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THE ADAPTED COMPLEXIFICATION OF THE TWO-SPHERE WITH A LIOUVILLE METRIC

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Abstract

We show that the two-sphere with a Riemannian metric that is Liouville with finite isometry group does not admit an unbounded adapted complexification in the sense of Lempert and Szőke and of Guillemin and Stenzel; that is, its Grauert tube cannot have infinite radius. We prove this by first extending a classical theorem valid for umbilical geodesics in a triaxial ellipsoid to general Liouville metrics. Furthermore, we derive an isometric rigidity result for the Monge–Ampère foliation of a two-dimensional Grauert tube with infinite radius.

1. Introduction

The geodesic flow of a Riemannian metric on the two-sphere $(\mathbf{S}^2, \mathbf{g})$ is said to be *integrable* if there is a smooth function $F\colon T^*\mathbf{S}^2 \to \mathbb{R}$ on the cotangent bundle, invariant by the co-geodesic flow, with $\mathrm{d}F \wedge \mathrm{d}H \neq 0$ in a dense subset of $T^*\mathbf{S}^2$, with H the Hamiltonian induced by $\mathbf{g}$. It is well known that an F as above and which is homogeneous quadratic in the cotangent fibers exists if and only if $\mathbf{g}$ can be put in Liouville form $\mathbf{g} = (f(x) + g(y))(\mathrm{d}x^2 + \mathrm{d}y^2)$ for coordinates (x, y) defined in a dense subset of $\mathbf{S}^2$ (see [**4**, **7**, **11**]). Such $\mathbf{g}$ is called a *Liouville metric*.

Consider a Liouville metric $\mathbf{g}$ with F given as above and such that F is not a constant linear combination of H and the square of a function that is both invariant by geodesic flow and linearly homogeneous in the cotangent fibers. Then (see [**4**, **11**] and Section 2) there are exactly four 'branch' points where F is proportional to H along the cotangent fibers. These points are contained in the fixed-point set, Σ_0, of an isometric involution σ^{Σ_0}. Another involution of $\mathbf{S}^2$, the 'antipodal involution' σ', which is an isometry if the metric $\mathbf{g}$ is real analytic, divides these branch points into two 'antipodal' pairs. With these facts we now describe the results in this paper.

We first prove in Theorem 5.1 that, given a branch point $\mathbf{p}$, there is a constant $C > 0$ such that for any geodesic γ passing through $\mathbf{p}$ and not supported by Σ_0, and for one of the two possible orientations of γ, the angle ω_k formed by γ with the curve Σ_0 the kth instance they meet follows the rule

$$\tan^2\left(\frac{\omega_k}{2}\right) = C^k \tan^2\left(\frac{\omega_0}{2}\right),$$

while for the other orientation of γ the constant C^{-1} applies.

This is an extension to Liouville metrics of a classical theorem of Hart for a triaxial ellipsoid (see [**2**, **9**, **17**]), $\{\sum_{i=1}^{3} x_i^2/a_i = 1\} \subset \mathbb{R}^3$, with $0 < a_1 < a_2 < a_3$, whose geodesic flow was explained by Jacobi in the nineteenth century. (In a triaxial ellipsoid such C exists and moreover $C \neq 1$. But, in general we may have $C = 1$ if the geodesics through $\mathbf{p}$ are closed.)

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Our interest in an extension of the Hart theorem to general Liouville metrics in $\mathbf{S}^2$ originates from [2] where we use the classical result to show that in a triaxial ellipsoid the adapted complexification in the sense of Lempert and Szőke [15] and Guillemin and Stenzel [8] cannot have infinite radius. We recall what this means.

Let X be a Stein manifold of dimension n over $\mathbb{C}$ with a smooth strictly plurisubharmonic exhaustion function $u : X \to \mathbb{R}$ such that off $M := u^{-1}(0)$ the complex homogeneous Monge–Ampère equation $(\partial\bar\partial\sqrt{u})^n = 0$ holds. For $r > 0$, $X_r := u^{-1}([0, r)) \subset X$ is a *Grauert tube* of radius r, which is regarded as a canonical complexification of the Riemannian manifold $(M, \mathbf{g})$, called the *center*, where $\mathbf{g}$ is the restriction of the Kähler metric induced from $\partial\bar\partial u$ and the complex structure on X. Here, the Riemannian metric $\mathbf{g}$ must be real analytic [14].

Conversely, given a compact real analytic Riemannian manifold $(M, \mathbf{g})$ there are an $r > 0$ and a unique complex structure $\mathbf{J}$ on $\mathcal{U}_r := E^{-1}([0, r)) \subset TM$, where E is the energy function on TM, such that $2E$ has the properties of the function u above. Such $\mathbf{J}$ is the *adapted complex structure* in $\mathcal{U}_r$ associated to $(M, \mathbf{g})$ (see [15, 18]).

DEFINITION 1.1 The adapted complex structure is said to have infinite radius if and only if it is defined on the entire tangent bundle TM. In this case the Grauert tube with center $(M, \mathbf{g})$ is said to be unbounded or to have infinite radius.

The question of which Riemannian manifolds admit adapted complexifications with infinite radius was raised in [15, 18], and is motivated by the problem of classifying Stein manifolds by exhaustion functions as in the case of $\mathbb{C}^n$ (see [6]). First examples of Grauert tubes with infinite radius and of dimension at least 2 are those whose centers are compact symmetric spaces [16] and certain metrics of revolution in $\mathbf{S}^2$ (see [18]); further examples are constructed by isometric actions [3].

It is known that in dimension two only the two-torus, the Klein bottle, $\mathbf{S}^2$ or the projective space $\mathbb{RP}^2$ can support a metric with adapted complexification with infinite radius [15]. But while it was established in [15] that in the two-torus and the Klein bottle only flat metrics have such unbounded complexifications, in $\mathbf{S}^2$ or $\mathbb{RP}^2$ the question is not yet settled.

It was shown by Szőke [18] that the set of all the metrics of revolution in $\mathbf{S}^2$ that have adapted complexification with infinite radius is a certain family depending on two real parameters that includes the Euclidean sphere.[1] Thus, it remains to study the case of metrics in $\mathbf{S}^2$ with finite isometry group.

Concerning this problem, using Theorem 5.1 we extend the result for a triaxial ellipsoid shown in [2]: we prove in Theorem 7.1 that a Liouville Riemannian metric with finite isometry group in $\mathbf{S}^2$ cannot have an adapted complexification with infinite radius (and as a consequence the only Liouville metrics with such complexifications are the metrics of revolution by Szőke mentioned earlier) and similarly for $\mathbb{RP}^2$, by Corollary 7.2.

The proof of Theorem 7.1 is based on the following. On the one hand the Jacobi field induced by X_F along a geodesic γ through a branch point vanishes only at branch points if γ is generic (that is, not supported by Σ_0) and vanishes identically if γ is otherwise, as explained in Lemma 3.9. This has implications, if the constant $C \neq 1$, for the length of the covariant derivative of J_γ as described in Theorem 6.1, and especially Corollary 6.2.

On the other hand, for any Riemannian manifold with adapted complexification with infinite radius, if a Jacobi field along a geodesic vanishes periodically the covariant derivative satisfies a

[1]The proof in [18] refers to surfaces of revolution in $\mathbb{R}^3$, but it can be easily adjusted to metrics in $\mathbf{S}^2$ with $\mathbf{S}^1$-symmetry that are not assumed *a priori* to be isometrically embedded in $\mathbb{R}^3$.

certain growth condition described by Proposition 7.3 which, when $C \neq 1$, is incompatible with the growth in Corollary 6.2.

In addition, the symmetry property of any Jacobi field vanishing periodically in the presence of an adapted complexification with infinite radius given in Proposition 7.4 is incompatible with the finiteness of the isometry group of $\mathbf{g}$, dealing with the case $C = 1$.

Finally, applying Theorem 7.1 we prove Theorem 8.1 which shows that the Kähler geometry of a Grauert tube of infinite radius and of complex dimension two is determined by the leaves, as sets, of the Monge–Ampère foliation. The proof uses the global version of a classical theorem of Dini [**19**] relating the existence of geodesically equivalent metrics with the existence of an F as above.

Sections 2 and 3 contain background material with proofs included, since the set-up is needed for the following sections; our main references for this part are [**4**, **11**].

2. Background

2.1. *Integrability and the induced Jacobi field*

Let $\pi: T^*\mathbf{S}^2 \to \mathbf{S}^2$ be the cotangent bundle of $\mathbf{S}^2$, and $\mathrm{d}\theta$ the canonical symplectic two-form on $T^*\mathbf{S}^2$, with $\theta = \sum p_i \mathrm{d}q_i$ in canonical local coordinates.

For any function $G: T^*\mathbf{S}^2 \to \mathbb{R}$ let X_G be the vector field in $T^*\mathbf{S}^2$ defined by

$$\mathrm{d}\theta(X_G, \cdot) = -\mathrm{d}G; \tag{1}$$

this is the symplectic gradient or Hamiltonian vector field of G (see [**1**, **10**]).

Give $\mathbf{S}^2$ the Riemannian metric $\mathbf{g}$ and consider the Hamiltonian $H: T^*\mathbf{S}^2 \to \mathbb{R}$ with $H = E \circ \lambda$, where $E: T\mathbf{S}^2 \to \mathbb{R}$ is the energy function with $2E(z) = \mathbf{g}(z, z) = \|z\|^2$, and

$$\lambda: T\mathbf{S}^2 \to T^*\mathbf{S}^2 \tag{2}$$

the Legendre transformation defined by $\lambda z = \mathbf{g}(z, \cdot)$ for all $z \in T\mathbf{S}^2$.

The flow of X_H is known as the co-geodesic flow. By (1) it leaves invariant the level sets of H, and the trajectories on the unit cotangent bundle $\{H = \frac{1}{2}\}$ projected on $\mathbf{S}^2$ via π are the unit-speed geodesics on $\mathbf{S}^2$ (see [**10**]).

PROPOSITION 2.1 *Let $F: T^*\mathbf{S}^2 \to \mathbb{R}$ be any function which is homogeneous quadratic along the fibers of $\pi: T^*\mathbf{S}^2 \to \mathbf{S}^2$. Then for all $z^* \in T^*\mathbf{S}^2$*

$$\mathbf{g}(\lambda^{-1}z^*, \pi_*(X_F|_{z^*})) = 2F(z^*). \tag{3}$$

Proof. Let Ξ be the vector field on $T^*\mathbf{S}^2$ generated by fiber-wise scaling, $\Xi = \sum p_i \partial/p_i$ in canonical coordinates; $\mathrm{d}\theta(\Xi, \cdot) = \theta$. By (1) and (2)

$$\begin{aligned} \mathbf{g}(\lambda^{-1}z^*, \pi_*(X_F|_{z^*})) &= z^*(\pi_* X_F) \\ &= \theta(X_F|_{z^*}) \\ &= \mathrm{d}\theta(\Xi|_{z^*}, X_F|_{z^*}) \\ &= \mathrm{d}F(\Xi|_{z^*}) \\ &= 2F(z^*), \end{aligned} \tag{4}$$

where we used in the last equality that F is a fiber-wise homogeneous quadratic.

Assume that the geodesic flow of the metric $\mathbf{g}$ in $\mathbf{S}^2$ is integrable. This means that there is a function $F\colon T^*\mathbf{S}^2 \to \mathbb{R}$, smooth and with $\mathrm{d}F \wedge \mathrm{d}E \neq 0$ on a dense subset of $T^*\mathbf{S}^2$, so that the Poisson bracket vanishes, $\mathrm{d}\theta(X_F, X_H) \equiv 0$, and thus the Lie bracket vanishes, $[X_F, X_H] = 0$, and the flows commute, $[\Phi^F, \Phi^H] = 0$.

Given $z^* \in \{H = \frac{1}{2}\} \subset T^*\mathbf{S}^2$ the map $\Psi\colon (-a, a) \times \mathbb{R} \to \mathbf{S}^2$, for some $a > 0$,

$$\Psi(t, s) := \pi(\Phi_t^F \Phi_s^H z^*),$$

is a variation of the geodesic γ defined by $s \mapsto \gamma(s) = \pi(\Phi_s^H z^*)$ via the geodesics γ_t given by $s \mapsto \gamma_t(s) := \pi(\Phi_t^F \Phi_s^H z^*)$; here $\gamma \equiv \gamma_0$. Then we have the following.

DEFINITION 2.2 Let $F\colon T^*\mathbf{S}^2 \to \mathbb{R}$ be invariant by the co-geodesic flow. For a unit-speed geodesic γ, $\gamma(s) = \pi(\Phi_s^H z^*)$, let J_γ be the Jacobi field along γ

$$J_\gamma(s) := \Psi_*\left(\frac{\partial}{\partial t}|_{t=0}\right) = \pi_*\left(X_F|_{\Phi_s^H z^*}\right). \tag{5}$$

PROPOSITION 2.3 *Let F be as in Definition 2.2 and, in addition, homogeneous quadratic along the cotangent fibers. For $z^* \in \{H = \frac{1}{2}\}$ let γ be the unit-speed geodesic $\gamma(s) = \pi(\Phi_s^H z^*)$. Then for all $s \in \mathbb{R}$*

$$\mathbf{g}(\dot\gamma(s), J_\gamma(s)) = 2F(z^*). \tag{6}$$

In particular, the left-hand side is a constant independent of s.

Proof. For all $s \in \mathbb{R}$, $\lambda\Phi_s^H z^* = \dot\gamma(s)$. Use Proposition 2.1 and the Φ^H-invariance of F.

REMARK 2.4 The vector field J_γ may be $\equiv 0$ along a geodesic γ, as the lift of γ may live where $\mathrm{d}E \wedge \mathrm{d}F = 0$. For instance, $J_\gamma \equiv 0$ along γ supported by Σ_0 in Lemma 3.9.

2.2. *Liouville metrics on* $\mathbf{S}^2$ *after* [**4**, **11**]

We describe smooth and real analytic Liouville metrics in $\mathbf{S}^2$ with $F \neq aF_l^2 + bH$ for $a, b \in \mathbb{R}$ and F_l homogeneous linear in the cotangent fiber. We follow [**4**]; see also [**11**].

Fix $0 < L \in \mathbb{R}$ and consider the lattice Γ on $\mathbb{C} = \{(x, y) \equiv x + \sqrt{-1}y \mid x, y \in \mathbb{R}\}$

$$\Gamma := \{(k, lL) \in \mathbb{C} \mid (k, l) \in \mathbb{Z} \times \mathbb{Z}\}.$$

The Weierstrass function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

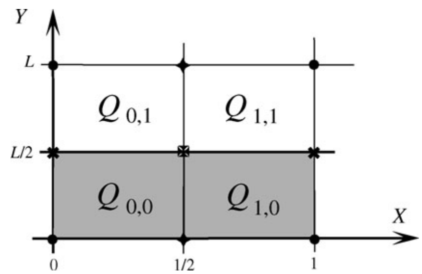

Figure 1. The shaded region covers $\mathbf{S}^2$ with branch points corresponding to $\frac{1}{2}\Gamma$.

is a meromorphic map with poles in Γ. Identify $\mathbb{C} \cup \infty \simeq \mathbf{S}^2$ to get the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\wp} & \mathbb{C} \cup \infty \simeq \mathbf{S}^2 \\ \downarrow{\scriptstyle \pi_\Gamma} & \nearrow{\scriptstyle \text{2-to-1}} & \\ \mathbb{C}/\Gamma \simeq T^2 & & \end{array} \tag{7}$$

with π_Γ induced by $\mathbb{C} \ni z \mapsto -z$; here, '2-to-1' is a branched double cover.

DEFINITION 2.5 $Q_{0,0} := \{(x, y) \in \mathbb{C} \mid 0 \leq x \leq 1/2,\ 0 \leq y \leq L/2\}$, $Q := Q_{0,0} \cup Q_{1,0}$,

$$Q_{k,l} := Q_{0,0} + (k/2, lL/2), \quad (k, l) \in \mathbb{Z} \times \mathbb{Z}. \tag{8}$$

The restriction of $\wp$ to $Q = Q_{0,0} \cup Q_{1,0}$ is onto, and one-to-one when restricted to $Q \setminus \partial Q$, where '∂' indicates boundary (see Fig. 1).

DEFINITION 2.6 Consider the set

$$\Sigma_0 := \wp(\partial Q_{0,0}) = \wp(\partial Q_{1,0}). \tag{9}$$

DEFINITION 2.7 The branch points of $\wp$ in $\mathbf{S}^2$ are $\{\mathbf{p}_1, \ldots, \mathbf{p}_4\} = \wp(\frac{1}{2}\Gamma) \subset \Sigma_0$,

$$\begin{aligned} \mathbf{p}_1 &:= \wp((0,0)), \quad \mathbf{p}_2 := \wp((1/2, 0)), \\ \mathbf{p}_3 &:= \wp((1/2, L/2)), \quad \mathbf{p}_4 := \wp((0, L/2)). \end{aligned} \tag{10}$$

We describe a class of metrics $\mathbf{g}$ on $\mathbf{S}^2$ (*Kolokol'tsov metrics* in [4]; see also [11]).

- *Data for* **g** *smooth*: On the complex plane $\mathbb{C}$ take the symmetric two-form

$$\hat{\mathbf{g}} = (f(x) + g(y))(\mathrm{d}x^2 + \mathrm{d}y^2), \tag{11}$$

where f and g are smooth functions satisfying $\forall x, y \in \mathbb{R}$

$$f(x) = f(x+1) = f(-x) \geq 0, \tag{12}$$

$$g(y) = g(y+L) = g(-y) \geq 0, \tag{13}$$

$$f^{-1}(0) = \left\{\frac{k}{2} \mid k \in \mathbb{Z}\right\}, \quad g^{-1}(0) = \left\{\frac{k}{2}L \mid k \in \mathbb{Z}\right\}. \tag{14}$$

In addition, there is a smooth function $h = h(u)$, $u \in \mathbb{R}$, defined near $u = 0$ with $h(0) = 0$ and $\mathrm{d}h/\mathrm{d}u|_{u=0} \neq 0$ such that for all $k \in \mathbb{Z}$ and small t

$$f\left(t + \frac{k}{2}\right) = h(t^2), \quad g\left(t + \frac{k}{2}L\right) = -h(-t^2). \tag{15}$$

- *Data for* **g** *real analytic*: Take a symmetric two-form in $\mathbb{C}$ as in (11) with f and g now real analytic, also satisfying (12) through (15). The real analyticity and (15) implies the additional $\frac{1}{2}\Gamma$-periodicity (see [**4**, p. 462]) $\forall x, y \in \mathbb{R}$

$$f(x) = f\left(x + \frac{1}{2}\right), \quad g(y) = g\left(y + \frac{L}{2}\right). \tag{16}$$

DEFINITION 2.8 Via $\wp$ the symmetric two-form $\hat{\mathbf{g}}$ on $\mathbb{C}$ in (11) induces a Riemannian metric on $\mathbf{S}^2$ denoted by $\mathbf{g}$ and satisfying $\wp^*\mathbf{g} = \hat{\mathbf{g}}$.

By the definitions $\wp\colon \left(\mathbb{C} \setminus \frac{1}{2}\Gamma,\ \hat{\mathbf{g}}\right) \to \left(\mathbf{S}^2 \setminus \{\mathbf{p}_1, \ldots, \mathbf{p}_4\},\ \mathbf{g}\right)$ is a local isometry and provides Liouville coordinates (x, y) in a dense subset of $\mathbf{S}^2$:

$$\mathbf{g} = (f(x) + g(y))(\mathrm{d}x^2 + \mathrm{d}y^2). \tag{17}$$

Then, by [**4**, Theorem 11.21; **11**, § 3], the metric $\mathbf{g}$ so constructed has an integrable geodesic flow with a quadratic F, where $F \neq aF_l^2 + bH$ for $a, b \in \mathbb{R}$ and F_l homogeneous linear in the cotangent fibers.

Conversely, a Liouville metric with a homogenous quadratic F (normalized so that $F = 0$ at the branch points) and $F \neq aF_l^2 + bH$ for $a, b \in \mathbb{R}$ and F_l homogeneous linear is isometric to $\mathbf{g}$ as above for appropriate choices of f, g and L.

REMARK 2.9 Note that $\mathbf{g}$ described above has an $F \neq aF_l^2 + bH$ for $a, b \in \mathbb{R}$ and F_l co-geodesic flow invariant and homogeneous linear. However this does not imply that no such linear F_l exists.[2]

[2] In a triaxial ellipsoid the metric $\mathbf{g}$, which has a finite isometry group, is written as above in terms of elliptic coordinates and the resulting $F \neq aF_l^2 + bH$; the branch points determined by F are inherently special for $\mathbf{g}$ since they are the four umbilical points. In contrast the standard metric in $\mathbf{S}^2$ is written in Liouville form in the sphero-conical coordinates λ_1, λ_2 ([**4**, p. 539]) which for $(x_1, x_2, x_3) \in \mathbf{S}^2$ are the roots of $x_1^2/(a+\lambda) + x_2^2/(b+\lambda) + x_3^2/(c+\lambda) = 0$, where $a < b < c$. The four branch points are located at $(s_1\sqrt{(b-a)/(c-a)}, 0, s_2\sqrt{(c-b)/(c-a)})$, $s_1, s_2 \in \{1, -1\}$, which lie in $\Sigma_0 = \{x_2 = 0\} \cap \mathbf{S}^2$. A rotation in $\mathbb{R}^3$ will locate Σ_0 in any great circle; so here the four branch points determined by F are special in relation to the chosen coordinate system.

ASSUMPTION 1 Henceforth $\mathbf{S}^2$ is endowed with the Riemannian metric $\mathbf{g}$ which is Liouville with a fixed F with $F \neq aF_l^2 + bH$ for $a, b \in \mathbb{R}$ and F_l a function homogeneous linear in the cotangent fibers; $\mathbf{g}$ is assumed as given in (17) for some f, g and L as above. Moreover, throughout $\mathbf{g}$ will be assumed smooth, or real analytic as indicated.

2.3. *Involutions σ^{Σ_0} and σ' ('antipodal map')*

(See also [**4**] or [**11**])

PROPOSITION 2.10 *There is an isometric involution σ^{Σ_0} on $(\mathbf{S}^2, \mathbf{g})$ with fixed-point set Σ_0.*

Proof. Γ is invariant by complex conjugation in $\mathbb{C}$, so $z = (x, y) \mapsto \bar{z} = (x, -y)$ in $\mathbb{C}$ induces the involution σ^{Σ_0} on $\mathbf{S}$ with fixed-point set Σ_0. It is an isometry by the Γ-periodicity of $\hat{\mathbf{g}}$ in (11).

PROPOSITION 2.11 *There is an involution σ' on $\mathbf{S}^2$ ('antipodal map') such that*

$$\sigma'(\mathbf{p}_1) = \mathbf{p}_3, \quad \sigma'(\mathbf{p}_2) = \mathbf{p}_4. \tag{18}$$

Moreover, if $\mathbf{g}$ is real analytic σ' is an isometry.

Proof. The involution σ' is the one induced by $(x, y) \mapsto (1/2 - x, L/2 + y)$ in $\mathbb{C}$. If $\mathbf{g}$ is real analytic the '$\frac{1}{2}\Gamma$-periodicity' of $\hat{\mathbf{g}}$ indicated in (16) renders σ' an isometry.

REMARK 2.12 There are two more involutions σ_1 and σ_2. They both leave Σ_0 invariant, and $\sigma_1(\mathbf{p}_1) = \mathbf{p}_4$, $\sigma_1(\mathbf{p}_2) = \mathbf{p}_3$, while $\sigma_2(\mathbf{p}_1) = \mathbf{p}_2$, $\sigma_2(\mathbf{p}_3) = \mathbf{p}_4$. Here, σ_1 is induced by $(x, y) \mapsto (x, L/2 - y)$ and σ_2 by $(x, y) \mapsto (\frac{1}{2} - x, y)$ in $\mathbb{C}$.

Note that from (16) when $\mathbf{g}$ is real analytic σ_1 and σ_2 are isometries.

3. Exponential map at a branch point

In this section we recall some known facts and provide proofs since we need the set up for later sections. The main result is Proposition 3.6 and Corollary 3.8; see, for example, [**11**, Theorem 2.1 and §3]

(*) *the exponential map based at a branch point $\mathbf{p}$-restricted to an appropriate disk is a diffeomorphism onto $\mathbf{S}^2 \setminus \sigma'(\mathbf{p})$.*

In addition to the statement (*) an important result for us in this section is Lemma 3.9 derived at the end.

3.1. *Geodesics in Liouville coordinates*

Take the canonical coordinates (x, y, P_x, P_y) associated with the coordinates (17) and trivializing $T^*(\mathbf{S}^2 \setminus \{\mathbf{p}_1, \dots, \mathbf{p}_4\})$, $z^* = (x(\pi z^*), y(\pi z^*), z^*(\partial/\partial x), z^*(\partial/\partial y))$.

The Hamiltonian reads $H = \frac{1}{2}(P_x^2 + P_y^2)(f(x) + g(y))^{-1}$, while the quadratic F is

$$F = \frac{1}{2}P_x^2 - f\ H = \frac{1}{2}\frac{gP_x^2 - fP_y^2}{f+g} = gH - \frac{1}{2}P_y^2. \tag{19}$$

Now $\mathrm{d}\theta = \mathrm{d}P_x \wedge \mathrm{d}x + \mathrm{d}P_y \wedge \mathrm{d}y$, hence

$$X_H = \frac{1}{f+g}\left(P_x\frac{\partial}{\partial x} + P_y\frac{\partial}{\partial y}\right) + \frac{H}{f+g}\left(f_x\frac{\partial}{\partial P_x} + g_y\frac{\partial}{\partial P_y}\right) \tag{20}$$

and

$$X_F = \frac{1}{f+g}\left(g\,P_x\frac{\partial}{\partial x} - f\,P_y\frac{\partial}{\partial y}\right) + \frac{H}{f+g}\left(gf_x\frac{\partial}{\partial P_x} - fg_y\frac{\partial}{\partial P_y}\right). \tag{21}$$

From $\mathrm{d}\theta(X_F, X_H) = 0$, a unit-speed geodesic γ has a constant value of F.

PROPOSITION 3.1 *For all $z^* \in T^*\mathbf{S}^2$, in the (x, y) Liouville coordinates with $\pi(z^*) = (x, y)$,*

$$\|\pi_*(X_F|_{z^*})\|^2 = 2f(x)g(y)H(z^*) + 2(g(y) - f(x))F(z^*). \tag{22}$$

Proof. By (19), on $\mathbf{S}^2 \setminus \{p_1, \ldots, p_4\}$,

$$P_x^2 - 2fH = 2F, \quad P_y^2 - 2gH = -2F. \tag{23}$$

Use this together with (17) and (21) to get (22) on $\mathbf{S}^2 \setminus \{p_1, \ldots, p_4\}$. This equality holds by continuity on $\{p_1, \ldots, p_4\}$, both sides being equal to zero.

PROPOSITION 3.2 *Let γ be a unit-speed geodesic in $(\mathbf{S}^2, \mathbf{g})$ and J_γ the Jacobi field induced by X_F as in Definition 2.2. Then, for all $s \in \mathbb{R}$, putting, in the (x, y) coordinates, $\gamma(s) = (x_s, y_s)$,*

$$\|J_\gamma(s)\|^2 = f(x_s)\,g(y_s) + 2\,(g(y_s) - f(x_s))\,F. \tag{24}$$

Proof. Use Proposition 3.1, the expression $J_\gamma(s) = \pi_*(X_F|_{\Phi_s^H z^*})$ as in Definition 2.2 and the Legendre transformation $\lambda\Phi_s^H z^* = \dot{\gamma}(s)$. Note that here $H(z^*) = 1/2$.

3.2. *Exponential at a branch point*

There follows a description in terms of F of the geodesics in $(\mathbf{S}^2, \mathbf{g})$ through any of the branch points defined in (10).

By Proposition 2.10 and the fact that any fixed-point set of an isometry is totally geodesic ([**13**, Theorem 5.1]), the set Σ_0 is the trajectory of a closed simple geodesic

$$\gamma^{\Sigma_0} : \mathbb{R} \to \mathbf{S}^2. \tag{25}$$

This geodesic (up to parametrization) is the *singular branch geodesic*.

PROPOSITION 3.3 *A unit-speed geodesic γ meets a branch point if and only if the corresponding F is equal to* 0.

Proof. For a geodesic $\gamma : \mathbb{R} \to \mathbf{S}^2$, consider a continuous $\wp$-lift $\hat{\gamma} : \mathbb{R} \to \mathbb{C}$ and put

$$\hat{x}_t := \Re(\hat{\gamma}(t)), \quad \hat{y}_t := \Im(\hat{\gamma}(t)) \tag{26}$$

for t the arc-length of γ.

(To show $\Rightarrow$) Let $\gamma(t_0) = \mathbf{p}$, a branch point. Since $\mathbf{p} \in \wp(\frac{1}{2}\Gamma)$, due to (14) $f(\hat{x}_{t_0}) = g(\hat{y}_{t_0}) = 0$. So, since $\wp^*\mathbf{g} = \hat{g}$, from (24) it follows that $J_\gamma(t_0) = 0$, where J_γ is the Jacobi field in Proposition 2.3; thus, by (6), $F = 0$.

(To show $\Leftarrow$) Let γ be a geodesic with $F = 0$. We show that there is a $t_0 \in \mathbb{R}$ for which $\hat{\gamma}(t_0) \in \frac{1}{2}\Gamma = \wp^{-1}(\{\mathbf{p}_1, \ldots, \mathbf{p}_4\})$.

(*) Assume γ is not supported by $\Sigma_0 \subset \mathbf{S}^2$, since we know that Σ_0 contains branch points. Thus, there are $t^* \in \mathbb{R}$ and $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ such that, with $Q_{k,l}$ as in (8),

$$\hat{\gamma}(t^*) \in Q_{k,l} \setminus \partial Q_{k,l}. \tag{27}$$

CLAIM 3.4 *The functions* (26) *are strictly monotone on $\mathcal{I}^*$, where $\mathcal{I}^* \subset \mathbb{R}$ is any connected component containing t^* of the set $\{t \in \mathbb{R} | \hat{\gamma}(t) \in Q_{k,l} \setminus \partial Q_{k,l}\}$.*

Proof. By translation in $\mathbb{C}$, if necessary identify $Q_{k,l}$ with $Q_{0,0}$ or $Q_{1,0}$, so that we may assume $\hat{\gamma}(t_1) \in Q = Q_{0,0} \cup Q_{1,0}$, and use the Liouville coordinates there. Then, for all $t \in \mathcal{I}^*$ we put $x_t = \hat{x}_t$ and $y_t = \hat{y}_t$. Since $F = 0$ we have, with an overdot indicating derivative with respect to arc-length t,

$$(f(x_t) + g(y_t))^2 \, \dot{x}_t^2 \overset{\text{(i)}}{=} P_x^2 \overset{\text{(ii)}}{=} f(x_t), \tag{28}$$

$$(f(x_t) + g(y_t))^2 \, \dot{y}_t^2 \overset{\text{(i)}}{=} P_y^2 \overset{\text{(ii)}}{=} g(y_t), \tag{29}$$

where we get (i) by (2), and (ii) by setting $F = 0$ in (23).

But $f^{-1}(0) \cup g^{-1}(0) = \frac{1}{2}\Gamma \subset \bigsqcup_{(k,l)\in\mathbb{Z}\times\mathbb{Z}} \partial Q_{k,l}$, by (8) and (14). Thus, by (28) and (29) we have $\dot{x}_t \neq 0$ and $\dot{y}_t \neq 0$ for all $t \in \mathcal{I}^*$. So the claim is proved.

Now Claim 3.4, (27) and the boundedness of $Q_{k,l}$ imply that there is a $t_0 \in \mathbb{R}$ such that $\hat{\gamma}(t_0) \in \partial Q_{k,l}$. Moreover, it follows that $\hat{\gamma}(t_0) \in \frac{1}{2}\Gamma$. Indeed, if $\hat{\gamma}(t_0) \notin \frac{1}{2}\Gamma$ then: $\hat{\gamma}(t_0) \in \left(\partial Q_{k,l} \cap f^{-1}(0)\right) \setminus \left(\partial Q_{k,l} \cap g^{-1}(0)\right)$, or $\hat{\gamma}(t_0) \in \left(\partial Q_{k,l} \cap g^{-1}(0)\right) \setminus \left(\partial Q_{k,l} \cap f^{-1}(0)\right)$. But, by (28) and (29), in either case $\dot{\gamma}(t_0)$ is tangential to Σ_0 at $\gamma(t_0)$, forcing γ to be the geodesic γ^{Σ_0} supported by Σ_0, contradicting the assumption (*). The proposition is proved.

DEFINITION 3.5 Put $T := I_x + I_y$, where

$$I_x := \int_0^{1/2} \sqrt{f(x)}\,\mathrm{d}x, \quad I_y := \int_0^{L/2} \sqrt{g(y)}\,\mathrm{d}y. \tag{30}$$

PROPOSITION 3.6 *Let $\gamma : \mathbb{R} \to \mathbf{S}^2$ be a unit-speed geodesic not supported by Σ_0 and such that $\gamma(t_0) = \mathbf{p}$, a branch point.*

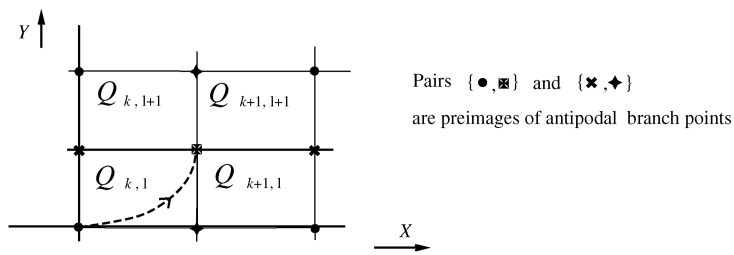

Figure 2. Inside $Q_{k,l}$ $x(\gamma(t))$ and $y(\gamma(t))$ are strictly monotone in t.

Then γ meets Σ_0 precisely at $\mathbf{p}$ and $\sigma'(\mathbf{p})$, alternating between the two, over and over, at integer multiples of T.

In other words, $\gamma(\mathbb{R}) \cap \Sigma_0 = \{\mathbf{p}, \sigma'(\mathbf{p})\}$ with

$$\gamma^{-1}(\mathbf{p}) = \{t_0 + 2kT \mid k \in \mathbb{Z}\}, \tag{31}$$
$$\gamma^{-1}(\sigma'(\mathbf{p})) = \{t_0 + (2k+1)T \mid k \in \mathbb{Z}\}.$$

Proof. Assume that $\gamma(t_0) = \mathbf{p} \in \{\mathbf{p}_1, \ldots, \mathbf{p}_4\}$ for some $t_0 \in \mathbb{R}$ and that the trajectory of γ is not Σ_0. Consider a lift $\hat{\gamma} : \mathbb{R} \to \mathbb{C}$ of γ. From the proof of Claim 3.4 in Proposition 3.3 we conclude that

- $\hat{\gamma}$ only meets a $\partial Q_{k,l}$ at a vertex ($\in \frac{1}{2}\Gamma = \wp$-preimage of branch points);
- the x and y components of $\hat{\gamma}$ are strictly monotone functions of arc length as $\hat{\gamma}$ traverses the interior of a $Q_{k,l}$.

Thus (see Fig. 2) the sequence according to which γ meets a branch point must be of the form $\{\ldots, \mathbf{p}, \sigma'(\mathbf{p}), \mathbf{p}, \sigma'(\mathbf{p}), \mathbf{p}, \sigma'(\mathbf{p}), \ldots\}$ with $\mathbf{p} = \mathbf{p}_-$ or $\mathbf{p} = \mathbf{p}_+$.

Thus, this defines a sequence $\{t_k \in \mathbb{R}, k \in \mathbb{Z}\}$ for which

$$\gamma(t) \text{ is not a branch point for } t \neq t_k, \text{ and } \gamma(t_k) = \begin{cases} \mathbf{p} & \text{for } k \text{ even,} \\ \sigma'(\mathbf{p}) & \text{for } k \text{ odd.} \end{cases}$$

Now, it just remains to show that

$$t_{k+1} - t_k = T \quad \forall k \in Z. \tag{32}$$

To prove this, for $t_k < s < t_{k+1}$ consider the functions $s \mapsto x_s$ and $s \mapsto y_s$ describing any $\wp$-lift of γ, once again as in the proof of Claim 3.4. Since γ is unit-speed,

$$1 = \|\dot{\gamma}(s)\|^2 = (f(x_s) + g(y_s))(\dot{x}_s^2 + \dot{y}_s^2). \tag{33}$$

But by (28) and (29)

$$\|\dot{\gamma}(s)\|^2 = |\dot{x}_s|\sqrt{f(x_s)} + |\dot{y}_s|\sqrt{g(y_s)}. \tag{34}$$

Thus, using (33) and (34),

$$t_{k+1} - t_k = \underbrace{\int_{t_k^+}^{t_{k+1}^-} |\dot{x}_s|\sqrt{f(x_s)}\,\mathrm{d}s}_{S_1} + \underbrace{\int_{t_k^+}^{t_{k+1}^-} |\dot{y}_s|\sqrt{g(y_s)}\,\mathrm{d}s}_{S_2},$$

where the two integrals S_1 and S_2 are improper as indicated. Now, recall that after translation in $\mathbb{C}$ if necessary, the portion of the $\wp$-lift of γ from t_k to t_{k+1} exists in $Q_{0,0} \setminus \partial Q_{0,0}$ or in $Q_{1,0} \setminus \partial Q_{1,0}$, depending on $\mathbf{p}$. In either case we have

$$S_2 = \int_0^{L/2} \sqrt{g(y)}\,\mathrm{d}y = I_y \tag{35}$$

and, since f is even and 1-periodic and hence $f(x) = f(-x) = f(1-x)$,

$$S_1 = \int_0^{1/2} \sqrt{f(x)}\,\mathrm{d}x = \int_{1/2}^{1} \sqrt{f(x)}\,\mathrm{d}x = I_x. \tag{36}$$

By Definition 3.5 this shows (32) and finishes the proof of the proposition.

COROLLARY 3.7 *The distance between* $\mathbf{p}$ *and* $\sigma'(\mathbf{p})$ *is* T *in Definition* 3.5.

Proof. This follows from the calculation above for geodesics not supported by Σ_0, together with that, by (35) and (36), $T = S_1 + S_2$ is also the length of the two segments from $\mathbf{p}$ to $\sigma'(\mathbf{p})$ along the geodesic γ^{Σ_0} supported by Σ_0.

COROLLARY 3.8 *The exponential map at a branch point* $\mathbf{p}$*,*

$$\exp_{\mathbf{p}} \colon \mathcal{B}(\mathbf{p}, T) = \{v \in T_{\mathbf{p}}\mathbf{S}^2 \mid \|v\| < T\} \to \mathbf{S}^2 \setminus \{\sigma'(\mathbf{p})\}, \tag{37}$$

is a diffeomorphism.

Proof. *Step* 1: We show first that the map (37) is one-to-one.

Assume there are geodesics γ_1 and γ_2 with $\gamma_1(0) = \gamma_2(0) = \mathbf{p}$ and $\gamma_1(t_1) = \gamma_2(t_2)$, where $0 < t_1 \le t_2 < T$, and get a contradiction to Corollary 3.7 as we now sketch.

Let $\epsilon > 0$ be small enough so that there is a unique geodesic segment γ_3 connecting $\gamma_1(t_1 - \epsilon)$ with $\gamma_2(t_2 + \epsilon)$. The segment of γ_1 from $\mathbf{p} = \gamma_1(0)$ to $\gamma_1(t_1 - \epsilon)$ has length $t_1 - \epsilon$; the segment γ_3 from $\gamma_1(t_1 - \epsilon)$ to $\gamma_2(t_2 + \epsilon)$ has length $a < 2\epsilon$ by the triangle inequality; and the segment of γ_2 from $\gamma_2(t_2 + \epsilon)$ to $\sigma'(\mathbf{p})$ has length $T - t_2 - \epsilon$.

By smoothing the broken curve of total length $(t_1 - \epsilon) + a + (T - t_2 - \epsilon) = T - (t_2 - t_1) - (2\epsilon - a) < T$ determined by those three segments, get a smooth curve from $\mathbf{p}$ to $\sigma'(\mathbf{p})$ of length less than T, violating Corollary 3.7. This shows step 1.

Step 2: Assume γ with $\gamma(0) = \mathbf{p}$ with $(\exp_{\mathbf{p}})_*|_{t\dot{\gamma}}$ singular for some $0 < c < T$ and get a contradiction to step 1. Now $\gamma(c)$ is a conjugate point of $\gamma(0)$ along γ, hence γ does not minimize the distance from $\mathbf{p}$ beyond $\gamma(c)$ ([**12**, Theorem 4.1]). So, for a c' with $c < c' < T$, there are a unit-speed geodesic $\tilde{\gamma}$ and a c'' with $0 < c'' < c' < T$ such that $\tilde{\gamma}(0) = \mathbf{p}$ and $\tilde{\gamma}(c'') = \gamma(c')$, contradicting step 1.

We close this section with a result used repeatedly later.

LEMMA 3.9 *Let* γ *be a unit-speed geodesic passing through a branch point* $\mathbf{p} = \gamma(0)$*. If* γ *is not supported by* Σ_0 *the Jacobi field in Definition* 2.2 *satisfies*

(i) J_γ *is normal, that is,* $\mathbf{g}(J_\gamma(t), \dot{\gamma}(t)) = 0$ *for all* $t \in \mathbb{R}$*;*

(ii) $J_\gamma(t) = 0 \Leftrightarrow t = kT, k \in \mathbb{Z}$.

On the other hand if γ *is the geodesic supported by* Σ_0*, that is,* $\gamma = \gamma^{\Sigma_0}$*, then* $J_{\gamma^{\Sigma_0}} \equiv 0$.

Proof. Since γ meets a branch point, $F_\gamma = 0$ by Proposition 3.3; so (i) follows from (6).

To show (ii) combine the formula for the length of J_γ in Proposition 3.2 with the fact that $f(x)g(y) = 0$ if and only if $(x, y) \in \Sigma_0 \supset \{\mathbf{p}_1, \ldots, \mathbf{p}_4\})$ by (14); then use Proposition 3.6. This argument also shows the vanishing of $J_{\gamma^{\Sigma_0}}$.

4. The one-form Ω on $\mathbf{S}^2 \setminus \Sigma_0$ (Theorem 4.5)

After the set-up from the previous sections we now take two systems of polar coordinates centered at two chosen non-antipodal branch points, $\mathbf{p}_-$ and $\mathbf{p}_+$, $\mathbf{p}_+ \neq \sigma'(\mathbf{p}_-)$, and define a one-form Ω on $\mathbf{S}^2 \setminus \Sigma_0$ that will be used in the proof of Theorem 5.1.

We will show that Ω can be integrated along geodesic segments connecting any pair $\mathbf{p}$ and $\sigma'(\mathbf{p})$ of antipodal branch points, and that the result of the integration does not depend on the geodesic segment. (In fact, the same holds for any path nowhere tangent to Σ_0 and meeting it only at $\mathbf{p}$ and $\sigma'(\mathbf{p})$, but we focus on geodesic segments.)

4.1. *Definition of* Ω

Consider a pair of (always distinct) branch points $\{\mathbf{p}_-, \mathbf{p}_+\} \subset \{\mathbf{p}_1, \ldots, \mathbf{p}_4\}$ with $\mathbf{p}_+ \neq \sigma'(\mathbf{p}_-)$. Henceforth such a pair will be called *non-antipodal*.

Let $\mathrm{d}\tilde{s}_-$ and $\mathrm{d}\tilde{\psi}_-$ be, respectively, the radial and angular one-forms on the punctured disk $\mathcal{B}(\mathbf{p}_-, T) \setminus \{0\} \subset T_{\mathbf{p}_-}\mathbf{S}^2$. Put

$$\mathrm{d}s_- := (\exp_{\mathbf{p}_-}^{-1})^* \mathrm{d}\tilde{s}_-, \quad \mathrm{d}\psi_- := (\exp_{\mathbf{p}_-}^{-1})^* \mathrm{d}\tilde{\psi}_-, \tag{38}$$

and denote the dual frame by $\partial/\partial s_-$ and $\partial/\partial\psi_-$. Similarly, using $\exp_{\mathbf{p}_+}$ on $T_{\mathbf{p}_+}\mathbf{S}^2 \setminus \{0\}$, define $\mathrm{d}s_+$, $\mathrm{d}\psi_+$, $\partial/\partial s_+$ and $\partial/\partial\psi_+$. We have the metric expressed as

$$\mathbf{g} = \mathrm{d}s_-^2 + \rho_-^2\,\mathrm{d}\psi_-^2 \quad \text{and} \quad \mathbf{g} = \mathrm{d}s_+^2 + \rho_+^2\,\mathrm{d}\psi_+^2, \tag{39}$$

respectively, on $\mathbf{S}^2 \setminus \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$ and $\mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$ for functions ρ_- and ρ_+.

Definition 4.1 Denote the orthonormal frames on $\mathbf{S}^2 \setminus \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$ by

$$U_- := \frac{\partial}{\partial s_-}, \quad V_- := \frac{1}{\rho_-}\frac{\partial}{\partial\psi}, \tag{40}$$

and analogously $\{U_+, V_+\}$ on $\mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$.

Then, on $\mathbf{S}^2 \setminus \{\mathbf{p}_1, \ldots, \mathbf{p}_4\}$, put

$$\cos\omega = \mathbf{g}\,(U_-, U_+)\,, \quad \sin\omega = \mathbf{g}\,(U_-, V_+). \tag{41}$$

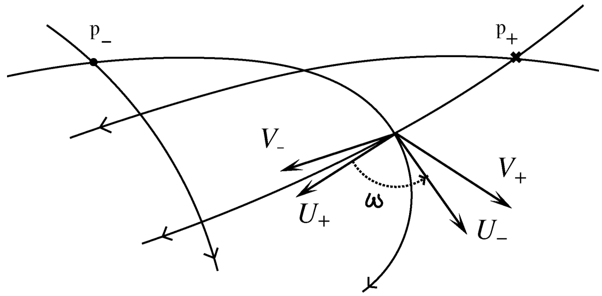

Figure 3. ω measures the angle between geodesic rays issued from two fixed non-antipodal branch points $\mathbf{p}_-$ and $\mathbf{p}_+$.

DEFINITION 4.2 *Given a pair of non-antipodal branch points* $\{\mathbf{p}_-, \mathbf{p}_+\}$ *define the one-form on* $\mathbf{S}^2 \setminus \Sigma_0$ (*see Fig.* 3) *by*

$$\Omega := \frac{\mathrm{d}\omega}{\sin \omega}. \tag{42}$$

CLAIM 4.3 Ω *is well defined and smooth on* $\mathbf{S}^2 \setminus \Sigma_0$. (*It depends on the choice of the pair* $\{\mathbf{p}_-, \mathbf{p}_+\}$ *up to sign, but we do not make that explicit in our notation.*)

Proof. Simply observe that

$$\sin \omega(q) \neq 0 \quad \text{for all } q \in \mathbf{S}^2 \setminus \Sigma_0. \tag{43}$$

Indeed, $\sin \omega(q) = 0$ implies by (41) that $U_+(q) = \pm U_-(q)$, which means that a geodesic passing through $\mathbf{p}_-$ is, at q, tangent (hence equal) to a geodesic passing through $\mathbf{p}_+ \neq \sigma'(\mathbf{p}_-)$; this is impossible unless $q \in \Sigma_0$, by Proposition 3.6.

The next property of Ω will be used in (71) during the proof of Theorem 4.5.

PROPOSITION 4.4 *Let* σ^{Σ_0} *be the isometric involution from Proposition* 2.10. *Then*

$$(\sigma^{\Sigma_0})^* \Omega = \Omega.$$

Proof. Once we show that

(a) $(\sigma^{\Sigma_0})^* \cos \omega = \cos \omega$,
(b) $(\sigma^{\Sigma_0})^* \sin \omega = -\sin \omega$,

the result follows by taking derivative in (a), and using (b) with (43).

Let $\{u_1, u_2\}$ be an orthonormal oriented basis of $T_{\mathbf{p}_-}\mathbf{S}^2$ with $u_1 \in T_{\mathbf{p}_-}\Sigma_0$, where we recall Σ_0 $(\supset \{\mathbf{p}_-, \mathbf{p}_+\})$ is the fixed-point set of σ^{Σ_0}. Put $\tilde{s}_-(v) = \|v\|$ and for $v \in \mathcal{B}(\mathbf{p}_-, T) \setminus \{tu_1, 0 \leq t \leq T\}$, let $\tilde{\psi}_-$ be the corresponding angle of v in $(0, 2\pi)$ measured from u_1. Then

$$\exp_{\mathbf{p}_-}^{-1} \circ \ \sigma^{\Sigma_0} \circ \exp_{\mathbf{p}_-}(\tilde{s}_-, \tilde{\psi}_-) = (\tilde{s}_-, 2\pi - \tilde{\psi}_-).$$

Hence, putting $q = \exp_{\mathbf{p}_-}(\tilde{s}_-, \tilde{\psi}_-)$, we have first on $\mathcal{B}(\mathbf{p}_-, T) \setminus \{tu_1, 0 \le t \le T\}$ and then, by continuity of σ^{Σ_0} and of $\exp_{\mathbf{p}_-}$, throughout $\mathcal{B}(\mathbf{p}_-, T)$,

$$\begin{aligned} \sigma_*^{\Sigma_0}(U_-|_q) &= \sigma_*^{\Sigma_0}\left((\exp_{\mathbf{p}_-})_* \frac{\partial}{\partial \tilde{s}_-}|_{(\tilde{s}_-, \tilde{\psi}_-)}\right) \\ &= (\exp_{\mathbf{p}_-})_* \frac{\partial}{\partial \tilde{s}_-}|_{(\tilde{s}_-, 2\pi - \tilde{\psi}_-)} \\ &= U_-|_{\sigma^{\Sigma_0}(q)}. \end{aligned} \tag{44}$$

Also,

$$\begin{aligned} \sigma_*^{\Sigma_0}(V_-|_q) &= \sigma_*^{\Sigma_0}\left((\exp_{\mathbf{p}_-})_* \frac{\partial}{\partial \tilde{\psi}_-}|_{(\tilde{s}_-, \tilde{\psi}_-)}\right) \\ &= -(\exp_{\mathbf{p}_-})_* \frac{\partial}{\partial \tilde{\psi}_-}|_{(\tilde{s}_-, 2\pi - \tilde{\psi}_-)} \\ &= -V_-|_{\sigma^{\Sigma_0}(q)}. \end{aligned} \tag{45}$$

Similarly, working from $\mathbf{p}_+$, we get $\sigma_*^{\Sigma_0} U_+ = U_+$ and $\sigma_*^{\Sigma_0} V_+ = -V_+$.

We now show (a):

$$\begin{aligned} \cos\omega(q) &= \mathbf{g}\left(U_-|_q, U_+|_q\right) \\ &\overset{\text{(i)}}{=} ((\sigma^{\Sigma_0})^* \mathbf{g})\left(U_-|_q, U_+|_q\right) \\ &= \mathbf{g}\left(\sigma_*^{\Sigma_0}(U_-|_q), \sigma_*^{\Sigma_0}(U_+|_q)\right) \\ &\overset{\text{(ii)}}{=} \mathbf{g}\left(U_-|_{\sigma^{\Sigma_0}(q)}, U_+|_{\sigma^{\Sigma_0}(q)}\right) \\ &= \cos\omega(\sigma^{\Sigma_0}(q)), \end{aligned} \tag{46}$$

where equality (i) uses that σ^{Σ_0} is an isometry, and (ii) uses (44).

Similarly one shows (b) using (41) and (45).

4.2. *Integral of Ω along segments connecting antipodal branch points*

THEOREM 4.5 *Let $\gamma : [0, T] \to \mathbf{S}^2$ be a unit-speed geodesic segment not supported by Σ_0 with $\gamma(0) = \mathbf{p}$, a branch point, and thus $\gamma(T) = \sigma'(\mathbf{p})$. Then*

$$\int_{0^+}^{T^-} \Omega(\dot{\gamma}(t))\mathrm{d}t < \infty, \tag{47}$$

and the result is independent of γ. (Here the integral is improper because Ω is not defined along $\Sigma_0 \supset \{\mathbf{p}, \sigma'(\mathbf{p})\}$.)

We give the proof of Theorem 4.5 at the end of this section after deriving the results needed to apply Stokes's theorem. (Of course, adjusting the proof, it would follow that (47) holds along any path that meets Σ_0 only at the end points $\mathbf{p}$ and $\sigma'(\mathbf{p})$ and does so non-tangentially.)

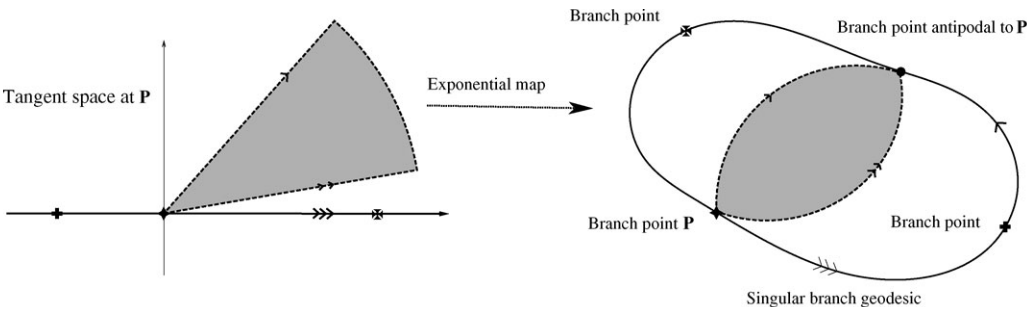

Figure 4. $\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)$ is open, disjoint from Σ_0 and enclosed by two geodesics through $\mathbf{p}$ not supported by Σ_0.

DEFINITION 4.6 (See Fig. 4) Let $\mathbf{p} \in \{\mathbf{p}_1, \dots, \mathbf{p}_4\}$ and $\psi_1 < \psi_2$ such that the interval

$$[\psi_1, \psi_2] \subset (0, \pi) \cup (\pi, 2\pi) \subset \mathbb{R}. \tag{48}$$

For $0 \neq v \in T_{\mathbf{p}}\mathbf{S}^2$, let $\mathrm{Arg}(v)$ be the angle of v with respect to $T_{\mathbf{p}}\Sigma_0$ for a fixed orientation of Σ_0. Put

$$\mathcal{O}(\mathbf{p}, \psi_1, \psi_2) := \exp_{\mathbf{p}}\left(\{0 < \|v\| < T,\ \psi_1 < \mathrm{Arg}(v) < \psi_2\}\right) \subset \mathbf{S}^2. \tag{49}$$

DEFINITION 4.7 Put $\mathcal{H}_0 \cup \mathcal{H}_1 = \mathbf{S}^2 \setminus \Sigma_0$, where $\mathcal{H}_i$ is a connected component of the complement of Σ_0, given with notation as in Definition 2.5, by

$$\mathcal{H}_i := \wp\left(Q_{i,0} \setminus \partial Q_{i,0}\right).$$

PROPOSITION 4.8

$$\mathcal{O}(\mathbf{p}, \psi_1, \psi_2) \subset \mathcal{H}_i \quad \textit{for } i = 1, 2. \tag{50}$$

Proof. With the notation as in Definition 4.12,

$$\mathcal{H}_0 = \exp_{\mathbf{p}}\left(\{0 < \|v\| < T,\ 0 < \mathrm{Arg}(v) < \pi\}\right),$$
$$\mathcal{H}_1 = \exp_{\mathbf{p}}\left(\{0 < \|v\| < T,\ \ \pi < \mathrm{Arg}(v) < 2\pi\}\right).$$

Thus, inclusion (50) is a consequence of Proposition 3.6 and the fact that the connected set $\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)$ is the union of all the geodesics that start at $\mathbf{p}$ with initial velocities restricted according to (48) and hence does not include the singular geodesic supported by Σ_0.

PROPOSITION 4.9 *The closure of* $\mathcal{O}$, $\overline{\mathcal{O}}$, *satisfies*

(i) $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \subset \bigcap_{\mathbf{p}^* \in \{\mathbf{p}_1, \dots, \mathbf{p}_4\} \setminus \{\mathbf{p}, \sigma'(\mathbf{p})\}} \exp_{\mathbf{p}^*}\left(\mathcal{B}(\mathbf{p}^*, T) \setminus \{0\}\right),$
(ii) $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \cap \Sigma_0 = \{\mathbf{p}, \sigma'(\mathbf{p})\}.$

Proof. This follows from the definitions and Corollary 3.8.

PROPOSITION 4.10

$$m(\mathbf{p}, \psi_1, \psi_2) := \inf_{q \in \mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} |\sin \omega(q)| > 0. \tag{51}$$

Proof. The proof is by contradiction. Assume inequality (51) is not true. Then there is a sequence $\{q_i\}_{i \in \mathbb{N}}$ with $q_i \in \mathcal{O}(\mathbf{p}, \psi_1, \psi_2)$ and

$$\lim_{i \to \infty} \sin \omega(q_i) = 0. \tag{52}$$

A subsequence of $\{q_i\}_{i \in \mathbb{N}}$ will have a limit $q^\sharp \in \overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}$, so that, by (43),

$$q^\sharp \in \overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \cap \Sigma_0 = \{\mathbf{p}, \sigma'(\mathbf{p})\}. \tag{53}$$

Now, choose a branch point $\tilde{\mathbf{p}} \neq \mathbf{p}$ with $\tilde{\mathbf{p}} \neq \sigma'(\mathbf{p})$ as well; that is, $\tilde{\mathbf{p}}$ is one of the two branch points not in $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}$.

Since each q_i belongs to $\mathcal{O}(\mathbf{p}, \psi_1, \psi_2) \subset \bigcap_{\mathbf{p}^* \in \{\mathbf{p}_1, \ldots, \mathbf{p}_4\}} \exp_{\mathbf{p}^*} (\mathcal{B}(\mathbf{p}^*, T) \setminus \{0\})$, by Corollary 3.8 applied to $\mathbf{p}$ and $\tilde{\mathbf{p}}$ there are two sequences of geodesics in $\mathbf{S}^2$, $\{\gamma_i\}_{i \in \mathbb{N}}$ and $\{\eta_i\}_{i \in \mathbb{N}}$ such that, for each $i \in \mathbb{N}$,

- $\gamma_i(0) = \mathbf{p}$ and $\eta_i(0) = \tilde{\mathbf{p}}$;
- γ_i and η_i are minimizing for arc length from 0 to T.

Let s_i be the distance from $\mathbf{p}$ to q_i, and t_i the distance from $\tilde{\mathbf{p}}$ to q_i. Thus $\gamma_i(s_i) = q_i$ and $\eta_i(t_i) = q_i$. Since $s_i, t_i \in [0, T]$, the two sequences of distances converge,

$$\{s_i\}_{i \in \mathbb{N}} \to s_\sharp, \quad \{t_i\}_{i \in \mathbb{N}} \to t_\sharp,$$

and by (53) they do so according to these two possibilities:

$$q^\# = \sigma'(\mathbf{p}), \quad \text{hence } s_\sharp = T \text{ and } t_\sharp = (\text{distance from } \tilde{\mathbf{p}} \text{ to } \sigma'(\mathbf{p})) \neq 0;$$
$$q^\# = \mathbf{p}, \quad \text{hence } s_\sharp = 0 \text{ and } t_\sharp = (\text{distance from } \tilde{\mathbf{p}} \text{ to } \mathbf{p}) \neq 0.$$

In either case, compactness of the set of unit vectors in $T_{\tilde{\mathbf{p}}}\mathbf{S}^2$ implies that a subsequence of $\{\dot{\eta}_i(0)\}_{i \in \mathbb{N}}$ has as limit a unit vector $\tilde{u} \in T_{\tilde{\mathbf{p}}}\mathbf{S}^2$, and the geodesic $\eta \colon \mathbb{R} \to \mathbf{S}^2$ defined by $\eta(0) = \tilde{\mathbf{p}}$ and $\dot{\eta}(0) = \tilde{u}$ satisfies

$$\eta(t_\sharp) \in \{\mathbf{p}, \sigma'(\mathbf{p})\}. \tag{54}$$

So, since by definition $\eta(0) = \tilde{\mathbf{p}} \notin \{\mathbf{p}, \sigma'(\mathbf{p})\}$, it follows from by Proposition 3.6 that η must be the geodesic supported by Σ_0, that is,

$$\eta(\mathbb{R}) \equiv \gamma^{\Sigma_0}(\mathbb{R}) = \Sigma_0. \tag{55}$$

But, from the definition of ω and by (52), we have the convergence of the sequence

$$\{|\cos \omega(q_i)| = |\mathbf{g}(\dot{\gamma}_i(s_i),\ \dot{\eta}_i(t_i))|\}_{i \in \mathbb{N}} \to 1. \tag{56}$$

From this and (55) it follows that in $T_{q^\#}\mathbf{S}^2$ we have the convergence

$$\{|\mathbf{g}(\dot{\gamma}_i(s_\#), \dot{\gamma}^{\Sigma_0})|\}_{i \in \mathbb{N}} \to 1.$$

This is incompatible with the definition (49) of $\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)$, which consists of geodesics issued from $q^{\#} \in \{\mathbf{p}, \sigma'(\mathbf{p})\}$, since the two geodesics segments starting at $\mathbf{p}$ and ending at $\sigma'(\mathbf{p})$ that form $\partial\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)$ are not tangent to Σ_0.

