t = f \tag{12}$$

(derivative is taken in the Banach topology of $C(\mathbf{R}^d)$).

Proof. Let f belong to the Schwartz space $S(\mathbf{R}^d)$. Then, its Fourier transform

$$g(p) = (Ff)(p) = \int_{\mathbf{R}^d} \mathrm{e}^{-\mathrm{i}px} f(x)\,\mathrm{d}x$$

also belongs to $S(\mathbf{R}^d)$. As the Fourier transform of (12) has the form

$$\dot{g}_s(p) = -\left[-\frac{1}{2}(G_s p, p) + \mathrm{i}(b_s, p) + \int (\mathrm{e}^{\mathrm{i}py} - 1 - \mathrm{i}py\mathbf{1}_{B_1})\nu_s(\mathrm{d}y)\right] g_s(p),$$

it has the obvious unique solution

$$g_s(p) = \exp\left\{\int_s^t \left[-\frac{1}{2}(G_\tau p, p) + i(b_\tau, p) + \int (\mathrm{e}^{\mathrm{i}py} - 1 - \mathrm{i}py\mathbf{1}_{B_1})\nu_\tau(\mathrm{d}y)\right]\mathrm{d}\tau\right\} g(p) \tag{13}$$

(the integral is defined both in Lebesgue and Riemann sense, as the discontinuity set of the integrand has measure zero), which belongs to $L^1(\mathbf{R}^d)$, so that $f_s = F^{-1}g_s = \Phi^{s,t} f$ belongs to $C_\infty(\mathbf{R}^d)$. As for any fixed s, t the operator $\Phi^{s,t}$ coincides with an operator from the semigroup of a certain homogeneous Lévy process, each $\Phi^{s,t}$ is a positivity-preserving contraction in $C_\infty(\mathbf{R}^d)$ preserving the spaces $(C_\infty \cap C^2)(\mathbf{R}^d)$ and $C^2_\infty(\mathbf{R}^d)$. Strong continuity is then obtained first for $f \in (C^2 \cap C_\infty)(\mathbf{R}^d)$ and then for general f by the density argument. Finally, the uniqueness follows from Theorem 1.

Let us define a time nonhomogeneous Lévy processes generated by the family $\{L_t\}$ as a time nonhomogeneous cadlag Markov process X_t such that

$$\mathbf{E}(f(X_t)|X_s = x) = (\Phi^{s,t} f)(x), \quad f \in C(\mathbf{R}^d),$$

where $\Phi^{s,t}$ is the propagator of positive linear contractions in $C_\infty(\mathbf{R}^d)$ from Proposition 3.1 (notice that a Markov process is defined uniquely up to a modification by its transition probabilities). The processes of this kind are sometimes referred to as additive processes (they are stochastically continuous and have independent increments), see, e.g., [16], pp. 51–68. We use the term "nonhomogeneous Lévy" stressing their translation invariance and the analytic properties of their propagators which represent the most straightforward time-nonhomogeneous extensions of the semigroups of the Lévy processes.

We like to interpret the nonhomogeneous Lévy processes as weak stochastic integrals. For this purpose, it will be notational more convenient to work with the generator families depending on time via a multidimensional parameter. Namely, let L_η be a family of the operators of form (11) with coefficients G_η, b_η, ν_η depending continuously on a parameter $\eta \in \mathbf{R}^n$ (ν_η is continuous as usual in the above specified weak sense). Let ξ_t be a curve in $\mathbf{R}^n$ with not more than countably many discontinuities and with left and right limits existing everywhere. Then, the family of operators L_{ξ_t} satisfies the assumptions of Proposition 3.1. Clearly, the resulting propagator $\{\Phi^{s,t}\}$ does not depend on the values of ξ_t at the points of discontinuity.

To go ahead, we shall need the following well known *randomization lemma* (see, e.g., Lemma 3.22 in [8]):

Lemma 1. *Let $\mu(x, \mathrm{d}z)$ be a probability kernel from a measurable space X to a Borel space Z. Then, there exists a measurable function $f : X \times [0,1] \to Z$ such that if θ is uniformly distributed on $[0,1]$, then $f(X,\theta)$ has distribution $\mu(x,.)$ for every $x \in X$.*

Applying Lemma 1 to the distributions of the family of the Lévy processes $Y_t(\eta)$ (corresponding to the generators L_η), we can define them on a single probability space (actually on the standard Lebesgue space) in such a way that they depend measurably on the parameter η.

Let ξ_s, α_s be piecewise constant left continuous functions (deterministic, to begin with) with values in $\mathbf{R}^n$ and $d \times d$-matrices, respectively, that is

$$\xi_s = \sum_{j=0}^{n} \xi^j \mathbf{1}_{(t_j,t_{j+1}]}(s), \quad \alpha_s = \sum_{j=0}^{n} \alpha^j \mathbf{1}_{(t_j,t_{j+1}]}(s), \tag{14}$$

where $0 = t_0 < t_1 < \dots . < t_{n+1}$. Then, it is natural to define the stochastic integral with respect to the nonlinear Lévy noise $Y_s(\xi_s)$ by the formula

$$\int_0^t \alpha_s \mathrm{d}Y_s(\xi_s) = \sum_{j=0}^{n} \alpha^j Y^j_{t\wedge t_{j+1}-t_j}(\xi^j)\mathbf{1}_{t_j<t}, \tag{15}$$

where $Y_t^j(\eta)$ are independent copies of the families of $Y_t(\eta)$ defined above via the randomization lemma. It is clear that so defined process $\int_0^t \alpha_s \mathrm{d}Y_s(\xi_s)$ is a nonhomogeneous Lévy process constructed by Proposition 3.1 from the generator family

$$L_t^{\alpha,\xi} f(x) = \frac{1}{2}((\alpha_t G_{\xi_t} \alpha_t')\nabla, \nabla) f(x) + (\alpha_t b_{\xi_t}, \nabla f)(x) + \int [f(x + \alpha_t y) - f(x) - (\alpha_t y, \nabla f(x)) \mathbf{1}_{B_1}(y)] \nu_{\xi_t}(\mathrm{d}y), \quad (16)$$

which coincides with L_{ξ_t} for $\alpha_t = 1$. Next, if ξ_t and α_t are arbitrary cadlag function, let us define its natural piecewise constant approximation as

$$\xi_t^\tau = \sum_{\tau j < t}^{n} \xi_{\tau j} \mathbf{1}_{(\tau j, \tau(j+1)]}, \quad \alpha_t^\tau = \sum_{\tau j < t}^{n} \alpha_{\tau j} \mathbf{1}_{(\tau j, \tau(j+1)]},$$

As usual, the integral $\int_0^t \alpha_s \mathrm{d}Y_s(\xi_s)$ should be defined as a limit (if it exists in some sense) of the integrals over its approximations $\int_0^t \alpha_s^\tau \mathrm{d}Y_s(\xi_s^\tau)$.

Theorem 3. *The distribution of the process of integrals $x + \int_0^t \alpha_s \mathrm{d}Y_s(\xi_s)$ is well defined as the weak limit, as $\tau \to 0$, of the distributions on the Skorohod space $D([0,T], \mathbf{R}^d)$, of the approximating simple integrals $x + \int_0^t \alpha_s^\tau \mathrm{d}Y_s(\xi_s^\tau)$, and is the distribution of the Lévy process started at x and generated by the family (16). This limit also holds in the sense of the convergence of the propagators of the corresponding nonhomogeneous Lévy processes.*

Proof. The right continuous versions ξ_{s+}^τ converge to ξ_t in the sense of the Skorohod topology. Hence, by Theorem 2, the corresponding processes $\int_0^t \mathrm{d}Y_s(\xi_s^\tau)$ converge to the nonhomogeneous Lévy process generated by the family (16) in the sense of the convergence of propagators. By the standard results of stochastic analysis, this implies the weak convergence as distributions on the Skorohod space.

In particular, we have constructed the probability kernel on the space $D([0,T], \mathbf{R}^d)$ of cadlag paths that takes a curve ξ_t to the distribution of the integral $x + \int_0^t \mathrm{d}Y_s(\xi_s)$. The main point is that the invariant measure for this kernel defines a weak solution to the stochastic equation $\xi_t = \xi_0 + \int_0^t \mathrm{d}Y_s(\xi_s)$, and its uniqueness is closely linked with the Markovianity of the corresponding process, which we shall discuss in the next section.

Now, let us describe a couple of more specific situations. We shall use the following elementary inequalities.

Proposition 3.2. *(i) For any $p \in [1,2]$ and $x \geq -1$ one has*

$$0 \leq (1+x)^p - 1 - px \leq (p-1)x^2, \quad (17)$$

(ii) For any $p \in [1,2]$, $d \in N$ and $A, B \in \mathbf{R}^d$

$$0 \le |A+B|^p - |A|^p - p(A,B)|A|^{p-2} \le c_p|B|^p, \tag{18}$$

where

$$c_p = \max_{x\in[0,1/2]}[(1-x)^p - x^p + px^{p-1}]. \tag{19}$$

The next statement describes the jump type processes when L_p-estimates are available.

Proposition 3.3. *Suppose $Y_s(\eta)$ is a family of Lévy processes in $\mathbf{R}^d$ with càdlàg paths, depending on a parameter $\eta \in \mathbf{R}^n$ and specified by their generators*

$$L_\eta f(x) = \int [f(x+y) - f(x) - (y,\nabla)f(x)]\nu_\eta(\mathrm{d}y), \tag{20}$$

where

$$\nu_\eta(\{0\}) = 0, \quad \sup_\eta \int |y|^p \nu_\eta(\mathrm{d}y) = \kappa < \infty \tag{21}$$

with a certain $p \in [1,2]$. Then, the process $\int_0^t \alpha_s \mathrm{d}Y_s(\xi_s)$ is a martingale and

$$\mathbf{E}\left|\int_0^t \alpha_s \mathrm{d}Y_s(\xi_s)\right|^p \le 2c_p\kappa_1 \int |\alpha_s|^p\,\mathrm{d}s. \tag{22}$$

Proof. In view of Theorem 3, it is enough to prove the statement for the approximations $\int_0^t \alpha_s^\tau \mathrm{d}Y_s(\xi_s^\tau)$. Then, the martingale property follows from the property of Lévy processes. Finally, by (19), one has

$$\mathbf{E}\left|\int_0^t \alpha_s^\tau \mathrm{d}Y_s(\xi_s^\tau)\right|^p \le c_p \sum_{j=0}^{[t/\tau]} \|(\alpha_{j\tau}\|^p \mathbf{E}|Y^j_{t\wedge\tau(j+1)-\tau j}(\xi_{\tau j})|^p,$$

implying (22). Alternatively, one can get this estimate from the martingale problem associated with the propagator of the corresponding non-homogeneous Lévy process, but the proof given extends straightforwardly to the case of random ξ.

As another situation of interest let consider the case of $Y_s(\eta)$ that are compound Poisson processes, i.e., they are generated by the family

$$L_\eta f(x) = \int [f(x+y) - f(x)]\nu_\eta(\mathrm{d}y) \tag{23}$$

with uniformly bounded measures ν_η.

Proposition 3.4. *For any cadlag curve ξ_t, the process $x + \int_0^t \mathrm{d}Y_s(\xi_s)$ has the following probabilistic description. Starting from the initial point x, it waits there a (non-homogeneous exponential) random time τ_1 with $\mathbf{P}(\tau_1 > t) = \exp\{-\int_0^t \|\nu_{\xi_s}\|\,\mathrm{d}s\}$ and then it jumps to a point y_1 distributed according to the law $\nu_{\xi_{\tau_1}}/\|\nu_{\xi_{\tau_1}}\|$. Then,* *it waits a random time* τ_2 *with* $\mathbf{P}(\tau_2 > t) = \exp\{-\int_{\tau_1}^{t+\tau_1} \|\nu_{\xi_s}\| \mathrm{d}s\}$ *and jumps to a point* $y_1 + y_2$ *with* y_2 *distributed according to the law* $\nu_{\xi_{\tau_2}}/\|\nu_{\xi_{\tau_2}}\|$*, etc.*

Proof. It follows from Theorem 3 and the basic series expansion of the propagator of a process generated by family (23), see, e.g., [8] for time-homogeneous case and [10] for time non-homogeneous case.

In particular, to link with a usual stochastic integration over a Poisson measure, let us note that if $N(\mathrm{d}s\mathrm{d}x)$ is the Poisson measure of a Levy process Z_t in $\mathbf{R}^d$ and $g(s,x)$ is a continuous bounded function on $\mathbf{R}_+ \times \mathbf{R}_d$, then

$$\int_0^t \int_A g(s,x) N(\mathrm{d}s\mathrm{d}x)$$

is the process of the type described in Proposition 3.4, specified by the family

$$L_t f(x) = \int [f(x + g(t,y)) - f(x)]\nu(\mathrm{d}y).$$

Remark 1. Similarly, one can define the integral $\int_0^t g_s(\mathrm{d}Y_s(\xi_s))$ for a cadlag family of nonlinear mappings $g_s : \mathbf{R}^n \to \mathbf{R}^d$ of the class $C^2(\mathbf{R}^n)$, as the limit of the approximating sums

$$\sum_{j=0}^n g_{j\tau}(Y^j_{t\wedge(j+1)\tau - j\tau}(\xi_{j\tau}))\mathbf{1}_{j\tau<t},$$

which converge to the non-homogeneous Lévy process generated by the family of the operators

$$\begin{aligned} L_t^{g,\xi} f(x) = & \frac{1}{2}\left(\left[\frac{\partial g_t}{\partial y}(0) G_{\xi_t} \left(\frac{\partial g_t}{\partial y}\right)'(0)\right]\nabla,\nabla\right) f(x) \\ & + \left(\frac{1}{2} G_{\xi_t} \frac{\partial^2 g_t}{\partial y^2}(0) + \frac{\partial g_t}{\partial y}(0) b_{\xi_t}, \nabla f\right)(x) \\ & + \int \left[f(x + g_t(y)) - f(x) - \left(\frac{\partial g_t}{\partial y}(0) y, \nabla f(x)\right) \mathbf{1}_{B_1}(y)\right] \nu_{\xi_t}(\mathrm{d}y). \end{aligned} \tag{24}$$

4 SDE Driven by Nonlinear Lévy Noise

In order to solve the equation

$$X_t = x + \int_0^t \mathrm{d}Y_s(X_s), \tag{25}$$

with x being a random variable with a given law μ, it is convenient to use Euler type approximations. For a $\tau > 0$, let the process $X_t^{\mu,\tau}$ be defined by the recursive equation

$$X_t^{\mu,\tau} = X_{l\tau}^{\mu,\tau} + Y_{t-l\tau}^l(X_{l\tau}^{\mu,\tau}), \quad \mathcal{L}(X_0^{\mu,\tau}) = \mu, \tag{26}$$

for $l\tau < t \leq (l+1)\tau$, where $\mathcal{L}(X)$ means the law of X. Clearly, these approximation processes are càdlàg. According to the above definition of the stochastic integral, this process satisfies the equation

$$X_t^{\mu,\tau} = X_0 + \int_0^t \mathrm{d}Y_s(X_{[s/\tau]}^{\mu,\tau}), \quad \mathcal{L}(X_0) = \mu. \tag{27}$$

By conditioning, one deduces that $X_t^{\mu,\tau}$ solves the following martingale problem. For any $f \in C_c^2(\mathbf{R}^d)$, the process

$$M_\tau(t) = f(X_t^{\mu,\tau}) - f(X_0) - \int_0^t L[X_{[s/\tau]\tau}^{\mu,\tau}]f(X_s^{\mu,\tau})\,\mathrm{d}s, \quad \mu = \mathcal{L}(X_0), \tag{28}$$

is a martingale, where we transferred the lower subscript to square brackets, i.e., $L[X] = L_X$ in previous notations. Due to the basic convergence criteria for martingale problem solutions (see e.g [5, 7, 8]), it follows that the family of processes $X_t^{\mu,\tau}$, $\tau > 0$, is tight (as was noted by many authors, see e.g [17] and [3]), and hence relatively compact and moreover, any limiting process X_t^μ solves the martingale problem for the family L_η: for any $f \in C_c^2(\mathbf{R}^d)$, the process

$$M_\tau(t) = f(X_t^\mu) - f(x) - \int_0^t L[X_s^\mu]f(X_s^\mu)\,\mathrm{d}s, \quad \mu = \mathcal{L}(x), \tag{29}$$

is a martingale. Moreover, using Skorohod's theorem, one can choose a probability space, where all theses approximations are simultaneously defined and converge a.s., and consequently, by Theorem 3, X_t^x also solves the stochastic equation (25). In particular, this equation has a solution.

Theorem 4. *Suppose for any x the probability kernel on the subset of $D([0,t],\mathbf{R}^d)$ of paths starting at x given by the integral constructed in Sect. 3 can have at most one invariant measure, i.e., (25) has at most one solution. Then, the approximations (26) converge weakly to the unique solution X_t^x of (25), which is a Markov process solving the martingale problem (29).*

Proof. Convergence follows from the uniqueness of the limit. Passing to the limit $\tau \to 0$ in the Markov property for the approximations

$$\mathbf{E}(f(X_t^{\mu,\tau}) \mid \sigma(X_u^{\mu,\tau})|_{u \leq j\tau}) = \mathbf{E}(f(X_t^{\mu,\tau}) \mid X_{j\tau}^{\mu,\tau})$$

yields the Markov property for the limit X_t^μ.

5 W_p-Estimates

In order to apply these results, we should be able to compare the Lévy measures. To this end, we introduce an extension of the Wasserstein–Kantorovich distance to unbounded measures. Namely, let $\mathcal{M}_p(\mathbf{R}^d)$ denote the class of Borel measures μ on $\mathbf{R}^d \backslash \{0\}$ (not necessarily finite) with finite pth moment (i.e., such that $\int |y|^p \mu(\mathrm{d}y) < \infty$). For a pair of measures ν_1, ν_2 in $\mathcal{M}_p(\mathbf{R}^d)$, we define the distance $W_p(\nu_1, \nu_2)$ by

$$W_p(\nu_1, \nu_2) = \left(\inf_\nu \int |y_1 - y_2|^p \nu(\mathrm{d}y_1 \mathrm{d}y_2) \right)^{1/p}, \tag{30}$$

where inf is taken over all $\nu \in \mathcal{M}_p(\mathbf{R}^{2d})$ such that condition

$$\int_{\mathbf{R}^{2d}} (\phi_1(x) + \phi_2(y)) \nu(\mathrm{d}x \mathrm{d}y) = (\phi_1, \nu_1) + (\phi_2, \nu_2)$$

holds for all ϕ_1, ϕ_2 satisfying $\phi_i(.)/|.|^p \in C(\mathbf{R}^d)$. It is easy to see that for finite measures this definition coincides with the usual one.

Moreover, by the same argument as for finite measures (see [15, 18]), we can show that whenever the distance $W_p(\nu_1, \nu_2)$ is finite, the infimum in (30) is achieved, i.e., there exists a measure $\nu \in \mathcal{M}_p(\mathbf{R}^{2d})$ such that

$$W_p(\mu_1, \mu_2) = \left(\int |y_1 - y_2|^p \nu(\mathrm{d}y_1 \mathrm{d}y_2) \right)^{1/p}. \tag{31}$$

To compare the distributions on Skorohod spaces, we shall use the corresponding Wasserstein–Kantorovich distances. These distances depend on the choice of distances between individual paths. The most natural choice of these distances is uniform leading to the distance on the distributions:

$$W_{p,T}(X^1, X^2) = \inf \left(\mathbf{E} \sup_{t \le T} |X_t^1 - X_t^2|^p \right)^{1/p}, \tag{32}$$

where inf is taken over all couplings of the distributions of the random paths X_1, X_2.

Let us now approach the *stochastic differential equation (SDE)* driven by *nonlinear Lévy noise* of the form

$$X_t = x + \int_0^t \mathrm{d}Y_s(g(X_{s-})) + \int_0^t b(X_{s-})\, \mathrm{d}s + \int_0^t G(X_{s-}) \mathrm{d}W_s, \tag{33}$$

where W_s is the standard Wiener process in $\mathbf{R}^d$ and $Y_t(\eta)$ is a family of pure-jump Lévy processes from Proposition 3.3.

Theorem 5. *Let (21), (34) hold for the family (23) with a* $p \in (1, 2]$. *Let* b, G *be bounded functions from* $\mathbf{R}^d$ *to* $\mathbf{R}^d$ *and to* $d \times d$*-matrices, respectively, and* b, G, ν *be Lipschitz continuous with a common Lipschitz constant* κ_2*, where* $\nu(x, .)$ *are equipped with* W_p*-metric, i.e.,*

$$W_p(\nu(x_1, .), \nu(x_2, .)) \leq \kappa_2 \|x_1 - x_2\|. \tag{34}$$

Finally, let x *be a random variable independent of all* $Y_s(z)$*. Then, the solution to (33) exists in the sense of distribution (i.e., the equation means the coincidence of the distributions) and is unique. Moreover, the solutions to this equation specify a Feller process in* $\mathbf{R}^d$*, whose generator contains the set* $C_c^2(\mathbf{R}^d)$*, where it is given by the formula*

$$\begin{aligned} Lf(x) = &\frac{1}{2}(G(x)\nabla, \nabla)f(x) + (b(x), \nabla f)(x) \\ &+ \int [f(x+y) - f(x) - (y, \nabla f(x))]\nu(x, \mathrm{d}y). \end{aligned} \tag{35}$$

Proof. It is based on the contraction principle in the complete metric space $M_p(t)$ of the distributions on the Skorohod space of càdlàg paths $\xi \in D([0, t], \mathbf{R}^d)$ with a finite pth moment $W_{p,t}(\xi, 0) < \infty$ and with the metric $W_{p,t}$. For any $\xi \in M_p(t)$, let $\Phi(\xi) = \Phi^1(\xi) + \Phi^2(\xi)$ with

$$\Phi^1(\xi)_t = x + \int_0^t \mathrm{d}Y_s(\xi_{s-}), \quad \Phi^2(\xi)_t = \int_0^t b(\xi_{s-})\,\mathrm{d}s + \int_0^t G(X_{s-})\mathrm{d}W_s.$$

One has

$$W_{p,t}^p(\Phi(\xi^1), \Phi(\xi^2)) = \inf_{\xi^1, \xi^2} W_{p,t,cond}^p(\Phi(\xi^1), \Phi(\xi^2)),$$

where the first infimum is over all couplings of ξ^1, ξ^2 and $W_{p,t,cond}$ denotes the distance (32) conditioned on the given values of ξ^1, ξ^2. Hence,

$$W_{p,t}^p(\Phi(\xi^1), \Phi(\xi^2)) \leq 2W_{p,t}^p(\Phi^1(\xi^1), \Phi^1(\xi^2)) + 2W_{p,t}^p(\Phi^2(\xi^1), \Phi^2(\xi^2)).$$

By Proposition 3.2, applied recursively to the increments of the discrete approximations of our stochastic integrals, one obtains

$$\sup_{s \leq t} \mathbf{E}|\Phi^1(\xi_s^1) - \Phi^1(\xi_s^2)|^p \leq 2t\kappa_2 c_p \sup_{s \leq t} |\xi_s^1 - \xi_s^2|^p$$

for the coupling of Φ^1, Φ^2 given by Proposition 5.1. Using now Doob's maximum inequality for martingales yields

$$W^p_{p,t,cond}(\Phi^1(\xi^1),\Phi^1(\xi^2)) \le 4\kappa_2 t c_p \sup_{s\le t}|\xi^1_s-\xi^2_s|^p,$$

implying

$$W^p_{p,t}(\Phi^1(\xi^1),\Phi^1(\xi^2)) \le 4\kappa_2 t c_p W^p_{p,t}(\xi^1,\xi^2).$$

On the other hand, by the standard properties of the Brownian motion,

$$\mathbf{E}|\Phi^2(\xi^1)-\Phi^2(\xi^2)|^2 \le t\kappa_2 \sup_{s\le t}|\xi^1_s-\xi^2_s|^2,$$

implying

$$W^2_{2,t}(\Phi^2(\xi^1),\Phi^2(\xi^2)) \le 4\kappa_2 t W^2_{2,t}(\xi^1,\xi^2)$$

and consequently

$$W^p_{p,t}(\Phi^2(\xi^1),\Phi^2(\xi^2)) \le (4\kappa_2 t)^{p/2} W^p_{p,t}(\xi^1,\xi^2).$$

Thus finally,

$$W^p_{p,t}(\Phi(\xi^1),\Phi(\xi^2)) \le c(t,\kappa_2) W^p_{p,t}(\xi^1,\xi^2).$$

Hence, the mapping $\xi\mapsto\Phi(\xi)$ is a contraction in $M_p(t)$ for small enough t. This implies the existence and uniqueness of a fixed point and hence of the solution to (33) for this t. For large t, this construction is extended by iterations.

