Representation of Lie Groups and Special Functions

Volume 3: Classical and Quantum Groups and Special Functions

by

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'Et moi, ..., si j'avait su comment en revenir, je n'y serais point allé.'

Jules Verne

The series is divergent; therefore we may be able to do something with it.

O. Heaviside

One service mathematics has rendered the human race. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled 'discarded non-sense'.

Eric T. Bell

Mathematics is a tool for thought. A highly necessary tool in a world where both feedback and non-linearities abound. Similarly, all kinds of parts of mathematics serve as tools for other parts and for other sciences.

Applying a simple rewriting rule to the quote on the right above one finds such statements as: 'One service topology has rendered mathematical physics ...'; 'One service logic has rendered computer science ...'; 'One service category theory has rendered mathematics ...'. All arguably true. And all statements obtainable this way form part of the raison d'être of this series.

This series, *Mathematics and Its Applications*, started in 1977. Now that over one hundred volumes have appeared it seems opportune to reexamine its scope. At the time I wrote

> "Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the 'tree' of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related. Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as 'experimental mathematics', 'CFD', 'completely integrable systems', 'chaos, synergetics and large-scale order', which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics."

By and large, all this still applies today. It is still true that at first sight mathematics seems rather fragmented and that to find, see, and exploit the deeper underlying interrelations more effort is needed and so are books that can help mathematicians and scientists do so. Accordingly MIA will continue to try to make such books available.

If anything, the description I gave in 1977 is now an understatement. To the examples of interaction areas one should add string theory where Riemann surfaces, algebraic geometry, modular functions, knots, quantum field theory, Kac-Moody algebras, monstrous moonshine (and more) all come together. And to the examples of things which can be usefully applied let me add the topic 'finite geometry'; a combination of words which sounds like it might not even exist, let alone be applicable. And yet it is being applied: to statistics via designs, to radar/sonar detection arrays (via finite projective planes), and to bus connections of VLSI chips (via difference sets). There seems to be no part of (so-called pure) mathematics that is not in immediate danger of being applied. And, accordingly, the applied mathematician needs to be aware of much more. Besides analysis and numerics, the traditional workhorses, he may need all kinds of combinatorics, algebra, probability, and so on.

In addition, the applied scientist needs to cope increasingly with the nonlinear world and the extra mathematical sophistication that this requires. For that is where the rewards are. Linear models are honest and a bit sad and depressing: proportional efforts and results. It is in the non-linear world that infinitesimal inputs may result in macroscopic outputs (or vice versa). To appreciate what I am hinting at: if electronics were linear we would have no fun with transistors and computers; we would have no TV; in fact you would not be reading these lines.

There is also no safety in ignoring such outlandish things as nonstandard analysis, superspace and anticommuting integration, p-adic and ultrametric space. All three have applications in both electrical engineering and physics. Once, complex numbers were equally outlandish, but they frequently proved the shortest path between 'real' results. Similarly, the first two topics named have already provided a number of 'wormhole' paths. There is no telling where all this is leading - fortunately.

Thus the original scope of the series, which for various (sound) reasons now comprises five sub-series: white (Japan), yellow (China), red (USSR), blue (Eastern Europe), and green (everything else), still applies. It has been enlarged a bit to include books treating of the tools from one subdiscipline which are used in others. Thus the series still aims at books dealing with:

- a central concept which plays an important role in several different mathematical and/or scientific specialization areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have, and have had, on the development of another.

Special functions are - well - special. They turn up all over the place in both theoretical and practical investigations and their importance is well illustrated by the fact that scores of them have received special names. For instance, Bessel functions; Jacobi, Legendre, Gegenbauer, Laguerre polynomials, Hamkel and Macdonald functions; Whittaker functions; Krawtchouk and Meixner polynomials; Chebyshev polynomials; Hahn and Racah polynomials; etc.

Both the ubiquity and the special properties of these functions were something of a mystery until the great discovery of Wigner, Miller, and Vilenkin, one of the authors of the present volume, that, especially, these functions arise as the coefficients of representations of groups. This tied two apparently rather disparate parts of mathematics tightly together and enormously stimulated developments in both fields. Since then (the 1960's) very much has happened: for instance, orthogonal polynomials in several variables, discrete analogues of special functions, and, quite recently, the discovery that q-special functions relate to quantum groups (Hopf algebras) in the same way as special functions to (Lie) groups.

The authors have undertaken the monumental task to survey and describe in three volumes all that is known in this area. This is the third volume of this complete, self-contained, and encyclopaedic treatise.

The shortest path between two truths in the real domain passes through the complex domain.

J. Hadamard

La physique ne nous donne pas seulement l'occasion de résoudre des problèmes ... elle nous fait pressentir la solution.

H. Poincaré

Never lend books, for no one ever returns them; the only books I have in my library are books that other folk have lent me.

Anatole France

The function of an expert is not to be more right than other people, but to be wrong for more sophisticated reasons.

David Butler

Amsterdam, August 1991

Michiel Hazewinkel

Table of Contents

List of Special Symbols

$A_\delta(\Lambda)$	Bessel function of a matrix argument (Sec. 17.4.2).
$[a]_q \equiv [a]$	q-number (Sec. 14.1.1).
$(a;q)_n$	expression for definition of basic hypergeometric functions (Sec. 14.1.1).
$\mathfrak{A}_n(\mathbf{F})$	the set of antisymmetric $n \times n$ matrices (Sec. 15.2.1).
$B_\delta(Z)$	Bessel function of a matrix argument (Sec. 17.4.4).
$(b-x)_q^n$	q-binomial expression (Sec. 14.1.4).
$C_\mu^{(\delta)}(Q)$	Gegenbauer polynomial of a matrix argument (Sec. 17.4.10).
$C_n(\cos\theta;\beta \mid q)$	continuous q-ultraspherical polynomials (Sec. 14.7.3).
$C_q(\boldsymbol{\ell};\mathbf{j})$	Clebsch-Gordan coefficients of the quantum algebra $U_q(\mathfrak{sl}_2)$ (Sec. 14.3.3).
c_λ^Λ	string function (Sec. 19.8.1).
$\mathrm{ch}\ T_\Lambda$	character of a representation T_Λ of an affine Lie algebra (Sec. 19.5.4).
$\cos_q x$	q-analogue of the function $\cos x$ (Sec. 14.1.6).
$D(n,\mathbf{F})$	group of diagonal matrices (Sec. 15.1.1).
D_q	q-differentiation operator (Sec. 14.1.5).
$\tilde{D}_\Lambda^{\mathbf{k}}$	differentiation operator (Sec. 17.1.1).
E_{ij}	matrix with entries $(E_{ij})_{st} = \delta_{is}\delta_{jt}$ (Sec. 15.1.2).
$E_k(\tau)$	Eisenstein series (Sec. 19.1.5).
$E_q(x)$	q-exponential function (Sec. 14.1.6).
$\langle \mathbf{e}_i', \mathbf{e}_j'' \mid \mathbf{e}_k^r \rangle$	Clebsch-Gordan coefficient (Sec. 18.2.1).
$e_q(x)$	q-exponential function (Sec. 14.1.6).
$\mathrm{etr} \equiv \exp\mathrm{tr}$	exponential function of trace (Sec. 17.1.1).
${}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta};\Lambda)$	hypergeometric function of a matrix argument (Sec. 17.4.1).
$\int_a^b f(x)d_qx$	q-integral (Sec. 14.1.5).
$G_k(\Gamma)$	the set of holomorphic modular forms of weight k (Sec. 19.1.4).
$\hat{\mathfrak{g}}$	affine Lie algebra corresponding to a simple Lie algebra $\mathfrak{g}$ (Sec. 19.4.1).
$\hat{\mathfrak{g}}(A)$	affine Lie algebra corresponding to generalized Cartan matrix A (Sec. 19.4.7).
$\hat{\mathfrak{g}}(\mu)$	twisted affine Lie algebra (Sec. 19.4.5).
$H \equiv \{z \in \mathbb{C} \mid \mathrm{Im}\ z > 0\}$	upper half-plane (Sec. 19.1.1).
$H_n(x \mid q)$	continuous q-Hermite polynomial (Sec. 14.7.4).
$\mathfrak{H}_n(\mathbf{F})$	the set of Hermitian $n \times n$ matrices (Sec. 15.2.1).
$\mathfrak{H}_n^*(\mathbf{F})$	the set connected with $\mathfrak{H}_n(\mathbf{F})$ (Sec. 15.2.1).
I_p	the unit $p \times p$ matrix (Sec. 15.1.10).
I_{pq}	the matrix diag $(I_p, -I_q)$ (Sec. 15.1.10).

J_{2p}	the matrix $\begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$ (Sec. 15.1.10).
$K_m(s \mid A, B)$	Macdonald function of a matrix argument (Sec. 17.4.5).
$K_n(x; b, N \mid q)$	q-Krawtchouk polynomial (Sec. 14.6.1).
K_p	the matrix $\begin{pmatrix} I_p & iI_p \\ iI_p & I_p \end{pmatrix}$ (Sec. 15.2.2).
$\mathcal{K}(G, K)$	subring of the group ring of group G (Sec. 17.2.1).
$L(\mathfrak{g})$	loop algebra (Sec. 19.4.1).
$L_{\mathbf{k}}^{\gamma}$	generalized Laguerre polynomial (Sec. 17.4.8).
$L_{\mu}^{(\gamma)}(Z)$	Laguerre function of a matrix argument (Sec. 17.4.6).
$M(k; \mu; Z)$	Whittaker function of a matrix argument (Sec. 17.4.7).
M_{Λ}	Verma module (Sec. 19.5.2).
$\mathfrak{M}_{mn}(\mathbf{F}) \equiv \mathfrak{M}(m, n; \mathbf{F})$	the set of $m \times n$ matrices over $\mathbf{F}$ (Sec. 15.1.1).
$\mathfrak{M}_n(\mathbf{F}) \equiv \mathfrak{M}(n; \mathbf{F})$	the set of $n \times n$ matrices over $\mathbf{F}$ (Sec. 15.1.1).
$N_{\pm}(n, \mathbf{F})$	groups of nilpotent triangular matrices (Sec. 15.1.1).
$\begin{bmatrix} N \\ m \end{bmatrix}_q$	q-binomial coefficient (Sec. 14.1.3).
$P_n^{(\alpha,\beta)}(x \mid q)$	continuous q-Jacobi polynomial (Sec. 14.7.6).
$P_n^{(\alpha,\beta)}(x; c, d \mid q)$	big q-Jacobi polynomial (Sec. 14.7.6).
$P_{\mu}^{(\gamma,\delta)}(\Lambda)$	Jacobi polynomial of a matrix argument (Sec. 17.4.10).
$p_n(x; a, b \mid q)$	little q-Jacobi polynomial (Sec. 14.4.4).
$p_n(x; a, b, c, d \mid q)$	q-Askey-Wilson polynomial (Sec. 14.7.1).
$p_n^{\alpha\beta\gamma}(\mathbf{x})$	generalized Jacobi polynomial (Sec. 14.3.9).
$\mathfrak{P}_n(\mathbf{F})$	the set of positive definite $n \times n$ matrices over $\mathbf{F}$ (Sec. 15.2.1).
$\mathfrak{P}_{pq}(\mathbf{F})$	the set of Hermitian $(p+q) \times (p+q)$ matrices X over $\mathbf{F}$ such that $X = YI_{pq}Y^*$ for some $Y \in \mathfrak{M}_n(\mathbf{F})$, $p + q = n$ (Sec. 15.2.1).
$Q_n(x; a, b; N \mid q)$	q-Hahn polynomials (Sec. 14.6.1).
$R_n(\mu(x); \alpha, \beta, \gamma; \delta \mid q)$	q-Racah polynomials (Sec. 14.6.5).
$R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell)$	Racah coefficients of the quantum algebra $U_q(\mathfrak{sl}_2)$ (Sec. 14.5.1).
$\mathfrak{R}_1(p, q)$	Hermitian symmetric space $U(p,q)/(U(p) \times U(q))$ (Sec. 15.2.2).
$\mathfrak{R}_2(p)$	Hermitian symmetric space $Sp(p, \mathbf{R})/U(p)$ (Sec. 15.2.2).
$\mathfrak{R}_3(p)$	Hermitian symmetric space $SO^*(2p)/U(p)$ (Sec. 15.2.2).
$S_{\pm}(n, \mathbf{F})$	group of triangular matrices over $\mathbf{F}$ (Sec. 15.1.1).
$S_k(\Gamma)$	the set of cusp forms of weight k (Sec. 19.1.4).
$SL(2, \mathbb{Z})$	group of unimodular 2×2 matrices over $\mathbb{Z}$ (Sec. 19.1.2).
$SL_q(2, \mathbb{C})$	quantum group (Sec. 14.8.1).
$S_q^2(c, d)$	quantum 2-sphere (Sec. 14.9.1).
S_q^3	quantum 3-sphere (Sec. 14.9.7).

$SU_q(2)$	compact quantum group (Sec. 14.8.1).
$\sin_q x$	q-analogue of the function $\sin x$ (Sec. 14.1.6).
$\mathfrak{S}_n(\mathbf{F})$	the set of symmetric $m \times n$ matrices over $\mathbf{F}$ (Sec. 15.2.1).
$\mathfrak{S}_n^*(\mathbf{F})$	the set related to $\mathfrak{S}_n(\mathbf{F})$ (Sec. 15.2.4).
$U_q(\mathfrak{sl}_2)$	quantum algebra (Sec. 14.3.1).
$U_q(\mathfrak{su}_2)$	real form of $U_q(\mathfrak{sl}_2)$ (Sec. 14.3.1).
$\left(\frac{u}{v}\right)$	Legendre-Jacobi symbol (Sec. 19.2.2).
$V_m(\mathbf{x})$	Vandermonde determinant (Sec. 17.2.4).
$V_m(\mathbf{f}, \mathbf{x})$	generalized Vandermonde determinant (Sec. 17.2.4).
$Z_{\mathbf{k}}(\Lambda)$	zonal spherical polynomial (Sec. 17.3.4).
$\mathrm{B}_m(\boldsymbol{\alpha}, \boldsymbol{\beta})$	multidimensional beta-function (Sec. 17.1.5).
$\Gamma_m(\boldsymbol{\alpha}, \mathbf{F}) \equiv \Gamma_m(\boldsymbol{\alpha})$	multidimensional gamma-function (Sec. 17.1.4).
$\Gamma_q(x)$	q-analogue of gamma-function (Sec. 14.1.7).
$\Delta(\Lambda)$	determinant of a matrix Λ (Sec. 17.1.1).
$\Delta^{\mathbf{a}}(\Lambda)$	power function of a matrix argument (Sec. 17.1.1).
$\theta(z, \tau)$	Jacobi theta function (Sec. 19.1.2).
$\theta(\mathbf{z}, \Omega)$	theta function of many variables (Sec. 19.2.6).
$\theta_{a,b}(z, \tau)$	theta function with characteristic (Sec. 19.2.5).
$\theta \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} (\mathbf{z}, \Omega)$	theta functions of many variables with characteristic (Sec. 19.2.9).
θ_μ^B	Hecke modular form (Sec. 19.8.5).
Θ^L, Θ_μ^L, $\Theta_{\mu,m}^L$	theta functions (Sec. 19.7.1).
$\varphi_\Lambda^{KL}(g)$	intertwining functions (Sec. 17.3.8).
$\varphi_T(g)$	zonal spherical function of a representation T (Sec. 17.2.2).

Chapter 14.
Quantum Groups, q-Orthogonal Polynomials and Basic Hypergeometric Functions

As we have seen in Chapter 13, representations of Chevalley groups lead to q-orthogonal polynomials with positive integral q. Basic hypergeometric functions and q-orthogonal polynomials for arbitrary (including complex) values of q are connected with quantum algebras and groups. They are considered in the present Chapter.

14.1. q-Analysis and Basic Hypergeometric Functions

14.1.1. q-Factorials. Some relations connected with q-analysis and with basic hypergeometric functions were examined in Chapter 13. Here we consider results of q-analysis in details.

As we have seen in Section 13.2.2, in the theory of basic hypergeometric functions the expressions

$$(a;q)_n = \prod_{j=1}^{n}(1-aq^{j-1}), \quad a \in \mathbb{C}, \quad n \in \mathbb{Z}_+,$$

where q is a fixed complex number and $(a,q)_0 = 1$, are utilized instead of ordinary factorials $n!$. If $|q| < 1$, then we may consider the expression

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{j=1}^{\infty}(1-aq^{j-1}).$$

It can be shown that $(a;q)_\infty$ is an analytic function of a. It is clear that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{1}$$

The formulas

$$(q;q)_{N+r} = (q;q)_N(q^{N+1};q)_r, \tag{2}$$

$$(q;q)_{N-r} = \frac{(-1)^r(q;q)_r}{(q^{-N};q)_r} q^{r(r-2N-1)/4}, \tag{3}$$

$$(q^{-1};q^{-1})_r = (-1)^r q^{-r(r+1)/2}(q;q)_r \tag{4}$$

are useful for passing from basic hypergeometric functions to series and backwards. They can be proved with the help of the expression

$$(q;q)_n = \prod_{j=1}^{n}(1-q^j)$$

for $(q;q)_n$.

The expressions

$$[a]_q \equiv [a] = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}} = q^{-(a-1)/2} \frac{1-q^a}{1-q}, \tag{5}$$

called *q-numbers*, are used in the theory of representations of quantum groups and algebras. One directly verifies that

$$[m] = q^{(m-1)/2} + q^{(m-3)/2} + \ldots + q^{-(m-1)/2},$$

$$[m+n] = q^{n/2}[m] + q^{-m/2}[n] = q^{-n/2}[m] + q^{m/2}[n], \tag{6}$$

$$[m-n] = q^{n/2}[m] - q^{m/2}[n] = q^{-n/2}[m] - q^{-m/2}[n], \tag{7}$$

$$[2m+1][2n+1] = [2|m-n|+1] + [(2|m-n|+1)+1] + \ldots + [2(m+n)+1], \tag{8}$$

$$[n_1][n_2+m] - [n_1+m][n_2] = [n_1-n_2][m] \tag{9}$$

for m, n, n_1, $n_2 \in \mathbb{Z}_+$. If $m \in \mathbb{Z}_+$, then *q-factorials* $[m]!$ are defined by the formulas

$$[m]! = [1][2]\ldots[m] = \frac{q^{-m(m-1)/4}}{(1-q)^m}(q;q)_m, \quad m \neq 0, \tag{10}$$

$$[0]! = 1.$$

Below we shall make use of the relations

$$[N+r]! = [N]! \frac{(q^{N+1};q)_r}{(1-q)^r} q^{-r(r+2N-1)/4}, \tag{11}$$

$$[N-r]! = [N]!(-1)^r \frac{(1-q)^r}{(q^{-N};q)_r} q^{r(r-2N-3)/4}, \tag{12}$$

$$(q^{-1};q^{-1})_r = [r]!(-1)^r(1-q)^r q^{-r(r+3)/4} \tag{13}$$

which are corollaries of formulas (2)–(4). By means of relations (10)–(12) one derives the following asymptotic relations for $N \to \infty$:

$$\frac{[N+r]!}{[N]!} \sim [N+1]^r q^{-r(r-1)/4}, \quad \text{where} \quad |q| < 1, \tag{14}$$

$$\frac{[N-r]!}{[N]!} \sim [N]^{-r} q^{r(r-1)/4}, \quad \text{where} \quad |q| > 1. \tag{15}$$

We also note the limit relations

$$\lim_{r\to\infty} \frac{[r+\alpha]!}{[r+\beta]!} = q^{-(\alpha-\beta)/2}, \quad \text{where} \quad |q| < 1, \tag{16}$$

$$\lim_{r\to\infty}\frac{[r+\alpha]!}{[r+\beta]!}=q^{(\alpha-\beta)/2},\qquad \text{where}\qquad |q|>1. \tag{17}$$

14.1.2. The function ${}_1\varphi_0(a;q,x)$. According to formula (14) of Section 13.2.2, for the basic hypergeometric function ${}_1\varphi_0(a;q,x)$ we have the expression

$${}_1\varphi_0(a;q,x)=\sum_{j=0}^{\infty}\frac{(a;q)_j}{(q;q)_j}x^j. \tag{1}$$

Since

$${}_1\varphi_0(a;q,x)=1+\frac{1-a}{1-q}x+\frac{(1-a)(1-qa)}{(1-q)(1-q^2)}x^2+\dots$$

and

$${}_1\varphi_0(a;q,qx)=1+\frac{1-a}{1-q}qx+\frac{(1-a)(1-qa)}{(1-q)(1-q^2)}q^2x^2+\dots\ ,$$

then

$${}_1\varphi_0(a;q,x)-\ {}_1\varphi_0(a;q,qx)=(1-a)x\ {}_1\varphi_0(aq;q,x) \tag{2}$$

and

$${}_1\varphi_0(a;q,x)-a\ {}_1\varphi_0(a;q,qx)=(1-a)\ {}_1\varphi_0(aq;q,x). \tag{3}$$

From formulas (2) and (3) we derive the recurrence relation

$$(1-x)\ {}_1\varphi_0(a;q,x)=(1-ax)\ {}_1\varphi_0(a;q,qx), \tag{4}$$

which leads to

$${}_1\varphi_0(a;q,x)=\frac{(ax;q)_n}{(x;q)_n}\ {}_1\varphi_0(a;q,q^nx). \tag{5}$$

If $|q|<1$, then for $n\to\infty$ we have ${}_1\varphi_0(a;q,q^nx)\to 1$. Therefore, from relation (5) we obtain

$${}_1\varphi_0(a;q,x)=\sum_{j=0}^{\infty}\frac{(a;q)_j}{(q;q)_j}x^j=\prod_{r=0}^{\infty}\frac{(1-axq^r)}{(1-xq^r)}=\frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{6}$$

where $|q|<1$. This formula is called *Heine's theorem.* It implies some interesting corollaries. For example,

$${}_1\varphi_0(a;q,x)\ {}_1\varphi_0(b;q,ax)=\ {}_1\varphi_0(ab;q,x). \tag{7}$$

Formula (6) yields ${}_1\varphi_0(1;q,x)=1$. Therefore, we obtain from (7) that

$${}_1\varphi_0(a^{-1};q,ax)=\{\ {}_1\varphi_0(a;q,x)\}^{-1}. \tag{7'}$$

Another corollary of formula (6) is the relation

$$ {}_2\varphi_1(a,b;c;q,t) = \frac{(b;q)_\infty (at;q)_\infty}{(c;q)_\infty (t;q)_\infty}\, {}_2\varphi_1\left(\frac{c}{b}, t; at; q, b\right), \tag{8} $$

where $|q|<1$, $|t|<1$, $|b|<1$. Really, using formula (1) of Section 14.1.1 and then relation (7) we have

$$ \begin{aligned} {}_2\varphi_1(a,b;c;q,t) &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n}\, t^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty}\, t^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}\, t^n\, \frac{(c/b;q)_m b^m q^{nm}}{(q;q)_m} \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m b^m}{(q;q)_m} \frac{(atq^m;q)_\infty}{(tq^m;q)_\infty} \\ &= \frac{(b;q)_\infty (at;q)_\infty}{(c;q)_\infty (t;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m (t;q)_m}{(at;q)_m (q;q)_m}\, b^m . \end{aligned} $$

This leads to (8).

The q-analog of formula (7) of Section 3.5.3

$$ {}_2\varphi_1\left(a,b;c;q,\frac{c}{ab}\right) = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}, \tag{9} $$

where $|c|<|ab|$, $|q|<1$, can be derived from relation (8). Indeed, using (8) and then formula (6) we have

$$ \begin{aligned} {}_2\varphi_1\left(a,b;c;q,\frac{c}{ab}\right) &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} \left(\frac{c}{ab}\right)^n \\ &= \frac{(b;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/ab;q)_n}{(q;q)_n}\, b^n \\ &= \frac{(b;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty} \frac{(c/a;q)_\infty}{(b;q)_\infty} \\ &= \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}. \end{aligned} $$

If $b = q^{-n}$, then formula (9) can be written as

$$ {}_2\varphi_1\left(a, q^{-n}; c; q, \frac{c}{a} q^n\right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{10} $$

14.1.3. Expressions for $(a;q)_n$, $(a;q)_n^{-1}$ and their corollaries. Putting $a = 0$ into formula (6) of Section 14.1.2 we obtain the expression for ${}_0\varphi_0(q, x)$:

$$ {}_0\varphi_0(q,x) \equiv 1 + \sum_{n=1}^{\infty} \frac{x^n}{(q;q)_n} = \prod_{n=0}^{\infty} (1 - xq^n)^{-1} = (x;q)_\infty^{-1}. \qquad (1) $$

In particular,

$$ (q;q)_\infty^{-1} = {}_0\varphi_0(q,q). \qquad (1') $$

We now replace x in formula (6) of Section 14.1.2 by x/a and tend a to infinity. The result is

$$ (x;q)_\infty \equiv \prod_{n=0}^{\infty} (1 - xq^n) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n q^{n(n-1)/2}}{(q;q)_n}. \qquad (2) $$

In particular,

$$ (q;q)_\infty = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n}. \qquad (2') $$

In formulas (1)–(2′) $|x| < 1$ and $|q| < 1$.

By setting $a = \alpha^{-1}$, $b = \beta^{-1}$ and $c = x$ into formula (9) of Section 14.1.2, we find

$$ 1 + \sum_{n=1}^{\infty} \frac{(\alpha - 1)(\alpha - q)(\alpha - q^{n-1})(\beta - 1)(\beta - q)(\beta - q^{n-1})x^n}{(q;q)_n (x;q)_n} $$
$$ = \frac{(x\alpha;q)_\infty (x\beta;q)_\infty}{(x;q)_\infty (x\alpha\beta;q)_\infty}. $$

We put $\alpha = \beta = 0$ here. As a result we obtain *Cauchy's formula*

$$ 1 + \sum_{n=1}^{\infty} \frac{q^{n^2-n} x^n}{(x;q)_n (q;q)_n} = \prod_{m=0}^{\infty} (1 - xq^m)^{-1} = (x;q)_\infty^{-1}. \qquad (3) $$

For $x = q$ it turns into *Euler's formula*

$$ 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \prod_{m=1}^{\infty} (1 - q^m)^{-1} = (q;q)_\infty^{-1}. \qquad (4) $$

In (3) and (4) $|q| < 1$.

We now prove that

$$ (x;q)_N = \sum_{j=1}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j x^j q^{j(j-1)/2}, \qquad (5) $$

where *q-binomial coefficients* are given by the formula

$$\begin{bmatrix} N \\ j \end{bmatrix}_q = \frac{(q;q)_N}{(q;q)_j(q;q)_{N-j}} = \frac{[N]!q^{(N-j)j/2}}{[j]![N-j]!}.$$

For this we make use of formulas (1) of Section 14.1.1 and (6) of Section 14.1.2:

$$\begin{aligned}(x;q)_N &= \frac{(x;q)_\infty}{(xq^N;q)_\infty} = \sum_{j=0}^{\infty} \frac{(q^{-N};q)_j x^j q^{jN}}{(q;q)_j} \\ &= \sum_{j=0}^{\infty} \frac{(1-q^{-N})(1-q^{-N+1})\dots(1-q^{-N+j-1})x^j q^{jN}}{(q;q)_j} \\ &= \sum_{j=0}^{N} \frac{(-1)^j q^{-jN+j(j-1)/2}(1-q^N)(1-q^{N-1})\dots(1-q^{N-j+1})x^j q^{jN}}{(q;q)_j} \\ &= \sum_{j=0}^{N} \frac{(q;q)_N}{(q;q)_j(q;q)_{N-j}} (-1)^j x^j q^{j(j-1)/2}.\end{aligned}$$

This leads to relation (5).

In the similar way we obtain

$$\begin{aligned}(x;q)_N^{-1} &= \frac{(xq^N;q)_\infty}{(x;q)_\infty} = \sum_{j=0}^{\infty} \frac{(q^N;q)_j}{(q;q)_j} x^j \\ &= \sum_{j=0}^{\infty} \frac{(q;q)_{N+j-1}}{(q;q)_j(q;q)_{N-j}} x^j\end{aligned}$$

and this gives the equality

$$(x;q)_N^{-1} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix}_q x^j. \tag{6}$$

In order to derive the following relation for $(q;q)_\infty$ we prove the equality

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{n=0}^{\infty} (1-q^{2n+2})(1+xq^{2n+1})(1+x^{-1}q^{2n+1}), \tag{7}$$

where $x \in \mathbb{C}$, $x \neq 0$, $|q| < 1$. We note that if $|x| > |q|$ and $|q| < 1$, then formula (2) implies that

$$\prod_{n=0}^{\infty} (1+xq^{2n+1}) = \sum_{m=0}^{\infty} \frac{x^m q^{m^2}}{(q^2;q^2)_m}.$$

Taking into account equality (1) of Section 14.1.1 and the fact that $(q^{2m+2};q^2)_\infty = 0$ for negative m, we obtain

$$\begin{aligned}\prod_{n=0}^{\infty}(1+xq^{2n+1}) &= (q^2;q^2)_\infty^{-1}\sum_{m=0}^{\infty} q^{m^2}(q^{2m+2};q^2)_\infty x^m \\ &= (q^2;q^2)_\infty^{-1}\sum_{m=-\infty}^{\infty} q^{m^2}(q^{2m+2};q^2)_\infty x^m \\ &= (q^2;q^2)_\infty^{-1}\sum_{m=-\infty}^{\infty} q^{m^2}x^m \sum_{r=0}^{\infty}\frac{(-1)^r q^{r^2+2mr+r}}{(q^2;q^2)_r} \\ &= (q^2;q^2)_\infty^{-1}\sum_{r=0}^{\infty}\frac{(-1)^r x^{-r}q^r}{(q^2;q^2)_r}\sum_{m=-\infty}^{\infty} q^{(m+r)^2}x^{m+r} \\ &= (q^2;q^2)_\infty^{-1}\sum_{r=0}^{\infty}\frac{(-q/x)^r}{(q^2;q^2)_r}\sum_{m=-\infty}^{\infty} q^{m^2}x^m \\ &= (q^2;q^2)_\infty^{-1}(-q/x;q^2)_\infty^{-1}\sum_{m=-\infty}^{\infty} q^{m^2}x^m .\end{aligned}$$

If $|x| > |q|$ and $|q| < 1$, then the series converge absolutely. After transferring $(q^2;q^2)_\infty(-q/x;q^2)_\infty$ to the left hand side and continuing analytically in x we obtain formula (7).

Equality (7) for $|q| < 1$ implies the relations

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2}q^{-jn} &= \sum_{n=0}^{\infty}(-1)^n q^{(2k+1)n(n+1)/2}q^{-jn}(1-q^{(2n+1)j}) \\ &= \prod_{n=0}^{\infty}(1-q^{(2k+1)(n+1)})(1-q^{(2k+1)n+j})(1-q^{(2k+1)(n+1)+j}).\end{aligned} \tag{8}$$

Setting $k=j=1$ here we derive that

$$\begin{aligned}(q;q)_\infty = \prod_{n=1}^{\infty}(1-q)^n &= \prod_{n=0}^{\infty}(1-q^{3n+3})(1-q^{3n+1})(1-q^{3n+2}) \\ &= (q^3;q^3)_\infty(q;q^3)_\infty(q^2;q^3)_\infty,\end{aligned} \tag{9}$$

$$\begin{aligned}(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n) &= 1+\sum_{m=1}^{\infty}(-1)^m q^{m(3m-1)}(1+q^m) \\ &= \sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m-1)/2}.\end{aligned} \tag{10}$$

Putting $x = -1$ into (7) we obtain the relation

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q^2;q^2)_\infty (q;q^2)_\infty (q;q^2)_\infty.$$

Using the definition of $(a;q)_\infty$ one can easily verify that

$$(q^2;q^2)_\infty = (q;q)_\infty(-q;q)_\infty,$$

$$(q^2;q^2)_\infty(q;q^2)_\infty = (q;q)_\infty.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty(q^2;q^2)_\infty(q;q^2)_\infty}{(q;q)_\infty(-q;q)_\infty} = \frac{(q;q)_\infty}{(-q;q)_\infty}.$$

Consequently,

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{m=1}^{\infty} \frac{1-q^m}{1+q^m}. \tag{11}$$

With the help of similar arguments we obtain

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{1}{2}(q;q)_\infty(-q;q)_\infty(-1;q)_\infty$$
$$= (q;q)_\infty(-q;q)_\infty(-q;q)_\infty = (q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

that is,

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty} \frac{1-q^{2m}}{1-q^{2m-1}}. \tag{12}$$

14.1.4. q-Analog of the binomial formula. Let $n \in \mathbb{Z}_+$. By the q-analog of binomial formula one means the relation

$$(1-x)_q^n = (1-q^{-1}x)(1-q^{-2}x)\dots(1-q^{-n}x) = (q^{-n}x;q)_n. \tag{1}$$

Using relation (5) of Section 14.1.3 we obtain that

$$(1-x)_q^n = \sum_{r=0}^{\infty} \begin{bmatrix} n \\ r \end{bmatrix}_q (-x)^r q^{r(r-2n-1)/2}. \tag{2}$$

Let us apply formula (3) of Section 14.1.1 and then formula (1) of Section 14.1.2 to this relation. We derive

$$(1-x)_q^n = \sum_{r=0}^{n} \frac{(q^{-n};q)_r}{(q;q)_r} x^r = {}_1\varphi_0(q^{-n};q,x). \tag{3}$$

Starting from this equality, for $|q| < 1$ one sets that

$$(1-x)_q^a = {}_1\varphi_0(q^{-a};q,x) \tag{4}$$

for arbitrary $a \in \mathbb{C}$.

For $(b-x)_q^n$ we have

$$\begin{aligned}(b-x)_q^n &= (b-q^{-1}x)(b-q^{-2}x)\dots(b-q^{-n}x)\\ &= b^n\left(1-\frac{x}{b}\right)_q^n = \sum_{r=0}^{n}\frac{(q^{-n};q)_r}{(q;q)_r} x^r b^{n-r}.\end{aligned} \tag{5}$$

It follows from formula (7) of Section 14.1.2 that

$$(1-x)_q^n(1-q^{-n}x)_q^m = (1-x)_q^{n+m}. \tag{6}$$

14.1.5. q-Differentiation and q-integration. For a fixed real or complex q the *q-differentiation operator* D_q is defined by the formula

$$D_q f(x) = \frac{f(x)-f(qx)}{x-qx}. \tag{1}$$

If $q \to 1$, then $D_q \to d/dx$. Applying the Taylor formula to the right hand side of (1) we obtain the following expression for the q-differentiation operator:

$$D_q f(x) = \sum_{n=0}^{\infty} \frac{(q-1)^n}{(n+1)!} x^n \frac{d^{n+1}}{dx^{n+1}} f(x), \tag{2}$$

provided that the expression on the right hand side exists.

One directly verifies that

$$\begin{aligned} D_q x^a &= [[a]]x^{a-1}, \quad a \in \mathbb{C},\\ D_q(1-bx)_q^a &= b[[a]](1-bqx)_q^{a-1}, \quad a,b \in \mathbb{C},\end{aligned} \tag{3}$$

where

$$[[a]] = \frac{1-q^a}{1-q} = q^{(a-1)/2}[a].$$

Applying D_q term by term to the series for the basic hypergeometric function ${}_n\varphi_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; q, x)$ and taking into account equality (3) we derive

$$D_q\ {}_n\varphi_{n-1}(a_1, \dots, a_n; b_1, \dots, b_{n-1}; q, x) = \frac{(1-a_1)\dots(1-a_n)}{(1-b_1)\dots(1-b_{n-1})(1-q)} \times {}_n\varphi_{n-1}(a_1q, \dots, a_nq; b_1q, \dots, b_{n-1}q; q, x). \tag{4}$$

Instead of ${}_n\varphi_{n-1}$ we shall often make use of the function

$${}_n\Phi_{n-1}(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n-1}; q, x) = {}_n\varphi_{n-1}(q^{\alpha_1}, \dots, q^{\alpha_n}; q^{\beta_1}, \dots, q^{\beta_{n-1}}; q, x). \tag{5}$$

For ${}_n\Phi_{n-1}$ formula (4) is rewritten as

$$D_q\ {}_n\Phi_{n-1}(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n-1}; q, x) = \frac{(1-q^{\alpha_1})\dots(1-q^{\alpha_n})}{(1-q^{\beta_1})\dots(1-q^{\beta_{n-1}})(1-q)} \times\ {}_n\Phi_{n-1}(\alpha_1+1, \dots, \alpha_n+1; \beta_1+1, \dots, \beta_{n-1}+1; q, x). \tag{6}$$

For the product $f_1(x)f_2(x)$ of functions we have

$$D_q\{f_1(x)f_2(x)\} = \frac{f_1(x)f_2(x) - f_1(qx)f_2(qx)}{x(1-q)}.$$

The right hand side can be represented in the form

$$\frac{f_2(x)\{f_1(x) - f_1(qx)\} + f_1(qx)\{f_2(x) - f_2(qx)\}}{x(1-q)}.$$

Therefore, the q-analog of the formula for differentiation by parts is of the form

$$D_q\{f_1(x)f_2(x)\} = f_2(x)D_qf_1(x) + f_1(qx)D_qf_2(x). \tag{7}$$

In the similar way one derives that

$$D_q\frac{f_1(x)}{f_2(x)} = \frac{f_2(x)D_qf_1(x) - f_1(x)D_qf_2(x)}{f_2(x)f_2(qx)}. \tag{8}$$

Using the method of mathematical induction we show that

$$D_q^n f(x) = (q-1)^{-n}x^{-n}q^{-n(n-1)/2} \times \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (-1)^m q^{m(m-1)/2} f(q^{n-m}x). \tag{9}$$

In particular,

$$D_q^2 f(x) = (q-1)^{-2} x^{-2} q^{-1} \{f(q^2 x) - (1+q) f(qx) + q f(x)\}. \tag{10}$$

q-Integration is the inverse operation to q-differentiation. If $D_q F(x) = f(x)$, then for $0 < q < 1$ we have

$$F(x) - F(0) = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x),$$

$$F(\infty) - F(x) = (q-1)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x).$$

Therefore, when $0 < q < 1$ the *q-integral* of a function $f(x)$ is defined by the formulas

$$\int_0^c f(x) d_q x = c(1-q) \sum_{j=0}^{\infty} q^j f(q^j c) = \sum_{r=0}^{\infty} (x_r - x_{r+1}) f(x_r), \tag{11}$$

where $x_r = cq^r$,

$$\int_c^{\infty} f(x) d_q x = c(1-q) \sum_{j=1}^{\infty} q^{-j} f(q^{-j} c). \tag{12}$$

For positive a and b we set

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Putting $c = 1$ into (11) and (12) we obtain the definition of the integral over the infinite interval

$$\int_0^{\infty} f(x) d_q x = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j). \tag{13}$$

It is easy to show that

$$\int_a^b f(sx) d_q x = \frac{1}{s} \int_{sa}^{sb} f(x) d_q x. \tag{14}$$

The formula for q-integration by parts is derived by q-integration of both sides of (7):

$$\int f_2(x) \{D_q f_1(x)\} d_q x = f_1(x) f_2(x) - \int f_1(qx) \{D_q f_2(x)\} d_q x. \tag{15}$$

The equality

$$f(x) = f(a) + \frac{(x-a)_q^1}{[[1]]!} D_q f(a) + \frac{(x-a)_q^2}{[[2]]!} D_q^2 f(a) + \dots , \tag{16}$$

where $(x-a)_q^n$ is understood in the sense of formula (5) of Section 14.1.4, $[[m]]!$ is the same as in (3) and

$$D_q^n f(a) = \{D_q^n f(x)\}_{x=a} ,$$

is the q-analog of the classical Taylor series

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots .$$

Formula (16) is valid for functions $f(x)$, representable in the form of a convergent series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

14.1.6. q-Analogs of the exponential and of the trigonometrical functions. The formula

$$E_q(x) = \sum_{r=0}^{\infty} \frac{x^r}{[[r]]!} \tag{1}$$

(see formula (3) of the previous section) defines the q-analog of the exponential function. When $|q| < 1$ this series converges uniformly and absolutely for $|x| < (1-q)^{-1}$ and diverges for $|x| > (1-q)^{-1}$. When $|q| > 1$, the convergence takes place for all finite x. For $|q| < 1$ the series in (1) can be written in the form $((1-q)x;q)_\infty^{-1}$. Thus, in this case

$$E_q(x) = ((1-q)x;q)_\infty^{-1} = \prod_{n=0}^{\infty} (1-(1-q)xq^j)^{-1}. \tag{2}$$

q-Differentiation of formula (1) term by term gives that

$$D_q E_q(ax) \equiv \frac{E_q(ax) - E_q(aqx)}{x - qx} = aE_q(ax). \tag{3}$$

Moreover, $E_q(ax)$ is a single function satisfying this equation and the initial condition $E_q(0) = 1$.

The function $E_q(x)$ does not have the semigroup property $E_q(x)E_q(y) = E_q(x+y)$. One can show that

$$E_q(x+y)E_q^{-1}(x)E_q^{-1}(y) = \prod_{n=0}^{\infty} \left(1 + \frac{q^{2n}(1-q)^2 xy}{1 - q^n(1-q)(x+y)}\right).$$

If follows directly from (1) that

$$E_{1/q}(x) = \sum_{r=0}^{\infty} \frac{x^r q^{r(r-1)/2}}{[[r]]!} . \qquad (1')$$

Multiplying out the series for $E_q(x)$ and $E_{1/q}(y)$ and then grouping corresponding terms we obtain

$$E_q(x)E_{1/q}(y) = 1 + \frac{(x+y)_q^1}{[[1]]!} + \frac{(x+y)_q^2}{[[2]]!} + \cdots .$$

Thus,

$$E_q(x)E_{1/q}(-x) = 1, \qquad (4)$$

that is,

$$E_{1/q}(-x) = E_q^{-1}(x) = ((1-q)x;q)_\infty . \qquad (5)$$

Let us note that if the operator D_q is defined by formula (3), that is $D_q f(x) = \{f(x) - f(qx)\}/(x - qx)$, then

$$D_q E_{1/q}(ax) = aE_{1/q}(qax). \qquad (6)$$

With the help of the q-exponential function one introduces the *q-trigonometrical functions*

$$\sin_q x = \frac{1}{2i}\{E_q(ix) - E_q(-ix)\}, \qquad (7)$$

$$\cos_q x = \frac{1}{2}\{E_q(ix) + E_q(-ix)\}. \qquad (8)$$

Consequently,

$$\sin_q x = x \sum_{r=0}^{\infty} \frac{(-x^2)^r}{[[2r+1]]!}, \qquad (9)$$

$$\cos_q x = \sum_{r=0}^{\infty} \frac{(-x^2)^r}{[[2r]]!}. \qquad (10)$$

When $q \to 1$, we have $\sin_q x \to \sin x$, $\cos_q x \to \cos x$. By means of formulas (4), (7) and (8) one directly verifies that

$$\cos_q x \cos_{1/q} x + \sin_q x \sin_{1/q} x = 1.$$

We also have

$$D_q \sin_q x = \cos_q x, \quad D_q \cos_q x = -\sin_q x.$$

The functions $\cos_q(\omega x)$ and $\sin_q(\omega x)$ are linearly independent solutions of the q-differential equation

$$D_q^2 f + \omega^2 f = 0.$$

The functions $\cos_{1/q}(\omega x)$ and $\sin_{1/q}(\omega x)$ are solutions of the equation

$$D_q^2 f(x) + \omega^2 f(q^2 x) = 0.$$

14.1.7. q-Analogs of the gamma- and beta-functions. The q-gamma function $\Gamma_q(\nu)$ is defined by the formula

$$\int_0^\infty x^{\nu-1} E_q(-x) d_q x = q^{-\nu(\nu-1)/2} \Gamma_q(\nu), \tag{1}$$

where Re $\nu > 1$ and the integral is understood in the sense of formula (13) of Section 14.1.5. Fulfilling q-integration by parts we have

$$\int_0^\infty x^{\nu-1} E_q(-x) d_q x = \frac{x^\nu}{[[\nu]]} E_q(-x) \Big|_0^\infty + \frac{q^\nu}{[[\nu]]} \int_0^\infty x^\nu E_q(-x) d_q x. \tag{2}$$

It is easy to derive from the equality $E_q(-x) E_{1/q}(x) = 1$ that $E_q(-x) \to 0$ when $x \to +\infty$. Therefore, the first summand on the right hand side of (2) vanishes and we obtain

$$\Gamma_q(\nu + 1) = [[\nu]] \Gamma_q(\nu). \tag{3}$$

In particular, if $\nu = n \in \mathbb{Z}_+$, then

$$\Gamma_q(n) = [[n-1]]! = \frac{(q;q)_{n-1}}{(1-q)^{n-1}}. \tag{3'}$$

When $q \to 1$ we have $\Gamma_q(\nu) \to \Gamma(\nu)$, where $\Gamma(\nu)$ is the ordinary gamma-function.

For $|q| < 1$ the q-gamma-function is represented as

$$\Gamma_q(\nu) = (1-q)^{1-\nu} \frac{\prod_{k=0}^\infty (1 - q^{k+1})}{\prod_{k=0}^\infty (1 - q^{\nu+k})} = (1-q)^{1-\nu} \frac{(q;q)_\infty}{(q^\nu;q)_\infty}. \tag{4}$$

For $|q| > 1$ we have

$$\Gamma_q(\nu) = q^{(\nu-1)(\nu-2)/2} \Gamma_{1/q}(\nu). \tag{5}$$

For studying Clebsch-Gordan coefficients of the quantum group $SU_q(2)$ it is convenient to use the modified q-gamma-function defined by the formula

$$\hat{\Gamma}_q(\nu) = q^{-(\nu-1)(\nu-2)/4}\Gamma_q(\nu). \tag{6}$$

This function has the properties

$$\hat{\Gamma}_q(\nu) = \hat{\Gamma}_{1/q}(\nu),$$

$$\hat{\Gamma}_q(\nu+1) = [\nu]\hat{\Gamma}_q(\nu) \equiv \frac{q^{\nu/2} - q^{-\nu/2}}{q^{1/2} - q^{-1/2}}\hat{\Gamma}_q(\nu), \tag{7}$$

$$\hat{\Gamma}_q(n) = [n-1]!, \quad n \in \mathbb{Z}_+. \tag{8}$$

By means of formula (7) one shows that if $\nu \to -m$, $m \in \mathbb{Z}_+$, then

$$\frac{\hat{\Gamma}_q(\nu+1)}{\hat{\Gamma}_q(\nu)} \to -\frac{\hat{\Gamma}_q(m+1)}{\hat{\Gamma}_q(m)}.$$

Consequently, for $\nu \to -m$, $m \in \mathbb{Z}_+$, and for $n \in \mathbb{Z}_+$ we have

$$\frac{\hat{\Gamma}_q(\nu+1)}{\hat{\Gamma}_q(\nu-n+1)} \to (-1)^n \frac{\hat{\Gamma}_q(m+n)}{\hat{\Gamma}_q(m)}. \tag{9}$$

The analog of the Stirling formula for the q-gamma-function is of the form

$$\hat{\Gamma}_q(\nu+1) \underset{\nu\to+\infty}{\sim} q^{-\nu(\nu-1)/4}(1-q)^{-\nu}\exp(-C_q), \tag{10}$$

where $0 < q < 1$ and

$$C_q = -\sum_{k=0}^{\infty} \ln(1-q^{k+1}).$$

For the quotient of q-gamma-functions one has the formula

$$\frac{\hat{\Gamma}_q(\nu+a)}{\hat{\Gamma}_q(\nu)} \underset{\nu\to+\infty}{\sim} q^{-a(a+2\nu-3)/4}(1-q)^{-a}. \tag{10'}$$

The q-beta-function $\mathrm{B}_q(\mu,\nu)$ is defined with the help of the q-integral:

$$\mathrm{B}_q(\mu,\nu) = \int_0^1 x^{\mu-1}(1-qx)_q^{\nu-1}d_qx. \tag{11}$$

In order to express $B_q(\mu,\nu)$ in terms of the q-gamma-function we calculate some q-integrals. For $0<q<1$ we have

$$\int_0^1 x^{\mu-1} d_q x = (1-q)\sum_{n=0}^{\infty} q^{n\mu} = \frac{1-q}{1-q^\mu}.$$

Let us prove by mathematical induction that

$$\int_0^1 x^{\mu-1}(xq;q)_n d_q x = \frac{(1-q)(q;q)_n}{(q^\mu;q)_{n+1}}. \tag{12}$$

We have

$$\int_0^1 x^{\mu-1}(xq;q)_{n+1} d_q x = \int_0^1 x^{\mu-1}(xq;q)_n d_q x - q^{n+1}\int_0^1 x^{\mu}(xq;q)_n d_q x$$
$$= \frac{(1-q)(q;q)_n}{(q^\mu;q)_{n+1}} - \frac{q^{n+1}(1-q)(q;q)_{n+1}}{(q^{\mu+1};q)_{n+1}} = \frac{(1-q)(q;q)_{n+1}}{(q^\mu;q)_{n+2}}.$$

Formula (12) can be rewritten in the form

$$\int_0^1 x^{\mu-1}\frac{(xq;q)_\infty}{(xq^n;q)_\infty} d_q x = \frac{(q^{n+\mu};q)_\infty (q;q)_\infty (1-q)}{(q^\mu;q)_\infty (q^n;q)_\infty}.$$

Replacing n by ν we see that both sides of this equality are analytic in $z=q^\nu$ inside the unit disc and coincide for $z=q,q^2,\ldots$. Since $q^n\to 0$ when $n\to\infty$, then we conclude that the equality

$$\int_0^1 x^{\mu-1}\frac{(xq;q)_\infty}{(xq^\nu;q)_\infty} d_q x = \frac{(1-q)(q^{\mu+\nu};q)_\infty (q;q)_\infty}{(q^\mu;q)_\infty (q^\nu;q)_\infty}, \tag{13}$$

where $\mathrm{Re}\,\mu\geq 0$, $y\neq 0,-1,-2,\ldots$, is valid. It can be written as

$$\sum_{n=0}^{\infty} q^{n\mu}\frac{(q^{n+1};q)_\infty}{(q^{n+\nu};q)_\infty} = \frac{(q^{\mu+\nu};q)_\infty (q;q)_\infty}{(q^\mu;q)_\infty (q^\nu;q)_\infty}. \tag{14}$$

It is clear that the left hand side of (13) coincides with the left hand side of (14), and that the right hand side coincides with $\Gamma_q(\mu)\Gamma_q(\nu)/\Gamma_q(\mu+\nu)$, that is, we have

$$B_q(\mu,\nu) = \frac{\Gamma_q(\mu)\Gamma_q(\nu)}{\Gamma_q(\mu+\nu)}. \tag{15}$$

We also note the values of three other integrals (see [10]):

$$\int_{-c}^{d} \frac{(qx/d;q)_\infty(-qx/c;q)_\infty}{(q^\alpha x/d;q)_\infty(-q^\beta x/d;q)_\infty} d_q x$$
$$= \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}(d+c)\frac{(-dq/c;q)_\infty(-cq/d;q)_\infty}{(-dq^\beta/c;q)_\infty(-cq^\alpha/d;q)_\infty}, \quad (16)$$

$$\int_0^\infty \frac{x^{\alpha-1}(-cq^{\alpha+\beta}x;q)_\infty}{(-cx;q)_\infty} d_q x = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \frac{(-cq^\alpha;q)_\infty(-q^{1-\alpha}/c;q)_\infty}{(-c;q)_\infty(-q/c;q)_\infty}, \quad (17)$$

$$\int_{-\infty}^{\infty} \frac{(cq^\alpha x;q)_\infty(-dq^\beta x;q)_\infty}{(cx;q)_\infty(-dx;q)_\infty} d_q x = \frac{2\Gamma_q(\alpha+\beta-1)}{\Gamma_q(\alpha)\Gamma_q(\beta)}$$
$$\times \frac{(-cq^\alpha/d;q)_\infty(-dq^\beta/c;q)_\infty(cd;q)_\infty(q/cd;q)_\infty(q^2;q^2)_\infty^2}{(c^2;q^2)_\infty(q^\alpha/c^2;q^2)_\infty(d^2;q^2)_\infty(q^2/d^2;q^2)_\infty}. \quad (18)$$

14.1.8. Properties of the basic hypergeometric function ${}_2\varphi_1$. The relation

$${}_2\varphi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z,q)_\infty}\, {}_2\varphi_1\left(\frac{c}{a},\frac{c}{b};c;q,\frac{ab}{c}z\right) \quad (1)$$

is an analog of formula (5) of Section 3.5.3 for the function ${}_2\varphi_1$. If $a=q^{-n}$, $n\in\mathbb{Z}_+$, then the relation

$${}_2\varphi_1(q^{-n},b;c;q,z) = (-1)^n q^{-n(n+1)/2}\frac{(b;q)_n}{(c;q)_n} z^n$$
$$\times\, {}_2\varphi_1\left(q^{-n},q^{-n+1}c^{-1};q^{-n+1}b^{-1};q,\frac{cq^{n+1}}{bz}\right) \quad (2)$$

is valid.

It follows from formulas (10) of Section 14.1.2 and (2) that

$${}_2\varphi_1(q^{-n},a;c;q,q) = \frac{(c/a;q)_n}{(c;q)_n}a^n. \quad (2')$$

Using term by term q-differentiation we find that

$$D_q\{x^a\, {}_2\varphi_1(a,b;c;q,x)\} = (1-a)x^{a-1}\, {}_2\varphi_1(aq,b;c;q,x), \quad (3)$$
$$D_q\{x^{c-1}\, {}_2\varphi_1(a,b;c;q,x)\} = (2-c)x^c\, {}_2\varphi_1(a,b;cq^{-1};q,x). \quad (4)$$

With the help of formulas (1) and (4) one derives that

$$D_q^n \left\{ x^{c+n-1} {}_2\varphi_1\left(\frac{c}{a}, \frac{c}{b}; cq^n; q, x\right)\right\}$$
$$= (c;q)_n x^{c-1} {}_2\varphi_1\left(\frac{c}{a}, \frac{c}{b}; c; q, x\right)$$
$$= \frac{(x;q)_n x^{c-1}}{{}_1\varphi_0(c/ab; q, xq^{a+b-c})} {}_2\varphi_1(a, b; c; q, xq^{a+b-c}).$$

Consequently,

$$D_q^n \left\{ \frac{x^{c+n-1} {}_2\varphi_1(aq^n, bq^n; cq^n; q, xq^{a+b-c+n})}{{}_1\varphi_0(cq^{-n}/ab; q, xq^{a+b-c+n})} \right\}$$
$$= \frac{(c;q)_n x^{c-1} {}_2\varphi_1(a, b; c; xq^{a+b-c})}{{}_1\varphi_0(c/ab; q, xq^{a+b-c})}. \tag{5}$$

If $a = q^{-n}$, then (5) implies the *q-analog of the Rodrigues formula*

$${}_1\varphi_0\left(\frac{cq^n}{b}; q, xq^{b-c-n}\right) D_q^n \frac{x^{c+n-1}}{{}_1\varphi_0(c/b; q, xq^{b-c})}$$
$$= (c;q)_n x^{c-1} {}_2\varphi_1(q^{-n}; b; c; q, xq^{b-c-n}). \tag{6}$$

Comparing coefficients at the same powers of the variable z one proves the recurrence formulas

$$(1-a)\, {}_2\varphi_1(qa, b; c; z) = {}_2\varphi_1(a, b; c; z) - a\, {}_2\varphi_1(a, b; c; qz), \tag{7}$$
$$(1-b)\, {}_2\varphi_1(a, qb; c; z) = {}_2\varphi_1(a, b; c; z) - b\, {}_2\varphi_1(a, b; c; qz), \tag{8}$$
$$(1-cq^{-1})\, {}_2\varphi_1(a, b; q^{-1}c; z) = {}_2\varphi_1(a, b; c; z) - cq^{-1}\, {}_2\varphi_1(a, b; c; qz), \tag{9}$$
$$\frac{z(1-a)(1-b)}{1-c}\, {}_2\varphi_1(qa, qb; qc; z) = {}_2\varphi_1(a, b; c; z) - {}_2\varphi_1(a, b; c; qz), \tag{10}$$
$$\frac{c(q-c) - z(ac - bc - ab - abq)}{c-1}\, {}_2\varphi_1(a, b; c; z) + (q-c)(1-c)(c-abz)$$
$$\times {}_2\varphi_1(a, b; q^{-1}c; z) - (a-c)(b-c)z\, {}_2\varphi_1(a, b; qc; z) = 0 \tag{11}$$

(for brevity we have omitted q in ${}_2\varphi_1(\alpha, \beta; \gamma; q, x)$). By combinations of relations (7)–(10) we obtain other relations for ${}_2\varphi_1$. With the help of the function

$${}_2\Phi_1(\alpha, \beta; \gamma; q, x) \equiv {}_2\varphi_1(q^\alpha, q^\beta; q^\gamma; q, z) \tag{11'}$$

they can be written in the form

$$[b-a]\, {}_2\Phi_1(a, b; c; z) + q^{b/2}[a]\, {}_2\Phi_1(a+1, b; c; z)$$
$$- q^{a/2}[b]\, {}_2\Phi_1(a, b+1; c; z) = 0, \tag{12}$$

$$\begin{aligned}&[b-a][c-1]\,{}_2\Phi_1(a,b;c-1;z)+[a][c-b-1]\,{}_2\Phi_1(a+1,b;c;z)\\&\qquad-[b][c-a-1]\,{}_2\Phi_1(a,b+1;c;z)=0,\end{aligned} \tag{13}$$

$$\begin{aligned}&{}_2\Phi_1(a,b-1;c;z)-{}_2\Phi_1(a-1,b;c;z)\\&\qquad=q^{(a+b-c-2)/2}\frac{[b-a]}{[c]}z\,{}_2\Phi_1(a,b;c+1;z)\end{aligned} \tag{14}$$

(here we also have omitted q). These relations will be useful for derivation of expressions for matrix elements of representations of the quantum algebra $U_q(\mathfrak{su}_2)$.

The q-analog of integral representation (13) of Section 3.5.2 for the ordinary hypergeometric function has the form

$$\begin{aligned}{}_2\varphi_1(a,b;c;q,z)&=\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)}\\&\quad\times\int_0^1 x^{a-1}(1-qx)_q^{c-a-1}(1-q^bzx)_q^{-b}d_qx,\end{aligned} \tag{15}$$

where Re $(c-a)>0$, Re $a>0$ and by $(1-y)_q^a$ one means a q-binomial expression (see Section 14.1.4). To prove this formula we have to replace $(1-q^bzx)_q^{-b}$ by its expression in the form of a series and carry out term by term q-integration according to formula (11) of Section 14.1.7. If we replace $(1-q^bzx)_q^{-b}$ on the right hand side by the basic hypergeometric function ${}_n\varphi_{n-1}(a_1,\dots,a_n;b_1,\dots,b_{n-1};q,zx)$ expressed in the form of the series, then we have

$$\begin{aligned}{}_{n+1}\varphi_n(a_1,\dots,a_n,a;b_1,\dots,b_{n-1},c;q,z)&=\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(c-a)}\\\times\int_0^1 x^{a-1}(1-qx)_q^{c-a-1}\;{}_n\varphi_{n-1}(a_1,\dots,a_n;b_1,\dots,b_{n-1};q,zx)d_qx.\end{aligned} \tag{16}$$

The hypergeometric function ${}_2\varphi_1(a,b;c;q,x)$ satisfies the second order q-differential equation

$$x(q^c-q^{a+b+1}x)D_q^2y+\{[[c]]-(q^a[[b+1]]+q^b[[a]])x\}D_qy-[[a]][[b]]y=0, \tag{17}$$

where, as above, $[[a]]=(1-q^a)/(1-q)$. Another solution of this equation is the function

$$y=x^{1-c}\,{}_2\varphi_1\left(\frac{aq}{c},\frac{bq}{c};\frac{q^2}{c};q,x\right). \tag{18}$$

The q-differential equation for ${}_2\varphi_1(a,b;c;q,x)$ can be represented in the form

$$(c-abqx)y(q^2x)-\{c+q-(a+b)qx\}y(qx)-q(1-x)y(x)=0. \tag{19}$$

14.2. Hopf Algebras, Their Representations and Corepresentations

14.2.1. Introduction. Investigation of nonlinear equations and development of the quantum method of the inverse scattering problem have led to appearance of new types of algebras. For example, the paper [182] contains the algebra U_h with generators H, E_+, E_- and the commutation relations

$$[H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = \frac{\sinh hH}{\sinh h}, \tag{1}$$

where h is fixed complex number interpreted as the Plank constant, and the article [44], devoted to the Liouville quantum model on a lattice, contains the algebra A_q generated by elements a, b, c, d and the relations

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \tag{2}$$

$$ad - da = (q - q^{-1})bc, \tag{2'}$$

where q is a non-zero complex number.

For $h \to 0$ relations (1) transform into the commutation relations for the generators of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Therefore, U_h can be regarded as a deformation (a quantization) of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. If $q \to 1$, then relations (2) and (2′) tend to the relations for elements of the matrix $g \equiv \begin{pmatrix} a'(g) & b'(g) \\ c'(g) & d'(g) \end{pmatrix}$ of the group $SL(2, \mathbb{C})$. Therefore, relations (2) and (2′) can be understood as a deformation $SL_q(2, \mathbb{C})$ of $SL(2, \mathbb{C})$. But elements of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

of this deformed group are non-commuting. Moreover, there is no dependence of a b, c, d on parameters. Matrices (3) have multiplicative property. Namely, if elements of the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ satisfy conditions (2) and (2′), then elements of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

also have these properties if elements of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ commute.

The attempts of investigation of these examples have led to new mathematical notions: quantum algebras and quantum groups. A quantum group is not a group in ordinary sense of this word. It is defined in terms of the algebra A_q of functions on it. Quantum algebras and algebras of functions on quantum groups are Hopf algebras.[1] Hopf algebras are very abstract objects. In order to understand the origin of the definition of Hopf algebras we shall consider at first an algebra of functions on an ordinary group G. This algebra is a commutative (and cocommutative if G is a commutative group) Hopf algebra.

14.2.2. Algebra of functions on a group. Let G be a group. In G we have a multiplication $(g_1, g_2) \to g_1 g_2$ (that is, a mapping $G \times G \to G$), an operation of taking the inverse $g \to g^{-1}$ (that is, a mapping $G \to G$), and a unit element e. Let $\mathcal{A} = \text{Fun}\, G$ be some complex associative algebra of functions on G with a unit element. The multiplication $(f_1, f_2) \to f_1 f_2$ and the unit I in $\mathcal{A}$ are assumed to be defined by the formula

$$(f_1, f_2)(g) = f_1(g) f_2(g), \quad I(g) \equiv 1. \tag{1}$$

The multiplication is a mapping $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. The algebra $\mathcal{A}$ is commutative.

The group operations allow us to introduce other operations on $\mathcal{A}$, namely,

1) the comultiplication $\Delta: \mathcal{A} \equiv \text{Fun}\, G \to \text{Fun}\,(G \times G)$,
2) the counit $\varepsilon: \mathcal{A} \to \mathbb{C}$,
3) the antipode $S: \mathcal{A} \to \mathcal{A}$.

They are defined by the formulas

$$(\Delta f)(g_1, g_2) = f(g_1 g_2), \quad g_1, g_2 \in G, \tag{2}$$

$$\varepsilon(f) = f(e), \tag{3}$$

$$(Sf)(g) = f(g^{-1}), \quad g \in G. \tag{4}$$

It is easy to verify that the mappings Δ, ε and S are homomorphisms of algebras.

The multiplication and the comultiplication on $\mathcal{A}$ can be defined by means of tensor product. Namely, if G is a finite group or a Lie group, then let $\text{Fun}\, G$ be the algebra of polynomials on G: $\text{Fun}\, G = \text{Pol}\, G$. Then $\text{Pol}\,(G \times G) = \text{Pol}\, G \otimes \text{Pol}\, G \equiv \mathcal{A} \otimes \mathcal{A}$. Elements of $\mathcal{A} \otimes \mathcal{A}$ are finite linear combinations $\sum_i f_i \otimes f_i'$. The multiplication in $\mathcal{A} \otimes \mathcal{A}$ is defined by the formula $(f_1 \otimes f_2)(h_1 \otimes h_2) = f_1 h_1 \otimes f_2 h_2$ and the unit element coincides with $I \otimes I$. In this case the multiplication m in $\mathcal{A}$ can be understood as a homomorphism from $\mathcal{A} \otimes \mathcal{A}$ into $\mathcal{A}$:

$$\left(m \sum_i f_i \otimes f_i' \right)(g) = \sum_i f_i(g) f_i'(g). \tag{5}$$

The comultiplication Δ can be understood as a mapping from $\mathcal{A}$ into $\mathcal{A} \otimes \mathcal{A}$. In the same way one defines m and Δ in the case when $\text{Fun}\,(G \times G) = \text{Fun}\, G \otimes \text{Fun}\, G$,

[1] In the papers [315–317] algebras of functions on quantum groups are defined as C^*-algebras.

where Fun $G \otimes$ Fun G means the completed tensor product (as, for example, in the case $\mathfrak{L}^2(G \times G) = \mathfrak{L}^2(G) \otimes \mathfrak{L}^2(G)$).

The group properties lead to some properties of the homomorphisms Δ, ε and S. Namely, the associativity $(g_1 g_2) g_3 = g_1(g_2 g_3)$ of the group operation implies the equality

$$f((g_1 g_2) g_3) = f(g_1(g_2 g_3)).$$

By means of the comultiplication we can represent the left hand side of this relation in the form

$$f((g_1 g_2) g_3) = \{[(\Delta \otimes \mathrm{id}) \circ \Delta] f\}(g_1, g_2, g_3) \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$$

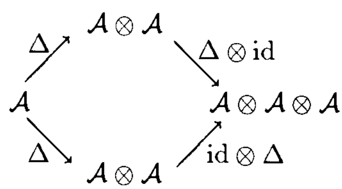
and the right hand side in the form

$$f(g_1(g_2 g_3)) = \{[(\mathrm{id} \otimes \Delta) \circ \Delta] f\}(g_1, g_2, g_3) \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A},$$

where id is the identity mapping of $\mathcal{A}$ onto $\mathcal{A}$. Therefore, the comultiplication satisfies the equality

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \tag{6}$$

which can be written in the form of the commutative diagram



(7)

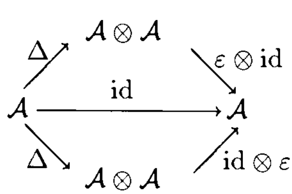
Since for every $g \in G$ we have $eg = ge = g$, then

$$f(eg) = f(ge) = f(g), \quad f \in \mathcal{A}.$$

This implies the following property of the counit ε:

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} \tag{8}$$

which can be written in the form of the commutative diagram



(9)

Since $g^{-1} g = g g^{-1} = e$, then

$$f(g^{-1} g) = f(g g^{-1}) = f(e). \tag{10}$$

Since $(\Delta f)(g_1, g_2) = f(g_1 g_2)$, then

$$\{[(S \otimes \mathrm{id}) \circ \Delta] f\}(g_1, g_2) = f(g_1^{-1} g_2).$$

For the multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ we have $(mF)(g) = F(g, g)$, $F \in \mathcal{A} \otimes \mathcal{A}$. Therefore,

$$\{[m \circ (S \otimes \mathrm{id}) \circ \Delta] f\}(g) = \{[(S \otimes \mathrm{id}) \circ \Delta] f\}(g, g) = f(g^{-1} g). \tag{11}$$

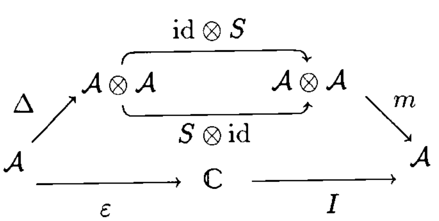
In the same way we obtain that

$$\{[m \circ (\mathrm{id} \otimes S) \circ \Delta] f\}(g) = f(g g^{-1}). \tag{12}$$

Since $f(e) = \varepsilon(f) I(g)$, then formulas (10)–(12) yield the relation

$$(m \circ (S \otimes \mathrm{id}) \circ \Delta)(f) = (m \circ (\mathrm{id} \otimes S) \circ \Delta)(f) = \varepsilon(f) I(g) \tag{13}$$

which can be written in the form of the commutative diagram



(14)

If G is a commutative group, then $f(g_1 g_2) = f(g_2 g_1)$, $f \in \mathcal{A}$. We define the linear mapping $\sigma: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by the formula $\sigma(f_1 \otimes f_2) = f_2 \otimes f_1$. From $f(g_1 g_2) = f(g_2 g_1)$ we obtain the commutativity property

$$\sigma \circ \Delta = \Delta. \tag{15}$$

14.2.3. Definition of a Hopf algebra. Let A_c be a linear space. It is called a *coalgebra* if A_c is equipped with a linear mapping $\Delta: A_c \to A_c \otimes A_c$, called a *comultiplication*, and with a linear mapping $\varepsilon: A_c \to \mathbb{C}$, called a *counit*, such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \quad (\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}. \tag{1}$$

The first relation is called the *coassociativity property*. A coalgebra A_c is said to be *cocommutative* if $\sigma \circ \Delta = \Delta$ where σ is a linear mapping from $A_c \otimes A_c$ onto $A_c \otimes A_c$ such that $\sigma(a \otimes b) = b \otimes a$, $\quad a, b \in A_c$.

Let now A be an associative algebra with unit I. The multiplication $(a_1, a_2) \to a_1 a_2$ in A is linearly extended to a mapping $m: A \otimes A \to A$. Then m is a homomorphism of $A \otimes A$ into A. By means of the unit we introduce the homomorphism

$\epsilon: \mathbb{C} \to A$ acting according to the formula $\epsilon(\alpha) = \alpha I$, $\alpha \in \mathbb{C}$. The algebra A is called a *bialgebra* if A is equipped with the coalgebra structure (that is, if a comultiplication Δ and a counit ε are defined in A), and if Δ and ε are homomorphisms of A into $A \otimes A$ and of A into $\mathbb{C}$ respectively.

A bialgebra A is said to be a *Hopf algebra* if it is equipped with a linear mapping $S: A \to A$ (called *antipode*) and if the equalities

$$(m \circ (S \otimes \mathrm{id}) \circ \Delta)(a) = (m \circ (\mathrm{id} \otimes S) \circ \Delta)(a) = \varepsilon(a)I, \quad a \in A, \tag{2}$$

hold. In general, $S^2 \equiv S \circ S \neq \mathrm{id}$ and $S(ab) \neq S(a)S(b)$. One can show that S possesses the properties

$$S(ab) = S(b)S(a), \quad S(I) = I, \quad \varepsilon \circ S = \varepsilon, \tag{3}$$

$$\sigma \circ (S \otimes S) \circ \Delta = \Delta \circ S. \tag{4}$$

Thus, S is an anti-homomorphism.

An associative algebra A with unit I is called a $*$-*algebra* if A is equipped with a $*$-operation with the properties

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^* \quad \text{(anti-linearity)}, \tag{5}$$

$$(a^*)^* = a \quad \text{(involutivity)}, \tag{6}$$

$$(ab)^* = b^* a^* \quad \text{(anti-multiplicativity)}, \tag{7}$$

$$I^* = I.$$

A Hopf algebra A is said to be a $*$-*Hopf algebra* if A is equipped with the $*$-operation with properties (5)–(7) and such that

$$S((S(a^*))^*) = a, \quad a \in A, \quad \text{that is} \quad S \circ * \circ S \circ * = \mathrm{id}, \tag{8}$$

and if Δ and ε are $*$-homomorphisms, that is

$$\varepsilon(a^*) = \overline{\varepsilon(a)}, \quad a \in A, \tag{9}$$

and

$$\Delta(a^*) = \sum_i b_i^* \otimes c_i^*, \tag{10}$$

where $\Delta(a) = \sum_i b_i \otimes c_i$. We can rewrite (10) in the form

$$\Delta \circ * = (* \otimes *) \circ \Delta. \tag{11}$$

Let us note that S and $*$ can be non-commuting.

The arguments of Section 14.2.2 imply that the algebra $\mathcal{A} =$ Pol G is a commutative Hopf algebra. If G is a non-commutative group, then $\mathcal{A}$ is a non-cocommutative Hopf algebra. Quantum algebras and algebras of functions on a quantum groups (see Sections 14.3.1 and 14.8.1) are examples of non-commutative Hopf algebras.

14.2.4. Coordinate functions.[2] Let us consider the Hopf algebra $\mathcal{A} =$ Pol G from Section 14.2.2. Let $G = SL(2,\mathbb{C})$. Then the functions π_{11}, π_{12}, π_{21}, π_{22} on matrices $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in SL(2,\mathbb{C})$ given by the formulas

$$\pi_{ij}(g) \equiv \pi_{ij} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{ij}, \quad i,j = 1,2, \tag{1}$$

belong to $\mathcal{A}$. They are called *coordinate functions* on $SL(2,\mathbb{C})$. Since every element from $\mathcal{A}$ can be represented as a polynomial in g_{11}, g_{12}, g_{21}, g_{22}, then π_{11}, π_{12}, π_{21}, π_{22} generate $\mathcal{A}$. In other words, elements of the algebra $\mathcal{A}$ are polynomials of the elements π_{11}, π_{12}, π_{21}, π_{22}, that is, if $f \in \mathcal{A}$, then

$$f = p(\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}).$$

This means that

$$f(g) \equiv f(g_{11}, g_{12}, g_{21}, g_{22}) = p(\pi_{11}(g), \pi_{12}(g), \pi_{21}(g), \pi_{22}(g)). \tag{2}$$

In fact, (1) and (2) means that $p = f$.

Let us evaluate the action of the comultiplication Δ upon the coordinate functions π_{11}, π_{12}, π_{21}, π_{22}. The definition of Δ implies that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,\mathbb{C})$ we have

$$\begin{aligned}
(\Delta(\pi_{11}))\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) &= \pi_{11}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \\
&= \pi_{11}\begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} = aa' + bc' \\
&= \pi_{11}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pi_{11}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} + \pi_{12}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pi_{21}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\
&= (\pi_{11} \otimes \pi_{11} + \pi_{12} \otimes \pi_{21})\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right),
\end{aligned} \tag{3}$$

that is

$$\Delta(\pi_{11}) = \pi_{11} \otimes \pi_{11} + \pi_{12} \otimes \pi_{21}.$$

[2] Reading of this section is not necessary to understand Sections 14.3-8.

In the same way we find

$$\Delta(\pi_{12}) = \pi_{11} \otimes \pi_{12} + \pi_{12} \otimes \pi_{22},$$
$$\Delta(\pi_{21}) = \pi_{21} \otimes \pi_{11} + \pi_{22} \otimes \pi_{21},$$
$$\Delta(\pi_{22}) = \pi_{21} \otimes \pi_{12} + \pi_{22} \otimes \pi_{22}.$$

In other words,

$$\Delta(\pi_{ij}) = \sum_{k=1,2} \pi_{ik} \otimes \pi_{kj}, \quad i,j = 1,2. \tag{4}$$

Since Δ is a homomorphism and π_{11}, π_{12}, π_{21}, π_{22} generate the whole algebra $\mathcal{A}$, then (4) defines the action of Δ on the whole $\mathcal{A}$.

These reasonings can be generalized. Let T be a finite dimensional irreducible representation of a group G by matrices $(t_{ij}(g))_{i,j=1}^n$ and let $t_{ij}(g)$ be polynomials on G. Then $t_{ij}(g) \in \mathcal{A} \equiv \text{Pol}\ G$. Let us find the actions of Δ, ε and S upon t_{ij}. Since $T(g_1g_2) = T(g_1)T(g_2)$, then $t_{ij}(g_1g_2) = \sum_k t_{ik}(g_1)t_{kj}(g_2)$. Therefore, according to formula (2) of Section 14.2.2 we have

$$\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}. \tag{5}$$

Since $T(e) = I$, then $t_{ij}(e) = \delta_{ij}$ and

$$\varepsilon(t_{ij}) = \delta_{ij}. \tag{6}$$

Since $T(g^{-1})T(g) = T(g)T(g^{-1}) = T(e)$, then

$$\sum_k t_{ik}(g^{-1})t_{kj}(g) = \sum_k t_{ik}(g)t_{kj}(g^{-1}) = t_{ij}(e).$$

Consequently,

$$\sum_k S(t_{ik})t_{kj} = \sum_k t_{ik}S(t_{kj}) = \delta_{ij}I. \tag{7}$$

14.2.5. Representations and corepresentations of Hopf algebras. By a representation of a Hopf algebra A we mean a representation of the associative algebra A. A comultiplication of a Hopf algebra appears under consideration of tensor products of representations. Detailed discussion of this appearance is in Section 14.3.3. For $*$-Hopf algebras one considers *$*$-representations*. By a $*$-representation we mean a representation T such that $T(a^*) = T(a)^*$ for all elements a of the $*$-Hopf algebra.

Along with representations one considers corepresentations of Hopf algebras. Let ε and Δ be the counit and the comultiplication of a Hopf algebra A and let V be a linear space. A linear mapping $T: V \to V \otimes A$, for which the diagrams

$$\begin{array}{ccc} & V\otimes A & \\ {}^{T}\nearrow & & \searrow^{\mathrm{id}\otimes\varepsilon} \\ V & \xrightarrow{\quad \mathrm{id} \quad} & V\otimes\mathbb{C} \end{array} \tag{1}$$

$$\begin{array}{ccc} & V\otimes A & \\ {}^{T}\nearrow & & \searrow^{\mathrm{id}\otimes\Delta} \\ V & & V\otimes A\otimes A \\ {}_{T}\searrow & & \nearrow_{T\otimes\mathrm{id}} \\ & V\otimes A & \end{array} \tag{2}$$

are commutative, is called a *right corepresentation* of the Hopf algebra A in the space V. The space V is said to be the *right A-comodule.* In (1) we identify $V \otimes \mathbb{C}$ with V by putting $v \otimes b = bv$, $v \in V$, $b \in \mathbb{C}$. A linear mapping $T: V \to A \otimes V$ of a linear space V into $A \otimes V$, for which the diagrams

$$\begin{array}{ccc} & A\otimes V & \\ {}^{T}\nearrow & & \searrow^{\varepsilon\otimes\mathrm{id}} \\ V & \xrightarrow{\quad \mathrm{id} \quad} & \mathbb{C}\otimes V \end{array} \tag{3}$$

$$\begin{array}{ccc} & A\otimes V & \\ {}^{T}\nearrow & & \searrow^{\Delta\otimes\mathrm{id}} \\ V & & A\otimes A\otimes V \\ {}_{T}\searrow & & \nearrow_{\mathrm{id}\otimes T} \\ & A\otimes V & \end{array} \tag{4}$$

are commutative, is called a *left corepresentation* of the Hopf algebra A in the linear space V. The space V is said to be the *left A-comodule.*

If T is a right corepresentation of a Hopf algebra A in a linear space V, then for $v \in V$ we have

$$T(v) = \sum_j v_j \otimes a_j, \quad v_j \in V, \quad a_j \in A. \tag{5}$$

Then by virtue of commutativity of diagrams (1) and (2) we can write

$$v = \sum_j v_j \varepsilon(a_j), \tag{6}$$

$$\sum_j T(v_j) \otimes a_j = \sum_j v_j \otimes \Delta(a_j). \tag{7}$$

Analogous relations are valid for left corepresentations.

Let $T: V \to V \otimes A$ be a right corepresentation of a Hopf algebra A. If W is a subspace of V such that $T(W) \subset W \otimes A$, then W is called a *right A-subcomodule* and the mapping $T: W \to W \otimes A$ is called a *right subcorepresentation* of the corepresentation $T: V \to V \otimes A$. One can easily formulate definitions of a direct (orthogonal) sum of right (left) corepresentations, of irreducibility and of complete reducibility of corepresentations.

Let $T: V \to V \otimes A$ and $Q: W \to W \otimes A$ be right corepresentations of a Hopf algebra A. If there exists a linear invertible mapping F from V onto W, for which the diagram

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ \downarrow T & & \downarrow Q \\ V \otimes A & \xrightarrow{F \otimes \mathrm{id}} & W \otimes A \end{array} \tag{8}$$

is commutative, that is, such that $Q \circ F(v) = (F \otimes \mathrm{id}) \circ T(v)$ for all $v \in V$, then T and Q are said to be *equivalent*. In the same way the notion of equivalence of left corepresentations can be given. In obvious way one can define unitary equivalence of corepresentations.

We choose a basis $e_1, \dots, e_n$ in the right A-comodule V in which a corepresentation T is realized. Then

$$T(e_i) = \sum_j e_j \otimes t_{ji}, \qquad \text{where} \qquad t_{ji} \in A. \tag{9}$$

Elements t_{ji} of the Hopf algebra A are called *matrix elements* of the corepresentation T. Taking into account relation (7) we have

$$\sum_j T(e_j) \otimes t_{ji} = \sum_j e_j \otimes \Delta(t_{ji}). \tag{10}$$

And by virtue of (9)

$$\sum_j T(e_j) \otimes t_{ji} = \sum_j \sum_k e_k \otimes t_{kj} \otimes t_{ji}. \tag{11}$$

It follows from (10) and (11) that

$$\Delta(t_{ki}) = \sum_{j=1}^{n} t_{kj} \otimes t_{ji}. \tag{12}$$

An $n \times n$ matrix (t_{ij}) with elements from the Hopf algebra A is called a *matrix corepresentation* of this algebra if relations (12) are fulfilled. Formulas (9)–(12) show that taking a basis in a right (left) A-comodule V we obtain a *matrix* *corepresentation* of the Hopf algebra. If T is a reducible corepresentation, then a basis can be taken in a such way that the corresponding matrix corepresentation is of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

14.3. Representations of the Quantum Algebra $U_q(\mathfrak{sl}_2)$ and Its Clebsch-Gordan Coefficients

14.3.1. The quantum algebra $U_q(\mathfrak{sl}_2)$ and its real forms. The matrices a_1, a_2, a_3 from formula (2) of Section 6.1.2 form a basis of the Lie algebra $\mathfrak{su}(2)$ and of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. The matrices

$$E_+ = a_1 - a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = a_1 + a_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = 2a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the commutation relations

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \quad [E_+, E_-] = H. \tag{1}$$

The elements H, E_+, E_-, satisfying these relations, generate the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$.

Let us fix a complex number $q = \exp h$ and deform relations (1) into the relations

$$[H, E_+] = 2E_+, \quad [H, E_-] = -2E_-, \tag{2}$$

$$[E_+, E_-] = \frac{\sinh(hH/2)}{\sinh(h/2)} = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}} \equiv [H], \tag{2'}$$

where $\sinh(hH/2)$ denotes the formal infinite series

$$\sinh \frac{h}{2}H = \frac{h}{2}H + \frac{1}{3!}\left(\frac{h}{2}H\right)^3 + \frac{1}{5!}\left(\frac{h}{2}H\right)^5 + \dots$$

which obtain the sense under consideration of representations of commutation relations (2) and (2′). An associative algebra with a unit element, generated by elements E_+, E_-, H, satisfying relations (2) and (2′), is called the *deformation* (or *q-deformation*) of the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ and is denoted by $U_q(\mathfrak{sl}_2)$. It consists of elements which are finite or infinite series of products of elements E_+, E_-, H.

In order to deal only with finite series one considers the elements

$$E_+,\ E_-,\ k = q^{H/4} \equiv \exp\frac{hH}{4},\ k^{-1} = q^{-H/4} \tag{3}$$

instead of E_+, E_-, H. For these new elements the commutation relations are of the form

$$kE_+k^{-1} = q^{1/2}E_+, \quad kE_-k^{-1} = q^{-1/2}E_-, \quad kk^{-1} = k^{-1}k = 1, \tag{4}$$

$$[E_+, E_-] = \frac{k^2 - k^{-2}}{q^{1/2} - q^{-1/2}}, \tag{4'}$$

and $U_q(\mathfrak{sl}_2)$ is an associative algebra generated by the elements E_+, E_-, k, k^{-1} satisfying relations (4) and (4′). In this case $U_q(\mathfrak{sl}_2)$ consists of finite sums of products of the elements E_+, E_-, k, k^{-1}.

It is clear that the monomials

$$E_+^\ell k^m E_-^n, \quad m \in \mathbb{Z}, \quad \ell, n \in \mathbb{Z}_+ \cup \{0\},$$

form a basis of $U_q(\mathfrak{sl}_2)$. The monomials

$$E_-^\ell k^m E_+^n, \quad n \in \mathbb{Z}, \quad \ell, n \in \mathbb{Z}_+ \cup \{0\},$$

are also a basis of $U_q(\mathfrak{sl}_2)$.

We equip $U_q(\mathfrak{sl}_2)$ with the structure of a Hopf algebra. For this we linearly extend the multiplication $(a, b) \to ab$ to the homomorphism

$$m: U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$$

and introduce the comultiplication

$$\Delta: U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2),$$

the antipode $S: U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ and the counit $\varepsilon: U_q(\mathfrak{sl}_2) \to \mathbb{C}$ in $U_q(\mathfrak{sl}_2)$ by the formulas

$$\Delta(E_\pm) = E_\pm \otimes q^{H/4} + q^{-H/4} \otimes E_\pm, \quad \Delta(H) = H \otimes 1 + 1 \otimes H, \tag{5}$$

$$S(E_\pm) = -q^{\pm 1/2}E_\pm, \quad S(H) = -H, \tag{6}$$

$$\varepsilon(E_\pm) = \varepsilon(H) = 0. \tag{7}$$

Since the operations Δ, S and ε are homomorphisms for the corresponding algebras, then they are defined on all elements from $U_q(\mathfrak{sl}_2)$. In order to show that Δ, defined by formula (5), can be extended to a homomorphism of the algebra $U_q(\mathfrak{sl}_2)$ into the algebra $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ it is sufficient to verify the relations

$$[\Delta(H), \Delta(E_\pm)] = \pm 2\Delta(E_\pm),$$

$$[\Delta(E_+), \Delta(E_-)] = \frac{\Delta(q^{H/2}) - \Delta(q^{-H/2})}{q^{1/2} - q^{-1/2}}.$$

We recommend to the reader to verify these relations. The associative algebra $U_q(\mathfrak{sl}_2)$ equipped with this structure of a Hopf algebra is called the *quantum algebra* and is denoted by the same symbol $U_q(\mathfrak{sl}_2)$.

The commutation relations (2) and (2′) do not change if we replace q by q^{-1}. This property is not valid for (5)–(7). In other words, the Hopf algebras $U_q(\mathfrak{sl}_2)$ and $U_{q^{-1}}(\mathfrak{sl}_2)$ do not coincide. Let σ be the linear operation in $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ acting upon $a \otimes b \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ as the permutation: $\sigma(a \otimes b) = b \otimes a$. Let us replace Δ and S by the operations

$$\Delta' = \sigma\Delta, \quad S' = S^{-1}.$$

It is easy to verify that Δ', S', and ε define the structure of a Hopf algebra on the associative algebra $U_q(\mathfrak{sl}_2)$. It is obvious from (5)–(7) that this Hopf algebra coincides with $U_{q^{-1}}(\mathfrak{sl}_2)$.

A simple (but awkward) verification shows that Δ and Δ' are connected by the relation

$$\Delta'(a) = R\Delta(a)R^{-1}, \quad a \in U_q(\mathfrak{sl}_2),$$

where the element R from $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ is defined by the formula

$$R = \exp\left(\frac{h}{4}H \otimes H\right) \sum_{n=0}^{\infty} \frac{(1-q^{-1})^n}{[n]!} \{(\exp \frac{h}{4}H)E_+\}^n \otimes \{(\exp \frac{-h}{4}H)E_-\}^n \tag{8}$$

and R^{-1} is related to R by the equality

$$(S \otimes \mathrm{id})R = R^{-1}.$$

Here $[n]!$ is given by formula (10) of Section 14.1.1 and id is the identity transformation in $U_q(\mathfrak{sl}_2)$.

The element R is called the *universal R-matrix*. In

$$U_q(\mathfrak{sl}_2)^3 \equiv U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$$

the relations

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \tag{9}$$

$$(\mathrm{id} \otimes \Delta)R = R_{13}R_{12} \tag{10}$$

hold, where the indices i and j in R_{ij} show that R acts in i-th and j-th factors.

Formulas (9) and (10) imply the relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{11}$$

called the *Yang-Baxter equation*. It is of a great importance in the quantization theory.

It can be directly verified that the element

$$C = (q^{1/2} - q^{-1/2})^{-2}(q^{(H+1)/2} - q^{-(H+1)/2})^2 + E_- E_+ \tag{12}$$

from $U_q(\mathfrak{sl}_2)$ commutes with E_+, E_-, H and, consequently, with all elements of $U_q(\mathfrak{sl}_2)$. It is called the *Casimir element* of this algebra. One can show that the center of $U_q(\mathfrak{sl}_2)$ is generated by this element.

For the elements E_+^n, E_-^n, $m \geq n$, of $U_q(\mathfrak{sl}_2)$ the relation

$$\begin{aligned} E_+^n E_-^m = \sum_{r=0}^{n} \frac{[n]![m]!}{[r]![n-r]![m-r]!} E_-^{m-r} E_+^{n-r} \\ \times [H-m+n][H-m+n-1]\dots[H-m+n-r+1] \end{aligned} \tag{13}$$

holds.

By introducing a $*$-operation into $U_q(\mathfrak{sl}_2)$ we distinguish real forms of this algebra (see Section 14.2.3). If q is real, then the formula

$$H^* = H, \quad E_+^* = E_-, \quad E_-^* = E_+ \tag{14}$$

define the $*$-structure which distinguishes in $U_q(\mathfrak{sl}_2)$ its compact real form, denoted by $U_q(\mathfrak{su}_2)$. For real q the formula

$$H^* = H, \quad E_+^* = -E_-, \quad E_-^* = -E_+ \tag{15}$$

define the non-compact real form, denoted by $U_q(\mathfrak{su}_{1,1})$.

Everywhere below we assume that q is not a root of unity. If q is a root of unity, then the theory, which is given below, must be changed.

14.3.2. Finite dimensional representations of $U_q(\mathfrak{sl}_2)$. By a *finite dimensional representation* T of the algebra $U_q(\mathfrak{sl}_2)$ we mean a homomorphism of the associative algebra $U_q(\mathfrak{sl}_2)$ into the algebra of linear operators in a finite dimensional complex linear space $\mathfrak{H}$. In order to give a representation T it is sufficient to give the operators $T(E_+)$, $T(E_-)$, $T(H)$ satisfying the relations

$$[T(H), T(E_\pm)] = \pm 2T(E_\pm), \tag{1}$$

$$[T(E_+), T(E_-)] = \frac{q^{T(H)/2} - q^{-T(H)/2}}{q^{1/2} - q^{-1/2}}, \tag{2}$$

where the commutator means the expression $[A, B] = AB - BA$. We shall consider finite dimensional representations T for which $T(H)^* = T(H)$ and $T(C)$, where C is the Casimir element, is a multiple of the identity operator.

With every integral and half-integral nonnegative number ℓ we associate a complex linear space $\mathfrak{H}_\ell$ with a basis $\{\mathbf{e}_m | m = -\ell, -\ell+1, \dots, \ell\}$ and the operators $T_\ell(E_\pm)$, $T_\ell(H)$ acting in $\mathfrak{H}_\ell$ according to the formulas

$$\left.\begin{aligned} T_\ell(E_+)\mathbf{e}_m &= ([\ell-m][\ell+m+1])^{1/2}\mathbf{e}_{m+1}, \\ T_\ell(E_-)\mathbf{e}_m &= ([\ell+m][\ell-m+1])^{1/2}\mathbf{e}_{m-1}, \\ T_\ell\left(\frac{H}{2}\right)\mathbf{e}_m &= m\mathbf{e}_m, \end{aligned}\right\} \tag{3}$$

where $[n]$ is given by formula (5) of Section 14.1.1. A direct verification shows that the operators $T_\ell(E_\pm)$, $T_\ell(H)$ satisfy relations (1) and (2) and, hence, define a representation of $U_q(\mathfrak{sl}_2)$ denoted by T_ℓ. As in the classical case, one can prove that the *representations* T_ℓ, $\ell \in \frac{1}{2}\mathbb{Z}_+$, *are irreducible.*

The condition $T(H)^* = T(H)$ means that the operator $T(H)$ is self-adjoint in every finite dimensional representation of the algebra $U_q(\mathfrak{sl}_2)$. Therefore, there is a basis in the carrier space $\mathfrak{H}$ of T consisting of eigenvectors of this operator. Making use of this fact we can solve the commutation relations (1) and (2) as it was done for the case of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ (see, for example, [119]). As a result, we conclude that *up to equivalence the representations* T_ℓ, $\ell = 0, 1/2, 1, 3/2, \dots$, *exhaust all finite dimensional irreducible representations of* $U_q(\mathfrak{sl}_2)$.

Operators (3) satisfy the conditions

$$T_\ell(E_\pm)^* = T_\ell(E_\mp), \quad T_\ell(H)^* = T_\ell(H). \tag{4}$$

Comparing (4) with formulas (14) of Section 14.3.1 we conclude that the restriction of the operator function $T_\ell(a)$, $a \in U_q(\mathfrak{sl}_2)$ onto $U_q(\mathfrak{su}_2)$ is a finite dimensional representation of the algebra $U_q(\mathfrak{su}_2)$ which will be denoted by the same symbol T_ℓ. The representations T_ℓ, $\ell \in \frac{1}{2}\mathbb{Z}_+$, of $U_q(\mathfrak{su}_2)$ are also irreducible and pairwise nonequivalent (as representations of different dimensions).

Thus, there is a one-to-one correspondence between finite dimensional irreducible representations of the algebra $U_q(\mathfrak{sl}_2)$ (and of the algebra $U_q(\mathfrak{su}_2)$) and unitary irreducible representations of the group $SU(2)$.

The Casimir operator $T_\ell(C)$ is a multiple of the identity operator. We have

$$T_\ell(C)\mathbf{e}_m = \left(\frac{q^{(2\ell+1)/4} - q^{-(2\ell+1)/4}}{q^{1/2} - q^{-1/2}}\right)^2 \mathbf{e}_m = \left[\ell + \frac{1}{2}\right]^2 \mathbf{e}_m. \tag{5}$$

By means of formula (3) we find that

$$T_\ell(E_\pm^n)\mathbf{e}_m = \left(\frac{[\ell \mp m]![\ell \pm m + n]!}{[\ell \mp m - n]![\ell \pm n]!}\right)^{1/2} \mathbf{e}_{m\pm n}, \tag{6}$$

where $|m \pm n| \le \ell$.

The representation T_ℓ can be realized in the space $\hat{\mathfrak{H}}_\ell$ of homogeneous polynomials of degree 2ℓ in two variables s and t. Set

$$T_\ell(E_+) = s\hat{D}_t, \quad T_\ell(E_-) = t\hat{D}_s, \quad T_\ell(H) = s\partial_s - t\partial_t, \tag{7}$$

where ∂_s and ∂_t are ordinary derivatives and $\hat{D}_x$ is defined by the formula

$$\hat{D}_x f(x) = \frac{f(q^{1/2}x) - f(q^{-1/2}x)}{(q^{1/2} - q^{-1/2})x}.$$

A direct verification shows that operators (7) satisfy relations (1) and (2) and, hence, define a representation of $U_q(\mathfrak{sl}_2)$.

Let us introduce the positive definite scalar product

$$\langle f, g\rangle \equiv f(\hat{D}_s, \hat{D}_t)g(s,t)|_{s=t=0}$$

in $\hat{\mathfrak{H}}_\ell$. Since $\hat{D}_x x^n = [n]x^{n-1}$, then the monomials

$$\mathbf{e}_m(s,t) = \frac{s^{\ell+m}t^{\ell-m}}{([\ell+m]![\ell-m]!)^{1/2}}, \quad m = -\ell, -\ell+1, \ldots, \ell, \tag{8}$$

form an orthonormal basis of $\hat{\mathfrak{H}}_\ell$. The action of operators (7) upon the basis functions (8) is given by formulas (3).

14.3.3. The tensor product of representations. Let T_{ℓ_1} and T_{ℓ_2} be two representations of the algebra $U_q(\mathfrak{sl}_2)$ acting in spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. If $q = 1$ (that is, in the classical case), then the tensor product $T_{\ell_1} \otimes T_{\ell_2}$ of the representations T_{ℓ_1} and T_{ℓ_2} acts in the space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, and the operators $E_\pm^\otimes = (T_{\ell_1} \otimes T_{\ell_2})(E_\pm)$ and $H^\otimes = (T_{\ell_1} \otimes T_{\ell_2})(H)$ are given by the formulas

$$E_\pm^\otimes = T_{\ell_1}(E_\pm) \otimes 1_{\ell_2} + 1_{\ell_1} \otimes T_{\ell_2}(E_\pm), \tag{1}$$
$$H^\otimes = T_{\ell_1}(H) \otimes 1_{\ell_2} + 1_{\ell_1} \otimes T_{\ell_2}(H), \tag{2}$$

where 1_{ℓ_i} is the identity operator in $\mathfrak{H}_i$. If $q \neq 1$, then operators (1) and (2) do not satisfy relations (1) and (2) of Section 14.3.2. To satisfy these relations we have to define $E_\pm^\otimes$, $H^\otimes$ by the formulas compatible with formulas (5) of Section 14.3.1, that is, by the formulas

$$E_\pm^\otimes = T_{\ell_1}(E_\pm) \otimes q^{H_2/4} + q^{-H_1/4} \otimes T_{\ell_2}(E_\pm), \tag{3}$$
$$H^\otimes = T_{\ell_1}(H) \otimes 1_{\ell_2} + 1_{\ell_1} \otimes T_{\ell_2}(H), \tag{4}$$

where $H_i \equiv T_{\ell_i}(H)$.

Since operator (4) is the same as in the case of the classical Lie algebra $\mathfrak{su}(2)$, then the spectrum of $H^\otimes$ is the same as in the classical case. In other words, the spectrum of the operator H in the representation $T_{\ell_1} \otimes T_{\ell_2}$ coincides with union of the spectra of H in the representations T_ℓ, $\ell = |\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1, \ldots, \ell_1 + \ell_2$. Since the values of the Casimir operator on the last representations are different and the spectrum of the operator H uniquely (up to equivalence) defines the irreducible representation, then the representation $T_{\ell_1} \otimes T_{\ell_2}$ is completely reducible and decomposes into the direct sum of the irreducible representations T_ℓ, $\ell = |\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1, \ldots, \ell_1 + \ell_2$:

$$T_{\ell_1} \otimes T_{\ell_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \oplus T_\ell. \tag{5}$$

We denote by $\mathfrak{H}_\ell$ the subspaces of $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ in which the representations T_ℓ are realized.

Let $\{\mathbf{e}_j\}$, $\{\mathbf{e}'_k\}$, $\{\mathbf{e}^\ell_m\}$ be orthonormal bases in $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_\ell$ respectively such that the actions of the operators $T(E_\pm)$, $T(H)$ upon them are given by formulas of type (1) and (2) of Section 14.3.2. According to decomposition (5) we have

$$\mathbf{e}^\ell_m = \sum_{j,k} C_q(\ell_1, \ell_2, \ell; j, k, m) \mathbf{e}_j \otimes \mathbf{e}'_k. \tag{6}$$

This formula defines *Clebsch-Gordan coefficients* (CGC's)

$$C_q(\ell_1, \ell_2, \ell; j, k, m) \equiv C_q(\boldsymbol{\ell}; \mathbf{j}) \tag{7}$$

of tensor product (5). The arguments of Section 8.1.2 remain valid for them. In particular, $C_q(\boldsymbol{\ell}; \mathbf{j}) = 0$ if $j + k \neq m$. Therefore, below we assume that $j + k = m$. The orthogonality relations

$$\sum_{\mathbf{j}} C_q(\boldsymbol{\ell}; \mathbf{j}) \overline{C_q(\boldsymbol{\ell}'; \mathbf{j})} = \delta_{\boldsymbol{\ell}\boldsymbol{\ell}'}, \tag{8}$$

$$\sum_{\ell} C_q(\boldsymbol{\ell}; \mathbf{j}) \overline{C_q(\boldsymbol{\ell}; \mathbf{j}')} = \delta_{\mathbf{j}\mathbf{j}'}, \tag{9}$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell)$, $\boldsymbol{\ell}' = (\ell_1, \ell_2, \ell')$, $\mathbf{j} = (j, k, m)$, $\mathbf{j}' = (j', k', m)$, are valid.

We introduce the operator $R^{\ell_1 \ell_2} = (T_{\ell_1} \otimes T_{\ell_2})(R)$ where R is the universal R-matrix. By means of formula (6) of Section 14.3.2 one calculates matrix elements of $R^{\ell_1 \ell_2}$ in the basis $\{\mathbf{e}_j \otimes \mathbf{e}'_k\}$. Nonvanishing matrix elements are of the form

$$\begin{aligned} R^{\ell_1 \ell_2}_{(n_1+n, n_2-n)(n_1, n_2)} &\equiv (R^{\ell_1 \ell_2} \mathbf{e}_{n_1} \otimes \mathbf{e}'_{n_2}, \mathbf{e}_{n_1+n} \otimes \mathbf{e}'_{n_2-n}) \\ &= \frac{(1-q^{-1})^n}{[n]!} \left(\frac{[\ell_1 - n_1]! [\ell_1 + n_1 + n]!}{[\ell_1 - n_1 - n]! [\ell_1 + n_1]!} \right)^{1/2} \\ &\times \left(\frac{[\ell_2 + n_2]! [\ell_2 + n - n_2]!}{[\ell_2 + n_2 - n]! [\ell_2 - n_2]!} \right)^{1/2} q^{(n_1 - n_2 + 2n)/2}. \end{aligned} \tag{10}$$

14.3.4. Calculation of CGC's. According to the formula (6) of Section 14.3.3 we have

$$\mathbf{e}_\ell^\ell = \sum_{j+k=\ell} C_q(\ell_1, \ell_2, \ell; j, k, m)\mathbf{e}_j \otimes \mathbf{e}'_k.$$

Applying the operator $T_\ell(E_+) \equiv (E_+)_\ell$ to both sides of this relation and taking into account that $(E_+)_\ell \mathbf{e}_\ell^\ell = 0$ and that

$$(E_+)_\ell \left(\sum_{j+k=\ell} C_q(\boldsymbol{\ell};\mathbf{j})\mathbf{e}_j \otimes \mathbf{e}'_k \right) = \sum_{j+k=m} C_q(\boldsymbol{\ell};\mathbf{j}) \times \{q^{k/2}((E_+)_{\ell_1}\mathbf{e}_j) \otimes \mathbf{e}'_k + q^{-j/2}\mathbf{e}_j \otimes (E_+)_{\ell_2}\mathbf{e}'_k\}$$

we obtain the recurrence relation for $C_q(\boldsymbol{\ell};\mathbf{j})$, $j+k=\ell$, which gives

$$C_q(\boldsymbol{\ell};\mathbf{j}) = (-1)^{\ell_1-j} q^{-(j+1)(\ell_1-j)/2} \left(\frac{[\ell_1+j]![\ell_2+k]!}{[\ell_1-j]![\ell_2-k]!} \right)^{1/2} A,$$

where $j+k=\ell$. Here A is independent on j and k. By means of the orthogonality relation

$$\sum_{j+k=\ell} |C_q(\boldsymbol{\ell};\mathbf{j})|^2 = 1$$

we find that

$$\sum_{j+k=\ell} q^{-(j+1)(\ell_1-j)} \frac{[\ell_1+j]![\ell_2+k]!}{[\ell_1-j]![\ell_2-k]!} = A^{-2}.$$

Making use of relations (11) and (12) of Section 14.1.1 and the definition of the basic hypergeometric function

$${}_2\varphi_1(q^a, q^b; q^c; q, x) \equiv {}_2\Phi_1(a, b; c; q, x)$$

we obtain

$$A^{-2} = [2\ell_1]![\ell+\ell_2-\ell_1]![\ell_1+\ell_2-\ell]!^{-1} \times {}_2\Phi_1(\ell+\ell_2-\ell_1+1, \ell-\ell_1-\ell_2; -2\ell_1; q, q^{-2\ell-1}).$$

This hypergeometric series is summed with the help of formula (10) of Section 14.1.2. As in the classical case, CGC's $C_q(\boldsymbol{\ell};\mathbf{j})$ are defined uniquely up to a number $c(\ell)$ which does not depend on j, k and such that $|c(\ell)| = 1$. Hence, we can assume that $A > 0$. Thus,

$$A^{-1} = q^{-(\ell_1+\ell_2-\ell)(\ell+\ell_2-\ell_1+1)/2} \left(\frac{[2\ell+1]![\ell_1+\ell_2-\ell]!}{[\ell_1+\ell_2+\ell+1]![\ell+\ell_1-\ell_2]![\ell-\ell_1+\ell_2]!} \right)^{1/2}$$

and for $j+k=\ell$ we obtain

$$C_q(\boldsymbol{\ell};\mathbf{j}) = (-1)^{\ell_1-j}q^a \times \left(\frac{[2\ell+1]![\ell_1+\ell_2-\ell]![\ell_1+j]![\ell_2+k]!}{[\ell_1+\ell_2+\ell+1]![\ell+\ell_1-\ell_2]![\ell-\ell_1+\ell_2]![\ell_1-j]![\ell_2-k]!}\right)^{1/2}, \quad (1)$$

where $a=\frac{1}{4}\{\ell_2(\ell_2+1)-\ell_1(\ell_1+1)-\ell(\ell+1)+2j(\ell+1)\}$.

Now from the equality

$$(E_-)_\ell^{\ell-m}\mathbf{e}_\ell^\ell = \left(\frac{[2\ell]![\ell-m]!}{[\ell+m]!}\right)^{1/2}\mathbf{e}_m^\ell$$

(see formula (6) of Section 14.3.2) we derive

$$\begin{aligned} C_q(\boldsymbol{\ell};\mathbf{j}) &= \left(\frac{[\ell+m]!}{[2\ell]![\ell-m]!}\right)^{1/2}(\mathbf{e}_j\otimes\mathbf{e}'_k,(E_-)_\ell^{\ell-m}\mathbf{e}_\ell^\ell) \\ &= \left(\frac{[\ell+m]!}{[2\ell]![\ell-m]!}\right)^{1/2}((E_+)_\ell^{\ell-m}(\mathbf{e}_j\otimes\mathbf{e}'_k),\mathbf{e}_\ell^\ell), \end{aligned} \quad (2)$$

where $(E_+)_\ell=(E_+)_{\ell_1}\otimes q^{H_2/4}+q^{-H_1/4}\otimes(E_+)_{\ell_2}$. Taking into account the equality

$$(E_+)_\ell^{\ell-m} = \sum_{k=0}^{\ell-m}\begin{bmatrix}\ell-m\\k\end{bmatrix}_{q^{-1}}((E_+)_{\ell_1}\otimes q^{H_2/4})^k(q^{-H_1/4}\otimes(E_+)_{\ell_2})^{\ell-m-k}$$

(see formula (4) of Section 13.2.2) we obtain from (2) that

$$\begin{aligned} C_q(\boldsymbol{\ell};\mathbf{j}) = \sum_{r=0}^{\ell-m} q^A\frac{[\ell-m]!}{[r]![\ell-m-r]!} &\times\left(\frac{[\ell_1-j]![\ell_2-k]![\ell_2+\ell-j-r]![\ell_1+j+k]!}{[\ell_1+j]![\ell_2+k]![\ell_2-\ell+j+r]![\ell_1-j-k]!}\right)^{1/2} \\ &\times C_q(\ell_1,\ell_2,\ell;j+r,k+\ell-m-r,\ell), \end{aligned}$$

where $A=\frac{1}{2}\{rk-j(\ell-m-r)\}$. From here and from (1) we derive a q-analog of formula (2) of Section 8.2.5 for CGC's:

$$\begin{aligned} C_q(\boldsymbol{\ell};\mathbf{j}) &= \frac{(-1)^{\ell_1-j}q^B([2\ell+1][\boldsymbol{\ell};\mathbf{j}])^{1/2}[\ell_1+\ell_2-\ell]!}{\Delta(\boldsymbol{\ell})[\ell_1+\ell_2+\ell+1]![\ell_1+j]![\ell_2+k]!} \\ &\times\sum_r\frac{(-1)^r q^{r(\ell+m+1)/2}[\ell+j+r]![\ell_2+\ell-j-r]!}{[r]![\ell-m-r]![\ell_1-j-r]![\ell_2-\ell+j+r]!}, \end{aligned} \quad (3)$$

where $m = j + k$, $B = \frac{1}{4}\{\ell_2(\ell_2+1) - \ell_1(\ell_1+1) - \ell(\ell+1) + 2j(m+1)\}$,

$$\Delta(\boldsymbol{\ell}) \equiv \Delta(\ell_1, \ell_2, \ell) = \left(\frac{[\ell_1+\ell_2-\ell]![\ell_1-\ell_2+\ell]![\ell-\ell_1+\ell_2]!}{[\ell_1+\ell_2+\ell+1]!}\right)^{1/2}, \tag{4}$$

$$[\boldsymbol{\ell};\mathbf{j}] = [\ell_1+j]![\ell_1-j]![\ell_2+k]![\ell_2-k]![\ell+m]![\ell-m]!, \tag{5}$$

and the summation is over integral r for which numbers under q-factorials are nonnegative.

14.3.5. Expressions for CGC's in terms of the function ${}_3\Phi_2$. By means of formulas (11) and (12) of Section 14.1.1 one reduces the sum in the formula (3) of Section 14.3.4 to the basic hypergeometric series (14) of Section 13.2.2 for ${}_3\varphi_2$. Using the notation ${}_3\Phi_2$ for this series (see formula (5) of Section 14.1.5), we have

$$C_q(\boldsymbol{\ell};\mathbf{j}) = \frac{(-1)^{\ell_1-j}q^B\Delta(\boldsymbol{\ell})[\ell+\ell_2-j]!([\boldsymbol{\ell};\mathbf{j}][2\ell+1])^{1/2}}{[\ell_1-\ell_2+\ell]![\ell+\ell_2-\ell_1]![\ell_2-\ell+j]![\ell_1-j]![\ell_2+k]![\ell-m]!} \times {}_3\Phi_2(j-\ell_1, \ell_1+j+1, -\ell+m; \ell_2-\ell+j+1, -\ell-\ell_2+j; q, q), \tag{1}$$

where B is the same as in formula (3) of Section 14.3.4. Other expressions for CGC's in terms of functions ${}_3\Phi_2$ are derived with the help of equality (17) of Section 13.2.2 and the equality

$${}_3\varphi_2(q^{-n}, abcdq^{n-1}, bz; bc, dc; q, q) = \frac{b^n(ac;q)_n(ad;q)_n}{c^n(bc;q)_n(bd;q)_n} \times {}_3\varphi_2(q^{-n}, abcdq^{n-1}, az; ac, ad; q, q) \tag{2}$$

which can be obtained by repeated application of formula (17) of Section 13.2.2.

Putting

$$n = \ell_1 - j, \quad a = q^{-\ell+m}, \quad b = q^{\ell_1+j+1}, \quad c = q^{\ell_2-\ell+j+1}, \quad d = q^{j-\ell-\ell_2}$$

into (1) and making use of relation (17) of Section 13.2.2, we derive

$$C_q(\boldsymbol{\ell};\mathbf{j}) = \frac{(-1)^{\ell_1-j}q^D[\ell+\ell_2-j]![\ell_1+\ell_2-m]![\ell_1+j]![\ell+m]![2\ell+1]^{1/2}}{\Delta(\boldsymbol{\ell})[\boldsymbol{\ell};\mathbf{j}]^{1/2}[\ell_1+\ell_2+\ell+1]!} \times {}_3\Phi_2(-\ell_1-\ell_2-\ell-1, -\ell+j, -\ell+m; -\ell-\ell_2+j, -\ell_1-\ell_2+m; q, q^{\ell_1+\ell-\ell_2+1}) \tag{3}$$

where $D = \frac{1}{4}\{(\ell_2-\ell_1-\ell)(\ell_1+\ell_2+\ell+1) + 2j(\ell+\ell_2+1) + 2k\ell_1\}$. Setting here

$$n = \ell_1+\ell_2+\ell+1, \quad a = q^{m-\ell}, \quad b = q^{\ell_1-\ell_2-\ell}, \quad c = q^{-2\ell}, \quad d = q^{j-\ell-\ell_2}$$

and utilizing the same relation, we obtain

$$C_q(\boldsymbol{\ell};\mathbf{j}) = (-1)^{\ell+\ell_1-k} q^F \frac{[2\ell]![\ell_1+\ell_2-\ell]![\ell+\ell_2-j]![\ell_1+j]![2\ell+1]^{1/2}}{\Delta(\boldsymbol{\ell})[\boldsymbol{\ell};\mathbf{j}]^{1/2}[\ell_1+\ell_2+\ell+1]!} \times {}_3\Phi_2(\ell_1-\ell_2+\ell, -\ell+m, -\ell_1-\ell_2-\ell-1; -2\ell, -\ell_2-\ell+j; q, q), \tag{4}$$

where $F = \frac{1}{4}\{(\ell+\ell_2-\ell_1)(\ell+\ell_1+\ell_2+1) - 2\ell_2 m - 2k(\ell+1)\}$.

By means of formula (2) (or by repeated application of formula (17) of Section 13.2.2) we derive from (1) that

$$C_q(\boldsymbol{\ell};\mathbf{j}) = \frac{q^E \Delta(\boldsymbol{\ell})[\boldsymbol{\ell};\mathbf{j}]^{1/2}}{[\ell_1+\ell_2-\ell]![\ell-\ell_1-k]![\ell-\ell_2+j]![\ell_1-j]![\ell_2+k]!} \times {}_3\Phi_2(\ell-\ell_1-\ell_2, -\ell_2-k, -\ell_1+j; \ell-\ell_2+j+1, \ell-\ell_1-k+1; q, q), \tag{5}$$

where $E = \frac{1}{4}\{(\ell_1+\ell_2-\ell)(\ell_1+\ell_2+\ell+1) + 2k\ell_1 - 2j\ell_2\}$.

14.3.6. Special cases of CGC's. Formula (1) of the previous section for $\ell = 0$ gives

$$C_q(\ell_1, \ell_2, 0; j, -j, 0) = \frac{(-1)^{\ell_1-j} q^{j/2}}{[2\ell_1+1]^{1/2}} \delta_{\ell_1\ell_2}. \tag{1}$$

For $\ell = \ell_1 + \ell_2$ formula (5) of Section 14.3.5 yields

$$C_q(\ell_1, \ell_2, \ell_1+\ell_2; j, k, m) = q^{(k\ell_1 - j\ell_2)/2} \times \left(\frac{[2\ell_1]![2\ell_2]![\ell_1+\ell_2-m]![\ell_1+\ell_2+m]!}{[2\ell_1+2\ell_2]![\ell_1-j]![\ell_1+j]![\ell_2-k]![\ell_2+k]!} \right)^{1/2}. \tag{2}$$

Setting $\ell = \ell_1 - \ell_2$ into formula (4) of Section 14.3.5 we have

$$C_q(\ell_1, \ell_2, \ell_1-\ell_2; j, k, m) = (-1)^{\ell_2+k} q^{-(\ell_2 j + \ell_1 k + k)/2} \times \left(\frac{[2\ell_2]![2\ell_1-2\ell_2+1]![\ell_1-j]![\ell_1+j]!}{[2\ell_1+1]![\ell_2-k]![\ell_2+k]![\ell_1-\ell_2+m]![\ell_1-\ell_2-m]!} \right)^{1/2}. \tag{2'}$$

It follows from formula (5) of Section 14.3.5 that

$$C_q(\ell_1, \ell_2, \ell_1+\ell_2-1; j, k, m) = q^{(\ell_1 k - \ell_2 j)/2} (q^{(\ell_2+\ell_1)/2}[\ell_1+j][\ell_2-k] - q^{-(\ell_1+\ell_2)/2}[\ell_1-j][\ell_2+k]) \times \left(\frac{[2\ell_1+2\ell_2-1]![2\ell_1-1]![2\ell_2-1]![\ell_1+\ell_2-m-1]![\ell_1+\ell_2+m-1]!}{[2\ell_1+2\ell_2]![\ell_1-j]![\ell_1+j]![\ell_2-k]![\ell_2+k]!} \right)^{1/2}. \tag{3}$$

If $\ell = \ell_1 - \ell_2 + 1$, then formula (4) of Section 14.3.5 yields

$$C_q(\ell_1, \ell_2, \ell_1 - \ell_2 + 1; j, k, m) = (-1)^{\ell_2+k} q^{-\{j(\ell_2-1)+k(\ell_1+1)\}/2}$$
$$\times (q^{-\ell_2/2}[\ell_1 + j + 1][\ell_1 - \ell_2 - m + 1] - q^{\ell_2/2}[\ell_1 - j + 1][\ell_1 - \ell_2 + m + 1])$$
$$\times \left(\frac{[2\ell_1 - 2\ell_2 + 3]![2\ell_2 - 1]![2\ell_1 - 2\ell_2 + 1]![\ell_1 - j]![\ell_1 + j]!}{[2\ell_1 + 2]![\ell_2 - k]![\ell_2 + k]![\ell_1 - \ell_2 - m + 1]![\ell_1 - \ell_2 + m + 1]!} \right)^{1/2} . \quad (3')$$

In (2)–(3′) we have $j + k = m$.

It is easy to derive from (2) and (2′) the expressions for CGC's $C_q(\ell, \frac{1}{2}, \ell \pm \frac{1}{2}; j, \pm\frac{1}{2}, m)$, $m = j \pm \frac{1}{2}$:

$$C_q\left(\ell, \frac{1}{2}, \ell + \frac{1}{2}; j, \pm\frac{1}{2}, j \pm \frac{1}{2}\right) = q^{\pm(\ell \mp m + 1/2)} \left(\frac{[\ell \pm m + 1/2]}{[2\ell + 1]} \right)^{1/2} , \quad (4)$$

$$C_q\left(\ell, \frac{1}{2}, \ell - \frac{1}{2}; j, \pm\frac{1}{2}, j \pm \frac{1}{2}\right) = \mp q^{\mp(\ell \pm m + 1/2)} \left(\frac{[\ell \mp m + 1/2]}{[2\ell + 1]} \right)^{1/2} . \quad (4')$$

14.3.7. Symmetries of CGC's. It follows from formula (5) of Section 14.3.5 that

$$C_q(\boldsymbol{\ell}; \mathbf{j}) \equiv C_q(\ell_1, \ell_2, \ell; j, k, m) = C_q(\ell_2, \ell_1, \ell; -k, -j, -m). \quad (1)$$

From formula (3) of Section 14.3.5 we derive that

$$C_q(\boldsymbol{\ell}; \mathbf{j}) = (-1)^{\ell - \ell_1 - k} q^{-k/2} \left(\frac{[2\ell + 1]}{[2\ell_1 + 1]} \right)^{1/2} C_q(\ell, \ell_2, \ell_1; m, -k, j). \quad (2)$$

Expression (5) of Section 14.3.5 for CGC's is symmetric with respect to the transformation

$$C_q(\boldsymbol{\ell}; \mathbf{j})$$
$$= C_q\left(\frac{\ell_1 + \ell_2 + m}{2}, \frac{\ell_1 + \ell_2 - m}{2}, \ell; \frac{\ell_1 - \ell_2 + j - k}{2}, \frac{\ell_1 - \ell_2 - j + k}{2}, \ell_1 - \ell_2 \right) . \quad (3)$$

With the help of relation (13) of Section 14.1.1 we derive from formula (3) of Section 14.3.4 that

$$C_q(\boldsymbol{\ell}; \mathbf{j}) = (-1)^{\ell_1 + \ell_2 - \ell} C_{q^{-1}}(\ell_1, \ell_2, \ell; -j, -k, -m). \quad (4)$$

Classical analogs of relations (1)–(4) (they correspond to the case $q = 1$) generate the symmetry group for CGC's which contains 72 elements (see Section 8.2.2). Relations (1)–(4) with $q \neq 1$ also generate the symmetry group for CGC's of the quantum algebra $U_q(\mathfrak{sl}_2)$, which contains 72 elements. But now relation (4) changes q by q^{-1}.

Besides above mentioned symmetries there are symmetries of CGC's related to the transition $\ell \to \overline{\ell} \equiv -\ell-1$. This substitution into expressions for CGC's leads to expressions of the type $[-k]!/[-m]!$, where $k, m \in \mathbb{Z}_+$. According to formula (9) of Section 14.1.7 such expressions must be replaces by $(-1)^{k-m}[m+1]!/[k-1]!$. This leads to a large number of supplementary symmetries. We present some of them:

$$\begin{aligned} C_q(\boldsymbol{\ell};\mathbf{j}) &= (-1)^{\ell_2+k} C_q(\overline{\ell}_1,\ell_2,\overline{\ell};j,k,m) = (-1)^{\ell_1-j} C_q(\ell_1,\overline{\ell}_2,\overline{\ell};j,k,m) \\ &= (-1)^{\ell_2-\ell-j} C_q(\overline{\ell}_1,\ell_2,\ell;j,k,m) = (-1)^{\ell_1+\ell_2-\ell} C_q(\overline{\ell}_1,\overline{\ell}_2,\ell;j,k,m). \end{aligned} \tag{5}$$

By means of the substitution $\ell = \ell_1 + s$ the first part of this chain yields the relation

$$C_q(\ell_1,\ell_2,\ell_1+s;j,k,m) = (-1)^{\ell_2+s} C_q(\overline{\ell}_1,\ell_2,\ell_1-s;j,k,m). \tag{6}$$

Combinations of (5) with other symmetries give a large number of new symmetries. For example, we have

$$C_q(\overline{\ell}_1,\overline{\ell}_2,\ell;j,k,m) = C_{q^{-1}}(\ell_1,\ell_2,\ell;-j,-k,-m). \tag{7}$$

14.3.8 Generating functions for CGC's. For brevity we shall use indices a, b, c, α, β, γ in notations for CGC's instead of ℓ_1, ℓ_2, ℓ, j, k, m and the notation $C^{abc}_{\alpha\beta\gamma}$ instead of $C_q(\boldsymbol{\ell};\mathbf{j})$. Let

$$F \equiv (1+x)_q^{b+c-a}\ {}_2\Phi_1(c-b-a,-a+\alpha;c-b+\alpha+1;q,-q^{a-b-c}x),$$

where $(1-y)_q^n$ is defined by formula (2) of Section 14.1.4. We represent $(1+x)_q^{b+c-a}$ in the form of a sum over n and ${}_2\Phi_1$ in the form of a sum over m. Further we put $m+n=b+\beta$ and replace the summations over n and m by the summations over β and m. After some manipulations we obtain

$$F = \sum_\beta \frac{(q^{a-b-c};q)_{b+\beta}}{(q;q)_{b+\beta}} x^{b+\beta}\ {}_3\Phi_2\left(\begin{matrix} c-b-a,\ -a+\alpha,\ -b-\beta \\ c-b+\alpha+1,\ c-a-\beta+1 \end{matrix}\,\middle|\, q,q\right).$$

Making use formula (5) of Section 14.3.5 for CGC's and relation (10) of Section 14.1.1 we derive that

$$\begin{aligned} &\left(\frac{[a-b+c]![\alpha+a]![2c+1]}{[b+c-a]![a+b-c]![a+b+c+1]![a-\alpha]!}\right)^{1/2} \frac{(1+x)_q^{b+c-a}}{[c-b+\alpha]!} \\ &\quad\times\ {}_2\Phi_1(c-b-a,-a+\alpha;c-b+\alpha+1;q,-q^{a-b-c}x) \\ &\quad= \sum_\beta C^{abc}_{\alpha,\beta,\alpha+\beta} \frac{q^A x^{b+\beta}}{([b+\beta]![b-\beta]![c+\alpha+\beta]![c-\alpha-\beta]!)^{1/2}}, \end{aligned} \tag{1}$$

where $A = \frac{1}{4}\{b(2a-2\alpha-2c-3b-3)-(a-c)(a+c+1)\}$. Thus, the expression on the left hand side is a generating function for CGC's $C^{abc}_{\alpha\beta\gamma}$, $\alpha+\beta=\gamma$. Other similar formulas are obtained with the help of symmetry relations.

Considering formula (1) as an expansion of the function on the left into a series in powers of x, we obtain that

$$\begin{aligned} C^{abc}_{\alpha\beta\gamma} &= M\frac{d^{b+\beta}}{dx^{b+\beta}}\{(1+x)_q^{b+c-a} \\ &\times {}_2\Phi_1(c-b-a,-a+\alpha;c-b+\alpha+1;q,-q^{a-b-c}x)\}\big|_{x=0}, \end{aligned} \tag{2}$$

where $\gamma=\alpha+\beta$ and

$$M = \frac{q^{-A}}{(b+\beta)!}\left(\frac{[b-\beta]![b+\beta]![c+\gamma]![c-\gamma]![a-b+c]![a+\alpha]![2c+1]}{[b+c-a]![a+b-c]![a+b+c+1]![a-\alpha]![c-b+\alpha]!^2}\right)^{1/2}. \tag{3}$$

Considering formula (1) as expansion (16) of Section 14.1.5 we derive that

$$\begin{aligned} C^{abc}_{\alpha\beta\gamma} &= M'D_q^{b+\beta}\{(1+x)_q^{b+c-a} \\ &\times {}_2\Phi_1(c-b-a,-a+\alpha;c-b+\alpha+1;q,-q^{a-b-c}x)\}\big|_{x=0}, \end{aligned} \tag{4}$$

where M' is obtained from M by replacing $(b+\beta)!$ by $[b+\beta]!$ and the q-derivative is applied to the variable x.

There is a relation which can be understood, in some sense, as the inverse of (1). To obtain it we put

$$n=\ell+\ell_2-\ell_1,\quad a=q^{-\ell_1-\ell_2-\ell-1},\quad b=q^{m-\ell},\quad c=q^{j-\ell-\ell_2},\quad d=q^{-2\ell} \tag{5}$$

into formula (4) of Section 14.3.5, apply formula (17) of Section 13.2.3 and then replace in the obtained expression the summation index r by $s=\ell+m-r$. After simple manipulations we derive the following expression for CGC's:

$$\begin{aligned} C^{abc}_{\alpha\beta\gamma} &= (-1)^{b+c+\beta+\gamma}q^{A} \\ &\times \frac{\Delta(a,b,c)[a+b+c+1]!([a-\alpha]![c-\gamma]![b-\beta]![2c+1])^{1/2}}{[a-b+c]![a+b-\gamma+1]![b-a-\gamma]!([a+\alpha]![c+\gamma]![b+\beta]!)^{1/2}} \\ &\times {}_3\Phi_2(-c-\gamma,c-\gamma+1,b-\beta+1;a+b-\gamma+2,b-a-\gamma+1;q,q), \end{aligned} \tag{6}$$

where $A=\frac{1}{4}\{a(a+1)-b(b+1)+c(c+1)+2\beta(\gamma-1)\}$. Applying formula (16) of Section 14.1.8 to the basic hypergeometric function from (6), we obtain

$$\begin{aligned} BC^{abc}_{\alpha\beta\gamma} &= \int_0^1 u^{b-\beta}(1-qu)_q^{a-\alpha}\,{}_2\Phi_1(-c-\gamma,c-\gamma+1;b-a-\gamma+1;q,u)d_qu \\ &= (1-q)\sum_{r=0}^{\infty} q^{r(b-\beta+1)}(1-q^{r+1})_q^{a-\alpha}\,{}_2\Phi_1(-c-\gamma,c-\gamma+1;b-a-\gamma+1;q,q^r), \end{aligned} \tag{7}$$

where

$$B = \frac{(-1)^{b+c+\beta+\gamma}}{q^A} \frac{[a-b+c]![a+b-\gamma+1]!}{\Delta(a,b,c)[a+b+c+1]!}$$
$$\times ([a+\alpha]![a-\alpha]![b+\beta]![b-\beta]![c+\gamma]!)^{1/2}([c-\gamma]![2c+1])^{-1/2}$$

(A is the same as in (6)).

Now we show that

$$F \equiv (t_1 - qt_2)_q^{a+b-c}(t_2 - t_3)_q^{b+c-a}(t_3 - t_1)_q^{a-b+c}$$

is a generating function for CGC's. For this we represent every its factor as the sum (5) of Section 14.1.4. The result is

$$F = \sum_{k_1=0}^{a+b-c} \sum_{k_2=0}^{b+c-a} \sum_{k_3=0}^{a-b+c} \frac{(q^{c-a-b};q)_{k_1}(q^{a-b-c};q)_{k_2}(q^{b-c-a};q)_{k_3}}{(q;q)_{k_1}(q;q)_{k_2}(q;q)_{k_3}}$$
$$\times q^{k_1} t_1^{k_3-k_1+a+b-c} t_2^{k_1-k_2+b+c-a} t_3^{k_2-k_3+a-b+c}.$$

Now we put

$$k_3 - k_1 + a + b - c = a + \alpha, \quad k_1 - k_2 + b + c - a = b + \beta, \quad k_2 - k_3 + a - b + c = c - \gamma$$

and replace the summations over k_1, k_2, k_3 by those over k_1, α, β. We obtain

$$F = \sum_{\alpha,\beta} t_1^{a+\alpha} t_2^{b+\beta} t_3^{c-\gamma} q^A \frac{(q^{a-b-c};q)_{c-a-\beta}(q^{b-a-c};q)_{c-b+\alpha}}{(q;q)_{c-a-\beta}(q;q)_{c-b+\alpha}} \tag{7'}$$
$$\times\ {}_3\Phi_2(c-a-b, -b-\beta, \alpha-a; c-a-\beta+1, c-b+\alpha+1; q, q),$$

where A can be easily found. Taking into account expression (5) of Section 14.3.5 for CGC's we derive from (7′) that

$$(t_1 - qt_2)_q^{a+b-c}(t_2 - t_3)_q^{b+c-a}(t_3 - t_1)_q^{a-b+c}$$
$$= \frac{\Delta(\mathbf{a})[a+b+c+1]!}{[2c+1]^{1/2}} \sum_{\alpha,\beta} \frac{t_1^{a+\alpha} t_2^{b+\beta} t_3^{c-\gamma} q^D}{[\mathbf{a};\boldsymbol{\alpha}]^{1/2}} C^{abc}_{\alpha,\beta,\alpha+\beta}, \tag{8}$$

where $\mathbf{a} = (a,b,c)$, $\boldsymbol{\alpha} = (\alpha, \beta, \alpha+\beta)$, $\Delta(\mathbf{a})$ and $[\mathbf{a};\boldsymbol{\alpha}]$ are the same as in formula (3) of Section 14.3.4 and

$$D = \frac{1}{2}\{(c-a-\beta)(a-b-c-1) + (c-b+\alpha)(b-a-c-2) + \alpha b - \beta a\}$$
$$- \frac{1}{4}(a+b-c)(a+b+c+1).$$

Another generating function for CGC's is given by the formula

$$(v_1u_2 - qu_1v_2)_q^{a+b-c}(w_1v_2 - w_2v_1)_q^{b+c-a}(u_1w_2 - w_1u_2)_q^{a-b+c}$$
$$= \Delta(\mathbf{a})[a+b+c+1]![2c+1]^{-1/2}$$
$$\times \sum_{\alpha,\beta} \frac{u_1^{a-\alpha}u_2^{a+\alpha}v_1^{b-\beta}v_2^{b+\beta}w_1^{c-\gamma}w_2^{c+\gamma}}{[\mathbf{a};\boldsymbol{\alpha}]^{1/2}} q^D C^{abc}_{\alpha,\beta,\alpha+\beta}, \tag{9}$$

where D is the same as in (8). To prove this formula it is sufficient to represent the left hand side in the form

$$u_1^{2a}v_1^{2b}w_1^{2c}\left(\frac{u_2}{u_1} - q\frac{v_2}{v_1}\right)_q^{a+b-c}\left(\frac{v_2}{v_1} - \frac{w_2}{w_1}\right)_q^{b+c-a}\left(\frac{w_2}{w_1} - \frac{u_2}{u_1}\right)_q^{a-b+c}$$

and to apply formula (8).

Replace in (8) t_1 by $q^{2/3}t_1$ and t_3 by $q^{1/3}t_3$. By virtue of relation (5) of Section 14.1.4 we conclude that (8) remains valid under the replacement of the left hand side by

$$(t_1 - q^{1/3}t_2)_q^{a+b-c}(t_2 - q^{1/3}t_3)_q^{b+c-a}(t_3 - q^{1/3}t_1)_q^{a-b+c}$$

and under multiplication of the expression under the sum sign by $q^{(2c-a-b+2\alpha-\gamma)/3}$. The corresponding statement is valid for relation (9).

The formula

$$\frac{E_q(x_1t_2)}{E_q(x_1t_3)}\frac{E_q(x_2t_3)}{E_q(x_2t_1)}\frac{E_q(x_3t_1)}{E_q(qx_3t_2)}$$
$$= \sum_{\substack{a,b,c\\ \alpha,\beta}} \frac{x_1^{b+c-a}x_2^{a-b+c}x_3^{a+b-c}t_1^{a+\alpha}t_2^{b+\beta}t_3^{c-\gamma}(1-q)^Aq^B}{\Delta(\mathbf{a})[\mathbf{a};\boldsymbol{\alpha}]^{1/2}} C^{abc}_{\alpha,\beta,\alpha+\beta}, \tag{10}$$

where $A = a+b-2c-\alpha+\beta$,

$$B = \frac{1}{2}\{-a(3a-1)-b(3b-1)-c(3c+1)-\alpha(a+c-3b+1)+\beta(b+c-2a+1)\}+ab+ac+bc,$$

also gives a generating function for CGC's. In order to prove formula (10) we have to represent the function $E_q(x_1t_2)$ as

$$E_q(x_1t_2) = \sum_{n=0}^{\infty} \frac{(t_2-t_3)_q^n}{[[n]]!}\{D^n_{q,t_2}E_q(x_1t_2)\}_{t_2=t_3},$$

then to represent $(t_2 - t_3)_q^n$ in the form (5) of Section 14.1.4, and to take into account that

$$D_{q,t}f(t) = \frac{f(tq)-f(q)}{tq-t}, \{D^n_{q,t_2}E_q(x_1t_2)\}_{t_2=t_3} = x_1^nE_q(x_1t_3).$$

The result is

$$\frac{E_q(x_1t_2)}{E_q(x_1t_3)} = \sum_{n=0}^{\infty} \frac{(1-q)^n x_1^n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q)^{-n};q)_k}{(q;q)_k} t_3^k t_2^{n-k}.$$

In a similar way one can represent other factors of the left hand side of (10). After appropriate replacement of the summation indices and simple transformations we receive (10).

14.3.9. The difference equation for CGC's. As in Section 8.2.7, the equality

$$\langle (E_-)_\ell (E_+)_\ell (\mathbf{e}_{j+1} \otimes \mathbf{e}'_{k-1}), \mathbf{e}^\ell_m \rangle = \langle \mathbf{e}_{j+1} \otimes \mathbf{e}'_{k-1}, (E_-)_\ell (E_+)_\ell \mathbf{e}^\ell_m \rangle,$$

where $(E_\pm)_\ell \equiv T_\ell(E_\pm)$, leads to the recurrence relation

$$aC^{\ell_1\ell_2\ell}_{j+2,m-(j+2),m} + bC^{\ell_1\ell_2\ell}_{j+1,m-(j+1),m} + cC^{\ell_1\ell_2\ell}_{j,m-j,m} = 0, \tag{1}$$

where

$$a = q^{(2j-m-1)/2}([\ell_1 - j - 1][\ell_1 + j + 2][\ell_2 + m - j - 1][\ell_2 - m + j + 2])^{1/2},$$

$$b = q^j[\ell_1 - j - 1][\ell_2 + j + 2] + q^{m-j}[\ell_2 - m + j + 1][\ell_2 + m - j] - q^{2j-m+1}[\ell - m][\ell + m + 1],$$

$$c = q^{(2j-m+1)/2}([\ell_1 - j][\ell_1 + j + 1][\ell_2 + m - j][\ell_2 - m + j + 1])^{1/2}.$$

Introducing the operator Δ_j acting upon a function $f(j)$ of discrete variable j according to the formula $\Delta_j f(j) = f(j+1) - f(j)$, we can replace (1) by the second order difference equation

$$\{a\Delta_j^2 + (b + 2a)\Delta_j + (a + b + c)\} C^{\ell_1\ell_2\ell}_{j,m-j,m} = 0. \tag{2}$$

14.3.10 Recurrence relations for CGC's. General recurrence relations for CGC's of the algebra $U_q(\mathfrak{sl}_2)$ are derived with the help of the generating functions of Section 14.3.8. By means of the connection (4) of Section 14.1.4 between $(1-x)_q^a$ and ${}_1\varphi_0(q^{-a}; q, x)$ and of relation (7) of Section 14.1.2 we obtain that

$$\begin{aligned}
&(t_1 - qt_2)_q^{a+b-c}(t_2 - t_3)_q^{b+c-a}(t_3 - t_1)_q^{a-b+c} \\
&\quad = (t_1 - qt_2)_q^{a'+b'-c'}(t_2 - t_3)_q^{b'+c'-a'}(t_3 - t_1)_q^{a'-b'+c'} \\
&\quad \times (t_1 - q^{c'-a'-b'+1}t_2)_q^{a''+b''-c''}(t_2 - q^{a'-b'-c'}t_3)_q^{b''+c''-a''} \\
&\quad \times (t_3 - q^{b'-a'-c'}t_1)_q^{a''-b''+c''},
\end{aligned} \tag{1}$$

where $a = a' + a''$, $b = b' + b''$, $c = c' + c''$. Repeating the derivation of formula (7) of Section 14.3.8, we have

$$
\begin{aligned}
&(t_1 - q^{c-a-b+1}t_2)_q^{a''+b''-c''}(t_2 - q^{a-b-c}t_3)_q^{b''+c''-a''}(t_3 - q^{b-a-c}t_1)_q^{a''-b''+c''} \\
&\quad = (-1)^{a''+b''-2c''}[a'' - b'' + c'']![b'' - a'' + c'']! \\
&\quad \times \sum_{\alpha'',\beta''} \frac{(-1)^{\beta''-\alpha''} q^D t_1^{a''+\alpha''} t_2^{b''+\beta''} t_3^{c''-\gamma''}}{[a''-\alpha'']![b''-\beta'']![c''-b''+\alpha'']!} \frac{1}{[c''-a''-\beta'']!} \\
&\quad \times {}_3\Phi_2\left(\begin{matrix} c''-b''-a'', & -a''+\alpha'', & -b''-\beta'' \\ c''-b''+\alpha''+1, & c''-a''-\beta''+1 & \end{matrix}\middle| q, q^{-a-b-c+1}\right),
\end{aligned} \tag{2}
$$

where $\gamma'' = \alpha'' + \beta''$ and

$$
\begin{aligned}
D = &\frac{1}{2}\{(c''+1)(a''+b''-2c'') + (b''-a'')(a''-b''+\gamma'') + (c''+1)(\beta''-\alpha'')\} \\
&+ c(a''+b''-2c'') - (a-b)(a''-b'') + \gamma''(b-a) + c(\beta''-\alpha'').
\end{aligned}
$$

Let us now substitute expressions (7′) of Section 14.3.8 and (2) for generating functions into (1) and compare coefficients at the same powers of t_1, t_2 and t_3. As a result we obtain

$$
\begin{aligned}
&A(\mathbf{a},\boldsymbol{\alpha})\, {}_3\Phi_2\left(\begin{matrix} c-a-b, & -a+\alpha, & -b-\beta \\ c-a-\beta+1, & c-b+\alpha+1 & \end{matrix}\middle| q, q\right) \\
&\quad = \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} q^p A(\mathbf{a}',\boldsymbol{\alpha}')A(\mathbf{a}'',\boldsymbol{\alpha}'')\, {}_3\Phi_2\left(\begin{matrix} c'-a'-b', & -a'+\alpha', & -b'-\beta' \\ c'-b'+\alpha'+1, & c'-a'-\beta'+1 & \end{matrix}\middle| q, q\right) \\
&\quad \times {}_3\Phi_2\left(\begin{matrix} c''-a''-b'', & -a''+\alpha'', & -b''-\beta'' \\ c''-b''+\alpha''+1, & c''-a''-\beta''+1 & \end{matrix}\middle| q, q^{-a'-b'-c'+1}\right),
\end{aligned} \tag{3}
$$

where $p = c'(a''+b''-2c''+\beta''-\alpha'') + (b'-a')(a''-b''+\alpha''+\beta'')$ and

$$
\begin{aligned}
A(\mathbf{a},\boldsymbol{\alpha}) \equiv A(a,b,c;\alpha,\beta,\gamma) = &\frac{[a-b+c]![b+c-a]!}{[a-\alpha]![b+\beta]![c-b+\alpha]![c-a-\beta]!} \\
&\times q^{(b+a)(a-b+\gamma)+(1/2)(c+1)(a+b-2c)+(1/2)(c+1)(\beta-\alpha)}.
\end{aligned}
$$

Due to formula (5) of Section 14.3.5 this equality is reduced to the recurrence relation

$$
\begin{aligned}
B(\mathbf{a},\boldsymbol{\alpha})C^{abc}_{\alpha,\beta,\alpha+\beta} = &\sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta}} q^r B(\mathbf{a}',\boldsymbol{\alpha}')A(\mathbf{a}'',\boldsymbol{\alpha}'') \\
&\times {}_3\Phi_2\left(\begin{matrix} c''-a''-b'', & -a''+\alpha'', & -b''-\beta'' \\ c''-b''+\alpha''+1, & c''-a''-\beta''+1 & \end{matrix}\middle| q, q^{-a'-b'-c'+1}\right) C^{a'b'c'}_{\alpha',\beta',\alpha'+\beta'}
\end{aligned} \tag{4}
$$

where, as above, $a = a' + a''$, $b = b' + b''$, $c = c' + c''$,

$$r = c'(a'' + b'' - 2c'' + \beta'' - \alpha'') + (b' - a')(a'' - b'' + \alpha'' + \beta''),$$
$$B(\mathbf{a}, \boldsymbol{\alpha}) = q^{2ab+(1/2)\alpha(2b-a-c-1)-(3/4)(a+b-c)(a+b-c+1)+(1/2)(2c+1)(a+b)}$$
$$\times q^{-(1/2)\beta(2a-b-c-1)}$$
$$\times \left(\frac{[a+b-c]![a-b+c]![b+c-a]![a+b+c+1]!}{[a-\alpha]![a+\alpha]![b-\beta]![b+\beta]![c+\alpha+\beta]![c-\alpha-\beta]![2c+1]} \right)^{1/2}.$$

For special values of parameters we obtain from (4) three-term recurrence relations for CGC's. For example, for $a'' = 0$, $b'' = 1/2$, $c'' = 1/2$ formula (4) turns into

$$\left(\frac{[2c][b-a+c][a+b+c+1]}{[2c+1]} \right)^{1/2} C^{abc}_{\alpha\beta\gamma}$$
$$= q^{(b+c-\beta-\gamma)/4}([b+\beta][c+\gamma])^{1/2} C^{a,b-1/2,c-1/2}_{\alpha,\beta-1/2,\gamma-1/2}$$
$$+ q^{-(b+c+\beta+\gamma)/4}([b-\beta][c-\gamma])^{1/2} C^{a,b-1/2,c-1/2}_{\alpha,\beta+1/2,\gamma+1/2}.$$

14.4. Matrix Elements of Representations of $U_q(\mathfrak{sl}_2)$

14.4.1. Introduction. Let T_ℓ be an irreducible finite dimensional representation of the quantum algebra $U_q(\mathfrak{sl}_2)$. We choose in the carrier space $\mathfrak{H}_\ell$ of this representation a basis $\mathbf{e}_m$, $m = -\ell, -\ell+1, \ldots, \ell$. Assume that the action of the operators $T_\ell(E_\pm)$, $T_\ell(H)$ upon this basis is given by the formulas (3) of Section 14.3.2. In the basis $\{\mathbf{e}_m\}$ the representation T_ℓ is given by the matrix with entries t^ℓ_{mn} depending on elements of $U_q(\mathfrak{sl}_2)$. Let us remind that elements of this algebra are sums of the products $E^k_+ H^m E^n_-$.

The arguments of Section 14.3.3 show that the tensor product $T_{\ell_1} \otimes T_{\ell_2}$ of representations of $U_q(\mathfrak{sl}_2)$ is non-commutative, that is,

$$T_{\ell_1}(a) \otimes T_{\ell_2}(a) \neq T_{\ell_2}(a) \otimes T_{\ell_1}(a).$$

Therefore,

$$t^{\ell_1}_{m_1 n_1} t^{\ell_2}_{m_2 n_2} \neq t^{\ell_2}_{m_2 n_2} t^{\ell_1}_{m_1 n_1}.$$

As in the classical case, every irreducible representation T_ℓ of the algebra $U_q(\mathfrak{sl}_2)$ is obtained by tensor multiplication of 2ℓ copies of the simplest irreducible nontrivial representation, that is, of the representation $T_{1/2}$. The process of successive tensor multiplication of $T_{1/2}$ by means of CGC's leads to expressions for the matrix elements t^ℓ_{mn} of the representation T_ℓ in the form of a polynomial of the matrix elements $t^{1/2}_{1/2,1/2}$, $t^{1/2}_{1/2,-1/2}$, $t^{1/2}_{-1/2,1/2}$, $t^{1/2}_{-1/2,-1/2}$ of the representation $T_{1/2}$. For brevity in notations we shall use the symbols π_{11}, π_{12}, π_{21}, π_{22} for last matrix elements.

In what follows we shall find expressions for the matrix elements t^{ℓ}_{mn} in terms of π_{11}, π_{12}, π_{21}, π_{22}. For this we shall use relations between matrix elements and CGC's which are corollaries of the decompositions

$$C_q(T_{\ell_1} \otimes T_{\ell_2})C_q^* = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \otimes T_\ell,$$

$$C_q^* \left(\sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \otimes T_\ell \right) C_q = T_{\ell_1} \otimes T_{\ell_2},$$

where C_q is the matrix whose elements are CGC's of the tensor product $T_{\ell_1} \otimes T_{\ell_2}$. As in the classical case, these relations are of the form

$$\sum_{j_1,j_2} C^{\ell_1\ell_2\ell}_{j_1,m_1-j_1,m_1} C^{\ell_1\ell_2\ell}_{j_2,m_2-j_2,m_2} t^{\ell}_{j_1j_2} t^{\ell_2}_{m_1-j_1,m_2-j_2} = t^{\ell}_{m_1m_2}, \tag{1}$$

$$\sum_{\ell} C^{\ell_1\ell_2\ell}_{j_1,k_1,j_1+k_1} C^{\ell_1\ell_2\ell}_{j_2,k_2,j_2+k_2} t^{\ell}_{j_1+k_1,j_2+k_2} = t^{\ell_1}_{j_1j_2} t^{\ell_2}_{k_1k_2} \tag{2}$$

(we assume that CGC's are real).

14.4.2. Relations for π_{ij}. According to formulas (4) and (4′) of Section 14.3.6, CGC's of the tensor product $T_{1/2} \otimes T_{1/2}$ of $U_q(\mathfrak{sl}_2)$ are of the form

$$C^{1/2,1/2,1}_{1/2,1/2,1} = C^{1/2,1/2,1}_{-1/2,-1/2,-1} = 1, \quad C^{1/2,1/2,1}_{1/2,-1/2,0} = -C^{1/2,1/2,0}_{-1/2,1/2,0} = \frac{q^{-1/4}}{[2]^{1/2}},$$

$$C^{1/2,1/2,1}_{-1/2,1/2,0} = C^{1/2,1/2,0}_{1/2,-1/2,0} = \frac{q^{1/4}}{[2]^{1/2}}.$$

Setting $\ell_1 = \ell_2 = 1/2$ into relation (2) of the previous section and using these CGC's, we find that

$$\pi_{22}\pi_{11} = \frac{q^{1/2}}{[2]}(t^1_{00} + q^{-1}I), \quad \pi_{11}\pi_{22} = \frac{q^{-1/2}}{[2]}(t^1_{00} + qI),$$

where t^1_{00} is the matrix element of the representation T_1 and $I = t^0_{00}$. Hence,

$$\pi_{22}\pi_{11} - q\pi_{11}\pi_{22} = (1-q)I. \tag{1}$$

In the same way, using formulas (1) and (2) of Section 14.4.1, we conclude that π_{11}, π_{12}, π_{21}, π_{22} satisfy the relations

$$\pi_{12}\pi_{21} = \pi_{21}\pi_{12}, \; \pi_{12}\pi_{11} = \sqrt{q}\pi_{11}\pi_{12}, \; \pi_{21}\pi_{11} = \sqrt{q}\pi_{11}\pi_{21}, \tag{2}$$

$$\pi_{22}\pi_{12} = \sqrt{q}\pi_{12}\pi_{22}, \quad \pi_{22}\pi_{21} = \sqrt{q}\pi_{21}\pi_{22}, \tag{3}$$

$$\pi_{22}\pi_{11} - \pi_{11}\pi_{22} = (q^{1/2} - q^{-1/2})\pi_{21}\pi_{12}, \tag{4}$$

$$\pi_{11}\pi_{22} - q^{-1/2}\pi_{21}\pi_{12} = I. \tag{5}$$

It can be shown that other relations between π_{11}, π_{12}, π_{21}, π_{22} are consequences of these ones. The elements π_{11}, π_{12}, π_{21}, π_{22} satisfying relations (1)–(5) generate the algebra called the *algebra of functions on the quantum group* $SL_q(2,\mathbb{C})$. Section 14.8.1 contains a detailed discussion of this algebra.

14.4.3 Calculation of matrix elements. Let us derive expressions for the matrix elements t^{ℓ}_{mn} of the representations T_ℓ of $U_q(\mathfrak{sl}_2)$ in terms of π_{11}, π_{12}, π_{21}, π_{22}. For this we use mathematical induction and CGC's of the tensor product $T_\ell \otimes T_{1/2}$. Putting $\ell_2 = 1/2$ into formula (1) of Section 14.4.1 and taking into account that

$$C^{\ell,1/2,\ell+1/2}_{\ell,1/2,\ell+1/2} = C^{\ell,1/2,\ell+1/2}_{-\ell,-1/2,-\ell-1/2} = 1$$

we obtain for $\ell = \ell_1 + 1/2$ that

$$t^{\ell+1/2}_{\ell+1/2,\ell+1/2} = t^{\ell}_{\ell\ell}\pi_{11}, \quad t^{\ell+1/2}_{-\ell-1/2,\ell+1/2} = t^{\ell}_{-\ell,\ell}\pi_{21},$$
$$t^{\ell+1/2}_{\ell+1/2,-\ell-1/2} = t^{\ell}_{\ell,-\ell}\pi_{12}, \quad t^{\ell+1/2}_{-\ell-1/2,-\ell-1/2} = t^{\ell}_{-\ell,-\ell}\pi_{22}.$$

Consequently,

$$t^{\ell}_{\ell\ell} = \pi_{11}^{2\ell}, \; t^{\ell}_{-\ell,\ell} = \pi_{21}^{2\ell}, \; t^{\ell}_{\ell,-\ell} = \pi_{12}^{2\ell}, \; t^{\ell}_{-\ell,-\ell} = \pi_{22}^{2\ell}. \tag{1}$$

Now we set $\ell_1 = 1/2$ into formula (2) of Section 14.4.1 and utilize expressions for CGC's of the tensor product $T_{1/2} \otimes T_\ell$ (they are obtained from formulas (4) and (4′) of Section 14.3.6 with the help of the symmetry relation (1) of Section 14.3.7). This leads to the relations

$$\begin{aligned}[2\ell+1]\pi_{11}t^{\ell}_{kj} &= q^{-(2\ell-k-j)/4}([\ell+k+1][\ell+j+1])^{1/2}t^{\ell+1/2}_{k+1/2,j+1/2} \\ &\quad + q^{(2\ell+k+j+2)/4}([\ell-k][\ell-j])^{1/2}t^{\ell-1/2}_{k+1/2,j+1/2},\end{aligned} \tag{2}$$

$$\begin{aligned}[2\ell+1]\pi_{12}t^{\ell}_{kj} &= q^{(k+j)/4}([\ell+k+1][\ell-j+1])^{1/2}t^{\ell+1/2}_{k+1/2,j-1/2} \\ &\quad - q^{(k+j)/4}([\ell-k][\ell+j])^{1/2}t^{\ell-1/2}_{k+1/2,j-1/2},\end{aligned} \tag{3}$$

$$\begin{aligned}[2\ell+1]\pi_{21}t^{\ell}_{kj} &= q^{(k+j)/4}([\ell-k+1][\ell+j+1])^{1/2}t^{\ell+1/2}_{k-1/2,j+1/2} \\ &\quad - q^{(k+j)/4}([\ell+k][\ell-j])^{1/2}t^{\ell-1/2}_{k-1/2,j+1/2},\end{aligned} \tag{4}$$

$$\begin{aligned}[2\ell+1]\pi_{22}t^{\ell}_{kj} &= q^{(2\ell+k+j)/4}([\ell-k+1][\ell-j+1])^{1/2}t^{\ell+1/2}_{k-1/2,j-1/2} \\ &\quad + q^{-(2\ell-k-j+2)/4}([\ell+k][\ell+j])^{1/2}t^{\ell-1/2}_{k-1/2,j-1/2}.\end{aligned} \tag{5}$$

Setting $\ell_2 = 1/2$ into formula (1) of Section 14.4.1, we have the relations

$$[2\ell+1]t^\ell_{kj}\pi_{11} = q^{(2\ell-k-j)/4}([\ell+k+1][\ell+j+1])^{1/2}t^{\ell+1/2}_{k+1/2,j+1/2} + q^{-(2\ell+k+j+2)/4}([\ell-k][\ell-j])^{1/2}t^{\ell-1/2}_{k+1/2,j+1/2}, \tag{6}$$

$$[2\ell+1]t^\ell_{kj}\pi_{12} = q^{-(k+j)/4}([\ell+k+1][\ell-j+1])^{1/2}t^{\ell+1/2}_{k+1/2,j-1/2} - q^{-(k+j)/4}([\ell-k][\ell+j])^{1/2}t^{\ell-1/2}_{k+1/2,j-1/2}, \tag{7}$$

$$[2\ell+1]t^\ell_{kj}\pi_{21} = q^{-(k+j)/4}([\ell-k+1][\ell+j+1])^{1/2}t^{\ell+1/2}_{k-1/2,j+1/2} - q^{-(k+j)/4}([\ell+k][\ell-j])^{1/2}t^{\ell-1/2}_{k-1/2,j+1/2}, \tag{8}$$

$$[2\ell+1]t^\ell_{kj}\pi_{22} = q^{-(2\ell+k+j)/4}([\ell-k+1][\ell-j+1])^{1/2}t^{\ell+1/2}_{k-1/2,j-1/2} + q^{(2\ell-k-j+2)/4}([\ell+k][\ell+j])^{1/2}t^{\ell-1/2}_{k-1/2,j-1/2}. \tag{9}$$

For $k=\ell$ we derive from formulas (2) and (3) that

$$[2\ell+1]^{1/2}\pi_{11}t^\ell_{\ell j} = q^{-(\ell-j)/4}[\ell+j+1])^{1/2}t^{\ell+1/2}_{\ell+1/2,j+1/2}, \tag{10}$$

$$[2\ell+1]^{1/2}\pi_{12}t^\ell_{\ell j} = q^{(\ell-j)/4}[\ell-j+1])^{1/2}t^{\ell+1/2}_{\ell+1/2,j-1/2}. \tag{11}$$

Formulas (4) and (5) for $k=-\ell$ yield

$$[2\ell+1]^{1/2}\pi_{21}t^\ell_{-\ell,j} = q^{-(\ell-j)/4}[\ell+j+1])^{1/2}t^{\ell+1/2}_{-\ell-1/2,j+1/2}, \tag{12}$$

$$[2\ell+1]^{1/2}\pi_{22}t^\ell_{-\ell,j} = q^{(\ell+j)/4}[\ell-j+1])^{1/2}t^{\ell+1/2}_{-\ell-1/2,j-1/2}. \tag{13}$$

Analogous relations follow from formulas (6)–(9). Considering these relations as recurrence formulas for the matrix elements $t^\ell_{\pm\ell,j}$ and $t^\ell_{j,\pm\ell}$ and making use of equalities (1), we derive

$$t^\ell_{n\ell} = q^{(\ell^2-n^2)/4}\left(\frac{[2\ell]!}{[\ell+n]![\ell-n]!}\right)^{1/2}\pi_{11}^{\ell+n}\pi_{21}^{\ell-n}, \tag{14}$$

$$t^\ell_{\ell n} = q^{(\ell^2-n^2)/4}\left(\frac{[2\ell]!}{[\ell+n]![\ell-n]!}\right)^{1/2}\pi_{11}^{\ell+n}\pi_{12}^{\ell-n}, \tag{15}$$

$$t^\ell_{n,-\ell} = q^{(\ell^2-n^2)/4}\left(\frac{[2\ell]!}{[\ell+n]![\ell-n]!}\right)^{1/2}\pi_{12}^{\ell+n}\pi_{22}^{\ell-n}, \tag{16}$$

$$t^\ell_{-\ell,n} = q^{(\ell^2-n^2)/4}\left(\frac{[2\ell]!}{[\ell+n]![\ell-n]!}\right)^{1/2}\pi_{21}^{\ell+n}\pi_{22}^{\ell-n}. \tag{17}$$

Now we can find expressions for general matrix elements. They are of the form

$$t^\ell_{mn} = \left(\frac{[\ell-n]![\ell+m]!}{[\ell+n]![\ell-m]!}\right)^{1/2} \frac{\pi_{11}^{m+n}\pi_{12}^{m-n}}{[m-n]!} q^{(m+n)(m-n)/4} \times {}_2\Phi_1(-\ell+m,\ell+m+1;m-n+1;q,-\sqrt{q}\pi_{12}\pi_{21}), \tag{18}$$

$$t^\ell_{nm} = \left(\frac{[\ell-n]![\ell+m]!}{[\ell+n]![\ell-m]!}\right)^{1/2} \frac{\pi_{11}^{m+n}\pi_{21}^{m-n}}{[m-n]!} q^{(m+n)(m-n)/4} \times {}_2\Phi_1(-\ell+m,\ell+m+1;m-n+1;q,-\sqrt{q}\pi_{12}\pi_{21}), \tag{19}$$

$$t^\ell_{-m,n} = \left(\frac{[\ell+m]![\ell+n]!}{[\ell-m]![\ell-n]!}\right)^{1/2} \frac{q^{-(m+n)(m-n)/4}}{[m+n]!} \times {}_2\Phi_1(-\ell+m,\ell+m+1;m+n+1;q,-\sqrt{q}\pi_{12}\pi_{21})\pi_{22}^{m-n}\pi_{21}^{m+n}, \tag{20}$$

$$t^\ell_{n,-m} = \left(\frac{[\ell+m]![\ell+n]!}{[\ell-m]![\ell-n]!}\right)^{1/2} \frac{q^{-(m+n)(m-n)/4}}{[m+n]!} \times {}_2\Phi_1(-\ell+m,\ell+m+1;m+n+1;q,-\sqrt{q}\pi_{12}\pi_{21})\pi_{22}^{m-n}\pi_{12}^{m+n} \tag{21}$$

if $m \geq |n|$. Really, it follows from relations (14)–(17) that formulas (18)–(21) are valid when $m = \ell$. In order to show their validity for arbitrary m we use mathematical induction. Namely, we substitute expression (18) for t^ℓ_{mn} into equalities (2)–(4). After simple manipulations these equalities are reduced to formulas (12)–(14) of Section 14.1.8. Since relations (3) and (4) allow us to go over from t^ℓ_{mn} to $t^{\ell+1}_{mn}$, then expression (18) for t^ℓ_{mn} is valid. By means of relations (2)–(4) one similarly proves that formula (19) is valid. Formulas (20) and (21) are proved with the help of relations (6)–(8).

In conclusion of this section we present the matrix of the representation T_1:

$$\begin{pmatrix} \pi_{11}^2 & q^{1/4}[2]^{1/2}\pi_{11}\pi_{12} & \pi_{12}^2 \\ q^{1/4}[2]^{1/2}\pi_{11}\pi_{21} & q^{1/4}[2]^{1/2}\pi_{11}\pi_{22} - qI & q^{1/4}[2]^{1/2}\pi_{12}\pi_{22} \\ \pi_{21}^2 & q^{1/4}[2]^{1/2}\pi_{21}\pi_{22} & \pi_{22}^2 \end{pmatrix}.$$

14.4.4. Expressions in terms of q-Jacobi polynomials. The formula

$$p_n(x;a,b\,|\,q) = {}_2\varphi_1(q^{-n},abq^{n+1};aq;q,qx) \tag{1}$$

defines a polynomial in x called the *little q-Jacobi polynomial.* Comparing (1) with formulas (18)–(21) of the previous section, we obtain expressions for matrix elements in terms of polynomials $p_n(x;a,b|q)$. Namely,

$$t^\ell_{mn} = \left(\frac{[\ell-n]![\ell+m]!}{[\ell+n]![\ell-m]!}\right)^{1/2} \frac{\pi_{11}^{m+n}\pi_{12}^{m-n}}{[m-n]!} q^{(m+n)(m-n)/4} \times p_{\ell-m}(-q^{-1/2}\pi_{12}\pi_{21};q^{m-n},q^{m+n}|q), \tag{2}$$

$$t^{\ell}_{nm}=\left(\frac{[\ell-n]![\ell+m]!}{[\ell+n]![\ell-m]!}\right)^{1/2}\frac{\pi_{11}^{m+n}\pi_{21}^{m-n}}{[m-n]!}q^{(m+n)(m-n)/4}$$
$$\times\, p_{\ell-m}(-q^{-1/2}\pi_{12}\pi_{21};q^{m-n},q^{m+n}|q), \tag{3}$$

$$t^{\ell}_{-m,n}=\left(\frac{[\ell+m]![\ell+n]!}{[\ell-m]![\ell-n]!}\right)^{1/2}\frac{q^{-(m+n)(m-n)/4}}{[m+n]!}$$
$$\times\, p_{\ell-m}(-q^{-1/2}\pi_{12}\pi_{21};q^{m+n},q^{m-n}|q)\pi_{22}^{m-n}\pi_{21}^{m+n}, \tag{4}$$

$$t^{\ell}_{n,-m}=\left(\frac{[\ell+m]![\ell+n]!}{[\ell-m]![\ell-n]!}\right)^{1/2}\frac{q^{-(m+n)(m-n)/4}}{[m+n]!}$$
$$\times\, p_{\ell-m}(-q^{-1/2}\pi_{12}\pi_{21};q^{m+n},q^{m-n}|q)\pi_{22}^{m-n}\pi_{12}^{m+n}, \tag{5}$$

where $m \geq |n|$.

14.5. Racah Coefficients of the Algebra $U_q(\mathfrak{sl}_2)$

14.5.1. Properties of Racah coefficients. Racah coefficients of the quantum algebra $U_q(\mathfrak{sl}_2)$ are defined in the same way as in the case of the group $SU(2)$ (see Section 8.4.1). Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ be carrier spaces of irreducible finite dimensional representations T_{ℓ_1}, T_{ℓ_2}, T_{ℓ_3} of $U_q(\mathfrak{sl}_2)$, and $\{\mathbf{e}_j\}$, $\{\mathbf{f}_k\}$ and $\{\mathbf{h}_m\}$ be the orthonormal bases of these spaces, for which the operators $T_{\ell_i}(H)$, $T_{\ell_i}(E_\pm)$, $i=1,2,3$, are given by formulas of the form (3) of Section 14.3.2. The tensor product $\mathcal{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2\otimes\mathfrak{H}_3$ can be considered either as the product $(\mathfrak{H}_1\otimes\mathfrak{H}_2)\otimes\mathfrak{H}_3$ or as the product $\mathfrak{H}_1\otimes(\mathfrak{H}_2\otimes\mathfrak{H}_3)$. As in the classical case (see Section 8.4.1), this leads to two orthonormal bases in $\mathcal{H}$ whose elements are respectively of the form

$$\mathbf{e}_p^{(\ell_1\ell_2)\ell_{12},\ell_3,\ell}=\sum_{i,j,k}C^{\ell_1\ell_2\ell_{12}}_{ijm}C^{\ell_{12}\ell_3\ell}_{mkp}(\mathbf{e}_i\otimes\mathbf{f}_j)\otimes\mathbf{h}_k, \tag{1}$$

$$e_p^{\ell_1,(\ell_2\ell_3)\ell_{23},\ell}=\sum_{i,j,k}C^{\ell_2\ell_3\ell_{23}}_{jkn}C^{\ell_1\ell_{23}\ell}_{inp}\mathbf{e}_i\otimes(\mathbf{f}_j\otimes\mathbf{h}_k), \tag{2}$$

where $i+j=m$, $m+k=p$, $i+n=p$, $j+k=n$ and $C^{abc}_{\alpha\beta\gamma}$ are CGC's of $U(\mathfrak{sl}_2)$. These basis elements are connected by a unitary matrix R:

$$\mathbf{e}_p^{(\ell_1\ell_2)\ell_{12},\ell_3,\ell}=\sum_{\ell_{23}}R_q(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)\mathbf{e}_p^{\ell_1,(\ell_2\ell_3)\ell_{23},\ell}. \tag{3}$$

Elements $R_q(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)$ of this matrix do not depend on indices i, j, k, m, n, p of basis elements from formulas (1) and (2). They are called *Racah coefficients* (RC's) of the quantum algebra $U(\mathfrak{sl}_2)$ or *q-Racah coefficients.* Unitarity of the matrix R means that the orthogonality relations

$$\sum_{\ell_{23}}R_q(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)R_q(\ell_1\ell_2\ell_3,\ell'_{12}\ell_{23},\ell)=\delta_{\ell_{12}\ell'_{12}}, \tag{4}$$

$$\sum_{\ell_{12}} R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell) R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell'_{23}, \ell) = \delta_{\ell_{23}\ell'_{23}} \tag{5}$$

hold.

The relation

$$\mathbf{e}_i \otimes (\mathbf{f}_j \otimes \mathbf{h}_k) = \sum_{\ell_{23}} \sum_{\ell} C^{\ell_2\ell_3\ell_{23}}_{jkn} C^{\ell_1\ell_{23}\ell}_{inp} \mathbf{e}^{\ell_1,(\ell_2\ell_3)\ell_{23},\ell}_p$$

is inverse to equality (2). Substituting this expression for $\mathbf{e}_i \otimes \mathbf{f}_j \otimes \mathbf{h}_k$ into formula (1) and comparing the obtained relation with (3) we derive

$$\begin{aligned} &R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell) \\ &\qquad = \sum_{i=-\ell_1}^{\ell_1} \sum_{j=-\ell_2}^{\ell_2} \sum_{k=-\ell_3}^{\ell_3} C^{\ell_1\ell_2\ell_{12}}_{ijm} C^{\ell_{12}\ell_3\ell}_{mkp} C^{\ell_2\ell_3\ell_{23}}_{jkn} C^{\ell_1\ell_{23}\ell}_{inp}, \end{aligned} \tag{6}$$

where $i+j=m$, $m+k=p$, $j+k=n$, $i+n=p$. It follows from (6) that RC's are real if CGC's are real.

As in the classical case, symmetry properties of RC's follow from those of CGC's by means of formula (6). It is more convenient to formulate symmetry properties for $6j$ Wigner symbols which are defined by the formula

$$\begin{aligned} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q &= (-1)^{\ell_1+\ell_2+\ell_3+\ell}([2\ell_{12}+1][2\ell_{23}+1])^{-1/2} \\ &\qquad \times R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell). \end{aligned} \tag{7}$$

Symmetry relations for $\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q$ have the same form as in the case of RC's of the group $SU(2)$ from Section 8.4.3. Besides, as it follows from formula (5) of Section 14.3.7, we have new symmetry relation

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q = \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_{q^{-1}}. \tag{8}$$

CGC $C^{\ell_1\ell_2\ell}_{ijk}$ vanishes if the triple (ℓ_1, ℓ_2, ℓ) does not satisfy the triangle condition $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$. Hence, as it follows from (6), Rc $R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell)$ vanishes if even one of the triples $(\ell_1, \ell_2, \ell_{12})$, $(\ell_{12}, \ell_3, \ell)$, $(\ell_2, \ell_3, \ell_{23})$, $(\ell_1, \ell_{23}, \ell)$ does not satisfy this condition.

14.5.2. Calculation of RC's. In just the same way as in the classical case (see Section 8.4.2) one proves that CGC's and RC's of the quantum algebra $U(\mathfrak{sl}_2)$ satisfy the relation

$$\begin{aligned} &\sum_{i=-\ell_1}^{\ell_1} \sum_{j=-\ell_2}^{\ell_2} C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} \\ &\qquad = R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell) C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k}, \end{aligned} \tag{1}$$

where the summation is over those admissible values of i and j for which $i+j=$ const. This relation is used to calculate RC's. Let us carry out this calculation. For this we put $i+j=\ell_{12}$, $k=\ell-\ell_{12}$ into formula (1). We have

$$R_q(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell)=(C^{\ell_{12}\ell_3\ell}_{\ell_{12},\ell-\ell_{12},\ell})^{-1}$$
$$\times\sum_{\substack{i,j\\ i+j=\text{const}}} C^{\ell_1\ell_2\ell_{12}}_{ij\ell_{12}}C^{\ell_2\ell_3\ell_{23}}_{j,\ell-\ell_{12},j+\ell-\ell_{12}}C^{\ell_1\ell_{23}\ell}_{i,j+\ell-\ell_{12},\ell}.$$

Utilizing here the expression for special cases of CGC's

$$C^{\ell_1\ell_2\ell}_{jk\ell}=(-1)^{\ell_1-j}q^A\frac{\Delta(\ell_1,\ell_2,\ell)}{[\ell_1-\ell_2+\ell]!}\left(\frac{[\ell_1+j]![\ell_2+k]![2\ell+1]}{[\ell_2-\ell_1+\ell]!^2[\ell_1-j]![\ell_2-k]!}\right)^{1/2},$$

where $j+k=\ell$, $A=\frac{1}{4}\{\ell_2(\ell_2+1)-\ell_1(\ell_1+1)-\ell(\ell+1)+2j(\ell+1)\}$, we obtain

$$\begin{Bmatrix}\ell_1 & \ell_2 & \ell_{12}\\ \ell_3 & \ell & \ell_{23}\end{Bmatrix}_q$$
$$=(-1)^{\ell-\ell_1+\ell_2+\ell_3}\left(\frac{[\ell+\ell_3+\ell_{12}+1]![\ell_1+\ell_2-\ell_{12}]![\ell_1+\ell_{23}-\ell]!}{[\ell_1+\ell_2+\ell_{12}+1]![\ell_1-\ell+\ell_{23}]![\ell_1-\ell_2+\ell_{12}]!}\right.$$
$$\times\left.\frac{[\ell-\ell_3+\ell_{12}]![2\ell_{23}+1]^{-1}}{[\ell_2-\ell_1+\ell_{12}]![\ell_1+\ell-\ell_{23}]![\ell-\ell_1+\ell_{23}]!}\right)^{1/2}q^B\sum_i q^{i(\ell+\ell_{12}+2)/2} \tag{2}$$
$$\times\left(\frac{[\ell+\ell_{23}-i]![\ell_2+\ell_{12}-i]!}{[\ell_{23}-\ell+i]![\ell_2-\ell_{12}+i]!}\right)^{1/2}\frac{[\ell_1+i]!}{[\ell_1-i]!}C^{\ell_2\ell_3\ell_{23}}_{\ell_{12}-i,\ell-\ell_{12},\ell-i},$$

where

$$B=\frac{1}{2}\{\ell_2(\ell_2+1)+\ell_{23}(\ell_{23}+1)-\ell_3(\ell_3+1)-2\ell_1(\ell_1+1)-2\ell_{12}(\ell+1)\}.$$

Substituting into (2) expression (5) of Section 14.3.5 for CGC in terms of a sum, we derive the relation

$$\begin{Bmatrix}\ell_1 & \ell_2 & \ell_{12}\\ \ell_3 & \ell & \ell_{23}\end{Bmatrix}_q$$
$$=\frac{(-1)^{\ell-\ell_1+\ell_2+\ell_3-\ell_{12}+\ell_{23}}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_1,\ell_{23},\ell)\Delta(\ell_2,\ell_3,\ell_{23})}{[\ell+\ell_3-\ell_{12}]![\ell_1-\ell_2+\ell_{12}]![\ell_2-\ell_1+\ell_{12}]![\ell_1-\ell_{23}+\ell]![\ell_{23}-\ell_1+\ell]!}$$
$$\times\frac{\Delta(\ell_{12},\ell_3,\ell)[\ell_{12}+\ell_3+\ell+1]!}{[\ell_2-\ell_3+\ell_{23}]![\ell_3-\ell_2+\ell_{23}]!}\sum_{i,r}\frac{(-1)^{i+r}q^C[\ell_1+i]![\ell_2+\ell_{12}-i+r]!}{[r]![\ell_1-i]![\ell_2-\ell_{12}+i-r]!}$$
$$\times\frac{[\ell_{23}+\ell-i]![\ell_3+\ell_{23}-\ell_{12}+i-r]!}{[\ell_3+\ell_{12}-\ell_{23}-i+r]![\ell_{23}-\ell+i-r]!},$$

where $C = \frac{1}{2}\{i(i+1) + r(\ell - i + \ell_{23} + 1) - \ell_1(\ell_1 + 1)\}$. Let us go over from the summation over r and i to the summation over $k = i - r$ and i. The sum over i is reduced to

$$_2\Phi_1(-\ell_1 + k, \ell_1 + k + 1; -\ell_{23} - \ell + k; q, q).$$

Application of formula (2′) of Section 14.1.8 to this function yields

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q$$
$$= \frac{(-1)^{\ell-\ell_1+\ell_2+\ell_3}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_1,\ell_{23},\ell)\Delta(\ell_2,\ell_3,\ell_{23})\Delta(\ell_{12},\ell_3,\ell)}{[\ell_3+\ell-\ell_{12}]![\ell_1-\ell_2+\ell_{12}]![\ell_2-\ell_1+\ell_{12}]![\ell_1-\ell_{23}+\ell]![\ell_2-\ell_3+\ell_{23}]!}$$
$$\times \frac{[\ell_{12}+\ell_3+\ell+1]![\ell_1+\ell+\ell_{23}+1]!}{[\ell_3-\ell_2+\ell_{23}]!} \sum_s \frac{(-1)^s[2\ell_2-s]!}{[s]![\ell_1+\ell_2-\ell_{12}-s]!}$$
$$\times \frac{[\ell_1-\ell_2+\ell_{12}+s]![\ell_3+\ell_{23}-\ell_2+s]!}{[\ell_2+\ell_3-\ell_{23}-s]![\ell_{23}+\ell_{12}-\ell_2-\ell+s]![\ell_{23}+\ell_{12}-\ell_2+\ell+s+1]!}, \quad (3)$$

where $s = \ell_2 - \ell_{12} + k$. The sum over s can be expressed in terms of the q-hypergeometric series $_4\varphi_3(\dots; q, q)$. Using the symmetry relation

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q = (-1)^{2(\ell_{12}+\ell_{23})} \begin{Bmatrix} \ell_1 & \ell_2 & \overline{\ell}_{12} \\ \ell_3 & \ell & \overline{\ell}_{23} \end{Bmatrix}_q$$

where $\overline{\ell} = -\ell - 1$ we obtain

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q$$
$$= \frac{(-1)^{\ell_2+\ell_{12}+\ell+\ell_{23}}\Delta(\ell_1,\ell_2,\ell_{12})\Delta(\ell_{12},\ell_3,\ell)\Delta(\ell_1,\ell_{23},\ell)}{[\ell_2-\ell_1+\ell_{12}]![\ell_1+\ell_2-\ell_{12}]![\ell-\ell_3+\ell_{12}]![\ell_{12}-\ell+\ell_3]!} \quad (4)$$
$$\times \frac{\Delta(\ell_2,\ell_3,\ell_{23})[2\ell_2]![\ell_2+\ell_{12}-\ell+\ell_{23}]![\ell_2+\ell+\ell_{12}+\ell_{23}+1]!}{[\ell_1-\ell+\ell_{23}]![\ell-\ell_1+\ell_{23}]![\ell_2+\ell_3-\ell_{23}]![\ell_2-\ell_3+\ell_{23}]!}$$
$$\times\ {}_4\Phi_3\left(\begin{matrix} \ell_1-\ell_2-\ell_{12},\ \ell_3-\ell_2-\ell_{23},\ -\ell_1-\ell_2-\ell_{12}-1,\ -\ell_2-\ell_3-\ell_{23}-1 \\ -2\ell_2,\ -\ell_2-\ell_{12}+\ell-\ell_{23},\ -\ell_2-\ell_{12}-\ell-\ell_{23}-1 \end{matrix}\middle|\, q, q\right).$$

Other expressions for RC's of $U_q(\mathfrak{sl}_2)$ can be derived from (3) and (4) with the help of symmetry formulas. Symmetries of expressions for RC's are equivalent to the relation

$$_4\varphi_3\left(\begin{matrix} q^{-n},\ a,\ b,\ c \\ d,\ e,\ f \end{matrix}\middle|\, q, q\right) = \left(\frac{bc}{d}\right)^n \frac{(aq^{1-n}/e;q)_n(aq^{1-n}/f;q)_n}{(e;q)_n(f;q)_n}$$
$$\times\ {}_4\varphi_3\left(\begin{matrix} q^{-n},\ a,\ d/b,\ d/c \\ d,\ aq^{1-n}/e,\ aq^{1-n}/f \end{matrix}\middle|\, q, q\right) \quad (5)$$

for the function ${}_4\varphi_3$ where $q^{1-n}abc = def$. Applying this relation to the right hand side of (4), we obtain different expressions for RC's in terms of the function ${}_4\Phi_3$. In particular, in this way one receives q-analogs of formulas (5), (6) of Section 8.4.4 and of other known expressions for RC's of $SU(2)$. All these q-analogs are obtained from classical expressions by replacements of all factorials $m!$ by the corresponding q-factorials $[m]!$ and (if RC's are expressed in terms of hypergeometric series) of

$$ {}_4F_3(a, b, c, d; e, f, g; 1) \quad \text{by} \quad {}_4\Phi_3(a, b, c, d; e, f, g; q, q). $$

We write out two other relations, analogous to equality (3), which will be useful below. Putting $k = \ell_3$, $i + j + k = \ell$ in (1) we find

$$ \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q = \frac{([\ell+\ell_3-\ell_{12}]![\ell+\ell_3+\ell_{12}+1]!)^{1/2}}{\Delta(\ell_1,\ell,\ell_{23})\Delta(\ell_2,\ell_3,\ell_{23})[2\ell_{12}+1]^{1/2}}(-1)^{\ell_{12}+\ell_3+\ell_{23}} q^A \sum_m (-1)^m $$
$$ \times q^{m(\ell+\ell_3+1)/2} \frac{[\ell+\ell_{23}+m]!}{[\ell_{23}-\ell+m]!} \left(\frac{[\ell_1+m]![\ell_2-\ell+\ell_3+m]!}{[\ell_1-m]![\ell_2+\ell-\ell_3-m]!} \right)^{1/2} C^{\ell_1\ell_2\ell_{12}}_{m,\ell-\ell_3-m,\ell-\ell_3}, \tag{6} $$

where $A = \frac{1}{4}\{\ell_{12}(\ell_{12}+1) - \ell_1(\ell_1+1) - \ell_2(\ell_2+1) - 2\ell(\ell+1) + 2\ell_{23}(\ell_{23}+1)\}$. Another relation is of the form

$$ \begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q = \frac{\Delta(\ell_1,\ell,\ell_{23})\Delta(\ell_2,\ell_3,\ell_{23})}{([\ell+\ell_3-\ell_{12}]![\ell+\ell_3+\ell_{12}+1]![2\ell_{12}+1])^{1/2}}(-1)^{\ell+\ell_1+\ell_2+\ell_3} $$
$$ \times q^A \sum_m \frac{q^{-m(\ell+\ell_3+1)/2}}{[\ell-\ell_{23}+m]![\ell+\ell_{23}+m+1]!} \left(\frac{[\ell_1+m]![\ell_2+\ell-\ell_3+m]!}{[\ell_1-m]![\ell_2-\ell+\ell_3-m]!} \right)^{1/2} $$
$$ \times C^{\ell_1\ell_2\ell_{12}}_{m,\ell_3-\ell-m,\ell_3-\ell}, \tag{7} $$

where A is the same as in (6).

14.5.3. Special values of RC's. If $\ell_{12} = 0$ in formula (4) of previous section, then $\ell_1 = \ell_2$ and $\ell_1 - \ell_2 - \ell_{12} = 0$. As a result we have

$$ \begin{Bmatrix} \ell_1 & \ell_2 & 0 \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q = \frac{(-1)^{\ell+\ell_1-\ell_{23}}}{([2\ell_1+1][2\ell_3+1])^{1/2}} \delta_{\ell_1\ell_2}\delta_{\ell\ell_3}. \tag{1} $$

Making use of formula (4) of Section 14.5.2 and symmetry relation for RC's, for $\ell_1 + \ell_2 = \ell_{12}$ we obtain

$$ \begin{Bmatrix} a & b & a+b \\ d & e & f \end{Bmatrix}_q = (-1)^{a+b+d+e} \left(\frac{[2a]![2b]![a+b+d+e+1]![a+b-d+e]!}{[2a+2b+1]![d+e-a-b]![a+e-f]!} \right. $$
$$ \times \left. \frac{[a+b+d-e]!e+f-a]![d+f-b]!}{[a-e+f]![a+e+f+1]![b+d-f]![b-d+f]![b+d+f+1]!} \right)^{1/2}, \tag{2} $$

where we have used obvious renotations.

We also give RC's with $\ell_3 = 1/2$:

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ 1/2 & \ell+1/2 & \ell_2 \pm 1/2 \end{Bmatrix}_q = (-1)^{\ell_1+\ell_2+1/2\pm 1/2} \times \left(\frac{[\ell_1+\ell_2+\ell\pm 1][\mp\ell_1 \pm \ell_2 + \ell + 1]}{[2\ell_2+1][2\ell+1][2\ell_2+1\pm 1][2\ell+2]} \right)^{1/2}, \tag{3}$$

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ 1/2 & \ell-1/2 & \ell_2 \pm 1/2 \end{Bmatrix}_q = (-1)^{\ell_1+\ell_2+\ell} \times \left(\frac{[\ell_1+\ell_2\mp\ell+1][\pm\ell_1 \mp \ell_2 + \ell]}{[2\ell_2+1][2\ell+1][2\ell][2\ell_2+1\pm 1]} \right)^{1/2}. \tag{4}$$

14.5.4. The Biedenharn-Elliott identity. For RC's of $U_q(\mathfrak{sl}_2)$ the Biedenharn-Elliott identity

$$\sum_{\ell_{23}} R_q(\ell_1\ell_2\ell_3, \ell_{12}\ell_{23}, \ell_{123}) R_q(\ell_1\ell_{23}\ell_4, \ell_{123}\ell_{234}, \ell) R_q(\ell_2\ell_3\ell_4, \ell_{23}\ell_{34}, \ell_{234}) = R_q(\ell_{12}\ell_3\ell_4, \ell_{123}\ell_{34}, \ell) R_q(\ell_1\ell_2\ell_{34}, \ell_{12}\ell_{234}, \ell) \tag{1}$$

is valid. Its proof coincides with the proof of the Biedenharn-Elliott identity for the group $SU(2)$ (see Section 8.4.8) and we omit it. In terms of $6j$ symbols it takes the form

$$\begin{Bmatrix} c' & a & a' \\ e & b' & b \end{Bmatrix}_q \begin{Bmatrix} a' & e & b' \\ d & d' & c \end{Bmatrix}_q = \sum_f (-1)^p [2f+1] \times \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q \begin{Bmatrix} c' & b & b' \\ d & d' & f \end{Bmatrix}_q \begin{Bmatrix} c' & a & a' \\ c & d' & f \end{Bmatrix}_q, \tag{2}$$

where $p = a' + b' - c' - d' - a - b - c - d + e - f$.

By means of the orthogonality relation for RC's we derive from (1) the equality which can be written down in the form

$$R_q(aa'f, c'c, d') = \sum_{e,e'} R_q(abd, ef, c) R_q(c'bd, e'f, d') \times R_q(a'ab, c'c, e') R_q(a'ed, e'c, d'). \tag{3}$$

Putting here $d = 1/2$, $b = f - 1/2$ and using RC's (3) and (4) of previous section, we obtain the recurrence relation for RC's which, with the help of the notation

$$R^{abc}_{jkm}(r) = R_q(r-m, a, b; r-k, c; r), \quad j + k = m, \tag{4}$$

can be written in the form

$$R^{abc}_{jkm}(r)$$

$$\left(\frac{[a+b-c][a-b+c+1][b-k][2r-b-k+1][c+m+1][2r+c-m+2]}{[2b]^2[2c+1][2r+1]^2[2c+2]}\right)^{1/2}$$
$$\times R^{a,b-1/2,c+1/2}_{j,k+1/2,m+1/2}(r+\tfrac{1}{2})$$
$$+\left(\frac{[b-a+c][a+b+c+1][b-k][2r-b-k+1][c-m][2r-c-m+1]}{[2b]^2[2c][2c+1][2r+1]^2}\right)^{1/2}$$
$$\times R^{a,b-1/2,c-1/2}_{j,k+1/2,m+1/2}(r+\tfrac{1}{2})$$
$$+\left(\frac{[a-b+c][a-b+c+1][b+k][2r+b-k+1][c-m+1][2r-c-m]}{[2b]^2[2c+1][2c+2][2r+1]^2}\right)^{1/2}$$
$$\times R^{a,b-1/2,c+1/2}_{j,k-1/2,m-1/2}(r-\tfrac{1}{2})$$
$$+\left(\frac{[b-a+c][a+b+c+1][b+k][2r+b-k+1][c+m][2r+c-m+1]}{[2b]^2[2c][2c+1][2r+1]^2}\right)^{1/2}$$
$$\times R^{a,b-1/2,c-1/2}_{j,k-1/2,m-1/2}(r-\tfrac{1}{2}). \tag{5}$$

If we start from RC's $R^{a,1/2,a\pm 1/2}_{j,\pm 1/2,j\pm 1/2}$, then by means of (5) we can find expressions for all RC's.

14.5.5. The addition theorem for RC's. An analog of addition theorem (8) of Section 8.4.7 in the case of RC's of $U_q(\mathfrak{sl}_2)$ is of the form

$$\sum_r (-1)^{c+f+r} q^A [2r+1] \begin{Bmatrix} a & d & r \\ e & b & c \end{Bmatrix}_q \begin{Bmatrix} a & d & r \\ b & e & f \end{Bmatrix}_q = q^B \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_q, \tag{1}$$

where

$$A = -\frac{1}{2}\{f(f+1)+c(c+1)+r(r+1)\},$$
$$B = -\frac{1}{2}\{a(a+1)+b(b+1)+d(d+1)+e(e+1)\}.$$

To prove this formula we replace the first RC on the left hand side by expression (6) of Section 14.5.2, the second RC by expression (7) of the same section and take into account the orthogonality relation for CGC's of $U_q(\mathfrak{sl}_2)$. As a result, the left hand side of (1) is reduced to the form

$$q^B(-1)^{d+e+c}\frac{\Delta(a,b,c)\Delta(c,d,e)}{\Delta(a,c,f)\Delta(b,d,f)}$$
$$\times \sum_n \frac{(-1)^{a-n}[a+n]![e+f-n]![b-e+d+n]!}{[a-n]![b-c+n]![b+c+n+1]![f-e+n]![d+e-b-n]!}.$$

This expression without q^B gives expression (3) of Section 14.5.2 for $6j$ Wigner symbol. Thus, formula (1) is proved.

If $f=0$, $a=d$, $b=e$, then relation (1) leads to the summation formula

$$\sum_r q^{-\{c(c+1)+r(r+1)\}/2}[2r+1]\begin{Bmatrix} a & b & c \\ a & b & r \end{Bmatrix}_q = (-1)^{2(a+b)}q^{-a(a+1)-b(b+1)}. \qquad (2)$$

14.5.6. Generalization of the Biedenharn-Elliott identity. Let us introduce the expression

$$\begin{Bmatrix} a & b & c \\ e & d & f \\ h & k & g \end{Bmatrix} = \sum_n (-1)^{2n} q^A [2n+1] \begin{Bmatrix} a & e & h \\ k & g & n \end{Bmatrix}_q \begin{Bmatrix} b & d & k \\ e & n & f \end{Bmatrix}_q \begin{Bmatrix} c & f & g \\ n & a & b \end{Bmatrix}_q, \qquad (1)$$

where $A = -\frac{1}{2}\{n(n+1)+h(h+1)+d(d+1)+c(c+1)\}$. It is called the $9j$ *symbol* of $U_q(\mathfrak{sl}_2)$. One has the symmetry relation

$$\begin{Bmatrix} a & b & c \\ e & d & f \\ h & k & g \end{Bmatrix} = (-1)^B q^{-\chi(B)} \begin{Bmatrix} a & c & b \\ e & f & d \\ h & g & k \end{Bmatrix}, \qquad (2)$$

where

$$B = \sum_{i=1}^{9} a_i \equiv a+b+c+d+e+f+h+k+g, \quad \chi(B) = \sum_{i=1}^{9} a_i(a_i+1).$$

In order to prove it we replace the first RC by the expression for it in terms of a sum of products of RC's (see formula (1) of Section 14.5.5) in which q^{-1} is used instead of q, and carry out summation over n with the help of the Biedenharn-Elliott identity. As a result we obtain the expression for the right hand side of (2).

Representing both sides of (2) in terms of RC's, we derive the relation

$$\sum_n (-1)^{2n}[2n+1]q^E \begin{Bmatrix} a & g & n \\ k & e & h \end{Bmatrix}_q \begin{Bmatrix} a & c & b \\ f & n & g \end{Bmatrix}_q \begin{Bmatrix} k & b & d \\ f & e & n \end{Bmatrix}_q = \sum_m (-1)^{2m}[2m+1]q^F \begin{Bmatrix} k & c & m \\ f & h & g \end{Bmatrix}_q \begin{Bmatrix} a & m & d \\ f & e & h \end{Bmatrix}_q \begin{Bmatrix} a & c & b \\ k & d & m \end{Bmatrix}_q, \qquad (3)$$

where

$$E = -\frac{1}{2}\{n(n+1)+h(h+1)+c(c+1)+d(d+1)\},$$

$$F = -\frac{1}{2}\{m(m+1) + g(g+1) + e(e+1) + b(b+1)\}.$$

14.5.7. CGC's as a limit of RC's. Let us prove that CGC's of the algebra $U_q(\mathfrak{sl}_2)$ are the limit case of its RC's, i.e. that

$$\lim_{r\to\infty} R^{abc}_{jkm}(r) = C^{abc}_{jkm}, \quad j+k=m, \quad |q|>1. \tag{1}$$

For this we note that the relation

$$\lim_{r\to\infty} \frac{[r+\alpha]}{[r+\beta]} = q^{(\alpha-\beta)/2}, \quad |q|>1, \tag{2}$$

and formulas (4), (4′) of Section 14.3.6 and (3), (4) of Section 14.5.3 implies validity of relation (1) for $b = 1/2$:

$$\lim_{r\to\infty} R^{a,1/2,a+m}_{j,k,j+k}(r) = C^{a,1/2,a+m}_{j,k,j+k}, \quad |q|>1, \tag{3}$$

where $m = \pm 1/2$, $k = \pm 1/2$. In order to derive relation (1) for the general case we make the limit $r \to \infty$ in formula (5) of Section 14.5.4. By virtue of formula (2) we have

$$\begin{aligned}
&\bar{C}^{abc}_{j,k,m} \\
&= q^{(t-s+1)/4}\left(\frac{[a+b-c][a-b+c+1][b-k][c+m+1]}{[2b]^2[2c+1][2c+2]}\right)^{1/2} \bar{C}^{a,b-1/2,c+1/2}_{j,k+1/2,m+1/2} \\
&\quad + q^{-(p+s)/4}\left(\frac{[b-a+c][a+b+c+1][b-k][c-m]}{[2b]^2[2c][2c+1]}\right)^{1/2} \bar{C}^{a,b-1/2,c-1/2}_{j,k+1/2,m+1/2} \\
&\quad - q^{(t-s-1)/4}\left(\frac{[a+b-c][a-b+c+1][b+k][c-m+1]}{[2b]^2[2c+1][2c+2]}\right)^{1/2} \bar{C}^{a,b-1/2,c+1/2}_{j,k-1/2,m-1/2} \\
&\quad + q^{(p-s)/4}\left(\frac{[b-a+c][a+b+c+1][b+k][c+m]}{[2b]^2[2c][2c+1]}\right)^{1/2} \bar{C}^{a,b-1/2,c-1/2}_{j,k-1/2,m-1/2},
\end{aligned} \tag{4}$$

where $\bar{C}^{abc}_{jkm} = \lim_{r\to\infty} R^{abc}_{jkm}(r)$, $s = j+2k$, $t = c-b$, $p = b+c$.

Let us show that CGC C^{abc}_{jkm} satisfies relation (4). For this we note that, as in the classical case, formula (1) of Section 14.5.2 implies

$$C^{afe}_{j,k,j+k} = \sum_{i,p,c} R_q(abd, cf, e) C^{abc}_{j,i,j+i} C^{cde}_{j+i,p,j+k} C^{bdf}_{ipk}.$$

Substituting $d = 1/2$, $b = f - 1/2$ here and making use of expressions (4), (4′) of Section 14.3.6 and (3), (4) of Section 14.5.3 for special values of CGC's and RC's, we obtain (up to notations) relation (4) for CGC's. Since relation (4) defines all

$\bar{C}^{abc}_{jkm}$ in terms of $\bar{C}^{a,1/2,c}_{jkm}$ and since, according to (3), we have $\bar{C}^{a,1/2,c}_{jkm} = C^{a,1/2,c}_{jkm}$, then relation (1) is valid for all values of indices. In ordinary notations it can be written as

$$\lim_{r\to\infty} R_q(a,b,d+r;c,f+r;e+r) = C^{abc}_{f-e,d-f,d-e}, \quad |q|>1. \tag{5}$$

This relation is equivalent to the asymptotic formula

$$\begin{Bmatrix} a & b & c \\ d+r & e+r & f+r \end{Bmatrix}_q \sim \frac{(-1)^{a+b+d+e}q^{-f/2}}{\sqrt{[2r+1][2c+1]}} C^{abc}_{f-e,d-f,d-e}, \quad |q|>1, \tag{6}$$

for large r.

If $|q|<1$, then instead of (2) we use the relation

$$\lim_{r\to\infty} \frac{[r+\alpha]}{[r+\beta]} = q^{-(\alpha-\beta)/2}, \quad |q|<1.$$

Repeating the above arguments, we obtain

$$\lim_{r\to\infty} R^{abc}_{jkm}(r) = (-1)^{a+b-c} C^{abc}_{-j,-k,-m}, \quad |q|<1, \tag{7}$$

instead of (1). Since $C^{abc}_{jkm} \equiv C^{abc}_{jkm}(q) \equiv C_q(\mathbf{a},\mathbf{j})$ is connected with $C^{abc}_{jkm}(q^{-1})$ by formula (4) of Section 14.3.7, then

$$\lim_{r\to\infty} R^{abc}_{jkm}(r) = C^{abc}_{jkm}(q^{-1}), \quad |q|<1. \tag{8}$$

For $6j$ Wigner symbols relation (7) takes the form of the asymptotic formula

$$\begin{Bmatrix} a & b & c \\ d+r & e+r & f+r \end{Bmatrix}_q \sim \frac{(-1)^{a+b-c}q^{f/2}}{\sqrt{[2c+1][2r+1]}} C^{abc}_{f-e,d-f,d-e}, \tag{9}$$

where $|q|<1$ and $r\to\infty$.

14.5.8. Other asymptotic formulas for RC's. By means of relation (5) of Section 14.5.2 it is easy to derive from formula (4) of the same section that the following expression for $6j$ Wigner symbols is valid:

$$\begin{Bmatrix} a & b & c \\ b+m & a+n & f \end{Bmatrix}_q = (-1)^{m+n-a-b-c}\Delta(a,b,c)$$
$$\times \left(\frac{[f-m]![f+m]![f-n]![f+n]![2a+n-f]![2b+m-f]!}{[2a+n+f+1]![2b+m+f+1]!} \right)^{1/2}$$
$$\times \left(\frac{[a+b-c+m+n]![a-b+c+n-m]![b-a+c+m-n]!}{[a+b+c+m+n+1]!} \right)^{1/2}$$
$$\times \sum_r \frac{(-1)^r [a+b+c+m+n+r+1]!([r]!)^{-1}}{[m+n+r]![f-m-r]![f-n-r]![a-b+c+n-f+r]![a+b-c-r]!}.$$

With the help of the formula

$$\frac{[N-r]!}{[N]!} \underset{N\to\infty}{\sim} [N[^{-r} q^{r(r-1)/4}, \quad |q|>1,$$

we can find asymptotics of the expressions

$$\frac{[2a+n-f]!}{[2a+n+f+1]!}, \frac{[2b+m-f]!}{[2b+m+f+1]!}, \frac{[a+b-c+m+n]!}{[a+b-c]!}, \frac{[a-b+c+n-m]!}{[a-b+c]!},$$

$$\frac{[b-a+c+m-n]!}{[b-a+c]!}, \frac{[a+b+c+1]!}{[a+b+c+m+n+1]!}, \frac{[a+b+c+m+n+r+1]!}{[a+b+c+1]!},$$

$$\frac{[b-a+c]!}{[b-a+c+m-f+r]!}, \frac{[a-b+c]!}{[a-b+c+n-f+r]!}, \frac{[a+b-c]!}{[a+b-r]!}$$

for large a, b, c. They allow us to derive that

$$\begin{aligned}
&\begin{Bmatrix} a & b & c \\ b+m & a+n & f \end{Bmatrix}_q \\
&\sim (-1)^{a+b+c-m-n} \left(\frac{[m+n]!^{-2}[f+m]![f+n]!}{[2a+1][2b+1][f-m]![f-n]!} \right)^{1/2} \\
&\times \left(\frac{[a+b-c][a+b+c+1]}{[2a+1][2b+1]} \right)^{(m+n)/2} \left(\frac{[a-b+c][b-a+c]}{[2a+1][2b+1]} \right)^{f-(m+n)/2} \\
&\times {}_2\Phi_1 \left(-f+m, -f+n; m+n+1; \frac{[a+b-c][a+b+c+1]}{[a-b+c][b-a+c]} \right), \quad |q|>1,
\end{aligned} \tag{1}$$

for large a, b, c (that is, for $a,b,c \gg m,n,f$). For $|q|<1$ formula (1) is also valid.

In the similar way one proves that

$$\begin{aligned}
&\lim_{r\to\infty} (-1)^{2r} \frac{[2r+c+f+1]!}{[2r+b+e]!} \begin{Bmatrix} a & b+r & c+r \\ d & e+r & f+r \end{Bmatrix}_q \\
&= \begin{cases} (-1)^{a+b+d+e} T(a,d,f-e,b-f,b-c,c-e) & \text{if } b+e \le c+f, \\ 0 & \text{if } b+e > c+f, \end{cases}
\end{aligned} \tag{2}$$

where

$$T(a,d,a',d',a'',d'') = \frac{1}{[a'-a'']!} \left(\frac{[a+a']![a-a'']![d-d']![d+d'']!}{[a-a']![a+a'']![d+d']![d-d'']!} \right)^{1/2}. \tag{3}$$

Let us replace ℓ_3, ℓ, ℓ_{23} in formula (6) of Section 14.5.2 by ℓ_3+r, $\ell+r$, $\ell_{23}+r$ respectively, and tend r to infinity. Making use of relations (6) and (9) of Section 14.5.7, we derive the equality

$$\begin{aligned}
C^{abc}_{jkm} &= (-1)^{a+b-c} \sum_{k'\ge k} (-1)^{j-j'} (q^{1/2}-q^{-1/2})^{j-j'} q^A \\
&\times T(a,b,j,k,j',k') C^{abc}_{-j',-k',-m},
\end{aligned} \tag{4}$$

where $j + k = j' + k' = m$ and

$$A = -\frac{1}{2}\{a(a+1) + b(b+1) - c(c+1)\} + \frac{1}{4}\{j - j' - (j+j')(k+k')\}.$$

With the help of the orthogonality relation for CGC's we obtain from (4) the equality

$$\sum_c (-1)^{a+b-c} q^B C^{abc}_{jkm} C^{abc}_{-j',-k',-m}$$
$$= \begin{cases} (-1)^{j-j'} q^A (q^{1/2} - q^{-1/2})^{j-j'} T(a,b,j,k,j',k') & \text{if } j \geq j', \\ 0 & \text{if } j < j', \end{cases}$$

where $B = \frac{1}{2}\{a(a+1) + b(b+1) - c(c+1)\}$, $A = \frac{1}{4}\{j - j' - (j+j')(k+k')\}$, which is valid both for $|q| > 1$ and for $|q| < 1$.

14.5.9. Recurrence relations and the second order difference equation. Three-term recurrence relations for RC's of the quantum algebra $U_q(\mathfrak{sl}_2)$ follow from the Biedenharn-Elliott identity of Section 14.5.4. Putting $c = 1/2$ into this identity and utilizing special values of RC's from formulas (3) and (4) of Section 14.5.3, we derive the recurrence relations, which are obtained from recurrence formulas (1)–(3) of Section 8.4.9 by replacement of all expressions in parentheses and of the number c in formula (1) by the corresponding q-numbers, that is, $(a+b+c+1)$ by $[a+b+c+1]$ and so on.

In the same way one can derive recurrence relations in which indices are changed by 1. For example,

$$A_d \begin{Bmatrix} a & b & c \\ d+1 & e & f \end{Bmatrix}_q - B_d \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_q + C_d \begin{Bmatrix} a & b & c \\ d-1 & e & f \end{Bmatrix}_q = 0, \qquad (1)$$

where

$$\begin{aligned} A_d &= [2d]([b+f-d][c+e-d][d+e-c+1][d+c-e+1][b+d-f+1] \\ &\quad \times [d+f-b+1][c+e+d+2][b+d+f+2])^{1/2}, \\ B_d &= [2d+2][b+d-f][d+c-e][b+d+f+1][d+e+c+1] \\ &\quad + [2d][b+f-d][c+e-d][d+f-b+1][d+e-c+1] \\ &\quad - [2d][2d+1][2d+2][b+c-a][a+b+c+1], \\ C_d &= [2d+2]([b+d-f][d+c-e][e-c+d][f-b+d][b+d+f+1] \\ &\quad \times [d+e+c+1][c+e-d+1][b+f-d+1])^{1/2}. \end{aligned}$$

In fact, formula (1) is the second order difference equation for RC's. Introducing the operator Δ_d acting upon functions $f(d)$ as $\Delta_d f(d) = f(d+1) - f(d)$ we can rewrite formula (1) in the form

$$\{A_{d+1}\Delta_d^2 + (2A_{d+1} - B_{d+1})\Delta_d + (A_{d+1} - B_{d+1} + C_{d+1})\} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_q = 0. \quad (2)$$

By means of limit procedure (1) or (8) of Section 14.5.7 we obtain from relation (1) the recurrence formula for CGC's of the algebra $U_q(\mathfrak{sl}_2)$. It can be written in the form

$$\tilde{A}_a C_{jkm}^{a+1,b,c} - \tilde{B}_a C_{jkm}^{abc} + \tilde{C}_a C_{jkm}^{a-1,b,c} = 0, \tag{3}$$

where

$$\begin{aligned} \tilde{A}_a &= [2a]([a+c-b+1][a+b+c+2][a-j+1][a+j+1] \\ &\quad \times [a+b-c+1][b+c-a])^{1/2}, \\ \tilde{B}_a &= q^{(c+k)/2}[2a][2a+1][2a+2][c-m] \\ &\quad - q^{-(a+1)/2}[2a][a+j+1][a+b-c+1][b+c-a] \\ &\quad - q^{a/2}[2a+2][a-j][a-b+c][a+b+c+1], \\ \tilde{C}_a &= [2a+2][b+c-a+1][a+b-c] \\ &\quad \times [a+j][a-j][a-b+c][a+b+c+1])^{1/2}. \end{aligned}$$

14.6. Representations of the Quantum Algebra $U_q(\mathfrak{sl}_2)$ and q-Orthogonal Polynomials

14.6.1. Matrix elements of representations and q-Krawtchouk polynomials. In Section 13.2.2 we introduced two types of q-Krawtchouk polynomials. The formula

$$K_n(x;b,N\,|\,q) = {}_2\varphi_1(q^{-n},x;q^{-N};q,bq^{n+1}), \tag{1}$$

where $N \in \mathbb{Z}_+$ and $n \in \{0,1,2,\ldots,N\}$, defines the third type of q-Krawtchouk polynomials.

Polynomials (1) can be obtained from the q-Hahn polynomials

$$Q_n(x;a,b;N\,|\,q) = {}_3\varphi_2\left(\begin{matrix} q^{-n},\ abq^{n+1},\ x \\ aq,\ q^{-N} \end{matrix}\,\middle|\, q,q\right) \tag{2}$$

with the help of the limit procedure. Namely, a direct calculation shows that

$$K_n(x;b,N\,|\,q) = \lim_{a\to\infty} Q_n(x;a,b;N\,|\,q). \tag{3}$$

Polynomials (1) can be expressed in terms of little q-Jacobi polynomials. We apply relation (2) of Section 14.1.8 to the right hand side of expression (1) of Section 14.4.4 and then rewrite the hypergeometric series, obtained, in terms of $K_n(x;a,N|q)$. As a result we have the relation

$$\begin{aligned} p_n(q^x;q^\alpha,q^\beta|q) &= (-q^{x+1})^n q^{-n(n+1)/2}\frac{(q^{n+\alpha+\beta+1};q)_n}{(q^{\alpha+1};q)_n} \\ &\quad \times K_n(q^{-n-\alpha};q^{-n-\beta-x-1},2n+\alpha+\beta|q), \end{aligned} \tag{4}$$

where $\alpha, \beta \in \mathbb{Z}_+$.

Let us express matrix elements (18)–(21) of Section 14.4.3 for the representations T_ℓ of $U_q(\mathfrak{sl}_2)$ in terms of the q-Krawtchouk polynomials (1). By means of formula (2) of Section 14.1.8 we represent formula (18) of Section 14.4.3 in the form

$$t^\ell_{mn} = q^a \frac{[2\ell]!\pi_{12}^{\ell-n}\pi_{21}^{\ell-m}}{([\ell-n]![\ell-m]![\ell+m]![\ell+n]!)^{1/2}} \times {}_2\Phi_1\left(-\ell+m, -\ell+n; -2\ell; q, \frac{-\sqrt{q}}{\pi_{12}\pi_{21}}\right)\pi_{11}^{m+n}, \tag{5}$$

where $a = (m+n)(m+n-2\ell)/4$. Analogous representations are valid for formulas (19)–(21) of Section 14.4.3. Note that because of the factor $\pi_{12}^{\ell-n}\pi_{21}^{\ell-m}$ the expression $\pi_{12}\pi_{21}$ in denominators of summands of the finite series (5) is absent. Moreover, as we shall show in Section 14.8.1, the algebra A, generated by the elements π_{11}, π_{12}, π_{21}, π_{22} and relations (1)–(5) of Section 14.4.2, has linear representations T_φ (defined by a continuous number φ) for which the operators $T_\varphi(\pi_{12}\pi_{21})$ are diagonal. Thus, expressions with $\pi_{12}\pi_{21}$ in denominator can be justified by considering matrix elements of these representations. In what follows we shall freely use expressions with $\pi_{12}\pi_{21}$ in denominators taking into account that these operations can be justified with the help of the representations of the algebra A. We recommend to the reader to fulfil the corresponding justifications.

By means of formula (1) we obtain from (5) that

$$t^\ell_{mn} = \frac{[2\ell]!q^{(m+n)(m+n-2\ell)/4}}{([\ell-m]![\ell-n]![\ell+m]![\ell+n]!)^{1/2}}\pi_{12}^{\ell-n}\pi_{21}^{\ell-m} \times K_{\ell-m}(q^{-\ell+n}; -q^{-\ell+m-1}\sqrt{q}/\pi_{12}\pi_{21}, 2\ell|q)\pi_{11}^{m+n}, \tag{6}$$

where $m \geq n \geq -m$. Analogous formulas are valid for other values of m and n.

14.6.2. The product and the addition theorems for q-Krawtchouk polynomials. Using symmetry relations and expression (4) of Section 14.3.5 for CGC's we easily obtain the expression

$$C_q(\boldsymbol{\ell};\mathbf{j}) = \left(\frac{[\ell_1-\ell_2+\ell]![\ell+m]![\ell-m]![\ell_1-j]![2\ell+1]}{[\ell_1+\ell_2-\ell+1]![\ell_1+\ell_2-\ell]![\ell+\ell_2-\ell_1]![\ell_1+j]![\ell_2+k]![\ell_2-k]!}\right)^{1/2} \times \frac{q^A(-1)^{\ell-\ell_1-\ell_2}[2\ell_2]!}{[\ell-\ell_2-j]!}\,{}_3\Phi_2\left(\begin{matrix}\ell-\ell_1-\ell_2,\ \ell+\ell_1-\ell_2+1,\ -\ell_2+k\\ \ell-\ell_2-j+1,\ -2\ell_2\end{matrix}\middle| q, q\right), \tag{1}$$

where $A = -\frac{1}{4}\{(\ell+\ell_1-\ell_2+1)(\ell_1+\ell_2-\ell)+2k(\ell-\ell_2)-2j\ell_2\}$. Substituting this expression for CGC's and expression (6) of Section 14.6.1 for the matrix elements t^ℓ_{mn} into formula (2) of Section 14.4.1 and introducing notations

$$N_1 = 2\ell_1, \quad N_2 = 2\ell_2, \quad n_1 = \ell_1 + j_1, \quad n_2 = \ell_2 - k_1,$$

$$x_1=\ell_1+j_2,\quad x_2=\ell_2-k_2,\quad b_1=q^{-\ell_1+j_2-1}z,\quad z=-\sqrt{q}/\pi_{12}\pi_{21}$$

after simple transformations we derive the *product theorem* for q-Krawtchouk polynomials

$$\begin{aligned}
&K_{n_1}(q^{-x_1};b_1,N_1|q)K_{n_2}(q^{-x_2};tb_1,N_2|q)\\
&\quad=\frac{(b_1;q)_{n_1+1}}{(b_1;q)_{N_1-x_1+1}}\sum_{r=0}^{N}\frac{(q^{n_1-N_1};q)_r(q^{x_1-N_1};q)_r(q^{-N_2};q)_r}{(q;q)_r(q^{-N_1};q)_r(q^{r-N_1-N_2-1};q)_r}\\
&\quad\times q^{r(r-x_1)}b_1^r\,{}_3\Phi_2\left(\begin{matrix}-r,\ N_1-r+1,\ -n_1\\ -N_1,\ N_1-n_1-r+1\end{matrix}\middle|\,q,q\right)\\
&\quad\times\,{}_3\Phi_2\left(\begin{matrix}-r,\ N_1-r+1,\ -x_2\\ -N_2,\ N_1-x_1-r+1\end{matrix}\middle|\,q,q\right)\\
&\quad\times K_{N_1+n_2-n_1-r}(q^{x_1-x_2-N_1+r};q^{n_1-x_1-n_2+r}b_1,N_1+N_2-2r|q),
\end{aligned}\tag{2}$$

where $t=q^{n_1-n_2}$, $N=\frac{1}{2}(N_1+N_2)-\max(|\ell_1-\ell_2|,|j_1+k_1|,|j_2+k_2|)$.

