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Preface

Tauberian operators were introduced to investigate a problem in summability theory from an abstract point of view. Since that introduction, they have made a deep impact on the isomorphic theory of Banach spaces. In fact, these operators have been useful in several contexts of Banach space theory that have no apparent or obvious connections. For instance, they appear in the famous factorization of Davis, Figiel, Johnson and Pełczyński [49] (henceforth the *DFJP factorization*), in the study of exact sequences of Banach spaces [174], in the solution of certain summability problems of tauberian type [63,115], in the problem of the equivalence between the Krein-Milman property and the Radon-Nikodým property [151], in certain sequels of James' characterization of reflexive Banach spaces [135], in the construction of hereditarily indecomposable Banach spaces [13], in the extension of the principle of local reflexivity to operators [27], in the study of certain Calkin algebras associated with the weakly compact operators [16], etc. Since the results concerning tauberian operators appear scattered throughout the literature, in this book we give a unified presentation of their properties and their main applications in functional analysis. We also describe some questions about tauberian operators that remain open.

This book has six chapters and an appendix. In Chapter 1 we show how the concept of tauberian operator was introduced in the study of a classical problem in summability theory–the characterization of conservative matrices that sum no bounded divergent sequences–by means of functional analysis techniques. One of those solutions is due to Crawford [45], who considered the second conjugate of the operator associated with one of those matrices. Crawford's solution led Kalton and Wilansky to introduce tauberian operators in [115] as those operators $T\colon X \longrightarrow Y$ acting between Banach spaces for which $T^{**}(X^{**}\backslash X) \subset Y^{**}\backslash Y$, where T^{**} denotes the second conjugate of T.

Chapter 2 displays the basic structural properties of the class of tauberian operators; in particular, the links between tauberian operators, weakly compact operators and reflexivity. We present some basic examples and describe the most important characterizations of tauberian operators: the sequential characterization of Kalton and Wilansky [115], the geometrical characterizations obtained by Neidinger and Rosenthal [135], the characterization in terms of reflexivity of the kernel of the compact perturbations given in [92], the algebraic characterization obtained in [92], and some characterizations in terms of the action of tauberian operators upon basic sequences proved by Holub [103].

We begin Chapter 3 by introducing the cotauberian operators as those operators T such that T^* is tauberian. Next we give the main properties of these operators. Several results show that cotauberian operators form the right class to be taken as the dual class of the tauberian operators. However, this relationship of duality is not perfect: we give an example, obtained in [8], of a tauberian operator T such that T^* is not cotauberian. We also include a perturbative characterization and an algebraic characterization for the cotauberian operators similar to those obtained for the tauberian operators in the previous chapter.

We describe an improved version of the DFJP factorization, obtained in [68], which allows us to show plenty of examples of tauberian and cotauberian operators: every operator $T\colon X \longrightarrow Y$ can be factorized as $T = jUk$, with j tauberian, k cotauberian and U a bijective isomorphism. Moreover, this version behaves well under duality.

The DFJP factorization has received a lot of attention. In particular, several variations of it have been introduced. We describe an isometric variation and a conditional variation. The first one was introduced by Lima, Nygaard and Oja [119] to study the approximation property of Banach spaces, and the second one was introduced by Argyros and Felouzis [13] to construct examples of hereditarily indecomposable Banach spaces. Moreover, following Beauzamy's exposition [21], we show that the intermediate space in the DFJP factorization can be identified with a real interpolation space for certain values of the interpolation parameters.

We treat other situations in which tauberian operators appear. For example, following [32, 35], we show that the natural embedding of certain Orlicz function spaces $L_\Phi(\mu)$ into $L_1(\mu)$ is a tauberian operator if and only if for every Banach space X the natural embedding of the space of vector-valued functions $L_\Phi(\mu, X)$ into $L_1(\mu, X)$ is a tauberian operator.

The aforementioned characterizations show that the tauberian and the cotauberian operators are closely linked to the operator ideal of the weakly compact operators. Following [89, 90, 92], we consider four other operator ideals that admit sequential characterizations. We show that each one has two classes of associated operators similar to the tauberian and the cotauberian operators. The first of these classes is defined in terms of sequences and the second one is defined by duality. We show that both classes admit a perturbative characterization and an algebraic characterization.

Chapter 4 is devoted to the study of tauberian operators $T\colon L_1(\mu) \longrightarrow Y$, where μ is a finite measure and Y is a Banach space [75]. The characterizations of relatively weakly compact subsets of $L_1(\mu)$ are applied to obtain some useful characterizations of these tauberian operators and show that their properties are better than those of the general tauberian operators. For example, the set of tauberian operators from $L_1(\mu)$ into Y is open in the set of all operators, and one of these operators T is tauberian if and only if so is its second conjugate.

In Chapter 5 we describe the main applications of tauberian operators in Banach space theory. Following Schachermayer in [151] and the exposition in [67], we show that, for a Banach space X for which there exists a tauberian operator $T: X \times X \longrightarrow X$, the Radon-Nikodým property and the Krein-Milman property are equivalent. We also show that tauberian operators preserve some isomorphic properties: following Neidinger's thesis [133], we show that, given a tauberian operator $T: X \longrightarrow Y$ and a bounded subset C of X, some isomorphic properties of the set $T(C)$ are inherited by C and some isomorphic properties of the space Y are inherited by X.

Using the version of the DFJP factorization presented in Chapter 3, we show that some operator ideals $\mathcal{A}$ possess the factorization property: each operator in $\mathcal{A}$ factors through a Banach space whose identity belongs to $\mathcal{A}$. Here we include some results of Heinrich [100] and some extensions of these results obtained in [68]. We also show that these factorization results can be extended in two directions: one of them by showing that we can obtain a uniform factorization of this kind for the operators of a compact set of operators [73], and the other one (see [71, 72]) by showing that the definition of some operator ideals can be extended to holomorphic mappings $f: X \longrightarrow Y$ acting between Banach spaces X and Y, and that in some cases these maps can be written as $f = T \circ g$ or $f = g \circ T$, where g is another holomorphic mapping and T is an operator that belongs to the same operator ideal as f.

We also give some applications of the isometric variation of the DFJP factorization to study the approximation property of Banach spaces, due to Lima, Nygaard and Oja [119], and following Astala and Tylli [16], we characterize the weakly compact approximation property of Banach spaces in terms of the weak Calkin algebra.

In Chapter 6 we consider some classes of operators that have a similar behavior to that of tauberian operators. Some of these classes were named *semigroups* in [89,90,92], following Lebow and Schechter [118] who did it for the semi-Fredholm operators. Finally, the notion of an *operator semigroup* was axiomatized in [1] as a counterpart to Pietsch's concept of an operator ideal [139].

Every operator ideal $\mathcal{A}$ has two semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ associated in a similar way as the weakly compact operators have the tauberian and the cotauberian operators. We summarize the main properties of these two operator semigroups and show other general constructions that provide semigroups.

We describe a strongly tauberian operator and its dual class, introduced by Rosenthal [147]. Moreover, we show how tauberian operators have been useful in distinguishing between the different concepts of local representability of operators that have appeared in the literature.

We study in some detail the ultrapower-stable operator semigroups. For that purpose, we consider two different types of finite representability for operators: local representability and local supportability. As an application, we investigate the class of supertauberian operators, which is the largest ultrapower-stable semigroup contained in the class of tauberian operators, and their dual class: the cosupertauberian operators.

Each chapter ends with a section of Notes and Remarks where we include some comments, complementary results and bibliographical references.

This book is addressed to graduate students and researchers interested in functional analysis and operator theory. The prerequisites for reading this book are a basic knowledge of functional analysis, including the consequences of the Hahn-Banach theorem and the open mapping theorem. Familiarity with the rudiments of Fredholm theory for operators and some parts of Banach space theory, like criteria for the existence of basic subsequences from a given sequence, Rosenthal's ℓ_1-theorem, ultraproducts and the principle of local reflexivity would be helpful. For the convenience of the reader, a brief exposition of these prerequisites has been included in Appendix A.

Our intention has been to present a self-contained exposition of the fundamental results of the subject. When describing the applications, sometimes we give a reference instead of a complete proof.

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Manuel González and Antonio Martínez-Abejón
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Notation

Henceforth, capital letters X, Y and Z denote Banach spaces. Most of the time we work with real scalars but, in a few places, we need complex scalars. Moreover, B_X and S_X are the closed unit ball and the unit sphere of X, X^* is the first dual of X, X^{**} the second dual (or bidual), and $X^{*(n)}$ the n-th dual.

Given a Banach space X, its elements will be denoted by small letters x, y, z; the elements of its dual X^* by x^*, y^*, and the elements of X^{**} by x^{**}, y^{**}, etc. Given $x \in X$ and $x^* \in X^*$, $\langle x^*, x\rangle$ denotes the value attained by x^* at x. We denote by $J_X \colon X \longrightarrow X^{**}$ the canonical embedding of X into X^{**}. In most cases we identify X with $J_X(X)$.

The symbol w will stand for the weak topology $\sigma(X, X^*)$ on X. Thus, in X^* w is $\sigma(X^*, X^{**})$ and w^* is $\sigma(X^*, X)$ when this notation is sufficiently clear. For instance, if we say that x^{**} is a w^*-cluster point of a subset A of X, w^* stands for the topology $\sigma(X^{**}, X^*)$ of X^{**}.

The norm closure of a subset A of X is denoted by $\overline{A}$; its closure with respect to w is represented by $\overline{A}^{\sigma(X,X^*)}$ or $\overline{A}^{w}$; the annihilator of A in X^* is

$$A^\perp := \{x^* \in X^* \colon \langle x^*, x\rangle = 0,\ \forall x \in A\}.$$

Analogously, given a subset B of X^*, its closure with respect to the weak* topology of X^* is denoted by $\overline{B}^{\sigma(X^*,X)}$ or $\overline{B}^{w^*}$. Moreover, $B_\perp$ denotes the annihilator $\{x \in X \colon \langle x^*, x\rangle = 0,\ \forall x^* \in B\}$.

The subspaces of a Banach space X that we consider are not necessarily closed; given a nonempty subset A of X, $\operatorname{span}\{A\}$ represents the subspace generated by A and $\overline{\operatorname{span}}\{A\}$ is the norm-closure of $\operatorname{span}\{A\}$.

Given a pair of Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear maps –henceforth operators– acting between X and Y.

An isomorphism is an injective operator $T \in \mathcal{L}(X, Y)$ with closed range (not necessarily bijective). Note that for every isomorphism $T \colon X \longrightarrow Y$, there exists a constant $d > 0$ such that $d^{-1} \le \|T(x)\| \le d$ for all $x \in S_X$. So we shall say that T is a *d-injection*, or a *metric injection* if $d = 1$.

We will say that *we identify two Banach spaces* X and Y when there is a bijective isomorphism $A \colon X \longrightarrow Y$. Similarly, we will say that *we identify two* *operators* $S \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(V,X)$ when there are two bijective isomorphisms $A\colon Y \longrightarrow V$ and $B\colon X \longrightarrow Z$ so that $S = BTA$.

The null operator and the identity on X are denoted by 0_X and I_X. Given $T \in \mathcal{L}(X,Y)$, its kernel and range are $N(T)$ and $R(T)$, its co-kernel is $Y/\overline{R(T)}$, its *conjugate operator* is $T^*\colon Y^* \longrightarrow X^*$, its second conjugate is T^{**} and $T^{*(n)}$ represents the n-th conjugate operator of T.

The class of all operators is denoted by $\mathcal{L}$. Given a class of operators $\mathcal{A}$, its component of operators acting between X and Y is

$$\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y).$$

In the case $X = Y$ we usually write $\mathcal{A}(X)$ instead of $\mathcal{A}(X,X)$.

Given a closed subspace E of a Banach space X, $J_E\colon E \longrightarrow X$ denotes the natural embedding of E into X, and $Q_E\colon X \longrightarrow X/E$ represents the quotient operator; we recall that ${Q_E}^*$ maps $(X/E)^*$ onto $E^\perp$ isometrically; moreover, since $N({J_E}^*) = E^\perp$, the operator ${J_E}^*$ induces an isometry from $X^*/E^\perp$ onto E^* that maps $x^* + E^\perp$ to $x^* \circ J_E$; thus, we identify $(X/E)^*$ with $E^\perp$, $X^*/E^\perp$ with E^*, $(X/E)^{**}$ with $X^{**}/E^{\perp\perp}$ and E^{**} with $E^{\perp\perp}$.

Given a set I of indices, $\ell_p(I)$ denotes the Banach space of all families of real numbers $(x_i)_{i\in I}$ endowed with the norm $\|(x_i)_{i\in I}\|_p := \left(\sum_{i\in I} |x_i|^p\right)^{1/p}$ if $1 \leq p < \infty$, and $\|(x_i)_{i\in I}\|_\infty := \sup_{i\in I} |x_i|$. Given a family of Banach spaces $\{X_i : i \in I\}$, we denote by $\ell_p(I,X)$ the Banach space of all families $(x_i)_{i\in I}$ with $x_i \in X_i$ endowed with the norm $\|(x_i)_{i\in I}\|_p := \|(\|x_i\|)_{i\in I}\|_p$. However, in the case $I = \mathbb{N}$ and $X_i = X$ for all i, we just write $\ell_p(X)$, and given a couple X_1 and X_2 of Banach spaces, $\ell_p(\{1,2\}, X_i)$ is denoted by $X_1 \oplus_p X_2$.

Chapter 1

The origins of tauberian operators

In 1976, Kalton and Wilansky [115] coined the term *tauberian* to designate those bounded operators $T\colon X \longrightarrow Y$ acting between Banach spaces that satisfy

$$T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y\,. \tag{1.1}$$

In this chapter we intend to answer the two following questions:

Question 1. Why are they called *tauberian*?

Question 2. When and why did those operators first appear?

1.1 Tauberian conditions in summability theory

In order to answer Question 1, we need to go back in time to 1897, when Tauber proved that if

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = \lambda \tag{1.2}$$

and

$$\lim_n a_n/n = 0, \tag{1.3}$$

then

$$\sum_{n=0}^{\infty} a_n = \lambda. \tag{1.4}$$

This is a conditioned converse of Abel's theorem which states that (1.2) is a consequence of (1.4) without the mediation of any hypothesis such as (1.3). Since then, it has been customary to classify certain types of theorems into abelian (or direct) or tauberian according to the following abstract and rather vague scheme: consider a category $\mathcal{A}$ and let p_1 and p_2 denote a pair of properties. Suppose that the following statement holds:

(1.5) *let f be a fixed morphism in the category $\mathcal{A}$; if x verifies p_1, then $f(x)$ verifies p_2.*

Let us also assume that its converse fails but it becomes true when an additional condition (t), like (1.3) in Tauber's theorem, is satisfied. In that case, condition (t) is a *tauberian condition*, the statement

(1.6) *if the condition (t) holds and $f(x)$ verifies p_2, then x verifies p_1;*

is a *tauberian theorem*, and statement (1.5) is an *abelian theorem.* Indeed, Hardy [97] described the above classification with the following words:

> "A tauberian theorem may be defined as the corrected form of the false converse of an abelian theorem. An abelian theorem asserts that, if a sequence or function behaves regularly, then some average of it behaves regularly."

It is not simple at all to provide a more precise definition of a tauberian theorem in regard to the variety of fields where tauberian theorems occur: [37], [55], [167], [168], and so on.

Let us now fix an operator $T\colon X \longrightarrow Y$ (henceforth, when we say *operator* we mean bounded linear operator) and consider the following statement:

(1.7) *(x_n) contains a weakly convergent subsequence if (Tx_n) is convergent and the tauberian condition of boundedness of (x_n) holds.*

The main result in [115] establishes that statement (1.1) is satisfied by T if and only if (1.7) is so. The formal similitude between statements (1.6) and (1.7) demonstrate the tauberian character of those operators satisfying (1.1), which answers Question 1.

1.2 Tauberian matrices

With regard to Question 2, we shall see that the concept of tauberian operator deepens its roots in summability theory, a branch of mathematics whose original purpose was assigning limits to sequences that are not convergent in the usual sense. One of the typical techniques in summability theory is the matrix method: consider an infinite matrix $A = (a_{ij})_{i=1}^{\infty}{}_{j=1}^{\infty}$. A sequence of complex numbers $x = (x_i)_i$ is said to be A-summable (or A-limitable) if the sequence $Ax := (\sum_{j=1}^{\infty} a_{ij}x_j)_i$ is well defined and convergent. In that case, $\lim_i Ax$ is denoted $\lim_A x_i$ and assigned to the sequence x. Thus, denoting by c the set of all convergent sequences of real numbers, answers to the following questions are required:

- What is the set ω_A formed by all the sequences x for which Ax exists?
- What is the set c_A formed by all the A-summable sequences?
- Does c_A contain c?
- If $c \subset c_A$, does A preserve limits?

Remark 1.2.1. When $c \subset c_A$, matrix A is called *conservative*. Moreover, if $\lim_i x_i = \lim_A x_i$ for all $(x_i) \in c$, then A is called *regular*.

A genuine example of the interest in regular matrices that sum bounded divergent sequences is provided by Féjer's theorem, which uses the Cesàro matrix to recover any function $f \in L_p(0, 2\pi)$ from its Fourier series.

Intensive research on matrix methods was only possible after the discovery in 1911 of the classical Toeplitz-Silverman conditions which assert that a matrix $A = (a_{ij})_{i=1}^{\infty}{}_{j=1}^{\infty}$ is conservative if and only if

(i) $\|A\| := \sup_i \sum_j |a_{ij}| < \infty$;

(ii) *there exists* $s := \lim_i s_i$, *where* $s_i := \sum_j a_{ij}$;

(iii) *there exists* $a_j := \lim_i a_{ij}$ *for each* j.

Indeed, the Toeplitz-Silverman conditions allow us to identify every conservative matrix A with an operator $S_A\colon c \longrightarrow c$ and also with an operator $T_A\colon \ell_\infty \longrightarrow \ell_\infty$, both of them defined by the expression Ax when x belongs respectively to the domains c or ℓ_∞, so $\|S_A\| = \|T_A\| = \|A\|$.

Searching for criteria to decide whether or not a conservative matrix sums a bounded divergent sequence became an engaging activity during the 1950s: [125], [161], [171], [172], etc. The next decade brought new characterizations with an undoubtedly algebraic character. Thus, Copping [44] obtained the following result:

(1.8) *Let A be a conservative matrix such that T_A is injective. Then A sums no bounded divergent sequence if and only if there is a conservative matrix B which is a left inverse of A.*

In 1964, Wilansky [168] improved Copping's result by replacing the injectivity of T_A with the weaker condition of injectivity of S_A. For the same matrices that same year, Berg [28] obtained the following characterization:

(1.9) *Let A be a conservative matrix such that S_A is injective. Then A sums no bounded divergent sequence if and only if A is not a left-topological divisor of zero, that is, there exists $\varepsilon > 0$ such that for every norm one element $x \in c$, $\|Ax\| \geq \varepsilon$.*

Obviously, if S_A is injective, then A is a left-topological divisor of zero if and only if the range of S_A is not closed. A definitive improvement dropped the hypothesis of injectivity of S_A in (1.9):

(1.10) *A conservative matrix A sums no bounded divergent sequence if and only if the operator $S_A\colon c \longrightarrow c$ has closed range and finite-dimensional null-space.*

Wilansky called *tauberian* the conservative matrices that sum no bounded divergent sequence [170].

Statement (1.10) was obtained with different proofs by Crawford in 1966 [45], Whitley in 1967 [166], and Garling and Wilansky in 1972 [63]. Each of the above mentioned papers meant a new stage in the increasing presence of functional analysis in summability theory, which paved the way for the first appearance of tauberian operators. Crawford's main contribution to the attainment of (1.10) is the introduction of duality techniques by means of the following result:

(1.11) *Given a conservative matrix A, we have ${T_A}^{-1}(c) \subset c$ if and only if ${S_A^{**}}^{-1}(c) \subset c$.*

Note that, in general, the operators T_A and S_A^{**} are not equal. Indeed, T_A is represented by the matrix A, but since the canonical embedding of c into its bidual space, ℓ_∞, maps every sequence (x_i) to $(\lim_i x_i, x_1, x_2, \ldots)$, the operator S_A^{**} is represented by the matrix

$$P = \begin{pmatrix} s & a_1 & a_2 & \cdots \\ s_1 - s & a_{11} - a_1 & a_{12} - a_2 & \cdots \\ s_2 - s & a_{21} - a_1 & a_{22} - a_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Crawford overcomes that difficulty by substitution of c for an isomorphic space, c_0, and taking advantage of the fact that for every operator $L\colon c_0 \longrightarrow c_0$, both L and L^{**} are representable by the same matrix. Thus, he considers the surjective isomorphism $U\colon c_0 \longrightarrow c$ that maps e_1 to the constant sequence $(1, 1, \ldots)$ and e_i to e_{i-1} for $i > 1$, and takes the operator $R := U^{-1} S_A U$ which is matrix representable by P. Next, by means of classical techniques of matrix summability, Crawford obtains the following result:

(1.12) *${T_A}^{-1}(c) \subset c$ if and only if $(R^{**})^{-1}(c_0) \subset c_0$;*

and since R is an isomorphism, statement (1.12) yields (1.11).

1.3 Tauberian operators

Garling and Wilansky's innovation with respect to Crawford's proof is that they study a general operator $T\colon X \longrightarrow Y$ satisfying ${T^{**}}^{-1}(Y) \subset X$ prior to consideration of the particular case $X = Y = c$. Thus they deduce the following results:

(1.13) *Let X and Y be a pair of Banach spaces, and $T\colon X \longrightarrow Y$ an operator. Consider the conditions*

(i) $T^{**-1}(Y) \subset X$,

(ii) $N(T^{**}) \subset X$,

(iii) $N(T)$ *is reflexive.*

Then (i)⇒(ii)⇒(iii) *and neither implication can be reversed.*

(1.14) *Moreover, for T with closed range the three conditions are equivalent.*

Garling and Wilansky obtained (1.10) with the following argument: if A is a conservative matrix that sums no bounded divergent sequence, then Crawford's result (1.11) yields $S_A^{**-1}(c) \subset c$, and by condition (i) in (1.13) it follows that $N(S_A^{**})$ is reflexive, and therefore finite-dimensional because c contains no infinite-dimensional reflexive subspace. They offer no new proof of the fact that $R(S_A)$ is closed. Conversely, if $R(S_A)$ is closed and $N(S_A)$ is finite-dimensional, then $N(S_A)$ is trivially reflexive, so (1.14) shows that $S_A^{**-1}(c) \subset c$, hence (1.11) yields A sums no bounded divergent sequence.

As far as we know, Crawford's statement (1.11) contains the first application of tauberian operators, but condition (i) in (1.13) is the first appearance of tauberian operators with the same level of generality given in (1.1). Garling and Wilansky stimulated interest in tauberian operators posing the following questions:

Question 1.3.1. For which pairs of non-reflexive Banach spaces X and Y can the assumption "closed range" be dropped in (1.14)?

Question 1.3.2. For which non-reflexive Banach spaces X and Y does condition (i) in (1.13) imply $R(T)$ closed?

Sufficient and necessary conditions for the equivalence between the three clauses of (1.13) were found by Kalton and Wilansky in [115], published in 1976. Their paper, which only uses functional analysis and Banach space theory, popularized the term *tauberian* for the operators defined in (1.1).

Full answers to Questions 1.3.1 and 1.3.2 are still unknown. However, the following sufficient condition was shown in [115]:

(1.15) *If X contains no reflexive infinite-dimensional subspace and $T\colon X \longrightarrow Y$ is tauberian, then T is upper semi-Fredholm.*

Let us recall that an operator $T\colon X \longrightarrow Y$ is said to be *upper semi-Fredholm* if it has closed range and finite dimensional kernel.

The reader will realize that (1.15), combined with Crawford's result (1.11), yields an immediate proof of (1.10). This observation was made by Wilansky in [170, Section 17.6]. But the most important fact concerning [115] is that it led to further research focused on tauberian operators. In fact, Kalton and Wilansky suggested that Statement 1.15 could be extended to more Banach spaces X other than those with no reflexive infinite-dimensional subspaces. In particular, as c_0 is isomorphic to a Banach space of continuous functions, they posed the following question:

Question 1.3.3. Given a pair of spaces of continuous functions, $C(K)$ and $C(L)$, is a tauberian map $T\colon C(K) \longrightarrow C(L)$ an isomorphism in some sense?

Kalton and Wilansky also asked in [115] about duality of tauberian operators:

Question 1.3.4. When is it true that an operator $T\colon X \longrightarrow Y$ is tauberian if and only if T^{**} is so?

Question 1.3.4 was suggested by the fact that its answer is positive when T has a closed range.

Besides, an operator $T\colon X \longrightarrow Y$ is tauberian if and only if the operator $T^{co}\colon X^{**}/X \longrightarrow Y^{**}/Y$, given by $T^{co}(x^{**} + X) := T^{**}(x^{**}) + Y$, is injective. So Kalton and Wilansky asked:

Question 1.3.5. Given an operator $T \in \mathcal{L}(X, Y)$, when is T^{co} an isomorphism?

Answers to these questions and subsequent results have been collected and organized in the chapters indicated in the next section.

1.4 Notes and Remarks

As we have already said, the first work entirely devoted to tauberian operators is [115], which came to light in 1976 from the hands of Kalton and Wilansky. But there are two other papers concerning tauberian operators, [49] and [174], published respectively in 1974 and in 1976. The authors of [115] and [174], prior to submission, were acquainted with the contents of the three mentioned papers, but a closer look at them reveals that actually [49], [115] and [174] are mathematically independent and pursue different ends. Thus, in [174], Yang extends the theory of Fredholm operators to the case of tauberian operators with closed range. His results lead to a presentation of reflexivity in Banach spaces from a homological point of view. In [49], Davis, Figiel, Johnson and Pełczyński obtain their famous factorization for weakly compact operators, which is the main source of examples of tauberian operators. It shall be the subject of further study in Chapter 3. Finally, as has been thoroughly explained in Chapter 1, paper [115] can be regarded as the continuation of the work of Garling and Wilansky [63] published in 1972, putting an endpoint to a longstanding problem in summability theory: the characterization of tauberian matrices. These arguments have led us to consider [63] and [115] as the seminal papers in the study of tauberian operators. Let us notice that the role played by tauberian operators in the solution of the aforementioned problem of tauberian matrices has been recognized by some summability theorists [116, p. 262].

Since this book is not primarily concerned with summability theory, the reader interested in that subject should consult [37], [116] or [170]. The first two references are very exhaustive monographs, while the third one is concise but contains most of the material about summability dealt with in Chapter 1, including the results in Crawford's Ph.D. thesis. The historical exposition about tauberian operators described in this chapter has been borrowed from [86].

Proofs for statements (1.13) and (1.15), as well as sufficient and necessary conditions for the equivalence between the statements (1.13), can be found in Chapter 2.

Question 1.3.3 was partially solved by Lotz, Peck and Porta [124], who proved that a compact space K is scattered if and only if every injective tauberian operator from $C(K)$ into a Banach space Y is an isomorphism.

Regarding Question 1.3.4, it is immediate, after Proposition 2.1.3, that T is tauberian provided that T^{**} is so as well. However, we shall see in Chapter 3 that the converse fails.

A partial answer to Question 1.3.1 is given in Proposition 2.1.12, which states that if X is a weakly sequentially complete Banach space, then every operator $T\colon X \longrightarrow Y$ with property (N) is tauberian. Moreover, if X is contained in a space L-embedded in its bidual, then T^{co} is an isomorphism. This fact, proved by Bermúdez and Kalton [29] and included in Chapter 6, means a partial answer to Question 1.3.5.

The operators T for which T^{co} is an isomorphism have been studied by Yang [175] and by Rosenthal, who called them *strongly tauberian* [147]. The most important structural properties and applications of strongly tauberian operators are dealt with in Chapter 6.

Chapter 2

Tauberian operators. Basic properties

This chapter is devoted to the general properties and characterizations of tauberian operators, with special emphasis on their relationship to reflexivity.

Tauberian operators and their most elementary properties are formally introduced in Section 2.1. One of them is the following: an operator $T \in \mathcal{L}(X,Y)$ is tauberian if and only if $T(B_X)$ is closed and $N(T^{**}) = N(T)$, which implies that $N(T)$ is reflexive.

Section 2.2 exhibits the main characterizations of tauberian operators which will be used throughout this book, sometimes without explicit mention. In particular, it contains Kalton and Wilansky sequential characterizations for tauberian operators (Theorem 2.2.4) and for operators T with $N(T^{**}) = N(T)$ (Theorem 2.2.2), which are derived from the Eberlein-Smulian theorem. A sequel is given in Theorem 2.2.7, which proves that an operator $T \in \mathcal{L}(X,Y)$ is tauberian if and only if, for every compact operator $K \in \mathcal{L}(X,Y)$, the kernel $N(T+K)$ is reflexive.

Section 2.3 pays particular attention to the research of Neidinger and Rosenthal on the action of tauberian operators over closed convex sets, which has a significant impact on the study of the Radon-Nikodým and the Krein-Milman properties, as we shall see in Chapter 5. Its main result states that $T \in \mathcal{L}(X,Y)$ is a tauberian operator if and only if $T(B_E)$ is closed for every closed subspace E of X. This characterization is a consequence of a fundamental theorem of James, which asserts that a Banach space X is reflexive if and only if every functional $x^* \in X^*$ attains its norm on B_X.

Finally, Section 2.4 describes some results, due to Holub, on the action of tauberian operators over shrinking basic sequences and boundedly complete basic sequences. Note that the closed linear span of a basic sequence is a reflexive subspace if and only if that sequence is both shrinking and boundedly complete.

2.1 Basic facts about tauberian operators

Let us start by recalling the definition of a tauberian operator formally introduced by Kalton and Wilansky in [115].

Definition 2.1.1. An operator $T \in \mathcal{L}(X,Y)$ is said to be *tauberian* whenever $T^{**-1}(Y) \subset X$.

The notion of weakly compact operator is inseparable from that of tauberian operator. As working definition, we adopt the following:

Definition 2.1.2. An operator $T \in \mathcal{L}(X,Y)$ is said to be *weakly compact* whenever $T^{**}(X^{**}) \subset Y$.

The action of a tauberian operator in its domain is, to some degree, opposite to the action of a weakly compact operator. Indeed, let us agree to call non-trivial any element $x \in X^{**} \setminus X$. Thus, an operator T is tauberian if no non-trivial element is mapped by T^{**} to a trivial element, while T is weakly compact if T^{**} maps each non-trivial element to a trivial one.

Henceforth, the class of all tauberian operators and that of all weakly compact operators will be respectively denoted by $\mathcal{T}$ and $\mathcal{W}$. According to our notation, their respective components of operators acting between the spaces X and Y will be represented by $\mathcal{T}(X,Y)$ and $\mathcal{W}(X,Y)$.

The most basic properties regarding the interaction between the classes $\mathcal{T}$ and $\mathcal{W}$ are included in the following result. Its proof is straightforward.

Proposition 2.1.3. *Let T and S be a pair of operators in $\mathcal{L}(X,Y)$, and U an operator in $\mathcal{L}(Y,Z)$. Then the following statements hold:*

(i) *if both T and U are tauberian, then UT is tauberian;*

(ii) *if UT is tauberian, then T is tauberian;*

(iii) *T is tauberian and weakly compact if and only if X is reflexive;*

(iv) *if T is tauberian and S is weakly compact, then $T+S$ is tauberian.*

Note that, unlike $\mathcal{W}$, the class $\mathcal{T}$ is far from being an operator ideal. In particular, for each Banach space X, the identity operator $I_X : X \longrightarrow X$ is tauberian, while the null operator $0_X : X \longrightarrow X$ is weakly compact.

Proposition 2.1.4. *Let Z be a closed subspace of X. Then the following statements hold:*

(i) *the natural embedding $J_Z : Z \longrightarrow X$ is tauberian;*

(ii) *the quotient operator $Q_Z : X \longrightarrow X/Z$ is tauberian if and only if Z is reflexive.*

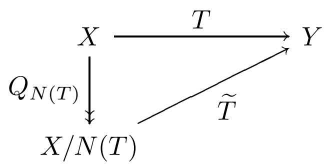
Proof. (i) Since $Z^{\perp\perp} \cap X = Z$ and $Z^{\perp\perp}$ is identified with Z^{**}, the proof of the statement is easy.

(ii) It is enough to observe that $(Q_Z)^{**}$ can be identified with the quotient operator $Q_{Z^{\perp\perp}}$, and that Z is reflexive if and only if $Z = Z^{\perp\perp}$. $\square$

Every operator $T \in \mathcal{L}(X, Y)$ can be factorized as

$$T = \widetilde{T} \circ Q_{N(T)} \tag{2.1}$$

where $\widetilde{T}\colon X/N(T) \longrightarrow Y$ is given by $\widetilde{T}(x + N(T)) := Tx$ for every $x \in X$. That yields the following commutative diagram:



Theorem 2.1.5. *For every $T \in \mathcal{L}(X, Y)$, the following statements hold:*

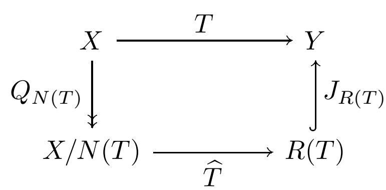
(i) *the operator T is tauberian if and only if $\widetilde{T}$ is tauberian and $N(T)$ is reflexive;*

(ii) *assume that $R(T)$ is closed; then T is tauberian if and only if $N(T)$ is reflexive.*

Proof. (i) Let us assume that T is tauberian. Thus, as $T = \widetilde{T} \circ Q_{N(T)}$, Proposition 2.1.3 shows that $Q_{N(T)}$ is tauberian, hence $N(T)$ is reflexive by Proposition 2.1.4. In order to prove that $\widetilde{T}$ is tauberian, note that $N(T) = N(T)^{\perp\perp}$, so we identify $(X/N(T))^{**}$ with $X^{**}/N(T)$, and consequently, $(\widetilde{T})^{**}$ can be regarded as a map between $X^{**}/N(T)$ and Y^{**}. Thus, given $x^{**} + N(T) \in X^{**}/N(T)$ such that $(\widetilde{T})^{**}(x^{**}+N(T)) = T^{**}x^{**} \in Y$, we have $x^{**} \in X$, so $x^{**}+N(T) \in X/N(T)$, concluding that $\widetilde{T}$ is tauberian.

For the converse, if $\widetilde{T}$ is tauberian and $N(T)$ is reflexive, then $Q_{N(T)}$ is tauberian by Proposition 2.1.4, and by Proposition 2.1.3 we see that $T = \widetilde{T} \circ Q_{N(T)}$ is tauberian.

(ii) The 'only if' implication is a consequence of (i). For the 'if' part, since $R(T)$ is closed, T factorizes as



where $\widehat{T}$ maps every $x + N(T)$ to Tx. Note that $\widehat{T}$ is tauberian because it is a surjective isomorphism. Moreover, since $R(T)$ is closed and $N(T)$ is reflexive, Proposition 2.1.4 yields that $J_{R(T)}$ and $Q_{N(T)}$ are both tauberian. Therefore, $T = J_{R(T)} \circ \widehat{T} \circ Q_{N(T)}$ is tauberian. $\square$

The argument of the following lemma will be applied on many occasions.

Lemma 2.1.6. *For every $T \in \mathcal{L}(X,Y)$ and every bounded subset A of X, we have:*

(i) $T^{**}(\overline{A}^{w^*}) = \overline{T(A)}^{w^*}$;

(ii) *if A is convex, then* $T^{**}(\overline{A}^{w^*}) \cap Y = \overline{T(A)}$.

In particular, $T^{**}(B_{X^{**}}) = \overline{T(B_X)}^{w^*}$ *and* $T^{**}(B_{X^{**}}) \cap Y = \overline{T(B_X)}$.

Proof. (i) Since T^{**} is weak* continuous and $\overline{A}^{w^*}$ is weak* compact, we have

$$T^{**}(\overline{A}^{w^*}) = \overline{T(A)}^{w^*}.$$

(ii) The weak closure of $T(A)$ equals $\overline{T(A)}^{w^*} \cap Y$, so statement (i) yields $\overline{T(A)}^{w} = T^{**}(\overline{A}^{w^*}) \cap Y$, and since the weak closure of any convex set equals its norm closure, we get $\overline{T(A)} = T^{**}(\overline{A}^{w^*}) \cap Y$.

The remaining results are a consequence of Goldstine's theorem, which states that $B_{X^{**}} = \overline{B_X}^{w^*}$. □

The following characterizations are fundamental in the study of tauberian operators.

Theorem 2.1.7. *For every operator $T \in \mathcal{L}(X,Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *$N(T^{**}) = N(T)$ and $T(B_X)$ is closed;*

(c) *$N(T^{**}) = N(T)$ and $\overline{T(B_X)}$ is contained in $R(T)$.*

Proof. (a)⇒(b) The equality $N(T^{**}) = N(T)$ is immediate. In order to prove that $T(B_X)$ is closed, take $y \in \overline{T(B_X)}$. By Lemma 2.1.6, there exists $x^{**} \in B_{X^{**}}$ so that $y = T^{**}x^{**}$. But T is tauberian, so $x^{**} \in B_X$, hence $y \in T(B_X)$.

(b)⇒(c) Trivial.

(c)⇒(a) Let x^{**} be a norm-one element in X^{**} such that $y := T^{**}x^{**} \in Y$. By Lemma 2.1.6, $y \in \overline{T(B_X)}$, and by hypothesis, $\overline{T(B_X)}$ is contained in $R(T)$, so $y = Tz$ for some $z \in X$. Thus $x^{**} - z \in N(T^{**})$, and as $N(T^{**}) = N(T)$ by assumption, it follows that $x^{**} \in X$, which proves that T is tauberian. □

It is convenient to name those operators T for which $N(T)$ equals $N(T^{**})$. We adopt the following notation introduced by Kalton and Wilansky in [115].

Definition 2.1.8. An operator $T \in \mathcal{L}(X,Y)$ is said to have *property* (N) whenever $N(T^{**}) = N(T)$.

Proposition 2.1.9. *An operator $T \in \mathcal{L}(X,Y)$ has property (N) if and only if $N(T)$ is reflexive and* $\overline{R(T^*)}^{w^*} = \overline{R(T^*)}$.

Proof. For every operator T, $N(T)^\perp = \overline{R(T^*)}^{w^*}$ (see Theorem 4.14 in [148]). Moreover, $N(T)$ is reflexive if and only if $N(T)$ equals $N(T)^{\perp\perp}$. Thus the result is a consequence of:

$$N(T) \subset N(T)^{\perp\perp} = \left(\overline{R(T^*)}^{w^*}\right)^\perp \subset \overline{R(T^*)}^\perp = N(T^{**}). \qquad \square$$

Theorem 2.1.7 and Proposition 2.1.9 show that the following implications hold for every operator T:

'*T tauberian $\Rightarrow$ T has property (N)*'
'*T has property (N) $\Rightarrow$ $N(T)$ is reflexive*'.

Theorem 2.1.5 and the examples below show that the converse implications are valid when T has closed range, but fail in general.

Example 2.1.10. Let $C \in \mathcal{L}(c_0, c_0)$ be the Cesàro operator, defined by

$$C(x_n)_n := \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n .$$

The operator C has property (N) but is not tauberian.

Proof. Indeed, C^{**} is injective and $C^{**}((1,-1,1,-1,\ldots)) \in c_0$. $\square$

Example 2.1.11. The operator $T\colon c_0 \longrightarrow \ell_2$ defined by $T(x_n) := (x_n/n)$ has property (N) but it is not tauberian. Moreover, T is weakly compact.

Proof. In fact, T is weakly compact because ℓ_2 is reflexive. Moreover, since c_0 is not reflexive, T cannot be tauberian. However, T^{**} maps every $(x_n) \in \ell_\infty$ to (x_n/n). So T^{**} is injective, which implies that T has property (N). $\square$

The context of Example 2.1.11 describes very well the opposite character of tauberian operators and weakly compact operators. Indeed, $\mathcal{L}(c_0, \ell_2) = \mathcal{W}(c_0, \ell_2)$ and $\mathcal{T}(c_0, \ell_2) = \emptyset$. Therefore, having property (N) is much weaker than being tauberian. However, every operator $T\colon X \longrightarrow Y$ with property (N) is tauberian if X is weakly sequentially complete.

Proposition 2.1.12. *Let X be a weakly sequentially complete Banach space, and let $T\colon X \longrightarrow Y$ be an operator. If T has property (N), then T is tauberian.*

Proof. According to Theorem 2.1.7, we only need to prove that the identity $N(T) = N(T^{**})$ implies that $T(B_X)$ is norm closed. To do so, take an element $y \in \overline{T(B_X)}$ and choose a sequence (x_n) in B_X so that $T(x_n) \underset{n}{\longrightarrow} y$. By Rosenthal's ℓ_1-theorem (Theorem A.3.10), (x_n) contains a weakly Cauchy subsequence or a subsequence (u_n) equivalent to the unit vector basis of ℓ_1.

In the latter case there would exist $u^{**} \in \overline{\{u_{2n} - u_{2n+1}\}}^{w^*} \setminus X$, and therefore $T^{**}(u^{**}) = 0$, in contradiction with $N(T^{**}) = N(T)$. Hence, the only possibility is that (x_n) contains a weakly Cauchy subsequence (v_n). But X is sequentially weakly complete, so there exists $z \in B_X$ such that $v_n \xrightarrow[n]{w} z$. Thus, we have a convex block sequence (z_n) of (v_n) such that $z_n \xrightarrow[n]{} z$. But $\|z_n\| \leq 1$ for all n, so $y = T(z) \in T(B_X)$. $\square$

The following examples show that property (N) is stronger than having reflexive kernel.

Example 2.1.13. The operator $T \in \mathcal{L}(c_0, c_0)$ that maps every element $(x_n)_n$ to $(x_n - x_{n+1})_n$ has reflexive kernel but fails property (N). Moreover, $T(B_{c_0})$ is not closed.

Proof. Indeed, T is injective, but $N(T^{**})$ is the space of all constant sequences, so T fails property (N). In order to show that $\overline{T(B_{c_0})} \not\subset R(T)$, let us take $(z_i) \in c_0$ so that $|\sum_{i=1}^n z_i| \leq 1/2$ for all n and $\sum_{i=1}^\infty z_i$ does not converge. Thus $(z_i) \notin R(T)$. In fact,

$$R(T) = \{(y_i) \in c_0 \colon \sum_{i=1}^\infty y_i \text{ converges}\}.$$

Moreover, given $(y_i) \in R(T)$,

$$T^{-1}\big((y_i)\big) = \Big(\sum_{j=1}^\infty y_j, \sum_{j=2}^\infty y_j, \sum_{j=3}^\infty y_j, \dots\Big).$$

For every $n \in \mathbb{N}$, let $P_n \colon c_0 \longrightarrow c_0$ be the projection with $R(P_n) = \operatorname{span}\{e_i\}_{i=1}^n$ and $N(P_n) = \overline{\operatorname{span}}\{e_i\}_{i=n+1}^\infty$, where $\{e_i\}_{i=1}^\infty$ is the unit vector basis of c_0. Thus $P_n\big((z_i)\big) \in T(B_{c_0})$ for all n, so $(z_i) \in \overline{T(B_{c_0})}$. $\square$

The following example exhibits an operator $T \colon X \longrightarrow Y$ which has reflexive kernel but is not tauberian, despite $T(B_X)$ being closed.

Example 2.1.14. The operator $T \in \mathcal{L}(\ell_1, \ell_2)$, defined by $T(x_n) := (x_n)$, maps B_{ℓ_1} onto a closed set and has reflexive kernel, but fails property (N).

Proof. Let $L \colon \ell_2 \longrightarrow c_0$ be the operator given by $L(x_n) := (x_n)$. Thus T is the conjugate of L, so $T(B_{\ell_1})$ is weak* compact, hence norm closed.

Clearly T is injective, so $N(T)$ is trivially reflexive. However, $R(T^*)$ is a separable subspace of ℓ_∞, hence $N(T^{**}) \neq \{0\}$. $\square$

The following example shows that, given an operator $T \colon X \longrightarrow Y$, the conditions $T(B_X)$ closed and $\overline{R(T^*)} = \overline{R(T^*)}^{w^*}$ are not enough to assure that T is tauberian.

Example 2.1.15. The null operator $0_{\ell_1} \colon \ell_1 \longrightarrow \ell_1$ maps B_{ℓ_1} onto a closed set and satisfies the identity $\overline{R({0_{\ell_1}}^*)} = \overline{R({0_{\ell_1}}^*)}^{w^*}$, but its kernel is not reflexive, so 0_{ℓ_1} is not tauberian.

By virtue of Theorem 2.1.5, the first examples of non-trivial tauberian operators are the operators with closed range and finite dimensional kernel, usually called *upper semi-Fredholm operators* (see Section A.1). Tauberian operators with closed range were studied by Yang [174], who called them generalized semi-Fredholm transformations.

Since the class $\mathcal{T}$ contains Φ_+, the following question arises naturally and establishes a general pattern followed by many researchers:

Question 2.1.16. Which properties of the operators in Φ_+ can be transferred to the operators in $\mathcal{T}$, and vice versa?

For instance, all the statements in Proposition 2.1.3 are valid if the words *'tauberian'* and *'weakly compact'* are respectively replaced by *'upper semi-Fredholm'* and *'compact'*.

Of course, there are reasonable properties which cannot be transferred from Φ_+ to $\mathcal{T}$. The topological structure of $\mathcal{T}$ offers an example in that direction. In fact, it is well known that the components of the class Φ_+ are always open. That assertion follows from the fact that if $T \in \Phi_+(X,Y)$, then X can be decomposed as $X = N(T) \oplus X_1$ where $T|_{X_1}$ is an isomorphism; thus, denoting $\beta := \inf\{\|Tx\| : x \in S_{X_1}\}$, given any operator $S \in \mathcal{L}(X,Y)$ such that $\|T - S\| < \beta$, it follows that $N(S) \subset N(T)$ and that $S|_{X_1}$ is an isomorphism, so $S \in \Phi_+$. Nevertheless, the following example shows that the components of $\mathcal{T}$ are not always open.

Example 2.1.17. Given a non-reflexive space X, the operator $T\colon \ell_2(X) \longrightarrow \ell_2(X)$ defined by

$$T\big((x_n)\big) := (x_n/n), \quad (x_n) \in \ell_2(X)$$

is tauberian and belongs to the topological boundary of $\mathcal{T}\big(\ell_2(X)\big)$.

Proof. We can identify the bidual of $\ell_2(X)$ with $\ell_2(X^{**})$ and T^{**} maps every (x_n^{**}) to (x_n^{**}/n). So it is clear that T is tauberian.

In order to prove that T belongs to the boundary of $\mathcal{T}\big(\ell_2(X)\big)$, it is enough to realize that for every positive integer k, the operator $T_k\colon \ell_2(X) \longrightarrow \ell_2(X)$ defined by

$$T_k(x_n) := \Big(x_1, \frac{x_2}{2}, \ldots, \frac{x_k}{k}, 0, 0, \ldots\ldots\Big)$$

satisfies $\|T - T_k\| = 1/(k+1)$ and it is not tauberian because its kernel is not reflexive. □

Nevertheless, Example 2.1.17 can still be used to trace an analogy between Φ_+ and $\mathcal{T}$. Indeed, the set of all upper semi-Fredholm operators acting between X and Y with complemented range in Y equals the set

$$\mathcal{K}_l(X,Y) := \{T \in \mathcal{L}(X,Y)\colon I_X - LT \in \mathcal{K} \text{ for some } L \in \mathcal{L}(Y,X)\}$$

where $\mathcal{K}$ denotes the class of all compact operators (see [160, IV.13 Problems]). Note that the inclusion of $\mathcal{K}_l(X,Y)$ in $\Phi_+(X,Y)$ is strict in general because every Banach space non-isomorphic to a Hilbert space contains non-complemented closed subspaces.

Analogously, but for different reasons, the set

$$\mathcal{W}_l(X,Y) := \{T \in \mathcal{L}(X,Y)\colon I_X - LT \in \mathcal{W} \text{ for some } L \in \mathcal{L}(Y,X)\}$$

is strictly contained in $\mathcal{T}(X,Y)$.

Proof. By Proposition 2.1.3, $I_X - LT \in \mathcal{W}$ implies LT tauberian, hence T is tauberian, which proves $\mathcal{W}_l \subset \mathcal{T}$.

Conversely, $\mathcal{W}_l(X,Y)$ is open. Indeed, given $T \in \mathcal{W}_l(X,Y)$, if we take an operator $A \in \mathcal{L}(Y,X)$ so that $I_X - AT = K \in \mathcal{W}$, then for every $S \in \mathcal{L}(X,Y)$ with $\|S\| < \|A\|^{-1}$, we find that $I_X + AS$ is invertible, hence

$$A \circ (T+S) = I_X + AS - K \in \mathcal{W}_l,$$

which immediately yields $T + S \in \mathcal{W}_l$; hence $\mathcal{W}_l(X,Y)$ is open for all Banach spaces X and Y.

Thus $\mathcal{W}_l \not\supset \mathcal{T}$ because $\mathcal{T}(X,Y)$ is not open in general as Example 2.1.17 shows. $\square$

Let us close this section with an example of a tauberian operator which is far from being upper semi-Fredholm, and yet its domain is not reflexive.

Example 2.1.18. Let J be the James space, which is formed by all null sequences (x_n) of real numbers for which the expression

$$\|(x_n)\|_J := \sup\left\{ \sum_{i=1}^{k-1} |x_{n_{i+1}} - x_{n_i}|^2 \,+\, |x_{n_k}|^2 \colon k \in \mathbb{N},\, \{n_1 < \ldots < n_k\} \subset \mathbb{N} \right\}^{1/2}$$

is finite. Then the natural inclusion operator $\iota\colon J \longrightarrow c_0$ is tauberian.

Proof. In fact, the bidual of J [122, 1.d.2] admits the representation

$$J^{**} = J \oplus \operatorname{span}\{(1,1,1,\ldots)\}.$$

Since $\iota^{**}\big((1,1,1,\ldots)\big) = (1,1,1,\ldots) \in \ell_\infty \setminus c_0$, it follows that ι is tauberian. $\square$

We point out that J does not contain any closed subspace isomorphic to c_0. Therefore, a restriction $\iota|_E$ is upper semi-Fredholm if and only if E is a finite-dimensional subspace of J.

A Banach space X for which $\dim X^{**}/X < \infty$ is said to be *quasi-reflexive*. The main feature of J is that it is isomorphic to its bidual, yet $\dim J^{**}/J = 1$.

2.2 Main characterizations of tauberian operators

The main result of this section is Kalton and Wilansky's sequential characterizations given in Theorems 2.2.2 and 2.2.4. Their proofs are strongly based upon the Eberlein-Smulian theorem. With regard to its importance, and for the sake of further purposes, we state it here following the version of [51, page 41].

Theorem 2.2.1 (Eberlein-Smulian theorem). *Given a bounded subset B of a given Banach space, the following statements are equivalent:*

(a) $\overline{B}^w$ *is not weakly compact;*

(b) *there is a basic sequence (x_n) in B with no weakly convergent subsequence;*

(c) $\overline{B}^w$ *is not weakly sequentially compact.*

The following result is fundamental in this chapter.

Theorem 2.2.2. *For each operator $T \in \mathcal{L}(X, Y)$ the following statements are equivalent:*

(a) *T has property (N);*

(b) *if (x_n) is a bounded sequence in X and (Tx_n) is weakly null, then (x_n) contains a weakly convergent subsequence;*

(c) *if (x_n) is a bounded sequence in X and (Tx_n) is null, then (x_n) contains a weakly convergent subsequence.*

Proof. (a)⇒(b) Let (x_n) be a bounded sequence contained in X such that (Tx_n) is weakly null, and consider the set $A := \{x_n \colon n \in \mathbb{N}\}$. Thus

$$\overline{T(A)}^{w^*} = \{Tx_n \colon n \in \mathbb{N}\} \cup \{0\} \subset Y.$$

Therefore, if $x^{**} \in \overline{A}^{w^*} \setminus A$, then $x^{**} \in N(T^{**}) = N(T)$. Since A is bounded and $\overline{A}^{w^*} = \overline{A}^w \subset X$, we conclude that A is relatively weakly compact, and by Theorem 2.2.1, A contains a weakly convergent sequence.

(b)⇒(c) Trivial.

(c)⇒(a) Let $x^{**} \in N(T^{**})$ such that $\|x^{**}\| \leq 1$. For every weak* neighborhood V of x^{**} in X^{**}, we choose a convex weak* neighborhood U of x^{**} so that $\overline{U}^{w^*} \subset V$. Thus $x^{**} \in \overline{U \cap B_X}^{w^*}$, so Lemma 2.1.6 yields

$$0 \in \overline{T(U \cap B_X)}^{w^*} \cap Y = \overline{T(U \cap B_X)}.$$

Hence, we can take a sequence (x_n) in $U \cap B_X$ so that $Tx_n \underset{n}{\longrightarrow} 0$. In accordance with condition (c), the sequence (x_n) contains a subsequence (x_{n_i}) which is weakly convergent to some $x \in V \cap N(T)$, so $V \cap N(T) \neq \emptyset$. Therefore, $x^{**} \in \overline{N(T)}^{w^*}$. However, statement (c) and the Eberlein-Smulian theorem imply that $N(T)$ is reflexive, so $\overline{N(T)}^{w^*}$ equals $N(T)$, which leads to $x^{**} \in N(T)$, hence $N(T^{**}) = N(T)$. □

The following result will be useful to derive some consequences regarding the existence of an injective tauberian operator between two Banach spaces.

Corollary 2.2.3. *For an operator $T \in \mathcal{L}(X,Y)$ the following statements are equivalent:*

(a) $N(T^{**}) = \{0\}$;

(b) *if (x_n) is a bounded sequence in X and (Tx_n) is weakly null, then (x_n) is weakly null.*

Now we give the sequential characterization of tauberian operators.

Theorem 2.2.4. *Let $T \in \mathcal{L}(X,Y)$ be an operator. The following statements are equivalent:*

(a) *T is tauberian;*

(b) *if (x_n) is a bounded sequence in X such that (Tx_n) is weakly convergent, then (x_n) contains a weakly convergent subsequence;*

(c) *if (x_n) is a bounded sequence in X such that (Tx_n) is convergent, then (x_n) contains a weakly convergent subsequence.*

Proof. (a)⇒(b) Let (x_n) be a bounded sequence in X such that (Tx_n) is weakly convergent to $y \in \overline{T(B_X)}$. By Theorem 2.1.7, we have $y \in R(T)$. Let $x \in X$ such that $y = Tx$. Thus $T(x_n - x) \xrightarrow[n]{w} 0$, and since T has property (N), Theorem 2.2.2 yields the existence of a weakly convergent subsequence of (x_n).

(b)⇒(c) Trivial.

(c)⇒(a) Theorem 2.2.2 shows that T has property (N). So, by Theorem 2.1.7, we only need to show that $\overline{T(B_X)}$ is contained in $R(T)$. For that purpose, let $y \in \overline{T(B_X)}$ and take a sequence (x_n) in B_X so that $Tx_n \xrightarrow[n]{} y$. By hypothesis, (x_n) contains a subsequence (x_{k_n}) which converges weakly to some $x \in B_X$. Therefore, $y = Tx$. □

An immediate consequence is the following characterization of tauberian operators in terms of their action on bounded sets.

Corollary 2.2.5. *Let $T \in \mathcal{L}(X,Y)$ be an operator. The following statements are equivalent:*

(a) *T is tauberian;*

(b) *for every bounded subset C of X such that $T(C)$ is relatively weakly compact, C is relatively weakly compact;*

(c) *for every bounded subset C of X such that $T(C)$ is relatively compact, C is relatively weakly compact.*

Proof. It is sufficient to observe that statements (b) and (c) are respectively equivalent to statements (b) and (c) of Theorem 2.2.4. Indeed, the first equivalence is a direct consequence of the aforementioned Eberlein-Smulian theorem. For the second equivalence, observe that a subset of a Banach space is relatively compact if and only if it is sequentially compact. □

An application of the Eberlein-Smulian theorem allows us to strengthen Theorem 2.2.4. We recall that a sequence (x_n) in a Banach space is said to be *semi-normalized* if there is a constant $C > 0$ for which $C^{-1} \leq \|x_n\| \leq C$ for all n.

Corollary 2.2.6. *Given an operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *if (x_n) is a semi-normalized basic sequence in X such that (Tx_n) is weakly convergent, then (x_n) is weakly null;*

(c) *if (x_n) is a semi-normalized basic sequence in X such that (Tx_n) is convergent, then (x_n) is weakly null.*

Proof. (a)⇒(b) Let us assume that T is tauberian, and let (x_n) be a semi-normalized basic sequence such that (Tx_n) is weakly convergent. Then, by Theorem 2.2.4, every subsequence (x_{n_i}) of (x_n) contains a weakly convergent subsequence which must be weakly null since (x_{n_i}) is basic. Therefore, (x_n) is weakly null.

(b)⇒(c) Trivial.

(c)⇒(a) Let us assume that (c) holds, and consider any bounded sequence (x_n) in X so that (Tx_n) is convergent. If (x_n) has no weakly convergent subsequence, then by the Eberlein-Smulian theorem (x_n) has a basic subsequence without weakly convergent subsequences, which contradicts (c). Hence (x_n) contains a weakly convergent subsequence and, by Theorem 2.2.4, T is tauberian. □

A classical theorem, included in Section A.1 as Theorem A.1.9, establishes that an operator $T: X \longrightarrow Y$ is upper semi-Fredholm if and only if for every compact operator $K: X \longrightarrow Y$ the kernel $N(T + K)$ is finite-dimensional. Corollary 2.2.6 yields an analogous characterization for tauberian operators.

Theorem 2.2.7. *An operator $T \in \mathcal{L}(X, Y)$ is tauberian if and only if for each compact operator $K \in \mathcal{L}(X, Y)$ the kernel $N(T + K)$ is reflexive.*

Proof. Let us assume that T is tauberian. Then, given any compact operator $K: X \longrightarrow Y$, the operator $T + K$ is tauberian by Proposition 2.1.3. Hence, the kernel $N(T + K)$ is reflexive.

For the converse, let us assume that T is not tauberian. Then, by Corollary 2.2.6, there is a normalized basic sequence (x_n) in X with no weakly convergent subsequence such that (Tx_n) converges to some element $y \in Y$. Thus the sequence (g_n) of coefficient functionals associated with (x_n) is bounded, and consequently, every g_n admits a Hahn-Banach extension $f_n \in X^*$ so that (f_n) is bounded. Since $Tx_n - y \underset{n}{\longrightarrow} 0$, passing to a subsequence if necessary, we can assume that $\|Tx_n - y\| \cdot \|f_n\| \leq 2^{-n}$ for each n. Therefore, the formula

$$Kx := \sum_{i=1}^{\infty} \langle f_i, x \rangle (y - Tx_i)$$

defines a compact operator $K \in \mathcal{L}(X, Y)$.

Clearly $(T + K)x_i = y$ for every i. Thus $N(T + K)$ contains the closed subspace generated by the sequence $(x_i - x_1)_{i=2}^{\infty}$ which is not reflexive, hence $N(T + K)$ is not reflexive. □

At this point, it should be clear that there is a deep connection between tauberian operators and reflexivity. Perhaps Theorem 2.2.7 is the clearest exposition of that fact because it characterizes tauberian operators in terms of its restrictions to non-reflexive closed subspaces. The reader should compare Theorem 2.2.7 with Theorem 2.1.7 and Proposition 2.1.9, which characterize tauberian operators $T\colon X \longrightarrow Y$ in terms of the reflexivity of $N(T)$ and some additional conditions such as the closedness of $T(B_X)$.

We also remark that all the information provided by Theorem 2.2.4 and Proposition 2.1.3 is now easily obtained from Theorem 2.2.7.

Corollary 2.2.8. *A Banach space X has no reflexive infinite-dimensional subspaces if and only if $\Phi_+(X, Y) = \mathcal{T}(X, Y)$ for every Y.*

Proof. Assume that every reflexive subspace of X is finite-dimensional. Then, by Theorems A.1.9 and 2.2.7, every operator $T \in \mathcal{T}(X, Y)$ is upper semi-Fredholm.

Conversely, if X contains a reflexive infinite-dimensional subspace R, then the quotient operator $Q_R\colon X \longrightarrow X/R$ is tauberian (Proposition 2.1.4) but it is not upper semi-Fredholm. □

Corollary 2.2.8 offers a partial answer to Question 1.3.2.

Another consequence of Theorem 2.2.7 is the following result of algebraic character:

Proposition 2.2.9. *Given an operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *every operator $S \in \mathcal{L}(W, X)$ is weakly compact whenever TS is weakly compact;*

(c) *any closed subspace $E \subset X$ is reflexive whenever the restriction $T|_E$ is weakly compact.*

Proof. (a)⇒(b) Let $S\colon W \longrightarrow X$ be an operator such that TS is weakly compact. Thus, for each bounded sequence (w_n) in W, (TSw_n) has a weakly convergent subsequence. But T is tauberian, so Theorem 2.2.4 shows that (Sw_n) must contain a weakly convergent subsequence. Therefore, S is weakly compact.

(b)⇒(c) Let $E \subset X$ be a closed subspace and let $J_E\colon E \longrightarrow X$ be the natural embedding of E into X. If $T \circ J_E$ is weakly compact, then J_E is weakly compact by hypothesis (b), so E must be reflexive by Proposition 2.1.3.

(c)⇒(a) Let us assume that T is not tauberian. Then Theorem 2.2.7 provides us with a compact operator $K \in \mathcal{L}(X, Y)$ so that $N(T+K)$ is not reflexive. Thus, the fact that $T|_{N(T+K)}$ is compact leads to the negation of (c). □

2.3 Preservation of the closedness of closed convex sets

The main result of this section is Theorem 2.3.4, which was obtained by Neidinger and Rosenthal [135]. It shows that an operator T is tauberian if and only if $T(C)$ is closed for each closed convex bounded set C. Its proof is a consequence and a generalization of the following fundamental theorem obtained by James in [104] and [105].

Theorem 2.3.1 (James' theorem). *A weakly closed subset C of a Banach space X is weakly compact if and only if each continuous linear functional on X attains a maximum on C.*

In particular, X is reflexive if and only if every continuous linear functional on X attains its maximum on B_X.

Note that the next result is only valid when X is not reflexive. When X is reflexive, statements (a) and (c) are trivially true, but statement (b) may fail.

Proposition 2.3.2. *Let X be a non-reflexive Banach space and let $T \in \mathcal{L}(X, Y)$ be an operator. The following statements are equivalent:*

(a) *T is tauberian;*

(b) *$R(T)$ is infinite-dimensional and $\overline{T(B_E)} \subset T(E)$ for every closed subspace E of X;*

(c) *$N(T)$ is reflexive and $\overline{T(B_E)} \subset T(E)$ for every closed subspace E of X.*

Proof. (a)⇒(b) In fact, $R(T)$ is infinite dimensional. If this were not the case, since $N(T)$ is reflexive, X would be reflexive. The inclusion of $\overline{T(B_E)}$ in $R(T)$ is a consequence of Theorem 2.1.7.

(b)⇒(c) Assume that $N(T)$ is not reflexive and $R(T)$ is infinite dimensional. Then we can construct a closed subspace E of X with $\overline{T(B_E)} \not\subset T(E)$ as follows: consider the quotient operator $Q\colon X \longrightarrow X/N(T)$. Since $X/N(T)$ is infinite-dimensional, we can take a normalized sequence $(x_n)_{n=0}^\infty$ in X so that $(Qx_n)_{n=0}^\infty$ is basic. Let $(z_n)_{n=1}^\infty$ be a normalized basic sequence in $N(T)$ with no weakly convergent subsequence. Thus, the sequence $(e_n)_{n=1}^\infty$ defined by

$$e_n := z_n + \frac{1}{n}x_n + x_0, \text{ for } n = 1, 2, 3, \ldots$$

is bounded and has no weakly convergent subsequence, so it contains a basic subsequence (e_{n_i}) by the Eberlein-Smulian theorem.

Let $E = \overline{\text{span}}\{x_{n_i}\}_{i=1}^\infty$. Since

$$Te_{n_i} = \frac{1}{n_i}Tx_{n_i} + Tx_0 \xrightarrow[i]{} Tx_0$$

and $\|e_n\| \leq 3$ for all n, $Tx_0 \in 3 \cdot \overline{T(B_E)}$. But $Tx_0 \notin T(E)$; otherwise, if $Tx_0 = Te$ for some $e = \sum_{i=1}^{\infty} c_i e_{n_i} \in E$, we would have $x_0 - e \in N(T) = N(Q)$, so

$$\begin{aligned} 0 = Q(x_0 - e) &= Qx_0 - \sum_{i=1}^{\infty} c_i \left(\frac{1}{n_i} Qx_{n_i} + Qx_0 \right) \\ &= \left(1 - \sum_{i=1}^{\infty} c_i \right) Qx_0 - \sum_{i=1}^{\infty} \frac{c_i}{n_i} Qx_{n_i}, \end{aligned}$$

and since (Qx_{n_i}) is basic, we would get $1 = \sum_{i=1}^{\infty} c_i$ and $c_i = 0$ for all i, a contradiction.

(c)⇒(a) Let us assume that $N(T)$ is reflexive but T is not tauberian. As in the proof of (b)⇒(c), we will find a closed subspace E of X so that $\overline{T(B_E)} \not\subset T(E)$. If $\overline{T(B_X)} \not\subset T(X)$, then the choice for E is X. In the case when $\overline{T(B_X)} \subset T(X)$, Theorem 2.1.7 gives $N(T^{**}) \neq N(T)$. Thus, Theorem 2.2.2 yields a normalized sequence $(x_n)_{n=0}^{\infty}$ in X with no weakly convergent subsequence such that $Tx_n \underset{n}{\longrightarrow} 0$. Moreover, the reflexivity of $N(T)$ and Theorem 2.1.5 imply that the quotient operator $Q\colon X \longrightarrow X/N(T)$ is tauberian. Thus, as $(x_n + x_0)_{n=1}^{\infty}$ has no weakly convergent subsequence, neither has $(Qx_n + Qx_0)_{n=1}^{\infty}$ by virtue of Theorem 2.2.4. So, passing to a subsequence of (x_n) if necessary, the three sequences $(Qx_n)_{n=0}^{\infty}$, $(x_n + x_0)_{n=1}^{\infty}$ and $(Qx_n + Qx_0)_{n=1}^{\infty}$ can be assumed to be basic. Let $E = \overline{\text{span}}\{x_n + x_0\}_{n=1}^{\infty}$. Note that $T(x_n + x_0) \underset{n}{\longrightarrow} Tx_0$, so $Tx_0 \in 2 \cdot \overline{T(B_E)}$. Let us prove that $Tx_0 \notin T(E)$.

Indeed, if there exists $e = \sum_{n=1}^{\infty} c_n(x_n - x_0)$ such that $Tx_0 = Te$, then $x_0 - e \in N(T) = N(Q)$, which yields

$$0 = Q(x_0 - e) = \left(1 + \sum_{n=1}^{\infty} c_n \right) Qx_0 - \sum_{n=1}^{\infty} c_n Qx_n,$$

so, since $(Qx_n)_{n=0}^{\infty}$ is basic, we obtain the contradictory identities $0 = 1 + \sum_{n=1}^{\infty} c_n$ and $c_n = 0$ for all n. Thus $Tx_0 \notin T(E)$. □

Given a Banach space X, the Bishop-Phelps theorem [30] establishes that the set

$$F_X := \{ f \in X^* \colon \langle f, x \rangle = \|f\| \text{ for some } x \in B_X \}$$

is dense in X^*. Moreover, by James' theorem, the space X is reflexive if and only if $X^* = F_X$ [105].

Despite Bishop-Phelps' result, we will see that given a non-reflexive Banach space X, every non-zero functional $f \in X^*$ admits a restriction $f|_Y$ which does not attain its norm on B_Y. The subspace Y may be chosen with $\dim(X/Y) = 1$. Its proof is based upon the aforementioned James' theorem.

Proposition 2.3.3. *Let X be a non-reflexive Banach space, and let $f \in S_{X^*}$. Then for every $\lambda \in (0, 1)$ there exists a subspace Y of X with co-dimension one such that $\|f|_Y\| = \lambda$ and $f|_Y$ does not attain its norm on B_Y.*

Proof. Let $W := \{x \in B_X : \langle f, x\rangle = \lambda\}$. Since $0 < \lambda < 1$, we get $W \neq \emptyset$. So picking $x_0 \in W$, we have $W = (x_0 + N(f)) \cap B_X$, so there exists $\delta > 0$ so that $x_0 + \delta B_{N(f)} \subset W$. Therefore, W is not relatively weakly compact because $N(f)$ is not reflexive. In virtue of Theorem 2.3.1, there exists $g \in X^*$ which does not attain its supremum on W; that is, the

$$A := \{\langle g, w\rangle : w \in W\}$$

does not have a maximum. Without loss of generality, we can assume that $\sup A = \lambda$. Let $h := f - g$. Note that $h \neq 0$ because f is constant on W. Hence, the subspace $Y := N(h)$ has co-dimension 1. Take $x^{**} \in \overline{W}^{w^*}$ so that $\langle x^{**}, g\rangle = \lambda$. Thus, as $\|x^{**}\| \leq 1$ and $\langle x^{**}, h\rangle = 0$, we get $x^{**} \in \overline{Y}^{w^*}$ and therefore

$$\|f|_Y\| \geq \langle x^{**}, h\rangle = \lambda.$$

Now, let us assume that there exists $y \in B_Y$ such that $\langle f, y\rangle = \lambda$. Then $y \in W$ and $\langle g, y\rangle = \lambda$, which means that g attains its supremum on W, in contradiction to the choice of g. We conclude that $\|f|_Y\| = \lambda$ and $f|_Y$ does not attain its norm on B_Y. □

The following theorem can be regarded as an extension of Proposition 2.3.3 to the general setting of operators.

Theorem 2.3.4. *Let X be a non-reflexive Banach space and let $T \in \mathcal{L}(X, Y)$ be a non-zero operator. Then the following statements are equivalent:*

(a) *T is tauberian;*

(b) *for all weakly closed bounded subsets C of X, $T(C)$ is weakly closed;*

(c) *for all closed bounded convex subsets C of X, $T(C)$ is closed;*

(d) *for all closed subspaces E of X, $T(B_E)$ is closed.*

Proof. (a)⇒(b) Let C be a weakly closed bounded subset of X, and $y \in \overline{T(C)}^w$. By Lemma 2.1.6, $T^{**}(\overline{C}^{w^*}) = \overline{T(C)}^{w^*}$. Thus we can take $x^{**} \in \overline{C}^{w^*}$ so that $T^{**}x^{**} = y$. But T is tauberian, so

$$x^{**} \in X \cap \overline{C}^{w^*} = \overline{C}^w = C$$

hence $y \in T(C)$.

(b)⇒(c) It is sufficient to bear in mind the fact that any convex closed subset is weakly closed.

(c)⇒(d) Trivial.

(d)⇒(a) In order to show that T is tauberian, by Proposition 2.3.2 we only need to prove that $N(T)$ is reflexive.

Let us assume $N(T)$ is not reflexive. The proof will be done as soon as we find a closed subspace E of X such that $T(B_E)$ is not closed. Let $x_0 \in X$ such that $\|Tx_0\| = 1$ and define

$$Z := \operatorname{span}\{x_0\} \oplus N(T).$$

Let $J\colon \operatorname{span}\{Tx_0\} \longrightarrow \mathbb{R}$ be the isometry defined by $J(Tx_0) = 1$, and consider the functional $f := J \circ T \in Z^*$. Since Z is not reflexive, Proposition 2.3.3 provides a closed subspace E of Z such that $f|_E$ does not attain its norm. But

$$\{\langle f, e\rangle \colon e \in B_E\} = \{\langle J, Te\rangle \colon e \in B_E\}$$

and as J is an isometry, we see that $T(B_E)$ is not closed, in contradiction with hypothesis (d). □

The reader should compare statements (d) in Theorem 2.3.4 and (b) in Theorem 2.1.7.

The following definition was introduced by Lotz and Porta in [124].

Definition 2.3.5. An operator $T \in \mathcal{L}(X, Y)$ is said to be a *semi-embedding* if it is injective and $T(B_X)$ is closed.

It is not difficult to find semi-embeddings that are not tauberian. Actually, for every $T \in \mathcal{L}(X, Y)$ with dense range, T^* is a semi-embedding. However, it follows from Theorem 2.3.4 that an injective operator is tauberian if and only if it is a semi-embedding hereditarily.

It is also worth comparing Theorem 2.3.4 with the following characterizations of the isomorphisms and the upper semi-Fredholm operators in terms of preservation of closedness of some sets.

Proposition 2.3.6. *Let $T \in \mathcal{L}(X, Y)$ be a non-zero operator.*

(i) *T is an isomorphism if and only if for all closed subsets C of X, $T(C)$ is closed;*

(ii) *T is upper semi-Fredholm if and only if for all bounded closed subsets C of X, $T(C)$ is closed.*

Proof. (i) The 'only if' direction is trivial. For the 'if' part, notice that $R(T)$ is closed by hypothesis. Moreover, if T were not injective, taking a non-null element $z \in N(T)$ and $x \in X \setminus N(T)$, we would see that

$$T(\{tz + (\arctan t)x \colon t \in \mathbb{R}\})$$

is not closed.

(ii) For the 'only if' part, consider the factorization $T = \widetilde{T} \circ Q_{N(T)}$ given in formula (2.1), and denote $Q := Q_{N(T)}$. Let C be a bounded closed subset of X. By virtue of (i), it is sufficient to prove that $Q(C)$ is closed in order to conclude that $T(C)$ is closed.

Let (x_n) be a sequence in C such that $Qx_n \underset{n}{\longrightarrow} Qx$. We shall prove that $Qx \in Q(C)$. Since (x_n) is bounded, there exists a bounded sequence (z_n) in $N(T)$ such that $x_n - x + z_n \underset{n}{\longrightarrow} 0$. But $N(T)$ is finite-dimensional, so (z_n) contains a convergent subsequence (z_{n_i}), and since C is closed, $x_{n_i} \underset{i}{\longrightarrow} v \in C$. Hence $Qx = Qv \in Q(C)$.

For the 'if' part, let us assume that T is not upper semi-Fredholm. Take $x \in X$ such that $Tx \neq 0$, and let (x_n) be a bounded sequence in X without convergent subsequences such that (Tx_n) converges to some z with

$$z \in \{0\} \cup \big(\overline{R(T)} \setminus R(T)\big).$$

Thus

$$A := \{x_n + n^{-1}x \colon n \in \mathbb{N}\}$$

is closed, but $z \in \overline{T(A)} \setminus T(A)$. □

2.4 Action of tauberian operators on basic sequences

A classical result asserts that a basic sequence (x_n) of a Banach space X spans a reflexive subspace if and only if (x_n) is both shrinking and boundedly complete [122]. A delicate improvement, due to Zippin [177], states that a Banach space with a basis is reflexive if all its bases are shrinking or all its bases are boundedly complete.

In this section, we characterize the tauberian operators and the operators with property (N) in terms of their action on boundedly complete basic sequences and shrinking basic sequences.

We notice that most of the results in the previous sections are of a subsequential nature in the sense that, if there is a sequence satisfying a certain property P, then those results assert that it contains a subsequence that possesses a certain property Q. However, the results presented in this section do not. Roughly speaking, they are of the following type: if a sequence satisfies a certain property P, then the same sequence satisfies a certain property Q.

For the definitions of basis, basic sequence, boundedly complete basic sequence and shrinking basic sequence we refer to Section A.3.

Recall that a sequence (y_n) is said to be a *block basis* of a basis (x_n) if there exists an increasing sequence (n_i) in $\mathbb{N}$ and a sequence (α_n) of scalars such that $y_j := \sum_{i=n_j+1}^{n_{j+1}} \alpha_i x_i$ and $y_j \neq 0$ for all j. Note that if both (x_n) and (y_n) are semi-normalized, then (α_n) is bounded.

Obviously, a shrinking basis (x_n) is weakly null. The following lemma is standard, but its proof is included for the sake of completeness.

Lemma 2.4.1. *A semi-normalized basic sequence (x_n) is shrinking if and only if all its semi-normalized block basic sequences are weakly null.*

Proof. Let (x_n) be a semi-normalized basic sequence and write $X_n := \overline{\text{span}}\{x_i\}_{i=n}^{\infty}$ for every $n \in \mathbb{N}$.

Let us assume that (x_n) is shrinking, and consider a semi-normalized block basic sequence (b_n) of (x_n), with

$$b_j = \sum_{i=n_j+1}^{n_{j+1}} \alpha_i x_i \quad \text{for every } j \in \mathbb{N}.$$

Thus, as $\|f|_{X_n}\| \underset{n}{\longrightarrow} 0$, for each $f \in X_1^*$,

$$\|\langle f, b_j\rangle\| \leq \|f|_{X_{n_j+1}}\| \cdot \|b_j\| \xrightarrow[j]{} 0$$

hence (b_j) is weakly null.

For the converse, let us assume that (x_n) is not shrinking. Then there is $f \in X_1^*$ such that $s := \limsup_n \|f|_{X_n}\| > 0$. Thus, for every positive integer j, we inductively obtain blocks

$$b_j := \sum_{i=n_j+1}^{n_{j+1}} \beta_i x_i$$

with $n_{j-1} < n_j$, so that (b_j) is semi-normalized and $\langle f, b_j\rangle \underset{j}{\longrightarrow} s$. □

Taking into account that a basic sequence in a Banach space spans a reflexive subspace if and only if it is shrinking and boundedly complete (Proposition A.3.16), it is easy to prove after Theorem 2.2.7 that an operator $T\colon X \longrightarrow Y$ is tauberian if and only if every semi-normalized basic sequence (x_n) of X is both shrinking and boundedly complete whenever $\sum_{n=1}^{\infty} \|Tx_n\|$ is convergent.

Actually, Proposition 2.4.5 reveals that, in order to prove that T is tauberian, it is sufficient to check that (x_n) is boundedly complete. If we only have the fact that (x_n) is shrinking, then T has just property (N), as the following proposition shows.

Proposition 2.4.2. *Given $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*

(a) *T has property (N);*

(b) *if (x_n) is a semi-normalized basic sequence in X and $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$, then (x_n) is shrinking.*

Proof. (a)⇒(b) Let (x_n) be a semi-normalized basic sequence in X such that $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$. In order to prove that (x_n) is shrinking, by Lemma 2.4.1, it is sufficient to check that any semi-normalized block basic sequence of (x_n) is weakly null. In order to do so, let

$$b_i = \sum_{n=n_i+1}^{n_{i+1}} \alpha_n x_n \quad \text{for all } i \in \mathbb{N}$$

be a semi-normalized block basic sequence of (x_n). Note that, since (x_n) and (b_i) are semi-normalized, (α_n) must be bounded. Therefore, for every i,

$$\begin{aligned}\|Tb_i\| &\leq \sum_{n=n_i+1}^{n_{i+1}} |\alpha_n| \cdot \|Tx_n\| \\ &\leq \sup_n |\alpha_n| \cdot \sum_{n=n_i+1}^{n_{i+1}} \|Tx_n\| \xrightarrow[i]{} 0,\end{aligned}$$

and since T has property (N), Theorem 2.2.2 yields that (b_i) is weakly null.

(b)⇒(a) Let us assume that T fails property (N). Then Theorem 2.2.2 provides us with a bounded sequence (x_n) with no weakly convergent subsequence such that $Tx_n \xrightarrow[n]{} 0$. Thus, by Theorem 2.2.1, (x_n) contains a basic subsequence (x_{n_i}) so that $\sum_{i=1}^{\infty} \|Tx_{n_i}\| < \infty$. Since (x_{n_i}) is not weakly null, (x_{n_i}) is not shrinking, so (b) fails. □

The following result weakens the hypothesis on the sequence (Tx_n) in statement (b) of Proposition 2.4.2.

Proposition 2.4.3. *Let $T \in \mathcal{L}(X, Y)$ be an operator satisfying property (N) and let (x_n) be a semi-normalized basic sequence in X. If (Tx_n) is a shrinking seminormalized basic sequence, then (x_n) is shrinking.*

Proof. Let us assume that (x_n) is not shrinking, but (Tx_n) is basic and shrinking. Then, by Lemma 2.4.1, there is a semi-normalized block basic sequence (b_n) of (x_n) which has no weakly null subsequence. As (Tb_n) is a block basic sequence of (Tx_n), it follows that (Tb_n) is weakly null. But T has property (N) so, by Theorem 2.2.2, (b_n) has a weakly null subsequence, and thus we obtain a contradiction. □

Note that Proposition 2.4.3, unlike Proposition 2.4.2, does not characterize property (N). Indeed, the null operator on a non-reflexive Banach space X satisfies the thesis of Proposition 2.4.3 but fails property (N).

A basic sequence without weakly convergent subsequences contains a subsequence (z_n) whose difference sequence $(z_n - z_{n+1})$ is also basic. This fact is a direct consequence of the following lemma, which will enable us to state Proposition 2.4.5 concerning tauberian operators and boundedly complete bases, in a similar spirit to that of Proposition 2.4.2.

Lemma 2.4.4. *Let (x_n) be a semi-normalized basis of X for which there exists $f \in X^*$ so that $\langle f, x_n \rangle = 1$ for all n. Then $(x_n - x_{n+1})_{n=1}^{\infty}$ is basic.*

Proof. For each $n \in \mathbb{N}$, we denote $y_n := x_n - x_{n+1}$ and $Y_n := \operatorname{span}\{y_i\}_{i=1}^{n}$. Since (y_n) is linearly independent, given any pair of positive integers m and n with $m \leq n$, the projections $Q_m^n \colon Y_n \longrightarrow Y_n$ given by

$$Q_m^n\left(\sum_{i=1}^{n} \alpha_i y_i\right) := \sum_{i=1}^{m} \alpha_i y_i$$

are well defined. We only need to see that $\sup_{m,n} \|Q_m^n\| < \infty$ to prove that (y_n) is basic. In order to do so, let K be the basis constant of (x_n), and let (f_i) be its sequence of coefficient functionals. For every $j \in \mathbb{N}$, we define

$$g_j := \sum_{i=1}^{j} f_i,$$

so (g_n) is a sequence of coefficient functionals for (y_n). Fix a pair of positive numbers m and n with $m \leq n$, and let $P_m \colon X \longrightarrow X$ be the projection

$$P_m\left(\sum_{i=1}^{\infty} \beta_i x_i\right) := \sum_{i=1}^{m} \beta_i x_i.$$

Thus $\sum_{i=1}^{m} f_i = f \circ P_m$, hence

$$\|g_m\| \leq \|f \circ P_m\| \leq \|f\| \cdot K.$$

Therefore, since $Q_m^n(x) = P_m(x) - \langle g_m, x\rangle x_{m+1}$ for all $x \in Y_n$, it follows that

$$\|Q_m^n\| \leq K + \|x_{m+1}\| \cdot \|f\| \cdot K,$$

and since (x_n) is semi-normalized, we get $\sup_{m,n} \|Q_m^n\| < \infty$. □

The equivalence between statements (a) and (b) in the following proposition can be easily derived from Theorem 2.2.7, but the implication (c)⇒(a) is not so simple.

Proposition 2.4.5. *Given $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *if (x_n) is a semi-normalized basic sequence in X and $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$, then (x_n) is boundedly complete and shrinking;*

(c) *if (x_n) is a semi-normalized basic sequence in X and $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$, then (x_n) is boundedly complete.*

Proof. (a)⇒(b) Let (x_n) be a semi-normalized basic sequence in X and suppose that $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$. Let us denote $Z := \overline{\operatorname{span}}\{x_n\}_{n=1}^{\infty}$ and let $(f_n) \subset Z^*$ be the sequence of coefficient functionals associated to (x_n). Since

$$Tz = \sum_{n=1}^{\infty} \langle f_n, z\rangle \cdot Tx_n \quad \text{for all } z \in Z,$$

the restriction $T|_Z$ is compact. Thus, by Proposition 2.2.9, Z is reflexive, and consequently, (x_n) is shrinking and boundedly complete [122, 1.b.5].

(b)⇒(c) Trivial.

(c)⇒(a) Let us assume that (c) holds but T is not tauberian. Then, by Theorem 2.2.4, there exist $y \in Y$ and a bounded sequence (x_n) with no weakly convergent subsequence such that $Tx_n \underset{n}{\longrightarrow} y$. Choose a functional $f \in X^*$ so that $\lambda := \limsup_n \langle f, x_n \rangle > 0$, and select a subsequence (x_{n_i}) of (x_n) so that

$$0 < \alpha_i := \langle f, x_{n_i} \rangle \xrightarrow[i]{} \lambda.$$

Thus, by Theorem 2.2.1, $(\alpha_i^{-1} x_{n_i})$ contains a basic subsequence (z_n) with no weakly convergent subsequence. Note that $Tz_n \underset{n}{\longrightarrow} \lambda^{-1} y$, so we can pick a subsequence (z_{n_i}) in (z_n) so that $\|Tz_{n_i} - Tz_{n_{i+1}}\| \leq 1/i^2$. Lemma 2.4.4 yields that $(u_i) := (z_{n_i} - z_{n_{i+1}})$ is a basic sequence. But $\sum_{i=1}^{\infty} \|Tu_i\| < \infty$, so (u_i) is boundedly complete by hypothesis (c). However, the sequence $(\sum_{i=1}^{j} u_i)_j$ equals $(z_{n_1} - z_{n_j})_j$, a bounded non-convergent sequence. So we get a contradiction. □

The following characterization for reflexive spaces is well known. We prove it as an application of the main results of this section.

Corollary 2.4.6. *Let X be a Banach space satisfying at least one of the following conditions:*

(i) *every semi-normalized basic sequence in X is shrinking;*

(ii) *every semi-normalized basic sequence in X is boundedly complete.*

Then X is reflexive.

Proof. Suppose that X is not reflexive.

The null operator $0_X : X \longrightarrow X$ fails property (N), and by Proposition 2.4.2, X must contain a semi-normalized basic sequence which is not shrinking; hence (i) fails.

Similarly, 0_X is not tauberian. So Proposition 2.4.5 supplies a semi-normalized non-boundedly complete basic sequence in X; hence (ii) fails. □

The next result parallels Proposition 2.4.3.

Proposition 2.4.7. *Let $T \in \mathcal{L}(X, Y)$ be a tauberian operator and (x_n) a basic sequence in X. If (Tx_n) is basic and boundedly complete, then (x_n) is boundedly complete.*

Proof. Let us assume that (Tx_n) is basic and boundedly complete. Take a sequence of scalars (α_n) so that $\sup_n \|\sum_{i=1}^{n} \alpha_i x_i\| < \infty$. Thus

$$\sup_n \left\| \sum_{i=1}^{n} \alpha_i T x_i \right\| \leq \|T\| \sup_n \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| < \infty.$$

Hence, since (Tx_n) is boundedly complete, $\sum_{i=1}^{\infty} \alpha_i T x_i$ is convergent. Now, since T is tauberian, by Theorem 2.2.4 there is $x \in X$ and an increasing sequence (n_j)

of positive integers such that

$$\sum_{i=1}^{n_{j+1}} \alpha_i x_i \xrightarrow[j]{w} x.$$

Therefore, x belongs to $\overline{\text{span}}\{x_n\}_{n=1}^{\infty}$ and, since (x_n) is basic, $x = \sum_{i=1}^{\infty} \alpha_i x_i$, which shows that (x_n) is boundedly complete. □

Remark 2.4.8. Proposition 2.4.7 does not characterize tauberian operators. Indeed, if X is a non-reflexive Banach space, then the null operator 0_X is not tauberian, but satisfies the thesis of Proposition 2.4.7.

The following example of Holub [103] will allow us to give further remarks on the scope of Proposition 2.4.7.

Example 2.4.9. There exist a Banach space X with a semi-normalized basis (x_n) and a non-tauberian operator $T\colon X \longrightarrow \ell_2$ with property (N) such that (Tx_n) is the unit vector basis of ℓ_2.

Proof. For every $n \in \mathbb{N}$ and every sequence $(x_i) \in c_0$, we denote by $(\hat{x}_i)$ any decreasing rearrangement of $(|x_i|)$, and we consider the semi-norm

$$\|(x_i)\|_n := \frac{\hat{x}_1 + \hat{x}_2 + \cdots + \hat{x}_n}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}.$$

Let X_1 be the largest linear subspace of c_0 normed by the expression

$$\|(x_i)\|_L := \sup_{n\in\mathbb{N}} \|(x_i)\|_n, \quad (x_i) \in c_0$$

and let X be the subspace formed by all the elements (x_i) of X_1 for which $\|(x_i)\|_n \xrightarrow[n]{} 0$.

In addition, let X_0 be the subspace of c_0 formed by all the elements (y_i) for which

$$\|(y_i)\|_T := \sum_{i=1}^{\infty} \frac{\hat{y}_i}{i} < \infty.$$

It turns out that the spaces X, X_0 and X_1, endowed with their respective norms, are Banach spaces, and that the unit vector basis of c_0, denoted by (e_n), is a basis of X. Moreover, Abel's identity

$$\sum_{k=1}^{n} u_k(v_k - v_{k-1}) = \sum_{k=1}^{n} (u_k - u_{k+1})v_k$$

is valid for all pairs of finite sequences $(u_k)_{k=1}^{n+1}$ and $(v_k)_{k=0}^{n}$ of real numbers with $v_0 = u_{n+1} = 0$ and leads to the inequality

$$\sum_{i=1}^{\infty} x_i y_i \le \|(x_i)\|_L \cdot \|(y_i)\|_T \quad \text{for all } (x_i) \in X_1 \text{ and all } (y_i) \in X_0 \tag{2.2}$$

which yields the duality relationships $X^* = X_0$ and $X_0^* = X_1$.

From inequality (2.2) we derive that, given $(x_i) \in X_1$, we have

$$\sum_{i=1}^{\infty} x_i^2 \leq \|(x_i)\|_L \cdot \|(x_i)\|_T \leq \|(x_i)\|_L^2 \cdot \|(1/i)\|_T = \frac{\pi^2}{6} \cdot \|(x_i)\|_L^2.$$

Hence the natural inclusion $S\colon X_1 \longrightarrow \ell_2$ is a bounded operator with norm less than or equal to $\pi^2/6$.

Let us see that the operator $T := S|_X$ satisfies our requirements.

Indeed, since $X^{**} = X_1$, standard arguments show that $T^{**} = S$. Since S is injective, T has property (N). But T cannot be tauberian because X is non-reflexive. □

For additional information on the construction of the spaces X, X_0 and X_1, refer to the section of Notes and Remarks.

Remark 2.4.10. Example 2.4.9 shows that in Proposition 2.4.7 we cannot replace the assumption T tauberian by T having property (N).

Indeed, since T has property (N), Proposition 2.4.3 shows that (x_n) is shrinking. However, (x_n) may not be boundedly complete because in that case X would be reflexive, and therefore T would be tauberian.

2.5 Notes and Remarks

Obviously, the intersection of Chapter 1 with Chapter 2 are the seminal papers of Garling and Wilansky [63] and Kalton and Wilansky [115].

Section 2.1 combines several papers. Theorems 2.1.5 and 2.1.7 and Proposition 2.1.9 include most of the results of general character about tauberian operators in [63]. Moreover, Theorem 2.1.7 and Proposition 2.1.9 contain the necessary and sufficient conditions given in [115] under which the three clauses of (1.13) are equivalent.

Tacon was the first to discover that the class $\mathcal{T}$ is not open [157]. Example 2.1.17 has been taken from [67], and the class $\mathcal{W}_l$ has been studied by Yang and the authors in [175] and [2]. A quantitative approach to some classes of operators related with $\mathcal{T}$ was done by Astala and Tylli, and it can be found in [16].