Consequently, the main condition of Theorem 4 is satisfied. The Feller property is easy to check (see [12]) and the form of the generator reads out from the martingale problem formulation discussed in the previous section.

Theorem 5 reduces the problem of constructing a Feller processes from a given pre-generator to a Monge–Kantorovich mass transportation (or optimal coupling) problem, where essential progress was made recently, see books [15] and [18]. The usual approach of stochastic analysis works in the case when all measures $\nu(x,\mathrm{d}y)$ can be expressed as images of regular enough family of mappings F_x of a certain given Lévy measure ν, see, e.g., [1]. To find such a family, an optimal solution (or its approximation) to the Monge problem is required. Our extension allows one to use instead the solutions of its more easy to handle extension called the Kantorovich problem (the introduction of which in the last century signified a major breakthrough in dealing with mass transportation problems). It is well known (and easy to see on examples with Dirac's measures) that the optimal coupling of probability measures (Kantorovich problem) can not always be realized via a mass transportation (a solution to the Monge problem), thus leading to the examples when the construction of the process via standard stochastic calculus would not work. Let us stress also that our coupling condition is quite different from non-degeneracy or monotonicity assumptions usually required by the analytic approaches to the construction of Markov semigroups, see [6, 9, 10, 13].

Concrete examples (including stable-like processes) are discussed in [12] together with the extensions to time nonhomogeneous and nonlinear (in distributions) evolutions, as well as some additional regularity criteria. Let us note only that the main assumption on ν is satisfied if one can decompose the Lévy measures $\nu(x;.)$ in the countable sums $\nu(x;.) = \sum_{n=1}^{\infty} \nu_n(x;.)$ of probability measures so that $W_p(\nu_i(x;.), \nu_i(z;.)) \leq a_i|x-z|$ and the series $\sum a_i^p$ converges.

Appendix

Proposition 5.1. *Let Y_s^i, $i = 1, 2$, be two Lévy processes in $\mathbf{R}^d$ specified by their generators*

$$L_i f(x) = \int (f(x+y) - f(x) - (\nabla f(x), y))\nu_i(\mathrm{d}y) \tag{36}$$

with $\nu_i \in \mathcal{M}_p(\mathbf{R}^d)$, $p \in [1, 2]$. Let $\nu \in \mathcal{M}_p(\mathbf{R}^{2d})$ be a coupling of ν_1, ν_2, i.e.,

$$\int\int (\phi_1(y_1) + \phi_2(y_2))\nu(\mathrm{d}y_1\mathrm{d}y_2) = (\phi_1, \nu_1) + (\phi_2, \nu_2) \tag{37}$$

holds for all ϕ_1, ϕ_2 satisfying $\phi_i(.)/|.|^p \in C(\mathbf{R}^d)$. Then, the operator

$$\begin{aligned} Lf(x_1, x_2) = \int [&f(x_1 + y_1, x_2 + y_2) - f(x_1, x_2) - ((y_1, \nabla_1) \\ &+(y_2, \nabla_2))f(x_1, x_2)]\nu(\mathrm{d}y_1\mathrm{d}y_2) \end{aligned} \tag{38}$$

(where ∇_i means the gradient with respect to x_i) specifies a Lévy process Y_s in $\mathbf{R}^{2d}$ with the characteristic exponent

$$\eta_{x_1,x_2}(p_1, p_2) = \int (\mathrm{e}^{\mathrm{i}y_1p_1 + \mathrm{i}y_2p_2} - 1 - i(y_1p_1 + y_2p_2))\nu(\mathrm{d}y_1\mathrm{d}y_2),$$

that is a coupling of Y_s^1, Y_s^2 in the sense that the components of Y_s have the distribution of Y_s^1 and Y_s^2, respectively. Moreover, if $f(x_1, x_2) = h(x_1 - x_2)$ with a function $h \in C^2(\mathbf{R}^d)$, then

$$Lf(x_1, x_2) = \int [h(x_1 - x_2 + y_1 - y_2) - h(x_1 - x_2) - (y_1 - y_2, \nabla h(x_1 - x_2)]\nu(\mathrm{d}y_1\mathrm{d}y_2). \tag{39}$$

And finally

$$\mathbf{E}|\xi + Y_t^1 - Y_t^2|^p \leq |\xi|^p + tc_p \int\int |y_1 - y_2|^p \nu(\mathrm{d}y_1\mathrm{d}y_2). \tag{40}$$

Proof. It is straightforward to see from the definition of coupling that if f depends only on x_1 (resp. x_2), then the operator (38) coincides with L_1 (resp. L_2). Similarly, one sees that the characteristic exponent of Y_s coincides with the characteristic exponent of Y^1 (resp. Y^2) for $p_2 = 0$ (resp. $p_1 = 0$). Formula (39) is a consequence of (38). To get (40), one uses Dynkin's formula for the function $f(x_1, x_2) = |\xi + x_1 - x_2|^p$ observing that by (39) and (19) one has

$$|Lf(x_1, x_2)| \leq c_p \int\int |y_1 - y_2|^p \nu(dy_1 dy_2).$$

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Discrete Algorithms for Multivariate Financial Calculus

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Abstract Quantitative financial calculus is dominated by calculations of integrals related to various moments of probability distributions used for modelling. Here, we develop a general technique that facilitates the numerical calculations of options, prices for the difficult case of multi-assets, for the majority of European payoff contracts. The algorithms proposed here rely on known weak convergence results, hence making use of the gaussian probability kernel even when modelling with non-gaussian distributions. In addition, this technique can be employed for calculating greek parameters. We prove that the weak convergence characterizing condition can still be applied under some mild assumption on the payoff function of financial options.

Keywords Approximation algorithms · Greeks · Multi-asset options · Weak convergence

MSC (2010): 65D32, 62P05, 91G60

1 Introduction

This paper develops a general method for generating deterministic algorithms useful in financial calculus. The range of applications cover the valuation of multi-asset European contingent claims including Asian options and spread options, and risk management calculations such as the sensitivity greeks parameters. However, the same algorithm can be applied to any other areas where the calculation of high-dimensional integrals is difficult. The main idea on which this paper is based is to use general results related to weak convergence of probability measures and derive numerical approximation schemes in closed form. The underpinning theory behind our approximations is probabilistic in nature but the formulae are deterministic, thus

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avoiding common pitfalls related to Monte Carlo simulations techniques. In addition, the method presented here is very general and it can be adapted to a wide range of modeling situations.

The payoffs of contingent claims generally considered in practice can be seen as functions Π of finitely many positive random variables $\{S_k\}_{1\leq k\leq d}$, representing asset prices. Under no-arbitrage principle, risk-neutral valuation of contingent claims is centred upon calculating expectations of the form $\mathcal{E}[\Pi(S_1,\ldots,S_d)]$ under an equivalent martingale probability measure $\mathbf{Q}$. One of the strengths of our methodology is that it can be applied to any general finite payoff Π usually encountered in financial markets. While the results presented here are drawn for asset processes following generically a geometric Brownian motion, the techniques outlined in this paper offer a numerical mathematical solution for general models and underlying processes.

The probability theory concerning the convergence of probability measures requires bounded and almost everywhere continuous test functions. However, in finance, there are many contingent claims with unbounded payoffs. In Sect. 4, we show that, under some suitable mild assumptions on the payoff functions, we can still apply the condition defining weak convergence for the chosen probability measure.

There is a price to pay for the high level of generality offered by this new technique and that is the computational effort required for implementation. However, since no simulation is required, parts of the calculations can be reused for pricing different options contingent on the same securities simultaneously. Moreover, for specific payoffs involving the maximum function, one can improve the calculations by imposing natural boundaries on the underlying asset.

Without reduction of generality, we therefore focus on the risk-neutral valuation of the contingent claims on one or several asset under nonstochastic interest rates. This is the same framework outlined in [10] and [34]. Moreover, while the focus here in terms of applications is on European style options, our methods can be useful also in the context of pricing American style derivatives since their calculation can be reduced to European options, see [21] for further details.

As a preview of the developments in this paper, consider an Asian call option on the arithmetic average over the d fixings dates $t_1,\ldots,t_d$. If $\{S_t\}_{t\geq 0}$, the underlying asset follows a geometric Brownian motion, then the average is given by the expression

$$A=\frac{S_0}{d}\sum_{i=1}^{d}\exp((r-\sigma^2/2)t_i+\sigma W_{t_i}) \tag{1}$$

where $\{W_t\}_{t\geq 0}$ is the associated Wiener process. The no-arbitrage risk-neutral price of the Asian call option with fixed strike price K is given by the integral

$$c=\mathrm{e}^{-rT}\int_0^\infty (a-K)^+\rho_A(a)\mathrm{d}a \tag{2}$$

where K is the fixed strike price and $\rho_A(\cdot)$ denotes the risk-neutral density of the arithmetic average A considered as a random variable. Since ρ_A is not known in exact form, we need to consider then the multivariate distribution of all stock prices contributing to the Asian underlying. Because the randomness comes only from the Wiener process $\{W_t\}_{t\geq 0}$ considered at fixing dates $t_1, \ldots, t_d$, the calculations are determined by ρ_Y, the joint probability density of the gaussian multivariate vector $Y = (Y_1, \ldots, Y_d)^\top$ with mean zero and covariance matrix

$$\Sigma = [\min(t_i, t_j)]_{i,j\in\{1,\ldots,d\}}. \tag{3}$$

For Asian option pricing, it is important then to calculate the integral of the type

$$\int_{\mathbb{R}^d} g(y)\rho_Y(y; 0_d, \Sigma)\mathrm{d}y \tag{4}$$

where the function $g : \mathbb{R}^d \to \mathbb{R}$ is defined through

$$g(Y_1, \ldots, Y_d) = \max\left[\frac{S_0}{d}\sum_{i=1}^{d}\exp\left((r-\sigma^2/2)t_i + \sigma\sqrt{t_i}Y_i\right) - K, 0\right].$$

The multidimensional integral in (4) appears for other exotic options and therefore methods for calculating this type of integral for general payoffs Π are very valuable in financial markets.

2 Short Review of Numerical Methods Used in Finance

Given the complexity of the models and payoffs involved, the vast majority of financial derivatives are priced using numerical methods. An excellent review of Monte Carlo and deterministic methods that can be used to approximate integrals is contained in [13]. In general, there is a clear classification of numerical methods applied in finance. One class is represented by those numerical methods originated in solving PDE. Finite difference approximations, implicit and explicit schemes, are widely used in pricing financial products and calculating risk management parameters, see, for instance, [8] for some excellent examples. A second class applies stochastic control techniques for problems in finance. This second class proves to be useful for situations when the Bellman equation, appearing frequently in decision making in finance, might not be formally given by a PDE or variational inequality. The most used stochastic control method is the Markov chain approximation method; a review on this topic is provided in [20]. Third, the more direct Monte Carlo simulation is very popular due to the generality of applications and relative straightforward implementation, as illustrated in [2]. Remarkable improvements on this direction are associated with the introduction of importance sampling and variance reduction methods such as control variates. The series of papers authored by Boyle, Broadie and Glasserman[1] contain some of the seminal ideas in this area. In spite of its widespread popularity, the Monte Carlo simulation has a major criticism related to the practical inability to find proper random numbers from specified distributions. This generated the development of quasi-Monte Carlo methods based on equidistributed sequences and low-discrepancy methods. A recent revision of applications of these methods in finance has been given in [12]. Furthermore, it is known that many nominally high-dimensional integrals arising from the pricing of options are of low effective dimension, see [37] and [38] for a very useful discussion. As opposed to Monte Carlo methods, quasi-Monte Carlo simulation benefits from low effective dimensions and work with substantially smaller errors than Monte Carlo methods even if the nominal dimension is high.

A fourth class of methods is spanned by approximations of some sort. This is a rich and varied class containing diverse methods. Jarrow and Rudd [17] proposed a technique for option pricing applying Edgeworth series expansion. This method has some valuable benefits in explaining some of the smile associated with options pricing but does not apply when it is used to approximate the density of the arithmetic average of a lognormal process as pointed out in [18]. The methodology underpinned by the Fast Fourier transform (FFT) was advocated by Carr and Madan [5] as a general tool for option pricing problems. One excellent application is in pricing spread options, as illustrated in [9]. The Laplace method is the cousin transformation of the Fourier transform. A great description of this technique in relation to Asian option pricing is given in [7] and [15]. Laplace transform was also successfully used in [26] for analytical investigations of stochastic volatility models of the Ornstein–Uhlenbeck type. Borovkov and Novikov [4] present an ingenious approach to calculate expectations for option pricing by simply integrating the respective moment generating function with a certain weight. In [22], Lee extended and unified Fourier-analytic methods for pricing a wide class of options on any underlying state variable whose characteristic function is known. Laguerre series generalize the Fourier expansions. They were masterfully applied by Dufresne in [10] who utilized the concept of ladder height densities for improving pricing of Asian options. These ideas were further expanded with excellent mathematical computational results in [34].

Saddlepoint approximations are often applied for computing the distribution of a random variable whose moment generating function is known. This technique has been elegantly applied in [33] to the calculation of prices for European put options under various Levy processes. Their results compare favorably with those using numerical integration (FFT).

Other expansion type methods are developed in [24] where the density of the average rate is approximated with a reciprocal gamma distribution by matching the first two moments, while in [25] the density function is approximated by matching the first four moments. Considering the Taylor expansion of the ratio of the characteristic function of the average to that of the approximating lognormal random variable around zero volatility, it is possible to unify the treatment of basket and

[1] See [16] and the references therein.

Asian option as illustrated in [18]. A method based on small disturbance asymptotics applicable to pricing contingent claims where underlying forward rates follow a continuous Ito process was developed in [19]. Using the moment expansion of the probability distribution function underlying the model, a new quadrature method, with applications in option pricing, has been proposed in [1].

The link between discrete time modelling and continuous time modelling has been emphasized at various points in the past and to a certain level of complexity. It has been proved in [6] that the limit of the discrete-binomial option pricing formula converges to the Black–Scholes formula as the number of time steps per unit of real time approaches zero. Duffie and Protter [11] reviewed conditions under which the discrete-time security market models converge in the limit to continuous time models. Moreover, they provide some pathological examples where a sequence of security price processes $S^{(n)}$ converges in distribution to a limit S and a sequence of associated trading strategies $\theta^{(n)}$ converges also in distribution to a strategy θ but the discrete financial gain process fails to converge in distribution to the financial gain process defined by the limits. This line of research has been thoroughly investigated in [32] where the most important results related to the convergence of discrete pricing schemes to continuous time limits are provided.

The results developed in this paper, however, make a contribution in a different direction. Rather than trying to *establish* weak convergence results of *known* discrete pricing schemes, we *apply* known weak convergence results from probability theory and *determine* discrete pricing schemes. The approximation formulae proposed here are deterministic.

A general formula for approximating the multi-dimensional integral appearing in the risk-neutral price of a general multi-asset option has been proposed in [3], where there is a similar aim to the research presented here, but a different approach has been chosen. The methods presented in this chapter are in a sense a simplified version of the quantization methodology introduced in the early 1990's in numerical probability to determine some quadrature integration formulae with respect to the distribution $\mathbf{P}_X$ of a random variable X on $\mathbb{R}^d$ using that $\mathcal{E}[\Pi(X)] \approx \mathcal{E}[\Pi(\widehat{X})]$ if the length of the approximating sequence is large enough. This approach is efficient in medium dimensions (see [27–29]) especially when many integrals need to be computed with respect to the same distribution $\mathbf{P}_X$. Lately, optimal quantization has been used to define tree based methods for solving multi-dimensional non-linear problems involving the computation of many conditional expectations: American option pricing, non-linear filtering for stochastic volatility models, portfolio optimization. Examples of quantization applications to computational finance problems are illustrated in [30, 31].

3 Framework and Notations

In this section, we outline the framework for the main results presented in the next section. In addition, we specify the notation used throughout the chapter. A sequence of probability measures $\{\mu_n\}_{n\geq 0}$ is said to converge weakly to a probability measure μ, denoted by $\mu_n \to \mu$, if the following equivalent condition is satisfied

$$\lim_{n\to\infty} \int f \,\mathrm{d}\mu_n = \int f \,\mathrm{d}\mu \tag{5}$$

for any bounded and μ- almost everywhere continuous function f. Other equivalent conditions are provided by Alexandrov's theorem [23], see also [14] for a more recent reference.

The generic notation for a probability density function is $\rho()$. For example, the probability density of a multivariate gaussian d-dimensional vector with mean vector m and covariance matrix Υ is

$$\rho(y; m, \Upsilon) = \frac{1}{\sqrt{(2\pi)^d \det(\Upsilon)}} \exp\left[-\frac{1}{2}(y-m)^\top \Upsilon^{-1}(y-m)\right].$$

As you may have already observed, the transpose of a matrix or vector is denoted by $\top$.

We assume that our primary assets have the following dynamics under an equivalent martingale measure

$$\mathrm{d}S_t^i = rS_t^i \mathrm{d}t + \sigma_i S_t^i \mathrm{d}W_t^i, \quad \text{for all } i \in \{1, \ldots, d\} \tag{6}$$

$$\mathrm{d}B_t = rB_t \mathrm{d}t \tag{7}$$

where $\mathcal{E}(\mathrm{d}W_t^i \mathrm{d}W_t^j) = \rho_{ij}\mathrm{d}t$ for any $i \neq j$. The last equation is associated with the money account with the short rate r.

Our objective is to price a European multi-asset option with payoff $\Pi(S_T^1, S_T^2, \ldots, S_T^d)$ at maturity T, contingent on the assets $\{S_T^1, S_T^2, \ldots S_T^d\}$. For simplicity, we shall drop the index T from our notations. There are many exotic options that require powerful computational methods for pricing determination. Here, we focus only on European style derivatives and for concreteness we consider the basket options and the discrete Asian arithmetic options as these two are representative of multivariate contingent claim contracts.

The value of each asset at maturity S^i can be calculated with

$$S^i = S_0^i \mathrm{e}^{(r-\sigma_i^2/2)T + \sigma_i\sqrt{T}Y_i}, \tag{8}$$

where $Y = (Y_1, \ldots, Y_d)$ is distributed $\mathcal{N}_d(0_d, \Upsilon)$ with the covariance matrix $\Upsilon = \left(\rho_{ij}\sigma_i\sigma_j T\right)_{i,j=1}^d$. The relevant probability density function is therefore

$$\rho_d(y; 0, \Upsilon) = \frac{1}{\sqrt{(2\pi)^d \det(\Upsilon)}} \exp\left[-\frac{1}{2}y^\top \Upsilon^{-1} y\right] \tag{9}$$

From (8), for all $i \in \{1, \ldots, d\}$

$$Y_i = \frac{1}{\sigma_i \sqrt{T}} \left[\ln\left(\frac{S^i}{S_0^i}\right) - \left(r - \frac{\sigma_i^2}{2}\right) T \right]. \tag{10}$$

For any integrable payoff $\Pi(S^1, \ldots, S^d)$, the value of the multi-asset claim at time 0 is given by

$$V_0 = \eta \int_0^\infty \cdots \int_0^\infty \Pi(S^1, \ldots, S^d) \frac{1}{S^1 \cdots S^d} \exp\left[-\frac{1}{2} Y^\top \Upsilon^{-1} Y\right] \mathrm{d}S^1 \cdots \mathrm{d}S^d \tag{11}$$

where $\eta = \frac{\mathrm{e}^{-rT}}{\sigma_1 \cdots \sigma_d \sqrt{(2\pi T)^d \det(\Upsilon)}}$ and $Y = (Y_1, \ldots, Y_d)^\top$ is determined from (10).

If e_i is the d-dimensional vector that has the value one on the i-th position and zero elsewhere, one can calculate the delta parameters of the multi-asset option as follows. Since $\Delta_i = \frac{\partial V}{\partial S_0^i}$, by derivation on the right side of formula (11), we obtain

$$\Delta_i = \eta \int_0^\infty \cdots \int_0^\infty \frac{1}{S_0^i \sigma_i \sqrt{T}} \mathrm{e}_i^\top \Upsilon^{-1} Y \Pi(S^1, \ldots, S^d) \frac{1}{S^1 \cdots S^d} \times \exp\left[-\frac{1}{2} Y^\top \Upsilon^{-1} Y\right] \mathrm{d}S^1 \cdots \mathrm{d}S^d. \tag{12}$$

The calculation of integrals in formulae (11) and (12) may become difficult as dimension d increases. For small dimensions d, the integrals appearing in formulae (11) and (12) can be calculated with cubature or quadrature methods, see [13] for a good description of such methods. Nevertheless, in financial calculus, the dimension d can be quite large. For example, pricing a basket option may lead easily to a d between 5 and 20. Moreover, we would like to take advantage of the gaussian kernel as this appear quite naturally in financial mathematics.

