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# The topological nature of Krasnoselskii's cone fixed point theorem

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**Abstract**

In recent years, the Krasnoselskii fixed point theorem for cone maps and its many generalizations have been successfully applied to establish the existence of multiple solutions in the study of boundary value problems of various types. In this article we discuss the topological nature of the Krasnoselskii theorem and show that it can be restated in a more general form without reference to a cone structure or the norm of the underlying Banach space. This new perspective brings out a closer relation between the Krasnoselskii Theorem and the classical Brouwer fixed point theorem. It also points to some obvious extensions of the cone theorem.
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*Keywords:* Krasnoselskii; Cone fixed point theorem

## 1. Introduction

The classical Brouwer–Schauder fixed point theorem is an undeniably important tool in the study of the existence of solutions to mathematical problems (see e.g. [4,6,14]). In recent years, another fixed point theorem due to Krasnoselskii [7,8] and its generalizations have been successfully applied to obtain existence results for multiple positive solutions of various types of boundary value problems, notably in the case of ordinary differential equations and their discrete versions. Krasnoselskii himself [8] has applied his result to study the existence of periodic solutions of periodic systems of ordinary differential equations.

The main impetus for seeking new cone fixed point theorems is to apply them to obtain better criteria for the existence of solutions, for whatever problems the authors are currently interested in. Krasnoselskii's theorem has two parts (to be described in Section 2). The first part, called the compressive form, bears resemblance to the Brouwer–Schauder theorem. In fact, in [9], we show that the former is a special case of a generalized Brouwer–Schauder theorem. The second part, the expansive form, complements the compressive form. A proof of the expansive form from the compressive form has also been found, but it has not yet been published.

I believe that one of the reasons why the close relationship between the Krasnoselskii's theorem and the Brouwer–Schauder theorem has been overlooked is that the former is usually stated in the setting of a cone embedded in a Banach space with a given norm. In this setting, the norm functional plays a couple of important roles: in defining the region of points we are interested in, and in stating the properties of the images under the given map. When

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attempting to extend Krasnoselskii's theorem, one naturally focuses on finding similar functionals to replace the norm while still preserving these roles. On the other hand, the Brouwer–Schauder theorem is more topological in nature, being free from the concept of a metric. One can easily be misguided by this fact to think that the Brouwer–Schauder theorem is not adequate to deal with the metric aspects of cone maps.

The goal of this paper is to point out that Krasnoselskii's theorem can indeed be interpreted in a non-metric framework. The norm function is more of a convenience rather than a necessity. There are simpler ways to generalize the theorem without using functionals.

In Section 2, we state a simplified version of Krasnoselskii's theorem and discuss several generalizations, especially the Krasnoselskii–Benjamin theorem. In Section 3, we discuss the topological nature of the simplified Krasnoselskii theorem and show that it is equivalent to a fixed point theorem for cylinder maps. This perspective makes it clear how we can formulate a generalized expansive cone result, which incidentally reads more like a Brouwer-type theorem than a cone theorem.

Proofs of the results stated in this paper are omitted, but they can be found in the preprint [10].

## 2. Krasnoselskii's theorem

The excellent expository article by Amann [1] (Chapter 11) has a discussion and proof of the Krasnoselskii theorem, with the general boundary conditions (2.7) and (2.8). See also [5,10].

Let $X$ be a (finite- or infinite-dimensional) Banach space with a given norm $\|.\|$, and $K \subset X$ be a closed convex cone defined in the usual way, namely, that $K$ satisfies the following conditions.

(K1) If $x \in K$, then $\lambda x \in K$ for all real numbers $\lambda > 0$.
(K2) If $x, y \in K$, then $x + y \in K$.
(K3) If both $x$ and $-x \in K$, then $x = 0$.
(K4) $K$ is closed.

For visualization, we can use the special case where $X$ is the three-dimensional space $R^3$ with the Euclidean norm, and $K$ is an infinite circular cone with its vertex at the origin, or, even more simply, use the case where $X$ is the two-dimensional plane $R^2$ and $K$ is the wedge-shaped region AOB in Fig. 1 or 2.