PROPOSITION 4.11 *Let γ be a unit-speed geodesic not supported by Σ_0 with $\gamma(0) = \mathbf{p}$, a branch point, and $\gamma(T) = \sigma'(\mathbf{p})$. Then*

$$\lim_{a\to 0^+, b\to T^-} \int_a^b \Omega(\dot\gamma(t))\, \mathrm{d}t < \infty. \tag{57}$$

Proof. Note that the segment of γ for $0 \le t \le T$ is contained in $\partial\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}$ for some ψ_1 and ψ_2 as in Definition 4.6. Now, to define Ω we fixed two branch points $\mathbf{p}_- \neq \mathbf{p}_+ \in \{\mathbf{p}_1, \dots, \mathbf{p}_4\}$ with $\mathbf{p}_+ \neq \sigma'(\mathbf{p}_-)$. We express $\mathrm{d}\omega$ in (42) in terms of

$$\{\Phi_1^- := \mathrm{d}s_-,\ \Phi_2^- := \rho_-\mathrm{d}\psi_-\}, \quad \{\Phi_1^+ := \mathrm{d}s_+,\ \Phi_2^+ := \rho_+\mathrm{d}\psi_+\}, \tag{58}$$

the co-frames dual of $\{U_-, V_-\}$ and of $\{U_+, V_+\}$, respectively, on $\mathbf{S}^2 \setminus \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$ and on $\mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$.

To do this consider the connections $\mathbf{K}_-$ and $\mathbf{K}_+$,

$$\begin{pmatrix} \mathrm{d}\Phi_1^- \\ \mathrm{d}\Phi_2^- \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \dfrac{\partial \ln \rho_-}{\partial s_-}\Phi_2^- \\ -\dfrac{\partial \ln \rho_-}{\partial s_-}\Phi_2^- & 0 \end{pmatrix}}_{:=\mathbf{K}_-} \begin{pmatrix} \Phi_1^- \\ \Phi_2^- \end{pmatrix},$$

and

$$\begin{pmatrix} \mathrm{d}\Phi_1^+ \\ \mathrm{d}\Phi_2^+ \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \dfrac{\partial \ln \rho_+}{\partial s_+}\Phi_2^+ \\ -\dfrac{\partial \ln \rho_+}{\partial s_+}\Phi_2^+ & 0 \end{pmatrix}}_{:=\mathbf{K}_+} \begin{pmatrix} \Phi_1^+ \\ \Phi_2^+ \end{pmatrix}.$$

Since, on $\mathbf{S}^2 \setminus \{\mathbf{p}_1, \dots, \mathbf{p}_4\}$, $\begin{pmatrix} \Phi_1^- \\ \Phi_2^- \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}}_{:=\mathbf{M}} \begin{pmatrix} \Phi_1^+ \\ \Phi_2^+ \end{pmatrix}$, from the change of connection equation $\mathbf{K}_- = \mathrm{d}\mathbf{M}\mathbf{M}^{-1} + \mathbf{M}\,\mathbf{K}_+\mathbf{M}^{-1}$ it follows that

$$\mathrm{d}\omega = \frac{\partial \ln \rho_-}{\partial s_-}\Phi_2^- - \frac{\partial \ln \rho_+}{\partial s_+}\Phi_2^+.$$

So, using again the matrix $\mathbf{M}$, on $\mathbf{S}^2 \setminus \{\mathbf{p}_1, \dots, \mathbf{p}_4\}$, we get the following useful expressions (note that the $+$ and $-$ labels turn up mixed):

$$\mathrm{d}\omega = -\frac{\partial \ln \rho_+}{\partial s_+}\sin\omega\Phi_1^- + \left(\frac{\partial \ln \rho_-}{\partial s_-} - \frac{\partial \ln \rho_+}{\partial s_+}\cos\omega\right)\Phi_2^-, \tag{59}$$

$$\mathrm{d}\omega = -\frac{\partial \ln \rho_-}{\partial s_-}\sin\omega\,\Phi_1^+ - \left(\frac{\partial \ln \rho_+}{\partial s_+} - \frac{\partial \ln \rho_-}{\partial s_-}\cos\omega\right)\Phi_2^+. \tag{60}$$

Now, according to our definition of the frames $U_{\pm}$, there are two possibilities:

(I) $\mathbf{p} \in \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$, in which case, for $0 \le t \le T$,

$$U_-|_{\gamma(t)} = \begin{cases} \dot\gamma(t) & \text{if } \mathbf{p} = \mathbf{p}_-, \\ -\dot\gamma(T-t) & \text{if } \mathbf{p} = \sigma'(\mathbf{p}_-); \end{cases} \tag{61}$$

(II) $\mathbf{p} \in \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$, in which case

$$U_+|_{\gamma(t)} = \begin{cases} \dot\gamma(t) & \text{if } \mathbf{p} = \mathbf{p}_+, \\ -\dot\gamma(T-t) & \text{if } \mathbf{p} = \sigma'(\mathbf{p}_+). \end{cases} \tag{62}$$

If (I) holds, for $0 < a < b < T$, putting $\tau(t) = t$ if $\mathbf{p} = \mathbf{p}_-$, or $\tau(t) = T - t$ if $\mathbf{p} = \sigma'(\mathbf{p}_-)$, we have using (61) in (59) that

$$\begin{aligned} \left| \int_a^b \Omega(\dot\gamma(t))\, \mathrm{d}t \right| &\le \int_a^b \left| \frac{\partial \ln \rho_+}{\partial s_+} \right|_{\gamma(\tau(t))} \mathrm{d}t \\ &\le (b-a)\, M_+^{(\psi_1,\psi_2)}, \end{aligned} \tag{63}$$

where

$$M_+^{(\psi_1,\psi_2)} := \max \left\{ \left| \frac{\partial \ln \rho_+}{\partial s_+} \right|_q \mid q \in \overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \right\} < \infty; \tag{64}$$

M_+ is finite since, as remarked in Proposition 4.9, whenever $\mathbf{p} \in \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$ we have $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \subset \mathbf{S}^2 \setminus \{\sigma'(\mathbf{p}_+)\} = \{\text{Domain of } \ln \rho_+\}$.

Taking $a \to 0^+$ and $b \to T^-$ in (63) shows (57).

If (II) holds, use (62) in (60); the inequality analogous to (63) now involves

$$M_-^{(\psi_1,\psi_2)} := \max \left\{ \left| \frac{\partial \ln \rho_-}{\partial s_-} \right|_q \mid q \in \overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \right\} < \infty, \tag{65}$$

finite since $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \subset \mathbf{S}^2 \setminus \{\sigma'(\mathbf{p}_-)\} = \{\text{Domain of } \ln \rho_-\}$.

DEFINITION 4.12 For $0 < r < T$ consider the set of points of $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}$ at a distance r from $\mathbf{p}$, that is, with the notation as in Definition 4.6, the arc

$$\alpha(\mathbf{p}, \psi_1, \psi_2, r) := \exp_{\mathbf{p}}(\{\|v\| = r, \psi_1 \le \mathrm{Arg}(v) \le \psi_2\}),$$

oriented in the direction from ψ_1 to ψ_2 (see Fig. 5).

PROPOSITION 4.13

$$\lim_{r \to 0^+} \int_{\alpha(\mathbf{p},\psi_1,\psi_2,r)} \Omega = 0, \quad \lim_{r \to T^-} \int_{\alpha(\mathbf{p},\psi_1,\psi_2,r)} \Omega = 0. \tag{66}$$

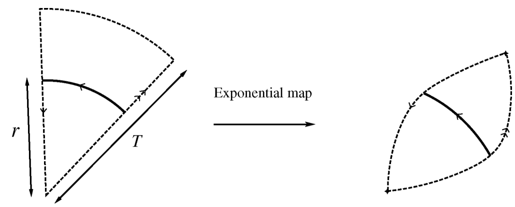

Figure 5. Set of points in $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}$ at a distance r from $\mathbf{p}$.

Proof. Let

$$R := \max_{q \in \mathbf{S}^2} \left\{ \rho_- \, , \, \rho_+ \, , \, \left| \frac{\partial \rho_-}{\partial s_-} \right| \, , \, \left| \frac{\partial \rho_+}{\partial s_+} \right| \right\}. \tag{67}$$

By Proposition 3.6 and Corollary 3.8, R is well defined and finite since ρ_- (respectively ρ_+) along any unit-speed geodesic from $\mathbf{p}_-$ to $\sigma'(\mathbf{p}_-)$ (respectively from $\mathbf{p}_+$ to $\sigma'(\mathbf{p}_+)$) is a smooth solution to the Jacobi field equation $\partial^2 \rho_- / \partial s_-^2 + \varrho \rho_- = 0$ (respectively $\partial^2 \rho_+ / \partial s_+^2 + \varrho \rho_+ = 0$), where ϱ is the Gauss curvature of $\mathbf{g}$, with ρ_- and ρ_+ vanishing with period T.

Now, consider the two possibilities as before:

(I) $\{\mathbf{p}, \sigma'(\mathbf{p})\} = \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$;
(II) $\{\mathbf{p}, \sigma'(\mathbf{p})\} = \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$.

In case (I) by (59) and (42) we have along $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}$

$$\begin{aligned} \left| \Omega \left(\frac{\partial}{\partial \psi_-} \right) \right| &= \left| \frac{\rho_-}{\sin \omega} \left(\frac{\partial \ln \rho_-}{\partial s_-} - \frac{\partial \ln \rho_+}{\partial s_+} \cos \omega \right) \right| \\ &\leq \frac{1}{|\sin \omega|} \left(\left| \frac{\partial \rho_-}{\partial s_-} \right| + \rho_- \left| \frac{\partial \ln \rho_+}{\partial s_+} \right| \right) \\ &\leq \frac{R}{m(\mathbf{p}, \psi_1, \psi_2)} \left(1 + M_+^{(\psi_1, \psi_2)} \right), \end{aligned} \tag{68}$$

with $m(\mathbf{p}, \psi_1, \psi_2)$ as in Proposition 4.10, R as in (67) and $M_+^{(\psi_1, \psi_2)}$ as in (64).

In case (II) use(60) and (42) to get on $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}$

$$\left| \Omega \left(\frac{\partial}{\partial \psi_+} \right) \right| \leq \frac{R}{m(\mathbf{p}, \psi_1, \psi_2)} \left(1 + M_-^{(\psi_1, \psi_2)} \right), \tag{69}$$

with $M_-^{(\psi_1, \psi_2)}$ from (65).

So, there is a constant $R^* = R^*(\psi_1, \psi_2)$ such that for $0 < r < T$,

$$\left| \int_{\alpha(\mathbf{p}, \psi_1, \psi_2, r)} \Omega \right| \leq R^* \times (\text{ length of } \alpha(\mathbf{p}, \psi_1, \psi_2, r)).$$

But the length of $\alpha(\mathbf{p}, \psi_1, \psi_2, r)$ goes to 0 as r goes to 0 and also as r goes to T, by Corollary 3.8.

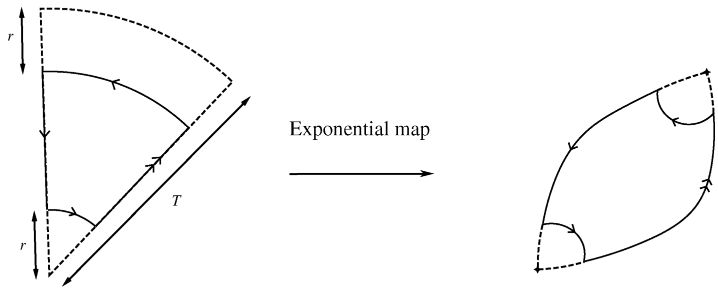

Figure 6. As $r \to 0$ we recover the closure of $\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)$.

4.3. *Proof of Theorem* 4.5

Let γ_1 and γ_2 be two unit-speed geodesic segments not supported by Σ_0, connecting the branch points $\gamma_i(0) = \mathbf{p}$ and $\gamma_i(T) = \sigma'(\mathbf{p})$. In light of Proposition 4.11, to prove Theorem 4.5 it only remains to show that

$$\int_0^T \Omega(\dot{\gamma}_1)\,\mathrm{d}t = \int_0^T \Omega(\dot{\gamma}_2)\,\mathrm{d}t. \tag{70}$$

CLAIM 4.14 *There is no loss in generality if both open segments* $\gamma_1(\{0 < t < T\})$ *and* $\gamma_2(\{0 < t < T\})$ *are assumed contained in the same connected component of* $\mathbf{S}^2 \setminus \Sigma_0$*,* $\mathcal{H}_0$ *or* $\mathcal{H}_1$ *as in Definition* 7.7.

Proof. Assume $\gamma_1(\{0 < t < T\}) \subset \mathcal{H}_1$ and $\gamma_2(\{0 < t < T\}) \subset \mathcal{H}_0$. Then, by applying the isometric involution σ^{Σ_0}, since $\sigma^{\Sigma_0}(\mathcal{H}_0) = \mathcal{H}_1$,

$$\gamma_3 := \sigma^{\Sigma_0} \circ \gamma_1 : [0, T] \to \mathbf{S}^2$$

is a geodesic segment not supported by Σ_0 with $\gamma_3(0) = \mathbf{p}$, $\gamma_3(T) = \sigma'(\mathbf{p})$ such that $\gamma_3(\{0 < t < T\}) \subset \mathcal{H}_0$. Moreover, since by Proposition 4.4 Ω is σ^{Σ_0}-invariant,

$$\int_0^T \Omega(\dot{\gamma}_3)\,\mathrm{d}t = \int_0^T \Omega\big(\big(\sigma^{\Sigma_0}\big)_* \dot{\gamma}_1\big)\,\mathrm{d}t = \int_0^T (\sigma^{\Sigma_0})^* \Omega(\dot{\gamma}_1)\,\mathrm{d}t = \int_0^T \Omega(\dot{\gamma}_1)\,\mathrm{d}t, \tag{71}$$

and the claim follows.

So, $\gamma_i : [0, T] \to \mathcal{H}_0 \cup \{\mathbf{p}, \sigma'(\mathbf{p})\}, i = 1, 2,$ bound $\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)} \subset \mathcal{H}_0$ where, after renaming $1 \rightleftarrows 2$ if necessary, $\psi_i = \operatorname{Arg}(\dot{\gamma}_i(0))$. Now we show that

$$\oint_{\partial \overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)}} \Omega = 0. \tag{72}$$

To do this, consider for $0 \leq r < T$ the truncation of $\mathcal{O}(\mathbf{p}, \psi_1, \psi_2)$ (see Fig. 6),

$$\overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2, r)} := \exp_{\mathbf{p}}(\{r \leq \|v\| \leq T - r, \psi_1 \leq \operatorname{Arg}(v) \leq \psi_2\}).$$

Since $\mathrm{d}\Omega = 0$ on $\mathbf{S}^2 \setminus \Sigma_0 \supset \overline{\mathcal{O}(\mathbf{p}, \psi_1, \psi_2, r)}$, by Stokes's theorem, for all $0 < r < T$, $\oint_{\partial\overline{\mathcal{O}(\mathbf{p},\psi_1,\psi_2,\epsilon)}} \Omega = \int_{\overline{\mathcal{O}(\mathbf{p},\psi_1,\psi_2,r)}} \mathrm{d}\Omega = 0$. Thus, we have

$$0 = \int_{\gamma_1([r,T-r])} \Omega + \int_{\alpha(\mathbf{p},\psi_1,\psi_2,T-r)} \Omega - \int_{\gamma_2([r,T-r])} \Omega - \int_{\alpha(\mathbf{p},\psi_1,\psi_2,r)} \Omega.$$

As $r \to 0$, according to Proposition 4.13 the second and fourth integrals above go to zero, while by Proposition 4.11 the first and the third integrals have a limit each; so, (72) holds. Consequently, $\int_{\gamma_1([0,T])} \Omega = \int_{\gamma_2([0,T])} \Omega$ showing (70).

COROLLARY 4.15 *Let $\gamma : \mathbb{R} \to \mathbf{S}^2$ be a unit-speed geodesic not supported by Σ_0 and such that $\gamma(0) = \mathbf{p}$, a branch point. Then, for all $k \in \mathbb{Z}$,*

$$\int_0^T \Omega(\dot\gamma)\,\mathrm{d}t = (-1)^k \int_k^{(k+1)T} \Omega(\dot\gamma)\,\mathrm{d}t. \tag{73}$$

Proof. By Proposition 3.6, for k in $\mathbb{Z}$ the restrictions $\gamma : [kT, (k+1)T] \to \mathbf{S}^2$ are geodesic segments not supported by Σ_0 going from $\mathbf{p}$ to $\sigma'(\mathbf{p})$ or vice-versa, depending on the parity of k, hence the $(-1)^k$ factor, when we apply Theorem 4.5.

5. The angle formed by a geodesic through a branch point with Σ_0

THEOREM 5.1 *Let $(\mathbf{S}^2, \mathbf{g})$ be a smooth Liouville metric on $\mathbf{S}^2$ with F homogeneous quadratic along the fibers of $T^*\mathbf{S}^2$, where $F \neq aF_l^2 + H$ for $a, b \in \mathbb{R}$ for any F_l homogeneous linear, and let $\mathbf{p}$ be a branch point. Orient the singular geodesic (25) supported by Σ_0. Then, there is a constant $0 < C$ with the following property.*

Let γ be any unit-speed geodesic, not supported by Σ_0, that meets $\mathbf{p}$ at $t = t_0$, and hence $\sigma'(\mathbf{p})$ at $t_0 + T$. Then, for one of the two orientations of γ, if we let ω_k be the angle that γ forms with Σ_0 at time $t_0 + kT$, for $k \in \mathbb{Z}$, then

$$\tan^2\left(\frac{\omega_k}{2}\right) = C^k \tan^2\left(\frac{\omega_0}{2}\right). \tag{74}$$

For the other orientation of γ the valid constant is C^{-1}.

Theorem 5.1 is proved after establishing Propositions 5.5 and 5.7 below.

REMARK 5.2 The constant C *is the same for all geodesics meeting the branch points* $\mathbf{p}$ *and* $\sigma'(\mathbf{p})$. *In addition*, *if* $\mathbf{g}$ *is real analytic*, C *is the same* for all geodesics *meeting any of the four branch points*. *This follows from Remark* 2.12 *since the involutions act transitively on the set of branch* points.

REMARK 5.3 It might be the case that $C = 1$, if the geodesics through $\mathbf{p}$ are closed.

5.1. *The angle ω_γ along a geodesic γ*

Let $\gamma : \mathbb{R} \to \mathbf{S}^2$ be a unit-speed geodesic not supported by Σ_0, with $\gamma(0) = \mathbf{p}_-$, a branch point, and thus $\gamma(T) = \sigma'(\mathbf{p}_-)$.

We will consider the angle that $\dot{\gamma}(t)$ makes with the geodesic rays radiating from a non-antipodal branch point $\mathbf{p}_+$, chosen in $\{\mathbf{p}_1, \ldots, \mathbf{p}_4\} \setminus \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$ according to the following remarks leading to (76).

- The two branch points $\mathbf{p}_-$ and $\mathbf{p}_+ \neq \sigma'(\mathbf{p})$ are used to define the two sets of frames $\{U_-, V_-\}$ and $\{U_+, V_+\}$, and also the one-form Ω as before. Now, by Proposition 3.6, since γ passes through $\mathbf{p}_-$ it misses the pair of branch points $\{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$, so

$$\gamma(\mathbb{R}) \subset \mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\} = \{\text{Domain of } \{U_+, V_+\}\}. \tag{75}$$

- For all $t \in \mathbb{R}$, $|\mathbf{g}(\dot{\gamma}(t), U_+)_{\gamma(t)}| \neq 1$; equality would mean that γ is tangent and hence equal to a geodesic passing through $\mathbf{p}_+$, but since γ goes through $\mathbf{p}_-$ by hypothesis, this is not possible because γ is not supported by Σ_0.

So, in light of these facts, we can take the one $\mathbf{p}_+$ for which, for all $t \in \mathbb{R}$

$$\mathbf{g}(\dot{\gamma}(t), V_+)_{\gamma(t)} > 0. \tag{76}$$

Now we make the following definition.

Definition 5.4 Let $\omega_\gamma : \mathbb{R} \to (0, \pi)$ be given by

$$\omega_\gamma(t) := \arccos \mathbf{g}\left(\dot{\gamma}(t),\ U_+\right)_{\gamma(t)}. \tag{77}$$

The relation between ω_γ and ω_k in Theorem 5.1 is as follows.

Proposition 5.5 *For all $k \in \mathbb{Z}$,*

$$|\omega_k| = |\omega_\gamma(kT)|. \tag{78}$$

Proof. For $\mathbf{p}_-$ and $\mathbf{p}_+$ as above the singular geodesic $\gamma^{\Sigma_0} : \mathbb{R} \to \Sigma_0$ is parametrized and oriented according to increasing arc length so that

$$\gamma^{\Sigma_0}(0) = \mathbf{p}_-, \quad \gamma^{\Sigma_0}(T) = \sigma'(\mathbf{p}_-), \quad \mathbf{p}_+ \in \gamma^{\Sigma_0}(\{0 < t < T\}).$$

By definition of the frame $\{U_+, V_+\}$ in (40), for all $k \in \mathbb{Z}$ (see Fig. 7),

$$\begin{aligned} U_+|_{\mathbf{p}_-} &= -\dot{\gamma}^{\Sigma_0}(kT) \in T_{\mathbf{p}_-}\Sigma_0 \quad \text{for } k \text{ even}, \\ U_+|_{\sigma'(\mathbf{p}_-)} &= \dot{\gamma}^{\Sigma_0}(kT) \in T_{\sigma'(\mathbf{p}_-)}\Sigma_0 \quad \text{for } k \text{ odd}. \end{aligned} \tag{79}$$

Thus, for $k \in \mathbb{Z}$, from the expression of ω_γ in Definition 5.4,

$$\omega_k = (\text{angle formed by } \dot{\gamma}(kT) \text{ and } \Sigma_0) = (-1)^{k+1}\omega_\gamma(kT).$$

Proposition 5.6 *For all $k \in \mathbb{Z}$, assuming $\gamma(0) = \mathbf{p}_-$, for $kT < t < (k+1)T$,*

$$\Omega(\dot{\gamma}(t)) = \frac{(-1)^k}{\sin \omega_\gamma(t)} \frac{\mathrm{d}\omega_\gamma}{\mathrm{d}t}. \tag{80}$$

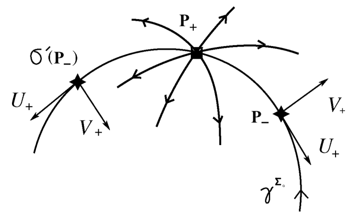

Figure 7. $U_+|_{\mathbf{p}_-} \in T_{\mathbf{p}_-}\Sigma_0$ and $U_+|_{\sigma'(\mathbf{p}_-)} \in T_{\sigma'(\mathbf{p}_-)}\Sigma_0$.

Proof. By Proposition 3.6, for all $k \in \mathbb{Z}$,

$$\gamma(2kT) = \mathbf{p}_-, \quad \gamma((2k+1)T) = \sigma'(\mathbf{p}_-). \tag{81}$$

Hence, in light of Corollary 3.8, for all $k \in \mathbb{Z}$,

$$\forall t \in (kT, (k+1)T) \begin{cases} \gamma(t) \text{ moves away from } \mathbf{p}_- \text{ if } k \text{ is even;} \\ \gamma(t) \text{ moves towards } \mathbf{p}_- \text{ if } k \text{ is odd.} \end{cases}$$

So, by (38) and (40) (see Fig. 8),

$$\dot\gamma(t) = (-1)^k U_-|_{\gamma(t)}. \tag{82}$$

Thus, for all $k \in \mathbb{Z}$, whenever $t \in (kT, (k+1)T)$,

$$\begin{aligned} \cos\omega(\gamma(t)) &\overset{\text{by (41)}}{=} \mathbf{g}(U_-,\ U_+)_{\gamma(t)} \\ &\overset{\text{by (82)}}{=} (-1)^k\ \mathbf{g}(\dot\gamma(t),\ U_+)_{\gamma(t)} \\ &\overset{\text{by (77)}}{=} (-1)^k\ \cos\omega_\gamma(t). \end{aligned} \tag{83}$$

Similarly,

$$\begin{aligned} \sin\omega(\gamma(t)) &= \mathbf{g}\,(U_-,\ V_+)_{\gamma(t)} \\ &= (-1)^k\ \mathbf{g}\,(\dot\gamma(t),\ V_+)_{\gamma(t)} \\ &= (-1)^k\ \sin\omega_\gamma(t). \end{aligned} \tag{84}$$

Use the t-derivative in (83) and the last equality in (84) to get (80).

5.2. *Proof of Theorem* 5.1

Let $\gamma : \mathbb{R} \to \mathbf{S}^2$ be a unit-speed geodesic not supported by Σ_0 with $\gamma(kT) = \mathbf{p}_-$, a branch point, for $k \in \mathbb{Z}$ even, and $\gamma(kT) = \sigma'(\mathbf{p}_-)$ for $k \in Z$ odd.

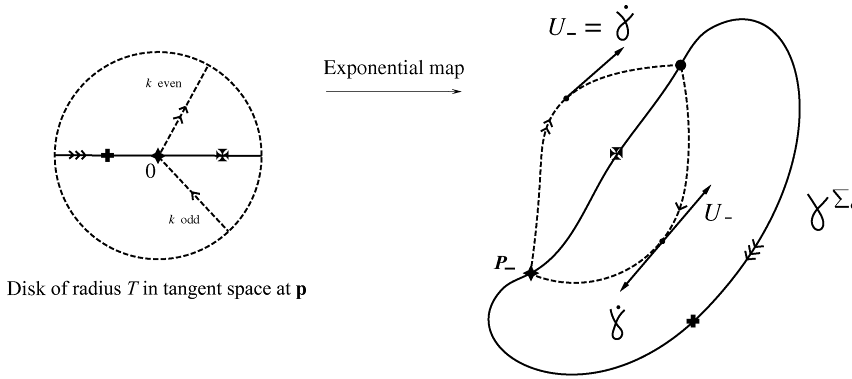

Figure 8. Illustrating (82), segments of γ towards and away from $\mathbf{p}_-$.

Let $\omega_\gamma : \mathbb{R} \to (0, \pi)$ as before, and recall $\mathrm{d} \ln \tan(x/2) \overset{*}{=} \mathrm{d}x/\sin x$. Then, $\forall\, k \in \mathbb{Z}$,

$$\begin{aligned} \ln \left| \frac{\tan(\omega_{k+1}/2)}{\tan(\omega_k/2)} \right| &\overset{\text{by Prop. 5.5}}{=} \ln \left| \frac{\tan(\omega_\gamma((k+1)T)/2)}{\tan(\omega_\gamma(kT)/2)} \right| \\ &\overset{\text{by } (*)}{=} \int_{kT}^{(k+1)T} \frac{1}{\sin \omega_\gamma(t)} \frac{\mathrm{d}\omega_\gamma}{\mathrm{d}t} \,\mathrm{d}t \\ &\overset{\text{by Prop. 5.6}}{=} (-1)^k \int_{kT}^{(k+1)T} \Omega(\dot\gamma(t)) \,\mathrm{d}t \\ &\overset{\text{by (73)}}{=} \int_0^T \Omega(\dot\gamma(t)) \,\mathrm{d}t \\ &\overset{\text{by Prop. 5.6}}{=} \int_0^T \frac{1}{\sin \omega_\gamma(t)} \frac{\mathrm{d}\omega_\gamma}{\mathrm{d}t} \,\mathrm{d}t \\ &=: A_0. \end{aligned} \tag{85}$$

Now, take $C = \mathrm{e}^{2A_0}$. Reversing the orientation of γ gives C^{-1}.

6. The derivative at branch points of the Jacobi field induced by *F*

THEOREM 6.1 *Let* $(\mathbf{S}^2, \mathbf{g})$ *with* F, $\mathbf{p}$, γ *and* C *be as in Theorem* 5.1. *Put*

$$A_\gamma := \tan^2\left(\frac{\omega_0}{2}\right), \tag{86}$$

where ω_0 ($\neq 0$ *for* $\gamma \neq \gamma^{\Sigma_0}$) *is the angle formed by* $\dot\gamma(0)$ *and* Σ_0; *also, for* $k \in \mathbb{Z}$, *let*

$$\alpha_k := \frac{(1 + A_\gamma)^2\, C^k}{(1 + A_\gamma\, C^k)^2}.$$

Let J_γ *be the Jacobi field induced by* X_F, J'_γ *its covariant derivative along* γ. *Then*

- *when* **g** *is smooth*

$$\|J'_\gamma(kT)\|^2 = \begin{cases} \alpha_k \|J'_\gamma(0)\|^2 & \text{for } k \in \mathbb{Z} \text{ even}, \\ \alpha_k \|J'_\gamma(T)\|^2 & \text{for } k \in \mathbb{Z} \text{ odd}; \end{cases} \tag{87}$$

- *when* **g** *is real analytic*

$$\|J'_\gamma(kT)\|^2 = \alpha_k \|J'_\gamma(0)\|^2 \quad \text{for all } k \in \mathbb{Z}. \tag{88}$$

COROLLARY 6.2 *For $\gamma \neq \gamma^{\Sigma_0}$ and if $C \neq 1$*

$$\sum_{k=-\infty}^{\infty} \frac{\|J'_\gamma(kT)\|^{-2}}{1+k^2T^2} = \infty. \tag{89}$$

Corollary 6.2 is used in Theorem 7.3.

Proof. (Assuming Theorem 6.1 and for **g** real analytic.)
Using (88)

$$\sum_{k=-\infty}^{\infty} \frac{\|J'_\gamma(kT)\|^{-2}}{1+k^2T^2} = \frac{\|J'_\gamma(0)\|^{-2}}{(1+A_\gamma)^2} \sum_{k=-\infty}^{\infty} \frac{\left(C^{-k/2} + A_\gamma C^{k/2}\right)^2}{1+k^2T^2},$$

which diverges, since $0 < C \neq 1$, and $A_\gamma > 0$ for γ is not supported by Σ_0.

The proof of Theorem 6.1 is given at the end of this section after we relate the Jacobi field J_γ along γ with the angle ω_γ.

Fix the branch point $\mathbf{p}_- := \mathbf{p} = \gamma(0)$, and a second branch point $\mathbf{p}_+ \neq \sigma'(\mathbf{p}_-)$. Consider the co-frames $\{\Phi_1^-, \Phi_2^-\}$ and $\{\Phi_1^+, \Phi_2^+\}$ as in (58).

PROPOSITION 6.3 *The images of Φ_1^- and Φ_1^+ live in $\{F = 0\} \cap \{H = 1/2\} \subset T^*\mathbf{S}^2$.*

Proof. The unit vector field U_- is, by definition of $\exp_{\mathbf{p}_-}$, tangential to the geodesics through $\mathbf{p}_-$ which all have a value of F equal to zero by Proposition 3.3. Similarly for U_+ using $\exp_{\mathbf{p}_+}$.

PROPOSITION 6.4 *There is a smooth function $\Upsilon : \mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\} \to \mathbb{R}$ such that the diagram below commutes.*

$$\begin{array}{ccc} \mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\} & \xrightarrow{\Phi_1^+} & T^*\mathbf{S}^2 \\ \Big\downarrow{\scriptstyle \Upsilon V_+} & & \Big\downarrow{\scriptstyle X_F} \\ T\mathbf{S}^2 & \xleftarrow{\pi_*} & T\left(T^*\mathbf{S}^2\right) \end{array}$$

Proof. We can always write $\pi_* \circ X_F \circ \Phi_1^+ = \alpha\, U_+ + \Upsilon\, V_+$ for appropriate smooth functions α and Υ on $\mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$. To show $\alpha \equiv 0$, first use that, by Proposition 2.1, for every

$z^* \in \{F = 0\} \cap \{H = 1/2\} \subset T^*\mathbf{S}^2$

$$\mathbf{g}(\lambda^{-1} z^*, \pi_* X_F|_{z^*}) = 0. \tag{90}$$

Then, putting $z^* = \Phi_1^+|_q = \lambda U_+|_q$, $q \in \mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$, with λ the Legendre transformation (2), in light of Proposition 6.3, we get

$$\alpha = \mathbf{g}(U_+|_q\ ,\ \pi_*(X_F|_{\Phi_1^+|_q})) = 0. \tag{91}$$

PROPOSITION 6.5 *Let $\gamma : \mathbb{R} \to \mathbf{S}^2 \setminus \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$ be any unit-speed geodesic that passes through a branch point $\mathbf{p}$ and J_γ the Jacobi field induced by X_F as in Definition* 2.2. *Then, for all $t \in \mathbb{R}$,*[3]

$$|\Upsilon(\gamma(t))| = \|J_\gamma(t)\|. \tag{92}$$

Proof. By Proposition 3.1, for all $z^* \in \{F = 0\} \cap \{H = \frac{1}{2}\} \subset T^*\mathbf{S}^2$,

$$\|\pi_*(X_F|_{z^*})\|^2 = f(x)g(y),$$

where $z^* = (x, y)$ in Liouville coordinates. Using this, first with $z^* = \Phi_1^+|_q$ and then with $z^* = \Phi_1^-|_q$ for $q \in \mathbf{S}^2 \setminus \{\mathbf{p}_1, \ldots, \mathbf{p}_4\}$, we get due to Proposition 6.3,

$$\begin{aligned} (\Upsilon(q))^2 &\overset{\text{(i)}}{=} \|\pi_*(X_F|_{\Phi_1^+|_q})\|^2 \\ &\overset{\text{(ii)}}{=} f(x)g(y) \\ &\overset{\text{(iii)}}{=} \|\pi_*(X_F|_{\Phi_1^-|_q})\|^2. \end{aligned} \tag{93}$$

where $q = (x, y)$. From these equalities we derive the following.

(1) By (ii), Υ is extended continuously to $\mathbf{S}^2$ (as zero at the branch points), and

$$\Sigma_0 = \{q \in \mathbf{S}^2 | \Upsilon(q) = 0\}, \tag{94}$$

since from (8) and (14), $\Sigma_0 = \wp(\partial Q_{0,0})$ and $f(x)$ or $g(y)$ vanishes on $\partial Q_{0,0}$.

(2) Let γ through $\mathbf{p}$ be a unit-speed geodesic not supported by Σ_0.
   - If $\mathbf{p} \in \{\mathbf{p}_+, \sigma'(\mathbf{p}_+)\}$ use (i) and put for $t \in \mathbb{R}$, with λ as in (2), $q = \gamma(t)$ and $\lambda\dot{\gamma}(t)\Phi_1^+|_{\gamma(t)}$.
   - If $\mathbf{p} \in \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$ use (iii) and put instead $q = \gamma(t)$ and $\lambda\dot{\gamma}(t)\Phi_1^-|_{\gamma(t)}$.

   In either case, in light of Definition 2.2 we get for all $t \in \mathbb{R}$ $(\Upsilon(\gamma(t)))^2 = \|J_\gamma(t)\|^2$.

(3) If $\gamma = \gamma^{\Sigma_0}$, the one supported by Σ_0, by Lemma 3.9, $J_\gamma \equiv 0$. Now use (94). Thus (92) is proved.

[3]To illustrate, a calculation similar to that in [2] for the triaxial ellipsoid shows that in the case of the round metric in $\mathbf{S}^2$ written in Liouville form in the sphero-conical coordinates (see footnote for Remark 2.9) we have, up to a multiplicative constant, $\|X_F|_{z^*}\|^2 = x_2^2$ for z^* in the unit cotangent bundle of $\mathbf{S}^2$. Let γ be a geodesic issued from a branch point $\gamma(0) = \mathbf{p}$, α the angle formed by γ and Σ_0 at $\mathbf{p}$, s the distance of $\mathbf{S}^2$ from $\mathbf{p}$ and $J_\gamma(s)$ the Jacobi field along γ induced by X_F evaluated at $\gamma(s)$. Then $\|J_\gamma(s)\|^2 = \sin^2\alpha\, \sin^2 s$, and $|\Upsilon(\gamma(t))| = |\sin\alpha\, \sin s|$.

6.1. *Proof of Theorem* 6.1

Let γ be a unit-speed geodesic passing through the branch point $\mathbf{p}$ at $t = 0$ not supported by Σ_0 and oriented so that the constant C in Theorem 5.1 applies. The Jacobi field J_γ is normal along γ and $J_\gamma(t) = 0$ if and only if $t = kT$ for $k \in \mathbb{Z}$ by Lemma 3.9. Thus we write

$$J_\gamma(t) = h_\gamma(t) P_\gamma(t),$$

where $h_\gamma\colon \mathbb{R} \to \mathbb{R}$ is a smooth function and $P_\gamma\colon \mathbb{R} \to T\mathbf{S}^2$ is a parallel vector field so that $\{\dot\gamma(t), P_\gamma(t)\}$ is an orthonormal oriented frame of $T_{\gamma(t)}\mathbf{S}^2$ for all $t \in \mathbb{R}$.

Moreover, since, by Proposition 6.5, for all $t \in \mathbb{R}$ $\|J_\gamma(t)\| = |\Upsilon(\gamma(t))|$, there is a sequence $\{\epsilon_k\}_{k\in\mathbb{Z}}$, with $\epsilon_k \in \{1, -1\}$ such that for $kT < t < (k+1)T$,

$$h_\gamma(t) = \epsilon_k\, \Upsilon(\gamma(t)).$$

Thus,

$$\begin{aligned}
\|J'_\gamma(kT)\| &= |\lim_{s\to 0} \frac{h_\gamma(kT+s) - h_\gamma(kT)}{s}| \\
&= |\lim_{s\to 0^+} \frac{h_\gamma(kT+s)}{s}| \\
&= |\lim_{s\to 0^+} \frac{\epsilon_k \Upsilon(\gamma(kT+s))}{s}| \\
&= |\mathrm{d}\Upsilon(\dot\gamma(kT))| \\
&\overset{(*)}{=} \Big|\cos\omega_\gamma(kT)\ \mathrm{d}\Upsilon\ (U_+)_{\gamma(kT)}\ +\ \sin\omega_\gamma(kT)\ \mathrm{d}\Upsilon\ (V_+)_{\gamma(kT)}\Big| \\
r &\overset{(**)}{=} \Big|\sin\omega_\gamma(kT)\ \mathrm{d}\Upsilon_+\ (V_+)_{\gamma(kT)}\Big|.
\end{aligned}$$

Here we used Definition 5.4 in (*), and in (**) that for all $k \in \mathbb{Z}$

$$\mathrm{d}\Upsilon(U_+)_{\gamma(kT)} = 0, \tag{95}$$

which holds since (see Fig. 9)

- $\gamma(kT) \in \{\mathbf{p}_-, \sigma'(\mathbf{p}_-)\}$,
- $U_+|_{\mathbf{p}_-} \in T_{\mathbf{p}_-}\Sigma_0$ and $U_+|_{\sigma'(\mathbf{p}_-)} \in T_{\sigma'(\mathbf{p}_-)}\Sigma_0$, and
- Υ_+ vanishes along Σ_0, by Corollary 6.9.

Consequently

$$\|J'_\gamma(kT)\| = \begin{cases} a\ \big|\sin\omega_\gamma(kT)\big| & \text{for } k \text{ even}, \\ b\ \big|\sin\omega_\gamma(kT)\big| & \text{for } k \text{ odd}, \end{cases} \tag{96}$$

where

$$\begin{aligned} a &:= |\mathrm{d}\Upsilon(V_+)_{\mathbf{p}_-}|, \\ b &:= |\mathrm{d}\Upsilon(V_+)_{\sigma'(\mathbf{p}_-)}|. \end{aligned} \tag{97}$$

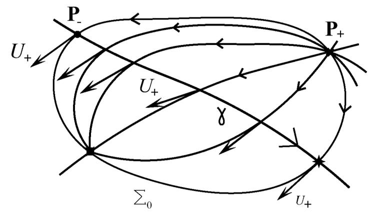

Figure 9. $U_+|_{\gamma(2rK)} \in T_{\mathbf{p}_-}\Sigma_0$, $U_+|_{\gamma((2r+1)K)} \in T_{\sigma'(\mathbf{p}_-)}\Sigma_0$. Illustrates (95).

Both $a > 0$ and $b > 0$, since $J_\gamma(kT) = 0$ and the Jacobi field J_γ is not identically zero, by Lemma 3.9, since γ is not supported by Σ_0.

But, from Theorem 5.1,

$$\tan^2\left(\frac{\omega_\gamma(kT)}{2}\right) = C^k \tan^2\left(\frac{\omega_\gamma(0)}{2}\right) \overset{(*)}{=} C^k A_\gamma, \tag{98}$$

where the equality $(*)$ uses both (86) and that, by the definitions and parametrizations,

$$|\cos\omega_0| = |\mathbf{g}\left(\dot\gamma(0), \dot\gamma^{\Sigma_0}(0)\right)| = |\mathbf{g}\left(\dot\gamma(0), U_+(\mathbf{p}_-)\right)| = |\cos\omega_\gamma(0)\big)|.$$

Now, use (96), (98) and the identity $\cos x = (1 - \tan^2(x/2))(1 + \tan^2(x/2))^{-1}$ to get, for all $r \in \mathbb{Z}$,

$$\begin{aligned} \|J'_\gamma(2rT)\|^2 &= \frac{4\,A_\gamma\,C^{2r}}{(1 + A_\gamma\,C^{2r})^2}\;\; a^2, \\ \|J'_\gamma((2r+1)T)\|^2 &= \frac{4\,A_\gamma\,C^{2r+1}}{(1 + A_\gamma\,C^{2r+1})^2}\;\; b^2. \end{aligned} \tag{99}$$

Use these formulae to calculate $\|J'_\gamma(0)\|^{-2}$ and $\|J'_\gamma(T)\|^{-2}$ first, and use them again to eliminate a^2 and b^2 to obtain (87).

To show (88), note that the constants a and b in (99) are defined by (97). Now, if $\mathbf{g}$ is real analytic σ' is an isometry (by Proposition 2.11) that leaves Σ_0 invariant; so

$$\sigma'_*\left(V_+|_{\mathbf{p}}\right) = \pm V_+|_{\sigma'(\mathbf{p})},$$

for $V_+|_{\mathbf{p}}$ and $V_+|_{\sigma'(\mathbf{p})}$ are unit vectors orthogonal to Σ_0 at $\mathbf{p}$ and $\sigma'(\mathbf{p})$, respectively. Thus, since Υ is invariant by the isometry σ',

$$a = |\mathrm{d}\Upsilon(V_+|_{\mathbf{p}_-})| = |\mathrm{d}\Upsilon(\sigma'_*(V_+|_{\mathbf{p}_-}))| = |\mathrm{d}\Upsilon(V_+|_{\sigma'(\mathbf{p}_-)})| = b,$$

and (88) follows.

7. Non-existence of adapted complexifications with infinite radius

Recall that the *adapted complex structure* $\mathbf{J}$ on the open set $\mathcal{U} \subset TM$ with $M \subset \mathcal{U}$ is the unique complex structure such that for every unit-speed geodesic $\gamma: \mathbb{R} \to M$

$$\gamma_*: (T\mathbb{R} \simeq \mathbb{C}) \cap \gamma_*^{-1}(\mathcal{U}) \to (TM, \mathbf{J}), \quad x + \sqrt{-1}y \to y\dot{\gamma}(x) \tag{100}$$

is holomorphic [15]. $\mathbf{J}$ *we call unbounded if the maximal* $\mathcal{U}$ *equals* TM*, in which case* $\mathcal{U}$ *is said to have infinite radius.*

In this section we prove the following.

THEOREM 7.1 *A real analytic Liouville Riemannian metric on the two-sphere* $\mathbf{S}^2$ *with finite isometry group cannot have an adapted complexification with infinite radius.*

COROLLARY 7.2 *Similarly for the real projective space* $\mathbb{RP}^2$.

Theorem 7.1 and Corollary 7.2 are proved after the next two propositions.

PROPOSITION 7.3 *Let* $\mathbf{g}$ *be a real analytic Riemannian metric on* $\mathbf{S}^2$ *whose adapted complexification has infinite radius. Let* γ *be a unit-speed geodesic in* $\mathbf{S}^2$ *and* J *a Jacobi field normal along* γ *(that is, point-wise orthogonal to* $\dot{\gamma}$*). Assume that there is a constant* $0 < T \in \mathbb{R}$ *such that*

$$J(t) = 0 \iff t \in \{kT \mid k \in \mathbb{Z}\}. \tag{101}$$

Then, denoting by J' *the covariant derivative of* J *in the direction of* $\dot{\gamma}$*,*

$$\sum_{k=-\infty}^{\infty} \frac{\|J'(kT)\|^{-2}}{1 + k^2T^2} < \infty. \tag{102}$$

Proof. Let G be the Jacobi field normal along γ and linearly independent from J such that the Wronskian between J and G,

$$\mathbf{g}\left(J(t), G'(t)\right) - \mathbf{g}\left(G(t), J'(t)\right) \equiv 1.$$

Since J is normal to γ then off the zeros of J, that is, for all $t \in \mathbb{R} \setminus \{kT \mid k \in \mathbb{Z}\}$,

$$G(t) = m(t)J(t), \tag{103}$$

where $m: \mathbb{R} \setminus \{kT \mid k \in \mathbb{Z}\} \to \mathbb{R}$ is real analytic with

$$m'(t) \|J(t)\|^2 = 1. \tag{104}$$

As a particular case of the results in [15] it follows that if the adapted complex structure exists on the whole $T\mathbf{S}^2$ then

(a) any Jacobi field canonically extends as a holomorphic section of the pull-back bundle $\gamma_*^{-1}T^{1,0}(T\mathbf{S}^2)$ over $\mathbb{C}$ (cf. 100), and

(b) any such extension does not vanish on $\mathbb{C} \setminus \mathbb{R}$.[4]

Hence the canonical holomorphic extension of G divided by that of J gives a meromorphic extension $m_{\mathbb{C}}$ in the complex plane $\mathbb{C}$ of the real-valued function m in (103),

$$m_{\mathbb{C}} : \mathbb{C} \to \mathbb{C} \cup \infty, \qquad m \equiv m_{\mathbb{C}}|_{\mathbb{R}\setminus\{kT \mid k\in\mathbb{Z}\}},$$

whose poles, in light of (104) and (b) in the previous paragraph, correspond to the zeros of J, namely $\{z = kT + 0\sqrt{-1} \mid k \in \mathbb{Z}\} \subset \mathbb{R} \subset \mathbb{C}$.

More is shown in [**15**] about $m_{\mathbb{C}}$ (see also [**18**]): (up to switching to $-G$ and adjusting the Wronskian condition and the right-hand side of (104)) we have

$$\Im m_{\mathbb{C}}(z) > 0$$

for $y = \Im z > 0$. Consequently the classical Fatou representation gives

$$\Im m_{\mathbb{C}}(z) = yD_{\gamma,J} + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_{\gamma,J}}{(x-t)^2 + y^2}, \tag{105}$$

where

$$0 \leq D_{\gamma,J} \in \mathbb{R} \tag{106}$$

is a constant and $\mathrm{d}\mu_{\gamma,J} \geq 0$ a Borel measure on $\mathbb{R}$, both depending on γ and J, with

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_{\gamma,J}(t)}{1+t^2} < \infty. \tag{107}$$

Now, since $m \equiv m_{\mathbb{C}}|_{\mathbb{R}\setminus\{kT \mid k\in\mathbb{Z}\}}$ and m is real-valued,

$$\Im m_{\mathbb{C}}|_{\mathbb{R}\setminus\{kT \mid k\in\mathbb{Z}\}} \equiv 0,$$

so the support of $\mathrm{d}\mu_{\gamma,J}$, being defined by means of the weak limit of $\Im m_{\mathbb{C}}$ as $\Im z \to 0^+$, lies in $\{t = kT \mid k \in \mathbb{Z}\}$; thus there is a sequence $\{(r_{\gamma,J})_k\}_{k\in\mathbb{Z}}$ with $(r_{\gamma,J})_k \geq 0$ so that (107) becomes

$$\sum_{k=-\infty}^{\infty} \frac{(r_{\gamma,J})_k}{1+k^2T^2} < \infty. \tag{108}$$

To identify each $(r_{\gamma,J})_k$ calculate, always with $z = x + \sqrt{-1}y$,

$$m'_{\mathbb{C}} = \frac{\mathrm{d}m_{\mathbb{C}}}{\mathrm{d}z} = \frac{\partial m_{\mathbb{C}}}{\partial x}$$

[4] For any Riemannian manifold M, a Jacobi field along a geodesic γ extends canonically as a section of $\gamma_*^{-1}T(TM)$ invariant by geodesic flow and rescaling along the fibers of TM; when the adapted structure $\mathbf{J}$ is defined on the whole TM, and as a consequence of (100), the corresponding holomorphic extension is obtained by taking the $(1,0)$ component with respect to $\mathbf{J}$ in $\gamma_*^{-1}T(TM) \otimes \mathbb{C}$; see [**15**].

on $\mathbb{C} \setminus \{x = kT \mid k \in \mathbb{Z}\}$ using the Cauchy–Riemann equations and (105) to obtain

$$m'_{\mathbb{C}}(z) = D_{\gamma,J} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{(z-t)^2}, \tag{109}$$

to get

$$m'_{\mathbb{C}}(z) = D_{\gamma,J} + \frac{1}{\pi}\sum_{k=-\infty}^{\infty} \frac{(r_{\gamma,J})_k}{(z-kT)^2}, \tag{110}$$

and thus, from (104), applying l'Hôpital's rule twice,

$$\begin{aligned} \frac{1}{\pi}\,(r_{\gamma,J})_k &= \lim_{t\to kT} \frac{(t-kT)^2}{\mathbf{g}\,(J(t), J(t))} \\ &= \lim_{t\to kT} \frac{(t-kT)}{\mathbf{g}\,(J'(t), J(t))} \\ &= \lim_{t\to kT} \frac{1}{\mathbf{g}\,(J'(t), J'(t)) + \mathbf{g}\,(J''(t), J(t))} \\ &= \|J'(kT)\|^{-2}, \end{aligned} \tag{111}$$

the last equality by (101). This used in (108) implies (102).

PROPOSITION 7.4 *Let* $\mathbf{g}$ *be a real analytic Riemannian metric on* $\mathbf{S}^2$ *whose adapted complexification has infinite radius. Let the geodesic* γ*, the Jacobi field* J*, and* T *be as in Proposition* 7.3*; in particular* (101) *holds. If, in addition, for all* $k \in \mathbb{Z}$

$$\|J'(0)\| = \|J'(kT)\|, \tag{112}$$

then there is a constant E *depending only on* $\gamma(0)$ *such that for all* $t \in \mathbb{R}$,

$$\frac{\|J(t)\|^2}{\|J'(0)\|^2} = \frac{T^2}{\pi^2}\,\frac{\sin^2(\pi t/T)}{1 + E\,\sin^2(\pi t/T)} \tag{113}$$

where, denoting by $\varrho\colon M \to \mathbb{R}$ *the Gauss curvature,*

$$3E = \frac{T^2}{\pi^2}\,\varrho\,(\gamma(0)) - 1. \tag{114}$$

Proof. By (111) and hypothesis (112), for all $k \in \mathbb{Z}$,

$$(r_{\gamma,J})_k = \pi\ \|J'(kT)\|^{-2} = \pi\ \|J'(0)\|^{-2} = (r_{\gamma,J})_0. \tag{115}$$

Thus, with m as in the proof of Proposition 7.3, for all $z \in \mathbb{C}$

$$\begin{aligned} m'_{\mathbb{C}}(z) &\overset{\text{by (110)}}{=} D_{\gamma,J} + \frac{1}{\pi}\sum_{k=-\infty}^{\infty}\frac{(r_{\gamma,J})_k}{(z-kT)^2} \\ &\overset{\text{by (115)}}{=} D_{\gamma,J} + \frac{(r_{\gamma,J})_0}{\pi\, T^2}\sum_{k=-\infty}^{\infty}\frac{1}{(z/T-k)^2} \\ &\overset{(*)}{=} D_{\gamma,J} + \frac{(r_{\gamma,J})_0\,\pi}{T^2}\,\frac{1}{\sin^2(\pi\, z/T)} \\ &\overset{(**)}{=} \frac{(r_{\gamma,J})_0\,\pi}{T^2}\left(E_{\gamma,J} + \frac{1}{\sin^2(\pi\, z/T)}\right), \end{aligned} \tag{116}$$

where in $(*)$ we used the classical identity $\sum_{k\in\mathbb{Z}}(x-k)^{-2} = \pi^2/\sin^2(\pi\, x)$, and in $(**)$ we are defining the quantity

$$E_{\gamma,J} := \frac{D_{\gamma,J}\, T^2}{\pi\, (r_{\gamma,J})_0}. \tag{117}$$

(Of course, $(r_{\gamma,J})_0 \neq 0$ since J is not the identically zero Jacobi field.)

But, for $t \in \mathbb{R} \setminus \{kT \mid k \in \mathbb{Z}\}$

$$m'_{\mathbb{C}}(t) = m'(t) \overset{\text{by (104)}}{=} \|J(t)\|^{-2},$$

and thus by (115) and (116) we get (113).

We now show that $E_{\gamma,J}$ in (117) is the constant E in (114); in particular independent of γ and of J.

Since by (103) m is the quotient of two solutions of $y''(t) + \varrho\,(\gamma(t))\, y(t) = 0$, where ϱ is the Gaussian curvature, the Schwarzian derivative of m equals 2ϱ.

Explicitly, $(m''(t)/m'(t))' - 1/2(m''(t)/m'(t))^2 = 2\,\varrho\,(\gamma(t))$. A calculation shows

$$\varrho\,(\gamma(t)) = \frac{\pi^2}{T^2}\left(1 + E_{\gamma,J} + 2E_{\gamma,J}\cos^2\left(\frac{\pi t}{T}\right)\right)\left(1 + E_{\gamma,J}\sin^2\left(\frac{\pi t}{T}\right)\right)^{-2};$$

setting $t = 0$ and solving for $E_{\gamma,J}$ yields (114) if we put $E := E_{\gamma,J}$.

7.1. *Proof of Theorem* 7.1

Let $(\mathbf{S}^2, \mathbf{g})$ be real analytic Liouville with finite isometry group. In particular, since the isometry group does not contain any $\mathbf{S}^1$ subgroup, it follows that $F \neq aF_l^2 + bH$ for $a, b \in \mathbb{R}$ and F_l linear homogenous along the fibers of $T^*\mathbf{S}^2$. Then, for any unit-speed geodesic γ not supported by Σ_0, with $\gamma(0) = \mathbf{p}$ one of the four branch points associated to F, the Jacobi field J_γ induced by F as in Definition 2.2 has the following properties (with $T \in \mathbb{R}$ the distance between antipodal branch points).

- By Theorem 5.1 and (88) in Theorem 6.1, there is a real constant $0 < A_\gamma$ that depends only on $\dot\gamma(0)$, and a real constant $C > 0$ independent of $\mathbf{p}$ and of γ, such that for all $k \in \mathbb{Z}$

$$\|J'_\gamma(kT)\|^2 = \frac{(1 + A_\gamma)^2\, C^k}{(1 + A_\gamma\, C^k)^2}\, \|J'_\gamma(0)\|^2. \tag{118}$$

- By Lemma 3.9, J_γ satisfies all the hypotheses on the Jacobi field needed in Proposition 7.3; hence, if $(\mathbf{S}^2, \mathbf{g})$ has adapted complexification with infinite radius,

$$\sum_{k=-\infty}^{\infty} \frac{\|J'_\gamma(kT)\|^{-2}}{1 + k^2T^2} < \infty. \tag{119}$$

Armed with these facts we treat the two possible cases (a) $C \neq 1$ and (b) $C = 1$.

(a) *Case* $(\mathbf{S}^2, \mathbf{g})$ *with* $C \neq 1$: Pick any unit-speed geodesic γ not supported by Σ_0, and $\gamma(0) = \mathbf{p}$, a branch point. By (118), since $C \neq 1$ (this is Corollary 6.2),

$$\sum_{k=-\infty}^{\infty} \frac{\|J'_\gamma(kT)\|^{-2}}{1 + k^2T^2} = \infty,$$

violating (119) if the adapted complexification of $(\mathbf{S}^2, \mathbf{g})$ has infinite radius.

(b) *Case* $(\mathbf{S}^2, \mathbf{g})$ *with* $C = 1$: Pick a branch point $\mathbf{p}$. Consider an oriented orthonormal frame of $T_\mathbf{p}\mathbf{S}^2$ given by $\{\dot\gamma^{\Sigma_0}, \dot\gamma_\perp^{\Sigma_0}\}$, where $\dot\gamma^{\Sigma_0}$ is tangent to Σ_0 at $\mathbf{p}$. Express the metric on $\mathbf{S}^2 \setminus \{\sigma'(\mathbf{p})\}$ in polar coordinates centered at $\mathbf{p}$ as done in (39),

$$\mathbf{g} = \mathrm{d}s^2 + \rho^2\, \mathrm{d}\psi^2. \tag{120}$$

We will show that, under the hypotheses, ρ does not depend on ψ, which implies an $\mathbf{S}^1$-symmetry contrary to the finiteness of the isometry group.

Take two families of Jacobi fields normal along the geodesic rays $\{\gamma^\psi\}_{\psi\in[0,2\pi)}$ emitted from $\mathbf{p}$, parametrized as follows: for $\psi \in [0, 2\pi)$, put $v_\psi := \cos\psi\; \dot\gamma^{\Sigma_0} + \sin\psi\; \dot\gamma_\perp^{\Sigma_0} \in T_\mathbf{p}\mathbf{S}^2$, and let γ^ψ be the unit-speed geodesic

$$t \mapsto \gamma^\psi(t) := \exp_\mathbf{p}\left(t\, v_\psi\right). \tag{121}$$

The two families of normal Jacobi fields along the geodesic segments γ^ψ restricted to $t \in [0, T]$ are $\{J_{\gamma^\psi}\}_{\psi\in[0,2\pi)}$ and $\{G_{\gamma^\psi} := \partial/\partial\psi|_{\gamma^\psi}\}_{\psi\in[0,2\pi)}$ defined as follows.

- For each ψ let J_{γ^ψ} be the field induced by X_F along γ^ψ as in Definition 2.2,

$$J_{\gamma^\psi}(t) := \pi_*\left(X_F(\Phi_t^H \lambda\, v_\psi\right).$$

  By (6), J_{γ^ψ} is point-wise orthogonal to $\dot\gamma^\psi$ since the geodesic γ^ψ passes through the branch point $\mathbf{p}$ and hence has F-value equal to zero by Proposition 3.3.

- For each ψ take the vector field along γ^ψ given by $G_{\gamma^\psi}(t) := \exp_{\mathbf{p}}|_* \big(t v_{\psi+\pi/2}\big)$. Each G_{γ^ψ} is a Jacobi field normal along γ^ψ, by Gauss lemma. In terms of the polar coordinates and the function ρ in (120), for $0 \le t \le T$,

$$G_{\gamma^\psi}(t) = \frac{\partial}{\partial \psi}|_{\gamma^\psi(t)} = \rho \; P_{\gamma^\psi}(t), \tag{122}$$

where $P_{\gamma^\psi} : \mathbb{R} \to T_{\gamma^\psi(t)}\mathbf{S}^2$ is parallel and orthogonal to $\dot{\gamma}^\psi$.

Now, for all $\psi \in (0, \pi) \cup (\pi, 2\pi)$, from (122) and Lemma3.9,

$$G_{\gamma^\psi}(0) = J_{\gamma^\psi}(0) = 0, \quad G_{\gamma^\psi}(T) = J_{\gamma^\psi}(T) = 0, \tag{123}$$

while for $0 < t < T$ we have both that $G_{\gamma^\psi}(t) \neq 0$ and $J_{\gamma^\psi}(t) \neq 0$. So, given that $\exp_{\mathbf{p}}$ is non-singular on $\{v \in T_{\mathbf{p}}\mathbf{S}^2 \; \|v\| < T\}$, from the initial conditions on these Jacobi fields it follows that for $0 \le t \le T$,

$$\begin{aligned} \|J_{\gamma^\psi}(t)\|^2 &= \|J'_{\gamma^\psi}(0)\|^2 \; \|G'_{\gamma^\psi}(0)\|^{-2} \; \|G_{\gamma^\psi}(t)\|^2 \\ &= \|J'_{\gamma^\psi}(0)\|^2 \; \rho^2, \end{aligned} \tag{124}$$

where we used (122) and that $\|G'_{\gamma^\psi}(0)\| = \|v_{\psi+\pi/2}\| = 1$.