4 Methods for Multivariate Contingent Claim Pricing

For the technique we propose in this paper, we would like to work on the $(-\infty, \infty)$ scale rather than $(0, \infty)$ scale resulted from the log-normal distribution of the geometric Brownian motion. Thus, we prefer to work with a formula using the gaussian kernel. For simplicity, we denote

$$g(Y_1, \ldots, Y_d) = \Pi\left(S_0^1 \mathrm{e}^{(r_f - \sigma_1^2/2)T + \sigma_1 \sqrt{T} Y_1}, \ldots, S_0^d \mathrm{e}^{(r_f - \sigma_d^2/2)T + \sigma_d \sqrt{T} Y_d}\right). \tag{13}$$

Hence, we can express the value of the multi-asset derivative as

$$V_0 = \frac{\mathrm{e}^{-rT}}{\sqrt{(2\pi)^d \det \Upsilon}} \int_{\mathbb{R}^d} g(Y_1, \ldots, Y_d) \exp\left[-\frac{1}{2} Y^\top \Upsilon^{-1} Y\right] \mathrm{d}Y_1 \cdots \mathrm{d}Y_d \tag{14}$$

and

$$\Delta_i = \frac{e^{-rT}}{\sqrt{(2\pi)^d \det \Upsilon}} \int_{\mathbb{R}^d} \frac{1}{S_0^i \sigma_i \sqrt{T}} e_i^\top \Upsilon^{-1} Y g(Y_1, \dots, Y_d)$$
$$\times \exp\left[-\frac{1}{2} Y^\top \Upsilon^{-1} Y\right] dY_1 \cdots dY_d. \tag{15}$$

Applying the Cholesky decomposition to the correlation matrix Υ, we get the square root matrix Δ such that $\Upsilon = \Delta\Delta^\top$. Making the change of variables $Y = \Delta x$ leads to

$$e^{rT} V_0 = \mathcal{E}[g(Y)] = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} g(\Delta x) \exp\left[-\frac{1}{2} x^\top x\right] dx. \tag{16}$$

This is now the expectation with respect to the standard multivariate gaussian distribution. Since we are going to employ the multivariate central limit theorem, the probability measure μ is going to be a gaussian probability measure while the measure μ_n is corresponding to a discrete multivariate distribution such as the multinomial. Then the calculations of (16) will follow by choosing an appropriate test function mainly driven by the payoff function Π and other subsequent transformations. Hence, there are three steps in our methodology of generating algorithms for calculations of integrals representing financial measures. First, one needs to identify the limiting measure or distribution. Since it can be envisaged that a central limit theorem results will be applied, it is natural to think that the limiting distribution will be gaussian. If the financial modelling is done with a gaussian kernel the normal distribution appears naturally; if not, it can be forced to appear. Second, upon applying a well-known weak convergence result, a sequence of discrete probability distributions defining the measures μ_n is identified. This is straightforward as the multinomial distribution is the only multivariate discrete distribution widely used. The problems here may appear in finding put the correct probability weights defined by this distribution. The third and final step consists in finding out the test function on which the weak convergence condition will be applied to. The test function basically absorbs all factors outside the kernel of the limiting distribution. This step looks the easiest but it is the test function that presents the first challenge.

4.1 A Solution to Payoff Unboundedness

One may think that all that is left to do now is to apply the weak convergence resulting from the central limit theorem for a discrete multivariate distribution such as the multinomial distribution and a suitable test function, and the discrete approximation algorithm would be established. Payoff functions in finance by default cannot exhibit an uncountable set of discontinuities so, from the continuity point of view, we are safe. However, one major inconvenience is that payoffs quite often are unbounded. Nevertheless, the payoffs cannot exhibit explosive growth so the theoretical weak convergence results underpinning our algorithms still work when the test functions satisfy some mild assumptions, as proved next.

We shall prove that for a given test function we have uniform convergence. More precisely, let $B(0,\delta)$ be a ball of radius $\delta \in \mathbb{R}^d$ and let $B^c(0,\delta)$ be its complementary set. For a given test function ψ, if the following two conditions are true

$$\lim_{n\to\infty} \mu_n = \mu, \tag{17}$$

$$\lim_{\delta\to\infty} \sup_n \int_{B^c(0,\delta)} \psi \mathrm{d}\mu_n = 0 \tag{18}$$

then it follows once again that

$$\lim_{n\to\infty} \int \psi \mathrm{d}\mu_n = \int \psi \mathrm{d}\mu. \tag{19}$$

The key is to prove that condition (18) holds. Applying Holder's inequality for $0 < v, s < \infty$ with $\frac{1}{v} + \frac{1}{s} = 1$, implies that

$$\begin{aligned} \int_{B^c(0,\delta)} \psi \mathrm{d}\mu_n &= \int \psi 1_{B^c(0,\delta)} \mathrm{d}\mu_n \\ &\leq \left(\int \psi^v \mathrm{d}\mu_n \right)^{\frac{1}{v}} (\mu_n(B^c(0,\delta))^s)^{\frac{1}{s}} \end{aligned} \tag{20}$$

If one can prove that

$$\lim_{\delta\to\infty} \sup_n \mu_n(B^c(0,\delta)) = 0 \tag{21}$$

and that

$$\sup_n \left(\int \psi^v \mathrm{d}\mu_n \right) < \infty \tag{22}$$

are then the condition (18) follows which ensures that the convergence in (19) works for the unbounded test function ψ. In order to fix the notation, we consider the real valued random variables $Y_n = \sqrt{n}\left(\frac{\xi^{(n)}}{n} - p\right)$ and the associated probability measure $\mu_n = \mathbf{P} \circ Y_n^{\leftarrow}$.

Proposition 1.

$$\lim_{\delta\to\infty} \sup_n \mu_n(B^c(0,\delta)) = 0$$

Proof. Considering the function

$$h_\delta(x) = \begin{cases} \frac{\|x\|^2}{\delta^2}, & \text{if } \|x\| > \delta, \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

with $\| \|$ the euclidian norm, it follows that

$$\begin{aligned}\mu_n(B^c(0,\delta)) &= \int 1_{B^c(0,\delta)} \mathrm{d}\mu_n \\ &\leq \int h_\delta \mathrm{d}\mu_n = \frac{1}{\delta^2}\int \|x\|^2 \mathrm{d}\mu_n \\ &= \frac{1}{\delta^2}\int (x_1^2 + x_2^2 + \ldots x_d^2)\mathrm{d}\mu_n \end{aligned} \tag{24}$$

The probability measure μ_n corresponds to the random variable Y_n that has component-wise mean zero and variance equal to $p_i(1-p_i)$ for the i-th variable of the vector. Therefore,

$$\mu_n(B^c(0,\delta)) \leq \frac{1}{\delta^2}\sum_{i=1}^{d} p_i(1-p_i)$$

$$0 \leq \sup_n \mu_n(B^c(0,\delta)) \leq \frac{1}{\delta^2}\sum_{i=1}^{d} p_i(1-p_i)$$

Taking the limit when δ goes to infinity concludes the proof. □

For the payoff function $\Pi : (0,\infty)^d \to \mathbb{R}$, we make the general assumption that there exist some positive real numbers $0 < m_1, m_2, \ldots, m_d$ such that

$$\Pi(u_1,\ldots,u_d) \leq u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d} \tag{25}$$

for any large positive real numbers $(u_1,\ldots,u_d)$. This condition prevents explosive growth and pathological cases.

Proposition 2.

$$\sup_n \int \psi^v \mathrm{d}\mu_n < \infty \tag{26}$$

Proof. Consider the application $U : \mathbb{R}^d \to \mathbb{R}, \;\; U(u) = \sqrt{n}\left(\frac{u}{n} - p\right)$. Making the notation $\alpha_i = S_0^i \mathrm{e}^{\mu_i T}$

$$\begin{aligned}\int \psi^v \mathrm{d}\mu_n &= \int \psi^v \mathrm{d}\mathbf{P} \circ (U(X^{(n)})^{\leftarrow}) \\ &= \sum_{\substack{0\leq k_1,\ldots,k_d\leq n \\ k_1+\ldots+k_d=n}} \psi^v(U(k_1,\ldots,k_d))\mathbf{P}(X^{(n)} = (k_1,\ldots,k_d)) \\ &= \sum_{\substack{0\leq k_1,\ldots,k_d\leq n \\ k_1+\ldots+k_d=n}} P_n^{k_1,\ldots,k_d}\, \Pi^v\left(\alpha_1 \mathrm{e}^{\sigma_1\sqrt{T}\frac{k_1-np_1}{\sqrt{n}}}, \ldots, \alpha_d \mathrm{e}^{\sigma_d\sqrt{T}\frac{k_d-np_d}{\sqrt{n}}}\right)\end{aligned}$$

Then

$$\begin{aligned}
\sup_n \left(\int \psi^v \mathrm{d}\mu_n \right) &= \sup_n \sum_{0 \le k_1,\dots,k_d \le n}^{k_1+\dots+k_d=n} P_n^{k_1,\dots,k_d} \left[\Pi \left(\alpha_1 \mathrm{e}^{\sigma_1 \sqrt{T} \frac{k_1 - np_1}{\sqrt{n}}}, \dots, \alpha_d \mathrm{e}^{\sigma_d \sqrt{T} \frac{k_d - np_d}{\sqrt{n}}} \right) \right]^v \\
&\le \sup_n \sum_{0 \le k_1,\dots,k_d \le n}^{k_1+\dots+k_d=n} P_n^{k_1,\dots,k_d} \prod_{i=1}^d \alpha_i^{vm_i} \mathrm{e}^{vm_i\sigma_i \sqrt{T} \frac{k_i - np_i}{\sqrt{n}}} \\
&= \alpha \sup_n \sum_{0 \le k_1,\dots,k_d \le n}^{k_1+\dots+k_d=n} \frac{n!}{k_1! \cdots k_d!} \mathrm{e}^{-v\sqrt{nT}(\sigma_1 m_1 p_1 + \dots \sigma_d m_d p_d)} \prod_{i=1}^d \left(p_i \mathrm{e}^{v\sigma_i m_i \sqrt{T/n}} \right)^{k_i} \\
&= \alpha \sup_n \left\{ \mathrm{e}^{-v\sqrt{nT}(\sigma_1 m_1 p_1 + \dots \sigma_d m_d p_d)} \left(p_1 \mathrm{e}^{v\sigma_1 m_1 \sqrt{T/n}} + \dots p_d \mathrm{e}^{v\sigma_d m_d \sqrt{T/n}} \right)^n \right\}
\end{aligned}$$

where $\alpha = \prod_{i=1}^d \alpha_i^{vm_i}$. Denoting

$$a = v\sqrt{T}(p_1\sigma_1 m_1 + \dots p_d \sigma_d m_d), \; a_i = v\sigma_i m_i \sqrt{T}, \;\; \text{for all } i = 1, \dots, d,$$

we can calculate

$$\lim_{n\to\infty} \mathrm{e}^{-a\sqrt{n}} \mathrm{e}^{n \ln\left[p_1 \mathrm{e}^{a_1/\sqrt{n}} + \dots p_d \mathrm{e}^{a_d/\sqrt{n}}\right]} = \lim_{n\to\infty} \mathrm{e}^{n\left\{\ln\left[p_1 \mathrm{e}^{a_1/\sqrt{n}} + \dots p_d \mathrm{e}^{a_d/\sqrt{n}}\right] - a/\sqrt{n}\right\}}$$

Making use of l'Hopital rule

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\ln\left[p_1 \mathrm{e}^{a_1/\sqrt{n}} + \dots p_d \mathrm{e}^{a_d/\sqrt{n}}\right] - a/\sqrt{n}}{1/n} \\
&= \lim_{n\to\infty} \left[\frac{\frac{1}{2}\sum_{i=1}^d p_i a_i \frac{1}{n\sqrt{n}} \mathrm{e}^{a_i/\sqrt{n}}}{\sum_{i=1}^d p_i \mathrm{e}^{a_i/\sqrt{n}}} - \frac{a}{2n\sqrt{n}} \right] n^2 \\
&= \lim_{n\to\infty} \left[\frac{n^2}{2} \sum_{i=1}^d (a_i - a) p_i \mathrm{e}^{a_i/\sqrt{n}} \right].
\end{aligned}$$

Applying l'Hopital rule four more times we get that

$$\begin{aligned}
\lim_{n\to\infty} \left[\frac{\sum_{i=1}^d (a_i - a) p_i \mathrm{e}^{a_i/\sqrt{n}}}{n^{-2}} \right] &= \lim_{n\to\infty} \left[\frac{\sum_{i=1}^d (a_i - a) p_i a_i \mathrm{e}^{a_i/\sqrt{n}}}{4n^{-3/2}} \right] \\
&= \dots \\
&= \lim_{n\to\infty} \left[\frac{\sum_{i=1}^d (a - a_i) p_i a_i^4 \mathrm{e}^{a_i/\sqrt{n}}}{24} \right]
\end{aligned}$$

The last limit is equal to $\frac{\sum_{i=1}^d (a_i - a) p_i a_i^4}{24}$ when n goes to infinite, and this is evidently a finite real number. Therefore, its exponential would be finite too and consequently $\sup_n \left(\int \psi^v \mathrm{d}\mu_n \right) < \infty$. □

4.2 Eliminating the Singularity of the Covariance Matrix

The set-up described here is quite general and encompasses a large variety of situations. For practical applications, we consider here the multinomial distribution. More specifically let $\xi^{(n)} = (\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_{d+1}^{(n)})$ be a random vector with a multinomial distribution with parameters n and $p = (p_1, \ldots, p_{d+1})$, where by definition $\sum_{j=1}^{d+1} p_j = 1$. The reason for taking the order $d + 1$ will become evident immediately.

The central limit theorem ensures that

$$\sqrt{n}\left(\frac{\xi^{(n)}}{n} - p\right) \xrightarrow{\mathcal{D}} Z \tag{27}$$

where $Z \sim \mathcal{N}_{d+1}[0_{d+1}, \Sigma(p)]$ and

$$\Sigma(p) = [Diag(p_1, \ldots, p_{d+1}) - pp^\top] \tag{28}$$

is the variance-covariance matrix of the multivariate gaussian limit distribution.

The CLT applied here results in the limit probability measure μ given by the probability density $\rho_d(y; 0_{d+1}, \Sigma(p))$. There are two problems with this limiting distribution. The first one is only apparent. The covariance matrix $\Sigma(p)$ has negative correlations off the diagonal, and therefore, it seems that it is not suitable for financial calculus. However, by making an adjustment reminiscent of the Radon-Nikodym derivative, we shall see later that this problem is easily solved. The second problem is more serious as it concerns the singularity of the covariance matrix

$$\Sigma(p) = \begin{pmatrix} p_1(1-p_1) & -p_1 p_2 & \ldots & -p_1 p_{d+1} \\ -p_1 p_2 & p_2(1-p_2) & \ldots & -p_2 p_{d+1} \\ \vdots & \vdots & \vdots & \vdots \\ -p_{d+1} p_1 & -p_{d+1} p_2 & \ldots & p_{d+1}(1-p_{d+1}) \end{pmatrix}$$

for *any* choice of probability vector $p = (p_1, \ldots, p_{d+1})$. This means that there is no probability density function for the limiting gaussian probability measure and calculus is impossible.

Nevertheless, we are going to show that we can still perform calculations by switching to a lower d-dimensional space. First, the delta-continuity theorem is applied for the multivariate case of the central limit theorem.

Lemma 1. *Suppose that* $\sqrt{n}\left(\frac{\xi^{(n)}}{n} - p\right) \xrightarrow{\mathcal{D}} Z$ *where* $Z \sim \mathcal{N}_d[0_{d+1}, \Sigma(p)]$ *and the vectors* $\xi^{(n)}$ *and* p *are* $d + 1$*-dimensional. Let* $h : \mathbb{R}^{d+1} \to \mathbb{R}^d$ *be defined by* $h(x) = (h_1(x), \ldots h_d(x))$ *and assume that each function* $h_i(\cdot)$ *has continuous partial derivatives at* $x = 0_d$*. Then*

$$\sqrt{n}\left(h(\frac{\xi^{(n)}}{n}) - h(p)\right) \xrightarrow{\mathcal{D}} Z^* \tag{29}$$

where $Z^* \sim \mathcal{N}_d[0_d, \nabla h(p)\Sigma(p)(\nabla h(p))^\top]$ *and* $(\nabla h(p))_{ij} := \left(\frac{\partial h_i(p)}{\partial p_j}\right)$ *for all* $1 \le i \le d, 1 \le j \le d+1$.

Taking $h_i(p_1, \ldots, p_{d+1}) = p_i$, the projection on the i-th direction, it is obvious that $(\nabla h(p))_{ij}$ equals 1 for $i = j$ and zero otherwise. Moreover, all h_i are continuous at any given probability vector p. The new covariance matrix is

$$\nabla h(p)\Sigma(p)(\nabla h(p))^\top = \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \ldots & -p_1p_d \\ -p_1p_2 & p_2(1-p_2) & \ldots & -p_2p_d \\ \vdots & \vdots & \vdots & \vdots \\ -p_1p_d & -p_2p_d & \ldots & p_d(1-p_d) \end{pmatrix} \tag{30}$$

which is the matrix obtained from $\Sigma(p)$ after removing the last row and the last column. It can be shown that this matrix is always invertible. With mathematical induction the following result can be proved.

Lemma 2. *For any probability vector* $p = (p_1, \ldots, p_{d+1})$ *and for the projection operators* $h_i(p_1, \ldots, p_{d+1}) = p_i$*, the following formula is true*

$$\det\left\{\left(\nabla h(p)\Sigma(p)(\nabla h(p))^\top\right)\right\} = \left(1 - \sum_{i=1}^{d} p_i\right)\prod_{i=1}^{d} p_i. \tag{31}$$

Since all $p_i > 0$ and the probability vector p is $d+1$ dimensional, the above theorem proves that $\det\left\{\left(\nabla h(p)\Sigma(p)(\nabla h(p))^\top\right)\right\} \neq 0$. Consequently, the probability distribution $\mathcal{N}_d[0_d, \nabla h(p)\Sigma(p)(\nabla h(p))^\top]$ has the probability density function $\rho_d(x; 0_d, \nabla h(p)\Sigma(p)(\nabla h(p))^\top)$.

4.3 Pricing Multi-Asset Options

After this digression, we recall that pricing multi-asset options could be done with formula (16). We aim to apply the central limit theorem for a multinomial distribution to generate a deterministic algorithm for approximating the expression on the right side of (16). However, since the covariance matrix of the limit gaussian distribution is not the identity matrix, we need to make some adjustments.

$$\mathcal{E}(g(Y)) = \frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d} g(\Delta x)\exp\left(-\frac{1}{2}x^\top x\right)\mathrm{d}x \tag{32}$$

$$= \int_{\mathbb{R}^d} g(\Delta x)\rho(x; 0_d, I_d)\mathrm{d}x \tag{33}$$

$$= \int_{\mathbb{R}^d} g(\Delta x) \frac{\rho(x; 0_d, I_d)}{\rho(x; 0_d, M)} \rho(x, 0_d, M) \mathrm{d}x \tag{34}$$

$$= \int_{\mathbb{R}^d} \psi(x) \rho(x; 0_d, M) \mathrm{d}x \tag{35}$$

where

$$\psi(x) = g(\Delta x) \frac{\rho(x; 0_d, I_d)}{\rho(x; 0_d, M)}, \quad M = \nabla h(p) \Sigma(p) (\nabla h(p))^{\top}. \tag{36}$$

After determining the functional expression of the test function, we need to identify the probabilities associated with the test function over the grid generated by the multinomial distribution.

Recalling that $\xi^{(n)} = (\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_{d+1}^{(n)})$ has a multinomial distribution with parameters n and $(p_1, \dots, p_{d+1})$, the sequence of probability measure μ_n is given by $\mu_n = \mathbb{P} \circ v_n \left(\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_{d+1}^{(n)} \right)^{\leftarrow}$ where

$$v_n(\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_{d+1}^{(n)}) = \left(\frac{\xi_1^{(n)} - np_1}{\sqrt{n}}, \dots, \frac{\xi_d^{(n)} - np_d}{\sqrt{n}} \right). \tag{37}$$

The vector used on the right side makes use of only a subset of the full multinomial vector. However, its conditional distribution is still multinomial

$$\left(\xi_1^{(n)}, \dots, \xi_d^{(n)} | \xi_{d+1}^{(n)} = j \right) \sim Multin \left(n - j; \frac{p_1}{p_1 + \ldots + p_d}, \dots, \frac{p_d}{p_1 + \ldots + p_d} \right) \tag{38}$$

In addition, the marginal distribution of $\xi_{d+1}^{(n)}$ is binomial with parameters n and p_{d+1}. Therefore, we can calculate the marginal distribution of the subset vector $(\xi_1^{(n)}, \dots, \xi_d^{(n)})$ as

$$\begin{aligned} &P(\xi_1^{(n)} = k_1, \dots, \xi_d^{(n)} = k_d) \\ &\quad = \sum_{j=0}^{j=n} P(\xi_1^{(n)} = k_1, \dots, \xi_d^{(n)} = k_d | \xi_{d+1}^{(n)} = j) P(\xi_{d+1}^{(n)} = j). \end{aligned} \tag{39}$$

Making the notation $\widetilde{p_i} = \frac{p_i}{p_1 + \ldots + p_d}$, for all $i = 1, \dots, d$, it follows that

$$P(\xi_1^{(n)} = k_1, \dots, \xi_d^{(n)} = k_d) = \sum_{j=0}^{j=n} \binom{n}{j} p_{d+1}^j (1 - p_{d+1})^{(n-j)} \frac{(n-j)!}{k_1! \dots k_d!} \prod_{i=1}^{d} \widetilde{p_i}^{k_i}. \tag{40}$$

Remarking that

$$\psi(x) = g(\Delta x)\sqrt{\det(M)}\exp\left[-\frac{1}{2}x^\top(I_d - M^{-1})x\right] \tag{41}$$

and applying the weak convergence of μ_n to μ for the test function ψ leads to our main result

Theorem 1.

$$\int_{\mathbb{R}^d}\psi(x)\rho(x;0_d,M)\mathrm{d}x = \lim_{n\to\infty}\sum_{j=0}^{j=n}\sum_{0\le k_1,\ldots,k_d\le n-j}^{k_1+\ldots+k_d=n-j} \times\psi\left(\frac{k_1-np_1}{\sqrt{n}},\ldots,\frac{k_d-np_d}{\sqrt{n}}\right)\frac{n!}{k_1!\ldots k_d!j!}p_{d+1}^j\prod_{i=1}^d p_i^{k_i}. \tag{42}$$

To recap, what we have done is to apply the probability convergence equivalent condition for the test function ψ

$$\lim_{n\to\infty}\int\psi\mathrm{d}\mu_n = \int\psi\mathrm{d}\mu \tag{43}$$

taking into account now that μ_n is the probability measure corresponding to a discrete probability function, so the integral is transformed into a sum where calculations depend on $\psi(v_n(k_1,k_2,\ldots,k_{d+1}))$ and the associated probabilities as determined in (40).