Proposition 2.1.3 collects results from several papers; so it is almost impossible to quote them accurately. They have been arranged together in order to show that $\mathcal{T}$ is an operator semigroup in the sense of [1]. This concept of operator semigroup will be discussed in depth in Chapter 6.

Section 2.2 owes much to the Eberlein–Smulian characterization of reflexivity, whose original proof was published in [57] and [154]. Theorem 2.2.1 follows Pełczyński's version of the Eberlein-Smulian theorem, which can be found in [51] and [53].

The proofs of Theorems 2.2.2 and 2.2.4 and Corollary 2.2.5 follow closely the arguments given in [115], but the proof of Theorem 2.2.2 has been simplified.

The argument of Corollary 2.2.8, applied to the particular case when $X = c_0$, is exactly the same as that used by Garling and Wilansky in [63] in order to obtain their proof of (1.10). Corollary 2.2.8 was first proved by Kalton and Wilansky [115], but the proof displayed here was obtained in [92] as a neat sequel of Theorem 2.2.7. This theorem, as well as Proposition 2.2.9, have been borrowed from [92].

In several papers, Cross investigated the concept of tauberian operators for unbounded operators [46, 47] and for linear relations between normed spaces. For an exposition of these results, we refer to the monograph [48]. Bonet and Ramanujan [36] investigated this concept for operators between Fréchet spaces.

Section 2.3 is devoted to the Neidinger and Rosenthal works [134], [133] and [135], but our presentation differs at several points. Indeed, Neidinger and Rosenthal give, in [135, Theorem 2.3], a theorem with six statements and, by means of James' theorem [105], prove that they are equivalent. Instead, we have put in Proposition 2.3.2 the statements whose proof do not require James' theorem The other statements remain together in Theorem 2.3.4. Notice that the proof of implication (c)⇒(a) in Proposition 2.3.2 simplifies the original proof [135, (5)⇒(1) in Theorem 2.3] by means of a suitable use of the semigroup properties of tauberian operators.

Neidinger and Rosenthal prove that, given any non-reflexive Banach space X, for every $f \in S_{X^*}$ and every $\lambda \in (0, 1]$, there exists a subspace Y of X of co-dimension 1 such that $f|_Y$ does not attain its norm on B_Y. In 2.3.3, we only include their proof for the case $\lambda \in (0, 1)$. The proof of Proposition 2.3.6 appears partially in [169] and [135].

Section 2.4 is devoted to the action of tauberian operators on basic sequences. Schauder bases and basic sequences are a fundamental topic treated by many texts on Banach space theory ([24], [51], [122], [152], etc.). From our point of view, a very suggestive presentation of that subject is given in [4] because that book clearly exhibits the links between the Eberlein-Smulian theorem and the possibility of extraction of basic subsequences from a given sequence (see Proposition A.3.7 and Theorem A.3.8).

Propositions 2.4.2, 2.4.3, 2.4.5 and 2.4.7 are due to Holub [103], although their proofs have been slightly modified. A proof for Lemma 2.4.4 and for its converse appears in [152, Theorem 9.2, II]. The proof given here follows an argument in [146]. The original proof of Corollary 2.4.6 appears in [152, Theorem 3 and Corollary 1].

A semi-normalized basic sequence (x_n) in a Banach space X is said to be a P^**-sequence* by Singer [153] [152, Proposition 3], or a *wide-(s)* by Rosenthal [147], if there exists $f \in X^*$ such that $\langle f, x_n \rangle = 1$ for all n.

The following result is due to H. Rosenthal [147].

Proposition 2.5.1. *An operator* $T \in \mathcal{L}(X, Y)$ *is not tauberian if and only if there is a* P^**-sequence* (x_n) *in* X *such that* (Tx_n) *is convergent.*

The original proof of Proposition 2.5.1 follows from the celebrated Rosenthal's ℓ_1 and c_0 theorems published respectively in [145] and [146]. A less sophisticated proof can be directly obtained from the definition of P^*-sequence and from Corollary 2.2.6.

Example 2.4.9 was given by Holub in [103], but his description is too brief. The space X_0 can be isometrically identified with the Lorentz sequence space $d\big((1/n)_n, 1\big)$, and X_1 with $d\big((1/n)_n, 1\big)^*$. We refer to [122, Section 4.e] and [62] for details. Observe that although Garling [62] focuses on reflexive symmetric Banach spaces of sequences, the steps of their construction are valid for the spaces X, X_0 and X_1 of Example 2.4.9.

Chapter 3

Duality and examples of tauberian operators

The cotauberian operators are the operators T such that T^* is tauberian. As could be expected from this definition, many results satisfied by cotauberian operators are dual versions of those satisfied by tauberian operators. However, this relationship of duality is not perfect: there are tauberian operators T such that T^* is not cotauberian.

Of course, it is desirable to have at hand characterizations of cotauberian operators which do not depend on duality. One such characterization is exhibited in Theorem 3.1.20: an operator T is cotauberian if and only if $Y/\overline{R(T+K)}$ is reflexive for every compact operator K. Note that this result and Theorem 2.2.7 show that cotauberian operators are the right choice to be taken as the dual class of tauberian operators.

The *DFJP factorization,* obtained by Davis, Figiel, Johnson and Pełczyński in [49], establishes that every operator T admits a factorization $T = jA$, where j is an injective tauberian operator. In the second section we present a refined version of the DFJP factorization. This version allows us to show that the operator A is cotauberian and behaves well under duality. The DFJP factorization has been extensively applied in Banach space theory and it is the main source of examples of tauberian and cotauberian operators.

The third section presents some variations on the DFJP factorization that have appeared in print, most of them having in mind concrete applications of the factorization. We also describe the relationship between the intermediate space in the DFJP factorization and the intermediate space in the real method of interpolation for Banach spaces.

In the fourth section we describe some examples of tauberian operators that are obtained as natural inclusions between some Banach spaces of vector-valued integrable functions. For example, if $L_\Phi(X)$ denotes the Orlicz space of X-valued functions associated with a Young function Φ, we show that the natural inclusion of $L_\Phi(X)$ into $L_1(X)$ is tauberian when the Orlicz space of scalar-valued functions L_Φ is reflexive.

Tauberian and cotauberian operators are closely connected to the operator ideal of weakly compact operators. In the fifth section we consider four other operator ideals which admit a characterization in terms of sequences and, for each one of them, we introduce two classes of operators that are similar to the tauberian and cotauberian operators respectively. We wish to point out that all the classes we introduce here admit a perturbative characterization and, for this reason, share many of the properties of tauberian operators.

3.1 Cotauberian operators

We showed in Chapter 2 that there are many formal similarities between the properties satisfied by upper semi-Fredholm operators and tauberian operators. These similarities suggest that we introduce the cotauberian operators as follows:

Definition 3.1.1. An operator $T \in \mathcal{L}(X,Y)$ is said to be *cotauberian* when T^* is tauberian.

Because of its definition, it is natural to denote by $\mathcal{T}^d$ the class of all cotauberian operators. Thus $\mathcal{T}^d(X,Y)$ denotes the cotauberian operators acting between the spaces X and Y.

Remark 3.1.2. Since the conjugate operator $T^*\colon Y^* \longrightarrow X^*$ is weak*-continuous and B_{Y^*} is a weak*-compact set, $T^*(B_{Y^*})$ is always closed. Therefore, by Theorem 2.1.7, an operator T is cotauberian if and only if $N(T^{***}) = N(T^*)$; i.e., if and only if T^* has property (N).

Remark 3.1.3. In general, the set $\mathcal{T}^d(X,Y)$ is not open in $\mathcal{L}(X,Y)$.

Indeed, if T and T_k are the operators considered in Example 2.1.17, we can show in a similar way that T^* is tauberian and the operators T_k^* are not tauberian. Thus T is cotauberian, the operators T_k are not cotauberian and (T_k) converges to T.

Some basic properties of the class $\mathcal{T}^d$ of cotauberian operators are included in the following result. Their proof can be derived from the corresponding properties of the operators in $\mathcal{T}$.

Proposition 3.1.4. *Let T and S be operators in $\mathcal{L}(X,Y)$ and U in $\mathcal{L}(Y,Z)$. Then the following statements hold:*

(i) *if both T and U are cotauberian, then UT is cotauberian;*

(ii) *if UT is cotauberian, then U is cotauberian;*

(iii) *T is cotauberian and weakly compact if and only if Y is reflexive;*

(iv) *if T is cotauberian and S is weakly compact, then $T+S$ is cotauberian.*

The cotauberian operators with closed range can be characterized in terms of their cokernel.

Proposition 3.1.5. *Let $T \in \mathcal{L}(X,Y)$ be an operator with closed range. Then T is cotauberian if and only if $Y/R(T)$ is reflexive.*

Proof. Since T has closed range if and only if T^* also does, the result can be proved as a direct application of the characterization of tauberian operators with closed range given in Theorem 2.1.5. □

Corollary 3.1.6. *Let $T \in \mathcal{L}(X,Y)$ be an operator with closed range. Then T is tauberian if and only if T^{**} is tauberian.*

Proof. Observe that $R(T)$ is closed if and only if $R(T^{**})$ is closed. Moreover, in this case we can identify isometrically

$$N(T^{**}) \equiv N(T)^{\perp\perp} \equiv N(T)^{**}.$$

So the result follows from Proposition 3.1.5 and the fact that a Banach space E is reflexive if and only if E^{**} is too. □

The residuum operator

Given a Banach space X, we denote by X^{co} the quotient space X^{**}/X and $Q_X \colon X^{**} \longrightarrow X^{co}$ is the quotient map.

The following concept will be very useful.

Definition 3.1.7. Given an operator $T \in \mathcal{L}(X,Y)$, the map $T^{co} \colon X^{co} \longrightarrow Y^{co}$ defined by

$$T^{co}(z+X) := T^{**}(z) + Y, \quad z + X \in X^{co}$$

is called the *residuum operator* of T.

Note that the operator T^{co} is determined by the equality $T^{co}Q_X = Q_Y T^{**}$.

The following two results are an immediate consequence of the definition of residuum operator.

Proposition 3.1.8. *For an operator $T \in \mathcal{L}(X,Y)$, the following results hold.*

(i) *$T^{co} \in \mathcal{L}(X^{co}, Y^{co})$ and $\|T^{co}\| \leq \|T\|$;*

(ii) *T is weakly compact if and only if $T^{co} = 0$;*

(iii) *T is tauberian if and only if T^{co} is injective.*

Proposition 3.1.9. *The map $T \in \mathcal{L}(X,Y) \longrightarrow T^{co} \in \mathcal{L}(X^{co}, Y^{co})$ is linear, and given $S \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(X,Y)$, $(ST)^{co} = S^{co}T^{co}$.*

Let us describe the behavior of the operation $(\cdot)^{co}$ under duality.

Proposition 3.1.10. *We can identify* $(X^*)^{co}$ *and* $(X^{co})^*$. *More precisely,*

$$Q_{X^*} \circ {Q_X}^* \colon (X^{co})^* \longrightarrow (X^*)^{co}$$

is a bijective isomorphism.

Proof. Note that ${Q_X}^* \colon (X^{co})^* \longrightarrow X^{***}$ is a linear isometry that maps $(X^{co})^*$ onto the subspace $X^\perp$ of X^{***}. Since we have the topological decomposition

$$X^{***} = X^* \oplus X^\perp,$$

the operator $Q_{X^*} \colon X^{***} \longrightarrow (X^*)^{co}$ defines an isomorphism from $X^\perp$ onto $(X^*)^{co}$; hence $Q_{X^*} \circ {Q_X}^*$ is a bijective isomorphism. □

Let us write $U_X := Q_{X^*} \circ {Q_X}^*$.

Proposition 3.1.11. *For an operator* $T \in \mathcal{L}(X, Y)$, *we can identify the operators* $(T^{co})^*$ *and* $(T^*)^{co}$. *More precisely,*

$$(T^*)^{co} = U_X\, (T^{co})^*\, U_Y^{-1}.$$

Proof. We have to prove that the diagram

$$\begin{array}{ccc} (Y^{co})^* & \xrightarrow{\;(T^{co})^*\;} & (X^{co})^* \\ {\scriptstyle U_Y}\big\downarrow & & \big\downarrow{\scriptstyle U_X} \\ (Y^*)^{co} & \xrightarrow[\;(T^*)^{co}\;]{} & (X^*)^{co} \end{array}$$

is commutative. Indeed, from the fundamental equality $T^{co} Q_X = Q_Y T^{**}$, we get

$$\begin{aligned} (T^*)^{co} U_Y &= (T^*)^{co} Q_{Y^*} {Q_Y}^* \\ &= Q_{X^*} T^{***} {Q_Y}^* \\ &= Q_{X^*} \big(Q_Y T^{**}\big)^* \\ &= Q_{X^*} \big(T^{co} Q_X\big)^* \\ &= Q_{X^*} {Q_X}^* (T^{co})^* = U_X (T^{co})^*, \end{aligned}$$

and the equality is proved. □

Let us see a direct consequence of this result.

Corollary 3.1.12. *For an operator* $T \in \mathcal{L}(X, Y)$, *the following assertions are equivalent:*

(a) T *is cotauberian;*

(b) T^{co} *has dense range;*

(c) $R(T^{**}) + Y$ *is dense in* Y^{**}.

Proof. It is enough to observe that T^{co} has dense range if and only if $(T^{co})^*$ is injective. $\square$

The following technical result will allow us to study the behavior of the operation $(\cdot)^{co}$ under passing subspaces or quotients.

Proposition 3.1.13. *Let Z be a closed subspace of a Banach space X. Then we can identify Z^{co} with a closed subspace of X^{co} and $(X/Z)^{co}$ with a quotient of X^{co}; more precisely, $J_Z^{co}\colon Z^{co} \longrightarrow X^{co}$ is an isomorphism (into) and $Q_Z^{co}\colon X^{co} \longrightarrow Z^{co}$ is surjective.*

Proof. Since Q_Z^{**} is surjective, so is Q_Z^{co}. Moreover, by Lemma A.5.1, for every $z^{**} \in Z^{\perp\perp}$,

$$\operatorname{dist}(z^{**}, Z) \leq 2 \operatorname{dist}(z^{**}, X); \tag{3.1}$$

hence $J_Z^{co}\colon Z^{co} \longrightarrow X^{co}$ is bounded below. $\square$

Corollary 3.1.14. *Let Z be a closed subspace of a Banach space X. Then $Z^{\perp\perp} + X$ is a closed subspace of X^{**}.*

Proof. Note that $R(J_Z^{co}) = (Z^{\perp\perp} + X)/X$ is closed in X^{co} by Proposition 3.1.13; hence $Z^{\perp\perp} + X$ is closed in X^{**}. $\square$

It follows from Proposition 3.1.13 that the diagram below is commutative and all its rows and columns are exact sequences; i.e. each arrow is an operator and the kernel of an arrow coincides with the range of the previous arrow. This diagram encodes a description of the spaces Z^{co}, X^{co} and $(X/Z)^{co}$.

$$\begin{array}{ccccccccc}
 & & \{0\} & & \{0\} & & \{0\} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & Z & \xrightarrow[J_Z]{} & X & \xrightarrow[Q_Z]{} & X/Z & \longrightarrow & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & Z^{**} & \xrightarrow[J_Z^{**}]{} & X^{**} & \xrightarrow[Q_Z^{**}]{} & (X/Z)^{**} & \longrightarrow & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & Z^{co} & \xrightarrow[J_Z^{co}]{} & X^{co} & \xrightarrow[Q_Z^{co}]{} & (X/Z)^{co} & \longrightarrow & \{0\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \{0\} & & \{0\} & & \{0\} & &
\end{array} \tag{3.2}$$

Proposition 3.1.15. *Let $T \in \mathcal{L}(X, Y)$ be an operator with closed range. Then T^{co} has also closed range.*

Proof. Since $R(T)$ is closed, $R(T^{**}) = R(T)^{\perp\perp}$; hence

$$R(T^{co}) = \frac{R(T^{**}) + Y}{Y} = \frac{R(T)^{\perp\perp} + Y}{Y}$$

is closed, by Corollary 3.1.14. $\square$

As a consequence, we can obtain an improvement of Corollary 3.1.6.

Corollary 3.1.16. *Let $T \in \mathcal{L}(X,Y)$ be a tauberian operator such that T^{co} has closed range. Then T^{**} is tauberian and $(T^{**})^{co}$ has closed range.*

Proof. It is enough to observe that, by Proposition 3.1.11, we can identify $(T^{**})^{co}$ and $(T^{co})^{**}$. □

A counterexample

Since T can be identified with a restriction of T^{**}, it is clear that T^{**} tauberian implies T tauberian; equivalently, T^* cotauberian implies T tauberian. Next we are going to show that the converse implications fail in general. The counterexample will be obtained using the following idea:

We have seen that T is tauberian if and only if T^{co} is injective, and T^* is cotauberian if and only if $(T^{co})^*$ has dense range. Therefore, in order to find a tauberian operator T such that T^* is not cotauberian, it is enough to find an injective operator S such that $R(S^*)$ is not dense, for which there exists an operator T such that $T^{co} = S$.

The following construction of a Banach space $J(X_n)$ with $J(X_n)^{co} \equiv \ell_1$ is a special case of a general construction of Bellenot inspired by the definition of James' quasi-reflexive space J.

Let $(e_k)_{k=1}^{\infty}$ be the unit vector basis of ℓ_1 and let X_n denote the subspace of ℓ_1 generated by $\{e_1, \dots, e_n\}$. We denote by $\|\cdot\|_1$ the norm in ℓ_1.

For a sequence (x_n) with $x_n \in X_n$ for each n, we define

$$\|(x_n)\|_J := \sup\left\{ \sum_{i=1}^{k-1} \|x_{n_{i+1}} - x_{n_i}\|_1^2 + \|x_{n_k}\|_1^2 \,:\, k \geq 2,\, n_1 < \cdots < n_k \right\}^{1/2},$$

and we consider the space

$$J(X_n) := \{(x_n) \,:\, x_n \in X_n,\, \|x_n\|_1 \underset{n}{\longrightarrow} 0,\, \|(x_n)\|_J < \infty\}.$$

Theorem 3.1.17. *The following results hold:*

(i) *$(J(X_n), \|\cdot\|_J)$ is a Banach space;*

(ii) *$J(X_n)^{**} = \{(x_n) \,:\, x_n \in X_n, \|(x_n)\|_J < \infty\}$;*

(iii) *$J(X_n)^{co}$ is linearly isometric to ℓ_1.*

Proof. (i) The proof is similar to that of the corresponding result for James' space J. See, for example, [4, Section 3.4].

(ii) It follows from the fact that $\{X_n : n \in \mathbb{N}\}$ is a shrinking Schauder decomposition of the space $J(X_n)$. Indeed, for each $k \in \mathbb{N}$, we consider the map

$$P_k \colon J(X_n) \longrightarrow J(X_n)$$

defined by

$$P_k\big((x_n)\big) := (x_1, \ldots, x_k, 0, 0, \ldots); \quad (x_n) \in J(X_n).$$

P_k is clearly a norm-one projection onto a finite dimensional subspace and, for every $(x_n) \in J(X_n)$,

$$\lim_{k\to\infty} P_k\big((x_n)\big) = (x_n).$$

To show that $\{X_n : n \in \mathbb{N}\}$ is a shrinking Schauder decomposition of $J(X_n)$ we have to prove that for each $f \in J(X_n)^*$,

$$\lim_{k\to\infty} \big\| f|_{N(P_k)} \big\| = 0. \tag{3.3}$$

Suppose that there exists $f \in J(X_n)^*$ failing (3.3). Then we could select integers $m_1 < n_1 < m_2 < n_2 < \cdots$ and a bounded sequence of vectors (v_i) in $J(X_n)$ with $v_i = (x_{in})_{n=1}^{\infty}$, so that $x_{in} \neq 0$ if and only if $m_i \leq n \leq n_i$ and $\langle f, v_i \rangle > 1$ for each $i \in \mathbb{N}$. Thus the series $\sum_{k=1}^{\infty} v_k/k$ converges in $J(X_n)$ and

$$\Big\langle f, \sum_{k=1}^{\infty} \frac{v_k}{k} \Big\rangle = \infty,$$

contradicting $f \in J(X_n)^*$.

Equation (3.3) implies that $\lim_k P_k^*(f) = f$ for each $f \in J(X_n)^*$. Now, if $\alpha \in J(X_n)^{**}$, then $P_k^{**}(\alpha) \in J(X_n)$ and

$$\lim_{k\to\infty} \big\langle P_k^{**}(\alpha), f \big\rangle = \langle \alpha, f \rangle \quad \text{for each } f \in J(X_n)^*.$$

Since $(P_k^{**}(\alpha))$ is a bounded sequence in $J(X_n)$, it is clear that we can identify α with a sequence (x_n), with $x_n \in X_n$ for each n and $\|(x_n)\|_J < \infty$.

Conversely, if $x_n \in X_n$ for each n and $\|(x_n)\|_J < \infty$, then the sequence

$$\big((x_1, \ldots, x_k, 0, 0, \ldots)\big)_{k=1}^{\infty}$$

is bounded in $J(X_n)$ and weak*-convergent to some $\alpha \in J(X_n)^{**}$.

(iii) Let $(x_n) \in J(X_n)^{**}$. Since $\|(x_n)\|_J < \infty$, the sequence (x_n) is convergent in ℓ_1. We consider the operator Q defined by

$$Q : (x_n) \in J(X_n)^{**} \longrightarrow \lim_{n\to\infty} x_n \in \ell_1.$$

Clearly, $\|Q\| \leq 1$. Moreover, if we denote by Q_n the natural projection from ℓ_1 onto X_n, then $(Q_n(x)) \in J(X_n)^{**}$ for every $x \in \ell_1$, and $Q(Q_n(x)) = x$. Therefore, Q is a surjective operator and its kernel coincides with $J(X_n)$; hence Q induces an isometry from $J(X_n)^{co}$ onto ℓ_1. □

Theorem 3.1.18. *There exists an operator $T \in \mathcal{L}\big(J(X_n)\big)$ such that T is tauberian and cotauberian, but T^* is not cotauberian (equivalently, T^{**} is not tauberian) and T^{**} is not cotauberian.*

Proof. Let $S\colon \ell_1 \longrightarrow \ell_1$ be the operator defined by $S(a_n) := (a_n/n)$. Since S is compact, the range of S^* is a separable subspace of ℓ_∞; thus $R(S^*)$ is not dense, hence S^{**} is not injective. Similarly, $R(S^{**})$ is not dense.

We consider the operator $T\colon J(X_n) \longrightarrow J(X_n)$ defined by

$$T\big((x_n)\big) := (Sx_n) \quad \text{for every } (x_n) \in J(X_n).$$

Clearly $T \in \mathcal{L}\big(J(X_n)\big)$ and the operator T^{co} can be identified with S. Thus T is tauberian and cotauberian, T^* is not cotauberian (hence T^{**} is not tauberian) and T^{**} is not cotauberian. □

The following lemma will be the key to proving the perturbative characterization of cotauberian operators.

Lemma 3.1.19. *Let (g_n) be a bounded sequence in a dual space Y^*. Suppose that $\inf_n \|g_n\| > 0$ and that 0 is a weak*-cluster point of $\{g_n : n \in \mathbb{N}\}$. Then (g_n) has a subsequence (g_{n_k}) for which there exists a bounded sequence (y_k) in Y so that $\langle g_{n_i}, y_j\rangle = \delta_{ij}$ for all $i, j \in \mathbb{N}$.*

Proof. Observe that 0 is a weak*-cluster point of $\{g_n/\|g_n\| : n \in \mathbb{N}\}$ too. So it is enough to prove the case in which the sequence (g_n) is normalized.

We shall find a basic subsequence (g_{n_k}) of (g_n) such that, denoting

$$F := \overline{\text{span}}\{g_{n_k} : k \in \mathbb{N}\}$$

and considering the operator $U\colon Y \longrightarrow F^*$ defined by

$$\langle U(y), g\rangle := \langle g, y\rangle \quad \text{for every } y \in Y, g \in F$$

the set $U(B_Y)$ contains the open unit ball of the closed subspace of F^* generated by the sequence of coefficient functionals corresponding to the basis (g_{n_k}) of F. Thus there exists a bounded sequence (y_i) in Y so that (Uy_i) is that sequence of coefficient functionals. In particular, $\langle g_{n_i}, y_j\rangle = \delta_{ij}$ for all $i, j \in \mathbb{N}$.

Let (ε_n) be a sequence in the open unit interval $(0, 1)$ such that $\sum_{n=1}^\infty \varepsilon_n < \infty$ and $\prod_{n=1}^\infty (1 - \varepsilon_n)^{-1} < \infty$. We claim that we can select a subsequence (g_{n_k}) of (g_n) and an increasing sequence $C_1 \subset C_2 \subset \cdots$ of finite subsets of the unit sphere of Y so that, denoting $F_k := \overline{\text{span}}\{g_{n_1}, \ldots, g_{n_k}\}$ for each $k \in \mathbb{N}$, the following conditions are satisfied:

(a) For each α in F_k^* with $\|\alpha\| = 1$, there exists $y \in C_k$ such that

$$|\langle g, y\rangle - \langle \alpha, g\rangle| \le (\varepsilon_k/3)\|g\| \quad \text{for all } g \in F_k.$$

(b) $|\langle g_{n_{k+1}}, y\rangle| < \varepsilon_k/3$; for all $y \in C_k$.

Indeed, take $g_{n_1} = g_1$ and assume that, for some $k \geq 1$, C_{i-1} and g_{n_i} have been chosen for $i \leq k$.

Since F_k is finite dimensional, there is a natural isometry between F_k^* and $Y/(F_k)_\perp$, which is given by

$$\langle y + (F_k)_\perp, \alpha \rangle := \langle \alpha, y \rangle \quad \text{for every } \alpha \in F_k \text{ and } y \in Y.$$

Moreover, the unit sphere of F_k^* is compact. So we can find a finite subset C_k in S_Y satisfying (a). Now, since 0 is a weak*-cluster point of $\{g_n : n \in \mathbb{N}\}$, we can choose $n_{k+1} > n_k$ so that (b) is satisfied.

Let us show that (g_{n_k}) is a basic sequence. Given $g \in F_k$ with $\|g\| = 1$ and a scalar number λ, we take $\alpha \in Y^{**}$ such that $\|\alpha\| = \|\alpha|_{F_k}\| = 1$ and $\langle \alpha, g \rangle = 1$. Condition (a) provides a vector $y \in C_k$ such that $|\langle g, y \rangle| > 1 - \varepsilon_k/3$; thus

$$\|g + \lambda g_{n_{k+1}}\| \geq |\langle g, y \rangle| - |\lambda \langle g_{n_{k+1}}, y \rangle| > 1 - \varepsilon_k/3 - |\lambda|\varepsilon_k/3;$$

hence $\|g + \lambda g_{n_{k+1}}\| > 1 - \varepsilon_k$ if $|\lambda| \leq 2$. Moreover, $\|g + \lambda g_{n_{k+1}}\| > 1$ if $|\lambda| > 2$.

From these inequalities, it follows that for every sequence (a_k) of scalars we have

$$\Big\| \sum_{i=1}^{k} a_i g_{n_i} \Big\| \leq (1 - \varepsilon_k)^{-1} \Big\| \sum_{i=1}^{k+1} a_i g_{n_i} \Big\|$$

and a repeated application of this inequality gives, for each $k < l$ in $\mathbb{N}$,

$$\Big\| \sum_{i=1}^{k} a_i g_{n_i} \Big\| \leq \prod_{n=k}^{\infty} (1 - \varepsilon_n)^{-1} \Big\| \sum_{i=1}^{l} a_i g_{n_i} \Big\|;$$

this implies that (g_{n_k}) is a basic sequence (see Proposition A.3.6).

Note that, denoting by P_k the projection from $F := \overline{\text{span}}\{g_{n_i} : i \in \mathbb{N}\}$ onto $F_k := \text{span}\{g_{n_1}, \ldots, g_{n_k}\}$, we have $\|P_k\| \leq \prod_{n=k}^{\infty} (1 - \varepsilon_n)^{-1}$; hence $\lim_k \|P_k\| = 1$.

We denote by (α_i) the sequence of coefficient functionals corresponding to the basis (g_{n_i}) of F. Note that (α_i) is a bounded sequence in F^*. Moreover, we consider the operator $U : Y \longrightarrow F^*$ defined by

$$\langle U(y), g \rangle := \langle g, y \rangle \quad \text{for every } y \in Y \text{ and } g \in F.$$

Observe that, for every $g \in F_k$,

$$\langle U(y), g \rangle = \Big\langle \sum_{i=1}^{k} \langle \alpha_i, g \rangle g_{n_i}, y \Big\rangle = \Big\langle \sum_{i=1}^{k} \langle g_{n_i}, y \rangle \alpha_i, g \Big\rangle.$$

Therefore, $\langle U(y), g \rangle = \sum_{i=1}^{\infty} \langle g_{n_i}, y \rangle \langle \alpha_i, g \rangle$ for every $y \in Y$ and $g \in F$. Moreover, if $y \in C_k$ for some $k \in \mathbb{N}$, then

$$\sum_{i=k+1}^{\infty} |\langle g_{n_i}, y \rangle| < \sum_{i=k+1}^{\infty} \varepsilon_i/3 < \infty;$$

hence $\sum_{i=1}^{\infty} \langle g_{n_i}, y\rangle \alpha_i$ converges in $\overline{\operatorname{span}}\{\alpha_i : i \in \mathbb{N}\}$.

By a standard approximation result (see [148, 4.13 Theorem]), in order to show that $U(B_Y)$ contains the open unit ball of the closed subspace of F^* generated by (α_i), it is enough to show that for each α in the unit sphere of $\operatorname{span}\{\alpha_j : j \in \mathbb{N}\}$ and every $\varepsilon' > 0$ there exists y in the unit sphere of Y such that $\|Uy - \alpha\| < 3\varepsilon'$.

Let $\alpha \in \operatorname{span}\{\alpha_j : j \in \mathbb{N}\}$ with $\|\alpha\| = 1$ and let $0 < \varepsilon' < 1$. We choose m so that $\alpha \in \operatorname{span}\{\alpha_1, \dots, \alpha_m\}$, $\sum_{i=m}^{\infty} \varepsilon_i < \varepsilon'$ and $\|P_n\| < 1 + \varepsilon'$ for each $n \geq m$.

We write $\|\gamma\|_0 := \|\gamma|_{F_m}\|$ for $\gamma \in Y^{**}$. It follows that

$$\|\alpha\|_0 \leq \|\alpha\| \leq \|P_m\| \|\alpha\|_0 \leq 2\|\alpha\|_0.$$

Let us take $\beta := (\|\alpha\|_0)^{-1}\alpha$. By condition (a) we can choose $y \in C_m$ in such a way that

$$\Big\| \sum_{i=1}^{m} \langle g_{n_i}, y\rangle \alpha_i - \beta \Big\|_0 \leq \varepsilon_m/3;$$

hence $\| \sum_{i=1}^{m} \langle g_{n_i}, y\rangle \alpha_i - \beta\| \leq 2\varepsilon_m/3 \leq 2\varepsilon'/3$.

We also have $\|\alpha_i\| \leq \|P_i - P_{i-1}\| \leq 4$ for $i > m$. Therefore, by condition (b),

$$\Big\| \sum_{i=m+1}^{\infty} \langle g_{n_i}, y\rangle \alpha_i \Big\| < 4 \sum_{i=m}^{\infty} \varepsilon_i/3 < 4\varepsilon'/3.$$

Thus $\|Uy - \beta\| < 2\varepsilon'$. Since

$$1 = \|\alpha\| \leq \|P_m\| \cdot \|\alpha\|_0 \leq (1 + \varepsilon')\|\alpha\|_0,$$

we have $\|\beta - \alpha\| = \|\alpha\|_0^{-1} - 1 \leq 1 + \varepsilon' - 1 = \varepsilon'$; hence $\|Uy - \alpha\| < 3\varepsilon'$. □

The cotauberian operators have been defined in terms of the conjugate operator. The following result is a perturbative characterization of the cotauberian operators in whose statement conjugate operators do not appear. This fact and the corresponding perturbative characterizations of the semi-Fredhom operators suggest that Definition 3.1.1 is the right choice for cotauberian operators.

Theorem 3.1.20. *An operator $T \in \mathcal{L}(X, Y)$ is cotauberian if and only if the cokernel $Y/\overline{R(T+K)}$ is reflexive for every compact operator $K \in \mathcal{K}(X, Y)$.*

Proof. The direct implication is easy: if T is cotauberian and K is compact, then T^* is tauberian and K^* is compact. By Proposition 2.1.3, $(T + K)^* = T^* + K^*$ is tauberian; hence $N(T^* + K^*)$ is reflexive, and therefore, so is $Y/\overline{R(T + K)}$.

For the converse implication, suppose that T is not cotauberian; hence T^* is not tauberian. By Remark 3.1.2, T^* fails property (N). So, by Theorem 2.2.2, there exists a bounded sequence (g_n) in Y^* containing no weakly convergent subsequence such that $(T^* g_n)$ converges in norm to 0. Observe that if g is a weak*-cluster point of $\{g_n : n \in \mathbb{N}\}$, then $T^*(g) = 0$; thus we can assume that 0 is a weak*-cluster point of $\{g_n : n \in \mathbb{N}\}$.

By Lemma 3.1.19, passing to a subsequence if necessary, we can assume that there exists a bounded sequence (y_n) in Y so that $\langle g_i, y_j\rangle = \delta_{ij}$ for each $i, j \in \mathbb{N}$, and $\|y_n\| \cdot \|T^* g_n\| < 2^{-n}$ for every n.

Now the expression

$$K(x) := -\sum_{n=1}^{\infty} \langle g_n, Tx\rangle y_n \quad \text{for all } x \in X,$$

defines a compact operator $K \in \mathcal{K}(X, Y)$. Moreover,

$$(T + K)^*(g_k) = T^* g_k - \sum_{n=1}^{\infty} \langle g_k, y_n\rangle\, T^* g_n = 0.$$

Since the kernel $N((T+K)^*)$ contains the sequence (g_n), it is non-reflexive; hence the cokernel $Y/\overline{R(T+K)}$, which is isomorphic to the predual of $N((T+K)^*)$, is not reflexive. □

Corollary 3.1.21. *A Banach space Y has no infinite dimensional reflexive quotients if and only if $\mathcal{T}^d(X, Y) = \Phi_-(X, Y)$ for every space X.*

Proof. The direct implication is a consequence of the perturbative characterizations of $\mathcal{T}^d$ (Theorem 3.1.20) and Φ_- (Theorem A.1.9).

For the converse, suppose that N is a closed subspace of Y such that Y/N is infinite dimensional and reflexive. Then the embedding $J_N : N \longrightarrow Y$ is cotauberian (Proposition 3.1.5), but it is not a lower semi-Fredholm operator. □

Next we give a result which is a dual version of Proposition 2.2.9. It provides additional characterizations of the operators $T \in \mathcal{T}^d$ for which T^* does not appear in the statement.

Proposition 3.1.22. *For an operator $T \in \mathcal{L}(X, Y)$, the following statements are equivalent:*

(a) *T is cotauberian;*

(b) *every operator $S \in \mathcal{L}(Y, W)$ is weakly compact whenever ST is weakly compact;*

(c) *any quotient Y/F of Y is reflexive whenever $Q_F T$ is weakly compact.*

Proof. (a) ⇒ (b) Suppose that $T \in \mathcal{T}^d(X, Y)$, $S \in \mathcal{L}(Y, W)$ and ST is weakly compact. Then $T^* S^*$ is weakly compact and $T^* \in \mathcal{T}$. By Proposition 2.2.9, S^* is weakly compact; hence S is weakly compact.

(b) ⇒ (c) Note that a quotient Y/F is reflexive if and only if the quotient map Q_F is weakly compact.

(c) ⇒ (a) Suppose that T is not cotauberian. By Theorem 3.1.20, we can find a compact operator $K \in \mathcal{L}(X, Y)$ so that the quotient $Y/\overline{R(T + K)}$ is not reflexive.

Let us write $F := \overline{R(T+K)}$. Then $Q_F T = -Q_F K$ is weakly compact, but Y/F is not reflexive. □

3.2 The DFJP factorization of operators

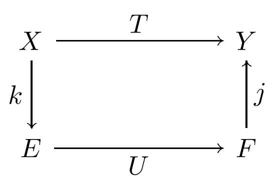
The DFJP factorization, obtained by Davis, Figiel, Johnson and Pełczyński in [49], is an important tool in Banach space theory and it is also the main source of non-trivial examples of tauberian operators. Next we give the statement of the original result:

Theorem 3.2.1 (DFJP factorization). *For every operator $T \in \mathcal{L}(X,Y)$ there exist a Banach space F and operators $A \in \mathcal{L}(X,F)$ and $j \in \mathcal{L}(F,Y)$ such that j is tauberian and $T = jA$.*

For the proof of Theorem 3.2.1, we refer to [49, Lemma 1, Corollary 1], or [52, Lemma 7.4.8].

In this section we are going to prove the following result, obtained in [68], which gives a refined version of the DFJP factorization and is essentially equivalent to it (see Remark 3.2.5), but also uncovers a richer structure and behaves well under duality, as we shall see later in this section.

Theorem 3.2.2. *For every operator $T \in \mathcal{L}(X,Y)$ there exist Banach spaces E and F, and operators $k \in \mathcal{L}(X,E)$, $U \in \mathcal{L}(E,F)$ and $j \in \mathcal{L}(F,Y)$ such that k is cotauberian and has dense range, j is tauberian and injective, U is a bijective isomorphism, and $T = jUk$.*



Proof. In order to construct the spaces E and F, we consider two sequences (p_T^n) and (q_T^n) of norms in X and Y respectively, which are equivalent to the original ones:

$$p_T^n(x) := 2^n\|Tx\| + 2^{-n}\|x\|; \quad x \in X,$$

and

$$q_T^n(y) := \inf\left\{s > 0 : y \in s\left(2^n T(B_X) + 2^{-n} B_Y\right)\right\}; \quad y \in Y.$$

Clearly $2^{-n}\|x\| \le p_T^n(x) \le (2^n\|T\|+1)\|x\|$, for each $x \in X$. Moreover, it is not difficult to check that $(2^n\|T\|+1)^{-1}\|y\| \le q_T^n(y) \le 2^n\|y\|$, for each $y \in Y$.

We denote by X_n the space X endowed with the norm $p_T^n(\cdot)$, and by Y_n the space Y endowed with the norm $q_T^n(\cdot)$. Clearly,

$$\ell_2(X_n) := \Big\{(x_n) \subset X : \|(x_n)\| = \Big(\sum_{n=1}^{\infty} p_T^n(x_n)^2\Big)^{1/2} < \infty\Big\}$$

is a Banach space, and we can define $\ell_2(Y_n)$ in a similar way.

The spaces E and F and the operators j, U and k will be obtained in several steps:

Step 1: For every element $(x_n) \in \ell_2(X_n)$, the series $\sum_{k=1}^{\infty} Tx_k$ is absolutely convergent in Y.

Indeed, it follows from the definition of the norm in $\ell_2(X_n)$ that

$$\|Tx_k\| \le 2^{-k}\|(x_n)\| \quad \text{for every } k \in \mathbb{N}.$$

As a consequence,

$$N_T := \Big\{(x_n) \in \ell_2(X_n) : \sum_{k=1}^{\infty} Tx_k = 0\Big\}$$

is a closed subspace of $\ell_2(X_n)$.

Step 2: We define the space E as the quotient space $\ell_2(X_n)/N_T$ and the operator $k\colon X \longrightarrow E$ by

$$k(x) := (x, 0, 0, 0, \dots) + N_T \quad \text{for every } x \in X.$$

Obviously, $k \in \mathcal{L}(X, E)$. Moreover, in Step 7 we will show that k^* is injective; hence k has dense range.

Step 3: We define the space F as the diagonal subspace

$$\{(y_n) \in \ell_2(Y_n) : y_n = y_1 \text{ for all } n\}$$

of $\ell_2(Y_n)$ and the operator $j\colon F \longrightarrow Y$ by

$$j(y, y, y, \dots) := y \quad \text{for every } (y, y, y, \dots) \in F.$$

Obviously, $j \in \mathcal{L}(F, Y)$ and it is injective.

The most technical part of the proof is contained in the next step.

Step 4: The map $U\colon E \longrightarrow F$ defined by

$$U((x_n) + N_T) := \Big(\sum_{k=1}^{\infty} Tx_k, \sum_{k=1}^{\infty} Tx_k, \sum_{k=1}^{\infty} Tx_k, \dots\Big); \quad (x_n) \in \ell_2(X_n)$$

is an isomorphism from E onto F.

First we show that U is well-defined. Given $(x_n) \in \ell_2(X_n)$, for every $m \in \mathbb{N}$ we write

$$c_m := \Big\|2^{-m}\sum_{k=1}^{m} x_k\Big\| \quad \text{and} \quad d_m := \Big\|2^m \sum_{k=m+1}^{\infty} Tx_k\Big\|.$$

Writing $c_m = \| \sum_{k=0}^{m-1} 2^{-k}2^{-(m-k)}x_{m-k}\|$, we get

$$\Big(\sum_{n=1}^{\infty} c_n^2\Big)^{1/2} \leq \sum_{k=0}^{\infty} 2^{-k}\Big(\sum_{n=1}^{\infty} \|2^{-n}x_n\|^2\Big)^{1/2} \leq 2\|(x_n)\|.$$

Similarly, writing $d_m = \| \sum_{k=1}^{\infty} 2^{-k}2^{m+k}Tx_{m+k}\|$, we get

$$\Big(\sum_{n=1}^{\infty} d_n^2\Big)^{1/2} \leq \sum_{k=1}^{\infty} 2^{-k}\Big(\sum_{n=1}^{\infty} \|2^{n}Tx_n\|^2\Big)^{1/2} \leq \|(x_n)\|.$$

Now observe that $\sum_{k=1}^{m} x_k \in 2^m c_m B_X$; hence $\sum_{k=1}^{m} Tx_k \in 2^m c_m T(B_X)$, and similarly, $\sum_{k=m+1}^{\infty} Tx_k \in 2^{-m} d_m B_Y$. Then

$$q_T^m\Big(\sum_{n=1}^{\infty} Tx_n\Big) \leq \max\{c_m, d_m\} \quad \text{for every } m;$$

hence

$$\Big(\sum_{n=1}^{\infty} q_T^m\Big(\sum_{n=1}^{\infty} Tx_n\Big)^2\Big)^{1/2} \leq 2\|(x_n)\|,$$

and we conclude that $U \in \mathcal{L}(E, F)$ with $\|U\| \leq 2$.

Clearly U is injective. So it remains to show that U is surjective.

Given $(y, y, y, \ldots) \in F$, for each $\varepsilon > 0$ we have

$$y \in (1+\varepsilon)q_T^n(y)\left(2^n T(B_X) + 2^{-n}B_Y\right).$$

Thus $y = Tu_n + v_n$ with $\|u_n\| \leq 2^n(1+\varepsilon)q_T^n(y)$ and $\|v_n\| \leq 2^{-n}(1+\varepsilon)q_T^n(y)$.

Since $q_T^n(y) \xrightarrow[n]{} 0$, the sequence (Tu_n) converges to y. We take $x_1 := u_1$ and $x_n := u_n - u_{n-1}$ for $n > 1$. Obviously the series $\sum_{n=1}^{\infty} Tx_n$ converges to y.

Note that $2^{-n}\|x_n\| \leq 2(1+\varepsilon)q_T^n(y)$ and, for $n > 1$,

$$2^n\|Tx_n\| \leq 2^n\|v_{n-1} - v_n\| \leq 3(1+\varepsilon)q_T^n(y).$$

Therefore,

$$\Big(\sum_{n=1}^{\infty} p_T^n(x_n)^2\Big)^{1/2} \leq 4(1+\varepsilon)\Big(\sum_{n=1}^{\infty} q_T^n(y)^2\Big)^{1/2};$$

hence $(x_n) \in \ell_2(X_n)$ and $U\big((x_n) + N_T\big) = (y, y, y, \ldots)$. Thus U is a bijective isomorphism.

Step 5: The operator j is tauberian and $T = jUk$.

The equality $T = jUk$ is obvious. In order to show that j is tauberian, note that $\ell_2(Y_n)^{**}$ can be identified with $\ell_2(Y_n^{**})$, F^{**} can be identified with the diagonal subspace $\{(\omega_n) \in \ell_2(Y_n^{**}) : \omega_n = \omega_1 \text{ for all } n\}$ of $\ell_2(Y_n^{**})$ and

$$j^{**}(\omega, \omega, \omega, \ldots) = \omega \qquad \text{for all } (\omega, \omega, \omega, \ldots) \in F^{**}.$$

So it is clear that j satisfies the definition of tauberian operator.

It remains to be shown that k is cotauberian. In order to do so we need a description of the dual spaces of X_n and Y_n.

We consider the sequences $(p^n_{T^*})$ and $(q^n_{T^*})$ of norms in Y^* and X^* associated to $T^* \in \mathcal{L}(Y^*, X^*)$:

$$p^n_{T^*}(g) := 2^n \|T^* g\| + 2^{-n}\|g\|; \quad g \in Y^*,$$

and

$$q^n_{T^*}(f) := \inf\left\{s > 0 : f \in s\left(2^n T^*(B_{Y^*}) + 2^{-n} B_{X^*}\right)\right\}; \quad f \in X^*.$$

Step 6: We have $X_n^* \equiv (X^*, q^n_{T^*}(\cdot))$ and $Y_n^* \equiv (Y^*, p^n_{T^*}(\cdot))$ (isometrically).

Indeed, we denote by Z the product space $X \times Y$, endowed with the norm $\|(x, y)\| := \|x\| + \|y\|$. We also consider the auxiliary operator $S \in \mathcal{L}(X, Z)$ defined by $Sx := (2^{-n}x, 2^n Tx)$. Note that the conjugate operator $S^* \in \mathcal{L}(Z^*, X^*)$ is given by $S^*(f, g) = 2^{-n} f + 2^n T^* g$.

Observe that the unit ball of X_n is $S^{-1}(B_Z)$; hence, the unit ball of X_n^* is

$$\begin{aligned}\{f \in X^* : \forall x \in S^{-1}(B_Z),\ |\langle f, x\rangle| \le 1\} &= S^*(B_{Z^*}) \\ &= 2^n T^*(B_{Y^*}) + 2^{-n} B_{X^*},\end{aligned}$$

which is the unit ball of $(X^*, q^n_{T^*}(\cdot))$; hence $X_n^* \equiv (X^*, q^n_{T^*}(\cdot))$.

The other identification can be proved in a similar manner.

Step 7: The operator k is cotauberian.

Note that we can identify E^* with $\{(f_n) \in \ell_2(X_n^*) : f_n = f_1 \text{ for all } n\}$, the diagonal subspace of $\ell_2(X_n^*)$, and the operator k^* is given by

$$k^*(f, f, f, \ldots) := f; \quad (f, f, f, \ldots) \in X^*.$$

Indeed, since $E = \ell_2(X_n)/N_T$, its dual space E^* can be identified with the annihilator $N_T^\perp$ in $\ell_2(X_n^*)$, and it is easy to check that this annihilator coincides with the diagonal subspace of $\ell_2(X_n^*)$.

Note that we have showed that the operator k^* has the same form as j. Thus the same argument that proved that j is tauberian shows that k^* is tauberian; hence k is cotauberian. □

Next we show that, from Theorem 3.2.2 and the basic properties of tauberian and cotauberian operators, we can derive an easy proof of the main result in [49].

Corollary 3.2.3. *An operator $T \in \mathcal{L}(X,Y)$ is weakly compact if and only if it factorizes through a reflexive Banach space.*

Proof. Suppose that T is weakly compact ($T \in \mathcal{W}$). Let $T = jUk$ be the decomposition in Theorem 3.2.2. Since j is tauberian, k is cotauberian and U is an isomorphism, the following implications hold:

$$jUk \in \mathcal{W} \Rightarrow Uk \in \mathcal{W} \Rightarrow U \in \mathcal{W} \Rightarrow \ E \text{ and } F \text{ reflexive.}$$

The converse implication is trivial. □

Definition 3.2.4. The factorization $T = jUk$ given in Theorem 3.2.2 is called the *tauberian decomposition of T*.

Remark 3.2.5. Let $T = jUk$ be the tauberian decomposition of T. Then $A = Uk$ and j give the DFJP factorization of T in Theorem 3.2.1.

Remark 3.2.6. Some versions of the DFJP construction start with an absolutely convex bounded subset K of a Banach space Y and, denoting by $q_n(\cdot)$ the caliber of the set $2^n K + 2^{-n} B_Y$,

$$q_n(y) := \inf\{t > 0 \colon\ y \in t(2^n K + 2^{-n} B_Y)\},$$

which is an equivalent norm on Y, and letting $Y_n := (Y, q_n(\cdot))$, they introduce a Banach space F_K as the diagonal subspace of $\ell_2(Y_n)$; i.e.,

$$F_K := \Big\{ y \in Y \colon\ \sum_{n=1}^{\infty} q_n(y)^2 < \infty \Big\}.$$

The expression $\|y\|_K := (\sum_{n=1}^{\infty} q_n(y)^2)^{1/2}$ defines a norm on F_K for which it is a Banach space, and arguments similar to those given for j in the proof of Theorem 3.2.2 show that the natural inclusion of F_K into Y is a tauberian operator.

This construction can be obtained as a particular case of Theorem 3.2.1. Indeed, taking a dense subset $\{y_i \colon\ i \in I\}$ of K, the expression

$$T(a_i)_{i\in I} := \sum_{i\in I} a_i y_i$$

defines an operator $T \colon \ell_1(I) \longrightarrow Y$ such that $\overline{T(B_{\ell_1(I)})} = \overline{K}$, and the space F_K and the tauberian inclusion of F_K into Y coincide with the intermediate space and the tauberian factor in the DFJP factorization of T.

In the case in which Y is a dual space and K is a weak*-closed subset, the space F_K is also a dual space. This fact was proved in [136].

Remark 3.2.7. The tauberian decomposition of operators provides us with plenty of non-trivial examples of tauberian and cotauberian operators. Note that j (or k) has closed range if and only if T has too.

It is remarkable that the tauberian decomposition behaves well under duality, as we can see in the following result.

Theorem 3.2.8. *Let $T = jUk$ be the tauberian decomposition of T. Then*

(i) *$T^* = k^*U^*j^*$ is equivalent to the tauberian decomposition of T^*.*

(ii) *$T^{co} = j^{co}U^{co}k^{co}$ is equivalent to the tauberian decomposition of T^{co}.*

Proof. (i) We saw in the proof of Theorem 3.2.2 (Step 7) that k^* can be identified with the operator j in the tauberian decomposition of T^*. The arguments for the other two identifications are similar.

(ii) Note that we can identify $\ell_2(X_n)^{**}/\ell_2(X_n)$ with $\ell_2(X_n^{**}/X_n)$. Indeed, the map

$$(\alpha_n) + \ell_2(X_n) \in \ell_2(X_n)^{**}/\ell_2(X_n) \longrightarrow (\alpha_n + X_n) \in \ell_2(X_n^{**}/X_n)$$

is a bijective isometry. Moreover, if F is the diagonal subspace of $\ell_2(Y_n)$, then F^{**} can be identified with $F^{\perp\perp}$, which is the diagonal subspace of $\ell_2(Y_n^{**})$, and similarly, F^{**}/F can be identified with the diagonal subspace of $\ell_2(X_n^{**}/X_n)$.

These and other similar arguments provide the identification of $j^{co}U^{co}k^{co}$ with the tauberian decomposition of T^{co}. □

Theorem 3.2.8 will be applied in Chapter 5 to find factorizations for operators in some operator ideals. Here we give some other consequences.

Corollary 3.2.9. *Let $T = jUk$ denote the tauberian decomposition of T and let $n \in \mathbb{N}$.*

(i) *The successive conjugates operators $j^{*(2n)}$ and $k^{*(2n-1)}$ are tauberian.*

(ii) *The successive conjugates operators $j^{*(2n-1)}$ and $k^{*(2n)}$ are cotauberian.*

Proof. It is a direct consequence of Theorem 3.2.8. □

Next we shall see that, in most cases, the intermediate space in the DFJP factorization contains copies of ℓ_2. We will need the following auxiliary result.

Lemma 3.2.10. *Let (X_k) be a sequence of Banach spaces, let M be an infinite dimensional subspace of $\ell_2(X_k)$, and for each $n \in \mathbb{N}$, let $P_n\colon \ell_2(X_k) \longrightarrow \ell_2(X_k)$ be the projection defined by $P_n(x_i) := (x_1, \ldots, x_n, 0, 0, \ldots)$.*

Suppose that for each $n \in \mathbb{N}$ and each $\varepsilon > 0$, there exists $x \in M$ with $\|x\| = 1$ and $\|P_n x\| < \varepsilon$. Then M contains a subspace isomorphic to ℓ_2.

Proof. It is a sliding hump argument. Note that, for every $x \in \ell_2(X_k)$, the sequence $(P_n x)$ converges to x.

First we choose $x_1 \in M$ with $\|x_1\| = 1$, and select $n_1 \in \mathbb{N}$ such that $\|P_{n_1} x_1\| > 1 - 2^{-2}$. Then we choose $x_2 \in M$ with $\|x_2\| = 1$ and $\|P_{n_1} x_2\| < 2^{-2}$, and select $n_2 \in \mathbb{N}$ such that $\|P_{n_2} x_2\| > 1 - 2^{-4}$.

Continuing the process, we obtain a normalized sequence (x_k) in M and a strictly increasing sequence (n_k) in $\mathbb{N}$ so that, for all $k \in \mathbb{N}$ and all $i < k$,

$$\|P_{n_k} x_k\| > 1 - 2^{-2k} \quad \text{and} \quad \|P_{n_i} x_k\| < 2^{-2k}.$$

It is easy to show that (x_k) generates a subspace isomorphic to ℓ_2 in M. □

Theorem 3.2.11. *Let $j\colon F \longrightarrow Y$ be the tauberian factor in the tauberian decomposition of $T \in \mathcal{L}(X, Y)$, and let M be a non-zero subspace of F. If the restriction $j|_M$ is not an isomorphism, then M contains a subspace isomorphic to ℓ_2.*

Proof. In the construction of the decomposition we saw that F is a subspace of $\ell_2(Y_n)$, with $Y_n = (Y, q_T^n(\cdot))$. We are going to obtain the result as an application of Lemma 3.2.10. In order to do that, let $n \in \mathbb{N}$ and $\varepsilon > 0$.

Since $j|_M$ is not an isomorphism, there exists $y \in M$ such that $\|y\| = 1$ and $\|j(y)\| \leq \varepsilon/2^n$.

We saw that $q_T^k(y) \leq 2^k \|j(y)\|$ for each $y \in F$ and $k \in \mathbb{N}$. Hence,

$$\|P_n y\|^2 = \sum_{k=1}^{n} q_T^k(y)^2 \leq \sum_{k=1}^{n} \varepsilon^2/2^{2(n-k)} < \varepsilon^2.$$

Thus Lemma 3.2.10 implies that M contains a subspace isomorphic to ℓ_2. □

Let us see a consequence of this theorem. Recall that a Banach space X is said to be *hereditarily ℓ_2* if every infinite dimensional closed subspace of X contains a subspace isomorphic to ℓ_2.

Corollary 3.2.12. *For every compact operator $T\colon X \longrightarrow Y$, the intermediate space F in the tauberian decomposition of T is hereditarily ℓ_2.*

Proof. If T is compact, then the tauberian factor $j\colon F \longrightarrow Y$ in the tauberian decomposition of T is also compact (see Proposition 5.3.3); hence, it is enough to apply Theorem 3.2.11. □

3.3 Variations of the DFJP factorization

In Section 3.2 we have described a refinement of the celebrated DFJP factorization. Several other variations of this factorization have been studied. The general structure is similar: every operator is factorized through an intermediate Banach space, and the second factor is a tauberian operator. However, they have been constructed for specific purposes, like getting a Banach lattice or a Banach algebra as intermediate space, satisfying certain conditions, and so forth. In this section we give a brief description of some of these variations. Later, in Chapter 5, we will show some applications.

An isometric variation of the DFJP factorization

Lima, Nygaard and Oja [119] introduced an isometric variation of the DFJP factorization and applied it to obtain some characterizations of the approximation property of Banach spaces.

In order to describe this construction, let $a > 1$ and let K be a closed absolutely convex subset of the unit ball B_Y of a Banach space Y. For each $n \in \mathbb{N}$, let $\|\cdot\|_n$ denote the gauge of the set

$$B_n := a^{n/2}K + a^{-n/2}B_Y.$$

Then $\|\cdot\|_n$ is a norm on Y equivalent to the original one. We define

$$\|y\|_K := \left(\sum_{n=1}^{\infty} \|y\|_n^2\right)^{1/2}.$$

We write $F_K := \{y \in Y : \|y\|_K < \infty\}$, $J_K : F_K \longrightarrow Y$ the natural embedding, and $C_K := \{y \in Y : \|y\|_K \leq 1\}$, the unit ball of F_K.

Remark 3.3.1. Given $T \in \mathcal{L}(X, Y)$, if we take $a = 4$ and $K = \overline{T(B_X)}$ in the previous construction, J_K and F_K are the tauberian operator and the intermediate space in the DFJP factorization of T.

To introduce the isometric bent, we consider a function $f : (1, \infty) \longrightarrow \mathbb{R}$ defined as follows:

$$f(a) := \left(\sum_{n=1}^{\infty} \frac{a^n}{(a^n + 1)^2}\right)^{1/2}.$$

Remark 3.3.2. It is easy to see that there exists $\tilde{a} > 1$ such that $f(\tilde{a}) = 1$. A good estimate of this $\tilde{a}$ is $e^{4/9}$. For this $\tilde{a}$, one has $K \subset C_K \subset B_Y$.

The proof of the following result is not difficult.

Lemma 3.3.3. *Let K be a closed absolutely convex subset of the unit ball of a Banach space Y and let $a > 1$. Then, with the notation we have just introduced in the previous comments, the following results hold:*

(i) *F_K is a Banach space;*

(ii) *$K \subset f(a)C_K$.*

Now, given a non-zero operator $T \in \mathcal{L}(X, Y)$, we take $K = \|T\|^{-1}\overline{T(B_X)}$ and, for $a > 1$, construct the corresponding space F_K.

It is not difficult to see that $A_K(x) := Tx$ defines an operator $A_K : X \longrightarrow F_K$. Let us see some properties of the factorization obtained with this scheme.

Theorem 3.3.4. *Let $T \in \mathcal{L}(X, Y)$ and let $K = \|T\|^{-1}\overline{T(B_X)}$. Then J_K is a tauberian injective operator and $T = J_K A_K$.*

If, additionally, $f(a) = 1$, then $\|T\| = \|A_K\|$ and $\|J_K\| = 1$.

Proof. The first part is similar to the proof of the result for the DFJP factorization. The isometric part of the proof is not difficult. □

Connections with real interpolation methods

In [123, Theorem 2.g.11], the factorization of weakly compact operators through reflexive spaces (Corollary 3.2.3) is obtained as an application of one of the versions of the real interpolation method for Banach spaces. Moreover, in Beauzamy's exposition [21], we can see that there is a similarity between the construction of the intermediate spaces in the DFJP factorization and in the tauberian decomposition, and the construction of intermediate spaces in the real interpolation method. Here we describe these connections.

First we recall the definitions of the intermediate spaces in two of the discrete versions of the real interpolation method.

Let A_0 and A_1 be two Banach spaces (endowed with the norms $\|\cdot\|_0$ and $\|\cdot\|_1$) which are subspaces of a certain vector space, so that the sum

$$A_0 + A_1 := \{x_0 + x_1 \colon\ x_0 \in A_0,\ x_1 \in A_1\}$$

is well-defined.

To follow this description, it could be useful to have a concrete case in mind: the spaces $A_0 = L_1(0,\infty)$ and $A_1 = L_\infty(0,\infty)$ of scalar measurable functions on $(0,\infty)$.

It is not difficult to show that $A_0 + A_1$, endowed with the norm

$$\|x\|_S := \inf\{\|x_0\|_0 + \|x_1\|_1 \colon\ x_0 \in A_0,\ x_1 \in A_1, x = x_0 + x_1\},$$

and $A_0 \cap A_1$, endowed with the norm

$$\|x\|_I := \max\{\|x\|_0, \|x\|_1\},$$

are Banach spaces.

Now, following [21, Section 1.4], we consider three fixed real numbers p, ξ_0 and ξ_1 satisfying $1 \le p < \infty$, $\xi_0 < 0$ and $\xi_1 > 0$, and we define the space

$$S_\alpha(p;\xi_0,\xi_1) := \Big\{x \in A_0 + A_1 \colon \exists (x_n)_{n\in\mathbb{Z}} \subset A_0 \cap A_1 \text{ satisfying } \sum_{n\in\mathbb{Z}} e^{\xi_0 np}\|x_n\|_0^p < \infty, \sum_{n\in\mathbb{Z}} e^{\xi_1 np}\|x_n\|_1^p < \infty \text{ and } x = \sum_{n\in\mathbb{Z}} x_n\Big\},$$

and the norm

$$\|x\|_\alpha := \inf\ \max\Big\{\sum_{n\in\mathbb{Z}} e^{\xi_0 np}\|x_n\|_0^p, \sum_{n\in\mathbb{Z}} e^{\xi_1 np}\|x_n\|_1^p\Big\}^{1/p},$$

where the infimum is taken over all the sequences $(x_n)_{n\in\mathbb{Z}} \subset A_0 \cap A_1$ for which $x = \sum_{n\in\mathbb{Z}} x_n$.

Proposition 3.3.5. *The space $S_\alpha(p;\xi_0,\xi_1)$, endowed with the norm $\|\cdot\|_\alpha$, is a Banach space.*

Proof. It is rather technical. We refer to [21, Section 1.4] for details. □

Definition 3.3.6. The space $S_\alpha(p;\xi_0,\xi_1)$ is the *real interpolation space between* A_0 *and* A_1, with parameters p, ξ_0 and ξ_1.

Remark 3.3.7. The space $S_\alpha(p;\xi_0,\xi_1)$ is one of the discrete versions of the real interpolation spaces between A_0 and A_1. There are other equivalent definitions of this interpolation space in terms of spaces of vector-valued measurable functions, instead of spaces of sequences. See [21, Chapter 1].

Next we describe another discrete version, which was introduced by Beauzamy, inspired by the DFJP factorization.

Let $j_0\colon A_0 \longrightarrow A_0 + A_1$ and $j_1\colon A_1 \longrightarrow A_0 + A_1$ be the natural inclusions. We write

$$B_0 := j_0(B_{A_0}) \quad \text{and} \quad B_1 := j_1(B_{A_1}).$$

Moreover, for every $n \in \mathbb{Z}$, we consider the subsets

$$U_n := e^{-\xi_0 n} B_0 + e^{-\xi_1 n} B_1$$

of $A_0 + A_1$, and denote by $q_n(\cdot)$ their calibers:

$$q_n(x) := \inf\{t > 0\colon\ x \in tU_n\}.$$

We define the space

$$S_\gamma(p;\xi_0,\xi_1) := \Big\{x \in A_0 + A_1\colon\ \sum_{n\in\mathbb{Z}} q_n(x)^p < \infty\Big\},$$

endowed with the norm $\|x\|_\gamma := \big(\sum_{n\in\mathbb{Z}} q_n(x)^p\big)^{1/p}$.

Proposition 3.3.8. *The spaces* $S_\alpha(p;\xi_0,\xi_1)$ *and* $S_\gamma(p;\xi_0,\xi_1)$ *coincide algebraically and the norms* $\|\cdot\|_\alpha$ *and* $\|\cdot\|_\gamma$ *are equivalent.*

For the proof we refer to [21, Section 1.4].

In the special case in which A_0 is continuously embedded in A_1; i.e., $A_0 \subset A_1$ and there exists $C > 0$ such that $\|x\|_1 \le C\|x\|_0$ for every $x \in A_0$, we have an equivalent expression for the norm of $S_\alpha(p;\xi_0,\xi_1)$.

Proposition 3.3.9. *Suppose that* A_0 *is continuously embedded in* A_1. *Then the expression* $\|z\|_{\gamma+} := \Big(\sum_{n\ge 0} q_n(z)^p\Big)^{1/p}$ *defines a norm on* $S_\gamma(p;\xi_0,\xi_1)$ *which is equivalent to* $\|\cdot\|_\gamma$.

For the proof we refer to [21, Section 1.5].

Now we have the tools we need to show that we can identify the intermediate space F of the tauberian decomposition of an operator $T\colon X \longrightarrow Y$ with a real interpolation space $S_\gamma(p;\xi_0,\xi_1)$ for special values of the parameters.

First, observe that it is not difficult to show that T and the associated injective operator $\widetilde{T}\colon X/N(T) \longrightarrow Y$ produce the same intermediate space F in the tauberian decomposition. So we can assume that the operator T is injective.

Now we can consider the space X as a subspace of Y, which is continuously embedded through the operator T. So we identify each $x \in X$ with $Tx \in Y$.

We take $A_0 := T(X)$, with the norm $\|Tx\|_0 = \|x\|$, and $A_1 := Y$, with its original norm.

We select as parameters the values $p = 2$, $\xi_0 = -\log 2$ and $\xi_1 = \log 2$. Therefore,

$$U_n = e^{-\xi_0 n} B_0 + e^{-\xi_1 n} B_1 = 2^n T(B_X) + 2^{-n} B_Y.$$

Now, comparing this with the construction in the proof of Theorem 3.2.2, it is clear that the map

$$V(y, y, y, \ldots) := y$$

defines an isomorphism from F onto $S_\gamma(2; -\log 2, \log 2)$.

A conditional variation of the DFJP factorization

In the DFJP factorization, the intermediate space is constructed as the diagonal subspace of a space $\ell_2(Y_n)$. We can loosely say that the norms of the spaces Y_n are averaged by means of an ℓ_2-sum. Since the unit vector basis of ℓ_2 is an unconditional basis, the intermediate space inherits some unconditional character. For example, in most cases the intermediate space in the factorization contains subspaces isomorphic to ℓ_2, as can be seen in Theorem 3.2.11. This is not convenient if our aim is to obtain a hereditarily indecomposable Banach space (see Definition 3.3.16), because these spaces do not contain unconditional basic sequences. We refer to the paper of Gowers and Maurey [95], in which the first Banach space of this kind is constructed, for additional information.

Argyros and Felouzis showed in [13] that certain operators factor through an hereditarily indecomposable Banach space. The factorization they construct is inspired in the DFJP factorization, but they have to take the intermediate space as a diagonal subspace of a conditional sum of Banach spaces, in order to avoid the appearance of unconditional basic sequences. Let us describe this construction, that can be considered as a generalization of the DFJP factorization.

Let (X_n) be a sequence of Banach spaces, let $\|\cdot\|_n$ denote the respective norms, and let $\prod_{n=1}^{\infty} X_n$ denote their cartesian product.

For $x = (x_n) \in \prod_{n=1}^{\infty} X_n$, the *support* $\operatorname{supp}(x)$ is the set of all $n \in \mathbb{N}$ such that $x_n \neq 0$. We denote

$$\Big(\prod_{n=1}^{\infty} X_n\Big)_{00} := \Big\{x \in \prod_{n=1}^{\infty} X_n \colon \ \operatorname{supp}(x) \text{ is finite } \Big\}.$$

We also denote by $P_n\colon \prod_{n=1}^\infty X_n \longrightarrow \prod_{n=1}^\infty X_n$ the projection defined by

$$P_n\big((x_i)\big) := (x_1, \ldots, x_n, 0, 0, \ldots) \quad \text{for every } (x_i) \in \prod_{i=1}^\infty X_i.$$

Given a finite subset $A \subset \mathbb{N}$ and $x = (x_n) \in \prod_{n=1}^\infty X_n$, we denote by $P_A(x)$ the element in $\prod_{n=1}^\infty X_n$ obtained from x by replacing x_n by 0 for $n \notin A$. In this way we obtain a projection

$$P_A\colon \prod_{n=1}^\infty X_n \longrightarrow \prod_{n=1}^\infty X_n.$$

Given a pair A, B of finite subsets of $\mathbb{N}$, we write $A < B$ if $\max A < \min B$. A sequence (x_n) of non-zero vectors in $(\prod_{n=1}^\infty X_n)_{00}$ is said to be a *block sequence* if $\operatorname{supp}(x_n) < \operatorname{supp}(x_{n+1})$ for each $n \in \mathbb{N}$.

Definition 3.3.10. Let $((X_n, \|\cdot\|_n))$ be a sequence of Banach spaces. A Banach space $(Z, \|\cdot\|)$ is said to be a *d-product of the sequence* (X_n) if it satisfies the following conditions:

(i) $(\prod_{n=1}^\infty X_n)_{00} \subset Z \subset \prod_{n=1}^\infty X_n$ (algebraically);

(ii) $(\prod_{n=1}^\infty X_n)_{00}$ is dense in Z;

(iii) the natural map from X_n into Z is an isometry, for each $n \in \mathbb{N}$;

(iv) the projection P_n is bounded on Z for each $n \in \mathbb{N}$, and $z = \lim_{n\to\infty} P_n z$ for each $z \in Z$.

Remark 3.3.11. The conditions of the previous definition imply that (X_n) is a Schauder decomposition of Z. Note that, by the uniform boundedness principle, $\sup_n \|P_n\| < \infty$.

We refer to [122, Section 1.9] for additional information on Schauder decompositions of Banach spaces.

Definition 3.3.12. Let Z be a d-product of a sequence (X_n) of Banach spaces, as in Definition 3.3.10.

(i) The d-product Z is *boundedly complete* if given $x \in \prod_{n=1}^\infty X_n$,

$$\sup_{n\in\mathbb{N}} \|P_n(x)\| < \infty \ \text{ implies } \ x \in Z;$$

(ii) the d-product Z is *shrinking* if $\alpha = \lim_{n\to\infty} P_n^* \alpha$, for every $\alpha \in Z^*$;

(iii) the d-product Z is *bimonotone* if $\|P_A\| = 1$ for every finite interval A of integers.

The proof of the following proposition is similar to that of the corresponding result for Banach spaces with a Schauder basis (see [122, Section 1.b]).

Proposition 3.3.13. *Let Z be a shrinking d-product of a sequence (X_n) of Banach spaces. Then the following assertions hold:*

(i) *Z^* is a d-product of the spaces (X_n^*), embedded through the natural maps;*

(ii) *we can identify Z^{**} with the space of all sequences (x_n^{**}) satisfying $x_n^{**} \in X_n^{**}$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \|x_1^{**} + \cdots + x_n^{**}\| < \infty$.*

We are interested in a special kind of d-product.

Definition 3.3.14. Let $(\|\cdot\|_n)$ be a sequence of equivalent norms on a Banach space X and let Z be a d-product of the sequence $\big((X, \|\cdot\|_n)\big)$. The *diagonal space ΔZ of Z* is defined by

$$\Delta Z := \{(x_i) \in Z\colon\ x_n = x_1 \text{ for all } n\},$$

and $J\colon \Delta Z \longrightarrow X_1 := (X, \|\cdot\|_1)$ is the map defined by

$$J\big((x, x, x, \dots)\big) := x \quad \text{for each } (x, x, x, \dots) \in \Delta Z.$$

Let us see that the diagonal space in Definition 3.3.14 has similar properties to that of the intermediate space in the DFJP factorization.

Proposition 3.3.15. *Let $(\|\cdot\|_n)$ be a sequence of equivalent norms on a Banach space X and let Z be a boundedly complete and shrinking d-product of the sequence $\big((X, \|\cdot\|_n)\big)$. Then $J\colon \Delta Z \longrightarrow X$ is an injective tauberian operator.*

Proof. Clearly J is injective. Moreover,

$$\|J(x, x, x, \dots)\| = \|P_1(x, x, x, \dots)\| \le \|P_1\| \cdot \|(x, x, x, \dots)\|;$$

hence J is a bounded operator.

Let $\alpha \in (\Delta Z)^{**}$. Since α can be attained as the weak*-limit of a net in ΔZ and Z is a shrinking d-product, by Proposition 3.3.13 there exists $x^{**} \in X^{**}$ so that

$$\alpha = (x^{**}, x^{**}, x^{**}, x^{**}, \dots).$$

Suppose that $J^{**}\alpha \in X$. Then $x^{**} = x \in X$ and

$$\sup_{n\in\mathbb{N}} \|P_n(x, x, x, \dots)\| = \sup_{n\in\mathbb{N}} \|P_n^{**}\alpha\| < \infty.$$

Since Z is boundedly complete, we have $\alpha = (x, x, x, \dots) \in Z$; hence $\alpha \in \Delta Z$. Therefore J is tauberian. $\square$

Definition 3.3.16. A Banach space X is *hereditarily indecomposable (H.I., for short)* if no subspace of X can be decomposed as the topological direct sum of two infinite dimensional closed subspaces.

Remark 3.3.17. It follows from Gowers' dichotomy [94] that an infinite dimensional Banach space contains an infinite dimensional H.I. closed subspace or an unconditional basic sequence.

Definition 3.3.18. A d-product space Z is *block-H.I.* if every closed subspace of Z generated by a block sequence is a H.I. space.

The following result is the key to obtain factorizations of operators through H.I. spaces. The proof is not difficult. However, finding d-products which are block-H.I. is rather tricky. See [13, Section 7].

Proposition 3.3.19. *Let $(\|\cdot\|_n)$ be a sequence of equivalent norms on a Banach space X and let Z be a d-product of the sequence $\big((X, \|\cdot\|_n)\big)$. If Z is block-H.I. and the operator $J\colon \Delta Z \longrightarrow X_1$ is strictly singular, then ΔZ is a H.I. space.*

For the proof, we refer to [13, Proposition 2.1].

The following result is an application of Proposition 3.3.19. We should compare it with Corollary 3.2.12.

Theorem 3.3.20. *Every compact operator $T\colon X \longrightarrow Y$ can be factorized through a H.I. Banach space.*

For the proof, we refer to [13, Theorem 8.5].

Remark 3.3.21. Theorem 3.3.20 is also true for any strictly singular operator $T\colon \ell_p \longrightarrow \ell_q$ with $q < \infty$ and for the natural inclusion $i\colon L_\infty(0,1) \longrightarrow L_1(0,1)$.
We refer to [13] for the details.

3.4 Inclusions of vector-valued function spaces

Here we show that some natural inclusions between Banach spaces of vector-valued measurable functions are tauberian operators. In order to introduce these spaces, we need some notation.

Let $\phi\colon [0,\infty] \to [0,\infty]$ be a non-decreasing, left-continuous, non-zero function satisfying $\phi(0) = 0$.

The left-inverse ψ of ϕ is given by $\psi(0) := 0$ and

$$\psi(v) := \sup\{u\colon \phi(u) < v\}, \quad \text{for } v > 0.$$

From ϕ and ψ we obtain the *Young function* $\Phi\colon [0,\infty] \to [0,\infty]$ and its *conjugate function* Ψ, as follows:

$$\Phi(u) := \int_0^u \phi(t)\,dt \quad \text{and} \quad \Psi(u) := \int_0^u \psi(t)\,dt.$$

Definition 3.4.1. We say that a Young function $\Phi\colon [0,\infty] \to [0,\infty]$ *satisfies the Δ_2-condition* if it is finite on $[0,\infty)$ and there exist $K > 0$ and $t_0 \geq 0$ so that $\Phi(2t) \leq K\Phi(t)$, for $t \geq t_0$.

Let (Ω, Σ, μ) be a finite measure space with no atoms, let Φ be a Young function and let X be a Banach space. For every Bochner measurable function $f\colon \Omega \to X$ we write

$$M_\Phi(f) := \int_\Omega \Phi(\|f(t)\|)\, d\mu(t).$$

The Orlicz space $L_\Phi(X)$ associated with Φ is defined as the space of all Bochner measurable functions $f\colon \Omega \to X$ such that $M_\Phi(kf) < \infty$ for some $k > 0$.

In the case when X is the scalar field $\mathbb{K}$, we write L_Φ instead of $L_\Phi(\mathbb{K})$.

The space $L_\Phi(X)$ coincides with the set of all Bochner measurable functions $f\colon \Omega \to X$ such that

$$\|f\|_\Phi := \sup\Big\{\int_\Omega \|f(t)\| h(t)\, d\mu(t)\colon h \in L_\Phi, M_\Phi(h) \le 1\Big\} < \infty.$$

Moreover, $(L_\Phi(X), \|\cdot\|_\Phi)$ is a Banach space contained in $L_1(X)$ and the natural embedding J_Φ of $L_\Phi(X)$ into $L_1(X)$ is a continuous operator.

Proposition 3.4.2. *Let Φ be a Young function such that Φ and its conjugate satisfy the Δ_2 condition. Then the embedding $J_\Phi\colon L_\Phi(X) \longrightarrow L_1(X)$ is a tauberian operator.*

Proof. Let (f_n) be a bounded sequence in $L_\Phi(X)$ which is weakly convergent to some f in $L_1(X)$.

From the fact that Φ and its conjugate satisfy the Δ_2 condition, it follows that we can identify $L_1(X)^*$ with a dense subspace of $L_\Phi(X)^*$. Now, since (f_n) is bounded in $L_\Phi(X)$, from

$$\lim_{n\to\infty} \langle f_n, g\rangle = \langle f, g\rangle$$

for each $g \in L_1(X)^*$, it follows that the same is true for each $g \in L_\Phi(X)^*$. Thus (f_n) is weakly convergent to f in $L_\Phi(X)$ and by Theorem 2.2.4 we conclude that J_Φ is tauberian. $\square$

Remark 3.4.3. Under the hypothesis of Proposition 3.4.2, we can identify $L_\Phi(X)^{**}$ with $L_\Phi(X^{**})$ and $L_\Phi(X)^{co}$ with $L_\Phi(X^{co})$. Thus we can give an alternative proof by showing that J_Φ^{co} can be identified with the natural inclusion of $L_\Phi(X^{co})$ into $L_1(X^{co})$.

It is not difficult to derive a slight extension of Proposition 3.4.2.

Corollary 3.4.4. *Let Φ_1 and Φ_2 be Young functions such that both of them and their respective conjugates satisfy the Δ_2 condition. Suppose that the associated Orlicz spaces satisfy $L_{\Phi_1}(X) \subset L_{\Phi_2}(X)$. Then the natural embedding*

$$J_{1,2}\colon L_{\Phi_1}(X) \longrightarrow L_{\Phi_2}(X)$$

is a tauberian operator.

Proof. Since we can write $J_{\Phi_1} = J_{\Phi_2} J_{1,2}$ and J_{Φ_1} is tauberian, it follows from Proposition 2.1.3 that $J_{1,2}$ is tauberian. □

The following result is a good complement to Proposition 3.4.2.

Proposition 3.4.5. *Given a Young function Φ, the following conditions are equivalent:*

(a) *The embedding operator $J_\Phi : L_\Phi \longrightarrow L_1$ is tauberian.*

(b) *For each Banach space X, the embedding operator $J_\Phi : L_\Phi(X) \longrightarrow L_1(X)$ is tauberian.*

Since the proof is rather technical, we omit it and refer the interested reader to [35, Theorem 3.1].

3.5 Tauberian-like classes of operators defined in terms of sequences

Inspired by semi-Fredholm operators and by tauberian and cotauberian operators, several classes of operators defined in terms of sequences have been studied (see [34], [79], [89], [90], [92], [102] and [126]). In this study, four operator ideals have been considered, and for each one of them, two classes have been introduced: one corresponds to tauberian operators and the other to cotauberian operators. The properties of the corresponding classes are similar to those of $\mathcal{T}$ and $\mathcal{T}^d$. In particular, they admit a perturbative characterization. This is important because it allows us to find relations of inclusion between some of these classes. We refer to Section A.2 for the fundamentals of the theory of operator ideals. Here we describe the main properties of these classes. Later, in Chapter 6, we will see that they are operator semigroups associated with some operator ideals. Thus we will introduce here a notation consistent with that in Chapter 6.

First we give the definitions of some well-known classes of operators. Recall that a series $\sum_{n=1}^\infty x_n$ in a Banach space X is called *weakly unconditionally Cauchy* if $\sum_{n=1}^\infty |\langle f, x_n\rangle| < \infty$ for every $f \in X^*$.

Definition 3.5.1. Let $T \in \mathcal{L}(X, Y)$.

(i) We say that T is *weakly precompact, or Rosenthal*, and we write $T \in \mathcal{R}$, if (Tx_n) has a weakly Cauchy subsequence for every bounded sequence (x_n) in X.

(ii) We say that T is *completely continuous*, and we write $T \in \mathcal{C}$, if (Tx_n) is convergent for every weakly Cauchy sequence (x_n) in X.

(iii) We say that T is *weakly completely continuous*, and we write $T \in \mathcal{WC}$, if (Tx_n) is weakly convergent for every weakly Cauchy sequence (x_n) in X.

(iv) We say that T is *unconditionally convergent*, and we write $T \in \mathcal{U}$, if the series $\sum_{n=1}^{\infty} Tx_n$ is unconditionally convergent for every weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ in X.

Remark 3.5.2. It is well-known that the classes $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ and $\mathcal{U}$ are operator ideals.

Remark 3.5.3. It is not difficult to show that $T \in \mathcal{L}(X,Y)$ is completely continuous if and only if it takes weakly convergent sequences into convergent sequences. So this action over sequences does not produce a new operator ideal.

Two of the operator ideals in Definition 3.5.1 admit a characterization in terms of restrictions.

Proposition 3.5.4. *For every operator $T \in \mathcal{L}(X,Y)$, the following statements hold:*

(i) *$T \in \mathcal{R}$ if and only if there is no subspace M of X isomorphic to ℓ_1 so that the restriction $T|_M$ is an isomorphism;*

(ii) *$T \in \mathcal{U}$ if and only if there is no subspace M of X isomorphic to c_0 so that the restriction $T|_M$ is an isomorphism.*

Proof. (i) Suppose that $T \in \mathcal{R}$. Since the unit vector basis of ℓ_1 has no weakly Cauchy subsequences, if a subspace M of X is isomorphic to ℓ_1, then $T|_M$ cannot be an isomorphism.

Conversely, suppose that $T \notin \mathcal{R}$. Then we can find a bounded sequence (x_n) in X such that (Tx_n) has no weakly Cauchy subsequences. By Rosenthal's ℓ_1-theorem (Theorem A.3.10), passing to a subsequence, we can assume that both (x_n) and (Tx_n) are equivalent to the unit vector basis of ℓ_1. Then

$$M := \overline{\text{span}}\{x_n \colon n \in \mathbb{N}\}$$

is isomorphic to ℓ_1 and $T|_M$ is an isomorphism.

(ii) Suppose that $T \in \mathcal{U}$. Let (e_n) denote the unit vector basis of c_0. Since $\sum_{n=1}^{\infty} e_n$ is a weakly unconditionally Cauchy series which is not unconditionally converging, if a subspace M of X is isomorphic to c_0, then $T|_M$ cannot be an isomorphism.

Conversely, suppose that $T \notin \mathcal{U}$. Then we can find a weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ in X such that $\sum_{n=1}^{\infty} Tx_n$ is not unconditionally converging.

By reordering the sequence (x_n), we can assume that the series $\sum_{n=1}^{\infty} Tx_n$ is not convergent. Then we can take $1 \leq k_1 \leq m_1 < k_2 \leq m_2 < \cdots$ in $\mathbb{N}$ so that the vectors

$$y_n := x_{k_n} + x_{k_n+1} + \cdots + x_{m_n} \quad (n \in \mathbb{N})$$

satisfy $\inf_{n\in\mathbb{N}} \|Ty_n\| > 0$.

Observe that $\sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty} Ty_n$ are weakly unconditionally Cauchy series. By the Bessaga-Pełczyński selection principle (Proposition A.3.7), passing to a subsequence, we can assume that both (y_n) and (Ty_n) are basic sequences.

Therefore, by Proposition A.3.12, both sequences are equivalent to the unit vector basis of c_0. Then

$$M := \overline{\operatorname{span}}\{y_n :\ n \in \mathbb{N}\}$$

is isomorphic to c_0 and $T|_M$ is an isomorphism. □

Recall that for every operator ideal $\mathcal{A}$, the expression

$$\mathcal{A}^d := \{T \in \mathcal{L} :\ T^* \in \mathcal{A}\}$$

defines the dual operator ideal $\mathcal{A}^d$ of $\mathcal{A}$ and

$$Sp(\mathcal{A}) = \{X :\ I_X \in \mathcal{A}\}$$

is the associated space ideal. See Section A.2.

Remark 3.5.5. The space ideals $Sp(\mathcal{W})$, $Sp(\mathcal{R})$, $Sp(\mathcal{C})$, $Sp(\mathcal{WC})$ and $Sp(\mathcal{U})$ are the reflexive spaces, the spaces containing no copies of ℓ_1, the spaces with the Schur property, the weakly sequentially complete spaces and the spaces containing no copies of c_0, respectively. Moreover,

$$Sp(\mathcal{A}^d) = \{X :\ X^* \in Sp(\mathcal{A})\}.$$

First we introduce the classes which are similar to tauberian operators (and to upper semi-Fredholm operators). The definition in each case is a property which is the opposite of the definition of the corresponding operator ideal.

Definition 3.5.6. Let $T \in \mathcal{L}(X, Y)$.

(i) We write $T \in \mathcal{R}_+$ if a sequence (x_n) in X has a weakly Cauchy subsequence, whenever (x_n) is bounded and (Tx_n) is weakly Cauchy in Y.

(ii) We write $T \in \mathcal{C}_+$ if a sequence (x_n) in X is convergent, whenever (x_n) is weakly Cauchy and (Tx_n) is convergent in Y.

(iii) We write $T \in \mathcal{WC}_+$ if a sequence (x_n) in X is weakly convergent, whenever (x_n) is weakly Cauchy and (Tx_n) is weakly convergent in Y.

(iv) We write $T \in \mathcal{U}_+$ if a series $\sum_{n=1}^{\infty} x_n$ in X is unconditionally convergent, whenever $\sum_{n=1}^{\infty} x_n$ is weakly unconditionally Cauchy and $\sum_{n=1}^{\infty} Tx_n$ unconditionally convergent in Y.

Next we introduce, using duality, the classes which are similar to cotauberian operators and lower semi-Fredholm operators.

Definition 3.5.7. Let $\mathcal{A}$ be one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$ and let $T \in \mathcal{L}(X, Y)$. We write $T \in \mathcal{A}^d{}_-$ whenever the conjugate $T^* \in \mathcal{A}_+$.

Remark 3.5.8. In Definition 3.5.7, for $\mathcal{A}$ equal to $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, we have denoted the dual class of $\mathcal{A}_+$ by $\mathcal{A}^d{}_-$. We have done that to be consistent with Definition 6.1.7, where we introduce the operator semigroups associated with an operator ideal $\mathcal{A}$.

Remark 3.5.9. According to Schauder's theorem and Gantmacher's theorem (see [4, A.4 and G.5]), the operator ideals $\mathcal{K}$ and $\mathcal{W}$ are self-dual, i.e., $\mathcal{K}^d = \mathcal{K}$ and $\mathcal{W}^d = \mathcal{W}$.

Recall that an operator $T \in \mathcal{L}(X, Y)$ is upper semi-Fredholm if and only if a sequence (x_n) in X is convergent, whenever (x_n) is bounded and (Tx_n) is convergent in Y (Proposition A.1.4).

The scheme of Definitions 3.5.6 and 3.5.7 applied to the operator ideal $\mathcal{K}$ gives as $\mathcal{K}_+$ and $\mathcal{K}_-$ the upper semi-Fredholm Φ_+ and the lower semi-Fredholm operators Φ_- respectively.

Similarly, the scheme of Definitions 3.5.6 and 3.5.7 applied to $\mathcal{W}$ gives as $\mathcal{W}_+$ and $\mathcal{W}_-$ the tauberian and the cotauberian operators respectively.

The following two results describe some basic properties of the classes $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$. Their proofs are left to the interested reader. In the first one we describe the behavior of $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$ under products.

Proposition 3.5.10. *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, and let $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(X, Y)$.*

(i) $S, T \in \mathcal{A}_+ \Rightarrow ST \in \mathcal{A}_+$.

(ii) $ST \in \mathcal{A}_+ \Rightarrow T \in \mathcal{A}_+$.

(iii) $S, T \in \mathcal{A}^d{}_- \Rightarrow ST \in \mathcal{A}^d{}_-$.

(iv) $ST \in \mathcal{A}^d{}_- \Rightarrow S \in \mathcal{A}^d{}_-$.

The second one shows some properties of $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$ that are stable under perturbation.

Proposition 3.5.11. *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, and let $T, K \in \mathcal{L}(X, Y)$.*

(i) $T \in \mathcal{A}_+ \Rightarrow N(T) \in Sp(\mathcal{A})$.

(ii) $T \in \mathcal{A}_+, K \in \mathcal{A} \Rightarrow T + K \in \mathcal{A}_+$.

(iii) $T \in \mathcal{A}^d{}_- \Rightarrow Y/\overline{R(T)} \in Sp(\mathcal{A}^d)$.

(iv) $T \in \mathcal{A}^d{}_-, K \in \mathcal{A}^d \Rightarrow T + K \in \mathcal{A}^d{}_-$.

The operators in $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$ with closed range admit a very simple characterization. Compare with part (ii) in Theorem 2.1.5 and Proposition 3.1.5.

Proposition 3.5.12. *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, and let $T \in \mathcal{L}(X, Y)$ be an operator with closed range. Then*

(i) $T \in \mathcal{A}_+$ *if and only if* $N(T) \in Sp(\mathcal{A})$.

(ii) $T \in \mathcal{A}^d{}_-$ *if and only if* $Y/R(T) \in Sp(\mathcal{A}^d)$.

Proof. (i) To prove the non-trivial implication, suppose that $R(T)$ is closed and $N(T) \in Sp(\mathcal{A})$.

Case $\mathcal{A} = \mathcal{R}$: Suppose that $T \notin \mathcal{R}_+$. So we can take a bounded sequence with no weakly Cauchy subsequences (x_n) in X such that (Tx_n) is weakly Cauchy. By Rosenthal's ℓ_1-theorem, passing to a subsequence if necessary, we can assume that (x_n) is equivalent to the unit vector basis of ℓ_1.

Note that $(Tx_{2n} - Tx_{2n-1})$ is weakly null. Since the norm-closure and the weak-closure coincide for convex sets, we can take a sequence (y_n) formed by successive convex combinations of $(x_{2n} - x_{2n-1})$ such that $T(y_n)$ converges in norm to 0, and yet (y_n) is equivalent to the unit vector basis of ℓ_1.

Since $R(T)$ is closed, there exists $C > 0$ such that

$$\|Tx\| \geq C \operatorname{dist}\big(x, N(T)\big),$$

for every $x \in X$. Hence

$$\lim_{n\to\infty} \operatorname{dist}\big(y_n, N(T)\big) = 0.$$

It follows from this fact that $N(T)$ contains a sequence equivalent to the unit vector basis of ℓ_1. Thus we obtain a contradiction.

Case $\mathcal{A} = \mathcal{C}$: Suppose that (x_n) is a weakly Cauchy sequence in X such that (Tx_n) is convergent. Since $R(T)$ is closed, (Tx_n) converges to Tx for some $x \in X$. As in the previous case, from $(Tx_n - Tx)$ converging in norm to 0, we get

$$\lim_{n\to\infty} \operatorname{dist}(x_n - x, N(T)) = 0.$$

Therefore we can choose a bounded sequence (y_n) in $N(T)$ which satisfies

$$\lim_{n\to\infty} \|y_n - x_n - x\| = 0.$$

Now, since weakly Cauchy sequences in $N(T)$ are convergent, we conclude that (x_n) is convergent in X; hence $T \in \mathcal{C}_+$.