Using the orthogonality relation for CGC's in the similar way we obtain the *addition theorem* for q-Krawtchouk polynomials:

$$\begin{aligned}
&\sum_{x=0}^{M}\sum_{m=0}^{N}\frac{(b;q)_{M-x+1}(q^{\sigma-M};q)_{M-n}(q^{-N-\sigma};q)_m(q^{\tau-M};q)_{M-x}(q^{-x-M};q)_x}{(b;q)_{m+1}(q;q)_{M-n}(q;q)_m(q;q)_{M-x}(q;q)_x q^{r_1^2+m(M-\sigma)+x(M-\tau)}}\\
&\times b_1^{-r_1}K_m(q^{-x};b,M|q)K_n(q^{-y};tb,N|q)Q_{r_1}(q^{m-M};q^{\sigma-M-1},q^{-N-\sigma-1},M|q)\\
&\times Q_{r_2}(q^{x-M};q^{-M+\tau-1},q^{-N-\tau-1},M|q)\\
&=\delta_{r_1r_2}AK_{M-\sigma-r_1}(q^{\tau-M+r_1};q^{\sigma-M+r_1}b_1,M+N-2r_1|q),
\end{aligned}\tag{3}$$

where

$$A=\frac{(q^{-M-N};q)_M^2(q;q)_{r_1}(q^{-M-\sigma};q)_{r_1}(q^{-N-\tau};q)_{r_1}(q^{-N};q)_{r_1}q^{r_1(\sigma+\tau-3M)}}{(q;q)_M^2(q^{\tau-M-N-1};q)_{r_1}(q^{-M-N};q)_{r_1-1}^2(q^{-N};q)_{r_1}(1-q^{2r_1-M-N-1})^2}.$$

Here r_1 and r_2 are nonnegative integers such that

$$M+N-2r_i\ge\max(|N-M|,|2\sigma-M+N|,|2\tau-M+N|).$$

14.6.3. A q-analog of the Burchnall-Chaundy formula. Let us substitute expression (1) of Section 14.6.2 for CGC's and expression (5) of Section 14.6.1 for the matrix elements t^{ℓ}_{mn} into formula (2) of Section 14.4.1. After renotations

$$r=\ell_1+\ell_2-\ell,\quad a=-\ell_1+j,\quad b=-\ell_1+j_2,\quad c=-2\ell_1,$$

$$\alpha=-\ell_2+k_1,\quad \beta=-\ell_2+k_2,\quad \gamma=-2\ell_2,\quad z=-\sqrt{q}/\pi_{12}\pi_{21}$$

and simple manipulations we obtain the relation

$$\begin{aligned}
&{}_2\Phi_1(a,b;c;q,z)\,{}_2\Phi_1(\alpha,\beta;\gamma;q,q^{a+b-c}z)\\
&= \sum_{r=0}^{\ell_1+\ell_2-M} q^{r(r+c-1)} \frac{(q^a;q)_r(q^b;q)_r(q^\gamma;q)_r}{(q;q)_r(q^c;q)_r(q^{c+\gamma+r-1};q)_r} z^r\\
&\times {}_3\Phi_2\left(\begin{matrix}-r,\ 1-r-c,\ \alpha\\ \gamma,\ 1-r-a\end{matrix}\middle| q,q\right) {}_3\Phi_2\left(\begin{matrix}-r,\ 1-r-c,\ \beta\\ \gamma,\ 1-r-b\end{matrix}\middle| q,q\right)\\
&\times {}_3\Phi_2(a+\alpha+r, b+\beta+r; c+\gamma+2r; q, z),
\end{aligned} \tag{1}$$

where $M = \max(|\ell_1-\ell_2|, |j_1+k_1|, |j_2+k_2|)$ and a, b, c, α, β, γ are negative integers. This formula implies a *q-analog of the Burchnall-Chaundy formula* (3) of Section 8.3.8:

$$\begin{aligned}
&{}_2\Phi_1(a,b;c;q,z)\,{}_2\Phi_1(\alpha,\beta;\gamma;q,q^{a+b-c}z)\\
&= \sum_{r=0}^{\infty} q^{r(r+c-1)} \frac{(q^a;q)_r(q^b;q)_r(q^\gamma;q)_r}{(q;q)_r(q^c;q)_r(q^{c+\gamma+r-1};q)_r} z^r\\
&\times {}_3\Phi_2\left(\begin{matrix}-r,\ 1-r-c,\ \alpha\\ \gamma,\ 1-r-a\end{matrix}\middle| q,q\right) {}_3\Phi_2\left(\begin{matrix}-r,\ 1-r-c,\ \beta\\ \gamma,\ 1-r-b\end{matrix}\middle| q,q\right)\\
&\times {}_3\Phi_2(a+\alpha+r, b+\beta+r; c+\gamma+2r; q, z),
\end{aligned} \tag{2}$$

where now a, b, c, α, β, γ are complex numbers and $|q| < 1$. In order to prove this formula it is sufficient to expand the functions ${}_2\varphi_1$ into series and to compare coefficients at the same powers z^n on the left and on the right. It follows from (1) and from Carlson's theorem[3] that these coefficients coincide. Thus, formula (2) is proved.

We also mention the formula (see reference [121] of the second volume)

$$\begin{aligned}
&{}_2\varphi_1(q^{-n}, abq^{n+1}; aq; q, xq)\,{}_2\varphi_1(q^{-n}, abq^{n+1}; aq; q, yq)\\
&= \frac{(bq;q)_n}{(aq;q)_n}(-b)^n q^{-n(n+1)/2} \sum_{k=0}^{n} \frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(bq;q)_k}(-byq^2)^k\\
&\times q^{k(k-1)/2} \sum_{m=0}^{k} \frac{(q^{-k};q)_m(abq^{k+1};q)_m}{(q;q)_m(aq;q)_m}(xq)^m\ {}_2\varphi_1\left(q^{m-k}, bxq; 0; q, \frac{1}{by}\right)
\end{aligned} \tag{3}$$

which can be derived from (1).

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[3] Carlson's theorem states that if for Re $z \geq 0$ a function $f(z)$ is regular and $f(z) = O(e^{k|z|})$, $k < \pi$, then the condition $f(z) = 0$ for $z = 0, 1, 2, \ldots$ implies that $f(z) \equiv 0$.

14.6.4. CGC's and q-Hahn polynomials. q-Hahn polynomials are defined by the formula

$$Q_n(q^{-x}; a, b; N|q) = {}_3\varphi_2 \left(\begin{matrix} q^{-n}, \ abq^{n+1}, \ q^{-x} \\ aq, \ q^{-N} \end{matrix} \middle| q, q \right) \tag{1}$$

(see Section 13.2.2), where $n \leq N$. Setting

$$x = \ell - m, \quad n = \ell_1 - j, \quad a = q^{\ell_2 - \ell + j}, \quad b = q^{\ell - \ell_2 + j}, \quad N = \ell + \ell_2 - j$$

into expression (1) of Section 14.3.5 for CGC's of $U_q(\mathfrak{sl}_2)$ we obtain the expression for CGC's in terms of q-Hahn polynomials:

$$\begin{aligned} C_q(\boldsymbol{\ell}; \mathbf{j}) &= (-1)^{\ell_1 - j} q^B \\ &\times \frac{\Delta(\boldsymbol{\ell})[\ell_2 + \ell - j]!([\ell_1 + j]![\ell_2 - k]![\ell + m]![2\ell + 1])^{1/2}}{[\ell_1 - \ell_2 + \ell]![\ell + \ell_2 - \ell_1]![\ell_2 - \ell + j]!([\ell_1 - j]![\ell_2 + k]![\ell - m]!)^{1/2}} \\ &\times Q_{\ell_1 - j}(q^{-\ell + m}; q^{\ell_2 - \ell + j}, q^{\ell - \ell_2 + j}; \ell + \ell_2 - j \,|\, q) \end{aligned} \tag{2}$$

where B is the same as in formula (1) of Section 14.3.5. This relation can be rewritten as

$$\begin{aligned} & C_q\left(\frac{N}{2}, \frac{N}{2} + \alpha + \beta, n + \frac{1}{2}(\alpha + \beta); \frac{N}{2} - x, \frac{\alpha - \beta - N}{2} + x, \frac{\alpha - \beta}{2} \right) \\ &= (-1)^{N + n - x} \frac{(q;q)_N}{(q;q)_\beta} \\ &\times \left(\frac{(q;q)_{N-x+\beta}(q;q)_{x+\alpha}(q;q)_{n+\beta}(q;q)_{n+\alpha+\beta}(1 - q^{2n+\alpha+\beta+1})}{(q;q)_{N-n}(q;q)_{N+n+\alpha+\beta+1}(q;q)_{N-x}(q;q)_x(q;q)_{n+\alpha}} \right)^{1/2} \\ &\times q^{\{(N-x)(\alpha+1)+n(a+n)\}/2} \frac{(q^{-n-\alpha};q)_n}{(q^{\beta+1};q)_n} Q_n(q^{-x}; q^\alpha, q^\beta; N|q). \end{aligned} \tag{3}$$

The orthogonality relation (8) of Section 14.3.4 for CGC's implies the orthogonality relation for q-Hahn polynomials:

$$\sum_{x=0}^{N} \frac{(q;q)_{N+\beta-x}(q;q)_{x+\alpha}}{(q;q)_x(q;q)_{N-x}} q^{-x(\alpha+1)} Q_n(q^{-x}) Q_m(q^{-x}) = c_n \delta_{mn}, \tag{4}$$

where $Q_n(q^{-x}) \equiv Q_n(q^{-x}; q^\alpha, q^\beta; N \,|\, q)$ and

$$\begin{aligned} c_n &= \frac{(q;q)_{N-n}(q;q)_{N+n+\alpha+\beta+1}(q;q)_n(q;q)_{n+\alpha} q^{-N(\alpha+1)-n(n+\alpha)}}{(q;q)_{n+\beta}(q;q)_{n+\alpha+\beta}(1 - q^{2n+\alpha+\beta+1})} \\ &\times \frac{(q;q)_\beta^2 (q^{\beta+1};q)_n^2}{(q;q)_N^2 (q^{-n-\alpha};q)_n^2}. \end{aligned}$$

By means of simple transformations this relation turns into the orthogonality relation

$$\sum_{x=0}^{N} Q_n(q^{-x})Q_m(q^{-x})\frac{(aq;q)_x(bq;q)_{N-x}}{(q;q)_x(q;q)_{N-x}}(aq)^{-x} = \delta_{mn}(-aq)^n \\ \times q^{(n(n-1)/2)-nN}\frac{(aq)^{-N}(abq^2;q)_N(1-abq)(q;q)_n(bq;q)_n(abq^{N+2};q)_n}{(1-q^{2n+1}ab)(q;q)_N(aq;q)_n(abq;q)_n(q^{-N};q)_n} \tag{5}$$

for the polynomials $Q_n(q^{-x}) \equiv Q_n(q^{-x};a,b;N|q)$.

The formula

$$E_n(\mu(x);a,b;N|q) = {}_3\varphi_2\left(\begin{matrix} q^{-n},\ abq^{x+1},\ q^{-x} \\ aq,\ q^{-N} \end{matrix}\middle| q,q\right) \tag{6}$$

defines polynomials of $\mu(x) = q^{-x} + abq^{x+1}$ called dual *q-Hahn polynomials* (or *q-Aberlane polynomials*). Comparison of formulas (1) and (6) gives

$$E_n(\mu(x);a,b;N|q) = Q_x(q^{-n};a,b;N|q) \tag{7}$$

and, hence, CGC's are also expressed in terms of dual q-Hahn polynomials. The orthogonality relation (9) of Section 14.3.4 for CGC's leads to the orthogonality relation for E_n which can be written in the form

$$\sum_{x=0}^{N} E_n(\mu(x))E_m(\mu(x))\frac{(1-abq^{2x+1})(aq;q)_x(abq;q)_x(q^{-N};q)_x}{(1-abq)(q;q)_x(bq;q)_x(abq^{N+2};q)_x}(-aq)^{-x} \\ \times q^{Nx-x(x-1)/2} = \delta_{mn}\frac{(abq^2;q)_N(aq)^{-N}(q;q)_n(b^{-1}q^{-N};q)_n}{(bq;q)_N(aq;q)_n(q^{-N};q)_n}(abq)^n, \tag{8}$$

where $E_n(\mu(x)) \equiv E_n(\mu(x);a,b;N|q)$.

The recurrence formula (3) of Section 14.5.9 for CGC's leads to the recurrence relation for the q-Hahn polynomials $Q_n(q^{-x}) \equiv Q_n(q^{-x};a,b;N|q)$:

$$A_nQ_{n+1}(q^{-x}) - \{A_n + C_n - (1-q^{-x})\}Q_n(q^{-x}) + C_nQ_{n-1}(q^{-x}) = 0, \tag{9}$$

where

$$A_n = \frac{(1-abq^{n+1})(1-aq^{n+1})(1-q^{n-N})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$
$$C_n = \frac{(1-q^n)(1-bq^n)(q^{-N-1}-abq^n)aq^{n+1}}{(1-abq^{2n})(1-abq^{2n+2})}.$$

Formula (1) of Section 14.3.9 (after application of the symmetry relation (2) of Section 14.3.7) gives the recurrence relation for the dual q-Hahn polynomials $E_n(\mu(x))$ $\equiv E_n(\mu(x);a,b;N|q)$:

$$A_nE_{n+1}(\mu(x)) - \{A_n + C_n + (1-q^{-x})(1-q^{x+1})\}E_n(\mu(x)) + C_nE_{n-1}(\mu(x)) = 0, \tag{10}$$

where

$$A_n = (1 - q^{-N+n})(1 - aq^{n+1}),$$
$$C_n = q(1 - q^n)(ab - aq^{-N+n-1}).$$

14.6.5. RC's and q-Racah polynomials. We set

$$R_n(\mu(x)); \alpha, \beta, \gamma, \delta|q) = {}_4\varphi_3 \left(\begin{matrix} q^{-n}, \ \alpha\beta q^{n+1}, \ q^{-x}, \ \gamma\delta q^{x+1} \\ \alpha q, \ \beta\delta q, \ \gamma q \end{matrix} \middle| q, q \right), \qquad (1)$$

where $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$. Since

$$R_n(\mu(x); \alpha, \beta, \gamma, \delta|q) = 1 + \sum_{k=1}^{n} \frac{(q^{-n};q)_k(\alpha\beta q^{n+1};q)_k q^k}{(\alpha q;q)_k(\beta\delta q;q)_k(\gamma q;q)_k(q;q)_k}$$
$$\times \prod_{j=0}^{k-1} \{1 + \gamma\delta q^{2j+1} - q^j \mu(x)\},$$

then function (1) is a polynomial of the variable $\mu(x)$, called the *q-Racah polynomial.* We shall also use another notation:

$$\tilde{R}_n(\mu(x); a, b, c, d|q) \equiv R_n(\mu(x); q^a, q^b, q^c, q^d|q). \qquad (2)$$

Putting

$$\left. \begin{aligned} x = \ell_2 - \ell_1 + \ell_{12}, \ n = \ell_2 - \ell_3 + \ell_{23}, \ \alpha = q^{-2\ell_2-1}, \ \beta = q^{-2\ell_{23}-1}, \\ \gamma = q^{-\ell_2-\ell-\ell_{12}-\ell_{23}-2}, \quad \delta = q^{-\ell_2-\ell_{12}+\ell_{23}+\ell}, \\ \mu(x) = q^{-x} + \gamma\delta q^{x+1} = q^{\ell_1-\ell_2-\ell_{12}}(1 + q^{-2\ell_1-1}) \end{aligned} \right\} \qquad (3)$$

into expression (4) of Section 14.5.2 for RC's and comparing the resulting formula with (1), we obtain the relation between RC's and q-Racah polynomials:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}_q =$$
$$= (-1)^{b+c+f+e} \frac{\Delta(a,b,c)\Delta(a,e,f)\Delta(c,d,e)\Delta(d,b,f)[2b]!}{[a+b-c]![b+c-a]![d+c-e]![c+e-d]![a+f-e]!}$$
$$\times \frac{[b+f+c-e]![b+f+c+e+1]!}{[f+e-a]![b+d-f]![b+f-d]!}$$
$$\times \tilde{R}_{b+f-d}(q^{a-b-c}(1-q^{-2a-1}); -2b-1, -2f-1, -b-c-e-f-2,$$
$$-b-c+e+f\,|\,q). \qquad (4)$$

The orthogonality relations (4) and (5) of Section 14.5.1 for RC's lead to those for q-Racah polynomials. They depend on conditions imposed on the parameters. Repeating the arguments of Section 8.5.4, we obtain the following result. For

$$\max(|\ell-\ell_{23}|,|\ell_2-\ell_{12}|)=|\ell-\ell_{23}|=\ell-\ell_{23},\quad \ell_2+\ell_{12}\le\ell+\ell_{23}$$

we have

$$0\le x,n\le \ell_{23}-\ell+\ell_2+\ell_{12}=-\beta-\gamma-1\equiv N,$$

and $-\alpha-1$ and $-\gamma-1$ do not coincide with any of the numbers $0,1,\ldots,N$ (where x, n, α, β, γ, δ are defined by formulas (3)), and the orthogonality relation for $R_n(\mu(x))\equiv R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$ is of the form

$$\sum_{x=0}^{N} R_n(\mu(x))R_m(\mu(x))w(x)=\delta_{mn}h_n, \tag{5}$$

where

$$w(x)=\frac{(1-\gamma\delta q^{2x+1})(\alpha\beta q)^{-x}(\gamma\delta q;q)_x(\alpha q;q)_x(\beta\delta q;q)_x(\gamma q;q)_x}{(1-\gamma\delta q)(q;q)_x(\gamma\delta q/\alpha;q)_x(\gamma\delta/\beta;q)_x(\delta q;q)_x}, \tag{6}$$

$$h_n=\frac{(1-\alpha\beta q)(\gamma\delta q)^n(q;q)_n(\beta q;q)_n(\alpha q/\delta;q)_n(\alpha\beta q/\gamma;q)_n}{(\alpha\beta q;q)_n(\alpha q;q)_n(\beta\delta q;q)_n(\gamma q;q)_n}\times\frac{(\beta/\gamma;q)_N(\alpha\beta q^2;q)_N}{(\alpha\beta q/\gamma;q)_N(\beta q;q)_N}. \tag{7}$$

But if

$$\max(|\ell-\ell_{23}|,|\ell_2-\ell_{12}|)=\ell-\ell_{23},\quad \ell+\ell_{23}\le\ell_2+\ell_{12},$$

then

$$0\le x,n<2\ell_3=-\alpha-1\equiv N;\quad -\gamma-1,-\beta-\delta-1\overline{\in}\{0,1,2,\ldots,N\}$$

and the orthogonality relation has the form (5) with the only difference that in h_n the factor

$$\frac{(\beta/\gamma;q)_N(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha\beta q/\gamma;q)_N}\quad\text{is replaced by}\quad\frac{(\gamma\delta q^2;q)_N(\alpha/\beta;q)_N}{(\gamma q/\beta;q)_N(\delta q;q)_N}.$$

One can analogously write out the orthogonality relation for $0\le x,n\le-\gamma-1\equiv N$. In all three cases $w(x)$ has the same form and h_n can be represented as

$$h_n=\frac{(1-\alpha\beta q)(\gamma\delta q)^n(q;q)_n(\beta q;q)_n(\alpha q/\delta;q)_n(\alpha\beta q/\gamma;q)_n}{(1-\alpha\beta q^{2n+1})(\alpha\beta q;q)_n(\alpha q;q)_n(\beta\delta q;q)_n(\gamma q;q)_n}\times\frac{(\gamma\delta q^2;q)_\infty(\gamma/\alpha\beta;q)_\infty(\delta/\alpha;q)_\infty(\beta^{-1};q)_\infty}{(\gamma\delta q/\alpha;q)_\infty(\gamma q/\beta;q)_\infty(\delta q;q)_\infty(\alpha^{-1}\beta^{-1}q^{-1};q)_\infty}. \tag{8}$$

Since one of the values αq, $\beta\delta q$, γq coincides with q^{-N}, then the infinite products of the form $(\epsilon; q)_\infty$ in (8) are finite.

The recurrence formula (1) of Section 14.5.9 for RC's implies the recurrence relation for the q-Racah polynomials $R_n(\mu(x)) \equiv R_n(\mu(x); \alpha, \beta, \gamma, \delta | q)$:

$$\begin{aligned} A_n R_{n+1}(\mu(x)) - \{A_n + C_n - (1-q^x)(1-\gamma\delta q^{x+1})\} R_n(\mu(x)) \\ + C_n R_{n-1}(\mu(x)) = 0, \end{aligned} \tag{9}$$

where

$$A_n = \frac{(1-\alpha\beta q^{n+1})(1-\alpha q^{n+1})(1-\beta\gamma q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})},$$

$$C_n = \frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}.$$

14.6.6. The addition theorem for q-Hahn polynomials. To derive the addition theorem for q-Hahn polynomials we utilize formula (1) of Section 14.5.1. Let us replace in it RC's by $6j$ Wigner symbols, CGC's $C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k}$ by

$$C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} = (-1)^{\ell_3-\ell_{23}-j} q^{j/2} \left(\frac{[2\ell_{23}+1]}{[2\ell_3+1]}\right)^{1/2} C^{\ell_2\ell_{23}\ell_3}_{-j,j+k,k}$$

(see relation (2) of Section 14.3.7) and make use of the symmetry relation

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_{12} \\ \ell_3 & \ell & \ell_{23} \end{Bmatrix}_q = \begin{Bmatrix} \ell_{12} & \ell_3 & \ell \\ \ell_{23} & \ell_1 & \ell_2 \end{Bmatrix}_q .$$

As a result we obtain the relation

$$\begin{aligned} &\sum_{i=-\ell_1}^{\ell_1} \sum_{\substack{j=-\ell_2 \\ i+j=\text{const}}}^{\ell_2} (-1)^{-j} q^{j/2} C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} C^{\ell_2\ell_{23}\ell_3}_{-j,j+k,k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} \\ &= (-1)^{\ell_{23}-\ell_1-\ell_2-\ell-2\ell_3} ([2\ell_{12}+1][2\ell_3+1])^{1/2} \begin{Bmatrix} \ell_{12} & \ell_3 & \ell \\ \ell_{23} & \ell_1 & \ell_2 \end{Bmatrix}_q C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k}. \end{aligned} \tag{1}$$

By means of relation (17) of Section 13.2.2 we derive from formula (1) of Section 14.3.5 the following expression for CGC's:

$$\begin{aligned} C^{\ell_1\ell_2\ell}_{jkm} &= (-1)^{\ell_1-j} q^A \frac{\Delta(\ell_1,\ell_1,\ell)[2\ell_1]!}{[\ell_1-\ell_2+\ell]![\ell_2-\ell_1+m]!} \\ &\times \left(\frac{[\ell_2-k]![\ell_2+k]![\ell+m]![2\ell+1]}{[\ell_1+\ell_2-\ell]!^2[\ell_1-j]![\ell_1+j]![\ell-m]!}\right)^{1/2} \\ &\times {}_3\Phi_2\left(\left.\begin{matrix} -\ell_1+j,\ -\ell-\ell_1+\ell_2,\ \ell+\ell_2-\ell_1+1 \\ -2\ell_1,\ -\ell_1+\ell_2+m+1 \end{matrix}\right| q,q\right), \end{aligned}$$

where $m = j + k$, $A = \frac{1}{2}j(\ell_2+1) - \frac{1}{4}(\ell+\ell_1-\ell_2)(\ell-\ell_1+\ell_2+1) + \frac{1}{2}k\ell_1$. Setting here

$$a = q^{\ell_2-\ell_1+m}, \quad b = q^{\ell_2-\ell_1-m}, \quad N = 2\ell_1, \quad n = \ell+\ell_1-\ell_2, \quad x = \ell_1 - j$$

and taking into account expression (1) of Section 14.6.4 for q-Hahn polynomials, we obtain

$$\begin{aligned} C_{jkm}^{\ell_1\ell_2\ell} &= (-1)^{\ell_1-j} q^A \frac{\Delta(\ell_1,\ell_1,\ell)[2\ell_1]!}{[\ell_1-\ell_2+\ell]![\ell_2-\ell_1+m]!} \\ &\times \left(\frac{[\ell_2-k]![\ell_2+k]![\ell+m]![2\ell+1]}{[\ell_1+\ell_2-\ell]!^2[\ell_1-j]![\ell_1+j]![\ell-m]!} \right)^{1/2} \\ &\times \tilde{Q}_{\ell_1-\ell_2+\ell}(\ell_1-j;\ell_2-\ell_1+m,\ell_2-\ell_1-m;2\ell_1\,|\,q), \end{aligned} \tag{2}$$

where the notation

$$\tilde{Q}_n(x;a,b;N\,|\,q) \equiv Q_n(q^{-x};q^a,q^b;N\,|\,q)$$

has been used.

To obtain the addition formula for q-Hahn polynomials we substitute expressions (2) for CGC's and expression (4) of Section 14.6.5 for RC's into (1). After simple (but awkward) transformations we derive the *addition formula*

$$\begin{aligned} &\sum_{x=0}^{2\ell_1}{}' \sum_{y=0}^{2\ell_2}{}' (-1)^{x+y} q^{\{(2\ell_1+\ell_2-\ell_{23})-x(\ell_{23}+\ell_2+2)\}/2} \\ &\quad\times \frac{[\ell_{23}+\ell_2-k-y]![\ell_{23}-\ell_2+k+y]!}{[x]![2\ell_1-x]!} \\ &\quad\times \tilde{Q}_{\ell_1-\ell_2+\ell_{12}}(x;\ell_2-\ell_1+m,\ell_2-\ell_1-m;2\ell_1\,|\,q) \\ &\quad\times \tilde{Q}_{\ell_2+\ell_3-\ell_{23}}(y;\ell_{23}-\ell_2+k,\ell_{23}-\ell_2-k;2\ell_2\,|\,q) \\ &\quad\times \tilde{Q}_{\ell_1+\ell-\ell_{23}}(x;\ell_{23}-\ell_1+m+k,\ell_{23}-\ell-m-k;2\ell_1\,|\,q) \\ &= M\tilde{R}_{\ell_2+\ell_3-\ell_{23}}(\mu(\ell+\ell_3-\ell_{12});-2\ell_3-1,-2\ell_2-1,-\ell_1-\ell_2-\ell_3-\ell-2, \\ &\qquad\qquad \ell_1+\ell_2-\ell_3-\ell\,|\,q) \\ &\quad\times \tilde{Q}_{\ell-\ell_3+\ell_{12}}(\ell_{12}-m;\ell_3-\ell_{12}+m+k,\ell_3-\ell_{12}-m-k;2\ell_{12}\,|\,q) \end{aligned} \tag{3}$$

(we have replaced $\ell_1 - i$ by x, $\ell_2 + j$ by y and $i + j$ by m), where primes at the sum signs mean that the summation is over those values of x and of y for which $y - x = m - \ell_1 + \ell_2$ (that is, the summation is, in fact, over one index) and

$$\begin{aligned} M &= (-1)^{\ell_{23}+\ell_{12}-\ell_1-\ell_3} q^{k(\ell_{12}-\ell_1-\ell_2)} \\ &\times q^{\{\ell_3(\ell_3+1)-\ell_1(\ell_1+1)-\ell_{23}(\ell_{23}+1)+2\ell_1\ell_2+\ell_{12}(\ell_{12}-\ell_3)+m(\ell_3+1)\}/2} \frac{[2\ell_3]![2\ell_{12}]!}{[2\ell_1]!^2} \\ &\times \frac{[\ell_1+\ell_2+\ell_3+\ell+1]![\ell_2+\ell_3+\ell-\ell_1]![\ell_1-\ell_2+\ell_{12}]![\ell_1+\ell_{23}-\ell]!}{[2\ell_2]![\ell_{12}+\ell+\ell_3+1]![\ell_{12}+\ell_2-\ell_1]![\ell_{12}+\ell_3-\ell]![\ell_{23}+\ell-\ell_1]!} \\ &\times \frac{[\ell_2-\ell_3+\ell_{23}]![\ell_2-\ell_1+m]![\ell_{23}-\ell_2+k]![\ell_{23}-\ell_1+m+k]![\ell_3-k]!}{[\ell_3+\ell_{23}-\ell_2]![\ell_3-\ell_{12}+m+k]![\ell_{12}+m]!}. \end{aligned}$$

Formula (3) is awkward. But it contains many simple special cases. For example, let us put $\ell_1 + \ell_2 = \ell_{12}$. In this case we have one-term expressions for $\begin{Bmatrix} \ell_{12} & \ell_3 & \ell \\ \ell_{23} & \ell_1 & \ell_2 \end{Bmatrix}_q$ and $C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j}$ and the addition theorem takes the form

$$\begin{aligned}
&\sum_{x=0}^{2\ell_1}{}' \sum_{y=0}^{2\ell_2}{}' (-1)^x q^{\{y(2\ell_1+\ell_2-\ell_{23})-x(\ell_{23}-\ell_2+1)\}/2} \\
&\quad \times \frac{[\ell_2+\ell_{23}-k-y]![\ell_{23}-\ell_2+k+y]!}{[x]![y]![2\ell_1-x]![2\ell_2-y]!} \\
&\quad \times \tilde{Q}_{\ell_1+\ell-\ell_{23}}(x;\ell_{23}-\ell_1+m+x,\ell_{23}-\ell_1-m-x;2\ell_1\,|\,q) \\
&\quad \times \tilde{Q}_{\ell_2+\ell_3-\ell_{23}}(y;\ell_{23}-\ell_2+k,\ell_{23}-\ell_2-k;2\ell_2\,|\,q) \\
&= M\tilde{Q}_n(\ell_1+\ell_2-m;\ell_3-\ell_1-\ell_2+m+k,\ell_3-\ell_1-\ell_2-m-k; \\
&\qquad 2\ell_1+2\ell_2\,|\,q),
\end{aligned} \tag{4}$$

where $n = \ell_1 + \ell_2 + \ell - \ell_3$, the summation and indices are the same as in (3) and

$$\begin{aligned}
M &= (-1)^{\ell_1-\ell_3+\ell_{23}-m} \\
&\quad \times q^{\{\ell_2(\ell_2+1)+\ell_3(\ell_3+1)-\ell_{23}(\ell_{23}+1)-(\ell_1+\ell_2)(\ell_3+1)+4\ell_1\ell_2+m(\ell_2+1)\}/2} \\
&\quad \times \frac{[2\ell_1+2\ell_2]![\ell_{23}-\ell_1+m+k]![\ell_{23}-\ell_2+k]![\ell_3-k]!}{[2\ell_1]![2\ell_2]![\ell_1+\ell_2-m]![\ell_1+\ell_2+m]![\ell_3-\ell_1-\ell_2+m+k]!}.
\end{aligned}$$

If we substitute expressions for CGC's and RC's into the relation

$$\begin{aligned}
C^{\ell_2\ell_3\ell_{23}}_{j,k,j+k} C^{\ell_1\ell_{23}\ell}_{i,j+k,i+j+k} = &\sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} R_q(\ell_1\ell_2\ell_3,\ell_{12}\ell_{23},\ell) \\
&\times C^{\ell_1\ell_2\ell_{12}}_{i,j,i+j} C^{\ell_{12}\ell_3\ell}_{i+j,k,i+j+k}
\end{aligned}$$

which is a corollary of formula (6) of Section 14.5.1, then we obtain the *product formula* for q-Hahn polynomials:

$$\begin{aligned}
&\tilde{Q}_{\ell_1+\ell+\ell_{23}}(x;\ell_{23}-\ell_1+m+k,\ell_{23}-\ell_1-m-k;2\ell_1\,|\,q) \\
&\quad \times \tilde{Q}_{\ell_2+\ell_3-\ell_{23}}(y;\ell_{23}-\ell_2+k,\ell_{23}-\ell_2-k;2\ell_2\,|\,q) = B\sum_{\ell_{12}=n}^{\ell_1+\ell_2} D(\ell_{12}) \\
&\quad \times \tilde{R}_{\ell_1+\ell_2-\ell_{23}}(\mu(\ell+\ell_3-\ell_{12});-2\ell_3-1,-2\ell_2-1,-\ell_1-\ell_2-\ell_3-\ell-2, \\
&\qquad \ell_1+\ell_2-\ell_3-\ell\,|\,q) \\
&\quad \times \tilde{Q}_{\ell_{12}+\ell-\ell_3}(\ell_{12}-m;\ell_3-\ell_{12}+m+k,\ell_3-\ell_{12}-m-k;2\ell_{12}\,|\,q) \\
&\quad \times \tilde{Q}_{\ell_{12}+\ell_1-\ell_2}(x;\ell_2-\ell_1+m,\ell_2-\ell_1-m;2\ell_1\,|\,q)
\end{aligned} \tag{5}$$

where $n = \max(|\ell_1 - \ell_2|, |m|, |\ell_2 - \ell|)$, $x = \ell_1 - i$, $y = \ell_2 + j$ and

$$D(\ell_{12}) = \frac{(-1)^{\ell_{12}} q^{-\ell_{12}(\ell_3-k+1)/2}}{[\ell_{12}-m]![\ell_1+\ell_2-\ell_{12}]![\ell_1+\ell_2+\ell_{12}+1]![\ell_{12}+\ell_3-\ell]!} \times \frac{[2\ell_{12}+1]!}{[\ell_{12}+\ell_3+\ell+1]![\ell_3-\ell_{12}+m+k]!},$$

$$\begin{aligned} B = &(-1)^{\ell_2+\ell_3-\ell_{23}+m} \\ &\times q^{\{\ell_3(\ell_3+1)+\ell_{23}(\ell_1+\ell_2+\ell_{23}+1)-\ell_2(\ell_1-1)+(\ell_1-x)(\ell_2+\ell_3-\ell_{23})+(y-\ell_2)(\ell_3-\ell_2+\ell_{23})\}/2} \\ &\times \frac{q^{\{m-k(\ell_1+\ell_2)\}/2}[2\ell_3]![\ell_1+\ell_2+\ell_3+1]![\ell_1-\ell+\ell_{23}]![\ell_2-\ell_3+\ell_{23}]!}{[2\ell_2]![\ell_2-\ell_1+m]![\ell-\ell_1+\ell_{23}]![\ell_{23}-\ell_2+k+y]![\ell_{23}+\ell_2+k-y]!} \\ &\times \frac{[\ell_2-\ell_3+\ell-\ell_1]![\ell_{23}-\ell_2+k]![\ell_{23}-\ell_1+m+k]![\ell_3-k]![2\ell_2-y]![y]!}{[\ell_3-\ell_2+\ell_{23}]!}. \end{aligned}$$

If $\ell_1 + \ell = \ell_{23}$, then (5) gives the equality

$$\begin{aligned} &B_1 \sum_{\ell_{12}=n}^{\ell_1+\ell_2} D_1(\ell_{12}) \tilde{Q}_{\ell_1-\ell_2+\ell_{12}}(x; \ell_2-\ell_1+m, \ell_2-\ell_1-m; 2\ell_1 \,|\, q) \\ &\times \tilde{Q}_{\ell-\ell_3+\ell_{12}}(\ell_{12}-m; \ell_3-\ell_{12}+m+k, \ell_3-\ell_{12}-m-k; 2\ell_{12} \,|\, q) \\ &= \tilde{Q}_{\ell_2+\ell_3-\ell_{23}}(x+m-\ell_1-\ell_2; \ell_1+\ell-\ell_2+k, \ell_1+\ell-\ell_2-k; 2\ell_2 \,|\, q), \end{aligned} \quad (6)$$

where

$$D_1(\ell_{12}) = \frac{(-1)^{\ell_{12}} q^a [2\ell_{12}+1]![\ell_{12}+\ell_2-\ell]!}{[\ell_1+\ell_2+\ell_{12}+1]![\ell_1-\ell_2+\ell_{12}]![\ell_1+\ell_2-\ell_{12}]!} \times \frac{1}{[\ell_{12}+\ell-\ell_3]![\ell_3-\ell_{12}+m+k]![\ell_{12}+\ell+\ell_3+1]![\ell_{12}-m]!},$$

$$B_1 = \frac{(-1)^b q^c [2\ell_1]![\ell_1+\ell_2+\ell_3+\ell+1]![\ell_1+\ell_2+\ell-\ell_3]![\ell_3-k]!}{[2\ell_2]![\ell_1+\ell_2+\ell-k-y]![\ell_1-\ell_2+\ell+k+y]![\ell_2-\ell_1+m]!} \times [\ell_1-\ell_2+\ell+k]![[\ell+m+k]![2\ell_2-y]![y]!,$$

$$a = -\ell_{12}(\ell_3-k+1), \quad b = \ell_3-\ell_2-\ell_1-\ell,$$

$$\begin{aligned} b = &\frac{1}{2}\{\ell_3(\ell_3+1) - \ell(\ell+1) - (\ell_1+\ell)(\ell_1+\ell+1) + \ell_2(\ell+1) + \ell_1(\ell_1+1) + m \\ &- k(\ell_1+\ell_2)\} + (y-\ell_2)(\ell_1+\ell-\ell_2+\ell_3) + (\ell_1-x)(\ell_2+\ell_3-\ell_1-\ell). \end{aligned}$$

Note that Gasper and Rahman (see reference [121] of the second volume) proved the following formulas for q-Hahn polynomials:

$$\sum_{n=0}^{z} \frac{(abq;q)_n(1-abq^{2n+1})(aq;q)_n(q^{-z};q)_n}{(q;q)_n(1-abq)(bq;q)_n(abq^{M+2};q)_n} \left(\frac{-1}{a}\right)^n q^{Mn-n(n+1)/2}$$
$$\times Q_n(q^{-x};a,b;M\,|\,q)Q_n(q^{-y};a,b;N\,|\,q)$$
$$= \sum_{r=0}^{z}\sum_{s=0}^{z-r} \frac{(q^{-z};q)_{r+s}(abq^2;q)_{2r+2s}(q^{-x};q)_r(q^{-y};q)_r(q^{x-M};q)_s(q^{y-N};q)_s}{(q;q)_r(q;q)_s(q^{-M};q)_{r+s}(q^{-N};q)_{r+s}(abq^{M+2};q)_{r+s}(aq;q)_r(bq;q)_s}$$
$$\times \frac{(q^{M-z};q)_{z-r-s}}{(abq^{M+r+s+2};q)_{z-r-s}}(-1)^{r+s}a^{-s}q^{M(r+s)-s(x+y+1)-(r+s)(r+s-1)/2}$$
$$\times\ {}_2\varphi_1(q^{r+s-z},q^{r+s-z+1};q^{M-z+1};q,abq^{M+z+2}), \qquad (7)$$

$$\sum_{n=0}^{M} \frac{(abq;q)_n(1-abq^{2n+1})(aq;q)_n(q^{-M};q)_n}{(q;q)_n(1-abq)(bq;q)_n(abq^{M+2};q)_n} \left(\frac{-t}{a}\right)^n q^{Mn-n(n+1)/2}$$
$$\times Q_n(q^{-x};a,b;M\,|\,q)Q_n(q^{-y};a,b;N\,|\,q)$$
$$= \sum_{r=0}^{x}\sum_{s=0}^{M-x} \frac{(abq^2;q)_{2r+2s}(q^{-x};q)_r(q^{-y};q)_r(q^{x-M};q)_s(q^{y-N};q)_s}{(q;q)_s(q;q)_r(q^{-N};q)_{r+s}(abq^{M+2};q)_{r+s}(aq;q)_r(bq;q)_r}$$
$$\times (t;q)_{M-r-s}(abq^{2r+2s+2};q)_{M-r-s}(-t)^{r+s}a^{-s}q^{M(r+s)-s(x+y+1)-(r+s)(r+s-1)/2}$$
$$\times\ {}_2\varphi_1(q^{M-r-s},tq^{M-r-s};tq;q,abq^{2r+2s+2}) \qquad (8)$$

which are valid for $x=0,1,2,\ldots,M$; $y=0,1,2,\ldots,N$; $z=0,1,2,\ldots,M$; $0\le t<1$, when $0<q<1$, $0<aq<1$, $0\le bq<1$, $M\le N$. They also proved the product formula

$$Q_n(q^{-x};a,b;N\,|\,q)Q_n(q^{-y};a,b;N\,|\,q)$$
$$= \frac{(bq;q)_n(abq^{N+2};q)_n}{(aq;q)_n(q^{-N};q)_n}(bq^{N+1})^{-n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(abq^{n+1};q)_k(bq^{N-x-y+1};q)_k}{(q;q)_k(bq;q)_k(abq^{N+2};q)_k}$$
$$\times q^{(x+1)k}\sum_{m=0}^{k}\frac{(q^{-k};q)_m(abq^{k+1};q)_m(q^{-x};q)_m(q^{-y};q)_m}{(q;q)_m(aq;q)_m(q^{-N};q)_m(bq^{N-x-y+1};q)_m}$$
$$\times\ {}_3\varphi_2(q^{m-k},q^{m-x},bq^{N-x+1};bq^{N+m-x-y+1},0;q,q). \qquad (9)$$

14.6.7. The addition formula for q-Racah polynomials. The $6j$ Wigner symbols satisfy relation (1) of Section 14.5.5. Replacing $6j$ symbols in this relation by suitable expressions, we obtain the equality

$$\sum_n \frac{(-1)^n q^{-n(n-2\ell_1-2\ell_3-1)}[s-2\ell-n]![2\ell_1+2\ell_3-2n+1]!}{[n]![s-n+1]![s-2\ell_1-2\ell_3+n]![s-2\ell_2-n]![2\ell_1+2\ell_3-n+1]!}$$
$$\times\ {}_4\Phi_3(-n, n-2\ell_1-2\ell_3-1, -x, x-2\ell_1-2\ell-1; -2\ell_1, -s-1, 2\ell_2-s; q, q)$$
$$\times\ {}_4\Phi_3(-n, n-2\ell_1-2\ell_3-1, -x', x'-2\ell_3-2\ell-1; -2\ell_3, -s-1, 2\ell_2-s; q, q)$$
$$= D\ {}_4\Phi_3(-x, x-2\ell_1-2\ell-1, -x', x'-2\ell_3-2\ell-1; -2\ell, -s-1, 2\ell_2-s; q, q), \tag{1}$$

where $s = \ell_1+\ell_2+\ell_3-\ell$,

$$D = \frac{(-1)^{-2\ell+x+x'} q^A [2\ell]![2\ell_1-x]![2\ell_3-x']!}{[2\ell_1]![2\ell_3]![2\ell-x]![2\ell-x']![s-2\ell_2]![s+1]!},$$

$$A = (\ell_1-\ell_2)(\ell_1+\ell_2+1)+(\ell+\ell_3)(\ell+\ell_3+1)+2\ell_1(\ell+\ell_3)$$
$$+ x(x-2\ell-2\ell_1-1)+x'(x'-2\ell-2\ell_3-1)$$

and the summation is over the values of n for which q-factorials have a sense. Introducing the notations

$$\alpha = -2\ell_1-1, \quad \beta = -2\ell_3-1, \quad \gamma = -\ell_1-\ell_2-\ell_3-\ell-2, \quad \delta = \ell_2+\ell_3-\ell_1-\ell$$

we obtain the addition formula for q-Racah polynomials:

$$\sum_n (-1)^n \frac{q^{n(n+\alpha+\beta+1)}[\gamma-\alpha-\beta-n-1]!(-\alpha-\beta-2n-1)}{[n]![-\beta-\delta-n-1]![\alpha-\delta+n]![-\gamma-n-1]![-\alpha-\beta-n-1]!}$$
$$\times \tilde{R}_n(\mu(x); \alpha, \beta, \gamma, \delta\,|\,q)\tilde{R}_n(\mu(x'); \beta, \alpha, \gamma, \beta+\delta-\alpha\,|\,q)$$
$$= \frac{(-1)^{x+x'+\delta+\gamma-\alpha+1} q^A [\alpha-\delta-\gamma-1]![-\alpha-x-1]![-\beta-x'-1]!}{[-\alpha-1]![-\beta-1]![\alpha-\delta-\gamma-x-1]![\alpha-\gamma-\delta-x'-1]![-\delta-\beta-1]!}$$
$$\times \tilde{R}_x(\mu(x'); \delta+\gamma-\alpha, \gamma, \beta+\delta-\alpha\,|\,q), \tag{2}$$

where

$$A = (\gamma+1)(\beta+\delta+1)+x(x+1)+x'(x'+1)+x(\delta+\gamma+2)+x'(\beta+\gamma+\delta-\alpha+2).$$

14.6.8. RC's and properties of basic hypergeometric functions. Symmetry relations for RC's imply the following relation for ${}_4\varphi_3$:

$${}_4\varphi_3\left(\begin{matrix} q^{-n}, & a, & b, & c \\ & d, & e, & f \end{matrix}\,\middle|\, q, q\right) = \left(\frac{bc}{d}\right)^n \frac{(aq^{1-n}/e; q)_n (aq^{1-n}/f; q)_n}{(e; q)_n (f; q)_n}$$
$$\times\ {}_4\varphi_3\left(\begin{matrix} q^{-n}, & a, & d/b, & d/c \\ d, & aq^{1-n}/e, & aq^{1-n}/f \end{matrix}\,\middle|\, q, q\right), \tag{1}$$

where $q^{1-n}abc = def$. Since $(1;q)_n = 0$, then for $d = c$ we have

$$ {}_3\varphi_2\left(\begin{matrix} q^{-n},\ a,\ b \\ e,\ f \end{matrix}\,\middle|\, q, q\right) = b^n \frac{(aq^{1-n}/e;q)_n(aq^{1-n}/f;q)_n}{(e;q)_n(f;q)_n}, \tag{2} $$

where $q^{1-n}ab = ef$.

Tending c and f in (1) to infinity, we derive

$$ {}_3\varphi_2\left(\begin{matrix} q^{-n},\ a,\ b \\ d,\ e \end{matrix}\,\middle|\, q, q\right) = a^n \frac{(e/a;q)_n}{(e;q)_n}\, {}_3\varphi_2\left(\begin{matrix} q^{-n},\ a,\ d/b \\ d,\ aq^{1-n}/e \end{matrix}\,\middle|\, q, \frac{bq}{e}\right) \tag{3} $$

(see formula (17) of Section 13.2.2). If $n \to \infty$, then (1) yields

$$ \begin{aligned} {}_3\varphi_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix}\,\middle|\, q, \frac{de}{abc}\right) &= \frac{(de/bc;q)_\infty(e/a;q)_\infty}{(de/abc;q)_\infty(e;q)_\infty} \\ &\quad\times {}_3\varphi_2\left(\begin{matrix} a,\ d/b,\ d/c \\ d,\ de/bc \end{matrix}\,\middle|\, q, \frac{e}{c}\right). \end{aligned} \tag{4} $$

And if $n \to \infty$ in (2), then we have

$$ {}_2\varphi_1\left(b, c; d; q, \frac{d}{bc}\right) = \frac{(d/c;q)_\infty(d/b;q)_\infty}{(d;q)_\infty(d/bc;q)_\infty}. \tag{5} $$

For $e \to \infty$ formula (2) gives

$$ {}_2\varphi_1\left(q^{-n}, b; d; q, \frac{dq^n}{b}\right) = \frac{(d/b;q)_n}{(d;q)_n}. \tag{5'} $$

When $e \to 0$ in (2), then

$$ {}_2\varphi_1(q^{-n}, b; d; q, q) = \frac{b^n(d/b;q)_n}{(d;q)_n}. \tag{5''} $$

The Biedenharn-Elliott formula for ordinary RC's implies the Whipple formula (5) of Section 8.4.13. By means of similar calculations we derive from formula (2) of Section 14.5.4 the *Jackson formula*

$$ \begin{aligned} &{}_8\varphi_7\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-N} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{N+1} \end{matrix}\,\middle|\, q, q\right) \\ &\quad= \frac{(aq;q)_N(aq/cd;q)_N(aq/bd;q)_N(aq/bc;q)_N}{(aq/b;q)_N(aq/c;q)_N(aq/d;q)_N(aq/bcd;q)_N}, \end{aligned} \tag{6} $$

where $N \in \mathbb{Z}_+$ and $a^2q^{N+1} = bcde$. Putting here $e = a^2q^{N+1}/bcd$ and tending N to infinity, we obtain

$$
{}_6\varphi_5\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d \end{matrix}\,\middle|\, q, \frac{aq}{bcd}\right)
= \frac{(aq;q)_\infty (aq/cd;q)_\infty (aq/bd;q)_\infty (aq/bc;q)_\infty}{(aq/b;q)_\infty (aq/c;q)_\infty (aq/d;q)_\infty (aq/bcd;q)_\infty}. \tag{7}
$$

For $d = q^{-N}$, $N \in \mathbb{Z}_+$, this equality turns into

$$
{}_6\varphi_5\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ q^{-N} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq^{N+1} \end{matrix}\,\middle|\, q, \frac{aq^{N+1}}{bc}\right)
= \frac{(a;q)_N (a/bc;q)_N}{(a/b;q)_N (a/c;q)_N}. \tag{8}
$$

If $d = \sqrt{a}$, then (7) yields

$$
{}_4\varphi_3\left(\begin{matrix} a,\ -q\sqrt{a},\ b,\ c \\ -\sqrt{a},\ aq/b,\ aq/c \end{matrix}\,\middle|\, q, \frac{q\sqrt{a}}{bc}\right)
= \frac{(aq;q)_\infty (\sqrt{a}q/c;q)_\infty (\sqrt{a}q/b;q)_\infty (aq/bc;q)_\infty}{(aq/b;q)_\infty (aq/c;q)_\infty (q\sqrt{a};q)_\infty (\sqrt{a}q/bc;q)_\infty}. \tag{9}
$$

14.6.9. Relations for little q-Jacobi polynomials and CGC's. With the help of the formula

$$
p_n(x;a,b|q) = {}_2\varphi_1(q^{-n}, abq^{n+1}; aq; q, qx)
$$

we express matrix elements (5) of Section 14.6.1 in terms of little q-Jacobi polynomials and then substitute them into formula (2) of Section 14.4.1. As a result we obtain the following relation for little q-Jacobi polynomials:

$$
\begin{aligned}
&A p_{\ell_1 - j_1}(z; q^{-2\ell_1-1}, q^{j_1+j_2}|q) p_{\ell_2-k_1}(z; q^{-2\ell_2-1}, q^{k_1+k_2}|q) \\
&= \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} (-1)^{\ell_1+\ell_2-\ell} \frac{q^{\{-\ell(m_1+m_2)-(\ell_1+\ell_2-\ell)\}/2}[2\ell_1]!}{([\ell-m_1]![\ell+m_1]![\ell-m_2]![\ell+m_2]!)^{1/2}} \\
&\quad \times C^{\ell_1\ell_2\ell}_{j_1k_1m_1} C^{\ell_1\ell_2\ell}_{j_2k_2m_2} z^{\ell_1+\ell_2-\ell} p_{\ell-m_1}(z; q^{-2\ell-1}, q^{m_1+m_2}|q),
\end{aligned} \tag{1}
$$

where $m_1 = j_1 + k_1$, $m_2 = j_2 + k_2$ and

$$
A = \frac{q^{\{(j_1+j_2)(j_1+j_2-2\ell_1)+(k_1+k_2)(k_1+k_2-2\ell_2)-(m_1+m_2)^2\}/4}[2\ell_1]![2\ell_2]!}{([\ell_1-j_1]![\ell_1+j_1]![\ell_1-j_2]![\ell_1+j_2]![\ell_2-k_1]![\ell_2+k_1]![\ell_2-k_2]![\ell_2+k_2]!)^{1/2}}.
$$

Due to the orthogonality relation

$$\frac{(q^{\alpha+1};q)_\infty(q^{\beta+1};q)_\infty}{(1-q)(q;q)_\infty(q^{\alpha+\beta+2};q)_\infty}\int_0^1 p_n(t;q^\alpha,q^\beta\,|\,q)p_m(t;q^\alpha,q^\beta\,|\,q)$$
$$\times\frac{(qt;q)_\infty}{(q^{\beta+1}t;q)_\infty}d_qt=\frac{q^{n(\alpha+1)}(1-q^{\alpha+\beta+1})(q^{\beta+1};q)_n(q;q)_n}{(1-q^{2n+\alpha+\beta+1})(q^{\alpha+1};q)_n(q^{\alpha+\beta+1};q)_n}\delta_{mn}$$

(for integral α and β it will be proved in Section 14.8.11), where $\alpha > -1$, $\beta > -1$, we derive from (1) that

$$B\int_0^1 p_{\ell_1-j_1}(zq^{k_1+k_2};q^{j_1-j_2},q^{j_1+j_2}\,|\,q)p_{\ell_2-k_1}(z;q^{k_1-k_2},q^{k_1+k_2}\,|\,q)z^{\ell_1+\ell_2-\ell}$$
$$\times p_{\ell-m_2}(q^{m_1+m_2}z;q^{m_1-m_2},q^{m_1+m_2}\,|\,q)d_qz=C^{\ell_1\ell_2\ell}_{j_1k_1m_1}C^{\ell_1\ell_2\ell}_{j_2k_2m_2}, \tag{2}$$

where $m_1-m_2\geq 0$, $m_1+m_2\geq 0$, $j_1-j_2\geq 0$, $j_1+j_2\geq 0$, $k_1-k_2\geq 0$, $k_1+k_2\geq 0$,

$$B=\frac{q^{\{(m_1+m_2)(j_1-j_2)+(k_1+k_2)(m_1-m_2)+(m_1^2-m_2^2)\}/4-\ell+m_1}}{(1-q)[j_1-j_2]![k_1-k_2]![m_1-m_2]!}(1-q^{2\ell+1})$$
$$\times\left(\frac{[\ell_1+j_1]![\ell_1-j_2]![\ell_2+k_1]![]\ell_2-k_2![\ell+m_1]![\ell-m_2]!}{[\ell_1-j_1]![\ell_1+j_2]![\ell_2-k_1]![]\ell_2+k_2![\ell-m_1]![\ell+m_2]!}\right)^{1/2}. \tag{3}$$

As in the classical case (see Sections 8.3.6 and 8.3.7), formulas (1) and (2) imply a series of interesting special cases.

We now rewrite formula (2) of Section 14.4.1 as

$$t^{\ell_1}_{j-j_2,j'+\ell_2}t^{\ell_2-k}_{j_2-k_1,k-\ell_2}=\sum_{\ell'=|\ell_1-\ell_2+k|}^{\ell_1+\ell_2-k}C^{\ell_1,\ell_2-k,\ell'}_{j-j_2,j_2-k,j-k}$$
$$\times C^{\ell_1,\ell_2-k,\ell'}_{j'+\ell_2,k-\ell_2,j'+k}t^{\ell'}_{j-k,j'+k}. \tag{4}$$

Let us multiply both sides of this relation by

$$q^{-k(\ell_2-j_2)/2}\left(\frac{[2\ell_2]![\ell_2+j_2-2k]!}{[2\ell_2-2k]![\ell_2+j_2]!}\right)^{1/2}t^k_{k,-k}$$

and take into account the relation

$$t^\ell_{m,-\ell}=q^{-k(\ell-m)/2}\left(\frac{[2\ell]![\ell+m-2k]!}{[2\ell-2k]![\ell+m]!}\right)^{1/2}t^{\ell-k}_{m-k,k-\ell}t^k_{k,-k}.$$

As a result we have

$$t^{\ell_1}_{j-j_2,j'+\ell_2}t^{\ell_2}_{j_2,-\ell_2} = q^{-k(\ell_2-j_2)/2}\left(\frac{[2\ell_2]![\ell_2+j_2-2k]!}{[2\ell_2-2k]![\ell_2+j_2]!}\right)^{1/2} \times \sum_{\ell'} C^{\ell_1,\ell_2-k,\ell'}_{j-j_2,j_2-k,j-k} C^{\ell_1,\ell_2-k,\ell'}_{j'+\ell_2,k-\ell_2,j'+k} t^{\ell'}_{j-k,j'+k} t^{k}_{k,-k}.$$

We now apply relation (4) to both sides of this formula and compare coefficients at $t^{\ell}_{jj'}$. We have

$$C^{\ell_1\ell_2\ell}_{j-j_2,j_2,j} C^{\ell_1\ell_2\ell}_{j'+\ell_2,-\ell_2,j'} = q^{-k(\ell_2-j_2)/2}\left(\frac{[2\ell_2]![\ell_2+j_2-2k]!}{[2\ell_2-2k]![\ell_2+j_2]!}\right)^{1/2} \times \sum_{\ell'=\ell-k}^{\ell+k} C^{\ell_1,\ell_2-k,\ell'}_{j-j_2,j_2-k,j-k} C^{\ell_1,\ell_2-k,\ell'}_{j'+\ell_2,k-\ell_2,j'+k} C^{\ell' k\ell}_{j-k,k,j} C^{\ell' k\ell}_{j'+k,-k,j'} .$$

Using here expressions for special CGC's, we derive the recurrence formula for CGC's:

$$C^{\ell_1\ell_2\ell}_{j-j_2,j_2,j} = \sum_{\ell'=\ell-k}^{\ell+k} A(\ell') C^{\ell_1,\ell_2-k,\ell'}_{j-j',j_2-k,j-k}, \tag{5}$$

where

$$A(\ell') = (-1)^{\ell+3\ell'+2j'+2j-k} q^{\{\ell(\ell+1)-\ell'(\ell'+1)+k(2k-2j+2j_2-\ell_2-2)\}/4} \times \frac{[2k]![\ell'+\ell-k]!}{[\ell'+k+\ell+1]![k+\ell'-\ell]![k-\ell'+\ell]!} \times \left(\frac{[\ell_1+\ell_2+\ell+1]![\ell_2+\ell-\ell_1]![\ell_2-\ell+\ell_1]![\ell_2+j_2-2k]!}{[\ell+\ell_1-\ell_2]![\ell_2+j_2]![\ell-j]![\ell'-k+j]!} \times \frac{[2\ell+1][2\ell'+1][\ell+j]![\ell'+k-j]![\ell_1-\ell_2+\ell'+k]!}{[\ell'+\ell_1+\ell_2-k+1]![\ell_2-k+\ell'-\ell_1]![\ell_2-k-\ell'+\ell_1]!}\right)^{1/2}.$$

14.7. q-Askey-Wilson Polynomials and their Special Cases

In this section we discuss the results of the theory of q-orthogonal polynomials. For some of them there are no explanations from point of view of the theory of representations of quantum groups and algebras. This theory is now at starting stage of development. Undoubtedly, its development will give an approach to many formulas mentioned in this section.

14.7.1. q-Askey-Wilson polynomials. The functions

$$p_n(x;a,b,c,d\,|\,q) = a^{-n}(ab;q)_n(ac;q)_n(ad;q)_n \times {}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n-1}abcd,\ ae^{i\theta},\ ae^{-i\theta} \\ ab,\quad ac,\quad ad \end{matrix}\,\middle|\, q,q\right), \quad x=\cos\theta, \tag{1}$$

are directly related to q-Racah polynomials. They are q-analogs of Wilson polynomials (1) of Section 8.5.5. Since

$$(ae^{i\theta};q)_k(ae^{-i\theta};q)_k=\prod_{j=0}^{k-1}(1-2aq^j\cos\theta+a^2q^{2j}),$$

then $p_n(x;a,b,c,d\,|\,q)$ are polynomials of degree n in $x=\cos\theta$. They are called *q-Askey-Wilson polynomials.*

The polynomials $p_n(x;a,b,c,d\,|\,q)$ are symmetric with respect to permutations of all the parameters a, b, c, d. The relations

$$p_n(-x;a,b,c,d\,|\,q)=(-1)^n p_n(x;-a,-b,-c,-d\,|\,q) \tag{2}$$

and

$$\begin{aligned}&{}_4\varphi_3\left(\begin{matrix}q^{-n},\ abcdq^{n-1},\ ae^{i\theta},\ ae^{-i\theta}\\ ab,\quad ac,\quad ad\end{matrix}\,\middle|\,q,q\right)\\ &= {}_4\varphi_3\left(\begin{matrix}q^{n},\ (abcd)^{-1}q^{1-n},\ a^{-1}e^{i\theta},\ a^{-1}e^{-i\theta}\\ (ab)^{-1},\quad (ac)^{-1},\quad (ad)^{-1}\end{matrix}\,\middle|\,q^{-1},q^{-1}\right)\end{aligned} \tag{3}$$

also hold. In (3) we assume that $abcd\neq 0$.

For q-Askey-Wilson polynomials the recurrence relation

$$2xp_n(x)=A_np_{n+1}(x)+B_np_n(x)+C_np_{n-1}(x), \tag{4}$$

where

$$A_n=\frac{1-abcdq^{n-1}}{(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$C_n=(1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})$$
$$\times\frac{(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})},$$

$$B_n=\frac{q^{n-1}\{(1+abcdq^{2n-1})(sq+s'abcd)-q^{n-1}(1+q)abcd(s+s'q)}{(1-abcdq^{2n-2})(1-abcdq^{2n})},$$

$$s=a+b+c+d,\quad s'=a^{-1}+b^{-1}+c^{-1}+d^{-1},$$

is valid.

R. Askey and J. Wilson (see reference [24] of the second volume) analyzed orthogonality relations for $p_n(x;a,b,c,d\,|\,q)$. They showed that if $a,b,c,d\in\mathbf{R}$, or if a, b, c, d are complex and constitute complex conjugate pairs and $\max(|a|,|b|,|c|,|d|)<1$, $-1<q<1$, then

$$\frac{1}{2\pi}\int_{-1}^{1}p_n(x;a,b,c,d\,|\,q)p_m(x;a,b,c,d\,|\,q)\frac{w(x)dx}{\sqrt{1-x^2}}=\delta_{mn}h_n, \tag{5}$$

where

$$w(x) = \frac{\prod_{k=0}^{\infty}(1-2(2x^2-1)q^k+q^{2k})}{h(x,a)h(x,b)h(x,c)h(x,d)}, \tag{6}$$

$$h(x,a) = \prod_{k=0}^{\infty}(1-2axq^k+a^2q^{2k}) = (ae^{i\theta};q)_\infty(ae^{-i\theta};q)_\infty, \tag{6'}$$

$$h_n = \frac{(abcdq^{2n};q)_\infty(abcdq^{n-1};q)_n(a^{n+1};q)_\infty^{-1}(abq^n;q)_\infty^{-1}}{(acq^n;q)_\infty(adq^n;q)_\infty(bcq^n;q)_\infty(bdq^n;q)_\infty(cdq^n;q)_\infty} \tag{7}$$

and $\cos\theta = x$.

But if $a > 1$ and b, c, d are real or one of them is real and two other are complex conjugate, and, besides, $\max(|b|,|c|,|d|) < 1$, $0 < q < 1$, and the pairwise products of a, b, c, d have absolute value less than one, then the orthogonality relation is of the form

$$\frac{1}{2\pi}\int_{-1}^{1} p_n(x)p_m(x)\frac{w(x)dx}{\sqrt{1-x^2}} + \sum_k p_n(x_k)p_m(x_k)w_k = \delta_{mn}h_n, \tag{8}$$

where the summation is over integral k for which $1 < aq^k \le a$; h_n and $w(x)$ are defined by formulas (6) and (7), $x_k = \frac{1}{2}(aq^k + a^{-1}q^{-k})$ and

$$w_k = \frac{(a^2;q)_\infty}{(q;q)_\infty(ab;q)_\infty(b/a;q)_\infty(ac;q)_\infty(c/a;q)_\infty(ad;q)_\infty(d/a;q)_\infty} \times \frac{(a^2;q)_k(1-aq^{2k}(ab;q)_k(ac;q)_k(ad;q)_k q^k}{(q;q)_k(1-a)(aq/b;q)_k(aq/c;q)_k(aq/d;q)_k(abcd)^k}. \tag{8'}$$

If any one of the numbers a^2, b^2, c^2, d^2, ab, $ac, \ldots, cd$ does not coincide with any of the numbers q^j; $j = 0,-1,-2,\ldots$, then

$$\frac{1}{2\pi i}\int_C p_n\left(\frac{z+z^{-1}}{2};a,b,c,d\,\middle|\,q\right)p_m\left(\frac{z+z^{-1}}{2};a,b,c,d\,\middle|\,q\right)f(z)dz = 2\delta_{mn}h_n, \tag{9}$$

where h_n is given by (7),

$$f(z) = \frac{(z^2;q)_\infty(z^{-1};q)_\infty}{z(az;q)_\infty(a/z;q)_\infty(bz;q)_\infty(b/z;q)_\infty(cz;q)_\infty(c/z;q)_\infty(dz;q)_\infty(d/z;q)_\infty}$$

and the contour C is the unit circle, traversed in the positive direction, with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to infinity. Such a contour exists by the assumptions above.

When $ab = q^{-N}$ the orthogonality relation (5) of Section 14.6.5 is obtained from (9).

The polynomials $p_n(x; \alpha, \beta, \gamma, d \,|\, q)$ can be expanded in the polynomials $p_k(x; a, b, c, d \,|\, q)$. R. Askey and J. Wilson have showed that

$$p_n(x; \alpha, \beta, \gamma, d \,|\, q) = \sum_{k=0}^{n} c_{kn} p_k(x; a, b, c, d \,|\, q), \tag{10}$$

where

$$\begin{aligned} c_{kn} &= (-1)^k q^{k(k+1)/2} d^{k-n} \\ &\times \frac{(q^{-n}; q)_k (\alpha\beta\gamma d q^{n-1}; q)_k (\alpha d; q)_n (\beta d; q)_n (\gamma d; q)_n}{(q; q)_k (abcdq^{k-1}; q)_k (\alpha d; q)_k (\beta d; q)_k (\gamma d; q)_k} \\ &\times {}_5\varphi_4 \left(\begin{matrix} q^{k-n},\ q^{n+k-1}\alpha\beta\gamma d,\ adq^k,\ bdq^k,\ cdq^k \\ abcdq^{2k},\ \alpha dq^k,\ \beta dq^k,\ \gamma dq^k \end{matrix} \,\middle|\, q; q \right). \end{aligned} \tag{11}$$

For integral a, b, c, d, α, β, γ this formula is a corollary of relation (2) of Section 14.5.4. If in (11) $\beta = b$, $\gamma = c$, then ${}_5\varphi_4$ turns into ${}_3\varphi_2$ and we can apply formula (2) of Section 14.6.8 to the last function. The result is

$$p_n(x; \alpha, b, c, d \,|\, q) = \sum_{k=0}^{n} c_{kn} p_k(x; a, b, c, d \,|\, q), \tag{12}$$

where

$$\begin{aligned} c_{kn} = a^{n-k} q^{nk-(k+2)(k+1)/2} (-1)^k &\frac{(q^{-n}; q)_k (\alpha/a; q)_{n-k} (\alpha bcdq^{n-1}; q)_k}{(abcdq^{k-1}; q)_k (abcdq^{2k}; q)_{n-k}} \\ \times &\frac{(bcq^k; q)_{n-k} (bdq^k; q)_{n-k} (cdq^k; q)_{n-k}}{(q; q)_k}. \end{aligned} \tag{13}$$

One can show that when $a = e^{i\theta}$ the right hand side of (12) is reduced to one sum which is expressed in terms of ${}_8\varphi_7$. We obtain *Watson's formula*

$$\begin{aligned} {}_4\varphi_3 &\left(\begin{matrix} q^{-n},\ aq/bc,\ d,\ e \\ deq^{-n}/a,\ aq/b,\ aq/c \end{matrix} \,\middle|\, q, q \right) = \frac{(aq/d; q)_n (aq/e; q)_n}{(aq; q)_n (aq/de; q)_n} \\ \times {}_8\varphi_7 &\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-n} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ cq^{n+1} \end{matrix} \,\middle|\, q, \frac{a^2 q^{n+2}}{bcde} \right) \end{aligned} \tag{14}$$

which is a q-analog of the Whipple formula (5) of Section 8.4.13.

If b, c, d, e diverge to infinity, then this formula turns into the equality

$$1 + \sum_{r=1}^{N} \frac{(a; q)_{r-1} (1 - aq^{2r}) a^{2r} q^{2r^2} q^{Nr} (q^{-N}; q)_r}{(q; q)_r (aq^{N+1}; q)_r} =$$

$$= \prod_{n=1}^{N} (1 - aq^n) \left\{ 1 + \sum_{r=1}^{N} \frac{(-1)^r q^{r(r+1)/2} (q^{-N}; q)_r a^r q^{Nr}}{(q; q)_r} \right\}.$$

If $N \to \infty$ here, then we have

$$1 + \sum_{r=1}^{\infty} (-1)^r a^{2r} q^{r(2r-1)/2} \frac{(1-aq^{2r})(a;q)_r}{(1-a)(q;q)_r} = \left(\prod_{k=0}^{\infty}(1-aq^k)\right) \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n}.$$

Setting $a = 1$ and then $a = q$ after some transformations (see, for example, reference [43] of the first volume), we obtain the *Rogers-Ramanujan identities*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \tag{15}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{16}$$

In conclusion of this section we note that formula (10) implies the formula for expansion of the little q-Jacobi polynomials

$$p_n(x;\gamma,\delta\,|\,q) = {}_2\varphi_1(q^{-n}, \gamma\delta q^{n+1}; \gamma q; q, qx)$$

in the polynomials $p_k(x;\alpha,\beta\,|\,q)$. We have

$$p_n(x;\gamma,\delta\,|\,q) = \sum_{k=0}^{n} a_{kn} p_k(x;\alpha,\beta\,|\,q), \tag{17}$$

where

$$\begin{aligned} a_{kn} &= (-1)^k q^{k(k+1)} \frac{(\gamma\delta q^{n+1};q)_k (q^{-n};q)_k (\alpha q;q)_k}{(q;q)_k (\gamma q;q)_k (\alpha\beta q^{k+1};q)_k} \\ &\times {}_3\varphi_2 \left(\begin{matrix} q^{-n+k}, \ \gamma\delta q^{n+k+1}, \ \alpha q^{k+1} \\ \gamma q^{k+1}, \ \alpha\beta q^{2k+2} \end{matrix} \,\middle|\, q, q \right). \end{aligned} \tag{18}$$

14.7.2. Properties of q-Askey-Wilson polynomials. Ismail and Wilson [118] derived the generating function for the q-Askey-Wilson polynomials $p_n(x) \equiv p_n(x;a,b,c,d\,|\,q)$:

$$\sum_{n=0}^{\infty} \frac{t^n a^{-n} (ac;q)_n (ad;q)_n}{(cd;q)_n (q;q)_n} p_n(x) = F(x,t), \tag{1}$$

where

$$F(x,t) = {}_2\varphi_1 \left(\begin{matrix} a/z, \ b/z \\ ab \end{matrix} \,\middle|\, q, zt \right) {}_2\varphi_1 \left(\begin{matrix} cz, \ dz \\ cd \end{matrix} \,\middle|\, q, \frac{t}{z} \right). \tag{2}$$

and $z = x - \sqrt{x^2-1}$ (in particular, $x = \cos\theta$ if $z = e^{-\theta}$). Since $|z| \le |z^{-1}|$, then the functions ${}_2\varphi_1$ in (2) converge for $|t| < |z|$. Relation (1) implies the integral representation for $p_n(x)$:

$$p_n(x) = \frac{(cd;q)_n(q;q)_n}{(ac;q)_n(ad;q)_n}\frac{a^n}{2\pi i}\int_C t^{-n-1}F(x,t)dt, \tag{3}$$

where $F(x,t)$ is given by formula (2) and C is the circle $|t| = p$ where $p < |z|$.

The formula

$$\int_{-1}^{1} w(y;a,b,\mu e^{i\theta},\mu e^{-i\theta})p_n(y;a,b,c,d\,|\,q)dy$$
$$= \frac{2\pi(ab\mu^2;q)_\infty}{(ab;q)_\infty(\mu^2;q)_\infty(q;q)_\infty|(a\mu e^{i\theta};q)_\infty(b\mu e^{i\theta};q)_\infty|^2}$$
$$\times\, p_n(\cos\theta; a\mu, b\mu, c\mu^{-1}, d\mu^{-1}\,|\,q), \tag{4}$$

where $\max(|a|,|b|,|\mu|) < 1$,

$$w(x;a,b,c,d) = \frac{h(x,1)h(x,-1)h(x,\sqrt{q})h(x,-\sqrt{q})}{(1-x^2)^{1/2}h(x,a)h(x,b)h(x,c)h(x,d)}, \tag{5}$$

$h(x,a)$ is given by formula (6′) of Section 14.7.1, and the formula

$$\int_{-1}^{1} K_\mu(x,y;q)p_n(y;a,b,c,d\,|\,q) = \lambda_n p_n(x;a,b,c,d\,|\,q), \tag{6}$$

where for $\max(|a|,|b|,|c\mu^{-1}|,|d\mu^{-1}|,|\mu|,|q|) < 1$ we have

$$K_\mu(x,y;q) = \frac{(ab;q)_\infty(cd\mu^{-2};q)_\infty}{(cd;q)_\infty(ab\mu^2;q)_\infty}\left|\frac{(q;q)_\infty(\mu^2;q)_\infty(ce^{i\theta};q)_\infty(de^{i\theta};q)_\infty}{2\pi}\right|^2$$
$$\times \int_{-1}^{1} w(z;c\mu^{-1},d\mu^{-1},\mu e^{i\theta},\mu e^{-i\theta})w(y;a,b,\mu e^{i\psi},\mu e^{-i\psi})$$
$$\times\, |(a\mu e^{i\psi};q)_\infty(b\mu e^{i\psi};q)_\infty|^2 dz, \quad z = \cos\psi,$$

$$\lambda_n = \frac{(ab;q)_n(cd\mu^{-2};q)_n}{(cd;q)_n(ab\mu^2;q)_n}\mu^{2n}$$

are proved in [257].

Gasper and Rahman (see reference [121] of the second volume) have proved the following product formula for q-Askey-Wilson polynomials:

$$
{}_4\varphi_3\left(\begin{matrix} q^{-n},\ aq^n,\ b,\ c \\ d,\ e,\ abcq/de \end{matrix}\,\middle|\, q,q\right)\ {}_4\varphi_3\left(\begin{matrix} q^{-n},\ aq^n,\ g,\ h \\ e,\ dgh/bc,\ abcq/de \end{matrix}\,\middle|\, q,q\right)
$$

$$
= \frac{(aq/e;q)_n(de/bc;q)_n}{(abcq/de;q)_n(e;q)_n}\left(\frac{bc}{d}\right)^n \sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(q^{-n};q)_{r+s}(aq^n;q)_{r+s}(g;q)_r(h;q)_r}{(q;q)_r(q;q)_s(d;q)_{r+s}(dgh/bc;q)_{r+s}} \tag{7}
$$

$$
\times \frac{(b;q)_r(c;q)_r(d/b;q)_s(d/c;q)_s(dg/bc;q)_s(dh/bc;q)_s(1-bcd^{-1}q^{r-s})q^{r+s}}{(e;q)_r(bcq/d;q)_r(abcq/de;q)_r(aq/e;q)_s(d/bc;q)_s(de/bc;q)_s(1-bcd^{-1}q^{-s})}.
$$

14.7.3. q-Gegenbauer polynomials. For $c = a$, $b = -d = q^{1/2}a$ polynomials (1) of Section 14.7.1 define polynomials which can be written in the form

$$
C_n(\cos\theta;\beta\,|\,q) = \frac{(\beta^2;q)_n}{\beta^{n/2}(q;q)_n}\ {}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^n\beta^2,\ \beta^{1/2}e^{i\theta},\ \beta^{1/2}e^{-i\theta}, \\ \beta q^{1/2},\ -\beta q^{1/2},\ -\beta \end{matrix}\,\middle|\, q,q\right). \tag{1}
$$

The recurrence relation for them has the form

$$
2x(1-\beta q^n)C_n(x;\beta\,|\,q) = (1-q^{n+1})C_{n+1}(x;\beta\,|\,q) + (1-\beta^2 q^{n-1})C_{n-1}(x;\beta\,|\,q). \tag{2}
$$

By means of this relation the generating function for these polynomials

$$
\frac{(\beta y e^{i\theta};q)_\infty(\beta y e^{-i\theta};q)_\infty}{(ye^{i\theta};q)_\infty(ye^{-i\theta};q)_\infty} = \sum_{n=0}^{\infty} C_n(\cos\theta;\beta\,|\,q)y^n, \quad |y|<1, \tag{3}
$$

is obtained (see reference [448] of the second volume). From (3) one can derive another expression for $C_n(\cos\theta;\beta\,|\,q)$:

$$
C_n(\cos\theta;\beta\,|\,q) = \frac{(\beta;q)_n}{(q;q)_n}e^{in\theta}\ {}_2\varphi_1\left(\begin{matrix} q^{-n},\ \beta \\ \beta^{-1}q^{1-n} \end{matrix}\,\middle|\, q, \frac{qe^{-2i\theta}}{\beta}\right). \tag{4}
$$

Askey and Ismail (see reference [448] of the second volume) obtained one more expression for these polynomials:

$$
C_n(\cos\theta;\beta\,|\,q) = \frac{(\beta^2;q)_n e^{-in\theta}}{\beta^n(q;q)_n}\ {}_3\varphi_2\left(\begin{matrix} q^{-n},\ \beta,\ \beta e^{2i\theta} \\ \beta^2,\ 0 \end{matrix}\,\middle|\, q,q\right). \tag{5}
$$

If $q \to 1^-$, then formula (2) turns into recurrence formula for ordinary Gegenbauer polynomials $C_n^\lambda(x)$ where $\beta = q^\lambda$. Consequently,

$$
\lim_{q\to 1^-} C_n(x;q^\lambda\,|\,q) = C_n^\lambda(x).
$$

Therefore, $C_n(x;\beta\,|\,q)$ are called *continuous q-ultraspherical polynomials* or *continuous q-Gegenbauer polynomials.* We shall omit the word "continuous" below.

One can show that[4]

$$C_n(x;\beta\,|\,q) = \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} T_{n-2k}(x), \tag{6}$$

where $T_r(x)$ is the Chebyshev polynomial of the first kind, that is, $T_r(\cos\theta) = \cos n\theta$. Setting here $\beta = q$ we obtain

$$C_n(\cos\theta;q\,|\,q) = \sum_{k=0}^{n} \cos(n-2k)\theta = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{7}$$

We also have

$$\lim_{\beta\to 1}(1-q^n)\frac{C_n(\cos\theta;\beta\,|\,q)}{2(1-\beta)} = \cos n\theta. \tag{7'}$$

The orthogonality relation for $C_n(x;\beta|q)$ follows from that for q-Askey-Wilson polynomials. If $-1 < q, \beta < 1$, then

$$\int_{-1}^{1} C_n(x;\beta\,|\,q)C_m(x;\beta\,|\,q)\frac{w_\beta(x)dx}{\sqrt{1-x^2}} = \delta_{mn}h_n, \tag{8}$$

where

$$h_n = 2\pi\frac{(1-\beta)(\beta^2;q)_n}{(1-\beta q^n)(q;q)_n}\frac{(\beta;q)_\infty(\beta q;q)_\infty}{(\beta^2;q)_\infty(q;q)_\infty}.$$

$$w_\beta(x) = \prod_{k=0}^{\infty}\frac{1-2(1x^2-1)q^k+q^{2k}}{1-2(2x^2-1)\beta q^k+\beta^2q^{2k}} = \frac{(e^{2i\theta};q)_\infty(e^{-2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty(\beta e^{-2i\theta};q)_\infty}$$

and $x = \cos\theta$.

For $C_n(x;\beta\,|\,q)$ one has the formulas

$$C_n(x;\gamma\,|\,q) = \sum_{k=0}^{[n/2]}\frac{\beta^k(\gamma/\beta;q)_k(\gamma;q)_{n-k}(1-\beta q^{n-2k})}{(q;q)_k(\beta q;q)_{n-k}(1-\beta)}C_{n-2k}(x;\beta\,|\,q), \tag{9}$$

---

[4] We erroneously defined Chebyshev polynomials of the first kind $T_n(z)$ on the p. 156 of the first volume as the polynomials $\frac{n}{2}C_n^0(z)$. Since $C_n^0 \equiv 0$ for $n > 0$, then we have to define $T_n(z)$ as

$$T_n(z) = \frac{n!\Gamma(1/2)}{\Gamma(n+1/2)}P_n^{(-1/2,-1/2)}(z),$$

where $P_n^{(\alpha,\beta)}(z)$ is the Jacobi polynomial. It is necessary to change respectively the reasoning at the end of p. 357 of the first volume.

$$C_n(x;\beta\,|\,q)C_m(x;\beta\,|\,q)=\sum_{k=0}^{\min(m,n)} A_{kmn}C_{m+n-2k}(x;\beta\,|\,q), \tag{10}$$

where

$$A_{kmn}=\frac{(q;q)_{m+n-2k}(\beta;q)_{n-k}(\beta;q)_{m-k}}{(\beta^2;q)_{m+n-2k}(q;q)_{n-k}(q;q)_{m-k}}\frac{(\beta;q)_k(\beta^2;q)_{m+n-k}(1-\beta q^{m+n-2k})}{(q;q)_k(\beta q;q)_{m+n-k}(1-\beta)} \tag{11}$$

which for $q\to 1^-$ yield the well-known formulas for Gegenbauer polynomials. In (9) $[n/2]$ means the integral part of $n/2$.

Formulas (9) and (10) can be inverted. The inversion of formula (9) is of the form

$$w_\beta(x)C_n(x;\beta\,|\,q)=\sum_{k=0}^{\infty} a(k,n)w_\gamma(x)C_{n+2k}(x;\gamma\,|\,q), \tag{12}$$

where $w_\beta(x)$ is the same as in (8) and

$$a(k,n)=\frac{\beta^k(\gamma/\beta;q)_k(q^{n+1};q)_{2k}(\gamma^2q^{n+2k};q)_\infty(\beta q^{n+k+1};q)_\infty(\beta;q)_\infty(1-\gamma q^{n+2k})}{(q;q)_k(\gamma q^{n+k};q)_\infty(\beta^2q^n;q)_\infty(\gamma;q)_\infty}.$$

The formula

$$\begin{aligned}&\frac{(\beta;q)_m(\beta;q)_n}{(q;q)_m(q;q)_n}C_{m+n}(x;\beta\,|\,q)\\ &=\frac{(\beta;q)_{m+n}}{(q;q)_{m+n}}\sum_{k=0}^{\min(m,n)} b(k,m,n)C_{m-k}(x;\beta\,|\,q)C_{n-k}(x;\beta\,|\,q),\end{aligned} \tag{13}$$

where

$$b(k,m,n)=\frac{(\beta^{-2}q^{-m-n};q)_k(1-\beta^{-2}q^{2k-m-n})(\beta^{-1};q)_k}{(q;q)_k(1-\beta^{-2}q^{-m-n})(\beta^{-1}q^{1-m-n};q)_k}\left(\frac{\beta^2}{q}\right)^k,$$

is an inversion of (10).

Rahman and Verma [258] have proved the product formula for q-Gegenbauer polynomial

$$C_n(x;a^2\,|\,q)C_n(y;a^2\,|\,q)=\frac{(a^2;q)_n}{(q;q)_n}a^{-n}\int_{-1}^{1}K(x,y,z;q)C_n(z;a^2\,|\,q)dz, \tag{14}$$

where

$$\begin{aligned}K(x,y,z;q)&=\frac{(q;q)_\infty(a^2;q)_\infty^2\left|(a^2e^{2i\theta};q)_\infty(a^2e^{2i\varphi};q)_\infty\right|^2}{2\pi(a^4;q)_\infty}\\ &\times(1-z^2)^{-1/2}\frac{h(z,1)h(z,-1)h(z,\sqrt{q})h(z,-\sqrt{q})}{h(z,ae^{i\theta+i\varphi})h(z,ae^{-i\theta-i\varphi})h(z,ae^{i\theta-i\varphi})h(z,ae^{i\varphi-i\theta})},\end{aligned}$$

$h(z,a)$ is given by formula (6′) of Section 14.7.1, $x=\cos\theta$, $y=\cos\varphi$, $0\le\theta,\varphi\le\pi$, $\max(|a|,|q|)<1$. Note that the kernel $K(x,y,z;q)$ is positive for all $z\in(-1,1)$ and $\theta,\varphi\in[0,\pi]$.

Rahman and Verma [258] have also derived the addition theorem for the polynomials $C_n(x;a^2\,|\,q)$:

$$C_n(z;a^2\,|\,q)=\sum_{m=0}^{n} c_{mn}(a)e^{-im\theta}(a^2e^{2i\theta};q)_m C_{n-m}(x;a^2q^m\,|\,q)$$
$$\times\, e^{-im\varphi}(a^2e^{2i\varphi};q)_m C_{n-m}(y;a^2q\,|\,q)$$
$$\times\, p_m(z;ae^{i\theta+i\varphi},ae^{-i\theta-i\varphi},ae^{i\theta-i\varphi},ae^{i\varphi-i\theta}\,|\,q), \tag{15}$$

where p_m is the q-Askey–Wilson polynomial, $x=\cos\theta$, $y=\cos\varphi$ and

$$c_{mn}(a)=\frac{(a^2;q)_m(a^2;q)_n(a^4q^n;q)_m(a^4q^{-1};q)_m(q;q)_{n-m}a^{n-2m}}{(q;q)_m(a^2\sqrt{q};q)_m(-a^2\sqrt{q};q)_m(-a^2;q)_m(a^4q^{-1};q)_{2m}(a^2q^m;q)^2_{n-m}}.$$

14.7.4. Continuous q-Hermite polynomials. The formula

$$H_n(x\,|\,q)=(q;q)_nC_n(x;0\,|\,q) \tag{1}$$

defines polynomials called *continuous q-Hermite polynomials.* The recurrence relation for them is of the form

$$2xH_n(x\,|\,q)=H_{n+1}(x\,|\,q)+(1-q^n)H_{n-1}(x\,|\,q).$$

Formula (3) of Section 14.7.3 leads to the generating function for $H_n(x\,|\,q)$:

$$|(xe^{i\theta};q)_\infty|^{-2}=\sum_{n=0}^{\infty}\frac{H_n(\cos\theta\,|\,q)}{(q;q)_n}x^n. \tag{2}$$

This relation can be written as

$$\prod_{n=0}^{\infty}(1-2xzq^n+z^2q^{2n})^{-1}=\sum_{n=0}^{\infty}\frac{H_n(x\,|\,q)}{(q;q)_n}z^n. \tag{3}$$

The formula

$$\frac{(z^2;q)_\infty}{|(ze^{i(\theta+\varphi)};q)_\infty(ze^{i(\theta-\varphi)};q)_\infty|^2}=\sum_{n=0}^{\infty}\frac{H_n(\cos\theta\,|\,q)H_n(\cos\varphi\,|\,q)}{(q;q)_n}z^n \tag{4}$$

provides a generating function for the product of q-Hermite polynomials.

The orthogonality relation for q-Hermite polynomials has the form

$$\int_{-1}^{1} H_n(x\,|\,q)H_m(x\,|\,q)\left(\prod_{k=0}^{\infty}(1-2(2x^2-1)q^k+q^{2k})\right)(1-x^2)^{-1/2}dx$$
$$= \frac{2\pi(q;q)_n}{(q;q)_\infty}\delta_{mn}. \tag{5}$$

14.7.5. Continuous q-Jacobi polynomials. Let

$$a = q^{(2\alpha+1)/4}, \quad b = -q^{(2\beta+1)/4}, \quad c = q^{(2\alpha+3)/4}, \quad d = -q^{(2\beta+3)/4}$$

in formula (1) of Section 14.7.1. We obtain polynomials which coincide up to a factor with

$$P_n^{(\alpha,\beta)}(x\,|\,q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \times {}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n+\alpha+\beta+1},\ q^{(2\alpha+1)/4}e^{i\theta},\ q^{(2\alpha+1)/4}e^{-i\theta} \\ q^{\alpha+1},\ -q^{(\alpha+\beta+1)/2},\ -q^{(\alpha+\beta+2)/2} \end{matrix}\;\middle|\; q,q\right), \tag{1}$$

where $x = \cos\theta$. If $0 < q < 1$, $\alpha \geq -1/2$, $\beta \geq -1/2$, then the orthogonality relation

$$\int_0^{\pi} P_n^{(\alpha,\beta)}(\cos\theta\,|\,q)P_m^{(\alpha,\beta)}(\cos\theta\,|\,q)w(\theta;q)d\theta = \delta_{mn}h_n, \tag{2}$$

where

$$w(\theta;q^2) = \left|\frac{(e^{i\theta};q)_\infty(-e^{i\theta};q)_\infty}{(q^{\alpha+1/2}e^{i\theta};q)_\infty(-q^{\beta+1/2}e^{i\theta};q)_\infty}\right|^2,$$

is valid. These polynomials are called *continuous q-Jacobi polynomials.*

Rahman [255] has introduced the polynomials

$$\tilde{P}_n^{(\alpha,\beta)}(x\,|\,q) = \frac{(q^{\alpha+1};q)_n(-q^{\beta+1};q)_n}{(q;q)_n(-q;q)_n} \times {}_4\varphi_3\left(\begin{matrix} q^{-n},\ q^{n+\alpha+\beta+1},\ q^{1/2}e^{i\theta},\ q^{1/2}e^{-i\theta} \\ q^{\alpha+1},\ -q^{\beta+1},\ -q \end{matrix}\;\middle|\; q,q\right) \tag{3}$$

which are a special case of q-Askey-Wilson polynomials. Comparing the orthogonality relations for polynomials (1) and (3), Askey and Wilson (see reference [24] of the second volume) have proved that

$$P_n^{(\alpha,\beta)}(x\,|\,q^2) = \frac{q^{\alpha n}(-q;q)_n}{(-q^{\alpha+\beta+1};q)_n}\tilde{P}_n^{(\alpha,\beta)}(x\,|\,q). \tag{4}$$

This formula implies the quadratic transformation for ${}_4\varphi_3$ which has the form

$$\begin{aligned}&{}_4\varphi_3\left(\begin{matrix}q^{-2n},\ a^2q^{2n},\ b^2q,\ c^2\\ -a,\ -aq,\ b^2c^2q^2\end{matrix}\,\middle|\,q^2,q^2\right)\\ &\qquad=(bc)^n\frac{(-a/bc;q)_n(-q;q)_n}{(-bcq;q)_n(-a;q)_n}\,{}_4\varphi_3\left(\begin{matrix}q^{-n},\ q^na,\ bq/c,\ c/b\\ -q,\ -a/bc,\ bcq\end{matrix}\,\middle|\,q,q\right).\end{aligned}\tag{5}$$

Setting $b=c$ into this formula we obtain the summation formula

$${}_4\varphi_3\left(\begin{matrix}q^{-2n},\ a^2q^{2n},\ b^2,\ b^2q\\ -a,\ -aq,\ b^4q^2\end{matrix}\,\middle|\,q^2,q^2\right)=\frac{b^{2n}(-a/b^2;q)_n(-q;q)_n}{(-b^2q;q)_n(-a;q)_n}.\tag{6}$$

One can rewrite (5) in terms of q-Askey-Wilson polynomials:

$$\begin{aligned}&p_n(x;b,bq,-c,-cq\,|\,q^2)=(bq^{-1/2})^n\\ &\qquad\times\frac{(-cq^{1/2};q)_n(-q;q)_n}{(-bq^{1/2};q)_n(-bc;q)_n}p_n(x;q^{1/2},b,-c,-q^{1/2}\,|\,q).\end{aligned}\tag{7}$$

Comparing formula (1) of Section 14.7.3 with (3) we see that $C_n(\cos\theta,\beta\,|\,q)$ is a multiple of $\tilde{P}_n^{(\beta,\beta)}(x\,|\,q)$.

Rahman [255] has proved the product formula for continuous q-Jacobi polynomials which can be written down in the form

$$\begin{aligned}&p_n(x;b,bq,-c,-cq\,|\,q^2)p_n(y;b,bq,-c,-cq\,|\,q^2)\\ &\qquad=\int_{-1}^{1}K(x,y,z)p_n(z;b,bq,-c,-cq\,|\,q^2)dz,\end{aligned}\tag{8}$$

where

$$\begin{aligned}K(x,y,z)&=\frac{(q^2;q^2)_\infty(b^2;q^2)_\infty(c^2;q^2)_\infty(q;q)_\infty(bc;q)_\infty(b/c;q)_\infty}{4\pi^2(b^2c^2;q^2)_\infty(b^2;q)_\infty|(be^{i\theta};q)_\infty(be^{i\varphi};q)_\infty|^{-2}}\\ &\times\frac{(1-z^2)^{-1/2}h_{q^2}(z,1)h_{q^2}(z,-1)h_{q^2}(z,q)h_{q^2}(z,-q)}{h_{q^2}(z,be^{i\theta+i\varphi})h_{q^2}(z,be^{-i\theta-i\varphi})}\\ &\times\int_{-1}^{1}\frac{h_q(\tau,1)h_q(\tau,-1)h_q(\tau,q^{1/2})h_q(\tau,-q^{1/2})}{h_q(\tau,\sqrt{c}e^{i\psi/2})h_q(\tau,\sqrt{c}e^{-i\psi/2})h_q(\tau,-\sqrt{c}e^{i\psi/2})h_q(\tau,-\sqrt{c}e^{-i\psi/2})}\\ &\times\frac{h_q(\tau,-\sqrt{bc}e^{(i\theta+i\varphi)/2})h_q(\tau,-\sqrt{bc}e^{-(i\theta+i\varphi)/2})}{h_q(\tau,\sqrt{b/c}e^{(i\theta-i\varphi)/2})h_q(\tau,\sqrt{b/c}e^{(i\varphi-i\theta)/2})}\frac{d\tau}{\sqrt{1-\tau^2}}.\end{aligned}\tag{9}$$

Here $x = \cos\theta$, $y = \cos\varphi$, $z = \cos\psi$, $0 \le \theta, \varphi, \psi \le \pi$, $0 \le b < c < 1$, $0 < q < 1$.

Replace in formula (1) of Section 14.7.1 a by $ae^{i\varphi}$, c by $ce^{i\varphi}$, set $b = ae^{i\varphi}$, $d = ce^{-i\varphi}$ and shift θ by φ. Then we obtain the polynomials

$$P_n(\cos(\theta+\varphi); a, c\,|\,q) = a^{-n} e^{-in\varphi} (ace^{2i\varphi}; q)_n (a^2; q)_n (ac; q)_n \times {}_4\varphi_3 \left(\begin{matrix} q^{-n},\ q^{n-1}a^2c^2,\ ae^{2i\varphi}e^{i\theta},\ ae^{-i\theta} \\ ace^{2i\varphi},\ a^2,\ ac \end{matrix} \middle|\, q, q \right). \tag{10}$$

If $|a| < 1$, $|c| < 1$, then these polynomials satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(\cos(\theta+\varphi); a, c\,|\,q) P_m(\cos(\theta+\varphi); a, c\,|\,q) w(\theta) d\theta = \delta_{mn} h_n, \tag{11}$$

where $a, c \in \mathbb{R}$ and

$$w(\theta) = \left| \frac{(e^{2i(\theta+\varphi)}; q)_\infty}{(ae^{i\theta}; q)_\infty (ce^{i\theta}; q)_\infty (ae^{i(\theta+2\varphi)}; q)_\infty (ce^{i(\theta+2\varphi)}; q)_\infty} \right|^2 .$$

For $P_n(x; a, b\,|\,q)$, $x = \cos(\theta+\varphi)$, one has the recurrence relation

$$2xP_n(x) = A_n P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x), \tag{12}$$

where

$$A_n = \frac{(1 - a^2c^2q^{n-1})}{(1 - a^2c^2q^{2n-1})(1 - a^2c^2q^{2n})},$$

$$B_n = \frac{2q^{n-1}(a+c)\cos\varphi\{(1 + a^2c^2q^{2n-1})(q + ac) - q^{n-1}(1+q)a^2c^2(1 + acq)\}}{(1 - a^2c^2q^{2n-2})(1 - a^2c^2q^{2n})},$$

$$C_n = \frac{(1-q^n)(1 - acq^{n-1})(1 - a^2q^{n-1})(1 - c^2q^{n-1})}{(1 - a^2c^2q^{2n-1})(1 + acq^{n-1})} \times (1 - 2acq^{n-1}\cos 2\varphi + a^2c^2q^{2n-2}).$$

In conclusion of this section we note the special cases of q-Askey–Wilson polynomials:

$$p_n(\cos\theta; 1, -1, q^{1/2}, -q^{1/2}\,|\,q) = 2(q^n; q)_n \cos n\theta,$$

$$p_n(\cos\theta; q, -1, q^{1/2}, -q^{1/2}\,|\,q) = (q^{n+1}; q)_n \frac{\sin((2n+1)\theta/2)}{\sin(\theta/2)},$$

$$p_n(\cos\theta; 1, -q, q^{1/2}, -q^{1/2}\,|\,q) = (q^{n+1}; q)_n \frac{\cos((2n+1)\theta/2)}{\cos(\theta/2)}.$$

14.7.6. Big q-Jacobi polynomials. Replacing q^{-N} in expression (1) of Section 14.6.4 by cq we obtain polynomials which can be written down in the form

$$P_n(x;a,b,c\,|\,q) = {}_3\varphi_2\left(\begin{matrix} q^{-n},\ q^{n+1}ab, x \\ aq,\ cq \end{matrix}\,\middle|\, q,q\right), \quad n=0,1,2,\ldots . \tag{1}$$

The formula

$$\sum_{n=0}^{\infty} \frac{(cq;q)_n t^n}{(q;q)_n (bq;q)_n} P_n(x;a,b,c\,|\,q) = \left(\sum_{n=0}^{\infty} \frac{(aq/x;q)_n (tx)^n}{(q;q)_n (aq;q)_n}\right)\left(\sum_{k=0}^{\infty} \frac{(bx/c;q)_k (-ct)^k q^{k(k+1)/2}}{(q;q)_k (bq;q)_k}\right) \tag{2}$$

provides a generating function for polynomials (1).

In fact, polynomials (1) are a special case of the so-called *big q-Jacobi polynomials*

$$P_n^{(\alpha,\beta)}(x;c,d\,|\,q) = c^n q^{-(\alpha+1)n} \frac{(q^{\alpha+1};q)_n (-q^{\alpha+1}d/c;q)_n}{(q;q)_n (-q;q)_n} \times {}_3\varphi_2\left(\begin{matrix} q^{-n},\ q^{n+\alpha+\beta+1},\ xq^{\alpha+1}/c \\ q^{\alpha+1},\ -q^{\alpha+1}d/c \end{matrix}\,\middle|\, q,q\right). \tag{3}$$

Askey and Wilson have proved the orthogonality relation

$$\int_{-d}^{c} P_n^{(\alpha,\beta)}(x;c,d\,|\,q) P_m^{(\alpha,\beta)}(x;c,d\,|\,q) w(x,q^\alpha,q^\beta,c,d)\, d_q x = \delta_{mn} h_n, \tag{4}$$

where

$$w(x,q^\alpha,q^\beta,c,d) = \frac{(qx/c;q)_\infty (-qx/d;q)_\infty}{(q^{\alpha+1}x/c;q)_\infty (-q^{\beta+1}x/d;q)_\infty}, \tag{5}$$

$$h_n = \frac{(cd)^{n+1} q^{n(n-1)/2} (q^{\alpha+1};q)_n (q^{\beta+1};q)_n (1-q^{\alpha+\beta+1})(-q^{\beta+1}c/d;q)_n}{(q^{\alpha+\beta+1};q)_n (q;q)_n (1-q^{2n+\alpha+\beta+1})(-q;q)_n^2} \times \frac{(-q^{\alpha+1}d/c;q)_n (1-q)(q;q)_\infty (q^{\alpha+\beta+2};q)_\infty (-d/c;q)_\infty (-c/d;q)_\infty}{(c+d)(q^{\alpha+1};q)_\infty (q^{\beta+1};q)_\infty (-q^{\alpha+1}d/c;q)_\infty (-q^{\beta+1}c/d;q)_\infty}. \tag{6}$$

For integral α, β, c, d this relation will be derived in Section 14.9.8.

The weight function (5) satisfies the symmetry condition

$$w(-x,\alpha,\beta,c,d) = w(x;\beta,\alpha,d,c).$$

This leads to the symmetry relation for polynomials (3) (see reference [24] of the second volume)

$$P_n^{(\alpha,\beta)}(-x;c,d\,|\,q) = (-1)^n P_n^{(\beta,\alpha)}(x;d,c\,|\,q). \tag{7}$$

If $c = d = 1$, then

$$\lim_{q\to 1} P_n^{(\alpha,\beta)}(x; c, d\,|\,q) = P_n^{(\alpha,\beta)}(x)$$

with the classical Jacobi polynomial on the right hand side.

For $c = 1$, $d = 0$ we have

$$P_n^{(\alpha,\beta)}(x; 1, 0\,|\,q) = (-1)^n \frac{q^{n(n-1)/2}(q^{\beta+1}; q)_n}{(q; q)_n(-q; q)_n} \times {}_2\varphi_1(q^{-n}, q^{n+\alpha+\beta+1}; q^{\beta+1}; q, xq).$$

Here the function ${}_2\varphi_1$ coincides with the little q-Jacobi polynomial $p_n(x; q^\beta, q^\alpha\,|\,q)$ (see formula (1) of Section 14.4.4).

Let us note also the formulas

$$P_{2n}^{(\alpha,\alpha)}(x; 1, 1\,|\,q) = c_n p_n(x^2; q^{-1/2}, q^\alpha\,|\,q^2),$$

$$P_{2n+1}^{(\alpha,\alpha)}(x; 1, 1\,|\,q) = c'_n x p_n(x^2; q^{1/2}, q^\alpha\,|\,q^2),$$

where c_n and c'_n can be found from these formulas at $x = 1$.

Let

$$\tilde{P}_n(x; a, b, c, d\,|\,q) = {}_3\varphi_2\left(\begin{matrix} q^{-n},\ abq^{n+1},\ axq/c \\ aq,\ -adq/c \end{matrix}\,\middle|\, q, q\right). \tag{8}$$

We prove that these polynomials satisfy the recurrence formula

$$D_q T_q^{-1}\{w(x; a, b, c, d)\tilde{P}_n(x; a, b, c, d\,|\,q)\} = -\frac{(1-\alpha)(1+ad/c)}{(1-q)ad} \times w(x; aq^{-1}, bq^{-1}, c, d)\tilde{P}_{n+1}(x; aq^{-1}, bq^{-1}, c, d\,|\,q), \tag{9}$$

where $w(x; a, b, c, d)$ is given by formula (5) and

$$D_q F(x) = \frac{F(qx) - F(x)}{x(q-1)}, \quad T_q F(x) = F(qx).$$

For this we note that

$$D_q T_q^{-1}\{w(x; a, b, c, d)(axq/c; q)_k q^k)\} = \frac{1}{(1-q)ad} w(x; aq^{-1}, bq^{-1}, c, d) \times\{(ax/c; q)_{k+1}(1 - abq^k) - (ax/c; q)_k(1 - aq^k)(1 + adq^k/c)\}.$$

Hence

$$\frac{D_qT_q^{-1}\{w(x;a,b,c,d)\tilde{P}_n(x;a,b,c,d\,|\,q)\}}{w(x;aq^{-1},bq^{-1},c,d)}$$
$$= \frac{1}{(1-q)ad}\left\{\sum_k \frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(aq;q)_k(-adq/c;q)_k}(ax/c;q)_{k+1}(1-abq^k)\right.$$
$$\left.-\sum_k \frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(aq;q)_{k-1}(-adq/c;q)_{k-1}}(ax/c;q)_k\right\}$$
$$= \frac{(1-a)(1+ad/c)}{(1-q)ad}\left\{\sum_k \frac{(q^{-n};q)_{k-1}(abq^{n+1};q)_{k-1}}{(q;q)_{k-1}(a;q)_k(-ad/c;q)_k}(ax/c;q)_k(1-abq^{k-1})\right.$$
$$\left.-\sum_k \frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(a;q)_k(-ad/c;q)_k}(ax/c;q)_k\right\}$$
$$= \frac{(1-a)(1+ad/c)}{(1-q)ad}\sum_k \frac{(q^{-n};q)_{k-1}}{(q;q)_k}\,\frac{(abq^{n+1};q)_{k-1}(ax/c;q)_k}{(a;q)_k(-ad/c;q)_k}$$
$$\times\{(1-q^k)(1-abq^{k-1})-(1-q^{k-n-1})(1-abq^{n+k})\}$$
$$= -\frac{(1-a)(1+ad/c)}{(1-q)ad}\sum_k \frac{(q^n;q)_k(abq^n;q)_k}{(q;q)_k(a;q)_k(-ad/c;q)_k}(ax/c;q)_kq^k$$
$$= -\frac{(1-a)(1+ad/c)}{(1-q)ad}\tilde{P}_n(x;aq^{-1},bq^{-1},c,d\,|\,q).$$

Thus, relation (9) is proved.

Formula (9) implies the Rodrigues formula for polynomials (8):

$$w(x;a,b,c,d)\tilde{P}_n(x;a,b,c,d\,|\,q) = \frac{q^{n(n+1)/2}(1-q^n)(-aq)^n}{(aq;q)_n(-adq/c;q)_n} \times (D_qT_q^{-1})^n w(x;aq^n,bq^n,c,d). \tag{10}$$

It is equivalent to the Rodrigues formula for the polynomials

$$\tilde{P}_n^{(\alpha,\beta)}(x;c,d\,|\,q) = {}_3\varphi_2\left(\begin{matrix} q^{-n},\ q^{n+\alpha+\beta+1},\ xq^{\alpha+1}/c \\ q^{\alpha+1},\ -q^{\alpha+1}d/c \end{matrix}\,\middle|\, q,q\right) \tag{11}$$

which is of the form

$$(D_qT_q^{-1})^n\left(\frac{(qx/c;q)_\infty(-qx/d;q)_\infty}{(q^{\alpha+n+1}x/c;q)_\infty(-q^{\beta+n+1}x/d;q)_\infty}\right)$$
$$= \frac{(-1)^nq^{-\alpha n-n(n+1)/2}}{(1-q)^nd^n}(q^{\alpha+1};q)_n(-q^{-\alpha+1}d/c;q)_n \tag{12}$$
$$\times \frac{(qx/c;q)_\infty(-qx/d;q)_\infty}{(q^{\alpha+1}x/c;q)_\infty(-q^{\beta+1}x/d;q)_\infty}\tilde{P}_n^{(\alpha,\beta)}(x;c,d\,|\,q).$$

14.8. Analysis on the Quantum Group $SL_q(2, \mathbb{C})$ and Little q-Jacobi Polynomials

14.8.1. The algebra of functions on the quantum group $SL_q(2, \mathbb{C})$. Let us return to relations (1)–(5) of Section 14.4.2. For simplification we use the notations x, u, v, y instead of π_{11}, π_{12}, π_{21}, π_{22} respectively, and q instead of $\sqrt{q}$. Then relations (1)–(5) of Section 14.4.2 can be written as

$$uv = vu, \quad ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy, \tag{1}$$

$$xy - q^{-1}uv = I, \quad yx - quv = I \tag{2}$$

(relations (2) are equivalent to relations (1), (4) and (5) of Section 14.4.2). The elements x, u, v, y, satisfying conditions (1) and (2), generate the algebra A. Finite linear complex combinations of the products

$$x^n u^m v^k y^r, \quad n, m, k, r, \in \mathbb{Z}_+ \cup \{0\}, \tag{3}$$

are elements of this algebra.

In order to find a basis of A we note that, due to relations (1) and (2), the elements $x^m y^m$ and $y^m x^m$ can be represented as linear combinations of the elements $(uv)^k$, $k = 0, 1, 2, \ldots, m$. Making use of the binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (see formula (3) of Section 13.2.2) and of their property

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k-n} + \begin{bmatrix} n \\ k+1 \end{bmatrix}_q,$$

with the help of mathematical induction one proves that

$$x^m y^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{k^2-2km}(uv)^k = (-q^{-1}uv; q^{-2})_m, \tag{4}$$

$$y^m x^m = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{k^2}(uv)^k = (-quv; q^2)_m. \tag{5}$$

By virtue of relations (1) one can represent elements (3) in the form $cu^m v^k x^n y^r$ or in the form $c'x^n y^r u^m v^k$ where c and c' are numbers. Let $n \le r$. Then the element $u^m v^k x^n y^r$ is represented as $u^m v^k (x^n y^n) y^{r-n}$ and by means of formula (4) we can represent it in the form of a linear combination of elements $u^p v^\ell y^{r-n}$, $p \ge 0$, $\ell \ge 0$. In just the same way with the help of formula (5) we can show that for $n \ge r$ element (3) is represented as a linear combination of the elements $x^{n-r} u^p v^\ell$, $p \ge 0$, $\ell \ge 0$. There is no linear dependence between the elements $u^p v^\ell y^r$, $x^r u^p v^\ell$, $r \ge 0$, $p \ge 0$, $\ell \ge 0$, of the algebra A. Hence, the elements

$$u^p v^\ell y^r, \quad x^r u^p v^q, \quad r \ge 0, \quad p \ge 0, \quad q \ge 0, \tag{6}$$

form a basis in A.

Let us equip A with the structure of a Hopf algebra. For this we represent the set of elements x, u, v, y in form of a matrix $\begin{pmatrix} x & u \\ v & y \end{pmatrix}$ and define a comultiplication $\Delta: A \to A \otimes A$, a counit $\varepsilon: A \to \mathbb{C}$ and an antipode $S: A \to A$. They are given by the formulas

$$\Delta \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} x \otimes 1 & u \otimes 1 \\ v \otimes 1 & y \otimes 1 \end{pmatrix} \begin{pmatrix} 1 \otimes x & 1 \otimes u \\ 1 \otimes v & 1 \otimes y \end{pmatrix}, \tag{7}$$

$$\varepsilon \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} y & -qu \\ -q^{-1}v & x \end{pmatrix}. \tag{8}$$

The right hand side of (7) contains the product of matrices, that is

$$\Delta(x) = x \otimes x + u \otimes v, \quad \Delta(u) = x \otimes u + u \otimes y, \tag{9}$$

$$\Delta(v) = v \otimes x + y \otimes v, \quad \Delta(y) = v \otimes u + y \otimes y. \tag{9'}$$

Since Δ and ε are homomorphisms and S is an anti-homomorphism (see Section 14.2.3), then formulas (7) and (8) define their action on all elements of the algebra A. For example, since $\Delta(x) = x \otimes x + u \otimes v$, then due to formula (4′) of Section 13.2.2 we have

$$\Delta(x^{2\ell}) = (\Delta x)^{2\ell} = (x \otimes x + u \otimes v)^{2\ell} = \sum_{i=-\ell}^{\ell} \begin{bmatrix} 2\ell \\ \ell + i \end{bmatrix}_{q^2} x^{\ell-i} u^{\ell+i} \otimes x^{\ell-i} v^{\ell+i}, \tag{10}$$

where $\ell \in \frac{1}{2}\mathbb{Z}_+$ and the summation is over integral (half-integral) values of i if ℓ is an integer (half-integer). In the same way we find that

$$\Delta(v^{2\ell}) = \sum_{i=-\ell}^{\ell} \begin{bmatrix} 2\ell \\ \ell + i \end{bmatrix}_{q^2} v^{\ell-i} y^{\ell+i} \otimes x^{\ell-i} v^{\ell+i}, \tag{11}$$

$$\Delta(x^{\ell-i} v^{\ell+i}) = \sum_{j=-\ell}^{\ell} \sum_{\mu \geq 0} \begin{bmatrix} \ell - i \\ \mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell + j \\ \ell + j - \mu \end{bmatrix}_{q^2} \times q^{\mu(\mu+i-j)} x^{\ell-i-\mu} u^{\mu} v^{i-j+\mu} y^{\ell+j-\mu} \otimes x^{\ell-j} v^{\ell+j}. \tag{12}$$

It is easy to verify that the operators Δ, ε and S satisfy all conditions of the definition of a Hopf algebra. The Hopf algebra, constructed above, is called the *algebra of functions on the quantum group* $SL_q(2, \mathbb{C})$ and is denoted by $A \equiv A(SL_q(2, \mathbb{C}))$.

In what follows we assume that q does not coincide with a root from the unit, that is, $q \neq \exp(2\pi i m/n)$, where $m, n \in \mathbb{Z}_+$. If q is real, then $A \equiv A(SL_q(2, \mathbb{C}))$ can ba equipped with a $*$-structure by the formula

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix}^* \equiv \begin{pmatrix} x^* & v^* \\ u^* & y^* \end{pmatrix} = \begin{pmatrix} y & -qu \\ -q^{-1}v & x \end{pmatrix}. \tag{13}$$

Since the operation $*$ is an anti-linear anti-automorphism, then it can be extended on all elements of the algebra A. The $*$-Hopf algebra, obtained, is denoted by A^*. It is called the *algebra of functions on the quantum group* $SU_q(2)$: $A^* \equiv A(SU_q(2))$, and $SU_q(2)$ is said to be a *real form of the complex quantum group* $SL_q(2,\mathbb{C})$.

The comultiplication $\Delta: A \to A \otimes A$ defines both right and left corepresentations of the Hopf algebra A in A. They are called, respectively, the *right and the left regular representations of* $SL_q(2,\mathbb{C})$ or the *right and the left regular corepresentations of the Hopf algebra* $A \equiv A(SL_q(2,\mathbb{C}))$.

Along with corepresentations of the Hopf algebra A one can consider its representations in Hilbert spaces by bounded operators. In order to give such a representation it is sufficient to give the operators $\pi(x)$, $\pi(u)$, $\pi(v)$, $\pi(y)$ satisfying relations (1) and (2). For $*$-Hopf algebras we shall consider only $*$-representations, that is, representations π, for which $\pi(a^*) = \pi(a)^*$ for all elements a of a Hopf algebra. To give $*$-representations of $A(SU_q(2))$ it is sufficient to define the operators $\pi(x)$ and $\pi(v)$. It is clear that the *correspondence*

$$x \to e^{i\psi}, \quad v \to 0, \quad u \to 0, \quad y \to e^{-i\psi} \tag{14}$$

gives an one-dimensional representation of $A(SL_q(2,\mathbb{C}))$ *for every fixed* ψ, $0 \le \psi < 2\pi$.

In order to give infinite dimensional $*$-representations of $A(SL_q(2,\mathbb{C}))$ we fix the Hilbert space $\mathfrak{H}$ with an orthonormal basis $\mathbf{e}_k$, $k = 0, 1, 2, \ldots$, and assume that $-1 < q < 1$, $q \neq 0$. A direct verification shows that *for a fixed* φ, $0 \le \varphi < 2\pi$, *the formulas*

$$\pi_\varphi(x)\mathbf{e}_0 = 0, \quad \pi_\varphi(x)\mathbf{e}_k = (1-q^{2k})^{1/2}\mathbf{e}_{k-1}, \quad k > 0, \tag{15}$$

$$\pi_\varphi(u)\mathbf{e}_k = e^{-i\varphi}q^k\mathbf{e}_k, \tag{16}$$

$$\pi_\varphi(v)\mathbf{e}_k = e^{i\varphi}q^k\mathbf{e}_k, \tag{17}$$

$$\pi_\varphi(y)\mathbf{e}_k = (1-q^{2k+2})^{1/2}\mathbf{e}_{k+1} \tag{18}$$

define a $*$-representation of the Hopf algebra $A(SL_q(2,\mathbb{C}))$. We suggest to the reader to prove that these representations are irreducible. One can show (see, for example, [300]) that up to equivalence the representations, given above, exhaust all irreducible $*$-representations of A.

14.8.2. Decomposition of the Hopf algebra A. A *quantum subgroup* H of the quantum group $SL_q(2,\mathbb{C})$ is defined by the Hopf algebra $A(H)$ (that is, by the algebra of functions on H) and by a homomorphism φ_H from $A(SL_q(2,\mathbb{C}))$ onto $A(H)$. Let us introduce the quantum subgroup K of the quantum group $SL_q(2,\mathbb{C})$ corresponding to the subgroup of diagonal matrices of the classical group $SL(2,\mathbb{C})$. By $A(K)$ we mean the ring $\mathbb{C}[t,t^{-1}]$ of polynomials of t and t^{-1}. The operations Δ_K, ε_K and S_K are introduced in $A(K)$ by the formulas

$$\Delta_K(t^{\pm 1}) = t^{\pm 1} \otimes t^{\pm 1}, \quad \varepsilon_K(t^{\pm 1}) = 1, \quad S_K(t^{\pm 1}) = t^{\mp 1}.$$

The homomorphism $\varphi_K: A(SL_q(2,\mathbb{C})) \to A(K)$ is given as

$$\varphi_K \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}. \tag{1}$$

If q is real, then the $*$-operation is transferred from $A(SU_q(2))$ into $A(K)$ and K *can be considered as a quantum subgroup of* $SU_q(2)$.

The formulas

$$L_K = (\varphi_K \otimes \mathrm{id}_A) \circ \Delta, \quad R_K = (\mathrm{id}_A \otimes \varphi_K) \circ \Delta \tag{2}$$

define the homomorphisms

$$L_K: A \to A(K) \otimes A, \quad R_K: A \to A \otimes A(K)$$

which are the left and the right corepresentations of the Hopf algebra $A(K)$ in A.

By means of the mappings L_K and R_K one introduces the subspaces $A[m,n]$, $m,n \in \mathbb{Z}$, of A:

$$A[m,n] = \{a \in A \mid L_K(a) = t^m \otimes a,\ R_K(a) = a \otimes t^n\}. \tag{3}$$

We have

$$L_K(x) = (\varphi_K \otimes \mathrm{id}_A)\Delta(x) = (\varphi_K \otimes \mathrm{id}_A)(x \otimes x + u \otimes v) = t \otimes x,$$
$$R_K(x) = (\mathrm{id}_A \otimes \varphi_K)\Delta(x) = x \otimes t,$$

that is, $x \in A[1,1]$. Similarly,

$$u \in A[1,-1], \quad v \in A[-1,1], \quad y \in A[-1,-1].$$

Since L_K and R_K are homomorphisms, then

$$A[m,n] \cdot A[p,r] \subset A[m+p, n+r]. \tag{4}$$

Consequently, for basis elements (6) of Section 14.8.1 of the algebra A we have

$$u^p v^\ell y^r \in A[p-\ell-r, \ell-p-r], \tag{5}$$

$$x^r u^p v^\ell \in A[r+p-\ell, r+\ell-p]. \tag{6}$$

Hence,

$$A \equiv A(SL_q(2,\mathbb{C})) = \sum_{m,n\in\mathbb{Z}} \oplus A[m,n]. \tag{7}$$

It follows from formulas (5)–(7) that the subspace $A[0,0]$ consists of linear combinations of the elements $u^m v^m$, $m = 0, 1, 2, \ldots$. We can write down this statement as

$$A[0,0] = \mathbb{C}[\zeta] \quad \text{where} \quad \zeta = -q^{-1}uv. \tag{8}$$

Here $\mathbb{C}[\zeta]$ denotes the space of polynomials of ζ and the factor $-q^{-1}$ is taken for convenience (see below).

Let us introduce elements $e_{mn} \in A[m,n]$ by the formulas

$$e_{mn} = x^{(n+m)/2} v^{(n-m)/2} \quad \text{if} \quad m+n \geq 0, \quad m \leq n, \tag{9}$$

$$e_{mn} = x^{(m+n)/2} u^{(m-n)/2} \quad \text{if} \quad m+n \geq 0, \quad m \geq n, \tag{10}$$

$$e_{mn} = u^{(m-n)/2} y^{-(m+n)/2} \quad \text{if} \quad m+n \leq 0, \quad m \geq n, \tag{11}$$

$$e_{mn} = v^{(n-m)/2} y^{-(m+n)/2} \quad \text{if} \quad m+n \leq 0, \quad m \leq n. \tag{12}$$

Since elements (6) of Section 14.8.1 form a basis of the Hopf algebra A we obtain the following statement. If $m \equiv n (\mathrm{mod}\ 2)$, then

$$A[m,n] = \mathbb{C}[\zeta] e_{mn} = e_{mn} \mathbb{C}[\zeta], \tag{13}$$

where $\mathbb{C}[\zeta]$ is given by formula (8). If $m \not\equiv n (\mathrm{mod}\ 2)$, then $A[m,n] = \{0\}$.

14.8.3. Finite dimensional corepresentations of A. For every half-integral or integral non-negative ℓ we introduce in $A \equiv A(SL_q(2,\mathbb{C}))$ the subspaces

$$V_\ell^L = \sum_{i=-\ell}^{\ell} \oplus \mathbb{C} e_i^{(\ell)}, \quad V_\ell^R = \sum_{i=-\ell}^{\ell} \oplus \mathbb{C} f_i^{(\ell)}, \tag{1}$$

where

$$e_i^{(\ell)} = \begin{bmatrix} 2\ell \\ \ell+i \end{bmatrix}_{q^2}^{1/2} e_{-2i,2\ell} = \begin{bmatrix} 2\ell \\ \ell+i \end{bmatrix}_{q^2}^{1/2} x^{\ell-i} v^{\ell+i}, \tag{2}$$

$$f_i^{(\ell)} = \begin{bmatrix} 2\ell \\ \ell+i \end{bmatrix}_{q^2}^{1/2} e_{2\ell,-2i} = \begin{bmatrix} 2\ell \\ \ell+i \end{bmatrix}_{q^2}^{1/2} x^{\ell-i} u^{\ell+i} \tag{3}$$

(see formulas (9) and (10) of Section 14.8.2). It follows from formula (12) of Section 14.8.1 that the comultiplication Δ maps V_ℓ^L into $A \otimes V_\ell^L$:

$$\Delta : V_\ell^L \to A \otimes V_\ell^L; \tag{4}$$

one shows similarly that

$$\Delta : V_\ell^R \to V_\ell^R \otimes A. \tag{5}$$

Since Δ is a homomorphism, then formulas (4) and (5) define finite dimensional left and right corepresentations T_ℓ^L and T_ℓ^R of A. They are called *representations of the quantum group* $SL_q(2,\mathbb{C})$.

We can write down the action of Δ upon the basis element $e_i^{(\ell)}$ of the space V_ℓ^L in the form

$$\Delta e_i^{(\ell)} = \sum_{j=-\ell}^{\ell} t_{ij}^{(\ell)} \otimes e_j^{(\ell)}. \tag{6}$$

The elements $t_{ij}^{(\ell)} \in A$ are said to be matrix elements of the representation T_ℓ^L of the quantum group $SL_q(2,\mathbb{C})$. Applying both sides of the equality $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ (see Section 14.2.3) to $e_i^{(\ell)}$ we obtain the relation

$$\Delta(t_{ij}^{(\ell)}) = \sum_{k=-\ell}^{\ell} t_{ik}^{(\ell)} \otimes t_{kj}^{(\ell)}. \tag{7}$$

By means of the equality $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$ (see Section 14.2.3) we derive that

$$\varepsilon(t_{ij}^{(\ell)}) = \delta_{ij}. \tag{8}$$

Since $e_i^{(\ell)} \in A[-2i, 2\ell]$, then formulas (1) and (3) of Section 14.8.2 yield

$$R_K(e_i^{(\ell)}) = e_i^{(\ell)} \otimes t^{2\ell}, \quad \varphi_K(e_i^{(\ell)}) = \delta_{i,-2\ell} t^{2\ell}.$$

It follows from these relations, from the definition of R_K (see formula (2) of Section 14.8.2), and from (6) that

$$e_i^{(\ell)} \otimes t^{2\ell} = t_{i,-\ell}^{(\ell)} \otimes t^{2\ell}.$$

Hence,

$$e_i^{(\ell)} = t_{i,-\ell}^{(\ell)}. \tag{9}$$

Since $x^{2\ell} = e_{-\ell}^{(\ell)}$, then formula (10) of Section 14.8.1 implies

$$\Delta(e_{-\ell}^{(\ell)}) = \sum_{j=-\ell}^{\ell} \begin{bmatrix} 2\ell \\ \ell + j \end{bmatrix}_{q^2}^{1/2} x^{\ell-j} u^{\ell+j} \otimes e_j^{(\ell)},$$

that is,

$$f_j^{(\ell)} = t_{-\ell,j}^{(\ell)}. \tag{10}$$

It follows from (7) and (10) that

$$\Delta(f_j^{(\ell)}) = \sum_{i=-\ell}^{\ell} f_i^{(\ell)} \otimes t_{ij}^{(\ell)}, \quad j = -l, -l+1, \dots, l.$$

Thus, the representations T_ℓ^L and T_ℓ^R in bases (2) and (3) respectively are given by the same matrix.

Now we show that $t_{ij}^\ell \in A[-2i,-2j]$. For this we note that (as it follows from the definition (2) of Section 14.8.3 for the operator L_K) the equality

$$(L_K \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ L_K$$

is valid on every subspace $A[m,n]$. But then (because of relation (7) of Section 14.8.2) this equality holds on the whole algebra A. With the help of this equality we find that

$$(L_K \otimes \mathrm{id}) \circ \Delta(e_i^{(\ell)}) = t^{-2i} \otimes \Delta(e_i^{(\ell)}) = \sum_j t^{-2i} \otimes t_{ij}^{(\ell)} \otimes e_j^{(\ell)}.$$

On the other hand, making use of formula (9) we obtain

$$(L_K \otimes \mathrm{id}) \circ \Delta(e_i^{(\ell)}) = \sum_j L_K(t_{ij}^{(\ell)}) \otimes e_j^{(\ell)}.$$

Hence, $L_K(t_{ij}^{(\ell)}) = t^{-2i} \otimes t_{ij}^{(\ell)}$. In the same way one shows that $R_K(t_{ij}^{(\ell)}) = (t_{ij}^{(\ell)}) \otimes t^{-2j}$. Thus,

$$t_{ij}^{(\ell)} \in A[-2i,-2j].$$

14.8.4. Calculation of matrix elements. The basis elements $e_i^{(\ell)}$ of the carrier space V_ℓ^L of the representation T_ℓ^L are multiples of $x^{\ell-i}v^{\ell+i}$. Therefore, the matrix elements $t_{ij}^{(\ell)}$ can be calculated with the help of formula (12) of Section 14.8.1. Assume that $i+j \le 0$ and $i \ge j$ in this formula. Applying relation (4) of Section 14.8.1 to $x^{\ell+j-\mu}v^{\ell+j-\mu}$ we conclude that the factor at $x^{\ell-j}v^{\ell+j}$ in formula (12) of Section 14.8.1 has the form

$$\tilde{t}_{ij}^{(\ell)} = \sum_{\mu \ge 0}\sum_{k \ge 0} x^{-i-j}v^{i-j} \begin{bmatrix} \ell-i \\ \mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell+i \\ \ell+j-\mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell+j-\mu \\ k \end{bmatrix}_{q^2} \times q^{\mu(\mu+i-j)-(\ell+j-\mu)(2\mu+i-j)+k^2-2k(\ell+j-\mu)}(uv)^{\mu+k}$$

(up to a number factor, $\tilde{t}_{ij}^{(\ell)}$ coincides with $t_{ij}^{(\ell)}$). Going over from the summation over μ and k to the summation over μ and $r = \mu + k$ we reduce this expression to

$$\tilde{t}_{ij}^{(\ell)} = x^{-i-j}v^{i-j}q^{(\ell+j)(j-i)} \sum_{\mu,r} \begin{bmatrix} \ell-i \\ \mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell+i \\ \ell+j-\mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell+j-\mu \\ r-\mu \end{bmatrix}_{q^2} \times q^{r^2-2r(\ell+j)+2\mu(\mu+i-j)}(uv)^r.$$

Regrouping terms in q-binomial coefficients, we obtain for the sum I_r over μ the expression

$$I_r = \sum_\mu \begin{bmatrix} \ell - i \\ \mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell + i \\ \ell + j - \mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell + j - \mu \\ r - \mu \end{bmatrix}_{q^2} q^{2\mu(\mu+i-j)}$$
$$= \sum_\mu \begin{bmatrix} \ell - i \\ \mu \end{bmatrix}_{q^2} \begin{bmatrix} i - j + r \\ r - \mu \end{bmatrix}_{q^2} \begin{bmatrix} \ell + i \\ \ell + j - r \end{bmatrix}_{q^2} q^{2\mu(\mu+i-j)}.$$

This sum is reduced to a basic hypergeometric function of the type (10) of Section 14.1.2 and hence, we have

$$I_r = \begin{bmatrix} \ell + i \\ \ell - j \end{bmatrix}_{q^2} \frac{(q^{-2(\ell+j)};q^2)_r (q^{2(\ell-j+1)};q^2)_r}{(q^2;q^2)_r (q^{2(i-j+1)};q^2)_r} (-q)^r q^{2(\ell+j)r-r^2}.$$

Therefore

$$\tilde{t}_{ij}^{(\ell)} = x^{-i-j} v^{i-j} q^{(\ell+j)(j-i)} \begin{bmatrix} \ell + i \\ i - j \end{bmatrix}_{q^2}$$
$$\times\ {}_2\varphi_1(q^{-2(\ell+j)}, q^{2(\ell-j+1)}; q^{2(i-j+1)}; q^2, -q^2 uv).$$

Going over to $t_{ij}^{(\ell)}$ we obtain

$$t_{ij}^{(\ell)} = q^{(\ell+j)(j-i)} \begin{bmatrix} \ell + i \\ i - j \end{bmatrix}_{q^2}^{1/2} \begin{bmatrix} \ell - j \\ i - j \end{bmatrix}_{q^2}^{1/2} x^{-i-j} v^{i-j}$$
$$\times\ {}_2\Phi_1(-\ell - j, \ell - j + 1, i - j + 1; q^2, q\zeta), \tag{1}$$

where $\zeta = -quv$, $i + j \le 0$, $i \ge j$ and the notation (11) of Section 14.1.8 is used for the basic hypergeometric function ${}_2\varphi_1$. Utilizing formula (1) of Section 14.4.1 for little q-Jacobi polynomials, we obtain

$$t_{ij}^{(\ell)} = a_{ij}^\ell x^{-i-j} v^{i-j} p_{\ell+j}(\zeta; q^{i-j}, q^{-i-j} \mid q^2) \tag{2}$$

if $i + j \le 0$, $i \ge j$, where

$$a_{ij}^\ell = q^{(\ell+j)(j-i)} \begin{bmatrix} \ell + i \\ i - j \end{bmatrix}_{q^2}^{1/2} \begin{bmatrix} \ell - j \\ i - j \end{bmatrix}_{q^2}^{1/2}.$$

In just the same way we derive that

$$t_{ij}^{(\ell)} = a_{ji}^\ell x^{-i-j} u^{j-i} p_{\ell+i}(\zeta; q^{j-i}, q^{-i-j} \mid q^2) \tag{3}$$

if $i+j \le 0$, $j \ge i$,

$$t_{ij}^{(\ell)} = a_{-i,-j}^{\ell} p_{\ell-j}(\zeta; q^{j-i}, q^{i+j} \,|\, q^2) u^{j-i} y^{i+j} \tag{4}$$

if $i+j \ge 0$, $j \ge i$,

$$t_{ij}^{(\ell)} = a_{-j,-i}^{\ell} p_{\ell-i}(\zeta; q^{i-j}, q^{i+j} \,|\, q^2) v^{i-j} y^{i+j} \tag{5}$$

if $i+j \ge 0$, $i \ge j$.

It follows from expressions for the matrix elements $t_{ij}^{(\ell)}$ that

$$t_{ij}^{(\ell)} = e_{-2i,-2j} F_{ij}^{(\ell)}(\zeta) = \tilde{F}_{ij}^{(\ell)}(\zeta) e_{-2i,-2j}, \tag{6}$$

where $F_{ij}^{(\ell)}(\zeta)$ and $\tilde{F}_{ij}^{(\ell)}(\zeta)$ belong to $A[0,0] = \mathbb{C}[\zeta]$. We derive from (6) and from decomposition (7) of Section 14.8.2 that for a fixed ℓ the matrix elements $t_{ij}^{(\ell)}$, $-\ell \le i,j \le \ell$, are linearly independent.

It is obvious from formulas (2)–(5) that the function $F_{ij}^{(\ell)}(\zeta)$ is a polynomial of degree $\ell - \max(|i|,|j|)$. Therefore, the functions $F_{ij}^{(\ell)}$, $\ell = \max(|i|,|j|) + n$, $n = 0,1,2\dots$, form a basis of the space $A[0,0] = \mathbb{C}[\zeta]$. It follows from formulas (13) of Section 14.2.8 and (6) that

$$A[-2i,-2j] = \sum_{\ell=\max(|i|,|j|)}^{\infty} \oplus\, \mathbb{C} t_{ij}^{(\ell)}. \tag{7}$$

Consequently, by virtue of formula (7) of Section 14.8.2 we have

$$A \equiv A(SL_q(2,\mathbb{C})) = \sum_{\ell,i,j} \oplus\, \mathbb{C} t_{ij}^{(\ell)} = \sum_{l} \oplus\, W_\ell, \tag{8}$$

where $W_\ell = \sum_{i,j} \oplus\, \mathbb{C} t_{ij}^{(\ell)}$ is the space in which both the left and the right corepresentations T_ℓ^L and T_ℓ^R of the algebra A are realized.

14.8.5. Irreducibility of the representations T_ℓ. Matrix corepresentations $T \equiv (t_{mn})$ and $R \equiv (r_{mn})$ of a Hopf algebra are said to be *equivalent* if their dimensions coincide and if there is an invertible matrix B such that $B(t_{mn}) = (r_{mn})B$. A matrix corepresentation $T \equiv (t_{mn})$ of a Hopf algebra is said to be *irreducible* if there is no matrix corepresentation R of this Hopf algebra with the matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ (where A, B, C, 0 are matrix blocks with $A \ne 0$, $C \ne 0$), which is equivalent to T.

Let $T_\ell \equiv (t_{ij}^{(\ell)})$ be the matrix corepresentation of the algebra $A \equiv A(SL_q(2,$ $\mathbb{C}))$, considered in Section 14.8.3. Existence of a matrix corepresentation $R =$

$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ of A, equivalent to T_ℓ, that is, such that $R = B^{-1}T_\ell B$ implies linear dependence of the matrix elements $t_{ij}^{(\ell)}$. Since these matrix elements are linearly independent (see the previous section), then the corepresentation $T_\ell \equiv (t_{mn}^{(\ell)})$ is irreducible. Thus, *the representations* T_ℓ^L *and* T_ℓ^R *of the quantum group* $SL_q(2, \mathbb{C})$ *are irreducible.*

The corepresentations T_ℓ^L and T_ℓ^R have the same dimension $2\ell+1$, since bases of their carrier spaces contain $2\ell + 1$ elements. Representations of different dimensions are nonequivalent. Therefore, *the corepresentations* T_ℓ^L, $\ell = 0, 1/2, 1, 3/2, \ldots$ *(respectively,* T_ℓ^R, $\ell = 0, 1/2, 1, 3/2, \ldots$*) are pairwise nonequivalent.* One can also show (see, for example, [207]) that *every finite dimensional irreducible left (right) corepresentation of* $A = A(SL_q(2, \mathbb{C}))$ *is equivalent to one of the corepresentations* T_ℓ^L, $\ell = 0, 1/2, 1, 3/2, \ldots$ *(respectively,* $T_\ell^R, \ell = 0, 1/2, 1, 3/2, \ldots$*).*

14.8.6. Invariant integral on A. Let $\mathcal{A}$ be the Hopf algebra introduced in Section 14.2.2. It is algebra of functions on a group G. A left-invariant integral on G is a linear functional φ on $\mathcal{A}$ such that

$$\varphi(f) = \int_G f(g)dg = \int_G f(g_0 g)dg, \quad f \in \mathcal{A}.$$

This equality can be rewritten by means of the comultiplication:

$$(\mathrm{id} \otimes \varphi) \circ \Delta(f) = \varphi(f) \cdot I, \tag{1}$$

where I is the identity function from $\mathcal{A}$.

In the same way we can introduce invariant integrals on the Hopf algebra $A \equiv A(SL_q(2, \mathbb{C}))$. A *left-invariant (right-invariant)* integral on A is a linear functional φ satisfying the condition

$$(\mathrm{id} \otimes \varphi) \circ \Delta(a) = I \cdot \varphi(a), \quad a \in A \tag{2}$$

(respectively, the condition

$$(\varphi \otimes \mathrm{id}) \circ \Delta(a) = \varphi(a) \cdot I, \quad a \in A). \tag{3}$$

Here I denotes the identity element of A. A linear functional which is simultaneously left- and right-invariant is called an *invariant integral.*

Let φ be the linear functional on A such that

$$\varphi(t_{00}^{(0)}) \equiv \varphi(I) = 1, \quad \varphi(t_{ij}^{(\ell)}) = 0, \quad \ell > 0. \tag{4}$$

With the help of formulas (2) and (3) one easily verifies that φ is an invariant integral on A.

Let now ψ be an invariant integral on A such that $\psi(I)=1$. Then, by means of the operations K_L and R_K (see Section 14.8.2), we obtain from the equalities

$$(\mathrm{id}\otimes\psi)\circ\Delta(a)=I\cdot\psi(a),\quad(\psi\otimes\mathrm{id})\circ\Delta(a)=\psi(a)\cdot I \tag{5}$$

that

$$I\cdot\psi(a)=t^m\cdot\psi(a),\quad\psi(a)\cdot I=\psi(a)\cdot t^n$$

for $a\in A[m,n]$. Thus, $\psi(a)=0$ for all $a\in A[m,n]$ such that $(m,n)\neq(0,0)$. Now we show that

$$\psi(\zeta^n)=\frac{1-q^2}{1-q^{2(n+1)}},\quad n=0,1,2,\dots\ . \tag{6}$$

Let $P\colon A\to A[0,0]$ be the projection operator with respect to decomposition (7) of Section 14.8.2. Due to formulas (4), (5), (9) and (9$'$) of Section 14.8.1 we find that

$$(\mathrm{id}\otimes P)\circ\Delta(\zeta^n)=\sum_{i+j=n}\begin{bmatrix}n\\ i\end{bmatrix}_{q^2}^2 q^{-2ij}\zeta^j(\zeta;q^{-2})_i\otimes\zeta^i(q^2\zeta;q^2)_j.$$

Therefore, the first formula in (5) assumes that

$$I\cdot\psi(\zeta^n)=\sum_{i+j=n}\begin{bmatrix}n\\ i\end{bmatrix}_{q^2}^2 q^{-2ij}\zeta^j(\zeta;q^{-2})_i\psi(\zeta^i(q^2\zeta;q^2)_j).$$

This equality can be understood as an equality of polynomials in ζ. Equating coefficients at ζ^1 we obtain the recurrence relation

$$\psi(\zeta^n)=\frac{1-q^{2n}}{1-q^{2(n+1)}}\psi(\zeta^{n-1}),\quad n\geq 1.$$

Since $\psi(I)=1$, then we derive formula (6) from this relation.

Due to formula (6) and to the fact that $\psi(a)=0$ for $a\in A[m,n]$ with $m\neq 0$ or $n\neq 0$ we conclude that the invariant integral ψ is uniquely defined. Thus, we have proved the following theorem.

Theorem. *There is a unique invariant integral φ on A such that $\varphi(I)=1$. This integral is given by formula* (4). *Its values on the polynomials ζ^n, $n=0,1,2,\dots$, are given by formula* (6).

It follows from formula (8) of Section 14.8.1 that

$$S\colon A[m,n]\to A[-n,-m],\quad S(\zeta)=\zeta.$$

Hence, for all $a\in A$ we have

$$\varphi(S(a))=\varphi(a). \tag{7}$$

If $q \in \mathbf{R}$, then the equalities

$$\zeta^* = \zeta, \quad A[m,n]^* \subset A[-m,-n]$$

analogously imply that

$$\varphi(a^*) = \varphi(a). \tag{8}$$

Relation (6) allows us to represent the invariant integral as a q-integral (see formula (11) of Section 14.1.5). Namely, if $f(\zeta) \in \mathbb{C}[\zeta]$, then

$$\varphi(f) = \int_0^1 f(\zeta) d_{q^2}\zeta = (1-q^2)\sum_{j=0}^{\infty} q^{2j} f(q^{2j}) \tag{9}$$

for $0 < |q| < 1$ and

$$\varphi(f) = \int_0^1 f(q^{-2}\zeta) d_{q^{-2}}\zeta = (1-q^{-2})\sum_{j=0}^{\infty} q^{-2j} f(q^{-2j-2}) \tag{10}$$

for $|q| > 1$.

It is proved in the theory of Hopf algebras that if there exists a left-invariant (right-invariant) integral on a Hopf algebra, then every its left (right) finite dimensional corepresentation is completely reducible. Thus, *every finite dimensional left (right) corepresentation of the Hopf algebra $A(SL_q(2,\mathbb{C}))$ is completely reducible.*

14.8.7. Scalar products on $A(SU_q(2))$. The formula

$$\tau\begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} q^{-2}x & u \\ v & q^2 y \end{pmatrix} \tag{1}$$

defines an automorphism of the Hopf algebra $A = A(SL_q(2,\mathbb{C}))$. In particular,

$$\tau(a) = q^{-(m+n)}a \quad \text{if} \quad a \in A[m,n]. \tag{1'}$$

Let φ be the invariant integral on A such that $\varphi(I) = 1$. We shall prove that

$$\varphi(ab) = \varphi(\tau(b)a) \tag{2}$$

for all $a, b \in A$. For this we define a bilinear mapping $F: A \otimes A \to \mathbb{C}$ by the formula

$$F(a,b) = \varphi(ab) - \varphi(\tau(b)a). \tag{3}$$

A direct verification shows that F satisfies the equality

$$F(a,bc) = F(ab,c) + F(\tau(c)a,b). \tag{4}$$

It follows from relations (2)–(4) that if formula (2) is valid for $b = x, u, v$ and y, then it is valid for all $b \in A$. Let us prove it for $b = x$. Since $x \in A[1,1]$ and $\varphi(a) = 0$ for $a \in A[m,n]$, $(m,n) \neq (0,0)$, then it is sufficient to consider the case $a \in A[-1,-1]$. Let $a = \zeta^n y$, $n \in \mathbb{Z}_+ \cup \{0\}$. Then

$$ax = \zeta^n(1 - q^2\zeta), \quad xa = q^{-2n-2}\zeta^n(1-\zeta)$$

and equality (2) for $\varphi(ax)$ is equivalent to

$$\varphi(\zeta^n(1-q^2\zeta)) = q^{-2n-2}\varphi(\zeta^n(1-\zeta)).$$

Validity of this relation follows from formula (6) of Section 14.8.6. In the same way one proves (2) for $b = u, v, y$. Thus, we have proved equality (2).

Below we consider the Hopf algebra $A(SU_q(2))$. The invariant integral φ and the automorphism τ considered on $A(SU_q(2))$ are the invariant integral and the automorphism of $A(SU_q(2))$. Let us introduce on $A(SU_q(2))$ the Hermitian forms

$$\langle a, b\rangle_R = \varphi(ab^*), \quad \langle a, b\rangle_L = \varphi(a^*b), \quad a, b \in A(SU_q(2)). \tag{5}$$

It is obvious that

$$\langle ac, b\rangle_R = \langle a, bc^*\rangle_R, \quad \langle ca, b\rangle_L = \langle a, c^*b\rangle_L. \tag{6}$$

By virtue of formula (2) we have

$$\langle a, b\rangle_L = \langle \tau(b), a\rangle_R. \tag{7}$$

According to properties of the invariant integral φ we obtain

$$\langle a, b\rangle_R = \langle a, b\rangle_L = 0,$$

where $a \in A[m,n]$, $b \in A[r,s]$ and $(m,n) \neq (r,s)$. In other words, *decomposition* (5) *of Section* 14.8.2 *for the algebra* $A = A(SL_q(2,\mathbb{C}))$ (*and, consequently, for the algebra* $A(SU_q(2))$) *is orthogonal with respect to the Hermitian forms* (5). Let

$$a = f_1(\zeta)e_{mn}, \quad b = f_2(\zeta)e_{mn}$$

(see Section 14.8.2). Then

$$\langle a, b\rangle_R = \varphi(f_1(\zeta)\Phi_{mn}(\zeta)f_2(\zeta)^*), \quad \Phi_{mn}(\zeta) = e_{mn}e_{mn}^*. \tag{8}$$

With the help of formulas (4), (5) of Section 14.8.1 and (9)–(12) of Section 14.8.2 it is easy to find that the weight function Φ from (8) is given by the formulas

$$\Phi_{mn}(\zeta) = q^{(n-m)(2-m-n)/2}\zeta^{(n-m)/2}(\zeta; q^{-2})_{(m+n)/2} \tag{9}$$

if $m+n \geq 0$, $m \leq n$,

$$\Phi_{mn}(\zeta) = \zeta^{(m-n)/2}(q^2\zeta; q^2)_{(m+n)/2} \tag{10}$$

if $m+n \geq 0$, $m \geq n$,

$$\Phi_{mn}(\zeta) = \zeta^{(m-n)/2}(q^2\zeta; q^2)_{-(m+n)/2} \tag{11}$$

if $m+n \leq 0$, $m \geq n$,

$$\Phi_{mn}(\zeta) = q^{n-m}\zeta^{(n-m)/2}(q^2\zeta; q^2)_{-(m+n)/2} \tag{12}$$

if $m+n \leq 0$, $m \leq n$. These formulas, expressions (9) and (10) of Section 14.8.6 for $\varphi(f)$ and the fact, that there is no non-zero polynomials $f(\zeta) \in \mathbb{C}[\zeta]$ with $f(q^{2k}) = 0$ (respectively, with $f(q^{-2k-2}) = 0$) for all integral non-negative k, imply that $\langle a, a\rangle_R > 0$ for $a \neq 0$. In other words, the Hermitian forms $\langle\cdot,\cdot\rangle_R$ and $\langle\cdot,\cdot\rangle_L$ are strictly positive definite on $A(SU_q(2))$. We choose them as a scalar product on $A(SU_q(2))$.

By means of formula (6) of Section 14.8.6 it is easy to find that

$$\varphi(\zeta^r(\zeta; q^{-2})_s) = q^{2(r+1)}\frac{(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_1}{(q^2;q^2)_{r+s+1}},$$

$$\varphi(\zeta^r(q^2\zeta; q^2)_s) = \frac{(q^2;q^2)_r(q^2;q^2)_s(q^2;q^2)_1}{(q^2;q^2)_{r+s+1}}.$$

14.8.8. Unitary representations of the quantum group $SU_q(2)$. Let T_L be a left finite dimensional corepresentation of the Hopf algebra $A(SU_q(2))$ in a space V with a scalar product $(\cdot,\cdot)$, conjugate linear in the second argument. We have $T_L\colon V \to A(SU_q(2)) \otimes V$. Let us extend $(\cdot,\cdot)$ to a form $\{\cdot,\cdot\}_R$ which maps

$$(A(SU_q(2)) \otimes V) \times (A(SU_q(2)) \otimes V) \quad \text{into} \quad A(SU_q(2))$$

by setting

$$\{a\otimes\xi, b\otimes\eta\}_R = ab^*(\xi,\eta), \quad a, b \in A(SU_q(2)), \quad \xi,\eta \in V. \tag{1}$$

The corepresentation T_L of the Hopf algebra $A(SU_q(2))$ is said to be *unitary* if for all $\xi,\eta \in V$ we have

$$\{T_L(\xi), T_L(\eta)\}_R = I\cdot(\xi,\eta), \tag{2}$$

where I is the unit in $A(SU_q(2))$.

Let e_i, $i = 1, 2, \ldots, \ell$, be an orthonormal basis of V and let $T \equiv (t_{ij})_{i,j=1}^{\ell}$ be the matrix of the corepresentation T_L in this basis. Then

$$T_L(e_i) = \sum_{j=1}^{\ell} t_{ij} \otimes e_j.$$

Calculating $\{T_L(e_i), T_L(e_j)\}_R$ by means of formula (2), we conclude that

$$\sum_{k=1}^{\ell} t_{ik} t_{kj}^* = I, \quad I \in A(SU_q(2)). \tag{3}$$

Thus, the unitarity of the left finite dimensional corepresentation T_L of the Hopf algebra $A(SU_q(2))$ is equivalent to the condition $TT^* = 1$, where $T^* \equiv (t_{ij}^*)_{j,i=1}^{\ell}$. It is easy to show that this condition is equivalent to the equality $T^*T = 1$. It is also equivalent to the equality $S(T) = T^*$ where $S(T) = (S(t_{ij}))_{i,j=1}^{\ell}$.

The unitarity of a right finite dimensional corepresentation T_R of $A(SU_q(2))$ in a space V with a scalar product $(\cdot,\cdot)$, conjugate linear in the first argument, is defined in the same way. In this case instead of (1) we define the form $\{\cdot,\cdot\}_L$ by

$$\{\xi \otimes a, \eta \otimes b\}_L = (\xi, \eta) a^* b, \quad a, b \in A(SU_q(2)), \quad \xi, \eta \in V. \tag{4}$$

The unitarity condition (2) is replaced by

$$\{T_R(\xi), T_R(\eta)\}_L = (\xi, \eta) I. \tag{5}$$

Theorem. *Every finite dimensional subcorepresentation of the left (right) regular corepresentation of the Hopf algebra $A(SU_q(2))$ is unitary with respect to the Hermitian form $\{\cdot,\cdot\}_R$ (respectively, $\{\cdot,\cdot\}_L$).*

Proof. Let T_L be a left corepresentation of the algebra $A(SU_q(2))$ in a subspace $V \subset A(SU_q(2))$ and let e_i, $i = 1, 2, \ldots, \ell$, be a basis in V. Let

$$T_L(e_i) = \Delta(e_i) = \sum_{j=1}^{\ell} t_{ij} \otimes e_j.$$

Then according to formula (1) we have

$$\begin{aligned} \{T_L(e_i), T_L(e_j)\}_R &= \sum_{r,s=1}^{\ell} t_{ir} t_{js}^* \langle e_r, e_s \rangle_R \\ &= \sum_{r,s=1}^{\ell} t_{ir} t_{js}^* \varphi(e_r e_s^*). \end{aligned}$$

The last expression coincides with $(\mathrm{id} \otimes \varphi) \circ \Delta(e_i e_j^*)$. But according to the definition of the invariant integral φ, this expression is equal to $I \cdot \langle e_i, e_j \rangle_R$, that is,

$$\{T_L(e_i), T_L(e_j)\}_R = I \cdot \langle e_i, e_j \rangle_R.$$

Theorem is proved.

Since every irreducible right (left) corepresentation of $A(SU_q(2))$ is equivalent to one of the corepresentations in the spaces $V_\ell^R \subset A(SU_q(2))$ (respectively, in the spaces $V_\ell^L \subset A(SU_q(2))$), then *every irreducible right (left) corepresentation of $A(SU_q(2))$ is equivalent to a unitary right (left) corepresentation.*

14.8.9. An analog of the Peter-Weyl theorem. The following lemma is a q-analog of the well-known classical lemma of averaging.

Lemma. *Let $T^{(\ell)} \equiv (t_{mn}^{(\ell)})$ be the matrix of the finite dimensional irreducible corepresentation from Section* 14.8.3 *and let M be a constant complex $(2\ell+1)\times(2k+1)$ matrix. Let*

$$\tilde{M} = \varphi(T^{(\ell)}MT^{(k)*}), \quad \tilde{M}' = \varphi(T^{(\ell)*}MT^{(k)}),$$

where φ is the invariant integral on $A(SU_q(2))$. Then $\tilde{M} = 0$, $\tilde{M}' = 0$ if $\ell \neq k$, and $\tilde{M} = c1$, $\tilde{M}' = c1$ (1 is the diagonal matrix with units from $A(SU_q(2))$ on the main diagonal) if $\ell = k$ where $c \in \mathbb{C}$.

Proof. We consider the case of the matrix $\tilde{M}$. For $\tilde{M}'$ the proof is analogous. Let us show that $\tilde{M}$ intertwines the representations $T^{(\ell)}$ and $T^{(k)}$, i.e., that $T^{(\ell)}\tilde{M} = \tilde{M}T^{(k)}$. For this we denote the matrix $(t_{ij}^{(\ell)} \otimes I)$ by $T_1^{(\ell)}$ and the matrix $(I \otimes t_{ij}^{(\ell)})$ by $T_2^{(\ell)}$. The definition of invariant integral φ implies that

$$\begin{aligned} T(\ell)\tilde{M}T^{(k)*} &= (\mathrm{id}\otimes\varphi)(T_1^{(\ell)}T_2^{(\ell)}MT_2^{(k)*}T_1^{(k)*}) \\ &= (\mathrm{id}\otimes\varphi)(T_1^{(\ell)}T_2^{(\ell)}M(T_1^{(k)}T_2^{(k)})^*) \\ &= (\mathrm{id}\otimes\varphi)\circ\Delta(T^{(\ell)}MT^{(k)*}) \\ &= \varphi(T^{(\ell)}MT^{(k)*}) = \tilde{M}, \end{aligned}$$

that is, $T^{(\ell)}\tilde{M} = \tilde{M}T^{(k)}$ since $T^{(k)}$ is an unitary corepresentation. Now it is sufficient to apply Schur's lemma which remains valid for finite dimensional corepresentations of Hopf algebras. Lemma is proved.

Theorem. *The matrix elements $t_{mn}^{(\ell)}$ of irreducible corepresentations $T^{(\ell)}$ of the Hopf algebra $A(SU_q(2))$ satisfy the orthogonality conditions*

$$\langle t_{mn}^{(\ell)}, t_{m'n'}^{(\ell')}\rangle_R = \frac{(1-q^2)q^{2(\ell-j)}}{1-q^{2(2\ell+1)}}\delta_{\ell\ell'}\delta_{mm'}\delta_{nn'}, \tag{1}$$

$$\langle t_{mn}^{(\ell)}, t_{m'n'}^{(\ell')}\rangle_L = \frac{(1-q^2)q^{2(\ell+i)}}{1-q^{2(2\ell+1)}}\delta_{\ell\ell'}\delta_{mm'}\delta_{nn'}. \tag{2}$$

Proof. Let E_{ij} be the $(2\ell+1)\times(2m+1)$-matrix with 1 on the intersection of ith row and jth column and with zeroes on other positions. We construct the matrix $\tilde{E}_{ij}$ as it is indicated in Lemma. Then

$$(\tilde{E}_{ij})_{rs} = \varphi(t_{ri}^{(\ell)}t_{sj}^{(m)*}) = \langle t_{ri}^{(\ell)}, t_{sj}^{(\ell)}\rangle_R.$$

By Lemma we have $\langle t_{ri}^{(\ell)}, t_{sj}^{(m)} \rangle_R = 0$ if $\ell \neq m$. Let $\ell = m$. Since $t_{ri}^{(\ell)} t_{sj}^{(\ell)*} \in A[2s-2r, 2j-2i]$ and $\varphi(a) = 0$ for $a \in A[m,n]$, $(m,n) \neq (0,0)$, then $\langle t_{ri}^{(\ell)}, t_{sj}^{(\ell)} \rangle_R = 0$ for $(r,i) \neq (s,j)$. The orthogonality with respect to $\langle \cdot, \cdot \rangle_L$ follows from the equalities

$$\langle t_{ij}^{(\ell)}, t_{rs}^{(m)} \rangle_L = \langle \tau(t_{rs}^{(m)}), t_{ij}^{(\ell)} \rangle_R = q^{2(r+s)} \langle t_{rs}^{(m)}, t_{ij}^{(\ell)} \rangle_R. \tag{3}$$

Let now $\tilde{E}_{jj} = \varphi(T^{(\ell)} E_{jj} T^{(\ell)*})$. Then according to Lemma there exists a constant $c_j \in \mathbb{C}$ such that

$$(\tilde{E}_{jj})_{ii} = \varphi(t_{ij}^{(\ell)} t_{ij}^{(\ell)*}) = \langle t_{ij}^{(\ell)}, t_{ij}^{(\ell)} \rangle_R = c_j$$

for all i,j $(-\ell \le i,j \le \ell)$. In just the same way, considering $\tilde{E}'_{ij} = \varphi(T^{(\ell)*} E_{ij} T^{(\ell)})$ we find that there exists a constant $c'_i \in \mathbb{C}$ such that

$$\langle t_{ij}^{(\ell)}, t_{ij}^{(\ell)} \rangle_L = c'_i.$$

for all i,j $(-\ell \le i,j \le \ell)$. But according to (3) we have $c'_i = q^{2(i+j)} c_j$. Consequently, there exists a constant $c \in \mathbb{C}$ such that

$$c = q^{-2i} c'_i = q^{2j} c_j$$

for all i and j. Since $t_{i,-\ell}^{(\ell)} = e_i^{(\ell)}$, then by virtue of formula (13) of Section 14.8.7 we have

$$c_{-\ell} = \frac{1-q^2}{1-q^{2(2\ell+1)}} q^{4\ell}.$$

Therefore, $c = q^{2\ell}(1-q^2)/(1-q^{2(2\ell+1)})$. Thus

$$c_j = \frac{(1-q^2) q^{2(\ell-j)}}{1-q^{2(2\ell+1)}}, \quad c'_i = \frac{(1-q^2) q^{2(\ell+i)}}{1-q^{2(2\ell+1)}}.$$

Theorem is proved.

14.8.10. The Fourier transform on the quantum group $SU_q(2)$. By virtue of formula (8) of Section 14.8.4 and of Theorem from the previous section the matrix elements $t_{ij}^{(\ell)}$ form a complete orthogonal system in $A(SU_q(2))$. Let $F \in A(SU_q(2))$. Define the numbers

$$\varphi(F t_{nm}^{(\ell)*}) \equiv \hat{F}_{mn}^{(\ell)}, \quad \ell = 0, 1/2, 1, 3/2, \dots, \quad -\ell \le m,n \le \ell. \tag{1}$$

The mapping $F \to \{\hat{F}_{mn}^{(\ell)}\}$ is called the *Fourier transform on the Hopf algebra* $A(SU_q(2))$ (or the *Fourier transform on the quantum group* $SU_q(2)$). By means of formula (1) of Section 14.8.9 it is easy to show that the inverse transform is of the form

$$F = \sum_{\ell} \frac{(1-q^{2(2\ell+1)}) q^{-2\ell}}{1-q^2} \sum_{i,j=-\ell}^{\ell} q^{2i} \hat{F}_{ij}^{(\ell)} t_{ji}^{(\ell)}, \tag{2}$$

where the first summation is over the values $0, 1/2, 1, 3/2, \ldots$ of the index ℓ. In addition, the Plancherel formula

$$\langle F, F' \rangle_R = \sum_\ell \frac{(1-q^{2(2\ell+1)})q^{-2\ell}}{1-q^2} \sum_{i,j=-\ell}^{\ell} q^{2i} \hat{F}_{ij}^{(\ell)} \overline{\hat{F}'^{(\ell)}_{ij}} \tag{3}$$

holds.

Formulas (1)–(3) can be rewritten in the matrix form. For this we introduce the matrices $T^{(\ell)} \equiv (t_{ij}^{(\ell)})$, $\hat{F} \equiv (\hat{F}_{ij}^{(\ell)})$, the q-trace Tr_q for an matrix $D \equiv (d_{ij})_{i,j=-\ell}^{\ell}$ with elements from $A(SU_q(2))$:

$$\mathrm{Tr}_q D = \sum_{i=-\ell}^{\ell} q^{2i} d_{ii} \in A(SU_q(2))$$

and the form

$$\langle D, D' \rangle_{R,q} = \mathrm{Tr}_q DD'^*.$$

Then formulas (1)–(3) are represented as

$$\varphi(FT^{(\ell)*}) = \hat{F}, \tag{4}$$

$$F = \sum_\ell \left[\ell + \frac{1}{2}\right]_{q^2} \mathrm{Tr}_q(\hat{F}T^{(\ell)}), \tag{5}$$

$$\langle F, F' \rangle_R = \sum_\ell \left[\ell + \frac{1}{2}\right]_{q^2} \langle \hat{F}, \hat{F}' \rangle_{R,q}, \tag{6}$$

where, according to formula (5) of Section 14.1.1,

$$[m]_{q^2} = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

14.8.11. Orthogonality of little q-Jacobi polynomials. q-Legendre and Wall polynomials. According to formulas (2)–(5) of Section 14.8.4 the matrix elements $t_{mn}^{(\ell)}$ are expressed in terms of the little q-Jacobi polynomials $p_k(z; a, b \,|\, q)$. These matrix elements satisfy the orthogonality relation (1) of Section 14.8.9. Taking into account expression (8) of Section 14.8.7 for the scalar product and expression (9) of Section 14.8.6 for the invariant integral, from the orthogonality relation for $t_{mn}^{(\ell)}$ we derive the orthogonality relation for little q-Jacobi polynomials. For $|q| < 1$ it is of the form

$$\int_0^1 p_m(z; q^\alpha, q^\beta \,|\, q) p_n(z; q^\alpha, q^\beta \,|\, q) z^\alpha (qz; q)_\beta d_q z$$
$$= \delta_{mn} q^{(\alpha+1)n} \frac{(1-q)(q;q)_\alpha^2 (q,q)_{\beta+n} (q;q)_n}{(1-q^{\alpha+\beta+2n+1})(q;q)_{\alpha+n}(q;q)_{\alpha+\beta+n}}, \tag{1}$$

where α, β are non-negative integers.

It follows from the results of Section 14.8.10 that the *polynomials*

$$p_n(z;q^\alpha,q^\beta\,|\,q), \quad n=0,1,2,\dots,$$

form a complete orthonormal system in the Hilbert space of functions $F(z)$, $z = 1, q, q^2, \dots$, *with the scalar product*

$$(F_1,F_2) = \int_0^1 F_1(z)\overline{F_2(z)}z^\alpha(qz;q)_\beta d_qz. \tag{2}$$

The q-polynomials $p_n(z;q^\alpha,q^\beta\,|\,q)$ with $\alpha=\beta=0$, that is, the polynomials $p_n(z;1,1\,|\,q)$ are called *little q-Legendre polynomials* (we shall call them *q-Legendre polynomials*), and the polynomials $p_n(z;q^\alpha,q^\infty\,|\,q)$, that is, $p_n(z;q^\alpha,0\,|\,q)$ (we assume that $0<q<1$), are called *Wall polynomials.* We have

$$\lim_{q\to 1^-} p_n((1-q)z;q^\alpha,0\,|\,q) = \frac{L_n^\alpha(z)}{L_n^\alpha(0)},$$

where $L_n^\alpha(z)$ are Laguerre polynomials. The orthogonality relation for Wall polynomials can be obtained from (1). We have

$$\begin{aligned}\int_0^1 p_n(z;q^\alpha,0\,|\,q)p_m(z;q^\alpha,0\,|\,q)z^\alpha(qz;q)_\infty d_qz \\ = \delta_{mn}\frac{(1-q)(q;q)_\infty q^{n(\alpha+1)}(q;q)_n}{(q^{\alpha+1};q)_\infty(q^{\alpha+1};q)_n}.\end{aligned} \tag{3}$$

The recurrence relation for Wall polynomials

$$\begin{aligned}q^n(1-aq^{n+1})p_{n+1}(z;a,0\,|\,q)+\{z-q^n(1+a-aq^n-aq^{n+1})\}p_n(z;a,0\,|\,q) \\ +\, q^n(a-aq^n)p_{n-1}(z;a,0\,|\,q)=0\end{aligned} \tag{4}$$

is valid.

With the help of the notations

$$\begin{aligned}P_n(q^k;q^\alpha\,|\,q) = \left(\frac{(q^{\alpha+1};q)_\infty(q^{k+1};q)_\infty(q^{\alpha+1};q)_n}{(q;q)_\infty(q;q)_n}\right)^{1/2} \\ \times(-1)^n q^{(k-n)(\alpha+1)/2}p_n(q^k;q^\alpha,0\,|\,q)\end{aligned} \tag{5}$$

relation (3) is written in the form

$$\sum_{k=0}^\infty P_n(q^k;q^\alpha\,|\,q)P_m(q^k;q^\alpha\,|\,q)=\delta_{mn}. \tag{6}$$

The polynomials $P_n(q^k; q^\alpha \,|\, q)$, $n = 0, 1, 2, \ldots$, form a complete orthonormal system in the Hilbert space l^2 of sequences $\mathbf{a} = (a_1, a_2, \ldots)$ with the scalar product $(\mathbf{a}, \mathbf{b}) = \sum_i a_i b_i$. Therefore, one has the dual orthogonality relation

$$\sum_{n=0}^{\infty} P_n(q^k; q^\alpha \,|\, q) P_n(q^r; q^\alpha \,|\, q) = \delta_{kr}. \tag{7}$$

14.8.12. Addition and product formulas for q-Legendre polynomials. In order to prove the addition formula for the q-Legendre polynomials $p_n(z; 1, 1 \,|\, q)$ we make use of the equality

$$\Delta(t_{00}^{(\ell)}) = \sum_{k=-\ell}^{\ell} t_{0k}^{(\ell)} \otimes t_{k0}^{(\ell)}. \tag{1}$$

By means of the notations

$$p_{\ell,k}(z) = \begin{bmatrix} \ell \\ k \end{bmatrix}_{q^2}^{1/2} \begin{bmatrix} \ell + k \\ k \end{bmatrix}_{q^2}^{1/2} q^{-k(\ell-k)} p_{\ell-k}(z; q^{2k}, q^{2k} \,|\, q^2)$$

we can rewrite matrix elements from (1) in the form

$$\begin{aligned} t_{k0}^{(\ell)} &= (x^*)^k p_{\ell,k}(vv^*) v^k, \\ t_{0k}^{(\ell)} &= (x^*)^k p_{\ell,k}(vv^*)(-qv^*)^k, \\ t_{-k,0}^{(\ell)} &= (-qv^*)^k p_{\ell,k}(vv^*) x^k, \\ t_{0,-k}^{(\ell)} &= v^k p_{\ell,k}(vv^*) x^k, \end{aligned}$$

where $k \geq 0$ and $x^* = y$, $v^* = -q^{-1}u$ (see formula (13) of Section 14.8.1). For $t_{0,0}^{(\ell)}$ we have

$$t_{0,0}^{(\ell)} = p_{\ell,0}(vv^*) = p_\ell(vv^*; 1, 1 \,|\, q^2). \tag{2}$$

Substituting this expression for $t_{0,0}^{(\ell)}$ into (1) and taking into account that Δ is a homomorphism, we have

$$\begin{aligned} p_{\ell,0}(\Delta(vv^*)) &= p_{\ell,0}(vv^*) \otimes p_{\ell,0}(vv^*) \\ &\quad + \sum_{k=1}^{\ell} (x^*)^k p_{\ell,k}(vv^*)(-qv^*)^k \otimes (x^*)^k p_{\ell,k}(vv^*) v^k \\ &\quad + \sum_{k=1}^{\ell} v^k p_{\ell,k}(vv^*) x^k \otimes (-qv^*)^k p_{\ell,k}(vv^*) x^k. \end{aligned} \tag{3}$$

Now let us consider the representation π_0 of the Hopf algebra $A(SU_q(2))$ (see formulas (15)–(18) of Section 14.8.1) and assume that x, x^*, v and v^* in (3) are operators of this representation. Apply both sides of relation (3) to the vectors $\mathbf{e}_{s+t} \otimes \mathbf{e}_t$ and consider the scalar products of each side with the vector[5]

$$\mathbf{h}_z^s = \sum_{t=0}^{\infty} P_t(q^{2z}; q^{2s} \,|\, q^2) \mathbf{e}_{s+t} \otimes \mathbf{e}_t,$$

where $P_t(q^k; q^\alpha \,|\, q)$ is given by formula (5) of Section 14.8.11. We obtain

$$\begin{aligned}
&\langle p_{\ell,0}(\Delta(vv^*))\mathbf{e}_{s+t} \otimes \mathbf{e}_t, \mathbf{h}_z^s \rangle \\
&\quad = p_{\ell,0}(q^{2s+2t}) p_{\ell,0}(q^{2t}) \langle \mathbf{e}_{s+t} \otimes \mathbf{e}_t, \mathbf{h}_z^s \rangle \\
&\qquad + \sum_{k=1}^{\ell} (-1)^k q^{k(s+2t+1)} (q^{2(s+t+1)}; q^2)_k^{1/2} (q^{2(t+1)}; q^2)_k^{1/2} \\
&\qquad\qquad \times p_{\ell,k}(q^{2s+2t}) p_{\ell,k}(q^{2t}) \langle \mathbf{e}_{s+t+k} \otimes \mathbf{e}_{t+k}, \mathbf{h}_z^s \rangle \\
&\qquad + \sum_{k=1}^{\ell} (-1)^k q^{k(s+2t-2k+1)} (q^{2(s+t)}; q^{-2})_k^{1/2} (q^{2t}; q^{-2})_k^{1/2} \\
&\qquad\qquad \times p_{\ell,k}(q^{2s+2t-2k}) p_{\ell,k}(q^{2t-2k}) \langle \mathbf{e}_{s+t-k} \otimes \mathbf{e}_{t-k}, \mathbf{h}_z^s \rangle. \qquad (4)
\end{aligned}$$

Let us find the left hand side of this equality. Since

$$\Delta(vv^*) = (v \otimes x + x^* \otimes v)(v^* \otimes x^* + x \otimes v^*),$$

then

$$\begin{aligned}
&\Delta(vv^*)\mathbf{e}_{s+t} \otimes \mathbf{e}_t \\
&\quad = q^{s+2t+1}(1 - q^{2s+2t+2})^{1/2}(1 - q^{2t+2})^{1/2} \mathbf{e}_{s+t+1} \otimes \mathbf{e}_{t+1} \\
&\qquad + (q^{2s+2t} + q^{2t} - q^{2s+4t} - q^{2s+4t+2}) \mathbf{e}_{s+t} \otimes \mathbf{e}_t \\
&\qquad + q^{s+2t-1}(1 - q^{2s+2t})^{1/2}(1 - q^{2t})^{1/2} \mathbf{e}_{s+t-1} \otimes \mathbf{e}_{t-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta(vv^*)\mathbf{h}_z^s = \sum_{t=0}^{\infty} \{ & P_{t-1}(q^{2z}; q^{2s} \,|\, q^2) q^{s+2t-1} (1 - q^{2s+2t})^{1/2} (1 - q^{2t})^{1/2} \\
& + P_t(q^{2z}; q^{2s} \,|\, q^2)(q^{2s+2t} + q^{2t} - q^{2s+4t} - q^{2s+4t+2}) \\
& + P_{t+1}(q^{2z}; q^{2s} \,|\, q^2) q^{s+2t+1} (1 - q^{2s+2t+2})^{1/2} (1 - q^{2t+2})^{1/2} \} \mathbf{e}_{s+t} \otimes \mathbf{e}_t.
\end{aligned}$$

[5] Because of the orthogonality relation (6) of Section 14.8.11 the vectors $\mathbf{h}_z^s$, $z = 0, 1, 2, \ldots$, form an orthonormal basis in the subspace $\sum_{t=0}^{\infty} \mathbb{C}\mathbf{e}_{s+t} \otimes \mathbf{e}_t$ of the space $\mathfrak{H} \otimes \mathfrak{H}$, where $\mathfrak{H}$ is the carrier space of the representation π_0.

Due to formulas (4) and (5) of Section 14.8.11 we conclude that the expression in braces is equal to $q^{2z}P_t(q^{2z};q^{2s}\,|\,q^2)$, that is,

$$\Delta(vv^*)\mathbf{h}_z^s = q^{2z}\mathbf{h}_z^s.$$

Taking into account this relation and the self-adjointness of the operator $\Delta(vv^*)$ in the carrier space $\mathfrak{H}$ of the representation (see Section 14.8.1), we conclude that the left hand side of (4) is equal to $p_{\ell,0}(q^{2z})P_t(q^{2z};q^{2s}\,|\,q^2)$. Hence, after replacement of q^2 by q we derive from relation (4) the following *addition formula* for q-Legendre polynomials (see [172])

$$\begin{aligned} p_\ell(q^z;1,1\,|\,q)p_t(q^z;q^s,0\,|\,q) &= p_\ell(q^{s+t};1,1\,|\,q)p_\ell(q^t;1,1\,|\,q)p_\ell(q^z;q^s,0\,|\,q) \\ &+\sum_{k=1}^{\ell}\frac{(q;q)_{s+t+k}(q;q)_{\ell+k}q^{k(t-\ell+k)}}{(q;q)_{s+t}(q;q)_{\ell-k}(q;q)_k^2} \\ &\times p_{\ell-k}(q^{s+t};q^k,q^k\,|\,q)p_{\ell-k}(q^t;q^k,q^k\,|\,q)p_{t+k}(q^z;q^s,0\,|\,q) \\ &+\sum_{k=1}^{\ell}\frac{(q;q)_t(q;q)_{\ell+k}q^{k(s+t-\ell+1)}}{(q;q)_{t-k}(q;q)_{\ell-k}(q;q)_k^2} \\ &\times p_{\ell-k}(q^{s+t-k};q^k,q^k\,|\,q)p_{\ell-k}(q^{t-k};q^k,q^k\,|\,q)p_{t-k}(q^z;q^s,0\,|\,q). \end{aligned} \tag{5}$$

When $q\to 1^-$, this formula turns into the addition formula for classical Legendre polynomials.

Using the orthogonality relation (3) of Section 14.8.11 for the polynomials $p_n(x;q^\alpha,0\,|\,q)$ we derive from (5) the *product formula* for q-Legendre polynomials:

$$\begin{aligned} &p_\ell(q^x;1,1\,|\,q)p_\ell(q^y;1,1\,|\,q) \\ &\quad = (1-q)\sum_{z=0}^{\infty} q^z p_\ell(q^z;1,1\,|\,q)K(q^x,q^y,q^z), \end{aligned} \tag{6}$$

where

$$\begin{aligned} K(q^x;q^y,q^z) = {} & \frac{(q^{x+1};q)_\infty(q^{y+1};q)_\infty(q^{z+1};q)_\infty}{(q;q)_\infty^2(1-q)}(q^{y-x+1};q)_x^2 \\ & \times q^{yz-xy-xz}q^{x(x-1)}\{p_x(q^z;q^{y-x},0\,|\,q)\}^2. \end{aligned} \tag{7}$$

The polynomials $p_n(x;q^\alpha,0\,|\,q)$ can be represented as (see [172])

$$p_n(x;q^\alpha,0\,|\,q) = \frac{(-1)^n q^{n(n+2\alpha+1)/2}x^n}{(q^{\alpha+1};q)_n}\,{}_3\varphi_2\left(\begin{matrix} q^{-n},\ q^{-n-\alpha},\ x^{-1} \\ 0,\ 0 \end{matrix}\,\middle|\, q,q\right).$$

Therefore, kernel (7) can be written in the symmetric form

$$\begin{aligned} K(q^x;q^y,q^z) = {} & \frac{(q^{x+1};q)_\infty(q^{y+1};q)_\infty(q^{z+1};q)_\infty q^{xy+xz+yz}}{(q;q)_\infty^2(1-q)} \\ & \times\{\,{}_3\varphi_2(q^{-x};q^{-y},q^{-z};0,0;q,q)\}^2. \end{aligned} \tag{8}$$

14.8.13. The differential form of the quantum group $SL_q(2, \mathbb{C})$. Starting from representations of the quantum algebra $U_q(\mathfrak{sl}_2)$ we have constructed the algebra A of functions on the quantum group $SL_q(2, \mathbb{C})$. Let us now show how one can go over from the Hopf algebra A to the quantum algebra $U_q(\mathfrak{sl}_2)$.

Let A' be the dual linear space to A. With the help of the formula

$$(\varphi \cdot \psi)(a) = (\varphi \otimes \psi) \circ \Delta(a), \quad \varphi, \psi \in A', \quad a \in A, \tag{1}$$

we equip A' with the multiplication, and A' turns into an algebra with a unit. The counit of A (see formula (8) of Section 14.8.1) is used as a unit in A'. By means of formula (1) one easily verifies that $(\varphi \cdot \psi) \cdot \eta = \varphi \cdot (\psi \cdot \eta)$, i.e., A' is an associative algebra.

With the help of formulas

$$k^{\pm} \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} q^{\pm 1/2} & 0 \\ 0 & q^{\mp 1/2} \end{pmatrix}, \tag{2}$$

$$E_+ \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- \begin{pmatrix} x & u \\ v & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3}$$

we introduce the elements k^+, k^-, E_+, E_- of A' assuming that k^+ and $k^-$are homomorphisms of A into $\mathbb{C}$, i.e., $k^{\pm}(ab) = k^{\pm}(a)k^{\pm}(b)$, and that the action of E_+ and E_- is extended onto the whole algebra A by the formula

$$E_{\pm}(ab) = E_{\pm}(a)k^+(b) + k^-(a)E_{\pm}(b), \quad E_{\pm}(1) = 0. \tag{4}$$

The elements k^+, k^-, E_+ and E_- satisfy the relations

$$k^+k^- = k^-k^+ = 1, \quad k^+E_+k^- = qE_-, \quad k^+E_-k^- = q^{-1}E_-, \tag{5}$$

$$[E_+, E_-] \equiv E_+E_- - E_-E_+ = \frac{k^{+2} - k^{-2}}{q - q^{-1}}. \tag{6}$$

We verify that relation (6) is valid. The relations (5) are proved in the same way. By means of equality (4) we check that both $\varphi = E_+E_- - E_-E_+$ and $\varphi = (k^{+2} - k^{-2})/(q - q^{-1})$ satisfy the relation

$$\varphi(ab) = \varphi(a)k^{+2}(b) + k^{-2}(a)\varphi(b), \quad a, b \in A.$$

Hence, it is sufficient to prove relation (6) for the elements x, u, v, y of A. This proof is carried out with the help of formulas (2) and (3).

The elements k^+, k^-, E_+, E_- generate a subalgebra with a unit which is the quantum algebra $U \equiv U_q(\mathfrak{sl}(2, \mathbb{C}))$. We write

$$U_q(\mathfrak{sl}(2, \mathbb{C})) = \mathbb{C}[k^+, k^-, E_+, E_-].$$

Comparing formulas (5) and (6) with formulas (4) and (4′) of Section 14.3.1, we conclude that, in fact, the algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ constructed here is isomorphic to the algebra $U_{q^2}(\mathfrak{sl}_2)$ from Section 14.3.1.

As in the case of $U_{q^2}(\mathfrak{sl}_2)$ the algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is equipped with the structure of a Hopf algebra. Namely, the comultiplication Δ_U is given by the formula

$$\Delta_U(k^{\pm}) = k^{\pm}\otimes k^{\pm}, \quad \Delta_U(E_{\pm}) = E_{\pm}\otimes k^{+} + k^{-}\otimes E_{\pm}, \tag{7}$$

and the antipode S_U and the counit ε_U are defined as

$$S_U(k^{\pm}) = k^{\mp}, \quad S_U(E_+) = -qE_+, \quad S_U(E_-) = -q^{-1}E_-, \tag{8}$$

$$\varepsilon_U(k^{\pm}) = 1, \quad \varepsilon_U(E_{\pm}) = 0. \tag{9}$$

In addition, for any ψ, φ from $U_q(\mathfrak{sl}(2,\mathbb{C}))$ and for any a, b from $A \equiv A(SL_q(2,\mathbb{C}))$ we have

$$(\varphi\cdot\psi)(a) = (\varphi\otimes\psi)(\Delta(a)), \quad 1_U(a) = \varepsilon(a),$$
$$\Delta_U(\varphi)(a\otimes b) = \varphi(ab), \quad \varepsilon_U(\varphi) = \varphi(I),$$
$$S_U(\varphi)(a) = \varphi(S(a)),$$

where Δ, S, ε, I are correspondingly the comultiplication, the antipode, the counit and the unit in A and 1_U is the unit in $U \equiv U_q(\mathfrak{sl}(2,\mathbb{C}))$.

14.8.14. The differential form of corepresentations. Let L be a left corepresentation of the Hopf algebra $A \equiv A(SL_q(2,\mathbb{C}))$ in a space V, that is, let a mapping $L\colon V \to A\otimes V$ be defined. Then to every $\varphi\in U$ there corresponds the operator $\mathcal{R}(\varphi)$ in V, given by the formula

$$\mathcal{R}(\varphi)(e) = (\varphi\otimes \mathrm{id}_V)\circ L(e), \quad e\in V, \tag{1}$$

where id_V is the identity operator in V. By means of this formula it is easy to verify that the mapping $\mathcal{R}\colon U \to \mathrm{Lin}(V)$ is linear and anti-homomorphic, that is, $\mathcal{R}(\varphi\cdot\psi) = \mathcal{R}(\psi)\mathcal{R}(\varphi)$. This means that $\mathcal{R}$ defines a right representation of $U_q(\mathfrak{sl}(2,\mathbb{C}))$ which is a representation such that the right action of operators upon vectors of a carrier space is defined.

Analogously, for a given right corepresentation $R\colon V\to V\otimes A$ the formula

$$\mathcal{L}(\varphi)(e) = (\mathrm{id}_V\otimes\varphi)\circ R(e), \quad e\in V, \tag{2}$$

defines a left representation of the algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$.

In particular, if $V = A$, and L and R are respectively the left and the right regular representations of the quantum group $SL_q(2,\mathbb{C})$, then for $\mathcal{R}$ and $\mathcal{L}$ we have

$$\mathcal{R}(\varphi)(a) = (\varphi\otimes\mathrm{id}_A)\circ\Delta(a), \quad a\in A,$$

$$\mathcal{L}(\varphi)(a) = (\mathrm{id}_A \otimes \varphi) \circ \Delta(a), \quad a \in A.$$

In addition, as it is easy to check,

$$\Delta \circ \mathcal{R}(\varphi) = (\mathcal{R}(\varphi) \otimes \mathrm{id}_A) \circ \Delta, \quad \Delta \circ \mathcal{L}(\varphi) = (\mathrm{id}_A \otimes \mathcal{L}(\varphi)) \circ \Delta \tag{3}$$

for any $\varphi \in U_q(\mathfrak{sl}(2,\mathbb{C}))$. These properties are called respectively the *right* and the *left invariances.*

Below we shall consider only the operators $\mathcal{R}(\varphi)$, $\varphi \in U_q(\mathfrak{sl}(2,\mathbb{C}))$ and for brevity denote them by $\hat{\varphi}$. It is clear that

$$\hat{E}_\pm(ab) = \hat{E}_\pm(a)\hat{k}^+(b) + \hat{k}^-(a)\hat{E}_\pm(b), \quad \hat{E}_\pm(I) = 0. \tag{4}$$

Let $V_\ell^L = \sum_{i=-\ell}^{\ell} \oplus \, \mathbb{C}e_i^{(\ell)}$ be the subspace in A introduced in Section 14.8.3. Then $\Delta(e_i^{(\ell)}) = \sum_j t_{ij}^{(\ell)} \otimes e_j^{(\ell)}$. The right action of elements $\varphi \in U_q(\mathfrak{sl}(2,\mathbb{C}))$ on V_ℓ^L is defined by

$$\hat{\varphi}(e_i^{(\ell)}) = \sum_j \varphi(t_{ij}^{(\ell)})e_j^{(\ell)} \tag{5}$$

and the matrix $\hat{\varphi}_\ell$ of the operator $\hat{\varphi}$ in the subspace V_ℓ^L has the form $\hat{\varphi}_\ell \equiv (\varphi(t_{ij}^{(\ell)}))_{i,j=-\ell}^{\ell}$. Taking the subspace $W_{\ell r}^L$, spanned by the basis elements $t_{ir}^{(\ell)}$, $i = -\ell, -\ell+1, \dots, \ell$, instead of V_ℓ^L, we obtain the formula

$$\hat{\varphi}(t_{ir}^{(\ell)}) = \sum_{j=-\ell}^{\ell} \varphi(t_{ij}^{(\ell)})t_{jr}^{(\ell)}, \tag{6}$$

that is, in V_ℓ^L and in $W_{\ell r}^L$ the operators $\hat{\varphi}$ are given by the same matrices (it is a corollary of the invariance relations (3)).

Using expression (2) of Section 14.8.3 for $e_j^{(\ell)}$, formulas (2) and (3) of Section 14.8.13 and relation (4), we directly find the action formulas for the operators $\hat{k}^+$, $\hat{k}^- = (\hat{k}^+)^{-1}$, $\hat{E}_+$ and $\hat{E}_-$:

$$\hat{k}^+ e_i^{(\ell)} = q^{-i} e_i^{(\ell)}, \tag{7}$$

$$\begin{aligned}\hat{E}_+ e_i^{(\ell)} &= ([\ell+i+1]_{q^2}[\ell-i]_{q^2})^{1/2} e_{i+1}^{(\ell)} \\ &= q^{-\ell+1/2}(1-q^{2(\ell+i+1)})^{1/2}(1-q^{2(\ell-i)})^{1/2}(1-q^2)^{-1} e_{i+1}^{(\ell)},\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{E}_- e_i^{(\ell)} &= ([\ell-i+1]_{q^2}[\ell+i]_{q^2})^{1/2} e_{i-1}^{(\ell)} \\ &= q^{-\ell+1/2}(1-q^{2(\ell-i+1)})^{1/2}(1-q^{2(\ell-i)})^{1/2}(1-q^2)^{-1} e_{i-1}^{(\ell)}.\end{aligned} \tag{9}$$

(Remind that $[m]_q = (q^{m/2} - q^{-m/2})/(q^{1/2} - q^{-1/2})$). Comparing (7)–(9) with formulas (3) of Section 14.3.2 we see that representation (5) is equivalent to the representation T_ℓ, constructed in Section 14.3.2.

14.8.15. The difference equation for little q-Jacobi polynomials. The element

$$C = \frac{(q^{1/2}k^+ - q^{-1/2}k^-)^2}{(q-q^{-1})^2} + E_-E_+$$

belongs to the center of the Hopf algebra $U \equiv U_q(\mathfrak{sl}(2,\mathbb{C}))$ (see Section 14.3.1). Applying the operator $\hat{C}$ to the vector $e_\ell^{(\ell)}$ we find that it is a multiple of the identity operator on V_ℓ^L and

$$\hat{C} = \frac{(q^{\ell+1/2} - q^{-\ell-1/2})^2}{(q-q^{-1})^2} \mathrm{id}_{V_\ell^L} \equiv \left[\ell + \frac{1}{2}\right]_{q^2}^2 \mathrm{id}_{V_\ell^L}. \tag{1}$$

Thus,

$$\hat{C}t_{ij}^{(\ell)} = \left[\ell + \frac{1}{2}\right]_{q^2}^2 t_{ij}^{(\ell)}. \tag{2}$$

In order to derive from (2) the second order difference equation for little q-Jacobi polynomials we evaluate how the operators E_+ and E_- act upon elements from $A[m,n]$. Introduce the operators

$$T_qF(z) = F(qz), \quad D_qF(z) = \frac{F(z) - F(qz)}{z - qz}. \tag{3}$$

By means of formula (4) of Section 14.8.14 we obtain the action of E_+ and E_- upon $x^\beta v^\alpha F(\zeta)$, $\zeta = -quv$, $\alpha = (n-m)/2$, $\beta = (m+n)/2$, $m+n \geq 0$, $n \geq 0$ (see Section 14.8.2). We have

$$\hat{E}_+(x^\beta v^\alpha F(\zeta)) = x^{\beta-1}v^{\alpha+1}q^{-(\alpha+\beta-1)/2}\left\{(1-\zeta)D_{q^2}T_{q^2}^{-1} + \frac{1-q^{2\beta}}{1-q^2}\right\}F(\zeta) \tag{4}$$

if $\alpha \geq 0$, $\beta > 0$, and

$$\hat{E}_-(x^\beta v^\alpha F(\zeta)) = x^{\beta+1}v^{\alpha+1}q^{-(\alpha+\beta-1)/2}\left\{q^{2\alpha}\zeta D_{q^2} + \frac{1-q^{2\alpha}}{1-q^2}\right\}F(\zeta) \tag{5}$$

if $\alpha > 0$, $\beta \geq 0$. For other conditions for m and n action formulas are of the similar form. With the help of these formulas we find that

$$\hat{C}(e_{mn}F(\zeta)) = e_{mn}\mathcal{D}_{\alpha\beta}F(\zeta) \qquad \text{if} \quad m+n \geq 0, \tag{6}$$

$$\hat{C}(F(\zeta)e_{mn}) = \{\mathcal{D}_{\alpha\beta}F(\zeta)\}e_{mn} \quad \text{if} \quad m+n \leq 0, \tag{7}$$

where $\alpha = |(m-n)/2|$, $\beta = |(m+n)/2|$ and

$$\mathcal{D}_{\alpha\beta} = \frac{-q^2}{(1-q^2)^2\zeta}\{(q^{\alpha-\beta-1} - q^{\alpha+\beta+1}\zeta)T_{q^2} + 2\zeta - q^{\alpha-\beta-1} \\ -q^{-\alpha-\beta-1} + q^{-\alpha-\beta-1}(1-\zeta)T_{q^2}^{-1}\}.$$

Substituting expressions (2)–(5) of Section 14.8.4 for $t_{ij}^{(\ell)}$ into relations (6) and (7) and taking into account formula (2), we obtain the *second order difference equation* for little q-Jacobi polynomials

$$\{q^{\alpha+\beta}(q^{-2\beta} - q^2\zeta)T_{q^2} + q(q^{2n+\alpha+\beta+1} + q^{-2n-\alpha-\beta-1})\zeta \\ - q^{-\alpha-\beta}(1+q^{2\alpha}) + q^{-\alpha-\beta}(1-\zeta)T_{q^2}^{-1}\}p_n(\zeta; q^\alpha, q^\beta \,|\, q^2) = 0. \tag{8}$$

It is equivalent to equation (19) of Section 14.1.8.

14.8.16. The Rodrigues formula for little q-Jacobi polynomials. The operator $\hat{E}_-$ acts upon the element $F(\zeta)v^\alpha y^\beta \in A[m,n]$, $F(\zeta) \in \mathbb{C}[\zeta]$, by the formula

$$\hat{E}_-(F(\zeta)v^\alpha y^\beta) = \tilde{F}(\zeta)v^{\alpha-1}y^{\beta-1}, \tag{1}$$

where

$$\tilde{F}(\zeta) = q^{(\alpha-\beta-1)/2}\zeta^{1-\alpha}(q^2\zeta;q)_{\beta-1}^{-1}T_{q^2}^{-1}D_{q^2}\{\zeta^\alpha(q^2\zeta;q^2)_\beta F(\zeta)\}. \tag{2}$$

Indeed, it follows from (1) and from formulas of Section 14.8.1 that

$$\hat{E}_-(F(\zeta)v^\alpha y^\beta)x^\beta u^\alpha = xu(-1)^{\alpha-1}T_{q^2}\{\tilde{F}(\zeta)\zeta^{\alpha-1}(q^2\zeta;q^2)_{\beta-1}\}. \tag{3}$$

On the other hand, $\hat{E}_-(x^\beta u^\alpha) = 0$. Therefore,

$$\begin{aligned}\hat{E}_-\{F(\zeta)v^\alpha y^\beta\}x^\beta u^\alpha &= q^{-(\alpha+\beta)/2}\hat{E}_-\{F(\zeta)v^\alpha y^\beta x^\beta u^\alpha\} \\ &= (-1)^\alpha q^{(\alpha-\beta)/2}\hat{E}_-\{F(\zeta)\zeta^\alpha(q^2\zeta;q^2)_\beta\} \\ &= (-1)^{\alpha+1}q^{(\alpha-\beta-1)/2}xuD_{q^2}\{F(\zeta)\zeta^\alpha(q^2\zeta;q^2)_\beta\}.\end{aligned}$$

Comparing this relation with (3) we obtain formulas (1) and (2).

According to formulas (6) and (9) of Section 14.8.14 the equality

$$\hat{E}_-t_{ij}^{(\ell)} = q^{-\ell+1/2}\frac{(1-q^{2(\ell-i+1)})^{1/2}(1-q^{2(\ell+i)})^{1/2}}{1-q^2}t_{i-1,j}^{(\ell)}$$

is valid. Replacing here $t_{ij}^{(\ell)}$ by its expression (5) of Section 14.8.4, we obtain

$$\hat{E}_-\{p_n(\zeta;q^\alpha,q^\beta \,|\, q^2)v^\alpha y^\beta\} = q^{(3-3\alpha-\beta)/2} \\ \times \frac{1-q^{2\alpha}}{1-q^2}p_{n+1}(\zeta;q^{\alpha-1},q^{\beta-1} \,|\, q^2)v^{\alpha-1}y^{\beta-1}.$$

Therefore, according to (1) and (2) we have

$$D_{q^2}T_{q^2}^{-1}\{\zeta^\alpha(q^2\zeta;q^2)_\beta p_n(\zeta;q^\alpha,q^\beta\,|\,q^2)$$
$$=q^{-2\alpha}\frac{1-q^{2\alpha}}{1-q^2}\zeta^{\alpha-1}(q^2\zeta;q^2)_{\beta-1}p_{n+1}(\zeta;q^{\alpha-1},q^{\beta-1}\,|\,q^2).$$

This equality leads to the *Rodrigues formula*

$$p_n(\zeta;q^\alpha,q^\beta\,|\,q^2)=q^{2\alpha n+n(n+1)}\frac{(1-q^2)^n}{(q^{2(\alpha+1)};q^2)_n\zeta^\alpha(q^2\zeta;q^2)_\beta} \times(D_{q^2}T_{q^2}^{-1})^n\{\zeta^{\alpha+n}(q^2\zeta;q^2)_{\beta+n}\}, \tag{4}$$

where α and β are non-negative integers.

14.9. Representations of the Quantum Group $SU_q(2)$ on Quantum Spheres and q-Orthogonal Polynomials

14.9.1. The algebra of functions on a quantum 2-sphere. Let c and d be fixed real numbers. We associate with them the so-called *quantum* 2-*sphere* $S_q^2\equiv S_q^2(c,d)$ which is defined, as the quantum group $SL_q(2,\mathbb{C})$, by the algebra $A(S_q^2)$ of functions on it. The algebra $A(S_q^2)$ is generated by three elements α, z and β satisfying the relations

$$z\alpha=q^2\alpha z,\quad \beta z=q^2z\beta, \tag{1}$$

$$q\alpha\beta=-(c-z)(d+z),\quad q\beta\alpha=-(c-q^2z)(d+q^2z). \tag{2}$$

Thus, $A(S_q^2)$ consists of finite linear combinations of products of α, z, β with complex coefficients.

The formulas

$$\alpha^*=-q^{-1}\beta,\quad \beta^*=-q\alpha,\quad z^*=z \tag{3}$$

define a $*$-structure on $A(S_q^2)$. The action of the operation $*$ is extended on elements $b\in A(S_q^2)$ by considering $*$ as a anti-linear anti-automorphism of the algebra $A(S_q^2)$.

Let us define a left corepresentation of the Hopf algebra $A(SU_q(2))$ (that is, a left representation of the quantum group $SU_q(2)$) on $A(S_q^2)$. For this we have to define a homomorphism

$$L\colon A(S_q^2)\to A(SU_q(2))\otimes A(S_q^2).$$

It is sufficient to give the action of L on the elements α, z, β. We define this action as

$$L\begin{pmatrix}e_{-1}\\ e_0\\ e_1\end{pmatrix}=T^{(1)}\otimes\begin{pmatrix}e_{-1}\\ e_0\\ e_1\end{pmatrix}, \tag{4}$$

where $T^{(1)}$ denotes the matrix of the corepresentation of the Hopf algebra $A(SU_q(2))$ from Section 14.8.3 corresponding to the number 1, that is

$$T^{(1)} = \begin{pmatrix} x^2 & (1+q^2)^{1/2}xu & u^2 \\ (1+q^2)^{1/2}xv & 1+(q+q^{-1})uv & (1+q^2)^{1/2}uy \\ v^2 & (1+q^2)^{1/2}vy & y^2 \end{pmatrix} \tag{5}$$

and

$$e_{-1} = \alpha, \quad e_0 = (1+q^2)^{-1/2}(c-d-(1+q^2)z), \quad e_1 = \beta.$$

Formula (4) is the matrix notation of the equalities

$$Le_{-1} = x^2 \otimes e_{-1} + (1+q^2)^{1/2}xu \otimes e_0 + u^2 \otimes e_1, \tag{6}$$

$$Le_0 = (1+q^2)^{1/2}xv \otimes e_{-1} + \{1+(q+q^{-1})uv\} \otimes e_0 + (1+q^2)^{1/2}uy \otimes e_1, \tag{7}$$

$$Le_1 = v^2 \otimes e_{-1} + (1+q^2)^{1/2}vy \otimes e_0 + y^2 \otimes e_1. \tag{8}$$

In the same way as in Section 14.8.1, one can define the action of the quantum algebra $U_q(\mathfrak{sl}(2,\mathbb{C})) = \mathbb{C}[k^+, k^-, E_+, E_-]$ on $A(S_q^2)$. Namely, with an element $a \in U_q(\mathfrak{sl}(2,\mathbb{C}))$ we associate the operator $\hat{a}\colon A(S_q^2) \to A(S_q^2)$, defined by the formula

$$\hat{a} = (a \otimes \mathrm{id}) \circ L. \tag{9}$$

It follows from equalities (6)–(8) that

$$\hat{E}_+\alpha = q^{-1/2}\{c-d-(1+q^2)z\}, \quad \hat{E}_+z = -q^{-1/2}\beta, \quad \hat{E}_+\beta = 0, \tag{10}$$

$$\hat{E}_-\alpha = 0, \quad \hat{E}_-z = -q^{-1/2}\alpha, \quad \hat{E}_-\beta = q^{-1/2}\{c-d-(1+q^2)z\}, \tag{11}$$

$$\hat{k}^+\alpha = q\alpha, \quad \hat{k}^+z = z, \quad \hat{k}^+\beta = q^{-1}\beta. \tag{12}$$

The operator $\hat{k}^+$ is an automorphism of $A(S_q^2)$. For $\hat{E}_+$ and $\hat{E}_-$ we have

$$\hat{E}_\pm(ab) = \hat{E}_\pm(a)\hat{k}^+(b) + \hat{k}^-(a)\hat{E}_\pm(b), \quad a, b \in A(S_q^2). \tag{13}$$

By means of formulas, given above, it is easy to show that if $(c,d) = (1,0)$, then the algebra $A(S_q^2)$ together with the structure of the corepresentation of $A(SU_q(2))$ is isomorphic to the subalgebra $\sum\limits_{m=-\infty}^{\infty} \oplus A[2m,0]$ of $A(SU_q(2))$ consisting of all functions, right invariant with respect to the quantum subgroup K. *In this sense the quantum 2-sphere $S_q^2(1,0)$ is identified with the quantum homogeneous space $SU_q(2)/K$.*

14.9.2. Decomposition of the algebra $A(S_q^2)$. In Section 14.8.2 we defined the homomorphism $\varphi_K\colon A(SL_q(2,\mathbb{C})) \to A(K)$. In the same way as in formula (2) of those section we define with the help of φ_K the homomorphism $L_K\colon A(S_q^2) \to A(K) \otimes A(S_q^2)$ which is given as

$$L_K = (\varphi_K \otimes \mathrm{id}) \circ L, \tag{1}$$

where L is determined by formula (4) of Section 14.9.1. Now for every $n \in 2\mathbb{Z}$ we introduce the subspace

$$A(S_q^2)[n] = \{a \in A(S_q^2) \mid L_K a = t^n \otimes a\}. \tag{2}$$

Repeating arguments of Section 14.8.2, we obtain

$$A(S_q^2)[n] \cdot A(S_q^2)[m] \subset A(S_q^2)[m+n], \tag{3}$$

$$A(S_q^2) = \sum_{m=-\infty}^{\infty} \oplus\, A(S_q^2)[2m]. \tag{4}$$

In particular,

$$z \in A(S_q^2)[0], \quad \alpha \in A(S_q^2)[2], \quad \beta \in A(S_q^2)[-2].$$

The subspace $A(S_q^2)[0]$ coincides with the ring $\mathbb{C}[z]$ of polynomials in one variable z. For $A(S_q^2)[2n]$, $n \in \mathbb{Z}$, we have

$$A(S_q^2)[2n] = \begin{cases} \alpha^n \mathbb{C}[z] & \text{if } n \geq 0, \\ \mathbb{C}(z)\beta^{-n} & \text{if } n \leq 0. \end{cases} \tag{5}$$

Let W_r denotes the subspace of $A(S_q^2)$ coinciding with

$$W_r = \sum_{i+j+k\leq r} \mathbb{C}\alpha^i z^j \beta^k.$$

It follows from formulas (6)–(8) of Section 14.9.1 that W_r is invariant with respect to the corepresentation L of the algebra $A(SU_q(2))$, that is,

$$LW_r \subset A(SU_q(2)) \otimes W_r.$$

Since W_r is finite dimensional and since finite dimensional corepresentations of the Hopf algebra $A(SU_q(2))$ are completely reducible, then the corepresentation L_{W_r} of $A(SU_q(2))$ in W_r decomposes into the direct sum of irreducible corepresentations. In order to find these irreducible components we have to find all elements $a \in W_r$

for which $\hat{E}_+ a = 0$. It follows from formulas (10) and (13) of Section 14.9.1 that these elements coincide with $c\beta^\ell$ where $c \in \mathbb{C}$ and $\ell = 0, 1, 2, \ldots, r$. Hence,

$$W_r = \sum_{\ell=0}^{r} \oplus V_\ell,$$

where V_ℓ is the space of an irreducible corepresentation. Since r is arbitrary we have the following decomposition of $A(S_q^2)$ into the direct sum of irreducible subspaces:

$$A(S_q^2) = \sum_{\ell=0}^{\infty} \oplus V_\ell. \tag{6}$$

For V_0 we have $V_0 = \mathbb{C} \cdot I$, where I denotes the unit of $A(S_q^2)$.

14.9.3. An invariant integral on S_q^2. A linear functional $\psi: A(S_q^2) \to \mathbb{C}$ is said to be an *invariant integral on* S_q^2 (or an invariant integral on $A(S_q^2)$) if

$$(\mathrm{id} \otimes \psi) \circ L(b) = I \cdot \psi(b) \tag{1}$$

for all $b \in A(S_q^2)$. We obtain the invariant integral φ on $A(S_q^2)$ by setting

$$\varphi(I) = 1 \text{ and } \varphi(a) = 0 \text{ if } a \in V_\ell, \quad \ell > 0. \tag{2}$$

In the same way as in Section 14.8.6, one proves that any invariant integral ψ on $A(S_q^2)$ vanishes on the subspaces $A(S_q^2)[2m]$, $m \neq 0$. It also follows from (1) that $\psi(\hat{E}_+ b) = 0$ for every $b \in A(S_q^2)$. Set $b = z^n \alpha$. Then formulas (10) and (13) of Section 14.9.1 imply that

$$\begin{aligned} q^{1/2}(1-q^2)\hat{E}_+(z^n\alpha) = & -(1-q^{2(n+2)})z^{n+1} + (1-q^{2(n+1)})(c-d)z^n \\ & + (1-q^{2n})cdz^{n-1}. \end{aligned}$$

Applying ψ to both sides of this equality, we obtain the recurrence relation

$$(1-q^{2(n+2)})\psi(z^{n+1}) - (c-d)(1-q^{2(n+1)})\psi(z^n) - cd(1-q^{2n})\psi(z^{n-1}) = 0.$$

Setting $\psi(I) = 1$ we derive from here that

$$\psi(z^n) = \frac{c^{n+1} - (-d)^{n+1}}{c+d} \frac{1-q^2}{1-q^{2(n+1)}}, \quad n = 0, 1, 2, \ldots . \tag{3}$$

Consequently, we have proved the following theorem.

Theorem. *There is a unique invariant integral φ on S_q^2 such that $\varphi(I) = 1$. The integral φ vanishes on all subspaces $A(S_q^2)[2m]$, $m \neq 0$. The value of φ on z^n is given by formula* (3).

As in the case of invariant integral on the quantum group $SU_q(2)$ (see Section 14.8.6), the invariant integral φ from Theorem can be represented as a q-integral. Namely, if $F(z) \in \mathbb{C}[z]$, then

$$\varphi(F(z)) = \frac{1}{c+d}\int_{-d}^{c} F(z)d_{q^2}z, \tag{4}$$

where, remind,

$$\int_{-d}^{c} F(z)d_q z = \int_{0}^{c} F(z)d_q z - \int_{0}^{-d} F(z)d_q z.$$

By means of the invariant integral φ we introduce the Hermitian form $\langle\cdot,\cdot\rangle$ on $A(S_q^2)$. It is given as

$$\langle a,b\rangle = \varphi(ab^*), \quad a,b \in A(S_q^2).$$

Repeating arguments of Section 14.8.7 we obtain the following statements:

1) *The Hermitian form $\langle\cdot,\cdot\rangle$ is invariant with respect to the action of the quantum group $SU_q(2)$, that is*

$$\langle La, Lb\rangle = \langle a,b\rangle, \quad a,b \in A(S_q^2).$$

2) *The subspaces V_ℓ from formula* (6) *of Section* 14.9.2 *are mutually orthogonal with respect to* $\langle\cdot,\cdot\rangle$.

3) *If the numbers c and d are such that $cd \geq 0$, $(c,d) \neq (0,0)$, then the Hermitian form $\langle\cdot,\cdot\rangle$ is strictly positive definite, that is, it defines a scalar product on $A(S_q^2)$.*

To prove the last statement one has to use the equalities

$$\alpha^\ell(\alpha^\ell)^* = q^{-2\ell}(cd)^\ell(z/c;q^{-2})_\ell(-z/d;q^{-2})_\ell, \tag{5}$$

$$\beta^\ell(\beta^\ell)^* = (cd)^\ell(q^2z/c;q^2)_\ell(-q^2z/d;q^2)_\ell. \tag{6}$$

14.9.4. Spherical functions on $A(S_q^2)$. There is a unique basis $e_{-\ell}^{(\ell)}$, $e_{-\ell+1}^{(\ell)},\dots,e_\ell^{(\ell)}$ in the subspace V_ℓ (see formula (6) of Section 14.9.2) of $A(S_q^2)$ such that $e_\ell^{(\ell)} = \beta^\ell$ and

$$Le_i^{(\ell)} = \sum_{j=-\ell}^{\ell} t_{ij}^{(\ell)} \otimes e_j^{(\ell)}, \quad i = -\ell, -\ell+2, \dots, \ell, \tag{1}$$

where $t_{ij}^{(\ell)}$ are matrix elements of the corepresentation $T^{(\ell)}$ of the Hopf algebra $A(SU_q(2))$ from Section 14.8.4. The elements $e_i^{(\ell)}$, $-\ell \le i \le \ell$, are called *spherical functions on the quantum 2-sphere* $S_q^2 \equiv S_q^2(c,d)$. Let us show that they are expressed in terms of the big q-Jacobi polynomials

$$\tilde{P}_n^{(\alpha,\beta)}(x;c,d\,|\,q) = {}_3\varphi_2\left(\begin{matrix} q^{-n},\ q^{\alpha+\beta+n+1},\ q^{\alpha+1}x/c \\ q^{\alpha+1},\ -q^{\alpha+1}d/c \end{matrix}\,\middle|\, q,q\right) \tag{2}$$

(see Section 14.7.6), namely,

$$\begin{aligned} e_i^{(\ell)} &= (-c)^{\ell-i}q^{-(\ell-i)(\ell+3i+3)/2}\begin{bmatrix} 2\ell \\ \ell-i \end{bmatrix}_{q^2}^{-1/2}\begin{bmatrix} \ell \\ \ell-i \end{bmatrix}_{q^2} \\ &\quad\times(-q^{2(i+1)}d/c;q^2)_{\ell-i}\tilde{P}_{\ell-i}^{(i,i)}(z;c,d\,|\,q^2)\beta^i \end{aligned} \tag{3}$$

if $0 \le i \le \ell$, and

$$\begin{aligned} e_i^{(\ell)} &= (-c)^{\ell+i}q^{-(\ell+i)(\ell-3i+3)/2}\begin{bmatrix} 2\ell \\ \ell+i \end{bmatrix}_{q^2}^{-1/2}\begin{bmatrix} \ell \\ \ell+i \end{bmatrix}_{q^2} \\ &\quad\times(-q^{2(1-i)}d/c;q^2)_{\ell+i}\alpha^{-i}\tilde{P}_{\ell+i}^{(-i,-i)}(z;c,d\,|\,q^2) \end{aligned} \tag{4}$$

if $-\ell \le i \le 0$.

At first we shall prove (3). According to formula (5) of Section 14.9.2 we have

$$e_i^{(\ell)} = F_{\ell-i}^{(\ell)}(z)\beta^i \quad \text{where} \quad F_i^{(\ell)} \in \mathbb{C}[z],\ 0 \le i \le \ell.$$

Since corepresentation (1) of the Hopf algebra $A(SU_q(2))$ is isomorphic to the corepresentation $T^{(\ell)}$ from Section 14.8.3, then by virtue of the results of Section 14.8.14 we obtain

$$\hat{E}_-\big(F_{\ell-i}^{(\ell)}(z)\beta^i\big) = q^{-\ell+1/2}\frac{(1-q^{2(\ell-i+1)})^{1/2}(1-q^{2(\ell+i)})^{1/2}}{1-q^2}F_{\ell-i+1}^{(\ell)}(z)\beta^{i-1}.$$

And since $\hat{E}_-(\alpha) = 0$, then due to formula (14) of Section 14.9.1 we receive

$$\hat{E}_-\big(F_{\ell-i}^{(\ell)}\beta^i\alpha^i\big) = q^{-\ell+i+1/2}\frac{(1-q^{2(\ell-i+1)})^{1/2}(1-q^{2(\ell+i)})^{1/2}}{1-q^2}F_{\ell-i+1}^{(\ell)}(z)\beta^{i-1}\alpha^i.$$

Since $\beta\alpha$ is expressed in terms of z by formula (2) of Section 14.9.1, then

$$\begin{aligned} &cd\hat{E}_-\big\{F_{\ell-i}^{(\ell)}(z)(q^2z/c;q^2)_i(-q^2z/d;q^2)_i\big\} \\ &\quad= q^{-\ell+i-3/2}\frac{(1-q^{2(\ell-i+1)})^{1/2}(1-q^{2(\ell+i)})^{1/2}}{1-q^2} \\ &\quad\quad\times F_{\ell-i+1}^{(\ell)}(z)(q^2z/c;q^2)_{i-1}(-q^2z/d;q^2)_{i-1}\alpha. \end{aligned} \tag{5}$$

It follows from formulas (11) and (13) of Section 14.9.1 that

$$\hat{E}_-(F(z)) = -q^{-1/2}\alpha D_{q^2}F(z)$$

for every $F(z) \in \mathbb{C}[z]$, where the q-differentiation operator is defined by $D_qF(z) = \{F(qz) - F(z)\}/(qz - z)$ (see Section 14.1.5). Consequently, we obtain the following recurrence formula for $F_i^{(\ell)}$:

$$\begin{aligned} &cdD_{q^2}\{F_{\ell-i}^{(\ell)}(z)(q^2z/c;q^2)_i(-q^2z/d;q^2)_i\} \\ &\quad = q^{-\ell+i+2}\frac{(1-q^{2(\ell-i+1)})^{1/2}(1-q^{2(\ell+i)})^{1/2}}{1-q^2} \\ &\qquad \times T_{q^2}\{F_{\ell-i+1}^{(\ell)}(z)(q^2z/c;q^2)_{i-1}(-q^2z/d;q^2)_{i-1}\} \end{aligned}$$

where T_q is given by $T_qF(z) = F(qz)$. Therefore,

$$\begin{aligned} &(cd)^{\ell-i}(T_{q^2}^{-1}D_{q^2})^{\ell-i}\{(q^2z/c;q^2)_\ell(-q^2z/d;q^2)_\ell\} \\ &\quad = q^{(\ell-i)(i-\ell+5)/2}\frac{(q^2;q^2)_{\ell-i}^{1/2}(q^2;q^2)_{2\ell}^{1/2}}{(1-q^2)^{\ell-i}(q^2;q^2)_{\ell+i}^{1/2}} \\ &\qquad \times F_{\ell-i}^{(\ell)}(z)(q^2z/c;q^2)_i(-q^2z/d;q^2)_i. \end{aligned}$$

Comparing this formula with the Rodrigues formula (12) of Section 14.7.6 we derive expression (3) for $e_i^{(\ell)}$.

In order to prove formula (4) we note that the matrix elements $t_{mn}^{(\ell)}$ satisfy the condition $(t_{-m,-n}^{(\ell)})^* = (-q)^{n-m}t_{mn}^{(\ell)}$. Hence, as it follows from (1), we have $e_{-i}^{(\ell)} = (-q)^{-i}(e_i^{(\ell)})^*$. This leads to formula (4).

It follows from formula (11) of Section 14.7.6 that for $c + q^{2k}d = 0$, where $k \in \mathbb{Z}$, $k \geq 0$, the polynomials $\tilde{P}$ in formulas (3) and (4) turn into q-Hahn polynomials. For example, if $0 \leq \ell < k$ and $0 \leq i \leq \ell$, then

$$\begin{aligned} e_i^{(\ell)} = (-c)^{\ell-i}q^{(\ell-i)(\ell+3i+3)/2}\begin{bmatrix} 2\ell \\ \ell-i \end{bmatrix}_{q^2}^{-1/2}\begin{bmatrix} \ell \\ \ell-i \end{bmatrix}_{q^2}(q^{2(i-k+1)};q^2)_{\ell-i} \\ \times Q_{\ell-i}(q^{2(i+1)}z/c;q^{2i},q^{2i},k-i-1\,|\,q^2)\beta^i. \end{aligned} \tag{6}$$

14.9.5. The orthogonality relation. Let us prove that the spherical functions $e_i^{(\ell)}$ satisfy the orthogonality relation

$$\langle e_i^{(\ell)}, e_j^{(m)}\rangle = \delta_{\ell m}\delta_{ij}\frac{1-q^2}{1-q^{2(2\ell+1)}}\begin{bmatrix} 2\ell \\ \ell \end{bmatrix}_{q^2}^{-1}\prod_{r=1}^{\ell}(c+q^{2r}d)(q^{2r}c+d). \tag{1}$$

For this we consider the matrix $M = (\langle e_i^{(\ell)}, e_j^{(m)}\rangle)_{ij}$. Since the Hermitian form $\langle \cdot, \cdot \rangle$ satisfies condition (4) of Section 14.9.3, then the matrices $T^{(\ell)} \equiv (t_{ij}^{(\ell)})$ have the property $T^{(\ell)}M = MT^{(m)}$ Because of the irreducibility of corepresentations $T^{(\ell)}$ and $T^{(m)}$ we obtain that $M = 0$ if $\ell \neq m$, and that M is a scalar matrix if $\ell = m$. This proves relation (1) for $(\ell, i) \neq (m, j)$ and shows independence of $\langle e_i^{(\ell)}, e_i^{(\ell)}\rangle$ on i. The expression $\langle e_i^{(\ell)}, e_i^{(\ell)}\rangle = \langle e_\ell^{(\ell)}, e_\ell^{(\ell)}\rangle = \langle \beta^\ell, \beta^\ell\rangle$ is given by formula (6) of Section 14.9.3:

$$\begin{aligned}\langle \beta^\ell, \beta^\ell\rangle &= (cd)^\ell \varphi((q^2z/c;q^2)_\ell(-q^2z/d;q^2)_\ell) \\ &= \frac{(cd)^\ell}{c+d}\int_{-d}^{c}(q^2z/c;q^2)_\ell(-q^2z/d;q^2)_\ell d_{q^2}z.\end{aligned}$$

This q-integral can be evaluated by means of formula (16) of Section 14.1.7. We have

$$\begin{aligned}\langle e_i^{(\ell)}, e_i^{(\ell)}\rangle &= \langle \beta^\ell, \beta^\ell\rangle \\ &= \frac{(1-q^2)(q^2;q^2)_\ell^2}{(q^2;q^2)_{2\ell+1}}(cd)^\ell(-q^2d/c;q^2)_\ell(-q^2c/d;q^2)_\ell.\end{aligned} \tag{2}$$

Replacing in (1) $e_i^{(\ell)}$ by its expression in terms of the big q-Jacobi polynomials $\tilde{P}_n^{(\alpha,\alpha)}(x;c,d\,|\,q)$ we derive the orthogonality relation for $\tilde{P}_n^{(\alpha,\alpha)}$:

$$\begin{aligned}&\int_{-d}^{c}\tilde{P}_m^{(\alpha,\alpha)}(x;c,d\,|\,q)\tilde{P}_n^{(\alpha,\alpha)}(x;c,d\,|\,q)(qx/c;q)_\alpha(-qx/d;q)_\alpha d_qx \\ &\qquad = \delta_{mn}\frac{q^{2n\alpha+n(n+3)/2}(1-q)(q;q)_\alpha^2(q;q)_n}{(1-q^{2n+2\alpha+1})(q;q)_{n+2\alpha}}(c+d) \\ &\qquad\quad \times \frac{d^n(-qd/c;q)_\alpha(-qc/d;q)_{n+\alpha}}{c^n(-q^{\alpha+1}d/c;q)_n},\end{aligned} \tag{3}$$

where α is a non-negative integer. If $c + q^kd = 0$ for some $k \in \mathbb{Z}$, $k \neq 0$, then the polynomial $\tilde{P}_n^{(\alpha,\alpha)}(x;c,d\,|\,q)$ turns into a q-Hahn polynomial and the q-integral in (3) turns into a finite sum. As a result we obtain the orthogonality relation for q-Hahn polynomials of the form $Q_n(q^x;q^\alpha,q^\alpha;N\,|\,q)$.