In other words, we have proved the following multi-asset option pricing approximation formula.

Theorem 2. *The no-arbitrage price of the European option with payoff function $\Pi = \Pi(S_T^1,\ldots,S_T^d)$ can be calculated with the formula*

$$\mathrm{e}^{-rT}\lim_{n\to\infty}\sum_{j=0}^{j=n}\sum_{0\le k_1,\ldots,k_d\le n-j}^{k_1+\ldots+k_d=n-j}\psi\left(\frac{k_1-np_1}{\sqrt{n}},\ldots,\frac{k_d-np_d}{\sqrt{n}}\right)\frac{n!}{k_1!\ldots k_d!j!}p_{d+1}^j\prod_{i=1}^d p_i^{k_i} \tag{44}$$

where $\psi:\mathbb{R}^d\to\mathbb{R}$

$$\psi(u) = \sqrt{\det(M)}g(\Delta u)\exp\left[-\frac{1}{2}u^\top(I_d - M^{-1})u\right]$$

with $M = \nabla h(p)\Sigma(p)(\nabla h(p))^\top$ and $u = (u_1,\ldots,u_d)^\top$.

Remark that $\frac{n!}{k_1!\ldots k_d!j!}p_{d+1}^j\prod_{i=1}^d p_i^{k_i}$ is exactly the multinomial probability for the $d+1$ multinomial vector.

Furthermore, since the result is true for any choice of multinomial probability vector values, it is convenient to choose all of them equal, that is, $p_1 = p_2 = \ldots = p_{d+1} = \frac{1}{d+1}$. The matrix M becomes in this particular case

$$M = \begin{pmatrix} \frac{d}{(d+1)^2} & -\frac{1}{(d+1)^2} & \cdots & -\frac{1}{(d+1)^2} \\ -\frac{1}{(d+1)^2} & \frac{d}{(d+1)^2} & \cdots & -\frac{1}{(d+1)^2} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{(d+1)^2} & -\frac{1}{(d+1)^2} & \cdots & \frac{d}{(d+1)^2} \end{pmatrix} \tag{45}$$

which has the determinant $\det(M) = \frac{1}{(d+1)^{d+1}}$.

The approximation result can be rewritten simpler as:

Corollary 1. *The no-arbitrage price of the European option with payoff function $\Pi = \Pi(S_T^1, \ldots, S_T^d)$ can be calculated with the formula*

$$e^{-rT} \lim_{n\to\infty} \sum_{j=0}^{j=n} \sum_{0\le k_1,\ldots,k_d \le n-j}^{k_1+\ldots+k_d=n-j} \psi\left(\frac{k_1 - np_1}{\sqrt{n}}, \ldots, \frac{k_d - np_d}{\sqrt{n}}\right) \frac{n!}{k_1!\ldots k_d! j!} \frac{1}{(d+1)^n} \tag{46}$$

where $\psi : \mathbb{R}^d \to \mathbb{R}$

$$\psi(u) = \frac{1}{(d+1)^{\frac{d+1}{2}}} g(\Delta u) \exp\left[-\frac{1}{2} u^\top (I_d - M^{-1}) u\right]$$

with M in (45).

Similarly, for the greek parameter Δ, recall the formula given in (15)

$$\begin{aligned} \Delta_i = & \frac{e^{-rT}}{\sqrt{(2\pi)^d \det \Upsilon}} \int_{\mathbb{R}^d} \frac{1}{S_0^i \sigma_i \sqrt{T}} e_i^\top \Upsilon^{-1} Y g(Y_1, \ldots, Y_d) \\ & \times \exp\left[-\frac{1}{2} Y^\top \Upsilon^{-1} Y\right] dY_1 \cdots dY_d \end{aligned}$$

All that needs to be done is to redefine the g function inside the integral. For example,

$$\begin{aligned} \Delta_i = & \frac{e^{-rT}}{\sqrt{(2\pi)^d \det \Upsilon}} \int_{\mathbb{R}^d} \frac{1}{S_0^i \sigma_i \sqrt{T}} \underbrace{e_i^\top \Upsilon^{-1} Y g(Y_1, \ldots, Y_d)}_{g^*(Y)} \\ & \times \exp\left[-\frac{1}{2} Y^\top \Upsilon^{-1} Y\right] dY_1 \cdots Y_d \end{aligned} \tag{47}$$

5 Some Univariate Considerations

In practice, the univariate case is more thoroughly studied due to safety regarding the numerical dimensionality curse. The univariate counterpart of the technique revealed above for the multidimensional case has been illustrated in [36] using the binomial distribution as the workhorse of the numerical scheme. From a mathematical point of view, other probability distributions can be used. The first example refers to a scheme of non-identically distributed random variables, this case presenting more of a mathematical interest.

Approximation with Poisson Generalized Binomial Distribution

For example, consider $\{X_n\}_{n\geq 1}$ a sequence of independent random variables such that

$$\mathbf{P}([\omega : X_j(\omega) = 1]) = p_j; \quad \mathbf{P}([\omega : X_j(\omega) = 0]) = 1 - p_j = q_j \tag{48}$$

for all $j \geq 1$. The distribution of the random variable $X^{(n)} = X_1 + \ldots + X_n$ is sometimes called the Poisson generalized binomial. The random variable $X^{(n)}$ takes only the values $0, 1, 2, \ldots, n$, and $\widetilde{P}_n^k = \mathbf{P}(X_{(n)} = k)$ is equal to the coefficient of t^k in the polynomial $(p_1 t + q_1)(p_2 t + q_2)\ldots(p_n t + q_n)$. Applying Liapounov's theorem, the following version of De Moivre–Laplace theorem can be easily proved.

Theorem 3. *Let $\{X^{(n)}\}_{n\geq 1}$ be a sequence of random variables such that for each positive integer n, $X_{(n)}$ has a Poisson generalized binomial distribution. Suppose that $\sum_{n\geq 1} p_n(1-p_n)$ is divergent. Then*

$$\lim_{n\to\infty} \mathbf{P} \circ \left(\frac{X_{(n)} - \sum_{i=1}^{i=n} p_i}{\sqrt{\sum_{i=1}^{i=n} p_i(1-p_i)}} \right)^{\leftarrow} = \mathbf{N}(0,1). \tag{49}$$

As usual, $\mathbf{N}(0,1)$ denotes the probability measure corresponding to the Gaussian distribution with mean 0 and variance 1. An example when the series $\sum_{n\geq 1} p_n(1 - p_n)$ is divergent is easily obtained for $p_n = 1/n$. Applying the weak convergence condition for

$$\mu_n = \mathbf{P} \circ \left(\frac{X_{(n)} - \sum_{i=1}^{i=n} p_i}{\sqrt{\sum_{i=1}^{i=n} p_i(1-p_i)}} \right)^{\leftarrow} \text{ and } \mu = \mathbf{N}(0,1) \tag{50}$$

leads to the next result indicated in [35].

Theorem 4. *Let $\varphi : [a,b] \to \mathbb{R}$ be Riemann integrable and $\{p_n\}_{n\geq 1}$ a sequence of real numbers such that $0 \leq p_n \leq 1$ and the series $\sum_{n\geq 1} p_n(1-p_n)$ is divergent.*

Then

$$\int_a^b \varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} a_n, \tag{51}$$

$$a_n = \sum_{k=0}^{k=n} \widetilde{P}_n^k \sqrt{2\pi}\,\varphi_1\left(\frac{k-\sum_{i=1}^{i=n} p_i}{\sqrt{\sum_{i=1}^{i=n} p_i(1-p_i)}}\right)\exp\left[\frac{1}{2}\left(\frac{k-\sum_{i=1}^{i=n} p_i}{\sqrt{\sum_{i=1}^{i=n} p_i(1-p_i)}}\right)^2\right],$$

and $\varphi_1 : \mathbb{R} \to \mathbb{R}$, *where* $\varphi_1(x) = \varphi(x)$ *if* $x \in [a,b]$ *and* $\varphi_1(x) = 0$ *otherwise.*

When $p_i \equiv p$ for all i, one obtains the binomial distribution case and the approximation scheme is then rewritten as

$$\int_a^b \varphi(x)\,\mathrm{d}x = \lim_{n\to\infty} a_n \tag{52}$$

$$a_n = \sum_{k=0}^{k=n} P_n^k \sqrt{2\pi}\,\varphi_1\left(\frac{k-np}{\sqrt{np(1-p)}}\right)\exp\left[\frac{1}{2}\left(\frac{k-np}{\sqrt{np(1-p)}}\right)^2\right]$$

where $P_n^k = \binom{n}{k} p^k(1-p)^{n-k}$ are the well known binomial probabilities. In [36], it is illustrated why the binomial approximation algorithm is still valid for unbounded payoffs that satisfy some mild growth constraints.

Similarly, another approximation scheme can be proposed for the Poisson generalized binomial distribution. Recalling that $\widetilde{P}_n^k$ is the coefficient of t^k in the polynomial $(p_1 t + q_1)(p_2 t + q_2)\ldots(p_n t + q_n)$ we have

Theorem 5. *The price at time zero of an European option with payoff* $\Pi(S_T)$ *at maturity* T, *contingent on an underlying asset* $\{S_t\}_{0\le t\le T}$ *that follows a geometric Brownian motion with parameters* μ *and* σ, *can be calculated as*

$$\mathrm{e}^{-rT} \lim_{n\to\infty} \sum_{k=0}^{k=n} \widetilde{P}_n^k \Pi\left[S_0 \exp\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\frac{k-\sum_{i=1}^{i=n} p_i}{\sqrt{\sum_{i=1}^{i=n} p_i q_i}}\right)\right] \tag{53}$$

Approximation with Negative Binomial Kernel

One can adapt the method discussed above to discrete distributions that take an infinity of possible values. Let $\{X_n\}_{n\ge 1}$ be a sequence of iid random variables, each variable taking only the values $1, 2, \ldots, k, \ldots$ with probabilities

$$\mathbf{P}([\omega : X_j(\omega) = k]) = p(1-p)^{k-1}. \tag{54}$$

Then, the random variable $X^{(n)} = X_1 + \ldots + X_n$ has the negative binomial distribution with parameters $-n$ and p, where $0 < p < 1$ and n obviously positive integer. Applying the Central Limit Theorem leads to

$$\mathbf{P} \circ \left(\frac{X_{(n)} - \frac{n}{p}}{\sqrt{n \frac{1-p}{p^2}}} \right)^{\leftarrow} \longrightarrow \mathbf{N}(0,1) \tag{55}$$

Following a similar route with the binomial case, we get the next approximation scheme

Theorem 6. *The price at time zero of a European option with payoff $\Pi(S_T)$ at maturity T, contingent on an underlying asset $\{S_t\}_{0 \le t \le T}$ that follows a geometric Brownian motion with parameters μ and σ, can be calculated as*

$$\mathrm{e}^{-rT} \lim_{n\to\infty} \sum_{k \ge n} \binom{k-1}{n-1} p^n (1-p)^{k-n} \Pi \left[S_0 \exp\left(\left(r - \frac{\sigma^2}{2} \right) T + \sigma\sqrt{T} \frac{kp-n}{\sqrt{n(1-p)}} \right) \right] \tag{56}$$

In [36], it is proved that the univariate approximation grids obtained in this manner are dense sets in the domain of the target integral for calculation. This is an important property, since it guarantees a uniform covering of the domain of integration, similar to low-discrepancy sequences, that proved their superiority to Monte Carlo methods.

6 Examples of Applications

Asian Options

Here, we consider the pricing of Asian options with discrete fixings at times $t_1, t_2, \ldots, t_d$. The underlying is only one asset described by a geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t.$$

The payoff of an Asian call options with strike K is

$$\max \left[\frac{1}{d} \sum_{i=1}^{i=d} S_{t_i} - K, 0 \right] \tag{57}$$

and for a put is

$$\max \left[K - \frac{1}{d} \sum_{i=1}^{i=d} S_{t_i}, 0 \right]. \tag{58}$$

The formula (44) can be used here if we make the identification

$$S_T^1 \doteq S_{t_1}, S_T^2 \doteq S_{t_2}, \ldots, S_T^d \doteq S_{t_d}. \tag{59}$$

In other words, the value of the stock at the grid pre-specified time t_i is viewed as the value of one asset at maturity T. The correlation between the components of the basket is the path correlation of *one* Brownian motion governing the dynamics of the underlying asset referencing the Asian option. The payoff for the arithmetic Asian put can be written as

$$g(S_T^1, \ldots, S_T^d) = \max\left[K - \frac{1}{d}\sum_{i=1}^{i=d} S_T^i, 0\right] \tag{60}$$

while for the geometric Asian put would be

$$g(S_T^1, \ldots, S_T^d) = \max\left[K - \sqrt[d]{\prod_{i=1}^{d} S_T^i}, 0\right]. \tag{61}$$

Application to Pricing Spread Options

Spread options can be viewed as particular cases of basket options that have the general payoff at maturity

$$\max\left(\sum_{i=1}^{d} \alpha_i S_i - K_{\text{basket}}\right)$$

where K_{basket} is the strike price defined in terms of the aggregate value of the basket and the coefficients α could be proper weights for standard basket contracts or simply taking $+1$ and -1 values for spread options when $d = 2$.

Options on the Maximum or the Minimum of Risky Assets

The most common case is that, of two assets, the case of a large number of assets is not tractable analytically and not very used. The payoff for options on the maximum (or the minimum) of two risky assets is given by

$$\max[\alpha(\beta \min(\beta S_T^1, \beta S_T^2) - K), 0] \tag{62}$$

where $\alpha = 1$ for a call, $\alpha = -1$ for a put and the option if on the maximum for $\beta = -1$ and on the minimum for $\beta = +1$. Also, S_T^1, S_T^2 are the two assets at maturity T and K is the exercise price.

We consider that each asset follows a geometric Brownian motion

$$\mathrm{d}S_t^i = rS_t^i \mathrm{d}t + \sigma_i S_t^i \mathrm{d}W_t^i$$

where $i = 1, 2$ and $W_t^1, \ldots, W_t^d$ are correlated Wiener processes with $\mathcal{E}(\mathrm{d}W_t^1, \mathrm{d}W_t^2) = \rho \mathrm{d}t$. Here, we should price a put on the maximum so the payoff is

$$g(S_T^1, S_T^2) = \max[\min(-S_T^1, -S_T^2) + K), 0] = \max[K - \max(S_T^1, S_T^2), 0]. \quad (63)$$

What we need to calculate is

$$\mathcal{E}[g(Y)] = \frac{1}{\sqrt{(2\pi)^d \det(\Upsilon)}} \int_{\mathbb{R}^d} g(y) \exp\left[-\frac{1}{2} y^\top \Upsilon^{-1} y\right] \mathrm{d}y.$$

Taking $\sigma_1 = 0.25, \sigma_2 = 0.4$, and correlation coefficient 0.10, the risk-free rate 10%, exercise price $K = 110$ and initial stock prices $S_0^1 = 100$ and $S_0^2 = 120$ and applying formula (44), we get the price equal to 2.071, which compares well with the Monte Carlo price of 2.067.

With the payoff g in (63), the calculation algorithm goes through the following steps

1. Determine the Cholesky square root matrix Δ of covariance matrix Υ.
2. Initialise the vector of probabilities (p_1, p_2, p_3), for our example $p_1 = p_2 = p_3 = \frac{1}{3}$.
3. Calculate the matrix M from formula (30).
4. Initialise n.
5. Calculate the series on the right side of (46), a simplified and more direct version of (44).
6. The calculations are done for increasing n until two consecutive calculations are close to a pre-specified tolerance level.

Pricing Basket Options

Basket options are very popular instruments. The payoff that applies in this case is specified as

$$g(S_T^1, S_T^2, \ldots, S_T^d) = N \times \max\left[K - \sum_{i=1}^{d} w_i \frac{S_T^i}{S_0^i}\right] \quad (64)$$

and it is evident that this case also falls under the methodology presented above in this chapter.

7 Conclusions and Further Research

Numerical methods are less developed for multivariate applications due to computational problems with increased dimensionality. At the same time, it is desirable to have general methods able to cover a wide spectrum of applied problems in economics, finance and other sciences. The algorithms presented here clearly satisfy the second objective. Moreover, due to advances in computer science related to parallel computation and multi-thread technology, one can envisage the future developments in this area, particularly for situations where similar calculations are required frequently in a multi-dimensional set-up. Investment banks having exotic options trading desks are the perfect example. The portfolio of products is usually large, hence the dimensionality problem. Given that the products that make the portfolio are not taken in and out of the portfolio very often, considerable computation savings can be made by saving the probability weights associated with the approximation grid suggested here.

An important line of further research is improving the speed of convergence and designing measures to ensure convergence has been reached. Currently, this can be done on a case by case basis where one can take advantage of the characteristics of the specific payoff.

While the discussion here has been focused on using the gaussian distribution as the limiting distribution, other cases may appear and are possible. For example, the χ^2 distribution is readily available after applying the delta-continuity theorem and a suitable transformation to a central limit result. Hence, a connection can be made in this way, if required, to non-gaussian models.

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Credit Risk, Market Sentiment and Randomly-Timed Default

Dorje C. Brody, Lane P. Hughston, and Andrea Macrina

Abstract We propose a model for the credit markets in which the random default times of bonds are assumed to be given as functions of one or more independent "market factors". Market participants are assumed to have partial information about each of the market factors, represented by the values of a set of market factor information processes. The market filtration is taken to be generated jointly by the various information processes and by the default indicator processes of the various bonds. The value of a discount bond is obtained by taking the discounted expectation of the value of the default indicator function at the maturity of the bond, conditional on the information provided by the market filtration. Explicit expressions are derived for the bond price processes and the associated default hazard rates. The latter are not given *a priori* as part of the model but rather are deduced and shown to be functions of the values of the information processes. Thus the "perceived" hazard rates, based on the available information, determine bond prices, and as perceptions change so do the prices. In conclusion, explicit expressions are derived for options on discount bonds, the values of which also fluctuate in line with the vicissitudes of market sentiment.

Keywords Information-based asset pricing · Credit derivatives · Random times

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1 Credit-Risk Modeling

In this paper, we consider a simple model for defaultable securities and, more generally, for a class of financial instruments for which the cash flows depend on the default times of a set of defaultable securities. For example, if τ is the time of default of a firm that has issued a simple credit-risky zero-coupon bond that matures at time T, then the bond delivers a single cash-flow H_T at T, given by

$$H_T = N\,\mathbb{1}\{\tau > T\}, \tag{1}$$

where N is the principal, and $\mathbb{1}\{\tau > T\} = 1$ if $\tau > T$ and $\mathbb{1}\{\tau > T\} = 0$ if $\tau \leq T$. By a "simple" credit-risky zero-coupon bond we mean the case of an idealised bond where there is no recovery of the principal if the firm defaults. If a fixed fraction R of the principal is paid at time T in the event of default, then we have

$$H_T = N\mathbb{1}\{\tau > T\} + RN\mathbb{1}\{\tau \leq T\}. \tag{2}$$

More realistic models can be developed by introducing random factors that determine the amount and timing of recovery levels. See, e.g., Brody, Hughston and Macrina [7, 10], and Macrina and Parbhoo [24].

As another example, let $\tau_1, \tau_2 \ldots, \tau_n$ denote the default times of a set of n discount bonds, each with maturity after T. Write $\bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_n$ for the "order statistics" of the default times. Hence $\bar{\tau}_1$ is the time of the first default (among $\tau_1, \tau_2, \ldots, \tau_n$), $\bar{\tau}_2$ is the time of the second default, and so on. Then a structured product that pays

$$H_T = K\,\mathbb{1}\{\bar{\tau}_k \leq T\} \tag{3}$$

is a kind of insurance policy that pays K at time T if there have been k or more defaults by time T.

One of the outstanding problems associated with credit-risk modeling is the following. What counts in the valuation of credit-risky products is not necessarily the "actual" or "objective" probability of default (even if this can be meaningfully determined), but rather the "perceived" probability of default. This can change over time, depending on shifts in market sentiment and the flow of relevant market information. How do we go about modeling the dynamics of such products?

2 Modeling the Market Filtration

We introduce a probability space with a measure $\mathbb{Q}$ which, for simplicity, we take to be the risk-neutral measure. Thus, the price process of any non-dividend-paying asset, when the price is expressed in units of a standard money-market account, is a $\mathbb{Q}$-martingale. We do not assume that the market is necessarily complete; rather, we merely assume that there is an established pricing kernel. We assume, again for simplicity, that the default-free interest-rate term structure is deterministic. Time 0 denotes the present, and we write P_{tT} for the price at t of a default-free discount bond that matures at T. To ensure absence of arbitrage, we require that $P_{tT} = P_{0T}/P_{0t}$, where $\{P_{0t}\}_{0\leq t<\infty}$ is the initial term structure. No attempt will be made in the present investigation to examine the case of stochastic interest rates: to keep credit-related issues in the foreground, we suppress considerations relating to the default-free interest rate term structure. See Rutkowski and Yu [25], Hughston and Macrina [20], Brody and Friedman [4], and Macrina and Parbhoo [24] for discussions of the stochastic interest rate case in an information-based setting.