Case $\mathcal{A} = \mathcal{WC}$: Suppose that (x_n) is a weakly Cauchy sequence in X such that (Tx_n) is weakly convergent. Since $R(T)$ is closed, (Tx_n) is weakly convergent to Tx for some $x \in X$.

Now, a similar argument to that in the case $\mathcal{A} = \mathcal{C}$ allows us to conclude that $T \in \mathcal{WC}_+$.

Case $\mathcal{A} = \mathcal{U}$: Suppose that $T \notin \mathcal{U}_+$. Therefore there exists a weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ which is not unconditionally converging, but $\sum_{n=1}^{\infty} Tx_n$ is unconditionally converging.

After reordering, we can suppose that $\sum_{n=1}^{\infty} x_n$ is not convergent. Now we can take $1 \leq k_1 \leq m_1 < k_2 \leq m_2 < \cdots$ in $\mathbb{N}$, so that the vectors

$$y_n := x_{k_n} + x_{k_n+1} + \cdots + x_{m_n} \quad (n \in \mathbb{N})$$

satisfy $\inf_{n\in\mathbb{N}} \|y_n\| > 0$.

Note that $\sum_{n=1}^{\infty} y_n$ is weakly unconditionally Cauchy and $\sum_{n=1}^{\infty} Ty_n$ is unconditionally converging. By the Bessaga-Pełczyński selection principle (Proposition A.3.7), (y_n) has a basic subsequence (x_n) which, by Proposition A.3.12, is equivalent to the unit vector basis of c_0.

Since (Ty_n) is norm convergent to 0 and $R(T)$ is closed, $\mathrm{dist}(y_n, N(T)) \to 0$. Thus we conclude that $N(T)$ contains a sequence equivalent to the unit vector basis of c_0, which gives a contradiction.

(ii) It follows from (i) and the fact that the dual space of $Y/\overline{R(T)}$ can be identified with $N(T^*)$. □

Remark 3.5.13. Let $\mathcal{A}$ be one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$ and let $T \in \mathcal{L}(X,Y)$. In all cases,

$$T \in \mathcal{A}_+ \not\Rightarrow T^* \in \mathcal{A}^d{}_-.$$

Therefore $T^{**} \in \mathcal{A}_+$ implies $T \in \mathcal{A}_+$, but the converse implication fails.

This is similar to the relation between the tauberian operators $\mathcal{T}$ and the cotauberian operators $\mathcal{T}^d$. In the case of $\mathcal{T}$ and $\mathcal{T}^d$, the counterexample is nontrivial (see Theorem 3.1.18) because a Banach space X is reflexive if and only if so is X^{**}. However, in the cases we are considering now, it is much easier to show a counterexample.

Indeed, it is well-known that for each of the mentioned operator ideals $\mathcal{A}$, there exists a Banach space X with a subspace E such that $E \in Sp(\mathcal{A})$, but $E^{**} \notin Sp(\mathcal{A})$. So, by Proposition 3.5.12, the quotient map $Q_E \colon X \longrightarrow X/E$ satisfies $Q_E \in \mathcal{A}_+$ and $Q_E^{**} \notin \mathcal{A}_+$.

Let us show that all the classes $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$ admit a perturbative characterization.

Theorem 3.5.14. *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, and let $T \in \mathcal{L}(X,Y)$. Then*

(i) *$T \in \mathcal{A}_+$ if and only if $N(T+K) \in Sp(\mathcal{A})$ for every compact operator $K \in \mathcal{K}(X,Y)$.*

(ii) *$T \in \mathcal{A}^d{}_-$ if and only if $Y/\overline{R(T+K)} \in Sp(\mathcal{A}^d)$ for every compact operator $K \in \mathcal{K}(X,Y)$.*

Proof. Since the compact operators $\mathcal{K}$ are contained in $\mathcal{A}$ and $\mathcal{A}^d$ for all the operator ideals $\mathcal{A}$ we are considering, the direct implications are a consequence of Proposition 3.5.11.

In order to prove the converse implications in (i), suppose that $T \notin \mathcal{A}_+$. For each $\mathcal{A}$, we will show the existence of bounded sequences (x_n) in X and (f_n) in X^* such that

1. $\langle f_i, x_j \rangle = \delta_{ij}$, for all $i, j \in \mathbb{N}$,

2. $\overline{\text{span}}\{x_n : n \in \mathbb{N}\} \notin Sp(\mathcal{A})$,
3. (Tx_n) converges to some $y \in Y$,
4. $\sum_{n=1}^{\infty} \|f_n\| \, \|Tx_n - y\| < \infty$.

Then, the expression

$$K(x) := -\sum_{n=1}^{\infty} \langle f_n, x\rangle (Tx_n - y); \quad x \in X$$

defines an operator $K \in \mathcal{K}(X, Y)$ such that $(T + K)x_n = y$ for all n. Since $N(T + K)$ contains a 1-codimensional closed subspace of $\overline{\text{span}}\{x_n : n \in \mathbb{N}\}$, we conclude $N(T + K) \notin Sp(\mathcal{A})$.

Case $\mathcal{A} = \mathcal{R}$: Since $T \notin \mathcal{R}_+$, there is a bounded sequence (z_n) in X having no weakly Cauchy subsequence and such that (Tz_n) is weakly Cauchy. By Rosenthal's ℓ_1-theorem, we can assume that (z_n) is equivalent to the unit vector basis of ℓ_1.

Now, taking $y_n := z_{2n} - z_{2n-1}$, the sequence (y_n) is equivalent to the unit vector basis of ℓ_1 and (Ty_n) is weakly null. Since the norm-closure and the weak-closure coincide for convex sets, we can take a sequence (x_n) formed by successive convex combinations of (y_n) such that $T(x_n)$ converges in norm to 0, and yet (x_n) is equivalent to the unit vector basis of ℓ_1. Clearly the sequence (f_n) exists, and (x_n) and (f_n) satisfy 1, 2 and 3. And, passing to subsequence if necessary, (x_n) and (f_n) satisfy 4 too.

Case $\mathcal{A} = \mathcal{C}$: Since $T \notin \mathcal{C}_+$, there exists a weakly Cauchy sequence (z_n) in X having no convergent subsequence and such that (Tz_n) is convergent to some $y \in Y$. By the Kadec-Pełczyński criterion (Theorem A.3.8), passing to a subsequence, we can assume that (z_n) is a basic sequence. Clearly, we can finish the argument as in the case $\mathcal{A} = \mathcal{R}$.

Case $\mathcal{A} = \mathcal{WC}$: Since $T \notin \mathcal{WC}_+$, there exists a weakly Cauchy sequence (y_n) in X having no weakly convergent subsequence, and such that (Ty_n) is weakly convergent to some $y \in Y$. As in the case $\mathcal{A} = \mathcal{R}$, we can take a sequence (x_n) formed by convex combinations of (y_n) such that $T(x_n)$ converges in norm to y, and (x_n) is a basic sequence having no weakly convergent subsequences. So the proof can be finished in a similar way.

Case $\mathcal{A} = \mathcal{U}$: Since $T \notin \mathcal{U}_+$, proceeding as in the proof of case $\mathcal{A} = \mathcal{U}$ in Proposition 3.5.12, we can get a sequence (y_n) in X equivalent to the unit vector basis of c_0 such that $\sum_{n=1}^{\infty} Ty_n$ is unconditionally converging. So the proof can be finished as in the case $\mathcal{A} = \mathcal{R}$.

In order to prove the converse implications in (ii), suppose that $T \notin \mathcal{A}^d{}_-$; hence $T^* \notin \mathcal{A}_+(Y^*, X^*)$. Proceeding as in the proof of the converse of (i), we obtain bounded sequences (g_n) in Y^* and (G_n) in Y^{**} satisfying conditions 1, 2, 3 and 4. In particular, (T^*g_n) converges to some $g \in Y^*$ and

$$\sum_{n=1}^{\infty} \|G_n\| \, \|T^*g_n - g\| < \infty.$$

The key point here is that an application of Lemma 3.1.19 allows us to take $G_n = y_n \in Y$, for every $n \in \mathbb{N}$. Therefore,

$$K(x) := -\sum_{n=1}^{\infty} \langle T^*g_n - g, x\rangle y_n; \quad x \in X$$

defines an operator $K \in \mathcal{K}(X,Y)$ such that $(T^* + K^*)g_n = g$ for all n.

Thus $N(T^* + K^*) \notin Sp(\mathcal{A})$; hence $Y/\overline{R(T+K)} \notin Sp(\mathcal{A}^d)$. □

From perturbative characterizations, like those in Theorem 3.5.14, we can derive characterizations of classes of Banach spaces similar to those characterized in Corollaries 2.2.8 and 3.1.21.

Proposition 3.5.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be two of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$ and let X be a Banach space.*

(i) *$\mathcal{A}_+(X,Z) \subset \mathcal{B}_+(X,Z)$ for every Banach space Z if and only if the closed subspaces of X in $Sp(\mathcal{A})$ belong to $Sp(\mathcal{B})$;*

(ii) *${\mathcal{A}^d}_-(Z,X) \subset {\mathcal{B}^d}_-(Z,X)$ for every Banach space Z if and only if the quotients of X in $Sp(\mathcal{A}^d)$ belong to $Sp(\mathcal{B}^d)$.*

Proof. (i) The direct implications are immediate consequences of the corresponding results in Theorem 3.5.14.

For the converse implications, suppose that M is a closed subspace of X which belongs to $Sp(\mathcal{A})$ but not to $Sp(\mathcal{B})$. Then, the quotient map $Q_M : X \longrightarrow X/M$ belongs to $\mathcal{A}_+$ but not to $\mathcal{B}_+$.

(ii) The direct implications are also immediate consequences of the corresponding results in Theorem 3.5.14.

For the converse implications, suppose that N is a closed subspace of X such that X/N belongs to $Sp(\mathcal{A})$ but not to $Sp(\mathcal{B})$. Then, the embedding map $J_N : N \longrightarrow X$ belongs to ${\mathcal{A}^d}_-$ but not to ${\mathcal{B}^d}_-$. □

Next, we show how to apply Theorem 3.5.14 and the corresponding results for $\mathcal{K}$ and $\mathcal{W}$ to characterize the classes of operators considered in Definition 3.5.6, Definition 3.5.7 and Remark 3.5.9.

Proposition 3.5.16. *Let $\mathcal{A}$ be one of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, and $T \in \mathcal{L}(X,Y)$. Then*

(i) *$T \in \mathcal{A}_+$ if and only if for every Banach space Z and every $A \in \mathcal{L}(Z,X)$, $TA \in \mathcal{A}$ implies $A \in \mathcal{A}$.*

(ii) $T \in \mathcal{A}^d{}_-$ *if and only if for every Banach space* Z *and every* $B \in \mathcal{L}(Y, Z)$, $BT \in \mathcal{A}^d$ *implies* $B \in \mathcal{A}^d$.

Proof. (i) The direct implication follows immediately from the sequential characterizations of $\mathcal{A}$ and $\mathcal{A}_+$.

For the converse, suppose that $T \notin \mathcal{A}_+$. By Theorem 3.5.14, there is a compact operator $K \in \mathcal{L}(X, Y)$ such that $N(T + K) \notin Sp(\mathcal{A})$.

Let us denote by A the inclusion of $N(T + K)$ into X. Since TA is compact, $TA \in \mathcal{A}$; however, $A \notin \mathcal{A}$.

The proof of (ii) can be obtained from (i) by using a duality argument. □

3.6 Notes and Remarks

The cotauberian operators were introduced by Tacon [157] as those operators $T \in \mathcal{L}(X, Y)$ such that $R(T^{**}) + Y$ is dense in Y^{**}. He also proved that T is cotauberian if and only if T^* is tauberian.

Yang [174] investigated cotauberian operators with closed range in his attempt to develop a generalized Fredholm theory in which the weakly compact operators play the role of the compact operators in the classical Fredholm theory. He also studied the class of operators with closed range which are tauberian and cotauberian, referring to them as the *weakly Fredholm* operators.

Diagram 3.2 was considered by Yang in [173]. He proves by diagram chasing that all its rows and columns are exact. As a consequence, he derives that reflexivity is a three-space property for Banach spaces: If M is a closed subspace of X and both M and X/M are reflexive, then so is X.

Diagram chasing is also the main technique applied in [174] to studying the aforementioned weakly Fredholm operators.

The example showing that T tauberian does not imply T^* cotauberian (Theorem 3.1.18) was obtained in [8]. It gives a negative answer to Question 1.3.4 using a construction of Bellenot in [25]).

The perturbative characterization for cotauberian operators was obtained in [92]. Lemma 3.1.19, which is the key to proving that characterization, can be found without proof in [109, Remark III.1]. The proof we present here is an adaptation of the techniques used in [109].

Given Banach spaces X and Y, the map

$$T \in \mathcal{L}(X, Y) \longrightarrow T^{co} \in \mathcal{L}(X^{co}, Y^{co})$$

can be seen as a representation of the quotient $\mathcal{L}(X, Y)/\mathcal{W}(X, Y)$. The properties of this map are studied in [93].

In most cases, this map is not surjective. For example, in the case in which $X = Y$ is c_0, $C[0, 1]$, ℓ_1, $L_1(0, 1)$ or ℓ_∞, its range does not contain non-zero inessential operators [93, Theorem 2.2]. Moreover, there exists a Banach space E with $E^{co} \equiv \ell_2$ such that

$$\{T^{co} \colon T \in \mathcal{L}(E)\}$$

coincides with the regular operators in $\mathcal{L}(\ell_2)$ [93, Theorem 2.6]. Thus it is a proper dense subalgebra of $\mathcal{L}(E)$.

Aliprantis and Burkinshaw [5] obtained some results dealing with factorization of positive weakly compact operators through reflexive Banach lattices, and asked whether or not every positive weakly compact operator between Banach lattices factors through a reflexive Banach lattice. In [159], Talagrand gave a negative answer by exhibiting a positive weakly compact operator between Banach lattices that does not factor through any reflexive Banach lattice.

Answering a question raised in [61], Blanco, Kaijser and Ransford proved in [31] that every weakly compact homomorphism between Banach algebras factors through a reflexive Banach algebra, with Banach algebra homomorphisms as factors. We refer to [43] for a variant of this result.

In order to prove the factorization result for homomorphisms, the authors of [31] develop a variant of the DFJP factorization which is better suited to dealing with Banach algebras. Essentially, it is a version of the real interpolation method in which they use a Banach space E with a normalized 1-unconditional basis $(e_i)_{i\in\mathbb{Z}}$ and some algebra weights with respect to $(e_i)_{i\in\mathbb{Z}}$, which are maps $\rho\colon \mathbb{Z} \longrightarrow \mathbb{R}_+$ which allow them to define a kind of continuous convolution product on E.

The intermediate space in the DFJP factorization of $T \in \mathcal{L}(X, Y)$ is obtained as the diagonal subspace of the space $\ell_2(Y_n)$, where each Y_n is the space Y endowed with an equivalent norm. In this construction we can replace $\ell_2(Y_n)$ by $\ell_p(Y_n)$, with $1 < p < \infty$, and most of the properties of the factorization are preserved. We refer to Neidinger's thesis [133] for details.

In [7], a version of the DFJP factorization for an unbounded linear operator was introduced. It was applied to studying the relation between certain classes of unbounded operators.

The examples of tauberian operators among the natural inclusions of Orlicz spaces of vector-valued measurable functions are taken from [32] and [35].

The class of operators $\mathcal{R}_+$ was introduced by Martin and Swart in [126]. Moreover, this class was studied by Bombal and Hernando ([34] and [102]), in terms of the set $B_1(X)$ of elements of X^{**} which are weak*-limits of weakly Cauchy sequence in X. In the case in which the space X is separable, they proved that an operator $T \in \mathcal{L}(X, Y)$ belongs to $\mathcal{R}_+$ if and only if $N(T^{**}) \subset B_1(X)$; and this is equivalent to $(T^{**})^{-1}B_1(Y) \subset B_1(X)$. In the general case, they obtained similar results using other subsets of X^{**} and Y^{**}.

The concept of operator in $\mathcal{R}_+$ has been extended to the case of unbounded operators and, more generally, to the case of linear relations between normed linear spaces in [9].

A systematic study of the classes $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$, with $\mathcal{A}$ one of the operator ideals $\mathcal{K}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, was made in [89], [90], [92] and [79]. Most of our results in Section 3.5 are taken from these papers.

In the case of the classes $\mathcal{A}_+$, the results in Proposition 3.5.11 can be interpreted as lifting results for certain classes of sequences. We refer to [91] for details.

Comparing this chapter with the previous one, it is evident that tauberian operators have received greater attention than cotauberian operators. This is clear in terms of the variety of techniques used in the study and in the number of concrete examples that have been analyzed. Thus there is an unexplored field of study in the development of a theory of cotauberian operators parallel to that of tauberian operators.

Chapter 4

Tauberian operators on spaces of integrable functions

The set $\mathcal{T}(L_1(m), Y)$ of tauberian operators from $L_1(m)$, where (Ω, Σ, m) is a σ-finite measure space, deserves attention for two reasons. First, because the properties of $\mathcal{T}(L_1(m), Y)$ are similar to those of $\Phi_+(L_1(m), Y)$; and second, because $L_1(m)$ supports many tauberian operators which are not upper semi-Fredholm when m is not purely atomic measure.

Since the theory of tauberian operators is essentially isomorphic in character, we shall only consider the case in which $L_1(m)$ is infinite dimensional.

Note that $L_1(m)$ is isomorphic to $L_1(\nu)$ for some finite measure ν. Indeed, let $\{\Omega_n\}_{n=1}^{\infty} \subset \Sigma$ be a countable partition of Ω with $0 < m(\Omega_n) < \infty$ for every $n \in \mathbb{N}$. Then

$$g := \sum_{n=1}^{\infty} 2^{-n} m(\Omega_n)^{-1} \chi_{\Omega_n}$$

defines a strictly positive, measurable function $g \colon \Omega \longrightarrow \mathbb{R}$ such that $\int_\Omega g\, d\mu = 1$.

Thus, we define the measure ν by $d\nu = g\, dm$, and the map $f \mapsto f/g$ defines an isometry from $L_1(m)$ onto $L_1(\nu)$.

It is well-known that $L_1(\nu)$ is isomorphic to one of the following spaces:

(i) ℓ_1, if ν is a purely atomic measure; or

(ii) $L_1(\mu)$, where the σ-algebra associated to μ has no atoms.

In case (ii), if $L_1(\nu)$ is separable, then it is isomorphic to $L_1[0,1]$. In the general case, it is isomorphic to an uncountable ℓ_1-sum of copies of $L_1[0,1]$. For further details on this matter, we refer to [117].

Case (i) is very simple: after Corollary 2.2.8, every tauberian operator on ℓ_1 is upper semi-Fredholm because every reflexive subspace of ℓ_1 is finite dimensional. Consequently, only case (ii) will be considered.

In this chapter, (Ω, Σ, μ) will be a finite measure space with no atoms. The following properties of $L_1(\mu)$ will be used repeatedly:

(i) the relatively weakly compact subsets of $L_1(\mu)$ coincide with the equi-integrable subsets (Proposition 4.1.1);

(ii) the *subsequence splitting property* (Corollary A.6.12); i.e., every bounded sequence in $L_1(\mu)$ has a subsequence which can be decomposed as the sum of a weakly convergent sequence and a disjointly supported sequence.

Another important tool will be Rosenthal's ℓ_1-theorem (Theorem A.3.10). This result is, in some sense, more general than the subsequence splitting property and it will be the key in Section 4.3 to identify the weakly precompact operators from $L_1(\mu)$ to Y with the perturbation class of $\mathcal{T}(L_1(\mu), Y)$.

Ultraproducts of Banach spaces are essential in order to study the tauberian operators on $L_1(\mu)$. Moreover, a representation of the ultrapowers on $L_1(\mu)$ allows us to obtain a proof of the aforementioned property (ii). This proof can be found in Section A.6.

4.1 Tauberian operators on $L_1(\mu)$ spaces

In this section, we assume that (Ω, Σ, μ) is a finite measure space and that Σ contains no atoms. We will describe the basic properties of the operators in $\mathcal{T}(L_1(\mu), Y)$. In particular, we will give characterizations of these operators in terms of their action on disjointly supported sequences.

First we give some auxiliary results.

Proposition 4.1.1. [4, Theorem 5.2.9] *A subset F of $L_1(\mu)$ is relatively weakly compact if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ so that*

$$\mu(A) < \delta \Rightarrow \int_A |f| \, d\mu < \varepsilon, \quad \textit{for every } f \in F;$$

i.e., if and only if F is equi-integrable.

Proposition 4.1.2. *An operator $T\colon L_1(\mu) \longrightarrow Y$ is tauberian if and only if T has property (N).*

Proof. Since weakly Cauchy sequences in $L_1(\mu)$ are weakly convergent [24, Proposition VI.2.6], the result is a direct consequence of Proposition 2.1.12. □

A sequence (f_n) in $L_1(\mu)$ is *disjointly supported* if $f_m \cdot f_m = 0$ for $m \neq n$. The following theorem is the main result of this section.

Theorem 4.1.3. *For an operator $T \in \mathcal{L}(L_1(\mu), Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *for every normalized, disjointly supported sequence* (f_n) *contained in* $L_1(\mu)$,

$$\liminf_n \|Tf_n\| > 0;$$

(c) *there exists a number* $r > 0$ *such that for every normalized disjointly supported sequence* (f_n) *in* $L_1(\mu)$,

$$\liminf_n \|Tf_n\| > r.$$

Proof. (a)⇒(b) Let (f_n) be a normalized disjointly supported sequence in $L_1(\mu)$ and assume that $\|T(f_n)\| \underset{n}{\longrightarrow} 0$. Since (f_n) is equivalent to the unit vector basis of ℓ_1, there exists an element $z^{**} \in \overline{\{f_n\}_{n=1}^{\infty}}^{w^*} \setminus L_1(\mu)$.

Note that $T^{**}(z^{**}) = 0$, which proves that $N(T) \neq N(T^{**})$; hence T is not tauberian.

(b)⇒(c) Let us suppose that (c) fails. Clearly, we can assume that $\|T\| = 1$.

For each $k \in \mathbb{N}$ there exists a normalized disjointly supported sequence $(f_n^k)_n$ in $L_1(\mu)$ such that $\|T(f_n^k)\| < 1/k$ for all n. We will find a disjointly supported sequence (f_n) satisfying $1/2 \leq \|f_n\| \leq 1$ and $\|T(f_n)\| \leq 2/n$ for all n. In order to do that, we recursively select a sequence in $(f_{k_n}^n)$ as follows:

First, we take $f_{k_1}^1 := f_1^1$. Since $\int_{\operatorname{supp} f_k^2} |f_1^1|\, d\mu \underset{k}{\longrightarrow} 0$, we can select $k_2 \in \mathbb{N}$ so that

$$\int_{\operatorname{supp} f_{k_2}^2} |f_1^1|\, d\mu < 1/2^2.$$

Let us assume that we have chosen functions $f_{k_1}^1, \dots, f_{k_n}^n$ satisfying

$$\int_{\operatorname{supp} f_{k_m}^m} |f_{k_l}^l|\, d\mu < \frac{1}{2^{2m}} \quad \text{for } 1 \leq l < m \leq n.$$

Thus, since

$$\lim_k \int_{\operatorname{supp} f_k^{n+1}} |f_{k_l}^l|\, d\mu = 0 \ \text{ for all } l \in \{1, \dots, n\},$$

we can select k_{n+1} so that $\int_{\operatorname{supp} f_{k_{n+1}}^{n+1}} |f_{k_l}^l|\, d\mu < 1/2^{2(n+1)}$ for $1 \leq l \leq n$.

This procedure yields a normalized sequence $(g_n) := (f_{k_n}^n)$ in $L_1(\mu)$ such that $\|T(g_n)\| < 1/n$ and

$$\int_{\operatorname{supp} g_k} |g_n|\, d\mu < 1/2^{2k} \quad \text{for } k > n.$$

Now, we define

$$A_n := \operatorname{supp} g_n \setminus \bigcup_{k=n+1}^{\infty} \operatorname{supp} g_k,$$

and $f_n := g_n \cdot \chi_{A_n}$. Clearly (f_n) is disjointly supported. Moreover,

$$1 \geq \|f_n\| \geq 1 - \sum_{k=n+1}^{\infty} \int_{\operatorname{supp} g_k} |g_n| \, d\mu \geq 1 - \sum_{k=n+1}^{\infty} 2^{-2k} \geq 1/2,$$

and

$$\|T(f_n)\| \leq \|T(g_n)\| + \|T\| \sum_{k=n+1}^{\infty} \int_{\operatorname{supp} g_k} |g_n| \, d\mu \leq \frac{1}{n} + 2^{-n} \leq 2/n.$$

Hence (b) fails.

(c)⇒(a) Let us assume that T is not tauberian. Thus, by Proposition 4.1.2, T fails property (N); hence, by Theorem 2.2.2, there exists a sequence (h_n) in the unit ball of $L_1(\mu)$ with no weakly convergent subsequence, such that $\lim_n T(h_n) = 0$.

By the subsequence splitting property (Corollary A.6.12), there is a subsequence (g_n) of (h_n) and a pair of sequences (u_n) and (v_n) in $L_1(\mu)$ such that (v_n) is weakly convergent to some function v, (u_n) is disjointly supported and $g_n = u_n + v_n$.

Note that $\liminf_n \|u_n\| > 0$, hence (u_n) is equivalent to the unit vector basis of ℓ_1. Since $w\text{-}\lim_n v_n = v$, there exists a sequence of non-negative real numbers (α_n) and an increasing sequence (k_n) in $\mathbb{N}$ such that $\sum_{i=k_n+1}^{k_{n+1}} \alpha_i = 1$ and

$$z_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i v_i \xrightarrow[n]{} v \quad \text{(in norm)}.$$

In addition, we consider the corresponding convex block subsequences

$$x_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i g_i \quad \text{and} \quad y_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i u_i$$

that satisfy $x_n = z_n + y_n$ for all n.

Since the sequence (y_n) is disjointly supported and $\liminf_n \|y_n\| > 0$, it is equivalent to the unit vector basis of ℓ_1. Moreover, both sequences

$$\big(T(x_{2n} - x_{2n-1})\big) \quad \text{and} \quad \big(T(z_{2n} - z_{2n-1})\big)$$

converge to 0; hence $T(y_{2n} - y_{2n-1}) \xrightarrow[n]{} 0$. We define

$$f_n := \frac{y_{2n} - y_{2n-1}}{\|y_{2n} - y_{2n-1}\|}.$$

The sequence (f_n) is normalized, disjointly supported and satisfies $T(f_n) \xrightarrow[n]{} 0$. Hence (c) fails, as we had sought to prove. □

Theorem 4.1.3 allows us to show some remarkable similarities between the tauberian operators and the upper semi-Fredholm operators on $L_1(\mu)$ that do not hold in the general case. For instance, it is well known that $\Phi_+(X,Y)$ is always open, but $\mathcal{T}(X,Y)$ fails to be open in some cases (Example 2.1.17). Let us show that it is open when $X = L_1(\mu)$.

Corollary 4.1.4. *For every operator $T\colon L_1(\mu) \longrightarrow Y$, let β_T be defined by*

$$\beta_T := \inf\{\liminf_n \|T(f_n)\|\colon (f_n) \subset L_1(\mu) \text{ normalized and disjointly supported}\}.$$

Then the following statements hold:

(i) *T is tauberian if and only if $\beta_T > 0$;*

(ii) *if T is tauberian and $S\colon L_1(\mu) \longrightarrow Y$ satisfies $\|T - S\| < \beta_T$, then S is tauberian.*

Therefore, $\mathcal{T}(L_1(\mu), Y)$ is open in $\mathcal{L}(L_1(\mu), Y)$.

Proof. (i) It is a straightforward consequence of Theorem 4.1.3 (c).

(ii) Let $S\colon L_1(\mu) \longrightarrow Y$ be an operator such that $\|T - S\| = \alpha < \beta_T$. Then, for every normalized disjointly supported sequence (f_n) in $L_1(\mu)$, we have $\liminf_n \|S(f_n)\| \geq \beta_T - \alpha > 0$, which implies that S is tauberian, according to Theorem 4.1.3.

From statement (ii), it trivially follows that $\mathcal{T}(L_1(\mu), Y)$ is open. □

Remark 4.1.5. The space $L_1(\mu)$ contains many infinite dimensional reflexive subspaces. Indeed, for every $p \in (1, 2]$, $L_1[0,1]$ contains an isomorphic copy of $L_p[0,1]$ (see [41] or [123, Corollary 2.f.5]). Moreover, every reflexive subspace of $L_1[0,1]$ is isomorphic to a subspace of $L_p[0,1]$ for some $p \in (1, 2]$ [144].

Given a reflexive subspace R of $L_1(\mu)$, it follows from Theorem 2.1.5 that the quotient map $Q_R\colon L_1(\mu) \longrightarrow L_1(\mu)/R$ is a tauberian operator. Let us see that the constant β_{Q_R} is easily computable.

Proposition 4.1.6. *For every reflexive subspace R of $L_1(\mu)$, $\beta_{Q_R} = 1$.*

Proof. Take $0 < \varepsilon < 1$ and let (f_n) be a normalized disjointly supported sequence in $L_1(\mu)$. Since R is reflexive, the set $3B_R$ is equi-integrable. Thus there exists $\delta > 0$ such that $\int_A |g|\, d\mu < \varepsilon$ for all $g \in 3B_R$ and all measurable subsets A with $\mu(A) < \delta$.

We select $n_0 \in \mathbb{N}$ such that $\mu(\operatorname{supp} f_n) < \delta$ for all $n \geq n_0$. Then for every $g \in 3B_R$,

$$\|f_n - g\| \geq \int_{\operatorname{supp} f_n} |f_n|\, d\mu - \int_{\operatorname{supp} f_n} |g|\, d\mu \geq 1 - \varepsilon$$

for $n \geq n_0$. Now, since $\|Q(f_n)\| = \inf\{\|f_n - g\|\colon g \in 3B_R\} \leq 1$, it follows that $\lim_n \|Q(f_n)\| = 1$, hence $\beta_Q = 1$. □

Given a measurable subset $C \subset \Omega$ with $\mu(C) > 0$, we denote by $L_1(C)$ the subspace of $L_1(\mu)$ consisting of all functions f with support contained in C:

$$L_1(C) := \{f \in L_1(\mu)\colon\ f = f \cdot \chi_C\}.$$

Corollary 4.1.7. *For every tauberian operator $T\colon L_1(\mu) \longrightarrow Y$ and every measurable set $A \subset \Omega$ with $\mu(A) > 0$ there exists a measurable subset $C \subset A$ with $\mu(C) > 0$ such that the restriction $T|_{L_1(C)}$ is an isomorphism.*

Proof. Suppose that the result fails. Then we can find a sequence (C_n) of disjoint measurable subsets of A with $\mu(C_n) > 0$ such that none of the restrictions $T|_{L_1(C_n)}$ is an isomorphism. Thus for every n there exists $f_n \in L_1(C_n)$ with $\|f_n\| = 1$ and $\|T(f_n)\| < 1/n$, in contradiction with Theorem 4.1.3. □

Sometimes the following version of Theorem 4.1.3 is more convenient:

Proposition 4.1.8. *For every operator $T\colon L_1(\mu) \longrightarrow Y$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *for every normalized sequence (f_n) in $L_1(\mu)$ such that $\lim_n \mu(\operatorname{supp} f_n) = 0$, we have $\liminf_n \|T(f_n)\| > 0$;*

(c) *there exists a real number $r > 0$ such that for every $f \in L_1(\mu)$ with $\|f\| = 1$ and $\mu(\operatorname{supp} f) < r$, we have $\|T(f)\| > r$.*

Proof. (a)⇒(b) Suppose that (b) fails. Then there exists a normalized sequence (f_n) in $L_1(\mu)$ such that $\lim_n \mu(\operatorname{supp} f_n) = 0$ and $\lim_n \|T(f_n)\| > 0$. Note that the sequence (f_n) has no equi-integrable subsequences; hence it has no weakly convergent subsequences. Thus, by Theorem 2.2.4, T is not tauberian.

(b)⇒(c) Suppose that (c) fails. Then we can select a normalized sequence (f_n) so that $\mu(\operatorname{supp} f_n) < 1/n$ and $\|T(f_n)\| < 1/n$; hence (b) fails.

(c)⇒(a) Let us assume T is not tauberian. Then, by Theorem 4.1.3, there exists a normalized disjointly supported sequence (f_n) so that $\lim_n \|T(f_n)\| = 0$. Since $\lim_n \mu(\operatorname{supp} f_n) = 0$, (c) fails. □

Observe that, since μ has no atoms, for every finite sequence $(\varepsilon_i)_{i=1}^n$ of positive numbers with $\sum_{i=1}^n \varepsilon_i = 1$ there exists a partition $\{\Omega_i\}_{i=1}^n$ of Ω into measurable sets such that $\mu(\Omega_i) = \varepsilon_i$ for all i. Moreover, it is clear that $L_1(\mu)$ is isomorphic to $L_1(\Omega_i)$ for some $i \in \{1, \dots, n\}$.

Corollary 4.1.9. *For every operator $T \in \mathcal{T}(L_1(\mu), Y)$, there exists a finite partition $\{\Omega_1, \dots, \Omega_n\}$ of Ω into measurable subsets such that each restriction $T|_{L_1(\Omega_i)}$ is an isomorphism.*

Proof. Let $r > 0$ be the number provided by statement (c) in Proposition 4.1.8. Thus it is enough to consider any finite partition of Ω into measurable sets of measure smaller than r. □

Corollary 4.1.10. *The class $\mathcal{T}(L_1(\mu), Y)$ is non-empty if and only if Y contains a subspace isomorphic to $L_1(\mu)$. In particular, if M is a reflexive subspace of $L_1(\mu)$, then $L_1(\mu)/M$ contains a subspace isomorphic to $L_1(\mu)$.*

Given $n \in \mathbb{N} \cup \{0\}$ and $k \in \{1, \dots, 2^n\}$, let χ_k^n denote the characteristic function of the interval I_k^n, where associated to the interval

$$I_k^n := \begin{cases} [(k-1)/2^n\,,\, k/2^n) & \text{if } 1 \leq k \leq 2^n - 1, \\ [(2^n-1)/2^n\,,\, 1] & \text{if } k = 2^n. \end{cases}$$

The *n-th Rademacher function* on the interval $[0, 1]$ is defined as

$$r_n := \sum_{k=1}^{2^n} (-1)^{k-1} \chi_k^n.$$

By the classical Khintchine inequalities [24, Proposition VI.1.1], the sequence (r_n) generates a subspace isomorphic to ℓ_2 in $L_1[0, 1]$. This fact allows us to illustrate Corollary 4.1.9 with the following example:

Example 4.1.11. Let R be the closed subspace of $L_1[0, 1]$ generated by the sequence of Rademacher functions $\big(r_n(t)\big)_{n=2}^{\infty}$ and let $Q_R\colon L_1[0, 1] \longrightarrow L_1[0, 1]/R$ be the quotient operator.

The operator Q_R is tauberian. Let us see that the restrictions $Q|_{L_1[0\,,\,1/2]}$ and $Q|_{L_1[1/2\,,\,1]}$ are isometries.

Indeed, let $f \in L_1[0\,,\,1/2]$ and $g = \sum_{i=2}^{\infty} a_i r_i \in R$. Then

$$\|f - g\|_1 = \int_0^{1/2} |f - g| + \int_{1/2}^1 |g| \geq \int_0^{1/2} |f| - \int_0^{1/2} |g| + \int_{1/2}^1 |g|.$$

Since $r_i(t) = r_i(t\,+\,1/2)$ for all $t \in [0,\,1/2]$, we have $\int_0^{1/2} |g| - \int_{1/2}^1 |g| = 0$; hence $\|f - g\| \geq \int_0^{1/2} |f| = \|f\|$. Therefore,

$$\|f\| \geq \|Q(f)\| = \inf_{g \in R} \|f - g\| \geq \|f\|,$$

and we conclude that $Q|_{L_1[0\,,\,1/2]}$ is an isometry.

The case of $Q|_{L_1[1/2\,,\,1]}$ is similar.

4.2 Ultrapowers of tauberian operators on $L_1(\mu)$ spaces

Tauberian operators on $L_1(\mu)$ can be described in terms of ultrapowers. Further information on ultrapowers of Banach spaces, measures and $L_1(\mu)$ spaces can be found in Section A.6.

In this section, I is an infinite set and $\mathfrak{U}$ is a countably incomplete ultrafilter on I. So there exists a partition $\{I_n\}_{n=1}^{\infty}$ of I disjoint with $\mathfrak{U}$.

Among other results, we can see in Section A.6 that the space of integrable functions with respect to the measure $\mu_{\mathfrak{U}}$ embeds isometrically into the ultrapower space $L_1(\mu)_{\mathfrak{U}}$ via a linear isometry

$$J_{\mu_{\mathfrak{U}}}\colon L_1(\mu_{\mathfrak{U}}) \longrightarrow L_1(\mu)_{\mathfrak{U}}.$$

Moreover, the subspace $J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$ is complemented in $L_1(\mu)_{\mathfrak{U}}$. The corresponding projection $P_{\mu_{\mathfrak{U}}}$ yields a decomposition

$$L_1(\mu)_{\mathfrak{U}} = J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big) \oplus_1 N(P_{\mu_{\mathfrak{U}}}) \quad \text{(Theorem A.6.5).}$$

The elements of the summands $J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$ and $N(P_{\mu_{\mathfrak{U}}})$ can be characterized as follows: let $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$.

(i) $\mathbf{f}$ belongs to $J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$ if and only if $\mathbf{f}$ admits a relatively weakly compact representative (Theorem A.6.6);

(ii) $\mathbf{f}$ belongs to $N(P_{\mu_{\mathfrak{U}}})$ if and only if $\mathbf{f}$ has a representative $(f_i)_{i\in I}$ such that the family $\big(\mu(\operatorname{supp} f_i)\big)_{i\in I}$ is null following $\mathfrak{U}$ (Theorem A.6.7).

The following result characterizes the tauberian operators on $L_1(\mu)$ in terms of the actions of their ultrapowers on the component $J_{\mu_{\mathfrak{U}}}\big(L_1(\mu)\big)$.

Proposition 4.2.1. *Let $\mathfrak{U}$ be an ultrafilter on I. An operator $T\colon L_1(\mu) \longrightarrow Y$ is tauberian if and only if $N(T_{\mathfrak{U}}) \subset J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$.*

Proof. Suppose T is not tauberian. By Theorem 4.1.3, there exists a normalized disjointly supported sequence (f_n) in $L_1(\mu)$ such that $T(f_n) \underset{n}{\longrightarrow} 0$.

Let $\{I_n\}_{n=1}^{\infty}$ be a partition of I disjoint with $\mathfrak{U}$ and define $g_i := f_n$ for every $i \in I_n$ and every $n \in \mathbb{N}$. Thus $0 \neq [g_i] \in N(T_{\mathfrak{U}})$ and, since $\lim_n \mu(\operatorname{supp} f_n) = 0$, we also have $\lim_{\mathfrak{U}} \mu(\operatorname{supp} g_i) = 0$. Therefore, by Theorem A.6.7, $[g_i] \in N(P_{\mu_{\mathfrak{U}}})$; hence $N(T_{\mathfrak{U}})$ is not contained in $L_1(\mu_{\mathfrak{U}})$.

Conversely, assume that there exists an element $[f_i] \in N(T_{\mathfrak{U}}) \setminus J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$. By Theorems A.6.5, A.6.6 and A.6.7, $[f_i]$ admits a decomposition $[f_i] = [g_i] + [h_i]$ where $[g_i] \in J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$ and the set $\{g_i\colon i \in I\}$ is relatively weakly compact, $[h_i] \in N(P_{\mu_{\mathfrak{U}}}) \setminus \{0\}$ and $\lim_{\mathfrak{U}} \mu(\operatorname{supp} h_i) = 0$.

Let us denote $A_i := \operatorname{supp} h_i$. Then we have

$$\lim_{\mathfrak{U}} \int_{A_i} |g_i|\, d\mu = 0 \quad \text{and} \quad \lim_{\mathfrak{U}} \int_{A_i} |h_i|\, d\mu = \|[h_i]\| > 0.$$

Let ε be a real number satisfying $0 < \varepsilon < \|[h_i]\|$. Since

$$\lim_{\mathfrak{U}} \int_{A_i} |f_i|\, d\mu \geq \lim_{\mathfrak{U}} \int_{A_i} |h_i|\, d\mu - \lim_{\mathfrak{U}} \int_{A_i} |g_i|\, d\mu > \varepsilon,$$

for every $n \in \mathbb{N}$, there exists $J_n \in \mathfrak{U}$ such that

$$\int_{A_i} |f_i|\, d\mu > \varepsilon \ \text{ and } \ \mu(A_i) < 1/n \ \text{ for all } i \in J_n.$$

Moreover, since $[f_i] \in N(T_{\mathfrak{U}})$, we have

$$K_n := \{i \in I : \|T(f_i)\| < 1/n\} \in \mathfrak{U} \text{ for all } n \in \mathbb{N}.$$

For each n, we select an index $j_n \in J_n \cap K_n$. Then,

$$\int_{A_{j_n}} |f_{j_n}|\, d\mu > \varepsilon,\ \|T(f_{j_n})\| < 1/n \text{ and } \mu(A_{j_n}) < 1/n.$$

Therefore the sequence (f_{j_n}) is not equi-integrable. So it contains a subsequence (g_n) with no weakly convergent subsequence. Obviously, $\lim_n \|T(g_n)\| = 0$ and this shows that T is not tauberian. □

The following result characterizes the tauberian operators on $L_1(\mu)$ in terms of the action of their ultrapowers on $N(P_{\mu_{\mathfrak{U}}})$.

Proposition 4.2.2. *Given an ultrafilter $\mathfrak{U}$ on I and an operator $T : L_1(\mu) \longrightarrow Y$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *$T_{\mathfrak{U}}|_{N(P_{\mu_{\mathfrak{U}}})}$ is an isomorphism;*

(c) *$T_{\mathfrak{U}}|_{N(P_{\mu_{\mathfrak{U}}})}$ is injective.*

Proof. (a)⇒(b) Assume that $T_{\mathfrak{U}}|_{N(P_{\mu_{\mathfrak{U}}})}$ is not an isomorphism. Then, for every positive integer n, there is a norm-one element $[f_i^n] \in N(P_{\mu_{\mathfrak{U}}})$ such that $\|T_{\mathfrak{U}}([f_i^n])\| < 1/2n$. By Theorem A.6.7, we may consider that $\lim_{\mathfrak{U}} \mu(\operatorname{supp} f_i^n) = 0$ for all n. Therefore, for each n, there is $i_n \in I$ such that $\|T(f_{i_n}^n)\| < 1/n$ and $\mu(\operatorname{supp} f_{i_n}^n) < 1/n$. Hence, according to Theorem 4.1.3, the operator T cannot be tauberian.

(b)⇒(c) Trivial.

(c)⇒(a) Assume that T is not tauberian. Then there is a normalized disjointly supported sequence (f_n) in $L_1(\mu)$ such that $\lim_n \|T(f_n)\| = 0$.

Let $\{I_n\}_{n=1}^{\infty}$ be a countable partition of I disjoint with $\mathfrak{U}$. Following a standard ultraproduct procedure, for every $n \in \mathbb{N}$ and every $i \in I_n$, we take $g_i := f_n$. It follows that $\lim_{\mathfrak{U}} \mu(\operatorname{supp} g_i) = 0$ and $\lim_{\mathfrak{U}} \|T(g_i)\| = 0$. So $[g_i]$ is a norm-one element of $N(P_{\mu_{\mathfrak{U}}}) \cap N(T_{\mathfrak{U}})$; hence $T_{\mathfrak{U}}|_{N(P_{\mu_{\mathfrak{U}}})}$ is not injective. □

4.3 The perturbation class of $\mathcal{T}(L_1(\mu), Y)$

Given a pair of Banach spaces X and Y satisfying $\mathcal{T}(X,Y) \neq \emptyset$, the *perturbation class* of $\mathcal{T}(X,Y)$ is defined as the set

$$P\big(\mathcal{T}(X,Y)\big) := \{S \in \mathcal{L}(X,Y) \colon \forall T \in \mathcal{T}(X,Y),\ T+S \in \mathcal{T}(X,Y)\}.$$

We observe that there is no explicit description of the set $P\big(\mathcal{T}(X,Y)\big)$ for X and Y arbitrary Banach spaces. However, the main result of this section will show that, in the case $X = L_1(\mu)$, it coincides with the weakly precompact operators $\mathcal{R}(L_1(\mu), Y)$ (see Definition 3.5.1).

Definition 4.3.1. A Banach space X is said to be *primary* if for every decomposition of X as a direct sum of two closed subspaces, at least one of them is isomorphic to X.

The proof of the main result of this section involves the subsequence splitting property, Rosenthal's ℓ_1-theorem and the primariness of $L_1(\mu)$. The latter result is due to Enflo and Starbird [60, Corollary 5.4].

Lemma 4.3.2. *If $T \colon X \longrightarrow Y$ is a tauberian operator and $(x_n) \subset X$ is a sequence equivalent to the unit vector basis of ℓ_1, then there exists $n_0 \in \mathbb{N}$ such that the restriction of T to $\overline{\text{span}}\{x_n \colon n \geq n_0\}$ is an isomorphism.*

Proof. Let us denote $L := \overline{\text{span}}\{x_n \colon n \in \mathbb{N}\}$. For every $m \in \mathbb{N}$, let $P_m \colon L \longrightarrow L$ be the operator that maps $\sum_{n=1}^{\infty} \lambda_n x_n$ to $\sum_{n=1}^{m} \lambda_n x_n$.

Since L is isomorphic to ℓ_1 and every reflexive subspace of ℓ_1 is finite-dimensional, Corollary 2.2.8 implies that the restriction $T|_L$ is an upper semi-Fredholm operator; that is, it has closed range and finite-dimensional kernel.

Observe that there exists $m_0 \in \mathbb{N}$ such that

$$N(T|_L) \cap \overline{\text{span}}\{x_n\}_{n=m_0+1}^{\infty} = \{0\}.$$

Indeed, it is enough to take a basis $\{y_i\}_{i=1}^{k}$ of $N(T|_L)$ and pick m_0 so that the set $\{P_{m_0}(y_i) \colon i = 1, \dots, k\}$ is linearly independent.

Clearly the restriction of T to $\overline{\text{span}}\{x_n\}_{n=m_0+1}^{\infty}$ is injective; hence it is an isomorphism. ☐

The following lemma mixes the principle of small perturbations for basic sequences and the fact that the closed subspace spanned by a disjointly supported sequence in $L_1(\mu)$ is complemented.

Lemma 4.3.3. *Let (f_n) and (h_n) be a pair of sequences in $L_1(\mu)$. Suppose that (h_n) is disjointly supported, $\inf_{n\in\mathbb{N}} \|h_n\| > 0$ and $\sum_{n=1}^{\infty} \|f_n - h_n\| < 1$. Then the closed subspace spanned by (f_n) is complemented in $L_1(\mu)$.*

Proof. Let us take $H := \overline{\text{span}}\{h_n\}_{n=1}^\infty$ and $F := \overline{\text{span}}\{f_n\}_{n=1}^\infty$. For every $n \in \mathbb{N}$, we define $g_n \in L_\infty(\mu)$ by

$$g_n(t) := \begin{cases} h_n(t)/|h_n(t)| & \text{if } h_n(t) \neq 0, \\ 0 & \text{if } h_n(t) = 0. \end{cases}$$

The sequence (h_n) is disjointly supported, the operator $P\colon L_1(\mu) \longrightarrow L_1(\mu)$, defined by

$$P(f) := \sum_{n=1}^\infty \|h_n\|^{-1}\Big(\int_\Omega f \cdot g_n \, d\mu\Big) h_n,$$

is a norm-one projection and $R(P)$ is the closed subspace spanned by (h_n). Let us write $y_n := h_n - f_n$.

Since $\sum_{n=1}^\infty \|y_n\| < 1$, the expression $K(f) := \sum_{n=1}^\infty \big(\int_\Omega f \cdot g_n \, d\mu\big) y_n$ defines an operator $K\colon L_1(\mu) \longrightarrow L_1(\mu)$ such that $\|K\| < 1$ and $k(h_n) = h_n - f_n$ for every $n \in \mathbb{N}$.

Now, if I denotes the identity operator on $L_1(\mu)$, $I - K$ is an automorphism that maps H onto F. Hence F is also complemented in $L_1(\mu)$. □

The following result is a well-known consequence of the subsequence splitting property.

Proposition 4.3.4. *Every non-reflexive subspace of $L_1(\mu)$ contains a subspace isomorphic to ℓ_1 and complemented in $L_1(\mu)$.*

Proof. Let M be a non-reflexive closed subspace of $L_1(\mu)$. We can select a bounded sequence (g_n) in M having no weakly convergent subsequence. By the subsequence splitting property of $L_1(\mu)$, passing to a subsequence if necessary, we can assume that $g_n = u_n + v_n$, where (u_n) is disjointly supported and (v_n) is weakly convergent.

Observe that $(v_{2n} - v_{2n-1})$ is weakly convergent to 0. So there exists a sequence of successive convex blocks of $(v_{2n} - v_{2n-1})$ which converges in norm to 0. Therefore, arguing as in the proof of (c)⇒(a) in Theorem 4.1.3, we can obtain from (u_n) a disjointly supported sequence (h_n) with $\inf_n \|h_n\| > 0$ and $\lim_n \text{dist}(h_n, M) = 0$. Therefore, the result follows from Lemma 4.3.3. □

Let us show that the perturbation class of $\mathcal{T}(L_1(\mu), Y)$ coincides with the weakly precompact operators. Observe that, for the set $P\big(\mathcal{T}(L_1(\mu), Y)\big)$ to be well-defined, we have to assume that Y contains a subspace isomorphic to $L_1(\mu)$ (see Corollary 4.1.10).

Theorem 4.3.5. *Let $K \in \mathcal{L}(L_1(\mu), Y)$ and suppose that $\mathcal{T}(L_1(\mu), Y) \neq \emptyset$. Then K is weakly precompact if and only if for every $T \in \mathcal{T}(L_1(\mu), Y)$, the operator $T + K$ is tauberian.*

Proof. For the direct implication, we suppose that $T\colon L_1(\mu) \longrightarrow Y$ is a tauberian operator and $T + K$ is not tauberian.

By Theorem 4.1.3, there exists a normalized disjointly supported sequence (f_n) in $L_1(\mu)$ for which $\lim_n \|(T+K)(f_n)\| = 0$. Since (f_n) is equivalent to the unit vector basis of ℓ_1, by Lemma 4.3.2 we can take $n_0 \in \mathbb{N}$ such that $(T(f_n))_{n=n_0}^{\infty}$ is also equivalent to the unit vector basis of ℓ_1. In particular, $(T(f_n))$ has no weakly Cauchy subsequences. Therefore $(K(f_n))$ has no weakly Cauchy subsequences and we conclude that K is non-weakly precompact.

For the converse, we assume that K is not weakly precompact and we then will find a tauberian operator $T \in \mathcal{L}(L_1(\mu), Y)$ such that $T + K$ is not tauberian.

By Rosenthal's ℓ_1-theorem, there exists a subspace H of $L_1(\mu)$ isomorphic to ℓ_1 such that the restriction $K|_H$ is an isomorphism. By Proposition 4.3.4, we can assume that H is a complemented subspace of $L_1(\mu)$.

Let M be a closed subspace of $L_1(\mu)$ such that $L_1(\mu) = H \oplus M$. Note that M is isomorphic to $L_1(\mu)$, because $L_1(\mu)$ is primary.

Since $\mathcal{T}(L_1(\mu), Y)$ is non-empty, by Corollary 4.1.10, Y contains a subspace L isomorphic to $L_1(\mu)$. One of the following cases then occurs:

(1) $K(H) + L$ is closed and $K(H) \cap L$ is finite-dimensional,

(2) $K(H) \cap L$ is infinite-dimensional,

(3) $K(H) + L$ is non-closed and $K(H) \cap L$ is finite-dimensional.

(1) Passing to a finite co-dimensional subspace of L if necessary, we may assume that $K(H) \cap L = \{0\}$. Let $U\colon M \longrightarrow L$ be a surjective isomorphism. Thus, the operator

$$T\colon L_1(\mu) = H \oplus M \longrightarrow K(H) \oplus L \subset Y$$

given by $T(x, y) := -K(x) + U(y)$ is also an isomorphism. In particular, T is tauberian. Moreover, $N(T + K)$ is not reflexive because it contains the subspace H, which is isomorphic to ℓ_1. Therefore, $T + K$ cannot be tauberian.

(2) Since $K(H)$ is isomorphic to ℓ_1, $K(H) \cap L$ contains a subspace N_1 isomorphic to ℓ_1. By Proposition 4.3.4, N_1 contains a subspace N_2 isomorphic to ℓ_1 and complemented in L. Since $L_1(\mu)$ is primary, the complement of N_2 in L is isomorphic to $L_1(\mu)$. So, replacing L by this complement and H by a smaller complemented subspace, we can assume that the sum $K(H) + L$ is direct and closed; hence we are in the conditions of case (1).

(3) In this case we are going to find a compact operator $K_1\colon L_1(\mu) \longrightarrow Y$ such that $(K + K_1)(H) \cap L$ is infinite-dimensional. Note that $(K + K_1)|_H$ is upper semi-Fredholm. So, passing to a finite codimensional subspace, we can assume that $(K + K_1)|_H$ is an isomorphism. Therefore, the argument of case (2) provides a tauberian operator $T\colon L_1(\mu) \longrightarrow Y$ such that $T + K + K_1$ is not tauberian; hence $T + K$ is not tauberian.

As in case (1), we can assume that $K(H) \cap L = \{0\}$. From the fact that $K(H) + L$ is not closed, it follows that there exists a normalized sequence (y_n) in $K(H)$ such that $\operatorname{dist}(y_n, L) < 2^{-n}$. Note that (y_n) cannot have convergent subsequences, because if y were the limit of such a subsequence, we would have $\|y\| = 1$ and $y \in K(H) \cap L$, which is not possible.

Since weakly Cauchy sequences in ℓ_1 are convergent and $K(H)$ is isomorphic to ℓ_1, by Rosenthal's ℓ_1-theorem we can assume that (y_n) is equivalent to the unit vector basis of ℓ_1.

Let (h_n) be a sequence in H such that $K(h_n) = y_n$ for every $n \in \mathbb{N}$. Since (h_n) is equivalent to the unit vector basis of ℓ_1, we can select a bounded sequence (h_n^*) in $L_1(\mu)^*$ such that $\langle h_i^*, h_j \rangle = \delta_{ij}$.

Now we select a sequence (z_n) in L such that $\|y_n - z_n\| < 2^{-n}$ for every $n \in \mathbb{N}$. Clearly, the expression

$$K_1(f) = \sum_{i=1}^{\infty} \langle h_i^*, f \rangle (z_i - y_i)$$

defines a compact operator such that $(K + K_1)(h_n) = z_n$ for every $n \in \mathbb{N}$. Hence $(K + K_1)(H) \cap L$ is infinite-dimensional and the proof is complete. □

Remark 4.3.6. There are pairs of Banach spaces X and Y for which $\mathcal{T}(X, Y)$ is non-empty and the set $\mathcal{R}(X, Y)$ of weakly precompact operators is not contained in $P(\mathcal{T}(X, Y))$.

Indeed, let J denote the James quasi-reflexive sequence space. Thus the natural inclusion $I\colon J \longrightarrow c_0$ is both tauberian (Example 2.1.18) and weakly precompact. However, the null operator from J into c_0 is not tauberian.

Let us give some additional characterizations for the perturbation class of $\mathcal{T}(L_1(\mu), Y)$:

Proposition 4.3.7. *For an operator $T\colon L_1(\mu) \longrightarrow Y$, the following statements are equivalent:*

(a) *T is weakly precompact;*

(b) *for every Banach space Z and every operator $A \in \mathcal{L}(Z, L_1(\mu))$ for which TA is tauberian, A is weakly compact;*

(c) *if H is a subspace of $L_1(\mu)$ such that $T|_H$ is an isomorphism, then H is reflexive.*

Proof. (a)⇒(b) Let us assume that T is weakly precompact, $A \in \mathcal{L}(Z, L_1(\mu))$ and TA is tauberian. Since T is weakly precompact, so is TA. Moreover, by Corollary 2.2.8 every tauberian operator on ℓ_1 is upper semi-Fredholm. Thus, Z cannot contain any subspace isomorphic to ℓ_1 and, by Rosenthal's ℓ_1-theorem, every bounded sequence in Z must have a weakly Cauchy subsequence. But $L_1(\mu)$ is sequentially weakly complete, so A is weakly compact.

(b)⇒(c) Suppose $T|_H$ is an isomorphism. Then (b) implies that the embedding $J\colon H \longrightarrow L_1(\mu)$ is weakly compact and hence H is reflexive.

(c)⇒(a) If T is not weakly precompact, by Rosenthal's ℓ_1-theorem there exists a sequence (f_n) in $L_1(\mu)$ equivalent to the unit vector basis of ℓ_1 such that the restriction of T to the closed linear span of (f_n) is an isomorphism. □

On the operators in $\mathcal{T}(L_1(\mu))$

We have mentioned before that for some Banach spaces Y there exist operators in $\mathcal{T}(L_1(\mu), Y)$ that are not upper semi-Fredholm. However, this is not clear in the case $Y = L_1(\mu)$. Here we include some comments to emphasize the relevance of the following open problem:

Problem 1. Let $T\colon L_1(\mu) \longrightarrow L_1(\mu)$ be a tauberian operator. Is T upper semi-Fredholm?

We observe that both sets $\Phi_+(L_1(\mu))$ and $\mathcal{T}(L_1(\mu))$ are open in $\mathcal{L}(L_1(\mu))$. Moreover, they have the same perturbation class:

$$P\big(\Phi_+(L_1(\mu))\big) = P\big(\mathcal{T}(L_1(\mu))\big) = \mathcal{R}(L_1(\mu)).$$

The result for $\mathcal{T}\big(L_1(\mu)\big)$ is a consequence of Theorem 4.3.5, and the result for $\Phi_+\big(L_1(\mu)\big)$ is due to V. D. Milman (see [163]).

Note that upper semi-Fredholm operators have closed range. Therefore, we can consider a weaker problem:

Problem 2. Let $T\colon L_1(\mu) \longrightarrow L_1(\mu)$ be a tauberian operator with closed range. Is T upper semi-Fredholm?

Clearly, the last problem is equivalent to the following one:

Problem 2′. Let R be an infinite dimensional reflexive subspace of $L_1(\mu)$. Is the quotient $L_1(\mu)/R$ isomorphic to a subspace of $L_1(\mu)$?

Remark 4.3.8. Observe that an infinite dimensional reflexive subspace R of $L_1(\mu)$ is never complemented. What is more, the quotient $L_1(\mu)/R$ is not a $\mathcal{L}_1$-space [120, Proposition 5.2]. However, $L_1(\mu)/R$ satisfies Grothendieck's theorem; i.e., it is a G.T. space (see [141, Chapter 6]).

Positive answers to Problem 2′ would be interesting because they provide new examples of G.T. spaces among the closed subspaces of $L_1(\mu)$.

Let us now show that Problem 2′ has a negative answer for some reflexive subspaces of $L_1(\mu)$.

Example 4.3.9. [141, Remark on page 117] There exists a subspace F of $L_1[0,1]$ isomorphic to ℓ_2 such that the quotient $L_1[0,1]/F$ is not isomorphic to a subspace of $L_1[0,1]$.

In order to construct F, we consider a Kašin orthogonal decomposition

$$L_2[0,1] = E_1 \oplus E_2 \oplus E_3$$

where each E_1, E_2 and E_3 are infinite dimensional closed subspaces such that the L_2-norm and the L_1-norm are equivalent on

$$E_1 \oplus E_2,\ E_2 \oplus E_3 \text{ and } E_3 \oplus E_1.$$

We refer to [141, Theorem 8.22] for the existence of this decomposition. We take as F the image of E_3 under the natural injection of $L_2[0,1]$ into $L_1[0,1]$.

Then the quotient $L_1[0,1]/F$ fails the Gordon-Lewis property (see [141, Definition 8.13]), which is a property satisfied by the subspaces of $L_1[0,1]$ (see [141, Remark in page 107]).

In relation to Problems 2 and 2′, we do not know the answer to the following even weaker problem:

Problem 3. Let R_0 be the closed subspace of $L_1[0,1]$ generated by the Rademacher functions. Is the quotient $L_1[0,1]/R_0$ isomorphic to a subspace of $L_1[0,1]$?

In [164], Weis claims that the answer to Problem 1 is positive in some special cases, although the paper does not include a proof of the claim. In order to describe this positive result, we need to introduce some concepts and notation. For simplicity, let us consider the space $L_1[0,1]$.

It was proved in [113] that for every operator $T\colon L_1[0,1] \longrightarrow L_1[0,1]$ there are measures μ_t; $t \in [0,1]$, with $t \to \mu_t$ weak*-measurable, such that

$$Tf(t) = \int_0^1 f(s)\, d\mu_t(s) \quad \text{for almost every } t \in [0,1].$$

Now we consider the canonical decompositions of each measure μ_t

$$\mu_t = \mu_t^{at} + \mu_t^{c} = \mu_t^{at} + \mu_t^{ac} + \mu_t^{sc},$$

where μ_t^{at} and μ_t^{c} are the atomic part and the continuous part of the measure μ_t, and $\mu_t^{c} = \mu_t^{ac} + \mu_t^{sc}$ is the decomposition of μ_t^{c} in absolutely continuous part plus singular part with respect to the Lebesgue measure on $[0,1]$.

Clearly the decomposition of the measures μ_t induces a decomposition of the operator $T = T^{at} + T^{ac} + T^{sc}$.

We observe that T^{at} corresponds to a countable sum of Riesz isomorphisms; i.e., we can write

$$T^{at}f(t) = \sum_{n=1}^{\infty} a_n(t) f\big(\sigma_n(t)\big)$$

while T^{ac} is a representable operator (Definition 5.3.15). We refer to [164] for details.

Proposition 4.3.10. [164, 6.4 Remark] *Let* $T\colon L_1[0,1] \longrightarrow L_1[0,1]$ *be an operator with* $T^{sc} = 0$*. Then* T *is tauberian if and only if it is upper semi-Fredholm.*

We observe that the hypothesis $T^{sc} = 0$ is very strong. Proposition 4.3.10 is far from a positive answer to Problem 1.

4.4 Biconjugates of tauberian operators on $L_1(\mu)$

One remarkable difference between tauberian and upper semi-Fredholm operators is that if T is upper semi-Fredholm, then so is T^{**}, while there are tauberian operators whose biconjugates are not tauberian, as was observed in Theorem 3.1.18. However, the main result of this section establishes that, when T is a tauberian operator on $L_1(\mu)$, then so is T^{**}.

First, we give a technical lemma.

Lemma 4.4.1. *Let $T\colon X \longrightarrow Y$ be an operator, $z^{**} \in \operatorname{int} B_{X^{**}}$, $y \in Y$ and $\varepsilon > 0$. Suppose that $\|T^{**}(z^{**}) - y\| < \varepsilon$ and denote $L := \{x \in B_X : \|T(x) - y\| < \varepsilon\}$. Then $z^{**} \in \overline{L}^{\sigma(X^{**},X^*)}$.*

Proof. First observe that the set L cannot be empty. Otherwise we would have $\varepsilon \leq \|T(x) - y\|$ for all $x \in B_X$, and by the Hahn-Banach theorem, there would exists $y^* \in S_{Y^*}$ so that

$$\varepsilon \leq \langle y^*, T(x) - y\rangle \quad \text{for all } x \in B_X. \tag{4.1}$$

Since $T^{**}(z^{**}) - y \in \overline{T(B_X) - y}^{\sigma(X^{**},X^*)}$, formula (4.1) would lead to

$$\varepsilon \leq \langle T^{**}(z^{**}) - y, y^*\rangle \leq \|T^{**}(z^{**}) - y\|$$

which contradicts the hypotheses of the statement.

Let us assume that $z^{**} \notin \overline{L}^{\sigma(X^{**},X^*)}$. By the Hahn-Banach theorem, there exists $x^* \in S_{X^*}$ and a real number a such that

$$-1 \leq \sup_{x\in L}\langle x^*, x\rangle \leq a < b := \langle z^{**}, x^*\rangle < 1.$$

Thus, $W := \{x \in \operatorname{int} B_X : 2^{-1}(a+b) < \langle x^*, x\rangle\}$ is non-empty and $z^{**} \in \overline{W}^{\sigma(X^{**},X^*)}$. Hence

$$T^{**}(z^{**}) - y \in \overline{T(W) - y}^{\sigma(Y^{**},Y^*)}. \tag{4.2}$$

Moreover, $W \cap L = \emptyset$, which implies that $\varepsilon \leq \|T(w) - y\|$ for all $w \in W$. Thus, by the Hahn-Banach theorem and (4.2), there is $y^* \in S_{Y^*}$ such that

$$\varepsilon \leq \inf_{w\in W}\langle y^*, T(w) - y\rangle \leq \langle y^*, T^{**}(z^{**}) - y\rangle,$$

and subsequently, $\varepsilon \leq \|T^{**}(z^{**}) - y\|$, a contradiction. □

We saw in Section 3.1 that an operator $T \in \mathcal{L}(X, Y)$ is tauberian if and only if T^{co} is injective. In the case $X = L_1(\mu)$, it is possible to say more, as can be seen in the following result.

Theorem 4.4.2. *For an operator $T \in \mathcal{L}(L_1(\mu), Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) *T^{co} is an isomorphism;*

(c) *T^{**} is tauberian.*

Proof. (a)⇒(b) Assume that T is tauberian but T^{co} is not an isomorphism. Then, by Proposition 4.1.8, there exists a real number $r > 0$ such that

$$\liminf_n \|T(f_n)\| > r \tag{4.3}$$

for all normalized sequences (f_n) for which $\mu(\operatorname{supp} f_n) \underset{n}{\longrightarrow} 0$.

Let us fix a real number $6/7 < \varepsilon < 1$. The assumption that T^{co} is not an isomorphism gives an element $x^{**} \in L_1(\mu)^{**}$ such that $\varepsilon < \|x^{**} + L_1(\mu)\|$, $\|x^{**}\| < 1$ and $\|T^{co}(x^{**}+L_1(\mu))\| < r/4$. Take $y \in Y$ so that $\|T^{**}(x^{**})+y\| < r/4$ and consider the set

$$A := \{x \in B_{L_1(\mu)} \colon \|T(x) - y\| < r/4\}.$$

By Lemma 4.4.1, x^{**} is a w^*-cluster point of A, where w^* stands for the weak* topology of $L_1(\mu)^{**}$. Thus, by Proposition A.5.3, the set A contains an ε-triangular sequence (g_n) (see Definition 6.2.12).

By the subsequence splitting property, there exists a subsequence (x_n) of (g_n), a weakly convergent sequence (w_n) and a disjointly supported sequence (v_n) such that $x_n = w_n + v_n$.

Since the subsequence (x_n) is also ε-triangular, there exists a sequence (f_n) in $B_{L_1(\mu)^*}$ satisfying

$$\begin{aligned} \langle f_i, x_j\rangle &> \varepsilon \quad \text{if } 1 \le i \le j,\\ \langle f_i, x_j\rangle &= 0 \quad \text{if } 1 \le j < i. \end{aligned}$$

Let (k_n) and (α_n) be an increasing sequence of positive integers and a sequence of non-negative real numbers with $\sum_{i=k_n+1}^{k_{n+1}} \alpha_i = 1$ for all n so that the sequence $w'_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i w_n$ is norm convergent. Consider also the induced block sequences

$$x'_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i x_n \quad \text{and} \quad v'_n := \sum_{i=k_n+1}^{k_{n+1}} \alpha_i v_n,$$

and note that $x'_n = w'_n + v'_n$ for all n.

Let $n_0 \in \mathbb{N}$ such that $\|w'_n - w'_m\| < \varepsilon/8$ and $\|T(w'_n - w'_m)\| < r/4$ for all $m, n \ge n_0$. Given the subsequence $(h_n) := (f_{k_n+1})$, we have $\langle h_i, x'_j\rangle \ge \varepsilon$ for $i \le j$ and $\langle h_i, x_j\rangle = 0$ for $j < i$, hence $\|x'_n - x'_m\| > \varepsilon$ when $m \ne n$. Thus, on the one hand, we obtain

$$\|v'_{2n+1} - v'_{2n}\| \ge \|x'_{2n+1} - x'_{2n}\| - \|w'_{2n+1} - w'_{2n}\| > (7/8)\varepsilon \text{ for all } n \ge n_0. \tag{4.4}$$

On the other hand, $\|T(g_n)+y\| < r/4$ implies $\|T(x_{2n+1}-x_{2n})\| < r/2$. So, for $n \geq n_0$, we have

$$\|T(v'_{2n+1}-v'_{2n})\| \leq \|T(x'_{2n+1}-x'_{2n})\| + \|T(w'_{2n+1}-w'_{2n})\| \leq (3/4)r.$$

By inequality (4.4), we have

$$\lambda_n := \|v'_{2n+1}-v'_{2n}\|^{-1} \leq (8/7)\varepsilon^{-1} < 4/3.$$

Thus, the normalized disjoint sequence (z_n) defined by $z_n := \big(\lambda_n(v'_{2n+1}-v'_{2n})\big)$ satisfies $\|Tz_n\| < r$ for all n, which contradicts inequality (4.3).

(b)⇒(c) Since, by Proposition 3.1.11, we can identify $(T^{co})^{**}$ and $(T^{**})^{co}$, if T^{co} is an isomorphism, then so is $(T^{co})^{**}$. In particular, $(T^{**})^{co}$ is injective and we conclude that T^{**} is tauberian.

(c)⇒(a) It is immediate. □

Remark 4.4.3. From Theorem 4.4.2 we can derive an alternative proof of the fact that $\mathcal{T}(L_1(\mu), Y)$ is open in $\mathcal{L}(L_1(\mu), Y)$.

Indeed, it is enough to observe that isomorphisms form an open subset of $\mathcal{L}(L_1(\mu), Y)$ and that $\|S^{co}-T^{co}\| \leq \|S-T\|$.

4.5 Notes and Remarks

Chapter 4 is almost exclusively devoted to results concerning tauberian operators on $L_1(\mu)$ spaces obtained in [75] and [78].

Using ultraproduct techniques, Bretagnolle et al. [41] proved that, for every $p \in (1,2]$, $L_p[0,1]$ is isomorphic to a closed subspace of $L_1[0,1]$. Therefore, when μ is not purely atomic, $L_1(\mu)$ contains many reflexive subspaces, which implies the existence of many tauberian operators acting on $L_1(\mu)$.

The characterization of tauberian operators on $L_1(\mu)$ in terms of their action on normalized disjoint sequences (Theorem 4.1.3) and all its sequels in Section 4.1 was obtained in [75].

An operator $T\colon L_1(\mu) \longrightarrow Y$ is called *Enflo* if $L_1(\mu)$ contains a subspace X isomorphic to $L_1(\mu)$ so that $T|_X$ is an isomorphism [60]. Corollary 4.1.7 means that a tauberian operator T on $L_1(\mu)$ may be regarded as a hereditarily Enflo operator. The problems on the operators in $\mathcal{T}(L_1(\mu))$ were raised in [77].

Section 4.2 includes the most relevant results regarding the ultrapowers of tauberian operators on $L_1(\mu)$ obtained in [78]: the characterizations of tauberian operators on $L_1(\mu)$ in terms of the action of their ultrapowers on the components $J_{\mu_\mathfrak{U}}\big(L_1(\mu_\mathfrak{U})\big)$ and $N(P_{\mu_\mathfrak{U}})$, given in Propositions 4.2.1 and 4.2.2. This paper includes an alternative proof of Corollary 4.1.4 based on ultrapower techniques.

The main result of Section 4.3 is Theorem 4.3.5. It was obtained in [75]. The notion of *perturbation class* was introduced by Lebow and Schechter in [118] in order to study semi-Fredholm operators.

Section 4.4 is exclusively devoted to demonstrating that an operator T on $L_1(\mu)$ is tauberian if and only if T^{**} is so too. This fact is proved in Theorem 4.4.2 as a consequence of the separation Lemma 4.4.1, which has been taken from [147], although its proof has been rearranged. The proof of Theorem 4.4.2 is basically the one given in [75].

In Chapter 6 we can find another two proofs of the fact that T^{**} is tauberian whenever T is a tauberian operator on $L_1(\mu)$. These proofs apply the facts that tauberian operators on $L_1(\mu)$ are strongly tauberian (see Section 6.2) and supertauberian (see Appendix A.6). Note that if an operator $T\colon X \longrightarrow Y$ is strongly tauberian or supertauberian, then T^{**} is so too.

Chapter 5

Some applications

In the previous chapters we have already described some applications of tauberian operators. For example, in Section 2.3 we can find some extensions of James' characterization of reflexive Banach spaces and Section 2.4 shows that some properties of basic sequences can be characterized in terms of the action of the tauberian operators over them. In this chapter we present some other applications. Several of them consist of results that have been derived from the existence of a tauberian operator acting between two Banach spaces. Some others are constructions of tauberian operators that are useful in the proof of some results or provide us with counterexamples.

In the first section we show that the Radon-Nikodým property and the Krein-Milman property are equivalent for a Banach space X for which there exists an injective tauberian operator $T\colon X \times X \longrightarrow X$.

The second section describes some isomorphic properties of Banach spaces or bounded sets that are preserved by tauberian (or injective tauberian) operators. These results are then applied in the next section, where we show that some operator ideals $\mathcal{A}$ have the *factorization property*: every $T \in \mathcal{A}$ factors through a Banach space $Z \in Sp(\mathcal{A})$ (i.e., $I_Z \in \mathcal{A}$). In the majority of cases, this is proved by showing that, for every $T \in \mathcal{A}$, the intermediate space in the DFJP factorization belongs to $Sp(\mathcal{A})$.

In the fourth section we give a uniform factorization result for a compact subset H of $\mathcal{A}(X, Y)$, where $\mathcal{A}$ is a closed, injective and surjective operator ideal. We show that there are Banach spaces X_H and Y_H, operators $j_H \in \mathcal{A}(Y_H, Y)$ and $k_H \in \mathcal{A}(X, X_H)$, and a relatively compact subset H_0 of $\mathcal{A}(X_H, Y_H)$ such that every $T \in H$ may be written as $T = j_H T_0 k_H$, with $T_0 \in H_0$. Moreover, in the fifth section we characterize the holomorphic mappings f between Banach spaces that can be written in the form either $f = T \circ g$ or $f = g \circ T$, where g is another holomorphic mapping and T is an operator belonging to a closed injective or a closed surjective operator ideal, respectively.

In the last section of this chapter we describe some applications of the DFJP factorization related to the approximation property of Banach spaces and we show the relation between the properties of the weak Calkin algebras –associated with the weakly compact operators– and the weak approximation property.

5.1 Equivalence of the Radon-Nikodým property and the Krein-Milman property

The Radon-Nikodým property and the Krein-Milman property of Banach spaces have a close relation because both of them admit geometric characterizations. Moreover, it is known that the Radon-Nikodým property implies the Krein-Milman property, but the question whether the converse implication is true or not remains open, despite the efforts of many mathematicians, including Bourgain, James and Schachermayer.

In [151], Schachermayer proved that a Banach space X isomorphic to $X \times X$ has the Radon-Nikodým property if and only if it has the Krein-Milman property. In fact, his proof shows that for the result being true it is enough to assume that there is an injective tauberian operator from $X \times X$ into X.

In this section, we include an exposition of Schachermayer's result. The key for the proof is Theorem 2.3.4: a tauberian operator takes closed bounded convex sets to closed sets.

Definition 5.1.1. A Banach space X is said to have the *Radon-Nikodým property* if, given a finite measure space (Ω, Σ, μ), for every μ-continuous X-valued measure $F : \Sigma \to X$ with bounded variation, there is an X-valued Bochner integrable function $f : \Omega \to X$, called the *Radon-Nikodým derivative* of F, such that

$$F(A) = \int_A f \, d\mu; \quad \text{for all } A \in \Sigma.$$

So, loosely stated, X has the Radon-Nikodým property if and only if the Radon-Nikodým theorem holds for X-valued measures. In particular, the scalar field has the Radon-Nikodým property as a consequence of the classical Radon-Nikodým theorem.

Examples of Banach spaces with the Radon-Nikodým property are reflexive spaces and separable dual spaces. Moreover, c_0 and $L_1(0,1)$ are well-known examples of spaces failing the Radon-Nikodým property. We refer to [52] for a classical description of the properties and applications of this class of Banach spaces and to [54] for a more modern and reader-friendly exposition.

We are going to need several concepts and techniques of a geometric flavor.

Definition 5.1.2. Let C be a convex subset of a vector space X. We say that a point $x \in C$ is an *extreme point of* C if the set $C \setminus \{x\}$ is convex.

We denote by $\text{ext}(C)$ the set of extreme points of C.

In general, $\mathrm{ext}(C)$ may be empty. However, the well-known Krein-Milman theorem guarantees that, if K is a compact convex subset of a locally convex space, then K coincides with the closed convex hull of $\mathrm{ext}(K)$. This result is the origin of the following concept:

Definition 5.1.3. A Banach space X is said to have the *Krein-Milman property* if every closed bounded convex subset of X is the closed convex hull of its extreme points.

Using the Hahn-Banach and the Bishop-Phelps theorems, we can derive a useful characterization of this property.

Proposition 5.1.4. *A Banach space X has the Krein-Milman property if and only if for every non-empty closed bounded convex subset C of X, the set* $\mathrm{ext}(C)$ *is non-empty.*

For the proof we refer to [54, Lemma 5.11].

Definition 5.1.5. Let C be a closed bounded convex subset of a Banach space X. A point $x \in C$ is called an *exposed point* of C if there exists $f \in X^*$ such that $f(x) > f(y)$ for each $y \in C \setminus \{x\}$.

In this case we say that f *exposes* x.

For $\alpha > 0$ and $f \in X^*$, we consider the *slice* $S(C, f, \alpha)$ *of* C, defined by

$$S(C, f, \alpha) := \{y \in C : f(y) > \sup_{z \in C} f(z) - \alpha\}.$$

A point $x \in C$ is called a *strongly exposed point* of C if there exists $f \in X^*$ that exposes x and has the further property that

$$\lim_{\alpha \to 0+} \mathrm{diam}\big(S(C, f, \alpha)\big) = 0,$$

where $\mathrm{diam}(A)$ stands for the *diameter* of the set A.

Banach spaces with the Radon-Nikodým property admit the following beautiful and highly non-trivial geometric characterization.

Theorem 5.1.6. *A Banach space has the Radon-Nikodým property if and only if every closed bounded convex subset of X is the closed convex hull of its strongly exposed points.*

Since every strongly exposed point of a convex set is an extreme point, the following result is an immediate consequence of Theorem 5.1.6.

Corollary 5.1.7. *If a Banach space X has the Radon-Nikodým property, then it also has the Krein-Milman property.*

Remark 5.1.8. It follows from Theorem 5.1.6 and the definitions given that both the Radon-Nikodým property and the Krein-Milman property are inherited by closed subspaces.

It is not known if the converse implication to that in Corollary 5.1.7 is valid. Here we will prove it under certain restrictions. The key for the proof is the following result:

Theorem 5.1.9. *A Banach space X has the Radon-Nikodým property if and only if $\ell_2(X)$ has the Krein-Milman property.*

Proof. Suppose that X has the Radon-Nikodým property. Then it is not difficult to prove that $\ell_2(X)$ has the Radon-Nikodým property; hence it has the Krein-Milman property.

The proof of the converse implication is rather technical. So we refer to [151, Proof of Theorem 2.3].

We observe that if X fails the Radon-Nikodým property, then it has a separable subspace that also fails it [52, Theorem II.3.2]. Therefore, it is enough to prove the case in which X is separable. □

Let us now show that the Krein-Milman property is preserved by injective tauberian operators.

Proposition 5.1.10. *Let X and Y be Banach spaces for which there is an injective tauberian operator $T\colon X \longrightarrow Y$. If Y has the Krein-Milman property, then X also has this property.*

Proof. Suppose that X fails the Krein-Milman property. By Proposition 5.1.4, X contains a non-empty closed bounded convex subset C for which $\text{ext}(C)$ is empty.

Note that $T(C)$ is a non-empty bounded convex subset of Y that has no extreme points. Moreover, by Theorem 2.3.4, $T(C)$ is closed. Hence, Y fails the Krein-Milman property by Proposition 5.1.4. □

The following stability result for the existence of tauberian operators is another critical ingredient in the proof of the main result of this section.

Theorem 5.1.11. *Let X be a Banach space for which there exists an injective tauberian operator $T\colon X \times X \longrightarrow X$. Then there exists an injective tauberian operator $S\colon \ell_2(X) \longrightarrow X$.*

Proof. We can assume that

$$T(x, y) = U(x) + V(y), \quad (x, y) \in X \times X,$$

with $U, V \in \mathcal{L}(X)$ and $r := \max\{\|U\|, \|V\|\} < 1$.

Now, for a finitely non-zero sequence $x = (x_1, \dots, x_n, 0, 0, \dots) \in \ell_2(X)$, we define $S(x)$ recursively by

$$S(0, \dots, 0, \dots) := 0,$$

$$S(x_1, \dots, x_n, 0, 0, \dots) := T\big(x_1, S(x_2, \dots, x_n, 0, 0, \dots)\big).$$

It is not difficult to check that

$$S(x_1, \ldots, x_n, 0, 0, \ldots) = \sum_{i=1}^{n} V^{i-1} U x_i.$$

Clearly, S is a linear map. Moreover,

$$\|S(x_1, \ldots, x_n, 0, 0, \ldots)\| \le \sum_{i=1}^{n} r^i \|x_i\| \le \Big(\sum_{i=1}^{n} r^{2i}\Big)^{1/2} \Big(\sum_{i=1}^{n} \|x_i\|^2\Big)^{1/2}.$$

Hence S can be extended to a continuous operator $S\colon \ell_2(X) \longrightarrow X$ which, by continuity, satisfies

$$S(x_i) = T\big(x_1, S(x_{i+1})\big) \qquad \text{for every } (x_i) \in \ell_2(X),$$

where (x_{i+1}) denotes the sequence $(x_2, x_3, \ldots)$.

Since T is injective, the last equality allows us to derive that S is injective. Indeed,

$$S(x_i) = 0 \Rightarrow x_1 = 0, S(x_{i+1}) = 0 \Rightarrow x_2 = 0, S(x_{i+2}) = 0 \Rightarrow \cdots$$

Similarly, the biconjugate operators

$$T^{**}\colon X^{**} \times X^{**} \longrightarrow X^{**} \qquad \text{and} \qquad S^{**}\colon \ell_2(X^{**}) \longrightarrow X^{**}$$

satisfy

$$S^{**}(z_i) = T^{**}\big(z_1, S^{**}(z_{i+1})\big) \qquad \text{for every } (z_i) \in \ell_2(X^{**}) \equiv \ell_2(X)^{**}.$$

Now, since T is tauberian, given $(z_i) \in \ell_2(X^{**})$, we have

$$S^{**}(z_i) \in X \Rightarrow z_1 \in X, S^{**}(z_{i+1}) \in X \Rightarrow z_2 \in X, S(z_{i+2}) \in X \Rightarrow \cdots$$

Hence $(z_i) \in \ell_2(X)$, and we conclude that S is tauberian. □

Now we can prove the main result of this section.

Theorem 5.1.12. *Let X be a Banach space for which there exists an injective tauberian operator $T\colon X \times X \longrightarrow X$. Then X has the Radon-Nikodým property if and only if X has the Krein-Milman property.*

Proof. For the non-trivial part, assume that X has the Krein-Milman property. By Theorem 5.1.11, there is an injective tauberian operator $S\colon \ell_2(X) \longrightarrow X$. Therefore, Proposition 5.1.10 implies that $\ell_2(X)$ has the Krein-Milman property; hence Theorem 5.1.9 allows us to conclude that X has the Radon-Nikodým property. □

5.2 Preservation of isomorphic properties

The property of X being isomorphic to a subspace of Y is strictly stronger than the existence of a tauberian (or an injective tauberian) operator $T\colon X \longrightarrow Y$. However, as we shall see in this section, in the latter cases some isomorphic properties of Y are inherited by X. Similarly, given a tauberian operator $T\colon X \longrightarrow Y$ and a bounded subset A of X, some isomorphic properties of the set $T(A)$ are inherited by the set A. These are the kind of results we refer to as preservation of isomorphic properties by tauberian operators.

Properties of spaces preserved by tauberian operators

We begin by describing some isomorphic properties preserved by injective tauberian operators.

Theorem 5.2.1. *Let X and Y be Banach spaces, and let $\mathcal{P}$ be one of the following properties of Banach spaces:*

(i) *Krein-Milman property;*

(ii) *Radon-Nikodým property;*

(iii) *reflexivity;*

(iv) *quasi-reflexivity; i.e.,* $\dim Y^{**}/Y < \infty$*;*

(v) *somewhat reflexivity; i.e., each infinite dimensional subspace contains an infinite dimensional reflexive subspace;*

(vi) *weak sequential completeness;*

(vii) *containing no copies of ℓ_1;*

(viii) *containing no copies of c_0;*

(ix) *all the weakly sequentially complete subspaces are reflexive;*

(x) *separability;*

(xi) *separability of the dual space.*

Suppose that there is an injective tauberian operator $T\colon X \longrightarrow Y$. Then X satisfies $\mathcal{P}$ whenever Y satisfies $\mathcal{P}$.

Proof. (i) It was proved in Proposition 5.1.10.

(ii) The proof is similar to that of Proposition 5.1.10, using the characterization of the Radon-Nikodým property given in Theorem 5.1.6.

(iii) It is a direct consequence of the definition.

(iv) Since $T^{co}\colon X^{**} \longrightarrow Y^{**}$ is injective, $\dim X^{**}/X \leq \dim Y^{**}/Y < \infty$.

(v) It is a consequence of (iii) and the fact that the restrictions of T are also tauberian operators.

(vi) Let (x_n) be a weakly Cauchy sequence in X. Since Y is weakly sequentially complete, (Tx_n) is weakly convergent. Since T is injective and tauberian, we conclude that (x_n) is also weakly convergent.

(vii) Since ℓ_1 contains no reflexive infinite dimensional subspaces, the restriction of T to a subspace of X isomorphic to ℓ_1 is an isomorphism; hence Y contains a copy of ℓ_1 if X does.

(viii) It is similar to the proof of (vii).

(ix) It is a consequence of (iii) and (vi), arguing as in the proof of (v).

(x) Since $T(B_X)$ is separable, we can take a sequence (x_n) in B_X such that (Tx_n) is dense in $T(B_X)$.

Let $x \in B_X$ and select (x_{n_k}) such that (Tx_{n_k}) converges to Tx. By Corollary 2.2.3, $(x_{n_k} - x)$ is weakly null. Since the weak closure and the norm closure coincide for convex sets, we conclude that the set of convex combinations with rational coefficients of (x_n) is dense in B_X; hence X is separable.

(xi) Suppose that Y^* is separable. Since $N(T^{**}) = N(T) = \{0\}$, the range of T^* is dense in X^*; hence X^* is separable. □

Corollary 5.2.2. *Let $\mathcal{P}$ be one of the first nine properties in Theorem* 5.2.1. *Suppose there exists a tauberian operator $T\colon X \longrightarrow Y$. Then X satisfies $\mathcal{P}$ whenever Y satisfies $\mathcal{P}$.*

Proof. We consider the decomposition $T = \widetilde{T}Q_{N(T)}$, where $\widetilde{T}\colon X/N(T) \longrightarrow Y$ is the injective operator associated with T.

Since $\widetilde{T}$ is tauberian, Theorem 5.2.1 implies that $X/N(T)$ satisfies $\mathcal{P}$. So it is enough to observe that each of the properties we consider in this result is a *three-space property:*

$$M \text{ reflexive, } X/M \text{ satisfies } \mathcal{P} \Rightarrow X \text{ satisfies } \mathcal{P}.$$

We refer to [42] for these implications. □

Remark 5.2.3. The previous corollary fails for the last two properties in Theorem 5.2.1 because the kernel of T or its dual space could be non-separable.

Properties of bounded subsets preserved by tauberian operators

We will show that some isomorphic properties of bounded subsets are preserved by injective tauberian operators. These results can be seen as "localizations" of some of the results given in Theorem 5.2.1. Their proofs are similar to those of the corresponding results for spaces.

Definition 5.2.4. Let A be a bounded closed convex subset of a Banach space X. We say that A has the *Krein-Milman property* if for each closed convex subset K of A, the closed convex hull of $\text{ext}(K)$ coincides with K.

We say that A has the *Radon-Nikodým property* if for each nonempty closed convex subset K of A and each $\varepsilon > 0$ there exists $x_0 \in K$ which is not in the convex hull of $\{y \in K : \|x_0 - y\| \geq \varepsilon\}$.

Note that X has the Krein-Milman property or the Radon-Nikodým property if and only if B_X has the same property.

Theorem 5.2.5. *Let X and Y be Banach spaces, let A be a bounded closed convex subset of X and let $\mathcal{P}$ be one of the following properties of bounded sets:*

(i) *Krein-Milman property;*

(ii) *Radon-Nikodým property;*

(iii) *relative weak compactness;*

(iv) *weak sequential completeness;*

(v) *containing no sequences equivalent to the unit vector basis of ℓ_1;*

(vi) *containing no sequences equivalent to the unit vector basis of c_0;*

(vii) *separability.*

Suppose that there is an injective tauberian operator $T\colon X \longrightarrow Y$. Then $T(A)$ satisfies $\mathcal{P}$ implies A satisfies $\mathcal{P}$.

For a proof of Theorem 5.2.5, we refer to Neidinger's thesis [133].

5.3 Operator ideals and factorization

One of the main applications of the DFJP factorization is a proof of the fact that every weakly compact operator admits a factorization through a reflexive Banach space. Afterwards, the same technique has been used to prove factorization results for other classes of operators.

Definition 5.3.1. We say that an operator ideal $\mathcal{A}$ has the *factorization property* if every operator $T \in \mathcal{A}$ factors through a Banach space in the space ideal $Sp(\mathcal{A})$ of $\mathcal{A}$.

Using the properties of tauberian operators, we are going to show that some operator ideals have the factorization property. In fact, we are going to show that they satisfy a stronger property related to the DFJP factorization.

We will need the following result relating an operator and its factors in the tauberian decomposition given in Theorem 3.2.2.

Proposition 5.3.2. *Let $T \in \mathcal{L}(X, Y)$ and let $T = jUk$ be its tauberian decomposition. For every $n \in \mathbb{N}$, we have*

(i) $\|k\,x\| \leq 2^n\|Tx\| + 2^{-n}\|x\|$ *for every $x \in X$;*

(ii) $j(B_F) \subset 2^n T(B_X) + 2^{-n} B_Y$.

Proof. (i) Recall that k acts from X into $E = \ell_2(X_n)/N_T$ and we have

$$k(x) = (0, \overset{(n-1)}{\ldots}, 0, x, 0, 0, \ldots) + N_T$$

for every $n \in \mathbb{N}$; hence

$$\|k(x)\| \leq \|(0, \overset{(n-1)}{\ldots}, 0, x, 0, 0, \ldots)\| = p_T^n(x) \leq 2^n\|Tx\| + 2^{-n}\|x\|$$

for every $x \in X$ and every $n \in \mathbb{N}$.