A cone map on $K$ is a completely continuous map $T : K \to K$ (of $K$ into itself). When $X$ is finite-dimensional, any continuous map is completely continuous. A point $x \in K$ is a fixed point of $T$ if $T(x) = x$.

Let $0 < a < b$ be two given numbers. We are interested in conditions which guarantee that $T$ has a fixed point in the annular region $K(a, b) = \{x \in K : a \le \|x\| \le b\}$. Note that $K(a, b)$ is in general not convex, even though $K$ is. We denote by $K_a = \{x \in K : \|x\| = a\}$ and $K_b = \{x \in K : \|x\| = b\}$ the inner and outer boundaries, respectively, of $K(a, b)$. We can extend the notation to define $K(0, a)$ and $K(b, \infty)$ in the obvious way. Theorem 1 is a simplified version of Krasnoselskii's original theorem. An illustration of this result in dimension 2 is depicted in Figs. 1 and 2.

**Theorem 1** (*Krasnoselskii 1960 [7]*). *Let $K(a, b)$, $T$, $K_a$, and $K_b$ be as defined above.*

1. *(Compressive Form) $T$ has a fixed point in $K(a, b)$ if*

$$\|T(x)\| \ge \|x\| \quad \text{for all } x \in K_a, \tag{2.1}$$

*and*

$$\|T(x)\| \le \|x\| \quad \text{for all } x \in K_b. \tag{2.2}$$

2. *(Expansive Form) $T$ has a fixed point in $K(a, b)$ if*

$$\|T(x)\| \le \|x\| \quad \text{for all } x \in K_a, \tag{2.3}$$

*and*

$$\|T(x)\| \ge \|x\| \quad \text{for all } x \in K_b. \tag{2.4}$$

Note that conditions (2.1)–(2.4) are imposed only on points on the two curved boundaries of $K(a, b)$. Interior points and points on the sides of the cone can be moved in any direction (as long as the image remains inside $K$). Also it is not stipulated that any particular image point $T(x)$ must lie inside $K(a, b)$.

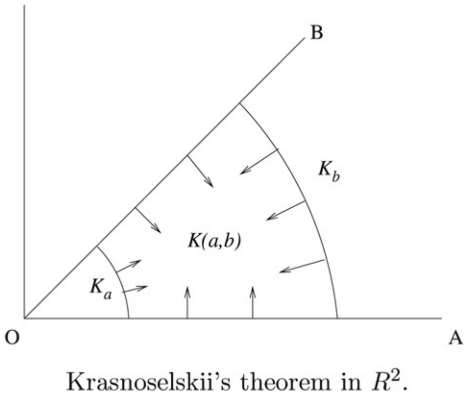


Fig. 1. Compressive form.

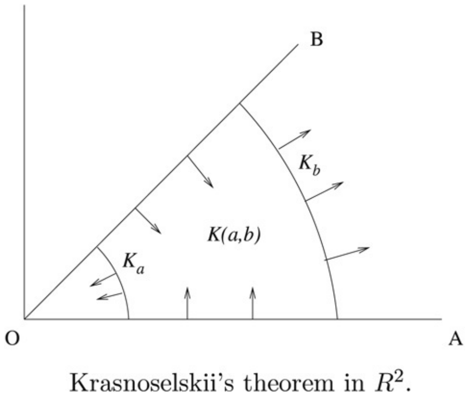


Fig. 2. Expansive form.

The adjectives "compressive" and "expansive" in the names of the two forms of the theorem are conventional, and they are not meant to correctly describe the behavior of $T$ under all circumstances. For instance, in the "compressive" case, it may happen that the inner boundary $K_a$ is pushed by $T$ far beyond the outer boundary $K_b$, resulting in a much large image $T(K(a, b))$ than $K(a, b)$.

When (2.1) (or (2.2)) holds, we say that $T$ is compressive on $K_a$ (or $K_b$) with respect to $K(a, b)$. The phrase "with respect to $K(a, b)$" may be omitted if it is obvious from the context. If the inequality in (2.1) (or (2.2)) is strict, we say that $T$ is strictly compressive on $K_a$ (or $K_b$). Likewise when (2.3) (or (2.4)) holds, $T$ is expansive on $K_a$ (or $K_b$), and $T$ is strictly expansive if the inequality in (2.3) (or (2.4)) is strict.