The probability space comes equipped with a filtration $\{\mathcal{G}_t\}$ which we take to be the market filtration. Our first objective is to define $\{\mathcal{G}_t\}$ in such a way that market sentiments concerning the default times can be modeled explicitly. We let $\tau_1, \tau_2, \ldots, \tau_n$ be a collection of non-negative random times such that $\mathbb{Q}(\tau_\alpha = 0) = 0$ and $\mathbb{Q}(\tau_\alpha > t) > 0$ for $t > 0$ and $\alpha = 1, 2, \ldots, n$. We set

$$\tau_\alpha = f_\alpha(X_1, X_2, \ldots, X_N), \qquad (\alpha = 1, 2, \ldots, n). \tag{4}$$

Here $X_1, X_2, \ldots, X_N$ are N independent, continuous, real-valued *market factors* that determine the default times, and f_α for each α is a smooth function of N variables that determines the dependence of τ_α on the market factors. We note that if two default times share an X-factor in common, then they will in general be correlated. With each τ_α, we associate a "survival" indicator process $\mathbb{1}\{\tau_\alpha > t\}$, $t \geq 0$, which takes the value unity until default occurs, at which time it drops to zero. Additionally, we introduce a set of N information processes $\{\xi_t^k\}_{t\geq 0}$ in association with the market factors, which in the present investigation we take to be of the form

$$\xi_t^k = \sigma_k t X_k + B_t^k. \tag{5}$$

Here, for each k, σ_k is a parameter ("information flow rate") and $\{B_t^k\}_{t\geq 0}$ is a Brownian motion ("market noise"). We assume that the X-factors and the market noise processes are all independent of one another.

We take the market filtration $\{\mathcal{G}_t\}_{t\geq 0}$ to be generated jointly by the information processes and the survival indicator processes. Therefore, we have:

$$\mathcal{G}_t = \boldsymbol{\sigma}\left[\{\xi_s^k\}_{0\leq s\leq t}^{k=1,\ldots,N}, \mathbb{1}\{\tau^\alpha > s\}_{0\leq s\leq t}^{\alpha=1,\ldots,n}\right]. \tag{6}$$

It follows that at t the market knows the information generated up to t and the history of the indicator processes up to time t. For the purpose of calculations, it is useful also to introduce the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the information processes:

$$\mathcal{F}_t = \boldsymbol{\sigma}\left[\{\xi_s^k\}_{0\leq s\leq t}^{k=1,\ldots,N}\right]. \tag{7}$$

Then clearly $\mathcal{F}_t \subset \mathcal{G}_t$. We do not require as such the notion of a "background" filtration for our theory. Indeed, we can think of the ξ's and the $\mathbb{1}$'s as providing two related but different types of information about the X's.

3 Credit-Risky Discount Bond

As an example, we study in more detail the case $n = 1$, $N = 1$. We have a single random market factor X and an associated default time $\tau = f(X)$. We assume that X is continuous and that $f(X)$ is monotonic. The market filtration $\{\mathcal{G}_t\}$ is generated jointly by an information process of the form

$$\xi_t = \sigma t X + B_t, \qquad t \geq 0, \tag{8}$$

and the indicator process $\mathbb{1}\{\tau > t\}$, $t \geq 0$. The Brownian motion $\{B_t\}_{t>0}$ is taken to be independent of X. The value of a defaultable T-maturity discount bond, when there is no recovery on default, is then given by

$$B_{tT} = P_{tT}\,\mathbb{E}\left[\mathbb{1}\{\tau > T\}\,|\,\mathcal{G}_t\right] \tag{9}$$

for $0 \leq t \leq T$. We shall write out an explicit expression for B_{tT}. First, we use the identity (see, e.g., Bielecki, Jeanblanc and Rutkowski [1]):

$$\mathbb{E}\left[\mathbb{1}\{\tau > T\}\,|\,\mathcal{G}_t\right] = \mathbb{1}\{\tau > t\}\frac{\mathbb{E}\left[\mathbb{1}\{\tau > T\}|\mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{\tau > t\}|\mathcal{F}_t\right]}, \tag{10}$$

where $\mathcal{F}_t$ is as defined in (7). It follows that

$$B_{tT} = P_{tT}\mathbb{1}\{\tau > t\}\frac{\mathbb{E}\left[\mathbb{1}\{\tau > T\}|\mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{\tau > t\}|\mathcal{F}_t\right]}. \tag{11}$$

We note that the information process $\{\xi_t\}$ has the Markov property with respect to its own filtration. To see this, it suffices to check that

$$\mathbb{Q}\left(\xi_t \leq x|\xi_s, \xi_{s_1}, \xi_{s_2}, \ldots, \xi_{s_k}\right) = \mathbb{Q}\left(\xi_t \leq x|\xi_s\right) \tag{12}$$

for any collection of times $t \geq s \geq s_1 \geq s_2 \geq \cdots \geq s_k > 0$. We observe that for $s > s_1 > s_2 > s_3 > 0$, the random variables $\{B_s/s - B_{s_1}/s_1\}$ and $\{B_{s_2}/s_2 - B_{s_3}/s_3\}$ are independent. This follows directly from a calculation of their covariance. Hence from the relation

$$\frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1} = \frac{B_s}{s} - \frac{B_{s_1}}{s_1} \tag{13}$$

we conclude that

$$\begin{aligned}\mathbb{Q}(\xi_t \leq x|\,\xi_s, \xi_{s_1}, \ldots, \xi_{s_k}) &= \mathbb{Q}\left(\xi_t \leq x\,\middle|\,\xi_s, \frac{\xi_s}{s} - \frac{\xi_{s_1}}{s_1}, \frac{\xi_{s_1}}{s_1} - \frac{\xi_{s_2}}{s_2}, \ldots, \frac{\xi_{s_{k-1}}}{s_{k-1}} - \frac{\xi_{s_k}}{s_k}\right)\\ &= \mathbb{Q}\left(\xi_t \leq x\,\middle|\,\xi_s, \frac{B_s}{s} - \frac{B_{s_1}}{s_1}, \frac{B_{s_1}}{s_1} - \frac{B_{s_2}}{s_2}, \ldots, \frac{B_{s_{k-1}}}{s_{k-1}} - \frac{B_{s_k}}{s_k}\right).\end{aligned} \tag{14}$$

However, since ξ_t and ξ_s are independent of $B_s/s - B_{s_1}/s_1$, $B_{s_1}/s_1 - B_{s_2}/s_2, \ldots,$ $B_{s_{k-1}}/s_{k-1} - B_{s_k}/s_k$, the result (12) follows.

As a consequence of the Markovian property of $\{\xi_t\}$ and the fact that X is $\mathcal{F}_\infty$-measurable, we therefore obtain

$$B_{tT} = P_{tT}\mathbb{1}\{\tau > t\} \frac{\mathbb{E}\left[\mathbb{1}\{\tau > T\} \,|\, \xi_t\right]}{\mathbb{E}\left[\mathbb{1}\{\tau > t\} \,|\, \xi_t\right]} \tag{15}$$

for the defaultable bond price. Thus we can write

$$B_{tT} = P_{tT}\,\mathbb{1}\{\tau > t\} \frac{\int_{-\infty}^{\infty} \mathbb{1}\{f(x) > T\}\rho_t(x)\mathrm{d}x}{\int_{-\infty}^{\infty} \mathbb{1}\{f(x) > t\}\rho_t(x)\mathrm{d}x}. \tag{16}$$

Here

$$\rho_t(x) = \mathbb{E}\left[\delta(X - x) \,|\, \xi_t\right] \tag{17}$$

is the conditional density for X given ξ_t, and a calculation using the Bayes law shows that:

$$\rho_t(x) = \frac{\rho_0(x) \exp\left[\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right]}{\int_{-\infty}^{\infty} \rho_0(x) \exp\left[\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right] \mathrm{d}x}, \tag{18}$$

where $\rho_0(x)$ is the *a priori* density of X. Thus for the bond price we obtain:

$$B_{tT} = P_{tT}\mathbb{1}\{\tau > t\} \frac{\int_{-\infty}^{\infty} \rho_0(x)\mathbb{1}\{f(x) > T\} \exp\left[\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right] \mathrm{d}x}{\int_{-\infty}^{\infty} \rho_0(x)\mathbb{1}\{f(x) > t\} \exp\left[\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t\right] \mathrm{d}x}. \tag{19}$$

It should be apparent that the value of the bond fluctuates as ξ_t changes. This reflects the effects of changes in market sentiment concerning the possibility of default. Indeed, if we regard τ as the default time of an obligor that has issued a number of different bonds (coupon bonds can be regarded as bundles of zero-coupon bonds), then similar formulae will apply for each of the various bond issues.

When the default times of two or more distinct obligors depend on a common market factor, the resulting bond price dynamics are correlated and so are the default times. The modelling framework presented here therefore provides a basis for a number of new constructions in credit-risk management, including explicit expressions for the volatilities and correlations of credit-risky bonds.

Our methodology is to consider models for which each independent X-factor has its own information process. Certainly, we can also consider the situation where there may be two or more distinct information processes available concerning the same X-factor. This situation is relevant to models with asymmetric information, where some traders may have access to "more" information about a given market factor than other traders; see Brody, Davis, Friedman and Hughston [3], Brody, Brody, Meister and Parry [2], and Brody, Hughston and Macrina [10].

In principle, a variety of different types of information processes can be considered. We have in (5) used for simplicity what is perhaps the most elementary type of information process, a Brownian motion with a random drift. The linearity of the drift term in the time variable ensures on the one hand that the information process has the Markov property, and on the other hand, since the Brownian term grows in magnitude on average like the square-root of time, that the drift term eventually comes to dominate the noise term, thus allowing for the release of information concerning the likely time of default. Information processes based variously on the Brownian bridge (Brody, Hughston and Macrina [7, 8]), the gamma bridge (Brody, Hughston and Macrina [9]), and the Lévy random bridge (Hoyle, Hughston and Macrina [18, 19]) have been applied to problems in finance and insurance.

Our work can be viewed in the context of the growing body of literature on the role of information in finance and its application to credit risk modeling in particular. No attempt will be made here at a systematic survey of material in this line. We refer the reader, for example, to Kusuoka [22], Föllmer, Wu and Yor [14], Duffie and Lando [13], Jarrow and Protter [21], Çetin, Jarrow, Protter and Yildirim [11], Giesecke [17], Geman, Madan and Yor [16], Coculescu, Geman and Jeanblanc [12], Frey and Schmidt [15], Bielecki, Jeanblanc and Rutkowski [1], and works cited therein.

4 Discount Bond Dynamics

Further insight into the nature of the model can be gained by working out the dynamics of the bond price. An application of Ito calculus gives the following dynamics over the time interval from 0 to T:

$$\mathrm{d}B_{tT} = (r_t + h_t)B_{tT}\mathrm{d}t + \sigma\Sigma_{tT}\, B_{tT}\mathrm{d}W_t + B_{t^-T}\mathrm{d}\mathbb{1}\{\tau > t\}. \tag{20}$$

Here

$$r_t = -\partial_t \ln(P_{0t}) \tag{21}$$

is the deterministic short rate of interest, and

$$h_t = \frac{\mathbb{E}\left[\delta(f(X)-t)\,|\,\mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X)>t\}\,|\,\mathcal{F}_t\right]} \tag{22}$$

is the so-called hazard rate. It should be evident that if $\tau \leq T$, then when the default time is reached the bond price drops to zero. The defaultable discount bond volatility Σ_{tT} is given by

$$\Sigma_{tT} = \frac{\mathbb{E}\left[\mathbb{1}\{f(X)>T\}X\,|\,\mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X)>T\}\,|\,\mathcal{F}_t\right]} - \frac{\mathbb{E}\left[\mathbb{1}\{f(X)>t\}X\,|\,\mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X)>t\}\,|\,\mathcal{F}_t\right]}. \tag{23}$$

The process $\{W_t\}$ appearing in the dynamics of $\{B_{tT}\}$ is defined by the relation:

$$W_t = \int_0^t \mathbb{1}\{f(X) > s\} \left(\mathrm{d}\xi_s - \sigma \mathbb{E}\left[X | \mathcal{G}_s\right] \mathrm{d}s \right). \tag{24}$$

To deduce the formulae above, we define a one-parameter family of $\{\mathcal{F}_t\}$-adapted processes $\{F_{tu}\}$ by setting

$$F_{tu} = \int_{-\infty}^{\infty} \rho_0(x) \mathbb{1}\{f(x) > u\} \exp\left[\sigma x \xi_t - \tfrac{1}{2}\sigma^2 x^2 t\right] \mathrm{d}x. \tag{25}$$

Then the bond price can be written in the form

$$B_{tT} = P_{tT} \mathbb{1}\{\tau > t\} \frac{F_{tT}}{F_{tt}}. \tag{26}$$

An application of Ito's lemma gives

$$\frac{\mathrm{d}F_{tT}}{F_{tT}} = \sigma \frac{\mathbb{E}\left[\mathbb{1}\{f(X) > T\} X \,|\, \mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X) > T\} \,|\, \mathcal{F}_t\right]} \mathrm{d}\xi_t \tag{27}$$

and

$$\frac{\mathrm{d}F_{tt}}{F_{tt}} = -\frac{\mathbb{E}\left[\delta(f(X) - t) \,|\, \mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X) > t\} \,|\, \mathcal{F}_t\right]} \mathrm{d}t + \sigma \frac{\mathbb{E}\left[\mathbb{1}\{f(X) > t\} X \,|\, \mathcal{F}_t\right]}{\mathbb{E}\left[\mathbb{1}\{f(X) > t\} \,|\, \mathcal{F}_t\right]} \mathrm{d}\xi_t. \tag{28}$$

The desired results then follow by use of the relation

$$\mathrm{d}\left(\frac{F_{tT}}{F_{tt}}\right) = \frac{F_{tT}}{F_{tt}} \left[\frac{\mathrm{d}F_{tT}}{F_{tT}} - \frac{\mathrm{d}F_{tt}}{F_{tt}} + \left(\frac{\mathrm{d}F_{tt}}{F_{tt}}\right)^2 - \frac{\mathrm{d}F_{tT}}{F_{tT}} \frac{\mathrm{d}F_{tt}}{F_{tt}} \right]. \tag{29}$$

The process $\{W_t\}_{0 \le t < \tau}$ defined by (24) is a $\{\mathcal{G}_t\}$-Brownian motion. This can be seen by use of Lévy's characterisation. Specifically, we have $\mathrm{d}W_t^2 = \mathrm{d}t$ and $\mathbb{E}[W_u \,|\, \mathcal{G}_t] = W_t$. To see that $\{W_t\}_{0 \le t < \tau}$ is a $\{\mathcal{G}_t\}$-martingale, we observe that

$$\begin{aligned} W_u &= \int_0^u \mathbb{1}\{\tau > s\} \left(\mathrm{d}\xi_s - \sigma \mathbb{E}\left[X \,|\, \mathcal{G}_s\right] \mathrm{d}s \right), \\ &= W_t + \int_t^u \mathbb{1}\{\tau > s\} \left(\mathrm{d}\xi_s - \sigma \mathbb{E}\left[X \,|\, \mathcal{G}_s\right] \mathrm{d}s \right), \end{aligned} \tag{30}$$

and hence

$$\mathbb{E}\left[W_u \,|\, \mathcal{G}_t\right] = W_t + \mathbb{E}\left[\int_t^u \mathbb{1} \left(\mathrm{d}\xi_s - \sigma \mathbb{E}[X \,|\, \mathcal{G}_s] \mathrm{d}s \right) \middle| \, \mathcal{G}_t \right]. \tag{31}$$

Then by inserting $\mathrm{d}\xi_s = \sigma X \mathrm{d}s + \mathrm{d}B_s$ and using the tower property, we find that the terms involving X cancel and we are left with

$$\begin{aligned}\mathbb{E}\left[W_u \,|\, \mathcal{G}_t\right] &= W_t + \mathbb{E}\left[\int_t^u \mathbb{1}\{\tau > s\}\mathrm{d}B_s \,\middle|\, \mathcal{G}_t\right], \\ &= W_t + \mathbb{E}\left[\mathbb{E}\left[\int_t^u \mathbb{1}\{\tau > s\}\mathrm{d}B_s \,\middle|\, \sigma\left\{X, \mathcal{G}_t\right\}\right] \,\middle|\, \mathcal{G}_t\right], \\ &= W_t. \end{aligned} \tag{32}$$

The Brownian motion that "drives" the defaultable bond is not adapted to a pre-specified background filtration. Rather, it is directly associated with information about the factors determining default. In this respect, the information-based approach is closer in spirit to a structural model, even though it retains the economy of a reduced-form model.

5 Hazard Rates and Forward Hazard Rates

Let us now examine more closely properties of the intensity process $\{h_t\}$ given by the expression (22). We remark first that the intensity at time t is a function of ξ_t. This shows that in the present model, the default intensity is determined by "market perceptions." Our model can thus be characterised as follows:

> The market does not know the "true" default intensity; rather, from the information available to the market a kind of "best estimate" for the default intensity is used for the pricing of bonds—but the market is "aware" of the fact that this estimate is based on perceptions, and hence as the perceptions change, so will the estimate, and so will the bond prices. Thus, in the present model (and unlike the majority of credit models hitherto proposed), there is no need for "fundamental changes" in the state of the obligor, or the underlying economic environment, as the basis for improvement or deterioration of credit quality: it suffices simply that the information about the credit quality should change—whether or not this information is actually representative of the true state of affairs.

In the present example, we can obtain a more explicit expression for the intensity by transforming variables as follows. Since f is invertible, we can introduce the inverse function $\phi(\tau) = f^{-1}(\tau) = X$ and write

$$\xi_t = \sigma t \phi(\tau) + B_t \tag{33}$$

for the information process. As before, we assume that the Brownian motion $\{B_t\}_{t\geq 0}$ is independent of the default time τ. Writing $p(u)$ for the *a priori* density of the random variable $\tau = f(X)$, we deduce from (22) that the hazard process is given by the expression

$$h_t = \frac{p(t)\,\mathrm{e}^{\sigma\phi(t)\xi_t - \frac{1}{2}\sigma^2\phi^2(t)t}}{\int_t^\infty p(u)\,\mathrm{e}^{\sigma\phi(u)\xi_t - \frac{1}{2}\sigma^2\phi^2(u)t}\,\mathrm{d}u}. \tag{34}$$

This expression manifestly reminds us the fact that $\{h_t\}$ is a function of the information ξ_t, and thus its value moves up and down according to the market perception of the timing of default.

In the present context, it is also natural to consider the forward hazard rate defined by the expression

$$h_{tu} = \frac{\mathbb{E}\,[\delta(f(X)-u)\,|\,\mathcal{F}_t]}{\mathbb{E}\,[\mathbb{1}\{f(X)>u\}\,|\,\mathcal{F}_t]}. \tag{35}$$

We observe that $h_{tu}\mathrm{d}u$ represents the *a posteriori* probability of default in the infinitesimal interval $[u, u+\mathrm{d}u]$, conditional on no default until time u. More explicitly, we have

$$h_{tu} = \frac{p(u)\,\mathrm{e}^{\sigma\phi(u)\xi_t - \frac{1}{2}\sigma^2\phi^2(u)t}}{\int_u^\infty p(v)\,\mathrm{e}^{\sigma\phi(v)\xi_t - \frac{1}{2}\sigma^2\phi^2(v)t}\,\mathrm{d}v} \tag{36}$$

for the forward hazard rate.

It is a straightforward matter to simulate the dynamics of the bond price and the associated hazard rates by Monte Carlo methods. First we simulate a value for X by use of the *a priori* density $\rho_0(x)$. From this, we deduce the corresponding value of τ. Then we simulate an independent Brownian motion $\{B_t\}$, and thereby also the information process $\{\xi_t\}$. Putting these ingredients together we have a simulation for the bond price and the associated hazard rate. In Fig. 1, we sketch several sample paths resulting from such a simulation study.

6 Options on Defaultable Bonds

We consider the problem of pricing an option on a defaultable bond with bond maturity T. Let K be the option strike price and let t $(< T)$ be the option maturity. The payoff of the option is then $(B_{tT} - K)^+$. Let us write the bond price in the form

$$B_{tT} = \mathbb{1}\{\tau > t\}B(t, \xi_t), \tag{37}$$

where the function $B(t, y)$ is defined by

$$B(t,y) = P_{tT}\,\frac{\int_T^\infty p(u)\exp\left[\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t\right]\mathrm{d}u}{\int_t^\infty p(u)\exp\left[\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t\right]\mathrm{d}u}. \tag{38}$$

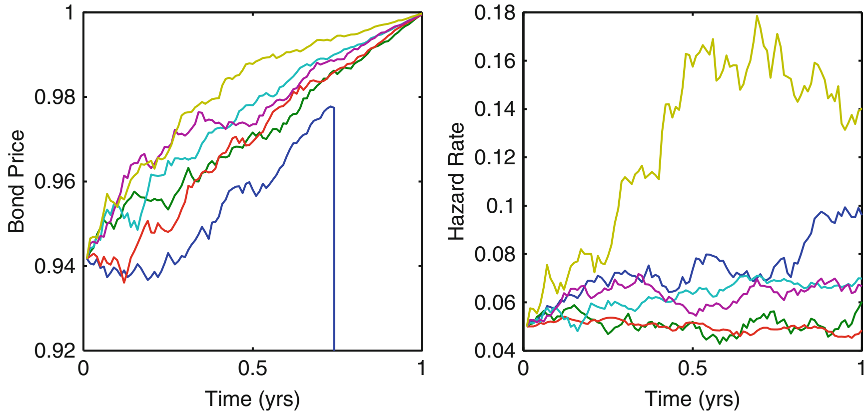

Fig. 1 Sample paths of the defaultable discount bond and the associated hazard rate. We choose $\phi(t) = \mathrm{e}^{-0.025t}$, and the initial term structure is assumed to be flat with $P_{0t} = \mathrm{e}^{-0.02t}$. The information flow rate parameter is set to be $\sigma = 0.3$, and the bond maturity is one year

We make note of the identity

$$\big(\mathbb{1}\{\tau > t\}B(t,\xi_t) - K\big)^+ = \mathbb{1}\{\tau > t\}\big(B(t,\xi_t) - K\big)^+, \tag{39}$$

satisfied by the option payoff. The price of the option is thus given by

$$C_0 = P_{0t}\,\mathbb{E}\Big[\mathbb{1}\{\tau > t\}\big(B(t,\xi_t) - K\big)^+\Big]. \tag{40}$$

We find, in particular, that the option payoff is a function of the random variables τ and ξ_t. To determine the expectation (40), we need to work out the joint density of τ and ξ_t, defined by

$$\rho(u,y) = \mathbb{E}\left[\delta(\tau - u)\delta(\xi_t - y)\right] = -\frac{\mathrm{d}}{\mathrm{d}u}\,\mathbb{E}\left[\mathbb{1}\{\tau > u\}\delta(\xi_t - y)\right]. \tag{41}$$

Note that the expression

$$A_0(u,y) = P_{0t}\,\mathbb{E}\left[\mathbb{1}\{\tau > u\}\delta(\xi_t - y)\right] \tag{42}$$

appearing on the right side of (41), with an additional discounting factor, can be regarded as representing the price of a "defaultable Arrow-Debreu security" based on the value at time t of the information process.