(ii) Note that $(y, y, y, \ldots) \in B_F$ implies $q_T^n(y) < 1$ for every $n \in \mathbb{N}$; hence $j(B_F) \subset 2^n T(B_X) + 2^{-n} B_Y$ for every $n \in \mathbb{N}$. □

As a consequence of the previous result, some properties of an operator are inherited by some of its factors in the tauberian decomposition.

Proposition 5.3.3. *Let $\mathcal{A}$ be an operator ideal, let $T \in \mathcal{L}(X, Y)$ and let $T = jUk$ be its tauberian decomposition. Then the following assertions hold:*

(i) *if $\mathcal{A}$ is closed injective and $T \in \mathcal{A}$, then $k \in \mathcal{A}$;*

(ii) *if $\mathcal{A}$ is closed surjective and $T \in \mathcal{A}$, then $j \in \mathcal{A}$.*

Proof. It is a direct consequence of Lemma A.2.6 and Proposition 5.3.2. □

The following concept will be useful in our exposition.

Definition 5.3.4. We say that an operator ideal $\mathcal{A}$ has the *interpolation property* if for every operator $T \in \mathcal{A}$, the identity of the intermediate space in the DFJP factorization of T belongs to $\mathcal{A}$.

Remark 5.3.5. Clearly, an operator ideal $\mathcal{A}$ has the interpolation property if and only if for every $T \in \mathcal{A}$, the intermediate spaces in the tauberian decomposition of T belong to $Sp(\mathcal{A})$.

Let us see some examples of operator ideals with this property.

Theorem 5.3.6. *The following operator ideals have the interpolation property:*

(i) *$\mathcal{F}$, the operators with finite dimensional range;*

(ii) *$\mathcal{W}$, the weakly compact operators;*

(iii) *$\mathcal{R}$, the Rosenthal operators.*

Proof. Let $T \in \mathcal{L}(X, Y)$ and let $T = jUk$ be its tauberian decomposition.

(i) Observe that $k\colon X \longrightarrow E$ has dense range and $j\colon F \longrightarrow Y$ is injective. Thus, if $R(T)$ is finite dimensional, then $\dim(F) = \dim(R(j)) = \dim(R(T))$.

(ii) is Corollary 3.2.3.

(iii) Since $\mathcal{R}$ is surjective and closed, Proposition 5.3.3 implies $j \in \mathcal{R}$; i.e, there is no subspace M of F isomorphic to ℓ_1 such that jJ_M is an isomorphism. Moreover, since j is tauberian and injective, by Corollary 2.2.8, the restriction of j to any subspace isomorphic to ℓ_1 is an isomorphism. Hence F contains no subspace isomorphic to ℓ_1; i.e., $F \in Sp(\mathcal{R})$. □

Proposition 5.3.3 can be applied to study products of operator ideals.

Definition 5.3.7. Let $\mathcal{A}$ and $\mathcal{D}$ be operator ideals. The *product of* $\mathcal{A}$ *and* $\mathcal{D}$ is the class $\mathcal{A} \circ \mathcal{D}$ of all operators T such that $T = AD$, with $A \in \mathcal{A}$ and $D \in \mathcal{D}$.

It is not difficult to check that the product $\mathcal{A} \circ \mathcal{D}$ of two operator ideals $\mathcal{A}$ and $\mathcal{D}$ is an operator ideal.

Corollary 5.3.8. *Let* $\mathcal{A}$ *and* $\mathcal{D}$ *be closed operator ideals. Suppose that* $\mathcal{A}$ *is surjective and* $\mathcal{D}$ *is injective. Then*

$$\mathcal{A} \circ \mathcal{D}(X, Y) = \mathcal{A}(X, Y) \cap \mathcal{D}(X, Y) \ \textit{ for all Banach spaces } X \textit{ and } Y.$$

Proof. Suppose that $T \in \mathcal{A}(X, Y) \cap \mathcal{D}(X, Y)$ and let $T = jUk$ be the tauberian decomposition of T. Applying Proposition 5.3.3, we get $j \in \mathcal{A}$ and $k \in \mathcal{D}$; hence $T \in \mathcal{A} \circ \mathcal{D}$.

The converse implication is trivial. □

Let (X_n) and (Y_n) be sequences of Banach spaces. It is not difficult to see that every operator

$$T \colon \ell_2(X_n) \longrightarrow \ell_2(Y_n)$$

admits a matrix representation with components $T_{mn} \colon X_n \longrightarrow Y_m$; $m, n \in \mathbb{N}$. Moreover, if $\mathcal{A}$ is an operator ideal and $T \in \mathcal{A}$, then $T_{mn} \in \mathcal{A}$ for all $m, n \in \mathbb{N}$.

Definition 5.3.9. We say that an operator ideal $\mathcal{A}$ is ℓ_2*-stable* if an operator $T \colon \ell_2(X_n) \longrightarrow \ell_2(Y_n)$ belongs to $\mathcal{A}$ when all its components T_{mn} belong to $\mathcal{A}$.

Remark 5.3.10. It is easy to show that every ℓ_2-stable operator ideal is closed.

The following result provides a sufficient condition for an operator ideal to have the interpolation property.

Theorem 5.3.11. *Let* $\mathcal{A}$ *be an injective, surjective and* ℓ_2*-stable operator ideal. Then* $\mathcal{A}$ *has the interpolation property.*

Proof. Let $T \in \mathcal{L}(X, Y)$ and let $T = jUk$ be its tauberian decomposition, where $j \colon F \longrightarrow Y$, $k \colon X \longrightarrow E$, F is the diagonal subspace of $\ell_2(Y_n)$ and $E = \ell_2(X_n)/N_T$.

We denote by $J \colon F \longrightarrow \ell_2(Y_n)$ and $Q \colon \ell_2(X_n) \longrightarrow \ell_2(X_n)/N_T$ the embedding and the quotient map, respectively.

$$\begin{array}{ccc}
X & \xrightarrow{\quad T \quad} & Y \\
{\scriptstyle k}\big\downarrow & & \big\uparrow{\scriptstyle j} \\
E & \xrightarrow[\quad U \quad]{} & F \\
{\scriptstyle Q}\big\uparrow & & \big\downarrow{\scriptstyle J} \\
\ell_2(X_n) & \longrightarrow & \ell_2(Y_n)
\end{array}$$

Recall that each space X_n is isomorphic to X and each space Y_n is isomorphic to Y. It is easy to check that the components $(JUQ)_{mn}$ of the matricial representation of the operator $JUQ\colon \ell_2(X_n) \longrightarrow \ell_2(Y_n)$ coincide with T acting from X_n to Y_m. Therefore, the hypothesis implies that $JUQ \in \mathcal{A}$. Since $\mathcal{A}$ is injective and surjective, $U \in \mathcal{A}$; hence E and F belong to $Sp(\mathcal{A})$ because U is an isomorphism. □

As an application, we give further examples of operator ideals with the interpolation property. First, we define some classes of operators.

Definition 5.3.12. Let $T \in \mathcal{L}(X, Y)$.

(i) T is a *Banach-Saks operator* if every bounded sequence (x_n) in X admits a subsequence (x_{n_k}) such that $\big(T(x_{n_1} + \cdots + x_{n_k})/k\big)_{k\in\mathbb{N}}$ is convergent; i.e., (Tx_{n_k}) is Cesàro convergent.

(ii) T is a *decomposing operator* if, for each finite measure space (Ω, Σ, μ) and each operator $S\colon Y \longrightarrow L_\infty(\mu)$, there exists a μ-measurable function $g\colon \Omega \longrightarrow X^*$ such that $(STx)(t) = \langle g(t), x\rangle$ a.e. for every $x \in X$.

We denote by $\mathcal{BS}$ and $\mathcal{D}$ the classes of Banach-Saks operators and decomposing operators, respectively.

Proposition 5.3.13. *The following operator ideals are injective, surjective and ℓ_2-stable:*

(i) $\mathcal{S}$*, the operators with separable range;*

(ii) $\mathcal{W}$*, the weakly compact operators;*

(iii) $\mathcal{BS}$*, the Banach-Saks operators;*

(iv) $\mathcal{D}$*, the decomposing operators.*

Therefore, by Theorem 5.3.11*, they have the interpolation property.*

Proof. (i) Clearly, $\mathcal{S}$ is injective and surjective. Moreover, since the ℓ_2-sum of separable spaces is separable, it is also ℓ_2-stable.

(ii) It is well-known that $\mathcal{W}$ is injective and surjective. Moreover, we can identify $\ell_2(X_n)^{**}$ with $\ell_2(X_n^{**})$. Therefore, the components of the second conjugate

$$T^{**}\colon \ell_2(X_n^{**}) \longrightarrow \ell_2(Y_n^{**})$$

can be identified with the operators T_{mn}^{**}.

From this it is clear that $T \in \mathcal{W}$ if and only if $T_{mn} \in \mathcal{W}$ for all $m, n \in \mathbb{N}$; i.e., $\mathcal{W}$ is ℓ_2-stable.

For the proof of (iii) and (iv), we refer to [100, Theorem 2.3]. □

The duality properties of the tauberian decomposition allow us to derive new examples of operator ideals with the interpolation property from given ones with this property.

Recall that given an operator ideal $\mathcal{A}$, the dual operator is denoted by $\mathcal{A}^d$. Moreover, it is not difficult to show that the expression

$$\mathcal{A}^{co}(X,Y) := \{T \in \mathcal{L}(X,Y)\colon\ T^{co} \in \mathcal{A}\}$$

defines an operator ideal $\mathcal{A}^{co}$.

Theorem 5.3.14. *Let $\mathcal{A}$ be an operator ideal with the interpolation property. Then both $\mathcal{A}^d$ and $\mathcal{A}^{co}$ also have the interpolation property.*

Proof. It is a direct consequence of Theorem 3.2.8: if $T = jUk$ is the tauberian decomposition of T, then $T^* = k^*U^*j^*$ and $T^{co} = j^{co}U^{co}k^{co}$ are the respective tauberian decompositions of T^* and T^{co}. □

Let us see that the converse of the first implication in Theorem 5.3.14 fails.

Definition 5.3.15. We say that an operator $A\colon L_1(\mu) \longrightarrow X$ is *representable* if there exists a measurable function $g\colon \Omega \longrightarrow Y$ so that

$$A(f) = \int_\Omega f(t)g(t)dt \quad \text{for every } f \in L_1(\mu).$$

We say that $T \in \mathcal{L}(X,Y)$ is a *Radon-Nikodým operator* if for every finite measure space (Ω,Σ,μ) and every operator $S\colon L_1(\mu) \longrightarrow X$, the operator TS is representable.

It is proved in [139, 24.2.6] that the class $\mathcal{RN}$ of all Radon-Nikodým operators is an operator ideal.

Remark 5.3.16. The dual operator ideal $\mathcal{RN}^d$ coincides with $\mathcal{D}$, the decomposing operators, which has the interpolation property (see Proposition 5.3.13). However, $\mathcal{RN}$ does not have the interpolation property.

Indeed, for some time the question whether the unconditionally converging operators $\mathcal{U}$ and the Radon-Nikodým operators $\mathcal{RN}$ have the factorization property was open. Observe that $\mathcal{RN}$ is properly contained in $\mathcal{U}$ (See [139]).

It was proved in [65] that there exists a Banach lattice X_2 and a surjective operator $T \in \mathcal{RN}(X_2, c_0)$ such that if T is written as the product of two operators, then one of them preserves a copy of c_0; i.e., it does not belong to $\mathcal{U}$. In particular, both $\mathcal{U}$ and $\mathcal{RN}$ fail the factorization property.

5.4 Uniform factorization for compact sets of operators

Here, given a closed injective and surjective operator ideal $\mathcal{A}$ and a compact subset H of $\mathcal{A}(X,Y)$, we will apply a variation of the construction of the tauberian decomposition of an operator to obtain a uniform factorization for the operators in H. This is stated with more precision in the following result:

Theorem 5.4.1. *Let X and Y be Banach spaces and let $\mathcal{A}$ be a closed, injective and surjective operator ideal. Then, for every compact subset H of $\mathcal{A}(X,Y)$, there is a pair of Banach spaces X_H and Y_H, a pair of operators $j_H \in \mathcal{A}(Y_H, Y)$ and $k_H \in \mathcal{A}(X, X_H)$, and a compact subset H_0 of $\mathcal{A}(X_H, Y_H)$ so that every $T \in H$ may be decomposed as*

$$T = j_H T_0 k_H \quad \text{with } T_0 \in H_0.$$

In order to prove Theorem 5.4.1, we need some notation and auxiliary results. We will give the proof later in this section.

Given a bounded subset H of $\mathcal{L}(X,Y)$, for each $T \in H$ we denote its tauberian decomposition by $T = j_T U_T k_T$.

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle k_T}\big\downarrow & & \big\uparrow{\scriptstyle j_T} \\ E_T & \xrightarrow[\quad U_T \quad]{} & F_T \end{array}$$

Also, given a subset A of a Banach space, we denote by $\overline{\text{aco}}(A)$ the closed absolutely convex hull of A.

We are going to make a construction similar to the one we made in Section 3.2. We let

$$W_H := \overline{\text{aco}}\Big(\bigcup_{T\in H} T(B_X)\Big) \quad \text{and} \quad W_{H^*} := \overline{\text{aco}}\Big(\bigcup_{T\in H} T^*(B_{Y^*})\Big).$$

The gauge $q_n(\cdot)$ of the set $2^n W_H + 2^{-n} B_Y$ is a norm on Y equivalent to the original one.

We write $Y_n := (Y, q_n(\cdot))$ and define

$$Y_H := \{(y_n) \in \ell_2(Y_n) \colon\ y_n = y_1 \text{ for all } n\},$$

the diagonal subspace of $\ell_2(Y_n)$.

Moreover, we denote by $j_H \colon Y_H \longrightarrow Y$ the operator defined by

$$j_H(y, y, y, \ldots) := y \quad \text{for each } (y, y, y, \ldots) \in Y_H.$$

Clearly, j_H is continuous and injective.

Remark 5.4.2. We can identify Y_H with a linear subspace of Y, which is not closed in general. Moreover, since

$$q_n(y) < 2^{-n} \quad \text{for every } y \in W_H,$$

we can identify W_H with a subset of the unit ball of Y_H.

Let us show that, with the identification in the previous remark, the norm of Y and the norm of Y_H are equivalent on W_H.

Lemma 5.4.3. *A sequence* $\big((y_k, y_k, y_k, \ldots)\big) \subset W_H$ *is convergent in* Y_H *if and only if* (y_k) *is convergent in* Y.

Proof. In order to prove the non-trivial implication, suppose that (y_k) converges to $y \in Y$ and let $\varepsilon > 0$.

Note that $y_k - y \in 2W_H$ for every $k \in \mathbb{N}$. We select $n_0 \in \mathbb{N}$ such that $2^{n_0}\varepsilon \geq 2$. Then $y_k - y \in 2^{n_0}\varepsilon\, W_H$ for every $k \in \mathbb{N}$; hence

$$q_n(y_k - y) < 2^{n_0 - n}\varepsilon, \qquad \text{for all } k, n \in \mathbb{N}.$$

We select $k_0 \in \mathbb{N}$ such that $y_k - y \in 2^{-n_0}\varepsilon\, B_Y$ for $k \geq k_0$. Then

$$q_n(y_k - y) < 2^{n_0 - n}\varepsilon \qquad \text{for all } n \in \mathbb{N} \text{ and all } k \geq k_0.$$

Hence, for $k \geq k_0$, we have

$$\begin{aligned} \|(y_k - y, y_k - y, y_k - y, \ldots)\|^2 &= \sum_{n=1}^{n_0} q_n(y_k - y)^2 + \sum_{n=n_0+1}^{\infty} q_n(y_k - y)^2 \\ &\leq \varepsilon^2 \sum_{n=1}^{n_0} 2^{2(n-n_0)} + \varepsilon^2 \sum_{n=n_0+1}^{\infty} 2^{n_0-n} < 4\varepsilon^2, \end{aligned}$$

and we conclude that $\big((y_k, y_k, y_k, \ldots)\big)$ converges in Y_H to $(y, y, y, \ldots)$. □

Now observe that the gauge $q_n^*(\cdot)$ of the set $2^n W_{H^*} + 2^{-n} B_{X^*}$ is a dual norm (since this set is weak*-compact) on X^*, which is equivalent to the original one. We denote by $p_n(\cdot)$ the corresponding predual norm on X.

It is not difficult to check that the norm $p_n(\cdot)$ coincides with the gauge of the following set:

$$\{x \in X \colon\ |\langle f, x\rangle| \leq 1, \ \text{ for every } \ f \in 2^n W_{H^*} + 2^{-n} B_{X^*}\}.$$

Observe that, letting $X_n := (X, p_n(\cdot))$, we can identify $\ell_2(X_n)^* \equiv \ell_2(X_n^*)$ isometrically. Therefore, denoting by M_H the diagonal subspace of $\ell_2(X_n^*)$, we can define

$$X_H := \ell_2(X_n)/(M_H)_\perp$$

and consider the operator $k_H \colon X \longrightarrow X_H$ defined by

$$k_H(x) := (x, 0, 0, 0, \ldots) + (M_H)_\perp \quad x \in X.$$

Remark 5.4.4. As in the case of the space associated with W_H, we can identify

(i) $X_H^* \equiv M_H$ with a subspace of X^*,

(ii) W_{H^*} with a subset of the unit ball of X_H^*.

Moreover, the norms of X^* and X_H^* are equivalent on W_{H^*}.

The following result will allow us to show that the operators j_H and k_H inherit some properties of the operators in H.

Lemma 5.4.5. *Let H be a bounded subset of $\mathcal{L}(X,Y)$ and let $j_H\colon Y_H \longrightarrow Y$ and $k_H\colon X \longrightarrow X_H$ be the operators we have defined. Then*

$$j_H(B_{Y_H}) \subset 2^n W_H + 2^{-n} B_Y \quad \text{for all } n \in \mathbb{N},$$

and

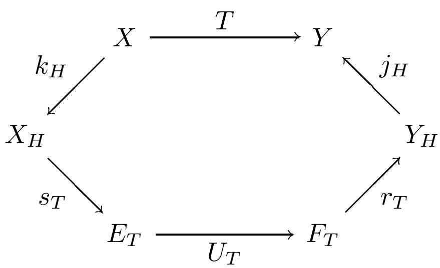
$$k_H(x) \leq \sup_{T\in H} 2^n \|Tx\| + 2^{-n}\|x\| \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}.$$

Proof. It is similar to the proof of Proposition 5.3.2. □

Let us now show that the construction in Lemma 5.4.5 allows us to obtain a uniform tauberian factorization for bounded subsets of $\mathcal{L}(X,Y)$.

Proposition 5.4.6. *Let H be a bounded subset of $\mathcal{L}(X,Y)$. Then there are operators $r_T \in \mathcal{L}(F_T, Y_H)$ and $s_T \in \mathcal{L}(X_H, E_T)$ so that*

$$j_T = j_H r_T \quad \text{and} \quad k_T = s_T k_H \quad \text{for every } T \in H.$$



Proof. We define the operator $r_T\colon F_T \longrightarrow Y_H$ by

$$r_T(y,y,y,\ldots) := (y,y,y,\ldots).$$

Clearly, $r_T \in \mathcal{L}(F_T, Y_H)$ and $\|r_T\| \leq 1$.

We also define the operator $s_T\colon X_H \longrightarrow E_T$ by

$$s_T((x_n) + (M_H)_\perp) := (x_n) + N_T.$$

We refer to Section 3.2 for the definition of N_T.

In a similar manner as we did in the construction of the tauberian decomposition, we can show that the conjugate operator of s_T has the same form as r_T; hence $s_T \in \mathcal{L}(X_H, E_T)$ and $\|s_T\| \leq 1$.

Finally, it is immediate to check that $j_T = j_H r_T$ and $k_T = s_T k_H$ for every $T \in H$. □

Next, we give a technical result that can be considered as a refinement of Lemma A.2.6.

Lemma 5.4.7. *Let $\mathcal{A}$ be a closed operator ideal and let H be a compact subset of $\mathcal{A}(X,Y)$.*

(i) *Suppose that $\mathcal{A}$ is injective, $S \in \mathcal{L}(X,Z)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|Sx\| \le \delta \sup_{T\in H} \|Tx\| + \varepsilon\|x\| \quad \textit{for every } x \in X.$$

Then $S \in \mathcal{A}$.

(ii) *Suppose that $\mathcal{A}$ is surjective, $S \in \mathcal{L}(Z,Y)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$S(B_Z) \subset \delta\, \overline{\mathrm{aco}}\,(\cup_{T\in H} T(B_X)) + \varepsilon B_Y.$$

Then $S \in \mathcal{A}$.

Proof. Let us fix $\varepsilon > 0$. We choose $T_1, \dots, T_n \in H$ so that for each $T \in H$ there exists $i \in \{1, \dots, n\}$ with $\|T - T_i\| < \varepsilon/\delta$.

(i) Clearly, for each $x_0 \in B_X$, there exists $i \in \{1, \dots, n\}$ such that

$$\sup_{T\in H} \|Tx_0\| < \|T_i x_0\| + 2\varepsilon\|x_0\|/\delta.$$

We consider the operator

$$U\colon X \longrightarrow \big(Y \times \overset{(n)}{\cdots} \times Y, \|\cdot\|_\infty\big)$$

defined by $Ux := (T_1x, \dots, T_nx)$. Then, for every $x \in X$,

$$\delta \sup_{T\in H} \|Tx\| < \delta\|Ux\| + 2\varepsilon\|x\|;$$

hence $\|Sx\| \le \delta\|Ux\| + 2\varepsilon\|x\|$ for every $x \in X$. Since $U \in \mathcal{A}$, which is closed and injective, Lemma A.2.6 allows us to conclude $S \in \mathcal{A}$.

(ii) Given $T \in H$, there exists $i \in \{1, \dots, n\}$ such that

$$T(B_X) \subset T_i(B_X) + (\varepsilon/\delta)B_Y.$$

We consider the operator

$$V\colon \big(X \times \overset{(n)}{\cdots} \times X, \|\cdot\|_1\big) \longrightarrow Y$$

defined by $V(x_1, \dots, x_n) := T_1x_1 + \dots + T_nx_n$. Then,

$$\delta \cup_{T\in H} T(B_X) \subset \delta V(B_{X \times \overset{(n)}{\cdots} \times X}) + \varepsilon B_Y.$$

Therefore, $S(B_Z) \subset \delta V(B_{X \times \overset{(n)}{\cdots} \times X}) + 2\varepsilon B_Y$. Since $V \in \mathcal{A}$, which is closed and surjective, Lemma A.2.6 allows us to conclude $S \in \mathcal{A}$. □

The properties of the previous construction and Lemmas 5.4.7 and 5.4.5 allow us to derive the following result:

Proposition 5.4.8. *Let $\mathcal{A}$ be a closed, injective and surjective operator ideal and let H be a compact subset of $\mathcal{A}(X,Y)$. Then the following assertions hold:*

(i) $j_H \in \mathcal{A}(Y_H, Y)$ *and* $k_H \in \mathcal{A}(X, X_H)$.

(ii) *For every* $T \in H$, $r_T \in \mathcal{A}(F_T, Y_H)$ *and* $s_T \in \mathcal{A}(X_H, Y_T)$.

Now we have the tools to prove the main result in this section.

Proof of Theorem 5.4.1. Recall that $\mathcal{A}$ is a closed, injective and surjective operator ideal and H is a compact subset of $\mathcal{A}(X,Y)$.

We consider the spaces X_H and Y_H, and the operators $r_T \in \mathcal{L}(F_T, Y_H)$ and $s_T \in \mathcal{L}(X_H, E_T)$ provided by Proposition 5.4.6. We have

$$T = j_H\, r_T\, U_T\, s_T\, k_H \quad \text{for every } T \in H.$$

We define the set H_0 as the closure of

$$\{T_0 := r_T U_T s_T \colon\ T \in H\} \subset \mathcal{L}(X_H, Y_H).$$

Clearly, $T = j_H T_0 k_H$ for every $T \in H$. Moreover, by Proposition 5.4.8, the three factors j_H, T_0 and k_H belong to $\mathcal{A}$. It remains to show that the set H_0 is compact.

Let (T_n) be a sequence in H. We have to show that $\big(r_{T_n} U_{T_n} s_{T_n}\big)$ has a convergent subsequence.

Since H is compact, we can assume that (T_n) converges to some $T \in \mathcal{A}(X,Y)$. Observe that $T(B_X) \subset W_H$ and $T^*(B_{Y^*}) \subset W_{H^*}$. Hence, by Lemma 5.4.3 and Remark 5.4.4, there exists $T_0 \in \mathcal{L}(X_H, Y_H)$ so that $T = j_H T_0 k_H$.

Now, since

$$T_n = j_H r_{T_n} U_{T_n} s_{T_n} k_H \quad \text{for each } n \in \mathbb{N}$$

the sequence $\big(r_{T_n} U_{T_n} s_{T_n} k_H\big)$ converges to $T_0 k_H$ by Lemma 5.4.3 and $\big(r_{T_n} U_{T_n} s_{T_n}\big)$ converges to T_0 by Remark 5.4.4. □

5.5 Factorization of holomorphic mappings

Some of the results of factorization for operators in certain operator ideals, given in Section 5.3, have been extended to the case of homogeneous polynomials, or to the case of holomorphic maps between Banach spaces, belonging to the corresponding classes of maps. In this section we describe briefly some of these extensions.

We begin by describing in some detail the results of Ryan for weakly compact holomorphic mappings, and afterwards we take a more general point of view. Following [71] and [72], we give a characterization of the holomorphic mappings f between Banach spaces that can be written in the form either $f = T \circ g$ or $f = g \circ T$, where g is another holomorphic mapping and T is an operator belonging to a closed injective or a closed surjective operator ideal. When the operator T belongs to an operator ideal with the factorization property, we derive characterizations in terms of factorization.

Let us introduce some concepts: given Banach spaces X and Y and an integer $n \in \mathbb{N}$, we say that a mapping $P\colon X \longrightarrow Y$ is a *homogeneous polynomial of order* n if there exists a (continuous) n-linear mapping

$$\hat{P} : X \times \overset{(n)}{\cdots} \times X \longrightarrow Y$$

such that $P(x) = \hat{P}(x, \dots, x)$ for every $x \in X$.

We denote by $\mathcal{P}(^n X, Y)$ the set of all homogeneous polynomials of order n from X into Y and we identify the space Y with $\mathcal{P}(^0 X, Y)$, the constant maps from X into Y.

Definition 5.5.1. A mapping $f\colon X \longrightarrow Y$ is said to be *holomorphic* if for each $z \in X$ there are a neighborhood U_z of z and a sequence of polynomials

$$d^n f[z] \in \mathcal{P}(^n X, Y); \quad n \in \mathbb{N} \cup \{0\},$$

so that the series

$$\sum_{n=0}^{\infty} (n!)^{-1} d^n f[z](x - z)$$

converges uniformly to $f(x)$ for $x \in U_z$.

The polynomial $d^n f[z]$ is the *differential of order n of f at z.*

We denote by $\mathcal{H}(X, Y)$ the vector space of holomorphic mappings from X into Y and by $\mathcal{H}_b(X, Y)$ the subspace of all mappings in $\mathcal{H}(X, Y)$ which are bounded on bounded sets.

Obviously, $\mathcal{P}(^n X, Y) \subset \mathcal{H}_b(X, Y)$ for all $n \in \mathbb{N} \cup \{0\}$.

Definition 5.5.2. We say that a mapping $f \in \mathcal{H}(X, Y)$ is *weakly compact* if each $z \in X$ has a neighborhood U_z such that the set $f(U_z)$ is relatively weakly compact in Y.

The following result gives an idea of what it means for a holomorphic mapping to be weakly compact.

Theorem 5.5.3. ([149, Theorem 3.2]) *For a mapping $f \in \mathcal{H}(X, Y)$, the following assertions are equivalent:*

(a) *f is weakly compact;*

(b) *f maps some neighborhood of 0 onto a relatively weakly compact subset of Y;*

(c) *$d^n f[z]$ is a weakly compact polynomial for every $n \in \mathbb{N}$ and every $z \in X$;*

(d) $d^n f[0]$ *is a weakly compact polynomial for every* $n \in \mathbb{N}$.

Let us see that, like the weakly compact operators, the weakly compact holomorphic mappings can be characterized in terms of factorization through a reflexive Banach space.

Theorem 5.5.4. [149, Theorem 3.7] *For a mapping* $f \in \mathcal{H}(X,Y)$*, the following assertions are equivalent:*

(a) f *is weakly compact;*

(b) *there are a reflexive Banach space* E*, an operator* $T \in \mathcal{L}(E,Y)$ *and a holomorphic mapping* $g \in \mathcal{H}(X,E)$ *such that* $f = T \circ g$.

Proof. The proof of the non-trivial part consists of two steps: first, we show that there are a Banach space Z, a weakly compact operator $S \in \mathcal{L}(Z,Y)$ and a holomorphic mapping $h \in \mathcal{H}(X,Z)$ such that $f = S \circ h$. Second, we apply the DFJP factorization of S.

We refer to [149, Proof of Theorem 3.7] for further details. □

Now we shall show that, for operator ideals satisfying certain conditions, we can prove results similar to Theorems 5.5.3 and 5.5.4.

The surjective case

It is well-known that surjective operator ideals correspond to certain families of bounded sets. Let us give a more precise description of this relation.

Let $\mathcal{A}$ be a surjective operator ideal. For every Banach space Y we denote by $\mathcal{C}_{\mathcal{A}}(Y)$ the family of bounded subsets of Y, defined by

$$\mathcal{C}_{\mathcal{A}}(Y) := \{T(B_Z) : \quad Z \text{ Banach space}, T \in \mathcal{A}(Z,Y)\}.$$

It is not difficult to see that an operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathcal{A}$ if and only if $T(B_X)$ belongs to $\mathcal{C}_{\mathcal{A}}(Y)$.

The following concept is inspired by the definition of weakly compact holomorphic mapping.

Definition 5.5.5. Let $\mathcal{A}$ be a closed surjective operator ideal. We say that a holomorphic mapping $f\colon X \longrightarrow Y$ *belongs to* $\mathcal{H}_{\mathcal{A}}(X,Y)$ if each $z \in X$ admits a neighborhood U_z such that $f(U_z)$ belongs to $\mathcal{C}_{\mathcal{A}}(Y)$.

Let us see that the mappings in $\mathcal{H}_{\mathcal{A}}(X,Y)$ have similar properties to those of the weakly compact holomorphic mappings given in Theorem 5.5.3.

Theorem 5.5.6. *Let* $\mathcal{A}$ *be a closed surjective operator ideal. For* $f \in \mathcal{H}(X,Y)$*, the following assertions are equivalent:*

(i) $f \in \mathcal{H}_{\mathcal{A}}(X,Y)$*;*

(ii) f *maps some neighborhood of* 0 *into a set in* $\mathcal{C}_{\mathcal{A}}(Y)$*;*

(iii) $d^n f[z] \in \mathcal{H}_{\mathcal{A}}$ *for every* $n \in \mathbb{N}$ *and every* $z \in X$;

(iv) $d^n f[0] \in \mathcal{H}_{\mathcal{A}}$ *for every* $n \in \mathbb{N}$.

This result is Proposition 5 in [72]. We refer to this paper for a proof.

As a consequence, we can derive the following factorization result, where $\mathcal{H}_b(X,Y)$ denotes the set of all maps in $\mathcal{H}(X,Y)$ which are bounded on bounded sets.

Theorem 5.5.7. *Let* $f \in \mathcal{H}(X,Y)$ *and let* $\mathcal{A}$ *be a closed surjective operator ideal. Then* $f \in \mathcal{H}_{\mathcal{A}}(X,Y)$ *if and only if there exist a Banach space* Z*, an operator* $T \in \mathcal{A}(Z,Y)$ *and a mapping* $g \in \mathcal{H}_b(X,Z)$ *so that* $f = T \circ g$.

The proof can be obtained as an application of Theorem 5.5.6, in a similar manner as Theorem 5.5.4 is derived from Theorem 5.5.3. We refer to [72] and [149] for the details.

As a consequence, we can characterize some classes of holomorphic maps in terms of factorization.

Corollary 5.5.8. *Let* $f \in \mathcal{H}(X,Y)$ *and let* $\mathcal{A}$ *be a closed surjective operator ideal with the factorization property. Then* $f \in \mathcal{H}_{\mathcal{A}}(X,Y)$ *if and only if there exist a Banach space* $Z \in Sp(\mathcal{A})$*, an operator* $T \in \mathcal{L}(Z,Y)$ *and a mapping* $g \in \mathcal{H}(X,Z)$ *so that* $f = T \circ g$.

The injective case

The injective operator ideals can be described in terms of locally convex topologies weaker than the norm topology. Let us give a more precise description of this relation.

Let $\mathcal{A}$ be an injective operator ideal. On every Banach space X we consider the topology $\tau_{\mathcal{A}}$ generated by the seminorms defined by

$$p_T(x) := \|Tx\| \quad x \in X,$$

where Z is a Banach space and $T \in \mathcal{A}(X,Z)$.

Clearly $\tau_{\mathcal{A}}$ is a locally convex topology on the space X and it was proved in [107, Section 3] that, for every Banach space Y,

$$\mathcal{A}(X,Y) = \mathcal{L}_c\big((X,\tau_{\mathcal{A}}),Y\big), \tag{5.1}$$

where $\mathcal{L}_c\big((X,\tau_{\mathcal{A}}),Y\big)$ is the set of all continuous linear maps from the locally convex space $(X,\tau_{\mathcal{A}})$ into Y.

Let $\mathcal{A}$ be a closed injective operator ideal. Then $\tau_{\mathcal{A}}$ is the finest locally convex topology on X that agrees with $\tau_{\mathcal{A}}$ on bounded subsets (see [107, Proposition 4.2]). Therefore, it follows from Formula 5.1 that an operator $T \in \mathcal{L}(X,Y)$ belongs to the operator ideal $\mathcal{A}$ if and only if it is $\tau_{\mathcal{A}}$-continuous on bounded subsets of X.

Let us see that the topology $\tau_{\mathcal{A}}$ allows us to extend some properties of the operators in a closed injective operator ideal to holomorphic mappings.

Definition 5.5.9. Let $\mathcal{A}$ be a closed injective operator ideal. We say that a holomorphic mapping $f\colon X \longrightarrow Y$ *belongs to* $\mathcal{H}^{\mathcal{A}}(X,Y)$ if it is uniformly $\tau_{\mathcal{A}}$-continuous on bounded subsets of X.

First we consider the special case of homogeneous polynomials.

Proposition 5.5.10. *Let $n \in \mathbb{N}$, let $P \in \mathcal{P}(^nX,Y)$ and let $\mathcal{A}$ be a closed injective operator ideal. Then the following assertions are equivalent:*

(a) *P is uniformly $\tau_{\mathcal{A}}$-continuous on bounded subsets of X;*

(b) *there exist a Banach space Z and an operator $T \in \mathcal{A}(X,Z)$ so that*

$$\|P(x)\| \leq \|Tx\|^n \quad \text{for all } x \in X;$$

(c) *there exist a Banach space Z, an operator $T \in \mathcal{A}(X,Z)$ and a polynomial $Q \in \mathcal{P}(^nZ,Y)$ so that $P = Q \circ T$.*

This result is Corollary 5 in [71]. It is derived from the corresponding result for n-linear continuous maps from X into Y.

Theorem 5.5.11. *Let $f \in \mathcal{H}(X,Y)$ and let $\mathcal{A}$ be a closed injective operator ideal. Then $f \in \mathcal{H}^{\mathcal{A}}$ if and only if there exist a Banach space Z, an operator $T \in \mathcal{A}(X,Z)$ and a mapping $g \in \mathcal{H}_b(Z,Y)$ so that $f = g \circ T$.*

For the proof we refer to [71, Proof of Theorem 8].

As in the injective case, from Theorem 5.5.11 we can derive characterizations for some classes of holomorphic maps in terms of factorization.

Corollary 5.5.12. *Let $f \in \mathcal{H}(X,Y)$ and let $\mathcal{A}$ be a closed injective operator ideal with the factorization property. Then $f \in \mathcal{H}^{\mathcal{A}}(X,Y)$ if and only if there exist a Banach space $Z \in Sp(\mathcal{A})$, an operator $T \in \mathcal{L}(X,Z)$ and a mapping $g \in \mathcal{H}_b(Z,Y)$ so that $f = T \circ g$.*

5.6 Approximation properties and Calkin algebras

Let us recall that a Banach space X has the *approximation property (A.P. for short)* if, for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator $T\colon X \longrightarrow X$ such that $\sup_{x\in K} \|Tx - x\| < \varepsilon$.

This concept was introduced in connection with the basis problem (whether every separable Banach space has a Schauder basis). It is easy to see that every X with a Schauder basis has the A.P. Finally, Enflo [59] gave a negative answer to the basis problem by constructing a separable Banach space failing the A.P. However, the A.P. has remained as an inspiring concept in Banach space theory.

Using the isometric variation of the DFJP factorization described in Section 3.3, it is possible to get characterizations of the approximation property.

Definition 5.6.1. We say that a closed subspace M of a Banach space X is an *ideal in* X if $M^{\perp}$ is the kernel of a norm-one projection in X^*.

Recall that $\mathcal{F}(X,Y)$ denotes the finite rank operators in $\mathcal{L}(X,Y)$. Here we denote by $\mathcal{K}_Y$ and $\mathcal{W}_Y$ the families of all absolutely convex subsets of the unit ball B_Y which are compact and weakly compact, respectively.

Given $K \in \mathcal{W}_Y$, the space F_K and the operator $J_K \colon F_K \longrightarrow Y$ are the ones introduced in the isometric variation of the DFJP factorization with $f(a) = 1$ described in Section 3.3.

We say that a net of operators (A_α) in $\mathcal{L}(X,Y)$ is *strongly convergent* to $A \in \mathcal{L}(X,Y)$ if the net $(A_\alpha x)$ converges to Ax for every $x \in X$.

Theorem 5.6.2. *For a Banach space Y, the following assertions are equivalent:*

(a) *Y has the A.P.*

(b) *$\mathcal{F}(Y_K, Y)$ is an ideal in $\mathcal{L}(Y_K, Y)$, for every $K \in \mathcal{W}_Y$.*

(c) *For every $K \in \mathcal{W}_Y$, there is a net (A_α) in $\mathcal{F}(Y_K, Y)$ with $\sup_\alpha \|A_\alpha\| \le \|J_K\|$, which is strongly convergent to J_K.*

(d) *For every $K \in \mathcal{K}_Y$, there is a net (A_α) in $\mathcal{F}(Y_K, Y)$ which is norm convergent to J_K.*

Proof. See [119, Theorem 1.2]. □

As an application of Theorem 5.6.2, we obtain the following characterizations of the A.P. for a Banach space or its dual space:

Theorem 5.6.3. *For a Banach space Y, the following assertions are equivalent:*

(a) *Y has the A.P.*

(b) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$, for every Banach space X.*

(c) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$, for every separable reflexive space X.*

(d) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$, for every closed subspace X of c_0.*

Proof. See [119, Theorem 3.3]. □

Theorem 5.6.4. *For a Banach space X, the following assertions are equivalent:*

(a) *X^* has the A.P.*

(b) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for every Banach space Y.*

(c) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for every separable reflexive space Y.*

(d) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$, for every closed subspace Y of c_0.*

Proof. See [119, Theorem 3.4]. □

Remark 5.6.5. In Theorems 5.6.3 and 5.6.4, we could have added three other equivalent items, replacing $\mathcal{W}(X,Y)$ by $\mathcal{K}(X,Y)$ in (b), (c) and (d).

For every Banach space X, the *Calkin algebra* is defined as the quotient algebra $\mathcal{L}(X)/\mathcal{K}(X)$.

The Calkin algebra was introduced in Fredholm theory because an operator $T \in \mathcal{L}(X)$ is Fredholm if and only if the class of T in $\mathcal{L}(X)/\mathcal{K}(X)$ is invertible.

More generally, for each $T \in \mathcal{L}(X,Y)$ we can consider the so-called *essential norm of* T, defined by

$$\|T\|_e := \operatorname{dist}(T, \mathcal{K}(X,Y));$$

i.e., $\|T\|_e$ is the norm of the class of T in $\mathcal{L}(X,Y)/\mathcal{K}(X,Y)$.

In [118], Lebow and Schechter studied the semi-Fredholm operators by means of the essential norm of an operator and a non-compactness measure for bounded subsets of a Banach space. Its aim was to extend the concepts of operator in Φ_+ or Φ_- (see Section A.1) to elements of Banach algebras.

Definition 5.6.6. Let C be a bounded subset of a Banach space X. The *measure of non-compactness* $\gamma(C)$ *of* C is defined by

$$\gamma(C) := \inf\{\varepsilon > 0\colon\ C \subset D + \varepsilon B_X\},$$

where the infimum is taken over all the compact subsets D of X.

Using the quantity $\gamma(\cdot)$, we can define a measure of non-compactness for operators.

Definition 5.6.7. For an operator $T \in \mathcal{L}(X,Y)$, we define $\gamma(T) := \gamma(T(B_X))$.

Obviously,

$$T \text{ compact} \iff \|T\|_e = 0 \iff \gamma(T) = 0.$$

Lebow and Schechter proved the following result:

Theorem 5.6.8. *For an operator $T \in \mathcal{L}(X,Y)$, the following assertions hold:*

(i) $T \in \Phi_+$ *if and only if there is a constant $C > 0$ such that, for each Banach space Z and each $S \in \mathcal{L}(Z,X)$, $\gamma(S) \leq C \cdot \gamma(TS)$.*

(ii) $T \in \Phi_-$ *if and only if there is a constant $C > 0$ such that, for each Banach space Z and each $S \in \mathcal{L}(Y,Z)$, $\gamma(S) \leq C \cdot \gamma(ST)$.*

For the proof we refer to Theorems 3.1, 4.8 and 5.5 in [118].

It is not difficult to see that $\gamma(T) \leq \|T\|_e$ for each $T \in \mathcal{L}(X,Y)$. Lebow and Schechter raised the question whether $\gamma(\cdot)$ and $\|\cdot\|_e$ are equivalent on $\mathcal{L}(X,Y)$; i.e., whether a constant $C > 0$ exists such that $\|T\|_e \leq C \cdot \gamma(T)$ for each $T \in \mathcal{L}(X,Y)$. They were able to give a positive answer when the second space satisfies the following variant of the A.P.

Definition 5.6.9. We say that a Banach space Y has the *bounded compact approximation property (B.C.A.P., for short)* if there exists $\lambda \geq 1$ such that for every compact subset D of X and every $\varepsilon > 0$, there exists a compact operator $K\colon X \longrightarrow X$ so that

$$\|K - I\| \leq \lambda \quad \text{and} \quad \sup_{x\in D} \|Kx - x\| < \varepsilon.$$

Later, Astala and Tylli [15] showed that this condition is also necessary. Let us state the complete result.

Theorem 5.6.10. *A Banach space Y has the B.C.A.P. if and only if there is a constant $C > 0$ such that, for every Banach space X, every operator $T \in \mathcal{L}(X, Y)$ satisfies $\|T\|_e \leq C \cdot \gamma(T)$.*

Proof. See [118, Theorem 3.6] and [15, Theorem 2.3]. □

Astala and Tylli [16] investigated tauberian operators in terms of the measure of weak non-compactness of a bounded set, the weak essential norm of an operator and a concept of weak approximation property of Banach spaces. Next, we describe their results.

Definition 5.6.11. Let C be a bounded subset of a Banach space X. The *measure of weak non-compactness $\omega(C)$ of C* is defined by

$$\omega(C) := \inf\{\varepsilon > 0\colon\ C \subset D + \varepsilon B_X\},$$

where the infimum is taken over all the weakly compact subsets D of X.

Once again, we can define two measures of weak non-compactness for operators.

Definition 5.6.12. Let $T \in \mathcal{L}(X, Y)$. We define the *weak essential norm $\|T\|_w$ of* T as the norm of the class of T in the quotient space $\mathcal{L}(X, Y)/\mathcal{W}(X, Y)$.

We also define $\omega(T) := \omega(T(B_X))$.

It is easy to show that $\omega(T) \leq \|T\|_w$, for every $T \in \mathcal{L}(X, Y)$.

Definition 5.6.13. Let $\lambda \geq 1$. We say that a Banach space Y has the *λ-weakly compact approximation property (λ-W.A.P. for short)* if for every weakly compact subset $D \subset Y$ and every $\varepsilon > 0$, there exists an operator $K \in \mathcal{W}(Y)$ such that

$$\sup_{y\in D} \|y - Ky\| < \varepsilon \ \text{ and } \ \|I - K\| \leq \lambda.$$

We say that Y has the *W.A.P.* if it has the λ-W.A.P. for some $\lambda \geq 1$.

Let us show that the equivalence between $\omega(\cdot)$ and $\|\cdot\|_w$ in $\mathcal{L}(X, Y)$ characterizes the W.A.P. of the second space.

Theorem 5.6.14. *For a Banach space Y, the following assertions hold:*

(i) *Suppose that Y has the λ-W.A.P. Then $\|T\|_w \le \lambda\omega(T)$ for every Banach space X and every $T \in \mathcal{L}(X,Y)$.*

(ii) *Suppose that Y fails the λ-W.A.P. for each λ. Then there exist a space X and a sequence (T_n) of norm-one operators in $\mathcal{L}(X,Y)$ such that $\omega(T_n) \underset{n}{\longrightarrow} 0$.*

Proof. (i) Suppose that Y has the λ-W.A.P. Given $T \in \mathcal{L}(X,Y)$ and $\delta > \omega(T)$, select a weakly compact subset D of Y such that $T(B_X) \subset D + \delta\, B_Y$.

By the λ-W.A.P. of Y, there exists an operator $K \in \mathcal{W}(Y)$ such that $\sup_{y\in D} \|y - Ky\| < \varepsilon$ and $\|I - K\| \le \lambda$. Now, given $x \in B_X$ we can choose $z \in D$ such that $\|Tx - z\| \le \delta$. Hence

$$\|(T - KT)x\| \le \|(I-K)(Tx - z)\| + \|z - Kz\| \le \lambda\delta + \varepsilon.$$

Since $KT \in \mathcal{W}(X,Y)$, we conclude $\|T\|_w \le \lambda\omega(T)$.

(ii) Suppose that Y fails the λ-W.A.P. for each λ. Then we can find a sequence (D_n) of weakly compact subsets of Y and a sequence (ε_n) of positive numbers so that

$$K \in \mathcal{L}(Y),\ \sup_{y\in D_n} \|y - Ky\| < \varepsilon_n,\ \|I - K\| \le n \Longrightarrow K \notin \mathcal{W}.$$

By the Krein-Smulian theorem [51, page 29], we can assume that each set D_n is closed and absolutely convex. Thus, the expressions

$$\|y\|_n := \inf\{t > 0\colon\ y \in t\big((\varepsilon_n/n)B_Y + D_n\big)\} \quad (n \in \mathbb{N})$$

define norms on Y for which there are numbers $r_n > 0$ so that

$$\frac{\varepsilon_n}{n}\|y\|_n \le \|y\| \le r_n\|y\|_n \quad \text{for each } y \in Y.$$

Let us write $Y_n := (Y, \|\cdot\|_n)$ and let us denote by U_n the identity map from Y_n onto Y. We are going to show that

$$0 < \omega(U_n) \le \frac{\|U_n\|_w}{n} \quad \text{for each } n \in \mathbb{N}.$$

Indeed, on the one hand, if $A \in \mathcal{L}(Y_n, Y)$ satisfies $\|U_n - A\| \le \varepsilon_n$, then $\sup_{x\in D_n} \|x - Ax\| \le \varepsilon_n$ and

$$\|I_Y - AU_n^{-1}\| \le (n/\varepsilon_n)\|U_n - A\| \le n;$$

hence A is not weakly compact. Therefore $\|U_n\|_w \ge \varepsilon_n$.

On the other hand, since Y is not reflexive, $\omega(B_Y) = 1$. Thus,

$$\omega(U_n) = \omega\big((\varepsilon_n/n)B_Y + D_n\big)\big) = \varepsilon_n/n.$$

Now, taking $X := c_0(Y_n)$ and denoting by $P_n\colon X \longrightarrow Y_n$ the natural projections, it is not difficult to see that the operators $T_n := \|U_n\|_w^{-1}U_nP_n$ satisfy the required conditions. □

As we observed earlier on, the first space failing the A.P. was discovered by Enflo in 1973. This example is highly non-trivial and it is very different from the classical Banach spaces. Therefore, it is remarkable that there are classical Banach spaces among the ones failing the W.A.P.

Proposition 5.6.15. *Let Y be a Banach space containing a non-convergent weakly convergent sequence. Suppose that every weakly compact operator $T\colon Y \longrightarrow Y$ is completely continuous. Then Y fails the W.A.P.*

Proof. Let (y_n) be a non-convergent weakly convergent sequence in Y. Clearly, we can assume that (y_n) is a weakly null sequence contained in B_Y and that, for some constant $c > 0$, $\|y_m - y_n\| > c$ for $m \neq n$.

Note that $D := \{y_n : n \in \mathbb{N}\} \cup \{0\}$ is a weakly compact subset in Y and that, for every weakly compact operator $K\colon Y \longrightarrow Y$, the sequence (Ky_n) converges in norm to 0. Thus $\sup_{y\in D} \|y - Ky\| > c$, and we conclude that Y fails the W.A.P. □

Let us recall that a Banach space X is said to have the *Dunford-Pettis property* if $\mathcal{W}(X,Y) \subset \mathcal{C}(X,Y)$ for every Banach space Y. Since $C[0,1]$ and $L_1(0,1)$ are examples of spaces with the Dunford-Pettis property [4, Theorem 5.4.5] and contain sequences equivalent to the unit vector basis of ℓ_2, the following result is an immediate consequence of Proposition 5.6.15.

Corollary 5.6.16. *The spaces $C[0,1]$ and $L_1(0,1)$ fail the W.A.P.*

Additional examples of Banach spaces with the W.A.P. have been described in [137] and [150].

5.7 Notes and Remarks

The equivalence of the Radon-Nikodým property and the Krein-Milman property have been obtained under conditions milder than those considered in Section 5.1, as we can see in [64]. However, the general problem remains open.

Section 5.2 includes many results from Neidinger's thesis [133], as well as results from [134] and [135]. Some related results were proved in [56]. For example, a Banach space X contains an infinite dimensional reflexive subspace if and only if there exists a non-trivial injective tauberian operator form X into a Banach space. See also [12].

In [139] Pietsch studies the factorization of operators belonging to an operator ideal from an abstract point of view. Given a space ideal $\mathbb{A}$ (see Definition A.2.2), he considers the operator ideal $Op(\mathbb{A})$ formed by those operators which factorize through a Banach space in $\mathbb{A}$. The operator ideals $\mathcal{A}$ with the interpolation property can be obtained in this way. Indeed, they satisfy $\mathcal{A} = Op(\mathbb{A})$, where $\mathbb{A}$ is the class of those Banach spaces X for which the identity I_X belongs to $\mathcal{A}$.

Beauzamy [22, 23] proved factorization results for Banach-Saks and other related classes of operators. The operators which factor through a quasi-reflexive Banach space were studied in [10].

In [66], the authors introduced a proper subclass of the Radon-Nikodým operators, the controllable R.N.P. operators, and also the controllable P.C.P. operators. Both classes form operator ideals with the interpolation property: the intermediate space of the DFJP factorization of an operator T has the Radon-Nikodým property, if T is controllable R.N.P, and the point of continuity property, if T is controllable P.C.P.

The uniform factorization for compact sets of operators given in Theorem 5.4.1 was proved in [73], extending some results of [14] and [96]. In [131] and [132], some related results were obtained, relying on the isometric variation of the DFJP factorization.

A Banach space X is said to be *Asplund* if every separable subspace of X has separable dual. This is equivalent to X^* having the Radon-Nikodým property [156]. As a consequence, an operator is decomposing if and only if it factorizes through an Asplund space. For this reason, the decomposing operators are called *Asplund operators* by some authors. In [142], Robertson investigates the Asplund holomorphic mappings, obtaining for them a characterization in terms of factorization.

In [33], holomorphic mappings of bounded type admitting a factorization similar to that in Theorem 5.5.11 are characterized in terms of its derivative.

The quantities $\gamma(\cdot)$ and $w(\cdot)$ are the Hausdorff measure of non-compactness and the De Blasi measure of weak non-compactness.

The results we have given in Section 5.6 for upper semi-Fredholm operators are very different from those given for tauberian operators. For example, the version of Theorem 5.6.8 with $\omega(\cdot)$ instead of $\gamma(\cdot)$ are not valid. Indeed, it is enough to observe that in some cases $\mathcal{T}(X, Y)$ is not open in $\mathcal{L}(X, Y)$. Moreover, the version of Theorem 5.6.8 fails even for tauberian operators with closed range [16]. These facts led Astala and Tylli to introduce some classes of operators $\mathcal{T}_\omega$, $\mathcal{T}_\gamma$ and $\mathcal{T}_c$ that are smaller than $\mathcal{T}$, but have a better behavior.

Definition 5.7.1. Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$.

(a) $T \in \mathcal{T}_\omega$ if $\omega(T(A)) \geq c \cdot \omega(A)$ for some $c > 0$ and each bounded set A in X;

(b) $T \in \mathcal{T}_\gamma$ if $\gamma(T(A)) \geq c \cdot \omega(A)$ for some $c > 0$ and each bounded set A in X;

(c) $T \in \mathcal{T}_c$ if $R(T)$ is closed and $N(T)$ is reflexive.

By Corollary 2.2.5, $T \in \mathcal{L}(X, Y)$ is tauberian if and only if for each bounded set A in X $\omega(T(A)) = 0$ implies $\omega(A) = 0$. Moreover, in Chapter 2 we introduced $\mathcal{W}_l$ as the class of all operators which are left-invertible modulo a weakly compact operator. In [162], the asymptotic behavior of $\omega(T^n)^{1/n}$ for $T \in \mathcal{L}(X)$ was analyzed.

The following result shows that the relations among $\mathcal{T}$, $\mathcal{W}_l$ and the classes introduced in Definition 5.7.1 are not satisfactory. We refer to [16] for the arguments and examples that prove it.

Proposition 5.7.2. *The following inclusions hold and are strict, in general.*

$$\mathcal{W}_l \subset \mathcal{T}_\omega \subset \mathcal{T}_\gamma \subset \mathcal{T} \quad \text{and} \quad \mathcal{T}_c \subset \mathcal{T}_\gamma.$$

Moreover, the classes $\mathcal{T}_c$ and T_ω are not comparable.

In [11], the authors introduced a set-measure of weak non-compactness that have a better behavior under duality than those considered in [16]. It would be interesting to investigate the properties of the semigroup that can be defined using this measure.

Chapter 6

Tauberian-like classes of operators

This chapter is concerned with some classes of operators that, as tauberian operators do, preserve isomorphic properties. It is convenient to introduce these classes axiomatically in order to systematize their study and understand their relationship with operator ideals. This is done in Section 6.1 by introducing the concept of *operator semigroup.* We describe some concrete examples and give some general constructions that provide operator semigroups.

The semigroup $\mathcal{ST}$ of *strongly tauberian operators* is analyzed in Section 6.2. It is a subclass of $\mathcal{T}$: an operator T is strongly tauberian if T^{co} is an isomorphism. We show that $T \in \mathcal{ST}$ implies $T^{**} \in \mathcal{ST}$ and that, when X is a closed subspace of a Banach space L-embedded in its bidual, an operator $T\colon X \longrightarrow Y$ has property (N) if and only if $T \in \mathcal{ST}$. We also study the dual class $\mathcal{ST}^d$ of *strongly cotauberian* operators.

Sections 6.3 and 6.4 are preliminary steps to the study of the supertauberian operators and their dual counterpart, the cosupertauberian operators. Section 6.3 is focused on two types of finite representability for operators: local representability, introduced by Pietsch in order to study ultrapower-stable operator ideals [140], and local supportability, aimed at the study of operator semigroups. The latter notion is a generalization of the finite representability for operators in the sense of Bellenot, whose original purpose was to investigate some topics related to the principle of local reflexivity [26]. We show that, given an operator T and an ultrafilter $\mathfrak{U}$, the operators $T_{\mathfrak{U}}$, T^{**} and T^{co} are both locally representable in and locally supportable by T, and $(T_{\mathfrak{U}})^*$ is both locally representable in and locally supportable by $(T^*)_{\mathfrak{U}}$. As in Section 4.2, all ultrafilters occurring in this chapter are assumed to be countably incomplete.

Section 6.4 studies ultrapower-stable ideals and semigroups, with a particular emphasis on those of the form $\mathcal{A}^{up}$, where $\mathcal{A}$ is an ideal or a semigroup and $\mathcal{A}^{up}$

is the class of the operators in $\mathcal{A}$ whose ultrapowers also belong to $\mathcal{A}$. We provide sufficient conditions for an ultrapower-stable semigroup to be closed under local supportability. In combination with the results of Section 6.3, these conditions are proved to be sufficient to show that T^{**} belongs to $\mathcal{A}$ whenever so does T. Similar results are given for ultrapower-stable operator ideals.

Section 6.5 is devoted to the semigroup $\mathcal{T}^{up}$ of *supertauberian operators.* These operators are characterized in terms of their action on finite ε-triangular sequences and in terms of the kernels of their ultrapowers. Next, these characterizations lead to a perturbative characterization similar to Theorem 2.2.7: an operator T is supertauberian if and only if for every compact operator K, the kernel $N(T+K)$ is super-reflexive. This result yields a pair of sequels. One of them is that $\mathcal{T}^{up}$ coincides with $(\mathcal{W}^{up})_+$. The other one is a characterization for the spaces X for which every tauberian operator $T\colon X \longrightarrow Y$ is supertauberian: each reflexive subspace of X is super-reflexive. This is the case when X is a $L_1(\mu)$ space. So we complete the study of the sets $\mathcal{T}(L_1(\mu), Y)$ begun in Chapter 4.

The class $(\mathcal{T}^{up})^d$ of the cosupertauberian operators is introduced in Section 6.6 in order to study the duality of the class $\mathcal{T}^{up}$. Most of the results in that section evidence a beautiful symmetry between the classes $\mathcal{T}^{up}$ and $(\mathcal{T}^{up})^d$, even richer than that between $\mathcal{ST}$ and $\mathcal{ST}^d$. Moreover, a characterization free of duality is offered: an operator $T\colon X \longrightarrow Y$ is cosupertauberian if and only if for every compact operator $K\colon X \longrightarrow Y$, the cokernel $Y/\overline{R(T+K)}$ is super-reflexive. As an application of cosupertauberian operators, we show that the notions of local representability and local supportability are independent. The arguments rely on the theory of operator semigroups.

6.1 Operator ideals and semigroups

As has been suggested in Chapter 1, the adjective *tauberian* can be assigned in a wide sense to any class of operators that preserves isomorphic properties. Most of these classes fit into the notion of *operator semigroup*, whose definition parallels the notion of operator ideal.

Given a pair of operators $S \in \mathcal{L}(V,W)$ and $T \in \mathcal{L}(X,Y)$, we denote by $S\times T$ the operator

$$S \times T\colon V \times X \longrightarrow W \times Y$$

that maps (v,x) to (Sv,Tx).

Definition 6.1.1. A class $\mathcal{S}$ of operators is said to be an *operator semigroup* (or a *semigroup* for short) if it satisfies the following conditions:

(i) every Fredholm operator belongs to $\mathcal{S}$;

(ii) given $T \in \mathcal{L}(V,W)$ and $U \in \mathcal{L}(X,Y)$, the operator $T\times U$ belongs to $\mathcal{S}$ if and only if both T and U belong to $\mathcal{S}$;

(iii) given $S \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(X, Y)$, if S and T belong to $\mathcal{S}$, then $ST \in \mathcal{S}$.

Remark 6.1.2. Condition (iii) is the reason for the name *semigroup*. The term *operator semigroup* is introduced to emphasize the parallelism with the operator ideals and to avoid confusion with the use of the term *semigroup* in other areas of operator theory.

Note that, for any Banach space X, the identity operator I_X belongs to every operator semigroup, while the null operator 0_X belongs to each operator ideal.

We refer to Appendix A.2 for the definitions of operator ideal and space ideal.

Examples 6.1.3. The following classes are operator semigroups:

(i) Φ, Φ_+ and Φ_-;

(ii) $\mathcal{T}$ and $\mathcal{T}^d$;

(iii) the classes $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$ introduced in Definitions 3.5.6 and 3.5.7.

Below, we give plenty of concrete examples and constructions that provide operator semigroups.

Given a class of operators $\mathcal{C}$, we consider the dual class

$$\mathcal{C}^d := \{T \in \mathcal{L} \colon T^* \in \mathcal{C}\}$$

and the residuum class

$$\mathcal{C}^{co} := \{T \in \mathcal{L} \colon T^{co} \in \mathcal{C}\}.$$

Proposition 6.1.4. *Let $\mathcal{S}$ be an operator semigroup. Then both $\mathcal{S}^d$ and $\mathcal{S}^{co}$ are operator semigroups.*

Proof. The classes $\mathcal{S}^d$ and $\mathcal{S}^{co}$ satisfy the three properties in the definition of semigroup because:

(i) $T \in \Phi$ implies that T^* and T^{co} belong to Φ,

(ii) we can identify $(T \times U)^*$ with $T^* \times U^*$ and $(T \times U)^{co}$ with $T^{co} \times U^{co}$,

(iii) $(ST)^* = T^*S^*$ and $(ST)^{co} = S^{co}T^{co}$. □

Proposition 6.1.5. *Given two operator semigroups $\mathcal{S}_1$ and $\mathcal{S}_2$, the class $\mathcal{S}_1 \cap \mathcal{S}_2$ defined by*

$$(\mathcal{S}_1 \cap \mathcal{S}_2)(X, Y) := \mathcal{S}_1(X, Y) \cap \mathcal{S}_2(X, Y)$$

is an operator semigroup.

The proof is an exercise.

One link between space ideals and operator semigroups is given in the following result. The proof is straightforward.

Proposition 6.1.6. *Given an operator semigroup $\mathcal{S}$, $Sp(\mathcal{S}) := \{X \colon 0_X \in \mathcal{S}\}$ is a space ideal.*

The following procedure to obtain semigroups is relevant because it yields the classes $\mathcal{T}$ and $\mathcal{T}^d$ from the weakly compact operators.

Definition 6.1.7. Given an operator ideal $\mathcal{A}$, we define two classes of operators $\mathcal{A}_+$ and $\mathcal{A}_-$ as follows:

(i) an operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathcal{A}_+$ if for every space W and every operator $S \in \mathcal{L}(W,X)$, S belongs to $\mathcal{A}(W,X)$ if $TS \in \mathcal{A}(W,Y)$:

$$\mathcal{A}_+ := \{T \in \mathcal{L} \colon S \in \mathcal{L} \text{ and } TS \in \mathcal{A} \Rightarrow S \in \mathcal{A}\}.$$

(ii) an operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathcal{A}_-$ if for every space Z and every operator $S \in \mathcal{L}(Y,Z)$, S belongs to $\mathcal{A}$ if $ST \in \mathcal{A}$:

$$\mathcal{A}_- := \{T \in \mathcal{L} \colon S \in \mathcal{L} \text{ and } ST \in \mathcal{A} \Rightarrow S \in \mathcal{A}\}.$$

Proposition 6.1.8. *For every operator ideal $\mathcal{A}$, the classes $\mathcal{A}_+$ and $\mathcal{A}_-$ are operator semigroups.*

The proof is a simple exercise.

Remark 6.1.9. Obviously, Propositions 2.2.9 and 3.1.22 show respectively that $\mathcal{T}$ equals $\mathcal{W}_+$ and $\mathcal{T}^d$ equals $\mathcal{W}_-$.

Classical theorems in Fredholm theory show that the class Φ_+ equals the semigroup $\mathcal{K}_+$ associated with the ideal of compact operators, and Φ_- equals $\mathcal{K}_-$ (see Section A.1).

Let $\mathcal{A}$ be one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ and $\mathcal{U}$ introduced in Definition 3.5.1. Then Proposition 3.5.16 says that the classes $\mathcal{A}_+$ and $\mathcal{A}^d{}_-$, introduced in Definitions 3.5.6 and 3.5.7, coincide with the operator semigroups associated with $\mathcal{A}$ and $\mathcal{A}^d$ according to Definition 6.1.7.

In order to classify the different types of semigroups, it is necessary to introduce some definitions.

Definition 6.1.10. Let $\mathcal{S}$ be an operator semigroup:

(i) $\mathcal{S}$ is said to be *injective* if it contains all upper semi-Fredholm operators,

(ii) $\mathcal{S}$ is said to be *surjective* if it contains all lower semi-Fredholm operators.

Proposition 6.1.11. *Let $\mathcal{S}$ be an operator semigroup.*

(i) *if $\mathcal{S}$ is injective, then $\mathcal{S}^d$ is surjective and $\mathcal{S}^{co}$ is injective;*

(ii) *if $\mathcal{S}$ is surjective, then $\mathcal{S}^d$ is injective and $\mathcal{S}^{co}$ is surjective.*

Proof. It follows from the classical identities $\Phi_- = (\Phi_+)^d$ and $\Phi_+ = (\Phi_-)^d$. □

Note that Definition A.2.5 can be reformulated by saying that an operator ideal $\mathcal{A}$ is injective if and only if $J_M \in \mathcal{A}_+$ for every closed subspace M of a Banach space X, and that $\mathcal{A}$ is surjective if and only if $Q_N \in \mathcal{A}_-$ for every closed subspace N of a Banach space X. This observation is crucial in the proof of the following result. Recall that, for every $T \in \mathcal{L}(X,Y)$, we denote by $\overline{T}\colon X \longrightarrow \overline{R(T)}$ the operator that maps $x \in X$ to Tx.

Proposition 6.1.12. *For every operator ideal $\mathcal{A}$, the following statements hold:*

(i) *$\mathcal{A}$ is injective if and only if $\mathcal{A}_+$ is injective;*

(ii) *$\mathcal{A}$ is surjective if and only if $\mathcal{A}_-$ is surjective.*

Proof. (i) Suppose that $\mathcal{A}$ is injective. Since every $T \in \Phi_+$ can be decomposed as $T = J_{R(T)}\overline{T}$, with $\overline{T} \in \Phi(X, R(T))$, it follows that $T \in \mathcal{A}_+$.

For the converse, it is enough to observe that $J_M \in \Phi_+$ for every closed subspace M of a Banach space X.

The proof of statement (ii) is similar: if $T \in \Phi_-$, then $T = \widetilde{T}Q_{N(T)}$ with $\widetilde{T} \in \Phi(X/N(T), Y)$. □

Definition 6.1.13. Let $\mathcal{S}$ be an operator semigroup.

(i) $\mathcal{S}$ is said to be *left-stable* if $ST \in \mathcal{S}$ implies $T \in \mathcal{S}$;

(ii) $\mathcal{S}$ is said to be *right-stable* if $TS \in \mathcal{S}$ implies $T \in \mathcal{S}$.

Proposition 6.1.14. *For every operator ideal $\mathcal{A}$, the semigroup $\mathcal{A}_+$ is left-stable and $\mathcal{A}_-$ is right-stable.*

Proof. In order to prove that $\mathcal{A}_+$ is left-stable, let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,Z)$ be a pair of operators such that $ST \in \mathcal{A}_+$. Let $U \in \mathcal{L}(W,X)$ be a third operator such that $TU \in \mathcal{A}$. Thus $STU \in \mathcal{A}(W,Z)$, and by definition of $\mathcal{A}_+$, the operator U belongs to $\mathcal{A}$, which proves that $T \in \mathcal{A}_+$, and therefore, $\mathcal{A}_+$ is left-stable.

A similar argument shows that $\mathcal{A}_-$ is right-stable. □

Since both operator ideals $\mathcal{W}$ and $\mathcal{K}$ are injective and surjective, the semigroups $\mathcal{W}_+$ and $\mathcal{K}_+$ are injective and left-stable, while $\mathcal{W}_-$ and $\mathcal{K}_-$ are surjective and right-stable.

Similarly, if $\mathcal{A}$ is one of the operator ideals $\mathcal{R}$, $\mathcal{C}$, $\mathcal{WC}$ or $\mathcal{U}$, then $\mathcal{A}_+$ is injective and left-stable, while ${\mathcal{A}^d}_-$ is surjective and right-stable.

Remark 6.1.15. Studying semigroups that are both injective and right-stable does not make much sense. Actually, since the only interesting semigroups are those whose elements preserve some isomorphic property, it is natural to call *trivial* any semigroup that contains the null operator 0_X for every Banach space X.

It is immediate that if a semigroup $\mathcal{S}$ is injective and right-stable, then $\mathcal{S}$ is trivial. Similarly, if $\mathcal{S}$ is surjective and left-stable, then $\mathcal{S}$ is trivial.

Definition 6.1.16. Let $\mathcal{S}$ be an operator semigroup:

(i) $\mathcal{S}$ is said to be an *upper* semigroup if it is injective and left-stable;

(ii) $\mathcal{S}$ is said to be a *lower* semigroup if it is surjective and right-stable.

Just after Proposition 6.1.14, we have given several examples of upper semigroups and lower semigroups.

Proposition 6.1.17. *Let $\mathcal{S}$ be an operator semigroup.*

(i) *If $\mathcal{S}$ is upper, then $\mathcal{S}^d$ is lower and $\mathcal{S}^{co}$ is upper;*

(ii) *If $\mathcal{S}$ is lower, then $\mathcal{S}^d$ is upper and $\mathcal{S}^{co}$ is lower.*

Proof. (i) We have seen before that $\mathcal{S}^d$ and $\mathcal{S}^{co}$ are operator semigroups. Moreover, Proposition 6.1.11 relates the injectivity or surjectivity of $\mathcal{S}$ with those of $\mathcal{S}^d$ and $\mathcal{S}^{co}$.

Let $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(Y,Z)$ such that $TS \in \mathcal{S}^d$. Then $S^*T^* \in \mathcal{S}$ and $\mathcal{S}$ left-stable implies $T^* \in \mathcal{S}$, hence $T \in \mathcal{S}^d$. We conclude that $\mathcal{S}^d$ is right-stable. Similarly, we can show that $\mathcal{S}^{co}$ is left-stable.

(ii) The proof is similar to that of part (i). □

Let us introduce another procedure for associating semigroups to an operator ideal.

Definition 6.1.18. Given an operator ideal $\mathcal{A}$, we define two classes of operators $\mathcal{A}_l$ and $\mathcal{A}_r$ as follows:

(i) an operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathcal{A}_l$ if there exists $A \in \mathcal{L}(Y,X)$ such that $I_X - AT \in \mathcal{A}$;

(ii) an operator $T \in \mathcal{L}(X,Y)$ belongs to $\mathcal{A}_r$ if there exists $B \in \mathcal{L}(Y,X)$ such that $I_Y - TB \in \mathcal{A}$.

In other words, $\mathcal{A}_l$ is the class of operators that are left-invertible modulo $\mathcal{A}$ and $\mathcal{A}_r$ is the class of operators that are right-invertible modulo $\mathcal{A}$.

The main structural properties of those classes of operators invertible modulo an operator ideal are described in the following results:

Proposition 6.1.19. *Given an operator ideal $\mathcal{A}$, the following statements hold:*

(i) *the class $\mathcal{A}_l$ is a left-stable operator semigroup contained in $\mathcal{A}_+$;*

(ii) *the class $\mathcal{A}_r$ is a right-stable operator semigroup contained in $\mathcal{A}_-$.*

Proof. We can prove that $\mathcal{A}_l$ is a left-stable semigroup and $\mathcal{A}_r$ is a right-stable semigroup in a similar way as we did for $\mathcal{A}_+$ and $\mathcal{A}_-$.

To prove the inclusions, let $T \in \mathcal{A}_l(X,Y)$ and choose $S \in \mathcal{L}(W,X)$ such that $TS \in \mathcal{A}(W,Y)$. Since there exists $A \in \mathcal{L}(Y,X)$ satisfying $I_X - AT \in \mathcal{A}$, we have $S - ATS \in \mathcal{A}$, hence $S \in \mathcal{A}$, which proves that $\mathcal{A}_l \subset \mathcal{A}_+$.

The proof of the inclusion $\mathcal{A}_r \subset \mathcal{A}_-$ is similar. □

Remark 6.1.20. In general, the operator semigroup $\mathcal{A}_l$, unlike $\mathcal{A}_+$, is not injective when so is $\mathcal{A}$: for every closed subspace M of a Banach space X, the operator $J_M \in \mathcal{A}_+$. However, $J_M \in \mathcal{K}_l$ if and only if M is complemented in X. This is a consequence of a result of Yood [176] that characterizes the operators in $\mathcal{K}_l$ as those operators in Φ_+ with complemented range.

Similarly $\mathcal{A}_r$ is not always surjective: $\mathcal{K}_r$ consists of all operators in Φ_- with complemented kernel.

The following example [16] shows that $\mathcal{W}_l$ is not injective. In particular, $\mathcal{W}_l$ is a proper subclass of $\mathcal{W}_+$. Let us recall that the Hardy space $H_1(\mathbb{T})$ can be identified with the closed subspace of the space of complex-valued functions $L_1(\mathbb{T})$ generated by $\{e^{int}\colon n \geq 0\}$, where $\mathbb{T} := \{z \in \mathbb{C}\colon |z| = 1\}$.

Example 6.1.21. The embedding operator $J\colon H_1(\mathbb{T}) \longrightarrow L_1(\mathbb{T})$ does not belong to the semigroup $\mathcal{W}_l$.

Proof. For simplicity, we will prove the result for the embedding of the one-codimensional closed subspace

$$H_1^0(\mathbb{T}) := \{zf(z)\colon f \in H_1(\mathbb{T})\}.$$

Suppose that $J \in \mathcal{W}_l$ and select an operator $R\colon L_1(\mathbb{T}) \longrightarrow H_1^0(\mathbb{T})$ such that with $RJ = I + V$, where I is the identity on $H_1^0(\mathbb{T})$ and V is weakly compact.

Let $T_s\big(f(e^{i\theta})\big) := f(e^{i(\theta+s)})$ be the translation by s. As J commutes with T_s, the expression

$$Q(f) := \frac{1}{2\pi}\int_0^{2\pi} T_s R T_{-s} f \, ds \quad \text{for } f \in L_1(\mathbb{T})$$

defines an operator from $L_1(\mathbb{T})$ into $H_1^0(\mathbb{T})$ that satisfies $QJ = I + W$, where W is a weakly compact operator given by

$$W(f) = \frac{1}{2\pi}\int_0^{2\pi} T_s V T_{-s} f \, ds \quad \text{for } f \in H_1^0(\mathbb{T}).$$

For each integer $n \in \mathbb{Z}$, we set $e_n(z) := z^n$ $(z \in \mathbb{T})$. Since Q commutes with T_s, $Q(e_n) = c_n e_n$, where $c_n = 0$ for $n \leq 0$ and $|c_n| \leq \|Q\|$ otherwise. Consider the distribution $g \in \mathcal{D}'(\mathbb{T})$ with Fourier coefficients $\hat{g}(n) = c_n$. Let us show that $g \in L_1(\mathbb{T})$.

Indeed, denoting by P_r the Poisson kernel,

$$\|g * P_r\|_1 = \|QP_r\|_1 \leq \|Q\| \quad \text{for all } r \in (0,1).$$

Hence the standard duality argument shows that g is a measure and, since the negative Fourier coefficients of g are null, $g \in L_1(\mathbb{T})$ by the F. and M. Riesz theorem.

Thus, the operator $Q\colon L_1(\mathbb{T}) \longrightarrow H_1^0(\mathbb{T})$ has the form $Q(f) = g * f$, with $g \in L_1(\mathbb{T})$. Since for every measurable subset E of $\mathbb{T}$ and every $f \in B_{L_1(\mathbb{T})}$,

$$\int_E |g * f| ds \leq \sup\Big\{\int_F |g| ds\colon m(F) = m(E)\Big\},$$

where $m(E)$ is the measure of E, $Q\big(B_{L_1(\mathbb{T})}\big)$ is equi-integrable. Hence Q is a weakly compact operator and therefore so is $I = QJ - W$, hence $H_1^0(\mathbb{T})$ is reflexive, which gives a contradiction. $\square$

A second example showing that $\mathcal{W}_l$ is a proper subclass of $\mathcal{W}_+$ is provided by a construction of Bourgain:

Example 6.1.22. $\mathcal{W}_+(\ell_1) \not\subset \mathcal{W}_l(\ell_1)$.

Proof. Let $J\colon \ell_1 \longrightarrow \ell_1$ an isomorphism such that $R(J)$ is not complemented in ℓ_1 (see [40]). Then $J \in \mathcal{K}_+(\ell_1) \setminus \mathcal{K}_l(\ell_1)$. But weakly convergent sequences in ℓ_1 are norm-convergent, so $\mathcal{K}(\ell_1) = \mathcal{W}(\ell_1)$. Thus $\mathcal{K}_+(\ell_1) = \mathcal{W}_+(\ell_1)$ and $\mathcal{K}_l(\ell_1) = \mathcal{W}_l(\ell_1)$, and therefore, $J \in \mathcal{W}_+(\ell_1) \setminus \mathcal{W}_l(\ell_1)$. □

Some other properties of the semigroups $\mathcal{K}_l$ and $\mathcal{W}_l$ used in Example 2.1.17 are valid in general for $\mathcal{A}_l$ and $\mathcal{A}_r$.

Proposition 6.1.23. *Given an operator ideal $\mathcal{A}$ and an operator $K \in \mathcal{A}(X,Y)$, the following statements hold:*

(i) *for every $T \in \mathcal{A}_l(X,Y)$, $T+K$ belongs to $\mathcal{A}_l(X,Y)$;*

(ii) *for every $S \in \mathcal{A}_r(X,Y)$, $S+K$ belongs to $\mathcal{A}_r(X,Y)$.*

Proof. We only prove part (i) because the proof of (ii) is analogous.

Let $A \in \mathcal{L}(Y,X)$ such that $C := I_X - AT \in \mathcal{A}(X)$. Thus, as $AK \in \mathcal{A}$, we get $I_X - A(T+K) = I_X - AT - AK \in \mathcal{A}(X)$, hence $T + K \in \mathcal{A}_l$. □

We say that a semigroup $\mathcal{S}$ is *open* when $\mathcal{S}(X,Y)$ is an open subset of $\mathcal{L}(X,Y)$ for every pair of spaces X and Y.

Proposition 6.1.24. *For every operator ideal $\mathcal{A}$, the semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ are open.*

Proof. Let $T \in \mathcal{A}_l(X,Y)$ and consider an operator $A \in \mathcal{L}(Y,X)$ so that $K := AT - I_X \in \mathcal{A}$. We claim that for every $S \in \mathcal{L}(X,Y)$ with $\|S\| < \|A\|^{-1}$, $T+S \in \mathcal{A}_l$.

Indeed, $I_X + AS$ is invertible because $\|AS\| < 1$. Thus Proposition 6.1.23 yields $A(T+S) = (I_X + AS) + K \in \mathcal{A}_l$. Since $\mathcal{A}_l$ is left-stable, we conclude $T + S \in \mathcal{A}_l$.

A similar argument shows that $\mathcal{A}_r$ is open. □

One important problem in operator theory is the identification of the perturbation class of the semi-Fredholm operators. Let us introduce this notion for operator semigroups.

Definition 6.1.25. Given an operator semigroup $\mathcal{S}$, the *perturbation class* $P\mathcal{S}$ of $\mathcal{S}$ is defined by its components:

$$P\mathcal{S}(X,Y) := \{K \in \mathcal{L}(X,Y) \colon T + K \in \mathcal{S}(X,Y) \quad \text{for all } T \in \mathcal{S}(X,Y)\},$$

where X and Y are Banach spaces for which $\mathcal{S}(X,Y)$ is non-empty.

Obviously, $P\mathcal{S}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$.

Remark 6.1.26. For every semigroup $\mathcal{S}$, the components $\mathcal{S}(X)$ are non-empty because the identity I_X belongs to $\mathcal{S}(X)$. It is not difficult to show that $P\mathcal{S}(X)$ is a bilateral ideal of $\mathcal{L}(X)$ (see [118]).

Definition 6.1.27. An operator $K \in \mathcal{L}(X,Y)$ is said to be *inessential* if $I_X - AK$ is a Fredholm operator for every $A \in \mathcal{L}(Y,X)$.

We denote by $\mathcal{I}$ the class of inessential operators.

Remark 6.1.28. The following two examples prevent us from defining $P\mathcal{S}(X,Y)$ as $\mathcal{L}(X,Y)$ when the component $\mathcal{S}(X,Y)$ is empty:

(i) When it can be defined, $P\Phi_+(X,Y)$ is contained in $\mathcal{I}(X,Y)$, the inessential operators. The same can be said for $P\Phi_-(X,Y)$ [118].

(ii) For $1 < p < \infty$, $p \neq 2$, both sets $\Phi_+(L_p(0,1),\ell_p)$ and $\Phi_-(\ell_p, L_p(0,1))$ are empty because $L_p(0,1)$ contains complemented subspaces isomorphic to ℓ_2. However, $\mathcal{I}(L_p(0,1),\ell_p) \neq \mathcal{L}(L_p(0,1),\ell_p)$ because $L_p(0,1)$ contains complemented subspaces isomorphic to ℓ_p.

In general, the perturbation classes for $\mathcal{A}_+$ or $\mathcal{A}_-$ are not well-known, even in the case $\mathcal{A} = \mathcal{K}$ [70]. However, some of their components are known. For example, $P\mathcal{W}_+(L_1(\mu),Y)$ was identified in Theorem 4.3.5 as the weakly precompact operators from $L_1(\mu)$ into Y.

Next we will show that the perturbation class for both $\mathcal{A}_l$ and $\mathcal{A}_r$ admit a good description.

Definition 6.1.29. Given an operator ideal $\mathcal{A}$, its *radical* $\mathcal{A}^{\mathrm{rad}}$ is the class of operators whose components are

$$\mathcal{A}^{\mathrm{rad}}(X,Y) := \{K \in \mathcal{L}(X,Y)\colon \forall\, S \in \mathcal{L}(Y,X), \\ \exists U \in \mathcal{L}(X) \quad \text{such that} \quad I_X - U(I_X - SK) \in \mathcal{A}\}.$$

Remark 6.1.30. Pietsch proved that for every operator ideal $\mathcal{A}$, the class $\mathcal{A}^{\mathrm{rad}}$ is a closed operator ideal that contains $\mathcal{A}$ [139].

Remark 6.1.31. In Definition 6.1.29, the expression

$$I_X - U(I_X - SK) \in \mathcal{A}$$

can be replaced by

$$I_X - (I_X - SK)U \in \mathcal{A}.$$

Proof. In fact, if $K \in \mathcal{A}^{\mathrm{rad}}$ and $L_1 := I_X - U(I_X - SK) \in \mathcal{A}$, then $I_X - U \in \mathcal{A}^{\mathrm{rad}}$. Thus there exists $W \in \mathcal{L}(X)$ so that $I_X - WU \in \mathcal{A}$, and

$$(I_X - SK)U = WU(I_X - SK)U - L_2 = W(I_X - L_1)U - L_2 = I_X - L_3,$$

where L_2 and L_3 belong to $\mathcal{A}$, hence $I_X - (I_X - SK)U \in \mathcal{A}$.

The converse implication admits a similar proof. □

The semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ allow us to give a simpler description of the radical of an operator ideal.

Proposition 6.1.32. *Let $\mathcal{A}$ be an operator ideal. For every $T \in \mathcal{L}(X,Y)$, the following statements are equivalent:*

(a) $T \in \mathcal{A}^{\mathrm{rad}}(X,Y)$;

(b) *for every* $S \in \mathcal{L}(Y,X)$, $I_X - ST \in \mathcal{A}_l(X)$;

(c) *for every* $S \in \mathcal{L}(Y,X)$, $I_Y - TS \in \mathcal{A}_l(Y)$;

(d) *for every* $S \in \mathcal{L}(Y,X)$, $I_X - ST \in \mathcal{A}_r(X)$;

(e) *for every* $S \in \mathcal{L}(Y,X)$, $I_Y - TS \in \mathcal{A}_r(Y)$.

Proof. (a)⇒(b) Assume $T \in \mathcal{A}^{\mathrm{rad}}(X,Y)$, take $S \in \mathcal{L}(Y,X)$ and select $U \in \mathcal{L}(X)$ such that $K_1 := I_X - U(I_X - ST) \in \mathcal{A}$. Thus $U(I_X - ST) = I_X - K_1 \in \mathcal{A}_l$, and as $\mathcal{A}_l$ is left-stable, $I_X - ST \in \mathcal{A}_l$.

(b)⇒(a) If $I_X - ST \in \mathcal{A}_l$ for all $S \in \mathcal{L}(Y,X)$, then any left-inverse modulo $\mathcal{A}$ of $I_X - ST$ can be taken as U in Definition 6.1.29.

The proof of (a)⇔(d) is similar, with the help of Remark 6.1.31.

The equivalences (b)⇔(c) and (d)⇔(e) can be proved with the following argument: if $U \in \mathcal{L}(X)$ is a left (respectively right) inverse of $I_X - ST$ modulo $\mathcal{A}$, then $I_Y + TUS$ is a left (respectively right) inverse of $I_Y - TS$ modulo $\mathcal{A}$. □

Theorem 6.1.33. *Let $\mathcal{A}$ be an operator ideal, and let $\mathcal{S}$ be any of the semigroups $\mathcal{A}_l$, $\mathcal{A}_r$ or $\mathcal{A}_l \cap \mathcal{A}_r$. Suppose that $\mathcal{S}(X,Y)$ is non-empty. Then $P\mathcal{S}(X,Y) = \mathcal{A}^{\mathrm{rad}}(X,Y)$.*

Proof. We only prove the result for $\mathcal{S} = \mathcal{A}_l$. The other cases are similar.

Let $K \in \mathcal{A}^{\mathrm{rad}}(X,Y)$ and $T \in \mathcal{A}_l(X,Y)$. Select $A \in \mathcal{L}(Y,X)$ so that

$$D_1 := I_X - AT \in \mathcal{A}(X).$$

For the operator $-A \in \mathcal{L}(Y,X)$, the definition of $\mathcal{A}^{\mathrm{rad}}$ gives another operator $U \in \mathcal{L}(X)$ such that $D_2 := I_X - U(I_X + AK) \in \mathcal{A}$. Since $\mathcal{A} \subset P\mathcal{A}_l$,

$$UA(T+K) = U(I_X - D_1 + AK) = I_X - D_2 - UD_1 \in \mathcal{A}_l.$$

Hence $T + K \in \mathcal{A}_l$, and $\mathcal{A}^{\mathrm{rad}}(X,Y) \subset P\mathcal{A}_l(X,Y)$ is proved.

For the converse inclusion, we first show that

$$K \in P\mathcal{A}_l(X,Y) \text{ and } A \in \mathcal{L}(Y) \Rightarrow AK \in P\mathcal{A}_l(X,Y). \tag{6.1}$$

Since each $A \in \mathcal{L}(Y)$ can be written as the sum of two invertible operators, it is enough to prove the result for A invertible. In this case, $A^{-1}U \in \mathcal{A}_l$ for every $U \in \mathcal{A}_l(X,Y)$. Thus $U + AK = A(A^{-1}U + K) \in \mathcal{A}_l$ and we conclude $AK \in P\mathcal{A}_l$.

Now, let $K \in \mathcal{L}(X,Y)$ such that $K \notin \mathcal{A}^{\mathrm{rad}}$. By Proposition 6.1.32, there exists $A \in \mathcal{L}(Y,X)$ such that $I_X - AK \notin \mathcal{A}_l(X)$.

Let $U \in \mathcal{A}_l(X,Y)$. Then $U(I_X - AK) = U - (UA)K \notin \mathcal{A}_l(X,Y)$. Therefore $(UA)K \notin P\mathcal{A}_l$, and (6.1) implies $K \notin P\mathcal{A}_l(X)$. □

Remark 6.1.34. Since $\Phi = \mathcal{K}_l \cap \mathcal{K}_r$, it is an immediate consequence of Theorem 6.1.33 that the radical $\mathcal{K}^{\mathrm{rad}}$ coincides with $\mathcal{I}$, the inessential operators. Therefore, $\mathcal{I}$ is an operator ideal.

6.2 Strongly tauberian operators

Rosenthal named *strongly tauberian* the operators T for which T^{co} is an isomorphism. He proved that the class of strongly tauberian operators is open and that T is strongly tauberian if and only if so is T^{**}.

In this section, we study the strongly tauberian operators and their dual counterpart, the strongly cotauberian operators. We show that the corresponding classes form an upper semigroup and a lower semigroup. Moreover, if $T\colon X \longrightarrow Y$ has property (N) and X is a closed subspace of a space which is L-embedded in its bidual, then T is strongly tauberian. Note that $L_1(\mu)$-spaces are L-embedded in their bidual spaces.

In the previous sections, we have identified a Banach space X with its canonical copy $J_X(X)$ contained by X^{**} in order to avoid cumbersome notation. However, the profusion of bidual, third dual and fourth dual spaces in this section asks for a more formal notation. Thus, the canonical copy of X in X^{**} is denoted by $J_X(X)$ throughout this section.

Note that the third dual and the fourth dual of X admit the following decompositions:

$$X^{*(3)} = J_{X^*}(X^*) \oplus J_X(X)^{\perp}, \tag{6.2}$$

$$X^{*(4)} = J_{X^{**}}(X^{**}) \oplus J_{X^*}(X^*)^{\perp}. \tag{6.3}$$

Moreover, given an operator $T\colon X \longrightarrow Y$, the decompositions of the third dual spaces reduce $T^{*(3)}$:

$$T^{*(3)}\big(J_{Y^*}(Y^*)\big) \subset J_{X^*}(X^*) \quad \text{and} \quad T^{*(3)}\big(J_Y(Y)^{\perp}\big) \subset J_X(X)^{\perp} \tag{6.4}$$

and the decompositions of the fourth dual spaces reduce $T^{*(4)}$:

$$T^{*(4)}\big(J_{X^{**}}(X^{**})\big) \subset J_{Y^{**}}(Y^{**}) \quad \text{and} \quad T^{*(4)}\big(J_{X^*}(X^*)^{\perp}\big) \subset J_{Y^*}(Y^*)^{\perp}. \tag{6.5}$$

Note that the isometry ${J_X}^{**}\colon X^{**} \longrightarrow X^{*(4)}$ maps X^{**} onto $J_X(X)^{\perp\perp}$, but the subspaces $J_{X^{**}}(X^{**})$ and $J_X(X)^{\perp\perp}$ are placed in different positions in $X^{*(4)}$. Indeed,

$$J_{X^{**}}(X^{**}) \cap J_X(X)^{\perp\perp} = J_{X^{**}} \circ J_X(X).$$

Definition 6.2.1. An operator $T\colon X \longrightarrow Y$ is said to be *strongly tauberian* if the operator $T^{co}\colon X^{**}/J_X(X) \longrightarrow Y^{**}/J_Y(Y)$ is an isomorphism.

The class of all strongly tauberian operators is denoted by $\mathcal{ST}$.

Remark 6.2.2. Obviously, if $T\colon X \longrightarrow Y$ is a tauberian operator and X is quasi-reflexive, then T is strongly tauberian. This is the case of the operator $\iota\colon J \longrightarrow c_0$ given in Example 2.1.18.

In the proof of the following theorem we apply the well known fact that an operator S is an isomorphism if and only if S^* is surjective, and S is surjective if and only if S^* is an isomorphism.