Since $C = 1$, by (118), for all $\psi \in (0, \pi) \cup (\pi, 2\pi)$ and all $k \in \mathbb{Z}$

$$\|J'_{\gamma^\psi}(kT)\| = \|J'_{\gamma^\psi}(0)\|.$$

Consequently, for each $\psi \in (0, \pi) \cup (\pi, 2\pi)$ all the hypotheses on the Jacobi field in Proposition 7.3 as well as those in Proposition 7.4 are met when we put $J = J_{\gamma^\psi}$ and $\gamma = \gamma^\psi$. Thus, for all $t \in [0, T]$ and all $\psi \in (0, \pi) \cup (\pi, 2\pi)$,

$$\rho^2|_{\gamma^\psi(t)} = \frac{\|J_{\gamma^\psi}(t)\|^2}{\|J'_{\gamma^\psi}(0)\|^2} = \frac{T^2}{\pi^2} \; \frac{\sin^2(\pi t/T)}{1 + E \; \sin^2(\pi t/T)}, \tag{125}$$

where E is the constant in (114) that depends only on the Gauss curvature at $\gamma^\psi(0) = \mathbf{p}$. So, in principle on $\mathbf{S}^2 \setminus \Sigma_0$ and then, by smoothness, on $\mathbf{S}^2 \setminus \{\sigma'(\mathbf{p})\}$, we have $\mathrm{d}\rho\,(\partial/\partial\psi) \equiv 0$.

Thus, the vector field given by $X_\psi = \partial/\partial\psi$ on $\mathbf{S}^2 \setminus \{\sigma'(\mathbf{p})\}$ and $X_\psi = 0$ at $\sigma'(\mathbf{p})$ is a complete Killing vector field on $(\mathbf{S}^2, \mathbf{g})$, contrary to the hypothesis on $\mathbf{g}$.

7.2. *Proof of Corollary* 7.2

Let $(\mathbb{RP}^2, \mathbf{g}_0)$ be real analytic, Liouville, with finite isometry group. Lift $\mathbf{g}_0$ to $\mathbf{g}$ in $\mathbf{S}^2$ to get the 2-to-1 Riemannian cover $\pi_0\colon (\mathbf{S}^2, \mathbf{g}) \to \big(\mathbb{RP}^2, \mathbf{g}_0\big)$. The lift $\mathbf{g}$ will be Liouville real analytic with finite isometry group. Now, from (100) it follows that if $\big(\mathbb{RP}^2, \mathbf{g}_0\big)$ has adapted complexification with infinite radius so will $(\mathbf{S}^2, \mathbf{g})$, contrary to Theorem 7.1.

8. Isometric rigidity of the foliation

As an application of Theorem 7.1 we prove the following.

THEOREM 8.1 *Let $\varphi \colon X_1 \to X_2$ be a real analytic bijection between* unbounded *Grauert tubes with centers $M_i := u_i^{-1}(0) \simeq \mathbf{S}^2$. Assume that φ sends, as sets, the leaves of the Monge–Ampère foliation defined by the exhaustion u_1 in X_1 to those defined by u_2 in X_2. Endow X_i with the Kähler metric from $\partial\bar{\partial}u_i$. Then X_1 is, up to rescaling of u_1, biholomorphically isometric to X_2.*

Proof. The Monge–Ampère foliation of X_i consists of leaves that are Riemann surfaces; this foliation is singular precisely along M_i: each leaf intersects M_i along a geodesic of the metric $\mathbf{g}_i$ induced by restriction of the Kähler metric defined by $\partial\bar{\partial}u_i$ (see [15]).

It follows that φ maps M_1 onto M_2, and that $\varphi^*\mathbf{g}_2$ is a real analytic Riemannian metric on M_1 geodesically equivalent to $\mathbf{g}_1$.

Now, the global version of a theorem of Dini ([4; 19, Theorem 15.13]; also [7, pp. 62–65], for the local version) states that if a two-dimensional manifold has two, non-homothetic, geodesically equivalent Riemannian metrics (that is, one metric is not a constant multiple of the other, and they have the same geodesics as sets) then both metrics are Liouville.

In light of this we have

(1) $\mathbf{g}_1 = c\varphi^*\mathbf{g}_2$ for a constant $c > 0$, and hence, by uniqueness of the adapted complex structure [8, 15], up to rescaling of the function u_1 the tubes X_1 and X_2 are biholomorphically isometric with respect to their Kähler metrics, or
(2) $\mathbf{g}_1$ is a Liouville metric.

But possibility (2) is ruled out by Theorem 7.1 in case $(M_1, \mathbf{g}_1)$ has finite isometry group for X_1 is assumed a Grauert tube with infinite radius.

On the other hand, if $(M_1, \mathbf{g}_1)$ has $\mathbf{S}^1$-symmetry, so does $\varphi^*\mathbf{g}_2$ (see for instance [19]); thus both $\mathbf{g}_1$ and $\varphi^*\mathbf{g}_2$ are in the two-parameter family of Szöke's metrics of revolution from [18], and their geodesic equivalence implies that they are homothetic; that is, they must conform to option (1). Indeed, after a rescaling of the metrics by constants if necessary, $\mathbf{g}_1 = \mathbf{g}_\epsilon$ and $\varphi^*\mathbf{g}_2 = \mathbf{g}_{\epsilon'}$, where, for $0 \le s \le \pi, 0 \le \psi < 2\pi$

$$\mathbf{g}_\epsilon = \mathrm{d}s^2 + {\rho_\epsilon}^2 \mathrm{d}\psi^2, \tag{126}$$

${\rho_\epsilon}^2 := \sin^2 s/(1 + \epsilon^2 \sin^2 s)$, which are geodesically inequivalent for $\epsilon \neq \epsilon'$.

To show this last fact[5] let ∇^ϵ denote the Levi-Civita connection for $\mathbf{g}_\epsilon$; then if $\mathbf{g}_\epsilon$ is geodesically equivalent to $\mathbf{g}_{\epsilon'}$ there is a one-form L such that (see [5] for instance)

$$A(U, V) := \nabla^{\epsilon'}_U V - \nabla^\epsilon_U V = L(U)V + L(V)U.$$

Here A is symmetric and tensorial in U and V which are vector fields.

Since the meridians are geodesics with s the arc-length for any value of ϵ, $\nabla^\epsilon_{\partial/\partial s}\partial/\partial s = 0$, and thus

$$0 = A\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 2L\left(\frac{\partial}{\partial s}\right)\frac{\partial}{\partial s},$$

hence $L(\partial/\partial s) \equiv 0$.

[5]We thank the referee for suggesting the infinitesimal argument given here which replaces our previous longer proof in an earlier version, in which we explicitly described geodesics of $\mathbf{g}_\epsilon$ that were not geodesics of $\mathbf{g}_{\epsilon'}$ when $\epsilon' \neq \epsilon$.

Similarly the vector field $(1/\rho_\epsilon)(\partial/\partial\psi)$ is parallel with regard to the metric $\mathbf{g}_\epsilon$ along meridians, hence $\nabla^\epsilon_{\partial/\partial s}\partial/\partial\psi = (\partial \ln\rho_\epsilon/\partial s)(\partial/\partial\psi)$ and thus, using $L(\partial/\partial s) \equiv 0$,

$$\frac{\partial \ln(\rho_{\epsilon'}/\rho_\epsilon)}{\partial s}\frac{\partial}{\partial\psi} = A\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial\psi}\right) = L(\frac{\partial}{\partial\psi})\frac{\partial}{\partial s},$$

implying $L(\partial/\partial\psi) \equiv 0$, $L \equiv 0$, and that the metrics must be identical, that is, $\epsilon = \epsilon'$.

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DIVISIBILITY OF EXPONENTIAL SUMS AND SOLVABILITY OF CERTAIN EQUATIONS OVER FINITE FIELDS

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Abstract

Carlitz [Solvability of certain equations in a finite field, *Quart. J. Math.* (*Oxford*) **7** (1956), 3–4] determined conditions under which infinite families of polynomials have solutions in a finite field. In this paper we extend some of Carlitz's results by computing the exact p-divisibility of certain exponential sums. As a by-product we obtain an upper bound for the Waring number for polynomials over extensions of finite fields.

1. Introduction

Exponential sums over finite fields have many applications to different areas of mathematics [7]. One of the problems of interest is the estimation of its p-divisibility. By computing the exact p-divisibility of the exponential sum associated to a polynomial, one can determine if families of equations have solutions over a finite field.

A common tool for the estimation of this divisibility is the well-known theorem of Stickelberger [8]. If the exponential sum is expressed as the sum of Gauss sums, then Stickelberger's theorem gives the exact divisibility of each of the Gauss sums. Of the many known results on the divisibility of exponential sums, we use results presented in [9] that give conditions to obtain exact divisibility.

In [2], Carlitz determined conditions under which infinite families of polynomials have solutions in a finite field. In this paper we compute the exact p-divisibility of certain exponential sums under some natural conditions and extend some of Carlitz's results.

We also apply our results to generalizations of Waring's problem over finite fields, where we consider sums of polynomials. These generalizations lead to examples where the degree of the polynomial can be arbitrarily large but its Waring number is small.

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2. Preliminaries

Let $q = p^f$, p a prime, $\mathbb{F}_q$ be the finite field with q elements and $\mathbb{F}_q^* = \mathbb{F}_q \backslash 0$. Given $0 \leq j$, j_i integers such that $0 \leq j_i < p$ and $j = \sum_{i=0}^{r-1} j_i p^i$, we define the p-weight of j by $\sigma_p(j) = \sum_{i=0}^{r-1} j_i$. The p-weight degree of a monomial $X_1^{e_1} \cdots X_n^{e_n}$ is defined by $w_p(X_1^{e_1} \cdots X_n^{e_n}) = \sigma_p(e_1) + \cdots + \sigma_p(e_n)$. Sometimes we use $\mathbf{X}$ to denote the variables $X_1, \ldots, X_n$. The p-weight degree of a polynomial $F(X_1, \ldots, X_n) = \sum_i a_i X_1^{e_{1i}} \cdots X_n^{e_{ni}}, a_i \neq 0$ over $\mathbb{F}_{p^f}$ is defined by $w_p(F) = \max_i \; w_p(X_1^{e_{1i}} \cdots X_n^{e_{ni}})$. The p-weight degree with respect to the variable X_i of the monomial $X_1^{e_1} \cdots X_n^{e_n}$ is denoted by $w_{p,X_i}(X_1^{e_1} \cdots X_n^{e_n}) = \sigma_p(e_i)$. We denote the p-weight degree with respect to the variable X_i of a polynomial F over $\mathbb{F}_q$ by $w_{p,X_i}(F) = \max_i w_{p,X_i}(X_1^{e_{1i}} \cdots X_n^{e_{ni}})$. From now on, we assume that a polynomial $F(X_1, \ldots, X_n)$ contains all the variables $X_1, \ldots, X_n$.

$\mathbb{Q}_p$ is the p-adic field with ring of integers $\mathbb{Z}_p$. Let K be the extension over $\mathbb{Q}_p$ obtained by adjoining a primitive $(p^f - 1)$th root of unity in $\overline{\mathbb{Q}_p}$, the algebraic closure of $\mathbb{Q}_p$. The residue class field is isomorphic to $\mathbb{F}_q$, and let $\mathcal{T}$ denote the Teichmüller representatives of $\mathbb{F}_q$ in K. Let ξ be a primitive pth root of unity in $\overline{\mathbb{Q}_p}$. Define $\theta = 1 - \xi$ and denote by v_θ the valuation over θ. Note that $v_\theta(p) = p - 1$ and $v_p(x) = v_\theta(x)/(p-1)$.

Let $\phi : \mathbb{F}_q \to \mathbb{Q}(\xi)$ be a non-trivial additive character. The exponential sum associated to F is defined as follows:

$$S(F) = \sum_{x_1,\ldots,x_n \in \mathbb{F}_q} \phi(F(x_1, \ldots, x_n)).$$

Note that, since ϕ is an additive character, if $F = \sum_i a_i X_1^{e_{1i}} \cdots X_n^{e_{ni}} = G(X_1, \ldots, X_n) + \beta$, where $G(0, \cdots, 0) = 0$, then

$$\sum_{x_1,\ldots,x_n \in \mathbb{F}_q} \phi(F) = \sum_{x_1,\ldots,x_n \in \mathbb{F}_q} \phi(G + \beta) = \phi(\beta) \sum_{x_1,\ldots,x_n \in \mathbb{F}_q} \phi(G).$$

Since $\phi(\beta)$ is a unit, the constant term does not affect the p-divisibility of the sum.

The following theorem [9] gives a bound for the valuation of an exponential sum with respect to θ.

THEOREM 2.1 *Let* $F(\mathbf{X}) = \sum_{i=1}^N a_i X_1^{e_{1i}} \cdots X_n^{e_{ni}},\; a_i \neq 0$. *If* $S(F)$ *is the exponential sum*

$$S(F) = \sum_{x_1,\ldots,x_n \in \mathbb{F}_q} \phi(F(x_1, \ldots, x_n)), \tag{1}$$

then $v_\theta(S(F)) \geq L$, *where* $L = \min_{(j_1,\ldots,j_N)} \left\{ \sum_{i=1}^N \sigma_p(j_i) \mid 0 \leq j_i < q \right\}$, *and* $(j_1, \ldots, j_N)$ *is a solution to the system*

$$\begin{aligned} e_{11} j_1 + e_{12} j_2 + \ldots + e_{1N} j_N &\equiv 0 \bmod q - 1 \\ \vdots \qquad\qquad &\quad \vdots \\ e_{n1} j_1 + e_{n2} j_2 + \ldots + e_{nN} j_N &\equiv 0 \bmod q - 1, \end{aligned} \tag{2}$$

where $\sum_{i=1}^N e_{li} j_i \neq 0$, *for* $l = 1, \ldots, n$.

To obtain this bound, the authors in [9] use the Teichmüller representatives $a_i' \in \mathcal{T}$ of the coefficients a_i of F to lift and expand the exponential sum $S(F)$:

$$S(F) = \sum_{j_1=0}^{q-1} \cdots \sum_{j_N=0}^{q-1} \left[\prod_{i=1}^{N} c(j_i)\right] \left[\sum_{\mathbf{t}\in\mathcal{T}^n} \mathbf{t}^{j_1\mathbf{e}_1+\cdots+j_N\mathbf{e}_N}\right] \left[\prod_{i=1}^{N} {a'}_i^{j_i}\right]. \tag{3}$$

For each term $T = \prod_{i=1}^{N} c(j_i) \sum_{\mathbf{t}} \mathbf{t}^{j_1\mathbf{e}_1+\cdots+j_N\mathbf{e}_N} \prod_{i=1}^{N} {a'}_i^{j_i}$ in the sum (3),

$$\begin{aligned} v_\theta &\left(\left[\prod_{i=1}^{N} c(j_i)\right] \left[\sum_{\mathbf{t}} \mathbf{t}^{j_1\mathbf{e}_1+\cdots+j_N\mathbf{e}_N}\right] \left[\prod_{i=1}^{N} {a'}_i^{j_i}\right]\right) \\ &= \sum_{i=1}^{N} \sigma_p(j_i) + f(p-1)s, \end{aligned} \tag{4}$$

where s is the number of expressions in (2) that are equal to zero for the vector $(j_1, \ldots, j_N)$ associated to the term. Since the polynomial F contains all the variables, we can always find a solution $(j_1, \ldots, j_N)$ such that $\sum_{i=1}^{N} \sigma_p(j_i)$ is minimum and all the expressions in (2) are positive multiples of $q-1$, giving $f(p-1)s = 0$. The triangle inequality is then used to obtain the bound.

Note that one does not have equality on the valuation of $S(F)$ because it could happen that there are more than one solutions $(j_1, \ldots, j_N)$ that give the minimum value and when the associated terms are added they could produce higher powers of θ dividing the exponential sum. In section 4 we prove that, for many infinite families of polynomials, there is a unique solution $(j_1, \ldots, j_N)$ that gives the minimum. This implies that $v_\theta(S(F)) = L$. To prove that there is a unique minimum solution one has to consider all the possible solutions to the modular system, including those that make some of the equations in (2) equal to zero.

In our computations we will be using the following lemma, which was proved in [9].

LEMMA 2.1 *For any natural number k, $\sigma_p((p^f-1)k) \geq \sigma_p(p^f-1) = f(p-1)$.*

The relation between an exponential sum $S(F) = \sum_{\mathbf{x}\in\mathbb{F}_q{}^n} \phi(F(\mathbf{x}))$ and the number of zeros of a system of polynomials $P_1(\mathbf{X}), \ldots, P_t(\mathbf{X})$ is given by the following lemma that can be found in [1].

LEMMA 2.2 *Let $q = p^f$, $P_1(\mathbf{X}), \ldots, P_t(\mathbf{X}) \in \mathbb{F}_q[\mathbf{X}]$ and let N be the number of common zeros of $P_1, \ldots, P_t$. Then*

$$N = p^{-tf} \sum_{\mathbf{x}\in\mathbb{F}_q{}^n, \mathbf{y}\in\mathbb{F}_q{}^t} \phi(y_1 P_1(\mathbf{x}) + \cdots + y_t P_t(\mathbf{x})).$$

In this paper we use p-divisibility of exponential sums to prove solvability of polynomial equations. Another common method to prove solvability of equations is to estimate the absolute value of the corresponding exponential sum. Usually, for the absolute value method, the solvability only depends on having $q > d^\epsilon$, where d is the degree of the polynomial and q is the size of the field (see [6, 7, 10]). The results presented here cover cases that are not covered by the absolute value method.

3. Carlitz's results

There are many results on divisibility properties of the number of solutions of systems of polynomial equations over finite fields. Our work generalizes the following results of Carlitz [2].

THEOREM 3.1 *If $F(X_1, \ldots, X_n)$ is homogeneous of degree n while $G(X_1, \ldots, X_n)$ is of degree less than n, and*

$$\sum_{x_1,\ldots,x_n \in \mathbb{F}_q} F^{q-1}(x_1, \ldots, x_n) \neq 0,$$

then the equation $F(X_1, \ldots, X_n) = G(X_1, \ldots, X_n)$ has at least one solution over $\mathbb{F}_q$.

This is a very general theorem but the condition $\sum_{\mathbf{x} \in \mathbb{F}_q{}^n} F^{q-1}(\mathbf{x}) \neq 0$ could be hard to verify. The following results do not assume this condition.

THEOREM 3.2 *Let d be a divisor of $p-1$, and $a_i \in \mathbb{F}_q{}^*$ for $i = 1, \ldots, d$. If $G(X_1, \ldots, X_d)$ is a polynomial over $\mathbb{F}_q$ with $\deg(G) < d$, then the equation $a_1 X_1^d + \cdots + a_d X_d^d + G(X_1, \ldots, X_d) = 0$ has at least one solution over $\mathbb{F}_q$.*

COROLLARY 3.1 *Let d be a divisor of $p-1$ and $F_1(X_1), \ldots, F_d(X_d)$ be polynomials over $\mathbb{F}_q$ of degree d. Then the equation $F_1(X_1) + \cdots + F_d(X_d) = 0$ has at least one solution over $\mathbb{F}_q$.*

4. Exact divisibility of exponential sums and solvability of equations

In this section we compute the exact divisibility of certain exponential sums and of the number of solutions of the related equations. With this we guarantee that these equations are solvable and obtain generalizations of Carlitz's results.

THEOREM 4.1 *Let d_i be a divisor of $p-1$ and $a_i \in \mathbb{F}_q{}^*$ for $i = 1, \ldots, t$. Consider the monomials*

$$(X_{i_1} \cdots X_{i_{n_1}})^{d_1}, (X_{i_{n_1+1}} \cdots X_{i_{n_2}})^{d_2}, \ldots, (X_{i_{n_{t-1}+1}} \cdots X_{i_n})^{d_t} \tag{5}$$

all with the same degree $d > 1$, disjoint support and $1 \leq i_j \leq n = n_t$. If $G(X_1, \ldots, X_n)$ is a polynomial over $\mathbb{F}_q$ with $w_p(G) < d$, and

$$F(X_1, \ldots, X_n) = a_1(X_{i_1} \cdots X_{i_{n_1}})^{d_1} + a_2(X_{i_{n_1+1}} \cdots X_{i_{n_2}})^{d_2} + \cdots$$
$$+ a_t(X_{i_{n_{t-1}+1}} \cdots X_{i_n})^{d_t} + G(X_1, \ldots, X_n),$$

then

$$v_\theta(S(F)) = f(p-1) \sum_{i=1}^{t} \frac{1}{d_i}.$$

Proof. Without loss of generality, we can assume that the monomials in (5) are of the form

$$(X_1 \cdots X_{n_1})^{d_1}, (X_{n_1+1} \cdots X_{n_2})^{d_2}, \ldots, (X_{n_{t-1}+1} \cdots X_n)^{d_t},$$

and $F(0, \ldots, 0) = 0$. Let $G(X_1, \ldots, X_n) = \sum_{r=1}^{N} b_r X_1^{e_{1r}} \cdots X_n^{e_{nr}}$. As in Theorem 2.1, we associate a modular system to the polynomial F. The following is the modular system associated to F:

$$1)\begin{cases} d_1h_1 + e_{1,1}s_1 + e_{1,2}s_2 + \cdots + e_{1,N}s_N & \equiv 0 \bmod q-1 \\ \quad\vdots & \quad\vdots \\ d_1h_1 + e_{n_1,1}s_1 + \cdots + e_{n_1,N}s_N & \equiv 0 \bmod q-1 \end{cases} \tag{6}$$

$$2)\begin{cases} d_2h_2 + e_{n_1+1,1}s_1 + \cdots + e_{n_1+1,N}s_N & \equiv 0 \bmod q-1 \\ \quad\vdots & \quad\vdots \\ d_2h_2 + e_{n_2,1}s_1 + \cdots + e_{n_2,N}s_N & \equiv 0 \bmod q-1 \end{cases}$$

$$\vdots$$

$$t)\begin{cases} d_th_t + e_{n_{t-1}+1,1}s_1 + \cdots + e_{n_{t-1}+1,N}s_N & \equiv 0 \bmod q-1 \\ \quad\vdots & \quad\vdots \\ d_th_t + e_{n,1}s_1 + \cdots + e_{n,N}s_N & \equiv 0 \bmod q-1 \end{cases}$$

Let $(h_1, \ldots, h_t, s_1, \ldots, s_N)$ be any solution to system (6), and T be the term in (3) associated to this solution. Then

$$v_\theta(T) = \sum_{i=1}^{t} \sigma_p(h_i) + \sum_{i=1}^{N} \sigma_p(s_i) + f(p-1)s,$$

where s is the number of equations in (6) that are equal to zero. Let $n_0 = 0$ and, for $i = 1, \ldots, t$, let r_i be the number of equations that are equal to zero in each block i of $n_i - n_{i-1}$ equations in (6). Since $d_i \leq p-1$, $\sigma_p(d_i) = d_i$. Applying σ_p to (6), using Lemma 2.1, adding the first $n_1 - r_1$ non-zero inequalities, and dividing by d_1n_1, we obtain

$$\sigma_p(h_1) + \frac{\sigma_p(e_{1,1}) + \cdots + \sigma_p(e_{n_1,1})}{d_1n_1}\sigma_p(s_1) + \cdots + \frac{\sigma_p(e_{1,N}) + \cdots + \sigma_p(e_{n_1,N})}{d_1n_1}\sigma_p(s_N) \geq \frac{f(p-1)(n_1-r_1)}{n_1d_1}.$$

We repeat the same to each block i of modular equations in (6) to obtain:

$$\sigma_p(h_i) + \frac{\sigma_p(e_{n_{i-1}+1,1}) + \cdots + \sigma_p(e_{n_i,1})}{d_i(n_i - n_{i-1})}\sigma_p(s_1) + \cdots + \frac{\sigma_p(e_{n_{i-1}+1,N}) + \cdots + \sigma_p(e_{n_i,N})}{d_i(n_i - n_{i-1})}\sigma_p(s_N) \geq \frac{f(p-1)(n_i - n_{i-1} - r_i)}{(n_i - n_{i-1})d_i} \text{for} 1 \leq i \leq t.$$

Recall that $d_i(n_i - n_{i-1}) = d$, and add the above inequalities to get

$$\sum_{i=1}^{t} \sigma_p(h_i) + \frac{\sigma_p(e_{1,1}) + \cdots + \sigma_p(e_{n,1})}{d}\sigma_p(s_1) + \cdots + \frac{\sigma_p(e_{1,N}) + \cdots + \sigma_p(e_{n,N})}{d}\sigma_p(s_N)$$
$$\geq f(p-1)\sum_{i=1}^{t} \frac{n_i - n_{i-1} - r_i}{(n_i - n_{i-1})d_i}.$$

Since $\sigma_p(e_{1,k}) + \cdots + \sigma_p(e_{n,k})$ is the p-weight degree of the kth monomial of G, and $w_p(G) < d$, we have that $(\sigma_p(e_{1,k}) + \cdots + \sigma_p(e_{n,k}))/d < 1$. Therefore,

$$\sum_{i=1}^{t} \sigma_p(h_i) + \sum_{i=1}^{N} \sigma_p(s_i) \geq \sum_{i=1}^{t} \sigma_p(h_i) + \frac{\sigma_p(e_{1,1}) + \cdots + \sigma_p(e_{n,1})}{d}\sigma_p(s_1)$$
$$+ \cdots + \frac{\sigma_p(e_{1,N}) + \cdots + \sigma_p(e_{n,N})}{d}\sigma_p(s_N) \;\geq\; f(p-1)\sum_{i=1}^{t} \frac{n_i - n_{i-1} - r_i}{(n_i - n_{i-1})d_i},$$

and

$$\begin{aligned} v_\theta(T) &\geq f(p-1)\left[\sum_{i=1}^{t} \frac{n_i - n_{i-1} - r_i}{(n_i - n_{i-1})d_i} + \sum_{i=1}^{t} r_i\right] \\ &= f(p-1)\left[\sum_{i=1}^{t} \frac{1}{d_i} + \sum_{i=1}^{t} \frac{r_i\left[(n_i - n_{i-1})d_i - 1\right]}{(n_i - n_{i-1})d_i}\right]. \end{aligned} \tag{7}$$

Note that if $s_i \neq 0$ for some i, we have strict inequality in (7). Also note that since $r_i\left[(n_i - n_{i-1})d_i - 1\right] \geq 0$, any solution with $v_\theta(T) = f(p-1)\sum_{i=1}^{t} 1/d_i$ is minimal and has $s_i = 0$ for all $i = 1, \ldots, N$.

Consider

$$\left(\frac{\lambda_1(q-1)}{d_1}, \ldots, \frac{\lambda_t(q-1)}{d_t}, 0, \ldots, 0\right), \tag{8}$$

where $0 \leq \lambda_i \leq d_i$ for all $i = 1, \ldots, t$. This is a solution to (6) with

$$\sum_{i=1}^{t} \sigma_p(h_i) + \sum_{i=1}^{N} \sigma_p(s_i) = \sum_{i=1}^{t} \sigma_p\left(\frac{\lambda_i(q-1)}{d_i}\right)$$
$$= \sum_{i=1}^{t} \sigma_p\left(\frac{\lambda_i(p-1)}{d_i}\sum_{k=1}^{f} p^{f-k}\right) = f(p-1)\sum_{i=1}^{t} \frac{\lambda_i}{d_i},$$

and,

$$v_\theta(T) = f(p-1)\left[\sum_{i=1}^{t} \frac{\lambda_i}{d_i} + s\right],$$

where s is the number of equations in (6) that are equal to zero for this solution.

If $\lambda_i = 1$ for $i = 1, \ldots, t$, then none of the equations in (6) are equal to zero and $v_\theta(T) = f(p-1)\sum_{i=1}^{t} 1/d_i$. Therefore, $((q-1)/d_1, \ldots, (q-1)/d_t, 0, \ldots, 0)$ is a minimal solution. Any other minimal solution must have the form (8) and $\sum_{i=1}^{t}(r_i[(n_i - n_{i-1})d_i - 1])/(n_i - n_{i-1})d_i = 0$. Since $r_i[(n_i - n_{i-1})d_i - 1] \geq 0$, this sum is zero if and only if $n_i - n_{i-1} = d_i = 1$ or $r_i = 0$ for each $i = 1, \ldots, t$. If $r_i \neq 0$ for some i, then $n_i - n_{i-1} = d_i = 1$, and this implies that the polynomial F has degree 1, which is a contradiction. If $r_i = 0$ for all $i = 1, \ldots, t$, then $\lambda_i \geq 1$ for all i, $v_\theta(T) = f(p-1)\sum_{i=1}^{t} \lambda_i/d_i$, and this is a minimal solution if and only if $\lambda_i = 1$ for all i.

Therefore $((q-1)/d_1, \ldots, (q-1)/d_t, 0, \ldots, 0)$ is the unique minimal solution and $v_\theta(F) = f(p-1)\sum_{i=1}^{t} 1/d_i$.

REMARK 4.1 This result is false if we allow $d = 1$. For example, consider $G(X_1, \ldots, X_n) = c$, a constant, and $F(X_1, \cdots, X_n) = a_1X_1 + \cdots + a_nX_n + c$. Then $S(F) = 0$, and the formula for $v_\theta(S(F))$ is not correct.

Even though the above theorem includes the case $p = 2$, because of the importance of Boolean functions, we include this case as a corollary.

COROLLARY 4.1 *Let $p = 2$ and consider the monomials*

$$(X_{i_1} \cdots X_{i_{n_1}}), (X_{i_{n_1+1}} \cdots X_{i_{n_2}}), \ldots, (X_{i_{n_{t-1}+1}} \cdots X_{i_n})$$

all with the same degree $d > 1$, disjoint support, and $1 \leq i_j \leq n = n_t$. If $G(X_1, \ldots, X_n)$ is a polynomial over $\mathbb{F}_q$ with $w_p(G) < d$, and

$$F(X_1, \ldots, X_n) = X_{i_1} \cdots X_{i_{n_1}} + X_{i_{n_1+1}} \cdots X_{i_{n_2}} + \cdots + X_{i_{n_{t-1}+1}} \cdots X_{i_n} + G(X_1, \ldots, X_n),$$

then

$$v_2(S(F)) = tf.$$

The next corollaries give information about $S(F)$ when $t = 1$.

COROLLARY 4.2 *Let d be a divisor of $p-1$, $nd > 1$ and $a \in \mathbb{F}_q^*$. If $G(X_1, \ldots, X_n)$ is a polynomial over $\mathbb{F}_q$ with $w_p(G) < dn$, and*

$$F(X_1, \ldots, X_n) = aX_1^d \cdots X_n^d + G(X_1, \ldots, X_n),$$

then $v_\theta(S(F)) = f(p-1)/d$.

COROLLARY 4.3 *Let $d \neq 1$ be a divisor of $p-1$ and $a \in \mathbb{F}_q^*$ If $F(X) = aX^d + b_1X^{d_1} + \cdots + b_rX^{d_r}$ is a polynomial over $\mathbb{F}_q$, where $\sigma_p(d_i) < d$ for every i; then $v_\theta(S(F)) = f(p-1)/d$. Furthermore $S(F) \neq 0$.*

As a consequence of Theorem 4.1 we can compute the exact divisibility of the number of solutions of families of polynomial equations.

THEOREM 4.2 *Let d_i be a divisor of $p-1$ and $a_i \in \mathbb{F}_q^*$ for $i = 1, \ldots, t$. Suppose that $\sum_{i=1}^{t} 1/d_i$ is an integer, and consider the monomials*

$$(X_{i_1} \cdots X_{i_{n_1}})^{d_1}, (X_{i_{n_1+1}} \cdots X_{i_{n_2}})^{d_2}, \ldots, (X_{i_{n_{t-1}+1}} \cdots X_{i_n})^{d_t} \tag{9}$$

all with the same degree $d > 1$, disjoint support and $1 \le i_j \le n = n_t$. If $G(X_1, \ldots, X_n)$ is a polynomial over $\mathbb{F}_q$ with $w_p(G) < d$, and

$$F(X_1, \ldots, X_n) = a_1(X_{i_1} \cdots X_{i_{n_1}})^{d_1} + a_2(X_{i_{n_1+1}} \cdots X_{i_{n_2}})^{d_2} + \cdots$$
$$+ a_t(X_{i_{n_{t-1}+1}} \cdots X_{i_n})^{d_t} + G(X_1, \ldots, X_n),$$

then $p^{f(\sum_{i=}^{t} 1/d_i - 1)}$ is the exact divisibility of the number of solutions of $F = 0$. In particular, F has at least one solution over $\mathbb{F}_q$.

Proof. Consider

$$F' = y\left(a_1(X_{i_1} \cdots X_{i_{n_1}})^{d_1} + \cdots + a_t(X_{i_{n_{t-1}+1}} \cdots X_{i_n})^{d_t} + G(X_1, \ldots, X_n)\right).$$

By Lemma 2.2, the number of solutions of $F = 0$ is $p^{-f} S(F')$. To compute $S(F')$ we follow the proof of Theorem 4.1 with F' instead of F. The modular system associated to F' is the same as that associated to F but with the additional equation

$$h_1 + \cdots + h_t + s_1 + \cdots + s_N \equiv 0 \bmod q-1.$$

If $G(0, \ldots, 0) = 0$, then the proof is almost the same as that for Theorem 4.1. If G has a constant term $b_N X_1^{e_{1N}} \cdots X_n^{e_{nN}} \neq 0$, then the term s_N appears in the last new equation of system (6), but $e_{i,N} s_N$ do not appear in the other equations. In any case, we follow the proof of Theorem 4.1 and note that expression (7) becomes

$$v_\theta(T) \ge f(p-1)\left[\sum_{i=1}^{t} \frac{1}{d_i} + \sum_{i=1}^{t} \frac{r_i\left[(n_i - n_{i-1})d_i - 1\right]}{(n_i - n_{i-1})d_i} + \alpha\right],$$

where $\alpha = 1$ if $h_1 + \cdots + h_t + s_1 + \cdots + s_N = 0$ and $\alpha = 0$ otherwise. Again, note that if $s_i \neq 0$ for some i, then we have a strict inequality. Also note that

$$\sum_{i=1}^{t} \frac{r_i[(n_i - n_{i-1})d_i - 1]}{(n_i - n_{i-1})d_i} + \alpha \ge 0,$$

and any solution with $v_\theta(T) = f(p-1)\sum_{i=1}^{t} 1/d_i$ is minimal and has $s_i = 0$ for all $i = 1, \ldots, N$.

The vector

$$\left(\frac{\lambda_1(q-1)}{d_1}, \ldots, \frac{\lambda_t(q-1)}{d_t}, 0, \ldots, 0\right), \tag{10}$$

is a solution to the new modular system if and only if $\sum_{i=1}^{t} \lambda_i/d_i$ is an integer. By hypothesis, $\sum_{i=1}^{t} 1/d_i$ is an integer, and therefore $((q-1)/d_1, \ldots, (q-1)/d_t, 0, \ldots, 0)$ is a minimal solution.

Any other minimal solution must have the form (10) and

$$\sum_{i=1}^{t} \frac{r_i[(n_i - n_{i-1})d_i - 1]}{(n_i - n_{i-1})d_i} + \alpha = 0.$$

This implies that

$$v_\theta(T) = f(p-1)\left[\sum_{i=1}^{t} \frac{\lambda_i}{d_i}\right],$$

and this is minimal if and only if $\lambda_i = 1$ for all i. Therefore,

$$\left(\frac{q-1}{d_1}, \ldots, \frac{q-1}{d_t}, 0, \ldots, 0\right)$$

is the only minimal solution and the result follows.

Using the above theorem one can generalize Theorem 3.2 by substituting the degree by the p-weight degree.

COROLLARY 4.4 *Let $d > 1$ be a divisor of $p-1$ and $a_i \in \mathbb{F}_q{}^*$ for $i = 1, \ldots, d$. If $G(X_1, \ldots, X_d)$ is polynomial over $\mathbb{F}_q$ with $w_p(G) < d$, then the equation $a_1 X_1^d + \cdots + a_d X_d^d + G(X_1, \ldots, X_d) = 0$ has at least one solution over $\mathbb{F}_q$.*

EXAMPLE 4.1 Let

$$F(X_1, \ldots, X_7) = X_1^7 + X_2^7 + \cdots + X_7^7 + \sum_{i<j} a_{i,j} X_i X_j + X_1^{29^i+1} + \cdots + X_7^{29^i+1}$$

over $\mathbb{F}_{29^f}$. The equation $F = \beta$ has at least one solution for any $\beta \in \mathbb{F}_{29^f}$.

In the following theorem we impose conditions on the partial weight of the monomials to compute the exact divisibility of exponential sums. With the exact divisibility we prove that the related equation is solvable.

THEOREM 4.3 *Let $d_i > 1$ be a divisor of $p-1$ and $a_i \in \mathbb{F}_q{}^*$ for $i = 1, \ldots, n$. Let $G_1, \ldots, G_N$ be monomials. If $G(X_1, \ldots, X_n) = G_1 + \cdots + G_N$ is polynomial over $\mathbb{F}_q$ with $\sum_{i=1}^{n} w_{p,X_i}(G_j)/d_i < 1$ for $j = 1, \ldots, N$, and $F = a_1 X_1^{d_1} + \cdots + a_n X_n^{d_n} + G(X_1, \ldots, X_n)$, then*

$$v_\theta(S(F)) = f(p-1)\sum_{i=1}^{n} \frac{1}{d_i}.$$

Proof. We can assume that

$$G(X_1, \ldots, X_n) = b_1 X_1^{d_{11}} \cdots X_n^{d_{n1}} + \cdots + b_N X_1^{d_{1N}} \cdots X_n^{d_{nN}}$$

and $G(0, \ldots, 0) = 0$.

The following is the system of modular equations associated to F:

$$\begin{aligned} d_1h_1 + d_{11}s_1 + \cdots + d_{1N}s_N &\equiv 0 \bmod q-1 \\ \vdots \qquad\qquad & \qquad \vdots \\ d_nh_n + d_{n1}s_1 + \cdots + d_{nN}s_N &\equiv 0 \bmod q-1. \end{aligned} \tag{11}$$

Let $(h_1, \ldots, h_n, s_1, \ldots, s_N)$ be any solution to system (11). As in Theorem 4.1, applying σ_p, we obtain:

$$\sigma_p(d_i)\sigma(h_i) + \sigma_p(d_{i1})\sigma_p(s_1) + \cdots + \sigma_p(d_{iN})\sigma_p(s_N) \geq \alpha_i\sigma_p(q-1) = \alpha_i f(p-1),$$

where $\alpha_i = 0$ or $\alpha_i = 1$.

Since $d_i \leq p-1$, $\sigma_p(d_i) = d_i$. Dividing by d_i we obtain

$$\sigma_p(h_i) + \frac{\sigma_p(d_{i1})}{d_i}\sigma_p(s_1) + \cdots + \frac{\sigma_p(d_{iN})}{d_i}\sigma_p(s_N) \geq \frac{\alpha_i f(p-1)}{d_i}, \tag{12}$$

for $i = 1, \ldots, n$.

Note that $\sum_{i=1}^n \sigma_p(d_{ij}) = \sum_{i=1}^n w_{p,X_i}(G_j)$. Adding all the inequalities and using the fact that $\sum_{i=1}^n w_{p,X_i}(G_j)/d_i < 1$, we get

$$\sum_{i=1}^n \sigma_p(h_i) + \sum_{i=1}^N \sigma_p(s_i) \geq$$

$$\sum_{i=1}^n \sigma_p(h_i) + \sum_{i=1}^n \frac{\sigma_p(d_{i1})}{d_i}\sigma_p(s_1) + \cdots + \sum_{i=1}^n \frac{\sigma_p(d_{iN})}{d_i}\sigma_p(s_N) \geq f(p-1)\sum_{i=1}^n \frac{\alpha_i}{d_i}.$$

If T is the term in (3) associated to this solution then

$$\begin{aligned} v_\theta(T) &\geq f(p-1)\left[\sum_{i=1}^n \frac{\alpha_i}{d_i} + \sum_{i=1}^n (1-\alpha_i)\right] \\ &= f(p-1)\left[\sum_{i=1}^n \frac{1}{d_i} + \sum_{i=1}^n \frac{(1-\alpha_i)(d_i-1)}{d_i}\right], \end{aligned}$$

where $\sum_{i=1}^n (1-\alpha_i)$ is the number of equations in (11) that are equal to zero. Note that if $s_i \neq 0$ for some i we have strict inequality and, since $(1-\alpha_i)(d_i-1) \geq 0$, any solution with $v_\theta(T) = f(p-1)\sum_{i=1}^n 1/d_i$ is minimal and has $s_i = 0$ for all i.

As in Theorem 4.1, taking $h_i = (q-1)/d_i$ and $s_1 = \cdots = s_N = 0$, we obtain a solution to the system with $\sum_{i=1}^n \sigma_p(h_i) + \sum_{i=1}^N \sigma_p(s_i) = f(p-1)\sum_{i=1}^n 1/d_i$ and therefore is minimal. Any other solution with $s_1 = \cdots = s_N = 0$ has $h_i = \lambda_i(q-1)/d_i$ for $0 \leq \lambda_i \leq d_i$ and

$$v_\theta(T) = \sum_{i=1}^n \sigma_p(h_i) + \sum_{i=1}^N \sigma_p(s_i) + f(p-1)s = f(p-1)\left[\sum_{i=1}^n \frac{\lambda_i}{d_i} + s\right],$$

where s is the number of equations in (11) that are equal to zero for this solution. For this solution to be minimal we must have $\sum_{i=1}^n (1-\alpha_i)(d_i-1)/d_i = 0$. If $1-\alpha_i \neq 0$ for some i, then $d_i = 1$, which is a contradiction. If $1-\alpha_i=0$ for all i, then $\lambda_i \geq 1$, $v_\theta(T) = f(p-1)\sum_{i=1}^n \lambda_i/d_i$, and this is minimal if and only if $\lambda_i = 1$ for all i.

Therefore, $((q-1)/d_1, \dots, (q-1)/d_n, 0, \dots, 0)$ is the unique minimal solution and $v_\theta(T) = f(p-1)\sum_{i=1}^n 1/d_i$.

Note that in Theorem 3.2 the polynomial has the same degree on each variable and the degree is equal to the number of variables. Applying Theorem 4.3 we can get another improvement to Theorem 3.2 that has less restrictions on the degrees. In section 5 this improvement will be applied to generalizations of Waring's problem.

THEOREM 4.4 *Let $d_i > 1$ be a divisor of $p-1$ and $a_i \in \mathbb{F}_q^*$ for $i = 1, \dots, n$. Suppose that $\sum_{i=1}^n 1/d_i$ is an integer, and let $G_1, \dots, G_N$ be monomials. If $G(X_1, \dots, X_n) = G_1 + \cdots + G_N$ is a polynomial over $\mathbb{F}_q$ with $\sum_{i=1}^n w_{p,X_i}(G_j)/d_i < 1$ for $j = 1, \dots, N$, and $F(X_1, \dots, X_n) = a_1X_1^{d_1} + \cdots + a_nX_n^{d_n} + G(X_1, \dots, X_n)$, then the exact divisibility of the number of solutions of $F = 0$ is $p^{f(\sum_{i=1}^n 1/d_i - 1)}$. In particular, the equation has at least one solution over $\mathbb{F}_q$.*

Proof. The proof is similar to the proof of Theorem 4.2.

EXAMPLE 4.2 Let $F(X_1, \dots, X_7) = X_1^{10} + \cdots + X_4^{10} + X_5^5 + X_6^5 + X_7^5 + X_1X_2X_3 + X_4X_5X_6X_7$ over $\mathbb{F}_{31}$. Then $F = \beta$ has solution for every $\beta \in \mathbb{F}_{31^f}$.

The following corollary improves Corollary 3.1.

COROLLARY 4.5 *Let $d > 1$ be a divisor of $p-1$, and suppose that n/d is an integer. If $F_i(X_i) = a_iX_i^d + G_i(X_i)$ is a polynomial over $\mathbb{F}_q$ with $w_p(G_i) < d$ for every i, then the exact divisibility of the number of solutions $F_1(X_1) + \cdots + F_n(X_n) = 0$ is $p^{f(n/d-1)}$. In particular, the equation has at least one solution over $\mathbb{F}_q$.*

EXAMPLE 4.3 Let $p > 5$, $d = (p-1)/2$, and consider the polynomial

$$F(X_1, \dots, X_d) = X_1^d + \cdots + X_d^d + X_1^{p^i+1} + \cdots + X_d^{p^i+1}$$

over $\mathbb{F}_q$. Then $F = \beta$ has at least one solution over $\mathbb{F}_q$ for each $\beta \in \mathbb{F}_q$.

If $q = p^{2i}$, then the equation $X_1^{p^i+1} + \cdots + X_d^{p^i+1} = \beta$ does not have solution for all $\beta \in \mathbb{F}_{p^{2i}} \setminus \mathbb{F}_{p^i}$. The above corollary implies that the extra terms force $X_1^d + \cdots + X_d^d + X_1^{p^i+1} + \cdots + X_d^{p^i+1} = \beta$ to have solutions for all $\beta \in \mathbb{F}_{p^{2i}}$.

5. Applications to generalizations of Waring's Problem

The original Waring's problem is to find the minimum number of variables such that the equation $X_1^d + \cdots + X_n^d = \beta$ has solutions for any natural number β. This minimum number is called the *Waring number* associated to d. Waring's problem has also been considered for equations over finite fields and there are many bounds for their Waring numbers ([6, 7, 11]). Many of these bounds are consequences of good estimates of the absolute value of Gauss sums [5, 7].

In this section we consider a generalization of Waring's problem: Given a polynomial $F(X)$ over $\mathbb{F}_q$, find the minimum number of variables such that

$$F(X_1) + \cdots + F(X_n) = \beta \tag{13}$$

has solution over $\mathbb{F}_q$ for any $\beta \in \mathbb{F}_q$. We denote this number by $\gamma(F, q)$. The above problem can be related to the following problem: Given polynomials $F_1(X_1), \ldots, F_n(X_n)$ over $\mathbb{F}_q$, find conditions such that every $\beta \in \mathbb{F}_q$ can be written as

$$\beta = F_1(x_1) + \cdots + F_n(x_n), \tag{14}$$

where $x_1, \ldots, x_n \in \mathbb{F}_q$. This problem was considered by Carlitz *et al.* [**3**] and Cochrane *et al.* [**4**] for the prime field. Carlitz *et al.* proved that given $F_1(X_1), \ldots, F_n(X_n)$ polynomials over $\mathbb{F}_p$ of degree $d_1, \ldots, d_n$, every element $\beta \in \mathbb{F}_p$ can be written as $\beta = F_1(x_1) + \cdots + F_n(x_n)$, provided that

$$\sum_{i=1}^{n} \left[\frac{p-1}{d_i} \right] + t > p,$$

where t is the number of F_is which are neither of degree $p-1$ nor of the form $\alpha(X_i - \beta)^{1/2(p-1)} + \lambda$. If $F = F_1 = \cdots = F_d$, the above result implies that $\gamma(F, p) \le d$, where d is the degree of F and $d \ne p-1, (p-1)/2$ (see [**4**]). In [**4**], Cochrane *et al.* use estimates for exponential sums to prove that (13) has at least one solution for every $\beta \in \mathbb{F}_p$, whenever $r_1 + \cdots + r_{\gamma(F,p)} \ge \log p$, where the absolute value of the exponential sum corresponding to each r_i is less than or equal to $p(1 - r_i)$. Note that these results are for polynomials over $\mathbb{F}_p$.

We now apply Theorem 4.4 and obtain some natural conditions so that the elements $\beta \in \mathbb{F}_q$ can be written as $\beta = F_1(x_1) + \ldots + F_n(x_n)$. This gives an upper bound on $\gamma(F, q)$ for polynomials F that satisfy certain natural conditions.

Our results apply to extension fields, while the above results only apply for polynomials over $\mathbb{F}_p$.

THEOREM 5.1 *Let $d_i > 1$ be a divisor of $p-1$, $a_i \in \mathbb{F}_q{}^*$ and $F_i(X) = a_i X_i^{d_i} + G_i(X_i)$ be polynomials over $\mathbb{F}_q$ for $i = 1, \ldots, n$. Suppose that $\sum_{i=1}^{n} 1/d_i$ is an integer. If $w_p(G_i) < d_i$, then every $\beta \in \mathbb{F}_q$ can be written as*

$$\beta = F_1(x_1) + \cdots + F_n(x_n),$$

for some $x_1, \ldots, x_n \in \mathbb{F}_q$.

EXAMPLE 5.1 Let $F_1(X_1) = X_1^6 + X_1^{18}$, $F_2(X_2) = X_2^3 + X_2^{14}$, $F_3(X_3) = X_3^2 + X_3$ over $\mathbb{F}_{13^f}$. Then every $\beta \in \mathbb{F}_{13^f}$ can be written as $\beta = x_1^6 + x_1^{18} + x_2^3 + x_2^{14} + x_3^2 + x_3$.

COROLLARY 5.1 *Let $F(X) = aX^d + G(X)$ be a polynomial over $\mathbb{F}_q$, where $d \ne 1$ divides $p-1$. If $w_p(G) < d$, then $\gamma(F, q) \le d$.*

EXAMPLE 5.2 Let $F(X) = X^3 + aX^{p^i+1}$ over $\mathbb{F}_q$, where 3 divides $p-1$. Then $\gamma(F, q) \le 3$. Using Maple, we obtain that $\gamma(x^3 + x^8, 49) = 2$ and $\gamma(x^3 + x^{14}, 169) = 2$. Note that $\gamma(x^8, 49)$ and $\gamma(x^{14}, 169)$ do not exist.

EXAMPLE 5.3 Let $F(X) = X^4 + a_1 X^{p^{i_1} + p^{j_1} + 1} + a_2 X^{p^{i_2} + p^{j_2} + 1} + \cdots + a_n X^{p^{i_n} + p^{j_n} + 1}$ over $\mathbb{F}_{29^f}$, then $\gamma(F, 29^f) \le 4$.

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MODULI SPACES OF PARABOLIC U(p, q)-HIGGS BUNDLES

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Abstract

Using the L^2-norm of the Higgs field as a Morse function, we study the moduli space of parabolic U(p, q)-Higgs bundles over a Riemann surface with a finite number of marked points, under certain genericity conditions on the parabolic structure. When the parabolic degree is zero this space is homeomorphic to the moduli space of representations of the fundamental group of the punctured surface in U(p, q), with fixed compact holonomy classes around the marked points. By means of this homeomorphism we count the number of connected components of this moduli space of representations. Finally, we apply our results to the study of representations of the fundamental group of elliptic surfaces of general type.

1. Introduction

A parabolic vector bundle over a compact Riemann surface with marked points consists of a vector bundle, equipped with a weighted flag structure on the fibre over each marked point. These objects were introduced by Seshadri [29] in relation to certain desingularizations of the moduli space of semistable vector bundles. It turns out that, similarly to the Narasimhan and Seshadri correspondence [14, 25] between stable vector bundles and representations of the fundamental group of the surface in the unitary group U(n), there is an analogous correspondence, proved by Metha and Seshadri [23] (see also [3]), relating stable parabolic bundles to unitary representations of the fundamental group of the punctured surface with a fixed holonomy class around each marked point.

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In order to study representations of the fundamental group of the punctured surface in $\mathrm{GL}(n,\mathbb{C})$ one has to consider parabolic Higgs bundles. These are pairs consisting of a parabolic vector bundle and a meromorphic endomorphism valued 1-form with a simple pole along each marked point, whose residue is nilpotent with respect to the flag. Moduli spaces of parabolic Higgs bundles provide interesting examples of hyperkähler manifolds. This theory, studied by Simpson in [31] and others [7, 20, 24, 26], generalizes the non-parabolic Higgs bundle theory studied by Hitchin [19], Donaldson [15], Simpson [30] and Corlette [12].

In this paper we study parabolic $\mathrm{U}(p,q)$-Higgs bundles. These are the objects that correspond to representations of the fundamental group of the punctured surface in $\mathrm{U}(p,q)$, with fixed compact holonomy classes around the marked points. Our approach combines the techniques used in [10] in the study of $\mathrm{U}(p,q)$-Higgs bundles in the non-parabolic case as well as those used in [18] to study the topology of moduli spaces of $\mathrm{GL}(n,\mathbb{C})$-parabolic Higgs bundles.

For a parabolic $\mathrm{U}(p,q)$-Higgs bundle there is an invariant, similar to the Toledo invariant in the non-parabolic case. We show that this parabolic Toledo invariant has a bound provided by a generalization of the Milnor–Wood inequality. Our main result in the paper is to show that if the genus of the surface and the number of marked points are both at least one, then the moduli space of parabolic $\mathrm{U}(p,q)$-Higgs bundles with fixed topological type, generic parabolic weights and full flags is non-empty and connected if and only if the parabolic Toledo invariant satisfies a generalized Milnor–Wood inequality (see Theorem 6.13).

As in [10, 18], the main strategy is to use the Bott–Morse-theoretic techniques introduced by Hitchin [19]. The connectedness properties of our moduli space reduce to the connectedness of a certain moduli space of parabolic triples introduced in [4] in connection with the study of the parabolic vortex equations and instantons of infinite energy. Much of the paper is devoted to a thorough study of these moduli spaces of triples and their connectedness properties.

After spelling out the correspondence between parabolic $\mathrm{U}(p,q)$-Higgs bundles and representations of the fundamental group of the punctured surface in $\mathrm{U}(p,q)$, we transfer our results on connectedness of the moduli space of parabolic $\mathrm{U}(p,q)$-Higgs bundles to the moduli space of representations (see Theorems 13.2 and 13.3). We then apply this to the study of representations of the fundamental group of certain complex elliptic surfaces of general type (see Theorem 14.4). These are complex surfaces whose fundamental group is isomorphic to the orbifold fundamental group of an orbifold Riemann surface.

We should point out that our main results do not apply when the genus of the Riemann surface is zero. This is not surprising if we have in mind that on $\mathbb{P}^1$ the parabolic weights must satisfy certain inequalites in order for parabolic bundles to exist [2, 5]. Presumably, something similar must be true also in the case of parabolic $\mathrm{U}(p,q)$-Higgs bundles. We plan to come back to this problem in a future paper.

In the process of finishing our paper we have come across several papers [9, 21, 22] that seem to be related to our work in the case of $\mathrm{U}(p,1)$. It would be interesting to investigate further the relationship between these different approaches.

2. Parabolic Higgs bundles

Let X be a closed, connected, smooth Riemann surface of genus $g \geq 0$ together with a finite set of marked points $x_1,\dots,x_s$. Denote by D the effective divisor $D = x_1 + \cdots + x_s$ defined by the marked points. A *parabolic vector bundle E* over X consists of a holomorphic vector bundle together with a parabolic structure at each $x \in D$, that is, a weighted flag on the fibre E_x,

$$E_x = E_{x,1} \supset E_{x,2} \supset \cdots \supset E_{x,r(x)+1} = \{0\},$$
$$0 \le \alpha_1(x) < \cdots < \alpha_{r(x)}(x) < 1.$$

We denote by $k_i(x) = \dim(E_{x,i}/E_{x,i+1})$ the *multiplicity* of the weight $\alpha_i(x)$. It will sometimes be convenient to repeat each weight according to its multiplicity, that is, we set $\tilde{\alpha}_1(x) = \cdots = \tilde{\alpha}_{k_1(x)}(x) = \alpha_1(x)$, etc. We then have weights $0 \le \tilde{\alpha}_1(x) \le \cdots \le \tilde{\alpha}_n(x) < 1$, where $n = \operatorname{rk} E$. Denote also by $\alpha(x) = (\tilde{\alpha}_1(x), \ldots, \tilde{\alpha}_n(x))$ the system of weights at x of E and by $\alpha = (\alpha(x))_{x\in D}$ the *weight type* of E. We say that the flags are *full* if $k_i(x) = 1$ for all i and $x \in D$. Note that in this case $\alpha(x) = (\tilde{\alpha}_1(x), \ldots, \tilde{\alpha}_n(x)) = (\alpha_1(x), \ldots, \alpha_n(x))$. A holomorphic map $f : E \to E'$ between parabolic bundles is called *parabolic* if $\alpha_i(x) > \alpha'_j(x)$ implies $f(E_{x,i}) \subset E'_{x,j+1}$ for all $x \in D$, and f is *strongly parabolic* if $\alpha_i(x) \ge \alpha'_j(x)$ implies $f(E_{x,i}) \subset E'_{x,j+1}$ for all $x \in D$, where we denote by $\alpha'_j(x)$ the weights on E'. We denote by $\operatorname{ParHom}(E, E')$ and $\operatorname{SParHom}(E, E')$ the sheaves of parabolic and strongly parabolic morphisms from E to E', respectively. If $E' = E$ we denote these sheaves by $\operatorname{ParEnd}(E)$ and $\operatorname{SParEnd}(E)$, respectively.

We define the *parabolic degree* and *parabolic slope* of E by

$$\operatorname{pardeg}(E) = \deg(E) + \sum_{x\in D}\sum_{i=1}^{r(x)} k_i(x)\alpha_i(x), \tag{1}$$

$$\operatorname{par}\mu(E) = \frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)}. \tag{2}$$

A parabolic bundle E is said to be (*semi*)-*stable* if for every non-trivial proper parabolic subbundle E' of E we have $\operatorname{par}\mu(E') < \operatorname{par}\mu(E)$ (resp. $\operatorname{par}\mu(E') \le \operatorname{par}\mu(E)$).

In the following we will use the following construction for parabolic bundles, called the *parabolic direct sum*. Let V and W be two parabolic bundles with weight types α and α'; we say that E is the parabolic direct sum of V and W if and only if $E = V \oplus W$ as holomorphic bundles, the system of weights, $\tilde{\alpha}$, on E consists of the ordered collection of the weights in α and α', and the corresponding filtration is such that

$$E_{x,k} = V_{x,i} \oplus W_{x,j},$$

where i (resp. j) is the smallest integer such that $\tilde{\alpha}_k(x) \le \alpha_i(x)$ (resp. $\tilde{\alpha}_k(x) \le \alpha'_j(x)$).

A *parabolic Higgs bundle* is a pair (E, Φ) consisting of a parabolic bundle E and $\Phi \in H^0(\operatorname{SParEnd}(E) \otimes K(D))$, that is, Φ is a meromorphic endomorphism valued 1-form with simple poles along D whose residue at $x \in D$ is nilpotent with respect to the flag. A parabolic Higgs bundle is called (semi-) stable if for every Φ-invariant subbundle E' of E, its parabolic slope satisfies $\operatorname{par}\mu(E') < \operatorname{par}\mu(E)$ (resp. $\operatorname{par}\mu(E) \le \operatorname{par}\mu(E)$), and it is said to be polystable if it is the direct sum of stable parabolic Higgs bundles of the same parabolic slope.

Fixing the topological invariants $n = \operatorname{rk} E$ and $d = \deg E$ and the weight type α, the moduli space $\mathcal{M} = \mathcal{M}(n, d; \alpha)$ is defined as the set of isomorphism classes of polystable parabolic Higgs bundles of type $(n, d; \alpha)$. Using geometric invariant theory, Yokogawa [34, 35] has shown that $\mathcal{M}$ is a complex quasi-projective variety, which is smooth at the stable points.

A *parabolic* $\mathrm{U}(p,q)$*-Higgs bundle* on X is a parabolic Higgs bundle (E,Φ) such that $E = V \oplus W$, where V and W are parabolic vector bundles of ranks p and q, respectively, and

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : (V \oplus W) \to (V \oplus W) \otimes K(D),$$

where $\beta : W \to V \otimes K(D)$ and $\gamma : V \to W \otimes K(D)$ are strongly parabolic morphisms. A parabolic $\mathrm{U}(p,q)$-Higgs bundle $(E = V \oplus W, \Phi)$ is (semi-) stable if the slope stability condition $\operatorname{par}\mu(E') < \operatorname{par}\mu(E)$ (resp. $\operatorname{par}\mu(E') \le \operatorname{par}\mu(E)$) is satisfied for all Φ-invariant parabolic subbundles of the form $E' = V' \oplus W'$, that is, for all parabolic subbundles $V' \subset V$ and $W' \subset W$ such that $\beta(W') \subseteq V' \otimes K(D)$ and $\gamma(V') \subseteq W' \otimes K(D)$. Note that, a priori, this definition of stability seems to be weaker than the stability definition for parabolic Higgs bundles (we ask for $V' \subset V$ and $W' \subset W$). But this is not the case, since for any Φ-invariant $E' \subset E$, we apply the $\mathrm{U}(p,q)$-stability condition to $V' \oplus W'$ and to $V'' \oplus W''$, where $V' = V \cap E'$, $W' = W \cap E'$, $V'' = \pi_V(E')$, $W'' = \pi_W(E')$ (where π_V, π_W are the projections of $V \oplus W$ onto V, W, respectively). Then using the exact sequences $V' \to E' \to W''$ and $W' \to E' \to V''$, one gets easily that $\operatorname{par}\mu(E') \le \operatorname{par}\mu(E)$).

Fix the topological invariants $a = \deg V$ and $b = \deg W$ and the weight types α and α' for V and W, respectively. This determines a system of weights $\tilde{\alpha}$ and a flag structure, given by the parabolic direct sum construction, on $E = V \oplus W$. Let

$$\mathcal{U} = \mathcal{U}(p, q, a, b; \alpha, \alpha')$$

be the moduli space of polystable parabolic $\mathrm{U}(p,q)$-Higgs bundles of degrees (a,b) and weights (α, α').