To work out the expectation appearing here we shall use the Fourier representation for the delta function:

$$\delta(\xi_t - y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\mathrm{i}y\lambda + \mathrm{i}\xi_t\lambda\right)\mathrm{d}\lambda, \tag{43}$$

which of course has to be interpreted in an appropriate way with respect to integration against a class of test functions. We have

$$\mathbb{E}\left[\mathbb{1}\{\tau > u\}\delta(\xi_t - y)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}y\lambda}\mathbb{E}\left[\mathbb{1}\{\tau > u\}\mathrm{e}^{\mathrm{i}\xi_t\lambda}\right]\mathrm{d}\lambda. \tag{44}$$

The expectation appearing in the integrand is given by

$$\mathbb{E}\left[\mathbb{1}\{\tau > u\}\mathrm{e}^{\mathrm{i}\xi_t\lambda}\right] = \int_{-\infty}^{\infty} \mathbb{1}\{x > u\}\, p(x) \exp\left(\mathrm{i}\sigma\lambda t\phi(x) - \tfrac{1}{2}\lambda^2 t\right) \mathrm{d}x, \tag{45}$$

where we have made use of the fact that the random variables τ and B_t appearing in the definition of the information process (33) are independent. We insert this intermediate result in (44) and rearrange terms to obtain

$$\mathbb{E}\left[\mathbb{1}\{\tau > u\}\delta(\xi_t - y)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{1}\{x > u\}\, p(x)\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}y\lambda+\mathrm{i}\sigma\lambda t\phi(x)-\frac{1}{2}\lambda^2 t}\mathrm{d}\lambda\mathrm{d}x. \tag{46}$$

Performing the Gaussian integration associated with the λ variable, we deduce that the price of the defaultable Arrow–Debreu security is

$$A_0(u, y) = \frac{P_{0t}}{\sqrt{2\pi\, t}}\int_{-\infty}^{\infty} \mathbb{1}\{x > u\}p(x) \exp\left[-\frac{(\sigma t\phi(x) - y)^2}{2t}\right]\mathrm{d}x. \tag{47}$$

For the calculation of the price of a call option written on a defaultable discount bond, it is convenient to rewrite (47) in the following form:

$$A_0(y) = \frac{P_{0t}}{\sqrt{2\pi\, t}} \exp\left(-\frac{y^2}{2t}\right)\int_{u}^{\infty} p(x) \exp\left(\sigma\phi(x)y - \tfrac{1}{2}\,\sigma^2\phi^2(x)t\right)\mathrm{d}x. \tag{48}$$

It follows that for the joint density function we have

$$\rho(u, y) = \frac{1}{\sqrt{2\pi t}}\, p(u)\, \mathrm{e}^{-\frac{1}{2t}y^2}\, \mathrm{e}^{\sigma\phi(u)y-\frac{1}{2}\,\sigma^2\phi^2(u)t}. \tag{49}$$

The price of the call option can therefore be worked out as follows:

$$\begin{aligned} C_0 &= P_{0t}\int_{-\infty}^{\infty}\mathrm{d}u\int_{-\infty}^{\infty}\mathrm{d}y\, \rho(u, y)\mathbb{1}\{u > t\}(B(t, y) - K)^+ \\ &= \frac{P_{0t}}{\sqrt{2\pi t}}\int_{-\infty}^{\infty}\mathrm{d}y\, \mathrm{e}^{-\frac{1}{2t}y^2}(B(t, y)-K)^+\left[\int_{t}^{\infty} p(u)\mathrm{e}^{\sigma\phi(u)y-\frac{1}{2}\sigma^2\phi^2(u)t}\mathrm{d}u\right]. \end{aligned} \tag{50}$$

We notice that the term inside the square brackets is identical to the denominator of the expression for $B(t, y)$. Therefore, we have

$$C_0 = \frac{P_{0t}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{1}{2t}y^2} \left(P_{tT} \int_T^{\infty} p(u) \mathrm{e}^{\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t} \mathrm{d}u \right.$$
$$\left. - K \int_t^{\infty} p(u) \mathrm{e}^{\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t} \mathrm{d}u \right)^+. \quad (51)$$

An analysis similar to the one carried out in Brody and Friedman [4] shows the following result: Provided that $\phi(u)$ is a decreasing function, there exists a unique critical value y^* for y such that

$$P_{tT} \int_T^{\infty} p(u) \mathrm{e}^{\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t} \mathrm{d}u - K \int_t^{\infty} p(u) \mathrm{e}^{\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t} \mathrm{d}u > 0 \quad (52)$$

if $y < y^*$. On the other hand, if $\phi(u)$ is increasing, then there is likewise a unique critical value $y^\dagger$ of y such that for $y > y^\dagger$ we have

$$P_{tT} \int_T^{\infty} p(u) \mathrm{e}^{\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t} \mathrm{d}u - K \int_t^{\infty} p(u) \mathrm{e}^{\sigma\phi(u)y - \frac{1}{2}\sigma^2\phi^2(u)t} \mathrm{d}u > 0. \quad (53)$$

Therefore, we can perform the y-integration in (51) to obtain the value

$$C_0 = P_{0T} \int_T^{\infty} p(u)\, N\left(\frac{y^* - \sigma\phi(u)t}{\sqrt{t}} \right) \mathrm{d}u$$
$$- P_{0t} K \int_t^{\infty} p(u)\, N\left(\frac{y^* - \sigma\phi(u)t}{\sqrt{t}} \right) \mathrm{d}u \quad (54)$$

when $\phi(u)$ is decreasing. If $\phi(u)$ is increasing, we have

$$C_0 = P_{0T} \int_T^{\infty} p(u)\, N\left(\frac{\sigma\phi(u)t - y^\dagger}{\sqrt{t}} \right) \mathrm{d}u$$
$$- P_{0t} K \int_t^{\infty} p(u)\, N\left(\frac{\sigma\phi(u)t - y^\dagger}{\sqrt{t}} \right) \mathrm{d}u. \quad (55)$$

The critical values $y^*_\phi(t, T, K, \sigma)$ and $y^\dagger_\phi(t, T, K, \sigma)$ can be determined numerically to value the prices of call options. An example of the call price as a function of its strike and maturity, when $\phi(u)$ is decreasing, is shown in Fig. 2. If the function $\phi(u)$ is not monotonic, then there is in general more than one critical value of y for which the argument of the max function in (51) is positive. Therefore, in this case there will be more terms in the option-valuation formula.

The case represented by a simple discount bond is merely an example and as such cannot be taken as a completely realistic model. Nevertheless, it is interesting that modelling the information available about the default time leads to a dynamical model for the bond price, in which the Brownian fluctuations driving the price process arise in a natural way as the innovations associated with the flow of

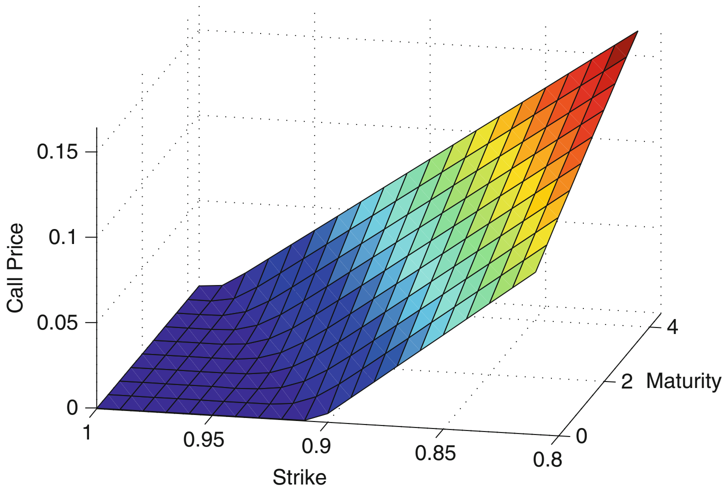

Fig. 2 Price of a call option on a defaultable discount bond. The bond maturity is five years. The information-adjusting function is set to be $\phi(t) = \mathrm{e}^{-0.05t}$, and the initial term structure is assumed to be flat so that $P_{0t} = \mathrm{e}^{-0.02t}$. The information flow rate is set to be $\sigma = 0.25$

information to the market concerning the default time. Thus no "background filtration" is required for the analysis of default in models here proposed. The information flow-rate parameter σ is not directly observable, but rather can be backed out from option-price data on an "implied" basis. The extension of the present investigation, which can be regarded as a synthesis of the "density" approach to interest-rate modeling proposed in Brody and Hughston [5,6] and the information-based asset pricing framework developed in Brody, Hughston and Macrina [7–9] and Macrina [23], to multiple asset situations with portfolio credit risk will be pursued elsewhere.

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Continuity of Mutual Entropy in the Limiting Signal-To-Noise Ratio Regimes

Mark Kelbert and Yuri Suhov

Abstract This article addresses the issue of the proof of the entropy power inequality (EPI), an important tool in the analysis of Gaussian channels of information transmission, proposed by Shannon. We analyse continuity properties of the mutual entropy of the input and output signals in an additive memoryless channel and discuss assumptions under which the entropy-power inequality holds true.

Keywords Mutual entropy · Gaussian channel · Entropy power inequality

MSC (2010): 62B10, 94A15

1 Introduction

The aim of this paper is to analyse continuity properties of mutual and conditional entropies, between the input and output of a channel with additive noise. Our attention is focused mainly on a distinctly nonGaussian situation, for both large and small signal-to-noise ratio. To our knowledge, this nontrivial aspect has not been discussed before at the level of generality adopted in this paper. A complex character of the continuity properties of various entropies was acknowledged as early as

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in the 1950s; see, e.g., paper [1] where a number of important (and elegant) results have been proven, about limiting behaviour of various entropies.

An additional motivation was provided by the short note [9] suggesting an elegant method of deriving the so-called entropy-power inequality (EPI). The way of reasoning in [9] is often referred to as the direct probabilistic method, as opposite to the so-called analytic method; see [2, 5, 6, 8]. The results of this paper (Lemmas 2.1–2.4 and Lemma 3.1) provide additional insight on the assumptions under which the direct probabilistic method can be used to establish the EPI in a rigorous manner. For completeness, we give in Sect. 4 a short derivation of the EPI in which we follow [9] but clarify a couple of steps thanks to our continuity lemmas. However, without rigorously proven continuity properties of mutual and conditional entropies in both signal-to-noise ratio regimes, the derivation of the EPI via the direct probabilistic method cannot be accomplished.

Another approach to EPI, for discrete random variables (RVs) where it takes a different form, is discussed in [4], see also references therein. For the history of the question, see [2]; for reader's convenience, the statement of the EPI is given at the end of this section.

To introduce the entropy power inequality, consider two independent RVs X_1 and X_2 taking values in $\mathbb{R}^d$, with probability density functions (PDFs) $f_{X_1}(x)$ and $f_{X_2}(x)$, respectively, where $x \in \mathbb{R}^d$. Let $h(X_i)$, $i = 1, 2$, stand for the differential entropies

$$h(X_i) = -\int_{\mathbb{R}^d} f_{X_i}(x) \ln f_{X_i}(x) \mathrm{d}x := -\mathbb{E} \ln f_{X_i}(X_i),$$

and assume that $-\infty < h(X_1), h(X_2) < +\infty$. The EPI states that

$$\mathrm{e}^{\frac{2}{d}h(X_1+X_2)} \geq \mathrm{e}^{\frac{2}{d}h(X_1)} + \mathrm{e}^{\frac{2}{d}h(X_2)}, \tag{1}$$

or, equivalently,

$$h(X_1 + X_2) \geq h(Y_1 + Y_2), \tag{2}$$

where Y_1 and Y_2 are *any* independent normal RVs with $h(Y_1) = h(X_1)$ and $h(Y_2) = h(X_2)$. This inequality was first proposed by Shannon [7]; as was mentioned earlier, it is used in the analysis of (memoryless) Gaussian channels of signal transmission. A rigorous proof of (1), (2) remains a subject of a growing amount of literature; see, e.g., references cited above. In particular, the question under what conditions upon PDFs f_{X_i} (1), (2) hold true remains largely open.

It is not hard to check that (1) is violated for discrete random variables (a trivial case is where (1) is wrong is when X_1, X_2 take one or two values). Nevertheless, continuity properties of joint entropy remain true (although look slightly different) when one or both of RVs X_1, X_2 have atoms in their distributions, i.e., admit values with positive probabilities. In our opinion, these properties can be useful in a number of situations.

When variables X_1 or X_2 have atoms, the corresponding differential entropies $h(X_1)$ and $h(X_2)$ are replaced with "general" entropies:

$$\begin{aligned} h(X_i) &= -\sum_x p_{X_i}(x) \ln\ p_{X_i}(x) - \int f_{X_i}(x) \ln\ f_{X_i}(x) \mathrm{d}x \\ &= -\int g_{X_i}(x) \ln\ g_{X_i}(x) m\big(\mathrm{d}x\big) := -\mathbb{E} \ln\ g_{X_i}(X_i). \end{aligned}$$

Here $\sum\limits_x$ represents summation over a finite or countable set $\mathbb{D}(=\mathbb{D}(X_I))$ of points $x \in \mathbb{R}^d$. Further, given an RV X, $p_X(x)$ stands for the (positive) probability assigned: $p_X(x) = \mathbb{P}(X = x) > 0$, with the total sum $\eta(X) := \sum\limits_x p_X(x) \leq 1$. Next, f_X, as before, denotes the PDF for values forming an absolutely continuous part of the distribution of X (with $\int f_X(x)\mathrm{d}x = 1 - \eta(X)$, so when $\eta(X) = 1$, the RV X has a discrete distribution, and $h(X) = -\sum\limits_x p_X(x) \ln\ p_X(x)$). Further, $m(=m_X)$ is a reference measure (a linear combination of the counting measure on the discrete part and the Lebesgue measure on the absolutely continuous part of the distribution of X) and g_X is the respective Radon–Nikodym derivative:

$$g_X(x) = p_X(x)\mathbf{1}(x \in \mathbb{D}) + f_X(x), \quad \text{with} \quad \int g_X(x) m(\mathrm{d}x) = 1.$$

We will refer to g_X as a probabiliy mass function (PMF) of RV X (with a slight abuse of traditional termonology). It is also possible to incorporate an (exotic) case where a RV X_i has a singular continuous component in its distribution, but we will not bother about this case in the present work.

It is worth noting that the scheme of proving (1) for a discrete case fails in Lemma 4.1 (see below).

2 Continuity of the Mutual Entropy

Throughout the paper, all random variables are taking values in $\mathbb{R}^d$ (i.e., are d-dimensional real random vectors). If Y is such an RV then the notation $h(Y)$, $f_Y(x)$, $p_Y(x)$, $g_Y(x)$ and $m(\mathrm{d}x)$ have the same meaning as in Sect. 1 (it will be clear from the local context which particular form of the entropy $h(Y)$ we refer to).

Similarly, $f_{X,Y}(x, y)$ and, more generally, $g_{X,Y}(x, y)$, $x, y \in \mathbb{R}^d$, stand for the joint PDF and joint PMF of two RVs X, Y (relative to a suitable reference measure $m(\mathrm{d}x \times \mathrm{d}y)\big(=m_{X,Y}(\mathrm{d}x \times \mathrm{d}y)\big)$ on $\mathbb{R}^d \times \mathbb{R}^d$). Correspondingly, $h(X, Y)$ denotes the joint entropy of X and Y and $I(X : Y)$ their mutual entropy:

$$\begin{aligned} h(X, Y) &= -\int g_{X,Y}(x, y) \ln\ g_{X,Y}(x, y) m\big(\mathrm{d}x \times \mathrm{d}y\big), \\ I(X : Y) &= h(X) + h(Y) - h(X, Y). \end{aligned}$$

We will use representations involving conditional entropies:

$$I(X:Y) = h(X) - h(X|Y) = h(Y) - h(Y|X),$$

where

$$h(X|Y) = h(X,Y) - h(Y),\ h(Y|X) = h(X,Y) - h(X).$$

In this section, we deal with various entropy-continuity properties related to the so-called additive channel where a RV X (a signal) is transformed into the sum $X + U$, with RV U representing "noise" in the channel. In fact, we will adopt a slightly more general scheme where X is compared with $X\sqrt{\gamma} + U$, $\gamma > 0$ being a parameter (called sometimes the signal-to-noise ratio), and study limits where $\gamma \to +\infty$ or $\gamma \to 0+$. We will assume that RVs X and U are independent (though this assumption may be relaxed), and that the "noise" U has a PDF $f_U(x)$ with $\int f_U(x)\mathrm{d}x = 1$. However, the signal X may have a general distribution including a discrete and an absolutely continuous part.

We begin with the analysis of behaviour of the mutual entropy $I(X : X\sqrt{\gamma}+U)$ when $\gamma \to +\infty$: this analysis will be used in Sect. 4, in the course of proving (1). We begin with the case where X has a PDF $f_X(x)$, with $\int f_X(x)\mathrm{d}x = 1$. Here and below, we use the (standard) notation $(b)_+ = \max\,[0,b]$ and $(b)_- = \min\,[0,b]$, $b \in \mathbf{R}$.

Lemma 2.1. *Let X, U be independent RVs with PDFs f_X and f_U where $\int f_X(x)$ $\mathrm{d}x = \int f_U(x)\mathrm{d}x = 1$. Suppose that* (A) $\int f_X(x)\big|\ln\ f_X(x)\big|\mathrm{d}x < +\infty$ *and that* (B) *for any $\varepsilon > 0$ there exists a domain $\mathbf{D}_\varepsilon \subseteq \mathbf{R}^d \times \mathbf{R}^d$ such that for all $\gamma > \gamma_0(\varepsilon)$*

$$-\int\limits_{(\mathbb{R}^d\times\mathbb{R}^d)\setminus\mathbf{D}_\varepsilon} \mathrm{d}x\mathrm{d}y\mathbf{1}\big(f_X(x)>0\big)f_X(x)f_U(y)\left(\ln\left[\int f_X\left(x+\frac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v\right]\right)_- < \varepsilon \tag{3}$$

and for $(x,y) \in \mathbf{D}_\varepsilon$ uniformly in $\gamma > \gamma_0(\varepsilon)$, the following inequality holds true:

$$-\left(\ln\left[\int f_X\left(x+\frac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v\right]\right)_- \le \Psi_\varepsilon(x,y) \tag{4}$$

where $\Psi_\varepsilon(x,y)$ is a function not depending on γ, with $\int \mathrm{d}x \int \mathrm{d}y f_X(x)f_U(y)\Psi_\varepsilon$ $(x,y) < \infty$.

Also assume that (C) *PDF f_X is piece-wise continuous (that is, f_X is continuous on each of open, pair-wise disjoint domains $\mathbf{C}_1,\dots,\mathbf{C}_N \subseteq \mathbf{R}^d$ with piece-wise smooth boundaries $\partial\mathbf{C}_1,\dots,\partial\mathbf{C}_N$, with dimension* $\dim\partial\mathbf{C}_j < d$, *and $f_X = 0$ on the complement $\mathbf{R}^d \setminus \cup_{1\le j\le N}\big(\mathbf{C}_j \cup \partial\mathbf{C}_j\big)$). Furthermore, let f_X be bounded:* $\sup\,[f_X(x):\ x \in \mathbb{R}] = b < +\infty$. *Then*

$$h(X) = \lim_{\gamma\to\infty}\left[I\big(X : X\sqrt{\gamma}+U\big) + h\big(U/\sqrt{\gamma}\big)\right]. \tag{5}$$

Proof of Lemma 2.1. Set: $Y := X\sqrt{\gamma} + U$. The problem is obviously equivalent to proving that

$$\left[h(X|Y) - h\big(U/\sqrt{\gamma}\big)\right] \to 0.$$

Writing $h\big(U/\sqrt{\gamma}\big) = -\ln\sqrt{\gamma} - \int f_U(u)\ln f_U(u)\mathrm{d}u$, we obtain

$$\begin{aligned}
&h(X|Y) - h(U/\sqrt{\gamma})\\
&= -\int \mathrm{d}x\mathbf{1}\big(f_X(x) > 0\big) f_X(x)\int f_U\big(y - x\sqrt{\gamma}\big)\ln \frac{f_X(x) f_U\big(y - x\sqrt{\gamma}\big)}{\int \mathrm{d}u f_X(u) f_U\big(y - u\sqrt{\gamma}\big)}\mathrm{d}y\\
&\quad + \ln\sqrt{\gamma} + \int f_U(u)\ln f_U(u)\mathrm{d}u\\
&= \int \mathrm{d}x\mathbf{1}\big(f_X(x) > 0\big) f_X(x)\int f_U(y)\ln\left[\frac{\sqrt{\gamma}\int \mathrm{d}u f_X(u) f_U\big(y + (x-u)\sqrt{\gamma}\big)}{f_X(x)}\right]\mathrm{d}y\\
&= \int \mathrm{d}x\mathbf{1}\big(f_X(x) > 0\big) f_X(x)\int f_U(y)\ln\left[\frac{\int f_X\left(x + \frac{y-v}{\sqrt{\gamma}}\right) f_U(v)\mathrm{d}v}{f_X(x)}\right]\mathrm{d}y := I(\gamma).
\end{aligned} \tag{6}$$

Next, we decompose the last integral:

$$I(\gamma) = I_+(\gamma) + I_-(\gamma) \tag{7}$$

where

$$I_+(\gamma) = \int \mathrm{d}x\mathbf{1}\big(f_X(x) > 0\big) f_X(x)\int f_U(y)\left(\ln\left[\frac{\int f_X\left(x + \frac{y-v}{\sqrt{\gamma}}\right) f_U(v)\mathrm{d}v}{f_X(x)}\right]\right)_+ \mathrm{d}y \tag{8}$$

and

$$I_-(\gamma) = \int \mathrm{d}x\mathbf{1}\big(f_X(x) > 0\big) f_X(x)\int f_U(y)\left(\ln\left[\frac{\int f_X\left(x + \frac{y-v}{\sqrt{\gamma}}\right) f_U(v)\mathrm{d}v}{f_X(x)}\right]\right)_- \mathrm{d}y. \tag{9}$$

The summand $I_+(\gamma)$ is dealt with by using Lebesgue's dominated convergence theorem. In fact, as $\gamma \to +\infty$, for almost all $x, y \in \mathbf{R}^d$,

$$\mathbf{1}\big(f_X(x) > 0\big)\left(\ln\left[\frac{\int f_X\left(x + \frac{y-v}{\sqrt{\gamma}}\right) f_U(v)\mathrm{d}v}{f_X(x)}\right]\right)_+ \to 0, \tag{10}$$

because (a) $f_X\left(x+\frac{y-v}{\sqrt{\gamma}}\right)\to f_X(x)\ \forall\ x,y,v\in\mathbf{R}^d$ by continuity of f_X, and (b) $\int f_X\left(x+\frac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v\to f_X(x)\ \forall\ x,y$ since f_X is bounded.