14.9.6. The difference equation. The second order difference equation for $\tilde{P}_n^{(\alpha,\alpha)}(x;c,d\,|\,q)$ is derived by means of the Casimir operator

$$\hat{C} = \frac{(q^{1/2}k^+ - q^{-1/2}k^-)^2}{(q-q^{-1})^2} + E_-E_+$$

acting in the space $A(S_q^2)$. The eigenvalue of $\hat{C}$ on the carrier space V_ℓ of the irreducible corepresentation is equal to $[\ell+1/2]_{q^2}^2$ (see Section 14.8.15). Therefore,

$$\hat{C}e_i^{(\ell)} = \frac{(q^{\ell+1/2} - q^{-\ell-1/2})^2}{(q-q^{-1})^2} e_i^{(\ell)}.$$

Direct calculation shows that

$$\hat{C}(\alpha^m F(z)) = \alpha^m R_m F(z), \quad m \geq 0,$$
$$\hat{C}(F(z)\beta^{-m}) = \{R_{-m}F(z)\}\beta^{-m}, \quad m \leq 0,$$

where

$$R_m = \frac{-q^{-2m-1}}{(1-q^2)^2 z^2}\{(c - q^{2(m+1)}z)(d + q^{2(m+1)}z)T_{q^2} - (1+q^2)cd - (1+q^{2m})q^2(c-d)z + 2q^{2m+3}z^2 + q^2(c-z)(d+z)T_{q^2}^{-1}\}$$

and T_q is defined by formula (3) of Section 14.8.15.

Using these formulas and formulas for $e_i^{(\ell)}$ we conclude that $\tilde{P}_n^{(\alpha,\alpha)}(x;c,d\,|\,q)$ satisfies the second order difference equation

$$\{(c - q^{\alpha+1}x)(d + q^{\alpha+1}x)T_q - (1+q)cd - q(1+q^\alpha)(c-d)x + q^{-n+1}(1+q^{2\alpha+2n+1})x^2 + q(c-x)(d+x)T_q^{-1}\}\tilde{P}_n^{(\alpha,\alpha)}(x;c,d\,|\,q) = 0. \tag{1}$$

14.9.7. The algebra of functions on a quantum 3-sphere. Big q-Jacobi polynomials of general form are obtained under consideration of corepresentations of the Hopf algebra $A(SU_q(2))$ on quantum 3-spheres. This case is analogous to the previous one. Therefore, we shall omit details.

Let us consider six elements α, β, γ, δ, c, d satisfying the relations

$$\beta\alpha = q\alpha\beta, \quad \gamma\alpha = q\alpha\gamma, \quad \delta\beta = q\beta\delta, \quad \delta\gamma = q\gamma\delta, \tag{1}$$

$$\gamma\beta - \beta\gamma = (q-q^{-1})d, \quad \alpha\delta - q^{-1}\beta\gamma = \delta\alpha - q\gamma\beta = c + d, \tag{2}$$

$$d\alpha = q^2\alpha d, \quad d\gamma = q^2\gamma d, \quad \beta d = q^2 d\beta, \quad \delta d = q^2 d\delta. \tag{3}$$

The associative algebra, generated by these elements, is called the *algebra of functions on the quantum sphere* S_q^3 and is denoted by $A = A(S_q^3)$. The formulas

$$\alpha^* = \delta, \quad \beta^* = -q^{-1}\gamma, \quad \gamma^* = -q\beta, \quad \delta^* = \alpha, \quad c^* = c, \quad d^* = d \tag{4}$$

give a $*$-structure on $A(S_q^3)$. The element c belongs to the center of $A(S_q^3)$.

If $c = 1$, $d = 0$, then relations (1) and (2) coincide up to notations with relations (1) and (2) of Section 14.8.1. Therefore,

$$A(S_q^3)\big|_{c=1,d=0} = A(SU_q(2)). \tag{5}$$

If

$$\xi_0 = \frac{1}{2}(\alpha + \alpha^*), \quad \xi_1 = \frac{1}{2i}(\alpha - \alpha^*), \quad \xi_2 = \frac{1}{2}(\beta + \beta^*), \quad \xi_3 = \frac{1}{2i}(\beta - \beta^*),$$

then formulas (1)–(3) imply the equation

$$\xi_0^2 + \xi_1^2 + \frac{1}{2}(1+q^2)(\xi_2^2 + \xi_3^2) = c + \frac{(q+q^{-1})^2}{4} d \tag{6}$$

of "sphere". The values $\xi_0, \xi_1, \xi_2, \xi_3$ play a role of real coordinates of the "sphere".

The left action (left corepresentation) of the Hopf algebra $A(SU_q(2))$ and the right action (right corepresentation) of the Hopf algebra $A(K)$ (see Section 14.8.2) can be defined on $A(S_q^3)$. These actions

$$L_G\colon A(S_q^3) \to A(G) \otimes A(S_q^3), \quad R_K\colon A(S_q^3) \to A(S_q^3) \otimes A(K),$$

where $G = SU_q(2)$, are algebraic homomorphisms with the property

$$(L_G \otimes \mathrm{id}) \circ R_K = (\mathrm{id} \otimes R_K) \circ L_G. \tag{7}$$

They are defined by values on α, β, γ, δ, c, d:

$$L_G \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x & u \\ v & y \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad L_G(c) = I \otimes c, \quad L_G(d) = I \otimes d, \tag{8}$$

$$R_K \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad R_K(c) = c \otimes I, \quad R_K(d) = d \otimes I. \tag{9}$$

As in Section 14.8.1, the first formula in (8) means that

$$L_G(\alpha) = x \otimes \alpha + u \otimes \gamma, \quad L_G(\beta) = x \otimes \beta + u \otimes \delta \quad \text{and so on.}$$

The first formula in (9) has an analogous sense:

$$R_K(\alpha) = \alpha \otimes t, \quad R_K(\beta) = \beta \otimes t^{-1}, \quad R_K(\gamma) = \gamma \otimes t, \quad R_K(\delta) = \delta \otimes t^{-1}.$$

For every integral or half-integral non-negative number ℓ we define the elements

$$\tilde{e}_j^{(\ell)} = \begin{bmatrix} 2\ell \\ \ell + j \end{bmatrix}_{q^2}^{1/2} \alpha^{\ell - j} \beta^{\ell + j} \in A(S_q^3), \quad j = -\ell, -\ell+1, \dots, \ell \tag{10}$$

(compare with formula (2) of Section 14.8.3). Then there exists a unique set of elements $e_{ij}^{(\ell)} \in A(S_q^3)$, $-\ell \leq i, j \leq \ell$, such that

$$L_G(e_{ij}^{(\ell)}) = \sum_{k=-\ell}^{\ell} t_{ik}^{(\ell)} \otimes e_{kj}^{(\ell)}, \quad e_{-\ell,j}^{(\ell)} = \tilde{e}_j^{(\ell)}, \tag{11}$$

where $t_{ik}^{(\ell)}$ are matrix elements from Section 14.8.3. This means that an irreducible left corepresentation of $A(SU_q(2))$ is realized on the subspace $V_{\ell j} = \sum_{i=-\ell}^{\ell} \oplus \mathbb{C} e_{ij}^{(\ell)}$.

The elements $e_{ij}^{(\ell)}$ are called *spherical functions on the quantum sphere* S_q^3. They are not matrix elements of a corepresentation. Only in the case $c = 1$, $d = 0$ we have $e_{ij}^{(\ell)} = t_{ij}^{(\ell)}$.

Let $\mathcal{A}$ denotes the commutative subalgebra of $A(S_q^3)$ generated by elements c and d. Then $\mathcal{A} = \mathbb{C}[c, d]$ (the algebra of polynomials in commuting elements c and d). It is easy to prove that

$$A(S_q^3) = \sum_{2\ell=0}^{\infty} \sum_{j=-\ell}^{\ell} \oplus (\mathcal{A} \otimes V_{\ell j}). \tag{12}$$

The right corepresentation (9) of the quantum group K provides the decomposition of $A(S_q^3)$ into the sum of the subspaces

$$A_m(S_q^3) = \{a \in A(S_q^3) \mid R_K(a) = a \otimes t^k\}.$$

The subspace $A_0(S_q^3)$ is denoted by $A(S_q^3/K)$.

Restricting the corepresentation L_G from $G = SU_q(2)$ onto the quantum subgroup K we define, as in Section 14.8.3, the left action L_K of K on $A(S_q^3)$ and, hence, on $A_m(S_q^3)$. Let

$$A_{nm}(S_q^3)]\{b \in A_m(S_q^3) \mid L_K(b) = t^n \otimes b\}.$$

It is easy to show that $e_{kj}^{(\ell)} \in A_{kj}(S_q^3)$. Making use of this fact and of expansion (12) one proves that

$$A_{00}(S_q^3) = \mathbb{C}[c, d, z] \quad \text{where} \quad z = c - \alpha\delta.$$

Since z commutes with c and d, then $\mathbb{C}[c, d, z]$ is a commutative subalgebra of $A(S_q^3)$.

One proves that there exists a unique invariant homomorphism $\varphi\colon A(S_q^3) \to \mathcal{A}$ such that $\varphi(I) = 1$. It is called the *invariant integral* on $A(S_q^3)$ or on S_q^3. Using this integral we introduce the Hermitian forms

$$\langle a, b\rangle_L = \varphi(a^* b), \quad \langle a, b\rangle_R = \varphi(ab^*)$$

on $\mathcal{A}(S_q^3)$ with values in $\mathcal{A}$. One proves that $\varphi(a) = 0$ if $a \in A_{mn}(S_q^3)$ and $(m,n) \neq (0,0)$. The invariant integral is defined uniquely on $A_{00}(S_q^3)$ by its values on z^n, $n = 0, 1, 2, \dots$. We have

$$\varphi(z^n) = \frac{c^{n+1} - (-d)^{n+1}}{c+d} \frac{1-q^2}{1-q^{2(n+1)}}.$$

This formula allows us to rewrite the functional φ on $\mathbb{C}[c,d,z]$ as the q-integral

$$\varphi(F(z)) = \frac{1}{c+d} \int_{-d}^{c} F(z) d_{q^2} z, \quad F(z) \in \mathbb{C}[c,d,z].$$

14.9.8. Spherical functions on S_q^3 and big q-Jacobi polynomials. By means of arguments analogous to those of Section 14.9.5 we prove that

$$\langle e_{ij}^{(\ell)}, e_{mn}^{(\ell')} \rangle_L = \langle e_{ij}^{(\ell)}, e_{mn}^{(\ell')} \rangle_R = 0 \tag{1}$$

if $(\ell, i, j) \neq (\ell', m, n)$. For $(\ell, i, j) = (\ell', m, n)$ we have

$$\langle e_{ij}^{(\ell)}, e_{ij}^{(\ell)} \rangle_L = q^{2(\ell+i)} \frac{1-q^2}{1-q^{2(2\ell+1)}} \prod_k (c + q^{2k} d), \tag{2}$$

where the product is over the values $-\ell - j, -\ell - j + 1, \dots, \ell - j$ without zero.

It follows from the unitarity of corepresentation (11) of Section 14.9.7 that

$$\sum_{k=-\ell}^{\ell} (e_{ki}^{(\ell)})^* e_{kj}^{(\ell)} = \delta_{ij} \prod_k (c + q^{2k} d), \tag{3}$$

where the product is the same as in formula (2).

Arguments, similar to those from Section 14.9.4, lead to explicit expressions for the spherical functions $e_{ij}^{(\ell)}$. We have

$$e_{ij}^{(\ell)} = \alpha^{-i-j} a_{-i-j,i-j}^{(\ell+j)} \tilde{P}_{\ell+j}^{(-i-j,i-j)}(z; c, d \,|\, q^2) \gamma^{i-j} \tag{4}$$

if $i + j \leq 0$, $i \geq j$,

$$e_{ij}^{(\ell)} = \alpha^{-i-j} \beta^{j-i} a_{-i-j,j-i}^{(\ell+j)} \tilde{P}_{\ell+i}^{(-i-j,j-i)}(z; c, d \,|\, q^2) \tag{5}$$

if $i + j \leq 0$, $i \leq j$,

$$e_{ij}^{(\ell)} = \beta^{j-i} a_{i+j,j-i}^{(\ell-j)} \tilde{P}_{\ell-j}^{(i+j,j-i)}(z; c, d \,|\, q^2) \delta^{i+j} \tag{6}$$

if $i+j \geq 0$, $i \leq j$,

$$e_{ij}^{(\ell)} = a_{i+j,i-j}^{(\ell-i)} \tilde{P}_{\ell-i}^{(i+j,i-j)}(z; c, d \,|\, q^2)\gamma^{i-j}\delta^{i+j} \tag{7}$$

if $i+j \geq 0$, $i \geq j$, where

$$a_{ks}^{(n)} = (-1)^n q^{-n(2k+s+n+1)/2} \begin{bmatrix} n+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n+k+s \\ k \end{bmatrix}_{q^2} c^n(-q^{k+1}d/c; q)_n.$$

Substituting these expressions into formulas (1) and (2), we obtain the orthogonality relation for the polynomials $\tilde{P}_n^{(a,b)}$:

$$\int_{-d}^{c} \tilde{P}_n^{(a,b)}(z; c, d \,|\, q)\tilde{P}_m^{(a,b)}(z; c, d \,|\, q)(qz/c; q)_a \times (-qz/d; q)_b d_q z = \delta_{mn} h_n, \tag{8}$$

where a, b are non-negative integers and

$$h_n = \frac{cd^{n+1}q^{n(a+1)+n(n-1)/2}(1-q^{a+b+1})}{(c+d)(1-q^{2n+b+a+1})} \times \frac{(q^{b+1}; q)_n(-q^{b+1}c/d; q)_n(q; q)_a(-d/c; q)_{a+1}(-c/d; q)_{b+1}}{(q^{a+b+1}; q)_n(-q; q)_n(q^{b+1}; q)_{a+1}}.$$

Chapter 15.
Semisimple Lie Groups and Related Homogeneous Spaces

In the first two volumes of this book we have considered special functions related to most degenerate series of irreducible representations of classical simple Lie groups and of corresponding inhomogeneous groups. In the next chapters we study special functions related to non-degenerate series of representations. These special functions depend on many variables and in some cases it is convenient to consider them as functions of matrix argument or as functions with matrix indices. In this chapter we give necessary information from the theory of semisimple Lie groups and related homogeneous spaces. We describe also invariant measures and invariant differential operators on these groups and spaces.

15.1. Decompositions of Semisimple Lie Algebras and Groups

15.1.1. Decompositions of $\mathfrak{sl}(n,\mathbb{C})$ and $SL(n,\mathbb{C})$. Below $\mathbb{F}$ will denote one of the fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. Then $GL(n,\mathbb{F})$ is the group of invertible $n\times n$ matrices with entries from $\mathbb{F}$. Every quaternion $x+yi+tj+sk$ may be written as the matrix $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z}\end{pmatrix}$ where $z=x+iy$, $w=t+is$ (see Example 11 of Section 1.0.3). By making use of this representation we write down matrices g from $GL(n,\mathbb{H})$ as elements of the group $GL(2n,\mathbb{C})$ and $\det g$ will denote determinants of the corresponding complex matrices.

The subgroups $S_\pm(n,\mathbb{F})$, $N_\pm(n,\mathbb{F})$, $D(n,\mathbb{F})$ of the group $GL(n,\mathbb{F})$ were introduced in Section 1.0.2. Let us remind that

$$S_\pm(n,\mathbb{F})=\{g\in GL(n,\mathbb{F})\mid g_{ij}=0 \quad \text{for} \quad i\gtrless j\},$$

$$N_\pm(n,\mathbb{F})=\{g\in S_\pm(n,\mathbb{F})\mid g_{ii}=1,\ i=1,2,\dots,n\},$$

$$D(n,\mathbb{F})=\{\operatorname{diag}(g_{11},g_{22},\dots,g_{nn})\mid g_{kk}\in\mathbb{F},\ g_{kk}\neq 0\}.$$

We shall also use the subgroups $S_\pm^+(n,\mathbb{F})$ and $A(n)$ which are defined as

$$S_\pm^+(n,\mathbb{F})=\{g\in S_\pm(n,\mathbb{F})\mid g_{kk}>0,\ k=1,2,\dots,n\},$$

$$A(n)=\{\operatorname{diag}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)\mid \varepsilon_k>0,\ k=1,2,\dots,n\}.$$

The set of matrices $g\in GL(n,\mathbb{F})$ with $\det g=1$ forms the group $SL(n,\mathbb{F})$. The intersection of the groups $S_\pm^+(n,\mathbb{F})$, $S_\pm(n,\mathbb{F})$, $A(n)$, $D(n,\mathbb{F})$ with $SL(n,\mathbb{F})$ will be denoted by $\tilde{S}_\pm^+(n,\mathbb{F})$, $\tilde{S}_\pm(n,\mathbb{F})$, $\tilde{A}(n)$, $\tilde{D}(n,\mathbb{F})$ respectively.

We shall need some decompositions of semisimple Lie groups and algebras. In this section we consider these decompositions for the simple Lie group $SL(n,\mathbb{C})$ and for its Lie algebra $\mathfrak{sl}(n,\mathbb{C})$.

The Cartan decomposition $\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{su}(n) + \mathfrak{p}$ was considered in Example 1 of Section 1.2.3, where $\mathfrak{p}$ is the space of Hermitian matrices with $\mathrm{Tr}\ g = 0$. The corresponding decomposition of the Lie group $SL(n,\mathbb{C})$ is of the form[1] $SL(n,\mathbb{C}) = SU(n)\mathcal{P}$, where $\mathcal{P}$ is the set of unimodular Hermitian positive definite matrices. Every matrix $p \in \mathcal{P}$ can be represented as $p = kak^{-1}$, where $k \in SU(n)$ and $a \in \tilde{A}(n)$. Since each element $g \in SL(n,\mathbb{C})$ may be written as the product $g = k_1 p$, $k_1 \in SU(n)$, then for g we obtain the decomposition $g = k_1 kak^{-1}$, where $a \in \tilde{A}(n)$. Thus, we have proved that every matrix $g \in SL(n,\mathbb{C})$ can be represented in the form $g = k_1 a k_2$, where $k_1, k_2 \in SU(n)$ and $a \in \tilde{A}(n)$. The decomposition $g = k_1 a k_2$ is not unique. If we require for $a = \mathrm{diag}\ (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ to satisfy the condition $\varepsilon_1 \geq \varepsilon_2 \geq \dots \geq \varepsilon_n$, then a is uniquely defined in this decomposition. The matrices k_1 and k_2 in $g = k_1 a k_2$ can be replaced by $k_1 h$ and $h^{-1} k_2$, respectively, where h belongs to $SU(n)$ and commutes with a (in particular, h is diagonal if $\varepsilon_1, \dots, \varepsilon_n$ are pairwise different).

Every matrix $X \in \mathfrak{sl}(n,\mathbb{C})$ can be represented as $X = Y + H + E_-$, where Y is a skew-Hermitian matrix with vanishing trace, H is a diagonal real matrix with vanishing trace and E_- is a lower triangular matrix with zeros on the main diagonal. With the help of the exponential mapping we obtain from here that the decomposition $g = n_- ak$ is valid in some neighborhood of the unit element of the group $SL(n,\mathbb{C})$, where $n_- \in N_-(n,\mathbb{C})$, $a \in \tilde{A}(n)$ and $k \in SU(n)$. By making use of the Schmidt-Gram orthogonalization procedure one can represent matrices $g \in SL(n,\mathbb{C})$ in the form $g = sk$, where $s \in \tilde{S}^+_-(n,\mathbb{C})$ and $k \in SU(n)$. It follows from here that every element $g \in SL(n,\mathbb{C})$ can be represented as $g = n_- ak$, $k \in SU(n)$, $a \in \tilde{A}(n)$, $n_- \in N_-(n,\mathbb{C})$ and the matrices n_-, a, k are uniquely determined by g. The decompositions

$$g = nak, \quad g = kan_-, \quad g = kan'$$

are also valid, where $a \in \tilde{A}(n)$, $k \in SU(n)$, $n_- \in N_-(n,\mathbb{C})$, $n, n' \in N_+(n,\mathbb{C})$.

Clearly, every matrix $X \in \mathfrak{sl}(n,\mathbb{C})$ can be represented as $X = E_+ + H + E_-$, where H is a diagonal matrix with vanishing trace and $E_\pm \in \mathfrak{n}_\pm(n,\mathbb{C})$ ($\mathfrak{n}_\pm(n,\mathbb{C})$ are the Lie algebras of the groups $N_\pm(n,\mathbb{C})$). Therefore, the exponential mapping leads to the decomposition $g = n_+ \delta n_-$, $n_\pm \in N_\pm(n,\mathbb{C})$, $\delta \in \tilde{D}(n,\mathbb{C})$, in some neighborhood of the identity element of the group $SL(n,\mathbb{C})$. One can show that this decomposition is valid for all matrices of $SL(n,\mathbb{C})$ except for a submanifold of smaller dimension. In this case we say about decomposition for almost all matrices from $SL(n,\mathbb{C})$. For almost all matrices $g \in SL(n,\mathbb{C})$ we also have the decomposition $g = n_- \delta w_0 n'_-$, where $n_-, n'_- \in N_-(n,\mathbb{C})$, $\delta \in \tilde{D}(n,\mathbb{C})$ and w_0 is the matrix with all zero elements except for the elements $w_{k,n-k+1} = \pm 1$, $w_0 = SL(n,\mathbb{C})$.

Some of the decompositions, considered above, can be generalized for matrices g from $\mathfrak{M}(m,n;\mathbb{C})$, $m \leq n$ (see Section 1.0.1). They are represented as $g = k_1 b k_2$,

[1] Let us note that in Section 1.2.3 we used the notation P for the set $\mathcal{P}$. In this section we shall use the letter P for notation of other set.

where $k_1 \in U(m)$, $k_2 \in U(n)$ and $b = (a, 0)$, $a \in A(m)$, 0 is the $m \times (n-m)$ matrix with zero entries. Almost all matrices from $\mathfrak{M}(m, n; \mathbb{C})$, $m \leq n$ can be represented in the form $g = n_-\delta n_+$, $n_- \in N_-(m, \mathbb{C})$, $n_+ \in N_+(n, \mathbb{C})$, $\delta = (\delta', 0)$, $\delta' \in D(m, \mathbb{C})$, and in the form $g = nbk$, $n \in N_-(m, \mathbb{C})$, $k \in U(n)$, $b = (a, 0)$, $a \in A(m)$. Along with these decompositions there are intermediate ones. The nilpotent subgroups $N_\pm(n, \mathbb{C})$ and the unitary subgroups $U(m)$ and $U(n)$ are replaced in them by semidirect product subgroups. The subgroup of matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $b \in \mathfrak{M}(p, n-p; \mathbb{C})$, $a \in U(p)$, $c \in N_+(n-p, \mathbb{C})$, is an example of such semidirect product. There are also decompositions in which the subgroup $U(n)$ is replaced by one of the real forms $U(p, q)$, $p + q = n$, of the group $GL(n, \mathbb{C})$.

15.1.2. Cartan subgroups and subalgebras. Roots and root subspaces. The decompositions of the Lie group $SL(n, \mathbb{C})$ and of its Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ can be generalized for all semisimple Lie groups and algebras. In order to introduce them we have to consider the subgroups analogous to $N_\pm(n, \mathbb{C})$, $\tilde{A}(n)$, $SU(n)$, $\tilde{D}(n, \mathbb{C})$ and the corresponding subalgebras.

Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra (see Section 1.1.7). Let $d(X)$ be the dimension of the centralizer $Z(X)$ of the element $X \in \mathfrak{g}$ in $\mathfrak{g}$. Since $[X, X] = 0$, then $d(X) > 0$. If $d(X) = \min_{Y \in \mathfrak{g}} d(Y)$, then the element X is called *regular*.

Example 1. Diagonal matrices X of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ with pairwise different diagonal elements are regular. For these matrices the centralizer $Z(X)$ coincides with the subalgebra of all diagonal elements and $d(X) = n - 1$.

Let G be the connected Lie group corresponding to the Lie algebra $\mathfrak{g}$. The commutation relations in $\mathfrak{g}$ are conserved by the conjugations Ad g, $g \in G$. Therefore, the elements $(\text{Ad } g)X$, $g \in G$, are regular if X is regular. A maximal commutative Lie subalgebra $\mathfrak{h}$ of the semisimple Lie algebra $\mathfrak{g}$, containing regular elements, is called a *Cartan subalgebra.* If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then for each $g \in G$ the set $(\text{Ad } g)\mathfrak{h}$ is also a Cartan subalgebra.

There is other (equivalent) definition of a Cartan subalgebra which uses the notion of semisimple elements of $\mathfrak{g}$. An element $X \in \mathfrak{g}$ is said to be *semisimple* if the operator ad X is diagonal in some basis of $\mathfrak{g}$. A maximal commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a Cartan subalgebra if and only if it consists of semisimple elements.

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then the closed subgroup of the Lie group G which is the centralizer C of $\mathfrak{h}$ in G (that is, which consists of all elements $h \in G$ such that $(\text{Ad } h)H = H$ for all $H \in \mathfrak{h}$) is called a *Cartan subgroup* of G. If C is a Cartan subgroup of G, then for each $g \in G$ the set gCg^{-1} is also a Cartan subgroup.

Example 2. The set $\mathfrak{h}$ of all diagonal traceless matrices is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$. All other Cartan subalgebras of $\mathfrak{sl}(n, \mathbb{C})$ are obtained from $\mathfrak{h}$ with the help of automorphisms Ad g, $g \in G$. The Cartan subgroup $\tilde{D}(n, \mathbb{C})$ of the group

$SL(n, \mathbb{C})$ corresponds to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n, \mathbb{C})$. All other Cartan subgroups of $SL(n, \mathbb{C})$ are received with the help of acting by inner automorphisms.

Example 3. If the group $SO(n, \mathbb{C})$ is realized as a group of transformations conserving the form $z_1 w_n + \ldots + z_n w_1$, then one of the Cartan subalgebras of $\mathfrak{so}(n, \mathbb{C})$ is generated by the matrices[2]

$$H_i = E_{ii} - E_{n-i+1,n-i+1}, \quad 1 \leq i \leq [n/2],$$

where $[n/2]$ is the integral part of $n/2$. If $n = 2\ell$, then the corresponding Cartan subgroup of $SO(n, \mathbb{C})$ consists of the matrices

$$\begin{pmatrix} A & 0 \\ 0 & SA^{-1}S \end{pmatrix},$$

where $A = \mathrm{diag}\,(e^{t_1}, \ldots, e^{t_\ell})$, $t_j \in \mathbb{C}$, and the entries of the matrix S are $S_{ij} = \delta_{i,\ell-j+1}$. If $n = 2\ell + 1$, then this Cartan subgroup consists of the matrices

$$\mathrm{diag}\,(A, 1, SA^{-1}S),$$

where A and S are of the same forms.

Example 4. The diagonal matrices $E_{ii} - E_{2\ell-i+1,2\ell-i+1}$, $i = 1, 2, \ldots, \ell$, generate the Cartan subalgebra of the Lie algebra $\mathfrak{sp}(\ell, \mathbb{C})$. The corresponding Cartan subgroup of $Sp(\ell, \mathbb{C})$ consists of the matrices $\mathrm{diag}(A, SA^{-1}S)$ where A and S are the same as in Example 3.

All Cartan subalgebras of a complex semisimple Lie algebra are pairwise conjugate, that is, they are connected by the operators Ad g, $g \in G$. Therefore, Cartan subalgebras of $\mathfrak{g}$ are of the same dimension. This dimension is called the *rank* of the Lie algebra $\mathfrak{g}$ and of the corresponding Lie group G. For example,

$$\mathrm{rank}\; SL(n, \mathbb{C}) = n - 1, \quad \mathrm{rank}\; SO(n, \mathbb{C}) = [n/2], \quad \mathrm{rank}\; Sp(n, \mathbb{C}) = n.$$

All Cartan subalgebras of a real semisimple Lie algebra also are of the same dimension. In this case they may be non-conjugate.

Example 5. The set $\mathfrak{h}$ of all diagonal matrices of the real simple Lie algebra $\mathfrak{sl}(n, \mathbf{R})$ is its Cartan subalgebra. This Lie algebra has other Cartan subalgebras which are not conjugate to $\mathfrak{h}$. They are given by integer q, $1 \leq q \leq [n/2]$, and consist of the traceless matrices

$$H_q = \mathrm{diag}\,(R(t_1, s_1), \ldots, R(t_q, s_q), r_1, \ldots, r_{n-2q}),$$

[2] Let us recall that E_{ij} denotes the matrix with entries $(E_{ij})_{st} = \delta_{is}\delta_{jt}$.

where

$$R(t,s) = tI_2 + sJ_2, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Example 6. The real simple Lie algebra $\mathfrak{su}(p,q)$, $p+q=n$, $p \le q$, has p Cartan subalgebras which are pairwise non-conjugate. They are defined by integer s, $1 \le s \le p$, and consist of traceless matrices of the form $A_s + B_s$, where

$$A_s = \mathrm{diag}\,(it_1, \dots, it_s, ir_{s+1}, \dots, ir_p, ir_p, \dots, ir_{s+1}, it'_s, \dots, it'_1),$$

$$B_s = \begin{array}{c} \begin{matrix} s & p-s & p-s & s \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & C_s & 0 \\ 0 & SC_sS & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{matrix} s \\ p-s \\ p-s \\ s \end{matrix}, \quad C_s = \begin{pmatrix} 0 & 0 & \dots & 0 & c_1 \\ 0 & 0 & \dots & c_2 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & c_{p-s-1} & \dots & 0 & 0 \\ c_{p-s} & 0 & \dots & 0 & 0 \end{pmatrix}$$

and S is the matrix with the entries $S_{ij} = \delta_{i,p-s-j+1}$.

Example 7. The real simple Lie algebra $\mathfrak{so}(p,q)$, $p+q=n$, $p \le q$, has $[p/2]$ Cartan subalgebras which are pairwise non-conjugate. They are defined by integer s, $1 \le s \le [p/2]$, and consist of traceless matrices of the form $\tilde{A}_s + \tilde{B}_s$, where

$$\tilde{A}_s = \mathrm{diag}\,(t_1J_2, \dots, t_sJ_2, 0, \dots, 0, t'_sJ_2, \dots, t'_1J_2),$$

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $\tilde{B}_s$ are obtained from matrices B_s of Example 6 with the help of replacement of s by $2s$.

We recommend to the reader to find non-conjugate Cartan subalgebras of other real simple Lie algebras.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be its Cartan subalgebra. Then ad H, $H \in \mathfrak{h}$, form the set of commuting operators. Consequently, there is a basis in $\mathfrak{g}$ consisting of eigenvectors of the operators ad H. Let $X_1, X_2, \dots, X_n$ be such basis. Then for all $H \in \mathfrak{h}$ we have

$$(\mathrm{ad}\, H)X_k = \alpha_k(H)X_k, \quad 1 \le k \le n.$$

It is easy to derive from here that

$$\alpha_k(\lambda H_1 + \mu H_2) = \lambda\alpha_k(H_1) + \mu\alpha_k(H_2), \quad \lambda, \mu \in \mathbb{C}, \quad H_1, H_2 \in \mathfrak{h}.$$

Thus, α_k, $1 \le k \le n$, are linear functionals on $\mathfrak{h}$. The space $\mathfrak{g}$ decomposes into a direct sum of eigenspaces of the operators ad H. This decomposition is of the form

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \ne 0} \mathfrak{g}_\alpha, \tag{1}$$

where α are non-vanishing linear functionals on $\mathfrak{h}$ and $\mathfrak{g}_\alpha$ are the corresponding eigenspaces. The functionals α are called *roots* of the Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. The set of all roots will be denoted by Δ. The eigenspaces $\mathfrak{g}_\alpha$ are called *root subspaces.*

Example 8. The traceless matrices $\sum_{k=1}^{\ell+1} a_k E_{kk}$ form a Cartan subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{sl}(\ell+1, \mathbb{C})$. Since

$$\left[\sum_{k=1}^{\ell+1} a_k E_{kk}, E_{ij}\right] = (a_i - a_j)E_{ij},$$

then the sets $\mathbb{C}E_{ij}$, $i \neq j$, are root subspaces. They correspond to the roots α_{ij} such that

$$\alpha_{ij}\left(\sum_k a_k E_{kk}\right) = a_i - a_j, \quad i \neq j. \tag{2}$$

The decomposition (1) in this case is of the form

$$\mathfrak{sl}(\ell+1, \mathbb{C}) = \mathfrak{h} + \sum_{i \neq j} \mathbb{C}E_{ij}. \tag{1'}$$

For this Lie algebra the Killing form $B(X, Y) = \mathrm{Tr}\,(\mathrm{ad}\, X)(\mathrm{ad}\, Y)$ can be written as

$$B(X, Y) = 2(\ell+1)\mathrm{Tr}\, XY, \quad X, Y \in \mathfrak{sl}(\ell+1, \mathbb{C})$$

(see Section 1.1.6).

Example 9. The matrices

$$H_i = E_{ii} - E_{2\ell-i+2, 2\ell-i+2}, \quad i = 1, 2, \ldots, \ell, \tag{3}$$

form a basis of the Cartan subalgebra $\mathfrak{h}$ of the simple Lie algebra $\mathfrak{so}(n, \mathbb{C})$, $n = 2\ell+1$ (see Example 3). The root subspaces are one-dimensional and can be defined by basis matrices

$$E^{+j,-i} = E_{ji} - E_{2\ell-i+2, 2\ell-j+2}, \quad i \neq j, \quad 1 \leq i, j \leq \ell, \tag{4}$$

$$E^{+i,+j} = E_{2\ell-i+2, j} - E_{2\ell-j+2, i}, \quad 1 \leq i < j \leq \ell, \tag{5}$$

$$E^{-i,-j} = E_{j, 2\ell-i+2} - E_{i, 2\ell-j+2}, \quad 1 \leq i < j \leq \ell, \tag{6}$$

$$E^{+i} = E_{i, \ell+1} - E_{\ell+1, 2\ell-i+2}, \quad 1 \leq i \leq \ell, \tag{7}$$

$$E^{-i} = E_{\ell+1, i} - E_{2\ell-i+2, \ell+1}, \quad 1 \leq i \leq \ell. \tag{8}$$

The matrices (3)-(8) form a basis of the Lie algebra $\mathfrak{so}(2\ell+1,\mathbb{C})$. Since for $H = \sum_{i=1}^{\ell} a_i H_i \in \mathfrak{h}$ we have

$$[H, E^{\pm i,\pm j}] = (\pm a_i \pm a_j)E^{\pm i,\pm j},$$
$$[H, E^{\pm i}] = \pm a_i E^{\pm i},$$

then the linear forms

$$\alpha_{\pm i,\pm j}(H) = \pm a_i \pm a_j, \quad i \neq j, \quad \alpha_{\pm i}(H) = \pm a_i, \quad 1 \leq i \leq \ell$$

(all possible combinations of signs are taken and signs at a_i and at a_j coincide with signs at i and at j respectively) constitute the set of roots of the Lie algebra $\mathfrak{so}(2\ell+1,\mathbb{C})$. The Killing form of this algebra is

$$B(X,Y) = (2\ell-1)\mathrm{Tr}\, XY.$$

Example 10. The matrices

$$H_i = E_{ii} - E_{2\ell-i+1,2\ell-i+1}, \quad 1 \leq i \leq \ell, \tag{9}$$

form a basis of the Cartan subalgebra $\mathfrak{h}$ of the simple Lie algebra $\mathfrak{so}(n,\mathbb{C})$, $n = 2\ell$. The root subspaces are one-dimensional and have basis matrices

$$E^{+i,-j} = E_{ji} - E_{2\ell-i+1,2\ell-j+1}, \quad i \neq j, \quad 1 \leq i,j \leq \ell, \tag{10}$$
$$E^{+i,+j} = E_{2\ell-i+1,j} - E_{2\ell-j+1,i}, \quad 1 \leq i < j \leq \ell, \tag{11}$$
$$E^{-i,-j} = E_{j,2\ell-i+1} - E_{i,2\ell-j+1}, \quad 1 \leq i < j \leq \ell. \tag{12}$$

The matrices (9)-(12) form a basis of the Lie algebra $\mathfrak{so}(2\ell,\mathbb{C})$. Since for $H = \sum_{i=1}^{\ell} a_i H_i \in \mathfrak{h}$ we have

$$[H, E^{\pm i,\pm j}] = (\pm a_i \pm a_j)E^{\pm i,\pm j},$$

then the linear forms

$$\alpha_{\pm i,\pm j}(H) = \pm a_i \pm a_j,$$

constitute the set of roots of the algebra $\mathfrak{so}(2\ell,\mathbb{C})$. The Killing form of this algebra is

$$B(X,Y) = (2\ell-2)\mathrm{Tr}\, XY.$$

Example 11. We realize the simple Lie algebra $\mathfrak{sp}(\ell,\mathbb{C})$ by matrices from $\mathfrak{sl}(2\ell,\mathbb{C})$. The matrices

$$H_i = E_{ii} - E_{2\ell-i+1,2\ell-i+1}, \quad i = 1,2,\ldots,\ell, \tag{13}$$

form a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sp}(\ell, \mathbb{C})$. The root subspaces are one-dimensional and have the basis elements

$$E^{+j,-i} = E_{ij} - E_{2\ell-j+1,2\ell-i+1}, \quad i \neq j, \quad 1 \leq i,j \leq \ell, \tag{14}$$

$$E^{-i,-j} = E_{i,2\ell-j+1} + E_{j,2\ell-i+1}, \quad 1 \leq i < j \leq \ell, \tag{15}$$

$$E^{+i,+j} = E_{2\ell-i+1,j} + E_{2\ell-j+1,i}, \quad 1 \leq i < j \leq \ell. \tag{16}$$

$$E^{+2i} = E_{2\ell-i+1,i}, \quad E^{-2i} = E_{i,2\ell-i+1}, \quad 1 \leq i \leq \ell. \tag{17}$$

The matrices (13)-(17) constitute a basis of $\mathfrak{sp}(\ell, \mathbb{C})$. For the elements $H = \sum_{i=1}^{\ell} a_i H_i \in \mathfrak{h}$ we have

$$[H, E^{\pm i,\pm j}] = (\pm a_i \pm a_j)E^{\pm i,\pm j},$$
$$[H, E^{\pm 2i}] = \pm 2a_i E^{\pm 2i}.$$

Therefore, the linear functionals

$$\alpha_{\pm i,\pm j}(H) = \pm a_i \pm a_j, \quad \alpha_{\pm 2i}(H) = \pm 2a_i$$

(all possible combinations of signs are taken) form the set of roots of the Lie algebra $\mathfrak{sp}(\ell, \mathbb{C})$. The Killing form of this algebra is

$$B(X, Y) = (2\ell + 2)\mathrm{Tr}\, XY.$$

Let us formulate the main properties of roots and root subspaces of complex semisimple Lie algebras:

(a) *If α is a root of the algebra $\mathfrak{g}$, then $-\alpha$ is also a root of this algebra. There are no other roots of the form $c\alpha$, $c \in \mathbb{C}$.*

(b) *If $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$, $\alpha + \beta \neq 0$, are root subspaces of $\mathfrak{g}$ and if E_α, E_β are elements of these subspaces, then*

$$[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha+\beta} \quad \textit{if } \alpha + \beta \textit{ is a root},$$

$$[E_\alpha, E_\beta] = 0 \quad \textit{otherwise}.$$

For $\alpha = -\beta$ we have $[E_\alpha, E_\beta] \in \mathfrak{h}$.

(c) *All root subspaces are one-dimensional.*

(d) *If $\alpha + \beta \neq 0$, then the corresponding root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the Killing form $B(\cdot\,,\cdot)$.*

(e) *The Killing form $B(\cdot\,,\cdot)$ is non-degenerate on the Cartan subalgebra $\mathfrak{h}$, that is, if $B(H_0, H) = 0$ for all $H \in \mathfrak{h}$, then $H_0 = 0$. For every root α there exists the unique element $H_\alpha \in \mathfrak{h}$ such that*

$$B(H_\alpha, H) = \alpha(H) \quad \textit{for all} \quad H \in h.$$

Let $\mathfrak{h}_R$ *be the subalgebra*

$$\mathfrak{h}_R = \sum_{\alpha\in\Delta} \mathbf{R} H_\alpha$$

of $\mathfrak{h}$. *Then the complex dimension of* $\mathfrak{h}$ *coincides with the real dimension of* $\mathfrak{h}_R$. *The Killing form is real and strictly positive definite on* $\mathfrak{h}_R$.

(f) *One can choose elements* E_α *and* $E_{-\alpha}$ *of the root subspaces* $\mathfrak{g}_\alpha$ *and* $\mathfrak{g}_{-\alpha}$ *such that* $B(E_\alpha, E_{-\alpha}) = 1$. *In this case*

$$[E_\alpha, E_{-\alpha}] = H_\alpha.$$

Proofs of these properties are not complicated. They can be found in [119].

Let us choose a basis $H_1, \dots, H_\ell$ in the Cartan subalgebra $\mathfrak{h}$ such that $\alpha(H_j) \in \mathbf{R}$ for all roots $\alpha \in \Delta$. We take a basis vector E_α in each of the root subspaces $\mathfrak{g}_\alpha$. The elements

$$H_k, \quad k = 1, 2, \dots, \ell; \quad E_\alpha, \quad \alpha \in \Delta, \tag{18}$$

form a basis of the Lie algebra $\mathfrak{g}$. If all the pairs E_α, $E_{-\alpha}$ are such that $[E_\alpha, E_{-\alpha}] = H_\alpha$, then we have the following commutation relations for these basis elements of $\mathfrak{g}$:

$$\left.\begin{array}{ll} [H_i, H_j] = 0, & 1 \le i, j \le \ell, \\ [H_i, E_\alpha] = \alpha(H_i) E_\alpha, & 1 \le i \le \ell, \quad \alpha \in \Delta, \\ [E_\alpha, E_{-\alpha}] = H_\alpha, & E_\alpha \in \Delta, \\ [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta}, & \alpha \ne -\beta, \end{array}\right\} \tag{19}$$

where $N_{\alpha\beta} = 0$ if $\alpha + \beta$ is not a root. The relation $[E_\alpha, E_\beta] = H_\alpha$ does not fix elements E_α and $E_{-\alpha}$ in $\mathfrak{g}$. It is proved that these elements may be chosen in such way that

$$N_{\alpha\beta}^2 = -\frac{q(p+1)}{2} B(H_\alpha, H_\beta),$$

where p (correspondingly q) is the non-negative integer such that $\beta - p\alpha$ (correspondingly $\beta + q\alpha$) is a root of $\mathfrak{g}$ and $\beta - (p+1)\alpha$ (correspondingly $\beta + (q+1)\alpha$) is not a root.

The set of elements (18), satisfying commutation relations (19), is called the *Cartan-Weyl basis* of the complex semisimple Lie algebra $\mathfrak{g}$. With the help of these basis the compact real form $\mathfrak{g}_k$ of the complex algebra $\mathfrak{g}$ is easily found. It consists of real linear combinations of the elements

$$iH_k, \quad 1 \le k \le \ell, \quad E_\alpha - E_{-\alpha}, \ i(E_\alpha + E_{-\alpha}), \quad \alpha \in \Delta, \quad i = \sqrt{-1}.$$

If an ordered basis $H_1, \dots, H_\ell$ of the Cartan subalgebra is chosen, then we may divide the set Δ of roots into two parts. A root α from Δ is called *positive* (*negative*) if the first non-zero number in the sequence $\alpha(H_1), \alpha(H_2), \ldots, \alpha(H_\ell)$ is positive (negative). The set of positive (negative) roots will be denoted by Δ_+ (by Δ_-). Clearly, $\Delta_+ = -\Delta_-$,

Let $\mathfrak{h}'_R$ be the subspace

$$\mathfrak{h}'_R = \sum_{\alpha\in\Delta} \mathbf{R}\alpha$$

of the dual space $\mathfrak{h}'$ for the Cartan subalgebra $\mathfrak{h}$. It is shown that dim $\mathfrak{h}'_R =$ dim $\mathfrak{h} = \ell$. If $\gamma \in \mathfrak{h}'_R$, then the formula $\gamma(H) = B(H_\gamma, H)$, $H \in \mathfrak{h}$, defines uniquely the corresponding element $H_\gamma \in \mathfrak{h}$. We have the one-to-one correspondence $\gamma \longleftrightarrow H_\gamma$ between elements from $\mathfrak{h}'_R$ and from $\mathfrak{h}_R$. We can introduce the scalar product

$$(\gamma, \gamma') = B(H_\gamma, H'_\gamma), \quad \gamma, \gamma' \in \mathfrak{h}'_R,$$

into $\mathfrak{h}'_R$. The reflection

$$S_\alpha : \gamma \to \gamma - \frac{2(\gamma, \alpha)}{(\alpha, \alpha)}\alpha, \quad \gamma \in \mathfrak{h}'_R, \tag{20}$$

of the space $\mathfrak{h}'_R$ is defined for every root $\alpha \in \Delta$. It is clear that $S_\alpha = S_{-\alpha}$.

The group W, generated by all reflections S_α, $\alpha \in \Delta$, is called the *Weyl group* of the algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. It is proved that W is a finite group. The Weyl group transforms Δ into Δ. Moreover, it is a maximal symmetry group of the root system Δ.

To every root α there corresponds the hyperplane $\mathfrak{h}'_\alpha = \{\gamma \in \mathfrak{h}'_R \mid (\alpha, \gamma) = 0\}$ in $\mathfrak{h}'_R$. The hyperplanes $\mathfrak{h}'_\alpha$, $\alpha \in \Delta_+$, divide the space $\mathfrak{h}'_R$ into connected parts which are called *Weyl chambers*. Elements of the Weyl group W permute transitively these chambers.

In the root system Δ there is a minimal set of roots which generates Δ with the help of action by elements of the Weyl group W. The set of so called simple roots possesses this property.

A root $\alpha \in \Delta_+$ is said to be *simple* if it cannot be represented as a sum of two positive roots. This definition shows that every positive (negative) root α may be represented in a form of a sum of simple roots with non-negative (non-positive) integral coefficients. A linear combination of simple roots with positive and negative coefficients simultaneously cannot be a root. Simple roots possess the following properties:

(a) *A number of simple roots coincides with rank ℓ of the Lie algebra $\mathfrak{g}$, that is, with the dimension of the Cartan subalgebra $\mathfrak{h}$.*

(b) *Simple roots are linearly independent.*

(c) *The reflections S_{α_i}, $i = 1, 2, \ldots, \ell$, where $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are simple roots of the algebra $\mathfrak{g}$, generate the Weyl group W of $\mathfrak{g}$.*

(d) *If the set $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ of simple roots of $\mathfrak{g}$ can be divided into two orthogonal (with respect to the scalar product $(\alpha, \beta) = B(H_\alpha, H_\beta)$) parts $\alpha_{i_1}, \ldots, \alpha_{i_n}$ and $\alpha_{j_1}, \ldots, \alpha_{j_m}$, $m+n = \ell$, then the Lie algebra $\mathfrak{g}$ is a direct sum of two semisimple Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ (which are ideals of $\mathfrak{g}$) such that $\alpha_{i_1}, \ldots, \alpha_{i_n}$ and $\alpha_{j_1}, \ldots, \alpha_{j_m}$ are the sets of their simple roots respectively.*

(e) *Multiplying all roots of the semisimple complex Lie algebra $\mathfrak{g}$ by a fixed number we obtain the root system of a Lie algebra isomorphic to $\mathfrak{g}$.*

Remark. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let G be the group of automorphisms of $\mathfrak{g}$. There exists the finite subgroup W of G which retain invariant the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. By making use of the mapping $\alpha \to H_\alpha$ we can imbed the set of roots Δ into $\mathfrak{h}$. The group W acts upon H_α in the same way as the Weyl group does. In other words, the Weyl group of $\mathfrak{g}$ can be imbedded into G.

Example 12. Let $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$. We choose the ordered basis $E_{ii} - E_{i+1,i+1}$, $i = 1, 2, \ldots, \ell$, of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(\ell+1, \mathbb{C})$. Then the roots $\alpha_{ij} = a_i - a_j$, $i < j$ (see Example 8), are positive. The roots

$$\alpha_{12}, \alpha_{23}, \ldots, \alpha_{\ell,\ell+1}$$

are simple. Rank of $\mathfrak{sl}(\ell+1, \mathbb{C})$ is equal to ℓ. It is convenient to embed roots of $\mathfrak{sl}(\ell+1, \mathbb{C})$ into $\ell+1$-dimensional real space. Namely, with the roots α_{ij} we associate the vectors

$$(\alpha_{ij}(E_{11}), \alpha_{ij}(E_{22}), \ldots, \alpha_{ij}(E_{\ell+1,\ell+1})).$$

They belong to the hyperplane $a_1 + a_2 + \ldots + a_{\ell+1} = 0$ of the $\ell+1$-dimensional space $\mathbf{R}^{\ell+1}$ of vectors $(a_1, a_2, \ldots, a_{\ell+1})$. The unit vectors $(0, \ldots, 0, 1, 0, \ldots, 0)$ of $\mathbf{R}^{\ell+1}$ are denoted as $\mathbf{e}_i$. The roots α_{ij} are represented as $\alpha_{ij} = \mathbf{e}_i - \mathbf{e}_j$. The Weyl group W of the algebra $\mathfrak{sl}(\ell+1, \mathbb{C})$ acts on the hyperplane

$$\mathfrak{h}'_R = \{a_1\mathbf{e}_1 + \ldots + a_{\ell+1}\mathbf{e}_{\ell+1} \mid a_1 + \ldots + a_{\ell+1} = 0\}.$$

With the help of formula (20) we find that to the root α_{ij} the reflection $S_{\alpha_{ij}} \equiv S_{ij}$ corresponds for which $S_{ij}(a_1, \ldots, a_{\ell+1})$ is obtained from $(a_1, \ldots, a_{\ell+1})$ by permutation of the numbers a_i and a_j. Thus, the Weyl group W of $\mathfrak{sl}(\ell+1, \mathbb{C})$ consists of all permutations of the numbers $a_1, \ldots, a_{\ell+1}$. That is, W coincides with the symmetric group $S_{\ell+1}$ (see Section 13.1.1).

Example 13. Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, $n = 2\ell+1$. We choose the ordered basis $H_i = E_{ii} - E_{2\ell-i+2,2\ell-i+2}$, $i = 1, 2, \ldots, \ell$, in the Cartan subalgebra $\mathfrak{h}$. Then the roots

$$\alpha_{j,-i}, \quad j < i; \quad \alpha_{i,j}, \quad i < j; \quad \alpha_i, \quad 1 \le i \le \ell,$$

are positive. The roots

$$\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{\ell-1,-\ell}, \alpha_\ell$$

are simple. With each root α of $\mathfrak{so}(2\ell+1,\mathbb{C})$ we associate the vector $(\alpha(H_1),\alpha(H_2),\ldots,\alpha(H_\ell))$ of ℓ-dimensional real vector space $\mathbf{R}^\ell$. Now roots can be represented as

$$\alpha_{j,-i}=\mathbf{e}_j-\mathbf{e}_i,\quad \alpha_{-i,-j}=-\mathbf{e}_i-\mathbf{e}_j,\quad \alpha_{ij}=\mathbf{e}_i+\mathbf{e}_j,\quad \alpha_{\pm i}=\pm\mathbf{e}_i.$$

With the help of formula (20) we find that reflection S_α, corresponding to a root α, acts upon vectors $(a_1,\ldots,a_\ell)$ as the permutation of the coordinates a_i and a_j if $\alpha=\pm(\mathbf{e}_j-\mathbf{e}_i)$, as the permutation of a_i and a_j and change of their signs if $\alpha=\pm(\mathbf{e}_i+\mathbf{e}_j)$, and as replacement of a_i by $-a_i$ if $\alpha=\pm\mathbf{e}_i$. Thus, the Weyl group W of $\mathfrak{so}(2\ell+1,\mathbb{C})$ consists of permutations of coordinates $a_1,\ldots,a_\ell$ with sign alternations of some of them.

Example 14. Let $\mathfrak{g}=\mathfrak{so}(n,\mathbb{C})$, $n=2\ell$. We choose the ordered basis $H_i=E_{ii}-E_{2\ell-i+1,2\ell-i+1}$, $i=1,2,\ldots,\ell$, of the Cartan subalgebra $\mathfrak{h}$. Then the roots $\alpha_{j,-i}$, α_{ij}, $i<j$, are positive. The roots

$$\alpha_{1,-2},\alpha_{2,-3},\ldots,\alpha_{\ell-1,-\ell},\alpha_{\ell-1,\ell}$$

are simple. If the vector

$$(\alpha(H_1),\alpha(H_2),\ldots,\alpha(H_\ell))$$

is associated with the root α, then

$$\alpha_{j,-i}=\mathbf{e}_j-\mathbf{e}_i,\quad \alpha_{ji}=\mathbf{e}_j+\mathbf{e}_i,\quad \alpha_{-j,-i}=-\mathbf{e}_j-\mathbf{e}_i.$$

Using formula (20) it is easy to find that elements of the Weyl group W of $\mathfrak{so}(2\ell,\mathbb{C})$ act upon vectors $(a_1,a_2,\ldots,a_\ell)$ as permutations of coordinates $a_1,a_2,\ldots,a_\ell$ with sign alternations of even number of coordinates.

Example 15. Let $\mathfrak{g}=\mathfrak{sp}(\ell,\mathbb{C})$. We choose the ordered basis of the Cartan subalgebra $\mathfrak{h}$ of this algebra in the form $H_i=E_{ii}-E_{2\ell-i+1,2\ell-i+1}$, $i=1,2,\ldots,\ell$. Then the roots $\alpha_{i,j}$, $\alpha_{i,-j}$, $i<j$, α_{2i}, $i=1,2,\ldots,\ell$, are positive. The roots

$$\alpha_{1,-2},\alpha_{2,-3},\ldots,\alpha_{\ell-1,-\ell},\alpha_{2\ell}$$

are simple. The roots can be represented as vectors of ℓ-dimensional vector space $\mathbf{R}^\ell$:

$$\alpha_{i,-j}=\mathbf{e}_i-\mathbf{e}_j,\quad \alpha_{i,j}=\mathbf{e}_i+\mathbf{e}_j,\quad \alpha_{-i,-j}=-\mathbf{e}_i-\mathbf{e}_j,\quad \alpha_{\pm 2i}=\pm 2\mathbf{e}_i.$$

Using this information and formula (20) we easily find that the Weyl group W of $\mathfrak{sp}(\ell,\mathbb{C})$ coincides with that of the Lie algebra $\mathfrak{so}(2\ell,\mathbb{C})$.

The Weyl chamber C_+ of $\mathfrak{h}$, whose elements λ satisfy the conditions $\lambda(H_i)\geq 0$ for all basis elements $H_1,\ldots,H_\ell$ of the Cartan subalgebra, is said to be *dominant*.

We have $C_+ = \{\varphi \in \mathfrak{h}'_R \mid (\varphi, \alpha_i) \geq 0, i = 1, 2, \dots, \ell\}$. The domain $C_{++} = \{\varphi \in \mathfrak{h}'_R \mid (\varphi, \alpha_i) > 0, i = 1, 2, \dots, \ell\}$ is called *strictly dominant*. It is possible to prove that if $\varphi \in \mathfrak{h}'_R$, then there exists element $w \in W$ such that $w\varphi \in C_+$. Let us represent $\mathfrak{h}'_R$ as the space of vectors $(a_1, \dots, a_{\ell+1})$ if $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$ and as the space of vectors $(a_1, \dots, a_\ell)$ if $\mathfrak{g}$ is one of the Lie algebras $\mathfrak{so}(2\ell, \mathbb{C})$, $\mathfrak{so}(2\ell+1, \mathbb{C})$, $\mathfrak{sp}(\ell, \mathbb{C})$. Using the descriptions of the Weyl groups, given above, we easily find that if $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$, then $(a_1, \dots, a_{\ell+1}) \in C_+$ if and only if

$$a_1 \geq a_2 \geq \dots \geq a_{\ell+1},$$

if $\mathfrak{g} = \mathfrak{so}(2\ell+1, \mathbb{C})$ or $\mathfrak{g} = \mathfrak{sp}(\ell, \mathbb{C})$, then $(a_1, \dots, a_\ell) \in C_+$ if and only if

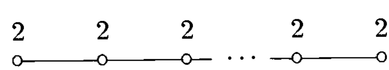
$$a_1 \geq a_2 \geq \dots \geq a_\ell \geq 0,$$

if $\mathfrak{g} = \mathfrak{so}(2\ell, \mathbb{C})$, then $(a_1, \dots, a_\ell) \in C_+$ if and only if

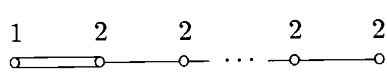
$$a_1 \geq a_2 \geq \dots \geq a_{\ell-1} \geq |a_\ell|.$$

By studying admissible systems of simple roots we can obtain the classification of complex simple Lie algebras (see, for example, [242]). As we saw in Section 1.1.6, up to an isomorphism there are three series $\mathfrak{sl}(\ell+1, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(\ell, \mathbb{C})$ of classical complex simple Lie algebras and five exceptional complex simple Lie algebras. Below we give diagrams of simple roots for each of these Lie algebras. They are called *Dynkin diagrams*. They uniquely (up to an isomorphism) determine the corresponding Lie algebras. On the diagrams simple roots are shown by circles. The roots are connected by one, two or three lines if the angle between them is equal to $2\pi/3$, $3\pi/4$ or $5\pi/6$ respectively. If the angle is equal to π, then roots are not connected by lines. The numbers at the simple roots means their squared lengths. The Dynkin diagrams are of the form

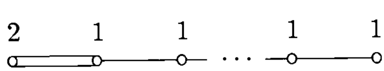
$\mathfrak{sl}(\ell+1, \mathbb{C})$:



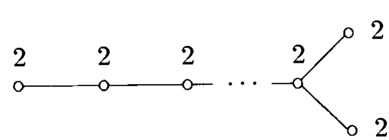
$\mathfrak{so}(2\ell+1, \mathbb{C})$:



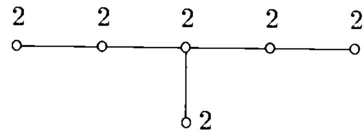
$\mathfrak{sp}(\ell, \mathbb{C})$:



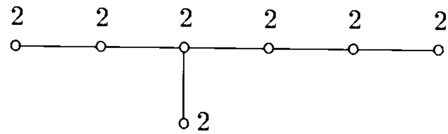
$\mathfrak{so}(2\ell, \mathbb{C})$:



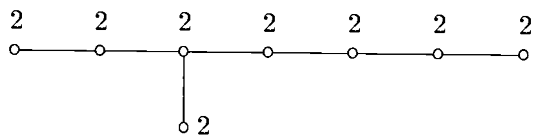
E_6:



E_7:

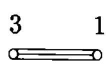


E_8:



F_4:

G_2:



The Lie algebras $\mathfrak{sl}(\ell+1,\mathbb{C})$, $\mathfrak{so}(2\ell,\mathbb{C})$, $\mathfrak{sp}(\ell,\mathbb{C})$, $\mathfrak{so}(2\ell+1,\mathbb{C})$ are also denoted by A_ℓ, B_ℓ, C_ℓ, D_ℓ respectively.

15.1.3. Generating elements of complex semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

be its root decomposition. We introduce the subspaces

$$\mathfrak{n}_+ = \sum_{\alpha\in\Delta_+} \oplus\, \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \sum_{\alpha\in\Delta_-} \oplus\, \mathfrak{g}_\alpha.$$

Since $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha+\beta}$, then these subspaces are Lie subalgebras of $\mathfrak{g}$. The sets $\Delta_\pm$ have a finite number of elements. This fact and the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ show that $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are nilpotent Lie algebras. It is shown that they are maximal nilpotent subalgebras of $\mathfrak{g}$ and that every nilpotent subalgebra of $\mathfrak{g}$ can be transformed by an inner automorphism into a subalgebra of $\mathfrak{n}_+$ and also into a subalgebra of $\mathfrak{n}_-$. The subalgebras $\mathfrak{h}+\mathfrak{n}_+$ and $\mathfrak{h}+\mathfrak{n}_-$ are maximal solvable Lie subalgebras of $\mathfrak{g}$.

It is clear that $\mathfrak{g}$ is the direct sum of the subalgebras $\mathfrak{n}_+$, $\mathfrak{n}_-$ and $\mathfrak{h}$: $\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_-$. The elements E_α, $\alpha \in \Delta_+$, from (18), Section 15.1.2, form a basis of $\mathfrak{n}_+$.

Let $\alpha_1, \alpha_2 \ldots , \alpha_\ell$ be the set of simple roots of the algebra $\mathfrak{g}$ and let $E_{\alpha_1}, \ldots ,$ E_{α_ℓ} be basis elements of the corresponding root subspaces $\mathfrak{g}_{\alpha_1}, \ldots , \mathfrak{g}_{\alpha_\ell}$. By using the commutator $[\cdot\ ,\cdot]$, given in $\mathfrak{g}$, we can obtain from $E_{\alpha_1}, \ldots , E_{\alpha_\ell}$ elements E_α corresponding to positive roots of $\mathfrak{g}$. It is possible to show (see, for example, [119]) that in this way we receive all root elements E_α from $\mathfrak{n}_+$. In other words, the elements $E_{\alpha_1}, \ldots , E_{\alpha_\ell}$ generate the Lie algebra $\mathfrak{n}_+$. Analogously, the elements $E_{-\alpha_1}, \ldots , E_{-\alpha_\ell}$ generate the Lie algebra $\mathfrak{n}_-$. Besides, as we saw,

$$[E_{\alpha_i}, E_{-\alpha_i}] = H_{\alpha_i}, \quad i = 1, 2, \ldots , \ell, \tag{1}$$

and ℓ elements $H_{\alpha_i} \equiv H'_i$ are a basis of the Cartan subalgebra $\mathfrak{h}$. Therefore, the root elements $E_{\alpha_1}, \ldots , E_{\alpha_\ell}, E_{-\alpha_1}, \ldots , E_{-\alpha_\ell}$, corresponding to the simple roots $\alpha_1, \ldots , \alpha_\ell$, generate the Lie algebra $\mathfrak{g}$.

If $\alpha_1, \ldots , \alpha_\ell$ are simple roots of the algebra $\mathfrak{g}$, then we introduce the numbers

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad i, j = 1, 2, \ldots , \ell.$$

The matrix $(a_{ij})_{i,j=1}^{\ell}$ is called the *Cartan matrix* of the Lie algebra $\mathfrak{g}$. It is proved (see, for example, [119]) that the numbers a_{ij}, $i \neq j$, are either non-positive integers or vanish. Moreover, $a_{ij} = 0$ if $a_{ji} = 0$. Besides, $a_{ii} = 2$, $i = 1, 2, \ldots , \ell$. The Cartan matrices of complex semisimple Lie algebras are non-singular.

The Cartan matrix and the generating elements $E_{\alpha_1}, \ldots , E_{\alpha_\ell}, E_{-\alpha_1}, \ldots ,$ $E_{-\alpha_\ell}$ completely define (up to an isomorphism) the Lie algebra $\mathfrak{g}$. Really, the following theorem is valid.

Theorem 1. *Let $E_i \equiv E_{\alpha_i}$, $F_i \equiv E_{-\alpha_i}$, $i = 1, 2, \ldots , \ell$, be the root elements corresponding to simple roots $\alpha_1, \ldots , \alpha_\ell$ of a complex semisimple Lie algebra $\mathfrak{g}$, where ℓ is the rank of $\mathfrak{g}$. Let $H_i \equiv H_{\alpha_i}$, $i = 1, 2, \ldots , \ell$, be the elements (1) of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $(a_{ij})_{i,j=1}^{\ell}$ be the Cartan matrix of $\mathfrak{g}$. Then*

$$[H_i, H_j] = 0, \quad [E_i, F_j] = 0, \quad i \neq j, \quad [E_i, F_i] = H_i, \tag{2}$$

$$[H_i, E_j] = a_{ij} E_j, \quad [H_i, E_j] = -a_{ij} F_j, \tag{3}$$

$$(ad\ E_i)^{-a_{ij}+1} E_j = 0, \tag{4}$$

$$(ad\ F_i)^{-a_{ij}+1} F_j = 0. \tag{5}$$

The elements E_i, F_i, H_i, $i = 1, 2, \ldots , \ell$, and these relations uniquely define (up to an isomorphism) the Lie algebra $\mathfrak{g}$.

Proof of this theorem can be found in [273].

Using Theorem 1 it is easy to prove that there is the automorphism ω of the Lie algebra $\mathfrak{g}$ such that

$$\omega H_i = -H_i, \quad \omega E_i = -F_i, \quad \omega F_i = -E_i. \tag{6}$$

We have $\omega^2 = 1$.

15.1.4. Restricted roots and root subspaces. We have defined roots and the corresponding root subspaces for complex semisimple Lie algebras. Now let $\mathfrak{g}$ be a real noncompact semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its Cartan decomposition (see Section 1.2.3). We can separate a maximal commutative subalgebra in the space $\mathfrak{p}$. It is possible to show (see, for example, reference [21] of the first volume) that all such subalgebras are of the same dimension (which is called the *real rank* of $\mathfrak{g}$). Moreover, they are conjugate by inner automorphisms of $\mathfrak{g}$.

Let us fix a maximal commutative subalgebra in $\mathfrak{p}$ and denote it by $\mathfrak{a}$. The algebra $\mathfrak{g}$ decomposes into the direct sum of eigenspaces of the operators ad H, $H \in \mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\gamma} \mathfrak{g}_{\gamma}, \tag{1}$$

where

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid (\mathrm{ad}\, H)X = 0 \quad \text{for all} \quad H \in \mathfrak{a}\},$$

$$\mathfrak{g}_{\gamma} = \{X \in \mathfrak{g} \mid (\mathrm{ad}\, H)X = \gamma(H)X \quad \text{for all} \quad H \in \mathfrak{a}\}.$$

The linear forms γ on $\mathfrak{a}$ are called *restricted roots* of the pair $(\mathfrak{g}, \mathfrak{a})$. The eigenspaces $\mathfrak{g}_{\gamma}$ are said to be *root subspaces* of $\mathfrak{g}$. Root subspaces $\mathfrak{g}_{\gamma}$ and $\mathfrak{g}_{\beta}$, $\gamma \neq -\beta$, are orthogonal with respect to the Killing form.

Properties of restricted roots are analogous to those of roots of complex Lie algebras. The essential difference is that root subspaces, corresponding to restricted roots, can be many-dimensional. The number dim $\mathfrak{g}_{\gamma}$ is called *multiplicity* of the restricted root γ. For some of restricted roots γ there may exist the multiple roots -2γ, $-\gamma$, γ, 2γ (or $-\gamma$, $-\gamma/2$, $\gamma/2$, γ).

To each restricted root γ there corresponds the reflection

$$S_{\gamma}: \varphi \to \varphi - \frac{2(\varphi, \gamma)}{(\gamma, \gamma)}\gamma$$

of the space $\mathfrak{a}'$ dual to $\mathfrak{a}$. The reflections S_{γ} generate the Weyl group $W(\mathfrak{a})$ of the pair $(\mathfrak{g}, \mathfrak{a})$. Properties of the group $W(\mathfrak{a})$ are the same as those of Weyl groups of complex semisimple Lie algebras.

As in the complex case, we fix a basis of $\mathfrak{a}$ and define positive and negative restricted roots, as well as the system of simple roots. The system of simple roots possesses the same properties as in the case of simple roots of a complex semisimple Lie algebra. Moreover, the system of simple roots of the Lie algebra $\mathfrak{g}$ coincides with that of some complex semisimple Lie algebra. It is clear that their Weyl groups coincide too.

In order to obtain all restricted roots from the set of simple roots of the pair $(\mathfrak{g}, \mathfrak{a})$ we have to act upon them by elements of the Weyl group $W(\mathfrak{a})$ and then to take doubled roots if it is necessary. (Simple roots give no information for which roots γ there exist roots 2γ.) This procedure does not give multiplicities of roots.

In order to obtain full information about roots we have to indicate multiplicities of all simple roots γ and of the doubled roots 2γ. (Multiplicities of 2γ is equal to 0 if 2γ is not a root). The Weyl group $W(\mathfrak{a})$ retains invariant multiplicities of roots.

As in the complex case, the Weyl group divides the space $\mathfrak{a}'$ (and the subalgebra $\mathfrak{a}$) into Weyl chambers. There is one chamber which is called *dominant* one.

The set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ may be obtained from the set of roots of the complex Lie algebra $\mathfrak{g}_c$ which is a complexification of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_c$ containing the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. We choose an ordered basis of the subalgebra $\mathfrak{h}$ containing the basis $H_1, \dots, H_\ell$ of $\mathfrak{a}$. Moreover, we consider that basis elements $H_1, \dots, H_\ell$ of $\mathfrak{a}$ are placed before other basis elements of $\mathfrak{h}$. Let Δ_+ be the set of positive roots of the Lie algebra $\mathfrak{g}_c$ with respect to this basis of $\mathfrak{h}$. Let Δ_+^- be the subset of roots which vanish on $\mathfrak{a}$ and let Δ_+^+ be the subset of other roots from Δ_+. Considering the roots α of Δ_+^+ on the subalgebra $\mathfrak{a}$ we obtain the set of all positive restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ with their multiplicities.

Decomposition (1) can also be obtained from decomposition of $\mathfrak{g}_c$ into a sum of root subspaces $(\mathfrak{g}_c)_\alpha$. Namely, $\mathfrak{g}_0$ coincides with

$$\mathfrak{g} \bigcap \left(\mathfrak{h} + \sum_{\alpha \in \Delta_+^-} (\mathfrak{g}_c)_\alpha + \sum_{\alpha \in \Delta_+^-} (\mathfrak{g}_c)_{-\alpha} \right).$$

The subspaces $\mathfrak{g}_\gamma$ coincides with an intersection of $\mathfrak{g}$ with a sum of those subspaces $(\mathfrak{g}_c)_\alpha$ for which restriction of α onto $\mathfrak{a}$ is equal to γ.

Example 1. From decomposition (1) of Section 15.1.2 for the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ we obtain the decomposition

$$\mathfrak{sl}(n, \mathbf{R}) = \mathfrak{h}_R + \sum_{i \neq j} \mathbf{R} E_{ij}$$

of $\mathfrak{sl}(n, \mathbf{R})$ into a sum of root subspaces.

Example 2. Let $\mathfrak{g}_r$ be a complex semisimple Lie algebra $\mathfrak{g}$, considered as a real Lie algebra. Then $\mathfrak{g}_r$ is a real semisimple Lie algebra. A real rank of $\mathfrak{g}_r$ is equal to a rank of $\mathfrak{g}$. A system of restricted roots of $\mathfrak{g}_r$ coincides with a root system of $\mathfrak{g}$. Multiplicities of all restricted roots are equal to 2. The Weyl groups of $\mathfrak{g}_r$ and $\mathfrak{g}$ also coincide.

15.1.5. Real simple Lie groups and algebras. There are the following classes of real simple Lie algebras:

(a) compact real forms of complex simple Lie algebras;

(b) noncompact real forms of complex simple Lie algebras;

(c) simple complex Lie algebras, considered as real Lie algebras.

Every real simple Lie algebra is isomorphic to an algebra of one of these classes. In order to obtain all (up to a local isomorphism) real simple Lie groups, we have to find for every of indicated real simple Lie algebras its Lie group. We have described all real forms of classical complex simple Lie groups in Section 1.1.7.

The systems of restricted roots of classical real simple noncompact Lie algebras $\mathfrak{g}$ are described in Table 15.1. Simple roots α_i of the complex algebras $\mathfrak{g}_c$, simple restricted roots λ_j of the pairs $(\mathfrak{g}, \mathfrak{a})$ and multiplicities $m(\lambda_j)$, $m(2\lambda_j)$ of the roots λ_j $2\lambda_j$ are given there. Simple roots of the Lie algebras $\mathfrak{g}_c$ from Δ_+^- (from Δ_+^+) are denoted by shaded (by non-shaded) circles. Simple roots α and α', coinciding on $\mathfrak{a}$, are connected by two-sided arrows. Lengths of roots are determined in the following way. If two roots are connected by one straight line, then the angle between them is equal to $2\pi/3$ and they are of the same length. If two roots are connected by doubled straight line, then the angle between them is $3\pi/4$ and squared length of one of the roots is doubled squared length of the other one. The lines connecting roots of different lengths are equipped with arrow which is directed to the shorter root.

15.1.6. The Iwasawa decomposition. Let $\mathfrak{g}$ be a linear real semisimple noncompact Lie algebra and let $\mathfrak{k}$ be its maximal compact subalgebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ and let $\mathfrak{a}$ be a maximal commutative Lie subalgebra in $\mathfrak{p}$. If Δ_+ is a system of positive restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$, then we set

$$\mathfrak{n} = \sum_{\gamma\in\Delta_+} \mathfrak{g}_\gamma, \quad \mathfrak{n}_- = \sum_{\gamma\in\Delta_+} \mathfrak{g}_{-\gamma}, \tag{1}$$

where $\mathfrak{g}_\gamma$ are the root subspaces of $\mathfrak{g}$ corresponding to the restricted roots γ. It is proved that $\mathfrak{n}$ and $\mathfrak{n}_-$ are maximal nilpotent Lie subalgebras of $\mathfrak{g}$. The decompositions

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{k} + \mathfrak{n}, \quad \mathfrak{g} = \mathfrak{a} + \mathfrak{k} + \mathfrak{n}_- \tag{2}$$

are valid (see, for example, reference [21] of the first volume), where sums are direct. Besides, $\mathfrak{a} + \mathfrak{n}$ is a maximal solvable Lie subalgebra in $\mathfrak{g}$. Formulas (2) are called *Iwasawa decompositions of the Lie algebra* $\mathfrak{g}$.

Let G be the linear connected Lie group for which $\mathfrak{g}$ is its Lie algebra, and let K, A, $N_+ \equiv N$, N_- be the analytic subgroups in G with the Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{n}_-$ respectively. These subgroups are closed in G. The subgroups A, N, N_- are simply connected and K is a maximal compact subgroup in G. Any element $g \in G$ can be uniquely decomposed as $g = khn$, where $k \in K$, $h \in A$, $n \in N$, and the mapping $(k, h, n) \to khn$ is an analytic diffeomorphism of the manifold $K \times A \times N$ onto G (see reference [21] of the first volume). The product $g = khn$ is called the *Iwasawa decomposition of an element* $g \in G$, and $G = KAN$ is said to be the *Iwasawa decomposition of the group* G.

The set $S = AN$ is a subgroup of G. It is a maximal solvable subgroup of the group G. Since

$$AN = (AN)^{-1} = N^{-1}A^{-1} = NA,$$

then instead of $G = KAN$ we can consider the decomposition $G = KNA$. We also have $G = NAK = ANK$. Besides, the decompositions

$$G = KAN_{-} = KN_{-}A = N_{-}AK = AN_{-}K$$

are valid for G. They correspond to negative roots of the pair $(\mathfrak{g}, \mathfrak{a})$.

Example 1. Let us consider $SL(n, \mathbb{C})$ as a real simple Lie group. Then $K = SU(n)$, $A = \tilde{A}(n)$, $N = N_{+}(n, \mathbb{C})$. For the group $SL(n, \mathbf{R})$ we have $K = SO(n)$, $A = \tilde{A}(n)$, $N = N_{+}(n, \mathbf{R})$.

Example 2. Let us realize $SO(n, \mathbb{C})$ as the group of matrices conserving the form $z_1 w_n + \ldots + z_n w_1$. Then the subgroups K, A and N are obtained as intersections of $SO(n, \mathbb{C})$ with $SU(n)$, $A = \tilde{A}(n)$ and $N_{+}(n, \mathbb{C})$ respectively. We recommend to the reader to describe these subgroups when $SO(n, \mathbb{C})$ is realized as the group of matrices conserving the form $z_1 w_1 + \ldots + z_n w_n$.

Let θ be the Cartan involutive automorphism of the Lie algebra $\mathfrak{g}$ related to the Cartan decomposition[3] $G = K\mathcal{P}$ of the group G considered in Section 1.2.3. The involutive automorphism $\tilde{\theta}$ of the group G corresponds to θ. With every element $s \in AN$ we associate the element $\psi(s) = \tilde{\theta}(s^{-1})s$. Then $\tilde{\theta}(s^{-1})s \in \mathcal{P}$ and ψ is a diffeomorphism of the group AN onto $\mathcal{P}$.

Example 3. If $G = SL(n, \mathbf{R})$ then the diffeomorphism ψ transforms triangular matrices $n \in N$ into symmetric matrices $n^t n$, where t denotes the transposition.

Remark 1. The subgroups N and AN in the Iwasawa decomposition $G = KAN$ are simply connected. Therefore, the group $G \sim K \times A \times N$ is simply connected if and only if the subgroup K has this property.

Remark 2. For reductive Lie groups and algebras the Iwasawa decomposition is defined in the following way. Let G' be a reductive real noncompact Lie group with Lie algebra $\mathfrak{g}'$ and let $G' = G \times V \times C$ be the decomposition of G' into the direct product of the semisimple, of the vector, and of the commutative compact subgroups. The corresponding decomposition of the Lie algebra $\mathfrak{g}'$ is denoted as $\mathfrak{g}' = \mathfrak{g} + \mathfrak{v} + \mathfrak{c}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and $G = KAN$ be the Iwasawa decompositions of $\mathfrak{g}$ and of G respectively. Then the Iwasawa decompositions of $\mathfrak{g}'$ and of G' are given by the formulas

$$\mathfrak{g}' = (\mathfrak{k} + \mathfrak{c}) + (\mathfrak{a} + \mathfrak{v}) + \mathfrak{n}, \quad G' = (KC)(AV)N.$$

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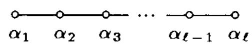
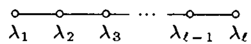
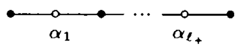
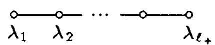
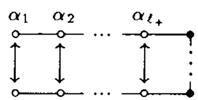
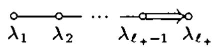
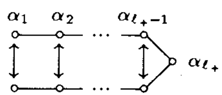
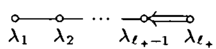
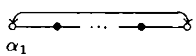
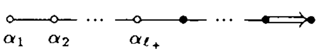
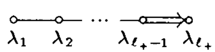
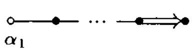
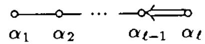
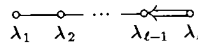
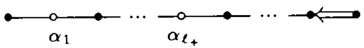
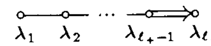
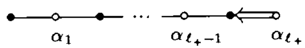
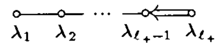
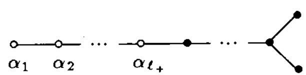
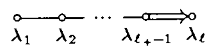
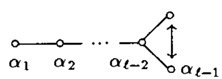
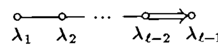
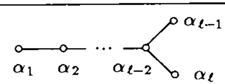
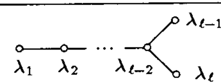
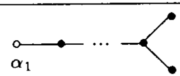
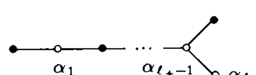
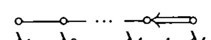
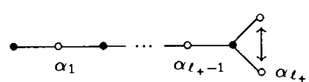
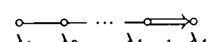
[3] See the footnote on p. 138.

Table 15.1

Real simple Lie algebra $\mathfrak{g}$	Simple roots of the complex Lie algebra $\mathfrak{g}_c$	Simple restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$	$m(\lambda_i)$	$m(2\lambda_i)$	The corresponding real simple Lie group G
AI			1	0	$SL(\ell+1, \mathbb{R})$
AII		 $(\ell = 2\ell_+ + 1,\ \ell_+ \geq 0)$	4	0	$SU^*(2\ell_+ + 2)$
AIII		 $(2 \leq \ell_+ \leq \ell/2)$	2 if $i < \ell_+$ $2(\ell - 2\ell_+ + 1)$ if $i = \ell_+$	0 1	$SU(\ell_+, \ell - \ell_+ + 1)$
		 $(\ell = 2\ell_+ - 1,\ \ell \geq 2)$	2 if $i < \ell_+$ 1 if $i = \ell_+$	0 0	$SU(\ell_+, \ell_+)$
AIV			$2(\ell - 1)$	1	$SU(1, \ell)$
BI		 $(\ell \geq 2,\ 2 \leq \ell_+ \leq \ell)$	1 if $i < \ell_+$ $2(\ell - \ell_+) + 1$ if $i = \ell_+$	0 0	$SO_0(\ell_+, 2\ell - \ell_+ - 1)$
BII			$2\ell - 1$	0	$SO_0(1, 2\ell)$

CI			1	0	$Sp(\ell, \mathbb{R})$
CII		 $(\ell \geq 3,\ 1 \leq \ell_+ \leq \frac{\ell-1}{2})$	4 if $i < \ell_+$ $4(\ell - 2\ell_+)$ if $i = \ell_+$	0 3	$Sp(\ell_+, \ell - \ell_+)$
		 $(\ell = 2\ell_+,\ \ell_+ \geq 2)$	4 if $i < \ell_+$ 3 if $i = \ell_+$	0 0	$Sp(\ell_+, \ell_+)$
DI		 $(\ell \geq 4,\ 2 \leq \ell_+ \leq \ell - 2)$	1 if $i < \ell_+$ $2(\ell - \ell_+)$ if $i = \ell_+$	0 0	$SO_0(\ell_+, 2\ell - \ell_+)$
			1 if $i < \ell - 1$ 2 if $i = \ell - 1$	0 0	$SO_0(\ell - 1, \ell + 1)$
			1	0	$SO_0(\ell, \ell)$
DII			$2(\ell - 1)$	0	$SO_0(1, 2\ell - 1)$
DIII		 $(\ell = 2\ell_+,\ \ell_+ \geq 2)$	4 if $i < \ell_+$ 1 if $i = \ell_+$	0 0	$SO^*(4\ell_+)$
		 $(\ell = 2\ell_+ + 1,\ \ell_+ \geq 2)$	4 if $i < \ell_+$ 4 if $i = \ell_+$	0 1	$SO^*(4\ell_+ + 2)$

15.1.7. The Gauss decomposition. Let $\mathfrak{m} = \mathfrak{k} \cap \mathfrak{z}(\mathfrak{a})$ and $M = K \cap Z(A)$, where $\mathfrak{z}(\mathfrak{a})$ and $Z(A)$ are the centralizers of $\mathfrak{a}$ in $\mathfrak{g}$ and of A in G respectively. Then $\mathfrak{m}$ is a Lie subalgebra in $\mathfrak{k}$ and M is a closed subgroup in K. It is possible to prove that $\mathfrak{m}$ is a reductive Lie algebra. If

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\gamma \in \Delta} \mathfrak{g}_\gamma \tag{1}$$

is the root decomposition of the Lie algebra $\mathfrak{g}$, then $\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a}$. We also have

$$\mathfrak{n} = \sum_{\gamma \in \Delta_+} \mathfrak{g}_\gamma, \quad \mathfrak{n}_- = \sum_{\gamma \in \Delta_+} \mathfrak{g}_{-\gamma},$$

where $\mathfrak{n}$ and $\mathfrak{n}_-$ are the nilpotent Lie subalgebras introduced in Section 15.1.6. Due to decomposition (1) we have

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} + \mathfrak{n}_-, \tag{2}$$

where the sum is direct. This formula is called the *Gauss decomposition of the Lie algebra* $\mathfrak{g}$. The global analogue of this decomposition is valid. *If G is a linear semisimple real connected Lie group, then the set $NAMN_-$ is open and everywhere dense in G.* Therefore, $G = \overline{NAMN_-}$, where the bar denotes the closure. This means that the decomposition $g = nhmn_-$, $n \in N$, $h \in A$, $m \in M$, $n_- \in N_-$, is valid for almost all elements from G. This decomposition is said to be the *Gauss decomposition of the group G*.

Example 1. For the simple Lie group $G = SL(n, \mathbb{C})$ we have $N = N_+(n, \mathbb{C})$, $N_- = N_-(n, \mathbb{C})$, $A = \tilde{A}(n)$. The subgroup M coincides with the set of all diagonal unimodular matrices with diagonal elements a_{ii} such that $|a_{ii}| = 1$. Therefore, the Gauss decomposition of the group $SL(n, \mathbb{C})$ is of the form

$$G = N_+(n, \mathbb{C}) D N_-(n, \mathbb{C}), \qquad \text{where} \qquad D = AM = \tilde{D}(n, \mathbb{C}).$$

This decomposition means that almost every unimodular matrix is a product of upper triangular and lower triangular matrices. This decomposition was used by Gauss for solving systems of linear equations. This is why it was called "the Gauss decomposition".

For complex semisimple Lie groups the subgroups AM are commutative. In this case G may be realized as a group of matrices, and N and N_- can be taken to be subgroups of $N_+(n, \mathbb{C})$ and $N_-(n, \mathbb{C})$ respectively. Then AM is a subgroup of $\tilde{D}(n, \mathbb{C})$. A real form G_r of a complex semisimple Lie group G is said to be *normal* if it possesses these properties. For example, the groups $SL(n, \mathbf{R})$, $SO_0(\ell, \ell)$, $SO_0(\ell + 1, \ell)$, $Sp(n, \mathbf{R})$ are normal real forms of the complex groups $SL(n, \mathbb{C})$, $SO(2\ell, \mathbb{C})$, $SO(2\ell + 1, \mathbb{C})$, $Sp(n, \mathbb{C})$ respectively.

Let us show that the subset $P = MAN$ of the real noncompact connected semisimple Lie group G is a subgroup in $G = KAN$. Really, MA is a direct product of the subgroups M and A. Thus, MA is a subgroup. Elements of MA commute with elements of A. It follows from this assertion that for $X \in \mathfrak{g}_\gamma \subset \mathfrak{n}$, $m \in M$, $h \in A$ we have $\mathrm{Ad}\,(mh)X \in \mathfrak{g}_\gamma$. Then $mhNh^{-1}m^{-1} \subset N$. Consequently, $mhN = Nmh$. In this reason

$$(MAN)(MAN) = (MA)(MA)NN = MAN.$$

Since $(MAN)^{-1} = N^{-1}A^{-1}M^{-1} = NAM = MAN$, then P is a subgroup of G. One can show that this subgroup is closed in G. It is called a *minimal parabolic subgroup* of G. One can prove that *all minimal parabolic subgroups are conjugate by inner automorphism of G.*

Example 2. Let $G = SL(n, \mathbf{R})$. Then A consists of diagonal matrices with positive elements on the main diagonal. The subgroup M consists of matrices of the form $m = \mathrm{diag}\,(\pm 1, \pm 1, \ldots, \pm 1)$ with all combinations of signs such that $\det m = 1$. Therefore, the set of all upper real triangular unimodular matrices is a minimal parabolic subgroup in $SL(n, \mathbf{R})$. It is evident that this subgroup is not connected. It consists of 2^{n-1} connected parts.

Let G' be a reductive connected Lie group and let $G' = G \times V \times C$ be the decomposition of G' into the direct product of the semisimple, of the vector, and of the commutative compact subgroups. If $P = MAN$ is a minimal parabolic subgroup of the group $G = KAN$, then $P' = (MC)(AV)N$ is a minimal parabolic subgroup of G'. It is clear that $M'' = MC$ is the centralizer of the subgroup $A' = AV$ in the maximal compact subgroup $K' = KC$ of G'. The Gauss decomposition is also defined for reductive connected Lie groups.

15.1.8. The Bruhat decomposition. We set $M' = K \cap N(A)$, where $N(A)$ is the normalizer of the subgroup A. The formula $m' \cdot h \equiv m'h(m')^{-1}$ defines the action of M' on A. The kernel of this action coincides with M. Therefore, M is a normal subgroup in M' and the action of the quotient group M'/M is isomorphic to the Weyl group $W(\mathfrak{a})$ of the pair $(\mathfrak{g}, \mathfrak{a})$. The action of M'/M on A defines its action on $\mathfrak{a}$ and, consequently, on $\mathfrak{a}'$ (the dual space for $\mathfrak{a}$). This action on $\mathfrak{a}$ and on $\mathfrak{a}'$ coincides with the action of the Weyl group $W(\mathfrak{a})$ on these spaces. In other words, for every element w from $W(\mathfrak{a})$ there exists the element $m_w \in M'/M$ which acts on $\mathfrak{a}$ and on $\mathfrak{a}'$ in the same way as w does. Below we identify $W(\mathfrak{a})$ and M'/M. Bruhat proved the following statement: *The group G is the union of the sets Pw_jP, where P is a fixed minimal parabolic subgroup and w_j runs over representatives of the cosets of M'/M. If $j \neq i$, then $Pw_jP \cap Pw_iP$ is the empty set.* The union

$$g = \bigcup_{w \in W(\mathfrak{a})} PwP$$

is called the *Bruhat decomposition of the group G.*

Example 1. Let $G = SL(n, \mathbb{C})$. Then $A = \tilde{A}(n)$ and the subgroup M consists of unitary unimodular diagonal matrices. The subgroup M' is generated by the subgroup M and by the matrices

$$w_{i,i+1} = \text{diag}\,(1, \ldots, 1, s, 1, \ldots, 1),$$

where the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is placed on i-th and $i+1$-th rows. It is easy to verify that the matrix $w_{i,i+1} \in M'$ corresponds to the reflection with respect to the hyperplane orthogonal to the simple root $\alpha_{i,i+1}$ (see Example 9 of Section 15.1.2). Therefore, the matrices $w_{i,i+1}$, $i = 1, 2, \ldots, n-1$, generate the Weyl group and are representatives of the cosets of M'/M. Since the matrix $w_{i,i+1}$ permutes the elements a_i and a_{i+1} of the matrices

$$H = \text{diag}\,(a_i, a_2, \ldots, a_n) \in \mathfrak{a},$$

then the Weyl group $W(\mathfrak{a})$ coincides with the symmetric group S_n of permutations of diagonal elements of matrices $H \in \mathfrak{a}$. Therefore, the group $W(\mathfrak{a}) = M'/M$ has $n!$ elements $w_1, w_2, \ldots, w_{n!}$. The set P of all upper triangular matrices of $SL(n, \mathbb{C})$ is a minimal parabolic subgroup in $SL(n, \mathbb{C})$. Consequently, every matrix from $SL(n, \mathbb{C})$ is represented in the form $p_1 w_j p_2$, $p_1, p_2 \in P$, $j = 1, 2, \ldots, n!$.

The notion of length for elements of the Weyl group $W(\mathfrak{a})$ can be introduced. The identity element is of length 0. If $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ is the set of simple restricted roots, then the reflections $S_{\gamma_i} \equiv S_i$, $i = 1, 2, \ldots, \ell$, are of length 1. Every element $w \in W(\mathfrak{a})$ is representable in the form of a product $w = S_{i_1} S_{i_2} \ldots S_{i_r}$. This representation is not unique. There is a representation $w = S_{i_1} S_{i_2} \ldots S_{i_r}$ with minimal $r \equiv \ell(w)$. This number is said to be the *length* of the element $w \in W(\mathfrak{a})$. It is clear that $\ell(w_1 w_2) \le \ell(w_1) + \ell(w_2)$.

There is the single element w_0 in $W(\mathfrak{a})$ which is of maximal length. We have $w_0 \mathfrak{a}^+ = \mathfrak{a}^-$, where $\mathfrak{a}^+$ is the dominant Weyl chamber and $\mathfrak{a}^- = \{H \in \mathfrak{a} \mid \gamma(H) < 0 \quad \text{for all} \quad \gamma \in \Delta_+\}$. Therefore, $w_0 \Delta_+ = -\Delta_+$ and $w_0^{-1} = w_0$. Replacing reflections by their representatives in M' we have $m_0 N m_0^{-1} = N_-$ if m_0 corresponds to w_0.

Since for $w \in W(\mathfrak{a})$ we have $w(MA)w^{-1} \subset MA$, then every component of the Bruhat decomposition can be represented in the form

$$PwP = MANwMAN = MANwN = NwMAN.$$

Thus, the Bruhat decomposition can be written as

$$G = \bigcup_{w \in W(\mathfrak{a})} PwN = \bigcup_{w \in W(\mathfrak{a})} NwP.$$

Therefore,

$$G = \bigcup_{w \in W(\mathfrak{a})} PwNw_0.$$

Let us consider the component Pw_0Nw_0. Since $w_0Nw_0^{-1} = N_-$, then it coincides with $PN_- = NAMN_-$. This expression is given by the Gauss decomposition of the group G. In this reason PN_- (and consequently, Pw_0N) has the dimension coinciding with the dimension of the group G, and PN_- is everywhere dense in G. Thus, there is only one component, namely Pw_0P, in the Bruhat decomposition which is of the dimension $\dim G$.

15.1.9. The Cartan decomposition. In Section 1.2.3 the decomposition[4] $G = K\mathcal{P}$ of semisimple connected Lie group was considered. It is possible to show that every element $p \in \mathcal{P}$ is represented in the form $p = khk^{-1}$, $k \in K$, $h \in A$. Therefore,

$$g = k_0p = k_0khk^{-1}, \quad k_0, k \in K, \quad h \in A.$$

Thus, every element $g \in G$ can be represented in the form $g = k_1hk_2$, where $k_1, k_2 \in K$, $h \in A$. This is the *Cartan decomposition* of elements of the group G. We have $G = KAK$.

The representation $g = k_1hk_2$ of elements $g \in G$ is not unique. Really, let $k_1 = k_1'm$, $k_1 = m^{-1}k_2'$, where $m \in M$. Then

$$g = k_1hk_2 = k_1'mhm^{-1}k_2' = k_1'hk_2'$$

and we obtain other representation $g = k_1'hk_2'$.

The transformation $h \to m'h(m')^{-1}$, $m' \in M'$, retains invariant the subgroup A. We construct the set $A^+ = \exp \mathfrak{a}^+$, where $\mathfrak{a}^+$ is the dominant Weyl chamber. Then $A^+ \subset A$. For elements $w \in W(\mathfrak{a})$, $w \neq 1$, we have $w \cdot h \overline{\in} A^+$ if $h \in A^+$. We construct the set $G' = KA^+K$. For elements $g \in G'$ the decomposition $g = k_1hk_2$, $k_1, k_2 \in K$, $h \in A^+$, is unique up to the replacement of k_1 and k_2 by k_1m and $m^{-1}k$, where $m \in M$. Moreover, $G = K\overline{A^+}K$, where $\overline{A^+}$ is the closure of A^+ in G. In other words, G' is everywhere dense in G.

Let G_c be the complexification of the group G and let G_k be its compact real form. Along with $A^+ = \exp \mathfrak{a}^+ \subset G$ we consider the set $A_k^+ = \exp i\mathfrak{a}^+ \subset G_k$, where $i = \sqrt{-1}$. We have $K \subset G_k$. One can show that for every element $g \in G_k$ the decomposition $g = k_1hk_2$, $k_1, k_2 \in K$, $h \in \overline{A_k^+}$, is valid. If $K \cap A_k = \{e\}$, then for elements from $G_k' = KA_k^+K$ this decomposition is unique up to the replacement of k_1 and k_2 by k_1m and $m^{-1}k_2$, where $m \in M$. If $K \cap A_k = J \neq \{e\}$, then J is a finite group. Then the transformation $h \to bhb^{-1}$, $b \in W(\mathfrak{a})J$, of the group A_k divides A_k into $r|W(\mathfrak{a})|$ non-overlapping parts, where r is the order of the group J. If we take instead of A_k^+ in the set $G_k' = KA_k^+K$ one of these parts, then we obtain

[4] See footnote on p. 138.

everywhere dense set of G_k and h is uniquely determined in the decomposition $g = k_1 h k_2$. In any case we have $G_k = KA_kK$.

The Cartan decompositions $G = KAK$ and $G_k = KA_kK$ establish close relations between these groups. The subgroups $A = \exp \mathfrak{a}$ and $A_k = \exp i\mathfrak{a}$ are contained in the subgroup $A_c = \exp \mathfrak{a}_c \subset G_c$, where $\mathfrak{a}_c$ is the complexification of the subalgebra $\mathfrak{a}$. Therefore, A and A_k can be parametrized by real and imaginary parts of complex parameters which parametrize elements of the subgroup A_c. This allows us to pass from the group $G = KAK$ to the group $G_k = KA_kK$ inside of G_c by the replacement of real parameters of A by pure imaginary parameters.

Example 1. Let $G_c = SL(n, \mathbb{C})$, $G = SL(n, \mathbf{R})$, $G_k = SU(n)$. In this case A^+ consists of diagonal matrices

$$\mathrm{diag}(e^{t_1}, \ldots, e^{t_n},), \quad t_1 > t_2 > \ldots > t_n, \quad \sum_{k=1}^{n} t_k = 0.$$

The set $\overline{A^+}$ consists of matrices of the same form with $t_1 \geq t_2 \geq \ldots \geq t_n$. Since $K = SO(n)$, then $G = SO(N)\overline{A^+}SO(N)$. The subset $A_k^+ \subset SU(n)$ consists of diagonal unimodular matrices

$$\mathrm{diag}(e^{it_1}, \ldots, e^{it_n},), \quad 2\pi > t_1 > \ldots > t_n \geq 0, \quad \sum_{k=1}^{n} t_k = 0 \ (\mathrm{mod}\ 2\pi).$$

The set $\overline{A_k}$ consists of matrices of the same form with $2\pi \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$. We have $SU(n) = SO(n)\overline{A_k^+}SO(n)$.

Example 2. If $G = SL(n, \mathbb{C})$, then the Cartan decomposition is of the form $G = SU(n)\overline{A^+}SU(n)$, where A^+ is the same as in Example 1.

If $G = KAK$ and $G = \tilde{K}\tilde{A}\tilde{K}$ are two Cartan decompositions of the group G, then there is an inner automorphism φ of G such that $K = \varphi(\tilde{K})$ and $A = \varphi(\tilde{A})$. This assertion is not valid for the compact group G_k. The group G_k is dual by Cartan (see Section 1.2.3) to different noncompact real semisimple Lie groups. In this reason there can exist different decompositions $G_k = KA_kA$ of G_k which are not connected by an automorphism.

Example 3. Let $G = SL(n, \mathbf{R})$. Then $G_c = SL(n, \mathbb{C})$. The groups $SU(p, q)$, $p + q = n$, and $SU^*(2m)$, if $n = 2m$ is even, are also real forms of $SL(n, \mathbb{C})$. We have the Cartan decompositions

$$SL(n, \mathbf{R}) = SO(n)ASO(n),$$

$$SU(p, q) = S(U(p) \times U(q))A_1 S(U(p) \times U(q)),$$

$$SU^*(2m) = Sp(m)A_2 Sp(m).$$

Therefore, we obtain the following decompositions of the compact group $SU(n)$:

$$SU(n) = SO(n)A_k SO(n),$$

$$SU(n) = S(U(p)\times U(q))(A_1)_k S(U(p)\times U(q)), \quad p+q=n,$$

$$SU(n) = Sp(m)(A_2)_k Sp(m), \quad n=2m.$$

where A_k, $(A_1)_k$, $(A_2)_k$ are the corresponding commutative subgroups of $SU(n)$. We recommend to the reader to describe these subgroups.

The analogous decompositions are valid for the groups $SO(n)$ and $Sp(n)$. In particular, for any ℓ, $1\le \ell\le [n/2]$, every element $g\in SO(n)$ is representable in the form

$$g = k_1 a_\ell(\boldsymbol{\theta})k_2, \quad k_1,k_2\in K_\ell \equiv S(O(\ell)\times O(n-\ell)),$$

$$a_\ell(\boldsymbol{\theta}) = \begin{pmatrix} \cos\boldsymbol{\theta} & -\sin\boldsymbol{\theta} & 0\\ \sin\boldsymbol{\theta} & \cos\boldsymbol{\theta} & 0\\ 0 & 0 & I_{n-2\ell}\end{pmatrix},$$

where $\cos\boldsymbol{\theta} = \operatorname{diag}(\cos\theta_1,\dots,\cos\theta_\ell)$, $\sin\boldsymbol{\theta} = \operatorname{diag}(\sin\theta_1,\dots,\sin\theta_\ell)$, $0\le\theta_k\le\pi/2$. Along with the decompositions $K_\ell A_\ell K_\ell$ the groups $SO(n)$, $SU(n)$, $Sp(n)$ have decompositions with different subgroups on the left and on the right. If $0<\ell\le m\le[n/2]$, then in the group $SO(n)$ the set of elements $k_\ell a_{\ell m}(\theta)k_m$ is open and everywhere dense, where $k_\ell\in S(O(\ell)\times O(n-\ell))$, $k_m\in S(O(m)\times O(n-m))$ and

$$a_{\ell m} = \begin{pmatrix} \cos\boldsymbol{\theta} & 0 & \sin\boldsymbol{\theta} & 0\\ 0 & I_{m-\ell} & 0 & 0\\ -\sin\boldsymbol{\theta} & 0 & \cos\boldsymbol{\theta} & 0\\ 0 & 0 & 0 & I_{n-\ell-m}\end{pmatrix}, \quad 0\le\theta_k\le\frac{\pi}{2}. \tag{1}$$

In the decomposition $g=k_\ell a_{\ell m}k_m$ the matrices k_ℓ and k_m are defined uniquely up to the replacement of k_ℓ and k_m by $k_\ell m$ and $m^{-1}k_m$ respectively, where

$$m = \operatorname{diag}(I_\ell,\omega_{m-\ell},I_\ell,\omega_{n-m-\ell}), \quad \omega_k\in SO(k).$$

Really, let $g\in SO(n)$ and let $\mathbf{e}_1,\dots,\mathbf{e}_n$ be the corresponding orthonormal basis in $\mathbf{R}^n$. Let X_ℓ and X_m be the subspaces, spanned by the vectors $\mathbf{e}_1,\dots,\mathbf{e}_\ell$ and by the vectors $\mathbf{e}_1,\dots,\mathbf{e}_m$ respectively, and let $X_\ell^\perp$ and $X_m^\perp$ be their orthogonal compliments. We denote gX_ℓ, gX_m, $gX_\ell^\perp$, $gX_m^\perp$ by Y_ℓ, Y_m, $Y_\ell^\perp$, $Y_m^\perp$ respectively. Let P (correspondingly, P^*) be the operator of orthogonal projection of X_ℓ into Y_m (of Y_m into X_ℓ). Then P^*P is a Hermitian positive definite operator in X_ℓ. Its eigenvalues lie between 0 and 1, and can be denoted by $\lambda_k=\cos^2\theta_k$, $1\le k\le\ell$, $0\le\theta_k\le\pi/2$. The set Ω of $g\in SO(n)$, for which all θ_k are pairwise different and are not equal to 1, is open and everywhere dense in $SO(n)$. For $g\in\Omega$ eigenvectors $\boldsymbol{\xi}_k$, $1\le k\le\ell$, of the operator P^*P are uniquely defined. Let Q be the operator of orthogonal projection of $X_m^\perp$ into $Y_\ell^\perp$ and let $\boldsymbol{\xi}_{m+1},\dots,\boldsymbol{\xi}_{m+\ell}$ be such that $Q^*Q\boldsymbol{\xi}_k=$

$\lambda_k \xi_k$. We choose $m-\ell$ vectors $\xi_{\ell+1}, \dots, \xi_m$ from the subspace X_m, invariant with respect to PP^* and $n-m-\ell$ vectors $\xi_{\ell+m+1}, \dots, \xi_n$ from $X_m^{\perp}$, invariant with respect to Q^*Q. The vectors ξ_k, $1 \le k \le n$, form an orthonormal basis in $\mathbf{R}^n$ in which matrix of the linear transform g is of the form (1). The matrices k_ℓ and k_m give the transition to the initial basis.

The analogous decompositions of the groups $SU(n)$ and $Sp(n)$ are valid. We also have the similar decompositions for the groups $SO_0(p,q)$, $SU(p,q)$, $Sp(p,q)$. For $SO_0(p,q)$ the left and the right multipliers are of the form

$$S(O(p_1,q_1) \times O(p_2,q_2)) \quad \text{and} \quad S(O(p_3,q_3) \times O(p_4,q_4))$$

respectively, where $p_1+p_2 = p_3+p_4 = p$, $q_1+q_2 = q_3+q_4 = q$. In these cases additional multipliers appear in the decompositions which belong to the Weyl group W. We do not consider detailly these decompositions.

15.1.10. Decompositions of classical groups. Above we have mainly illustrated decompositions of semisimple Lie groups with the help of the group $SL(n,\mathbf{C})$. Here we shall give the corresponding decompositions for other classical complex Lie groups and for their real forms. In Example 2 of Section 15.1.6 we have constructed the subgroups K, A, N of the group $SO(n,\mathbf{C})$. In the analogous way these subgroups are constructed for the group $Sp(n,\mathbf{C})$. Namely, we realize this group as the group of complex matrices conserving the form $\mathbf{z}\begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}\mathbf{w}^t$, where s is the $n \times n$ matrix with units on the non-main diagonal and zeroes on other entries (that is, as the group of matrices (g_{ij}) such that

$$\sum_j \text{sign}\,(jq) g_{jp} g_{-j,-q} = \delta_{pq}).$$

Then the subgroups K, A, N, N_-, M of $Sp(n,\mathbf{C})$ are obtained by intersection of $Sp(n,\mathbf{C})$ with the corresponding subgroups of the group $SL(2n,\mathbf{C})$. By means of these subgroups Iwasawa, Bruhat and Cartan decompositions are constructed for the groups $SO(n,\mathbf{C})$ and $Sp(n,\mathbf{C})$. Besides, every element $g \in G$ can be decomposed as $g = n_- s_+ n_-^{-1}$, where $n_- \in N_-$ and $s_+ \in AN$. This decomposition is unique for matrices with pairwise different eigenvalues if instead AN the set A^+N is taken. Let us note that in the Cartan decomposition $g = k_1 h k_2$ the diagonal elements of the matrix h are square roots from the eigenvalues $\varepsilon_{-p}^2, \varepsilon_{-p+1}^2, \dots, \varepsilon_p^2$ of the matrix gg^* ordered in such way that $\varepsilon_k \varepsilon_{-k} = 1$, $1 \le k \le p$.

Example 1. The group $SL(n,\mathbf{C})$ has the noncompact real forms $SL(n,\mathbf{R})$, $SU(p,q)$, $p+q=n$, $pq \ne 0$, $p \le q$, and also the real form $SU^*(2p)$ if $n=2p$ is an even number. The involutive automorphism $\tilde{\theta}$ of the group $SU(p,q)$ is of the form

$$g \to I_{pq}(g^t)^{-1} I_{pq}, \quad I_{pq} = \text{diag}\,(I_p, -I_q),$$

where I_r is the identity $r \times r$ matrix. This automorphism separates the maximal compact subgroup $K = S(U(p) \times U(q))$ in $SU(p,q)$ and the manifold $\mathcal{P}$ consisting of Hermitian matrices of the form $g = \begin{pmatrix} X & Y \\ Y^* & T \end{pmatrix}$, where

$$X = X^*, \quad T = T^*, \quad X^2 - YY^* = I_p, \quad YY^* - T^2 = I_q, \quad XY = Y^*T.$$

If $p \leq q$, then the subgroup A consists of matrices of the form

$$\begin{pmatrix} \cosh \mathbf{t} & \sinh \mathbf{t} & 0 \\ \sinh \mathbf{t} & \cosh \mathbf{t} & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix},$$

where $\cosh \mathbf{t}$ and $\sinh \mathbf{t}$ are diagonal $p \times p$ matrices of the form

$$\cosh \mathbf{t} = \operatorname{diag}(\cosh t_1, \dots, \cosh t_p), \quad \sinh \mathbf{t} = (\sinh t_1, \dots, \sinh t_p).$$

The subgroup N consists of matrices of the form analogous to matrices (2) of Section 11.1.3. The subgroup N_- is obtained by application of the automorphism $\tilde{\theta}$ to N. The subgroup M consists of unimodular matrices of the form

$$\operatorname{diag}(e^{i\varphi_1}, \dots, e^{i\varphi_p}, e^{i\varphi_1}, \dots, e^{i\varphi_p}, g), \quad g \in U(q-p).$$

The Weyl group $W(\mathfrak{a})$ permutes the parameters $t_1, \dots, t_p$ and changes their signs. As we can see from Table 15.1, this Weyl group coincides with that of the complex simple Lie algebra B_p or C_p.

Example 2. The group $SU^*(n)$, $n = 2p$, can be realized as the subgroup of $SL(n,\mathbb{C})$ consisting of unimodular matrices $\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix}$, $X, Y \in \mathfrak{M}(p,\mathbb{C})$, or as the group $SL(p,\mathbf{H})$ of quaternion matrices. For the second realization, the subgroup K is the set of matrices $g \in SL(p,\mathbf{H})$ such that $gg^* = I_p$. This subgroup is isomorphic to $Sp(p)$. The subset $\mathcal{P}$ of $SL(p,\mathbf{H})$ consists of positive definite unimodular Hermitian quaternion matrices. Besides,

$$M = \{\operatorname{diag}(a_1, \dots, a_p) \mid a_i \in \mathbf{H}, |a_i| = 1\}, \quad A = \tilde{A}(p), \quad N = N_+(p,\mathbf{H}),$$

$N_- = N_-(p,\mathbf{H})$, and $W(\mathfrak{a})$ is the group of permutations of diagonal elements with changes of their signs. We recommend to the reader to describe these subgroups when $SU^*(n)$ is realized as the subgroup of the group $SL(n,\mathbb{C})$.

Example 3. The complex simple Lie group $SO(n,\mathbb{C})$ has the noncompact real forms $SO_0(p,q)$, $p+q=n$, $pq \neq 0$, and also the real form $SO^*(n)$ if $n = 2p$ is an even number. The subgroups K, A, M, N, N_- of $SO_0(p,q)$ are obtained as intersections of the corresponding subgroups of $SU(p,q)$ with $SO_0(p,q)$. The group

$SO^*(n)$, $n = 2p$, can be realized either as intersection of $SO(n, \mathbb{C})$ with the group of non-singular matrices of the form

$$\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix} \quad \text{where} \quad X, Y \in \mathfrak{M}(p, \mathbb{C})$$

or as the orthogonal quaternion group, that is, as the group of matrices $g \in SL(p, \mathbf{H})$ such that $g^t g = I_p$. For the first realization of the group $SO^*(n)$ the involutive automorphism $\tilde{\theta}$ is of the form $\tilde{\theta}(g) = J_n g J_n^{-1}$, where $J_n = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$. Therefore, the maximal compact subgroup K in $SO^*(n)$ consists of matrices of the form $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$, $X, Y \in \mathfrak{M}(p, \mathbf{R})$, for which $X + iY \in U(p)$. This subgroup is isomorphic to $U(p)$ and this isomorphism is given by the formula $\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \longleftrightarrow X + iY$. The space $\mathcal{P}$ consists of the matrices $\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix}$, where $X, Y \in \mathfrak{M}(p, \mathbb{C})$ and $X^t = \bar{X}$, $Y = -Y^t$. The subgroup A consists of matrices $\begin{pmatrix} X(\mathbf{t}) & 0 \\ 0 & \bar{X}(\mathbf{t}) \end{pmatrix}$, where

$$X(\mathbf{t}) = \operatorname{diag}\,(\tilde{X}(t_1), \dots, \tilde{X}(t_k)), \quad \tilde{X}(t) = \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix}$$

and k is equal to the integral part $[p/2]$ of the number $p/2$. The subgroup N consists of matrices of the form $\begin{pmatrix} X & Y \\ -\bar{Y} & X \end{pmatrix} \in SO^*(n)$ such that X and $I_p + Y$ belong to $S_+(p, \mathbb{C})$. The subgroup N_- is obtained from N with the help of action by the automorphism $\tilde{\theta}$. Th subgroup $W(\mathfrak{a})$ consists of permutations of $t_1, \dots, t_k$ with changing their signs. The noncompact symmetric space $SO^*(2p)/U(p)$ is of rank $k = [p/2]$. We recommend to the reader to construct the subgroups for the second realization of $SO^*(n)$.

Example 4. The complex simple Lie group $Sp(n, \mathbb{C})$ has the noncompact real forms $Sp(n, \mathbf{R})$ and $Sp(p, q) = Sp(n, \mathbb{C}) \cap SU(2p, 2q)$, $p + q = n$, $pq \neq 0$. The subgroups of the group $Sp(n-1, 1)$ are described in Section 11.7.2. In the case of the groups $Sp(p, q)$, $p > 1$, $q > 1$, they can be described in the similar manner. The subgroups K, A, N, N_- are obtained by intersections of the corresponding subgroups of $SU(2p, 2q)$ with $Sp(n, \mathbb{C})$.

The subgroups and the decompositions of semisimple noncompact connected Lie groups, considered above, can be generalized. Part of these generalizations is related to considerations of commutative Lie subalgebras of special types (for example, the Lie subalgebras of $\mathfrak{sl}(n, \mathbb{C})$ consisting of matrices of the form $\operatorname{diag}\,(tI_p, sI_{n-p})$, where p is fixed and $p(t - s) + ns = 0$). They lead to decompositions in which the subgroup A of diagonal matrices is replaced by the subgroup of block-diagonal matrices, the subgroup N of triangular matrices by the subgroup of block-triangular matrices and so on. Other decompositions are related to replacement of maximal compact subgroups by real noncompact subgroups of the same dimension (for example, of the subgroup $SU(n)$ by $SU(p,q)$, $p+q=n$).

15.1.11. Noncompact analogues of the Iwasawa and Cartan decompositions. The following analogue of the Iwasawa decomposition is valid for the group $SL(n,\mathbb{C})$. Almost every matrix $g \in SL(n,\mathbb{C})$ can be represented in the form $g = n_- dwh$, where $h \in SU(p,q)$, $p+q=n$, $d \in D(n,\mathbb{C})$, $n_- \in N_-(n,\mathbb{C})$, $w \in W$. Here W is the Weyl group of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$, imbedded into the subgroup $K = U(n)$. Really, let $g = (\mathbf{z}_1, \dots, \mathbf{z}_n) \in SL(n,\mathbb{C})$, where $\mathbf{z}_j$ are vector-rows of the matrix g. These vectors form a basis in $\mathbb{C}^n$. With g we associate the matrix $\tilde{g} = gI_{pq}g^*$. The matrix g is called *non-exceptional* if all main minors of $\tilde{g}$ are non-singular. Almost all matrices of $SL(n,\mathbb{C})$ are non-exceptional.

If g is a non-exceptional matrix, then by applying to the vectors $\mathbf{z}_1, \dots, \mathbf{z}_n$ the orthogonalization and normalization procedure with respect to the form $\mathbf{z}^* I_{pq}\mathbf{w}$ we obtain the orthonormal basis $\mathbf{w}_1, \dots, \mathbf{w}_n$ of $\mathbb{C}^n$. Due to the inertia law of quadratic forms, there are exactly p vectors in this basis such that $\mathbf{w}_j^* I_{pq}\mathbf{w}_j = 1$ and exactly q vectors such that $\mathbf{w}_j^* I_{pq}\mathbf{w}_j = -1$. This means that the matrix $\hat{g} = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ is of the form $\hat{g} = \check{g}\, w$, where $\check{g} \in SU(p,q)$, $w \in W$. Since the orthonormalization procedure, applied to the vectors $\mathbf{z}_1, \dots, \mathbf{z}_n$, reduces to left multiplication of the matrix g by the matrix $n_- d$, $n_- \in N_-(n,\mathbb{C})$, $d \in D(n,\mathbb{C})$, then our assertion is proved.

Analogous decomposition is valid for the group $SL(n,\mathbf{R})$. In this case we have to replace $SU(p,q)$, $N_-(n,\mathbb{C})$, $D(n,\mathbb{C})$ by $SO_0(p,q)$, $N_-(n,\mathbf{R})$, $D(n,\mathbf{R})$ respectively. Let us briefly describe the general method of construction of such decompositions. Let $\mathfrak{g}$ be a noncompact simple real Lie algebra and let θ be its Cartan automorphism. Let σ be an involutive automorphism of $\mathfrak{g}$ commuting with θ. We continue linearly σ and θ onto the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ and set

$$\mathfrak{h} = \{H \in \mathfrak{g} \mid \sigma(H) = H\}, \quad \mathfrak{h}_c = \{H \in \mathfrak{g}_c \mid \sigma(H) = H\},$$

$$\mathfrak{q} = \{X \in \mathfrak{g} \mid \sigma(X) = -X\}, \quad \mathfrak{q}_c = \{X \in \mathfrak{g}_c \mid \sigma(X) = -X\}.$$

Then along with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and with the decomposition $\mathfrak{g}_c = \mathfrak{k}_c + \mathfrak{p}_c$ we have the direct decompositions $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$, $\mathfrak{g}_c = \mathfrak{h}_c + \mathfrak{q}_c$. We suppose that the center of $\mathfrak{q} \cap \mathfrak{k}$ is nontrivial.

Let G_c be the simply connected complex Lie group with the Lie algebra $\mathfrak{g}_c$ and let G be the real analytical subgroup of G_c with the Lie algebra $\mathfrak{g}$. The group G has a finite center. The automorphisms σ and θ of $\mathfrak{g}$ define the automorphisms of G which will be denoted by the same symbols. We continue the automorphisms σ and θ onto G_c. Let $K \equiv G_\theta$ and $H \equiv G_\sigma$ be the subgroups of G consisting of θ-invariant and σ-invariant elements respectively. The subgroup K is compact and connected. Let H_c be the analytical subgroup of G_c with the Lie algebra $\mathfrak{h}_c$ and let $G_{c,\sigma}$ be the subgroup of σ-invariant elements of G_c. Then $G_{c,\sigma} = H_c$. The group $H = G_\sigma$ is θ-invariant and consists of finite number of connected parts.

Let $\hat{\mathfrak{a}}$ be a maximal commutative Lie subalgebra in $\mathfrak{p} \cap \mathfrak{q}$ and let $\Sigma \equiv \Sigma(\mathfrak{g}, \mathfrak{a})$ be a system of restricted roots of $\mathfrak{g}$ with respect to $\hat{\mathfrak{a}}$. These roots are obtained by restriction of the roots of the complex Lie algebra $\mathfrak{g}_c$ onto $\hat{\mathfrak{a}}$. As in the case of the restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ above, we introduce the system Σ^+ of positive roots for Σ. We have $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\gamma\in\Sigma} \mathfrak{g}_\gamma$, where $\mathfrak{g}_0$ and $\mathfrak{g}_\gamma$ are the corresponding root subspaces. Let

$$\hat{\mathfrak{n}} = \sum_{\gamma\in\Sigma^+} \mathfrak{g}_\gamma, \quad \hat{\mathfrak{n}}_- = \sum_{\gamma\in\Sigma^+} \mathfrak{g}_{-\gamma} = \theta(\hat{\mathfrak{n}}) = \sigma(\hat{\mathfrak{n}}).$$

Under the above assumptions the decomposition $G = H\hat{A}K$ is valid where $\hat{A} = \exp\hat{\mathfrak{a}}$. It is a generalization of the Cartan decomposition. The analogues of the Iwasawa decompositions are also valid (see, for example, [210] and [241]).

15.1.12. Block (partial) decompositions of groups and parabolic subgroups. Let $n = p + q$. For matrices $g \in GL(n, \mathbb{C})$ we have the following *block* (or *partial*) analogue of the Gauss decomposition:

$$g = \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix} \begin{pmatrix} A & 0 \\ O & B \end{pmatrix} \begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix}, \tag{1}$$

where $Y \in \mathfrak{M}(p,q,\mathbb{C})$, $X \in \mathfrak{M}(q,p,\mathbb{C})$, $A \in GL(p,\mathbb{C})$, $B \in GL(q,\mathbb{C})$. It is clear that we may also consider the analogous decompositions corresponding to the partitions $n = n_1 + \ldots + n_r$ of the number n.

Now we describe the *block Iwasawa decompositions of the group* $GL(n,\mathbb{C})$. Let $n = p + q$. A matrix $g \in GL(n,\mathbb{C})$ can be represented in the form

$$g = n_- a u \equiv \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix} \begin{pmatrix} A & 0 \\ O & B \end{pmatrix} u, \tag{2}$$

where $u \in U(n)$ and A, B, X are of the same sense as in formula (1). The matrices A, B, u are uniquely defined up to their replacement by $A\Gamma_1$, $B\Gamma_2$, $\Gamma^{-1}u$, respectively, where $\Gamma_1 \in U(p)$, $\Gamma_1 \in U(q)$, $\Gamma = \mathrm{diag}\,(\Gamma_1, \Gamma_2)$. The block Iwasawa decompositions of $GL(n,\mathbb{C})$, corresponding to the partitions $n = n_1 + \ldots + n_r$ of the number n, are analogously constructed. Moreover, these constructions are easily generalized for other simple Lie groups. To every block Iwasawa decomposition there corresponds the parabolic subgroup of G and its decomposition. Every subgroup of G, which contains a minimal parabolic subgroup $P = MAN$, is called *parabolic subgroup* of G. If $\gamma_1, \ldots, \gamma_m$ are simple restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$, then every set $\Theta = \{\gamma_{i_1}, \ldots, \gamma_{i_r}\}$ of the simple roots defines the parabolic subgroup which will be denoted by P_Θ. Moreover, every parabolic subgroup is of this form.

Every parabolic subgroup P_Θ is a semidirect product of the reductive subgroup L_Θ (the *Levi subgroup*) and the nilpotent invariant subgroup N_Θ. For example, the subgroup of matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ of $GL(n, \mathbb{C})$ is parabolic. In this case the subgroup L_Θ consists of matrices diag (A, C), and N_Θ consists of matrices $\begin{pmatrix} I_p & X \\ 0 & I_q \end{pmatrix}$. Any two Levi subgroups of a given parabolic subgroup P_Θ are conjugate by element from N_Θ.

A commutative subgroup $\hat{A} \subset G$ is called a *split* subgroup if for any $a \in \hat{A}$ the operator Ad a (acting in $\mathfrak{g}$) can be diagonalized over $\mathbf{R}$. Let P_Θ be a parabolic subgroup of G and L_Θ be its Levi subgroup. Let A_Θ be the maximal connected commutative split subgroup, lying in the center of L_Θ. The subgroup A_Θ is uniquely defined and L_Θ is the centralizer of A_Θ in G. Let X_Θ be the group of continuous homomorphisms of L_Θ into the multiplicative group of real numbers. We set

$$M_\Theta = \bigcap_{\chi \in X_\Theta} \operatorname{Ker} \chi.$$

Then $L_\Theta = M_\Theta A_\Theta$ and $M_\Theta \cap A_\Theta = \{e\}$. The decomposition $P_\Theta = M_\Theta A_\Theta N_\Theta$ is called *Langlands decomposition* of P_Θ, and A_Θ is called a split component of P_Θ. Any two split components are conjugate by element from N_Θ. The dimension of the subgroup A_Θ is called *parabolic rank* of P_Θ and is denoted by prk P_Θ. If a parabolic subgroup is minimal, then its parabolic rank coincides with real rank of the group G. Parabolic subgroups P_Θ with prk $P_\Theta = 1$ are called *maximal*.

Let us note that if $P = MAN$ is a minimal parabolic subgroup of G, then any parabolic subgroup $P_\Theta = M_\Theta A_\Theta N_\Theta$ can be represented as $P_\Theta = \hat{M}AN$, where $A_\Theta \subset A$, $N_\Theta \subset N$ and $\hat{M}$ is an extension of M, that is $K \supset \hat{M} \supset M$.

15.1.13. Limits and contractions of Lie algebras. Let G be a Lie group with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Lie subalgebra in $\mathfrak{g}$ and let $g(t)$ be a one-parameter subgroup of G. Then for any value of t the set $\mathfrak{h}_t = \text{Ad}\,(g(t))\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{h}_\infty$ the set of elements X for which there exists a line $X(t)$ such that $\lim_{t \to \infty} \text{Ad}\,(g(t))(X(t)) = X$. Evidently, $\mathfrak{h}_\infty$ is a Lie subalgebra of $\mathfrak{g}$. It is called the *limit of the Lie subalgebras* $\mathfrak{h}_t$ when $t \to \infty$.

Example 1. Matrices of the form $g(s) = \text{diag}\,(I_{n-1}, e^{-s})$ constitute a one-parameter subgroup in the group $GL(n, \mathbf{R})$. We choose in $\mathfrak{so}(n)$ the line

$$X(s) = \begin{pmatrix} \omega_{n-1} & e^s \mathbf{x} \\ -e^s \mathbf{x}^t & 0 \end{pmatrix}, \quad \omega_{n-1} \in \mathfrak{so}(n-1), \quad \mathbf{x} \in \mathbf{R}^{n-1}.$$

Then

$$\lim_{s \to \infty} g(s) X(s) g^{-1}(s) = \lim_{s \to \infty} \begin{pmatrix} \omega_{n-1} & \mathbf{x} \\ e^{-2s} \mathbf{x}^t & 0 \end{pmatrix} = \begin{pmatrix} \omega_{n-1} & \mathbf{x} \\ \mathbf{0}^t & 0 \end{pmatrix}.$$

Thus, $\mathfrak{h}_\infty$ consists of matrices of the form $\begin{pmatrix} \omega_{n-1} & \mathbf{x} \\ \mathbf{0}^t & 0 \end{pmatrix}$, where $\omega_{n-1} \in \mathfrak{so}(n-1)$, $\mathbf{x} \in \mathbf{R}^{n-1}$. The corresponding Lie group is the group of motion of the $(n-1)$-dimensional Euclidean space.

Example 2. By using the matrices $g(t) = \operatorname{diag}(I_p, e^{-t}, \dots, e^{-t})$ of $SL(p+q;\mathbf{R})$ we obtain with the help of the limit procedure the Lie algebra consisting of block-triangular matrices of the form $\begin{pmatrix} \omega_p & x_{pq} \\ 0 & \omega_q \end{pmatrix}$, where $\omega_p \in \mathfrak{so}(p)$, $\omega_q \in \mathfrak{so}(q)$, $x_{pq} \in \mathfrak{M}(p,q;\mathbf{R})$. It corresponds to the group of matrices $\begin{pmatrix} g_p & x_{pq} \\ 0 & g_q \end{pmatrix}$, where $g_p \in SO(p)$ and $g_q \in SO(q)$.

Limit forms for compact semisimple Lie algebra $\mathfrak{k}$ are obtained with the help of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $X \in \mathfrak{p}$ and let $g(t) = \exp tX$ be the corresponding one-parameter subgroup of the group G with the Lie algebra $\mathfrak{g}$. We set $\mathfrak{k}(X) = \{Y \in \mathfrak{k} \mid [X,Y] = 0\}$ and denote by Z_j, $j = 1, 2, \dots, q$, the eigenvectors for the operator $\operatorname{ad} X$ corresponding to positive eigenvalues:

$$(\operatorname{ad} X)(Z_j) = \lambda_j Z_j, \quad \lambda_j > 0.$$

Since $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, then $Z_j = X_j + Y_j$, where $X_j \in \mathfrak{k}$ and $Y_j \in \mathfrak{p}$. We have $[X, X_j] \in \mathfrak{p}$ and $[X, Y_j] \in \mathfrak{k}$. Therefore, $[X, X_j] = \lambda_j Y_j$ and $[X, Y_j] = \lambda_j X_j$. It is evident that $\mathfrak{k}(X)$ and $X_1, \dots, X_q$ generate $\mathfrak{k}$.

It is easy to verify that $(\operatorname{Ad} g(t))(Z) = Z$ if $Z \in \mathfrak{k}(X)$ and $(\operatorname{Ad} g(t))(X_j) = \cosh(\lambda_j t) X_j + \sinh(\lambda_j t) Y_j$. Then

$$\lim_{t\to\infty} \frac{(\operatorname{Ad} g(t))(X_j)}{\cosh(\lambda_j t)} = \lim_{t\to\infty} (X_j + \tanh(\lambda_j t) Y_j) = X_j + Y_j = Z_j.$$

Thus, if $X(t)$ is a line in $\mathfrak{k}$ for which limit $Y = \lim_{t\to\infty} (\operatorname{Ad} g(t))(X(t))$ exists, then Y belongs to the subalgebra, generated by $\mathfrak{k}(X)$ and $Z_1, \dots, Z_q$. Thus $\lim_{t\to\infty} (\operatorname{Ad} g(t))\mathfrak{k} = \mathfrak{k}(X) + \mathfrak{n}_+(X)$ where $\mathfrak{n}_+(X)$ is the nilpotent Lie subalgebra in $\mathfrak{g}$, generated by eigenvectors of the operator $\operatorname{ad} X$ corresponding to positive eigenvalues.

Now we consider limits in semidirect sums of Lie algebras. Let $\mathfrak{g}$ be a semidirect sum of a commutative ideal $\mathfrak{n}$ and a Lie subalgebra $\mathfrak{h}$, and let $X \in \mathfrak{n}$, $g(t) = \exp(tX)$. Let us prove that

$$\lim_{t\to\infty} (\operatorname{Ad} g(t))\mathfrak{h} = \mathfrak{h}(X) + \mathfrak{n} \quad \text{where} \quad \mathfrak{h}(X) = \{Y \in \mathfrak{h} \mid [X,Y] = 0\}.$$

Really, since the subalgebra $\mathfrak{n}$ is commutative, then for $Z \in \mathfrak{n}$ we have $(\operatorname{Ad} g(t))(Z) = Z$. Since for $Y \in \mathfrak{h}$ we have $[X,Y] \in \mathfrak{n}$, then $[X,[X,Y]] = 0$. Therefore, for all $Y \in \mathfrak{g}$ we have

$$(\operatorname{Ad} g(t))(Y) = Y + t[X,Y].$$

Suppose that $\lim_{t\to\infty}(\mathrm{Ad}\, g(t))(Y(t)) = Y$ where $Y(t) \in \mathfrak{h}$. Then

$$Y = \lim_{t\to\infty}(Y(t) + [X, Y(t)]).$$

Since $Y(t) \in \mathfrak{h}$ and $[X, Y(t)] \in \mathfrak{n}$, then the limits

$$\lim_{t\to\infty} Y(t), \quad \lim_{t\to\infty}[X, Y(t)]$$

exist and the relation $[X, \lim_{t\to\infty} Y(t)] = 0$ is valid. It follows from here that $\lim_{t\to\infty} Y(t) \in \mathfrak{h}(X)$. Thus, $Y \in \mathfrak{h}(X) + \mathfrak{n}$ and our assertion is proved.

The notion of contraction of Lie algebras is connected with limits of Lie algebras. Let $g(t)$ be a one-parameter family of linear transformations of a Lie algebra $\mathfrak{g}$. If the limit

$$[X, Y]_\infty = \lim_{t\to\infty} g^{-1}(t)[g(t)X, g(t)Y] \tag{1}$$

exists for any $X, Y \in \mathfrak{g}$, then $[X, Y]_\infty$ defines the new commutation relations in the linear space $\mathfrak{g}$. This new Lie algebra (which is denoted by $\mathfrak{g}_\infty$) is called a *contraction of the Lie algebra* $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ and let $\mathfrak{l}$ be a linear subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$. Putting

$$g(t)X = \begin{cases} X & \text{if} \quad X \in \mathfrak{h}, \\ X/t & \text{if} \quad X \in \mathfrak{l}, \end{cases}$$

after contraction we obtain semidirect sum of the Lie subalgebra $\mathfrak{h}$ and the commutative ideal $\mathfrak{l}$. Really, $[X, Y]_\infty = [X, Y]$ if $X, Y \in \mathfrak{h}$, and $[X, Y] = 0$ if $X, Y \in \mathfrak{l}$. Now let $X \in \mathfrak{h}$ and $Y \in \mathfrak{l}$. We denote by φ_1 and by φ the projectors of $\mathfrak{g}$ onto $\mathfrak{h}$ and onto $\mathfrak{l}$ respectively. Then

$$\begin{aligned} [X, Y]_\infty &= \lim_{t\to\infty} \frac{g^{-1}(t)}{t}[X, Y] = \lim_{t\to\infty} \frac{g^{-1}(t)}{t}(\varphi_1([X, Y]) + \varphi([X, Y])) \\ &= \lim_{t\to\infty} \frac{\varphi_1([X, Y])}{t} + \varphi([X, Y]) = \varphi([X, Y]) \in \mathfrak{l}. \end{aligned}$$

These relations show that $\mathfrak{l}$ is a commutative ideal in $\mathfrak{g}_\infty$.

More general construction leads to semidirect sums in which an ideal is nilpotent. In this construction we take a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}_1 + \ldots + \mathfrak{l}_n$, where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, set

$$g(t)X = \begin{cases} X & \text{if } X \in \mathfrak{h}, \\ X/t^k & \text{if } X \in \mathfrak{l}_k, \end{cases}$$

and consider the corresponding limit. In particular, if $\mathfrak{h} = 0$, then the algebra $\mathfrak{g}_\infty$ is nilpotent.

With the help of contractions we can obtain block-triangular Lie algebras. Block-triangular Lie groups can be obtained by means of maximal nilpotent subgroups of semisimple Lie groups. For example, we take the simple Lie group $SU(p,q)$, $p \leq q$, $p+q=n$, realized as the group of unimodular linear transformations in $\mathbb{C}^{p+q}$, conserving the Hermitian form

$$\frac{1}{2}\sum_{j=1}^{p}(z_j\bar{z}_{n-j+1}+\bar{z}_j z_{n-j+1})+\sum_{j=p+1}^{q}|z_j|^2.$$

The group of matrices of the form

$$\begin{matrix} p & q-p & p & \\ \left(\begin{matrix} A \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} B^* \\ I_{p-q} \\ 0 \end{matrix} & \left.\begin{matrix} Z(A^*)^{-1}S \\ B(A^*)^{-1}S \\ S(A^*)^{-1}S \end{matrix}\right) & \begin{matrix} p \\ q-p \\ p \end{matrix} \end{matrix} \tag{2}$$

where $A \in N_+(p,\mathbb{C})$, $B \in \mathfrak{M}(q-p,p,\mathbb{C})$, $Z = \frac{1}{2}(B^*B+iH)$, $H \in \mathfrak{M}(p,\mathbb{C})$, $H=H^*$, $S=(s_{ij})=(\delta_{i,p-j+1})$, is a maximal nilpotent subgroup in $SU(p,q)$. The normalizer of this subgroup consists of matrices (2) in which I_{p-q} is replaced by $U \in SU(q-p)$ and $B(A^*)^{-1}S$ by $UB(A^*)^{-1}S$. The analogous subgroups are obtained with the help of maximal nilpotent subgroups of the groups $SO_0(p,q)$ and $Sp(p,q)$.

15.2. Homogeneous Spaces with Semisimple Motion Groups

With every decomposition of a semisimple Lie group G we associate homogeneous spaces for which G is a motion group. For example, the homogeneous spaces $K\backslash G$, G/MN, G/MAN correspond to the Iwasawa decomposition $G=KAN$ of G. These homogeneous spaces, as well as other ones, will be considered in this section.

15.2.1. Homogeneous self-adjoint cones. The symbol $\mathbf{F}$ will denote one of the fields $\mathbf{R}$, $\mathbb{C}$, $\mathbf{H}$. The real dimensions of these fields are denoted by ν, that is $\nu(\mathbf{R})=1$, $\nu(\mathbb{C})=2$, $\nu(\mathbf{H})=4$. The space $\mathfrak{M}_{mn}(\mathbf{F}) \equiv \mathfrak{M}(m,n;\mathbf{F})$ of $m\times n$ matrices over the field $\mathbf{F}$ is of real dimension νmn. In particular, $\dim \mathfrak{M}_n(\mathbf{F}) = \nu n^2$, where $\mathfrak{M}_n(\mathbf{F}) \equiv \mathfrak{M}_{nn}(\mathbf{F})$.

We shall consider the following subsets of the space $\mathfrak{M}_n(\mathbf{F})$:

a) the set $\mathfrak{S}_n(\mathbf{F})$ of symmetric matrices,
b) the set $\mathfrak{A}_n(\mathbf{F})$ of antisymmetric (i.e. skew-symmetric) matrices,
c) the set $\mathfrak{H}_n(\mathbf{F})$ of Hermitian matrices,
d) the set $\mathfrak{P}_n(\mathbf{F})$ of positive definite matrices,
e) the set $\mathfrak{P}_{pq}(\mathbf{F})$, $p+q=n$, of Hermitian matrices X such that $X=YI_{pq}Y^*$ for some $Y \in \mathfrak{M}_n(\mathbf{F})$.

Let us note that $\mathfrak{S}_n(\mathbf{R}) = \mathfrak{H}_n(\mathbf{R})$ and that matrices from $i\mathfrak{H}_n(\mathbf{F})$ are anti-Hermitian. The subsets of unimodular matrices from $\mathfrak{S}_n(\mathbf{F})$, $\mathfrak{A}_n(\mathbf{F})$, $\mathfrak{H}_n(\mathbf{F})$, $\mathfrak{P}_n(\mathbf{F})$, $\mathfrak{P}_{pq}(\mathbf{F})$ will be denoted by $\tilde{\mathfrak{S}}_n(\mathbf{F})$, $\tilde{\mathfrak{A}}_n(\mathbf{F})$, $\tilde{\mathfrak{H}}_n(\mathbf{F})$, $\tilde{\mathfrak{P}}_n(\mathbf{F})$, $\tilde{\mathfrak{P}}_{pq}(\mathbf{F})$ respectively.

In this section we denote matrices by capital Greek or italic roman letters and sets of matrices by capital Gotic or script letters. However, if a matrix is considered as an element of a group, then it is denoted by a lowercase letter.

The maximal compact subgroup of the group $GL(n, \mathbf{F})$ is denoted by $U(n, \mathbf{F})$. Therefore, $U(n, \mathbf{R}) = SO(n)$, $U(n, \mathbb{C}) = U(n)$, $U(n, \mathbf{H}) = Sp(n)$.

The sets $\mathfrak{P}_n(\mathbf{F})$ are *real cones*, that is, they possess the properties

a) *if* $X, Y \in \mathfrak{P}_n(\mathbf{F})$, *then* $\alpha X + \beta Y \in \mathfrak{P}_n(\mathbf{F})$ *for* $\alpha > 0$, $\beta > 0$,

b) $\mathfrak{P}_n(\mathbf{F})$ *contains no real straight lines.*

A cone $\mathcal{V}$ in a linear space $\mathfrak{L}$ is called *convex* if it is open in $\mathfrak{L}$. The cone $\mathfrak{P}_n(\mathbf{F})$ is open in $\mathfrak{H}_n(\mathbf{F})$. Let $\mathcal{V}$ be an open cone in $\mathfrak{L}$. The set of real linear functionals from conjugate space $\mathfrak{L}'$ for $\mathfrak{L}$, taking positive values on $\mathcal{V}$, is a cone in $\mathfrak{L}'$. This cone is called *conjugate* to $\mathcal{V}$ and is denoted by $\mathcal{V}'$. If the space $\mathfrak{L}$ is Euclidean, then $\mathfrak{L}'$ can be identified with $\mathfrak{L}$. If this identification leads to coincidence of $\mathcal{V}$ and $\mathcal{V}'$, then $\mathcal{V}$ is called *self-adjoint.* The cone $\mathfrak{P}_n(\mathbf{F})$ is self-adjoint with respect to the scalar product $(X, Y) = \operatorname{Tr} XY^*$ in $\mathfrak{H}_n(\mathbf{F})$.

A cone $\mathcal{V}$ in $\mathfrak{L}$ is called *irreducible* if it cannot be represented as a direct sum of cones of smaller dimension. It is possible to show that except for the cones $\mathfrak{P}_n(\mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbb{C}, \mathbf{H}$, there is only one infinite series of self-adjoint irreducible cones. Cones of this series consist of vectors $\mathbf{y} \in \mathbf{R}^n$ such that

$$y_1 y_2 - y_3^2 - \ldots - y_n^2 > 0, \quad y_1 > 0.$$

These cones are denoted by $\hat{C}_+^n$. Along with four infinite series of cones there is one exceptional irreducible self-adjoint cone. It consists of positive definite matrices of the third order over the algebra of octaves. This non-commutative algebra is of real dimension 8. It is non-associative.

All irreducible self-adjoint cones are homogeneous spaces. The group $GL(n,\mathbf{F})$ acts on $\mathfrak{P}_n(\mathbf{F})$ as

$$X \to gXg^*, \quad g \in GL(n, \mathbf{F}).$$

The stationary subgroup for the identity matrix $I_n \in \mathfrak{P}_n(\mathbf{F})$ coincides with $U(n, \mathbf{F})$. The motion group of the cone $\hat{C}_+^n$ is $\mathbf{R}_+ \times SO_0(n-1, 1)$, and $SO(n-1)$ is the stationary subgroup for the point $(0, \ldots, 0, 1)$.

Matrix X of $\mathfrak{M}_n(\mathbf{H})$ can be realized in the form of matrices $X = \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix}$, where $Z, W \in \mathfrak{M}_n(\mathbb{C})$ (see Section 1.0.3). Under this realization, $\mathfrak{P}_n(\mathbf{H})$ is the set of positive definite matrices of the indicated type.

The cones $\mathfrak{P}_n(\mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbb{C}, \mathbf{H}$, and $\hat{C}_+^n$ are homogeneous symmetric spaces. The groups $U(n, \mathbf{F})$, imbedded "anti-diagonally" into their motion groups $U(n, \mathbf{F}) \times$

$U(n, \mathbf{F})$ (that is, imbedded as the set of elements (g, g^{-1}), $g \in U(n, \mathbf{F})$), are the compact symmetric spaces related to $\mathfrak{P}_n(\mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbb{C}, \mathbf{H}$. Motions on these spaces are defined as

$$(g_1, g_2) \circ (g, g^{-1}) = (g_1 g g_2^{-1}, g_2 g^{-1} g_1^{-1}).$$

The stationary subgroup of the point (e, e) is $H = \{(g, g) \mid g \in U(n, \mathbf{F})\}$.

The third space, corresponding to the spaces $\mathfrak{P}_n(\mathbf{F})$ and $U(n, \mathbf{F})$, is

$$\mathfrak{H}_n(\mathbf{F}) = IU(n, \mathbf{F})/U(n, \mathbf{F}),$$

where $IU(n, \mathbf{F})$ is the semidirect product of the group $U(n, \mathbf{F})$ with the additive group $\mathfrak{H}_n(\mathbf{F})$. Motions in this space are given as

$$(g, X) \circ Y = gYg^* + X, \quad g \in U(n, \mathbf{F}), \quad X, Y \in \mathfrak{H}_n(\mathbf{F}).$$

The spaces corresponding to the cones $\hat{C}_+^n$ are constructed in analogous way.

The motion groups of the cones $\mathfrak{P}_n(\mathbf{F})$ act on $\mathfrak{P}_n(\mathbf{F})$ transitively. These actions are not simply transitive since the corresponding stationary subgroups do not coincide with $\{e\}$. The subgroups $S_+^+(n, \mathbf{F})$ (see Section 15.1.1) act simply transitively on $\mathfrak{P}_n(\mathbf{F})$. Namely, any matrix $g \in \mathfrak{P}_n(\mathbf{F})$ can be uniquely represented in the form $g = ss^*$, where $s \in S_+^+(n, \mathbf{F})$.

Applying the Cartan decomposition $g = k_1 a k_2$ to the elements of the product $p = gg^*$ we obtain the following statement. *Every element p of the cone $\mathfrak{P}_n(\mathbf{F})$ can be represented in the form $p = kak^*$, where $k \in U(n, \mathbf{F})$ and $a \in A(n)$.* In this decomposition the matrix a is uniquely defined up to permutations of diagonal elements and k is unique up to replacement of k by $k\gamma$, where $\gamma \in U(n, \mathbf{F})$ and is permutable with element a. In particular, if all diagonal elements of a are pairwise different, then $\gamma \in U(n, \mathbf{F}) \cap D(n, \mathbf{F})$.

15.2.2. Hermitian symmetric space. Let $\mathfrak{X}$ be a connected complex space with a Hermitian structure. It is called *Hermitian symmetric space* if every its element is isolated fixed point for some involutive holomorphic isometry of this space. The class of Hermitian symmetric spaces of noncompact type coincides with the class of symmetric bounded domains in many-dimensional complex spaces $\mathbb{C}^n$. Any simply connected globally symmetric Hermitian space is a direct product of Hermitian symmetric spaces of the form G/K or of the form U/K_0. Here G is a connected noncompact simple Lie group with the center $\{e\}$, K is its maximal compact subgroup, U is a connected compact simple Lie group with the center $\{e\}$, and K_0 is its maximal connected non-trivial subgroup with non-discrete center.

There are four infinite classes of noncompact Hermitian symmetric spaces with simple motion groups:

$$\mathfrak{R}_1(p, q) = SU(p, q)/S(U(p) \times U(q)) \approx U(p, q)/(U(p) \times U(q)), \quad p > 0, \quad q > 0,$$

$$\mathfrak{R}_2(p) = Sp(p, \mathbf{R})/U(p),$$

$$\mathfrak{R}_3(p) = SO^*(2p)/U(p),$$
$$\mathfrak{R}_4(p) = SO(p,2)/S(O(p) \times O(2)).$$

Except for these four infinite classes, there are two noncompact Hermitian symmetric spaces connected with exceptional simple Lie groups. They are of dimension 32 and 54.

In this section we construct realizations of Hermitian symmetric spaces in the form of domains of the spaces $\mathbb{C}^n$. The space $\mathfrak{R}_1(p,q) = U(p,q)/(U(p) \times U(q))$ can be imbedded into $U(p,q)$. Then $\mathfrak{R}_1(p,q)$ is realized as the set of matrices $g = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$, where $A > 0$, $D > 0$, $B \in \mathfrak{M}_{pq}(\mathbb{C})$ and $gI_{pq}g^* = I_{pq}$. It follows from here that the relations

$$A^2 - BB^* = I_p, \quad D^2 - B^*B = I_q, \quad AB = BD$$

are fulfilled. We set $A^{-1}B = BD^{-1} = Z$. Then we have

$$I_p - ZZ^* = A^{-2}, \quad I_q - Z^*Z = D^{-2}, \quad B = AZ$$

and, therefore, the matrix g is uniquely defined by the matrix Z. Thus, we realize $\mathfrak{R}_1(p,q)$ as the space of matrices $Z \in \mathfrak{M}_{pq}(\mathbb{C})$ such that $I_p - ZZ^* > 0$. It is a many-dimensional analog of the disk. Motions in this space are given by the formula

$$g \circ Z = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(p,q). \tag{1}$$

The stationary subgroup of the zero point coincides with the set of matrices diag (A, D), $A \in U(p)$, $D \in U(q)$.

In order to construct an analogous realization of the space $\mathfrak{R}_2(p)$ we replace matrices $g \in Sp(p, \mathbf{R})$ by $\hat{g} = K_p g K_p^{-1}$, where $K_p = \begin{pmatrix} I_p & iI_p \\ iI_p & I_p \end{pmatrix}$. The set of these matrices form the group $\hat{Sp}(p, \mathbf{R})$ isomorphic to $Sp(p, \mathbf{R})$. The matrices $\hat{g}$ are of the form $\hat{g} = \begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix}$. Since $g^t J_p g = J_p$ for $g \in Sp(p, \mathbf{R})$, then $\hat{g}^* I_{pp} \hat{g} = I_{pp}$. It follows from here that

$$U^*U - V^*V = I_p, \quad U^tV = V^tU. \tag{2}$$

We set $Z = VU^{-1}$. Then it follows from (2) that $Z^t = Z$. Moreover, $I_p - Z^*Z = (U^{-1})^*U^{-1} > 0$. Since Z is uniquely defined by the matrix $\hat{g}$, then $\mathfrak{R}_2(p)$ is realized as the space of matrices $Z \in \mathfrak{S}_p(\mathbb{C})$ such that $I_p - \bar{Z}Z > 0$. Motions in $\mathfrak{R}_2(p)$ are given as

$$g \circ Z = (UZ + \bar{V})(VZ + \bar{U})^{-1}, \quad g = \begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix} \in \hat{Sp}(p, \mathbf{R}). \tag{3}$$

It is easy to see that $g \circ Z \in \mathfrak{S}_p(\mathbb{C})$. The stationary subgroup of the point $Z = 0$ coincides with $\{\text{diag}\,(U, \bar{U}) \mid U \in U(p)\}$.

We construct analogously the realization of $\mathfrak{R}_3(p)$ in the form of the set of matrices $Z \in \mathfrak{A}_p(\mathbb{C})$ such that $I_p - \bar{Z}Z > 0$. Motions in this space are given by the formula

$$\hat{g} \circ Z = (UZ + \bar{V})(-VZ + \bar{U})^{-1}, \tag{4}$$

where $\hat{g} \in \hat{SO}^*(2p) = K_p SO^*(2p) K_p^{-1}$. The elements of $\hat{SO}^*(2p)$ are of the form $\hat{g} = \begin{pmatrix} U & \bar{V} \\ -V & \bar{U} \end{pmatrix}$, where

$$U^*U - V^*V = I_p, \quad U^tV = -V^tU.$$

The stationary subgroup of the matrix $Z = 0$ coincides with $\{\text{diag}\,(U, \bar{U}) \mid U \in U(p)\}$.

The space $\mathfrak{R}_4(p)$ is analogously realized as the space of vectors $\mathbf{z} = (z_1, \dots, z_p) \in \mathbb{C}^p$ such that

$$\mathbf{z}\mathbf{z}^* < \frac{1}{2}(1 + |\mathbf{z}^t\mathbf{z}|^2) < 1. \tag{5}$$

Let us describe the compact symmetric spaces dual (by Cartan) to $\mathfrak{R}_1(p,q)$, $\mathfrak{R}_2(p)$, $\mathfrak{R}_3(p)$, $\mathfrak{R}_4(p)$. The space

$$\tilde{\mathfrak{R}}_1(p,q) = U(p+q)/(U(p) \times U(q))$$

is dual to $\mathfrak{R}_1(p,q)$. It can be realized as the set $\mathfrak{M}_{pq}(\mathbb{C})$, where $U(p+q)$ acts as

$$g \circ Z = (AZ + B)(CZ + D)^{-1}. \tag{6}$$

Here $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a matrix from $U(p+q)$. We have

$$AA^* + BB^* = I_p, \quad CC^* + DD^* = I_q, \quad AC^* + BD^* = 0.$$

The third space corresponding to the spaces $\mathfrak{R}_1(p,q)$ and $\tilde{\mathfrak{R}}_1(p,q)$ is the same space $\mathfrak{M}_{pq}(\mathbb{C})$, where the group of block-triangular matrices $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, $A \in U(p)$, $D \in U(q)$, acts as

$$g \circ Z = AZD^{-1} + BD^{-1}. \tag{7}$$

The compact symmetric space $\tilde{\mathfrak{R}}_2(p) = Sp(p)/U(p)$ is dual (by Cartan) to $\mathfrak{R}_2(p)$. It can be realized as the space $\mathfrak{S}_p(\mathbb{C})$, where the group $\hat{Sp}(n) = K_p Sp(p) K_p^{-1}$ acts. The motion

$$g \circ Z = (UZ - \bar{V})(VZ + \bar{U})^{-1} \tag{8}$$

corresponds to the matrix $g = \begin{pmatrix} U & -\bar{V} \\ V & \bar{U} \end{pmatrix} \in \hat{Sp}(p)$. Let us note that $\begin{pmatrix} U & -\bar{V} \\ V & \bar{U} \end{pmatrix} \in \hat{Sp}(p)$ if and only if

$$U^*U + V^*V = I_p, \quad V^tU = U^tV.$$

The third space, corresponding to $\mathfrak{R}_2(p)$ and $\tilde{\mathfrak{R}}_2(p)$ is the same space $\mathfrak{S}_p(\mathbb{C})$, where the group of matrices $g = \begin{pmatrix} U & B \\ 0 & \bar{U} \end{pmatrix}$, $BU^t \in \mathfrak{S}_p(\mathbb{C})$, $U \in U(p)$, acts. We have

$$g \circ Z = UZU^t + BU^t. \tag{9}$$

The compact symmetric space $\tilde{\mathfrak{R}}_3(p) = SO(2p)/U(p)$ is dual to $\mathfrak{R}_3(p)$. It can be realized as the set $\mathfrak{A}_p(\mathbb{C})$, where the group $\hat{SO}(2p) = K_pSO(2p)K_p^{-1}$ acts as

$$g \circ Z = (UZ + \bar{V})(VZ + \bar{U})^{-1}, \quad g = \begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix} \in \hat{SO}(2p). \tag{10}$$

Let us note that a matrix $\begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix}$ belongs to $\hat{SO}(2p)$ if and only if

$$U^*U + V^*V = I_p, \quad U^tV = -V^tU.$$

The third space corresponds to the spaces $\mathfrak{R}_3(p)$ and $\tilde{\mathfrak{R}}_3(p)$. It coincides with the same space $\mathfrak{A}_p(\mathbb{C})$, where the group of matrices $g = \begin{pmatrix} U & B \\ 0 & \bar{U} \end{pmatrix}$, $U \in U(p)$, $BU^t \in \mathfrak{A}_p(\mathbb{C})$, acts as

$$g \circ Z = UZU^t + BU^t. \tag{11}$$

The compact symmetric space $\tilde{\mathfrak{R}}_4(p) = SO(p+2)/S(O(p) \times O(2))$ is dual to $\mathfrak{R}_4(p)$. It can be realized as the set $\mathfrak{M}_{p2}(\mathbf{R})$. The third space is also realized as $\mathfrak{M}_{p2}(\mathbf{R})$. The group of matrices $g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, $A \in SO(p)$, $D \in SO(2)$, $B \in \mathfrak{M}_{p2}(\mathbf{R})$, acts in $\mathfrak{M}_{p2}(\mathbf{R})$ as

$$g \circ X = AXD^{-1} + BD^{-1}. \tag{12}$$

The notion of a *characteristic manifold* $\mathcal{E}$ for a bounded simply connected domain $\mathfrak{R}$ is introduced in the theory of analytical functions. It is the manifold situated on the bound of $\mathfrak{R}$ and possessing the properties:

a) module of any function, holomorphic in $\mathfrak{R}$, has its maximal value on $\mathcal{E}$,
b) for every point $a \in \mathcal{E}$ there exists a holomorphic (in $\mathfrak{R}$) function f_a with maximal value of its module at a.

It is possible to show that in this definition linear functions may be considered instead of holomorphic functions. The characteristic manifold $\mathcal{E}_1(p,q)$ for the domain $\mathfrak{R}_1(p,q)$ consists of matrices $X \in \mathfrak{M}_{pq}(\mathbb{C})$ such that $XX^* = I_{pq}$. The characteristic manifold $\mathcal{E}_2(p)$ for the domain $\mathfrak{R}_2(p)$ consists of all symmetric unitary matrices of order p. The characteristic manifold $\mathcal{E}_3(p)$ for $\mathfrak{R}_3(p)$ is separately defined for odd and even values of p. If p is even, then $\mathcal{E}_3(p)$ consists of all antisymmetric unitary matrices of order p. If p is odd, then $\mathcal{E}_3(p)$ consists of all matrices of the form UDU^t, where $U \in U(p)$ and

$$D = \begin{pmatrix} 0 & I_q & \mathbf{0} \\ -I_q & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix}, \quad p = 2q+1.$$

The characteristic manifold $\mathcal{E}_4(p)$ for $\mathfrak{R}_4(p)$ consists of vectors of the form $e^{i\theta}\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^p$, $\|\mathbf{x}\| = 1$.

Characteristic manifolds are homogeneous spaces. Moreover, every point $a \in \mathcal{E}$ can be transferred into any other point $b \in \mathcal{E}$ by a transformation leaving invariant any fixed point of $\mathfrak{R}$.

Along with Riemannian Hermitian symmetric spaces, pseudo-Riemannian Hermitian symmetric spaces are considered. We use the notations of Section 15.1.11. Let $\mathfrak{l}$ be a θ-invariant subspace in $\mathfrak{g}$ or in $\mathfrak{g}_c$. The intersection of $\mathfrak{l}$ with $\mathfrak{k}$ (with $\mathfrak{p}$) will be denoted by $\mathfrak{l}_k$ (by $\mathfrak{l}_p$). We have the decompositions

$$\mathfrak{g} = \mathfrak{h}_k + \mathfrak{q}_k + \mathfrak{h}_p + \mathfrak{q}_p,$$
$$\mathfrak{g}_c = \mathfrak{h}_{ck} + \mathfrak{q}_{ck} + \mathfrak{h}_{cp} + \mathfrak{q}_{cp}.$$

The pseudo-Riemannian symmetric space $\mathfrak{X} = G/H$ is called *Hermitian* if $\mathfrak{q}_{ck}$ has non-trivial center. An analogous definition is given in the case when G is semisimple. In this case we demand that $\mathfrak{h}_c$ does not contain non-trivial ideals of the Lie algebra $\mathfrak{g}_c$ and that $\mathfrak{z}_{\mathfrak{q}_c}(\mathfrak{c}) = \mathfrak{q}_{ck}$, where $\mathfrak{c}$ is the centralizer of $\mathfrak{q}_{ck}$ in $\mathfrak{q}_{ck}$ and $\mathfrak{z}_{\mathfrak{q}_c}(\mathfrak{c})$ is the centralizer of $\mathfrak{c}$ in $\mathfrak{q}_c$. It is possible to show that in this case $\mathfrak{c}$ does not vanishes and is contained in the center of $\mathfrak{k}$. Moreover, $\mathfrak{z}_{\mathfrak{q}_c} = \mathfrak{k}$ and rank G/H = rank $(K/K \cap H)$.

The homogeneous spaces

$$SU(p,q)/SO_0(p,q),$$
$$SU(n,n)/(SL(n,\mathbb{C}) \times \mathbf{R}),$$
$$SO^*(2n)/SO(n,\mathbb{C}),$$
$$SO_0(2,q)/(SO_0(1,q-k) \times SO(1,k)),$$
$$Sp(n,\mathbf{R})/(SL(n,\mathbf{R}) \times \mathbf{R}),$$
$$SU(2p,2q)/Sp(p,q),$$
$$SO^*(4n)/(SU^*(2n) \times \mathbf{R}),$$

$$Sp(2n,\mathbf{R})/Sp(n,\mathbb{C})$$

are pseudo-Riemannian Hermitian spaces with classical simple Lie groups as motion groups.

15.2.3. Tube domains. Let $\mathcal{V}$ be a self-adjoint convex cone in a real linear space $\mathfrak{L}$. The set of vectors $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x} \in \mathfrak{L}$ and $\mathbf{y} \in \mathcal{V}$, is called a *tube domain of the first kind*, corresponding to the cone $\mathcal{V}$, and is denoted by $\mathfrak{T}(\mathcal{V})$. Let us show that $\mathfrak{R}_1(p,p)$, $\mathfrak{R}_2(p)$, $\mathfrak{R}_3(p)$ and $\mathfrak{R}_4(p)$ can be realized as tube domains. We shall denote these tube domains by $\mathfrak{T}_1(p,p)$, $\mathfrak{T}_2(p)$, $\mathfrak{T}_3(p)$ $\mathfrak{T}_4(p)$ respectively. We shall need some statements.

Statement 1. *If $Z \in \mathfrak{M}_{pq}(\mathbb{C})$, then the inequalities $Z^*Z < I_q$ and $ZZ^* < I_p$ are equivalent.*

This statement follows from the identity

$$\begin{pmatrix} I_p & 0 \\ 0 & I_q - Z^*Z \end{pmatrix} = \begin{pmatrix} I_p & -Z \\ Z^* & I_q - Z^*Z \end{pmatrix}^* \begin{pmatrix} I_p - ZZ^* & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} I_p & -Z \\ Z^* & I_q - Z^*Z \end{pmatrix}.$$

Statement 2. *If $T \in \mathfrak{M}_p(\mathbf{F})$ and $T^*T < I_p$, then the matrices $I_p - T$ and $I_p + T$ are non-singular.*

Really, if $(I_p - T)\mathbf{x} = \mathbf{0}$, $\mathbf{x} \in \mathbf{F}^p$, then $\mathbf{x} = T\mathbf{x}$, $\mathbf{x}^* = \mathbf{x}^*T^*$. Therefore, $\mathbf{x}^*(I_p - T^*T)\mathbf{x} = 0$. Since $I_p - T^*T > 0$, then $\mathbf{x} = 0$. Thus, $I_p - T$ is non-singular. In the same way one proves that $I_p + T$ is non-singular.

Statement 3. *If $T \in \mathfrak{M}_p(\mathbf{F})$ and*

$$Z = i(I_p + T)(I_p - T)^{-1} = i(I_p - T)^{-1}(I_p + T),$$

then the inequalities $T^*T < I_p$ and $(1/2i)(Z - Z^*) > 0$ are equivalent.

Proof. We have

$$\begin{aligned} \frac{1}{2i}(Z - Z^*) &= \frac{1}{2}[(I_p - T)^{-1}(I_p + T) + (I_p + T^*)(I_p - T^*)^{-1}] \\ &= \frac{1}{2}(I_p - T)^{-1}[(I_p + T)(I_p - T^*) + (I_p - T)(I_p + T^*)](I_p - T^*)^{-1} \\ &= (I_p - T)^{-1}(I_p - TT^*)(I_p - T^*)^{-1}. \end{aligned}$$

If $I_p - TT^* > 0$, then $(1/2i)(Z - Z^*) > 0$. Moreover,

$$Z - iI_p = \frac{2iT}{I_p - T}, \quad Z + iI_p = \frac{2iI_p}{I_p - T}.$$

Therefore, $Z + iI_p$ is non-singular and

$$T = (Z - iI_p)(Z + iI_p)^{-1} = (Z + iI_p)^{-1}(Z - iI_p) \equiv \frac{Z - iI_p}{Z + iI_p}.$$

Now let $\frac{1}{2i}(Z - Z^*) > 0$. Then the matrix $Z + iI_p$ is non-singular. Really, if $(Z+iI_p)\mathbf{x} = \mathbf{0}$, then $Z\mathbf{x} = -i\mathbf{x}$, $\mathbf{x}^*Z^* = i\mathbf{x}^*$. Therefore, $\frac{1}{2i}\mathbf{x}^*(Z - Z^*)\mathbf{x} = -\mathbf{x}^*\mathbf{x} \leq 0$, and in this reason $\mathbf{x} = \mathbf{0}$. Thus, $Z + iI_p$ is non-singular. We may consider the matrix

$$T = (Z - iI_p)(Z + iI_p)^{-1} = (Z + iI_p)^{-1}(Z - iI_p) \equiv \frac{Z - iI_p}{Z + iI_p}.$$

We have

$$\begin{aligned} I_p - TT^* &= I_p - (Z + iI_p)^{-1}(Z - iI_p)(Z^* + iI_p)(Z^* - iI_p)^{-1} \\ &= (Z + iI_p)^{-1}[(Z + iI_p)(Z^* - iI_p) - (Z - iI_p)(Z^* + iI_p)](Z^* - iI_p)^{-1} \\ &= (Z + iI_p)^{-1}\frac{2}{i}(Z - Z^*)(Z^* - iI_p)^{-1}. \end{aligned}$$

Consequently, the condition $(1/2i)(Z - Z^*) > 0$ implies that $TT^* < I_p$. Finally, we have

$$I_p - T = \frac{2iI_p}{Z + iI_p}, \quad I_p + T = \frac{2Z}{Z + iI_p}.$$

In this reason the matrix $I_p - T$ is non-singular and

$$Z = i(I_p + T)(I_p - T)^{-1} = i(I_p - T)^{-1}(I_p + T) \equiv i\frac{I_p + T}{I_p - T}.$$

The statement is proved.

Let us apply Statement 3 to the domain $\mathfrak{R}_1(p,p)$ and make the Cayley transform $Z = i(I_p + T)(I_p - T)^{-1}$, $T \in \mathfrak{R}_1(p,p)$. We obtain that $\mathfrak{R}_1(p,p)$ can be realized as the Siegel half-space $\mathfrak{T}_1(p,p) \equiv \mathfrak{T}(\mathfrak{P}_p(\mathbb{C}))$, that is as the space of matrices $Z = X + iY$, where $X \in \mathfrak{H}_p(\mathbb{C})$ and $Y \in \mathfrak{P}_p(\mathbb{C})$. Since $T = T^t$ implies $Z = Z^t$, then we derive that the domain $\mathfrak{R}_2(p)$ is realized as the space $\mathfrak{T}_2(p)$ of matrices $Z = X + iY$, where $X \in \mathfrak{H}_p(\mathbf{R})$ and $Y \in \mathfrak{P}_p(\mathbf{R})$. For the domain $\mathfrak{R}_3(p)$, $p = 2k$, we obtain the realization in the form of the space $\mathfrak{T}_3(p)$ of matrices $Z = X + iY$, where $X \in \mathfrak{H}_k(\mathbf{H})$, $Y \in \mathfrak{P}_k(\mathbf{H})$. The domain $\mathfrak{R}_4(p)$ is realized as the space $\mathfrak{T}_4(p)$ of vectors $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ such that $\mathbf{x} \in \mathbf{R}^p$, $\mathbf{y} \in \hat{C}^p_+$. The domain $\mathfrak{R}_3(p)$, $p = 2k + 1$, is realized as the upper Siegel half-plane of the second kind. We do not describe it here.

The motion groups for the tube domains are obtained from those for the corresponding Hermitian symmetric spaces with the help of the Cayley transform. In particular, for $\mathfrak{T}_1(p,p)$ the motion group $U(p,p)$ acts as

$$g \circ Z = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(p,p).$$

For $\mathfrak{T}_2(p)$ the motion group $Sp(n, \mathbf{R})$ acts as

$$g \circ Z = (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbf{R}).$$

The motion group $SO^*(2p)$ acts by the same formula on $\mathfrak{T}_3(p)$, where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO^*(2p)$.

15.2.4. Parametrizations of the space $\mathfrak{P}_m(\mathbf{F})$. Matrices $\Lambda = (\lambda_{ij}) \in \mathfrak{H}_m(\mathbf{F})$ can be parametrized by the real numbers λ_{jj}, $1 \le j \le m$, $\lambda_{ij,\zeta}$, $1 \le i < j \le m$, $1 \le \zeta \le \nu$; $\nu(\mathbf{R}) = 1$, $\nu(\mathbb{C}) = 2$, $\nu(\mathbf{H}) = 4$, where $\lambda_{ij,\zeta}$ are real coordinates of the numbers λ_{ij}, $1 \le i < j \le m$. The other parametrization is given by the numbers $z_{ij,\zeta}$, where $\lambda_{ij} = \eta_{ij} z_{ij}$, $\eta_{ii} = 1$ for $1 \le i \le m$ and $\eta_{ij} = \frac{1}{2}$ for $i \ne j$. The space of matrices (z_{ij}) is denoted by $\mathfrak{H}_m^*(\mathbf{F})$. It is evident that if $\Lambda \in \mathfrak{H}_m(\mathbf{R})$, $Z \in \mathfrak{H}_m^*(\mathbf{F})$, then

$$(\Lambda, Z) = \operatorname{Tr} \Lambda Z^* = \sum_{1 \le i \le j \le m} \frac{\lambda_{ij} z_{ij}}{\eta_{ij}}.$$

Since $\mathfrak{P}_m(\mathbf{F}) \subset \mathfrak{H}_m(\mathbf{F})$, then we obtain the parametrizations of $\mathfrak{P}_m(\mathbf{F})$.

Other parametrizations of $\mathfrak{P}_m(\mathbf{F})$ are related to the Gauss and Cartan decompositions of the group $GL(m, \mathbf{F})$.

Every matrix $\Lambda \in \mathfrak{P}_m(\mathbf{F})$ is representable in the form $\Lambda = T^* A T$, where $T \in N_+(m, \mathbf{F})$, $A \in A(m)$. The parametrization of $\mathfrak{P}_m(\mathbf{F})$ by positive diagonal elements $a_1, \dots, a_m$ of matrices A and by real numbers $t_{ij,\zeta}$, $1 \le i < j \le m$, $1 \le \zeta \le \nu$, corresponds to this decomposition. The formula

$$\Lambda = \begin{pmatrix} I_p & 0 \\ X^* & I_q \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_p & X \\ 0 & I_q \end{pmatrix}, \tag{1}$$

where $X \in \mathfrak{M}_{pq}(\mathbf{F})$, $A \in \mathfrak{P}_p(\mathbf{F})$, $B \in \mathfrak{P}_q(\mathbf{F})$, $p + q = m$, gives the *partial* (*block*) *Gauss decomposition* of $\mathfrak{P}_m(\mathbf{F})$. It defines the parametrization by the real numbers $x_{ij,\zeta}$, where $X = (x_{ij})$, $1 \le i \le p$, $1 \le j \le q$, and by parameters giving the matrices A and B.

The Cartan decomposition in $\mathfrak{P}_m(\mathbf{F})$ is of the form $\Lambda = \Omega^* A \Omega$, where $\Omega \in U(m, \mathbf{F})$, $A \in A(m)$. We denote by $\Lambda^{1/2}$ the matrix $\Omega^* A^{1/2} \Omega$. It is evident that $(\Lambda^{1/2})^2 = \Lambda$. The diagonal elements $a_1, \dots, a_m$ of A and parameters giving the matrix Ω are parameters of the matrix Λ.

The formula

$$\Lambda = \begin{pmatrix} \Omega_1^* & 0 \\ 0 & \Omega_2^* \end{pmatrix} \begin{pmatrix} A_1 & X \\ X^* & A_2 \end{pmatrix} \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}, \tag{2}$$

where $\Omega_1 \in U(p, \mathbf{F})$, $\Omega_2 \in U(q, \mathbf{F})$, $A_1 \in A(p)$, $A_2 \in A(q)$, $X \in \mathfrak{M}_{pq}(\mathbf{F})$, gives the *partial* (*block*) *Cartan decomposition* of Λ. The numbers $a_1, \dots, a_p, a_{p+1}, \dots, a_{p+q}$ (diagonal elements of the matrices A_1 and A_2), $x_{ij,\zeta}$ and parameters, defining the matrices Ω_1 and Ω_2, give a parametrization of the matrix Λ.

In some cases (see Section 17.2.6 below) the elliptic coordinates in $\mathfrak{P}_m(\mathbf{F})$ are useful. They are determined in the following way. Let $\Lambda = \Omega A \Omega^*$, where $A \in A(m)$ and $\Omega \in U(m, \mathbf{F})$. Let $\mathbf{f}_1, \dots, \mathbf{f}_m$ be the eigenvectors of the matrix Λ corresponding to the eigenvalues $a_1, \dots, a_m$. We set $\mathbf{e}_k = \Omega \mathbf{f}_k$, $1 \le k \le m$. Let $\mathfrak{L}$ be the subspace spanned by the vectors $\mathbf{e}_1, \dots, \mathbf{e}_{m-1}$ and let P be the corresponding orthogonal projection onto $\mathfrak{L}$. Let us find eigenvalues of the operator PAP. For this we note that if $PAP\mathbf{f} = b\mathbf{f}$, then $PA\mathbf{f} = b\mathbf{f}$. Therefore,

$$A\mathbf{f} = b\mathbf{f} + \xi \mathbf{e}_m. \tag{3}$$

Consequently, if $\xi_1, \dots, \xi_m$ and $u_1, \dots, u_m$ are the coordinates of the vectors $\mathbf{f}$ and $\mathbf{e}_m$ respectively, then it follows from (3) that $a_k \xi_k = b\xi_k + \xi u_k$ and $\xi_k = \xi u_k/(a_k - b)$. The vector $\mathbf{f}$ is orthogonal to $\mathbf{e}_m$. Thus, $\sum_k \xi_k \bar{u}_k = 0$. We obtain from here the relation

$$\sum_k \frac{|u_k|^2}{a_k - b} = 0,$$

connecting eigenvalues $b_1, b_2, \dots, b_{m-1}$ of the operator PAP with the coordinates of the vector $\mathbf{e}_m$. We set $\eta_k = u_k/|u_k|$, $|u_k|^2 = t_k$. Since

$$\sum_k |u_k|^2 = \|\mathbf{e}_m\|^2 = 1, \quad \text{then} \quad \sum_k t_k = 1, \quad t_k > 0, \quad 1 \le k \le m.$$

Let us find the domain of definition of the variables $b_1, \dots, b_{m-1}$ and the Jacobian of transition from $t_1, \dots, t_{m-1}$ to $b_1, \dots, b_{m-1}$. We situate the numbers $a_1, \dots, a_m$ in the increasing order. Let $a_1 < a_2 < \dots < a_m$. Since the numbers $b_1, \dots, b_{m-1}$ are solutions of the equation $\sum_k t_k/(a_k - x) = 0$, where $\sum_k t_k = 1$ and all t_k are positive, the inequalities

$$a_1 < b_1 < a_2 < b_2 < \dots < a_{m-1} < b_{m-1} < a_m \tag{3$'$}$$

have to be fulfilled.

We prove that if the numbers $b_1, \dots, b_{m-1}$ satisfy inequalities (3′), then there exist $t_1, \dots, t_m$ such that $\sum_k t_k = 1$, $t_k > 0$, $1 \le k \le m$, and for every j the equality $\sum_k t_k/(a_k - b_j) = 0$ is fulfilled. It is sufficient to show that the system of equations

$$\sum_k t_k/(a_k - b_j) = 0, \quad 1 \le j \le m - 1, \quad \sum_k t_k = 1 \tag{4}$$

is consistent. To show it we set $c_j = b_j^{-1}$, $1 \le j \le m-1$, $c_m = 0$. Then the determinant of system (4) is equal to $\det\left((1 - c_j a_k)^{-1}\right)_{j,k=1}^m$. According to the Cauchy lemma (see Section 17.2.5 below) we have

$$\det\left((1 - c_j a_k)^{-1}\right)_{j,k=1}^m = \prod_{1 \le k < j \le m} (c_j - c_k)$$
$$\times \prod_{1 \le k < j \le m} (a_j - a_k) \prod_{j,k=1}^m (1 - c_j a_k)^{-1} \ne 0.$$

Solving the system (4), after simple transformations we find that

$$t_k = \prod_{j\neq k}(a_j - a_k)^{-1} \prod_{j=1}^{m-1}(b_j - a_k). \tag{5}$$

It follows from (3′) that all t_k are positive. Thus, inequalities (3′) give the domain of definition of variables $b_1, \dots, b_{m-1}$.

We find from (5) that Jacobian of transition from $t_1, \dots, t_{m-1}$ to $b_1, \dots, b_{m-1}$ is equal to

$$\prod_{1\le k<j\le m-1}(b_j - b_k) \prod_{1\le p<j\le m}(a_j - a_p)^{-1}. \tag{6}$$

Let $\mathbf{f}_1^{(1)}, \dots, \mathbf{f}_{m-1}^{(1)}$ be eigenvectors of the operator $A^{(1)} = PAP$, corresponding to the eigenvalues $b_1, \dots, b_{m-1}$. Now we apply to the operator $P^{(1)} = \Omega^{(1)}A^{(1)}\Omega^{(1)*}$ the same procedure. Continuing application of this procedure further we obtain the system of coordinates in $\mathfrak{P}_m(\mathbf{F})$ which can be situated (after the renotations $a_k = a_k^{(0)}$, $1 \le k \le m$, $b_k = a_k^{(1)}$, $1 \le k \le m-1$, and so on) in the form of two triangular tables[5]

$$\begin{pmatrix} a_1^{(0)} & a_2^{(0)} & \dots & a_{m-1}^{(0)} & a_m^{(0)} \\ & a_1^{(1)} & \dots\dots\dots\dots & a_{m-1}^{(1)} & \\ & & \dots\dots\dots\dots & & \\ & & a_1^{(m-1)} & & \end{pmatrix}, \quad \begin{pmatrix} u_1^{(1)} & \dots & u_{m-1}^{(1)} \\ & \dots\dots\dots & \\ & u_1^{(m-1)} & \end{pmatrix}. \tag{7}$$

The elements of the first one satisfy the inequalities

$$a_j^{(k)} < a_j^{(k+1)} < a_{j+1}^{(k)}. \tag{8}$$

The second table consists of numbers ± 1 if $\mathbf{F} = \mathbf{R}$ and of complex numbers (quaternions) of unit module if $\mathbf{F} = \mathbb{C}$ (if $\mathbf{F} = \mathbf{H}$). For $\mathbf{F} = \mathbb{C}$ or $\mathbf{H}$ it is necessary to do obvious transition to real parameters for $u_j^{(k)}$.

Let us note that relations (8) are analogous to the betweenness relations for the Gel'fand-Tsetlin patterns (see Section 18.1.1 below). This is a reflection of the mysterious duality, observed by I. M. Gel'fand. This duality displays in other branches of the theory of representations (for example, for expressions for matrix elements of irreducible representations of the group $SU(2)$ and its Clebsch-Gordan coefficients).

15.2.5. Spherical, conic and flag spaces. The homogeneous space $\mathfrak{P} = K\backslash G$ is connected with the Cartan decomposition $G = KAK$ of G. We call it a

[5] The parameters $u_1^{(0)}, \dots, u_m^{(0)}$ are omitted since the matrix Ω in the decomposition $\Lambda = \Omega A \Omega^*$ is defined up to the right multiplier $\Gamma \in U(m, \mathbf{F}) \cap D(m, \mathbf{F})$.

spherical space. If $G = SO_0(n-1,1)$ and $K = SO(n-1)$, then $\mathfrak{P}$ is the upper sheet of two-sheeted hyperboloid. The space $\mathfrak{P}$ can be parametrized by system of parameters for the group A and for the homogeneous space $B = M\backslash K$. There are other parametrizations of $\mathfrak{P}$. For example, the Iwasawa decomposition supplies the system of parameters giving elements of the subgroups A and N. As it was noted in Section 15.1.1, there exist "intermediate" systems of parameters for which $M\backslash K$ and N are replaced by semidirect products of compact and nilpotent subgroups. There are parametrizations related to decompositions of the type $G = KAH$, where the subgroup H is separated by an involutive automorphism commuting with the Cartan automorphism (see Section 15.1.11). In this case we have the parametrization of $\mathfrak{P}$ by parameters giving A and $M\backslash H$.

The homogeneous space $\mathfrak{X} = N_{-}M\backslash G$ is connected with the Gauss decomposition $G = N_{-}MAN$. We call it a *conic space.* If $G = SO_0(n-1,1)$, then it is realized as upper sheet of the cone $[\mathbf{x},\mathbf{x}] = 0$ of the pseudo-Euclidean space $E_{n-1,1}$. We can also consider the space $N_{-}AM\backslash G = \mathfrak{F}$. If $G = SO_0(n-1,1)$, then this space is realized as the sphere. In the general case it can be identified with the homogeneous space $B = M\backslash K$. For $G = GL(n,\mathbf{F})$ it can be realized as follows. A sequence $P_{n_1}, P_{n_2}, \ldots, P_{n_k}$ of linear subspaces in $\mathbf{F}$ such that $P_{n_1} \subset P_{n_2} \subset \ldots \subset P_{n_k}$ and $\dim P_{n_i} = n_i$ (therefore, $0 \le n_1 < n_1 < \ldots < n_k \le n$) are called a *flag of type* $(n_1, n_2, \ldots, n_k)$ over the field $\mathbf{F}$. Flags of type $(1, 2, \ldots, n)$ are called of the *principal type.* A space of flags (a flag space) of a given type is homogeneous with respect to action of the group $GL(n,\mathbf{F})$. It is easy to verify that the stationary subgroup of the flag of type $(1, 2, \ldots, n)$, for which P_k are the subspaces spanned by the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$, coincides with $D(n,\mathbf{F})N_{-}(n,\mathbf{F})$. Therefore, the space of flags of the principal type can be realized as the homogeneous space $N_{-}AM\backslash G$. The space of flags of the type (n_1) is the Grassmann space. In the case of a general semisimple connected group, a quotient space $P\backslash G$, where P is its parabolic subgroup, is called a *flag space.*

We can also consider different parametrizations on conic homogeneous spaces. For example, the Iwasawa decomposition $G = N_{-}AK$ shows that $N_{-}M\backslash G$ can be parametrized by parameters of elements from A and from $B = M\backslash K$. Instead of $M\backslash K$ we may take $M\backslash H$. In this case we have to take into account that $M\backslash H$ is not connected and its connected parts are indexed by elements of the Weyl group. Using the Bruhat decomposition (see Section 15.1.8) we obtain other parametrization of $N_{-}M\backslash G$ which is defined by parameters of elements from A and N and by elements of the Weyl group. As in the case of spherical spaces, "intermediate" systems of coordinates exist on conic spaces.

Let χ be a multiplicative character on the subgroup A, that is $\chi(a_1, a_2) = \chi(a_1)\chi(a_2)$. A function f on the conic space $\mathfrak{X} = N_{-}M\backslash G$ is called *homogeneous of degree* χ if for every $a \in A$ and $X \in \mathfrak{X}$ we have $f(aX) = \chi(a)f(X)$. It will be shown below that with the help of generalization of the Mellin transform any "good" function on $\mathfrak{X}$ can be expanded in homogeneous functions.

15.3. Invariant Metrics, Measures, and Differential Operators on Lie Groups and on Homogeneous Spaces

15.3.1. Relations between invariant measures on Lie groups. The integral relations between invariant measures on the groups are connected with the decompositions of Lie groups, considered in Section 15.1. At first we consider these relations for the group $GL(n, \mathbb{R})$. We shall use the following statements:

a) *the matrix* $X^{-1}dX$ (*the matrix* $dX \cdot X^{-1}$) *is invariant with respect to left (right) multiplication of* X *by* g;
b) *if* $\mathbf{w} = f(\mathbf{z})$ *is an analytical mapping* $\mathbb{C}^m \to \mathbb{C}^m$ *and* $J = D(\mathbf{w})/D(\mathbf{z})$ *is its Jacobian, then the Jacobian of the corresponding mapping for real and imaginary parts is equal to* $|J|^2$.

The first statement is obvious. The second statement for $n = 1$ follows from Cauchy-Riemann conditions. The general case is proved by the induction.

Using these statements it is easy to prove that the relation

$$d_*g = \beta^{-1}(g)d_*n_-d_*\delta d_*n_+ \tag{1}$$

corresponds to the Gauss decomposition $G = N_-DN_+$ of the group $G = GL(m, \mathbb{R})$. Here $g = n_-\delta n_+$ and d_*g, $d_*n_\pm$, $d_*\delta$ are the invariant measures on the groups $GL(m, \mathbb{R})$, $N_\pm(m, \mathbb{R})$, $D(m, \mathbb{R})$, given by the formulas[6]

$$d_*g = (\det g)^{-m} \prod_{i,j=1}^{m} dg_{ij}, \quad d_*n = \prod_{1 \le i < j \le m} dn_{ij} \text{ (or } dn_{ji}),$$

$$d_*\delta = \frac{d\delta_1 \dots d\delta_m}{|\delta_1| \dots |\delta_m|}, \quad \text{where} \quad \delta = \operatorname{diag}(\delta_1, \dots, \delta_m).$$

The function $\beta(\delta)$ from (1) coincides with

$$\beta(\delta) = \prod_{j=1}^{m} |\delta_j|^{2j-m-1}. \tag{2}$$

The analogous equality is valid for the Iwasawa decomposition of $GL(m, \mathbb{R})$. Namely, if $g = kan$, then

$$\int_G f(g)d_*g = \int_K \int_A \int_N f(kan)\beta^{-1}(a)d_*nd_*ad_*k, \tag{3}$$

[6] We denote by dX the matrix consisting of differentials of elements of X and by d_*X the invariant measure.

where the measure d_*k on K is normalized as $\int\limits_K d_*k = 1$.

Let us derive relations connected with the Cartan decomposition $G = KAK$ of the group $GL(m, \mathbf{R})$. In the decomposition $g = k_1 a k_2$ the matrix a is uniquely defined up to permutations of diagonal elements. The matrices k_1 and k_2 in this decomposition are determined uniquely up to replacement of k_1 and k_2 by $k_1 m$ and $m^{-1}k_2$, where m is a diagonal matrix with diagonal elements equal to ± 1. Therefore, the mapping

$$O(m) \times A(m) \times O(m) \to GL(m, \mathbf{R})$$

covers multiply the group $GL(m, \mathbf{R})$, namely, $2^m m!$ times.

If $g = k_1 a k_2$, then the invariant measure d_*g can be represented as

$$d_*g = \varphi(k_1, a, k_2) d_*k_1 d_*a d_*k_2.$$

Using the invariance of the measure d_*k on K it is easy to verify that the function $\varphi(k_1, a, k_2)$ does not depend on k_1 and k_2. Therefore, $d_*g = \omega(a) d_*k_1 d_*a d_*k_2$. It is necessary to evaluate the function $\omega(a)$. Since $\omega(a)$ is independent on k_1 and k_2, then it is sufficient to evaluate the Jacobian of transition from d_*g to $d_*k_1 d_*a d_*k_2$ at $k_1 = k_2 = I_m$.

We write down $g = k_1 a k_2$ in the form $g = k^{-1}k_2^{-1} a k_2$, where $k_1 = k^{-1}k_2^{-1}$. Then $k_2 k g = a k_2$ and in this reason

$$dk_2 \cdot kg + k_2 dk \cdot g + k_2 k dg = da \cdot k_2 + a dk_2. \tag{4}$$

We use the right multiplication of both sides of this relation by $g^{-1}k^{-1}k_2^{-1} = k_2^{-1}a^{-1}$ and set $k = k_2 = I_m$. We obtain

$$d_*k + d_*k_2 + d_*g = d_*a + a d_*k_2 \cdot a^{-1},$$

and, therefore

$$d_*g = d_*a - d_*k - d_*k_2 + a d_*k_2 \cdot a^{-1}.$$

Since $kk^t = I_m$, then $k(dk)^t + (dk)k^t = 0$, that is $(d_*k)^t = -d_*k$. Analogously, $(d_*k_2)^t = -d_*k_2$. Therefore, the matrix elements $(d_*k)_{pq}$, $p < q$, may be taken as parameters of the matrix $d_*k = dk \cdot k^{-1}$. The numbers $(d_*k_2)_{pq}$, $p < q$, and $(d_*a)_{pp}$, $1 \le p \le m$, are used as parameters of d_*k_2 and d_*a. For these parameters we have

$$(d_*g)_{pp} = (d_*a)_{pp}, \quad 1 \le p \le m,$$

$$\left.\begin{aligned} (d_*g)_{pq} &= -(d_*k)_{pq} + \left(\frac{a_p}{a_q} - 1\right)(d_*k_2)_{pq}, \\ (d_*g)_{qp} &= (d_*k)_{pq} - \left(\frac{a_q}{a_p} - 1\right)(d_*k_2)_{pq}, \end{aligned}\right\} \tag{5}$$

where $p < q$. It is easy to see from here that the Jacobian of transition from $(d_*g)_{pp}$ to $(d_*a)_{pp}$ is equal to 1. If p and q are fixed and $p < q$, then the Jacobian of transition from $(d_*g)_{pq}$ and $(d_*g)_{qp}$ to $(d_*k)_{pq}$ and $(d_*k_2)_{pq}$ is equal to $\frac{a_q}{a_p} - \frac{a_p}{a_q}$. It follows from here that

$$d_*g = c_m(a_1 a_2 \dots a_m)^{-m-1} \prod_{p<q} (a_p^2 - a_q^2) d_*k d_*a d_*k_2, \tag{6}$$

where d_*k, d_*a and d_*k_2 are invariant measures on K, A and K, respectively. Taking into account the normalization of the measure on K and the multiple cover of G by KAK, we obtain that

$$c_m = \pi^{m(m+1)} \prod_{j=1}^{m} \Gamma^{-1}\left(\frac{j}{2}\right) \Gamma^{-1}\left(\frac{j}{2} + 1\right). \tag{7}$$

Setting $a_k^2 = \exp t_k$, $1 \le k \le m$, in (6), we receive for $SL(m, \mathbf{R})$ that

$$d_*g = c'_m \left(\prod_{p<q} |\sinh(t_p - t_q)| \right) d_*k d_*k_2 dt_1 dt_2 \dots dt_{m-1}, \tag{6'}$$

where

$$c'_m = 2^{(m+1)(m-2)/2} c_m. \tag{7'}$$

The similar relations are valid for all connected semisimple (and reductive) Lie groups. Let $G = KAK$ and $G_k = KA_kK$ are the Cartan decompositions of dual noncompact and compact semisimple Lie groups. Let us introduce the functions

$$D(a) = \prod_{\gamma \in \Delta_+} |\sinh \gamma(H)|^{m(\gamma)}, \quad a = \exp H, \quad H \in \mathfrak{a}, \tag{8}$$

$$D_k(a') = \prod_{\gamma \in \Delta_+} |\sin \gamma(iH)|^{m(\gamma)}, \quad a' = \exp H, \quad H \in i\mathfrak{a}, \tag{9}$$

on the subgroups $A = \exp \mathfrak{a}$ and $A_k = \exp i\mathfrak{a}$, where Δ_+ is the set of positive restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$ (see Section 15.1.4). The relations

$$\int_G f(g) d_*g = c_1 \int_K \int_A \int_K f(k_1 a k_2) D(a) d_*k_1 d_*a d_*k_2, \tag{10}$$

$$\int_{G_k} f(g) d_*g = c_2 \int_K \int_{A_k} \int_K f(k_1 a k_2) D_k(a) d_*k_1 d_*a d_*k_2 \tag{11}$$

hold, where c_1 and c_2 are real coefficients.

Since $k_1 a k_2 = k_1 m a m^{-1} k_2$, $m \in M$, then formula (10) can be written as

$$\int_G f(g) d_* g = c_1 \int_K \int_A \int_{M\backslash K} f(k_1 a \tilde{k}_2) D(a) d_* k_1 d_* a d_* \tilde{k}_2, \qquad (10')$$

where $\tilde{k}_2 \in M\backslash K$. Formula (11) may be represented in the analogous way.

Let us note that $\gamma(H)$ is real if $\gamma \in \Delta_+$ and $H \in \mathfrak{a}$. Analogously, if $a = \exp H \in A_k$, then $H \in i\mathfrak{a}$ and $\gamma(iH)$ is real for $\gamma \in \Delta_+$. In this reason we have in (9) the product of sines of real arguments.

If $G = N_- A M N_+$ is the Gauss decomposition of the group G, then

$$d_* g = \beta^{-1}(a) d_* n_- d_* a d_* m d_* n_+, \qquad (12)$$

where $d_* g$, $d_* n_\pm$, $d_* a$, $d_* m$ are invariant measures on the groups G, $N_\pm$, A, M, respectively, and

$$\beta(a) = e^{-2\rho(H)}, \quad \rho = \frac{1}{2} \sum_{\gamma \in \Delta_+} m(\gamma)\gamma, \quad a = \exp H. \qquad (13)$$

If $G = KAN$ is the Iwasawa decomposition of the group G, then we have the relations

$$\begin{aligned} \int_G f(g) d_* g &= \int_K \int_N \int_A f(ank) d_* a d_* n d_* k \\ &= \int_N \int_A \int_K f(kan) \beta^{-1}(a) d_* k d_* a d_* n \\ &= \int_K \int_A \int_N f(nak) \beta^{-1}(a) d_* n d_* a d_* k, \end{aligned} \qquad (14)$$

where $\beta(a)$ is given by formula (13). The measure $d_* k$ is usually normalized as $\int d_* k = 1$, the invariant measure da on A is taken as $d_* a = dt_1 \dots dt_\ell$, where $a = \text{diag}\,(e^{t_1}, \dots, e^{t_\ell})$, and the invariant measure $d_* n$ on N is normalized by the condition

$$\int_N e^{-2\rho(H(n))} d_* n = 1,$$

where $H(g) \in \mathfrak{a}$ is defined by the Iwasawa decomposition

$$g = k(\exp H(g)) n', \quad k \in K, \quad \exp H(g) \in A, \quad n' \in N.$$

The invariant measure $d_* n_-$ on N_- is obtained from that on N with the help of the equality $d_* n_- = \tilde{\theta}(d_* n)$, where $\tilde{\theta}$ is the Cartan isomorphism.

Example 1. For the Iwasawa decomposition $G = ANK$ of the group $GL(n, \mathbb{C})$ we have

$$d_*g = d_*a d_*n d_*k, \tag{15}$$

where $d_*g = \prod_{i,j=1}^{n} dg_{ij} d\overline{g_{ij}}$, d_*k is the normalized invariant measure on $K = U(n)$, and

$$d_*n = \prod_{1 \le j < i \le n} dn_{ij} d\overline{n_{ij}}. \tag{16}$$

Let G_c be a complex semisimple Lie group with the Lie algebra $\mathfrak{g}_c$ and let Δ_+^c be the set of positive roots of $\mathfrak{g}_c$. We may consider G_c and $\mathfrak{g}_c$ as the real semisimple Lie group and algebra. Then positive restricted roots of $\mathfrak{g} \equiv \mathfrak{g}_c$ coincides with roots from Δ_+^c. Moreover, their multiplicities are equal to 2. Therefore, in this case

$$\int_{G_c} f(g) d_*g = c \int_K \int_A \int_K f(k_1 a k_2) |D(a)|^2 d_*k_1 d_*a d_*k_2, \tag{17}$$

where K is the maximal compact subgroup of G_c and

$$D(a) = \prod_{\alpha \in \Delta_+^c} \sinh \alpha(H), \quad a = \exp H.$$

Example 2. Using formula (17) for the group $SL(m, \mathbb{C})$ we obtain

$$\begin{aligned} \int_{SL(m,\mathbb{C})} f(g) d_*g = c \int_K \int_{\mathbb{R}^m} \int_K f(u_1 a u_2) \\ \times \prod_{j<k} |\sinh(t_j - t_k)|^2 d_*u_1 dt_1 \dots dt_{m-1} d_*u_2, \end{aligned} \tag{18}$$

where $a = \operatorname{diag}(e^{t_1}, \dots, e^{t_m})$ and $c = 2^{m-1} \pi^{m^2-1} \prod_{j=1}^{m-1} (j!)^{-2}$.

Example 3. Let $G = SU(p, q)$ and $G_k = SU(p+q)$, $K = S(U(p) \times U(q))$, $p+q = m$, $p \le q$. Then

$$\rho(H) = \sum_{j=1}^{p} \rho_j t_j, \quad \rho_j = 2(p-j) + q - p + 1, \quad H \in \mathfrak{a},$$

where $t_1, \dots, t_p$ are the Cartan coordinates. Therefore, $D(a) = \sigma\omega^2$, where

$$\sigma(a) = 2^{2p(q-p+1)} \prod_{j=1}^{p} \sinh^{2q-2p} t_j \sinh 2t_j, \tag{19}$$

$$\omega(a) = 2^{p(p-1)/2} \prod_{j<i} (\cosh 2t_j - \cosh 2t_i). \tag{20}$$

For $D_k(a)$ we have $D_k(a) = \sigma_k \omega_k^2$, where

$$\sigma_k(a) = 2^{2p(q-p+1)} \prod_{j=1}^{n} \sin^{2q-2p} t_j \sin 2t_j, \tag{21}$$

$$\omega_k(a) = 2^{p(p-1)/2} \prod_{j<i} (\cos 2t_j - \cos 2t_i). \tag{22}$$

15.3.2. Invariant metrics and measures on homogeneous cones. The Riemannian metric on the cone $\mathfrak{P}_m(\mathbf{F})$, invariant with respect to the transformations $\Lambda \to g^*\Lambda g$, $g \in GL(m, \mathbf{F})$, is given by the formula

$$ds^2 = \mathrm{Tr}\,((\Lambda^{-1} d\Lambda)^2), \tag{1}$$

where, as before, $\Lambda = (\lambda_{ij})$, $d\Lambda = (d\lambda_{ij})$ and for $\mathbf{F} = \mathbf{H}$ the matrices are replaced by the corresponding complex matrices. Let

$$\Lambda = \begin{pmatrix} I_p & X^* \\ 0 & I_q \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix} \tag{1'}$$

be the block Gauss decomposition for Λ (see Section 15.1.12). Then the formula

$$ds_\Lambda^2 = ds_V^2 + ds_W^2 + 2\mathrm{Tr}\,(V^{-1} dX \cdot W dX) \tag{2}$$

is valid, where ds_V (correspondingly ds_W) is an element of arc length in $\mathfrak{P}_p(\mathbf{F})$ (in $\mathfrak{P}_q(\mathbf{F})$) and $X = (x_{ij}) \in \mathfrak{M}_{qp}(\mathbf{F})$, $dX = (dx_{ij})$. We omit the proof of this formula. It is based on direct evaluations.

Let for $\Lambda \in \mathfrak{P}_m(\mathbf{F})$ we have $\Lambda = \Omega A \Omega^*$, $\Omega \in U(m, \mathbf{F})/(U(m, \mathbf{F}) \cap D(m, \mathbf{F}))$, $A \in A(m)$. Then

$$ds_\Lambda^2 = \sum_{j=1}^{m} a_j^{-2} da_j^2 + \mathrm{Tr}\,((A^{-1} d\Omega \cdot \Omega^{-1} A - d\Omega \cdot \Omega^{-1})^2). \tag{3}$$

The proof of this formula is also omitted.

The invariant Riemannian metric on $\mathfrak{P}_m(\mathbf{F})$ allows us to find the corresponding invariant measure. It is of the form

$$d_*\Lambda = \Delta^{-\theta}(\Lambda) \prod_{1 \le i < j \le m} \prod_{\zeta=1}^{\nu} d\lambda_{ij,\zeta} \prod_{i=1}^{m} d\lambda_{ii}, \tag{4}$$

where $\theta = \frac{1}{2}(2+(m-1)\nu)$ and $\Delta(\Lambda) = \det \Lambda$ for $\mathbf{F} = \mathbf{R}, \mathbb{C}$ and $\Delta(\Lambda) = \det^{1/2}\Lambda$ for $\mathbf{F} = \mathbf{H}$. (We shall show below that $\det \Lambda > 0$ for $\mathbf{F} = \mathbf{H}$.) In what follows we shall often write down formulas of the type (4) in the form $d_*\Lambda = \Delta^{-\theta}(\Lambda)d\Lambda$.

It is easy to verify that if Λ is represented in form (1′), then the Jacobian $\partial\Lambda/\partial(V, W, X)$ is equal to $\Delta^{p\nu}(W)$. If Λ is represented in the form

$$\Lambda = \begin{pmatrix} I_p & 0 \\ Y^* & I_q \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix}, \tag{4'}$$

then $\partial\Lambda/\partial(M, N, Y) = \Delta^{q\nu}(M)$. It follows from here that for the decomposition (1′)

$$d_*\Lambda = \Delta^{-q\nu/2}(V)\Delta^{p\nu/2}(W)d_*Vd_*WdX \tag{5}$$

and for the decomposition (4′)

$$d_*\Lambda = \Delta^{q\nu/2}(M)\Delta^{-p\nu/2}(N)d_*Md_*NdY. \tag{5'}$$

Every matrix $\Lambda \in \mathfrak{P}_m(\mathbf{F})$ can be represented as $\Lambda = SS^* = N^*AN$, where $S \in S_+^+(m, \mathbf{F})$, $N \in N_+(m, \mathbf{F})$, $A \in A(m)$. For this representation we have

$$d_*\Lambda = 2^m d_*Nd_*A. \tag{6}$$

Let $g = \Omega\Lambda \in GL(m, \mathbf{F})$, where $\Omega \in U(m, \mathbf{F})$, $\Lambda \in \mathfrak{P}_m(\mathbf{F})$, and let $\Lambda^*\Lambda = R$. Then

$$d_*g = \beta_m d_*\Omega d_*R, \tag{7}$$

where β_m is given by the formula

$$\beta_m = \pi^{\nu m(m+1)/4} \prod_{j=1}^{m} \left(\Gamma\left(\nu\frac{m-j+1}{2}\right)\right)^{-1}. \tag{8}$$

To the decomposition $\Lambda = \Omega^* A\Omega$ the formula

$$d_*\Lambda = j(A)d_*\Omega d_*A \tag{9}$$

corresponds, where

$$j(A) = \beta_m|\Delta(A)|^{-(m-1)\nu} \prod_{1\le i<j\le m} |a_i - a_j|^{\mu} \tag{10}$$

and β_m is given by formula (8), $\mu = \nu$ for $\mathbf{F} = \mathbf{R}, \mathbb{C}$ and $\mu = \nu/2$ for $\mathbf{F} = \mathbf{H}$.

Using the substitution $a_i = e^{t_i}$, $1 \le i \le m$, in formulas (9) and (10), after simple transformations we obtain for $\mathfrak{P}_m(\mathbf{R})$ that

$$d_*\Lambda = c \prod_{1\le i<j\le m} \left|\sinh\frac{t_i - t_j}{2}\right|^{\nu} d_*\Omega dt_1 \dots dt_m, \tag{9'}$$

where $c = 2^{m(m-1)/2}\beta_m$. This formula is generalized for wide class of homogeneous spaces.

Let G and G_k be dual noncompact and compact connected semisimple Lie groups such that G has a finite center. Let K be a maximal compact subgroup of G. We consider the homogeneous spaces $\mathfrak{X} = G/K$ and $\mathfrak{X}_k = G_k/K$. Every element $X \in \mathfrak{X}$ can be represented in the form $X = kak^{-1}$, where $k \in K$, $a \in A$. The invariant measure on $\mathfrak{X}$ is represented as

$$d_*X = c_1 j(a) d_*a d_*\hat{k}, \tag{11}$$

where $d_*\hat{k}$ is the invariant measure on K/M, c_1 is a constant, and

$$j(a) = \prod_{\gamma\in\Delta_+} \left|\sinh \frac{\gamma(H)}{2}\right|^{m(\gamma)}, \quad a = \exp H \tag{12}$$

($m(\gamma)$ is multiplicity of the restricted root γ of the pair $(\mathfrak{g}, \mathfrak{a})$). Analogously, every element $X \in \mathfrak{X}_k$ can be represented in the form $X = kak^{-1}$, where $k \in K$, $a \in A_k$. The invariant measure on $\mathfrak{X}_k$ can be written as

$$d_*X = c_2 j_k(a) d_*a d_*\hat{k}, \tag{13}$$

where $d_*\hat{k}$ is the invariant measure on K/M, c_2 is a constant, and

$$j_k(a) = \prod_{\gamma\in\Delta_+} \left|\sin \frac{\gamma(H)}{2}\right|^{m(\gamma)}, \quad a = \exp iH. \tag{14}$$

Let us note that functions (12) and (14) are obtained from functions (8) and (9) of Section 15.3.1 with the help of transition from $H \in \mathfrak{a}$ to $H/2$.

Formulas (11)-(14) allows us to obtain analogues of formulas (9) and (10) for homogeneous spaces.

a) Let $U = V\Theta V^{-1}$, where $U, V \in U(m)$, $\Theta = \mathrm{diag}\,(e^{i\theta_1}, \dots, e^{i\theta_m})$, $2\pi \geq \theta_1 \geq \theta_2 \geq \dots \geq \theta_m \geq 0$. We set $\tilde{U}(m) = U(m)/(U(m) \cap D(m, \mathbb{C}))$. Then

$$d_*U = (2\pi)^{-m} d_*\tilde{V} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \dots d\theta_m, \quad \tilde{V} \in \tilde{U}(m). \tag{15}$$

b) Let $H = \tilde{U}\Lambda\tilde{U}^*$, where $H \in \mathfrak{H}_m(\mathbb{C})$, $\Lambda = \mathrm{diag}\,(\lambda_1, \dots, \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$, $\tilde{U} \in \tilde{U}(m)$. Then

$$dH = \tilde{c} d_*\tilde{U} \prod_{j<k} (\lambda_j - \lambda_k)^2 d\lambda_1 \dots d\lambda_m, \quad \tilde{c} = (2\pi)^{m(m-1)/2} \prod_{j=1}^{m-1} (j!)^{-1}. \tag{16}$$

c) Any matrix $Z \in \mathfrak{M}_m(\mathbb{C})$ can be represented in the form $Z = U\Lambda V$, where $U, V \in U(m)$, $\Lambda \in A(m)$, $\Lambda = \text{diag}\,(\lambda_1, \dots, \lambda_m)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0$. We have

$$dZ = c d_* U d_* \tilde{V} \prod_{i \leq j} (\lambda_i - \lambda_j)^2 d\lambda_1 \dots d\lambda_m, \quad c = (\pi)^{m^2} \prod_{j=1}^{m-1} (j!)^{-2}. \tag{17}$$

d) Any matrix Z from $\mathfrak{S}_m(\mathbb{C})$ can be represented in the form $Z = U\Lambda U^{-1}$, where $U \in U(m)$ and Λ is such as in the previous case. Then

$$dZ = (2\pi)^{m(m-1)/2} \prod_{j=1}^{m-1} (j!)^{-1} \prod_{j<k} |\lambda_j^2 - \lambda_k^2| \lambda_1 \lambda_2 \dots \lambda_m d\lambda_1 \dots d\lambda_m d_* \tilde{U}. \tag{18}$$

For $T \in \mathfrak{S}_m(\mathbb{R})$ we have $T = \Gamma\Lambda\Gamma^t$, where $\Gamma \in SO(m)$ and Λ is of the same form as before. For the invariant measure d_*T we have

$$d_*T = \pi^{m(m+1)/4} 2^{m(m-1)/2} \prod_{j=1}^{m} \Gamma\left(\frac{j}{2}\right)^{-2} d_*\tilde{\Gamma} \prod_{i<j} |\lambda_i - \lambda_j| d\lambda_1 \dots d\lambda_m, \tag{19}$$

where $\tilde{\Gamma} \in SO(m)/W$, $W = \{\text{diag}\,(\gamma_1, \dots, \gamma_m) \mid \gamma_j = \pm 1\}$.

e) Any symmetric unitary matrix S is of the form $S = \Gamma\Lambda\Gamma^t$, where $\Gamma \in SO(m)$ and

$$\Lambda = \text{diag}\,(e^{i\theta_1}, \dots, e^{i\theta_m}), \quad \pi \geq \theta_i \geq \theta_2 \geq \dots \geq \theta_m \geq -\pi.$$

We have

$$d_*S = (2\pi^{1/2})^{m+1} \Gamma\left(\frac{m+1}{2}\right) \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}| d\theta_1 \dots d\theta_m d_*\tilde{\Gamma}, \tag{20}$$

where $\tilde{\Gamma} \in SO(m)/(SO(m) \cap D(m, \mathbb{R}))$.

f) Any matrix $Z \in \mathfrak{A}_m(\mathbb{C})$ can be represented as $Z = UMU^t$, where $U \in U(m)$ and

$$M = \begin{cases} \text{diag}\,(M_1, M_2, \dots, M_p, 0) & \text{if } m = 2p+1, \\ \text{diag}\,(M_1, M_2, \dots, M_p) & \text{if } m = 2p, \end{cases}$$

$$M_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}, \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0.$$

If $m = 2p+1$, then any matrix $K \in \mathfrak{A}_m(\mathbb{C}) \cap U(m)$ is of the form $K = \Gamma F \Gamma^t$, where $\Gamma \in SO(m)$ and

$$F = \text{diag}\,(F_1, F_2, \dots, F_p, 0), \quad F_k = \begin{pmatrix} 0 & \exp i\theta_k \\ -\exp i\theta_k & 0 \end{pmatrix},$$

$$\det F = 1, \quad \pi \geq \theta_1 \geq \theta_2 \geq \ldots \geq \theta_p \geq 0.$$

If $m = 2p$, then $K \in \mathfrak{A}_m(\mathbb{C}) \cap U(m)$ is represented in the same form $K = \Gamma F \Gamma^t$ and now the case $\det F = -1$ is admissible. We have

$$dK = c_m d_* \tilde{\Gamma} \prod_{1 \leq i < j \leq m} \sin^2(\theta_i - \theta_j) d\theta_1 \ldots d\theta_m, \tag{21}$$

where

$$\tilde{\Gamma} \in SO(m)/\Delta(m), \quad \Delta(m) = \{\operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_m)\},$$

$$\varepsilon_k = \begin{pmatrix} \cos\delta_k & \sin\delta_k \\ -\sin\delta_k & \cos\delta_k \end{pmatrix},$$

$$c_m = \begin{cases} 2^{2p(p-1)+p/2} a & \text{if } m = 2p, \\ 2^{2p(p-1)+3p/2} a & \text{if } m = 2p+1, \end{cases}$$

where a is the full volume of $\tilde{\Gamma}$.

Let us also give the invariant measure on the upper Siegel half-place $\mathfrak{T}_2(p)$ (see Section 15.2.3). Let $W \in \mathcal{R}_2(p)$ (see Section 15.2.2). Then

$$W = \begin{pmatrix} I_p & 0 \\ X & I_p \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} I_p & X \\ 0 & I_p \end{pmatrix},$$

where $V \in \mathfrak{P}_p(\mathbf{R})$ and $X \in \mathfrak{S}_p(\mathbf{R})$. The direct calculation shows that the invariant Riemannian metric on $\mathfrak{R}_2(p)$ is given by the formula

$$ds^2 = 2\operatorname{Tr}\,[(V^{-1}dV)^2 + (V^{-1}dX)^2]. \tag{22}$$

Passing from $\mathfrak{R}_2(p)$ to $\mathfrak{T}_2(p)$ (see Section 15.2.3) we obtain the invariant metric on $\mathfrak{T}_2(p)$:

$$ds^2 = 2\operatorname{Tr}\,(Y^{-1}dZ \cdot Y^{-1}d\bar{Z}), \tag{22'}$$

where $Z = X + iY$. It follows from here that the invariant measure on $\mathfrak{T}_2(p)$ is given by the formula

$$d_* Z = \Delta^{-p-1}(Y) \prod_{1 \leq i < j \leq p} dx_{ij} dy_{ij}, \quad Z = X + iY, \tag{23}$$

where $\Delta(Y)$ is determinant of the matrix Y.

15.3.3. Laplace operators on semisimple Lie groups and their radial parts. The ring of integral operators, commuting with the operators of the left and of the right shifts in the space of functions (that is with the operators $f(g) \to f(g_1^{-1} g g_2)$, $g_1, g_2 \in G$) is of a great importance for the theory of representations of compact Lie groups. These operators are defined by kernels of the type $K(g_1, g_2) =$

$\varphi(g_1 g_2^{-1})$, where φ is a function on the group G , constant on classes of conjugate elements.

If the group G is not compact, then functions, constant on classes of conjugate elements, generally speaking, can be non-integrable and the corresponding convolution can be divergent. In this case it is reasonable to consider the ring of differential operators, commuting with the operators of the left and of the right shifts. These operators are called *Laplace operators.* If differential operators are considered on spaces of functions, constant on cosets with respect to some subgroup H, then we obtain differential operators on the homogeneous space $\mathfrak{X} = G/H$. If they are considered on the functions, constant on the two-sided cosets HgH, then we receive *radial parts* of differential operators.

The Laplace-Beltrami operator

$$\Delta \equiv \Delta_{\mathfrak{X}} = g^{-1/2} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} g^{1/2} \sum_{j=1}^{n} g^{ij} \frac{\partial}{\partial x_j}$$

is defined on any Riemannian homogeneous space $\mathfrak{X}$, where

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j, \quad (g^{ij}) = (g_{ij})^{-1}, \quad g = \det(g_{ij}).$$

It is invariant with respect to the motion group G.

Example 1. Operators of the form

$$\operatorname{Tr}\left(\left(X\frac{\partial}{\partial X}\right)^j\right), \quad j \in \mathbb{Z}_+, \quad \text{where} \quad \frac{\partial}{\partial X} = \left(\eta_{ij}\frac{\partial}{\partial x_{ij}}\right), \quad \eta_{ij} = \frac{1}{2}(1+\delta_{ij}),$$

are invariant differential operators in the space of smooth functions on $\mathfrak{P}_m(\mathbf{R})$. The operators $\operatorname{Tr}\left(\left(X\frac{\partial}{\partial X}\right)^j\right)$, $1 \le j \le m$, form a basis in the space of invariant differential operators on $\mathfrak{P}_m(\mathbf{R})$. Using the expression for the Riemannian metric on $\mathfrak{P}_m(\mathbf{R})$ in the block Gauss coordinates we obtain the following result. Let

$$X = \begin{pmatrix} I_{m-1} & \mathbf{0} \\ \mathbf{x}^t & 1 \end{pmatrix} \begin{pmatrix} V & \mathbf{0} \\ \mathbf{0}^t & w \end{pmatrix} \begin{pmatrix} I_{m-1} & \mathbf{x} \\ \mathbf{0}^t & 1 \end{pmatrix},$$

where $V \in \mathfrak{P}_{m-1}(\mathbf{R})$, $w > 0$, $\mathbf{x} \in \mathbf{R}^{m-1}$. Then the invariant differential operator of the second order (the Laplace-Beltrami operator) on $\mathfrak{P}_m(\mathbf{R})$ is of the form

$$\Delta = \left(w\frac{\partial}{\partial w}\right)^2 + \frac{1-m}{2} w \frac{\partial}{\partial w} + \frac{1}{2} w \sum_{i,j=1}^{m-1} v^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$
$$+ \Delta_V + \frac{1}{2}\operatorname{Tr}\left(V\frac{\partial}{\partial V}\right),$$

where $(v^{ij}) = V^{-1}$. In the coordinates, defined by the decomposition $X = \Omega^t A \Omega$, $\Omega \in SO(m)$, $A = \mathrm{diag}\,(a_1, \ldots, a_m)$, of matrix $X \in \mathfrak{P}_m(\mathbf{R})$, this operator is of the form

$$\Delta = \sum_{k=1}^{m} a_k^2 \frac{\partial^2}{\partial a_k^2} - \frac{m-3}{2} \sum_{k=1}^{m} a_k \frac{\partial}{\partial a_k}$$
$$+ \sum_{k=1}^{m} \left(\sum_{\substack{j=1 \\ j \neq k}}^{m} \frac{a_k^2}{a_k - a_j} \right) \frac{\partial}{\partial a_k} + \frac{1}{2} \sum_{1 \le i < j \le m} \frac{a_j a_i}{(a_i - a_j)^2} \frac{\partial^2}{\partial x_{ij}^2}.$$

Let us formulate the main results on radial parts $\overset{\circ}{\Delta}$ of Laplace operators Δ.