We say that the weights are *generic* when every semistable parabolic Higgs bundle is automatically stable, that is, there are no properly semistable parabolic Higgs bundles. We will keep the following assumption on the weights all throughout the paper (although some of the results hold in more general situations).

ASSUMPTION 2.1 The weights of (E,Φ) are generic and (E,Φ) has full flags at each parabolic point. This means that all the weights of V and W are different and of multiplicity one.

Note that the set of weights such that, for fixed degree and rank of E, make (E,Φ) strictly semistable has positive codimension. This justifies the term generic for the weights which do not allow strict semistability.

The construction of $\mathcal{U}$ follows the same arguments given in the non-parabolic case (see [**10**]).

PROPOSITION 2.2 *Let $n = p+q$, $d = a+b$, and let $\tilde{\alpha}$ be the system of weights defined by α and α' as above. Then $\mathcal{U}(p,q,a,b;\alpha,\alpha')$ embeds as a closed subvariety in $\mathcal{M}(n,d;\tilde{\alpha})$.*

Proof. The proof is similar to that in the non-parabolic case (see [**10**, Proposition 3.11]). One only notices that in the case $p = q$, the parabolic bundles V and W cannot be parabolically isomorphic since they have different weights.

REMARK 2.3 Sometimes we refer to elements $(E,\Phi) \in \mathcal{M}$ as parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles, since the structure group of the frame bundle of E is $\mathrm{GL}(n,\mathbb{C})$.

3. Deformation theory

The results of Yokogawa [10, 34] can be readily adapted to describe the deformation theory of parabolic $\mathrm{U}(p,q)$-Higgs bundles.

Let $(E = V \oplus W, \Phi)$ be a parabolic $\mathrm{U}(p,q)$-Higgs bundle. We introduce the following notation:

$$U = \mathrm{ParEnd}\,(E), \quad \hat{U} = \mathrm{SParEnd}\,(E),$$

$$U^+ = \mathrm{ParEnd}\,(V) \oplus \mathrm{ParEnd}\,(W), \quad \hat{U}^+ = \mathrm{SParEnd}\,(V) \oplus \mathrm{SParEnd}\,(W),$$

$$U^- = \mathrm{ParHom}\,(W, V) \oplus \mathrm{ParHom}\,(V, W), \quad \hat{U}^- = \mathrm{SParHom}\,(W, V) \oplus \mathrm{SParHom}\,(V, W).$$

With this notation, $U = U^+ \oplus U^-$, $\hat{U} = \hat{U}^+ \oplus \hat{U}^-$, $\Phi \in H^0(\hat{U}^- \otimes K(D))$, and $\mathrm{ad}(\Phi)$ sends U^+ to $\hat{U}^-$ and U^- to $\hat{U}^+$. We consider the complex of sheaves

$$C^\bullet : U^+ \xrightarrow{\mathrm{ad}(\Phi)} \hat{U}^- \otimes K(D). \tag{3}$$

LEMMA 3.1 *Let (E, Φ) be a stable parabolic $\mathrm{U}(p,q)$-Higgs bundle. Then*

$$\ker\big(\mathrm{ad}(\Phi)\colon H^0(U^+) \to H^0(\hat{U}^- \otimes K(D))\big) = \mathbb{C}, \tag{4}$$

$$\ker\big(\mathrm{ad}(\Phi)\colon H^0(U^-) \to H^0(\hat{U}^+ \otimes K(D))\big) = 0. \tag{5}$$

Proof. Since (E, Φ) is stable as a parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle, it is simple, that is, its only endomorphisms are the non-zero scalars. Thus

$$\ker\big(\mathrm{ad}(\Phi)\colon H^0(U) \to H^0(\hat{U} \otimes K(D))\big) = \mathbb{C}.$$

Since $U = U^+ \oplus U^-$ and $\mathrm{ad}(\Phi)$ sends U^+ to $\hat{U}^-$ and U^- to $\hat{U}^+$, the statements of the lemma follow.

PROPOSITION 3.2 (i) *The space of endomorphisms of (E, Φ) is isomorphic to the zeroth hypercohomology group $\mathbb{H}^0(C^\bullet)$.*

(ii) *The space of infinitesimal deformations of (E, Φ) is isomorphic to the first hypercohomology group $\mathbb{H}^1(C^\bullet)$.*

(iii) *There is a long exact sequence*

$$0 \longrightarrow \mathbb{H}^0(C^\bullet) \longrightarrow H^0(U^+) \longrightarrow H^0(\hat{U}^- \otimes K(D)) \longrightarrow \mathbb{H}^1(C^\bullet)$$
$$\longrightarrow H^1(U^+) \longrightarrow H^1(\hat{U}^- \otimes K(D)) \longrightarrow \mathbb{H}^2(C^\bullet) \longrightarrow 0, \tag{6}$$

where the maps $H^i(U^+) \longrightarrow H^i(\hat{U}^- \otimes K(D))$ are induced by $\mathrm{ad}(\Phi)$.

PROPOSITION 3.3 *Let (E, Φ) be a stable parabolic $\mathrm{U}(p,q)$-Higgs bundle, then*

(a) $\mathbb{H}^0(C^\bullet) = \mathbb{C}$ *(in other words (E, Φ) is simple), and*

(b) $\mathbb{H}^2(C^\bullet) = 0$.

Proof. (a) This follows immediately from Lemma 3.1 and (iii) of Proposition 3.2.

(b) For parabolic bundles E and F the sheaves ParHom (E, F) and SParHom $(F, E) \otimes \mathcal{O}(D)$ are naturally dual to each other (see for example [7]) and we thus have that

$$\mathrm{ad}(\Phi)\colon H^1(U^+) \to H^1(\hat{U}^- \otimes K(D))$$

is Serre dual to $\mathrm{ad}(\Phi)\colon H^0(U^-) \to H^0(\hat{U}^+ \otimes K(D))$. Hence Lemma 3.1 and (iii) of Proposition 3.2 show that $\mathbb{H}^2(C^\bullet) = 0$.

PROPOSITION 3.4 *Assuming Assumption* 2.1*, the moduli space* $\mathcal{U}$ *of stable parabolic* $\mathrm{U}(p, q)$*-Higgs bundles is a smooth complex variety of dimension*

$$1 + (g-1)(p+q)^2 + \frac{s}{2}\big((p+q)^2 - (p+q)\big), \tag{7}$$

where g *is the genus of* X*, and* s *is the number of marked points.*

REMARK 3.5 The formula (7) is also valid in the case $s = 0$ and genus $g \geq 2$. In such cases we recover the formula for the dimension of the moduli space of non-parabolic $\mathrm{U}(p, q)$-Higgs bundles given in [10]. As expected, this dimension is half the dimension of the moduli space $\mathcal{M}$ of parabolic $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles of rank $n = p + q$. Observe also that, in order to have a non-empty moduli space we need $s \geq 3$ when $g = 0$.

Proof. Our assumption on the genericity of the weights implies that there are no properly semistable parabolic $\mathrm{U}(p, q)$-Higgs bundles and hence every point in $\mathcal{U}$ is stable. Smoothness follows from Propositions 3.2 and 3.3. Now, our assumptions on having full flags and different weights on V and W imply that

$$\mathrm{SParHom}\,(V, W) = \mathrm{ParHom}\,(V, W)$$

and

$$\begin{aligned} \dim \mathrm{ParHom}\,(V, W)_x + \dim \mathrm{ParHom}\,(W, V)_x &= pq, \\ \dim \mathrm{ParEnd}\,(V)_x &= \frac{p(p+1)}{2}, \\ \dim \mathrm{ParEnd}\,(W)_x &= \frac{q(q+1)}{2}. \end{aligned}$$

Also, the short exact sequence

$$0 \to \mathrm{ParHom}\,(V, W) \to \mathrm{Hom}\,(V, W) \to \bigoplus_{x \in D} \frac{\mathrm{Hom}\,(V_x, W_x)}{\mathrm{ParHom}\,(V_x, W_x)} \to 0$$

implies that

$$\deg(\mathrm{ParHom}\,(V, W)) = p\deg(W) - q\deg(V) + \sum_{x \in D}(\dim \mathrm{ParHom}\,(V_x, W_x) - pq).$$

Using the above information and Proposition 3.2 we have that the dimension of the tangent space of $\mathcal{U}$ at a point (E, Φ) is

$$\begin{aligned}
\dim \mathbb{H}^1(C^\bullet) &= \dim \mathbb{H}^0(C^\bullet) + \dim \mathbb{H}^2(C^\bullet) - \chi(C^\bullet) \\
&= 1 - \chi(\mathrm{ParEnd}\,(V) \oplus \mathrm{ParEnd}\,(W)) \\
&\quad + \chi((\mathrm{SParHom}\,(V, W) \oplus \mathrm{SParHom}\,(W, V)) \otimes K(D)) \\
&= 1 - (p^2 + q^2)(1-g) - \deg(\mathrm{ParEnd}\,(V)) - \deg(\mathrm{ParEnd}\,(W)) + 2pq((1-g) \\
&\quad + \deg(\mathrm{ParHom}\,(V, W)) + \deg(\mathrm{ParHom}\,(W, V)) + 2pq(2g-2) + 2pqs \\
&= 1 + (g-1)(p+q)^2 + 2pqs + (p^2 + q^2 - 2pq)s + \sum_{x \in D} \Big(\dim \mathrm{ParHom}\,(V, W)_x \\
&\quad + \dim \mathrm{ParHom}\,(W, V)_x - \dim \mathrm{ParEnd}\,(V)_x - \dim \mathrm{ParEnd}\,(W)_x \Big) \\
&= 1 + (g-1)(p+q)^2 + \frac{s}{2}((p+q)^2 - (p+q)).
\end{aligned}$$

4. Parabolic Toledo invariant

In analogy with the non-parabolic case [10], one can associate a Toledo invariant to a parabolic $\mathrm{U}(p, q)$-Higgs bundle.

DEFINITION 4.1 The *parabolic Toledo invariant* corresponding to the parabolic Higgs bundle $(E = V \oplus W, \Phi)$ is

$$\tau = 2\frac{pq}{p+q}(\mathrm{par}\,\mu(V) - \mathrm{par}\,\mu(W)). \tag{8}$$

The Toledo invariant will give us a way to classify components of the moduli space of parabolic $\mathrm{U}(p, q)$-Higgs bundles. So we first determine the possible values that it can take.

PROPOSITION 4.2 *Let* $\left(E = V \oplus W, \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right)$ *be a semistable parabolic* $\mathrm{U}(p, q)$*-Higgs bundle. Then*

$$p(\mathrm{par}\,\mu(V) - \mathrm{par}\,\mu(E)) \le \mathrm{rk}(\gamma)\left(g - 1 + \frac{s}{2}\right),$$

$$q(\mathrm{par}\,\mu(W) - \mathrm{par}\,\mu(E)) \le \mathrm{rk}(\beta)\left(g - 1 + \frac{s}{2}\right).$$

Proof. Consider the parabolic bundles $N = \ker(\gamma)$ and $I = \mathrm{im}\,(\gamma) \otimes K(D)^{-1}$. We have an exact sequence of parabolic bundles

$$0 \longrightarrow N \longrightarrow V \longrightarrow I \otimes K(D) \longrightarrow 0$$

and

$$\begin{aligned}
\mathrm{pardeg}\,(V) &= \mathrm{pardeg}\,(N) + \mathrm{pardeg}\,(I \otimes K(D)) \\
&= \mathrm{pardeg}\,(N) + \mathrm{pardeg}\,(I) + \mathrm{rk}(I)(2g - 2 + s).
\end{aligned} \tag{9}$$

Note that I is a subsheaf of W and the map $I \hookrightarrow W$ is a parabolic map. Let $\tilde{I} \subset W$ be its saturation, which is a subbundle of W, and endow it with the induced parabolic structure. So $N, V \oplus \tilde{I} \subset E$ are Φ-invariant parabolic subbundles of E. The semistability of (E, Φ) implies that

$$\begin{aligned} &\operatorname{par}\mu(N) \leq \operatorname{par}\mu(E), \\ &\operatorname{par}\mu(V \oplus I) \leq \operatorname{par}\mu(V \oplus \tilde{I}) \leq \operatorname{par}\mu(E). \end{aligned} \tag{10}$$

This yields

$$\begin{gathered} \operatorname{pardeg}(N) \leq \operatorname{rk}(N) \operatorname{par}\mu(E), \\ \operatorname{pardeg}(V) + \operatorname{pardeg}(I) \leq (p + \operatorname{rk}(I)) \operatorname{par}\mu(E). \end{gathered}$$

Adding both and using (9) we have the inequality

$$2 \operatorname{pardeg}(V) \leq 2p \operatorname{par}\mu(E) + \operatorname{rk}(I)(2g - 2 + s),$$

and hence

$$p(\operatorname{par}\mu(V) - \operatorname{par}\mu(E)) \leq \operatorname{rk}(\gamma)\left(g - 1 + \frac{s}{2}\right).$$

The other case is analogous.

REMARK 4.3 The inequalities in Proposition 4.2 are not sharp. This is due to the fact that (10) can be improved by assigning to I the weights induced by the inclusion $I \subset W$.

One has the following bound for the Toledo invariant.

PROPOSITION 4.4 *Let* (E, Φ) *be a semistable parabolic* $\mathrm{U}(p, q)$*-Higgs subbundle. Then*

$$|\tau| \leq \tau_M = \min\{p, q\}(2g - 2 + s).$$

Proof. Noting that

$$\operatorname{par}\mu(E) = \frac{p}{p+q} \operatorname{par}\mu(V) + \frac{q}{p+q} \operatorname{par}\mu(W), \tag{11}$$

Proposition 4.2 may be rewritten as

$$\begin{gathered} q(\operatorname{par}\mu(E) - \operatorname{par}\mu(W)) \leq \operatorname{rk}(\gamma)\left(g - 1 + \frac{s}{2}\right), \\ p(\operatorname{par}\mu(E) - \operatorname{par}\mu(V)) \leq \operatorname{rk}(\beta)\left(g - 1 + \frac{s}{2}\right). \end{gathered}$$

By (11) we also have $\tau = 2p(\operatorname{par}\mu(V) - \operatorname{par}\mu(E)) = 2q(\operatorname{par}\mu(E) - \operatorname{par}\mu(W))$. The result follows.

5. Hitchin equations and parabolic Higgs bundles

In order to study the topology of $\mathcal{U}$ we need a gauge-theoretic interpretation of this moduli space in terms of solutions to the Hitchin equations. One can adapt the arguments given by Simpson [31] for the case of parabolic $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles to the $\mathrm{U}(p,q)$ situation, along the lines of what is done in [10] in the non-parabolic case. Similarly, to construct the moduli space from this point of view, one can adapt the construction given by Konno [20] (see also [26]) in the parabolic $\mathrm{GL}(n,\mathbb{C})$ case.

A parabolic structure on a smooth vector bundle is defined in a similar way to what is done in the holomorphic category. Let E be a smooth parabolic vector bundle of rank n and fix a hermitian metric h on E which is smooth in $X \setminus D$ and whose (degenerate) behaviour around the marked points is given as follows. We say that a local frame $\{e_1,\dots,e_n\}$ for E around x *respects the flag at* x if $E_{x,i}$ is spanned by the vectors $\{e_{M_i+1}(x),\dots,e_n(x)\}$, where $M_i=\sum_{j\le i}k_j(x)$. Let z be a local coordinate around x such that $z(x)=0$. We require that h be of the form

$$h=\begin{pmatrix}|z|^{2\tilde\alpha_1} & & 0\\ & \ddots & \\ 0 & & |z|^{2\tilde\alpha_n}\end{pmatrix}$$

with respect to some local frame around x which respects the flag at x, where $\tilde\alpha_i=\tilde\alpha_i(x)$.

A unitary connection d_A associated to a smooth $\bar\partial$ operator $\bar\partial_E$ on E via the hermitian metric h is singular at the marked points: if we write $z=\rho\exp(\sqrt{-1}\theta)$ and $\{e_i\}$ is the local frame used in the definition of h, then with respect to the local frame $\{\epsilon_i=e_i/|z|^{\tilde\alpha_i}\}$, the connection is of the form

$$d_A=d+\sqrt{-1}\begin{pmatrix}\tilde\alpha_1 & & 0\\ & \ddots & \\ 0 & & \tilde\alpha_r\end{pmatrix}d\theta+A',$$

where A' is regular. We denote the space of smooth $\bar\partial$-operators on E by $\mathscr{C}_E$, the space of associated h-unitary connections by $\mathscr{A}_E$, the group of complex parabolic gauge transformations by $\mathscr{G}_E^{\mathbb{C}}$ and the subgroup of h-unitary parabolic gauge transformations by $\mathscr{G}_E$.

Let V and W be smooth parabolic vector bundles equipped with hermitian metrics h_V and h_W adapted to the parabolic structures in the sense explained above. We denote $\mathscr{C}:=\mathscr{C}_V\times\mathscr{C}_W$, $\mathscr{G}^{\mathbb{C}}:=\mathscr{G}_V^{\mathbb{C}}\times\mathscr{G}_W^{\mathbb{C}}$, $\mathscr{G}:=\mathscr{G}_V\times\mathscr{G}_W$. The space of Higgs fields is $\mathbf{\Omega}=\mathbf{\Omega}^+\oplus\mathbf{\Omega}^-$, where $\mathbf{\Omega}^+=\Omega^{1,0}(\mathrm{SParHom}\,(W,V)\otimes\mathcal{O}(D))$ and $\mathbf{\Omega}^-=\Omega^{1,0}(\mathrm{SParHom}\,(V,W)\otimes\mathcal{O}(D))$. Here we regard $\mathrm{SParHom}\,(W,V)$ and $\mathrm{SParHom}\,(V,W)$ as smooth vector bundles defined as in the holomorphic category.

Following Biquard [3] and Konno [20], we introduce certain weighted Sobolev norms and denote the corresponding Sobolev completions of the spaces defined above by $\mathscr{C}_1^k$, $\mathbf{\Omega}_1^k$, $(\mathscr{G}^{\mathbb{C}})_2^k$ and $\mathscr{G}_2^k$. Let

$$\mathscr{H}=\{(\bar\partial_E,\Phi)\in\mathscr{C}\times\mathbf{\Omega}\mid\bar\partial_E\Phi=0\}$$

and let $\mathscr{H}_1^k$ be the corresponding subspace of $\mathscr{C}_1^k\times\mathbf{\Omega}_1^k$.

Let $\bar\partial_E=(\bar\partial_V,\bar\partial_W)$, where $\bar\partial_V\in\mathscr{C}_V$ and $\bar\partial_W\in\mathscr{C}_W$, and $\Phi=\begin{pmatrix}0&\beta\\ \gamma&0\end{pmatrix}$ with $\beta\in\mathbf{\Omega}^+$ and $\gamma\in\mathbf{\Omega}^-$. Let $F(A_V)$ and $F(A_W)$ be the curvatures of the h_V- and h_W-unitary connections corresponding to $\bar\partial_V$

and $\bar{\partial}_W$, respectively. Let β^* and γ^* be the adjoints with respect to h_V and h_W. Fix a Kähler form ω on X with volume of X normalized to 2π. We consider the moduli space $\mathcal{S}$ defined by the subspace of elements in $\mathscr{H}_1^k$ satisfying *Hitchin equations*

$$F(A_V) + \beta\beta^* + \gamma^*\gamma = -\sqrt{-1}\mu \,\mathrm{Id}_V\, \omega,$$
$$F(A_W) + \gamma\gamma^* + \beta^*\beta = -\sqrt{-1}\mu \,\mathrm{Id}_W\, \omega,$$

modulo gauge transformations in $\mathscr{G}_2^k$, where the equations are only defined on $X \setminus D$. Taking the traces of the equations, adding them, integrating over $X \setminus D$, and using the Chern–Weil formula for parabolic bundles, we find that $\mu = \mathrm{par}\,\mu(V \oplus W)$.

The subspace of smooth points in $\mathscr{H}_1^k$ carries a Kähler metric induced by the complex structure of X and the hermitian metrics h_V and h_W. The Hitchin equations are moment map equations for the action of $\mathscr{G}_2^k$ on this subspace. In particular, the smooth part of $\mathcal{S}$, which corresponds to irreducible solutions, is obtained as a Kähler quotient. Under the genericity assumptions on the parabolic weights in Assumption 2.1, all the solutions are irreducible and the moduli space $\mathcal{S}$ is a smooth Kähler manifold.

Fix the topological invariants $p = \mathrm{rk}\, V$, $q = \mathrm{rk}\, W$, $a = \deg V$, $b = \deg W$ and the weight types α and α' of V and W, respectively. Then

$$\mathcal{U}(p, q, a, b; \alpha, \alpha') \cong (\mathscr{H}^s)_1^k / (\mathscr{G}^{\mathbb{C}})_2^k,$$

where $\mathscr{H}^s$ are the stable elements in $\mathscr{H}$. Moreover, if $\mathcal{S}(p, q, a, b; \alpha, \alpha')$ is the moduli space of solutions for these fixed invariants, we have the following.

THEOREM 5.1 *There is a homeomorphism*

$$\mathcal{U}(p, q, a, b; \alpha, \alpha') \cong \mathcal{S}(p, q, a, b; \alpha, \alpha').$$

6. Morse theory on the moduli space of parabolic U(p, q)-Higgs bundles

In this section we recall the Bott–Morse theory used already in the study of parabolic Higgs bundles in [7, 18]. There is an action of $\mathbb{C}^*$ on $\mathcal{U}$ given by

$$\psi : \mathbb{C}^* \times \mathcal{U} \to \mathcal{U}$$
$$(\lambda, (E, \Phi)) \mapsto (E, \lambda\Phi).$$

This restricts to a Hamiltonian action of the circle on the moduli space $\mathcal{S}$ of solutions to the Hitchin equations, which is isomorphic to $\mathcal{U}$ (Theorem 5.1), with associated moment map

$$[(E, \Phi)] \mapsto -\frac{1}{2}\|\Phi\|^2 = -\sqrt{-1}\int_X \mathrm{Tr}\,(\Phi\Phi^*).$$

We choose to use the positive function, $f : \mathcal{U} \to \mathbb{R}$

$$f([E, \Phi]) = \|\Phi\|^2. \tag{12}$$

Clearly f is bounded below since it is non-negative. It is also proper; this follows from the properness of the moment map associated to the circle action on $\mathcal{M}$ [5] (see also [18]) and the fact that $\mathcal{U} \subset \mathcal{M}$ is a closed subset.

To study the connectedness properties of $\mathcal{U}$, we use the following basic result: if Z is a Hausdorff space and $f : Z \to \mathbb{R}$ is proper and bounded below then f attains a minimum on each connected component of Z. Therefore, if the subspace of local minima of f is connected then so is Z. We thus have the following.

LEMMA 6.1 *The function* $f : \mathcal{U} \to \mathbb{R}$ *defined in* (12) *has a minimum on each connected component of* $\mathcal{U}$. *Moreover, if the subspace of local minima of* f *is connected then so is* $\mathcal{U}$.

Now we will describe the minima of f. For this we introduce the subset of $\mathcal{U}$ defined by

$$\mathcal{N} = \mathcal{N}(p, q, a, b; \alpha, \alpha') = \{(E, \Phi) \in \mathcal{U}(p, q, a, b; \alpha, \alpha') \text{ such that } \beta = 0 \text{ or } \gamma = 0\}. \quad (13)$$

PROPOSITION 6.2 *For every* $(E, \Phi) \in \mathcal{U}$

$$f(E, \Phi) \geq \frac{|\tau|}{2},$$

with equality if and only if $(E, \Phi) \in \mathcal{N}$.

Proof. The proof is similar to the one for [10, Proposition 4.5] apart from the fact that we are using adapted metrics on the bundle.

We will prove that $\mathcal{N}$ is the subvariety of local minima of f. For this we have to describe the critical points of f and characterize the local minima. By a theorem of Frankel [16], the critical points of f are exactly the fixed points of the circle action.

For a fixed point (E, Φ) of the circle action, we have an isomorphism $(E, \Phi) \cong (E, e^{\sqrt{-1}\theta}\Phi)$ which yields the following commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E \otimes K(D) \\ \psi_\theta \downarrow & & \downarrow \psi_\theta \otimes 1_{K(D)} \\ E & \xrightarrow{e^{\sqrt{-1}\theta}\Phi} & E \otimes K(D) \end{array}$$

PROPOSITION 6.3 [31, Theorem 8] *The equivalence class of a stable parabolic Higgs bundle* (E, Φ) *is fixed under the action of* S^1 *if and only if it is a* parabolic Hodge bundle. *This means that* E *decomposes as a direct sum*

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_m$$

of parabolic bundles, such that $\Phi_l = \Phi|_{E_l}$ *belongs to* $H^0(\mathrm{SParHom}\,(E_l, E_{l+1}) \otimes K(D))$. *If* $\Phi_l \neq 0$, *then the weight of the isomorphism* $\psi_\theta : E \longrightarrow E$ *on* E_{l+1} *is one plus the weight of* ψ_θ *on* E_l.

The decomposition of E is given by the eigenbundles corresponding to the eigenvalues of the circle action on (E, Φ).

COROLLARY 6.4 *In the situation of Proposition* 6.3*, if* (E, Φ) *is stable, then each* Φ_l *is non-zero and the* E_l *are alternately contained in* V *and* W.

Proof. The proof goes similarly to the non-parabolic case (see [10, Proposition 4.10]).

Now we want to compute the index of a critical point (E, Φ). For this we need to write the complex in (3) in terms of the eigenbundle decomposition provided by Proposition 6.3. Hence

$$\mathrm{ParEnd}\,(V) \oplus \mathrm{ParEnd}\,(W) = \bigoplus_{-m \le 2k \le m} U_{2k}$$

$$\mathrm{SParHom}\,(V, W) \oplus \mathrm{SParHom}\,(W, V) = \bigoplus_{-m \le 2k+1 \le m} \hat{U}_{2k+1}.$$

where

$$\begin{aligned} U_l &= \bigoplus_{i-j=l} \mathrm{ParHom}\,(E_j, E_i), \\ \hat{U}_l &= \bigoplus_{i-j=l} \mathrm{SParHom}\,(E_j, E_i). \end{aligned} \tag{14}$$

Therefore the deformation complex (3) for a parabolic $\mathrm{U}(p, q)$-Higgs bundle (E, Φ) can be written as

$$C^\bullet\colon \bigoplus_{-m \le 2k \le m} U_{2k} \xrightarrow{\mathrm{ad}(\Phi)} \bigoplus_{-m \le 2k+1 \le m} \hat{U}_{2k+1} \otimes K(D).$$

Each piece of this complex gives a subcomplex whose hypercohomology gives an eigenspace of the tangent space $T_{(E,\Phi)}\mathcal{U}$ for the circle action.

PROPOSITION 6.5 *Let* (E, Φ) *be a stable parabolic* $\mathrm{U}(p, q)$*-Higgs bundle which represents a fixed point of the circle action on* $\mathcal{U}$*. Then the eigenspace of the Hessian of* f *corresponding to the eigenvalue* $-2k$ *is* $\mathbb{H}^1$ *of the following complex:*

$$C^\bullet_{2k} : U_{2k} \xrightarrow{\mathrm{ad}(\Phi)} \hat{U}_{2k+1} \otimes K(D)\,.$$

Proof. This is similar to the non-parabolic case (see [10, Proposition 4.11]).

COROLLARY 6.6 (E, Φ) *is a local minimum of* f *if and only if* $\mathbb{H}^1(C^\bullet_{2k}) = 0$ *for all* $k \ge 1$.

PROPOSITION 6.7 *Let* (E, Φ) *be a stable parabolic* $\mathrm{U}(p, q)$*-Higgs bundle which is a fixed point of the* S^1*-action on* $\mathcal{U}$*. Then* $\chi(C^\bullet_{2k}) \le 0$ *for all* $k \ge 1$*, and equality holds if and only if*

$$\mathrm{ad}(\Phi)|_{U_{2k}} : U_{2k} \to \hat{U}_{2k+1} \otimes K(D)$$

is an isomorphism of bundles.

Proof. We want to get a bound for

$$\chi(C^\bullet_{2k}) = \chi(U_{2k}) - \chi(\hat{U}_{2k+1} \otimes K(D)). \tag{15}$$

The dual of each U_l is

$$U_l^\vee = \bigoplus_{i-j=l} (\mathrm{ParHom}\,(E_j, E_i))^\vee = \bigoplus_{i-j=l} \mathrm{SParHom}\,(E_i, E_j(D)) = \hat{U}_{-l}(D).$$

The dual of $\mathrm{ad}(\Phi)|_{U_{2k}}$ is

$$(\mathrm{ad}(\Phi)|_{U_{2k}})^t = \mathrm{ad}(\Phi)|_{U_{-2k-1}} \otimes 1_{K^{-1}} : U_{-2k-1} \otimes K^{-1} \to \hat{U}_{-2k}(D).$$

The vector bundle $\mathrm{ParEnd}\,(E)$ has a natural parabolic structure induced by the parabolic structure of E. In fact $\mathrm{ParEnd}\,(E)$ as a parabolic bundle is the parabolic tensor product of the parabolic bundle E and the parabolic dual of E (see [34]), and hence its parabolic degree is 0. With respect to this parabolic structure $(\mathrm{ParEnd}\,(E), \mathrm{ad}(\Phi))$, where $\mathrm{ad}(\Phi) : \mathrm{ParEnd}\,(E) \to \mathrm{SParEnd}\,(E) \otimes K(D)$, is a parabolic Higgs bundle. Now, the stability of (E, Φ) implies the polystability of $(\mathrm{ParEnd}\,(E), \mathrm{ad}(\Phi))$. This can be seen by producing a solution to the Hitchin equations on $(\mathrm{ParEnd}\,(E), \mathrm{ad}(\Phi))$ out of the solution on (E, Φ), which exists by Theorem 5.1. Since the solution on $(\mathrm{ParEnd}\,(E), \mathrm{ad}(\Phi))$ may not be irreducible, we only have polystability (in particular, semistability) of $(\mathrm{ParEnd}\,(E), \mathrm{ad}(\Phi))$. The subbundles $\ker(\mathrm{ad}(\Phi)|_{U_{2k}})$ and $\ker(\mathrm{ad}(\Phi)|_{U_{-2k-1}})$ of $\mathrm{ParEnd}\,(E)$ are $\mathrm{ad}(\Phi)$-invariant and hence we can apply the stability condition on the parabolic slopes. Since the ordinary degree is smaller than the parabolic degree, we have $\deg(\ker(\mathrm{ad}(\Phi)|_{U_{2k}})) \le 0$ and $\deg(\ker(\mathrm{ad}(\Phi)|_{U_{-2k-1}})) \le 0$. Therefore we have the following chain of inequalities:

$$\begin{aligned}
\deg(U_{2k}) &= \deg(\ker(\mathrm{ad}(\Phi)|_{U_{2k}})) + \deg(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{2k}})) \\
&\le \deg(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{2k}})) \le -\deg(\mathrm{im}\,((\mathrm{ad}(\Phi)|_{U_{2k}})^t)) \\
&= -\deg(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{-2k-1}} \otimes 1_{K^{-1}})) \\
&= -\deg(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{-2k-1}})) + \mathrm{rk}(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{-2k-1}}))(2g-2) \\
&= \deg(\ker(\mathrm{ad}(\Phi)|_{U_{-2k-1}})) - \deg(U_{-2k-1}) + \mathrm{rk}(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{-2k-1}}))(2g-2) \\
&\le -\deg(U_{-2k-1}) + \mathrm{rk}(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{-2k-1}}))(2g-2) \\
&= \deg(\hat{U}_{2k+1}(D)) + \mathrm{rk}(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{-2k-1}}))(2g-2),
\end{aligned} \tag{16}$$

where we have used that $\mathrm{rk}(\mathrm{im}\,(h)) = \mathrm{rk}(\mathrm{im}\,(h^t))$ and that $\deg(\mathrm{im}\,(h)) \le -\deg(\mathrm{im}\,(h^t))$ for any morphism of sheaves h.

Using this we have that

$$\begin{aligned}
\chi(C^\bullet_{2k}) &= \deg(U_{2k}) + \mathrm{rk}(U_{2k})(1-g) - \deg(U_{2k+1} \otimes K(D)) - \mathrm{rk}(U_{2k+1})(1-g) \\
&= \deg(U_{2k}) + \mathrm{rk}(U_{2k})(1-g) - \deg(U_{2k+1}) - \mathrm{rk}(U_{2k+1})(g-1+s) \\
&\le \deg(\hat{U}_{2k+1}(D)) + \mathrm{rk}(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{2k}}))(2g-2) + \mathrm{rk}(U_{2k})(1-g) - \deg(U_{2k+1}) \\
&\quad - \mathrm{rk}(U_{2k+1})(g-1+s) \\
&= (g-1)(2\,\mathrm{rk}(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{2k}})) - \mathrm{rk}(U_{2k}) - \mathrm{rk}(U_{2k+1})),
\end{aligned}$$

where we have used that $\hat{U}_{2k+1} = U_{2k+1}$ since all the weights are different and of multiplicity 1, and hence for $i \neq j$ it is $\mathrm{SParHom}\,(E_i, E_j) = \mathrm{ParHom}\,(E_i, E_j)$, since E_i and E_j are different pieces in the decomposition of Proposition 6.3. We thus have $\chi(C^\bullet_{2k}) \leq 0$. If equality holds then $\mathrm{rk}(\mathrm{im}\,(\mathrm{ad}(\Phi)|_{U_{2k}})) = \mathrm{rk}(U_{2k}) = \mathrm{rk}(U_{2k+1})$, and also equality holds in (16), showing that $\mathrm{ad}(\Phi)|_{U_{2k}}$ is an isomorphism as claimed.

COROLLARY 6.8 *Let* (E, Φ) *be a stable parabolic* $\mathrm{U}(p, q)$*-Higgs bundle which represents a critical point of the Morse function* f*. This critical point is a minimum if and only if*

$$\mathrm{ad}(\Phi)|_{U_{2k}} : U_{2k} \to \hat{U}_{2k+1} \otimes K(D)$$

is an isomorphism for all $k \geq 1$.

Proof. By Corollary 6.6, (E, Φ) is a local minimum if and only if

$$\mathbb{H}^1(C^\bullet_{2k}) = 0 \quad \forall k \geq 1. \tag{17}$$

Note that by Proposition 3.3, $\mathbb{H}^0(C^\bullet_{2k}) = 0$ and $\mathbb{H}^2(C^\bullet_{2k}) = 0$, for $k \geq 1$. Hence (E, Φ) is a local minimum if and only if

$$\chi(C^\bullet_{2k}) = \sum (-1)^i \dim \mathbb{H}^i(C^\bullet_{2k}) = 0 \quad \forall k \geq 1.$$

By Proposition 6.7, this is equivalent to requiring that

$$\mathrm{ad}(\Phi) : U_{2k} \to \hat{U}_{2k+1} \otimes K(D)$$

be an isomorphism of sheaves.

Finally, we show that all these minima are in $\mathcal{N}$.

PROPOSITION 6.9 *Let* $(E, \Phi) = (E_0 \oplus \cdots \oplus E_m, \Phi)$ *be stable and a fixed point of the circle action, with* $m \geq 2$*. Then* (E, Φ) *is not a local minimum.*

Proof. First note that $U_l = \hat{U}_l = 0$ for $l > m$, and note also that for $l = m$, $U_m = \mathrm{ParHom}\,(E_0, E_m)$. Now we divide the proof conforming the different possibilities for U_l and $\hat{U}_l$ as the number m of terms in the bundle decomposition of E is even or odd.

If m is even then $2k = m$ and

$$\mathrm{ad}(\Phi)|_{U_m} : \mathrm{ParHom}\,(E_0, E_m) \to 0$$

does not satisfy Corollary 6.8, hence (E, Φ) is not a local minimum.

If $m \geq 2$ is odd, then $2k = m - 1$ and

$$\mathrm{ad}(\Phi)|_{U_{m-1}} : \mathrm{ParHom}\,(E_0, E_{m-1}) \oplus \mathrm{ParHom}\,(E_1, E_m) \to \mathrm{SParHom}\,(E_0, E_m) \otimes K(D).$$

We will show that this is not an injective map of sheaves, and therefore $(E_0 \oplus \cdots \oplus E_m, \Phi)$ is not a minimum. We prove this in a small open set where all the bundles trivialize. We need to find

$\zeta = (\zeta_1, \zeta_2) \in U_{m-1}$, $\zeta \neq 0$ such that $\mathrm{ad}(\Phi)|_{U_{m-1}}(\zeta) = 0$, that is, we need to find ζ_1 and ζ_2 making the following diagram commutative.

$$\begin{array}{ccc} E_0 & \xrightarrow{\Phi} & E_1 \otimes K(D) \\ \Big\downarrow{\scriptstyle \zeta_1} & & \Big\downarrow{\scriptstyle \zeta_2 \otimes 1_{K(D)}} \\ E_{m-1} & \xrightarrow{\Phi} & E_m \otimes K(D) \end{array}$$

For this, take $\zeta_2 \neq 0$ such that $\zeta_2 \otimes 1_{K(D)}(E_1 \otimes K(D)) \subset \Phi(E_{m-1})$; this is possible by taking ζ_2 as the composition of Φ_l in Proposition 6.3 tensor the appropriate power of $K(D)$ (note that they are non-zero by Corollary 6.4). Now take ζ_1 such that

$$\Phi \circ \zeta_1 = (\zeta_2 \otimes 1_{K(D)}) \circ \Phi;$$

therefore $\Phi_{m-1}(\zeta) = (\zeta_2 \otimes 1_{K(D)}) \circ \Phi - \Phi \circ \zeta_1 = 0$ with $\zeta \neq 0$. So Φ_{m-1} is not injective.

COROLLARY 6.10 *The subvariety of local minima of* $f : \mathcal{U}(p, q, a, b; \alpha, \alpha') \to \mathbb{R}$ *coincides with the set* $\mathcal{N}(p, q, a, b; \alpha, \alpha')$ *defined in* (13).

Proof. By Proposition 6.9, for (E, Φ) to be a minimum it must have a decomposition of the form $E = E_0 \oplus E_1$ with Φ mapping E_0 into E_1. But by definition the only possible decompositions are $E = V \oplus W$ with $\Phi = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}$ and $E = W \oplus V$ with $\Phi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$. So $(E, \Phi) \in \mathcal{N}$.

Conversely, if $(E, \Phi) \in \mathcal{N}$ then $m = 1$ and $U_{2k} = \hat{U}_{2k+1} = 0$ for $k \geq 1$. So Corollary 6.8 applies and (E, Φ) is a minimum.

Which of the two components of the Higgs field vanishes is given by the following.

LEMMA 6.11 *Let* $(E, \Phi) \in \mathcal{N}$. *Then the Toledo invariant* $\tau \neq 0$ *and*

(i) $\gamma = 0$ *if and only if* $\tau < 0$,
(ii) $\beta = 0$ *if and only if* $\tau > 0$.

Proof. Observe that τ cannot be equal to zero because this implies $\gamma = \beta = 0$ and then (E, Φ) cannot be stable. The rest follows directly from the definition of the Toledo invariant.

Our main goal in the rest of the paper is to show the following.

THEOREM 6.12 *Suppose* $g > 0$. *Then there is a value*

$$\tau_L = \min\{p, q\}(2g - 2 + s) - \frac{|p - q|}{p + q}\,\epsilon,$$

with $\epsilon > 0$ *explicitly computable* (*see Remark* 11.9)*, such that the subvariety* $\mathcal{N}(p, q, a, b; \alpha, \alpha')$ *is non-empty and connected if and only if the parabolic Toledo invariant* τ *satisfies the bound* $|\tau| \leq \tau_L$. *The moduli space of parabolic* U(p, q)*-Higgs bundles* $\mathcal{U}(p, q, a, b; \alpha, \alpha')$ *is empty for* $|\tau| > \tau_L$.

Proof. In the case $p \neq q$, the result will follow from Proposition 7.4 and Theorem 11.8. In the case $p = q$, the result will follow from Propositions 7.4 and 7.7, Corollary 12.12 and Remark 12.13. Note that $\tau_L = \tau_M$ for $p = q$.

Combining Theorem 6.12, Corollary 6.10 and Lemma 6.1, we have the main result of our paper.

THEOREM 6.13 *Suppose $g > 0$ and $s > 0$. The moduli space of parabolic* $\mathrm{U}(p, q)$*-Higgs bundles* $\mathcal{U}(p, q, a, b; \alpha, \alpha')$ *is non-empty and connected if and only if* $|\tau| \leq \tau_L$.*The moduli space is empty whenever* $|\tau| > \tau_L$.

REMARK 6.14 It is likely that Theorem 6.13 holds more generally than under Assumption 2.1. It should be enough to assume that $V \oplus W$ have full flags, but arbitrary (non-generic) weights. The reason is that the assumption of full flags is strong enough to avoid the type of problem that comes up in [**10**, Theorem 3.32], since all the weights are distinct. One way to prove this would be to show that the moduli spaces for different choices of weights are related by flips as with the moduli spaces of triples (as in [**32**]).

REMARK 6.15 Actually, in both Theorems 6.12 and 6.13, the case $|\tau| = \tau_L$ does not occur under Assumption 2.1. This is true since $\sigma = 2g - 2$ is not a critical value for the appropriate moduli space of triples appearing in Proposition 7.4 (see Remark 7.5). For $p = q$, it cannot happen that $|\tau| = \tau_M$, as pointed out in Remark 12.13.

7. Parabolic triples

In the previous section, we have concluded that it is necessary to study the connectedness of the subspace $\mathcal{N}$ of $\mathcal{U}$. This subset consists of parabolic $\mathrm{U}(p, q)$-Higgs bundles with $\gamma = 0$ or $\beta = 0$, and hence gives rise in a natural way to objects called parabolic triples.

We recall the basics of parabolic triples from [**4**, **18**]. A *parabolic triple* is a holomorphic triple $T = (E_1, E_2, \phi)$, where E_1 and E_2 are parabolic bundles and $\phi : E_2 \to E_1(D)$ is a strongly parabolic homomorphism, that is, $\phi \in H^0(\mathrm{SParHom}\,(E_2, E_1(D)))$. We denote by $\alpha = (\alpha^1, \alpha^2)$ the parabolic system of weights for the triple (E_1, E_2, ϕ), where α^i is the system of weights of E_i with $i = 1, 2$.

For $\sigma \in \mathbb{R}$ the parabolic *σ-degree* and *σ-slope* of T are defined as

$$\begin{aligned} \mathrm{pardeg}_{\sigma}(T) &= \mathrm{pardeg}\,(E_1) + \mathrm{pardeg}\,(E_2) + \sigma\, \mathrm{rk}(E_2), \\ \mathrm{par}\,\mu_{\sigma}(T) &= \frac{\mathrm{pardeg}\ E_1 + \mathrm{pardeg}\ E_2}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} + \sigma \frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)}. \end{aligned} \tag{18}$$

A parabolic triple $T' = (E_1', E_2', \phi')$ is a *parabolic subtriple* of $T = (E_1, E_2, \phi)$ if $E_i' \subset E_i$ are parabolic subbundles for $i = 1, 2$ and $\phi' = \phi|_{E_2'}$ being $\phi(E_2') \subset E_1'(D)$. As usual, T is called σ-stable (resp. σ-semistable) if for any non-zero proper subtriple T' we have $\mathrm{par}\,\mu_{\sigma}(T') < \mathrm{par}\,\mu_{\sigma}(T)$ (resp. $\mathrm{par}\,\mu_{\sigma}(T') \leq \mathrm{par}\,\mu_{\sigma}(T)$). The triple T is called σ-polystable if it is the direct sum of parabolic triples with the same parabolic σ-slope.

Let

$$\mathcal{N}_{\sigma} = \mathcal{N}_{\sigma}(r_1, r_2, d_1, d_2; \alpha^1, \alpha^2)$$

be the moduli space of isomorphism classes of σ-polystable triples with fixed system of weights (α^1, α^2) and $r_1 = \mathrm{rk}(E_1)$, $r_2 = \mathrm{rk}(E_2)$, $d_1 = \deg(E_1)$, $d_2 = \deg(E_2)$. Let

$$\mathcal{N}_\sigma^s \subset \mathcal{N}_\sigma$$

be the open subset consisting of σ-stable triples.

PROPOSITION 7.1 *A necessary condition for $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2, \alpha^1, \alpha^2)$ to be non-empty is*

$$\sigma_m < \sigma < \sigma_M \quad \text{if } r_1 \neq r_2,$$
$$\sigma_m < \sigma \quad \text{if } r_1 = r_2,$$

where

$$\sigma_m = \operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2),$$
$$\sigma_M = \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right)(\operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2)) + s\frac{r_1 + r_2}{|r_1 - r_2|}, \quad \textit{if } r_1 \neq r_2.$$

Proof. See [**18**, Proposition 4.3].

REMARK 7.2 We will see later on that there is an effective upper bound σ_L given by equation (38) which in general is strictly smaller than σ_M.

The correspondence between parabolic triples and parabolic U(p,q)-Higgs bundles goes as follows. Let (E, Φ) be a parabolic U(p,q)-Higgs bundle with $\Phi = \beta : W \to V \otimes K(D)$. This defines a triple $T = (E_1, E_2, \phi)$, where $E_1 = V \otimes K$, $E_2 = W$, $\phi = \beta$. Conversely, given a parabolic triple $T = (E_1, E_2, \phi)$ we get a parabolic U(p,q)-Higgs bundle with $\Phi = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$ by defining $(E = V \oplus W, \Phi)$ where $V = E_1 \otimes K^{-1}$, $W = E_2$ and $\beta = \phi$. When (E, Φ) is a parabolic U(p,q)-Higgs bundle with $\Phi = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} : V \to W \otimes K(D)$ we have an analogous correspondence. That is, the corresponding triple to (E, Φ) is $T = (W \otimes K, V, \gamma)$.

LEMMA 7.3 *A parabolic* U(p,q)*-Higgs bundle (E, Φ) with $\beta = 0$ or $\gamma = 0$ is parabolically (semi-)stable if and only if the corresponding parabolic triple is σ-(semi-)stable for $\sigma = 2g - 2$.*

Proof. Let $T = (E_1, E_2, \phi)$ be the triple defined by (E, Φ) (without loss of generality we assume that $\gamma = 0$). Therefore if we set $\sigma = 2g - 2$ we have

$$\begin{aligned} \operatorname{par}\mu_\sigma(T) &= \frac{\operatorname{pardeg}(E_1) + \operatorname{pardeg}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} + \sigma\frac{\mathrm{rk}(E_2)}{\mathrm{rk}(E_1) + \mathrm{rk}(E_2)} \\ &= \frac{\operatorname{pardeg}(V) + \operatorname{pardeg}(W) + p(2g-2)}{p+q} + \sigma\frac{q}{p+q} \\ &= \operatorname{par}\mu(E) + 2g - 2. \end{aligned} \tag{19}$$

Note that the correspondence between parabolic triples and U(p,q) parabolic bundles with $\beta = 0$ or $\gamma = 0$ gives also a correspondence between parabolic subtriples and parabolic subbundles. That is, given a subtriple T' of T the corresponding parabolic $\mathrm{U}(p,q)$-Higgs bundle is a Φ-invariant subbundle of (E,Φ), and conversely given (E',Φ') the corresponding triple gives a parabolic subtriple of T. Hence equation (19) gives that $\operatorname{par}\mu_{2g-2}(T') < \operatorname{par}\mu_{2g-2}(T)$ if and only if $\operatorname{par}\mu(E') < \operatorname{par}\mu(E)$ (and analogously for the semistability condition).

Combining the arguments above and Lemma 6.11, we have the following correspondence.

PROPOSITION 7.4 *Let $\mathcal{N}(p,q,a,b;\alpha,\alpha')$ be the submanifold of local minima of $\mathcal{U}(p,q,a,b;\alpha,\alpha')$ and let τ be the Toledo invariant.*

(i) *If $\tau<0$ then $\mathcal{N}(p,q,a,b;\alpha,\alpha') = \mathcal{N}_{2g-2}(p,q,a+p(2g-2),b;\alpha,\alpha')$.*
(ii) *If $\tau>0$ then $\mathcal{N}(p,q,a,b;\alpha,\alpha') = \mathcal{N}_{2g-2}(q,p,b+q(2g-2),a;\alpha',\alpha)$.*

Proof. This follows immediately from Lemma 6.11.

REMARK 7.5 Note that the genericity condition on the weights implies that there are no properly σ-semistable triples for $\sigma = 2g-2$, that is, $\mathcal{N}^s_{2g-2} = \mathcal{N}_{2g-2}$.

So we state the following assumption that we shall use during the rest of the paper, and which is a translation of Assumption 2.1 via Proposition 7.4.

ASSUMPTION 7.6 We consider moduli spaces of σ-stable triples $\mathcal{N}_\sigma(r_1,r_2,d_1,d_2;\alpha^1,\alpha^2)$ satisfying that there are no properly $(2g-2)$-semistable triples and such that all the weights are of multiplicity one, and the weights of E_1 and E_2 are all different.

It is clear that in order for $\mathcal{N}(p,q,a,b,\alpha,\alpha')$ to be non-empty, $2g-2$ must be in the range for σ given by Proposition 7.1, where σ_m and σ_M are determined by the correspondence given in Proposition 7.4. In fact, one has the following comparison of such a necessary condition with the Milnor–Wood inequality for the parabolic Toledo invariant τ given in Proposition 4.4

PROPOSITION 7.7 *Let σ_m and σ_M be the bounds for σ defined in Proposition 7.1 for the moduli space of parabolic triples identified in Proposition 7.4 with the subvariety $\mathcal{N}(p,q,a,b,\alpha,\alpha')$. Recall that $\tau_M = \min\{p,q\}(2g-2+s)$. Then*

$$0 \le |\tau| \le \tau_M \Leftrightarrow \begin{cases} \sigma_m \le 2g-2 \le \sigma_M & \text{if } p\ne q,\\ \sigma_m \le 2g-2 & \text{if } p=q. \end{cases}$$

Proof. Write σ_m and σ_M in terms of τ, that is,

$$\begin{cases} \sigma_m = \dfrac{(p+q)}{2pq}\tau + 2g-2 & \text{if } \tau<0,\\ \sigma_m = -\dfrac{(p+q)}{2pq}\tau + 2g-2 & \text{if } \tau>0,\\ \sigma_M = \left(1+\dfrac{p+q}{|p-q|}\right)\left(\dfrac{(p+q)}{2pq}\tau+2g-2\right) + s\dfrac{p+q}{|p-q|} & \text{if } \tau<0,\\ \sigma_M = \left(1+\dfrac{p+q}{|p-q|}\right)\left(-\dfrac{(p+q)}{2pq}\tau+2g-2\right) + s\dfrac{p+q}{|p-q|} & \text{if } \tau>0. \end{cases}$$

From these equalities, the result is clear.

REMARK 7.8 Proposition 7.7 gives conditions for the number of marked points in order for $\mathcal{N}$ to be non-empty. Namely

(i) if $g = 0$ then $s \geq 3$,
(ii) if $g = 1$ then $s \geq 1$,

and no extra condition when $g \geq 2$.

8. Extensions and deformations of parabolic triples

In order to study the differences between the moduli spaces $\mathcal{N}_\sigma$ as σ changes, we need to study extensions and deformations of parabolic triples. This study is done in [18]. We summarize the main results.

Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two parabolic triples. Let Hom (T'', T') denote the vector space of homomorphisms from T'' to T', and $\mathrm{Ext}^1(T'', T')$ be the vector space of extensions of the form

$$0 \to T' \to T \to T'' \to 0,$$

that is, commutative diagrams.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi''} & & \\ 0 & \longrightarrow & E'_1(D) & \longrightarrow & E_1(D) & \longrightarrow & E''_1(D) & \longrightarrow & 0 \end{array}$$

In order to study extensions of parabolic triples, we consider the following complex of sheaves:

$$\begin{aligned} C^\bullet(T'', T') : \mathrm{ParHom}\,(E''_1, E'_1) \oplus \mathrm{ParHom}\,(E''_2, E'_2) &\to \mathrm{SParHom}\,(E''_2, E'_1(D)) \\ (\psi_1, \psi_2) &\mapsto \phi'\psi_2 - \psi_1\phi''. \end{aligned} \tag{20}$$

PROPOSITION 8.1 [18, Proposition 4.7] *There are natural isomorphisms*

$$\mathrm{Hom}\,(T'', T') \cong \mathbb{H}^0(C^\bullet(T'', T')),$$
$$\mathrm{Ext}^1(T'', T') \cong \mathbb{H}^1(C^\bullet(T'', T')),$$

and a long exact sequence

$$\begin{aligned} 0 &\to \mathbb{H}^0 \to H^0(\mathrm{ParHom}\,(E''_1, E'_1) \oplus \mathrm{ParHom}\,(E''_2, E'_2)) \to H^0(\mathrm{SParHom}\,(E''_2, E'_1(D))) \\ &\to \mathbb{H}^1 \to H^1(\mathrm{ParHom}\,(E''_1, E'_1) \oplus \mathrm{ParHom}\,(E''_2, E'_2)) \to H^1(\mathrm{SParHom}\,(E''_2, E'_1(D))) \\ &\to \mathbb{H}^2 \to 0. \end{aligned} \tag{21}$$

We denote

$$\begin{aligned} h^i(T'', T') &= \dim \mathbb{H}^i(C^\bullet(T'', T')), \\ \chi(T'', T') &= h^0(T'', T') - h^1(T'', T') + h^2(T'', T'). \end{aligned} \tag{22}$$

PROPOSITION 8.2 [18, Proposition 4.8] *For parabolic triples T' and T''*

$$\chi(T'', T') = \chi(\mathrm{ParHom}\,(E_1'', E_1')) + \chi(\mathrm{ParHom}\,(E_2'', E_2')) - \chi(\mathrm{SParHom}\,(E_2'', E_1'(D))).$$

COROLLARY 8.3 [18, Corollary 4.9] *For any extension $0 \to T' \to T \to T'' \to 0$ of parabolic triples we have that*

$$\chi(T, T) = \chi(T', T') + \chi(T'', T'') + \chi(T'', T') + \chi(T', T'').$$

Using the same arguments as in [11, Proposition 3.5] one can prove the following.

PROPOSITION 8.4 *Suppose that T' and T'' are σ-semistable.*

(i) *If* $\mathrm{par}\,\mu_\sigma(T') < \mathrm{par}\,\mu_\sigma(T'')$*, then* $\mathbb{H}^0(C^\bullet(T'', T')) \cong 0$.
(ii) *If* $\mathrm{par}\,\mu_\sigma(T') = \mathrm{par}\,\mu_\sigma(T'')$ *and T', T'' are σ-stable, then*

$$\mathbb{H}^0(C^\bullet(T'', T')) \cong \begin{cases} \mathbb{C} & \text{if } T' \cong T'', \\ 0 & \text{if } T' \not\cong T''. \end{cases} \tag{23}$$

THEOREM 8.5 *Let $T = (E_1, E_2, \phi)$ be a σ-stable parabolic triple.*

(i) *The Zariski tangent space at the point defined by T in the moduli space $\mathcal{N}_\sigma^s$ of σ-stable triples is isomorphic to* $\mathbb{H}^1(C^\bullet(T, T))$.
(ii) *If* $\mathbb{H}^2(C^\bullet(T, T)) = 0$*, then the moduli space $\mathcal{N}_\sigma^s$ of σ-stable parabolic triples is smooth in a neighbourhood of the point defined by T.*
(iii) $\mathbb{H}^2(C^\bullet(T, T)) = 0$ *if and only if the homomorphism*

$$H^1(\mathrm{ParEnd}\,(E_1)) \oplus H^1(\mathrm{ParEnd}\,(E_2)) \to H^1(\mathrm{SParHom}\,(E_2, E_1(D)))$$

is surjective.
(iv) *At the smooth point in $\mathcal{N}_\sigma^s$ represented by T, the dimension of the moduli space of σ-stable parabolic triples is*

$$\begin{aligned} \dim \mathcal{N}_\sigma^s &= h^1(T, T) = 1 - \chi(T, T) \\ &= 1 - \chi(\mathrm{ParEnd}\,(E_1)) - \chi(\mathrm{ParEnd}\,(E_2)) + \chi(\mathrm{SParHom}\,(E_2, E_1(D))). \end{aligned}$$

(v) *If ϕ is injective or surjective then T defines a smooth point in the moduli space.*

Proof. The proof is analogous to the non-parabolic situation (see [11, Proof of Theorem 3.8]).

9. Critical values

A parabolic triple $T=(E_1, E_2, \phi)$ is *strictly σ-semistable* if and only if there is a proper subtriple $T'=(E_1', E_2', \phi')$ such that $\operatorname{par}\mu_\sigma(T)=\operatorname{par}\mu_\sigma(T')$, that is,

$$\operatorname{par}\mu(T')+\sigma\frac{r_2'}{r_1'+r_2'}=\operatorname{par}\mu(T)+\sigma\frac{r_2}{r_1+r_2}, \tag{24}$$

where $r_1'=\operatorname{rk}(E_1')$, $r_2'=\operatorname{rk}(E_2')$. There are two ways in which this can happen. One is that there exists a parabolic subtriple such that

$$\frac{r_2'}{r_1'+r_2'}=\frac{r_2}{r_1+r_2},$$

therefore this implies

$$\operatorname{par}\mu(T')=\operatorname{par}\mu(T).$$

In this case T is strictly σ-semistable for all σ (or at least for an interval of values of σ) and it is called *σ-independent semistable*. The other way in which strict σ-semistability can happen is if equality holds for (24) but with

$$\frac{r_2'}{r_1'+r_2'}\neq\frac{r_2}{r_1+r_2}.$$

DEFINITION 9.1 The values of σ such that there exists a strictly σ-semistable triple T with a subtriple T' such that $\operatorname{par}\mu_\sigma(T')=\operatorname{par}\mu_\sigma(T)$ and

$$\frac{r_2'}{r_1'+r_2'}\neq\frac{r_2}{r_1+r_2}$$

are called *critical values*.

PROPOSITION 9.2 [**18**, Proposition 5.2] (i) *The critical values of σ form a discrete subset of $[\sigma_m, \sigma_M]$ if $r_1\neq r_2$, and of $[\sigma_m,\infty)$ if $r_1=r_2$.*

(ii) *The stability criteria for two values of σ between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.*

(iii) *For generic weights, $\sigma=2g-2$ is not a critical value.*

Let σ_c be a critical value such that $\sigma_m<\sigma_c<\sigma_M$. Here we adopt the convention that $\sigma_M=\infty$ when $r_1=r_2$. Set

$$\sigma_c^+=\sigma_c+\epsilon,\quad \sigma_c^-=\sigma_c-\epsilon,$$

where $\epsilon>0$ is small enough so that σ_c is the only critical value in the interval (σ_c^-,σ_c^+).

LEMMA 9.3 *Let $\sigma_c\in(\sigma_m,\sigma_M)$ be a critical value. We define the flip loci $\mathcal{S}_{\sigma_c^\pm}$ as the set of triples in $\mathcal{N}^s_{\sigma_c^\pm}$ which are $\sigma_c^\pm$-stable but not $\sigma_c^\mp$-stable. Then*

$$\mathcal{N}^s_{\sigma_c^+}-\mathcal{S}_{\sigma_c^+}=\mathcal{N}^s_{\sigma_c}=\mathcal{N}^s_{\sigma_c^-}-\mathcal{S}_{\sigma_c^-}.$$

The following result is analogous to [11, Proposition 5.4].

PROPOSITION 9.4 *Let $\sigma_c \in (\sigma_m, \sigma_M)$ be a critical value. Let $T = (E_1, E_2, \phi)$ be a triple which is σ_c-semistable.*

(1) *Suppose that T represents a point in $\mathcal{S}_{\sigma_c^+}$, that is, suppose that T is σ_c^+-stable but not σ_c^--stable. Then T has a description as the middle term in an extension*

$$0 \to T' \to T \to T'' \to 0 \tag{25}$$

in which

(a) *T' and T'' are both σ_c^+-stable, with* $\operatorname{par}\mu_{\sigma_c^+}(T') < \operatorname{par}\mu_{\sigma_c^+}(T)$,

(b) *T' and T'' are both σ_c-semistable with* $\operatorname{par}\mu_{\sigma_c}(T') = \operatorname{par}\mu_{\sigma_c}(T)$.

(2) *Similarly, if T represents a point in $\mathcal{S}_{\sigma_c^-}$, that is, if T is σ_c^--stable but not σ_c^+-stable, then T has a description as the middle term in an extension (25) in which*

(a) *T' and T'' are both σ_c^--stable with* $\operatorname{par}\mu_{\sigma_c^-}(T') < \operatorname{par}\mu_{\sigma_c^-}(T)$,

(b) *T' and T'' are both σ_c-semistable with* $\operatorname{par}\mu_{\sigma_c}(T') = \operatorname{par}\mu_{\sigma_c}(T)$.

The following lemma is proved with arguments analogous to those in [11, Proposition 3.6].