The conventional technique to apply the cone fixed point theorem to obtain existence results for a boundary value problem is to rewrite the problem as an integral equation, usually via the use of a Green's function. The Banach space is the space of continuous functions with an appropriate norm, and the positive cone is the set of continuous positive functions or some suitable subset of it. The integral operator is a completely continuous cone map and if one can find suitable constants $a$ and $b$ such that the hypotheses of the cone theorem are satisfied, then the annular region has a fixed point that is equivalent to a positive solution of the boundary value problem.

Many generalizations of Theorem 1 are known. The first direction of extension is to relax conditions (2.1)–(2.4). Krasnoselskii's original result is actually stated with weaker assumptions. In the compressive form, instead of (2.1) and (2.2), it is only required that

$$x - T(x) \notin K \quad \text{for all } x \in K_a, \tag{2.5}$$

and

$$T(x) - x \notin K \quad \text{for all } x \in K_b. \tag{2.6}$$

This allows part (but not all) of the inner boundary $K_a$ to be pushed nearer the origin, and part of the outer boundary $K_b$ to be pushed away from the origin. Similar conditions are used by Krasnoselskii in place of (2.3) and (2.4) in the expansive form.

In [1], it is shown that these conditions can be further weakened. Amann attributes this result to Benjamin [3] (also established later independently by Nussbaum [12]). More precisely, conditions (2.1) and (2.2) can be replaced by

$$K \setminus 0 \quad \text{such that } x - T(x) \neq \lambda p,\ \forall \lambda \geq 0,\ x \in K_a, \tag{2.7}$$

and

$$T(x) \neq \lambda x, \quad \text{for any } \lambda > 1,\ x \in K_b, \tag{2.8}$$

and conditions (2.3) and (2.4) can be replaced by

$$T(x) \neq \lambda x, \quad \text{for any } \lambda > 1,\ x \in K_a, \tag{2.9}$$

and

$$K \setminus 0 \quad \text{such that } x - T(x) \neq \lambda p,\ \forall \lambda \geq 0,\ x \in K_b. \tag{2.10}$$

In the literature (although not in [1]), condition (2.8) is called the Leray–Schauder condition. Petryshyn [13] has used it to extend the Brouwer–Schauder theorem. Some authors have thus referred to the above result as the Petryshyn–Krasnoselskii theorem. Following [1], we will refer to it as the Krasnoselskii–Benjamin theorem. A generalized Leray–Schauder condition is introduced in [9] to further extend Brouwer's theorem. In Section 4, we will show how this technique can also be used to extend the Krasnoselskii theorem.

Geometrically, (2.8) means that no point on $K_b$ is pushed by $T$ away from the origin "radially". In other words, pushing a point $x$ on $K_b$ above $K_b$ is allowed as long as the image point $T(x)$ is not collinear with $x$ and the origin. Geometrically, (2.7) means that no point on $K_a$ is pushed by $T$ towards the origin in a direction parallel to $p$; pushing it in the opposite direction away from the origin is allowed.

There is an apparent asymmetry in the pair of conditions (2.7) and (2.8), when compared to (2.1) and (2.2), or (2.5) and (2.6). An explanation will be given in Section 4 and the symmetry will be restored in our generalization of the Krasnoselskii–Benjamin result.

A second direction of extension is to look at regions more general than $K(a, b)$. A result due to Guo, see [5], replaces $K(a, b)$ in Theorem 1 by the more general region

$$J = K \cap (\overline{\Omega}_2 \setminus \Omega_1), \tag{2.11}$$

where $\Omega_1$ and $\Omega_2$ are two bounded open sets in $X$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $\overline{A}$ denotes the closure of a set $A$. We will also use $\partial A$ to denote the boundary of $A$. Conditions (2.1), (2.2) or (2.3), (2.4) are assumed to hold, but now for points on $K \cap \partial\Omega_1$ and $K \cap \partial\Omega_2$, instead of on $K_a$ and $K_b$, respectively. The hypotheses that $\Omega_1$ and $\Omega_2$ are open but otherwise arbitrarily means that we can apply the result to fairly general regions $J$. For instance, $J$ may contain holes. Most applications to differential equations, however, do not require such generalities. The new results in this paper are formulated for regions more general than $K(a, b)$, but not as general as in Guo's theorem.