A. Radial parts of Laplace operators on compact semisimple Lie groups. Let G_k be a compact semisimple Lie group. We imbed "anti-diagonally" G_k into $G_k \times G_k$: $G_k \ni g \to (g, g^{-1}) \in G_k \times G_k$. Motions in G_k are given by the formula

$$(g_1, g_2) \circ (g, g^{-1}) = (g_1^{-1} g g_2, g_2^{-1} g^{-1} g_1).$$

The subgroup $H = \{(g, g)\} \in G_k \times G_k$ is stationary for the element (e, e). Therefore, classes of conjugate elements in G_k are "spheres" for these motions. Intersection of every of these classes with the Cartan subgroup $A_k \subset G_k$ has finite number of elements. Moreover, these elements are transformed each into other by elements of the Weyl group. In this reason radial parts of Laplace operators on G_k are differential operators in the space of W-invariant functions on A_k.

The exponential mapping allows us to transfer these operators onto W-invariant functions on the Cartan subalgebra $\mathfrak{a}_k$. Let us consider on $\mathfrak{a}_k$ two coordinate systems $t^1, \ldots, t^\ell$ and $t_1, \ldots, t_\ell$, related by the relation $t_p = \sum_q g_{pq} t^q$, where (g_{pq}) is the matrix of the Cartan scalar product on $\mathfrak{a}_k$. Radial parts of Laplace operators on the group G_k are given by the formula

$$\overset{\circ}{\Delta}_k\,(P) = \frac{1}{j_k(\mathbf{t})} P\left(\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^\ell}\right) j_k(\mathbf{t}), \tag{1}$$

where $P(t_1, \ldots, t_\ell)$ is an arbitrary W-invariant polynomial on the Cartan subalgebra $\mathfrak{a}_k$ and $j_k(\mathbf{t}) \equiv j_k(t_1, \ldots, t_\ell)$ is defined by the formula

$$j_k(h) = \prod_{\alpha \in \Delta_+^c} \sin \frac{\alpha(H)}{2},$$

where $h = \exp H \in A_k$ and Δ_+^c is the set of positive roots of the complex Lie algebra $\mathfrak{g}_c$ (which is the complexification of the Lie algebra of the group G_k).

B. Radial parts of Laplace operators on complex semisimple Lie groups. Radial parts of Laplace operators on a complex semisimple Lie group G_c act in the space of complex analytical functions on G_c. They are of the form

$$\overset{\circ}{\Delta}_c(P) = \frac{1}{j_c(\mathbf{t})} P\left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^\ell}\right) j_c(\mathbf{t}), \tag{2}$$

where $P(t_1, \dots, t_\ell)$ is a W-invariant polynomial, t_p, $p = 1, 2, \dots, \ell$, and t^p, $p = 1, 2, \dots, \ell$, are complex coordinates on the Cartan subalgebra $\mathfrak{a}_c$ and $\mathfrak{g}_c$, which are defined as in the previous case, and $j_c(\mathbf{t})$ is determined by the formula

$$j_c(a) = \prod_{\alpha \in \Delta_+^c} \sinh \frac{\alpha(H)}{2}, \quad h = \exp H \in \mathfrak{a}_c.$$

The group G_c can be considered as a real Lie group with doubled number of parameters. Instead of real coordinates σ^k, τ^k, $k = 1, 2, \dots, \ell$, it is convenient to consider the coordinates t^k, $\bar{t}^k$, $k = 1, 2, \dots, \ell$, where $t^k = \tau^k + i\sigma^k$, $\bar{t}^k = \tau^k - i\sigma^k$, and the derivatives

$$\frac{\partial}{\partial t^k} = \frac{1}{2}\left(\frac{\partial}{\partial \tau^k} - i\frac{\partial}{\partial t^k}\right), \quad \frac{\partial}{\partial \bar{t}^k} = \frac{1}{2}\left(\frac{\partial}{\partial \tau^k} + i\frac{\partial}{\partial \sigma^k}\right).$$

In these coordinates we have the following formulas for radial parts of Laplace operators on G_c:

$$\overset{\circ}{\Delta}_c(P) = \frac{1}{j_c(\mathbf{t})} P\left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^\ell}\right) j_c(\mathbf{t}), \tag{3}$$

$$\overset{\circ}{\Delta}_{\bar{c}}(P) = \frac{1}{\overline{j_c(\mathbf{t})}} P\left(\frac{\partial}{\partial \bar{t}^1}, \dots, \frac{\partial}{\partial \bar{t}^\ell}\right) \overline{j_c(\mathbf{t})}, \tag{4}$$

where $j_c(\mathbf{t})$ and $P(t_1, \dots, t_\ell)$ are of the same sense as in formula (2). Operators (3) and (4) act in spaces of functions of real variables on the complex group G_c.

C. Radial parts of Laplace operators on homogeneous spaces G/K. Let G be semisimple noncompact Lie group with a finite center, K be its maximal compact subgroup, and $\mathfrak{X} = G/K$. Let A be the subgroup $\exp \mathfrak{a}$ and W be the corresponding Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. We introduce on $\mathfrak{a}$ two coordinate systems $t_1, \dots, t_\ell$ and $t^1, \dots, t^\ell$ which are related in the same way as above. Radial parts of Laplace operators on $\mathfrak{X}$ are given by the formula

$$\overset{\circ}{\Delta}_{\mathfrak{X}}(P) = \frac{1}{j(\mathbf{t})} P\left(\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^\ell}\right) j(\mathbf{t}), \tag{5}$$

where $P(t^1, \dots, t^\ell)$ is a W-invariant polynomial on $\mathfrak{a}$ and $j(\mathbf{t})$ is given by the formula (14) of Section 15.3.2.

Let $G = KAN$ be the Iwasawa decomposition of the group G. We denote by $\mathfrak{R}$ the space of smooth functions on G such that $f(kan) = f(a)$, $k \in K$, $a \in A$, $n \in N$. It can be considered as the space of functions on $\mathfrak{X} = K\backslash G$, constant on trajectories of the subgroup N. These trajectories and their shifts are called *orispheres* in $\mathfrak{X}$. Since Laplace operators commute with shift operators, then $f \in \mathfrak{R}$ leads to $\Delta f \in \mathfrak{R}$ for all Laplace operators Δ. Therefore, we may restrict Laplace operators onto $\mathfrak{R}$. These restrictions are called *orispherical radial parts* of Laplace operators Δ and are denoted by $\overset{\circ}{\Delta}_h$.

Orispherical radial parts of Laplace operators can be considered as differential operators in the space of W-invariant functions on $\mathfrak{a}$. It is shown that in the Cartan coordinates $t_1, \ldots, t_\ell$ they are differential operators with constant coefficients. They are given by the formula

$$\overset{\circ}{\Delta}_h = e^{i(\rho,H)} P e^{-i(\rho,H)}, \quad H \in \mathfrak{a}, \tag{6}$$

where ρ is the half-sum of positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and P is arbitrary differential operator with constant coefficients in the Cartan coordinates on the space of W-invariant smooth functions on $\mathfrak{a}$.

Orispherical radial parts of Laplace operators can be obtained from their radial parts by the limit procedure when center of the sphere is removed to the infinity along some geodesic line on $\mathfrak{X}$. This procedure transforms spheres into orispheres and eigenfunctions of radial parts of Laplace operators into eigenfunctions of their orispherical radial parts.

Chapter 16.
Representations of Semisimple Lie Groups and Their Matrix Elements

16.1. Irreducible Finite Dimensional Representations of Lie Groups

16.1.1. Representations of Lie groups with normal Gauss decompositions. One of fundamental results of the theory of finite dimensional representations is the following theorem (see, for example, reference [58] of the first volume).

Theorem (S. Lie). *Every irreducible finite dimensional representation of connected solvable Lie group in a complex vector space is of dimension* 1.

Let a semisimple connected Lie group G has the normal Gauss decomposition $G = N_-MAN_+$ (that is, the subgroup $D = MA$ is commutative). Then according to the Lie theorem a restriction of an irreducible finite dimensional representation T of G onto the subgroup $S_- = N_-MA$ can be reduced to the triangular form

$$\begin{pmatrix} \mu_1(s_-) & & & \\ & \mu_2(s_-) & & * \\ 0 & & \ddots & \\ & & & \mu_r(s_-) \end{pmatrix}, \tag{1}$$

where $\mu_1, \mu_2 \ldots, \mu_r$ are characters of the solvable group S_-, that is functions on S_- such that $\mu_j(s_1 s_2) = \mu_j(s_1)\mu_j(s_2)$. Therefore, these characters are uniquely defined by their values on the commutative subgroup $D = MA$, that is, they are characters on this subgroup:

$$\mu_j(s_-) = \mu_j(\delta), \quad \text{where} \quad s_- = n_-\delta, \quad n_- \in N_-, \quad \delta \in D.$$

If the representation T is realized by right shifts on the linear span V of the functions $t_{1j}(g)$, $1 \le j \le r$ (that is, T is imbedded into regular representation of G), then for all $f \in V$ we have

$$f(n_-\delta g) = \mu_1(\delta) f(g). \tag{2}$$

In particular, $f(n_-\delta n_+) = \mu_1(\delta) f(n_+)$, $n_+ \in N_+$. Therefore, the functions f from the carrier space V of the representation T are uniquely defined by their values on the subgroup N_+ and representation operators are given by the formula

$$(T(g)f)(n) = \alpha(\delta) f(\tilde{n}), \tag{3}$$

where $\alpha(\delta) = \mu_1(\delta)$, $ng = n_-\delta\tilde{n}$, $n_- \in N_-$, $n, \tilde{n} \in N_+$, $\delta \in D$. In particular, for $n_1 \in N_+$ we have

$$(T(n_1)f)(n) = f(nn_1). \tag{4}$$

If $\delta \in D$, then $n\delta = \delta(\delta^{-1}n\delta)$ and $\delta^{-1}n\delta \in N_+$. Thus,

$$(T(\delta)f)(n) = \alpha(\delta)f(\delta^{-1}n\delta).$$

Let us apply the Lie theorem to the solvable subgroup $S_+ = DN_+$ of G. It follows from this theorem that there is a vector in V which is an eigenvector for all operators $T(s_+)$, $s_+ \in S_+$. This vector is invariant with respect to operators $T(n_+)$, $n_+ \in N_+$. According to formula (4), there is a unique N_+-invariant vector in V. Up to normalization it coincides with $f(n) \equiv 1$.

Thus, we prove that *every finite dimensional irreducible representation T of a semisimple Lie group G with a normal Gauss decomposition can be realized by formula* (3) *in a space of functions on N_+*. Not for every character α of the subgroup D formula (3) defines a finite dimensional representation of G. If α defines a finite dimensional representation of G, then it is called an *inductive character*. We say that representation (3) of G is *induced* by the character α. It follows from (3) that an inductive character is uniquely defined by the corresponding representation. Moreover, representations of G are non-equivalent if they are induced by different characters.

Since T is a finite dimensional representation, then its restriction onto the commutative subgroup D decomposes into a direct sum of one-dimensional representations of D. Non-zero vectors from the corresponding one-dimensional subspaces are called *weight vectors*. The functions $\beta(\delta)$ such that $T(\delta)\mathbf{x} = \beta(\delta)\mathbf{x}$ for some vectors $\mathbf{x} \in V$ are called *weights* of these vectors.

It follows from the above reasonings that *in a carrier space of an irreducible finite dimensional representation T of a semisimple Lie group G with a normal Gauss decomposition there is one and (up to a normalization) only one vector $\mathbf{f}_0$ invariant with respect to the subgroup N_+: $T(n)\mathbf{f}_0 = \mathbf{f}_0$. It is a weight vector, that is $T(\delta)\mathbf{f}_0 = \alpha_0(\delta)\mathbf{f}_0$*. The character α_0 is called the *highest weight* of T and $\mathbf{f}_0$ is called the *vector of highest weight* (or the *highest weight vector*).

Thus, the problem of decomposition of a finite dimensional reducible representation T of the group G into irreducible components is reduced to finding the maximal subspace V_0 of the carrier space V of T, consisting of invariant vectors for the operators $T(n_+)$, $n_+ \in N_+$, and to evaluation of the "weight spectrum" of V_0. The irreducible representation, corresponding to a given highest weight vector $\mathbf{f}_0 \in V_0$, acts in the linear span of the vectors $V(g)\mathbf{f}_0$, $g \in G$.

The following statement is useful for finding inductive characters α. *If G_0 is a semisimple subgroup of G and if intersection of components of the Gauss decomposition of G with G_0 gives components of normal Gauss decomposition of this subgroup, then the restriction α_0 of an inductive character α for G onto the corresponding commutative Gauss component D_0 of G_0 is an inductive character for G_0.*

Really, if the linear span of the functions $f_g(n) = \alpha(ng)$, $n \in N_+$, $g \in G$, is of finite dimension, then it is the case for the linear span V_0 of the functions $f_{g_0}(n_0) = \alpha(n_0 g_0)$, $n \in N_+ \cap G_0$, $g_0 \in G_0$. The representation of G_0 in V_0 is irreducible, since V_0 is cyclically spanned by the unique highest weight vector $f(n_0) \equiv 1$. $n_0 \in N_+ \cap G_0$. Therefore, the character α_0 is inductive.

16.1.2. Finite dimensional irreducible representations of classical complex Lie groups. Almost for all $g \in GL(n,\mathbb{C})$ the Gauss decomposition $g = n_-\delta n_+$ (see Section 15.1.1) is valid, where $n_\pm \in N_\pm(n,\mathbb{C})$, $\delta \in D(n,\mathbb{C})$. Let $\Delta_{\mathbf{ij}}$, where $\mathbf{i} = (i_1, \dots, i_p)$, $\mathbf{j} = (j_1, \dots, j_p)$, be the minor of a matrix g, situated on the intersections of rows with indices $i_1, \dots, i_p$ and columns with indices $j_1, \dots, j_p$. Let Δ_p be the p-th main minor of g. The minor, obtained by adding to Δ_p the μ-th row below and ν-th column on the right, will be denoted by $\Delta_p^{\mu\nu}$. It is simple to verify that if $g = n_-\delta n_+$, then

$$(n_-)_{pq} = \frac{\Delta_{q-1}^{pq}}{\Delta_q}, \quad \delta_p = \frac{\Delta_p}{\Delta_{p-1}}, \quad (n_+)_{pq} = \frac{\Delta_{p-1}^{pq}}{\Delta_p}, \tag{1}$$

where all minors are taken for the matrix g. We have $\Delta_p(g) = \Delta_p(\delta)$ and, consequently, the main minors $\Delta_p(g)$ depend only on the diagonal elements $\delta_1, \dots, \delta_n$ of the matrix δ.

Every complex-analytical character α of the group $D(n,\mathbb{C})$ is of the form $\alpha(\delta) = \delta_1^{m_1} \dots \delta_n^{m_n}$, where $\delta = \operatorname{diag}(\delta_1, \dots, \delta_n)$. Let us prove that this character is inductive for the group $GL(n,\mathbb{C})$ if and only if the *order relations*

$$m_1 \geq m_2 \geq \dots \geq m_n \tag{2}$$

are satisfied. In order to prove it we have to restrict the character of D onto the subgroup D_0 of matrices of the form $\operatorname{diag}(\lambda, \lambda^{-1}, 1, \dots, 1)$ and to take into account that the obtained character have to be inductive for the group G_0 of matrices of the form $\operatorname{diag}(g_0, 1, \dots, 1)$, $g_0 \in SL(2,\mathbb{C})$. This character is of the form $\alpha_0(\lambda) = \lambda^{m_1 - m_2}$. It is inductive only for $m_1 \geq m_2$ (we omit the trivial proof of this statement based on the fact that for $m < 0$ the space of linear combinations of the binomials $(\beta z + \delta)^m$, $\beta, \delta \in \mathbb{C}$, is infinite dimensional). Thus, $m_1 \geq m_2$. Shifting the subgroup $SL(2,\mathbb{C})$ along the main diagonal, we are convinced that $m_1 \geq m_2 \geq \dots, \geq m_n$. Below we write down the character α in the form $\alpha(\delta) = \boldsymbol{\delta}^{\mathbf{m}}$, where $\boldsymbol{\delta} = (\delta_1, \dots, \delta_n)$, $\mathbf{m} = (m_1, \dots, m_n)$. We shall also use the notation $\alpha(\delta) = \Delta_1^{r_1}\Delta_2^{r_2} \dots \Delta_n^{r_n}$, where $\Delta_p = \delta_1\delta_2 \dots \delta_p$, $r_p = m_p - m_{p+1}$ (supposing that $m_{n+1} = 0$). If $g = n_-\delta n_+$, then

$$\alpha(g) = \Delta_1^{r_1}(\delta) \dots \Delta_n^{r_n}(\delta). \tag{3}$$

Therefore, if all r_p are non-negative integers, then the functions $f_g(n_+) = \alpha(n_+ g)$ are polynomials in elements of the matrix n_+. Moreover, powers of these polynomials are uniformly bounded in $g \in GL(n,\mathbb{C})$. Consequently, the linear span of the functions $f_g(n_+)$ is a finite dimensional space and, therefore, the character α is inductive. Collections of numbers $\mathbf{m} \equiv (m_1, \dots, m_n)$ defining irreducible finite dimensional representations of $GL(n,\mathbb{C})$ are called *signatures* of these representations. They are also called *highest weights* of representations. The representation with signature $\mathbf{m}$ is denoted by $T_{\mathbf{m}}$.

We have constructed (up to an equivalence) all complex-analytical irreducible representations of the group $GL(n,\mathbb{C})$. Anti-analytical irreducible representations of $GL(n,\mathbb{C})$ are constructed in the same way (we have to replace the subgroup $N_+(n,\mathbb{C})$ by $\bar{N}_+(n,\mathbb{C})$, that is, the elements n_{ij} by $\overline{n_{ij}}$). Every irreducible finite dimensional representation of the group $GL(n,\mathbb{C})$ is a tensor product of analytical and anti-analytical irreducible representations, that is, it is of the form $T_{\mathbf{m}} \otimes \bar{T}_{\mathbf{m}'}$.

The analogous statements are valid for the group $SL(n,\mathbb{C})$ which is a subgroup of $GL(n,\mathbb{C})$. The difference is that now the numbers $m_1,\ldots,m_n$ are defined up to a common summand, that is, only the numbers $r_p = m_p - m_{p+1}$, $p = 2,3,\ldots,n$, are fixed by representations. Therefore, we may take the numbers $m_1,\ldots,m_n$ in such way that $m_1 \geq m_2 \geq \ldots \geq m_n = 0$.

Irreducible finite dimensional representations of the groups $SO(n,\mathbb{C})$ and $Sp(\ell,\mathbb{C})$ can be described in the similar manner. We realize the group $Sp(\ell,\mathbb{C})$ as it was described in Section 15.1.2 and index the basis vectors as $\mathbf{e}_{-\ell}, \mathbf{e}_{-\ell+1},\ldots,\mathbf{e}_{-1}$, $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_\ell$. Then the Gauss decomposition of the group $GL(2\ell,\mathbb{C})$ gives the corresponding Gauss decomposition for the group $Sp(\ell,\mathbb{C})$ (by intersecting the subgroups $N_\pm(n,\mathbb{C})$ and $D(n,\mathbb{C})$, $n = 2\ell$, with $Sp(\ell,\mathbb{C})$). A diagonal matrix $\delta = \mathrm{diag}\,(\delta_1,\ldots,\delta_{2\ell}) \in GL(2\ell,\mathbb{C})$ belongs to $Sp(\ell,\mathbb{C})$ if and only if $\delta_k = \delta^{-1}_{2\ell-k+1}$, $1 \leq k \leq \ell$. These diagonal matrices form the subgroup $D_0(2\ell,\mathbb{C}) = D(2\ell,\mathbb{C}) \cap Sp(\ell,\mathbb{C})$. Every complex-analytical character of this subgroup is of the form

$$\alpha(\delta) = \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_\ell^{m_\ell}. \tag{4}$$

This character is inductive for the group $Sp(\ell,\mathbb{C})$ if and only if all m_k are nonnegative integers such that $m_1 \geq m_2 \geq \ldots \geq m_\ell \geq 0$. The corresponding finite dimensional irreducible representation of $Sp(\ell,\mathbb{C})$ is given by formula (3) of Section 16.1.1 and has signature (highest weight) $(m_1,m_2,\ldots,m_\ell)$.

The situation is somewhat more complicated in the case of the orthogonal group. We index the basis vectors as $\mathbf{e}_{-\ell},\mathbf{e}_{-\ell+1},\ldots,\mathbf{e}_{-1},\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_\ell$ if $n = 2\ell$ and as $\mathbf{e}_{-\ell},\ldots,\mathbf{e}_{-1},\mathbf{e}_0,\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_\ell$ if $n = 2\ell+1$ (see Section 15.1.2). The diagonal matrix $\delta = \mathrm{diag}\,(\delta_{-\ell},\ldots,\delta_\ell)$ belong to $SO(n,\mathbb{C})$ if and only if $\delta_k = \delta_{-k}^{-1}$, $1 \leq k \leq \ell$. We denote by $D_0(n,\mathbb{C})$ the set of such diagonal matrices. A complex-analytical character α of $D_0(n,\mathbb{C})$ is of the form

$$\alpha(\delta) = \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_\ell^{m_\ell}. \tag{5}$$

This character is a highest weight of an one-valued irreducible finite dimensional representation of the group $SO(n,\mathbb{C})$ if and only if all numbers m_k are integers and satisfy the ordering relations

$$m_1 \geq m_2 \geq \ldots \geq m_{\ell-1} \geq |m_\ell| \quad \text{for } n = 2\ell,$$
$$m_1 \geq m_2 \geq \ldots \geq m_\ell \geq 0 \quad \text{for } n = 2\ell+1.$$

The group $SO(n,\mathbb{C})$ is not simply connected. Therefore, it possesses two-valued irreducible representations. Signatures $\mathbf{m} = (m_1, \ldots, m_\ell)$ of these representations consists of half-integral numbers satisfying the same ordering conditions.

These statements are proved in the same way as in the case of the group $GL(n,\mathbb{C})$. Using the subgroups, isomorphic to $SL(2,\mathbb{C})$, we obtain that signatures have to satisfy the condition $m_1 \geq m_2 \geq \ldots \geq m_\ell$. Considering the representations $\tilde{T}(g) = T(s_0 g s_0^{-1})$, where s_0 is an element of the group $O(n,\mathbb{C})$ which does not belong to $SO(n,\mathbb{C})$ (for example, $s_0 = -I_n$ if $n = 2\ell+1$ and $s_0 = \operatorname{diag}(1,\ldots,1,s,1,\ldots,1)$, $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, if $n = 2\ell$), we obtain the condition $m_1 \geq m_2 \geq \ldots \geq m_{\ell-1} \geq -m_\ell$. These conditions lead to the above ordering relations. The numbers $m_k - m_\ell$ and $m_k + m_\ell$ must be integers. Therefore, $m_1, \ldots, m_\ell$ are simultaneously integers or simultaneously half-integers.

The characters, described above, are inductive. Really, considering, if necessary, the representation $\tilde{T}$ instead of T, we can assume that $m_\ell \geq 0$. The character with the parameters $(m_1, \ldots, m_\ell, 0, \ldots, 0)$ is inductive with respect to the group $GL(n,\mathbb{C})$. Therefore, the restriction of this character onto $D_0(n,\mathbb{C})$ is inductive for the group $SO(n,\mathbb{C})$.

If in formula (5) for the character the numbers m_k are half-integers, then we can write this character in the form $\alpha(\delta) = \Delta_1^{r_1}\Delta_2^{r_2}\ldots\Delta_\ell^{r_\ell}$, where r_ℓ is half-integer and other r_k are integers if $n = 2\ell+1$. If $n = 2\ell$, then the numbers $r_{\ell-1}$ and r_ℓ may be half-integers. It follows from here that for $n = 2\ell+1$ the ℓ-th main minor $\Delta_\ell(n_+, g)$ of the matrix $n_+ g$, $n \in N_+$, is a square of some polynomial in elements of the matrix n_+:

$$\Delta_\ell(n_+, g) = \mathfrak{S}_0^2(n_+, g). \tag{6}$$

The formula

$$S(g)f(n_+) = f(n_+ g), \quad n_+ \in N, \tag{7}$$

gives a representation of the group $SO(n,\mathbb{C})$ in the linear span of the functions $f_g(n_+) = \mathfrak{S}_0(n_+, g)$. This representation is two-valued:

$$S(g_1 g_2) = \pm S(g_1)S(g_2). \tag{8}$$

Moreover, this two-valued property can not be avoided. The corresponding linear objects are called *spinors*. If $n = 2\ell$, then there are two polynomials $\mathfrak{S}_+$ and $\mathfrak{S}_-$ on the subgroup N_+ with the properties

$$\Delta_{\ell-1}(n_+, g) = \mathfrak{S}_-(n_+, g)\mathfrak{S}_+(n_+, g), \quad \Delta_\ell(n_+, g) = \mathfrak{S}_+^2(n_+, g). \tag{9}$$

The corresponding linear objects are *called spinors of the first and of the second types* respectively.

With the help of the introduced polynomials the characters can be written as

$$\Delta_1^{r_1}\Delta_2^{r_2}\ldots\Delta_{\ell-1}^{r_{\ell-1}}\mathfrak{S}_0^{r_\ell} \tag{10}$$

if $n = 2\ell + 1$, and as

$$\Delta_1^{r_1}\Delta_2^{r_2}\dots\Delta_{\ell-2}^{r_{\ell-1}}\mathfrak{S}_+^{r_+}\mathfrak{S}_-^{r_-} \tag{11}$$

if $n = 2\ell$.

It we go over from the group $SO(2\ell, \mathbb{C})$ to the group $O(2\ell, \mathbb{C})$, then conjugate pairs of irreducible representations (that is, the representations with signatures $(r_1, \dots, r_{\ell-2}, r_+, r_-)$ and $(r_1, \dots, r_{\ell-2}, r_-, r_+)$ become irreducible representations of $O(2\ell, \mathbb{C})$. If $n = 2\ell + 1$, then an irreducibility of representations for $SO(n, \mathbb{C})$ is equivalent to an irreducibility for $O(n, \mathbb{C})$.

Along with finite dimensional irreducible analytical representations of $SO(n, \mathbb{C})$ we can consider anti-analytical representations. Any irreducible finite dimensional representation of $SO(n, \mathbb{C})$ is a tensor product of analytical and anti-analytical irreducible representations.

16.1.3. Block Gauss decompositions and representations. In order to obtain degenerate series of finite dimensional representations by the methods of Section 16.1.2 we have to use block Gauss decompositions. For elements $g \in GL(n, \mathbb{C})$ and for partition $n = p + q$ we have the decomposition

$$g = \begin{pmatrix} I_p & 0 \\ X & I_q \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix}, \tag{1}$$

where $X \in \mathfrak{M}_{qp}(\mathbb{C})$, $Y \in \mathfrak{M}_{pq}(\mathbb{C})$, $A \in GL(p, \mathbb{C})$, $B \in GL(q, \mathbb{C})$. Multiplying the matrices from the right hand side, we obtain

$$g \equiv \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} A & AY \\ XA & XAY + B \end{pmatrix},$$

where g_{ij} are block matrices. Therefore, if $\det g_{11} \neq 0$, then

$$A = g_{11}, \quad X = g_{21}g_{11}^{-1}, \quad Y = g_{11}^{-1}g_{12}, \quad B = g_{22} - g_{21}g_{11}^{-1}g_{12}. \tag{2}$$

Let Q be the irreducible representation of $GL(p, \mathbb{C})$ with signature $\mathbf{m} = (m_1, m_2, \dots, m_p)$ in the space L. Then the formula

$$(T_Q(g)\mathbf{f}(Y) = Q(\tilde{A})\mathbf{f}(\tilde{Y}) \tag{3}$$

define an irreducible degenerate representation of $GL(n, \mathbb{C})$ with signature $(m_1, m_2, \dots, m_p, 0, \dots, 0)$, where $\tilde{A}$ and $\tilde{Y}$ are determined by the block Gauss decomposition (1) of the matrix $\begin{pmatrix} I_p & Y \\ 0 & I_q \end{pmatrix} g$ and $\mathbf{f}$ are functions with values in L. It is easy to verify that

$$\tilde{A} = g_{11} + Yg_{21}, \quad \tilde{Y} = (g_{11} + Yg_{21})^{-1}(g_{12} + Yg_{22}). \tag{4}$$

Degenerate finite dimensional irreducible representations of the groups $SO(n, \mathbb{C})$ and $Sp(p,\mathbb{C})$ are constructed analogously. The analogue of decomposition (1) for these groups is of the form

$$g = \begin{pmatrix} I_p & 0 \\ X & I_p \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} \begin{pmatrix} I_p & Y \\ 0 & I_p \end{pmatrix}. \tag{5}$$

We recommend to the reader to verify that the matrices diag $(A,(A^*)^{-1})$, $A \in GL(p,\mathbb{C})$, form a subgroup in $SO(n,\mathbb{C})$ and in $Sp(p,\mathbb{C})$.

16.1.4. The Kostant theorem on separation of variables. Let X be a complex linear space and let G be a connected Lie group which is a complexification of a compact Lie group of linear transformations of X. Let $\mathcal{P}(X)$ be the ring of polynomials in coordinates of vectors from X with respect to some basis, and let $\mathcal{I}(X)$ be the subring of G-invariant polynomials from $\mathcal{P}(X)$. The set of polynomials from $\mathcal{I}(X)$ with vanishing numerical summand is denoted by $\mathcal{I}^+(X)$. In what follows we shall omit X in $\mathcal{P}(X)$, $\mathcal{I}(X)$, $\mathcal{I}^+(X)$.

The following assertion is proved with the help of induction in powers of polynomials. *If $\mathcal{L}$ is a subspace of $\mathcal{P}$ such that $\mathcal{P} = \mathcal{I}\mathcal{P} + \mathcal{L}$ is a direct sum decomposition, then $\mathcal{P} = \mathcal{I}\mathcal{L}$.*

Statement 1. *Under the same conditions the following assertions are equivalent:*

a) *The mapping $\mathcal{I}\otimes\mathcal{L} \to \mathcal{P}$, given by the formula $j\otimes\ell \to j\ell$, is an isomorphism;*

b) *the right $\mathcal{P}$ is free over $\mathcal{I}$;*

c) *if $\mathcal{M}$ is a subspace of $\mathcal{P}$ such that $\mathcal{M}\cap\mathcal{I}^+\mathcal{P} = \{0\}$, then linear independence of elements from $\mathcal{M}$ over $\mathbb{C}$ is equivalent to their linear independence over $\mathcal{I}$.*

It is evident that assertion b) follows from a). The assertion, formulated before this statement, shows that a) follows from c). Let $\{e_i\}$ be a $\mathcal{I}$-basis in $\mathcal{P}$ and let f_j, $1 \le j \le k$, be linearly independent elements of $\mathcal{M}$. Let $f_i = \sum_i a_{ij} e_i$. Then rank of the matrix (a_{ij}) is equal to k. Therefore, assertion c) follows from b). Statement is proved.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{U}$ be its universal enveloping algebra. Then any representation T of G defines the corresponding representation of the algebra $\mathfrak{U}$. Therefore, the action of $\mathfrak{U}$ on $\mathcal{P}$ is defined. We denote this action as $u \cdot P$, $P \in \mathcal{P}$, $u \in \mathfrak{U}$. The restriction of polynomial $P \in \mathcal{P}$ onto a subset $A \subset X$ will be denoted by $P \mid A$.

Statement 2. *Let $P_j \in \mathcal{P}$, $1 \le j \le k$, and let u_i, $i = 1,2,\dots$, be a basis of $\mathfrak{U}$. We set $D = (d_{ij})$, where $d_{ij} = u_i \cdot P_j$. Elements of the matrix D belong to $\mathcal{P}$. Replacing polynomials by their values at fixed point $x \in X$ we transform D into the numerical matrix $D(x)$ with complex elements. If O_x is the G-orbit of a point $x \in X$, then the functions $P_j \mid O_x$, $1 \le j \le k$, are linearly independent if and only if rank of the matrix $D(x)$ is equal to k.*

We omit the detailed proof of this statement. It is based on the fact that an analytical function f on the complex homogeneous space G/G_x (where G_x is the stationary subgroup of the point x) is identically equal to zero on G/G_x if and only if $(u \cdot f)(a) = 0$ for all $u \in \mathfrak{U}$ (here a is the point from G/G_x which corresponds to the coset coinciding with the subgroup G_x).

Statement 2 implies the following criterion on linear independence of polynomials over $\mathcal{I}$.

Statement 3. *Let $P_j \in \mathcal{P}$, $1 \leq j \leq k$. If the functions $P_j\,|\,O$ are linearly independent for some G-orbit $O \subset X$, then P_j, $1 \leq j \leq k$, are linearly independent over $\mathcal{I}$.*

Really, let $\sum\limits_j g_j P_j = 0$, where $g_i \in \mathcal{I}$. Let D be the matrix from Statement 2 and let $x \in O$. There is a minor of this matrix of order k such that $\alpha(x) \neq 0$, where $\alpha \in \mathcal{P}$ is the determinant of this minor. Then this determinant is non-vanishing in some neighborhood W of the point x. Therefore, $D(y)$ is of rank k in W and $P_j\,|\,O_y$ are linearly independent for all $y \in W$. The polynomials $g_i \in \mathcal{I}$ are constant on every orbit. Consequently, it follows from $\sum\limits_j g_j P_j = 0$ that all polynomials g_j vanish in the neighborhood W. In this reason they vanish on X. Statement is proved.

Let Y be a subset of X and let $I(Y)$ be the ideal of $\mathcal{P}$, consisting of polynomials vanishing on Y. Let C be the cone in X consisting of points x in which all polynomials from $\mathcal{I}^+$ vanish. Since C is defined by the ideal $\mathcal{I}^+\mathcal{P}$, than $I(C)$ is the radical of this ideal and the cone C is irreducible in the sense of algebraic geometry. Moreover, $\mathcal{I}^+\mathcal{P} = I(C)$ if and only if the ideal $\mathcal{I}^+\mathcal{P}$ is simple (that is, the quotient space $\mathcal{P}/\mathcal{I}^+\mathcal{P}$ has no non-trivial ideals).

Statement 4. *Let the ideal $\mathcal{I}^+\mathcal{P}$ be simple and let an orbit O exist which is everywhere dense in C. Let $\mathcal{P} = \mathcal{I}^+\mathcal{P} + \mathcal{L}$ be the direct sum decomposition. Then the mapping $\mathcal{I} \otimes \mathcal{L} \to \mathcal{P}$, given by the formula $j \otimes \ell \to j\ell$, is an isomorphism.*

Really, let $\mathcal{M}$ be a subspace in $\mathcal{P}$ such that $\mathcal{M} \cap \mathcal{I}^+\mathcal{P} = \{0\}$. Since $\mathcal{I}^+\mathcal{P} = I(C)$, then it is clear that linear independence of polynomials $P_j \in \mathcal{M}$, $1 \leq j \leq k$, implies linear independence of $P_j\,|\,C$, $1 \leq j \leq k$. Since $\overline{O} = C$, then $P_j\,|\,O$, $1 \leq j \leq k$, are also linearly independent. Then according to Statement 3 the polynomials P_j, $1 \leq j \leq k$, are linearly independent over $\mathcal{I}$ and our statement follows from assertions b) and c) of Statement 1.

Let $B(\cdot,\cdot)$ be a non-degenerate G-invariant bilinear form on $\mathcal{P}$. This form defines the isomorphism of the rings $\mathcal{P}$ and $\mathcal{P}_*$ (polynomial differential operators) (see Section 1.0.5). This isomorphism is also denoted by B. It is evident that[1] $B(\mathcal{I}_*^+) = \mathcal{I}^+$, $B(\mathcal{I}_*) = \mathcal{I}$. A polynomial $P \in \mathcal{P}$ is called *G-harmonic* if $\partial P = 0$ for

[1] If $\mathcal{A} \subset \mathcal{P}$, then $\mathcal{A}_*$ is obtained by replacement of coordinates by their derivatives in all $P \subset \mathcal{A}$.

all G-invariant operators $\partial \in \mathcal{I}_*^+$. The space of all G-harmonic polynomials from $\mathcal{P}$ is denoted by $\mathcal{H}$.

The polynomials $B(\partial_x)$ corresponds to a point $x \in C$. The cone of such polynomials from $\mathcal{P}$ is denoted by C'. Let $\mathcal{H}_c$ be the subspace of $\mathcal{P}$ spanned by all powers z^m, $m = 0, 1, 2, \dots$, where $z \in C'$.

Statement 5. *The subspace $\mathcal{H}_c$ belongs to $\mathcal{H}$. The direct decomposition $\mathcal{P} = \mathcal{I}^+\mathcal{P} + \mathcal{H}$ of $\mathcal{P}$ into the sum of G-modules is valid and, therefore, we have $\mathcal{P} = \mathcal{I}\mathcal{H}$.*

The inclusion $\mathcal{H}_c \subset \mathcal{H}$ is proved by direct calculations. We have $B(\mathcal{I}_*^+\mathcal{P}_*) = \mathcal{I}^+\mathcal{P}$, and in order to prove that $\mathcal{P} = \mathcal{I}^+\mathcal{P} + \mathcal{H}$ it is sufficient to show (due to dimension equality) that $\mathcal{H} \cap \mathcal{I}^+\mathcal{P} = 0$. To show it we have to prove that B defines non-degenerate bilinear form on $\mathcal{I}^+\mathcal{P}_*$. Let K be a maximal compact subgroup in G. Then there exists the real subspace $X_R \subset X$ such that

(1) X_R is invariant with respect to K,

(2) the form B is positive definite on X_R,

(3) $X = X_R + iX_R$ and the sum is direct.

It follows from (2) that B gives a positive definite bilinear form on $\mathcal{P}_{*R}(X_R) \equiv \mathcal{P}_R$. Since G is the complexification of K, then due to (1) and (3) $\mathcal{I}_*^+$ is the complexification of $\mathcal{I}_*^+ \cap \mathcal{P}_R$. Consequently, $\mathcal{I}_*^+\mathcal{P}_*$ is the complexification of $\mathcal{I}_*^+\mathcal{P}_* \cap \mathcal{P}_R$. In this reason B induces a non-degenerate bilinear form on $\mathcal{I}_*^+\mathcal{P}_*$. Statement is proved.

Statements 4 and 5 imply the theorem on separation of variables.

Theorem. *Let the group G conserves the symmetric non-degenerate bilinear form on X and let $\mathcal{P}$, $\mathcal{H}$, $\mathcal{I}$ be respectively the ring of all polynomials on X, the ring of G-harmonic polynomials on X and the ring of G-invariant polynomials on X. Let C be the homogeneous affine manifold, define by the ideal $\mathcal{I}^+\mathcal{P}$ of $\mathcal{P}$. We suppose that there is G-orbit O in X such that $\overline{O} = C$ and that the ideal $\mathcal{I}^+\mathcal{P}$ is simple. Then the mapping $\mathcal{I} \otimes \mathcal{H} \to \mathcal{P}$, given by the formula $j \otimes h \to jh$, is an isomorphism of G-modules. If $C' \subset \mathcal{P}$ is the cone conjugate (with respect to the bilinear form) to the cone C and if $\mathcal{H}_c$ is the subspace of $\mathcal{P}$, generated by z^m, $m = 0, 1, 2, \dots, z \in C'$, then $\mathcal{H} = \mathcal{H}_c$.*

Proof. Since G acts as an algebra of automorphisms on $\mathcal{P}$, then $\mathcal{I} \otimes \mathcal{H} \to \mathcal{P}$ is a mapping of G-modules. According to Statements 4 and 5 this mapping is an isomorphism. Let $\mathcal{H}_* \subset \mathcal{P}_*$ be the space generated by all ∂_x^m, $x \in C$, and let B be the mapping of $\mathcal{P}_*$ onto $\mathcal{P}$. Then $B(\mathcal{H}_*) = \mathcal{H}$. If[2] $\mathcal{H}_{m,c} \neq \mathcal{H}_m$, then due to the dimension equality and Statement 5 there is non-zero polynomial $H \in \mathcal{H}_m$ such that $\partial H = 0$ for all $\partial \in \mathcal{H}_*$. We set $\partial = \partial_x^m/m!$, where $x \in X$. It follows from $(\partial_x^m/m!)H = 0$ that $H(x) = 0$ for all $x \in C$. Since $\mathcal{I}^+\mathcal{P}$ is a simple ideal, we have

[2] Here $\mathcal{H}_m$ denotes the set of homogeneous polynomials of power m from $\mathcal{H}$.

$H \in \mathcal{I}^+\mathcal{P}$. Since $\mathcal{I}^+\mathcal{P} \cap \mathcal{H} = \{0\}$, then we obtain a contradiction. Theorem is proved.

The theorem on separation of variables is valid (with the corresponding reformulation) if a skew-symmetric non-degenerate form is given instead of symmetric one.

16.1.5. Realization of finite dimensional representations on spaces of polynomials in minors. Let $\mathcal{P}(\mathbf{m})$, $\mathbf{m} = (m_1, \dots, m_n)$, $m_1 \geq m_2 \geq \dots \geq m_n \geq 0$, $m_j \in \mathbb{Z}$, be the space of polynomials in elements of matrices $g \in GL(n, \mathbb{C})$, which are sums of monomials of the form $A_1(g) \dots A_n(g)$, where $A_k(g)$, $1 \leq k \leq n$, are products of powers of minors of k-th order, constructed from the first k rows of matrices g. Moreover, we suppose that sum of all exponents in $A_k(g)$ is equal to $r_k = m_k - m_{k+1}$, where $m_{n+1} = 0$. In other words $\mathcal{P}(\mathbf{m})$ is the space of polynomials P in elements of matrices g such that

$$P(n_-\delta g) = \alpha(\delta)P(g) \tag{1}$$

for all $n_- \in N_-(n, \mathbb{C})$, $\delta \in D(n, \mathbb{C})$, where $\alpha(\delta) = \delta_1^{m_1}\delta_2^{m_2} \dots \delta_n^{m_n}$. It follows from the Cauchy-Binet identity

$$\Delta_{ij}(gg_0) = \sum_k \Delta_{ik}(g)\Delta_{kj}(g) \tag{2}$$

that the space $\mathcal{P}(\mathbf{m})$ is invariant with respect to right shifts $P(g) \to P(gg_0)$, $g_0 \in GL(n, \mathbb{C})$. Therefore, the formula

$$T_{\mathbf{m}}(g_0)P(g) = P(gg_0) \tag{3}$$

gives a representation of the group $GL(n, \mathbb{C})$ in the space $\mathcal{P}(\mathbf{m})$. Moreover, $\mathcal{P}(\mathbf{m})$ contains one and only one (up to a normalization) element, invariant with respect to right shifts $P(g) \to P(gn_+)$. This element coincides with $\Delta_1^{r_1}\Delta_2^{r_2} \dots \Delta_n^{r_n}$, where, remind, $\Delta_1, \Delta_2, \dots, \Delta_n$ are the main minors of g. Thus, the finite dimensional representation $T_{\mathbf{m}}$ of the group $GL(n, \mathbb{C})$ is irreducible and it is induced by the character $\alpha(\delta) = \delta_1^{m_1}\delta_2^{m_2} \dots \delta_n^{m_n}$.

Let us note that due to the Weyl unitary trick (see the reference [53] of the first volume) the described construction defines all irreducible representations of the group $U(n)$ which is the compact form of the complex group $GL(n, \mathbb{C})$.

Now we give the analogous realization for finite dimensional irreducible representations of the groups $SO(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$. Let $2k \leq n$. The set of matrices $X \in \mathfrak{M}_{kn}(\mathbb{C})$ such that $XX^t = 0$ and rank $X = k$ is called *complex $k \times n$ matrix cone*. We denote it by $C^{kn}(\mathbb{C})$. It is evident that $gX \in C^{kn}(\mathbb{C})$ if $X \in C^{kn}(\mathbb{C})$, $g \in GL(k, \mathbb{C})$. We also have $X\omega \in C^{kn}(\mathbb{C})$ if $X \in C^{kn}(\mathbb{C})$ and $\omega \in O(n, \mathbb{C})$.

We fix $\mathbf{m} = (m_1, \dots, m_k)$, $m_1 \geq \dots \geq m_k \geq 0$ and denote by $\hat{\mathcal{P}}(\mathbf{m})$ the set of polynomials in elements of matrix $X \in C^{kn}(\mathbb{C})$ such that

$$P(n_-\delta X) = \alpha_m(\delta)P(X), \quad n_- \in N_-(k, \mathbb{C}), \quad \delta \in D(k, \mathbb{C}), \tag{4}$$

where

$$\alpha_{\mathbf{m}}(\delta) = \boldsymbol{\delta}^{\mathbf{m}} \equiv \delta_1^{m_1}\delta_2^{m_2}\dots\delta_k^{m_k}. \tag{5}$$

It follows from the Cauchy-Binet identity that the formula

$$T_{\mathbf{m}}(g)P(X) = P(Xg) \tag{6}$$

gives a finite dimensional analytical representation of the group $O(n,\mathbb{C})$. It is possible to show that this representation is irreducible. It is equivalent to the representation with signature $(m_1,\dots,m_k,0,\dots,0)$. If $n \neq 2k$, then restriction of it onto $SO(n,\mathbb{C})$ is also irreducible. If $n = 2k$, then the restriction decomposes into a sum of two reflexively symmetric irreducible representations. Namely, if $n = 2k$, then the manifold $C^{kn}(\mathbb{C})$ splits into two parts, which are invariant with respect to the transformations $X \to X\omega$, $\omega \in SO(n,\mathbb{C})$. One of these parts contains the matrix (I_k, I_k) and the other contains the matrix $(I_k, I_k)s$, where s is the matrix of reflexive symmetry. Considering polynomials from $\hat{\mathcal{P}}(m)$ on these parts, we obtain reflexively symmetric invariant subspaces in which the irreducible representations with signatures $(m_1,\dots,m_{k-1},m_k)$ and $(m_1,\dots,m_{k-1},-m_k)$ are realized.

We have realized irreducible finite dimensional representations of the group $SO(n,\mathbb{C})$ in the spaces of polynomials on $C^{kn}(\mathbb{C})$ satisfying conditions (4). Applying the Kostant theorem on separation of variables and description of vector invariants for the group $SO(n,\mathbb{C})$, we conclude that these representations can be realized in spaces of harmonic polynomials, satisfying conditions of the same type. Really, it follows from definition of the cone $C^{kn}(\mathbb{C})$ that for any j and m we have $(\mathbf{z}_j, \mathbf{z}_m) = 0$, where $\mathbf{z}_j = (z_{j1},\dots,z_{jn})$, $\mathbf{z}_m = (z_{m1},\dots,z_{mn})$. Since any $SO(n,\mathbb{C})$-invariant polynomial is a polynomial of such scalar products, then it vanishes on $C^{kn}(\mathbb{C})$. Therefore, according to the Kostant theorem, every polynomial in elements z_{jm}, $1 \leq j \leq k$, $1 \leq m \leq n$, of matrix (z_{jm}) takes on $C^{kn}(\mathbb{C})$ the same values as some $SO(n,\mathbb{C})$-harmonic polynomial $P(z)$ does, that is a polynomial for which $\Delta_{\ell m}P(z) = 0$ for all ℓ and m, where

$$\Delta_{\ell m} = \sum_{j=1}^{n} \frac{\partial^2}{\partial z_{j\ell}\partial z_{jm}}.$$

The character $\alpha_{\mathbf{m}}(\delta n) = \alpha_{\mathbf{m}}(\delta)$ of the subgroup $S_-(k,\mathbb{C})$ induces also the irreducible representation $Q_{\mathbf{m}}$ of the group $GL(k,\mathbb{C})$. Let us denote by $\mathcal{A}_{\mathbf{m}}$ the space of vector functions on $C^{kn}(\mathbb{C})$ taking values in the space $\mathcal{L}_{\mathbf{m}}$ of the representation $Q_{\mathbf{m}}$ and having polynomial components. The equality $T_{\mathbf{m}}(\omega)P(X) = P(X\omega)$ gives a representation of the group $O(n,\mathbb{C})$ in this space. It is possible to show that this representation is equivalent to the irreducible representation of $O(n,\mathbb{C})$ with signature $(m_1,\dots,m_k,0,\dots,0)$. Its restriction onto the compact subgroup $O(n)$ is also irreducible.

The representation $T(\omega)P(X) = P(X\omega)$ of the group $O(n)$ in the space $\mathcal{P}(C^{kn}(\mathbb{C}))$ of all polynomials on $C^{kn}(\mathbb{C})$ is unitary with respect to the scalar product introduced in Section 16.1.8 below. This representation is reducible. It decomposes into a direct sum of irreducible representations $T_{\mathbf{m}}$ with signatures $\mathbf{m} = (m_1, \dots, m_k, 0, \dots, 0)$ (with signatures $\mathbf{m} = (m_1, \dots, m_{k-1}, \pm m_k)$ if $2k = n$). Moreover, multiplicity of the representation $T_{\mathbf{m}}$ in this decomposition is equal to dimension $d_{\mathbf{m}}$ of the representation $Q_{\mathbf{m}}$ of the group $GL(k, \mathbb{C})$. This dimension is given by the formula

$$d_{\mathbf{m}} = \frac{\prod\limits_{1 \le i < j \le k} (m_i - m_j + j - i)}{\prod\limits_{1 \le i < j \le k} (j - i)}$$

(if $n = 2k$, then m_k has to be replaced by $|m_k|$).

As it was noted above, every polynomial from $\mathcal{P}(\mathfrak{M}_{kn}(\mathbb{C}))$ takes on $C^{kn} \equiv C^{kn}(\mathbb{C})$ values coinciding with values of some harmonic polynomial. This shows that with every irreducible invariant (with respect to operators of representation T) subspace of $\mathcal{P}(C^{kn})$ we can associate the certain space $\mathcal{H}^{kn}_{\mathbf{m}\rho}$, $1 \le \rho \le d_{\mathbf{m}}$, of harmonic polynomials in $\mathfrak{M}_{kn}(\mathbb{C})$. These polynomials are uniquely defined by their values on $C^{kn}(\mathbb{C})$. Let us restrict these polynomials onto the Stiefel manifold $\mathrm{St}(k,n) = SO(n)/SO(n-k)$, that is onto the set of matrices $X \in \mathfrak{M}_{kn}(\mathbf{R})$ such that $XX^t = I_k$. We obtain the subspaces $\mathcal{H}_{\mathbf{m}\rho} \equiv \mathcal{H}_{\mathbf{m}\rho}(\mathrm{St}(k,n))$, $1 \le \rho \le d_{\mathbf{m}}$. It is possible to show that the space $\mathfrak{L}^2(\mathrm{St}(k,n))$ of square integrable functions on $\mathrm{St}(k,n)$ is the orthogonal sum of the subspaces $\mathcal{H}_{\mathbf{m}\rho}$, $\mathbf{m} = (m_1, \dots, m_k, 0, \dots, 0)$ $1 \le \rho \le d_{\mathbf{m}}$. The scalar product

$$(P_1, P_2) = \int\limits_{\mathrm{St}(k,n)} P_1(X)\overline{P_2(X)}dX$$

in $\mathfrak{L}^2(\mathrm{St}(k,n))$ (where dX is the invariant measure) is invariant with respect to operators $T(\omega)$, $\omega \in O(n)$.

Thus, the spaces $\mathfrak{L}^2(C^{kn})$ and $\mathfrak{L}^2(\mathrm{St}(k,n))$ have the same decompositions (up to an equivalence) into irreducible components. The intertwining operators for the corresponding irreducible representations of $O(n)$ coincide with the operator of the analytical continuation of harmonic polynomials.

Irreducible finite dimensional representations of the group $Sp(n, \mathbb{C})$ are analogously realized in spaces of polynomials. Namely, we introduce *symplectic complex $k \times n$ matrix cone* $\hat{C}^{kn}(\mathbb{C})$ consisting of matrices X from $\mathfrak{M}_{k,2n}(\mathbb{C})$ such that $XJ_nX^t = 0$ and rank $X = k$, where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Let $\mathbf{m} = (m_1, \dots, m_k)$ $m_1 \ge m_2 \ge \dots \ge m_k \ge 0$, and let $\alpha_{\mathbf{m}}$ be character (5) of the group $S_-(k, \mathbb{C})$. We denote by $\mathcal{P}_{\mathbf{m}}$ the space of polynomials P on $\hat{C}^{kn}(\mathbb{C})$ such that

$$P(s_- X) = \alpha_{\mathbf{m}}(s_-)P(X), \quad s_- \in S_-(k, \mathbb{C}).$$

The formula $(T_{\mathbf{m}}(g)P)(X) = P(Xg)$ gives the representation $T_{\mathbf{m}}$ of the group $Sp(n, \mathbb{C})$ which is equivalent to finite dimensional irreducible representation with signature $(m_1, \ldots, m_k, 0, \ldots, 0)$. Restriction of the representation $T_{\mathbf{m}}$ onto the compact subgroup $Sp(n)$ is irreducible.

Let R be the right quasi-regular representation of $Sp(n, \mathbb{C})$ in the space $\mathfrak{L}^{kn}$ consisting of polynomials in coordinates of matrices $X \in \mathfrak{M}_{k,2n}(\mathbb{C})$ restricted onto $\hat{C}^{kn}(\mathbb{C})$. It follows from the Kostant theorem on separation of variables that instead of $\mathfrak{L}^{kn}$ we can take the space $Sp(n, \mathbb{C})$-harmonic polynomials on $\mathfrak{M}_{k,2n}(\mathbb{C})$, that is the space of polynomials P such that $\Delta_{jm} P = 0$ for all j and m, $1 \le j, m \le k$, where

$$\Delta_{jm} = \sum_{\ell=1}^{n} \left(\frac{\partial^2}{\partial z_{j\ell} \partial z_{m,\ell+n}} - \frac{\partial^2}{\partial z_{j,\ell+n} \partial z_{m\ell}} \right).$$

The restriction of the representation R onto the subgroup $Sp(n)$ is unitary with respect to the scalar product from Section 16.1.8. This representation decomposes into an orthogonal sum of irreducible components which are equivalent to representations $T_{\mathbf{m}}$, constructed above. Thus, we obtain the realization of the representations $T_{\mathbf{m}}$ of $Sp(n)$ in the spaces of $Sp(n, \mathbb{C})$-harmonic polynomials of coordinates in $\mathfrak{M}_{k,2n}(\mathbb{C})$.

The space of matrices X from $\mathfrak{M}_{k,2n}(\mathbb{C})$ such that $XX^* = I_k$ and $XJ_nX^t = J_k$ is called *symplectic Stiefel manifold* and is denoted by $\hat{\mathrm{St}}(k, n)$. This space is identified with the homogeneous space $Sp(n)/Sp(n-k)$. The right quasi-regular representation of $Sp(n)$ in $\mathfrak{L}^2(\hat{\mathrm{St}}(k, n))$ decomposes into the same irreducible representations (and with the same multiplicities) as in the case of the quasi-regular representation R, constructed above, in the space $\mathfrak{L}^2(\hat{C}^{kn}(\mathbb{C}))$. The scalar product

$$(P_1, P_2) = \int\limits_{\hat{\mathrm{St}}(k,n)} P_1(X)\overline{P_2(X)} dX$$

(where dX is the invariant measure) is invariant with respect to the quasi-regular representation. Therefore, we obtain the realization of the unitary representations $T_{\mathbf{m}}$ of $Sp(n)$ in subspaces of $\mathfrak{L}^2(\hat{\mathrm{St}}(k, n))$.

16.1.6. Decomposition of symmetric powers of finite dimensional irreducible representations. Let T be a finite dimensional representation of a group G in a linear space X. Let $\mathcal{P}_m(X)$ be the subspace of the space $\mathcal{P}(m)$ of polynomials on X, consisting of homogeneous polynomials of power m. The formula

$$(Q(g)P)(x) = P(T(g^{-1})x), \quad x \in X, \tag{1}$$

gives the representation Q of G in $\mathcal{P}(X)$. The subspace $\mathcal{P}_m(X)$ is invariant with respect to the representation Q. The restriction of the representation Q onto $\mathcal{P}_m(X)$ is called *m-th symmetric power of the representation* T and is denoted by $\sigma_m(T)$.

In this section we construct decompositions of some symmetric powers of irreducible representations. The group $G = GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C})$, $n_1 \leq n_2$, has the normal Gauss decomposition. Therefore, every its irreducible finite dimensional representation is induced by highest weight of the form $\alpha(g) = \alpha_1(g_1)\alpha_2(g_2)$, where $\alpha_k(g_k)$ are highest weights for $GL(n_k, \mathbb{C})$, $k = 1, 2$. The double signatures

$$\begin{gathered} \boldsymbol{\mu} = (m_1, \dots, m_{n_1} \mid \ell_1, \dots, \ell_{n_2}), \\ m_1 \geq m_2 \geq \dots \geq m_{n_1}, \quad \ell_1 \geq \ell_2 \geq \dots \geq \ell_{n_2}, \end{gathered} \tag{2}$$

correspond to these representations.

Now we consider the following three representations of the group $G = GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C})$ in the space $\mathfrak{M}_{n_1 n_2}(\mathbb{C})$:

$$T(g_1, g_2)X = g_1 X g_2^t, \tag{3}$$

$$T'(g_1, g_2)X = g_1 X g_2^*, \tag{4}$$

$$T''(g_1, g_2)X = g_1 X g_2^{-1}, \tag{5}$$

and their m-th symmetric powers $\sigma_m(T)$, $\sigma_m(T')$, $\sigma_m(T'')$. We shall prove that they decompose into irreducible components in the following way:

$$\sigma_m(T) = \sum (m_1, \dots, m_{n_1} \mid m_1, \dots, m_{n_1}, 0, \dots, 0), \tag{6}$$

$$\sigma_m(T') = \sum (\overline{m_1, \dots, m_{n_1}} \mid m_1, \dots, m_{n_1}, 0, \dots, 0), \tag{7}$$

$$\sigma_m(T'') = \sum (-m_{n_1}, \dots, -m_1 \mid m_1, \dots, m_{n_1}, 0, \dots, 0), \tag{8}$$

where summations are over those $\mathbf{m} = (m_1, \dots, m_{n_1})$, for which

$$m_1 \geq m_2 \geq \dots \geq m_{n_1} \geq 0, m_1 + m_2 + \dots + m_{n_1} = m,$$

and a bar indicates that a representation is anti-analytical. Irreducible representations are contained with unit multiplicities in the decompositions (6)-(8).

In order to prove formulas (6) and (7) we use the analogue of the Gauss decomposition for matrices from $\mathfrak{M}_{n_1 n_2}(\mathbb{C})$ (see Section 15.1.1). Let $X = n_- \lambda n_+$, where $n_- \in N_-(n_1, \mathbb{C})$, $n_+ \in N_+(n_2, \mathbb{C})$, $\lambda = (\delta, 0)$, $\delta \in D(n_1, \mathbb{C})$, $0 \in \mathfrak{M}_{n_1, n_2 - n_1}$. The functions $\delta_p(X) = \Delta_p / \Delta_{p-1}$ form a functional basis in the algebra Ω of all N_+-invariants. Let $P(X)$ be a polynomial in matrix elements x_{ij} of X which is invariant with respect to all operators $T(n_-^{(1)}, n_-^{(2)})$. Then if $X^0 = n_-^0 \lambda^0 n_+^0$, then $P(X^0) = \tilde{P}(\lambda^0)$ and, therefore, $P(X)$ is a rational function in the main minors $\Delta_i(X)$. Thus, invariant polynomials are sums of monomials of the form

$$\omega_{\boldsymbol{\nu}}(X) = \delta_1^{\nu_1} \delta_2^{\nu_2} \dots \delta_{n_1}^{\nu_{n_1}}, \tag{9}$$

where $\delta_i = \lambda_j(X)$. It is easy to see that $\omega_{\boldsymbol{\nu}}(X)$ is a weight vector with respect to the subgroup D corresponding to the weight

$$(\nu_1, \dots, \nu_{n_1} \mid \nu_1, \dots, \nu_{n_1}, 0, \dots, 0). \tag{10}$$

Since all such weights are different, then every highest weight vector has to be a monomial of the form (9). Since such vector generates the irreducible representation of G with signature (10), then all ν_j are integers and $\nu_1 \geq \nu_2 \geq \dots \geq \nu_{n_1} \geq 0$.

The case of the representation T' is analogously proved. Instead of analyticity of the right hand side of the weight we have to take its anti-analyticity. For the case of the representation T'' we have to modify the Gauss decomposition. Namely, we apply the Gauss decomposition to matrices $Y = S^{-1}X$, where $S \equiv (s_{ij}) = (\delta_{i,n-j+1})$. For X we have the decomposition $X = Sn_-S^{-1}S\lambda n_+ = \tilde{n}_-S\lambda n_+$. The remaining part of the proof is the same as for the representation T.

If $n_1 = n_2 = n$, then the formulas

$$(T_1 \otimes T_1)(g)X = gXg^t, \tag{11}$$

$$(T_1 \otimes \bar{T}_1)(g)X = gXg^*, \tag{12}$$

$$(T_1 \otimes \hat{T}_1)(g)X = gXg^{-1} \tag{13}$$

give representation of the group $GL(n, \mathbb{C})$ in $\mathfrak{M}_n(\mathbb{C})$. Here T_1 is the representation $g \to T_1(g) \equiv g$ of the group $GL(n, \mathbb{C})$ (that is the irreducible representation with highest weight $(1, 0, \dots, 0)$) and $\bar{T}_1$ ($\hat{T}_1$) is complex conjugate (contragradient) to T_1. The reasonings, similar to above ones, show that

$$\sigma_m(T_1 \otimes T_1) = \sum_{\mathbf{m}} T_{\mathbf{m}} \otimes T_{\mathbf{m}}, \tag{14}$$

$$\sigma_m(T_1 \otimes \bar{T}_1) = \sum_{\mathbf{m}} T_{\mathbf{m}} \otimes \bar{T}_{\mathbf{m}}, \tag{15}$$

$$\sigma_m(T_1 \otimes \hat{T}_1) = \sum_{\mathbf{m}} T_{\mathbf{m}} \otimes \hat{T}_{\mathbf{m}}, \tag{16}$$

where $T_{\mathbf{m}}$ is an irreducible representation with highest weight $\mathbf{m} = (m_1, \dots, m_n)$ such that $m_1 + \dots + m_n = m$. Formula (15) gives the decomposition of $\sigma_m(T_1 \otimes \bar{T}_1)$ into irreducible components. In formulas (14) and (16) we additionally must decompose the tensor products $T_{\mathbf{m}} \otimes T_{\mathbf{m}}$ and $T_{\mathbf{m}} \otimes \hat{T}_{\mathbf{m}}$ into irreducible components.

Replacing $\mathfrak{M}_n(\mathbb{C})$ by the spaces of all symmetric or all skew-symmetric matrices we obtain the decompositions

$$\sigma_m(T_{(2,0,\dots,0)}) = \sum (2m_1, \dots, 2m_n), \tag{17}$$

$$\sigma_m(T_{(1,1,0,\dots,0)}) = \sum (m_1, m_1, \dots, m_k, m_k), \tag{18}$$

where the summations are over all $(m_1, m_2, \dots)$, for which $m_1 \geq m_2 \geq \dots \geq 0$. The proofs are analogous to the previous ones. In the second case we have to reduce matrices to block-diagonal form diag $(g_1, \dots, g_n)$, where, due to the skew symmetry, the blocks are of the form $g_k = \begin{pmatrix} 0 & x_k \\ -x_k & 0 \end{pmatrix}$ (if dimension is odd, then the last block coincides with number 0).

16.1.7. Restrictions of irreducible representations of classical groups. Restricting an irreducible representation of a group G onto a subgroup G_0, we obtain, generally speaking, reducible representation of G_0. Let $G = GL(n, \mathbb{C})$, $G_0 = GL(n-1, \mathbb{C})$ and G_0 is imbedded into $GL(n, \mathbb{C})$ as the subgroup of matrices $\begin{pmatrix} h & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$, $h \in GL(n-1, \mathbb{C})$. The Gauss decomposition $G = \overline{N_- D N_+}$ of the group $GL(n, \mathbb{C})$ defines the corresponding decomposition $G_0 = \overline{N_-^0 D^0 N_+^0}$ of G_0, where $N_-^0 = N_- \cap G_0$ and so on. Let us prove the following statement.

Restriction of the irreducible finite dimensional representation $T_{\mathbf{m}}$, $\mathbf{m} = (m_1, \dots, m_n)$ of the group $G = GL(n, \mathbb{C})$ onto the subgroup $G_0 = GL(n-1, \mathbb{C})$ decomposes into a direct sum of all irreducible representations $Q_{\boldsymbol{\ell}}$ of this subgroup, highest weights (signatures) $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{n-1})$ of which satisfy the betweenness conditions

$$m_1 \geq \ell_1 \geq m_2 \geq \ell_2 \geq \dots \geq m_{n-1} \geq \ell_{n-1} \geq m_n. \tag{1}$$

Every of these representations of G_0 is contained in the decomposition with unit multiplicity.

In order to prove this statement we use the realization of the representation $T_{\mathbf{m}}$, described in Section 16.1.5. Any matrix $g \in GL(n, \mathbb{C})$ is representable in the form $g = n_- h(\delta, \mathbf{p}) n_+^0$, where $n_- \in N_-(n, \mathbb{C})$,

$$n_+^0 = \begin{pmatrix} n_+ & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}, \quad n_+ \in N_+(n-1, \mathbb{C}), \quad h(\delta, \mathbf{p}) = \begin{pmatrix} \delta & \mathbf{p} \\ \mathbf{0}^t & \delta_n \end{pmatrix},$$

$$\delta = \operatorname{diag}(\delta_1, \dots, \delta_{n-1}) \in D(n-1, \mathbb{C}), \quad \mathbf{p} \in \mathbb{C}^{n-1}, \quad \delta_n \in \mathbb{C}.$$

Let us find in the space $\mathcal{P}(\mathbf{m})$ the highest weight vectors for the representations of the subgroup G_0. They have to be polynomials in the minors, indicated in Section 16.1.5, which are invariant vectors with respect to right shifts by $n_+^0 \in N_+^0$ and eigenfunctions for right shifts by $\delta \in D(n-1, \mathbb{C})$.

A monomial $A_1 \dots A_n$ (see Section 16.1.5) is right N_+^0-invariant if it is a product of powers of minors of the form

$$\Delta_k = \det(g_{ij})_{1 \leq i,j \leq k}, \quad 1 \leq k \leq n; \quad \tilde{\Delta}_k = \det(h_{ij})_{1 \leq i,j \leq k}, \quad 1 \leq k \leq n-1,$$

$$h_{ij} = g_{ij} \quad \text{for} \quad 1 \leq j \leq k-1, \quad h_{ik} = g_{in}.$$

If $g = f(\delta, \mathbf{p})$, then $\Delta_k = \delta_1\delta_2\ldots\delta_k$, $\tilde{\Delta}_k = \delta_1\delta_2\ldots\delta_{k-1}p_k$. Since the minors Δ_k and $\tilde{\Delta}_k$ are invariant for left shifts by $n_- \in N_-$, then for $g = n_-h(\delta,\mathbf{p})n_+^0$ the corresponding monomials are of the form

$$\begin{aligned} A_1A_2\ldots A_n &= \Delta_1^{\nu_1}\Delta_2^{\nu_2}\ldots\Delta_{n-1}^{\nu_{n-1}}\Delta_n^{r_n}\tilde{\Delta}_1^{r_1-\nu_1}\tilde{\Delta}_2^{r_2-\nu_2}\ldots\tilde{\Delta}_{n-1}^{r_{n-1}-\nu_{n-1}} \\ &= \delta_1^{\nu_1+r_2+\ldots+r_n}\delta_2^{\nu_2+r_3+\ldots+r_n}\ldots\delta_n^{r_n}p_1^{r_1-\nu_1}p_2^{r_2-\nu_2}\ldots p_{n-1}^{r_{n-1}-\nu_{n-1}} \\ &= \delta_1^{m_2+\nu_1}\delta_n^{m_n}p_1^{r_1-\nu_1}\ldots p_{n-1}^{r_{n-1}-\nu_{n-1}}, \end{aligned}$$

where $r_k = m_k - m_{k+1}$ and exponents are non-negative integers.

The right multiplication by $\begin{pmatrix}\tilde{\delta} & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix} \in D(n,\mathbb{C})$ leads to the multiplication of the monomial by expression $\tilde{\delta}_1^{m_2+\nu_1}\ldots\tilde{\delta}_{n-1}^{m_n+\nu_{n-1}}$ and, therefore, the corresponding highest weight (the signature of the representation Q_ℓ) is $(m_2+\nu_1,\ldots,m_n+\nu_{n-1})$. Moreover, $0 \le \nu_k \le m_k - m_{k+1}$, $1 \le k \le n-1$. Thus, this highest weight satisfies betweenness conditions (1).

The problem of restriction of irreducible finite dimensional representations of the group $SO(n,\mathbb{C})$ onto the subgroup of matrices $\begin{pmatrix}\omega & \mathbf{0}\\ \mathbf{0} & 1\end{pmatrix}$, $\omega \in SO(n-1,\mathbb{C})$, are solved analogously. If $T_{\mathbf{m}}$, $\mathbf{m} = (m_1,\ldots,m_\ell)$, is an irreducible finite dimensional representation of $SO(2\ell+1,\mathbb{C})$, then

$$T_{\mathbf{m}}\Big\downarrow^{SO(2\ell+1,\mathbb{C})}_{SO(2\ell,\mathbb{C})} = \sum_{\mathbf{k}} \oplus T_{\mathbf{k}},$$

where the summation is over all highest weights $\mathbf{k} = (k_1,k_2,\ldots,k_\ell)$ for which the betweenness conditions

$$m_1 \ge k_1 \ge m_2 \ge k_2 \ge \ldots \ge k_{\ell-1} \ge m_\ell \ge k_\ell \ge -m_\ell \tag{2}$$

are satisfied. If $T_{\mathbf{m}}$, $\mathbf{m} = (m_1,\ldots,m_\ell)$ is an irreducible representation of $SO(2\ell,\mathbb{C})$, then

$$T_{\mathbf{m}}\Big\downarrow^{SO(2\ell,\mathbb{C})}_{SO(2\ell-1,\mathbb{C})} = \sum_{\mathbf{k}} \oplus T_{\mathbf{k}},$$

where the summation is over all highest weights $\mathbf{k} = (k_1,k_2,\ldots,k_{\ell-1})$ for which

$$m_1 \ge k_1 \ge m_2 \ge k_2 \ge \ldots \ge k_{\ell-2} \ge m_{\ell-1} \ge k_{\ell-1} \ge |m_\ell|. \tag{3}$$

Formulas for representations of finite dimensional irreducible representations of the group $Sp(n,\mathbb{C})$ onto the subgroup $Sp(n-1,\mathbb{C})$ are more complicated (see reference [58] of the first volume).

The successive restriction of irreducible representation $T_{\mathbf{m}}$ of $GL(n,\mathbb{C})$ onto the subgroups $GL(n-1,\mathbb{C}) \supset GL(n-2,\mathbb{C}) \supset \ldots \supset GL(1,\mathbb{C})$ is used for construction of orthonormal bases (so called the *Gel'fand-Tsetlin bases*) of the carrier space of $T_{\mathbf{m}}$. The detailed consideration of these bases is given in Chapter 18.

16.1.8. The scalar product in the space $\mathcal{P}(\mathfrak{M}_{mn}(\mathbf{F}))$. We consider $\mathfrak{M}_{mn}(\mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbb{C}, \mathbf{H}$, as real spaces $\mathbf{R}^N$, where $N = \nu mn$, $\nu(\mathbf{R}) = 1$, $\nu(\mathbb{C}) = 2$, $\nu(\mathbf{H}) = 4$. The elements $x_{ij,\zeta}$, $1 \le i \le m$, $1 \le j \le n$, $1 \le \zeta \le \nu$, of matrices are taken as coordinates in $\mathfrak{M}_{mn}(\mathbf{F})$. Using some ordering for values of (i, j, ζ), we associate with matrices $X \in \mathfrak{M}_{mn}(\mathbf{F})$ the vectors $\mathbf{x} = (x_1, \dots, x_N)$. If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)$, $\alpha_j \in Z_+ \cup \{0\}$, then we set

$$\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}, \quad \boldsymbol{\alpha}! = \alpha_1!\alpha_2!\dots\alpha_N!. \tag{1}$$

Polynomials $P \in \mathcal{P} \equiv \mathcal{P}(\mathfrak{M}_{mn}(\mathbf{F}))$ will be written as

$$P(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}, \quad a_{\boldsymbol{\alpha}} \in \mathbf{F}. \tag{2}$$

The polynomial $\sum_{\boldsymbol{\alpha}} \overline{a_{\boldsymbol{\alpha}}} \mathbf{x}^{\boldsymbol{\alpha}}$ is denoted by $\bar{P}(\mathbf{x})$, where the bar means complex conjugation.

With every polynomial $P(\mathbf{x})$ we associate differential operator $P(D)$, which is obtained by replacing $x_1, \dots, x_N$ by $\partial/\partial x_1, \dots, \partial/\partial x_N$. In Section 16.1.4 the space of such operators was denoted by $\mathcal{P}_*$. The formula

$$(P_1, P_2) = P_1(D)(P_2)|_{\mathbf{x}=\mathbf{0}} \tag{3}$$

defines a non-degenerate bilinear form on $\mathcal{P}$. If P_1 and P_2 are represented in the form (2) with coefficients $a_{\boldsymbol{\alpha}}$ and $b_{\boldsymbol{\alpha}}$ respectively, then

$$(P_1, P_2) = \sum_{\boldsymbol{\alpha}} \boldsymbol{\alpha}! a_{\boldsymbol{\alpha}} b_{\boldsymbol{\alpha}}. \tag{4}$$

It follows from here that

$$\langle P_1, P_2 \rangle \equiv (P_1, \bar{P}_2) = \sum_{\boldsymbol{\alpha}} \boldsymbol{\alpha}! a_{\boldsymbol{\alpha}} \bar{b}_{\boldsymbol{\alpha}} \tag{5}$$

is a scalar product on $\mathcal{P}$. The subspaces $\mathcal{P}_m$ of $\mathcal{P}$, consisting of homogeneous polynomials of powers m, are orthogonal with respect to this scalar product.

The formula $g \cdot X = Xg$, $X \in \mathfrak{M}_{mn}(\mathbf{F})$, defines an action of the group $GL(n, \mathbf{F})$ on $\mathfrak{M}_{mn}(\mathbf{F})$. This action determines the representation

$$(R(g)P)(X) = P(Xg), \quad P \in \mathcal{P}, \quad g \in GL(n, \mathbf{F}), \tag{6}$$

of the group $GL(n, \mathbf{F})$ in $\mathcal{P} \equiv \mathcal{P}(\mathfrak{M}_{mn}(\mathbf{F}))$.

Let us show that for $P \in \mathcal{P}$ and $g \in GL(n, \mathbf{F})$ we have

$$R(g)P(D)R(g^{-1}) = (R(g^\vee)P)(D), \tag{7}$$

where $g^\vee = (g^t)^{-1}$. For this we note that the linear functionals X^*, $X^*(Y) = \mathrm{Tr}\,(XY^t)$, correspond to matrices $X \in \mathfrak{M}_{mn}(\mathbf{F})$. We have $R(g)X^* = (Xg^t)^*$. If $X \in \mathfrak{M}_{mn}(\mathbf{F})$, then

$$(X^*(D)f)(Y) = \sum_{i,j} x_{ij}(D_{ij}f)(Y) = \left(\frac{d}{dt}f(Y + tX)\right)\bigg|_{t=0}, \tag{8}$$

where $D_{ij} = \partial/\partial y_{ij}$. Consequently,

$$\begin{aligned}(R(g)X^*(D)R(g^{-1})f)(Y) &= (X^*(D)R(g^{-1})f)(Yg)\\ &= \left(\frac{d}{dt}R(g^{-1})f(Yg + tX)\right)\bigg|_{t=0} = \left(\frac{d}{dt}f(Y + tXg^{-1})\right)\bigg|_{t=0}.\end{aligned} \tag{9}$$

Thus, $R(g)X^*(D)R(g^{-1}) = (Xg^{-1})^*(D)$. For $Y \in \mathfrak{M}_{mn}(\mathbf{F})$ we have

$$(Xg^{-1})^*(Y) = \mathrm{Tr}\,(Xg^{-1}Y^t) = \mathrm{Tr}\,(X(Y(g^{-1})^t)^t) = (R(g^\vee)X^*)(Y).$$

It follows from here that formula (7) is valid for all $g \in GL(n, \mathbb{C})$ and for all $P \in \mathcal{P}$. Really, this formula is valid for linear functions X^*. And if it is fulfilled for polynomials P_1 and P_2, then it is valid for $P_1 + P_2$ and for P_1P_2. Since any polynomial from $\mathcal{P}$ is a linear combination of products of linear functionals, then formula (7) is proved.

It follows from formula (7) and from the relation $u^\vee = u$, $u \in U(n, \mathbf{F})$, that the scalar product (5) is invariant with respect to the representation R of the group $U(n, \mathbf{F})$:

$$\langle P_1, P_2\rangle = \langle R(u)P_1, R(u)P_2\rangle, \quad u \in U(m, \mathbf{F}). \tag{10}$$

16.2. The Principal Series Representations of Classical Lie Groups and Their Matrix Elements

16.2.1. The principal series representations of the group $GL(n, \mathbb{C})$. Infinite dimensional representations of the group $GL(n, \mathbb{C})$ are constructed in the following way. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. We set $\chi = (\boldsymbol{\lambda}, \mathbf{m})$. The formula

$$\chi(\delta) = \prod_{k=1}^{n} \delta_k^{(\lambda_k + m_k)/2}\bar{\delta}_k^{(\lambda_k - m_k)/2} = \prod_{k=1}^{n} |\delta_k|^{\lambda_k - m_k}\delta_k^{m_k}, \tag{1}$$

where $\delta = \mathrm{diag}\,(\delta_1, \ldots, \delta_n) \in D(n, \mathbb{C})$, gives the character χ of the group $D(n, \mathbb{C})$. Let $\mathfrak{D}_\chi$ be the space of smooth functions f on $GL(n, \mathbb{C})$ such that

$$f(n_-\delta g) = \chi(\delta)f(g). \tag{2}$$

We define the representation T_χ of $GL(n,\mathbb{C})$ in this space by the formula

$$(T_\chi(g_0)f)(g) = f(gg_0), \quad g \in GL(n,\mathbb{C}). \tag{3}$$

Due to the Iwasawa decomposition (see Section 15.1.1) functions f, satisfying relation (2), are uniquely defined by their values on the maximal compact subgroup $K = U(n)$ and

$$f(\gamma u) = \chi(\gamma)f(u), \quad u \in U(n), \quad \gamma \in M \equiv U(n) \cap D(n,\mathbb{C}). \tag{4}$$

Therefore, the representation T_χ can be realized in the space of smooth functions φ on $K = U(n)$, satisfying relation (4). The operators $T_\chi(g)$ are given by the formula

$$(T_\chi(g)\varphi)(u) = \chi(\delta)\varphi(\tilde{u}), \quad g \in GL(n,\mathbb{C}), \quad u \in U(n), \tag{5}$$

where $\delta = \text{diag}\,(\delta_1, \ldots, \delta_n) \in D(n,\mathbb{C})$, $\delta_j > 0$, and $\tilde{u} \in U(n)$ are defined by the relation $ug = n_-\delta\tilde{u}$.

Another realization of the representation T_χ is obtained by using the Gauss decomposition (see Section 15.1.1). It follows from this decomposition that every function $f \in \mathfrak{D}_\chi$ is uniquely defined by its values on the subgroup $N_+(n,\mathbb{C})$. Therefore, the representation T_χ is realized in the space of smooth functions on $N_+(n,\mathbb{C})$ by the operators

$$(T_\chi(g)f)(n_+) = \chi(\delta)f(\tilde{n}_+), \tag{6}$$

where $\delta \in D(n,\mathbb{C})$ and $\tilde{n}_+ \in N_+(n,\mathbb{C})$ are defined by the decomposition $n_+g = n_-\delta\tilde{n}_+$, where $n_- \in N_-(n,\mathbb{C})$. There are "intermediate" realizations of the representation T_χ in spaces of functions in matrices of the form

$$\begin{pmatrix} k & b \\ 0 & \hat{n}_+ \end{pmatrix}, \quad k \in U(j), \quad b \in \mathfrak{M}_{j,n-j}(\mathbb{C}), \quad \hat{n}_+ \in N_+(n-j,\mathbb{C}). \tag{7}$$

The set of the representations T_χ of the group $GL(n,\mathbb{C})$ is called the *principal non-degenerate series* of representations. There are unitary representations in this set. Namely, if all λ_k, $k = 1, 2, \ldots, n$, are of the form $\lambda_k = n - 1 - 2(k-1) + i\rho_k$, $\rho_k \in \mathbf{R}$, then the operators $T_\chi(g)$ are unitary with respect to the scalar product

$$(\varphi, \psi) = \int \varphi(u)\overline{\psi(u)}du \tag{8}$$

of the space $\mathfrak{L}^2(U(n))$. The corresponding unitary representations constitute the so-called *principal unitary series* of representations.

There are other series of unitary representations of the group $GL(n,\mathbb{C})$. They are connected with other scalar products in the space $\mathfrak{D}_\chi$. Let $1 \le k < n/2$ and

$$\chi(\delta) = \prod_{j=1}^{n} |\delta_j|^{n-1-2(j-1)} \prod_{j=1}^{n-2k} |\delta_j|^{m_j+i\rho_j}\delta_j^{-m_j}$$
$$\times \prod_{j=1}^{k} |\lambda_j|^{m_j'+i\rho_j'+\tau_j}\lambda_j^{-m_j'}|\mu_j|^{m_j'+i\rho_j'-\tau_j}\mu_j^{-m_j'},$$

where $\lambda_j = \delta_{n-2k+2j-1}$, $\mu_j = \delta_{n-2k+2j}$; $m_j, m'_j \in \mathbb{Z}$, $\rho'_j \in \mathbf{R}$, $0 < \tau_j < 1$. Then the representation T_χ is unitary with respect to the scalar product

$$(f_1, f_2) = \int_{\dot{N}_+} \int_{N_+(n,\mathbb{C})} A(\dot{n}_+) f_1(n_+) f_2(\dot{n}_+ n_+) dn_+ d\dot{n}_+, \tag{9}$$

where $\dot{N}_+$ is the set of block-triangular matrices $\dot{n}_+ = (n_{ij})$, $1 \le i,j \le k+1$, such that $n_{ij} \in \mathfrak{M}_{k_i k_j}(\mathbb{C})$ ($k_1 = n-2k, k_2 = k_3 = \ldots = k_{k+1} = 2$), $n_{ij} = 0$ for $i > j$, $n_{11} = I_{n-2k}$, $n_{jj} = \begin{pmatrix} 1 & n_{j-1} \\ 0 & 1 \end{pmatrix}$, $n_{j-1} \in \mathbb{C}$, $j > 1$. The function $A(\dot{n}_+)$ from (9) is defined by the formula

$$A(\dot{n}_+) = \prod_{j=1}^{k} |n_j|^{2\tau_j - 2}.$$

These representations belong to the *complementary non-degenerate series of representations.*

The group $GL(n,\mathbb{C})$ has degenerate series of representations. They are obtained with the help of replacement of the subgroups $D(n,\mathbb{C})$, $N_\pm(n,\mathbb{C})$ by the corresponding subgroups of block-triangular matrices. For the most degenerate series of representations the subgroup $D(n,\mathbb{C})$ is replaced by the subgroup of matrices $\begin{pmatrix} h & \mathbf{0} \\ \mathbf{0} & \delta_n \end{pmatrix}$, where $h \in GL(n-1,\mathbb{C})$, $\delta_n \in \mathbb{C}$. These representations are realized in the space of functions in $\mathbf{z} \in \mathbb{C}^n$ such that

$$f(\alpha \mathbf{z}) = \alpha^{(\lambda+m)/2} \bar{\alpha}^{(\lambda-m)/2} f(\mathbf{z}), \quad \alpha \in \mathbb{C},$$

where $\lambda \in \mathbb{C}$ and $m \in \mathbb{Z}$ are parameters defining representations. The corresponding operators $T_{\lambda m}(g)$ are determined by the formula

$$(T_{\lambda m}(g) f)(\mathbf{z}) = f(\mathbf{z} g), \quad g \in GL(n,\mathbb{C}). \tag{10}$$

These representations are partial cases of representations, realized in spaces of determinant-homogeneous functions. A function f on the set of matrices of rank p from $\mathfrak{M}_{pn}(\mathbb{C})$, $p \le n$, is called *determinant-homogeneous* of power $\chi = (\lambda, m)$ if for any $h \in \mathfrak{M}_p(\mathbb{C})$ we have

$$f(hX) = |\det h|^{\lambda - m} (\det h)^m f(X).$$

Every such function is uniquely determined by its values on the set of matrices (Y, I_p), $Y \in \mathfrak{M}_{p,n-p}(\mathbb{C})$. The formula

$$(T_\chi(g) f)(X) = f(Xg)$$

defines a representation of $GL(n,\mathbb{C})$ in the space of determinant-homogeneous functions. It is the degenerate series representation which is determined by signature $(\boldsymbol{\lambda}, \mathbf{m})$, $\boldsymbol{\lambda} = (\lambda, \dots, \lambda, 0, \dots, 0)$, $\mathbf{m} = (m, \dots, m, 0, \dots, 0)$, where λ and m are repeated p times. We set $f(Y, I_p) = F(Y)$. Then for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(n,\mathbb{C})$, $A \in \mathfrak{M}_{n-p}(\mathbb{C})$, $D \in \mathfrak{M}_p(\mathbb{C})$, we have

$$(T_\chi(g)F)(Y) = |\det(YB + D)|^{\lambda - m}(\det(YB + D))^m F((YB + D)^{-1}(YA + C)).$$

There are unitary representations in the degenerate series. They form the principal and the complementary series. Another series of unitary representations of $GL(n,\mathbb{C})$ were found by E. Stein.

Almost all representations of the principal series are irreducible. The carrier spaces of these representations have non-trivial invariant subspaces only for some integral values of representation parameters. For example, if $q_j = (1/2)(\lambda_j + m_j)$, $r_j = (1/2)(\lambda_j - m_j)$ are non-negative integers, then there is the invariant subspace $\mathcal{P}(\mathbf{q}) \otimes \bar{\mathcal{P}}(\mathbf{r})$ in the space $\mathfrak{D}_\chi$, where $\mathcal{P}(\mathbf{p})$ is the space of polynomials in matrix elements g_{ij} of $g \in GL(n,\mathbb{C})$, which is defined by $\mathbf{p}$, and $\bar{\mathcal{P}}(\mathbf{p})$ is the corresponding space of polynomials in $\overline{g_{ij}}$. In this case there are other invariant subspaces in $\mathfrak{D}_\chi$. We do not describe them.

Infinite dimensional irreducible representations of the group $SL(n,\mathbb{C})$ are described in a similar way. We propose to the reader to make the corresponding changes in our above reasonings.

The same reasonings are used to describe irreducible representations of the Lie groups $SO(n,\mathbb{C})$ and $Sp(n,\mathbb{C})$. We have the principal unitary series and the complementary series of representations for these groups. Their detailed description can be found in [76].

16.2.2. Representations of real semisimple Lie groups. Description of irreducible representations for real semisimple Lie groups is more complicated than for complex semisimple Lie groups. This is explained by two reasons. Firstly, a real semisimple Lie group G have several non-conjugate Cartan subgroups. To every class of Cartan subgroups there corresponds the series of representations of G. Secondly, except for the principal unitary series and the complementary series, real semisimple Lie groups can have the discrete series of unitary representations.

A unitary irreducible representation T of a semisimple (or reductive) Lie group G belongs to the *discrete series* if it is a direct summand in a decomposition of the regular representation of G into irreducible components. In this case the function $(T(g)\mathbf{x}, \mathbf{y})$ is square integrable with respect to the invariant measure on G for every vectors $\mathbf{x}$ and $\mathbf{y}$ from the carrier space of T. In particular, all matrix elements of the discrete series representations are square integrable.

A semisimple noncompact Lie group G has the discrete series of unitary representations if and only if G has a compact Cartan subgroup. In this case G is called a *cuspidal group*.

Example 1. The group $SO_0(p,q)$ is cuspidal if and only if at least one of the numbers p and q is even. If $p = 2s$, then the set of matrices

$$\text{diag}\,(\omega_1, \dots, \omega_s, \omega_{s+1}, \dots, \omega_{s+t}, 1), \quad t = [q/2], \quad \omega_j \in SO(2),$$

(the number 1 is absent on the main diagonal if $p+q$ is even) is its compact Cartan subgroup.

Example 2. The group $U(p,q)$, $p+q = n$, $p \leq q$, is cuspidal. Some of representations of the discrete series of this group are constructed in the following way. Let $\mathfrak{L}$ be the space of functions $f(Z^{(1)}, Z^{(2)}, W)$, $Z^{(1)} \in \mathfrak{M}_p(\mathbb{C})$, $Z^{(2)} \in \mathfrak{M}_q(\mathbb{C})$, $W \in \mathfrak{R}_1(p,q)$ (see Section 15.2.2), satisfying the conditions:

1) At fixed W and $Z^{(2)}$ (respectively, $Z^{(1)}$) functions f belong to $\mathcal{P}_{\mathbf{m}}(\mathfrak{M}_p(\mathbb{C}))$ (to $\mathcal{P}_{\boldsymbol{\ell}}(\mathfrak{M}_q(\mathbb{C}))$), where $\mathbf{m} = (m_1, \dots, m_p)$, $\boldsymbol{\ell} = (\ell_1, \dots, \ell_q)$.
2) At fixed $Z^{(1)}$ and $Z^{(2)}$ functions f analytically depend on pq elements of the matrix W in the domain $\mathfrak{R}_1(p,q)$.
3) The integral

$$\begin{aligned}\|f\|^2 = \int |f(U_p(I_p - W^*W)^{1/2},\ V_q(I_q - \bar{W}W^t)^{1/2},\ W)|^2 \\ \times \det{}^{-n}(I_q - W^*W) d_*U_p d_*V_q dW\end{aligned} \tag{1}$$

converges for $U_p \in U(p)$ and $V_q \in U(q)$. Here d_*U_p and d_*V_q are the invariant measures on the groups $U(p)$ and $U(q)$, and for $W = (w_{ij})$ we have $dW = \prod_{i,j} dx_{ij}dy_{ij}$, $w_{ij} = x_{ij} + iy_{ij}$. If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(p,q)$, where $A \in \mathfrak{M}_p(\mathbb{C})$, $D \in \mathfrak{M}_q(\mathbb{C})$, then the operator $T_{\mathbf{m}\boldsymbol{\ell}}(g)$ in the space $\mathfrak{L}$ corresponds to the matrix g which is given by the formula

$$\begin{aligned}(T_{\mathbf{m}\boldsymbol{\ell}}(g)f)(Z^{(1)}, Z^{(2)}, W) = f(Z^{(1)}(A^* + C^*W)^{-1}, Z^{(2)}(B^tW + D^t)^{-1}, \\ (WC + A)^{-1}(WD + B)).\end{aligned} \tag{2}$$

The discrete series representations $T_{\mathbf{m}\boldsymbol{\ell}}$ are defined by the discrete set of numbers $m_1, \dots, m_p, \ell_1, \dots, \ell_q$. These representations are unitary with respect to the scalar product in $\mathfrak{L}$ corresponding to norm (1). Other set of representations of the discrete series $U(p,q)$ are obtained by replacement of the spaces $\mathfrak{L}$ of analytical functions in W by the corresponding spaces of anti-analytical functions in W. The groups $U(p,q)$ have the “strange” discrete series of unitary representations which may be realized in the spaces of cohomologies.

In the analogous way (replacing $\mathfrak{R}_1(p,q)$ by other Hermitian symmetric manifolds) representations of the discrete series of the groups $SO^*(2n)$, $SO_0(p,2)$ and $Sp(n, \mathbf{R})$ are constructed.

Description of irreducible representations of a semisimple Lie group G can be reduced to description of the discrete series representations of some reductive subgroups of G. Let us give this construction. Let $G = NAK$ be the Iwasawa decomposition of G and $P = NAM$ be its minimal parabolic subgroup (see Section 15.1.7). All minimal parabolic subgroups are conjugate in G. A subgroup $\tilde{P}$, containing some minimal parabolic subgroup, is called *parabolic subgroup* of G. Let $\tilde{P} = \tilde{N}\tilde{A}\tilde{M}$ be the Langlands decomposition of the subgroup $\tilde{P}$ (see Section 15.1.12). Then $\tilde{N} \subset N$, $\tilde{A} \subset A$, $\tilde{M} \supset M$. The reductive subgroup $\tilde{M}$ may be noncompact. If the subgroup $\tilde{M}$ is cuspidal, then the parabolic subgroup $\tilde{P}$ is called *cuspidal.*

Let $\tilde{P} = \tilde{N}\tilde{A}\tilde{M}$ be a cuspidal parabolic subgroup of a connected semisimple Lie group G with finite center. We fix an irreducible representation Q of $\tilde{M}$, which either belongs to the discrete series or is a limiting point of the discrete series, and a character λ of the commutative subgroup $\tilde{A}$. Let $\mathfrak{D}_{\lambda Q}$ be the space of smooth vector functions $\mathbf{f}$ on G, taking values in the carrier space $\mathfrak{L}$ of the representation Q and such that

$$\mathbf{f}(\tilde{n}\tilde{a}\tilde{m}g) = \lambda(\tilde{a})Q(\tilde{m})f(g), \quad \tilde{n}\tilde{a}\tilde{m} \in \tilde{P} = \tilde{N}\tilde{A}\tilde{M}. \tag{3}$$

The formula

$$(T_{\lambda Q}(g_0)\mathbf{f})(g) = \mathbf{f}(gg_0), \quad g_0 \in G, \tag{4}$$

defines the representation $T_{\lambda Q}$ of the group G. Generally speaking, this representation may be reducible. Let w_0 be the element of the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$ such that $w_0\Delta_+ = \Delta_-$, where Δ_+ and Δ_- are the sets of positive and negative roots of this pair respectively. We set $N_- = w_0N$, $\bar{P} = N_-AM$. Then the formula

$$(A_{\bar{P}P}(Q,\lambda)\mathbf{f})(g) = \int_{N_-} \mathbf{f}(ng)dn \tag{5}$$

gives an integral operator on the space $\mathfrak{D}_{\lambda Q}$. Let $\mathcal{E}_{\lambda Q}$ be the kernel of the operator $A_{\bar{P}P}$ and let $\tilde{T}_{\lambda Q}$ be the representation of G, defined by the representation $T_{\lambda Q}$ in the quotient space $\mathfrak{D}_{\lambda Q}/\mathcal{E}_{\lambda Q}$. If the representation $T_{\lambda Q}$ is irreducible, then $\tilde{T}_{\lambda Q} = T_{\lambda Q}$. It is possible to show that every irreducible representation of the group G is equivalent to one of the representations $\tilde{T}_{\lambda Q}$ obtained with the help of this construction (Langlands-Knapp-Zuckerman theorem). More detailed description of this construction can be found in reference [195] of the second volume).

If $\tilde{P}$ is a minimal parabolic subgroup of G, then formula (4) defines representations $T_{\lambda Q}$ of the *principal non-degenerate series* of G. These representations are given by irreducible unitary representations Q of the compact subgroup M and by characters λ of the commutative subgroup $A = \exp \mathfrak{a}$. The representation $T_{\lambda Q}$ of this series acts in the space of vector functions $\mathbf{f}$ on G with values in the carrier space $\mathfrak{L}$ of Q, satisfying the condition

$$\mathbf{f}(namg) = \lambda(a)Q(m)\mathbf{f}(g), \quad namg \in P = NAM, \tag{6}$$

and is given by the formula

$$(T_{\lambda Q}(g_0)\mathbf{f})(g) = \mathbf{f}(gg_0). \tag{7}$$

The representations $T_{\lambda Q}$ may be reducible only in the integral points. In these points hierarchy of invariant subspaces appears. In the corresponding quotient spaces irreducible representations of G act.

Let us note that vector functions from $\mathfrak{D}_{\lambda Q}$ are uniquely defined by their values on the subgroup K and their values on K satisfy the relation

$$\varphi(mk) = Q(m)\varphi(k), \quad m \in M. \tag{8}$$

Let $\mathfrak{L}_Q$ be the space of smooth vector functions on K with values in the carrier space $\mathfrak{L}$ of the representation Q satisfying relations (8). The representation $T_{\lambda Q}$ of G is defined in $\mathfrak{L}_Q$ by the formula

$$(T_{\lambda Q}(g)\varphi)(k) = \lambda(a)\varphi(\tilde{k}), \tag{9}$$

where $kg = na\tilde{k}$, $n \in N$. There are realizations of the representations $T_{\lambda Q}$, connected with "intermediate" decompositions of G.

Example 3. Let us describe representations of the group $U(p,q)$, $p+q=n$, $p \le q$, which do not belong to the discrete series. This group has p non-discrete series of representations. We denote them by $d_1, d_2, \dots, d_p$. Representations T_{QR} of the series d_r, $1 \le r \le p$, are given by representations Q of the discrete series of the subgroup $U(p-r, q-r)$ and by irreducible representations R of the principal series of the subgroup $GL(r,\mathbb{C})$. They correspond to the Cartan subgroup of $U(p,q)$ which is isomorphic to the group $\mathbf{R}^r \times U(1)^{n-r}$.

In order to describe these representations we realize $U(p,q)$ as a group of matrices, conserving the Hermitian form defined by the matrix

$$\begin{pmatrix} 0 & 0 & S \\ 0 & I & 0 \\ S & 0 & 0 \end{pmatrix},$$

where S is the $r \times r$ matrix $S = (s_{ij}) \equiv (\delta_{j,r-i+1})$, $I = \mathrm{diag}\,(I_{p'}, -I_{q'})$, $p' = p - r$, $q' = q - r$. Matrices of $U(p,q)$ in this realization will be written in the form $g = (g_{ij})_{i,j=1,2,3}$. Let $N_-^r(n,\mathbb{C})$ be the subgroup of $U(p,q)$ consisting of matrices of the form

$$\begin{pmatrix} I_r & 0 & 0 \\ A & I_{n-2r} & 0 \\ B & C & I_r \end{pmatrix}.$$

The subgroup $N_+^r(n,\mathbb{C})$ is obtained by transposition of matrices from $N_-^r(n,\mathbb{C})$. The set of matrices $\mathrm{diag}\,(g_{-1}, g_0, g_1) \in U(p,q)$ forms the subgroup which is denoted by $D^r(n,\mathbb{C})$. It is easy to see that the conditions $g_{-1}Sg_1^*S = I_r$, $g_1 \in GL(r,\mathbb{C})$, $g_0 \in U(p-r, q-r)$ are fulfilled for matrices from $D^r(n,\mathbb{C})$.

We have the generalized Gauss decomposition

$$g = n_-\delta n_+,\ n_\pm \in N^r_\pm(n,\mathbb{C}),\ \delta \in D^r(n,\mathbb{C}), \tag{10}$$

of elements of the group $U(p,q)$. Let $\tilde{\mathfrak{L}}_Q$ be the space of vector functions $\mathbf{f}(g')$ on $U(p',q') \equiv U(p-r,q-r)$, where a representation Q of the discrete series of $U(p',q')$ is realized, and let $\tilde{\mathfrak{L}}_R$ be the space of functions on $GL(r,\mathbb{C})$, where a representation R of the principal series of $GL(r,\mathbb{C})$ is realized. Let $\tilde{\mathfrak{L}}$ be the space of smooth functions on $N^r_+(n,\mathbb{C})$.

The representation T_{QR} of the group $U(p,q)$ defined by the representations Q and R of the subgroups $U(p',q')$ and $GL(r,\mathbb{C})$ acts in the space $\mathfrak{H}_{QR} = \tilde{\mathfrak{L}}_Q \otimes \tilde{\mathfrak{L}}_R \otimes \tilde{\mathfrak{L}}$. Elements of $\mathfrak{H}_{QR}$ are vector functions $\mathbf{f}(g', g_r, n_+)$. Let $n_+ \in N^r_+(n,\mathbb{C})$ and $g \in U(p,q)$. We represent the matrix n_+g in the form

$$n_+g = n_-\delta\tilde{n}_+, \quad n_- \in N^r_-(n,\mathbb{C}), \quad \delta \in D(n,\mathbb{C}), \quad \tilde{n}_+ \in N^r_+(n,\mathbb{C}).$$

Here $\delta = \mathrm{diag}\,(g_{-1}, g_0, g_1)$, where $g_0 \in U(p',q')$, $g_1 \in GL(r,\mathbb{C})$. We have

$$T_{QR}(g)\mathbf{f}(g', g_r, n_+) = Q(g_0)R(g_1)\mathbf{f}(g', g_r, \tilde{n}_+)J(\tilde{n}_+, n_+), \tag{11}$$

where $J(\tilde{n}_+, n_+)$ is the Jacobian of transformation from parameters of the matrix n_+ to parameters of the matrix $\tilde{n}_+$, the operator $Q(g_0)$ acts upon $\mathbf{f}$ as upon a function of g', and the operator $R(g_1)$ acts upon $\mathbf{f}$ as upon a function of g_r. We see that representations of the series d_r are given by $n-r$ integers and r complex numbers. Under certain conditions onto representation parameters a representation T_{QR} is unitary with respect to the scalar product

$$\langle \mathbf{f}_1, \mathbf{f}_2 \rangle = \int (\mathbf{f}_1(g_1, g_r, n_+), \mathbf{f}_2(g_1, g_r, n_+))d_*g_1 d_*g_r d_*n_+, \tag{12}$$

where d_*g_1, d_*g_r, d_*n_+ are the invariant measures on the corresponding subgroups.

Example 4. The group $GL(n,\mathbf{R})$, $n > 2$, has $[n/2]+1$ non-conjugate Cartan subgroups. Since $GL(n,\mathbf{R})$ has no compact Cartan subgroups, then this group does not have discrete series representations. Representations of the series d_m, $0 \le m \le [n/2]$, are given by m non-negative integers, $m+\tau$ real numbers ($\tau = n-2m$), and $\tau - 1$ numbers taking the values 0 and 1. They are constructed in spaces of functions in m complex and $\frac{1}{2}n(n-1) - m$ real variables, which are analytical in complex variables separately in upper and in lower half-planes. The generalized Gauss decomposition, related to the corresponding representations of the series d_m, is of the form

$$G = N_-DN_+, \quad D = \{\mathrm{diag}\,(g_1, \cdots, g_m, \lambda_{2m+1}, \cdots, \lambda_n),$$
$$g_k \in GL(2,\mathbf{R}), \quad \lambda_j \in \mathbf{R}.$$

Representations of this series are induced by representations of the block-diagonal subgroup, which are of the form

$$Q_1 \otimes \ldots \otimes Q_m \otimes R_1 \otimes \ldots \otimes R_{m+r},$$

where Q_j are representations of the discrete series of the group $GL(2, \mathbf{R})$ and R_k are characters of the multiplicative subgroup $\mathbf{R}\backslash\{0\}$.

Representations of Examples 3 and 4 are irreducible if continuous parameters, giving representations, do not satisfy the certain integrity conditions[3]. If these integrity conditions are satisfied, then invariant subspaces appear in the carrier spaces of representations and we obtain irreducible representations of other series.

Example 5. Let $G = SO_0(n,1)$, $K = SO(n)$, $M = SO(n-1)$. In this case the subgroup $A = \exp \mathfrak{a}$ is one-dimensional. The representation $T_{Q\lambda}$ of the principal series of $SO_0(n,1)$ is given by irreducible unitary representation Q of the subgroup M and by character λ of the subgroup A. This representation acts in the space $\mathfrak{X}_Q$ of vector functions $\mathbf{f}$ on $K = SO(n)$ with values in the carrier space $\mathcal{V}$ of the representation Q, satisfying the relation $\mathbf{f}(mk) = Q(m)\mathbf{f}(k)$ for all $m \in M$. The representation $T_{Q\lambda}$ is given by the formula

$$(T_{Q\lambda}(g)\mathbf{f})(k) = \mathbf{f}(kg) = \lambda(a)\mathbf{f}(\tilde{k}), \quad g \in SO_0(n,1), \tag{13}$$

where $a \in A$ and $\tilde{k} \in K$ are defined by the Iwasawa decomposition $kg = n_- a\tilde{k}$, $n_- \in N_-$, of the element kg. The space $\mathfrak{X}_Q$ can be described as the space of vector functions on the sphere S^{n-1} with values in $\mathcal{V}$, that is we can consider $\mathbf{f}(k)$ as functions $\mathbf{F}(\theta_1, \ldots, \theta_{n-1})$ in angles on S^{n-1} (see Section 9.2.1). For $g = g'_n(t) \in SO_0(n,1)$ (see Section 9.1.1) the element $a = g'_n(t')$ in (13) is defined by the angle t' which is defined by the formula $e^{t'} = \cosh t + \sinh t \cos \theta_{n-1}$. We have

$$T_{Q\lambda}(g'_n(t))\mathbf{F}(\theta_1, \ldots, \theta_{n-1}) = (\cosh t + \sinh t \cos \theta_{n-1})^{\lambda} \mathbf{F}(\theta_1, \ldots, \theta_{n-2}, \theta'_{n-1}),$$

where θ'_{n-1} is determined by the formula (7) of Section 9.2.1. More detailed consideration of these representations is given in Chapter 18.

16.2.3. Realization of the principal series representations in spaces of functions on matrix cones and hyperboloids. For realizations of infinite dimensional representations of the principal series of classical Lie groups we can apply the same construction which was used for realizations of finite dimensional irreducible representations of these groups (see Section 16.1.5). Let $C^{p,2p}(\mathbf{R})$ be the space of matrices Ξ of rank p from $\mathfrak{M}_{p,2p}(\mathbf{R})$ such that $\Xi I_{pp} \Xi^t = 0$, where

[3] Except for representations of the series d_m, $m = n/2$, of the group $GL(n, \mathbf{R})$. These representations decompose into two irreducible components.

$I_{pp} = \text{diag}\,(I_p, -I_p)$. It is evident that if $\Xi \in C^{p,2p}(\mathbf{R})$, $g \in GL(p, \mathbf{R})$, $\omega \in SO_0(p,p)$, then $g\Xi\omega \in C^{p,2p}(\mathbf{R})$. Let

$$\chi(s_-) = \chi(n_-\delta) = \prod_{k=1}^{p} |\delta_k|^{\lambda_k} (\text{sign}\ \delta_k)^{\varepsilon_k} \tag{1}$$

be a character of the group $S_-(p, \mathbf{R})$, where $\lambda_k \in \mathbb{C}$, $\varepsilon_k \in \{0,1\}$, $1 \le k \le p$. Let $\mathfrak{D}_\chi$ be the space of smooth functions f on $C^{p,2p}(\mathbf{R})$ such that $f(s_-\Xi) = \chi(s_-)f(\Xi)$ for all $s_- \in S_-(p, \mathbf{R})$. Then the formula

$$(T_\chi(\Omega)f)(\Xi) = f(\Xi\Omega), \quad \Omega \in SO_0(p,p), \tag{2}$$

defines a representation of the group $SO_0(p,p)$ in the space $\mathfrak{D}_\chi$. This representation is equivalent to one of representations of the principal series of $SO_0(p,p)$.

Almost every matrix Ξ from $C^{p,2p}(\mathbf{R})$ can be written in the form $\Xi = s_-\Xi_0$, where $s_- \in S_-(p, \mathbf{R})$, $\Xi_0 = (\omega_1, \omega_2)$, $\omega_1, \omega_2 \in SO(p)$. Almost every matrix $\Omega \in SO_0(p,p)$ is represented as $\Omega = \Omega_1 A \Omega_2$, where

$$\Omega_i = \text{diag}\,(\omega_{i1}, \omega_{i2}), \quad \omega_{i1}, \omega_{i2} \in SO(p), \quad i = 1,2,$$

$$A = \begin{pmatrix} \cosh \mathbf{t} & \sinh \mathbf{t} \\ \sinh \mathbf{t} & \cosh \mathbf{t} \end{pmatrix}, \quad \begin{matrix} \cosh \mathbf{t} = \text{diag}\,(\cosh t_1, \dots, \cosh t_p), \\ \sinh \mathbf{t} = \text{diag}\,(\sinh t_1, \dots, \sinh t_p). \end{matrix}$$

Therefore, functions $f \in \mathfrak{D}_\chi$ are uniquely defined by their values on the set $\Gamma = \{(\omega_1, \omega_2) \mid \omega_1, \omega_2 \in SO(m)\}$. If $\Omega = \text{diag}\,(\tilde{\omega}_1, \tilde{\omega}_2)$, then

$$(T_\chi(\Omega)f)(\omega_1, \omega_2) = f(\omega_1\tilde{\omega}_1, \omega_2, \tilde{\omega}_2). \tag{3}$$

Therefore, in formula (2) it is sufficient to indicate the expression for $(T_\chi(A)f)(\Xi_0)$. We have

$$(\omega_1, \omega_2) \begin{pmatrix} \cosh \mathbf{t} & \sinh \mathbf{t} \\ \sinh \mathbf{t} & \cosh \mathbf{t} \end{pmatrix} = (\hat{\omega}_1, \hat{\omega}_2),$$

where

$$\hat{\omega}_1 = \omega_1 \cosh \mathbf{t} + \omega_2 \sinh \mathbf{t}, \quad \hat{\omega}_2 = \omega_1 \sinh \mathbf{t} + \omega_2 \cosh \mathbf{t}.$$

Thus, if $(\hat{\omega}_1, \hat{\omega}_2) = s_-(\tilde{\omega}_1, \tilde{\omega}_2)$, where $s_- \in S_-(p, \mathbf{R})$, $\tilde{\omega}_1, \tilde{\omega}_2 \in SO(p)$, then

$$(T_\chi(A)f)(\Xi_0) = \chi(s_-)f(\tilde{\omega}_1, \tilde{\omega}_2). \tag{4}$$

In order to define the invariant measure on $C^{p,2p}(\mathbf{R})$ we note that $C^{p,2p}(\mathbf{R})$ is separated in $\mathfrak{M}_{p,2p}(\mathbf{R})$ by the condition $\Xi I_{pp} \Xi^t = 0$. It follows from here that $\dim C^{p,2p}(\mathbf{R}) = (1/2)(3p^2 - p)$. The invariant measure on $C^{p,2p}(\mathbf{R})$ is given by the formula

$$d_*\Xi = \delta(\Xi I_{pp} \Xi^t) d\Xi, \tag{5}$$

where for $\Xi = (\xi_{ij})$ we have $d\Xi = \prod_{i=1}^{p}\prod_{j=1}^{2p} d\xi_{ij}$.

Almost every matrix $\Xi \in C^{p,2p}(\mathbf{R})$ can be decomposed in the form $\Xi = s_-\Xi_0$, where $s_- \in S_-(p,\mathbf{R})$, $\Xi_0 = (\omega_1, \omega_2)$, $\omega_1, \omega_2 \in SO(p)$. For the invariant measure $d_*\Xi$ we have

$$d_*\Xi = \gamma(s_-, \omega_1, \omega_2) d_*s_- d_*\omega_1 d_*\omega_2,$$

where $\gamma(s_-, \omega_1, \omega_2)$ is a scalar multiplier, d_*s_- is the left invariant measure on $S_-(p,\mathbf{R})$, and $d_*\omega$ is the invariant measure on $SO(p)$. If $\Omega = \text{diag}\,(\tilde{\omega}_1, \tilde{\omega}_2)$, then $\Xi_0\Omega = (\omega_1\tilde{\omega}_1, \omega_2\tilde{\omega}_2)$. It follows from here that $\gamma(s_-, \omega_1, \omega_2)$ does not depend on ω_1 and ω_2, that is $\gamma(s_-, \omega_1, \omega_2) = \gamma(s_-)$. Taking into account that the function $\delta(x)$ is homogeneous of degree -1, we derive from (5) that $\gamma(s_-) = \gamma(n_-\delta) = c\Delta^{p-1}(\delta)$ and therefore,

$$d_*\Xi = \frac{4\pi^2}{\Gamma^2\left(\frac{p}{2}\right)} \Delta^{p-1}(\delta) d_*s_- d_*\omega_1 d_*\omega_2, \tag{6}$$

where $\Delta(\delta)$ is determinant of the matrix δ, and the measures $d_*\omega_1$ and $d_*\omega_2$ are normalized.

Subsets of the form

$$\{s_-\Xi \mid s_- \in S_-(p,\mathbf{R})\},$$

where Ξ is fixed, are called *generatrices* of the matrix cone $C^{p,2p}(\mathbf{R})$. A set $\Gamma \subset C^{p,2p}(\mathbf{R})$, intersecting one time every generatrix, is called a *contour*.

As in the case of the usual cone C^{n-1} in the space $\mathbf{R}^n$ (see Chapter 10), on every contour Γ we can take the measure, consistent with the invariant measure on $C^{p,2p}(\mathbf{R})$, that is the measure for which the integral over a contour Γ of a homogeneous functions of homogeneity degree

$$\chi_0(\delta) = \delta_1^{2-2p}\delta_2^{4-2p} \ldots \delta_{p-1}^{-2} \tag{7}$$

does not depend on a choice of a contour Γ. This allows us to choose in the set of the representations T_χ those representations which are unitary with respect to the scalar product

$$(f_1, f_2) = \int_\Gamma f_1(\Xi)\overline{f_2(\Xi)} d\gamma,$$

to derive conditions of equivalence of pairs of representations and so on.

Representations of the principal series of the group $SO_0(p,p)$ can be realized in the space of $\Box_p$-harmonic functions. Namely, we denote by $\mathfrak{Z}^{p,2p}(\mathbf{R})$ the set of matrices $X \in \mathfrak{M}_{p,2p}(\mathbf{R})$ such that $XI_{pp}X^t \in \mathfrak{P}_p(\mathbf{R})$. If $X \in \mathfrak{M}_{p,2p}(\mathbf{R})$, then by $\partial/\partial X$ we denote the matrix X in which elements x_{ij} are replaced by $\partial/\partial x_{ij}$. Let $\Box_p$ be the matrix $\partial/\partial X \cdot I_{pp} \cdot \partial/\partial X$. A function $f(X)$ on $\mathfrak{Z}^{p,2p}(\mathbf{R})$ is called $\Box_p$-*harmonic* if for every i and j, $1 \le i, l \le p$, we have

$$\sum_{k=1}^{p} \frac{\partial^2 f}{\partial x_{ik}\partial x_{jk}} - \sum_{k=p+1}^{2p} \frac{\partial^2 f}{\partial x_{ik}\partial x_{jk}} = 0. \tag{8}$$

Let χ be a character of the group $S_-(p,\mathbf{R})$. A function $f(X)$ on $\mathfrak{F}^{p,2p}(\mathbf{R})$ is called *χ-homogeneous* if

$$f(s_-X) = \chi(s_-)f(X) \quad \text{for all} \quad s_- \in S_-(p,\mathbf{R}). \tag{9}$$

The set of matrices $X \in \mathfrak{M}_{p,2p}(\mathbf{R})$ such that $XI_{p,p}X^t = I_p$ will be called the *matrix hyperboloid.* We denote it by $H^{p,2p}(\mathbf{R})$. Any matrix $X \in \mathfrak{F}^{p,2p}(\mathbf{R})$ can be represented in the form $X = s_-X_0$, where $s_- \in S_-(p,\mathbf{R})$, $X_0 \in H^{p,2p}(\mathbf{R})$. Therefore, every χ-homogeneous functions f on $\mathfrak{F}^{p,2p}(\mathbf{R})$ is uniquely defined by its values on $H^{p,2p}(\mathbf{R})$. Moreover, for every function f on $H^{p,2p}(\mathbf{R})$ there exists $\Box_p$-harmonic χ-homogeneous function h, coinciding on $H^{p,2p}(\mathbf{R})$ with f.

If a function h is $\Box_p$-harmonic and χ-homogeneous, then for every $g \in SO_0(p,p)$ the function $h_g(X) \equiv h(Xg)$ is also $\Box_p$-harmonic and χ-homogeneous. (It follows from the fact that shifts by elements $g \in SO_0(p,p)$ commute with $\Box_p$.) Therefore, in the space of functions on $H^{p,2p}(\mathbf{R})$ the representation Q_χ of the group $SO_0(p,p)$ is defined, which is given by the formula $Q_\chi(g)h(X) = h(Xg)$. For non-integral character χ the representation Q_χ is equivalent to the representation T_χ.

There is the integral transform intertwining the representations T_χ and $Q_{\chi'}$ of the group $SO_0(p,p)$ for which χ and χ' are connected by the relation $\chi(s_-)\chi'(s_-) = \chi_0(s_-)$, where χ_0 is given by formula (7). This integral transform is a generalization of the Poisson transform.

Representations T_χ of the principal series of the group $SU(p,p)$ are realized in the space of χ-homogeneous functions on the Hermitian matrix cone, that is on the manifold

$$C_*^{p,2p}(\mathbb{C}) = \{Z \in \mathfrak{M}_{p,2p}(\mathbb{C}) \mid \operatorname{rank} Z = p, ZI_{p,p}Z^* = 0\}.$$

Every χ-homogeneous function f on $C_*^{p,2p}(\mathbb{C})$ is uniquely defined by its values on the set of matrices (u_1,u_2), where $u_1,u_2 \in SU(p)$. If $U = (\tilde{u}_1,\tilde{u}_2)$, $u_1,u_2 \in SU(p)$, then $(T_\chi(U)f)(u_1,u_2) = f(u_1\tilde{u}_1,u_2\tilde{u}_2)$. We recommend to the reader to write down the expression for

$$(T_\chi(A)f)(u_1,u_2) \quad \text{if} \quad A = \begin{pmatrix} \cosh \mathbf{t} & \sinh \mathbf{t} \\ \sinh \mathbf{t} & \cosh \mathbf{t} \end{pmatrix}.$$

In the analogous way representations of the principal series of the groups $Sp(n,\mathbf{R})$ and $SO^*(2n)$ can be constructed.

16.2.4. Relations between finite and infinite dimensional representations of classical groups. Let G be a complex classical Lie group. Let G_0 and $\tilde{G}$ be its compact and noncompact real forms. It was mentioned above that if a representation T_χ of the principal series of $\tilde{G}$ satisfies the certain integrity conditions, then it is reducible and one of its irreducible subrepresentations is finite dimensional and is realized in a space of polynomials. Suppose that this finite dimensional subrepresentation is analytical. Then it can be analytically continued to a complex analytical irreducible representation of the complex group G. We may restrict it onto the compact subgroup G_0. In this way we obtain finite dimensional irreducible representation of G_0. Thus, connection between analytical irreducible finite dimensional representations of the noncompact real form $\tilde{G}$ of G and irreducible representations of the compact real form G_0 of G is established.

Let $\tilde{K}$ be the maximal compact subgroup of the group $\tilde{G}$. The principal series representations of $\tilde{G}$ can be constructed in the space of vector functions $\mathbf{f}$ on $\tilde{K}$ such that $\mathbf{f}(mk) = Q(m)\mathbf{f}(k)$, $m \in M$. This leads to different realizations of irreducible representations of the compact Lie group G_0.

Example 1. Let $G = SL(n, \mathbb{C})$, $\tilde{G} = SL(n, \mathbf{R})$. Then $G_0 = SU(n)$ and $\tilde{K} = SO(n)$. Therefore, in this case we obtain realization of irreducible representations of the group $SU(n)$ in the space of functions on the group $SO(n)$. In this realization the decomposition $u = \omega_1 \delta \omega_2$ of elements of $SU(n)$ is used, where $\omega_1, \omega_2 \in SO(n)$ and $\delta = \text{diag}\,(e^{i\varphi_1}, \dots, e^{i\varphi_n})$. Other realizations of irreducible representations of the group $SU(n)$ are connected with other real forms of the group $SL(n, \mathbb{C})$, namely with the forms $SU(p,q)$, $p+q=n$ and $SU^*(2p)$ if $2p = n$.

Example 2. If $G = SO(n, \mathbb{C})$, then we have the noncompact real forms $SO_0(p,q)$ $p+q=n$, and $SO^*(2p)$ if $2p=n$, as well as compact real form $SO(n)$. To the groups $SO_0(p,q)$ the realizations of irreducible representations of $SO(n)$ correspond in the spaces of functions on the manifolds $SO(p) \times i\text{St}(p,q)$ where $\text{St}(p,q) = \{X \in \mathfrak{M}_{p,q}(\mathbf{R}) \mid XX^t = I_p\}$.

16.2.5. Representations of nilpotent groups and of semidirect products. Every nilpotent Lie group is isomorphic to some subgroup of one of the groups $N_+(n, \mathbb{C})$. But classification of nilpotent groups is not obtained. Nevertheless, there exists a general method of description of irreducible representations of these groups. Let G be a nilpotent Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{g}^*$ be the linear space conjugate to $\mathfrak{g}$. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *subordinated to a functional* $f \in \mathfrak{g}^*$ if $\langle f, [x,y]\rangle = 0$ for all $x, y \in \mathfrak{h}$, where $\langle f, x\rangle$ is the value of f on the element x. The mapping $x \to \langle f, x\rangle$ is a one-dimensional real representation of the Lie algebra $\mathfrak{h}$. The formula

$$U_{f,H}(\exp x) = \exp(2\pi i \langle f, x\rangle) \tag{1}$$

gives a one-dimensional unitary representation (character) of the group $H = \exp \mathfrak{h}$. A. A. Kirillov has proved that *any irreducible unitary representation T of a connected simply connected nilpotent Lie group G is induced by one of representations* (1) *of some connected subgroup H*. However, some of these representations of G are reducible. *The representation T is irreducible if and only if the Lie algebra $\mathfrak{h}$ of the corresponding subgroup H is a subalgebra of maximal dimension in the set of Lie subalgebras, subordinated to the functional f. The irreducible representations, induced by characters U_{f_1,H_1} and U_{f_2,H_2} are equivalent if and only if the functionals f_1 and f_2 belong to the same orbit in $\mathfrak{g}^*$ with respect to the action of the adjoint representation of the group G.*

In some cases a nilpotent group can be represented as a semidirect product of a nilpotent group of lower dimension and a commutative group. In these cases it is useful to apply the method of construction of representations of semidirect products, introduced by E. Wigner (we have used this method for construction of irreducible representations of groups of inhomogeneous linear transformations).

Let a Lie group G be a semidirect product of an invariant subgroup N and a subgroup H with respect to the automorphism $h \to hnh^{-1}$, $n \in N$, $h \in H$. Let Q be a representation of the group N in a linear space $\mathfrak{L}''$. We denote by H_Q the subgroup of elements $h \in H$ such that $Q(hnh^{-1}) = Q(n)$ for all $n \in N$. We set $P = H_Q N$. It is a subgroup of G. Let S be a representation of H_Q in a linear space $\mathfrak{L}'$ and let $\mathfrak{L} = \mathfrak{L}' \otimes \mathfrak{L}''$. Then

$$R(hn) = S(h) \otimes Q(n), \quad h \in H_Q, \quad h \in N,$$

is a representation of the group $P = H_Q N$ by operators acting in $\mathfrak{L}$. Really, since $(h')^{-1}nh' \in N$, then

$$R(hnh'n') = R(hh'(h')^{-1}nh'n') = S(hh') \otimes Q((h')^{-1}nh'n').$$

Since $h' \in H_Q$, then $Q((h')^{-1}nh') = Q(n)$ and

$$\begin{aligned} R(hnh'n') &= S(h)S(h') \otimes Q(n)Q(n') \\ &= (S(h) \otimes Q(n))(S(h') \otimes Q(n')) = R(hn)R(h'n'). \end{aligned}$$

Thus, R is a representation of P.

We denote by T_R the representation of G induced by the representation R of the subgroup P. It acts in the space of vector functions $\mathbf{f}$ on G with values in $\mathfrak{L}$ such that $\mathbf{f}(hng) = R(hn)\mathbf{f}(g)$ and is given by the formula

$$(R_R(g_0)\mathbf{f})(g) = \mathbf{f}(gg_0).$$

If the representations Q and S are irreducible, then T_R is also irreducible.

The method, described above, is used for construction of irreducible representations of the principal series of motion groups G of symmetric spaces, triple to dual compact and noncompact symmetric spaces with semisimple groups of motions. Recall (see Section 1.2 of Volume 1) that such group G is a semidirect product of the maximal compact subgroup K and the additive group of the linear space $\mathfrak{p}$ which touches in the point e the subset $\mathcal{P}$ from the Cartan decomposition $K\mathcal{P}$. Namely, these representations are given by one-dimensional representations λ of the additive subgroup $\mathfrak{a} \subset \mathfrak{p}$ (which is contained in the Iwasawa decomposition of the corresponding semisimple Lie algebra) and by irreducible representations S of the compact subgroup K_λ consisting of elements $k \in K$ such that $\lambda(k^{-1}Xk) = \lambda(X)$ for all $X \in \mathfrak{a}$. This representations determine the representations $\lambda \otimes S$ of the subgroup $K_\lambda \mathfrak{a}$ which induce a principal series representations of the semidirect product of K and $\mathfrak{p}$.

In this way irreducible representations of the principal series of the semidirect product of the compact group $U(m, \mathbf{F})$ and the additive group $\mathfrak{P}_m(\mathbf{F})$ are constructed. Representations of the principal series of the group of triangular matrices are analogously constructed.

Example 1. Let $m = p + q$, $1 \leq q \leq p \leq m - 1$. Every matrix $g \in N_+(m, \mathbf{R})$ is representable in the form

$$g = \begin{pmatrix} a & Xb \\ 0 & b \end{pmatrix}, \quad \text{where} \quad a = N_+(p, \mathbf{R}), \quad b \in N_+(q, \mathbf{R}), \quad X \in \mathfrak{M}_{pq}(\mathbf{R}).$$

We have

$$\begin{pmatrix} a_1 & X_1 b_1 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & X_2 b_2 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & (a_1 X_2 b_1^{-1} + X_1) b_1 b_2 \\ 0 & b_1 b_2 \end{pmatrix}.$$

Thus, the group $N_+(m, \mathbf{R})$ is the semidirect product of the subgroup $H = N_+(p, \mathbf{R}) \times N_+(q, \mathbf{R})$ and of the commutative invariant subgroup $N = \left\{ \begin{pmatrix} I_p & X \\ 0 & I_q \end{pmatrix} \,\middle|\, X \in \mathfrak{M}_{pq}(\mathbf{R}) \right\}$ which is isomorphic to the additive group $\mathfrak{M}_{pq}(\mathbf{R})$. Every one-dimensional representation of the group N is of the form

$$Q\left(\begin{pmatrix} I_p & X \\ 0 & I_q \end{pmatrix}\right) = \exp(\operatorname{Tr} MX), \qquad \text{where} \quad M \in \mathfrak{M}_{qp}(\mathbb{C}).$$

It is unitary if the matrix M is imaginary. Let us put $M = (I_q, 0)$. Then the subgroup H_Q consists of matrices of the form diag (I_q, C, I_q), $C \in N_+(p - q, \mathbf{R})$. Let S be a one-dimensional representation of $N_+(p - q, \mathbf{R})$. For example, let

$$S(C) = \exp i(c_{q+1,q+2} + \ldots + c_{p-1,p}).$$

Then the formula $R(hn) = S(h) \otimes Q(n)$ gives a representation of the group $H_Q N$. If S and Q are unitary representations, then the induced representation is also unitary. This construction gives $m - 1$ series of irreducible representations of the group $N_+(m, \mathbf{R})$.

Irreducible representations of the groups, consisting of block-triangular matrices, are constructed in the same manner.

Example 2. Let $p + q = m$, $1 \leq q \leq p \leq m - 1$. We denote by G_{pq} the group of matrices of the form $\begin{pmatrix} \omega_1 & X\omega_2 \\ 0 & \omega_2 \end{pmatrix}$, where $\omega_1 \in SO(p)$, $\omega_1 \in SO(q)$, $X \in \mathfrak{M}_{pq}(\mathbf{R})$. We have

$$\begin{pmatrix} \omega_1 & X\omega_2 \\ 0 & \omega_2 \end{pmatrix} \begin{pmatrix} \tilde{\omega}_1 & \tilde{X}\tilde{\omega}_2 \\ 0 & \tilde{\omega}_2 \end{pmatrix} = \begin{pmatrix} \omega_1\tilde{\omega}_1 & (\omega_1 \tilde{X} \omega_2^{-1} + X)\omega_2\tilde{\omega}_2 \\ 0 & \omega_2\tilde{\omega}_2 \end{pmatrix}.$$

Therefore, G_{pq} is the semidirect product of the subgroup $H = SO(p) \times SO(q)$ and the commutative invariant subgroup

$$N = \left\{ \begin{pmatrix} I_p & X \\ 0 & I_q \end{pmatrix} \middle| \, X \in \mathfrak{M}_{pq}(\mathbf{R}) \right\}.$$

The formula $Q(X) = \exp(\mathrm{Tr}\, MX)$ gives a one dimensional representation of N. If $M = (I_q, 0)$, then the corresponding subgroup H_Q consists of matrices of the form diag (I_q, ω, I_q), where $\omega \in SO(p-q)$. Let S be an irreducible representation of the subgroup $SO(p-q)$. Then $R(hn) = S(h) \otimes Q(n)$ is an irreducible representation of the subgroup $H_Q N$ which induces an irreducible representation of the group G_{pq}.

In an analogous way representations of the groups of matrices $\begin{pmatrix} \omega_1 & X\omega_2 \\ 0 & \omega_2 \end{pmatrix}$, where $\omega_1 \in U(p, \mathbf{F})$, $\omega_2 \in U(q, \mathbf{F})$, $X \in \mathfrak{M}_{pq}(\mathbf{F})$, and of groups of the same matrices such that $p = q$, $X^* = X$ (or $X^* = -X$) are constructed.

16.2.6. Block splittings of matrices of irreducible representations. Let $\hat{G}$ be the set of classes of equivalent irreducible unitary representations of the group G and let H be a compact subgroup of G. Let Q be an irreducible representation of H. The set of representations T from $\hat{G}$ such that $T\downarrow^G_H$ contain the representation Q will be denoted by $(\hat{G}, Q)$. If $T \in \hat{G}$, then $(\hat{H}, T)$ will denote the set of representations Q from $\hat{H}$ which are contained in the representation $T\downarrow^G_H$ of H. Thus,

$$T\downarrow^G_H = \sum_{\alpha \in (\hat{H},T)} \sum_{\lambda=1}^{p(\alpha)} \oplus Q_{\alpha\lambda}, \tag{1}$$

where $p(\alpha)$ is multiplicity of the representation Q_α of H in $T\downarrow^G_H$ and λ is the index which distinguishes multiple representations.

Let $\mathfrak{L}$ be the carrier space of the representation $T \in \hat{G}$. We choose in $\mathfrak{L}$ a scalar product and an orthonormal basis which is agreed with decomposition (1). We denote vectors of this basis as

$$\mathbf{e}_{\alpha\lambda i}, \quad \alpha \in (\hat{H}, T), \quad 1 \le \lambda \le p(\alpha), \quad 1 \le i \le \dim Q_\alpha. \tag{2}$$

The matrix $T(g)$ of the representation T is splitted in this basis into blocks of the form

$$T^{\alpha\beta,\lambda\mu}(g) = (t_{ij}^{\alpha\beta,\lambda\mu}), \quad \alpha, \beta \in (\hat{H}, T), \tag{3}$$

$$1 \le \lambda \le p(\alpha), \quad 1 \le \mu \le p(\beta), \quad 1 \le i \le \dim Q_\alpha, \quad 1 \le j \le \dim Q_\beta.$$

If $h \in H$, then $T^{\alpha\beta,\lambda\mu}(h) = \delta_{\alpha\beta}\delta_{\lambda\mu}Q_\alpha(h)$.

The blocks $T^{\alpha\beta,\lambda\mu}(g)$ can be considered as a generalization of matrix elements of irreducible representations of the group G. The relation $T(g_1 g_2) = T(g_1)T(g_2)$

implies that

$$T^{\alpha\beta,\lambda\mu}(g_1g_2) = \sum_{\gamma\in(\hat{H},T)}\sum_{\nu=1}^{p(\gamma)} T^{\alpha\gamma,\lambda\nu}(g_1)T^{\gamma\beta,\nu\mu}(g_2). \tag{4}$$

If $h_1, h_2 \in H$, then

$$T^{\alpha\beta,\lambda\mu}(h_1gh_2) = Q_\alpha(h_1)T^{\alpha\beta,\lambda\mu}(g)Q_\beta(h_2). \tag{5}$$

We say that a complex matrix function $X(g)$ on G *belongs to a representation Q_α on the left (on the right)* if $X(hg) = Q_\alpha(h)X(g)$ ($X(gh) = X(g)Q_\alpha(h)$) for all h from H. Formula (5) shows that $T^{\alpha\beta,\lambda\mu}(g)$ belongs to the representation Q_α on the left and to the representation Q_β on the right. The columns $\mathbf{t}_j^{\alpha\beta,\lambda\mu}(g)$ of $T^{\alpha\beta,\lambda\mu}(g)$ with elements $\mathbf{t}_{ij}^{\alpha\beta,\lambda\mu}(g)$, $1 \le i \le \dim Q_\alpha$, belong to the representation Q_β on the left. The corresponding rows belong to the representation Q_β on the right.

We fix Q_α and Q_β and denote by $\mathfrak{N}^{\alpha\beta}$ the set of matrices consisting of $p(\alpha)p(\beta)$ blocks $F^{\alpha\beta,\lambda\mu}(g)$ with $\dim Q_\alpha$ rows and $\dim Q_\beta$ columns. Let $\mathfrak{R}^{\alpha\beta}$ be the subset of $\mathfrak{N}^{\alpha\beta}$ consisting of matrices F for which every block $F^{\alpha\beta,\lambda\mu}$ belongs to Q_α on the left and to Q_β on the right. For matrices $F_1(g) \in \mathfrak{N}^{\alpha\gamma}$, $F_2(g_\in\mathfrak{N}^{\gamma\beta}$ we define the *convolution*

$$F(g) = (F_1 * F_2)(g) = \int_G F_1(g_1)F_2(g_1^{-1}g)dg_1, \tag{6}$$

where the integral is taken elementwise. It is easy to show that if $F_1 \in \mathfrak{R}^{\alpha\gamma}$, $F_2 \in \mathfrak{R}^{\gamma\beta}$, then $F_1 * F_2 \in \mathfrak{R}^{\alpha\beta}$. In particular, if $F_1, F_2 \in \mathfrak{R}^{\alpha\alpha}$, then $F_1 * F_2 \in \mathfrak{R}^{\alpha\alpha}$. Therefore, $\mathfrak{R}^{\alpha\alpha}$ is a ring.

With every matrix $F^{\alpha\beta} \in \mathfrak{R}^{\alpha\beta}$ we associate the $p^2(\alpha) \times p^2(\beta)$ matrix Φ with elements

$$\Phi^{\lambda\mu,\rho\tau} = \operatorname{Tr} \int_G F^{\alpha\beta,\lambda\mu}(g)(T^{\alpha\beta,\rho\tau}(g))^* dg, \tag{7}$$

$$1 \le \lambda, \rho \le p(\alpha), \quad 1 \le \mu, \tau \le p(\beta),$$

where $F^{\alpha\beta,\lambda\mu}$ are blocks which constitute the matrix $F^{\alpha\beta}$. It is possible to show that if $F_1(g) \in \mathfrak{R}^{\alpha\gamma}$, $F_2(g) \in \mathfrak{R}^{\gamma\beta}$, $F = F_1 * F_2$, then

$$\Phi_1\Phi_2 = (\dim Q_\gamma)\Phi, \tag{8}$$

where Φ_1, Φ_2, Φ are the matrices corresponding to F_1, F_2, F respectively. If $p(\alpha) = p(\beta) = 1$ and $F \in \mathfrak{R}^{\alpha\alpha}$, then the corresponding matrix Φ contains only one element

$$\operatorname{Tr} \int F(g)(T^{\alpha\alpha}(g))^* dg.$$

In this case the mapping

$$F(g) \to (\dim Q_\alpha)\Phi = (\dim Q_\alpha)\mathrm{Tr} \int F(g)(T^{\alpha\alpha}(g))^* dg$$

is a homomorphism of the ring $\mathfrak{R}^{\alpha\alpha}$ onto $\mathbb{C}$.

Let T be an irreducible representation of the group G and let H be a compact subgroup of G. We set

$$A(g) = \int_H T(hgh^{-1})dh, \tag{9}$$

$$J = \int_H \int_H T(h_1 g_1 h_1^{-1} h_2^{-1} g_2 h_2) dh_1 dh_2, \tag{10}$$

and denote by $A^{\alpha\beta,\lambda\mu}(g)$ (by $J^{\alpha\beta,\lambda\mu}$) the blocks which constitute the matrix $A(g)$ (the matrix J). Let $\varphi_\alpha^{\lambda\mu}(g) = \mathrm{Tr}\ T^{\alpha\alpha,\lambda\mu}(g)$ and let $\Phi_\alpha(g)$ be the matrix with elements $\varphi_\alpha^{\lambda\mu}(g)$, $1 \le \lambda \le p(\alpha)$, $1 \le \mu \le p(\alpha)$. Then the relation

$$\Phi_\alpha(g_1)\Phi_\alpha(g_2) = (\dim Q_\alpha) \int_H \Phi_\alpha(g_1 h^{-1} g_2 h) dh \tag{11}$$

is valid. In particular, if $p(\alpha) = 1$, then

$$\varphi_\alpha(g_1)\varphi_\alpha(g_1) = (\dim Q_\alpha) \int_H \varphi_\alpha(g_1 h^{-1} g_2 h) dh, \tag{12}$$

where

$$\varphi_\alpha(g) = \mathrm{Tr}\ T^{\alpha\alpha}(g). \tag{13}$$

Let $T^{\alpha\beta}(g)$ be the block matrix consisting of the blocks $T^{\alpha\beta,\lambda\mu}(g)$, $1 \le \lambda \le p(\alpha)$, $1 \le \mu \le p(\beta)$. If $h \in H$ and $\alpha \ne \beta$ or $\lambda \ne \mu$, then $T^{\alpha\beta,\lambda\mu}(h)$ is the zero matrix. The matrix $T^{\alpha\alpha}(h)$ is block-diagonal with the blocks $T^{\alpha\alpha,\lambda\lambda}(h)$ on the main diagonal. For $g_1, g_2 \in G$ and $h \in H$ we have the relation

$$T^{\alpha\beta}(g_1 h g_2) = \sum_\gamma T^{\alpha\gamma}(g_1) T^{\gamma\gamma}(h) T^{\gamma\beta}(g_2). \tag{14}$$

It is an *analogue of the addition theorem.* Let us multiply both sides of (14) by $\mathrm{Tr}\ T^{\gamma\gamma}(h)$ and integrate over the subgroup H. Using the orthogonality relations for matrix elements we obtain the analogue of the product formula:

$$\int_H T^{\alpha\beta}(g_1 h g_2) \overline{\mathrm{Tr}\ T^{\gamma\gamma}(h)} dh = (\dim Q_\gamma)^{-1} T^{\alpha\gamma}(g_1) T^{\gamma\beta}(g_2). \tag{15}$$

16.2.7. The orthogonality relation for rows and columns. If the group G is compact, then the orthogonality relation for matrix elements of irreducible representations of G shows that the vector functions $\mathbf{t}_j^{\nu,\alpha\beta,\lambda\mu}(g)$ with the components $t_{ij}^{\nu,\alpha\beta,\lambda\mu}(g)$ (which are the matrix elements of the irreducible representation T_ν of G) are orthogonal with respect to the scalar product

$$(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{\dim Q_\alpha} \int_G x_i(g)\overline{y_i(g)}dg = \int_G \langle \mathbf{x}(g), \mathbf{y}(g)\rangle dg. \tag{1}$$

Here $\mathbf{x}(g)$, $\mathbf{y}(g)$ are vector functions on G with components $x_1(g),\dots,x_d(g)$ and $y_1(g),\dots,y_d(g)$ respectively, where $d = \dim Q_\alpha$. We have

$$\left\|\mathbf{t}_j^{\nu,\alpha\beta,\lambda\mu}\right\|^2 = \frac{\dim Q_\alpha}{\dim T_\nu}. \tag{2}$$

The similar statement for rows is valid.

Let $X(g)$ be a matrix function on G which belongs to Q_α on the left and to Q_β on the right. We introduce the scalar product in the space of such functions:

$$(X,Y) = \int_G \mathrm{Tr}\,(X(g)Y^*(g))dg. \tag{3}$$

The blocks $(T_\nu)^{\alpha\beta,\lambda\mu}(g) \equiv T^{\nu,\alpha\beta,\lambda\mu}(g)$ belong to this space. We have

$$\begin{aligned}(T^{\nu,\alpha\beta,\lambda\mu}, T^{\nu',\alpha\beta,\rho\tau}) &= (\dim Q_\alpha)(\dim Q_\beta)(\dim T_\nu)^{-1} \\ &\quad \times \delta_{\nu\nu'}\delta_{\lambda\rho}\delta_{\mu\tau}.\end{aligned} \tag{4}$$

The vector functions $\mathbf{t}_j^{\nu,\alpha\beta,\lambda\mu}(g)$ (the block functions $T^{\nu,\alpha\beta,\lambda\mu}(g)$) form orthogonal basis in the Hilbert space consisting of vector functions (block functions) on G. Namely, the following statement, generalizing the Peter-Weyl theorem, is valid.

Theorem 1. *Let $\mathfrak{L}_\alpha^2(G)$ be the Hilbert space of square integrable vector functions on the compact group G belonging on the left to the irreducible representation Q_α of the subgroup $H \subset G$. Then for $\mathbf{x}(g) \in \mathfrak{L}_\alpha^2(G)$ we have*

$$\begin{aligned}\mathbf{x}(g) = \sum_{\nu\in(\hat{G},Q_\alpha)} \sum_{\beta\in(\hat{H},T_\nu)} \sum_{\lambda=1}^{p(\alpha)}\sum_{\mu=1}^{p(\beta)}\sum_{j=1}^{\dim Q_\beta} a_j^{\nu,\alpha\beta,\lambda\mu} \\ \times \mathbf{t}_j^{\nu,\alpha\beta,\lambda\mu}(g),\end{aligned} \tag{5}$$

where

$$a_j^{\nu,\alpha\beta,\lambda\mu} = \frac{\dim T_\nu}{\dim Q_\alpha}(\mathbf{x}, \mathbf{t}_j^{\nu,\alpha\beta,\lambda\mu}). \tag{6}$$

The analogue of the Plancherel formula is valid:

$$\|\mathbf{x}\|^2 = \sum_{\nu\in(\hat{G},Q_\alpha)} \sum_{\beta\in(\hat{H},T_\nu)} \sum_{\lambda,\mu,j} \frac{\dim Q_\alpha}{\dim T_\nu} |a_j^{\nu,\alpha\beta,\lambda\mu}|^2. \tag{7}$$

A similar theorem for rows is also valid. For matrix functions we have the following theorem.

Theorem 2. *Let $\mathfrak{L}^2_{\alpha\beta}(G)$ be the Hilbert space of square integrable matrix functions $X(g)$ on the compact group G belonging to the irreducible representation Q_α on the left and to the irreducible representation Q_β on the right, where Q_α and Q_β belong to $\hat{H}$. Then the expansion*

$$X(g) = \sum_{\nu\in(\hat{G},Q_\alpha)\cap(\hat{G},Q_\beta)} \sum_{\lambda=1}^{p(\alpha)} \sum_{\mu=1}^{p(\beta)} a^{\nu,\alpha\beta,\lambda\mu} T^{\nu,\alpha\beta,\lambda\mu}(g) \tag{8}$$

is valid, where

$$a^{\nu,\alpha\beta,\lambda\mu} = \frac{\dim T_\nu}{(\dim Q_\alpha)(\dim Q_\beta)} (X, T^{\nu,\alpha\beta,\lambda\mu}). \tag{9}$$

Moreover, one has the analogue of the Plancherel formula:

$$\|X\|^2 = \sum_{\nu} \sum_{\lambda,\mu} \frac{(\dim Q_\alpha)(\dim Q_\beta)}{\dim T_\nu} |a^{\nu,\alpha\beta,\lambda\mu}|^2. \tag{10}$$

In order to prove formula (5) we write down the vector function $\mathbf{x}(g)$ in the coordinate form and represent the coordinates $x_1(g), \ldots, x_d(g)$, $d = \dim Q_\alpha$, according to formula (7) of Section 2.3.5. We obtain

$$\mathbf{x}(g_0) = \sum_{\nu\in\hat{G}} (\dim T_\nu) \int_G \chi_\nu(g_0 g^{-1}) \mathbf{x}(g) dg. \tag{11}$$

If $h \in H$ then $\mathbf{x}(g) = \mathbf{x}(hh^{-1}g) = Q_\alpha(h)\mathbf{x}(h^{-1}g)$. Substituting this expression for $\mathbf{x}(g)$ into (11) and integrating both parts over the subgroup H we obtain

$$\mathbf{x}(g_0) = \sum_{\nu\in\hat{G}} (\dim T_\nu) \int_G \int_H \chi_\nu(g_0 g^{-1} h^{-1}) Q_\alpha(h) \mathbf{x}(g) dh dg. \tag{12}$$

Since

$$\begin{aligned} I &\equiv \int_H \chi_\nu(g_0 g^{-1} h^{-1}) Q_\alpha(h) dh \\ &= \int_H \operatorname{Tr}\, [T_\nu(g_0 g^{-1}) T_\nu(h^{-1})] Q_\alpha(h) dh, \end{aligned}$$

then due to the orthogonality relation for matrix elements of irreducible representations of H we have

$$I = \sum_{\beta\in(\hat{H},T_\nu)} \sum_{\lambda=1}^{p(\beta)} \int_H \mathrm{Tr}\,[T^{\nu,\beta\beta,\lambda\lambda}(g_0 g^{-1}) T^{\nu,\beta\beta,\lambda\lambda}(h^{-1})]$$

$$\times\, Q_\alpha(h)dh = \frac{1}{\dim Q_\alpha} \sum_{\lambda=1}^{p(\alpha)} T^{\nu,\alpha\alpha,\lambda\lambda}(g_0 g^{-1}) \tag{13}$$

$$= \frac{1}{\dim Q_\alpha} \sum_{\lambda=1}^{p(\alpha)} \sum_{\gamma\in(\hat{H},T_\nu)} \sum_{\rho=1}^{p(\gamma)} T^{\nu,\alpha\gamma,\lambda\rho}(g_0) T^{\nu,\gamma\alpha,\rho\lambda}(g^{-1}).$$

Therefore,

$$\mathbf{x}(g_0) = \sum_{\nu\in(\hat{G},Q_\alpha)} \sum_{\lambda=1}^{p(\alpha)} \frac{\dim T_\nu}{\dim Q_\alpha}$$

$$\times \sum_{\gamma\in(\hat{H},T_\nu)} \sum_{\rho=1}^{p(\gamma)} T^{\nu,\alpha\gamma,\lambda\rho}(g_0) \int_G T^{\nu,\gamma\alpha,\rho\lambda}(g^{-1}) \mathbf{x}(g) dg.$$

Using the fact that the representation T_ν is unitary, we have

$$\mathbf{x}(g_0) = \sum_\nu \sum_\lambda \sum_\gamma \sum_\rho T^{\nu,\alpha\gamma,\lambda\rho}(g_0) \mathbf{a}^{\nu,\alpha\gamma,\lambda\rho},$$

where

$$\mathbf{a}^{\nu,\alpha\gamma,\lambda\rho} = \frac{\dim T_\nu}{\dim Q_\alpha} \int_G \overline{T^{\nu,\alpha\gamma,\lambda\rho}(g)} \mathbf{x}(g) dg.$$

Splitting the matrix $T^{\nu,\alpha\gamma,\lambda\rho}(g)$ into rows $\mathbf{t}_j^{\nu,\alpha\gamma,\lambda\rho}(g)$ we obtain formulas (5), (6) and then (7). Theorem 1 is proved. Theorem 2 is analogously proved.

Let us fix T_ν, Q_α and denote by $\mathcal{A}(\nu,\alpha)$ the space spanned by the vectors $\mathbf{t}_j^{\nu,\alpha\beta,\lambda\mu}(g)$. Since

$$\mathbf{t}_j^{\nu,\alpha\beta,\lambda\mu}(g)(gg_0) = \sum_{\gamma,\rho,i} \mathbf{t}_i^{\nu,\alpha\gamma,\lambda\rho}(g) t_{ij}^{\nu,\gamma\beta,\rho\mu}(g_0),$$

then right shifts define the representation of the group G in $\mathcal{A}(\nu,\alpha)$ having the matrices $(t_{ij}^{\alpha,\beta,\lambda\mu}(g_0))$. This representation appears in the decomposition of the representation T_{Q_α} of the group G, induced by the representation Q_α of H, with multiplicity $p(\alpha)$. The *Frobenius reciprocity principle* follows from here:

Let G be a compact group and let H be its subgroup. Multiplicity of an irreducible representation T of G in the representation T_Q of G, induced by an irreducible representation Q of H, is equal to multiplicity of Q in the representation $T\downarrow_H^G$.

Let us give the important particular case of this principle:

Multiplicity of an irreducible representation T of a compact group G in the quasi-regular representation of this group in the space of functions on the homogeneous space G/H is equal to multiplicity of the unit representation of H in $T\downarrow_H^G$.

It follows from here that *if H is a massive subgroup of a compact group G* (see Section 2.3.8), *then the quasi-regular representation of G in the space of functions on G/H contains irreducible representations of G not more than with unit multiplicity.*

16.2.8. Block matrix elements of irreducible representations of semisimple Lie groups. Let G be a real connected semisimple Lie group with finite center and let K, A, W, M be such as in Chapter 15. The blocks $T^{\alpha\beta,\lambda\mu}(g)$ of an irreducible representation T of the group G, constructed with respect to irreducible representations Q_α of K, are called *reduced block matrix elements.* Applying the Cartan decomposition to the element a_1ka_2 we obtain

$$a_1ka_2 = k_1ak_2, \quad k_1,k_2,k \in K, \quad a_1,a_2,a \in A.$$

Since matrices $T(k)$, $k \in K$, are block-diagonal, then we have the addition theorem

$$\begin{aligned} &Q_\alpha(k_1)T^{\alpha\beta,\lambda\mu}(a)Q_\beta(k_2) \\ &= \sum_{\gamma\in(\hat{H},T)}\sum_{\nu=1}^{p(\gamma)} T^{\alpha\gamma,\lambda\nu}(a_1)Q_\gamma(k)T^{\gamma\beta,\nu\mu}(a_2). \end{aligned} \tag{1}$$

If $p(\gamma)=1$, then this formula takes the simpler form:

$$Q_\alpha(k_1)T^{\alpha\beta,\lambda\mu}(a)Q_\beta(k_2) = \sum_\gamma T^{\alpha\gamma,\lambda\nu}(a_1)Q_\gamma(k)T^{\gamma\beta,\nu\mu}(a_2). \tag{1'}$$

If $a \in A$, $w \in W$, then $w^{-1}aw \in A$. Since $W \subset K$, then the blocks $T^{\alpha\beta,\gamma\mu}(a)$ satisfy the symmetry relation

$$T^{\alpha\beta,\lambda\mu}(w^{-1}aw) = Q_\alpha(w^{-1})T^{\alpha\beta,\lambda\mu}(a)Q_\beta(w). \tag{2}$$

Let us multiply both sides of (1) by $\overline{\mathrm{Tr}\ Q_\gamma(k)}$ and integrate over K. Using the Peter-Weyl theorem for the compact group K we obtain the product formula

$$\begin{aligned} &\int_K Q_\alpha(k_1)T^{\alpha\beta,\lambda\mu}(a)Q_\beta(k_2)\overline{\mathrm{Tr}\ Q_\gamma(k)}dk = \\ &= \sum_{\nu=1}^{p(\gamma)}(\dim Q_\gamma)^{-1}T^{\alpha\gamma,\lambda\nu}(a_1)T^{\gamma\beta,\nu\mu}(a_2), \end{aligned} \tag{3}$$

where $k_1 a k_2 = a_1 k a_2$.

From the decomposition $g = k_1 a k_2$, $a \in A$, $k_1, k_2 \in K$, we have the equality

$$T^{\alpha\beta,\lambda\mu}(k_1 a k_2) = Q_\alpha(k_1) T^{\alpha\beta,\lambda\mu}(a) Q_\beta(k_2). \tag{4}$$

More exact structure of the blocks $T^{\alpha\beta,\lambda\mu}(a)$ can be obtained by restriction of the representations Q_α and Q_β onto the subgroup M which is the centralizer of A in K. Let

$$\begin{aligned} Q_\alpha{\downarrow}^K_M &= \sum_{\gamma\in(\hat{M},Q_\alpha)} \sum_{\nu=1}^{q(\gamma)} \oplus\, R^\alpha_{\gamma\nu}, \\ Q_\beta{\downarrow}^K_M &= \sum_{\delta\in(\hat{M},Q_\beta)} \sum_{\rho=1}^{q(\delta)} \oplus\, R^\beta_{\delta\rho}. \end{aligned} \tag{5}$$

Then the blocks $T^{\alpha\beta,\lambda\mu}(g)$ split into subblocks of the form $T^{\alpha\beta,\lambda\mu}_{\gamma\delta,\nu\rho}(g)$. Since $am = ma$, then

$$T^{\alpha\beta,\lambda\mu}_{\gamma\delta,\nu\rho}(a) R_\delta(m) = R_\gamma(m) T^{\alpha\beta,\lambda\mu}_{\gamma\delta,\nu\rho}(a). \tag{6}$$

Due to the Schur lemma we have $T^{\alpha\beta,\lambda\mu}_{\gamma\delta,\nu\rho}(a) = 0$ if $\gamma \neq \delta$ and $T^{\alpha\beta,\lambda\mu}(a)$ consists of $q(\gamma)\dim R_\gamma \times q(\delta)\dim R_\delta$ blocks. These blocks are the zero matrices if $\gamma \neq \delta$ and consist of $q^2(\gamma)$ scalar $d(\gamma) \times d(\gamma)$ matrices if $\gamma = \delta$. This construction is simplified if multiplicities of irreducible representations do not exceed 1.

16.2.9. Integral expression for matrix elements of the principal series representations. Representations of the principal series of semisimple non-compact Lie group G with finite center are described in Section 16.2.2. They can be realized in the space $\mathfrak{D}_Q$ of smooth vector functions $\mathbf{f}$ on K with values in the carrier space $\mathfrak{L}$ of the representation Q of M such that

$$\mathbf{f}(mk) = Q(m)\mathbf{f}(k), \quad m \in M, \quad k \in K. \tag{1}$$

We introduce the scalar product

$$(\mathbf{f}_1, \mathbf{f}_2) = \int_K \langle \mathbf{f}_1(k), \mathbf{f}_2(k)\rangle dk \tag{2}$$

in $\mathfrak{D}_Q$, where $\langle \cdot\ , \cdot\rangle$ is the scalar product in $\mathfrak{L}$, and complete $\mathfrak{D}_Q$ to form the Hilber space $\mathfrak{L}^2_Q$. In accordance with Theorem 1 of Section 16.2.7 we choose an orthonormal basis in $\mathfrak{L}^2_Q$ which consists of columns

$$\left(\frac{\dim T_\nu}{\dim Q_\alpha}\right)^{1/2} \mathbf{q}^A_j(k), \quad A = (\nu, \alpha\beta, \lambda\mu), \quad \nu \in (\hat{K}, Q_\alpha), \quad \beta \in (\hat{M}, T_\nu), \tag{3}$$

$$1 \le j \le \dim Q_\beta, \quad 1 \le \lambda \le p(\alpha), \quad 1 \le \mu \le p(\beta).$$

We have

$$(T_{\lambda Q}(g)\mathbf{q}_j^A)(k) = \lambda(\tilde{a})\mathbf{q}_j^A(\tilde{k}) = \sum_{\beta,i} t_{ij}^{AB}(g)\mathbf{q}_i^B(k),$$

where $kg = n_-\tilde{a}\tilde{k}$, $n_- \in N_-$, $\tilde{a} \in A$, $\tilde{k} \in K$. Therefore, we receive for matrix elements of the representations $T_{\lambda Q}$ of G the expression

$$\begin{aligned} t_{ij}^{AB}(g) &= N_{ij}^{AB}(\lambda(\tilde{a})\mathbf{q}_j^A(\tilde{k}), \mathbf{q}_i^B(k)) \\ &= \sum_{m=1}^{\dim Q} N_{ij}^{AB} \int_K \lambda(\tilde{a})\mathbf{q}_{mj}^A(\tilde{k})\overline{\mathbf{q}_{mi}^B(k)}dk, \end{aligned} \tag{4}$$

where N_{ij}^{AB} are the corresponding normalization multipliers. Applying formula (4) for integral dominant points λ we obtain expressions for matrix elements of finite dimensional representations of G. By analytical continuation we can obtain from them matrix elements of finite dimensional representations of the complex group G_c which is the complexification of G, and then of the corresponding compact group G_k which is the compact real form of G_c. More detailed description of this procedure one can find in reference [24] of the first volume.

16.3. Hypergeometric Functions of Many Variables and Representations of the Group $GL(n, \mathbf{R})$

16.3.1. The Lauricella functions. We introduce the notations

$$\begin{aligned} &\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \quad \mathbf{m} = (m_1, \dots, m_n), \quad |\mathbf{m}| = m_1 + \dots + m_n, \\ &\mathbf{m}! = m_1! \dots m_n!, \quad \mathbf{x}^{\mathbf{m}} = x_1^{m_1} \dots x_n^{m_n}, \quad \Gamma(\boldsymbol{\alpha}) = \Gamma(\alpha_1)\dots\Gamma(\alpha_n), \\ &(\boldsymbol{\alpha})_{\mathbf{m}} = (\alpha_1)_{m_1} \dots (\alpha_n)_{m_n} = \frac{\Gamma(\alpha_1 + m_1)\dots\Gamma(\alpha_n + m_n)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)}, \\ &(\mathbf{x}, \mathbf{y}) = x_1y_1 + \dots + x_ny_n, \quad d\mathbf{x} = dx_1 \dots dx_n. \end{aligned} \tag{1}$$

Multiplying n hypergeometric series $F(\alpha_k, \beta_k, \gamma_k; x_k)$, $1 \le k \le n$, we obtain the series

$$\sum_{\mathbf{m}} \frac{(\boldsymbol{\alpha})_{\mathbf{m}}(\boldsymbol{\beta})_{\mathbf{m}}}{(\boldsymbol{\gamma})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}. \tag{2}$$

If one successively replaces here products $(\alpha_j)_{m_j} \dots (\alpha_{j+p-1})_{m_{j+p-1}}$ by $(\alpha)_m$, $m = m_j + \dots + m_{j+p-1}$, and makes the same operations with $(\beta_j)_{m_j}$ and $(\gamma_j)_{m_j}$, then new classes of hypergeometric functions are obtained.

G. Lauricella considered four types of such series:

$$F_A(\alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}; \mathbf{x}) = \sum_{\mathbf{m}} \frac{(\alpha)_{|\mathbf{m}|}(\boldsymbol{\beta})_{\mathbf{m}}}{(\boldsymbol{\gamma})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}, \tag{3}$$

$$F_B(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma; \mathbf{x}) = \sum_{\mathbf{m}} \frac{(\boldsymbol{\alpha})_{\mathbf{m}}(\boldsymbol{\beta})_{\mathbf{m}}}{(\gamma)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}, \tag{4}$$

$$F_C(\alpha, \beta, \boldsymbol{\gamma}; \mathbf{x}) = \sum_{\mathbf{m}} \frac{(\alpha)_{|\mathbf{m}|}(\beta)_{|\mathbf{m}|}}{(\boldsymbol{\gamma})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}, \tag{5}$$

$$F_D(\alpha, \boldsymbol{\beta}, \gamma; \mathbf{x}) = \sum_{\mathbf{m}} \frac{(\alpha)_{|\mathbf{m}|}(\boldsymbol{\beta})_{\mathbf{m}}}{(\gamma)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}. \tag{6}$$

It is possible to obtain many other similar series using the method of branching which can be graphically represented with the help of trees. We do not consider these series here.

The convergence domains of the functions F_A, F_B, F_C, F_D are

for F_A: $|x_1| + \ldots + |x_n| < 1$,

for F_B and F_D: $|x_1| < 1, \ldots, |x_n| < 1$,

for F_C: $\sqrt{|x_1|} + \ldots + \sqrt{|x_n|} < 1$.

Sums of these series can be expressed by the integrals:

$$F_A(\alpha, \boldsymbol{\beta}, \boldsymbol{\gamma}; \mathbf{x}) = \frac{\Gamma(\boldsymbol{\gamma})}{\Gamma(\boldsymbol{\beta})\Gamma(\boldsymbol{\gamma} - \boldsymbol{\beta})} \times \int_0^1 \cdots \int_0^1 \mathbf{u}^{\boldsymbol{\beta}-\mathbf{1}}(\mathbf{1} - \mathbf{u})^{\boldsymbol{\gamma}-\boldsymbol{\beta}-\mathbf{1}}(1 - (\mathbf{u}, \mathbf{x}))^{-\alpha} d\mathbf{u}, \tag{7}$$

$$F_B(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma; \mathbf{x}) = \frac{\Gamma(\gamma)}{\Gamma(\boldsymbol{\alpha})\Gamma(\gamma - |\boldsymbol{\alpha}|)} \times \int_0^1 \cdots \int_0^1 \mathbf{u}^{\boldsymbol{\alpha}-\mathbf{1}}(1 - |\mathbf{u}|)^{\gamma-|\boldsymbol{\alpha}|-1}(\mathbf{1} - \mathbf{u}\mathbf{x})^{-\boldsymbol{\beta}} d\mathbf{u}, \tag{8}$$

$$F_D(\alpha, \boldsymbol{\beta}, \gamma; \mathbf{x}) = \frac{\Gamma(\gamma)}{\Gamma(\boldsymbol{\beta})\Gamma(\gamma - |\boldsymbol{\beta}|)} \times \int_0^1 \cdots \int_0^1 \mathbf{u}^{\boldsymbol{\beta}-\mathbf{1}}(1 - |\mathbf{u}|)^{\gamma-|\boldsymbol{\beta}|-1}(1 - (\mathbf{u}, \mathbf{x}))^{-\alpha} d\mathbf{u}, \tag{9}$$

where $(1 - \mathbf{u}\mathbf{x})^{-\boldsymbol{\beta}} = (1 - u_1 x_1)^{-\beta_1} \ldots (1 - u_n x_n)^{-\beta_n}$. In the formulas for F_B and F_D we have $\mathbf{u} \geq 0$, $1 - |\mathbf{u}| \geq 0$.

The function F_D can be given by a usual integral:

$$F_D(\alpha, \boldsymbol{\beta}, \gamma; \mathbf{x}) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(\mathbf{1}-u\mathbf{x})^{-\boldsymbol{\beta}} du. \tag{10}$$

From the integral representations one can obtain different transformation formulas for the Lauricella functions. For example,

$$\begin{aligned} F_A(\alpha, \boldsymbol{\beta}; \boldsymbol{\gamma}; \mathbf{x}) &= (1-x_1)^{-\alpha} F_A(\alpha, \hat{\boldsymbol{\beta}}; \boldsymbol{\gamma}; \hat{\mathbf{x}}) \\ &= (1-|\mathbf{x}|)^{-\alpha} F_A\left(\alpha, \boldsymbol{\gamma}-\boldsymbol{\beta}; \boldsymbol{\gamma}; \frac{\mathbf{x}}{|\mathbf{x}|-1}\right), \end{aligned} \tag{11}$$

where $\hat{\boldsymbol{\beta}} = (\gamma_1 - \beta_1, \beta_2, \dots, \beta_n)$, $\hat{\mathbf{x}} = \left(\frac{x_1}{x_1-1}, \frac{x_2}{1-x_1}, \dots, \frac{x_n}{1-x_1}\right)$,

$$\begin{aligned} F_D(\alpha, \boldsymbol{\beta}; \gamma; \mathbf{x}) &= (\mathbf{1}-\mathbf{x})^{-\boldsymbol{\beta}} F_D\left(\gamma-\alpha, \boldsymbol{\beta}; \gamma; \frac{\mathbf{x}}{|\mathbf{x}|-\mathbf{1}}\right) \\ &= (1-x_1)^{-\alpha} F_D(\alpha, \check{\boldsymbol{\beta}}; \gamma; \check{\mathbf{x}}) \\ &= (1-x_1)^{\gamma-\alpha}(\mathbf{1}-\mathbf{x})^{-\boldsymbol{\beta}} F_D(\gamma-\alpha, \tilde{\boldsymbol{\beta}}; \gamma; \tilde{\mathbf{x}}), \end{aligned} \tag{12}$$

where $\check{\boldsymbol{\beta}} = (\gamma - |\boldsymbol{\beta}|, \beta_2, \dots, \beta_n)$, $\check{\mathbf{x}} = (x_1-1)^{-1}(x_1, x_1-x_2, \dots, x_1-x_n)$,

$$\tilde{\boldsymbol{\beta}} = (\gamma - |\boldsymbol{\beta}|, \beta_2, \dots, \beta_n), \quad \tilde{\mathbf{x}} = \left(x_1, \frac{x_2-x_1}{x_2-1}, \dots, \frac{x_n-x_1}{x_n-1}\right).$$

The reduction formulas for the Lauricella functions follows from the integral representations. For example,

$$F_D(\alpha, \boldsymbol{\beta}; \gamma; x \cdot \mathbf{1}) = F(\alpha, |\boldsymbol{\beta}|; \gamma; x). \tag{13}$$

In particular,

$$F_D(\alpha, \boldsymbol{\beta}; \gamma; \mathbf{1}) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-|\boldsymbol{\beta}|)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-|\boldsymbol{\beta}|)}. \tag{13'}$$

Each of four Lauricella functions satisfies the system of linear partial differential equations. For F_A this system is of the form

$$\begin{aligned} & x_j(1-x_j)\frac{\partial^2 F}{\partial x_j^2} - x_j \sum_{k\neq j} x_k \frac{\partial^2 F}{\partial x_k \partial x_j} - \beta_j \sum_{k \neq j} x_k \frac{\partial F}{\partial x_k} \\ & + [\gamma_j - (\alpha+\beta_j+1)x_j]\frac{\partial F}{\partial x_j} - \alpha\beta_j F = 0, \quad 1 \le j \le n. \end{aligned} \tag{14}$$

For F_B we have the system

$$x_j(1-x_j)\frac{\partial^2 F}{\partial x_j^2}+\sum_{k\neq j} x_k \frac{\partial^2 F}{\partial x_k \partial x_j}+[\gamma-(\alpha_j+\beta_j+1)x_j]\frac{\partial F}{\partial x_j} \\ -\alpha_j\beta_j F=0, \quad 1\leq j\leq n. \tag{15}$$

For F_C this system is of the form

$$x_j(1-x_j)\frac{\partial^2 F}{\partial x_j^2}-\sum_{r\neq j} x_r \sum_{s\neq j} x_j \frac{\partial^2 F}{\partial x_r \partial x_s}+[\gamma_j-(\alpha+\beta+1)x_j]\frac{\partial F}{\partial x_j} \\ -(\alpha+\beta+1)\sum_{k\neq j} x_k \frac{\partial F}{\partial x_k}-\alpha\beta F=0, \quad 1\leq j\leq n. \tag{16}$$

For F_D we have the system

$$x_j(1-x_j)\frac{\partial^2 F}{\partial x_j^2}+(1-x_j)\sum_{k\neq j} \frac{\partial^2 F}{\partial x_k \partial x_j}+[\gamma-(\alpha+\beta_j+1)x_j]\frac{\partial F}{\partial x_j} \\ -\beta_j \sum_{k\neq j} x_k \frac{\partial F}{\partial x_k}-\alpha\beta_j F=0, \quad 1\leq j\leq n. \tag{17}$$

The fundamental system of solutions is constructed for every of these systems of equations. All solutions of these systems are linear combinations of fundamental solutions. These solutions are expressed in terms of the Lauricella functions. For example, the following 2^n functions are solutions of the system (14):

$$\begin{aligned}
&\{F_A(\alpha,\boldsymbol{\beta};\boldsymbol{\gamma};\mathbf{x}),\\
&\begin{cases} x_k^{1-\gamma_k}F_A(\alpha-\gamma_k+1,\boldsymbol{\beta}^{(k)};\boldsymbol{\gamma}^{(k)};\mathbf{x}), \quad 1\leq k\leq n,\\ \text{where} \quad \boldsymbol{\beta}^{(k)}=(\beta_1,\ldots,\beta_k-\gamma_k+1,\ldots,\beta_n),\\ \qquad\quad \boldsymbol{\gamma}^{(k)}=(\gamma_1,\ldots,2-\gamma_k,\ldots,\gamma_n),\end{cases}\\
&\begin{cases} x_j^{1-\gamma_j}x_k^{1-\gamma_k}F_A(\alpha+2-\gamma_j-\gamma_k,\boldsymbol{\beta}^{(j,k)};\boldsymbol{\gamma}^{(j,k)};\mathbf{x}), \quad \leq j<k\leq n,\\ \text{where} \quad \boldsymbol{\beta}^{(j,k)}=(\beta_1,\ldots,\beta_j-\gamma_j+1,\beta_{j+1},\ldots,\beta_k-\gamma_k+1,\ldots,\beta_n),\\ \qquad\quad \boldsymbol{\gamma}^{(j,k)}=(\gamma_1,\ldots,2-\gamma_j,\gamma_{j+1},\ldots,2-\gamma_k,\ldots,\gamma_n),\end{cases}
\end{aligned} \tag{18}$$

. . . .

$$\{x^{1-\gamma}F_A(\alpha-|\gamma|+n,\boldsymbol{\beta}-\boldsymbol{\gamma}+\mathbf{1};\mathbf{2}-\boldsymbol{\gamma};\mathbf{x}).$$

The system of equations (15) is reduced to the system (14) by the transformation

$$\mathbf{x}=\frac{1}{\boldsymbol{\xi}}, \quad F=\boldsymbol{\xi}^\alpha F', \qquad \text{where} \quad \frac{1}{\boldsymbol{\xi}}=\left(\frac{1}{\xi_1},\ldots,\frac{1}{\xi_n}\right).$$

This fact allows us to write down 2^n partial solutions of the system of equations (15) for the function F_B. The following functions are solutions of equations (16):

$$\begin{aligned}
&\{F_C(\alpha,\beta;\boldsymbol{\gamma};\mathbf{x}),\\
&\begin{cases} x_j^{1-\gamma_j} F_C(\alpha-\gamma_j+1,\beta-\gamma_j+1;\boldsymbol{\gamma}^{(j)};\mathbf{x}), \quad 1\le j\le n,\\ \text{where}\quad \boldsymbol{\gamma}^{(j)}=(\gamma_1,\dots,2-\gamma_j,\dots,\gamma_n),\end{cases}\\
&\begin{cases} x_j^{1-\gamma_j}x_k^{1-\gamma_k} F_C(\alpha+2-\gamma_j-\gamma_k,\beta+2-\gamma_j-\gamma_k;\boldsymbol{\gamma}^{(j,k)};\mathbf{x}),\\ \qquad\qquad 1\le j<k\le n,\\ \text{where}\quad \boldsymbol{\gamma}^{(j,k)}=(\gamma_1,\dots,2-\gamma_j,\gamma_{j+1},\dots,2-\gamma_k,\dots,\gamma_n),\end{cases}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\{\mathbf{x}^{1-\boldsymbol{\gamma}}F_C(\alpha+n-|\boldsymbol{\gamma}|,\beta+n-|\boldsymbol{\gamma}|;\mathbf{2}-\boldsymbol{\gamma};\mathbf{x})\,.
\end{aligned} \tag{19}$$

We do not write down the fundamental solutions for the system of equations (17).

16.3.2. The most degenerate series representations of the group $SL(n,\mathbf{R})$. These representations are given by two numbers $\chi=(a,\varepsilon)$, where $a\in\mathbb{C}$ and $\varepsilon\in\{0,1\}$. They are constructed in the spaces $\mathfrak{D}_\chi$ of homogeneous infinitely differentiable functions f on $\mathbf{R}^n$ with homogeneity degree χ:

$$f(t\mathbf{x})=|t|^a(\text{sign }t)^\varepsilon f(\mathbf{x}),\quad \mathbf{x}\in\mathbf{R}^n,\quad t\in\mathbf{R}.$$

The representations T_χ are given by the formula

$$(T_\chi(g)f)(\mathbf{x})=f(\mathbf{x}g)=f(\tilde{\mathbf{x}}), \tag{1}$$

where $\mathbf{x}=(x_1,\dots,x_n)$ is a vector-row.

Homogeneous functions are given by their values on any hypersurface in $\mathbf{R}^n$, intersecting one time every ray going from origin of coordinates. Realizing the representation T_χ in the space of functions on the hyperplane $x_n=1$, we obtain

$$(T_\chi(g)\varphi)(\boldsymbol{\xi})=|\tilde{\mathbf{x}}_n|^a(\text{sign }\tilde{x}_n)^\varepsilon\varphi(\tilde{\boldsymbol{\xi}}/\tilde{x}_n), \tag{1'}$$

where $\boldsymbol{\xi}=(x_1,\dots,x_{n-1})$, $\tilde{\boldsymbol{\xi}}=(\tilde{x}_1,\dots,\tilde{x}_{n-1})$. The functions $f=(x_1,\dots,x_{n-1},1)$ $=\varphi(\boldsymbol{\xi})$ are infinitely differentiable in $x_1,\dots,x_{n-1}$ and at every fixed $x_1,\dots,x_{n-1}$ the functions $\Phi(t)=|t|^a(\text{sign }t)^\varepsilon\varphi(\boldsymbol{\xi}/t)$ are infinity differentiable in t.

The other realization of representations T_χ is obtain with the help of the Mellin transform. Let A be a sequence, consisting of $n-1$ signs $+$ and $-$, that is $A = (\alpha_1, \dots, \alpha_{n-1})$, $\alpha_k \in \{+, -\}$. With every function $\varphi(\boldsymbol{\xi})$ we associate the vector function $F(\lambda)$ with 2^{n-1} components

$$F_A(\boldsymbol{\lambda}) = \int\limits_{\mathbb{R}^{n-1}} \boldsymbol{\xi}_A^{\boldsymbol{\lambda}-1} \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi}, \qquad (2)$$

where $\boldsymbol{\xi}_A^{\boldsymbol{\mu}} = (x_1^{\mu_1})_{\alpha_1} \dots (x_{n-1}^{\mu_{n-1}})_{\alpha_{n-1}}$. These integrals are convergent for $0 < \operatorname{Re} \lambda_k$, $1 \le k \le n-1$, $\operatorname{Re} |\boldsymbol{\lambda}| < -\operatorname{Re} a$. The inverse Mellin transform shows that

$$\varphi(\boldsymbol{\xi}) = \frac{1}{(2\pi i)^{n-1}} \int\limits_{\mathbf{b}-i\infty}^{\mathbf{b}+i\infty} (\mathbf{F}(\boldsymbol{\mu}), \boldsymbol{\xi}^{-\boldsymbol{\mu}}) d\boldsymbol{\mu}, \qquad (3)$$

where $0 < \operatorname{Re} \mu_k$, $1 \le k \le n-1$, $\operatorname{Re} |\boldsymbol{\mu}| < -\operatorname{Re} a$, and

$$(\mathbf{F}(\boldsymbol{\mu}), \boldsymbol{\xi}^{-\boldsymbol{\mu}}) = \sum_A F_A(\boldsymbol{\mu}) \boldsymbol{\xi}_A^{-\boldsymbol{\mu}}.$$

It follows from formulas (1)-(3) that

$$\mathbf{F}^{(g)}(\boldsymbol{\lambda}) = \int\limits_{\mathbf{b}-i\infty}^{\mathbf{b}+i\infty} (\mathbf{K}(\boldsymbol{\lambda}, \boldsymbol{\mu}; \chi, g) \mathbf{F}(\boldsymbol{\mu}) d\boldsymbol{\mu}, \qquad (4)$$

where $\mathbf{F}^{(g)}(\boldsymbol{\lambda})$ is the vector function corresponding to the function $(T_\chi(g)\varphi)(\boldsymbol{\xi})$, and $\mathbf{K}(\boldsymbol{\lambda}, \boldsymbol{\mu}; \chi, g)$ is the matrix with the elements

$$K_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; \chi, g) = K_{AB}^+(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g) + (-1)^\varepsilon K_{AB}^-(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g). \qquad (5)$$

Here

$$K_{AB}^\gamma(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g) = \frac{1}{(2\pi i)^{n-1}} \int\limits_{\mathbb{R}^{n-1}} \boldsymbol{\xi}_A^{\boldsymbol{\lambda}-1} \tilde{\boldsymbol{\xi}}_{\gamma B}^{-\boldsymbol{\mu}} (\tilde{x}_n)_\gamma^{a+\mu} d\boldsymbol{\xi}, \quad \mu = |\boldsymbol{\mu}|. \qquad (6)$$

16.3.3. Generalized beta-functions and the kernels K_{AB}^λ. Almost every matrix from the group $SL(n, \mathbf{R})$ is a product of diagonal matrices of the form $d = \operatorname{diag}(d_1, \dots, d_n)$ and of matrices of the forms

$$u_k = \begin{pmatrix} I_{k-1} & \mathbf{0} & 0 \\ \boldsymbol{\ell}^t & 1 & \mathbf{0}^t \\ 0 & \mathbf{0} & I_{n-k} \end{pmatrix}, \quad v_k = \begin{pmatrix} I_{k-1} & \boldsymbol{\ell} & 0 \\ \mathbf{0}^t & 1 & \mathbf{0}^t \\ 0 & \mathbf{0} & I_{n-k} \end{pmatrix}, \qquad (1)$$

where I_p is the unit matrix and $\boldsymbol{\ell} = (\ell_1, \dots, \ell_{k-1} \in \mathbf{R}^{k-1}$. Therefore, it is sufficient to evaluate the kernels $K_{AB}^\gamma(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g)$ for the matrices u_k, v_k, d. It is easy to verify that for $d = \operatorname{diag}(d_1, \dots, d_n)$, $d_1 d_2 \dots d_n = 1$, $d_k \in \mathbf{R}$, we have

$$K_{AB}^\gamma(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g) = |d_1|^{-\mu_1 - a} \dots |d_{n-1}|^{-\mu_{n-1} - a} \delta_{CD}(\boldsymbol{\lambda}, \boldsymbol{\mu}), \qquad (2)$$

where $C = (\alpha_1, \dots, \alpha_{n-1}, \gamma)$, $D = (\gamma_1\beta_1, \dots, \gamma_{n-1}\beta_{n-1}, \gamma_n)$, $\gamma_j = \text{sign}\ d_j$, and $\delta_{CD}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is defined by the formula

$$\int \delta_{CD}(\boldsymbol{\lambda}, \boldsymbol{\mu})\psi(\boldsymbol{\mu})d\boldsymbol{\mu} = \begin{cases} \psi(\boldsymbol{\lambda}) & \text{if } C = D, \\ 0 & \text{if } C \neq D. \end{cases}$$

For given A, B let Z_1, Z_2, Z_3 and Z_4 be the sets of indices s ($s = 1, 2, \dots, n$) for which $\alpha_s = \beta_s = +$ (for Z_1), $\alpha_s = +, \beta_s = -$ (for Z_2), $\alpha_s = \beta_s = -$ (for Z_3), $\alpha_s = -, \beta_s = +$ (for Z_4). We introduce the generalized beta-function $\mathrm{B}_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ by setting $\mathrm{B}_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$ if Z_4 is not empty set and

$$\begin{aligned} \mathrm{B}_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= \prod_{s=1}^{n-1} [\pi^{-1}\Gamma(\lambda_s)\Gamma(\mu_s - \lambda_s)\Gamma(1 - \mu_s) \\ &\times \prod_{s \in Z_1} \sin \pi\mu_s \prod_{s \in Z_2} \sin \pi(\mu_s - \lambda_s) \prod_{s \in Z_3} \sin \pi\lambda_s \end{aligned} \tag{3}$$

if Z_4 is empty. Then

$$K^{+}_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, u_n) = \frac{1}{(2\pi i)^{n-1}} \mathrm{B}_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}), \tag{4}$$

where $0 < \text{Re}\ \lambda_s$ for $s \in Z_1 \cup Z_2$, $\text{Re}\ \lambda_s < \text{Re}\ \mu_s$ for $s \in Z_1 \cup Z_3$, $\text{Re}\ \mu_s < 1$ for $s \in Z_2 \cup Z_3$. Let us note that

$$K^{-}_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, u_n) = 0, \tag{5}$$

and

$$K^{-\gamma}_{BA}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, v_n) = 0 \tag{6}$$

if $B \neq \gamma A$. We also have

$$\begin{aligned} K^{+}_{AA}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, v_n) &= \frac{1}{(2\pi i)^{n-1}} \Gamma(1 - a + |\boldsymbol{\mu}|)\Gamma(-a - |\boldsymbol{\lambda}|) \\ &\times \Gamma^{-1}(1 + a + \lambda' + \mu'')\Gamma(-a - \lambda' - \mu'') \prod_{s=1}^{n-1} \Gamma(\lambda_s - \mu_s), \end{aligned} \tag{7}$$

where $\lambda' = \sum_{\alpha_s = -} \lambda_s$, $\mu'' = \sum_{\alpha_s = +} \mu_s$, $\text{Re}\ \mu_k < \text{Re}\ \lambda_k$ for $1 \leq k \leq n - 1$ and $-1 + \text{Re}\ |\boldsymbol{\lambda}| < -1 - \text{Re}\ a < \text{Re}\ |\boldsymbol{\mu}|$. We derive that

$$K^{-}_{A,-A}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, v_n) = 0 \tag{8}$$

if $\alpha_s = +$ for all s and

$$K^-_{A,-A}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, v_n) = \frac{1}{(2\pi i)^{n-1}} \Gamma(1 + a + |\boldsymbol{\mu}|)\Gamma(-a - |\boldsymbol{\lambda}|) \times \Gamma^{-1}(1 + \mu' - \lambda')\Gamma^{-1}(\lambda' - \mu') \prod_{s=1}^{n-1} \Gamma(\lambda_s - \mu_s) \tag{9}$$

in other case. Here $\mu' = \sum_{\alpha_s = -} \mu_s$. Let $\mathbf{t} = (t_1, \dots, t_{n-1})$. If all t_k are non-vanishing and $w_n \equiv w_n(\mathbf{t}) = \begin{pmatrix} I_{n-1} & -\mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix}$, then

$$K^\gamma_{A,\gamma B}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, w_n) = \prod_{k=1}^{n-1} |t_k|^{\mu_k - \lambda_k} \cdot K^\gamma_{A',\gamma A'}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, v_n), \tag{10}$$

where $\alpha'_s = -\alpha_s \operatorname{sign} t_s$. If $B \neq \gamma A$, then

$$K^\gamma_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, w_n) = 0. \tag{11}$$

16.3.4. The Lauricella functions and the kernels K^γ_{AB}.

Let

$$g_n(\mathbf{t}) = \begin{pmatrix} I_{n-1} & -\mathbf{t} \\ \mathbf{1} & 1 - |\mathbf{t}| \end{pmatrix}, \tag{1}$$

where $\mathbf{1} = (1, \dots, 1)$, $\mathbf{t} = (t_1, \dots, t_{n-1})$. We obtain from formula (6) of Section 16.3.2 that

$$K^\gamma_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t})) = \frac{1}{(2\pi i)^{n-1}} \int_{\mathbb{R}^{n-1}} (1 - (\mathbf{t}, \mathbf{x}))_\gamma^{a+|\boldsymbol{\mu}|} \times (\mathbf{x} - \mathbf{1})_{\boldsymbol{\alpha}}^{\boldsymbol{\lambda}-1} \mathbf{x}_{\boldsymbol{\beta}}^{-\boldsymbol{\mu}} d\mathbf{x}. \tag{2}$$

In some cases this integral can be expressed in terms of the Lauricella functions F_A and F_D. If $\alpha_s = -$, $\beta_s = +$ for all s and $|t_1| + \dots + |t_{n-1}| < 1$, then

$$K^+_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t})) = \frac{1}{(2\pi i)^{n-1}} \prod_{s=1}^{n-1} \mathrm{B}(\lambda_s, 1 - \mu_s) \times F_A(-a - |\boldsymbol{\mu}|, \mathbf{1} - \boldsymbol{\mu}; \mathbf{1} + \boldsymbol{\lambda} - \boldsymbol{\mu}; \mathbf{t}). \tag{3}$$

If $\alpha_s = -$, $\beta_s = +$, $t_s > 0$ for all s and $|t_1| + \dots + |t_{n-1}| < 1$, then

$$K^-_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t})) = \frac{1}{(2\pi i)^{n-1}} \frac{\Gamma(1 + a + |\boldsymbol{\mu}|) \sin \pi(|\boldsymbol{\mu}| - |\boldsymbol{\lambda}|)}{\Gamma(1 + a + |\boldsymbol{\lambda}|) \sin \pi(a + |\boldsymbol{\lambda}|)} \times \Gamma(\boldsymbol{\lambda} - \boldsymbol{\mu}) \mathbf{t}^{\boldsymbol{\mu} - \boldsymbol{\lambda}} F_A(-a - |\boldsymbol{\lambda}|, \mathbf{1} - \boldsymbol{\lambda}; \mathbf{1} - \boldsymbol{\lambda} - \boldsymbol{\mu}; \mathbf{t}), \tag{4}$$

$$K^{-}_{A,-B}(\boldsymbol{\lambda},\boldsymbol{\mu};a,g_n(\mathbf{t}))=0. \tag{5}$$

If $\alpha_s=-$, $\beta_s=+$ and $t_s>1$, then

$$\begin{gathered}K^{+}_{AB}(\boldsymbol{\lambda},\boldsymbol{\mu};a,g_n(\mathbf{t}))=\frac{1}{(2\pi i)^{n-1}}\frac{\Gamma(1+a+|\boldsymbol{\mu}|)\Gamma(\mathbf{1}-\boldsymbol{\mu})}{\Gamma(a+n)}\\ \times\mathbf{t}^{\boldsymbol{\mu}-\mathbf{1}}F_B(\mathbf{1}-\boldsymbol{\lambda},\mathbf{1}-\boldsymbol{\mu};a+n;1/\mathbf{t}),\end{gathered} \tag{6}$$

Using in (2) the substitutions $x_k=1-y_k$, $1\le k\le n-1$, we obtain

$$\begin{aligned}K^{\gamma}_{AB}(\boldsymbol{\lambda},\boldsymbol{\mu};a,g_n(\mathbf{t}))&=(1-|\mathbf{t}|)_+^{a+|\boldsymbol{\mu}|}\\ &\times\{K^{\gamma}_{-\gamma B,-\gamma A}(\mathbf{1}-\boldsymbol{\mu},\mathbf{1}-\boldsymbol{\lambda};a',g_n(\mathbf{t}'))\\ &+K^{-\gamma}_{-\gamma B,\gamma A}(\mathbf{1}-\boldsymbol{\mu},\mathbf{1}-\boldsymbol{\lambda};a',g_n(\mathbf{t}'))\},\end{aligned} \tag{7}$$

where $a'=a+|\boldsymbol{\mu}|-|\boldsymbol{\lambda}|+n-1$, $\mathbf{t}'=\mathbf{t}/(|\mathbf{t}|-1)$.

Integral (2) is an analogue of the Gauss integral representation for the function ${}_2F_1$. In order to construct an analogue of the Barnes integral representation we note that $g_n(\mathbf{t})=u_nv_n$, where u_n and v_n are matrices (1) of Section 16.3.3 with $k=n$, $\boldsymbol{\ell}=\mathbf{t}$. Since $T_\chi(g_n(\mathbf{t}))=T_\chi(u_n)T_\chi(v_n)$, then

$$\mathbf{K}(\boldsymbol{\lambda},\boldsymbol{\mu};\chi,g_n(\mathbf{t}))=\int_{\mathbf{b}-i\infty}^{\mathbf{b}+i\infty}\mathbf{K}(\boldsymbol{\lambda},\boldsymbol{\nu};\chi,u_n)\mathbf{K}(\boldsymbol{\nu},\boldsymbol{\mu};\chi,v_n)d\boldsymbol{\nu}.$$

Comparing the corresponding elements of matrices on the left and on the right and taking into account formulas (3)-(6), we obtain that

$$\begin{gathered}K^{\gamma}_{AB}(\boldsymbol{\lambda},\boldsymbol{\mu};a,g_n(\mathbf{t}))\\ =\int_{\mathbf{b}-i\infty}^{\mathbf{b}+i\infty}K^{+}_{A,\gamma B}(\boldsymbol{\lambda},\boldsymbol{\nu};a,u_n)K^{\gamma}_{\gamma B,B}(\boldsymbol{\nu},\boldsymbol{\mu};a,v_n)d\boldsymbol{\nu},\end{gathered} \tag{8}$$

where $0<\operatorname{Re}\lambda_s$ for $s\in Z_1\cup Z_2$; $\operatorname{Re}\lambda_s<\operatorname{Re}\nu_s$ for $s\in Z_1\cup Z_3$; $\operatorname{Re}\nu_s<1$ for $s\in Z_2\cup Z_3$; $\operatorname{Re}\mu_s<\operatorname{Re}\nu_s$, $1\le s<n-1$; $-1+\operatorname{Re}|\boldsymbol{\nu}|<-1-\operatorname{Re}a<\operatorname{Re}|\boldsymbol{\mu}|$. It follows from here that

$$\begin{gathered}K^{+}_{AB}(\boldsymbol{\lambda},\boldsymbol{\mu};a,g_n(\mathbf{t}))=\frac{(-1)^n2\Gamma(1+a+|\boldsymbol{\mu}|)}{(2\pi)^{2n-1}}\int_{\mathbf{b}-i\infty}^{\mathbf{b}+i\infty}\prod_{s=1}^{n-1}|t_s|^{\mu_s-\nu_s}\\ \times\mathrm{B}_{AB}(\boldsymbol{\lambda},\boldsymbol{\nu})\Gamma(-a-|\boldsymbol{\nu}|)\Gamma(-\boldsymbol{\nu}-\boldsymbol{\mu})\sin\pi(a+\nu'+\mu'')d\boldsymbol{\nu},\end{gathered} \tag{9}$$

where $\nu' = \sum_{s\in I_+} \nu_s$, $\mu'' = \sum_{s\in I_-} \mu_s$, where $I_\pm = \{s \mid \beta_s t_s \gtrless 0\}$. We also have

$$K^-_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t})) = \frac{(-1)^n 2\Gamma(1+a+|\boldsymbol{\mu}|)}{(2\pi)^{2n-1}} \int_{b-i\infty}^{b+i\infty} \prod_{s=1}^{n-1} |t_s|^{\mu_s-\nu_s} \tag{10}$$
$$\times \mathrm{B}_{A,-B}(\boldsymbol{\lambda}, -\boldsymbol{\nu})\Gamma(-a-|\boldsymbol{\nu}|)\Gamma(\boldsymbol{\nu}-\boldsymbol{\mu}) \sin\pi(\mu''-\nu'')d\boldsymbol{\nu},$$

where $\nu'' = \sum_{s\in I_-} \nu_s$. By evaluating the integrals (9) and (10) with the help of the residue theorem, we obtain formulas (3) and (4). In the same way we can derive new formulas. Let $\alpha_s = -$ and $\beta_s = +$ for all s, and let $t_s > 1$ for $1 \le s \le m$ and $t_s < 0$ for $m+1 \le s \le n-1$. Then

$$\begin{aligned} K^+_{AB}(\boldsymbol{\lambda},\boldsymbol{\mu}; a, g_n(\mathbf{t})) = {} & \frac{1}{(2\pi)^{n-1}}\Gamma(1+a+|\boldsymbol{\mu}|) \prod_{s=m+1}^{n-1} \mathrm{B}(\lambda_s, 1-\mu_s) \\ & \times \sum_{k=1}^{\infty}\cdots\sum_{k_m=1}^{\infty} (-1)^{k'}\Gamma^{-1}(1+a+m+\mu''+k') \\ & \times \left[\prod_{s=1}^{m} \Gamma(1-\mu_s+k_s)\mathrm{B}^{-1}(k_s, \lambda_s-k_s)t_s^{\mu_s-k_s-1}\right] \\ & \times F_A(-a-m-\mu''-k'; 1-\mu_{m+1}, \dots, 1-\mu_{n-1}; \\ & \qquad \lambda_{m+1}-\mu_{m+1}+1, \dots, \lambda_{n-1}-\mu_{n-1}+1; t_{m+1}, \dots, t_{n-1}), \end{aligned} \tag{11}$$

where $k' = k_1 + \ldots + k_m$, $\mu'' = \mu_{m+1} + \ldots + \mu_{n-1}$. If we also have $|t_{m+1}| + \ldots + |t_{n-1}| < 1$, then F_A can be expanded into a power series. As a result, we obtain the corresponding expansion for the kernel K^+_{AB}.

As in the case of formula (4), we prove that if $\alpha_s = -$, $\beta_s = +$ for all s, $t_s > 1$ for $1 \le s \le m$ and $t_s < 0$ for $s \ge m+1$, then

$$\begin{aligned} K^-_{A,-B}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t})) = {} & [\sin\pi(a+\mu'')]^{-1} \sin\pi\mu' \\ & \times K^+_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t})). \end{aligned} \tag{12}$$

Since $g_n(\mathbf{t})w_n(\mathbf{x}) = g_n(\mathbf{t}+\mathbf{x})$, then

$$K^\gamma_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t}+\mathbf{x})) = \int_{b-i\infty}^{b+i\infty} [K^+_{A,\gamma B}(\boldsymbol{\lambda}, \boldsymbol{\nu}; a, g_n(\mathbf{t}))K^\gamma_{\gamma B,B}(\boldsymbol{\nu}, \boldsymbol{\mu}; a, w_n(\mathbf{x}))$$
$$+ K^-_{A,-\gamma B}(\boldsymbol{\lambda}, \boldsymbol{\nu}; a, g_n(\mathbf{t}))K^{-\gamma}_{-\gamma B,B}(\boldsymbol{\nu}, \boldsymbol{\mu}; a, w_n(\mathbf{x}))]d\boldsymbol{\nu}.$$

Substituting here the expressions for $K^{-\gamma}_{\gamma B,B}(\dots; w_n(\mathbf{x}))$ and $K^{-\gamma}_{-\gamma B,B}(\dots; w_n(\mathbf{x}))$ we express $K^\gamma_{AB}(\boldsymbol{\lambda}, \boldsymbol{\mu}; a, g_n(\mathbf{t}+\mathbf{x}))$ as the Mellin transform of a linear combinations of the functions $K^+_{A,\gamma B}(\boldsymbol{\lambda}, \boldsymbol{\nu}; a, g_n(\mathbf{t}))$ and $K^-_{A,\gamma B}(\boldsymbol{\lambda}, \boldsymbol{\nu}; a, g_n(\mathbf{t}))$. The relation

$$g_n(\mathbf{t}+\mathbf{x})w_n(-\mathbf{x}) = g_n(\mathbf{t})$$

leads to the inverse formula for this integral transform. We recommend to the reader to write down relations for the Lauricella functions which follows from these integral transforms.

Chapter 17. Group Representations and Special Functions of a Matrix Argument

17.1. Elementary Functions of a Matrix Argument. Gamma-Function and Beta-Function

17.1.1. Elementary functions of a matrix argument. A product of two positive definite matrices can be not a positive definite matrix. Therefore, another product operation is defined in $\mathfrak{P}_m(\mathbf{F})$. To define it let us note that every matrix $\Lambda \in \mathfrak{P}_m(\mathbf{F})$ is representable in the form $\Lambda = TT^*$, where $T \in N_-(m,\mathbf{F})A$ and A is the group of diagonal matrices diag $(a_1, \dots, a_m)$ with $a_i > 0$. We transfer the operation of group multiplication, defined in $N_-(m,\mathbf{F})A$, into the set $\mathfrak{P}_m(\mathbf{F})$. Namely, for $\Lambda = T_1T_1^*$, $M = T_2T_2^*$ we put

$$\Lambda \vartriangle M = T_1T_2(T_1T_2)^* = T_1MT_1^*. \tag{1}$$

The inverse for the element $\Lambda = TT^*$ with respect to the operation $\vartriangle$ is ${}^{-1}\Lambda = T^{-1}(T^*)^{-1}$. We have $\Lambda \vartriangle I_m = I_m \vartriangle \Lambda = \Lambda$.

We define *power functions of a matrix argument* $\Lambda \in \mathfrak{P}_m(\mathbf{F})$ as solutions of the functional equation $f(\Lambda \vartriangle M) = f(\Lambda)f(M)$. Every such function is a product of powers of main minors of the matrix Λ. (Since Λ is a positive definite matrix, then these minors are positive definite.) We denote these products as

$$\Delta^{\mathbf{a}}(\Lambda) \equiv \Lambda^{\mathbf{a}} = \Delta_1^{a_1-a_2}(\Lambda)\Delta_2^{a_2-a_3}(\Lambda)\dots\Delta_m^{a_m}(\Lambda), \tag{2}$$

where $\mathbf{a} = (a_1, \dots, a_m)$, $a_k \in \mathbb{C}$, $1 \le k \le m$. Here $\Delta_k(\Lambda)$ denotes the k-th main minor of the matrix Λ if $\mathbf{F} = \mathbf{R}, \mathbb{C}$ and the square root of this minor if $\mathbf{F} = \mathbf{H}$. Let us note that determinant of any matrix from $\mathfrak{P}_m(\mathbf{H})$ is equal to P^2, where P is a certain polynomial in elements of this matrix. This assertion is proved by the induction in n. We omit this proof.

Evidently, if $n_- \in N_-(m,\mathbf{F})$ and $M = n_-\Lambda n_-^*$, then $\Lambda^{\mathbf{a}} = M^{\mathbf{a}}$. If $M = \delta\Lambda\delta^*$, where $\delta = \text{diag}\,(\delta_1, \dots, \delta_m)$, then

$$M^{\mathbf{a}} = |\delta_1|^{2a_1}\dots|\delta_m|^{2a_m}\Lambda^{\mathbf{a}}.$$

Let us consider in $\mathfrak{H}_m(\mathbf{F})$ two parametrizations (λ_{ij}) and (z_{ij}), where for $\Lambda \in \mathfrak{H}_m(\mathbf{F})$ we have

$$\Lambda = (\lambda_{ij}), \quad \lambda_{ij} \in \mathbf{F}; \quad \Lambda = (\eta_{ij}z_{ij}), \quad \eta_{ij} = \frac{1}{2}(1+\delta_{ij})$$

(if $\mathbf{F} = \mathbb{C}, \mathbf{H}$, then we usually go over to real components $\lambda_{ij,\zeta}$ and $z_{ij,\zeta}$ of λ_{ij} and z_{ij}). The space of matrices $Z = (z_{ij})$ is denoted by $\mathfrak{H}_m^*(\mathbf{F})$. If $\mathbf{F} = \mathbb{C}, \mathbf{R}$, then we have

$$\exp \operatorname{tr} \Lambda Z^* = \exp\left(\sum_{i,j=1}^{m} \lambda_{ij}\bar{z}_{ij}\right) \tag{3}$$

(below exp tr will be denoted by etr: exp tr $\Lambda Z \equiv$ etr ΛZ). If $\mathbf{F} = \mathbf{H}$, then tr ΛZ is equal to $\frac{1}{2}$ tr $\hat{\Lambda}\hat{Z}$, where $\hat{\Lambda}$ and $\hat{Z}$ are obtained respectively from Λ and Z by replacement of them by the corresponding complex matrices.

We also define the differential operators $\tilde{D}^{\mathbf{k}}_{\Lambda}$ and $\tilde{D}^{\mathbf{k}}_{Z}$, $\mathbf{k} = (k_1, \ldots, k_m)$, $k_j \in \mathbb{Z}$, $0 \leq k_1 \leq k_2 \leq \ldots \leq k_m$, on spaces of functions on $\mathfrak{H}_m(\mathbf{F})$. Namely, we set

$$\tilde{D}^{\mathbf{k}}_{\Lambda} = \tilde{\Delta}_1^{k_m - k_{m-1}}\left(\eta_{ij}\frac{\partial}{\partial \lambda_{ij}}\right)\ldots\tilde{\Delta}_m^{k_1}\left(\eta_{ij}\frac{\partial}{\partial \lambda_{ij}}\right), \tag{4}$$

$$\tilde{D}^{\mathbf{k}}_{Z} = \tilde{\Delta}_1^{k_m - k_{m-1}}\left(\frac{\partial}{\partial z_{ij}}\right)\ldots\tilde{\Delta}_m^{k_1}\left(\frac{\partial}{\partial z_{ij}}\right), \tag{5}$$

where $\tilde{\Delta}_j(\Lambda)$ is the algebraic complement to the minor $\Delta_{m-j}(\Lambda)$. We have $\Delta_m(\Lambda) = \tilde{\Delta}_m(\Lambda)$. Sometimes $\Delta_m(\Lambda)$ will be denoted by $\Delta(\Lambda)$. The relations

$$\tilde{D}^{\mathbf{k}}_{\Lambda}(\text{etr } \Lambda Z) = \tilde{\Delta}^{\mathbf{k}}(Z)\text{etr } \Lambda Z, \tag{6}$$

$$\tilde{D}^{\mathbf{k}}_{Z}(\text{etr } \Lambda Z) = \tilde{\Delta}^{\mathbf{k}}(\Lambda)\text{etr } \Lambda Z \tag{7}$$

are valid.

17.1.2. The Fourier and the Laplace transforms of functions of a matrix argument. The Fourier transform of functions f on $\mathfrak{M}_{km}(\mathbf{F})$ is defined by the formula

$$F(Z) = \int\limits_{\mathfrak{M}_{km}(\mathbf{F})} f(\Lambda)\text{etr}(i\Lambda Z^*)d\Lambda, \tag{1}$$

where $Z \in \mathfrak{M}_{km}(\mathbf{F})$. The inverse transform is of the form

$$f(\Lambda) = \frac{1}{(2\pi)^N}\int\limits_{\mathfrak{M}_{km}(\mathbf{F})} F(Z)\text{etr}(-i\Lambda Z^*)dZ, \tag{2}$$

where $N = km\nu$ and $\nu = 1$ for $\mathbf{F} = \mathbf{R}$, $\nu = 2$ for $\mathbf{F} = \mathbb{C}$, and $\nu = 4$ for $\mathbf{F} = \mathbf{H}$. The Plancherel equality

$$\int\limits_{\mathfrak{M}_{km}(\mathbf{F})} |f(\Lambda)|^2 d\Lambda = \frac{1}{(2\pi)^N}\int\limits_{\mathfrak{M}_{km}(\mathbf{F})} |F(Z)|^2 dZ \tag{3}$$

is fulfilled. Formulas (1)-(3) are direct corollaries of the corresponding formulas for the usual Fourier transform.

Let V be a cone in a real space $\mathfrak{L}$ and let U be an analytical involution of this cone, that is an analytical transformation $\mathbf{y} = U(\mathbf{x})$, $\mathbf{x}, \mathbf{y} \in V$, such that $U^2 = E$. Let $R(\mathbf{z})$ be a holomorphic function in the tube domain $\mathfrak{T}(V)$ such that

$R(U(\mathbf{z})) = R(\mathbf{z})$, $R(\mathbf{z}) \neq 0$ for $\mathbf{z} \in \mathfrak{T}(V)$ and $R(\mathbf{x}) \in \mathbf{R}$ for $\mathbf{x} \in V$. For example, if $\mathfrak{L} = \mathfrak{S}_m(\mathbf{R})$, then we can set $V = \mathfrak{P}_m(\mathbf{R})$ and $R(Z) = Z^{-1}$.

Let V' be the cone conjugate to V. For $\boldsymbol{\mu} \in V'$, $\boldsymbol{\lambda} \in \mathfrak{L}'$, $n = \dim V$ we set

$$S(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \frac{1}{(2\pi i)^n} \int\limits_{\mathbf{x}+i\mathfrak{L}} R^{-1}(\mathbf{z}) e^{-(\boldsymbol{\mu}, U(\mathbf{z}))+(\boldsymbol{\lambda},\mathbf{z})} d\mathbf{z}, \tag{4}$$

$$M_p(\mathbf{x}) = \left(\int\limits_{\mathfrak{L}} |R(\mathbf{x} + i\boldsymbol{\alpha})|^{-p} d\boldsymbol{\alpha} \right)^{1/p}. \tag{5}$$

If $M_1(\mathbf{x}) < +\infty$, then integral (4) converges absolutely and for every compact subset $A \subset V$ we have

$$\sup_{\mathbf{x} \in A} |M_1(\mathbf{x})| < \infty. \tag{6}$$

In this case value of integral (4) is independent on $\mathbf{x} = (x_1, \ldots, x_n)$. It is proved by changing successively values of the variables $x_1, \ldots, x_n$ and by using the one-dimensional Cauchy theorem. If we also have $M_2(\mathbf{x}) < +\infty$, then according to the Plancherel equality for many-dimensional Fourier integral we obtain

$$\int\limits_{\mathfrak{L}} |S(\boldsymbol{\mu}, \boldsymbol{\lambda})|^2 e^{-2(\boldsymbol{\lambda},\mathbf{x})} d\mathbf{x} = \frac{1}{(2\pi)^n} \int\limits_{\mathfrak{L}} |R(\mathbf{x} + i\boldsymbol{\alpha})|^{-2} \times e^{-2(\boldsymbol{\mu}, \operatorname{Re} U(\mathbf{z}))} d\boldsymbol{\alpha}. \tag{7}$$

It follows from here that if for any $\mathbf{x}^0 \in V$ we have

$$\sup_{t \geq 1} M_2(t\mathbf{x}^0) < +\infty, \tag{8}$$

then due to the inequality $(\boldsymbol{\mu}, \operatorname{Re} U(\mathbf{z})) \geq 0$, the function $S(\boldsymbol{\mu}, \boldsymbol{\lambda})$ vanishes if λ lies outside of the cone V'. Thus, we proved the following statement: *If conditions* (6) *and* (8) *are fulfilled and* $\boldsymbol{\mu} \in V'$, *then*

$$\frac{1}{(2\pi i)^n} \int\limits_{\mathbf{x}+i\mathfrak{L}} R^{-1}(\mathbf{z}) \exp[-(\boldsymbol{\mu}, U(\mathbf{z})) + (\boldsymbol{\lambda}, \mathbf{z})] d\mathbf{z} = \begin{cases} S(\boldsymbol{\mu}, \boldsymbol{\lambda}) & \text{if } \boldsymbol{\lambda} \in V', \\ 0 & \text{if } \boldsymbol{\lambda} \overline{\in} V' \end{cases} \tag{9}$$

and this integral is independent on $\mathbf{x} \in V$.

This statement shows that if $p(z_1, \ldots, z_n)$ and $q(z_1, \ldots, z_n)$ are polynomials such that $q(U(\mathbf{z}))p(\mathbf{z}) = 1$ and the polynomial $q^*(\mathbf{z})$ is conjugate to $q(\mathbf{z})$, then for any function $R(\mathbf{z})$ the function $S(\boldsymbol{\mu}, \boldsymbol{\lambda})$ satisfies the differential equation

$$q^* \left(\frac{\partial}{\partial \boldsymbol{\mu}} \right) p \left(\frac{\partial}{\partial \boldsymbol{\lambda}} \right) S(\boldsymbol{\mu}, \boldsymbol{\lambda}) = S(\boldsymbol{\mu}, \boldsymbol{\lambda}),$$

where $p\left(\frac{\partial}{\partial \boldsymbol{\lambda}}\right) \equiv p\left(\frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_n}\right)$.

Applying the inverse Fourier transform to relation (9) we obtain that for $\mathbf{z} \in \mathfrak{T}(V)$ the equality

$$\int_{V'} S(\boldsymbol{\mu}, \boldsymbol{\lambda}) e^{-(\boldsymbol{\lambda}, \mathbf{z})} d\boldsymbol{\lambda} = R^{-1}(\mathbf{z}) e^{-(\boldsymbol{\mu}, U(\mathbf{z}))} \tag{10}$$

is fulfilled. Replacing $\mathbf{z}$ by $U(\mathbf{z})$ we derive that

$$R^{-1}(\mathbf{z}) e^{-(\boldsymbol{\mu}, \mathbf{z})} = \int_{V'} S(\boldsymbol{\mu}, \boldsymbol{\lambda}) e^{-(\boldsymbol{\lambda}, U(\mathbf{z}))} d\boldsymbol{\lambda}. \tag{10'}$$

Let us apply these results to the cone $V = \mathfrak{P}_m(\mathbf{R})$ in the space $\mathfrak{L} = \mathfrak{S}_m(\mathbf{R})$. We set

$$n = m(m+1)/2, \quad \Lambda = (\lambda_{ij}), \quad 1 \le i, j \le m. \tag{11}$$

If the integral

$$\int_{\Lambda > 0} \operatorname{etr}(-\Lambda Z) f(\Lambda) d\Lambda \tag{12}$$

converges absolutely in the right half-plane Re $Z > X_0$, $Z = X + iY$, then it gives an analytical function $g(Z)$ in this half-plane. This function is said to be the *Laplace transform* of a function f. If

$$\int_{\mathfrak{S}_m^*(\mathbf{R})} |g(X+iY)| dY < \infty$$

for some $X_0 > 0$, then this inequality is fulfilled for all $X > X_0$. Moreover, the inverse formula

$$\frac{1}{(2\pi i)^n} \int_{\operatorname{Re} Z = X_0} \operatorname{etr}(\Lambda Z) g(Z) dZ = \begin{cases} f(\Lambda) & \text{if } \Lambda > 0, \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

is valid. Here the integration is over $Z \in X_0 + i\mathfrak{S}_m^*(\mathbf{R})$. The integral (13) is independent on X_0. Conversely, if a function $g(Z)$ is given which is analytical in the half-plane Re $Z > X_0$ and such that

$$\int_{\mathfrak{S}_m^*(\mathbf{R})} |g(X+iY)| dY < \infty$$

for all $X > X_0$ and

$$\lim_{X \to +\infty} \int_{\mathfrak{S}_m^*(\mathbf{R})} |g(X+iY)| dY = 0,$$

then formula (13) gives a function f on $\mathfrak{P}_m(\mathbf{R})$ such that the Laplace transform of f is $g(Z)$.

The analogous statement for half-planes defined for $\mathfrak{H}_m(\mathbb{C})$ and for $\mathfrak{H}_m(\mathbf{H})$ are valid.

17.1.3. The Fourier transform of harmonic polynomials. It was show in Example 1 of Section 3.2.3 that the Fourier transform of the function $\exp(-x^2/2)$ coincides with $\sqrt{2\pi}\exp(-\lambda^2/2)$. We generalize this assertion for functions of a matrix argument. The following statement is valid: *Let* $Q = (q_{j\alpha}) \in \mathfrak{M}_{km}(\mathbf{R})$, $dQ = \prod\limits_{j,\alpha} dq_{j\alpha}$ *and* $Z \in \mathfrak{M}_m(\mathbf{R})$. *If* $\operatorname{Re} Z > 0$, $S, Q \in \mathfrak{M}_{km}(\mathbf{R})$, *then*

$$\int\limits_{\mathfrak{M}_{km}(\mathbf{R})} \operatorname{etr}(-2\pi i S^t Q)\operatorname{etr}(-\pi Q Z Q^t) dQ = \operatorname{etr}(-\pi S Z^{-1} S^t)\Delta_m^{-k/2}(Z), \tag{1}$$

or else

$$\int\limits_{\mathfrak{M}_{km}(\mathbf{R})} \operatorname{etr}(-i S^t Q)\operatorname{etr}(-Q Z Q^t) dQ = \pi^{km/2}\operatorname{etr}\left(-\frac{1}{4} S Z^{-1} S^t\right)\Delta_m^{-k/2}(Z). \tag{1'}$$

To prove this statement we note that the Jacobian of the transformation $T \to TZ$ is equal to $\Delta_m^k(Z)$. We first prove formula (1) for $Z > 0$ and then analytically continue it in Z. If $Z > 0$ we make the substitutions $Q \to QZ^{-1/2}$, $S \to SZ^{-1/2}$. Then formula (1) takes the form[1]

$$\operatorname{etr}(-\pi S^t S) = \int\limits_{\mathfrak{M}_{km}(\mathbf{R})} \operatorname{etr}(-2\pi i S^t Q)\operatorname{etr}(-\pi Q^t Q) dQ = \sum_{j,\alpha} \int\limits_{-\infty}^{\infty} \exp(-2\pi i s_{j\alpha} q_{j\alpha}) \exp(-\pi q_{j\alpha}^2) dq_{j\alpha}.$$

This formula follows directly from the assertion on the Fourier transform of the function $\exp(-x^2/2)$.

To generalize formula (1) we define harmonic polynomials of a matrix argument. The operator

$$\Delta_{\mathfrak{M}} = \sum_{j,\alpha} \frac{\partial^2}{\partial t_{j\alpha}^2}$$

[1] Remind that if $Z = \Omega D \Omega^t$, $\Omega \in SO(m)$, $D = \operatorname{diag}(\delta_1, \dots, \delta_m)$, $\delta_j > 0$, then $Z^{1/2} = \Omega D^{1/2} \Omega^t$, where $D^{1/2} = \operatorname{diag}(\delta_1^{1/2}, \dots, \delta_m^{1/2})$, and $Z^{-1/2} = (Z^{1/2})^{-1}$. It is evident that $(Z^{1/2})^2 = Z$.

is called the *Laplace operator* on $\mathfrak{M}_{km}(\mathbf{R})$. A function f on $\mathfrak{M}_{km}(\mathbf{R})$ for which $\Delta_{\mathfrak{M}} f = 0$ is called *harmonic.* A harmonic polynomial $P(T)$, $T \in \mathfrak{M}_{km}(\mathbf{R})$, is called *$H$-harmonic of degree* λ if for every $Z \in \mathfrak{M}_m(\mathbf{R})$ the relation $P(TZ) = \Delta_m^\lambda(Z)P(T)$ is fulfilled.