LEMMA 9.5 *Let T' and T'' be triples which are σ-stable and of the same σ-slope, for some $\sigma \geq 2g-2$. Then*

$$\mathbb{H}^2(C^\bullet(T'', T')) = 0.$$

COROLLARY 9.6 *$\mathcal{N}_\sigma$ is smooth of the expected dimension, for any $\sigma \geq 2g-2$.*

PROPOSITION 9.7 *If $\sigma_c > 2g-2$ then the loci $\mathcal{S}_{\sigma_c^\pm} \subset \mathcal{N}^s_{\sigma_c^\pm}$ have codimension bigger than or equal to $-\chi(T', T'')$.*

Proof. Let us do the case of σ_c^+. For simplicity we denote

$$\mathcal{N}'_{\sigma_c^\pm} = \mathcal{N}^s_{\sigma_c^\pm}(r_1', r_2', d_1', d_2'; \alpha^{1'}, \alpha^{2'}),$$
$$\mathcal{N}''_{\sigma_c^\pm} = \mathcal{N}^s_{\sigma_c^\pm}(r_1'', r_2'', d_1'', d_2''; \alpha^{1''}, \alpha^{2''}).$$

It is known from [35] that $\mathcal{N}'_{\sigma_c^\pm}$ and $\mathcal{N}''_{\sigma_c^\pm}$ are fine moduli spaces. That is, there are universal parabolic triples $\mathcal{T}' = (\mathcal{E}_1', \mathcal{E}_2', \Phi')$ and $\mathcal{T}'' = (\mathcal{E}_1'', \mathcal{E}_2'', \Phi)$ over $\mathcal{N}'_{\sigma_c^+} \times X$ and $\mathcal{N}''_{\sigma_c^+} \times X$, respectively. Thus we consider the complex $C^\bullet(\mathcal{T}'', \mathcal{T}')$ as defined in (20) and take relative hypercohomology with respect to the projection

$$\pi : X \times \mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+} \to \mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+}.$$

We define $W^+ := \mathbb{H}^1_\pi(C^\bullet(\mathcal{T}'', \mathcal{T}'))$. By Proposition 9.4, $\mathcal{S}_{\sigma_c^+}$ is a subset of the projective fibration $\mathbb{P}W^+$ over $\mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+}$. The fibres of this fibration are projective spaces of dimension

$$\begin{aligned}\dim \mathbb{P}(\mathrm{Ext}^1(T'', T')) &= \dim \mathrm{Ext}^1(T'', T') - 1\\ &= h^0(T'', T') + h^2(T'', T') - \chi(T'', T') - 1\\ &= -\chi(T'', T') - 1,\end{aligned}$$

using Lemma 9.5 and Proposition 8.4 to substitute $h^0(T'', T') = h^2(T'', T') = 0$. Therefore

$$\begin{aligned}\dim \mathcal{S}_{\sigma_c^+} &\leq -\chi(T'', T') + \dim(\mathcal{N}'_{\sigma_c^+} \times \mathcal{N}''_{\sigma_c^+})\\ &= -\chi(T'', T') - 1 + 1 - \chi(T', T') + 1 - \chi(T'', T'')\\ &= \dim \mathcal{N}_{\sigma_c^+} + \chi(T', T''),\end{aligned}$$

since the moduli spaces $\mathcal{N}'_{\sigma_c^+}$ and $\mathcal{N}''_{\sigma_c^+}$ are smooth of the expected dimension. Therefore $\dim \mathcal{N}^s_{\sigma_c^+} - \dim \mathcal{S}_{\sigma_c^+} \geq -\chi(T', T'')$.

Hence, if we prove that this codimension is positive then the moduli spaces $\mathcal{N}^s_\sigma$ for different values of $\sigma \geq 2g-2$ are birational, and in particular have the same number of irreducible components.

10. Codimension of the flip loci

Let σ_c be a critical value in the interval (σ_m, σ_M) such that $\sigma_c \geq 2g-2$. Let T' and T'' be two $\sigma_c^{\pm}$-stable (and σ_c-semistable) parabolic triples with $\mathrm{par}\,\mu_{\sigma_c}(T') = \mathrm{par}\,\mu_{\sigma_c}(T'')$. Changing the roles of T' and T'', we may compute the bound $\chi(T'', T')$ for the codimension of the flip locus (Proposition 9.7) using the complex (20). Under our Assumption 7.6, we have $\mathrm{SParHom}\,(E_2'', E_1'(D)) = \mathrm{ParHom}\,(E_2'', E_1'(D))$, and hence the complex (20) is

$$\begin{aligned}C^\bullet(T'', T') : C_1 = \mathrm{ParHom}\,(E_1'', E_1') \oplus \mathrm{ParHom}\,(E_2'', E_2') &\xrightarrow{a_1} C_0(D) = \mathrm{ParHom}\,(E_2'', E_1'(D))\\ (\xi_1, \xi_2) &\longmapsto \phi'\xi_2 - \xi_1\phi''.\end{aligned}$$

Our task is to bound the Euler characteristic of the complex $C^\bullet(T'', T')$, that is,

$$\chi(C^\bullet(T'', T')) = (1-g)(\mathrm{rk}(C_1) - \mathrm{rk}(C_0)) + \deg(C_1) - \deg(C_0(D)).$$

In order to obtain bounds for $\deg(C_1)$ and $\deg(C_0)$, we follow a similar strategy to that used in [**10**] in the non-parabolic case, exploiting the existence theorem for parabolic vortex equations.

THEOREM 10.1 [**4**, Theorem 3.4] *Let $T = (E_1, E_2, \phi)$ be a parabolic triple. Let τ_1 and τ_2 satisfy $\tau_1\,\mathrm{rk}(E_1) + \tau_2\,\mathrm{rk}(E_2) = \mathrm{pardeg}\,(E_1) + \mathrm{pardeg}\,(E_2)$, and let $\sigma = \tau_1 - \tau_2$. Then E_1 and E_2 admit* *hermitian metrics, adapted to the parabolic structures, satisfying*

$$\sqrt{-1}\Lambda F(E_1) + \phi\phi^* = \tau_1 \operatorname{Id}_{E_1},$$
$$\sqrt{-1}\Lambda F(E_2) - \phi^*\phi = \tau_2 \operatorname{Id}_{E_2},$$

if and only if T *is* σ*-polystable. Here* $F(E_i)$ *is the curvature of the hermitian metric of* E_i *and* Λ *is the contraction with a Kähler form on* X *with volume normalized to* 2π.

One can easily show that

$$\tau_1 = \operatorname{par}\mu_\sigma(T),$$
$$\tau_2 = \operatorname{par}\mu_\sigma(T) - \sigma.$$

Moreover, adding up the equations in Theorem 10.1, integrating, and using the Chern–Weil formula for parabolic bundles, we have that

$$r_1\tau_1 + r_2\tau_2 = \operatorname{pardeg}(E_1) + \operatorname{pardeg}(E_2).$$

In our situation, the triples T' and T'' are σ-stable for $\sigma = \sigma_c^{\pm}$, and hence, by Theorem 10.1, there exist adapted hermitian metrics such that

$$\sqrt{-1}\Lambda F(E_1') + \phi'(\phi')^* = \tau_1' \operatorname{Id}_{E_1'}, \quad \sqrt{-1}\Lambda F(E_2') - (\phi')^*\phi' = \tau_2' \operatorname{Id}_{E_2'},$$
$$\sqrt{-1}\Lambda F(E_1'') + \phi''(\phi'')^* = \tau_1'' \operatorname{Id}_{E_1''}, \quad \sqrt{-1}\Lambda F(E_2'') - (\phi'')^*\phi'' = \tau_2'' \operatorname{Id}_{E_2''},$$

where $\sigma = \tau_1' - \tau_2' = \tau_1'' - \tau_2''$. In particular, $\tau_1' - \tau_1'' = \tau_2' - \tau_2''$.

Let us consider the induced adapted hermitian metrics on C_0 and C_1. The corresponding curvatures are given by

$$F(C_0) = -F(E_2'')^t \otimes \operatorname{Id}_{E_1'} + \operatorname{Id} \otimes F(E_1'),$$
$$F(C_1) = \left(-F(E_1'')^t \otimes \operatorname{Id}_{E_1'} + \operatorname{Id}_{E_1''} \otimes F(E_1'), -F(E_2'')^t \otimes \operatorname{Id}_{E_2'} + \operatorname{Id}_{E_2''} \otimes F(E_2')\right).$$

Actually, we have defined C_0 and C_1 as holomorphic bundles, but they admit parabolic structures in a natural way: given parabolic bundles E and F, there are parabolic duals E^{*p} and parabolic tensor products $E \otimes^p F$ (see [**18**, **34**]). Then the parabolic structure on $\operatorname{ParHom}(E, F)$ is given by $E^{*p} \otimes^p F$. In the formulae for $F(C_0)$ and $F(C_1)$ we have to consider the adapted metrics for the parabolic structures on each $(E_j'')^{*p} \otimes^p E_i'$, induced by the adapted metrics on the bundles E_k' and E_k'', for $k = 1, 2$.

Consider the homomorphism a_2 defined by

$$\operatorname{ParHom}(E_1'', E_2')(-D) \xrightarrow{a_2} \operatorname{ParHom}(E_1'', E_1') \oplus \operatorname{ParHom}(E_2'', E_2')$$
$$\xi \longrightarrow (\phi'\xi, \xi\phi'').$$

The connections on C_0 and C_1 satisfy

$$\begin{aligned}\sqrt{-1}\Lambda F(C_0) + a_1 a_1^* &= (\tau_1' - \tau_2'') \operatorname{Id}_{C_0},\\ \sqrt{-1}\Lambda F(C_1) - a_1^* a_1 + a_2 a_2^* &= (\tau_1' - \tau_1'') \operatorname{Id}_{C_1}.\end{aligned} \tag{26}$$

LEMMA 10.2 *Let K and $Q(D)$ denote the kernel and the torsion-free part of the cokernel, respectively, of the homomorphism a_1. Then*

$$\operatorname{par}\mu(K) \leq \operatorname{par}\mu_\sigma(T') - \operatorname{par}\mu_\sigma(T''),$$
$$\operatorname{par}\mu(Q) \geq \operatorname{par}\mu_\sigma(T'') - \operatorname{par}\mu_\sigma(T') + \sigma.$$

Proof. The kernel K is a subbundle of the hermitian bundle C_1, so that we may take the $\mathcal{C}^\infty$ orthogonal splitting $C_1 = K \oplus S$. Since K is a holomorphic subbundle, the induced connection D_K on K satisfies $D_{C_1}|_K = D_K + A$, where D_{C_1} is the connection on C_1 and $A \in \Omega^{1,0}(\operatorname{Hom}(K, S))$ is the second fundamental form of $K \subset C_1$. Therefore the curvature $F(K)$ of the connection on K satisfies $F(C_1)|_K = F(K) + \bar{A}^t \wedge A$.

We now use the second equation in (26) restricted to K, take the trace and integrate on $X \setminus D$, to get

$$\int_{X\setminus D} \operatorname{Tr}(\sqrt{-1}\Lambda(F(K) + \bar{A}^t \wedge A) - a_1^* a_1|_K + a_2 a_2^*|_K) = \int_{X\setminus D} \operatorname{Tr}((\tau_1' - \tau_1'')\operatorname{Id}_{C_1}|_K).$$

That is,

$$\operatorname{pardeg}(K) + \|A\|^2_{L^2} + \int_{X\setminus D} \operatorname{Tr}(a_2 a_2^*|_K) = (\tau_1' - \tau_1'')\operatorname{rk}(K),$$

obtaining

$$\operatorname{pardeg}(K) \leq (\tau_1' - \tau_1'')\operatorname{rk}(K)$$

as desired, since $\tau_1' = \operatorname{par}\mu_\sigma(T')$ and $\tau_1'' = \operatorname{par}\mu_\sigma(T'')$.

To get the second inequality, let $S'(D)$ be the saturation of the image of a_1, which is a holomorphic subbundle of $C_0(D)$. Then there is a $\mathcal{C}^\infty$ orthogonal splitting $C_0 = S' \oplus Q$. The curvature of the induced connection on Q satisfies $F(C_0)|_Q = F(Q) + B \wedge \bar{B}^t$ with $B \in \Omega^{0,1}(\operatorname{Hom}(Q, S'))$. If we consider the first equation in (26) restricted to Q, take the trace and integrate, we get

$$\int_{X\setminus D} \operatorname{Tr}(\sqrt{-1}\Lambda(F(Q) + B \wedge \bar{B}^t) + a_1 a_1^*)|_Q = \int_{X\setminus D} \operatorname{Tr}((\tau_1' - \tau_2'')\operatorname{Id}_{C_0}|_Q).$$

That is,

$$\operatorname{pardeg}(Q) - \|B\|^2_{L^2} = (\tau_1' - \tau_2'')(\operatorname{rk}(C_0) - \operatorname{rk}(\operatorname{im}(a_1))).$$

Hence

$$\operatorname{pardeg}(Q) \geq (\tau_1' - \tau_2'')(\operatorname{rk}(C_0) - \operatorname{rk}(\operatorname{im}(a_1))) \tag{27}$$

as stated.

THEOREM 10.3 *Let T' and T'' be $\sigma_c^\pm$-stable parabolic triples over a punctured Riemann surface of genus $g > 0$ such that* $\operatorname{par}\mu_{\sigma_c}(T') = \operatorname{par}\mu_{\sigma_c}(T'')$ *for $\sigma_c \geq 2g - 2$. Suppose that the morphism a_1 is not an isomorphism of bundles. Then*

$$\chi(C^\bullet(T'', T')) < 0.$$

Proof. We have

$$\begin{aligned}\chi(C^\bullet(T'',T')) &= (1-g)(\mathrm{rk}(C_1)-\mathrm{rk}(C_0))+\deg(C_1)-\deg(C_0(D))\\ &= (1-g)(\mathrm{rk}(C_1)-\mathrm{rk}(C_0))+\deg(K)+\deg(\mathrm{im}\,(a_1))-\deg(C_0(D))\\ &\leq (1-g)(\mathrm{rk}(C_1)-\mathrm{rk}(C_0))+\deg(K)-\deg(Q)\\ &= (1-g)(\mathrm{rk}(C_1)-\mathrm{rk}(C_0))+\deg(K)-\deg(Q(-D)(D)). \end{aligned} \tag{28}$$

Observe that for any (non-zero) parabolic bundle E, $\deg(E(D)) > \mathrm{pardeg}\,(E) \geq \deg(E)$, where the strict inequality is given by the fact that the weights on E always satisfy $0 \leq \alpha_i(x) < 1$ for all i and all $x \in D$. Using this, the hypothesis $\sigma \geq 2g-2$, and Lemma 10.2, we have

$$\begin{aligned}\chi(C^\bullet(T'',T')) &\leq (1-g)(\mathrm{rk}(C_1)-\mathrm{rk}(C_0))+\mathrm{pardeg}\,(K)-\mathrm{pardeg}\,(Q(-D))\\ &= (1-g)(\mathrm{rk}(C_1)-\mathrm{rk}(C_0))-\sigma(\mathrm{rk}(C_0(D))-\mathrm{rk}(\mathrm{im}\,(a_1))\\ &\leq (1-g)(\mathrm{rk}(C_1)-\mathrm{rk}(C_0))+2(1-g)(\mathrm{rk}(C_0)-\mathrm{rk}(\mathrm{im}\,(a_1))\\ &= (1-g)(\mathrm{rk}(C_1)+\mathrm{rk}(C_0)-2\,\mathrm{rk}(\mathrm{im}\,(a_1))\\ &\leq 0, \end{aligned} \tag{29}$$

using that $g \geq 1$. If either K or Q is a non-zero bundle, then the first line of (29) is a strict inequality. If both are zero and a_1 is not an isomorphism, then the third line of (28) is a strict inequality since $\mathrm{im}\,(a_1) \neq C_0(D)$. In both cases,

$$\chi(C^\bullet(T'',T')) < 0.$$

REMARK 10.4 Note that this theorem does not cover the case $g = 0$. This is not so surprising if we recall that, in order for parabolic bundles to exist on $\mathbb{P}^1$, the parabolic weights must satisfy certain inequalites [**2**, **5**]. Presumably something similar must be true also in the case of parabolic $\mathrm{U}(p,q)$-Higgs bundles.

The following result will be useful in the next sections.

LEMMA 10.5 *If a_1 is generically an isomorphism of bundles, then either*

(a) $E_1'' = 0$ *and* $\phi' : E_2' \to E_1'$ *is generically an isomorphism, in which case* $r_2 > r_1$*, or*
(b) $E_2' = 0$ *and* $\phi'' : E_2'' \to E_1''$ *is generically an isomorphism, in which case* $r_2 < r_1$.

Proof. One may look at a generic point $x \in X \setminus D$, that is, a point where the maps ϕ' and ϕ'' are generic. We have

$$\begin{aligned}(a_1)_x : \mathrm{ParHom}\,(\mathbb{C}^{r_1''},\mathbb{C}^{r_1'})\oplus\mathrm{ParHom}\,(\mathbb{C}^{r_2''},\mathbb{C}^{r_2'}) &\longrightarrow \mathrm{ParHom}\,(\mathbb{C}^{r_2''},\mathbb{C}^{r_1'})\\ (\alpha,\beta) &\longmapsto \phi_x'\,\beta-\alpha\,\phi_x''.\end{aligned}$$

If ϕ_x'' is not surjective, take $\beta = 0$ and $\alpha \neq 0$ with $\alpha|_{\mathrm{im}\,(\phi_x'')} = 0$. Then $(a_1)_x(\alpha,\beta) = 0$. If ϕ_x' is not injective, take $\alpha = 0$ and $\beta \neq 0$ with $\mathrm{im}\,(\beta) \subset \ker\phi_x'$, to get $(a_1)_x(\alpha,\beta) = 0$. Both possibilities contradict the injectivity of $(a_1)_x$. Therefore ϕ_x'' is surjective and ϕ_x' is injective.

If neither of ϕ'_x and ϕ''_x is an isomorphism, then take a map $\mathbb{C}^{r''_2} \to \mathbb{C}^{r'_1}$ which induces a non-zero map $\ker(\phi''_x) \to \operatorname{coker}(\phi'_x)$. This cannot be in the image of $(a_1)_x$, contradicting our assumption. So either ϕ'_x or ϕ''_x is an isomorphism. In the first case $r'_1 r''_1 + r'_2 r''_2 = r''_2 r'_1$ gives $r''_1 = 0$ and we are in case (a). In the second, we are in case (b).

11. Irreducibility of the moduli space of triples for $r_1 \neq r_2$

This section is devoted to study the irreducibility and non-emptiness of the moduli space of σ-stable parabolic triples for ranks $r_1 \neq r_2$.

Given a triple $T = (E_1, E_2, \phi)$ one has the dual triple $T^* = (E_2^{*p}, E_1^{*p}, \phi^t)$, where E_i^{*p} is the parabolic dual of E_i and ϕ^t is the dual of ϕ.

PROPOSITION 11.1 *The σ-stability of T is equivalent to the σ-stability of T^*. The map $T \mapsto T^*$ defines an isomorphism of the corresponding moduli spaces of σ-stable triples.*

This allows us to restrict to the case $r_1 > r_2$ and appeal to duality for the case $r_1 < r_2$. So throughout this section we assume that $r_1 > r_2$.

LEMMA 11.2 *Let X be a Riemann surface with a finite number of marked points and let E, F be parabolic bundles on X. Let $p \in X$ be a parabolic point. Then there is a natural exact sequence*

$$0 \longrightarrow \frac{\operatorname{Hom}(E_p, F_p)}{\operatorname{ParHom}(E_p, F_p)} \otimes \mathcal{O}(-p) \longrightarrow \operatorname{ParHom}(E, F)_p \longrightarrow \operatorname{ParHom}(E_p, F_p) \longrightarrow 0.$$

The second map is induced by restriction to p. The first map is multiplication by a holomorphic function vanishing once at p.

Proof. We have a defining exact sequence for the bundle of parabolic homomorphisms from E to F given by

$$0 \longrightarrow \operatorname{ParHom}(E, F) \longrightarrow \operatorname{Hom}(E, F) \longrightarrow \bigoplus_{x \in D} \frac{\operatorname{Hom}(E_x, F_x)}{\operatorname{ParHom}(E_x, F_x)} \longrightarrow 0.$$

Now we tensor with the skyscraper sheaf $\mathbb{C}(p)$, to get

$$0 \longrightarrow \operatorname{Tor}\left(\frac{\operatorname{Hom}(E_p, F_p)}{\operatorname{ParHom}(E_p, F_p)}, \mathbb{C}(p)\right) \longrightarrow \operatorname{ParHom}(E, F)_p$$
$$\longrightarrow \operatorname{Hom}(E, F)_p \longrightarrow \frac{\operatorname{Hom}(E_p, F_p)}{\operatorname{ParHom}(E_p, F_p)} \longrightarrow 0.$$

This is because $\operatorname{Tor}\left(\frac{\operatorname{Hom}(E_x, F_x)}{\operatorname{ParHom}(E_x, F_x)}, \mathbb{C}(p)\right) = 0$ for $p \neq x$, and the fact that if Θ is a torsion sheaf supported scheme-theoretically at p (that is, supported at p and with no infinitesimal information), we have that Tor $(\Theta, \mathbb{C}(p)) \cong \Theta \otimes \mathcal{O}(-p)$ naturally (to see this, tensor the exact sequence $\mathcal{O}(-p) \to \mathcal{O} \to \mathbb{C}(p)$ with Θ). Hence

$$0 \longrightarrow \frac{\mathrm{Hom}\,(E_p, F_p)}{\mathrm{ParHom}\,(E_p, F_p)} \otimes \mathcal{O}(-p) \longrightarrow \mathrm{ParHom}\,(E, F)_p \longrightarrow \mathrm{Hom}\,(E_p, F_p) \longrightarrow \frac{\mathrm{Hom}\,(E_p, F_p)}{\mathrm{ParHom}\,(E_p, F_p)} \longrightarrow 0,$$

which yields

$$0 \longrightarrow \frac{\mathrm{Hom}\,(E_p, F_p)}{\mathrm{ParHom}\,(E_p, F_p)} \otimes \mathcal{O}(-p) \longrightarrow \mathrm{ParHom}\,(E, F)_p \longrightarrow \mathrm{ParHom}\,(E_p, F_p) \longrightarrow 0.$$

Locally, with a local coordinate z vanishing at p, the second map is given by $(f_0 + f_1 z + \cdots)_p \mapsto f_0$. The first map is $f_1 \mapsto (f_1 z)_p$.

To clarify the Lemma, let us see an example, where E has rank 3 and weights β_i, F has rank 4 and weights α_j and $\beta_1 < \alpha_1 < \alpha_2 < \alpha_3 < \beta_2 < \beta_3 < \alpha_4$. Then a typical parabolic homomorphism from E to F has matrix of the form

$$\phi(z) = \left(\begin{array}{c|cc} \phi_{11}(z) & \phi_{12}(z) & \phi_{13}(z) \\ \phi_{21}(z) & \phi_{22}(z) & \phi_{23}(z) \\ \phi_{31}(z) & \phi_{32}(z) & \phi_{33}(z) \\ \hline \phi_{41}(z) & \phi_{42}(z) & \phi_{43}(z) \end{array}\right)$$

around p. The parabolicity of ϕ means that for $z = 0$, the only non-zero entries are those below the broken line. The line in the matrix is easy to construct: starting by the upper-left corner, draw a horizontal line for each β_j, and a vertical line for each α_i, considering the αs and βs in increasing order. The sheaf ParHom (E, F) is actually a bundle (since it is torsion-free) of rank $\mathrm{rk}(E)\,\mathrm{rk}(F)$. Its stalk at p, ParHom $(E, F)_p$, is formed by the matrices with entries which are complex numbers below the broken line, and which are complex numbers times z above the line.

PROPOSITION 11.3 *Assume that $g > 0$, $\sigma_c \geq 2g - 2$ and $r_1 > r_2$. Let T', T'' be $\sigma_c^{\pm}$-stable triples with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T'')$. Then $\chi(C^\bullet(T'', T')) = 0$ if and only if the following conditions hold:*

(1) *$E_2' = 0$;*
(2) *$\phi'' : E_2'' \to E_1''(D)$ is a fibre bundle isomorphism at $X \setminus D$. In particular, $r_2'' = r_1''$;*
(3) *at any point $p \in D$, write $\phi'' = z^{-1}(\phi_0 + \phi_1 z + \phi_2 z^2 + \cdots)$, where z is a local holomorphic coordinate around p in X. Then* ParHom $(E_{1,p}'', E_{1,p}') \to$ ParHom $(E_{2,p}'', E_{1,p}')$, *$f \mapsto -f \circ \phi_0$, is surjective;*
(4) *at any $p \in D$, consider the induced homomorphism $\phi_1 : \ker \phi_0 \to \operatorname{coker} \phi_0$. Then* ParHom (coker $\phi_0, E_{1,p}'$) $\to$ Hom (ker $\phi_0, E_{1,p}'$), *$f \mapsto -f \circ \phi_1$, is surjective.*

Proof. By Theorem 10.3, $\chi(C^\bullet(T'', T')) = 0$ if and only if a_1 is an isomorphism. By Lemma 10.5, if a_1 is generically an isomorphism and $r_1 > r_2$ then $E_2' = 0$. This proves (1). Also $\phi'' : E_2'' \to E_1''(D)$

is generically an isomorphism. Moreover the two bundles involved in the complex $C^\bullet(T'', T')$ must be of the same rank and of the same degree. The complex $C^\bullet(T'', T')$ reduces to

$$\mathrm{ParHom}\,(E_1'', E_1') \xrightarrow{a_1} \mathrm{ParHom}\,(E_2'', E_1'(D)),$$

where $a_1(f) = -f \circ \phi''$ is an isomorphism of bundles. Restricting a_1 to the open subset $U = X \setminus D$, we have that $\mathrm{Hom}\,(E_1'', E_1')|_U \to \mathrm{Hom}\,(E_2'', E_1'(D))|_U$ is an isomorphism. Hence $E_2''|_U \to E_1''(D)|_U$ is an isomorphism of bundles, and (2) follows.

Now let $p \in D$, take a neighbourhood U of p, and a coordinate z vanishing at p. Hence we may write $\phi'' = \phi_0 z^{-1} + \phi_1 + \phi_2 z + \cdots$, where $\phi_i \in \mathrm{Hom}\,(E_{2,p}'', E_{1,p}'')$ and $\phi_0 \in \mathrm{ParHom}\,(E_{2,p}'', E_{1,p}'')$, on U. We want to characterize when

$$\mathrm{ParHom}\,(E_1'', E_1')_p \to \mathrm{ParHom}\,(E_2, E_1'(D))_p = \mathrm{ParHom}\,(E_2'', E_1'(p))_p$$

is an isomorphism of vector spaces. It is enough to analyse when this map is surjective. Using Lemma 11.2, we have a commutative diagram whose rows are short exact sequences.

$$\begin{array}{ccccc}
\dfrac{\mathrm{Hom}\,(E_{1,p}'', E_{1,p}')}{\mathrm{ParHom}\,(E_{1,p}'', E_{1,p}')} \otimes \mathcal{O}(-p) & \xrightarrow{\cdot z} & \mathrm{ParHom}\,(E_1'', E_1')_p & \longrightarrow & \mathrm{ParHom}\,(E_{1,p}'', E_{1,p}') \\
\big\downarrow b_0 & & \big\downarrow b_1 & & \big\downarrow b_2 \\
\dfrac{\mathrm{Hom}\,(E_{2,p}'', E_{1,p}')}{\mathrm{ParHom}\,(E_{2,p}'', E_{1,p}')} & \xrightarrow{\cdot z} & \mathrm{ParHom}\,(E_2'', E_1'(p))_p & \longrightarrow & \mathrm{ParHom}\,(E_{2,p}'', E_{1,p}') \otimes \mathcal{O}(p)
\end{array}$$

The middle vertical arrow is induced by $f \mapsto -f \circ \phi''$. Thus the right vertical arrow is induced by $f_0 \mapsto -(f_0 \circ \phi_0)z^{-1}$. The left vertical arrow is thus given by $f_1 \mapsto -(f_1 \circ \phi_0)z^{-1}$.

We want to characterize the cases where the middle vertical arrow is surjective. Using the long exact sequence produced by the snake lemma, we see that b_1 being surjective is equivalent to b_2 being surjective and the connecting homomorphism $\ker b_2 \to \mathrm{coker}\ b_0$ also being surjective. The condition that b_2 is surjective is exactly (3).

For the remaining condition, we need to spell out the connecting homomorphism. Take $f_0 \in \mathrm{ParHom}\,(E_{1,p}'', E_{1,p}')$ lying in

$$\ker b_2 = \mathrm{ParHom}\,(E_{1,p}''/\phi_0(E_{2,p}''), E_{1,p}').$$

Lift f_0 to a local section of $\mathrm{ParHom}\,(E_1'', E_1')$ on U, for example, taking $f(z) \equiv f_0$. Compose with ϕ'' to get $-(f \circ \phi_0 + f \circ \phi_1 z + \cdots)z^{-1}$. Recalling that $f \circ \phi_0 = 0$, the leading term is

$$-f_0 \circ \phi_1 \in \mathrm{coker}\ b_0 = \frac{\mathrm{Hom}\,(E_{2,p}'', E_{1,p}')}{\mathrm{ParHom}\,(E_{2,p}'', E_{1,p}') + b_0(\mathrm{Hom}\,(E_{1,p}'', E_{1,p}'))}.$$

Assuming that (3) holds already, we have that $\mathrm{ParHom}\,(E_{2,p}'', E_{1,p}') \subset b_0(\mathrm{ParHom}\,(E_{1,p}'', E_{1,p}')) \subset b_0(\mathrm{Hom}\,(E_{1,p}'', E_{1,p}'))$, since the maps b_0 and b_2 are both composition with ϕ_0. Hence the image of

f_0 under the connecting homomorphism is

$$-f_0 \circ \phi_1 \in \text{coker } b_0 = \frac{\text{Hom}\,(E''_{2,p}, E'_{1,p})}{b_0(\text{Hom}\,(E''_{1,p}, E'_{1,p}))} = \text{Hom}\,(\ker \phi_0, E'_{1,p})\,.$$

Therefore the surjectivity of the connecting homomorphism is equivalent to (4).

LEMMA 11.4 *Condition* (4) *of Proposition* 11.3 *holds if and only if all the weights of* $E'_{1,p}$ *are bigger than those of* coker ϕ_0, *and* $\phi_1 : \ker \phi_0 \to$ coker ϕ_0 *is an isomorphism.*

Proof. The condition (4) says that

$$\text{ParHom}\left(\frac{E''_{1,p}}{\phi_0(E''_{2,p})}, E'_{1,p}\right) \to \text{Hom}\,(\ker \phi_0, E'_{1,p}), \quad f \mapsto -f \circ \phi_1,$$

is surjective. Since $E''_{1,p}/\phi_0(E''_{2,p})$ and $\ker \phi_0$ are vector spaces of the same dimension, this is equivalent to the following two conditions:

- $\phi_1 : E''_{2,p} \to E''_{1,p} \subset E_{1,p}$ satisfies that $\phi_1 : \ker \phi_0 \to \text{coker } \phi_0$ is an isomorphism;
- $\text{ParHom}\,(E''_{1,p}/\phi_0(E''_{2,p}), E'_{1,p}) = \text{Hom}\,(E''_{1,p}/\phi_0(E''_{2,p}), E'_{1,p})$. Hence all the weights of $E''_{1,p}/\phi_0(E''_{2,p})$ are smaller than those of $E'_{1,p}$.

Let $\sigma_c \in (\sigma_m, \sigma_M)$ be a critical value with $\sigma_c \geq 2g-2$. We aim to characterize when $\mathcal{N}^s_{\sigma_c^-}$ and $\mathcal{N}^s_{\sigma_c^+}$ are birational by using Proposition 9.7. Let us deal with either of $\mathcal{S}_{\sigma_c^\pm}$. Suppose that T' and T'' are σ_c-semistable, $\sigma_c^\pm$-stable triples with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T'')$. We consider extensions

$$0 \longrightarrow T'' \longrightarrow T \longrightarrow T' \longrightarrow 0 \tag{30}$$

(note that we have changed the role of T' and T'' in the computation of the codimension of the flip loci in Section 10, so that now T'' is the subtriple), where $\mu_{\sigma_c^\pm}(T'') < \mu_{\sigma_c^\pm}(T)$, by Proposition 9.4. The first conclusion to infer from Proposition 11.3 is that, if $\chi(C^\bullet(T'', T')) = 0$ then $r'_2 = 0$ and $r''_2 = r''_1$. So $\mu_{\sigma_c^+}(T'') > \mu_{\sigma_c^+}(T)$. Therefore $\mathcal{S}_{\sigma_c^+}$ cannot be of zero codimension. So our study is limited to $\mathcal{S}_{\sigma_c^-}$: the only situation we may encounter when $\chi(C^\bullet(T'', T')) = 0$ is that $\mathcal{N}^s_{\sigma_c^-}$ has more irreducible components than $\mathcal{N}^s_{\sigma_c^+}$.

To analyse when $\chi(C^\bullet(T'', T')) = 0$ we have to check when conditions (3) and (4) of Proposition 11.3 are satisfied. Let $p \in D$ be a parabolic point. We need to understand the *parabolic vector spaces* $E_{2,p}$ and $E_{1,p}$. These have parabolic weights of multiplicity one and all weights are different, by Assumption 7.6. We shall keep the following notation for the rest of the section: α_i denote the weights of $E_{1,p}$ and β_j denote the weights of $E_{2,p}$ (we drop p from the notation in the weights when this causes no confusion).

Since T is a triple which is an extension (30) with $r_2' = 0$ and $r_2'' = r_1''$, then $\phi : E_2 \to E_1(D)$ comes from a map $\phi'' : E_2 \to E_1''(D)$ as follows.

$$\begin{array}{ccccc} E_2 & = & E_2 & \longrightarrow & 0 \\ {\scriptstyle \phi''}\downarrow & & {\scriptstyle \phi}\downarrow & & \downarrow \\ E_1''(D) & \longrightarrow & E_1(D) & \longrightarrow & E_1'(D) \end{array}$$

Take a neighbourhood U of p where $E_1|_U = E_1'|_U \oplus E_1''|_U$. Then $\phi = (\phi_0 + \phi_1 z + \cdots)z^{-1}$ and $\phi_0 : E_{2,p} \to E_{1,p}''$ is a parabolic map. This gives decompositions of the parabolic vector spaces

$$\begin{aligned} E_{1,p} &= E_{1,p}' \oplus E_{1,p}'', \\ E_{1,p}'' &= \text{im } \phi_0 \oplus \text{coker } \phi_0, \end{aligned} \tag{31}$$

as direct sums of parabolic vector subspaces (the splitting is non-canonical, but the weights of the different subspaces are well determined).

Let us see that there is a 'canonical' distribution of weights in (31) such that conditions (3) and (4) hold. Note that ParHom $(E_{2,p}, E_{1,p})$ is a vector space, in particular an irreducible affine variety. We may consider the action of ParAut $(E_{2,p}) \times$ ParAut $(E_{1,p})$ on this space (this corresponds to lower triangular changes of bases). Then there is a unique open dense orbit, which is the only orbit of maximal dimension. We shall call an element of such orbit a *generic* parabolic homomorphism of $E_{2,p}$ to $E_{1,p}$. For instance, if $E_{2,p}$ is 7-dimensional with weights β_j and $E_{1,p}$ is 9-dimensional with weights α_i, and

$$\alpha_1 < \beta_1 < \beta_2 < \alpha_2 < \beta_3 < \beta_4 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_6 < \beta_5 < \alpha_7 < \alpha_8 < \beta_6 < \beta_7 < \alpha_9,$$

then the generic elements are the orbit of the element

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

Lemma 11.5 *Suppose that* $\phi_0 : E_{2,p} \to E_{1,p}$ *is a generic parabolic homomorphism, and let* $E_{1,p} = E_{1,p}' \oplus E_{1,p}''$ *be any parabolic splitting with* im $\phi_0 \subset E_{1,p}''$. *Then condition* (3) *in Proposition* 11.3 *is satisfied.*

Proof. Suppose that ϕ_0 is a generic element in ParHom $(E_{2,p}, E_{1,p})$, and let us see that the map ParHom $(E_{1,p}'', E_{1,p}') \to$ ParHom $(E_{2,p}, E_{1,p}')$, $f \mapsto -f \circ \phi_0$, is surjective. Take

$g \in \mathrm{ParHom}\,(E_{2,p}, E'_{1,p})$. Consider the map $\phi_\epsilon = \phi_0 \oplus \epsilon g : E_{2,p} \to E''_{1,p} \oplus E'_{1,p}$. For ϵ small we have that ϕ_ϵ also lives in the generic open set, so it is equivalent to ϕ_0 by the action of $\mathrm{ParAut}\,(E_{2,p}) \times \mathrm{ParAut}\,(E_{1,p})$. This means that

$$\begin{pmatrix} a_\epsilon & b_\epsilon \\ c_\epsilon & d_\epsilon \end{pmatrix} \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} M_\epsilon = \begin{pmatrix} \phi_0 \\ \epsilon g \end{pmatrix}.$$

Both matrices, $\begin{pmatrix} a_\epsilon & b_\epsilon \\ c_\epsilon & d_\epsilon \end{pmatrix}$ and M_ϵ, are the identity for $\epsilon = 0$, so a_ϵ is invertible for small ϵ. Therefore $\phi_0 M_\epsilon = a_\epsilon^{-1}\phi_0$ and $c_\epsilon \phi_0 M_\epsilon = \epsilon g$. This yields

$$g = \epsilon^{-1} c_\epsilon a_\epsilon^{-1} \phi_0,$$

as required.

Recall that we have fixed topological data (fixed ranks, degrees and parabolic weights) for the triples T we are studying. When we write such a triple T as an extension $T'' \to T \to T'$, there are different possible topological types for T' and T''. By the above discussion, our best chance to obtain $\chi(C^\bullet(T', T'')) = 0$ is to arrange the topological types as follows.

- Fix the ranks $r'_2 = 0$, $r''_2 = r_2$, $r''_1 = r_2$, $r'_1 = r_1 - r_2$. This is necessary for conditions (1) and (2) to hold. So $\phi : E_2 \to E_1(D)$ should be induced by $\phi'' : E''_2 \to E''_1(D)$ by means of the inclusion $E''_1(D) \to E_1(D)$.
- At each $p \in D$, consider a generic element $\phi_p \in \mathrm{ParHom}\,(E_{2,p}, E_{1,p})$. This determines the weights of $\mathrm{im}\ \phi_p \subset E''_{1,p}$. By Lemma 11.5 condition (3) is satisfied.
- Choose the weights of $\mathrm{coker}\ \phi''_p$ in the unique way such that Lemma 11.4 is satisfied. This gives the weights of $E''_{1,p} = \mathrm{im}\ \phi_p \oplus \mathrm{coker}\ \phi''_p$ at each $p \in D$, and hence the weights of $E'_{1,p}$.
- $d''_2 = d_2$. Now condition (2) determines the degree of E''_1, since the map $\phi'' : E_2 \to E''_1(D)$ is an isomorphism on $X \setminus D$ and it is of a specified form at each $p \in D$. Namely, introduce the number
$$r_p = \min\{\dim \mathrm{coker}\ \psi_0 \mid \psi_0 \in \mathrm{ParHom}\,(E_{2,p}, E_{1,p})\} - (r_1 - r_2). \tag{33}$$

  Obviously this minimum is obtained for a generic parabolic morphism. Moreover $r_p = \dim \mathrm{coker}\ \phi_0$, where $\phi_p : E_{2,p} \to E_{1,p}$ is generic, and $\phi_0 = \phi_p : E_{2,p} \to E''_{1,p}$, using that $E''_{1,p} \subset E_{1,p}$. With this notation, $E_2 \to E''_1(D) \to \bigoplus_{p\in D} \mathbb{C}(p)^{r_p}$ is an exact sequence of sheaves, so $d''_1 = d_2 - r_2 s + \sum_{p \in D} r_p$.

This does not guarantee the existence or uniqueness of the topological types of T' and T'' to have $\chi(C^\bullet(T', T'')) = 0$, but tells us in which direction to look for such distributions of topological types.

Let us see this discussion in the particular example (32). For a generic $\phi_p : E_{2,p} \to E_{1,p}$, the weights of $\mathrm{im}\ \phi_0$ are $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_9$, and the weight of $\mathrm{coker}\ \phi_0$ is α_1. Thus the weights of

$E'_{1,p}$ are α_6, α_8. The map ϕ takes the form

$$\phi = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + O(z^2) \right) \cdot z^{-1}$$

around $p \in D$. Note that such $\phi : E_2 \to E_1(D)$ is injective for $z \neq 0$, as required by condition (2).

REMARK 11.6 The definitions of generic parabolic map and of r_p given in (33) are also valid in the case $r_1 = r_2$.

PROPOSITION 11.7 *Assume $g > 0$, $r_1 > r_2$ and $\sigma_c \geq 2g - 2$. Let T', T'' be σ_c^--stable triples with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T'')$. If $\chi(C^\bullet(T'', T')) = 0$ then the following hold:*

(i) *$r'_2 = 0$, $r''_2 = r_2 = r''_1$, $d''_2 = d_2$;*
(ii) *for each $p \in D$, the parabolic map $\phi''_p : E_{2,p} \to E''_{1,p}$ has rank $r_2 - r_p$, with r_p defined in (33);*
(iii) *$d''_1 = d_2 - r_2 s + \sum_{p \in D} r_p$;*
(iv) *for each $1 \leq k \leq r_2 - r_p$, define*

$$i_k = \min\{ j \mid 1 \leq j \leq r_1, \beta_k < \alpha_j, j > i_{k-1} \} \tag{34}$$

and let $I = \{i_1, \ldots, i_{r_2 - r_p}\}$. Let $J \subset \{1, \ldots, r_1\} - I$ be the set of the lowest r_p elements of $\{1, \ldots, r_1\} - I$. Then the weights of $E''_{1,p}$ are exactly $\{\alpha_i \mid i \in I \cup J\}$.

In particular, the ranks, degrees and weights of T' and T'' are univocally determined. Thus there is at most one possible value of σ_c for which $\chi(C^\bullet(T'', T')) = 0$.

Proof. Item (i) follows from Proposition 11.3 (1).

Item (iii) follows once we know item (ii), and using Proposition 11.3 (2), since in this case we have an exact sequence of sheaves

$$E_2 = E''_2 \xrightarrow{\phi''} E''_1(D) \to \bigoplus_{p \in D} \mathbb{C}(p)^{r_p}.$$

Next, note that the increasing sequence of numbers $i_1, i_2, \ldots \in \{1, \ldots, r_1\}$ is well defined for $1 \leq k \leq r_2 - r_p$. Actually, looking at a generic parabolic map $\psi_0 : E_{2,p} \to E_{1,p}$, the weights of $\operatorname{im} \psi_0$ are $\alpha_{i_1}, \ldots, \alpha_{i_{r_2 - r_p}}$, with $r_2 - r_p = \dim \operatorname{im} \psi_0$ (see (32) for a specific example).

Now we shall prove (2) and (4) using Proposition 11.3 (3), that is, that

$$\begin{aligned} \mathrm{ParHom}\,(E''_{1,p}, E'_{1,p}) &\longrightarrow \mathrm{ParHom}\,(E_{2,p}, E'_{1,p}) \\ f &\longmapsto f \circ \phi_0 \end{aligned} \tag{35}$$

is surjective, denoting as before, $\phi_0 = \phi''_p$. Let $\{e_1, \ldots, e_{r_1}\}$ be a basis for $E_{1,p}$ adapted to its parabolic structure (and adapted to the splitting $E'_{1,p} \oplus E''_{1,p}$, that is, each e_i belongs either to $E'_{1,p}$ or $E''_{1,p}$), and let $\{v_1, \ldots, v_{r_2}\}$ be a basis for $E_{2,p}$ adapted to its parabolic structure.

Now let $t_0 \in \{1, \ldots, r_1\}$ such that α_{t_0} is the lowest weight of $E'_{1,p}$. Let $0 \le a \le r_2 - r_p$ such that $i_a < t_0 \le i_{a+1}$ (introducing the notation $i_0 = 0$, $i_{r_2 - r_p + 1} = r_1 + 1$). Let us see that $\alpha_{i_{a+1}}, \ldots, \alpha_{r_2 - r_p}$ are weights of im ϕ_0 (if $a = r_2 - r_p$ then there is nothing to prove). Actually, they cannot be weights of coker ϕ_0, since by Lemma 11.4 all the weights of coker ϕ_0 are smaller than α_{t_0}. So they are weights of im ϕ_0 or of $E'_{1,p}$ by (31). Suppose that $\alpha_{i_{a+1}}, \ldots, \alpha_{i_{b-1}}$ are weights of im ϕ_0 but α_{i_b} is the first weight of $E'_{1,p}$ in the list. Then take $V = \langle v_1, \ldots, v_b \rangle \subset E_{2,p}$. The surjectivity of (35) gives that

$$\mathrm{ParHom}\,(E''_{1,p}, \langle e_{i_b} \rangle) \twoheadrightarrow \mathrm{ParHom}\,(E_{2,p}, \langle e_{i_b} \rangle) \twoheadrightarrow \mathrm{ParHom}\,(V, \langle e_{i_b} \rangle) = \mathrm{Hom}\,(V, \langle e_{i_b} \rangle)$$

is surjective (the last equality follows from $\alpha_{i_b} > \beta_b$). Therefore $\phi_0|_V : V \to E''_{1,p}$ must be injective, and all the weights of $\phi_0(V) \subset E''_{1,p}$ should be smaller than α_{i_b}. So there are weights $\alpha_{x_1} < \cdots < \alpha_{x_b} < \alpha_{i_b}$ with $\beta_j < \alpha_{x_j}$. This implies that $i_j \le x_j$, $j = 1, \ldots, b$, which contradicts that $x_b < i_b$.

The next step is to see that there are $y_1 < \cdots < y_a < t_0$ such that $i_j \le y_j$, $j = 1, \ldots, a$ and α_{y_j} are weights of im ϕ_0. As before, take $V = \langle v_1, \ldots, v_a \rangle \subset E_{2,p}$. The surjectivity of (35) gives that

$$\mathrm{ParHom}\,(E''_{1,p}, \langle e_{t_0} \rangle) \twoheadrightarrow \mathrm{ParHom}\,(E_{2,p}, \langle e_{t_0} \rangle) \twoheadrightarrow \mathrm{ParHom}\,(V, \langle e_{t_0} \rangle) = \mathrm{Hom}\,(V, \langle e_{t_0} \rangle)$$

is surjective. So $\phi_0|_V : V \to E''_{1,p}$ must be injective, and all the weights or $\phi_0(V) \subset E''_{1,p}$ should be smaller than α_{t_0}. So there are weights $\alpha_{y_1} < \cdots < \alpha_{y_a} < \alpha_{t_0}$ with $\beta_j < \alpha_{y_j}$. This implies that $i_j \le y_j$, $j = 1, \ldots, a$.

The elements

$$\{y_1, \ldots, y_a, i_{a+1}, \ldots, i_{r_2 - r_p}\} \tag{36}$$

are weights of im ϕ_0. So $\dim \mathrm{im}\, \phi_0 \ge r_2 - r_p$. As obviously $\dim \mathrm{im}\, \phi_0 \le r_2 - r_p$, it must be $\dim \mathrm{im}\, \phi_0 = r_2 - r_p$, implying item (2). Thus the weights of im ϕ_0 are exactly those in (36). The elements

$$\{1, \ldots, t_0 - 1\} - \{y_1, \ldots, y_a\} \tag{37}$$

are the subindices of the weights of coker ϕ_0, by Lemma 11.4. So $t_0 - 1 - a = r_p$, that is, $t_0 = r_p + a + 1$. Finally (the subindices of) the weights of $E''_{1,p}$ are

$$\begin{aligned} (\{1, \ldots, t_0 - 1\} - \{y_1, \ldots, y_a\}) &\cup \{y_1, \ldots, y_a, i_{a+1}, \ldots, i_{r_2 - r_p}\} \\ &= \{1, \ldots, t_0 - 1\} \cup \{i_{a+1}, \ldots, i_{r_2 - r_p}\} = I \cup J, \end{aligned}$$

as required.

Our final result in this section completes the proof of Theorem 6.12. We have to use Theorem 12.10, which will be proved in the next section. First, consider the distribution of weights and degrees given by Proposition 11.7, and consider the critical value associated to it, which is

$$\begin{aligned}\sigma_L &= \left(1+\frac{r_1+r_2}{r_1-r_2}\right)(\mathrm{par}\,\mu(E_1)-\mathrm{par}\,\mu(E_2))+s\frac{r_1+r_2}{r_1-r_2}-\frac{\mathrm{pardeg}\,(E_1''(D))-\mathrm{pardeg}\,(E_2)}{r_2}\\ &= \sigma_M-\frac{1}{r_2}\epsilon\,,\end{aligned} \tag{38}$$

where

$$\epsilon = \mathrm{pardeg}\,(E_1''(D)) - \mathrm{pardeg}\,(E_2) > 0, \tag{39}$$

and the weights and degree of E_1'' are given by Proposition 11.7. For instance, in the example worked out in (32), $\epsilon = \sum_{i\neq 6,8}\alpha_i - \sum \beta_j + 1$.

The value of σ_L is very close to σ_M but strictly smaller, as expected.

THEOREM 11.8 *Assume $r_1 > r_2$ and $g > 0$. If $\sigma_L > 2g-2$ then $\mathcal{N}_\sigma^s$ is irreducible and non-empty for all $2g-2 \leq \sigma < \sigma_L$. If $\sigma_L < 2g-2$ then $\mathcal{N}_\sigma^s$ is empty for all $\sigma \geq 2g-2$.*

Proof. First, note that for $\sigma > \sigma_M$, $\mathcal{N}_\sigma$ is empty by Proposition 7.1. Assume for a while that $\mathcal{N}_\sigma^s$ is non-empty for some value of $\sigma \geq 2g-2$, then there must exist the minimum value $\tilde{\sigma}_L \in (2g-2, \sigma_M)$ of σ such that $\mathcal{N}^s_{\tilde{\sigma}_L^+} = \emptyset$ and $\mathcal{N}^s_{\tilde{\sigma}_L^-} \neq \emptyset$. Clearly this $\tilde{\sigma}_L$ is a critical value and must correspond to a set of extensions $T'' \to T \to T'$ with $\chi(C^\bullet(T'', T')) = 0$.

By Proposition 11.7 there is at most one (topological) possibility for T' and T'' to have $\chi(C^\bullet(T'', T')) = 0$. This implies that $\tilde{\sigma}_L = \sigma_L$. For any other critical value σ_c, the moduli spaces $\mathcal{N}^s_{\sigma_c^+}$ and $\mathcal{N}^s_{\sigma_c^-}$ are birational, by Proposition 9.7. So all moduli spaces $\mathcal{N}_\sigma^s$ are birational for $2g-2 \leq \sigma < \sigma_L$.

Moreover there may be different distributions of weights, ranks and degrees giving rise to the critical value σ_L, but only the one given by Proposition 11.7 gives critical subsets $\mathcal{S}_{\sigma_L^-}$ of codimension zero. So the number of irreducible components is given by the number of irreducible components of a subset of the space of extensions $T'' \to T \to T'$ with the distribution of weights, ranks and degrees given by Proposition 11.7. Let us see that this space of extensions is non-empty and irreducible: the triples T' have $r_2' = 0$, $r_1' = r_1 - r_2$, so they are parametrized by a moduli space of parabolic bundles E_1', which is non-empty, irreducible and of the expected dimension by [**8**]. The triples T'' have $r_1'' = r_2'' = r_2$, and $d_1'' + r_1''s - d_2'' - \sum r_p = 0$, so they are parametrized by a moduli space of σ_L^--stable triples which is non-empty, irreducible and of the expected dimension by Theorem 12.10. Now the dimension of the projective fibres of the space of extensions $T'' \to T \to T'$ is

$$-\chi(C^\bullet(T', T'')) - 1 \geq 0,$$

since $\chi(C^\bullet(T', T'')) < 0$, by Theorem 10.3. Therefore there is a non-empty space of extensions. Moreover, a generic triple T' is σ_L-stable. In that case, any non-trivial extension $T'' \to T \to T'$ is σ_L^--stable (see Proposition 9.4). So the space $\mathcal{S}_{\sigma_L^-}$ is non-empty, and irreducible.

Finally, if $\sigma_L > 2g-2$, the argument above proves that $\mathcal{N}^s_{\sigma_L^-}$ is non-empty, so there is some non-empty $\mathcal{N}_\sigma$ with $\sigma > 2g-2$ and the statement of the theorem follows. Conversely, if some $\mathcal{N}_\sigma$ with $\sigma > 2g-2$ is non-empty, then it must be $\sigma_L > 2g-2$ completing the argument.

Now Proposition 7.7 transfers the inequalities $\sigma_m \leq 2g-2 < \sigma_L$ into a Milnor–Wood type inequality $0 \leq |\tau| < \tau_L$, where

$$\tau_L = \min\{p,q\}(2g-2+s) - \frac{|p-q|}{p+q}\epsilon\,, \tag{40}$$

where ϵ is given in (39).

REMARK 11.9 One can spell out the process for computing ϵ, by using the procedure of Proposition 11.7 and the identification of Proposition 7.4. Let $p = \mathrm{rk}(V)$, $q = \mathrm{rk}(W)$, α the system of weights of V and β the system of weights of W. Suppose that $q \leq p$ (the other case is similar, interchanging the roles of V and W). Define, at each $x \in D$, $\alpha_{i+pl}(x) = \alpha_i(x) + l$, for any $l \geq 1$. Put $i_0 = 0$ and define, for $1 \leq k \leq q$,

$$i_k = \min\{j \mid j > i_{k-1}, \alpha_j > \beta_k\}.$$

Then

$$\epsilon = \sum_{x\in D}\sum_{k=1}^{p}(\alpha_{i_k}(x) - \beta_k(x)).$$

12. The moduli space of triples for $r_1 = r_2$ and large σ

In this section, we study the moduli space of triples with equal ranks $r_1 = r_2$. We prove that some of them are irreducible and non-empty for $\sigma \geq 2g-2$. The results here are enough for the proof of Theorem 11.8 to work, but we also analyse some other cases. It is likely that the result holds in general.

PROPOSITION 12.1 *Suppose that $r_1 = r_2$ and $g > 0$. Then all the moduli spaces $\mathcal{N}_\sigma$ for $\sigma \geq 2g-2$ are birational to each other.*

Proof. This is a consequence of Theorem 10.3 and Proposition 9.7. For $\chi(C^\bullet(T', T''))$ to vanish, a_1 must be an isomorphism. But this is impossible if $r_1 = r_2$ by Lemma 10.5.

Now let us see that the moduli spaces $\mathcal{N}_\sigma$ stabilize for σ large.

PROPOSITION 12.2 *Suppose that $r_1 = r_2$. Then there is a value σ_1 such that any σ-stable parabolic triple $T = (E_1, E_2, \phi)$ with $\sigma > \sigma_1$ satisfies that ϕ is injective. Hence*

$$0 \longrightarrow E_2 \longrightarrow E_1(D) \longrightarrow S \longrightarrow 0, \tag{41}$$

where S is a torsion sheaf.

Proof. Denote $N = \ker\phi$ and consider the parabolic subtriple $(0, N, \phi)$. Suppose that $k = \mathrm{rk}(N) > 0$. The σ-stability of T implies that

$$\text{pardeg } N + k\sigma < k\left(\frac{\text{pardeg }(E_1 \oplus E_2)}{2r_1} + \frac{1}{2}\sigma\right).$$

Now consider the subtriple (I, E_2, ϕ) where $I(D)$ is the parabolic image sheaf of ϕ, with rank $\mathrm{rk}(I) = r_1 - k$. The σ-stability of T gives us

$$\text{pardeg }(I \oplus E_2) + r_1\sigma < (2r_1 - k)\left(\frac{\text{pardeg }(E_1 \oplus E_2)}{2r_1} + \frac{1}{2}\sigma\right).$$

Adding up both equations, and noting that pardeg N + pardeg $I(D)$ = pardeg E_2, we get

$$2\,\text{pardeg } E_2 - (r_1 - k)s + (r_1 + k)\sigma < \text{pardeg }(E_1 \oplus E_2) + r_1\sigma,$$

which is rewritten as

$$\sigma \le \frac{\text{pardeg } E_1 - \text{pardeg } E_2 + (r_1 - k)s}{k}.$$

So for $\sigma_1 = \text{pardeg } E_1 - \text{pardeg } E_2 + (r_1 - 1)s$ the result follows.

LEMMA 12.3 *Suppose that $r_1 = r_2$ and $\sigma > \sigma_1$. Let T be a σ-stable triple and T' a subtriple of T with $r_1' = r_2'$. Write $E_2 \to E_1(D) \to S$, $E_2' \to E_1'(D) \to S'$, t = length S, t' = length S'. Then*

$$\text{par}\,\mu(E_1') < \text{par}\,\mu(E_1) + \frac{1}{2}\left(\frac{t'}{r_1'} - \frac{t}{r_1}\right) + s,$$
$$\text{par}\,\mu(E_2') < \text{par}\,\mu(E_2) - \frac{1}{2}\left(\frac{t'}{r_1'} - \frac{t}{r_1}\right) + s.$$

Proof. From Proposition 12.2, as $\sigma > \sigma_1$, ϕ is an injective morphism. So ϕ' is injective for any subtriple T' of T. Hence for a subtriple T' with $r_1' = r_2'$ we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_2' & \longrightarrow & E_1'(D) & \longrightarrow & S' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_2 & \longrightarrow & E_1(D) & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

where S and S' are torsion sheaves. Let t and t' denote the lengths of S and S' respectively, as in the statement. By stability,

$$\begin{aligned} 0 &> \text{par}\,\mu_\sigma(T') - \text{par}\,\mu_\sigma(T) \\ &= \frac{1}{2}\big(\text{par}\,\mu(E_1') + \text{par}\,\mu(E_2') - \text{par}\,\mu(E_1) - \text{par}\,\mu(E_2)\big) \\ &= \text{par}\,\mu(E_1') - \text{par}\,\mu(E_1) - \frac{1}{2}(\text{par}\,\mu(E_1') - \text{par}\,\mu(E_2')) + \frac{1}{2}(\text{par}\,\mu(E_1) - \text{par}\,\mu(E_2)) \\ &= \text{par}\,\mu(E_2') - \text{par}\,\mu(E_2) + \frac{1}{2}(\text{par}\,\mu(E_1') - \text{par}\,\mu(E_2')) - \frac{1}{2}(\text{par}\,\mu(E_1) - \text{par}\,\mu(E_2)). \end{aligned}$$

Now at each point $p \in D$, $|\sum \beta_j(p) - \sum \alpha_i(p)| \leq r_1$, so $t - r_1 s \leq \operatorname{pardeg} E_1(D) - \operatorname{pardeg} E_2 \leq t + r_1 s$, equivalently $t - 2r_1 s \leq \operatorname{pardeg} E_1 - \operatorname{pardeg} E_2 \leq t$ or

$$\frac{t}{r_1} - 2s \leq \operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2) \leq \frac{t}{r_1}.$$

Analogously, for T' we have

$$\frac{t}{r_1'} - 2s \leq \operatorname{par}\mu(E_1') - \operatorname{par}\mu(E_2') \leq \frac{t}{r_1'}.$$

Substituting into the formulae above, we get the result in the statement.

PROPOSITION 12.4 *Suppose that $r_1 = r_2$. Then there is a value $\sigma_2 \geq \sigma_1$ such that $\mathcal{N}_\sigma^s = \mathcal{N}_{\sigma'}^s$ for any $\sigma, \sigma' \geq \sigma_2$, that is, there are no critical values above σ_2.*

Proof. Consider a σ-stable triple $T = (E_1, E_2, \phi)$ with $\sigma > \sigma_1$. Suppose that T is properly σ_c-semistable for some σ_c, and let $T' \subset T$ be a σ_c-destabilizing subtriple. Clearly $r_2' \leq r_1'$, since ϕ being injective implies that ϕ' is also injective. On the other hand, if $r_1' = r_2'$ then T is σ-semistable for generic values of σ and could not be σ-stable for some σ. Therefore $r_2' < r_1'$. In the formula

$$\sigma_c = 2\operatorname{par}\mu(E_1')\frac{r_1'}{r_1' - r_2'} + 2\operatorname{par}\mu(E_2')\frac{r_2'}{r_1' - r_2'} - (\operatorname{par}\mu(E_1) + \operatorname{par}\mu(E_2))\frac{r_1' + r_2'}{r_1' - r_2'}, \tag{42}$$

we want to bound the values of $\operatorname{par}\mu(E_1')$ and $\operatorname{par}\mu(E_2')$ in order to get a bound for the critical value σ_c which is independent of T.

Apply Lemma 12.3 to the subtriples $(\phi'(E_2')(-D), E_2', \phi')$ and $(E_1', (\phi')^{-1}(E_1'(D)), \phi')$, both of which satisfy the equal rank condition. The first one has no torsion, the second has torsion with $0 \leq t' \leq t$. We get

$$\begin{aligned}
&\operatorname{par}\mu(E_2') < \operatorname{par}\mu(E_2) + \frac{t}{2r_1} + s,\\
&\operatorname{par}\mu(E_1') < \operatorname{par}\mu(E_1) + \frac{1}{2}\left(\frac{t'}{r_1'} - \frac{t}{r_1}\right) + s \leq \operatorname{par}\mu(E_1) + \frac{t(r_1 - r_1')}{2r_1 r_1'} + s.
\end{aligned}$$

Using that $(t/r_1) \leq \operatorname{par}\mu(E_1) - \operatorname{par}\mu(E_2) + 2s$, by the exact sequence (41) and $1 \leq r_1' \leq r_1 - 1$, we get bounds on $\operatorname{par}\mu(E_1')$ and $\operatorname{par}\mu(E_2')$. Substituting these bounds into (42) and using that $r_1' - r_2' \geq 1$ and $r_1', r_2' \leq r_1 = r_2$, we get a bound on σ_c, as required.