Theorem 6.2.3. *For every operator $T\colon X \longrightarrow Y$, the following statements are equivalent:*

(a) *T is strongly tauberian;*

(b) $T^{*(3)}\big(J_Y(Y)^\perp\big) = J_X(X)^\perp$*;*

(c) *T^{*co} is surjective;*

(d) *$T^{*(4)}|_{J_{X^*}(X^*)^\perp}$ is an isomorphism;*

(e) *T^{**} is strongly tauberian.*

Proof. (a)⇔(c)⇔(e) It is enough to observe that, by Proposition 3.1.11, we can identify T^{*co} with T^{co*} and T^{**co} with T^{co**}.

(b)⇔(c) It is a consequence of Equations (6.2) and (6.4).

(d)⇔(e) It follows from Equations (6.3) and (6.5). □

Next, we introduce the dual class associated with $\mathcal{ST}$.

Definition 6.2.4. An operator T is said to be *strongly cotauberian* if T^* is strongly tauberian.

Accordingly, $\mathcal{ST}^d$ denotes the class of all strongly cotauberian operators.

The next results show that the duality relationship between $\mathcal{ST}$ and $\mathcal{ST}^d$ are better than that between $\mathcal{T}$ and $\mathcal{T}^d$.

Proposition 6.2.5. *An operator T is strongly tauberian if and only if T^* is strongly cotauberian.*

Proof. It is a direct consequence of the equivalence (a)⇔(e) in Theorem 6.2.3. □

We provide a similar result to Theorem 6.2.3 for the class $\mathcal{ST}^d$.

Theorem 6.2.6. *Given an operator $T\colon X \longrightarrow Y$, the following statements are equivalent:*

(a) *T is strongly cotauberian;*

(b) *$T^{*(3)}|_{J_Y(Y)^\perp}$ is an isomorphism;*

(c) $T^{*(4)}(J_{X^*}(X^*)^\perp) = J_{Y^*}(Y^*)^\perp$.

Proof. It is similar to the proof of some of the equivalences in Theorem 6.2.3. □

The following result reveals that the classes $\mathcal{ST}$ and $\mathcal{ST}^d$ share some regularity properties with Φ_+ and Φ_-.

Proposition 6.2.7. *Both classes $\mathcal{ST}$ and $\mathcal{ST}^d$ are open.*

Proof. Observe that the map $T \in \mathcal{L}(X,Y) \longrightarrow T^{co} \in \mathcal{L}(X^{co}, Y^{co})$ is continuous, because $\|T^{co}\| \leq \|T\|$. Since the isomorphisms and the surjective operators constitute open sets in $\mathcal{L}(X^{co}, Y^{co})$, both results are immediate. □

Remark 6.2.8. The classes $\mathcal{ST}$ and $\mathcal{ST}^d$ are strictly contained in $\mathcal{T}$ and $\mathcal{T}^d$ respectively. Indeed, it is enough to take into account that $\mathcal{ST}$ and $\mathcal{ST}^d$ are open classes, but $\mathcal{T}$ and $\mathcal{T}^d$ are not, as it was shown in Example 2.1.17 and in Remark 3.1.3.

A second proof can be obtained from the the fact that $\mathcal{ST}$ and $\mathcal{ST}^d$ are stable under biduality, while $\mathcal{T}$ and $\mathcal{T}^d$ are not. Indeed, Theorem 3.1.18 exhibits a tauberian and cotauberian operator T such that T^{**} is neither tauberian nor cotauberian.

Let us look into the structure of $\mathcal{ST}$ and $\mathcal{ST}^d$.

Proposition 6.2.9. *The class $\mathcal{ST}$ is an upper operator semigroup and the class $\mathcal{ST}^d$ is a lower operator semigroup.*

Proof. We begin with $\mathcal{ST}$.

(i) Let $T \in \Phi_+(X,Y)$. Then $R(T)$ is closed and $N(T)$ is finite dimensional. Therefore T is tauberian and, by Proposition 3.1.15, $R(T^{co})$ is closed. Hence T is strongly tauberian and we have proved that Φ_+ is contained in $\mathcal{ST}$.

(ii) Given two operators $S \in \mathcal{L}(V,W)$ and $T \in \mathcal{L}(X,Y)$, it is not difficult to see that we can identify $(S \times T)^{co}$ with $S^{co} \times T^{co}$.

Since $S^{co} \times T^{co}$ is an isomorphism if and only if both S^{co} and T^{co} are isomorphisms, $S \times T \in \mathcal{ST}$ if and only if both S and T are in $\mathcal{ST}$.

(iii) Let $S \in \mathcal{ST}(Y,Z)$ and $T \in \mathcal{ST}(X,Y)$. Since $(ST)^{co} = S^{co}T^{co}$, we conclude $ST \in \mathcal{ST}(X,Z)$.

We have just shown that $\mathcal{ST}$ is an injective semigroup. It only remains to see that it is left-stable.

Let $S \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(X,Y)$ such that $ST \in \mathcal{ST}(X,Z)$. Since $(ST)^{co}$ is an isomorphism and $(ST)^{co} = S^{co}T^{co}$, T^{co} is an isomorphism, hence $T \in \mathcal{ST}(X,Y)$ and the proof for $\mathcal{ST}$ is done.

The proof for $\mathcal{ST}^d$ is similar. □

Remark 6.2.10. Since an operator K is weakly compact if and only if K^{co} is null, both semigroups $\mathcal{ST}$ and $\mathcal{ST}^d$ are stable under weakly compact perturbations. However, neither of them admits a perturbative characterization like Theorems 2.2.7 or 3.1.20. Indeed, every reflexive subspace E of a Banach space X is the kernel of $Q_E \colon X \longrightarrow X/E$, which is strongly tauberian, and every reflexive quotient X/F is the cokernel of $J_F \colon F \longrightarrow X$, which is strongly cotauberian.

The following technical result will be applied to obtain a sequential characterization of the operators in $\mathcal{ST}$.

Lemma 6.2.11. *Given an operator $T\colon X \longrightarrow Y$, the following statements are equivalent:*

(a) *there is a normalized sequence $\big(\xi_n^{**} + J_X(X)\big)_{n=1}^{\infty} \subset X^{**}/J_X(X)$ such that*

$$\|T^{co}\big(\xi_n^{**} + J_X(X)\big)\| \xrightarrow[n]{} 0;$$

(b) *there is a sequence $(x_n^{**})_{n=1}^{\infty}$ in X^{**} with $\|x_n^{**}\| < 3$ and* $\operatorname{dist}\big(x_n^{**}, J_X(X)\big) = 1$ *for every n such that*

$$\|T^{**}(x_n^{**})\| \xrightarrow[n]{} 0.$$

Proof. (a)⇒(b) Assume the existence of a normalized sequence $\big(\xi_n^{**} + J_X(X)\big)$ in $X^{**}/J_X(X)$ for which $\|T^{co}\big(\xi_n^{**} + J_X(X)\big)\| \xrightarrow[n]{} 0$. For every positive integer n, pick $w_n^{**} \in \frac{3}{4}\xi_n^{**} + J_X(X)$ such that $\|w_n^{**}\| < 1$, denote $\varepsilon_n := \|T^{co}(\xi_n^{**} + J_X(X))\| + 1/n$, and select $u_n \in Y$ so that $\|T^{**}(w_n^{**}) - u_n\| < \varepsilon_n$. Let

$$L := \{x \in B_X \colon \|T(x) - u_n\| < \varepsilon_n\}.$$

Then, since $\|w_n^{**}\| < 1$, Lemma 4.4.1 shows that $w_n^{**} \in \overline{L}^{\sigma(X^{**},X^*)}$, hence there exists $v_n \in B_X$ satisfying $\|T(v_n) - u_n\| < \varepsilon_n$. Thus, defining $z_n^{**} := w_n^{**} - v_n$, we get

$$\|T^{**}(z_n^{**})\| \leq \|T^{**}(w_n^{**}) - u_n\| + \|T(v_n) - u_n\| < 2\varepsilon_n \xrightarrow[n]{} 0,$$

$\|z_n^{**}\| < 2$ and $\operatorname{dist}\big(z_n^{**}, J_X(X)\big) = 3/4$ for every positive integer n. Therefore, the elements $x_n^{**} := (4/3)z_n^{**}$ satisfy the statement.

(b)⇒(a) The proof is straightforward. □

James proved that a Banach space is reflexive if and only if it does not have any ε-triangular sequence (see Appendix A.5 for more information). Tauberian operators and strongly tauberian operators can be characterized in terms of their action on ε-triangular sequences.

Definition 6.2.12. Given a real number $\varepsilon > 0$, a sequence (x_n) in a Banach space X is said to be *ε-triangular* if $\|x_n\| \leq 1$ for all n and there exists a sequence of norm-one functionals (x_n^*) in X^* such that $\langle x_i^*, x_j\rangle > \varepsilon$ for all $1 \leq i \leq j$ and $\langle x_i^*, x_j\rangle = 0$ for all $1 \leq j < i$.

Note that if (x_n) is ε-triangular, then $\varepsilon \leq \|x_n\| \leq 1$ for all n.

In the following result, the hypothesis $\|T\| = 1$ is introduced in order to deal with ε-triangular sequences. It is a minor technical restriction since all non-zero multiples of a strongly tauberian operator are also strongly tauberian.

Theorem 6.2.13. *Given a norm-one operator $T\colon X \longrightarrow Y$, the following statements are equivalent:*

(a) *T is strongly tauberian;*

(b) *for every $0 < \varepsilon < 1$ there exists $\eta > 0$ such that every ε-triangular sequence in X contains a subsequence (z_n) such that $\big(T(z_n)\big)$ is η-triangular;*

(c) *there exist $0 < \varepsilon < 1$ and $0 < \eta < 1$ such that every ε-triangular sequence in X contains a subsequence (z_n) such that $\big(T(z_n)\big)$ is η-triangular.*

Proof. (a)⇒(b) Assume T is strongly tauberian and let $0 < \varepsilon < 1$. Then there exists a constant $0 < \lambda \leq 1$ such that $\lambda \leq \big\|T^{co}\big(x^{**} + J_X(X)\big)\big\|$ for all $x^{**} \in X^{**}$ with $\|x^{**} + J_X(X)\| = 1$.

Let (x_n) be an ε-triangular sequence in X. By Proposition A.5.2, there exists a $\sigma(X^{**}, X^*)$-cluster point z^{**} of (x_n) such that $\operatorname{dist}\big(z^{**}, J_X(X)\big) \geq \varepsilon/2$. Therefore,

$$\big\|T^{co}\big(z^{**} + J_X(X)\big)\big\| \geq \lambda\varepsilon/2,$$

and as $T^{**}(z^{**})$ is a $\sigma(Y^{**}, Y^*)$-cluster point of $\big(T(x_n)\big)$, by Proposition A.5.3, (x_n) contains a subsequence (z_n) such that $\big(T(z_n)\big)$ is η-triangular with $\eta = \lambda\varepsilon/2$.

(b)⇒(c) It is trivial.

(c)⇒(a) Let us assume that T is not strongly tauberian and take any pair of real numbers $0 < \varepsilon < 1$ and $0 < \eta < 1$. By hypothesis, there exists $x^{**} \in X^{**}$ such that $\|x^{**}\| < 1$, $\operatorname{dist}(x^{**}, X) > \varepsilon$ and $\operatorname{dist}(T^{**}(x^{**}), Y) < \eta/4$. Take $y \in Y$ so that $\|T^{**}(x^{**}) - y\| < \eta/2$. Thus, denoting

$$A := \{x \in B_X\colon \|T(x) - y\| < \eta/2\},$$

by Lemma 4.4.1, x^{**} is a $\sigma(X^{**}, X^*)$-cluster point of A, and therefore, by Proposition A.5.3, A has an ε-triangular sequence (x_n). Obviously, $\|T(x_n) - T(x_m)\| < \eta$ for all n and m, so $\big(T(x_n)\big)$ cannot contain any η-triangular subsequence. □

Remark 6.2.14. As a consequence of Corollary 2.2.5 and Proposition A.5.3, a norm-one operator $T \in \mathcal{L}(X, Y)$ is tauberian if and only if for every ε-triangular sequence (x_n) in X, (Tx_n) contains a λ-triangular subsequence; but if for a fixed ε a same value of λ works for all ε-triangular sequences (x_n), then T is moreover strongly tauberian, as follows from Theorem 6.2.13.

For the following result, given a Banach space X and a number $1 \leq p < \infty$, let $L_p(X)$ denote the space of Bochner measurable functions $f\colon [0,1] \longrightarrow X$ endowed with the norm

$$\|f\|_p := \left(\int_0^1 \|f(t)\|^p dt\right)^{1/p}.$$

It follows from Proposition 3.4.5 that the natural embedding of $L_p(X)$ into $L_1(X)$ is tauberian. Let us see that it is strongly tauberian only in the trivial cases:

Proposition 6.2.15. *Let X be a Banach space and let $1 < p < \infty$. The natural embedding $J_p\colon L_p(X) \longrightarrow L_1(X)$ is strongly tauberian if and only if X is reflexive.*

Proof. If X is reflexive, then so is $L_p(X)$ [21]. Therefore, J_p is strongly tauberian.

Suppose that X is not reflexive. Note that J_p is a norm-one operator. Fix a pair of real numbers ε and η in $(0, 1)$. By Theorem 6.2.13, we will have proved that J_p is not strongly tauberian as soon as we find an ε-triangular sequence (f_i) in $L_p(X)$ such that $\big(J_p(f_i)\big)$ does not contain any η-triangular subsequence. In order to do that, take a positive integer n so that $2^{(\frac{1}{p}-1)n} < \eta$ and let (x_i) be any ε-triangular sequence in X (see Proposition A.5.3). For every $i \in \mathbb{N}$, define the vector-valued function f_i as $f_i(t) := 2^{n/p}x_i$ if $t \in [0, 2^{-n}]$, and $f_i(t) = 0$ otherwise. Thus (f_i) is an ε-triangular sequence in $L_p(X)$, but $\big(J_p(f_i)\big)$ does not contain any η-triangular subsequence because

$$\|f_i\|_1 = \frac{1}{2^n} 2^{n/p} < \eta \quad \text{for all } i \in \mathbb{N}.$$

So the result is proved. □

Strongly tauberian operators on L-embedded spaces

The following results are aimed at proving that if a Banach space X is L-embedded in its bidual, an operator $T \in \mathcal{L}(X, Y)$ with property (N) is strongly tauberian.

Definition 6.2.16. A Banach space X is said to be *L-embedded in its bidual* if $X^{**} = J_X(X) \oplus_1 N$ for some subspace N of X^{**}.

Lemma 6.2.17. *Let X be a space L-embedded in its bidual, and let R be a reflexive subspace of X. Then X/R is L-embedded in $(X/R)^{**}$.*

Proof. Since R is reflexive, there exists an isometric bijection from $(X/R)^{**}$ onto X^{**}/R that maps each F to an element that will be denoted $x_F^{**} + R$. Moreover, if $F \in J_{X/R}(X/R)$, then $F = J_{X/R}(x+R)$ for some $x \in X$, so $x_F^{**} - J_X(x) \in J_X(R)$.

By hypothesis, X^{**} contains a subspace N such that

$$X^{**} = J_X(X) \oplus_1 N. \tag{6.6}$$

Let $P\colon (X/R)^{**} \longrightarrow (X/R)^{**}$ be the operator that maps every F to $J_{X/R}(x_F+R)$, where $x_F^{**} = J_X(x_F) + u_F$ is the decomposition of x_F^{**} with $x_F \in X$ and $u_F \in N$. It is immediate that P is a projection, that its range is $J_{X/R}(X/R)$, and that its kernel is $\{F\colon x_F \in R\}$. In addition, as a consequence of (6.6),

$$\begin{aligned}\|F\| &= \inf_{r\in R} \|J_X(x_F) + u_F + J_X(r)\| \\ &= \inf_{r\in R} \|x_F + r\| + \inf_{r\in R} \|u_F + J_X(r)\| = \|P(F)\| + \|F - P(F)\|\end{aligned}$$

which shows that X/R is L-embedded in its bidual. □

We observe that every closed subspace of a space L-embedded in its bidual is weakly sequentially complete [98]. Therefore, the following result should be compared with Proposition 2.1.12, where we proved that if $T\colon X \longrightarrow Y$ has property (N) and X is weakly sequentially complete, then T is tauberian.

Theorem 6.2.18. *Let V be a space L-embedded in its bidual and let X be a closed subspace of V. Then every operator $T\colon X \longrightarrow Y$ with property (N) is strongly tauberian.*

Proof. The result is trivial when X is reflexive. So, without loss of generality, we assume that X is non-reflexive, $T\colon X \longrightarrow Y$ has property (N) and $\|T\| = 1$. Consider the factorization $T = \widetilde{T} \circ Q_{N(T)}$ given in Equation (2.1). Since $N(T)$ is reflexive, $Q_{N(T)}$ is strongly tauberian and $\widetilde{T}$ has property (N). Moreover, $V/N(T)$ is L-embedded in its bidual by Lemma 6.2.17. Therefore, it is enough to prove the result when $N(T) = \{0\}$.

Let us suppose that T is injective but not strongly tauberian. The proof will be finished as soon as we find a pair of sequences (x_n) in B_X and (x_n^*) in $\mathrm{int}(B_{X^*})$ satisfying the following conditions for each $m \in \mathbb{N}$:

$$\|T(x_i)\| \le 1/i \quad \text{for each } 1 \le i \le m, \tag{6.7}$$

$$|\langle x_j^*, x_i\rangle| \ge 1/16 \quad \text{if } 1 \le i \le j \le m. \tag{6.8}$$

Indeed, given any $\sigma(X^*, X)$-cluster point x^* of (x_n^*), condition (6.8) implies that $|\langle x^*, x_i\rangle| \ge 1/16$ for all i, so (x_n) cannot contain any weakly null subsequence. However, as T has property (N), condition (6.7) and Theorem 2.2.2 imply that (x_n) must contain a weakly null subsequence, a contradiction.

Let us find recursively the wished sequences (x_n) and (x_n^*). First, by Lemma 6.2.11, there exists a sequence (x_n^{**}) in X^{**} such that $\mathrm{dist}\big(x_n^{**}, J_X(X)\big) = 1/3$, $\|x_n^{**}\| < 1$ and $\|T^{**}(x_n^{**})\| < 1/n$ for all n.

Let us denote by $\mathcal{J}\colon X \longrightarrow V$ the operator that embeds X into V, so $\mathcal{J}^{**}$ embeds X^{**} into V^{**} isometrically and

$$\mathcal{J}^{**}(X^{**}) = \overline{\mathcal{J}(X)}^{\sigma(V^{**},V^*)} = \mathcal{J}(X)^{\perp\perp} :$$

In order to avoid confusion, an element x of X will be denoted by $\mathcal{J}x$ when it is regarded as an element of V, and any $x^{**} \in X^{**}$ will be denoted by $\mathcal{J}^{**}(x^{**})$ when it is regarded as an element of V^{**}.

Our hypothesis implies the existence of a closed subspace N of V^{**} such that $V^{**} = J_V(V) \oplus_1 N$. Therefore, for every x_n^{**}, there exist $v_n \in V$ and v_n^{**} in N so that

$$\mathcal{J}^{**}(x_n^{**}) = J_V(v_n) + v_n^{**}.$$

Since $\mathrm{dist}\big(\mathcal{J}^{**}(x_n^{**}), \mathcal{J}(X)\big) = 1/3$, Lemma A.5.1 yields $\mathrm{dist}\big(\mathcal{J}^{**}(x_n^{**}), V\big) \ge 1/6$; but $\|x_n^{**}\| = \|v_n\| + \|v_n^{**}\| < 1$, hence $1/6 \le \|v_n^{**}\| < 1$ and $\|v_n\| < 5/6$.

Pick any $x_1 \in B_X$ and choose $x_1^* \in \mathrm{int}\, B_{X^*}$ so that $\langle x_1^*, x_1\rangle \ge 1/8$. Observe that $\|T(x_1)\| \le 1$.

Assume that, for $n \geq 2$, the finite sequences

$$(x_i)_{i=1}^{n-1} \subset B_X \quad \text{and} \quad (x_i^*)_{i=1}^{n-1} \subset \operatorname{int} B_{X^*}$$

satisfying (6.7) and (6.8) for $m = n-1$ have been already chosen. In order to find x_n and x_n^*, consider the finite dimensional subspaces

$$\begin{aligned} F &:= \operatorname{span}\{\{\mathcal{J}(x_i)\}_{i=1}^{n-1} \cup \{v_n\}\} \subset V, \\ G &:= J_V(F) \oplus \operatorname{span}\{v_n^{**}\} \subset V^{**}. \end{aligned}$$

Next, choose $\psi \in F^*$ as follows. In the case when $v_n \in \mathcal{J}(X)$ and $\langle x_{n-1}^*, v_n\rangle \geq 0$, let $\psi := x_{n-1}^*|_F$, and if $\langle x_{n-1}^*, v_n\rangle < 0$, let $\psi := -x_{n-1}^*|_F$; and in the case when $v_n \notin \mathcal{J}(X)$, ψ is a Hahn-Banach extension of $x_{n-1}^*|_{F\cap\mathcal{J}(X)}$ such that $\langle \psi, v_n\rangle > 0$. Thus, in all cases $\|\psi\| < 1$ and

$$0 \leq r := \langle \psi, v_n\rangle < 1.$$

Observe that the extension $\varphi \in G^*$ of ψ, given by $\langle \varphi, v_n^{**}\rangle := 1/8$ satisfies $\|\varphi\| < 1$. Indeed, let

$$\eta := \max\{\|\psi\|, 6/8\} < 1.$$

For every $u^{**} \in G$, consider the decomposition $u^{**} = J_V(u) + \mu v_n^{**}$ with $u \in F$ and $\mu \in \mathbb{R}$. Then,

$$\begin{aligned} |\langle \varphi, u^{**}\rangle| &= |\langle \varphi, J_V(u) + \mu v_n^{**}\rangle| \leq |\langle \psi, u\rangle| + \frac{1}{8}|\mu| \\ &\leq \|\psi\|\|u\| + \frac{6}{8}|\mu|\|v_n^{**}\| \leq \max\{\|\psi\|, 6/8\} \cdot \|u^{**}\| = \eta\|u^{**}\|, \end{aligned}$$

which proves that $\|\varphi\| < 1$.

Let $\phi \in V^{*(3)}$ be a Hahn-Banach extension of φ. By the principle of local reflexivity, there exists $v^* \in V^*$ with $\|v^*\| < 1$ such that

$$\langle u^{**}, v^*\rangle = \langle \phi, u^{**}\rangle \quad \text{for all } u^{**} \in G.$$

Thus, the choice for x_n^* is

$$x_n^* := \mathcal{J}^*(v^*) = v^*|_X.$$

Moreover, as $\|x_n^{**}\| < 1$ and $\|T^{**}(x_n^{**})\| < 1/n$, Lemma 4.4.1 yields an element $x_n \in B_X$ satisfying

$$\begin{aligned} &|\langle x_n^*, x_n\rangle| > |\langle x_n^{**}, x_n^*\rangle| - \frac{1}{16}, \\ &\|T(x_n)\| < 1/n. \end{aligned} \tag{6.9}$$

Obviously, our choice of x_n^* and x_n satisfies condition (6.7) for $m = n$. In order to verify condition (6.8),

$$\begin{aligned}\langle x_n^{**}, x_n^* \rangle &= \langle x_n^{**}, \mathcal{J}^*(v^*) \rangle = \langle \mathcal{J}^{**}(x_n^{**}), v^* \rangle \\ &= \langle v^*, v_n \rangle + \langle v_n^{**}, v^* \rangle = \langle \phi, v_n \rangle + \langle \phi, v_n^{**} \rangle \\ &= \langle \varphi, v_n \rangle + \frac{1}{8} = r + \frac{1}{8} \geq \frac{1}{8}.\end{aligned}$$

Thus, by (6.9),

$$|\langle x_n^*, x_n \rangle| > |\langle x_n^{**}, x_n^* \rangle| - \frac{1}{16} \geq \frac{1}{8} - \frac{1}{16} \geq \frac{1}{16},$$

and for $1 \leq k < n$, as $x_k \in F$,

$$|\langle x_n^*, x_k \rangle| = |\langle \psi, x_k \rangle| = |\langle x_{n-1}^*, x_k \rangle|.$$

So (6.8) is satisfied for $m = n$ by the finite sequences $(x_i)_{i=1}^n$ and $(x_i^*)_{i=1}^n$ and the proof is complete. □

Corollary 6.2.19. *Let μ be a finite measure, R a reflexive subspace of $L_1(\mu)$ and Y a Banach space. Then every operator $T \in \mathcal{L}\big(L_1(\mu)/R, Y\big)$ with property (N) is strongly tauberian.*

Proof. Since $L_1(\mu)$ is L-embedded in its bidual, the remark is a consequence of Theorem 6.2.18 and Lemma 6.2.17. □

We observe that if R is infinite dimensional, then $L_1(\mu)/R$ is not isomorphic to a $\mathcal{L}_1$-space [121].

A proof of the fact that $L_1(\mu)$ is an L-summand of $L_1(\mu)^{**}$ can be found in Corollary A.6.11.

6.3 Finite representability of operators

In order to develop the remaining sections of this chapter, we need some notions of finite representability of operators.

While the notion of finite representability of Banach spaces is well established (see Definition A.4.13), there are several non-equivalent definitions for finite representability of operators. Roughly speaking, all these definitions say that an operator T is finitely representable in an operator S if they are locally similar. Here, we only consider two of those definitions, one of them oriented to the study of operator ideals and the other to operator semigroups: the *local representability* (Definition 6.3.1) and the *local supportability* (Definition 6.3.4).

In this section, for any operator T and any ultrafilter $\mathfrak{U}$, we show that the operators $T_{\mathfrak{U}}$, T^{**} and T^{co} are locally representable in and locally supportable by T, and that ${T_{\mathfrak{U}}}^*$ is locally representable and locally supportable by ${T^*}_{\mathfrak{U}}$.

Recall that for every isomorphism $T\colon X \longrightarrow Y$, there exists a constant $d > 0$ such that

$$d^{-1} \leq \|T(x)\| \leq d \text{ for all } x \in S_X.$$

In this case, we say that T is a *d-injection.* Note that $T^{-1}\colon T(X) \longrightarrow X$ is a d-injection too.

An isometry (or 1-injection) is also named *metric injection.* Note that any ε-isometry is a $(1-\varepsilon)^{-1}$-injection.

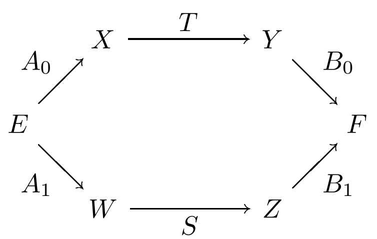
An operator $T \in \mathcal{L}(X,Y)$ is said to be a *metric surjection* if T^* is a metric injection or, equivalently, if $T(\operatorname{int} B_X) = \operatorname{int} B_Y$.

The following definition was introduced by Pietsch.

Definition 6.3.1. Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(W,Z)$ be operators and let $c > 0$ be a real number.

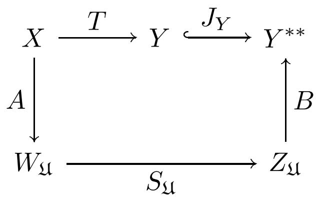
We say that T is *locally c-representable* in S if for every $\varepsilon > 0$ and every pair of operators $A_0 \in \mathcal{L}(E,X)$ and $B_0 \in \mathcal{L}(Y,F)$, with E and F finite dimensional spaces, there is a pair of operators $A_1 \in \mathcal{L}(E,W)$ and $B_1 \in \mathcal{L}(Z,F)$ satisfying $\|A_1\| \cdot \|B_1\| \leq (c+\varepsilon)\|A_0\| \cdot \|B_0\|$ and $B_0 T A_0 = B_1 S A_1$.

We say that T is *locally representable* in S when it is locally c-representable for some $c > 0$.



Let us translate the notion of local representability to the language of ultraproducts.

Proposition 6.3.2. *Given a real number $c > 0$, an operator $T \in \mathcal{L}(X,Y)$ is locally c-representable in $S \in \mathcal{L}(W,Z)$ if and only if there is an ultrafilter $\mathfrak{U}$ and operators $A \in \mathcal{L}(X, W_{\mathfrak{U}})$ and $B \in \mathcal{L}(Z_{\mathfrak{U}}, Y^{**})$ such that $BS_{\mathfrak{U}}A = J_Y T$ and $\|B\|\|A\| \leq c$.*



Proof. Let us assume that T is locally c-representable in S. Consider the set of indices I formed by all the triples $i \equiv (E_i, F_i, \varepsilon_i)$ where E_i is a finite dimensional subspace of X, F_i is a finite co-dimensional subspace of Y, and ε_i is a real number greater than zero. The set I is endowed with the order $\preceq$, where $i \preceq j$ means that $E_j \supset E_i$, $F_j \subset F_i$ and $\varepsilon_j \leq \varepsilon_i$. Let $\mathfrak{U}$ be an ultrafilter on I containing the $\preceq$-order filter.

For every index i, consider the subspace operator $J_i \colon E_i \longrightarrow X$ and the quotient operator $Q_i \colon Y \longrightarrow Y/F_i$. By hypothesis, there exists a pair of operators $A_i \in \mathcal{L}(E_i, W)$ and $B_i \in \mathcal{L}(Z, Y/F_i)$ such that $B_i S A_i = Q_i T J_i$, $\|A_i\| \leq 1$ and $\|B_i\| \leq c + \varepsilon_i$.

For every $x \in X$, let

$$x_i := \begin{cases} A_i(x), & \text{if } x \in E_i, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the expression $A(x) := [x_i]$ defines an operator $A \in \mathcal{L}(X, W_{\mathfrak{U}})$ such that $\|A\| \leq 1$. Besides, for every family $(y_i)_{i \in I} \in \ell_\infty(I, F_i)$,

$$y_i \xrightarrow[\mathfrak{U}]{\sigma(Y^{**}, Y^*)} 0.$$

Indeed, let y^{**} be the $\sigma(Y^{**}, Y^*)$-limit of $(y_i)_{i \in I}$ following $\mathfrak{U}$. Given any $y^* \in Y^*$, let $F := N(y^*)$; thus $\{i \in I \colon y_i \in F\} \in \mathfrak{U}$ and $\langle y^{**}, y^* \rangle = 0$, hence $y^{**} = 0$. Therefore, the operator $Q \colon (Y/F_i)_{\mathfrak{U}} \longrightarrow Y^{**}$ that maps every $[y_i + F_i]$ to $\sigma(Y^{**}, Y^*)$-$\lim_{\mathfrak{U}} y_i$ is well defined and $\|Q\| = 1$. Note that

$$Q \circ (Q_i)_{\mathfrak{U}}|_Y = J_Y.$$

Consider the operator $B := Q \circ (B_i)_{\mathfrak{U}} \in \mathcal{L}(Z_{\mathfrak{U}}, Y^{**})$. Obviously,

$$\|B\| \leq \lim_{\mathfrak{U}} \|B_i\| \leq c + \lim_{\mathfrak{U}} \varepsilon_i = c.$$

It only remains to prove that $BS_{\mathfrak{U}}A = J_Y T$. Fix $x \in X$ and $J \in \mathfrak{U}$ so that $x \in E_i$ for all $i \in J$. Thus,

$$B_i S A_i(x) = Q_i T J_i(x) \quad \text{for all } i \in J$$

hence

$$\begin{aligned} BS_{\mathfrak{U}}A(x) &= Q \circ (B_i)_{\mathfrak{U}} \circ S_{\mathfrak{U}} \circ A(x) = Q\big([B_i S(x_i)]\big) \\ &= Q\big([Q_i T J_i(x)]\big) = Q \circ (Q_i)_{\mathfrak{U}}\big([T(x)]\big) = J_Y T(x). \end{aligned}$$

For the converse, let us assume that there exists a pair of operators A in $\mathcal{L}(X, W_{\mathfrak{U}})$ and B in $\mathcal{L}(Z_{\mathfrak{U}}, Y^{**})$ so that $J_Y T = BS_{\mathfrak{U}}A$ and $\|A\|\|B\| \leq c$, where $\mathfrak{U}$ is an ultrafilter on a certain set I of indices. Let E and F be a pair of finite dimensional subspaces, $A_0 \in \mathcal{L}(E, X)$ and $B_0 \in \mathcal{L}(Y, F)$ a pair of operators, and a real number $\varepsilon > 0$.

Moreover, consider the operators $A' := AA_0$ and $B' := B_0^{**} B$, and choose a real number $0 < \varepsilon' < 1$ small enough so that $(1 + \varepsilon')^2 < 1 + \varepsilon$.

Let $N := A'(E) \cap N(S_{\mathfrak{U}})$ and choose a basis $\{e_k\}_{k=1}^n$ of $A'(E)$ such that $N = \text{span}\{A'(e_k)\}_{k=l+1}^n$ for some $0 \leq l \leq n$. Fix a representative $(w_i^k)_{i\in I}$ of every e_k, and for each $i \in I$, denote $N_i := N(S) \cap \text{span}\{w_i^k\}_{k=1}^n$. As $(N_i)_{\mathfrak{U}} \subset N$, we may assume that there are a positive integer $l \leq m \leq n$ and a subset $J_1 \in \mathfrak{U}$ such that for every $i \in J_1$, N_i is spanned by $\{w_i^k\}_{k=m+1}^n$. For every $j \in J_1$, consider the operator $L_j \colon A'(E) \longrightarrow W$ that sends each e_k to w_j^k, and the operator $R_j \colon \text{span}\{Sw_j^k\}_{k=1}^m \longrightarrow Z_{\mathfrak{U}}$ that sends Sw_j^k to $[Sw_i^k]$. By Lemma A.4.12 and its arguments, there is $j \in J_1$ such that L_j is a $(1+\varepsilon')$-injection and $\|R_j\| \leq 1+\varepsilon'$. Moreover, $S_{\mathfrak{U}}|_{A'(E)} = R_j S L_j$. Thus, defining $A_1 := L_j A'$ and $B_1 := B' R_j$, it is straightforward that $B_1 S A_1 = B_0 T A_0$. Besides, $\|A_1\| \leq (1+\varepsilon')\|A\| \cdot \|A_0\|$ and $\|B_1\| \leq (1+\varepsilon')\|B\| \cdot \|B_0\|$. Thus, the inequality $\|B\|\|A\| \leq c$ and the choice of ε' lead to

$$\|B_1\|\|A_1\| \leq c(1+\varepsilon)\|B_0\|\|A_0\|$$

and the proof is done. □

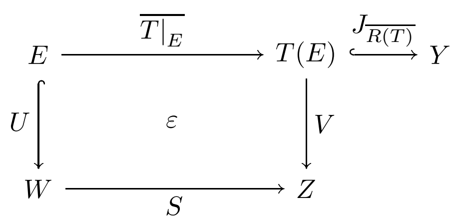
The following result follows immediately from Proposition 6.3.2.

Corollary 6.3.3. *Given an operator T and an ultrafilter $\mathfrak{U}$, the ultrapower $T_{\mathfrak{U}}$ is locally representable in T.*

Next we introduce the second type of finite representability for operators, oriented to the study of operator semigroups.

Definition 6.3.4. Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(W,Z)$ be operators and let $d > 1$ be a real number.

We say that T is *locally d-supportable* by S if for every $\varepsilon > 0$ and every finite dimensional subspace E of X there are a $(d+\varepsilon)$-injection $U \in \mathcal{L}(E,W)$ and an operator $V \in \mathcal{L}(T(E),Z)$ verifying $\|V\| \leq d+\varepsilon$ and $\|(SU - VT)x\| \leq \varepsilon$ for all $x \in S_E$. We say that T is *locally supportable* by S when it is locally d-supportable for some $d > 1$.

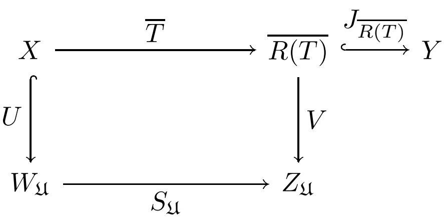


Henceforth, the notation $T \prec_{\text{ls}} S$ means that the operator T is locally supportable by S, and $T \prec_{\text{lr}} S$ means that T is locally representable in S.

There is a striking difference between the definitions of local supportability and local representability: the diagram corresponding to Definition 6.3.4 is commutative up to ε, while the diagram corresponding to the definition of local representability is commutative. The reason for this is that if we asked for exact commutativity in Definition 6.3.4 ($\varepsilon = 0$), then T would be injective whenever so was S, reducing the power of this type of operator finite representability: for instance, S^{**} could not be locally supportable by S in general, failing Theorem 6.3.8, one of the main results of this section.

Nevertheless, when Definitions 6.3.1 and 6.3.4 are translated to the ultraproduct language (Propositions 6.3.2 and 6.3.5), the relation between both notions becomes clearer.

Proposition 6.3.5. *An operator* $T \in \mathcal{L}(X,Y)$ *is locally* d*-supportable by* $S \in \mathcal{L}(W,Z)$ *if and only if there is an ultrafilter* $\mathfrak{U}$*, a* d*-injection* $U \in \mathcal{L}(X, W_{\mathfrak{U}})$ *and an operator* $V \in \mathcal{L}(\overline{R(T)}, Z_{\mathfrak{U}})$ *such that* $S_{\mathfrak{U}}U = V\overline{T}$ *and* $\|V\| \leq d$.



Proof. Assume that T is locally d-supportable by S. Let $\mathcal{F}$ be the class of all finite dimensional subspaces of X and consider the order filter on $\mathcal{F}$, which consists of all sets

$$\{E \in \mathcal{F} : E \supset F\}, \quad F \in \mathcal{F}.$$

Let $\mathfrak{U}$ be an ultrafilter on $\mathcal{F}$ containing the order filter and, for each $E \in \mathcal{F}$, let $\varepsilon_E := (\dim E)^{-1}$. By hypothesis, there is a $(d+\varepsilon_E)$-injection $U_E \in \mathcal{L}(E, W)$ and an operator $V_E \in \mathcal{L}(T(E), Z)$ such that $\|V_E\| \leq d+\varepsilon_E$ and $\|(SU_E - V_E T)x\| \leq \varepsilon_E$ for all $x \in S_E$.

We define an operator $U \in \mathcal{L}(X, W_{\mathfrak{U}})$ by $U(x) = [x_E]$ where

$$x_E := \begin{cases} U_E(x), & \text{if } x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for every $x \in S_X$,

$$\lim_{E \to \mathfrak{U}} (d+\varepsilon_E)^{-1} \leq \|U(x)\| \leq \lim_{E \to \mathfrak{U}} (d+\varepsilon_E),$$

and as $\lim_{E \to \mathfrak{U}} \varepsilon_E = 0$, we obtain that U is a d-injection.

Analogously, for each $y \in T(X)$ and every $E \in \mathcal{F}$, we define

$$y_E := \begin{cases} V_E(x), & \text{if } y \in T(E), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the expression $V(y) = [y_E]$ defines an operator from $\overline{R(T)}$ into $Z_{\mathfrak{U}}$ such that $\|V\| \leq d$. The identity $S_{\mathfrak{U}}U - V\overline{T} = 0$ follows from the fact that

$$\|(S_{\mathfrak{U}}U - VT)x\| \leq \lim_{E \to \mathfrak{U}} \varepsilon_E \|x\| \quad \text{for all } x \in X.$$

For the converse, let us assume that there are an ultrafilter $\mathfrak{U}$ on a set I, a d-injection $U \in \mathcal{L}(X, W_{\mathfrak{U}})$ and an operator $V \in \mathcal{L}(\overline{R(T)}, Z_{\mathfrak{U}})$ satisfying $\|V\| \leq d$ and $S_{\mathfrak{U}}U - V\overline{T} = 0$.

Fix a finite dimensional subspace E of X, a real number $\varepsilon > 0$ and a basis $\{e_j\}_{j=1}^n$ of E such that $\{e_j\}_{j=m+1}^n$ spans $N(T|_E)$. For each $j \in \{1, \ldots, n\}$, let $Ue_j = [x_i^j]_i$, and for every i, let $y_i^j := Sx_i^j$. By Proposition A.4.12, there is $J \in \mathfrak{U}$ such that, for every $i \in J$, the operators $U_i \in \mathcal{L}(E, X)$ and $V_i \in \mathcal{L}(T(E), Y)$, defined by $U_i(e_j) := x_i^j$ for $1 \leq j \leq n$ and $V_i(Te_j) := y_i^j$ for $1 \leq j \leq m$, satisfy that $\|V_i\| \leq d + \varepsilon$ and that U_i is a $(d+\varepsilon)$-injection. Moreover, the identity $S_{\mathfrak{U}}U - V\overline{T} = 0$ and the fact that E is finite dimensional allow us to select $i \in J$ so that $\|(SU_i - V_iT)x\| \leq \varepsilon$ for all $x \in S_E$. Thus, T is locally d-supportable by S. □

Proposition 6.3.5 yields the following corollary. Its proof is immediate.

Corollary 6.3.6. *Given an operator T and an ultrafilter $\mathfrak{U}$, the ultrapower $T_{\mathfrak{U}}$ is locally supportable by T.*

In order to simplify the proof of Theorem 6.3.8, we isolate the central step in the following lemma.

Lemma 6.3.7. *Let E be an n-dimensional subspace of a dual space X^*, $\{e_i\}_{i=1}^p$ an η-net in S_E with $0 < \eta < 1$ and $\delta > 0$ real numbers. If $(L_\alpha)_{\alpha \in A}$ is a net of operators from E into X^* such that $\|L_\alpha(e_i)\| \leq 1 + \delta$ and $L_\alpha(e_i) \xrightarrow[\alpha]{w^*} e_i$ for all $1 \leq i \leq p$, then there is $\beta \in A$ such that for every $\beta \geq \alpha$, L_α is a $(\eta+\delta)(1-\eta)^{-1}$-isometry.*

Moreover, if $\mathcal{V}$ is an absolutely convex w^*-neighborhood of $0 \in X^*$ and η and δ are small enough so that

$$\delta\mathcal{V} + \left(\eta + \frac{1+\delta}{1-\eta}\eta\right) B_{X^*} \subset \mathcal{V},$$

then $L_\alpha(e) \in e + \mathcal{V}$ for all $e \in S_E$ and all $\alpha \geq \beta$.

Proof. Since $L_\alpha(e_i) \xrightarrow[\alpha]{w^*} e_i$, we can select β so that, for every $\alpha \geq \beta$ and every $1 \leq i \leq p$, $\|L_\beta(e_i)\| \geq 1 - \delta$ and $L(e_i) \in e_i + \delta\mathcal{V}$. Thus, by virtue of Lemma A.4.10, L_α is an $(\eta + \delta)(1-\eta)^{-1}$-isometry.

Moreover, assume that the inclusion $\delta\mathcal{V} + (\eta + \frac{1+\delta}{1-\eta}\eta)B_{X^*} \subset \mathcal{V}$ holds. Thus, for $\alpha \geq \beta$, given any $e \in S_E$ and picking an element e_j in the given η-net so that

$\|e - e_j\| \le \eta$, we get

$$\begin{aligned} L_\alpha(e) &= L_\alpha(e_j) + L_\alpha(e - e_j) \\ &\in e_j + \delta\mathcal{V} + \frac{1+\delta}{1-\eta}\eta B_{X^*} \\ &\subset e - e + e_j + \delta\mathcal{V} + \frac{1+\delta}{1-\eta}\eta B_{X^*} \\ &\subset e + \delta\mathcal{V} + \left(\eta + \frac{1+\delta}{1-\eta}\eta\right) B_{X^*} \subset e + \mathcal{V}, \end{aligned}$$

finishing the proof. □

Next we state the main result in this section. Given an operator T, it will allow us to prove that T^{**} and T^{co} are locally supportable and locally representable in T.

Theorem 6.3.8. *Let $T \in \mathcal{L}(X,Y)$ be an operator, E a finite dimensional subspace of X^{**} and F a finite dimensional subspace of Y^{**} verifying $F \cap T^{**}(E) = \{0\}$. Fix a pair of weak* neighborhoods $\mathcal{U}$ of $0 \in X^{**}$ and $\mathcal{V}$ of $0 \in Y^{**}$ and $0 < \varepsilon < 1$. Then there exists a pair of ε-isometries $U \in \mathcal{L}(E,X)$ and $V \in \mathcal{L}(T^{**}(E) \oplus F, Y)$ verifying the following statements:*

(i) $U(x) = x$ *for all* $x \in E \cap X$;

(ii) $V(y) = y$ *for all* $y \in (T^{**}(E) \oplus F) \cap Y$;

(iii) $\|(TU - VT^{**})x\| < \varepsilon$ *for all* $x \in S_E$;

(iv) $U(e) \in e + \mathcal{U}$ *for all* $e \in S_E$;

(v) $V(f) \in f + \mathcal{V}$ *for all* $f \in S_{T^{**}(E)\oplus F}$.

Proof. Without loss of generality, we assume that the neighborhoods $\mathcal{U}$ and $\mathcal{V}$ are absolutely convex and $\|T\| = 1$.

Choose real numbers $0 < \eta < 1$ and $\delta > 0$ so that

$$\begin{aligned} &\frac{\eta+\delta}{1-\eta} < \varepsilon, \\ &\delta\mathcal{U} + \left(\eta + \frac{1+\delta}{1-\eta}\eta\right) B_{X^*} \subset \mathcal{U}, \\ &\delta\mathcal{V} + \left(\eta + \frac{1+\delta}{1-\eta}\eta\right) B_{Y^*} \subset \mathcal{V}. \end{aligned}$$

Let $\{x_i^1\}_{i=1}^p \cup \{x_i^2\}_{i=1}^q \cup \{x_i^3\}_{i=1}^t$ be a basis of E contained in $\operatorname{int} B_E$ and satisfying:

$\{x_i^1\}_{i=1}^p$ is a basis of $E \cap X$,
$\{x_i^1\}_{i=r+1}^p$ spans $N(T|_{E\cap X})$,
$\{x_i^1\}_{i=1}^p \cup \{x_i^2\}_{i=1}^q$ is a basis of $(T^{**}|_E)^{-1}Y$,
$\{x_i^1\}_{i=r+1}^p \cup \{x_i^2\}_{i=s+1}^q$ spans $N(T^{**}|_E)$.

Let us denote $y_i^k := T^{**}(x_i^k)$ for every $1 \leq k \leq 3$ and every i. Take a basis $\{y_i^4\}_{i=1}^u \cup \{y_i^5\}_{i=1}^v$ in $\operatorname{int} B_F$ such that $\{y_i^4\}_{i=1}^u$ spans $F \cap Y$. Let $(h_i)_{i=1}^q$ be the coordinate functionals associated with $(x_i^2)_{i=1}^q$ and let $\xi := (1+\delta)\sum_{i=1}^q \|h_i\|$.

Take a pair of η-nets $(e_j)_{j=1}^n$ in S_E and $(f_j)_{j=1}^m$ in $S_{T^{**}(E)\oplus F}$, and consider the representations:

$$e_j = \sum_{i=1}^p \lambda_{1i}^j x_i^1 + \sum_{i=1}^q \lambda_{2i}^j x_i^2 + \sum_{i=1}^t \lambda_{3i}^j x_i^3,$$
$$f_j = \sum_{i=1}^r \mu_{1i}^j y_i^1 + \sum_{i=1}^s \mu_{2i}^j y_i^2 + \sum_{i=1}^t \mu_{3i}^j y_i^3 + \sum_{i=1}^u \mu_{4i}^j y_i^4 + \sum_{i=1}^v \mu_{5i}^j y_i^5.$$

Let the operator

$$S\colon \ell_\infty^q(X) \oplus_\infty \ell_\infty^t(X) \oplus_\infty \ell_\infty^v(Y) \longrightarrow \ell_\infty^n(X) \oplus_\infty \ell_\infty^m(Y) \oplus_\infty \ell_\infty^q(Y)$$

be given by $S = (S_1, S_2, S_3)$, with

$$S_1\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) = \Big(\sum_{i=1}^q \lambda_{2i}^j a_i + \sum_{i=1}^t \lambda_{3i}^j b_i\Big)_{j=1}^n \in \ell_\infty^n(X),$$
$$S_2\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) = \Big(\sum_{i=1}^t \mu_{3i}^j T b_i + \sum_{i=1}^v \mu_{5i}^j c_i\Big)_{j=1}^m \in \ell_\infty^m(Y),$$
$$S_3\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) = (\varepsilon^{-1}\xi\, T a_i)_{i=1}^q \in \ell_\infty^q(Y),$$

where $a_i \in X$, $b_i \in X$ and $c_i \in Y$ for all i. Consider the element

$$z = \bigg(\Big(\sum_{i=1}^p \lambda_{1i}^j x_i^1\Big)_{j=1}^n, \Big(\sum_{i=1}^r \mu_{1i}^j y_i^1 + \sum_{i=1}^s \mu_{2i}^j y_i^2 + \sum_{i=1}^u \mu_{4i}^j y_i^4\Big)_{j=1}^m, (-\varepsilon^{-1}\xi y_i^2)_{i=1}^q\bigg).$$

Then $S\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) + z$ equals

$$\bigg(\Big(\sum_{i=1}^p \lambda_{1i}^j x_i^1 + \sum_{i=1}^q \lambda_{2i}^j a_i + \sum_{i=1}^t \lambda_{3i}^j b_i\Big)_{j=1}^n,$$
$$\Big(\sum_{i=1}^r \mu_{1i}^j y_i^1 + \sum_{i=1}^s \mu_{2i}^j y_i^2 + \sum_{i=1}^t \mu_{3i}^j T b_i + \sum_{i=1}^u \mu_{4i}^j y_i^4 + \sum_{i=1}^v \mu_{5i}^j c_i\Big)_{j=1}^m,$$
$$\big(\varepsilon^{-1}\xi(T(a_i) - y_i^2)\big)_{i=1}^q\bigg).$$

Therefore,

$$S^{**}\big((x_i^2)_{i=1}^q, (x_i^3)_{i=1}^t, (y_i^5)_{i=1}^v\big) + z = \big((e_j)_{j=1}^n, (f_j)_{j=1}^m, (0)_{i=1}^q\big)$$

is a norm-one element, and Lemma 4.4.1 gives us a net

$$\big((a_i^\alpha)_{i=1}^q, (b_i^\alpha)_{i=1}^t, (c_i^\alpha)_{i=1}^v\big)$$

contained in the unit ball of $\ell_\infty^q(X) \oplus_\infty \ell_\infty^t(X) \oplus_\infty \ell_\infty^v(Y)$ such that

$$\big((a_i^\alpha)_{i=1}^q, (b_i^\alpha)_{i=1}^t, (c_i^\alpha)_{i=1}^v\big) \xrightarrow[\alpha]{w^*} \big((x_i^2)_{i=1}^q, (x_i^3)_{i=1}^t, (y_i^5)_{i=1}^v\big)$$

and

$$\big\|S\big((a_i^\alpha)_{i=1}^q, (b_i^\alpha)_{i=1}^t, (c_i^\alpha)_{i=1}^v\big) + z\big\| < 1 + \delta \text{ for all } \alpha.$$

In particular,

$$\|Ta_i^\alpha - y_i^2\| \le \varepsilon \xi^{-1}(1+\delta) \text{ for all } 1 \le i \le q. \tag{6.10}$$

Now, we define $U_\alpha \in \mathcal{L}(E, X)$ and $V_\alpha \in \mathcal{L}(T^{**}(E) \oplus F, Y)$ by

$$\begin{array}{llll}
U_\alpha(x_i^1) := x_i^1 & \text{for all } i \in \{1,\dots,p\}; & V_\alpha(y_i^1) := y_i^1 & \text{for all } i \in \{1,\dots,r\}; \\
U_\alpha(x_i^2) := a_i^\alpha & \text{for all } i \in \{1,\dots,q\}; & V_\alpha(y_i^2) := y_i^2 & \text{for all } i \in \{1,\dots,s\}; \\
U_\alpha(x_i^3) := b_i^\alpha & \text{for all } i \in \{1,\dots,t\}; & V_\alpha(y_i^3) := T(b_i^\alpha) & \text{for all } i \in \{1,\dots,t\}; \\
 & & V_\alpha(y_i^4) := y_i^4 & \text{for all } i \in \{1,\dots,u\}; \\
 & & V_\alpha(y_i^5) := c_i^\alpha & \text{for all } i \in \{1,\dots,v\};
\end{array}$$

the ε-isometries U and V we are looking for will be chosen among these operators U_α and V_α.

Note that for all $x \in E \cap X$, $y \in (T^{**}(E) \oplus F) \cap Y$ and α, $U_\alpha(x) = x$ and $V_\alpha(y) = y$. So conditions (i) and (ii) are satisfied by all U_α and V_α.

Besides, for every $e \in S_E$,

$$(TU_\alpha - V_\alpha T^{**})(e) = \sum_{i=1}^q \langle h_i, e\rangle (T(a_i^\alpha) - y_i^2).$$

Hence, formula (6.10) and the value of ξ yields, for every $e \in S_E$,

$$\|(TU_\alpha - V_\alpha T^{**})(e)\| \le \sum_{i=1}^q |\langle h_i, e\rangle| \cdot \|Ta_i^\alpha - y_i^2\| \le \varepsilon.$$

So $\|TU_\alpha - V_\alpha T^{**}|_E\| \le \varepsilon$ and condition (iii) holds for any U_α and V_α. Moreover, since

$$\|U_\alpha(e_j)\| \le 1 + \delta \text{ and } U_\alpha(e_j) \xrightarrow[\alpha]{w^*} e_j \text{ for all } 1 \le j \le n,$$

$$\|V_\alpha(f_j)\| \le 1 + \delta \text{ and } V_\alpha(f_j) \xrightarrow[\alpha]{w^*} f_j \text{ for all } 1 \le j \le m,$$

Lemma 6.3.7 implies the existence of an index β such that both U_β and V_β are $(\eta+\delta)(1-\eta)^{-1}$-isometries — hence ε-isometries, by the choice of η and δ — and also satisfy conditions (iv) and (v). So the proof is complete. □

Let us translate Theorem 6.3.8 to the language of ultraproducts.

Theorem 6.3.9. *For every operator $T \in \mathcal{L}(X,Y)$, there exist an ultrafilter $\mathfrak{U}$, metric injections $U \in \mathcal{L}(X^{**}, X_{\mathfrak{U}})$ and $V \in \mathcal{L}(Y^{**}, Y_{\mathfrak{U}})$ and metric surjections $P \in \mathcal{L}(X_{\mathfrak{U}}, X^{**})$ and $Q \in \mathcal{L}(Y_{\mathfrak{U}}, Y^{**})$ so that*

(i) $T_{\mathfrak{U}} \circ U = V \circ T^{**}$;

(ii) $T^{**} \circ P = Q \circ T_{\mathfrak{U}}$;

(iii) $T^{**} = Q \circ T_{\mathfrak{U}} \circ U$.

Moreover, $U(x) = [x]$ and $P([x]) = x$ for all $x \in X$ and $V(y) = [y]$ and $Q([y]) = y$ for all $y \in Y$.

Proof. Let J be the set of all tuples $j \equiv (E_j, F_j, \varepsilon_j, \mathcal{U}_j, \mathcal{V}_j)$ where E_j and F_j are finite dimensional subspaces of X^{**} and Y^{**} respectively, $\varepsilon_j \in (0,1)$, $\mathcal{U}_j$ is a weak* neighborhood of $0 \in X^{**}$ and $\mathcal{V}_j$ is a weak* neighborhood of $0 \in Y^{**}$. We define an order $\preceq$ in J by $i \preceq j$ if $E_i \subset E_j$, $F_i \subset F_j$, $\varepsilon_i \geq \varepsilon_j$, $\mathcal{U}_i \supset \mathcal{U}_j$ and $\mathcal{V}_i \supset \mathcal{V}_j$. Let $\mathfrak{U}$ be an ultrafilter refining the order filter on J.

For every $j \in J$, Theorem 6.3.8 provides us with a couple of $(1+\varepsilon_j)$-injections $U_j \in \mathcal{L}(E_j, X)$ and $V_j \in \mathcal{L}(T^{**}(E_j) + F_j, Y)$ such that

$$\begin{aligned}
&U_j(e) = e \ \text{ for all } e \in E_j \cap X,\\
&V_j(f) = f \ \text{ for all } f \in (T^{**}(E_j) + F_j) \cap Y,\\
&\|(TU_j - V_jT^{**})(e)\| < \varepsilon \ \text{ for all } e \in S_{E_j},\\
&U_j(e) \in e + \mathcal{U}_j \ \text{ for all } e \in S_{E_j},\\
&V_j(f) \in f + \mathcal{V}_j \ \text{ for all } f \in S_{T^{**}(E_j)+F_j}.
\end{aligned}$$

The operators U, V, P and Q are defined as follows:

$$\begin{aligned}
&U(x^{**}) = [x_j] \text{ where } x_j := U_j(x^{**}) \text{ if } x^{**} \in E_j, \text{ and } x_j := 0 \text{ otherwise},\\
&V(y^{**}) = [y_j] \text{ where } y_j := V_j(y^{**}) \text{ if } y^{**} \in T^{**}(E_j) + F_j, \text{ and } y_j := 0 \text{ otherwise},\\
&P([x_j]) = w^*\text{-}\lim_{j\to\mathfrak{U}} x_j \in X^{**},\\
&Q([y_j]) = w^*\text{-}\lim_{j\to\mathfrak{U}} y_j \in Y^{**}.
\end{aligned}$$

Since $\varepsilon_i \underset{\mathfrak{U}}{\longrightarrow} 0$, Lemma A.4.11 shows that both U and V are metric injections.

The fact that P is a metric surjection follows from $P(\operatorname{int} B_{X_{\mathfrak{U}}}) = \operatorname{int} B_{X^{**}}$. The same argument applies for Q.

To prove (i), take $x^{**} \in S_{X^{**}}$ and $\delta > 0$. Select $j_0 \in J$ such that $\varepsilon_{j_0} \leq \delta$ and $x^{**} \in E_{j_0}$. Thus

$$\{j \in J : \|(TU_j - V_jT^{**})(x^{**})\| \leq \delta\} \supset \{j \in J : j_0 \preceq j\} \in \mathfrak{U}$$

which shows that $(T_{\mathfrak{U}}U - VT^{**})x^{**} = 0$.

For statement (ii), take $[x_j] \in X_{\mathfrak{U}}$. Then

$$T^{**}P([x_j]) = T^{**}(w^*\text{-}\lim_{j\to\mathfrak{U}} x_j) = w^*\text{-}\lim_{j\to\mathfrak{U}} T(x_j) = QT_{\mathfrak{U}}([x_j]).$$

Part (iii) is achieved by using similar arguments. The equalities $U(x) = [x]$ and $P([x]) = x$ for all $x \in X$ and $V(y) = [y]$ and $P([y]) = y$ for all $y \in Y$ are trivial. □

Theorem 6.3.9 and the translations to ultrapower language of the operator local representability and local supportability given in Propositions 6.3.2 and 6.3.5 immediately yield the following result:

Corollary 6.3.10. *For every $T \in \mathcal{L}(X,Y)$, T^{**} is both locally 1-representable in and locally 1-supportable by T.*

Our next target is to prove that, for every operator T and every ultrafilter $\mathfrak{U}$, ${T_{\mathfrak{U}}}^*$ is locally representable and locally supportable by ${T^*}_{\mathfrak{U}}$. First, we give a preliminary result.

Proposition 6.3.11. *For every operator $T \in \mathcal{L}(X,Y)$ and every ultrafilter $\mathfrak{U}$, the set $B = \{\mathbf{h} \in B_{{Y_{\mathfrak{U}}}^*} : \|{T_{\mathfrak{U}}}^*(\mathbf{h})\| \le 1\}$ is the weak* closure in ${Y_{\mathfrak{U}}}^*$ of*

$$A = \{\mathbf{h} \in B_{{Y^*}_{\mathfrak{U}}} : \|{T^*}_{\mathfrak{U}}(\mathbf{h})\| \le 1\}.$$

Proof. Let I be the set of indices on which $\mathfrak{U}$ is taken, and let w^* denote the $\sigma({Y_{\mathfrak{U}}}^*, Y_{\mathfrak{U}})$ topology of ${Y_{\mathfrak{U}}}^*$.

The inclusion $\overline{A}^{w^*} \subset B$ is immediate. For the converse inclusion, take $\mathbf{f} \notin \overline{A}^{w^*}$ and prove that $\mathbf{f} \notin B$. By the Hahn-Banach theorem, there is $\mathbf{y}_0 = [y_i] \in Y_{\mathfrak{U}}$ and a pair of real numbers a, b such that $\langle \mathbf{h}, \mathbf{y}_0\rangle \le a < b < \langle \mathbf{f}, \mathbf{y}_0\rangle$ for all $\mathbf{h} \in A$. For every $i \in I$, let

$$V_i := \{f \in B_{Y^*} : b < \langle f, y_i\rangle\}.$$

Since ${Y^*}_{\mathfrak{U}}$ is weak* dense in ${Y_{\mathfrak{U}}}^*$ (see Proposition A.4.23),

$$\{i \in I : V_i \ne \emptyset\} \in \mathfrak{U}.$$

Let $W := (V_i)_{\mathfrak{U}}$ and note that $\mathbf{f} \in \overline{W}^{w^*}$ and $A \cap W = \emptyset$. Thus ${T_{\mathfrak{U}}}^*(\mathbf{f}) \in \overline{{T^*}_{\mathfrak{U}}(W)}^{w^*}$ and $\|{T^*}_{\mathfrak{U}}(\mathbf{w})\| > 1$ for all $\mathbf{w} \in W$. Therefore there exist $\theta > 1$ and $J \in \mathfrak{U}$ such that

$$\|T^*(v)\| \ge \theta \text{ for all } i \in J \text{ and all } v \in V_i. \tag{6.11}$$

Otherwise, for every $n \in \mathbb{N}$ and $J \in \mathfrak{U}$, we would have

$$J_n := \{i \in J : \text{there is } v_i \in V_i \text{ such that } \|T^*(v_i)\| < 1 + n^{-1}\} \in \mathfrak{U}; \tag{6.12}$$

since $\mathfrak{U}$ is $\aleph_0$-incomplete, for each $n \in \mathbb{N}$ we could take subsets $G_n \subset J_n$ such that $G_n \in \mathfrak{U}$, $G_n \supset G_{n+1}$ and $\cap_{n=1}^{\infty} G_n = \emptyset$.

For every $i \in G_1$, let $n_i \in \mathbb{N}$ be the unique integer such that $i \in G_{n_i} \setminus G_{n_i+1}$. By formula (6.12), there exist $v_i \in V_i$ such that $\|T^*(v_i)\| < 1 + n_i^{-1}$. So defining $v_i := 0$ when $i \in I \setminus G_1$, we would get

$$\|T^*{}_{\mathfrak{U}}([v_i])\| = \lim_{i\to\mathfrak{U}} \|T^*(v_i)\| \leq 1;$$

hence $[v_i] \in A \cap W$, in contradiction with $A \cap W = \emptyset$. Therefore (6.11) holds.

Next, we choose $\theta > \eta > 1$. Since each $\overline{V_i}^{w^*}$ is w^*-compact, by formula (6.11) there exists $x_i \in B_X$ such that $\langle T^*(v), x_i\rangle > \eta$ for all $v \in V_i$. Hence, for $\mathbf{x} := [x_i]$, we get $\langle T^*{}_{\mathfrak{U}}(\mathbf{w}), \mathbf{x}\rangle \geq \eta$ for all $\mathbf{w} \in W$. Moreover $T_{\mathfrak{U}}{}^*(\mathbf{f}) \in \overline{T_{\mathfrak{U}}{}^*(W)}^{w^*}$, so

$$\|T_{\mathfrak{U}}{}^*(\mathbf{f})\| \geq \langle T_{\mathfrak{U}}{}^*(\mathbf{f}), \mathbf{x}\rangle \geq \eta,$$

hence $\mathbf{f} \notin B$. □

The following theorem is another of the central results in this section. In its proof, an operator L is identified with the adjoint of a certain operator A. That identification is based upon the fact that, for any Banach space Z, any positive integer m and any ultrafilter $\mathfrak{U}$, the dual of $\ell_1^m(Z_{\mathfrak{U}})$ is isometrically identified with the space $\ell_\infty^m(Z_{\mathfrak{U}}{}^*)$.

Theorem 6.3.12. *Let $T \in \mathcal{L}(X,Y)$, $\mathfrak{U}$ an ultrafilter on I, F and G a pair of finite dimensional subspaces of $Y_{\mathfrak{U}}{}^*$ and of $X_{\mathfrak{U}}{}^*$ such that $T_{\mathfrak{U}}{}^*(F) \cap G = \{0\}$, and let $H := T_{\mathfrak{U}}{}^*(F) \oplus G$. Then, given a weak* neighborhood $\mathcal{U}$ of 0 in $Y_{\mathfrak{U}}{}^*$, a weak* neighborhood $\mathcal{V}$ of 0 in $X_{\mathfrak{U}}{}^*$ and $\varepsilon > 0$, there is a pair of $(1+\varepsilon)$-isometries $U \in \mathcal{L}(F, Y^*{}_{\mathfrak{U}})$ and $V \in \mathcal{L}(H, X^*{}_{\mathfrak{U}})$ verifying*

(i) $\|(T^*{}_{\mathfrak{U}}U - VT_{\mathfrak{U}}{}^*)\mathbf{f}\| \leq \varepsilon$ *for all* $\mathbf{f} \in S_F$,

(ii) $U(\mathbf{f}) \in \mathbf{f} + \mathcal{U}$ *for all* $\mathbf{f} \in S_F$,

(iii) $V(\mathbf{h}) \in \mathbf{h} + \mathcal{V}$ *for all* $\mathbf{h} \in S_H$.

Proof. Without loss of generality, we assume that $\mathcal{U}$ and $\mathcal{V}$ are absolutely convex and $\|T\| = 1$. Choose real numbers $0 < \eta < 1$ and $\delta > 0$ small enough so that

$$\frac{\eta+\delta}{1-\eta} < \varepsilon,$$
$$\delta\mathcal{U} + \left(\eta + \frac{1+\delta}{1-\eta}\eta\right) B_{Y_{\mathfrak{U}}{}^*} \subset \mathcal{U},$$
$$\delta\mathcal{V} + \left(\eta + \frac{1+\delta}{1-\eta}\eta\right) B_{X_{\mathfrak{U}}{}^*} \subset \mathcal{V}.$$

Let $\{\mathbf{f}_i\}_{i=1}^k$ be a normalized basis of the kernel $N(T_{\mathfrak{U}}{}^*|_F)$, which is completed up to a normalized basis $\{\mathbf{f}_i\}_{i=1}^l$ of F. Take a normalized basis $\{\mathbf{h}_i\}_{i=l+1}^m$ of G and write $\mathbf{h}_i := T_{\mathfrak{U}}{}^*(\mathbf{f}_i)$ for $i = k+1, \dots, l$. Select η-nets $\{\mathbf{e}_i\}_{i=1}^n$ in S_F and $\{\mathbf{c}_i\}_{i=1}^n$ in S_H and real numbers λ_i^j and μ_i^j so that

$$\mathbf{e}_j = \sum_{i=1}^{l} \lambda_i^j \mathbf{f}_i \quad \text{and} \quad \mathbf{c}_j = \sum_{i=k+1}^{m} \mu_i^j \mathbf{h}_i \quad \text{for all } j.$$

Consider the operator

$$L\colon \ell_\infty^l(Y_{\mathfrak{U}}{}^*) \oplus_\infty \ell_\infty^{m-l}(X_{\mathfrak{U}}{}^*) \longrightarrow \ell_\infty^n(Y_{\mathfrak{U}}{}^*) \oplus_\infty \ell_\infty^n(X_{\mathfrak{U}}{}^*) \oplus_\infty \ell_\infty^k(X_{\mathfrak{U}}{}^*)$$

that maps each $((\mathbf{v}_i)_{i=1}^l, (\mathbf{w}_i)_{i=l+1}^m)$ to

$$\left(\Big(\sum_{i=1}^{l} \lambda_i^j \mathbf{v}_i\Big)_{j=1}^n, \Big(\sum_{i=k+1}^{l} \mu_i^j T_{\mathfrak{U}}{}^*(\mathbf{v}_i) + \sum_{i=l+1}^{m} \mu_i^j \mathbf{w}_i \Big)_{j=1}^n, \Big(k\varepsilon^{-1} T_{\mathfrak{U}}{}^*(\mathbf{v}_i)\Big)_{i=1}^k \right).$$

Note that L maps the subspace $\ell_\infty^l(Y^*{}_{\mathfrak{U}}) \oplus_\infty \ell_\infty^{m-l}(X^*{}_{\mathfrak{U}})$ into

$$\ell_\infty^n(Y^*{}_{\mathfrak{U}}) \oplus_\infty \ell_\infty^n(X^*{}_{\mathfrak{U}}) \oplus_\infty \ell_\infty^k(X^*{}_{\mathfrak{U}}).$$

Also note that L can be identified with the adjoint of some operator

$$A\colon \ell_1^n(Y_{\mathfrak{U}}) \oplus_1 \ell_1^n(X_{\mathfrak{U}}) \oplus_1 \ell_1^k(X_{\mathfrak{U}}) \longrightarrow \ell_1^l(Y_{\mathfrak{U}}) \oplus_1 \ell_1^{m-l}(X_{\mathfrak{U}}).$$

Therefore, as

$$\big\| L\big((\mathbf{f}_i)_{i=1}^l, (\mathbf{h}_i)_{i=l+1}^m\big) \big\| = \big\| \big((\mathbf{e}_j)_{j=1}^n, (\mathbf{c}_j)_{j=1}^n, (0)_{j=1}^k\big) \big\| \leq 1,$$

Proposition 6.3.11 gives a net $((\mathbf{f}_i^\alpha)_{i=1}^l, (\mathbf{h}_i^\alpha)_{i=l+1}^m)_\alpha$ contained in the unit ball of $\ell_\infty^l(Y^*{}_{\mathfrak{U}}) \oplus_\infty \ell_\infty^{m-l}(X^*{}_{\mathfrak{U}})$ which is weak* convergent to $((\mathbf{f}_i)_{i=1}^l, (\mathbf{h}_i)_{i=l+1}^m)$ and satisfies $\|L((\mathbf{f}_i^\alpha)_{i=1}^l, (\mathbf{h}_i^\alpha)_{i=l+1}^m)_\alpha\| \leq 1$ for all α.

For each α, we define operators $U_\alpha \in \mathcal{L}(F, Y^*{}_{\mathfrak{U}})$ and $V_\alpha \in \mathcal{L}(H, X^*{}_{\mathfrak{U}})$ by

$$U_\alpha(\mathbf{f}_i) := \mathbf{f}_i^\alpha \text{ for all } i \in \{1, \dots, l\},$$
$$V_\alpha(\mathbf{h}_i) := \begin{cases} T^*{}_{\mathfrak{U}}(\mathbf{f}_i^\alpha), & \text{if } i \in \{k+1, \dots, l\}, \\ \mathbf{h}_i^\alpha, & \text{if } i \in \{l+1, \dots, m\}. \end{cases}$$

We obtain

$$w^*\text{-}\lim_\alpha U_\alpha(\mathbf{e}_i) = \mathbf{e}_i \text{ and } \|U_\alpha(\mathbf{e}_i)\| \leq 1 \text{ for all } i \in \{1, \dots, n\}, \tag{6.13}$$

$$w^*\text{-}\lim_\alpha V_\alpha(\mathbf{c}_i) = \mathbf{c}_i \text{ and } \|V_\alpha(\mathbf{c}_i)\| \leq 1 \text{ for all } i \in \{1, \dots, n\}, \tag{6.14}$$

$$\|T^*{}_{\mathfrak{U}}(U_\alpha \mathbf{f}_i)\| \leq k^{-1}\varepsilon \text{ for all } i \in \{1, \dots, k\}. \tag{6.15}$$

The choice of δ and η, Lemma 6.3.7 and formulas (6.13) and (6.14) allow us to choose an index β such that U_β and V_β are $(1+\varepsilon)$-isometries satisfying statements

(ii) and (iii). For statement (i), given $\mathbf{v} = \sum_{i=1}^{l} \nu_i \mathbf{f}_i \in S_F$, formula (6.15) and the fact that $\{\mathbf{f}_i\}_{i=1}^k$ is a normalized basis lead to

$$\|(T^*{}_{\mathfrak{U}} U_\alpha - V_\alpha T_{\mathfrak{U}}{}^*)(\mathbf{v})\| = \Big\|\sum_{i=1}^{k} \nu_i T^*{}_{\mathfrak{U}} U_\alpha(\mathbf{f}_i)\Big\| \le k^{-1}\varepsilon \sum_{i=1}^{k} |\nu_i| \le \varepsilon,$$

as we wanted to prove. □

The following theorem is a translation of Theorem 6.3.12 to ultraproduct language. It allows us to show that, given an operator T and an ultrafilter $\mathfrak{U}$, $T_{\mathfrak{U}}{}^*$ is both locally representable in and locally supportable by $T^*{}_{\mathfrak{U}}$.

Theorem 6.3.13. *For every operator $T \in \mathcal{L}(X, Y)$ and every ultrafilter $\mathfrak{U}$ there are an ultrafilter $\mathfrak{V}$, metric injections $U \in \mathcal{L}(Y_{\mathfrak{U}}{}^*, (Y^*{}_{\mathfrak{U}})_{\mathfrak{V}})$ and $V \in \mathcal{L}(X_{\mathfrak{U}}{}^*, (X^*{}_{\mathfrak{U}})_{\mathfrak{V}})$, and metric surjections $P \in \mathcal{L}((Y^*{}_{\mathfrak{U}})_{\mathfrak{V}}, Y_{\mathfrak{U}}{}^*)$ and $Q \in \mathcal{L}((X^*{}_{\mathfrak{U}})_{\mathfrak{V}}, X_{\mathfrak{U}}{}^*)$ verifying*

(i) $(T^*{}_{\mathfrak{U}})_{\mathfrak{V}} \circ U = V \circ T_{\mathfrak{U}}{}^*$;

(ii) $T_{\mathfrak{U}}{}^* \circ P = Q \circ (T^*{}_{\mathfrak{U}})_{\mathfrak{V}}$;

(iii) $T_{\mathfrak{U}}{}^* = Q \circ (T^*{}_{\mathfrak{U}})_{\mathfrak{V}} \circ U$.

Proof. Let J be the set of all tuples $j \equiv (F_j, E_j, \varepsilon_j, \mathcal{U}_j, \mathcal{V}_j)$, where F_j and E_j are finite dimensional subspaces of $Y_{\mathfrak{U}}{}^*$ and $X_{\mathfrak{U}}{}^*$, $\varepsilon_j \in (0,1)$, $\mathcal{U}_j$ is a weak* neighborhood of 0 in $Y_{\mathfrak{U}}{}^*$ and $\mathcal{V}_j$ is a weak* neighborhood of $0 \in X_{\mathfrak{U}}{}^*$. We endow J with the order $\preceq$, where $i \preceq j$ means $F_i \subset F_j$, $E_i \subset E_j$, $\varepsilon_i \ge \varepsilon_j$, $\mathcal{U}_i \supset \mathcal{U}_j$ and $\mathcal{V}_i \supset \mathcal{V}_j$. Let $\mathfrak{V}$ be an ultrafilter on J containing the $\preceq$-order filter.

For every index $j \in J$, Theorem 6.3.12 gives a pair of $(1+\varepsilon_j)$-injections $U_j \in \mathcal{L}(F_j, Y^*{}_{\mathfrak{U}})$ and $V_j \in \mathcal{L}(T_{\mathfrak{U}}{}^*(F_j) + E_j, X^*{}_{\mathfrak{U}})$ verifying

$$\begin{aligned} &\|T^*{}_{\mathfrak{U}} U_j - V_j T_{\mathfrak{U}}{}^*|_{F_j}\| \le \varepsilon_j, \\ &U_j(\mathbf{v}) \in \mathbf{v} + \mathcal{U}_j \text{ for all } \mathbf{v} \in S_{F_j}, \\ &V_j(\mathbf{w}) \in \mathbf{w} + \mathcal{V}_j \text{ for all } \mathbf{w} \in S_{T_{\mathfrak{U}}{}^*(F_j)+E_j}. \end{aligned}$$

The operators U, V, P and Q are defined as follows:

$$\begin{aligned} U(\mathbf{v}) &:= [\mathbf{f}_j], \text{ where } \mathbf{f}_j := U_j(\mathbf{v}) \text{ if } \mathbf{v} \in F_j \text{ and } \mathbf{f}_j := 0 \text{ otherwise}, \\ V(\mathbf{w}) &:= [\mathbf{g}_j], \text{ where } \mathbf{g}_j := V_j(\mathbf{w}) \text{ if } \mathbf{w} \in T_{\mathfrak{U}}{}^*(F_j) + E_j \text{ and } \mathbf{g}_j := 0 \text{ otherwise}, \\ P([\mathbf{v}_j]) &:= w^*\text{-}\lim_{j\to\mathfrak{V}} \mathbf{v}_j \in Y_{\mathfrak{U}}{}^* \text{ for all } [\mathbf{v}_j] \in (Y^*{}_{\mathfrak{U}})_{\mathfrak{V}}, \\ Q([\mathbf{w}_j]) &:= w^*\text{-}\lim_{j\to\mathfrak{V}} \mathbf{w}_j \in X_{\mathfrak{U}}{}^* \text{ for all } [\mathbf{w}_j] \in (X^*{}_{\mathfrak{U}})_{\mathfrak{V}}. \end{aligned}$$

Since $\lim_{j\to\mathfrak{V}} \varepsilon_j = 0$, Lemma A.4.11 yields that U and V are metric injections.

In order to prove that P is a metric surjection, take any $\mathbf{v} \in S_{Y_{\mathfrak{U}}{}^*}$. By Proposition 6.3.11, we can choose a family $\{\mathbf{v}_j\}_{j\in J}$ in $B_{Y^*{}_{\mathfrak{U}}}$ such that $w^*\text{-}\lim_{j\to\mathfrak{V}} \mathbf{v}_j = \mathbf{v}$ and $\lim_{j\to\mathfrak{V}} \|\mathbf{v}_j\| = 1$; hence, since $\|P\| \le 1$, we have $P(\operatorname{int} B_{(Y^*{}_{\mathfrak{U}})_{\mathfrak{V}}}) = \operatorname{int} B_{Y_{\mathfrak{U}}{}^*}$ hence P is a metric surjection. The same argument applies for Q.

To prove (i), take $\mathbf{v} \in S_{Y_{\mathfrak{U}}{}^*}$ and $\varepsilon > 0$. Let $j_0 \in J$ such that $\mathbf{v} \in F_{j_0}$ and $\varepsilon_{j_0} < \varepsilon$. Then

$$\{j \in J : \|(T^*{}_{\mathfrak{U}} U_j - V_j T_{\mathfrak{U}}{}^*)\mathbf{v}\| < \varepsilon\} \supset \{j \in J : j_0 \preceq j\} \in \mathfrak{V},$$

so $((T^*{}_{\mathfrak{U}})_{\mathfrak{V}} U - V T_{\mathfrak{U}}{}^*)\mathbf{v} = 0$. For statement (ii), take $[\mathbf{v}_j] \in (Y^*{}_{\mathfrak{U}})_{\mathfrak{V}}$. Then

$$\begin{aligned} T_{\mathfrak{U}}{}^* P([\mathbf{v}_j]) &= T_{\mathfrak{U}}{}^*(w^*\text{-}\lim_{j\to\mathfrak{V}} \mathbf{v}_j) \\ &= w^*\text{-}\lim_{j\to\mathfrak{V}} T_{\mathfrak{U}}{}^*(\mathbf{v}_j) = Q(T^*{}_{\mathfrak{U}})_{\mathfrak{V}}([\mathbf{v}_j]). \end{aligned}$$

The proof of statement (iii) is similar to that of (i) and (ii). □

The next result is a direct consequence of Theorem 6.3.13 and the characterizations of local representability and local supportability in terms of ultrapowers given in Propositions 6.3.2 and 6.3.5.

Corollary 6.3.14. *Given an operator $T \in \mathcal{L}(X, Y)$ and an ultrafilter $\mathfrak{U}$, $T_{\mathfrak{U}}{}^*$ is locally 1-representable and locally 1-supportable by $T^*{}_{\mathfrak{U}}$.*

Some of the subsequent results need the fact that, for any Banach space Y and any ultrafilter $\mathfrak{U}$, the space $(Y^{**})_{\mathfrak{U}}$ embeds in $(Y_{\mathfrak{U}})^{**}$.

Proposition 6.3.15. *Let Y be a Banach space, $J_Y : Y \longrightarrow Y^{**}$ the natural embedding and $\mathfrak{U}$ an ultrafilter on a set I. Then there is an isometric operator $K : (Y^{**})_{\mathfrak{U}} \longrightarrow (Y_{\mathfrak{U}})^{**}$ so that $J_{Y_{\mathfrak{U}}} = K \circ (J_Y)_{\mathfrak{U}}$.*

Proof. Let $\mathcal{J} : (Y^*)^*{}_{\mathfrak{U}} \longrightarrow (Y^*)_{\mathfrak{U}}{}^*$ be the isometry that maps each $[y_i^{**}]$ to the functional $\mathbf{y}$ defined by

$$\langle \mathbf{y}, [y_i^*] \rangle := \lim_{\mathfrak{U}} \langle y_i^{**}, y_i^* \rangle \quad \text{for all } [y_i^*] \in (Y^*)_{\mathfrak{U}}. \tag{6.16}$$

Since $(Y^*)_{\mathfrak{U}}$ is a local dual of $Y_{\mathfrak{U}}$ (see Example A.4.19), Theorem A.4.20 provides an isometric extension operator $L : (Y^*)_{\mathfrak{U}}{}^* \longrightarrow (Y_{\mathfrak{U}})^{**}$ so that $J_{Y_{\mathfrak{U}}}(Y_{\mathfrak{U}}) \subset R(L)$.

Let us check that

$$J_{Y_{\mathfrak{U}}} = L \circ \mathcal{J} \circ (J_Y)_{\mathfrak{U}}. \tag{6.17}$$

Indeed, each $[y_i] \in Y_{\mathfrak{U}}$ is mapped by $\mathcal{J} \circ (J_Y)_{\mathfrak{U}}$ to the element $\mathbf{y}$ defined as in (6.16). Moreover, L is an extension operator, so $L(\mathbf{y})|_{(Y^*)_{\mathfrak{U}}} = \mathbf{y}$. Since $(Y^*)_{\mathfrak{U}}$ is $\sigma\big((Y_{\mathfrak{U}})^*, Y_{\mathfrak{U}}\big)$-dense in $(Y_{\mathfrak{U}})^*$, it follows that $L(\mathbf{y}) = J_{Y_{\mathfrak{U}}}([y_i])$. Thus, defining the isometry K as $L \circ \mathcal{J}$, identity (6.17) shows that $J_{Y_{\mathfrak{U}}} = K \circ (J_Y)_{\mathfrak{U}}$. □

A binary relation in a set A is said to be a *preorder* if it satisfies both the reflexive and the transitive properties.

Proposition 6.3.16. *Local representability is a preorder.*

Proof. Let $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$ and $T_3 \in \mathcal{L}(X_3, Y_3)$ be three operators such that $T_1 \prec_{\mathrm{lr}} T_2$ and $T_2 \prec_{\mathrm{lr}} T_3$, and prove that $T_1 \prec_{\mathrm{lr}} T_3$.

By Proposition 6.3.2, there exist an ultrafilter $\mathfrak{U}$ and a pair of operators $A_1 \in \mathcal{L}\big(X_1, (X_2)_{\mathfrak{U}}\big)$ and $B_1 \in \mathcal{L}\big((Y_2)_{\mathfrak{U}}, Y_1^{**}\big)$ so that

$$J_{Y_1} \circ T_1 = B_1 \circ (T_2)_{\mathfrak{U}} \circ A_1. \tag{6.18}$$

Also, there are an ultrafilter $\mathfrak{V}$ and a pair of operators $A_2 \in \mathcal{L}\big(X_2, (X_3)_{\mathfrak{V}}\big)$ and $B_2 \in \mathcal{L}\big((Y_3)_{\mathfrak{V}}, Y_2^{**}\big)$ such that

$$J_{Y_2} \circ T_2 = B_2 \circ (T_3)_{\mathfrak{V}} \circ A_2. \tag{6.19}$$

Let $P \in \mathcal{L}(Y_1^{*(4)}, Y_1^{**})$ be the operator that maps each F to $F|_{Y_1^*}$, so

$$B_1 = P \circ B_1^{**} \circ J_{(Y_2)_{\mathfrak{U}}}. \tag{6.20}$$

We consider the isometry $K \in \mathcal{L}\big((Y_2^{**})_{\mathfrak{U}}, ((Y_2)_{\mathfrak{U}})^{**}\big)$ supplied by Proposition 6.3.15, which satisfies $J_{(Y_2)_{\mathfrak{U}}} = K \circ (J_{Y_2})_{\mathfrak{U}}$. Taking ultrapowers following $\mathfrak{U}$ in (6.19), we get

$$K \circ (J_{Y_2})_{\mathfrak{U}} \circ (T_2)_{\mathfrak{U}} = K \circ (B_2)_{\mathfrak{U}} \circ \big((T_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} \circ (A_2)_{\mathfrak{U}}.$$

Thus,

$$J_{(Y_2)_{\mathfrak{U}}} \circ (T_2)_{\mathfrak{U}} = K \circ (B_2)_{\mathfrak{U}} \circ \big((T_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} \circ (A_2)_{\mathfrak{U}},$$

and composing with $P \circ B_1^{**}$ on the left and A_1 on the right, formula (6.20) yields

$$B_1 \circ (T_2)_{\mathfrak{U}} \circ A_1 = P \circ B_1^{**} \circ K \circ (B_2)_{\mathfrak{U}} \circ \big((T_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} \circ (A_2)_{\mathfrak{U}} \circ A_1.$$

Thus, by (6.18),

$$J_{Y_1} \circ T_1 = P \circ B_1^{**} \circ K \circ (B_2)_{\mathfrak{U}} \circ \big((T_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} \circ (A_2)_{\mathfrak{U}} \circ A_1. \tag{6.21}$$

Moreover, by the iteration theorem for ultrapowers (see Proposition A.4.7), there are surjective isometries

$$U : \big((X_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} \longrightarrow (X_3)_{\mathfrak{U}\times\mathfrak{V}}$$
$$V : (Y_3)_{\mathfrak{U}\times\mathfrak{V}} \longrightarrow \big((Y_3)_{\mathfrak{V}}\big)_{\mathfrak{U}}$$

such that $\big((T_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} = V \circ (T_3)_{\mathfrak{U}\times\mathfrak{V}} \circ U$. Therefore, defining

$$A := U \circ (A_2)_{\mathfrak{U}} \circ A_1,$$
$$B := P \circ B_1^{**} \circ K \circ (B_2)_{\mathfrak{U}} \circ V,$$

formula (6.21) yields $J_{Y_1} \circ T_1 = B \circ (T_3)_{\mathfrak{U}\times\mathfrak{V}} \circ A$, which proves that $T_1 \prec_{\mathrm{lr}} T_3$. □

The analogous result for local supportability is the following one:

Proposition 6.3.17. *Local supportability is a preorder.*

Proof. Let $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$ and $T_3 \in \mathcal{L}(X_3, Y_3)$ be three operators such that $T_1 \prec_{\mathrm{ls}} T_2$ and $T_2 \prec_{\mathrm{ls}} T_3$, and prove that $T_1 \prec_{\mathrm{ls}} T_3$. By Proposition 6.3.5, there are ultrafilters $\mathfrak{U}$ and $\mathfrak{V}$, embeddings $U_1 \in \mathcal{L}(X_1, (X_2)_{\mathfrak{U}})$ and $U_2 \in \mathcal{L}(X_2, (X_3)_{\mathfrak{V}})$ and operators $V_1 \in \mathcal{L}(\overline{R(T_1)}, (Y_2)_{\mathfrak{U}})$ and $V_2 \in \mathcal{L}(\overline{R(T_2)}, (Y_3)_{\mathfrak{V}})$ so that

$$V_1\overline{T_1} = (T_2)_{\mathfrak{U}} \circ U_1 \quad \text{and} \quad V_2\overline{T_2} = (T_3)_{\mathfrak{V}} \circ U_2.$$

Thus, $(V_2)_{\mathfrak{U}} V_1 \overline{T_1} = ((T_3)_{\mathfrak{V}})_{\mathfrak{U}} \circ (U_2)_{\mathfrak{U}} \circ U_1$. Besides, by the theorem of iteration for ultrapowers, there exists a pair of surjective isometries

$$A\colon \big((X_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} \longrightarrow (X_3)_{\mathfrak{U}\times\mathfrak{V}},$$
$$B\colon \big((Y_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} \longrightarrow (Y_3)_{\mathfrak{U}\times\mathfrak{V}}$$

satisfying $B \circ \big((T_3)_{\mathfrak{V}}\big)_{\mathfrak{U}} = (T_3)_{\mathfrak{U}\times\mathfrak{V}} \circ A$. Thus,

$$B \circ (V_2)_{\mathfrak{U}} \circ V_1 \circ \overline{T_1} = (T_3)_{\mathfrak{U}\times\mathfrak{V}} \circ A \circ (U_2)_{\mathfrak{U}} \circ U_1,$$

and Proposition 6.3.5 proves that $T_1 \prec_{\mathrm{ls}} T_3$. □

The following lemma will be necessary to show that, for any operator T, the residuum T^{co} is locally representable in and locally supportable by T. We observe that the existence of the operator L in the statement is a consequence of the principle of local reflexivity.

Lemma 6.3.18. *Let X be a Banach space, $Q_X\colon X^{**} \longrightarrow X^{co}$ the associated quotient operator, M a finite dimensional subspace of X^{co} and $0 < \varepsilon < 1$. Let $Z := Q_X^{-1}(M)$, take any projection $Q\colon Z \longrightarrow Z$ with $R(Q) = X$ and denote its kernel by G. Then we have:*

(i) *there exists a finite dimensional subspace F of X such that, for each element $g \in (I_Z - Q)B_Z$, there is $e \in F$ verifying $\|g - e\| \le 1 + \varepsilon$;*

(ii) *let $L\colon F \oplus G \longrightarrow X$ be a $(1+\varepsilon)$-injection verifying $L(x) = x$ for all $x \in F$, and define $P := Q + L(I_Z - Q)$. Then $P\colon Z \longrightarrow Z$ is a projection onto X with $\|P\| \le 3 + 4\varepsilon$;*

(iii) *the operator $U := Q_X|_{N(P)}$ is an isomorphism onto M such that $\|U\| = 1$, $\|U^{-1}\| \le 1 + \|P\|$ and $U^{-1}(g + X) = g - Lg$ for all $g \in G$.*

Proof. (i) Since $(I_Z - Q)(B_Z)$ is compact, we can choose a finite set $\{z_i\}_{i=1}^n$ in B_Z so that for $g_i := (I_Z - Q)(z_i)$, the family $\{g_i\}_{i=1}^n$ is an ε-net of $(I_Z - Q)(B_Z)$. Let $x_i := Q(z_i)$ for all $1 \le i \le n$ and prove that $F := \operatorname{span}\{x_i\}_{i=1}^n$ is the wanted subspace. Indeed, given $g \in (I_Z - Q)B_Z$, take g_i so that $\|g - g_i\| \le \varepsilon$. Thus

$$\|g + x_i\| \le \|g - g_i\| + \|g_i + x_i\| \le \varepsilon + \|z_i\| \le \varepsilon + 1.$$

(ii) It is straightforward that $P^2 = P$, so P is a projection. To evaluate $\|P\|$, take $z \in B_Z$ and write $g := (I_Z - Q)(z)$. By part (i), there is $h \in F$ verifying $\|g - h\| \leq 1 + \varepsilon$. Thus

$$\|L(g) - h\| = \|L(g - h)\| \leq \|L\|\|g - h\| \leq (1 + \varepsilon)^2$$

and

$$\|Q(z) + h\| \leq \|Q(z) + g\| + \|g - h\| = \|z\| + \|g - h\| \leq 2 + \varepsilon.$$

Therefore,

$$\begin{aligned}\|P(z)\| &= \|Q(z) + L(g)\| \\ &\leq \|Q(z) + h\| + \|L(g) - h\| \leq 3 + 4\varepsilon.\end{aligned}$$

The identity $R(P) = X$ follows from the fact that Q is a projection onto X. So $(I_Z - Q)(x) = 0$ for every $x \in X$, hence $P(x) = x$.