The usual technique to obtain multiple solutions to a boundary value problem is to stack two or more annular regions together and apply the alternative forms of Krasnoselskii's theorem to each of the regions to get a fixed point. For example, take three positive numbers $0 < a < b < c$, and define the corresponding regions $K(a, b)$ and $K(b, c)$. Let us assume that (2.1) and (2.2) hold for $K_a$ and $K_b$, and (2.4) holds for $K_c$ (replace $b$ in (2.4) by $c$). Then there must be one fixed point in $K(a, b)$ and one fixed point in $K(b, c)$. There is a possibility that these two fixed points are one and the same. If so, it must lie on the common boundary $K_b$. In order to exclude this situation, we have to make the stronger assumption that $T$ maps $K_b$ strictly away from $K_b$, in other words, $T$ is strictly compressive on $K_b$ with respect to $K(a, b)$. In another example, if we assume that $T$ is strictly expansive on $K(a, b)$, and strictly compressive on $K(b, c)$, then we get at least three fixed points, one in each of $K(0, a)$, $K(a, b)$, and $K(b, c)$.

Therefore, a third way to extend the cone theorem is to look for more general ways to construct such stacked-annulus structures. For instance, one may use the same inner and outer boundaries $K_a$ and $K_c$ as in the example above, but replace $K_b$ by a set of points defined by some given continuous functional. Conditions (2.1)–(2.4) will, of course, have to be adjusted accordingly. Leggett and Williams [11] use a concave functional for this purpose. Avery [2] applies similar ideas to the boundaries $K_a$ and $K_c$, resulting in a five-functional theorem. In Section 4, we will see that stronger forms of both of these results are corollaries of our general result.

## 3. The topological nature of the fixed point property

Let $B$ denote the closed unit ball in the Banach space $X$. The Brouwer–Schauder theorem is often stated in the form:

*Any completely continuous map of $B$ into itself has a fixed point.*

However, it is well known that this result can be applied to much more general sets. Let $A$ be a subset of $X$ that is topologically isomorphic (homeomorphic) to $B$. There exists a one-to-one topological map $F$, such that $F(B) = A$. If $S : A \to A$ is a completely continuous map, then the composite map $F^{-1}SF : B \to B$ is a completely continuous map, so that there is a fixed point, $F^{-1}SF(x) = x$. It follows that, $F(x)$ is a fixed point of $S$.

Suppose $K$ is a bounded closed subset of $X$ with the following star-shaped properties: there exists an interior point $O$, which has a neighborhood contained inside $K$, and for every point $A$ on the boundary of $K$, the line segment $OA$ is contained in the interior of $K$, except the end-point $A$. Then it is obvious that $K$ is homeomorphic to the unit ball, via the topological map $F$ that scales every line $OA$ radially towards $O$ to be of unit length. Hence, the Brouwer–Schauder theorem holds for $K$.

It is obvious that any bounded closed convex set with a non-empty interior satisfies the above star-shaped property. Therefore, the Brouwer–Schauder theorem holds for any bounded closed convex set with a non-empty interior. (In fact, it can shown that this is true for any bounded closed convex set, but the weaker assertion suffices for our purpose in this paper.) In particular, this applies to the cylinder $C(0, 1)$ that is used in Theorem 3 below. The cylinder $C(0, 1)$ is defined as the cross product of the unit interval $[0, 1]$ and the unit ball $B^*$ in the reduced space of codimension 1, and is therefore convex.

An implication of the above observation is that the role played by the norm of the Banach space is not really that essential to the fixed point property (other than being used in the definition of bounded sets in $X$).

The same arguments can be applied to the Krasnoselskii theorem. We can topologically deform the cone $K$ and the annular region $K(a, b)$ in any way and still have a fixed point result. In the rest of this section we give two applications of this principle.

First let us deform $K(a, b)$ by moving every point on $K_a$ radially (and continuously) to a new point, while avoiding a neighborhood of the origin 0. Likewise we can move every point on $K_b$ radially and continuously, while keeping it strictly "greater" than the corresponding point on the deformed $K_a$. The set $K(a, b)$ is now transformed to a new set $L$, which we can think of as a "finite segment" of the cone $K$ with continuous boundaries. Let us define this transformation more precisely and apply the above principle to a generalization of Theorem 1.