If $Z \in \mathfrak{H}_m(\mathbb{C})$, Re $Z > 0$, and a polynomial $P(Q)$ is H-harmonic of degree λ, then

$$\int\limits_{\mathfrak{M}_{km}(\mathbf{R})} \operatorname{etr}(-2\pi i S^t Q)\operatorname{etr}(-\pi Q Z Q^t) P(Q) dQ$$
$$= \Delta_m^{-\lambda-k/2}(Z)\operatorname{etr}(-\pi S Z^{-1} S^t) P(-iS). \tag{2}$$

To prove this formula we use the substitutions $Q \to QZ^{1/2}$, $S \to SZ^{-1/2}$ and reduce it to the case $Z = E$. Since $Q \in \mathfrak{M}_{km}(\mathbf{R})$ we may assume that $Q \in \mathbf{R}^\ell$, $\ell = km$, and $P(Q)$ is a usual harmonic polynomial of degree $m\lambda$. Therefore, it is sufficient to show that

$$\int\limits_{\mathbf{R}^\ell} \exp(-2\pi i S^t Q)\exp(-\pi Q^t Q) P(Q) dQ = \exp(-\pi S^t S) P(-iS),$$

where $S, Q \in \mathbf{R}^\ell$ and a polynomial $P(Q)$ is homogeneous and harmonic. For this we set

$$R(-iS) = \exp(\pi S^t S) \int\limits_{\mathbf{R}^\ell} \exp(-2\pi i S^t T)\exp(-\pi T^t T) P(T) dT.$$

We have to prove that $R = P$. The polynomials R and P have the same degree μ. Replacing S by iS we obtain

$$R(S) = \int\limits_{\mathbf{R}^\ell} \exp(-\pi T^t T) P(T + S) dT.$$

It follows from here that polynomial R is harmonic and $R = P + R_{\mu-2} + R_{\mu-4} + \dots$, where $R_{\mu-2j}$ are homogeneous harmonic polynomials. Thus, the mappings $P \to R_{\mu-2}, P \to R_{\mu-4}, \dots$ are defined. It is easy to verify that these mappings commute with representations of the group $SO(\ell)$ given in the spaces of homogeneous harmonic polynomials by the right shifts $S \to Sg$, $g \in SO(\ell)$. Since these representations are irreducible and non-equivalent, then the mappings $P \to R_{\mu-2j}$ are vanishing and, therefore, $P = R$.

In the conclusion we describe some H-harmonic polynomials (we call them H-polynomials). Evidently, if $k' \geq k$, then every H-polynomial in $\mathfrak{M}_{km}(\mathbf{R})$ gives an H-polynomial in $\mathfrak{M}_{k'm}(\mathbf{R})$. If $k = m$, then up to a constant multiplier the polynomials $P_0(Q) \equiv 1$, $P_1(Q) = \Delta_m(Q)$ exhaust all H-polynomials. If $k = 2m$, then a matrix $Q \in \mathfrak{M}_{km}$ can be written in the form $\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, where $Q_j \in \mathfrak{M}_m(\mathbf{R})$, $j = 1, 2$. For any $\lambda \in \mathbb{Z}$, $\lambda \geq 0$, the functions

$$\operatorname{Re} \Delta_m(Q_1 + iQ_2)^\lambda \quad \text{and} \quad \operatorname{Im} \Delta_m(Q_1 + iQ_2)^\lambda \tag{3}$$

are H-polynomials of degree λ.

The relation $P(QZ) = \Delta_m^\lambda(Z)P(Q)$ shows that for any i, j every H-polynomial satisfies the relation

$$\hat{\Delta}_{ij}P(Q) = 0, \quad \text{where} \quad \hat{\Delta}_{ij} = \sum_\alpha \frac{\partial^2}{\partial q_{i\alpha}\partial q_{j\alpha}}.$$

In the domain, where $\Delta_k(QQ^t) \neq 0$, the function $\Delta_k^{-(k-m-1)/2}(QQ^t)$ is canceled by every operator $\hat{\Delta}_{ij}$. For $k = m+1$ this assertion is also valid for the function $\ln \Delta_k(QQ^t)$. Thus, the indicated functions are analogues of the fundamental solutions for the Laplace equation for functions of a matrix argument. We do not discuss the corresponding integral representations of solutions of this equation.

17.1.4. Generalized gamma-functions. Analogously to the definition of the usual gamma-function from Section 3.4.3, the function

$$\Gamma_m(\boldsymbol{\alpha}; \mathbf{F}) = \int\limits_{\mathfrak{P}_m(\mathbf{F})} \operatorname{etr}(-\Lambda)\Delta^\alpha(\Lambda)d_*\Lambda \tag{1}$$

is called the *multidimensional gamma-function* over the field $\mathbf{F}$. This function can be expressed in terms of the usual gamma-function:

$$\Gamma_m(\boldsymbol{\alpha}; \mathbf{F}) = \pi^{m(m-1)\nu/4}\Gamma(\alpha_1)\Gamma\left(\alpha_2 - \frac{\nu}{2}\right)\Gamma(\alpha_3 - \nu)\ldots\Gamma\left(\alpha_m - \frac{\nu(m-1)}{2}\right), \tag{2}$$

where ν is as above.

To prove formula (2) we note that every matrix Λ from $\mathfrak{P}_m(\mathbf{F})$ can be represented in the form $\Lambda = \delta n n^* \delta$, where $\delta \in A(m, \mathbf{R})$, $n \in N_-(m, \mathbf{F})$. Taking into account formula (4) of Section 15.3.2 for the invariant measure on $\mathfrak{P}_m(\mathbf{F})$ and the fact that $A(m, \mathbf{R})$ splits into 2^m connected parts corresponding to signs of the diagonal elements, we obtain

$$\Gamma_m(\boldsymbol{\alpha}; \mathbf{F}) = 2^m \int\limits_{N_-}\int\limits_{A} \operatorname{etr}(-\delta n n^* \delta)\Delta^\alpha(\delta n n^* \delta)d_*\delta d_* n$$

(remind that $A \equiv A(m, \mathbf{R})$ consists of matrices from $D(m, \mathbf{R})$ with positive diagonal elements). Due to the properties of the function $\Delta^{2\alpha}$ we derive from here that

$$\Gamma_m(\boldsymbol{\alpha}; \mathbf{F}) = 2^m \int\limits_{N_-}\int\limits_{A} \operatorname{etr}(-\delta n n^* \delta)\Delta^\alpha(\delta^2)d_*\delta d_* n.$$

Inverting the order of integrations, using the substitution $n = I_m + W$ and taking into account the equality

$$\int\limits_0^\infty e^{-x^2}x^\lambda d\lambda = \frac{1}{2}\Gamma\left(\frac{\lambda+1}{2}\right),$$

we obtain formula (2).

Let us note that the integrals

$$\int_0^\infty e^{-c_j^2} c_j^{2(\alpha_j-\alpha_{j+1})-(j-1)\nu-1} dc_j ,$$

which are used for evaluation of integral (1), converge in the domains Re $(\alpha_j - \alpha_{j+1}) > \frac{1}{2}(j-1)\nu$. The formula (2) gives the analytical continuation for $\Gamma_m(\boldsymbol{\alpha}; \mathbf{F})$. It follows from (2) that $(\Gamma_m(\boldsymbol{\alpha}; \mathbf{F}))^{-1}$ is an entire function.

The similar reasonings allow us to prove the formula

$$\int_{\mathfrak{P}_m(\mathbf{F})} \operatorname{etr}(-T\Lambda)\Delta^{\boldsymbol{\alpha}}(\Lambda)d_*\Lambda = \Gamma_m(\boldsymbol{\alpha}; \mathbf{F})\Delta^{\boldsymbol{\alpha}}(T^{-1}), \tag{3}$$

where $T \in \mathfrak{P}_m(\mathbf{F})$ (proving it we have to take into account that $\operatorname{etr}(-T\Lambda) = \operatorname{etr}(-T^{1/2}\Lambda T^{1/2})$ and to use the substitution $T^{1/2}\Lambda T^{1/2} = M$). Formula (3) can be analytically continued onto $\mathfrak{H}_m(\mathbf{F})$.

The cone $\hat{C}_{m,2}$ (see Section 15.2.1) is also self-adjoint. The analogue of formula (3) for $\hat{C}_{m,2}$ is

$$\int_{\hat{C}_{m,2}} e^{-(\mathbf{y},\mathbf{t})}\Delta^{\boldsymbol{\alpha}}(\mathbf{y})d_*\mathbf{y} = \hat{\Gamma}_m(\boldsymbol{\alpha})\Delta^{\boldsymbol{\alpha}}(\mathbf{t}^{-1}), \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha, \beta)$, $d_*\mathbf{y} = \lambda(\mathbf{y})^{-m/2}dy_1 \dots dy_m$, $\lambda(\mathbf{y}) = y_1y_2 - y_3^2 - \dots - y_m^2$,

$$(\mathbf{y},\mathbf{t}) = y_1t_1 + y_2t_2 + 2\sum_{k=1}^m y_kt_k, \quad \hat{\Gamma}_m(\boldsymbol{\alpha}) = \pi^{(m/2)-1}\Gamma(\alpha+\beta)\Gamma\left(\beta - \frac{m}{2} + 1\right),$$

$$\Delta^{\boldsymbol{\alpha}}(\mathbf{y}) = y_1^\alpha(\lambda(\mathbf{y}))^\beta, \quad \mathbf{t}^{-1} = \left(\frac{t_2}{\lambda(\mathbf{t})}, \frac{t_1}{\lambda(\mathbf{t})}, -\frac{t_3}{\lambda(\mathbf{t})}, \dots, -\frac{t_m}{\lambda(\mathbf{t})}\right). \tag{5}$$

Below we often write $\Gamma_m(\boldsymbol{\alpha})$ instead of $\Gamma_m(\boldsymbol{\alpha}; \mathbf{F})$. If $\boldsymbol{\alpha} = (\alpha, \dots, \alpha, \alpha)$, then instead of $\Gamma_m(\boldsymbol{\alpha})$ we write $\Gamma_m(\alpha)$. It follows from (2) that

$$\Gamma_m(\alpha) = \pi^{m(m-1)\nu/4}\Gamma(\alpha)\Gamma\left(\alpha - \frac{\nu}{2}\right)\dots\Gamma\left(\alpha - \frac{1}{2}\nu(m-1)\right). \tag{2'}$$

Applying to relation (3) the inversion formula for the Laplace transform we obtain that in the domain of absolute convergence of the integral the equality

$$\frac{1}{(2\pi i)^{m\theta}} \int_{\text{Re } Z = X_0 > 0} (\operatorname{etr} \Lambda Z)\Delta^{\boldsymbol{\alpha}}(Z^{-1})d_*Z = \begin{cases} \Gamma_m^{-1}(\boldsymbol{\alpha}+\theta)\Delta^{\boldsymbol{\alpha}}(\Lambda) & \text{if } \Lambda > 0, \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

is valid, where $\theta = \frac{1}{2}[(m-1)\nu + 2]$, $\boldsymbol{\alpha} + \theta = (\alpha_1 + \theta, \dots, \alpha_{m-1} + \theta, \alpha_m + \theta)$.

Since the Laplace transform commutes with the differentiation operator, then we obtain from (6) the equality

$$\tilde{D}_\Lambda^{\mathbf{k}}(\Delta^{\boldsymbol{\alpha}}(\Lambda)) = \frac{\Gamma_m(\boldsymbol{\alpha}+\theta)}{(2\pi i)^{m\theta}} \int\limits_{\mathrm{Re}\ Z = X_0} (\mathrm{etr}\ \Lambda Z) \tilde{\Delta}^{\mathbf{k}}(Z) \Delta^{\boldsymbol{\alpha}}(Z^{-1}) d_* Z.$$

Since $\tilde{\Delta}_j(Z) = \Delta_{m-j}(Z^{-1})/\Delta_m(Z^{-1})$, then

$$\tilde{D}_\Lambda^{\mathbf{k}}(\Delta^{\boldsymbol{\alpha}}(\Lambda)) = \frac{\Gamma_m(\boldsymbol{\alpha}+\theta)}{(2\pi i)^{m\theta}} \int\limits_{\mathrm{Re}\ Z = X_0} (\mathrm{etr}\ \Lambda Z) \Delta^{\boldsymbol{\alpha}-\mathbf{k}}(Z^{-1}) d_* Z.$$

Applying again formula (6) we derive that

$$\tilde{D}_\Lambda^{\mathbf{k}}(\Delta^{\boldsymbol{\alpha}}(\Lambda)) = \frac{\Gamma_m(\boldsymbol{\alpha}+\theta)}{\Gamma_m(\boldsymbol{\alpha}-\mathbf{k}+\theta)} \Delta^{\boldsymbol{\alpha}-\mathbf{k}}(\Lambda). \tag{7}$$

In particular,

$$\tilde{D}_\Lambda(\Delta^{\boldsymbol{\alpha}}(\Lambda)) = \frac{\Gamma_m(\boldsymbol{\alpha}+\theta)}{\Gamma_m(\boldsymbol{\alpha}+\theta-1)} \Delta^{\boldsymbol{\alpha}-1}(\Lambda). \tag{7$'$}$$

Let us introduce the notation

$$[\boldsymbol{\alpha}; m]_{\mathbf{k}} \equiv [\boldsymbol{\alpha}; \mathbf{F}, m]_{\mathbf{k}} = \frac{\Gamma_m(\boldsymbol{\alpha}+\theta, \mathbf{F})}{\Gamma_m(\boldsymbol{\alpha}-\mathbf{k}+\theta, \mathbf{F})}. \tag{8}$$

Then

$$\tilde{D}_\Lambda^{\mathbf{k}}(\Delta^{\boldsymbol{\alpha}}(\Lambda)) = [\boldsymbol{\alpha}; \mathbf{F}, m]_{\mathbf{k}} \Delta^{\boldsymbol{\alpha}-\mathbf{k}}(\Lambda). \tag{9}$$

It follows from formula (2) that

$$[\boldsymbol{\alpha}; m]_{\mathbf{k}} = (\tilde{\alpha}_1)_{k_1} \left(\tilde{\alpha}_2 - \frac{\nu}{2}\right)_{k_2} \cdots \left(\tilde{\alpha}_m - \frac{\nu(m-1)}{2}\right)_{k_m}, \tag{10}$$

where $(\alpha)_k = \Gamma(\alpha)/\Gamma(\alpha - k)$ and $\tilde{\alpha}_i = \alpha_i + \theta$. Below we also use the notation

$$[\alpha]_{\mathbf{k}} \equiv [\alpha; \mathbf{F}]_{\mathbf{k}} = \prod_{j=1}^{m} \left(\alpha - \frac{1}{2}(j-1)\nu\right)_{k_j}. \tag{10$'$}$$

As for usual functions (see Section 3.5.10), the operations of fractional differentiation and integration can be constructed for functions of a matrix argument. Namely, we set

$$(P_m^{\boldsymbol{\rho}} f)(\Lambda) = \frac{1}{\Gamma_m(\boldsymbol{\rho})} \int\limits_{\Lambda}^{\infty} \Delta^{\boldsymbol{\rho}-\theta}(M - \Lambda) f(M) d_* M. \tag{11}$$

Then

$$P_m^{\rho} P_m^{\sigma} = P_m^{\sigma} P_m^{\rho} = P_m^{\rho+\sigma}, \tag{12}$$

$$P_m^{\mathbf{0}} = E \quad \text{(the identity operator)},$$

$$P_m^{\rho} \Delta^{\sigma}(\Lambda) = \frac{\Gamma_m(\sigma+\theta)}{\Gamma_m(\sigma+\rho+\theta)} \Delta^{\sigma}(\Lambda)\tilde{\Delta}^{\sigma}(\Lambda). \tag{13}$$

We do not construct the theory of these operators. It is connected with the theory of generalized functions.

17.1.5. Generalized beta-functions. The theorem on a convolution is valid for the Laplace transform of functions of a matrix argument. *If g_1 and g_2 are the Laplace transforms of functions f_1 and f_2, then the Laplace transforms of the function*

$$f(\Lambda) = (f_1 * f_2)(\Lambda) = \int_0^{\Lambda} f_1(R) f_2(\Lambda - R)\, dR \tag{1}$$

is the function $g_1 g_2$. Here the integration is over all matrices R such that $0 \le R \le \Lambda$ (that is such that $R \in \mathfrak{P}_m(\mathbf{F})$ and $\Lambda - R \in \mathfrak{P}_m(\mathbf{F})$).

We set

$$\mathrm{B}_m(\alpha, \beta; \mathbf{F}) = \int_0^{I_m} \Delta^{\alpha}(R) \Delta^{\beta-\theta}(I_m - R)\, d_* R, \tag{2}$$

where, remind, $\theta = \frac{1}{2}[(m-1)\nu + 2]$ and $d_*R = \Delta^{-\theta}(R)dR$. The function (2) is called the *multidimensional beta-function.* The substitution $R \to \Lambda^{1/2} R \Lambda^{1/2}$, $\Lambda > 0$, shows that

$$\mathrm{B}_m(\alpha, \beta; \mathbf{F}) = \Delta^{-\alpha-\beta+\theta}(\Lambda) \int_0^{\Lambda} \Delta^{\alpha}(R) \Delta^{\beta-\theta}(\Lambda - R)\, d_* R.$$

Taking into account formula (3) of Section 17.1.4 and the theorem on a convolution we obtain the relation

$$\mathrm{B}_m(\alpha, \beta; \mathbf{F}) \Gamma_m(\alpha+\beta; \mathbf{F}) \Delta^{-\alpha-\beta}(Z) = \Gamma_m(\alpha) \Delta^{-\alpha}(Z) \Gamma_m(\beta) \Delta^{-\beta}(Z).$$

Therefore,

$$\mathrm{B}_m(\alpha, \beta; \mathbf{F}) \equiv \mathrm{B}_m(\alpha, \beta) = \frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\alpha+\beta)}. \tag{3}$$

Some other integrals of functions of a matrix argument can be expressed in terms of beta-function (2). For example,

$$\mathrm{B}_m(\rho, \sigma) = \int_0^{\infty} \Delta^{-\rho-\sigma}(I_m + \Lambda) \Delta^{\sigma-\theta}(\Lambda)\, d\Lambda. \tag{4}$$

Really, let $\Lambda = TT^*$, where $T \in N_-(m, \mathbf{F})A$. We set ${}^{-1}\Lambda = T^{-1}(T^*)^{-1}$. Evidently, $\Lambda^{\vartriangle}\ {}^{-1}\Lambda = I_m$. The mapping $\Lambda \to\ {}^{-1}\Lambda$ transforms the interval $(0, I_m)$ (that is the set of matrices Λ such that $0 \leq \Lambda \leq I_m$) into the ray $(I_m, +\infty)$. We also have $({}^{-1}\Lambda)^{\boldsymbol{\rho}} = \Lambda^{-\boldsymbol{\rho}}$.

Making in (2) the substitution $R \to\ {}^{-1}R$ after some transforms we obtain

$$\mathrm{B}_m(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \int_{I_m}^{\infty} \Delta^{-\boldsymbol{\rho}-\boldsymbol{\sigma}+2\theta}(\Lambda)\Delta^{\boldsymbol{\sigma}-\theta}(\Lambda - I_m)d({}^{-1}\Lambda),$$

where $d\Lambda = \Delta^{\theta}(\Lambda)d_*\Lambda$. The density of the measure $d({}^{-1}\Lambda)$ with respect to the Euclidean measure $d\Lambda$ is a generalized power function, that is $d({}^{-1}\Lambda) = c\Delta^{\mu}(\Lambda)d\Lambda$. Since $d(M\Lambda) = \Delta^{-\theta}(M)d\Lambda$, then $\mu = -2\theta$ and, therefore,

$$\mathrm{B}_m(\boldsymbol{\rho}, \boldsymbol{\sigma}) = c\int_{I_m}^{\infty} \Delta^{-\boldsymbol{\rho}-\boldsymbol{\sigma}}(\Lambda)\Delta^{\boldsymbol{\sigma}-\theta}(\Lambda - I_m)d\Lambda.$$

Using the substitution $\Lambda \to I_m + \Lambda$ we derive from here that

$$\mathrm{B}_m(\boldsymbol{\rho}, \boldsymbol{\sigma}) = c\int_{0}^{\infty} \Delta^{-\boldsymbol{\rho}-\boldsymbol{\sigma}}(I_m + \Lambda)\Delta^{\boldsymbol{\sigma}-\theta}(\Lambda)d\Lambda.$$

Let us prove that $c = 1$. According to formula (3) of Section 17.1.4 we have

$$\Delta^{-\boldsymbol{\rho}-\boldsymbol{\sigma}}(I_m + \Lambda) = \frac{1}{\Gamma_m(\boldsymbol{\rho}+\boldsymbol{\sigma})}\int_{0}^{\infty} \mathrm{etr}(-M^{\vartriangle}(I_m + \Lambda))\Delta^{\boldsymbol{\rho}+\boldsymbol{\sigma}}(M)d_*M.$$

Therefore,

$$\mathrm{B}_m(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{c}{\Gamma_m(\boldsymbol{\rho}+\boldsymbol{\sigma})}\int_{0}^{\infty} \Delta^{\boldsymbol{\sigma}-\theta}(\Lambda)d\Lambda \int_{0}^{\infty} \mathrm{etr}(-M^{\vartriangle}(I_m + \Lambda))\Delta^{\boldsymbol{\rho}+\boldsymbol{\sigma}}(M)d_*M.$$

Inverting the order of integrations we obtain

$$\mathrm{B}_m(\boldsymbol{\rho}, \boldsymbol{\sigma}) = \frac{c}{\Gamma_m(\boldsymbol{\sigma}+\boldsymbol{\rho})}\int_{0}^{\infty} \mathrm{etr}(-M)\Delta^{\boldsymbol{\rho}+\boldsymbol{\sigma}}(M)d_*M$$

$$\times \int_{0}^{\infty} \mathrm{etr}(-\Lambda^{\vartriangle}M)\Delta^{\boldsymbol{\sigma}}(\Lambda)d_*\Lambda = \frac{c\Gamma_m(\boldsymbol{\sigma})}{\Gamma_m(\boldsymbol{\rho}+\boldsymbol{\sigma})}$$

$$\times \int_{0}^{\infty} \mathrm{etr}(-M)\Delta^{\boldsymbol{\rho}}(M)d_*M = \frac{c\Gamma_m(\boldsymbol{\rho})\Gamma_m(\boldsymbol{\sigma})}{\Gamma_m(\boldsymbol{\rho}+\boldsymbol{\sigma})}.$$

Using formula (3) we have $c = 1$. Thus, we proved that in the domain of absolute convergence formula (4) is valid. We also proved that $d({}^{-1}\Lambda) = \Delta^{-2\theta}(\Lambda)d\Lambda$. Let us note that the equality

$$\int_0^M \Delta^{\rho}(\Lambda)\Delta^{\sigma-\theta}(M-\Lambda)d_*\Lambda = \mathrm{B}_m(\boldsymbol{\rho},\boldsymbol{\sigma})\Delta^{\rho+\sigma-\theta}(M) \tag{3'}$$

follows from (2).

Below we shall need the integral

$$\int_{\mathbf{F}^k} \prod_{j=1}^{k}\left(1+\sum_{i=1}^{j}|z_i|^2\right)^{\alpha_{j+1}-\alpha_j} d\mathbf{z} = \pi^{k\nu/2}\Gamma_k\left(\tilde{\boldsymbol{\alpha}}-\frac{\nu}{2}\right)\Gamma_k^{-1}(\tilde{\boldsymbol{\alpha}}), \tag{5}$$

where $d\mathbf{z} = \prod_{j=1}^{k}\prod_{\zeta=1}^{\nu} dz_{j\zeta}$, $\tilde{\boldsymbol{\alpha}} = (\alpha_k,\dots,\alpha_1)$, $\alpha_{k+1} = 0$. To prove this formula one has to fulfil successive integration over $z_k, z_{k-1},\dots,z_1$ with the help of the relation

$$\int_{\mathbf{F}}(1+a^2+|z|^2)^{-\lambda}dz = \pi^{\nu/2}\Gamma\left(\lambda-\frac{\nu}{2}\right)\Gamma^{-1}(\lambda)(1+a^2)^{-\lambda+\nu/2}.$$

We shall also need the integral

$$\int_{B^k(\mathbf{F})} \prod_{j=1}^{k}\left(1-\sum_{i=1}^{j}|z_i|^2\right)^{\alpha_j-\alpha_{j+1}} dz = \pi^{k\nu/2}\Gamma_k(\boldsymbol{\alpha}+\theta_k)\Gamma_k^{-1}\left(\boldsymbol{\alpha}+\theta_k+\frac{\nu}{2}\right). \tag{6}$$

Here $B^k(\mathbf{F}) = \{\mathbf{z} \mid |\mathbf{z}|^2 < 1\}$, $\theta_k = \frac{1}{2}\nu(k-1)+1$. Formula (6) is proved by making use of the relation

$$\int_{0\le|z|<1} (1-a^2-|z|^2)^{\lambda}dz = \frac{\pi^{\nu/2}\Gamma(\lambda+1)}{\Gamma(\lambda+1+\nu/2)}(1-a^2)^{\lambda+\nu/2}.$$

17.1.6. Matrix analogues of the integral $\int_{-\infty}^{\infty}(1+x^2)^{-\alpha}dx$. We set

$$J_m(\boldsymbol{\alpha}_m;\mathbf{F}) = \int_{\mathfrak{H}_m(\mathbf{F})} \Delta^{-\alpha_m}(T_m+T^2)dT, \tag{1}$$

where $dT = \prod_{j=1}^{m} dt_{jj} \prod_{j<k} \prod_{\zeta=1}^{\nu} dt_{jk,\zeta}$, $\boldsymbol{\alpha}_m = (\alpha_1, \dots, \alpha_m)$. We shall prove that the recurrence relation

$$J_m(\boldsymbol{\alpha}_m; \mathbf{F}) = \pi^{((m-1)\nu+1)/2} \Gamma_m(\tilde{\boldsymbol{\alpha}}_m) \Gamma^{-1}(\hat{\boldsymbol{\alpha}}_m) J_{m-1}(\boldsymbol{\alpha}^*_{m-1}; \mathbf{F}) \tag{2}$$

is valid, where $\tilde{\boldsymbol{\alpha}}_m = (\alpha_m - 1/2, \alpha_m + \alpha_{m-1} - 1, \dots, \alpha_m + \alpha_1 - 1)$, $\hat{\boldsymbol{\alpha}}_m = (\alpha_m, \alpha_m + \alpha_{m-1} + \nu/2 - 1, \dots, \alpha_m + \alpha_1 + \nu/2 - 1)$, $\boldsymbol{\alpha}^*_{m-1} = (\alpha_1, \dots, \alpha_{m-2}, \alpha_{m-1} - \nu/2)$. To prove this recurrence relation we need the following statement: *If $a > 0$ and $ac - b^2 > 0$, then*

$$\int_{-\infty}^{\infty} (ax^2 + 2bx + c)^{-\alpha} dx = a^{\alpha-1}(ac - b^2)^{-\alpha+1/2} \sqrt{\pi} \frac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha)} \tag{3}$$

(here and below we do not indicate the convergence domain of integrals). To prove this formula it is sufficient to take full square expression, and then to use the integral representation for the beta-function and the equality $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

We now prove formula (2). Let

$$T = \begin{pmatrix} T_1 & \mathbf{v} \\ \mathbf{v}^* & t \end{pmatrix} \in \mathfrak{H}_m(\mathbf{F}),$$

where $T_1 \in \mathfrak{H}_{m-1}(\mathbf{F})$, $\mathbf{v} \in \mathbf{F}^{m-1}$, $t \in \mathbf{R}$. Then

$$I_m + T^2 = \begin{pmatrix} I_{m-1} + T_1^2 + \mathbf{v}\mathbf{v}^* & T_1\mathbf{v} + \mathbf{v}t \\ \mathbf{v}^*T + t\mathbf{v}^* & 1 + \mathbf{v}^*\mathbf{v} + t^2 \end{pmatrix}.$$

If $A = A^*$, then

$$\begin{pmatrix} I_m & \mathbf{0} \\ -\mathbf{b}^*A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^* & \gamma \end{pmatrix} \begin{pmatrix} I_m & \mathbf{0} \\ -\mathbf{b}^*A^{-1} & 1 \end{pmatrix}^* = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \alpha \end{pmatrix}, \tag{3'}$$

where $\alpha = \gamma - \mathbf{b}^*A^{-1}\mathbf{b}$. Therefore,

$$\Delta_m(I_m + T^2) = \Delta_{m-1}(I_{m-1} + T_1^2 + \mathbf{v}\mathbf{v}^*)\{1 + \mathbf{v}^*\mathbf{v} + t^2 - \\ -(\mathbf{v}^*T_1 + t\mathbf{v}^*)(I_{m-1} + T_1^2 + \mathbf{v}\mathbf{v}^*)^{-1}(T_1\mathbf{v} + \mathbf{v}t)\}.$$

The second multiplier of this expression can be written in the form $at^2 + 2bt + c$. Let us show that $ac - b^2 = 1$.

Since the matrix $I_{m-1} + T_1^2$ is Hermitian, then it can be represented as $I_{m-1} + T_1^2 = \Gamma\Gamma^*$, where $\Gamma \in N_-(m-1, \mathbf{F})A$. We set $\mathbf{v} = \Gamma\mathbf{w}$. Then $I_{m-1} + T_1^2 + \mathbf{v}\mathbf{v}^* = \Gamma(I_{m-1} + \mathbf{w}\mathbf{w}^*)\Gamma^*$. It follows from here that

$$d\mathbf{v} = \Delta_{m-1}^{\nu/2}(I_{m-1} + T_1^2) d\mathbf{w},$$

$$\Delta_k(I_{m-1}+T_1^2+\mathbf{v}\mathbf{v}^*)=\Delta_k(I_{m-1}+T_1^2)\Delta_k(I_{m-1}+\mathbf{w}\mathbf{w}^*).$$

Using the equality $(\mathbf{w}\mathbf{w}^*)^2=\mathbf{w}\mathbf{w}^*(\mathbf{w}\mathbf{w}^*)$ and the relations

$$\mathbf{u}^*(I_{m-1}+\mathbf{w}\mathbf{w}^*)^{-1}\mathbf{u}=\mathbf{u}^*\mathbf{u}-\frac{(\mathbf{u}^*\mathbf{w})^2}{1+\mathbf{w}^*\mathbf{w}},\quad \mathbf{u}\in\mathbf{F}^{m-1},$$

$$\mathbf{w}^*(I_{m-1}+\mathbf{w}\mathbf{w}^*)^{-1}=\frac{\mathbf{w}}{1+\mathbf{w}^*\mathbf{w}},$$

which follow from it, after some transformations we obtain

$$a=(1+\mathbf{w}^*\mathbf{w})^{-1},\quad b=-(1+\mathbf{w}^*\mathbf{w})^{-1}\mathbf{w}^*\Gamma^*T_1\Gamma\mathbf{w},$$

$$c=1+\mathbf{w}^*\mathbf{w}+(1+\mathbf{w}^*\mathbf{w})^{-1}(\mathbf{w}^*\Gamma^*T_1\Gamma\mathbf{w})^2.$$

Therefore, $ac-b^2=1$.

Let us use in integral (1) the parameters T_1, $\mathbf{w}$, t. We obtain

$$\begin{aligned}J_m(\boldsymbol{\alpha}_m;\mathbf{F})&=\int\Delta^{-\boldsymbol{\alpha}_{m-1}}(I_{m-1}+T_1^2+\mathbf{v}\mathbf{v}^*)(at^2+2bt+c)^{-\alpha_m}\,dt\,d\mathbf{v}\,dT_1\\&=\frac{\sqrt{\pi}\Gamma(\alpha_m-1/2)}{\Gamma(\alpha_m)}\int_{\mathbf{F}^{m-1}}\Delta^{-\boldsymbol{\alpha}_{m-1}-\alpha_m+1}(I_{m-1}+\mathbf{w}\mathbf{w}^*)d\mathbf{w}\\&\times\int_{\mathfrak{H}_{m-1}(\mathbf{F})}\Delta^{-\boldsymbol{\alpha}_{m-1}+\nu/2}(T_1)dT_1,\end{aligned}\tag{4}$$

where $\boldsymbol{\alpha}_{m-1}=(\alpha_1,\ldots,\alpha_{m-1})$ and $\boldsymbol{\alpha}_{m-1}+c=(\alpha_1+c,\ldots,\alpha_{m-1}+c)$.

Using the Cartan decomposition it is easy to show that for $X\in\mathfrak{M}_{km}(\mathbf{R})$ we have $\Delta_k(I_k+XX^*)=\Delta_m(I_m+X^*X)$. Therefore, $\Delta_k(1+\mathbf{w}\mathbf{w}^*)=1+|w_1|^2+\ldots+|w_k|^2$. Using this equality and formula (5) of Section 17.1.5 we evaluate the first integral of the right hand side of relation (4) and obtain recurrence relation (2). Thus, evaluation of integral (1) is reduced to the integral

$$\int_0^\infty(1+x^2)^{-\alpha}dx=\sqrt{\pi}\Gamma\left(\alpha-\frac{1}{2}\right)\Gamma^{-1}(\alpha).$$

If $\boldsymbol{\alpha}_m=\alpha$, then

$$J_m(\alpha;\mathbf{F})=\int_{\mathfrak{H}_m(\mathbf{F})}\Delta^{-\alpha}(I_m+T^2)dT$$

and the recurrence relation (2) takes the form

$$J_m(\alpha;\mathbf{F})=2^{2-2\alpha}\pi^{\theta_m}\Gamma(2\alpha-\theta_m)\Gamma^{-2}(\alpha)J_{m-1}\left(\alpha-\frac{\nu}{2};\mathbf{F}\right),\tag{5}$$

where $\theta_k = \frac{1}{2}[(k-1)\nu + 2]$ (we applied formula (4) of Section 3.4.3). In this case we have

$$J_m(\alpha; \mathbf{F}) = 2^{m(1+\theta_m-2\alpha)}\pi^{m\theta_m}\Gamma_m(2\alpha-\theta_m)\Gamma_m^{-2}(\alpha). \tag{6}$$

The similar formula can be derived for the integral

$$K_m(\boldsymbol{\alpha}_m) = \int\limits_{\mathfrak{A}_m(\mathbf{F})} \Delta^{-\alpha_m}(I_m + KK^*)dK, \tag{7}$$

where $\mathfrak{A}_m(\mathbf{F})$ is the space of skew-symmetric $m \times m$ matrices over the field $\mathbf{F}$ and $dK = \prod\limits_{i<j}\prod\limits_{\zeta=1}^{\nu} dk_{ij,\zeta}$. Using reasonings similar to those used for proving formula (2) the recurrence relation

$$\begin{aligned} K_m(\boldsymbol{\alpha}_m) &= \pi^{(m-1)\nu/2}\Gamma_{m-1}\left(\tilde{\boldsymbol{\alpha}}_{m-1} + \alpha_m - \frac{\nu}{2}\right)\Gamma_{m-1}^{-1}(\tilde{\boldsymbol{\alpha}}_{m-1} + \alpha_m) \\ &\times K_{m-1}\left(\boldsymbol{\alpha}_{m-1} - \frac{\nu}{2}\right) \end{aligned} \tag{8}$$

is derived, where $\boldsymbol{\alpha}_{m-1} = (\alpha_1, \dots, \alpha_{m-1})$, $\tilde{\boldsymbol{\alpha}}_{m-1} = (\alpha_{m-1}, \dots, \alpha_2, \alpha_1)$.

By making use of formula (8) and the equality $K_2(\alpha) = \pi^{\nu/2}\Gamma(\alpha-\nu/2)\Gamma^{-1}(\alpha)$ we can write down the expression for $K_m(\boldsymbol{\alpha}_m)$. In particular, we have

$$K_m(\alpha) = \frac{\pi^{m(m-1)\nu/4}\Gamma_{m-1}\left(2\alpha - \frac{(m-1)\nu}{2}\right)}{\Gamma(2\alpha)\Gamma(2\alpha-\nu)\dots\Gamma(2\alpha-(m-2)\nu)}. \tag{9}$$

Let us note that integrals of the forms, considered above, are used for evaluation of the Plancherel measure for the spherical transforms (see Section 17.2.9).

17.1.7. Matrix analogues of the integral $\int\limits_0^1 (1-x^2)^\lambda dx$. Let $\varphi(z)$ be a function on $\mathfrak{M}_{km}(\mathbf{F})$ taking values in $\mathfrak{H}_m(\mathbf{F})$. We denote by $\Delta_+^\alpha(\varphi(Z))$ the function on $\mathfrak{M}_{km}(\mathbf{F})$ taking the value $\Delta^\alpha(\varphi(Z))$ if $\varphi(Z) > 0$ and the value 0 otherwise. The following statement is valid: *If $Z \in \mathfrak{M}_{km}(\mathbf{F})$ and*

$$J_{km}(\boldsymbol{\alpha}; \mathbf{F}) = \int\limits_{ZZ^* < I_k} \Delta_+^\alpha(I_k - ZZ^*)dZ, \tag{1}$$

where $dZ = \prod\limits_{i,j}\prod\limits_{\zeta=1}^{\nu} dz_{ij,\zeta}$, then

$$J_{km}(\boldsymbol{\alpha}; \mathbf{F}) = \pi^{km\nu/2}\Gamma_m(\boldsymbol{\alpha} + \theta_m)/\Gamma_m(\boldsymbol{\alpha} + \theta_{k+m}). \tag{2}$$

To prove this statement we first prove the recurrence formula for the integral $\int f(Z)dZ$ over the domain $\{Z \mid I_k - ZZ^* > 0\}$.

We represent the matrix Z in the form $Z = (W, \mathbf{x})$, where $W \in \mathfrak{M}_{k,m-1}(\mathbf{F})$ and $\mathbf{x} \in \mathbf{F}^k$. Evidently, we have $I_k - ZZ^* = I_k - WW^* - \mathbf{x}\mathbf{x}^*$. Moreover, it follows from the inequalities $I_k - ZZ^* > 0$ and $\mathbf{x}\mathbf{x}^* \geq 0$ that $I_k - WW^* > 0$. Therefore, there exists a matrix $\Gamma \in N_-(k, \mathbf{F})A$ such that $I_k - WW^* = \Gamma\Gamma^*$. We use the substitution $\mathbf{x} = \Gamma\mathbf{y}$ and obtain

$$d\mathbf{x} = |\Delta_k(\Gamma)|^\nu d\mathbf{y} = \Delta_k^{\nu/2}(\Gamma\Gamma^*)d\mathbf{y} = \Delta_k^{\nu/2}(I_k - WW^*)d\mathbf{y}.$$

In this reason

$$\int\limits_{ZZ^*<I_k} f(Z)dZ = \int\limits_{WW^*<I_k} \Delta_k^{\nu/2}(I_k - WW^*)dW \int\limits_{\mathbf{y}\mathbf{y}^*<I_k} f(Z)d\mathbf{y}, \tag{3}$$

where we have used the formula

$$I_k - ZZ^* = I_k - WW^* - \mathbf{x}\mathbf{x}^* = \Gamma(I_k - \mathbf{y}\mathbf{y}^*)\Gamma^*. \tag{4}$$

It is easy to prove with the help of the Cartan decomposition for matrices from $\mathfrak{M}_{km}(\mathbf{F})$ that

$$\Delta_k(I_k - ZZ^*) = \Delta_m(I_m - Z^*Z)$$

and that the conditions $I_k - ZZ^* > 0$ and $I_m - Z^*Z > 0$ are equivalent. In particular, the inequality $I_k - \mathbf{y}\mathbf{y}^* > 0$ is equivalent to the inequality $1 - \mathbf{y}^*\mathbf{y} > 0$ which gives in $\mathbf{F}^k \sim \mathbf{R}^{k\nu}$ the usual unit ball. Therefore,

$$\int\limits_{ZZ^*<I_k} f(Z)dZ = \int\limits_{WW^*<I_k} \Delta^{\nu/2}(I_k - WW^*)dW \int\limits_{\mathbf{y}^*\mathbf{y}<1} f(Z)d\mathbf{y}. \tag{5}$$

In an analogous way it is proved that if $Z = \begin{pmatrix} U \\ \mathbf{w}^t \end{pmatrix}$, then

$$\int\limits_{ZZ^*<I_k} f(Z)dZ = \int\limits_{UU^*<I_{k-1}} \Delta^{\nu/2}(I_{k-1} - UU^*)dU \int\limits_{\mathbf{v}\mathbf{v}^*<I_m} f(Z)d\mathbf{v}, \tag{6}$$

where $\mathbf{w} = \Gamma\mathbf{v}$, $I_{k-1} - UU^* = \Gamma^*\Gamma$. The inequality $\mathbf{v}\mathbf{v}^* < I_m$ is equivalent to $\mathbf{v}^*\mathbf{v} < 1$.

The successive applications of relation (5) give for the left hand side of (6) the expression

$$\begin{aligned}\int\limits_{ZZ^*<I_k} f(Z)dZ &= \int\limits_{\mathbf{w}_1^*\mathbf{w}_1<1} (1 - \mathbf{w}_1^*\mathbf{w}_1)^{(n-1)\nu/2}d\mathbf{w}_1 \\ &\times \int\limits_{\mathbf{w}_2^*\mathbf{w}_2<1} (1 - \mathbf{w}_2^*\mathbf{w}_2)^{(n-2)\nu/2}d\mathbf{w}_2 \ldots \int\limits_{\mathbf{w}_m^*\mathbf{w}_m<1} f(Z)d\mathbf{w}_m,\end{aligned} \tag{7}$$

where $\mathbf{w}_j \in \mathbf{F}^k$, $1 \le j \le m$.

It follows from (4) that for $j \le k$ we have

$$\Delta_j(I_k - ZZ^*) = \Delta_j(I_k - WW^*)\Delta_j(I_k - \mathbf{y}\mathbf{y}^*).$$

Therefore, for $f(Z) = \Delta_+^{\boldsymbol{\alpha}}(I_k - ZZ^*)$ we obtain from (7) that

$$J_{km}(\boldsymbol{\alpha}; \mathbf{F}) = \int\limits_{\mathfrak{M}_{k,m-1}} \Delta_+^{\boldsymbol{\alpha}+\nu/2}(I_k - WW^*)dW \int\limits_{\mathbf{F}^k} \Delta_+^{\boldsymbol{\alpha}}(I_k - \mathbf{y}\mathbf{y}^*)d\mathbf{y}.$$

Using the equality $\Delta_j(I_k - \mathbf{y}\mathbf{y}^*) = 1 - |y_1|^2 - \ldots - |y_j|^2$, where $\mathbf{y} = (y_1, \ldots, y_k)$ and formula (6) of Section 17.1.5 we derive from here the recurrence relation

$$J_{km}(\boldsymbol{\alpha}; \mathbf{F}) = \pi^{k\nu/2}\Gamma_k(\boldsymbol{\alpha} + \theta_k)\Gamma_k^{-1}\left(\boldsymbol{\alpha} + \theta_k + \frac{\nu}{2}\right) J_{k,m-1}\left(\boldsymbol{\alpha} + \frac{\nu}{2}, \mathbf{F}\right) \tag{8}$$

which leads to formula (2).

For $\boldsymbol{\alpha} = \alpha$ formula (8) take the form

$$J_{km}(\alpha; \mathbf{F}) = \pi^{k\nu/2}\Gamma(\alpha + 1)\Gamma^{-1}(\alpha + \theta_{k+1})J_{k,m-1}\left(\alpha + \frac{\nu}{2}, \mathbf{F}\right),$$

and, therefore,

$$J_{km}(\alpha; \mathbf{F}) = \pi^{km\nu/2}\Gamma_m^{-1}(\alpha + \theta_{k+m}) \prod_{j=1}^{m} \Gamma(\alpha + \theta_j). \tag{9}$$

It follows from formula (6) that if $Z \in \mathfrak{M}_{km}(\mathbf{F})$, $\ell < k$, and a function $f(Z)$ does not depend on the last ℓ rows of Z, then

$$\begin{aligned}\int\limits_{ZZ^*<I_k} f(Z)dZ &= \pi^{m(k-\ell)\nu/2}\Gamma_m\left(\frac{1}{2}(k-\ell)\nu\right)\Gamma_m^{-1}\left(\frac{1}{2}(m+k-\ell)\nu\right) \\ &\times \int\limits_{Z_\ell Z_\ell^*<I_{k-\ell}} f(Z)\Delta^{(k-\ell)\nu/2}(I_{k-\ell} - Z_\ell Z_\ell^*)dZ_\ell,\end{aligned} \tag{10}$$

where Z_ℓ is the matrix consisting of the first $k - \ell$ rows of the matrix Z. The analogous formula is valid for the case when $f(Z)$ is of the form $f(Z) = \Delta^{\alpha}(Z)\varphi(Z)$, where $\varphi(Z)$ does not depend on the last ℓ rows of Z.

Formulas analogous to (2) are valid if Z runs over the set $\mathfrak{S}_m(\mathbf{F})$ or over the set $\mathfrak{A}_m(\mathbf{F})$.

We now prove the following generalization of formula (3) of Section 17.1.6. Let $a, c \in \mathbf{R}$, $b \in \mathbf{F}$, $a < 0$, $|b|^2 - ac > 0$, $\lambda > -1$. Then

$$\int\limits_{zz^*<1} (az\bar{z} + \bar{b}z + b\bar{z} + c)_+^{\lambda} dz = \frac{\pi^{\nu/2}\Gamma(\lambda+1)}{\Gamma(\lambda+1+\nu/2)} \frac{(|b|^2 - ac)^{\lambda+\nu/2}}{|a|^{\lambda+\nu}}, \tag{11}$$

where $dz = \prod_{\zeta=1}^{\nu} dz_\zeta$. Really, making the substitution $w = (z - ba^{-1})\sqrt{a^2(|b|^2 - ac)^{-1}}$ we obtain $dz = |a|^{-\nu}(|b|^2 - ac)^{\nu/2}dw$ and

$$az\bar{z} + \bar{b}z + b\bar{z} + c = a^{-1}(ac - |b|^2)(1 - w\bar{w}).$$

Therefore, we have for the left hand side of (11) the expression

$$|a|^{-\nu-\lambda}(|b|^2 - ac)^{\lambda+\nu/2} \int\limits_{w\bar{w}<1} (1 - w\bar{w})^\lambda dw$$

which gives the right hand side of (11).

For $Z \in \mathfrak{S}_m(\mathsf{F})$, $\boldsymbol{\alpha}_m = (\alpha_1, \dots, \alpha_m)$, $dZ = \prod_{j\le k} \prod_{\zeta=1}^{\nu} dz_{jk,\zeta}$ we set

$$L_m(\boldsymbol{\alpha}_m; \mathsf{F}) = \int \Delta_+^{\alpha_m}(I_m - Z\bar{Z})dZ = \int\limits_{Z\bar{Z}<I_m} \Delta^{\alpha_m}(I_m - Z\bar{Z})dZ. \qquad (12)$$

Let us prove the recurrence relation

$$L_m(\boldsymbol{\alpha}_m; \mathsf{F}) = \frac{\pi^{m\nu/2}\Gamma(\alpha_m + 1)\Gamma_{m-1}(\hat{\boldsymbol{\alpha}}_{m-1})}{\Gamma(\alpha_m + 1 + \nu/2)\Gamma_{m-1}(\hat{\boldsymbol{\alpha}}_{m-1} + \nu/2)} L_{m-1}\left(\boldsymbol{\alpha}_{m-1} + \frac{\nu}{2}; \mathsf{F}\right), \qquad (13)$$

where $\boldsymbol{\alpha}_{m-1} = (\alpha_1, \dots, \alpha_{m-1})$, $\hat{\boldsymbol{\alpha}}_{m-1} = \boldsymbol{\alpha}_{m-1} + \alpha_m + \boldsymbol{\theta}_{m-1} + \nu$. Setting $Z = \begin{pmatrix} Z_1 & \mathbf{v} \\ \mathbf{v}^t & z \end{pmatrix}$, where $Z_1 \in \mathfrak{S}_{m-1}(\mathsf{F})$, $\mathbf{v} \in \mathsf{F}^{m-1}$, $z \in \mathsf{F}$, we have

$$I_m - Z\bar{Z} = \begin{pmatrix} I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^* & -(Z_1\bar{\mathbf{v}} + \mathbf{v}\bar{z}) \\ -(\mathbf{v}^t\bar{Z}_1 + z\mathbf{v}^*) & 1 - \mathbf{v}^t\bar{\mathbf{v}} - z\bar{z} \end{pmatrix}.$$

With the help of a triangular transformation the matrix $I_m - Z\bar{Z}$ can be reduced to the form $\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & a \end{pmatrix}$ (see formula (3′) of Section 17.1.6), and we conclude that the condition $I_m - Z\bar{Z} > 0$ is equivalent to the following two relations:

$$I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^* > 0,$$

$$1 - \mathbf{v}^t\bar{\mathbf{v}} - z\bar{z} - (\mathbf{v}^t\bar{Z}_1 + z\mathbf{v}^*)(I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^*)^{-1}(\mathbf{v}^t\bar{Z}_1 + z\mathbf{v}^*)^* > 0.$$

Moreover, we have $(\mathbf{v}^t\bar{Z}_1 + z\mathbf{v}^*)^* > 0$ and

$$\Delta_m(I_m - Z\bar{Z}) = \Delta_{m-1}(I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^*)\{1 - \mathbf{v}^t\bar{\mathbf{v}} - z\bar{z}$$
$$-(\mathbf{v}^t\bar{Z}_1 + z\times^*)(I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^*)^{-1}(\mathbf{v}^t\bar{Z}_1 + z\mathbf{v}^*)^*\}.$$

It follows from here that

$$L_m(\boldsymbol{\alpha}_m; \mathbf{F}) = \int \Delta_+^{\boldsymbol{\alpha}_{m-1}}(I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^*)dZ_1 d\mathbf{v} \times \int (az\bar{z} + \bar{b}z + b\bar{z} + c)_+^{\alpha_m} dz,$$

where

$$a = -1 - \mathbf{v}^*(I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^*)^{-1}\mathbf{v}, \quad (a < 0),$$
$$b = -\mathbf{v}^t\bar{Z}_1(I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^*)^{-1}\mathbf{v},$$
$$c = 1 - \mathbf{v}^t\bar{\mathbf{v}} - \mathbf{v}^t\bar{Z}_1(I_{m-1} - Z_1\bar{Z}_1 - \mathbf{v}\mathbf{v}^*)^{-1}Z_1\bar{\mathbf{v}}.$$

Since $I_{m-1} - Z_1\bar{Z}_1 > 0$, then there exists the lower triangular matrix Γ such that $I_{m-1} - Z_1\bar{Z}_1 = \Gamma\Gamma^*$. We make the substitution $\mathbf{v}^t = \mathbf{u}^t\Gamma^t$. Then

$$d\mathbf{v} = |\Delta_{m-1}(\Gamma)|^\nu d\mathbf{u} = \Delta_{m-1}^{\nu/2}(I_{m-1} - Z_1\bar{Z}_1)d\mathbf{u}.$$

Direct evaluation shows that $a = -(1 - \mathbf{u}^*\mathbf{u})^{-1}$, $|b|^2 - ac = 1$ and

$$\Delta_k(I_m - Z\bar{Z}) = \Delta_k(I_{m-1} - Z_1\bar{Z}_1)\Delta_k(I_{m-1} - \mathbf{u}\mathbf{u}^*), \quad 1 \le k \le m-1.$$

Substituting this expression into integral (12) and using formulas (5) of Section 17.1.5 and (11) after some transformations we obtain recurrence relation (13).

It follows from (13) that

$$L_m(\alpha; \mathbf{F}) = \frac{\pi^{m(m+1)\nu/4}\Gamma_m(\alpha + 1; \mathbf{F})\Gamma_m(2\alpha + \nu + 1; \mathbf{F})}{\Gamma_m\left(\alpha + \frac{\nu+2}{2}; \mathbf{F}\right)\Gamma_m\left(2\alpha + \frac{m+1}{2}\nu + 1; \mathbf{F}\right)}. \tag{14}$$

It is proved in an analogous way that if

$$M_m(\boldsymbol{\alpha}_m; \mathbf{F}) = \int_{\mathfrak{A}_m(\mathbf{F})} \Delta_+^{\boldsymbol{\alpha}_m}(I_m + Z\bar{Z})dZ, \quad dZ = \prod_{i<j}\prod_{\zeta=1}^{\nu} dz_{ij,\zeta}, \tag{15}$$

then we have the recurrence relation

$$M_m(\boldsymbol{\alpha}_m; \mathbf{F}) = \frac{\pi^{(m-1)\nu/2}\Gamma_{m-1}(\hat{\boldsymbol{\alpha}}_{m-1} - \nu/2)}{\Gamma_{m-1}(\hat{\boldsymbol{\alpha}}_{m-1})} M_{m-1}\left(\boldsymbol{\alpha}_{m-1} + \frac{\nu}{2}; \mathbf{F}\right), \tag{16}$$

where $\hat{\alpha}_{m-1}$ is as in formula (13). If $\boldsymbol{\alpha}_m = \alpha$, then

$$M_m(\alpha; \mathbf{F}) = \pi^{\nu m(m-1)/4}\Gamma_{m-1}^{-1}(2\alpha + (m-1)\nu) \prod_{j=1}^{m-2} \Gamma\left(2\alpha + \frac{\nu}{2} + j\nu\right). \tag{17}$$

The expressions for volumes of the domains from $\mathfrak{M}_{km}(\mathbf{F})$, $\mathfrak{S}_m(\mathbf{F})$ and $\mathfrak{A}_m(\mathbf{F})$, defined by the inequalities $I_k - ZZ^* > 0$, $I_m - Z\bar{Z} > 0$ and $I_m + Z\bar{Z} > 0$ respectively, are special cases of formulas (9), (14) and (17). They are obtained by setting $\alpha = 0$ in these formulas.

Methods, used in this section, allows us to evaluate matrix analogues of integrals of the form $\int_1^\infty (x^2-1)^\alpha dx$, namely, the integrals

$$\int_{\mathfrak{M}_{km}(\mathbf{F})} \Delta^{\alpha_k}(ZZ^* - I_k)dZ,$$

$$\int_{\mathfrak{S}_m(\mathbf{F})} \Delta^{\alpha_k}(Z\bar{Z} - I_m)dZ,$$

$$\int_{\mathfrak{A}_m(\mathbf{F})} \Delta^{\alpha_k}(K\bar{K} + I_m)dK.$$

We recommend to the reader to evaluate them.

17.2. Zonal Spherical Functions and Characters

17.2.1. Gel'fand pairs. Let G be a unimodular Lie group and let K be its compact subgroup. If the subring $\mathcal{K} \equiv \mathcal{K}(G, K)$ of the group ring of the group G, consisting of continuous finite functions on G constant on two-sided cosets with respect to K, is commutative, then (G, K) is called *Gel'fand pair.* The following commutativity criterion was proved by I.M. Gel'fand.

Let Θ be an involutive automorphism of a unimodular group G and let $K = \{k \in G \mid \Theta k = k\}$ be a compact subgroup of G. We set $\mathcal{P} = \{p \in G \mid \Theta p = p^{-1}\}$. If every element $g \in G$ can be decomposed in the form $g = kp$, $k \in K$, $p \in \mathcal{P}$, then (G, K) is a Gel'fand pair.

Really, the Haar measure on G is invariant with respect to Θ. Since the subgroup K is Θ-invariant, then the subring $\mathcal{K}$ is also Θ-invariant. This means that $f^\Theta \in \mathcal{K}$ if $f \in \mathcal{K}$, where $f^\Theta(g) = f(\Theta^{-1}g)$. Moreover, $(f * \varphi)^\Theta = f^\Theta * \varphi^\Theta$. Let $g = kp$, $k \in K$, $p \in \mathcal{P}$. Then

$$\Theta g = \Theta(kp) = (\Theta k)(\Theta p) = kp^{-1} = k(p^{-1}k^{-1})k.$$

Therefore, for $f \in \mathcal{K}$ we have $f(\Theta g) = f(p^{-1}k^{-1}) = f((kp)^{-1})$, that is $f^\Theta(g) = \check{f}(g)$, where $\check{f}(g) = f(g^{-1})$. Since G is an unimodular group, then the Haar measure is invariant with respect to the transformation $g \to g^{-1}$. Therefore, $(\check{f} * \check{\varphi})^\vee = \varphi * f$. This means that for $f, \varphi \in \mathcal{K}$ we have $f^\Theta * \varphi^\Theta = \check{f} * \check{\varphi} = (\varphi * f)^\vee = (\varphi * f)^\Theta = \varphi^\Theta * f^\Theta$. Since Θ is involutive, then $f * \varphi = \varphi * f$, that is, the subring $\mathcal{K}$ is commutative and (G, K) is a Gel'fand pair.

The Gel'fand criterion can be formulated in the following way: *Let G be a unimodular Lie group and let K be its compact subgroup. If there is an involutive automorphism Θ of G such that for every $g \in G$ we have $\Theta g \in KgK$, then (G, K) is a Gel'fand pair.*

A compact subgroup H of a unimodular Lie group G is called *large* if the restriction of every irreducible representation T of G onto H has simple spectrum (that is multiplicities of irreducible representations of H in $T\downarrow_H^G$ do not exceed 1). We omit the proof of the following statement. *A compact subgroup H of a unimodular Lie group G is large if and only if $(G \times H, \tilde{H})$ is a Gel'fand pair, where $\tilde{H}$ is the image of H under the mapping $h \to (h, h)$.*

Due to the assertion of Section 16.1.7 the subgroup $SU(n-1)$ is large in $SU(n)$ and the subgroup $SO(n-1)$ is large in $SO(n)$.

If G is a connected noncompact simple Lie group and K is its maximal compact subgroup, then it follows from the Cartan decomposition of G (see Section 1.2.3) that (G, K) is a Gel'fand pair. If U is a compact group dual to G (that is such that the symmetric pairs G/K and U/K are dual by Cartan), then (U, K) is also a Gel'fand pair.

Let us prove that *if G is a semidirect product of a compact group K and of a commutative invariant subgroup A such that from $a^2 = e$, $a \in A$, it follows that $a = e$, then (G, K) is a Gel'fand pair.* Really, we set $\Theta(ka) = ka^{-1}$, $k \in K$, $a \in A$. Evidently, Θ is an continuous involutive mapping of G onto G. Moreover, since for $k' \in K$, $a \in A$ we have $(k')^{-1}ak' \in A$, then

$$\Theta(kak'a') = \Theta((kk')(k')^{-1}ak'a') = kk'((a')^{-1}(k')^{-1}a^{-1}k').$$

On the other hand

$$\Theta(ka)\Theta((k'a') = ka^{-1}k'(a')^{-1} = (kk')((k')^{-1}a^{-1}k'(a')^{-1}).$$

Since A is commutative, then $\Theta(kak'a') = \Theta(ka)\Theta(k'a')$. Thus, Θ is an involutive automorphism of G. Now we have to prove that if $\Theta g = g$, then $g \in K$. If $g = ka$, $k \in K$, $a \in A$, then $\Theta g = g$ means that $ka^{-1} = ka$. Therefore, $a^2 = e$. Then $a = e$ and $g = k \in K$. Our assertion is proved. It shows that

$$(ISO(n), SO(n)), \quad (ISU(n), SU(n))$$

are Gel'fand pairs.

Let (G, K) be a Gel'fand pair and let T be an irreducible representation of G. If $\mathbf{e}_1, \ldots, \mathbf{e}_p$ are K-invariant linearly independent vectors from the carrier space of T and $p > 1$, then linear combinations of the matrix elements $t_{ij}(g) = (T(g)\mathbf{e}_j, \mathbf{e}_i)$, $1 \le i, j \le p$, form a noncommutative subring of $\mathcal{K}(G, K)$. Therefore, $p \le 1$ and K is a massive subgroup of G.

Let (G, H) be a Gel'fand pair. Since $\mathcal{K} \equiv \mathcal{K}(G, H)$ is a commutative ring, then we can apply to it the general theory of commutative Banach algebras. According to this theory the ring $\mathcal{K}$ can be realized as the ring of functions on the space $\mathfrak{M}$ of maximal ideals[2]. There is close relationship between maximal ideals of $\mathcal{K}$ and irreducible representations of the group G. Transition to the space of functions on $\mathfrak{M}$ is related to the transform

$$f \to \int_G f(g)\varphi_T(g)dg,$$

where T runs over the set of class 1 (with respect to H) irreducible representations of the group G and $\varphi_T(g)$ are zonal spherical functions of these representations.

17.2.2. Zonal spherical functions and their properties. Let us remind that an irreducible unitary representation T of a group G is of class 1 with respect to a compact subgroup H if in the carrier space $\mathfrak{L}$ of T there is a normed vector $\mathbf{e}_0$ invariant with respect to all operators $T(h)$, $h \in H$. If (G, H) is a Gel'fand pair, then up to a constant factor this vector is unique and its shifts generate the whole space $\mathfrak{L}$. In this case the matrix element

$$\varphi_T(g) \equiv t_{00}(g) = (T(g)\mathbf{e}_0, \mathbf{e}_0) \tag{1}$$

is called a *zonal spherical function* of the representation T with respect to H.

Zonal spherical functions have the following properties:

(a) *Zonal spherical functions $\varphi_T(g)$ are constant on two-sided cosets with respect to H: $\varphi_T(h_1 g h_2) = \varphi_T(g)$, $h_1, h_2 \in H$.*

(b) *Functions $\varphi_T(g)$ for unitary irreducible representations T are positive definite, that is, for each $n \in \mathbb{Z}_+$ and for every elements $g_1, \dots, g_n$ from G and every complex numbers $a_1, \dots, a_n$ we have*

$$\sum_{j,k=1}^{n} \varphi_T(g_j^{-1} g_k) a_j \overline{a_k} \geq 0. \tag{2}$$

(c) *The mapping $f \to \int_G f(g)\varphi_T(g)dg$ is a non-trivial homomorphism of the ring $\mathcal{K}(G, H)$ into $\mathbb{C}$.*

[2] It is known from I.M. Gel'fand's private information that the connection of the theory of representations with the theory of commutative Banach algebras was established by him before the theory of representations of the Lorentz group was constructed. This connection gave an orientation in the studying of infinite dimensional representations. In particular, it showed that problems arising under construction of the theory are rather of analytical nature than of set-theoretical one. However, this result was published later. Let us also note that when constructing the theory of infinite dimensional representations, I.M. Gel'fand and M.A. Naimark in fact used the ideas of the theory of generalized functions. They did not become to construct the complete theory of such functions and transferred results, obtained with application of this theory, in the language of classical analysis.

(d) *A function $\varphi_T(g)$ is an eigenfunction of every Laplace operator on G.*

(e) *If G is a compact subgroup, then zonal spherical functions of non-equivalent irreducible representations are orthogonal.*

Property (a) follows from the equalities $T(h_1 g h_2) = T(h_1)T(g)T(h_2)$ and $T(h_1)\mathbf{e}_0 = T(h_2)\mathbf{e}_0 = \mathbf{e}_0$, where $h_1, h_2 \in H$. The equalities

$$\sum_{j,k=1}^{n} \varphi_T(g_j^{-1} g_k) a_j \overline{a_k} = \sum_{j,k}^{n} (T(g_k)\mathbf{e}_0, T(g_j)\mathbf{e}_0) a_j \overline{a_k} = (\boldsymbol{\eta}, \boldsymbol{\eta}),$$

where $\boldsymbol{\eta} = \sum_{k=1}^{n} T(g_k)\mathbf{e}_0 \overline{a_k}$, prove property (b). Property (c) has been proved in Section 2.3.10. Property (d) is a corollary of irreducibility of representations T and of the fact that T is realized by shift operators in the space of functions on G, spanned by the functions $\varphi_T(gg_0)$, $g_0 \in G$. The function $\varphi_T(g)$ belongs to this space and Laplace operators Δ are multiples of the identity operator on it. Therefore,

$$\Delta \varphi_T(g) = \alpha_T \varphi_T(g). \tag{3}$$

Property (e) follows from the orthogonality relation for matrix elements of irreducible representations of compact groups.

If T_1 and T_2 are class 1 irreducible unitary representations of the group G and their zonal spherical functions coincide, then these representations are equivalent. Really, these representations are equivalent to the representation by shift operators in the space generated by shifts of zonal spherical functions. Therefore, T_1 and T_2 are equivalent representations.

If two class 1 irreducible unitary representations have coinciding eigenvalues for all G-left-invariant and H-right-invariant differential operators on G, then their zonal spherical functions coincide. This assertion follows from analyticity of zonal spherical functions on G and from coincidence of two analytical functions on G having the same values of derivatives of any order at $g = e$.

It follows from the assertions, formulated above, that *there is a one-to-one correspondence between positive definite zonal spherical functions on G and classes of equivalent irreducible unitary representations of class 1 with respect to the subgroup H.*

If G is a compact Lie group and (G, H) is a Gel'fand pair, then any function f on G which is constant on two-sided cosets with respect to H can be expanded into the series

$$f(g) = \sum_T a_T \varphi_T(g), \tag{4}$$

where the sum is over all class 1 irreducible representations of G. Moreover,

$$a_T = \int_G f(g) \overline{\varphi_T(g)} dg \tag{5}$$

(see Section 2.3.9).

It follows from formula (14) of Section 16.2.6 that

$$\int_H \varphi_T(g_1 h g_2) dh = \varphi_T(g_1)\varphi_T(g_2), \quad g_1, g_2 \in G. \tag{6}$$

If G is a compact group and the space $X = G/H$ is symmetric, then zonal spherical functions of G can be expressed in terms of characters of the corresponding representations of G. Really, we realize a class 1 irreducible unitary representation T of G in the space $\mathfrak{L}$ obtained from $\varphi_T(g)$ by right shifts. Let P be the operator $\int_H T(h)dh$. Evidently, $P^2 = P$ and all functions from $P\mathfrak{L}$ are invariant with respect to P. Therefore, they are proportional to φ_T. Thus, $PT(g_0)P$ transforms $\mathfrak{L}$ into $\mathbb{C}\varphi_T$. Moreover, the function φ_T is transformed into $\varphi_T(g_0)\varphi_T$. Consequently,

$$\varphi_T(g) = \mathrm{Tr}\,(PT(g)P) = \mathrm{Tr}\,(PT(g)) = \mathrm{Tr}\int_H T(h)T(g)dh = \int_H \chi(hg)dh, \tag{7}$$

where χ_T is the character of T.

Formula (7) expresses the zonal spherical function φ_T in terms of the character χ_T of T. The converse statement is also valid: *If T is a class 1 unitary irreducible representation of the group G in a Hilbert space $\mathfrak{H}$, then the right hand side of formula (7) is the zonal spherical function of T.* Really, if

$$P = \int_H T(h)dh, \quad \text{then} \quad P^2 = P \quad \text{and} \quad \mathrm{Tr}\,(PT(g)P) = \int_H \chi(hg)dh.$$

If $\mathbf{e}_0$ is a vector in $\mathfrak{H}$ invariant with respect to the operators $T(h)$, $h \in H$, then for every $g \in G$ the vector $PT(g)\mathbf{e}_0$ is also invariant with respect to these operators. Therefore, $P\mathfrak{H} = \{\lambda \mathbf{e}_0 \mid \lambda \in \mathbb{C}\}$. Let $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ be an orthogonal basis of $\mathfrak{H}$. Since $P\mathbf{e}_j = 0$ for $j \geq 1$, then $\mathrm{Tr}\,(PT(g)P) = (T(g)\mathbf{e}_0, \mathbf{e}_0) = \varphi_T(g)$. Thus, $\int_H \chi_T(hg)dh$ is the zonal spherical function of T.

Let G be a noncompact semisimple Lie group with finite center and let K be its maximal compact subgroup. In this case a representation $T_{\lambda Q}$ of the principal unitary series of G is of class 1 with respect to K if Q is the unit representation of the subgroup M. We consider that elements $a \in A$ are given by parameters $e^{t_1}, \ldots, e^{t_\ell}$ and $\lambda(a) = \exp(i\boldsymbol{\lambda} - \boldsymbol{\rho}, \mathbf{t})$, where $\mathbf{t} = (t_1, \ldots, t_\ell)$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\ell) \in \mathbf{R}^\ell$ and $\boldsymbol{\rho}$ is the half-sum of positive restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$. The corresponding class 1 representation of G is denoted by $T_{\boldsymbol{\lambda}}$. This representation is realized in the space $\mathfrak{L}_0^2(K)$ of functions on K constant on left cosets with respect to M. It is given by the formula

$$(T_{\boldsymbol{\lambda}}(g_0)f)(k) = \exp(i\boldsymbol{\lambda} - \boldsymbol{\rho}, \mathbf{t})f(\tilde{k}), \tag{8}$$

where $kg_0 = n a \tilde{k}$, $n \in N$ and a is given by the parameters $\mathbf{t}$.

It follows from properties of zonal spherical functions and from the Cartan decomposition $G = KAK$ that the zonal spherical function $\varphi_{T_\lambda}(g) \equiv \varphi_{\boldsymbol{\lambda}}(g)$ is uniquely defined by its values on the subgroup A. Since the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{a})$ belongs to K, then

$$\varphi_{\boldsymbol{\lambda}}(w \circ a) \equiv \varphi_{\boldsymbol{\lambda}}(waw^{-1}) = \varphi_{\boldsymbol{\lambda}}(a). \tag{9}$$

If the representation $T_{\boldsymbol{\lambda}}$ is realized by formula (8), then the function $f_0(k) \equiv 1$ is K-invariant and $(T_{\boldsymbol{\lambda}}(g) f_0)(k) = \exp(-i\boldsymbol{\lambda} - \boldsymbol{\rho}, \mathbf{t})$, where $\mathbf{t}$ is such as in (8). Thus, we obtain the following integral representation of $\varphi_{\boldsymbol{\lambda}}$:

$$\varphi_{\boldsymbol{\lambda}}(g) = \int \exp(-i\boldsymbol{\lambda} - \boldsymbol{\rho}, \mathbf{t}) dk, \quad \text{where} \quad kg = na\tilde{k}. \tag{10}$$

Property (d) of zonal spherical functions shows that if $\overset{\circ}{\Delta}$ is a radial part of a Laplace operator Δ and $\overset{\circ}{\varphi}_{\boldsymbol{\lambda}}(a)$ is restriction of $\varphi_{\boldsymbol{\lambda}}(g)$ onto A, then

$$(\overset{\circ}{\Delta} \overset{\circ}{\varphi}_{\boldsymbol{\lambda}})(a) = \alpha_{\boldsymbol{\lambda}} \overset{\circ}{\varphi}_{\boldsymbol{\lambda}}(a). \tag{11}$$

17.2.3. Characters of representations as spherical functions. Every group G can be considered as a homogeneous space with the motion group $G \times G$ and with the action $(g_1, g_2) \circ g = g_1 g g_2^{-1}$. The subgroup $H = \{(g, g) \mid g \in G\} \sim G$ is the stationary subgroup of the point e. The homogeneous space $X = G$ can be imbedded into $G \times G$ as $X = \{(g, g^{-1}) \mid g \in G\}$.

Let G be a compact Lie group. Let us find irreducible representations of the group $G \times G$ having class 1 with respect to the subgroup $H = G$. Every irreducible representation T of $G \times G$ is defined by two irreducible representations T_1 and T_2 of G and is given by the formula

$$T(g_1, g_2) C = T_1(g_1) C T_2(g_2^{-1}), \tag{1}$$

where $T_1(g)$ and $T_2(g)$ are matrices of the corresponding representations and $C \in \mathfrak{M}_{pq}(\mathbb{C})$, $p = \dim T_1$, $q = \dim T_2$. Since G is compact, then we may consider that representations T_1 and T_2 are unitary. Then $(C_1, C) = \operatorname{Tr} C_1 C^*$ gives the scalar product in $\mathfrak{M}_{pq}(\mathbb{C})$ invariant for the representation T.

To elements $(g, g) \in H$ there correspond transformations of the form $T(g, g) C = T_1(g) C T_2(g^{-1})$. A matrix C is invariant for these transformations if and only if for all $g \in G$ we have $T_1(g) C = C T_2(g)$. Since T_1 and T_2 are irreducible representations, then it is a case if and only if the representations T_1 and T_2 of G are equivalent. Without losing the generality, we may consider that $T_1(g) = T_2(g)$, $g \in G$. Then it follows from $T_1(g) C = C T_2(g)$ that C is a scalar matrix, $C = \lambda I_p$.

The matrix λI_p is a vector in the space of the representation T invariant with respect to the operators $T(g,g)$, $g \in G$.

Thus, *class* 1 *irreducible representations of the group* $G \times G$ *are of the form* $T(g_1, g_2)C = T_1(g_1)CT_1(g_2^{-1})$, *where* T_1 *is an irreducible representation of the group* G. The zonal spherical functions of such representations are

$$\varphi_T(g_1, g_2) = (T(g_1, g_2)\lambda I_p, \lambda I_p) = |\lambda|^2 \mathrm{Tr}\,(T_1(g_1)T_1^*(g_2)).$$

Normalizing λ by the condition $(\lambda I_p, \lambda I_p) = 1$ we obtain

$$\varphi_T(g_1, g_2) = \frac{1}{\dim T_1} \mathrm{Tr}\,(T_1(g_1)T_1^*(g_2)).$$

This function is constant on the cosets with respect to H. Every such coset contains an element of the form (g, e), $g \in G$. Thus, the zonal spherical function $\varphi_T(g, e) \equiv \psi(g)$ is really given on G and is defined by the formula

$$\varphi_T(g, e) = \psi(g) = \frac{1}{\dim T_1} \mathrm{Tr}\,(T_1(g)),$$

that is, it is the character of the representation T_1 of the group G divided by $\dim T_1$.

17.2.4. Evaluation of characters of irreducible representations of classical Lie groups. Let G_k be a compact real form of one of the classical complex Lie groups, that is, let G_k be one of the groups $U(n)$, $SO(n)$, $Sp(n)$. Any element $g \in G_k$ is conjugate to a diagonal matrix of G_k and, therefore, can be represented as $g = g_1^{-1}ag_1$, where $a \in A_k$, $g_1 \in G_k$ (A_k is the Cartan subgroup of G_k). For any character χ of G_k we have $\chi(g) = \chi(g_1^{-1}ag_1) = \chi(a)$. Therefore, χ is uniquely defined by its values on the subgroup A_k. Moreover, characters of G_k are W-invariant functions on A_k.

Restricting an irreducible representation T of the group G_k onto the subgroup A_k we obtain a representation of a compact commutative group. The group A_k is isomorphic to m-dimensional torus, $A_k = \mathbf{T}^m$, where m is the rank of the group G_k. Therefore, any irreducible representation λ of A_k is of the form

$$\lambda(a) = \lambda(e^{i\varphi_1}, \dots, e^{i\varphi_m}) = \prod_{k=1}^{m} e^{if_k\varphi_k} = e^{i(\mathbf{f}, \boldsymbol{\varphi})}, \tag{1}$$

where $\mathbf{f} = (f_1, \dots, f_m)$, $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)$. Thus, a character of a representation T of the group G_k is a W-invariant trigonometrical polynomial of the form

$$P(e^{i\varphi_1}, \dots, e^{i\varphi_m}) = \sum_{\mathbf{f}} a_{\mathbf{f}} e^{i(\mathbf{f}, \boldsymbol{\varphi})}, \tag{2}$$

where coefficients are non-negative and only finite number of coefficients are non-vanishing. These coefficients are multiplicities of irreducible representations (1) of A_k in $T\downarrow_{A_k}^{G_k}$.

Taking into account that characters of irreducible representations of a compact group are orthonormalized (see Section 2.3.11) we obtain that evaluation of characters of irreducible representations of the group G_k is reduced to finding orthonormal functions on G_k which are constant on classes of conjugate elements and such that their restrictions onto A_k are orthogonal W-invariant functions of the form (2). Let us first solve this problem for the group $G_k = U(m)$.

Let $\mathbf{f} = (f_1, \dots, f_m)$, $f_1 \geq \dots \geq f_m$, $\mathbf{x} = (x_1, \dots, x_m)$. We assume that either all f_k are integers or all $x_k \geq 0$. We set $\boldsymbol{\rho} = (m-1, \dots, 1, 0)$ (the half-sum of positive roots) and

$$V_m(\mathbf{f}, \mathbf{x}) = \Delta_m((x_j^{f_k})_{j,k=1}^m). \tag{3}$$

The function $V_m(\mathbf{f}, \mathbf{x})$ is called the *generalized Vandermonde determinant.*

The function $V_m(\mathbf{f}+\boldsymbol{\rho}, \mathbf{x})V_m^{-1}(\mathbf{x})$, where $V_m(\mathbf{x}) \equiv V_m(\boldsymbol{\rho}, \mathbf{x})$ is the Vandermonde determinant, is W-invariant. If all f_k are non-negative integers, then it is a polynomial in $x_1, x_2, \dots, x_m$. Really, at $x_j = x_k$, $j \neq k$, this function vanishes and, therefore, $V_m(\mathbf{f}+\boldsymbol{\rho}, x)$ is divisible by $V_m(\mathbf{x}) = \prod\limits_{j<k}(x_j - x_k)$. Replacing x_k by $e^{i\varphi_k}$ in $V_m(\mathbf{f}+\boldsymbol{\rho}, \mathbf{x})V_m^{-1}(\mathbf{x})$ we obtain the W-invariant trigonometrical polynomial. Thus, the equality

$$X_{\mathbf{f}}(e^{i\boldsymbol{\varphi}}) = V_m(\mathbf{f}+\boldsymbol{\rho}, e^{i\boldsymbol{\varphi}})V_m^{-1}(e^{i\boldsymbol{\varphi}}) \tag{4}$$

defines the W-invariant trigonometrical polynomial on the diagonal subgroup of $U(m)$ which can be continued to give the function $X_{\mathbf{f}}(u)$ on $U(m)$ constant on classes of conjugate elements. Let us prove that these functions are orthogonal on $U(m)$. For evaluation of the integral

$$\int\limits_{U(m)} X_{\mathbf{f}}(u)\overline{X_{\mathbf{g}}(u)}d_*u$$

we apply formulas (15) of Section 15.3.2 and (4). After substitution of expressions for $X_{\mathbf{f}}(u)$ and for the measure from these formulas, we cancel $|V_m(e^{i\boldsymbol{\varphi}})|^2$ and obtain the integral

$$\frac{c}{m!(2\pi)^m}\int\limits_0^{2\pi}\dots\int\limits_0^{2\pi} V_m(\mathbf{f}+\boldsymbol{\rho}, e^{i\boldsymbol{\varphi}})\overline{V_m(\mathbf{g}+\boldsymbol{\rho}, e^{i\boldsymbol{\varphi}})}d\boldsymbol{\varphi}.$$

Since the system of functions $\{e^{ik\boldsymbol{\varphi}}\}$ is orthogonal, then this integral is equal to $c\delta_{\mathbf{fg}}$. If $\mathbf{f} = \mathbf{g} = \mathbf{0}$, then this integral equals c. According to formula (15) of Section 15.3.2, for $\mathbf{f} = \mathbf{g} = \mathbf{0}$ the value of this integral coincides with the volume of $U(m)$, that is with 1. Therefore, we have

$$\int\limits_{U(m)} X_{\mathbf{f}}(u)\overline{X_{\mathbf{g}}(u)}d_*u = \delta_{\mathbf{fg}}. \tag{5}$$

Let us show that the constructed system of trigonometrical polynomials is complete in the space of central functions on $U(m)$. For this it is sufficient to prove that any trigonometrical polynomial is a linear combination of the polynomials $X_{\mathbf{f}}(e^{i\varphi})$. We order lexicographically the monomials $e^{i(\mathbf{f},\varphi)}$. Evidently, the highest term in $X_{\mathbf{f}}(e^{i\varphi})$ is equal to $e^{i(\mathbf{f},\varphi)}$. Therefore, if the highest term of a polynomial $P(e^{i\varphi})$ is $a_{\mathbf{f}}e^{i(\mathbf{f},\varphi)}$, then $P(e^{i\varphi}) = a_{\mathbf{f}}X_{\mathbf{f}}(e^{i\varphi}) + P_1(e^{i\varphi})$, where P_1 is W-invariant polynomial lower than $e^{i(\mathbf{f},\varphi)}$. With the help of induction we show that

$$P(e^{i\varphi}) = \sum_{\mathbf{k}\le\mathbf{f}} a_{\mathbf{k}} X_{\mathbf{k}}(e^{i\varphi}). \tag{6}$$

Moreover, if coefficients in $P(e^{i\varphi})$ are integral, then the coefficients $a_{\mathbf{k}}$ are integral.

We now apply expansion (6) to the characters $\chi_{\mathbf{f}}$ of irreducible representations $T_{\mathbf{f}}$ of the group $U(m)$ (these representations are described in Section 16.1.5). We have

$$\int_{U(m)} |\chi_{\mathbf{f}}(u)|^2 d_*u = 1.$$

On the other hand, using expansion (6) and the orthogonality of the functions $X_{\mathbf{k}}(u)$ on $U(m)$ we have

$$\int_{U(m)} |\chi_{\mathbf{f}}(u)|^2 d_*u = \sum_{\mathbf{k}} |a_{\mathbf{k}}|^2,$$

where $a_{\mathbf{k}}$ are non-negative integers. Therefore, only one coefficient $a_{\mathbf{k}}$ is non-vanishing and it is equal to 1. Applying the realization of the representations $T_{\mathbf{f}}$, given in Section 16.1.5, to diagonal matrices from $U(m)$ we easily find that the highest term in $\chi_{\mathbf{f}}$ is $e^{i(\mathbf{f},\varphi)}$. Thus, $a_{\mathbf{f}} = 1$ and $X_{\mathbf{f}}(u) = \chi_{\mathbf{f}}(u)$.

Consequently, we proved that *the characters of the irreducible representations $T_{\mathbf{f}}$ of the group $U(m)$ are given by the formula*

$$\chi_{\mathbf{f}}(\delta) = V_m(\mathbf{f}+\boldsymbol{\rho}, e^{i\varphi})V_m^{-1}(e^{i\varphi}), \tag{7}$$

where $\delta = \mathrm{diag}\,(e^{i\varphi_1}, \ldots, e^{i\varphi_m})$. Moreover, every expression of the form $V_m(\mathbf{f} + \boldsymbol{\rho}, e^{i\varphi})V_m^{-1}(e^{i\varphi})$, where $f_1 \ge \ldots \ge f_m$ and $f_k \in \mathbb{Z}$, is a character of one of irreducible representations of the group $U(m)$.

Formula (7) can be applied to evaluate the dimension of the representation $T_{\mathbf{f}}$ of the group $U(m)$ (and therefore of the group $GL(m,\mathbb{C})$). Since $\chi_{\mathbf{f}}(e) = \dim T_{\mathbf{f}}$, then we have to find the value of the right hand side of (7) at $\varphi_1 = \ldots = \varphi_m = 0$. For this we put $\varphi_k = k\varphi$, $1 \le k \le m$ and take the limit $\varphi \to 0$. Using the relation $e^{ik\varphi} \sim 1 + ik\varphi$ we obtain

$$\dim T_{\mathbf{f}} = V_m(\mathbf{f}+\boldsymbol{\rho})V_m^{-1}(\boldsymbol{\rho}). \tag{8}$$

The analogous formulas are valid for characters and dimensions of finite dimensional irreducible representations of the groups $Sp(m)$ and $SO(m)$. For the group $Sp(m)$ the Cartan (diagonal) subgroup consists of the matrices

$$\delta = \mathrm{diag}\,(e^{i\varphi_1}, \dots, e^{i\varphi_m}, e^{-i\varphi_m}, \dots, e^{-i\varphi_1}).$$

We set $e^{i\boldsymbol{\varphi}} = (e^{i\varphi_1}, \dots, e^{i\varphi_m})$ and

$$\tilde{V}_m(\mathbf{f}, e^{i\boldsymbol{\varphi}}) = \Delta_m((e^{if_j\varphi_k} - e^{-if_j\varphi_k})_{j,k=1}^m).$$

Then the character of the representation[3] $\tilde{T}_{\mathbf{f}}$ of the group $Sp(m)$ with highest weight $\mathbf{f} = (f_1, \dots, f_m)$ is given by the formula

$$\tilde{\chi}_{\mathbf{f}}(\delta) = \tilde{V}_m(\mathbf{f} + \tilde{\boldsymbol{\rho}}, e^{i\boldsymbol{\varphi}})\tilde{V}_m^{-1}(e^{i\boldsymbol{\varphi}}), \tag{9}$$

where $\tilde{\boldsymbol{\rho}} = (m, m-1, \dots, 1)$ and $\tilde{V}_m(e^{i\boldsymbol{\varphi}}) = \tilde{V}_m(\tilde{\boldsymbol{\rho}}, e^{i\boldsymbol{\varphi}})$.

The dimension of the representation $\tilde{T}_{\mathbf{f}}$ is

$$\dim \tilde{T}_{\mathbf{f}} = \prod_{j=1}^m \frac{f_j + \rho_j}{\rho_j} \prod_{\substack{j,k=1\\ j<k}}^m \frac{(f_j+\rho_j)^2 - (f_k+\rho_k)^2}{\rho_j^2 - \rho_k^2}. \tag{10}$$

For the group $SO(2m)$ we have $\delta = \mathrm{diag}\,(e^{i\varphi_1}, \dots, e^{i\varphi_m}, e^{-i\varphi_m}, \dots, e^{-i\varphi_1})$. We set $\tilde{\mathbf{f}} = (f_1, f_2, \dots, f_{n-1}, |f_m|)$ for $\mathbf{f} = (f_1, \dots, f_{n-1}, f_m)$,

$$\hat{V}_{m0}(\mathbf{f}, e^{i\boldsymbol{\varphi}}) = \Delta_m(e^{if_j\varphi_k} + e^{-if_j\varphi_k})_{j,k=1}^m, \tag{11}$$

$$\hat{V}_{m1}(\mathbf{f}, e^{i\boldsymbol{\varphi}}) = \Delta_m(e^{if_j\varphi_k} - e^{-if_j\varphi_k})_{j,k=1}^m, \tag{11'}$$

and $\hat{\boldsymbol{\rho}} = (m-1, m-2, \dots, 1, 0)$. For characters $\hat{\chi}_{\mathbf{f}}$ of the irreducible representations $\hat{T}_{\mathbf{f}}$ of $SO(2m)$ with highest weights $\mathbf{f}$ we have

$$\hat{\chi}_{\mathbf{f}}(\delta) = \hat{V}_{m0}(\tilde{\mathbf{f}} + \hat{\boldsymbol{\rho}}, e^{i\boldsymbol{\varphi}})\hat{V}_{m0}^{-1}(e^{i\boldsymbol{\varphi}}) \tag{12}$$

if $f_m = 0$ and

$$\hat{\chi}_{\mathbf{f}}(\delta) = \frac{1}{2}[\hat{V}_{m0}(\tilde{\mathbf{f}} + \hat{\boldsymbol{\rho}}, e^{i\boldsymbol{\varphi}}) + (\mathrm{sign}\, f_m)\hat{V}_{m1}(\tilde{\mathbf{f}} + \hat{\boldsymbol{\rho}}, e^{i\boldsymbol{\varphi}})]\hat{V}_{m0}^{-1}(e^{i\boldsymbol{\varphi}}) \tag{12'}$$

if $f_m \neq 0$. The dimension of the representation $\hat{T}_{\mathbf{f}}$ is

$$\dim \hat{T}_{\mathbf{f}} = \frac{2^{m-1}}{(2m-2)!(2m-4)!\dots 4!2!} \prod_{\substack{j,k=1\\ j<k}}^m [(f_j + \hat{\rho}_j)^2 - (f_k + \hat{\rho}_k)^2]. \tag{13}$$

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[3] In this section we use the notations $\tilde{T}_{\mathbf{f}}$, $\hat{T}_{\mathbf{f}}$, $T_{\mathbf{f}}^*$ to differ irreducible representations with highest weight $\mathbf{f}$ of different groups. Here these notations have no relations to contragradient and other types of representations.

For the group $SO(2m+1)$ we have

$$\delta = \mathrm{diag}\,(e^{i\varphi_1}, \dots, e^{i\varphi_m}, 1, e^{-i\varphi_m}, \dots, e^{-i\varphi_1})$$

and set $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)$, $\boldsymbol{\rho}^* = (m-1/2, m-3/2, \dots, 1/2)$. The character $\chi_{\mathbf{f}}^*$ of the irreducible representation $T_{\mathbf{f}}{}^*$ of $SO(2m+1)$ with highest weight $\mathbf{f}$ is given by the formula

$$\chi_{\mathbf{f}}^*(\delta) = \hat{V}_{m1}(\mathbf{f} + \boldsymbol{\rho}^*, e^{i\boldsymbol{\varphi}})\hat{V}_{m1}^{-1}(e^{i\boldsymbol{\varphi}}) \tag{14}$$

and its dimension by the formula

$$\dim T_{\mathbf{f}}{}^* = \frac{2^m \prod\limits_{k=1}^{m} (f_k + \rho_k^*)}{(2m-1)! \dots 3!1!} \prod_{\substack{j,k=1 \\ j<k}}^{m} [(f_j + \rho_j^*)^2 - (f_k + \rho_k^*)^2]. \tag{15}$$

The general formula for characters of irreducible representations of the semisimple compact groups can be derived. It is of the form

$$\chi_{\mathbf{f}}(e^{i\boldsymbol{\varphi}}) = \frac{\sum\limits_{w \in W} (-1)^{\sigma(w)} e^{i(w(\mathbf{f}+\boldsymbol{\rho}), \boldsymbol{\varphi})}}{\sum\limits_{w \in W} (-1)^{\sigma(w)} e^{i(w\boldsymbol{\rho}, \boldsymbol{\varphi})}}, \tag{16}$$

where $\sigma(w)$ is the length of the element $w \in W$.

The formulas for characters, given above, can be used for characters of irreducible finite dimensional representations of classical complex Lie groups. For example, for group $GL(m, \mathbb{C})$

$$\chi_{\mathbf{f}}(\boldsymbol{\delta}) = V_m(\mathbf{f} + \boldsymbol{\rho}, \boldsymbol{\delta}) V_m^{-1}(\boldsymbol{\delta}), \tag{17}$$

where $\boldsymbol{\delta} = (\delta_1, \delta_2, \dots, \delta_m)$.

17.2.5. Identities for characters of irreducible representations of $GL(n, \mathbb{C})$. Decompositions for symmetric powers of some representations of the groups $G = GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C})$ and $GL(n, \mathbb{C})$ are given in Section 16.1.6. It follows from formula (6) of Section 16.1.6 that for $g \in GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C})$, $n_1 \le n_2$, we have

$$\chi_{T,f}(g) \equiv \chi_{T,f}(g_1, g_2) = \sum_{|\mathbf{f}|=f} \chi_{\mathbf{f}}(g_1) \chi_{\hat{\mathbf{f}}}(g_2), \tag{1}$$

where $\chi_{T,f}$ is the character of the representation $\sigma_f(T)$, $T(g_1, g_2)X = g_1 X g_2^t$, of the group G, the sum on the right hand side is over all $\mathbf{f} = (f_1, \dots, f_{n_1})$, $\hat{\mathbf{f}} = (f_1, \dots, f_{n_1}, 0, \dots, 0)$ such that $f_1 \ge \dots \ge f_{n_1} \ge 0$, $f_k \in \mathbb{Z}$, $|\mathbf{f}| = f_1 + \dots + f_{n_1} = f$.

We multiply both sides of (1) by t^f and sum over all integral non-negative values of f. We obtain

$$\sum_{f=0}^{\infty} \chi_{T,f}(g_1, g_2) t^f = \sum_{f=0}^{\infty} t^f \sum_{|\mathbf{f}|=f} \chi_{\mathbf{f}}(g_1)\chi_{\hat{\mathbf{f}}}(g_2). \tag{2}$$

Almost all $g \in GL(n, \mathbb{C})$ are conjugate to diagonal matrices and characters are constant on classes of conjugate elements. Therefore, it is sufficient to evaluate the left hand side of (2) for $g_1 = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{n_1})$ $g_2 = \mathrm{diag}\,(\mu_1, \ldots, \mu_{n_2})$. We take monomials of the form $\prod_{i=1}^{n_1} \prod_{j=1}^{n_2} x_{ij}^{r_{ij}}$, where

$$\sum_{i,j} r_{ij} = f, \quad \sum_{j} r_{ij} = r_i, \quad \sum_{i} r_{ij} = s_j,$$

as a basis of the carrier space of the representation $\sigma_f(T)$. The action of diagonal matrices g_1 and g_2 multiples these monomials by $\lambda_1^{r_1}, \ldots, \lambda_{n_1}^{r_{n_1}} \mu_1^{s_1}, \ldots, \mu_{n_2}^{s_{n_2}}$. Therefore, for such g_1 and g_2 relation (2) takes the form

$$\sum_{f=0}^{\infty} \chi_{T,f}(g_1, g_2) t^f = \sum_{f=0}^{\infty} t^f \sum \lambda_1^{r_1}, \ldots, \lambda_{n_1}^{r_{n_1}} \mu_1^{s_1}, \ldots, \mu_{n_2}^{s_{n_2}},$$

where for the internal sum we have $\sum_{i=1}^{n_1} r_j = \sum_{j=1}^{n_2} s_j = f$. For $|t| < 1$, $|\lambda_i| < 1$, $|\mu_j| < 1$, $1 \le i \le n_1$, $1 \le j \le n_2$, this relation can be written as

$$\sum_{f=0}^{\infty} t^f \sum_{|\mathbf{f}|=f} \chi_{\mathbf{f}}(g_1)\chi_{\hat{\mathbf{f}}}(g_2) = \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1 - t\lambda_i\mu_j)^{-1},$$

where $g_1 = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_{n_1})$ $g_2 = \mathrm{diag}\,(\mu_1, \ldots, \mu_{n_2})$. We replace here $\chi_{\mathbf{f}}$ and $\chi_{\hat{\mathbf{f}}}$ by their expressions (7) of Section 17.2.4 and set $t = 1$. As a result, we obtain the following identity for generalized Vandermonde determinants:

$$\begin{aligned} V_{n_1}(\boldsymbol{\lambda}) V_{n_2}(\boldsymbol{\mu}) \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1 - \lambda_i\mu_j)^{-1} \\ = \sum_{\mathbf{f}} V_{n_1}(\mathbf{f} + \boldsymbol{\rho}_1, \boldsymbol{\lambda}) V_{n_2}(\widehat{\mathbf{f}} + \boldsymbol{\rho}_2, \boldsymbol{\mu}). \end{aligned} \tag{3}$$

In particular, if $n_1 = n_2 = n$, then

$$V_n(\boldsymbol{\lambda}) V_n(\boldsymbol{\mu}) \prod_{i,j=1}^{n} (1 - \lambda_i\mu_j)^{-1} = \sum_{\mathbf{f}} V_n(\mathbf{f} + \boldsymbol{\rho}, \boldsymbol{\lambda}) V_n(\mathbf{f} + \boldsymbol{\rho}, \boldsymbol{\mu}).$$

Applying to this relation the Cauchy lemma, which states that

$$\Delta_n(((1-\lambda_i\mu_j)^{-1})_{i,j=1}^n) = V_n(\boldsymbol{\lambda})V_n(\boldsymbol{\mu}) \prod_{i,j=1}^n (1-\lambda_i\mu_j)^{-1} \tag{3'}$$

(the proof of this lemma can be found in reference [53] of the first volume, Chap. 7, Sect. 6), we obtain the identity

$$\sum_{\mathbf{f}} V_n(\mathbf{f}+\boldsymbol{\rho},\boldsymbol{\lambda})V_n(\mathbf{f}+\boldsymbol{\rho},\boldsymbol{\mu}) = \Delta_n(((1-\lambda_i\mu_j)^{-1})_{ij}). \tag{4}$$

From decomposition (17) of Section 16.1.6 we derive that

$$\chi_{\Delta_1^2,f}(g) = \sum_{|\mathbf{f}|=f} \chi_{2\mathbf{f}}(g), \tag{5}$$

where $\chi_{\Delta_1^2,f}$ is the character of the representation $\sigma_f(\Delta_1^2)$ of the group $GL(n,\mathbb{C})$, $2\mathbf{f} = (2f_1,\dots,2f_n)$, $f_1 \geq \dots \geq f_n \geq 0$, $|\mathbf{f}| = f_1 + \dots + f_n$. Thus,

$$\sum_{f=0}^{\infty} \chi_{\Delta_1^2,f}(g)t^f = \sum_{f=0}^{\infty} t^f \sum_{|\mathbf{f}|=f} \chi_{2\mathbf{f}}(g). \tag{6}$$

Substituting the expression for $\chi_{2\mathbf{f}}(g)$ from formula (7) of Section 17.2.4 and evaluating $\chi_{\Delta_1^2,f}(g)$ for diagonal matrices we derive that

$$\sum_{\mathbf{f}} V_n(2\mathbf{f}+\boldsymbol{\rho},\boldsymbol{\lambda})t^{|\mathbf{f}|} = V_n(\boldsymbol{\lambda}) \prod_{1\leq i\leq j} (1-t\lambda_i\lambda_j)^{-1}, \tag{7}$$

where $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n)$, $|\lambda_i| < 1$, $|t| < 1$. Substituting into the left hand side the expression for $V_n(2\mathbf{f}+\boldsymbol{\rho},\boldsymbol{\lambda})$, setting $t=1$ and using the evident relation

$$\sum_{\mathbf{f}} a_1^{f_1}\dots a_n^{f_n} = (1-a_1)^{-1}(1-a_1a_2)^{-1}\dots(1-a_1a_2\dots a_n)^{-1},$$

we obtain the identity

$$\begin{aligned} \sum_{(i_1,\dots,i_n)} & \frac{\delta(i_1,\dots,i_n)\lambda_{i_1}^{n-1}\lambda_{i_2}^{n-2}\dots\lambda_{i_{n-1}}}{(1-\lambda_{i_1}^2)(1-\lambda_{i_1}^2\lambda_{i_2}^2)\dots(1-\lambda_{i_1}^2\dots\lambda_{i_n}^2)} \\ &= V_n(\boldsymbol{\lambda}) \prod_{1\leq i\leq j} (1-\lambda_i\lambda_j)^{-1}. \end{aligned} \tag{8}$$

Here $(i_1,i_2,\dots,i_n)$ is a permutation of the numbers $(1,2,\dots,n)$ and $\delta(i_1,\dots,i_n) = (-1)^\sigma$, where σ is the evenness of this permutation.

In the same way from decomposition (18) of Section 16.1.6 we obtain for $|\lambda_i| < 1$, $|t| < 1$ the relation

$$\sum_{\mathbf{f}} V_n(\tilde{\mathbf{f}} + \boldsymbol{\rho}, \boldsymbol{\lambda}) t^{|\mathbf{f}|} = V_n(\boldsymbol{\lambda}) \prod_{1 \le i < j} (1 - t\lambda_i \lambda_j)^{-1}, \tag{9}$$

where the sum is over $\mathbf{f} = (f_1, \dots, f_\nu)$, $\nu = [n/2]$, $f_1 \ge f_2 \ge \dots \ge f_\nu$, $\tilde{\mathbf{f}} = (f_1, f_1, \dots, f_\nu, f_\nu)$. It follows from (9) that

$$\sum_{(i_1, \dots, i_n)} \frac{\delta(i_1, \dots, i_n) \lambda_{i_1}^{n-1} \lambda_{i_2}^{n-2} \dots \lambda_{i_{n-1}}}{(1 - \lambda_{i_1}\lambda_{i_2})(1 - \lambda_{i_1}\lambda_{i_2}\lambda_{i_3}\lambda_{i_4}) \dots (1 - \lambda_{i_1} \dots \lambda_{i_{2\nu}})} = V_n(\boldsymbol{\lambda}) \prod_{1 \le i < j} (1 - \lambda_i \lambda_j)^{-1}. \tag{10}$$

Substituting $g_1 = e$, $g_2 = e$ into (3) we obtain the equality

$$\sum_{|\mathbf{f}| = f} V_m(\mathbf{f} + \boldsymbol{\rho}_m) V_n(\hat{\mathbf{f}} + \boldsymbol{\rho}_n) = \frac{(mn + f - 1)!}{(mn - 1)! f!} V_m(\boldsymbol{\rho}_m) V_n(\boldsymbol{\rho}_n), \tag{11}$$

where $n \ge m \ge 0$, $\mathbf{f} = (f_1, \dots, f_m)$, $\widehat{\mathbf{f}} = (f_1, \dots, f_m, 0, \dots, 0)$, $|\mathbf{f}| = f_1 + \dots + f_m$, $\boldsymbol{\rho}_k = (k-1, k-2, \dots, 0)$

Analogously, from (7) and (9) we derive the relations

$$\sum_{|\mathbf{f}| = f} V_n(2\mathbf{f} + \boldsymbol{\rho}) = \frac{(f + \frac{1}{2}n(n+1) - 1)!}{(\frac{1}{2}n(n+1))! f!} V_n(\boldsymbol{\rho}), \tag{12}$$

$$\sum_{|\mathbf{f}| = f} V_n(\tilde{\mathbf{f}} + \boldsymbol{\rho}) = \frac{(f + \frac{1}{2}n(n-1) - 1)!}{(\frac{1}{2}n(n-1))! f!} V_n(\boldsymbol{\rho}), \tag{13}$$

where $\mathbf{f} = (f_1, \dots, f_\nu)$, $\tilde{\mathbf{f}} = (f_1, f_1, \dots, f_\nu, f_\nu)$, $\nu = [n/2]$.

From these formulas we obtain the identities

$$\sum_{\mathbf{f}} V_m(\mathbf{f} + \boldsymbol{\rho}_m) V_n(\hat{\mathbf{f}} + \boldsymbol{\rho}_n) t^{|\mathbf{f}|} = (1 - t)^{-mn} V_m(\boldsymbol{\rho}_m) V_n(\boldsymbol{\rho}_n), \tag{14}$$

$$\sum_{\mathbf{f}} V_n(2\mathbf{f} + \boldsymbol{\rho}) t^{|\mathbf{f}|} = (1 - t)^{-n(n+1)/2} V_n(\boldsymbol{\rho}), \tag{15}$$

$$\sum_{\mathbf{f}} V_n(\tilde{\mathbf{f}} + \boldsymbol{\rho}) t^{|\mathbf{f}|} = (1 - t)^{-n(n-1)/2} V_n(\boldsymbol{\rho}). \tag{16}$$

Identities for determinants follow also from decompositions of representations obtained in Section 16.1.7. The number 1 is one of the eigenvalues of a matrix $\begin{pmatrix} h & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$, $h \in GL(m-1, \mathbb{C})$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{m-1})$. We set $\tilde{\boldsymbol{\lambda}} = (\lambda_1, \ldots, \lambda_{m-1}, 1)$. Due to assertions of Section 16.1.7 we have

$$V_m(\mathbf{f} + \boldsymbol{\rho}, \tilde{\boldsymbol{\lambda}}) = V_m(\tilde{\boldsymbol{\lambda}}) \sum_{\mathbf{g}} V_{m-1}(\mathbf{g} + \boldsymbol{\rho}', \boldsymbol{\lambda}) V_{m-1}^{-1}(\boldsymbol{\lambda}), \tag{17}$$

where $\boldsymbol{\rho} = (m-1, m-2, \ldots, 0)$, $\boldsymbol{\rho}' = (m-2, m-3, \ldots, 0)$ and the sum is over those $\mathbf{g} = (g_1, \ldots, g_{m-1})$, for which

$$f_1 \geq g_1 \geq f_2 \geq g_2 \geq \ldots \geq g_{m-1} \geq f_m.$$

Analogous results are valid for $\hat{V}_{m0}$, $\hat{V}_{m1}$, V_{m0}^*, V_{m1}^*. To write down them it is necessary to use statements of Sections 16.1.7 and 17.2.4.

In conclusion of this section we derive the generating function for characters of irreducible finite dimensional representations of the group $GL(n, \mathbb{C})$ and another expression for these characters. We set $\boldsymbol{\lambda} = \boldsymbol{\varepsilon}$, $\boldsymbol{\mu} = \mathbf{z}$ in formula (3′) and obtain

$$V_n(\boldsymbol{\varepsilon}) V_n(\mathbf{z}) \prod_{k=1}^{n} \varphi^{-1}(z_k) = \Delta_n \left(\left(\frac{1}{1 - \varepsilon_j z_k} \right)_{j,k} \right),$$

where for $a = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ we have

$$\varphi(z) = \prod_{j=1}^{n} (1 - \varepsilon_j z) = \Delta_n(I_n - za) = 1 - q_1 z + \ldots + q_n z^n.$$

Due to the Cauchy-Binet theorem and the relation

$$(1 - \varepsilon z)^{-1} = 1 + \varepsilon z + \ldots + \varepsilon^k z^k + \ldots ,$$

we have

$$V_n(\boldsymbol{\varepsilon}) V_n(\mathbf{z}) \prod_{k=1}^{n} \varphi^{-1}(z_k) = \sum_{\boldsymbol{\ell}} V_n(\boldsymbol{\ell}, \boldsymbol{\varepsilon}) \mathbf{z}^{\boldsymbol{\ell}},$$

where the sum is over all $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots, \ell_n)$, $\ell_1 > \ell_2 > \ldots > \ell_n \geq 0$. Thus,

$$V_n(\mathbf{z}) \prod_{k=1}^{n} \varphi^{-1}(z_k) = \sum_{\boldsymbol{\ell}} \chi_{\mathbf{f}}(\boldsymbol{\varepsilon}) \mathbf{z}^{\boldsymbol{\ell}},$$

where $\boldsymbol{\ell} = \mathbf{f} + \boldsymbol{\rho}$.

We set $\varphi(z)^{-1} = \Delta_n^{-1}(I_n - za) = p_0 + p_1 z + \ldots + p_k z_k + \ldots$, $p_{-1} = p_{-2} = \ldots = 0$. Then

$$\chi_{\mathbf{f}}(\boldsymbol{\varepsilon}) = \Delta_n(p_{\ell_k - \rho_k}(\varepsilon_j))_{j,k=1}^{n} \cdot$$

17.2.6. Evaluation of zonal spherical functions of classical complex Lie groups. To guess formulas for zonal spherical functions of classical complex Lie groups we use the following heuristic method. Let $X = G_k/K$ and $Y = G/K$ be the dual symmetric Riemannian spaces and let $\varphi_{\mathbf{f}}$ and $\varphi_{\boldsymbol{\lambda}}$ be zonal spherical functions of class 1 irreducible representations of the groups G_k and G. We denote by $e^{i\tau_k}$, $1 \le k \le \ell$, the eigenvalues of $g_k \in G_k$ and by $t_1, \dots, t_\ell$ the Cartan coordinates of an element $a \in A \subset G$. Then $\varphi_{\boldsymbol{\lambda}}(g)$ are obtained from $\varphi_{\mathbf{f}}(g_k)$ by the replacement of $f_1, \dots, f_\ell$ by $i\lambda_1, \dots, i\lambda_\ell$ and of $e^{i\tau_k}$, $1 \le k \le \ell$, by e^{t_k}, $1 \le k \le \ell$, respectively.

Zonal spherical functions of the space $(K \times K)/K$ are $\chi_{\mathbf{f}}(k)/\chi_{\mathbf{f}}(e)$, where $\chi_{\mathbf{f}}$ is the character of the irreducible representation $T_{\mathbf{f}}$ of K. Let G_c be a complex semisimple Lie group with a maximal compact subgroup K. Then G_c (considered as a real Lie group) and $K \times K$ are noncompact and compact real forms of the same complex Lie group. These assertions allows us to guess formula for zonal spherical functions $\varphi_{\boldsymbol{\lambda}}$ of the space G_c/K, where G_c is a complex semisimple Lie group. This formula is

$$\varphi_{\boldsymbol{\lambda}}(g) = c\frac{\sum\limits_{w\in W}(-1)^{\sigma(w)}e^{i(\boldsymbol{\lambda}, w\mathbf{t})}}{\prod\limits_{\alpha\in\Delta_+}(\boldsymbol{\alpha}, \boldsymbol{\lambda})\prod\limits_{\alpha\in\Delta_+}\sinh\frac{(\boldsymbol{\alpha},\mathbf{t})}{2}}, \tag{1}$$

where $g = k_1ak_2$, $a \in A$, $k_1, k_2 \in K$, $\mathbf{t} = (t_1, \dots, t_\ell)$ are the Cartan coordinates of a, $\sigma(w)$ is the length of w, c does not depend on $\boldsymbol{\lambda}$ and $\mathbf{t}$ and is defined by the condition $\varphi_{\boldsymbol{\lambda}}(e) = 1$.

For example, if $G_c = SL(n, \mathbb{C})$, then formula (1) gives

$$\varphi_{\boldsymbol{\lambda}}(g) = V_n(i\boldsymbol{\lambda}, e^{\mathbf{t}})V_n^{-1}(e^{\mathbf{t}})V_n^{-1}(i\boldsymbol{\lambda})V_n(\rho). \tag{2}$$

Zonal spherical functions for the group $G_s = K \times \mathfrak{p}$, triple to the groups G_c and $K \times K$, are given by the similar formula. For the representation, given by the formula

$$T_{\boldsymbol{\lambda}}((k, \mathbf{p}_0))f(\mathbf{p}) = e^{i(\boldsymbol{\lambda}, \mathbf{p}_0)}f(k^{-1}\mathbf{p})$$

on the space of functions on the orbit $\mathcal{O}_{\mathbf{p}}$ in $\mathfrak{p}$, the zonal spherical function is

$$\varphi_{\boldsymbol{\lambda}}(\mathbf{t}) = c\frac{\sum\limits_{w\in W} i(-1)^{\sigma(w)}(\boldsymbol{\lambda}, w\mathbf{t})}{\prod\limits_{\alpha\in\Delta_+}(\boldsymbol{\alpha}, \boldsymbol{\lambda})\prod\limits_{\alpha\in\Delta_+}(\boldsymbol{\alpha}, \mathbf{t})}. \tag{3}$$

In particular, if $K = U(n)$ and $\mathfrak{p}$ is the space of Hermitian matrices, then for zonal spherical functions we have the formula

$$\varphi_{\boldsymbol{\lambda}}(\mathbf{t}) = V_n(i\boldsymbol{\lambda}, \mathbf{t})V_n^{-1}(i\boldsymbol{\lambda})V_n^{-1}(\mathbf{t})V_n(\rho). \tag{4}$$

In order to prove the formulas obtained by the heuristic method we have to evaluate integral (10) of Section 17.2.2 and to show the coincidence of results. For the group $GL(n, \mathbb{C})$ this evaluation is fulfilled in the following way. We express $e^{(i\boldsymbol{\lambda}-\boldsymbol{\rho},\mathbf{t})}$ in terms of minors of the matrix $X = u\delta^2 u^*$, $u \in U(n)$, $\delta \in A(n, \mathbb{C})$. Now the integral is reduced to

$$\int\limits_{U(n)} \Delta^{i\boldsymbol{\lambda}-\boldsymbol{\rho}}(\Xi(u))d_*u, \tag{5}$$

where $\Xi(u)$ is the matrix with elements $(\delta^2\mathbf{e}_j, \mathbf{e}_k)$, where $\mathbf{e}_j = u^*\mathbf{f}_j$ and $\mathbf{f}_1, \ldots, \mathbf{f}_n$ is a fixed orthonormal basis. Integration in (5) is over the group $U(n)$, that is over all orthonormal bases. The integral (5) is evaluated with the help of induction in n. For this the elliptic coordinates, defined in Section 15.2.4, are used.

17.2.7. The Green functions. Let G be a semisimple compact Lie group with a Lie algebra $\mathfrak{g}$ and let $G = KA_kK$ be the Cartan decomposition of G. We set $A_k = \exp \mathfrak{a}_k$. Let $\mathbf{t} = (t_1, \ldots, t_n)$ be the Cartan coordinates on A_k and let $T_\mathbf{f}$ be an irreducible unitary representation of G with highest weight $\mathbf{f}$. We set

$$\psi_{\boldsymbol{\ell}}(\mathbf{t}) = \frac{1}{\dim T_\mathbf{f}} \sum_{w \in W} (-1)^{\sigma(w)} \exp(\mathbf{f} + \boldsymbol{\rho}, w\mathbf{t}), \tag{1}$$

where $\boldsymbol{\ell} = \mathbf{f} + \boldsymbol{\rho}$, $\boldsymbol{\rho}$ is the half-sum of the positive roots and W is the Weyl group. The sum

$$G(\mathbf{t}, \boldsymbol{\tau}) = \sum_{\mathbf{f}} \sum_{w \in W} e^{i(w(\mathbf{f}+\boldsymbol{\rho}),\mathbf{t})} \psi_{\mathbf{f}+\boldsymbol{\rho}}(\boldsymbol{\tau}), \quad \boldsymbol{\tau}, \mathbf{t} \in \mathfrak{a}_k, \tag{2}$$

is called the *Green function* for the group G. The summation in (2) is over all highest weights $\mathbf{f}$ of irreducible unitary representations of G. The equality

$$\psi_{\boldsymbol{\ell}}(\mathbf{t}) = (2\pi)^{-n} \int\limits_{D'} \sum_{w \in W} e^{i(w\boldsymbol{\ell},\mathbf{x})} G(\mathbf{x}, \mathbf{t}) d\mathbf{x} \tag{3}$$

is valid, where D' is the fundamental domain generated by the Weyl group W and by shifts by vectors of lattice Γ_1 consisting of the inverse image of the identity element of G under the canonical mapping of $\mathfrak{a}_k$ onto A_k.

Let L be the differential operator which is the product of derivatives in directions of all positive roots of $\mathfrak{g}_c$ (the complexification of $\mathfrak{g}$). For example, for $\mathfrak{g}_c = \mathfrak{gl}(n, \mathbb{C})$ we have

$$L = L_\mathbf{t} = \frac{1}{1!2!\ldots(n-1)!} \prod_{i<j} \left(\frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_j} \right). \tag{4}$$

Then

$$L_{\boldsymbol{\tau}} G(\mathbf{t}, \boldsymbol{\tau}) = \frac{1}{|W|} \sum_{w \in W} \delta(\mathbf{t} - w\boldsymbol{\tau}), \tag{5}$$

where $|W|$ is the order of the group W. Let us note that the function $G(\mathbf{t}, \boldsymbol{\tau})$ is skew-symmetric in $\boldsymbol{\tau}$. Formula (5) is proved by substitution of the expression for $\psi_{\boldsymbol{\ell}}$ and by differentiation and summation with the help of the formula

$$\sum_{\mathbf{m}\in Z^n} e^{(\mathbf{m},\mathbf{t})} = (2\pi)^n \sum_{\mathbf{k}\in Z^n} \delta(\mathbf{t} + 2\pi i\mathbf{k}).$$

Let $\tilde{a}(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ be the function obtained from $G(\mathbf{t} - \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ with the help of alternation in $\mathbf{t}$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$. We set

$$a(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = j(\boldsymbol{\tau}_1)j(\boldsymbol{\tau}_2)j^{-1}(\mathbf{t})\tilde{a}(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2), \tag{6}$$

where $j(\mathbf{t}) = \prod_{\alpha\in\Delta_+} \sin\frac{(\alpha,\mathbf{t})}{2}$. We shall use the notation $\hat{f}(\mathbf{t}) = f(\exp\mathbf{t})$. I.M. Gel'fand (see reference [125] of the second volume) proved the following statement: *Let functions* $f_1(g)$ *and* $f_2(g)$ *belong to the ring* $\mathcal{K} \equiv \mathcal{K}(G, K)$ (*see Section* 17.2.1). *Then*

$$(f_1 * f_2)(\exp\mathbf{t}) = \int_{D'}\int_{D'} a(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)\hat{f}_1(\boldsymbol{\tau}_1)\hat{f}_2(\boldsymbol{\tau}_2)d\boldsymbol{\tau}_1 d\boldsymbol{\tau}_2. \tag{7}$$

The formula for the convolution of functions from the ring $\mathcal{K} \equiv \mathcal{K}(G, K)$ is of an analogous form when G is the corresponding noncompact semisimple Lie group and K is its maximal compact subgroup. In this case zonal spherical functions and Green functions are connected by the formulas

$$G(\mathbf{t}, \boldsymbol{\tau}) = (2\pi)^{-n/2} \int \sum_{w\in W} e^{i(\boldsymbol{\lambda}+\boldsymbol{\rho}, w\mathbf{t})}\psi_{\boldsymbol{\lambda}}(\boldsymbol{\tau})d\boldsymbol{\lambda}, \tag{8}$$

$$\psi_{\boldsymbol{\lambda}}(\mathbf{t}) = (2\pi)^{-n/2} \int \sum_{w\in W} e^{i(\boldsymbol{\lambda}+\boldsymbol{\rho}, w\mathbf{x})} G(\mathbf{t}, \mathbf{x})d\mathbf{x}. \tag{9}$$

The integral in formula (8) is over all $\boldsymbol{\lambda}$ defining class 1 irreducible unitary representations of the group G and in formula (9) over a fundamental domain D in $\mathfrak{a}$ with respect to the action of the Weyl group (for example, over one of the Weyl chambers). The function $\tilde{a}(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is obtained from $G(\mathbf{t} - \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ as before and

$$a(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2) = j_1(\boldsymbol{\tau}_1)j_1(\boldsymbol{\tau}_2)j_1^{-1}(\mathbf{t})\tilde{a}(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2), \tag{10}$$

where $j_1(\mathbf{t}) = \prod_{\alpha\in\Delta_+} \sinh\frac{(\alpha,\mathbf{t})}{2}$. For the convolution we have

$$(f_1 * f_2)(\exp\mathbf{t}) = \int_D\int_D a(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)\hat{f}_1(\boldsymbol{\tau}_1)\hat{f}_2(\boldsymbol{\tau}_2)d\boldsymbol{\tau}_1 d\boldsymbol{\tau}_2. \tag{11}$$

For compact groups the function $G(\mathbf{t},\boldsymbol{\tau})$ is periodic in $\mathbf{t}$. For noncompact groups it has the following properties. We fix $\boldsymbol{\tau}$ and define in the space of variables $\mathbf{t}$ convex hull $\Delta(\boldsymbol{\tau})$ of the points $w\boldsymbol{\tau}$, $w \in W$. In the analogous way we define $\Delta(\mathbf{t})$. Then

(a) $G(\mathbf{t},\boldsymbol{\tau})$, *as a function of* $\mathbf{t}$ *with fixed* $\boldsymbol{\tau}$, *vanishes outside of the domain* $\Delta(\boldsymbol{\tau})$;

(b) $G(\mathbf{t},\boldsymbol{\tau})$, *as a function of* $\boldsymbol{\tau}$ *with fixed* $\mathbf{t}$, *vanishes outside of the domain* $\Delta(\mathbf{t})$.

17.2.8. Spherical transforms. Let G be a connected noncompact semisimple Lie group with a finite center and let K be its maximal compact subgroup. Let $G = NAK$ be the Iwasawa decomposition of G. We defined in Section 15.1.9 the subset A^+ of the subgroup $A = \exp \mathfrak{a}$ and the subset $\mathfrak{a}^+$ of the subalgebra $\mathfrak{a}$. We have $A^+ = \exp \mathfrak{a}^+$. Let Δ and Δ_+ be respectively the sets of restricted roots and of positive restricted roots of the pair $(\mathfrak{g},\mathfrak{a})$. We denote by $\mathfrak{a}'$ the set of all linear functionals on $\mathfrak{a}$ and by $\mathfrak{a}'_R$ the subset $\sum_{\alpha\in\Delta_+} \mathbf{R}\alpha$ of $\mathfrak{a}'$.

Let $T_{\boldsymbol{\lambda}}$ be a class 1 representation of the principal nonunitary series of G and let $\varphi_{\boldsymbol{\lambda}}$ be its zonal spherical function. The transform

$$f \to F(\boldsymbol{\lambda}) = \int_G f(g)\varphi_{\boldsymbol{\lambda}}(g)d_*g, \tag{1}$$

where $f \in \mathcal{K} \equiv \mathcal{K}(G,K)$ is called the *spherical transform* on $\mathcal{K}$. It is possible to show that the inverse transform is given by the formula

$$f(g) = \int F(\boldsymbol{\lambda})\overline{\varphi_{\boldsymbol{\lambda}}(g)}d\mu(\boldsymbol{\lambda}), \tag{2}$$

where the integral is over the set $\hat{G}_0$ of non-equivalent class 1 unitary irreducible representations $T_{\boldsymbol{\lambda}}$ of G and $\mu(\boldsymbol{\lambda})$ is the measure on $\hat{G}_0$ called the *Plancherel spherical measure.* The Plancherel equality

$$\int_G |f(g)|^2 dg = \int_{\hat{G}_0} |F(\boldsymbol{\lambda})|^2 d\mu(\boldsymbol{\lambda}) \tag{3}$$

is valid. Up to a set of zero measure the set $\hat{G}_0$ can be identified with $\{T_{\boldsymbol{\lambda}} \mid i\boldsymbol{\lambda} \in \mathfrak{a}'_R, i\boldsymbol{\lambda}(H) \geq 0 \text{ for all } H \in \mathfrak{a}^+\}$. The Plancherel measure $d\mu(\boldsymbol{\lambda})$ is given by the formula

$$d\mu(\boldsymbol{\lambda}) = |c(\boldsymbol{\lambda})|^{-2} d\boldsymbol{\lambda}, \tag{4}$$

where

$$c(\boldsymbol{\lambda}) = \int_{N_-} \exp(-(\boldsymbol{\lambda}+\boldsymbol{\rho}), H(n))dn. \tag{5}$$

Here $(\mu, H) \equiv \mu(H)$, $\boldsymbol{\rho}$ is the half-sum of roots from Δ_+ and $H(n) \in \mathfrak{a}$ is defined from the decomposition $n = k \exp H(n)\tilde{n}$ of an element $n \in N_-$, where $k \in K$, $\exp H(n) \in A$, $\tilde{n} \in N$. The invariant measure dn in (5) has to be normalized in such way that

$$\int\limits_{N_-} \exp(-\boldsymbol{\rho}, H(n))dn = 1. \tag{6}$$

The function $c(\lambda)$ has the property $|c(w\boldsymbol{\lambda})| = |c(\boldsymbol{\lambda})|$, $w \in W$. It can be represented in the form

$$c(\boldsymbol{\lambda}) = I(i\boldsymbol{\lambda})/I(\boldsymbol{\rho}),$$

where

$$I(\boldsymbol{\lambda}) = \prod_{\boldsymbol{\alpha}\in\Delta_+} \mathrm{B}\left(\frac{m(\boldsymbol{\alpha})}{2}, \frac{m(\boldsymbol{\alpha}/2)}{4} + \frac{(\boldsymbol{\lambda},\boldsymbol{\alpha})}{(\boldsymbol{\alpha},\boldsymbol{\alpha})}\right). \tag{7}$$

Here the product is over all positive restricted roots (without multiplicities), $m(\boldsymbol{\alpha})$ is the multiplicity of the root $\boldsymbol{\alpha}$ and B is the classical beta-function.

For the group $SL(n, \mathbf{R})$ (and therefore for the symmetric space $\mathfrak{X} = SL(n, \mathbf{R})/SO(n)$) the Plancherel measure is of the form

$$d\mu(\boldsymbol{\lambda}) = \pi^{-n(n+1)/2} j_1^{-2} \prod_{1\le p<q\le n} \frac{\lambda_p - \lambda_q}{2} \left(\tanh \pi \frac{\lambda_p - \lambda_q}{2}\right) d\boldsymbol{\lambda}, \tag{8}$$

where

$$j_1 = \prod_{1\le p<q\le n} \mathrm{B}\left(\frac{q-p}{2}, \frac{1}{2}\right). \tag{8'}$$

For the group $Sp(n, \mathbf{R})$ (and for the symmetric space $\mathfrak{X}_2 = Sp(n, \mathbf{R})/U(n)$) we have

$$\begin{aligned} d\mu(\boldsymbol{\lambda}) = \pi^{-n^2} j_2^{-2} \prod_{1\le p<q\le n} \frac{\lambda_p^2 - \lambda_q^2}{4} \left(\tanh \pi \frac{\lambda_p - \lambda_q}{2}\right) \left(\tanh \pi \frac{\lambda_p + \lambda_q}{2}\right) \\ \times \prod_{p=1}^{n} \frac{\lambda_p}{2} \left(\tanh \frac{\pi \lambda_p}{2}\right) d\boldsymbol{\lambda}, \end{aligned} \tag{9}$$

where

$$j_2 = \prod_{1\le p<q\le n} \mathrm{B}\left(\frac{q-p}{2}, \frac{1}{2}\right) \mathrm{B}\left(n+1-\frac{p+q}{2}, \frac{1}{2}\right) \prod_{p=1}^{n} \mathrm{B}\left(\frac{n-p+1}{2}, \frac{1}{2}\right). \tag{9'}$$

For the group $SO(n, n)$ the Plancherel measure is of the form

$$\begin{aligned} d\mu(\boldsymbol{\lambda}) = \pi^{-n(n-1)} j_3^{-2} \prod_{1\le p<q\le n} \frac{\lambda_p^2 - \lambda_q^2}{4} \left(\tanh \pi \frac{\lambda_p - \lambda_q}{2}\right) \\ \times \left(\tanh \pi \frac{\lambda_p + \lambda_q}{2}\right) d\boldsymbol{\lambda}, \end{aligned} \tag{10}$$

where

$$j_3 = \prod_{1 \le p < q \le n} \mathrm{B}\left(\frac{q-p}{2}, \frac{1}{2}\right) \mathrm{B}\left(n - \frac{p+q}{2}, \frac{1}{2}\right). \tag{10'}$$

Formulas (8)-(10) are the particular cases of the general formula

$$d\mu(\boldsymbol{\lambda}) = \mathrm{const} \prod_{\alpha \in \Delta_+} \frac{(\boldsymbol{\alpha}, \boldsymbol{\lambda})}{2} \left(\tanh \pi \frac{(\boldsymbol{\alpha}, \boldsymbol{\lambda})}{2}\right) d\boldsymbol{\lambda}. \tag{11}$$

The function $c(\boldsymbol{\lambda})$ is related to an asymptotic behavior of zonal spherical functions. We say that element $a = \exp H \in A$ tends to infinity, $a \to +\infty$, if $(\boldsymbol{\alpha}, H) \to +\infty$ for all $\boldsymbol{\alpha} \in \Delta_+$. If $a \to +\infty$, then for the zonal spherical functions $\varphi_{\boldsymbol{\lambda}}$ we have

$$\varphi_{\boldsymbol{\lambda}}(a) \sim \exp(-\boldsymbol{\rho}, H) \sum_{w \in W} c(w\boldsymbol{\lambda}) \exp i(\boldsymbol{\lambda}, wH). \tag{12}$$

Proofs of the statements formulated in this section can be found, for example, in the reference [152] of the second volume.

17.2.9. Average values and Laplace operators. Let $\mathfrak{X} = G/K$ be a symmetric Riemannian space with a noncompact connected semisimple motion group G and let $y = gK \in \mathfrak{X}$. There exists the element $a \in A$ such that $g = k_1 a k_2$, where $k_1, k_2 \in K$. Let $\mathbf{t} = (t_1, \dots, t_n)$ be the Cartan coordinates of a. The numbers $e^{\mathbf{t}} \equiv (e^{t_1}, \dots, e^{t_n})$ are called the *composite distance* from the initial point $x_0 = K \in \mathfrak{X}$ to y. The orbit KaK of the element $y \in \mathfrak{X}$ is called the *K-sphere* of composite radius $e^{\mathbf{t}}$ with the center in the point x_0. (Let us note that the numbers $t_1, t_2, \dots, t_n$ are determined up to their permutations.) If the motion $g \in G$ transforms the point x_0 into x, then this sphere is transformed into the sphere with the center in the point x. We denote the last sphere by $S(\mathbf{t}, x)$. The invariant measure on K defines the normalized invariant measure μ on $S(\mathbf{t}, x)$.

The operator

$$V(\mathbf{t}, x)f = \int_{S(\mathbf{t},x)} f(y) d\mu(y) \tag{1}$$

is called the *operator of averaging* on the sphere $S(\mathbf{t}, y)$. Let $\Delta_x^{(1)}, \dots, \Delta_x^{(n)}$ be Laplace operators on $\mathfrak{X}$ which are generating elements in the ring of differential operators on $\mathfrak{X}$ commuting with all motions of $\mathfrak{X}$. The operator, induced by $\Delta_x^{(k)}$ in the space $\mathcal{K} \equiv \mathcal{K}(G, K)$, is denoted by $\Delta_{\mathbf{t}}^{(k)}$. The following statement is fulfilled: *A function $\psi(\mathbf{t}, x) = V(\mathbf{t}, x)f$ satisfies the system of differential equations*

$$\Delta_x^{(k)} \psi(\mathbf{t}, x) = \Delta_{\mathbf{t}}^{(k)} \psi(\mathbf{t}, x), \quad 1 \le k \le n. \tag{2}$$

We firstly prove this statement for zonal spherical functions on $\mathfrak{X}$. Fundamental equation (6) of Section 17.2.2 shows that for $\varphi_{\boldsymbol{\lambda}}(x)$ the equation

$$V(\mathbf{t}_1, x)\varphi_{\boldsymbol{\lambda}} = \varphi_{\boldsymbol{\lambda}}(\mathbf{t}_1)\varphi_{\boldsymbol{\lambda}}(\mathbf{t}_2) \tag{3}$$

is fulfilled where $\mathbf{t}_2$ is the composite distance from the initial point x_0 to the point x. The function $\varphi_{\boldsymbol{\lambda}}(\mathbf{t})$ is an eigenfunction for the Laplace operators $\Delta_{\mathbf{t}}^{(k)}$. Therefore,

$$\Delta_x^{(k)} V(\mathbf{t}_1, x)\varphi_{\boldsymbol{\lambda}} = \Delta_x^{(k)} \varphi_{\boldsymbol{\lambda}}(\mathbf{t}_1)\varphi_{\boldsymbol{\lambda}}(\mathbf{t}_2) = \alpha_{\boldsymbol{\lambda}}^{(k)} \varphi_{\boldsymbol{\lambda}}(\mathbf{t}_1)\varphi_{\boldsymbol{\lambda}}(\mathbf{t}_2).$$

We also have

$$\Delta_{\mathbf{t}_1}^{(k)} V(\mathbf{t}_1, x)\varphi_{\boldsymbol{\lambda}} = \Delta_{\mathbf{t}_1}^{(k)} \varphi_{\boldsymbol{\lambda}}(\mathbf{t}_1)\varphi_{\boldsymbol{\lambda}}(\mathbf{t}_2) = \alpha_{\boldsymbol{\lambda}}^{(k)} \varphi_{\boldsymbol{\lambda}}(\mathbf{t}_1)\varphi_{\boldsymbol{\lambda}}(\mathbf{t}_2).$$

Thus, formula (2) is fulfilled for zonal spherical functions.

Using the inversion formula for the spherical transform (see Section 17.2.8) we obtain that any function on $\mathfrak{X}$ can be approximated by zonal spherical functions and their shifts. Since equations (2) are linear and Laplace operators commute with shifts, then these equations are valid for all smooth functions on $\mathfrak{X}$.

One can show that the operators $\Delta_x^{(k)}$ and $\Delta_{\mathbf{t}}^{(k)}$ have the same spectra. Therefore, the zonal spherical functions $\varphi_{\boldsymbol{\lambda}}(x)$ are uniquely defined by their eigenvalues $\alpha^{(1)}, \dots, \alpha^{(n)}$ of the operators $\Delta_x^{(1)}, \dots, \Delta_x^{(n)}$ and we can write

$$\varphi_{\boldsymbol{\lambda}}(x) = \varphi(\mathbf{t}, \alpha^{(1)}, \dots, \alpha^{(n)}).$$

The operators $\Delta_x^{(k)}$, $1 \le k \le n$, commute with each other and the function $\varphi(\mathbf{t}, \alpha^{(1)}, \dots, \alpha^{(n)})$ is defined on their spectrum. Therefore, we may consider the operator $\varphi_{\boldsymbol{\lambda}}(\mathbf{t}, \Delta_x^{(1)}, \dots, \Delta_x^{(n)})$. The following statement is valid: *Let $\varphi(\mathbf{t}, \alpha^{(1)}, \dots, \alpha^{(n)})$ be the zonal spherical function. Then $\varphi(\mathbf{t}, \Delta_x^{(1)}, \dots, \Delta_x^{(n)})$ is the operator of averaging on the sphere $S(\mathbf{t}, x)$.*

Really, let $\mathbf{t}$ be the composite distance from x_0 to x. Applying the operator $\varphi(\mathbf{t}, \Delta_x^{(1)}, \dots, \Delta_x^{(n)})$ to the function $\varphi(\mathbf{t}, \alpha^{(1)}, \dots, \alpha^{(n)})$ we obtain

$$\begin{aligned} &\varphi(\mathbf{t}_1, \Delta_x^{(1)}, \dots, \Delta_x^{(n)})\varphi(\mathbf{t}, \alpha^{(1)}, \dots, \alpha^{(n)}) \\ &= \varphi(\mathbf{t}_1, \alpha^{(1)}, \dots, \alpha^{(n)})\varphi(\mathbf{t}, \alpha^{(1)}, \dots, \alpha^{(n)}). \end{aligned}$$

Due to formula (3), the right hand side of this relation is equal to $V(\mathbf{t}_1, x)\varphi(\mathbf{t}, \alpha^{(1)}, \dots, \alpha^{(n)})$. Therefore,

$$\varphi(\mathbf{t}, \Delta_x^{(1)}, \dots, \Delta_x^{(n)})\varphi_{\boldsymbol{\lambda}} = V(\mathbf{t}, x)\varphi_{\boldsymbol{\lambda}}$$

(we replaced $\mathbf{t}_1$ by $\mathbf{t}$).

17.2.10. The algebra of representations. The duality between characters of irreducible representations of a compact semisimple Lie group and zonal spherical functions of class 1 irreducible representations of its complexification is connected with the duality between the ring $\mathcal{K} \equiv \mathcal{K}(G, K)$ and the ring of functions on the algebra of representations for a compact group.

Let G be a compact semisimple Lie group and let Ψ be the set of representations of G, considered up to an equivalence. We define in Ψ the algebraic operations, taking the direct sum $T_1 + T_2$ and the tensor product $T_1 \otimes T_2$ of representations as an addition and a multiplication. It is easy to verify that this operations are associative and commutative. We also have the distributivity property $T \otimes (T_1 + T_2) = T \otimes T_1 + T \otimes T_2$. The trivial representation T_0 is the unit element in the algebra obtained: $T \otimes T_0 = T_0 \otimes T = T$.

The operation $T \to \hat{T}$, where $\hat{T}$ is the contragradient representation to T, defines an involution in Ψ. Evidently, that

$$\hat{\hat{T}} = T, \quad (T_1 + T_2)^\wedge = \hat{T}_1 + \hat{T}_2, \quad (T_1 \otimes T_2)^\wedge = \hat{T}_1 \otimes \hat{T}_2$$

and that there is the trivial representation T_0 in decompositions of all tensor products $T \otimes \hat{T}$. It is possible to prove that if T_1 and T_2 are irreducible representations, then $T_1 \otimes T_2$ contains the trivial representation T_0 if and only if $T_2 = \hat{T}_1$. Moreover, if T is irreducible, then $T \otimes \hat{T}$ contains T_0 with the unit multiplicity. Therefore, if T is irreducible, then the multiplicity of T_0 in $\hat{T} \otimes T_1$ is equal to the multiplicity of T in T_1.

Let Ξ be the ring of complex additive functions ξ on Ψ, that is, such that $\xi(T) = \xi(T_1) + \xi(T_2)$ if $T = T_1 + T_2$. These functions are uniquely determined by their values on irreducible representations of G. The set of non-equivalent unitary irreducible representations of G is denoted by $\hat{G}$.

Below T means an irreducible representation of G. We define a multiplication in Ξ setting

$$(\xi_1 \cdot \xi_2)(T) = \sum_{T_1 \in \hat{G}} \xi_1(\hat{T}_1 \otimes T)\xi(T_1). \tag{1}$$

We also define a norm

$$\|\xi\| = \sum_{T \in \hat{G}} |\xi(T)| \dim T \tag{2}$$

in Ξ. It is easy to see that $\|\xi\|$ satisfies the usual properties of a norm, that is,

(a) $\|\xi\| \geq 0$, and if $\|\xi\| = 0$ then $\xi \equiv 0$,

(b) $\|\xi_1 + \xi_2\| \leq \|\xi_1\| + \|\xi_2\|$,

(c) $\|\xi_1 \cdot \xi_2\| \leq \|\xi_1\| \cdot \|\xi_2\|$.

Let $(T_1, T_2; T)$ be the multiplicity of T in $T_1 \otimes T_2$. We have

$$(T_1, T_2; T) = \delta(\hat{T}_1 \otimes T)\delta(\hat{T}_2 \otimes T), \tag{3}$$

where the additive function δ is defined by the values $\delta(T) = 1$ for $T = T_0$ and $\delta(T) = 0$ otherwise. Simple transformations show that the multiplication in Ξ is given by the formula

$$(\xi_1 \cdot \xi_2)(T) = \sum_{T_1, T_2 \in \hat{G}} \xi_1(T_1)\xi_2(T_2)(T_1, T_2; T). \tag{4}$$

We saw that homomorphisms of the ring $\mathcal{K}$ into $\mathbb{C}$ are given by zonal spherical functions. In the same way, homomorphisms of the ring Ξ into $\mathbb{C}$ are given by characters of irreducible representations of G. Really, since characters are constant on classes of conjugate elements, we denote them by $\chi(T, \mathbf{t})$, where $\mathbf{t}$ parametrizes these classes. With every function $\xi \in \Xi$ and with every $\mathbf{t}$ we associate the number

$$M(\xi, \mathbf{t}) = \sum_{T \in \hat{G}} \xi(T)\chi(T, \mathbf{t}). \tag{5}$$

Let us firstly show that for any fixed $\mathbf{t}$ the mapping $\xi \to M(\xi, \mathbf{t})$ is a homomorphism of Ξ into $\mathbb{C}$. Really,

$$\begin{aligned} M(\xi_1 \cdot \xi_2, \mathbf{t}) &= \sum_T (\xi_1 \cdot \xi_2)(T)\chi(T, \mathbf{t}) \\ &= \sum_T \chi(T, \mathbf{t}) \sum_{T_1, T_2 \in \hat{G}} \xi_1(T_1)\xi_2(T_2)(T_1, T_2; T) \\ &= \sum_{T_1, T_2 \in \hat{G}} \xi_1(T_1)\xi_2(T_2) \sum_T \chi(T, \mathbf{t})(T_1, T_2; T). \end{aligned}$$

Due to the definition of $(T_1, T_2; T)$ we have

$$\sum_T \chi(T, \mathbf{t})(T_1, T_2; T) = \chi(T_1 \otimes T_2; \mathbf{t}) = \chi(T_1, \mathbf{t})\chi(T_2, \mathbf{t})$$

and, therefore,

$$M(\xi_1 \cdot \xi_2, \mathbf{t}) = \sum_{T_1, T_2 \in \hat{G}} \xi_1(T_1)\xi_2(T_2)\chi(T_1, \mathbf{t})\chi(T_2, \mathbf{t}) = M(\xi_1, \mathbf{t})M(\xi_2, \mathbf{t}).$$

Thus, $\xi \to M(\xi, \mathbf{t})$ is a homomorphism of Ξ into $\mathbb{C}$. Let us show that any homomorphism of Ξ into $\mathbb{C}$ is of such form. If this statement is not true, then there exists a non-vanishing function ξ such that

$$M(\xi, \mathbf{t}) \equiv f(\mathbf{t}) = \sum_T \xi(T)\chi(T, \mathbf{t}) = 0.$$

Due to completeness of the system of characters in the space $\mathfrak{L}_0^2$ of central functions (see Section 2.3.11) we have $\|f\|^2 = \sum_T |\xi(T)|^2$. Therefore, $\xi(T) \equiv 0$ and every homomorphism of Ξ into $\mathbb{C}$ is of the form $\xi \to M(\xi, \mathbf{t})$.

Characters of irreducible representations of compact semisimple Lie groups G are given by signatures (highest weights) $\mathbf{f} = (f_1, \dots, f_n)$ of the corresponding representations. We set $\boldsymbol{\ell} = \mathbf{f} + \boldsymbol{\rho}$, where $\boldsymbol{\rho}$ is the half-sum of positive roots. Classes of conjugate elements of G are parametrized by real numbers $t_1, \dots, t_n$

and consist of elements conjugate to elements $a(\mathbf{t}) = a(t_1, \ldots, t_n) \in A_k^+$. For example, if $G = U(n)$, then $a(\mathbf{t})$ are the matrices diag $(e^{it_1}, \ldots, e^{it_n})$, $t_1 \geq \ldots \geq t_n$. The parameters $\mathbf{t} = (t_1, \ldots, t_n)$ runs over dominant Weyl chamber D. If an irreducible representations are given by signatures $\mathbf{f}$ and classes of conjugate elements by parameters $\mathbf{t} = (t_1, \ldots, t_n)$, then instead of $\chi(T, \mathbf{t})$ we write $\chi(\boldsymbol{\ell}, \mathbf{t})$, $\boldsymbol{\ell} = \mathbf{f} + \boldsymbol{\rho}$.

We define the function

$$G(\boldsymbol{\lambda}, \boldsymbol{\ell}) = \frac{1}{(2\pi)^n} \int_D e^{-i(\boldsymbol{\lambda}, \mathbf{t})} \chi(\boldsymbol{\ell}, \mathbf{t}) d\mathbf{t}.$$

Then

$$\chi(\boldsymbol{\ell}, \mathbf{t}) = \sum_{\boldsymbol{\lambda}} G(\boldsymbol{\lambda}, \boldsymbol{\ell}) e^{i(\boldsymbol{\lambda}, \mathbf{t})}.$$

It is seen from here that $G(\boldsymbol{\lambda}, \boldsymbol{\ell})$ is equal to the multiplicity of the weight $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ in the representation $T_{\mathbf{f}}$.

Let us define the difference operator $\hat{\Lambda}$ dual to the operator L from Section 17.2.7. For this we define the operator $\exp\left(h\frac{\partial}{\partial x}\right)$:

$$\exp\left(h\frac{\partial}{\partial x}\right) f(x) = f(x+h)$$

and set

$$\hat{\Lambda} \equiv \hat{\Lambda}_{\boldsymbol{\lambda}} = \frac{2^N}{|W|} \prod_{\alpha \in \Delta_+} \sinh\left(\frac{1}{2}\frac{\partial}{\partial \lambda_\alpha}\right),$$

where N is the number of positive roots and $|W|$ is the order of the Weyl group. The operator $\hat{\Lambda}$ has the property

$$\hat{\Lambda}_{\boldsymbol{\lambda}} G(\boldsymbol{\lambda}, \boldsymbol{\ell}) = \frac{2^N}{|W|} \sum_{w \in W} (-1)^{\sigma(w)} \delta(\boldsymbol{\lambda} - w\boldsymbol{\ell}).$$

Therefore, it is invertible in the set of symmetric functions and in the set of skew-symmetric functions. Moreover,

$$(\hat{\Lambda}^{-1}\varphi)(\boldsymbol{\ell}) = \sum_{\boldsymbol{\lambda}} G(\boldsymbol{\lambda}, \boldsymbol{\ell}) \varphi(\boldsymbol{\lambda})$$

if φ is a symmetric function and

$$(\hat{\Lambda}^{-1}\varphi)(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\ell}} G(\boldsymbol{\lambda}, \boldsymbol{\ell}) \varphi(\boldsymbol{\ell})$$

if φ is a skew-symmetric function.

Between the functions $G(\boldsymbol{\lambda}, \boldsymbol{\ell})$ and $(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2; \boldsymbol{\ell})$ there is a relationship analogous to the relationship between the functions $G(\mathbf{t}, \boldsymbol{\tau})$ and $a(\mathbf{t}; \boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ from Section 17.2.7.

Duality between the functions $G(\mathbf{t}, \boldsymbol{\tau})$ and $G(\boldsymbol{\lambda}, \boldsymbol{\ell})$ includes also the fact that they are expressed by similar formulas. Namely, the space of variables $(\mathbf{t}, \boldsymbol{\tau})$ can be splitted into domains inside of which $G(\mathbf{t}, \boldsymbol{\tau})$ is a polynomial of $t_1, \dots, t_n, \tau_1, \dots, \tau_n$. In the same domains the space of variables $(\boldsymbol{\lambda}, \boldsymbol{\ell})$ can be splitted. Inside of these domains $G(\boldsymbol{\lambda}, \boldsymbol{\ell})$ is expressed by the same polynomials in which the powers x^k are replaced by the "symbolic powers" $x^{(k)} = x(x-1)\dots(x-k+1)$.

17.3. Zonal and Intertwining Polynomials

17.3.1. Recurrence formulas. Let $\mathfrak{X} = G_0/K$ and $\mathfrak{Y} = G/K$ be dual symmetric Riemannian spaces with real semisimple connected motion groups, where G is a compact group. Let $\mathfrak{g}_0$ and $\mathfrak{g}$ be the Lie algebras of the Lie groups G_0 and G, and let $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}_0$ and $\mathfrak{g} = \mathfrak{k} + i\mathfrak{p}_0$ be their Cartan decompositions. We denote by $\mathfrak{g}_c$ the complexification of $\mathfrak{g}$ and of $\mathfrak{g}_0$ and by G_c the corresponding Lie group. We choose in $\mathfrak{g}_0$ a maximal abelian Lie subalgebra $\mathfrak{h}_0$ such that $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$ is a maximal abelian subalgebra in $\mathfrak{p}_0$. Then $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. We have $\mathfrak{h}_0 = \mathfrak{a}_0 + \mathfrak{h}_k$, where $\mathfrak{h}_k = \mathfrak{h}_0 \cap \mathfrak{k}$. The set $\mathfrak{h} = \mathfrak{h}_k + i\mathfrak{a}_0$ is a maximal abelian subalgebra in $\mathfrak{g}$. Let $\mathfrak{h}_c$ and $\mathfrak{a}_c$ be the complexifications of the Lie subalgebras $\mathfrak{h}_0$ and $\mathfrak{a}_0$. We denote by Σ the set of all roots of the complex Lie algebra $\mathfrak{g}_c$ with respect to the Cartan subalgebra $\mathfrak{h}_c$ and by Δ the set of restricted roots of the pair $(\mathfrak{g}_0, \mathfrak{a}_0)$. The sets of positive roots from Σ and Δ are denoted by Σ_+ and Δ_+ respectively. With the help of the Killing form $B(\cdot\ ,\cdot)$ we shall identify the roots and linear forms on $\mathfrak{h}_c$ (and on $\mathfrak{a}_0$) with elements from $\mathfrak{h}_c$ (from $\mathfrak{a}_c$). Let $n = \dim \mathfrak{h}_c$ and $\ell = \dim \mathfrak{a}_0$. Then n is the rank of the complex Lie algebra $\mathfrak{g}_c$ and ℓ is the real rank of the real Lie algebra $\mathfrak{g}_0$.

Since $\mathfrak{g}_c = \mathfrak{g}_0 + i\mathfrak{g}_0$, then the formula $\sigma(X+iY) = X - iY$, $X, Y \in \mathfrak{g}_0$, defines an involutive automorphism of $\mathfrak{g}_c$. The formula $\tilde{\mathbf{\Lambda}} = (\mathbf{\Lambda} + \sigma\mathbf{\Lambda})/2$ gives the restriction of linear forms, defined on $\mathfrak{h}_0$, onto $\mathfrak{a}_0$. Let $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n$ be the system of simple roots in Σ_+. We may renumerate the roots $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n$ in such a way that

$$\sigma\boldsymbol{\alpha}_i = \begin{cases} \boldsymbol{\alpha}_k - \sum\limits_{j=m+1}^{n} n_{ij}\boldsymbol{\alpha}_j & \text{if} \quad 1 \le i \le m, \\ -\boldsymbol{\alpha}_i & \text{if} \quad m+1 \le i \le n, \end{cases} \tag{1}$$

where

$$k = \begin{cases} i & \text{if} \quad 1 \le i \le \ell_1, \\ i + \ell_2 & \text{if} \quad \ell_1 + 1 \le i \le \ell_1 + \ell_2 = \ell, \\ i - \ell_2 & \text{if} \quad \ell_1 + \ell_2 + 1 \le i \le \ell_1 + 2\ell_2 = m. \end{cases} \tag{1'}$$

The formulas (1) and (1′) uniquely define the positive integers m, ℓ_1 and ℓ_2, where $\ell_1 + \ell_2 = \ell$ is the real rank of $\mathfrak{g}_0$. The restricted roots $\tilde{\boldsymbol{\alpha}}_1, \dots, \tilde{\boldsymbol{\alpha}}_\ell$ form the system of simple roots in Δ_+.

The groups G_0 and G have the same irreducible finite dimensional representations. Their analytical continuations give complex-analytical irreducible finite dimensional representations of G_c. Highest weights $\mathbf{\Lambda}$ of these representations are dominant integral forms on $\mathfrak{h}_c$, that is, such that the numbers $2B(\mathbf{\Lambda}, \boldsymbol{\alpha}_i)/B(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i)$, $i = 1, 2, \ldots, n$, are non-negative integers. Let $\hat{\mathbf{\Lambda}}_i$, $i = 1, 2, \ldots, n$, be the highest weights for which $2B(\hat{\mathbf{\Lambda}}, \boldsymbol{\alpha}_j)/B(\boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j) = \delta_{ij}$. They are called the *fundamental weights.* The corresponding irreducible representations of G_c (and of G) are called the *fundamental representations.* A linear form $\mathbf{\Lambda}$ is a highest weight of some irreducible finite dimensional representation if and only if

$$\mathbf{\Lambda} = \sum_{i=1}^{n} m_i \hat{\mathbf{\Lambda}}, \quad m_i \in \mathbb{Z}_+ \cup \{0\}. \tag{2}$$

It is easy to prove that an irreducible finite dimensional representation $T_{\mathbf{\Lambda}}$ of G is of class 1 with respect to the subgroup K if and only if its highest weight (2) satisfies the condition $\mathbf{\Lambda} = \tilde{\mathbf{\Lambda}}$ (that is $\mathbf{\Lambda} = \sigma\mathbf{\Lambda}$). We set

$$\boldsymbol{\mu}_i = \begin{cases} \hat{\mathbf{\Lambda}}_i & \text{if} \quad 1 \le i \le \ell_1 \quad \text{and} \quad \sigma\boldsymbol{\alpha}_i \ne \boldsymbol{\alpha}_i, \\ 2\hat{\mathbf{\Lambda}}_i & \text{if} \quad 1 \le i \le \ell_1 \quad \text{and} \quad \sigma\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_i, \\ \hat{\mathbf{\Lambda}}_i + \hat{\mathbf{\Lambda}}_{i+\ell_2} & \text{if} \quad \ell_1 + 1 \le i \le \ell_1 + \ell_2 = \ell. \end{cases} \tag{3}$$

Then the irreducible representations $T_{\mathbf{\Lambda}}$ of G with highest weights $\mathbf{\Lambda} = \sum_{i=1}^{\ell} m_i \boldsymbol{\mu}_i$ and only they are of class 1 with respect to K. The proof of this statement see in [305]. The set of these representations is denoted by $\hat{G}_0$.

Let $T_{\mathbf{\Lambda}} \in \hat{G}_0$ and let $\mathbf{e}_{\mathbf{\Lambda}}$ be the normalized vector in the carrier space $V_{\mathbf{\Lambda}}$ of $T_{\mathbf{\Lambda}}$ such that $T_{\mathbf{\Lambda}}(k)\mathbf{e}_{\mathbf{\Lambda}} = \mathbf{e}_{\mathbf{\Lambda}}$ for all $k \in K$. Then $\varphi_{\mathbf{\Lambda}}(g) = (T_{\mathbf{\Lambda}}(g)\mathbf{e}_{\mathbf{\Lambda}}, \mathbf{e}_{\mathbf{\Lambda}})$ is the zonal spherical function of $T_{\mathbf{\Lambda}}$. Let $\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_q$ be an orthonormal weight basis of $V_{\mathbf{\Lambda}}$, that is, such that $T_{\mathbf{\Lambda}}(\exp H)\mathbf{f}_i = e^{\mathbf{\Lambda}_i(H)}\mathbf{f}_i$, where $H \in \mathfrak{h}$, $\mathbf{\Lambda}_0 = \mathbf{\Lambda}$. We have

$$\varphi_{\mathbf{\Lambda}}(\exp H) = \sum_{i=0}^{q} |\mathbf{e}_{\mathbf{\Lambda}}, \mathbf{f}_i|^2 \mathbf{e}^{\mathbf{\Lambda}_i(H)}, \quad H \in \mathfrak{h}. \tag{4}$$

It is possible to prove that only those coefficients $c_i = |\mathbf{e}_{\mathbf{\Lambda}}, \mathbf{f}_i|^2$ from (4) are non-vanishing for which $\mathbf{\Lambda}_i = \sum_{k=1}^{\ell} n_k \boldsymbol{\mu}_k$, where $n_k \in \mathbb{Z}$ and $\boldsymbol{\mu}_k$ are given by formula (3). (the proof can be found in the reference [51], volume 1, p. 210, of the first volume.)

In what follows we identify $(m_1, \ldots, m_\ell) \in \mathbb{Z}^\ell$ with the weights $\sum_{i=1}^{\ell} m_i \boldsymbol{\mu}_i$. We also define the *partial ordering* in $\mathbb{Z}^\ell$ setting

$$\mathbf{\Lambda}_1 \preceq \mathbf{\Lambda}_2 \quad \text{if} \quad B(\mathbf{\Lambda}_1, \boldsymbol{\mu}_i) \le B(\mathbf{\Lambda}_2, \boldsymbol{\mu}_i), \quad i = 1, 2, \ldots, \ell.$$

The set of points $(m_1, \dots, m_\ell) \in \mathbb{Z}^\ell$ for which all m_j are non-negative, is denoted by $\mathbb{Z}_+^\ell$. The following properties of our ordering are valid:

(a) *If $\mathbf{\Lambda}_i$ is a weight from the right hand side of* (4), *then* $\Lambda_i \preceq \mathbf{\Lambda}$.

(b) *For any* $\mathbf{\Lambda} \in \mathbb{Z}_+^\ell$ *the set* $\{\boldsymbol{\mu} \in \mathbb{Z}_+^\ell \mid \boldsymbol{\mu} \preceq \mathbf{\Lambda}\}$ *is finite.*

(c) *For any* $\boldsymbol{\mu} \in \mathbb{Z}_+^\ell$ *there is* $w \in W$ *such that* $w\boldsymbol{\mu} \in \mathbb{Z}_+^\ell$.

(d) $\mathbb{Z}_+^\ell$ *coincides with the set of highest weights of class 1 irreducible representations of* G.

(e) *Every* $\boldsymbol{\mu} \in \mathbb{Z}^\ell$ *is a weight of some irreducible representation of* G.

The zonal spherical functions $\varphi_{\mathbf{\Lambda}}$, $\mathbf{\Lambda} \in \mathbb{Z}_+^\ell$, form an orthonormal basis of the ring $\mathcal{K} = \mathcal{K}(G, K)$. Since product of functions from $\mathcal{K}$ belongs to $\mathcal{K}$, then for any two zonal spherical functions $\varphi_{\mathbf{\Lambda}_1}$ and $\varphi_{\mathbf{\Lambda}_2}$ we have

$$\varphi_{\mathbf{\Lambda}_1}(g)\varphi_{\mathbf{\Lambda}_2}(g) = \sum_{\mathbf{\Lambda}} c_{\mathbf{\Lambda}}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)\varphi_{\mathbf{\Lambda}}, \quad c_{\mathbf{\Lambda}}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) \in \mathbb{C}.$$

Using formula (10) of Section 18.2.1 it is easy to prove that $c_{\mathbf{\Lambda}}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2) = \sum_r |K_r(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)|^2$, where $K_r(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)$ are Clebsch-Gordan coefficients of the group G, and that $c_{\mathbf{\Lambda}}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)$ may be non-vanishing only for $\mathbf{\Lambda} \preceq \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2$. Thus, we have

$$\varphi_{\mathbf{\Lambda}_1}(g)\varphi_{\mathbf{\Lambda}_2}(g) = \sum_{\substack{\mathbf{\Lambda} \preceq \mathbf{\Lambda}_1 + \mathbf{\Lambda}_2 \\ \mathbf{\Lambda} \in \mathbb{Z}_+^\ell}} c_{\mathbf{\Lambda}}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)\varphi_{\mathbf{\Lambda}}, \tag{5}$$

where $c_{\mathbf{\Lambda}}(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2)$ are non-negative numbers.

For every $\mathbf{\Lambda} \in \mathbb{Z}_+^\ell$ we have $\overline{\varphi_{\mathbf{\Lambda}}(g)} = \varphi_{\bar{\mathbf{\Lambda}}}(g)$, where $\bar{\mathbf{\Lambda}}$ is the highest weight of the irreducible representation $T_{\bar{\mathbf{\Lambda}}}$ contragradient to $T_{\mathbf{\Lambda}}$. The tensor product $T_{\mathbf{\Lambda}} \otimes T_{\mathbf{\Lambda}_1}$ contains the trivial representation if and only if $\mathbf{\Lambda}_1 = \bar{\mathbf{\Lambda}}$. It follows from here, from the Peter-Weyl theorem and from (5) that

$$\int_G \varphi_{\mathbf{\Lambda}_1}(g)\varphi_{\mathbf{\Lambda}_2}(g)dg = 0 \quad \text{for} \quad \mathbf{\Lambda}_1 \neq \bar{\mathbf{\Lambda}}_2. \tag{6}$$

Let us consider the special case $\mathbf{\Lambda}_1 = \boldsymbol{\mu}_i$ of expansion (5). We have

$$(\varphi_{\boldsymbol{\mu}}\varphi_{\mathbf{\Lambda}}, \varphi_{\boldsymbol{\nu}}) \equiv \int_G \varphi_{\boldsymbol{\mu}}(g)\varphi_{\mathbf{\Lambda}}(g)\overline{\varphi_{\boldsymbol{\nu}}(g)}dg \neq 0$$

only for $\boldsymbol{\nu} \preceq \mathbf{\Lambda} + \boldsymbol{\mu}$. But $(\varphi_{\boldsymbol{\mu}}\varphi_{\mathbf{\Lambda}}, \varphi_{\boldsymbol{\nu}}) = \overline{(\varphi_{\bar{\boldsymbol{\mu}}}\varphi_{\boldsymbol{\nu}}, \varphi_{\mathbf{\Lambda}})}$ and $\mathbf{\Lambda} \preceq \bar{\boldsymbol{\mu}} + \boldsymbol{\nu}$. Thus, the coefficient $c_{\boldsymbol{\nu}}(\boldsymbol{\mu}, \mathbf{\Lambda})$ in expansion (5) for $\varphi_{\boldsymbol{\mu}}\varphi_{\mathbf{\Lambda}}$ is non-vanishing only for $-\bar{\boldsymbol{\mu}} \preceq \boldsymbol{\nu} - \mathbf{\Lambda} \preceq \boldsymbol{\mu}$. The set of such $\boldsymbol{\nu}$ is finite and the number of elements in this set does not depend on $\mathbf{\Lambda}$. Therefore, the following statement is fulfilled: *Let $\boldsymbol{\mu}$ be a fixed unit vector from $\mathbb{Z}_+^\ell$. There is a real-valued function $c_{\boldsymbol{\nu}}(\mathbf{\Lambda})$ defined on $\mathbb{Z}_+^\ell$ such that*

$$\varphi_{\boldsymbol{\mu}}(g)\varphi_{\mathbf{\Lambda}}(g) = \sum_{\substack{-\bar{\boldsymbol{\mu}}\preceq\boldsymbol{\nu}\preceq\boldsymbol{\mu} \\ \mathbf{\Lambda}+\boldsymbol{\nu}\in Z_+^\ell}} c_{\boldsymbol{\nu}}(\mathbf{\Lambda})\varphi_{\mathbf{\Lambda}+\boldsymbol{\nu}}(g), \quad g \in G. \tag{7}$$

The number of terms on the right hand side is independent on $\mathbf{\Lambda}$.

For every $\boldsymbol{\mu} = \boldsymbol{\mu}_i$, $i = 1, 2, \ldots, \ell$, formula (7) can be considered as a recurrence relation for zonal spherical functions.

The formulas, analogous to (7), are valid for the zonal spherical functions $\varphi_{\mathbf{\Lambda}}$ of class 1 (with respect to K) irreducible infinite dimensional representations of the noncompact semisimple real Lie group G_0. In this case $\mathbf{\Lambda}$ runs over continual set and $\varphi_{\boldsymbol{\mu}} = \varphi_{\boldsymbol{\mu}_i}$, $i = 1, 2, \ldots, \ell$, are the zonal spherical functions of the fundamental finite dimensional representations of G_0, which are obtained from the corresponding zonal spherical functions of the compact group G by an analytical continuation of the group parameters. For the noncompact group G_0 the coefficients $c_{\boldsymbol{\nu}}(\mathbf{\Lambda})$ are expressed in terms of function $c(\mathbf{\Lambda})$ from Section 17.2.8 or of its analytical continuations.

17.3.2. Spherical functions as orthogonal polynomials. Let $\mathbf{x}^{\mathbf{\Lambda}}$ denotes the monomial $x_1^{m_1}x_2^{m_2}\ldots x_\ell^{m_\ell}$, where $\mathbf{\Lambda} = \sum_i m_i\boldsymbol{\mu}_i$, $(m_1, \ldots, m_\ell) \in \mathbb{Z}_+^\ell$. We say that a polynomial $P(\mathbf{x})$ is of *degree* $\mathbf{\Lambda}$, $\mathbf{\Lambda} \in \mathbb{Z}_+^\ell$, if

$$P(\mathbf{x}) = \sum_{\substack{\boldsymbol{\nu}\preceq\mathbf{\Lambda} \\ \boldsymbol{\nu}\in Z_+^\ell}} a_{\boldsymbol{\nu}}\mathbf{x}^{\boldsymbol{\nu}}, \quad a_{\mathbf{\Lambda}} \neq 0.$$

Applying formula (5) of Section 17.3.1 we see that

$$P(\boldsymbol{\varphi}) \equiv P(\varphi_{\boldsymbol{\mu}_1}, \ldots, \varphi_{\boldsymbol{\mu}_\ell}) = \sum_{\boldsymbol{\nu}\preceq\mathbf{\Lambda}} b_{\boldsymbol{\nu}}\varphi_{\boldsymbol{\nu}}, \quad b_{\mathbf{\Lambda}} \neq 0. \tag{1}$$

It is clear that if $P(\boldsymbol{\varphi})$ vanishes on G, then P is the zero polynomial. Thus, we can talk about polynomials of $\varphi_i \equiv \varphi_{\boldsymbol{\mu}_i}$, $i = 1, 2, \ldots, \ell$.

Along with the partial ordering $\mathbf{\Lambda} \preceq \mathbf{\Lambda}_1$ we shall also use a *total ordering* $\mathbf{\Lambda} < \mathbf{\Lambda}_1$ in $\mathbb{Z}_+^\ell$ which has the properties:

(a) $\mathbf{\Lambda}_1 < \mathbf{\Lambda}_2$ if $\mathbf{\Lambda}_1 \prec \mathbf{\Lambda}_2$;

(b) the set $\{\boldsymbol{\nu} \in \mathbb{Z}_+^\ell \mid \boldsymbol{\nu} \leq \mathbf{\Lambda}\}$ is finite for every $\mathbf{\Lambda} \in \mathbb{Z}_+^\ell$.

For example, as a total ordering we can take the lexicographical ordering with respect to an orthonormal basis of $\mathfrak{a}_0$ with first element $\mathbf{v}$ such that $B(\mathbf{v}, \boldsymbol{\mu}_i) > 0$, $B(\mathbf{v}, \tilde{\boldsymbol{\alpha}}_i) > 0$, $i = 1, 2, \ldots, \ell$.

Every zonal spherical function $\varphi_{\mathbf{\Lambda}}$ is a polynomial of degree $\mathbf{\Lambda}$ in the functions $\varphi_1, \dots, \varphi_\ell$. Really, this statement is fulfilled for $\mathbf{\Lambda} = 0$. Let us assume that it is true for all $\boldsymbol{\nu}$, $\boldsymbol{\nu} < \mathbf{\Lambda}$. We choose an index i such that $\mathbf{\Lambda} - \boldsymbol{\mu}_i \in \mathbb{Z}_+^\ell$. Then according to formula (5) of Section 17.3.1 we have

$$\varphi_{\mathbf{\Lambda}} = c\varphi_{\boldsymbol{\mu}_i}\varphi_{\mathbf{\Lambda}-\boldsymbol{\mu}_i} + \sum_{\boldsymbol{\nu}<\mathbf{\Lambda}} c_{\boldsymbol{\nu}}\varphi_{\boldsymbol{\nu}}, \quad c \neq 0.$$

Therefore, our statement is fulfilled for $\varphi_{\mathbf{\Lambda}}$.

This statement shows that complex conjugation permutes the spherical functions $\varphi_1, \dots, \varphi_\ell$. Really, let $\varphi_{k'} \equiv \varphi_{\boldsymbol{\mu}'_k} = \bar{\varphi}_k$. Then $\varphi_{k'} = P(\varphi_1, \dots, \varphi_\ell)$ and

$$\varphi_k = \overline{P(\boldsymbol{\varphi})} = \bar{P}(\varphi_{1'}, \dots, \varphi_{\ell'}).$$

If P is of degree $\sum\limits_j n_j\boldsymbol{\mu}_j$, then $\bar{P}$ is of degree $\sum\limits_j n_j\boldsymbol{\mu}'_j$. This is possible only if for some j we have $\boldsymbol{\mu}_k = \boldsymbol{\mu}'_j$ and our assertion is proved.

Now we define the mapping $F\colon i\mathfrak{a}_0 \to \mathbb{C}^\ell$ by setting

$$F\colon iH \to (\varphi_1(\exp iH), \dots, \varphi_\ell(\exp iH)), \quad H \in \mathfrak{a}_0,$$

and denote by Ω the image of $i\mathfrak{a}_0$ under this mapping. We shall show that zonal spherical functions are orthogonal polynomials on Ω with respect to a positive weight function.

Let $\Gamma = \{\boldsymbol{\alpha} \in \Delta_+ \mid 2\boldsymbol{\alpha}\overline{\in}\Delta_+\}$. Then the determinant $\Delta(F)$ of the mapping F is $\Delta(F) = c\prod\limits_{\boldsymbol{\alpha}\in\Gamma} \sin\boldsymbol{\alpha}$. Really, expansion (4) of Section 17.3.1 shows that $\Delta(F)$ is a linear combination of exponentials $e^{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \mathbb{Z}^\ell$, $\boldsymbol{\nu} \preceq \mathbf{\Lambda}_0$, where $\mathbf{\Lambda}_0 = \sum\limits_{i=1}^{\ell} \boldsymbol{\mu}_i$. Since $\Delta(F)$ is W-invariant, then $\Delta F(wH) = \Delta(w)\Delta(F)$, $\Delta(w) = (-1)^{\sigma(w)}$. Therefore, $\Delta(F) = c\sum\limits_w \Delta(w)e^{w\mathbf{\Lambda}_0}$. We put $\mathbf{\Lambda}_1 = \sum\limits_{\boldsymbol{\alpha}\in\Gamma} \boldsymbol{\alpha}$. Let w_i be the reflection from W corresponding to a simple root $\tilde{\boldsymbol{\alpha}}_i$. Then w_i permutes the roots of Γ except for $\tilde{\boldsymbol{\alpha}}_i$ if $2\tilde{\boldsymbol{\alpha}}_i\overline{\in}\Delta_+$ and except for $2\tilde{\boldsymbol{\alpha}}_i$ if $2\tilde{\boldsymbol{\alpha}}_i \in \Delta_+$. Hence, $w_i\mathbf{\Lambda}_1 = \mathbf{\Lambda}_1 - n\tilde{\boldsymbol{\alpha}}_i$, where $n = 2$ or $n = 4$. But

$$w_i\mathbf{\Lambda}_0 = \mathbf{\Lambda}_0 - 2(B(\boldsymbol{\mu}_i, \tilde{\boldsymbol{\alpha}}_i)/(\tilde{\boldsymbol{\alpha}}_i, \tilde{\boldsymbol{\alpha}}_i))\tilde{\boldsymbol{\alpha}}_i.$$

Therefore, $\mathbf{\Lambda}_1 - \mathbf{\Lambda}_0$ is invariant for all reflections w_i, $1 \le i \le \ell$, and, hence, $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_0$. Since $\prod\limits_{\boldsymbol{\alpha}\in\Gamma} \sin\boldsymbol{\alpha}$ is also a skew-symmetric linear combination of $e^{\boldsymbol{\nu}}$, $\boldsymbol{\nu} \in \mathbb{Z}^\ell$, $\boldsymbol{\nu} \preceq \mathbf{\Lambda}_0$, then $\Delta(F) = c\prod\limits_{\boldsymbol{\alpha}\in\Gamma} \sin\boldsymbol{\alpha}$.

We put $A_k = \exp i\mathfrak{a}_0$ and

$$D = \{H \in i\mathfrak{a}_0 \mid \boldsymbol{\alpha}(H) \in \pi i\mathbb{Z} \quad \text{for some} \quad \boldsymbol{\alpha} \in \Delta\},$$

$$E = \{H \in i\mathfrak{a}_0 \mid \alpha(wH - H) \in \pi i\mathbb{Z} \quad \text{for some} \quad \alpha \in \Delta \quad \text{and} \quad w \in W\}.$$

Let A'_k be the complement of $\exp D \cup \exp E$ in A_k. It is easy to show that $F': a \to (\varphi_1(a), \ldots, \varphi_\ell(a))$, $a \in A'_k$, is a regular pq-to-one mapping of A'_k onto a subset Ω' of Ω, where p is the order of the Weyl group W and q is the number of elements in the set $K \cap A_k$.

Let, as before, $\overline{\varphi_{\mu}} = \varphi_{\mu'}$. We assume that

$$\mu'_j = \begin{cases} \mu_{j+k} & \text{for } j = 1, 2, \ldots, k, \\ \mu_{j-k} & \text{for } j = k+1, \ldots, 2k, \\ \mu_j & \text{for } j = 2k+1, \ldots, \ell, \end{cases}$$

where k is defined by the group G. For $(z_1, \ldots, z_\ell) \in \mathbb{Z}^\ell$ we set

$$x_j = \begin{cases} (z_j + z_{j+k})/2 & \text{for } 1 \le j \le k, \\ (z_{j-k} - z_j)/2i & \text{for } k+1 \le j \le 2k, \\ z_j & \text{for } 2k+1 \le j \le \ell. \end{cases}$$

Then $\mathfrak{F}: (z_1, \ldots, z_\ell) \to (x_1, \ldots, x_\ell)$ maps Ω into $\mathbf{R}^\ell$.

We define the function $w(\mathbf{x})$ on $\mathfrak{F}(\Omega)$ setting

$$w(\mathfrak{F}(F(iH))) = \left| \prod_{\alpha \in \Sigma_+} \sin \tilde{\alpha}(H) \prod_{\tilde{\alpha} \in \Gamma} \sin^{-1} \tilde{\alpha}(H) \right|, \quad H \in \mathfrak{a}_0.$$

Then for all continuous functions f on Ω one has

$$\int_{\mathfrak{F}(\Omega)} f(\mathfrak{F}^{-1}(\mathbf{x})) w(\mathbf{x}) d\mathbf{x} = c \int_G f(\varphi_1(u), \ldots, \varphi_\ell(u)) du.$$

It follows from this formula that *zonal spherical functions on G/K can be considered as orthogonal polynomials with respect to the positive weight function w determined on some domain on $\mathbf{R}^\ell$.*

17.3.3. Invariant polynomials. Any matrix $\Lambda \in \mathfrak{P}_m(\mathbf{F})$ can be represented in the form $\Lambda = X^*X$, where $X \in \mathfrak{M}_m(\mathbf{F})$. This decomposition is not unique. We have $X^*X = Y^*Y$ if and only if $Y = UX$, where $U \in U(m, \mathbf{F})$. A function $f(X)$, $X \in \mathfrak{M}_m(\mathbf{F})$, is called *left U-invariant* if for any $U \in U(m, \mathbf{F})$ we have $f(X) = f(UX)$. To every function $\varphi(\Lambda)$, $\Lambda \in \mathfrak{P}_m(\mathbf{F})$, there corresponds a left U-invariant function $f = A\varphi$ on $\mathfrak{M}_m(\mathbf{F})$ defined by the formula $f(X) = \varphi(X^*X)$. Conversely, to every left U-invariant function $f(X)$, $X \in \mathfrak{M}_m(\mathbf{F})$, there corresponds a single function $\varphi(\Lambda)$, $\Lambda \in \mathfrak{P}_m(\mathbf{F})$, defined by the formula $\varphi(X^*X) = f(X)$. We write $\varphi = A^{-1}f$.

If a left U-invariant function $P(X)$ is a homogeneous polynomial of degree $2k$ in elements of the matrix X, then $(A^{-1}P)(\Lambda)$ is a homogeneous polynomial of degree k in elements of the matrix $\Lambda = X^*X$. This follows from the fact that every U-invariant polynomial in coordinates of the vectors $\mathbf{x}, \mathbf{y} \in \mathbf{F}^m$ is a polynomial in real coordinates of the scalar products $(\mathbf{x},\mathbf{x})$, $(\mathbf{y},\mathbf{y})$, $(\mathbf{x},\mathbf{y})$, where $(\mathbf{x},\mathbf{y}) = x_1\bar{y}_1 + \ldots + x_m\bar{y}_m$.

Since $\mathfrak{P}_m(\mathbf{F})$ is an open subset of $\mathfrak{H}_m(\mathbf{F})$, then any polynomial, given on $\mathfrak{P}_m(\mathbf{F})$, can be uniquely continued onto $\mathfrak{H}_m(\mathbf{F})$. Therefore, the operator A establishes a one-to-one correspondence between the space $\mathcal{J}_{2k}(\mathbf{F})$ of left invariant homogeneous polynomials of degree $2k$ on $\mathfrak{M}_m(\mathbf{F})$ and the space $\mathcal{P}_k(\mathbf{F})$ of homogeneous polynomials of degree k on $\mathfrak{H}_m(\mathbf{F})$. This operator intertwines the representations

$$(Q_k(g)f)(X) = f(Xg), \tag{1}$$

$$(S_k(g)\varphi)(\Lambda) = \varphi(g^*\Lambda g) \tag{2}$$

of the group $GL(m,\mathbf{F})$ on the spaces $\mathcal{J}_{2k}(\mathbf{F})$ and $\mathcal{P}_k(\mathbf{F})$ respectively and realizes their equivalence. Decompositions of the representations S_k into irreducible components for $\mathbf{F} = \mathbb{C}$ were given in Section 16.1.6. For $\mathbf{F} = \mathbf{R}, \mathbb{C}, \mathbf{F}$ these decompositions are of the form

$$S_k(g) \sim Q_k(g) = \sum_{\mathbf{k}} T_{2\mathbf{k}}(g), \quad g \in GL(m,\mathbf{F}), \tag{3}$$

where $T_{2\mathbf{k}}$ are irreducible representations of the group $G = GL(m,\mathbf{F})$ of signature $2\mathbf{k}$ and the sum is over all $\mathbf{k} = (k_1,\ldots,k_m)$ $k_1 \geq \ldots \geq k_m \geq 0$ such that $k_1+\ldots+k_m = k$. For $\mathbf{F} = \mathbf{R}$ the signature $2\mathbf{k}$ denotes $(2k_1,\ldots,2k_m)$, for $\mathbf{F} = \mathbb{C}$ it denotes $(\overline{k_1,\ldots,k_m}; k_1,\ldots,k_m)$ and for $\mathbf{F} = \mathbf{H}$ it is $(k_1,k_1,k_2,k_2,\ldots,k_m,k_m)$.

The carrier space $\mathfrak{L}_{2\mathbf{k}}$, $\mathfrak{L}_{2\mathbf{k}} \subset \mathcal{P}_k(\mathbf{F})$, of the representation $T_{2\mathbf{k}}$ has the highest weight vector which coincides with the homogeneous polynomial

$$\Delta^{\mathbf{k}}(\Lambda) = \Delta_1^{k_1-k_2}\Delta_2^{k_2-k_3}\ldots\Delta_m^{k_m}, \tag{4}$$

where $\mathbf{k} = (k_1,\ldots,k_m)$ and $\Delta_1,\ldots,\Delta_m$ are the main minors of the matrix Λ for $\mathbf{F} = \mathbf{R}, \mathbb{C}$ and the square roots of these minors for $\mathbf{F} = \mathbf{H}$. The space $\mathfrak{L}_{2\mathbf{k}}$ is generated by shifts of the polynomial $\Delta^{\mathbf{k}}(\Lambda)$, that is, by the polynomials $\Delta^{\mathbf{k}}(\Lambda_g)$, where $\Lambda_g = g^*\Lambda g$, $g \in G = GL(m,\mathbf{F})$. In particular, all polynomials $\Delta^{\mathbf{k}}(U^{-1}\Lambda U)$, $U \in U(m,\mathbf{F})$, belong to $\mathfrak{L}_{2\mathbf{k}}$.

If a polynomial $Q(X)$, $X \in \mathfrak{M}_m(\mathbf{F})$, is right U-invariant, that is $Q(XU) = Q(X)$, $X \in \mathfrak{M}_m(\mathbf{F})$, $U \in U(m,\mathbf{F})$, then $A^{-1}Q$ is U-invariant on $\mathfrak{H}_m(\mathbf{F})$, that is, $(A^{-1}Q)(U^*\Lambda U) = (A^{-1}Q)(\Lambda)$ for $\Lambda \in \mathfrak{H}_m(\mathbf{F})$, $U \in U(m,\mathbf{F})$. Let us prove that in every irreducible invariant subspace $\mathfrak{L}_{2\mathbf{k}} \subset \mathcal{P}_k(\mathbf{F})$ there exists uniquely defined (up to a constant factor) non-vanishing U-invariant polynomial $P_{\mathbf{k}}(\Lambda)$.

Really, the polynomial

$$P_{\mathbf{k}}(\Lambda) = \int\limits_{U(m,\mathsf{F})} \Delta^{\mathbf{k}}(U^*\Lambda U)d_*U \tag{5}$$

obeys the properties indicated and this polynomial belongs to $\mathfrak{L}_{2\mathbf{k}}$. In order to prove that such polynomial is single in $\mathfrak{L}_{2\mathbf{k}}$ let us note that it is determined by its values on the subgroup of diagonal matrices and is invariant with respect to permutations of diagonal elements. But the dimension of the space of homogeneous symmetric polynomials of degree k in variables $\delta_1, \ldots, \delta_m$ is equal to the number of partitions of the positive integer k into m summands, that is, to the number of components in decomposition (3). This means that to every component of decomposition (3) there corresponds unique (up to a constant factor) invariant polynomial.

17.3.4. Zonal spherical polynomials and their properties. We denote by $P_{\mathbf{k}}(\Lambda)$ the single U-invariant polynomial on the space $\mathfrak{H}_m(\mathsf{F})$ belonging to $\mathfrak{L}_{2\mathbf{k}}$ and such that $P_{\mathbf{k}}(I_m) = 1$. It is clear that $(AP_{\mathbf{k}})(X)$ is the U-invariant polynomial on the homogeneous space $U(m,\mathsf{F})\backslash G$ belonging to the carrier space of the representation $T_{2\mathbf{k}}$, that is, it is the zonal spherical polynomial of this space. In this reason $P_{\mathbf{k}}(\Lambda)$ is called a *zonal spherical polynomial* on $\mathfrak{H}_m(\mathsf{F})$.

Let us give some properties of the polynomials $P_{\mathbf{k}}(\Lambda)$. As it was shown in Section 16.1.8, there exists the scalar product in the space $A\mathfrak{L}_{2\mathbf{k}}$ which is invariant with respect to operators $T_{2\mathbf{k}}(U)$, $U \in U(m,\mathsf{F})$. The Peter-Weyl theorem shows that the polynomials $P_{\mathbf{k}}(\Lambda)$ and $P_{\boldsymbol{\ell}}(\Lambda)$, $\mathbf{k} \neq \boldsymbol{\ell}$, are orthogonal with respect to this scalar product, transfered into $\mathcal{P}_k(\mathsf{F})$.

If $\mathsf{F} = \mathbb{C}$, then the zonal spherical polynomial $P_{\mathbf{k}}(\Lambda)$ is the zonal spherical function of the irreducible representation $T_{2\mathbf{k}}$. This representation can be analytically continued to the corresponding representation of the compact group $U(m) \times U(m)$. It was proved in Section 17.2.3 that such zonal spherical function is (up to a factor) the character $\chi_{2\mathbf{k}}$ of this representation. Since $\chi_{2\mathbf{k}}(I_m) = \dim T_{2\mathbf{k}}$, then for $\mathsf{F} = \mathbb{C}$ we have

$$P_{\mathbf{k}}(\Lambda) = (\dim T_{2\mathbf{k}})^{-1}\chi_{2\mathbf{k}}(\Lambda). \tag{1}$$

For $\mathsf{F} = \mathsf{R}, \mathsf{H}$ the explicit expressions for zonal spherical functions are not known.

Since $\operatorname{Tr}(U^*\Lambda U) = \operatorname{Tr}\Lambda$, $U \in U(m,\mathsf{F})$, then $(\operatorname{Tr}\Lambda)^d$ is a U-invariant polynomial of degree d. The function $\operatorname{etr}\Lambda$ is also U-invariant. We have

$$(\operatorname{Tr}\Lambda)^d = \sum_{|\mathbf{k}|=d} \alpha_{\mathbf{k}} P_{\mathbf{k}}(\Lambda), \tag{2}$$

where $|\mathbf{k}| = k_1 + \ldots + k_m$, $\alpha_{\mathbf{k}} = d!\|P_{\mathbf{k}}\|^{-2}$. Really, direct differentiation shows that

$$(D(P_{\mathbf{k}})e_\Lambda)(M) = P_{\mathbf{k}}(\Lambda)e_\Lambda(M), \tag{3}$$

where $e_\Lambda(M) = \mathrm{etr}(\Lambda M)$, $\Lambda, M \in \mathfrak{H}_m(\mathbf{F})$, and $D(P_\mathbf{k}) \equiv P_\mathbf{k}(D)$ (see Section 16.1.8). Therefore,

$$(D(P_\mathbf{k})e_I)(0) = P_\mathbf{k}(I_m) = 1. \tag{4}$$

Since the function e_I is U-invariant, then

$$e_I = \sum_\mathbf{k} \beta_\mathbf{k} P_\mathbf{k}, \tag{5}$$

where the coefficients $\beta_\mathbf{k}$ can be found by the substitution of this expression for e_I into (4). Due to (4) we have

$$\sum_{\mathbf{k}'} \beta_{\mathbf{k}'}(D(P_\mathbf{k})P_{\mathbf{k}'})(0) = \sum_{\mathbf{k}'} \beta_{\mathbf{k}'}(P_{\mathbf{k}'}, P_\mathbf{k}) = 1.$$

Since $(P_\mathbf{k}, P_{\mathbf{k}'}) = 0$ for $\mathbf{k} \neq \mathbf{k}'$, then $\beta_\mathbf{k} = \|P_\mathbf{k}\|^{-2}$. Evaluating both sides of (5) at $\xi\Lambda$, $\xi \in \mathbf{R}$, $\Lambda \in \mathfrak{H}_m(\mathbf{F})$, we obtain the equality

$$\sum_{d=0}^{\infty} \frac{1}{d!}\xi^d (\mathrm{Tr}\ \Lambda)^d = \sum_{d=0}^{\infty} \xi^d \left(\sum_{|\mathbf{k}|=d} \beta_\mathbf{k} P_\mathbf{k}(\Lambda) \right). \tag{6}$$

Comparing coefficients at ξ^d on both sides we obtain relation (2).

We normalized the zonal spherical polynomials $P_\mathbf{k}$ by the condition $P_\mathbf{k}(I_m) = 1$. The normalization, for which all coefficients of the expansion of $(\mathrm{Tr}\ \Lambda)^d$ into the series in $P_\mathbf{k}$ are equal to 1, is more convenient. We set

$$Z_\mathbf{k}(\Lambda) = d!\|P_\mathbf{k}\|^{-2} P_\mathbf{k}(\Lambda), \tag{7}$$

where $d = |\mathbf{k}|$. Then

$$(\mathrm{Tr}\ \Lambda)^d = \sum_{|\mathbf{k}|=d} Z_\mathbf{k}(\Lambda). \tag{8}$$

In particular, it follows from (8) that

$$Z_\mathbf{k}(I_m) = d!\|P_\mathbf{k}\|^{-2} > 0. \tag{9}$$

For $\mathbf{F} = \mathbf{R}$ the value $Z_\mathbf{k}(I_m)$ is

$$Z_\mathbf{k}(I_m) = 2^{2d} d! \left[\frac{m}{2}\right]_\mathbf{k} \prod_{i \le j \le p} (2k_i - 2k_j - i + j) \prod_{i=1}^{p} (2k_i + p - i)!^{-1}, \tag{10}$$

where $p = [m/2]$ and $[a]_\mathbf{k} \equiv [a; m]_\mathbf{k}$ is defined by formula (10) of Section 17.1.4.

Let $Z = Z_{\mathbf{k}}$. We denote the polynomial $Z(X^*X)$, $X \in \mathfrak{M}_m(\mathbf{F})$, by $p_Z(X)$. This polynomial is left and right U-invariant, that is

$$p_Z(U_1XU_2) = p_Z(X), \quad U_1, U_2 \in U(m, \mathbf{F}). \tag{11}$$

The Cartan decomposition shows that it is uniquely defined by values on the Cartan subgroup $A(m, \mathbf{F})$. Therefore, for $X \in \mathfrak{M}_m(\mathbf{F})$ we have

$$p_Z(X^*) = p_Z(X). \tag{12}$$

Thus,

$$p_Z(XY^*) = p_Z(YX^*), \quad X, Y \in \mathfrak{M}_m(\mathbf{F}). \tag{12'}$$

It follows from here that $p_Z(XY^*)$, as a function of X and as a function of Y, is a U-invariant polynomial of degree $2d$. Since $p_Z(X) = Z(X^*X)$, then we obtain the following properties of zonal spherical polynomials:

(a) *The value of $Z_{\mathbf{k}}$ at $\Lambda \in \mathfrak{H}_m(\mathbf{F})$ is uniquely defined by the eigenvalues $\lambda_1, \dots, \lambda_m$ of the matrix Λ.*

(b) *If $\Lambda = X^*X$, $M = Y^*Y$, then*

$$Z_{\mathbf{k}}(Y\Lambda Y^*) = Z_{\mathbf{k}}(XMX^*). \tag{13}$$

We obtain from (13) that

$$Z_{\mathbf{k}}(M^{1/2}\Lambda M^{1/2}) = Z_{\mathbf{k}}(\Lambda^{1/2}M\Lambda^{1/2}). \tag{14}$$

It gives

$$Z_{\mathbf{k}}(M \vartriangle \Lambda) = Z_{\mathbf{k}}(\Lambda \vartriangle M). \tag{14'}$$

The polynomials $Z_{\mathbf{k}}(\Lambda \vartriangle M)$ are extended onto the space $\mathfrak{H}_m(\mathbf{F}) \times \mathfrak{H}_m(\mathbf{F})$ and the functions obtained, as functions in Λ and as functions in M, belong to $\mathcal{P}_k(\mathfrak{H}_m)$. Since the matrices $\Lambda^{1/2}M\Lambda^{1/2}$, $M^{1/2}\Lambda M^{1/2}$, ΛM, $M\Lambda$ have the same eigenvalues, then below we write $Z_{\mathbf{k}}(\Lambda M)$ or $Z_{\mathbf{k}}(M\Lambda)$ instead of $Z_{\mathbf{k}}(\Lambda^{1/2}M\Lambda^{1/2})$. Note that for all $\Lambda, M \in \mathfrak{H}_m(\mathbf{F})$, $P \in \mathfrak{P}_m(\mathbf{F})$ we have

$$Z_{\mathbf{k}}(P^{1/2}\Lambda P^{1/2}M) = Z_{\mathbf{k}}(\Lambda P^{1/2}MP^{1/2})$$

Properties of zonal spherical functions (see Section 17.2.2) show that for all $\Lambda, M \in \mathfrak{H}_m(\mathbf{F})$ the equality

$$\int_{U(m,\mathbf{F})} Z_{\mathbf{k}}(\Lambda U^*MU)d_*U = \frac{Z_{\mathbf{k}}(\Lambda)Z_{\mathbf{k}}(M)}{Z_{\mathbf{k}}(I_m)} \tag{15}$$

is fulfilled.

17.3.5. Integral representations of zonal spherical polynomials. The formula (5) of Section 17.3.3 gives the following integral representation for zonal spherical polynomials $Z_{\mathbf{k}}(\Lambda)$:

$$Z_{\mathbf{k}}(\Lambda) = Z_{\mathbf{k}}(I_m) \int\limits_{U(m,\mathbf{F})} \Delta^{\mathbf{k}}(U^*\Lambda U) d_*U. \tag{1}$$

Formula (7) of Section 17.2.2 leads to another integral representation for $Z_{\mathbf{k}}(\Lambda)$:

$$Z_{\mathbf{k}}(X^*X) = Z_{\mathbf{k}}(I_m) \int\limits_{U(m,\mathbf{F})} \chi_{2\mathbf{k}}(XU) d_*U. \tag{2}$$

Here $\chi_{2\mathbf{k}}$ is the character of the representation $T_{2\mathbf{k}}$ of the group $GL(m, \mathbf{F})$. If $\mathbf{k}$ is a partition of an integer k containing odd terms, then

$$\int\limits_{U(m,\mathbf{F})} \chi_{\mathbf{k}}(XU) d_*U = 0. \tag{3}$$

Let $f(X)$, $X \in GL(m, \mathbf{F})$, be a symmetric function in eigenvalues $\xi_1 \dots, \xi_m$ of a matrix X. Then it can be expanded in the characters $\chi_{\mathbf{k}}$:

$$f(X) = \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} c_{\mathbf{k}} \chi_{\mathbf{k}}(X). \tag{4}$$

Due to formula (2) we obtain from here that

$$\int\limits_{U(m,\mathbf{F})} f(XU) d_*U = \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} c_{2\mathbf{k}} \frac{Z_{\mathbf{k}}(X^*X)}{Z_{\mathbf{k}}(I_m)}. \tag{5}$$

17.3.6. The Laplace transform of zonal polynomials. If M and Z are such that $M \in \mathfrak{H}_m(\mathbf{F})$, Re $Z > 0$, Im $Z \in \mathfrak{H}_m(\mathbf{F})$, then

$$\int\limits_{\Lambda>0} \operatorname{etr}(-\Lambda Z) \Delta^{\alpha}(\Lambda) Z_{\mathbf{k}}(\Lambda M) d_*\Lambda = [\alpha]_{\mathbf{k}} \Gamma_m(\alpha) \Delta^{-\alpha}(Z) Z_{\mathbf{k}}(MZ^{-1}), \tag{1}$$

where Re $\alpha > \frac{1}{2}(m-1)\nu - k_m$ and expressions for $[\alpha]_k$ and $\Gamma_m(\alpha, \mathbf{F})$ are given in Section 17.1.4. Really, denoting the left hand side of this relation by $J(M, Z)$ and using relation (1) of Section 17.3.5 we obtain

$$J(I_m, I_m) = \int\limits_{\Lambda>0} \operatorname{etr}(-\Lambda) \Delta^{\alpha}(\Lambda) Z_{\mathbf{k}}(\Lambda) d_*\Lambda$$

$$= Z_{\mathbf{k}}(I_m) \int\limits_{\Lambda>0} \int\limits_{U(m,\mathbf{F})} \operatorname{etr}(-\Lambda) \Delta^{\alpha}(\Lambda) \Delta^{\mathbf{k}}(U^*\Lambda U) d_*U d_*\Lambda.$$

Let us make the substitution $U^*\Lambda U = M$. Since $U^* = U^{-1}$, $|\Delta(U)| = 1$ and the measure of the group $U(m, \mathbf{F})$ is equal to 1, then

$$\begin{aligned} J(I_m, I_m) &= Z_{\mathbf{k}}(I_m) \int\limits_{\Lambda>0} \operatorname{etr}(-\Lambda)\Delta^{\alpha}(\Lambda)\Delta^{\mathbf{k}}(\Lambda)d_*\Lambda \\ &= Z_{\mathbf{k}}(I_m)[\alpha]_{\mathbf{k}}\Gamma_m(\alpha, \mathbf{F}). \end{aligned} \tag{2}$$

The function $J(M, I_m)$ is invariant with respect to $U(m, \mathbf{F})$ and belongs to the space $\mathcal{P}_k(\mathfrak{H}_m)$. Therefore, $J(M, I_m)$ is multiple of $Z_{\mathbf{k}}(M)$, that is, $J(M, I_m) = cZ_{\mathbf{k}}(M)$. Evidently, $c = J(I_m, I_m)/Z_{\mathbf{k}}(I_m)$ and

$$J(M, I_m) = [\alpha]_{\mathbf{k}}\Gamma_m(\alpha, \mathbf{F})Z_{\mathbf{k}}(M). \tag{3}$$

Let $Z > 0$. Making the substitution $\Lambda \to Z^{-1/2}\Lambda Z^{-1/2}$ in the integral (1) we obtain

$$J(M, Z) = \Delta^{-\alpha}(Z)J(Z^{-1/2}MZ^{-1/2}, I_m)$$

and, therefore,

$$J(M, Z) = [\alpha]_{\mathbf{k}}\Gamma_m(\alpha, \mathbf{F})\Delta^{-\alpha}(Z)Z_{\mathbf{k}}(MZ^{-1}).$$

Continuing analytically this equation in Z we derive formula (1) for Re $Z > 0$, Im $Z \in \mathfrak{H}_m(\mathbf{F})$.

Applying to (1) the inversion formula for the Laplace transform we have

$$\begin{aligned} \Delta^{\alpha}(\Lambda)Z_k(\Lambda M) &= [\alpha]_{\mathbf{k}}\Gamma_m(\alpha, \mathbf{F})(2\pi i)^{-m\theta} \\ &\times \int\limits_{X_0-i\infty}^{X_0+i\infty} \operatorname{etr}(\Lambda Z)\Delta^{-\alpha}(Z)Z_{\mathbf{k}}(MZ^{-1})d_*Z. \end{aligned} \tag{4}$$

Now we prove that

$$\begin{aligned} J(Z) &\equiv \int\limits_0^{I_m} \Delta^{\alpha}(\Lambda)\Delta^{\beta-\alpha-\theta}(I_m - \Lambda)Z_{\mathbf{k}}(\Lambda Z)d_*\Lambda \\ &= \Gamma_m(\alpha)\Gamma_m(\beta-\alpha)\Gamma_m^{-1}(\beta)[\alpha]_{\mathbf{k}}[\beta]_{\mathbf{k}}^{-1}Z_{\mathbf{k}}(Z), \end{aligned} \tag{5}$$

where Re $\alpha > (m-1)\nu/2$, Re $(\beta - \alpha) > (m-1)\nu/2$, $\theta = ((m-1)\nu+2)/2$ and the integral converges absolutely. Really, if $Z \in \mathfrak{H}_m$, then U-invariance of the measure $d_*\Lambda$ on the "interval" $[0, I_m]$ means that $J(Z)$ is a U-invariant polynomial from $A^{-1}\mathfrak{L}_{2\mathbf{k}}$. In this reason $J(Z)$ is multiple of Z_k:

$$J(Z) = J(I_m)Z_{\mathbf{k}}(Z)/Z_{\mathbf{k}}(I_m). \tag{6}$$

We analytically continue this relation onto the complexification $\mathfrak{H}_m^c$ of $\mathfrak{H}_m$. We have to evaluate $J(I_m)$. Due to formulas (1) and (6)

$$\Gamma_m(\beta)[\beta]_{\mathbf{k}} J(I_m) = \left(\int_0^\infty \operatorname{etr}(-\Lambda)\Delta(\Lambda)^\beta Z_{\mathbf{k}}(\Lambda) d_*\Lambda\right) J(I_m) Z_{\mathbf{k}}(I_m)^{-1}$$
$$= \int_0^\infty \operatorname{etr}(-\Lambda)\Delta(\Lambda)^\beta J(\Lambda) d_*\Lambda.$$

Substituting here expression (5) for $J(\Lambda)$ we have

$$[\beta]_{\mathbf{k}}\Gamma_m(\beta) J(I_m) = \int_0^\infty d_*\Lambda \int_0^{I_m} \operatorname{etr}(-\Lambda)\Delta(\Lambda)^\beta \Delta(M)^\alpha$$
$$\times \Delta(I_m - M)^{\beta-\alpha-\theta} Z_{\mathbf{k}}(\Lambda M) d_* M.$$

Then we make the substitution $M \to \Lambda^{-1/2} M \Lambda^{-1/2}$ in the internal integral, the substitution $\Lambda \to \Lambda + M$ in the external one and obtain

$$[\beta]_{\mathbf{k}}\Gamma_m(\beta) J(I_m) = \int_0^\infty d_*\Lambda \int_0^\infty \operatorname{etr}(-\Lambda - M)\Delta(M)^\alpha \Delta(\Lambda)^{\beta-\alpha} Z_{\mathbf{k}}(M) d_* M$$
$$= \int_0^\infty \operatorname{etr}(-\Lambda)\Delta(\Lambda)^{\beta-\alpha} d_*\Lambda \int_0^\infty \operatorname{etr}(-M)\Delta(M)^\alpha Z_{\mathbf{k}}(\Lambda) d_* M$$
$$= [\alpha]_{\mathbf{k}}\Gamma_m(\alpha)\Gamma_m(\beta-\alpha) Z_{\mathbf{k}}(I_m).$$

It leads to formula (5).

17.3.7. Evaluation of zonal spherical polynomials. We have mentioned above that the explicit expression for zonal spherical polynomials at $\mathbf{F} = \mathbf{R}$ is not known. They can be evaluated with the help of recurrence relation for their coefficients. This relation is based on the fact that zonal polynomials are eigenfunctions for the radial parts of the Laplace-Beltrami operator. The general formula for this operator on a Riemannian space X is

$$\Delta^{-1/2}((g_{st})) \sum_{k=1}^n \frac{\partial}{\partial x_k} \Delta^{1/2}((g_{st})) \sum_{i=1}^n g^{ik} \frac{\partial}{\partial x_i}, \tag{1}$$

where (g_{st}) is the metric tensor on X and $(g_{st})^{-1} = (g^{ij})$. For the space $\mathfrak{S}_m(\mathbf{R})$ the invariant metric form is

$$ds^2 = \operatorname{Tr}((X^{-1}dX)^2). \tag{2}$$

We write down this form in the parameters ω_{ij}, $i < j$; y_i, where $X = \Omega^{-1} Y \Omega$, $\Omega \in SO(m)$, $Y = \operatorname{diag}(y_1, \ldots, y_m)$. Simple evaluations show that

$$\Delta^{1/2}((g_{ij})) = \prod_{i=1}^{m} y_i^{-(m+1)/2} \prod_{i<j} (y_i - y_j), \quad g^{ij} = y_i^2 \delta_{ij}. \tag{3}$$

Therefore, in this case the Laplace-Beltrami operator is

$$\sum_{i=1}^{m} \left[y_i^2 \frac{\partial^2}{\partial y_i^2} - \frac{m-3}{2} y_i \frac{\partial}{\partial y_i} + \sum_{\substack{j=1 \\ j \neq i}}^{m} y_i^2 (y_i - y_j)^{-1} \frac{\partial}{\partial y_i} \right]. \tag{4}$$

Due to the Euler formula for homogeneous functions, zonal spherical polynomials are also eigenfunctions of the operator

$$\sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} y_i^2 (y_i - y_j)^{-1} \frac{\partial}{\partial y_i}. \tag{5}$$

The zonal spherical polynomial $Z_{\mathbf{k}}(Y)$ is of the form

$$Z_{\mathbf{k}}(Y) = c_k y_1^{k_1} \ldots y_m^{k_m} + \{\text{terms of lower weights}\}.$$

Applying operator (5) to the highest term $c_k y_1^{k_1} \ldots y_m^{k_m}$ and comparing the coordinates on both sides we derive that the corresponding eigenvalue is $\lambda_{\mathbf{k}} = \sum_{i=1}^{m} k_i(k_i + m - i - 1)$. Therefore, the zonal polynomial $Z_{\mathbf{k}}(Y)$ satisfies the differential equation

$$\left(\sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} y_i^2 (y_i - y_j)^{-1} \frac{\partial}{\partial y_i} \right) Z_{\mathbf{k}}(Y) = \lambda_{\mathbf{k}} Z_{\mathbf{k}}(Y). \tag{6}$$

The operator $\sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2}$ acts upon $y_i^{\ell_1} \ldots y_m^{\ell_m}$ as the multiplication by $\sum_{i=1}^{m} \ell_i(\ell_i - 1)$. Let us find the action formula for the operator $D = \sum_{i \neq j} y_i^2 (y_i - y_j)^{-1} \frac{\partial}{\partial y_i}$. For this we note that

$$\begin{aligned}
&(y_i - y_j)^{-1} \left(y_i^2 \frac{\partial}{\partial y_i} - y_j^2 \frac{\partial}{\partial y_j} \right) (y_i^{\ell_i} y_j^{\ell_j} + y_i^{\ell_j} y_j^{\ell_i}) \\
&\quad = \ell_i (y_i^{\ell_i} y_j^{\ell_j} + \text{symmetric term}) \\
&\qquad + (\ell_i - \ell_j)(y_i^{\ell_i - 1} y_j^{\ell_j + 1} + \text{symmetric term}) \\
&\qquad + (\ell_i - \ell_j)(y_i^{\ell_i - 2} y_j^{\ell_j + 2} + \text{symmetric term}).
\end{aligned}$$

It follows from here that the operator D multiplies the monomial $y_1^{\ell_1}\dots y_m^{\ell_m}$ by $\sum_{i=1}^{m}\ell_i(m-i)$ and add the sum of admissible monomials $y_1^{\ell_1}\dots y_i^{\ell_i-r}\dots y_j^{\ell_j+r}\dots y_m^{\ell_m}$ multiplied by $(\ell_i-\ell_j)$ (such monomial is admissible if $\ell_1\ge\ell_2\ge\dots\ge\ell_i-r\ge\dots\ge\ell_j+r\ge\dots\ge\ell_{m-1}\ge\ell_m$).

The recurrence relation for the coefficients $c_{\boldsymbol{\ell}}$ in the expansion of $Z_{\mathbf{k}}(Y)$ into summands, obtained by symmetrization of monomials, is of the form

$$c_{\boldsymbol{\ell}}=\sum_{\mathbf{p}}((\ell_i+r)-(\ell_j-r))c_{\mathbf{p}}(\rho_{\mathbf{k}}-\rho_{\boldsymbol{\ell}}), \tag{7}$$

where

$$\rho_{\boldsymbol{\ell}}=\sum_{j=1}^{m}\ell_j(\ell_j-j),\quad \boldsymbol{\ell}=(\ell_1,\dots,\ell_m),$$

$$\mathbf{p}=(\ell_1,\ell_2,\dots,\ell_i-r,\dots,\ell_j+r,\dots,\ell_{m-1},\ell_m) \tag{8}$$

and r runs over those values for which $\boldsymbol{\ell}<\mathbf{p}<\mathbf{k}$ (the ordering is lexicographic).

Another method of evaluation of zonal spherical polynomials is based on expansion of them in characters of irreducible representations of the group $GL(m,\mathbf{R})$. Namely, the following statement is valid: *Let $S\in\mathfrak{P}_m(\mathbf{R})$. Let $\mathbf{k}$ be a partition of a non-negative integer k into summands $k_1\ge k_2\ge\dots\ge k_m$ and let $Z_{\mathbf{k}}(S)$ be the corresponding zonal polynomial. Then*

$$Z_{\mathbf{k}}(S)=\sum_{\mathbf{j}\le\mathbf{k}}d_{\mathbf{kj}}\chi_{\mathbf{j}}(S), \tag{9}$$

where $\chi_{\mathbf{j}}$ are the characters of the irreducible finite dimensional representations $T_{2\mathbf{j}}$ of the group $GL(m,\mathbf{R})$ and the sum is over all $\mathbf{j}=(j_1,\dots,j_m)$ such that $\mathbf{j}\le\mathbf{k}$ with respect to the lexicographic ordering.

17.3.8. Intertwining functions. Let G be a compact Lie group and let K and L be its *Gel'fand subgroups* (that is, such that (G,K) and (G,L) are Gal'fand pairs). Let $T_{\mathbf{\Lambda}}$ be a unitary irreducible representation of G with highest weight $\mathbf{\Lambda}$ in a space $\mathcal{V}$. We suppose that $T_{\mathbf{\Lambda}}$ is of class 1 with respect to K and L. Then in $\mathcal{V}$ there are vectors $\mathbf{e}_K$ and $\mathbf{e}_L$ invariant with respect to K and L respectively. These vectors are uniquely defined up to constants α_K and α_L such that $|\alpha_K|=|\alpha_L|=1$. We set

$$\varphi_{\mathbf{\Lambda}}^{KL}(g)=(T_{\mathbf{\Lambda}}(g)\mathbf{e}_L,\mathbf{e}_K). \tag{1}$$

The operator with the kernel $K(g_1,g_2)=\varphi_{\mathbf{\Lambda}}^{KL}(g_1^{-1}g_2)$ intertwines realizations of the representation $T_{\mathbf{\Lambda}}$ in the spaces of functions on G/K and on G/L generated by shifts of the zonal spherical functions $\varphi_{\mathbf{\Lambda}}^{K}$ and $\varphi_{\mathbf{\Lambda}}^{L}$. The function $\varphi_{\mathbf{\Lambda}}^{KL}$ is called *intertwining.* Evidently, for $K=L$ it coincides with the zonal spherical function $\varphi_{\mathbf{\Lambda}}^{KK}=\varphi_{\mathbf{\Lambda}}^{K}$.

Let $\mathcal{C}^{KL}$ be the space of continuous functions on G, constant on two-sided cosets of the form KgL, $g \in G$. It follows from (1) that $\varphi_{\mathbf{\Lambda}}^{KL} \in \mathcal{C}^{KL}$. Direct verification shows that if K, L, M are Gel'fand subgroups in G and $f_1 \in \mathcal{C}^{KL}$, $f_2 \in \mathcal{C}^{LM}$, then $f_1 * f_2 \in \mathcal{C}^{KM}$. With a function $f_1 \in \mathcal{C}^{KL}$ we associate the number

$$a_{\mathbf{\Lambda}}^{KL}(f_1) = \int_G f_1(g)\varphi_{\mathbf{\Lambda}}^{KL}(g)dg. \tag{2}$$

In the analogous way we define the numbers $a_{\mathbf{\Lambda}}^{LM}(f_2)$ and $a_{\mathbf{\Lambda}}^{KM}(f)$, where $f_2 \in \mathcal{C}^{LM}$, $f \in \mathcal{C}^{KM}$. We have

$$a_{\mathbf{\Lambda}}^{KM}(f_1 * f_2) = a_{\mathbf{\Lambda}}^{KL}(f_1)a_{\mathbf{\Lambda}}^{LM}(f_2). \tag{3}$$

To prove this formula we choose an orthonormal basis $\mathbf{e}_0, \dots, \mathbf{e}_n$ in $\mathcal{V}$ corresponding to the decomposition of restriction of $T_{\mathbf{\Lambda}}$ onto L. Let $\mathbf{e}_0 = \mathbf{e}_L$. We have

$$\begin{aligned} a_{\mathbf{\Lambda}}^{KM}(f_1 * f_2) &= \int_G \int_G f_1(g) f_2(g_1^{-1}g_2)\varphi_{\mathbf{\Lambda}}^{KM}(g_2)dg_1dg_2 \\ &= \int_G \int_G f_1(g) f_2(g_2)(T_{\mathbf{\Lambda}}(g_1g_2)\mathbf{e}_M, \mathbf{e}_K)dg_1dg_2 \\ &= \sum_{j=0}^{n} \int_G f_1(g)(T_{\mathbf{\Lambda}}(g_1)\mathbf{e}_j, \mathbf{e}_K)dg_1 \\ &\quad \times \int_G f_2(g_2)(T_{\mathbf{\Lambda}}(g_2)\mathbf{e}_M, \mathbf{e}_j)dg_2. \end{aligned}$$

Using the reasonings of Section 16.2.2 we obtain from here formula (3).

We also have the relation

$$\int_L \varphi_{\mathbf{\Lambda}}^{KM}(g_1\ell g_2)d\ell = \varphi_{\mathbf{\Lambda}}^{KL}(g_1)\varphi_{\mathbf{\Lambda}}^{LM}(g_2) \tag{4}$$

generalizing formula (6) of Section 17.2.2. To prove it we use the equalities

$$\begin{aligned} \varphi_{\mathbf{\Lambda}}^{KM}(g_1\ell g_2) &= (T_{\mathbf{\Lambda}}(g_1\ell g_2)\mathbf{e}_M, \mathbf{e}_K) = (T_{\mathbf{\Lambda}}(\ell)T_{\mathbf{\Lambda}}(g_2)\mathbf{e}_M, T_{\mathbf{\Lambda}}(g_1^{-1})\mathbf{e}_K) \\ &= \sum_{j=0}^{n} (T_{\mathbf{\Lambda}}(\ell)T_{\mathbf{\Lambda}}(g_2)\mathbf{e}_M, \mathbf{e}_j)\overline{(T_{\mathbf{\Lambda}}(g_1^{-1})\mathbf{e}_K, \mathbf{e}_j)} \\ &= \sum_{j=0}^{n} (T_{\mathbf{\Lambda}}(g_2)\mathbf{e}_M, T_{\mathbf{\Lambda}}(\ell^{-1})\mathbf{e}_j)(T_{\mathbf{\Lambda}}(g_1)\mathbf{e}_j, \mathbf{e}_K). \end{aligned}$$

Integrating last term over L we obtain the single non-vanishing summand corresponding to $j = 0$, which is equal to $\varphi_{\mathbf{\Lambda}}^{KL}(g_1)\varphi_{\mathbf{\Lambda}}^{LM}(g_2)$. This leads to formula (4).

Now we assume that the compact Lie group G is semisimple. We say that a pair (K, L) of Gel'fand subgroups of G is a *Cartan pair* if there is an abelian subgroup A_{KL} in G such that $A_{KL} \cap K = A_{KL} \cap L = \{e\}$, every element $g \in G$ is representable in the form $g = ka\ell$, where $k \in K$, $\ell \in L$, $a \in A_{KL}$, and this decomposition is unique for almost all elements of G up to replacement of (k, a, ℓ) by $(k\gamma, \gamma^{-1}a\gamma, \gamma^{-1}\ell)$, where γ belongs to the normalizer N_{KL} of A_{KL} in $K \cap L$. The quotient group N_{KL}/Z_{KL}, where Z_{KL} is the centralizer of A_{KL} in $K \cap L$, is denoted by W_{KL}.

If (K, L) is a Cartan pair, then the intertwining function $\varphi_{\mathbf{\Lambda}}^{KL}$ is uniquely defined by its values on A_{KL}. The restriction of $\varphi_{\mathbf{\Lambda}}^{KL}$ onto A_{KL} is W_{KL}-invariant, that is

$$\varphi_{\mathbf{\Lambda}}^{KL}(waw^{-1}) = \varphi_{\mathbf{\Lambda}}^{KL}(a), \quad a \in A_{KL}, \quad w \in W_{KL}.$$

We choose in A_{KL} the Cartan coordinates $\mathbf{t} = (t_1, \ldots, t_s)$. If (K, L) and (L, M) are Cartan pairs, then the convolution of functions $f_1 \in \mathcal{C}^{KL}$ and $f_2 \in \mathcal{C}^{LM}$ can be written as

$$(f_1 * f_2)(a(\mathbf{t})) = \int a(\mathbf{t}_1, \mathbf{t}_2; \mathbf{t}) f_1(a(\mathbf{t}_1)) f_2(a(\mathbf{t}_2)) d\mathbf{t}_1 d\mathbf{t}_2, \tag{5}$$

where $a(\mathbf{t}) \in A_{KM}$, $a(\mathbf{t}_1) \in A_{KL}$, $a(\mathbf{t}_2) \in A_{LM}$. To prove this formula it is sufficient to write down the convolution of functions in the form

$$(f_1 * f_2)(a(\mathbf{t})) = \int\limits_{A_{KL}} \int\limits_{K} f_1(a(\mathbf{t}_1)) f_2(a(-\mathbf{t}_1) k a(\mathbf{t})) \omega(\mathbf{t}_1) dk d\mathbf{t}_1,$$

where $\omega(\mathbf{t}_1)$ is the Jacobian of transition from g to $(k_1, \mathbf{t}_1, \tilde{k}_2)$, and to define the new variables setting $a(-\mathbf{t}_1) k a(\mathbf{t}) = \ell a(\mathbf{t}_2) m$, $\ell \in L$, $m \in M$.

Let $G = SU(n, \mathbf{F})$, $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$. Then the subgroups K_ℓ and K_m, where $K_p = S(U(n-p, \mathbf{F}) \times U(p, \mathbf{F}))$, form a Cartan pair in G (for the case $\mathbf{F} = \mathbf{R}$ the corresponding decomposition is given in Section 15.1.9 and for other cases they are analogously constructed). We assume that $0 \le \ell \le m \le n/2$. In this case we can set $A_{K_\ell K_m} \equiv A_{\ell m} = A_{\ell\ell} \equiv A_\ell$.