Next, we derive from (8) that $I_+(\gamma)\to 0$. Here we write

$$\left[\ln\int f_X\left(x+\frac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v-\ln\ f_X(x)\right]_+\le|\ln b|+|\ln\ f_X(x)|$$

and again use the Lebesgue's theorem, in conjunction with the assumption that

$$\int f_X(x)|\ln\ f_X(x)|\mathrm{d}x<+\infty.$$

The summand $I_-(\gamma)$ requires a different approach. Here we write $I_-(\gamma)=I_-(\gamma,\mathbf{D}_\varepsilon)+I_-(\gamma,\overline{\mathbf{D}}_\varepsilon)$, by restricting integration in $\mathrm{d}x\mathrm{d}y$ to $\mathbf{D}_\varepsilon$ and $\overline{\mathbf{D}}_\varepsilon=(\mathbf{R}^d\times\mathbf{R}^d)\setminus\mathbf{D}_\varepsilon$, respectively. The summand $I_-(\gamma,\mathbf{D}_\varepsilon)\to 0$ by an argument similar to the above (i.e., with the help of the Lebesgue's theorem). For $I_-(\gamma,\overline{\mathbf{D}}_\varepsilon)$ we have that $\limsup_{\gamma\to+\infty}-I_-(\gamma,\overline{\mathbf{D}}_\varepsilon)\le\varepsilon$. Since ε can be made arbitrarily close to 0, the statement of Lemma 2.1 follows. ∎

In Sect. 3, we check conditions of Lemma 2.1 in a number of important cases.

Remark 2.1. An assertion of the type of Lemma 2.1 is crucial for deriving the EPI in (1) by a direct probabilistic method, and the fact that it was not provided in [9] made the proof of the EPI given in [9] incomplete (the same is true of some other papers on this subject).

In the discrete case where signal X takes finitely or countably many values, one has the following

Lemma 2.2. *Let X and U be independent RVs. Assume that X is non-constant, admits discrete values $x_1, x_2,\dots$ with probabilities $p_X(x_1), p_X(x_2),\dots,$ and has $h(X)=-\sum_{x_i}p_X(x_i)\ln\ p_X(x_i)<+\infty$. Next, assume that U has a bounded PDF $f_U(x)$ with $\int f_U(x)\mathrm{d}x=1$ and* $\sup\left[f_U(x):\ x\in\mathbb{R}^d\right]=a<+\infty$, *and*

$$\lim_{\alpha\to\pm\infty}f_U(x+\alpha x_0)=0,\quad\forall\ x,x_0\in\mathbb{R}^d\ \textit{ with } x_0\ne 0.$$

Finally, suppose that $\int f_U(x)\big|\ln\ f_U(x)\big|\mathrm{d}x<+\infty$. Then

$$h(X)=\lim_{\gamma\to\infty}I(X:X\sqrt{\gamma}+U).\tag{11}$$

Proof of Lemma 2.2. Setting as before, $Y = X\sqrt{\gamma} + U$, we again reduce our task to proving that $h(X|Y) \to 0$.

Now write

$$h(X|Y) = -\sum_{i\geq 1} p_X(x_i) \int f_U(y - x_i\sqrt{\gamma}) \ln \frac{p_X(x_i) f_U(y - x_i\sqrt{\gamma})}{\sum\limits_{j\geq 1} p_X(x_j) f_U(y - x_j\sqrt{\gamma})} \mathrm{d}y$$

$$= \sum_{i\geq 1} p_X(x_i) \int f_U(y) \ln \Bigg[1 + \sum_{j:j\neq i} p_X(x_j) p_X(x_i)^{-1} \times f_U\big(y - (x_j - x_i)\sqrt{\gamma}\big) f_U(y)^{-1} \Bigg] \mathrm{d}y. \tag{12}$$

The expression under the logarithm clearly converges to 1 as $\gamma \to +\infty$, $\forall\ i \geq 1$ and $y \in \mathbb{R}^d$. Thus, $\forall\ i \geq 1$ and $y \in \mathbb{R}^d$, the whole integrand

$$f_U(y) \ln \left[1 + \sum_{j:j\neq i} p_X(x_j) p_X(x_i)^{-1} f_U\big(y - (x_j - x_i)\sqrt{\gamma}\big) f_U(y)^{-1} \right] \to 0.$$

To guarantee the convergence of the integral we set $q_i = \sum\limits_{j:j\neq i} p_X(x_j) p_X(x_i)^{-1} = p_X(x_i)^{-1} - 1$ and $\psi(y) = \ln\ f_U(y)$ and use the bound

$$\begin{aligned}
&\ln \left[1 + \sum_{j:j\neq i} p_X(x_j) p_X(x_i)^{-1} f_U(y - (x_j - x_i)\sqrt{\gamma}) f_U(y)^{-1} \right] \\
&\leq \ln\ \big(1 + a q_i \mathrm{e}^{-\psi(y)}\big) \\
&\leq \mathbf{1}\big(a q_i \mathrm{e}^{-\psi(y)} > 1\big) \ln\ \big(2 a q_i \mathrm{e}^{-\psi(y)}\big) + \mathbf{1}\big(a q_i \mathrm{e}^{-\psi(y)} \leq 1\big) \ln\ 2 \\
&\leq 2 \ln\ 2 + \ln\ a + \ln\ (q_i + 1) + |\psi(y)|.
\end{aligned}$$

We then again apply Lebesgue's dominated convergence theorem and deduce that $\lim\limits_{\gamma\to+\infty} h(X|Y) = 0$. ■

In the general case, the arguments developed lead to the following continuity property:

Lemma 2.3. *Let RVs X and U be independent. Assume a general case where X may have discrete and absolutely continuous parts on its distribution while U has a PDF f_U with $\int f_U(x)\mathrm{d}x = 1$. Suppose the PDF f_X, with $\int f_X(x)\mathrm{d}x := 1 - \eta(X) \leq 1$, is continuous, bounded and satisfies assumption* (B) *from Lemma 2.1. Next, suppose that the PDF f_U is bounded and*

$$\lim_{\alpha\to\pm\infty} f_U(x + x_0\alpha) = 0, \ \ \forall\ x, x_0 \in \mathbb{R}^d, \ \ \textit{with}\ \ x_0 \neq 0.$$

Finally, assume that $\int g_X(u)\big|\ln\ g_X(u)\big|m_X(\mathrm{d}u) + \int f_U(u)\big|\ln\ f_U(u)\big|\mathrm{d}u < +\infty$. *Then*

$$h(X) = \lim_{\gamma\to\infty}\Big[I(X : X\sqrt{\gamma}+U) + [1-\eta(X)]h\left(U/\sqrt{\gamma}\right)\Big].$$

The proof of the EPI (1) in Sect. 4 requires an analysis of the behaviour of $I(X : X\sqrt{\gamma}+N)$ also when $\gamma \to 0$. Here we are able to cover a general case for RV X in a single assertion:

Lemma 2.4. *Let* X, U *be independent RVs. Assume that* U *has a bounded and continuous PDF* $f_U \in C^0(\mathbb{R}^d)$, *with* $\int f_U(x)\mathrm{d}x = 1$ *and* $\sup\,[f_U(x),\ x \in \mathbb{R}^d] = a < +\infty$ *whereas the distribution of* X *may have discrete and continuous parts. Next, assume, as in Lemma* 2.3, *that*

$$\int g_X(u)\big|\ln\ g_X(u)\big|m_X(\mathrm{d}u) + \int f_U(u)\big|\ln\ f_U(u)\big|\mathrm{d}u < +\infty.$$

Then

$$\lim_{\gamma\to 0} I\left(X : X\sqrt{\gamma}+U\right) = 0. \tag{13}$$

Proof of Lemma 2.4. Setting again $Y = X\sqrt{\gamma}+U$, we now reduce the task to proving that $h(X|Y) \to h(X)$. Here we write

$$\begin{aligned} h(X|Y) &= -\int g_X(x)\int f_U\left(y-x\sqrt{\gamma}\right)\ln\ \frac{g_X(x)f_U\left(y-x\sqrt{\gamma}\right)}{\int g_X(u)f_U\left(y-u\sqrt{\gamma}\right)m_X(\mathrm{d}u)}\mathrm{d}y\,m_X(\mathrm{d}x)\\ &= \int g_X(x)\int f_U(y)\ln\left[\frac{\int g_X(u)f_U\left(y+(x-u)\sqrt{\gamma}\right)m_X(\mathrm{d}u)}{g_X(x)f_U(y)}\right]\mathrm{d}y\,m_X(\mathrm{d}x). \end{aligned} \tag{14}$$

Due to continuity of f_U, the ratio under the logarithm converges to $(g_X(x))^{-1}$ as $\gamma \to 0$, $\forall\ x, y \in \mathbb{R}^d$. Hence, the integral in (14) converges to $h(X)$ as $\gamma \to 0$. Again, the proof is completed with the help of the Lebesgue-dominated convergence theorem. ■

Remark 2.2. Lemma 2.4 is another example of a missing step in earlier direct probabilistic proofs of the EPI.

3 Uniform Integrability

As was said before, in this section we discuss several cases where assumptions of Lemma 2.1 can be checked. We begin with the case where PDF f_X is lower-bounded by a multiple of the normal PDF. Let ϕ_Σ (or, briefly, ϕ) stand for the standard d-variate normal PDF with mean vector 0 and a $d \times d$ covariance matrix Σ, and we assume that Σ is strictly positive definite:

$$\phi_\Sigma(x) = \frac{1}{(2\pi)^{d/2}\det\,\Sigma^{1/2}}\exp\left[-\frac{1}{2}\left\langle x, \Sigma^{-1}x\right\rangle\right],\ x \in \mathbb{R}^d. \tag{15}$$

Here and below, $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the Euclidean scalar product in $\mathbb{R}^d$.

Proposition 3.1. *Assume that* $f_X(x) \geq \alpha\phi_\Sigma(x)$, $x \in \mathbf{R}^d$, *where* $\alpha \in (0,1]$, *and*

$$\int f_X(x)|\ln\, f_X(x)|\mathrm{d}x,\ \int f_X(x)\left\langle x, \Sigma^{-1}x\right\rangle\mathrm{d}x,\ \int f_U(y)\left\langle y, \Sigma^{-1}y\right\rangle\mathrm{d}y < +\infty. \tag{16}$$

Then assumptions (A) *and* (B) *of Lemma* 2.1 *hold, and the bound in* (4) *is satisfied, with* $\gamma_0(\varepsilon) = 1$, $\mathbf{D}_\varepsilon \equiv \mathbb{R}^d \times \mathbb{R}^d$ *and*

$$\Psi(x,y) = \left(\left\langle x, \Sigma^{-1}x\right\rangle + \left\langle y, \Sigma^{-1}y\right\rangle\right) + \left|\ln\int\exp\left[-2\left\langle v, \Sigma^{-1}v\right\rangle\right]f_U(v)\mathrm{d}v\right| + \rho \tag{17}$$

where $\rho = \left|\ln\,\frac{\alpha}{(2\pi)^{d/2}\det\,\Sigma^{1/2}}\right|$.

Proof of Proposition 3.1. Write:

$$\begin{aligned}
\int f_X\left(x + \frac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v &\geq \alpha\int\phi_\Sigma\left(x + \frac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v \\
&\geq \frac{\alpha}{(2\pi)^{d/2}\det\,\Sigma^{1/2}}\int\exp\left[-\frac{1}{2}\left\langle x + \frac{y-v}{\sqrt{\gamma}}, \Sigma^{-1}\left(x + \frac{y-v}{\sqrt{\gamma}}\right)\right\rangle\right]f_U(v)\mathrm{d}v \\
&\geq \frac{\alpha}{(2\pi)^{d/2}\det\,\Sigma^{1/2}}\exp\left[-\left\langle x, \Sigma^{-1}x\right\rangle\right]\exp\left[-\frac{1}{\gamma}\left\langle y, \Sigma^{-1}y\right\rangle\right] \\
&\quad\times\int\exp\left[-\frac{2}{\gamma}\left\langle v, \Sigma^{-1}v\right\rangle\right]f_U(v)\mathrm{d}v.
\end{aligned}$$

Hence, we obtain that $\forall\ x, y \in \mathbf{R}^d$,

$$\begin{aligned}
&-\left[\ln\int f_X\left(x + \frac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v - \ln\, f_X(x)\right]_- \\
&\quad\leq \left\langle x, \Sigma^{-1}x\right\rangle + \frac{1}{\gamma}\left\langle y, \Sigma^{-1}y\right\rangle + \left|\ln\int\exp\left[-\frac{2}{\gamma}\left\langle v, \Sigma^{-1}v\right\rangle\right]f_U(v)\mathrm{d}v\right| \\
&\qquad + \left|\ln\, f_X(x)\right| + \left|\ln\,\frac{\alpha}{(2\pi)^{d/2}\det\,\Sigma^{1/2}}\right|,
\end{aligned}$$

with the RHS of this bound decreasing with γ. For $\gamma = 1$, it gives the bound of (3). ∎

It is not difficult to extend the argument from the proof of Proposition 3.1 to the case where a more general lower bounds holds:

$$f_X(x) \geq \exp\big(-P(x)\big),\ x \in \mathbf{R}^d$$

where $P(x)$ is a polynomial in $x \in \mathbf{R}^d$ bounded from below. Of course, existence of finite polynomial moments should be assumed, for both PDFs f_X and f_U. Moreover, the lower bounds for f_X can be replaced by lower bounds for $f_U(y)$; in particular, this covers the case where $f_U(y) \geq \alpha\phi(y)$, $y \in \mathbf{R}^d$, $\alpha \in (0, 1]$, and (16) holds true.

An "opposite" case of substantial interest is where both PDFs f_X and f_U have compact supports. In this paper, we do not address this case in full generality, leaving this for future work. However, we will discuss a couple of examples to show what mechanisms are behind convergence. For $A, B \in \mathbf{R}^d$ denote

$$[[A, B]] = \underset{1\leq j\leq d}{\times}[A_j, B_j]$$

(tacitly assuming that $A_i < B_i\ \forall i$).

Proposition 3.2. *Let PDF f_X admit finitely many values. Further, assume that PDF f_U has a compact support $[[A, B]]$, and satisfies the lower bound*

$$f_U(y) \geq \frac{\alpha}{\prod\limits_{1\leq i\leq d}(B_i - A_i)}\mathbf{1}(A_i < y < B_i,\ i = 1, \ldots, d)$$

where $0 < \alpha < 1$ and $A = (A_1, \ldots, A_d)$, $B = (B_1, \ldots, B_d) \in \mathbf{R}^d$ obey $-\infty < A_i < B_i < +\infty$. Then assumption (B) *of Lemma* 2.1 *holds with function $\Psi_\varepsilon(x, y) \equiv 0$ and domains $\mathbf{D}_\varepsilon = \underset{1\leq j\leq d}{\times}\mathbf{D}_\varepsilon^{(i)}$ where the sets $\mathbf{D}_\varepsilon^{(i)}$ are defined in (21). Moreover, $\gamma_0(\varepsilon) = C\varepsilon^{-2/d}$.*

Proof of Proposition 3.2. Consider first a simplified scalar case where $f_X(x) = \frac{1}{b-a}\mathbf{1}(a < x < b)$, $-\infty < a < b < \infty$, while f_U has support $[A, B]$ and satisfies the bound $f_U(y) \geq \frac{\alpha}{B-A}\mathbf{1}(A < y < B)$, with $0 < \alpha < 1$ and $-\infty < A < B < +\infty$. Take $\gamma > 4(B-A)^2\big/(b-a)^2$. Then write

$$\begin{aligned} I &:= -\int \mathrm{d}x\mathbf{1}\big(f_X(x) > 0\big)f_X(x)\int \mathrm{d}y f_U(y) \\ &\quad\times\left(\ln\left[\int f_X\left(x + \tfrac{y-v}{\sqrt{\gamma}}\right)f_U(v)\mathrm{d}v\Big/ f_X(x)\right]\right)_- \\ &= -\tfrac{1}{(b-a)}\int_a^b \mathrm{d}x\int_A^B \mathrm{d}y\, f_U(y)\left(\ln\left[\int\limits_{A\vee(y+(x-b)\sqrt{\gamma})}^{B\wedge(y+(x-a)\sqrt{\gamma})} f_U(v)\mathrm{d}v\right]\right)_- \\ &:= I(0) + I(1). \end{aligned} \tag{18}$$

The decomposition $I = I(0) + I(1)$ in the RHS of (18) extracts an "interior" term $I(0)$, which vanishes, and a "boundary" term $I(1)$ which has to be estimated. More precisely, $I(0) = I_-(0) + I_0(0) + I_+(0)$ where

$$I_-(0) = -\frac{1}{(b-a)} \int\limits_{a}^{a+(B-A)/\sqrt{\gamma}} \mathrm{d}x \int\limits_{B-(x-a)\sqrt{\gamma}}^{B} \mathrm{d}y\, f_U(y) \left(\ln \left[\int\limits_{A}^{B} \mathrm{d}v\, f_U(v) \right] \right)_- = 0, \tag{19}$$

$$I_0(0) = -\frac{1}{(b-a)} \int\limits_{a+(B-A)/\sqrt{\gamma}}^{b-(B-A)/\sqrt{\gamma}} \mathrm{d}x \int\limits_{A}^{B} \mathrm{d}y\, f_U(y) \left(\ln \left[\int\limits_{A}^{B} \mathrm{d}v\, f_U(v) \right] \right)_- = 0, \tag{20a}$$

and

$$I_+(0) = -\frac{1}{(b-a)} \int\limits_{b-(B-A)/\sqrt{\gamma}}^{b} \mathrm{d}x \int\limits_{A}^{A+(b-x)\sqrt{\gamma}} \mathrm{d}y\, f_U(y) \left(\ln \left[\int\limits_{A}^{B} \mathrm{d}v\, f_U(v) \right] \right)_- = 0. \tag{20b}$$

Correspondingly, set $\mathbf{D}_\varepsilon$ in the case under consideration is the union of three sets $\mathbf{D}_\varepsilon = \mathbf{D}_{\varepsilon,-} \cup \mathbf{D}_{\varepsilon,0} \cup \mathbf{D}_{\varepsilon,+}$ where

$$\mathbf{D}_{\varepsilon,-} = \Big\{(x,y) :\ a < x < a + (B-A)\sqrt{\varepsilon},\ B - (x-a)/\sqrt{\varepsilon} < y < B\Big\},$$

$$\mathbf{D}_{\varepsilon,0} = \Big(a + (B-A)\sqrt{\varepsilon}, b - (B-A)\sqrt{\varepsilon}\Big) \times \big(A, B\big)$$

and

$$\mathbf{D}_{\varepsilon,+} = \Big\{(x,y) :\ b - (B-A)\sqrt{\varepsilon} < x < b,\ A < y < A + (b-x)/\sqrt{\varepsilon}\Big\}. \tag{21}$$

Observe that so far we have not used the upper bound for PDF f_U. See Fig. 1.

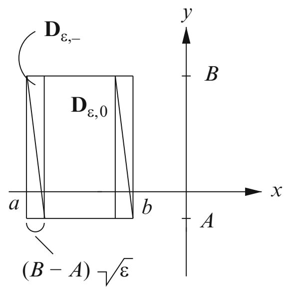

Fig. 1 Domain of integration for rectangular PDF f_U

Further, $I(1) = I_-(1) + I_+(1)$ where

$$I_-(1) = -\frac{1}{b-a} \int\limits_a^{a+(B-A)/\sqrt{\gamma}} \mathrm{d}x \int\limits_A^{B-(x-a)\sqrt{\gamma}} \mathrm{d}y\, f_U(y) \left(\ln \left[\int\limits_A^{y+(x-a)\sqrt{\gamma}} \mathrm{d}v\, f_U(v) \right] \right)_- \tag{22}$$

and

$$I_+(1) = -\frac{1}{b-a} \int\limits_{b-(B-A)/\sqrt{\gamma}}^{b} \mathrm{d}x \int\limits_{A+(b-x)\sqrt{\gamma}}^{B} \mathrm{d}y\, f_U(y) \left(\ln \left[\int\limits_{y+(x-b)\sqrt{\gamma}}^{B} \mathrm{d}v\, f_U(v) \right] \right)_- . \tag{23}$$

We have to upper-bound integrals $I_-(1)$ and $I_+(1)$. For definiteness, we focus on $I_-(1)$; the changes for $I_+(1)$ are automatic. We have that

$$\begin{aligned} I_-(1) &\leq - \int\limits_a^{a+(B-A)/\sqrt{\gamma}} \frac{\mathrm{d}x}{b-a} \int\limits_A^{B-(x-a)\sqrt{\gamma}} \mathrm{d}y\, f_U(y) \ln \left[\frac{\alpha}{B-A} \big(y + (x-a)\sqrt{\gamma} - A \big) \right] \\ &= - \int\limits_0^{(B-A)/\sqrt{\gamma}} \frac{\mathrm{d}x}{b-a} \int\limits_0^{B-A-x\sqrt{\gamma}} \mathrm{d}y\, f_U(y+A) \ln \left[\frac{\alpha}{B-A} \big(y + x\sqrt{\gamma} \big) \right] \\ &= -\frac{B-A}{b-a} \int\limits_0^{1/\sqrt{\gamma}} \mathrm{d}x \int\limits_0^{1-x\sqrt{\gamma}} \mathrm{d}y\, f_U(y(B-A)+A) \ln \left[\alpha \big(y + x\sqrt{\gamma} \big) \right] \\ &= -\frac{B-A}{b-a} \frac{1}{\sqrt{\gamma}} \int\limits_0^1 \mathrm{d}x \int\limits_0^{1-x} \mathrm{d}y\, f_U(y(B-A)+A) \ln \left[\alpha \big(y + x \big) \right]. \end{aligned} \tag{24}$$

It remains to check that the integral in the RHS of (24) is finite.