(iii) For every $z \in N(P)$, we have

$$\begin{aligned}\|U(z)\| &= \inf_{x \in X} \|z + x\| \\ &\geq \|I_Z - P\|^{-1} \inf_{x \in X} \|(I_Z - P)(z + x)\| = \|I_Z - P\|^{-1}\|z\|.\end{aligned}$$

Therefore, U is an isomorphism and $\|U^{-1}\| \leq 1 + \|P\|$.

Moreover, given $z \in X \oplus G$, let $z = x + g$ be its decomposition with $x \in X$ and $g \in G$. It is straightforward that $z \in N(P)$ if and only if $x + Lg = 0$, that is, if and only if $z = -L(g) + g$. Consequently, $U(N(P)) = M$ and $U^{-1}(g + X) = g - L(g)$ for all $g \in G$. □

Theorem 6.3.19. *For every $T \in \mathcal{L}(X, Y)$, the residuum operator T^{co} is locally supportable by T.*

Proof. Let M_0 be a finite dimensional subspace of X^{co} and $0 < \varepsilon < 1/2$. Let $Q_X : X^{**} \longrightarrow X^{co}$ and $Q_Y : Y^{**} \longrightarrow Y^{co}$ be the respective quotient operators.

Let $M_1 := T^{co}(M_0)$, $Z_0 := Q_X^{-1}(M_0)$ and $Z_1 := Q_Y^{-1}(M_1)$. We choose a finite dimensional subspace G_0 of X^{**} such that $Z_0 = X \oplus G_0$, and we denote $K_0 := \|Q_X|_{G_0}^{-1}\|$. We also consider a decomposition $T^{**}(G_0) = H_1 \oplus G_1$, where $H_1 \subset Y$ and $G_1 \cap Y = 0$. Obviously $Z_1 = Y \oplus G_1$.

Take the projections $Q_0 \in \mathcal{L}(Z_0, Z_0)$ and $Q_1 \in \mathcal{L}(Z_1, Z_1)$ with kernels G_0 and G_1 and ranges X and Y. By Lemma 6.3.18 (i), there are a pair of finite dimensional subspaces $F_0 \subset X$ and a subset F_1 of Y so that, for every $z_0 \in B_{Z_0}$ and every $z_1 \in B_{Z_1}$, there are $e_0 \in F_0$ and $e_1 \in F_1$ verifying $\|(I_{Z_0} - Q_0)(z_0) - e_0\| \leq 3/2$ and $\|(I_{Z_1} - Q_1)(z_1) - e_1\| \leq 3/2$.

By Theorem 6.3.8, there is a pair of 3/2-injections $L_0 : F_0 \oplus G_0 \longrightarrow X$ and $L_1 : (H_1 + F_1) \oplus G_1 \longrightarrow Y$ verifying $\|TL_0 - L_1 T^{**}|_{F_0 \oplus G_0}\| \leq \varepsilon K_0^{-1}$. Lemma 6.3.18 (ii) enables us to say that the operators

$$P_0 := Q_0 + L_0(I_{Z_0} - Q_0) \ \text{ and } \ P_1 := Q_1 + L_1(I_{Z_1} - Q_1)$$

are projections with norm smaller than or equal to 5 and their respective ranges are X and Y. Thus, Lemma 6.3.18 (iii) shows that the operators $U_0 := Q_X|_{N(P_0)}$ and $U_1 := Q_Y|_{N(P_1)}$ are 6-injections with $R(U_0) = M_0$ and $R(U_1) = M_1$. Let us denote by $U_0^{-1}\colon M_0 \longrightarrow N(P_0)$ and $U_1^{-1}\colon M_1 \longrightarrow N(P_1)$ the respective inverses of U_0 and U_1 on their ranges.

It only remains to show that $\|T^{**}U_0^{-1} - U_1^{-1}T^{co}|_{M_0}\| \leq \varepsilon$. In order to do that, take $g \in G_0$. Note that, by Lemma 6.3.18 (iii), $T^{**}U_0^{-1}(g+X) = T^{**}(g) - TL_0(g)$ and $U_1^{-1}T^{co}(g+X) = T^{**}(g) - L_1T^{**}(g)$, so

$$\|(T^{**}U_0^{-1} - U_1^{-1}T^{co})(g+X)\| \leq \varepsilon K_0^{-1}\|g\| \leq \varepsilon\|g+X\|.$$

Thus, $T^{co} \prec_{\text{ls}} T^{**}$. Since $T^{**} \prec_{\text{ls}} T$ and, by Proposition 6.3.17, $\prec_{\text{ls}}$ is a preorder, we conclude $T^{co} \prec_{\text{ls}} T$. □

Theorem 6.3.20. *For each $T \in \mathcal{L}(X,Y)$, the residuum operator T^{co} is locally representable in T.*

Proof. Let E and F be a pair of finite dimensional spaces, $A \in \mathcal{L}(E, X^{co})$ and $B \in \mathcal{L}(Y^{co}, F)$ a pair of operators, and $0 < \varepsilon < 1$.

We write $Q_X \in \mathcal{L}(X^{**}, X^{co})$ and $Q_Y \in \mathcal{L}(Y^{**}, Y^{co})$ the natural quotient operators. Let $Z := Q_X^{-1}(A(E))$, take a projection $Q \in \mathcal{L}(Z,Z)$ onto X and let $G := N(Q)$.

By Lemma 6.3.18 (i), there is a finite dimensional subspace F of X such that for every $z \in B_Z$ there is $e \in F$ so that $\|(I_Z - Q)(z) - e\| \leq 3/2$. By Theorem 6.3.8 there is a $(1+\varepsilon)$-injection $L \in \mathcal{L}(F \oplus G, X)$ verifying $L(x) = x$ for all $x \in F$. Hence, parts (ii) and (iii) of Lemma 6.3.18 show that $P := Q + L(I_Z - Q)$ is a projection with $\|P\| \leq 5$, $U := Q_X|_{N(P)}$ is a norm-one isomorphism with range $Q_X(N(P)) = A(E)$, the norm of the inverse $U^{-1}\colon A(E) \longrightarrow N(P)$ satisfies $\|U^{-1}\| \leq 6$ and $U^{-1}(g+X) = g - L(g)$ for all $g \in G$. Thus, defining operators $A_1 := U^{-1}A$ and $B_1 := BQ_Y$, we get

$$B_1T^{**}A_1 = BQ_YT^{**}U^{-1}A = BT^{co}Q_XU^{-1}A = BT^{co}A.$$

Moreover, $\|A_1\| \cdot \|B_1\| \leq \|U^{-1}\| \cdot \|A\| \cdot \|B\| \leq 6\|A\| \cdot \|B\|$. Therefore, T^{co} is locally 6-representable in T^{**}. Since $T^{**} \prec_{\text{lr}} T$ and $\prec_{\text{lr}}$ is a preorder (Proposition 6.3.16), we conclude $T^{co} \prec_{\text{lr}} T$. □

6.4 Ultrapower-stable classes of operators

For every class $\mathcal{C}$ of operators, we consider the class $\mathcal{C}^{up}$ defined by

$$\mathcal{C}^{up} := \{T \in \mathcal{C}\colon \text{ for each ultrafilter } \mathfrak{U},\ T_{\mathfrak{U}} \in \mathcal{C}\}.$$

In this section, we prove that, when $\mathcal{C}$ is an operator semigroup, $\mathcal{C}^{up}$ is the largest ultrapower-stable semigroup contained in $\mathcal{C}$. We also prove an analogous result when $\mathcal{C}$ is an operator ideal. Next, the notions of local supportability and local representability will be applied in the study of ultrapower-stable semigroups and ideals, obtaining results concerning the stability of those classes under duality.

Definition 6.4.1. A class of operators $\mathcal{C}$ is said to be *ultrapower-stable* if $\mathcal{C}^{up} = \mathcal{C}$.

Crucial examples of ultrapower-stable classes of operators are Φ_+, Φ_- and $\mathcal{K}$. Let us prove it.

Proposition 6.4.2. *The class Φ_+ of upper semi-Fredholm operators and the class Φ_- of lower semi-Fredholm operators are ultrapower-stable.*

Proof. The fact that both classes Φ_+ and Φ_- are operator semigroups has already been noted in Remark 6.1.9. Consider now an ultrafilter $\mathfrak{U}$.

Given an operator $T \in \Phi_+$, by Proposition A.4.22, $R(T_{\mathfrak{U}})$ is closed and $N(T_{\mathfrak{U}}) = N(T)_{\mathfrak{U}}$. Thus, as $N(T)$ is a finite dimensional subspace, $N(T) = N(T)_{\mathfrak{U}}$. Therefore, $T_{\mathfrak{U}}$ belongs to Φ_+.

Analogously, given $S \in \Phi_-$, Proposition A.4.22 gives that $R(T_{\mathfrak{U}})$ is closed and equals $R(T)_{\mathfrak{U}}$, so $Y_{\mathfrak{U}}/R(T_{\mathfrak{U}}) = Y_{\mathfrak{U}}/R(T)_{\mathfrak{U}}$. But by Proposition A.4.6, the quotient $Y_{\mathfrak{U}}/R(T)_{\mathfrak{U}}$ is isometric to $\big(Y/R(T)\big)_{\mathfrak{U}}$, hence $Y_{\mathfrak{U}}/R(T_{\mathfrak{U}})$ is finite dimensional, which yields that $T_{\mathfrak{U}} \in \Phi_-$. □

Proposition 6.4.3. *The class $\mathcal{K}$ of all compact operators is an ultrapower-stable operator ideal.*

Proof. Let $T \in \mathcal{L}(X,Y)$ be a compact operator, $\mathfrak{U}$ an ultrafilter on I, and prove that $T_{\mathfrak{U}}$ is also compact.

Since $T(B_X)$ is relatively compact, Propositions A.4.4 and A.4.5 yield that $\overline{T(B_X)}$ equals $T(B_X)_{\mathfrak{U}}$. But by Proposition A.4.21, $T_{\mathfrak{U}}(B_{X_{\mathfrak{U}}})$ equals $T(B_X)_{\mathfrak{U}}$, so $T_{\mathfrak{U}}(B_{X_{\mathfrak{U}}})$ is compact, hence $T_{\mathfrak{U}}$ is a compact operator. □

Let us state some of the structural properties of the class $\mathcal{S}^{up}$, where $\mathcal{S}$ is a semigroup.

Proposition 6.4.4. *Given an operator semigroup $\mathcal{S}$, its subclass $\mathcal{S}^{up}$ is the largest ultrapower-stable operator semigroup contained in $\mathcal{S}$.*

Proof. Let us first prove that $\mathcal{S}^{up}$ is a semigroup. By definition, $\mathcal{S}^{up}$ contains all ultrapower-stable subclasses of $\mathcal{S}$. In particular, since Proposition 6.4.2 has shown that Φ_+ and Φ_- are ultrapower-stable, the class $\Phi = \Phi_+ \cap \Phi_-$ is also ultrapower-stable, and therefore, $\Phi \subset \mathcal{S}^{up}$.

Consider now a pair of operators $S \in \mathcal{S}^{up}(V,W)$ and $T \in \mathcal{S}^{up}(X,Y)$ and an ultrafilter $\mathfrak{U}$. The isometric identifications between $(V \times X)_{\mathfrak{U}}$ and $V_{\mathfrak{U}} \times X_{\mathfrak{U}}$ and between $(W \times Y)_{\mathfrak{U}}$ and $W_{\mathfrak{U}} \times Y_{\mathfrak{U}}$ induce a natural identification of the operator $(S \times T)_{\mathfrak{U}}$ with $S_{\mathfrak{U}} \times T_{\mathfrak{U}}$. Thus, since $S_{\mathfrak{U}} \times T_{\mathfrak{U}} \in \mathcal{S}$, it follows that $(S \times T)_{\mathfrak{U}} \in \mathcal{S}$ as well. Therefore, $S \times T \in \mathcal{S}^{up}$.

Finally, let $S \in \mathcal{S}^{up}(W,X)$ and $T \in \mathcal{S}^{up}(X,Y)$. Thus $(TS)_{\mathfrak{U}} = T_{\mathfrak{U}}S_{\mathfrak{U}}$ so $TS \in \mathcal{S}^{up}$. We have just proved that $\mathcal{S}^{up}$ is an operator semigroup.

Let us prove now that $\mathcal{S}^{up}$ is ultrapower-stable, that is, prove that for every $T \in \mathcal{S}^{up}$ and every ultrafilter $\mathfrak{V}$, $T_{\mathfrak{V}}$ belongs to $\mathcal{S}^{up}$. In order to do so, take any ultrafilter $\mathfrak{U}$. The isometric identification of $X_{\mathfrak{U}\times\mathfrak{V}}$ with $(X_{\mathfrak{V}})_{\mathfrak{U}}$ and of $Y_{\mathfrak{U}\times\mathfrak{V}}$ with $(Y_{\mathfrak{V}})_{\mathfrak{U}}$ given in Proposition A.4.7 induce a natural identification of $(T_{\mathfrak{V}})_{\mathfrak{U}}$ with the operator $T_{\mathfrak{U}\times\mathfrak{V}}$. Thus, since $T_{\mathfrak{U}\times\mathfrak{V}} \in \mathcal{S}$, we get $(T_{\mathfrak{V}})_{\mathfrak{U}} \in \mathcal{S}$, such as we wanted. As a consequence, $\mathcal{S}^{up}$ is the largest ultrapower-stable semigroup contained in $\mathcal{S}$. □

Proposition 6.4.5. *Let $\mathcal{S}$ be an operator semigroup. Then the following statements hold:*

(i) *If $\mathcal{S}$ is upper, then $\mathcal{S}^{up}$ is upper too;*

(ii) *If $\mathcal{S}$ is lower, then $\mathcal{S}^{up}$ is lower too.*

Proof. (i) If $\mathcal{S}$ is an upper semigroup, then $\Phi_+ \subset \mathcal{S}$, and as Φ_+ is ultrapower-stable, it follows from Proposition 6.4.4 that $\Phi_+ \subset \mathcal{S}^{up}$, hence $\mathcal{S}^{up}$ is injective.

Besides, given a pair of operators $S \in \mathcal{L}(W,X)$ and $T \in \mathcal{L}(X,Y)$ such that $TS \in \mathcal{S}(W,Y)$, we have, for every ultrafilter $\mathfrak{U}$,

$$(TS)_{\mathfrak{U}} = T_{\mathfrak{U}} S_{\mathfrak{U}} \in \mathcal{S}.$$

Thus, since $\mathcal{S}$ is left-stable, then $S_{\mathfrak{U}} \in \mathcal{S}$, so $S \in \mathcal{S}^{up}$, which proves that $\mathcal{S}^{up}$ is left-regular, and therefore, $\mathcal{S}^{up}$ is an upper semigroup.

(ii) Let us assume that $\mathcal{S}$ is a lower semigroup. Then, with analogous arguments such as those given for part (i), the surjectivity of $\mathcal{S}^{up}$ is derived from the fact that Φ_- is ultrapower-stable, and the right-stability of $\mathcal{S}^{up}$ is a consequence of the right-stability of $\mathcal{S}$. □

The analogous result to Proposition 6.4.4 for operator ideals is the following result.

Proposition 6.4.6. *Let $\mathcal{A}$ be an operator ideal. Then $\mathcal{A}^{up}$ is the largest ultrapower-stable operator ideal contained in $\mathcal{A}$. Moreover, if $\mathcal{A}$ is regular, then so is $\mathcal{A}^{up}$.*

Proof. The fact that $\mathcal{A}^{up}$ is an operator ideal is a direct consequence of the following statements, which hold for any ultrafilter $\mathfrak{U}$:

(i) if T is an operator with $\dim R(T) < \infty$, then $R(T) = R(T_{\mathfrak{U}})$ (see Propositions A.4.5 and A.4.22);

(ii) for each pair of operators S and T in $\mathcal{L}(X,Y)$ and every pair of real numbers λ and μ, $(\lambda S + \mu T)_{\mathfrak{U}} = \lambda S_{\mathfrak{U}} + \mu T_{\mathfrak{U}}$;

(iii) for every pair of operators $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(Y,Z)$, $(TS)_{\mathfrak{U}} = T_{\mathfrak{U}} S_{\mathfrak{U}}$.

In order to prove that $\mathcal{A}^{up}$ is the largest ultrapower-stable ideal contained in $\mathcal{A}$, it is enough to apply the same argument given in Proposition 6.4.2 to proving that $\mathcal{S}^{up}$ is the largest ultrapower-stable semigroup contained in $\mathcal{S}$.

Next, assume that $\mathcal{A}$ is a regular ideal, and prove that $\mathcal{A}^{up}$ is also regular. Take an operator $T \in \mathcal{L}(X,Y)$, fix an ultrafilter $\mathfrak{U}$ on a set I and assume that $J_Y T \in \mathcal{A}^{up}(X,Y^{**})$. By Proposition 6.3.15, there exists an isometry $K\colon (Y^{**})_{\mathfrak{U}} \longrightarrow (Y_{\mathfrak{U}})^{**}$ such that

$$J_{Y_{\mathfrak{U}}} \circ T_{\mathfrak{U}} = K \circ (J_Y)_{\mathfrak{U}} \circ T_{\mathfrak{U}} = K \circ (J_Y T)_{\mathfrak{U}} \in \mathcal{A}.$$

Since $\mathcal{A}$ is regular, $T_{\mathfrak{U}} \in \mathcal{A}$, and therefore, $T \in \mathcal{A}^{up}$, proving that $\mathcal{A}^{up}$ is regular. □

Remark 6.4.7. Given a closed subspace E of a Banach space X and an ultrafilter $\mathfrak{U}$, it is immediate that $(J_E)_{\mathfrak{U}}$ is identified with $J_{(E_{\mathfrak{U}})}$ and $(Q_E)_{\mathfrak{U}}$ with $Q_{(E_{\mathfrak{U}})}$; therefore, if $\mathcal{A}$ is an injective (respectively surjective) operator ideal, then $\mathcal{A}^{up}$ is also injective (resp. surjective).

The main regular ultrapower-stable operator ideals occurring along the following sections are $\mathcal{K}$ and $\mathcal{W}^{up}$. The ultrapower-stability of $\mathcal{K}$ was proved in Proposition 6.4.3. Proposition 6.4.6 shows that $\mathcal{W}^{up}$ is a regular ideal as a consequence of the regularity of $\mathcal{W}$.

Definition 6.4.8. An operator $T\colon X \longrightarrow Y$ is said to be *super weakly compact* whenever $T \in \mathcal{W}^{up}$.

The elements of $\mathcal{W}^{up}$ are also called *uniform convexifying* operators (see [19] and [101]). Note that, since $\mathcal{K}$ is ultrapower-stable and $\mathcal{K} \subset \mathcal{W}$, then $\mathcal{K} \subset \mathcal{W}^{up}$ because $\mathcal{W}^{up}$ is the largest ultrapower-stable subclass contained in $\mathcal{W}$.

Although the DFJP factorization shows that every weakly compact operator factorizes through a reflexive space (see Corollary 3.2.3), Beauzamy ([19] found a super weakly compact operator that does not factorize through any super-reflexive space; in other words: $\mathcal{W}^{up}$ does not satisfy the interpolation property.

Let us see some applications of local supportability and local representability to the study of ultrapower-stable operator semigroups and ideals.

Definition 6.4.9. Given a class of operators $\mathcal{A}$ endowed with a preorder $\preceq$, $\mathcal{A}$ is said to be *$\preceq$-stable* if $T \in \mathcal{A}$ and $S \preceq T$ imply $S \in \mathcal{A}$.

Proposition 6.4.10. *Let $\mathcal{A}$ be an ultrapower-stable, regular ideal. Then $\mathcal{A}$ is $\prec_{\mathrm{lr}}$-stable.*

Proof. Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{A}(W,Z)$ be a pair of operators such that $T \prec_{\mathrm{lr}} S$. Then, by Proposition 6.3.2, there exist an ultrafilter $\mathfrak{U}$ and a pair of operators A in $\mathcal{L}(X,W_{\mathfrak{U}})$ and B in $\mathcal{L}(Z_{\mathfrak{U}},Y^{**})$ such that $BS_{\mathfrak{U}}A = J_Y T$. Since $\mathcal{A}$ is ultrapower-stable, it follows that $J_Y T \in \mathcal{A}$, and as $\mathcal{A}$ is regular, T belongs to $\mathcal{A}$. □

The following result shows that the role played by local supportability with respect to upper ultrapower-stable semigroups is similar to that played by local representability with respect to regular ultrapower-stable ideals.

Proposition 6.4.11. *Let $\mathcal{S}$ be an upper ultrapower-stable operator semigroup. Then $\mathcal{S}$ is $\prec_{\mathrm{ls}}$-stable.*

Proof. Assume $S \in \mathcal{S}$ and $T \prec_{\mathrm{ls}} S$. By Proposition 6.3.5 there are an ultrafilter $\mathfrak{U}$, an isomorphism U and an operator V such that $V\overline{T} = S_{\mathfrak{U}}U$. Since $\mathcal{S}$ is ultrapower-stable, we have $S_{\mathfrak{U}} \in \mathcal{S}$. Moreover, $\mathcal{S}$ is injective so $U \in \mathcal{S}$, therefore $S_{\mathfrak{U}}U \in \mathcal{S}$. Left-stability yields $\overline{T} \in \mathcal{S}$, and again, the injectivity of $\mathcal{S}$ leads to $T \in \mathcal{S}$. □

For lower semigroups, local supportability is also operative. In order to see this, let us first prove the following result about ultrapower-stability.

Proposition 6.4.12. *Given an ultrapower-stable semigroup $\mathcal{S}$, the following statements hold:*

(i) *if $\mathcal{S}$ is lower, then $\mathcal{S}^d$ is ultrapower-stable;*

(ii) *if $\mathcal{S}$ is upper, then $\mathcal{S}^d$ is ultrapower-stable.*

Proof. (i) Take $T \in \mathcal{S}^d$, that is, $T^* \in \mathcal{S}$. Given any ultrafilter $\mathfrak{U}$, we have ${T^*}_{\mathfrak{U}} \in \mathcal{S}$. By Theorem 6.3.13, there exist a pair of metric surjections P and Q and an ultrafilter $\mathfrak{V}$ such that ${T_{\mathfrak{U}}}^* \circ P = Q \circ ({T^*}_{\mathfrak{U}})_{\mathfrak{V}}$. Since $\mathcal{S}$ is ultrapower-stable and surjective, we have $Q \circ ({T^*}_{\mathfrak{U}})_{\mathfrak{V}} \in \mathcal{S}$. But $\mathcal{S}$ is also right-stable, so ${T_{\mathfrak{U}}}^* \in \mathcal{S}$ and $T_{\mathfrak{U}} \in \mathcal{S}^d$.

The proof of statement (ii) follows a similar argument to that of (i). □

Thus we get the subsequent result for ultrapower-stable lower semigroups.

Corollary 6.4.13. *Let $\mathcal{S}$ be a lower ultrapower-stable operator semigroup. Then $\mathcal{S}^d$ is $\prec_{\mathrm{ls}}$-stable.*

Proof. It follows from Propositions 6.4.11 and 6.4.12. □

Corollary 6.4.14. *Let $\mathcal{S}$ be either an upper or a lower ultrapower-stable operator semigroup and $T \in \mathcal{S}(X, Y)$. Then T^{**} and T^{co} belong to $\mathcal{S}$.*

Proof. Assume $\mathcal{S}$ is upper. By Corollary 6.3.10 and Theorem 6.3.20, both operators T^{**} and T^{co} are locally supportable by T, and by Proposition 6.4.11, $\mathcal{S}$ is $\prec_{\mathrm{ls}}$-stable. Thus, if $T \in \mathcal{S}$, then T^{**} and T^{co} also belong to $\mathcal{S}$.

Assume $\mathcal{S}$ is lower. By Theorem 6.3.9, there exist an ultrafilter $\mathfrak{U}$ and metric surjections $P \in \mathcal{L}(X_{\mathfrak{U}}, X^{**})$ and $Q \in \mathcal{L}(Y_{\mathfrak{U}}, Y^{**})$ such that $T^{**} \circ P = Q \circ T_{\mathfrak{U}}$. If $T \in \mathcal{S}$, since $\mathcal{S}$ is surjective and ultrapower-stable, we get $Q \circ T_{\mathfrak{U}} \in \mathcal{S}$, and therefore the right-stability of $\mathcal{S}$ yields that T^{**} belongs to $\mathcal{S}$.

In order to prove that $T^{co} \in \mathcal{S}$, let us denote by $Q_X \in \mathcal{L}(X^{**}, X^{co})$ and $Q_Y \in \mathcal{L}(Y^{**}, Y^{co})$ the natural quotient operators. Since $Q_Y \circ T^{**} = T^{co} \circ Q_X$ and $T^{**} \in \mathcal{S}$, a similar argument to that of part (i) leads to $T^{co} \in \mathcal{S}$. □

There is an analogous result to Corollary 6.4.14 for operator ideals.

Proposition 6.4.15. *Let $\mathcal{A}$ be an ultrapower-stable operator ideal and $T \in \mathcal{A}$. Then $T^{**} \in \mathcal{A}$. Moreover, if $\mathcal{A}$ is regular, then $T^{co} \in \mathcal{A}$.*

Proof. By Theorem 6.3.9, there is an ultrafilter $\mathfrak{U}$ and a pair of operators Q and U such that $T^{**} = QT_{\mathfrak{U}}U$. Thus, the hypothesis of ultrapower-stability of $\mathcal{A}$ yields that T^{**} belongs to $\mathcal{A}$.

On the other hand, it has been shown in Theorem 6.3.20 that T^{co} is locally representable in T, so Proposition 6.4.10 implies $T^{co} \in \mathcal{A}$. □

6.5 Supertauberian operators

The subject of this section is the study of the ultrapower-stable semigroup $\mathcal{T}^{up}$, also known as the class of supertauberian operators. Although the class $\mathcal{T}^{up}$ is smaller than the class $\mathcal{ST}$ of strongly tauberian operators, it turns out that, for a finite measure μ, all tauberian operators on $L_1(\mu)$ are supertauberian, as will be proved later.

For technical reasons, the class of supertauberian operators will be introduced via its original definition, given by Tacon, rather than introducing it as the ultrapower-stable class $\mathcal{T}^{up}$. Note that Tacon's definition is local and does not require the use of ultrafilters.

The central result of this section is the perturbative characterization of Theorem 6.5.16, which says that an operator T is supertauberian if and only if for every compact operator K, the kernel $N(T+K)$ is super-reflexive.

Let us recall that a Banach space X is super-reflexive if and only if every Banach space Y finitely representable in X is reflexive (see Appendix A.5). Super-reflexivity admits characterizations in terms of finite ε-triangular sequences. They are also used to define the notion of supertauberian operator.

Definition 6.5.1. Given a real number $\varepsilon > 0$, a finite sequence $(x_k)_{k=1}^n$ in a Banach space X is said to be a *finite ε-triangular sequence* if $\|x_k\| \leq 1$ for all k, and there exists a finite sequence of norm-one functionals $(x_k^*)_{k=1}^n$ in X^* such that $\langle x_i^*, x_j\rangle > \varepsilon$ for all $1 \leq i \leq j \leq n$ and $\langle x_i^*, x_j\rangle = 0$ for all $1 \leq j < i \leq n$.

Definition 6.5.2. An operator $T \in \mathcal{L}(X,Y)$ is said to be *supertauberian* if for every $0 < \varepsilon < 1$, there exists $\delta > 0$ and there exists a positive integer n for which there is not any finite ε-triangular sequence $\{x_k\}_{k=1}^n$ such that $\sup_{1\leq k\leq n} \|T(x_k)\| < \delta$.

The following proposition shows that every supertauberian operator is strongly tauberian, and consequently, also tauberian.

Proposition 6.5.3. *Every supertauberian operator $T \in \mathcal{L}(X,Y)$ is strongly tauberian.*

Proof. Without loss of generality, we will assume that $\|T\| = 1$.

Let $T \in \mathcal{L}(X,Y)$ be a non-strongly tauberian operator. In order to prove that T is not supertauberian, fix a pair of real numbers $0 < \varepsilon < 1/4$ and $\delta > 0$, and a positive integer n. Let

$$L := \{x \in B_X : \|T(x)\| < \delta\}.$$
$$A := \{x^{**} \in \operatorname{int} B_{X^{**}} : \|T^{**}(x^{**})\| < \delta\}.$$

By Lemma 4.4.1, $A \subset \overline{L}^{w^*}$. But T is not strongly tauberian, so, by Lemma 6.2.11, A has an element x^{**} with $\operatorname{dist}(x_n^{**}, X) = 1/3$. Thus, by Proposition A.5.3, L contains an ε-triangular sequence $(x_k)_{k=1}^{\infty}$ satisfying $\sup_{k\in\mathbb{N}} \|T(x_k)\| \le \delta$. Obviously, $\{x_k\}_{k=1}^{n}$ is a triangular sequence, so T is not supertauberian. □

The following results are aimed at characterizing the supertauberian operators in terms of their ultrapowers, and proving that the sum of a supertauberian operator plus a compact one is supertauberian.

Proposition 6.5.4. *Let $\mathfrak{U}$ be an ultrafilter and $T \in \mathcal{L}(X,Y)$ such that $N(T_{\mathfrak{U}})$ is reflexive. Then $N(T^{**}) \subset X$.*

Proof. Let us assume that there exists $x^{**} \in N(T^{**})$ with $\operatorname{dist}(x^{**}, X) > \varepsilon > 0$ and $\|x^{**}\| = 1$. For every positive integer n, let

$$A_n := \{x \in B_X : \|T(x)\| < 1/n\}.$$

Then, since $T^{**}(x^{**}) = 0$, Lemma 4.4.1 yields that $x^{**} \in \overline{A_n}^{w^*}$, hence, by Proposition A.5.3, A_n contains an ε-triangular sequence $(x_l^n)_{l=1}^{\infty}$. Take a sequence $(f_l^n)_{l=1}^{\infty}$ in B_{X^*} so that $\langle f_p^n, x_q^n\rangle > \varepsilon$ if $1 \le p \le q$ and $\langle f_p^n, x_q^n\rangle = 0$ if $1 \le q < p$.

Let $\{I_n\}_{n=1}^{\infty}$ be a partition of I disjoint with $\mathfrak{U}$, and for every $i \in I$, denote by n_i the only positive integer for which $i \in I_{n_i}$. Next, for every $l \in \mathbb{N}$, define

$$z_l^i := x_l^{n_i} \quad \text{and} \quad g_l^i := f_l^{n_i} \quad \text{for all } i \in I$$

and let $\mathbf{z}_l := [z_l^i]_i \in B_{X_{\mathfrak{U}}}$ and $\mathbf{g}_l := [g_l^i]_i \in B_{X_{\mathfrak{U}}^*}$. Thus, for every pair of positive integers p and q,

$$\langle \mathbf{g}_p, \mathbf{z}_q\rangle = \lim_{i\to\mathfrak{U}} \langle f_p^{n_i}, x_q^{n_i}\rangle = \begin{cases} \ge \varepsilon & \text{if } p \le q, \\ = 0 & \text{if } q < p. \end{cases}$$

That shows that $(\mathbf{z}_l)_{l=1}^{\infty}$ is an $\varepsilon/2$-triangular sequence. Moreover, given any $l \in \mathbb{N}$, for every $i \in I$, $\|T(z_l^i)\| \le 1/n_i$ so

$$\|T_{\mathfrak{U}}(\mathbf{z}_l)\| = \lim_{i\to\mathfrak{U}} \|T(z_l^i)\| \le \lim_{i\to\mathfrak{U}} 1/n_i = 0$$

hence $(\mathbf{z}_l)_{l=1}^{\infty}$ is contained in $N(T_{\mathfrak{U}})$, which proves, after Proposition A.5.2, that $N(T_{\mathfrak{U}})$ is not reflexive. □

An immediate consequence of the following theorem is that the kernel of each supertauberian operator is super-reflexive. Obviously, the converse fails.

Theorem 6.5.5. *Given an operator $T \in \mathcal{L}(X,Y)$ and an ultrafilter $\mathfrak{U}$ on a set I, the following statements are equivalent:*

(a) *T is supertauberian;*

(b) *$N(T_{\mathfrak{U}})$ is super-reflexive;*

(c) *$N(T_{\mathfrak{U}})$ is reflexive;*

(d) *there exist a real number $0 < \varepsilon < 1$, a real number $\delta > 0$ and a positive integer n for which there does not exist any finite ε-triangular sequence $(x_k)_{k=1}^n$ such that* $\sup_{1\le k\le n} \|T(x_k)\| < \delta$.

Proof. (a)⇒(d) It is immediate.

(d)⇒(b) Let us assume that $N(T_{\mathfrak{U}})$ is not super-reflexive and prove that statement (d) does not hold. Fix any pair of real numbers $0 < \varepsilon < 1$ and $\delta > 0$, and any positive integer n. Then, by Proposition A.5.6, $N(T_{\mathfrak{U}})$ contains an ε-triangular sequence $\{\mathbf{x}_l\}_{l=1}^n$. Thus, agreeing that $\operatorname{span}\{\mathbf{x}_l\}_{l=1}^0$ stands for the null subspace $\{0\}$, we have

$$\varepsilon < \operatorname{dist}\left(\operatorname{span}\{\mathbf{x}_l\}_{l=1}^{k-1}, \operatorname{conv}\{\mathbf{x}_l\}_{l=k}^n\right) \text{ for all } 1 \le k \le n. \tag{6.22}$$

Let us choose a representative $(x_i^l)_{i\in I}$ for every $\mathbf{x}_l$, so

$$\lim_{i\to\mathfrak{U}} \|T(x_i^l)\| = 0 \text{ for all } 1 \le l \le n. \tag{6.23}$$

Formulas (6.22) and (6.23) yield the existence of $J \in \mathfrak{U}$ such that, for every $j \in J$,

$$\varepsilon < \operatorname{dist}\left(\operatorname{span}\{x_j^l\}_{l=1}^{k-1}, \operatorname{conv}\{x_j^l\}_{l=k}^n\right) \text{ for all } 1 \le k \le n.$$
$$\|T(x_j^l)\| < \delta \text{ for all } 1 \le l \le n.$$

Thus, for every $j \in J$, the finite sequence $\{x_j^l\}_{l=1}^n$ is ε-triangular and satisfies $\|T(x_j^l)\| < \delta$ for all $1 \le l \le n$, which proves that statement (d) does not hold.

(b)⇒(c) It is immediate.

(c)⇒(a) Assume T is not supertauberian. Then there exists a real number $0 < \varepsilon < 1$ such that for every $n \in \mathbb{N}$, there exists an ε-triangular sequence $\{x_k^n\}_{k=1}^n$ such that $\|T(x_k^n)\| < 1/n$ for all $1 \le k \le n$. For each of those ε-triangular sequences, take a system $\{f_k^n\}_{k=1}^n$ in S_{X^*} so that $\langle f_l^n, x_m^n\rangle > \varepsilon$ if $1 \le l \le m \le n$, and $\langle f_l^n, x_m^n\rangle = 0$ if $1 \le m < l \le n$. Let us define $x_k^n := 0 \in X$ and $f_k^n := 0 \in X^*$ for $n < k$.

Let $\{I_n\}_{n=1}^\infty$ be a partition of I disjoint with $\mathfrak{U}$, and for every $i \in I$, let n_i denote the only positive integer for which $i \in I_{n_i}$. Next, for every $l \in \mathbb{N}$ and every $i \in I$, define

$$z_l^i := x_l^{n_i},$$
$$g_l^i := f_l^{n_i},$$
$$\mathbf{z}_l := [z_l^i],$$
$$\mathbf{g}_l := [g_l^i].$$

Thus,

$$\langle \mathbf{g}_l, \mathbf{z}_m \rangle = \lim_{i\to\mathfrak{U}} \langle g_l^i, z_m^i \rangle = \lim_{i\to\mathfrak{U}} \langle f_l^{n_i}, x_m^{n_i} \rangle = \begin{cases} \geq \varepsilon & \text{if } 1 \leq l \leq m, \\ = 0 & \text{if } 1 \leq m < l, \end{cases}$$

which shows that $\{\mathbf{z}_n\}_{n=1}^\infty$ is an $\varepsilon/2$-triangular sequence. Moreover, given any $\mathbf{z}_l$, for every $k \in \mathbb{N}$,

$$\{i \in I : \|T(z_l^i)\| < 1/k\} \supset \{i : n_i \geq k\} = \bigcup_{l=k}^{\infty} I_l \in \mathfrak{U},$$

so $\mathbf{z}_n \in N(T_\mathfrak{U})$. Therefore, $N(T_\mathfrak{U})$ contains an infinite $\varepsilon/2$-triangular sequence, which means that $N(T_\mathfrak{U})$ is not reflexive. □

Statement (d) in Theorem 6.5.5 is apparently stronger than the definition of supertauberian operator. It will be fundamental in the proof of the perturbative characterization given in Theorem 6.5.16. Another application of the aforementioned statement (d) is the following:

Corollary 6.5.6. *For every pair of Banach spaces X and Y, the set of all the supertauberian operators in $\mathcal{L}(X,Y)$ is open.*

Proof. Suppose the result is not true. Then there is a supertauberian operator $T \in \mathcal{L}(X,Y)$ and a sequence $(T_k)_{k=1}^\infty$ in $\mathcal{L}(X,Y)$ such that, for all k, $\|T_k\| < 1/k$ and $T + T_k$ is not supertauberian. Pick any pair of real numbers $0 < \varepsilon < 1$, $\delta > 0$, and any positive integer n. Choose $m \in \mathbb{N}$ so that $1/m < \delta/2$. Theorem 6.5.5 implies that there exists a finite ε-triangular sequence $(x_k)_{k=1}^n$ in B_X for which $\sup_{1\leq k\leq n} \|T + T_m(x_k)\| < \delta/2$. But $\|T_m\| < 1/m$, so $\|T(x_k)\| < \delta$ for all $1 \leq k \leq n$, in contradiction with T supertauberian. □

The following result proves that the class of the supertauberian operators equals the class $\mathcal{T}^{up}$.

Theorem 6.5.7. *Given an operator $T \in \mathcal{L}(X,Y)$ and any ultrafilter $\mathfrak{U}$, the following statements are equivalent:*

(a) *T is supertauberian;*

(b) *$T_\mathfrak{U}$ is supertauberian;*

(c) *$T_\mathfrak{U}$ is tauberian.*

Proof. (a)⇒(b) Suppose T is supertauberian. By Theorem 6.5.5, $N(T_{\mathfrak{U}\times\mathfrak{U}})$ is super-reflexive; equivalently, $N\big((T_\mathfrak{U})_\mathfrak{U}\big)$ is super-reflexive because of the theorem of iteration of ultrapowers. Therefore, $T_\mathfrak{U}$ is supertauberian.

(b)⇒(c) Assume $T_\mathfrak{U}$ is supertauberian. By Theorem 6.5.5, $N\big((T_\mathfrak{U})_\mathfrak{U}\big)$ is reflexive, and therefore, by Proposition 6.5.4, $N\big((T_\mathfrak{U})^{**}\big)$ is contained in $X_\mathfrak{U}$. Moreover, by Proposition A.4.21, $T_\mathfrak{U}(B_{X_\mathfrak{U}})$ is closed, and subsequently, by Theorem 2.1.7, $T_\mathfrak{U}$ is tauberian.

(c)⇒(a) Assume $T_{\mathfrak{U}}$ is tauberian. Then $N(T_{\mathfrak{U}})$ is reflexive, and by Theorem 6.5.5, it follows that T is supertauberian. □

Corollary 6.5.8. *The class of supertauberian operators coincides with the ultrapower-stable, upper semigroup $\mathcal{T}^{up}$.*

Proof. It is an immediate consequence of Theorem 6.5.7 and Proposition 6.4.5 applied to the upper semigroup $\mathcal{T}$ of tauberian operators. □

There is a subtle but important difference between Corollary 6.5.8 and Theorem 6.5.5 on the one hand, and Theorem 6.5.7 on the other. Indeed, the statements of the mentioned theorems are of the form *'certain property P holds if and only if a second property Q is valid for some ultrafilter, in which case, Q holds for any ultrafilter'*. However, the statement of Corollary 6.5.8 says that *'certain property P holds if and only if a second property Q is valid for all ultrafilters'*. The formal difference between these statements is precisely the reason why we have preferred to introduce the supertauberian operators as in Definition 6.5.2 rather than via the class $\mathcal{T}^{up}$.

Corollary 6.5.9. *The semigroup $\mathcal{T}^{up}$ is stable under local supportability.*

Proof. In fact, by Corollary 6.5.8, $\mathcal{T}^{up}$ is an upper ultrapower stable semigroup, and therefore, by Proposition 6.4.11, $\mathcal{T}^{up}$ is $\prec_{\text{ls}}$-stable. □

Proposition 6.5.10. *An operator $T \in \mathcal{L}(X,Y)$ is supertauberian if and only if so is T^{**}.*

Proof. The fact that $\mathcal{T}^{up}$ is an upper semigroup directly yields that if T^{**} is supertauberian, so is T. For the reverse implication, by Corollary 6.5.9, the upper semigroup $\mathcal{T}^{up}$ of supertauberian operators is $\prec_{\text{ls}}$-stable. Thus, given $T \in \mathcal{T}^{up}$, since $T^{**} \prec_{\text{ls}} T$, it follows that T^{**} belongs to $\mathcal{T}^{up}$. □

Remark 6.5.11. If $T \in \mathcal{L}(X,Y)$ is a supertauberian operator and X is non-reflexive, then there exists an ultrafilter $\mathfrak{U}$ so that $T_{\mathfrak{U}}$ is not strictly singular.

In fact, if T is supertauberian, by Proposition 6.5.3, T is strongly tauberian, that is, T^{co} is an isomorphism. But by Theorem 6.3.19, T^{co} is locally supportable by T, and consequently, by Proposition 6.3.5, there exist an ultrafilter $\mathfrak{U}$, an isomorphism $U \in \mathcal{L}(X^{co}, X_{\mathfrak{U}})$ and an operator $V \in \mathcal{L}(\overline{R(T^{co})}, Y_{\mathfrak{U}})$ so that $T_{\mathfrak{U}}U = V\overline{T^{co}}$. Since the operators U_0 and U_1 occurring in the proof of the aforementioned Theorem 6.3.19 are 6-injections, a closer look at the proof of Proposition 6.3.5 reveals that the operator V that we are considering here can be assumed to be an isomorphism. Thus, there is an infinite-dimensional subspace F of $X_{\mathfrak{U}}$ isomorphic to X^{co} such that $T_{\mathfrak{U}}|_F$ is an isomorphism. Hence, $T_{\mathfrak{U}}$ is not strictly singular.

Corollary 6.5.12. *Let $T \in \mathcal{L}(X,Y)$. If $R(T)$ is closed and $N(T)$ is super-reflexive, then T is supertauberian.*

Proof. Assume that $R(T)$ is closed and $N(T)$ is super-reflexive, and let $\mathfrak{U}$ be any ultrafilter. By Proposition A.4.14, $N(T)_{\mathfrak{U}}$ is finitely representable in $N(T)$, so $N(T)_{\mathfrak{U}}$ is reflexive. But since $R(T)$ is closed, by Proposition A.4.22, $N(T)_{\mathfrak{U}}$ equals $N(T_{\mathfrak{U}})$, so $N(T_{\mathfrak{U}})$ is super-reflexive, and then, by Theorem 6.5.5, T is supertauberian. □

Let us prove now that the class $\mathcal{T}^{up}$ is stable under super weakly compact perturbations.

Proposition 6.5.13. *If $T \in \mathcal{L}(X,Y)$ is supertauberian and $K \in \mathcal{L}(X,Y)$ is super weakly compact, then $T+K$ is supertauberian. In particular, if K is compact, then $T+K$ is supertauberian.*

Proof. Let $\mathfrak{U}$ be an ultrafilter. Then $T_{\mathfrak{U}}$ is supertauberian and $K_{\mathfrak{U}}$ is weakly compact, so $(T+K)_{\mathfrak{U}} = T_{\mathfrak{U}} + K_{\mathfrak{U}}$ is tauberian, and therefore, $T+K$ is supertauberian.

Assume now that K is compact. Then, since the ideal $\mathcal{K}$ of compact operators is ultrapower-stable (Proposition 6.4.3), it follows that $K_{\mathfrak{U}}$ is compact, and consequently, is weakly compact, so K is super weakly compact. Therefore, $T+K$ is supertauberian. □

Every finite ε-triangular sequence $(x_i)_{i=1}^n$ in a Banach space X admits a biorthogonal sequence $(h_i)_{i=1}^n$ in X^* so that the functionals f_i are all norm bounded above by a constant that only depends on the parameters n and ε. A proof of that fact is given along the two following lemmas, which may be sketched as follows. Take $(f_i)_{i=1}^n \subset B_{X^*}$ such that, letting $\varepsilon_{ij} := \langle f_i, x_j\rangle$, then $\varepsilon_{ij} > \varepsilon$ if $i \le j$, and $\varepsilon_{ij} = 0$ otherwise. Obviously, the matrix $(\varepsilon_{ij})_{i=1}^n{}_{j=1}^n$ is triangular, and therefore, can be transformed in a diagonal matrix by the usual procedure which consists of producing zeros in its last column by adding multiples of its last row to the other rows. Next, we do the same with the penultimate column, and so on. That procedure induces the construction of certain linear combinations of the functionals f_i which lead to the wanted biorthogonal functionals g_i. Lemma 6.5.14 corresponds to the first step, when zeros in the last column are produced, and Lemma 6.5.15 is a recursive application of Lemma 6.5.14 in order to produce zeros in the remaining columns.

Lemma 6.5.14. *Let X be a Banach space, a pair of real numbers, $0 < \varepsilon < 1$ and $\eta \ge 1$, and a positive integer n. Let $\{x_i\}_{i=1}^n \subset S_X$, $\{f_i\}_{i=1}^n \subset \eta B_{X^*}$ and write $\varepsilon_{ij} := \langle f_i, x_j\rangle$. If $\varepsilon < \varepsilon_{ij} \le 1$ for all $1 \le i \le j \le n$ and $\varepsilon_{ij} = 0$ for all $1 \le j < i \le n$, then* $\operatorname{span}\{f_i\}_{i=1}^n$ *contains a subset $\{g_i\}_{i=1}^n$ in $\eta(1+\varepsilon^{-1})B_{X^*}$ satisfying*

(i) $\|g_i\| \le \eta(1+\varepsilon^{-1})$ *for all* $1 \le i \le n$,

(ii) $\langle g_i, x_j\rangle = \varepsilon_{ij}$ *for all* $1 \le i \le n$ *and all* $1 \le j \le n-1$,

(iii) $\langle g_i, x_n\rangle = \delta_{in}$ *for all* $1 \le i \le n$.

Proof. Consider the functionals

$$g_i := f_i - \frac{\varepsilon_{in}}{\varepsilon_{nn}} f_n \quad \text{for all } 1 \leq i \leq n-1,$$
$$g_n := \frac{1}{\varepsilon_{nn}} f_n.$$

It is immediate that conditions (ii) and (iii) hold. Moreover, for $1 \leq i \leq n-1$,

$$\|g_i\| \leq \|f_i\| + \frac{1}{\varepsilon}\|f_n\| \leq \eta(1+\varepsilon^{-1}),$$

and $\|g_n\| \leq \varepsilon^{-1}\|f_n\| \leq \eta(1+\varepsilon^{-1})$, so condition (i) is satisfied, and the lemma is proved. □

Lemma 6.5.15. *Let X be a Banach space, a pair of real numbers $0 < \varepsilon < 1$ and $\eta \geq 1$, and a positive integer n. Let $\{x_i\}_{i=1}^n \subset S_X$ and $\{f_i\}_{i=1}^n \subset B_{X^*}$. Let us write $\varepsilon_{ij} := \langle f_i, x_j\rangle$, and assume that $\varepsilon < \varepsilon_{ij} \leq 1$ for all $1 \leq i \leq j \leq n$ and $\varepsilon_{ij} = 0$ for all $1 \leq j < i \leq n$. Then there exists a subset $\{h_i\}_{i=1}^n \subset \operatorname{span}\{f_i\}_{i=1}^n$ such that*

$$\|h_i\| \leq (1+\varepsilon^{-1})^n \quad \textit{for all } 1 \leq i \leq n,$$
$$\langle h_i, x_j\rangle = \delta_{ij} \quad \textit{for all } 1 \leq i \leq n \textit{ and all } 1 \leq j \leq n.$$

Proof. The proof consists of n consecutive applications of Lemma 6.5.14.

First, given the family $\{f_i\}_{i=1}^n$ in B_{X^*}, Lemma 6.5.14 yields a subset $\{g_i^n\}_{i=1}^n$ in $(1+\varepsilon^{-1})B_{X^*} \cap \operatorname{span}\{f_i\}_{i=1}^n$ such that, for every $1 \leq i \leq n$,

$$\langle g_i^n, x_j\rangle = \varepsilon_{ij} \quad \text{for all } 1 \leq j \leq n-1,$$
$$\langle g_i^n, x_n\rangle = \delta_{in}.$$

Next, Lemma 6.5.14, applied on the families $\{x_i\}_{i=1}^{n-1}$ and $\{g_i^n\}_{i=1}^{n-1}$, supplies a subset $\{g_i^{n-1}\}_{i=1}^{n-1}$ of $(1+\varepsilon^{-1})^2 B_{X^*} \cap \operatorname{span}\{f_i\}_{i=1}^n$ such that, for every $1 \leq i \leq n-1$,

$$\langle g_i^{n-1}, x_j\rangle = \varepsilon_{ij} \quad \text{for all } 1 \leq j \leq n-2,$$
$$\langle g_i^{n-1}, x_j\rangle = \delta_{ij} \quad \text{for all } n-1 \leq j \leq n.$$

Keeping this procedure for n times successively, we obtain the subsets

$$\{g_i^n\}_{i=1}^n, \{g_i^{n-1}\}_{i=1}^{n-1}, \ldots, \{g_i^1\}_{i=1}^1$$

of $\operatorname{span}\{f_i\}_{i=1}^n$ satisfying for each $1 \leq k \leq n$ and for each $1 \leq i \leq k$,

$$\|g_i^k\| \leq (1+\varepsilon^{-1})^{n-k+1},$$
$$\langle g_i^k, x_j\rangle = \delta_{ij} \quad \text{for all } k \leq j \leq n,$$
$$\langle g_k^k, x_j\rangle = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Thus, the wanted functionals are $h_i := g_i^i$ for all $1 \leq i \leq n$. □

The following result is the central theorem of this section. Most of the information about supertauberian operators can be derived from it.

Theorem 6.5.16. *An operator $T \in \mathcal{L}(X,Y)$ is supertauberian if and only if for every compact operator $K \in \mathcal{L}(X,Y)$, the kernel $N(T+K)$ is super-reflexive.*

Proof. If T is supertauberian and $K \in \mathcal{L}(X,Y)$ is compact, then, by Proposition 6.5.13, the operator $T+K$ is supertauberian. Therefore, Theorem 6.5.5 yields that $N(T+K)$ is super-reflexive.

For the converse implication, let us assume that T is not supertauberian and find a compact operator K so that $N(T+K)$ is not super-reflexive. In order to achieve that goal, given a real number $0 < \varepsilon < 1$, we shall recursively find, for every $n \in \mathbb{N}$, finite sets

$$\{x_i^n\}_{i=1}^n \subset S_X, \ \{f_i^n\}_{i=1}^n \subset S_{X^*}, \quad \text{and} \quad \{h_i^n\}_{i=1}^n \subset X^* \tag{6.24}$$

such that, for each $p \in \mathbb{N}$, the following conditions are satisfied:

$$\begin{aligned} \langle h_i^k, x_j^l \rangle = \delta_{ij}\delta_{kl} \ \ &\text{for all } \{k,l\} \subset \{1,\dots,p\}, \\ &\text{all } 1 \le i \le k \text{ and all } 1 \le j \le l, \end{aligned} \tag{6.25}$$

$$\sum_{i=1}^{k} \|h_i^k\| \|T(x_i^k)\| < 2^{-k} \ \ \text{for all } k \in \{1,\dots,p\}, \tag{6.26}$$

$$\langle f_i^k, x_j^k \rangle > \varepsilon \ \ \text{for all } k \in \{1,\dots,p\} \text{ and all } 1 \le i \le j \le k, \tag{6.27}$$

$$\langle f_i^k, x_j^k \rangle = 0 \ \ \text{for all } k \in \{1,\dots,p\} \text{ and all } 1 \le j < i \le k. \tag{6.28}$$

First, Theorem 6.5.5 supplies $x_1^1 \in S_X$ and $f_1^1 \in S_{X^*}$ such that $\langle f_1^1, x_1^1 \rangle > \varepsilon$ and $\|T(x_1^1)\| < 2^{-1}\varepsilon$. Thus, choosing

$$h_1^1 := \frac{1}{\langle f_1^1, x_1^1 \rangle} f_1^1,$$

we get $\langle h_1^1, x_1^1 \rangle = 1$ and $\|h_1^1\| \|T(x_1^1)\| < 2^{-1}$, so, the singletons $\{x_1^1\}$, $\{f_1^1\}$ and $\{h_1^1\}$ fulfill conditions (6.25), (6.26), (6.27) and (6.28) for $p = 1$.

Assume that the families $\{x_i^p\}_{i=1}^p$, $\{f_i^p\}_{i=1}^p$ and $\{h_i^p\}_{i=1}^p$ have been already obtained for all $p \in \{1,\dots,n-1\}$, and let us find the families

$$\{x_i^n\}_{i=1}^n, \ \{f_i^n\}_{i=1}^n \quad \text{and} \quad \{h_i^n\}_{i=1}^n.$$

Let P denote the projection on X whose kernel and range are

$$\begin{aligned} N(P) &= \operatorname{span}\{x_i^k \colon 1 \le k \le n-1, \ 1 \le i \le k\} \\ R(P) &= \bigcap_{k=1}^{n-1} \bigcap_{i=1}^{k} N(h_i^k), \end{aligned}$$

and let
$$\delta := \frac{1}{n2^n(1+\varepsilon^{-1})^n\|P\|}.$$
Since $R(P)$ is finite co-dimensional in X, by Proposition 6.5.13, $T|_{R(P)}$ is not supertauberian. Hence, by Theorem 6.5.5, there exist a pair of subsets $\{x_i^n\}_{i=1}^n$ in $S_{R(P)}$ and $\{f_i^n\}_{i=1}^n$ in S_{X^*} satisfying $\langle f_i^n, x_j^n\rangle > \varepsilon$ for all $1 \le i \le j \le n$, $\langle f_i^n, x_j^n\rangle = 0$ for all $1 \le j < i \le n$, and $\|T(x_i^n)\| \le \delta$ for all $1 \le i \le n$. Thus, conditions (6.27) and (6.28) hold for $p = n$. Next, by Lemma 6.5.15, we obtain a system
$$\{g_i\}_{i=1}^n \subset (1+\varepsilon^{-1})^n B_{X^*}$$
so that $\langle g_i, x_j^n\rangle = \delta_{ij}$ for all i and j. Define the functionals $h_i^n := g_i \circ P$ for all $1 \le i \le n$. Thus, since $\{x_i^n\}_{i=1}^n \subset R(P)$, then
$$\langle h_i^n, x_j^n\rangle = \langle g_i, x_j^n\rangle = \delta_{ij} \text{ for all } \{i,j\} \subset \{1,\dots,n\};$$
moreover, as every x_j^n belongs to $\cap_{k=1}^{n-1}\cap_{i=1}^k N(h_i^k)$, it follows that $\langle h_i^k, x_j^n\rangle = 0$ for all $1 \le k \le n-1$ and all $1 \le i \le k$; and for $k \le n-1$, every x_j^k belongs to $N(P)$, so $\langle h_i^n, x_j^k\rangle = 0$. Therefore, condition (6.25) holds for $p = n$.

In order to check condition (6.26) for $p = n$, taking into account the upper bounds for $\|h_i^k\|$ and $\|T(x_i^k)\|$, we get
$$\sum_{i=1}^n \|h_i^n\|\|T(x_i^n)\| < n(1+\varepsilon^{-1})^n\|P\|\delta = 2^{-n}.$$
Once the families $\{x_i^n\}_{i=1}^n$, $\{f_i^n\}_{i=1}^n$ and $\{h_i^n\}_{i=1}^n$ have been obtained for all $n \in \mathbb{N}$, we define the operator $K \in \mathcal{L}(X,Y)$ by
$$K(x) := -\sum_{n=1}^\infty \sum_{i=1}^n \langle h_i^n, x\rangle T(x_i^n).$$
Note that condition (6.26) yields
$$\|K\| \le \sum_{n=1}^\infty \sum_{i=1}^n \|h_i^n\|\|T(x_i^n)\| \le \sum_{n=1}^\infty 2^{-n} = 1,$$
hence K is well defined and compact. In order to finish, note that every finite sequence $\{x_i^n\}_{i=1}^n$ is ε-triangular and is contained in $N(T+K)$. Therefore, by Proposition A.5.6, $N(T+K)$ is not super-reflexive, concluding the proof. □

Observe that when T is not supertauberian, it is possible to find a compact operator Q with norm as small as we please so that $N(T+Q)$ is not super-reflexive. Indeed, in the proof of the above theorem, when T is not supertauberian, given any $q \in \mathbb{N}$, it is sufficient to substitute the compact operator K by
$$Q(x) := -\sum_{n=q+1}^\infty \sum_{i=1}^n \langle h_i^n, x\rangle T(x_i^n);$$

obviously, Q is compact, $\|Q\| < 2^{-q}$, and $N(T+Q)$ is not super-reflexive.

Theorem 6.5.16 in combination with the analogous perturbative characterizations for the semigroups Φ_+ and $\mathcal{T}$, yields important consequences.

Proposition 6.5.17. *Given a Banach space X, the following statements hold:*

(i) *every super-reflexive subspace of X is finite dimensional if and only if, for each Banach space Y, every supertauberian operator $T \in \mathcal{L}(X,Y)$ is upper semi-Fredholm;*

(ii) *every reflexive subspace of X is super-reflexive if and only if, for each Banach space Y, every tauberian operator $T \in \mathcal{L}(X,Y)$ is supertauberian.*

Proof. (i) Assume all super-reflexive subspaces of X are finite dimensional. Thus, by Theorem 6.5.16, if $T \in \mathcal{L}(X,Y)$ is supertauberian, then $N(T+K)$ is finite dimensional for every compact operator $K \in \mathcal{L}(X,Y)$. Therefore, by Theorem A.1.9, T is upper semi-Fredholm.

For the converse, let us assume that X contains an infinite dimensional super-reflexive subspace R. Then the quotient operator $Q_R \in \mathcal{L}(X, X/R)$ is not upper semi-Fredholm, but by Corollary 6.5.12, is supertauberian.

(ii) The proof is analogous to that of (i), but here, we need to use the perturbative characterizations for supertauberian operators and for tauberian operators given respectively in Theorem 6.5.16 and in Theorem 2.2.7. □

Situations described in the above theorem are not trivial. For instance, the original Tsirelson space X_{T} is reflexive but not super-reflexive, and any of its infinite dimensional closed subspaces or quotients contain an isomorphic copy of X_{T}. Then, every $T \in \mathcal{L}(X_{\mathsf{T}}, Y)$ is tauberian, but T is supertauberian if and only if T is upper semi-Fredholm.

The case when X is a $L_1(\mu)$ space, considered in the following result, is also very interesting and completes the study begun in Chapter 4.

Theorem 6.5.18. *Let μ be a finite, purely non-atomic measure. Thus, given any Banach space Y, every tauberian operator $T \in \mathcal{L}\big(L_1(\mu), Y\big)$ is supertauberian.*

Proof. This is an immediate consequence of Proposition 6.5.17 and of the fact, proved in Corollary A.6.14, that every reflexive subspace of $L_1(\mu)$ is super-reflexive. □

As a consequence, if $T\colon L_1(\mu) \longrightarrow Y$ is tauberian, then $T^{*(2n)}$ is supertauberian for all n.

Let us offer a further sequel of Theorem 6.5.16 concerning the ideal $\mathcal{W}^{up}$ introduced in Definition 6.4.8.

Proposition 6.5.19. *Given $T \in \mathcal{L}(X,Y)$, the following statements are equivalent:*

(a) *the operator T is supertauberian;*

(b) *for every space* Z *and every* $A \in \mathcal{L}(Z, X)$, *if* TA *is super weakly compact, then* A *is super weakly compact;*

(c) *any subspace* E *of* X *is super-reflexive whenever* $T|_E$ *is super weakly compact.*

Proof. (a)⇒(b) Assume T is supertauberian and TA is super weakly compact. Thus, given any ultrafilter $\mathfrak{U}$, by definition of $\mathcal{W}^{up}$, it follows that

$$(TA)_{\mathfrak{U}} = T_{\mathfrak{U}} A_{\mathfrak{U}} \in \mathcal{W}. \tag{6.29}$$

But by Theorem 6.5.7, $T_{\mathfrak{U}}$ is tauberian, so Proposition 2.2.9 yields that $A_{\mathfrak{U}}$ is weakly compact. Hence, $A \in \mathcal{W}^{up}$.

(b)⇒(c) Let E be a closed subspace of X so that $T|_E$ is super weakly compact, and consider the corresponding subspace operator $J_E \in \mathcal{L}(E, X)$. Then, since $T|_E = TJ_E$, by hypothesis (b), J_E is super weakly compact, that is, given any ultrafilter $\mathfrak{U}$, $(J_E)_{\mathfrak{U}} \in \mathcal{W}$. But $(J_E)_{\mathfrak{U}}$ equals the natural embedding $J_{E_{\mathfrak{U}}}$ of $E_{\mathfrak{U}}$ into $X_{\mathfrak{U}}$, so $J_{E_{\mathfrak{U}}}$ is weakly compact, and therefore, $E_{\mathfrak{U}}$ is reflexive, which implies that E is super-reflexive.

(c)⇒(a) Let us assume T is not supertauberian. Then, by Theorem 6.5.16, there exists a compact operator $K \in \mathcal{L}(X, Y)$ so that $N(T - K)$ is not super-reflexive. But the ideal $\mathcal{K}$ is ultrapower-stable (see Proposition 6.4.3) and moreover, $\mathcal{K} \subset \mathcal{W}^{up}$, so K is super weakly compact. Therefore, $T|_{N(T-K)} = K|_{N(T-K)}$ is super weakly compact, and since $N(T - K)$ is not super-reflexive, statement (c) does not hold. □

According to the notations concerning semigroups and ideals introduced in Section 6.1, it follows from Proposition 2.2.9 that $\mathcal{T} = \mathcal{W}_+$, that $Sp(\mathcal{W})$ is the space ideal of reflexive spaces, and $Sp(\mathcal{W}^{up})$ is the space ideal of super-reflexive spaces. Thus, as the class $\mathcal{T}^{up}$ of all supertauberian operators is $(\mathcal{W}_+)^{up}$, Proposition 6.5.19 yields immediately the following result:

Remark 6.5.20. The identity $(\mathcal{W}_+)^{up} = (\mathcal{W}^{up})_+$ holds.

Moreover, Theorem 6.5.16 can be rephrased as

$$T \in (\mathcal{W}_+)^{up}(X, Y) \Leftrightarrow \forall K \in \mathcal{K}(X, Y),\ N(T + K) \in Sp(\mathcal{W}^{up}).$$

Let us show that some of the examples of tauberian operators we have given before are not supertauberian.

Proposition 6.5.21. *The tauberian operator* $J_p \colon L_p(X) \longrightarrow L_1(X)$ *considered in Proposition* 6.2.15 *is supertauberian if and only if* X *is super-reflexive.*

Proof. Indeed, if X is super-reflexive, then $L_p(X)$ is also super-reflexive [21]. Therefore, J_p is trivially supertauberian.

For the converse, assume that X is not super-reflexive and apply the same argument given in Proposition 6.2.15, with the caution of substituting infinite triangular sequences for finite triangular ones. □

Proposition 6.5.22. *The tauberian operator* $\iota\colon J \longrightarrow c_0$ *given in Example* 2.1.18 *is not supertauberian.*

Proof. Fix $n \in \mathbb{N}$ and $\delta > 0$. We only need to find a finite 1-triangular sequence $(x_k)_{k=1}^n$ in J such that $\|\iota(x_k)\| < \delta$ for all k. Without loss of generality, we will assume that $1/\sqrt{2n} < \delta$.

Put $p := n + 1$ and, for every $k \in \{1, \dots, n\}$, let y_k be a sequence of real numbers defined by

$$y_k(i) := \begin{cases} \frac{1}{\sqrt{2n}} & \text{if } i = 1 + k, (p+1) + k, (2p+1) + k, \dots, ((n-1)p + 1) + k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x_k := \sum_{j=1}^k y_j$. Roughly speaking, every sequence x_k consists of a chain of n plateaux beginning at the position $i = 2$; each plateau has length equal to k and height equal to $1/\sqrt{2n}$, and each pair of consecutive plateaux is separated by a valley of length $p - k$. It is easy to check that $\|x_k\|_J = 1$ for all k.

For every $1 \leq k \leq n-1$, every $u \in \operatorname{span}\{x_l\}_{l=1}^k$ and every $v \in \operatorname{conv}\{x_l\}_{l=k+1}^n$, we have $\|u - v\|_J \geq 1$. Indeed, writing $y := u - v$, since

$$\begin{aligned} 0 &= y(1) = y(p+1) = y(2p+1) = \dots = y(np+1) \\ 1/\sqrt{2n} &= y(p) = y(2p) = \dots = y(np), \end{aligned}$$

it follows $\|y\|_J \geq 1$. We have just proved that $(x_k)_{k=1}^n$ is a finite 1-triangular sequence in J. Moreover, since $\|\iota(x_k)\|_\infty \leq 1/\sqrt{2n} < \delta$ for all k, it follows that ι is not supertauberian. □

6.6 Cosupertauberian operators

This section studies the class $(\mathcal{T}^{up})^d$, a subclass of $\mathcal{ST}^d$ whose elements are named *cosupertauberian* operators. The main result about $(\mathcal{T}^{up})^d$ asserts that an operator $T \in \mathcal{L}(X, Y)$ is cosupertauberian if and only if for every compact operator from X into Y, the cokernel $Y/\overline{R(T+K)}$ is super-reflexive.

It was noted in Chapter 3 that $\mathcal{T}$ and $\mathcal{T}^d$ are not symmetric. However, we will see throughout this section that the symmetries between $\mathcal{T}^{up}$ and $(\mathcal{T}^{up})^d$ are even richer than those between $\mathcal{ST}$ and $\mathcal{ST}^d$.

Definition 6.6.1. An operator $T \in \mathcal{L}(X, Y)$ is said to be *cosupertauberian* if T^* is supertauberian.

Following the notation used in this book, the class of all cosupertauberian operators is $(\mathcal{T}^{up})^d$. Thus, from the comments at the end of the preceding section, we have

$$(\mathcal{T}^{up})^d = \big((\mathcal{W}_+)^{up}\big)^d = \big((\mathcal{W}^{up})_+\big)^d.$$

The three following results about $(\mathcal{T}^{up})^d$ are obtained by duality.

Proposition 6.6.2. *Every cosupertauberian operator is strongly cotauberian.*

Proof. Let $T \in \mathcal{L}(X, Y)$ be a cosupertauberian operator. Thus, T^* is supertauberian, and by Proposition 6.5.3, T^* is strongly tauberian. Hence by Proposition 6.2.5, T is strongly cotauberian. □

Proposition 6.6.3. *Let $T \in \mathcal{L}(X, Y)$. If $R(T)$ is closed and $Y/R(T)$ is super-reflexive, then T is cosupertauberian.*

Proof. Assume that $R(T)$ is closed and $Y/R(T)$ is super-reflexive. Then $R(T^*)$ is closed and $N(T^*)$ is super-reflexive, so T^* is supertauberian by virtue of Corollary 6.5.12. Hence T is cosupertauberian. □

Proposition 6.6.4. *For every pair of Banach spaces X and Y, the set of all the supertauberian operators in $\mathcal{L}(X, Y)$ is open.*

Proof. The proof is directly obtained by duality from Corollary 6.5.6. □

Proposition 6.6.5. *An operator $T \in \mathcal{L}(X, Y)$ is cosupertauberian if and only if so is T^{**}.*

Proof. The proof is immediate from Proposition 6.5.10. □

A subtle application of duality techniques shows that $(\mathcal{T}^{up})^d$ is ultrapower-stable. The proof is reached as a consequence of the local supportability of $T_{\mathfrak{U}}{}^*$ by $T^*{}_{\mathfrak{U}}$ for any operator T and any ultrafilter $\mathfrak{U}$.

Theorem 6.6.6. *Given an operator $T \in \mathcal{L}(X, Y)$ and an ultrafilter $\mathfrak{U}$, the following statements are equivalent:*

(a) *T is cosupertauberian;*

(b) *$T^*{}_{\mathfrak{U}}$ is tauberian;*

(c) *$T_{\mathfrak{U}}{}^*$ is supertauberian;*

(d) *$T_{\mathfrak{U}}{}^*$ is tauberian;*

(e) *$T_{\mathfrak{U}}$ is cosupertauberian;*

(f) *$T_{\mathfrak{U}}$ is cotauberian.*

Proof. (a)⇒(b) By definition, if T is cosupertauberian, then T^* is supertauberian. Thus, by Theorem 6.5.7, $T^*{}_{\mathfrak{U}}$ is tauberian.

(b)⇒(c) Assume $T^*{}_{\mathfrak{U}}$ is tauberian. Then, by Theorem 6.5.7, $T^*{}_{\mathfrak{U}}$ is supertauberian. Next, on the one hand, Corollary 6.5.9 yields that $\mathcal{W}_+{}^{up}$ is stable under local supportability, and on the other hand, according to Theorem 6.3.12, $T_{\mathfrak{U}}{}^*$ is locally supportable by $T^*{}_{\mathfrak{U}}$, so $T_{\mathfrak{U}}{}^*$ is supertauberian.

(c)⇒(d) It follows from the inclusion $\mathcal{T}^{up} \subset \mathcal{T}$.

(d)⇒(a) If $T_{\mathfrak{U}}{}^*$ is tauberian, then $T^*{}_{\mathfrak{U}} = T_{\mathfrak{U}}{}^*|_{Y^*{}_{\mathfrak{U}}}$ is also tauberian, hence T^* is supertauberian, and therefore, T is cosupertauberian.

The equivalences (d)⇔(f) and (e)⇔(c) are trivial. □

Therefore, after Theorem 6.6.6, the words *cosupertauberian* and *supercotauberian* have equivalent meanings. Equivalently:

$$(\mathcal{W}_+{}^{up})^d = (\mathcal{W}_+{}^d)^{up}.$$

Theorem 6.6.7. *Given an operator $T \in \mathcal{L}(X,Y)$ and an ultrafilter $\mathfrak{U}$, the following statements are equivalent:*

(a) *T is cosupertauberian;*

(b) *$N(T_{\mathfrak{U}}{}^*)$ is super-reflexive;*

(c) *$N(T_{\mathfrak{U}}{}^*)$ is reflexive;*

(d) *$N(T^*{}_{\mathfrak{U}})$ is reflexive;*

(e) *$N(T^*{}_{\mathfrak{U}})$ is super-reflexive.*

Proof. (a)⇒(b) If T is cosupertauberian, then so is $T_{\mathfrak{U}}$ by virtue of Theorem 6.6.6. Thus, by definition, $T_{\mathfrak{U}}{}^*$ is supertauberian, and by Theorem 6.5.5, $N(T_{\mathfrak{U}}{}^*)$ is super-reflexive.

(b)⇒(c) Trivial.

(c)⇒(d) The proof follows easily from the fact that $N(T^*{}_{\mathfrak{U}})$ is a subspace of the kernel $N(T_{\mathfrak{U}}{}^*)$.

(d)⇒(e) If $N(T^*{}_{\mathfrak{U}})$ is reflexive, then Theorem 6.5.5 yields that it is also super-reflexive.

(d)⇒(a) If $N(T^*{}_{\mathfrak{U}})$ is super-reflexive, then, by Theorem 6.5.5, T^* is supertauberian; equivalently, T is cosupertauberian. □

We notice that some of the implications in the proof of Theorem 6.6.7, like (e)⇒(b), can be obtained by means of the following general result: for every operator T and every ultrafilter $\mathfrak{U}$, the kernel $N(T_{\mathfrak{U}}{}^*)$ is finitely representable in $N(T^*{}_{\mathfrak{U}})$. Indeed, by Theorem 6.3.13, there exist an ultrafilter $\mathfrak{V}$ and a pair of metric injections $U \in \mathcal{L}\big(Y_{\mathfrak{U}}{}^*, (Y^*{}_{\mathfrak{U}})_{\mathfrak{V}}\big)$ and $V \in \mathcal{L}\big(X_{\mathfrak{U}}{}^*, (X^*{}_{\mathfrak{U}})_{\mathfrak{V}}\big)$ so that

$$VT_{\mathfrak{U}}{}^* = (T^*{}_{\mathfrak{U}})_{\mathfrak{V}}U.$$

Therefore, the kernel of $T_{\mathfrak{U}}{}^*$ is isometrically mapped by U into that of $(T^*{}_{\mathfrak{U}})_{\mathfrak{V}}$, hence, by means of Proposition A.4.14, $N(T_{\mathfrak{U}}{}^*)$ is finitely representable in $N(T^*{}_{\mathfrak{U}})$.

Proposition 6.6.8. *Let $\mathfrak{U}$ be an ultrafilter on I. An operator $T \in \mathcal{L}(X,Y)$ is cosupertauberian if and only if the identity $N(T^*{}_{\mathfrak{U}}) = N(T_{\mathfrak{U}}{}^*)$ holds.*

Proof. In this proof, the $\sigma(Y_{\mathfrak{U}}{}^*, Y_{\mathfrak{U}})$ topology will be denoted by w^*.

Let us assume that T is cosupertauberian. Then, by Theorem 6.6.7, $N(T^*{}_{\mathfrak{U}})$ is a reflexive subspace of $N(T_{\mathfrak{U}}{}^*)$, so $\overline{N(T^*{}_{\mathfrak{U}})}^{w^*} = N(T^*{}_{\mathfrak{U}})$. But by Proposition A.4.23, $N(T^*{}_{\mathfrak{U}})$ is w^*-dense in $N(T_{\mathfrak{U}}{}^*)$, so the identity $N(T^*{}_{\mathfrak{U}}) = N(T_{\mathfrak{U}}{}^*)$ holds.

For the converse, assume that $N(T^*_{\mathfrak{U}})$ equals $N(T_{\mathfrak{U}}{}^*)$ and that T is not cosupertauberian. By Theorem 6.6.7, $N(T_{\mathfrak{U}}{}^*)$ is not reflexive. Therefore, $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$ is not reflexive either, and by virtue of Proposition A.5.3, there exist $0 < \varepsilon < 1$ and a pair of normalized sequences

$$(\mathbf{y}_n + \overline{R(T_{\mathfrak{U}})}) \subset Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})},$$
$$(\mathbf{f}_n) \subset [Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}]^* = N(T_{\mathfrak{U}}{}^*)$$

such that

$$\langle \mathbf{f}_k, \mathbf{y}_m \rangle = \begin{cases} > \varepsilon, & \text{if } 1 \leq k \leq m, \\ = 0, & \text{if } 1 \leq m < k. \end{cases} \tag{6.30}$$

Let $\mathbf{y}_n = [y_n^i]_i$ and $\mathbf{f}_n = [f_n^i]_i$ for all $n \in \mathbb{N}$. Let $\mathbf{g}$ be a w^*-cluster point of $\{\mathbf{f}_n : n \in \mathbb{N}\}$. Thus $\mathbf{g} \in N(T_{\mathfrak{U}}{}^*)$, but by hypothesis, $N(T_{\mathfrak{U}}{}^*) = N(T^*_{\mathfrak{U}})$, so there is a family $(g_i)_{i\in I}$ in B_{Y^*} so that $\mathbf{g} = [g_i]$.

From (6.30), it follows that

$$\langle \mathbf{g}, \mathbf{y}_n \rangle = 0 \quad \text{for all } n \in \mathbb{N}. \tag{6.31}$$

Formulas (6.30) and (6.31) allow us to define recursively the following decreasing set sequence:

$$A_1 := \{i \in I : \langle g_i, y_1^i \rangle < \varepsilon/2,\ \langle f_1^i, y_1^i \rangle > \varepsilon\} \in \mathfrak{U},$$
$$A_n := A_{n-1} \cap \{i \in I : \langle g_i, y_n^i \rangle < \varepsilon/2,\ \langle f_k^i, y_n^i \rangle > \varepsilon,\ 1 \leq k \leq n\} \in \mathfrak{U}.$$

Since $\mathfrak{U}$ is $\aleph_0$-incomplete, there exists another decreasing sequence $(C_n)_{n=1}^\infty \subset \mathfrak{U}$ such that $\cap_{n=1}^\infty C_n = \emptyset$ and such that $C_n \subset A_n$ for all n. Let $C_0 := I$, and define

$$s_i := \begin{cases} 0, & \text{if } i \in C_0 \setminus C_1, \\ y_k^i, & \text{if } i \in C_k \setminus C_{k+1} \text{ and } k \in \mathbb{N}. \end{cases}$$

Thus $[s_i]$ is a norm-one element of $Y_{\mathfrak{U}}$, and moreover, $\langle \mathbf{g}, [s_i] \rangle \geq \varepsilon$. Indeed, given $k \in \mathbb{N}$, for every positive integer $m > k$ and every $i \in C_m \setminus C_{m+1}$, we have

$$\langle f_k^i, s_i \rangle = \langle f_k^i, y_m^i \rangle > \varepsilon,$$

and since $\cup_{m=k}^\infty (C_m \setminus C_{m+1}) \in \mathfrak{U}$, it follows that

$$\langle \mathbf{f}_k, [s_i] \rangle = \lim_{\mathfrak{U}} \langle f_k^i, s_i \rangle \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

Therefore, since $\mathbf{g}$ is a w^*-cluster point of $\{\mathbf{f}_n : n \in \mathbb{N}\}$, we get

$$\langle \mathbf{g}, [s_i] \rangle \geq \varepsilon. \tag{6.32}$$

Moreover, for every $n \in \mathbb{N}$ and every $i \in C_n \setminus C_{n+1}$, we have

$$\langle g_i, s_i \rangle = \langle g_i, y_n^i \rangle < \varepsilon/2;$$

thus, as $\cup_{n=1}^\infty (C_n \setminus C_{n+1}) \in \mathfrak{U}$, we obtain $\langle \mathbf{g}, [s_i] \rangle = \lim_{\mathfrak{U}} \langle g_i, s_i \rangle \leq \varepsilon/2$, in contradiction with (6.32). □

The next result shows that $\mathcal{W}^{up} = (\mathcal{W}^{up})^d$.

Proposition 6.6.9. *An operator T is super weakly compact if and only if so is T^*.*

Proof. Let $T \in \mathcal{L}(X, Y)$ be a super weakly compact operator. Then, given any ultrafilter $\mathfrak{U}$, $T_{\mathfrak{U}}$ is weakly compact, and by the Gantmacher theorem, so is ${T_{\mathfrak{U}}}^*$. But the class $\mathcal{W}$ is an operator ideal, so ${T^*}_{\mathfrak{U}} = {T_{\mathfrak{U}}}^*|_{{Y^*}_{\mathfrak{U}}}$ is weakly compact, and by definition, T^* is super weakly compact.

Assume now T^* is super weakly compact. Then we have just proved that $T^{**} \in \mathcal{W}^{up}$, and since $\mathcal{W}^{up}$ is an operator ideal, we get $T = T^{**}|_X \in \mathcal{W}^{up}$. □

Proposition 6.6.10. *If $T \in \mathcal{L}(X, Y)$ is cosupertauberian and $K \in \mathcal{L}(X, Y)$ is super weakly compact, then $T + K$ is cosupertauberian. In particular, if K is compact, then $T + K$ is cosupertauberian.*

Proof. It follows directly via duality from Propositions 6.5.13 and 6.6.9. □

The following result about cosupertauberian operators parallels, but it is not a literal dual translation, of the perturbative characterization for supertauberian operators given in Theorem 6.5.16.

Theorem 6.6.11. *An operator $T \in \mathcal{L}(X, Y)$ is cosupertauberian if and only if for every compact operator $K \in \mathcal{L}(X, Y)$, the cokernel $Y/\overline{R(T + K)}$ is super-reflexive.*

Proof. Assume T is cosupertauberian, and let $K \in \mathcal{L}(X, Y)$ be a compact operator. Then by Proposition 6.6.10, $T + K$ is cosupertauberian, hence $T^* + K^*$ is supertauberian, and the kernel $N(T^* + K^*)$, which is isometrically identified with $\big(Y/\overline{R(T + K)}\big)^*$, is super-reflexive. Therefore, $Y/\overline{R(T + K)}$ is super-reflexive.

For the converse implication, let us assume that T is not cosupertauberian and find a compact operator K so that $Y/\overline{R(T + K)}$ is not super-reflexive.

In order to get our target, given a real number $0 < \varepsilon < 1$, we shall recursively find finite sets

$$\{f_i^n\}_{i=1}^n \subset S_{Y^*},\ \{z_i^n\}_{i=1}^n \subset S_Y, \text{ and } \{y_i^n\}_{i=1}^n \subset Y \text{ for all } n \in \mathbb{N} \tag{6.33}$$

such that, for each $p \in \mathbb{N}$,

$$\langle f_i^k, y_j^l \rangle = \delta_{ij}\delta_{kl} \text{ for all } \{k, l\} \subset \{1, \ldots, p\}, \text{ all } i \in \{1, \ldots, k\} \text{ and all } j \in \{1, \ldots, l\}, \tag{6.34}$$

$$\sum_{i=1}^{k} \|y_i^k\| \|T^*(f_i^k)\| < 2^{-k} \text{ for all } k \in \{1, \ldots, p\}, \tag{6.35}$$

$$\langle z_i^k, f_j^k \rangle > \varepsilon \text{ for all } k \in \{1, \ldots, p\} \text{ and all } 1 \le i \le j \le k, \tag{6.36}$$

$$\langle z_i^k, f_j^k \rangle = 0 \text{ for all } k \in \{1, \ldots, p\} \text{ and all } 1 \le j < i \le k. \tag{6.37}$$

By Theorem 6.5.5 (d), there exist $f_1^1 \in S_{Y^*}$ and $F_1^1 \in S_{Y^{**}}$ such that $\langle F_1^1, f_1^1\rangle > \varepsilon$ and $\|T^*(f_1^1)\| < 2^{-1}\varepsilon$. The principle of local reflexivity gives $z_1^1 \in S_Y$ satisfying $\langle z_1^1 - F_1^1, f_1^1\rangle = 0$. Thus, taking

$$y_1^1 := \frac{1}{\langle z_1^1, f_1^1\rangle} z_1^1,$$

the singletons $\{f_1^1\}$, $\{z_1^1\}$ and $\{y_1^1\}$ satisfy the conditions (6.34), (6.35), (6.36) and (6.37) for $p = 1$.

Assume the families $\{f_i^p\}_{i=1}^p$, $\{z_i^p\}_{i=1}^p$ and $\{y_i^p\}_{i=1}^p$ have been already chosen for all $p \in \{1, \dots, n-1\}$, and find the next families, $\{f_i^n\}_{i=1}^n$, $\{z_i^n\}_{i=1}^n$ and $\{y_i^n\}_{i=1}^n$.

Let $P \in \mathcal{L}(Y, Y)$ denote the projection whose kernel and range are

$$N(P) = \operatorname{span}\{y_i^k \colon 1 \le k \le n-1,\ 1 \le i \le k\},$$
$$R(P) = \bigcap_{k=1}^{n-1} \bigcap_{i=1}^{k} N(f_i^k),$$

and let

$$\delta := \frac{1}{n 2^n (1+\varepsilon^{-1})^n \|P\|}.$$

Since $N(P)$ is finite dimensional, then $I_{Y^*} - P^*$ has finite rank, hence the operator $T^* \circ (I_{Y^*} - P^*)$ is compact. Consequently, as T^* is not supertauberian and

$$T^* = T^* \circ P^* + T^* \circ (I_{Y^*} - P^*),$$

then T^*P^* cannot be supertauberian because of Proposition 6.5.13. But $R(P^*)$ equals $N(P)^\perp$, so $T^*P^*|_{N(P)^\perp} = T^*|_{N(P)^\perp}$, and therefore, by Theorem 6.5.5, there exists a pair of subsets $\{f_i^n\}_{i=1}^n$ in $S_{N(P)^\perp}$ and $\{F_i^n\}_{i=1}^n$ in $S_{Y^{**}}$ such that $\langle F_i^n, f_j^n\rangle > \varepsilon$ for all $1 \le i \le j \le n$, $\langle F_i^n, f_j^n\rangle = 0$ for all $1 \le j < i \le n$, and $\|T^*(f_i^n)\| \le \delta$ for all $1 \le i \le n$. Making use of the principle of local reflexivity, we may pick a subset $\{z_i^n\}_{i=1}^n$ in S_Y so that $\langle z_i^n, f_j^n\rangle > \varepsilon$ if $1 \le i \le j \le n$ and $\langle z_i^n, f_j^n\rangle = 0$ if $1 \le j < i \le n$. Thus, both conditions (6.36) and (6.37) are satisfied for $p = n$.

Next, by Lemma 6.5.15, there exists a subset

$$\{w_i\}_{i=1}^n \subset (1+\varepsilon^{-1})^n B_Y \tag{6.38}$$

such that $\langle f_i^n, w_j\rangle = \delta_{ij}$ for all i and j. Let $y_i^n := P(w_i)$ for all $1 \le i \le n$, and prove that condition (6.34) holds for the chosen y_i^n. In fact, if $k < n$, then $y_j^n \in R(P) \subset N(f_i^k)$, so $\langle f_i^k, y_j^n\rangle = 0$; and as $f_i^n \in N(P)^\perp$ and $y_i^k \in N(P)$, then $\langle f_i^n, y_j^k\rangle = 0$. Finally, as $f_i^n \in N(P)^\perp = R(P^*)$, then $f_i^n = P^*(f_i^n)$, so

$$\langle f_i^n, y_j^n\rangle = \langle f_i^n, P(w_j)\rangle = \langle f_i^n, w_j^n\rangle = \delta_{ij},$$

such as we wanted. It only remains to check condition (6.35) for $p = n$. Indeed, by formula (6.38), since $\|T^*(f_i^n)\| \leq \delta$ and $y_i^n = P(w_i^n)$,

$$\sum_{i=1}^{n} \|y_i^n\| \|T^*(f_i^n)\| \leq n\|P\|(1+\varepsilon^{-1})^n \delta = 2^{-n}.$$

Once the families $\{f_i^n\}_{i=1}^n$, $\{z_i^n\}_{i=1}^n$ and $\{y_i^n\}_{i=1}^n$ have been obtained for all $n \in \mathbb{N}$, we define an operator $Q \in \mathcal{L}(Y^*, X^*)$ by

$$Q(f) := -\sum_{n=1}^{\infty} \sum_{i=1}^{n} \langle f, y_i^n \rangle \cdot T^*(f_i^n).$$

Note that condition (6.35) yields

$$\|Q\| \leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} \|y_i^n\| \|T^*(f_i^n)\| \leq \sum_{n=1}^{\infty} 2^{-n} = 1,$$

hence Q is well defined and compact.

Notice that all the finite ε-triangular sequences $(f_i^n)_{i=1}^n$ are contained in $N(T^* + Q)$, so $N(T^* + Q)$ is not super-reflexive by virtue of Proposition A.5.6. But Q is the conjugate operator of $K \in \mathcal{L}(X, Y)$, where

$$K(x) = -\sum_{n=1}^{\infty} \sum_{i=1}^{n} \langle T^*(f_i^n), x \rangle y_i^n.$$

Therefore, as $N(T^* + K^*) = \big(Y/\overline{R(T+K)}\big)^*$, it follows that $Y/\overline{R(T+K)}$ cannot be super-reflexive. The proof is done. □

Note that, in the preceding theorem, the operator K can be chosen so that its norm is arbitrarily small. Indeed, given $\eta > 0$, it is enough to substitute the operator K in the proof of Theorem 6.6.11 for $K_\eta = -\sum_{n=k}^{\infty} \sum_{i=1}^{n} \langle T^*(f_i^n), \cdot \rangle y_i^n$, where k must be chosen sufficiently large so that $\|K_\eta\| < \eta$.

Theorem 6.6.11 has important consequences.

Proposition 6.6.12. *Given a Banach space Y, the following statements hold:*

(i) *every super-reflexive quotient of Y is finite dimensional if and only if, for each Banach space X, every cosupertauberian operator $T \in \mathcal{L}(X, Y)$ is lower semi-Fredholm;*

(ii) *every reflexive quotient of Y is super-reflexive if and only if, for each Banach space X, every cotauberian operator $T \in \mathcal{L}(X, Y)$ is cosupertauberian.*

Proof. (i) Assume all super-reflexive quotients of Y are finite dimensional. Thus, by Theorem 6.6.11, if $T \in \mathcal{L}(X, Y)$ is cosupertauberian, then the cokernel of $T + K$ is finite dimensional for all compact operators $K \in \mathcal{L}(X, Y)$. Therefore, T is lower semi-Fredholm because of Theorem A.1.9.

For the converse, assume that Y contains a subspace R so that Y/R is infinite-dimensional and super-reflexive. Then the embedding $J_R\colon R \longrightarrow Y$ is not lower semi-Fredholm, but by Proposition 6.6.3, it is cosupertauberian.

(ii) The proof is analogous to that of (i), but here we have to apply the perturbative characterizations for cosupertauberian operators and for cotauberian operators given respectively in Theorem 6.6.11 and in Theorem 3.1.20. □

As in Proposition 6.5.17, situations described in Proposition 6.6.12 are not trivial. For instance, given the original Tsirelson space X_{T} and any Banach space X, every operator $T \in \mathcal{L}(X, X_{\mathsf{T}})$ is cotauberian, but T is cosupertauberian if and only if T is lower semi-Fredholm.

Another consequence of Theorem 6.6.11 is the following algebraic characterization for cosupertauberian operators.

Proposition 6.6.13. *Given $T \in \mathcal{L}(X,Y)$, the following statements are equivalent:*

(a) *the operator T is cosupertauberian;*

(b) *for every space Z and every $A \in \mathcal{L}(Y,Z)$, if AT is super weakly compact, then A is super weakly compact;*

(c) *a quotient Y/E is super-reflexive whenever $Q_E T$ is super weakly compact.*

Proof. (a)⇒(b) Assume T is cosupertauberian and AT is super weakly compact. Thus, T^*A^* is also super weakly compact, and T^* is supertauberian, so Proposition 6.5.19 yields A^* is super weakly compact and hence so is A.