For every point $p$ on $K_1$, the ray (the half-infinite straight line) coming out from the origin towards $p$ intersects $L$ in a finite line segment $[\theta(p)p, \phi(p)p]$, where $0 < \theta(p) < \phi(p)$ are real numbers that depend continuously on $p$. In addition, we assume that there exist a positive constant $\epsilon$ such that $\epsilon \le \theta(p)$ for all $p$. $L$ is bounded from below by the inner boundary $L_a = \{\theta(p)p : p \in K_1\}$ and from above by the outer boundary $L_b = \{\phi(p)p : p \in K_1\}$, and on the side by the side of $K$. We keep the subscript $a$ and $b$ in the notation $L_a$ and $L_b$ to remind us that they are analogs of $K_a$ and $K_b$ in Theorem 1. They should have been named $L_\theta$ and $L_\phi$ instead.

If we take $\theta$ to be the constant function $\theta(p) = a$ and $\phi$ to be the constant function $\phi(p) = b$, then $L$, $L_a$, and $L_b$ coincide with $K(a, b)$, $K_a$, and $K_b$ in the classical case, respectively.

Like $K(a, b)$, $L$ is in general not convex, but both of them are "radially convex" in the sense that if two points in $L$ are collinear with the origin 0, then the line segment joining the two points is contained in $L$.

We can extend the functions $\theta$ and $\phi$ to all $p \in K$, $p \neq 0$, by defining

$$\theta(p) = \theta\left(\frac{p}{\|p\|}\right), \quad \phi(p) = \phi\left(\frac{p}{\|p\|}\right). \tag{3.12}$$

The geometric meaning of these functions are: a point $p$ lies "above" $K_a$ if and only if $\theta(p) \le \|p\|$ and it lies "below" $K_b$ if and only if $\|p\| \le \phi(p)$.

**Theorem 2.** *Let $L$, $L_a$, $L_b$, $\theta$, and $\phi$ be as described above, and $T : L \to K$ be a completely continuous map.*

1. *(Compressive Form) T has a fixed point in L if*

$$\|T(x)\| \geq \theta(Tx) \quad \text{for all } x \in L_a, \tag{3.13}$$

*and*

$$\|T(x)\| \leq \phi(Tx) \quad \text{for all } x \in L_b. \tag{3.14}$$

2. *(Expansive Form) T has a fixed point in L if*

$$\|T(x)\| \leq \theta(Tx) \quad \text{for all } x \in L_a, \tag{3.15}$$

*and*

$$\|T(x)\| \geq \phi(Tx) \quad \text{for all } x \in L_b. \tag{3.16}$$

Following the conventions used by some authors, we can also restate the result using some functionals. Let $\alpha : K \to [0, \infty)$ and $\beta : K \to [0, \infty)$ be two continuous functionals defined on the cone $K$, such that

$$\alpha(x) \geq \beta(x) \quad \text{for all } x \in K. \tag{3.17}$$

We also require that they are strictly increasing in the radial direction, namely, that (the same holds for $\beta$)

$$\alpha(x) > 0 \quad \text{for } x \neq 0, \quad \text{and} \quad \alpha(\lambda x) > \alpha(x) \quad \text{if } \lambda > 1. \tag{3.18}$$

Let $0 < a < b$ be two real numbers. Then $L = \{x \in K : \alpha(x) \geq a, \beta(x) \leq b\}$ is a region as in Theorem 2 with boundaries $L_a = \{x \in K : \alpha(x) = a\}$ and $L_b = \{x \in K : \beta(x) = b\}$. Conditions (3.13)–(3.16) are then replaced by

$$\alpha(T(x)) \geq a \quad \text{for all } x \in L_a, \tag{3.19}$$

and

$$\beta(T(x)) \leq b \quad \text{for all } x \in L_b, \tag{3.20}$$

and

$$\alpha(T(x)) \leq a \quad \text{for all } x \in L_a, \tag{3.21}$$

and

$$\beta(T(x)) \leq b \quad \text{for all } x \in L_b. \tag{3.22}$$

In our second application, we deform the annular region $K(a, b)$ into a cylinder