But the integrand in the RHS of (24) has singularity at $x = y = 0$ only, which is integrable. A similar calculation applies to $I_+(1)$. This yields that, in the simplified case under consideration, the integral I in (18) obeys $I \leq C/\sqrt{\gamma}$. In the multidimensional situation, if we assume that $X \sim \mathrm{U}([[A, B]])$, then a similar argument gives that $I \leq C\gamma^{-d/2}$. The extension of this argument to the general scalar case where f_X takes a finite number of values is straightforward. In the multidimensional case, when $X \sim U \sim \mathrm{U}([0, 1]^d)$, a similar argument gives that $I = C\gamma^{-d/2}$, finally a similar bounds holds when X takes a finite number of values. ■

Proposition 3.3. *Let both f_X and f_U have a pyramidal form*

$$f_X(x) = \prod_{i=1}^{d}(1-|x_i|)_+, \quad f_U(y) = \prod_{i=1}^{d}\frac{1}{a_i}\left(1-\frac{1}{a_i}|y_i-b_i|\right)_+, \quad x, y \in \mathbf{R}^d, \tag{25}$$

where $a_1, \dots, a_d > 0$ and $-\infty < b_i < \infty$, $i = 1, \dots, d$. Then assumption (B) *of Lemma* 2.1 *holds, with*

$$\mathbf{D}_\varepsilon = \Big([[-1+\underline{\varepsilon}, -\underline{\varepsilon}]] \cup [[-1+\underline{\varepsilon}, -\underline{\varepsilon}]]\Big) \times [[b-a, b+a]], \tag{26}$$

where $a = (a_1, \dots, a_d)$, $(b_1, \dots, b_d)$ and $\underline{\varepsilon} = (\varepsilon, \dots, \varepsilon) \in \mathbf{R}^d$ and $\varepsilon \in (0, 1/2)$. Further, $\gamma_0(\varepsilon) = 1/(4\varepsilon^2)$ and function $\Psi_\varepsilon(x, y)$ is given by

$$\Psi_\varepsilon(x, y, \gamma) = \prod_{i=1}^{d}\left|\ln\left[1 - 2\varepsilon\frac{a_i + b_i + y_i\big(\operatorname{sgn} x_i\big)}{1-|x_i|}\right]\right|. \tag{27}$$

Proof of Proposition 3.3. First, consider a scalar case (where both PDFs have a triangular form) and assume, w.l.o.g., that $b = 0$. It is convenient to denote the rectangle $(-1, 1) \times (-a, a)$ in the (x, y)-plane by $\mathcal{R}$. For $(x, y) \in \mathcal{R}$, set:

$$J(= J(x,y)) = \int \mathrm{d}v f_U(v) f_X\left(x + \frac{y-v}{\sqrt{\gamma}}\right) = \sqrt{\gamma}\int \mathrm{d}u f_X(u) f_U\big((x-u)\sqrt{\gamma}+y\big). \tag{28}$$

Then, for $\sqrt{\gamma} > a$, on the parallelogram

$$\mathcal{P}_+\big(=\mathcal{P}_+(\gamma)\big) = \big\{(x, y) \in \mathcal{R}:\ a - x\sqrt{\gamma} < y < \sqrt{\gamma}(1-x) - a\big\},$$

we have that

$$J = \frac{\sqrt{\gamma}}{a}\left[\int_{x+\frac{y}{\sqrt{\gamma}}}^{x+\frac{y+a}{\sqrt{\gamma}}}(1-u)\Big(1 - \tfrac{1}{a}(\sqrt{\gamma}(x-u)+y)\Big)\mathrm{d}u \right. \\ \left. + \int_{x+\frac{y-a}{\sqrt{\gamma}}}^{x+\frac{y}{\sqrt{\gamma}}}(1-u)\Big(\tfrac{1}{a}(\sqrt{\gamma}(x-u)+y) - 1\Big)\mathrm{d}u\right].$$

By the direct computation

$$J = 1 - x - \frac{1}{\sqrt{\gamma}}(a+y).$$

So, for $\sqrt{\gamma} > a$, in parallelogram $\mathcal{P}_+$, we have

$$J(x,y) = 1 - x - \frac{1}{\sqrt{\gamma}}(y+a) \text{ and } \ln \frac{J(x,y)}{f_X(x)} = \ln \left[1 - \frac{y+a}{\sqrt{\gamma}(1-x)}\right]. \quad (29)$$

Geometrically, parallelogram $\mathcal{P}_+$ corresponds to the case where the support of the scaled PDF

$$x \mapsto \sqrt{\gamma} f_U\big((x-u)\sqrt{\gamma} + y\big) \quad (30)$$

lies entirely in (0, 1), the right-hand half of the support for f_X. Cf. Fig. 2 below.

Similarly, on the symmetric parallelogram

$$\mathcal{P}_-\big(=\mathcal{P}_-\big) = \big\{(x,y) \in \mathcal{R}:\ -(1+x)\sqrt{\gamma} + a < y < -\sqrt{\gamma}x - a\big\},$$

we have

$$J(x,y) = 1 - x - \frac{1}{\sqrt{\gamma}}(a-y) \text{ and } \ln \frac{J(x,y)}{f_X(x)} = \ln \left[1 - \frac{a-y}{\sqrt{\gamma}(1+x)}\right]. \quad (31)$$

This parallelogram corresponds with case where the support of the PDF in (30) lies in $(-1,0)$. On the union $\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_-$,

$$H(x,y,\gamma) := \ln \frac{J(x,y)}{f_X(x)} = \ln \left[1 - \frac{a + y\big(\text{sgn } x\big)}{\sqrt{\gamma}(1-|x|)}\right]. \quad (32)$$

If we set $H(x,y;\gamma) = 0$ on $\mathcal{R} \setminus \mathcal{P}$, then function $H(x,y,\gamma)$ converges to 0 pointwise on the whole of $\mathcal{R}$. Given $\varepsilon \in (0, 1/2)$, take $\gamma > 1/(4\varepsilon^2)$. On the set $\mathbf{D}_\varepsilon$, we have that

$$|H(x,y,\gamma)| \le |H(x,y,1/(4\varepsilon^2))| := \Psi_\varepsilon(x,y).$$

The complement $\mathcal{R}\backslash\mathcal{P}$ is partitioned in six domains (a right triangle with a vertex at point $(-1,-a)$ plus an adjacent trapezoid on the left, a right triangle with

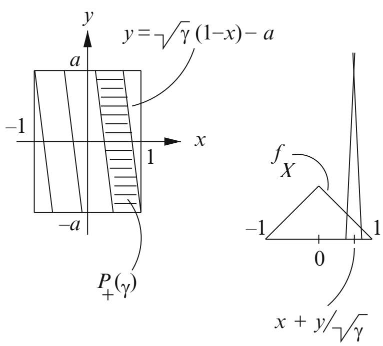

Fig. 2 Domain of integration for pyramidal PDF f_U

a vertex at point $(1, a)$ plus an adjacent trapezoid on the right, and two adjacent parallelograms in the middle). These domains correspond to various positions of the "centre" and the endpoints of support of PDF $x \mapsto \sqrt{\gamma} f_U\big((x-u)\sqrt{\gamma}+y\big)$ relative to $(-1,1)$ (excluding the cases covered by set $\mathcal{P}$). On each of these domains, function $J(x,y)$ is a polynomial, of degree ≤ 3. Viz., on the RHS triangle $\{(x,y)\in\mathcal{R}:\ \sqrt{\gamma}(1-x)<y\}$,

$$J(x,y)=\frac{\sqrt{\gamma}}{a}\int_{x+\frac{y-a}{\sqrt{\gamma}}}^{1}(1-u)\Big(\frac{1}{a}(\sqrt{\gamma}(x-u)+y)-1\Big)\mathrm{d}u.$$

Next, in the RHS trapezoid $\{(x,y)\in\mathcal{R}:\ \sqrt{\gamma}(1-x)-a<y<\sqrt{\gamma}(1-x)\}$

$$J(x,y)=\frac{\sqrt{\gamma}}{a}\Big[\int_{x+\frac{y}{\sqrt{\gamma}}}^{1}(1-u)\Big(1-\frac{1}{a}(\sqrt{\gamma}(x-u)+y)\Big)\mathrm{d}u$$

$$+\int_{x+\frac{y-a}{\sqrt{\gamma}}}^{x+\frac{y}{\sqrt{\gamma}}}(1-u)\Big(\frac{1}{a}(\sqrt{\gamma}(x-u)+y)-1\Big)\mathrm{d}u\Big].$$

Finally, on the RHS parallelogram $\{(x,y)\in\mathcal{R}:\ -x\sqrt{\gamma}<y<a-x\sqrt{\gamma}\}$,

$$J(x,y)=\frac{\sqrt{\gamma}}{a}\Big[\int_{x+\frac{y-a}{\sqrt{\gamma}}}^{0}(-1+u)\Big(1-\frac{1}{a}(\sqrt{\gamma}(x-u)+y)\Big)\mathrm{d}u$$

$$+\int_{0}^{x+\frac{y}{\sqrt{\gamma}}}(1-u)\Big(1-\frac{1}{a}(\sqrt{\gamma}(x-u)+y)\Big)\mathrm{d}u$$

$$+\int_{x+\frac{y}{\sqrt{\gamma}}}^{(x+\frac{y+a}{\sqrt{\gamma}}\wedge 1)}(1-u)\Big(\frac{1}{a}(\sqrt{\gamma}(x-u)+y)-1\Big)\mathrm{d}u\Big].$$

Similar formulas take place for LHS counterparts. The integrals

$$\int\int f_X(x)f_U(y)\Big|\ln\ \big[J(x,y)/f_X(x)\big]\Big|\mathrm{d}x\mathrm{d}y \tag{33}$$

over each of these domains are assessed by inspection and decay as $O(1/\sqrt{\gamma})$.

In addition, to cover the complement $\mathcal{R}\backslash\mathbf{D}_\varepsilon$, we have to consider the set $\mathcal{P}\backslash\mathbf{D}_\varepsilon$ and integrate the function $J(x,y)$ from (30) and (33). This again is done by inspection; the corresponding integral is assessed as $O(\varepsilon)$. Hence, the integral (19) over the entire complement $\mathcal{R}\backslash\mathbf{D}_\varepsilon$ is $\leq C\varepsilon$, for an appropriately chosen constant $C>0$.

The above argument can be easily extended to the d-dimensional case since we will be dealing with products of integrals. ∎

4 The Entropy-Power Inequality

In this section, we show how to deduce the EPI (1) from the lemmas established in Sect. 2. We begin with a convenient representation of the mutual entropy $I(X : X\sqrt{\gamma} + U)$ in the case where U is a d-variate normal RV, with a short (and elementary) proof. We do not consider this proof as new: it follows the same line as [3] but is more elementary.

Lemma 4.1. *Let X and N be two independent RV, where $N \sim N(0, \Sigma)$ while X has a PDF f_X. Suppose that $\int f_X(x)||x||^2 dx < +\infty$. Given $\gamma > 0$, write the mutual entropy between X and $X\sqrt{\gamma} + N$:*

$$\begin{aligned} I(X : X\sqrt{\gamma} + N) = &-\int f_X(x)\phi(u - x\sqrt{\gamma}) \ln \left[f_X(x)\phi(u - x\sqrt{\gamma})\right] du dx \\ &+\int f_X(x) \ln f_X(x) dx \\ &+\int f_{X\sqrt{\gamma}+N}(u) \ln f_{X\sqrt{\gamma}+N}(u) du \end{aligned}$$

where

$$f_{X\sqrt{\gamma}+N}(u) = \int f_X(x)\phi(u - x\sqrt{\gamma}) dx. \tag{34}$$

Then

$$\frac{d}{d\gamma}\Big[I(X : X\sqrt{\gamma} + N) + h\big(N/\sqrt{\gamma}\big)\Big] = \frac{1}{2}M(X;\gamma) - \frac{1}{2\gamma} \tag{35}$$

where

$$M(X;\gamma) = \mathbb{E}\Big[\big\|X - \mathbb{E}(X|X\sqrt{\gamma} + N)\big\|^2\Big]$$

and the norm $\|x\|^2 = \langle x, \Sigma^{-1}x\rangle$, $x \in \mathbb{R}^d$.

Proof of Lemma 4.1. Differentiate expression for $I(X : X\sqrt{\gamma} + N)$ given in (34), and observe that the derivative of the joint entropy $h(X, X\sqrt{\gamma} + N)$ vanishes, as $h(X, X\sqrt{\gamma} + N)$ does not change with $\gamma > 0$:

$$\begin{aligned} &h(X, X\sqrt{\gamma} + N) \\ &\quad = -\int f_X(x)\phi(u - x\sqrt{\gamma})\left[\ln f_X(x) + \ln \phi(u - x\sqrt{\gamma})\right] dx du \\ &\quad = h(X) + h(N). \end{aligned}$$

The derivative of the marginal entropy $h(X\sqrt{\gamma} + N)$ requires some calculations:

$$\begin{aligned} \frac{d}{d\gamma}h(X\sqrt{\gamma} + N) &= -\frac{d}{d\gamma}\int f_{X\sqrt{\gamma}+N}(u) \ln f_{X\sqrt{\gamma}+N}(u) du \\ &= \int \frac{1}{2\sqrt{\gamma}} \int f_X(y)\phi(u - \sqrt{\gamma}y) \big\langle (u - \sqrt{\gamma}y), \Sigma^{-1}y\big\rangle dy \end{aligned}$$

$$\times \ln \int f_X(z)\phi(u-\sqrt{\gamma}z)\mathrm{d}z\mathrm{d}u + \int \frac{1}{2\sqrt{\gamma}} \int f_X(y)\phi(u-\sqrt{\gamma}y)\mathrm{d}y$$

$$\times \frac{\int f_X(w)\phi(u-\sqrt{\gamma}w)\left\langle (u-\sqrt{\gamma}w), \Sigma^{-1}w \right\rangle \mathrm{d}w}{\int f_X(z)\phi(u-\sqrt{\gamma}z)\mathrm{d}z}\,\mathrm{d}u. \tag{36}$$

The second summand vanishes, as (i) the integrals

$$\int f_X(y)\phi(u-\sqrt{\gamma}y)\mathrm{d}y \text{ and } \int f_X(z)\phi(u-\sqrt{\gamma}z)\mathrm{d}z$$

cancel each other and (ii) the remaining integration can be taken first in du, which yields 0 for $\forall\ w$. The first integral we integrate by parts. This leads to the representation

$$\frac{\mathrm{d}}{\mathrm{d}\gamma} I(X : X\sqrt{\gamma}+N) = \frac{1}{2\sqrt{\gamma}} \int\int f_X(y)\phi(u-\sqrt{\gamma}y)$$

$$\times \frac{\int f_X(x)\phi(u-\sqrt{\gamma}x)\left\langle (u-\sqrt{\gamma}x), \Sigma^{-1}y \right\rangle \mathrm{d}x}{\int f_X(z)\phi(u-\sqrt{\gamma}z)\mathrm{d}z}\mathrm{d}y\mathrm{d}u$$

$$= \frac{1}{2\sqrt{\gamma}} \int\int \mathrm{d}y\mathrm{d}u\, f_X(y)\phi(u-\sqrt{\gamma}y)$$

$$\times \frac{\int f_X(x)\phi(u-\sqrt{\gamma}x)\left[\left\langle (u-\sqrt{\gamma}y, \Sigma^{-1}y \right\rangle + \sqrt{\gamma}\left\langle (y-x), \Sigma^{-1}y \right\rangle \right]\mathrm{d}}{\int f_X(z)\phi(u-\sqrt{\gamma}z)\mathrm{d}z}. \tag{37}$$

The integral arising from the summand $\left\langle (u-\sqrt{\gamma}y, \Sigma^{-1}y \right\rangle$ vanishes, because the mean vector in PDF ϕ is zero. The remaining contributions, from $\left\langle y, \Sigma^{-1}y \right\rangle - \left\langle x, \Sigma^{-1}y \right\rangle$, is equal to

$$\frac{1}{2}\mathbb{E}\Big[\left\| X - \mathbb{E}(X|X\sqrt{\gamma}+N) \right\|^2 \Big].$$

On the other hand, the first term in RHS of (36) equals

$$\int \left\| x - \frac{\int f_X(y)\phi(u-\sqrt{\gamma}y)\Sigma y \mathrm{d}y}{\int f_X(z)\phi(u-\sqrt{\gamma}z)\mathrm{d}z} \right\|^2 f_X(x)\phi(u-\sqrt{\gamma}x)\mathrm{d}x\mathrm{d}u$$
$$= \mathbb{E}\Big[\|X - \mathbb{E}(X|X\sqrt{\gamma}+N)\|^2 \Big] \equiv M(X;\gamma).$$

■

We are now going to derive the EPI (1) following the line of argument proposed in [9] and based on Lemma 4.1. First, suppose that X is an RV with a PDF f_X where $\int f_X(x)\mathrm{d}x = 1$. Then we assume that $f_X(x)$ satisfies the assumptions stated in Lemma 2.1 and Lemma 2.4 and use these lemmas with $U = N \sim \mathrm{N}(0, \Sigma)$. Consequently, $\forall\ \varepsilon > 0$,

$$\begin{aligned} h(X) &= \lim_{\gamma\to+\infty}\Big[I\big(X : X\sqrt{\gamma}+N\big)+h\big(N/\sqrt{\gamma}\big)\Big] \\ &= \int\limits_{\varepsilon}^{+\infty} \tfrac{\mathrm{d}}{\mathrm{d}\gamma}\left[I\big(X : X\sqrt{\gamma}+N\big)+h\big(N/\sqrt{\gamma}\big)\right]\mathrm{d}\gamma + I\big(X : X\sqrt{\varepsilon}+N\big)+h\big(N/\sqrt{\varepsilon}\big) \\ &= \tfrac{1}{2}\int_{\varepsilon}^{+\infty}\Big[M(X;\gamma)-\tfrac{1}{\gamma}\mathbf{1}(\gamma>1)\Big]\mathrm{d}\gamma + h(N) + I\big(X : X\sqrt{\varepsilon}+N\big). \end{aligned} \tag{38}$$

Here we use the identity $\int\limits_{\varepsilon}^{1}(1/\gamma)\mathrm{d}\gamma = -\ln\varepsilon$. By Lemma 2.4 the last term in (38) tends to 0 as $\varepsilon \to 0$. Hence, for an RV X with PDF $f_X \in C^0$ we obtain

$$h(X) = h(N) + \frac{1}{2}\int_0^{\infty}\Big[M(X;\gamma) - \mathbf{1}(\gamma > 1)\frac{1}{\gamma}\Big]\mathrm{d}\gamma. \tag{39}$$

Remark 4.1. A straightforward calculation shows that (39) can be written in an equivalent form

$$h(X) = \frac{1}{2}\ln\,\left(2\pi e\sigma_X^2\right) - \frac{1}{2}\int_0^{\infty}\left[\frac{\sigma_X^2}{1+\gamma\sigma_X^2} - M(X;\gamma)\right]\mathrm{d}\gamma \tag{40}$$

used in (4) from [9]. We thank the referee who pointed at this fact in his comments.

The proof of EPI is based on (39) and the following result from [5].

Lemma 4.2. *([5], Theorem 6) Let $\mathcal{X}$ be a given class of probability distributions on $\mathbb{R}^d$, closed under convex linear combinations and convolutions. The inequality*

$$h(X_1\cos\theta + X_2\sin\theta) \geq h(X_1)\cos^2\theta + h(X_2)\sin^2\theta, \tag{41}$$

for any $\theta \in [0, 2\pi]$ and any pair of independent RVs X_1, X_2 with distributions from $\mathcal{X}$, holds true if and only if the entropy power inequality is valid for any pair of RVs X_1, X_2 with distributions from $\mathcal{X}$.

Theorem 4.1. *Let U be $d-$variate normal $N(0, \mathbf{I})$. Assume that RV's X_1, X_2 take values in $\mathbb{R}^d$ and have continuous and bounded PDFs $f_{X_1}(x)$, $f_{X_2}(x)$, $x \in \mathbb{R}^d$ satisfying condition* (A)–(B) *of Lemma 2.1. Assume that the differential entropies $h(X_1)$ and $h(X_2)$ satisfy $-\infty < h(X_1), h(X_2) < +\infty$. Then the EPI (see* (1)–(2)*) holds true.*

Proof of Theorem 4.1. The proof follows Ref. [9] and is provided here only for completeness of presentation. The result of Theorem 4.1 may also be established for piece-wise continuous PDFs $f_{X_1}(x)$ and $f_{X_2}(x)$ as well (cf. Lemma 3.2). According to Lemma 4.2, it suffices to check bound (41) $\forall\ \theta \in (0, 2\pi)$ and $\forall$ pair of RVs X_1, X_2 with continuous and bounded PDFs $f_{X_i}(x)$, $i = 1, 2$. Take any such *pair and let N be $N(0, \mathbf{I})$ where $\mathbf{I}$ is the $d \times d$ unit matrix. Following the argument developed in [9], we apply formula (39) for the RV $X = X_1 \cos\phi + X_2 \sin\phi$:*

$$h(X_1 \cos\phi + X_2 \sin\phi) = h(N) + \frac{1}{2}\int_0^\infty \Big[M(X_1 \cos\phi + X_2 \sin\phi;\gamma) - \mathbf{1}(\gamma > 1)\frac{1}{\gamma}\Big]\mathrm{d}\gamma.$$

To verify (41), we need to check that

$$M(X_1 \cos\phi + X_2 \sin\phi;\gamma) \geq \cos\phi^2 M(X_1;\gamma) + \sin\phi^2 M(X_2;\gamma). \qquad (42)$$

To this end, we take two independent RVs $N_1, N_2 \sim N(0, \mathbf{I})$ and set

$$Z_1 = X_1\sqrt{\gamma} + N_1, Z_2 = X_2\sqrt{\gamma} + N_2, \ \ \textit{and} \ \ Z = Z_1 \cos\phi + Z_2 \sin\phi.$$

Then inequality (41) holds true because

$$\begin{aligned}&\mathbb{E}\Big[\, \|X - \mathbb{E}(X|Z)\|^2 \Big] \geq \mathbb{E}\Big[\, \|X - \mathbb{E}(X|Z_1, Z_2)\|^2 \Big] \\ &\quad = \mathbb{E}\Big[\, \|X_1 - \mathbb{E}(X_1|Z_1)\|^2 \Big]\cos\phi^2 + \mathbb{E}\Big[\, \|X_2 - \mathbb{E}(X_2|Z_2)\|^2 \Big]\sin\phi^2. \qquad \blacksquare\end{aligned}$$

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