(b)⇒(c) Let E be a closed subspace of Y so that $Q_E T$ is super weakly compact. Thus, by hypothesis, the quotient operator $Q_E \in \mathcal{L}(Y, Y/E)$ is super weakly compact. Therefore, given any ultrafilter $\mathfrak{U}$, $(Q_E)_{\mathfrak{U}}$ is weakly compact and surjective, so $(Y/E)_{\mathfrak{U}}$ is reflexive, that is, Y/E is super-reflexive.

(c)⇒(a) Let us assume T is not cosupertauberian. Then, by Theorem 6.6.11, there exists a compact operator $K \in \mathcal{L}(X,Y)$ so that Y/E is not super-reflexive, where $E := \overline{R(T-K)}$. Consider the quotient operator $Q_E \in \mathcal{L}(Y\ , Y/E)$. Thus $Q_E T = Q_E K$, and since the class of all compact operators is ultrapower-stable, then $Q_E T$ is super weakly compact, but Y/E is not super-reflexive. Therefore, (c) fails. □

Remark 6.6.14. Proposition 6.6.13 yields the identity

$$(\mathcal{W_+}^{up})^d = (\mathcal{W}^{up})_-.$$

and Theorem 6.6.11 can be expressed as

$$T \in (\mathcal{W}^{up})_-(X,Y) \Leftrightarrow \forall K \in \mathcal{K}(X,Y),\ Y/\overline{R(T+K)} \in Sp(\mathcal{W}^{up}).$$

After Proposition 3.1.22, it follows that the class of all cotauberian operators, $(\mathcal{W}_+)^d$, equals $\mathcal{W}_-$. But by virtue of Theorem 3.1.18, $(\mathcal{W}_-)^d$ is strictly smaller than $\mathcal{W}_+$, pointing out clearly that the duality between the classes of tauberian operators and cotauberian operators is not perfect.

Nevertheless, we have the following pair of results about duality concerning the class of supertauberian operators and that of cosupertauberian operators.

Proposition 6.6.15. *The identity* $(\mathcal{W}^{up})_- = (\mathcal{W}_-)^{up}$ *holds.*

Proof. On the one hand, Theorem 6.6.6 shows that the class of all cosupertauberian operators coincides with $(\mathcal{W}_-)^{up}$. On the other hand, Proposition 6.6.13 yields that the class of all cosupertauberian operators is $(\mathcal{W}^{up})_-$. □

Some of the main results of Sections 6.3, 6.5 and this section merge in the proof of the following proposition.

Proposition 6.6.16. *The following identities hold:*

$$(\mathcal{W}_+)^{up} = \big((\mathcal{W}_-)^d\big)^{up} = \big((\mathcal{W}_-)^{up}\big)^d = \big((\mathcal{W}^{up})_-\big)^d.$$

Proof. Throughout this proof, we will freely use the identity $(\mathcal{W}_+)^{up} = (\mathcal{W}^{up})_+$ (see Remark 6.5.20), the fact that the class of cosupertauberian operators equals $(\mathcal{W}^{up})_-$ (see Remark 6.6.14), and the identity $(\mathcal{W}^{up})_- = (\mathcal{W}_-)^{up}$ proved in Proposition 6.6.15.

Let $\mathfrak{U}$ be any ultrafilter.

In order to prove that $(\mathcal{W}_+)^{up} \subset \big((\mathcal{W}_-)^d\big)^{up}$, take $T \in (\mathcal{W}_+)^{up}$. By Theorem 6.5.7, $T_{\mathfrak{U}} \in (\mathcal{W}_+)^{up}$, and since the class of supertauberian operators is stable under biduality, then $(T_{\mathfrak{U}})^{**} \in (\mathcal{W}_+)^{up}$. Thus, $(T_{\mathfrak{U}})^{**} \in \mathcal{W}_+$, so $(T_{\mathfrak{U}})^* \in \mathcal{W}_-$. Hence $T_{\mathfrak{U}} \in (\mathcal{W}_-)^d$, and therefore, Theorem 6.6.6 yields $T \in \big((\mathcal{W}_-)^d\big)^{up}$.

To prove $\big((\mathcal{W}_-)^d\big)^{up} \subset \big((\mathcal{W}_-)^{up}\big)^d$, take $T \in \big((\mathcal{W}_-)^d\big)^{up}$. By definition, $(T_{\mathfrak{U}})^* \in \mathcal{W}_-$, but by Proposition 3.1.22, $\mathcal{W}_-$ equals $\mathcal{T}^d$, so $(T_{\mathfrak{U}})^{**} \in \mathcal{W}_+$. Thus $T_{\mathfrak{U}} \in \mathcal{W}_+$, and Theorem 6.5.7 yields $T_{\mathfrak{U}} \in (\mathcal{W}_+)^{up}$. Then $T \in (\mathcal{W}_+)^{up}$, and as

$$(T^{**})_{\mathfrak{U}} \prec_{\mathrm{ls}} T^{**} \prec_{\mathrm{ls}} T,$$

and $(\mathcal{W}_+)^{up}$ is stable under local supportability, we get $(T^{**})_{\mathfrak{U}} \in (\mathcal{W}_+)^{up}$. Thus, $(T^{**})_{\mathfrak{U}} \in \mathcal{W}_+$, and by Theorem 6.6.6, $({T^*}_{\mathfrak{U}})^* \in \mathcal{W}_+$, so $T^* \in (\mathcal{W}_-)^{up}$. Therefore, $T \in \big((\mathcal{W}_-)^{up}\big)^d$.

The inclusion $\big((\mathcal{W}_-)^{up}\big)^d \subset \big((\mathcal{W}^{up})_-\big)^d$ is trivial because $(\mathcal{W}_-)^{up} = (\mathcal{W}^{up})_-$.

Finally, let us prove that $\big((\mathcal{W}^{up})_-\big)^d \subset (\mathcal{W}_+)^{up}$. Let $T \in \big((\mathcal{W}^{up})_-\big)^d$. Thus $T^* \in (\mathcal{W}^{up})_-$, and by Remark 6.6.14, T^* is cosupertauberian, that is, T^{**} is supertauberian. Then T is also supertauberian, that is, $T \in (\mathcal{W}_+)^{up}$, and the proof is done. □

Local supportability versus local representability

Local supportability and local representability are closely related in situations like those of Theorem 6.3.9 or Theorem 6.3.13. Nevertheless, both notions are mutually independent. This is a consequence of an argument that can be outlined as follows: assume there exist an $\prec_{\mathrm{ls}}$-stable semigroup $\mathcal{S}$ and a pair of operators $T \notin \mathcal{S}$ and $S \in \mathcal{S}$ such that $T \prec_{\mathrm{lr}} S$. Then T is not locally supportable by S, which proves that local representability does not imply local supportability. The fact that local supportability does not imply local representability is proved analogously.

This section provides the corresponding examples of semigroups $\mathcal{S}$ and operators T and S that fulfill the arguments given above. In particular, Proposition 6.6.21 involves the semigroup of cosupertauberian operators.

Let us introduce some notation. The space of all continuous functions on the unit interval $I := [0, 1]$ will be denoted by $\mathcal{C}(I)$, and its dual, the space of all Radon measures on I, by $\mathcal{M}$. The space of all Lebesgue-integrable functions on I is isometrically identified with a subspace of $\mathcal{M}$, denoted $L_1(I)$. Moreover, let $\mathcal{N}$ be the subspace of $\mathcal{M}$ of all singular measures with respect to the Lebesgue measure on the unit interval. It is a classic result that

$$\mathcal{M} = L_1(I) \oplus_1 \mathcal{N}.$$

Given a function $f\colon I \longrightarrow \mathbb{R}$, and a positive integer $1 \le i \le 2^k$, let

$$\begin{aligned} m_i^k(f) &:= \inf f(\overline{I_i^k}), \\ M_i^k(f) &:= \sup f(\overline{I_i^k}), \\ \rho_k(f) &:= \max\{M_i^k(f) - m_i^k(f) : 1 \le i \le 2^k\}, \end{aligned}$$

where I_i^k is the dyadic interval defined as in Example 4.1.11 and χ_i^k is its characteristic function.

We consider a system of positive norm-one measures $\{\nu_i^k\}_{i=1}^{2^k}{}_{k=0}^{\infty}$ in $\mathcal{M}$ such that every ν_i^k is concentrated on I_i^k. Let $G_k \in \mathcal{L}(\mathcal{M})$ be the norm-one projection defined by

$$G_k(\lambda) := \sum_{i=1}^{2^k} \lambda(I_i^k)\nu_i^k. \tag{6.39}$$

Lemma 6.6.17. *Given $k \in \mathbb{N}$, $f \in \mathcal{C}(I)$ and $\lambda \in \mathcal{M}$, we have*

$$|\langle \lambda - G_k(\lambda), f\rangle| \le \|\lambda\|\rho_k(f).$$

Proof. It is sufficient to show the result for a positive measure λ. We consider the functions m_f and M_f defined on the unit interval by

$$m_f(t) := \sum_{i=1}^{2^k} m_i^k(f)\chi_i^k(t),$$

$$M_f(t) := \sum_{i=1}^{2^k} M_i^k(f)\chi_i^k(t).$$

Note that $\int_0^1 m_f\, d\lambda = \int_0^1 m_f\, dG_k(\lambda)$ and $\int_0^1 M_f\, d\lambda = \int_0^1 M_f\, dG_k(\lambda)$. Moreover,

$$\int_0^1 m_f\, d\lambda \leq \int_0^1 f\, d\lambda \leq \int_0^1 M_f\, d\lambda,$$

and

$$\int_0^1 m_f\, dG_k(\lambda) \leq \int_0^1 f\, dG_k(\lambda) \leq \int_0^1 M_f\, dG_k(\lambda),$$

therefore, we get

$$\left|\int_0^1 f\, d\lambda - \int_0^1 f\, dG_k(\lambda)\right| \leq \int_0^1 (M_f - m_f)\, d\lambda \leq \sum_{i=1}^{2^k} \rho_k(f)\lambda(I_i^k) = \|\lambda\|\rho_k(f). \qquad \square$$

Proposition 6.6.18. *Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{N}$ and define $G \in \mathcal{L}(\mathcal{M}, L_1(I)_{\mathfrak{U}})$ by $G(\lambda) := [G_n(\lambda)]_n$ (G_n defined as in formula* (6.39)*). Then the next statements hold:*

(i) $\lim_n \langle G_n(\lambda), f\rangle = \langle \lambda, f\rangle$ *for all $\lambda \in \mathcal{M}$ and all $f \in \mathcal{C}(I)$;*

(ii) $\lim_n \|G_n(\lambda)\| = \|\lambda\|$ *for all $\lambda \in \mathcal{M}$, so G is a metric injection.*

Proof. (i) Let $\lambda \in S_{\mathcal{M}}$ and $f \in \mathcal{C}(I)$. By uniform continuity of f, there is a positive integer n_0 verifying $\rho_{n_0}(f) < \varepsilon$. So, by Lemma 6.6.17, we have $|\langle \lambda - G_n(\lambda), f\rangle| < \varepsilon$ for all $n \geq n_0$.

(ii) Let $\lambda \in \mathcal{M}$ and $\varepsilon > 0$. Choose $f \in B_{\mathcal{C}(I)}$ so that $\langle \lambda, f\rangle > \|\lambda\| - 2^{-1}\varepsilon$. By statement (i), there is n_0 such that $|\langle G_n\lambda, f\rangle| > |\langle \lambda, f\rangle| - 2^{-1}\varepsilon$ for all $n \geq n_0$, so

$$\|\lambda\| - \varepsilon < |\langle G_n\lambda, f\rangle| \leq \|G_n(\lambda)\| \leq \|\lambda\|. \qquad \square$$

The following theorem offers an example of operator local representability concerning the space $L_1(I)$.

Theorem 6.6.19. *Let $T \in \mathcal{L}\big(Y, \mathcal{C}(I)\big)$ and $\{\nu_i^k\}_{i=1}^{2^k}{}_{k=0}^{\infty} \subset \mathcal{M}$ a system of positive, norm-one measures such that every ν_i^k is concentrated in I_i^k. Let Z be the closed subspace of $\mathcal{M}$ generated by $\{\nu_i^k\}_{i=1}^{2^k}{}_{k=0}^{\infty}$, consider the metric injection G given in Proposition* 6.6.18*, and the metric surjection $P \in \mathcal{L}(Y^*{}_{\mathfrak{U}}, Y^*)$ defined by*

$$P([y_n^*]) := w^*\text{-}\lim_{n\to\mathfrak{U}} y_n^*,$$

where w^* denotes the $\sigma(Y^*, Y)$ topology. Then $T^* = P \circ (T^*|_Z)_{\mathfrak{U}} \circ G$. Hence, T^* is locally representable in $T^*|_Z$.

Proof. Given $\lambda \in \mathcal{M}$, Proposition 6.6.18 shows

$$P \circ (T^*|_Z)_{\mathfrak{U}} \circ G(\lambda) = w^*\text{-} \lim_{n\to\mathfrak{U}} T^* G_n(\lambda) = T^*(\lambda),$$

so Proposition 6.3.2 proves that T^* is locally representable in $T^*|_Z$. □

Corollary 6.6.20. *For every $T \in \mathcal{L}(Y, \mathcal{C}(I))$, the conjugate T^* is locally representable in $T^*|_{L_1(I)}$ and in $T^*|_{\mathcal{N}}$.*

Proof. For every dyadic interval I_i^k, we define a measure μ_i^k by

$$\mu_i^k(A) := 2^k \mu(A \cap I_i^k) \ \text{ for every borelian set } A \subset I,$$

where μ is the Lebesgue measure on I. Let us denote by δ_i^k the Dirac delta associated with the middle point of I_i^k. Let

$$Z := \overline{\text{span}}\{\delta_i^k\}_{i=1}^{2^k}{}_{k=0}^{\infty} \subset \mathcal{N}$$

and note that $L_1(I) = \overline{\text{span}}\{\mu_i^k\}_{i=1}^{2^k}{}_{k=0}^{\infty}$. By Theorem 6.6.19, T^* is locally representable in $T^*|_{L_1(I)}$ and in $T^*|_Z$, hence in $T^*|_{\mathcal{N}}$. □

Proposition 4.2.2 shows that given an operator $T\colon L_1(\mu) \longrightarrow Y$, T is tauberian if and only if $T_{\mathfrak{U}}|_{N(P_{\mu_{\mathfrak{U}}})}$ is an isomorphism. This result contrasts with the following:

Proposition 6.6.21. *There is a non-tauberian operator $T^* \in \mathcal{L}(\mathcal{M})$ such that $T^*|_{\mathcal{N}}$ is an isomorphism. Hence, local representability does not imply local supportability.*

Proof. For every $n \in \mathbb{N}$, we denote

$$\begin{aligned} J_n &:= [1/2^n, 2/2^n], \\ J_n^+ &:= (2/2^{n+1}, 3/2^{n+1}), \\ J_n^- &:= (3/2^{n+1}, 4/2^{n+1}), \end{aligned}$$

and define the functions

$$\begin{aligned} f_n(t) &:= \sin(2^{n+1}\pi t)\chi_{J_n}(t) \in \mathcal{C}(I), \\ h_n(t) &:= 2^n(\chi_{J_n^+} - \chi_{J_n^-})(t) \in L_1(I). \end{aligned}$$

Note that $\|h_n\|_1 = 1$ and $\langle h_m, f_n\rangle = 2\pi^{-1}\delta_{mn}$.

Since $\lim_n \langle h_n, f\rangle = 0$ for all $f \in \mathcal{C}(I)$, given any null sequence $\alpha \equiv (\alpha_n)_{n\in\mathbb{N}}$ contained in the interval $(0,1)$, we can define $P_\alpha \in \mathcal{L}(\mathcal{C}(I))$ by

$$P_\alpha(f) := \sum_{n=1}^{\infty} (1-\alpha_n)\langle h_n, f\rangle f_n$$

and $T := I_{\mathcal{C}(I)} - 2\pi^{-1}P_\alpha$, so

$$T^*(\lambda) = \lambda - 2\pi^{-1}\sum_{n=1}^{\infty}(1-\alpha_n)\langle\lambda, f_n\rangle h_n.$$

Note that $P_\alpha^*(\lambda) \in L_1(I)$ for all $\lambda \in \mathcal{M}$, so the decomposition $\mathcal{M} = L_1(I) \oplus_1 \mathcal{N}$ yields $\|T^*\nu\| \geq \|\nu\|$ for all $\nu \in \mathcal{N}$, hence $T^*|_{\mathcal{N}}$ is an isomorphism. However, $T^*(h_n) = \alpha_n h_n$, so $\lim_n T^*(h_n) = 0$. Moreover, (h_n) is a normalized disjoint sequence, hence Theorem 4.1.3 yields that $T^*|_{L_1(I)}$ is not tauberian, so $T^*|_{L_1(I)}$ is not supertauberian either. As $(\mathcal{W}_+)^{up}$ is an injective semigroup, it follows that $T^* \notin (\mathcal{W}_+)^{up}$. Moreover, as ${\mathcal{W}_+}^{up}$ is an ultrapower-stable upper semigroup, Proposition 6.4.11 shows that T^* is not locally supportable by $T^*|_{\mathcal{N}}$, and by Corollary 6.6.20, we conclude that local representability does not imply local supportability. □

Let us prove finally that local supportability does not imply local representability.

Proposition 6.6.22. *Let $\Upsilon \in \mathcal{L}(\ell_2, \ell_\infty)$ be a metric injection, and let I_{ℓ_2} be the identity operator on ℓ_2. Then I_{ℓ_2} is locally supportable by Υ, but I_{ℓ_2} is not locally representable in Υ. Hence, local supportability does not imply local representability.*

Proof. It is immediate that I_{ℓ_2} is locally supportable by Υ. Let us assume that I_{ℓ_2} is locally c-representable in Υ. Then, by Proposition 6.3.2, there is a ultrafilter $\mathfrak{U}$ and operators $A \in \mathcal{L}(\ell_2, (\ell_2)_\mathfrak{U})$, $B \in \mathcal{L}((\ell_\infty)_\mathfrak{U}, \ell_2)$ so that $I_{\ell_2} = B\Upsilon_\mathfrak{U} A$. Since $(\ell_\infty)_\mathfrak{U}$ is isometric to a space of continuous functions on some compact set [99], it follows that $(\ell_\infty)_\mathfrak{U}$ has the Dunford-Pettis property. But ℓ_2 is reflexive so $(\ell_\infty)_\mathfrak{U}$ and ℓ_2 are essentially incomparable, and therefore, B is an inessential operator (see Theorem 1 in [69]). Besides, by Remark 6.1.34, the class $\mathcal{I}$ of all inessential operators is an operator ideal. Therefore, $0 = I_{\ell_2} - B\Upsilon_\mathfrak{U} A$ is a Fredholm operator, a contradiction. □

6.7 Notes and Remarks

Operator ideals in Banach space theory were popularized by Pietsch, but the notion of operator ideal, as well as that of operator semigroup, already occurred in Fredholm theory. It is important to point out that the notion of operator semigroup considered in this book (Definition 6.1.1) has nothing to do with the semigroups in the context of operator theory and differential equations, like in [55].

The paper of Lebow and Schechter [118] is one of the first references in which the term *semigroup* is considered in a sense similar to the one given here. Afterwards, there are more appearances of operator semigroups in the literature, sometimes implicitly, as in the case of the semigroup $\mathcal{RN}_+$ associated with the ideal $\mathcal{RN}$ of the Radon-Nikodym operators considered by Bourgain in [40] (see Definition 5.3.15).

The term *semigroup* reappears in [89] and it is finally axiomatized by Aiena et al. in [1]. Most of the concepts introduced in Section 6.1 have been taken from [1].

The notion of operator semigroup given in Definition 6.1.1 differs slightly from the original one given in [1]: in the first case, an operator semigroup is required to contain all Fredholm operators while in the second one it is enough to contain the bijective operators. This weakening is rather formal. The only difference is that the original definition admits the class of all bijective operators as a semigroup, while Definition 6.1.1 does not. From an algebraic point of view, it would have been desirable to accept the class of all bijective operators as a semigroup, but this makes sense only if the class of all null operators were an operator ideal itself. That would have meant a modification in the original definition of operator ideal given by Pietsch (see Definition A.2.1). But the theory of operator ideals is well established. So we have just preferred to keep the original definition of operator ideal and to change that of operator semigroup.

Semigroups of the form $\mathcal{A}_+$ and $\mathcal{A}_-$ have been studied for some operator ideals $\mathcal{A}$ other than $\mathcal{K}$, $\mathcal{W}$ or $\mathcal{W}^{up}$ (see [6], [34], [89], [92], [87], [102] and [126]).

Bourgain proved that the class of all separable $\mathcal{L}_1$-spaces with the property of Radon-Nikodym has no universal element by finding a convolution operator in $\mathcal{RN}_+$ [40].

The semigroup $\mathcal{RN}_+$ has been explicitly studied in [88] in order to prove the existence of $\mathcal{L}_1$-spaces with the Radon-Nikodym property containing infinite dimensional reflexive subspaces. See also [138]. Moreover, the semigroup $(\mathcal{RN}^d)_-$ was studied in [87] where, among other results, it is proved that $\mathcal{RN}^d{}_-$ coincides with $(\mathcal{RN}_+)^d$ and admits a perturbative characterization.

The semigroups $\mathcal{A}_l$ and $\mathcal{A}_r$ were introduced and studied in [2]. That paper contains a detailed description of the structural properties of $\mathcal{A}_l$ and $\mathcal{A}_r$, including their behavior under duality. Actually, that paper follows the pattern laid out by Yood, who studied the semigroups $\mathcal{K}_l$ and $\mathcal{K}_r$ [176], and by Yang, who studied the semigroups $\mathcal{W}_l$ and $\mathcal{W}_r$ [175].

Given an space ideal $\mathbb{A}$, we can define two classes of operators $\mathbb{A}$-$\mathcal{SS}$ and $\mathbb{A}$-$\mathcal{SC}$ that coincide with the strictly singular and the strictly cosingular operators in the case when $\mathbb{A}$ is the ideal of the finite dimensional spaces. When $\mathbb{A}$ satisfies certain incomparability conditions, $\mathbb{A}$-$\mathcal{SS}$ or $\mathbb{A}$-$\mathcal{SC}$ is an operator ideal. The corresponding semigroups $\mathbb{A}$-$\mathcal{SS}_+$ and $\mathbb{A}$-$\mathcal{SC}_-$ where studied in [3], where it is shown that they admit a perturbative characterization.

Example 6.1.21 is due to Astala and Tylli [16]. The notion of perturbation class of a semigroup is due to Lebow and Schechter [118]. The notion of radical of an operator ideal was introduced by Pietsch in [139].

The terminology *strongly tauberian* was introduced by Rosenthal [147], but these concepts were around before, such as in [115] (see Question 1.3.5) and in [175]. The semigroup $\mathcal{ST} \cap \mathcal{ST}^d$ also occurs in [93]. However, credit is certainly due to Rosenthal for making the first penetrating study of $\mathcal{ST}$ in his paper [147], which is a sequel of his c_0-dichotomy theorem [146].

Most of the results in Section 6.2 belong to [147] with the exceptions of Proposition 6.2.9, the results concerning strongly cotauberian operators, Lemma 6.2.17 (borrowed from [98]), Lemma 6.2.11 and Theorem 6.2.18, which were proved for complex Banach spaces by Bermúdez and Kalton [29].

Theorem 6.2.13 follows closely the original argument used by Rosenthal in [147], but it is necessary to point out that the definition of ε-triangular sequence used in this book and that of Rosenthal given in [147] are non-equivalent, although, from a technical point of view, both types of ε-triangular sequence are intended for the same purposes and are handled in a similar way.

As a consequence of Theorem 6.2.18, Bermúdez and Kalton proved that, given a complex von Neumann algebra X and an operator $T \in \mathcal{L}(X)$, T^* has non-void point spectrum. Indeed, let $S := T^*$. Non-reflexivity of X yields that S^{co} has non-void spectrum. Thus, choosing any λ in the boundary of $\sigma(S^{co})$, the operator $\lambda I_{X^{**}/X} - T^{co}$ is not strongly tauberian, and since the dual space of a von Neumann algebra is L-embedded in its bidual, the complex version of Theorem 6.2.18 proves that $N(\lambda I_{X^{*(3)}} - S^{**}) \not\subset X^*$. Thus, the authors conclude that T^* cannot be injective, so T^* must have some eigenvalue. As a consequence of this fact, Kalton and Bermúdez prove that a complex, non-reflexive von Neumann algebra cannot support any topologically transitive operator because the conjugate of such an operator does not have any eigenvalue (an operator $T \in \mathcal{L}(X)$ is said to be *topologically transitive* if for every pair of neighborhoods, U and V, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$).

In general, the operator j of the tauberian decomposition of an operator is not strongly tauberian. In fact, let T be an operator such that the range of T^{co} is not closed, and consider the tauberian decomposition $T = jUk$. By Theorem 3.2.8, $T^{co} = j^{co}U^{co}k^{co}$ is equivalent to the tauberian decomposition of T^{co}. Thus, the range of j^{co} is not closed, hence j cannot be strongly tauberian. Similar arguments show that the cotauberian factor k is not strongly cotauberian in general.

All the material in Section 6.3, devoted to operator finite representability, has been borrowed from [129], [130] and [140]. While the notion of finite representability is well established, the situation has been very different for their operators. The first definitions for operator finite representability were given in 1976 and 1977 by Beauzamy ([19] [20], [24]) in studying the ideal of uniform convexifying operators, which matches with $\mathcal{W}^{up}$. But his definitions were not very appropriate for looking into more operator ideals other than $\mathcal{W}^{up}$. This situation was ameliorated by Heinrich in 1980 [101], and later by Pietsch in 1999 [140], whose definition of operator finite representability has been adopted in this book as Definition 6.3.1. Heinrich finite representability is just a particular case of that of Pietsch. Indeed, after [100, Theorem 1.2], an operator T is finitely representable in S when the operators A and B in Proposition 6.3.2 are a metric injection and a metric surjection respectively. But those were not the only definitions of operator finite representability. Beauzamy gave another one in 1982 [24] with the sole intention of studying the ideal $\mathcal{W}^{up}$, and Bellenot introduced another one in 1984 looking for certain strengthening of the principle of local reflexivity [26]. These two last definitions are very different from that given by Heinrich. Quoting Bellenot's words in his paper:

> "This is a strange definition, but it is what we obtain from Corollary 7,... The definition above uses the domain as the point of reference where as Beauzamy [19] uses the range and Heinrich uses both."

This probably did not contribute much to clarifying what *operator finite representability* should mean. However, the operator semigroup axiomatization illuminated the situation. Indeed, rather than looking at technical differences, the paper [130] classifies all different types of operator finite representability into two types in accordance with their applications to either operator ideals or operator semigroups; therefore, all possible connections between these two types are just a reflection of the interplay between operator ideals and operator semigroups.

Thus, following the schema laid down by Pietsch for operator ideals, the authors of [130] introduced the notion of *local supportability* (see Definition 6.3.4) as a generalization of Bellenot's finite representability (his definition is just the particular case in Definition 6.3.4 when U and V are ε-isometries, which makes the operators U and V in Proposition 6.3.5 to be isometries), which is in turn a generalization of that of Beauzamy in [24] (indeed, Beauzamy's definition asks the additional condition that $\|SU - VT|_E\| = 0$). Let us say that all those successive generalizations are useful. Indeed, given an operator T, in general, T^{**} is not f.r. in T in the sense of Beauzamy in [24], but it is Bellenot f.r. in T; moreover, T^{co} is locally supportable by T (see Theorem 6.3.19), but it is not clear if T^{co} is Bellenot f.r. in T. Analogously, T^{**} is Heinrich finitely representable in T, but it is not well known if T^{co} is Heinrich f.r. or not in T, but it is locally representable in T (see Theorem 6.3.20). A comparison between all the mentioned types of operator finite representability can be found in [130], including all the results of Section 6.6 that prove the independence between local supportability and local representability.

Theorem 6.3.8, which comprises the results about the finite representability of T^{**} in T, and its sequels about the finite representability of T^{co} in T (Theorems 6.3.19 and 6.3.20) are proved in [129]. Restricted versions of Theorem 6.3.8 were obtained by Heinrich [101], Basallote and Díaz [18], Behrends [27] and Bellenot [26]. It is remarkable that Behrends proof [27, Corollary 5.4] is only achieved for tauberian operators. Lemma 6.3.18 is based upon a result of Kalton [114].

The proof of Theorem 6.3.12 about the finite representability of ${T_{\mathfrak{U}}}^*$ in ${T^*}_{\mathfrak{U}}$ has been taken from [129]. We point out that, with the hypotheses of Theorem 6.3.12, it is possible to obtain the following additional properties:

(iv) $U(\mathbf{f}) = \mathbf{f}$ for all $\mathbf{f} \in F \cap {Y^*}_{\mathfrak{U}}$,

(v) $V(\mathbf{h}) = \mathbf{h}$ for all $\mathbf{h} \in H$.

For that purpose, it is sufficient to use [85, Theorem 3.4] instead of Proposition 6.3.11. Heinrich gave an earlier proof of the fact that ${T_{\mathfrak{U}}}^*$ is locally representable in ${T^*}_{\mathfrak{U}}$, including theses (iv) and (v) [101].

All the results in Section 6.4 about applications of operator finite representability belong to [129], [130] and [140]. In particular, it is proved in Corollary 6.4.14 that if $\mathcal{S}$ is an upper (or lower) ultrapower-stable semigroup and $T \in \mathcal{S}$, then $T^{**} \in \mathcal{S}$. This fact was proved earlier in [80], but only for upper ultrapower-stable semigroups $\mathcal{Z}$ for which an operator T belongs to $\mathcal{Z}$ if and only if $N(T_{\mathfrak{U}}) \in Sp(\mathcal{Z})$ for all ultrafilters $\mathfrak{U}$. An earlier antecedent of this kind is due to Tacon [158], who proved the same result but only for the semigroup of supertauberian operators.

The facts that $T^{**} \prec_{\rm ls} T$ and ${T_{\mathfrak{U}}}^* \prec_{\rm ls} {T^*}_{\mathfrak{U}}$ for all operators T and all ultrafilters $\mathfrak{U}$ seem to be sufficient to study the duality properties of the semigroups $\mathcal{S}$ and $\mathcal{S}^d$ where $\mathcal{S}$ is any ultrapower-stable upper semigroup. However, after the counterexamples given by Basallote and Díaz in [17, Section 2], there is not much hope for the existence of an appropriate notion of operator finite representability for the study of non-dual lower ultrapower-stable semigroups.

Supertauberian operators and cosupertauberian operators were introduced by Tacon in order to find some classes of tauberian operators with tauberian biconjugates [157] [158]. His results are given in the language of non-standard analysis. A treatment of these semigroups in terms of ultrapowers was carried out in [127] and in [80], where it is proved that, given any operator T and any ultrafilter $\mathfrak{U}$, the kernel of ${T_{\mathfrak{U}}}^*$ is finitely representable in that of ${T^*}_{\mathfrak{U}}$. All the results about supertauberian operators displayed in Section 6.5 have been taken from [74]. The perturbative characterization for cosupertauberian operators, including its sequels, given in Theorem 6.6.11, also belong to [74].

The results of Proposition 6.5.21 about the natural inclusions of vector valued spaces $L_p(X)$ into $L_1(X)$ have been borrowed from [127]. The analysis of Proposition 6.5.22 about the inclusion of the James space into c_0 appears in [127] and in [81].

As it was shown in Chapter 3, the tauberian decomposition $T = jUk$ of an operator T supplies a tauberian operator j and a cotauberian operator k. In general, neither is j supertauberian nor is k cosupertauberian. It was proved in [127] that this is the case for the super-weakly compact operator T_0 that does not factorize through any super-reflexive space [19, 20]. The argument is as follows: by Proposition 5.3.3, the factor j for T_0 is super-weakly compact. If j were supertauberian, the initial space of j would be super-reflexive, which is not possible. The argument for the factor k is similar.

In general, given an ultrafilter $\mathfrak{V}$, $T_{\mathfrak{V}} = j_{\mathfrak{V}} U_{\mathfrak{V}} k_{\mathfrak{V}}$ is not the tauberian decomposition of $T_{\mathfrak{V}}$. Indeed, if T has non-closed range, by the properties of tauberian decompositions given in Theorem 3.2.1 and Proposition A.4.22, $j_{\mathfrak{V}}$ is not injective.

The semigroups $(\mathcal{U}^{up})_+$ and $(\mathcal{R}^{up})_+$ admit a parallel study to that carried out for the semigroups of supertauberian and cosupertauberian operators throughout

Sections 6.5 and 6.6. Indeed, it was proved in [81] that given an operator T in $Lc(X,Y)$, the following statements are equivalent:

(1) $T \in (\mathcal{U}^{up})_+$ (alt., $T \in (\mathcal{R}^{up})_+$);

(2) given any ultrafilter $\mathfrak{U}$, the kernel $N(T_\mathfrak{U})$ does not contain any copy of c_0 (alt., any copy of ℓ_1);

(3) given any ultrafilter $\mathfrak{U}$, c_0 (alt., ℓ_1) is not finitely representable in $N(T_\mathfrak{U})$;

(4) for every compact operator $K \in \mathcal{L}(X,Y)$, c_0 (alt., ℓ_1) is not finitely representable in $N(T+K)$.

As a consequence of the perturbative characterization given in statement (4), the aforementioned paper shows that the identities $(\mathcal{U}^{up})_+ = (\mathcal{U}_+)^{up}$ and $(\mathcal{R}^{up})_+ = (\mathcal{R}_+)^{up}$ hold, which yields that both semigroups are ultrapower-stable and $\prec_{\rm ls}$-stable upper semigroups.

The study of the dual semigroups $(\mathcal{U}^{up}{}_+)^d$ and $(\mathcal{R}^{up}{}_+)^d$ also parallels that of cosupertauberian operators, although it is necessary to bear in mind that although $\mathcal{W}^{up}$ is a self-dual ideal, $\mathcal{U}^{up}$ and $\mathcal{R}^{up}$ are not. Thus, it is proved in [81] that the following statements are equivalent for an operator $T \in \mathcal{L}(X,Y)$:

(1′) $T \in (\mathcal{U}^{up}{}_+)^d$ (alt., $T \in (\mathcal{R}^{up}{}_+)^d$);

(2′) given any ultrafilter $\mathfrak{U}$, $N(T^*{}_\mathfrak{U})$ does not contain any copy of c_0 (alt., any copy of ℓ_1);

(3′) given any ultrafilter $\mathfrak{U}$, c_0 (alt., ℓ_1) is not finitely representable in $N(T^*{}_\mathfrak{U})$;

(4′) for every compact operator $K \in \mathcal{L}(X,Y)$ and for every $n \in \mathbb{N}$, the quotient $Y/\overline{R(T+K)}$ does not contain uniformly complemented copies of the spaces ℓ_1^n (alt., of ℓ_∞^n).

Let $\mathcal{E}$ denote one of the ideals $\mathcal{U}$ or $\mathcal{R}$. From the perturbative characterization (4′) and the $\prec_{\rm ls}$-stability of the semigroups $\mathcal{E}^{up}{}_+$ it can be proved that $(\mathcal{E}^{up}{}_+)^d = (\mathcal{E}^d)^{up}{}_-$. For more information, consult [1] and [81].

Appendix A

Basic concepts

Here we include some definitions and results that are fundamental and appear throughout this book.

A.1 Semi-Fredholm operators

Fredholm theory was originated in the study of the existence of solutions for integral equations from an abstract point of view. It has found applications in Banach space theory, and some aspects of the study of tauberian operators have been inspired by this theory.

In this section we include the main definitions and some basic results of Fredholm theory.

Definition A.1.1. An operator $T \in \mathcal{L}(X, Y)$ is said to be *compact* if it takes bounded sets to relative compact subsets.

Definition A.1.2. Let $T \in \mathcal{L}(X, Y)$.

(i) T is said to be *upper semi-Fredholm* if its range is closed and its kernel is finite dimensional.

(ii) T is said to be *lower semi-Fredholm* if its range is finite codimensional (hence closed).

(iii) T is said to be *semi-Fredholm* if it is upper semi-Fredholm or lower semi-Fredholm.

We denote by $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$ the subsets of upper and lower semi-Fredholm operators in $\mathcal{L}(X, Y)$, respectively.

Definition A.1.3. The *index of a semi-Fredholm operator* $T \in \Phi_+(X, Y) \cup \Phi_-(X, Y)$ is defined by

$$\text{ind}(T) := \dim N(T) - \dim Y/R(T).$$

Note that $\operatorname{ind}(T) \in \mathbb{Z} \cup \{\pm\infty\}$.

An operator T is said to be a *Fredholm operator* if $\operatorname{ind}(T) \in \mathbb{Z}$. Therefore, the class Φ of Fredholm operators is given by

$$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

It is elementary that an operator $K\colon X \longrightarrow Y$ is compact if and only if for every bounded sequence (x_n) in X, (Tx_n) has a convergent subsequence. Upper semi-Fredholm operators admit a similar sequential characterization.

Proposition A.1.4. *Let $T \in \mathcal{L}(X,Y)$. Then $T \in \Phi_+$ if and only if a bounded sequence (x_n) in X has a convergent subsequence whenever (Tx_n) is convergent.*

Proof. Suppose that $T \in \Phi_+$. Let (x_n) be a bounded sequence in X such that (Tx_n) is convergent.

Since $N(T)$ is finite dimensional, there exists a closed subspace M of X so that $X = N(T) \oplus M$. Observe that the restriction $T|_M$ is an isomorphism. Therefore, if we write $x_n = y_n + z_n$ with $y_n \in N(T)$ and $z_n \in M$, then (z_n) is convergent and (y_n) has a convergent subsequence. Hence (x_n) has a convergent subsequence.

For the converse, suppose that $T \notin \Phi_+$. In the case when $N(T)$ is infinite dimensional, we can find a bounded sequence (x_n) in $N(T)$ without convergent subsequences and such that (Tx_n) converges. Otherwise, if $N(T)$ is finite dimensional, we can find a closed subspace M such that $X = N(T) \oplus M$, the restriction $T|_M$ is injective but it is not an isomorphism. Let us take a normalized sequence (z_n) in M such that $\lim_n \|Tz_n\| = 0$. Note that (z_n) cannot have convergent subsequences, because if z were a limit of a subsequence of z_n, we would have $z \in M \cap N(T)$ and $\|z\| = 1$, which is not possible. □

Next we describe some algebraic properties and the behavior under duality of the semi-Fredholm operators.

Proposition A.1.5. *Let $S \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(X,Y)$.*

(i) *if S and T belong to Φ_+, then $ST \in \Phi_+$ and* $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$*;*

(ii) *if $ST \in \Phi_+$, then $T \in \Phi_+$;*

(iii) *if S and T belong to Φ_-, then $ST \in \Phi_-$ and* $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$*;*

(iv) *if $ST \in \Phi_-$, then $S \in \Phi_-$;*

(v) *T is semi-Fredholm if and only if so is T^*. In this case,* $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

The classes Φ_+ and Φ_- are stable under compact perturbations, as follows from the next result:

Proposition A.1.6. *Consider an upper semi-Fredholm operator T and a compact operator K, both in $\mathcal{L}(X,Y)$. Then $T + K$ is a semi-Fredholm operator and* $\operatorname{ind}(T+K) = \operatorname{ind}(T)$.

Besides, the semi-Fredholm operators are stable under small norm perturbations:

Proposition A.1.7. *The set $\Phi_+(X,Y) \cup \Phi_-(X,Y)$ of semi-Fredholm operators is open in $\mathcal{L}(X,Y)$. Moreover, the index is constant on each connected component of $\Phi_+(X,Y) \cup \Phi_-(X,Y)$.*

Next we introduce an important subclass of the compact operators.

Definition A.1.8. An operator $K\colon X \longrightarrow Y$ is said to be *nuclear* if there exist sequences (f_n) in X^* and (y_n) in Y so that $\sum_{n=1}^{\infty} \|f_n\| \cdot \|y_n\| < \infty$ and

$$K(x) = \sum_{n=1}^{\infty} \langle f_n, x\rangle y_n, \quad \text{for every } x \in X.$$

Obviously, $\|K\| \leq \sum_{n=1}^{\infty} \|f_n\| \cdot \|y_n\|$.

Next we will give perturbative characterizations for the classes of semi-Fredholm operators. These are probably the most useful characterizations of the semi-Fredholm operators, which in turn inspired the corresponding characterizations for the tauberian operators and other classes of operators. The case $X = Y$ was considered in [118]. The proofs in the general case are not very different, but we include them for the sake of completeness.

Theorem A.1.9. *Let $T \in \mathcal{L}(X,Y)$.*

(i) *The operator T is upper semi-Fredholm if and only if $N(T+K)$ is finite dimensional for each compact operator $K \in \mathcal{L}(X,Y)$;*

(ii) *the operator T is lower semi-Fredholm if and only if $Y/\overline{R(T+K)}$ is finite dimensional for each compact operator $K \in \mathcal{L}(X,Y)$.*

Proof. (i) For the direct implication, let us assume that T is upper semi-Fredholm, and $K\colon X \longrightarrow Y$ is a compact operator. Then $X = X_1 \oplus X_2$, where X_1 is finite dimensional, $T|_{X_2}$ is an isomorphism and $\|K|_{X_2}\| < \|T\|$. Thus the restriction of $T+K$ to X_2 is an isomorphism, and since X_2 is finite co-dimensional in X, it follows that $T+K$ is upper semi-Fredholm.

For the converse implication, let T be an operator that is not upper semi-Fredholm, and let us find a compact operator $K\colon X \longrightarrow Y$ so that $N(T+K)$ is infinite dimensional. The case when $N(T)$ is infinite dimensional is trivial. If $N(T)$ is finite dimensional, then $X = N(T) \oplus X_1$, where $T|_{X_1}$ is injective and has non-closed range. We shall find a normalized sequence (x_n) in X_1 and a sequence (x_n^*) in X^* such that $\langle x_i^*, x_j\rangle = \delta_{ij}$ for all i and j, and such that the operator $K\colon X \longrightarrow Y$ given by

$$K(x) := -\sum_{n=1}^{\infty} \langle x_n^*, x\rangle T(x_n)$$

is well defined and compact. Therefore, since $T(x_n) = -K(x_n)$ for all n, $N(T+K)$ is infinite dimensional.

The obtention of (x_n) and (x_n^*) will be done recursively. First, since $T(X_1)$ is non-closed, there exists $x_1 \in S_{X_1}$ such that $\|T(x_1)\| \leq 2^{-1}$. Take $x_1^* \in S_{X^*}$ so that $\langle x_1^*, x_1 \rangle = 1$. Let us assume that the elements $x_1, \ldots, x_n$ in X_1 and $x_1^*, \ldots, x_n^*$ in X^* have already been chosen and satisfy

$$\langle x_i^*, x_j \rangle = \delta_{ij}, \quad \|x_i\| = 1, \quad \|x_i^*\| \leq \|P_{i-1}\|,$$
$$\text{and} \quad \|T(x_i)\| \leq \frac{1}{2^i \|P_{i-1}\|},$$

for all i and j, where P_0 is the identity operator on X, and for $i \geq 1$, $P_i \colon X \longrightarrow X$ is the projection that maps every x to $x - \sum_{j=1}^{i} \langle x_j^*, x \rangle x_j$.

In order to proceed, as $R(P_n)$ is finite co-dimensional, the range of $T|_{R(P_n)}$ is not closed, so there exists a norm-one element $x_{n+1} \in R(P_n)$ such that

$$\|T(x_{n+1})\| \leq 2^{-n-1} \|P_n\|^{-1}.$$

Choose a norm-one functional $f \in X^*$ so that $\langle f, x_{n+1} \rangle = 1$ and let $x_{n+1}^* := f \circ P_n$. Thus, since $R(P_n) = \bigcap_{i=1}^{n} N(x_i^*)$, we get $\langle x_i^*, x_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n+1$. Once the sequences (x_n) and (x_n^*) have been so chosen, we have

$$\|K\| \leq \sum_{n=1}^{\infty} \|x_n^*\| \|T(x_n)\| \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

Thus K is well defined and approximable by the finite rank operators

$$K_n(x) := -\sum_{i=1}^{n} \langle x_i^*, x \rangle T(x_i).$$

Hence K is compact.

(ii) For the direct implication, let us assume that T is lower semi-Fredholm and $K \colon X \longrightarrow Y$ is a compact operator. Then T^* is upper semi-Fredholm and K^* is compact. By part (i), $(T+K)^* = T^* + K^*$ is upper semi-Fredholm; hence $T + K$ is lower semi-Fredholm.

For the converse implication, suppose that T is not lower semi-Fredholm. In the case when $R(T)$ is closed we have $\dim Y/R(T) = \infty$; hence the proof is done taking $K = 0$.

Assume the case when $R(T)$ is not closed. Let (a_n) be a sequence of real numbers defined inductively by $a_1 := 2$ and

$$a_{n+1} := 2\left(1 + \sum_{k=1}^{n} a_k\right); \quad n \in \mathbb{N}.$$

We will find sequences (y_k) in Y and (y_k^*) in Y^* such that

$$\langle y_i^*, y_j \rangle = \delta_{ij}, \quad \|y_i\| \leq a_i, \quad \|y_i^*\| = 1,$$
$$\text{and} \quad \|T^*(y_i^*)\| \leq \frac{1}{2^i a_i},$$

for each $i, j \in \mathbb{N}$. These sequences will be obtained recursively.

Since $R(T)$ is not closed, neither is $R(T^*)$. Hence there exists $y_1^* \in Y^*$ such that $\|y_1^*\| = 1$ and $\|T^*(y_1^*)\| < 1/4$. We can also select $y_1 \in Y$ with $\|y_1\| < 2$ such that $\langle y_1^*, y_1 \rangle = 1$.

Suppose that $n > 1$ and that we have found y_k and y_k^* for all $k < n$. Since the restriction of T^* to $\{y_1, \ldots, y_{n-1}\}^\perp$ has non-closed range, we can find $y_n^* \in Y^*$ such that

$$\langle y_n^*, y_k \rangle = 0 \quad \text{for } k < n, \quad \|y_n^*\| = 1,$$
$$\text{and} \quad \|T^*(y_n^*)\| \leq \frac{1}{2^n a_n}.$$

We can also select $y \in Y$ with $\|y\| < 2$ such that $\langle y_n^*, y \rangle = 1$. Let us define

$$y_n := y - \sum_{k=1}^{n-1} \langle y_k^*, y \rangle y_k.$$

Then $\|y_n\| \leq \|y\| \left(1 + \sum_{k=1}^{n-1} \|y_k\|\right) \leq 2\left(1 + \sum_{k=1}^{n-1} a_k\right) = a_n$, and clearly the required sequences are obtained in this way.

The remainder of the proof is similar to that of part (i). Since

$$\sum_{n=1}^{\infty} \|T^*(y_n^*)\| \cdot \|y_n\| < \infty$$

the expression

$$K(x) := -\sum_{n=1}^{\infty} \langle T^*(y_n^*), x \rangle y_n$$

defines a compact operator $K \colon X \longrightarrow Y$, whose conjugate operator is given by

$$K^*(y^*) := -\sum_{n=1}^{\infty} \langle y^*, y_n \rangle T^*(y_n^*).$$

Now, since $T^*(y_n^*) = -K^*(y_n^*)$ for all n, we conclude that $N(T^* + K^*)$ is infinite dimensional; hence its predual space $Y/\overline{R(T+K)}$ is also infinite dimensional. □

A.2 Operator ideals

The theory of operator ideals began with the fundamental work of Grothendieck and is now a branch of functional analysis that has produced many results and problems of its own interest. One of its aims is the classification of Banach spaces in terms of the properties of the operators that act between them.

Next we define the operator ideals and space ideals.

Definition A.2.1. A class $\mathcal{A}$ of operators is said to be an *operator ideal* if it satisfies the following conditions:

(i) all finite rank operators belong to $\mathcal{A}$;

(ii) for every pair of Banach spaces X and Y, $\mathcal{A}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$;

(iii) given Banach spaces W, X, Y and Z, if $T \in \mathcal{L}(W,X)$, $S \in \mathcal{A}(X,Y)$ and $R \in \mathcal{L}(Y,Z)$, then $RST \in \mathcal{A}(W,Z)$.

The class $\mathcal{L}$ of all operators and the class $\mathcal{F}$ of all finite rank operators are the biggest and the smallest operator ideals.

It is easy to check that, if $\mathcal{A}$ is an operator ideal, then

$$\mathcal{A}^d = \{T : T^* \in \mathcal{A}\}$$

is also an operator ideal. It is called the *dual operator ideal* of $\mathcal{A}$.

Definition A.2.2. A class $\mathbb{A}$ of Banach spaces is said to be a *space ideal* if it satisfies the following conditions:

(i) all finite dimensional spaces belong to $\mathbb{A}$;

(ii) X and Y belong to $\mathbb{A}$ if and only if $X \times Y$ belongs to $\mathbb{A}$;

(iii) if X belongs to $\mathbb{A}$ and Y is isomorphic to X, then Y belongs to $\mathbb{A}$.

The class $\mathbb{B}$ of all Banach spaces and the class $\mathbb{F}$ of all finite dimensional Banach spaces are the biggest and the smallest space ideals.

A crucial link between operator ideals and space ideals is given in the following result.

Proposition A.2.3. *Given an operator ideal $\mathcal{A}$, the class of Banach spaces given by*

$$Sp(\mathcal{A}) := \{X : I_X \in \mathcal{A}\}$$

is a space ideal.

Let us introduce some relevant classes of operator ideals.

Definition A.2.4. Let $\mathcal{A}$ be an operator ideal.

(i) $\mathcal{A}$ is said to be *closed* if $\mathcal{A}(X,Y)$ is closed in $\mathcal{L}(X,Y)$ for each pair of Banach spaces X, Y;

(ii) $\mathcal{A}$ is said to be *regular* if given an operator $T \in \mathcal{L}(X,Y)$, $J_Y T \in \mathcal{A}(X,Y^{**})$ implies $T \in \mathcal{A}$.

Let us recall that, given a closed subspace N of a Banach Y, J_N denotes the inclusion operator, and Q_N denotes the quotient operator.

Definition A.2.5. Let $\mathcal{A}$ be an operator ideal.

(i) $\mathcal{A}$ is said to be *injective* if given Banach spaces X and Z and a closed subspace Y of Z, for each $T \in \mathcal{L}(X, Y)$, $J_Y T \in \mathcal{A}$ implies $T \in \mathcal{A}$.

(ii) $\mathcal{A}$ is said to be *surjective* if given Banach spaces X and Z and a closed subspace Y of Z, for each $T \in \mathcal{L}(Z/Y, X)$, $TQ_Y \in \mathcal{A}$ implies $T \in \mathcal{A}$.

It is well known that injective operator ideals can be characterized in terms of seminorms and the surjective operator ideals in terms of bounded subsets (see, for example, [106] and [108]). In the special case in which the operator ideals are closed, we have the following operative characterizations.

Lemma A.2.6. *Let $\mathcal{A}$ be a closed operator ideal.*

(a) *$\mathcal{A}$ is injective if and only if for an operator $T \in \mathcal{L}(X, Y)$ to belong to $\mathcal{A}$ it is both necessary and sufficient that for every $\varepsilon > 0$ there exist a Banach space Z_ε and an operator $S_\varepsilon \in \mathcal{A}(X, Z_\varepsilon)$ so that*

$$\|Tx\| \leq \|S_\varepsilon x\| + \varepsilon\|x\|; \quad \textit{for every } x \in X.$$

(b) *$\mathcal{A}$ is surjective if and only if for an operator $T \in \mathcal{L}(X, Y)$ to belong to $\mathcal{A}$ it is both necessary and sufficient that for every $\varepsilon > 0$ there exist a Banach space Z_ε and an operator $S_\varepsilon \in \mathcal{A}(Z_\varepsilon, Y)$ so that*

$$T(B_X) \subset S_\varepsilon(B_{Z_\varepsilon}) + \varepsilon B_Y.$$

Proof. For part (a), see [106, Theorem 20.7.3]; and for part (b), see [108, Proposition 2.9]. □

A.3 Bases and basic sequences

Here we recall the notion of Schauder basis of a Banach space and the corresponding notion of basic sequence. Moreover, we will state some basic principles that guarantee the existence of a basic subsequence for sequences satisfying certain conditions.

Definition A.3.1. A sequence of elements (x_n) in an infinite dimensional Banach space X is said to be a *basis* of X if for each $x \in X$ there is a unique sequence of scalars (a_n) such that $x = \sum_{n=1}^{\infty} a_n x_n$.

The unit vector basis (e_n) is a basis in the spaces c_0 and ℓ_p for $1 \leq p < \infty$.

Suppose that (x_n) is a basis of X. Then it follows from the basic principles of functional analysis that, for each $k \in \mathbb{N}$, the expression

$$\Big\langle x_k^*, \sum_{n=1}^{\infty} a_n x_n \Big\rangle := a_k$$

defines an element $x_k^* \in X^*$. Moreover

$$P_k\Big(\sum_{n=1}^{\infty} a_n x_n\Big) := \sum_{n=1}^{k} a_n x_n$$

defines a projection $P_k\colon X \longrightarrow X$ such that $\lim_k P_k(x) = x$ for each $x \in X$; hence, (P_k) is a bounded sequence of projections.

Definition A.3.2. Let (x_n) be a basis of X. Then (x_n^*) is called the *sequence of coefficient functionals* of the basis (x_n).

Moreover, $C := \sup_{k\in\mathbb{N}} \|P_k\|$ is called the *basis constant* of (x_n).

Observe that, given a basis (x_n) of X, the coefficient functionals (x_n^*) are determined by the equalities $\langle x_i^*, x_j\rangle = \delta_{ij}$ for all positive integers i and j.

Definition A.3.3. A sequence (x_n) in a Banach space X is said to be a *basic sequence* if it is a basis of the subspace $\overline{\operatorname{span}}\{x_n\}$.

Remark A.3.4. It is easy to see that if (x_n) is a basis in X, then the corresponding sequence of coefficient functionals (x_n^*) is a basic sequence in X^*. Indeed, for every $x^* \in \overline{\operatorname{span}}\{x_n^*\}$, $x^* = \sum_{n=1}^{\infty}\langle x^*, x_n\rangle x_n^*$.

Definition A.3.5. Let (x_n) be a basic sequence in a Banach space X. A sequence (y_n) is said to be a *block basis* of (x_n) if there exist an increasing sequence (n_i) in $\mathbb{N}$ and a sequence (α_n) of scalars such that $y_j := \sum_{i=n_j+1}^{n_{j+1}} \alpha_i x_i$ and $y_j \neq 0$ for all j.

It is easy to check that a block basis of a basic sequence is also a basic sequence.

The following result allows us to recognize a sequence in a Banach space as a basic sequence.

Proposition A.3.6. [4, Proposition 1.1.9] *A sequence (x_n) of nonzero elements of a Banach space X is basic if and only if there is a positive constant C such that*

$$\Big\|\sum_{k=1}^{m} a_k x_k\Big\| \le C\Big\|\sum_{k=1}^{n} a_k x_k\Big\|$$

for any sequence of scalars (a_k) and any integers m, n such that $m \le n$.

Using this criterion, it is easy to show that every infinite dimensional Banach space contains a basic sequence. See [122, Theorem 1.a.5] for a proof.

The following principle is very useful. For a proof we refer to [4, Proposition 1.5.4].

Proposition A.3.7 (Bessaga-Pełczyński selection principle). *If (x_n) is a weakly null sequence in a Banach space X such that $\inf_n \|x_n\| > 0$, then it contains a basic subsequence.*

Kadec and Pełczyński obtained the following criterion to know when a subset of a Banach space contains a basic sequence [112]. For a proof we refer to [4, Theorem 1.5.6].

Theorem A.3.8. *Let S be a bounded subset of a Banach space such that $0 \notin \overline{S}$. Then the following assertions are equivalent:*

(a) *S does not contain a basic sequence;*

(b) *$\overline{S}^w$ is weakly compact and does not contain 0.*

A direct consequence of the above theorem is that the only weakly convergent subsequences that a basic sequence may contain must be weakly null.

The notion of equivalent sequences is indispensable in the study of isomorphic theories like that of tauberian operators.

Definition A.3.9. Two sequences (x_n) and (y_n) in a Banach space X are said to be *equivalent* if there exists a bijective isomorphism $T\colon \overline{\text{span}}\{x_n\} \longrightarrow \overline{\text{span}}\{y_n\}$ such that $Tx_n = y_n$ for each n.

Note that if the sequences (x_n) and (y_n) in X are equivalent and one of them is basic, so is the other one.

The following result allows us to characterize the existence of subspaces isomorphic to ℓ_1 in a Banach space. It was obtained in [145]. For a proof, we refer to [4, Theorem 10.2.1].

Theorem A.3.10 (Rosenthal's ℓ_1-theorem). *Let (x_n) be a bounded sequence in a Banach space X. Then either*

(i) *(x_n) has a weakly Cauchy subsequence, or*

(ii) *(x_n) has a subsequence which is equivalent to the unit vector basis of ℓ_1.*

Let us give a characterization for basic sequences equivalent to the unit vector basis of c_0.

Definition A.3.11. A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is said to be *weakly unconditionally Cauchy* if $\sum_{n=1}^{\infty} |f(x_n)| < \infty$, for every $f \in X^*$; equivalently, if

$$\sup \Big\{ \Big\| \sum_{i \in A} x_i \Big\| \colon A \subset \mathbb{N} \text{ finite} \Big\} < \infty.$$

The following result is due to Pełczyński.

Proposition A.3.12. [51, Corollary V.7] *A basic sequence (x_n) in a Banach space is equivalent to the unit vector basis of c_0 if and only if $\inf_{n\in\mathbb{N}} \|x_n\| > 0$ and $\sum_{n=1}^{\infty} x_n$ is a weakly unconditionally Cauchy series.*

Basic sequences and duality

Some special classes of bases and basic sequences have been useful in the study of Banach spaces, in particular in the research on reflexive spaces.

Definition A.3.13. Let (x_n) be a basic sequence in a Banach space X.

(i) (x_n) is said to be *boundedly complete* if for each sequence of scalars (α_j), $\sum_{j=1}^{\infty} \alpha_j x_j$ is convergent whenever $\sup_n \| \sum_{j=1}^{n} \alpha_j x_j \| < \infty$;

(ii) (x_n) is said to be *shrinking* if $\|f|_{X_n}\| \underset{n}{\longrightarrow} 0$ for all $f \in X^*$, where $X_n := \overline{\text{span}}\{x_i\}_{i=n}^{\infty}$.

Proposition A.3.14. [122, Proposition 1.b.2] *A basis (x_n) in X is shrinking if and only if the corresponding sequence of coefficient functionals (x_n^*) is a basis of X^*.*

Proposition A.3.15. [122, Proposition 1.b.4] *A Banach space X with a boundedly complete basis (x_n) is isomorphic to a dual space. In fact, X is isomorphic to the dual of $\overline{\text{span}}\{x_n^*\}$.*

By combining the two previous propositions we obtain the following characterization of reflexivity in terms of bases:

Proposition A.3.16. [122, Proposition 1.b.5] *A Banach space X with a basis (x_n) is reflexive if and only if (x_n) is both shrinking and boundedly complete.*

Remark A.3.17. It was proved by Zippin [177] that a Banach space X with a basis is reflexive if every basis in X is shrinking or, alternatively, if every basis in X is boundedly complete.

Proposition A.3.18. [122, Proposition 1.b.6] *Let X be a Banach space such that X^* has a basis. Then X has a shrinking basis and therefore X^* has a boundedly complete basis.*

A.4 Ultraproducts in Banach space theory

The ultraproduct constructions are fundamental in model theory, and have been applied in functional analysis through the concepts of ultrapower of Banach spaces and ultrapowers of operators.

In this section, I is an infinite set of indices. We introduce the ultrafilters on I and define the limit on a topological space following an ultrafilter. This concept allows us to define the ultrapowers of Banach spaces and operators.

Filters and ultrafilters

A *filter* on a non-empty set I is a collection $\mathfrak{F}$ of subsets of I satisfying the following conditions:

(i) $\emptyset \notin \mathfrak{F}$,

(ii) *if* $A \in \mathfrak{F}$ *and* $B \in \mathfrak{F}$*, then* $A \cap B \in \mathfrak{F}$*,*

(iii) *if* $A \in \mathfrak{F}$ *and* $A \subset B$*, then* $B \in \mathfrak{F}$*.*

The elements of I are usually referred to as *indices*. Typical examples of filters on I are the following:

(1) the Fréchet filter, which consists of all co-finite subsets of I;

(2) the collection of all subsets of I containing a fixed non-empty subset A of I.

If the set I is endowed with an order $\preceq$, the collection $\{I_j : j \in I\}$, where $I_j := \{i \in I : j \preceq i\}$, is a filter on I called the $\preceq$*-order filter on* I.

An *ultrafilter* $\mathfrak{U}$ on I is a maximal filter on I. Equivalently, a filter $\mathfrak{U}$ on I is an ultrafilter if it satisfies the following additional condition:

(iv) *for each subset* A *of* I*, either* $A \in \mathfrak{U}$ *or* $I \setminus A \in \mathfrak{U}$*.*

The existence of an ultrafilter enlarging a given filter $\mathfrak{F}$ on I is ensured by Zorn's Lemma.

Any ultrafilter enlarging the Fréchet filter on I is called *non-trivial*. Any other ultrafilter $\mathfrak{U}$ on I is of the form $\{A \subset I : j \in I\}$ for some fixed element j. Such an ultrafilter is called *principal* or *trivial*.

A non-trivial ultrafilter $\mathfrak{U}$ on I is said to be *countably incomplete* or $\aleph_0$-*incomplete* if there exists a countable partition $\{I_n\}_{n=1}^{\infty}$ of I such that for every $n \in \mathbb{N}$, $I_n \notin \mathfrak{U}$.

Consider a topological space X, a filter $\mathfrak{F}$ on a set I and an element $x \in X$. A family $(x_i)_{i\in I} \subset X$ is said to be *convergent to* x *following* $\mathfrak{F}$ if for every neighborhood $\mathcal{V}$ of x,

$$\{i \in I : x_i \in \mathcal{V}\} \in \mathfrak{F};$$

that element x is called the *limit of* (x_i) *following* $\mathfrak{F}$, and it is denoted by $x = \lim_{\mathfrak{F}} x_i$, $x = \lim_{i\to\mathfrak{F}} x_i$, $x_i \underset{\mathfrak{F}}{\longrightarrow} x$, or $x_i \underset{i\to\mathfrak{F}}{\longrightarrow} x$.

Note that if I is the set $\mathbb{N}$ and $\mathfrak{F}$ is the Fréchet filter on $\mathbb{N}$, then the convergence following $\mathfrak{F}$ is the usual sequence convergence.

Lemma A.4.1. *Let* X *be a compact topological space, and let* $\mathfrak{U}$ *be an ultrafilter on* I*. Then every family* $(x_i)_{i\in I}$ *in* X *is convergent following* $\mathfrak{U}$*.*

Proof. Assume there is a family $(x_i)_{i\in I}$ in X which is not convergent following $\mathfrak{U}$. Then, for every $z \in X$, there exists a neighborhood $\mathcal{V}_z$ of z such that $I_z := \{i \in I : x_i \in \mathcal{V}_z\} \notin \mathfrak{U}$. But X is compact, so there exists a finite collection $\{z_k\}_{k=1}^{n}$ in X such that $X \subset \bigcup_{k=1}^{n} \mathcal{V}_{z_k}$. That yields $I = \bigcup_{k=1}^{n} I_{z_k} \in \mathfrak{U}$, which is a contradiction because none of the subsets I_{z_k} belongs to $\mathfrak{U}$. □

Let $\mathfrak{U}$ and $\mathfrak{V}$ be a pair of ultrafilters on I and J respectively. The product $\mathfrak{U} \times \mathfrak{V}$ is an ultrafilter on $I \times J$ formed by all subsets K of $I \times J$ for which

$$\big\{i \in I : \{j \in J : (i,j) \in K\} \in \mathfrak{V}\big\} \in \mathfrak{U}. \tag{A.1}$$

Thus, given a topological space X, a family $(r_{ij})_{(i,j)\in I\times J} \subset X$ is convergent following $\mathfrak{U}\times\mathfrak{V}$ if for every $i \in I$, there exists $\lim_{\mathfrak{V}} r_{ij} = x_i$ and there exists $\lim_{\mathfrak{U}} x_i$, in which case the following identity holds:

$$\lim_{\mathfrak{U}\times\mathfrak{V}} r_{ij} = \lim_{\mathfrak{U}} \lim_{\mathfrak{V}} r_{ij}. \tag{A.2}$$

Henceforth, all ultrafilters will be non-trivial and countably incomplete.

Set-theoretic ultraproducts

Let $\mathfrak{U}$ be an ultrafilter on a set I, and consider a collection of non-empty sets $(\Omega_i)_{i\in I}$. Let $\equiv$ be the equivalence relation on $\prod_{i\in I}\Omega_i$ defined by $(x_i)_{i\in I} \equiv (y_i)_{i\in I}$ if $\{i \colon x_i = y_i\} \in \mathfrak{U}$.

The *set-theoretic ultraproduct* of $\Omega_{i\in I}$ following $\mathfrak{U}$ is the set

$$(\Omega_i)^{\mathfrak{U}} := \frac{\prod_{i\in I}\Omega_i}{\equiv}.$$

The element of $(\Omega_i)^{\mathfrak{U}}$ whose representative is $(x_i)_{i\in I}$ is denoted by $(x_i)^{\mathfrak{U}}$. Given a family $(A_i)_{i\in I}$, where $A_i \subset \Omega_i$ for all i, its *set-theoretic ultraproduct following* $\mathfrak{U}$ is defined as

$$(A_i)^{\mathfrak{U}} := \{(x_i)^{\mathfrak{U}} \colon \exists J\in\mathfrak{U} \text{ such that } \forall i\in J,\ x_i\in A_i\}.$$

Notice that $(A_i)^{\mathfrak{U}}$ equals $(B_i)^{\mathfrak{U}}$ if and only if there exists $J \in \mathfrak{U}$ such that $A_i = B_i$ for all $i \in J$. Also notice that if there exists $J \in \mathfrak{U}$ such that $A_i = \emptyset$ for all $i \in J$, then $(A_i)^{\mathfrak{U}} = \emptyset$.

Proposition A.4.2. *Let $\mathfrak{U}$ be an ultrafilter on a set I, and $\{\Omega_i\}_{i\in I}$ a collection of non-empty sets. For every $i \in I$, let Σ_i be a set algebra of subsets of Ω_i, and consider the collection*

$$(\Sigma_i)^{\mathfrak{U}} := \{(A_i)^{\mathfrak{U}} \colon A_i \in \Sigma_i,\ i \in I\}.$$

Then $(\Sigma_i)^{\mathfrak{U}}$ is a set algebra.

Proof. We only need to prove that, for every pair of elements $(A_i)^{\mathfrak{U}}$ and $(B_i)^{\mathfrak{U}}$ of $(\Sigma_i)^{\mathfrak{U}}$, the sets $(A_i)^{\mathfrak{U}} \cup (B_i)^{\mathfrak{U}}$ and $(\Omega_i)^{\mathfrak{U}} \setminus (B_i)^{\mathfrak{U}}$ also belong to $(\Sigma_i)^{\mathfrak{U}}$.

First, notice that

$$(A_i)^{\mathfrak{U}} \cup (B_i)^{\mathfrak{U}} = (A_i \cup B_i)^{\mathfrak{U}} \in (\Sigma_i)^{\mathfrak{U}}. \tag{A.3}$$

Indeed, the inclusion of $(A_i)^{\mathfrak{U}} \cup (B_i)^{\mathfrak{U}}$ in $(A_i \cup B_i)^{\mathfrak{U}}$ is trivial. For the reverse inclusion, let $(x_i)^{\mathfrak{U}} \in (A_i \cup B_i)^{\mathfrak{U}}$. Thus there exists $J \in \mathfrak{U}$ such that $x_i \in A_i \cup B_i$ for all $i \in J$. Let

$$J_1 := \{i \in J \colon x_i \in A_i\},$$
$$J_2 := \{i \in J \colon x_i \in B_i\}.$$

Thus $J = J_1 \cup J_2$, so $J_1 \in \mathfrak{U}$ or $J_2 \in \mathfrak{U}$. If $J_1 \in \mathfrak{U}$, then $(x_i)^{\mathfrak{U}} \in (A_i)^{\mathfrak{U}}$, and if $J_2 \in \mathfrak{U}$, then $(x_i)^{\mathfrak{U}} \in (B_i)^{\mathfrak{U}}$, which proves (A.3). Analogously, it can be proved that

$$(\Omega_i)^{\mathfrak{U}} \setminus (B_i)^{\mathfrak{U}} = (\Omega_i \setminus B_i)^{\mathfrak{U}} \in (\Sigma_i)^{\mathfrak{U}}. \qquad \square$$

It is remarkable that, given a countable family $\{(A_i^n)^{\mathfrak{U}}\}_{n=1}^{\infty}$ in $(\Sigma_i)^{\mathfrak{U}}$, the inclusions $(\cup_{n=1}^{\infty} A_i^n)^{\mathfrak{U}} \subset \cup_{n=1}^{\infty} (A_i^n)^{\mathfrak{U}}$ and $(\cap_{n=1}^{\infty} A_i^n)^{\mathfrak{U}} \subset \cap_{n=1}^{\infty} (A_i^n)^{\mathfrak{U}}$ are strict in general, even if each Σ_i is a σ-algebra.

Let $\sigma(\Sigma)$ denote the minimal σ-algebra generated by the set algebra Σ. The following result is a consequence of the ultraproduct iteration theorem.

Proposition A.4.3. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be a pair of ultrafilters on I and J respectively, and for every pair $(i,j) \in I \times J$, consider a non-empty set Ω_{ij} and an algebra Σ_{ij} of subsets of Ω_{ij}. Thus the σ-algebra $\sigma\big((\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}\big)$ is isomorphic to $\sigma\Big(\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}\Big)$.*

Proof. From the definition of product of ultrafilters given in (A.1), it is immediate that the map $\mathcal{I}\colon (\Omega_{ij})^{\mathfrak{U}\times\mathfrak{V}} \longrightarrow \big((\Omega_{ij}^{\mathfrak{V}})\big)^{\mathfrak{U}}$ that sends every $(t_{ij})^{\mathfrak{U}\times\mathfrak{V}}$ to $\big((t_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$ is well defined and bijective. Thus, every set $(A_{ij})^{\mathfrak{U}\times\mathfrak{V}}$ is sent by $\mathcal{I}$ onto $\big((A_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$, which means that the algebra $(\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}$ is transformed by $\mathcal{I}$ into the algebra $\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$. Consequently, every σ-algebra of $(\Omega_{ij})^{\mathfrak{U}\times\mathfrak{V}}$ containing $(\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}$ is transformed by $\mathcal{I}$ into a σ-algebra of $\big((\Omega_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$ containing the algebra $\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$, concluding that the map from $\sigma\big((\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}\big)$ onto $\sigma\Big(\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}\Big)$ that sends every A to $\mathcal{I}(A)$ is a σ-algebra isomorphism. $\square$

Ultraproducts of Banach spaces and operators

Here we describe the construction of ultrapowers of Banach spaces and operators and some of their basic properties. In [99], we can find a more detailed description.

Let $\mathfrak{U}$ be a countably incomplete ultrafilter on a set I. Given a collection $(X_i)_{i\in I}$ of Banach spaces, let $\ell_\infty(I, X_i)$ be the Banach space that consists of all bounded families $(x_i)_{i\in I}$, endowed with the supremum norm $\|(x_i)_{i\in I}\|_\infty := \sup_{i\in I} \|x_i\|$. Let $N_{\mathfrak{U}}(X_i)$ be the closed subspace of all families $(x_i)_{i\in I}$ of $\ell_\infty(I, X)$ which converge to 0 following $\mathfrak{U}$.

The *ultraproduct of* $(X_i)_{i\in I}$ *following* $\mathfrak{U}$ is defined as the quotient

$$(X_i)_{\mathfrak{U}} := \frac{\ell_\infty(I, X_i)}{N_{\mathfrak{U}}(X_i)}.$$

The element of $(X_i)_{\mathfrak{U}}$ including the family $(x_i)_{i\in I}$ as a representative is denoted by $[x_i]$. Thus, $[x_i]$ equals $[y_i]$ if and only if $\lim_{\mathfrak{U}} \|x_i - y_i\| = 0$, which easily yields the usual identity to compute the norm of $[x_i]$:

$$\big\|[x_i]\big\| = \lim_{\mathfrak{U}} \|x_i\|;$$

the convergence of $(\|x_i\|)_{i\in I}$ following $\mathfrak{U}$ is provided by Lemma A.4.1.

Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be collections of Banach spaces, and, for every $i \in I$, let $T_i\colon X_i \longrightarrow Y_i$ be an operator. Suppose that the collection $(T_i)_{i\in I}$ is bounded; i.e., that $\sup_{i\in I} \|T_i\| < \infty$.

The *ultraproduct of* $(T_i)_{i\in I}$ *following* $\mathfrak{U}$ is the operator $(T_i)_{\mathfrak{U}}$ from $(X_i)_{\mathfrak{U}}$ into $(Y_i)_{\mathfrak{U}}$ defined by

$$(T_i)_{\mathfrak{U}}([x_i]) := [T_i x_i].$$

When $X_i = X$ for all i, then $(X)_{\mathfrak{U}}$ is called the *ultrapower of* X *following* $\mathfrak{U}$, and it is denoted by $X_{\mathfrak{U}}$. Usually, its elements are denoted by bold letters $\mathbf{x}$, $\mathbf{y}$, etc. Analogously, when all the operators T_i equal an operator T, their ultraproduct following $\mathfrak{U}$ is called the *ultrapower of* T *following* $\mathfrak{U}$ and is denoted by $T_{\mathfrak{U}}$.

The ultrapower $X_{\mathfrak{U}}$ contains a canonical copy of X via the isometry from X into $X_{\mathfrak{U}}$ that sends every x to the constant class $[x]$.

Given a Banach space X, an ultrafilter $\mathfrak{U}$ on I and a collection $\{C_i\}_{i\in I}$ of non-empty subsets of X, the ultraproduct of $\{C_i\}_{i\in I}$ following $\mathfrak{U}$ is defined as the subset

$$(C_i)_{\mathfrak{U}} = \{[x_i] \in X_{\mathfrak{U}}\colon x_i \in C_i\}.$$

Proposition A.4.4. *Let C be a subset of a Banach space X, and let $\mathfrak{U}$ be an ultrafilter on I. Thus the ultrapower $C_{\mathfrak{U}}$ is a closed subset of $X_{\mathfrak{U}}$, and $\overline{C} = C_{\mathfrak{U}} \cap X$.*

Proof. In order to prove that $C_{\mathfrak{U}}$ is closed, consider any element $\mathbf{y}$ in its closure, and take a sequence $(\mathbf{x}_n)$ in $C_{\mathfrak{U}}$ so that $\|\mathbf{x}_n - \mathbf{y}\| < 1/n$ for all n. Choose a representative $(x_i^n)_{i\in I}$ of every $\mathbf{x}_n$ such that each x_i^n belongs to C and such that the set $\{\|x_i^n\|\colon i \in I,\ n \in \mathbb{N}\}$ is bounded. Let $(y_i)_{i\in I}$ be a representative of $\mathbf{y}$.

Next, let $\{I_n\}_{n=1}^{\infty}$ be a sequence of subsets of I disjoint with $\mathfrak{U}$. For every positive integer n, let

$$J_n := (\cup_{k=n}^{\infty} I_k) \bigcap \{i \in I\colon \|x_i^n - y_i\| < 1/n\} \in \mathfrak{U},$$

and for every $n \in \mathbb{N}$ and every $i \in J_n \setminus J_{n+1}$, define $x_i := x_i^n$, so $(x_i)_{i\in I}$ is bounded and $[x_i]$ belongs to $C_{\mathfrak{U}}$. Moreover, since

$$\{i \in I\colon \|x_i - y_i\| < 1/n\} \supset J_n \in \mathfrak{U} \text{ for all } n \in \mathbb{N},$$

it follows that $\|[x_i] - [y_i]\| = 0$, hence $\mathbf{y} = [x_i] \in C_{\mathfrak{U}}$.

The inclusion $\overline{C} \subset C_{\mathfrak{U}} \cap X$ is immediate from the fact that $C_{\mathfrak{U}}$ is closed. For the reverse inclusion, note that if $x \in C_{\mathfrak{U}} \cap X$, then there exists a family $(c_i)_{i\in I}$ in C so that $[c_i] = [x]$. Thus $c_i \underset{\mathfrak{U}}{\longrightarrow} x$, hence $x \in \overline{C}$. □

Ultrafilters are useful in finding compact subsets.

Proposition A.4.5. *Let A be a subset of a Banach space X, and let $\mathfrak{U}$ be an ultrafilter on a set I. Then A is relatively compact if and only if $A_{\mathfrak{U}} \subset X$. As a consequence, X is finite dimensional if and only if $X = X_{\mathfrak{U}}$.*

Proof. Assume A is relatively compact. Then the inclusion $A_{\mathfrak{U}} \subset X$ follows easily from Lemma A.4.1.

For the reverse implication, let us assume that A is not relatively compact. Then there are $\delta > 0$ and a countable subset $\{x_n\}_{n=1}^{\infty}$ in A such that $\|x_m - x_n\| \geq \delta$ for all n and all $m \neq n$. Let $\{I_n\}_{n=1}^{\infty}$ be a partition of I disjoint with $\mathfrak{U}$, and for every n and every $i \in I_n$, define $z_i := x_n$. Trivially, $[z_i] \in A_{\mathfrak{U}}$. Moreover, $\operatorname{dist}([z_i], X) \geq \delta/2$. Indeed, given $x \in X$, either $\|x - x_n\| \geq \delta/2$ for all $n \in \mathbb{N}$ or there exists $m \in \mathbb{N}$ such that $\|x - x_m\| < \delta/2$. In the first case, we get $\|x - z_i\| \geq \delta/2$ for all $i \in I$, so $\|x - [z_i]\| \geq \delta/2$. In the second case, for every $n \in \mathbb{N} \setminus \{m\}$,

$$\|x - x_n\| \geq \|x_m - x_n\| - \|x - x_m\| \geq \delta - \delta/2 = \delta/2$$

hence, for every $i \in \bigcup_{n \neq m} I_n$, $\|x - z_i\| \geq \delta/2$, and since $\bigcup_{n \neq m} I_n \in \mathfrak{U}$, it follows that $\|x - [z_i]\| \geq \delta/2$, concluding the proof. □

Proposition A.4.6. *Given a Banach space X, a closed subspace E of X and an ultrafilter $\mathfrak{U}$ on I, the spaces $(X/E)_{\mathfrak{U}}$ and $X_{\mathfrak{U}}/E_{\mathfrak{U}}$ are canonically isometric via the operator that maps every $[x_i + E]$ to $[x_i] + E_{\mathfrak{U}}$.*

The proof is straightforward.

Proposition A.4.7. *Let $\{X_{ij}\}_{(i,j) \in I \times J}$ be a collection of Banach spaces and $\mathfrak{U}$ and $\mathfrak{V}$ be a pair of ultrafilters on I and J respectively. Thus, there exists a canonic isometrical isomorphism $U\colon (X_{ij})_{\mathfrak{U} \times \mathfrak{V}} \longrightarrow \big((X_{ij})_{\mathfrak{V}}\big)_{\mathfrak{U}}$ which maps each $[x_{ij}]_{(i,j) \in I \times J}$ to $\big[[x_{ij}]_j\big]_i$.*

Proof. It is a direct consequence of formulas (A.1) and (A.2). □

Lemma A.4.8. *Let X be a Banach space and take any α-net $\{x_i\}_{i \in I}$ in B_X with $0 < \alpha < 1$. Thus, for every $x \in S_X$, there are a sequence $(x_{i_n})_{n=1}^{\infty}$ in the net and a scalar sequence $(\lambda_n)_{n=1}^{\infty}$ such that, for every positive integer n,*

(i) $0 \leq \lambda_n \leq \alpha^{n-1}$, *and*

(ii) $\|x - \sum_{m=1}^{n} \lambda_m x_{i_m}\| < \alpha^n$.

Proof. The choice of the elements λ_n and x_{i_n} is carried out recursively. First, we take $\lambda_1 := 1$ and select x_{i_1} so that $\|x - x_{i_1}\| < \alpha$. Let us assume that $\{\lambda_1, \ldots, \lambda_{n-1}\}$ and $\{x_{i_1}, \ldots, x_{i_{n-1}}\}$ have been already chosen satisfying conditions (i) and (ii). Then we take

$$\lambda_n := \|x - (\lambda_1 x_{i_1} + \cdots + \lambda_{n-1} x_{i_{n-1}})\| < \alpha^{n-1}.$$

If $\lambda_n = 0$, this recursive procedure ends by setting $\lambda_p = 0$ for all $p \geq n$. If $\lambda_n \neq 0$, we select x_{i_n} so that $\|\lambda_n^{-1}(x - \lambda_1 x_{i_1} - \cdots - \lambda_{n-1} x_{i_{n-1}}) - x_{i_n}\| < \alpha$, so we get

$$\|x - \lambda_1 x_{i_1} - \cdots - \lambda_{n-1} x_{i_{n-1}} - \lambda_n x_{i_n}\| < \alpha \lambda_n < \alpha^n. \qquad \square$$

An important tool in local theory is the following one:

Definition A.4.9. Given $\varepsilon > 0$, an operator $L\colon X \longrightarrow Y$ is said to be an *ε-isometry* if

$$1-\varepsilon \le \|T(x)\| \le 1+\varepsilon \quad \text{for all } x \in S_X.$$

The difference between the concepts of ε-isometry and d-injection is just technical.

Lemma A.4.10. *Let E be a closed subspace of a Banach space X and $\{x_i\}_{i\in I}$ be an α-net in S_E with $0<\alpha<1$. Let $\delta>0$ and $L\colon E \longrightarrow X$ a bounded operator such that $1-\delta \le \|L(x_i)\| \le 1+\delta$ for all $i \in I$. Then L is an $(\alpha+\delta)(1-\alpha)^{-1}$-isometry.*

Proof. Let $x \in S_E$. By Lemma A.4.8, there are a scalar sequence $(\lambda_n)_{n=1}^\infty$ and a sequence $(x_{i_n})_{n=1}^\infty$ in the net $\{x_i\}_{i\in I}$ such that $0 \le \lambda_n < \alpha^{n-1}$ and $x = \sum_{n=1}^\infty \lambda_n x_{i_n}$. Thus

$$\|L(x)\| \le \sum_{n=1}^{\infty} \lambda_n \|L(x_{i_n})\| \le \frac{1+\delta}{1-\alpha} = 1 + \frac{\alpha+\delta}{1-\alpha}. \tag{A.4}$$

In order to bound $\|L(x)\|$ from below, choose x_j in the net so that $\|x-x_j\|<\alpha$. Thus, by (A.4),

$$\|L(x)\| \ge \|L(x_j)\| - \|L\|\cdot\|x-x_j\| \ge 1-\delta-\frac{1+\delta}{1-\alpha}\alpha = 1-\frac{\alpha+\delta}{1-\alpha}. \qquad \square$$

Most of the applications of ultraproduct techniques are obtained from the following pair of lemmas. Lemma A.4.11 is a sort of converse of Lemma A.4.12.

Lemma A.4.11. *Let $\mathfrak{U}$ be an ultrafilter on a set I, $\{\varepsilon_i\}_{i\in I}$ a family of positive real numbers such that $\varepsilon_i \underset{\mathfrak{U}}{\longrightarrow} 0$, and a family of ε_i-isometries $T_i\colon X_i \longrightarrow Y_i$. Then $(T_i)_\mathfrak{U}$ is an isometry from $(X_i)_\mathfrak{U}$ into $(Y_i)_\mathfrak{U}$.*

Proof. Let $[x_i]$ be a norm-one element in $(X_i)_\mathfrak{U}$. Given $\varepsilon>0$, choose $\varepsilon'>0$ so that $1-\varepsilon<(1-\varepsilon')^2$ and $(1+\varepsilon')^2<1+\varepsilon$, and select $J\in\mathfrak{U}$ so that $\varepsilon_i<\varepsilon'$ and $1-\varepsilon'<\|x_i\|<1+\varepsilon'$ for all $i\in J$. Thus,

$$1-\varepsilon<(1-\varepsilon')^2<\|T_i(x_i)\|<(1+\varepsilon')^2<1+\varepsilon \quad \text{for all } i\in J$$

which proves that $\lim_\mathfrak{U}\|T_i(x_i)\|=1$, that is, $T_\mathfrak{U}([x_i])$ is norm-one. $\square$

Lemma A.4.12. *Let $\varepsilon>0$, m a positive integer, $\mathfrak{U}$ an ultrafilter on a set I, and $\{E_i\}_{i\in I}$ be a collection of Banach spaces with $\dim E_i = m$ for all i. Thus, the following statements hold:*

(i) $\dim (E_i)_\mathfrak{U} = m$.

(ii) *Let $\{\mathbf{x}_k\}_{k=1}^m$ be a normalized basis of $(E_i)_\mathfrak{U}$ with $\mathbf{x}_k=[x_i^k]_i$ for every k. For each $j\in I$, let $L_j\colon (E_i)_\mathfrak{U} \longrightarrow E_j$ be the operator that maps every $\mathbf{x}_k$ to x_j^k. Then there exists $J\in\mathfrak{U}$ such that, for all $j\in J$, L_j is an ε-isometry.*

Proof. (i) By Auerbach's Lemma, there exists a biorthogonal system $(e_i^k, f_i^l)_{k=1}^m{}_{l=1}^m$ in every product $E_i \times E_i^*$ with $\|e_i^k\| = \|f_i^k\| = 1$ for all i and all k.

For every $1 \le k \le m$, consider the element $\mathbf{e}_k := [e_i^k]$ and let us prove that $\{\mathbf{e}_k\}_{k=1}^m$ is a basis of $(E_i)_{\mathfrak{U}}$.

First, given $\mathbf{x} \in S_{(E_i)_{\mathfrak{U}}}$, let $(x_i)_{i\in I}$ be one of its representatives with $\|x_i\| = 1$ for all i. Thus, for every x_i there are scalars λ_i^k such that $x_i = \sum_{k=1}^m \lambda_i^k e_i^k$. Note that $|\lambda_i^k| \le 1$. Therefore, there exists $\lambda_k := \lim_{i\to\mathfrak{U}} \lambda_i^k \in \mathbb{R}$. Thus,

$$\mathbf{x} = [x_i] = [\sum_{k=1}^m \lambda_i^k e_i^k] = \sum_{k=1}^m [\lambda_i^k e_i^k] = \sum_{k=1}^m [\lambda_k e_i^k] = \sum_{k=1}^m \lambda_k [e_i^k] = \sum_{k=1}^m \lambda_k \mathbf{e}_k,$$

which proves that $\{\mathbf{e}_k\}_{k=1}^m$ is a generator system of $(E_i)_{\mathfrak{U}}$.

In order to prove that $\{\mathbf{e}_k\}_{k=1}^m$ is free, for every $1 \le k \le m$, consider the functional $\mathbf{f}_k \in {(E_i)_{\mathfrak{U}}}^*$ that maps each $[x_i] \in (E_i)_{\mathfrak{U}}$ to $\lim_{i\to\mathfrak{U}}\langle f_i^k, x_i\rangle$. Clearly, $(\mathbf{e}_k, \mathbf{f}_l)_{k=1}^m{}_{l=1}^m$ is a biorthogonal system of $(E_i)_{\mathfrak{U}} \times {(E_i)_{\mathfrak{U}}}^*$, which shows that $\{\mathbf{e}_k\}_{k=1}^m$ is free, and therefore, a basis of $(E_i)_{\mathfrak{U}}$.

(ii) Let $\varepsilon' > 0$ be small enough so that $2\varepsilon'(1-\varepsilon')^{-1} < \varepsilon$, and take an ε'-net $\{\mathbf{z}_k\}_{k=1}^n$ in $S_{(E_i)_{\mathfrak{U}}}$. For each $\mathbf{z}_k$ there exists a finite sequence of scalars $(\lambda_l^k)_{l=1}^m$ so that $\mathbf{z}_k = \sum_{l=1}^m \lambda_l^k \mathbf{x}_l$. For every $j \in I$, let us write

$$z_j^k := \sum_{l=1}^m \lambda_l^k x_j^l \text{ for all } 1 \le k \le n,$$

so $L_j(\mathbf{z}_k) = z_j^k$ for all $\mathbf{z}_k$. Since

$$J := \{i \in I \colon \forall 1 \le k \le n,\ 1-\varepsilon' < \|z_i^k\| < 1+\varepsilon'\} \in \mathfrak{U},$$

Lemma A.4.10 yields that, for every $j \in J$, the operator L_j is a $2\varepsilon'(1-\varepsilon')^{-1}$-isometry, hence, is an ε-isometry because of the choice of ε'. □

One of the main notions concerning local theory is the following:

Definition A.4.13. Given a pair of Banach spaces X and Y, X is said to be *finitely representable in* Y if for every $\varepsilon > 0$ and every finite dimensional subspace E of X there is an ε-isometry $L\colon E \longrightarrow Y$.

Proposition A.4.14. *A Banach space Y is finitely representable in X if and only if there exists an ultrafilter $\mathfrak{U}$ such that Y is isometrically contained in $X_{\mathfrak{U}}$.*

Proof. Let Y be finitely representable in X. Let I be the set of all pairs $i \equiv (E_i, \varepsilon_i)$ where E_i is a finite dimensional subspace of Y and ε_i is a positive real number. The set I is endowed with an order relation $\preceq$, where $i \preceq j$ means that $E_i \subset E_j$ and $\varepsilon_j \le \varepsilon_i$. Let $\mathfrak{U}$ be an ultrafilter containing the $\preceq$-order filter.

By hypothesis, for every $i \in I$ there is an ε_i-isometry $L_i\colon E_i \longrightarrow X$. By Lemma A.4.11, the ultrapower $(L_i)_{\mathfrak{U}}$ is an isometry. Let $L\colon Y \longrightarrow (E_i)_{\mathfrak{U}}$ be given by the expression $L(y) = [y_i]$, where $y_i := y$ if $y \in E_i$, and $y_i := 0$ otherwise. Note that, for every $y \in Y$, the set $I_y := \{i : y \in E_i\}$ belongs to $\mathfrak{U}$, hence L is an isometry, and therefore, so is the operator $(L_i)_{\mathfrak{U}} \circ L : Y \longrightarrow X_{\mathfrak{U}}$.

For the reverse implication, let us assume that there exists an isometry $L : Y \longrightarrow X_{\mathfrak{U}}$ for a certain ultrafilter $\mathfrak{U}$. Let E be a finite dimensional subspace of Y, and fix a real number $\varepsilon > 0$. Let $\{e_i\}_{i=1}^n$ be a basis of E, and for each element e_k, let $(x_i^k)_{i\in I}$ be a representative of $L(e_k)$. For every $i \in I$, consider the finite dimensional subspace X_i spanned by $\{x_i^k\}_{k=1}^n$ and the operator $L_i : E \longrightarrow X_i$ defined by $L_i(e_k) := x_i^k$ for all $1 \leq k \leq n$. Thus $(L_i)_{\mathfrak{U}}$ equals $L|_E$, and therefore, by Lemma A.4.12, there exists $J \in \mathfrak{U}$ so that for every $j \in J$, L_j is an ε-isometry. □

The principle of local reflexivity establishes that the bidual of a Banach space X is finitely representable in X with certain additional properties.