With this result, we may introduce the notation $\mathcal{N}_L^s$ for the moduli space of σ-stable triples for any value $\sigma > \sigma_2$. We shall refer to this as the moduli space *for large values of* σ. There is an obvious condition for $\mathcal{N}_L^s$ to be non-empty. Let $\phi : E_2 \to E_1(D)$ be a parabolic morphism which is moreover injective. For any $p \in D$, it induces a parabolic map $\phi_p \in \mathrm{ParHom}\,(E_{2,p}, E_{1,p})$. This satisfies

$$\dim \mathrm{im}\ \phi_p \le r_1 - r_p,$$

with r_p defined in (33) (cf. Remark 11.6). Therefore for any parabolic map $\phi \in \mathrm{ParHom}\,(E_2, E_1(D))$, we have that

$$d_1 + r_1 s - d_2 \ge \sum_{p\in D} r_p. \tag{43}$$

Let us now see that this is a sufficient condition for non-emptiness and irreducibility of $\mathcal{N}_L^s$. First we need some preliminary results.

LEMMA 12.5 *If both E_2 and E_1 are parabolic stable bundles, and $\phi : E_2 \to E_1(D)$ is an injective parabolic map, then $T = (E_1, E_2, \phi)$ is a σ-stable triple for large values of σ.*

Proof. Any subtriple $T' \subset T$ should have $r_2' \le r_1'$. The stability of the bundles implies that $\mathrm{par}\,\mu(E_1') < \mathrm{par}\,\mu(E_1)$ and $\mathrm{par}\,\mu(E_2') < \mathrm{par}\,\mu(E_2)$, from where it follows that $\mathrm{par}\,\mu_\sigma(T') < \mathrm{par}\,\mu_\sigma(T)$, for any σ, and in particular for large values of σ.

LEMMA 12.6 *Let L be a fixed parabolic line bundle. Consider the moduli space $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha, \beta)$ of σ-stable parabolic triples $T = (E_1, E_2, \phi)$ of degrees (d_1, d_2) and weight types (α, β). Let $(\tilde{d}_1, \tilde{d}_2)$ and $(\tilde{\alpha}, \tilde{\beta})$ be the degrees and weight types of the triples of the form $(E_1 \otimes^p L, E_2 \otimes^p L, \phi)$. Then $(E_1, E_2, \phi) \mapsto (E_1 \otimes^p L, E_2 \otimes^p L, \phi)$ gives an isomorphism $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2; \alpha, \beta) \cong \mathcal{N}_\sigma(r_1, r_2, \tilde{d}_1, \tilde{d}_2; \tilde{\alpha}, \tilde{\beta})$.*

Let us see that tensoring with a suitable parabolic line bundle allows us to reduce to the case $r_p = 0$ for all $p \in D$. For this we need an alternative characterization of r_p. Fix $p \in D$, and denote by $\alpha_1 < \cdots < \alpha_{r_1}$ the weights of $E_{1,p}$ and by $\beta_1 < \cdots < \beta_{r_1}$ the weights of $E_{2,p}$, since $r_2 = r_1$. Extend the weights to an infinite sequence of real numbers by declaring $\alpha_{k+r_1 m} = \alpha_k + m$, $1 \le k \le r_1, m \in \mathbb{Z}$. This means that we have a sequence

$$\cdots < \alpha_{r_1} - 1 < \alpha_1 < \cdots < \alpha_{r_1} < \alpha_1 + 1 < \alpha_2 + 1 < \cdots .$$

In this strictly increasing sequence $\mathbb{Z} \to \mathbb{R}$, 1 is sent to α_1 characterized as the smallest non-negative number in the sequence. Similarly consider the infinite sequence β_k from the weights of $E_{2,p}$. Define the functions

$$\begin{aligned} f : [0, \infty) &\longrightarrow \mathbb{R}, \\ x &\longmapsto \#\{\alpha_k \mid 0 < \alpha_k < x\}, \\ g : [0, \infty) &\longrightarrow \mathbb{R}, \\ x &\longmapsto \#\{\beta_k \mid 0 < \beta_k \le x\}. \end{aligned} \tag{44}$$

Note that $f(x + 1) = f(x) + r_1$ and $g(x + 1) = g(x) + r_1$. Now we have the following.

LEMMA 12.7 $r_p = \max(f - g) = \max_{[0,1)}(f - g)$.

Proof. From the way f and g are defined, $f-g$ is a right-continuous step function, with jumps by $+1$ at the points α_k and -1 at the points β_k. As $f-g$ is 1-periodic, the existence of a maximum and the equality $\max(f-g)=\max_{[0,1)}(f-g)$ are clear. Let $M=\max(f-g)$ and $x_0\in[0,1)$ be a point that is not a weight and that satisfies $(f-g)(x_0)=M$. Then, writing $k=f(x_0)$, we have $\alpha_k<x_0<\alpha_{k+1}$ and $k-M=g(x_0)$, that is, $\beta_{k-M}<x_0<\beta_{k-M+1}$. The maximality of $f-g$ at x_0 implies that we have $\beta_{k-M}<\alpha_k<x_0<\beta_{k-M+1}<\alpha_{k+1}$. So any parabolic map $\phi_0: E_{2,p}\to E_{1,p}$ satisfies that $\phi_0(E_{2,p,k-M+1})\subset E_{1,p,k+1}$ and hence

$$\dim\ker\phi_0\geq\dim E_{2,p,k-M+1}-\dim E_{1,p,k+1}=(r_1-k+M)-(r_1-k)=M.$$

Conversely, let $\phi_0: E_{2,p}\to E_{1,p}$ be a map such that $\phi_0(E_{2,p,k-M+1})\subset E_{1,p,k+1}$ for each k. Then ϕ_0 is a parabolic map: for if $\beta_i>\alpha_j$, take $\beta_i>x>\alpha_j$. So $g(x)\leq i-1$ and $f(x)\geq j$. So $j-i+1\leq f(x)-g(x)\leq M$ and hence $i\geq j-M+1$. Thus $\phi_0(E_{2,p,i})\subset\phi_0(E_{2,p,j-M+1})\subset E_{1,p,j+1}$. On the other hand, it is clear that there are maps satisfying $\phi_0(E_{2,p,k-M+1})\subset E_{1,p,k+1}$ for each k with $\dim\ker\phi_0=M$. Hence there are parabolic maps ϕ_0 with $\dim\ker\phi_0=M$, completing the proof that $M=r_p$.

PROPOSITION 12.8 *There exists a suitable parabolic line bundle L such that the moduli space of σ-stable triples of the form $(E_1\otimes^p L, E_2\otimes^p L,\phi)$ has associated $\tilde r_p=0$, for all $p\in D$.*

Proof. We shall assume that there is only one point $p\in D$ and we shall tensor with a parabolic line bundle of the form $L=\mathcal{O}_{[x]}$, that is, the trivial line bundle with weight $x\in[0,1)$ at p. Take $x_0\in(0,1)$ which does not coincide with any weight and gives the maximum value of the function $f-g$. Let $L=\mathcal{O}_{[1-x_0]}$. Denoting by $k_0=f(x_0)$, the weights of $E_2\otimes^p L$ are

$$0\leq\alpha_{k_0+1}-x_0<\cdots<\alpha_{r_1}-x_0<\alpha_1-x_0+1<\cdots<\alpha_{k_0}-x_0+1<1$$

(see [**18**]). Put otherwise, if $\tilde\alpha_k$ is the infinite sequence associated with the weights of $\tilde E_2=E_2\otimes^p L$, then $\tilde\alpha_k=\alpha_{k+k_0}-x_0$. The function $\tilde f$ associated to $\tilde E_2$ as in (44) is

$$\begin{aligned}\tilde f(x)&=\#\{\tilde\alpha_k\mid 0<\tilde\alpha_k<x\}\\&=\#\{\alpha_k\mid 0<\alpha_k-x_0<x\}\\&=\#\{\alpha_k\mid x_0<\alpha_k<x+x_0\}\\&=f(x+x_0)-f(x_0),\end{aligned}$$

the last equality follows because x_0 is not a weight of $E_{2,p}$. Analogously for $\tilde E_1=E_1\otimes^p L$, the function $\tilde g$ associated to it is

$$\tilde g(x)=g(x+x_0)-g(x_0).$$

Then the number r_p associated to the moduli spaces of triples $(\tilde E_1,\tilde E_2,\phi)$ is

$$\tilde r_p=\max(\tilde f(x)-\tilde g(x))=\max(f(x+x_0)-g(x+x_0))-M=0.$$

PROPOSITION 12.9 *Assume that $r_1=r_2$ and $r_p=0$ for all $p\in D$. Then the moduli space of σ-stable triples for σ large and $d_2+r_2s=d_1$ is irreducible.*

Proof. Any triple $T=(E_1,E_2,\phi)$ in $\mathcal{N}_L^s$ satisfies that $\phi: E_2\to E_1(D)$ is generically an isomorphism by Proposition 12.2. So the condition on the degrees implies that it is an isomorphism of bundles. Moreover, by Lemma 12.3, the family $\mathcal{H}$ of bundles E_1 appearing as part of triples of $\mathcal{N}_L^s$ is a bounded family which is irreducible and the generic element is a stable bundle (see [11]).

Let us study the fibres of $\mathcal{N}_L^s \to \mathcal{H}$. Fix $E_1 \in \mathcal{H}$ and consider the fibre over E_1. Identifying E_2 with $E_1(D)$ (as bundles) via the isomorphism ϕ, an element $(E_1, E_2, \phi) = (E_1, E_1(D), \mathrm{Id})$ in the fibre consists of giving for each $p \in D$ a flag for $V = E_{1,p}$ and a flag for $V = E_{2,p}$ such that the identity map $\mathrm{Id} : V \to V$ is a parabolic map with respect to these flags. For simplicity, assume there is only one point $p \in D$. Let

$$\mathcal{F}_1 = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{r_1} = E_{1,p} \mid \dim V_i = i\}$$

be the space parametrizing (complete) flags at $E_{1,p}$, with fixed weights $\alpha_1 < \cdots < \alpha_{r_1}$. This is an irreducible variety. Analogously define the space

$$\mathcal{F}_2 = \{0 \subset W_1 \subset W_2 \subset \cdots \subset W_{r_1} = E_{2,p} \mid \dim W_i = i\}$$

of (complete) flags for $E_{2,p}$, with fixed weights $\beta_1 < \cdots < \beta_{r_1}$. The condition $r_p = 0$ means that $g(x) \le f(x)$, for all x, with the notation of (44). The identity map is parabolic if $W_i \subset V_{i+k(i)}$, $1 \le i \le r_1$, for some set of integers $k(i) \ge 0$ such that $0 < 1 + k(1) \le 2 + k(2) \le \cdots \le r_1 + k(r_1) = r_1$. The set of compatible flags is given by

$$\mathcal{F} = \{(F_1, F_2) \mid W_i \subset V_{i+k(i)}, 1 \le i \le r_1\} \subset \mathcal{F}_1 \times \mathcal{F}_2 \,. \tag{45}$$

This is also an irreducible variety, as $\mathcal{F} \to \mathcal{F}_1$ is a fibration with irreducible base and irreducible fibres. Note that the other projection $\mathcal{F} \to \mathcal{F}_2$ is also surjective.

A generic stable bundle E_1 satisfies that a generic flag $F_1 \in \mathcal{F}_1$ gives a parabolic stable bundle. Let $U_1 \subset \mathcal{F}_1$ be a (dense) open subset with this property. Analogously consider a dense open subset $U_2 \subset \mathcal{F}_2$ such that $E_2 = E_1(D)$ with a flag $F_2 \in \mathcal{F}_2$ is parabolically stable. If $\mathcal{F} \cap (U_1 \times U_2) = \emptyset$ then $\mathcal{F} \subset ((\mathcal{F}_1 - U_1) \times \mathcal{F}_2) \cup (\mathcal{F}_1 \times (\mathcal{F}_2 - U_2))$. Being irreducible, $\mathcal{F}$ should be contained in either $((\mathcal{F}_1 - U_1) \times \mathcal{F}_2)$ or $(\mathcal{F}_1 \times (\mathcal{F}_2 - U_2))$. This contradicts the surjectivity of both $\mathcal{F} \to \mathcal{F}_1$ and $\mathcal{F} \to \mathcal{F}_2$. This proves that $\mathcal{F} \cap (U_1 \times U_2) \neq \emptyset$, so the generic element of $\mathcal{F}$ gives parabolic stable bundles E_1 and E_2. By Lemma 12.5, such element is σ-stable for σ large. Therefore the generic stable bundle E_1 satisfies that the fibre of $\mathcal{N}_L^s \to \mathcal{H}$ is an open subset of the space of compatible flags $\mathcal{F}$. This shows that $\mathcal{N}_L^s$ is irreducible and non-empty.

THEOREM 12.10 *Suppose that $r_1 = r_2$ and that $d_1 + r_1 s - d_2 = \sum_{p\in D} r_p$. Then the moduli space $\mathcal{N}_L^s$ is irreducible, of the expected dimension and non-empty.*

Proof. By Proposition 12.8 there exists a parabolic line bundle L such that $(E_1, E_2, \phi) \mapsto (\tilde{E}_1 = E_1 \otimes^p L, \tilde{E}_2 = E_2 \otimes^p L, \phi)$ gives an isomorphism of moduli spaces of σ-stable triples $\mathcal{N}_\sigma(r_1, r_1, d_1, d_2; \alpha, \beta) \cong \mathcal{N}_\sigma(r_1, r_1, \tilde{d}_1, \tilde{d}_2; \tilde{\alpha}, \tilde{\beta})$ such that $\tilde{r}_p = 0$ for each $p \in D$. Then

$$\tilde{d}_1 + r_1 s - \tilde{d}_2 = d_1 + r_1 s - d_2 - \sum_{p\in D} r_p.$$

This is easily seen by computing the degrees $\tilde{d}_1$ and $\tilde{d}_2$. For instance, suppose that there is only one point $p \in D$. Then, with the notation of the proof of Proposition 12.8,

$$\begin{aligned}\tilde{d}_1 &= \deg \tilde{E}_1 = \operatorname{pardeg}(E_1 \otimes^p L) - \sum \tilde{\alpha}_k \\ &= \operatorname{pardeg}(E_1) + r_1 \operatorname{pardeg}(L) - \Big(\sum (\alpha_k - x_0) + k_0\Big) \\ &= d_1 + \sum \alpha_k + r_1(1 - x_0) - \sum \alpha_k + r_1 x_0 - k_0 \\ &= d_1 + r_1 - k_0 = d_1 + r_1 - f(x_0)\,.\end{aligned}$$

Analogously, $\tilde{d}_2 = d_2 + r_1 - g(x_0)$, so that $\tilde{d}_1 - \tilde{d}_2 = d_1 - d_2 - r_p$.

Now the moduli space $\mathcal{N}_L^s(r_1, r_1, \tilde{d}_1, \tilde{d}_2; \tilde{\alpha}, \tilde{\beta})$ is non-empty and irreducible by Proposition 12.9. So the same is true of our initial moduli space by using Lemma 12.6. The dimension statement follows from Corollary 9.6.

THEOREM 12.11 *Suppose that $r_1 = r_2$ and $d_1 + r_1 s - d_2 \geq \sum_{p\in D} r_p$. Then the moduli space $\mathcal{N}_L^s$ is non-empty, of the expected dimension and irreducible.*

Proof. The dimension statement follows from Corollary 9.6. Arguing as in the proof of Theorem 12.10, we may suppose that $r_p = 0$, for $p \in D$. Now, there exist triples $\phi : E_2 \to E_1(D)$, with ϕ injective, E_1 and E_2 stable bundles and satisfying that the torsion sheaf quotient of the map ϕ is generic (in particular, supported on $X \setminus D$). This follows from [**11**], where non-parabolic σ-stable triples for σ large are found by constructing σ-stable triples with these properties.

Now the argument of the proof of Proposition 12.9 works here to find parabolic structures on E_1 and E_2 such that (E_1, E_2, ϕ) is a σ-stable parabolic triple for σ large, since the only necessary fact is that $\phi_p : E_{2,p} \to E_{1,p}$ is an isomorphism for all $p \in D$. This gives the non-emptiness of $\mathcal{N}_L^s$.

For proving the irreducibility of $\mathcal{N}_L^s$, the main obstacle are the triples with quotient supported at points of D. We work as follows. Let $\mathcal{H}$ be the family of bundles E_1 appearing in triples $T = (E_1, E_2, \phi) \in \mathcal{N}_L^s$. This is a bounded and irreducible family whose generic element $E_1 \in \mathcal{H}$ is a generic stable bundle. Let $\mathcal{Q} = \operatorname{Quot}^t(\mathcal{H})$ be the Quot scheme parametrizing quotients $E_1(D) \to S$, with $E_1 \in \mathcal{H}$ and $t = \operatorname{length} S = d_1 + r_1 s - d_2 - \sum_{p\in D} r_p$. The kernel of a generic element in $\mathcal{Q}$ is a stable bundle E_2. If the support of S is contained in $X \setminus D$, then the fibre of the map $\mathcal{N}_L^s \to \mathcal{Q}$ over a quotient $E_1(D) \to S$ in $\mathcal{Q}$ is a subset of the set of compatible flags $\mathcal{F}$ defined in (45). For a generic element in $\mathcal{Q}$, this is actually an open subset of $\mathcal{F}$, as proved in the proof of Proposition 12.9. This produces an open subset $U \subset \mathcal{N}_L^s$, which is of dimension

$$\dim \mathcal{Q} + \dim \mathcal{F}\,.$$

Let us see the irreducibility of $\mathcal{N}_L^s$ by checking that $\dim(\mathcal{N}_L^s \setminus U) < \dim U$. Certainly, the only effect that we must take care of is the jumping in the dimension of the fibre of $\mathcal{N}_L^s \to \mathcal{Q}$ when the torsion sheaf is supported at some points of D. Let $p \in D$, and suppose that p is in the support of S, say $S_p = \mathbb{C}^l$. The set of quotients $E_{1,p} \to S_p$ is parametrized by the grassmannian $\operatorname{Gr}(l, r_1)$. The codimension of the space $\mathcal{Q}^l \subset \mathcal{Q}$ parametrizing such quotients is

$$r_1 \,\mathrm{length}\, S - (r_1(\mathrm{length}\, S - l) + l(r_1 - l)) = r_1 l - l r_1 + l^2 = l^2.$$

Now let us compute the dimension of the fibre of $\mathcal{N}_L^s \to \mathcal{Q}$ over a point in $\mathcal{Q}^l$. With the definition of $k(i)$ given in Proposition 12.9, such fibre is the space

$$\mathcal{F}_* = \{(W_i, V_i) \in \mathcal{F}_1 \times \mathcal{F}_2 \mid \phi(W_i) \subset V_{i+k(i)}\}.$$

Equivalently, $(W_i, V_i) \in \mathcal{F}_* \Leftrightarrow W_i \subset \phi^{-1}(V_{i+k(i)})$. It remains to see that

$$\dim \mathcal{F}_* - \dim \mathcal{F} < l^2.$$

The fibration $\mathcal{F} \to \mathcal{F}_1$ is surjective and the dimension of the fibre is

$$\sum_{i=1}^{r_1} k(i).$$

Let us compute the dimension of a fibre of $\mathcal{F}_* \to \mathcal{F}_1$. Such dimension depends on the flag $\{V_i\} \in \mathcal{F}_1$, so we need to stratify $\mathcal{F}_1$ as follows. The flag $\{V_i\}$ is determined by a collection of numbers $0 \le a_1 \le \cdots \le a_{r_1} = r_1 - l$ such that

$$\begin{array}{ccccccc} 0 & \subset & V_1 \cap \mathrm{im}\,(\phi) & \subset & \cdots & \subset & V_{r_1} \cap \mathrm{im}\,(\phi) = \mathrm{im}\,(\phi) \\ \| & & \| & & & & \| \\ 0 & \subset & \mathbb{C}^{a_1} & \subset & \cdots & \subset & \mathbb{C}^{a_{r_1}} = \mathbb{C}^{r_1 - l}. \end{array}$$

Clearly, $a_{i+1} = a_i + \delta_{i+1}$ ($a_0 = 0$), where there are uniquely defined $1 \le i_1 < \cdots < i_{r_1 - l} \le r_1$ such that $\delta_{i_k} = 1$ and $\delta_j = 0$ for $j \ne i_k$, $k = 1, \ldots, r_1 - l$. The codimension of the stratum $S_{a_1,\ldots,a_{r_1}} \subset \mathcal{F}_1$ defined by such $\{V_i\}$ is

$$\sum_{k=1}^{r_1 - l} (l - i_k + k).$$

The fibre of $\mathcal{F}_* \to \mathcal{F}_1$ over $\{V_i\} \in S_{a_1,\ldots,a_{r_1}}$ is given by flags $\{W_i\} \in \mathcal{F}_2$ such that $W_i \subset \tilde{V}_{i+k(i)}$, with $\tilde{V}_i = \phi^{-1}(V_i) \cong \mathbb{C}^{l+a_i}$. The dimension of such a fibre is thus

$$\begin{aligned} \sum_{i=1}^{r_1} (l + a_{i+k(i)} - i) &\le \sum_{i=1}^{r_1} (l + a_i - i) + \sum k(i) \\ &= \sum_{i=1}^{r_1} (l - i) + \sum_{k=1}^{r_1 - l} (r_1 - i_k + 1) + \sum k(i). \end{aligned}$$

So the dimension of the preimage of $S_{a_1,\ldots,a_{r_1}}$ by the map $\mathcal{F}_* \to \mathcal{F}_1$ is less than or equal to

$$\begin{aligned} \dim \mathcal{F}_1 &- \sum_{k=1}^{r_1 - l} (l - i_k + k) + \sum_{i=1}^{r_1} (l - i) + \sum_{k=1}^{r_1 - l} (r_1 - i_k + 1) + \sum k(i) \\ &= \dim \mathcal{F}_1 + \sum k(i) + \frac{l^2 - l}{2} = \dim \mathcal{F} + \frac{l^2 - l}{2}. \end{aligned}$$

Since this is true for any stratum, we have

$$\dim \mathcal{F}_* \leq \dim \mathcal{F} + \frac{l^2 - l}{2} < \dim \mathcal{F} + l^2$$

as required.

Combining Theorem 12.11 with Proposition 12.1 we have the following.

COROLLARY 12.12 *Let $g > 0$, $r_1 = r_2$ and $d_1 + r_1 s - d_2 \geq \sum_{p \in D} r_p$. Then the moduli spaces $\mathcal{N}_\sigma$ are non-empty, irreducible and of the expected dimension for any $\sigma \geq 2g - 2$.*

REMARK 12.13 Corollary 12.12 and the correspondence in Proposition 7.4 gives that the moduli space $\mathcal{U}(p, p, a, b; \alpha, \beta)$ is non-empty and connected if and only if the following is satisfied.

(i) In the case $\tau < 0$. It must be that $|\tau| \leq \tau_M$ by Proposition 7.7. Also, defining $r_x = \min\{\dim \operatorname{coker} \phi \mid \phi \in \operatorname{ParHom}(V_x, W_x)\}$, for $x \in D$, we must have $b + (2g - 2 + s)p - a \geq \sum_{x \in D} r_x$, by Corollary 12.12. But this last condition is redundant: $\tau < 0$ is equivalent to $\operatorname{par} \mu(V) < \operatorname{par} \mu(W)$, hence

$$a = \deg(V) \leq \operatorname{pardeg}(V) < \operatorname{pardeg}(W) < \deg(W) + ps = b + ps + (2g - 2)s,$$

since $g > 0$. Also, we may tensor with a suitable parabolic line bundle L to arrange $r_x = 0$, for all $x \in D$, by Proposition 12.8 (this does not change τ or the inequality that we need to check). So $b + (2g - 2 + s)p - a \geq 0$, as required.

(ii) The case $\tau > 0$ is worked out similarly, and the only condition we obtain is $|\tau| \leq \tau_M$.

Note that the genericity of the weights (Assumption 2.1) prevents the case $|\tau| = \tau_M$ from happening.

13. Representations of fundamental groups in $\mathrm{U}(p, q)$

Let X be a compact Riemann surface of genus $g \geq 0$ and let $S = \{x_1, \ldots, x_s\}$ be a set of distinct points of X. Let $\Gamma = \pi_1(X \setminus S)$ be the fundamental group of $X \setminus S$. The group Γ is generated by the usual generators a_i, b_i, $1 \leq i \leq g$, of $\pi_1(X)$, together with additional generators $\gamma_1, \ldots, \gamma_s$ corresponding to loops enclosing each x_i simply, not enclosing any x_j, $j \neq i$, and which are homotopic to zero relatively to the base point on X. There is also the relation $[a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_s = 1$, where $[a_i, b_i]$ is the commutator of a_i and b_i.

Parabolic Higgs bundles are related to representations of Γ. To be precise, let us fix integers $n = \operatorname{rk} E$, $d = \deg E$ and the weight type $\alpha = \{\alpha(x)\}_{x \in S}$, where $\alpha(x) = (\alpha_1(x), \ldots, \alpha_{r(x)}(x))$ are weights with multiplicities $k_i(x)$ for every $x \in S$. It is convenient to repeat each weight according to its multiplicity, by setting $\tilde{\alpha}_1(x) = \cdots = \tilde{\alpha}_{k_1(x)}(x) = \alpha_1(x)$, etc., thus having weights $0 \leq \tilde{\alpha}_1(x) \leq \cdots \leq \tilde{\alpha}_n(x) < 1$ (see Section 1).

For every $x_i \in S$ there is $C_i \in \mathrm{U}(n)$ defined by

$$C_i = \begin{pmatrix} \exp(2\pi\sqrt{-1}\tilde{\alpha}_1(x_i)) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi\sqrt{-1}\tilde{\alpha}_n(x_i)) \end{pmatrix}. \tag{46}$$

Consider the set of representations $\mathrm{Hom}^+_\alpha(\Gamma, \mathrm{GL}(n,\mathbb{C}))$ defined by semisimple homomorphisms $\rho : \Gamma \to \mathrm{GL}(n,\mathbb{C})$ such that $\rho(\gamma_i)$ is conjugated to C_i by an element in $\mathrm{GL}(n,\mathbb{C})$ for $1 \le i \le s$. Here by semisimple we mean that ρ is a direct sum of irreducible representations. The moduli space of representations of Γ in $\mathrm{GL}(n,\mathbb{C})$ with fixed holonomy in the conjugacy class of C_i is defined by the quotient

$$\mathcal{R}(n;\alpha) := \frac{\mathrm{Hom}^+_\alpha(\Gamma, \mathrm{GL}(n,\mathbb{C}))}{\mathrm{GL}(n,\mathbb{C})},$$

where $\mathrm{GL}(n,\mathbb{C})$ acts by conjugation. The set $\mathcal{R}(n;\alpha)$ has a natural structure of a complex algebraic variety. The following is proved by Simpson in [31].

THEOREM 13.1 *Let $(n, d; \alpha)$ be such that*

$$d + \sum_{x\in S}(\tilde{\alpha}_1(x) + \cdots + \tilde{\alpha}_n(x)) = 0,$$

that is, the parabolic degree vanishes. Then there is a homeomorphism

$$\mathcal{R}(n;\alpha) \cong \mathcal{M}(n, d; \alpha).$$

This generalizes the theorem of Metha and Seshadri [23] which identifies the moduli space of parabolic bundles of type (n, d, α) with vanishing parabolic degree with the moduli space of representations of Γ in $\mathrm{U}(n)$ with fixed holonomy conjugated to C_i around the marked points.

There is a similar correspondence between representations of Γ in $\mathrm{U}(p,q)$ and parabolic $\mathrm{U}(p,q)$-Higgs bundles. To explain this, let us come back to the notation in Section 2 and fix the types of the parabolic bundles V and W to be (p, a, α) and (q, b, α'), respectively. For every $x_i \in S$ there are matrices $C_i \in \mathrm{U}(p)$ and $C'_i \in \mathrm{U}(q)$ defined as in (46) by the weight systems α and α', respectively.

Consider now the set of representations $\mathrm{Hom}^+_{\alpha,\alpha'}(\Gamma, \mathrm{U}(p,q))$ defined by semisimple homomorphisms $\rho : \Gamma \to \mathrm{U}(p,q)$ such that $\rho(\gamma_i)$ is conjugated to $C_i \times C'_i \in \mathrm{U}(p) \times \mathrm{U}(q)$ (recall that $\mathrm{U}(p) \times \mathrm{U}(q)$ is the maximal compact subgroup of $\mathrm{U}(p,q)$) by an element in $\mathrm{U}(p,q)$ for $1 \le i \le s$. Define the moduli space of representations of Γ in $\mathrm{U}(p,q)$ with fixed holonomy $\mathrm{U}(p,q)$-conjugated to $C_i \times C'_i$ by the quotient

$$\mathcal{R}(p,q;\alpha,\alpha') := \frac{\mathrm{Hom}^+_{\alpha,\alpha'}(\Gamma, \mathrm{U}(p,q))}{\mathrm{U}(p,q)}.$$

The set $\mathcal{R}(p,q;\alpha,\alpha')$ is a real analytic variety. We can adapt the arguments of Simpson [31] to prove the following.

THEOREM 13.2 *Let (p, a, α) and (q, b, α') be such that*

$$\operatorname{pardeg}(V) + \operatorname{pardeg}(W) = a + b + \sum_{x \in S} (\tilde{\alpha}_1(x) + \cdots + \tilde{\alpha}_p(x) + \tilde{\alpha}'_1(x) + \cdots + \tilde{\alpha}'_q(x)) = 0.$$

Then there is a homeomorphism

$$\mathcal{R}(p, q; \alpha, \alpha') \cong \bigsqcup_{a,b} \mathcal{U}(p, q, a, b; \alpha, \alpha').$$

Note that $(p, q, a, b; \alpha, \alpha')$ must also satisfy the Milnor–Wood inequality, which in these cases reduces to

$$|\operatorname{pardeg}(V)| \leq \min\{p, q\} \left(g - 1 + \frac{s}{2}\right),$$

since $\operatorname{pardeg}(W) = -\operatorname{pardeg}(V)$.

Combining Theorem 13.2 and Theorem 6.12 we have the following.

THEOREM 13.3 *Under the genericity conditions given by Assumption* 2.1*, and for $g > 0$, the number of non-empty connected components of $\mathcal{R}(p, q; \alpha, \alpha')$ equals the number of integers a such that*

$$\left| a + \sum_{x \in S} (\tilde{\alpha}_1(x) + \cdots + \tilde{\alpha}_p(x)) \right| \leq \frac{\tau_L}{2},$$

where τ_L is given by (40).

REMARK 13.4 The condition on the genus g comes from Theorem 13.2.

As in [31, proof of Theorem 13.1], the main ingredients in the proof of Theorem 13.2 are, on the one hand, the correspondence given by Theorem 5.1 between polystable parabolic $\mathrm{U}(p, q)$-Higgs bundles and solutions to Hitchin equations, and, on the other, the existence of a harmonic adapted metric on a $\mathrm{U}(p, q)$-bundle with a semisimple meromorphic flat connection with simple poles. To see this, let us come back to the framework of Section 5, and consider smooth parabolic vector bundles V and W of types $(p, a; \alpha)$ and $(q, b; \alpha')$, respectively. On the bundle $V \oplus W$ we consider flat $\mathrm{U}(p, q)$-connections D on $X \setminus S$, meromorphic at $x_i \in S$ and whose residue at x_i is conjugated to $C_i \times C'_i$. We say that D is semisimple if the corresponding representation is semisimple. These connections are in correspondence with elements in $\operatorname{Hom}^+_{\alpha,\alpha'}(\Gamma, \mathrm{U}(p, q))$.

Let $h = (h_V, h_W)$, where h_V and h_W are adapted hermitian metrics on V and W, respectively. We decompose D as $D = d_A + \Psi$, where d_A is a $\mathrm{U}(p) \times \mathrm{U}(q)$ connection and Ψ takes values in $\mathfrak{m}$, where $\mathfrak{u}(p, q) = \mathfrak{u}(p) \oplus \mathfrak{u}(q) + \mathfrak{m}$ is the Cartan decomposition of the Lie algebra of $\mathrm{U}(p, q)$. We say that h is harmonic if $d_A^* \Psi = 0$. Then the following can be proved easily adapting the results in [12, 31].

THEOREM 13.5 *A connection D as above is semisimple if and only if there exists a harmonic hermitian metric $h = (h_V, h_W)$.*

The relation with parabolic $\mathrm{U}(p, q)$-Higgs bundle is given as follows. If D is a semisimple flat connection as above and h is a harmonic solution, then the pair (d_A, Φ), where Φ is determined by the equation $\Psi = \Phi + \Phi^*$, solves the $\mathrm{U}(p, q)$-Hitchin equations and hence, by Theorem 5.1, corresponds to a polystable parabolic $\mathrm{U}(p,q)$-Higgs bundle. Conversely, if we have a polystable parabolic $\mathrm{U}(p,q)$-Higgs bundle we can find a solution (d_A, Φ) to the Hitchin equations, and then out of it a solution to the harmonic equation on the flat connection $D = d_A + \Phi + \Phi^*$, which is then semisimple by Theorem 13.5.

14. Elliptic surfaces, orbifolds and parabolic Higgs bundles

Parabolic bundles have been related by several authors to unitary representations of the fundamental group of elliptic surfaces of general type [1, 28]. The key fact is that the fundamental group of such a surface is isomorphic to the orbifold fundamental group of an orbifold Riemann surface, whose unitary representations are, in turn, related to parabolic bundles by the Metha–Seshadri theorem [3, 6, 23, 27].

Let X be a compact Riemann surface of genus $g \geq 0$ and let $S = \{x_1, \ldots, x_s\}$ be a set of distinct points of X. Suppose that for each i we are given integers $m_i \geq 1$, such that $2g + \sum_{1\leq i\leq s}(1 - 1/m_i) > 2$. We call the data of X, S and m_i, $1 \leq i \leq g$, a 2-*orbifold*. As in Section 13, let $\Gamma = \pi_1(X \setminus S)$ be the fundamental group of $X \setminus S$. As we have seen in Section 13, Γ has $2g + s$ generators a_i, b_i, $1 \leq i \leq g$, and γ_j, $1 \leq j \leq s$, satisfying the relation

$$\prod_{1\leq i\leq g} [a_i, b_i] \cdot \prod_{1\leq j\leq s} \gamma_j = 1.$$

We define the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(X)$ as the quotient of Γ by the smallest normal subgroup containing $\gamma_i^{m_i}$. Thus $\pi_1^{\mathrm{orb}}(X)$ is freely generated by the elements a_i, b_i, $1 \leq i \leq g$, and γ_j, $1 \leq j \leq s$, subject to the relations

$$\prod_{1\leq i\leq g} [a_i, b_i] \cdot \prod_{1\leq j\leq s} \gamma_j = 1, \quad \text{and} \quad \gamma_j^{m_j} = 1, \ \ 1 \leq j \leq s.$$

The 2-orbifold Riemann surface ought to be thought of as a Riemann surface with singularities at the points x_i, which locally are of the form $\Delta/\mathbb{Z}_{m_i}$, where Δ is the unit disc in $\mathbb{C}$. The group $\pi_1^{\mathrm{orb}}(X)$ is clearly the fundamental group of this orbifold surface (see [6, 26] and references therein for basic facts on orbifold surfaces).

The following is proved in [13, 33] (see also [17, 28]).

THEOREM 14.1 *Given an orbifold fundamental group* $\pi_1^{\mathrm{orb}}(X)$ *and an integer* $\chi > 0$, *there is an elliptic surface* Y, *unique up to diffeomorphism, with*

$$\pi_1(Y) = \pi_1^{\mathrm{orb}}(X), \quad \textit{and} \quad \chi(\mathcal{O}_Y) = \chi.$$

Conversely, given an elliptic surface Y *with* $b_1(Y)$ *even,* $\chi(\mathcal{O}_Y) > 0$ *and* $\mathrm{kod}(Y) = 1$ *we have*

$$\pi_1(Y) = \pi_1^{\mathrm{orb}}(X),$$

for some 2*-orbifold Riemann surface* X.

To understand this result and the relation of Y to the 2-orbifold X, recall that an elliptic surface is a smooth compact complex surface Y with a fibration $f : Y \to X$ onto a Riemann surface X such that the generic fibre is an elliptic curve (the complex structure of the fibre may vary from point to point). In some special points the fibre may degenerate into nodal fibres. This is always the case for the elliptic surfaces we are dealing with. Technically this is the condition $\chi > 0$. The effect of these singularities is that they kill the extra generators of the fundamental group determined by the fibre. In addition to these nodal fibres there are multiple fibres, located over the marked points of X. They are defined analogously to orbifold singularities: a neighbourhood Y_m of such a multiple fibre in X is the quotient by a finite cyclic group,

$$f : Y_m \cong \frac{(\Delta \times E_{\tau(z)})}{\mathbb{Z}_m} \longrightarrow \frac{\Delta}{\mathbb{Z}_m \cong \Delta}$$

defined by $[(t, c)] \mapsto t^m = z$, where Δ is the unit disc in $\mathbb{C}$, E_τ is the torus $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ and the generator of $\mathbb{Z}_m$ acts as $(t, c) \mapsto (t \cdot \exp(2\pi\sqrt{-1}/m), c + 1/m)$. The crutial difference of a multiple fibre of Y and the orbifold point is, however, that this action is free and hence the quotient is smooth. Roughly speaking, the orbifold singularity is now hidden in the map f between two smooth manifolds Y and X.

To relate representations $\rho : \pi_1^{\mathrm{orb}}(X) \to \mathrm{GL}(n, \mathbb{C})$ to parabolic Higgs bundles, we observe that $\rho(\gamma_i)$ must be conjugated to a matrix of the form

$$C_i = \begin{pmatrix} \exp(2\pi\sqrt{-1}\frac{l_1(x_i)}{m_i}) & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi\sqrt{-1}\frac{l_n(x_i)}{m_i}) \end{pmatrix} \tag{47}$$

for integers $l_j(x_i)$ such that

$$0 \leq l_1(x_i) \leq \cdots \leq l_n(x_i) < m_i. \tag{48}$$

This follows from the fact that $\rho(\gamma_i)^{m_i} = I$. Such a representation of $\pi_1^{\mathrm{orb}}(X)$ lifts to a representation $\tilde{\rho} : \Gamma \to \mathrm{GL}(n, \mathbb{C})$. Conversely, if $\tilde{\rho} : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is such that $\rho(\gamma_i)$ is conjugated to a matrix C_i as above then $\tilde{\rho}$ descends to a representation $\rho : \pi_1(X^{\mathrm{orb}}) \to \mathrm{GL}(n, \mathbb{C})$. We thus have proved the following.

PROPOSITION 14.2 *There is a one-to-one correspondence between representations $\rho : \pi_1(X^{\mathrm{orb}}) \to \mathrm{GL}(n, \mathbb{C})$ and representations $\tilde{\rho} : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ such that $\tilde{\rho}(\gamma_i)$ is conjugated to a matrix of the form* (47) *for integers $l_j(x_i)$ satisfying* (48).

Similarly, we have the following.

PROPOSITION 14.3 *There is a one-to-one correspondence between representations $\rho : \pi_1^{\mathrm{orb}}(X) \to \mathrm{U}(p, q)$ and representations $\tilde{\rho} : \Gamma \to \mathrm{U}(p, q)$ such that $\tilde{\rho}(\gamma_i)$ is $\mathrm{U}(p, q)$-conjugated to an element of the form $C_i \times C_i' \subset \mathrm{U}(p) \times \mathrm{U}(q)$ with C_i and C_i' as in* (47)*, defined for integers $l_j(x_i)$ and $l_k'(x_i)$ satisfying*

$$0 \leq l_1(x_i) \leq \cdots \leq l_p(x_i) < m_i \quad \text{and} \quad 0 \leq l_1'(x_i) \leq \cdots \leq l_q'(x_i) < m_i. \tag{49}$$

Let

$$\lambda = \{\lambda(x_i) = (l_1(x_i), \ldots, l_n(x_i))\}_{x_i \in S}, \tag{50}$$

where $l_j(x_i)$ are integers satisfying (48). Let $\mathcal{R}_X^{\text{orb}}(n;\lambda)$ and $\mathcal{R}_Y(n;\lambda)$ be the moduli spaces of semi-simple representations of $\pi_1^{\text{orb}}(X)$ and $\pi_1(Y)$ in $\mathrm{GL}(n,\mathbb{C})$ such that $\rho(\gamma_i)$ is conjugated to the matrix (47). Similarly, let

$$\lambda = \{\lambda(x_i) = (l_1(x_i),\ldots,l_p(x_i))\}_{x_i\in S} \quad \text{and} \quad \lambda' = \{\lambda'(x_i) = (l_1'(x_i),\ldots,l_q'(x_i))\}_{x_i\in S} \tag{51}$$

satisfying (49). Let $\mathcal{R}_X^{\text{orb}}(p,q;\lambda,\lambda')$ and $\mathcal{R}_Y(p,q;\lambda,\lambda')$ be the moduli spaces of semisimple representations of $\pi_1^{\text{orb}}(X)$ and $\pi_1(Y)$ in $\mathrm{U}(p,q)$ such that $\rho(\gamma_i)$ is conjugated to a matrix $C_i \times C_i'$ as in Proposition 14.3. Of course, since $\pi_1^{\text{orb}}(X) \cong \pi_1(Y)$, $\mathcal{R}_X^{\text{orb}}(n;\lambda) \cong \mathcal{R}_Y(n;\lambda)$ and $\mathcal{R}_X^{\text{orb}}(p,q;\lambda,\lambda') \cong \mathcal{R}_Y(p,q;\lambda,\lambda')$.

Combining Propositions 14.2 and 14.3 and Theorems 13.1 and 13.2 we have the following.

THEOREM 14.4 *Let λ given by (50) satisfying (48) and let $\tilde{\alpha}(x_i) = \lambda(x_i)/m_i$. Let (n,d) be such that*

$$d + \sum_{x\in S}(\tilde{\alpha}_1(x) + \cdots + \tilde{\alpha}_n(x)) = 0.$$

Then

$$\mathcal{R}_X^{\text{orb}}(n;\lambda) \cong \mathcal{R}_Y(n;\lambda) \cong \mathcal{R}(n,d;\alpha) \cong \mathcal{M}(n,d;\alpha).$$

Similarly, let λ and λ' given by (51) satisfying (49) and let $\tilde{\alpha}(x_i) = \lambda(x_i)/m_i$ and $\tilde{\alpha}'(x_i) = \lambda'(x_i)/m_i$. Let (p,q,a,b) be such that

$$a + b + \sum_{x\in S}(\tilde{\alpha}_1(x) + \cdots + \tilde{\alpha}_p(x) + \tilde{\alpha}_1'(x) + \cdots + \tilde{\alpha}_q'(x)) = 0.$$

Then

$$\mathcal{R}_X^{\text{orb}}(p,q;\lambda,\lambda') \cong \mathcal{R}_Y(p,q;\lambda,\lambda') \cong \mathcal{R}(p,q;\alpha,\alpha') \cong \bigsqcup_{a,b}\mathcal{U}(p,q,a,b;\alpha,\alpha').$$

As established by Simpson and Corlette, higher-dimensional non-abelian Hodge theory [**12**, **30**] gives a correspondence between semisimple flat bundles or representations of the fundamental group of a compact Kähler manifold (Y,ω), and polystable Higgs bundles on (Y,ω) with vanishing first and second Chern classes (see [**30**] for the definition of stability). Now, a $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle on Y is defined as a pair (E,Φ) consisting of a holomorphic vector bundle E over Y and a homomorphism $\Phi : E \to E\otimes\Omega_Y^1$ such that $[\Phi,\Phi] = 0$, where Ω_Y^1 is the bundle of holomorphic 1-forms on Y. If $E = V\oplus W$, where V and W are holomorphic bundles of ranks p and q respectively, and

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : (V\oplus W) \to (V\oplus W)\otimes\Omega_Y^1,$$

then (E,Φ) is said to be a $\mathrm{U}(p,q)$-Higgs bundle. Of course, when Y is a Riemann surface we recover the original definition of Higgs bundle since Ω_Y^1 is the canonical bundle and the condition $[\Phi,\Phi] = 0$ is trivially satisfied.

If Y is a complex elliptic surface as above, equipped with a Kähler metric ω, non-abelian Hodge theory on (Y,ω) combined with Theorem 14.4 gives the following.

THEOREM 14.5 *There is a one-to-one correspondence between the moduli space of polystable* $\mathrm{GL}(n, \mathbb{C})$*-Higgs bundles on* (Y, ω) *with vanishing Chern classes and the moduli space of parabolic* $\mathrm{GL}(n, \mathbb{C})$*-Higgs bundles on* X *with parabolic structure on the orbifold points.*

Similarly, there is a one-to-one correspondence between the moduli space of polystable $\mathrm{U}(p, q)$*-Higgs bundles on* (Y, ω) *with vanishing Chern classes and the moduli space of parabolic* $\mathrm{U}(p, q)$*-Higgs bundles on* X *with parabolic structure on the orbifold points.*

REMARK 14.6 It would be very interesting to work out this correspondence directly in a similar fashion to that done by Bauer [1] for the case of moduli spaces of vector bundles. We plan to come back to this problem in a future paper.

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HODGE POLYNOMIALS OF THE MODULI SPACES OF TRIPLES OF RANK (2, 2)

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Abstract

Let X be a smooth projective curve of genus $g \geq 2$ over the complex numbers. A holomorphic triple (E_1, E_2, ϕ) on X consists of two holomorphic vector bundles E_1 and E_2 over X and a holomorphic map $\phi\colon E_2 \to E_1$. There is a concept of stability for triples which depends on a real parameter σ. In this paper, we determine the Hodge polynomials of the moduli spaces of σ-stable triples with $\text{rk}(E_1) = \text{rk}(E_2) = 2$, using the theory of mixed Hodge structures (in the cases that these moduli spaces are smooth and compact). This gives in particular the Poincaré polynomials of these moduli spaces. As a byproduct, we also give the Hodge polynomial of the moduli space of even degree rank 2 stable vector bundles.

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ over the field of complex numbers. A holomorphic triple $T = (E_1, E_2, \phi)$ on X of rank (n_1, n_2) consists of two holomorphic vector bundles E_1 and E_2 over X (of ranks n_1 and n_2, and degrees d_1 and d_2, respectively) and a holomorphic map $\phi\colon E_2 \to E_1$. There is a concept of stability for a triple which depends on the choice of a parameter $\sigma \in \mathbb{R}$. Let $\mathcal{N}_\sigma$ and $\mathcal{N}_\sigma^s$ denote the moduli spaces of σ-semistable and σ-stable triples, respectively. These have been widely studied in [**4**, **5**, **14**, **21**].

The range of the parameter σ is an interval $I \subset \mathbb{R}$ split by a finite number of *critical values* σ_c in such a way that when σ moves without crossing a critical value, then $\mathcal{N}_\sigma$ remains unchanged, but when σ crosses a critical value, $\mathcal{N}_\sigma$ undergoes a transformation which we call *flip*. The study of this

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process allows us to obtain information on the topology of all moduli spaces $\mathcal{N}_\sigma$, for any σ, once we know such information for one particular $\mathcal{N}_\sigma$ (usually the one corresponding to the minimum or maximum possible values of the parameter).

One of the main motivations to study the topology of the moduli spaces of triples is that they appear when looking at the topology of the moduli spaces of Higgs bundles [**14**, **15**, **18**] via Morse theory techniques. Higgs bundles are pairs (E, Φ), formed by a holomorphic vector bundle E of rank r and a holomorphic map $\Phi : E \to E \otimes K$, where K is the canonical bundle of the curve, and they are intimately related to the representation varieties of the fundamental group of the surface underlying the complex curve into the general Lie group $GL(r, \mathbb{C})$. The moduli spaces of triples and the more general moduli spaces of chains [**1**, **2**, **16**] appear as critical sets of a natural Morse–Bott function on the moduli space of Higgs bundles [**15**, **18**].

When the rank of E_2 is one, we have the so-called *pairs* [**3**, **13**, **21**]. The moduli spaces of pairs are smooth for any rank n_1, and in the case of rank $n_1 = 2$ and fixed determinant, they are very well-understood thanks to the work of Thaddeus [**24**]. In this case, the flips have a very nice geometrical interpretation, consisting of blowing up an embedded subvariety and then blowing down the exceptional divisor in a different way. Moreover, there are also very explicit descriptions of the moduli spaces of pairs for the minimum and maximum possible values of σ.

The flips do not have such a nice behavior for moduli spaces of triples of rank (n_1, n_2) with $n_1 + n_2 > 3$. The flip locus may have singularities, it may consist of several irreducible components intersecting in a non-transverse way, the moduli spaces themselves may have singularities for $n_1, n_2 \geq 2$, and the moduli spaces for σ large are difficult to handle in the situation when $n_1 = n_2$, since then they are described in terms of Quot schemes.

These difficulties can be overcome in two different ways. The first way is to introduce parabolic structures with generic weights. The moduli spaces of parabolic triples have been studied in [**14**], where the Poincaré polynomials have been given for the moduli of parabolic triples of ranks $(2, 1)$. The parabolic weights tend to prevent the singularities of the moduli spaces and flip loci. However, for obtaining information on the moduli space of non-parabolic triples, one should relate the parabolic and the non-parabolic situations.

The second route to compute the Poincaré polynomials of the moduli spaces of triples was introduced in [**21**]. It consists in using the theory of mixed Hodge structures of Deligne [**8**] to compute the Hodge polynomials of the moduli space. The Hodge polynomials recover the usual Poincaré polynomial when we deal with a smooth compact algebraic variety, but they can be defined for non-smooth and non-compact algebraic varieties as well. This allows to compute the Poincaré polynomials of the moduli spaces of triples which are smooth and compact, no matter if the flip loci have singularities.

In this paper, we use the mixed Hodge theory to compute the Hodge polynomials of some of the moduli spaces of triples of rank $(2, 2)$. By the results of [**5**], if $d_1 - d_2 > 4g - 4$ then $\mathcal{N}_\sigma^s$ is smooth. Moreover, when $d_1 + d_2$ is odd, the moduli spaces $\mathcal{N}_\sigma$ only consist of σ-stable triples for non-critical values of σ, therefore $\mathcal{N}_\sigma^s$ are projective varieties. Because of this, we shall compute the Hodge polynomials of the moduli spaces of triples of rank $(2, 2)$ in the case $d_1 - d_2 > 4g - 4$ and $d_1 + d_2$ odd. This gives in particular the Poincaré polynomials of these moduli spaces.

We start by reviewing the rudiments of mixed Hodge theory and the standard results on triples that we shall use throughout the paper, in Sections 2 and 3. Then Section 4 recalls the computations of the Hodge polynomials of the moduli spaces of triples of ranks $(2, 1)$ and $(1, 2)$, from [**21**]. In Section 5 we use the Hodge polynomial of the moduli spaces of triples to deduce the Hodge polynomials of the moduli spaces of rank 2 stable vector bundles. The case of odd degree rank 2 bundles is already known [7, 11, 21], but we do the case of even degree rank 2 stable bundles, proving the following result (see Theorem 5.2).

THEOREM 1.1 *Let $M^s(2,d)$ denote the moduli space of rank 2, degree d stable vector bundles on X. If d is even then the Hodge polynomial of $M^s(2,d)$ is*

$$e(M^s(2,d)) = \frac{1}{2(1-uv)(1-(uv)^2)}\big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\big).$$

Note that the moduli space $M^s(2,d)$ is smooth but non-compact. The formula for the Hodge polynomial of the moduli space of rank 2, even degree and *fixed determinant* stable vector bundles appears in [19, Section 6.2].

Next we move to the study of the moduli spaces of triples of rank (2, 2), which are the main focus of the paper. The critical values are computed in Section 6.

In Section 7 we compute the Hodge polynomial of the moduli space of stable triples of rank (2, 2) for the smallest allowable values of the parameter σ, proving the following result (see Theorem 7.2 and Corollary 7.3).

THEOREM 1.2 *Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2,2,d_1,d_2)$ be the moduli space of σ-stable triples of rank (2, 2). Assume that $d_1 - d_2 > 4g-4$ and that $d_1 + d_2$ is odd. Let $\sigma_m = (d_1/2) - (d_2/2)$ be the minimum value of the parameter σ and $\sigma_{\mathrm{m}}^+ = \sigma_m + \epsilon$ for $\epsilon > 0$ small. Then $\mathcal{N}_{\sigma_{\mathrm{m}}^+}$ is smooth and projective, it only consists of stable triples, and its Hodge polynomial is*

$$e(\mathcal{N}_{\sigma_{\mathrm{m}}^+}) = \frac{(1+u)^{2g}(1+v)^{2g}(1-(uv)^N)(u^gv^g(1+u)^g(1+v)^g - (1+u^2v)^g(1+uv^2)^g)}{(1-uv)^3(1-(uv)^2)^2}\cdot \Big((1+u)^g(1+v)^g(u^{g+1}v^{g+1}+u^{N+g-1}v^{N+g-1}) - (1+u^2v)^g(1+uv^2)^g(1+u^Nv^N)\Big),$$

where $N = d_1 - d_2 - 2g + 2$.

Under the condition $d_1 - d_2 > 4g-4$, the Hodge polynomial of $\mathcal{N}_{\sigma_{\mathrm{m}}^+}(2,2,d_1,d_2)$ when both d_1, d_2 are odd is easily given (see Theorem 7.1). When both d_1, d_2 are even, it may be computed with similar techniques to those of Theorem 7.2. However, to remove the condition $d_1 - d_2 > 4g-4$ is not possible with the current techniques.

The contribution of the flips to the Hodge polynomials of the moduli spaces of σ-stables triples of rank (2, 2) is computed in Section 8. This is added up to the information of the Hodge polynomial of the small parameter moduli space to get the Hodge polynomial of the moduli space of σ-stable triples of rank (2, 2) for the largest values of σ in Section 9. We get the following result (see Theorem 9.2 and Corollary 9.3).

THEOREM 1.3 *Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2,2,d_1,d_2)$ be the moduli space of σ-stable triples of rank (2, 2). Assume that $d_1 - d_2 > 4g-4$ and that $d_1 + d_2$ is odd. Let $\sigma_M = d_1 - d_2$. Then all the moduli spaces $\mathcal{N}_\sigma$ are* *isomorphic for $\sigma > \sigma_M$. Let $\sigma_M^+ = \sigma_M + \epsilon$ for $\epsilon > 0$. Then $\mathcal{N}_{\sigma_M^+}$ is smooth and projective, it consists only of stable triples, and*

$$
\begin{aligned}
e(\mathcal{N}_{\sigma_M^+}) = \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^3(1-(uv)^2)^2}\Big[&(1+u^2v)^{2g}(1+uv^2)^{2g}(1-(uv)^{2N}) \\
&- N\,(1+u^2v)^g(1+uv^2)^g(1+u)^g(1+v)^g(uv)^{N+g-1}(1-(uv)^2) \\
&+ (1+u)^{2g}(1+v)^{2g}(1+uv)^2(uv)^{2g-2+(N+1)/2}\Big(\big(1-(uv)^{N+1}\big) \\
&- \frac{N+1}{2}(1-uv)\big(1+(uv)^N\big)\Big) \\
&- g(1+u)^{2g-1}(1+v)^{2g-1}(1-(uv)^2)^2(uv)^{2g-2+(N+1)/2}\big(1-(uv)^N\big)\Big],
\end{aligned}
$$

where $N = d_1 - d_2 - 2g + 2$.

The computation of the contribution of the flips to the Hodge polynomials of the moduli spaces of σ-stables triples of rank (2, 2) is done under the assumptions $d_1 + d_2$ odd and $d_1 - d_2 > 2g - 2$. This can be extended to the case $d_1 + d_2$ even, keeping in mind that in this case we will find the Hodge polynomials of the moduli spaces $\mathcal{N}_\sigma^s$ which are non-compact and of the moduli spaces $\mathcal{N}_\sigma$ which have singularities at non-stable points. However the assumption $d_1 - d_2 > 2g - 2$ cannot be removed with the current techniques.

The Poincaré polynomials of the moduli spaces $\mathcal{N}_{\sigma_m^+}$ and $\mathcal{N}_{\sigma_M^+}$ are obtained from the Hodge polynomials, for $d_1 - d_2 > 4g - 4$ and $d_1 + d_2$ odd (see Corollaries 7.4 and 9.4), since they are smooth projective varieties.

2. Hodge polynomials

2.1. *Hodge–Deligne theory*

Let us start by recalling the Hodge–Deligne theory of algebraic varieties over $\mathbb{C}$. Let H be a finite-dimensional complex vector space equipped with a conjugation (that is, H is the complexification of a real vector space). A *pure Hodge structure of weight k* on H is a decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}$$

such that $H^{q,p} = \overline{H}^{p,q}$, the bar denoting complex conjugation in H. We denote

$$h^{p,q}(H) = \dim H^{p,q},$$

which is called the Hogde number of type (p, q). A Hodge structure of weight k on H gives rise to the so-called *Hodge filtration F* on H, where

$$F^p = \bigoplus_{s \geq p} H^{s,k-s},$$

which is a descending filtration. Note that $\mathrm{Gr}_F^p H = F^p/F^{p+1} = H^{p,q}$.

Let H be a finite-dimensional complex vector space equipped with a conjugation. A (*mixed*) *Hodge structure* over H consists of an ascending weight filtration W on H invariant by the conjugation, and a descending Hodge filtration F on H, such that F induces a pure Hodge filtration of weight k on each $\mathrm{Gr}_k^W H = W_k/W_{k-1}$. Again we define

$$h^{p,q}(H) = \dim H^{p,q}, \quad \text{where } H^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H.$$

A morphism of Hodge structures $L : H \to H'$, between two Hodge structures H and H', is a complex linear map which commutes with the conjugation and respects both the weight and Hodge filtrations.

Deligne has shown [8] that, for each complex algebraic variety Z, the cohomology $H^k(Z)$ and the cohomology with compact support $H_c^k(Z)$ both carry natural Hodge structures. If Z is a compact smooth projective variety (hence compact Kähler) then the Hodge structure $H^k(Z)$ is pure of weight k and coincides with the classical Hodge structure given by the Hodge decomposition of harmonic forms into (p, q) types.

DEFINITION 2.1 For *any* complex algebraic variety Z (not necessarily smooth, compact or irreducible), we define the Hodge numbers as

$$h_c^{k,p,q}(Z) = h^{p,q}(H_c^k(Z)) = \dim \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H_c^k(Z).$$

Introduce the Euler characteristic

$$\chi_c^{p,q}(Z) = \sum_k (-1)^k h_c^{k,p,q}(Z).$$

The *Hodge polynomial* of Z is defined [6] as

$$e(Z) = e(Z)(u, v) = \sum_{p,q} (-1)^{p+q} \chi_c^{p,q}(Z) u^p v^q.$$

If Z is smooth and projective then the mixed Hodge structure on $H_c^k(Z)$ is pure of weight k, so $\mathrm{Gr}_k^W H_c^k(Z) = H_c^k(Z) = H^k(Z)$ and the other pieces $\mathrm{Gr}_m^W H_c^k(Z) = 0$, $m \neq k$. So

$$\chi_c^{p,q}(Z) = (-1)^{p+q} h^{p,q}(Z),$$

where $h^{p,q}(Z)$ is the usual Hodge number of Z. In this case,

$$e(Z)(u, v) = \sum_{p,q} h^{p,q}(Z) u^p v^q$$

is the (usual) Hodge polynomial of Z. Note that in this case, the Poincaré polynomial of Z is

$$P_Z(t) = \sum_k b^k(Z) t^k = \sum_k \left(\sum_{p+q=k} h^{p,q}(Z) \right) t^k = e(Z)(t, t), \tag{1}$$

where $b^k(Z)$ is the kth Betti number of Z.

THEOREM 2.2 [10, Theorem 2] *Let Z be a complex algebraic variety. Suppose that Z is a finite disjoint union $Z = Z_1 \cup \cdots \cup Z_n$, where the Z_i are algebraic subvarieties. Then*

$$e(Z) = \sum_i e(Z_i).$$

Note that we can assign to *any* complex algebraic variety Z (not necessarily smooth, compact or irreducible) a polynomial

$$P_Z(t) = e(Z)(t,t) = \sum_m (-1)^m \chi_c^m(Z) t^m = \sum_{k,m} (-1)^{k+m} \dim \mathrm{Gr}_m^W H_c^k(Z) t^m,$$

where

$$\chi_c^m(Z) = \sum_{p+q=m} \chi_c^{p,q}(Z).$$

This is called the *virtual Poincaré polynomial* of Z, see [10, 12]. It satisfies an additive property analogous to that of Theorem 2.2, and it recovers the usual Poincaré polynomial when Z is a smooth projective variety.