Let $\mathfrak{g} = \mathfrak{k}_m + i\mathfrak{p}_m$ and $\mathfrak{g} = \mathfrak{k}_\ell + i\mathfrak{p}_\ell$, $i = \sqrt{-1}$, be the Cartan decompositions of the Lie algebra $\mathfrak{g} = \mathfrak{su}(n, \mathbf{F})$ with respect to the subalgebras $\mathfrak{k}_m$ and $\mathfrak{k}_\ell$ of the Lie subgroups K_m and K_ℓ respectively. Let $\mathfrak{a}_m$ and $\mathfrak{a}_\ell$ be maximal commutative subalgebras in $\mathfrak{p}_m$ and $\mathfrak{p}_\ell$ respectively. Since $\ell \le m \le n/2$, then we can assume that $\mathfrak{a}_\ell \subset \mathfrak{a}_m$ and $A_\ell = \exp i\mathfrak{a}_\ell$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing the subalgebra $\mathfrak{a}_\ell$. Then $\mathfrak{h} = \mathfrak{a}_\ell + \mathfrak{h}_\ell$, where $\mathfrak{h}_\ell \subset \mathfrak{k}_\ell$.

We choose a coordinate system in $\mathfrak{a}_\ell$ which corresponds to the unit vectors $\hat{\mathbf{e}}_1, \ldots, \hat{\mathbf{e}}_\ell$ such that the restricted roots are of the form $\pm\hat{\mathbf{e}}_i \pm \hat{\mathbf{e}}_j$, $\pm\hat{\mathbf{e}}_i$, $\pm 2\hat{\mathbf{e}}_i$ and the simple roots are of the form $\boldsymbol{\alpha}_i = \hat{\mathbf{e}}_i - \hat{\mathbf{e}}_{i+1}$, $i = 1, \ldots ; \ell - 1$, $\boldsymbol{\alpha}_\ell = \hat{\mathbf{e}}_\ell$. Then the vectors $\sum\limits_{i=1}^{\ell} m_i \hat{\mathbf{e}}_i$ with non-negative integral m_i, $i = 1, \ldots, \ell$, are highest weights of irreducible representations of G. The highest weights

$$\boldsymbol{\mu}_k = 2\sum_{i=1}^{k} \hat{\mathbf{e}}_i, \quad k = 1, \dots, \ell$$

are the *fundamental highest weights for class* 1 (with respect to K_ℓ) irreducible representations of G and every class 1 irreducible representation of G has highest weight of the form $\sum_{k=1}^{\ell} n_k \boldsymbol{\mu}_k$ with non-negative integral n_k.

Let us note that for $\mathsf{F} = \mathbf{R}$ the restricted root $2\mathbf{e}_\ell$ is absent. In this case we take $\boldsymbol{\mu}_\ell$ in the form $\boldsymbol{\mu}_\ell = \sum_{i=1}^{\ell} \hat{\mathbf{e}}_i$. Moreover, in this case we consider class 1 representations for $O(n)/(O(n-\ell) \times O(\ell))$ instead of $SO(n)/(SO(n-\ell) \times SO(\ell))$.

Below we shall use the partial ordering defined in Section 17.3.1. According to it we have $\Lambda_1 \preceq \Lambda_2$ if $B(\Lambda_2 - \Lambda_1, \boldsymbol{\mu}_k) \geq 0$ for $k = 1, \dots, \ell$. The restriction of the zonal spherical function $\varphi^\ell_{\mathbf{\Lambda}} \equiv \varphi^{\ell\ell}_{\mathbf{\Lambda}}$ onto $A_\ell = \exp i\mathfrak{a}_\ell$ is the trigonometric polynomial

$$\varphi^\ell_{\mathbf{\Lambda}}(\exp H) = \sum_{\boldsymbol{\lambda} \preceq \mathbf{\Lambda}} c(\boldsymbol{\lambda}, \mathbf{\Lambda}) \exp(-\boldsymbol{\lambda}, H), \quad H \in i\mathfrak{a}_\ell. \tag{6}$$

This polynomial is W_ℓ-invariant.

Let $\boldsymbol{\nu}$ be a highest weight of some irreducible representation of G. Suppose that $\boldsymbol{\nu}$ vanishes on $\mathfrak{h}_\ell$. Then there is an irreducible representation $T_{\mathbf{\Lambda}}$ of G of class 1 with respect to K_ℓ and K_m having the highest weight $\mathbf{\Lambda} = 2\boldsymbol{\nu}$. (Proof of this statement can be found in [307].) Consequently, to every highest weight of the form $\mathbf{\Lambda} = \sum_{i=1}^{\ell} n_k \boldsymbol{\mu}_k$ with non-negative integers n_k there corresponds an intertwining function $\varphi^{\ell m}_{\mathbf{\Lambda}}$. It is possible to show that for $\mathsf{F} = \mathbb{C}, \mathbf{H}$ they exhaust all intertwining functions. Note also that the restriction of $\varphi^{\ell m}_{\mathbf{\Lambda}}$ onto A_ℓ is W_ℓ-invariant.

Using the expansions of the zonal spherical functions $\varphi^\ell_{\mathbf{\Lambda}}$ and $\varphi^m_{\mathbf{\Lambda}}$ from Section 17.3.1

$$\varphi^\ell_{\mathbf{\Lambda}}(\exp H) = (T_{\mathbf{\Lambda}}(\exp H)\mathbf{e}_\ell, \mathbf{e}_\ell) = \sum_{j=0}^{q} |c_j|^2 e^{-2(\boldsymbol{\nu}_j, H)}, \quad c_0 \neq 0, \tag{7}$$

$$\varphi^m_{\mathbf{\Lambda}}(\exp H) = \sum_{j=0}^{q} |d_j|^2 e^{-2(\boldsymbol{\nu}_j, H)}, \quad d_0 \neq 0, \tag{7$'$}$$

where $\boldsymbol{\nu}_0 = \boldsymbol{\nu}$, $2\boldsymbol{\nu}_j \leq \mathbf{\Lambda}$, $H \in i\mathfrak{a}_\ell$, we obtain the expansion

$$\varphi^{\ell m}_{\mathbf{\Lambda}}(\exp H) = \sum_{j=0}^{q} c_j \bar{d}_j e^{-2(\boldsymbol{\nu}_j, H)}, \quad c_0 d_0 \neq 0. \tag{8}$$

Thus, the intertwining function $\varphi_{\Lambda}^{\ell m}$ is a W_ℓ-invariant trigonometric polynomial on A_ℓ of the form

$$\varphi_{\Lambda}^{\ell m}(\exp H) = \sum_{\boldsymbol{\lambda} \preceq \boldsymbol{\Lambda}} \tilde{c}(\boldsymbol{\lambda}, \boldsymbol{\Lambda}) e^{(-\boldsymbol{\lambda}, H)}, \quad H \in i\mathfrak{a}_\ell, \tag{8'}$$

where the sum is over weights of the form $\boldsymbol{\lambda} = \sum_{k=1}^{\ell} n_k \boldsymbol{\mu}_k$, $n_k \in \mathbb{Z}$.

17.3.9. Generalized Jacobi polynomials. In order to obtain expressions for the intertwining functions $\varphi_{\Lambda}^{\ell m}$ we define orthogonal polynomials in many variables which generalize Jacobi polynomials.

We take in $\mathbb{Z}^\ell$ the lexicographic and the partial orderings. The partial ordering $\mathbf{m} \prec \mathbf{n}$, $\mathbf{m} = (m_1, \dots, m_\ell)$, $\mathbf{n} = (n_1, \dots, n_\ell)$, means that $\mathbf{m} \neq \mathbf{n}$ and $\sum_{i=1}^{k} m_i \le \sum_{i=1}^{k} n_i$, $k = 1, \dots, \ell$. A polynomial of the form

$$p_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{m} \le \mathbf{n}} c_{\mathbf{m}} x^{\mathbf{m}}, \quad c_{\mathbf{n}} \neq 0, \tag{1}$$

where $\mathbf{m} \le \mathbf{n}$ is taken in the lexicographic ordering, is said to be of *degree* $\mathbf{n}$ *with leading term* $c_{\mathbf{n}} x^{\mathbf{n}} = c_{\mathbf{n}} x_1^{n_1} \dots x_\ell^{n_\ell}$. If $p_{\mathbf{n}}(\mathbf{x})$ is a symmetric polynomial, then for its leading term the condition $n_1 \ge n_2 \ge \dots \ge n_\ell$ is fulfilled.

For $\alpha > -1$, $\beta > -1$, $\gamma > -\frac{1}{2}$ we set

$$w(\mathbf{x}) = w^{\alpha\beta\gamma}(\mathbf{x}) = \prod_{i=1}^{\ell} (1 - x_i)^\alpha (1 + x_i)^\beta \prod_{i<j} (x_i - x_j)^{2\gamma+1} \tag{2}$$

and consider this function on the domain

$$\Omega = \{\mathbf{x} \mid -1 \le x_\ell \le x_{\ell-1} \le \dots \le x_1 \le 1\}. \tag{3}$$

We define the *generalized Jacobi polynomials* $p_{\mathbf{n}}(\mathbf{x}) = p_{\mathbf{n}}^{\alpha\beta\gamma}(\mathbf{x})$ as polynomials satisfying the conditions

(a) $p_0 = 1$;

(b) $p_{\mathbf{n}}(\mathbf{x})$ is a symmetric polynomial in $\mathbf{x} = (x_1, \dots, x_\ell)$ with leading term $\mathbf{x}^{\mathbf{n}}$;

(c) if $q(\mathbf{x})$ is a symmetric polynomial in $\mathbf{x}$ of degree $\mathbf{m}$, $\mathbf{m} < \mathbf{n}$, then

$$\int_\Omega p_{\mathbf{n}}^{\alpha\beta\gamma}(\mathbf{x}) q(\mathbf{x}) w^{\alpha\beta\gamma}(\mathbf{x}) d\mathbf{x} = 0. \tag{4}$$

It is easy to show that these conditions define polynomials $p_{\mathbf{n}}(\mathbf{x})$ uniquely up to constant multipliers.

Replacing in $p_{\mathbf{n}}^{\alpha\beta\gamma}(\mathbf{x})$ the variables x_i by $\cos 2\theta_i$, $1 \le i \le \ell$, we obtain trigonometrical polynomials which are orthogonal with respect to the weight function

$$\tilde{w} = \prod_{i=1}^{\ell} \sin^{2\alpha+1}\theta_i \cos^{2\beta+1}\theta_i \prod_{i<j}(\cos 2\theta_i - \cos 2\theta_j)^{2\gamma+1} \tag{5}$$

defined on the domain

$$\tilde{\Omega} = \left\{\boldsymbol{\theta} \mid 0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_\ell \le \frac{\pi}{2}\right\}. \tag{6}$$

We define in $\mathbb{Z}^\ell$ the scalar product corresponding to the Killing form $B(\cdot,\cdot)$ and set $r = B(\mathbf{e}_i, \mathbf{e}_i)$, where $\mathbf{e}_i = (0,\dots,0,1,0,\dots,0)$. Let $D \equiv D^{\alpha\beta\gamma}$ be the differential operator

$$D = 4r\sum_{i=1}^{\ell}(1-x_i^2)\frac{\partial^2}{\partial x_i^2} + (w^{\alpha\beta\gamma}(\mathbf{x}))^{-1}\frac{\partial}{\partial x_i}[w^{\alpha\beta\gamma}(\mathbf{x})(1-x_i^2)]\frac{\partial}{\partial x_i}. \tag{7}$$

The substitution $x_i = \cos 2t_i$, $1 \le i \le \ell$, transforms it into

$$\tilde{D} = r\sum_{i=0}^{\ell} \tilde{w}^{-1}\frac{\partial}{\partial t_i}\tilde{w}\frac{\partial}{\partial t_i}. \tag{8}$$

Let us prove that for polynomials f and g of $\mathbf{x}$ we have

$$\int_{\Omega}(Df)\bar{g}w\,d\mathbf{x} = \int_{\Omega} f(\overline{Dg})w\,d\mathbf{x}. \tag{9}$$

Really the substitution $x_i = \cos 2t_i$, $1 \le i \le \ell$, turns this formula into

$$\int_{\tilde{\Omega}}(\tilde{D}f)\bar{g}\tilde{w}\,d\mathbf{t} = \int_{\tilde{\Omega}} f(\overline{\tilde{D}g})\tilde{w}\,d\mathbf{t}. \tag{10}$$

According to the Gauss theorem we have

$$\int_{\tilde{\Omega}}(\tilde{D}f)\bar{g}\tilde{w}\,d\mathbf{t} - \int_{\tilde{\Omega}} f(\overline{\tilde{D}g})\tilde{w}\,d\mathbf{t}$$

$$= \sum_{i=1}^{\ell}\int_{\partial\tilde{\Omega}}\left(\frac{\partial f}{\partial t_i}\bar{g} - \frac{\partial \bar{g}}{\partial t_i}f\right)\tilde{w}\,dt_1\dots dt_{i-1}dt_{i+1}\dots dt_\ell,$$

where $\partial\tilde{\Omega}$ is the boundary of the domain $\tilde{\Omega}$. Since $\tilde{w}$ vanishes on the boundary, then formula (9) is proved for $\alpha, \beta, \gamma > -\frac{1}{2}$. For other values of α, β, γ this formula is obtained by analytical continuation.

The generalized Jacobi polynomials $p_{\mathbf{n}}^{\alpha\beta\gamma}(\mathbf{x})$ *are eigenfunctions of the operator* $D^{\alpha\beta\gamma}$ *with eigenvalues* $|\boldsymbol{\rho}|^2 - |2\mathbf{n} + \boldsymbol{\rho}|^2$, *where* $|\boldsymbol{\nu}|^2 = B(\boldsymbol{\nu}, \boldsymbol{\nu})$ *and*

$$\boldsymbol{\rho} = (\alpha + \beta + 1)\tilde{\mathbf{e}}_0 + (2\gamma + 1)\tilde{\mathbf{e}}_1, \tag{11}$$

$$\tilde{\mathbf{e}}_0 = (1, \dots, 1), \ \tilde{\mathbf{e}}_1 = (\ell - 1, \ell - 2, \dots, 1, 0).$$

Really, the operator $D^{\alpha\beta\gamma}$ can be represented in the form

$$D = 4r \sum_{i=1}^{\ell} \left\{ (1 - x_i^2)\frac{\partial^2}{\partial x_i^2} + [(1+\beta)(1 - x_i) - (1+\alpha)(1 + x_i)]\frac{\partial}{\partial x_i} \right.$$
$$\left. +(2\gamma + 1)(1 - x_i^2)\frac{1}{V_\ell(\mathbf{x})}\frac{\partial}{\partial x_i}V_\ell(\mathbf{x})\frac{\partial}{\partial x_i} \right\},$$

where $V_\ell(\mathbf{x}) = \prod\limits_{i<j}(x_i - x_j)$ is the Vandermonde determinant. It is evident that $Dp_{\mathbf{n}}(\mathbf{x})$ is a symmetric polynomial and

$$Dp_{\mathbf{n}}(\mathbf{x}) = 4r \sum_{i=1}^{\ell} \{-n_i(n_i - 1) - (1+\alpha)n_i - (1+\beta)n_i - (2\gamma_1 + 1)n_i(\ell - i)\}\, \mathbf{x}^{\mathbf{n}}$$
$$+ \text{lower terms.} \tag{12}$$

Besides, for $\mathbf{m} < \mathbf{n}$ we have

$$\int_\Omega (Dp_{\mathbf{n}})\overline{p_{\mathbf{m}}}w d\mathbf{x} = \int_\Omega p_{\mathbf{n}}(\overline{Dp_{\mathbf{m}}})w d\mathbf{x} = 0. \tag{13}$$

These conditions uniquely determine the polynomial $Dp_{\mathbf{n}}$ and show that $Dp_{\mathbf{n}}$ differs from $p_{\mathbf{n}}$ by a constant multiplier. Formula (12) indicate that this multiplier is equal to $|\boldsymbol{\rho}|^2 - |2\mathbf{n} + \boldsymbol{\rho}|^2$.

Let us give two special cases of parameter γ for which generalized Jacobi polynomials are expressed in terms of usual Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

Case 1: $\gamma = -\frac{1}{2}$. We have

$$p_{\mathbf{n}}^{\alpha,\beta,-1/2}(\mathbf{x}) = \sum p_{i_1}^{(\alpha,\beta)}(x_1)\dots p_{i_\ell}^{(\alpha,\beta)}(x_\ell), \tag{14}$$

where the sum is over the different permutations $(i_1, \dots, i_\ell)$ of $(n_1, \dots, n_\ell)$ and $p_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. Really, the right hand side of (14) is a symmetric polynomial with leading term $\mathbf{x}^{\mathbf{n}}$. All other terms are of the form $\mathbf{x}^{\mathbf{m}} = cx_1^{m_1} \dots x_\ell^{m_\ell}$ with $m_j \le n_j$. Therefore,

$$\sum_{j=1}^{k} m_j \le \sum_{j=1}^{k} n_j, \ 1 \le k \le \ell.$$

This means that $\mathbf{m} \prec \mathbf{n}$ and, hence, $\mathbf{m} < \mathbf{n}$. The orthogonality relation for $p_{\mathbf{n}}^{\alpha,\beta,-1/2}$ follows from that for usual Jacobi polynomials and from the equality

$$\int_{\Omega} f w d\mathbf{x} = \frac{1}{\ell!} \int_{\hat{\Omega}} f|w| d\mathbf{x},$$

where $\hat{\Omega} = \{\mathbf{x} \mid |x_i| \le 1, \ 1 \le k \le \ell\}$, which is fulfilled for symmetric functions.

Case 2: $\gamma = \frac{1}{2}$. In this case

$$p_{\mathbf{n}}^{\alpha,\beta,1/2} = V_\ell^{\alpha\beta}(\mathbf{n} + \boldsymbol{\rho}, \mathbf{x}) / V_\ell(\mathbf{x}), \tag{15}$$

where $\boldsymbol{\rho} = (\ell - 1, \ell - 2, \dots, 1, 0)$,

$$V_\ell^{\alpha\beta}(\mathbf{m}, \mathbf{x}) = \Delta_\ell((p_{m_k}^{(\alpha,\beta)}(x_j))). \tag{16}$$

Really, if $m_1 > m_2 \dots > m_\ell > 0$, then $V_\ell^{\alpha\beta}(\mathbf{m}, \mathbf{x})$ is a skew-symmetric polynomial of degree $\mathbf{m}$ with leading term $\mathbf{x}^{\mathbf{m}}$. In particular, $V_\ell^{\alpha\beta}(\boldsymbol{\rho}, \mathbf{x}) = V_\ell(\mathbf{x})$. Therefore, $p_{\mathbf{n}}^{\alpha,\beta,1/2}(\mathbf{x})$ is a symmetric polynomial with leading term $\mathbf{x}^{\mathbf{n}}$. Since $w^{\alpha,\beta,1/2}(\mathbf{x}) = (V_\ell(\mathbf{x}))^2 w^{\alpha,\beta,-1/2}(\mathbf{x})$, then the orthogonality relation is obtained as in the previous case.

For other values of γ the expressions for $p_{\mathbf{n}}^{\alpha\beta\gamma}(\mathbf{x})$ in terms of classical orthogonal polynomials are not known.

The generalized Jacobi polynomials $p_{\mathbf{n}}^{\alpha,\beta,1/2}(\mathbf{x})$ can be expanded in characters of irreducible finite dimensional representations of the group $GL(\ell, \mathbb{C})$. To obtain this expansion we replace in $V_\ell^{\alpha\beta}(\mathbf{m}, \mathbf{x})$ the Jacobi polynomials by the corresponding hypergeometric functions and obtain the determinant of the matrix with elements a_{jk}, $1 \le j, \ k \le \ell$, where

$$\begin{aligned} a_{jk} &= F\left(m_k + \alpha + \beta + 1, -m_k; \alpha + 1; \frac{1}{2}(1 - x_j)\right) \\ &= \frac{\Gamma(\alpha + 1) m_k!}{\Gamma(m_k + \alpha + \beta + 1)} \sum_{n=0}^{m_k} \frac{\Gamma(m_k + \alpha + \beta + n + 1)}{\Gamma(\alpha + n + 1)(m_k - n)! n!} \left(\frac{x_j - 1}{1}\right)^n . \end{aligned}$$

Applying the Cauchy-Binet theorem on minors and taking out from the rows and from the columns common multipliers we obtain that

$$V_\ell^{\alpha\beta}(\mathbf{m},\mathbf{x}) = \frac{\Gamma^\ell(\alpha+1)\prod_{k=1}^{\ell} m_k!}{\prod_{k=1}^{\ell}\Gamma(m_k+\alpha+\beta+1)} \sum_{\mathbf{n}} \frac{D^{\alpha\beta}(\mathbf{m},\mathbf{n})V_\ell\left(\mathbf{n},\frac{\mathbf{x}-\mathbf{1}}{2}\right)}{\prod_{q=1}^{\ell}\Gamma(\alpha+n_q+1)n_q!}, \tag{17}$$

where $n = (n_1, \dots, n_\ell)$, $n_1 > n_2 > \dots > n_\ell$, and $D^{\alpha\beta}(\mathbf{m},\mathbf{n})$ is the determinant of the matrix with elements $b_{kq} = 0$ if $m_k < n_q$ and

$$b_{kq} = \frac{\Gamma(m_k+\alpha+\beta+n_q+1)}{(m_k-n_q)!} \quad \text{if} \quad m_k \geq n_q.$$

The expression $V_\ell\left(\mathbf{n},\frac{\mathbf{x}-\mathbf{1}}{2}\right)$ from (17) is contained in the character of the irreducible representation of $GL(n,\mathbb{C})$, namely,

$$V_\ell\left(\mathbf{n},\frac{\mathbf{x}-\mathbf{1}}{2}\right)\Big/V_\ell\left(\boldsymbol{\rho},\frac{\mathbf{x}-\mathbf{1}}{2}\right) = \chi_{\mathbf{n}-\boldsymbol{\rho}}(\delta), \quad \delta = \mathrm{diag}\left(\frac{x_1-1}{2},\dots,\frac{x_\ell-1}{2}\right).$$

17.3.10. Generalized Jacobi polynomials and intertwining operators. It was shown in Section 17.3.8 that the intertwining functions $\varphi_\Lambda^{\ell m}$ are W_ℓ-invariant trigonometrical polynomials on the subgroup A_ℓ. The orthogonality relations for matrix elements of irreducible representations show that the functions $\varphi_\Lambda^{\ell m}$ and $\varphi_{\mathbf{M}}^{\ell m}$ are orthogonal for $\Lambda \neq \mathbf{M}$. Since they are left K_ℓ-invariant and right K_m-invariant, then the restrictions of these functions onto A_ℓ are orthogonal with respect to the weight $\hat{w}(\mathbf{t})$ such that $d_*g = \hat{w}(\mathbf{t})d_*k_\ell d\mathbf{t}d_*k_m$. Evaluating the weight $\hat{w}$ for the groups $U(n,\mathbf{F})$ (in the same way as it was done in Section 15.3.1) we obtain that

$$\hat{w}(\mathbf{t}) = \prod_{i=1}^{\ell}(\cos t_i)^{\nu(m-\ell+1)-1}(\sin t_i)^{\nu(n-\ell-m+1)-1}\prod_{1\leq i<j\leq\ell}|\cos 2t_i - \cos 2t_j|^\nu, \tag{1}$$

where $\nu(\mathbf{R}) = 1$, $\nu(\mathbb{C}) = 2$, $\nu(\mathbf{H}) = 4$. Moreover, W_ℓ-invariance shows that the corresponding trigonometrical polynomial is connected with symmetric algebraic polynomial in the variables $x_i = \cos 2t_i$, $i = 1, \dots, \ell$.

Comparing weight (1) with weight (5) of Section 17.3.9 we see that the intertwining polynomials are expressed in terms of generalized Jacobi polynomials. Namely, we have

$$\varphi_\Lambda^{\ell m}(\mathbf{t}) = p_{\mathbf{k}}^{\alpha\beta\gamma}(\mathbf{x})/p_{\mathbf{k}}^{\alpha\beta\gamma}(\mathbf{1}), \tag{2}$$

where $\mathbf{k} = \Lambda/2$, $\alpha = (\nu/2)(n-m-\ell+1)-1$, $\beta = (\nu/2)(m-\ell+1)-1$, $\gamma = (\nu-1)/2$.

Zonal spherical functions on some compact Riemannian symmetric spaces are also expressed in terms of generalized Jacobi polynomials. Namely, up to a normalization we have

for	$Sp(\ell)/U(\ell)$	$\varphi_{\mathbf{\Lambda}}(t) = p_{\mathbf{k}}^{0,0,0}(\mathbf{x})$,
for	$SO(4\ell)/U(2\ell)$	$\varphi_{\mathbf{\Lambda}}(t) = p_{\mathbf{k}}^{0,0,3/2}(\mathbf{x})$,
for	$SO(4\ell+2)/U(2\ell+1)$	$\varphi_{\mathbf{\Lambda}}(\mathbf{t}) = p_{\mathbf{k}}^{2,0,3/2}(\mathbf{x})$,
for	$(Sp(\ell)\times Sp(\ell))/Sp(\ell)$	$\varphi_{\mathbf{\Lambda}}(\mathbf{t}) = p_{\mathbf{k}}^{1/2,1/2,1/2}(\mathbf{x})$,
for	$(SO(2\ell+1)\times SO(2\ell+1))/SO(2\ell+1)$	$\varphi_{\mathbf{\Lambda}}(\mathbf{t}) = p_{\mathbf{k}}^{1/2,-1/2,1/2}(\mathbf{x})$.

For some of these cases intertwining and zonal functions are expressed in terms of usual Jacobi polynomials.

The Jacobi polynomials $P_n^{\alpha,\beta}(x)$ are eigenfunctions of the differential operator

$$D_{\alpha\beta} = (1-x^2)\frac{d^2}{dx^2} + [\beta-\alpha-(\alpha+\beta+2)x]\frac{d}{dx}$$

with eigenvalues $-n(n+\alpha+\beta+1)$. Using this fact it is possible to show that radial parts $\overset{\circ}{\Delta}$ of Laplace operators on the space $K_\ell\backslash G/K_\ell$, where $G = SU(n)$, $K_\ell = S(U(n-\ell)\times U(\ell))$, is of the form

$$\overset{\circ}{\Delta} = (\tilde{w}^{\alpha\beta\gamma}(\mathbf{t}))^{-1}P(L_1,\dots,L_\ell)w^{\alpha\beta\gamma}(\mathbf{t}),$$

where

$$L_p = \frac{1}{4}\frac{\partial^2}{\partial t_p^2} + \frac{1}{2}[(n-2\ell)\cot t_p + \cot 2t_p]\frac{\partial}{\partial t_p}$$

and $P(x_1,\dots,x_\ell)$ is a symmetric polynomial in $x_1,\dots,x_\ell$. Let $\lambda_{\mathbf{\Lambda}}(\overset{\circ}{\Delta})$ be the eigenvalue of the operator $\overset{\circ}{\Delta}$ corresponding to the irreducible representation $T_{\mathbf{\Lambda}}$. We have

$$\lambda_{\mathbf{\Lambda}}(\overset{\circ}{\Delta}) = P(\rho_1,\dots,\rho_\ell),$$

where $\rho_i = -r_i(r_i+n-2\ell+1)$, $r_i = p_i+\ell-i$ and p_i are the numbers giving the representation $T_{\mathbf{\Lambda}}$ (if $\lambda_1,\dots,\lambda_n$ are the coordinates of the highest weight $\mathbf{\Lambda}$ and $\sum\limits_i \lambda_i = 0$, then $p_i = \lambda_i = -\lambda_{n-i+1}$, $1\le i\le\ell$). Thus, the homomorphism $\overset{\circ}{\Delta}\to\lambda_{\mathbf{\Lambda}}(\overset{\circ}{\Delta})$ defines the highest weight of the representation $T_{\mathbf{\Lambda}}$.

17.3.11. Zonal spherical functions and generalized Jacobi and Bessel functions. Formula (2) of Section 17.3.10 for $\ell = m$ gives the expression for zonal spherical functions on the compact symmetric space $\mathfrak{X}_1 = SU(n)/S(U(n-\ell) \times U(\ell))$. We have

$$\varphi_{\mathbf{\Lambda}}^{\ell\ell}(\mathbf{t}) \equiv \varphi_{\mathbf{\Lambda}}(\mathbf{t}) = c j_1^{-1}(\mathbf{t}) \Delta_\ell ((P_{r_j}^{(n-2\ell,0)}(\cos 2t_i))_{i,j=1}^{\ell}),$$

where

$$j_1(\mathbf{t}) = \prod_{i<j} (\sin^2 t_i - \sin^2 t_j). \tag{1}$$

Expressing the Jacobi polynomials in terms of the hypergeometric function we obtain

$$\varphi_{\mathbf{\Lambda}}(\mathbf{t}) = A_1 j_1^{-1}(\mathbf{t}) \Delta_\ell ((F(n-\ell+\lambda_j-j+1, -\lambda_j-\ell+j; n-2\ell+1, \sin^2 t_i)_{i,j=1}^{\ell}), \tag{2}$$

where

$$A_1 = \prod_{j<k} (\rho_j - \rho_k)^{-1} \prod_{j=1}^{\ell-1} j!(n-2\ell+j)^{\ell-j}, \tag{3}$$

$\rho_j = r_j(n-2\ell+r_j+1)$ and r_j are as at the end of Section 17.3.10.

By analytical continuation we obtain from (2) the expression for zonal spherical functions on the noncompact Riemannian space $\mathfrak{X}_2 = SU(n-\ell,\ell)/S(U(n-\ell) \times U(\ell))$, dual to $\mathfrak{X}_1$. It is of the form

$$\begin{gathered} \varphi_{i\mathbf{\Lambda}-\boldsymbol{\rho}}(\mathbf{t}) = A_2 j_2^{-1}(t) \prod_{j<k} (\lambda_j^2 - \lambda_k^2)^{-1} \\ \times \Delta_\ell \left(\left(F\left(a_j, b_j; n-2\ell+1; -\sinh^2 t_k\right)\right)_{j,k=1}^{\ell} \right), \end{gathered} \tag{4}$$

where $a_j = \frac{1}{2}(n-2\ell+i\lambda_j+1)$, $b_j = \frac{1}{2}(n-2\ell-i\lambda_j+1)$ and

$$j_2(\mathbf{t}) = \prod_{j<k} (\sinh^2 t_j - \sinh^2 t_k), \tag{5}$$

$$A_2 = (-1)^{\ell(\ell-1)/2} 2^{n(n-1)} \prod_{j=1}^{\ell-1} j!(n-2\ell+j)^{\ell-j}.$$

This formula was given without proof by Berezin and Karpelevich [14] and was proved by Hoogenboom [112] with the help of Harish-Chandra's general theory of spherical functions. Namely, zonal spherical functions were considered as eigenfunctions of the radial parts of Laplace operators. For the space under consideration these operators are of the form

$$\mathring{\Delta} = j_2^{-1}(\mathbf{t}) P(M_1, \dots, M_\ell) j_2(\mathbf{t}), \tag{6}$$

where $P(x_1, \dots, x_\ell)$ is a symmetric polynomial in $x_1, \dots, x_\ell$ and

$$M_p = \frac{1}{4}\frac{\partial^2}{\partial t_p^2} + \frac{1}{2}[(n-2\ell)\coth t_p + \coth 2t_p]\frac{\partial}{\partial t_p}. \tag{7}$$

Formula (4) can be also obtained from the integral representation of zonal spherical functions.

The space $\mathfrak{X}_0$ of matrices $X \in \mathfrak{M}_{n-\ell,\ell}(\mathbb{C})$ is triple to the spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. The group G_s of block-triangular matrices $g = \begin{pmatrix} U_1 & YU_2 \\ 0 & U_2 \end{pmatrix}$, $U_1 \in U(n-\ell)$, $U_2 \in U(\ell)$, $Y \in \mathfrak{M}_{n-\ell,\ell}(\mathbb{C})$, acts upon $\mathfrak{X}_0$ by the formula $g \circ X = U_1 X U_2^{-1} + Y U_2^{-1}$. Irreducible representations of this group are described in Section 16.2.5. Zonal spherical functions of these representations are given by the formula

$$\varphi_{i\Lambda}(\mathbf{t}) = A_0 j_0^{-1}(\mathbf{t})\Delta_\ell(((r_j t_k)^{2\ell-n} J_{n-2\ell}(r_j t_k))_{j,k=1}^{\ell}), \tag{8}$$

where

$$j_0(\mathbf{t}) = \prod_{j<k}(t_j^2 - t_k^2), \tag{9}$$

$$A_0 = 2^{\ell(\ell-1)} m!(m+1)!\dots(m+\ell-1)! \prod_{j<k}(\rho_j - \rho_k)^{-1}, \quad \rho_j = -r_j^2.$$

They are eigenfunctions of radial parts of Laplace operators which are of the form

$$\mathring{\Delta} = j_0^{-1}(\mathbf{t}) P(N_1, \dots, N_\ell) j_0(\mathbf{t}), \tag{10}$$

where P is the same as in formula (6) and

$$N_p = \frac{1}{4}\frac{\partial^2}{\partial t_p^2} + (2n - 4\ell + 1)\frac{1}{t_p}\frac{\partial}{\partial t_p}. \tag{11}$$

The functions (4) and (8) are called *generalized Jacobi and Bessel functions* respectively.

As the generalized Jacobi polynomials, the generalized Jacobi and Bessel functions can be expressed in characters of irreducible finite dimensional representations of the group $GL(\ell, \mathbb{C})$. We recommend to the reader to write down the corresponding formulas.

The integral expressions for generalized Jacobi and Bessel functions can be obtained with the help of the general integral representation of zonal spherical functions. For generalized Bessel functions this expression is

$$\varphi_{iP}(T) = c \int_{U(\ell)} \int_{U(n-\ell)} \operatorname{etr}(iU_2 P U_1 T^*) d_* U_1 d_* U_2,$$

where

$$P = (P_1, 0), \quad T = (T_1, 0), \quad P_1 = \mathrm{diag}\,(r_1, \ldots, r_\ell), \quad T_1 = \mathrm{diag}\,(t_1, \ldots, t_\ell).$$

17.3.12. Generalized Gel'fand pairs. Let us generalize the notion of Gel'fand pairs (G, H) for the case when H is a noncompact subgroup of a group G. Let G be a unimodular Lie group and let H be its unimodular Lie subgroup. Let $\mathfrak{D}'(\mathfrak{X})$ be the space of generalized functions on the homogeneous space $\mathfrak{X} = G/H$. We consider these functions as anti-linear functionals on the space $\mathfrak{D}(\mathfrak{X})$ of smooth finite functions on $\mathfrak{X}$. A unitary representation T of the group G in a Hilbert space $\mathfrak{H}$ is called *realizable on* $\mathfrak{X}$ if there exists a continuous linear injective mapping $j: \mathfrak{H} \to \mathfrak{D}'(\mathfrak{X})$ such that

$$jT(g) = L(g)j, \tag{1}$$

where L denotes the action of G on $\mathfrak{D}'(\mathfrak{X})$ by shift operators. A pair (G, H) is called a *generalized Gel'fand pair* if all representations T of G, realizable on $\mathfrak{X}$, have no multiple subrepresentations, that is, the algebra of bounded linear operators on $\mathfrak{H}$ commuting with all operators $T(g)$, $g \in G$, is commutative.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let σ be its involutive automorphism. Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$ and let G_c be the simply connected complex Lie group with Lie algebra $\mathfrak{g}_c$. Let G be a connected Lie subgroup of G_c with Lie algebra $\mathfrak{g}$. The involutive automorphism σ defines the corresponding automorphism Ω in G_c which leaves invariant the subgroup G.

Let $H_\sigma = \{g \in G \mid \Omega g = g\}$ and let H_σ^0 be the connected component of the identity element of H_σ. We take a subgroup H of G such that $H_\sigma^0 \subset H \subset H_\sigma$. Then (G, H) is called a *semisimple symmetric pair.* The following statement is valid. *Let $\mathfrak{D}'(G, H)$ be the space of left H-invariant generalized functions on $\mathfrak{X} = G/H$ and let J be an anti-automorphism of this space. If $J\mathfrak{H} = \mathfrak{H}$ for all G-invariant (or for all minimal G-invariant) Hilbert subspaces of $\mathfrak{D}'(G, H)$, then (G, H) is a generalized Gel'fand pair.*

This statement can be formulated in another way. Let a subgroup H consists of fixed elements for an involutive automorphism τ. Let $F \in \mathfrak{D}'(G, H)$. We set $\langle F^\tau, f\rangle = \langle F, f^\tau\rangle$, where for $f \in \mathfrak{D}(G)$ we have $f^\tau(g) = f(\tau g)$, and define $\bar{F}$ by the formula $\langle \bar{F}, f\rangle = \overline{\langle F, \bar{f}\rangle}$. *If $JF = F$ for all left and right H-invariant positive definite generalized functions on G, where $JF = \bar{F}^\tau$, then (G, H) is a generalized Gel'fand pair.*

The pairs

$$(SO_0(p, q+1), SO_0(p, q)), \;\; p, q \geq 1,$$

$$(SU(p, q+1), S(U(p, q) \times U(1))), \;\; p, q \geq 1,$$

$$(Sp(p, q+1), Sp(p, q) \times Sp(1)), \;\; p, q \geq 1,$$

$$(SL(n+1, \mathbf{R}), S(GL_+(n, \mathbf{R}) \times GL_+(1, \mathbf{R}))), \;\; n \geq 2,$$

$$(Sp(n+1,\mathbf{R}), Sp(n,\mathbf{R}) \times Sp(1,\mathbf{R}))$$

are generalized Gel'fand pairs of rank 1.

17.3.13. Ordered symmetric spaces and Volterra algebras. Let $\mathfrak{X}$ be a locally compact space with an ordering, where the graph

$$\Gamma = \{(x,y) \mid x \geq y\} \tag{1}$$

is closed. We suppose that a locally compact group G acts transitively on $\mathfrak{X}$ and $\mathfrak{X} = G/H$. Let an ordering in $\mathfrak{X}$ be G-invariant, that is $gx \geq gy$ for all $g \in G$ if $x \geq y$. We set

$$S = \{g \in G \mid gx_0 \geq x_0\} \quad \text{where} \quad x_0 = eH. \tag{2}$$

Then S is a closed semi-group in G such that $S \cap S^{-1} = H$. Conversely, if S is a closed semi-group in G such that $S \cap S^{-1} = H$, then S define on $\mathfrak{X}$ a G-invariant ordering.

If $a, b \in \mathfrak{X}$, then we set

$$I_{ab} = \{x \in \mathfrak{X} \mid a \leq x \leq b\}. \tag{3}$$

We suppose that

(a) all intervals I_{ab}, $a, b \in \mathfrak{X}$, are compact;

(b) all compact sets of $\mathfrak{X}$ are bounded below and above;

(c) there exists a G-invariant measure d_*x on $\mathfrak{X}$ such that the measure of I_{ab} continuously depends on a and b.

A function $K(x,y)$ on $\mathfrak{X} \times \mathfrak{X}$ is called a *Volterra kernel* if it is continuous on Γ and vanishes outside of Γ. The equality

$$(K_1 * K_2)(x,y) = \int K_1(x,z)K_2(z,y)d_*z \tag{4}$$

defines the convolution of Volterra kernels. The space $\mathcal{V}(\mathfrak{X})$ of Volterra kernels forms an algebra with respect to the convolution. It is called the *Volterra algebra on $\mathfrak{X}$*. A kernel $K(x,y)$ is called *G-invariant* if $K(x,y) = K(gx,gy)$ for all $x, y \in \mathfrak{X}$ and $g \in G$. Invariant Volterra kernels form the subalgebra $\mathcal{V}_0(\mathfrak{X})$ of $\mathcal{V}(\mathfrak{X})$.

Let σ be an involutive automorphism of the group G such that $\sigma H = H$. We associate with σ the symmetry s of $\mathfrak{X}$ defined by the formula $s(gH) = \sigma(g)H$. This symmetry satisfies the condition $s(gx) = \sigma(g)s(x)$.

If for every $g \in S$ we have $\sigma(g) \in Hg^{-1}H$ and the symmetry s leaves invariant the measure d_*x on $\mathfrak{X}$, then the algebra $\mathcal{V}_0(\mathfrak{X})$ is commutative. Really, if $x \geq y$, then $s(x) \leq s(y)$. We set $K^\nu(x,y) = K(y,x)$. Then K^ν is a Volterra kernel corresponding to the inverse ordering. If K_1 and K_2 are Volterra kernels, then

$(K_1 * K_2)^\nu = K_2^\nu * K_1^\nu$. We set $K^\sigma(x,y) = K(s(x), s(y))$. Then $(K_1 * K_2)^\sigma = K_2^\sigma * K_1^\sigma$. Let K be a G-invariant Volterra kernel. If $g \in G$, then

$$\begin{aligned} K^\nu(x_0, gx_0) &= K(gx_0, x_0) = K(x_0, g^{-1}x_0) \\ &= K(x_0, \sigma(g)x_0) = K^\sigma(x_0, gx_0). \end{aligned}$$

Hence, $K^\nu = K^\sigma$ and

$$(K_1 * K_2)(x,y) = \int K_1(x,z)K_2(z,y)d_*z$$

$$= \int K_1(s(z), s(x))K_2(s(y), s(z))d_*z = (K_2 * K_1)(s(y), s(x))$$

$$= (K_2 * K_1)(x,y).$$

Thus, $K_1 * K_2 = K_2 * K_1$ and the algebra $\mathcal{V}_0(\mathfrak{X})$ is commutative.

The set $\mathfrak{H}_m(\mathbf{F})$ for $\mathbf{F} = \mathbf{R}$ or $\mathbb{C}$ is an example of an ordered space. We consider that $X > Y$ if and only if $X - Y > 0$ (that is $X - Y \in \mathfrak{P}_m(\mathbf{F})$) and $X \geq Y$ if $X - Y \in \overline{\mathfrak{P}_m(\mathbf{F})}$. This order is invariant with respect to the action $g \circ X = gXg^*$ of the group $GL(m, \mathbf{F})$.

If $p + q = m$, then $\mathfrak{P}_{pq}(\mathbf{F})$ denotes the space of matrices of the form $X = gI_{pq}g^*$, $g \in GL(m, \mathbf{F})$, that is the set of matrices of signature (p,q) from $\mathfrak{M}_m(\mathbf{F})$. It is evident that $\mathfrak{P}_{pq}(\mathbf{F}) = G/H$, where $G = GL(m, \mathbf{F})$ and $H = U(p,q;\mathbf{F})$. The invariant measure on $\mathfrak{P}_{pq}(\mathbf{F})$ is

$$d_*X = |\Delta_m(X)|^{-\theta}dX, \tag{5}$$

where dX is the Lebesgue measure on $\mathfrak{H}_m(\mathbf{F})$ and $\theta = \frac{1}{2}\nu(m-1) + 1$, $\nu(\mathbf{R}) = 1$, $\nu(\mathbb{C}) = 2$.

Let A be the subgroup in $GL(m, \mathbf{F})$ consisting of matrices $a = \operatorname{diag}(e^{t_1}, \dots, e^{t_m})$, $t_k \in \mathbf{R}$. Then

$$aI_{pq}a^* = \operatorname{diag}(e^{2t_1}, \dots, e^{2t_p}, -e^{2t_{p+1}}, \dots, -e^{2t_m}).$$

It is clear that $aI_{pq}a^* > I_{pq}$ if and only if $t_1 > 0, \dots, t_p > 0, t_{p+1} < 0, \dots, t_m < 0$. Let A_+ be the set of matrices $a \in A$ such that $aI_{pq}a^* > I_{pq}$. The following statement is fulfilled.

If $X \in \mathfrak{P}_{pq}(\mathbf{F})$, then $X > I_{pq}$ if and only if for some $h \in H$ and $a \in A_+$ we have $X = haI_{pq}a^*h^*$. Really, if $X = haI_{pq}a^*h^*$, $h \in H$, $a \in A_+$, then

$$X - I_{pq} = h(aI_{pq}a^* - I_{pq})h^* > 0.$$

To prove the converse assertion we note that the decompositions $GL(m, \mathbf{F}) = KAH = HAK$ with $K = U(m, \mathbf{F})$ are valid (since any Hermitian matrix can be diagonalized with respect to some orthonormal basis). Let $X > I_{pq}$. Then $X = I_{pq} + Y$, where $Y > 0$. We can find $g \in G \equiv GL(m, \mathbf{F})$ such that $Y = gI_m g^*$. Since $G = HAK$, then $g = hbk$, $h \in H$, $b \in A$, $k \in K$. Hence,

$$X = I_{pq} + hbI_m b^* h^* = h(I_{pq} + bI_m b^*)h^*.$$

The matrix $Z = I_{pq} + bI_m b^*$ is diagonal and belongs to $\mathfrak{P}_{pq}(\mathbf{F})$. Moreover, $Z > I_{pq}$. Therefore, $Z = aI_{pq}a^*$, where $a \in A_+$. Our statement is proved.

It follows from this statement that any element $g \in G$ such that $gI_{pq}g^* > I_{pq}$ can be represented in the form $g = h_1 a h_2$, where $h_1, h_2 \in H$, $a \in A$. Consequently, the set HA_+H is everywhere dense in the semi-group $S = \{g \in G \mid gI_{pq}g^* \geq I_{pq}\}$. Let us show that this assertion implies a commutativity of the Volterra algebra $\mathcal{V}_0(\mathfrak{X})$, $\mathfrak{X} = \mathfrak{P}_{pq}(\mathbf{F})$. Really, let σ be the involutive automorphism of G given by the formula $\sigma(g) = I_{pq}(g^*)^{-1}I_{pq}$. Then $H = \{h \in G \mid \sigma(h) = h\}$ and $\sigma(a) = a^{-1}$ for $a \in A$. Therefore, if $g = h_1 a h_2 \in HA_+H$, then $\sigma(g) = h_1 a^{-1} h_2 \in Hg^{-1}H$. As it was shown above, this means that the algebra $\mathcal{V}_0(\mathfrak{X})$ is commutative.

17.3.14. Zonal spherical functions on the space $\mathfrak{P}_{pq}(\mathbf{F})$ Let $\mathfrak{X} = G/H$ be an ordered homogeneous space and let S be the semi-group defining this ordering and such that $S \cap S^{-1} = H$. We assume that H is unimodular subgroup. The invariant measure on H is denoted by d_*h. A continuous function φ on the open kernel $\overset{\circ}{\Delta}$ of the semi-group S is called a *zonal spherical function* on $\mathfrak{X}$ if for all $x, y \in \overset{\circ}{S}$ the fundamental equation

$$\int_H \varphi(xhy)d_*h = \varphi(x)\varphi(y) \tag{1}$$

is fulfilled.

Let $G = GL(m, \mathbf{F})$, $H = U(p, q; \mathbf{F})$, $N = N_-(m, \mathbf{F})$, $\mathbf{F} = \mathbb{R}$ or $\mathbb{C}$. For any matrix $X \in \mathfrak{P}_{pq}(\mathbf{F})$, $X > I_{pq}$, there are matrices $n \in N$, $a \in A$ such that $X = naI_{pq}a^*n^*$. Really, if for all k, $1 \leq k \leq m$, we have $\Delta_k(X) \neq 0$, then there exist $n \in N$ and $d \in D(m, \mathbf{F})$, $d = \operatorname{diag}(d_1, \ldots, d_m)$, such that $X = ndn^*$. Since $X > I_{pq}$, then for all k, $1 \leq k \leq p$ we have $\Delta_k(X) > 0$. Therefore, $d_k > 0$ for $1 \leq k \leq p$. Since the matrices d and X have the same signature (p, q), we have $d_k < 0$ for $p + 1 \leq k \leq p + q = m$. Thus, the matrix d can be written as $d = aI_{pq}a^*$ and we have $(-1)^k \Delta_{p+k}(X) > 0$, $1 \leq k \leq q$.

Now we have to prove that if $X > I_{pq}$, then for all k, $1 \leq k \leq m$, the inequalities $\Delta_k(X) \neq 0$ are fulfilled. If $\Delta_{k_0}(X) = 0$, then we can find a matrix X_1 in a small heighborhood of X such that $\Delta_k(X_1) \neq 0$ for all k and $\Delta_{k_0}(X_1) < 0$ if $k_0 \leq p$, $(-1)^{k_0 - p}\Delta_{k_0}(X_1) < 0$ if $k_0 > p$. This leads to contradiction.

If $X > I_{pq}$, then $X = naI_{pq}a^*n^*$ is called the *Gauss decomposition* of X.

Let $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{C}^m$ and let

$$|\Delta(X)|^{\mathbf{s}} = \prod_{j=1}^{m} |\Delta_j(X)|^{s_j - s_{j+1}}. \tag{2}$$

If $n \in N$ and $a \in A$, then

$$|\Delta(naXa^*n^*)|^{\mathbf{s}} = |\Delta(naI_{pq}a^*n^*)|^{\mathbf{s}}|\Delta(X)|^{\mathbf{s}}. \tag{3}$$

Let us prove the following statement.

Let $\boldsymbol{\sigma} = \operatorname{Re} \mathbf{s}$. Suppose that for all $g \in G$ and for all $X \in \mathfrak{X} = G/H$ such that $X > I_{pq}$ we have

$$\int_H |\Delta(ghXh^*g^*)|^{\boldsymbol{\sigma}} d_*h < +\infty. \tag{4}$$

Then

$$\varphi_{\mathbf{s}}(X) = \int_H |\Delta(hXh^*)|^{\mathbf{s}} d_*h \tag{5}$$

is a zonal spherical function, that is, for all $g \in \overset{\circ}{S}$ and $X > I_{pq}$ the relation

$$\int_H \varphi_{\mathbf{s}}(ghXh^*g^*)d_*h = \varphi_{\mathbf{s}}(gI_{pq}g^*)\varphi_{\mathbf{s}}(X) \tag{6}$$

is fulfilled.

For $g \in \overset{\circ}{S}$ we set

$$F(g) = \int_H |\Delta(ghXh^*g^*)|^{\mathbf{s}} d_*h.$$

This function is right H-invariant and if $g = na$, $n \in N$, $a \in A$, then

$$F(na) = |\Delta(naI_{pq}a^*n^*)|^{\mathbf{s}} \int_H |\Delta(hXh^*)|^{\mathbf{s}} d_*h.$$

This means that

$$\int_H |\Delta(ghXh^*g^*)|^{\mathbf{s}} d_*h = |\Delta(gI_{pq}g^*)|^{\mathbf{s}} \varphi_{\mathbf{s}}(X).$$

Replacing g by $h'g$, $h' \in H$, and integrating in h' we obtain relation (6).

The function $\varphi_{\mathbf{s}}$ is uniquely defined by its values on the subgroup A. Let $a(\mathbf{t}) = \operatorname{diag}(e^{t_1}, \dots, e^{t_m}) \in A$, $\boldsymbol{\lambda} = \mathbf{s} + \boldsymbol{\rho}$, $\boldsymbol{\rho} = (m-1, m-2, \dots, 1, 0)$, and let W_p

and W_q be the symmetric groups S_p and S_q respectively. If Re $\lambda_j < 0$ for $1 \le j \le p$ and Re $\lambda_j > 0$ for $p+1 \le j \le m$, then integral (4) absolutely converges and

$$\varphi_{\mathbf{s}}(a(\mathbf{t})I_{pq}a(\mathbf{t})) = c\prod_{i<j}[(\lambda_i - \lambda_j)(e^{2t_i} - e^{2t_j})]^{-1} \sum_{w\in W_p} (-1)^{\sigma(w)} \times \exp\left(2\sum_{j=1}^{p} \lambda_j t_{w(j)}\right) \sum_{w\in W_q} (-1)^{\sigma(w)} \exp\left(2\sum_{j=1}^{q} \lambda_{p+j} t_{p+w(j)}\right), \tag{7}$$

where $\sigma(w)$ is the length of the element w and c is a constant depending on a normalization of the Haar measure on H. For $q = 0$ this formula was given in Section 17.2.6. For any q it was proved by J. Faraut with the help of the induction in q and the formula

$$\int_{U(p,q)} f(u)d_*u = \int_{\mathfrak{X}_{pq}} \int_{U(p,q-1)} f(\tilde{u}v)d_*v d_*\tilde{u},$$

where $\mathfrak{X}_{pq} = U(p,q)/U(p,q-1)$ and d_*v, $d_*\tilde{u}$ are the invariant measures on $U(p,q-1)$ and $\mathfrak{X}_{pq}$ respectively.

Zonal spherical functions on $\mathfrak{X} = \mathfrak{P}_{pq}(\mathbf{F})$ allow us to construct the homomorphism of algebra $\mathcal{V}_0(\mathfrak{X})$ into $\mathbb{C}$. It was shown above that any element $X \in \mathfrak{X}$ such that $X > I_{pq}$ can be represented in the form $X = haI_{pq}a^*h^*$, where $h \in H$, $a = \text{diag}\,(e^{t_1}, \dots, e^{t_m})$ and $t_k > 0$ for $1 \le k \le p$, $t_k < 0$ for $p+1 \le k \le m$. For the invariant measure d_*X on X we have

$$\int_{X > I_{pq}} f(X)d_*X = \int_{A_+} \left(\int_H f(haI_{pq}a^*h^*)d_*h \right) j(a)da, \tag{8}$$

where $da = dt_1 \dots dt_m$ and

$$j(a) = c\prod_{i<j} |\sinh(t_i - t_j)|^{\nu}. \tag{9}$$

For invariant Volterra kernels we define the *spherical transform*

$$\hat{K}(\mathbf{s}) = \int_{A_s} K(aI_{pq}a^*, I_{pq})\varphi_{\mathbf{s}}(aI_{pq}a^*)j(a)da \tag{10}$$

(we suppose that the integral converges). This formula can be written as

$$\hat{K}(\mathbf{s}) = \int K(X, I_{pq})|\Delta(X)|^{\mathbf{s}} d_*X. \tag{10'}$$

If K_1 and K_2 are invariant Volterra kernels for which integral (10) *absolutely converges, then*

$$(K_1 * K_2)^\wedge(\mathbf{s}) = \hat{K}_1(\mathbf{s})\hat{K}_2(\mathbf{s}). \tag{11}$$

To prove this statement we at first show that if K is an invariant Volterra kernel, then

$$\int K(X,Z)|\Delta(X)|^{\mathbf{s}} d_*X = |\Delta(Z)|^{\mathbf{s}} \hat{K}(s). \tag{12}$$

Really, we set $Z = gI_{pq}g^*$, $g \in G$. Then

$$\int K(X, gI_{pq}g^*)|\Delta(X)|^{\mathbf{s}} d_*X = \int K(g^{-1}X(g^{-1})^*, I_{pq})|\Delta(X)|^{\mathbf{s}} d_*X$$
$$= \int K(Y, I_{pq})|\Delta(gYg^*)|^{\mathbf{s}} d_*Y.$$

Using formula (8) we obtain that

$$\int K(X, gI_{pq}g^*)|\Delta(X)|^{\mathbf{s}} d_*X = \int_{A_+} K(aI_{pq}a^*, I_{pq})$$
$$\times \left(\int_H |\Delta(ghaI_{pq}a^*h^*g^*)|^{\mathbf{s}} d_*h \right) j(a)da.$$

Due to formula (6) we have

$$\int K(X,Z)|\Delta(X)|^{\mathbf{s}} d_*X$$
$$= |\Delta(Z)|^{\mathbf{s}} \int K(Y, I_{pq})|\Delta(Y)|^{\mathbf{s}} d_*Y = |\Delta(Z)|^{\mathbf{s}} \hat{K}(\mathbf{s}).$$

Using this relation we obtain

$$(K_1 * K_2)^\wedge(\mathbf{s}) = \int (K_1 * K_2)(X, I_{pq})|\Delta(X)|^{\mathbf{s}} d_*X$$
$$= \int\int K_1(X,Z)K_2(Z, I_{pq})|\Delta(X)|^{\mathbf{s}} d_*X d_*Z$$
$$= \int \left(\int K_1(X,Z)|\Delta(X)|^{\mathbf{s}} d_*X \right) K_2(Z, I_{pq}) d_*Z.$$

Applying formula (12) we derive relation (11).

17.4. Hypergeometric Functions of a Matrix Argument

17.4.1. Hypergeometric functions on $\mathfrak{H}_m(\mathsf{F})$. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of the matrix $\Lambda \in \mathfrak{H}_m(\mathsf{F})$. The number $\|\Lambda\| = \max_i \lambda_i$ will be called the *norm of the matrix* Λ. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)$, where $\alpha_j \in \mathbb{C}$, $\beta_j \in \mathbb{C}$. The sum of the series

$$ {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta}; \Lambda) = \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} \frac{[\alpha_1]_{\mathbf{k}} \ldots [\alpha_p]_{\mathbf{k}}}{[\beta_1]_{\mathbf{k}} \ldots [\beta_q]_{\mathbf{k}}} \frac{Z_{\mathbf{k}}(\Lambda)}{k!} \tag{1} $$

is called the *hypergeometric function of a matrix argument* Λ *with indices* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$. Remind that $[\alpha]_{\mathbf{k}} = \prod_{j=1}^{m} \left(\alpha - \frac{1}{2}(j-1)\nu\right)_{k_j}$. For example, it follows from formula (8) of Section 17.3.4 that

$$ {}_0F_0(\Lambda) = \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} \frac{Z_{\mathbf{k}}(\Lambda)}{k!} = \sum_{k=0}^{\infty} \frac{(\mathrm{Tr}\,\Lambda)^k}{k!} = \mathrm{etr}\,\Lambda. \tag{2} $$

If $p < q$, then the series (1) absolutely converges on $\mathfrak{H}_m^c(\mathsf{F})$.

Let us evaluate the sum of the series ${}_1F_0(\alpha; \Lambda)$. If $\mathrm{Re}\,\alpha > \nu(m-1)/2$ and $\|\Lambda\| < 1$, then according to formula (1) of Section 17.3.6 we obtain

$$ \begin{aligned} \Delta^{-\alpha}(I_m - \Lambda) &= \frac{1}{\Gamma_m(\alpha)} \int_{M>0} \mathrm{etr}(-(I_m - \Lambda)M) \Delta^{\alpha}(M) d_* M \\ &= \frac{1}{\Gamma_m(\alpha)} \int_{M>0} \mathrm{etr}(-M)\, {}_0F_0(M\Lambda) \Delta^{\alpha}(M) d_* M \\ &= \frac{1}{\Gamma_m(\alpha)} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\mathbf{k}|=k} \int_{M>0} \mathrm{etr}(-M) \Delta^{\alpha}(M) Z_k(M\Lambda) d_* M \\ &= \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} \frac{[\alpha]_{\mathbf{k}}}{k!} Z_{\mathbf{k}}(\Lambda) = {}_1F_0(\alpha; \Lambda). \end{aligned} $$

Thus, for $\mathrm{Re}\,\alpha > (m-1)\nu/2$, $\|\Lambda\| < 1$ we have

$$ {}_1F_0(\alpha; \Lambda) = \Delta^{-\alpha}(I_m - \Lambda). \tag{3} $$

This equality allows us to continue analytically this function onto the complexification $\mathfrak{H}_m^c(\mathsf{F})$ of the space $\mathfrak{H}_m(\mathsf{F})$. The right hand side of the formula obtained is defined for the principal part of argument values.

It is possible to show that ${}_pF_q(\Lambda)$ diverges if $p > q+1$. The series ${}_{q+1}F_q(\Lambda)$ converges for $\|\Lambda\| < 1$. This statement is proved with the help of the assertions on convergence of hypergeometric functions of a numerical argument and of the inequality

$$|Z_{\mathbf{k}}(\Lambda)| < m^k \|\Lambda\|^k, \quad |\mathbf{k}| = k, \quad \Lambda \in \mathfrak{H}_m(\mathbf{F}). \tag{4}$$

The Cauchy-Binet formula for determinants and conditions of congruence equalities for numerical series are also used.

The relation

$$\int_0^{I_m} \Delta^{\alpha}(M)\Delta^{\beta-\alpha-\theta}(I_m - M)\, {}_pF_{p-1}(\boldsymbol{\alpha};\boldsymbol{\beta}; M\Lambda) d_*M = \frac{\Gamma_m(\alpha)\Gamma_m(\beta-\alpha)}{\Gamma_m(\beta)}\, {}_{p+1}F_p(\tilde{\boldsymbol{\alpha}};\tilde{\boldsymbol{\beta}};\Lambda), \tag{5}$$

where $\tilde{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, \alpha)$, $\tilde{\boldsymbol{\beta}} = (\boldsymbol{\beta}, \beta)$, $\theta = \frac{1}{2}(m-1)\nu + 1$,

$$\operatorname{Re}\beta > \operatorname{Re}\alpha + \frac{1}{2}(m-1)\nu > (m-1)\nu, \quad \|\Lambda\| < 1,$$

is proved by replacement of ${}_pF_{p-1}$ by its expansion (1) and by term-by-term integration with the help of formula (5) of Section 17.3.6. For $p = 2$ we obtain the following analogue of the Euler integral for the usual hypergeometric function:

$${}_2F_1(a,b;c;\Lambda) = \frac{\Gamma_m(c)}{\Gamma_m(b)\Gamma_m(c-b)} \int_0^{I_m} \Delta^b(M)\Delta^{c-b-\theta}(I_m - M) \times \Delta^{-a}(I_m - M\Lambda) d_*M, \tag{6}$$

where $\operatorname{Re} c > \operatorname{Re} b + \frac{1}{2}(m-1)\nu > (m-1)\nu$ and $\|\lambda\| < 1$.

Let us derive the formula for the Laplace transform of the hypergeometric function. Let $p \le q$ and $\operatorname{Re}\alpha > \frac{1}{2}(m-1)\nu$. Then

$$\frac{1}{\Gamma_m(\alpha)} \int_{M>0} \operatorname{etr}(-MZ)\, {}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta};M)\Delta^{\alpha}(M) d_*M = \Delta^{-\alpha}(Z)\, {}_{p+1}F_q(\tilde{\boldsymbol{\alpha}};\boldsymbol{\beta};Z^{-1}), \tag{7}$$

where $\tilde{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, \alpha)$. Note that for $p < q$ integral (7) converges absolutely on the right half-plane

$$\Phi = \{Z = X + iY \mid X, Y \in \mathfrak{H}_m(\mathbf{F}), X > 0\}.$$

If $p = q$, then it converges for all $Z \in \mathfrak{H}_m^c$ such that $0 < (\operatorname{Re} Z)^{-1} < I_m$.

We set $a = \text{Re } \alpha$, $Z = X + iY$, $X, Y \in \mathfrak{H}_m(\mathbf{F})$, and denote by $c_{\mathbf{k}}$ the coefficient at $Z_{\mathbf{k}}$ in expression (1) for the function ${}_pF_q$. We suppose that $a > \frac{1}{2}(m-1)\nu$ and $X > 0$. The integrand function in (7) is of the form

$$F(M) = \sum_{\mathbf{k}} c_{\mathbf{k}} \text{etr}(-MZ)\Delta^{\alpha}(M)Z_{\mathbf{k}}(M).$$

Fulfilling formally term-by-term integration and using formula (1) of Section 17.3.6 we obtain relation (7). To justify this derivation we note that due to formula (1) of Section 17.3.6 and to the theorem on monotonic convergence we have

$$\begin{aligned}
&\int\limits_{M>0} \sum_{\mathbf{k}} |c_{\mathbf{k}} \text{etr}\,(-MZ)\Delta^{a}(M)Z_{\mathbf{k}}(M)|d_*M \\
&\qquad = \int\limits_{M>0} \sum_{\mathbf{k}} |c_{\mathbf{k}}| \text{etr}\,(-MZ)\Delta^{a}(M)Z_{\mathbf{k}}(M)d_*M \\
&\qquad = \sum_{\mathbf{k}} |c_{\mathbf{k}}| \int\limits_{M>0} \text{etr}\,(-MZ)\Delta^{a}(M)Z_{\mathbf{k}}(M)d_*M \\
&\qquad = \Gamma_m(a)\Delta^{-a}(Z)\sum_{\mathbf{k}} |c_{\mathbf{k}}|[a]_{\mathbf{k}} Z_{\mathbf{k}}(Z^{-1}).
\end{aligned}$$

Applying to this series the convergence conditions for hypergeometric series of the form ${}_{p+1}F_p(Z^{-1})$ we are convinced that for $p < q$ it converges for all Z^{-1}. Therefore, if $p < q$, then formula (7) is true for all $Z = X + iY \in \Phi$. For $p = q$ this series converges only in the domain $\|Z^{-1}\| < 1$. Therefore, if $p = q$, then formula (7) is true for all $Z = X + iY \in \Phi$ such that $\|(\text{Re } Z)^{-1}\| < 1$, that is, such that $0 < (\text{Re } Z)^{-1} < I_m$.

It follows from formula (1) of Section 17.3.6 and from the definition of the hypergeometric function of a matrix argument that for $p \leq q$, $\tilde{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, \gamma)$ we have

$$\begin{aligned}
{}_{p+1}F_q(\tilde{\boldsymbol{\alpha}}; \boldsymbol{\beta}; Z^{-1}) = \frac{\Delta^{\gamma}(Z)}{\Gamma_m(\gamma)} \int\limits_{\Lambda>0} \text{etr}(-\Lambda Z) \\
\times\ {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta}; \Lambda)\Delta^{\gamma}(\Lambda)d_*\Lambda.
\end{aligned} \tag{8}$$

But according to formula (1)

$${}_{p+1}F_{q+1}(\tilde{\boldsymbol{\alpha}}; \tilde{\boldsymbol{\beta}}; \Lambda) = {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta}; \Lambda), \quad \tilde{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, \gamma), \quad \tilde{\boldsymbol{\beta}} = (\boldsymbol{\beta}, \gamma).$$

Replacing ${}_pF_q$ by ${}_{p+1}F_{q+1}$ in formula (8) and applying the inversion formula for the Laplace transform of a function of a matrix argument we obtain the relation

$$\begin{aligned}
{}_pF_{q+1}(\boldsymbol{\alpha}; \tilde{\boldsymbol{\beta}}; \Lambda)\Delta^{\gamma-\theta}(\Lambda) = \frac{\Gamma_m(\gamma)}{(2\pi i)^{m\theta}} \\
\times \int\limits_{\text{Re } Z = X_0 > 0} \text{etr}(\Lambda Z)\ {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta}; Z^{-1})\Delta^{-\gamma}(Z)dZ.
\end{aligned} \tag{9}$$

This relation follows also from formula (4) of Section 17.3.6.

The transition from ${}_pF_q$ to ${}_{p+1}F_{q+1}$ can be fulfilled either as ${}_pF_q \to {}_{p+1}F_q \to {}_{p+1}F_{q+1}$ or as ${}_pF_q \to {}_pF_{q+1} \to {}_{p+1}F_{q+1}$. Both ways lead to the same result.

The formula

$$
\begin{aligned}
&{}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta};\Lambda,M)\\
&=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{|\mathbf{k}|=k}\frac{[\alpha_1]_{\mathbf{k}}\dots[\alpha_p]_{\mathbf{k}}}{[\beta_1]_{\mathbf{k}}\dots[\beta_q]_{\mathbf{k}}}\frac{Z_{\mathbf{k}}(\Lambda)Z_{\mathbf{k}}(M)}{Z_{\mathbf{k}}(I_m)}
\end{aligned}
\tag{10}
$$

defines the hypergeometric function of two matrix variables.

Formula (15) of Section 17.3.4 leads to the integral relation

$$
{}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta};\Lambda,M)=\int_{U(m,\mathbf{F})}{}_pF_q(\boldsymbol{\alpha};\boldsymbol{\beta};\Lambda UMU^*)d_*U. \tag{11}
$$

Formulas (8) and (9) indicate the ways of further generalizations of the hypergeometric function of a matrix argument. Namely, we set $A=(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_p)$, $B=(\boldsymbol{\beta}_1,\dots,\boldsymbol{\beta}_q)$, where $\boldsymbol{\alpha}_j=(\alpha_{j1},\dots,\alpha_{jm})$, $\boldsymbol{\beta}_k=(\beta_{k1},\dots,\beta_{km})$. Then the equalities

$$
\begin{aligned}
&\frac{1}{\Gamma_m(\boldsymbol{\alpha})}\int_{M>0}\operatorname{etr}(-M\Lambda)\,{}_pF_q(A;B;M)\Delta^{\boldsymbol{\alpha}}(M)d_*M\\
&\qquad=\Delta^{-\boldsymbol{\alpha}}(\Lambda)\,{}_{p+1}F_q(\tilde{A};B;\Lambda^{-1}),
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\frac{\Gamma_m(\boldsymbol{\beta})}{(2\pi i)^{m\theta}}\int_{\operatorname{Re} Z=X_0>0}\operatorname{etr}(\Lambda Z)\,{}_pF_q(A;B;Z^{-1})\Delta^{-\boldsymbol{\beta}}(Z)dZ\\
&\qquad=\Delta^{\boldsymbol{\beta}-\theta}(\Lambda)\,{}_pF_{q+1}(A;\tilde{B};-\Lambda),
\end{aligned}
\tag{13}
$$

where $\tilde{A}=(A,\boldsymbol{\alpha})$, $\tilde{B}=(B,\boldsymbol{\beta})$, allow us *to define recursively* the functions ${}_pF_q(A;B;\Lambda)$ for all p and q, $p\le q+1$. However, this generalization has some deficiencies. For example, it is not known whether the ways ${}_pF_q \to {}_{p+1}F_q \to {}_{p+1}F_{q+1}$ and ${}_pF_q \to {}_pF_{q+1} \to {}_{p+1}F_{q+1}$ lead to the same function. Besides, many other relations, valid for the functions (1), are not true for the functions ${}_pF_q(A;B;\Lambda)$. In this reason we below consider the functions (1). Moreover, for the sake of simplicity we confine ourselves to the case $\mathbf{F}=\mathbf{R}$. In this case we set $n=\frac{1}{2}m(m+1)$, $p=\frac{1}{2}(m+1)$.

17.4.2. Bessel functions of a matrix argument. We define the function $A_\delta(Z)$ of a matrix argument $Z=X+iY\in\mathfrak{S}_m(\mathbb{C})$ by the formula

$$
\begin{aligned}
A_\delta(\Lambda)&=\Pi_m^{-1}(\delta)\,{}_0F_1(\delta+p;-\Lambda)\\
&=(2\pi i)^{-n}\Delta^{-\delta}(\Lambda)\int_{\operatorname{Re} Z=X_0>0}\operatorname{etr}(\Lambda Z-Z^{-1})\Delta^{-\delta-p}(Z)dZ,
\end{aligned}
\tag{1}
$$

where[4] $\Pi_m(\delta) = \Gamma_m(\delta + p)$. This integral converges absolutely for $\Lambda > 0$, Re $\delta > p - 1$. It can be written as

$$A_\delta(\Lambda) = (2\pi i)^{-n} \int\limits_{\text{Re } Z = X_0 > 0} \text{etr}\,(Z - \Lambda Z^{-1}) \Delta^{-\delta - p}(Z) dZ. \tag{2}$$

For $m = 1$, $Z = x > 0$ the function $A_\delta(Z)$ turns into $\left(\frac{x}{2}\right)^{-\delta/2} J_\delta(\sqrt{x})$ and in this reason $A_\delta(Z)$ is called the *Bessel function of a matrix argument.*

Setting $X_0 = I_m$ into (2) we obtain that $\text{etr}(-\Lambda Z^{-1})$ is bounded in Z for any Λ. Moreover, boundedness is uniform on any compact subset in the domain Re $Z > I_m$. It follows from here that $A_\delta(\Lambda)$ is an entire function in Λ and analytical in δ for Re $\delta > p - 1$. Moreover, if Re $\delta > p - 1$ and $\Lambda > 0$, then the function $A_\delta(\Lambda)$ is bounded and tends to zero when $\Lambda \to \infty$.

When Re δ is sufficiently large, then we can apply to both sides of formula (1) the operator $\tilde{D}_\Lambda = \det(\eta_{ij}\partial/\partial\lambda_{ij})$ defined in Section 17.1.1. We obtain

$$\tilde{D}_\Lambda\{\Delta^\delta(\Lambda) A_\delta(\Lambda)\} = \Delta^{\delta-1}(\Lambda) A_{\delta-1}(\Lambda), \tag{3}$$

$$\tilde{D}_\Lambda\{A_\delta(\Lambda)\} = (-1)^\delta A_{\delta+1}(\Lambda). \tag{4}$$

These relations allow us to continue analytically the function $A_\delta(\Lambda)$ in δ. The expression $\Delta^{1-\delta}(\Lambda)\tilde{D}_\Lambda\{\Delta^\delta(\Lambda)\}$ is a linear combination of functions in δ and Λ with differential operator action. These functions are polynomials in δ. It follows from here and from (3) that $A_\delta(\Lambda)$ is an entire function in δ and in Λ. For Re $\delta > p-1$ the evident estimate $A_\delta(\Lambda) = O(\text{etr } |\Lambda|)$ is valid which is also fulfilled for all derivatives of this function. Therefore, the same estimate is valid for $A_{\delta-1}(\Lambda)$. This means that $A_\delta(\Lambda) = O(\text{etr } \Lambda)$ for all δ and, consequently, the integral

$$\int\limits_{\Lambda > 0} \text{etr}(-\Lambda Z) A_\delta(\Lambda M) \Delta^\delta(\Lambda) d\Lambda = \text{etr}(M Z^{-1}) \Delta^{-\delta-p}(Z) \tag{5}$$

converges absolutely for all M, Re $Z > 0$, Re $\delta > -1$. This formula shows that the Laplace transform of the function $\Delta^\delta(\Lambda) A_\delta(\Lambda M)$ is expressed in terms of the power and exponential functions.

[4] For the general case this function is defined by the formula

$$\begin{aligned} A_\delta(\Lambda; \mathsf{F}) &= \Gamma_m^{-1}(\delta + \theta)\, {}_0F_1(\delta + \theta; -\Lambda) \\ &= (2\pi i)^{-m\theta} \Delta^{-\delta}(\Lambda) \int\limits_{\text{Re } Z = X_0 > 0} \text{etr}\,(\Lambda Z - Z^{-1}) \Delta^{-\delta-\theta}(Z) dZ. \end{aligned}$$

Almost all results of this section can be generalized for this function.

Let us derive some functional relations for the functions $A_\delta(\Lambda)$. From the theorem on a convolution and from formula (5) we obtain that

$$\Delta^{\gamma+\delta+p}(\Lambda)A_{\gamma+\delta+p}((M_1+M_2)\Lambda)$$
$$=\int_0^\Lambda A_\gamma(M_1R)A_\delta(M_2(\Lambda-R))\Delta^\gamma(R)\Delta^\delta(\Lambda-R)dR, \tag{6}$$

where $\operatorname{Re}\gamma>-1$, $\operatorname{Re}\delta>-1$, $\Lambda\geq 0$ and M_1, M_2 are any complex matrices. Setting $M_1=M$, $M_2=0$, $\Lambda=I_m$ into (6) and taking into account that $A_\delta(0)=\Gamma_m^{-1}(\delta+p)$, after the replacement of δ by $\delta-p$ we obtain the equality

$$A_{\gamma+\delta}(M)=\Gamma_m^{-1}(\delta)\int_0^{I_m} A_\gamma(MR)\Delta^\gamma(R)\Delta^{\delta-p}(I_m-R)dR. \tag{7}$$

Another formula is obtained by setting $M_1=I_m$, $M_2=0$ and by replacement of δ by $\delta-p$. We have

$$\Delta^{\gamma+\delta}(\Lambda)A_{\gamma+\delta}(\Lambda)=\Gamma_m^{-1}(\delta)\int_0^\Lambda A_\gamma(R)\Delta^\gamma(R)\Delta^{\delta-p}(\Lambda-R)dR. \tag{8}$$

It can be considered as a matrix analogue of the formula of fractional differentiation.

17.4.3. Hankel transforms of functions of a matrix argument. In this and in next sections we assume that $\gamma>-1$. We denote by $\mathfrak{L}^2_\gamma$ the Hilbert space of functions $f(\Lambda)$, $\Lambda>0$, such that

$$\|f\|_\gamma^2=\int_{R>0}|f(R)|^2\Delta^\gamma(R)dR<\infty.$$

A self-adjoint unitary linear transformation of $\mathfrak{L}^2_\gamma$ is called a *Watson transform.* If W is a Watson transform, then W^2 is the identity operator.

We shall prove that *the transformation $f\to g$, where*

$$g(\Lambda)\sim\int_{R>0}A_\gamma(\Lambda R)\Delta^\gamma(R)f(R)dR \tag{1}$$

is a Watson transform in $\mathfrak{L}^2_\gamma$. Here $\sim$ means the equality if integral (1) absolutely converges and the corresponding limit in $\mathfrak{L}^2_\gamma$ otherwise.

We denote the function g from formula (1) by $U_\gamma f$. The operator U_γ is called the *Hankel transform* or the *Hankel γ-transform.*

To prove that U_γ is a Watson transform we note that due to formula (5) of Section 17.4.2 $U_\gamma e_Z = \Delta^{-\gamma-p}(Z)e_{Z^{-1}}$, where $e_Z(R) = \mathrm{etr}(-RZ)$. This equality means that U_γ is a real linear involutive transformation in the linear space $\mathfrak{L} \subset \mathfrak{L}^2_\gamma$ spanned by the functions e_Z, Re $Z > 0$. The equalities

$$(e_{Z_1}, e_{Z_2})_\gamma = \int\limits_{R>0} \mathrm{etr}(-RZ_1)\overline{\mathrm{etr}(-RZ_2)}\Delta^\gamma(R)dR$$
$$= \Gamma_m(\gamma+p)\Delta^{-\gamma-p}(Z_1+\bar{Z}_2),$$
$$(U_\gamma e_{Z_1}, U_\gamma e_{Z_2})_\gamma = \Delta^{-\gamma-p}(Z_1)\Delta^{-\gamma-p}(\bar{Z}_2)(e_{Z_1^{-1}}, e_{Z_2^{-1}})_\gamma$$
$$= \Delta^{-\gamma-p}(Z_1\bar{Z}_2)\Gamma_m(\gamma+p)\Delta^{-\gamma-p}(Z_1^{-1}+\bar{Z}_2^{-1}) = \Gamma_m(\gamma+p)\Delta^{-\gamma-p}(Z_1+\bar{Z}_2)$$

show that U_γ conserves the norm.

The subspace $\mathfrak{L}$ is everywhere dense in $\mathfrak{L}^2_\gamma$. Really, if $(e_Z, f)_\gamma = 0$ for all Z, Re $Z > 0$, then the Laplace transform of f vanishes and, therefore, $f \equiv 0$. This means that U_γ can be extended onto the whole space $\mathfrak{L}^2_\gamma$ and this extension gives a Watson transform.

The following analogue of the Tricomi theorem is valid. *Let* $f \in \mathfrak{L}^2_\gamma$ *and*

$$\Phi(Z) = \int\limits_{R>0} \mathrm{etr}(-RZ)f(R)\Delta^\gamma(R)dR, \quad \mathrm{Re}\, Z > 0. \tag{2}$$

If the integral

$$\Psi(Z) = \int\limits_{\Lambda>0} \mathrm{etr}(-\Lambda Z)g(\Lambda)\Delta^\gamma(\Lambda)d\Lambda$$

absolutely converges for Re $Z > 0$ *and* $g = U_\gamma f$, *then*

$$\Psi(Z) = \Delta^{-\gamma-p}(Z)\Phi(Z^{-1}). \tag{3}$$

Conversely, if the equality (3) *is fulfilled, then* $g \in \mathfrak{L}^2_\gamma$ *and* $g = U_\gamma f$.

Really, we set $g_1 = U_\gamma f$ and

$$\Psi_1(Z) = \int\limits_{\Lambda>0} \mathrm{etr}(-\Lambda Z)g_1(\Lambda)\Delta^\gamma(\Lambda)d\Lambda.$$

Since $g_1 \in \mathfrak{L}^2_\gamma$, then this integral absolutely converges for Re $Z > 0$. Moreover,

$$\Psi_1(Z) = (e_Z, U_\gamma\bar{f})_\gamma = (U_\gamma e_Z, \bar{f})_\gamma = \Delta^{-\gamma-p}(Z)(e_{Z^{-1}}, \bar{f})_\gamma$$
$$= \Delta^{-\gamma-p}(Z)\Phi(Z^{-1}).$$

Thus, setting $g_1 = g$ we obtain relation (3). Conversely, if relation (3) is valid, then due to the uniqueness of the Laplace transform we have $g = g_1$ almost everywhere.

We omitted the details of reasonings connected with the convergence in the space $\mathfrak{L}^2_\gamma$.

If Re $\delta > p - \frac{1}{2}$, then $\Delta^{\delta-p}(I_m - R) \in \mathfrak{L}^2_\gamma$. Consequently, it follows from formula (7) of Section 17.4.2 that $A_{\gamma+\delta}$ is the Hankel transform of a function from $\mathfrak{L}^2_\gamma$. Therefore, $A_{\gamma+\delta} \in \mathfrak{L}^2_\gamma$. Thus, we proved the following statement: *The function* $\Delta^\beta(\Lambda)A_\alpha(\Lambda)$ *belongs to* $\mathfrak{L}^2(\mathfrak{P}_m(\mathbf{R}))$ *if and only if* Re $\left(\alpha - p + \frac{1}{2}\right) >$ Re $(2\beta - 1)$.

We now establish the relationship between the Hankel and the Fourier transforms of "radial functions" which generalizes the Bochner theorem from Section 10.1.7. A function $F(Q)$, $Q \in \mathfrak{M}_{km}(\mathbf{R})$, $k \geq m$, is called *radial* if there exists a function $f(R)$, $R > 0$, such that $F(Q) = f(\pi Q^t Q)$.

If $F(Q)$ is a radial function on $\mathfrak{M}_{km}$, $k \geq m$, $P(Q)$ is an H-polynomial of degree μ and $F(Q)P(Q) \in \mathfrak{L}^2(\mathfrak{M}_{km}(\mathbf{R}))$, then $f \in \mathfrak{L}^2_\gamma$, $\gamma = \mu + \frac{1}{2}(k - m - 1)$, and

$$\int\limits_{\mathfrak{M}_{km}} \operatorname{etr}(-2\pi i S^t Q)F(Q)P(Q)dQ \sim G(S)P(-iS), \tag{4}$$

where $G(S) = g(\pi S^t S)$ and g is the Hankel γ-transform of the function f.

Really, setting $Q^tQ = R > 0$ we have $Q = VR^{1/2}$, where V belongs to the Stiefel manifold $\operatorname{St}(k, m)$. We obtain

$$\int\limits_{\mathfrak{M}_{km}} |F(Q)|^2|P(Q)|^2dQ = \frac{1}{2^m}\int\limits_{R>0} |f(\pi R)|^2\Delta^\gamma(R)dR \int\limits_{\operatorname{St}(k,m)} |P(V)|^2 d_*V$$
$$= \frac{1}{\pi^{m\mu}\Gamma_m\left(\frac{k}{2}\right)}\int\limits_{\operatorname{St}\,(k,m)} |P(V)|^2 d_*V \int\limits_{R>0} |f(R)|^2\Delta^\gamma(R)dR,$$

where d_*V is the normalized invariant measure on $\operatorname{St}(k, m)$. Therefore, $F(Q) \in \mathfrak{L}^2(\mathfrak{M}_{km})$ if and only if $f \in \mathfrak{L}^2_\gamma$. Since the linear span of the functions e_Z, Re $Z > 0$, is everywhere dense in $\mathfrak{L}^2_\gamma$, then the function $F(Q)P(Q)$ belongs to the subspace generated by the functions $\operatorname{etr}(-\pi QZQ^t)P(Q)$, Re $Z > 0$. Therefore, due to formula (1) of Section 17.1.3 we have

$$\int\limits_{\mathfrak{M}_{km}} \operatorname{etr}(-2\pi i S^t Q)\operatorname{etr}(-\pi QZQ^t)P(Q)dQ$$
$$= \Delta^{-\mu-k/2}(Z)\operatorname{etr}(-\pi SZ^{-1}S^t)P(-iS).$$

Since the Fourier transform is unitary, then the transform of the function $F(Q)P(Q)$ belongs to the closed subspace generated by the functions $\operatorname{etr}(-\pi SZ^{-1} \times S^t)P(-iS)$, Re $Z > 0$, and, hence, is of the form $G(S)P(-iS)$, where G is a radial function.

We set $G(S) = g(\pi S^t S)$. It follows from the Plancherel formula that

$$\int\limits_{\mathfrak{M}_{km}} \operatorname{etr}(-\pi Q Z Q^t) F(Q) |P(Q)|^2 dQ$$
$$= \Delta^{-\mu-k/2}(Z) \int\limits_{\mathfrak{M}_{km}} \operatorname{etr}(-\pi S Z^{-1}) G(S) |P(S)|^2 dS,$$

that is,

$$\int\limits_{R>0} \operatorname{etr}(-RZ) F(R) \Delta^{\gamma}(R) dR$$
$$= \Delta^{\gamma-p}(Z) \int\limits_{\Lambda>0} \operatorname{etr}(-\Lambda Z^{-1}) g(\Lambda) \Delta^{\gamma}(\Lambda) d\Lambda.$$

Now formula (4) follows from the formula $(U_\gamma e_Z, U_\gamma f)_\gamma = (e_Z, f)_\gamma$.

Formula (4) can be represented in the form

$$\int\limits_{\mathfrak{M}_{km}} \operatorname{etr}(iS^tQ) f(Q^tQ) P(Q) dQ \sim \left(\frac{i}{2}\right)^{m\mu} \pi^{mk/2} g\left(\frac{1}{4} S^t S\right) P(S). \qquad (4')$$

Let I^{km} be the matrix $\begin{pmatrix} I_m \\ 0 \end{pmatrix} \in \mathfrak{M}_{km}$. We take an H-polynomial P of degree μ such that $P(I^{km}) = 1$. Setting $S = I^{km}$ into (4′) we obtain

$$\int\limits_{\mathfrak{M}_{km}} \operatorname{etr}(2iQ) f(Q^tQ) P(Q) dQ \sim i^{m\mu} \pi^{mk/2} g(I_m). \qquad (5)$$

Let $f(R) = \operatorname{etr}(-RZ)$, $\operatorname{Re} Z > 0$. Due to the decomposition $Q = VR^{1/2}$ we obtain

$$\frac{\pi^{mk/2}}{\Gamma_m(k/2)} \int\limits_{R>0} \operatorname{etr}(-RZ) \Delta^{(\mu+k-m-1)/2}(R) dR$$
$$\times \int\limits_{\mathrm{St}(k,m)} \operatorname{etr}(2iVR^{1/2}) P(V) d_*V = i^{m\mu} \pi^{mk/2} \Delta^{\mu-k/2}(Z) \operatorname{etr}(-Z^{-1}).$$

Using the uniqueness of the Laplace transform we have from here that

$$\Delta^{\mu/2}(R) A_\gamma^{(m)}(R) = \frac{(-i)^{m\mu}}{\Gamma_m\left(\frac{k}{2}\right)} \int\limits_{\mathrm{St}(k,m)} \operatorname{etr}(2iVR^{1/2}) P(V) d_*V, \qquad (6)$$

where, as usual, $\gamma = \frac{1}{2}(k-m-1)+\mu$ and the index (m) at $A_\gamma(R)$ means the dimension. This formula implies that for all $S \in \mathfrak{M}_m(\mathbf{R})$ the relation

$$A^{(m)}_{\delta+\mu}\left(\frac{1}{4}S^tS\right)\Delta^\mu\left(\frac{1}{2}S\right) = \frac{(-i)^{m\mu}}{\Gamma_m(\delta+p)}\int\limits_{\mathrm{St}(k,m)} \mathrm{etr}(iVS)P(V)d_*V \tag{7}$$

is valid, where $\delta = \gamma - \mu$.

As it was mentioned in Section 17.1.3, if $k = m$, then there are two H-polynomials $P_0(Q) = 1$ and $P_1(Q) = \Delta(Q)$. We define the corresponding functions

$$\cos_m S = \int\limits_{O(m)} \mathrm{etr}(i\Omega S)d_*\Omega, \tag{8}$$

$$\sin_m S = i^{-m}\int\limits_{O(m)} \mathrm{etr}(i\Omega S)\Delta(\Omega)d_*\Omega. \tag{8'}$$

It is clear that

$$A^{(m)}_{-1/2}\left(\frac{1}{4}S^tS\right) = \Gamma_m^{-1}\left(p-\frac{1}{2}\right)\cos_m S, \tag{9}$$

$$A^{(m)}_{1/2}\left(\frac{1}{4}S^tS\right) = 2^m\Gamma_m^{-1}\left(p-\frac{1}{2}\right)\Delta^{-1}(S)\sin_m S. \tag{9'}$$

If $k \geq 2m$, then from (7) we obtain the analogue of the Poisson formula

$$\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)\left(\frac{z}{2}\right)^{-\nu}J_\nu(z) = \int\limits_{-1}^{1} q^{izt}(1-t^2)^{\nu-1/2}dt.$$

Namely, if $\mathrm{Re}\,\delta > p-1$ and $\gamma = \mu + \frac{1}{2}(k-m-1)$, then

$$\Gamma_m^{-1}(\delta)\int\limits_{\substack{QQ^t<I_k\\ Q\in\mathfrak{M}_{km}}} \mathrm{etr}(iS^tQ)\Delta^{\delta-p}(I_k-QQ^t)P(Q)dQ = \left(\frac{i}{2}\right)^{m\mu}\pi^{km/2}A^{(m)}_{\gamma+\delta}\left(\frac{1}{4}S^tS\right)P(S). \tag{10}$$

If $k = m$ and $P(s) = 1$, then we have

$$A_\delta\left(\frac{1}{4}S^tS\right) = \frac{\pi^{-m^2/2}}{\Gamma_m(\delta+1/2)}\int\limits_{\substack{T^tT<I_m\\ T\in\mathfrak{M}_{km}}} \mathrm{etr}(iS^tT)\Delta^{\delta-m^2/2}(I_m-T^tT)dT, \tag{11}$$

where Re $\delta > p - 3/2$.

In order to prove formula (10) we remind that the inequalities $QQ^t < I_k$ and $Q^tQ < I_m$ are equivalent and $\Delta(I_k - QQ^t) = \Delta(I_m - Q^tQ)$. It is sufficient to prove formula (10) for Re $\delta > p - \frac{1}{2}$ and make the analytical continuation in δ. Therefore, formula (10) follows from the analogue of the Tricomi theorem, proved in this section, and from the fact that (due to formula (7) of Section 17.4.2) $A_{\gamma+\delta}(M)$ is the Hankel γ-transform of $\Delta^{\delta-p}(I_m - R)$. Using formula (6) of Section 17.4.2 we derive from (11) that if $S_1 \in \mathfrak{M}_m(\mathbf{R})$, $S_2 \in \mathfrak{M}_{km}(\mathbf{R})$, then for every k we have

$$A_\delta\left(\frac{1}{2}(S_1^tS_1 + S_2^tS_2)\right) = \pi^{-m^2} \int\limits_{Q^tQ<I_m} \operatorname{etr}(iS_1^tQ) \times A_{\delta-m/2}\left(\frac{1}{4}S_2(I_m - Q^tQ)S_2^t\right)\Delta^{\delta-m/2}(I_m - Q^tQ)dQ. \tag{12}$$

Let us note that due to equality (7) the formula

$$A_\delta\left(\frac{1}{4}S^tS\right) = \frac{1}{\Gamma_m(\delta+p)} \int\limits_{\mathrm{St}(k,m)} \operatorname{etr}(iS^tV)d_*V \tag{13}$$

is valid for every matrix $S \in \mathfrak{M}_{km}$, where $\delta = \frac{1}{2}(k - m - 1)$.

If $k \geq 2m$, then matrices $V \in \mathrm{St}(k,m)$ can be represented in the form $V = \begin{pmatrix} Q \\ UR^{1/2} \end{pmatrix}$, where $Q \in \mathfrak{M}_m(\mathbf{R}) \equiv \mathfrak{M}_m$, $R = I_m - Q^tQ > 0$ and $U \in \mathrm{St}(k-m,m)$. This representation is uniquely defined in the domain $Q^tQ < I_m$. We have

$$d_*V = \Delta^{\delta-m/2}(I_m - Q^tQ)dQd_*U, \quad \delta = \frac{1}{2}(k - m - 1). \tag{14}$$

With the help of this formula we can write down formula (7) in the form

$$A_{\delta+\mu}\left(\frac{1}{4}S^tS\right)\Delta^\mu\left(\frac{1}{2}S\right) = (-i)^{m\mu}\pi^{-m^2/2}\Gamma_m^{-1}\left(\delta+\frac{1}{2}\right) \times \int\limits_{\substack{Q^tQ<I_m \\ Q\in\mathfrak{M}_m}} \operatorname{etr}(iS^tQ)\Delta^{\delta-m/2}(I_m - Q^tQ)dQ \int\limits_{\mathrm{St}(k-m,m)} P_\mu(V)d_*U, \tag{15}$$

where V is related to Q and U as above. If $P_\mu(I^{km}) = 1$, then

$$\int\limits_{\mathrm{St}(k-m,m)} P_\mu(V)d_*U$$

depends only on Q and does not depend on a choice of H-polynomial $P_\mu(V)$ of degree μ. We set

$$\int_{\mathrm{St}(k-m,m)} P_\mu(V)d_*U = C_\mu^{(\delta)}(Q). \tag{16}$$

It will be shown below that $C_\mu^{(\delta)}(Q)$ are orthogonal polynomials generalizing spherical functions. Thus,

$$A_{\delta+\mu}\left(\frac{1}{4}S^tS\right)\Delta^\mu\left(\frac{1}{2}S\right) = (-i)^{m\mu}\pi^{-m^2/2}\Gamma_m^{-1}\left(\delta+\frac{1}{2}\right) \times \int_{\substack{Q^tQ<I_m\\ Q\in\mathfrak{M}_m}} \mathrm{etr}(iS^tQ)C_\mu^{(\delta)}(Q)\Delta^{\delta-m/2}(I_m-Q^tQ)dQ. \tag{17}$$

It is possible to prove that formula (17) uniquely determine $C_\mu^{(\delta)}(Q)$, $\delta=\frac{1}{2}(k-m-1)$.

If $S\in\mathfrak{M}_{km}$, $V\in\mathrm{St}(k,m)$ and

$$\Delta_{ij}=\sum_{\alpha=1}^{k}\frac{\partial^2}{\partial s_{i\alpha}\partial s_{j\alpha}},$$

then $\Delta_{ij}\mathrm{etr}(iS^tV)=-\delta_{ij}\mathrm{etr}(iS^tV)$. Therefore, it follows from (13) that

$$\Delta_{ij}A_\delta\left(\frac{1}{4}S^tS\right) = -\delta_{ij}A_\delta\left(\frac{1}{4}S^tS\right), \tag{18}$$

where $S\in\mathfrak{M}_{km}$, $\delta=\frac{1}{2}(k-m-1)$. One analogously prove that

$$\Delta_{ij}\cos_k S = -\delta_{ij}\cos_k S. \tag{19}$$

We can show that the single function $F(S)$, $S\in\mathfrak{M}_{km}$, satisfying the conditions

(a) $F(S)=F(\Omega S)$ $\Omega\in O(k)$,

(b) $\Delta_{ii}F=-F$, $1\le i\le m$,

(c) $F(0)=1$

is $F(S)=\cos_k S$.

Let us note without proof that if $\Lambda\in\mathfrak{H}_k(\mathbb{C})$, rank $\Lambda=m$, then

$$A_\delta^{(k)}(\Lambda)=\Gamma_m\left(\delta+p+\frac{1}{2}(k-m)\right)\Gamma_k^{-1}(\delta+p)A_{\delta+(k-m)/2}^{(m)}(\Lambda), \quad \delta\in\mathbb{C}, \tag{20}$$

where the index (p) at A_γ denotes the dimension of the space. In particular, for $S \in \mathfrak{M}_{km}(\mathbf{R})$, $k \geq m$, one has

$$\cos_k S = \Gamma_m(\delta + p) A_\delta^{(m)} \left(\frac{1}{4} S^t S \right), \quad \delta = \frac{1}{2}(k - m - 1). \tag{21}$$

We also note that if $F(S) \in \mathfrak{L}^2(\mathfrak{M}_{km}(\mathbf{R}))$ and the Fourier transform of F is absolutely convergent, then

$$2^{-k} \pi^{-mk/2} \int\limits_{R>0} \Delta^\gamma(R) dR \int\limits_{\mathfrak{M}_{km}} A_\gamma \left(\frac{1}{4}(S - Q) R (S - Q)^t \right) F(Q) dQ = F(S), \tag{22}$$

where $\gamma = \frac{1}{2}(k - m - 1)$.

17.4.4. Bessel functions of the second kind in a matrix argument. These functions are defined by the formula

$$B_\delta(Z) = \int\limits_{\Lambda>0} \operatorname{etr}(-\Lambda - \Lambda^{-1} Z) \Delta^{-\delta-p}(\Lambda) d\Lambda. \tag{1}$$

This integral absolutely converges for all δ in the domain $\operatorname{Re} Z > 0$. Using the substitution $\Lambda \to \Lambda^{-1}$ we obtain

$$B_{-\delta}(Z) = \Delta^\delta(Z) B_\delta(Z). \tag{2}$$

It follows from here that

$$B_\delta(Z) = \int\limits_{\Lambda>0} \operatorname{etr}(-\Lambda Z - \Lambda^{-1}) \Delta^{\delta-p}(\Lambda) d\Lambda. \tag{3}$$

This integral absolutely converges for $\operatorname{Re} Z = 0$ if and only if $-\operatorname{Re} \delta > p - 1$. We derive from here that for $\delta \in \mathbf{R}$ the function B_δ has a singularity at zero.

Repeating reasonings used for the derivation of formulas (3) and (4) of Section 17.4.2 we obtain the equalities

$$D_Z(\Delta^\delta(Z) B_\delta(Z)) = (-1)^m \Delta^{\delta-1}(Z) B_{\delta-1}(Z), \tag{4}$$

$$D_Z(B_\delta(Z)) = (-1)^m B_{\delta+1}(Z), \tag{5}$$

where $D_Z = \det(\partial/\partial z_{ij})$. The hyperbolic differential equation

$$D_Z[\Delta^{1-\delta}(Z) D_Z(\Delta^\delta(Z) B_\delta(Z))] = B_\delta(Z) \tag{6}$$

for the functions $B_\delta(Z)$ follows from them.

It is easy to prove that

$$\int\limits_{R>0} A_\gamma(\Lambda R)B_{-\delta}(ZR)\Delta^\gamma(R)dR = \Gamma_m(\gamma+\delta+p)\Delta^\delta(Z)\Delta^{-\gamma-\delta-p}(Z+\Lambda), \tag{7}$$

where the integral absolutely converges for Re $Z > 0$, Re $(\gamma+\delta) > -1$. This integral can be considered as the Hankel transform for $\gamma > -1$, Re $\delta > -\frac{1}{2}(1+\gamma)$. From the inversion formula for this transform we derive that

$$B_{-\delta}(MZ)\Delta^{-\delta}(Z) \sim \Gamma_m(\gamma+\delta+p)\int\limits_{\Lambda>0} \Delta^{-\gamma-\delta-p}(Z+\Lambda)\Delta^\gamma(\Lambda)A_\gamma(\Lambda M)d\Lambda, \tag{8}$$

where the integral converges in the topology of the Hilbert space $\mathfrak{L}^2_\gamma$. From here we obtain

$$B_\gamma(MZ)\Delta^\gamma(Z) \sim \Gamma_m(p)\int\limits_{\Lambda>0} \Delta^{-p}(Z+\Lambda)\Delta^\gamma(\Lambda)A_\gamma(\Lambda M)d\Lambda, \tag{9}$$

where $-1 < \gamma < 1$.

The convergence in the mean for all Z in the domain Re $Z > 0$ means that for all Z from this domain the integral converges almost for all $M > I_m$. Unfortunately, it is not proved that convergence has place for $M = I_m$. If it would be so, then we would have the equality

$$B_\gamma(Z)\Delta^\gamma(Z) = \Gamma_m(p)\int\limits_{\Lambda>0} \Delta^{-p}(Z+\Lambda)\Delta^\gamma(\Lambda)A_\gamma(\Lambda)d\Lambda, \tag{10}$$

which expresses $B_\gamma(Z)\Delta^\gamma(Z)$ as the Stieltjes transform of the function $A_\gamma(\Lambda)$.

Applying the analogue of the Tricomi theorem to equality (7) we derive the analogues of formulas (10) and (11) of Section 17.4.3. The first one is of the form

$$\begin{aligned}\Gamma_m(\delta+p)&\int\limits_{\mathfrak{M}_{km}} \operatorname{etr}(iS^tQ)\Delta^{-\delta-p}(I_k+QQ^t)P(Q)dQ \\ &= \frac{i^m\pi^{mk/2}}{2^{m\mu}}B^{(m)}_{\gamma-\delta}\left(\frac{1}{4}S^tS\right)P(S), \quad \gamma=\mu+\frac{1}{2}(k-m-1),\end{aligned} \tag{11}$$

where the integral absolutely converges for Re $\delta > \gamma+p-1$ and converges in the mean for Re $\delta > \frac{1}{2}(\gamma-1)$. If $k = m$ and $P(Q) = 1$, then we obtain the analogue of the Poisson formula:

$$B_{-\delta}\left(\frac{1}{4}S^tS\right) = \frac{\Gamma_m\left(\delta+p-\frac{1}{2}\right)}{\pi^{m^2/2}}\int\limits_{\mathfrak{M}_m} \operatorname{etr}(iS^tQ)\Delta^{-\delta-m/2}(I_m+Q^tQ)dQ. \tag{12}$$

This integral absolutely converges for Re $\delta > p-1$ and converges in the mean if Re $\delta > -\frac{1}{4}$.

There are many relations for the function B_δ. We mention the product formula

$$\int_{R>0} \operatorname{etr}(-RZ-(\Lambda+M)R^{-1})B_\delta(\Lambda R^{-1}MR^{-1})\Delta^{-\delta-p}(R)dR \\ = B_\delta(\Lambda Z)B_\delta(MZ)\Delta^\delta(Z) \tag{13}$$

which is analogous to the product formula

$$\int_{R>0} \operatorname{etr}(-RZ)A_\delta(\Lambda R)A_\delta(MR)\Delta^\delta(R)dR \\ = \operatorname{etr}(-(\Lambda+M)Z^{-1})A_\delta(-\Lambda Z^{-1}MZ^{-1})\Delta^{-\delta-p}(Z) \tag{14}$$

for the Bessel functions of the first kind in a matrix argument. In (13) $\Lambda > 0$, $M > 0$, Re $Z > 0$ and in (14) Re $\delta > -1$, Re $Z > 0$.

To prove formula (14) we note that it is sufficient to derive it for $Z = I_m$ and $\delta = \gamma > -1$. In this case the left hand side of (14) is the Hankel γ-transform of the function $\operatorname{etr}(-R)A_\gamma(MR)$ and our result follows from the analogue of the Tricomi theorem. To prove formula (13) we consider the integrals

$$B_\delta(\Lambda Z) = \Delta^{-\delta}(\Lambda)\int_{R>0} \operatorname{etr}(-RZ-\Lambda R^{-1})\Delta^{\delta-p}(R)dR,$$

$$B_\delta(MZ)\Delta^\delta(Z) = \int_{R>0} \operatorname{etr}(-RZ-MR^{-1})\Delta^{-\delta-p}(R)dR.$$

Applying the theorem on a convolution for the Laplace transform we derive from these formulas that

$$B_\delta(\Lambda Z)B_\delta(MZ)\Delta^\delta(Z) = \Delta^{-\delta}(\Lambda)\int_{R>0} \operatorname{etr}(-RZ) \\ \times \int_0^R \operatorname{etr}(-\Lambda(R-Q)^{-1})\Delta^{\delta-p}(R-Q)\operatorname{etr}(-MQ^{-1})\Delta^{-\delta-p}(Q)dQdR. \tag{15}$$

Making here the substitution $Q = (R^{-1}+X)^{-1}$ we have $dQ = \Delta^{-2p}(R^{-1}+X)dX$. Note that $Q = (R^{-1}+X)^{-1}$ is in fact the reduced writing for $Q = R^{1/2}(I_m + R^{1/2}XR^{1/2})^{-1}R^{1/2}$. We have

$$\Delta^{-\delta}(\Lambda)\Delta^{-p}(R)\int_0^\infty \operatorname{etr}\{-\Lambda R^{-1}(I_m+X^{-1})\}\operatorname{etr}\{-MR^{-1}(I_m+X)\} \\ \times\Delta^{\delta-p}(X)dX = \operatorname{etr}(-(\Lambda+M)R^{-1})B_\delta(\Lambda R^{-1}MR^{-1})\Delta^{-\delta-p}(R).$$

The analogue of formula (20) of Section 17.4.3 is also valid. Namely, if $\Lambda \in \mathfrak{M}_k$, rank $\Lambda = m$ and $-\text{Re}\,\delta > \frac{1}{2}(k-m-1)$, then

$$B^{(k)}_{-\delta}(\Lambda) = \Gamma_k(\delta)\Gamma_m^{-1}\left(\delta - \frac{1}{2}(k-m)\right) B^{(m)}_{-\delta+(k-m)/2}(\Lambda). \tag{16}$$

17.4.5. Macdonald functions of a matrix argument. The classical Macdonald function $K_\nu(z)$ can be defined by the integral

$$K_\nu(z) = \frac{1}{2}\int_0^\infty w^\nu \exp\left[-\frac{z}{2}\left(w + \frac{1}{w}\right)\right]\frac{dw}{w}, \tag{1}$$

where $\nu, z \in \mathbb{C}$, Re $z > 0$. The Macdonald function of a matrix argument is analogously defined by the formula[5]

$$K_m(\mathbf{s} \mid A, B) = \int_{Y>0} \Delta^{\mathbf{s}}(Y)\text{etr}(-AY - BY^{-1})d_*Y, \tag{2}$$

where $\mathbf{s} \in \mathbb{C}^m$, $\quad A, B \in \mathfrak{S}_m(\mathbf{R})$.

The formula

$$\Gamma_m(\mathbf{s}) \int_{\mathfrak{M}_{km}} \Delta^{\mathbf{s}}((A + X^tX)^{-1})\text{etr}(2iR^tX)dX = \pi^{km/2}K_m\left(\mathbf{s} - \frac{k}{2}\,\middle|\, A, R^tR\right) \tag{3}$$

is valid, where $A > 0$, $R \in \mathfrak{M}_{km}(\mathbf{R})$, $\mathbf{s} = (s_1, \ldots, s_m)$. To prove it we express the function $\Gamma_m(\mathbf{s})$ from the left hand side by the integral (1) of Section 17.1.4 and invert the integration order. Denoting the left hand side of (3) by I we obtain

$$I = \int_{\mathfrak{M}_{km}}\int_{Y>0} \Delta^{\mathbf{s}}(Y)\Delta^{\mathbf{s}}((A + X^tX)^{-1})\text{etr}(-Y) \times \text{etr}(2iR^tX)d_*Y dX.$$

Since $A > 0$, then $A + X^tX > 0$ and there is $T \in S_+(m, \mathbf{R})$ such that $A + X^tX = TT^t$. Using the substitution $Y \to T^tYT$ we have

$$I = \int_{\mathfrak{M}_{km}}\int_{Y>0} \Delta^{\mathbf{s}}(T^tYT)\Delta^{\mathbf{s}}((TT^t)^{-1})\text{etr}(-T^tYT) \times \text{etr}(2iR^tX)d_*Y dX.$$

[5] In this section we consider the general case of the function $\Delta^{\mathbf{s}}$ for $\mathbf{F} = \mathbf{R}$.

Since the matrix T is triangular, then

$$\begin{aligned}\Delta^{\mathbf{s}}(T^tYT)\Delta^{\mathbf{s}}((TT^t)^{-1}) &= \Delta^{\mathbf{s}}(Y)\Delta^{\mathbf{s}}(T^tT)\Delta^{\mathbf{s}}((T^{-1})^tT^{-1})\\ &= \Delta^{\mathbf{s}}(Y)\Delta^{\mathbf{s}}(T^t(T^{-1})^tT^{-1}T) = \Delta^{\mathbf{s}}(Y).\end{aligned}$$

Besides, Tr (T^tYT) = Tr (TT^tY). Therefore, the left hand side of (3) is of the form

$$\int\limits_{Y>0} \Delta^{\mathbf{s}}(Y)\mathrm{etr}(-AY)d_*Y \int\limits_{\mathfrak{M}_{km}} \mathrm{etr}(-X^tXY + 2iR^tX)dX. \tag{4}$$

We make the substitution $X \to XY^{-1/2}$. Since $dX \to \Delta^{-m/2}(Y)dX$, then

$$\int\limits_{\mathfrak{M}_{km}} \mathrm{etr}(-X^tXY + 2iR^tX)dX$$

$$= \int\limits_{\mathfrak{M}_{km}} \mathrm{etr}(-(XY^{-1/2})^t(XY^{-1/2})Y + 2iR^tXY^{-1/2})\Delta^{-m/2}(Y)dX.$$

We have Tr $((XY^{-1/2})^t(XY^{-1/2})Y)$ = Tr (X^tX). Setting $C = iRY^{-1/2}$ and taking into account the relation

$$\mathrm{Tr}\,((X-C)^t(X-C)) + \mathrm{Tr}\,(RY^{-1}R^t) = \mathrm{Tr}\,(X^tX - 2iR^tXY^{-1/2})$$

we now can represent the integral in X in the form

$$\Delta^{-m/2}(Y) \int\limits_{\mathfrak{M}_{km}} \mathrm{etr}(-(X-C)^t(X-C)) + \mathrm{etr}(-RY^{-1}R^t)dX.$$

Therefore, the left hand side of (3) is of the form

$$\int\limits_{Y>0} \Delta^{\mathbf{s}-m/2}(Y)\mathrm{etr}(-AY - R^tRY^{-1})dY$$
$$\times \int\limits_{\mathfrak{M}_{km}} \mathrm{etr}(-(X-C)^t(X-C))dX.$$

The substitution $X \to X + C$ shows that the integral in X is equal to $\pi^{mk/2}$. Thus, the left hand side of (3) is reduced to the integral determining its right hand side and formula (3) is proved.

Now we define k-Bessel function setting

$$k_{\ell,m-\ell}(\mathbf{s} \mid Y, N) = \int\limits_{\mathfrak{M}_{\ell,m-\ell}} \Delta^{\mathbf{s}}(Z^tYZ)\mathrm{etr}(2iN^tX)dX, \tag{5}$$

where $Y > 0$, $\mathbf{s} \in \mathbb{C}^m$, $N \in \mathfrak{M}_{\ell,m-\ell}$, $Z = \begin{pmatrix} I_\ell & 0 \\ X^t & I_{m-\ell} \end{pmatrix}$ and dX is the Euclidean measure. Using the inversion formula for the Fourier transform in $\mathbf{R}^{\ell(m-\ell)}$ we obtain

$$\Delta^{\mathbf{s}}(Z^tYZ) = \left(\frac{2}{\pi}\right)^{\ell(m-\ell)} \int\limits_{\mathfrak{M}_{\ell,m-\ell}} k_{\ell,m-\ell}(\mathbf{s} \mid Y, N)\mathrm{etr}(-2iN^tX)dN.$$

It follows from the definition that under the action of the abelian subgroup $\left\{\begin{pmatrix} I_\ell & 0 \\ X^t & I_{m-\ell} \end{pmatrix}\right\}$ of $GL(m, \mathbf{R})$ the k-Bessel function transforms as

$$k_{\ell,m-\ell}(\mathbf{s} \mid Z^tYZ, N) = \mathrm{etr}(-2iN^tX)k_{\ell,m-\ell}(\mathbf{s} \mid Y, N), \tag{6}$$

where Z and X are such as above.

We also have the relation

$$\Gamma_\ell(\tilde{\mathbf{s}})k_{\ell,m-\ell}\left(\mathbf{s}_{\mathrm{ext}} \middle| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, N\right)$$
$$= \pi^{\ell(m-\ell)}\Delta^{-\ell/2}(B)K_\ell\left(\hat{\mathbf{s}} \mid NB^{-1}N^t, A\right), \tag{7}$$

where $\mathbf{s} = (s_1, \ldots, s_\ell) \in \mathbb{C}^\ell$, $\mathbf{s}_{\mathrm{ext}} = (s_1, \ldots, s_\ell, 0, \ldots, 0) \in \mathbb{C}^m$, $\tilde{\mathbf{s}} = (-s_\ell, \ldots, -s_1)$, $\hat{\mathbf{s}} = \mathbf{s} + (m-\ell)/2$. Really, it follows from the previous assertion that

$$\Gamma_\ell(\mathbf{s}) \int\limits_{\mathfrak{M}_{\ell,m-\ell}} \Delta^{\mathbf{s}}((A + X^tX)^{-1})\mathrm{etr}(2iR^tX)dX$$
$$= K_\ell\left(\mathbf{s} - \frac{m-\ell}{2} \middle| A, RR^t\right)\pi^{\ell(m-\ell)/2}, \tag{8}$$

where $A > 0$, $R \in \mathfrak{M}_{\ell,m-\ell}$, $\mathbf{s} \in \mathbb{C}^\ell$. A simple evaluation shows that $\Delta^{\tilde{\mathbf{s}}}(Y^{-1}) = \Delta^{\mathbf{s}}(SYS)$, where $s_{ki} = (\delta_{k,\ell-i+1})$. Replacing $\mathbf{s}$ by $\tilde{\mathbf{s}}$ in (8) we obtain

$$\Gamma_\ell(\tilde{\mathbf{s}}) \int\limits_{\mathfrak{M}_{\ell,m-\ell}} \Delta^{\mathbf{s}}(SAS + SX^tXS)\mathrm{etr}(2iR^tX)dX$$
$$= \pi^{\ell(m-\ell)/2}K_\ell\left(\tilde{\mathbf{s}} - \frac{m-\ell}{2} \middle| A, RR^t\right).$$

Since

$$\Delta^{\mathbf{s}_{\mathrm{ext}}}\left(Z^t\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}Z\right) = \Delta^{\mathbf{s}}(A + XBX^t),\ Z = \begin{pmatrix} I_\ell & 0 \\ X^t & I_{m-\ell} \end{pmatrix},$$

then making the substitutions $X \to SXB^{1/2}$, $dX = \Delta^{m/2}(B)dX$ and then the substitutions $R^t \to B^{-1/2}R^tS$, $A \to SAS$ we obtain formula (7).

Since $\Delta^{\mathbf{s}}(\Lambda)$ is an eigenfunction of invariant differential operators on $\mathfrak{P}_m(\mathbf{R})$, then the k-Bessel function also is an eigenfunction of these operators. Using relation (7) between the Bessel functions K and k we can derive from here the differential equation for the K-Bessel function.

Let us give some properties of the Bessel functions. We have

$$\int\limits_{A>0} \Delta^{\mathbf{s}}(A) K_m(\mathbf{t} \mid I_m, A) d_* A = \Gamma_m(\mathbf{s}) \Gamma_m(\mathbf{t}+\mathbf{s}). \tag{9}$$

Really, the left hand side can be written as

$$\int\limits_{A>0} \int\limits_{Y>0} \Delta^{\mathbf{s}}(A) \Delta^{\mathbf{t}}(Y) \operatorname{etr}(-Y - AY^{-1}) d_* Y d_* A.$$

Let $Y = Q^t Q$, where $Q \in S_+(m, \mathbf{R})$. Making the substitution $A \to Q^t A Q$ after simple transformations we obtain (9).

The function $k_{m-1,1}(\mathbf{s} \mid Y, \mathbf{n})$ can be reduced to the function of one matrix argument:

$$\begin{aligned} k_{m-1,1}(\mathbf{s} \mid Y, \mathbf{n}) = \operatorname{etr}(-2i\mathbf{n}^t \mathbf{q}) \Delta^{\hat{\mathbf{s}}}(A) \\ \times k_{m-1,1}(\mathbf{s} \mid I_m, T\mathbf{n} h^{-1/2}). \end{aligned} \tag{10}$$

Here $\hat{\mathbf{s}} = \mathbf{s} + \left(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{m-1}{2}\right)$ and

$$Y = \begin{pmatrix} I_{m-1} & \mathbf{q} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} G & \mathbf{0} \\ \mathbf{0} & h \end{pmatrix} \begin{pmatrix} I_{m-1} & \mathbf{0} \\ \mathbf{q}^t & 1 \end{pmatrix}, \quad A = \begin{pmatrix} G & \mathbf{0} \\ \mathbf{0} & h \end{pmatrix}, \quad G > 0, \quad h > 0,$$

$$\mathbf{q} \in \mathbf{R}^{m-1}, \; G = T^t T, \; T \in S_+(m-1, \mathbf{R}).$$

For the function $k_{m-1,1}(\mathbf{s} \mid I_m, \mathbf{n})$ we have

$$\begin{aligned} k_{m-1,1}(\mathbf{s} \mid I_m, \mathbf{n}) = \int\limits_{-\infty}^{\infty} (1+u^2)^r e^{2in_1 u} \\ \times k_{m-2,1}(\mathbf{s}^* \mid I_{m-1}, (1+u^2)^{1/2} \mathbf{n}_1) du, \end{aligned} \tag{11}$$

where $\mathbf{n} = (n_1, \dots, n_{m-1}) \in \mathbf{R}^{m-1}$, $\mathbf{n}_1 = (n_2, \dots, n_{m-1})$, $r = s_1 + (m-2)/2$, $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{C}^m$, $\mathbf{s}^* = (s_2, \dots, s_m)$. We omit the proofs of formulas (10) and (11) which are not complicated.

17.4.6. The confluent hypergeometric function of a matrix argument. The function ${}_1F_1(\alpha; \beta; M)$ can be defined in two ways. According to formula (9) of Section 17.4.1 we have

$$\begin{aligned} {}_1F_1(\alpha; \beta; M) \Delta^{\beta - p}(M) = \Gamma_m(\beta)(2\pi i)^{-n} \\ \times \int\limits_{\operatorname{Re} Z = X_0 > 0} \operatorname{etr}(MZ) \Delta^{-\alpha}(I_m - Z^{-1}) \Delta^{-\beta}(Z) dZ, \end{aligned} \tag{1}$$

where the integral absolutely converges for any complex M and α in the domain $X_0 > 0$, $X_0 > \text{Re } M$, $\text{Re } \beta > m$. It is easy to show that the function ${}_1F_1$ is entire both in M and in α and holomorphic in the half-plane $\text{Re } \beta > p - 1$. Formula (1) has the form of the inverse Laplace transform. Therefore,

$$\frac{1}{\Gamma_m(\beta)} \int\limits_{\Lambda>0} \text{etr}(-\Lambda Z)\, {}_1F_1(\alpha;\beta;\Lambda)\Delta^{\beta-p}(\Lambda)d\Lambda = \Delta^{-\alpha}(I_m - Z^{-1})\Delta^{-\beta}(Z). \tag{2}$$

This integral absolutely converges for $\text{Re } Z > 0$, $\text{Re } \beta > p - 1$. The Laplace transform of the function

$$\Gamma_m^{-1}(\beta)\text{etr}(-\Lambda)\Delta^{\beta-p}(\Lambda)\, {}_1F_1(\alpha;\beta;\Lambda)$$

is

$$\Delta^{-\alpha}(I_m - (I_m + Z)^{-1})\Delta^{-\beta}(I_m + Z) = \Delta^{-(\beta-\alpha)}(I_m + Z^{-1})\Delta^{-\beta}(Z). \tag{3}$$

Comparing it with the right hand side of formula (2) we obtain the Kummer formula

$$\text{etr}(-\Lambda)\, {}_1F_1(\alpha;\beta;\Lambda) = {}_1F_1(\beta-\alpha;\beta;-\Lambda). \tag{4}$$

Using the analytical continuation we convince that this formula is true for $\Lambda \in \mathfrak{S}_m(\mathbb{C})$.

For $\text{Re } \alpha > p - 1$, $\text{Re } \beta > p - 1$, $\text{Re } (\beta - \alpha) > p - 1$ we have the equality

$${}_1F_1(\alpha;\beta;Z) = \Gamma_m(\beta)[\Gamma_m(\alpha)\Gamma_m(\beta-\alpha)]^{-1} \times \int\limits_0^{I_m} \text{etr}(RZ)\Delta^{\alpha-p}(R)\Delta^{\beta-\alpha-p}(I_m - R)dR. \tag{5}$$

Really, the Laplace transform of the function $\Gamma_m^{-1}(\beta)\, {}_1F_1(\alpha;\beta;-\Lambda)\Delta^{\beta-p}(\Lambda)$ is equal to

$$\Delta^{-\alpha}(I_m - Z^{-1})\Delta^{-\beta}(Z) = \Delta^{-\alpha}(I_m + Z)\Delta^{\beta-\alpha}(Z),$$

that is to product of the Laplace transforms of the functions

$$\Gamma_m^{-1}(\alpha)\text{etr}(-\Lambda)\Delta^{\alpha-p}(\Lambda) \quad \text{and} \quad \Gamma_m^{-1}(\beta-\alpha)\Delta^{\beta-\alpha-p}(\Lambda).$$

Therefore, using the theorem on a convolution we obtain for $\Lambda > 0$ that

$$\frac{\Delta^{\beta-p}(\Lambda)}{\Gamma_m(p)}\, {}_1F_1(\alpha;\beta;-\Lambda) = \Gamma_m^{-1}(\alpha)\Gamma_m^{-1}(\beta-\alpha) \times \int\limits_0^{\Lambda} \text{etr}(-M)\Delta^{\alpha-p}(M)\Delta^{\beta-\alpha-p}(\Lambda - M)dM. \tag{5'}$$

The replacement $M = \Lambda^{1/2}R\Lambda^{1/2}$ shows that formula (5) is valid for all $Z < 0$. Using the analytical continuation we derive that formula (5) is true for all $Z \in \mathfrak{S}_m(\mathbb{C})$.

The expression

$$L_\mu^{(\gamma)}(Z) = \frac{\Gamma_m(\gamma+\mu+p)}{\Gamma_m(\gamma+p)}\, {}_1F_1(-\mu;\gamma+p;Z), \tag{6}$$

where $\mathrm{Re}\,\gamma > -1$, $\mathrm{Re}\,(\gamma+\mu) > -1$, is called the *Laguerre function* of Z. The Kummer formula (4) shows that

$$L_\mu^{(\gamma)}(Z) = \frac{\Gamma_m(\gamma+\mu+p)}{\Gamma_m(\gamma+p)}(\mathrm{etr}\,Z)\, {}_1F_1(\gamma+\mu+p;\gamma+p;-Z). \tag{7}$$

If $Z > 0$, $\mathrm{Re}\,\mu > p-1$, then according to the theorem on a convolution one has

$$L_\mu^{(\gamma)}(Z) = \frac{(\mathrm{etr}\,Z)\Delta^{-\gamma}(Z)}{\Gamma_m(-\mu)}\int_0^Z \mathrm{etr}(-R)\Delta^{\gamma+\mu}(R)\Delta^{-\mu-p}(Z-R)dR.$$

Using the fractional differentiation we can write this formula in the form

$$L_\mu^{(\gamma)}(Z) = (\mathrm{etr}\,Z)\Delta^{-\gamma}(Z)D^\mu\{\mathrm{etr}(-Z)\Delta^{\gamma+\mu}(Z)\}. \tag{8}$$

It is an analogue of the formula of the Rogrigues type for the Laguerre function of a scalar argument.

With the help of formula (8) of Section 17.4.1 the function ${}_1F_1$ can be defined in the other way:

$${}_1F_1(\alpha;\beta;-Z^{-1}) = \frac{\Gamma_m(\beta+p)}{\Gamma_m(\alpha)}\int_{\Lambda>0}\mathrm{etr}(-\Lambda Z)A_\beta(\Lambda)\Delta^\alpha(\Lambda)d_*\Lambda. \tag{9}$$

Due to formula (7) we have

$$\int_{R>0}\mathrm{etr}(-RZ)A_\gamma(\Lambda R)\Delta^{\gamma+\mu}(R)dR$$
$$= \mathrm{etr}(-\Lambda Z^{-1})L_\mu^{(\gamma)}(\Lambda Z^{-1})\Delta^{-\gamma-\mu-p}(Z). \tag{10}$$

17.4.7. Whittaker functions of a matrix argument. We define the Whittaker function $M(\kappa;\mu;Z)$, $Z \in \mathfrak{S}_m(\mathbf{R})$, by the formula

$$M(\kappa;\mu;Z) = \frac{\Gamma_m(2\mu+p)}{\Gamma_m\left(\mu+\kappa+\frac{p}{2}\right)}\frac{\Delta^{\mu+p/2}(Z)\mathrm{etr}\left(-\frac{1}{2}Z\right)}{\Gamma_m\left(\mu-\kappa+\frac{p}{2}\right)}$$
$$\times\int_0^{I_m}\mathrm{etr}(RZ)\Delta^{\mu-\kappa-p/2}(R)\Delta^{\mu+\kappa-p/2}(I_m-R)dR, \tag{1}$$

where, remind, $p = \frac{1}{2}(m+1)$. Making the substitution $R \to I_m - R$ one has

$$M(\kappa;\mu;Z) = \frac{\Gamma_m(2\mu+p)}{\Gamma_m\left(\mu+\kappa+\frac{p}{2}\right)} \frac{\Delta^{\mu+p/2}(Z)\mathrm{etr}\left(\frac{1}{2}Z\right)}{\Gamma_m\left(\mu-\kappa+\frac{p}{2}\right)} \times \int_0^{I_m} \mathrm{etr}(-RZ)\Delta^{\mu+\kappa-p/2}(R)\Delta^{\mu-\kappa-p/2}(I_m-R)dR. \tag{1'}$$

The integral (1) absolutely converges for Re $\mu > -\frac{1}{2}$ and Re $(\mu \pm \kappa) > \frac{p}{2} - 1$. It follows from (1) and (1′) that

$$M(\kappa;\mu;-Z) = (-1)^{m(\mu+p/2)}(\mathrm{etr}\ Z)M(-\kappa;\mu;Z). \tag{2}$$

Another integral representation of the function $M(\kappa;\mu;Z)$ can be obtained from formulas (1) and (1′) with the help of the substitution $R^{1/2}ZR^{1/2} \to R$, where $Z = X + iY$, $R > 0$, $X > 0$, $X, Y \in \mathfrak{S}_m(\mathbf{R})$. We receive

$$\mathrm{B}_m\left(\mu+\kappa+\frac{p}{2},\mu-\kappa+\frac{p}{2}\right)\Delta^{\mu-p/2}(Z)M(\kappa;\mu;Z)$$
$$= \mathrm{etr}\left(-\frac{1}{2}Z\right)\int_0^Z (\mathrm{etr}\ R)\Delta^{\mu-\kappa-p/2}(R)\Delta^{\mu+\kappa-p/2}(Z-R)dR$$
$$= \mathrm{etr}\left(\frac{1}{2}Z\right)\int_0^Z \mathrm{etr}(-R)\Delta^{\mu+\kappa-p/2}(R)\Delta^{\mu-\kappa-p/2}(Z-R)dR.$$

The substitution $2R - Z \to RZ$ leads to the formula

$$M(\kappa;\mu;Z) = \frac{2^{-2\mu m}\Delta^{\mu+1/2}(Z)}{\mathrm{B}_m\left(\mu+\kappa+\frac{p}{2},\mu-\kappa+\frac{p}{2}\right)} \times \int_{-I_m}^{I_m} \mathrm{etr}\left(\frac{1}{2}RZ\right)\Delta^{\mu-\kappa-p/2}(I_m+R)\Delta^{\mu+\kappa-p/2}(I_m-R)dR. \tag{3}$$

Applying to both sides of formula (1) the operator $D_Z = \left(\frac{\partial}{\partial z_{ij}}\right)_{i,j=1}^m$ and inverting the order of differentiation and integration we derive the equality

$$\mathrm{B}_m\left(\mu+\kappa+\frac{p}{2},\mu-\kappa+\frac{p}{2}\right)D_Z\left[\mathrm{etr}\left(\frac{1}{2}Z\right)\Delta^{-\mu-1/2}(Z)M(\kappa;\mu;Z)\right] = \int_0^{I_m} \mathrm{etr}(RZ)\Delta^{\mu-\kappa+1-p/2}(R)\Delta^{\mu+\kappa-p/2}(I_m-R)dR, \tag{4}$$

which leeds to the formula

$$D_Z^{(2)}\left(\frac{\Gamma_m\left(\mu-\kappa+\frac{p}{2}\right)}{\Gamma_m(2\mu+p)}\mathrm{etr}\left(\frac{1}{2}Z\right)\Delta^{-\mu-p/2}(Z)M(\kappa;\mu;Z)\right)$$
$$=\frac{\Gamma_m\left(\mu+2-\kappa+\frac{p}{2}\right)}{\Gamma_m(2\mu+2+p)}\mathrm{etr}\left(\frac{1}{2}Z\right)\Delta^{-\mu-1-p/2}(Z)M(\kappa-1;\mu+1;Z). \quad (5)$$

Thus we proved that one of the solutions of the equation

$$D_Z^{(2)}\tilde{F}(\kappa;\mu;Z)=\tilde{F}(\kappa-1;\mu+1;Z) \quad (6)$$

is the function

$$\tilde{F}(\kappa;\mu;Z)=\frac{\Gamma_m\left(\mu-\kappa+\frac{p}{2}\right)}{\Gamma_m(2\mu+p)}\mathrm{etr}\left(\frac{1}{2}Z\right)\Delta^{-\mu-p/2}(Z)M(\kappa;\mu;Z), \quad (7)$$

where $Z\in\mathfrak{S}_m(\mathbb{C})$, Re $Z>0$, Re $(\mu\pm\kappa)>\frac{p}{2}-1$, Re $\mu>-\frac{1}{2}$. Analogous results are obtained by application of the operator D_Z to other integral representations of $M(\kappa;\mu;Z)$.

Using operators of fractional differentiation of functions in a matrix argument we derive from (4) that

$$D_Z^{(\alpha)}\left[\mathrm{etr}\left(\frac{1}{2}Z\right)\Delta^{\mu-p/2}(Z)M(\kappa;\mu;Z)\right]=\frac{\Gamma_m(2\mu+p)}{\Gamma_m(2\mu-\alpha+p)}$$
$$\times\Delta^{\mu-(\alpha+p)/2}(Z)\mathrm{etr}\left(\frac{1}{2}Z\right)M\left(\kappa-\frac{\alpha}{2};\mu-\frac{\alpha}{2};Z\right), \quad (8)$$

where Re $(\mu+\kappa)>\alpha+\frac{p}{2}-1$, Re $(\mu-\kappa)>\frac{p}{2}-1$.

From formula (4) we analogously obtain that

$$D_Z^{(\alpha)}\left[\mathrm{etr}\left(-\frac{1}{2}Z\right)\Delta^{\mu-p/2}(Z)M(\kappa;\mu;Z)\right]=\frac{\Gamma_m(2\mu+p)}{\Gamma_m(2\mu-\alpha+p)}$$
$$\times\Delta^{\mu-(\alpha+p)/2}(Z)\mathrm{etr}\left(-\frac{1}{2}Z\right)M\left(\kappa+\frac{\alpha}{2};\mu-\frac{\alpha}{2};Z\right). \quad (8')$$

It follows from formula (1) that

$$\int\limits_{W>0}\mathrm{etr}(-WZ)\Delta^{\mu-p/2}(W)M(\kappa;\mu;CW)dW$$
$$=\Gamma_m(2\mu+p)\Delta^{\kappa-\mu-p/2}(I_m-C(Z+\tfrac{1}{2}C)^{-1})\Delta^{-2\mu-p}(Z+\tfrac{1}{2}C), \quad (9)$$

where $Z = X + iY \in \mathfrak{H}_m(\mathbb{C})$, $X > X_0$, $C > 0$, Re $\mu > -\frac{1}{2}$, Re $(\mu \pm \kappa) > \frac{p}{2} - 1$. Therefore, setting

$$f_r(W) = \Delta^{\mu_r - p/2}(W) M(\kappa_r; \mu_r; CZ), \quad r = 1, 2,$$

and applying the theorem on a convolution for the Laplace transform we derive that for

$$\mathrm{Re}\, \mu_r > -\frac{1}{2},\ \mathrm{Re}\, (\mu_r \pm \kappa_r) > \frac{p}{2} - 1,\ C > 0,\ r = 1, 2, \tag{10}$$

the Laplace transform of the function

$$\Delta^{\mu_1 - p/2}(W) M(\kappa_1; \mu_1; CW) * \Delta^{\mu_2 - p/2}(W) M(\kappa_2; \mu_2; CW) \tag{11}$$

ia equal to

$$\begin{aligned}\Gamma_m(2\mu_1 + p)\Gamma_m(2\mu_2 + p)\Delta^{-p+\kappa_1+\kappa_2-\mu_1-\mu_2}(I_m - C(Z + \tfrac{1}{2}C)^{-1}) \\ \times \Delta^{-2p-2\mu_1-2\mu_2}(Z + \tfrac{1}{2}C).\end{aligned} \tag{11'}$$

Thus, if conditions (10) are fulfilled, then

$$\begin{aligned}&\int_0^W \Delta^{\mu_1 - p/2}(R)\Delta^{\mu_2 - p/2}(W - R) M(\kappa_1; \mu_1; CR) M(\kappa_2; \mu_2; C(W - R)) dR \\ &\quad = \mathrm{B}_m(2\mu_1 + p, 2\mu_2 + p)\Delta^{\mu_1+\mu_2}(W) M\left(\kappa_1 + \kappa_2; \mu_1 + \mu_2 + \frac{p}{2}; CW\right).\end{aligned} \tag{12}$$

In an analogous way the relation

$$\begin{aligned}&\int_0^W \mathrm{etr}\left(-\frac{1}{2}(W - R)\right) \Delta^{\alpha - p}(W - R)\Delta^{\mu - p/2}(R) M(\kappa; \mu; R) dR \\ &\quad = \mathrm{B}(\alpha, 2\mu + p)\Delta^{\mu + (\alpha - p)/2}(W) M\left(\kappa + \frac{\alpha}{2}; \mu + \frac{1}{2}; W\right),\end{aligned}$$

is proved, where Re $\alpha > p - 1$, Re $(\mu \pm \kappa) > \frac{p}{2} - 1$.

17.4.8. Generalized Laguerre polynomials. Let $\sigma(R)$, $R \in \mathfrak{M}_m$, be a homogeneous symmetric polynomial, that is $\sigma(\Omega R \Omega^t) = \sigma(R)$, $\Omega \in O(m)$. We set

$$\mathrm{etr}(-S) L_\sigma^\gamma(S) = \int_{R > 0} \mathrm{etr}(-R)\Delta^\gamma(R)\sigma(R) A_\gamma(RS) dR, \tag{1}$$

where Re $\gamma > -1$. Evidently, $L_\sigma^\gamma(S)$ and $\sigma(R)$ are polynomials of the same degree. Moreover, if σ runs over a basis of the space $\mathfrak{A}$ of homogeneous symmetric polynomials, then L_σ^γ generates everywhere dense subspace of polynomials in the space $\mathfrak{L}^2(w)$. The space $\mathfrak{L}^2(w)$ consists of functions $f(S)$, $S>0$, such that

$$\int_{S>0} |f(S)|^2 w(S) dS < \infty, \qquad \text{where} \qquad w(S) = \operatorname{etr}(-S)\Delta^\gamma(S).$$

If polynomials σ_1 and σ_2 are of different degrees, then polynomials $L_{\sigma_1}^\gamma$ and $L_{\sigma_2}^\gamma$ are orthogonal in $\mathfrak{L}^2(w)$.

The zonal spherical polynomials $Z_\mathbf{k}(R)$ form a basis in the space $\mathfrak{A}$. We set

$$\operatorname{etr}(-S)L_\mathbf{k}^\gamma(S) = \int_{R>0} \exp(-R)\Delta^\gamma(R)Z_\mathbf{k}(R)A_\gamma(RS)dR. \tag{2}$$

Using integral representation (1) of Section 17.4.2 of the function $A_\gamma(R)$ and setting $p = \frac{1}{2}(m+1)$ we obtain that

$$\begin{aligned} L_\mathbf{k}^\gamma(S) = \Gamma_m(\gamma+p+k)&\frac{2^{m(m-1)/2}}{(2\pi i)^{m(m+1)/2}} \\ \times \int_{\text{Re } Z>0} &(\operatorname{etr} Z)\Delta^{-\gamma-p}(Z)Z_\mathbf{k}(I_m - SZ^{-1})dZ. \end{aligned} \tag{3}$$

This formula has a form of the inverse Laplace transform. Therefore, we have

$$\int_{S>0} \operatorname{etr}(-SZ)\Delta^\gamma(S)L_\mathbf{k}^\gamma(S)dS = \Gamma_m(\gamma+p+k)\Delta^{-\gamma-p}(Z)Z_\mathbf{k}(I_m - Z^{-1}). \tag{4}$$

We substitute into formula (3) the expansion of $Z_\mathbf{k}(I_m - SZ^{-1})$ in terms of $Z_\mathbf{n}(S \times Z^{-1})$:

$$Z_\mathbf{k}(I_m - SZ^{-1}) = Z_\mathbf{k}(I_m)\sum_{n=0}^\infty \sum_{|\mathbf{n}|=n} (-1)^n a_{\mathbf{kn}} \frac{Z_\mathbf{n}(SZ^{-1})}{Z_\mathbf{n}(I_m)}. \tag{5}$$

Taking into account formula (1) of Section 17.3.6 we obtain that

$$L_\mathbf{k}^\gamma(S) = (\gamma+p)_\mathbf{k} Z_\mathbf{k}(I_m)\sum_{n=0}^{|\mathbf{k}|}(-1)^n \frac{a_{\mathbf{kn}}}{(\gamma+p)_\mathbf{n}} \frac{Z_\mathbf{n}(S)}{Z_\mathbf{n}(I_m)}. \tag{6}$$

The explicit expression for the coefficients $a_{\mathbf{kn}}$ is not known. However, they can be recursively evaluated with the help of formula (6). It follows from (6) that $L_\mathbf{k}^\gamma(S)$ is a polynomial of degree $|\mathbf{k}|$ in S and

$$L_\mathbf{k}^\gamma(0) = (\gamma+p)_\mathbf{k} Z_\mathbf{k}(I_m). \tag{7}$$

The generating function for the Laguerre polynomials $L_{\mathbf{k}}^{\gamma}$ is given by the formula

$$\Delta^{-\gamma-p}(I_m - Z) \int\limits_{O(m)} \operatorname{etr}(-S\Omega^t Z(I_m - Z)^{-1}\Omega)d_*\Omega$$
$$= \Delta^{-\gamma-p}(I_m - Z)\sum_{k=0}^{\infty}\sum_{|\mathbf{k}|=k}[Z_{\mathbf{k}}(-S)Z_{\mathbf{k}}(I_m - Z)]^{-1}[k!Z_{\mathbf{k}}(I_m)]^{-1} \tag{8}$$
$$= \sum_{k=0}^{\infty}\sum_{|\mathbf{k}|=k} L_{\mathbf{k}}^{\gamma}(S)Z_{\mathbf{k}}(Z)[k!Z_{\mathbf{k}}(I_m)]^{-1}.$$

To prove this formula we multiply its both sides by $\operatorname{etr}(-SW)\Delta^{\gamma}(S)$ and integrate in $S > 0$. Due to (4) we have

$$\Gamma_m(\gamma+p)\Delta^{-\gamma-p}(W)\sum_{k=0}^{\infty}\sum_{|\mathbf{k}|=k}(\gamma+p)_{\mathbf{k}}Z_{\mathbf{k}}(Z)\frac{Z_{\mathbf{k}}(I_m - W^{-1})}{k!Z_{\mathbf{k}}(I_m)}$$
$$= \Gamma_m(\gamma+p)\Delta^{-\gamma-p}(W)\int\limits_{O(m)}\Delta^{-\gamma-p}(I_m - Z\Omega^t(I_m - W^{-1})\Omega)d_*\Omega$$
$$= \Gamma_m(\gamma+p)\Delta^{-\gamma-p}(W)\Delta^{-\gamma-m}(I_m - Z)$$
$$\times\int\limits_{O(m)}\Delta^{-\gamma-p}(I_m - Z(I_m - Z)^{-1}\Omega^t W^{-1}\Omega)d_*\Omega.$$

Applying the relation (1) of Section 17.3.6 we derive for the left hand side the expression

$$\Gamma_m(\gamma+p)\Delta^{-\gamma-p}(W)\Delta^{-\gamma-p}(I_m - Z)\sum_{k=0}^{\infty}\sum_{|\mathbf{k}|=k}(\gamma+p)_{\mathbf{k}}$$
$$\times Z_{\mathbf{k}}(Z(I_m - Z)^{-1})Z_{\mathbf{k}}(-W^{-1})[k!Z_{\mathbf{k}}(I_m)]^{-1}$$

which coincides with the expression for the right hand side obtained above. Since a function is uniquely defined by its Laplace transform, then formula (8) is proved.

By using the generating function we prove that the polynomials $L_{\mathbf{k}}^{\gamma}$ and $L_{\mathbf{n}}^{\gamma}$, $\mathbf{k} \neq \mathbf{n}$, are orthogonal, that is,

$$\int\limits_{S>0} L_{\mathbf{k}}^{\gamma}(S)L_{\mathbf{n}}^{\gamma}(S)\operatorname{etr}(-S)\Delta^{\gamma}(S)dS = 0. \tag{9}$$

For this we multiply both sides of formula (8) by $\mathrm{etr}(-S)\Delta^{\gamma}(S)Z_{\mathbf{n}}(S)$, where $\mathbf{n}$ is a partition of n, and integrate in $S > 0$. Due to formula (1) of Section 17.3.6 we obtain for the left hand side the expression

$$\Delta^{-\gamma-p}(I_m - Z) \int\limits_{O(m)} \int\limits_{S>0} \mathrm{etr}(-S(I_m + \Omega^t Z(I_m - Z)^{-1}\Omega))$$
$$\times \Delta^{\gamma}(S)Z_{\mathbf{n}}(S)dSd_*\Omega = \Gamma_m(\gamma + p)Z_{\mathbf{n}}(I_m - Z).$$

For the right hand side we have the expression

$$\sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} \frac{Z_{\mathbf{k}}(Z)}{k! Z_{\mathbf{k}}(I_m)} \int\limits_{S>0} \mathrm{etr}(-S)\Delta^{\gamma}(S)Z_{\mathbf{n}}(S)L_{\mathbf{k}}^{\gamma}(S)dS.$$

Since

$$Z_{\mathbf{n}}(I_m - Z) = (-1)^n Z_{\mathbf{n}}(Z) + \text{terms of lower degree},$$

then comparing coefficients at $Z_{\mathbf{n}}(Z)$ we derive for $\mathbf{k} > \mathbf{n}$ that

$$\int\limits_{S>0} \mathrm{etr}(-S)\Delta^{\gamma}(S)Z_{\mathbf{n}}(S)L_{\mathbf{k}}^{\gamma}(S)dS = 0.$$

Therefore, $L_{\mathbf{k}}^{\gamma}(S)$ is orthogonal to all polynomials $Z_{\mathbf{n}}(S)$ of lower degree. Since

$$L_{\mathbf{k}}^{\gamma}(S) = (-1)^{|\mathbf{k}|} Z_{\mathbf{k}}(S) + \text{terms of lower degree},$$

then $L_{\mathbf{n}}^{\gamma}$ and $L_{\mathbf{k}}^{\gamma}$ are orthogonal for $\mathbf{n} \neq \mathbf{k}$.

Let us give without proof the estimate

$$|L_{\mathbf{k}}^{\beta}(S)| \leq (\beta + p)_{\mathbf{k}} Z_{\mathbf{k}}(I_m) \mathrm{etr}\, S. \tag{10}$$

17.4.9. The Gauss hypergeometric function of a matrix argument. The Gauss hypergeometric function of a matrix argument ${}_2F_1(\alpha, \beta; \gamma; Z)$ is defined by formula (1) of Section 17.4.1. Due to formula (6) of Section 17.4.1 we have the integral representation

$${}_2F_1(\alpha, \beta; \gamma; Z)$$
$$= \frac{\Gamma_m(\gamma)}{\Gamma_m(\beta)\Gamma_m(\gamma - \alpha)} \int\limits_0^{I_m} \Delta^{\beta-p}(R)\Delta^{\gamma-\beta-p}(I_m - R)\Delta^{-\alpha}(I_m - RZ)dR. \tag{1}$$

This integral converges for $\mathrm{Re}\,\beta > p - 1$, $\mathrm{Re}\,(\gamma - \beta) > p - 1$ and for all α in the domain $\mathrm{Re}\, Z < I_m$.

Integral (1) is not symmetric with respect to the permutation of α and β. But the hypergeometric function has this symmetry:

$$ {}_2F_1(\alpha,\beta;\gamma;Z) = {}_2F_1(\beta,\alpha;\gamma;Z). \tag{2}$$

To prove this formula it is sufficient to apply formula (1) of Section 17.4.1 which for our case is of the form

$$ {}_2F_1(\alpha,\beta;\gamma;\Lambda) = \sum_{k=0}^{\infty}\sum_{|\mathbf{k}|=k} \frac{[\alpha]_{\mathbf{k}}[\beta]_{\mathbf{k}}}{[\gamma]_{\mathbf{k}}}\frac{Z_{\mathbf{k}}(\Lambda)}{k!}. \tag{3}$$

Using in (1) the substitution $Z \to -Z(I_m - Z)^{-1}$ we obtain the Euler formula

$$ {}_2F_1(\alpha,\beta;\gamma;Z) = \Delta^{-\alpha}(I_m - Z)\, {}_2F_1(\alpha,\gamma-\beta;\gamma;-Z(I_m-Z)^{-1}). \tag{4}$$

This formula shows that ${}_2F_1(\alpha,\beta;\gamma;Z)$ is defined in the domain Re $Z > 0$ for all α if Re $\beta > p-1$ and Re $\gamma > p-1$. With the help of formula (4) we derive that

$$ {}_2F_1(\alpha,\beta;\gamma;Z) = \Delta^{\gamma-\alpha-\beta}(I_m - Z)\, {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;Z). \tag{5}$$

Let us prove the integral formula for ${}_2F_1(\alpha,\beta;\gamma;Z)$ containing Bessel functions:

$$\begin{aligned}\int\limits_{\Lambda>0} B_{\alpha-\beta}(\Lambda Z)A_{\gamma-p}(\Lambda)\Delta^{\alpha-p}(\Lambda)d\Lambda &= \frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\gamma)} \\ &\times \Delta^{-\alpha}(Z)\, {}_2F_1(\alpha,\beta;\gamma;-Z^{-1}).\end{aligned} \tag{6}$$

Here Re $Z > 0$, Re $\alpha > p-1$, Re $\beta > p-1$. For this we note that $B_\delta(\Lambda Z)\Delta^\gamma(\Lambda)$, Re $Z > 0$, is absolutely integrable for values of parameters defined by the inequalities Re $\gamma > -1$, Re $(\gamma-\delta) > -1$. Therefore, if Re $\alpha > p-1$, Re $\beta > p-1$, then the function $B_{\beta-\alpha}(\Lambda Z)\Delta^{\beta-p}(\Lambda)$ is absolutely integrable. From other side, the function $A_{\gamma-p}(\Lambda)$ is entire and for $\Lambda \to +\infty$ it is of a polynomial growth. The function $B_{\beta-\alpha}(\Lambda Z)$ decreases for $\Lambda \to +\infty$ faster that any function $P^{-1}(\Lambda)$, where P is a polynomial. Therefore, the left hand side of (6) is defined for Re $Z > 0$, Re $\alpha > p-1$, Re $\beta > p-1$.

Suppose that Re $\gamma > p-1$. Due to integral representation (1) the function ${}_2F_1(\alpha,\beta;\gamma;Z)$ is defined in the domain Re $Z < I_m$ for all α and for β, γ such that Re $\gamma > p-1$, Re $(\gamma-\beta) > p-1$. Formula (4) shows that ${}_2F_1(\alpha,\beta;\gamma;-Z^{-1})$ is defined in the domain Re $Z > 0$ for all α if Re $\beta > p-1$, Re $\gamma > p-1$. Thus, the right hand side of (6) is determined and both parts of this formula give an analytical function of α, β, γ, Z in the simply connected domain defined by the inequalities Re $\alpha > p-1$, Re $\beta > p-1$, Re $\gamma > p-1$, Re $Z > 0$. This means that we have to prove formula (6) for any part of this domain which allows a unique analytical continuation.

If $\alpha \in \mathbf{R}$, then for Re $\beta > \frac{1}{2}(\alpha+p-1)$ we have $B_{\alpha-\beta} \in \mathfrak{L}^2_{\alpha-p}$. Analogously, if Re $(\gamma-\alpha) > p-\frac{1}{2}$, then $A_{\gamma-p} \in \mathfrak{L}^2_{\alpha-p}$. If both these conditions are fulfilled, then the left hand side is the scalar product of the functions $B_{\alpha-\beta}(\Lambda Z)$ and $A_{\gamma-p}(\Lambda)$ in $\mathfrak{L}^2_{\alpha-p}$. Due to formula (8) of Section 17.4.4 $B_{\alpha-\beta}(\Lambda Z)$ is the $(\alpha-p)$-Hankel transform of the function

$$\Gamma_m(\beta)\Delta^{-\alpha}(Z)\Delta^{-\beta}(I_m+RZ^{-1})$$

and due to formula (7) of Section 17.4.2 $A_{\gamma-p}(\Lambda)$ is the $(\alpha-p)$-Hankel transform of the function which is equal to $\Gamma_m^{-1}(\gamma-\alpha)\Delta^{\gamma-\alpha-p}(I_m-R)$ for $R<I_m$ and to 0 otherwise. Since the Hankel transform is unitary, then the left hand side of (6) is equal to

$$\frac{\Gamma_m(\beta)}{\Gamma_m(\gamma-\alpha)}\Delta^{-\alpha}(Z)\int_0^{I_m}\Delta^{-\beta}(I_m+RZ^{-1})\Delta^{\gamma-\alpha-p}(I_m-R)\Delta^{\alpha-p}(R)dR$$
$$=\frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\gamma)}\Delta^{-\alpha}(Z)\,{}_2F_1(\alpha,\beta;\gamma;-Z^{-1})$$

and formula (6) is proved.

The hypergeometric function ${}_2F_1$ can be represented in the form of fractional derivative. To obtain this representation we write down formula (1) as

$${}_2F_1(\gamma+\mu+p,-\beta;\gamma+p;Z)=\frac{\Gamma_m(\gamma+p)\Delta^{-\gamma}(Z)}{\Gamma_m(\gamma+\mu+p)\Gamma_m(-\mu)}\int_0^Z\Delta^{\gamma+\mu}(R)$$
$$\times\Delta^{\beta}(I_m-R)\Delta^{-\mu-p}(Z-R)dR.$$

It follows from here that

$${}_2F_1(\gamma+\mu+p,-\beta;\gamma+p;Z)=\frac{\Gamma_m(\gamma+p)}{\Gamma_m(\gamma+\mu+p)}\Delta^{-\gamma}(Z)$$
$$\times D^{\mu}[\Delta^{\gamma+\mu}(Z)\Delta^{\beta}(I_m-Z)], \tag{7}$$

where Re $(\gamma+\mu)>-1$, $0<Z<I_m$. If μ is a non-negative integer, then

$$\int_{\Lambda>0}\mathrm{etr}(-\Lambda Z)(D^{\mu}f)(\Lambda)d\Lambda=\Delta^{\mu}(Z)\int_{\Lambda>0}\mathrm{etr}(-\Lambda Z)f(\Lambda)d\Lambda. \tag{8}$$

Therefore, due to formula (5) of Section 17.4.6 we have

$$\int_0^{I_m}\mathrm{etr}(-\Lambda Z)\,{}_2F_1(\gamma+\mu+p,-\beta;\gamma+p;\Lambda)\Delta^{\gamma}(\Lambda)d\Lambda=\frac{\Gamma_m(\beta+p)\Gamma_m(\gamma+p)}{\Gamma_m(\beta+\gamma+\mu+2p)}$$
$$\times\Delta^{\mu}(Z)\,{}_1F_1(\gamma+\mu+p;\beta+\gamma+\mu+2p;-Z),$$

where the integral absolutely converges for Re $Z > 0$, Re $\beta > -1$, Re $\gamma > -1$.

17.4.10. Jacobi and Gegenbauer functions of a matrix argument. If Re $\gamma > -1$, then the function

$$P_\mu^{(\gamma,\delta)}(\Lambda) = \frac{\Gamma_m(\gamma+\mu+p)}{\Gamma_m(\mu+p)}\, {}_2F_1(-\mu, \gamma+\delta+\mu+p; \gamma+p; \Lambda) \tag{1}$$

is defined in the strip $0 < \mathrm{Re}\,\Lambda < I_m$. Due to formula (5) of Section 17.4.9 we have

$$P_\mu^{(\gamma,\delta)}(\Lambda) = \frac{\Gamma_m(\gamma+\mu+p)}{\Gamma_m(\gamma+p)}\Delta^{-\delta}(I_m-\Lambda)\, {}_2F_1(\gamma+\mu+p, -\delta-\mu; \gamma+p; \Lambda). \tag{1'}$$

Therefore, applying formula (7) of Section 17.4.9 we obtain the analogue of the Rodrigues formula for $P_\mu^{(\gamma,\delta)}$:

$$P_\mu^{(\gamma,\delta)}(\Lambda) = \Delta^{-\gamma}(\Lambda)\Delta^{-\delta}(I_m-\Lambda)D^\mu(\Delta^{\gamma+\mu}(\Lambda)\Delta^{\delta+\mu}(I_m-\Lambda)). \tag{2}$$

It follows from (2) that if $\mu \in \mathbb{Z}_+ \cup \{0\}$, then $P_\mu^{(\gamma,\delta)}(\Lambda)$ is a symmetric polynomial of degree $m\mu$. With the help of integration by parts we derive that $P_\mu^{(\gamma,\delta)}(\Lambda)$ is orthogonal on the "interval" $0 < \Lambda < I_m$ to every polynomial of lower degree with respect to the weight $\Delta^\gamma(\Lambda)\Delta^\delta(I_m-\Lambda)$. These polynomials are an analogue of Jacobi polynomials of a real variable. However, for $m > 1$ this set of polynomials is not complete.

Now we define Gegenbauer polynomials of a matrix argument. For $Q \in \mathfrak{M}_m(\mathbf{R})$ they are given by the formulas

$$C_{2\mu}^{(\delta)}(Q) = (-1)^{m\mu}\Gamma_m\left(\delta+\frac{1}{2}\right)\Gamma_m^{-1}\left(\delta+\mu+\frac{1}{2}\right)P_\mu^{\left(-\frac{1}{2},\delta-\frac{m}{2}\right)}(Q^tQ), \tag{3}$$

$$C_{2\mu+1}^{(\delta)}(Q) = (-1)^{m\mu}\Gamma_m\left(\delta+\frac{1}{2}\right)\Gamma_m^{-1}\left(\delta+\mu+\frac{1}{2}\right)\Delta(Q)P_\mu^{\left(\frac{1}{2},\delta-\frac{m}{2}\right)}(Q^tQ). \tag{3'}$$

The polynomials $C_\mu^{(\delta)}$ are orthogonal on the "matrix disk" $Q^tQ < I_m$ with respect to the weight $\Delta^{\delta-m/2}(I_m-Q^tQ)$ (the condition Re $\delta > p - \frac{3}{2}$ guarantees the integrability of the weight function). We have $C_\mu^{(\delta)}(I_m) = 1$.

Let us write down the inversion formula for the Laplace transform (8) of Section 17.4.1 setting $p = q = 1$ and expressing ${}_2F_1$ and ${}_1F_1$ in terms of $P_\mu^{(\gamma,\delta)}$ and $L_\delta^{(\gamma)}$ respectively. After simple transformations we obtain

$$\begin{aligned}\int_0^{I_m} &\mathrm{etr}(-\Lambda Z)P_\mu^{(\gamma,\delta)}(\Lambda)\Delta^\gamma(\Lambda)\Delta^\delta(I_m-\Lambda)d\Lambda \\ &= \Gamma_m(\delta+\mu+p)\mathrm{etr}(-Z)L_{-\delta-\mu-p}^{(\gamma+\delta+2\mu+p)}(Z)\Delta^\mu(Z).\end{aligned} \tag{4}$$

This integral absolutely converges for Re $\gamma > -1$, Re $\delta > -1$ and the function $P_\mu^{(\gamma,\delta)}(\Lambda)\Delta^\delta(I_m - \Lambda)$ belongs to the space $\mathfrak{L}^2_\gamma$, $\gamma > -1$, if Re $\delta > -\frac{1}{2}$. Taking into account formula (10) of Section 17.4.6 and applying the analogue of the Tricomi theorem from Section 17.4.3 we derive the generalization of equality (7) of Section 17.4.2:

$$A_{\gamma+\delta+2\mu+p}(M)\Delta^\mu(M) = \Gamma_m^{-1}(\delta+\mu+p) \times \int_0^{I_m} A_\gamma(\Lambda M)P_\mu^{(\gamma,\delta)}(\Lambda)\Delta^\gamma(\Lambda)\Delta^\delta(I_m-\Lambda)d\Lambda. \tag{5}$$

This integral absolutely converges for Re $\gamma > -1$, Re $\delta > -1$. Considering formula (5) as the Hankel transform and writing down the inversion transform one obtains the relation

$$\int_{R>0} A_\gamma(\Lambda R)A_{\gamma+\delta+2\mu+p}(R)\Delta^{\gamma+\mu}(R)dR = \Gamma_m^{-1}(\delta+\mu+p)P_\mu^{(\gamma,\delta)}(\Lambda)\Delta^\delta(I_m-\Lambda), \quad 0 \le \Lambda \le I_m. \tag{6}$$

This integral absolutely converges for Re $\gamma > -1$, Re $\delta > p-1$, Re$(\gamma+\mu) > -1$, Re$(\delta+\mu) > -1$. Formula (6) is also valid for non-integral μ. The proof in this case is analogous to that of formula (3) of Section 17.4.9.

Formula (4) of Section 17.4.3 can be considered as a generalization of the Poisson integral for the case of a matrix variable. Namely, if δ is fixed, then for any $\mu \in \mathbb{Z}_+ \cup \{0\}$ there exists a polynomial $C_\mu^{(\delta)}(T)$ of degree $m\mu$ in elements of a matrix $T \in \mathfrak{M}_m(\mathbf{R})$ such that

$$A_{\delta+\mu}\left(\frac{1}{4}S^tS\right)\Delta^\mu\left(\frac{1}{2}S\right) = (-i)^{m\mu}\pi^{-m^2/2}\Gamma_m^{-1}\left(\alpha+\frac{1}{2}\right) \times \int_{\substack{T^tT<I_m\\ T\in\mathfrak{M}_m}} \operatorname{etr}(iS^tT)C_\mu^{(\delta)}(T)\Delta^{\delta-m/2}(I_m-T^tT)dT. \tag{7}$$

This integral absolutely converges for Re $\delta > p - \frac{3}{2}$ and converges in the mean for Re $\delta > p-1$. In particular, it follows from here that the function $A_{\delta+\mu}(R)\Delta^{\mu/2}(R)$ is bounded on the "ray" $R > 0$ and tends to 0 if $R \to +\infty$.

Formula (15) of Section 17.4.3 leads to the following important statement: *Let P_μ be a H-polynomial of degree μ in elements of a matrix from $\mathfrak{M}_{km}$, $k \ge 2m$, and let $P_\mu\left(\begin{pmatrix} I_m \\ 0 \end{pmatrix}\right) = 1$. Let $\delta = \frac{1}{2}(k-m-1)$ and*

$$V = \begin{pmatrix} T \\ U(I_m - T^tT)^{1/2} \end{pmatrix},$$

where $T \in \mathfrak{M}_m(\mathbf{R})$, $T^tT \leq I_m$ *and* $U \in St(k-m,m)$. *Averaging the polynomial* P_μ *over the Stiefel manifold* $St(k-m,m)$ *we obtain the orthogonal polynomials* $C_\mu^{(\delta)}(T)$:

$$C_\mu^{(\delta)}(S) = \int_{\mathrm{St}(k-m,m)} P_\mu(V) d_*U. \tag{8}$$

Substituting here the expression for $P_\mu(V)$ given by results of Section 17.1.3 we have

$$C_\mu^{(\delta)}(S) = \int_{\mathrm{St}(k-m,m)} \Delta^\mu(S + i\hat{U}(I_m - Q^tQ)^{1/2}) d_*U, \tag{9}$$

where $\hat{U}$ is any submatrix of U of order m.

Integral representation (8) of Section 17.4.1 for ${}_{p+1}F_q$ can be written as

$${}_{p+1}F_q(\tilde{\boldsymbol{\alpha}}; \boldsymbol{\beta}; Z) = \Gamma_m^{-1}(\gamma) \int_{\Lambda>0} \mathrm{etr}(-\Lambda)\ {}_pF_q(\boldsymbol{\alpha}; \boldsymbol{\beta}; \Lambda Z) \Delta^{\gamma-p}(\Lambda) d\Lambda. \tag{10}$$

Setting here $p = q = 1$ and substituting Jacobi and Laguerre functions instead of ${}_2F_1$ and ${}_1F_1$ one derives that

$$P_\mu^{(\gamma,\delta)} = \Gamma_m^{-1}(\gamma + \delta + \mu + p) \int_{R>0} \mathrm{etr}(-R) L_\mu^{(\gamma)}(\Lambda R) \Delta^{\gamma+\delta+\mu}(R) dR. \tag{11}$$

For $\lambda, \mu \in \mathbb{Z}_+ \cup \{0\}$ we have

$$\begin{aligned} &\int_{R>0} A_{\gamma+p+2\lambda}(R) A_{\gamma+p+2\mu}(R) \Delta^{\gamma+\lambda+\mu}(R) dR \\ &\qquad = \frac{\Gamma_m(\gamma + 2\mu + p)}{\Gamma_m(p)\Gamma_m(\gamma + 2\mu + 2p)} \delta_{\lambda\mu}. \end{aligned} \tag{12}$$

Really, we set $\delta = 0$ in (5). Due to orthogonality of the polynomials $P_\mu^{(\gamma,0)}$ and to the Plancherel equality for the Hankel transform we have orthogonality in $\mathfrak{L}_\gamma^2$ of the functions $A_{\gamma+p+2\mu}(R)\Delta^\mu(R)$ and $A_{\gamma+p+2\lambda}(R)\Delta^\lambda(R)$, $\mu \neq \lambda$. From other side, it is possible to show that for any $\mu \geq 0$ we have

$$\int_{R>0} [A_{\gamma+p+2\mu}(R)]^2 \Delta^{\gamma+2\mu}(R) dR = \frac{\Gamma_m(\gamma + 2\mu + p)}{\Gamma_m(p)\Gamma_m(\gamma + 2\mu + 2p)}.$$

If $\alpha > -1$, then it follows from formula (7) of Section 17.4.2 that

$$A_{\alpha+p}(M) = \Gamma_m^{-1}(p) \int_0^{I_m} A_\alpha(MR) \Delta^\alpha(R) dR. \tag{13}$$

Since this formula can be considered as the Hankel transform and it conserves the norm, then we obtain

$$\int\limits_{M>0} [A_{\alpha+p}(M)]^2\Delta^\alpha(M)dM = \Gamma_m^{-2}(p)\int\limits_0^{I_m}\Delta^\alpha(R)dR$$
$$= \frac{\Gamma_m(\alpha+p)}{\Gamma_m(p)\Gamma_m(\alpha+2p)}. \tag{14}$$

Setting $\delta = m/2$ we derive from (7) that

$$\int\limits_{\mathfrak{M}_m} A_{\mu+m/2}\left(\frac{1}{4}S^tS\right)A_{\lambda+m/2}\left(\frac{1}{4}S^tS\right)\Delta^{\mu+\lambda}\left(\frac{1}{2}S\right)dS$$
$$= \frac{(2\sqrt{\pi})^{m^2}\Gamma_m\left(\lambda+p-\frac{1}{2}\right)}{\Gamma_m(p)\Gamma_m\left(p-\frac{1}{2}\right)\Gamma_m\left(\lambda+p+\frac{m}{2}\right)}\delta_{\mu\lambda}. \tag{12'}$$

It follows from (12) and (12′) that

$$\int\limits_0^{I_m}[P_\mu^{(\gamma,0)}(R)]^2\Delta^\gamma(R)dR = \frac{[\Gamma_m(\mu+p)]^2\Gamma_m(\gamma+p+2\mu)}{\Gamma_m(p)\Gamma_m(\gamma+2p+2\mu)},$$

$$\int\limits_{\substack{T^*T<I_m\\ T\in\mathfrak{M}_m}}\left[C_\mu^{\left(\frac{m}{2}\right)}(T)\right]^2 dT = \pi^{m^2/2}\frac{\Gamma_m(p)\Gamma_m\left(\mu+p-\frac{1}{2}\right)}{\Gamma_m\left(p-\frac{1}{2}\right)\Gamma_m\left(\mu+p+\frac{m}{2}\right)}.$$

Functions from $\mathfrak{L}_\gamma^2$ can be expanded into the Neumann series

$$f(\Lambda)\sim\sum_{\mu=0}^{\infty}C_\mu A_{\gamma+p+2\mu}(\Lambda)\Delta^\mu(\Lambda) \tag{15}$$

which converges in the mean. The coefficients C_μ are given by the formula

$$C_\mu = \frac{\Gamma_m(p)\Gamma_m(\gamma+2\mu+2p)}{\Gamma_m(\gamma+2\mu+p)}\int\limits_{R>0} f(R)A_{\gamma+p+2\mu}(R)\Delta^{\gamma+\mu}(R)dR. \tag{16}$$

If f has expansion (15) and $g = U_\gamma f$ is the γ-Hankel transform for f, then according to formula (5)

$$g(M) = \begin{cases}\sum\limits_{\mu=0}^{\infty} C_\mu\Gamma_m^{-1}(\mu+p)P_\mu^{(\gamma,0)}(M) & \text{if } 0\le M\le I_m,\\ 0 & \text{otherwise.}\end{cases} \tag{17}$$

Conversely, if a function g vanishes outside of $[0, I_m]$ and has the expansion (17) on the interval $[0, I_m]$, then f has the expansion (15). Thus, the following statement is proved: *Let $f\in\mathfrak{L}_\gamma^2$ and let $g = U_\gamma f$ be the γ-Hankel transform for f. Then we have the Neumann expansion* (15) *for f if and only if g vanishes outside of* $[0, I_m]$ *and has the expansion* (17) *on* $[0, I_m]$. *The coefficients of expansion* (15) *are given by formula* (16).

Chapter 18.
Representations in the Gel'fand-Tsetlin Basis and Special Functions

18.1. Infinitesimal Operators of Representations of the Groups $U(n)$ and $SO(n)$

18.1.1. The Gel'fand-Tsetlin basis. Let $T_{\mathbf{m}}$ be the irreducible finite dimensional representation of the group $GL(n,\mathbb{C})$ with highest weight $\mathbf{m} = (m_{1n}, m_{2n}, \ldots, m_{nn})$ and let $\mathfrak{H}_{\mathbf{m}}$ be the space of this representation. The restriction of $T_{\mathbf{m}}$ onto the subgroup $GL(n-1,\mathbb{C})$ is reducible. It decomposes into the direct sum of all irreducible representations $T_{\mathbf{m}'}$ of $GL(n-1,\mathbb{C})$ with highest weights $\mathbf{m}' = (m_{1,n-1}, \ldots, m_{n-1,n-1})$ for which the betweenness conditions

$$m_{jn} \geq m_{j,n-1} \geq m_{j+1,n}, \ 1 \leq j \leq n-1, \tag{1}$$

are satisfied. Each of these representations of $GL(n-1,\mathbb{C})$ is contained in this decomposition with the unit multiplicity (see Section 16.1.7).

If we continue this procedure of restriction of irreducible representations successively onto the subgroups

$$GL(n-2,\mathbb{C}), GL(n-3,\mathbb{C}), \ldots, GL(1,\mathbb{C}), \tag{2}$$

then we finally get one-dimensional representations of the group $GL(1,\mathbb{C})$. The basis vectors of these one-dimensional representations (which are basis vectors of $\mathfrak{H}_{\mathbf{m}}$) can be denoted by the tables

$$M = \begin{bmatrix} m_{1n} \quad m_{2n} \ \cdots \ m_{n-1,n} \quad m_{nn} \\ m_{1,n-1} \ \cdots\cdots \ m_{n-1,n-1} \\ \cdots\cdots\cdots\cdots\cdots \\ m_{12} \quad m_{22} \\ m_{11} \end{bmatrix}, \tag{3}$$

where the numbers m_{ij} satisfy the betweenness conditions of type (1). They are called the *Gel'fand-Tsetlin patterns.* We shall also denote them as

$$M = (\mathbf{m}_n, \mathbf{m}_{n-1}, \ldots, \mathbf{m}_2, \mathbf{m}_1),$$

where $\mathbf{m}_k = (m_{1k}, \ldots, m_{kk})$. The vectors, corresponding to the patterns M, are denote by $\mathbf{v}(M)$. Let us emphasize that the first row of patterns (3) is fixed and coincides with highest weight of the representation $T_{\mathbf{m}}$, that is $\mathbf{m} \equiv \mathbf{m}_n$.

Since the restriction $Q_{\mathbf{m}}$ of the representation $T_{\mathbf{m}}$ of $GL(n,\mathbb{C})$ onto the subgroup $U(n)$ is also irreducible (see Section 2.2.6), then considering irreducible representations of $U(n)$ we have the same basis elements $\mathbf{v}(M)$ which are labelled by the Gel'fand-Tsetlin patterns. The space $\mathfrak{H}_{\mathbf{m}}$ can be equipped with a scalar product for which the representation $Q_{\mathbf{m}}$ of $U(n)$ is unitary (see Section 2.2.3). It is evident that for different Gel'fand-Tsetlin patterns L and M the vectors $\mathbf{v}(L)$ and $\mathbf{v}(M)$ are orthogonal. We shall assume that the vectors $\mathbf{v}(M)$ are taken to be orthonormal. The orthonormal basis $\{\mathbf{v}(M)\}$ of $\mathfrak{H}_{\mathbf{m}}$ is called the *Gel'fand-Tsetlin basis.* The subgroup chain $U(n-1) \supset U(n-2) \supset \ldots \supset U(1)$ defines the orthonormal basis $\{\mathbf{v}(M)\}$ of $\mathfrak{H}_{\mathbf{m}}$ uniquely up to multiplication of the basis elements by numbers $\beta(M)$ such that $|\beta(M)| = 1$.

In an analogous way orthonormal bases of carrier spaces $\mathfrak{H}_{\mathbf{m}}$ of irreducible representations $T_{\mathbf{m}}$ of the group $SO(n)$ are constructed. Highest weights $\mathbf{m}$ of these representations are given by the numbers $\mathbf{m} \equiv \mathbf{m}_n = (m_{1n}, m_{2n}, \ldots, m_{kn})$ for which

$$m_{1n} \geq m_{2n} \geq \ldots \geq m_{k-1,n} \geq |m_{kn}| \quad \text{if} \quad n = 2k,$$

$$m_{1n} \geq m_{2n} \geq \ldots \geq m_{kn} \geq 0 \quad \text{if} \quad n = 2k+1,$$

and m_{jk} are all integers or all half-integers.

Restriction of the representation $T_{\mathbf{m}}$ of the group $SO(2k+1)$ onto $SO(2k)$ decomposes into the direct sum of all irreducible representations $T_{\mathbf{m}'}$, $\mathbf{m}' \equiv \mathbf{m}_{n-1} = (m_{1,n-1}, \ldots, m_{k,n-1})$ for which the betweenness conditions

$$m_{1,2k+1} \geq m_{1,2k} \geq m_{2,2k+1} \geq m_{2,2k} \geq \cdots \geq m_{k,2k+1} \geq m_{k,2k} \geq -m_{k,2k+1} \quad (4)$$

are satisfied, where the numbers are all integers or all half-integers. For restriction of the representation $T_{\mathbf{m}}$ of $SO(2k)$ onto the subgroup $SO(2k-1)$ instead of (4) we have betweenness conditions

$$m_{1,2k} \geq m_{1,2k-1} \geq m_{2,2k} \geq m_{2,2k-1} \geq \cdots \geq m_{k-1,2k} \geq m_{k-1,2k-1} \geq |m_{k,2k}|. \quad (5)$$

All multiplicities in the decompositions are equal to 1 (see Section 16.1.7).

As in the case of the group $U(n)$, conditions (4) and (5) define the Gel'fand-Tsetlin basis $\{\mathbf{v}(M)\}$ of the space $\mathfrak{H}_{\mathbf{m}}$ of the representation $T_{\mathbf{m}}$ of $SO(n)$ which is labelled by the Gel'fand-Tsetlin patterns

$$M = (\mathbf{m}_n, \mathbf{m}_{n-1}, \ldots, \mathbf{m}_3, \mathbf{m}_2)$$
$$= \begin{bmatrix} m_{1n} \quad m_{2n} \quad \cdots\cdots\cdots \quad m_{kn} \\ m_{2,n-1} \quad \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ m_{15} \quad m_{25} \\ m_{14} \quad m_{24} \\ m_{13} \\ m_{12} \end{bmatrix}. \quad (6)$$

18.1.2. Infinitesimal operators of irreducible representations. Let E_{ik} be the $n \times n$ matrix with all elements equal to zero except for the element e_{ik}

which is equal to 1. The matrices E_{ik}, $1 \le i, k \le n$, form a basis of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of the group $GL(n, \mathbb{C})$. In order to give explicitly the representation $T_{\mathbf{m}}$ of $\mathfrak{gl}(n, \mathbb{C})$, it is sufficiently to define action of the operators $T_{\mathbf{m}}(E_{ik})$ onto the basis elements $\mathbf{v}(M)$. For brevity these operators will be denoted as E_{ik}. They have to satisfy the commutation relations

$$[E_{ik}, E_{js}] = 0 \quad \text{if} \quad i \neq s, \quad k \neq j,$$

$$[E_{ik}, E_{k\ell}] = E_{i\ell} \quad \text{if} \quad i \neq \ell, \quad [E_{ik}, E_{ki}] = E_{ii} - E_{kk}.$$

The explicit form for the operators E_{ik} was predicted by I. M. Gel'fand using some arguments of duality. After that M. L. Tsetlin proved that these explicit formulas really satisfy the necessary commutation relations.

Let us denote $m_{pq} - p$ by ℓ_{pq} and set

$$a^j_{p-1} = \left| \frac{\prod_{i=1}^{p} (\ell_{ip} - \ell_{j,p-1}) \prod_{i=1}^{p-2} (\ell_{i,p-2} - \ell_{j,p-1} - 1)}{\prod_{i \neq j} (\ell_{i,p-1} - \ell_{j,p-1})(\ell_{i,p-1} - \ell_{j,p-1} - 1)} \right|^{1/2}, \tag{1}$$

$$b^j_{p-1} = \left| \frac{\prod_{i=1}^{p} (\ell_{ip} - \ell_{j,p-1} + 1) \prod_{i=1}^{p-2} (\ell_{i,p-2} - \ell_{j,p-1})}{\prod_{i \neq j} (\ell_{i,p-1} - \ell_{j,p-1})(\ell_{i,p-1} - \ell_{j,p-1} + 1)} \right|^{1/2}. \tag{2}$$

The Gel'fand-Tsetlin pattern M, in which $m_{j,p-1}$ is replaced by $m_{j,p-1} + 1$ (by $m_{j,p-1} - 1$), will be denoted by M^{+j}_{p-1} (by M^{-j}_{p-1}). The Gel'fand-Tsetlin formulas for the operators $E_{p-1,p}$, $E_{p,p-1}$, $2 \le p \le n$, E_{pp}, $1 \le p \le n$, are of the form

$$E_{p-1,p}\mathbf{v}(M) = \sum_{j=1}^{p} a^j_{p-1} \mathbf{v}(M^{+j}_{p-1}), \tag{3}$$

$$E_{p,p-1}\mathbf{v}(M) = \sum_{j=1}^{p} b^j_{p-1} \mathbf{v}(M^{-j}_{p-1}), \tag{4}$$

$$E_{p,p}\mathbf{v}(M) = \left(\sum_{i=1}^{p} m_{ip} - \sum_{j=1}^{p-1} m_{j,p-1} \right) \mathbf{v}(M). \tag{5}$$

The other operators E_{pq} are obtained with the help of commutations of operators (3)-(5).

Remark 1. In order to prove that formulas (1)-(5) define a representation of the algebra $\mathfrak{gl}(n, \mathbb{C})$, it is sufficient to use Theorem 1 of Section 15.1.3. We have to verify the relations of this theorem, written down for the algebra $\mathfrak{gl}(n, \mathbb{C})$.

Remark 2. Expressions (1) and (2) depend on entries of the Gel'fand-Tsetlin pattern M, that is, $a^j_{p-1} \equiv a^j_{p-1}(M)$, $b^j_{p-1} \equiv b^j_{p-1}(M)$. It is easy to verify that

$$b^j_{p-1}(M) = a^j_{p-1}(M^{-j}_{p-1}). \tag{6}$$

The Lie algebra $\mathfrak{u}(n)$ of $U(n)$ is generated by the elements

$$iE_{kk}, \quad 1 \le k \le n,$$

$$E_{k,k-1} - E_{k-1,k}, \quad i(E_{k,k-1} + E_{k-1,k}), \quad 1 < k \le n, \quad i = \sqrt{-1}.$$

The one-parameter subgroups

$$\exp i\varphi E_{kk}, \quad \exp \varphi(E_{k,k-1} - E_{k-1,k}), \quad \exp i\varphi(E_{k,k-1} + E_{k-1,k})$$

correspond to these elements of $\mathfrak{u}(n)$. Therefore, the representation $T_{\mathbf{m}}$ of $U(n)$ is unitary, that is $T^*_{\mathbf{m}}(g) = T^{-1}_{\mathbf{m}}(g)$, $g \in U(n)$, if and only if

$$E^*_{kk} = E_{kk}, \quad E^*_{k,k-1} = E_{k-1,k}.$$

It can be easily verified that the operators (3)-(5) satisfy these conditions.

Now let us write down explicit formulas for the infinitesimal operators of the irreducible representations $T_{\mathbf{m}}$ of the group $SO(n)$. The matrices

$$I_{ki} = E_{ik} - E_{ki}, \quad 1 \le i < k \le n,$$

form a basis of the Lie algebra $\mathfrak{so}(n)$ of the group $SO(n)$. These matrices satisfy the commutation relations (2) of Section 9.1.1. It is sufficient to give the operators corresponding to the matrices $I_{p,p-1}$. In the Gel'fand-Tsetlin basis of the carrier space of the representation $T_{\mathbf{m}}$ these operators are given by the formulas

$$I_{2p+1,2p}\mathbf{v}(M) = \sum_{j=1}^{p} A^j_{2p}(M)\mathbf{v}(M^{+j}_{2p}) - \sum_{j=1}^{p} A^j_{2p}(M^{-j}_{2p})\mathbf{v}(M^{-j}_{2p}), \tag{7}$$

$$\begin{aligned} I_{2p+2,2p+1}\mathbf{v}(M) &= \sum_{j=1}^{p} B^j_{2p+1}(M)\mathbf{v}(M^{+j}_{2p+1}) \\ &- \sum_{j=1}^{p} B^j_{2p+1}(M^{-j}_{2p+1})\mathbf{v}(M^{-j}_{2p+1}) + iC_{2p}(M)\mathbf{v}(M), \end{aligned} \tag{8}$$

where

$$A^j_{2p}(M) = \frac{1}{2} \times \left| \frac{\prod_{r=1}^{p-1} \left[\left(\ell_{r,2p-1} - \frac{1}{2}\right)^2 - \left(\ell_{j,2p} + \frac{1}{2}\right)^2 \right] \prod_{r=1}^{p} \left[\left(\ell_{r,2p+1} - \frac{1}{2}\right)^2 - \left(\ell_{j,2p} + \frac{1}{2}\right)^2 \right]}{\prod_{r \ne j} (\ell^2_{r,2p} - \ell^2_{j,2p})(\ell^2_{r,2p} - (\ell_{j,2p} + 1)^2)} \right|^{1/2}, \tag{9}$$

$$B^j_{2p+1}(M) = \left| \frac{\prod_{r=1}^{p} (\ell^2_{r,2p} - \ell^2_{j,2p+1}) \prod_{r=1}^{p+1} (\ell^2_{r,2p+2} - \ell^2_{j,2p+1})}{\ell^2_{j,2p+1}(4\ell^2_{j,2p+1} - 1) \prod_{r\neq j} (\ell^2_{r,2p+1} - \ell^2_{j,2p+1})(\ell^2_{j,2p+1} - (\ell_{r,2p+1} - 1)^2)} \right|^{1/2}, \quad (10)$$

$$C_{2p}(M) = \frac{\prod_{r=1}^{p} \ell_{r,2p} \prod_{r=1}^{p+1} (\ell_{r,2p+2}}{\prod_{r=1}^{p} \ell_{r,2p+1}(\ell_{r,2p+1} - 1)} \quad (11)$$

and $\ell_{j,2p} = m_{j,2p} + p - j$, $\ell_{j,2p+1} = m_{j,2p+1} + p - j + 1$. Since $\ell_{p+1,2p+2} = m_{p+1,2p+2}$, then the last summand of (8) vanishes if the last coordinate $m_{p+1,2p+2}$ of the highest weight $\mathbf{m}$ equals zero.

As in the case of the group $U(n)$, the operators $I_{q,q-1}$, given by formulas (7)-(11), satisfy the unitarity condition $I^*_{q,q-1} = -I_{q,q-1}$.

Since any operator I_{ks}, $k > s$, can be obtained with the help of commutation relations from the operators (7) and (8), then formulas (7)-(11) determine the representation $T_{\mathbf{m}}$ of the Lie algebra $\mathfrak{so}(n)$.

18.2. Clebsch-Gordan Coefficients for the Gel'fand-Tsetlin Basis

18.2.1. Definition of Clebsch-Gordan coefficients. Let $T_{\mathbf{m}'}$ and $T_{\mathbf{m}''}$ be two irreducible unitary representations of the group $U(n)$ or of the group $SO(n)$ in the spaces $\mathfrak{H}_{\mathbf{m}'}$ and $\mathfrak{H}_{\mathbf{m}''}$ respectively. The tensor product of these representations decomposes into a direct sum of irreducible components:

$$T_{\mathbf{m}'} \otimes T_{\mathbf{m}''} = \sum_{\mathbf{m}} \oplus\, \nu(\mathbf{m}) T_{\mathbf{m}}. \quad (1)$$

The integer $\nu(\mathbf{m})$ indicates multiplicity of the irreducible representation $T_{\mathbf{m}}$ in this decomposition. The decomposition (1) means that

$$\mathfrak{H}_{\mathbf{m}'} \otimes \mathfrak{H}_{\mathbf{m}''} = \sum_{\mathbf{m}} \oplus\, \nu(\mathbf{m}) \mathfrak{H}_{\mathbf{m}}, \quad (2)$$

where $\nu(\mathbf{m})\mathfrak{H}_{\mathbf{m}} \equiv \mathfrak{H}_{\mathbf{m}} \oplus \ldots \oplus \mathfrak{H}_{\mathbf{m}}$ ($\nu(\mathbf{m})$ times). In order to distinguish multiple spaces $\mathfrak{H}_{\mathbf{m}}$ they are supplied by additional index r: $\mathfrak{H}^r_{\mathbf{m}}$, $r = 1, 2, \ldots, \nu(\mathbf{m})$.

We choose two orthonormal bases in the space (2):

$$\mathbf{e}'_i \otimes \mathbf{e}''_j, \quad i = 1, 2, \ldots, \dim \mathfrak{H}_{\mathbf{m}'}, \quad j = 1, 2, \ldots, \dim \mathfrak{H}_{\mathbf{m}''}, \quad (3)$$

$$\mathbf{e}^r_k(\mathbf{m}) \equiv \mathbf{e}^r_k, \quad k = 1, 2, \ldots, \dim \mathfrak{H}_{\mathbf{m}}, \quad r = 1, 2, \ldots, \nu(\mathbf{m}), \quad (4)$$

where $\{\mathbf{e}'_i\}$, $\{\mathbf{e}''_j\}$, $\{\mathbf{e}^r_k\}$ are orthonormal bases of the spaces $\mathfrak{H}_{\mathbf{m}'}$, $\mathfrak{H}_{\mathbf{m}''}$, $\mathfrak{H}^r_{\mathbf{m}}$, respectively. The bases (3) and (4) are connected by the unitary matrix C:

$$\mathbf{e}^r_k = \sum_{i,j} \langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k \rangle \mathbf{e}'_i \otimes \mathbf{e}''_j. \tag{5}$$

The numbers $\langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k \rangle$ are called *Clebsch-Gordan coefficients* (CGC's) of the tensor product $T_{\mathbf{m}'} \otimes T_{\mathbf{m}''}$. The numbers $\overline{\langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k \rangle}$ are denoted by $\langle \mathbf{e}^r_k \mid \mathbf{e}'_i, \mathbf{e}''_j \rangle$:

$$\mathbf{e}'_i \otimes \mathbf{e}''_j = \sum_{\mathbf{m},k,r} \langle \mathbf{e}^r_k \mid \mathbf{e}'_i, \mathbf{e}''_j \rangle \mathbf{e}^r_k. \tag{6}$$

Properties of the CGC's $\langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k \rangle$ are similar to those of the CGC's of the group $SU(2)$ (see Chapter 8). Namely, the orthogonality relations

$$\sum_{i,j} \langle \mathbf{e}^r_k(\mathbf{m}) \mid \mathbf{e}'_i, \mathbf{e}''_j \rangle \langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^t_s(\tilde{\mathbf{m}}) \rangle = \delta_{\mathbf{m}\tilde{\mathbf{m}}} \delta_{rt} \delta_{ks}, \tag{7}$$

$$\sum_{\mathbf{m},k,r} \langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k(\mathbf{m}) \rangle \langle \mathbf{e}^r_k(\mathbf{m}) \mid \mathbf{e}'_s, \mathbf{e}''_t \rangle = \delta_{is} \delta_{jt} \tag{8}$$

are valid. We have the following relations between CGC's and matrix elements of irreducible representations:

$$t^{\mathbf{m}}_{k\ell}(g) = \sum_{i,s,j,t} \langle \mathbf{e}^r_k \mid \mathbf{e}'_i, \mathbf{e}''_j \rangle t^{\mathbf{m}'}_{is}(g) t^{\mathbf{m}''}_{jt}(g) \langle \mathbf{e}'_s, \mathbf{e}''_t \mid \mathbf{e}^r_\ell \rangle, \tag{9}$$

$$t^{\mathbf{m}'}_{is}(g) t^{\mathbf{m}''}_{jt}(g) = \sum_{\mathbf{m},k,\ell} \left(\sum_r \langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k(\mathbf{m}) \rangle \langle \mathbf{e}^r_\ell(\mathbf{m}) \mid \mathbf{e}'_s, \mathbf{e}''_t \rangle \right) t^{\mathbf{m}}_{k\ell}(g). \tag{10}$$

It follows from (10) and from the orthogonality relations for matrix elements of representations that

$$\begin{aligned} &\sum_r \langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k(\mathbf{m}) \rangle \langle \mathbf{e}^r_\ell(\mathbf{m}) \mid \mathbf{e}'_s, \mathbf{e}''_t \rangle \\ &= (\dim T_{\mathbf{m}}) \int t^{\mathbf{m}'}_{is}(g) t^{\mathbf{m}''}_{jt}(g) \overline{t^{\mathbf{m}}_{k\ell}(g)} dg. \end{aligned} \tag{11}$$

We are interested in CGC's for the Gel'fand-Tsetlin bases. In this reason CGC's $\langle \mathbf{e}'_i, \mathbf{e}''_j \mid \mathbf{e}^r_k \rangle$ will be written as

$$\langle \mathbf{m}', M'; \mathbf{m}'', M'' \mid \mathbf{m}, M \rangle^r,$$

where, as before, the index r distinguishes multiple representations $T_{\mathbf{m}}$ in the tensor product (1). If $T_{\mathbf{m}}$ is contained in the tensor product with the unit multiplicity, then index r will be omitted.

Let multiple representations be absent in decomposition (1). Then relation (11) can be written as

$$\begin{aligned}
&\langle \mathbf{m}', M'; \mathbf{m}'', M'' \mid \mathbf{m}, M\rangle \langle \mathbf{m}, N \mid \mathbf{m}', N'; \mathbf{m}'', N''\rangle \\
&= (\dim T_{\mathbf{m}}) \overline{\int t^{\mathbf{m}'}_{M'N'}(g) t^{\mathbf{m}''}_{M''N''}(g) t^{\mathbf{m}}_{MN}(g) dg} \\
&= \frac{\dim T_{\mathbf{m}}}{\dim T_{\mathbf{m}'}} \overline{\langle \mathbf{m}, M; \bar{\mathbf{m}}'', \bar{M}'' \mid \mathbf{m}', M'\rangle \langle \mathbf{m}', N' \mid \mathbf{m}, N; \bar{\mathbf{m}}'', \bar{N}''\rangle}.
\end{aligned} \tag{12}$$

Fixing the Gel'fand-Tsetlin patterns N, N', N'', we obtain the symmetry relation

$$\langle \mathbf{m}, M; \bar{\mathbf{m}}'', \bar{M}'' \mid \mathbf{m}', M'\rangle = \beta \left(\frac{\dim T_{\mathbf{m}}}{\dim T_{\mathbf{m}'}} \right)^{1/2} \overline{\langle \mathbf{m}', M'; \mathbf{m}'', M'' \mid \mathbf{m}, M\rangle}, \tag{13}$$

where β is a constant such that $|\beta| = 1$, and $\bar{\mathbf{m}}$ denotes the highest weight of the irreducible representation $T_{\bar{\mathbf{m}}}$, whose matrix is complex conjugate to the matrix of the representation $T_{\mathbf{m}}$. In other words, the representation $T_{\bar{\mathbf{m}}}$ is contragradient to the representation $T_{\mathbf{m}}$. For the unitary group $U(n)$ we have $\bar{\mathbf{m}} = (-m_{nn}, \dots, -m_{2n}, -m_{1n})$ if $\mathbf{m} = (m_{1n}, m_{2n}, \dots, m_{nn})$. For the group $SO(2k+1)$ the equality $\bar{\mathbf{m}} = \mathbf{m}$ holds. We also have $\bar{\mathbf{m}} = \mathbf{m}$ for the groups $SO(2k)$ if $m_{k,2k} = 0$. In formula (13) $\bar{M}$ denotes the Gel'fand-Tsetlin pattern which is obtained from $M = (\mathbf{m}_n, \mathbf{m}_{n-1}, \dots,)$ if we replace all $\mathbf{m}_j$ by $\bar{\mathbf{m}}_j$.

One can easily show that

$$\langle \bar{\mathbf{m}}', \bar{M}'; \mathbf{m}, M \mid \mathbf{m}'', M''\rangle = \gamma \left(\frac{\dim T_{\mathbf{m}}}{\dim T_{\mathbf{m}''}} \right)^{1/2} \overline{\langle \mathbf{m}', M'; \mathbf{m}'', M'' \mid \mathbf{m}, M\rangle}, \tag{14}$$

where $|\gamma| = 1$. Since CGC's are uniquely defined by representations up to constants, then we may assume that constants β in (13) and γ in (14) are equal to 1.

18.2.2. Scalar factors. Let us restrict the representations of the relation (1) of Section 18.2.1 onto the subgroup $G_1 = U(n-1)$ if we consider the group $G = U(n)$ and onto the subgroup $G_1 = SO(n-1)$ if $G = SO(n)$. Every of the spaces from formula (2) of Section 18.2.1 can be decomposed into a sum of invariant irreducible subspaces for the subgroup G_1:

$$\mathfrak{H}_{\mathbf{m}} \equiv \mathfrak{H}(\mathbf{m}_n) = \sum_{\mathbf{m}_{n-1}} \oplus\, \mathfrak{H}(\mathbf{m}_{n-1}).$$

The formula (2) of Section 18.2.1 may be considered as a sum of decompositions of the products $\mathfrak{H}(\mathbf{m}'_{n-1}) \otimes \mathfrak{H}(\mathbf{m}''_{n-1})$ of the spaces in which the representations of the subgroup G_1 are realized. In this case the numbers $\langle \mathbf{m}', M'; \mathbf{m}'', M'' \mid \mathbf{m}, M\rangle^r$ can be regarded as CGC's of these tensor products. In other words, they are multiple (with the sum over multiplicity index) of the CGC's for the tensor products of the irreducible representations of G_1. Therefore, the relation

$$\begin{aligned}\langle \mathbf{m}', M'; \mathbf{m}'', M'' \mid \mathbf{m}, M\rangle^r &\equiv \langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M\rangle^r \\ = \sum_t \left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array}\right)^r_t &\langle \mathbf{m}'_{n-1}, M'_1; \mathbf{m}''_{n-1}, M''_1 \mid \mathbf{m}_{n-1}, M_1\rangle^t \end{aligned} \tag{1}$$

is valid, where on the right hand side we have CGC's of the subgroup G_1 and M'_1, M''_1, M_1 are the Gel'fand-Tsetlin patterns which are obtained from M', M'', M, respectively, by removing the first row. The highest weights $\mathbf{m}'_{n-1}$, $\mathbf{m}''_{n-1}$, $\mathbf{m}_{n-1}$ from (1) coincide with the corresponding rows in M', M'', M, respectively. They are the first rows correspondingly in M'_1, M''_1, M_1. If multiple representations are absent in (1), then we can remove the indices r and t. In this case we have

$$\begin{aligned}&\langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M\rangle \\ &= \left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array}\right) \langle \mathbf{m}'_{n-1}, M'_1; \mathbf{m}''_{n-1}, M''_1 \mid \mathbf{m}_{n-1}, M_1\rangle. \end{aligned} \tag{2}$$

The coefficients $\left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array}\right)^r_t$ from (1) are called G_1-*scalar factors* of the CGC's $\langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M\rangle^r$.

It follows from (1) and from the orthogonality relations for CGC's that

$$\begin{aligned}\left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array}\right)^r_t &= \sum_{M'_1, M''_1} \langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}, M\rangle^r \\ &\times \langle \mathbf{m}'_{n-1}, M'_1; \mathbf{m}''_{n-1}, M''_1 \mid \mathbf{m}_{n-1}, M_1\rangle^t. \end{aligned} \tag{3}$$

If multiple representations are absent, then we have

$$\left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array}\right) = \frac{\langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M\rangle}{\langle \mathbf{m}'_{n-1}, M'_1; \mathbf{m}''_{n-1}, M''_1 \mid \mathbf{m}_{n-1}, M_1\rangle}. \tag{4}$$

Applying factorization (2) to the CGC's of the subgroups we obtain that

$$\begin{aligned}\langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M\rangle &= \left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array}\right) \\ \times \left(\begin{array}{cc|c} \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \\ \mathbf{m}'_{n-2} & \mathbf{m}''_{n-2} & \mathbf{m}_{n-2} \end{array}\right) &\left(\begin{array}{cc|c} \mathbf{m}'_{n-2} & \mathbf{m}''_{n-2} & \mathbf{m}_{n-2} \\ \mathbf{m}'_{n-3} & \mathbf{m}''_{n-3} & \mathbf{m}_{n-3} \end{array}\right) \cdots . \end{aligned} \tag{5}$$

Thus, CGC's of the group $U(n)$ in the Gel'fand-Tsetlin bases are factorized into product (or into a sum of products if multiple representations exist) of $U(j)$-scalar factors, $j = n-1, n-2, \dots, 1$, which depend only on two rows of the corresponding Gel'fand-Tsetlin patterns.

It follows from formula (1) or from formula (5) that the CGC

$$\langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M\rangle^r \equiv \left\langle \begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \\ \mathbf{m}'_{n-2} & \mathbf{m}''_{n-2} & \mathbf{m}_{n-2} \\ \dots & \dots & \dots \end{array} \right\rangle^r \tag{6}$$

vanishes if for some j ($j = n, n-1, \dots, 1$) the representation $T_{\mathbf{m}_j}$ of the subgroup $U(j)$ is not contained in the tensor product $T_{\mathbf{m}'_j} \otimes T_{\mathbf{m}''_j}$ where $\mathbf{m}_j$, $\mathbf{m}'_j$, $\mathbf{m}''_j$ are taken from $n-j+1$-th row of (6). The analogous assertions are valid for the group $SO(n)$.

Acting onto both sides of relation (6) of Section 18.2.1 by the operators E_{jj}, $j = 1, 2, \dots, n$, and taking into account the formula (5) of Section 18.1.2, we conclude that CGC (6) of the group $U(n)$ vanishes if some of the relations

$$\sum_{i=1}^{j} m'_{ij} + \sum_{i=1}^{j} m''_{ij} = \sum_{i=1}^{j} m_{ij}, \quad j = 1, 2, \dots, n, \tag{7}$$

are not fulfilled, where m_{kj}, m'_{kj}, m''_{kj} are the coordinates of the highest weights $\mathbf{m}_j$, $\mathbf{m}'_j$, $\mathbf{m}''_j$ respectively.

The orthogonality relations (7) and (8) of Section 18.2.1 lead to the orthogonality relations for scalar factors:

$$\sum_{\mathbf{m}'_{n-1}, \mathbf{m}''_{n-1}, t} \left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array} \right)^r_t \overline{\left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}'''_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}'''_{n-1} \end{array} \right)^s_t}$$
$$= \delta_{\mathbf{m}_n \mathbf{m}'''_n} \delta_{\mathbf{m}_{n-1} \mathbf{m}'''_{n-1}} \delta_{rs}, \tag{8}$$

$$\sum_{\mathbf{m}_n, r} \left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array} \right)^r_t \overline{\left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{p}'_{n-1} & \mathbf{p}''_{n-1} & \mathbf{m}_{n-1} \end{array} \right)^r_q}$$
$$= \delta_{\mathbf{m}'_{n-1} \mathbf{p}'_{n-1}} \delta_{\mathbf{m}''_{n-1} \mathbf{p}''_{n-1}} \delta_{tq}. \tag{9}$$

The symmetry relations for CGC's leads to the symmetries of scalar factors. If multiple representations are absent, then they are of the form

$$\left(\begin{array}{cc|c} \mathbf{m}'_n & \mathbf{m}''_n & \mathbf{m}_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} & \mathbf{m}_{n-1} \end{array} \right)$$
$$= \beta \left(\frac{\dim T(\mathbf{m}''_n) \dim T(\mathbf{m}_{n-1})}{\dim T(\mathbf{m}_n) \dim T(\mathbf{m}''_{n-1})} \right)^{-1/2} \overline{\left(\begin{array}{cc|c} \bar{\mathbf{m}}'_n & \mathbf{m}_n & \mathbf{m}''_n \\ \bar{\mathbf{m}}'_{n-1} & \mathbf{m}_{n-1} & \mathbf{m}''_{n-1} \end{array} \right)} \tag{10}$$
$$= \gamma \left(\frac{\dim T(\mathbf{m}'_n) \dim T(\mathbf{m}_{n-1})}{\dim T(\mathbf{m}_n) \dim T(\mathbf{m}'_{n-1})} \right)^{-1/2} \overline{\left(\begin{array}{cc|c} \mathbf{m}_n & \bar{\mathbf{m}}''_n & \mathbf{m}'_n \\ \mathbf{m}_{n-1} & \bar{\mathbf{m}}''_{n-1} & \mathbf{m}'_{n-1} \end{array} \right)},$$

where $\bar{\mathbf{m}}_n$ and $\bar{\mathbf{m}}_{n-1}$ are the highest weights of the representations contragradient to the representations with the highest weights $\mathbf{m}_n$ and $\mathbf{m}_{n-1}$, respectively, and $|\beta| = |\gamma| = 1$. If in relations (13) and (14) of Section 18.2.1 $\beta = \gamma = 1$, then in (10) we have $\beta = \gamma = 1$.

It follows from the definition of contragradient representations that

$$\left(\begin{matrix} \mathbf{m}'_n & \mathbf{m}''_n \\ \mathbf{m}'_{n-1} & \mathbf{m}''_{n-1} \end{matrix} \middle| \begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \end{matrix}\right) = \delta \left(\begin{matrix} \bar{\mathbf{m}}'_n & \bar{\mathbf{m}}''_n \\ \bar{\mathbf{m}}'_{n-1} & \bar{\mathbf{m}}''_{n-1} \end{matrix} \middle| \begin{matrix} \bar{\mathbf{m}}_n \\ \bar{\mathbf{m}}_{n-1} \end{matrix}\right), \tag{11}$$

where $|\delta| = 1$.

18.2.3. Tensor operators. Let $T_{\mathbf{m}}$ be a finite dimensional representation of the compact group G acting in the space $\mathfrak{H}_{\mathbf{m}}$ with the orthonormal basis $\{\mathbf{v}_M \mid M \in I\}$. Let $\{R_M \mid M \in I\}$ be a set of operators acting in the Hilbert space $\mathfrak{H}$. We say that $\mathbf{R}^{\mathbf{m}} \equiv \{R_M\}$ is a *tensor operator*, transforming under the representation $T_{\mathbf{m}}$ of G, if there is a representation T of G acting in $\mathfrak{H}$ such that

$$T(g)R_M T(g^{-1}) = \sum_{N \in I} t^{\mathbf{m}}_{NM}(g) R_N, \quad M \in I, \tag{1}$$

where $t^{\mathbf{m}}_{NM}(g)$ are matrix elements of the representation $T_{\mathbf{m}}$ with respect to the basis $\{\mathbf{v}_M\}$.

Let $g(t)$ be a one-parameter subgroup in G with the tangent vector $X \in \mathfrak{g}$. Setting into (1) $g(t)$ instead of g and differentiating with respect to t at $t = 0$, we obtain

$$[T(X), R_M] = \sum_{N \in I} t^{m}_{NM}(X) R_N. \tag{2}$$

This equality defines tensor operators in an infinitesimal form (with the help of representations of the Lie algebra $\mathfrak{g}$).

Infinitesimal operators of the representations of Lie algebras are the important examples of tensor operators. Let $\mathfrak{g}$ be a noncompact real semisimple Lie algebra and let $\mathfrak{k}$ be its maximal compact Lie subalgebra. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (see Section 1.2.3). Moreover, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. In other words, the operators ad X, $X \in \mathfrak{k}$, define the representation $T^{\rm ad}$ of the Lie algebra $\mathfrak{k}$ acting in the space $\mathfrak{p}$:

$$(\mathrm{ad}\, X)Y = [X, Y] \in \mathfrak{p}, \quad Y \in \mathfrak{p}.$$

Let us choose a basis $P_1, \ldots, P_r$ in $\mathfrak{p}$. For any representation T of the Lie algebra $\mathfrak{g}$ the operators $T(P_i)$, $i = 1, 2, \ldots, r$, form a tensor operator which transforms under the representation $T^{\rm ad}$ of the Lie algebra $\mathfrak{k}$.

Example 1. Let $\mathfrak{g} = \mathfrak{u}(n-1, 1)$, $\mathfrak{g}_k = \mathfrak{u}(n)$. Then $\mathfrak{k} = \mathfrak{u}(n-1) + \mathfrak{u}(1)$ and we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{g}_k = \mathfrak{k} + i\mathfrak{p}$, where $\mathfrak{p}$ is spanned by the matrices

$$E_{jn} + E_{nj}, \quad \sqrt{-1}(E_{jn} - E_{nj}), \quad j = 1, 2, \ldots, n-1.$$

In the complexification $\mathfrak{p}_c$ of the space $\mathfrak{p}$ we can choose the basis consisting of the matrices

$$E_{jn}, \quad E_{nj}, \quad j = 1, 2, \ldots, n-1. \tag{3}$$

It is easy to see that the operators $T^{\rm ad}(X)$, $X \in \mathfrak{k}$, leave invariant the subspace $\mathfrak{p}_1$, spanned by the matrices E_{jn}, $j = 1, 2, \ldots, n-1$, and the subspace $\mathfrak{p}_2$, spanned by the matrices E_{nj}, $j = 1, 2, \ldots, n-1$. In other words, for any representation of the Lie algebra $\mathfrak{u}(n-1, 1)$ (of the Lie algebra $\mathfrak{u}(n)$) the operators $\mathbf{E}_n \equiv \{E_{jn} \mid 1 \le j \le n-1\}$, as well as the operators $\mathbf{E}'_n \equiv \{E_{nj} \mid 1 \le j \le n-1\}$ are tensor operators transforming under representations of the compact Lie algebra $\mathfrak{k}$. Using formulas (3)-(5) of Section 18.1.2 it is easy to find that the tensor operator $\mathbf{E}_n$ (the tensor operator $\mathbf{E}'_n$) transforms under the irreducible representation of the Lie algebra $\mathfrak{k} = \mathfrak{u}(n-1) + \mathfrak{u}(1)$ with highest weight $(1, 0, \ldots, 0)$ (-1) (with highest weight $(0, \ldots, 0, -1)$ (1)), where (-1) (respectively, (1)) means the one-dimensional representation $\varphi \to e^{-i\varphi}$ (the representation $\varphi \to e^{i\varphi}$) of the group $U(1)$. The operators E_{jn} and E_{nj} correspond to the appropriate elements of the Gel'fand-Tsetlin bases of the representations of $U(n-1)$ with highest weights $(1, 0, \ldots, 0)$ and $(0, \ldots, 0, -1)$, respectively. This correspondence is

$$E_{jn} \to \begin{pmatrix} 1 \quad 0 \ldots\ldots 0 \\ \ldots\ldots\ldots \\ 1 \quad 0 \ldots 0 \\ 0 \ldots 0 \\ \ldots \\ 0 \end{pmatrix}, \qquad E_{nj} \to \begin{pmatrix} 0 \ldots\ldots 0 \quad -1 \\ \ldots\ldots\ldots\ldots \\ 0 \ldots 0 \quad -1 \\ 0 \ldots 0 \\ \ldots \\ 0 \end{pmatrix}, \tag{4}$$

where there are $n-j$ rows $(1, 0, \ldots, 0)$ and $n-j$ rows $(0, \ldots, 0, -1)$.

Example 2. For any representation of the Lie algebra $\mathfrak{so}(n)$ the matrices $I_{jn} \equiv E_{jn} - E_{nj}$, $j = 1, 2, \ldots, n-1$, form the tensor operator which transforms under the irreducible representation of $\mathfrak{so}(n-1)$ with highest weight $(1, 0 \ldots, 0)$.

18.2.4. The Wigner-Eckart theorem. Let us decompose the representation T of the group G from formula (1) of Section 18.2.3 into a sum of irreducible components:

$$T = \sum_{\mathbf{m}'} \oplus T_{\mathbf{m}'}. \tag{1}$$

For simplicity we assume that this decomposition has no multiple representations. Let us choose the orthonormal bases $\mathbf{e}_s^{\mathbf{m}'}$ $s = 1, 2, \ldots, \dim T_{\mathbf{m}'}$, in the carrier spaces of the representations $T_{\mathbf{m}'}$ from (1).

We derive the formula for matrix elements of the operators R_M which constitute the tensor operator $\mathbf{R}^{\mathbf{m}}$. For this we consider matrix elements of both parts of the relation

$$\sum_{N \in I} t^m_{NM}(g) R_N T(g) = T(g) R_M$$

with respect to the basis $\{\mathbf{e}_s^{\mathbf{m}'}\}$. We have

$$\sum_{N'} t^{\mathbf{m}'}_{M'N'}(g)(R_M \mathbf{e}^{\mathbf{m}''}_{M''}, \mathbf{e}^{\mathbf{m}'}_{N'})$$
$$= \sum_{N,N''} t^{\mathbf{m}}_{NM}(g) t^{\mathbf{m}''}_{N''M''}(g)(R_N \mathbf{e}^{\mathbf{m}''}_{N''}, \mathbf{e}^{\mathbf{m}'}_{M'}).$$

Let us apply relation (10) of Section 18.2.1 to the right hand side of this formula:

$$\sum_{N'} t^{\mathbf{m}'}_{M'N'}(g)(R_M \mathbf{e}^{\mathbf{m}''}_{M''}, \mathbf{e}^{\mathbf{m}'}_{N'}) = \sum_{N,N''} \sum_{\mathbf{m}'''} \sum_{N''',M'''} t^{\mathbf{m}'''}_{N'''M'''}(g)$$
$$\times \langle \mathbf{m}, N; \mathbf{m}'', N'' \mid \mathbf{m}''', N''' \rangle \langle \mathbf{m}''', M''' \mid \mathbf{m}, M; \mathbf{m}'', M'' \rangle (R_N \mathbf{e}^{\mathbf{m}''}_{N''}, \mathbf{e}^{\mathbf{m}'}_{M'}). \tag{2}$$

Here we again assume that multiple irreducible representations are absent in the tensor product $T_{\mathbf{m}} \otimes T_{\mathbf{m}''}$. Let us multiply both sides of formula (2) by $\overline{t^{\mathbf{m}'}_{N'M'_0}(g)}$ and integrate with respect to the invariant measure on G. Due to the orthogonality relations for matrix elements we obtain

$$(R_M \mathbf{e}^{\mathbf{m}''}_{M''}, \mathbf{e}^{\mathbf{m}'}_{M'}) = \sum_{N,N''} \langle \mathbf{m}, N; \mathbf{m}'', N'' \mid \mathbf{m}', N' \rangle$$
$$\times \langle \mathbf{m}', M' \mid \mathbf{m}, M; \mathbf{m}'', M'' \rangle (R_N \mathbf{e}^{\mathbf{m}''}_{N''}, \mathbf{e}^{\mathbf{m}'}_{N'}). \tag{3}$$

The expressions

$$\langle \mathbf{m}' \| \mathbf{R}^{\mathbf{m}} \| \mathbf{m}'' \rangle = (\dim T_{\mathbf{m}'})^{-1} \sum_{N,N',N''} \langle \mathbf{m}, N; \mathbf{m}'', N'' \mid \mathbf{m}', N' \rangle$$
$$\times (R_N \mathbf{e}^{\mathbf{m}''}_{N''}, \mathbf{e}^{\mathbf{m}'}_{N'}) \tag{4}$$

are called *reduced matrix elements* of the tensor operator $\mathbf{R}^{\mathbf{m}}$. They do not depend on indices of basis elements. Formula (3) can be written as

$$(R_M \mathbf{e}^{\mathbf{m}''}_{M''}, \mathbf{e}^{\mathbf{m}'}_{M'}) = \langle \mathbf{m}', M' \mid \mathbf{m}, M; \mathbf{m}'', M'' \rangle \langle \mathbf{m}' \| \mathbf{R}^{\mathbf{m}} \| \mathbf{m}'' \rangle. \tag{5}$$

This equality is called the *Wigner-Eckart theorem for tensor operators.*

If multiple irreducible representations appear in the tensor product $T_{\mathbf{m}} \otimes T_{\mathbf{m}''}$, then instead of formula (5) we have

$$(R_M \mathbf{e}^{\mathbf{m}''}_{M''}, \mathbf{e}^{\mathbf{m}'}_{M'}) = \sum_r \langle \mathbf{m}', M' \mid \mathbf{m}, M; \mathbf{m}'', M'' \rangle^r \langle \mathbf{m}' \| \mathbf{R}^{\mathbf{m}} \| \mathbf{m}'' \rangle_r. \tag{6}$$

Let us note that the Wigner-Eckart theorem is generalization of the Schur lemma (see Section 2.2.8). The Schur lemma coincides with the Wigner-Eckart theorem when the tensor operator $\mathbf{R}^{\mathbf{m}}$ transforms under the identity representation of the group G.

18.2.5. Matrix elements of the operators $E^k_{n-1,n}$ and $E^k_{n,n-1}$ of representations of $\mathfrak{gl}(n,\mathbb{C})$. We gave the matrix elements of the operators $E_{n-1,n}$ and $E_{n,n-1}$ for irreducible unitary representations of the group $U(n)$. Let us evaluate matrix elements for powers of these operators. For convenience the Gel'fand-Tsetlin basis elements $\mathbf{v}(M)$ will be denoted by $\mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1}, N)$, where N is the reminder part of the Gel'fand-Tsetlin pattern, or else by $\mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1}, \mathbf{m}_{n-2})$.

It follows from formula (3) of Section 18.1.2 that

$$E^k_{n-1,n}\mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1}, N) = \sum_{\boldsymbol{\rho}} A^k_{\boldsymbol{\rho}}(\mathbf{m}_n, \mathbf{m}_{n-1}, \mathbf{m}_{n-2})\mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1} + \boldsymbol{\rho}, N), \qquad (1)$$

where summation is over all integral values $\boldsymbol{\rho} = (\rho_1, \rho_2, \cdots, \rho_{n-1})$ such that $\rho_1 + \rho_2 + \ldots + \rho_{n-1} = k$, $\rho_i \geq 0$, $i = 1, 2, \ldots, n-1$. The coefficients $A^k_{\boldsymbol{\rho}}(\mathbf{m}_n, \mathbf{m}_{n-1}, \mathbf{m}_{n-2})$ from (1) are matrix elements of the operators $E^k_{n-1,n}$. Let us prove that for these operators the formula

$$\begin{aligned} A^k_{\boldsymbol{\rho}}(\mathbf{m}_n, \mathbf{m}_{n-1}, \mathbf{m}_{n-2}) = k! \Bigg[&\prod_{i<j}(\ell_{i,n} - \ell_{j,n-1})(\ell'_{i,n-1} - \ell'_{j,n-1}) \\ \times &\frac{\prod\limits_{i\leq j}(\ell'_{i,n-1} - \ell_{j,n-2})! \prod\limits_{i<j}(\ell_{i,n-2} - \ell_{j,n-1} - 1)! \prod\limits_{i\leq j}(\ell_{in} - \ell_{j,n-1})!}{\prod\limits_{i<j}(\ell_{i,n-2} - \ell'_{j,n-1} - 1)! \prod\limits_{i\leq j}(\ell_{i,n-1} - \ell_{j,n-2})! \prod\limits_{i<j}(\ell_{i,n-1} - \ell_{jn} - 1)!} \\ \times \times &\frac{\prod\limits_{i<j}(\ell'_{i,n-1} - \ell_{jn} - 1)!}{\prod\limits_{i\leq j}(\ell_{in} - \ell'_{j,n-1})!}\Bigg]^{1/2} \frac{\prod\limits_{i<j}(\ell_{i,n-1} - \ell'_{j,n-1} - 1)!}{\prod\limits_{i\leq j}(\ell'_{i,n-1} - \ell_{j,n-1})!} \end{aligned} \qquad (2)$$

is valid, where $\ell_{jk} = m_{jk} - j$, $\ell'_{i,n-1} = \ell_{i,n-1} + \rho_i$, $i - 1, 2, \ldots, n-1$.