Theorem A.4.15 (Principle of local reflexivity). *Let X be a Banach space, and let E and F be a pair of finite dimensional subspaces of X^{**} and X^* respectively. Thus, given $\varepsilon > 0$, there exists an ε-isometry $T : E \longrightarrow X$ satisfying the following conditions:*

(i) *$T(x) = x$ for all $x \in E \cap X$,*

(ii) *$\langle x^*, T(x^{**})\rangle = \langle x^{**}, x^*\rangle$ for all $x^{**} \in E$ and all $x^* \in F$.*

Its original proof can be found in [120] and [110]. Short proofs can be found in [50], [128] and [155], but it can be also proved from Theorem 6.3.8 in combination with an easy argument of approximation.

The principle of local reflexivity admits the following translation into the ultraproduct language:

Proposition A.4.16. *Let X be any Banach space. Then there exist an $\aleph_0$-incomplete ultrafilter $\mathfrak{V}$ on certain set I and an isometry $L : X^{**} \longrightarrow X_{\mathfrak{V}}$ satisfying the following properties:*

(i) *$L(x) = [x]$ for all $x \in X$,*

(ii) *$L(X^{**})$ is complemented in $X_{\mathfrak{V}}$,*

(iii) *for each $x^{**} \in X^{**}$ and each representative $(y_i)_{i\in I}$ of $L(x^{**})$, $y_i \xrightarrow[\mathfrak{V}]{w^*} x^{**}$.*

Proof. Let I be the set of all triples $i \equiv (E_i, \mathcal{V}_i, \varepsilon_i)$ where E_i is a finite dimensional subspace of X^{**}, $\mathcal{V}_i$ is a weak* neighborhood of 0 in X^{**}, and ε_i is a positive real number. The set I is endowed with the order $\preceq$, where $i \preceq j$ means that $E_i \subset E_j$, $\mathcal{V}_i \supset \mathcal{V}_j$ and $\varepsilon_i > \varepsilon_j$.

Let $\mathfrak{F}$ be the $\preceq$-order filter on I, and take an ultrafilter $\mathfrak{V}$ refining $\mathfrak{F}$. Note that $\mathfrak{V}$ is $\aleph_0$-incomplete. In fact, for every index i, let F_i be the smallest linear subspace of X^* such that $F_i^\perp \subset \mathcal{V}_i$. For every $n \in \mathbb{N}$, let I_n be the subset of I formed by all indices j for which $\dim E_j \geq n$, $\dim F_j \geq n$ and $\varepsilon_j \leq 1/n$, and let $I_0 := I$. Thus $\{I_n\}_{n=0}^\infty$ is a decreasing sequence of elements of $\mathfrak{U}$ such that

$\bigcap_{n=0}^{\infty} I_n = \emptyset$, and therefore, $\{I_{n+1} \setminus I_n\}_{n=0}^{\infty}$ is a countable partition of I such that $I_{n+1} \setminus I_n \notin \mathfrak{V}$ for all $n \in \mathbb{N}$.

Let us now define the isometry L. For every index $i \in I$, the principle of local reflexivity provides us with an ε_i-isometry $L_i \colon E_i \longrightarrow X$ so that $L_i(x^{**}) \in x^{**} + \mathcal{V}_i$ for all $x^{**} \in E_i$ and $L_i(x) = x$ for all $x \in X \cap E_i$. Given $x^{**} \in X^{**} \setminus \{0\}$, let

$$x_i := \begin{cases} L_i(x^{**}) & \text{if } x^{**} \in E_i \\ 0 & \text{if } x^{**} \notin E_i. \end{cases}$$

Thus, for every $\varepsilon > 0$,

$$\{i \in I \colon 1 - \varepsilon \leq \|x_i\| \cdot \|x^{**}\|^{-1} \leq 1 + \varepsilon\} \supset \{i \in I \colon x^{**} \in E_i,\ \varepsilon_i \leq \varepsilon\} \in \mathfrak{U},$$

so $\|L(x^{**})\| = \lim_{\mathfrak{V}} \|x_i\| = \|x^{**}\|$, that is, L is an isometry.

Notice that, given any weak* neighborhood $\mathcal{V}$ of 0, there exists $B \in \mathfrak{V}$ such that $\mathcal{V} \supset \mathcal{V}_i$ for all $i \in B$, so $x_i \in x^{**} + \mathcal{V}$ for all $i \in B$, and therefore, $x_i \xrightarrow[\mathfrak{V}]{w^*} x^{**}$. Thus, if $(y_i)_{i \in I}$ is any representative of $L(x^{**})$, since $\lim_{\mathfrak{V}} \|x_i - y_i\| = 0$, we get $w^*\text{-}\lim_{\mathfrak{V}} y_i = x^{**}$, which proves (iii).

Moreover, given $x \in X$, there exists $j \in I$ so that $x \in E_i$ for all $i \succeq j$; hence $x = L_i(x)$ for all $i \in \{i \colon i \succeq j\} \in \mathfrak{V}$. Thus $L(x) = [x]$, and (i) is done.

In order to prove (ii), consider the norm-one operator $Q \colon X_{\mathfrak{V}} \longrightarrow X^{**}$ given by $Q([z_i]) := w^*\text{-}\lim_{\mathfrak{V}} z_i$. Since $x^{**} = w^*\text{-}\lim_{\mathfrak{V}} x_i$ for every $x^{**} \in X^{**}$, where $[x_i] = L(x^{**})$, we find that QL is the identity operator on X^{**}, so $P := LQ$ is a norm-one projection from $X_{\mathfrak{V}}$ onto $L(X^{**})$. □

Definition A.4.17. If $\mathfrak{V}$ and L are respectively an ultrafilter and an isometry as in the preceding proposition, we will say that $\mathfrak{V}$ *and* L *are associated with the bidual of* X.

A concept in local theory concerning finite representability is that of local duality.

Definition A.4.18. Given a Banach space X, a closed subspace Z of X^* is said to be a *local dual of* X if for every pair of finite dimensional subspaces E and F of X^* and X respectively, and for every $\varepsilon > 0$, there exists an ε-isometry $L \colon E \longrightarrow Z$ satisfying the following conditions:

(i) $L(x^*) = x^*$ for all $x^* \in E \cap Z$;

(ii) $\langle L(x^*) - x^*, x \rangle = 0$ for all $x^* \in E$ and all $x \in F$.

The classical principle of local reflexivity (Theorem A.4.15) and the principle of local reflexivity for ultrapowers (see [99]) can be stated in terms of local duality:

Example A.4.19. Let X be a Banach space and let $\mathfrak{U}$ be an ultrafilter:

(i) the space X is a local dual of X^*;

(ii) the ultrapower $(X^*)_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$.

An *extension operator* is an operator $T \in \mathcal{L}(Z^*, X^{**})$, where Z is a closed subspace of X^* and $T(z^*)|_Z = z^*$. Local duality admits the following non-local characterizations:

Theorem A.4.20. *Given a closed subspace Z of the dual of a Banach space X, the following statements are equivalent:*

(a) *Z is a local dual of X;*

(b) *there exist an ultrafilter $\mathfrak{U}$ on a set I and an isometry T from X^* into $Z_{\mathfrak{U}}$ such that $T|_Z$ is the natural embedding of Z into X^* and $QT = I_{X^*}$, where Q maps each $[x_i^*] \in Z_{\mathfrak{U}}$ to the $\sigma(X^*, X)$-limit of $(x_i)_{i\in I}$ following $\mathfrak{U}$;*

(c) *there exists an isometric extension operator $T \in \mathcal{L}(Z^*, X^{**})$ with $X \subset R(P)$ and $Z^\perp = N(P)$.*

The equivalences (a)⇔(b) and (a)⇔(c) are respectively proved in [84] and in [83].

Let us see some basic properties of the ultrapowers of an operator. These and other related results can be found in [76].

Proposition A.4.21. *Let $T \in \mathcal{L}(X, Y)$ be an operator, and $\mathfrak{U}$ an ultrafilter. Then, the following statements hold:*

(i) *$T_{\mathfrak{U}}(B_{X_{\mathfrak{U}}})$ is closed and equals $T(B_X)_{\mathfrak{U}}$;*

(ii) *$N(T)_{\mathfrak{U}} \subset N(T_{\mathfrak{U}})$;*

(iii) *$R(T_{\mathfrak{U}}) \subset R(T)_{\mathfrak{U}}$.*

Proof. (i) The equality $T_{\mathfrak{U}}(B_{X_{\mathfrak{U}}}) = T(B_X)_{\mathfrak{U}}$ is straightforward, and in combination with Proposition A.4.4, we get $T_{\mathfrak{U}}(B_{X_{\mathfrak{U}}})$ is closed.

The proofs of (ii) and (iii) are immediate. □

Proposition A.4.22. *Let $T \in \mathcal{L}(X, Y)$ be an operator, and $\mathfrak{U}$ an ultrafilter on I. Then, the following statements are equivalent:*

(a) *T has closed range;*

(b) *$N(T)_{\mathfrak{U}} = N(T_{\mathfrak{U}})$;*

(c) *$R(T)_{\mathfrak{U}} = R(T_{\mathfrak{U}})$;*

(d) *$T_{\mathfrak{U}}$ has closed range.*

Proof. (a)⇒(b) and (c). As T has closed range, there is a constant $M > 0$ such that, for each $x \in X$, there exists $z \in X$ so that $x - z \in N(T)$ and $\|T(x)\| \geq M\|z\|$.

Let $[x_i] \in N(T_{\mathfrak{U}})$. For every x_i there exists a pair of elements $z_i \in X$ and $u_i \in N(T)$ so that $x_i = z_i + u_i$ and $\|T(x_i)\| \geq M\|z_i\|$. Therefore, $\{z_i\}_{i\in I}$ is a bounded family and $\|z_i\| \underset{\mathfrak{U}}{\longrightarrow} 0$, so $[z_i] = 0$. Thus, $\{u_i\}_{i\in I}$ is also bounded and $[x_i] = [u_i] \in N(T)_{\mathfrak{U}}$. This and Proposition A.4.21 show that (b) holds.

Take now $[T(x_i)] \in R(T)_{\mathfrak{U}}$. Since $\{T(x_i)\}_{i\in I}$ is bounded, choosing for every x_i an element $z_i \in X$ so that $x_i - z_i \in N(T)$ and $\|T(x_i)\| \geq M\|z_i\|$, we find that $\{z_i\}_{i\in I}$ is bounded and

$$[T(x_i)] = [T(z_i)] = T_{\mathfrak{U}}([z_i]) \in R(T_{\mathfrak{U}}).$$

This last equality combined with Proposition A.4.21 yields $R(T_{\mathfrak{U}})$ equals $R(T)_{\mathfrak{U}}$, proving (c).

(b)⇒(a). Suppose $R(T)$ is not closed. Then, for every $n \in \mathbb{N}$, there is $x_n \in S_X$ such that $0 < \|T(x_n)\| < 1/n$ and $\operatorname{dist}\big(x_n, N(T)\big) \geq 1/2$. Let $\{I_n\}_{n=1}^{\infty}$ be a partition of I disjoint with $\mathfrak{U}$. For every $i \in I$, denote $h_i := x_n$ if $i \in I_n$. For each family $(z_i) \in \ell_\infty\big(I, N(T)\big)$, we have $\|[h_i] - [z_i]\| = \lim_{\mathfrak{U}} \|h_i - z_i\| \geq 1/2$, hence $[h_i] \notin N(T)_{\mathfrak{U}}$.

On the other hand, for every $n \in \mathbb{N}$,

$$\{i \in I\colon \|T(h_i)\| < 1/n\} \supset \cup_{k=n}^{\infty} I_k \in \mathfrak{U},$$

which leads to $[h_i] \in N(T_{\mathfrak{U}})$.

(c)⇒(d). It follows directly from Proposition A.4.4.

(d)⇒(a). If $T_{\mathfrak{U}}$ has closed range, by the open mapping theorem, there is $a > 0$ such that

$$T_{\mathfrak{U}}(aB_{X_{\mathfrak{U}}}) \supset B_{R(T_{\mathfrak{U}})}. \tag{A.5}$$

In addition, for every $y \in B_{\overline{R(T)}}$, it is immediate that $[y]$ belongs to $\overline{R(T_{\mathfrak{U}})}$, hence $[y] \in R(T_{\mathfrak{U}})$, and therefore

$$Y \cap B_{R(T_{\mathfrak{U}})} = B_{\overline{R(T)}}. \tag{A.6}$$

On the other hand, by Propositions A.4.4 and A.4.21,

$$\overline{T(B_X)} = Y \cap T_{\mathfrak{U}}(B_{X_{\mathfrak{U}}}). \tag{A.7}$$

Consecutive applications of (A.6), (A.5) and (A.7) lead to $\overline{T(aB_X)} \supset B_{\overline{R(T)}}$, and then Lemma A.4.8 yields $R(T) = \overline{R(T)}$. □

Recall that a closed subspace Z of a dual space X^* is said to be 1-*norming* if, for every $x \in X$, $\|x\| = \sup_{f\in Z}\langle f, x\rangle$. As a consequence, B_Z is $\sigma(X^*, X)$-dense in B_{X^*}.

Proposition A.4.23. *For every operator $T \in \mathcal{L}(X, Y)$ and every ultrafilter $\mathfrak{U}$, the kernel $N(T^*{}_{\mathfrak{U}})$ is a* 1*-norming subspace of $N(T_{\mathfrak{U}}{}^*)$.*

Proof. Taking into account that $N(T_{\mathfrak{U}}{}^*)$ is the dual of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$, we only need to prove that, given $\mathbf{y} \in S_{Y_{\mathfrak{U}}}$ with $\operatorname{dist}\big(\mathbf{y}, \overline{R(T_{\mathfrak{U}})}\big) = 1$, and given $0 < \delta < 1$, there exists a norm-one element $\mathbf{g} \in N(T^*{}_{\mathfrak{U}})$ such that

$$\langle \mathbf{g}, \mathbf{y}\rangle \geq 1 - \delta.$$

Let I be the set of indices on which $\mathfrak{U}$ is taken, and choose a representative $(y_i)_{i\in I}$ of $\mathbf{y}$ such that $\|y_i\| = 1$ for all i. Since

$$\operatorname{dist}\big(\mathbf{y}, nT_{\mathfrak{U}}(B_{X_{\mathfrak{U}}})\big) \geq 1 \quad \text{for all } n \in \mathbb{N},$$

we have, for every positive integer n, that

$$J_n := \{i \in I\colon \operatorname{dist}\big(y_i, nT(B_X)\big) > 1 - \delta\} \in \mathfrak{U}.$$

Since $\mathfrak{U}$ is $\aleph_0$-incomplete, there exists a decreasing set sequence $(C_n)_{n=1}^{\infty} \subset \mathfrak{U}$ such that $\cap_{n=1}^{\infty} C_n = \emptyset$ and $C_n \subset J_n$ for all n. Put $C_0 := I$; for every $i \in I$, let m_i be the only non-negative integer for which $i \in C_{m_i} \setminus C_{m_i+1}$, and define

$$K_i := y_i + m_i T(B_X);$$

thus, $K_i \cap (1-\delta)B_Y = \emptyset$, and therefore, the Hahn-Banach theorem provides a functional $g_i \in S_{Y^*}$ such that

$$\inf_{y\in K_i} \langle g_i, y\rangle \geq 1 - \delta. \tag{A.8}$$

Let $\mathbf{g} := [g_i]$, and prove that $\mathbf{g}$ is the wanted element of $N({T^*}_{\mathfrak{U}})$. In fact, on the one hand, formula (A.8) gives $\langle \mathbf{g}, \mathbf{y}\rangle \geq 1 - \delta$.

Yet on the other hand, for each n, each $i \in C_n$ and each $x \in B_X$,

$$|\langle g_i, T(x)\rangle| \leq \delta/n;$$

hence, $\langle [g_i], [z_i]\rangle = 0$ for all $[z_i] \in T_{\mathfrak{U}}(B_{X_{\mathfrak{U}}})$, which means

$$\mathbf{g} := [g_i] \in R(T_{\mathfrak{U}})^{\perp} \cap {Y^*}_{\mathfrak{U}} = N({T^*}_{\mathfrak{U}}),$$

concluding the proof. □

In particular, for every Banach space Y and every ultrafilter $\mathfrak{U}$, ${Y^*}_{\mathfrak{U}}$ is $\sigma({Y_{\mathfrak{U}}}^*, Y_{\mathfrak{U}})$ dense in ${Y_{\mathfrak{U}}}^*$.

A.5 Reflexivity and super-reflexivity

Reflexivity and super-reflexivity are classical subjects in local theory of Banach spaces. Here we give some of the characterizations for reflexive and super-reflexive Banach spaces due to James.

Recall that, in Definitions 6.5.1 and 6.2.12, we introduced the *ε-triangular* sequences as follows: given a real number $\varepsilon > 0$, a (finite or infinite) sequence (x_n) in a Banach space X is ε-triangular if $\|x_n\| \leq 1$ for all n and there exists a sequence of norm-one functionals (x_n^*) in X^* such that $\langle x_i^*, x_j\rangle > \varepsilon$ for all $1 \leq i \leq j$ and $\langle x_i^*, x_j\rangle = 0$ for all $1 \leq j < i$.

Note that if a sequence (x_n) is ε-triangular, then $\varepsilon \leq \|x_n\| \leq 1$ for all $n \in \mathbb{N}$.

Lemma A.5.1. *Let Z be a closed subspace of a Banach space X. For every $z^{**} \in \overline{Z}^{\sigma(X^{**},X^*)}$, we have*

$$\operatorname{dist}(z^{**}, X) \geq \frac{1}{2}\operatorname{dist}(z^{**}, Z).$$

Proof. Let $\delta := \operatorname{dist}(z^{**}, Z)$. If $\delta = 0$, there is nothing to be proved. Thus, assume that $\delta > 0$, and take $x \in X$.

If $\operatorname{dist}(x, Z) < \delta/2$, choose $x_1 \in Z$ so that $\|x - x_1\| < \delta/2$. Then,

$$\begin{aligned} \|z^{**} - x\| &= \|z^{**} - x_1 + x_1 - x\| \\ &\geq \|z^{**} - x_1\| - \|x_1 - x\| > \delta - \delta/2 = \delta/2. \end{aligned} \tag{A.9}$$

And if $\operatorname{dist}(x, Z) \geq \delta/2$, by the Hahn-Banach theorem, there exists $x^* \in S_{X^*}$ such that $x^* \in Z^\perp$ and $\langle x^*, x\rangle \geq \delta/2$. Thus, as $\overline{Z}^{\sigma(X^{**},X^*)} = Z^{\perp\perp}$, it follows that $\langle z^{**}, x^*\rangle = 0$, so

$$\|z^{**} - x\| \geq \langle x - z^{**}, x^*\rangle = \langle x^*, x\rangle \geq \delta/2. \tag{A.10}$$

Thus, (A.9) and (A.10) yield $\operatorname{dist}(z^{**}, X) \geq \delta/2$, as we wanted. □

Proposition A.5.2. *Let (x_n) be an ε-triangular sequence in a Banach space X. Thus, every $\sigma(X^{**}, X^*)$-cluster point x^{**} of (x_n) satisfies* $\operatorname{dist}(x^{**}, X) \geq \varepsilon/2$.

Proof. Let Z be the closed span of (x_n) in X and consider a sequence of normalized functionals (x_n^*) in X^* satisfying

$$\langle x_i^*, x_j\rangle > \varepsilon \quad \text{if } 1 \leq i \leq j, \tag{A.11}$$

$$\langle x_i^*, x_j\rangle = 0 \quad \text{if } 1 \leq j < i. \tag{A.12}$$

Let x^{***} be a $\sigma(X^{*(3)}, X^{**})$-cluster point of (x_n^*). Thus, (A.11) yields $\langle x^{**}, x_i^*\rangle \geq \varepsilon$ for all i, hence $\langle x^{***}, x^{**}\rangle \geq \varepsilon$. But (A.12) shows that $\langle x^{***}, x_j\rangle = 0$ for all j, hence $\operatorname{dist}(x^{**}, Z) \geq \varepsilon$, and by Lemma A.5.1, $\operatorname{dist}(x^{**}, X) \geq \varepsilon/2$. □

As a consequence of Proposition A.5.2, an ε-triangular sequence in X cannot contain any weakly convergent subsequence. Therefore, a reflexive space cannot contain any ε-triangular sequence. The reverse is also true, as the following result proves:

Proposition A.5.3. *Let A be a subset of the unit closed ball of a Banach space X. If x^{**} is a $\sigma(X^{**}, X^*)$-cluster point of A with* $\operatorname{dist}(x^{**}, X) > \varepsilon > 0$, *then A contains an ε-triangular sequence.*

Proof. The wanted ε-triangular sequence (x_n) in A is obtained recursively as follows. Choose a functional $x_1^* \in S_{X^*}$ so that $\langle x^{**}, x_1^*\rangle > \varepsilon$ and consider the $\sigma(X^{**}, X^*)$-neighborhood of x^{**} given by

$$V_1 = \{y^{**} \in X^{**} \colon \langle y^{**}, x_1^*\rangle > \varepsilon\}.$$

Let $A_1 := A \cap V_1$ and pick $x_1 \in A_1$. By the Hahn-Banach theorem, there exists a functional $x_2^* \in S_{X^*}$ such that $\langle x_2^*, x_1\rangle = 0$ and $\langle x^{**}, x_2^*\rangle > \varepsilon$.

Given a positive integer $p \geq 2$, let us assume that there are two finite sequences $(x_n^*)_{n=1}^p \subset S_{X^*}$ and $(x_n)_{n=1}^{p-1} \subset (x_n)$ satisfying the conditions

$$\langle x_i^*, x_j\rangle > \varepsilon \text{ if } 1 \leq i \leq j \leq p-1, \tag{A.13}$$

$$\langle x_i^*, x_j\rangle = 0 \text{ if } 1 \leq j < i \leq p, \tag{A.14}$$

$$\langle x^{**}, x_i^*\rangle > \varepsilon \text{ for all } 1 \leq i \leq p. \tag{A.15}$$

Condition (A.15) ensures that

$$V_p = \{y^{**} \colon \langle y^{**}, x_i^*\rangle > \varepsilon,\ i = 1, \ldots, p\}$$

is a $\sigma(X^{**}, X^*)$-neighborhood of x^{**}. As x^{**} is a $\sigma(X^{**}, X^*)$-cluster point of A, then

$$A_p := V_p \cap A \neq \emptyset$$

so we may pick $x_p \in A_p$. Thus, by the Hahn-Banach theorem, there exists a functional $x_{p+1}^* \in S_{X^*}$ such that

$$\langle x_{p+1}^*, x_j\rangle = 0 \text{ for all } j = 1, \ldots, p,$$
$$\langle x^{**}, x_{p+1}^*\rangle > \varepsilon.$$

From (A.13), (A.14), and from the fact that $x_p \in V_p$, we get

$$\langle x_i^*, x_j\rangle > \varepsilon \text{ if } 1 \leq i \leq j \leq p,$$
$$\langle x_i^*, x_j\rangle = 0 \text{ if } 1 \leq j < i \leq p+1,$$

and repeating recursively the above argument, we prove that (x_n) is an ε-triangular sequence contained in A. □

Proposition A.5.3 admits the following sequential version:

Proposition A.5.4. *Let (z_n) be a sequence contained in the unit closed ball of a Banach space X and let x^{**} be a $\sigma(X^{**}, X^*)$-cluster point of (z_n) satisfying* $\operatorname{dist}(z^{**}, X) > \varepsilon > 0$*. Then (z_n) contains an ε-triangular subsequence.*

Proof. In the proof of Proposition A.5.3, take $A := \{z_n \colon n \in \mathbb{N}\}$ and $A_1 := A \cap V_1$, and for $p \geq 2$, once the finite subsequence $(x_n)_{n=1}^{p-1}$ of (z_n) and the finite sequence $(x_n^*)_{n=1}^p$ have been found, (where $x_n = z_{k_n}$ for $1 \leq n \leq p-1$), take

$$A_p := V_p \cap \{z_n \colon n > k_{p-1}\}.$$

Thus, x_p must be chosen as $z_{k_p} \in (z_n)$ for some $k_p > k_{p-1}$, which guarantees that the sequence (k_n) of positive integers is increasing, and therefore $(x_n) = (z_{k_n})$ is an ε-triangular subsequence of (z_n). □

Now, let us summarize some results concerning super-reflexivity.

Definition A.5.5. A Banach space X is said to be super-reflexive if every Banach space Y finitely representable in X is reflexive.

Super-reflexive Banach spaces can be characterized in terms of finite ε-triangular sequences.

Proposition A.5.6. *Given a Banach space X, the following statements are equivalent:*

(a) *the space X is not super-reflexive;*

(b) *for every $0 < \varepsilon < 1$ and every $n \in \mathbb{N}$, there exists a finite ε-triangular sequence $(x_k)_{k=1}^n$ in X;*

(c) *there exists $0 < \varepsilon < 1$ such that for every $n \in \mathbb{N}$, there is a finite ε-triangular sequence $(x_k)_{k=1}^n$ in X.*

Proof. (a)⇒(b) Let us assume that X is not super-reflexive and fix a real number $0 < \varepsilon < 1$ and a positive integer n, and take a second real number $0 < \varepsilon < \varepsilon' < 1$. By hypothesis, there exists a non-reflexive Banach space Y finitely representable in X, and by Proposition A.4.14, there exists an ultrafilter $\mathfrak{U}$ on a set I and an isometry $L\colon Y \longrightarrow X_{\mathfrak{U}}$, hence $X_{\mathfrak{U}}$ is not reflexive. Thus, by Proposition A.5.3, $X_{\mathfrak{U}}$ contains an ε'-triangular sequence $(\mathbf{x}_k)_{k=1}^{\infty}$, and subsequently,

$$\varepsilon' < \operatorname{dist}\left(\operatorname{span}\{\mathbf{x}_l\}_{l=1}^{k-1}, \operatorname{conv}\{\mathbf{x}_l\}_{l=k}^{n}\right) \quad \text{for all } 1 \le k \le n, \tag{A.16}$$

where we adopt the agreement that $\operatorname{span}\{\mathbf{x}_l\}_{l=1}^{0} := \{0\}$. Fix a representative $(x_i^k)_{i\in I} \subset B_X$ for every $\mathbf{x}_k$. Thus, since $\varepsilon < \varepsilon'$, formula (A.16) and Lemma A.4.12 provide us with an index $j \in I$ such that

$$\varepsilon < \operatorname{dist}\left(\operatorname{span}\{x_j^l\}_{l=1}^{k-1}, \operatorname{conv}\{x_j^l\}_{l=k}^{n}\right) \quad \text{for all } 1 \le k \le n,$$

which shows, with the help of the Hahn-Banach theorem, that $(x_j^k)_{k=1}^n$ is a finite ε-triangular sequence in X.

(b)⇒(c) Trivial.

(c)⇒(a) Let us assume that there exists $0 < \varepsilon < 1$ such that for every $n \in \mathbb{N}$, there is a finite ε-triangular sequence $(x_l^n)_{l=1}^n$ in X. Let $\{f_l^n\}_{l=1}^n$ be the corresponding family of normalized functionals in X^* so that

$$\langle f_i^n, x_j^n\rangle = \begin{cases} > \varepsilon & \text{if } 1 \le i \le j \le n, \\ = 0 & \text{if } 1 \le j < i \le n. \end{cases}$$

Let $\mathfrak{U}$ be an ultrafilter on $\mathbb{N}$, and for every positive integer l, consider the element $\mathbf{x}_l := [x_l^n]_n \in X_{\mathfrak{U}}$ and the functional $\mathbf{f}_l \in (X_{\mathfrak{U}})^*$ that maps every $[u_n] \in X_{\mathfrak{U}}$ to $\lim_{n\to\mathfrak{U}}\langle f_l^n, u_n\rangle$. Clearly, $\|\mathbf{x}_l\| \le 1$ and $\|\mathbf{f}_l\| = 1$ for all l. Moreover,

$$\langle \mathbf{f}_i, \mathbf{x}_j\rangle = \begin{cases} \ge \varepsilon & \text{if } 1 \le i \le j, \\ = 0 & \text{if } 1 \le j < i. \end{cases}$$

Thus, $X_{\mathfrak{U}}$ contains an ε-triangular sequence, so it is not reflexive. But by Proposition A.4.14, $X_{\mathfrak{U}}$ is finitely representable in X, hence X is not super-reflexive. □

A.6 Ultraproducts of $L_1(\mu)$ spaces

Here we describe the ultrapowers of $L_1(\mu)$ spaces with the aim of providing proofs of the following properties that are useful in the study of tauberian operators on $L_1(\mu)$ spaces:

(i) $L_1(\mu)$ is L-embedded in its bidual space,

(ii) $L_1(\mu)$ has the subsequence splitting property,

(iii) each reflexive subspace of $L_1(\mu)$ is super-reflexive.

We begin by introducing some concepts and notation in order to show that the ultraproduct of a family $\big(L_1(\mu_i)\big)_{i\in I}$ can be represented as an $L_1(\hat{\mu})$ space for a certain measure $\hat{\mu}$.

Let $\mathfrak{U}$ be an ultrafilter on a set I, and let $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i\in I}$ be a collection of finite positive measure spaces with no atoms such that $\sup_{i\in I}\mu_i(\Omega_i) < \infty$. Its ultraproduct following $\mathfrak{U}$ is a measure space on $(\Omega_i)^{\mathfrak{U}}$, which is defined as follows:

First, consider the set algebra

$$(\Sigma_i)^{\mathfrak{U}} := \{(A_i)^{\mathfrak{U}} \colon A_i \in \Sigma_i\},$$

where $(A_i)^{\mathfrak{U}}$ stands for the set-theoretic ultraproduct of $(A_i)_{i\in I}$ following $\mathfrak{U}$ (see Appendix A.4).

Let $\sigma\big((\Sigma_i)^{\mathfrak{U}}\big)$ be the smallest σ-algebra containing the algebra $(\Sigma_i)^{\mathfrak{U}}$. The measures μ_i induce a measure $(\mu_i)_{\mathfrak{U}}$ on $\sigma\big((\Sigma_i)^{\mathfrak{U}}\big)$ which is univocally defined by its values on $(\Sigma_i)^{\mathfrak{U}}$, as is proved in the following result:

Theorem A.6.1. *Given a collection of finite positive measure spaces* $\{(\Omega_i,\Sigma_i, \mu_i)\}_{i\in I}$ *with no atoms such that* $\sup_{i\in I}\mu_i(\Omega_i) < \infty$ *and an ultrafilter* $\mathfrak{U}$ *on* I*, let us consider the set mapping* $(\mu_i)_{\mathfrak{U}} \colon (\Sigma_i)^{\mathfrak{U}} \longrightarrow [0,\infty)$ *given by*

$$(\mu_i)_{\mathfrak{U}}\big((A_i)^{\mathfrak{U}}\big) := \lim_{\mathfrak{U}} \mu_i(A_i), \text{ for all } (A_i)^{\mathfrak{U}} \in (\Sigma_i)^{\mathfrak{U}}.$$

Then $(\mu_i)_{\mathfrak{U}}$ *admits a unique extension to a* σ*-additive measure on* $\sigma\big((\Sigma_i)^{\mathfrak{U}}\big)$.

Proof. It is straightforward that the mapping $(\mu_i)_{\mathfrak{U}}$ is well defined and finitely additive on $(\Sigma_i)^{\mathfrak{U}}$. In order to prove that it is also σ-additive, we consider a countable collection of disjoint sets $\{A^k\}_{k=1}^{\infty} \subset (\Sigma_i)^{\mathfrak{U}}$. Let $A^k = (A_i^k)^{\mathfrak{U}}$ with $A_i^k \in \Sigma_i$ for every i and every k. Thus, for each $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n}(\mu_i)_{\mathfrak{U}}(A^k) \le \sum_{k=1}^{n}(\mu_i)_{\mathfrak{U}}(A^k) + (\mu_i)_{\mathfrak{U}}\Big(\bigcup_{k=n+1}^{\infty} A^k\Big) = (\mu_i)_{\mathfrak{U}}\Big(\bigcup_{k=1}^{\infty} A^k\Big),$$

hence $\sum_{k=1}^{\infty}(\mu_i)_{\mathfrak{U}}(A^k) \le (\mu_i)_{\mathfrak{U}}\big(\bigcup_{k=1}^{\infty} A^k\big)$.

In order to obtain the converse inequality, take $\varepsilon > 0$ and, for every A^k, let us write

$$J_k := \big\{i \in I \colon \mu_i(A_i^k) \le (\mu_i)_{\mathfrak{U}}(A^k) + 2^{-k}\varepsilon\big\} \in \mathfrak{U}$$

and

$$B_i^k := \begin{cases} A_i^k & \text{if } i \in J_k, \\ \emptyset & \text{if } i \notin J_k. \end{cases}$$

Then $A^k = (B_i^k)^{\mathfrak{U}}$, and $\mu_i(B_i^k) \le (\mu_i)_{\mathfrak{U}}(A^k) + \varepsilon/2^k$ for all k and all i. Thus,

$$\bigcup_{k=1}^{\infty} A^k = \bigcup_{k=1}^{\infty} (B_i^k)^{\mathfrak{U}} \subset \left(\bigcup_{k=1}^{\infty} B_i^k\right)^{\mathfrak{U}},$$

and therefore,

$$\begin{aligned}(\mu_i)_{\mathfrak{U}}\left(\bigcup_{k=1}^{\infty} A^k\right) &\le (\mu_i)_{\mathfrak{U}}\left((\bigcup_{k=1}^{\infty} B_i^k)^{\mathfrak{U}}\right) \\ &= \lim_{\mathfrak{U}} \mu_i\left(\bigcup_{k=1}^{\infty} B_i^k\right) \le \sum_{k=1}^{\infty}(\mu_i)_{\mathfrak{U}}(A^k) \,+\, \varepsilon,\end{aligned}$$

which proves that $(\mu_i)_{\mathfrak{U}}(\bigcup_{k=1}^{\infty} A^k) = \sum_{k=1}^{\infty}(\mu_i)_{\mathfrak{U}}(A^k)$.

Once the σ-additivity of $(\mu_i)_{\mathfrak{U}}$ on $(\Sigma_i)^{\mathfrak{U}}$ has been proved, the Caratheodory extension theorem ensures that actually, the set function $(\mu_i)_{\mathfrak{U}}$ defined on $(\Sigma_i)^{\mathfrak{U}}$ can be extended to a unique σ-additive measure on the σ-algebra $\sigma\big((\Sigma_i)^{\mathfrak{U}}\big)$. □

The measure on $\sigma\big((\Sigma_i)^{\mathfrak{U}}\big)$ supplied by Theorem A.6.1 is also denoted by $(\mu_i)_{\mathfrak{U}}$, and turns $\big((\Omega_i)^{\mathfrak{U}}, (\Sigma_i)^{\mathfrak{U}}, (\mu_i)_{\mathfrak{U}}\big)$ into a finite measure space called *the ultraproduct of $(\Omega_i, \Sigma_i, \mu_i)_{i\in I}$ following* $\mathfrak{U}$.

According to the proof of Caratheodory's theorem, the value of $(\mu_i)_{\mathfrak{U}}$ on every $A \in \sigma\big((\Sigma_i)^{\mathfrak{U}}\big)$ is

$$(\mu_i)_{\mathfrak{U}}(A) := \inf\left\{\sum_{n=1}^{\infty}(\mu_i)_{\mathfrak{U}}(C_n) \colon A \subset \bigcup_{n=1}^{\infty} C_n,\ C_n \in (\Sigma_i)^{\mathfrak{U}}\right\}.$$

The following result allows us to simplify the computation of $(\mu_i)_{\mathfrak{U}}(A)$:

Proposition A.6.2. *Let $(\mu_i)_{\mathfrak{U}}$ be the ultraproduct measure of the finite positive measure spaces $(\Omega_i, \Sigma_i, \mu_i)_{i\in I}$ with no atoms and $\sup_{i\in I}\mu_i(\Omega_i) < \infty$. Let $\mathfrak{U}$ be an ultrafilter on I. Thus, for every $A \in \sigma\big((\Sigma_i)^{\mathfrak{U}}\big)$, the following identities hold:*

(i) $(\mu_i)_{\mathfrak{U}}(A) = \inf\{(\mu_i)_{\mathfrak{U}}(C) \colon C \in (\Sigma_i)^{\mathfrak{U}},\ A \subset C\}$;

(ii) $(\mu_i)_{\mathfrak{U}}(A) = \sup\{(\mu_i)_{\mathfrak{U}}(C) \colon C \in (\Sigma_i)^{\mathfrak{U}},\ A \supset C\}$.

Proof. Let $\mu_0 := (\mu_i)_{\mathfrak{U}}$.

(i) Let us write

$$m(A) := \inf\{\mu_0(B) \colon A \subset B \in (\Sigma_i)^{\mathfrak{U}}\}.$$

The inequality $m(A) \geq \mu_0(A)$ is trivial. For the converse inequality, given any $\varepsilon > 0$ and any countable covering $\{B_n\}_{n=1}^{\infty}$ of A picked in $(\Sigma_i)^{\mathfrak{U}}$, we only need to find a set $D \in (\Sigma_i)^{\mathfrak{U}}$ so that $\bigcup_{n=1}^{\infty} B_n \subset D$ and $\mu_0(D) \leq \varepsilon + \sum_{n=1}^{\infty} \mu_0(B_n)$.

If the series $\sum_{n=1}^{\infty} \mu_0(B_n)$ diverges, then it is enough to choose $D := (\Omega_i)^{\mathfrak{U}}$.

In the case when $\sum_{n=1}^{\infty} \mu_0(B_n)$ converges, take a positive integer k so that

$$\sum_{n=k+1}^{\infty} \mu_0(B_n) < \varepsilon/2.$$

Thus, if we get $C \in (\Sigma_i)^{\mathfrak{U}}$ such that $\bigcup_{n=k+1}^{\infty} B_n \subset C$ and $\mu_0(C) \leq \varepsilon$, then a sensible choice for D is $\bigcup_{n=1}^{k} B_n \cup C$.

In order to get C, note that for every B_n there exists a collection $(B_n^i)_{i\in I}$ so that $B_n = (B_n^i)^{\mathfrak{U}}$ and

$$J_n := \{i \in I \colon \mu_i(B_n^i) < \mu_0(B_n) + 2^{-n-1}\varepsilon\} \in \mathfrak{U},$$

so we may assume that $B_n^i = \emptyset$ for all n and all $i \in I \setminus J_n$, and therefore,

$$\mu_i(B_n^i) < \mu_0(B_n) + 2^{-n-1}\varepsilon, \text{ for all } n \text{ and all } i. \tag{A.17}$$

Thus, the wanted set C is $\left(\bigcup_{n=k+1}^{\infty} B_n^i\right)^{\mathfrak{U}} \in (\Sigma_i)^{\mathfrak{U}}$ because it contains the set $\bigcup_{n=k+1}^{\infty} (B_n^i)^{\mathfrak{U}}$ and satisfies

$$\begin{aligned} \mu_0(C) &= \lim_{\mathfrak{U}} \mu_i \left(\bigcup_{n=k+1}^{\infty} B_n^i \right) \\ &\leq \lim_{\mathfrak{U}} \sum_{n=k+1}^{\infty} \mu_i(B_n^i) \leq \lim_{\mathfrak{U}} \sum_{n=k+1}^{\infty} \mu_0(B_n) + \sum_{n=k+1}^{\infty} 2^{-n-1}\varepsilon < \varepsilon. \end{aligned}$$

(ii) We write

$$s(A) := \sup\{\mu_0(B) \colon A \supset B \in \Sigma^{\mathfrak{U}}\},$$

and prove that $s(A) = m(A)$.

The inequality $s(A) \leq m(A)$ is trivial. For the converse inequality, let $\varepsilon > 0$ and choose a set $B \in (\Sigma_i)^{\mathfrak{U}}$ so that $A \subset B$ and $\mu_0(B \setminus A) = \mu_0(B) - m(A) < \varepsilon/2$. Next, by (i), there exists $C \in (\Sigma_i)^{\mathfrak{U}}$ so that $B \setminus A \subset C$ and $\mu_0(C) < \varepsilon$. Thus $B \setminus C \subset A$, and since $(\Sigma_i)^{\mathfrak{U}}$ is a set algebra, we get $B \setminus C \in (\Sigma_i)^{\mathfrak{U}}$. Hence

$$s(A) \geq \mu_0(B \setminus C) \geq \mu_0(B) - \mu_0(C) \geq m(A) - \mu_0(C),$$

so $s(A) \geq m(A) - \varepsilon$ for all $\varepsilon > 0$, and the inequality $s(A) \geq m(A)$ follows. Thus, by part (i), $s(A) = \mu_0(A)$. □

If $(\Omega_i, \Sigma_i, \mu_i) = (\Omega, \Sigma, \mu)$ for all $i \in I$, then the measure space $(\Omega^{\mathfrak{U}}, \sigma(\Sigma^{\mathfrak{U}}), \mu_{\mathfrak{U}})$ is called the *ultrapower of* (Ω, Σ, μ) *following* $\mathfrak{U}$.

The notation $(\Omega, \tilde{\Sigma}, \tilde{\mu})$ denotes the completion of the measure space (Ω, Σ, μ) with respect to μ.

Proposition A.6.3. *Let $\{(\Omega_{ij}, \Sigma_{ij}, \mu_{ij})\}_{(i,j)\in I\times J}$ be a double indexed collection of finite positive purely measure spaces with no atoms such that $\sup_{(i,j)\in I\times J} \mu_{ij}(\Omega_{ij}) < \infty$, and let $\mathfrak{U}$ and $\mathfrak{V}$ be a pair of ultrafilters on I and J respectively. Thus the completions of the measure spaces*

$$\Big(\big((\Omega_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}, \sigma\big(\sigma\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}\big), \big((\mu_{ij})_{\mathfrak{V}}\big)_{\mathfrak{U}}\Big)$$

and

$$\Big((\Omega_{ij})^{\mathfrak{U}\times\mathfrak{V}}, \sigma\big((\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}\big), (\mu_{ij})_{\mathfrak{U}\times\mathfrak{V}}\Big)$$

are isomorphic.

Proof. Let $\mathcal{I}\colon (\Omega_{ij})^{\mathfrak{U}\times\mathfrak{V}} \longrightarrow \big((\Omega_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$ be the bijective mapping that sends every $(t_{ij})^{\mathfrak{U}\times\mathfrak{V}}$ to $\big((t_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$.

Let us also denote by $\mathcal{I}$ the induced σ-algebra isomorphism from $\wp\big((\Omega_{ij})^{\mathfrak{U}\times\mathfrak{V}}\big)$ onto $\wp\Big(\big((\Omega_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}\Big)$. As it is observed in Proposition A.4.3 (Appendix A.4), the σ-algebra $\sigma\big((\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}\big)$ is mapped isomorphically by $\mathcal{I}$ onto $\sigma\Big(\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}\Big)$. Therefore,

$$\mathcal{I}\Big(\sigma\big((\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}\big)\Big) \subset \sigma\Big(\sigma\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}\Big). \tag{A.18}$$

We shall prove that the completion $\mathcal{S}_1 := \widetilde{\sigma}\big((\Sigma_{ij})^{\mathfrak{U}\times\mathfrak{V}}\big)$ is mapped by $\mathcal{I}$ onto $\mathcal{S}_2 := \widetilde{\sigma}\big(\sigma\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}\big)$.

After (A.18), the inclusion $\mathcal{I}(\mathcal{S}_1) \subset \mathcal{S}_2$ is immediate. To obtain the converse inclusion, we only have to prove that for every $\big((\mu_{ij})_{\mathfrak{V}}\big)_{\mathfrak{U}}$-null set A, $\mathcal{I}^{-1}(A)$ is contained in a $(\mu_{ij})_{\mathfrak{U}\times\mathfrak{V}}$-null set.

Since Proposition A.6.2 shows that, for every $n \in \mathbb{N}$, A is contained in a set $A_n \in \sigma\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$ with $\big((\mu_{ij})_{\mathfrak{V}}\big)_{\mathfrak{U}}(A_n) < 1/n$, our goal will be achieved as soon as we prove that for each collection $\{A_i\}_{i\in I} \subset \sigma\big((\Sigma_{ij})^{\mathfrak{V}}\big)$, $\mathcal{I}^{-1}((A_i)^{\mathfrak{U}})$ is contained in a set of $\Sigma^{\mathfrak{U}\times\mathfrak{V}}$ of the same measure. Thus, given $(A_i)^{\mathfrak{U}} \in \sigma\big((\Sigma_{ij})^{\mathfrak{V}}\big)^{\mathfrak{U}}$, consider a partition $\{I_n\}_{n=1}^{\infty}$ of I disjoint with $\mathfrak{U}$. For each $n \in \mathbb{N}$ and each $i \in I_n$, by Proposition A.6.2, there is $B_i \in (\Sigma_{ij})^{\mathfrak{V}}$ so that $B_i \supset A_i$ and $(\mu_{ij})_{\mathfrak{V}}(B_i \backslash A_i) < n^{-1}$.

Let $B_i = (B_{ij})^{\mathfrak{V}}$, with $B_{ij} \in \Sigma_{ij}$. Note that $\mathcal{I}^{-1}\big((A_i)^{\mathfrak{U}}\big) \subset (B_{ij})^{\mathfrak{U}\times\mathfrak{V}}$ and

$$\big((\mu_{ij})_{\mathfrak{V}}\big)_{\mathfrak{U}}\big((A_i)^{\mathfrak{U}}\big) = \lim_{\mathfrak{U}}(\mu_{ij})_{\mathfrak{V}}(A_i) = \lim_{\mathfrak{U}}(\mu_{ij})_{\mathfrak{V}}(B_i).$$

Moreover, by the iteration theorem for products of ultrafilters (see Appendix A.4),

$$\lim_{\mathfrak{U}}(\mu_{ij})_{\mathfrak{V}}(B_i) = \lim_{\mathfrak{U}} \lim_{\mathfrak{V}} \mu_{ij}(B_{ij}) = \lim_{\mathfrak{U}\times\mathfrak{V}} \mu_{ij}(B_{ij}).$$

Thus

$$\big((\mu_{ij})_{\mathfrak{V}}\big)_{\mathfrak{U}}\big((A_i)^{\mathfrak{V}}\big) = (\mu_{ij})_{\mathfrak{U}\times\mathfrak{V}}\big(B_{ij}\big)^{\mathfrak{U}\times\mathfrak{V}},$$

as we wanted to prove. □

Let us now move on to the case when $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i\in I}$ is a set of finite real measure spaces with no atoms, such that $\sup_{i\in I} |\mu_i|(\Omega_i) < \infty$.

Following a standard procedure, we decompose each μ_i, into its positive part and its negative part, $\mu_i = \mu_i^+ - \mu_i^-$. Next we define the measure $(\mu_i)_{\mathfrak{U}}$ on $\sigma(\Sigma^{\mathfrak{U}})$ as

$$(\mu_i)_{\mathfrak{U}} = (\mu_i^+)_{\mathfrak{U}} - (\mu_i^-)_{\mathfrak{U}}.$$

It is immediate that the positive part and the negative part of $(\mu_i)_{\mathfrak{U}}$ are $(\mu_i)_{\mathfrak{U}}^+ = (\mu_i^+)_{\mathfrak{U}}$ and $(\mu_i)_{\mathfrak{U}}^- = (\mu_i^-)_{\mathfrak{U}}$.

In order to give the theorems that describe the structural properties of the ultrapowers of $L_1(\mu)$, it is necessary to introduce additional notations.

Let $\mathfrak{U}$ be an ultrafilter on I and (Ω, Σ, μ) a finite positive measure space with no atoms. Given $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$, consider the measures μ_i on Σ defined by $d\mu_i = f_i\, d\mu$, and denote by $(\mu_i)_{\mathfrak{U}}$ the ultrapower measure defined on $\sigma(\Sigma^{\mathfrak{U}})$ by

$$\mu_{\mathbf{f}} := (\mu_i)_{\mathfrak{U}}.$$

Its value on each $A = (A_i)^{\mathfrak{U}} \in \Sigma^{\mathfrak{U}}$ is given by

$$\mu_{\mathbf{f}}(A) := \lim_{\mathfrak{U}} \int_{A_i} f_i\, d\mu.$$

Let us also write

$$\mathbf{f}^+ := [f_i^+] \text{ and } \mathbf{f}^- := [f_i^-].$$

Thus, $\mathbf{f} = \mathbf{f}^+ - \mathbf{f}^-$ and $\mu_{\mathbf{f}} = \mu_{\mathbf{f}^+} - \mu_{\mathbf{f}^-}$. Consequently, $\mu_{\mathbf{f}}^+ = \mu_{\mathbf{f}^+}$ and $\mu_{\mathbf{f}}^- = \mu_{\mathbf{f}^-}$.

By the theorems of Lebesgue decomposition of measures and Radon-Nikodym, there exist unique measures $w_{\mathbf{f}}$ and $m_{\mathbf{f}}$, both on $\sigma(\Sigma^{\mathfrak{U}})$, and a function $g_{\mathbf{f}} \in L_1(\mu_{\mathfrak{U}})$, so that

$$m_{\mathbf{f}} \perp \mu_{\mathfrak{U}},\ w_{\mathbf{f}} \ll \mu_{\mathfrak{U}},\ \mu_{\mathbf{f}} = w_{\mathbf{f}} + m_{\mathbf{f}},$$

and

$$w_{\mathbf{f}}(A) = \int_A g_{\mathbf{f}}\, d\mu_{\mathfrak{U}},\ A \in \sigma(\Sigma^{\mathfrak{U}}).$$

We also write $|\mathbf{f}| := \mathbf{f}^+ + \mathbf{f}^-$. So $|\mu_{\mathbf{f}}| = \mu_{|\mathbf{f}|}$, $|w_{\mathbf{f}}| = w_{|\mathbf{f}|}$ and $|m_{\mathbf{f}}| = m_{|\mathbf{f}|}$.

Proposition A.6.4. *Let (Ω, Σ, μ) be a finite, positive measure space with no atoms, and let $\mathfrak{U}$ be an ultrafilter. Let $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$ and $A \in \sigma(\Sigma^{\mathfrak{U}})$. Thus, if $\mu_{\mathbf{f}}(C) = 0$ for all subsets $C \in \Sigma^{\mathfrak{U}}$ of A, then $\mu_{\mathbf{f}}(A) = 0$.*

Its proof is a direct consequence of Proposition A.6.2.

The following three theorems feature the main structural properties of the ultrapowers of $L_1(\mu)$:

Theorem A.6.5. *For each positive, finite, measure space (Ω, Σ, μ) with no atoms and every ultrafilter $\mathfrak{U}$, there exist a canonical isometry $J_{\mu_{\mathfrak{U}}} \colon L_1(\mu_{\mathfrak{U}}) \longrightarrow L_1(\mu)_{\mathfrak{U}}$ and a canonical projection $P_{\mu_{\mathfrak{U}}} \colon L_1(\mu)_{\mathfrak{U}} \longrightarrow L_1(\mu)_{\mathfrak{U}}$ with $R(P_{\mu_{\mathfrak{U}}}) = J_{\mu_{\mathfrak{U}}}(L_1(\mu_{\mathfrak{U}}))$, such that $L_1(\mu)_{\mathfrak{U}} = R(P_{\mu_{\mathfrak{U}}}) \oplus_1 N(P_{\mu_{\mathfrak{U}}})$ and $\|P_{\mu_{\mathfrak{U}}}\| = 1$.*

Proof. Let I be the set of indices on which $\mathfrak{U}$ is taken. The canonical isometry $J_{\mu_{\mathfrak{U}}} \colon L_1(\mu_{\mathfrak{U}}) \longrightarrow L_1(\mu)_{\mathfrak{U}}$ is given as follows. Let H be the subset of all finite sums

$$H := \left\{ \sum_{k=1}^{n} \alpha_k \chi_{A_k} : n \in \mathbb{N},\ \alpha_k \in \mathbb{R},\ A_k \in \Sigma^{\mathfrak{U}},\ A_k \cap A_l = \emptyset \text{ for all } k \neq l \right\}.$$

By Proposition A.6.2, H is a dense subset of $L_1(\mu_{\mathfrak{U}})$.

First, $J_{\mu_{\mathfrak{U}}}$ is defined on the characteristic functions χ_A, where $A = (A_i)^{\mathfrak{U}} \in \Sigma^{\mathfrak{U}}$, by

$$J_{\mu_{\mathfrak{U}}}(\chi_A) := [\chi_{A_i}] \in L_1(\mu)_{\mathfrak{U}}.$$

Since $\|\chi_A\| = \mu_{\mathfrak{U}}(A) = \lim_{\mathfrak{U}} \mu(A_i) = \|[\chi_{A_i}]\|$, we get $\|J_{\mu_{\mathfrak{U}}}(\chi_A)\| = \|\chi_A\|$. Thus, as H is dense in $L_1(\mu_{\mathfrak{U}})$, $J_{\mu_{\mathfrak{U}}}$ can be extended to an isometry on the whole $L_1(\mu_{\mathfrak{U}})$.

Let us prove now the existence of the projection $P_{\mu_{\mathfrak{U}}}$. For each $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$, let $w_{\mathbf{f}}$ and $m_{\mathbf{f}}$ be respectively the absolutely continuous part and the singular part of $\mu_{\mathbf{f}}$ with respect to $\mu_{\mathfrak{U}}$, and let $g_{\mathbf{f}} \in L_1(\mu_{\mathfrak{U}})$ be the Radon-Nikodym derivative of $w_{\mathbf{f}}$; that is

$$w_{\mathbf{f}}(A) = \int_A g_{\mathbf{f}}\, d\mu_{\mathfrak{U}}, \text{ for all } A \in \sigma(\Sigma^{\mathfrak{U}}).$$

Let $D \colon L_1(\mu)_{\mathfrak{U}} \longrightarrow L_1(\mu_{\mathfrak{U}})$ be the operator that sends every $\mathbf{f}$ to $g_{\mathbf{f}}$. Notice that for each set $A = (A_i)^{\mathfrak{U}} \in \Sigma^{\mathfrak{U}}$, the associated characteristic function $\mathbf{f} = \chi_A$ is sent by $D \circ J_{\mu_{\mathfrak{U}}}$ to itself, because $D \circ J_{\mu_{\mathfrak{U}}}(\chi_A) = D([\chi_{A_i}]) = \chi_{(A_i)^{\mathfrak{U}}}$.

Indeed, for every $C = (C_i)^{\mathfrak{U}} \in \Sigma^{\mathfrak{U}}$,

$$\mu_{\mathbf{f}}(C) = \lim_{\mathfrak{U}} \int_{A_i \cap C_i} d\mu = \lim_{\mathfrak{U}} \mu(A_i \cap C_i) = \mu_{\mathfrak{U}}(A \cap C) = \int_C \chi_A \, d\mu_{\mathfrak{U}}.$$

Thus, if $\mu_{\mathfrak{U}}(C) = 0$, then $\mu_{\mathbf{f}}(C) = 0$. Now, by Proposition A.6.4, we conclude that $\mu_{\mathbf{f}}$ is absolutely continuous with respect to $\mu_{\mathfrak{U}}$. Consequently, $\mu_{\mathbf{f}} = w_{\mathbf{f}}$ and $g_{\mathbf{f}} = \chi_A$. Thus $D \circ J_{\mu_{\mathfrak{U}}}(\chi_A) = \chi_A$, and since H is dense in $L_1(\mu_{\mathfrak{U}})$, it follows that $D \circ J_{\mu_{\mathfrak{U}}}$ is the identity operator on $L_1(\mu_{\mathfrak{U}})$. So $P_{\mu_{\mathfrak{U}}} := J_{\mu_{\mathfrak{U}}} \circ D$ is a projection whose range is $J_{\mu_{\mathfrak{U}}}(L_1(\mu_{\mathfrak{U}}))$. Moreover, since D is norm-one, we get $\|P_{\mu_{\mathfrak{U}}}\| = 1$.

Finally, for every $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$, since $w_{\mathbf{f}} \perp m_{\mathbf{f}}$, we have $\|\mu_{\mathbf{f}}\| = \|w_{\mathbf{f}}\| + \|m_{\mathbf{f}}\|$ or equivalently, $\|\mathbf{f}\| = \|P_{\mu_{\mathfrak{U}}}(\mathbf{f})\| + \|\mathbf{f} - P_{\mu_{\mathfrak{U}}}(\mathbf{f})\|$. Thus,

$$L_1(\mu)_{\mathfrak{U}} = J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big) \oplus_1 N(P_{\mu_{\mathfrak{U}}}).$$ □

The following two theorems characterize the elements in the summands of the decomposition $L_1(\mu)_{\mathfrak{U}} = J_{\mu_{\mathfrak{U}}}(L_1(\mu_{\mathfrak{U}})) \oplus_1 N(P_{\mu_{\mathfrak{U}}})$.

Theorem A.6.6. *An element* $\mathbf{f} \in L_1(\mu)_\mathfrak{U}$ *belongs to* $J_{\mu_\mathfrak{U}}\big(L_1(\mu_\mathfrak{U})\big)$ *if and only if it admits a relatively weakly compact representative. In particular, the canonical copy of* $L_1(\mu)$ *into* $L_1(\mu)_\mathfrak{U}$ *is contained in* $J_{\mu_\mathfrak{U}}\big(L_1(\mu_\mathfrak{U})\big)$.

Proof. Let $\mathbf{f}$ be an element of $L_1(\mu)_\mathfrak{U}$ with a relatively weakly compact representative $(f_i)_{i\in I}$, and take $\varepsilon > 0$. Since $\{f_i\}_{i\in I}$ is equi-integrable, there exists $\delta > 0$ so that $\int_A |f_i|\, d\mu < \varepsilon$ for all $i \in I$ and all $A \in \Sigma$ with $\mu(A) < \delta$. Let $C = (C_i)^\mathfrak{U} \in \Sigma^\mathfrak{U}$ with $\mu_\mathfrak{U}(C) = 0$. Then we have $K := \{i \in I \colon \mu(C_i) < \delta\} \in \mathfrak{U}$; hence

$$\int_{C_i} |f_i|\, d\mu < \varepsilon \quad \text{for all } i \in K.$$

Therefore, $\mu_{|\mathbf{f}|}(C) = \lim_\mathfrak{U} \int_{C_i} |f_i|\, d\mu = 0$. But Proposition A.6.4 yields $\mu_\mathbf{f} \ll \mu_\mathfrak{U}$, thus $m_\mathbf{f} = 0$, hence $\mathbf{f} \in J_{\mu_\mathfrak{U}}\big(L_1(\mu_\mathfrak{U})\big)$.

For the converse, take $\mathbf{f} = [f_i] \in J\big(L_1(\mu_\mathfrak{U})\big)$ with $\|f_i\| = 1$ for all i, and for every $k \in \mathbb{N}$, write

$$\begin{aligned} A_i^k &:= \{t \colon |f_i(t)| > k\}, \\ B_i^k &:= \Omega \setminus A_i^k, \\ f_i^k &:= f_i \cdot \chi_{B_i^k}, \\ \mathbf{f}_k &:= [f_i^k]. \end{aligned}$$

Notice that $\mu(A_i^k) \le 1/k$ and that $\{(A_i^k)^\mathfrak{U}\}_{k=1}^\infty$ is a decreasing sequence in $\Sigma^\mathfrak{U}$ for all i. So, denoting $A := \bigcap_{k=1}^\infty (A_i^k)^\mathfrak{U}$, we get

$$\mu_\mathfrak{U}(A) = \lim_k \lim_{i\to\mathfrak{U}} \mu(A_i^k) \le \lim_k \frac{1}{k} = 0.$$

But by hypothesis, $\mu_{|\mathbf{f}|} \ll \mu_\mathfrak{U}$, so

$$0 = \mu_{|\mathbf{f}|}(A) = \lim_k \mu_{|\mathbf{f}|}\big((A_i^k)^\mathfrak{U}\big) = \lim_k \|\mathbf{f} - \mathbf{f}_k\|.$$

Let $\{I_n\}_{n=1}^\infty$ be a decreasing sequence of elements of $\mathfrak{U}$ such that $\bigcap_{n=1}^\infty I_n = \emptyset$. Write $r_k := \|\mathbf{f} - \mathbf{f}_k\|$, take

$$\begin{aligned} &H_1 := \{i \in I \colon \|f_i - f_i^1\| < 2r_1\} \in \mathfrak{U}, \\ &H_k := I_k \cap H_{k-1} \cap \{i \in I \colon \|f_i - f_i^k\| < 2r_k\} \in \mathfrak{U}, \text{ for } k \ge 2 \end{aligned}$$

and let $J_0 := I \setminus H_1$, and $J_k := H_k \setminus H_{k+1}$ for $k \in \mathbb{N}$.

For every $i \in I$, let n_i be the unique positive integer for which $i \in J_{n_i}$, and let us prove that $[f_i] = [f_i^{n_i}]$. Indeed, given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ so that $2r_k < \varepsilon$ for all $k \ge n$. Thus

$$\{i \in I \colon \|f_i - f_i^{n_i}\| < \varepsilon\} \supset \bigcup_{k=n}^\infty J_k \in \mathfrak{U},$$

hence $\lim_{\mathfrak{U}} \|f_i - f_i^{n_i}\| = 0$; in other words, $[f_i] = [f_i^{n_i}]$.

Now we claim that $\{f_i^{n_i}\}_{i\in I}$ is equi-integrable. Indeed, given $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $r_k < \varepsilon/2$ for all $k \geq n$. Observe that i belongs either to $\cup_{k=1}^{n-1} J_k$ or to H_n. If $i \in \bigcup_{k=1}^{n-1} J_k$, then $|f_i^{n_i}(x)| < n$; hence

$$\int_{\{|f_i^{n_i}|>n\}} |f_i^{n_i}| \, d\mu = 0.$$

If $i \in H_n$, then

$$\int_{\{|f_i^{n_i}|>n\}} |f_i^{n_i}| \, d\mu \leq \int_{\{|f_i|>n\}} |f_i| \, d\mu = \|f_i - f_i^n\| < 2r_n < \varepsilon.$$

Thus $\{f_i^{n_i}\}_{i\in I}$ is equi-integrable; equivalently, it is relatively weakly compact in $L_1(\mu)$. □

Theorem A.6.7. *Consider the canonical projection* $P_{\mu_{\mathfrak{U}}} : L_1(\mu)_{\mathfrak{U}} \longrightarrow L_1(\mu)_{\mathfrak{U}}$. *An element* $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$ *belongs to* $N(P_{\mu_{\mathfrak{U}}})$ *if and only if* $\mathbf{f}$ *admits a representative* $(f_i)_{i\in I}$ *such that* $\lim_{\mathfrak{U}} \mu(\operatorname{supp} f_i) = 0$.

Proof. Let $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$ and assume that $\mu_{\mathfrak{U}}((\operatorname{supp} f_i)^{\mathfrak{U}}) = 0$. Then $\mu_{\mathbf{f}} \perp \mu_{\mathfrak{U}}$, so $w_{\mathbf{f}} = 0$, and consequently, $\mathbf{f} \in N(P_{\mu_{\mathfrak{U}}})$.

For the converse, let $\mathbf{f} = [f_i] \in N(P_{\mu_{\mathfrak{U}}})$. For every $i \in I$, consider the measurable sets $A_i := \{|f_i| < 1\}$, $B_i := \Omega \setminus A_i$, $A := (A_i)^{\mathfrak{U}}$ and $B := (B_i)^{\mathfrak{U}}$, the functions $g_i := f_i \cdot \chi_{A_i}$ and $h_i := f_i \cdot \chi_{B_i}$, and the elements $\mathbf{g} := [g_i]$ and $\mathbf{h} := [h_i]$, so $\mathbf{f} = \mathbf{g} + \mathbf{h}$.

Since $\mathbf{f} \in N(P_{\mu_{\mathfrak{U}}})$, we have $\mu_{\mathbf{f}} \perp \mu_{\mathfrak{U}}$. Moreover, the measures $\mu_{|\mathbf{f}|}$ and $\mu_{|\mathbf{g}|}$ are concentrated in $(\operatorname{supp} f_i)^{\mathfrak{U}}$ and in A respectively. Thus $\mu_{|\mathbf{g}|} \ll \mu_{|\mathbf{f}|}$, hence $\mu_{|\mathbf{g}|} \perp \mu_{\mathfrak{U}}$, which shows that $\mathbf{g} \in N(P_{\mu_{\mathfrak{U}}})$. Besides, since $|g_i(x)| < 1$ for all x and all i, $\{g_i\}_{i\in I}$ is relatively weakly compact. So Theorem A.6.6 shows that $\mathbf{g} \in J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$. Thus $\mathbf{g} = 0$; equivalently, $\mathbf{f} = \mathbf{h}$.

Since $(\operatorname{supp} h_i)^{\mathfrak{U}} \subset B$, in order to finish the proof we only need to show that $\mu_{\mathfrak{U}}(B) = 0$. Note that $\mu_{|\mathbf{f}|}$ is concentrated in a $\mu_{\mathfrak{U}}$-null subset L of B. Moreover, by the definition of B, for every subset $C \in \Sigma^{\mathfrak{U}}$ of $B \setminus L$, we have $\mu_{\mathfrak{U}}(C) \leq \mu_{|\mathbf{f}|}(C) = 0$. Therefore, by Proposition A.6.2, $\mu_{\mathfrak{U}}(B) = \mu_{\mathfrak{U}}(B \setminus L) = 0$. □

Remark A.6.8. Heinrich [99] proved that $L_1(\mu)_{\mathfrak{U}}$ is an L_1-space. His proof is based upon the theorem of Nakano and Bohnenblust [117].

An explicit representation of the projection $P_{\mu_{\mathfrak{U}}}$ can be derived from the following theorem:

Theorem A.6.9. *For each* $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$, *we have* $g_{\mathbf{f}}((t_i)^{\mathfrak{U}}) = \lim_{\mathfrak{U}} f_i(t_i)$ $\mu_{\mathfrak{U}}$-*a.e.*

Proof. In order to prove the statement, it is sufficient to show that

$$\mu_{\mathfrak{U}}\big(\{(t_i)^{\mathfrak{U}} : g_{\mathbf{f}}((t_i)^{\mathfrak{U}}) < \lim_{\mathfrak{U}} f_i(t_i)\}\big) = 0 \tag{A.19}$$

and

$$\text{(A.20)} \qquad \mu_{\mathfrak{U}}\big(\{(t_i)^{\mathfrak{U}} \colon g_{\mathbf{f}}((t_i)^{\mathfrak{U}}) > \lim_{\mathfrak{U}} f_i(t_i)\}\big) = 0.$$

In turn, equalities (A.19) and (A.20) are immediate consequences of the inclusions

$$\text{(A.21)} \qquad \{g_{\mathbf{f}} < c\} \subset (\{f_i < c\})^{\mathfrak{U}} \ \mu_{\mathfrak{U}}\text{-a.e.},$$

$$\text{(A.22)} \qquad (\{f_i \le c\})^{\mathfrak{U}} \subset \{g_{\mathbf{f}} \le c\} \ \mu_{\mathfrak{U}}\text{-a.e.}$$

for all $c \in \mathbb{R}$, which are going to be proved below.

Fix a real number c and, by means of Theorems A.6.5 and A.6.7, consider the decomposition $\mathbf{f} = [w_i] + [v_i]$, where $[w_i] \in J_{\mu_{\mathfrak{U}}}(g_{\mathbf{f}})$, $[v_i] \in N(P_{\mu_{\mathfrak{U}}})$, $f_i = w_i + v_i$ for all i and $\lim_{\mathfrak{U}} \mu_{\mathfrak{U}}(\operatorname{supp} v_i) = 0$.

The last condition implies that

$$(\{f_i < c\})^{\mathfrak{U}} = (\{w_i < c\})^{\mathfrak{U}} \ \mu_{\mathfrak{U}}\text{-a.e. and } (\{f_i \le c\})^{\mathfrak{U}} = (\{w_i \le c\})^{\mathfrak{U}} \ \mu_{\mathfrak{U}}\text{-a.e.}$$

To obtain the inclusion (A.21), since $\{g_{\mathbf{f}} < c\} = \cup_{n=1}^{\infty}\{g_{\mathbf{f}} < c - n^{-1}\}$, it is enough to prove that $\{g_{\mathbf{f}} < c - n^{-1}\} \subset (\{w_i < c\})^{\mathfrak{U}}$ $\mu_{\mathfrak{U}}$-a.e. for each n. By Proposition A.6.4, we only need to show that every $B = (B_i)^{\mathfrak{U}} \in \Sigma^{\mathfrak{U}}$ contained in $\{g_{\mathbf{f}} < c - n^{-1}\} \setminus (\{w_i < c\})^{\mathfrak{U}}$ is $\mu_{\mathfrak{U}}$-null. In order to do so, note that on the one hand,

$$w_{\mathbf{f}}(B) = \int_B g_{\mathbf{f}}\, d\mu_{\mathfrak{U}} \le (c - n^{-1})\mu_{\mathfrak{U}}(B).$$

On the other hand, since $B \subset (\{w_i \ge c\})^{\mathfrak{U}}$, we have

$$w_{\mathbf{f}}(B) = \lim_{\mathfrak{U}} \int_{B_i} w_i\, d\mu \ge c \cdot \mu_{\mathfrak{U}}(B);$$

hence $\mu_{\mathfrak{U}}(B) = 0$, as we wanted to prove.

The proof of inclusion (A.22) is analogous: it is enough to show that every $B = (B_i)^{\mathfrak{U}} \in \Sigma^{\mathfrak{U}}$ contained in $(\{w_i \le c\})^{\mathfrak{U}} \setminus \{g_{\mathbf{f}} < c + n^{-1}\}$ is $\mu_{\mathfrak{U}}$-null.

Thus, on the one hand, $w_{\mathbf{f}}(B) = \int_B g_{\mathbf{f}} \ge (c + n^{-1})\mu_{\mathfrak{U}}(B)$. On the other hand,

$$w_{\mathbf{f}}(B) = \lim_{\mathfrak{U}} \int_{B_i} w_i\, d\mu \le c \cdot \mu_{\mathfrak{U}}(B).$$

So we get $\mu_{\mathfrak{U}}(B) = 0$, concluding the proof. □

Notice that the expression for $g_{\mathbf{f}}$ provided by Theorem A.6.9 works for any representative of $\mathbf{f}$.

Theorem A.6.10. *Let L and $\mathfrak{V}$ be an isometry and an ultrafilter associated with the bidual of $L_1(\mu)$, and let us consider the canonical isometry $J_{\mu_{\mathfrak{V}}}$ and the canonical projection $P_{\mu_{\mathfrak{V}}}$. Then*

$$L\big(L_1(\mu)^{**}\big) = J_{\mu_{\mathfrak{V}}}\big(L_1(\mu)\big) \oplus_1 N,$$

where N is a closed subspace of $N(P_{\mu_{\mathfrak{V}}})$.

Proof. In order to avoid a cumbersome notation, let us denote $P := P_{\mu_{\mathfrak{V}}}$ and $J := J_{\mu_{\mathfrak{V}}}$, and let I be the set of indices corresponding to the ultrafilter $\mathfrak{V}$.

Let us define an operator $W\colon L_1(\mu)_{\mathfrak{V}} \longrightarrow L_1(\mu)$ as follows. Take $\mathbf{f} \in L_1(\mu)_{\mathfrak{V}}$ and, by means of Theorem A.6.6, choose a relatively weakly compact representative $\{f_i\}_{i\in I}$ of $P(\mathbf{f}) \in J\big(L_1(\mu_{\mathfrak{V}})\big)$. Note that for any other representative $\{g_i\}_{i\in I}$ of $P(\mathbf{f})$, we have $w\text{-}\lim_{\mathfrak{V}} f_i = w\text{-}\lim_{\mathfrak{V}} g_i$. So the operator W given by $W(\mathbf{f}) := w\text{-}\lim_{\mathfrak{V}} f_i$ is well defined and $\|W\| = 1$.

Let $\mathcal{J}$ be the canonical inclusion of $L_1(\mu)$ in $L_1(\mu)^{**}$, and consider the operator $S := \mathcal{J}WL$. We shall see that S is a norm-one projection with range $L_1(\mu)$.

$$L_1(\mu)^{**} \xhookrightarrow{\ L\ } L_1(\mu)_{\mathfrak{V}} \xrightarrow{\ W\ } L_1(\mu) \xhookrightarrow{\ \mathcal{J}\ } L_1(\mu)^{**}$$

Indeed, given $f \in L_1(\mu)$, since L and $\mathfrak{V}$ are associated with the bidual of $L_1(\mu)$, we get $L(f) = [f]$, and by Theorem A.6.6, it follows that $[f] \in J\big(L_1(\mu_{\mathfrak{V}})\big)$. Then $f = W([f])$, hence $f = S(f)$.

Thus, $R(S) = L_1(\mu)$ and $S^2(x^{**}) = S(x^{**})$ for all $x^{**} \in L_1(\mu)^{**}$. We have just proved that S is a projection from $L_1(\mu)^{**}$ onto $\mathcal{J}\big(L_1(\mu)\big)$. Moreover, as W and L are norm-one operators, we get $\|S\| = 1$.

In order to finish the proof, it is enough to prove that $N := L\big(N(S)\big)$ is contained in $N(P)$. To do so, take $x^{**} \in N(S)$ and, according to Theorems A.6.5, A.6.6 and A.6.7, consider the decomposition

$$L(x^{**}) = [g_i] + [h_i],$$

where $\{g_i\}_{i\in I}$ is relatively weakly compact, $\lim_{\mathfrak{V}} \mu(\operatorname{supp} h_i) = 0$ and

$$\|x^{**}\| = \lim_{\mathfrak{V}} \|g_i + h_i\| = \lim_{\mathfrak{V}} \|g_i\| + \lim_{\mathfrak{V}} \|h_i\|. \tag{A.23}$$

Since $x^{**} \in N(S)$, it follows that $g_i \xrightarrow[\mathfrak{V}]{w} 0$. Take any $\varepsilon > 0$ and consider $x^* \in S_{L_1(\mu)^*}$ so that $\|x^{**}\| - \varepsilon \leq \langle x^{**}, x^*\rangle$. Thus, since $g_i + h_i \xrightarrow[\mathfrak{V}]{w^*} x^{**}$,

$$\begin{aligned}\|x^{**}\| - \varepsilon \leq \langle x^{**}, x^*\rangle &= \lim_{\mathfrak{V}} \langle g_i + h_i, x^*\rangle \\ &= \lim_{\mathfrak{V}} \langle h_i, x^*\rangle \leq \lim_{\mathfrak{V}} \|h_i\| \leq \|x^{**}\|.\end{aligned}$$

So formula (A.23) gives $\|x^{**}\| = \lim_{\mathfrak{V}} \|h_i\|$ and $[g_i] = 0$, which proves that $L(x^{**}) \in N(P)$. □

The fact that $L_1(\mu)$ L-embeds in its bidual and the subsequence splitting property can be easily derived from the previous results.

Corollary A.6.11. *The Banach space $L_1(\mu)$ is L-embedded in its bidual space; that is, there is a projection $S\colon L_1(\mu)^{**} \longrightarrow L_1(\mu)^{**}$ such that $R(S) = L_1(\mu)$ and, for every $x^{**} \in L_1(\mu)^{**}$,*

$$\|x^{**}\| = \|S(x^{**})\| + \|x^{**} - S(x^{**})\|.$$

Proof. It is enough to consider the projection S given in Theorem A.6.10. In fact, keeping the same notation established in its proof, let us take an element $x^{**} \in L_1(\mu)^{**}$, and consider the corresponding decomposition

$$L(x^{**}) = [f_i] + [h_i]$$

with $[f_i] \in J\big(L_1(\mu_{\mathfrak{V}})\big)$ and $[h_i] \in N(P)$. Thus $f_i \xrightarrow[\mathfrak{V}]{w} f \in L_1(\mu)$, where $f = S(x^{**})$. Since L and $\mathfrak{V}$ are associated with the bidual of $L_1(\mu)$, $f_i + g_i \xrightarrow[\mathfrak{V}]{w^*} x^{**}$. Hence, as L is an isometry,

$$\|x^{**}\| = \lim_{\mathfrak{V}} \|f_i\| + \lim_{\mathfrak{V}} \|h_i\| \geq \|f\| + \|x^{**} - f\|.$$

Therefore, by the triangular inequality, $\|x^{**}\| = \|f\| + \|x^{**} - f\|$; that is, $\|x^{**}\| = \|S(x^{**})\| + \|x^{**} - S(x^{**})\|$. □

The following result is known as the subsequence splitting property.

Corollary A.6.12. *Let (f_n) be a bounded sequence in $L_1(\mu)$, and let x^{**} be a weak*-cluster point of $\{f_n\}_{n=1}^{\infty}$. Then (f_n) has a subsequence (f_{k_n}) such that*

(i) $f_{k_n} = w_n + v_n$ *for all* n*,*

(ii) (w_n) *is weakly convergent,*

(iii) (v_n) *is disjointly supported,*

(iv) $\lim_n \|w_n\| + \lim_n \|v_n\| = \lim_n \|f_{k_n}\|$*,*

(v) $\lim_n \|v_n\| \geq \operatorname{dist}\big(x^{**}, L_1(\mu)\big)$*.*

Proof. As usual, let w stand for the weak topology $\sigma(L_1(\mu), L_1(\mu)^*)$, and let w^* stand for $\sigma(L_1(\mu)^{**}, L_1(\mu)^*)$. For every w^* neighborhood $\mathcal{V}$ of x^{**}, let us set

$$N_{\mathcal{V}} := \{n \in \mathbb{N} \colon x_n \in \mathcal{V}\}.$$

For each pair of w^*-neighborhoods $\mathcal{V}$ and $\mathcal{W}$ of x^{**}, we have $N_{\mathcal{V}\cap\mathcal{W}} \subset N_{\mathcal{V}} \cap N_{\mathcal{W}}$. So the collection of all subsets $N_{\mathcal{V}}$ is a filter basis ordered by the set inclusion; hence it can be extended to an ultrafilter $\mathfrak{U}$ on $\mathbb{N}$. Note that for that ultrafilter,

$$f_n \xrightarrow[n\to\mathfrak{U}]{w^*} x^{**}.$$

Next, consider the element $[f_n] \in L_1(\mu)_{\mathfrak{U}}$. As $L_1(\mu)_{\mathfrak{U}} = J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big) \oplus_1 N(P_{\mu_{\mathfrak{U}}})$, by means of the characterizations given in Theorems A.6.6 and A.6.7, we may write

$$[f_n] = [b_n] + [h_n] \tag{A.24}$$

where $[b_n] \in J_{\mu_\mathfrak{U}}\big(L_1(\mu_\mathfrak{U})\big)$, $[h_n] \in N(P_{\mu_\mathfrak{U}})$, $\{b_n\}_{n=1}^\infty$ is relatively weakly compact, and

$$\mu(\operatorname{supp} h_n) \xrightarrow[\mathfrak{U}]{} 0. \tag{A.25}$$

Let $d_n := f_n - b_n - h_n$. Then $\|d_n\| \xrightarrow[n]{} 0$ and, denoting $g_n := b_n + d_n$, we get

$$f_n = g_n + h_n \quad \text{for all } n.$$

Since $\{b_n\}_{n=1}^\infty$ is relatively weakly compact, there exists $g := w\text{-}\lim_\mathfrak{U} g_n$. Therefore, $h_n \xrightarrow[\mathfrak{U}]{w^*} x^{**} - g$. Thus, denoting $h^{**} := x^{**} - g$, we get

$$\lim_\mathfrak{U} \|h_n\| \geq \|h^{**}\| \geq \operatorname{dist}\big(x^{**}, L_1(\mu)\big). \tag{A.26}$$

Moreover, equality (A.24) yields

$$\lim_\mathfrak{U} \|f_n\| = \lim_\mathfrak{U} \|g_n\| + \lim_\mathfrak{U} \|h_n\|. \tag{A.27}$$

Let (δ_n) be a decreasing sequence of positive real numbers so that, if $\mu(D) < \delta_n$, then

$$\int_D |h_n|\, d\mu < \frac{1}{n}. \tag{A.28}$$

Hence, formulas (A.25), (A.27) and (A.28) allow us to get recursively the decreasing subset sequence $\{J_n\}_{n=1}^\infty \subset \mathfrak{U}$ given as follows:

$$\begin{aligned} J_n := \big\{k > k_n \colon & \mu(\operatorname{supp} h_k) < 2^{-n}\delta_{k_n}, \\ & \|h_k\| > \|h^{**}\| - 1/n,\ \big|\|f_k\| - \|g_k\| - \|h_k\|\big| < 1/n\big\}, \end{aligned}$$

where $k_1 := 1$ and $k_{n+1} := \min J_n$.

For every $n \in \mathbb{N}$, write

$$\begin{aligned} D_n &:= \bigcup_{i=n}^\infty \operatorname{supp} h_{k_i}, \\ w_n &:= g_{k_n} + h_{k_n} \cdot \chi_{D_{n+1}}, \\ v_n &:= h_{k_n} - h_{k_n} \cdot \chi_{D_{n+1}}. \end{aligned}$$

Since

$$\mu(D_{n+1}) \leq \sum_{i=n+1}^\infty \mu(\operatorname{supp} h_{k_i}) \leq \sum_{i=n+1}^\infty \frac{\delta_{k_n}}{2^{i-1}} \leq \delta_{k_n},$$

it follows from (A.28) that

$$\|h_{k_n} \cdot \chi_{D_{n+1}}\| \leq \frac{1}{k_n} \leq \frac{1}{n} \xrightarrow[n]{} 0; \tag{A.29}$$

hence $\{w_n\}_{n=1}^{\infty}$ is relatively weakly compact and

$$f_{k_n} = w_n + v_n \quad \text{for all } n \in \mathbb{N}, \tag{A.30}$$

$$v_n \cdot v_m = 0 \ \mu\text{-a.e. if } n \neq m, \tag{A.31}$$

$$\lim_n \|f_{k_n}\| = \lim_n \|w_n\| + \lim_n \|v_n\|. \tag{A.32}$$

Besides, as a consequence of (A.29),

$$\begin{aligned} \lim_n \|v_n\| &\geq \lim_n \|h_{k_n}\| - \lim_n \|h_{k_n} \cdot \chi_{D_{k_n+1}}\| \\ &\geq \|h^{**}\| \geq \operatorname{dist}\big(x^{**}, L_1(\mu)\big). \end{aligned} \tag{A.33}$$

Thus, after formulas (A.30), (A.31), (A.32) and (A.33), we have a subsequence (f_{k_n}) of (f_n), a countable, relatively weakly compact subset $\{w_n\}_{n=1}^{\infty}$ and a disjointly supported sequence (v_n) such that $\lim_n \|f_{k_n}\| = \lim_n \|w_n\| + \lim_n \|v_n\|$ and $\lim_n \|v_n\| \geq \operatorname{dist}\big(x^{**}, L_1(\mu)\big)$. Passing to a subsequence if necessary, we may assume by the Eberlein-Smulian theorem that (w_n) is weakly convergent. So statements (i), (ii), (iii), (iv) and (v) hold, and the proof is finished. □

Remark A.6.13. The subsequence splitting property for L_1 spaces was originally obtained by Kadec and Pełczyński [111]. Independently, Rosenthal proved this result, but his proof was never published (see reference [11] in [39]).

We end this section with the following result.

Corollary A.6.14. *Every reflexive subspace of $L_1(\mu)$ is super-reflexive.*

Proof. Let $\mathfrak{U}$ be any ultrafilter on a set I, and consider the natural isometry $F\colon L_1(\mu_{\mathfrak{U}\times\mathfrak{U}}) \longrightarrow L_1\big((\mu_{\mathfrak{U}})_{\mathfrak{U}}\big)$ that maps every $\chi_{(A_{ij})^{\mathfrak{U}\times\mathfrak{U}}}$ to $\chi_{((A_{ij})^{\mathfrak{U}})^{\mathfrak{U}}}$. By Proposition A.6.3, the completions of the measure spaces $(\Omega^{\mathfrak{U}\times\mathfrak{U}}, \sigma(\Sigma^{\mathfrak{U}\times\mathfrak{U}}), \mu_{\mathfrak{U}\times\mathfrak{U}})$ and $\big((\Omega^{\mathfrak{U}})^{\mathfrak{U}}, \sigma(\sigma(\Sigma^{\mathfrak{U}})^{\mathfrak{U}}), (\mu_{\mathfrak{U}})_{\mathfrak{U}}\big)$ are isomorphic; hence F is surjective.

Moreover, the operator G from $L_1(\mu)_{\mathfrak{U}\times\mathfrak{U}}$ onto $\big(L_1(\mu)_{\mathfrak{U}}\big)_{\mathfrak{U}}$ that maps every $[f_{ij}]$ to $\big[[f_{ij}]_j\big]_i$ is also a surjective isometry by virtue of Proposition A.4.7.

Let J_1 and J_2 denote the isometries $J_{\mu_{\mathfrak{U}\times\mathfrak{U}}}$ and $J_{(\mu_{\mathfrak{U}})_{\mathfrak{U}}}$ defined as in Theorem A.6.5. Thus we get the next commutative diagram:

$$\begin{array}{ccc} L_1(\mu_{\mathfrak{U}\times\mathfrak{U}}) & \xrightarrow{\ F\ } & L_1\big((\mu_{\mathfrak{U}})_{\mathfrak{U}}\big) \\ J_1 \Big\downarrow & & \Big\downarrow J_2 \\ L_1(\mu)_{\mathfrak{U}\times\mathfrak{U}} & \xrightarrow[\ G\ ]{} & \big(L_1(\mu)_{\mathfrak{U}}\big)_{\mathfrak{U}} \end{array}$$

Let E be a reflexive subspace of $L_1(\mu)$. By Theorem A.6.6, $E_{\mathfrak{U}\times\mathfrak{U}}$ is contained in $J_1\big(L_1(\mu_{\mathfrak{U}\times\mathfrak{U}})\big)$. Therefore, since $J_2F = GJ_1$, it follows that $(E_{\mathfrak{U}})_{\mathfrak{U}} \subset J_2\big(L_1\big((\mu_{\mathfrak{U}})_{\mathfrak{U}}\big)\big)$. Thus, a second application of Theorem A.6.6 proves that $E_{\mathfrak{U}}$ is reflexive; hence E is super-reflexive. □

Remark A.6.15. A proof of Corollary A.6.14 can be derived from some results in [143], [122] and [58].

Indeed, [143] shows that every closed subspace of $L_1(\mu)$ either contains a subspace isomorphic to ℓ_1 or it is isomorphic to a closed subspace of $L_p(\mu)$ for some $p \in (1, 2]$. In [122], there is a proof of the uniform convexity of $L_p(\mu)$ for all $p \in (1, \infty)$. And finally, Enflo demonstrates in [58] that any Banach space is super-reflexive if and only if it is isomorphic to a uniformly convex Banach space. Since super-reflexivity is a hereditary property, it is evident that a combination of the three mentioned results proves that every reflexive subspace of $L_1(\mu)$ is super-reflexive.

Obviously, the extent of the three cited results goes far beyond the study of tauberian operators. Thus, the proof of Corollary A.6.14 via ultraproducts is sufficient to fulfill the purpose of this book.

Final remarks

All the results in this section are derived from the Lebesgue decomposition theorem and the Radon-Nikodým theorem. Indeed, all of them are deduced from Theorem A.6.5, due to Heinrich [99], who applies the Lebesgue decomposition theorem to prove that

$$L_1(\mu)_{\mathfrak{U}} = J_{\mu_{\mathfrak{U}}}(L_1(\mu_{\mathfrak{U}})) \oplus_1 N(P_{\mu_{\mathfrak{U}}})$$

and uses the Radon-Nikodým derivative to give a representation of the embedding $J_{\mu_{\mathfrak{U}}}$ of $L_1(\mu_{\mathfrak{U}})$ into $L_1(\mu)_{\mathfrak{U}}$. Next, Theorem A.6.6 provides a characterization for the elements of $J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$ and, as a sequel, Theorem A.6.7 does the same for the elements of $N(P_{\mu_{\mathfrak{U}}})$.

Theorem A.6.6 was proved by Weis only for ultrafilters on $\mathbb{N}$ [165], and was extended to all ultrafilters in [82]. The proofs of Theorems A.6.6 and A.6.7, as well as the representation for $J_{\mu_{\mathfrak{U}}}$ given in Theorem A.6.9, have been borrowed from [82].

It is remarkable that the subsequence splitting property allows us to recover the theorems of Lebesgue decomposition and Radon-Nikodym.

Indeed, for the sake of simplicity, let us restrict ourselves to the case of the space $\mathcal{M}$ of the Radon measures on $[0, 1]$. For every $\lambda \in C[0, 1]^*$, we may take a sequence (f_i) in $L_1(\mu)$ so that $f_i \xrightarrow[i]{w^*} \lambda$. By the subsequence splitting property, there is a subsequence (f_{i_k}) such that $f_{i_k} = g_k + h_k$, where (g_k) is a weakly convergent to a function $g \in L_1(\mu)$, and (h_k) is disjointly supported. Let λ_c be the measure generated by g, that is, $d\lambda_c = g\,d\mu$. Thus, (h_i) converges in the weak*-topology to $\lambda_s := \lambda - \lambda_c$, which is singular with respect to μ since (h_i) is disjointly supported. Obviously, λ_c is absolutely continuous with respect to μ, so $\lambda = \lambda_c + \lambda_s$ is the Lebesgue decomposition of λ, and g is the Radon-Nikodym derivative of the absolutely continuous part λ_c.

Recall that a *selective ultrafilter* $\mathfrak{W}$ is an ultrafilter on a countable set N satisfying that, for every countable partition $\{I_n\}_{n=1}^{\infty}$ of N disjoint with $\mathfrak{W}$, there is a countable collection of subsets $\{A_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{W}$, $\operatorname{card} A_n \leq 1$ and $A_n \subset I_n$ for all n.

The existence of selective ultrafilters on $\mathbb{N}$ is undecidable in the Zermelo-Fraenkel set theory, but it holds if some additional axiom, like the Continuum Hypothesis or the Martin axiom, is accepted [38].

When $\mathfrak{U}$ is a selective ultrafilter, Theorems A.6.6 and A.6.7 admit the following variant:

Theorem A.6.16. *Given a selective ultrafilter* $\mathfrak{U}$ *on a set* N, μ *a finite measure with no atoms and* $\mathbf{f}$ *an element of* $L_1(\mu)_{\mathfrak{U}}$, *the following statements hold:*

(i) $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$ *belongs to* $J_{\mu_{\mathfrak{U}}}\big(L_1(\mu_{\mathfrak{U}})\big)$ *if and only if any of its representatives is weakly convergent following* $\mathfrak{U}$*;*

(ii) $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$ *belongs to* $N(P_{\mu_{\mathfrak{U}}})$ *if and only if it admits a disjointly supported representative.*

Although the characterizations given in Theorem A.6.16 are very manageable, they fail if $\mathfrak{U}$ is not selective [82]. In particular, after Theorem A.6.7, it is straightforward that statement (ii) in Theorem A.6.16 does not make sense if the set N of indices is uncountable and every element of $\mathfrak{U}$ is also uncountable.

At this point, it should be noticed that there are some remarkable ultrafilters that are not selective. For instance, if $\mathfrak{U}$ and $\mathfrak{V}$ are a pair of ultrafilters on a countable set, then $\mathfrak{U} \times \mathfrak{V}$ is not selective. Besides, if L is an isometry and $\mathfrak{V}$ is an ultrafilter on I associated with the bidual of $L_1(\mu)$, then $\mathfrak{V}$ is not selective because none of its elements is countable. Indeed, if $\mathfrak{V}$ had some countable element, then $\operatorname{card} L_1(\mu)_{\mathfrak{V}} = 2^{\omega_0}$ because of the separability of $L_1(\mu)$, while $\operatorname{card} L_1(\mu)^{**} = 2^{2^{\omega_0}}$, which denies the existence of any isometry from $L_1(\mu)^{**}$ into $L_1(\mu)_{\mathfrak{V}}$.

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