The following Hodge polynomials will be needed later:

- Let $Z = \mathbb{P}^n$, then $e(Z) = 1 + uv + (uv)^2 + \cdots + (uv)^n = (1-(uv)^{n+1})/(1-uv)$. For future reference, we shall denote

$$e_n := e(\mathbb{P}^{n-1}) = e(\mathbb{P}(\mathbb{C}^n)) = \frac{1-(uv)^n}{1-uv}. \tag{2}$$

- Let $\mathrm{Jac}^d X$ be the Jacobian of (any) degree d of a (smooth, projective) complex curve X of genus g. Then

$$e(\mathrm{Jac}^d X) = (1+u)^g (1+v)^g. \tag{3}$$

LEMMA 2.3 [6, Corollary 1.9] *Suppose that $\pi : Z \to Y$ is an algebraic fiber bundle with fiber F which is locally trivial in the Zariski topology, then $e(Z) = e(F)\, e(Y)$. (In particular this is true for $Z = F \times Y$.)*

LEMMA 2.4 *Suppose that $\pi : Z \to Y$ is a map between quasi-projective varieties which is a locally trivial fiber bundle in the usual topology, with fibers projective spaces $F = \mathbb{P}^N$ for some $N > 0$. Then $e(Z) = e(F)\, e(Y)$.*

Proof. This follows from [9, 17]. For completeness we provide a proof. Let H be a hyperplane section of Z (here we use that Z is quasi-projective). Then H has degree $k > 0$ on the fiber $F = \mathbb{P}^N \subset Z$. Therefore $H \cdot F = k\, h$, where h is the hyperplane class of the projective space. We have a morphism of (mixed) Hodge structures:

$$\begin{aligned} L : H^*(\mathbb{P}^N) \otimes H_c^*(Y) &\longrightarrow H_c^*(Z) \\ k^i\, h^i \otimes \alpha &\longmapsto H^i \cap \pi^*(\alpha). \end{aligned} \tag{4}$$

Note that π is a proper map, so that $\pi^* : H_c^*(Y) \to H_c^*(Z)$ is a morphism of Hodge structures (see [8]). This easily yields that L is a morphism of Hodge structures.

Note that L is not multiplicative. Let us see that L is injective. If $x = \sum H^i \cap \pi^*(\alpha_i) = 0$, let i_0 be the maximum i for which $\alpha_i \neq 0$. Then

$$0 = \pi_*(H^{N-i_0} \cap x) = \alpha_{i_0}.$$

So L must be injective. On the other hand, the Leray spectral sequence of the fibration π has E_2-term isomorphic to $H^*(\mathbb{P}^N) \otimes H_c^*(Y)$ and converges to $H_c^*(Z)$. So $\dim H^*(\mathbb{P}^N) \otimes H_c^*(Y) \geq \dim H_c^*(Z)$ and L must be bijective. Therefore L is an isomorphism of Hodge structures, and the result follows.

LEMMA 2.5 *The Hodge polynomial of the Grassmannian* $\mathrm{Gr}(k, N)$ *is*

$$e(\mathrm{Gr}(k, N)) = \frac{(1-(uv)^{N-k+1}) \cdots (1-(uv)^{N-1})(1-(uv)^N)}{(1-uv) \cdots (1-(uv)^{k-1})(1-(uv)^k)}.$$

Proof. This is well known, but we provide a proof for completeness.

Let us review first the case of the projective space $\mathbb{P}^{N-1} = (\mathbb{C}^N - \{0\})/(\mathbb{C} - \{0\})$. Then $\mathbb{C}^N - \{0\} \to \mathbb{P}^{N-1}$ is a locally trivial fibration, since it is the restriction of the universal line bundle $U \to \mathbb{P}^{N-1}$ to the complement of the zero section. Using either Lemma 2.3 or Lemma 2.4, we have $e(\mathbb{C}^N - \{0\}) = e(\mathbb{C} - \{0\})\, e(\mathbb{P}^{N-1})$, that is $(uv)^N - 1 = (uv - 1)e(\mathbb{P}^{N-1})$, from where (2) is recovered. Now in the case of $k > 1$, denote

$$F(k, n) = \{(v_1, \dots, v_k) | v_i \text{ are linearly independent vectors of } \mathbb{C}^n\}.$$

Then $\mathrm{Gr}(k, N) = F(k, N)/GL(k, \mathbb{C})$ and there is a locally trivial fibration $F(k, N) \to \mathrm{Gr}(k, N)$ with fiber $GL(k, \mathbb{C}) \cong F(k, k)$ (again it is the principal bundle associated with the universal bundle $U \to \mathrm{Gr}(k, N)$). So by Lemma 2.3, $e(\mathrm{Gr}(k, N)) = e(F(k, N))/e(F(k, k))$. Now we use that the map

$$F(k, n) \longrightarrow F(k-1, n),$$

given by forgetting the last vector, is a locally trivial fibration, with fiber $\mathbb{C}^n - \mathbb{C}^{k-1}$. Using Lemma 2.3 and Theorem 2.2, we have $e(F(k, n)) = e(F(k-1, n))\, e(\mathbb{C}^n - \mathbb{C}^{k-1}) = e(F(k-1, n))\, ((uv)^n - (uv)^{k-1})$. By recursion this gives

$$e(F(k, n)) = ((uv)^n - (uv)^{k-1}) \cdots \big((uv)^n - uv\big)\big((uv)^n - 1\big).$$

So

$$\begin{aligned} e(\mathrm{Gr}(k, N)) &= \frac{((uv)^N - (uv)^{k-1}) \cdots \big((uv)^N - uv\big)\big((uv)^N - 1\big)}{((uv)^k - (uv)^{k-1}) \cdots ((uv)^k - uv)((uv)^k - 1)} \\ &= \frac{(1-(uv)^{N-k+1}) \cdots \big(1-(uv)^{N-1})(1-(uv)^N\big)}{(1-uv) \cdots (1-(uv)^{k-1})(1-(uv)^k)}. \end{aligned}$$

LEMMA 2.6 *Let M be a smooth projective variety. Consider the algebraic variety $Z = (M \times M)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as $(x, y) \mapsto (y, x)$. The Hodge polynomial of Z is*

$$e(Z) = \frac{1}{2}\Big(e(M)(u, v)^2 + e(M)(-u^2, -v^2)\Big).$$

Proof. The cohomology of Z is

$$H^*(Z) = H^*(M \times M)^{\mathbb{Z}_2} = (H^*(M) \otimes H^*(M))^{\mathbb{Z}_2}.$$

We claim that this is an equality of Hodge structures: certainly, $\sigma(x, y) = (y, x)$ acts algebraically on $M \times M$, so $\sigma^* : H^*(M \times M) \to H^*(M \times M)$ is a morphism of Hodge structures, and hence $H^*(M \times M)^{\mathbb{Z}_2}$ is a Hodge substructure of $H^*(M \times M)$. Also, the map $M \times M \to Z$ is algebraic, so $H^*(Z) \to H^*(M \times M)$ is a morphism of Hodge structures, which is moreover injective and with image $H^*(M \times M)^{\mathbb{Z}_2}$. The claim follows.

The Hodge structure of M is of pure type, therefore the Hodge structure of Z is also of pure type. Moreover,

$$H^{p,q}(Z) = \left(\bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H^{p_1,q_1}(M) \otimes H^{p_2,q_2}(M) \right)^{\mathbb{Z}_2}.$$

Therefore we have

$$h^{p,q}(Z) = \frac{1}{2} \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q \\ (p_1,q_1)\neq(p_2,q_2)}} h^{p_1,q_1}(M) h^{p_2,q_2}(M) + \epsilon_{p,q},$$

where

$$\epsilon_{p,q} = \begin{cases} 0, & p \text{ or } q \text{ odd}, \\ \dim\left(\mathrm{Sym}^2 H^{p_1,q_1}(M)\right), & p = 2p_1,\, q = 2q_1,\, p_1 + q_1 \text{ even}, \\ \dim\left(\bigwedge^2 H^{p_1,q_1}(M)\right), & p = 2p_1,\, q = 2q_1,\, p_1 + q_1 \text{ odd}. \end{cases}$$

If V is a vector space of dimension n, then $\dim\left(\mathrm{Sym}^2 V\right) = (1/2)(n^2 + n)$ and $\dim\left(\bigwedge^2 V\right) = (1/2)(n^2 - n)$, so

$$\epsilon_{p,q} = \begin{cases} 0, & p \text{ or } q \text{ odd}, \\ \frac{1}{2}(h^{p_1,q_1}(M)^2 + (-1)^{p_1+q_1} h^{p_1,q_1}(M)), & p = 2p_1, \quad q = 2q_1. \end{cases}$$

This yields

$$\begin{aligned} e(Z) &= \sum h^{p,q}(Z) u^p v^q \\ &= \frac{1}{2} \sum h^{p_1,q_1}(M) h^{p_2,q_2}(M) u^{p_1+p_2} v^{q_1+q_2} + \frac{1}{2} \sum (-1)^{p_1+q_1} h^{p_1,q_1}(M) u^{2p_1} v^{2q_1} \\ &= \frac{1}{2} e(M) \cdot e(M) + \frac{1}{2} e(M)(-u^2, -v^2). \end{aligned}$$

3. Moduli spaces of triples

3.1. *Holomorphic triples*

Let X be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. A *holomorphic triple* $T = (E_1, E_2, \phi)$ on X consists of two holomorphic vector bundles E_1 and E_2 over X, of ranks n_1 and n_2 and degrees d_1 and d_2, respectively, and a holomorphic map $\phi\colon E_2 \to E_1$. We refer to (n_1, n_2, d_1, d_2) as the *type* of T to (n_1, n_2) as the *rank* of T, and to (d_1, d_2) as the *degree* of T.

A homomorphism from $T' = (E'_1, E'_2, \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{array}{ccc} E'_2 & \xrightarrow{\phi'} & E'_1 \\ \downarrow & & \downarrow \\ E_2 & \xrightarrow{\phi} & E_1, \end{array}$$

where the vertical arrows are holomorphic maps. A triple $T' = (E'_1, E'_2, \phi')$ is a subtriple of $T = (E_1, E_2, \phi)$ if $E'_1 \subset E_1$ and $E'_2 \subset E_2$ are subbundles, $\phi(E'_2) \subset E'_1$ and $\phi' = \phi|_{E'_2}$. A subtriple $T' \subset T$ is called *proper* if $T' \neq 0$ and $T' \neq T$. The quotient triple $T'' = T/T'$ is given by $E''_1 = E_1/E'_1$, $E''_2 = E_2/E'_2$ and $\phi''\colon E''_2 \to E''_1$ being the map induced by ϕ. We usually denote by (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$ the types of the subtriple T' and the quotient triple T''.

DEFINITION 3.1 For any $\sigma \in \mathbb{R}$ the *σ-slope* of T is defined by

$$\mu_\sigma(T) = \frac{d_1 + d_2}{n_1 + n_2} + \sigma \frac{n_2}{n_1 + n_2}.$$

To shorten the notation, we define the *μ-slope* and *λ-slope* of the triple T as $\mu = \mu(E_1 \oplus E_2) = (d_1 + d_2)/(n_1 + n_2)$ and $\lambda = n_2/(n_1 + n_2)$, so that $\mu_\sigma(T) = \mu + \sigma\lambda$.

DEFINITION 3.2 We say that a triple $T = (E_1, E_2, \phi)$ is *σ-stable* if

$$\mu_\sigma(T') < \mu_\sigma(T)$$

for any proper subtriple $T' = (E'_1, E'_2, \phi')$. We define *σ-semistability* by replacing the above strict inequality with a weak inequality. A triple is called *σ-polystable* if it is the direct sum of σ-stable triples of the same σ-slope. It is *σ-unstable* if it is not σ-semistable, and *strictly σ-semistable* if it is σ-semistable but not σ-stable. A σ-destabilizing subtriple $T' \subset T$ is a proper subtriple satisfying $\mu_\sigma(T') \geq \mu_\sigma(T)$.

We denote by

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$$

the moduli space of (S-equivalence classes of) σ-semistable triples $T = (E_1, E_2, \phi)$ of type (n_1, n_2, d_1, d_2) and drop the type from the notation when it is clear from the context. This is identified as a set with the space formed by the σ-polystable triples of the given type. The open subset of σ-stable triples is denoted by $\mathcal{N}^s_\sigma = \mathcal{N}^s_\sigma(n_1, n_2, d_1, d_2)$. This moduli space is constructed in [4] by using dimensional reduction. A direct construction is given by Schmitt [23] using geometric invariant theory.

There are certain necessary conditions in order for σ-semistable triples to exist. Let $\mu_i = \mu(E_i) = d_i/n_i$ stand for the slope of E_i, for $i = 1, 2$. We write

$$\sigma_m = \mu_1 - \mu_2,$$
$$\sigma_M = \left(1 + \frac{n_1 + n_2}{|n_1 - n_2|}\right)(\mu_1 - \mu_2), \quad \text{if } n_1 \neq n_2.$$

PROPOSITION 3.3 [5] *The moduli space $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ is a complex projective variety. For $n_1, n_2 > 0$, let I denote the interval $I = [\sigma_m, \sigma_M]$ if $n_1 \neq n_2$, or $I = [\sigma_m, \infty)$ if $n_1 = n_2$. A necessary condition for $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ to be non-empty is that $\sigma \in I$.*

REMARK 3.4 It is not clear if the condition $\sigma \in I$ is also sufficient. This is a delicate issue studied in [5], where it is proved the following result: if $\sigma_m \geq 2g - 2$, then the condition $\sigma \in I$ is necessary and sufficient for $\mathcal{N}_\sigma(n_1, n_2, d_1, d_2)$ to be non-empty [5, Theorem A]. This is the best known result.

3.2. *Critical values*

To study the dependence of the moduli spaces $\mathcal{N}_\sigma$ on the parameter, we need to introduce the concept of critical value [4, 21].

DEFINITION 3.5 The values of $\sigma_c \in I$ for which there exist $0 \leq n_1' \leq n_1$, $0 \leq n_2' \leq n_2$, d_1' and d_2', with $n_1' n_2 \neq n_1 n_2'$, such that

$$\sigma_c = \frac{(n_1 + n_2)(d_1' + d_2') - (n_1' + n_2')(d_1 + d_2)}{n_1' n_2 - n_1 n_2'}, \tag{5}$$

are called the critical values.

Given a triple $T = (E_1, E_2, \phi)$, the condition of σ-(semi)stability for T can only change when σ crosses a critical value. If $\sigma = \sigma_c$ as in (5) and if T has a subtriple $T' \subset T$ of type (n_1', n_2', d_1', d_2'), then $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$ and

(1) if $\lambda' > \lambda$ (where λ' is the λ-slope of T'), then T is not σ-stable for $\sigma > \sigma_c$,
(2) if $\lambda' < \lambda$, then T is not σ-stable for $\sigma < \sigma_c$.

Note that $n_1' n_2 \neq n_1 n_2'$ is equivalent to $\lambda' \neq \lambda$.

Of course, it may happen that there is no triple T as above and hence that the moduli spaces $\mathcal{N}_\sigma$ and $\mathcal{N}_\sigma^s$ do not change when crossing σ_c (see Remark 6.6).

PROPOSITION 3.6 [5, Proposition 2.6] *Fix (n_1, n_2, d_1, d_2). Then*

(1) *The critical values are a finite number of values $\sigma_c \in I$.*
(2) *The stability and semistability criteria for two values of σ lying between two consecutive critical values are equivalent; thus the corresponding moduli spaces are isomorphic.*
(3) *If σ is not a critical value and* $\gcd(n_1, n_2, d_1 + d_2) = 1$, *then σ-semistability is equivalent to σ-stability, that is, $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$.*

Note that if $\gcd(n_1, n_2, d_1 + d_2) \neq 1$ then it may happen that there exists triples T which are strictly σ-semistable for non-critical values of σ.

3.3. *Extensions and deformations of triples*

The homological algebra of triples is controlled by the hypercohomology of a certain complex of sheaves which appears when studying infinitesimal deformations [5, Section 3]. Let $T' = (E'_1, E'_2, \phi')$ and $T'' = (E''_1, E''_2, \phi'')$ be two triples of types (n'_1, n'_2, d'_1, d'_2) and $(n''_1, n''_2, d''_1, d''_2)$, respectively. Let $\mathrm{Hom}(T'', T')$ denote the linear space of homomorphisms from T'' to T', and let $\mathrm{Ext}^1(T'', T')$ denote the linear space of equivalence classes of extensions of the form

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0,$$

where by this we mean a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_1 & \longrightarrow & E_1 & \longrightarrow & E''_1 & \longrightarrow & 0 \\ & & {\scriptstyle \phi'}\uparrow & & {\scriptstyle \phi}\uparrow & & {\scriptstyle \phi''}\uparrow & & \\ 0 & \longrightarrow & E'_2 & \longrightarrow & E_2 & \longrightarrow & E''_2 & \longrightarrow & 0. \end{array}$$

To analyze $\mathrm{Ext}^1(T'', T')$ one considers the complex of sheaves

$$C^\bullet(T'', T') \colon (E_1''^{*} \otimes E'_1) \oplus (E_2''^{*} \otimes E'_2) \xrightarrow{c} E_2''^{*} \otimes E'_1, \tag{6}$$

where the map c is defined by

$$c(\psi_1, \psi_2) = \phi'\psi_2 - \psi_1\phi''.$$

PROPOSITION 3.7 [5, Proposition 3.1] *There are natural isomorphisms*

$$\mathrm{Hom}(T'', T') \cong \mathbb{H}^0(C^\bullet(T'', T')),$$
$$\mathrm{Ext}^1(T'', T') \cong \mathbb{H}^1(C^\bullet(T'', T')),$$

and a long exact sequence associated with the complex $C^\bullet(T'', T')$*:*

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(C^\bullet(T'', T')) \longrightarrow H^0((E_1''^{*} \otimes E'_1) \oplus (E_2''^{*} \otimes E'_2)) \longrightarrow H^0(E_2''^{*} \otimes E'_1) \\ &\longrightarrow \mathbb{H}^1(C^\bullet(T'', T')) \longrightarrow H^1((E_1''^{*} \otimes E'_1) \oplus (E_2''^{*} \otimes E'_2)) \longrightarrow H^1(E_2''^{*} \otimes E'_1) \\ &\longrightarrow \mathbb{H}^2(C^\bullet(T'', T')) \longrightarrow 0. \end{aligned}$$

We introduce the following notation:

$$h^i(T'', T') = \dim \mathbb{H}^i(C^\bullet(T'', T')),$$
$$\chi(T'', T') = h^0(T'', T') - h^1(T'', T') + h^2(T'', T').$$

PROPOSITION 3.8 [5, Proposition 3.2] *For any holomorphic triples T' and T'', we have*

$$\begin{aligned} \chi(T'', T') &= \chi(E_1''^{*} \otimes E'_1) + \chi(E_2''^{*} \otimes E'_2) - \chi(E_2''^{*} \otimes E'_1) \\ &= (1-g)(n''_1 n'_1 + n''_2 n'_2 - n''_2 n'_1) + n''_1 d'_1 - n'_1 d''_1 + n''_2 d'_2 - n'_2 d''_2 - n''_2 d'_1 + n'_1 d''_2, \end{aligned}$$

where $\chi(E) = \dim H^0(E) - \dim H^1(E)$ *is the Euler characteristic of* E.

Since the space of infinitesimal deformations of T is isomorphic to $\mathbb{H}^1(C^\bullet(T,T))$, the previous results also apply to studying deformations of a holomorphic triple T.

THEOREM 3.9 *Let $T=(E_1,E_2,\phi)$ be a σ-stable triple of type (n_1,n_2,d_1,d_2). Then*

(1) *The triple T is simple. In particular, $\mathbb{H}^0(C^\bullet(T,T))=\mathbb{C}$.*
(2) *The Zariski tangent space at the point defined by T in the moduli space of stable triples is isomorphic to $\mathbb{H}^1(C^\bullet(T,T))$.*
(3) *If $\mathbb{H}^2(C^\bullet(T,T))=0$, then the moduli space of σ-stable triples is smooth in a neighbourhood of the point defined by T.*
(4) *At a smooth point $T\in\mathcal{N}_\sigma^s(n_1,n_2,d_1,d_2)$ the dimension of the moduli space of σ-stable triples is*

$$\begin{aligned}\dim\mathcal{N}_\sigma^s(n_1,n_2,d_1,d_2)&=h^1(T,T)=1-\chi(T,T)\\&=(g-1)(n_1^2+n_2^2-n_1n_2)-n_1d_2+n_2d_1+1.\end{aligned}$$

(5) *Let $T=(E_1,E_2,\phi)$ be a σ-stable triple. If T is injective or surjective (meaning that $\phi: E_2\to E_1$ is injective or surjective) then the moduli space is smooth at T.*

Proof. (1) follows from [**4**, Corollary 3.2]. (2)–(5) are the content of [**5**, Theorem 3.8].

3.4. *Crossing critical values*

Fix the type (n_1,n_2,d_1,d_2) for the moduli spaces of holomorphic triples. We want to describe the differences between the two spaces $\mathcal{N}_{\sigma_1}^s$ and $\mathcal{N}_{\sigma_2}^s$ when σ_1 and σ_2 are separated by a critical value. Let $\sigma_c\in I$ be a critical value and set

$$\sigma_c^+=\sigma_c+\epsilon,\quad \sigma_c^-=\sigma_c-\epsilon,$$

where $\epsilon>0$ is small enough so that σ_c is the only critical value in the interval (σ_c^-,σ_c^+).

DEFINITION 3.10 We define the *flip loci* as

$$\begin{aligned}\mathcal{S}_{\sigma_c^+}&=\{T\in\mathcal{N}_{\sigma_c^+}\,\|\,T\text{ is }\sigma_c^-\text{-unstable}\}\subset\mathcal{N}_{\sigma_c^+},\\ \mathcal{S}_{\sigma_c^-}&=\{T\in\mathcal{N}_{\sigma_c^-}\mid T\text{ is }\sigma_c^+\text{-unstable}\}\subset\mathcal{N}_{\sigma_c^-},\end{aligned}$$

and $\mathcal{S}_{\sigma_c^\pm}^s=\mathcal{S}_{\sigma_c^\pm}\cap\mathcal{N}_{\sigma_c^\pm}^s$ for the stable part of the flip loci.

Note that for $\sigma_c=\sigma_m$, $\mathcal{N}_{\sigma_m^-}$ is empty, hence $\mathcal{N}_{\sigma_m^+}=\mathcal{S}_{\sigma_m^+}$. Analogously, when $n_1\neq n_2$, $\mathcal{N}_{\sigma_M^+}$ is empty and $\mathcal{N}_{\sigma_M^-}=\mathcal{S}_{\sigma_M^-}$.

LEMMA 3.11 *Let σ_c be a critical value. Then*

(1) $\mathcal{N}_{\sigma_c^+}-\mathcal{S}_{\sigma_c^+}=\mathcal{N}_{\sigma_c^-}-\mathcal{S}_{\sigma_c^-}$.
(2) $\mathcal{N}_{\sigma_c^+}^s-\mathcal{S}_{\sigma_c^+}^s=\mathcal{N}_{\sigma_c^-}^s-\mathcal{S}_{\sigma_c^-}^s=\mathcal{N}_{\sigma_c}^s$.

Proof. Item (1) is an easy consequence of the definition of flip loci. Item (2) is the content of [5, Lemma 5.3].

Let us describe the flip loci $\mathcal{S}_{\sigma_c^\pm}$. Let σ_c be a critical value, and let (n_1', n_2', d_1', d_2') be such that $\lambda' \neq \lambda$ and (5) holds. Put $(n_1'', n_2'', d_1'', d_2'') = (n_1 - n_1', n_2 - n_2', d_1 - d_1', d_2 - d_2')$. Denote $\mathcal{N}_\sigma' = \mathcal{N}_\sigma(n_1', n_2', d_1', d_2')$ and $\mathcal{N}_\sigma'' = \mathcal{N}_\sigma(n_1'', n_2'', d_1'', d_2'')$.

LEMMA 3.12 [21, Lemma 4.7] *Let $T \in \mathcal{S}_{\sigma_c^+}$ (resp. $T \in \mathcal{S}_{\sigma_c^-}$). Then T sits in a non-split exact sequence*

$$0 \to T' \to T \to T'' \to 0, \tag{7}$$

where $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$, $\lambda' < \lambda$ (resp. $\lambda' > \lambda$) and T' and T'' are both σ_c-semistable.

Conversely, if $T' \in \mathcal{N}_{\sigma_c}'$ and $T'' \in \mathcal{N}_{\sigma_c}''$ are both σ_c-stable, and $\lambda' < \lambda$ (resp. $\lambda' > \lambda$). Then for any non-trivial extension (7), T lies in $\mathcal{S}_{\sigma_c^+}^s$ (resp. in $\mathcal{S}_{\sigma_c^-}^s$). Moreover, such T can be written uniquely as an extension (7) with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T)$.

In particular, suppose σ_c is not a critical value for the moduli spaces of triples of types (n_1', n_2', d_1', d_2') and $(n_1'', n_2'', d_1'', d_2'')$, $\gcd(n_1', n_2', d_1' + d_2') = 1$ and $\gcd(n_1'', n_2'', d_1'' + d_2'') = 1$. Then if $\lambda' < \lambda$ (resp. $\lambda' > \lambda$), there is a bijective correspondence between non-trivial extensions (7), with $T' \in \mathcal{N}_{\sigma_c}'$ and $T'' \in \mathcal{N}_{\sigma_c}''$ and triples $T \in \mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$).

THEOREM 3.13 [21, Theorem 4.8] *Let σ_c be a critical value with $\lambda' < \lambda$ (resp. $\lambda' > \lambda$). Assume*

(1) *σ_c is not a critical value for the moduli spaces of triples of types (n_1', n_2', d_1', d_2') and $(n_1'', n_2'', d_1'', d_2'')$, $\gcd(n_1', n_2', d_1' + d_2') = 1$ and $\gcd(n_1'', n_2'', d_1'' + d_2'') = 1$.*
(2) *$\mathbb{H}^0(C^\bullet(T'', T')) = \mathbb{H}^2(C^\bullet(T'', T')) = 0$, for every $(T', T'') \in \mathcal{N}_{\sigma_c}' \times \mathcal{N}_{\sigma_c}''$.*

Then $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) is the projectivization of a bundle of rank $-\chi(T'', T')$ over $\mathcal{N}_{\sigma_c}' \times \mathcal{N}_{\sigma_c}''$.

The construction of the flip loci can be used for the critical value $\sigma_c = \sigma_m$, which allows to describe the moduli space $\mathcal{N}_{\sigma_m^+}$. We refer to the value of σ given by $\sigma = \sigma_m^+ = \sigma_m + \epsilon$ as *small*.

Let $M(n, d)$ denote the moduli space of (S-equivalence classes of) semistable vector bundles of rank n and degree d over X. This moduli space is projective. We also denote by $M^s(n, d)$ the open subset of stable bundles, which is smooth and is of dimension $n^2(g - 1) + 1$. If $\gcd(n, d) = 1$, then $M(n, d) = M^s(n, d)$.

PROPOSITION 3.14 [21, Proposition 4.10] *There is a map*

$$\pi : \mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(n_1, n_2, d_1, d_2) \to M(n_1, d_1) \times M(n_2, d_2)$$

which sends $T = (E_1, E_2, \phi)$ to (E_1, E_2).

(1) *If $\gcd(n_1, d_1) = 1$, $\gcd(n_2, d_2) = 1$ and $\mu_1 - \mu_2 > 2g - 2$, then $\mathcal{N}_{\sigma_m^+}^s = \mathcal{N}_{\sigma_m^+}$ is a projective bundle over $M(n_1, d_1) \times M(n_2, d_2)$, whose fibers are projective spaces of dimension $n_2 d_1 - n_1 d_2 - n_1 n_2 (g - 1) - 1$.*
(2) *In general, if $\mu_1 - \mu_2 > 2g - 2$, then the open subset*

$$\pi^{-1}(M^s(n_1, d_1) \times M^s(n_2, d_2)) \subset \mathcal{N}_{\sigma_m^+}$$

is a projective bundle over $M^s(n_1, d_1) \times M^s(n_2, d_2)$, whose fibers are projective spaces of dimension $n_2 d_1 - n_1 d_2 - n_1 n_2 (g - 1) - 1$.

4. Hodge polynomials of the moduli spaces of triples of ranks (2,1) and (1,2)

4.1. *Moduli space of triples of rank* (2, 1)

In this section we recall the main results of [21]. Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ denote the moduli space of σ-semistable triples $T = (E_1, E_2, \phi)$ where E_1 is a vector bundle of degree d_1 and rank 2 and E_2 is a line bundle of degree d_2. By Proposition , σ is in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 4(\mu_1 - \mu_2)] = [d_1/2 - d_2, 2d_1 - 4d_2], \qquad \text{where } \mu_1 - \mu_2 \geq 0.$$

Otherwise $\mathcal{N}_\sigma$ is empty.

THEOREM 4.1 [21, Theorem 5.1] *For $\sigma \in I$, $\mathcal{N}_\sigma$ is a projective variety. It is smooth and of (complex) dimension $3g - 2 + d_1 - 2d_2$ at the stable points $\mathcal{N}_\sigma^s$. Moreover, for non-critical values of σ, $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ (hence it is smooth and projective).*

The critical values corresponding to $n_1 = 2$ and $n_2 = 1$ are given by Definition 3.5:

(1) $n_1' = 1$, $n_2' = 0$. The corresponding σ_c-destabilizing subtriple is of the form $0 \to E_1'$, where $E_1' = L_c$ is a line bundle of degree $\deg(L_c) = d_{L_c}$. The critical value is

$$\sigma_c = 3d_{L_c} - d_1 - d_2.$$

(2) $n_1' = 1$, $n_2' = 1$. The corresponding σ_c-destabilizing subtriple T' is of the form $E_2 \to E_1'$, where E_1' is a line bundle. Let $T'' = T/T'$ be the quotient bundle, which is of the form $0 \to E_1''$, where $E_1'' = L_c$ is a line bundle, and let $d_{L_c} = \deg(L_c)$ be its degree. Then $d_2' = d_2$, $d_1' = d_1 - d_{L_c}$ and

$$\sigma_c = -\big(3(d_1 - d_{L_c} + d_2) - 2(d_1 + d_2)\big) = 3d_{L_c} - d_1 - d_2.$$

(3) $n_1' = 2$, $n_2' = 0$. In this case, the only possible subtriple is $0 \to E_1$. This produces the critical value

$$\sigma_c = \frac{d_1 - 2d_2}{2} = \mu_1 - \mu_2 = \sigma_m,$$

that is, the minimum of the interval I for σ.

(4) $n_1' = 0$, $n_2' = 1$. The subtriple T' must be of the form $E_2 \to 0$. This forces $\phi = 0$ in $T = (E_1, E_2, \phi)$. So T is decomposable, of the form $T' \oplus T'' = (0, E_2, 0) \oplus (E_1, 0, 0)$, and T is σ-unstable for any $\sigma \neq \sigma_c$, where

$$\sigma_c = \frac{2d_2 - d_1}{-2} = \mu_1 - \mu_2 = \sigma_m.$$

LEMMA 4.2 [21, Lemma 5.3] *Let $\sigma_c = 3d_{L_c} - d_1 - d_2$ be a critical value. Then*

$$\mu_1 \leq d_{L_c} \leq d_1 - d_2, \tag{8}$$

and $\sigma_c = \sigma_m \Leftrightarrow d_{L_c} = \mu_1$.

The Hodge polynomials of the moduli spaces $\mathcal{N}_\sigma$ for non-critical values of σ are given in [21]. As this moduli space is projective and smooth, we may recover the Poincaré polynomial from the Hodge polynomial via the formula (1).

THEOREM 4.3 [21, Theorem 6.2] *Suppose that $\sigma > \sigma_m$ is not a critical value. Set $d_0 = [(1/3)(\sigma + d_1 + d_2)] + 1$. Then the Hodge polynomial of $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2, 1, d_1, d_2)$ is*

$$e(\mathcal{N}_\sigma) = \operatorname*{coeff}_{x^0}\left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}}\left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g-1+2d_0}}{1-(uv)^2x}\right)\right].$$

4.2. *Moduli space of triples of rank* (1, 2)

Triples of rank (1, 2) are of the form $\phi : E_2 \to E_1$, where E_2 is a rank 2 bundle and E_1 is a line bundle. By Proposition 3.3, σ is in the interval

$$I = [\sigma_m, \sigma_M] = [\mu_1 - \mu_2, 4(\mu_1 - \mu_2)] = [d_1 - d_2/2, 4d_1 - 2d_2], \quad \text{where } \mu_1 - \mu_2 \geq 0.$$

THEOREM 4.4 *For $\sigma \in I$, $\mathcal{N}_\sigma$ is a projective variety. It is smooth and of (complex) dimension $3g - 2 + 2d_1 - d_2$ at the stable points $\mathcal{N}_\sigma^s$. Moreover, for non-critical σ, $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ (hence it is smooth and projective).*

Proof. Given a triple $T = (E_1, E_2, \phi)$ one has the dual triple $T^* = (E_2^*, E_1^*, \phi^*)$, where E_i^* is the dual of E_i and ϕ^* is the transpose of ϕ. The map $T \mapsto T^*$ defines an isomorphism

$$\mathcal{N}_\sigma(1, 2, d_1, d_2) \cong \mathcal{N}_\sigma(2, 1, -d_2, -d_1).$$

The result now follows from Theorem 4.1.

Also from Lemma 4.2, we get

LEMMA 4.5 [21, Lemma 7.2] *The critical values for $\mathcal{N}_\sigma(1, 2, d_1, d_2)$ are the numbers $\sigma_c = 3d_{L_c} + d_1 + d_2$, where $-\mu_2 \leq d_{L_c} \leq d_1 - d_2$. Also $\sigma_c = \sigma_m \Leftrightarrow d_{L_c} = -\mu_2$.*

THEOREM 4.6 [21, Theorem 7.3] *Consider $\mathcal{N}_\sigma = \mathcal{N}_\sigma(1, 2, d_1, d_2)$. Let $\sigma > \sigma_m$ be a non-critical value. Set $d_0 = \Big[(1/3)(\sigma - d_1 - d_2)\Big] + 1$. Then the Hodge polynomial of $\mathcal{N}_\sigma$ is*

$$e(\mathcal{N}_\sigma) = \operatorname*{coeff}_{x^0}\left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_0}}\left(\frac{(uv)^{d_1-d_2-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{d_2+g-1+2d_0}}{1-(uv)^2x}\right)\right].$$

5. Hodge polynomial of the moduli space of rank 2 even degree stable bundles

Let $M(2, d)$ denote the moduli space of semistable vector bundles of rank 2 and degree d over X. As $M(2, d) \cong M(2, d + 2k)$, for any integer k, there are two moduli spaces, depending on whether the degree is even or odd. We are going to apply the results of Section 4 to compute the Hodge polynomials of these moduli spaces.

We first recall the Hodge polynomial of the moduli space of rank 2 odd degree stable bundles from [7, 11, 21].

THEOREM 5.1 [21, Proposition 8.1] *The Hodge polynomial of $M(2,d)$ with odd degree d is*

$$e(M(2,d)) = \frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}.$$

Now we compute the Hodge polynomial of the moduli space of rank 2 even degree stable bundles. Note that this moduli space is smooth but non-compact. It is irreducible and of dimension $4g-3$.

THEOREM 5.2 *The Hodge polynomial of $M^s(2,d)$ with even degree d is*

$$e(M^s(2,d)) = \frac{1}{2(1-uv)(1-(uv)^2)}\Big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\Big).$$

Proof. We compute this by relating $M^s(2,d)$ with the moduli space $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(2,1,d,d_2)$ of triples of rank $(2,1)$ for small σ. Choose $(n_1,d_1)=(2,d)$ and $(n_2,d_2)=(1,d_2)$. If d_2 is very negative so that $\mu_1-\mu_2 = d/2-d_2 > 2g-2$ then Proposition 3.14 (2) applies. We shall choose the maximum possible value of d_2 for this condition to hold, that is, $d-2d_2=4g-2$.

There is a decomposition $\mathcal{N}_{\sigma_m^+} = X_0 \sqcup X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$ into locally closed algebraic subsets, defined by the following strata:

(1) The open subset $X_0 \subset \mathcal{N}_{\sigma_m^+}$ consists of those triples of the form $\phi: L\to E$, where E is a stable rank 2 bundle of degree d, L is a line bundle of degree d_2 and ϕ is a non-zero map (defined up to multiplication by non-zero scalars). Actually, by Proposition 3.14 there is a map

$$\pi: \mathcal{N}_{\sigma_m^+} \to M(2,d)\times \operatorname{Jac}^{d_2} X,$$

and $X_0 = \pi^{-1}(M^s(2,d)\times \operatorname{Jac}^{d_2} X)$. Proposition 3.14 (2) says that X_0 is a projective bundle over $M^s(2,d)\times \operatorname{Jac}^{d_2} X$ with fibers isomorphic to $\mathbb{P}^{d-2d_2-2g+2-1} = \mathbb{P}^{2g-1}$. By Lemma 2.4,

$$e(X_0) = e(M^s(2,d))e(\operatorname{Jac} X)e_{2g},$$

where $e_{2g} = e(\mathbb{P}^{2g-1})$, following the notation in (2).

(2) The subset X_1 parameterizes triples $\phi: L\to E$ where E is a strictly semistable bundle of degree d which sits as a non-trivial extension

$$0\to L_1\to E\to L_2\to 0, \tag{9}$$

with $L_1 \not\cong L_2$, $L_1, L_2 \in \operatorname{Jac}^{d/2} X$ and $L\in \operatorname{Jac}^{d_2} X$.

Let Y_1 be the family which parameterizes such bundles E. For fixed L_1, L_2 with $L_1\not\cong L_2$, the extensions (9) are determined by $\mathbb{P}\operatorname{Ext}^1(L_2,L_1)$. As L_1, L_2 are non-isomorphic, $\dim \operatorname{Ext}^1(L_2,L_1) = \dim H^1(L_1\otimes L_2^*) = g-1$, so $\mathbb{P}\operatorname{Ext}^1(L_2,L_1)\cong \mathbb{P}^{g-2}$. Therefore Y_1 is a fiber bundle over $\mathrm{Jac}^{d/2} X \times \mathrm{Jac}^{d/2} X - \Delta$, where Δ is the diagonal, with fibers isomorphic to $\mathbb{P}^{g-2}$. Thus using Theorem 2.2 and Lemma 2.4,

$$e(Y_1) = \big(e(\mathrm{Jac}\, X)^2 - e(\mathrm{Jac}\, X)\big)e_{g-1}. \tag{10}$$

Now we want to describe X_1. For each fixed $E \in Y_1$ as in (9), and $L \in \mathrm{Jac}^{d_2} X$, there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}(L, L_1) \longrightarrow \mathrm{Hom}(L, E) \longrightarrow \mathrm{Hom}(L, L_2) \longrightarrow 0.$$

Here $\mathrm{Ext}^1(L, L_1) = 0$ since $\deg(L_1) - \deg(L) = d/2 - d_2 > 2g - 2$. So we may write $\mathrm{Hom}(L, E) \cong \mathrm{Hom}(L, L_1) \oplus \mathrm{Hom}(L, L_2)$, non-canonically. Let us see when $\phi \in \mathrm{Hom}(L, E)$ gives rise to a σ_{m}^+-stable triple $T = (E, L, \phi)$. First note that T is σ_m-semistable, since by Section 4.1, the only possibility for not being σ_m-semistable is to have a subtriple of rank $(0, 1)$, that is, a line subbundle $L_{\mathrm{c}} \subset E$, which by Lemma 4.2 should have degree $d_{L_{\mathrm{c}}} > \mu_1$, contradicting the semistability of E. If T is not σ_{m}^+-stable then it must have a σ_m-destabilizing subtriple T' of rank $(1, 1)$ by Section 4.1. Such subtriple is of the form $\phi : L \to L'$, with $L' \subset E$. As $\mu_{\sigma_m}(T') = \mu_{\sigma_m}(T) \implies \mu(L') = \mu(E)$, L' is a destabilizing subbundle of E. But the only destabilizing subbundle of E is L_1, so ϕ satisfies $\phi(L) \subset L_1$. Equivalently, $\phi = (\phi_1, 0) \in \mathrm{Hom}(L, E) = \mathrm{Hom}(L, L_1) \oplus \mathrm{Hom}(L, L_2)$ gives rise to σ_{m}^+-unstable triples.

This discussion implies that given $(E, L) \in Y_1 \times \mathrm{Jac}^{d_2} X$, the morphisms ϕ giving rise to σ_{m}^+-stable triples (E, L, ϕ) are those in

$$\mathrm{Hom}(L, E) - \mathrm{Hom}(L, L_1). \tag{11}$$

By Riemann–Roch, $\dim \mathrm{Hom}(L, E) = d - 2d_2 - 2g + 2 = 2g$ and $\dim \mathrm{Hom}(L, L_1) = d/2 - d_2 - g + 1 = g$. So the space (11) is isomorphic to $\mathbb{C}^{2g} - \mathbb{C}^g$.

The isomorphism class of the triple $T = (E, L, \phi)$ is determined up to multiplication by non-zero scalar $(E, L, \phi) \mapsto (E, L, \lambda\phi)$, since $\mathrm{Aut}(T) = \mathbb{C}^*$. This follows from the fact that $\mathrm{Aut}(E) = \mathbb{C}^*$ [since E is a non-trivial extension (9)] and $\mathrm{Aut}(L) = \mathbb{C}^*$. Taking into account the $\mathbb{C}^*$-action by automorphisms, the fibers of the map $\pi : X_1 \to Y_1 \times \mathrm{Jac}^{d_2} X$ are isomorphic to the projectivization of (11), that is, $\mathbb{P}^{2g-1} - \mathbb{P}^{g-1}$. Hence

$$e(X_1) = e(\mathrm{Jac}\, X)e(Y_1)(e_{2g} - e_g) = e(\mathrm{Jac}\, X)^2(e(\mathrm{Jac}\, X) - 1)e_{g-1}(e_{2g} - e_g).$$

(For this, write $X_1 = X_1' - X_1''$, where X_1' is a $\mathbb{P}^{2g-1}$-bundle over $Y_1 \times \mathrm{Jac}\, X$ and X_1'' is a $\mathbb{P}^{g-1}$-bundle over $Y_1 \times \mathrm{Jac}\, X$. By Theorem 2.2, $e(X_1) = e(X_1') - e(X_1'')$. Now use Lemma 2.4 to compute $e(X_1')$ and $e(X_1'')$.)

(3) The subset X_2 parameterizes triples $\phi : L \to E$ where E is a strictly semistable bundle of degree d which sits as a non-trivial extension

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_1 \longrightarrow 0$$

with $L_1 \in \mathrm{Jac}^{d/2} X$ and $L \in \mathrm{Jac}^{d_2} X$.

The family Y_2 parameterizing such bundles E is a fiber bundle over $\mathrm{Jac}^{d/2} X$ with fibers $\mathbb{P}\,\mathrm{Ext}^1(L_1, L_1) = \mathbb{P}H^1(\mathcal{O}) = \mathbb{P}^{g-1}$ (actually, this fiber bundle is trivial, so $Y_2 = \mathrm{Jac}^{d/2} X \times \mathbb{P}^{g-1}$). Thus by Lemma 2.3,

$$e(Y_2) = e(\mathrm{Jac}\, X)e_g. \tag{12}$$

For each $L_1 \in \mathrm{Jac}^{d/2} X$, there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}(L, L_1) \longrightarrow \mathrm{Hom}(L, E) \longrightarrow \mathrm{Hom}(L, L_1) \longrightarrow 0.$$

So we may write $\mathrm{Hom}(L, E) \cong \mathrm{Hom}(L, L_1) \oplus \mathrm{Hom}(L, L_1)$, non-canonically. In order to describe X_2, let us see when a triple $T = (E, L, \phi)$, with $E \in Y_2$, is σ_{m}^+-stable. As before, the morphisms ϕ giving rise to σ_{m}^+-stable triples (E, L, ϕ) are those in

$$\mathrm{Hom}(L, E) - \mathrm{Hom}(L, L_1) = \mathrm{Hom}(L, L_1) \times (\mathrm{Hom}(L, L_1) - \{0\}). \tag{13}$$

For a bundle E in Y_2, the automorphism group of E is $\mathbb{C} \times \mathbb{C}^*$, where $\mathbb{C} \times \mathbb{C}^*$ acts on $\mathrm{Hom}(L, E)$ by

$$(a, \lambda) \cdot (\phi_1, \phi_2) = (\lambda\phi_1 + a\phi_2, \lambda\phi_2).$$

Thus for any $(E, L) \in Y_2 \times \mathrm{Jac}^{d_2} X$, the morphisms ϕ giving rise to σ_{m}^+-stable triples (E, L, ϕ) are parameterized by

$$(\mathrm{Hom}(L, L_1) \times (\mathrm{Hom}(L, L_1) - \{0\}))/\mathbb{C} \times \mathbb{C}^*. \tag{14}$$

This is a fiber bundle over $\mathbb{P}\,\mathrm{Hom}(L, L_1) = (\mathrm{Hom}(L, L_1) - \{0\})/\mathbb{C}^*$ with fibers isomorphic to $\mathrm{Hom}(L, L_1)/\mathbb{C}\phi_2$ for every $[\phi_2] \in \mathbb{P}\,\mathrm{Hom}(L, L_1)$. As $\dim \mathrm{Hom}(L, E) = d - 2d_2 - 2g + 2 = 2g$ and $\dim \mathrm{Hom}(L, L_1) = d/2 - d_2 - g + 1 = g$, the space (14) is a $\mathbb{C}^{g-1}$-bundle over $\mathbb{P}^{g-1}$.

Therefore $X_2 \to Y_2 \times \mathrm{Jac}^{d_2} X$ is $\mathbb{C}^{g-1}$-bundle over a $\mathbb{P}^{g-1}$-bundle over $Y_2 \times \mathrm{Jac}^{d_2} X$. So

$$e(X_2) = e(\mathrm{Jac}\, X)e(Y_2)e_g(e_g - e_{g-1}) = e(\mathrm{Jac}\, X)^2 e_g^2(e_g - e_{g-1}).$$

(To apply Lemma 2.4, we write $X_2 \to P$, where P is the $\mathbb{P}^{g-1}$-bundle over $Y_2 \times \mathrm{Jac}^{d_2} X$. Then $X_2 = X_2' - X_2''$, where X_2' is a $\mathbb{P}^{g-1}$-bundle over P and X_2'' is a $\mathbb{P}^{g-2}$-bundle over P.)

(4) The subset X_3 parameterizes triples $\phi : L \to E$ where E is a decomposable bundle of the form $E = L_1 \oplus L_2$, $L_1 \ncong L_2$, $L_1, L_2 \in \mathrm{Jac}^{d/2} X$ and $L \in \mathrm{Jac}^{d_2} X$. The space parameterizing such bundles E is

$$Y_3 = \tilde{Y}_3/\mathbb{Z}_2, \quad \text{where } \tilde{Y}_3 = \mathrm{Jac}^{d/2} X \times \mathrm{Jac}^{d/2} X - \Delta, \tag{15}$$

with $\mathbb{Z}_2$ acting by permuting the two factors.

As before, the condition for $\phi \in \mathrm{Hom}(L, E)$ to give rise to a σ_{m}^+-unstable triple is that there is a subtriple $\phi : L \to L'$ where $\mu(L') = \mu(E)$. There are only two possible such choices for L', namely L_1 and L_2. So given $(E, L) \in Y_3 \times \mathrm{Jac}^{d_2} X$, the morphisms $\phi \in \mathrm{Hom}(L, E) = \mathrm{Hom}(L, L_1) \oplus \mathrm{Hom}(L, L_2)$ giving rise to σ_{m}^+-stable triples (E, L, ϕ) are those with both components non-zero, that is, lying in

$$(\mathrm{Hom}(L, L_1) - \{0\}) \times (\mathrm{Hom}(L, L_2) - \{0\}).$$

The automorphisms of E are $\mathrm{Aut}(E) = \mathbb{C}^* \times \mathbb{C}^*$, therefore the map $\phi \in \mathrm{Hom}(L, E) = \mathrm{Hom}(L, L_1) \oplus \mathrm{Hom}(L, L_2)$ is determined up to the action of $\mathbb{C}^* \times \mathbb{C}^*$ on both factors. So ϕ

are parameterized by

$$\mathbb{P}\operatorname{Hom}(L, L_1) \times \mathbb{P}\operatorname{Hom}(L, L_2).$$

Let $\tilde{X}_3 \to \tilde{Y}_3 \times \operatorname{Jac}^{d_2} X$ be the fiber bundle with fiber over (L_1, L_2, L) equal to $\mathbb{P}\operatorname{Hom}(L, L_1) \times \mathbb{P}\operatorname{Hom}(L, L_2)$. Then $X_3 = \tilde{X}_3/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by permuting $(\phi_1, \phi_2) \mapsto (\phi_2, \phi_1)$. This covers the action of $\mathbb{Z}_2$ on $\tilde{Y}_3$. Now $\tilde{X}_3 = X_3' - X_3''$, where $\pi : X_3' \to \operatorname{Jac}^{d/2} X \times \operatorname{Jac}^{d/2} X \times \operatorname{Jac}^{d_2} X$ is a $\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$-bundle and $X_3'' = \pi^{-1}(\Delta \times \operatorname{Jac}^{d_2} X)$. If $A \to \operatorname{Jac}^{d/2} X \times \operatorname{Jac}^{d_2} X$ is the $\mathbb{P}^{g-1}$-bundle with fiber over L_1 equal to $\mathbb{P}\operatorname{Hom}(L, L_1)$, then $X_3' = A \times_{\operatorname{Jac}^{d_2} X} A$. We apply Lemma 2.6 fiberwise: $A \to \operatorname{Jac}^{d_2} X$ is a fibration whose fiber is A_L, which in turn is a fibration over $\operatorname{Jac}^{d/2} X$ with fibers $\mathbb{P}\operatorname{Hom}(L, L_1)$. Then X_3' fibers over $\operatorname{Jac}^{d_2} X$ with fibers $(A_L \times A_L)/\mathbb{Z}_2$. Now

$$\begin{aligned} H^*((A \times_{\operatorname{Jac}^{d_2} X} A)/\mathbb{Z}_2) &= H^*(A \times_{\operatorname{Jac}^{d_2} X} A)^{\mathbb{Z}_2} \\ &= (H^*(A_L \times A_L) \otimes H^*(\operatorname{Jac}^{d_2} X))^{\mathbb{Z}_2} \\ &= (H^*(A_L \times A_L))^{\mathbb{Z}_2} \otimes H^*(\operatorname{Jac}^{d_2} X). \end{aligned}$$

So

$$\begin{aligned} e\left(\frac{X_3'}{\mathbb{Z}_2}\right) &= e((A_L \times A_L)/\mathbb{Z}_2) e(\operatorname{Jac} X) \\ &= \frac{1}{2}\left(e(\operatorname{Jac} X)^2 e_g^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2g}}{1-u^2v^2}\right) e(\operatorname{Jac} X). \end{aligned}$$

On the other hand, X_3'' is a $\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$-bundle over $\Delta \times \operatorname{Jac}^{d_2} X$, the action of $\mathbb{Z}_2$ is trivial on the base and acts by permutation on the fibers. So $X_3''/\mathbb{Z}_2$ is a bundle over $\Delta \times \operatorname{Jac}^{d_2} X$ with fibers

$$(\mathbb{P}\operatorname{Hom}(L, L_1) \times \mathbb{P}\operatorname{Hom}(L, L_1))/\mathbb{Z}_2 = (\mathbb{P}^{g-1} \times \mathbb{P}^{g-1})/\mathbb{Z}_2.$$

This fibration is locally trivial in the Zariski topology, since it is associated with a locally trivial (in the Zariski topology) vector bundle over $\Delta \times \operatorname{Jac}^{d_2} X$. Hence by Lemma 2.3 and Lemma 2.4,

$$e(X_3''/\mathbb{Z}_2) = e(\operatorname{Jac} X)^2 e(\mathbb{P}^{g-1} \times \mathbb{P}^{g-1}/\mathbb{Z}_2) = \frac{1}{2} e(\operatorname{Jac} X)^2 \left(e_g^2 + \frac{1-(uv)^{2g}}{1-u^2v^2}\right).$$

Finally using Theorem 2.2,

$$\begin{aligned} e(X_3) &= e(\tilde{X}_3/\mathbb{Z}_2) = e(X_3'/\mathbb{Z}_2) - e(X_3''/\mathbb{Z}_2) \\ &= \frac{1}{2}\left(e(\operatorname{Jac} X)^2 e_g^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2g}}{1-u^2v^2}\right) e(\operatorname{Jac} X) \\ &\quad - \frac{1}{2} e(\operatorname{Jac} X)^2 \left(e_g^2 + \frac{1-(uv)^{2g}}{1-u^2v^2}\right). \end{aligned}$$

(5) The subset X_4 parameterizes triples $\phi : L \to E$, where E is a decomposable bundle of the form $E = L_1 \oplus L_1$, $L_1 \in \mathrm{Jac}^{d/2} X$ and $L \in \mathrm{Jac}^{d_2} X$. Such bundles E are parameterized by $Y_4 = \mathrm{Jac}^{d/2} X$. The morphism ϕ lives in

$$\mathrm{Hom}(L, E) = \mathrm{Hom}(L, L_1) \oplus \mathrm{Hom}(L, L_1) = \mathrm{Hom}(L, L_1) \otimes \mathbb{C}^2. \tag{16}$$

The condition for a triple $T = (E, L, \phi)$ to be σ_{m}^+-unstable is that there is a destabilizing subbundle $L' \subset E$. A destabilizing subbundle of E is necessarily isomorphic to L_1 and there exists $(a, b) \neq (0, 0)$ such that $L' \cong L_1 \hookrightarrow E$ is given by $x \mapsto (ax, bx)$. This means that $\phi = (a\psi, b\psi) \in \mathrm{Hom}(L, L_1) \otimes \mathbb{C}^2$, for some $\psi \in \mathrm{Hom}(L, L_1)$. All this discussion implies that the set of ϕ giving rise to σ_{m}^+-stable triples are those of the form $\phi = (\phi_1, \phi_2) \in \mathrm{Hom}(L, L_1) \otimes \mathbb{C}^2$, with ϕ_1, ϕ_2 linearly independent.

The automorphisms of $T = (E, L, \phi)$ are $\mathrm{Aut}(T) \cong \mathrm{Aut}(E) = GL(2, \mathbb{C})$. This acts on (16) via the standard representation of $GL(2, \mathbb{C})$ on $\mathbb{C}^2$. So the morphisms ϕ are parameterized by the grassmannian $\mathrm{Gr}(2, \mathrm{Hom}(L, L_1))$. As $\dim \mathrm{Hom}(L, L_1) = g$, we have that $\mathrm{Gr}(2, \mathrm{Hom}(L, L_1)) \cong \mathrm{Gr}(2, g)$.

Moreover, $X_4 \to Y_4 \times \mathrm{Jac}^{d_2} X$ is a locally trivial fibration in the Zariski topology since it is associated with the (locally trivial in the Zariski topology) vector bundle over $Y_4 \times \mathrm{Jac}^{d_2} X$ with fibers $\mathrm{Hom}(L, L_1)$. Using Lemma 2.5,

$$e(X_4) = e(\mathrm{Jac}\, X)^2 e(\mathrm{Gr}(2, g)) = e(\mathrm{Jac}\, X)^2 \frac{(1 - (uv)^{g-1})(1 - (uv)^g)}{(1 - (uv)^2)(1 - uv)}.$$

Putting all together,

$$\begin{aligned} e(\mathcal{N}_{\sigma_{\mathrm{m}}^+}) &= e(X_0) + e(X_1) + e(X_2) + e(X_3) + e(X_4) \\ &= e(M^s(2, d)) e(\mathrm{Jac}\, X) e_{2g} + e(\mathrm{Jac}\, X)^2 (e(\mathrm{Jac}\, X) - 1) e_{g-1} (e_{2g} - e_g) \\ &\quad + e(\mathrm{Jac}\, X)^2 e_g^2 (e_g - e_{g-1}) \\ &\quad + \frac{1}{2}\left(e(\mathrm{Jac}\, X)^2 e_g^2 + (1 - u^2)^g (1 - v^2)^g \frac{1 - (uv)^{2g}}{1 - u^2 v^2} \right) e(\mathrm{Jac}\, X) \\ &\quad - \frac{1}{2} e(\mathrm{Jac}\, X)^2 \left(e_g^2 + \frac{1 - (uv)^{2g}}{1 - u^2 v^2} \right) + e(\mathrm{Jac}\, X)^2 e(\mathrm{Gr}(2, g)). \end{aligned} \tag{17}$$

To compute the left hand side, we use Theorem 4.3 for $\sigma = \sigma_{\mathrm{m}}^+ = \mu_1 - \mu_2 + \epsilon$, $\epsilon > 0$ small. It gives

$$d_0 = \left[\tfrac{1}{3}(\mu_1 - \mu_2 + \varepsilon + 2\mu_1 + \mu_2)\right] + 1 = [\mu_1] + 1 = \frac{d}{2} + 1.$$

Substituting into the formula for $e(\mathcal{N}_\sigma)$ with $d_1 = d/2$ and $d - 2d_2 = 4g - 2$, the Hodge polynomial of $\mathcal{N}_{\sigma_{\mathrm{m}}^+}$ equals

$$e(\mathcal{N}_{\sigma_{\mathrm{m}}^+}) = \operatorname*{coeff}_{x^0} \left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{2g-2}} \left(\frac{(uv)^{2g-2}}{1 - (uv)^{-1}x} - \frac{(uv)^{g+1}}{1 - (uv)^2 x} \right) \right].$$

Using the following equality (see the proof of [21, Proposition 8.1])

$$\operatorname*{coeff}_{x^0} \frac{(1+ux)^g(1+vx)^g}{(1-ax)(1-bx)(1-cx)x^{2g-2}} = \frac{(a+u)^g(a+v)^g}{(a-b)(a-c)} + \frac{(b+u)^g(b+v)^g}{(b-a)(b-c)} + \frac{(c+u)^g(c+v)^g}{(c-a)(c-b)},$$

one gets the following expression

$$e(\mathcal{N}_{\sigma_m^+}) = \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^2(1-(uv)^2)}\big[(1+u^2v)^g(1+v^2u)^g(1-(uv)^{2g})+ + (1+u)^g(1+v)^g\big((uv)^{3g-1}+(uv)^{2g+1}-(uv)^{2g-1}-(uv)^{g+1}\big)\big].$$

Finally we substitute this into (17) to get the Hodge polynomial $e(M^s(2,d))$ as in the statement.

COROLLARY 5.3 *The Hodge polynomial of the moduli space of semistable rank* 2 *even degree* d *vector bundles is*

$$e(M(2,d)) = \frac{1}{2(1-uv)(1-(uv)^2)}\big(2(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (1+u)^{2g}(1+v)^{2g}(1+2u^{g+1}v^{g+1}-u^2v^2) - (1-u^2)^g(1-v^2)^g(1-uv)^2\big) + \frac{1}{2}\big((1+u)^{2g}(1+v)^{2g}+(1-u^2)^g(1-v^2)^g\big).$$

Proof. We only need to compute $e(M^{ss}(2,d))$, where $M^{ss}(2,d) = M(2,d) - M^s(2,d)$ is the locus of non-stable and polystable rank 2 bundles of degree d. Such bundles are of the form $L_1 \oplus L_2$, where $L_1, L_2 \in \mathrm{Jac}^{d/2} X$. Therefore $M^{ss}(2,d) \cong (\mathrm{Jac}\, X \times \mathrm{Jac}\, X)/\mathbb{Z}_2$. By Lemma 2.6 and (3),

$$e((\mathrm{Jac}\, X \times \mathrm{Jac}\, X)/\mathbb{Z}_2) = \frac{1}{2}\big((1+u)^{2g}(1+v)^{2g}+(1-u^2)^g(1-v^2)^g\big).$$

Adding this to $e(M^s(2,d))$ in Theorem 5.2 we get the result.

For instance, the formula of Corollary 5.3 for $g=2$ gives

$$e(M(2,0)) = (1+u)^2(1+v)^2(1+uv+u^2v^2+u^3v^3).$$

This formula agrees with [20, Remark 4.11]. Note that the moduli space $M(2,0)$ is smooth for $g=2$ (see [22]).

6. Critical values for triples of rank (2,2)

Now we move to the analysis of the moduli spaces of σ-semistable triples of rank $(2,2)$. Let $\mathcal{N}_\sigma = \mathcal{N}_\sigma(2,2,d_1,d_2)$. By Proposition 3.3, σ takes values in the interval

$$I = [\sigma_m, \infty) = [\mu_1 - \mu_2, \infty), \quad \text{where } d_1 - d_2 \geq 0.$$

Otherwise $\mathcal{N}_\sigma$ is empty.