The formula (2) can be represented in another form. For this we introduce the notation

$$\begin{aligned} S_{pq}(\mathbf{m}_p, \mathbf{m}_q) &\equiv S_{pq}(m_{1p}, \ldots, m_{pp}, m_{1q}, \ldots, m_{qq}) \\ &= \prod_{i\leq j}(\ell_{ip} - \ell_{jq})!^{1/2} \prod_{i<j}(\ell_{iq} - \ell_{jp} - 1)!^{-1/2}. \end{aligned} \qquad (3)$$

It is easy to verify that if $\mathbf{m}_n$ is highest weight of the irreducible representation $T_{\mathbf{m}_n}$ of $U(n)$, then

$$[S_{nn}(\mathbf{m}_n, \mathbf{m}_n)]^2 = \prod_{i<j}(\ell_{in} - \ell_{jn}) = (\dim T_{\mathbf{m}_n}) \prod_{k=1}^{n-1} k!. \qquad (4)$$

With the help of the expressions (3) formula (2) can be written as

$$A^k_{\boldsymbol\rho}(\mathbf{m}_n,\mathbf{m}_{n-1},\mathbf{m}_{n-2}) = k!\frac{S_{n-1,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1})}{[S_{n-1,n-1}(\mathbf{m}'_{n-1},\mathbf{m}_{n-1})]^2} \times \frac{S_{n-1,n-1}(\mathbf{m}'_{n-1},\mathbf{m}'_{n-1})S_{n-1,n-2}(\mathbf{m}'_{n-1},\mathbf{m}_{n-2})S_{n,n-1}(\mathbf{m}_n,\mathbf{m}_{n-1})}{S_{n-1,n-2}(\mathbf{m}_{n-1},\mathbf{m}_{n-2})S_{n,n-1}(\mathbf{m}_n,\mathbf{m}'_{n-1})}, \tag{5}$$

where $\mathbf{m}'_{n-1} = \mathbf{m}_{n-1} + \boldsymbol\rho$.

In order to derive formula (5) we use the recurrence relation

$$\begin{aligned}(E^{k+1}_{n-1,n}\mathbf{v}(\mathbf{m}_n,\mathbf{m}_{n-1}-\boldsymbol\rho,\mathbf{m}_{n-2}),\mathbf{v}(\mathbf{m}_n,\mathbf{m}_{n-1},\mathbf{m}_{n-2}))\\ \equiv A^{k+1}_{\boldsymbol\rho}(\mathbf{m}_n,\mathbf{m}_{n-1}-\boldsymbol\rho,\mathbf{m}_{n-2})\\ = \sum_{s=1}^{n-1} a^s_{n-1}(M)(E^k_{n-1,n}\mathbf{v}(\mathbf{m}_n,\mathbf{m}^{+s}_{n-1}-\boldsymbol\rho,\mathbf{m}_{n-2}),\mathbf{v}(\mathbf{m}_n,\mathbf{m}_{n-1},\mathbf{m}_{n-2}))\end{aligned} \tag{6}$$

which follows from formula (3) of Section 18.1.2. We use the method of mathematical induction. For $k = 1$ formula (5) (or equivalent formula (2)) is valid. Let us suppose that it is valid for the operators $E^r_{n-1,n}$, $r \le k$. We have to prove that (5) is valid for the operator $E^{k+1}_{n-1,n}$. We substitute expression (5) for matrix elements of the operator $E^k_{n-1,n}$ into formula (6) and transform the expressions S_{ij} from the right hand side of the obtained formula. For example, we have

$$\begin{aligned}&S_{n-1,n-1}(\mathbf{m}^{+s}_{n-1}-\boldsymbol\rho,\mathbf{m}^{+s}_{n-1}-\boldsymbol\rho)\\ &= \left[\prod_{i<j}(\ell_{i,n-1}-\ell_{j,n-1}+\rho_j-\rho_i+\delta_{si}-\delta_{sj})\right]^{1/2}\\ &= \left[\prod_{\substack{i<j\\ i,j\neq s}}(\ell_{i,n-1}-\ell_{j,n-1}+\rho_j-\rho_i)\prod_{\substack{p=1\\ p\neq s}}|(\ell_{p,n-1}-\ell_{s,n-1}+\rho_s-\rho_p-1)|\right]^{1/2}\\ &= \left[\frac{\prod_{i<j}(\ell_{i,n-1}-\ell_{j,n-1}+\rho_j-\rho_i)\prod_{p\neq s}(\ell_{p,n-1}-\ell_{s,n-1}+\rho_s-\rho_p-1)}{\prod_{p\neq s}(\ell_{p,n-1}-\ell_{s,n-1}+\rho_s-\rho_p)}\right]^{1/2}\\ &= \left[\frac{\prod_{p\neq s}(\ell_{p,n-1}-\ell_{s,p-1}+\rho_s-\rho_p-1)}{\prod_{p\neq s}(\ell_{p,n-1}-\ell_{s,n-1}+\rho_s-\rho_p)}\right]^{1/2} S_{n-1,n-1}(\mathbf{m}_{n-1}-\boldsymbol\rho,\mathbf{m}_{n-1}-\boldsymbol\rho).\end{aligned}$$

In the same way we obtain

$$[S_{n-1,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1}^{+s}-\boldsymbol{\rho})]^2=\left[\prod_{p=1}^{s}(\ell_{p,n-1}-\ell_{s,n-1}+\rho_s)\right.$$
$$\left.\times\prod_{p=s+1}^{n-1}(\ell_{s,n-1}-\ell_{p,n-1}-\rho_s)\right]^{-1}[S_{n-1,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1}-\boldsymbol{\rho})]^2,$$

$$S_{n-1,n-2}(\mathbf{m}_{n-1}^{+s}-\boldsymbol{\rho},\mathbf{m}_{n-2})=\left[\prod_{p=1}^{s-1}(\ell_{p,n-2}-\ell_{s,n-1}+\rho_s-1)\right.$$
$$\left.\times\prod_{p=s}^{n-2}(\ell_{s,n-1}-\ell_{p,n-2}-\rho_s+1)\right]^{1/2}S_{n-1,n-2}(\mathbf{m}_{n-1}-\boldsymbol{\rho},\mathbf{m}_{n-2}),$$

$$S_{n,n-1}(\mathbf{m}_n,\mathbf{m}_{n-1}^{+s}-\boldsymbol{\rho})=\left[\prod_{p=1}^{s}(\ell_{pn}-\ell_{s,n-1}+\rho_s)\right.$$
$$\left.\times\prod_{p=s+1}^{n}(\ell_{s,n-1}-\ell_{pn}-\rho_s)\right]^{-1/2}S_{n,n-1}(\mathbf{m}_n,\mathbf{m}_{n-1}-\boldsymbol{\rho}).$$

Using this relations the right hand side of (6) can be represented as

$$\begin{aligned}&k!\frac{S_{n-1,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1})S_{n-1,n-1}(\mathbf{m}_{n-1}-\boldsymbol{\rho},\mathbf{m}_{n-1}-\boldsymbol{\rho})}{[S_{n-1,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1}-\boldsymbol{\rho})]^2S_{n-1,n-2}(\mathbf{m}_{n-1}-\boldsymbol{\rho},\mathbf{m}_{n-2})}\\&\quad\times\frac{S_{n-1,n-2}(\mathbf{m}_{n-1},\mathbf{m}_{n-2})S_{n,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1}-\boldsymbol{\rho})}{S_{n,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1})}\\&\quad\times\sum_{s=1}^{n-1}\rho_s\prod_{\substack{p=1\\p\neq s}}^{n-1}\frac{(\ell_{p,n-1}-\ell_{s,n-1}+\rho_s)}{(\ell_{p,n-1}-\ell_{s,n-1}+\rho_s-\rho_p)}.\end{aligned}\tag{7}$$

The sum from this expression can be written as

$$\begin{aligned}&\sum_{s=1}^{n-1}\rho_s\prod_{\substack{p=1\\p\neq s}}^{n-1}\left(1-\frac{\rho_p}{(\ell_{p,n-1}-\rho_p)-(\ell_{s,n-1}-\rho_s)}\right)\\&=\sum_{s=1}^{n-1}\sum_{\substack{i_1,i_2,\dots,i_s\\i_1<i_2<\dots<i_s}}\rho_{i_1}\rho_{i_2}\dots\rho_{i_s}\sum_r\prod_{q\neq r}\frac{1}{(\ell_{q,n-1}-\rho_q)-(\ell_{r,n-1}-\rho_r)},\end{aligned}\tag{8}$$

where r and q take the values $i_1, i_2, \ldots, i_s$. If we relabel the indices $i_1, i_2, \ldots, i_s$ by $1, 2, \ldots, s$, respectively, then the sum over r in (8) becomes

$$F(a_1, \ldots, a_s) = \sum_{r=1}^{s} \prod_{\substack{q=1 \\ q \neq r}}^{s} \frac{1}{a_r - a_q}, \quad a_j = \ell_{j,n-1} - \rho_j. \tag{9}$$

Let us prove that for any set $a_1, a_2, \ldots, a_s$ of different integers a_j the sum (9) is equal to zero if $s \geq 2$. We introduce the function $f(z) = \prod_{q=1}^{s} (z - a_q)^{-1}$. Then due to the theorem on residues we have

$$F(a_1, \ldots, a_s) = \frac{1}{2\pi i} \int_C f(z) dx,$$

where integration is over the closed curve containing inside all residues of the function $f(z)$. From the other side, the value of this integral is equal to the residue of the function $f(z)$ at the infinity. Since $s \geq 2$, then this residue is equal to zero. Therefore, the sum (9) is equal to zero.

Since the sum over r in (8) is a sum of the form (9), then in the sum over s only one summand, corresponding to $s = 1$, is non-vanishing. This summand is equal to $\rho_1 + \rho_2 + \ldots, \rho_{n-1} = k+1$. Substituting this value for the sum (8) into (7) we obtain that expression (7) coincides with expression (5) for $A_{\boldsymbol{\rho}}^{k+1}(\mathbf{m}_n, \mathbf{m}_{n-1} - \boldsymbol{\rho}, \mathbf{m}_{n-2})$. Thus, formula (2) is proved.

Since $E^*_{n,n-1} = E_{n-1,n}$ (see Section 18.1.2), then $E^k_{n,n-1} = (E^k_{n-1,n})^*$. Therefore, if $\boldsymbol{\rho}$ is such as in (2), then for matrix elements of the operator $E^k_{n,n-1}$ we have the expression

$$\begin{aligned}
&(E^k_{n,n-1} \mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1}, \mathbf{m}_{n-2}), \mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1} - \boldsymbol{\rho}, \mathbf{m}_{n-2})) \\
&= k! \Bigg[\prod_{i<j} (\ell_{i,n-1} - \ell_{j,n-1}) (\ell'_{i,n-1} - \ell'_{j,n-1}) \\
&\times \frac{\prod_{i \leq j} (\ell_{i,n-1} - \ell_{j,n-2})! \prod_{i<j} (\ell_{i,n-2} - \ell'_{j,n-1} - 1)! \prod_{i \leq j} (\ell_{in} - \ell'_{j,n-1})!}{\prod_{i<j} (\ell_{i,n-2} - \ell_{j,n-1} - 1)! \prod_{i<j} (\ell'_{i,n-1} - \ell_{j,n-2})! \prod_{i<j} (\ell'_{i,n-1} - \ell_{j,n} - 1)!} \\
&\times \frac{\prod_{i<j} (\ell_{i,n-1} - \ell_{jn} - 1)!}{\prod_{i \leq j} (\ell_{in} - \ell_{j,n-1})!} \Bigg]^{1/2} \frac{\prod_{i<j} (\ell'_{i,n-1} - \ell_{j,n-1} - 1)!}{\prod_{i \leq j} (\ell_{i,n-1} - \ell'_{j,n-1})!},
\end{aligned} \tag{10}$$

where $\ell'_{i,n-1} = \ell_{i,n-1} - \rho_i$, $i = 1, 2, \ldots, n-1$.

18.2.6. CGC's for the tensor product $T_{\mathbf{m}_n} \otimes T_{(p,\mathbf{0})}$. Let us consider the tensor product of the irreducible representations of the group $U(n)$ with the highest weights $\mathbf{m}_n = (m_{1n}, m_{2n}, \dots, m_{nn})$ and $(p, 0, \dots, 0) \equiv (p, \mathbf{0})$. It is possible to show with the help of characters of irreducible representations of $U(n)$ that

$$T_{\mathbf{m}_n} \otimes T_{(p,\mathbf{0})} = \sum_{\boldsymbol{\rho}} \oplus T_{\mathbf{m}_n + \boldsymbol{\rho}},$$

where summation is over those $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_n)$ for which ρ_j are non-negative integers such that

$$\rho_1 + \rho_2 + \dots + \rho_n = p,$$
$$m_{1n} + \rho_1 \geq m_{1n} \geq m_{2n} + \rho_2 \geq m_{2n} \geq \dots \geq m_{nn} + \rho_n \geq m_{nn}.$$

The CGC's of this tensor product can be factorized into the product of scalar factors of the form

$$\left(\begin{array}{cc|c} \mathbf{m}_k & (\tilde{p}, \mathbf{0}) & \mathbf{m}'_k \\ \mathbf{m}_{k-1} & (0, \mathbf{0}) & \mathbf{m}_{k-1} \end{array} \right), \tag{1}$$

$$\left(\begin{array}{cc|c} \mathbf{m}_k & (\tilde{p}, \mathbf{0}) & \mathbf{m}'_k \\ \mathbf{m}_{k-1} & (q, \mathbf{0}) & \mathbf{m}'_{k-1} \end{array} \right), \tag{2}$$

where

$$\sum_{i=1}^{k} m_{ik} + \tilde{p} = \sum_{i=1}^{k} m'_{ik}, \quad \sum_{i=1}^{k-1} m_{i,k-1} + q = \sum_{i=1}^{k-1} m'_{i,k-1}.$$

Using formulas (3), (4) and (6) of Section 18.1.2, we easily derive for CGC's of $U(n)$ the analogue of the recurrence relation (3) of Section 8.2.7:

$$\begin{aligned} &\sum_{s=1}^{n-1} a^s_{n-1}(M'^{-s}_{n-1}) \langle \mathbf{m}'_n, M'^{-s}_{n-1}; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M \rangle^r \\ &\quad + \sum_{s=1}^{n-1} a^s_{n-1}(M''^{-s}_{n-1}) \langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M''^{-s}_{n-1} \mid \mathbf{m}_n, M \rangle^r \\ &= \sum_{s=1}^{n-1} a^s_{n-1}(M) \langle \mathbf{m}'_n, M'; \mathbf{m}''_n, M'' \mid \mathbf{m}_n, M^{+s}_{n-1} \rangle^r. \end{aligned} \tag{3}$$

Setting here $\mathbf{m}''_n = (p, \mathbf{0})$, $\mathbf{m}''_{n-1} = (\mathbf{0}, \mathbf{0})$, $M' = M^{+s}_{n-1}$ we obtain the recurrence relation

$$a^s_{n-1}(M^{+s}_{n-1}) \left(\begin{array}{cc|c} \mathbf{m}_n & (p, \mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}^{+s}_{n-1} & (0, \mathbf{0}) & \mathbf{m}^{+s}_{n-1} \end{array} \right) = a^s_{n-1}(M) \left(\begin{array}{cc|c} \mathbf{m}_n & (p, \mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}_{n-1} & (0, \mathbf{0}) & \mathbf{m}_{n-1} \end{array} \right)$$

for the scalar factors (1), where $\mathbf{m}^{+s}_{n-1}$ is obtained from $\mathbf{m}_{n-1}$ if we replace $m_{s,n-1}$ by $m_{s,n-1} + 1$. It can be written as

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (p, \mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}^{+s}_{n-1} & (0, \mathbf{0}) & \mathbf{m}^{+s}_{n-1} \end{array} \right) = \left| \frac{\prod_{j=1}^{n} (\ell'_{jn} - \ell_{s,n-1})}{\prod_{j=1}^{n} (\ell_{jn} - \ell_{s,n-1})} \right|^{1/2} \left(\begin{array}{cc|c} \mathbf{m}_n & (p, \mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}_{n-1} & (0, \mathbf{0}) & \mathbf{m}_{n-1} \end{array} \right). \tag{4}$$

Let $(\mathbf{m}_{n-1})^k$ be the weight $\mathbf{m}_{n-1}$ in which $m_{k,n-1}$ is replaced by m_{kn}. It is easy to obtain from (4) that

$$\left(\begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \end{matrix} \quad \begin{matrix} (p,\mathbf{0}) \\ (0,\mathbf{0}) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ \mathbf{m}_{n-1} \end{matrix}\right) = F_k \left(\begin{matrix} \mathbf{m}_n \\ (\mathbf{m}_{n-1})^k \end{matrix} \quad \begin{matrix} (p,\mathbf{0}) \\ (0,\mathbf{0}) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ (\mathbf{m}_{n-1})^k \end{matrix}\right), \tag{5}$$

where

$$F_k = \left[\frac{\prod_{s=1}^{k} (\ell'_{sn} - \ell_{k,n-1})! \prod_{s=k+1}^{n} (\ell_{kn} - \ell'_{sn} - 1)! \prod_{s=1}^{k} (\ell_{sn} - \ell_{kn})!}{\prod_{s=1}^{k} (\ell'_{sn} - \ell_{kn})! \prod_{s=k+1}^{n} (\ell_{k,n-1} - \ell'_{sn} - 1)! \prod_{s=1}^{k} (\ell_{sn} - \ell_{k,n-1})!} \right.$$
$$\left. \times \frac{\prod_{s=k+1}^{n} (\ell_{k,n-1} - \ell_{sn} - 1)!}{\prod_{s=k+1}^{n} (\ell_{kn} - \ell_{sn} - 1)!} \right]^{1/2} \tag{6}$$

Applying relation (5) to other coordinates of the highest weight $\mathbf{m}_{n-1}$ we derive that

$$\left(\begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \end{matrix} \quad \begin{matrix} (p,\mathbf{0}) \\ (0,\mathbf{0}) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ \mathbf{m}_{n-1} \end{matrix}\right) = \left(\prod_{k=1}^{n-1} F_k\right) \left(\begin{matrix} \mathbf{m}_n \\ \mathbf{m}^0_{n-1} \end{matrix} \quad \begin{matrix} (p,\mathbf{0}) \\ (0,\mathbf{0}) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ \mathbf{m}^0_{n-1} \end{matrix}\right), \tag{7}$$

where $\mathbf{m}^0_{n-1} = (m_{1n}, \dots, m_{n-1,n})$ and F_k is given by formula (6). With the help of expressions for S_{pq} from Section 18.2.5 the coefficient from (7) can be represented as

$$\prod_{k=1}^{n-1} F_k = \frac{S_{n,n-1}(\mathbf{m}'_n, \mathbf{m}_{n-1}) S_{n,n-1}(\mathbf{m}_n, \mathbf{m}^0_{n-1})}{S_{n,n-1}(\mathbf{m}_n, \mathbf{m}_{n-1}) S_{n,n-1}(\mathbf{m}'_n, \mathbf{m}^0_{n-1})}. \tag{8}$$

In the following section we shall show that

$$\left(\begin{matrix} \mathbf{m}_n \\ \mathbf{m}_{n-1} \end{matrix} \quad \begin{matrix} (p,\mathbf{0}) \\ (0,\mathbf{0}) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ \mathbf{m}_{n-1} \end{matrix}\right) = \sqrt{p} \frac{S_{n,n-1}(\mathbf{m}'_n, \mathbf{m}_{n-1}) S_{nn}(\mathbf{m}'_n, \mathbf{m}'_n)}{S_{n,n-1}(\mathbf{m}_n, \mathbf{m}_{n-1}) S_{nn}(\mathbf{m}'_n, \mathbf{m}_n)}$$
$$= \prod_{i<j} (\ell'_{in} - \ell'_{jn}) \left[\frac{p \prod_{i \le j} (\ell'_{in} - \ell_{j,n-1})! \prod_{i<j} (\ell_{i,n-1} - \ell_{jn} - 1)! \prod_{i<j} (\ell_{in} - \ell'_{jn} - 1)!}{\prod_{i<j} (\ell_{i,n-1} - \ell'_{jn} - 1)! \prod_{i \le j} (\ell_{in} - \ell_{j,n-1})! \prod_{i \le j} (\ell'_{in} - \ell_{jn})!} \right]^{1/2}. \tag{9}$$

18.2.7. Evaluation of scalar factors. In order to evaluate scalar factors (1) and (2) of Section 18.2.6 we use the Wigner-Eckart theorem. Let T be the representation of $U(n)$ which is a direct sum of all pairwise non-equivalent irreducible representations $T_{\mathbf{m}_n}$ and let $\mathfrak{H}$ be the carrier space of this representation.

Let $\mathbf{R}^p \equiv \{R^p_P\}$ be the tensor operator transforming under the irreducible representation $T_{(p,\mathbf{0})}$ of the group $U(n)$:

$$T(g)R^p_P T(g^{-1}) = \sum_{P'} t^{(p,\mathbf{0})}_{P'P}(g) R^p_{P'}, \tag{1}$$

where P and P' are Gel'fand-Tsetlin patterns for the representation $T_{(p,\mathbf{0})}$. We assume that matrix elements of this tensor operator are

$$(R^p_P \mathbf{v}(M), \mathbf{v}(M')) = \langle \mathbf{m}_n, M' \mid \mathbf{m}_n, M; (p, \mathbf{0}), P\rangle, \tag{2}$$

that is, that all its reduced matrix elements are equal to 1 (see formula (5) of Section 18.2.4).

The components of the tensor operator $\mathbf{R}^p$ corresponding to the Gel'fand-Tsetlin patterns

$$\begin{pmatrix} p \quad 0 \ldots\ldots\ldots 0 \\ q \quad 0 \ldots\ldots 0 \\ 0 \ldots\ldots 0 \\ \cdots \\ 0 \end{pmatrix}, \quad p \geq q,$$

are denoted by R^p_q. Writing down definition (1) in the infinitesimal form (see formula (2) of Section 18.2.3) we have

$$[E_{n-1,n}, R^p_q] = [(p-q)(q+1)]^{1/2} R^p_{q+1}, \tag{3}$$

where $E_{n-1,n}$ is the corresponding operator in the representation T. We take q commutators of R^p_0 with the operator $E_{n-1,n}$. As a result, we receive

$$\begin{aligned} R^p_q &= \left(\frac{(p-q)!}{p!q!}\right)^{1/2} [E_{n-1,n}, [E_{n-1,n}, [\cdots, [E_{n-1,n}, R^p_0]\cdots]]] \\ &= \left(\frac{(p-q)!q!}{p!}\right)^{1/2} \sum_{t=0}^{q} \frac{(-1)^t}{t!(q-t)!} E^{q-t}_{n-1,n} R^p_0 E^t_{n-1,n}. \end{aligned} \tag{4}$$

Let us use this relation for derivation of expressions for scalar factors (1) and (2) of Section 18.2.6. We suppose that formula (9) of Section 18.2.6 is valid for the groups $U(n-1)$, $U(n-2)$, ... and prove that it is valid for the group $U(n)$. For the group $U(2)$ this formula is valid and coincides with the expression for the corresponding Clebsch-Gordan coefficients of the group $SU(2)$ from Chapter 8. Formulas (7)-(9) of Section 18.2.6 show that scalar factor (1) of the same section for the group $U(n)$ is equal to the right hand side of formula (9) of Section 18.2.6, multiplied by some expression (we denote it by $N(\mathbf{m}_n, \mathbf{m}'_n)$) which does not depend on $\mathbf{m}_{n-1}$. Let us evaluate this expression.

We consider matrix elements of both sides of relation (4) and divide them by expression (9) of Section 18.2.6 for the scalar factor for the subgroup $U(n-1)$. Due to relation (2) of Section 18.2.2 and formula (2) we obtain on the left hand side the scalar factor

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (p,\mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}_{n-1} & (q,\mathbf{0}) & \mathbf{m}'_{n-1} \end{array} \right). \tag{5}$$

In the last part of (4) we use the expressions of Section 18.2.5 for matrix elements of the operators $E^k_{n-1,n}$ and expression (9) of Section 18.2.6 for scalar factor (5) with $q=0$, multiplied by $N(\mathbf{m}_n,\mathbf{m}'_n)$. As a result, we have

$$\begin{aligned} \left(\begin{array}{cc|c} \mathbf{m}_n & (p,\mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}_{n-1} & (q,\mathbf{0}) & \mathbf{m}'_{n-1} \end{array} \right) &= \sqrt{p-q}\frac{S_{nn}(\mathbf{m}'_n,\mathbf{m}'_n)S_{n,n-1}(\mathbf{m}_n,\mathbf{m}_{n-1})}{S_{nn}(\mathbf{m}'_n,\mathbf{m}_n)S_{n,n-1}(\mathbf{m}'_n,\mathbf{m}'_{n-1})} \\ &\times S_{n-1,n-1}(\mathbf{m}'_{n-1},\mathbf{m}_{n-1})S_{n-1,n-1}(\mathbf{m}_{n-1},\mathbf{m}_{n-1}) \\ &\times \sum_{\rho_1,\dots,\rho_{n-1}} (-1)^{\rho_1+\dots+\rho_{n-1}} \left[\frac{S_{n,n-1}(\mathbf{m}'_n,\tilde{\mathbf{m}}_{n-1})}{S_{n,n-1}(\mathbf{m}_n,\tilde{\mathbf{m}}_{n-1})} \right. \\ &\times \left. \frac{S_{n-1,n-1}(\tilde{\mathbf{m}}_{n-1},\tilde{\mathbf{m}}_{n-1})}{S_{n-1,n-1}(\mathbf{m}'_{n-1},\tilde{\mathbf{m}}_{n-1})S_{n-1,n-1}(\tilde{\mathbf{m}}_{n-1},\mathbf{m}_{n-1})} \right]^2 N(\mathbf{m}_n,\mathbf{m}'_n), \end{aligned} \tag{6}$$

where $\tilde{\mathbf{m}}_{n-1} = \mathbf{m}_{n-1} + \boldsymbol{\rho}$ and summation is over all integral vectors $\boldsymbol{\rho} = (\rho_1,\dots,\rho_{n-1})$ such that $\rho_i \geq 0$, $\rho_1+\dots+\rho_{n-1} \leq q$ and $m_{i,n-1}+\rho_i \leq m'_{i,n-1}$, $i = 1,2,\dots,n-1$.

Now we show that $N(\mathbf{m}_n,\mathbf{m}'_n)=1$ in (6). We consider CGC

$$K^n_{\max} = \left\langle \begin{array}{cc|c} \mathbf{m}_n & (p,\mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}^0_{n-1} & (p,\mathbf{0}) & \mathbf{m}'^0_{n-1} \\ M_{\max} & P_{\max} & M'_{\max} \end{array} \right\rangle, \tag{7}$$

where $\mathbf{m}^0_{n-1} = (m_{1n},\dots,m_{n-1,n})$, $\mathbf{m}'^0_{n-1} = (m'_{1n},\dots,m'_{n-1,n})$, $M_{\max}$ is the lower part of the corresponding Gel'fand-Tsetlin pattern in which all entries take maximal admissible values, and the same sense $P_{\max}$ and $M'_{\max}$ have. Orthogonality relation (7) of Section 18.2.1 shows that $|K^n_{\max}|^2 = 1$. In the same way we obtain that $|K^{n-1}_{\max}| = 1$, where CGC $K^{n-1}_{\max}$ of the subgroup $U(n-1)$ is received from CGC (7) by removing the first row. CGC's are defined uniquely up to a common constant β such that $|\beta|=1$. Thus, we can suppose that

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (p,\mathbf{0}) & \mathbf{m}'_n \\ \mathbf{m}^0_{n-1} & (q,\mathbf{0}) & \mathbf{m}'^0_{n-1} \end{array} \right) = \frac{K^n_{\max}}{K^{n-1}_{\max}} = 1.$$

Equating to 1 expression (6) for this scalar factor, we find that $N(\mathbf{m}_n,\mathbf{m}'_n)=1$. Consequently, formula (9) of Section 18.2.6 is proved and scalar factor (5) is given by formula (6) with $N(\mathbf{m}_n,\mathbf{m}'_n)=1$.

Due to orthogonality relations (8) and (9) of Section 18.2.2, scalar factors (6) with fixed p and $\mathbf{m}_n$ are an orthogonal system of functions $Q(q, \mathbf{m}_{n-1} \mid \mathbf{m}'_{n-1})$ of discrete variables q and $\mathbf{m}_{n-1}$. The functions are labelled by highest weights $\mathbf{m}'_{n-1}$. The dual orthogonal system

$$R(\mathbf{m}'_{n-1} \mid q, \mathbf{m}_{n-1}) \equiv Q(q, \mathbf{m}_{n-1} \mid \mathbf{m}'_{n-1})$$

is also defined. As in the case of the group $SU(2)$ (see Chapter 8), these systems determine orthogonal polynomials of many variables which are a generalization of the Hahn polynomials.

18.2.8. CGC's of the tensor product $T_{\mathbf{m}} \otimes T_{(\mathbf{0},-p)}$. The irreducible unitary representation of the group $U(n)$ with highest weight $(0, \dots, 0, -p) = (\mathbf{0}, -p)$, $p \geq 0$, is contragradient to the irreducible representation with highest weight $(p, 0, \dots, 0)$. From the other side, unitary matrices of the irreducible representation $T_{\bar{\mathbf{m}}_n}$ (contragradient to the representation $T_{\mathbf{m}_n}$) is obtained from matrices of the representation $T_{\mathbf{m}_n}$ by complex conjugation. Therefore, CGC's of the tensor product $T_{\bar{\mathbf{m}}_n} \otimes T_{(p,\mathbf{0})}$ define CGC's for the tensor product $T_{\mathbf{m}_n} \otimes T_{(\mathbf{0},-p)}$. To obtain them we note that

$$T_{\mathbf{m}_n} \otimes T_{(\mathbf{0},-p)} = \sum_{\boldsymbol{\rho}} \oplus T_{\mathbf{m}_n - \boldsymbol{\rho}},$$

where $\boldsymbol{\rho}$ are such as in Section 18.2.6. CGC's of this tensor product factorize into products of scalar factors of the form

$$\left(\begin{matrix} \mathbf{m}_k & (\mathbf{0}, -\tilde{p}) \\ \mathbf{m}_{k-1} & (\mathbf{0}, 0) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_k \\ \mathbf{m}_{k-1} \end{matrix} \right), \quad \left(\begin{matrix} \mathbf{m}_k & (\mathbf{0}, -\tilde{p}) \\ \mathbf{m}_{k-1} & (\mathbf{0}, -q) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_k \\ \mathbf{m}'_{k-1} \end{matrix} \right), \tag{1}$$

where

$$\sum_{i=1}^{k} m_{ik} - \tilde{p} = \sum_{i=1}^{k} m'_{ik}, \quad \sum_{i=1}^{k-1} m_{i,k-1} - q = \sum_{i=1}^{k-1} m'_{i,k-1}. \tag{2}$$

We suppose that CGC's are real. Then

$$\left\langle \begin{matrix} \mathbf{m}_n & (\mathbf{0}, -p) \\ \mathbf{m}_{n-1} & (\mathbf{0}, -q) \\ \cdots & \cdots \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ \mathbf{m}'_{n-1} \\ \cdots \end{matrix} \right\rangle = \left\langle \begin{matrix} \bar{\mathbf{m}}_n & (p, \mathbf{0}) \\ \bar{\mathbf{m}}_{n-1} & (q, \mathbf{0}) \\ \cdots & \cdots \end{matrix} \middle| \begin{matrix} \bar{\mathbf{m}}'_n \\ \bar{\mathbf{m}}'_{n-1} \\ \cdots \end{matrix} \right\rangle, \tag{3}$$

where, remind, $\bar{\mathbf{m}}_r = (-m_{rr}, \dots, -m_{2r}, -m_{1r})$ if $\mathbf{m}_r = (m_{1r}, m_{2r}, \dots, m_{rr})$. It follows from here that for scalar factors we have

$$\left(\begin{matrix} \mathbf{m}_n & (\mathbf{0}, -p) \\ \mathbf{m}_{n-1} & (\mathbf{0}, -q) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ \mathbf{m}'_{n-1} \end{matrix} \right) = \left(\begin{matrix} \bar{\mathbf{m}}_n & (p, \mathbf{0}) \\ \bar{\mathbf{m}}_{n-1} & (q, \mathbf{0}) \end{matrix} \middle| \begin{matrix} \bar{\mathbf{m}}'_n \\ \bar{\mathbf{m}}'_{n-1} \end{matrix} \right). \tag{4}$$

In particular, by making use of formula (9) of Section 18.2.6 we obtain

$$\left(\begin{matrix} \mathbf{m}_n & (\mathbf{0}, -p) \\ \mathbf{m}_{n-1} & (\mathbf{0}, 0) \end{matrix} \middle| \begin{matrix} \mathbf{m}'_n \\ \mathbf{m}_{n-1} \end{matrix} \right) = \sqrt{p \frac{S_{n,n-1}(\bar{\mathbf{m}}'_n, \bar{\mathbf{m}}_{n-1}) S_{nn}(\bar{\mathbf{m}}'_n, \bar{\mathbf{m}}'_n)}{S_{n,n-1}(\bar{\mathbf{m}}_n, \bar{\mathbf{m}}_{n-1}) S_{nn}(\bar{\mathbf{m}}'_n, \bar{\mathbf{m}}_n)}}. \tag{5}$$

Since

$$\prod_{i<j}(\ell_{n-j+1,n}-\ell_{n-i+1,n})! = \prod_{s<k}(\ell_{sn}-\ell_{kn})!,$$

$$\prod_{i<j}(\ell_{n-j+1,n}-\ell_{n-i,n-1}-1)! = \prod_{s\le k}(\ell_{sn}-\ell_{k,n-1}-1)!,$$

$$\prod_{i\le j}(\ell_{n-j,n-1}-\ell'_{n-i+1,n})! = \prod_{s<k}(\ell_{s,n-1}-\ell'_{kn})!,$$

then (5) leads to the formula

$$\left(\begin{matrix}\mathbf{m}_n & (\mathbf{0},-p)\\ \mathbf{m}_{n-1} & (\mathbf{0},0)\end{matrix}\middle|\begin{matrix}\mathbf{m}'_n\\ \mathbf{m}_{n-1}\end{matrix}\right) = \prod_{s<k}(\ell'_{sn}-\ell'_{kn})$$
$$\times\left[\frac{p\prod\limits_{s<k}(\ell_{s,n-1}-\ell'_{kn})!\prod\limits_{s\le k}(\ell_{sn}-\ell_{k,n-1}-1)!\prod\limits_{s<k}(\ell'_{sn}-\ell_{kn}-1)!}{\prod\limits_{s\le k}(\ell'_{sn}-\ell_{k,n-1}-1)!\prod\limits_{s<k}(\ell_{s,n-1}-\ell_{kn})!\prod\limits_{s\le k}(\ell_{sn}-\ell'_{kn})!}\right]^{1/2}. \tag{6}$$

In the similar way expressions for scalar factors (4) with $q \neq 0$ are derived from formula (6) of Section 18.2.7.

18.2.9. CGC's of the tensor product of $T_{\mathbf{m}_n}$ with fundamental representations. The irreducible representations of the group $U(n)$ with highest weights $(1,\dots,1,0,\dots,0)$ are called *fundamental*. These highest weights will be denoted by $(1_k,0_{n-k})$ if k coordinates are equal to 1. Using characters of irreducible representations of the group $U(n)$ it is proved that

$$\begin{aligned} T_{\mathbf{m}_n}\otimes T_{(1_k,0_{n-k})} &\equiv T(\mathbf{m}_n)\otimes T(1_k,0_{n-k})\\ &= \sum_{0<i_1<i_2<\dots<i_k\le n}\oplus\, T(\mathbf{m}_n+\boldsymbol{\varepsilon}(i_1,i_2,\dots,i_k)), \end{aligned} \tag{1}$$

where summation is over integers $i_1,i_2,\dots,i_k$ and $\boldsymbol{\varepsilon}(i_1,i_2,\dots,i_k)$ is the vector whose i_1-th,..., i_k-th coordinates are equal to 1 and other $n-k$ coordinates are zero's. The summands of the right hand side, whose coordinates do not satisfy the condition $m'_{1n}\ge m'_{2n}\ge\dots\ge m'_{nn}$, have to be omitted.

CGC's of the tensor product (1) are factorized into product of scalar factors of the types

$$\left(\begin{matrix}\mathbf{m}_n & (1_k,0_{n-k})\\ \mathbf{m}_{n-1} & (1_k,0_{n-k-1})\end{matrix}\middle|\begin{matrix}\mathbf{m}_n+\boldsymbol{\varepsilon}(i_1,\dots,i_k)\\ \mathbf{m}_{n-1}+\boldsymbol{\varepsilon}(j_1,\dots,j_k)\end{matrix}\right), \tag{2}$$

$$\left(\begin{matrix}\mathbf{m}_n & (1_k,0_{n-k})\\ \mathbf{m}_{n-1} & (1_{k-1},0_{n-k})\end{matrix}\middle|\begin{matrix}\mathbf{m}_n+\boldsymbol{\varepsilon}(i_1,\dots,i_k)\\ \mathbf{m}_{n-1}+\boldsymbol{\varepsilon}(j_1,\dots,j_{k-1})\end{matrix}\right). \tag{3}$$

The scalar factor (2) is given by the expression

$$(-1)^{k(k-1)/2}\prod_{r,s=1}^{k} S(j_r,i_s)$$

$$\times\prod_{r=1}^{k}\left[\prod_{\substack{j=1\\ j\neq j_1,\dots,j_k}}^{n-1}\frac{\ell_{i_r n}-\ell_{j,n-1}+1}{\ell_{j_r,n-1}-\ell_{j,n-1}+1}\prod_{\substack{i=1\\ i\neq i_1,\dots,i_k}}^{n}\frac{\ell_{j_r,n-1}-\ell_{in}}{\ell_{i_r n}-\ell_{in}}\right]^{1/2} \tag{4}$$

and the scalar factor (3) by the expression

$$(-1)^{k(k-1)/2}\prod_{r=1}^{k-1}\prod_{s=1}^{k} S(j_r,i_s)$$

$$\times\prod_{r=1}^{k-1}\prod_{s=1}^{k}\left[\prod_{\substack{j=1\\ j\neq j_1,\dots,j_{k-1}}}^{n-1}\frac{\ell_{i_s n}-\ell_{j,n-1}+1}{\ell_{j_r,n-1}-\ell_{j,n-1}+1}\prod_{\substack{i=1\\ i\neq i_1,\dots,i_k}}^{n}\frac{\ell_{j_r,n-1}-\ell_{in}}{\ell_{i_s n}-\ell_{in}}\right]^{1/2}, \tag{5}$$

where $S(j,i)=1$ if $j\geq i$ and $S(j,i)=-1$ if $j<i$.

Proof of formulas (4) and (5) are analogous to that of expressions for the scalar factors of the tensor product $T_{\mathbf{m}_n}\otimes T_{(p,\mathbf{0})}$. In this reason we give only the sketch of the proof. We recommend to the reader to carry out necessary evaluations.

At the beginning the extreme scalar factors are evaluated. With the help of the orthogonality relations from Section 18.2.2 we find that for $i_k\neq n$ we have

$$|K_k^k|^2\equiv\left|\begin{pmatrix}\mathbf{m}_n & (1_k,0_{n-k}) \\ \mathbf{m}_{n-1}^0 & (1_k,0_{n-k-1})\end{pmatrix}\middle|\begin{matrix}\mathbf{m}_n+\varepsilon(i_1,\dots,i_k) \\ \mathbf{m}_{n-1}^0+\varepsilon(j_1,\dots,j_k)\end{matrix}\right)\right|^2=1. \tag{6}$$

If $i_k=n$, then

$$|K_{k-1}^k|^2\equiv\left|\begin{pmatrix}\mathbf{m}_n & (1_k,0_{n-k}) \\ \mathbf{m}_{n-1}^0 & (1_{k-1},0_{n-k})\end{pmatrix}\middle|\begin{matrix}\mathbf{m}_n+\varepsilon(i_1,\dots,i_k) \\ \mathbf{m}_{n-1}^0+\varepsilon(j_1,\dots,j_{k-1})\end{matrix}\right)\right|^2=1, \tag{7}$$

where $\mathbf{m}_{n-1}^0=(m_{1n},\dots,m_{n-1,n})$. Since CGC's are defined up to a common constant, then we can assume that K_k^k and K_{k-1}^k are equal to 1 or -1 according to values, given by formulas (4) and (5).

Now we form the tensor operator $\mathbf{R}^{(1_k,0_{n-k})}\equiv\{R_M\}$ transforming under the irreducible representation of the group $U(n)$ with highest weight $(1_k,0_{n-k})$. The symmetry of this tensor operator is realized by the same representation T of $U(n)$ as in Section 18.2.7. We suppose that all reduced matrix elements of our tensor operator are equal to 1, that is, its matrix elements coincide with the corresponding CGC's of the group $U(n)$. We have

$$[E_{n-1,n},R_{M'}]=t_{M''M'}^{(1_k,0_{n-k})}(E_{n-1,n})R_{M''}, \tag{8}$$

where M' is the Gel'fand-Tsetlin pattern

$$\begin{bmatrix} 1_k & 0_{n-k} \\ 1_{k-1} & 0_{n-k} \\ 1_{k-1} & 0_{n-k-1} \\ \dots & \dots \end{bmatrix}$$

and M'' is obtained from this pattern by replacement of the second row by $(1_k, 0_{n-k-1})$. Let us act by both sides of relation (8) onto the Gel'fand-Tsetlin basis element $\mathbf{v}(M) \equiv \mathbf{v}(\mathbf{m}_n, (\mathbf{m}_{n-1}^0)^{-1}, \dots)$ and equate coefficients with the same vectors. As a result, we receive the recurrence relations connecting products of scalar factors (2) and (3) (one for the group $U(n)$ and other for the subgroup $U(n-1)$).

Now the method of mathematical induction is applied. Namely, we assume that formulas (4) and (5) for the scalar factors are valid for all groups $U(m)$, $m < n$, and prove that they are true for the group $U(n)$. For $U(2)$ formulas (4) and (5) are valid. The recurrence relation for CGC's, analogous to relation (3) of Section 18.2.6, allows us to evaluate all scalar factors (2) for the group $U(n)$ with $i_k \neq n$, if we use the scalar factors K_k^k from formula (6) for the group $U(n)$ and the scalar factors (2) for the subgroup $U(n-1)$. With the help of recurrence relations, obtained from relation (8), we find scalar factors (3) of the group $U(n)$ with $i_k \neq n$. Then using initial scalar factor K_{k-1}^k from formula (7) instead of initial scalar factor K_k^k, in the same way we find scalar factors (2) and (3) with $i_k = n$.

Using the expressions for scalar factors (2) and (3) we can find the scalar factors for the tensor product $T_{\mathbf{m}_n} \otimes T_{(0_k,(-1)_{n-k})}$. We have

$$\left(\begin{matrix} \mathbf{m}_n & (0_k,(-1)_{n-k}) \\ \mathbf{m}_{n-1} & (0_k,(-1)_{n-k-1}) \end{matrix} \middle| \begin{matrix} \mathbf{m}_n - \varepsilon(i_1,\dots,i_{n-k}) \\ \mathbf{m}_{n-1} - \varepsilon(j_1,\dots,j_{n-k-1}) \end{matrix} \right)$$
$$= \left(\begin{matrix} \mathbf{m}_n & (1_k,0_{n-k}) \\ \mathbf{m}_{n-1} & (1_k,0_{n-k-1}) \end{matrix} \middle| \begin{matrix} \mathbf{m}_n + \tilde{\varepsilon}(i_1,\dots,i_{n-k}) \\ \mathbf{m}_{n-1} + \tilde{\varepsilon}(j_1,\dots,j_{n-k-1}) \end{matrix} \right), \tag{9}$$

$$\left(\begin{matrix} \mathbf{m}_n & (0_k,(-1)_{n-k}) \\ \mathbf{m}_{n-1} & (0_{k-1},(-1)_{n-k}) \end{matrix} \middle| \begin{matrix} \mathbf{m}_n - \varepsilon(i_1,\dots,i_{n-k}) \\ \mathbf{m}_{n-1} - \varepsilon(j_1,\dots,j_{n-k}) \end{matrix} \right)$$
$$= \left(\begin{matrix} \mathbf{m}_n & (1_k,0_{n-k}) \\ \mathbf{m}_{n-1} & (1_{k-1},0_{n-k}) \end{matrix} \middle| \begin{matrix} \mathbf{m}_n + \tilde{\varepsilon}(i_1,\dots,i_{n-k}) \\ \mathbf{m}_{n-1} + \tilde{\varepsilon}(j_1,\dots,j_{n-k}) \end{matrix} \right). \tag{10}$$

where $\varepsilon(i_1,\dots,i_{n-k})$ is the same as in formulas (2) and (3) and $\tilde{\varepsilon}(i_1,\dots,i_{n-k})$ is the vector whose i_1-th,..., i_{n-k}-th coordinates are equal to zero's and other coordinates are 1's.

18.2.10. CGC's of the tensor product $T_{\mathbf{m}_n} \otimes T_{(1,\mathbf{0})}$. According to formula (1) of Section 18.2.9 we have

$$T_{\mathbf{m}_n} \otimes T_{(1,\mathbf{0})} \equiv T(\mathbf{m}_n) \otimes T(1,\mathbf{0}) = \sum_{i=1}^{n} \oplus\, T(\mathbf{m}_n^{+i}), \tag{1}$$

$$T(\mathbf{m}_n) \otimes T(\mathbf{0}, -1) = \sum_{i=1}^{n} \oplus T(\mathbf{m}_n^{-i}), \tag{2}$$

where $\mathbf{m}_n^{+i}$ and $\mathbf{m}_n^{-i}$ are obtained from $\mathbf{m}_n$ by replacement of m_{in} by $m_{in}+1$ and $m_{in}-1$, respectively. On the right hand sides of (1) and (2) we have to omit the summands for which the conditions $m'_{1n} \geq m'_{2n} \geq \ldots \geq m'_{nn}$ are not fulfilled.

CGC's of tensor product (1) are factorized into products of scalar factors of the types

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (1,\mathbf{0}) & \mathbf{m}_n^{+i} \\ \mathbf{m}_{n-1} & (1,\mathbf{0}) & \mathbf{m}_n^{+j} \end{array}\right), \quad \left(\begin{array}{cc|c} \mathbf{m}_n & (1,\mathbf{0}) & \mathbf{m}_n^{+i} \\ \mathbf{m}_{n-1} & (0,\mathbf{0}) & \mathbf{m}_{n-1} \end{array}\right).$$

Using the results of Section 18.2.9 we can write

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (1,\mathbf{0}) & \mathbf{m}_n^{+i} \\ \mathbf{m}_{n-1} & (0,\mathbf{0}) & \mathbf{m}_{n-1} \end{array}\right) = \left| \frac{\prod_{j=1}^{n-1} (\ell_{j,n-1} - \ell_{in} - 1)}{\prod_{j \neq i} (\ell_{jn} - \ell_{in})} \right|^{1/2}, \tag{3}$$

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (1,\mathbf{0}) & \mathbf{m}_n^{+i} \\ \mathbf{m}_{n-1} & (1,\mathbf{0}) & \mathbf{m}_{n-1}^{+j} \end{array}\right) = S(i,j) \left| \frac{\prod_{k \neq j} (\ell_{k,n-1} - \ell_{in} - 1) \prod_{k \neq i} (\ell_{kn} - \ell_{j,n-1})}{\prod_{k \neq i} (\ell_{kn} - \ell_{in}) \prod_{k \neq j} (\ell_{k,n-1} - \ell_{j,n-1} - 1)} \right|^{1/2}, \tag{4}$$

where $\ell_{sk} = m_{sk} - s$, $S(i,j) = 1$ if $i \leq j$, and $S(i,j) = -1$ if $i > j$.

For the scalar factors of CGC's of tensor product (2) we have the expressions

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (\mathbf{0},-1) & \mathbf{m}_n^{-i} \\ \mathbf{m}_{n-1} & (\mathbf{0},0) & \mathbf{m}_{n-1} \end{array}\right) = \left| \frac{\prod_{j=1}^{n-1} (\ell_{j,n-1} - \ell_{in})}{\prod_{j \neq i} (\ell_{jn} - \ell_{in})} \right|^{1/2}, \tag{5}$$

$$\left(\begin{array}{cc|c} \mathbf{m}_n & (\mathbf{0},-1) & \mathbf{m}_n^{-i} \\ \mathbf{m}_{n-1} & (\mathbf{0},-1) & \mathbf{m}_{n-1}^{-j} \end{array}\right) = S(i,j) \left| \frac{\prod_{k \neq j} (\ell_{k,n-1} - \ell_{in}) \prod_{k \neq i} (\ell_{kn} - \ell_{j,n-1} + 1)}{\prod_{k \neq i} (\ell_{kn} - \ell_{in}) \prod_{k \neq j} (\ell_{k,n-1} - \ell_{j,n-1} + 1)} \right|^{1/2}. \tag{6}$$

If $\mathbf{m}_n = (m, 0, \ldots, 0, m') \equiv (m, \mathbf{0}, m')$, then we obtain from formulas (3) and (4) that

$$\begin{pmatrix} (m,\mathbf{0},m') & (1,\mathbf{0}) & (m+1,\mathbf{0},m') \\ (m_1,\mathbf{0},m_1') & (0,\mathbf{0}) & (m_1,\mathbf{0},m_1') \end{pmatrix} = \left(\frac{(m-m_1+1)(m-m_1'+n-1)}{(m+1)(m-m'+n-1)} \right)^{1/2}, \tag{7}$$

$$\begin{pmatrix} (m,\mathbf{0},m') & (1,\mathbf{0}) & (m,\mathbf{0},m'+1) \\ (m_1,\mathbf{0},m_1') & (0,\mathbf{0}) & (m_1,\mathbf{0},m_1') \end{pmatrix} = \left(\frac{(m_1-m'+n-2)(m_1'-m')}{(m-m'+n-1)(-m'+n-2)} \right)^{1/2}, \tag{8}$$

$$\begin{pmatrix} (m,\mathbf{0},m') & (1,\mathbf{0}) & (m+1,\mathbf{0},m') \\ (m_1,\mathbf{0},m_1') & (1,\mathbf{0}) & (m_1+1,\mathbf{0},m_1') \end{pmatrix} = \left(\frac{(m-m_1'+n-1)(m_1-m'+n-1)(m_1+1)}{(m+1)(m-m'+n-1)(m_1-m_1'+n-1)} \right)^{1/2}, \tag{9}$$

$$\begin{pmatrix} (m,\mathbf{0},m') & (1,\mathbf{0}) & (m+1,\mathbf{0},m') \\ (m_1,\mathbf{0},m_1') & (1,\mathbf{0}) & (m_1,\mathbf{0},m_1'+1) \end{pmatrix} = \left(\frac{(m-m_1+1)(m_1'-m'+1)(-m_1'+n-3)}{(m-m'+n-1)(m+1)(m_1-m_1'+n-3)} \right)^{1/2}, \tag{10}$$

$$\begin{pmatrix} (m,\mathbf{0},m') & (1,\mathbf{0}) & (m,\mathbf{0},m'+1) \\ (m_1,\mathbf{0},m_1') & (1,\mathbf{0}) & (m_1+1,\mathbf{0},m_1') \end{pmatrix} = -\left(\frac{(m_1'-m')(m_1+1)(m_1-m'+n-1)}{(m-m'+n-1)(-m'+n-2)(m_1-m_1'+n-1)} \right)^{1/2}, \tag{11}$$

$$\begin{pmatrix} (m,\mathbf{0},m') & (1,\mathbf{0}) & (m,\mathbf{0},m'+1) \\ (m_1,\mathbf{0},m_1') & (1,\mathbf{0}) & (m_1,\mathbf{0},m_1'+1) \end{pmatrix} = -\left(\frac{(m_1-m'+n-2)(m-m_1'+n-2)(-m_1+n-3)}{(m-m'+n-1)(-m'+n-2)(m_1-m_1'+n-3)} \right)^{1/2}. \tag{12}$$

Setting into formulas (9) and (10) of Section 18.2.1 $\mathbf{m}'' = (1, \mathbf{0})$, $\mathbf{m}' = (m, 0, m')$ and using CGC's with scalar factors (7)-(12) we receive recurrence relations for matrix elements of representations of the group $U(n)$, which are of class 1 with respect to the subgroup $U(n-1)$. In particular, we obtain recurrence relations for functions (7) and (9) of Section 11.6.8 and, consequently, relations for the hypergeometric functions ${}_5F_4$. Taking special cases (1) and (2) of Section 11.6.13 for matrix elements (7) and (9) of Section 11.6.8, recurrence relations for the hypergeometric functions ${}_3F_2$ are obtained.

Interesting relations for special functions can be obtained with the help of scalar factors of Section 18.2.8 if equality (10) of Section 18.2.1 with $\mathbf{m}'' = (p, 0, \ldots, 0)$ is used. For example, for $m' = (m, \mathbf{0}, m')$ we have

$$t^{n,m,-m'}_{(q,-q')0}(g_{n-1}(\theta))t^{n,p,0}_{(0,0)0}(g_{n-1}(\theta)) = \sum_{s,s'} \left(\begin{array}{cc|c} (m,\mathbf{0},m') & (p,\mathbf{0}) & (s,\mathbf{0},s') \\ (q,\mathbf{0},q') & (0,\mathbf{0}) & (q,\mathbf{0},q') \end{array} \right)$$
$$\times \left(\begin{array}{cc|c} (m,\mathbf{0},m') & (p,\mathbf{0}) & (s,\mathbf{0},s') \\ (0,\mathbf{0},0) & (0,\mathbf{0}) & (0,\mathbf{0},0) \end{array} \right) t^{n,s,-s'}_{(q,-q')0}(g_{n-1}(\theta)),$$

where $s+s' = m+m'+p$ and $t^{n,m,-m'}_{(q,-q')0}(g_{n-1}(\theta))$ are matrix elements (7) of Section 11.3.5 of representations of the group $U(n)$. Substituting here expressions for the matrix elements and for the scalar factors we obtain the relation

$$\begin{aligned} &\frac{(p+n-1)!(q'-m')}{p!(m-q)!} \\ &\times \sum_{k=0}^{p} \frac{(m-m'-p+k-1)!(m-m'+2k-p+n)(m-q+k)!}{k!(n-1)!(p-k)!(m-m'+k+n)!(q'-m'-p+k)!} \\ &\times P^{(q-q'+n-1,q+q'-m-m'-p)}_{m-q+k}(x) = P^{(n-1,-p)}_{p}(x)P^{(q-q'+n-1,q+q'-m-m')}_{m-q}(x), \end{aligned} \tag{13}$$

where $m \geq 0 \geq m'$, $q \geq 0 \geq q'$.

In the same way from the relation

$$t^{n,m0}_{(q,0)(q',0)(r,0)}(g_{n-1}(\theta))t^{n,p,0}_{(0,0)0}(g_{n-1}(\theta)) = \sum_{s,s'} \left(\begin{array}{cc|c} (m,\mathbf{0}) & (p,\mathbf{0}) & (s,\mathbf{0},s') \\ (q,\mathbf{0}) & (0,\mathbf{0}) & (q,\mathbf{0}) \end{array} \right)$$
$$\times \left(\begin{array}{cc|c} (m,\mathbf{0}) & (p,\mathbf{0}) & (s,\mathbf{0},s') \\ (q',\mathbf{0}) & (0,\mathbf{0}) & (q',\mathbf{0}) \end{array} \right) t^{n,s,-s'}_{(q,0)(q',0)(r,0)}(g_{n-1}(\theta))$$

we receive the equality

$$\begin{aligned} &\frac{(p+n-1)!(m-q')!(m+p+n)!}{p!(n-1)!(q-q')!(q+n-1)!} \sum_{k=0}^{p} \frac{(m+p-k+n-1)!(k+q+n-1)!}{k!(p-k)!(k+m+n)!} \\ &\qquad \times {}_3F_2\left(\left. \begin{array}{c} k+q+1,\ q-r,\ k+q-m-p \\ q+m,\ q-q'+1 \end{array} \right| x \right) \\ &\qquad = P^{(n-1,-p)}_{p}(2x-1)P^{(q-q',q+q'-r-m)}_{m-q}(2x-1). \end{aligned} \tag{14}$$

18.3. Matrix Elements of Representations of the Group $GL(n,\mathbb{C})$ and General Beta-Functions

18.3.1. Matrix elements of irreducible finite dimensional representations of $GL(n,\mathbb{C})$. The matrices E_{ij} form a basis of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$. The one-dimensional subgroups

$$\exp tE_{ii} = (\exp t)E_{ii} + (I_n - E_{ii}), \tag{1}$$

$$\exp tE_{ij} = I_n + tE_{ij}, \quad i \neq j, \tag{2}$$

of the group $GL(n,\mathbb{C})$ correspond to these matrices, where I_n is the unit matrix. If $T_{\mathbf{m}}$ is an irreducible finite dimensional representation of the group $GL(n,\mathbb{C})$ (and of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$), then

$$T_{\mathbf{m}}(\exp tE_{ij}) = \exp(tE_{ij}^{\mathbf{m}}), \tag{3}$$

where $E_{ij}^{\mathbf{m}} \equiv T_{\mathbf{m}}(E_{ij})$. Therefore, as it follows from formula (5) of Section 18.1.2, for the Gel'fand-Tsetlin basis elements $\mathbf{v}(M)$ we have

$$(T_{\mathbf{m}}(\exp tE_{ii})\mathbf{v}(M), \mathbf{v}(M)) = \exp rt, \tag{4}$$

where $r = m_{1i} + \ldots + m_{ii} - m_{1,i-1} - \ldots - m_{i-1,i-1}$. The non-diagonal matrix elements of the operator $T_{\mathbf{m}}(\exp +tE_{ii})$ vanish.

For the matrix $E_{n-1,n}$ we have

$$T_{\mathbf{m}}(\exp tE_{n-1,n}) = I + tE_{n-1,n}^{\mathbf{m}} + \frac{t^2}{2!}(E_{n,n-1}^{\mathbf{m}})^2 + \ldots .$$

It follows from formula (3) of Section 18.1.2 that $(E_{n,n-1}^{\mathbf{m}})^k$ transforms the basis element $\mathbf{v}(M)$ into a linear combination of basis elements $\mathbf{v}(M')$ for which

$$m'_{1,n-1} + \ldots + m'_{n-1,n-1} - m_{1,n-1} - \ldots - m_{n-1,n-1} = k. \tag{5}$$

Consequently, if the Gel'fand-Tsetlin patterns M and M' satisfy relation (5), then

$$\begin{aligned}(T_{\mathbf{m}}(\exp tE_{n-1,n})\mathbf{v}(M), \mathbf{v}(M')) &= \frac{t^k}{k!}((E_{n-1,n}^{\mathbf{m}})^k\mathbf{v}(M), \mathbf{v}(M')) \\ &= A_{\mathbf{m}'-\mathbf{m}}^k(\mathbf{m}_n, \mathbf{m}_{n-1}, \mathbf{m}_{n-2})\frac{t^k}{k!},\end{aligned} \tag{6}$$

where A_{ρ}^k is given by formula (2) of Section 18.2.5.

In the same way the matrix elements of the operator $T_{\mathbf{m}}(\exp tE_{n,n-1})$ are evaluated. They are given in terms of expression (10) of Section 18.2.6:

$$(T_{\mathbf{m}}(\exp tE_{n,n-1})\mathbf{v}(M), \mathbf{v}(M')) = \frac{t^k}{k!}((E_{n,n-1}^{\mathbf{m}})^k\mathbf{v}(M), \mathbf{v}(M')), \tag{7}$$

where $k = m_{1,n-1} + \ldots + m_{n-1,n-1} - m'_{1,n-1} - \ldots - m'_{n-1,n-1}$.

More simple expressions for operators $T_{\mathbf{m}}(\exp tE_{n,n-1})$ and $T_{\mathbf{m}}(\exp tE_{n,n-1})$ are obtained in the basis

$$\mathbf{e}(M) = \lambda(M)\mathbf{v}(M), \tag{8}$$

where $\lambda(M)$ are the numerical coefficients defined by the formula

$$\lambda(M) = \prod_{k=2}^{n} \left(\frac{\prod_{i>j}(\ell_{j,k-1} - \ell_{ik} - 1)! \prod_{i\le j}(\ell_{ik} - \ell_{j,k-1})!}{\prod_{i<j}(\ell_{i,k-1} - \ell_{j,k-1})} \right)^{1/2}. \tag{9}$$

In this basis we have

$$T_{\mathbf{m}}(\exp tE_{n,n-1})\mathbf{e}(M) = \sum_{M'} B^{(1)}_{n-1}(M', M) t^{k_{n-1} - k'_{n-1}} \mathbf{e}(M'), \tag{10}$$

$$T_{\mathbf{m}}(\exp tE_{n-1,n})\mathbf{e}(M) = \sum_{M'} B^{(2)}_{n-1}(M', M) t^{k'_{n-1} - k_{n-1}} \mathbf{e}(M'), \tag{11}$$

where $k_p = m_{1p} + m_{2p} + \ldots + m_{pp}$,

$$\begin{aligned} B^{(1)}_{n-1}(M', M) = &\prod_{i<j} \frac{(\ell_{i,n-1} - \ell_{jn} - 1)!}{(\ell'_{i,n-1} - \ell_{jn} - 1)!} \prod_{i\le j} \frac{(\ell_{i,n-1} - \ell_{j,n-2})!}{(\ell'_{i,n-1} - \ell_{j,n-2})!} \\ &\times \frac{\prod_{i<j}(\ell'_{i,n-1} - \ell_{j,n-1} - 1)!}{\prod_{i\le j}(\ell_{i,n-1} - \ell'_{j,n-1})!} \prod_{i<j}(\ell'_{i,n-1} - \ell'_{j,n-1}), \end{aligned} \tag{12}$$

$$\begin{aligned} B^{(2)}_{n-1}(M', M) = &\prod_{i\le j} \frac{(\ell_{in} - \ell_{j,n-1})!}{(\ell_{in} - \ell'_{j,n-1})!} \prod_{i<j} \frac{(\ell_{i,n-2} - \ell_{j,n-1} - 1)!}{(\ell_{i,n-2} - \ell'_{j,n-1} - 1)!} \\ &\times \frac{\prod_{i<j}(\ell_{i,n-1} - \ell'_{j,n-1} - 1)!}{\prod_{i\le j}(\ell'_{i,n-1} - \ell_{j,n-1})!} \prod_{i<j}(\ell'_{i,n-1} - \ell'_{j,n-1}). \end{aligned} \tag{13}$$

The operator $T_{\mathbf{m}}(\exp tE_{ii})$ is given in this basis by the same formula as in the basis $\{\mathbf{v}(M)\}$.

18.3.2. General beta-functions, related to the Gel'fand-Tsetlin basis. Formulas (10)-(13) of Section 18.3.1 define the operators $T_{n,n-1}(t) \equiv T_{\mathbf{m}}(\exp t \times E_{n,n-1})$ and $T_{n-1,n}(t) \equiv T_{\mathbf{m}}(\exp tE_{n-1,n})$. Let us consider more general operators. Let $\mathbf{z} \in \mathbb{C}^{n-1}$ and $\boldsymbol{\zeta} \in \mathbb{C}^{n-1}$. We denote by Z^- and Z^+ the subgroups of matrices

$$z = \begin{pmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{z} & 1 \end{pmatrix}, \quad \zeta = \begin{pmatrix} I_{n-1} & \boldsymbol{\zeta}^t \\ \mathbf{0} & 1 \end{pmatrix}, \tag{1}$$

respectively, and imbed $GL(n-1,\mathbb{C})$ into $GL(n,\mathbb{C})$:

$$GL(n-1,\mathbb{C}) \ni g_{n-1} \to \begin{pmatrix} g_{n-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \in GL(n,\mathbb{C}).$$

Almost every matrix $g_n \in GL(n,\mathbb{C})$ can be represented in the form

$$g_n = \lambda z g_{n-1}\zeta, \quad \lambda \neq 0, \quad g_{n-1} \in GL(n-1,\mathbb{C}), \quad z \in Z^-, \quad \zeta \in Z^+. \qquad (1')$$

Therefore,

$$T_{\mathbf{m}}(g_n) = T_{\mathbf{m}}(\lambda I)T_{\mathbf{m}}(z)T_{\mathbf{m}}(g_{n-1})T_{\mathbf{m}}(\zeta).$$

Let us introduce the notation

$$T_{\mathbf{m}}(z) = F^{(1)}(\mathbf{z}) \equiv F^{(1)}(z_1,\dots,z_{n-1}), \qquad (2)$$

$$T_{\mathbf{m}}(\zeta) = F^{(2)}(\boldsymbol{\zeta}) \equiv F^{(2)}(\zeta_1,\dots,\zeta_{n-1}). \qquad (3)$$

If the operator functions $F^{(1)}$ and $F^{(2)}$ are evaluated, then matrix elements of the representations of $GL(n,\mathbb{C})$ are reduced to those for the representations of the subgroup $GL(n-1,\mathbb{C})$.

We assume that the vector $\mathbf{z}$ from (2) is represented as a row and the vector $\boldsymbol{\zeta}$ from (3) as a column. Let us define action of matrices $g_{n-1} \in GL(n-1,\mathbb{C})$ onto $\mathbf{z}$ and $\boldsymbol{\zeta}$ by the formulas

$$\mathbf{z} = (z_1,\dots,z_{n-1}) \to \mathbf{z}g_{n-1} = (z'_1,\dots,z'_{n-1}),$$

$$\boldsymbol{\zeta} = (\zeta_1,\dots,\zeta_{n-1}) \to g_{n-1}^{-1}\boldsymbol{\zeta} = (\zeta'_1,\dots,\zeta'_{n-1}),$$

where

$$z'_i = \sum_{j=1}^{n-1} z_j (g_{n-1})_{ji}, \quad \zeta'_i = \sum_{j=1}^{n-1} (g_{n-1}^{-1})_{ij}\zeta_j.$$

A continuous functions $F^{(1)}(\mathbf{z})$ and $F^{(2)}(\boldsymbol{\zeta})$ taking values in a carrier space of a representation $T_{\mathbf{m}}$, are called *general beta-functions of rank* $n-1$ if the operators $F^{(1)}(\mathbf{0})$ and $F^{(2)}(\mathbf{0})$ are non-singular and for any $\mathbf{z}'$, $\mathbf{z}''$, $\boldsymbol{\zeta}'$, $\boldsymbol{\zeta}''$ and for $g_{n-1} \in GL(n-1,\mathbb{C})$ the relations

$$F^{(1)}(\mathbf{z}' + \mathbf{z}'') = F^{(1)}(\mathbf{z}')F^{(1)}(\mathbf{z}''), \qquad (4)$$

$$F^{(2)}(\boldsymbol{\zeta}' + \boldsymbol{\zeta}'') = F^{(2)}(\boldsymbol{\zeta}')F^{(2)}(\boldsymbol{\zeta}''), \qquad (4')$$

$$T_{\mathbf{m}}^{-1}(g_{n-1})F^{(1)}(\mathbf{z})T_{\mathbf{m}}(g_{n-1}) = F^{(1)}(\mathbf{z}g_{n-1}), \qquad (5)$$

$$T_{\mathbf{m}}^{-1}(g_{n-1})F^{(2)}(\boldsymbol{\zeta})T_{\mathbf{m}}(g_{n-1}) = F^{(2)}(g_{n-1}^{-1}\boldsymbol{\zeta}) \qquad (5')$$

are fulfilled. Properties (5) and (5′) show that the functions $F^{(1)}$ and $F^{(2)}$ are completely defined by their values on the vector $(0, \dots, 0, 1)$.

A pair of beta-functions $(F^{(1)}, F^{(2)})$ is called *conjugate* if in addition to the properties, listed above, the relation

$$F^{(2)}(\boldsymbol{\zeta})F^{(1)}(\mathbf{z}) = (1 + \mathbf{z} \cdot \boldsymbol{\zeta})^{-m_{1n} - \dots - m_{nn}} T_m((I + \zeta z) \\ \times F^{(1)}(\mathbf{z})F^{(2)}(\boldsymbol{\zeta})T_m((1 + \mathbf{z} \cdot \boldsymbol{\zeta})I) \tag{6}$$

is fulfilled, where I is the unit matrix, $\mathbf{z} \cdot \boldsymbol{\zeta} = z_1\zeta_1 + \dots + z_{n-1}\zeta_{n-1}$, and ζz is the product of matrices (1) corresponding to the vectors $\boldsymbol{\zeta}$ and $\mathbf{z}$.

Two pairs of conjugate general beta-functions $(F^{(1)}, F^{(2)})$ and $(\hat{F}^{(1)}, \hat{F}^{(2)})$ are called *equivalent* if they act in the same space $\mathfrak{H}_{\mathbf{m}}$ and are connected by the relations

$$\hat{F}^{(i)}(\mathbf{z}) = A^{-1}F^{(i)}(\mathbf{z})A, \quad i = 1, 2,$$

where A is a non-singular operator in $\mathfrak{H}_{\mathbf{m}}$. Simple reasonings show that to every irreducible finite dimensional representation $T_{\mathbf{m}}$ of the group $GL(n, \mathbb{C})$ there corresponds a conjugate pair of general beta-functions which is defined by $T_{\mathbf{m}}$ uniquely up to equivalence. Conversely, to every conjugate pair of general beta-functions corresponds an irreducible finite dimensional representation of the group $GL(n, \mathbb{C})$.

18.3.3. Matrix beta-functions. Let $F^{(1)}(\mathbf{z})$ and $F^{(2)}(\boldsymbol{\zeta})$ be a conjugate pair of general beta-functions. The matrices of the operators $F^{(1)}(\mathbf{z})$ and $F^{(2)}(\boldsymbol{\zeta})$ with respect to the basis $\{\mathbf{e}(M)\}$ are called *matrix beta-functions* and are denoted by $(B^{(1)}(M', M \mid \mathbf{z}))$ and $(B^{(2)}(M', M \mid \boldsymbol{\zeta}))$, respectively. Thus,

$$F^{(1)}(\mathbf{z})\mathbf{e}(M) = \sum_{M'} B^{(1)}(M', M \mid \mathbf{z})\mathbf{e}(M'), \tag{1}$$

$$F^{(2)}(\boldsymbol{\zeta})\mathbf{e}(M) = \sum_{M'} B^{(2)}(M', M \mid \boldsymbol{\zeta})\mathbf{e}(M'). \tag{2}$$

Statement 1. *The functions $B^{(1)}(M', M \mid \mathbf{z})$ and $B^{(2)}(M', M \mid \boldsymbol{\zeta})$ are monomials of the coordinates z_i and ζ_j of the vectors $\mathbf{z}$ and $\boldsymbol{\zeta}$, namely,*

$$B^{(1)}(M', M \mid \mathbf{z}) = B^{(1)}(M', M) \prod_{i=1}^{n-1} z_i^{r_i - r_i'}, \tag{3}$$

$$B^{(2)}(M', M \mid \boldsymbol{\zeta}) = B^{(2)}(M', M) \prod_{i=1}^{n-1} \zeta_i^{r_i' - r_i}, \tag{4}$$

where $r_i = k_i - k_{i-1}$, $k_i = m_{1i} + \dots + m_{ii}$, $1 \le i \le n$, $k_0 = 0$. Moreover,

$$B^{(1)}(M', M) = B^{(1)}(M', M \mid \mathbf{z}_1), \tag{5}$$

$$B^{(2)}(M',M) = B^{(2)}(M',M \mid \zeta_1), \tag{6}$$

where $\mathbf{z}_1 = \zeta_1 = (1,1,\ldots,1)$.

Proof. Let numbers $z_1, z_2, \ldots, z_{n-1}$ be non-zero and let $\delta = \operatorname{diag}(z_1,\ldots,z_{n-1}) \in GL(n-1,\mathbb{C})$. Due to formula (5) of Section 18.3.2 we have

$$F^{(1)}(z_1,\ldots,z_{n-1}) = T_{\mathbf{m}}^{-1}(\delta)F^{(1)}(\mathbf{z}_1)T_{\mathbf{m}}(\delta) \tag{7}$$

and

$$T_{\mathbf{m}}(\delta)\mathbf{e}(M) = \left(\prod_{i=1}^{n} z_i^{r_i}\right)\mathbf{e}(M).$$

Taking matrix elements for both sides of relation (7) we obtain (3). If some of the coordinates are zero's, then we have to fulfil analytic continuation. Formula (4) is proved in the same way. Statement is proved.

Below the matrices consisting of coefficients $B^{(1)}(M',M)$ and $B^{(2)}(M',M)$ from (3) and (4) will be also called matrix beta-functions. It follows from (3) that $B^{(1)}(M',M) = 0$ if $r_i - r'_i < 0$ for some i. Analogously, $B^{(2)}(M',M) = 0$ if $r'_i - r_i < 0$ for some i.

It is easy to show that the matrices $(B^{(1)}(M',M))$ and $(B^{(2)}(M',M))$ are of the triangular form, that is, $B^{(1)}(M',M)$ (correspondingly, $B^{(2)}(M',M)$) may be non-vanishing only if $m_{ij} \geq m'_{ij}$ (only if $m_{ij} \leq m'_{ij}$) for all i and j.

Formulas (4)-(6) of Section 18.3.2 can be written as conditions for matrix beta-functions. Namely, it follows from formulas (4) and (4′) of Section 18.3.2 that

$$\begin{aligned}\sum_M B^{(1)}(M',M)B^{(1)}(M,M'') \prod_{i=1}^{n-1} \alpha_i^{r_i-r'_i}\beta_i^{r''_i-r_i} \\ = B^{(1)}(M',M'') \prod_{i=1}^{n-1}(\alpha_i+\beta_i)^{r''_i-r'_i},\end{aligned} \tag{8}$$

$$\begin{aligned}\sum_M B^{(2)}(M',M)B^{(2)}(M,M'') \prod_{i=1}^{n-1} \alpha_i^{r'_i-r_i}\beta_i^{r_i-r''_i} \\ = B^{(2)}(M',M'') \prod_{i=1}^{n-1}(\alpha_i+\beta_i)^{r'_i-r''_i},\end{aligned} \tag{9}$$

where, as before, $r_i = k_i - k_{i-1}$, $k_i = m_{1i} + \ldots + m_{ii}$, $1 \leq i \leq n$, $k_0 = 0$ and α_i, β_i are arbitrary numbers.

Equating the coefficients at the same powers of α_i and of the β_i in equalities (8) and (9), we receive the relations

$$\sum_M B^{(1)}(M',M)B^{(1)}(M,M'') = B^{(1)}(M',M'') \prod_{i=1}^{n-1} \frac{(r''_i - r'_i)!}{(r''_i - r_i)!(r_i - r'_i)!}, \tag{10}$$

$$\sum_M B^{(2)}(M',M)B^{(2)}(M,M'') = B^{(2)}(M',M'')\prod_{i=1}^{n-1}\frac{(r_i'-r_i'')!}{(r_i'-r_i)!(r_i-r_i'')!}, \tag{11}$$

where summations are over all patterns M such that the sums $m_{1i}+\ldots+m_{ii}=k_i$ are fixed numbers.

From formulas (5) and (5') of Section 18.3.2 we obtain

$$\begin{aligned}\sum_{M',M''}(M^{(1)},M' \mid g_{n-1}^{-1})B^{(1)}(M',M'')(M'',M^{(2)} \mid g_{n-1})\prod_{i=1}^{n-1}\alpha_i^{r_i''-r_i'} \\ = B^{(1)}(M^{(1)},M^{(2)})\prod_{i=1}^{n-1}(\alpha_1 g_{1i}+\alpha_2 g_{2i}+\ldots+\alpha_{n-1}g_{n-1,i})^{r_i^{(2)}-r_i^{(1)}},\end{aligned} \tag{12}$$

$$\begin{aligned}\sum_{M',M''}(M^{(1)},M' \mid g_{n-1}^{-1})B^{(2)}(M',M'')(M'',M^{(2)} \mid g_{n-1})\prod_{i=1}^{n-1}\alpha_i^{r_i'-r_i''} \\ = B^{(2)}(M^{(1)},M^{(2)})\prod_{i=1}^{n-1}((g_{n-1}^{-1})_{i1}\alpha_1+\ldots+(g_{n-1}^{-1})_{i,n-1}\alpha_{n-1})^{r_i^{(1)}-r_i^{(2)}},\end{aligned} \tag{13}$$

where $g\equiv g_{n-1}\in GL(n-1,\mathbb{C})$ and $(M,M' \mid g_{n-1})$ are the matrix elements of the operator $T_m(g_{n-1})$.

The equality (6) of Section 18.3.2 leads to the relation

$$\begin{aligned}\sum_M B^{(2)}(M^{(1)},M)B^{(1)}(M,M^{(2)})\prod_{i=1}^{n-1}\alpha_i^{r_i^{(1)}-r_i}\beta_i^{r_i^{(2)}-r_i} \\ = \sum_{M,M'}(M^{(1)},M \mid g_{n-1})B^{(1)}(M,M')B^{(2)}(M',M^{(2)}) \\ \times(1+\alpha_1\beta_1+\ldots+\alpha_{n-1}\beta_{n-1})^{-r_n}\prod_{i=1}^{n-1}\alpha_i^{r_i'-r_i^{(2)}}\beta_i^{r_i'-r_i},\end{aligned} \tag{14}$$

where $g_{n-1}=I_{n-1}+\alpha\beta$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_{n-1})^t$, $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_{n-1})$.

Let us note that $(B_{n-1}^{(1)}(M',M))$ and $(B_{n-1}^{(2)}(M',M))$ from formulas (12) and (13) of Section 18.3.1 coincide with $F^{(1)}(0,\ldots,0,1)$ and $F^{(2)}(0,\ldots,0,1)$, respectively.

18.3.4. Recurrence formulas for general beta-functions. We shall obtain expressions for general beta-functions of rank $n-1$ in terms of general beta-functions of rank $n-2$. Applying the mathematical induction in a rank, it is seen that every conjugate pair of beta-functions $F_k^{(1)}$, $F_k^{(2)}$ of rank k $(1\le k\le n-1)$ can be considered as operators in a carrier space $\mathfrak{H}_{\mathbf{m}}$ of a representation $T_{\mathbf{m}}$ of the group $U(n)$.

According to the definition we have $F^{(1)}_{n-2}(\mathbf{z}_1) = T_{\mathbf{m}}(z_{n-1})$, where $\mathbf{z}_1 = (1,\dots,1)$ and z_{n-1} is the matrix with 1's on the main diagonal and on below row and with 0's on all other places. This matrix transforms the vector $\mathbf{z}_0 = (0,\dots,0,1) \in \mathbf{R}^{n-1}$ into the vector $\mathbf{z}_1$. Therefore, due to relation (5) of Section 18.3.2 we have the equality

$$F^{(1)}_{n-1}(\mathbf{z}_1) = (F^{(1)}_{n-2}(\mathbf{z}_1))^{-1} F^{(1)}_{n-1}(\mathbf{z}_0) F^{(1)}_{n-2}(\mathbf{z}_1).$$

We have derived the explicit expression for the operator $F^{(1)}_{n-1}(\mathbf{z}_0)$. Thus, we expressed the general beta-function of rank $n-1$ in terms of general beta-functions of rank $n-2$.

We derive analogously that

$$F^{(2)}_{n-1}(\boldsymbol{\zeta}_1) = F^{(2)}_{n-2}(\boldsymbol{\zeta}_1) F^{(2)}_{n-1}(\boldsymbol{\zeta}_0)(F^{(2)}_{n-2}(\boldsymbol{\zeta}_1))^{-1},$$

where $\boldsymbol{\zeta}_1 = (1,\dots,1)$ $\boldsymbol{\zeta}_0 = (0,\dots,0,1)$. In order to represent these relations in a matrix form, we note that for the matrix elements $B^{(1)}_{n-2}(M',M)$ and $B^{(2)}_{n-2}(M',M)$ of the operators $F^{(1)}_{n-2}(\mathbf{z}_1)$ and $F^{(2)}_{n-2}(\mathbf{z}_1)$ we have

$$B^{(1)}_{n-2}(M',M) = B^{(1)}_{n-2}(\tilde{M}',\tilde{M}),$$

$$B^{(2)}_{n-2}(M',M) = \tilde{B}^{(2)}_{n-2}(\tilde{M}',\tilde{M}),$$

where $\tilde{M}'$ and $\tilde{M}$ are the Gel'fand-Tsetlin patterns, obtained respectively from M' and M by removing the first row.

Let us note also that

$$(F^{(1)}_{n-2}(\mathbf{z}_1))^{-1} = T^{-1}(\delta) F^{(1)}_{n-2}(\mathbf{z}_1) T(\delta), \quad \mathbf{z}_1 = (1,\dots,1),$$

where $\delta = \mathrm{diag}(1,\dots,1,-1) \in GL(n-1,\mathbb{C})$. Due to formula (4) of Section 18.3.1 we receive from here that the matrix elements of $(F^{(1)}_{n-2}(\mathbf{z}_1))^{-1}$ are of the form

$$(-1)^{k'_{n-2}-k_{n-2}} B^{(1)}_{n-2}(M',M).$$

Thus, we proved the recurrence formula

$$\begin{aligned} B^{(1)}_{n-1}(M^{(1)},M^{(2)}) = \sum_{M,M'} \prod_{i<j} (\ell'_{i,n-1} - \ell'_{j,n-1}) \prod_{i<j} \frac{(\ell_{i,n-1} - \ell_{jn} - 1)!}{(\ell'_{i,n-1} - \ell_{jn} - 1)!} \\ \times \prod_{i\le j} \frac{(\ell_{i,n-1} - \ell_{j,n-2})!}{(\ell'_{i,n-1} - \ell_{j,n-2})!} \frac{\prod_{i<j} (\ell'_{i,n-1} - \ell_{j,n-1} - 1)!}{\prod_{i\le j} (\ell_{i,n-1} - \ell'_{j,n-1})!} \\ \times B^{(1)}_{n-2}(M^{(1)},M') B^{(1)}_{n-2}(M,M^{(2)}). \end{aligned} \tag{1}$$

In the analogous way the formula

$$B_{n-1}^{(2)}(M^{(1)},M^{(2)}) = \sum_{M,M'} \prod_{i<j}(\ell'_{i,n-1} - \ell'_{j,n-1}) \prod_{i\le j} \frac{(\ell_{in} - \ell_{j,n-1})!}{(\ell_{in} - \ell'_{j,n-1})!}$$
$$\times \prod_{i<j} \frac{(\ell_{i,n-2} - \ell_{j,n-1} - 1)!}{(\ell_{i,n-2} - \ell'_{j,n-1} - 1)!} \frac{\prod_{i<j}(\ell_{i,n-1} - \ell'_{j,n-1} - 1)!}{\prod_{i\le j}(\ell'_{i,n-1} - \ell_{j,n-1})!} \tag{2}$$
$$\times B_{n-2}^{(2)}(M^{(1)},M')B_{n-2}^{(2)}(M,M^{(2)})$$

is proved. It is easy to show that the summations in (1) and (2) are over those M and M' for which

$$m_{i,n-1} = m_{i,n-1}^{(2)}, \quad m'_{i,n-1} = m_{i,n-1}^{(1)}, \quad 1 \le i \le n-1,$$

$$m_{ij} = m'_{ij}, \quad j < n-1.$$

18.4. Representations of $U(n)$ in the Gel'fand-Tsetlin Bases and Special Functions

18.4.1. Matrix elements of the representations of the group $U(n)$. Let us evaluate the matrix elements $t^{\mathbf{m}}_{M'M}(g) = (T_{\mathbf{m}}(g)\mathbf{v}(M), \mathbf{v}(M'))$ of the representations $T_{\mathbf{m}}$ of the group $U(n)$ in the Gel'fand-Tsetlin basis $\{\mathbf{v}(M)\}$ for the matrices $g_{n-1}(\theta)$ (see Section 11.1.2). The tangent matrix to the one-parameter subgroup $\{g_{n-1}(\theta)\}$ is of the form $E_{n,n-1} - E_{n-1,n}$. Therefore,

$$T_{\mathbf{m}}(g_{n-1}(\theta)) = \exp\theta(E^{\mathbf{m}}_{n,n-1} - E^{\mathbf{m}}_{n-1,n}).$$

The explicit expressions for the operators $T_{\mathbf{m}}(E_{n,n-1}) \equiv E^{\mathbf{m}}_{n,n-1}$ and $T_{\mathbf{m}}(E_{n-1,n}) \equiv E^{\mathbf{m}}_{n-1,n}$ (see Section 18.1.2) show that $t^{\mathbf{m}}_{M'M}(g_{n-1}(\theta)) = 0$ if $M = (\mathbf{m} \equiv \mathbf{m}_n, \mathbf{m}_{n-1}, \mathbf{m}_{n-2}, \dots)$ and $M' = (\mathbf{m}_n, \mathbf{m}'_{n-1}, \mathbf{m}'_{n-2}, \dots)$ are such that for some i ($i = 2, 3, \dots, n-1$) we have $\mathbf{m}_{n-i} \ne \mathbf{m}'_{n-i}$. If $\mathbf{m}_{n-i} = \mathbf{m}'_{n-i}$, $i = 2, \dots, n-1$, then $t^{\mathbf{m}}_{M'M}(g_{n-1}(\theta))$ does not depend on $\mathbf{m}_{n-3}, \mathbf{m}_{n-4}, \dots$. In this reason these matrix elements will be denoted as

$$t^{\mathbf{m}}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(g_{n-1}(\theta)) \equiv t^{\mathbf{m}}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\theta) \tag{1}$$

(see Section 11.3.1).

They are evaluated with the help of the formulas for the operators $T_{\mathbf{m}}(\exp t \times E_{n,n-1})$ and $T_{\mathbf{m}}(\exp tE_{n-1,n})$ from Section 18.3.1 in exactly the same way as in the case of the representations $T^{n\ell\ell'}$ (see Section 11.6.7). Using decomposition (2) of Section 11.6.7 for the matrices $g_{n-1}(\theta)$ we derive that

$$t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) = (-1)^p \mu(\boldsymbol{\ell},\mathbf{m},\mathbf{j})\mu(\boldsymbol{\ell},\mathbf{p},\mathbf{j})(\cos\theta)^{-\ell-j}(\tan\theta)^{-m-p}\sum_{\mathbf{r}}(-1)^r$$
$$\times \frac{\prod_{i<s}(r_i-\ell_s-i+s-1)!\prod_{i\le s}(r_i-j_s+s-i)!\prod_{i<s}(m_i-r_s-i+s-1)!}{\prod_{i\le s}(\ell_i-r_s-i+s)!\prod_{i<s}(j_i-r_s-i+s-1)!\prod_{i\le s}(r_i-m_s-i+s)!}$$
$$\times \frac{\prod_{i<s}(p_i-r_s-i+s-1)!}{\prod_{i\le s}(r_i-p_s-i+s)!}\left(\prod_{i<s}(r_i-r_s-i+s)\right)\sin^{2r}\theta, \qquad (2)$$

where

$$\mu(\boldsymbol{\ell},\mathbf{m},\mathbf{j}) = \left(\prod_{i<s}(m_i-m_s-i+s)\frac{\prod_{i<s}(j_i-m_s-i+s-1)!\prod_{i\le s}(\ell_i-m_s-i+s)!}{\prod_{i<s}(m_i-\ell_s-i+s-1)!\prod_{i\le s}(m_i-j_s-i+s)!}\right)^{1/2} \qquad (3)$$

and summation is over those $\mathbf{r}=(r_1,\dots,r_{n-1})$ for which $\min(\ell_i,j_{i-1})\ge r_i\ge \max(m_i,p_i)$, $i=1,2,\dots,n-1$ (for brevity we introduced the renotations $\mathbf{m}_n\equiv\mathbf{m}=\boldsymbol{\ell}$, $\mathbf{m}'_{n-1}=\mathbf{m}$, $\mathbf{m}_{n-1}=\mathbf{p}$, $\mathbf{m}_{n-2}=\mathbf{j}$). Using decomposition (1) of Section 11.6.7 for $g_{n-1}(\theta)$ we find that

$$t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) = (-1)^m \nu(\boldsymbol{\ell},\mathbf{m},\mathbf{j})\nu(\boldsymbol{\ell},\mathbf{p},\mathbf{j})(\cos\theta)^{\ell+j}(\tan\theta)^{m+p}\sum_{\mathbf{r}}(-1)^r$$
$$\times \frac{\prod_{i\le s}(\ell_i-r_s-i+s)!\prod_{i<s}(j_i-r_s-i+s-1)!\prod_{i<s}(r_i-m_s-i+s-1)!}{\prod_{i<s}(r_i-\ell_s-i+s-1)!\prod_{i\le s}(r_i-j_s-i+s)!\prod_{i\le s}(m_i-r_s-i+s)!}$$
$$\times \frac{\prod_{i<s}(r_i-p_s-i+s-1)!}{\prod_{i\le s}(p_i-r_s-i+s)!}\left(\prod_{i<s}(r_i-r_s-i+s)\right)\sin^{-2r}\theta, \qquad (4)$$

where

$$\nu(\boldsymbol{\ell},\mathbf{m},\mathbf{j}) = \left(\prod_{i<s}(m_i-m_s-i+s)\frac{\prod_{i<s}(m_i-\ell_s-i+s-1)!\prod_{i\le s}(m_i-j_s-i+s)!}{\prod_{i<s}(j_i-m_s-i+s-1)!\prod_{i\le s}(\ell_i-m_s-i+s)!}\right)^{1/2}, \qquad (5)$$

summation is over those $\mathbf{r}=(r_1,\dots,r_{n-1})$ for which $\max(\ell_{i+1},j_i)\le r_i\le\min(m_i, p_i)$, $i=1,2,\dots,n-1$. In (2) and (4) $\ell=\ell_1+\dots+\ell_n$ and in the same way m, p, j, r are defined.

Expressions (2) and (4) for $t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta)$ can be represented in terms of the hypergeometric function ${}_{q+1}F_q$. Namely, using a hypergeometric series for the sum over r_1 in formula (2) we obtain for $m_1 \geq p_1$ the expression

$$t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) = (-1)^m \mu(\boldsymbol{\ell}, \mathbf{m}, \mathbf{j})\mu(\boldsymbol{\ell}, \mathbf{p}, \mathbf{j})(\cos\theta)^{-\ell-j}(\tan\theta)^{-m-p}$$
$$\times \sum_{r_2, r_3, \dots, r_{n-1}} (-\sin^2\theta)^{r_2+\dots+r_{n-1}}$$
$$\times \frac{\prod\limits_{1<i\leq s}(r_i - j_s - i + s)! \prod\limits_{1<i<s}(r_i - \ell_s - i + s - 1)! \prod\limits_{1<i<s}(r_i - r_s - i + s)}{\prod\limits_{1<i\leq s}(r_i - m_s - i + s)! \prod\limits_{i\leq s}(\ell_i - r_s - i + j)! \prod\limits_{1<i\leq s}(r_i - p_s - i + s)!}$$
$$\times \prod_{i<s} \frac{(m_i - r_s - i + s - 1)!(p_i - r_s - i + s - 1)!}{(j_i - r_s - i + s - 1)!} \cdot M^{\boldsymbol{\ell}}_{\mathbf{mp}}(\theta) \tag{6}$$
$$\times {}_{3n-4}F_{4n-5}\left(\begin{matrix} \{m_1 - \ell_i + i - 1\}_1^n, \{m_1 - j_i + i\}_1^{n-2}, \{m_1 - r_i + i\}_2^{n-1} \\ \{m_1 - r_i + i - 1\}_2^{n-1}, \{m_1 - m_i + i\}_2^{n-1}, \{m_1 - p_i + i\}_1^{n-1} \end{matrix} \middle| x \right),$$

where $x = \sin^2\theta$,

$$M^{\boldsymbol{\ell}}_{\mathbf{mp}}(\theta) = \frac{(m_1 - \ell_n + n - 2)!(\sin\theta)^{2m_1}}{(\ell_1 - m_1)!(m_1 - p_1)!}$$
$$\prod_{i=2}^{n-1} \frac{(m_1 - \ell_i + i - 2)!(m_1 - r_i + i - 1)!}{(m_1 - m_i + i - 1)!(m_1 - p_i + i - 1)!} \prod_{i=1}^{n-2}(m_1 - j_i + i - 1)!,$$

$\{a_i\}_s^t$, $s \leq t$, denotes the set $a_s, a_{s+1}, \dots, a_t$ and summation is over those $r_2, r_3, \dots, r_{n-1}$ for which $\min(\ell_i, j_{i-1}) \geq r_i \geq \max(m_i, p_i)$, $i = 2, 3, \dots, n-1$.

If $m_1 < p_1$, then in $M^{\boldsymbol{\ell}}_{\mathbf{mp}}(\theta)_{3n-4}F_{3n-5}(\dots)$ we have to permute m_1 and p_1. Expression (4) can be analogously represented in terms of ${}_{3n-4}F_{3n-5}$.

18.4.2. The symmetry relations. The symmetry relations for the matrix elements (1) of Section 18.4.1 follow from expressions (2) and (4) of the same section and from symmetry relations for matrix elements of the infinitesimal operators $E^{\mathbf{m}}_{n-1,n}$ and $E^{\mathbf{m}}_{n,n-1}$ if we use the formula $T_{\mathbf{m}}(g_{n-1}(\theta)) = \exp\theta(E^{\mathbf{m}}_{n,n-1} - E^{\mathbf{m}}_{n-1,n})$. We give these symmetry relations without proofs (we recommend to the reader to verify these relations).

1. Let us represent the highest weight $\mathbf{m}$ in the form $\mathbf{m} = (m_{1n}, \mathbf{m}'_{n-2}, m_{nn})$. Then we have

$$t^{(m_{1n}, \mathbf{m}'_{n-2}, m_{nn})}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\theta) = t^{(m_{1n}, \mathbf{m}_{n-2}, m_{nn})}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}'_{n-2}}(\theta), \tag{1}$$

that is, the matrix element is symmetric with respect to permutation of $\mathbf{m}'_{n-2}$ and $\mathbf{m}_{n-2}$.

2. Let $\mathbf{m}_n = (m_{1n}, \ldots, m_{in}, \mathbf{0})$, $\mathbf{m}'_{n-1} = (m'_{1,n-1}, \ldots, m'_{i,n-1}, \mathbf{0})$, $\mathbf{m}_{n-1} = (m_{1,n-1}, \ldots, m_{i,n-1}, \mathbf{0})$, $\mathbf{m}_{n-2} = (m_{1,n-2}, \ldots, {}_{i,n-2}, \mathbf{0})$, where $\mathbf{0}$ denotes other coordinates of $\mathbf{m}_j$ which are equal to zero. Then the matrix elements

$$t^{\mathbf{m}_{i+k}}_{\mathbf{m}'_{i+k-1}\mathbf{m}_{i+k-1}\mathbf{m}_{i+k-2}}(g_{i+k-1}(\theta)), \quad n = i+k, \quad k = 2, 3, \ldots, \tag{2}$$

do not depend on k. This property takes place for matrix elements of the infinitesimal operators $E^{\mathbf{m}}_{n-1,n}$ and $E^{\mathbf{m}}_{n,n-1}$.

3. Let

$$\mathbf{m}_n = (m_{1n}, \ldots, m_{in}, \mathbf{0}, \tilde{m}_{in}, \ldots, \tilde{m}_{1n}), \tag{3}$$

$$\mathbf{m}_{n-1} = (m_{1,n-1}, \ldots, m_{i,n-1}, \mathbf{0}, \tilde{m}_{i,n-1}, \ldots, \tilde{m}_{1,n-1}), \tag{4}$$

$$\mathbf{m}'_{n-1} = (m'_{1,n-1}, \ldots, m'_{i,n-1}, \mathbf{0}, \tilde{m}'_{i,n-1}, \ldots, \tilde{m}'_{1,n-1}), \tag{5}$$

$$\mathbf{m}_{n-2} = (m_{1,n-2}, \ldots, m_{i,n-2}, \mathbf{0}, \tilde{m}_{i,n-2}, \ldots, \tilde{m}_{1,n-2}), \tag{6}$$

where $\mathbf{0}$ denotes zero coordinates. Then for $k = 1, 2, 3, \ldots$, we have

$$t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(g_{n-1}(\theta)) = t^{\mathbf{m}_{n+k}+\boldsymbol{\varepsilon}_k}_{\mathbf{m}'_{n+k-1}+\boldsymbol{\varepsilon}_k, \mathbf{m}_{n+k-1}+\boldsymbol{\varepsilon}_k, \mathbf{m}_{n+k-2}+\boldsymbol{\varepsilon}_k}(g_{n+k-1}(\theta)), \tag{7}$$

where $\mathbf{m}_{n+k}$, $\mathbf{m}'_{n+k-1}$, $\mathbf{m}_{n+k-1}$, $\mathbf{m}_{n+k-2}$ are obtained from $\mathbf{m}_n$, $\mathbf{m}'_{n-1}$, $\mathbf{m}_{n-1}$, $\mathbf{m}_{n-2}$ by adding k zero coordinates and $\boldsymbol{\varepsilon}_k = (\mathbf{0}, -k, -k, \ldots, -k)$ with i coordinates equal to $-k$. In other words, the matrix elements $t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(g_{n-1}(\theta))$ are not changed if we add k additional zero coordinates in weights (3)-(6) and subtract the number k from each of the last i coordinates. It is clear that this operations turns matrix elements of representations of the group $U(n)$ into those for the group $U(n+k)$.

4. If $k_{n-1} = m_{1,n-1} + \ldots + m_{n-1,n-1}$, $k'_{n-1} = m'_{1,n-1} + \ldots + m'_{n-1,n-1}$, then

$$t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\theta) = (-1)^{k_{n-1}+k'_{n-1}} t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(-\theta). \tag{8}$$

5. Permuting $\mathbf{m}'_{n-1}$ and $\mathbf{m}_{n-1}$ in the matrix elements we obtain the relation

$$t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\theta) = (-1)^{k_{n-1}+k'_{n-1}} t^{\mathbf{m}_n}_{\mathbf{m}_{n-1}\mathbf{m}'_{n-1}\mathbf{m}_{n-2}}(\theta). \tag{9}$$

6. The matrix elements are not changed when we go over from the representations $T_{\mathbf{m}}$ to the contragradient representations $T_{\tilde{\mathbf{m}}}$:

$$t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\theta) = t^{\tilde{\mathbf{m}}_n}_{\tilde{\mathbf{m}}'_{n-1}\tilde{\mathbf{m}}_{n-1}\tilde{\mathbf{m}}_{n-2}}(\theta). \tag{10}$$

Combination of relations (7) and (10) gives the new symmetry formula. This formula and relation (7) show that the matrix element $t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\theta)$ with highest weights (3)-(6) is not changed if we add to the first i coordinates and to the last i coordinates of the weights $\mathbf{m}_n$, $\mathbf{m}'_{n-1}$, $\mathbf{m}_{n-1}$, $\mathbf{m}_{n-2}$ the same integer.

7. The expressions for matrix elements (1) of Section 18.4.1 are not changed if we make the same permutation in the sets of numbers $(\ell_{2n}, \ell_{3n}, \dots, \ell_{n-1,n})$ and $(\ell_{1,n-2}, \dots, \ell_{n-2,n-2})$, where $\ell_{in} = m_{in} - i$, $\ell_{j,n-2} = m_{j,n-2} - j$. The same permutations in the sets $(\ell_{1,n-1}, \dots, \ell_{n-1,n-1})$ and $(\ell'_{1,n-1}, \dots, \ell'_{n-1,n-1})$ can change only a sign.

8. The matrix element (1) of Section 18.4.1 is not changed if we add to all coordinates of the highest weights $\mathbf{m}_n$, $\mathbf{m}'_{n-1}$, $\mathbf{m}_{n-1}$, $\mathbf{m}_{n-2}$ the same integer.

The properties of matrix elements, given above, allow us to express many matrix elements of the irreducible representations $T_{\mathbf{m}}$ of the group $U(n)$, which are of class j with respect to the subgroup $U(n-1)$ (that is, the representations with highest weights $(m_{1n}, \dots, m_{jn}, 0, \dots, 0, m'_{jn}, \dots, m'_{1n})$) in terms of matrix elements of representations of lower classes. For example, it follows from property 6 that

$$\begin{aligned} &t^{(\ell,k,\mathbf{0},k',\ell')}_{(m,k,\mathbf{0},k',m')(p,k,\mathbf{0},k',p')(j,k,\mathbf{0},k',j')}(g_{n-1}(\theta)) \\ &\quad = t^{(\ell-k,\mathbf{0},\ell'-k')}_{(m-k,\mathbf{0},m'-k')(p-k,\mathbf{0},p'-k')(j-k,\mathbf{0},j'-k')}(g_{n+k+k'-1}(\theta)), \end{aligned} \tag{11}$$

where on the right hand side there is the matrix element of the representation of the group $U(n+k+k')$. Using properties of matrix elements of the class 1 representations (see Section 11.6.11) we obtain from (11) the relation

$$\begin{aligned} &t^{(\ell,k,\mathbf{0},k',\ell')}_{(m,k,\mathbf{0},k',m')(p,k,\mathbf{0},k',p')(j,k,\mathbf{0},k',j')}(g_{n-1}(\theta)) \\ &\quad = t^{(\ell,\mathbf{0},\ell')}_{(m,\mathbf{0},m')(p,\mathbf{0},p')(j,\mathbf{0},j')}(g_{n-1}(\theta)) \end{aligned} \tag{12}$$

for matrix elements of class 2 representations of $U(n)$. In the similar way it is shown that if $K = (k, \dots, k, \mathbf{0}, k', \dots, k')$, then the relation

$$t^{(\ell,K,\ell')}_{(m,K,m')(p,K,p')(j,K,j')}(g_{n-1}(\theta)) = t^{(\ell,\mathbf{0},\ell')}_{(m,\mathbf{0},m')(p,\mathbf{0},p')(j,\mathbf{0},j')}(g_{n-1}(\theta)) \tag{13}$$

for matrix elements of representations of $U(n)$ is valid. In particular,

$$t^{(\ell,K,\ell')}_{KKK}(g_{n-1}(\theta)) = t^{(\ell,\mathbf{0},\ell')}_{\mathbf{000}}(g_{n-1}(\theta)), \tag{14}$$

$$t^{K}_{KKK}(g_{n-1}(\theta)) = t^{0}_{\mathbf{000}}(g_{n-1}(\theta)) = 1. \tag{14'}$$

We derive from properties 1 and 6 for $k \geq j$, $k' \leq j'$ the relations

$$\begin{aligned} &t^{n,(\ell,k,\mathbf{0},k',\ell')}_{(m,j,\mathbf{0},j',m')(p,j,\mathbf{0},j',p')(j,j,\mathbf{0},j',j')}(g_{n-1}(\theta)) \\ &\quad = t^{n+j+j',(\ell-j,k-j,\mathbf{0},k'-j',\ell'-j')}_{(m-j,\mathbf{0},m'-j')(p-j,\mathbf{0},p'-j')(0,\mathbf{0},0)}(g_{n+j+j'-1}(\theta)) \\ &\quad = t^{n+j+j',(\ell-j,\mathbf{0},\ell'-j')}_{(m-j,\mathbf{0},m'-j')(p-j,\mathbf{0},p'-j')(k-j,\mathbf{0},k'-j')}(g_{n+j+j'-1}(\theta)) \\ &\quad = t^{n,(\ell,\mathbf{0},\ell')}_{(m,\mathbf{0},m')(p,\mathbf{0},p')(k,\mathbf{0},k')}(g_{n-1}(\theta)) \\ &\quad = t^{n,(\ell,k,\mathbf{0},k',\ell')}_{(m,\mathbf{0},m')(p,\mathbf{0},p')(0,\mathbf{0},0)}(g_{n-1}(\theta)) \end{aligned} \tag{15}$$

(here the matrix elements are equipped with the additional index which indicates onto dimension of the corresponding group ($U(s)$). This property can be generalized. For example,

$$\begin{aligned} & t^{n+k+k',(\ell-k,\mathbf{0},\ell'-k')}_{(m-k,\mathbf{0},m'-k')(0,\mathbf{0},0)(0,\mathbf{0},0)}(\theta) \\ &= t^{n,(\ell,k,\mathbf{0},k',\ell')}_{(m,j,\mathbf{0},j',m')(k,j,\mathbf{0},j',k')(j,j,\mathbf{0},j',j')}(\theta) \\ &= t^{n,(\ell,j,j,\mathbf{0},j',j',\ell')}_{(m,j,\mathbf{0},j',m')(k,j,\mathbf{0},j',k')(k,\mathbf{0},k')}(\theta) \\ &= t^{n,(\ell,j,j,\mathbf{0},j',j',\ell')}_{(m,j,i,\mathbf{0},i',j',m')(k,j,i,\mathbf{0},i',j',k')(k,i,i,\mathbf{0},i',i',k')}(\theta) \\ &= t^{n,(\ell,k,i,i,\mathbf{0},i',i',k',\ell')}_{(m,j,i,\mathbf{0},i',j',m')(k,i,j,\mathbf{0},i',j',k')(j,j,\mathbf{0},j',j')}(\theta) \\ &= t^{n,(\ell,k,i,i,\mathbf{0},i',i',k',\ell')}_{(m,j,i,s,\mathbf{0},s',i',j',m')(k,j,i,s,\mathbf{0},s',i',j',k')(j,j,s,s,\mathbf{0},s',s',j',j')}(\theta) \\ &= t^{n,(\ell,j,j,s,s,\mathbf{0},s',s',j',j',\ell')}_{(m,j,i,s,\mathbf{0},s',i',j',m')(k,j,i,s,\mathbf{0},s',i',j',k')(k,i,i,\mathbf{0},i',i',k')}(\theta) = \dots . \end{aligned} \tag{16}$$

18.4.3. Matrix elements of the fundamental representations. The symmetry relations of Section 18.4.2 can be used for evaluation of matrix elements of the fundamental representations of the group $U(n)$, that is, for the representations with highest weights $m = (1,\dots,1,0,\dots,0) = (1_s, 0_{n-s})$. It follows from the properties 1 and 6 of Section 18.4.2 that

$$t^{n,1_s}_{1_s1_s1_s}(g_{n-1}(\theta)) = t^{n,1_s}_{1_{s-1}1_{s-1}1_{s-2}}(g_{n-1}(\theta)) = 1 \tag{1}$$

(we write 1_s instead of $(1_s, \mathbf{0})$ and indicate dimension of the group $U(n)$). From the properties 2 and 8 we have

$$t^{n,1_s}_{1_s1_s1_{s-1}}(g_{n-1}(\theta)) = t^{s+1,1_s}_{1_s1_s1_{s-1}}(g_s(\theta)) = t^{s+1,(\mathbf{0},-1)}_{\mathbf{000}}(g_s(\theta)) = \cos\theta. \tag{2}$$

In the same way we find that

$$t^{n,1_s}_{1_{s-1}1_{s-1}1_{s-1}}(g_{n-1}(\theta)) = \cos\theta. \tag{3}$$

$$t^{n,1_s}_{1_{s-1}1_s1_{s-1}}(g_{n-1}(\theta)) = -t^{n,1_s}_{1_s1_{s-1}1_{s-1}}(g_{n-1}(\theta)) = \sin\theta. \tag{4}$$

The other non-diagonal matrix elements $t^{n,1_s}_{\dots}(g_{n-1}(\theta))$ are equal to zero and the other diagonal elements are equal to 1.

18.4.4. Matrix elements and CGC's. Let us consider relation (10) of Section 18.2.1 for matrix elements of representations and for CGC's of the group $U(n)$. In particular, we have

$$t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta)t^{(p,\mathbf{0})}_{\mathbf{000}}(\theta) = \sum_{\boldsymbol{\ell}'} \left(\begin{matrix} \boldsymbol{\ell} & (p,\mathbf{0}) \\ \mathbf{m} & \mathbf{0} \end{matrix}\middle| \begin{matrix} \boldsymbol{\ell}' \\ \mathbf{m} \end{matrix}\right) \left(\begin{matrix} \boldsymbol{\ell} & (p,\mathbf{0}) \\ \mathbf{p} & \mathbf{0} \end{matrix}\middle| \begin{matrix} \boldsymbol{\ell}' \\ \mathbf{p} \end{matrix}\right) t^{\boldsymbol{\ell}'}_{\mathbf{mpj}}(\theta), \tag{1}$$

where $\begin{pmatrix} \boldsymbol{\ell} & (p,\mathbf{0}) & \boldsymbol{\ell}' \\ \mathbf{m} & \mathbf{0} & \mathbf{m} \end{pmatrix}$ is the scalar factor and the summation is over all highest weights $\boldsymbol{\ell}' = (\ell'_1, \dots, \ell'_n)$ of the irreducible representations $T_{\boldsymbol{\ell}'}$ of $U(n)$, which contain the representations $T_{\mathbf{m}}$ and $T_{\mathbf{p}}$ of the subgroup $U(n-1)$ and satisfy the conditions

$$\begin{gathered} \ell'_1 + \ldots + \ell'_n - \ell_1 - \ldots - \ell_n = p; \\ \ell'_1 \geq \ell_1 \geq \ell'_2 \geq \ell_2 \geq \ldots \geq \ell'_n \geq \ell_n. \end{gathered} \tag{1'}$$

Since $T^{(p,\mathbf{0})}_{\mathbf{000}}(\theta) = \cos^p \theta$, then substituting expressions for the scalar factors from Section 18.2.7 into (1) we obtain the equality

$$\begin{aligned} \cos^p \theta t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) &= p \sum_{\boldsymbol{\ell}'} \prod_{i<k} (\ell'_i - \ell_k - i + k)^2 \\ &\times \left(\prod_{i \leq k} \frac{(\ell'_i - m_k - i + k)!(\ell'_i - p_k - i + k)!}{(\ell_i - m_k - i + k)!(\ell_i - p_k - i + k)!} \right. \\ &\times \left. \prod_{i<k} \frac{(m_i - \ell_k - i + k - 1)!(p_i - \ell_k - i + k - 1)!}{(m_i - \ell'_k - i + k - 1)!(p_i - \ell'_k - i + k - 1)!} \right)^{1/2} \\ &\times \frac{\prod\limits_{i<k} (\ell_i - \ell'_k - i + k - 1)!}{\prod\limits_{i \leq k} (\ell'_i - \ell_k - i + k)!} t^{\boldsymbol{\ell}'}_{\mathbf{mpj}}(\theta), \end{aligned} \tag{2}$$

where summation is such as in (1).

It also follows from relation (10) of Section 18.2.1 that

$$\begin{aligned} & t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) t^{(\mathbf{0},-p)}_{\mathbf{000}}(\theta) \\ &= \sum_{\boldsymbol{\ell}''} \begin{pmatrix} \boldsymbol{\ell} & (\mathbf{0},-p) & \boldsymbol{\ell}'' \\ \mathbf{m} & \mathbf{0} & \mathbf{m} \end{pmatrix} \begin{pmatrix} \boldsymbol{\ell} & (\mathbf{0},-p) & \boldsymbol{\ell}'' \\ \mathbf{p} & \mathbf{0} & \mathbf{p} \end{pmatrix} t^{\boldsymbol{\ell}''}_{\mathbf{mpj}}(\theta), \end{aligned} \tag{3}$$

where the summation is over all highest weights $\boldsymbol{\ell}'' = (\ell''_1, \dots, \ell''_n)$ of the irreducible representations $T_{\boldsymbol{\ell}''}$ of the group $U(n)$, which contain the representations $T_{\mathbf{m}}$ and $T_{\mathbf{p}}$ of the subgroup $U(n-1)$ and satisfy the conditions

$$\begin{gathered} \ell_1 + \ldots + \ell_n - \ell''_1 - \ldots - \ell''_n = p; \\ \ell_1 \geq \ell''_1 \geq \ell_2 \geq \ell''_2 \geq \ldots \geq \ell_n \geq \ell''_n. \end{gathered} \tag{3'}$$

We obtain

$$\cos^p \theta t^{\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) = p\sum_{\boldsymbol{\ell}''}\prod_{i<k}(\ell''_i - \ell''_k - i + k)^2$$
$$\times \left(\prod_{i<k}\frac{(m_i - \ell''_k - i + k)!(p_i - \ell''_k - i + k)!}{(m_i - \ell_k - i + k)!(p_i - \ell_k - i + k)!}\right.$$
$$\left.\times \prod_{i\le k}\frac{(\ell_i - m_k - i + k - 1)!(\ell_i - p_k - i + k - 1)!}{(\ell''_i - m_k - i + k - 1)!(\ell''_i - p_k - i + k - 1)!}\right)^{1/2} \tag{4}$$
$$\times \frac{\prod_{i<k}(\ell''_i - \ell_k - i + k - 1)!}{\prod_{i\le k}(\ell_i - \ell''_k - i + k)!} t^{\boldsymbol{\ell}''}_{\mathbf{mpj}}(\theta),$$

where summation is such as in (3).

Applying to (1) the orthogonality relation for matrix elements of irreducible representations of $U(n)$ and using formula (9) of Section 11.1.5 we have

$$\sum_{\mathbf{j}}(\dim T^{n-2,\mathbf{j}}) \int_0^{\pi/2} t^{n,\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) t^{n,(p,\mathbf{0})}_{\mathbf{000}}(\theta) t^{n,\boldsymbol{\ell}'}_{\mathbf{mpj}}(\theta)\sin^{2n-3}\theta\cos\theta d\theta$$
$$= \frac{1}{2n-2}\frac{(\dim T^{n-1,\mathbf{m}})(\dim T^{n-1,\mathbf{p}})}{(\dim T^{n\boldsymbol{\ell}'})}\left(\begin{matrix}\boldsymbol{\ell} & (p,\mathbf{0}) \\ \mathbf{m} & \mathbf{0}\end{matrix}\middle|\begin{matrix}\boldsymbol{\ell}' \\ \mathbf{m}\end{matrix}\right)\left(\begin{matrix}\boldsymbol{\ell} & (p,\mathbf{0}) \\ \mathbf{p} & \mathbf{0}\end{matrix}\middle|\begin{matrix}\boldsymbol{\ell}' \\ \mathbf{p}\end{matrix}\right), \tag{5}$$

where $\boldsymbol{\ell}'$ satisfies conditions (1′). Consequently,

$$\sum_j\left(\prod_{i<k}(j_i - j_k - i + k)\right)\int_0^{\pi/2} t^{n,\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) t^{n,\boldsymbol{\ell}'}_{\mathbf{mpj}}(\theta)\sin^{2n-3}\theta\cos^{\ell'-\ell+1}\theta d\theta$$
$$= \frac{\ell'-\ell}{2(n-2)!}\prod_{i<k}(\ell'_i - \ell'_k - i + k)\prod_{i<k}(m_i - m_k - i + k)(p_i - p_k - i + k)$$
$$\times \frac{\prod_{i<k}(\ell_i - \ell'_k - i + k - 1)!}{\prod_{i\le k}(\ell'_i - \ell_k - i + k)!}\left(\prod_{i\le k}\frac{(\ell'_i - m_k - i + k)!(\ell'_i - p_k - i + k)!}{(\ell_i - m_k - i + k)!(\ell_i - p_k - i + k)!}\right. \tag{6}$$
$$\left.\times \prod_{i<k}\frac{(m_i - \ell_k - i + k - 1)!(p_i - \ell_k - i + k - 1)!}{(m_i - \ell'_k - i + k - 1)!(p_i - \ell'_k - i + k - 1)!}\right)^{1/2},$$

where $\ell = \ell_1 + \ell_2 + \ldots + \ell_n$, $\ell' = \ell'_1 + \ell'_2 + \ldots + \ell'_n$.

The analogous relations can be obtained from the tensor products of the irreducible representations $T_{\boldsymbol{\ell}}$ and $T_{(1_s,0_{n-s})}$ of the group $U(n)$. We recommend to the reader to write down these relations.

18.4.5. Matrix elements of representations of $U(n)$ and generalizations of classical polynomials of a discrete variable. Let us fix $\mathbf{m}_n$, $\mathbf{m}_{n-2}$ and θ. The finite set of functions

$$F_{\mathbf{m}'_{n-1}}(\mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}, \theta) \equiv t^{\mathbf{m}_n}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\theta) \tag{1}$$

of discrete variables $\mathbf{m}_{n-1} \equiv (m_{1,n-1}, \dots, m_{n-1,n-1})$, labelled by indices $\mathbf{m}'_{n-1} \equiv (m'_{1,n-1}, \dots, m'_{n-1,n-1})$, forms an orthogonal complete system in the space of functions $f(m_{1,n-1}, \dots, m_{n-1,n-1})$, given on the set of those integral values of the variables for which

$$\min(m_{in}, m_{i-1,n-2}) \geq m_{i,n-1} \geq \max(m_{i+1,n}, m_{i,n-2}),$$

where $m_{0,n-2} = \infty$, $m_{n-1,n-2} = -\infty$. Really, since the representations $T_{\mathbf{m}}$ are unitary, we have

$$\sum_{\mathbf{m}_{n-1}} F_{\mathbf{p}_{n-1}}(\mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}, \theta) \times F_{\mathbf{p}'_{n-1}}(\mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}, \theta) = \delta(\mathbf{p}_{n-1}, \mathbf{p}'_{n-1}),$$

where $\delta(\mathbf{m}, \mathbf{p})$ is the Kronecker symbol.

With the help of the appropriate renotations of the variables we can obtain from functions (1) an orthogonal system of polynomials of many discrete variables. Because of their awkwardness we do not write down them here. They are generalizations of the Krawtchouk polynomials (see Section 6.8.1).

The matrix elements $t^{\mathbf{m}}_{\mathbf{m}'_{n-1}\mathbf{m}_{n-1}\mathbf{m}_{n-2}}(\pi/2)$ can be used for obtaining matrix elements of the operators $T_{\mathbf{m}}(g_{ij}(\theta))$ (see Section 9.5.7). For the representations of the group $U(n)$ with highest weights $(m_{1n}, 0, \dots, 0, m_{nn})$ these matrix elements are expressed in terms of the Racah polynomials of a discrete variable. The analogous matrix elements for the representations $T_{\mathbf{m}}$ of $U(n)$ lead to polynomials of many discrete variables which are generalizations of the Racah polynomials.

18.4.6. Representations of $U(n)$ and generalized Jacobi polynomials. Let us represent expression (4) of Section 18.4.1 for matrix elements (1) of Section 18.4.1 in the form

$$t^{\mathbf{m}_{p+1}}_{\mathbf{m}'_p\mathbf{m}_p\mathbf{m}_{p-1}}(\theta) = (-1)^{k'_p}(\cos\theta)^{k_{p+1}+k_{p-1}}(\tan\theta)^{k_p+k'_p} \times F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; \sin^{-2}\theta), \tag{1}$$

where $k_i = m_{1i} + \dots + m_{ii}$. Then

$$F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) = \nu(\mathbf{m}_{p+1}, \mathbf{m}'_p, \mathbf{m}_{p-1})\nu(\mathbf{m}_{p+1}, \mathbf{m}_p, \mathbf{m}_{p-1})$$
$$\times \sum_{\mathbf{m}''_p} (-1)^{k''_p} \prod_{i<j} (\ell''_{ip} - \ell''_{jp}) \frac{\prod\limits_{i\le j} (\ell_{i,p+1} - \ell''_{jp})! \prod\limits_{i<j} (\ell_{i,p-1} - \ell''_{jp} - 1)!}{\prod\limits_{i\le j} (\ell''_{ip} - \ell_{j,p-1})! \prod\limits_{i<j} (\ell''_{ip} - \ell_{j,p+1} - 1)!}$$
$$\times \frac{\prod\limits_{i<j} (\ell''_{ip} - \ell_{jp} - 1)!(\ell''_{ip} - \ell'_{jp} - 1)!}{\prod\limits_{i\le j} (\ell_{ip} - \ell''_{jp})!(\ell'_{ip} - \ell''_{jp})!} x^{k''_p}, \tag{2}$$

where $\nu(\dots)$ is given by formula (5) of Section 18.4.1 and $\ell_{ij} = m_{ij} - i$.

Formula (2) of Section 18.4.1 can be represented as

$$t^{\mathbf{m}_{p+1}}_{\mathbf{m}'_p \mathbf{m}_p \mathbf{m}_{p-1}}(\theta) = (-1)^{k_p} (\cos\theta)^{-k_{p+1} - k_{p-1}} (\cot\theta)^{-k_p - k'_p}$$
$$\times F_2(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; \sin^2\theta), \tag{3}$$

where

$$F_2(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) = \mu(\mathbf{m}_{p+1}, \mathbf{m}'_p, \mathbf{m}_{p-1})\mu(\mathbf{m}_{p+1}, \mathbf{m}_p, \mathbf{m}_{p-1})$$
$$\times \sum_{\mathbf{m}''_p} (-1)^{k''_p} \prod_{i<j} (\ell''_{ip} - \ell''_{jp}) \frac{\prod\limits_{i<j} (\ell''_{ip} - \ell_{j,p+1} - 1)! \prod\limits_{i\le j} (\ell''_{ip} - \ell_{j,p-1})!}{\prod\limits_{i<j} (\ell_{i,p-1} - \ell''_{jp} - 1)! \prod\limits_{i\le j} (\ell_{i,p+1} - \ell''_{jp})!}$$
$$\times \frac{\prod\limits_{i<j} (\ell_{ip} - \ell''_{jp} - 1)!(\ell'_{ip} - \ell''_{jp} - 1)!}{\prod\limits_{i\le j} (\ell''_{ip} - \ell_{jp})!(\ell''_{ip} - \ell'_{jp})!} x^{k''_p} \tag{4}$$

and $\mu(\dots)$ is given by formula (3) of Section 18.4.1.

If $\mathbf{m}_{p+1}$ is such that $m_{2,p+1} + m_{3,p+1} + \dots + m_{p+1,p+1} \ge 0$, then $F_1(\dots; x)$ and $F_2(\dots; x)$ are polynomials. For $n = 2$ they can be expressed in terms of Jacobi polynomials (see Section 6.3.7). In this reason we call $F_1(\dots; x)$ and $F_2(\dots; x)$ the generalized Jacobi polynomials. Comparing expressions (1) and (3) we have

$$F_2(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) = (-x)^{k_p + k'_p} (1 - x)^{k_{p+1} + k_{p-1} - k_p - k'_p}$$
$$\times F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x^{-1}). \tag{5}$$

Since the representations $T_{\mathbf{m}}$ of the group $U(n)$ are unitary, then the relations

$$F_i(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) = F_i(\mathbf{m}_p, \mathbf{m}'_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x), \tag{6}$$

$$\sum_{\mathbf{m}'_p} (x - 1)^{-k_p - k'_p} F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x)$$
$$\times F_1(\mathbf{m}'_p, \mathbf{m}''_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) = \left(\frac{x}{x-1}\right)^{k_{p+1} + k_{p-1}} \delta(\mathbf{m}_p, \mathbf{m}''_p) \tag{7}$$

are valid, where $\delta(\mathbf{m}_p, \mathbf{m}_p'')$ is the Kronecker symbol. The relation, analogous to equality (7), holds for F_2.

If $m_{ip} = \max(m_{i+1,p+1}, m_{i,p-1})$, $1 \le i \le p$, then it follows from $\mathbf{m}_p'' \le \mathbf{m}_p$ that $\mathbf{m}_p'' = \mathbf{m}_p'$ in (2). In this case

$$F_1(\mathbf{m}_p', \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) = \mu(\mathbf{m}_{p+1}, \mathbf{m}_p', \mathbf{m}_{p-1})\mu(\mathbf{m}_{p+1}, \mathbf{m}_p, \mathbf{m}_{p-1}) \times \frac{\prod\limits_{i<j}(\ell_{i,p+1} - \ell_{jp} - 1)! \prod\limits_{i\le j}(\ell_{i,p+1} - \ell_{jp})! \prod\limits_{i<j}(\ell_{ip} - \ell_{jp}' - 1)!}{\prod\limits_{i<j}(\ell_{ip} - \ell_{j,p+1} - 1)! \prod\limits_{i\le j}(\ell_{ip} - \ell_{j,p-1})! \prod\limits_{i\le j}(\ell_{ip}' - \ell_{jp})!}(-x)^{k_p}. \tag{8}$$

Analogously, if $m_{ip} = \min(m_{i+1,p}, m_{i-1,p-1})$, $1 \le i \le p$, then

$$F_2(\mathbf{m}_p', \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) = \nu(\mathbf{m}_{p+1}, \mathbf{m}_p', \mathbf{m}_{p-1})\nu(\mathbf{m}_{p+1}, \mathbf{m}_p, \mathbf{m}_{p-1}) \times \frac{\prod\limits_{i<j}(\ell_{ip} - \ell_{j,p+1} - 1)! \prod\limits_{i\le j}(\ell_{ip} - \ell_{j,p-1})! \prod\limits_{i<j}(\ell_{ip}' - \ell_{jp} - 1)!}{\prod\limits_{i<j}(\ell_{i,p-1} - \ell_{jp} - 1)! \prod\limits_{i\le j}(\ell_{i,p+1} - \ell_{jp})! \prod\limits_{i\le j}(\ell_{ip} - \ell_{jp}')!}(-x)^{k_p}. \tag{9}$$

If $m_{i,p+1} = 0$ for $i \ge 2$, then $\mathbf{m}_k = (m_k, 0, \dots, 0)$, $k = p+1, p, p-1$, and we have

$$\begin{aligned} F_1(\mathbf{m}_p', \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) &= (m_{p+1} - m_{p-1})![(m_{p+1} - m_p')!(m_{p+1} - m_p)! \\ &\times (m_p' - m_{p-1})!(m_p - m_{p-1})!]^{-1/2}(-x)^{m_{p-1}} \\ &\times {}_2F_1(m_p' - m_{p-1}, m_p - m_{p-1}; m_{p+1} - m_{p-1}; x). \end{aligned} \tag{10}$$

The similar formula for the function F_2 is valid.

Now we derive the formula for the Fourier series expansion of the generalized Jacobi polynomials. We have

$$g_{n-1}(\theta) = d_n(i)g_{n-1}\left(\frac{\pi}{4}\right)d_{n-1}(e^{-i\theta})d_{n-1}(-e^{-i\theta})g_{n-1}\left(\frac{\pi}{4}\right)d_n(i),$$

where $d_k(a) = \mathrm{diag}(1, \dots, 1, a, 1, \dots, 1)$ (a is situated on the k-th place). Writing this relation in the matrix form we obtain

$$\begin{aligned} &(\cos\theta)^{k_n + k_{n-2}}(i\tan\theta)^{k_{n-1}+k_{n-1}''}F_1(\mathbf{m}_{n-1}'', \mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}; \sin^{-2}\theta) \\ &\quad = (e^{i\theta}/2)^{k_n - k_{n-2}}\sum_{M'} F_1(\mathbf{m}_{n-1}'', \mathbf{m}_{n-1}'; \mathbf{m}_n, \mathbf{m}_{n-2}; 2) \\ &\qquad \times F_1(\mathbf{m}_{n-1}', \mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}; 2)\exp(-2ik_{n-1}'\theta). \end{aligned} \tag{11}$$

Consequently,

$$\begin{aligned} &F_1(\mathbf{m}_{n-1}'', \mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}; \sin^{-2}\theta) \\ &\quad = (1 - e^{-2i\theta})^{-k_{n-1}-k_{n-1}''}(1 + e^{-2i\theta})^{k_{n-1}+k_{n-1}''-k_n-k_{n-2}} \\ &\qquad \times \sum_{M'} F_1(\mathbf{m}_{n-1}'', \mathbf{m}_{n-1}'; \mathbf{m}_n, \mathbf{m}_{n-2}; 2) \\ &\qquad \times F_1(\mathbf{m}_{n-1}', \mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}; 2)\exp(-2ik_{n-1}'\theta). \end{aligned} \tag{12}$$

The analogous formula for F_2 is valid.

18.4.7. The addition theorem for the polynomials F_1 and F_2. The relation

$$\begin{aligned} &g_p(\varphi)d_p(e^{i\alpha})d_{p+1}(e^{-i\alpha})g_p(\psi) \\ &\quad = d_p(e^{i\beta})d_{p+1}(e^{-i\beta})g_p(\theta)d_p(e^{i\gamma})d_{p+1}(e^{-i\gamma}) \end{aligned} \tag{1}$$

has place in the group $U(n)$, where

$$\cos 2\theta = \cos 2\varphi \cos 2\psi - \sin 2\varphi \sin 2\psi \cos 2\alpha, \tag{2}$$

$$e^{i(\beta+\gamma)} = \cos^{-1}\theta(\cos\varphi\cos\psi e^{i\alpha} - \sin\varphi\sin\psi e^{-i\alpha}), \tag{3}$$

$$e^{i(\beta-\gamma)} = \sin^{-1}\theta(\cos\varphi\sin\psi e^{i\alpha} + \sin\varphi\cos\psi e^{-i\alpha}). \tag{4}$$

Writing this equality for the representation operators, we receive the addition theorem

$$\begin{aligned} &\exp[i(2k_p'' - k_{p-1} - k_{p+1})\beta + i(2k_p - k_{p-1} - k_{p+1})\gamma](\cos\theta)^{k_{p+1}+k_{p-1}}(\tan\theta)^{k_p+k_p''} \\ &\quad \times F_1(\mathbf{m}_p'', \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; \sin^{-2}\theta) \\ &\quad = \exp[-i(k_{p+1}+k_{p-1})\alpha](\cos\varphi\cos\psi)^{k_{p+1}+k_{p-1}}(\tan\varphi)^{k_p''}(\tan\psi)^{k_p} \\ &\quad \times \sum_{\mathbf{m}_p'} \exp(2ik_p'\alpha)(-\tan\varphi\tan\psi)^{k_p'} F_1(\mathbf{m}_p'', \mathbf{m}_p'; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; \sin^{-2}\varphi) \\ &\quad \times F_1(\mathbf{m}_p', \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; \sin^{-2}\psi), \end{aligned} \tag{5}$$

where the parameters α, β, γ, φ, ψ, θ are connected by relations (2)-(4). The analogous formula can be written for F_2.

The product formula for the functions F_1 follows from equality (5). We recommend to the reader to write down this formula.

The other addition formula for the functions F_1 and F_2 follows from the relation

$$g_{n-1}(\varphi)g_{n-2}(\alpha)g_{n-1}(\psi) = g_{n-2}(\beta)g_{n-1}(\theta)g_{n-2}(\gamma),$$

where

$$\begin{aligned} &\cos\theta = \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\alpha, \\ &\sin\beta = \sin^{-1}\theta\sin\psi\sin\alpha, \\ &\sin\gamma = \sin^{-1}\theta\sin\varphi\sin\alpha. \end{aligned}$$

18.4.8. Recurrence relations. Let us consider the representations $T_{\mathbf{m}}$ of the complex group $GL(n, \mathbb{C})$. We have the relation

$$(I + sE_{p+1,p})g_p(t) = d_p(a)d_{p+1}(a^{-1})g_p(\varphi)d_p(b)d_{p+1}(b^{-1}),$$

where I is the identity matrix, $d_k(a) = \mathrm{diag}\,(1, \dots, 1, a, 1, \dots, 1)$ (a is situated on the k-th place) and

$$\cos 2\varphi = \cos 2t - s \sin 2t, \ a^2 = \frac{\sin 2t}{\sin 2\varphi},$$
$$b^2 = \sin^{-1} 2\varphi [s(1 + \cos 2t) + \sin 2t].$$

Therefore,

$$T_{\mathbf{m}}(I + sE_{p+1,p}) T_{\mathbf{m}}(g_p(t))$$
$$= T_{\mathbf{m}}(d_p(a) d_{p+1}(a^{-1})) T_{\mathbf{m}}(g_p(\varphi)) T_{\mathbf{m}}(d_p(b) d_{p+1}(b^{-1})).$$

Differentiating both sides of this relation with respect to s and setting $s = 0$, we derive

$$E^{\mathbf{m}}_{p+1,p} T_{\mathbf{m}}(g_p(t)) = \frac{1}{2} \left[(E^{\mathbf{m}}_{pp} - E^{\mathbf{m}}_{p+1,p+1}) \tan t + \frac{d}{dt} \right] T_{\mathbf{m}}(g_p(t)).$$

Comparing matrix elements of the both sides of this equality for Gel'fand-Tsetlin basis $\{\mathbf{v}(M)\}$ we obtain the recurrence relation for F_1:

$$\begin{aligned} &- \left[k_p - k_{p-1} - k_{p+1} + \frac{1}{2}(k_p + k'_p) x \right] F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) \\ &+ \frac{1}{2} x(x-1) \frac{d}{dx} F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) \\ &= \sum_{i=1}^{p} D_i(\mathbf{m}_p, \mathbf{m}_{p+1}, \mathbf{m}_{p-1}) F_1(\mathbf{m}'_p + 1_i, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x), \end{aligned} \tag{1}$$

where 1_i denotes the vector $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the i-th place and

$$D_i(\mathbf{m}_p, \mathbf{m}_{p+1}, \mathbf{m}_{p-1})$$
$$= \left(\frac{\prod\limits_{i \le j} (\ell_{ip} - \ell_{j,p-1} - 1) \prod\limits_{i > j} (\ell_{j,p-1} - \ell_{ip} + 1) \prod\limits_{i < j} (\ell_{ip} - \ell_{j,p+1}) \prod\limits_{i \ge j} (\ell_{j,p+1} - \ell_{ip})}{\prod\limits_{i > j} (\ell_{jp} - \ell_{ip} - 1) \prod\limits_{i < j} (\ell_{ip} - \ell_{jp} + 1) \prod\limits_{i < j} (\ell_{ip} - \ell_{jp}) \prod\limits_{i > j} (\ell_{jp} - \ell_{ip})} \right)^{1/2}.$$

From the relation

$$(I + sE_{p,p+1}) g_p(t) = d_p(a) d_{p+1}(a^{-1}) g_p(\varphi) d_p(b) d_{p+1}(b^{-1}),$$

where

$$\cos 2\varphi = \cos 2t + s \sin 2t, \quad a^2 = \frac{\sin 2\varphi}{\sin 2t},$$

$$b^2 = \sin^{-1} 2\varphi[\sin 2t + s(1-\cos 2t)],$$

we obtain the recurrence relation

$$\begin{aligned}
&\left[k_p - \frac{1}{2}(k_p + k'_p)x\right] F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) \\
&\quad + \frac{1}{2}(x-1)x\frac{d}{dx}F_1(\mathbf{m}'_p, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x) \\
&\quad = (1-x)\sum_{i=1}^{p} E_i(\mathbf{m}_p, \mathbf{m}_{p+1}, \mathbf{m}_{p-1}) \\
&\quad \times F_1(\mathbf{m}'_p - \mathbf{1}_i, \mathbf{m}_p; \mathbf{m}_{p+1}, \mathbf{m}_{p-1}; x),
\end{aligned} \tag{2}$$

where

$$E_i(\mathbf{m}_p, \mathbf{m}_{p+1}, \mathbf{m}_{p-1})$$
$$= \left(\frac{\prod\limits_{j\le i}(\ell_{j,p+1} - \ell_{ip} + 1)\prod\limits_{j<i}(\ell_{j,p-1} - \ell_{ip})\prod\limits_{j>i}(\ell_{ip} - \ell_{j,p+1} - 1)\prod\limits_{j\ge i}(\ell_{ip} - \ell_{j,p-1})}{\prod\limits_{j<i}(\ell_{jp} - \ell_{ip} + 1)\prod\limits_{j>i}(\ell_{ip} - \ell_{jp} - 1)\prod\limits_{i<i}(\ell_{jp} - \ell_{ip})\prod\limits_{j>i}(\ell_{ip} - \ell_{jp})} \right)^{1/2}.$$

The similar recurrence relations are valid for the function F_2.

Other recurrence formulas for F_1 and F_2 can be derived from the results of Section 18.4.4.

18.4.9. Orthogonality relations. For the matrix elements $t^{n,\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta)$ we have the orthogonality relation

$$\begin{aligned}
&\sum_{\mathbf{j}}(\dim T^{n-2,\mathbf{j}}) \int_0^{\pi/2} t^{n,\boldsymbol{\ell}}_{\mathbf{mpj}}(\theta) t^{n,\boldsymbol{\ell}'}_{\mathbf{mpj}}(\theta) \sin^{2n-3}\theta\cos\theta d\theta \\
&\quad = \frac{\pi}{n-1}(\dim T^{n\boldsymbol{\ell}})^{-1}(\dim T^{n-1,\mathbf{m}})(\dim T^{n-1,\mathbf{p}})\delta_{\boldsymbol{\ell}\boldsymbol{\ell}'}.
\end{aligned} \tag{1}$$

Writing formula (1) for the function F_1, we obtain the orthogonality relation

$$\begin{aligned}
&\sum_{\mathbf{m}_{n-2}} (\dim T^{n-2,\mathbf{m}_{n-2}}) \int_1^{\infty} F(\mathbf{m}'_{n-1}, \mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}; x) \\
&\quad \times F_1(\mathbf{m}'_{n-1}, \mathbf{m}_{n-1}; \mathbf{m}'_n, \mathbf{m}_{n-2}; x) x^{-k_n - k_{n-2} - n} \\
&\quad \times (x-1)^{k_n + k_{n-2} - k_{n-1} - k_{n-1} - 1} dx \\
&\quad = \frac{\pi}{n-1}(\dim T^{n,\mathbf{m}_n})^{-1}(\dim T^{n-1,\mathbf{m}_{n-1}})(\dim T^{n-1,\mathbf{m}'_{n-1}})\delta_{\mathbf{m}_n\mathbf{m}'_n}.
\end{aligned} \tag{2}$$

This relation can be interpreted in the following way. We denote by $\mathbf{F}(\mathbf{m}'_{n-1}, \mathbf{m}_{n-1}; \mathbf{m}'_n; x)$ the vector function with the components

$$\left(\dim T^{n-2,\mathbf{m}_{n-2}}\right)\left(\frac{x-1}{x}\right)^{(k_n+k_{n-2})/2} F_1(\mathbf{m}'_{n-1}, \mathbf{m}_{n-1}; \mathbf{m}_n, \mathbf{m}_{n-2}; x).$$

Then the vector functions

$$F(\mathbf{m}'_{n-1}, \mathbf{m}_{n-1}; \mathbf{m}_n; x) \quad \text{and} \quad F(\mathbf{m}'_{n-1}, \mathbf{m}_{n-1}; \mathbf{p}_n; x), \quad \mathbf{m}_n \neq \mathbf{p}_n,$$

are orthogonal on $[1, +\infty)$ with respect to the weight

$$x^{-n}(x-1)^{-k_{n-1}-k'_{n-1}-1}.$$

18.5. Matrix Elements of Representations of the Groups $U(n-1,1)$, $IU(n-1)$ in the Gel'fand-Tsetlin Basis

18.5.1. Representations of the group $U(n-1,1)$. Let $T_\mathbf{r}$ be an irreducible representation of the group $U(n-2)$ with highest weight $\mathbf{r} = (r_1, \dots, r_{n-2})$, acting in the space $\mathcal{V}$. We fix an integer d and a complex number σ. Let $\mathfrak{D}_{\mathbf{r}d\sigma}$ be the linear space of infinitely differentiable vector functions $\mathbf{F}(\mathbf{x})$ on the complex cone $C_c^{n-1} \equiv \{\mathbf{x} \in E^c_{n-1,1} \mid [\mathbf{x}, \mathbf{x}] = 0, \mathbf{x} \neq \mathbf{0}\}$ (see Section 11.1.1) with values in the space $\mathcal{V}$, possessing the homogeneity and covariance properties[1]

$$\mathbf{F}(a\mathbf{x}) = a^{(\sigma-r+d)/2}\bar{a}^{(\sigma+r-d)/2}\mathbf{F}(\mathbf{x}), \quad r = r_1 + \dots + r_{n-2}, \quad a \in \mathbb{C}, \tag{1}$$

$$\mathbf{F}(\mathbf{x}k) = T_\mathbf{r}(k^{-1})\mathbf{F}(\mathbf{x}), \quad k \in U(n-2).$$

The space $\mathfrak{D}_{\mathbf{r}d\sigma}$ is invariant with respect to the operators

$$T^{\mathbf{r}d\sigma}(g)\mathbf{F}(\mathbf{x}) = \mathbf{F}(g^{-1}\mathbf{x}), \quad g \in U(n-1,1). \tag{2}$$

Therefore, the correspondence $g \to T^{\mathbf{r}d\sigma}(g)$ is a representation of the group $U(n-1,1)$. The representations $T^{\mathbf{r}d\sigma}$ form the *principal nonunitary series* of representations of $U(n-1,1)$. If $T_\mathbf{r}$ is the identity representation of the subgroup $U(n-2)$, then these representations coincide with representations $T^{n\sigma k}$ of Section 11.2.4.

As in Section 11.2.4, condition (1) allows us to consider the vector functions $\mathbf{f}(\boldsymbol{\xi})$ on S_C^{n-2}:

$$\mathbf{f}(\xi_1, \dots, \xi_{n-1}) = \mathbf{F}(\xi_1, \dots, \xi_{n-1}, 1)$$

[1] In fact, the covariance property means a possibility of continuation of functions F onto the group $U(n-1,1)$.

instead of the vector functions $\mathbf{F}(\mathbf{x}) \equiv \mathbf{F}(x_1, x_2, \ldots, x_n)$ on C_c^{n-1}. In the space $\mathfrak{D}_r$ of these vector functions $\mathbf{f}(\boldsymbol{\xi})$ we can introduce the scalar product

$$(\mathbf{f}_1, \mathbf{f}_2) = \int\limits_{S_C^{n-2}} \langle \mathbf{f}_1(\boldsymbol{\xi}), \mathbf{f}_2(\boldsymbol{\xi}) \rangle d\boldsymbol{\xi}, \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of the space $\mathcal{V}$. The operators $T^{\mathbf{r}d\sigma}(g'_{n-1}(t))$ act in the space $\mathfrak{D}_r$ of vector functions on S_C^{n-2} by formula (13) of Section 11.2.4. For the operators $T^{\mathbf{r}d\sigma}(k)$, $k \in U(n-1)$, we have

$$T^{\mathbf{r}d\sigma}(k)\mathbf{f}(\boldsymbol{\xi}) = \mathbf{f}(k^{-1}\boldsymbol{\xi}). \tag{4}$$

It is easy to obtain from here the decomposition of restriction of $T^{\mathbf{r}d\sigma}$ onto the subgroup $U(n-1)$. We have

$$T^{\mathbf{r}d\sigma}\downarrow_{U(n-1)} = \sum_{\mathbf{m}} \oplus T_{\mathbf{m}}, \tag{5}$$

where summation is over those irreducible representations $T_{\mathbf{m}}$, $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$, of the subgroup $U(n-1)$ which contain the irreducible representation $T_{\mathbf{r}}$ of the subgroup $U(n-2)$, that is, such that

$$m_1 \geq r_1 \geq m_2 \geq r_2 \geq \ldots \geq m_{n-2} \geq r_{n-2} \geq m_{n-1}. \tag{6}$$

Moreover, the representations $T_{\mathbf{m}}$ of $U(n-1)$ are contained in the decomposition (5) with the unit multiplicity.

According to formula (5) we have

$$\mathfrak{D}_{\mathbf{r}} = \sum_{\mathbf{m}} \oplus \mathfrak{H}_{\mathbf{m}}, \tag{7}$$

where $\mathfrak{H}_{\mathbf{m}}$ are the subspaces in which the irreducible representations $T_{\mathbf{m}}$ of $U(n-1)$ are realized. We choose the Gel'fand-Tsetlin basis $\{\mathbf{v}(M)\} \equiv \{\mathbf{v}(\mathbf{m} \equiv \mathbf{m}_{n-1}, \mathbf{m}_{n-2}, \mathbf{m}_{n-3}, \ldots)\}$ in every of the subspaces $\mathfrak{H}_{\mathbf{m}}$. Uniting the bases $\{\mathbf{v}(M)\}$ of the spaces $\mathfrak{H}_{\mathbf{m}}$ we receive the basis of the space $\mathfrak{D}_{\mathbf{r}}$.

It is easy to show that

$$T^{\mathbf{r}d\sigma}(d_n(e^{i\varphi}))\mathbf{f} = e^{i(r-d-m)\varphi}\mathbf{f}, \quad \mathbf{f} \in \mathfrak{H}_{\mathbf{m}}, \tag{8}$$

where $r = r_1 + \ldots + r_{n-2}$, $m = m_1 + \ldots + m_{n-1}$.

Using the reasonings of Section 11.2.4, it is easy to prove that the representations $T^{\mathbf{r}d\sigma}$ with $\sigma = i\rho - n + 1$, $\rho \in \mathbf{R}$, are unitary with respect to the scalar product (3). They form the *principal unitary series* of representations of the group $U(n-1, 1)$.

The representations $T^{\mathbf{r}d\sigma}$ of the group $U(n-1,1)$ determine the corresponding representations of the Lie algebra $\mathfrak{u}(n-1,1)$. We denote them by the same symbols. The representation $T^{\mathbf{r}d\sigma}$ of $\mathfrak{u}(n-1,1)$ can be continued to the representation $T^{\mathbf{r}d\sigma}$ of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ (which is the complexification of $\mathfrak{u}(n-1,1)$). It is proved (see reference [24] of the first volume) that the operators $E^{\mathbf{r}d\sigma}_{n-1,n} \equiv T^{\mathbf{r}d\sigma}(E_{n-1,n})$ and $E^{\mathbf{r}d\sigma}_{n,n-1} \equiv T^{\mathbf{r}d\sigma}(E_{n,n-1})$ corresponding to the matrices $E_{n-1,n}$ and $E_{n,n-1}$ from the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ act onto the basis elements $\mathbf{v}(M)$ by the formulas

$$E^{\mathbf{r}d\sigma}_{n,n-1}\mathbf{v}(\mathbf{m},\mathbf{m}_{n-2},\dots) = \sum_{i=1}^{n-1}\left(-m_i - \frac{\sigma+d}{2} - n + i + 1\right) \times \omega_i(\mathbf{m}^{-i},\mathbf{r},\mathbf{m}_{n-2})\mathbf{v}(\mathbf{m}^{-i},\mathbf{m}_{n-2},\dots), \tag{9}$$

$$E^{\mathbf{r}d\sigma}_{n-1,n}\mathbf{v}(\mathbf{m},\mathbf{m}_{n-2},\dots) = \sum_{i=1}^{n-1}\left(m_i - \frac{\sigma-d}{2} - i + 1\right) \times \omega_i(\mathbf{m},\mathbf{r},\mathbf{m}_{n-2})\mathbf{v}(\mathbf{m}^{+i},\mathbf{m}_{n-2},\dots), \tag{10}$$

where $\mathbf{m}^{\pm i}$ is obtained from $\mathbf{m}$ by replacing m_i by $m_i \pm 1$, and

$$\omega_i(\mathbf{m},\mathbf{r},\mathbf{p}) = \left(\frac{\dim T^{n-1,\mathbf{m}^{+i}}}{\dim T^{n-1,\mathbf{m}}}\right)^{1/2} \left(\begin{matrix}\mathbf{m} & (1,0)\\ \mathbf{p} & (0,0)\end{matrix}\middle|\begin{matrix}\mathbf{m}^{+i}\\ \mathbf{p}\end{matrix}\right)\left(\begin{matrix}\mathbf{m} & (1,0)\\ \mathbf{r} & (0,0)\end{matrix}\middle|\begin{matrix}\mathbf{m}^{+i}\\ \mathbf{r}\end{matrix}\right)$$
$$= \left|\frac{\prod_{s=1}^{n-2}(m_i - p_s - i + s + 1)(m_i - r_s - i + s + 1)}{\prod_{k=1, k\neq i}^{n-1}(m_i - m_k - i + k + 1)(m_i - m_k - i + k)}\right|^{1/2}$$

(here we used expressions for the $U(n-2)$-scalar factors of CGC's of the group $U(n-1)$).

18.5.2. Matrix elements of the representations of $U(n-1,1)$. Repeating the reasonings of Section 18.2.5 we evaluate matrix elements of the operators $(E^{\mathbf{r}d\sigma}_{n-1,n})^k$ and $(E^{\mathbf{r}d\sigma}_{n,n-1})^k$. Using these matrix elements we derive for $m_i \geq p_i$, $i = 1,2,\dots,n-1$, the formula

$$((\exp tE^{\mathbf{r}d\sigma}_{n-1,n})\mathbf{v}(M'),\mathbf{v}(M)) = \frac{t^{m-p}}{(m-p)!}((E^{\mathbf{r}d\sigma}_{n-1,n})^{m-p}\mathbf{v}(M'),\mathbf{v}(M))$$
$$= t^{m-p}\prod_{i\leq s}\frac{(r_{i-1}-p_s-i+s)!}{(r_{i-1}-m_s-i+s)!}\prod_{s=1}^{n-1}\frac{\Gamma\left(\frac{\sigma-d}{2}-p_s+s\right)}{\Gamma\left(\frac{\sigma-d}{2}-m_s+s\right)}\frac{\prod_{i<s}(p_i-m_s-i+s-1)!}{\prod_{i\leq s}(m_i-p_s-i+s)!}$$
$$\times\prod_{i<s}\frac{(j_i-p_s-i+s-1)!}{(j_i-m_s-i+s-1)!}\prod_{i<s}(m_i-m_s-i+s), \tag{1}$$

and for $p_i \geq m_i$, $i = 1, 2, \dots, n-1$, the formula

$$\left((\exp tE^{\mathbf{r}d\sigma}_{n,n-1})\mathbf{v}(M'), \mathbf{v}(M)\right) = t^{p-m} \prod_{i<s} \frac{(p_i - r_{s-1} - i + s - 1)!}{(m_i - r_{s-1} - i + s - 1)!}$$
$$\times \prod_{i<s} (m_i - m_s - i + s) \prod_{i=1}^{n-1} \frac{\Gamma\left(p_i + \frac{\sigma+d}{2} - i + n\right)}{\Gamma\left(m_i + \frac{\sigma+d}{2} - i + n\right)} \prod_{i\leq s} \frac{(p_i - j_s - i + s)!}{(m_i - j_s - i + s)!}$$
$$\times \frac{\prod\limits_{i<s} (m_i - p_s - i + s - 1)!}{\prod\limits_{i\leq s} (p_i - m_s - i + s)!}, \tag{2}$$

where $M' = (\mathbf{p}, \mathbf{j}, \dots)$, $M = (\mathbf{m}, \mathbf{j}, \dots)$, $m = \sum\limits_{i=1}^{n-1} m_i$, $p = \sum\limits_{i=1}^{n-1} p_i$.

In the same way as in the case of the class 1 representations of the group $U(n-1, 1)$ (see Section 11.6.8), we derive with the help of formulas (1) and (2) two expressions for the matrix elements

$$t^{\mathbf{r}d\sigma}_{\mathbf{mpj}}(t) = (T^{\mathbf{r}d\sigma}(g'_{n-1}(t))\mathbf{v}(\mathbf{p}, \mathbf{j}, \dots), \mathbf{v}(\mathbf{m}, \mathbf{j}, \dots)). \tag{3}$$

The first expression is of the form

$$t^{\mathbf{r}d\sigma}_{\mathbf{mpj}}(t) = \mu(\mathbf{r}, \mathbf{p}, \mathbf{j})\mu(\mathbf{r}, \mathbf{m}, \mathbf{j})(\cosh t)^{d-r-j}(\tanh t)^{-m-p}$$
$$\times \sum_{\mathbf{k}} \prod_{i=1}^{n-1} \frac{\Gamma\left(k_i + \frac{\sigma+d}{2} + n - i\right)\Gamma\left(\frac{\sigma-d}{2} - p_i + i\right)}{\Gamma\left(m_i + \frac{\sigma+d}{2} + n - i\right)\Gamma\left(\frac{\sigma-d}{2} - k_i + i\right)}$$
$$\times \frac{\prod\limits_{i\leq s} (k_i - r_s - i + s)!(k_i - j_s - i + s)!}{\prod\limits_{i\leq s} (k_i - m_s - i + s)!(k_i - p_s - i + s)!} \tag{4}$$
$$\times \frac{\prod\limits_{i<s} (m_i - k_s - i + s - 1)!(p_i - k_s - i + s - 1)!(k_i - k_s - i + s)}{\prod\limits_{i<s} (r_i - k_s - i + s - 1)!(j_i - k_s - i + s - 1)!} (\sinh t)^{2k},$$

where summation is over those $\mathbf{k} = (k_1, \dots, k_{n-1})$ for which $k_1 \geq \max(m_1, p_1)$, $\min(r_{i-1}, j_{i-1}) \geq k_i \geq \max(m_i, p_i)$, $i = 2, 3, \dots, n-1$, $\mu(\mathbf{r}, \mathbf{p}, \mathbf{j})$ is given by the formula

$$\mu(\mathbf{r}, \mathbf{p}, \mathbf{j}) = \left(\frac{\prod\limits_{i<s} (j_i - p_s - i + s - 1)!(r_i - p_s - i + s - 1)!(p_i - p_s - i + s)!}{\prod\limits_{i\leq s} (p_i - j_s - i + s)!(p_i - r_s - i + s)!} \right)^{1/2}, \tag{5}$$

$k = k_1 + \ldots + k_{n-1}$ and in the similar way r, m, p, j are defined. The second expression for the matrix elements is of the form

$$t^{\mathbf{r}d\sigma}_{\mathbf{mpj}}(t) = \nu(\mathbf{r},\mathbf{p},\mathbf{j})\nu(\mathbf{r},\mathbf{m},\mathbf{j})(\cosh t)^{r-d+j}(\tanh t)^{m+p}$$
$$\times \sum_{\mathbf{k}} \prod_{i=1}^{n-1} \frac{\Gamma\left(\frac{\sigma-d}{2} - k_i + i\right)\Gamma\left(\frac{\sigma+d}{2} + p_i - i + n\right)}{\Gamma\left(\frac{\sigma-d}{2} - m_i + i\right)\Gamma\left(\frac{\sigma+d}{2} + k_i - i + m\right)}$$
$$\times \frac{\prod\limits_{i<s}(r_i - k_s - i + s - 1)!(j_i - k_s - i + s - 1)!}{\prod\limits_{i\le s}(k_i - r_s - i + s)!(k_i - j_s - i + s)!} \tag{6}$$
$$\times \frac{\prod\limits_{i<s}(k_i - p_s - i + s - 1)!(k_i - m_s - i + s - 1)!(k_i - k_s - i + s)}{\prod\limits_{i\le s}(p_i - k_s - i + s)!(m_i - k_s - i + s)!}(\sinh t)^{-2k},$$

where summation is over those $\mathbf{k} = (k_1, \ldots, k_{n-1})$ for which $\min(m_i, p_i) \ge k_i \ge \max(k_i, j_i)$, $i = 1, 2, \ldots, n-2$, $\min(m_{n-1}, p_{n-1}) \ge k_{n-1}$ and

$$\nu(\mathbf{r},\mathbf{p},\mathbf{j}) = \left(\frac{\prod\limits_{i<s}(p_i - p_s - i + s)! \prod\limits_{i\le s}(p_i - j_s - i + s)!(p_i - r_s - i + s)!}{\prod\limits_{i<s}(j_i - p_s - i + s - 1)!(r_i - p_s - i + s - 1)!}\right)^{1/2}. \tag{7}$$

Expressions (4) and (6) for $t^{\mathbf{r}d\sigma}_{\mathbf{mpj}}(t)$ can be expressed in terms of the hypergeometric function ${}_{3n-4}F_{3n-5}$.

The formulas

$$t^{\mathbf{r}d\sigma}_{\mathbf{mpj}} = (\cosh t)^{r-d+j}(\tanh t)^{m+p}\mathfrak{F}_1(\mathbf{m},\mathbf{p};\mathbf{r},d,\mathbf{j};\sinh^{-2} t), \tag{8}$$

$$t^{\mathbf{r}d\sigma}_{\mathbf{mpj}} = (\cosh t)^{d-r-j}(\tanh t)^{-m-p}\mathfrak{F}_2(\mathbf{m},\mathbf{p};\mathbf{r},d,\mathbf{j};\sinh^{2} t) \tag{9}$$

define the generalized Jacobi functions $\mathfrak{F}_1$ and $\mathfrak{F}_2$. For $\sigma \in \mathbb{Z}_+$ the functions $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are reduced to the generalized Jacobi polynomials F_1 and F_2, introduced in Section 18.4.6. Using properties of the group $U(n-1,1)$, recurrence relations, addition and product theorems and other formulas are derived for the generalized Jacobi functions.

18.5.3. Representations of the group $IU(n-1)$. Representations of the group $IU(n-1)$ are constructed in the space $\mathfrak{D}_{\mathbf{r}}$ of vector functions $\mathbf{f}(\boldsymbol{\xi})$ on S_C^{n-2}, introduced in Section 18.5.1. Let $\mathbf{r}$ and $T_{\mathbf{r}}$ be such as in Section 18.5.1 and $\rho \in \mathbb{C}$. Then the operators

$$T^{\mathbf{r}\rho}(g(u,\mathbf{x}))\mathbf{f}(\boldsymbol{\xi}) = e^{\rho(\mathbf{x},\boldsymbol{\xi})}\mathbf{f}(u^{-1}\boldsymbol{\xi}), \tag{1}$$

where $(\mathbf{x},\boldsymbol{\xi}) = \sum\limits_{i=1}^{n-1}(\mathrm{Re}\,\mathbf{x}\;\mathrm{Re}\,\boldsymbol{\xi} + \mathrm{Im}\,\mathbf{x}\;\mathrm{Im}\,\boldsymbol{\xi})$ and

$$g(u,\mathbf{x}) \equiv \begin{pmatrix} u & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix}, \quad u \in U(n-1), \; \mathbf{x} \in \mathbb{C}^{n-1},$$

define the representation $T^{\mathbf{r}\rho}$ of $IU(n-1)$ in the space $\mathfrak{D}_{\mathbf{r}}$. If $\rho = iR$, $R \in \mathbf{R}$, then the operators (1) are unitary with respect to scalar product (3) of Section 18.5.1. Therefore, the representations $T^{\mathbf{r},iR}$, $R \in \mathbf{R}$, are unitary.

Let F_{n-1} and F^{n-1} be the elements of the Lie algebra $\mathrm{iu}(n-1)$ from Section 11.6.5. The operators $F^{\mathbf{r}\rho}_{n-1}$ and $F^{n-1,\mathbf{r}\rho}$ correspond to these elements in the representation $T^{\mathbf{r}\rho}$ which are given by the formulas

$$F^{\mathbf{r}\rho}_{n-1}\mathbf{v}(\mathbf{m},\mathbf{j},\dots) = \rho\sum_{i=1}^{n-1}\omega_i(\mathbf{m},\mathbf{r},\mathbf{j})\mathbf{v}(\mathbf{m}^{+i},\mathbf{j},\dots), \tag{2}$$

$$F^{n-1,\mathbf{r}\rho}\mathbf{v}(\mathbf{m},\mathbf{j},\dots) = \rho\sum_{i=1}^{n-1}\omega_i(\mathbf{m}^{-i},\mathbf{r},\mathbf{j})\mathbf{v}(\mathbf{m}^{-i},\mathbf{j},\dots), \tag{3}$$

where ω_i and $\mathbf{m}^{\pm i}$ are such as in Section 18.5.1.

18.5.4. Matrix elements of the representations of $IU(n-1)$. In the same way as in Section 18.2.5 we evaluate matrix elements of the operators $\left(F^{\mathbf{r}\rho}_{n-1}\right)^k$ and $\left(F^{n-1,\mathbf{r}\rho}\right)^k$. Using this results, it is possible to derive that for $m_i \ge p_i$, $i = 1,2,\dots,n-1$, we have

$$\left(\left(\exp\frac{r}{2}F^{\mathbf{r}\rho}_{n-1}\right)\mathbf{v}(M'),\mathbf{v}(M)\right) = \left(\frac{r\rho}{2}\right)^{m-p}\prod_{i\le s}\frac{(r_{i-1}-p_s+s-i-1)!}{(r_{i-1}-m_s+s-i-1)!}$$
$$\times\prod_{i<s}\frac{(j_i-p_s+s-i-1)!}{(j_i-m_s+s-i-1)!}\,\frac{\prod\limits_{i<s}(p_i-m_s+s-i-1)!(m_i-m_s+s-i)}{\prod\limits_{i\le s}(m_i-p_s+s-i)!}, \tag{1}$$

where $M' = (\mathbf{p},\mathbf{j},\dots)$, $M = (\mathbf{m},\mathbf{j},\dots)$, and for $p_i \ge m_i$, $i = 1,2,\dots,n-2$, we have

$$\left(\left(\exp\frac{r}{2}F^{n-1,\mathbf{r}\rho}\right)\mathbf{v}(M'),\mathbf{v}(M)\right) = \left(\frac{r\rho}{2}\right)^{p-m}\prod_{i<s}\frac{(p_i-r_{s-1}-i+s)!}{(m_i-r_{s-1}-i+s)!}$$
$$\times\prod_{i\le s}\frac{(p_i-j_s-i+s)!}{(m_i-j_s-i+s)!}\,\frac{\prod\limits_{i<s}(m_i-p_s-i+s-1)!(m_i-m_s-i+s)}{\prod\limits_{i\le s}(p_i-m_s-i+s)!}. \tag{2}$$

Repeating the reasonings of Section 11.6.9, with the help of formulas (1) and (2) we find two expressions for the matrix elements

$$t^{\mathbf{r}\rho}_{\mathbf{mpj}}(r) = (T^{\mathbf{r}\rho}(g_r)\mathbf{v}(\mathbf{p},\mathbf{j},\dots),\mathbf{v}(\mathbf{m},\mathbf{j},\dots)) \tag{3}$$

of the operators $T^{\mathbf{r}\rho}(g_r)$. The first one is of the form

$$t^{\mathbf{r}\rho}_{\mathbf{mpj}}(r) = \nu(\mathbf{r},\mathbf{p},\mathbf{j})\nu(\mathbf{r},\mathbf{m},\mathbf{j})\left(\frac{\rho r}{2}\right)^{p+m}\sum_{\mathbf{q}}\prod_{i<s}(q_i - q_s - i + s)$$
$$\times \frac{\prod\limits_{i<s}(r_i - q_s - i + s - 1)!(j_i - q_s - i + s - 1)!}{\prod\limits_{i\le s}(q_i - r_s - i + s)!(q_i - j_s - i + s)!}$$
$$\times \frac{\prod\limits_{i<s}(q_i - p_s - i + s - 1)!(q_i - m_s - i + s - 1)!}{\prod\limits_{i\le s}(p_i - q_s - i + s)!(m_i - q_s - i + s)!}\left(\frac{\rho r}{2}\right)^{-2q}, \qquad (4)$$

where the sum and $\nu(\mathbf{r},\mathbf{p},\mathbf{j})$ are such as in formula (6) of Section 18.5.2. The second expression is

$$t^{\mathbf{r}\rho}_{\mathbf{mpj}}(r) = \mu(\mathbf{r},\mathbf{p},\mathbf{j})\mu(\mathbf{r},\mathbf{m},\mathbf{j})\left(\frac{\rho r}{2}\right)^{-p-m}\sum_{\mathbf{q}}\prod_{i<s}(q_i - q_s - i + s)$$
$$\times \frac{\prod\limits_{i\le s}(q_i - r_s - i + s)!(q_i - j_s - i + s)!}{\prod\limits_{i\le s}(q_i - m_s - i + s)!(q_i - p_s - i + s)!}$$
$$\times \frac{\prod\limits_{i<s}(m_i - q_s - i + s - 1)!(p_i - q_s - i + s - 1)!}{\prod\limits_{i<s}(r_i - q_s - i + s - 1)!(j_i - q_s - i + s - 1)!}\left(\frac{\rho r}{2}\right)^{2q}, \qquad (5)$$

where the sum and $\mu(\mathbf{r},\mathbf{p},\mathbf{j})$ are such as in formula (4) of Section 18.5.2.

Matrix elements (4) and (5) can be expressed in terms of the hypergeometric function ${}_{3n-6}F_{3n-5}$.

The vector function $\mathbf{t}^{n-1,\mathbf{m}}_M(\boldsymbol{\xi}) \equiv \{t^{n-1,\mathbf{m}}_{MM'}(\boldsymbol{\xi})\}$ (the components are labelled by the Gel'fand-Tsetlin patterns M'), where $t^{n-1,\mathbf{m}}_{MM'}$ are the matrix elements of the representation $T_{\mathbf{m}}$ of $U(n-1)$, can be taken as the basis elements $\mathbf{v}(M)$ of the space $\mathfrak{D}_{\mathbf{r}}$. Repeating the reasonings of Section 11.6.9 we derive the integral representation

$$t^{\mathbf{r}\rho}_{\mathbf{mpj}}(r) = 2(n-2)\frac{(\dim T^{n-1,\mathbf{m}}\dim T^{n-1,\mathbf{p}})^{1/2}}{\dim T^{n-2,\mathbf{j}}}i^{m-p}\sum_{\mathbf{q}}(\dim T^{n-3,\mathbf{q}})$$
$$\times\int\limits_0^{\pi/2} J_{m-p}(-i\rho r\cos\theta)t^{n-1,\mathbf{m}}_{\mathbf{jrq}}(\theta)t^{n-1,\mathbf{p}}_{\mathbf{jrq}}(\theta)\sin^{2n-5}\theta\cos\theta d\theta \qquad (6)$$

for the matrix elements of the representations $T^{\mathbf{r}\rho}$, where $t^{n-1,\mathbf{p}}_{\mathbf{jkq}}(\theta)$ are matrix elements (1) of Section 18.4.1 for the representation $T_{\mathbf{m}}$ of the group $U(n-1)$.

The formulas

$$t^{\mathbf{r}\rho}_{\mathbf{mpj}}(r) = \left(\frac{\rho r}{2}\right)^{p+m} J_1(\mathbf{m},\mathbf{p};\mathbf{r},\mathbf{j};(\rho r/2)^{-2}), \tag{7}$$

$$t^{\mathbf{r}\rho}_{\mathbf{mpj}}(r) = \left(\frac{\rho r}{2}\right)^{-p-m} J_2(\mathbf{m},\mathbf{p};\mathbf{r},\mathbf{j};(\rho r/2)^{2}), \tag{8}$$

define the generalized Bessel functions. Using properties of the group $IU(n-1)$, recurrence relations, addition and product theorems and other formulas are derived for these functions. It is possible to show that the generalized Bessel functions from formulas (7) and (8) are the limiting cases of the corresponding generalized Jacobi polynomials and functions.

18.6. Representations of the Groups $SO(n)$, $SO_0(n-1,1)$, $ISO(n-1)$ and Special Functions with Matrix Indices[2]

18.6.1. Introduction. Now we discuss some problems related to studying matrix elements of representations of the groups of motions on the real spaces of a constant curvature, namely, of the groups $SO(n)$, $SO_0(n-1,1)$, $ISO(n-1)$. If $n=3$, then these matrix elements are well studied and are reduced to the classical Jacobi polynomials and functions and to the classical Bessel functions, respectively. When we go over from matrix elements for the groups of low dimensions to those for the groups of high dimension, then matrix elements become essentially complicated. Moreover, the difficulties are of principal nature. Obviously, for $n>3$ matrix elements of the operators, corresponding to the elements $g_{n-1}(\theta)$ $g'_{n-1}(\theta)$ and g_r (see Chapter 9), do not satisfy linear homogeneous differential equations of the second order. In this reason these matrix elements are not reduced to classical special functions of the hypergeometric type.

Below we propose a new approach to matrix elements of irreducible unitary representations, based on consideration of sets of matrix elements of equivalent and non-equivalent representations. These sets of matrix elements constitute square matrices which are called *matroms*. Matroms can be described as special functions with matrix indices. This means that with every matrom we associate some scalar special function of the hypergeometric type depending on one or two parameters (indices). Substituting constant matrices instead of these parameters, we obtain our matroms. These matrices, which are used as indices, are called *generators of matroms*.

If $n=3$, then this construction does not give new results. In this case matroms coincide with usual matrix elements of representations. When $n>3$, then evaluation of entries of matroms (which are usual matrix elements of representations) is reduced to a spectral decomposition of their generators.

Utilization of matroms instead of matrix elements allows us to avoid (in a certain sense) combersome and hardly visible formulas for individual matrix elements.

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[2] Sections 18.6 and 18.7 were written by A.V. Rozenblyum and L.V. Rozenblyum.

We can study matroms with the help of methods of the classical theory of special functions and of the theory of functions of ordered operators. Besides, the notion of matrom clarifies (to some extent) the origin of complicated expressions for matrix elements of representations for the groups of high dimension. These complications appear because of spectral analysis of generators of matroms. This analysis leads to decomposition of matrix elements in terms of classical special functions.

18.6.2. Representations of the groups $SO_0(n-1,1)$ and $ISO(n-1)$. In Chapter 9 we constructed irreducible representations of the groups $SO(n)$, $SO_0(n-1,1)$ and $ISO(n-1)$ which are of class 1 with respect to the subgroup $SO(n-1)$. In Sections 18.1.1 and 18.1.2 we gave all irreducible representations of the group $SO(n)$ and formulas for their infinitesimal operators. Now we construct representations of the groups $SO_0(n-1,1)$ and $ISO(n-1)$. Let $\mathbf{r} = (r_1, \dots, r_{n-2})$ be highest weight of the irreducible finite dimensional representation $T_\mathbf{r}$ of the subgroup $SO(n-2)$ acting in the space $\mathcal{V}$. We fix complex number σ. Let $\mathfrak{D}_{\mathbf{r}\sigma}$ be the space of infinitely differentiable vector functions $\mathbf{F}(\mathbf{x})$ on the cone $C^{n-1} \equiv \{\mathbf{x} \in \mathbf{R}^n \mid [\mathbf{x},\mathbf{x}] = 0, x_n > 0\}$ with values in the space $\mathcal{V}$, which possess homogeneity and covariance properties[3]

$$\mathbf{F}(a\mathbf{x}) = a^\sigma \mathbf{F}(\mathbf{x}), \quad a > 0, \tag{1}$$

$$\mathbf{F}(\mathbf{x}k) = T_\mathbf{r}(k^{-1})\mathbf{F}(\mathbf{x}), \quad k \in SO(n-2).$$

The operators

$$T^{\mathbf{r}\sigma}(g)\mathbf{F}(\mathbf{x}) = \mathbf{F}(g^{-1}\mathbf{x}), \quad g \in SO_0(n-1,1), \tag{2}$$

leave the space $\mathfrak{D}_{\mathbf{r}\sigma}$ invariant. Therefore, the correspondence $g \to T^{\mathbf{r}\sigma}(g)$ is a representation of the group $SO_0(n-1,1)$. They form the *principal nonunitary series* of representations of $SO_0(n-1,1)$. For $\mathbf{r} = \mathbf{0}$ they coincide with the corresponding representations of $SO_0(n-1,1)$ from Chapter 9.

As in Section 9.2.1, condition (1) allows us to consider the vector functions $\mathbf{f}(\boldsymbol{\xi})$ on S^{n-2}:

$$\mathbf{f}(\xi_1, \dots, \xi_{n-1}) = \mathbf{F}(\xi_1, \dots, \xi_{n-1}, 1)$$

instead of the vector functions $\mathbf{F}(\mathbf{x})$ on C^{n-1}. We introduce the scalar product

$$(\mathbf{f}_1, \mathbf{f}_2) = \int_{S^{n-2}} \langle \mathbf{f}_1(\boldsymbol{\xi}), \mathbf{f}_2(\boldsymbol{\xi}) \rangle d\boldsymbol{\xi} \tag{3}$$

in the space $\mathfrak{D}_\mathbf{r}$ of vector functions $\mathbf{f}(\boldsymbol{\xi})$, where $\langle \cdot , \cdot \rangle$ denotes the scalar product in $\mathcal{V}$. The operators $T^{\mathbf{r}\sigma}(g'_{n-1}(t))$ act in the space $\mathfrak{D}_\mathbf{r}$ by the formula (16) of Section 9.2.1. The operators $T^{\mathbf{r}\sigma}(k)$, $k \in SO(n-1)$ act in $\mathfrak{D}_\mathbf{r}$ as the left shifts by k^{-1}. It follows from here that

$$T^{\mathbf{r}\sigma}\downarrow_{SO(n-1)} = \sum_\mathbf{m} \oplus T_\mathbf{m}, \tag{4}$$

[3] In fact, the covariance property means that functions $\mathbf{F}$ can be continued onto the group $SO_0(n-1,1)$.

where summation is over highest weights $\mathbf{m} = (m_1, m_2, \ldots, m_s)$, $s = \left[\frac{n-1}{2}\right]$, of irreducible representations of $SO(n-1)$ containing the irreducible representation $T_{\mathbf{r}}$ of the subgroup $SO(n-2)$, that is, for which the corresponding betweenness conditions are fulfilled (see Section 18.1.1). The representations $T_{\mathbf{m}}$ of $SO(n-1)$ are contained in decomposition (4) with the unit multiplicity. The union of the Gel'fand-Tsetlin bases $\{\mathbf{v}(M)\}$ of the carrier spaces of the representations $T_{\mathbf{m}}$ from (4) is an orthonormal basis of the space $\mathfrak{D}_{\mathbf{r}}$ with scalar product (3). This basis will be called the *Gel'fand-Tsetlin basis* of the space $\mathfrak{D}_{\mathbf{r}}$.

The infinitesimal operator $J^{\mathbf{r}\sigma}_{n,n-1} \equiv T^{\mathbf{r}\sigma}(J_{n,n-1})$, $J_{n,n-1} = E_{n,n-1} + E_{n-1,n}$ acts onto the Gel'fand-Tsetlin basis by the formula

$$J^{\mathbf{r}\sigma}_{2k+1,2k}\mathbf{v}(M) = \sum_{s=1}^{k}\left(\sigma + \frac{n-1}{2} + \ell_{s,2k}\right)\omega_s(\mathbf{r}, \mathbf{m}_{2k}, \mathbf{m}_{2k-1})\mathbf{v}(M^{+s}_{2k})$$
$$+ \sum_{s=1}^{k}\left(\sigma + \frac{n-1}{2} - \ell_{s,2k}\right)\omega_s(\mathbf{r}, \mathbf{m}^{-s}_{2k}, \mathbf{m}_{2k-1})\mathbf{v}(M^{-s}_{2k}) \qquad (5)$$

if $n = 2k+1$, where $M \equiv (\mathbf{m}_{2k}, \mathbf{m}_{2k-1}, \ldots)$, $M^{\pm s}_n$ has the same sense as in Section 18.1.1, and

$$\omega_s(\mathbf{r}, \mathbf{m}_{2k}, \mathbf{m}_{2k-1})$$
$$= \left|\frac{\prod_{r=1}^{k-1}(\ell_{r,2k-1} + \ell_{s,2k})(\ell_{r,2k-1} - \ell_{s,2k} - 1)(\ell_r + \ell_{s,2k})(\ell_r - \ell_{s,2k} - 1)}{\prod_{\substack{r=1\\ r\neq s}}^{k}(\ell^2_{r,2k} - \ell^2_{s,2k})(\ell^2_{r,2k} - (\ell_{s,2k}+1)^2)}\right|^{1/2}, \qquad (6)$$

and by the formula

$$J^{\mathbf{r}\sigma}_{2k+2,2k+1}\mathbf{v}(M) = \sum_{s=1}^{k}\left(\sigma + \frac{n-2}{2} + \ell_{s,2k+1}\right)\omega_s(\mathbf{r}, \mathbf{m}_{2k+1}, \mathbf{m}_{2k})\mathbf{v}(M^{+s}_{2k+1})$$
$$+ \sum_{s=1}^{k}\left(\sigma + \frac{n-2}{2} - \ell_{s,2k+1} + 1\right)\omega_s(\mathbf{r}, \mathbf{m}^{-s}_{2k+1}, \mathbf{m}_{2k})\mathbf{v}(M^{-s}_{2k+1})$$
$$+ \left(\sigma + \frac{n-2}{2}\right)\prod_{s=1}^{k}\frac{\ell_s\ell_{s,2k}}{\ell_{s,2k+1}(\ell_{s,2k+1}-1)}\mathbf{v}(M) \qquad (7)$$

if $n = 2k+2$, where

$$\omega_s(\mathbf{r}, \mathbf{m}_{2k+1}, \mathbf{m}_{2k})$$
$$= \left|\frac{\prod_{r=1}^{k}(\ell^2_r - \ell^2_{s,2k+1})(\ell^2_{r,2k} - \ell^2_{s,2k+1})}{\ell^2_{s,2k+1}(4\ell_{s,2k+1} - 1)\prod_{\substack{r=1\\ r\neq s}}^{k}(\ell^2_{r,2k+1} - \ell^2_{s,2k+1})((\ell_{r,2k+1}-1)^2 - \ell^2_{s,2k+1})}\right|^{1/2}. \qquad (8)$$

In (5)-(8) $\ell_i = r_i + [n/2] - i$, $i = 1, 2, \dots, [(n-1)/2]$ ($[a/2]$ is the integral part of the number $a/2$) and $\ell_{in} = m_{in} + [(n+1)/2] - i$, $i = 1, 2, \dots, [n/2]$.

The representations $T^{\mathbf{r}\sigma}$, $\sigma = i\rho - \frac{n-2}{2}$, $\rho \in \mathbf{R}$, are unitary. They constitute the *principal unitary series* of representations of $SO_0(n-1,1)$. The classification of all unitary irreducible representations of this group is given in reference [24] of the first volume.

To every representation $T^{\mathbf{r}\sigma}$ of the group $SO_0(n-1,1)$ the representation $T^{\mathbf{r}\sigma}$ of the group $ISO(n-1)$ corresponds which acts in the same space $\mathfrak{D}_{\mathbf{r}}$ and is given by the formula

$$T^{\mathbf{r}\sigma}(g(u,\mathbf{x}))\mathbf{f}(\boldsymbol{\xi}) = e^{\sigma(\mathbf{x},\boldsymbol{\xi})}\mathbf{f}(u^{-1}\boldsymbol{\xi}),$$

where $(\mathbf{x},\boldsymbol{\xi}) = \sum_{i=1}^{n-1} x_i\xi_i$ and elements $g(u,\mathbf{x})$ of $ISO(n-1)$ are determined as

$$g(u,\mathbf{x}) = \begin{pmatrix} u & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix}, \quad u \in SO(n-1), \quad \mathbf{x} \in \mathbf{R}^{n-1}.$$

The one-parameter subgroup $g_r = g(e,\mathbf{r})$, $\mathbf{r} = (0,\dots,0,r)$, has the tangent matrix J_{n-1}. The operator $J^{\mathbf{r}\sigma}_{n-1} = T^{\mathbf{r}\sigma}(J_{n-1})$ acts onto basis elements $\mathbf{v}(M)$ as

$$\begin{aligned} J^{\mathbf{r}\sigma}_{2k}\mathbf{v}(M) &= \sigma\sum_{s=1}^{k}\omega_s(\mathbf{r},\mathbf{m}_{n-1},\mathbf{m}_{n-2})\mathbf{v}(M^{+s}_{2k}) \\ &+ \sigma\sum_{s=1}^{k}\omega_s(\mathbf{r},\mathbf{m}^{-s}_{n-1},\mathbf{m}_{n-2})\mathbf{v}(M^{-s}_{2k}) \end{aligned} \tag{9}$$

if $n = 2k+1$ and as

$$\begin{aligned} J^{\mathbf{r}\sigma}_{2k+1}\mathbf{v}(M) &= \sigma\sum_{s=1}^{k}\omega_s(\mathbf{r},\mathbf{m}_{n-1},\mathbf{m}_{n-2})\mathbf{v}(M^{+s}_{2k+1}) \\ &+ \sigma\sum_{s=1}^{k}\omega_s(\mathbf{r},\mathbf{m}^{-s}_{n-1},\mathbf{m}_{n-2})\mathbf{v}(M^{-s}_{2k+1}) \\ &+ \sigma\prod_{r=1}^{k}\frac{\ell_r\ell_{r,2k}}{\ell_{r,2k+1}(\ell_{r,2k+1}-1)}\mathbf{v}(M) \end{aligned} \tag{10}$$

if $n = 2k+2$, where $\omega_s(\mathbf{r},\mathbf{m}_{n-1},\mathbf{m}_{n-2})$ are given by formulas (6) and (8).

The representations $T^{\mathbf{r}\sigma}$, $\sigma = iR$, $R \in \mathbf{R}$, of $ISO(n-1)$ are unitary. They (except for the case when $\sigma = 0$) exhaust all infinite dimensional irreducible unitary representations of the group $ISO(n-1)$.

As in the case of the groups $U(n)$, $U(n-1,1)$ and $IU(n-1)$ (see Section 18.5), matrix elements of the operators $T^{\mathbf{m}}(g_{n-1}(\theta))$, $g_{n-1}(\theta) \in SO(n)$, $T^{\mathbf{r}\sigma}(g'_{n-1}(\theta))$, $g'_{n-1}(t) \in SO_0(n-1,1)$, $T^{\mathbf{r}\sigma}(g_r)$, $g_r \in ISO(n-1)$, with respect to the bases $\{\mathbf{v}(M)\}$ depend only on the corresponding highest weights of irreducible representations of the subgroups $SO(n-1)$ and $SO(n-2)$. We denote them by

$$t^{\mathbf{m}}_{\mathbf{kpj}}(g_{n-1}(\theta)), \quad t^{\mathbf{r}\sigma}_{\mathbf{kpj}}(g'_{n-1}(t)), \quad t^{\mathbf{r}\sigma}_{\mathbf{kpj}}(g_r), \tag{11}$$

respectively, where $\mathbf{k} = (k_1, k_2, \dots)$, $\mathbf{p} = (p_1, p_2, \dots)$ are highest weights of representations of $SO(n-1)$ and $\mathbf{j} = (j_1, j_2, \dots)$ is highest weight of representation of $SO(n-2)$.

In order to consider irreducible representations of the groups $SO(n)$, $SO_0(n-1,1)$ and $ISO(n-1)$ and their matrix elements simultaneously, we introduce new notations. The highest weights $\mathbf{m} = (m_{1n}, \dots, m_{kn})$, $k = [n/2]$, of irreducible representations of $SO(n)$ are written as $(\sigma, \mathbf{r})$, where

$$\sigma = m_{1n}, \quad \mathbf{r} = (r_1, r_2, \dots, r_s), \quad s = \left[\frac{n-1}{2}\right], \quad r_j = m_{j+1,n}. \tag{12}$$

Below we shall denote the representations $T^{\mathbf{r}\sigma}$ of the groups $SO_0(n-1,1)$ and $ISO(n-1)$ and the representations $T^{\mathbf{m}}$ of the group $SO(n)$ by the same symbol $T_{\mathbf{z}}$, where

$$\mathbf{z} = (a, \mathbf{y}), \quad \mathbf{y} = (y_1, y_2, \dots, y_s), \quad s = \left[\frac{n-1}{2}\right]; \quad a = \sigma + s, \quad y_k = r_k + s - k.$$

The matrix elements (11) will be denoted by $d^{a,\mathbf{y}}_{\mathbf{tt'x}}(\gamma) \equiv d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma)$, where $\gamma = t$ for $SO_0(n-1,1)$, $\gamma = r$ for $ISO(n-1)$, $\gamma = \theta$ for $SO(n)$ and $\mathbf{t} = (t_1, t_2, \dots)$, $\mathbf{t}' = (t'_1, t'_2, \dots)$, $\mathbf{x} = (x_1, x_2, \dots)$ are connected with $\mathbf{k}$, $\mathbf{p}$, $\mathbf{j}$ by the formulas

$$t_i = k_i + \left[\frac{n}{2}\right] - i, \quad t'_i = p_i + \left[\frac{n}{2}\right] - i, \quad i = 1, 2, \dots, \left[\frac{n-1}{2}\right], \tag{12'}$$

$$x_i = j_i + \left[\frac{n-1}{2}\right] - i, \quad i = 1, 2, \dots, \left[\frac{n-2}{2}\right]. \tag{12''}$$

According to formula (5) of Section 2.1.5 the matrix elements $d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma)$ satisfy the relation

$$\frac{d}{d\gamma} d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma) = \sum_{j=-q}^{q} B_j(\mathbf{z}, \mathbf{t}', \mathbf{x}) d^{\mathbf{z}}_{\mathbf{t}, A_j\mathbf{t}', \mathbf{x}}(\gamma), \tag{13}$$

where $B_j(\mathbf{z}, \mathbf{t}', \mathbf{x})$ are matrix elements of the corresponding infinitesimal operator from Section 18.1.2 or 18.6.2, $q = [n/2]$ and A_j is defined as

$$A_j\mathbf{t} \equiv A_j(t_1, t_2, \dots, t_p) = (t_1, \dots, t_j + \operatorname{sign} j, \dots, t_p), \quad p = \left[\frac{n-1}{2}\right].$$

The matrix elements $d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma)$, taken for the group $SO(n-1)$, will be denoted as $f^{\mathbf{t}}_{\mathbf{xx'm}}(\gamma)$, where $\mathbf{t}$, $\mathbf{x}$, $\mathbf{x}'$ are defined with the help of the corresponding highest weights by formulas (12) and $\mathbf{m}$ is determined in the same way. Instead of (13), for $f^{\mathbf{t}}_{\mathbf{xx'm}}(\gamma)$ we have the relation

$$\frac{d}{d\gamma} f^{\mathbf{t}}_{\mathbf{xx'm}}(\gamma) = \sum_{j=-p}^{p} J_j(\mathbf{t}, \mathbf{x}', \mathbf{m}) f^{\mathbf{t}}_{\mathbf{x}, A_j\mathbf{x}', \mathbf{m}}(\gamma), \tag{14}$$

where $J_j(\mathbf{t}, \mathbf{x}', \mathbf{m})$ are the matrix elements of the infinitesimal operator $I_{n-1,n-2}$ (see Section 18.1.2).

18.6.3. Matroms of representations. The groups $SO(n)$, $SO_0(n-1,1)$ and $ISO(n-1)$ will be denoted by the symbol G_n. Let us fix an irreducible representation $T_{\mathbf{z}} \equiv T_{(a,\mathbf{y})}$ of the group G_n and consider the matrix elements $d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma)$ of this representation corresponding to the one-parameter subgroup. The vectors

$$\mathbf{x} = (x_1, x_2, \dots, x_p), \quad \mathbf{y} = (y_1, y_2, \dots, y_p), \quad p = \left[\frac{n-2}{2}\right],$$

which are defined by the highest weights of the representations of the subgroup $SO(n-2)$ satisfy the betweenness conditions. These conditions are defined by the vectors $\mathbf{t}$ and $\mathbf{t}'$. The set of admissible values of $\mathbf{x}$ (and $\mathbf{y}$) consists of all integral points from the parallelepiped, which will be denoted by Ω. The vertices of the parallelepiped Ω are defined by the vectors $\mathbf{t}$ and $\mathbf{t}'$.

Now we consider the matrix elements $d^{(a,\mathbf{y})}_{\mathbf{tt'x}}(\gamma)$ for all admissible values of $\mathbf{x}$ and $\mathbf{y}$ and constitute the matrix $D^{a\mathbf{tt}'}(\gamma)$ with elements $d^{(a,\mathbf{y})}_{\mathbf{tt'x}}(\gamma)$:

$$(D^{a\mathbf{tt}'}(\gamma))_{\mathbf{xy}} = d^{(a,\mathbf{y})}_{\mathbf{tt'x}}(\gamma), \quad \mathbf{x}, \mathbf{y} \in \Omega. \tag{1}$$

The matrices $D^{a\mathbf{tt}'}(\gamma)$ are called *matroms* of the group G_n. It is easy to see that the elements of a fixed row of $D^{a\mathbf{tt}'}(\gamma)$ are matrix elements of non-equivalent representations of the group G_n. Elements of a fixed column, indexed by $\mathbf{y}$, are matrix elements of the representation $T_{(a,\mathbf{y})}$.

It is easy to evaluate the dimension of the matrom $D^{a\mathbf{tt}'}(\gamma)$. It is equal to $\prod_{k=1}^{p}(b_k - b'_{k+1} + 1)$, where

$$b_k = \min(t_k, t'_k) - 1, \quad b'_k = \max(t_k, t'_k) - 1, \quad k = 1, 2, \dots, p,$$

$$b'_{p+1} = -\min(t_p, t'_p) + 1$$

if $n = 2p+2$, and

$$b_k = \min(t_k, t'_k), \quad b'_k = \max(|t_k|, |t'_k|) + 1, \quad k = 1, 2, \dots, p+1,$$

if $n = 2p + 3$.

If $G_n = SO(n)$, then $g_{n-1}(\theta) = \exp(\theta I_{n,n-1})$, where $I_{n,n-1}$ is the infinitesimal generator of the one-parameter subgroup $\{g_{n-1}(\theta)\}$. For the representations $T_{\mathbf{z}}$ we have

$$T_{\mathbf{z}}(g_{n-1}(\theta)) = \exp(\theta T_{\mathbf{z}}(I_{n,n-1})).$$

Matrix elements of the operator $T_{\mathbf{z}}(I_{n,n-1})$ are given in Section 18.1.2. They are symmetrical with respect to the permutation of

$$(m_{2n}, m_{3n}, \ldots, m_{kn}) \quad \text{and} \quad (m_{1,n-2}, m_{2,n-2}, \ldots, m_{k-1,n-2}), \quad k = \left[\frac{n}{2}\right].$$

In this reason matrix elements of the operators $T_{\mathbf{z}}(g_{n-1}(\theta))$ possess this property, that is,

$$d^{(a,\mathbf{y})}_{\mathbf{tt}'\mathbf{x}}(\gamma) = d^{(a,\mathbf{x})}_{\mathbf{tt}'\mathbf{y}}(\gamma).$$

The formulas of Section 18.6.2 show that this relation is valid also for matrix elements of the representations of the groups $SO_0(n-1,1)$ and $ISO(n-1)$. Therefore, matroms $D^{a\mathbf{tt}'}(\gamma)$ of the group G_n are symmetrical matrices.

18.6.4. Differential relations of the first order. Let $d_{n-1}(\gamma)$ denote $g_{n-1}(\gamma)$ for $SO(n)$, $g'_{n-1}(\gamma)$ for $SO_0(n-1,1)$ and g_r, $r = \gamma$, for $ISO(n-1)$. We consider the element

$$h \equiv h(\varphi, \gamma, \psi) = g_{n-2}(\varphi) d_{n-1}(\gamma) g_{n-2}(\psi), \quad g_{n-2}(\varphi) \in SO(n-1). \tag{1}$$

We have

$$h(\varphi, \gamma, \psi) d_{n-1}(\mu) = h(\varphi', \gamma', \psi'), \tag{2}$$

where φ', γ', ψ' are functions of variables φ, γ, ψ, μ. If V is a space of functions $F(h(\varphi,\gamma,\psi)) \equiv F(\varphi,\gamma,\psi)$ on the set of elements (1), then we may consider the action of these elements on V by right shifts. If B_{n-1} is the corresponding infinitesimal operator associated with the one-parameter subgroup $\{d_{n-1}(\mu)\}$, then repeating the reasonings of Section 6.10.3 we obtain the differential form of this operator:

$$B_{n-1} = f_1 \frac{\partial}{\partial \gamma} + f_2 \frac{\partial}{\partial \psi} + f_3 \frac{\partial}{\partial \varphi}, \tag{3}$$

where

$$f_1 = \left| \frac{\partial \gamma'}{\partial \mu} \right|_{\mu=0}, \quad f_2 = \left| \frac{\partial \psi'}{\partial \mu} \right|_{\mu=0}, \quad f_3 = \left| \frac{\partial \varphi'}{\partial \mu} \right|_{\mu=0}.$$

Direct evaluations show that

$$\begin{array}{llll} f_1 = \cos\psi, & f_2 = -f_3 = -\sin\psi/\gamma & & \text{for } G_n = ISO(n-1), \\ f_1 = \cos\psi, & f_2 = -\tan^{-1}\gamma \sin\psi, & f_3 = \sin\psi/\sin\gamma & \text{for } G_n = SO(n), \\ f_1 = \cos\psi, & f_2 = -\tanh^{-1}\gamma \sin\psi, & f_3 = \sin\psi/\sinh\gamma & \text{for } G_n = SO_0(n-1,1). \end{array}$$

Now let $T_{\mathbf{z}} = T_{(a,\mathbf{y})}$ be the irreducible representation of the group G_n from Section 18.1.2 or 18.6.2. Then to the elements (1) there correspond the matrix elements

$$F^{\mathbf{z}}(\varphi,\gamma,\psi) \equiv \sum_{\mathbf{x}''} f^{\mathbf{t}}_{\mathbf{x}\mathbf{x}''\mathbf{m}}(\varphi) d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}''}(\gamma) f^{\mathbf{t}'}_{\mathbf{x}''\mathbf{x}'\mathbf{m}}(\psi) \tag{4}$$

of this representation with respect to the basis $\{\mathbf{v}(M)\}$. The action of operator (3) onto these matrix elements coincides with the action of the representation operator $T_{\mathbf{z}}(B_{n-1})$ onto the corresponding basis elements $\mathbf{v}(M)$. Acting by B_{n-1} onto the functions (4) and putting $\varphi = 0$ we obtain

$$\begin{aligned}(B_{n-1}F^{\mathbf{z}})(0,\gamma,\psi) &= f_1 \frac{\partial}{\partial\gamma} d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}}(\gamma) + f_2 d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}}(\gamma) \left.\frac{\partial}{\partial\varphi} f^{\mathbf{t}}_{\mathbf{x}\mathbf{x}'\mathbf{m}}(\varphi)\right|_{\varphi=0} \\ &- f_3 \sum_{j=-p}^{p} J_j(\mathbf{t},\mathbf{x},\mathbf{m}) d^{\mathbf{z}}_{\mathbf{t},\mathbf{t}',A_j\mathbf{x}}(\gamma) f^{\mathbf{t}'}_{A_j\mathbf{x},\mathbf{x}',\mathbf{m}}(\psi),\end{aligned} \tag{5}$$

where formula (14) of Section 18.6.2 was taken into account. Let us remind that $p = \left[\frac{n-2}{2}\right]$.

Now we act by the operator B_{n-1} onto the functions (4) with the help of formula (13) of Section 18.6.2 and put $\varphi = 0$. Comparing the formula obtained with formula (5), we obtain the relation between the matrix elements $d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}}(\gamma)$ and $f^{\mathbf{t}}_{\mathbf{x}\mathbf{x}'\mathbf{m}}(\psi)$. It is of the form

$$\begin{aligned} & f_1 \frac{\partial}{\partial\gamma} d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}}(\gamma) + f_2 d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}}(\gamma) \frac{\partial}{\partial\varphi} f^{\mathbf{t}}_{\mathbf{x}\mathbf{x}'\mathbf{m}}(\varphi) \Big|_{\varphi=0} \\ & - f_3 \sum_{j=-p}^{p} J_j(\mathbf{t},\mathbf{x},\mathbf{m}) d^{\mathbf{z}}_{\mathbf{t},\mathbf{t}',A_j\mathbf{x}}(\gamma) f^{\mathbf{t}'}_{A_j\mathbf{x},\mathbf{x}',\mathbf{m}}(\psi) \\ & = \sum_{j=-q}^{q} B_j(\mathbf{z},\mathbf{t}',\mathbf{x}') d^{\mathbf{z}}_{\mathbf{t},A_j\mathbf{t}',\mathbf{x}}(\gamma) f^{A_j\mathbf{t}'}_{\mathbf{x},\mathbf{x}',\mathbf{m}}(\psi),\end{aligned} \tag{6}$$

where $q = \left[\frac{n-1}{2}\right]$. For $\psi = 0$ and $\mathbf{x} = \mathbf{x}'$ it turns into formula (13) of Section 18.6.2. We put into (6) $\mathbf{x}' = A_k\mathbf{x}$, differentiate it with respect to ψ and put $\psi = 0$. As a result, we obtain the differential equation for $d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}}(\gamma)$:

$$\begin{aligned}&\left[\frac{d}{d\gamma} + u(\gamma)\right] d^{\mathbf{z}}_{\mathbf{t}\mathbf{t}'\mathbf{x}}(\gamma) + v(\gamma) J_k(\mathbf{t},\mathbf{x},\mathbf{m}) J_k^{-1}(\mathbf{t}',\mathbf{x},\mathbf{m}) d^{\mathbf{z}}_{\mathbf{t},\mathbf{t}',A_k\mathbf{x}}(\gamma) \\ &= \sum_{j=-q}^{q} B_j(\mathbf{z},\mathbf{t}',A_k\mathbf{x}) J_k(A_j\mathbf{t}',\mathbf{x},\mathbf{m}) J_s^{-1}(\mathbf{t}',\mathbf{x},\mathbf{m}) d^{\mathbf{z}}_{\mathbf{t},A_j\mathbf{t}',\mathbf{x}}(\gamma),\end{aligned} \tag{7}$$

where $s = -p, -p+1, \ldots, p$ and

$$\begin{aligned} u(\gamma) &= -v(\gamma) = -1/\gamma & &\text{if } G_n = ISO(n-1), \\ u(\gamma) &= -\tan^{-1}\gamma,\ v(\gamma) = \sin^{-1}\gamma & &\text{if } G_n = SO(n), \\ u(\gamma) &= -\tanh^{-1}\gamma,\ v(\gamma) = \sinh^{-1}\gamma & &\text{if } G_n = SO_0(n-1,1).\end{aligned}$$

By taking linear combinations of relations (13) of Section 18.6.2 and (7) we receive the differential-recurrence relations for $d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma)$:

$$\left[\frac{d}{d\gamma} + (\tau - p)u(\gamma)\right] d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma) + v(\gamma) \sum_{j=-p}^{p} F_j d^{\mathbf{z}}_{\mathbf{t},\mathbf{t}',A_j\mathbf{x}}(\gamma)$$
$$= \Phi^+ d^{\mathbf{z}}_{A_1\mathbf{t},\mathbf{t}',\mathbf{x}}(\gamma), \tag{8}$$

$$\left[\frac{d}{d\gamma} - (\tau + n - p - 3)u(\gamma)\right] d^{\mathbf{z}}_{\mathbf{tt'x}}(\gamma) + v(\gamma) \sum_{j=-p}^{p} R_j d^{\mathbf{z}}_{\mathbf{t},\mathbf{t}',A_j\mathbf{x}}(\gamma)$$
$$= \Phi^- d^{\mathbf{z}}_{A_{-1}\mathbf{t},\mathbf{t}',\mathbf{x}}(\gamma) \tag{9}$$

which can be written as relations for the matroms $D^{a\mathbf{tt}'}(\gamma)$:

$$\left[\frac{d}{d\gamma} + (\tau - p)u(\gamma) - v(\gamma)F\right] D^{a\mathbf{tt}'}(\gamma) = \nu T_\tau^{-1} D^{a,A_1\mathbf{t},\mathbf{t}'}(\gamma) T_\tau. \tag{10}$$

$$\left[\frac{d}{d\gamma} - (\tau + n - p - 3)u(\gamma) + v(\gamma)R\right] D^{a\mathbf{tt}'}(\gamma) = -\delta T_{\tau-1}^{-1} D^{a,A_{-1}\mathbf{t},\mathbf{t}'}(\gamma) T_{\tau-1}. \tag{10'}$$

The coefficients F_j, R_j and $\Phi^\pm$ in (8) and (9) are defined by the matrices F, R and T_τ from (10) and (10′). The matrices F and R are determined by their action onto vectors $\mathbf{f}$ with coordinates $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and are given by the formulas

$$(R\mathbf{f})(\mathbf{x}) = \prod_{j=1}^{p} [R_j^+(\mathbf{x})\mathbf{f}(A_j\mathbf{x}) + R_j^-(\mathbf{x})\mathbf{f}(A_{-j}\mathbf{x})] + R(\mathbf{x})\mathbf{f}(\mathbf{x}),$$

$$(F\mathbf{f})(\mathbf{x}) = \prod_{j=1}^{p} [F_j^+(\mathbf{x})\mathbf{f}(A_j\mathbf{x}) + F_j^-(\mathbf{x})\mathbf{f}(A_{-j}\mathbf{x})] + F(\mathbf{x})\mathbf{f}(\mathbf{x}).$$

The matrix T_τ, $\tau = t_1$, is diagonal with elements $T_\tau(\mathbf{x})$ on the main diagonal. If $n = 2p + 2$, then $F_j^\pm(\mathbf{x})$, $R_j^\pm(\mathbf{x})$, $F(\mathbf{x})$, $R(\mathbf{x})$, $T_\tau(\mathbf{x})$, ν, δ are given by the formulas

$$F_j^+(\mathbf{x}) = [2(\tau + x_j)]^{-1} A_j^+(\mathbf{x}), \;\; R_j^+(\mathbf{x}) = [2(\tau - x_j - 1)]^{-1} A_j^+(\mathbf{x}),$$

$$A_j^+(\mathbf{x}) = \prod_{i \neq j} (x_i^2 - x_j^2)^{-1} \prod_{k=1}^{p} [(a_k + x_j)(a_k - x_j - 1)(a_k' + x_j)(a_k' - x_j - 1)]^{1/2},$$

$$F_j^-(\mathbf{x}) = F_j^+(-\mathbf{x}), \;\; R_j^-(\mathbf{x}) = R_j^+(-\mathbf{x}), \;\; F(\mathbf{x}) = R(\mathbf{x}) = 0,$$

$$\nu \equiv \nu(\mathbf{t}) = \left[\frac{(2\tau - 1)(a^2 - \tau^2)}{(2\tau + 1)} \prod_{k=1}^{p} \frac{(\tau - t_k)(\tau + t_k - 1)}{(\tau + t_k)(\tau - t_k + 1)}\right]^{1/2},$$

$$\delta = \nu(\mathbf{e}-\mathbf{t}),\ \mathbf{e}=(1,1,\dots,1),\ T_\tau(\mathbf{x}) = \prod_{k=1}^{p}(\tau^2 - x_k^2)^{1/2},\ \tau = t_1.$$

If $n = 2p+3$, then

$$F_j^+(\mathbf{x}) = [2(\tau + x_j)]^{-1}A_j^+(\mathbf{x}),\ R_j^+(\mathbf{x}) = [2(\tau - x_j)]^{-1}A_j^+(\mathbf{x}),$$

$$A_j^+(\mathbf{x}) = 2\left[x_j(2x_j-1)\prod_{i\neq j}(x_i^{(2)} - x_j^{(2)})\right]^{-1}\prod_{k=1}^{p-1}[(a_k^2 - x_j^2)(a_k'^2 - x_j^2)]^{1/2},$$

$$F_j^-(\mathbf{x}) = F_j^+(\mathbf{e}-\mathbf{x}),\ R_j^-(\mathbf{x}) = R_j^+(\mathbf{e}-\mathbf{x}),\ \mathbf{e} = (1,1,\dots,1),$$

$$F(\mathbf{x}) = R(\mathbf{x}) = \prod_{k=1}^{p+1} a_k a_k'\left(\tau\prod_{i=1}^{p}x_i^{(2)}\right)^{-1};\ T_\tau(\mathbf{x}) = \prod_{k=1}^{p}[(\tau + x_k)(\tau - x_k + 1)]^{1/2},$$

$$\nu \equiv \nu(\mathbf{t}) = \left[(a+\tau)(a-\tau-1)\prod_{k=2}^{p+1}\frac{\tau^2 - t_k^2}{(\tau+1)^2 - t_k^2}\right]^{1/2},$$

$$\delta = \nu(-\mathbf{t}),\quad x^{(2)} = x(x-1).$$

The numbers a_k and a_k' are defined by the formulas

$$a_k = \min(t_k, t_k');\ a_k' = \max(t_k, t_k');\ k \neq p+1,$$

$$a_{p+1}' = \max(|t_{p+1}|, |t_{p+1}'|).$$

In the case of the group $G_n = SO(n)$ we can set $D^{a\mathbf{tt}'}(\text{arc cos }\mu) = P^{a\mathbf{tt}'}(\mu)$. The matrix function $P^{a\mathbf{tt}'}(\mu)$ satisfies the equalities

$$\left[(1-\mu^2)\frac{d}{d\mu} + (\tau - p)\mu - F\right]P^{a\mathbf{tt}'}(\mu) = -\sigma\sqrt{1-\mu^2}T_\tau^{-1}P^{a,A_1\mathbf{t},\mathbf{t}'}(\mu)T_\tau, \quad (11)$$

$$\left[(1-\mu^2)\frac{d}{d\mu} - (\tau + n - p - 3)\mu + R\right]P^{a\mathbf{tt}'}(\mu) = \delta\sqrt{1-\mu^2}T_{\tau-1}^{-1}P^{a,A_{-1}\mathbf{t},\mathbf{t}'}(\mu)T_{\tau-1}, \quad (11')$$

where the coefficients are the same as in formulas (10) and (10′). The matrices $P^{a\mathbf{tt}'}(\mu)$ will be also called matroms of the group $SO(n)$.

18.6.5. Differential equations of the second order. The differential operator equation of the second order for the matroms $D^{a\mathbf{tt}'}(\gamma)$ of the groups $SO(n)$, $SO_0(n-1,1)$, $ISO(n-1)$ are described with the help of the second order Casimir operators of these groups. We give these results, as well as most results of Sections

18.6.6 and 18.7 below, without proofs. The reader can find the corresponding proofs in [266].

The matroms $D^{\rho \mathbf{t}\mathbf{t}'}(\gamma)$ of the group $ISO(n-1)$ satisfy the differential equation

$$\left[\frac{d^2}{d\gamma^2}+\frac{n-2}{\gamma}\frac{d}{d\gamma}-\frac{1}{\gamma^2}\tilde{A}+\rho^2 E\right] D^{\rho \mathbf{t}\mathbf{t}'}(\gamma)=0, \tag{1}$$

where the matrix $\tilde{A}$ acts onto a vector $\mathbf{f}$ with components $\mathbf{f}(\mathbf{x})$, $\mathbf{x}\in\Omega$, as

$$(\tilde{A}\mathbf{f})(\mathbf{x})=\sum_{j=1}^{p}[A_j^+(\mathbf{x})\mathbf{f}(A_j\mathbf{x})+A_j^-(\mathbf{x})\mathbf{f}(A_{-j}\mathbf{x})]+\tilde{A}(\mathbf{x})\mathbf{f}(\mathbf{x}), \tag{2}$$

$A_j^+(\mathbf{x})$ are defined in Section 18.6.4 and

$$A_j^-(\mathbf{x})=A_j^+(-\mathbf{x}),\ \tilde{A}(\mathbf{x})=\sum_{k=1}^{p}(2x_k^2-a_k^{(2)}-a_k^{'(2)})+p(p-1)$$

for $n=2p+2$,

$$A_j^-(\mathbf{x})=A_j^+(\mathbf{e}-\mathbf{x}),\ \tilde{A}(\mathbf{x})=2\sum_{k=1}^{p}x_k^{(2)}-\sum_{k=1}^{p+1}(a_k^2+a_k^{'2})+p(p+1)$$

for $n=2p+3$.

The matroms $P^{a\mathbf{t}\mathbf{t}'}(\mu)=D^{a\mathbf{t}\mathbf{t}'}(\text{arc cos }\mu)$ of the group $SO(n)$ satisfy the differential equation

$$\begin{aligned}\Big\{(1-\mu^2)\frac{d^2}{d\mu^2}-(n-1)\mu\frac{d}{d\mu}+\frac{A\mu-B}{1-\mu^2}+\frac{1}{2}\text{ad }B\\ +(a+p)(a+p-n+2)\Big\}P^{a\mathbf{t}\mathbf{t}'}(\mu)=0,\end{aligned} \tag{3}$$

where $(\text{ad }B)P=BP-PB$, $B=\text{diag }\{-\tilde{A}(\mathbf{x})\}$ and $\tilde{A}(\mathbf{x})$ is such as in formula (2),

$$(A\mathbf{f})(\mathbf{x})=\sum_{j=1}^{p}[A_j^+(\mathbf{x})\mathbf{f}(A_j\mathbf{x})+A_j^-(\mathbf{x})\mathbf{f}(A_{-j}\mathbf{x})]+A(\mathbf{x})\mathbf{f}(\mathbf{x})$$

and $A_j^{\pm}(\mathbf{x})$ are defined in the same way as in the case of equation (1),

$$A(\mathbf{x})=2\prod_{k=1}^{p+1}a_k a_k'\left(\prod_{j=1}^{p}x_j(x_j-1)\right)^{-1}\quad \text{for}\quad n=2p+3,$$

$$A(\mathbf{x})=0\qquad\qquad \text{for}\quad n=2p+2.$$

By using relations (11) and (11') of Section 18.6.4 we can derive other differential equation of the second order for the matrom $D(\gamma) = D^{att'}(\gamma)$ of the group $SO(n)$:

$$\frac{d^2}{d\gamma^2}D(\gamma) + (n-2)\tan^{-1}\gamma\frac{d}{d\gamma}D(\gamma) + \sin^{-1}\gamma\left(F\frac{d}{d\gamma}D(\gamma) - \frac{dD(\gamma)}{d\gamma}G\right)$$
$$+ \sin^{-2}\gamma[(\tau+n-3)(F\cos\gamma - \tau E)D(\gamma) + (\tau E\cos\gamma - F)D(\gamma)G]$$
$$+ (a+p)(a+p-n+2)D(\gamma) = 0, \qquad (4)$$

where G is the transpose matrix for R, and F, R are such as in Section 18.6.4.

Matroms of the group $SO_0(n-1,1)$ satisfy the differential equation of the second order, similar to equation (4). In this equation hyperbolic functions are used instead of trigonometric ones.

Let us note that for $P(\mu) \equiv P^{att'}(\mu)$ the equation (4) take the form

$$(1-\mu^2)\frac{d^2}{d\mu^2}P(\mu) - [(n-1)\mu E + F]\frac{d}{d\mu}P(\mu) + \frac{dP(\mu)}{d\mu}G$$
$$+ (1-\mu^2)^{-1}[(\tau+n-3)(F\mu - \tau E)P(\mu) + (\tau\mu E - F)P(\mu)G]$$
$$+ (a+p)(a+p-n+2)P(\mu) = 0. \qquad (5)$$

18.6.6. Bessel and Jacobi functions with matrix indices. The differential matrix equations from Sections 18.6.4 and 18.6.5 are the matrix analogs of the well known differential equations related to Bessel and Jacobi functions and to spherical functions on the sphere S^{n-1}. Therefore, it is natural to expect that matroms, satisfying the matrix differential equations, are matrix analogs of these special functions. In this section we describe the scheme of solving the matrix differential equations of Section 18.6.5 and show that solutions are generalizations of classical special functions. A formulation of final results uses notions of the theory of functions of ordered operators.

Let f be an analytical function of two variables r and s:

$$f(r,s) = \sum_{i,j} a_{ij} r^i s^j.$$

Let R and S be square matrices of the same dimension. We set

$$f(R(2), S(1)) = \sum_{i,j} a_{ij} R^i S^j \qquad (1)$$

where the indices 1 and 2 at S and R means the order of arrangement of these matrices in (1). In the case of one variable a similar matrix series can be written with the help of Silvester's interpolation formula [67] in the form of a finite sum.

An analogous result is valid for functions of many variables. Let us consider the case when the matrices R and S can be diagonalized. We assume that the spectral decompositions of R and S are of the form

$$R = \sum_{i=0}^{N} r_i R_i, \quad S = \sum_{i=0}^{N} s_i S_i,$$

where r_i and s_i are eigenvalues of R and S, and R_i and S_i are the corresponding projection operators. Then

$$R_i = \prod_{\substack{k=0\\k\neq i}}^{N} \frac{R - r_k E}{r_i - r_k}, \quad S_i = \prod_{\substack{k=0\\k\neq i}}^{N} \frac{S - s_k E}{s_i - s_k}, \tag{2}$$

where E is the unit matrix. The series (1) can be represented in the form of the following finite sum:

$$f(R(2), S(1)) = \sum_{i,j=0}^{N} f(r_i, s_j) R_i S_j.$$

Now we consider matrix equations (1) and (4) of Section 18.6.5. They are equations of the form

$$\sum_{k=1}^{M} \varphi_k(F) L_k[D(\gamma)] \psi_k(G) = 0, \tag{3}$$

where $D(\gamma)$ is an unknown matrix, φ_k and ψ_k are scalar functions, F and G are constant square matrices, and L_k is a linear scalar operator (for example, a differential one). Along with matrix equation (3) we consider its scalar analog

$$\sum_{k=1}^{M} \varphi_k(r) \psi_k(s) L_k[d_{rs}(\gamma)] = 0, \tag{4}$$

where $d_{rs}(\gamma)$ is a scalar unknown function and r, s are parameters. Formula (4) is called the *symbol of the matrix equation* (3), and the function $d_{rs}(\gamma)$ is called the *symbol of the matrix* $D(\gamma)$.

Let C be a numerical matrix. It is easy to show that the matrix function

$$D(\gamma) = \sum_{i,j=0}^{N} F_i C G_j d_{r_i s_j}(\gamma)$$

satisfies equation (3). In the case of a non-singular matrix C the matrix $D(\gamma)$ will be called a *general non-degenerate solution of equation* (3).

Theorem 1. *A general non-degenerate solution $D(\gamma)$ of equation* (3) *can be represented in the form of function of two matrix indices:*

$$D(\gamma) = C d_{\bar{F}(2),G(1)}(\gamma) = d_{F(2),\bar{G}(1)}(\gamma) C, \tag{5}$$

where C is an arbitrary non-singular matrix, $\bar{F} = C^{-1}FC$, $\bar{G} = CGC^{-1}$, and $d_{rs}(\gamma)$ is the scalar function (the symbol of the operator), satisfying equation (4).