$$C(0, 1) = \left\{(t, x^*) : 0 \leq t \leq 1, x^* \in B^*\right\}, \tag{3.23}$$

where $B^*$ is the unit ball in the reduced space of co-dimension 1. To see this, first note that every point $x$ in $K(a, b)$ has the spherical coordinate $(\|x\|, x/\|x\|)$. Hence, $K(a, b)$ is isomorphic to $[a, b] \times K_1$. Here $K_1$ is the intersection of the unit sphere of the Banach space with the convex cone $K$. It is not the entire unit sphere, but rather, a proper "convex subset" of the unit sphere. It is "convex" in the sense that given any two points in $K_1$, the spherical "straight line" joining these two points is contained in $K_1$. We can then map $[a, b]$ linearly onto $[0, 1]$ and deform $K_1$ to $B^*$. We can easily extend the isomorphism between $K(a, b)$ and $C$ to an isomorphism between $K$ and the half-infinite cylinder

$$C = \left\{(t, x^*) : -1 \leq t, x^* \in B^*\right\}. \tag{3.24}$$

Theorem 1 is thus equivalent to the next theorem, which is shown to follow from the classical Brouwer–Schauder theorem in an elementary way.We thus have a new proof of the Krasnoselskii theorem.

**Theorem 3.** *Let $T : C(0, 1) \to C$ be a completely continuous map, with the cylindrical coordinate representation*

$$T(x) = (s, y), \quad 0 \leq s < \infty, \ y \in B^*. \tag{3.25}$$

1. *(Compressive Form) T has a fixed point in* $C(0, 1)$ *if*

$$s \geq 0 \quad \text{for all } x = (0, x^*), \tag{3.26}$$

*and*

$$s \leq 1 \quad \text{for all } x = (1, x^*). \tag{3.27}$$

2. *(Expansive Form) T has a fixed point in* $C(0, 1)$ *if*

$$s \leq 0 \quad \text{for all } x = (0, x^*), \tag{3.28}$$

*and*

$$s \geq 1 \quad \text{for all } x = (1, x^*). \tag{3.29}$$

The fact that the expansive form can be reduced to the compressive form opens up another direction of extension. However, we will not pursue this matter further in this paper, other than giving the following example. Let $A = \left\{x = (x_i)_{i=1,\ldots,n} \in R^n : 0 \leq x_i \leq 1\right\}$ be the unit cube in $R^n$. For each $i$, there are two faces $\{x \in A : x_i = 0\}$ and $\{x \in A : x_i = 1\}$ and there is an obvious way to define the concept of a compressive or expansive map on these faces in the $i$th direction. Suppose that $T : A \to R^n$ is a continuous map that is either compressive or expansive in each $i$th direction. Then $T$ has a fixed point.

In the special case when $A$ is a square, we have the interesting result: let $T : A \to R^2$ be a continuous function on a square A such that $T$ maps the upper edge to points above itself, the lower edge to points below itself, the left edge to points to its left, and the right edge to points to its right, then $T$ has a fixed point.

It is now obvious how the same arguments can be used to obtain the following generalization of the expansive cone theorem. It does not read like a cone theorem, but it does imply the expansive Krasnoselskii theorem.

**Theorem 4.** *Let* $K_1, K_2, \ldots, K_{2n}$ *be a collection of* $2n$ *non-overlapping subsets of another subset* $K$ *of a Banach space. Assume that* $K$ *and* $K_i$ *are each isomorphic to the unit ball. Denote by* $K_i^o$ *the interior of each* $K_i$*, and*

$$L = K \setminus \bigcup_{i=1}^{2n} K_i^o. \tag{3.30}$$

*Let* $T : L \to K$ *be a completely continuous map such that*

$$T(\partial K_i) \subset K_i, \quad \text{for } i = 1, \ldots, 2n, \tag{3.31}$$

*where* $\partial K_i$ *denotes the boundary of* $K_i$*. Then* $T$ *has a fixed point in* $L$*.*

A simple example will be the unit ball $B$ with an even number of spherical holes inside it. Note that the result is false if there is only an odd number of holes.

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