THEOREM 6.1 *For $\sigma \in I$, $\mathcal{N}_\sigma$ is a projective variety. It is smooth of dimension $4g + 2d_1 - 2d_2 - 3$ at any σ-stable point for $\sigma \geq 2g - 2$, or at any σ-stable injective triple. Moreover, if $d_1 + d_2$ is odd then $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$ for non-critical σ.*

Proof. Projectiveness follows from Proposition 3.3. The smoothness at injective triples follows from Theorem 3.9(4); the dimension follows from Theorem 3.9(3); the smoothness result for $\sigma \geq 2g - 2$ comes from [**5**, Theorem 3.8 (6)]. If $d_1 + d_2$ is odd then $\gcd(2, 2, d_1 + d_2) = 1$ and so, for non-critical σ, $\mathcal{N}_\sigma = \mathcal{N}_\sigma^s$, by Proposition 3.6(3). On the other hand, if $d_1 + d_2$ is even, then it may happen that there are strictly σ-semistable triples for non-critical values of σ.

Let us now compute the critical values for $\mathcal{N}_\sigma(2, 2, d_1, d_2)$. According to (5) we have the following possibilities for $n_1 = 2$, $n_2 = 2$:

(1) $n_1' = 1$, $n_2' = 0$. The corresponding σ_c-destabilizing subtriple is of the form $0 \to E_1'$ where $E_1' = L$ is a line bundle of degree d_L. The critical value is

$$\sigma_c = \frac{4d_L - (d_1 + d_2)}{2} = 2d_L - \mu_1 - \mu_2.$$

(2) $n_1' = 1$, $n_2' = 2$. The σ_c-destabilizing subtriple T' is of the form $E_2 \to E_1'$ where E_1', is a line bundle. The quotient triple $T'' = T/T'$ is of the form $0 \to E_1''$, where $E_1'' = L$ is a line bundle of degree d_L, and $d_1' = d_1 - d_L$. Note that $\phi : E_2 \to E_1$ is not injective. The critical value is

$$\sigma_c = \frac{4(d_1 - d_L + d_2) - 3(d_1 + d_2)}{-2} = 2d_L - \mu_1 - \mu_2.$$

(3) $n_1' = 2$, $n_2' = 1$. The σ_c-destabilizing subtriple T' is of the form $E_2' \to E_1$, where E_2' is a line bundle. Then the quotient triple $T'' = T/T'$ is of the form $E_2'' \to 0$, where $E_2'' = F$ is a line bundle of degree d_F, and $d_2' = d_2 - d_F$.

$$\sigma_c = \frac{4(d_1 + d_2 - d_F) - 3(d_1 + d_2)}{2} = \mu_1 + \mu_2 - 2d_F.$$

(4) $n_1' = 0$, $n_2' = 1$. The σ_c-destabilizing subtriple is of the form $E_2' \to 0$, where $E_2' = F$ is a line bundle of degree d_F. Again in this case ϕ is not injective. The corresponding critical value is

$$\sigma_c = \frac{4d_F - (d_1 + d_2)}{-2} = \mu_1 + \mu_2 - 2d_F.$$

(5) $n_1' = 2$, $n_2' = 0$. The subtriple is of the form $0 \to E_1$. The corresponding critical value is $\sigma_c = \mu_1 - \mu_2 = \sigma_m$.

(6) $n_1' = 0$, $n_2' = 2$. The subtriple is of the form $E_2 \to 0$. This only happens if $\phi = 0$, and so $T = (0, E_2, 0) \oplus (E_1, 0, 0)$. The critical value is $\sigma_c = \mu_1 - \mu_2 = \sigma_m$, and the triple is σ-unstable for any $\sigma \neq \sigma_m$.

Note that the case $n_1' = 1$, $n_2' = 1$ does not appear, since $\lambda' = \lambda$ and therefore this does not give a critical value. In the Cases (1), (3) and (5), we have $\lambda' < \lambda$, so the corresponding triples are σ-unstable for $\sigma < \sigma_c$. In the Cases (2), (4) and (6), we have $\lambda' > \lambda$, so the corresponding triples are σ-unstable for $\sigma > \sigma_c$.

PROPOSITION 6.2 (1) *Let* $\sigma_c = 2d_L - \mu_1 - \mu_2$ *be a critical value corresponding to the Case* (1) *or* (3)*. Then* $\mu_1 \leq d_L \leq (3\mu_1 - \mu_2)/2$*. Also* $d_L = \mu_1 \iff \sigma_c = \sigma_m$*. (2) Let* $\sigma_c = \mu_1 + \mu_2 - 2d_F$ *be a critical value corresponding to the Cases* (2) *or* (4)*. Then* $(3\mu_2 - \mu_1)/2 \leq d_F \leq \mu_2$*. Also* $d_F = \mu_2 \iff \sigma_c = \sigma_m$*.*

Proof. We shall do the first item, since the second is analogous. Fix the critical value $\sigma_c = 2d_L - \mu_1 - \mu_2$ and suppose that there is a strictly σ_c-semistable triple T in either Case (1) or (3) above. Then the subtriple T' and quotient triple T'' are both σ_c-semistable by Lemma 3.12. In either case, there exists a σ_c-semistable triple of type $(1, 2, d_1 - d_L, d_2)$. By Proposition 3.3 applied to this situation, we get

$$d_1 - d_L - \frac{d_2}{2} \leq \sigma_c = 2d_L - \frac{d_1}{2} - \frac{d_2}{2} \leq 4\left(d_1 - d_L - \frac{d_2}{2}\right).$$

We can write this inequality in the equivalent form

$$\frac{d_1}{2} \leq d_L \leq \frac{3d_1 - d_2}{4}.$$

THEOREM 6.3 *Let* $\sigma_M = 2(\mu_1 - \mu_2)$*. For* $\sigma > \sigma_M$ *the moduli spaces of* σ*-(semi)stable triples do not change, and all* σ*-semistable triples* $T = (E_1, E_2, \phi)$ *are injective, that is,* T *defines an exact sequence of the form*

$$0 \longrightarrow E_2 \xrightarrow{\phi} E_1 \longrightarrow S \longrightarrow 0,$$

where S *is a torsion sheaf of degree* $d_1 - d_2$*.*

Proof. If we are in the first situation in Proposition 6.2, then $\sigma_c = 2d_L - \mu_1 - \mu_2 \leq 3\mu_1 - \mu_2 - \mu_1 - \mu_2 = 2(\mu_1 - \mu_2)$. In the second situation, $\sigma_c = \mu_1 + \mu_2 - 2d_F \leq \mu_1 + \mu_2 - (3\mu_2 - \mu_1) = 2(\mu_1 - \mu_2)$.

Now let T be a σ-semistable triple for $\sigma > 2(\mu_1 - \mu_2)$. If $\phi : E_2 \to E_1$ were not injective, then T has a subtriple $T' = (0, \ker\phi, 0)$, which clearly has $\lambda' > \lambda$. This forces $\mu_\sigma(T') > \mu_\sigma(T)$ for σ large, and hence for σ bigger than the last critical value.

REMARK 6.4 Note that for any critical value σ_c, all the triples in $\mathcal{S}_{\sigma_c^-}$ are not injective.

REMARK 6.5 By [5, Proposition 6.5] there is a value σ_0 such that all σ-semistable triples for $\sigma > \sigma_0$ are injective. By [5, Theorem 8.6] there is a value σ_L such that the moduli spaces $\mathcal{N}_\sigma$ are isomorphic for all $\sigma > \sigma_L$. In our case, $n_1 = n_2 = 2$, both numbers are $2(\mu_1 - \mu_2)$.

REMARK 6.6 In Proposition 6.2 we see that, for the triples of rank (2,2), there are critical values for which the moduli spaces do not change (those corresponding to $d_L > (3\mu_1 - \mu_2)/2$ and those corresponding to $d_F < (3\mu_2 - \mu_1)/2$).

REMARK 6.7 If we have simultaneously $\sigma_c = 2d_L - \mu_1 - \mu_2$ and $\sigma_c = \mu_1 + \mu_2 - 2d_F$, then $2d_L - \mu_1 - \mu_2 = \mu_1 + \mu_2 - 2d_F \implies d_1 + d_2 = 2d_L + 2d_F$ is an even number.

Therefore, if $d_1 + d_2 \notin 2\mathbb{Z}$, then Cases (1) and (3) (resp. Cases (2) and (4)) do not happen simultaneously (for the same critical value). So the flip locus $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) will consist only of triples of one type for any $\sigma_c > \sigma_m$. In this situation the critical values $\sigma_c \in (\mu_1 + \mu_2 + \mathbb{Z}) \cap [\mu_1 - \mu_2, 2(\mu_1 - \mu_2)]$.

If $d_1 + d_2 \in 2\mathbb{Z}$, then Cases (1) and (3) (resp. Cases (2) and (4)) do happen simultaneously. The flip locus $\mathcal{S}_{\sigma_c^+}$ (resp. $\mathcal{S}_{\sigma_c^-}$) consists of two types of triples, which yields two components that must be considered independently. In this situation the critical values $\sigma_c \in (\mu_1 + \mu_2 + 2\mathbb{Z}) \cap [\mu_1 - \mu_2, 2(\mu_1 - \mu_2)]$. Moreover, it may happen that, for the same triple T, both Cases (1) and (3) (or Cases (2) and (4)) occur. So the two components intersects. This causes extra difficulties, and that is one of the reasons for us to restrict to the case $d_1 + d_2$ odd in Sections 8 and 9.

In the next section, it will be useful to have a vanishing result for the hypercohomology $\mathbb{H}^2$ to find the flip loci $\mathcal{S}_{\sigma_c^\pm}$ for the moduli spaces of triples of type $(2, 2, d_1, d_2)$.

PROPOSITION 6.8 *Let $T = (E_1, E_2, \phi)$ be a strictly σ_c-semistable triple of type $(2, 2, d_1, d_2)$ with $\sigma_c > \sigma_m$, $T' = (E_1', E_2', \phi')$ a destabilizing subtriple and $T'' = T/T' = (E_1'', E_2'', \phi'')$ the corresponding quotient triple.*

(1) *If $T \in \mathcal{S}_{\sigma_c^-}$ then $\mathbb{H}^2(C^\bullet(T'', T')) = 0$.*
(2) *If $T \in \mathcal{S}_{\sigma_c^+}$ then $\mathbb{H}^2(C^\bullet(T'', T')) = 0$, if $d_1 - d_2 > 2g - 2$.*

Proof. By Proposition 3.7 and Serre duality, the vanishing $\mathbb{H}^2(C^\bullet(T'', T')) = 0$ is equivalent to the injectivity of the map

$$\begin{array}{ccc} H^0(E_1'^* \otimes E_2'' \otimes K) & \stackrel{P}{\longrightarrow} & H^0(E_1'^* \otimes E_1'' \otimes K) \oplus H^0(E_2'^* \otimes E_2'' \otimes K) \\ \psi & \longmapsto & ((\phi'' \otimes Id) \circ \psi,\ \psi \circ \phi'). \end{array}$$

(1) If $T \in \mathcal{S}_{\sigma_c^-}$, then $H^0(E_1'^* \otimes E_2'' \otimes K)$ is trivial because either we are in Case (4) and so $E_1' = 0$ or we are in Case (2) and so $E_2'' = 0$.
(2) If $T \in \mathcal{S}_{\sigma_c^+}$, we may have two cases:
   (a) If we are in Case (3), then $E_1' = E_1$ and $E_1'' = 0$. The map P is

$$\begin{array}{ccc} H^0(E_1^* \otimes E_2'' \otimes K) & \stackrel{P}{\longrightarrow} & H^0(E_2'^* \otimes E_2'' \otimes K) \\ \psi & \longmapsto & \psi \circ \phi'. \end{array}$$

   If P is not injective, let $\psi : E_1 \to E_2'' \otimes K$ be a non-trivial homomorphism in $\ker P$. Then, as $\phi' : E_2' \to E_1$, ψ must factor through the quotient E_1/E_2'. Both E_1/E_2' and $E_2'' \otimes K$ are line bundles, hence $\deg(E_1/E_2') = d_1 - d_2' \leq \deg(E_2'' \otimes K) = d_2'' + 2g - 2$. This yields $d_1 - d_2 \leq 2g - 2$.
   (b) If we are in Case (1), then $E_2' = 0$ and $E_2'' = E_2$. Then the map P is

$$\begin{array}{ccc} H^0(E_1'^* \otimes E_2 \otimes K) & \stackrel{P}{\longrightarrow} & H^0(E_1'^* \otimes E_1'' \otimes K) \\ \psi & \longmapsto & (\phi'' \otimes Id) \circ \psi. \end{array}$$

   If P is not injective, let $\psi : E_1' \to E_2 \otimes K$ be a non-trivial homomorphism in $\ker P$. Denote by Q the kernel of $\phi'' : E_2 \to E_1''$, so ψ must factor through $Q \otimes K$. As E_1' and $Q \otimes K$ are line bundles, we have $\deg(E_1') = d_1' \leq \deg(Q \otimes K) = d_2 - d_1'' + 2g - 2$, which is rewritten as $d_1 - d_2 \leq 2g - 2$.

   In both cases, if P is not injective then $d_1 - d_2 \leq 2g - 2$. Therefore, if $d_1 - d_2 > 2g - 2$, then P must be injective.

REMARK 6.9 This result is a sort of improvement of [5, Proposition 3.6] for the case of triples of rank (2, 2). Here we prove the vanishing of $\mathbb{H}^2$ for *any* critical value σ_c under the condition $\sigma_m = \mu_1 - \mu_2 > g - 1$, whereas in [5, Proposition 3.6] it is proved the vanishing of $\mathbb{H}^2$ only for critical values $\sigma_c > 2g - 2$ (but without condition on σ_m).

7. Hodge polynomial of the moduli of triples of rank (2,2) and small σ

In this section we want to compute the Hodge polynomial of the moduli space

$$\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}(2, 2, d_1, d_2)$$

of σ-stable triples of types $(2, 2, d_1, d_2)$ for σ small, under the assumption $\mu_1 - \mu_2 > 2g - 2$. The study of $\mathcal{N}_{\sigma_m^+}$ is simpler when both d_1 and d_2 are odd, since in this case the bundles are automatically stable. However in this case $d_1 + d_2$ is even and hence $\gcd(2, 2, d_1 + d_2) \neq 1$. So there may be strictly σ-semistable triples in $\mathcal{N}_\sigma$ for non-critical values of σ, making the moduli space $\mathcal{N}_\sigma^s$ non-compact and the moduli space $\mathcal{N}_\sigma$ singular (this does not happen for $\sigma = \sigma_m^+$; see Theorem 7.1).

THEOREM 7.1 *Suppose that d_1 and d_2 are odd and that $\mu_1 - \mu_2 > 2g - 2$. Then $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$, it is smooth, compact and*

$$e(\mathcal{N}_{\sigma_m^+}) = \left(\frac{(1+u)^g(1+v)^g(1+u^2v)^g(1+uv^2)^g - (uv)^g(1+u)^{2g}(1+v)^{2g}}{(1-uv)(1-(uv)^2)}\right)^2 \times \frac{1-(uv)^{2d_1-2d_2-4g+4}}{1-uv}.$$

Proof. The equality $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$ is a consequence of Proposition 3.14(i). Next, since $\sigma_m = \mu_1 - \mu_2 > 2g - 2$, Theorem 6.1 implies that the moduli $\mathcal{N}_{\sigma_m^+}$ is smooth and compact. By Proposition 3.14(i), it is the projectivization of a fiber bundle over $M(2, d_1) \times M(2, d_2)$ of rank $2d_1 - 2d_2 - 4g + 4$. Therefore

$$e(\mathcal{N}_{\sigma_m^+}) = e(M(2, d_1))e(M(2, d_2))e_{2d_1-2d_2-4g+4}.$$

The result follows now applying Theorem 5.1.

The case where d_1 is odd and d_2 is even is more involved, since we have to deal with the presence of strictly semistable bundles in $M(2, d_2)$.

THEOREM 7.2 *Suppose that d_1 is odd and d_2 is even and that $\mu_1 - \mu_2 > 2g - 2$. Then $\mathcal{N}_{\sigma_m^+} = \mathcal{N}_{\sigma_m^+}^s$, it is smooth and compact and*

$$e(\mathcal{N}_{\sigma_m^+}) = \frac{(1+u)^{2g}(1+v)^{2g}(1-(uv)^N)\left(u^gv^g(1+u)^g(1+v)^g - (1+u^2v)^g(1+uv^2)^g\right)}{(1-uv)^3(1-(uv)^2)^2} \cdot \times \left((1+u)^g(1+v)^g(u^{g+1}v^{g+1} + u^{N+g-1}v^{N+g-1}) - (1+u^2v)^g(1+uv^2)^g(1+u^Nv^N)\right),$$

where $N = d_1 - d_2 - 2g + 2$.

Proof. As $d_1 + d_2$ is odd, Theorem 6.1 implies that $\mathcal{N}_{\sigma_m^+} = \mathcal{N}^s_{\sigma_m^+}$ and it is smooth and compact, since $\sigma_m = \mu_1 - \mu_2 > 2g - 2$. To compute $e(\mathcal{N}_{\sigma_m^+})$ we decompose $\mathcal{N}_{\sigma_m^+} = X_0 \sqcup X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4$, where:

(1) The open subset $X_0 \subset \mathcal{N}_{\sigma_m^+}$ consists of those triples of the form $\phi : E_2 \to E_1$, where E_1 and E_2 are both stable bundles, and ϕ is a non-zero map defined up to multiplication by scalars. By Proposition 13.4(2), $X_0 \to M(2, d_1) \times M^s(2, d_2)$ is a projective fibration whose fibers are projective spaces of dimension $2d_1 - 2d_2 - 4g + 4 - 1 = 2N - 1$. Therefore, and using the notation (2),

$$e(X_0) = e(M(2, d_1))e(M^s(2, d_2))e_{2N}.$$

(2) The subset X_1 parameterizes σ_m^+-stable triples of the form $\phi : E_2 \to E_1$ where E_2 is a strictly semistable bundle of degree d_2 which is a non-split extension

$$0 \longrightarrow L_1 \longrightarrow E_2 \longrightarrow L_2 \longrightarrow 0,$$

where $L_1, L_2 \in \mathrm{Jac}^{d_2/2} X$ are non-isomorphic and E_1 is a stable bundle. The space Y_1 parameterizing such bundles E_2 was described in (2) of the proof of Theorem 5.2 and its Hodge polynomial is given in (10).

Now in order to describe X_1, we must characterize when a triple $T = (E_1, E_2, \phi)$, with $E_2 \in Y_1$, is σ_m^+-stable. As T is σ_m-semistable, then the only possibility for T being σ_m^+-unstable is that it has a subtriple T' of rank $(1, 2)$ or $(0, 1)$, corresponding to Cases (2) or (4) of Section 6, respectively. If T' is of rank $(1, 2)$, then it is of the form $E_2 \to L$, where L is a line bundle of degree $d_L = \mu_1$, by Proposition 6.2. But this is impossible, since d_1 is odd. If T' is of rank $(0, 1)$, then it is of the form $F \to 0$, where F is a line bundle of degree $d_F = \mu_2$, by Proposition 6.2. Therefore F is a destabilizing subbundle for E_2. Since the only destabilizing subbundle of E_2 is L_1, we have $F = L_1$. So it must be $\phi(L_1) = 0$. Any such ϕ lies in the image of the inclusion $\mathrm{Hom}(L_2, E_1) \hookrightarrow \mathrm{Hom}(E_2, E_1)$, under the natural projection $E_2 \to L_2$. This discussion implies that given $(E_1, E_2) \in M(2, d_1) \times Y_1$, the morphisms ϕ giving rise to σ_m^+-stable triples (E_1, E_2, ϕ) are those in

$$\mathrm{Hom}(E_2, E_1) - \mathrm{Hom}(L_2, E_1).$$

Note that since the group of automorphisms of E_1 and E_2 are both equal to $\mathbb{C}^*$, ϕ is defined up to multiplication by non-zero scalars. So the map $\pi : X_1 \to M(2, d_1) \times Y_1$ is a fibration with fiber over (E_1, E_2) equal to

$$\mathbb{P}\,\mathrm{Hom}(E_2, E_1) - \mathbb{P}\,\mathrm{Hom}(L_2, E_1)\ . \tag{18}$$

By Riemman–Roch, $\dim \mathrm{Hom}(E_2, E_1) = 2d_1 - 2d_2 - 4g + 4 = 2N$, since $\mu_1 - \mu_2 > 2g - 2$ implies that $H^1(E_2^* \otimes E_1) = H^0(E_1 \otimes E_2^* \otimes K) = 0$, E_1 and E_2 being both semistable bundles. Also $\dim \mathrm{Hom}(L_2, E_1) = d_1 - 2(d_2/2) - 2g + 2 = d_1 - d_2 - 2g + 2 = N$, since $\mu_1 - \deg L_2 = \mu_1 - d_2/2 > 2g - 2$. Hence (18) is isomorphic to $\mathbb{P}^{2N-1} - \mathbb{P}^{N-1}$. Therefore as in (2) of the proof of Theorem 5.2,

$$\begin{aligned} e(X_1) &= e(M(2, d_1))e(Y_1)(e_{2N} - e_N) \\ &= e(M(2, d_1))e(\mathrm{Jac}\, X)(e(\mathrm{Jac}\, X) - 1)e_{g-1}(e_{2N} - e_N). \end{aligned}$$

(3) The subset X_2 parameterizes σ_{m}^+-stable triples of the form $\phi : E_2 \to E_1$, where E_2 is a strictly semistable bundle of degree d_2 which is non-split extension

$$0 \longrightarrow L_1 \longrightarrow E_2 \longrightarrow L_1 \longrightarrow 0,$$

where $L_1 \in \mathrm{Jac}^{d_2/2} X$ and E_1 is a stable bundle. The space Y_2 parameterizing such bundles E_2 was described in (3) of the proof of Theorem 5.2 and its Hodge polynomial is given in (12).

To describe X_2, we must characterize when a triple $T = (E_1, E_2, \phi)$, with $E_2 \in Y_2$, is σ_{m}^+-stable. As before, given $(E_1, E_2) \in M(2, d_1) \times Y_2$, the morphisms ϕ giving rise to σ_{m}^+-stable triples (E_1, E_2, ϕ) are those in

$$\mathrm{Hom}(E_2, E_1) - \mathrm{Hom}(L_1, E_1).$$

For a triple $T = (E_1, E_2, \phi) \in X_2$, $\mathrm{Aut}(E_1) = \mathbb{C}^*$, so $\mathrm{Aut}(T) \cong \mathrm{Aut}(E_2) = \mathbb{C} \times \mathbb{C}^*$. There is an exact sequence

$$0 \longrightarrow \mathrm{Hom}(L_1, E_1) \longrightarrow \mathrm{Hom}(E_2, E_1) \longrightarrow \mathrm{Hom}(L_1, E_1) \longrightarrow 0.$$

Under the (non-canonical) decomposition $\mathrm{Hom}(E_2, E_1) \cong \mathrm{Hom}(L_1, E_1) \oplus \mathrm{Hom}(L_1, E_1)$, $\mathrm{Aut}(E_2)$ acts as $(a, \lambda)(x, y) \mapsto (\lambda x + ay, \lambda y)$. So the fiber of $\pi : X_2 \to M(2, d_1) \times Y_2$ is

$$(\mathrm{Hom}(E_2, E_1) - \mathrm{Hom}(L_1, E_1))/\mathbb{C} \times \mathbb{C}^* \cong (\mathbb{C}^{2N} - \mathbb{C}^N)/\mathbb{C} \times \mathbb{C}^*,$$

which is a $\mathbb{C}^{N-1}$-bundle over $\mathbb{P}^{N-1}$. Therefore as in (3) of the proof of Theorem 5.2,

$$\begin{aligned} e(X_2) &= e(M(2, d_1))e(Y_2)(e_N - e_{N-1})e_N \\ &= e(M(2, d_1))e(\mathrm{Jac}\, X)e_g(e_N - e_{N-1})e_N. \end{aligned}$$

(4) The subset X_3 parameterizes σ_{m}^+-stable triples of the form $\phi : E_2 \to E_1$ where E_1 is a stable bundle and $E_2 = L_1 \oplus L_2$, $L_1 \ncong L_2$ are two line bundles of degree $d_2/2$. The space Y_3 parameterizing such bundles is described in (15).

As above, the condition for $\phi \in \mathrm{Hom}(E_2, E_1)$ to give rise to a σ_{m}^+-unstable triple is that there is a subtriple T' of the form $F \to 0$, with F a line bundle of degree $d_F = \mu_2$. Then it must be either $F = L_1$ or $F = L_2$. This means that $\phi \in (\mathrm{Hom}(L_1, E_1) \oplus \{0\}) \cup (\{0\} \oplus \mathrm{Hom}(L_2, E_1)) \subset \mathrm{Hom}(E_2, E_1)$. Therefore, given $(E_1, E_2) \in M(2, d_1) \times Y_3$, the morphisms ϕ giving rise to σ_{m}^+-stable triples (E_1, E_2, ϕ) are those in

$$(\mathrm{Hom}(L_1, E_1) - \{0\}) \times (\mathrm{Hom}(L_2, E_1) - \{0\}).$$

The group of automorphisms of E_2 is $\mathbb{C}^* \times \mathbb{C}^*$ acting on $L_1 \oplus L_2$ by diagonal matrices. Therefore $\phi \in (\mathrm{Hom}(L_1, E_1) - \{0\}) \times (\mathrm{Hom}(L_2, E_1) - \{0\})$ is defined up to the action of $\mathbb{C}^* \times \mathbb{C}^*$, where each $\mathbb{C}^*$ acts by multiplication on each of the two summands. So the map $\pi : X_3 \to M(2, d_1) \times Y_3$ has fiber

$$\mathbb{P}\,\mathrm{Hom}(L_1, E_1) \times \mathbb{P}\,\mathrm{Hom}(L_2, E_1). \tag{19}$$

By Riemann–Roch, $\dim \mathrm{Hom}(L_1, E_1) = \dim \mathrm{Hom}(L_2, E_1) = d_1 - d_2 - 2g + 2 = N$. Therefore (19) is isomorphic $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. To compute $e(X_3)$ we work as in (4) of the proof of

Theorem 5.2. Write $X_3 = \tilde{X}_3/\mathbb{Z}_2 = X_3'/\mathbb{Z}_2 - X_3''/\mathbb{Z}_2$, where X_3' is a fibration over $M(2, d_1)$ with fiber $(A_{E_1} \times A_{E_1})/\mathbb{Z}_2$, where A_{E_1} is a projective bundle over $\mathrm{Jac}^{d_2/2} X$ with fibers $\mathbb{P}\,\mathrm{Hom}(L, E_1) \cong \mathbb{P}^{N-1}$, and $\mathbb{Z}_2$ acts by permutation. X_3'' is a fibration over $M(2, d_1) \times \mathrm{Jac}^{d_2/2} X$ with fibers $(\mathbb{P}^{N-1} \times \mathbb{P}^{N-1})/\mathbb{Z}_2$. So using Theorem 2.2,

$$\begin{aligned}
e(X_3) &= e(\tilde{X}_3/\mathbb{Z}_2) = e(X_3'/\mathbb{Z}_2) - e(X_3''/\mathbb{Z}_2) \\
&= \frac{1}{2} e(M(2, d_1)) \Bigg(\bigg(e(\mathrm{Jac}\, X)^2 e_N^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2N}}{1-u^2v^2} \bigg) \\
&\quad - e(\mathrm{Jac}\, X) \bigg(e_N^2 + \frac{1-(uv)^{2N}}{1-u^2v^2} \bigg) \Bigg).
\end{aligned}$$

(5) The subset X_4 parameterizes triples $\phi : E_2 \to E_1$, where E_1 is a stable bundle and $E_2 = L_1 \oplus L_1$, $L_1 \in \mathrm{Jac}^{d_2/2} X$. Such bundles E_2 are parameterized by $Y_4 = \mathrm{Jac}^{d_2/2} X$. The map ϕ lies in

$$\mathrm{Hom}(E_2, E_1) = \mathrm{Hom}(L_1, E_1) \oplus \mathrm{Hom}(L_1, E_1) \cong \mathrm{Hom}(L_1, E_1) \otimes \mathbb{C}^2. \qquad (20)$$

The condition for a triple $T = (E_1, E_2, \phi)$ to be σ_{m}^+-unstable is that there is a line subbundle $F \subset E_2$ of degree $d_F = \mu_2$ such that $\phi(F) = 0$. A destabilizing subbundle of E_2 is necessarily isomorphic to L_1 and there exists $(a, b) \neq (0, 0)$ such that $F \cong L_1 \hookrightarrow E_2$ is given by $x \mapsto (ax, bx)$. So $\phi = (a\psi, b\psi) \in \mathrm{Hom}(L_1, E_1) \otimes \mathbb{C}^2$, for some $\psi \in \mathrm{Hom}(L_1, E_1)$. Therefore $T = (E_1, E_2, \phi)$ is σ_{m}^+-stable if $\phi = (\phi_1, \phi_2) \in \mathrm{Hom}(L_1, E_1) \otimes \mathbb{C}^2$ satisfies that ϕ_1 and ϕ_2 are linearly independent.

On the other hand, a triple $(E_1, E_2, \phi) \in X_4$ is determined up to the action of $\mathrm{Aut}(E_2) = GL(2, \mathbb{C})$. This acts on (20) via the standard representation on $\mathbb{C}^2$. Thus for $(E_1, E_2) \in M(2, d_1) \times Y_4$, the morphisms ϕ giving rise to σ_{m}^+-stable triples (E_1, E_2, ϕ) are parameterized by $\mathrm{Gr}(2, \mathrm{Hom}(L_1, E_1))$. But $\dim \mathrm{Hom}(L_1, E_1) = d_1 - d_2 - 2g + 2 = N$, so this fiber is isomorphic to $\mathrm{Gr}(2, N)$. So

$$e(X_4) = e(M(2, d_1)) e(Y_4) e(\mathrm{Gr}(2, N)) = e(M(2, d_1)) e(\mathrm{Jac}\, X) e(\mathrm{Gr}(2, N)).$$

Adding up all contributions together we get

$$\begin{aligned}
e(\mathcal{N}_{\sigma_{\mathrm{m}}^+}) &= e(X_0) + e(X_1) + e(X_2) + e(X_3) + e(X_4) \\
&= e(M(2, d_1)) \Bigg(e(M^s(2, d_2)) e_{2N} + e(\mathrm{Jac}\, X)(e(\mathrm{Jac}\, X) - 1) e_{g-1} (e_{2N} - e_N) \\
&\quad + e(\mathrm{Jac}\, X) e_g (e_N - e_{N-1}) e_N \\
&\quad + \frac{1}{2} \bigg(e(\mathrm{Jac}\, X)^2 e_N^2 + (1-u^2)^g (1-v^2)^g \frac{1-(uv)^{2N}}{1-u^2v^2} \bigg) \\
&\quad - \frac{1}{2} e(\mathrm{Jac}\, X) \bigg(e_N^2 + \frac{1-(uv)^{2N}}{1-u^2v^2} \bigg) + e(\mathrm{Jac}\, X) e(\mathrm{Gr}(2, N)) \Bigg)
\end{aligned}$$

$$= \frac{(1+u)^{2g}(1+v)^{2g}(1-(uv)^N)(u^g v^g(1+u)^g(1+v)^g - (1+u^2v)^g(1+uv^2)^g)}{(1-uv)^3(1-(uv)^2)^2} \cdot$$
$$\times \big((1+u)^g(1+v)^g(u^{g+1}v^{g+1} + u^{N+g-1}v^{N+g-1})$$
$$- (1+u^2v)^g(1+uv^2)^g(1+u^N v^N)\big).$$

COROLLARY 7.3 *Suppose that d_1 is even and d_2 is odd and that $\mu_1 - \mu_2 > 2g-2$. Then $\mathcal{N}_{\sigma_m^+} = \mathcal{N}^s_{\sigma_m^+}$, it is smooth and compact and its Hodge polynomial has the same formula as that of Theorem 7.2, where $N = d_1 - d_2 - 2g + 2$.*

Proof. We use the isomorphism $\mathcal{N}_\sigma(2,2,d_1,d_2) \cong \mathcal{N}_\sigma(2,2,-d_2,-d_1)$. Note that

$$d_1 - d_2 = (-d_2) - (-d_1),$$

so that the small value $\sigma_m^+ = \mu_1 - \mu_2$ and the condition on the slopes $\mu_1 - \mu_2 > 2g-2$ is the same for both moduli spaces $\mathcal{N}_\sigma(2,2,d_1,d_2)$ and $\mathcal{N}_\sigma(2,2,-d_2,-d_1)$. Now we apply Theorem 7.2 to get the stated formula where $N = -d_2 - (-d_1) - 2g + 2$.

COROLLARY 7.4 *Suppose that $d_1 + d_2$ is odd and $\mu_1 - \mu_2 > 2g-2$. Then the Poincaré polynomial of $\mathcal{N}_{\sigma_m^+}$ is*

$$P_t(\mathcal{N}_{\sigma_m^+}) = \frac{\begin{array}{c}(1+t)^{4g}(1-t^{2N})(t^{2g}(1+t)^{2g} - (1+t^3)^{2g})((1+t)^{2g} \\ (t^{2g+2} + t^{2N+2g-2}) - (1+t^3)^{2g}(1+t^{2N}))\end{array}}{(1-t^2)^3(1-t^4)^2},$$

where $N = d_1 - d_2 - 2g + 2$.

Proof. $\mathcal{N}_{\sigma_m^+}$ is smooth and projective, so $P_t(\mathcal{N}_{\sigma_m^+}) = e(\mathcal{N}_{\sigma_m^+})(t,t)$. The result follows from Theorem 7.2 and Corollary 7.3.

We could also deal with the case when d_1 and d_2 are both even and $d_1 - d_2 > 4g-4$. This is similar to the case just treated in Theorem 7.2, with further complication that there are semistable loci for both E_1 and E_2.

However, dealing with the case $d_1 - d_2 \le 4g-4$ is more complicated, since Proposition 3.14 does not apply as there is a Brill–Noether problem consisting on determining the loci of those (E_1, E_2) where $\dim \operatorname{Hom}(E_2, E_1 \otimes K)$ is constant.

8. Contribution of the flips to the Hodge polynomials

In this section, we shall compute the change in the Hodge polynomial of $\mathcal{N}_\sigma(2,2,d_1,d_2)$ when we cross a critical value σ_c. We restrict to the case $d_1 + d_2$ is odd, since in the case $d_1 + d_2$ even there may be strictly σ-semistable triples for non-critical values of σ (and in this case $\mathcal{N}^s_\sigma$ is non-compact and $\mathcal{N}_\sigma$ is non-smooth). For $d_1 + d_2$ odd, Theorem 6.1 guarantees that $\mathcal{N}_\sigma$ is compact and smooth for any non-critical $\sigma \ge 2g-2$. The critical values are given in Proposition 6.2. These are of two types. The following two propositions treat them separately.

PROPOSITION 8.1 *Let $\sigma_c = 2d_L - \mu_1 - \mu_2$ be a critical value for triples of type $(2, 2, d_1, d_2)$ with $d_1 + d_2$ odd, such that $\sigma_c > \sigma_m$. Suppose that $\mu_1 - \mu_2 > g - 1$. Then*

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = \operatorname*{coeff}_{x^0}\Bigg[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g\big((uv)^{g-1-d_1+2d_L} - (uv)^{1-g+d_1-d_2}\big)}{(1-uv)^2(1-x)(1-uvx)x^{[3\mu_1-\mu_2]-2d_L}} \times \left(\frac{(uv)^{(3d_1-d_2-1)/2-2d_L}}{1-(uv)^{-1}x} - \frac{(uv)^{2d_L-d_1+g}}{1-(uv)^2x}\right)\Bigg].$$

Proof. Theorem 6.1 implies that $\mathcal{N}_{\sigma_c^\pm} = \mathcal{N}^s_{\sigma_c^\pm}$. Then Lemma 3.11 and the properties of the Hodge polynomials give

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}).$$

Let us start by studying $\mathcal{S}_{\sigma_c^+}$. By Lemma 3.12, any $T \in \mathcal{S}_{\sigma_c^+}$ sits in a non-split extension

$$0 \to T' \to T \to T'' \to 0 \tag{21}$$

in which T' and T'' are σ_c-semistable, $\lambda' < \lambda$ and $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$. Since T corresponds to Case (1) in Section 6, we have $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\mathcal{N}'_{\sigma_c} = \mathcal{N}_{\sigma_c}(1, 0, d_L, 0) \cong \mathrm{Jac}^{d_L} X,$$
$$\mathcal{N}''_{\sigma_c} = \mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2).$$

The moduli space of triples of rank $(1, 0)$ has no critical values; and for the moduli space of triples of rank $(1, 2)$, the critical values are of the form $3d_{L_c} + d_1'' + d_2''$, by Lemma 4.5, and are in particular integers. But $\sigma_c = 2d_L - (d_1 + d_2)/2 \notin \mathbb{Z}$, so σ_c is not a critical value for $\mathcal{N}''_{\sigma_c}$.

By [5, Proposition 3.5], $\mathbb{H}^0(T'', T') = 0$ and by Proposition 6.8 (2), $\mathbb{H}^2(T'', T') = 0$. So Theorem 3.13 implies that $\mathcal{S}_{\sigma_c^+}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = 1 - g + d_1 - d_2.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^+}) = e(\mathrm{Jac}^{d_L} X)\, e(\mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2))e_{1-g+d_1-d_2}.$$

The case of $\mathcal{S}_{\sigma_c^-}$ is similar. Any $T \in \mathcal{S}_{\sigma_c^-}$ sits in an exact sequence (21) with $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\mathcal{N}'_{\sigma_c} = \mathcal{N}_{\sigma_c}(1, 2, d_1 - d_L, d_2),$$
$$\mathcal{N}''_{\sigma_c} = \mathcal{N}_{\sigma_c}(1, 0, d_L, 0) \cong \mathrm{Jac}^{d_L} X,$$

corresponding to the Case (2) in Section 6. The hypothesis of Theorem 3.13 are satisfied and so $\mathcal{S}_{\sigma_c^-}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = g - 1 - d_1 + 2d_L.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^-}) = e(\mathrm{Jac}^{d_L} X)\, e(\mathcal{N}_{\sigma_c}(1,2,d_1-d_L,d_2))e_{g-1-d_1+2d_L}.$$

Subtracting, we get

$$\begin{aligned} e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}) &= (e_{1-g+d_1-d_2} - e_{g-1-d_1+2d_L})(1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(1,2,d_1-d_L,d_2)) = \\ &= \frac{(uv)^{g-1-d_1+2d_L} - (uv)^{1-g+d_1-d_2}}{1-uv}(1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(1,2,d_1-d_L,d_2)). \end{aligned}$$

Being σ_c a non-critical value for the moduli of triples of rank $(1,2)$, we can apply Theorem 4.6 to compute the Hodge polynomial of $\mathcal{N}_\sigma(1,2,d_1-d_L,d_2)$. First,

$$\begin{aligned} d_0 &= \left[\frac{1}{3}(2d_L - \mu_1 - \mu_2 - (d_1-d_L) - d_2)\right] + 1 \\ &= d_L + [-\mu_1 - \mu_2] + 1. \end{aligned}$$

So $e(\mathcal{N}_\sigma(1,2,d_1-d_L,d_2))$ equals

$$\operatorname*{coeff}_{x^0}\left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2-d_L-d_0}}\left(\frac{(uv)^{d_1-d_2-d_L-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{d_2+g-1+2d_0}}{1-(uv)^2x}\right)\right],$$

where $d_1-d_2-d_L-d_0 = [3\mu_1-\mu_2] - 2d_L = (3d_1-d_2-1)/2 - 2d_L$ and $d_2+2d_0 = 2d_L - d_1 + 1$. The result follows from this.

PROPOSITION 8.2 *Let $\sigma_c = \mu_1+\mu_2-2d_F$ be a critical value for triples of type $(2,2,d_1,d_2)$ with d_1+d_2 odd, such that $\sigma_c > \sigma_m$. Suppose that $\mu_1-\mu_2 > g-1$. Then*

$$\begin{aligned} &e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) \\ &\quad = \operatorname*{coeff}_{x^0}\left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g\big((uv)^{g-1+d_2-2d_F} - (uv)^{1-g+d_1-d_2}\big)}{(1-uv)^2(1-x)(1-uvx)x^{2d_F-[3\mu_2-\mu_1]-1}}\right. \\ &\qquad \left.\times\left(\frac{(uv)^{2d_F+(d_1-3d_2-1)/2}}{1-(uv)^{-1}x} - \frac{(uv)^{d_2-2d_F+g}}{1-(uv)^2x}\right)\right]. \end{aligned}$$

Proof. This is very similar to the proof of Proposition 8.1. Again

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}).$$

We start with $\mathcal{S}_{\sigma_c^+}$. Any $T \in \mathcal{S}_{\sigma_c^+}$ sits in a non-split extension like (21), with $\mu_{\sigma_c}(T') = \mu_{\sigma_c}(T) = \mu_{\sigma_c}(T'')$, $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\begin{aligned} \mathcal{N}'_{\sigma_c} &= \mathcal{N}_{\sigma_c}(2,1,d_1,d_2-d_F), \\ \mathcal{N}''_{\sigma_c} &= \mathcal{N}_{\sigma_c}(0,1,0,d_F) \cong \mathrm{Jac}^{d_F} X, \end{aligned}$$

corresponding to the Case (3) in Section 6. The moduli space of triples of rank $(0,1)$ has no critical values; and for the moduli space of triples of rank $(2,1)$, the critical values are of the form

$3d_{L_c} - d'_1 - d'_2 \in \mathbb{Z}$, while $\sigma_c = ((d_1 + d_2)/2) - 2d_F \notin \mathbb{Z}$, so σ_c is not a critical value for $\mathcal{N}'_{\sigma_c}$. The other conditions of Theorem 3.13 are checked as before. So $\mathcal{S}_{\sigma_c^+}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = 1 - g + d_1 - d_2.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^+}) = e(\mathrm{Jac}^{d_F} X)\, e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F))\, e_{1-g+d_1-d_2}.$$

Moving to $\mathcal{S}_{\sigma_c^-}$, any $T \in \mathcal{S}_{\sigma_c^-}$ sits in an exact sequence (21) with $T' \in \mathcal{N}'_{\sigma_c}$ and $T'' \in \mathcal{N}''_{\sigma_c}$, where

$$\mathcal{N}'_{\sigma_c} = \mathcal{N}_{\sigma_c}(0, 1, 0, d_F) \cong \mathrm{Jac}^{d_F} X,$$
$$\mathcal{N}''_{\sigma_c} = \mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F),$$

corresponding to the Case (4) in Section 6. Arguing as before, we have that $\mathcal{S}_{\sigma_c^-}$ is the projectivization of a bundle over $\mathcal{N}'_{\sigma_c} \times \mathcal{N}''_{\sigma_c}$ of rank

$$-\chi(T'', T') = g - 1 + d_2 - 2d_F.$$

Therefore

$$e(\mathcal{S}_{\sigma_c^-}) = e(\mathrm{Jac}^{d_F} X)\, e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)) e_{g-1+d_2-2d_F}.$$

Subtracting, we get

$$\begin{aligned} e(\mathcal{S}_{\sigma_c^+}) - e(\mathcal{S}_{\sigma_c^-}) &= (e_{1-g+d_1-d_2} - e_{g-1+d_2-2d_F})(1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)) = \\ &= \frac{(uv)^{g-1+d_2-2d_F} - (uv)^{1-g+d_1-d_2}}{1-uv}(1+u)^g(1+v)^g e(\mathcal{N}_{\sigma_c}(2, 1, d_1, d_2 - d_F)). \end{aligned}$$

Being σ_c a non-critical value for the moduli of triples of rank (2, 1), we can apply Theorem 4.3 to compute the Hodge polynomial of $\mathcal{N}_\sigma(2, 1, d_1, d_2 - d_F)$. First,

$$\begin{aligned} d_0 &= \left[\frac{1}{3}(\mu_1 + \mu_2 - 2d_F + d_1 + d_2 - d_F\right] + 1 \\ &= [\mu_1 + \mu_2] - d_F + 1. \end{aligned}$$

So $e(\mathcal{N}_\sigma(2, 1, d_1, d_2 - d_F))$ equals

$$\operatorname*{coeff}_{x^0}\left[\frac{(1+u)^{2g}(1+v)^{2g}(1+ux)^g(1+vx)^g}{(1-uv)(1-x)(1-uvx)x^{d_1-d_2+d_F-d_0}}\left(\frac{(uv)^{d_1-d_2+d_F-d_0}}{1-(uv)^{-1}x} - \frac{(uv)^{-d_1+g-1+2d_0}}{1-(uv)^2x}\right)\right],$$

where $d_1 - d_2 + d_F - d_0 = 2d_F - [3\mu_2 - \mu_1] - 1 = 2d_F + (d_1 - 3d_2 - 1)/2$ and $-d_1 + 2d_0 = d_2 - 2d_F + 1$. The result follows from this.

We gather together Propositions 8.1 and 8.2 in a single result.

COROLLARY 8.3 *The critical values $\sigma_c > \sigma_m$ for triples of type $(2, 2, d_1, d_2)$ with $d_1 + d_2$ odd are of the form $\sigma_c = \mu_1 - \mu_2 + n$, $1 \le n \le [\mu_1 - \mu_2]$, $n \in \mathbb{Z}$. Suppose that $\mu_1 - \mu_2 > g - 1$. Then*

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) = \operatorname{coeff}_{x^0}\left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g\big((uv)^{g-1+n} - (uv)^{1-g+d_1-d_2}\big)}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]-n}}\right.$$
$$\left.\times\left(\frac{(uv)^{(d_1-d_2-1)/2-n}}{1-(uv)^{-1}x} - \frac{(uv)^{g+n}}{1-(uv)^2x}\right)\right].$$

Proof. For simplicity let us assume that d_1 is odd and d_2 is even (the other case is analogous). We have the following possibilities:

(a) If $\sigma_c = 2d_L - \mu_1 - \mu_2$, write $d_L = \mu_1 + 1/2 + m$ with m integer. Then $\sigma_c = \mu_1 - \mu_2 + 2m + 1$. As $\mu_1 < d_L \le (3\mu_1 - \mu_2)/2$ by Proposition 6.2 (1), we have $0 \le m \le (\mu_1 - \mu_2 - 1)/2$. Substituting the values $3d_1 - d_2 - 1 - 4d_L = d_1 - d_2 - 1 - 4m - 2, 2d_L - d_1 + g = g + 2m + 1$, $[3\mu_1 - \mu_2] - 2d_L = [\mu_1 - \mu_2] - 2m - 1$ and $g - 1 - d_1 + 2d_L = g + 2m$ into the formula of Proposition 8.1, one gets

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-})$$
$$= \operatorname{coeff}_{x^0}\left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g\big((uv)^{g+2m} - (uv)^{1-g+d_1-d_2}\big)}{(1-uv)^2(1-x)(1-uvx)(1-(uv)^{-1}x)x^{[\mu_1-\mu_2]-2m-1}}\right.$$
$$\left.\times\left(\frac{(uv)^{(d_1-d_2-1)/2-2m-1}}{1-(uv)^{-1}x} - \frac{(uv)^{g+2m+1}}{1-(uv)^2x}\right)\right].$$

(b) If $\sigma_c = \mu_1 + \mu_2 - 2d_F$, write $d_F = \mu_2 - m - 1$ with m an integer. Then $\sigma_c = \mu_1 - \mu_2 + 2m + 2$. As $(3\mu_2 - \mu_1)/2 \le d_F < \mu_2$ by Proposition 6.2 (1), we have $0 \le m \le (\mu_1 - \mu_2)/2 - 1$. Substituting the values $4d_F + d_1 - 3d_2 - 1 = d_1 - d_2 - 1 - 4m - 4, d_2 - 2d_F + g = g + 2m + 2$, $2d_F - [3\mu_2 - \mu_1] - 1 = [\mu_1 - \mu_2] - 2m - 2$ and $g - 1 + d_2 - 2d_F = g + 2m + 1$ into the formula of Proposition 8.2, we have

$$e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-})$$
$$= \operatorname{coeff}_{x^0}\left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g\big((uv)^{g+2m+1} - (uv)^{1-g+d_1-d_2}\big)}{(1-uv)^2(1-x)(1-uvx)(1-(uv)^{-1}x)x^{[\mu_1-\mu_2]-2m-2}}\right.$$
$$\left.\times\left(\frac{(uv)^{(d_1-d_2-1)/2-2m-2}}{1-(uv)^{-1}x} - \frac{(uv)^{g+2m+2}}{1-(uv)^2x}\right)\right].$$

Case (a) corresponds to $n = 2m + 1$ odd, and Case (b) to $n = 2m + 2$ even in the formula in the statement. The range for n is $1 \le n \le \mu_1 - \mu_2$. But, since $\mu_1 - \mu_2$ is not an integer, this range is actually $1 \le n \le [\mu_1 - \mu_2]$.

9. Hodge polynomial of the moduli of triples of rank (2,2) and large σ

Now we use all the information in Sections 6–8 to compute the Hodge polynomial of the $\mathcal{N}_\sigma(2, 2, d_1, d_2)$, for any non-critical $\sigma > \sigma_m$. Recall that by Theorem 6.3, there is a value

$\sigma_M = 2(\mu_1 - \mu_2)$ such that for $\sigma > \sigma_M$ all the moduli spaces $\mathcal{N}_\sigma$ are isomorphic. We refer to

$$\mathcal{N}_{\sigma_M^+} = \mathcal{N}_{\sigma_M^+}(2, 2, d_1, d_2)$$

as the *large* σ moduli space.

PROPOSITION 9.1 *Suppose that* d_1 *is even and* d_2 *is odd and that* $\mu_1 - \mu_2 > g - 1$. *Let* $\sigma > \sigma_m$ *be a non-critical value. Set* $n_0 = \min\{[\sigma - \mu_1 + \mu_2], [\mu_1 - \mu_2]\}$. *Then*

$$\begin{aligned} e(\mathcal{N}_\sigma) - e(\mathcal{N}_{\sigma_m^+}) = \operatorname*{coeff}_{x^0} \Bigg[& \frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}} \\ & \times \left(\frac{(uv)^{g-1+(d_1-d_2-1)/2} x(1-x^{n_0})}{(1-(uv)^{-1}x)(1-x)} - \frac{(uv)^{(3d_1-3d_2-1)/2-g} x(1-(uv)^{-n_0}x^{n_0})}{(1-(uv)^{-1}x)^2} \right. \\ & \left. - \frac{(uv)^{2g+1}x(1-(uv)^{2n_0}x^{n_0})}{(1-(uv)^2x)^2} + \frac{(uv)^{d_1-d_2+2}x(1-(uv)^{n_0}x^{n_0})}{(1-(uv)^2x)(1-uvx)} \right) \Bigg]. \end{aligned}$$

Proof. By Corollary 8.3, the critical values are of the form $\sigma_c = \mu_1 - \mu_2 + n$ with $1 \le n \le [\mu_1 - \mu_2]$. Now $\sigma_m < \sigma_c < \sigma$ is equivalent to $n \le [\sigma - \mu_1 + \mu_2]$ (note that $\sigma - \mu_1 + \mu_2 \notin \mathbb{Z}$ since σ is not critical). Therefore,

$$\begin{aligned} e(\mathcal{N}_\sigma) - e(\mathcal{N}_{\sigma_m^+}) &= \sum_{\sigma_m < \sigma_c < \sigma} e(\mathcal{N}_{\sigma_c^+}) - e(\mathcal{N}_{\sigma_c^-}) \\ &= \sum_{n=1}^{n_0} \operatorname*{coeff}_{x^0} \Bigg[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g\left((uv)^{g-1+n} - (uv)^{1-g+d_1-d_2}\right)}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]-n}} \\ &\quad \times \left(\frac{(uv)^{(d_1-d_2-1)/2-n}}{1-(uv)^{-1}x} - \frac{(uv)^{g+n}}{1-(uv)^2x} \right) \Bigg] \\ &= \operatorname*{coeff}_{x^0} \Bigg[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}} \\ &\quad \times \left(\frac{1}{1-(uv)^{-1}x} \sum_{n=1}^{n_0} (uv)^{g-1+(d_1-d_2-1)/2} x^n \right. \\ &\quad - \frac{1}{1-(uv)^{-1}x} \sum_{n=1}^{n_0} (uv)^{1-g+(3d_1-3d_2-1)/2-n} x^n \\ &\quad \left. - \frac{1}{1-(uv)^2x} \sum_{n=1}^{n_0} (uv)^{2g-1+2n} x^n + \frac{1}{1-(uv)^2x} \sum_{n=1}^{n_0} (uv)^{1+d_1-d_2+n} x^n \right) \Bigg] \\ &= \operatorname*{coeff}_{x^0} \Bigg[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}} \\ &\quad \times \left(\frac{(uv)^{g-1+(d_1-d_2-1)/2} x(1-x^{n_0})}{(1-(uv)^{-1}x)(1-x)} - \frac{(uv)^{1-g+(3d_1-3d_2-1)/2-1} x(1-(uv)^{-n_0}x^{n_0})}{(1-(uv)^{-1}x)^2} \right. \\ &\quad \left. - \frac{(uv)^{2g-1+2}x(1-(uv)^{2n_0}x^{n_0})}{(1-(uv)^2x)^2} + \frac{(uv)^{1+d_1-d_2+1}x(1-(uv)^{n_0}x^{n_0})}{(1-(uv)^2x)(1-uvx)} \right) \Bigg]. \end{aligned}$$

THEOREM 9.2 *Suppose that d_1 is odd and d_2 is even. Then the large σ moduli space $\mathcal{N}_{\sigma_M^+} = \mathcal{N}^s_{\sigma_M^+}$ is smooth and compact. If $\mu_1 - \mu_2 > 2g-2$, its Hodge polynomial is*

$$e(\mathcal{N}_{\sigma_M^+}) = \frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^3(1-(uv)^2)^2}\Big[(1+u^2v)^{2g}(1+uv^2)^{2g}(1-(uv)^{2N}) \\ - N(1+u^2v)^g(1+uv^2)^g(1+u)^g(1+v)^g(uv)^{N+g-1}(1-(uv)^2) \\ + (1+u)^{2g}(1+v)^{2g}(1+uv)^2(uv)^{2g-2+(N+1)/2} \\ \times \left((1-(uv)^{N+1}) - \frac{N+1}{2}(1-uv)(1+(uv)^N)\right) \\ - g(1+u)^{2g-1}(1+v)^{2g-1}(1-(uv)^2)^2(uv)^{2g-2+(N+1)/2}(1-(uv)^N)\Big],$$

where $N = d_1 - d_2 - 2g + 2$.

Proof. The first statement follows from Theorem 9.1. To compute $e(\mathcal{N}_{\sigma_M^+}) - e(\mathcal{N}_{\sigma_m^+})$ we use Proposition 9.1 for $\sigma = \sigma_M^+$. Note that in this case $n_0 = [\mu_1 - \mu_2]$. All the terms in the formula of Proposition 9.1 involving x^{n_0} yield positive powers of x, so they can be disregarded for computing coeff_{x^0}. Hence

$$e(\mathcal{N}_{\sigma_M^+}) = e(\mathcal{N}_{\sigma_m^+}) + \operatorname*{coeff}_{x^0}\left[\frac{(1+u)^{3g}(1+v)^{3g}(1+ux)^g(1+vx)^g}{(1-uv)^2(1-x)(1-uvx)x^{[\mu_1-\mu_2]}}\cdot \\ \times\left(\frac{(uv)^{g-1+(d_1-d_2-1)/2}x}{(1-(uv)^{-1}x)(1-x)} - \frac{(uv)^{(3d_1-3d_2-1)/2-g}x}{(1-(uv)^{-1}x)^2} \\ - \frac{(uv)^{2g+1}x}{(1-(uv)^2x)^2} + \frac{(uv)^{d_1-d_2+2}x}{(1-(uv)^2x)(1-uvx)}\right)\right].$$

As $\mu_1 - \mu_2 > 2g-2$, let $m \geq 0$ such that $[\mu_1 - \mu_2] = 2g-2+m$. Introduce the following function

$$F(a,b,c) = \operatorname*{coeff}_{x^0}\left(\frac{(1+ux)^g(1+vx)^g x^{3-2g-m}}{(1-ax)^2(1-bx)(1-cx)}\right) = \mathrm{Res}_{x=0}\left(\frac{(1+ux)^g(1+vx)^g x^{2-2g-m}}{(1-ax)^2(1-bx)(1-cx)}\right),$$

where $a, b, c \neq 0$. So

$$e(\mathcal{N}_{\sigma_M^+}) = e(\mathcal{N}_{\sigma_m^+}) + \frac{(1+u)^{3g}(1+v)^{3g}}{(1-uv)^2}\Big((uv)^{3g-3+m}F(1,uv,(uv)^{-1}) - (uv)^{5g-5+3m} \\ \times F((uv)^{-1},1,uv) - (uv)^{2g+1}F((uv)^2,1,uv) + (uv)^{4g-1+2m}F(uv,1,(uv)^2)\Big) \quad (22)$$

using $d_1 - d_2 = 4g-3+2m$.

The function

$$G(x)=\frac{(1+ux)^g(1+vx)^g x^{2-2g-m}}{(1-ax)^2(1-bx)(1-cx)}$$

is a meromorphic function on $\mathbb{C}\cup\{\infty\}$ with poles at $x=0$, $x=1/a$, $x=1/b$ and $x=1/c$. Note that there is no pole at ∞. So

$$F(a,b,c)=-\mathrm{Res}_{x=1/a}G(x)-\mathrm{Res}_{x=1/b}G(x)-\mathrm{Res}_{x=1/c}G(x).$$

An easy calculation, using that

$$\begin{aligned}\mathrm{Res}_{x=1/a}G(x)&=\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=1/\mathrm{a}}\big(G(x)(x-1/a)^2\big).\\ \mathrm{Res}_{x=1/b}G(x)&=G(x)(x-1/\mathrm{b})|_{x=1/b},\\ \mathrm{Res}_{x=1/c}G(x)&=G(x)(x-1/\mathrm{c})|_{x=1/c},\end{aligned}$$

yields

$$\begin{aligned}F(a,b,c)=&\frac{a^{m-1}b(a+u)^g(a+v)^g}{(a-b)^2(c-a)}+\frac{a^{m-1}c(a+u)^g(a+v)^g}{(b-a)(c-a)^2}\\ &+\frac{b^m(b+u)^g(b+v)^g}{(a-b)^2(b-c)}+\frac{c^m(c+u)^g(c+v)^g}{(c-a)^2(c-b)}\\ &+\frac{a^{m-1}(a+u)^{g-1}(a+v)^{g-1}}{(a-b)(a-c)}(ga(2a+u+v)+(m-2)(a+u)(a+v)).\end{aligned}$$

Using this in (22) and Theorem 7.2, we have

$$\begin{aligned}(\mathcal{N}_{\sigma_M^+})=&\frac{(1+u)^{2g}(1+v)^{2g}}{(1-uv)^3(1-(uv)^2)^2}\Big[(1+u^2v)^{2g}(1+uv^2)^{2g}(1-(uv)^{4g+4m-2})\\ &+(1-2m-2g)(1+u^2v)^g(1+uv^2)^g(1+u)^g(1+v)^g(uv)^{3g+2m-2}(1-(uv)^2)\\ &+(1+u)^{2g}(1+v)^{2g}(1+uv)^2(uv)^{3g+m-2}\big((1-(uv)^{2g+2m})\\ &-(m+g)(1-uv)(1+(uv)^{2g+2m-1})\big)\\ &-g(1+u)^{2g-1}(1+v)^{2g-1}(1-(uv)^2)^2(uv)^{3g+m-2}(1-(uv)^{2g+2m-1})\Big].\end{aligned}$$

As $N=d_1-d_2-2g+2=2m+2g-1$, we get the formula in the statement.

COROLLARY 9.3 *Suppose that d_1 is even and d_2 is odd. Then the large σ moduli space $\mathcal{N}_{\sigma_M^+}=\mathcal{N}^s_{\sigma_M^+}$ is smooth and compact. If $\mu_1-\mu_2>2g-2$, its Hodge polynomial has the same formula as that of Theorem 9.2.*

Proof. Use the isomorphism $\mathcal{N}_\sigma(2,2,d_1,d_2)\cong\mathcal{N}_\sigma(2,2,-d_2,-d_1)$ together with Theorem 9.2.

COROLLARY 9.4 *Suppose that $d_1 + d_2$ is odd and $\mu_1 - \mu_2 > 2g - 2$. Then the Poincaré polynomial of $\mathcal{N}_{\sigma_M^+}$ is*

$$P_t(\mathcal{N}_{\sigma_M^+}) = \frac{(1+t)^{4g}}{(1-t^2)^3(1-t^4)^2}\Big[(1+t^3)^{4g}(1-t^{4N}) - N(1+t^3)^{2g}(1+t)^{2g}t^{2N+2g-2}(1-t^4)$$
$$+ (1+t)^{4g}(1+t^2)^2 t^{N+4g-3}\left((1-t^{2N+2}) - \frac{N+1}{2}(1-t^2)(1+t^{2N})\right)$$
$$- g(1+t)^{4g-2}(1-t^4)^2 t^{N+4g-3}(1-t^{2N})\Big],$$

where $N = d_1 - d_2 - 2g + 2$.

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