Let us apply this theorem to equations (1) and (5) of Section 18.6.5. If $G = ISO(n-1)$, we have to set in (3) $F = \tilde{A}$ and $G = E$. The symbol of matrix equation (1) of Section 18.6.5 is of the form

$$\left[\frac{d^2}{d^2\gamma} + \frac{n-2}{\gamma}\frac{d}{d\gamma} - \frac{r}{\gamma^2} + \rho^2\right] d_r(\gamma) = 0.$$

It can be easily reduced to the Bessel equation. The function

$$d_r(\gamma) = (\rho\gamma)^{-(n-3)/2} J_{\sqrt{r+((n-3)/2)^2}}(\rho\gamma) C,$$

where J_ν is the Bessel function and C is an arbitrary constant, is a general solution of this equation, bounded at zero. By applying Theorem 1 we obtain the following theorem.

Theorem 2. *The matrom $D^{\rho \mathbf{t}\mathbf{t}'}(\gamma)$ of the group $ISO(n-1)$ is described in terms of the Bessel function with a matrix index:*

$$D^{\rho \mathbf{t}\mathbf{t}'}(\gamma) = (\rho\gamma)^{-(n-3)/2} J_{\tilde{B}}(\rho\gamma) C, \tag{6}$$

where $\tilde{B} = \sqrt{\tilde{A} + \left(\frac{n-3}{2}\right)^2 E}$, C is the constant matrix and $\tilde{A}$ is the matrix from Section 18.6.5.

The explicit form of the matrix C will be given in Section 18.7.

Now we consider equation (5) of Section 18.6.5 for the matrom of the group $SO(n)$. Its symbol (the scalar analog) is of the form

$$\begin{aligned}\Big\{(1-\mu^2)\frac{d^2}{d\mu^2} - [(n-1)\mu + r - s]\frac{d}{d\mu} + \frac{1}{1-\mu^2}[(\tau + n - 3)(r\mu - \tau) \\ + (\tau\mu - r)s] + (a+p)(a+p-n+2)\Big\} f_{rs}^{a,\tau}(\mu) = 0. \end{aligned} \tag{7}$$

This equation can be reduced to the differential equation for the Jacobi polynomial. The function

$$f_{rs}^{a,\tau}(\mu) = (1-\mu)^{(\tau-p-s)/2}(1+\mu)^{(\tau-p+s)/2} P_\nu^{(\alpha,\beta)}(\mu)\kappa, \tag{8}$$

where $P_\nu^{(\alpha,\beta)}(\mu)$ is the Jacobi polynomial, κ is a numerical constant and

$$\nu = a-\tau+2p-n+2,\ \alpha = \frac{1}{2}(2\tau+n-2p-r-s-3),\ \beta = \frac{1}{2}(2\tau+n-2p+r+s-3),$$

is a solution of equation (7). By applying Theorem 1 we obtain the following result.

Theorem 3. *A general non-degenerate solution $P(\mu)$ of equation* (5) *of Section* 18.6.5 *for the matrom of the group $SO(n)$ is of the form*

$$P(\mu) = f^{a,\tau}_{F(2),\bar{G}(1)}(\mu)C, \tag{9}$$

where F, G are the matrices from this equation, $\bar{G} = CGC^{-1}$ and C is an arbitrary non-singular matrix.

More detailed analysis of the equations (11) and (11′) of Section 18.6.4 allow us to define more exactly the expression for the matrom $P^{\mathbf{att}'}(\mu)$.

Theorem 4. *The matrom $P^{\mathbf{att}'}(\mu)$ of the group $SO(n)$ is representable in the form of the Jacobi function with two non-commuting matrix indices:*

$$P^{\mathbf{att}'}(\mu) = f^{a,\tau}_{R(2),S(1)}(\mu)C, \tag{10}$$

where the matrix S is of the same structure as in the case of the matrix R in Section 18.6.4, $S(\mathbf{x}) = R(\mathbf{x})$,

$$S_j^+(\mathbf{x}) = (a - x_j - 1)(a + p - q + x_j)^{-1}R_j^+(\mathbf{x}),$$

$$S_j^-(\mathbf{x}) = (a + p - q + x_j - 1)(a - x_j)^{-1}R_j^-(\mathbf{x}),$$

$$p = \left[\frac{n-2}{2}\right],\ q = \left[\frac{n-1}{2}\right],$$

and C is the constant matrix consisting of values of matrix elements at the point $\pi/2$ (which are called the Weyl coefficients).

It is possible to show that the matrom $P^{\mathbf{att}'}(\mu)$ can be reduced to a matrix polynomial. Namely, the matrix analog of formula (8) is valid:

$$P^{\mathbf{att}'}(\mu) = Q^{\mathbf{att}'}(\mu)(1-\mu)^{\frac{(\tau-p)E-S}{2}}(1+\mu)^{\frac{(\tau-p)E+S}{2}}, \tag{11}$$

where $Q^{\mathbf{att}'}(\mu)$ is a polynomial of power $\nu = a - \tau + 2p - n + 2$ in μ with matrix coefficients. This polynomial can be considered as a matrix analog of the usual Jacobi polynomial.

We can introduce in the space $\mathfrak{M}$ of matrix functions $f(\mu)$ the scalar product

$$\langle f_1, f_2\rangle = \int_{-1}^{1} f_1(\mu)W(\mu)f_2^*(\mu)d\mu, \qquad (12)$$

where $f^* = \bar{f}^t$ and $W(\mu)$ is some Hermitian positive definite (on $[-1,1]$) matrix weight function. It is proved that the polynomials $Q^{a\mathbf{tt}'}(\mu)$ are orthogonal with respect to the weight function $W(\mu)$, described in [264].

Formula (10) can be written as

$$P^{a\mathbf{tt}'}(\mu) = \sum_{i,j=0}^{N} f_{r_i s_j}^{a,\tau}(\mu)R_iS_jC, \qquad (13)$$

where r_k, s_k, $k = 0, 1, \ldots, N$, are different eigenvalues of the matrices R and S, and R_k, S_k are the corresponding projection matrices, defined by formulas (2). Therefore, the following algorithm of evaluation of elements of the matrom $P^{a\mathbf{tt}'}$ can be proposed:

(a) to find eigenvalues r_i, s_j of the matrices R and S;

(b) to construct the projection matrices R_i and S_j for R and S;

(c) to substitute these values to the right hand side of formula (13) and to write down this formula elementwise.

As a result, we obtain the expansion of matrix elements $P_{\mathbf{tt}'\mathbf{x}}^{a,\mathbf{y}}(\mu) = d_{\mathbf{tt}'\mathbf{x}}^{a,\mathbf{y}}(\text{arc cos }\mu)$ as a finite linear combination of Jacobi polynomials:

$$P_{\mathbf{tt}'\mathbf{x}}^{a,\mathbf{y}}(\mu) = \sum_{i,j=0}^{N} a_{ij} f_{r_i s_j}^{a,\tau}(\mu), \qquad (14)$$

where

$$a_{ij} = \sum_{\mathbf{x}',\mathbf{y}'\in\Omega} R_i(\mathbf{x},\mathbf{x}')S_j(\mathbf{x}',\mathbf{y}')C(\mathbf{y}',\mathbf{y})$$

and on the right hand side here there are elements of the matrices R_i, S_j and C.

The projection matrices R_i and S_j can be evaluated with the help of formulas (2). Other method of evaluation of these matrices will be described in Section 18.7.

Let us remark that connection of matrix elements of representations of the groups $SO(n)$, $SO_0(n-1,1)$ and $ISO(n-1)$ with the theory of functions of ordered operators, considered here, can be generalized for other Lie groups. This approach allows us to describe elements of representation matrices in terms of classical special functions and polynomials, depending on matrix indices.

18.7. Orthogonal Polynomials of Many Discrete and Continuous Variables

18.7.1. General procedure of spectral analysis of infinitesimal operators and matroms. There are different ways to obtain special functions (in particular, orthogonal polynomials) of many variables in the theory of group representations. One of these ways is related to the spectral analysis of infinitesimal operators of representations and of generators (matrix indices) of matroms. This way leads to different classes of orthogonal polynomials of many discrete and continuous variables. We shall consider infinitesimal operators of the groups $SO(n)$ and $U(n)$ in the Gel'fand-Tsetlin bases and generators of matroms of the groups $ISO(n-1)$ and $SO(n)$. Orthogonal polynomials, which will appear under these considerations, generalize three main classes of classical orthogonal polynomials of one variable, namely, Jacobi, Laguerre, and Hermite polynomials. In the discrete case we obtain many-dimensional generalization of orthogonal polynomials of one discrete variable (such as Krawtchouk, Racah, Hahn and dual Hahn polynomials.)

Let us consider the general procedure of spectral analysis, using the examples of the infinitesimal operators of representations of the groups $SO(n)$ and $U(n)$. Let G_n be one of these groups and let T be an irreducible representation of G_n in the space V. We take the Gel'fand-Tsetlin basis $\{\mathbf{v}(M)\}$ of V, where $M = (\mathbf{m}_n, \mathbf{m}_{n-1}, \beta)$. Here β is the remaining part of the Gel'fand-Tsetlin pattern M (see Section 18.1.1) and $\mathbf{m}_n$ is the highest weight of the representation T. The infinitesimal operator, corresponding to the one-parameter subgroup $\{g_{n-1}(\theta)\}$ will be denoted by I (it corresponds to the matrix $E_{n,n-1} - E_{n-1,n}$). It is given by formulas of Section 18.1.2.

We denote by V_β the subspace of V spanned by the Gel'fand-Tsetlin basis elements $\mathbf{v}(M) \equiv \mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1}, \beta)$ with fixed β. This subspace is invariant with respect to the operator I and $V = \sum\limits_\beta \oplus V_\beta$. Therefore, it is sufficient to study the operator I in each of the subspace V_β.

We denote the vectors $\mathbf{v}(\mathbf{m}_n, \mathbf{m}_{n-1}, \beta)$ of V_β by $\mathbf{v_m}$, where $\mathbf{m} \equiv \mathbf{m}_{n-1} = (m_1, m_2, \ldots, m_p)$. It is clear that $\mathbf{m}$ runs over the discrete set Π of integral points of the parallelepiped Ω which is defined by $\mathbf{m}_n$ and $\mathbf{m}_{n-2}$. Let $\mathfrak{F}(\Pi)$ be the set of all functions defined on Π. It is clear that the space V_β is isomorphic to $\mathfrak{F}(\Pi)$. This isomorphism $V_\beta \ni \mathbf{v} \to f \in \mathfrak{F}(\Pi)$ is given by the formula

$$\mathbf{v} = \sum_{\mathbf{m}\in\Pi} f(\mathbf{m})\mathbf{v_m}.$$

Identifying V_β with $\mathfrak{F}(\Pi)$, we consider elements of V_β as functions on the lattice Π (i.e. as functions of p discrete variables). The space $\mathfrak{F}(\Pi)$ is called the *discrete realization* of the space V_β. The action of the infinitesimal operator I in $\mathfrak{F}(\Pi)$ can be written as

$$(If)(\mathbf{m}) = \sum_{i=1}^{p}[a_i(\mathbf{m})f(A_i\mathbf{m}) + b_i(\mathbf{m})f(A_{-i}\mathbf{m})] + c(\mathbf{m})f(\mathbf{m}), \tag{1}$$

where $A_j\mathbf{m} = (m_1, \dots, m_{|j|} + \text{sign } j, \dots, m_p)$. The coefficients $a_i(\mathbf{x})$, $b_i(\mathbf{x})$ and $c(\mathbf{m})$ are determined by the formulas of Section 18.1.2. Thus, the problem of spectral analysis of the operator I is reduced to the Sturm-Liouville problem

$$(If)(\mathbf{m}) = \sigma f(\mathbf{m}), \ \mathbf{m} \in \Pi, \tag{2}$$

in the discrete space Π.

18.7.2. Partial difference equations connected with infinitesimal operators and matroms. Let us consider the infinitesimal operator I of the irreducible representation of the unitary group $U(p+1)$ with highest weight $\mathbf{m}_{p+1} = (m_{1,p+1}, \dots, m_{p+1,p+1})$. Then

$$\mathbf{m}_p = (m_1, m_2, \dots, m_p), \ \mathbf{m}_{p-1} = (k_1, k_2, \dots, k_{p-1}).$$

We introduce the notations

$$x_j = m_j - j, \ j = 1, 2, \dots, p; \ \mathbf{x} = (x_1, x_2, \dots, x_p),$$

$$a_j = \min(m_{j,p+1}, k_{j-1}) - j + 1, \ a'_j = \max(m_{j,p+1}, k_{j-1}) - j + 1, \ j = 2, 3, \dots, p,$$

$$a_1 = m_{1,p+1}, \ a'_{p+1} = m_{p+1,p+1} + p.$$

Then the space V_β is realized as the set of all functions of discrete variables $\mathbf{x} = (x_1, x_2, \dots, x_p)$, defined on the lattice

$$\Omega = \{(x_1, \dots, x_p) \mid a'_{p+1} \le x_j \le a_j - 1, \ j = 1, 2, \dots, p\}.$$

The generator I acts in V_β by the formula

$$(If)(\mathbf{x}) = \sum_{j=1}^{p} [I_j(\mathbf{x}) f(A_j\mathbf{x}) - I_j(A_{-j}\mathbf{x}) f(A_{-j}\mathbf{x})],$$

where

$$I_j(\mathbf{x}) = \left| \frac{\prod_{r=1}^{p} (\alpha_r - x_j - 1) \prod_{k=2}^{p+1} (a'_k - x_j - 1)}{\prod_{\substack{\ell=1 \\ \ell \ne j}}^{p} (x_\ell - x_j)(x_\ell - x_j - 1)} \right|^{1/2}.$$

Let $Q(\mathbf{x})$ be an eigenfunction of the operator I corresponding to an eigenvalue σ. We put

$$Q(\mathbf{x}) = i^{x_1 + \dots + x_p} \sqrt{\rho(\mathbf{x})} P(\mathbf{x}), \tag{1}$$

where

$$\rho(\mathbf{x}) = \frac{\prod_{i<j}(x_\ell - x_j)\prod_{k=2}^{p}\prod_{j=1}^{p}|a_k - x_j|^{(s_k)}}{\prod_{j=1}^{p}(a_1 - x_j - 1)!(x_j - a'_{p+1})!}, \ s_k = a'_k - a_k. \tag{2}$$

Here we have used the generalized power

$$[f(x_j)]^{(s)} = f(x_j)f(x_j - 1)\dots f(x_j - s + 1).$$

The function $P(\mathbf{x})$ satisfies the following difference equation of the second order:

$$\sum_{j=1}^{p}\frac{(a_1 - x_j - 1)!(x_j - a'_{p+1})!}{\prod_{\ell\neq j}(x_\ell - x_j)\prod_{k=2}^{p}(a_k - x_j)^{(s_k)}}$$

$$\times\,\Delta_j\left[\frac{\prod_{k=2}^{p}(a_k - x_j)^{(s_k+1)}}{(a_1 - x_j - 1)!(x_j - a'_{p+1} - 1)!}\nabla_j P(\mathbf{x})\right] = \lambda P(\mathbf{x}), \tag{3}$$

where $\mathbf{x}\in\Omega$, $\lambda = -i\sigma + s_2 + \dots + s_p - a'_{p+1} - a_1 + p$ and

$$\Delta_j P(\mathbf{x}) = P(A_j\mathbf{x}) - P(\mathbf{x}),\ \nabla_j P(\mathbf{x}) = \Delta_j P(A_{-j}\mathbf{x}).$$

When $p = 1$ (i.e. for the group $U(2)$) the equation (3) takes the form

$$(x - a'_2)\Delta\nabla P(x) - (2x - a_1 - a'_2 + 1)\Delta P(x) = \lambda P(x),\ a'_2 \le x \le a_1 - 1. \tag{4}$$

This equation is reduced to the equation for the Krawtchouk polynomials $K_n(t; s, N)$ with $s = \frac{1}{2}$. Thus, (3) can be considered as a many-dimensional analog of the equation for the Krawtchouk polynomials.

Now we consider the group $SO(2p+1)$. In this case highest weight of an irreducible representation is of the form $\mathbf{m}_{2p+1} = (m_{1,2p+1}, \dots, m_{p,2p+1})$. The next two rows of Gel'fand-Tsetlin patterns are

$$\mathbf{m}_{2p} \equiv \mathbf{m} = (m_1, m_2, \dots, m_p),\ \mathbf{m}_{2p-1} = (k_1, k_2, \dots, k_{p-1}).$$

We introduce the notations

$$x_j = m_j + p - j,\ j = 1, 2, \dots, p;\ \mathbf{x} = (x_1, x_2, \dots, x_p),$$

$$a_j = \min(m_{j,2p+1}, k_{j-1}) + p - j + 1,\ a'_j = \max(m_{j,2p+1}, k_{j-1}) + p - j + 1,\ j = 2, \dots, p,$$

$$a_1 = m_{1,2p+1} + p.$$

The space V_β is realized as the set of functions, defined on the lattice

$$\Omega = \{(x_1, \dots, x_p) \mid a'_{j+1} \le x_j \le a_j - 1,\ j = 1, 2, \dots, p-1;\ |x_p| \le a_p - 1\}. \quad (5)$$

Let $Q(\mathbf{x})$ be an eigenfunction of the operator I corresponding to an eigenvalue σ. We put

$$Q(\mathbf{x}) = i^{x_1 + \dots + x_p} \sqrt{\rho(\mathbf{x})} P(\mathbf{x}), \quad (6)$$

where

$$\rho(\mathbf{x}) = \frac{\prod\limits_{\ell<j} (x_\ell^2 - x_j^2) \prod\limits_{k=2}^{p} \prod\limits_{j=1}^{p} |(a_k - x_j)(a'_k + x_j - 1)|^{(s_k)}}{\prod\limits_{j=1}^{p} (a_1 - x_j - 1)!(a_1 + x_j - 1)!}.$$

The function $P(\mathbf{x})$ satisfies the equation

$$\sum_{j=1}^{p} \frac{(a_1 - x_j - 1)!(a_1 + x_j - 1)!}{\prod\limits_{\ell \ne j} (x_\ell^2 - x_j^2) \prod\limits_{k=2}^{p} [(a_k - x_j)(a'_k + x_j - 1)]^{(s_k)}}$$
$$\times \Delta_j \left[\frac{\prod\limits_{k=2}^{p} [(a_k - x_j)(a'_k + x_j - 1)]^{(s_k+1)}}{(a_1 - x_j - 1)!(a_1 + x_j - 2)!} \nabla_j P(\mathbf{x}) \right] = \lambda P(\mathbf{x}), \quad (7)$$

$$\mathbf{x} \in \Omega, \quad \lambda = 2(i\sigma - a_1 - s_2 - \dots - s_p - p), \quad s_k = a'_k - a_k, \quad k = 2, 3, \dots, p.$$

In particular, for $p = 1$ (i.e. for the group $SO(3)$) equation (7) takes the form

$$(a_1 - x - 1)!(a_1 + x - 1)!\Delta \left[\frac{1}{(a_1 - x - 1)!(a_1 + x - 2)!} \nabla P(x) \right] = \lambda P(x),$$

$$-(a_1 - 1) \le x \le a_1 - 1.$$

This equation can be reduced to the equation for the Krawtchouk polynomials $K_n(x; s, N)$ with $s = \frac{1}{2}$. Thus, equation (7) is other many-dimensional generalization of the equation for Krawtchouk polynomials.

Now we consider the group $SO(2n+2)$. Upper three rows of the Gel'fand-Tsetlin pattern are of the form

$$\mathbf{m}_{2p+2} = (m_{1,2p+2}, \dots, m_{p+1,2p+2}),$$

$$\mathbf{m}_{2p+1} \equiv \mathbf{m} = (m_1, \dots, m_p),\ \mathbf{m}_{2p} = (k_1, \dots, k_p).$$

We introduce the notations

$$x_j = m_j + p - j + 1,\ j = 1, 2, \dots, p;\ \mathbf{x} = (x_1, x_2, \dots, x_p),\ a_1 = m_{1,2p+2} + p,$$

$$a_j = \min(m_{j,2p+2}, k_{j-1})+p-j+1,\ a'_j = \max(m_{j,2p+2}, k_{j-1})+p-j+1,\ j = 2,\dots,p,$$

$$a_{p+1} = (\mathrm{sign}\, m_{p+1,2p+2})(\mathrm{sign}\, k_p)\min(|m_{p+1,2p+2}|, |k_p|),$$

$$a'_{p+1} = \max(|m_{p+1,2p+2}|, |k_p|).$$

The space V_β is realized as the set of functions of discrete variables $x_1, x_2, \dots, x_p$, defined on the lattice

$$\Omega = \{(x_1, x_2, \dots, x_p) \mid a'_{j+1} + 1 \le x_j \le a_j,\ j = 1, 2, \dots, p\}. \tag{8}$$

Let $(IQ)(\mathbf{x}) = \sigma Q(\mathbf{x})$. We put

$$Q(\mathbf{x}) = (-i)^{x_1+\dots+x_p}\sqrt{\rho(\mathbf{x})}P(\mathbf{x}), \tag{9}$$

where

$$\rho(\mathbf{x}) = \frac{\prod\limits_{\ell<j}(x_\ell^{(2)} - x_j^{(2)})\prod\limits_{k=2}^{p+1}\prod\limits_{j=1}^{p}|(a_k - x_j + 1)(a'_k + x_j - 1)|^{(s_k)}(2x_j - 1)}{\prod\limits_{j=1}^{p}(a_1 - x_j)!(a_1 + x_j - 1)!}, \tag{10}$$

$$s_k = a'_k - a_k,\ x_j^{(2)} = x_j(x_j - 1).$$

The function $P(\mathbf{x})$ satisfies the equation

$$\sum_{j=1}^{p}\frac{(a_1 - x_j)!(a_1 + x_j - 1)!}{(2x_j - 1)\prod\limits_{\ell\ne j}(x_\ell^{(2)} - x_j^{(2)})\prod\limits_{k=2}^{p+1}[(a_k - x_j + 1)(a'_k + x_j - 1)]^{(s_k)}}$$

$$\times\Delta_j\left[\frac{\prod\limits_{k=2}^{p+1}[(a_k - x_j + 1)(a'_k + x_j - 1)]^{(s_k+1)}}{(x_j - 1)(a_1 - x_j)!(a_1 + x_j - 2)!}\nabla_j P(\mathbf{x})\right] = \lambda P(\mathbf{x}), \tag{11}$$

where $\lambda = i\sigma + a_1 - s_2 - \dots - s_{p+1} - p$. For $p = 1$ (i.e. for the group $SO(4)$) this equation takes the form

$$\frac{(a_1 - x)!(a_1 + x - 1)!(a_2 + x - 1)!(x - a'_2 - 1)!}{(2x - 1)(x - a_2 - 1)!(a'_2 + x - 1)!}$$

$$\times\Delta\left[\frac{(x - a_2 - 1)!(a'_2 + x - 1)!}{(x - 1)(a_1 - x)!(a_1 + x - 2)!(a_2 + x - 1)!(x - a'_2 - 2)!}\nabla P(x)\right] = \lambda P(x).$$

It is equation for dual Hahn polynomials of one variable.

Let $D^{\rho tt'}(\gamma)$ be the matrom of the group $ISO(n-1)$. The matrix $\tilde{A}$ from equation (1) of Section 18.6.5. is called the generator of this matrom. Let $Q(\mathbf{x})$ be an eigenfunction of the matrix $\tilde{A}$ corresponding to an eigenvalue σ. We set

$$Q(\mathbf{x}) = \sqrt{\eta(\mathbf{x})}P(\mathbf{x}), \tag{12}$$

where

$$\begin{aligned} \eta(\mathbf{x}) &= \prod_{k,j=1}^{p} |(a_k - x_j)(a'_k + x_j - 1)|^{(s_k)} \qquad \text{if} \quad n = 2p+2, \\ \eta(\mathbf{x}) &= \prod_{k=1}^{p+1}\prod_{j=1}^{p} |(a_k - x_j + 1)(a'_k + x_j - 1)|^{(s_k)} \quad \text{if} \quad n = 2p+3. \end{aligned} \tag{13}$$

The function $P(\mathbf{x})$ satisfies the equation

$$\begin{aligned} &\sum_{j=1}^{p} \left\{ \prod_{\ell \neq j} (x_\ell^2 - x_j^2) \prod_{k=1}^{p} [(a_k - x_j)(a'_k + x_j - 1)]^{(s_k)} \right\}^{-1} \\ &\times \Delta_j \left\{ \prod_{k=1}^{p} [(a_k - x_j)(a'_k + x_j - 1)]^{(s_k+1)} \nabla_j P(\mathbf{x}) \right\} = \lambda P(\mathbf{x}), \end{aligned} \tag{14}$$

$$\lambda = -\sigma + (s_1 + \ldots + s_p)(s_1 + \ldots + s_p + 2p - 1), \ \mathbf{x} \in \Omega,$$

if $n = 2p+2$, where Ω is given by formula (5), and the equation

$$\begin{aligned} &\sum_{j=1}^{p} 2 \left\{ (2x_j - 1) \prod_{\ell \neq j} (x_\ell^{(2)} - x_j^{(2)}) \prod_{k=1}^{p+1} [(a_k - x_j + 1)(a'_k + x_j - 1)]^{(s_k)} \right\}^{-1} \\ &\times \Delta_j \left\{ \frac{1}{x_j - 1} \prod_{k=1}^{p+1} [(a_k - x_j + 1)(a'_k + x_j - 1)]^{(s_k+1)} \nabla_j P(\mathbf{x}) \right\} = \lambda P(\mathbf{x}), \end{aligned} \tag{15}$$

$$\lambda = -\sigma + (s_1 + \ldots + s_{p+1})(s_1 + \ldots + s_{p+1} + 2p - 2), \ \mathbf{x} \in \Omega,$$

if $n = 2p+3$, where Ω is given by formula (8). If $p = 1$ (i.e. for the groups $ISO(3)$ and $ISO(4)$) the equations (14) and (15) take the form

$$\begin{aligned} &\frac{(a-x-1)!(a+x-1)!}{(a'-x-1)!(a'+x-1)!} \\ &\times \Delta \left[\frac{(a'-x)!(a'+x-1)!}{(a-x-1)!(a+x-2)!} \nabla P(x) \right] = \lambda P(x), \ |x| \leq a-1, \end{aligned} \tag{16}$$

$$\frac{2(a_1-x)!(a_1+x-1)!(a_2+x-1)!(x-a_2'-1)!}{(2x-1)(a_1'-x)!(a_1'+x-1)!(x-a_2-1)!(x+a_2'-1)!}$$
$$\times\Delta\left[\frac{(a_1'-x+1)!(a_1'+x-1)!(x-a_2-1)!(a_2'+x-1)!}{(x-1)(a_1-x)!(a_1+x-2)!(x-a_2'-2)!(x-a_2-2)!}\nabla P(x)\right]$$
$$=\lambda P(x), \tag{17}$$

where $a_2'+1\le x\le a_1$. The equation (16) is reduced to the equation for the Hahn polynomials $Q_n(x;\alpha,\beta;N)$ with $\alpha=\beta$, and the equation (17) coincides with that for the Racah polynomials.

The generators R and S of the matrom $P^{\mathbf{att}'}(\mu)$ of the group $SO(n)$ lead to the equations which with the help of an appropriate substitution are reduced to the equations (7) and (11).

18.7.3. Spectral characteristics of discrete equations. In contrast to the equations for classical polynomials of one discrete variable, the equations of Section 18.7.2 are degenerate, i.e. eigenvalues are multiple. The rules of variation of multiplicities are complicated. The key, which allows us to understand their behavior, is a description of eigenvalues with the help of many-parameter collections. In a geometric sense this approach is equivalent to solving the problem of discrete programming on enumeration of integral points of some convex many-dimensional sets in many-dimensional spaces. In the same terms the problems on multiplicities of weights of irreducible representations and on multiplicities of irreducible components in tensor products of representations of the group $U(n)$ can be formulated.

The following results on the spectrum of many-dimensional discrete equations of Section 18.7.2 are valid.

Theorem 1. *The equations* (3), (11) *and* (15) *of Section* 18.7.2 *have* $N=a_1-a_{p+1}'-s_2-\dots,-s_p-p+1$ *different eigenvalues which are of the form*

$$\lambda=\lambda_k=\begin{cases}-2k & \text{for equation (3)},\\ 2k & \text{for equation (11)},\\ 4k(k+s_1+\ldots+s_{p+1}+p) & \text{for equation (15)},\end{cases}$$

where $k=0,1,2,\dots,N-1$. *The multiplicity of* λ_k *is equal to a number of integral points* $(q_1,\dots,q_p)\in\mathbf{R}^p$ *belonging to the section of the parallelepiped*

$$\Gamma=\{(q_1,\dots,q_p)\mid 0\le q_j\le a_j-a_{j+1}'-1,\ j=1,2,\dots,p\}$$

by the hyperplane $\Gamma_k=\{(q_1,\dots,q_p)\mid q_1+\ldots+q_p=k\}$. *The equations* (7) *and* (14) *of Section* 18.7.2 *have* $N=2(a_1-s_2-\ldots-s_p-p)+1$ *different eigenvalues which are of the form*

$$\lambda=\lambda_k=\begin{cases}-2k & \text{for equation (7)},\\ -k(k+2s_1+\ldots+2s_p+2p-1) & \text{for equation (14)},\end{cases}$$

where $k = 0, 1, 2, \ldots, N-1$. *The multiplicity of* λ_k *is equal to a number of integral points of the set* $\Gamma \cap \Gamma_k$, *where* Γ *is the parallelepiped*

$$\Gamma = \{(q_1, \ldots, q_p) \mid 0 \le q_j \le a_j - a'_{j+1} - 1,\ j = 1, 2, \ldots, p-1;\ 0 \le q_p \le 2a_p - 2\}$$

and Γ_k *is the hyperplane*

$$\Gamma_k = \{(q_1, \ldots, q_p) \mid 2(q_1 + \ldots + q_{p-1}) + q_p = k\}.$$

We do not give the proof of this theorem (the reader can find it in [266]). We only mention that the proof is based on studying the action of difference operators from the left hand sides of the equations in the space of symmetric polynomials of p variables.

Let ρ be a non-negative function, defined on a finite set Ω. It generates the discrete measure on Ω. We can define the space $\mathfrak{L}^2_\rho(\Omega)$ of all real functions on Ω. The scalar product

$$(f, g) = \sum_{\mathbf{x} \in \Omega} f(\mathbf{x}) g(\mathbf{x}) \rho(\mathbf{x})$$

turns $\mathfrak{L}^2_\rho(\Omega)$ into a Hilbert space.

Let L denote the difference operator from the left hand sides of the discrete equations of Section 18.7.2. Then L is self-adjoint in the appropriate space $\mathfrak{L}^2_\rho(\Omega)$, where Ω is the corresponding lattice from Section 18.7.2. For the cases of equations (14) and (15) of Section 18.7.2 the self-adjointness is valid for the weight function $\rho(\mathbf{x}) = \kappa(\mathbf{x})\eta(\mathbf{x})$, where $\eta(\mathbf{x})$ is given by formula (13) of Section 18.7.2 and

$$\kappa(\mathbf{x}) = \prod_{\ell < j} (x_\ell^2 - x_j^2) \qquad \text{for equation (14).}$$

$$\kappa(\mathbf{x}) = \prod_{k=1}^{p} (2x_k - 1) \prod_{\ell < j} (x_\ell^{(2)} - x_j^{(2)}) \quad \text{for equation (15).}$$

These reasonings show that solutions of our discrete equations are orthogonal. Namely, solutions corresponding to different eigenvalues are orthogonal on the set Ω with respect to the weight ρ.

A farther investigation of the discrete equations show that they have polynomial solutions of a certain structure. Let us consider equation (3) of Section 18.7.2 connected with the unitary group $U(p+1)$. We denote by W_0 the linear space of symmetric polynomials in p variables. It is possible to show that solutions of this equation belong to W_0. To construct an orthogonal basis of solutions, one can apply the orthogonalization procedure. Let $k_1, k_2, \ldots, k_p$ be non-negative integers such that $k_1 \ge k_2 \ge \ldots \ge k_p$. We suppose that in $\{k_1, k_2, \ldots, k_p\}$ there are s different numbers which have multiplicities $n_1, n_2, \ldots, n_s$, respectively. Let $U_{k_1 \ldots, k_p}(\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_p)$, be the symmetric polynomial

$$U_{k_1 \ldots k_p}(\mathbf{x}) = \frac{1}{n_1! \ldots n_s!} \sum_{\sigma \in S_p} x_1^{k_{\sigma(1)}} \ldots x_p^{k_{\sigma(p)}}, \tag{1}$$

where S_p is the symmetric group of permutations of p objects. It is evident that polynomials of this type generate W_0. The number k_1 is called the rank of the homogeneous polynomial (1). Any polynomial $f \in W_0$ can be uniquely decomposed into a sum of polynomials of type (1). The highest rank of these polynomials is called the rank of f. Let $\mathfrak{U}_k$ be the set of polynomials of type (1) having rank k. In some way we order basis polynomials in $\mathfrak{U}_k$. Then we arrange the sets $\mathfrak{U}_k$ according to increasing the number k. As a result, we obtain a set of symmetric polynomials. The orthogonalization procedure of this set on the lattice Ω with respect to the weight ρ of formula (2), Section 18.7.2, gives an orthonormal basis of solutions of equation (3) of Section 18.7.2.

Let us give one of bases of solutions of this equation. For this we consider the p-parameter collection of symmetric polynomials

$$f^{(\mathbf{q})}(\mathbf{x}) = \prod_{i,j=1}^{p} \prod_{k=1}^{q_j} (x_i - a_j + k), \quad \mathbf{q} = (q_1, \dots, q_p) \in \Gamma,$$

and form the set of polynomials

$$P^{(\mathbf{q})}(\mathbf{x}) = L_0 L_1 \dots L_{k-1} f^{(\mathbf{q})}(x), \quad \mathbf{q} \in \Gamma, \tag{2}$$

where $k = q_1 + q_2 + \dots + q_p$ and L_m, $m = 0, 1, 2, \dots, k-1$, are the difference operators

$$(L_m f)(\mathbf{x})$$
$$= \sum_{j=1}^{p} \prod_{\ell \neq j} (x_\ell - x_j)^{-1} \left[\prod_{r=1}^{p} (a_r - x_j - 1)\Delta_j + \prod_{r=2}^{p+1} (a'_r - x_j)\nabla_j \right] f(\mathbf{x}) + 2m f(\mathbf{x}).$$

Polynomials (2) form a basis of solutions of equation (3), Section 18.7.2. Moreover, solution (2) corresponds to eigenvalue λ_k if $\mathbf{q} \in \Gamma \cap \Gamma_k$.

The analogous results are valid for other discrete equations. Solutions of equations (11) and (15) of Section 18.7.2 belong to the space W_1 consisting of symmetric polynomials of generalized squares $x_1^{(2)}, x_2^{(2)}, \dots, x_p^{(2)}$. Solutions of equations (7) and (14) of the same section form the space W_2, generated by those polynomials of type (1) for which the integers $k_1, k_2, \dots, k_p$ are such that $k_j \equiv 0 (\mathrm{mod}\ 2)$, $j = 1, 2, \dots, p$.

18.7.4. Continuous analogs of discrete polynomials. The orthogonal polynomials, introduced above, are many-dimensional generalizations of the orthogonal polynomials of one discrete variable. The last polynomials approximate the classical orthogonal polynomials of an continuous variable, namely, the Jacobi, Laguerre and Hermite polynomials. Moreover, polynomials of an continuous variable are obtained from the corresponding polynomials of a discrete variable by the limiting procedure. We shall show that in the case of polynomials of many discrete variables we have a similar situation. There are limiting procedures which transform our discrete equations into differential equations with partial derivatives. Solutions of the last equations form the certain types of orthogonal polynomials of many continuous variables, generalizing classical orthogonal polynomials of one continuous variable.

Let us consider discrete equation (15) of Section 18.7.2 related to the generators of the matrom of $ISO(2p+2)$. We fix $h > 0$ and make the substitution $x_j = y_j/h$, $j = 1,2,\dots,p$. We obtain the functions $P^h(\mathbf{y}) = P(\mathbf{y}/h)$, $\mathbf{y} = (y_1, y_2, \dots, y_p)$, and the discrete equation for function $P^h(\mathbf{y})$, in which all finite differences and generalized powers are taken with step h. Now we take the limit $h \to 0$; $a_k, a'_k \to \infty$, $k = 1,2,\dots,p+1$, such that $a_k h \to b_k$, $a'_k - a_k \to s_k$, $k = 1,2,\dots,p+1$. As a result, the discrete equation transforms into the differential one. We make the substitutions $y_j = \sqrt{x_j}$, $j = 1,2,\dots,p$, $b_k = \sqrt{a_k}$, $k = 1,2,\dots,p+1$, and obtain the differential equation

$$\sum_{j=1}^{p} \prod_{\ell \neq j} (x_\ell - x_j)^{-1} \prod_{k=1}^{p+1} (a_k - x_j)^{-s_k} \frac{\partial}{\partial x_j} \left[\prod_{k=1}^{p+1} (a_k - x_j)^{s_k+1} \frac{\partial}{\partial x_j} P(\mathbf{x}) \right] = \lambda P(\mathbf{x}), \quad (1)$$

$$\mathbf{x} \in X = \{(x_1, x_2, \dots, x_p) \mid a_{j+1} \le x_j \le a_j, i = 1,2,\dots,p\}.$$

Analogously, equation (11) of Section 18.7.2 leads to the differential equation

$$\begin{aligned} &\sum_{j=1}^{p} \prod_{\ell \neq j} (x_\ell - x_j)^{-1} e^{x_j} \prod_{k=2}^{p+1} (a_k - x_j)^{-s_k} \\ &\qquad \times \frac{\partial}{\partial x_j} \left[e^{-x_j} \prod_{k=2}^{p+1} (a_k - x_j)^{s_k+1} \frac{\partial}{\partial x_j} P(\mathbf{x}) \right] = \lambda P(\mathbf{x}), \end{aligned} \quad (2)$$

$$\mathbf{x} \in X = \{\mathbf{x} \in \mathbf{R}^p \mid a_2 \le x_1 < \infty; \quad a_{j+1} \le x_j \le a_j, \quad j = 2,3,\dots,p\}.$$

The limiting procedure in equation (3) of Section 18.7.2 gives the differential equation

$$\begin{aligned} &\sum_{j=1}^{p} \prod_{\ell \neq j} (x_\ell - x_j)^{-1} e^{x_j^2} \prod_{k=2}^{p+1} (a_k - x_j)^{-s_k} \\ &\qquad \times \frac{\partial}{\partial x_j} \left[e^{-x_j^2} \prod_{k=2}^{p} (a_k - x_j)^{s_k+1} \frac{\partial}{\partial x_j} P(\mathbf{x}) \right] = \lambda P(\mathbf{x}), \end{aligned} \quad (3)$$

$$\mathbf{x} \in X = \{\mathbf{x} \in \mathbf{R}^p \mid a_2 \le x_1 < \infty; \quad a_{j+1} \le x_j \le a_j, \quad j \neq 1,p; \quad -\infty < x_p \le a_p\}.$$

For $p = 1$ equations (1)-(3) take the form

$$(a_1 - x)^{-s_1} (x - a_2)^{-s_2} \frac{d}{dx} \left[(a_1 - x)^{s_1+1} (x - a_2)^{s_2+1} \frac{d}{dx} P(x) \right] = \lambda P(x), \quad (4)$$

$$x \in [a_2, a_1];$$

$$e^{x}(x-a)^{-s}\frac{d}{dx}\left[e^{-x}(x-a)^{s+1}\frac{d}{dx}P(x)\right]=\lambda P(x),\quad x\in[a,\infty); \tag{5}$$

$$e^{x^2}\frac{d}{dx}\left[e^{-x^2}\frac{d}{dx}P(x)\right]=\lambda P(x),\quad x\in\mathbf{R}. \tag{6}$$

Equation (4) describes Jacobi polynomials on $[a_2, a_1]$. equation (5) leads to Laguerre polynomials on $[a,\infty)$, and (6) is equation for Hermite polynomials.

The limiting procedure can be also fulfilled in equations (14) and (7) of Section 18.7.2. We obtain the differential equations

$$\begin{aligned}\sum_{j=1}^{p}\prod_{\ell\neq j}(x_\ell^2-x_j^2)^{-1}\prod_{k=1}^{p}(a_k^2-x_j^2)^{-s_k}&\\ \times\frac{\partial}{\partial x_j}\left[\prod_{k=1}^{p}(a_k^2-x_j^2)^{s_k+1}\frac{\partial}{\partial x_j}P(\mathbf{x})\right]&=\lambda P(\mathbf{x}),\end{aligned} \tag{7}$$

$$\mathbf{x}\in X=\{\mathbf{x}\in\mathbf{R}^p\mid a_{j+1}\le x_j\le a_j,\quad j\neq p;\quad -a_p\le x_p\le a_p\},$$

$$\begin{aligned}\sum_{j=1}^{p}\prod_{\ell\neq j}(x_\ell^2-x_j^2)^{-1}e^{x_j^2}\prod_{k=2}^{p}(a_k^2-x_j^2)^{-s_k}&\\ \times\frac{\partial}{\partial x_j}\left[e^{-x_j^2}\prod_{k=2}^{p}(a_k^2-x_j^2)^{s_k+1}\frac{\partial}{\partial x_j}P(\mathbf{x})\right]&=\lambda P(\mathbf{x}),\end{aligned} \tag{8}$$

$$\begin{aligned}\mathbf{x}\in X=\{\mathbf{x}\in\mathbf{R}^p\mid a_2\le x_1<\infty;&\\ a_{j+1}\le x_j\le a_j,\quad j=2,3,\dots,p-1;\quad |x_p|\le a_p\}.&\end{aligned}$$

Equation (7) is a many-dimensional analog of the equation for Gegenbauer polynomials and equation (8) is an analog of that for Hermite polynomials.

Let us formulate the results on polynomial solutions of equations (1)-(3). Solutions of these equations, corresponding to different eigenvalues, are orthogonal in the domain X with respect to the weights

$$\rho_1(\mathbf{x})=\prod_{k=1}^{p+1}\prod_{j=1}^{p}|a_k-x_j|^{s_k}\prod_{\ell<j}(x_\ell-x_j),$$

$$\rho_2(\mathbf{x})=\exp\left(-\sum_{j=1}^{p}x_j\right)\prod_{k=2}^{p+1}\prod_{j=1}^{p}|a_k-x_j|^{s_k}\prod_{\ell<j}(x_\ell-x_j),$$

$$\rho_3(\mathbf{x})=\exp\left(-\sum_{j=1}^{p}x_j^2\right)\prod_{k=2}^{p}\prod_{j=1}^{p}|a_k-x_j|^{s_k}\prod_{\ell<j}(x_\ell-x_j).$$

Equations (1)-(3) have polynomial solutions and they belong to the space W_0 of symmetric polynomials of p continuous variables. The eigenvalues are

$$\lambda \equiv \lambda_k = \begin{cases} k(k+s_1+\ldots+s_{p+1}+p) & \text{for equation (1),} \\ k & \text{for equation (2),} \\ -2k & \text{for equation (3),} \end{cases}$$

where $k = 0, 1, 2, \ldots$. Multiplicity of the eigenvalue λ_k is $(k+p-1)!/k!(p-1)!$.

Basis solutions of equations (1)-(3) in the space W_0 can be obtained by the orthogonalization procedure from the same sequences which were constructed in Section 18.7.3 for equation (3) of Section 18.7.2.

Let us describe detailly the spaces V_k of polynomial eigenfunctions corresponding to eigenvalues λ_k. Let

$$t_j = \prod_{i\neq j}(a_i - a_j)^{-1}\prod_{\ell=1}^{p}(x_\ell - a_j), \quad j = 1, 2, \ldots, p+1, \tag{9}$$

and let $\mathfrak{F}$ be the space of all polynomials of $t_1, t_2, \ldots, t_{p+1}$. We define in $\mathfrak{F}$ the linear operator D which acts onto monomials as

$$D(t_1^{k_1}t_2^{k_2}\ldots t_{p+1}^{k_{p+1}}) = \frac{\Gamma(s_1+1)\ldots\Gamma(s_{p+1}+1)}{\Gamma(s_1+k_1+1)\ldots\Gamma(s_{p+1}+k_{p+1}+1)}t_1^{k_1}t_2^{k_2}\ldots t_{p+1}^{k_{p+1}}. \tag{10}$$

For each fix $k \in \mathbb{Z}_+$ and every set $\alpha_1, \alpha_2, \ldots, \alpha_{p+1}$ of real numbers such that $\alpha_1 + \alpha_2 + \ldots + \alpha_{p+1} = 0$ we construct the function

$$g^{(k)}_{\alpha_1\alpha_2\ldots\alpha_{p+1}}(\mathbf{x}) = D(\alpha_1 t_1 + \ldots + \alpha_{p+1}t_{p+1})^k. \tag{11}$$

Theorem 1. *The space V_k of polynomial eigenfunctions of equation (1), corresponding to eigenvalue λ_k $(k = 0, 1, 2, \ldots)$, is the linear span of functions of type (11).*

In the case of equation (2) the analogous result is valid. The difference is that instead of functions (11) we have to take the functions

$$g^{(k)}_{\alpha_2\alpha_3\ldots\alpha_{p+1}}(\mathbf{x})$$
$$= D[\alpha_2(t_2-1)+\alpha_3(t_3-1)+\ldots+\alpha_{p+1}(t_{p+1}-1)]^k, \quad \alpha_2, \ldots, \alpha_{p+1} \in \mathbf{R}.$$

Now we consider the case of equation (3). For each set of real numbers $\alpha_2, \alpha_3, \ldots, \alpha_p$ we construct the function

$$R^{(k)}_{\alpha_2,\alpha_3,\ldots,\alpha_p}(\mathbf{x}) = D[\alpha_2(2t_2+1)+\alpha_3(2t_3+1)+\ldots+\alpha_p(2t_p+1)]^k,$$

where t_j are defined by formula (9) and the operator D by formula (10) with $k_1 = k_{p+1} = 0$.

Theorem 2. *The space V_k of solutions of equation* (3), *corresponding to eigenvalue λ_k, decomposes into the orthogonal sum $V_k = W_k^0 + W_k^1 + \ldots + W_k^k$, where W_k^ℓ is the space spanned by the functions of the form*

$$P^{(k,\ell)}_{\alpha_2,\alpha_3,\ldots,\alpha_p}(\mathbf{x}) = H_{k-\ell}(z) R^{(\ell)}_{\alpha_2,\alpha_3,\ldots,\alpha_p}(\mathbf{x}),$$

where $\alpha_2, \alpha_3, \ldots, \alpha_p \in \mathbf{R}$, $z = x_1 + x_2 + \ldots + x_p - a_2 - a_3 - \ldots - a_p$ and $H_m(z)$ is the Hermite polynomial. In particular, if $p = 2$ (i.e. for the group $U(3)$) the space W_k^ℓ are one-dimensional and the function $R^{(\ell)}_{\alpha_2,\ldots,\alpha_p}(x)$ reduces to the Laguerre polynomial.

18.7.5. Expansions of representation matrix elements. The results on spectral analysis of infinitesimal operators and of generators of matroms allow us to represent matrix elements of representations of the groups under consideration in the form of finite linear combinations of classical special functions. Before to consider these representations we give precise result on matroms of the group $ISO(n-1)$ (see formula (6) of Section 18.6.6).

Theorem 1. *The matrom $D^{\rho\mathbf{t}\mathbf{t}'}(\gamma)$ of the group $ISO(n-1)$ is representable in the form*

$$D^{\rho\mathbf{t}\mathbf{t}'}(\gamma) = (i\rho\gamma)^{-(n-3)/3} i^{\tilde{B}} J_{\tilde{B}}(\rho\gamma)\tilde{C}, \tag{1}$$

where the matrix $\tilde{B}$ is given in Theorem 2 *of Section* 18.6.6 *and $\tilde{C}$ is the diagonal matrix $\tilde{C} = diag\,\{\tilde{C}(\mathbf{x})\}$ with*

$$\tilde{C}(\mathbf{x}) = 2^{\varepsilon(n-3)/2} i^s d\sigma(\mathbf{x}), \quad \varepsilon = (-1)^{n-1}, \quad s = s_1 + s_2 + \ldots + s_p.$$

Here d is the certain multiplier and

$$\sigma(\mathbf{x}) = \prod_{\ell<j} (x_\ell^2 - x_j^2)^{-1} \quad \textit{for} \quad ISO(2p+1),$$

$$\sigma(\mathbf{x}) = \left[\prod_{k=1}^{p} (2x_k - 1) \prod_{\ell<j} (x_\ell^{(2)} - x_j^{(2)}) \right]^{-1} \quad \textit{for} \quad ISO(2p+2).$$

This theorem allows us to obtain expansion of representation matrix elements for the group $ISO(n-1)$ in Bessel functions with coefficients which are described in terms of solutions of the corresponding discrete equations.

Theorem 2. *The representation matrix elements for the group $ISO(n+1)$ are represented as*

$$d^{\rho,\mathbf{y}}_{\mathbf{tt'x}}(\gamma) = d\sqrt{\eta(\mathbf{x})\eta(\mathbf{y})}(2\rho\gamma)^{-(2p-1)/2} \sum_{k=0}^{2(a'_1-s-p)} A_k(\mathbf{x},\mathbf{y}) J_{k+s+p-1/2}(\rho\gamma),$$

where

$$A_k(\mathbf{x},\mathbf{y}) = i^k \sum_{\ell} P_k^{(\ell)}(\mathbf{x}) P_k^{(\ell)}(\mathbf{y})$$

if $n = 2p+2$, and as

$$d^{\rho,\mathbf{y}}_{\mathbf{tt'x}}(\gamma) = d\sqrt{\eta(\mathbf{x})\eta(\mathbf{y})} \left(\frac{\rho\gamma}{2}\right)^{-p} \sum_{k=0}^{a'_1-a_p-s-p} A_k(\mathbf{x},\mathbf{y}) J_{2k+s+p}(\rho\gamma),$$

where

$$A_k(\mathbf{x},\mathbf{y}) = (-1)^k \sum_{\ell} P_k^{(\ell)}(\mathbf{x}) P_k^{(\ell)}(\mathbf{y})$$

if $n = 2p+3$. Here $\{P_k^{(\ell)}(\mathbf{y})\}$ is any orthonormal basis of eigenfunctions of discrete equations (6) *and* (7) *of Section* 18.7.4, *corresponding to the eigenvalue λ_k. For example, for class 1 representations of $ISO(n-1)$ we obtain expansion of matrix elements in Bessel functions with Racah polynomials as coefficients.*

Now we consider the elements $P^{a,\mathbf{y}}_{\mathbf{tt'x}}(\mu)$ of the matrom of the group $SO(n)$. The general expression for these functions is given by formula (14) of Section 18.6.6. Elements of the projection operators R_i and S_j are determined by the formulas

$$R_i(\mathbf{x},\mathbf{y}) = J(\mathbf{x})J^{-1}(\mathbf{y})\rho(\mathbf{y}) \sum_{\ell} P_i^{(\ell)}(\mathbf{x}) P_i^{(\ell)}(\mathbf{y}),$$

$$S_j(\mathbf{x},\mathbf{y}) = I(\mathbf{x})I^{-1}(\mathbf{y})\rho(\mathbf{y}) \sum_{\ell} P_j^{(\ell)}(\mathbf{x}) P_j^{(\ell)}(\mathbf{y}).$$

Here $\rho(\mathbf{y})$ is the weight function for discrete equations (7) and (9) of Section 18.7.2, given by formulas (6) and (10) of that section, $\{P_k^{(\ell)}\}$ is any orthonormal basis of solutions of these equations, corresponding to the eigenvalue λ_k, and the explicit expressions for the functions $J(\mathbf{x})$ and $I(\mathbf{x})$ are given in [266]. In particular, in the case of the group $SO(4)$ we obtain an expansion of matrix elements in Jacobi functions with Krawtchouk polynomials as coefficients.

The other expansion of representation matrix elements $d^{a,\mathbf{y}}_{\mathbf{tt'x}}(\gamma)$ for the group $SO(n)$ can be obtained on the base of spectral analysis of the infinitesimal operator. In this case we obtain expansion in exponential functions. An analogous expansion is obtained for matrix elements of representations of the group $U(n)$ using solutions of discrete equation (3) of Section 18.7.2.

Generators of matroms have some group-theoretical sense. So, generators R and S for the group $SO(n)$ are similar to the infinitesimal operators of representations of the group $SO(n-1)$. Eigenvectors of matroms for the group $ISO(3)$ are reduced to Clebsch-Gordan coefficients of the group $SO(3)$, and for the group $ISO(4)$ they are reduced to Racah coefficients of $SO(4)$. These reasonings and above considerations propose some non-solved problems:

1. To clarify the group-theoretical sense of relations of matrom generators to infinitesimal operators and tensor products of representations.

2. To fulfil spectral analysis of infinitesimal operators of representations of different Lie groups and to evaluate the corresponding eigenfunctions as functions (or as polynomials) of many discrete variables.

3. To clarify the group-theoretical sense of the limiting formulas which transform orthogonal polynomials of discrete variables (eigenfunctions of matrom generators and of infinitesimal operators) into polynomials of continuous variables.

4. To give a group-theoretical interpretation of the obtained orthogonal polynomials of many continuous variables. Probably, they are related to the theory of spherical functions on symmetric Riemannian spaces of rank r, $r > 1$.

Chapter 19.
Modular Forms, Theta Functions and Representations of Affine Lie Algebras

19.1. Modular Forms

19.1.1. Linear-fractional transformations of the upper half-plane. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$ and let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the extended (compactified) complex plane. The formula

$$g \cdot z = \frac{az+b}{cz+d}$$

defines a linear-fractional transformation of $\bar{\mathbb{C}}$. If g is a scalar matrix , then the corresponding transformation $g \cdot z$ is identical. Therefore, we consider that g does not coincide with a scalar matrix. A matrix g is conjugate to one of the matrices

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}. \tag{1}$$

The transformations

$$z \to z + \lambda^{-1}, \quad z = cz, \ c \neq 1, \tag{2}$$

correspond to matrices (1). In the first case the matrix g and the corresponding transformation are called *parabolic* and in the second case they are called *elliptic* if $|c| = 1$ and *hyperbolic* if c is real and positive. Otherwise, a matrix and a transformation are called *loxodromic*. It follows from (2) that a non-scalar transformation g has one or two fixed points. Moreover, only parabolic matrices have one fixed point.

Since the matrices λI_2, $\lambda \in \mathbb{C}$, do not move points of the plane $\bar{\mathbb{C}}$, then it is sufficient to consider the quotient group $GL(2,\mathbb{C})/\{\lambda I_2\} \sim SL(2,\mathbb{C})$. Therefore, we restrict ourselves to the group $SL(2,\mathbb{C})$. In this case the classification can be given in terms of traces of matrices. Namely, if $g \in SL(2,\mathbb{C})$, $g \neq \pm I_2$, then

(a) g is a parabolic transformation if and only if $\mathrm{Tr}\, g = \pm 2$,
(b) g is an elliptic transformation if and only if $\mathrm{Tr}\, g$ is real and $|\mathrm{Tr}\, g| < 2$,
(c) g is a hyperbolic transformation if and only if $\mathrm{Tr}\, g$ is real and $|\mathrm{Tr}\, g| > 2$,
(d) g is a loxodromic transformation if and only if $\mathrm{Tr}\, g$ is not real.

Really, since $\det g = 1$, then g is conjugate to one of the matrices

$$\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \neq \pm 1.$$

From here we derive the necessary and sufficient condition of parabolicity and the necessary conditions of ellipticity and hyperbolicity. Let $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and let the number Tr $g = \lambda + \lambda^{-1}$ be real. If λ is real, then g has to be hyperbolic. If λ is imaginary, then λ and $\bar{\lambda}$ are the roots of the equation $x^2 - (\text{Tr } g)x + 1 = 0$ and, therefore, $\lambda\bar{\lambda} = 1$. Hence, g is an elliptic matrix. We obtain that g can not be loxodromic if Tr g is real. Consequently, the sufficient conditions of ellipticity and hyperbolicity are proved. Since g belongs to one of the types of transformations, listed above, then the necessary and sufficient condition of loxodromicity is proved.

The real axis is invariant with respect to a transformation $g \in SL(2, \mathbb{C})$ if and only if $g \in SL(2, \mathbf{R})$. Besides, as it is easy to verify

$$\text{Im } (g \cdot z) = \frac{\text{Im } z}{|cz + d|^2}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}).$$

Therefore, *the upper half-plane* $H = \{z \in \mathbb{C} | \text{Im } z > 0\}$ *of* $\mathbb{C}$ *is invariant with respect to transformations from* $SL(2, \mathbf{R})$. *Moreover,* $g \in SL(2, \mathbf{R})$ *is a holomorphic transformation of* H *and the group of all holomorphic transformations of* H *is isomorphic to the quotient group* $PSL(2, \mathbf{R}) \sim SL(2, \mathbf{R})/\{\pm I_2\}$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $i = \sqrt{-1}$, then $g \cdot i = i$ if and only if $a = d$, $b = -c$ and $a^2 + b^2 = 1$. These conditions separate in $SL(2, \mathbf{R})$ the subgroup $SO(2)$. Thus, the stationary subgroup of the point i in $SL(2, \mathbf{R})$ is $SO(2)$. The action of $SL(2, \mathbf{R})$ in H is transitive. Really, for $a > 0$ the transformation $\begin{pmatrix} a^{1/2} & ba^{-1/2} \\ 0 & a^{-1/2} \end{pmatrix}$ transfers the point i into the point $ai + b \in H$. Therefore, $H \sim SL(2, \mathbf{R})/SO(2)$. The group $SL(2, \mathbf{R})$ has no loxodromic transformations.

Let g_z is a transformation from $SL(2, \mathbf{R})$ such that $g_z \cdot i = z$, $z \in H$. Then $g_z SO(2) g_z^{-1}$ is the stationary subgroup of the point z. Since characteristic roots of all matrices of the group $SL(2, \mathbf{R})$ have the same moduli, equal to 1, then an element $g \in SL(2, \mathbf{R})$ having a fixed point on H is either elliptic or $\pm I_2$.

Let $G_z \subset SL(2, \mathbf{R})$ be the stationary subgroup of the point $z \in H \cup \{\infty\} \cup \mathbf{R}$ and let P_z be the set of parabolic elements from G_z with elements $\pm I_2$ added. Then

$$G_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \neq 0 \right\}, \quad P_\infty = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \middle| h \neq \mathbf{R} \right\}.$$

Clearly, $P_\infty \sim \mathbf{R} \otimes \{\pm 1\}$. We have $G_z = g_0 G_\infty g_0^{-1}$, $P_z = g_0 P_\infty g_0^{-1}$, where g_0 is an element from $SL(2, \mathbf{R})$ such that $g_0 \cdot \infty = z$. These reasonings show that if a matrix $g \in SL(2, \mathbf{R})$ does not coincide with $\pm I_2$ and has a fixed point on $\mathbf{R} \cup \{\infty\}$, then g is a parabolic or a hyperbolic transformation. Thus, we have proved the following statement. *If* $g \in SL(2, \mathbf{R})$ *and* $g \neq \pm I_2$, *then*

(a) *g is a parabolic transformation if and only if g has only one fixed point on* $\mathbf{R} \cup \{\infty\}$;

(b) *g is an elliptic transformation if and only if g has one fixed point z on H; the second fixed point is* $\bar{z}$;

(c) *g is a hyperbolic transformation if and only if g has two fixed points on $\mathbf{R} \cup \{\infty\}$.*

Let Γ be a subgroup of the discrete group $SL(2, \mathbb{Z})$ (which may coincide with $SL(2, \mathbb{Z})$). A point z from H is called an *elliptic point* of the group Γ if there exists an elliptic element $g \in \Gamma$ such that $g \cdot z = z$. Analogously, a point x on $\mathbf{R} \cup \{\infty\}$ is called a *parabolic point* (or a *cusp*) of Γ if there exists a parabolic element $g \in \Gamma$ such that $g \cdot x = x$. It is easy to show that if w is a parabolic (elliptic) point of Γ and $g \in \Gamma$, then $g \cdot w$ is also a parabolic (elliptic) point of Γ.

Let us give without proofs several statements on elliptic and parabolic matrices and points:

(1) *If z is an elliptic element of a discrete group $\Gamma \subset SL(2, \mathbb{Z})$, then the set $\{g \in \Gamma | g \cdot z = z\}$ is a finite cyclic group.*

(2) *Elements of finite order in Γ are elliptic elements or the matrices $\pm I_2$.*

(3) *The set of elliptic points of Γ has no limit point in H.*

(4) *Let x be a parabolic point of the group Γ and let $\Gamma_x = \{g \in \Gamma | g \cdot x = x\}$. Then the quotient group Γ / Γ_x is isomorphic to the group $\mathbb{Z}$. Every element from Γ_x is parabolic or coincides with $\pm I_2$.*

(5) *An elliptic or a parabolic element $g \in SL(2,\mathbf{R})$ cannot be conjugate in $SL(2,\mathbf{R})$ to g^{-1}.*

19.1.2. The transformation group $SL(2, \mathbb{Z})$. The group $SL(2, \mathbb{Z})$ and its subgroups are of great importance in the theory of modular forms and theta functions. Elements of $SL(2, \mathbb{Z})$ are linear-fractional transformations of H. Two points z_1 and z_2 from H are called equivalent with respect to the group Γ, $\Gamma \subset SL(2, \mathbb{Z})$, if there is an element $A \in \Gamma$ such that $A z_1 = z_2$. It is clear that $A^{-1} z_2 = z_1$. An open connected set $D' \subset H$ is called a *fundamental domain* for the group Γ if it does not contain equivalent points and if its closure contains at least one point from every class of equivalent points. It is clear that acting upon a fundamental domain D' by all transformations from Γ we cover almost all points of H. Moreover, images of the domain D' either coincide or are mutually disjoint.

As in the case of the group $SL(2, \mathbf{R})$, two matrices g_1 and g_2 from $SL(2, \mathbb{Z})$ give the same transformation of H if and only if $g_1 = \pm g_2$. The quotient group $SL(2, \mathbb{Z}) / \{\pm I_2\}$ is called the *modular group*. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{1}$$

be the matrices from $SL(2, \mathbb{Z})$. Then

$$S \cdot z = -\frac{1}{z}, \quad T \cdot z = z + 1. \tag{2}$$

It is easy to see that

$$S^2 = I_2, \quad (ST)^3 = I_2. \tag{3}$$

Let

$$D = \left\{ z \in H \,\middle|\, |z| > 1, \quad |\mathrm{Re}\ z| < \frac{1}{2} \right\}.$$

We shall prove that D is a fundamental domain for the group $SL(2,\mathbb{Z})$.

Theorem. *For every $z \in H$ there exists a matrix $g \in SL(2,\mathbb{Z})$ such that $g \cdot z \in \bar{D}$, where $\bar{D}$ is the closure of D. Let z and z' be two different equivalent points from $\bar{D}$. Then either* $\mathrm{Re}\ z = \mathrm{Re}\ z' = \pm 1/2$, $z = z \pm 1$ *or* $|z| = 1$ *and* $z' = -1/z$. *Let G_z, $G_z \subset SL(2,\mathbb{Z})$, be a stationary subgroup of a point $z \in \bar{D}$. Then $G_z = \{I_2\}$ for all points z except for $z = i$, $\rho \equiv e^{2\pi i/3}$, $-\bar{\rho} = e^{\pi i/3}$. For these exceptional points we have*

$$G_i = \{I_2, S\}, \quad G_\rho = \{I_2, ST, (ST)^2\}, \quad G_{-\bar{\rho}} = \{I_2, TS, (TS)^2\}.$$

The group $SL(2,\mathbb{Z})$ is generated by the elements S and T.

Proof. Let G be the subgroup of $SL(2,\mathbb{Z})$ generated by the matrices S and T, and let $z \in H$. Let us show that there exists a matrix $g \in G$ such that $g \cdot z \in \bar{D}$. If $g' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, then $\mathrm{Im}\ (g' \cdot z) = \dfrac{\mathrm{Im}\ z}{|cz+d|^2}$. Since c and d are integers, then the number of pairs (c,d) for which $|cz+d|$ is bounded from above by fixed number, is finite. Consequently, there exists a matrix $g' \in G$ for which $\mathrm{Im}\ (g' \cdot z)$ is maximal. From the other side, there exists an integer n for which the real part of $T^n(g' \cdot z)$ is situated between $-1/2$ and $1/2$. The point $T^n(g' \cdot z)$ belongs to $\bar{D}$. Really, it is sufficient to show that $|z'| \geq 1$. If it would be $|z'| < 1$, then $\mathrm{Im}\ (-1/z') > \mathrm{Im}\ z'$. It is not possible. Thus, if $g = T^n g'$, then $g \cdot z \in \bar{D}$. This proves the first part of our theorem.

Let $z \in \bar{D}$ and let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ be such that $g \cdot z \in \bar{D}$. Replacing, if it is necessary, the pair (z, g) by $(g \cdot z, g^{-1})$ we can assume that $\mathrm{Im}\ (g \cdot z) \geq \mathrm{Im}\ z$, that is $|cz+d| \leq 1$. The last inequality is not possible if $|c| \geq 2$. Thus, $c = 0, \pm 1$. If $c = 0$, then $d = \pm 1$ and g is the shift by $\pm b$. Since $\mathrm{Re}\ z$ and $\mathrm{Re}\ (g \cdot z)$ are situated between $-1/2$ and $1/2$, then either $b = 0$ and $g = I_2$ or $b = \pm 1$ and one of the numbers $\mathrm{Re}\ z$, $\mathrm{Re}\ (g \cdot z)$ is equal to $-1/2$ and other one to $1/2$.

If $c = 1$, then it follows from $|z + d| \leq 1$ that $d = 0$ always except for the cases $z = \rho$ and $z = -\bar{\rho}$. The case $d = 0$ gives $g \cdot z = a - \frac{1}{z}$ and the first part of our reasonings shows that $a = 0$ always except for the case $\mathrm{Re}\ z = \pm 1/2$, where $z = \rho$ or $-\bar{\rho}$ (when $a = 0, -1$ or $a = 0, 1$). The case $z = \rho$, $d = 1$ gives $g \cdot z = a - \frac{1}{1+\rho} = a + \rho$. This leads to $a = 0, 1$. The case $z = -\bar{\rho}$, $d = -1$ is analogously investigated.

The case $c = -1$ is reduced to the case $c = 1$ by changing the signs of elements of the matrix g. We now have to show that $G = SL(2,\mathbb{Z})$. Let g be a matrix from $SL(2,\mathbb{Z})$ and let z_0 is an intrinsic point of the domain $\bar{D}$. We set $z = g \cdot z_0$. We above saw that there exists a matrix $g' \in G$ such that $g' \cdot z \in \bar{D}$. The points z_0 and

$g' \cdot z = g'g \cdot z_0$ from $\bar{D}$ are equivalent with respect to the group $SL(2,\mathbb{Z})$ and one of them is intrinsic in $\bar{D}$. Therefore, these points are coinciding and $g'g = I_2$, that is, $g' \in SL(2,\mathbb{Z})$. Theorem is proved.

The set of classes of equivalent points of H with respect to the subgroup Γ is denoted by H/Γ. If $\Gamma = SL(2,\mathbb{Z})$, then we equip H/Γ with the topology such that the natural mapping φ: $H \to H/\Gamma$ is continuous. Then H/Γ is equivalent to the domain $\bar{D}$ in which equivalent points are identified. We compactify the set H/Γ, $\Gamma = SL(2,\mathbb{Z})$, joining to it the point ∞. It is easy to define the complex analytical structure on the space obtained. As a result, we have the compact Riemann surface which can be identified with the sphere. It is a *compact Riemann surface of genus* 0.

Up to an equivalence the group $SL(2,\mathbb{Z})$ has in H three elliptic points i, $\rho = e^{2\pi i/3}$, $-\bar{\rho} = e^{\pi i/3}$ and one parabolic point ∞. Any rational point on the line $\mathbf{R}$ is parabolic and equivalent to ∞ with respect to $SL(2,\mathbb{Z})$. These points exhaust all parabolic points on $\mathbf{R}$. We recommend to the reader to prove these statements.

19.1.3. Congruence subgroups of $SL(2,\mathbb{Z})$. For every integer $N \in \mathbb{Z}_+$ we define the subgroup

$$\Gamma(N) = \{g \in SL(2,\mathbb{Z}) \mid g \equiv I_2 (\mathrm{mod}\, N)\}$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 (\mathrm{mod}\, N), \quad b \equiv c \equiv 0 (\mathrm{mod}\, N) \right\}. \tag{1}$$

One can easily verify that $\Gamma(N)$ is an invariant subgroup in $SL(2,\mathbb{Z})$. The subgroups $\Gamma(N)$ are called *main congruence subgroups* of $SL(2,\mathbb{Z})$. A subgroup $\Gamma \subset SL(2,\mathbb{Z})$ is called a *congruence subgroup* if it contain some subgroup $\Gamma(N)$.

It is possible to show (see, for example, [275]) that

$$SL(2,\mathbb{Z})/\Gamma(N) \sim SL(2,\mathbb{Z}/N\mathbb{Z}). \tag{2}$$

If $N = \prod_p p^s$ is the factorization of N into a product of simple numbers, then

$$\mathbb{Z}/N\mathbb{Z} \sim \prod_p (\mathbb{Z}/p^s\mathbb{Z}),$$
$$SL(2,\mathbb{Z}/N\mathbb{Z}) \sim \prod_p SL(2,\mathbb{Z}/p^s\mathbb{Z}). \tag{3}$$

One can easily evaluate that the order of the group $SL(2,\mathbb{Z}/p^s\mathbb{Z})$ is equal to $p^{3s}(1-p^{-2})$. Therefore, according to formulas (2) and (3) the index $(\Gamma(1){:}\,\Gamma(N))$ of the subgroup $\Gamma(N)$ in the group $\Gamma(1) \sim SL(2,\mathbb{Z})$ (that is the number of elements in the quotient group $\Gamma(1)/\Gamma(N)$) is equal to $N^3 \prod_{p|N}(1-p^{-2})$, where the product is over all simple divisors of the integer N.

If Γ is a discrete subgroup of the group $SL(2,\mathbb{Z})$, then $\bar{\Gamma}$ will denote the image of Γ under the natural mapping

$$\varphi\colon SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z})/\{\pm I_2\}.$$

For the index $(\bar{\Gamma}(1)\colon \bar{\Gamma}(N))$ of the subgroup $\bar{\Gamma}(N)$ in $\bar{\Gamma}(1) \sim PSL(2,\mathbb{Z})$ we have

$$(\bar{\Gamma}(1)\colon \bar{\Gamma}(N)) = \begin{cases} N^{3/2} \prod\limits_{p|N} (1-p^{-2}) & \text{if } N > 2, \\ 6 & \text{if } N = 2. \end{cases}$$

It is clear that the fundamental domain $D(N)$ in H of the group $\Gamma(N)$ coincides (up to a set of dimension 1) with the union of $(\bar{\Gamma}(1)\colon \bar{\Gamma}(N))$ different fundamental domains of the group $SL(2,\mathbb{Z})$.

The group $\Gamma(N)$, $N > 1$, does not contain elliptic elements. Really, every elliptic element of the group $\Gamma(1) \equiv SL(2,\mathbb{Z})$ is conjugate to one of the matrices

$$\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \pm\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

But no one of these matrices is conjugate to $I_2 (\text{mod } N)$. Since $\Gamma(N)$ is an invariant subgroup in $SL(2,\mathbb{Z})$, then this leads to our assertion.

Let $\bar{\Gamma}(N)_\infty$ be the set of elements $g \in \bar{\Gamma}(N)$ such that $g \cdot \infty = \infty$. It is easy to show that

$$\bar{\Gamma}(N)_\infty = \bar{\Gamma}(N) \cap \bar{\Gamma}(1)_\infty = \left\{ \begin{pmatrix} 1 & mN \\ 0 & 1 \end{pmatrix} \middle| \, m \in \mathbb{Z} \right\}.$$

We have $(\bar{\Gamma}(1)_\infty\colon \bar{\Gamma}(N)_\infty) = N$. Since up to an equivalence the group $\bar{\Gamma}(1)$ has only one parabolic point, then $\bar{\Gamma}(N)$ has $(\bar{\Gamma}(1)\colon \bar{\Gamma}(N))/N$ non-equivalent parabolic points.

Classes of equivalent parabolic points for $\Gamma(N)$ are described by the following statement: *Let $x = a/b$ and $x' = c/d$ be parabolic points of the group $\Gamma(N)$, where a, b, c, d are integers such that $(a,b) = 1$ and $(c,d) = 1$ ((m,n) means the greatest common divisor of m and n). Then x and x' are equivalent with respect to $\Gamma(N)$ if and only if $\pm\begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} (\text{mod } N)$. The group $\Gamma(2)$ has three classes of equivalent parabolic points. We can take 0, 1, ∞ as their representatives.*

Let $H^* = H \cup \mathbb{Q} \cup \{\infty\}$, where $\mathbb{Q}$ is the set of rational numbers, and let $H^*/\Gamma(N)$ be the set of classes of equivalent points in H^* with respect to $\Gamma(N)$. As in the case of the group $\Gamma(1)$, the space $H^*/\Gamma(N)$ is a Riemann surface. Genus g_N of this surface is given by the formula

$$g_N = 1 + \frac{1}{12N}(\bar{\Gamma}(1)\colon \bar{\Gamma}(N))(N-6), \quad N > 1.$$

For $N \in \mathbb{Z}_+$ we set

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; c \equiv 0 (\mathrm{mod}\ N) \right\}.$$

Then $\Gamma_0(N)$ is a subgroup of $SL(2,\mathbb{Z})$ containing the group $\Gamma(N)$. This means that $\Gamma_0(N)$ is a congruence subgroup. One can easily derive that

$$\Gamma_0(N) = g_0^{-1} SL(2,\mathbb{Z}) g_0 \cap SL(2,\mathbb{Z}), \qquad \text{where} \quad g_0 = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows that $-I_2 \in \Gamma_0(N)$. It is proved (see [275]) that

$$(\Gamma(1) \colon \Gamma_0(N)) = (\bar{\Gamma}(1) \colon \bar{\Gamma}_0(N)) = N \prod_{p|N} (1 - p^{-1}).$$

The group $\Gamma_0(2)$ is generated by three matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If N is a simple integer, then $\Gamma_0(N)$ has only two classes of equivalent parabolic points with representatives 0 and ∞.

19.1.4. Modular forms of an integral weight. Let Γ be a subgroup of a finite index in $SL(2,\mathbb{Z})$ and let $k \in \mathbb{Z}$. A complex function $f(\tau)$ on H is called a *modular form of weight* k with respect to the group Γ if

(a) f is meromorphic on H,

(b) for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f(g \cdot \tau) \equiv f\left(\frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau), \tag{1}$$

(c) f is meromorphic in every parabolic point of Γ.

The last condition is of the following sense. If x is a parabolic point of the group Γ, then there exists a matrix $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in SL(2,\mathbf{R})$ such that $A_0 \cdot \infty = x$. Let $\Gamma_x = \{g \in \Gamma \mid g \cdot x = x\}$. Then

$$A_0^{-1} \Gamma_x A_0 \cdot \{\pm I_2\} = \left\{ \pm \begin{pmatrix} 1 & mh \\ 0 & 1 \end{pmatrix} \middle| m \in \mathbb{Z} \right\},$$

where h is some fixed positive integer. Due to condition (b) the function $F(\tau) = (c_0\tau + d_0)^{-k} f((a_0\tau + b_0)/(c_0\tau + d_0))$ satisfies relation (1) for all $g \in A_0^{-1} \Gamma_x A_0$.

Let k be an even integer. Since F is invariant with respect to the shift $\tau \to \tau + h$, then there exists the function $\Phi(q)$ which is meromorphic in the domain $0 < |q| < r$ (for some $r > 0$) and such that

$$F(\tau) = \Phi(e^{2\pi i\tau/h}).$$

In this case condition (c) means that the function $\Phi(q)$ is meromorphic in a neiborhood of the point $q = 0$.

Let now k be an odd integer. If $-I_2 \in \Gamma$, then condition (b) means that $f(\tau) = -f(\tau)$ and, therefore, we have no non-vanishing modular forms of weight k. Assume that $-I_2 \bar{\in} \Gamma$. Then $A_0^{-1}\Gamma_x A_0$ is generated either by the matrix $\begin{pmatrix} 1 & mh \\ 0 & 1 \end{pmatrix}$ or by the matrix $-\begin{pmatrix} 1 & mh \\ 0 & 1 \end{pmatrix}$. In the first case the point x is called *regular* and in the second case it is called *irregular.* If x is regular the condition (c) means the same as in the previous case. If x is irregular, then the function $F(\tau)$ satisfies the relation $F(\tau + h) = -F(\tau)$. Consequently, $F(\tau + 2h) = F(h)$. Condition (c) means that there exists the function $\Psi(q)$ which is meromorphic in a neiborhood of the point $q = 0$ and such that

$$F(\tau) = \Psi(e^{\pi i\tau/h}).$$

The function Ψ is odd.

It is easy to show that this explanation does not depend on a choice of element A_0 for which $A_0 \cdot \infty = x$. Besides, if the described condition is fulfilled for a parabolic point x, then it is also fulfilled for every parabolic point y equivalent to x.

The expression for the function $F(\tau)$ in the form of a power series in $e^{2\pi i\tau/h}$ or in $e^{\pi i\tau/h}$ is called the *Fourier expansion* of the modular form f in the parabolic point x. It is of the form

$$F(\tau) = \sum_{n \geq n_0} c_n e^{2\pi i n\tau/h}.$$

The numbers c_n are called the *Fourier coefficients.*

Let $\mathfrak{M}_k(\Gamma)$ be the set of all modular forms of weight k for the group Γ and let $G_k(\Gamma)$ be the set of forms $f \in \mathfrak{M}_k(\Gamma)$, holomorphic on H, for which in every parabolic point the functions Φ and Ψ, defined above, are holomorphic at the zero point. The last condition means that the Fourier coefficients c_n vanish for $n < 0$. The functions $f \in G_k(\Gamma)$ are called *holomorphic modular forms.*

A form $f \in G_k(\Gamma)$, for which the Fourier coefficient c_0 vanishes at every parabolic point, is called a *cusp form of weight* k. The set of cups forms of weight k is denoted by $S_k(\Gamma)$. It is easy to see that if $f_1(\tau) \in \mathfrak{M}_k(\Gamma)$, $f_2(\tau) \in \mathfrak{M}_m(\Gamma)$, then $f(\tau) \equiv f_1(\tau)f_2(\tau)$ belongs to $\mathfrak{M}_{k+m}(\Gamma)$. This statement is also valid for the sets $G_k(\Gamma)$ and $S_k(\Gamma)$. Let us formulate without proofs the following statements (their proofs can be found in [275]):

(a) *Let Γ' be a subgroup of a finite index in Γ. Then $\mathfrak{M}_k(\Gamma)$ (and $G_k(\Gamma)$, $S_k(\Gamma)$) coincides with the set of all forms f from $\mathfrak{M}_k(\Gamma')$ (respectively from $G_k(\Gamma')$ and from $S_k(\Gamma')$) such that $f(g \cdot \tau) = (c\tau + d)^k f(\tau)$ for all $g \in \Gamma$.*

(b) *For every integer k we have $\mathfrak{M}_k(\Gamma) \neq \{0\}$.*

(c) *The set $\mathfrak{M}_0(\Gamma)$ is a field. The set $\mathfrak{M}_k(\Gamma)$ is a one-dimensional vector space over the field $\mathfrak{M}_0(\Gamma)$.*

The sets $G_k(\Gamma)$ and $S_k(\Gamma)$ also are complex linear spaces. Let m be the number of non-equivalent parabolic points of the group Γ and let $e_1, \dots, e_r$ be the orders of non-equivalent elliptic elements of Γ. Let g be the genus of the surface H^*/Γ. By complicated evaluations we can derive formulas for the dimensions of the spaces $G_k(\Gamma)$ and $S_k(\Gamma)$ in terms of g, e_j and k (see, for example, [275]). If $k < 0$, then $\dim G_k(\Gamma) = 0$ and $\dim S_k(\Gamma) = 0$. If k is even, then

$$\dim G_0(\Gamma) = 1, \quad \dim S_0(\Gamma) = \begin{cases} 0 & \text{if } m > 0, \\ 1 & \text{if } m = 0, \end{cases}$$

$$\dim G_2(\Gamma) = \begin{cases} g & \text{if } m = 0, \\ g + m - 1 & \text{if } m > 0, \end{cases}$$

$$\dim S_2(\Gamma) = 2,$$

$$\dim G_k(\Gamma) = (k-1)(g-1) + \frac{k}{2}m + \sum_{i=1}^{r} \left[\frac{k(e_i - 1)}{2e_i}\right], \quad k > 2, \tag{2}$$

$$\dim S_k(\Gamma) = \dim G_k(\Gamma) - m, \quad k > 2.$$

Let now k be positive and odd. Let u and u' be respectively the numbers of non-equivalent regular and irregular parabolic points of the group Γ. Then

$$\dim G_1(\Gamma) = u/2, \quad S_1(\Gamma) = \{0\} \quad \text{if} \quad u > 2g - 2,$$

$$\dim G_k(\Gamma) = (k-1)(g-1) + \frac{1}{2}uk + \frac{1}{2}u'(k-1) + \sum_{i=1}^{r} \left[\frac{k(e_i - 1)}{2e_i}\right], \ k > 1, \tag{3}$$

$$\dim S_k(\Gamma) = \dim G_k(\Gamma) - u, \quad k > 1.$$

In (2) and (3) $[a]$ means the integral part of the number $a \in \mathbf{R}$. In particular, if $\Gamma = SL(2, \mathbb{Z})$ and k is even, $k \geq 2$, then $g = 0$, $m = 1$, $e_1 = 2$, $e_2 = 3$. In this case

$$\dim G_k(\Gamma) = \begin{cases} [k/12] & \text{if } k \equiv 2 (\text{mod } 12), \\ [k/12] + 1 & \text{if } k \not\equiv 2 (\text{mod } 12), \end{cases}$$

$$\dim S_k(\Gamma) = \begin{cases} 0 & \text{if } k = 2, \\ [k/12] - 1 & \text{if } k \equiv 2 (\text{mod } 12), \quad k > 2, \\ [k/12] & \text{if } k \not\equiv 2 (\text{mod } 12), \quad k > 2, \end{cases}$$

where $[a]$ is the integral part of a.

19.1.5. Eisenstein series. Let τ be a point from the upper half-plane H. The set $\mathbb{Z}\tau \oplus \mathbb{Z}$ is a lattice in H which will be denoted by Λ_τ. Let k be an even integer such that $k > 2$. We consider the series

$$E_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \sum_{\substack{\lambda\in\Lambda_\tau \\ \lambda\neq 0}} \frac{1}{\lambda^k}. \tag{1}$$

It absolutely converges for all $\tau \in H$. Really, since points of the lattice Λ_τ are uniformly situated, then the sum $\sum\limits_{\substack{\lambda\in\Lambda_\tau \\ \lambda\neq 0}} \frac{1}{|\lambda|^k}$ converges if the integral $\iint\limits_{|x+iy|>1} |x+iy|^{-k}dxdy$ converges. This integral is proportional to the integral $\int\limits_1^\infty t^{1-k}dt$ and, consequently, converges for $k > 2$. The series (1) is a holomorphic function of τ on H.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, then

$$\begin{aligned} E_k(g\cdot\tau) = E_k\left(\frac{a\tau+b}{c\tau+d}\right) &= \sum_{(m,n)\neq(0,0)} \left[m\left(\frac{a\tau+b}{c\tau+d}\right)+n\right]^{-k} \\ &= (c\tau+d)^k \sum_{(m,n)\neq(0,0)} \frac{1}{[(am+cn)\tau+(bm+dn)]^k}. \end{aligned}$$

If (m,n) runs over $\mathbb{Z}^2$, then $(am+cn, bm+dn)$ for fixed $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ also runs $\mathbb{Z}^2$. This means that

$$E_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k E_k(\tau). \tag{2}$$

Let us evaluate the Fourier series expansion of the function $E_k(\tau)$. For this we rearrange terms in the sum (1):

$$\begin{aligned} E_k(\tau) &= \sum_{n\neq 0} \frac{1}{n^k} + \sum_{m\neq 0}\left(\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}\right) \\ &= 2\sum_{n=1}^{\infty} \frac{1}{n^k} + 2\sum_{m=1}^{\infty}\left(\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}\right). \end{aligned} \tag{3}$$

The series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{4}$$

converges for Re $z > 1$ and define the *Riemann zeta function.* The terms in the parantheses on the right hand side of formula (3) are periodic with respect to the shift $\tau \to \tau + 1$. To expand them into the Fourier series we use the formula

$$\pi \cot \pi z = \frac{1}{z} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} + \frac{1}{z-m} \right). \tag{5}$$

Setting here $q = e^{2\pi i z}$ we have

$$\begin{aligned} \pi \cot \pi z &= \frac{\pi \cos \pi z}{\sin \pi z} = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \\ &= \pi i \frac{q+1}{q-1} = \pi i \left(1 - 2 \sum_{n=0}^{\infty} q^n \right). \end{aligned} \tag{6}$$

Comparing the right hand sides of formulas (5) and (6) and successively differentiating the equation obtained one receive

$$\sum_{m \in \mathbf{Z}} \frac{1}{(z+m)^2} = (2\pi i)^2 \sum_{n=1}^{\infty} n q^2,$$

$$-2 \sum_{m \in \mathbf{Z}} \frac{1}{(z+m)^3} = (2\pi i)^3 \sum_{n=1}^{\infty} n^2 q^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(-1)^k (k-1)! \sum_{m \in \mathbf{Z}} \frac{1}{(z+m)^k} = (2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Therefore, it follows from (3) that if k is even and $k > 2$, then

$$E_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{k-1} q^{mn}. \tag{7}$$

This formula can be represented in the form

$$E_k(\tau) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{8}$$

where $q = e^{2\pi i z}$ and $\sigma_s(n)$ is the sum of the numbers d^s over all positive divisors d of the number n:

$$\sigma_s(n) = \sum_{d|n} d^s.$$

If Im $\tau \to \infty$, then $E_k(\tau) \to 2\zeta(k)$. Thus, we proved that

$$E_k(\tau) \in G_k(SL(2,\mathbb{Z})). \tag{9}$$

The modular forms $E_k(\tau)$, $k \in 2\mathbb{Z}_+$, $k > 2$, are called the *Eisenstein series.* Sometimes they are normalized in such way that the scalar summand in expansion (8) is equal to 1. Then we obtain the series

$$E_k^*(\tau) = 1 + \frac{(2\pi i)^k}{(k-1)!\zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{10}$$

The numbers

$$B_r = \frac{\zeta(2r)(2r)!}{2^{2r-1}\pi^{2r}}$$

are called the *Bernoulli numbers.* Therefore,

$$E_k^*(\tau) = 1 + (-1)^{k/2} \frac{2k}{B_{k/2}} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{10'}$$

Using the definition of the function $\zeta(z)$ one can evaluate that

$$\zeta(4) = \frac{(2\pi)^4}{12 \cdot 120}, \quad \zeta(6) = \frac{(2\pi)^6}{216 \cdot 280}.$$

We set

$$\begin{aligned} g_2(\tau) &= 60E_4(\tau), \quad g_3(\tau) = 140E_6(\tau), \\ \Delta(\tau) &= g_2(\tau)^3 - 27g_3(\tau)^2, \quad J(\tau) = \frac{12^3 g_2(\tau)^3}{\Delta(\tau)}. \end{aligned} \tag{11}$$

The following statement describe the properties of the functions $\Delta(\tau)$ and $J(\tau)$. *The function $\Delta(\tau)$ is a cusp form of weight 12 for the group $SL(2,\mathbb{Z})$ and $\Delta(\tau) \neq 0$ for all points $\tau \in H$. The function $J(\tau)$ is a modular form of weight 0 for the same group with the Fourier series expansion at the infinity of the form*

$$J(\tau) = q^{-1} + \sum_{n=0}^{\infty} c_n q^n, \quad q = e^{2\pi i \tau}, \tag{12}$$

where c_n are integral coefficients. Every modular form of weight 0 for $SL(2,\mathbb{Z})$ is a rational function of $J(\tau)$.

We give the proof of the first part of this statement. The proof of the second part can be found, for example, in [275]. Since $g_2 \in G_4(\Gamma)$, $g_3 \in G_6(\Gamma)$, $\Gamma =$

$SL(2, \mathbb{Z})$, and $G_m(\Gamma)G_n(\Gamma) \subset G_{m+n}(\Gamma)$, then $\Delta \in G_{12}(\Gamma)$. In the same way it is proved that $J \in \mathfrak{M}_0(\Gamma)$. We set

$$X = \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad Y = \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

Then

$$g_2(\tau) = (2\pi)^4 \left[\frac{1}{12} + 20X\right], \quad g_3(\tau) = (2\pi)^6 \left[\frac{1}{216} - \frac{7}{3}Y\right].$$

Therefore,

$$\begin{aligned}\frac{1}{(2\pi)^{12}}\Delta(\tau) &= \frac{1}{12}(5X + 7Y) + 100X^2 + 20^3X^3 - 3 \cdot 7^2Y^2 \\ &= \sum_{n=1}^{\infty}\sum_{d|n} \frac{5d^3 + 7d^5}{12} q^n + \sum_{n>1} a_n q^n\end{aligned}$$

with the integral coefficients a_n. Besides, $d^5 \equiv d^3 \pmod{12}$ for all integral d. Thus,

$$(2\pi)^{-12}\Delta(\tau) = \sum_{n=1}^{\infty} b_n q^n, \quad b_1 = 1, \quad b_n \in \mathbb{Z}.$$

Consequently, $\Delta \in S_{12}(\Gamma)$ and the Fourier series expansion for J has the form stated.

The form $\Delta(\tau)$ is representable in the form

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24} = (2\pi)^{12} q(q;q)_{\infty}^{24}. \tag{13}$$

This formula is proved in the following way. It is derived that the function $F(\tau) = q(q;q)_{\infty}^{24}$ is a cusp form of weight 12 for the group $SL(2, \mathbb{Z})$. The space of cusp forms of weight 12 for $SL(2, \mathbb{Z})$ is one-dimensional. Therefore, $\Delta(\tau)$ is multiple to $F(\tau)$. Evaluating the multiplier, which connects the coefficients at q^1, we derive formula (13). The details of this proof can be found in [274].

19.1.6. Modular forms with multiplier system. In Section 19.1.4 we defined modular forms of an integral weight. There are more general definition of modular forms. Let $k \in \mathbf{R}$ and let Γ be a subgroup of a finite index in $SL(2, \mathbb{Z})$. Let χ be a complex character of Γ, that is a function on Γ such that $|\chi(g)| = 1$ and $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$. A complex function f on H is called a *modular form of weight k and multiplier system χ* for the group Γ if

(a) f is holomorphic on H,

(b) for all $g \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f(g \cdot \tau) \equiv f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(g)(c\tau + d)^k f(\tau),$$

(c) f is meromorphic in every parabolic point of the group Γ.

A modular form f of weight k and multiplier system χ is called a *holomorphic modular form* (a *cups form*) if it is holomorphic (vanishes) at all parabolic points of the group Γ.

Modular forms with multiplier system can be obtained from theta functions $\theta(z, \tau)$ by setting $z = 0$. In particular, setting $z = 0$ in the Jacobi theta function $\theta_3(z, \tau)$ (see formula (3) of Section 19.2.1) we obtain the modular form of weight 1/2:

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2}. \tag{1}$$

Modularity of $\theta(\tau)$ and other properties of this function follow from properties of the theta function $\theta_3(z, \tau)$ which will be derived in Section 19.2. The form $\theta(\tau)$ can be constructed with the help of the representation of the discrete series of the group $\widetilde{SL}(2, \mathbb{Z})$ (the universal covering group for $SL(2, \mathbb{Z})$; see Section 6.1.5) with highest weight 1/4. The details of this construction can be found in [192].

The Mellin transform connects modular form (1) with the Riemann zeta function $\zeta(z)$. Really, we apply the Mellin transform

$$(\mathcal{M}f)(s) \equiv \tilde{f}(s) = \int_0^\infty f(x) x^{s-1} dx$$

to the function $f(x) = 2 \sum_{n=1}^{\infty} \exp(-\pi n^2 x)$. For this function we have

$$1 - f(-i\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} = \theta(\tau)$$

or $f(x) = \theta(ix) - 1$. It follows from the properties of the theta function $\theta_3(z, \tau)$ (see Section 19.2) that

$$|f(x)| \le c e^{-\pi x} \quad \text{for} \quad x \to \infty,$$
$$|f(x)| \le c' e^{-1/2} \quad \text{for} \quad x \to 0.$$

Therefore, if Re $s > 1$, then

$$\mathcal{M}\Big(\theta(ix) - 1\Big)\left(\frac{s}{2}\right) = 2\int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\pi n^2 x}\right) x^{(s-2)/2} dx$$

and this integral absolutely converges. Thus, we can invert the order of differentiation and summation. Making the substitution $y = \pi n^2 x$ and taking into account the definition $\Gamma(s) = \int\limits_0^\infty t^{s-1}e^{-t}dt$, Re $s > 0$, of the Γ-function one has

$$\mathcal{M}\left(\theta(ix) - 1\right)\left(\frac{s}{2}\right) = 2\pi^{-s/2}\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right)\int\limits_0^\infty e^{-y}y^{(s-2)/2}dy$$
$$= 2\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right).$$

Consequently, for Re $s > 1$ we receive

$$2\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \int\limits_0^\infty (\theta(ix) - 1)x^{(s-2)/2}dx. \tag{2}$$

Properties of the function $\zeta(s)$ follows from those of the function $\theta(\tau)$. Really, we represent formula (2) in the form

$$2\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \int\limits_1^\infty (\theta(ix) - 1)x^{(s-2)/2}dx + \int\limits_0^1 (\theta(ix) - 1)x^{(s-2)/2}dx.$$

The first integral converges for all $s \in \mathbb{C}$ and defines an entire function. With the help of the identity $\theta(ix) = x^{-1/2}\theta(i/x)$ the second integral can be represented in the form

$$\int\limits_0^1 (\theta(ix) - 1)x^{(s-2)/2}dx = \int\limits_0^1 \theta(i/x)x^{-1/2}x^{(s-2)/2}dx - \int\limits_0^1 x^{(s-2)/2}dx$$
$$= \int\limits_1^\infty \theta(iy)y^{(-s-1)/2}dy - \frac{2}{s}$$
$$= \int\limits_1^\infty (\theta(iy) - 1)y^{(-s-1)/2}dy + \int\limits_1^\infty y^{-(s+1)/2}dy - \frac{2}{s}$$
$$= \int\limits_1^\infty (\theta(iy) - 1)y^{(-s-1)/2}dy - \frac{2}{1-s} - \frac{2}{s}.$$

The last integral here gives entire function in s. We have

$$\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) = -\frac{1}{s} - \frac{1}{1-s} + \frac{1}{2}\int\limits_1^\infty (\theta(ix) - 1)(x^{s/2} - x^{(1-s)/2})\frac{dx}{x}. \tag{3}$$

It follows from here that $\pi^{-s/2}\zeta(s)\Gamma(s/2)$ is a meromorphic function with simple poles at $s = 0$ and at $s = 1$. Since $\Gamma(s)$ has a simple pole at $s = 0$, then the Riemann zeta function $\zeta(s)$ admits meromorphic continuation onto the whole complex plane with a simple pole at the point $s = 1$. Formula (3) shows that the function $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \pi^{(s-1)/2}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right). \tag{4}$$

These results can be generalized in the following way. Let f be a modular form of weight k for the group $\Gamma(N)$. For simplicity we assume that k is a non-negative integer. Since $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N)$ and, consequently, $f(z+N) = f(z)$, then f has the Fourier series expansion

$$f(\tau) = \sum_{m=0}^{\infty} a_m e^{2\pi i m\tau/N}.$$

We associate with f the series

$$Z_f(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \tag{5}$$

which is called the *Dirichlet series* of the function f.

The following statement describes properties of Dirichlet series. *The Dirichlet series* (5) *converges for Re $s > k$ and can be meromorphically continued onto the whole complex plane with one simple pole at $s = k$. Moreover, the space $G_k(\Gamma(N))$ is representable in the form $G_k^+ \oplus G_k^-$ such that if $f \in G_k^\varepsilon$, $\varepsilon = \pm$, then $Z_f(s)$ satisfies the functional equation*

$$\left(\frac{N}{2\pi}\right)^s \Gamma(s) Z_f(s) = \varepsilon \left(\frac{N}{2\pi}\right)^{k-s} \Gamma(k-s) Z_f(k-s).$$

We recommend to the reader to prove that the Dirichlet series of the Eisenstein series $E_k(\tau)$ coincides with

$$Z_{E_k}(s) = 2(-2\pi i)^k [(n-1)!]^{-1} \zeta(s)\zeta(s-k+1).$$

19.2. Theta Functions

19.2.1. The Jacobi theta functions. The Jacobi theta functions θ_1, θ_2, θ_3, θ_4 depend on two variables $z \in \mathbb{C}$ and $\tau \in H$, where H is the upper half-plane in $\mathbb{C}$. They are given by the series

$$\theta_1(z,\tau) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau(n-1/2)^2} e^{i\pi(2n-1)z}, \tag{1}$$

$$\theta_2(z,\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n-1/2)^2} e^{i\pi(2n-1)z}, \tag{2}$$

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{2\pi i n z}, \tag{3}$$

$$\theta_4(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\tau n^2} e^{2\pi i n z}. \tag{4}$$

It is evident that

$$\theta_1(z,\tau) = 2e^{i\pi\tau/4} \sum_{n=0}^{\infty} (-1)^n e^{i\pi\tau n(n+1)} \sin[(2n+1)\pi z], \tag{1'}$$

$$\theta_2(z,\tau) = 2e^{i\pi\tau/4} \sum_{n=0}^{\infty} e^{i\pi\tau n(n+1)} \cos[(2n+1)\pi z], \tag{2'}$$

$$\theta_3(z,\tau) = 1 + 2\sum_{n=1}^{\infty} e^{i\pi\tau n^2} \cos(2n\pi z), \tag{3'}$$

$$\theta_4(z,\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n e^{i\pi\tau n^2} \cos(2n\pi z). \tag{4'}$$

The functions θ_1, θ_2, θ_4 can be expressed in terms of the function θ_3. Really, it is easy to verify that

$$\theta_3\left(z + \frac{1}{2}, \tau\right) = \theta_4(z,\tau). \tag{5}$$

We also have

$$\begin{aligned} \theta_3\left(z + \frac{\tau}{2}, \tau\right) &= \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{2\pi i n(z+\tau/2)} \\ &= \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+1/2)^2} e^{-i\pi\tau/4} e^{2\pi i n z} \\ &= e^{-i\pi\tau/4} e^{-i\pi z} \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+1/2)^2} e^{\pi i(2n+1)z} \\ &= e^{-i\pi\tau/4} e^{-i\pi z} \theta_2(z,\tau). \end{aligned} \tag{6}$$

In the same way it is shown that

$$\theta_3\left(z+\frac{1}{2}+\frac{\tau}{2},\tau\right)=ie^{-i\pi\tau/4}e^{-i\pi z}\theta_1(z,\tau). \tag{7}$$

Therefore, we shall study the function

$$\theta(z,\tau)\equiv\theta_3(z,\tau)=\sum_{n=-\infty}^{\infty}e^{i\pi\tau n^2}e^{2\pi inz}. \tag{8}$$

Let us note that $\theta(-z,\tau)=\theta(z,\tau)$.

If $|\text{Im } z|<c$, $\text{Im }\tau>\varepsilon$, then

$$|e^{i\pi\tau n^2}e^{2\pi inz}|<[\exp(-\pi\varepsilon)]^{n^2}(\exp 2\pi c)^n.$$

Hence, if one choose n_0 such that $[\exp(-\pi\varepsilon)]^{n_0}\exp 2\pi c<1$, then it follows from the inequality

$$|e^{i\pi\tau n^2}e^{2\pi inz}|<[\exp(-\pi\varepsilon)]^{n^2-nn_0}$$

that the series in formula (8) converges rapidly. In this reason this series converges absolutely and uniformly on any compact subset of $\mathbb{C}\times H$.

It is easy to verify that

$$\theta(z+1,\tau)=\theta(z,\tau),$$

$$\theta(z+\tau,\tau)=e^{-i\pi\tau}e^{-2\pi iz}\theta(z,\tau),$$

that is, the *function* $\theta(z,\tau)$ *is quasi-periodic with respect to the lattice* $\Lambda_\tau\in\mathbb{C}$ *generated by the numbers* 1 *and* τ. A shift by any element of the lattice Λ_τ gives

$$\theta(z+m\tau+n,\tau)=e^{-i\pi\tau m^2}e^{-2\pi imz}\theta(z,\tau). \tag{9}$$

Conversely, *if any entire function* $f(z)$ *is quasi-periodic on the lattice* Λ_τ *and*

$$f(z+1)=f(z),\quad f(z+\tau)=\exp(-i\pi\tau-2\pi iz)f(z),$$

where 1 *is a least period and* τ *is a least quasi-period, then up to a constant we have* $f(z)=\theta(z,\tau)$. Really, because of the equality $f(z+1)=f(z)$ the Fourier series for $f(z)$ can be represented as

$$f(z)=\sum_{n\in\mathbb{Z}}a_n\exp 2\pi inz,\quad a_n\in\mathbb{C}.$$

Expressing $f(z+\tau+1)$ in terms of $f(z)$ we receive

$$f(z+\tau+1)=\exp[a(z+1)+b]f(z+1)=(\exp a)\exp(az+b)f(z),$$

where $a = -2\pi i$, $b = -i\pi\tau$. Substituting here the Fourier-series expansion for f we obtain

$$\sum_{n\in\mathbb{Z}} a_n \exp(2\pi in\tau)\exp(2\pi inz) = f(z+\tau)$$
$$= \exp(-2\pi iz + b)f(z) = \sum_{n\in\mathbb{Z}} a_n \exp[2\pi i(n-1)z]\exp b$$
$$= \sum_{n\in\mathbb{Z}} a_{n+1}(\exp b)\exp(2\pi inz).$$

This means that for all $n \in \mathbb{Z}$ we have

$$a_n = a_{n+1}\exp(b - 2\pi in\tau),$$

that is

$$a_n = a_0 \exp[-nb + i\pi n(n-1)\tau).$$

Therefore,

$$f(z) = a_0 \sum_{n\in\mathbb{Z}} \exp(-nb - i\pi n\tau)\exp(i\pi n^2\tau + 2\pi inz)$$
$$= a_0 \sum_{n\in\mathbb{Z}} \exp(i\pi n^2\tau + 2\pi inz) = a_0\theta(z,\tau).$$

It proves our assertion.

19.2.2. Functional equation. We have studied the properties of $\theta(z,\tau)$ considered as a function of z. The function $\theta(z,\tau)$ is also symmetric (up to a multiplier) with respect to some simultaneous transformations of z and τ. These transformations form the group isomorphic to the subgroup N of the group $SL(2,\mathbb{Z})$ which consists of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ with even ab and cd.

In order to derive the corresponding transformations we note that the replacement of z by $z+1$ leads to multiplication of the function $\theta((c\tau+d)z,\tau)$ by the exponential function. It is easy to select a compensating multiplier for $\theta((c\tau+d)z,\tau)$ which turns it into a periodic function. Namely, if

$$\psi(z,\tau) = \exp(i\pi c(c\tau+d)z^2)\theta((c\tau+d)z,\tau),$$

then $\psi(z+1,\tau) = \psi(z,\tau)$ since cd is even. The quasi-periodicity of the function θ with respect to the shift $z \to z+\tau$ turns ψ into a quasi-periodic function:

$$\psi\left(z + \frac{a\tau+b}{c\tau+d}, \tau\right) = \exp\left(-i\pi\frac{a\tau+b}{c\tau+d} - 2\pi iz\right)\psi(z,\tau). \qquad (1)$$

Really, according to the definition of ψ we have

$$\frac{\psi\left(z+\frac{a\tau+b}{c\tau+d},\tau\right)}{\theta((c\tau+d)z+a\tau+b,\tau)} = \exp\left[i\pi c(c\tau+d)z^2+2\pi icz(a\tau+b)+i\pi c\frac{(a\tau+b)^2}{c\tau+d}\right].$$

Moreover,

$$\frac{\theta((c\tau+d)z+a\tau+b,\tau)}{\psi(z,\tau)} = \exp[-i\pi\tau a^2-2\pi iaz(c\tau+d)-\pi ic(c\tau+d)z^2].$$

Let us multiply these equalities and take into account that $ad-cb=1$. We obtain

$$\frac{\psi\left(z+\frac{a\tau+b}{c\tau+d},\tau\right)}{\theta(z,\tau)} = \exp\left[-i\pi z-\frac{i\pi}{c\tau+d}(a^2\tau d-2abc\tau-b^2c)\right]. \tag{2}$$

Since

$$a^2\tau d-2abc\tau-b^2c=a(ad-bc)\tau-ab(c\tau+d)+b(ad-bc)=(a\tau+b)-ab(c\tau+d)$$

and the product ab is even, then the relation (1) follows from (2).

The function $\psi(z,\tau)$ is quasi-periodic on the lattice $\Lambda_{\tau'}$ where $\tau'=(a\tau+b)/(c\tau+d)$ (with an appropriate exponential multiplier). This quasi-periodicity coincides with the quasi-periodicity of the function $\theta(z,\tau')$. Then according to the above reasonings we have

$$\psi(z,\tau)=\varphi(\tau)\theta(z,(a\tau+b)/(c\tau+d)),$$

where $\varphi(\tau)$ is an appropriate function. In order to find this function let us note that it follows from the Fourier-series expansion for the function $\theta(z,\tau)$ that $\int_0^1\theta(z,\tau)dz=1$. In this reason

$$\varphi(\tau)=\int_0^1\theta(z,\tau)dz=\int_0^1\exp(i\pi c(c\tau+d)z^2)\theta((c\tau+d)z,\tau)dz. \tag{3}$$

We have $\varphi(\tau)=d=\pm1$ if $c=0$. Therefore, we can consider that $c>0$. Substituting into (3) the series for the function θ we obtain after regrouping the summands that

$$\varphi(\tau)=\int_0^1\sum_{n\in\boldsymbol{Z}}\exp\left[i\pi(cz+n)^2\left(\tau+\frac{d}{c}\right)-i\pi n^2\frac{d}{c}\right]dz$$
$$=\sum_{n\in\boldsymbol{Z}}\exp\left(-i\pi n^2\frac{d}{c}\right)\int_0^1\exp\left[i\pi(cz+n)^2\left(\tau+\frac{d}{c}\right)\right]dz.$$

Since cd is even, then

$$\exp(-i\pi d(n+c)^2/c) = \exp\left(-i\pi n^2 \frac{d}{c}\right)$$

and we obtain

$$\varphi(\tau) = \sum_{1\leq n\leq c} \exp\left(-i\pi n^2 \frac{d}{c}\right) \int_{-\infty}^{\infty} \exp\left[i\pi c^2 z^2 \left(\tau + \frac{d}{c}\right)\right] dz. \qquad (4)$$

Using the substitution $\tau = it - d/c$ we receive for this integral the expression

$$\int_{-\infty}^{\infty} \exp(-\pi c^2 z^2 t) dz = \frac{1}{ct^{1/2}} \int_{-\infty}^{\infty} \exp(-\pi u^2) du = \frac{1}{ct^{1/2}},$$

where $t > 0$. Analytic continuation in τ, $\text{Im}\,\tau > 0$, gives

$$\int_{-\infty}^{\infty} \exp\left[i\pi c^2 z^2 \left(\tau + \frac{d}{c}\right)\right] dy = \frac{1}{c[-i(\tau + d/c)]^{1/2}},$$

where the branch of the root with positive real part is taken.

The sum before the integral in formula (4) is the well known Gauss sum

$$S_{d,c} = \sum_{1\leq n\leq c} \exp\left(-i\pi n^2 \frac{d}{c}\right).$$

Taking the expression for this sum (see, for example, [192]) we obtain the functional equation for $\theta(z,\tau)$ which is of the form

$$\theta\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = \zeta(c\tau + d)^{1/2} \left(\exp \frac{i\pi c z^2}{c\tau + d}\right) \theta(z,\tau), \qquad (5)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are matrices from $SL(2,\mathbb{Z})$ for which ab and cd are even, and ζ is a root of unity of degree 8. In order to define ζ exactly, we suppose that either $c > 0$ or $c = 0$ and $d > 0$ (we multiply $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by -1 if it is necessary). Now $\text{Im}\,(c\tau + d) > 0$ and we can take the meaning of $(c\tau + d)^{1/2}$ from the first quadrant. Then

$$\zeta = i^{(d-1)/2} \left(\frac{c}{|d|}\right)$$

if c is even and d is odd,

$$\zeta = \exp(-i\pi c/4)\left(\frac{d}{c}\right)$$

if c is odd and d is even, where $\left(\frac{x}{y}\right)$ is the Legendre-Jacobi symbol[1].

In particular, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ where b is even, then

$$\theta(z, \tau + b) = \theta(z, \tau), \tag{6}$$

and if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$\theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = \exp\left(-\frac{i\pi}{4}\right)\tau^{1/2}\exp\frac{i\pi z^2}{\tau}\theta(z, \tau). \tag{7}$$

Let us define the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \qquad (z, \tau) \to \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right)$$

of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ on $\mathbb{C} \times H$. It is easy to show that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z, \tau) = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix}(z, \tau).$$

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ leaves the function $\theta(z, \tau)$ invariant (up to a multiplier) if ab and cd are even. The other matrices from $SL(2, \mathbb{Z})$ transform

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[1] *The Legendre-Jacobi symbol* $\left(\frac{k}{u}\right)$ is defined for all $k = 0, \pm 1, \pm 2, \ldots$ and for all $u = 1, 3, 5, 7 \ldots$. If $u = p$ is odd simple integer, then $\left(\frac{k}{p}\right) = 1$ if p is not a divisor of k and k is a quadratic residue (mod p) (that is, the congruence $x^2 \equiv k (\mathrm{mod}\ p)$ has an integral solution x), $\left(\frac{k}{p}\right) = -1$ if p is not a divisor of k and k is not a quadratic residue (mod p), and $\left(\frac{k}{p}\right) = 0$ if p is a divisor of k. If $u = p_1 p_2 \ldots p_r$ where p_j are odd simple numbers, then

$$\left(\frac{k}{u}\right) = \left(\frac{k}{p_1}\right)\left(\frac{k}{p_2}\right)\cdots\left(\frac{k}{p_r}\right).$$

It is possible to show that if u and v are odd positive integers and $(u, v) = 1$, then

$$\left(\frac{u}{v}\right)\left(\frac{v}{u}\right) = (-1)^{(u-1)(v-1)/4}, \quad \left(\frac{-1}{u}\right) = (-1)^{(u-1)/2}, \quad \left(\frac{2}{u}\right) = (-1)^{(u^2-1)/8}.$$

functions θ_1, θ_2, θ_3, θ_4 among themselves. Since the group $SL(2,\mathbb{Z})$ is generated by the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then it is sufficient to give the action of these elements onto θ_1, θ_2, θ_3, θ_4. We have

$$(T\theta_3)(z,\tau) = \theta_3(z,\tau+1) = \theta_4(z,\tau),$$

$$(T\theta_4)(z,\tau) = \theta_4(z,\tau+1) = \theta_3(z,\tau),$$

$$(T\theta_2)(z,\tau) = \theta_2(z,\tau+1) = e^{\pi i/4}\theta_2(z,\tau),$$

$$(T\theta_1)(z,\tau) = \theta_1(z,\tau+1) = e^{\pi i/4}\theta_1(z,\tau),$$

$$(S\theta_3)(z,\tau) = \theta_3\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{i\pi z^2/\tau}\theta_3(z,\tau),$$

$$(S\theta_4)(z,\tau) = \theta_4\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{i\pi z^2/\tau}\theta_2(z,\tau),$$

$$(S\theta_2)(z,\tau) = \theta_2\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{1/2}e^{i\pi z^2/\tau}\theta_4(z,\tau),$$

$$(S\theta_1)(z,\tau) = \theta_1\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -(i\tau)^{1/2}e^{i\pi z^2/\tau}\theta_1(z,\tau).$$

The functional equation (5) for $\theta(z,\tau)$ together with the limit property for $Im\ \tau \to +\infty$ characterizes the theta function completely. Namely, $\theta(z,\tau)$ is a single holomorphic function $f(z,\tau)$ on $\mathbb{C}\times H$ such that

$$\begin{aligned}
&(a)\quad f(z+1,\tau) = f(z,\tau),\\
&(b)\quad f(z+1,\tau) = \exp(-i\pi\tau - 2\pi i z)f(z,\tau),\\
&(c)\quad f\left(z+\frac{1}{2},\tau+1\right) = f(z,\tau),\\
&(d)\quad f\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-i\pi)^{1/2}\exp(i\pi z^2/\tau)f(z,\tau),\\
&(e)\quad \lim_{\mathrm{Im}\ \tau\to+\infty} f(z,\tau) = 1 \quad \text{for all} \quad z\in\mathbb{C}.
\end{aligned}$$

The property (c) of the theta function $\theta(z,\tau)$ is proved in the following way:

$$\theta\left(z+\frac{1}{2},\tau+1\right) = \sum_{n\in\mathbb{Z}}\exp\left[i\pi n^2(\tau+1) + 2\pi i n\left(z+\frac{1}{2}\right)\right]$$
$$= \sum_{n\in\mathbb{Z}}(-1)^{n^2+n}\exp(i\pi n^2\tau + 2\pi i n z) = \theta(z,\tau).$$

Other properties follow from formulas (3) and (3′) of Section 19.2.1. In order to show that properties (a)-(e) define the theta function $\theta(z,\tau)$ we take a function $f(z,\tau)$ which satisfies these properties. Then it follows from (a) and (b) that $f(z,\tau)=g(\tau)\theta(z,\tau)$ where $g(\tau)$ is a holomorphic function on H. From (c) and (d) we derive that

$$g(\tau+1)\equiv g(T\tau)=g(\tau), \quad g\left(-\frac{1}{\tau}\right)\equiv g(S\tau)=g(\tau),$$

that is, $g(\tau)$ is an invariant (with respect to $SL(2,\mathbb{Z})$) holomorphic function. According to property (e) $g(\tau)\to 1$ when $\operatorname{Im}\tau\to+\infty$. It means that $g(\tau)$ is bounded outside of some horizontal strip. Then the $SL(2,\mathbb{Z})$ invariance means that $g(\tau)$ is bounded everywhere. Therefore, if the module $|g(\tau)-1|$ is not identically equal to zero, then it has a positive maximum in some point of the fundamental domain D of the discrete group $SL(2,\mathbb{Z})$. It is not possible. It means that $g(\tau)\equiv 1$. Our assertion is proved.

19.2.3. The Jacobi theta functions and the heat equation. Putting $z=x\in\mathbf{R}$, $\tau=it$, $t>0$, into $\theta(z,\tau)$ we obtain

$$\theta(x,it)=1+2\sum_{n=1}^{\infty}\exp(-\pi n^2 t)\cos(2\pi n x). \tag{1}$$

Thus, $\theta(x,it)$ is a real function for which $\theta(x+1,it)=\theta(x,it)$. It follows from (1) that

$$\frac{\partial}{\partial t}\theta(x,it)=2\sum_{n=1}^{\infty}(-\pi n^2)\exp(-\pi n^2 t)\cos(2\pi n x),$$

$$\frac{\partial^2}{\partial x^2}\theta(x,it)=2\sum_{n=1}^{\infty}(-4\pi^2 n^2)\exp(-\pi n^2 t)\cos(2\pi n x),$$

that is, $\theta(x,it)$ satisfies the heat equation

$$\left(\frac{\partial}{\partial t}-\frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\right)y=0.$$

Therefore, *the theta function can be defined as a unique solution of the heat equation satisfying some periodic initial condition at* $t=0$. In order to find a behavior of $\theta(x,it)$ when $t\to 0$ we integrate $\theta(x,it)$ with the periodic weight function $f(x)=\sum_m a_m\exp(2\pi i m x)$. Then

$$\int_0^1\theta(x,it)f(x)dx=\int_0^1\sum_{m,n}a_m\exp(-\pi n^2 t)\exp[2\pi i(m+n)x]dx$$

$$=\sum_{m,n}a_m\exp(-\pi n^2 t)\int_0^1\exp[2\pi i(n+m)x]dx=\sum_n a_{-n}\exp(-\pi n^2 t).$$

Hence,

$$\lim_{t\to 0}\int_0^1 \theta(x,it)f(x)dx = \lim_{t\to 0}\sum_n a_n \exp(-\pi n^2 t) = \sum_n a_n = f(0).$$

It means that *when $t \to 0$, then $\theta(x,it)$ tends in the space of generalized functions to the sum of the delta-functions localized at all integral points $x \in \mathbb{Z}$. Thus, $\theta(x,it)$ is the fundamental solution of the heat equation on the circle $\mathbf{R}/\mathbb{Z}$.*

It follows from formula (7) of Section 19.2.2 that

$$\theta\left(\frac{x}{it},\frac{i}{t}\right) = t^{1/2}\exp\frac{\pi x^2}{t}\theta(x,it).$$

Therefore,

$$\begin{aligned}\theta(x,it) &= t^{-1/2}\exp\left(-\frac{\pi x^2}{t}\right)\sum_{n\in Z}\exp\left(-\frac{\pi n^2}{t}+\frac{2\pi n x}{t}\right)\\ &= t^{-1/2}\sum_{n\in Z}\exp\left(-\pi\frac{(x-n)^2}{t}\right).\end{aligned}$$

The function $t^{-1/2}\exp\left(-\frac{\pi x^2}{t}\right)$ is the fundamental solution of the heat equation on the real line satisfying at $t = 0$ the initial condition in the form of the delta-function localized at $x = 0$. Thus, $\theta(x,it)$ is a superposition of infinite set of such solutions, every of which corresponds to the initial condition in the form of the delta-functions, localized at one of the points $x = n$, $n \in \mathbb{Z}$.

19.2.4. Factorization of the theta function into infinite product. In order to obtain the expression for the Jacobi theta function $\theta(x,it)$ in the form of the infinite product we use the relation (7) of Section 14.1.3,

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+xq^{2n+1})(1+x^{-1}q^{2n+1}), \tag{1}$$

where $x \in \mathbb{C}$, $|q| < 1$, $x \neq 0$. Putting here $q = \exp i\pi\tau$ and $x = \exp 2\pi i z$ we obtain on the left hand side the expression for $\theta(z,\tau)$. It means that

$$\begin{aligned}\theta(z,\tau) = &\prod_{m=1}^{\infty}(1-\exp[\pi i(2m)\tau])\\ &\times\prod_{m=0}^{\infty}\{(1+\exp[\pi i(2m+1)\tau-2\pi i z])(1+\exp[\pi i(2m+1)\tau+2\pi i z])\}.\end{aligned}\tag{2}$$

Using the expressions of the type $(a;q)_\infty$ from Section 14.1.1 we derive from (2) that

$$\theta(z,\tau) = (e^{2\pi i\tau}; e^{2\pi i\tau})_\infty(-e^{2\pi iz+i\pi\tau}; e^{2\pi i\tau})_\infty \times (-e^{-2\pi iz+i\pi\tau}; e^{2\pi i\tau})_\infty. \tag{3}$$

It also follows from (2) that

$$\theta(z,\tau) = Q_0 \prod_{n=1}^{\infty}(1 + 2e^{i\pi\tau(2n-1)}\cos 2\pi z + e^{2\pi i\tau(n-1)}),$$

where

$$Q_0 = \prod_{m=1}^{\infty}(1 - \exp 2\pi i m\tau). \tag{4}$$

By using the relations of the functions θ_1, θ_2, θ_4 with $\theta_3 \equiv \theta$ we obtain the expressions for θ_1, θ_2, θ_4 in the form of infinite products:

$$\theta_1(z,\tau) = iQ_0\left(\exp\frac{i\pi\tau}{4}\right)[\exp \pi iz - \exp(-\pi iz)] \times \prod_{m=1}^{\infty}\{(1 - \exp[i\pi(2m)\tau - 2\pi iz])(1 - \exp[\pi i(2m)\tau + 2\pi iz])\}, \tag{5}$$

$$\theta_2(z,\tau) = Q_0\left(\exp\frac{i\pi\tau}{4}\right)[\exp \pi iz + \exp(-\pi iz)] \times \prod_{m=1}^{\infty}\{(1 + \exp[\pi i(2m)\tau - 2\pi iz])(1 + \exp[\pi i(2m)\tau + 2\pi iz])\}, \tag{6}$$

$$\theta_4(z,\tau) = Q_0\prod_{m=1}^{\infty}\{(1 - \exp[\pi i(2m+1)\tau - 2\pi iz]) \times (1 - \exp[\pi i(2m+1)\tau + 2\pi iz])\}. \tag{7}$$

From these expressions for the theta functions we easily find their zeroes. For $\theta(z,\tau)$ these points are

$$z = m + \frac{1}{2} + \left(n + \frac{1}{2}\right)\tau, \quad m,n \in \mathbb{Z}.$$

At $z = 0$ we have

$$\theta(0,\tau) = Q_0\prod_{m=0}^{\infty}[1 + \exp(\pi i(2m+1)\tau)]^2 = (e^{2\pi i\tau}; e^{2\pi i\tau})_\infty(-e^{\pi i\tau}; e^{2\pi i\tau})^2_\infty. \tag{8}$$

By using the expressions in the form of infinite product it is easy to find that

$$\theta^{-1}(z,\tau)\frac{d}{dz}\theta(z,\tau) = 4\pi\sum_{m=0}^{\infty}(-1)^m\frac{q^m}{1-q^{2m}}\sin 2\pi m z,$$

$$\ln\left(\frac{\theta(z,\tau)}{\theta(0,\tau)}\right) = 4\sum_{m=1}^{\infty}(-1)^m\frac{q^m}{1-q^{2m}}\frac{1}{m}\sin^2 \pi m z,$$

where $q = \exp i\pi\tau$. Similar formulas can be written for the functions θ_1, θ_2, θ_4.

The relations

$$\theta_1^2(z)\theta_2^2(0) = \theta_4^2(z)\theta_3^2(0) - \theta_3^2(z)\theta_4^2(0),$$
$$\theta_1^2(z)\theta_3^2(0) = \theta_4^2(z)\theta_2^2(0) - \theta_2^2(z)\theta_4^2(0),$$
$$\theta_1^2(z)\theta_4^2(0) = \theta_3^2(z)\theta_2^2(0) - \theta_2^2(z)\theta_3^2(0),$$
$$\theta_4^2(z)\theta_4^2(0) = \theta_3^2(z)\theta_3^2(0) - \theta_2^2(z)\theta_2^2(0),$$

are valid for the Jacobi functions (here we have omitted the parameter τ in $\theta_i(z,\tau)$ and in $\theta_i(0,\tau)$). They are proved in the following way. It is shown that the ratios of both sides of these relations have no zeros and poles and are doubly periodic functions with periods 1 and τ. According to the Liouville theorem such functions are constants. In order to calculate these constants a partial values of z are used. We recommend to the reader to carry out this proof.

There are many other interesting relations for the functions θ_1, θ_2, θ_3, θ_4. They can be found in special literature.

19.2.5. Theta functions with characteristics. Let us permute $\mathbf{a}$ and $\mathbf{b}$ in formula (5) of Section 12.1.1 and represent t in the form $e^{2\pi i\lambda}$. As a result, we obtain a realization of the Heisenberg group by elements $g(\mathbf{a},\mathbf{b},\lambda)$ for which the multiplication is given by the formula

$$g(\mathbf{a},\mathbf{b},\lambda)g(\mathbf{a}',\mathbf{b}',\lambda') = g(\mathbf{a}+\mathbf{a}',\mathbf{b}+\mathbf{b}',\lambda\lambda'\exp 2\pi i\mathbf{b}\cdot\mathbf{a}^t). \tag{1}$$

Suppose that $\mathbf{a}$ and $\mathbf{b}$ are one-dimensional vectors. Then the Heisenberg group is three-dimensional. We denote this Heisenberg group by N. We have the topological equivalence

$$N \simeq \mathbf{R}\times\mathbf{R}\times U(1).$$

In the same way as in Section 12.2, we can verify that the formula

$$U(a,b,\lambda)f(z) = \lambda\exp(i\pi a^2\tau + 2\pi i a z)f(z+a\tau+b) \tag{2}$$

defines a representation of the group N in the Hilbert space of entire functions $f(z)$ for which

$$\|f\|^2 = \int_{\mathbf{C}}\exp(-2\pi y^2/\mathrm{Im}\,\tau)|f(x+iy)|^2dxdy < \infty.$$

In particular, we have

$$U(0,b,0)f(z) = f(z+b), \tag{3}$$
$$U(a,0,0)f(z) = \exp(i\pi a^2\tau + 2\pi i a z) f(z + a\tau). \tag{4}$$

The set
$$\Gamma = \{(a,b,1) \in N \mid a,b \in \mathbb{Z}\}$$
of elements from N constitutes a subgroup in N. According to the results of Section 19.2.1 the Jacobi theta function $\theta(z,\tau)$ is a unique (up to a constant multiplier) entire function invariant with respect to the action of the operators $U(a,b,1)$, $(a,b,1) \in \Gamma$. Let ℓ be a positive integer. We form the subgroup

$$\ell\Gamma = \{(\ell a, \ell b, 1) \in N \mid a,b \in \mathbb{Z}\}$$

of the group Γ. Let $\mathfrak{S}_\ell$ be the space of entire functions $f(z)$ invariant with respect to the operator $U(a',b',1)$, $(a',b',1) \in \ell\Gamma$. An entire function $f(z)$ belong to the space $\mathfrak{S}_\ell$ if and only if

$$f(z) = \sum_{n \in \frac{1}{\ell}\mathbb{Z}} c_n \exp(i\pi n^2\tau + 2\pi i n z), \tag{5}$$

where $c_n = c_m$ for $n - m \in \ell\mathbb{Z}$. Really, the invariance of $f(z)$ with respect to the operators $U(0,\ell b,1)$, $b \in \mathbb{Z}$, means that

$$f(z) = \sum_{n \in \frac{1}{\ell}\mathbb{Z}} c'_n \exp(2\pi i n z).$$

Now we denote c'_n by $c_n \exp(i\pi n^2\tau)$ and take into account the invariance condition with respect to the operators $U(\ell a, 0, 1)$, $a \in \mathbb{Z}$. As a result, we find that $c_{n+\ell} = c_n$, $n \in \mathbb{Z}$, what proves formula (5). The inverse statement is evident. It follows from the proved assertion that

$$\dim \mathfrak{S}_\ell = \ell^2. \tag{6}$$

Let C_{ℓ^2} be the commutative group of the roots of 1 of degree ℓ^2. Then

$$C_{\ell^2} = \left\{ \exp\frac{2\pi i m}{\ell^2} \;\middle|\; m = 0,1,2,\dots,\ell^2 - 1 \right\}.$$

We define the finite group

$$N_\ell = \left\{ (a,b,\lambda) \;\middle|\; a,b \in \frac{1}{\ell}\mathbb{Z}, \quad \lambda \in C_{\ell^2} \right\} \pmod{\ell\Gamma}$$
$$= \left(\frac{1}{\ell}\mathbb{Z}\Big/\ell\mathbb{Z}\right) \times \left(\frac{1}{\ell}\mathbb{Z}\Big/\ell\mathbb{Z}\right) \times C_{\ell^2}$$

with the same group multiplication as in the case of the group N:

$$(a, b, \lambda)(a', b', \lambda') = (a + a', b + b', \lambda\lambda' \exp(2\pi i b a')).$$

The elements $T_{1/\ell} \equiv \left(\frac{1}{\ell}, 0, 1\right)$ and $S_{1/\ell} \equiv \left(0, \frac{1}{\ell}, 1\right)$ generate the group N_ℓ. With the help of formula (1) one can easily verify that $T_{1/\ell}$, $S_{1/\ell}$ (as elements of the group N) commute with all elements of the subgroup $\ell\Gamma$. Therefore, the action of the elements $T_{1/\ell}$, $S_{1/\ell}$ and of all elements from N_ℓ on the space $\mathfrak{S}_\ell$ is defined. Namely,

$$\begin{aligned} S_{1/\ell} &\left(\sum_{n \in \frac{1}{\ell}\mathbb{Z}} c_n \exp(i\pi n^2 \tau + 2\pi i n z) \right) \\ &= \sum_{n \in \frac{1}{\ell}\mathbb{Z}} c_n \exp \frac{2\pi i n}{\ell} \exp(i\pi n^2 \tau + 2\pi i n z), \end{aligned} \tag{7}$$

$$\begin{aligned} T_{1/\ell} &\left(\sum_{n \in \frac{1}{\ell}\mathbb{Z}} c_n \exp(i\pi n^2 \tau + 2\pi i n z) \right) \\ &= \sum_{n \in \frac{1}{\ell}\mathbb{Z}} c_{n-1/\ell} \exp(i\pi n^2 \tau + 2\pi i n z). \end{aligned} \tag{8}$$

The action of the group N_ℓ in $\mathfrak{S}_\ell$ gives its irreducible representation. Really, let V be an invariant subspace of $\mathfrak{S}_\ell$ with respect to this action and let f be a non-zero element from V having the form (5) with $c_{n_0} \neq 0$. Acting by powers of the operator $S_{1/\ell}$ onto f we find that

$$\begin{aligned} &\sum_{0 \le p \le \ell^2 - 1} \exp\left(-2\pi i n_0 \frac{p}{\ell}\right) \left(S_{1/\ell}^p f\right)(z) \\ &= \sum_{n \in \frac{1}{\ell}\mathbb{Z}} c_n \left[\sum_p \exp\left(2\pi i (n - n_0) \frac{p}{\ell}\right) \exp(i\pi n^2 \tau + 2\pi i n z) \right] \\ &= \ell^2 c_{n_0} \sum_{n \in n_0 + \ell\mathbb{Z}} \exp(i\pi n^2 \tau + 2\pi i n z). \end{aligned}$$

Since $c_{n_0} \neq 0$, then V contains the function

$$\sum_{n \in n_0 + \ell\mathbb{Z}} \exp(i\pi n^2 \tau + 2\pi i n z).$$

Repeating these reasonings for the operator $T_{1/\ell}$ we obtain that V contains analogous functions for all $n_0 \in \left(\frac{1}{\ell}\mathbb{Z}\right) / \ell\mathbb{Z}$, that is, $V = \mathfrak{S}_\ell$. This proves our statement.

Since the space $\mathfrak{S}_\ell$ contains the Jacobi theta function $\theta(z,\tau)$ and the representation of N_ℓ in $\mathfrak{S}_\ell$ is irreducible, we can introduce into $\mathfrak{S}_\ell$ the standard basis. Namely, ℓ^2 functions

$$\theta_{a/\ell,b/\ell} = S^b_{1/\ell}T^a_{1/\ell}\theta = U\left(\frac{a}{\ell},\frac{b}{\ell},1\right)\theta, \quad a,b\in\{0,1,2,\dots,\ell-1\} \tag{9}$$

are elements of this basis. These functions are of the form

$$\begin{aligned}\theta_{a,b}(z,\tau) &= \exp[i\pi a^2\tau + 2\pi i a(z+b)]\theta(z+a\tau+b,\tau)\\ &= \sum_{n\in\mathbb{Z}} \exp[i\pi(a^2+n^2)\tau + 2\pi i n(z+a\tau+b) + 2\pi i a(z+b)]\\ &= \sum_{n\in\mathbb{Z}} \exp[i\pi(a+n)^2\tau + 2\pi i(a+n)(z+b)].\end{aligned} \tag{10}$$

They are called *theta functions with rational characteristics* $a,b\in\frac{1}{\ell}\mathbb{Z}$. We have

$$U(0,b',1)\theta_{a,b} = \theta_{a,b+b'}, \quad a,b,b'\in\frac{1}{\ell}\mathbb{Z},$$

$$U(a',0,1)\theta_{a,b} = \exp(-2\pi i a'b)\theta_{a+a',b}, \quad a,a',b\in\frac{1}{\ell}\mathbb{Z},$$

$$\theta_{a+p,b+q} = \exp(2\pi i a q)\theta_{a,b}, \quad p,q\in\mathbb{Z}, \quad a,b\in\frac{1}{\ell}\mathbb{Z}.$$

Let us note that the Jacobi theta functions θ_1, θ_2, θ_3, θ_4 from Section 19.2.1 also are theta functions with characteristics. Namely,

$$\theta_1 = \theta_{\frac12,\frac12}, \quad \theta_2 = \theta_{\frac12,0}, \quad \theta_3 = \theta_{0,0}, \quad \theta_4 = \theta_{0,\frac12}.$$

These functions form a basis of the space $\mathfrak{S}_2$.

19.2.6. Theta functions of many variables. The theta function $\theta(z,\tau)$, considered above, depends on the complex variable z and on the variable $\tau\in H$. In order to obtain more general class of theta functions we replace the variable z by the vector $\mathbf{z}=(z_1,z_2,\dots,z_r)\in\mathbb{C}^r$ and the upper half-plane H by the upper Siegel half-space $\mathfrak{H}_r$. Remind that $\mathfrak{H}_r$ consists of all symmetric complex $r\times r$-matrices Ω with positive definite imaginary part. In other words, $\mathfrak{H}_r$ consists of the matrices $X+iY$ where X and Y are symmetric real matrices, for which Y is positive definite. The set $\mathfrak{H}_r$ is an open subset of $\mathbb{C}^{r(r+1)/2}$. The *theta function of many variables* is defined by the formula

$$\theta(\mathbf{z},\Omega) = \sum_{\mathbf{n}\in\mathbb{Z}^r} \exp(\pi i \mathbf{n}^t\Omega\mathbf{n} + 2\pi i\mathbf{n}^t\cdot\mathbf{z}), \tag{1}$$

where we suppose that $\mathbf{n}$ and $\mathbf{z}$ are written in the form of columns, $\mathbf{n}^t \cdot \mathbf{z} = n_1 z_1 + n_2 z_2 + \ldots + n_r z_r$, and t denotes a transposition.

The series on the right hand side of the formula (1) converges absolutely and uniformly in $\mathbf{z}$ and Ω on every subset of the form

$$\max_j |\text{Im } z_j| < c_1/2\pi, \quad \text{Im } \Omega \geq c_2 I_r,$$

where I_r is the unit $r \times r$ matrix and the notation $A \geq B$ means that $A - B$ is positive definite matrix. Really,

$$|\exp(\pi i \mathbf{n}^t \Omega \mathbf{n} + 2\pi i \mathbf{n}^t \cdot \mathbf{z})| \leq \exp(-\pi c_2 \sum_i n_i^2 + c_1 \sum_i |n_i|)$$
$$= \prod_{j=1}^{r} \exp(-\pi c_2 n_j^2 + c_1 |n_j|).$$

Therefore, $\left(\sum_{n \geq 0} \exp(-\pi c_2 n^2 + c_1 n) \right)^r$ majorizes the series on the right hand side of (1). And

$$\sum_{n \geq 0} \exp(-\pi c_2 n^2 + c_1 n) = c \sum_{n \geq 0} \exp \left[-\pi c_2 \left(n - \frac{c_1}{2\pi c_2} \right)^2 \right],$$

where c is a constant. The last series converges since the integral $\int_0^\infty \exp(-x^2) dx$ converges. From here our statement on the absolute and uniform convergence follows. We can conclude from this statement that the right hand side of (1) defines a holomorphic function on $\mathbb{C}^r \times \mathfrak{H}_r$. We shall show below that all main properties of the theta function $\theta(z, \tau)$ are valid for the function $\theta(\mathbf{z}, \Omega)$.

The Fourier series for the theta function $\theta(\mathbf{z}, \Omega)$ has the form

$$\theta(\mathbf{z}, \Omega) = \sum_{\mathbf{n} \in \mathbb{Z}^r} a_{\mathbf{n}} \exp(2\pi i \mathbf{n}^t \cdot \mathbf{z}), \quad a_{\mathbf{n}} = \exp(i\pi \mathbf{n}^t \Omega \mathbf{n}). \tag{2}$$

Since $a_{\mathbf{n}} = \exp(i\pi \mathbf{n}^t \Omega \mathbf{n}) \neq 0$, then for every $\Omega \in \mathfrak{H}_r$ we can find $\mathbf{z}$ such that $\theta(\mathbf{z}, \Omega) \neq 0$.

It is easy to see from (2) that for every $\mathbf{m} \in \mathbb{Z}^r$ we have

$$\theta(\mathbf{z} + \mathbf{m}, \Omega) = \theta(\mathbf{z}, \Omega). \tag{3}$$

Let us show that

$$\theta(\mathbf{z} + \Omega \mathbf{m}, \Omega) = \exp(-i\pi \mathbf{m}^t \Omega \mathbf{m} - 2\pi i \mathbf{m}^t \cdot \mathbf{z}) \theta(\mathbf{z}, \Omega) \tag{4}$$

for all $\mathbf{m} \in \mathbb{Z}^r$. For summands of the series for $\theta(\mathbf{z} + \Omega\mathbf{m}, \Omega)$ we have

$$\begin{aligned}&\exp[i\pi\mathbf{n}^t\Omega\mathbf{n} + 2\pi i\mathbf{n}^t \cdot (\mathbf{z} + \Omega\mathbf{m})] \\ &\quad = \exp(i\pi\mathbf{n}^t\Omega\mathbf{n} + i\pi\mathbf{n}^t\Omega\mathbf{m} + \pi i\mathbf{m}^t\Omega\mathbf{n} + 2\pi i\mathbf{n}^t \cdot \mathbf{z}) \\ &\quad = \exp[i\pi(\mathbf{m}+\mathbf{n})^t\Omega(\mathbf{m}+\mathbf{n}) + 2\pi i(\mathbf{m}+\mathbf{n})^t \cdot \mathbf{z} - i\pi\mathbf{m}^t\Omega\mathbf{m} - 2\pi i\mathbf{m}^t \cdot \mathbf{z}].\end{aligned}$$

Since $\mathbf{m} + \mathbf{n}$ and $\mathbf{n}$ rum over $\mathbb{Z}^r$, then

$$\begin{aligned}\theta(\mathbf{z} + \Omega\mathbf{m}, \Omega) &= \sum_{\mathbf{n}\in Z^r} \exp(i\pi\mathbf{n}^t\Omega\mathbf{n} + 2\pi i\mathbf{n}^t \cdot (\mathbf{z} + \Omega\mathbf{m})] \\ &= \exp(-i\pi\mathbf{m}^t\Omega\mathbf{m} - 2\pi i\mathbf{m}^t \cdot \mathbf{z}) \sum_{\mathbf{n}\in Z^r} \exp(i\pi\mathbf{n}^t\Omega\mathbf{n} + 2\pi i\mathbf{n}^t \cdot \mathbf{z})\end{aligned}$$

what proves formula (4).

The inverse statement is also valid. Namely, *if* $f(\mathbf{z})$ *is an entire function of* $\mathbf{z} = (z_1, \ldots, z_r)$ *such that*

$$f(\mathbf{z} + \mathbf{m}) = f(\mathbf{z}), \tag{5}$$

$$f(\mathbf{z} + \Omega\mathbf{m}) = \exp(-i\pi\mathbf{m}^t\Omega\mathbf{m} - 2\pi i\mathbf{m}^t \cdot \mathbf{z})f(\mathbf{z}), \tag{6}$$

then $f(\mathbf{z}) = c\theta(\mathbf{z}, \Omega)$ *where* c *is a constant.* Really, it follows from (5) that the Fourier series expansion for $f(\mathbf{z})$ have the form

$$f(\mathbf{z}) = \sum_{\mathbf{n}\in Z^r} c_{\mathbf{n}} \exp(2\pi i\mathbf{n}^t \cdot \mathbf{z}).$$

Substituting this expression for $f(\mathbf{z})$ into formula (6) and putting $\mathbf{m} = (0, \ldots, 0, 1, 0, \ldots, 0) \equiv \varepsilon_k$ we have

$$\begin{aligned}f(\mathbf{z} + \Omega_k) &= \sum_{\mathbf{n}\in Z^r} c_{\mathbf{n}} \exp[2\pi i\mathbf{n}^t \cdot (\mathbf{z} + \Omega_k)] \\ &= \sum_{\mathbf{n}\in Z^r} c_{\mathbf{n}} \exp(2\pi i\mathbf{n}^t \cdot \Omega_k) \exp(2\pi i\mathbf{n}^t \cdot \mathbf{z}),\end{aligned}$$

where Ω_k is k-th column of the matrix Ω. Comparing the last two expressions we obtain

$$c_{\mathbf{n}+\varepsilon_k} = c_{\mathbf{n}} \exp(2\pi i\mathbf{n}^t \cdot \Omega_k + i\pi\Omega_{kk}).$$

Hence, the coefficients $c_{\mathbf{n}}$ are completely defined by the coefficient c_0. Expressing $c_{\mathbf{n}}$ in terms of c_0 we obtain our assertion.

Formulas (3) and (4) (and also formulas (5) and (6)) define quasi-periodicity in $\mathbf{z}$ of the theta function $\theta(\mathbf{z}, \Omega)$ with respect to the lattice

$$L_\Omega = \mathbb{Z}^r + \Omega\mathbb{Z}^r.$$

It is clear that this lattice in $\mathbb{C}^r$ is generated by the unit vectors $\varepsilon_k = (0, \dots, 0, 1, 0, \dots, 0)$ and by the columns Ω_k, $k = 1, 2, \dots, r$, of the matrix Ω.

19.2.7. The symplectic group. Let us remind some facts on the symplectic group $Sp(r, \mathbf{R})$. This group consists of real linear transformations of the space $\mathbf{R}^{2r}$ conserving the skew-symmetric form

$$B((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2)) = \mathbf{x}_1^t \cdot \mathbf{y}_2 - \mathbf{x}_2^t \cdot \mathbf{y}_1, \quad \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbf{R}^r.$$

Since the matrix of this form is $\begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$ where I_r is the unit $r \times r$ matrix, then the matrices $g \in Sp(r, \mathbf{R})$ are represented in the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (where A, B, C, D are $r \times r$ matrices) for which

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}. \tag{1}$$

It follows from here that $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to $Sp(r, \mathbf{R})$ if and only if

$$A^t C = C^t A, \quad B^t D = D^t B, \quad A^t D - C^t B = I_r. \tag{2}$$

We obtain from (1) that if

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t, \quad \text{then} \quad g^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}. \tag{3}$$

If $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(r, \mathbf{R})$, then the action

$$g\Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

of g on the Siegel upper half space $\mathfrak{H}_r$ is given. Moreover, these actions are one-to-one and mutually holomorphic.

The group $Sp(r, \mathbf{R})$ is generated by the matrices

$$g(A) = \begin{pmatrix} A^* & 0 \\ 0 & A \end{pmatrix}, \quad A \in GL(r, \mathbf{R}), \quad A^* \equiv (A^t)^{-1}, \tag{5}$$

$$g'(S) = \begin{pmatrix} I_r & S \\ 0 & I_r \end{pmatrix}, \quad J \equiv J_r = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \tag{6}$$

where S belong to the space $\mathfrak{S}_r(\mathbf{R})$ of symmetric real $r \times r$ matrices. It is obvious that

$$g(A)\Omega = (A^t)^{-1}\Omega A^{-1} \in \mathfrak{H}_r, \quad g(S)\Omega = \Omega + S \in \mathfrak{H}_r.$$

The action of the group $Sp(r,\mathbf{R})$ *on* $\mathfrak{H}_r$ *is transitive.* Really, if $\Omega = X + iY \in \mathfrak{H}_r$, then there is a matrix $A \in GL(r,\mathbf{R})$ such that $A^tA = Y$. Then $g = g(A)g'(A^{-1}) \in Sp(r,\mathbf{R})$ and action of this matrix onto element $iI_r \in \mathfrak{H}_r$ gives matrices Ω, that is, $Sp(r,\mathbf{R})$ acts on $\mathfrak{H}_r$ transitively.

A transitivity of the action of $Sp(r,\mathbf{R})$ on $\mathfrak{H}_r$ means that $\mathfrak{H}_r$ is a quotient space of $Sp(r,\mathbf{R})$ with respect to stationary subgroup of some point. The stationary subgroup of the matrix $iI_r \in \mathfrak{H}_r$ is $U(r)$. This subgroup is imbedded into $Sp(r,\mathbf{R})$ as

$$U(n) \ni A + iB \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(r,\mathbf{R}).$$

Thus,

$$\mathfrak{H}_r = Sp(r,\mathbf{R})/U(r) \tag{7}$$

and $\mathfrak{H}_r$ *is a noncompact Riemannian symmetric space* (see Section 1.2.2). In $\mathfrak{H}_r$ there is the measure invariant with respect to the group $Sp(r,\mathbf{R})$. If matrices Ω of $\mathfrak{H}_r$ are written down in the form

$$\Omega = X + iY = (x_{\alpha\beta} + iy_{\alpha\beta}),$$

then the invariant measure is given by the formula

$$d\Omega = (\det Y)^{-n-1} dXdY \equiv (\det Y)^{-n-1} \prod_{1\le\alpha\le\beta\le n} dx_{\alpha\beta}dy_{\alpha\beta}. \tag{8}$$

19.2.8. The functional equation for the theta function of many variables. We have defined the action of the group $Sp(n,\mathbf{R})$ on $\mathfrak{H}_r$. Therefore, we can define the action of $Sp(n,\mathbf{R})$ on $\mathbb{C}^r \times \mathfrak{H}_r$. Namely, if $(\mathbf{z},\Omega) \in \mathbb{C}^r \times \mathfrak{H}_r$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$g \cdot (\mathbf{z},\Omega) = (((C\Omega + D)^t)^{-1}\mathbf{z}, (A\Omega + B)(C\Omega + D)^{-1}). \tag{1}$$

The theta function $\theta(\mathbf{z},\Omega)$ admits a symmetry relation with respect to elements of some subgroup of the group $Sp(r,\mathbf{R})$. Namely, if $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ belongs to $Sp(n,\mathbb{Z})$ and diagonal elements in A^tC and in B^tD are even, then

$$\begin{aligned} &\theta(((C\Omega + D)^t)^{-1}\mathbf{z}, (A\Omega + B)(C\Omega + D)^{-1}) \\ &\quad = \zeta_g \det\,(C\Omega + D)^{1/2} \exp[i\pi\mathbf{z}^t(C\Omega + D)^{-1}C\mathbf{z}]\theta(\mathbf{z},\Omega), \end{aligned} \tag{2}$$

where ζ_g is a root of 1 of power 8. Matrices $g \in Sp(r,\mathbb{Z})$ for which this relation is valid, constitute the group which is denoted by $\Gamma_{1,2}$. It can be described as the subgroup of $Sp(r,\mathbb{Z})$ whose elements conserve (up to mod 2) the quadratic form

$$Q(\mathbf{n}_1,\mathbf{n}_2) = \mathbf{n}_1^t \cdot \mathbf{n}_2 \in \mathbb{Z}/2\mathbb{Z}$$

over $\mathbb{Z}/2\mathbb{Z}$. Therefore, we have

$$\Gamma_{1,2} = \{g \in Sp(r, \mathbb{Z}) \mid Q(g\mathbf{n}_1, g\mathbf{n}_2) = Q(\mathbf{n}_1, \mathbf{n}_2) \bmod 2\}.$$

The group $Sp(r, \mathbb{Z})$ is generated by matrices

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \quad \begin{pmatrix} I_r & B \\ 0 & I_r \end{pmatrix}, \quad \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

for all $A \in GL(r, \mathbb{Z})$ and for all symmetric $r \times r$ matrices B with integral elements. Using this fact it is possible to prove that the subgroup $\Gamma_{1,2}$ is generated by the matrices

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \quad \begin{pmatrix} I_r & B \\ 0 & I_r \end{pmatrix}, \quad \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \tag{3}$$

for all $A \in GL(r, \mathbb{Z})$ and for all symmetric matrices B with integral entries whose diagonal elements are even. It is sufficient to prove symmetry relation (2) for matrices of the types (3). For matrices of the first type this relation has the form

$$\theta(A\mathbf{z}, A\Omega A^t) = \theta(\mathbf{z}, \Omega) \tag{4}$$

(in this case $\zeta = \det A^{1/2}$), for matrices of the second type the form

$$\theta(\mathbf{z}, \Omega + B) = \theta(\mathbf{z}, \Omega) \tag{5}$$

(in this case $\zeta = 1$), and for $\begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$ the form

$$\theta(\Omega^{-1}\mathbf{z}, -\Omega^{-1}) = (\det \Omega/i)^{1/2} \exp(i\pi \mathbf{z}^t \Omega^{-1} \mathbf{z}) \theta(\mathbf{z}, \Omega), \tag{6}$$

where for $(\det \Omega/i)^{1/2}$ those branch of the root is used which gives positive values for pure imaginary Ω.

The relation (4) is proved as follows

$$\begin{aligned}
\theta(A\mathbf{z}, A\Omega A^t) &= \sum_{\mathbf{z} \in Z^r} \exp(i\pi \mathbf{n}^t A\Omega A^t \mathbf{n} + 2\pi i \mathbf{n}^t \cdot A\mathbf{z}) \\
&= \sum_{\mathbf{n} \in Z^r} \exp[i\pi (A^t\mathbf{n})^t \Omega (A^t\mathbf{n}) + 2\pi i (A^t\mathbf{n})^t \cdot \mathbf{z}] \\
&= \sum_{\mathbf{m} \in Z^r} \exp(i\pi \mathbf{m}^t \Omega \mathbf{m} + 2\pi i \mathbf{m}^t \cdot \mathbf{z}) = \theta(\mathbf{z}, \Omega).
\end{aligned}$$

From the equality

$$\begin{aligned}
\theta(\mathbf{z}, \Omega + B) &= \sum_{\mathbf{n} \in Z^r} \exp[i\pi \mathbf{n}^t (\Omega + B)\mathbf{n} + 2\pi i \mathbf{n}^t \cdot \mathbf{z}] \\
&= \sum_{\mathbf{n} \in Z^r} \exp(i\pi \mathbf{n}^t B \mathbf{n}) \exp(i\pi \mathbf{n}^t \Omega \mathbf{n} + 2\pi i \mathbf{n}^t \cdot \mathbf{z}) = \theta(\mathbf{z}, \Omega)
\end{aligned}$$

follows relation (5).

The relation (6) is proved with the help of the Poisson summation formula

$$\sum_{\mathbf{n}\in Z^r} f(\mathbf{n}) = \sum_{\mathbf{n}\in Z^r} \hat{f}(\mathbf{n}),$$

where

$$\hat{f}(\mathbf{y}) = \int_{\mathbf{R}^r} f(\mathbf{x})\exp(2\pi i \mathbf{x}^t\cdot\mathbf{y})dx_1\dots dx_r.$$

Applying this formula to the function $f(\mathbf{x}) = \exp(i\pi\mathbf{x}^t\Omega\mathbf{x} + 2\pi i\mathbf{x}^t\cdot\mathbf{z})$ we have

$$\sum_{\mathbf{n}\in Z^r} f(\mathbf{n}) = \theta(\mathbf{z},\Omega).$$

In order to evaluate $\hat{f}$ we show that

$$\int_{\mathbf{R}^r} \exp(i\pi\mathbf{x}^t\Omega\mathbf{x} + 2\pi i\mathbf{x}^t\cdot\mathbf{z})d\mathbf{x} = (\det\,\Omega/i)^{-1/2}\exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z}), \tag{7}$$

where $d\mathbf{x} = dx_1\dots dx_r$. For this we write down this integral in the form

$$I = \exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})\int_{\mathbf{R}^r}\exp[\pi i(\mathbf{x}+\Omega^{-1}\mathbf{z})^t\Omega(\mathbf{x}+\Omega^{-1}\mathbf{z})]d\mathbf{x}.$$

Since both sides of equality (7) are holomorphic in Ω and $\mathbf{z}$, then it is sufficient to prove equality (7) for pure imaginary Ω and $\mathbf{z}$. Let us set $\mathbf{z} = i\mathbf{y}$, $\mathbf{y}\in\mathbf{R}^r$, and $\Omega = iA^tA$, where A is a real positive definite symmetric matrix. Then

$$I = \exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})\int_{\mathbf{R}^r}\exp[-\pi(\mathbf{x}+(A^tA)^{-1}\mathbf{y})^t \times A^tA(\mathbf{x}+(A^tA)^{-1}\mathbf{y})]d\mathbf{x}.$$

Replacing $\mathbf{x}$ by $\mathbf{x}-(A^tA)^{-1}\mathbf{y}$ we have

$$I = \exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})\int_{\mathbf{R}^r}\exp(-\pi(\mathbf{x}^tA^tA\mathbf{x})d\mathbf{x}.$$

Using the substitution $\mathbf{w} = A\mathbf{x}$ we obtain

$$\begin{aligned} I &= \exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})\int_{\mathbf{R}^r}\exp(-\pi(\mathbf{w}^t\cdot\mathbf{w})(\det\,A)^{-1}d\mathbf{w} \\ &= \exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})(\det\,A^tA)^{-1/2}\prod_{i=1}^{r}\int_{\infty}^{\infty}e^{-\pi w_i^2}dw_i \\ &= \exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})(\det\,\Omega/i)^{-1/2}. \end{aligned}$$

The relation (7) is proved. Using it for $\hat{f}(\mathbf{y})$ we have

$$\hat{f}(\mathbf{y}) = \int_{\mathbb{R}^r} \exp(i\pi\mathbf{x}^t\Omega\mathbf{x} + 2\pi i\mathbf{x}^t \cdot \mathbf{z})\exp(2\pi i\mathbf{x}^t \cdot \mathbf{y})d\mathbf{x}$$
$$= (\det\,\Omega/i)^{-1/2}\exp(-i\pi(\mathbf{z}+\mathbf{y})^t\Omega^{-1}(\mathbf{z}+\mathbf{y})).$$

Hence,

$$\sum_{\mathbf{n}\in Z^r} \hat{f}(\mathbf{n}) = (\det\,\Omega/i)^{-1/2}\exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})$$
$$\times \sum_{\mathbf{n}\in Z^r} \exp(-i\pi\mathbf{n}^t\Omega^{-1}\mathbf{n} + 2\pi i\mathbf{n}^t\Omega^{-1}\mathbf{z})$$
$$= (\det\,\Omega/i)^{-1/2}\exp(-i\pi\mathbf{z}^t\Omega^{-1}\mathbf{z})\theta(\Omega^{-1}\mathbf{z}, -\Omega^{-1}).$$

With the help of the Poisson summation formula it leads to relation (6).

19.2.9. Theta functions of many variables with characteristics. Theta functions $\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z},\Omega)$ with rational characteristics $\mathbf{a},\mathbf{b}\in\mathbb{Q}^r$ ($\mathbb{Q}$ is the field of rational numbers) are obtained from $\theta(\mathbf{z},\Omega)$ by shifts of the arguments and by multiplication by exponential functions:

$$\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z},\Omega) = \exp[i\pi\mathbf{a}^t\Omega\mathbf{a} + 2\pi i\mathbf{a}^t\cdot(\mathbf{z}+\mathbf{b})]\theta(\mathbf{z}+\Omega\mathbf{a}+\mathbf{b},\Omega). \quad (1)$$

This expression can be written in the form

$$\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z},\Omega) = \sum_{\mathbf{n}\in Z^r}\exp[i\pi(\mathbf{n}+\mathbf{a})^t\Omega(\mathbf{n}+\mathbf{a}) + 2\pi i(\mathbf{n}+\mathbf{a})^t\cdot(\mathbf{z}+\mathbf{b})]. \quad (2)$$

It is clear that $\theta(\mathbf{z},\Omega) = \theta\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}(\mathbf{z},\Omega)$. It is easy to see that if $\mathbf{m},\mathbf{n}\in\mathbb{Z}^r$, then

$$\theta\begin{bmatrix}\mathbf{a}+\mathbf{n}\\ \mathbf{b}+\mathbf{m}\end{bmatrix}(\mathbf{z},\Omega) = \exp(2\pi i\mathbf{a}^t\cdot\mathbf{m})\,\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z},\Omega). \quad (3)$$

A direct verification shows that $\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}$ has the following quasi-periodicity properties:

$$\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z}+\mathbf{m},\Omega) = \exp(2\pi i\mathbf{a}^t\cdot\mathbf{m})\,\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z},\Omega), \quad (4)$$

$$\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z}+\Omega\mathbf{m},\Omega) = \exp(-2\pi i\mathbf{b}^t\cdot\mathbf{m})$$
$$\times\exp(-i\pi\mathbf{m}^t\Omega\mathbf{m} - 2\pi i\mathbf{m}^t\cdot\mathbf{z})\,\theta\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(\mathbf{z},\Omega), \quad (4')$$

where $\mathbf{m} \in \mathbb{Z}^r$.

Let us fix $\Omega \in \mathfrak{H}_r$ and a positive integer ℓ. An entire function $\varphi(\mathbf{z})$ on $\mathbb{C}^r$ is called L_Ω*-quasi-periodic of weight* ℓ if for all $\mathbf{m} \in \mathbb{Z}^r$ the relations

$$f(\mathbf{z} + \mathbf{m}) = f(\mathbf{z}), \tag{5}$$

$$f(\mathbf{z} + \Omega\mathbf{m}) = \exp(-i\pi\ell\mathbf{m}^t\Omega\mathbf{m} - 2\pi i\ell\mathbf{m}^t \cdot \mathbf{z})f(\mathbf{z}) \tag{5'}$$

are fulfilled. The linear space of such functions is denoted by $R_{\ell,\Omega}$.

Theorem 1. *The theta functions*

$$f_{\mathbf{a}}(\mathbf{z}) = \theta\begin{bmatrix}\mathbf{a}/\ell\\0\end{bmatrix}(\ell\mathbf{z}, \ell\Omega), \quad a_i \in \mathbb{Z},\ 0 \le a_i < \ell, \tag{6}$$

as well as the theta functions

$$g_{\mathbf{b}}(\mathbf{z}) = \theta\begin{bmatrix}0\\\mathbf{b}/\ell\end{bmatrix}(\mathbf{z}, \ell^{-1}\Omega), \quad b_i \in \mathbb{Z},\ 0 \le b_i < \ell, \tag{7}$$

form a basis of the space $R_{\ell,\Omega}$. *If* $\ell = k^2$, $k \in \mathbb{Z}$, *then the functions*

$$h_{\mathbf{a},\mathbf{b}}(z) = \theta\begin{bmatrix}\mathbf{a}/k\\\mathbf{b}/k\end{bmatrix}(\ell\mathbf{z}, \Omega), \quad a_i, b_i \in \mathbb{Z},\ 0 \le a_i, b_i < k, \tag{8}$$

also constitute a basis of $R_{\ell,\Omega}$

Proof. Expanding functions from $R_{\ell,\Omega}$ into the Fourier series in z we find that quasi-periodicity with respect to $\Omega \cdot \mathbb{Z}^r$ means that a function f belongs to $R_{\ell,\Omega}$ if and only if it can be represented in the form

$$f_\chi(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^r} \chi(\mathbf{n}) \exp\left(i\pi\frac{1}{\ell}\mathbf{n}^t\Omega\mathbf{n} + 2\pi i\mathbf{n}^t \cdot \mathbf{z}\right),$$

where χ is constant on cosets with respect to $\ell\mathbb{Z}^r$. If the characteristic function of the coset $\mathbf{a} + \ell\mathbb{Z}^r$ is taken as χ, then f_χ turns into $f_{\mathbf{a}}$. If χ is the character $\mathbf{n} \to \exp(2\pi i(1/\ell)\mathbf{n}^t \cdot \mathbf{b})$, then f_χ turns into $g_{\mathbf{b}}$. If $\ell = k^2$, then we take the restriction of the function $\mathbf{n} \to \exp(2\pi i(1/\ell)\mathbf{n}^t \cdot \mathbf{b})$ onto $\mathbf{a} + k\mathbb{Z}^r$ as χ. Then f_χ coincides with $h_{\mathbf{a},\mathbf{b}}$ Theorem is proved.

19.2.10. Relations for products of theta functions. For the Jacobi theta function $\theta(z, \tau)$ the *Riemann relation*

$$\begin{aligned}\sum_{\eta=0,\frac{1}{2},\frac{1}{2}\tau,\frac{1}{2}(1+\tau)} & e_\eta\theta(x+\eta,\tau)\theta(y+\eta,\tau)\theta(u+\eta,\tau)\theta(v+\eta,\tau)\\ &= 2\theta(x_1,\tau)\theta(y_1,\tau)\theta(u_1,\tau)\theta(v_1,\tau)\end{aligned} \tag{1}$$

is valid where

$$x_1 = \frac{1}{2}(x+y+u+v), \quad y_1 = \frac{1}{2}(x+y-u-v),$$

$$u_1 = \frac{1}{2}(x-y+u-v), \quad v_1 = \frac{1}{2}(x-y-u+v),$$

$$e_\eta = 1 \quad \text{for} \quad \eta = 0, \frac{1}{2},$$

$$e_\eta = \exp[i\pi\tau + i\pi(x+y+u+v)] \quad \text{for} \quad \eta = \frac{1}{2}\tau, \frac{1}{2}(1+\tau).$$

For the theta function $\theta(\mathbf{z}, \Omega)$ the Riemann relation is of the form

$$\begin{aligned} &\theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y}+\mathbf{u}+\mathbf{v}), \Omega\right)\theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y}-\mathbf{u}-\mathbf{v}), \Omega\right)\theta\left(\frac{1}{2}(\mathbf{x}-\mathbf{y}+\mathbf{u}-\mathbf{v}), \Omega\right) \\ &\times \theta\left(\frac{1}{2}(\mathbf{x}-\mathbf{y}-\mathbf{u}+\mathbf{v}), \Omega\right) \\ &= 2^{-r} \sum_{\alpha,\beta\in\frac{1}{2}\mathbf{Z}^r/\mathbf{Z}^r} \exp[4\pi i\boldsymbol{\alpha}^t\Omega\boldsymbol{\alpha} + 2\pi i\boldsymbol{\alpha}^t(\mathbf{x}+\mathbf{y}+\mathbf{u}+\mathbf{v})] \\ &\times \theta(\mathbf{x}+\Omega\boldsymbol{\alpha}+\boldsymbol{\beta}, \Omega)\theta(\mathbf{y}+\Omega\boldsymbol{\alpha}+\boldsymbol{\beta}, \Omega)\theta(\mathbf{u}+\Omega\boldsymbol{\alpha}+\boldsymbol{\beta}, \Omega)\theta(\mathbf{v}+\Omega\boldsymbol{\alpha}+\boldsymbol{\beta}, \Omega). \end{aligned} \tag{2}$$

Proofs of these relations are not complicated but take much place. In this reason we do not give them here (they can be found in the book [230]).

Let us also give the formula

$$\begin{aligned} &\theta\begin{bmatrix}\mathbf{a}/n \\ \mathbf{0}\end{bmatrix}(\mathbf{z}_1, n_1\Omega)\theta\begin{bmatrix}\mathbf{b}/n \\ \mathbf{0}\end{bmatrix}(\mathbf{z}_2, n_2\Omega) \\ &= \sum_{\mathbf{d}\in\mathbf{Z}^r/(n_1+n_2)\mathbf{Z}^r} \theta\begin{bmatrix}\frac{n_1\mathbf{d}+\mathbf{a}+\mathbf{b}}{n_1+n_2} \\ \mathbf{0}\end{bmatrix}(\mathbf{z}_1+\mathbf{z}_2, (n_1+n_2)\Omega) \\ &\times \theta\begin{bmatrix}\frac{n_1n_2\mathbf{d}+n_2\mathbf{a}-n_1\mathbf{b}}{n_1n_2(n_1+n_2)} \\ \mathbf{0}\end{bmatrix}(n_2\mathbf{z}_1 - n_1\mathbf{z}_2, n_1n_2(n_1+n_2)\Omega). \end{aligned} \tag{3}$$

The special case of this formula when $\mathbf{z}_1 = n_1\mathbf{z}$ and $\mathbf{z}_2 = n_2\mathbf{z}$ is of great profit. In this case we have

$$\begin{aligned} &\theta\begin{bmatrix}\mathbf{a}/n_1 \\ \mathbf{0}\end{bmatrix}(n_1\mathbf{z}, n_1\Omega)\theta\begin{bmatrix}\mathbf{b}/n_2 \\ \mathbf{0}\end{bmatrix}(n_2\mathbf{z}, n_2\Omega) \\ &= \sum_{\mathbf{d}\in\mathbf{Z}^r/(n_1+n_2)\mathbf{Z}^r} \theta\begin{bmatrix}\frac{n_1n_2\mathbf{d}+n_2\mathbf{a}-n_1\mathbf{b}}{n_1n_2(n_1+n_2)} \\ \mathbf{0}\end{bmatrix}(\mathbf{0}, n_1n_2(n_1+n_2)\Omega) \\ &\times \theta\begin{bmatrix}\frac{n_1\mathbf{d}+\mathbf{a}+\mathbf{b}}{n_1+n_2} \\ \mathbf{0}\end{bmatrix}((n_1+n_2)\mathbf{z}, (n_1+n_2)\Omega). \end{aligned} \tag{4}$$

The first multiplier of the summands of the right hand sides does not depend on $\mathbf{z}$. Therefore, this formula gives an expansion of a product of theta functions in theta functions of the space $R_{\ell,\Omega'}$ with certain values of ℓ and Ω'.

If $n_1 = n_2$, then with the help of the simple identity

$$\sum_{\mathbf{e}\in Z^r/nZ^r} \theta\begin{bmatrix}\frac{\mathbf{e}+\mathbf{a}}{n}\\ \mathbf{0}\end{bmatrix}(n\mathbf{z}, n^2\Omega) = \theta\begin{bmatrix}\mathbf{a}\\ \mathbf{0}\end{bmatrix}(\mathbf{z},\Omega)$$

the relation (3) can be reduced to the form

$$\begin{aligned}\theta\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{z}_1, n\Omega)\theta\begin{bmatrix}\mathbf{b}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{z}_2, n\Omega) &= \sum_{\mathbf{d}\in Z^r/2Z^r} \theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{a}+\mathbf{b}}{2n}\\ \mathbf{0}\end{bmatrix}(\mathbf{z}_1+\mathbf{z}_2, 2n\Omega) \\ &\times \theta\begin{bmatrix}\frac{\mathbf{d}}{2}+\frac{\mathbf{a}-\mathbf{b}}{2n}\\ \mathbf{0}\end{bmatrix}(\mathbf{z}_1-\mathbf{z}_2, 2n\Omega).\end{aligned} \tag{5}$$

19.3. Theta Functions and the Decomposition of Quasi-Regular Representation of the Heisenberg Group on the Cube

19.3.1. Auxiliary theta functions. In Section 19.2.9 we defined theta functions with characteristics

$$\theta\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{z},\Omega) = \sum_{\mathbf{m}\in Z^r} \exp\left[i\pi n\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right) + 2\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\cdot\mathbf{z}\right], \tag{1}$$

where $\mathbf{a} \in \mathbb{Z}^r/n\mathbb{Z}^r$. In order to study the quasi-regular representation of the Heisenberg group we need auxiliary theta functions depending on $2r$ complex parameters $\mathbf{z} = (z_1, z_2, \dots, z_r)$ and $\mathbf{u} = (u_1, u_2, \dots, u_r)$:

$$\begin{aligned}\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{u},\mathbf{z}\,|\,\Omega) &= (2\pi i n)^{|\mathbf{j}|}\sum_{\mathbf{m}\in Z^r}\left(\mathbf{u}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}} \\ &\times \exp\left\{i\pi n\left[\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right) + 2\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\cdot\mathbf{z}\right]\right\},\end{aligned} \tag{2}$$

where $\mathbf{j} \in \mathbb{Z}^r$ and $(\mathbf{b})^{\mathbf{j}} = b_1^{j_1}\dots b_r^{j_r}$, $|\mathbf{j}| = j_1 + \dots + j_r$.

Let us introduce the notation

$$\left(\mathbf{b}+\frac{\partial}{\partial\mathbf{z}}\right)^{\mathbf{j}} \equiv \left(b_1+\frac{\partial}{\partial z_1}\right)^{j_1}\cdots\left(b_r+\frac{\partial}{\partial z_r}\right)^{j_r}.$$

It is easy to verify that

$$\left(2\pi i n\mathbf{u}+\frac{\partial}{\partial\mathbf{z}}\right)^{\mathbf{j}}\exp\left[i\pi n\left(\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)+2\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\cdot\mathbf{z}\right)\right]$$
$$=(2\pi i n)^{|\mathbf{j}|}\left(\mathbf{u}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}}\exp\left[i\pi n\left(\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)+2\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\cdot\mathbf{z}\right)\right].$$

Summing both sides of these relations over $\mathbf{m}\in\mathbb{Z}^r$ we obtain the connections between the functions $\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}$ and $\theta\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}$:

$$\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{u},\mathbf{z}\,|\,\Omega)=\left(2\pi i n\mathbf{u}+\frac{\partial}{\partial\mathbf{z}}\right)^{\mathbf{j}}\theta\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{z},\Omega). \tag{3}$$

It is easy to prove the quasi-periodicity

$$\begin{aligned}&\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{u}+\mathbf{b},\mathbf{z}+\hat{\mathbf{b}}+\Omega\mathbf{b}\,|\,\Omega)\\ &\quad=\exp[-i\pi n(\mathbf{b}^t\Omega\mathbf{b}+2\mathbf{b}^t\cdot\mathbf{z})]\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{u},\mathbf{z}\,|\,\Omega),\end{aligned} \tag{4}$$

where $\mathbf{b},\hat{\mathbf{b}}\in\mathbb{Z}^r$.

We denote by $\Theta_{n,\Omega}$ the space of functions $f(\mathbf{u},\mathbf{z})$ of $2r$ complex variables under condition that they are polynomials in $\mathbf{u}=(u_1,\dots,u_r)$ with coefficients, which are entire analytic functions of $\mathbf{z}=(z_1,\dots,z_r)$ and such that for all $\mathbf{b},\hat{\mathbf{b}}\in\mathbb{Z}^r$ the relation

$$f(\mathbf{u}+\mathbf{b},\mathbf{z}+\hat{\mathbf{b}}+\Omega\mathbf{b})=\exp[-i\pi n(\mathbf{b}^t\Omega\mathbf{b}+2\mathbf{b}^t\cdot\mathbf{z})]f(\mathbf{u},\mathbf{z})$$

is fulfilled.

Theorem 1. *The functions*

$$\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{u},\mathbf{z}\,|\,\Omega),\quad \mathbf{j}\equiv(j_1,\dots,j_r)\in\mathbb{Z}^r,\quad j_k\geq 0;\quad \mathbf{a}\in\mathbb{Z}^r/n\mathbb{Z}^r, \tag{5}$$

form a basis of the space $\Theta_{n,\Omega}$.

Proof. It follows from relation (4) that the functions (5) belong to $\Theta_{n,\Omega}$. It is clear that they are linearly independent. Let $f(\mathbf{u},\mathbf{z})=\sum_{\mathbf{j}}\mathbf{u}^{\mathbf{j}}f_{\mathbf{j}}(\mathbf{z})$ be a function from $\Theta_{n,\Omega}$ and let $\mathbf{u}^{\mathbf{k}}f_{\mathbf{k}}(\mathbf{z})$ is one of the summands of maximal degree $\mathbf{k}$ with respect to $\mathbf{u}$. Comparing coefficients at $\mathbf{u}^{\mathbf{k}}$ from both sides of the relation

$$\sum_{\mathbf{j}}(\mathbf{u}+\mathbf{b})^{\mathbf{j}}f_{\mathbf{j}}(\mathbf{z}+\hat{\mathbf{b}}+\Omega\mathbf{b})=\exp[-i\pi n(\mathbf{b}^t\Omega\mathbf{b}+2\mathbf{b}^t\cdot\mathbf{z})]\sum_{\mathbf{j}}\mathbf{u}^{\mathbf{j}}f_{\mathbf{j}}(\mathbf{z})$$

we have

$$f_{\mathbf{k}}(\mathbf{z}+\hat{\mathbf{b}}+\Omega\mathbf{b}) = \exp[-i\pi n(\mathbf{b}^t\Omega\mathbf{b}+2\mathbf{b}^t\cdot\mathbf{z})]f_{\mathbf{k}}(\mathbf{z}).$$

Due to the results of Section 19.2.9 there exists a set of numbers $c_{\mathbf{m}}$, $m \in \mathbb{Z}^r/n\mathbb{Z}^r$, such that

$$f_{\mathbf{k}}(\mathbf{z}) = \sum_{\mathbf{m}} c_{\mathbf{m}}\theta\begin{bmatrix}\mathbf{m}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{z},\Omega).$$

Therefore, as it follows from formula (3), the function

$$f(\mathbf{u},\mathbf{z}) - \sum_{\mathbf{m}} c_{\mathbf{m}}\theta_{\mathbf{k}}\begin{bmatrix}\mathbf{m}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{u},\mathbf{z}\,|\,\Omega).$$

belongs to $\Theta_{n,\Omega}$ and it is without summand with $\mathbf{u}^{\mathbf{k}}$, and degrees of $\mathbf{u}$ for all its summands are smaller than $\mathbf{k}$. By repeating these subtractions we obtain an expression for $f(\mathbf{u},\mathbf{z})$ in the form of a linear combination of functions (5). Theorem is proved.

Now we introduce the functions

$$\begin{aligned}\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) &= (2\pi i n)^{-|\mathbf{j}|}\exp[i\pi n(\mathbf{x}^t\Omega\mathbf{x}+2\hat{\mathbf{x}}^t\cdot\mathbf{x}-2x_0)]\\ &\times\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\mathbf{x},\hat{\mathbf{x}}+\Omega\mathbf{x}\,|\,\Omega).\end{aligned} \tag{6}$$

Taking into account the expressions for $\theta_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}$ we conclude that

$$\begin{aligned}\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) &= \exp(-2\pi i n x_0)\sum_{\mathbf{m}\in\mathbb{Z}^r}\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}}\\ &= \exp\left[i\pi n\left(\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)+2\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\cdot\hat{\mathbf{x}}\right)\right].\end{aligned} \tag{7}$$

The parameters $(\hat{\mathbf{x}},\mathbf{x},x_0)$ will be considered as the elements

$$(\hat{\mathbf{x}},\mathbf{x},x_0) = \begin{pmatrix}1 & \hat{x}_1 & \dots & \hat{x}_r & x_0\\ 0 & 1 & \dots & 0 & x_1\\ 0 & 0 & \dots & 0 & x_2\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & x_r\\ 0 & 0 & \dots & 0 & 1\end{pmatrix}$$

of the Heisenberg group $H(r,\mathbf{R})$ (see Section 12.1.1). The group multiplication in $H(r,\mathbf{R})$ is given by the formula

$$(\hat{\mathbf{x}},\mathbf{x},x_0)(\hat{\mathbf{y}},\mathbf{y},y_0) = (\hat{\mathbf{x}}+\hat{\mathbf{y}},\mathbf{x}+\mathbf{y},x_0+y_0+\hat{\mathbf{x}}\cdot\mathbf{y}^t). \tag{8}$$

Considering $\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}$ as a function on $H(r,\mathbf{R})$ we find that

$$\begin{aligned}\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) &= \exp(2\pi i\mathbf{a}\cdot\hat{\mathbf{x}}^t)\\ &\times\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}\left((\hat{\mathbf{x}},\mathbf{x},x_0)\left(\mathbf{0},\frac{\mathbf{a}}{n},0\right)\Big|\,\Omega\right).\end{aligned} \tag{9}$$

For elements $(\hat{\mathbf{b}},\mathbf{b},b_0)$ of the subgroup $H(r,\mathbb{Z})$ consisting of matrices with integral entries from $H(r,\mathbf{R})$, we have

$$\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}((\hat{\mathbf{b}},\mathbf{b},b_0)(\hat{\mathbf{x}},\mathbf{x},x_0)\,|\,\Omega)=\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega). \tag{10}$$

In other words, functions $\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}$ are invariant with respect to left shifts by elements of the subgroup $H(r,\mathbb{Z})$. This means that these functions can be considered as functions on

$$H(r,\mathbb{Z})\backslash H(r,\mathbf{R})\simeq[0,1)^{2r+1},$$

where $[0,1)^n=[0,1)\times[0,1)\times\ldots\times[0,1)$ (n times). Validity of formula (10) follows from the equalities

$$\begin{aligned}\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}((\hat{\mathbf{b}},\mathbf{b},b_0)(\hat{\mathbf{x}},\mathbf{x},x_0)\,|\,\Omega) &= \phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{b}}+\hat{\mathbf{x}},\mathbf{b}+\mathbf{x},b_0+x_0+\hat{\mathbf{b}}\cdot\mathbf{x}^t\,|\,\Omega)\\ &= \exp[2\pi i(b_0+x_0+\hat{\mathbf{b}}\cdot\mathbf{x}^t)]\sum_{\mathbf{m}\in\mathbb{Z}^r}\left(\mathbf{x}+\mathbf{b}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}}\\ &\times\exp\left[i\pi n\left(\left(\mathbf{x}+\mathbf{b}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{x}+\mathbf{b}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)\right)\right.\\ &\left.+2\left(\mathbf{x}+\mathbf{b}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\cdot(\hat{\mathbf{x}}+\hat{\mathbf{b}})\right]=\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega).\end{aligned}$$

It is clear that $\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)$ is a real analytic function on $H(r,\mathbb{Z})\backslash H(r,\mathbf{R})$.

The Lie algebra $\mathfrak{h}(r,\mathbf{R})$ of left invariant vector fields on $H(r,\mathbf{R})$ has a basis

$$\hat{D}_i=\frac{\partial}{\partial\hat{x}_i},\quad D_i=\frac{\partial}{\partial x_i}+\hat{x}_i\frac{\partial}{\partial x_0},\quad D_0=-\frac{\partial}{\partial x_0},\quad 1\le i\le r,$$

for which the commutation relations

$$\begin{gathered}[D_0,\hat{D}_i]=[D_0,D_i]=[D_i,D_k]=[\hat{D}_i,\hat{D}_k]=0,\\ [D_i,\hat{D}_i]=D_0,\quad [D_i,\hat{D}_j]=0,\quad i\neq j,\end{gathered}$$

are fulfilled. The operators $\hat{D}_k$, D_k, D_0 act onto the functions $\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}$ by the formulas

$$\hat{D}_k\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = 2\pi i\phi_{\mathbf{j}+\boldsymbol{\varepsilon}_k}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega), \tag{11}$$

$$D_k\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = j_k\phi_{\mathbf{j}-\boldsymbol{\varepsilon}_k}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)$$
$$+2\pi in\sum_{p=1}^{r}\Omega_{kp}\phi_{\mathbf{j}+\boldsymbol{\varepsilon}_p}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega), \tag{12}$$

$$D_0\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = 2\pi i\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega), \tag{13}$$

where $\boldsymbol{\varepsilon}_k = (0,\ldots,0,1,0,\ldots,0)$ (the unit is on the kth place). The formula (11) is proved in the following way:

$$\hat{D}_k\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = \frac{\partial}{\partial\hat{x}_k}\left\{\exp(-2\pi inx_0)\sum_{\mathbf{m}\in\mathbf{Z}^r}\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}}\right.$$
$$\left.\times\exp\left[i\pi n\left(\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)+2\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\cdot\hat{\mathbf{x}}\right)\right]\right\}$$
$$= 2\pi in\exp(-2\pi inx_0)\sum_{\mathbf{m}}\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}+\boldsymbol{\varepsilon}_k}$$
$$\times\exp\left\{i\pi n\left[\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)+2\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{m}\right)^t\cdot\hat{\mathbf{x}}\right]\right\}$$
$$= 2\pi in\phi_{\mathbf{j}+\boldsymbol{\varepsilon}_k}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega).$$

The formulas (12) and (13) are proved in the same way.

It follows from (11)-(13) that

$$(D_i+\sum_{p=1}^{r}\Omega_{ip}\hat{D}_p)\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = j_i\phi_{\mathbf{j}-\boldsymbol{\varepsilon}_i}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega), \tag{14}$$

$$(\hat{D}_iD_i-\sum_{p=1}^{r}\Omega_{ip}\hat{D}_i\hat{D}_p)\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = 2\pi ij_i\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega). \tag{15}$$

Let us prove the orthogonality relations

$$\begin{aligned}&\int\overline{\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)}\phi_{\mathbf{k}}\begin{bmatrix}\mathbf{c}/s\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)d\hat{\mathbf{x}}d\mathbf{x}dx_0\\ &= \int_{\mathbf{R}^r}\mathbf{y}^{\mathbf{j}+\mathbf{k}}\exp(-2\pi n\mathbf{y}^t\Omega''\mathbf{y})d\mathbf{y}\cdot\delta_{ns}\delta_{\mathbf{a},\mathbf{c}(\mathrm{mod}\ n)},\end{aligned} \tag{16}$$

where the first integration is over $H(r,\mathbb{Z})\backslash H(r,\mathbf{R})$ and $\Omega = \Omega' + i\Omega''$. The integral on the left hand side is equal to

$$\sum_{\mathbf{m},\mathbf{m}'\in Z^r} \int\limits_{Z^r\backslash\mathbf{R}^r} \int\limits_{Z^r\backslash\mathbf{R}^r} \Bigg\{ \int\limits_{Z\backslash\mathbf{R}} \exp[-2\pi i(s-n)x_0]dx_0$$
$$\times \left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}} \cdot \left(\mathbf{x}+\mathbf{m}'+\frac{\mathbf{c}}{m}\right)^{\mathbf{k}} \exp i\pi \left[-n\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t \Omega \left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)\right.$$
$$\left.+s\left(\mathbf{m}'+\frac{\mathbf{c}}{s}\right)^t \Omega\left(\mathbf{m}'+\frac{\mathbf{c}}{s}\right)\right] \exp 2\pi i \left[-n\left(\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t \cdot \hat{\mathbf{x}} + s\left(\mathbf{m}'+\frac{\mathbf{c}}{s}\right)^t \cdot \hat{\mathbf{x}}\right]\Bigg\} d\hat{\mathbf{x}}d\mathbf{x}.$$

It follows from here that the left hand side of (16) is equal to 0 if $n \neq s$ or if $\mathbf{a} \not\equiv \mathbf{c} (\operatorname{mod} n)$. If $n = s$ and $\mathbf{a} \equiv \mathbf{c} (\operatorname{mod} n)$, then

$$\sum_{\mathbf{m}\in Z^r} \int\limits_{Z^r\backslash\mathbf{R}^r} \left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^{\mathbf{j}+\mathbf{k}} \exp\left[-2\pi n\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t \Omega''\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)\right] d\mathbf{x}$$
$$= \int\limits_{\mathbf{R}^r} \mathbf{y}^{\mathbf{j}+\mathbf{k}} \exp(-2\pi n \mathbf{y}^t \Omega'' \mathbf{y}) d\mathbf{y}.$$

This proves formula (16).

Monomials $\mathbf{y}^{\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}^r$, $j_k \geq 0$, $k = 1,2,\ldots,r$, form a basis of the Hilbert space $\mathfrak{L}^2_\rho(\mathbf{R}^r)$ with the scalar product

$$\langle f_1, f_2\rangle = \int\limits_{\mathbf{R}^r} f_1(\mathbf{y})\overline{f_2(\mathbf{y})}\rho(\mathbf{y})d\mathbf{y}, \quad \rho(\mathbf{y}) = \exp(-2\pi n \mathbf{y}^t \Omega'' \mathbf{y})$$

(see Section 9.6). The transformation $f(\mathbf{y}) \to \exp(-\pi n \mathbf{y}^t \Omega'' \mathbf{y}) f(\mathbf{y})$ is an isomorphism of $\mathfrak{L}^2_\rho(\mathbf{R}^r)$ onto $\mathfrak{L}^2(\mathbf{R}^r)$. Therefore, the functions

$$\exp(-\pi n \mathbf{y}^t \Omega'' \mathbf{y}) \mathbf{y}^{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^r, \quad j_k \geq 0, \quad k = 1,2,\ldots,r,$$

form a basis of the Hilbert space $\mathfrak{L}^2(\mathbf{R}^r)$.

19.3.2. The space $H_\Omega(\mathbf{a}/n)$. Let us span on the functions

$$\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}}, \mathbf{x}, x_0 \,|\, \Omega), \quad \mathbf{j} \in \mathbb{Z}^r, \quad j_k \geq 0, \quad k = 1,2,\ldots,r, \tag{1}$$

a linear space and close it with respect to the scalar product (16) of Section 19.3.1. We obtain the Hilbert space which will be denoted by $H_\Omega(\mathbf{a}/n)$. It is a closed subspace of the space $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R})) \simeq \mathfrak{L}^2(I^{2r+1})$ where $I = [0,1)$.

The formula

$$R(\hat{\mathbf{y}}, \mathbf{y}, y_0) f(\hat{\mathbf{x}}, \mathbf{x}, x_0) = f((\hat{\mathbf{x}}, \mathbf{x}, x_0)(\hat{\mathbf{y}}, \mathbf{y}, y_0)), \quad (\hat{\mathbf{y}}, \mathbf{y}, y_0) \in H(r, \mathbf{R}), \tag{2}$$

define the *quasi-regular representation* R of the Heisenberg group $H(r, \mathbf{R})$ in the space $\mathfrak{L}^2(H(r, \mathbb{Z}) \backslash H(r, \mathbf{R}))$, where the action of $(\hat{\mathbf{y}}, \mathbf{y}, y_0) \in H(r, \mathbf{R})$ onto $(\hat{\mathbf{x}}, \mathbf{x}, x_0)$ is understood in the sense of the formula (8) of Section 19.3.1:

$$(\hat{\mathbf{x}}, \mathbf{x}, x_0)(\hat{\mathbf{y}}, \mathbf{y}, y_0) = (\hat{\mathbf{x}} + \hat{\mathbf{y}}, \mathbf{x} + \mathbf{y}, x_0 + y_0 + \hat{\mathbf{x}} \cdot \mathbf{y}^t).$$

The action of the operators $R(\hat{\mathbf{y}}, \mathbf{y}, y_0)$ onto functions of the space $H_\Omega(\mathbf{a}/n)$ are also defined. The operators, corresponding to elements of the Lie algebra $\mathfrak{h}(r, \mathbf{R})$ of the group $H(r, \mathbf{R})$ act onto these functions by formulas (11)-(13) of Section 19.3.1. It is easy to see from these formulas that the space $H_\Omega(\mathbf{a}/n)$ is invariant with respect to the operators $\hat{D}_i$, D_i, $i = 1, 2, \ldots, r$, D_0. Therefore, $H_\Omega(\mathbf{a}/n)$ is invariant with respect to operators $R(\hat{\mathbf{y}}, \mathbf{y}, y_0)$, $(\hat{\mathbf{y}}, \mathbf{y}, y_0) \in H(r, \mathbf{R})$. In other words, these operators define the representation ρ_n of the group $H(r, \mathbf{R})$ on the space $H_\Omega(\mathbf{a}/n)$. With the help of the formulas (11)-(13) of Section 19.3.1 it is easy to prove that this representation is irreducible. Since for any $\phi \in H_\Omega(\mathbf{a}/n)$ we have

$$D_0 \phi = -\frac{\partial}{\partial x_0} \phi = 2\pi n \phi$$

(see formula (13) of Section 19.3.1), then

$$\rho_n(\mathbf{0}, \mathbf{0}, y_0) \phi = \exp(-2\pi i n y_0) \phi$$

for all $y_0 \in \mathbf{R}$. As we know, irreducible unitary representations of the Heisenberg group are uniquely determined (up to equivalence) by their values on elements of the center, that is, on the elements $(\mathbf{0}, \mathbf{0}, y_0)$. Therefore, *the representation ρ_n of the group $H(r, \mathbf{R})$ in $H_\Omega(\mathbf{a}/n)$ is equivalent to the representation $R^{-2\pi n}$ from Section* 12.1.4.

The formula (16) of Section 19.3.1 shows that the correspondence

$$\Phi_\Omega^{\mathbf{a}/n} \colon \exp(i\pi n \mathbf{y}^t \Omega \mathbf{y}) \mathbf{y}^{\mathbf{j}} \to \phi_{\mathbf{j}} \begin{bmatrix} \mathbf{a}/n \\ \mathbf{0} \end{bmatrix} (\hat{\mathbf{x}}, \mathbf{x}, x_0 \,|\, \Omega) \tag{3}$$

can be continued to an isomorphism of the space $\mathfrak{L}^2(\mathbf{R}^r)$ onto $H_\Omega(\mathbf{a}/n)$. The representation $R^{-2\pi n}$ acts in the space $\mathfrak{L}^2(\mathbf{R}^r)$ by the formula

$$R^{-2\pi n}(\hat{\mathbf{y}}, \mathbf{y}, y_0) f(\mathbf{u}) = \exp[-2\pi i n(y_0 + \hat{\mathbf{y}} \cdot \mathbf{u}^t)] f(\mathbf{u} + \mathbf{y}).$$

Let us show that the operator $\Phi_\Omega^{\mathbf{a}/n} \circ \mathfrak{F}$, where $\mathfrak{F}$ is the Fourier transform on $\mathfrak{L}^2(\mathbf{R}^r)$, intertwines the representations ρ_n and $R^{-2\pi n}$, that is, for all $(\hat{\mathbf{y}}, \mathbf{y}, y_0) \in H(r, \mathbf{R})$ we have

$$\rho_n(\hat{\mathbf{y}}, \mathbf{y}, y_0) \cdot \Phi_\Omega^{\mathbf{a}/n} \circ \mathfrak{F} = \Phi_\Omega^{\mathbf{a}/n} \circ \mathfrak{F} \cdot R^{-2\pi n}(\hat{\mathbf{y}}, \mathbf{y}, y_0). \tag{4}$$

It is sufficient to prove formula (4) for operators, corresponding to elements of the Lie algebra $\mathfrak{h}(r,\mathbf{R})$ of the group $H(r,\mathbf{R})$. Moreover, it is sufficient to verify validity of the relation (4) on basis elements of the space $\mathfrak{L}^2(\mathbf{R}^r)$.

Since

$$R^{-2\pi n}(\hat{\mathbf{x}},\mathbf{x},x_0)[\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}] \\ = \exp[-2\pi in(x_0+\hat{\mathbf{x}}\cdot\mathbf{u}^t)]\exp[i\pi n(\mathbf{u}+\hat{\mathbf{x}})^t\Omega(\mathbf{u}+\hat{\mathbf{x}})](\mathbf{u}+\hat{\mathbf{x}})^{\mathbf{j}},$$

then

$$R^{-2\pi n}(D_0)[\exp[i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}] \\ = -\frac{\partial}{\partial x_0}\{R^{-2\pi n}(\mathbf{0},\mathbf{0},x_0)[\exp i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}]\}\big|_{x_0=0} = -2\pi in\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}},$$

$$R^{-2\pi n}(D_k)[\exp[i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}] = -2\pi in\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}+\epsilon_k},$$

$$R^{-2\pi n}(\hat{D}_k)[\exp[i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}] \\ = 2\pi i\sum_{p=1}^{r}\Omega_{kp}p\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}+\epsilon_p} + j_k\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}-\epsilon_k}.$$

Consequently,

$$\begin{aligned}&\{(\Phi_\Omega^{\mathbf{a}/n}\cdot R^{-2\pi n}(D_0))\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}\}(\hat{\mathbf{x}},\mathbf{x},x_0)\\ &= 2\pi in\phi_i\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)\\ &= \{(\rho_n(D_0)\cdot\Phi_\Omega^{\mathbf{a}/n})\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}\}(\hat{\mathbf{x}},\mathbf{x},x_0),\end{aligned} \tag{5}$$

$$\begin{aligned}&\{(\Phi_\Omega^{\mathbf{a}/n}\cdot R^{-2\pi n}(\hat{D}_k))\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}\}(\hat{\mathbf{x}},\mathbf{x},x_0)\\ &= j_k\{\Phi_\Omega^{\mathbf{a}/n}\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}-\epsilon_k}\}(\hat{\mathbf{x}},\mathbf{x},x_0)\\ &+ 2\pi in\sum_{p=1}^{r}\Omega_{kp}\{\Phi_\Omega^{\mathbf{a}/n}\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}+\epsilon_p}\}(\hat{\mathbf{x}},\mathbf{x},x_0)\\ &= j_k\phi_{\mathbf{j}+\epsilon_k}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) + 2\pi in\sum_{p=1}^{r}\Omega_{kp}\phi_{\mathbf{j}+\epsilon_p}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)\\ &= \rho_n(D_k)\phi_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)\\ &= \{(\rho_n(D_k)\cdot\Phi_\Omega^{\mathbf{a}/n})\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}\}(\hat{\mathbf{x}},\mathbf{x},x_0).\end{aligned} \tag{6}$$

In the same way we verify that

$$\begin{aligned}&\{(\Phi_\Omega^{\mathbf{a}/n}\cdot R^{-2\pi n}(-D_k))\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}\}(\hat{\mathbf{x}},\mathbf{x},x_0)\\ &\quad= \{(\rho_n(\hat{D}_k)\cdot\Phi_\Omega^{\mathbf{a}/n})\exp(i\pi n\mathbf{u}^t\Omega\mathbf{u})\mathbf{u}^{\mathbf{j}}\}(\hat{\mathbf{x}},\mathbf{x},x_0).\end{aligned} \tag{7}$$

According to the results of Example 1 of Section 12.1.4 we have

$$\mathfrak{F}D_0\mathfrak{F}^{-1} = D_0, \quad \mathfrak{F}\hat{D}_k\mathfrak{F}^{-1} = D_k, \quad \mathfrak{F}D_k\mathfrak{F}^{-1} = -\hat{D}_k.$$

These relations together with formulas (5)-(7) give relation (4).

We also introduce the space $\bar{H}_\Omega(\mathbf{a}/n)$ consisting of all functions which are complex conjugate to functions of the space $H_\Omega(\mathbf{a}/n)$. We can symbolically write

$$\bar{H}_\Omega(\mathbf{a}/n) = \overline{H_\Omega(\mathbf{a}/n)}.$$

The formula (16) of Section 19.3.1 gives a scalar product in the space $\bar{H}_\Omega(\mathbf{a}/n)$ which turns it into a Hilbert space. The space $\bar{H}_\Omega(\mathbf{a}/n)$ is a subspace of the Hilbert space $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$. The operators (2), restricted onto $\bar{H}_\Omega(\mathbf{a}/n)$, define the irreducible representation $\bar{\rho}_n$ of the group $H(r,\mathbf{R})$ which is equivalent to the representation $R^{2\pi n}$ of Section 12.1.4. The mapping

$$\bar{\Phi}_\Omega^{\mathbf{a}/n} : \exp(i\pi n \mathbf{y}^t \Omega \mathbf{y})\mathbf{y}^{\mathbf{j}} \to \overline{\phi_{\mathbf{j}} \begin{bmatrix} \mathbf{a}/n \\ \mathbf{0} \end{bmatrix} (\hat{\mathbf{x}}, \mathbf{x}, x_0 \,|\, \Omega)}$$

determines equivalence of these representations.

19.3.3. Decomposition of the quasi-regular representation. In the previous section we have defined the quasi-regular representation R of the Heisenberg group in the Hilbert space $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$. In order to decompose it into irreducible representations of $H(r,\mathbf{R})$ we need the following lemma.

Lemma 1. *Let $f(\hat{\mathbf{x}},\mathbf{x},x_0)$ be a real analytic function on $H(r,\mathbb{Z})\backslash H(r,\mathbf{R})$ such that $\exp(2\pi i n x_0) f(\hat{\mathbf{x}},\mathbf{x},x_0)$ does not depend on x_0 and*

$$(D_k - \sum_{p=1}^{r} \Omega_{kp}\hat{D}_p) f(\hat{\mathbf{x}},\mathbf{x},x_0) = 0, \quad k = 1,2,\ldots,r.$$

Then

$$\psi(\hat{\mathbf{x}},\mathbf{x}) = \exp[-i\pi n(\mathbf{x}^t\Omega\mathbf{x} + 2\hat{\mathbf{x}}\cdot\mathbf{x}^t - 2x_0)] f(\hat{\mathbf{x}},\mathbf{x},x_0)$$

as function of $\mathbf{z} = \hat{\mathbf{x}} + \Omega\mathbf{x}$ belongs to the space $R_{n,\Omega}$ (see Section 19.2.9).

Proof. For every $(\hat{\mathbf{b}},\mathbf{b},b_0) \in H(r,\mathbf{R})$ we have

$$\begin{aligned}&\exp[-i\pi n((\mathbf{x}+\mathbf{b})^t\Omega(\mathbf{x}+\mathbf{b}) + 2(\hat{\mathbf{x}}+\hat{\mathbf{b}})\cdot(\mathbf{x}+\mathbf{b})^t - 2b_0 - 2x_0 - 2\hat{\mathbf{b}}\cdot\mathbf{x}^t)]\\ &\quad\times f((\hat{\mathbf{b}},\mathbf{b},b_0)(\hat{\mathbf{x}},\mathbf{x},x_0))\\ &= \exp[-i\pi n(\mathbf{b}^t\Omega\mathbf{b} + 2(\hat{\mathbf{x}}+\mathbf{x}\Omega)\cdot\mathbf{b}^t)]\exp[-i\pi n(\mathbf{x}^t\Omega\mathbf{x} + 2\hat{\mathbf{x}}\cdot\mathbf{x}^t - 2x_0)] f(\hat{\mathbf{x}},\mathbf{x},x_0).\end{aligned}$$

Consequently, the function $\psi(\hat{\mathbf{x}},\mathbf{x})$ satisfies the relation

$$\psi(\hat{\mathbf{x}}+\hat{\mathbf{b}},\mathbf{x}+\mathbf{b}) = \exp[-i\pi n(\mathbf{b}^t\Omega\mathbf{b} + 2(\hat{\mathbf{x}}+\Omega\mathbf{x})\cdot\mathbf{b}^t)]\psi(\hat{\mathbf{x}},\mathbf{x}).$$

If we consider $\psi(\hat{\mathbf{x}},\mathbf{x})$ as a function of $\mathbf{z}=\hat{\mathbf{x}}+\Omega\mathbf{x}$, then this relation is equivalent to relations (5) and (5′) of Section 19.2.9.

Since for $\mathbf{z}=\hat{\mathbf{x}}+\Omega\mathbf{x}$ we have $\bar{\mathbf{z}}=\hat{\mathbf{x}}+\bar{\Omega}\mathbf{x}$ and

$$\frac{\partial}{\partial\hat{\mathbf{x}}}=\frac{\partial}{\partial\mathbf{z}}+\frac{\partial}{\partial\bar{\mathbf{z}}},\quad \frac{\partial}{\partial\mathbf{x}}=\Omega\frac{\partial}{\partial\mathbf{z}}+\bar{\Omega}\frac{\partial}{\partial\bar{\mathbf{z}}},$$

then

$$\Omega\frac{\partial}{\partial\hat{\mathbf{x}}}-\frac{\partial}{\partial\mathbf{x}}=(\Omega-\bar{\Omega})\frac{\partial}{\partial\bar{\mathbf{z}}}.$$

In order to prove that $\psi(\hat{\mathbf{x}},\mathbf{x})$ is an entire function of $\mathbf{z}=\hat{\mathbf{x}}+\Omega\mathbf{x}$ it is sufficient to show that $\frac{\partial}{\partial\bar{\mathbf{z}}}\psi(\hat{\mathbf{x}},\mathbf{x})=0$, that is,

$$\left(\Omega\frac{\partial}{\partial\hat{\mathbf{x}}}-\frac{\partial}{\partial\mathbf{x}}\right)\psi(\hat{\mathbf{x}},\mathbf{x})=0.$$

Writing down this equality for i-th component we have

$$\left(\sum_{p=1}^{r}\Omega_{kp}\frac{\partial}{\partial\hat{x}_p}-\frac{\partial}{\partial x_k}\right)\psi(\hat{\mathbf{x}},\mathbf{x})=\left(\sum_{p=1}^{r}\Omega_{kp}\frac{\partial}{\partial\hat{x}_p}-\frac{\partial}{\partial x_k}\right.$$
$$\left.-\hat{x}_k\frac{\partial}{\partial x_0}\right)\psi(\hat{\mathbf{x}},\mathbf{x})=\left(\sum_{p=1}^{r}\Omega_{kp}\frac{\partial}{\partial\hat{x}_p}-\frac{\partial}{\partial x_k}-\hat{x}_k\frac{\partial}{\partial x_0}\right)$$
$$\times[\exp(-i\pi n(\mathbf{x}^t\Omega\mathbf{x}+2\hat{\mathbf{x}}\cdot\mathbf{x}^t-2x_0))]f(\hat{\mathbf{x}},\mathbf{x},x_0)$$
$$+\exp[-i\pi n(\mathbf{x}^t\Omega\mathbf{x}+2\hat{\mathbf{x}}\cdot\mathbf{x}^t-2x_0)]\left(\sum_{p=1}^{r}\Omega_{kp}\hat{D}_p-D_k\right)f(\hat{\mathbf{x}},\mathbf{x},x_0)=0.$$

Thus, $\psi(\hat{\mathbf{x}},\mathbf{x})$ is an entire function of $\mathbf{z}=\hat{\mathbf{x}}+\Omega\mathbf{x}$. Lemma is proved.

Theorem 1. *For the space $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$ we have the decomposition*

$$\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))=\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathbb{Z}^r/n\mathbb{Z}^r}\oplus H_{\Omega}(\mathbf{a}/n)$$
$$\oplus\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathbb{Z}^r/n\mathbb{Z}^r}\oplus\bar{H}_{\Omega}(\mathbf{a}/n)\tag{1}$$
$$\oplus\sum_{\mathbf{t},\mathbf{s}\in\mathbb{Z}^r}\oplus\mathbb{C}\exp[2\pi i(\mathbf{t}\cdot\hat{\mathbf{x}}^t+\mathbf{s}\cdot\mathbf{x}^t)].$$

Proof. According to formula (8) of Section 19.3.1 for every $b_0\in\mathbb{Z}$ and for every function $f\in\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$ we have

$$f(\hat{\mathbf{x}},\mathbf{x},x_0)=f((\mathbf{0},\mathbf{0},b_0)(\hat{\mathbf{x}},\mathbf{x},x_0))=f((\hat{\mathbf{x}},\mathbf{x},x_0)(\mathbf{0},\mathbf{0},b_0)),$$

that is,

$$R(\mathbf{0},\mathbf{0},b_0)f(\hat{\mathbf{x}},\mathbf{x},x_0) = f(\hat{\mathbf{x}},\mathbf{x},x_0).$$

It means that the decomposition of the quasi-regular representation R into irreducible representations contains the representations $R^{2\pi k}$ only with integral values of k.

Every of the spaces from the right hand side of formula (1) is contained in the space $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$. Therefore, it is sufficient to prove that in $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$ there are no other subspaces orthogonal to the subspaces of formula (1).

The space W of real analytic functions in $\hat{\mathbf{x}}$, $\mathbf{x}$, x_0 on $H(r,\mathbb{Z})\backslash H(r,\mathbf{R})$ is everywhere dense in the space $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$. Besides, W is invariant with respect to the operators D_i, $\hat{D}_i$, $i = 1,2,\dots,r$, D_0. In this reason it is sufficient to fulfil decomposition (1) for the subspace W. Let V be an irreducible invariant (with respect to the representation R of the group $H(r,\mathbf{R})$) subspace of W such that for all $f \in V$ we have

$$R(\mathbf{0},\mathbf{0},y_0)f(\hat{\mathbf{x}},\mathbf{x},x_0) = \exp(-2\pi i n y_0)f(\hat{\mathbf{x}},\mathbf{x},x_0).$$

If $n = 0$, then there exist $\mathbf{t}$ and $\mathbf{s}$ from $\mathbb{Z}^r$ such that

$$V = \mathbb{C}\exp[2\pi i(\mathbf{t}\cdot\hat{\mathbf{x}}^t + \mathbf{s}\cdot\mathbf{x}^t)].$$

Let $n > 0$. Then R determines in V the representation R_n which is equivalent to the representation $R^{-2\pi n}$. Let Q be the operator intertwining the representations in V and in $H_\Omega(\mathbf{a}/n)$. Then

$$R_n(D_0)Q = Q\rho_n(D_0), \quad R_n(D_i)Q = Q\rho_n(D_i), \quad R_n(\hat{D}_i)Q = Q\rho_n(\hat{D}_i).$$

Since $H_\Omega(\mathbf{a}/n)\cap W$ contains the function ϕ_0 satisfying the relation

$$\left(D_i - \sum_{p=1}^{r}\Omega_{ip}\hat{D}_p\right)\phi_0\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = 0, \quad i = 1,2,\dots,r,$$

then V contains a function $\varphi(\hat{\mathbf{x}},\mathbf{x},x_0)$ which satisfies the same system of differential equations. Since

$$\varphi(\hat{\mathbf{x}},\mathbf{x},x_0) = R(\mathbf{0},\mathbf{0},x_0)\varphi(\hat{\mathbf{x}},\mathbf{x},0) = \exp(-2\pi i n x_0)\varphi(\hat{\mathbf{x}},\mathbf{x},0),$$

then $\varphi(\hat{\mathbf{x}},\mathbf{x},x_0)$ satisfies the conditions of our lemma. Thus,

$$\varphi(\hat{\mathbf{x}},\mathbf{x},x_0) = \exp[i\pi n(\mathbf{x}^t\Omega\mathbf{x} + 2\hat{\mathbf{x}}\cdot\mathbf{x}^t - 2x_0)] \times \sum_{\mathbf{b}\in\mathbb{Z}^r/n\mathbb{Z}^r} a_{\mathbf{b}}\theta\begin{bmatrix}\mathbf{b}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}}+\Omega\mathbf{x},\Omega),$$

where $a_{\mathbf{b}} \in \mathbb{C}$. Therefore,

$$\varphi \in H_n \equiv \sum_{\mathbf{b} \in \mathbb{Z}^r / n\mathbb{Z}^r} \oplus H_\Omega(\mathbf{b}/n).$$

On the other hand, V is spanned by $\hat{D}^{\mathbf{j}}\varphi$, $\mathbf{j} \in \mathbb{Z}^r$, $j_k \geq 0$. Consequently, $V \subset H_n$. From the relation

$$\begin{aligned}
&\phi_{\mathbf{j}} \begin{bmatrix} \mathbf{a}/n \\ \mathbf{0} \end{bmatrix} \left(\hat{\mathbf{x}} + \frac{\hat{\mathbf{a}}}{n}, \mathbf{x}, x_0 \,|\, \Omega \right) \\
&\qquad = \exp \left[2\pi i \left(\hat{\mathbf{a}} \cdot \mathbf{x}^t + \frac{1}{n} \hat{\mathbf{a}} \cdot \mathbf{a}^t \right) \right] \phi_{\mathbf{j}} \begin{bmatrix} \mathbf{a}/n \\ \mathbf{0} \end{bmatrix} (\hat{\mathbf{x}}, \mathbf{x}, x_0 \,|\, \Omega)
\end{aligned}$$

we see that a function φ from H_n belongs to $H_\Omega(\mathbf{a}/n)$ if and only if

$$\varphi \left(\hat{\mathbf{x}} + \frac{\hat{\mathbf{a}}}{n}, \mathbf{x}, x_0 \right) = \exp \left[2\pi i \left(\hat{\mathbf{a}} \cdot \mathbf{x}^t + \frac{1}{n} \hat{\mathbf{a}} \cdot \mathbf{a}^t \right) \right] \varphi(\hat{\mathbf{x}}, \mathbf{x}, x_0).$$

In the same way it is proved that if $n < 0$, then $V \subset H_n$. Theorem is proved.

As we have seen above, the irreducible representations $R^{-2\pi n}$ of the Heisenberg group $H(r, \mathbf{R})$ are realized in the spaces $H_\Omega(\mathbf{a}/n)$, $\mathbf{a} \in \mathbb{Z}^r / n\mathbb{Z}^r$. A number of these spaces is n^r. The same assertion is valid for the spaces $\tilde{H}_\Omega(\mathbf{a}/n)$. Therefore, we obtain from formula (1) the following decomposition of the quasi-regular representation R of $H(r, \mathbf{R})$ into irreducible components:

$$R \simeq \sum_{n=-\infty}^{\infty} \oplus |n|^r R^{2\pi n} \oplus \sum_{\mathbf{s}, \mathbf{t} \in \mathbb{Z}^r} \oplus T_{\mathbf{st}}, \tag{2}$$

where $T_{\mathbf{st}}$ are the one-dimensional representations

$$(\hat{\mathbf{x}}, \mathbf{x}, x_0) \to \exp[2\pi i (\mathbf{t} \cdot \hat{\mathbf{x}}^t + \mathbf{s} \cdot \mathbf{x}^t)]$$

of the group $H(r, \mathbf{R})$.

It is easy to show that the spaces H_n, introduced above, do not depend on Ω.

19.3.4. The orthonormal basis of the space $H_\Omega(\mathbf{a}/n)$. Let $H_n(x)$ be Hermite polynomials (see Section 5.3.6). We denote by $H_{\mathbf{j}}(\mathbf{x})$, $\mathbf{j} = (j_1, \dots, j_r)$, $j_k \geq 0$, $\mathbf{x} = (x_1, \dots, x_r)$, the products of the Hermite polynomials:

$$H_{\mathbf{j}}(\mathbf{x}) = H_{j_1}(x_1) \dots H_{j_r}(x_r).$$

Due to the orthogonality relations for Hermite polynomials we have

$$\int_{\mathbf{R}^r} H_{\mathbf{j}}(\mathbf{x}) H_{\mathbf{k}}(\mathbf{x}) \exp(-\mathbf{x} \cdot \mathbf{x}^t) d\mathbf{x} = 2^{|\mathbf{j}|} \mathbf{j}! \pi^{r/2} \delta_{\mathbf{j},\mathbf{k}}.$$

Since $\Omega'' = (i/2)(\Omega - \bar{\Omega})$ is a positive definite symmetric matrix, then we can determine the matrix $\sqrt{\Omega''}$. The functions $H_{\mathbf{j}}(\mathbf{x}\sqrt{2\pi\Omega''})$ satisfy the orthogonality relation

$$\int_{\mathbf{R}^r} H_{\mathbf{j}}(\mathbf{x}\sqrt{2\pi\Omega''})H_{\mathbf{k}}(\mathbf{x}\sqrt{2\pi\Omega''})\exp(-2\pi n\mathbf{x}^t\Omega''\mathbf{x})d\mathbf{x} = \frac{2^{|\mathbf{j}|}\mathbf{j}!}{2^{r/2}\sqrt{\det\,\Omega''}}\delta_{\mathbf{j},\mathbf{k}}. \tag{1}$$

Now we constitute the functions

$$H_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega) = \frac{2^{r/4}(\det\,\Omega'')^{1/4}}{2^{|\mathbf{j}|/2}\sqrt{\mathbf{j}!}}\exp(-2\pi nix_0)$$
$$\times\sum_{\mathbf{m}\in Z^r} H_{\mathbf{j}}\left(\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)\sqrt{2\pi\Omega''}\right)\exp\left[i\pi n\left(\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)^t\Omega\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)\right.\right.$$
$$\left.\left.+\,2\hat{\mathbf{x}}^t\cdot\left(\mathbf{x}+\mathbf{m}+\frac{\mathbf{a}}{n}\right)\right)\right], \tag{2}$$
$$\mathbf{j} = (j_1,\dots,j_r),\quad j_k \geq 0.$$

With the help of formulas (16) of Section 19.3.1 and (1) we show that

$$\int H_{\mathbf{j}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)H_{\mathbf{k}}\begin{bmatrix}\mathbf{a}/n\\ \mathbf{0}\end{bmatrix}(\hat{\mathbf{x}},\mathbf{x},x_0\,|\,\Omega)dx_0d\hat{\mathbf{x}}d\mathbf{x} = \delta_{\mathbf{jk}}, \tag{3}$$

where the integration is over $H(r,\mathbb{Z})\backslash H(r,\mathbf{R}) \simeq [0,1)^{2r+1}$.

Therefore, *the set of functions* (2) *constitutes an orthonormal basis of the space* $H_\Omega(\mathbf{a}/n)$. *The set of functions* (2) *for all* n, $\mathbf{a}$ *together with complex conjugate functions and with the exponential functions from the right hand side of formula* (1) *of Section* 19.3.3 *form a basis of the space* $\mathfrak{L}^2(H(r,\mathbb{Z})\backslash H(r,\mathbf{R}))$.

19.4. Affine Lie Algebras

19.4.1. Non-twisted affine Lie algebras. Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{h}$ be its Cartan subalgebra. If Δ is the set of roots of the algebra $\mathfrak{g}$ with respect to the subalgebra $\mathfrak{h}$, then let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta}\mathfrak{g}_\alpha \tag{1}$$

be the decomposition of $\mathfrak{g}$ into the sum of root subspaces. With the help of the algebra $\mathfrak{g}$ we shall construct an infinite dimensional Lie algebra. For this we use the algebra $\mathbb{C}[t,t^{-1}]$ whose elements are expressions (Laurent polynomials)

$$P(t,t^{-1}) = \sum_{i\geq 0} a_it^i + \sum_{j>0} b_jt^{-j},\quad a_i,b_j\in\mathbb{C}, \tag{1'}$$

where all but a finite number of coefficients are zeros. The usual multiplication and addition are algebraic operations in $\mathbb{C}[t, t^{-1}]$.

Now we construct the *loop algebra*

$$L(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \tag{2}$$

which is the Lie algebra with commutation relations

$$[P \otimes X, Q \otimes Y]_0 = PQ \otimes [X, Y], \quad P, Q \in \mathbb{C}[t, t^{-1}], \quad X, Y \in \mathfrak{g}. \tag{3}$$

If $X_1, \ldots, X_n$ is a basis of the Lie algebra $\mathfrak{g}$, then the elements

$$1 \otimes X_i, \quad t^k \otimes X_i, \quad t^{-k} \otimes X_i, \quad i = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, \tag{4}$$

constitute a basis of the loop algebra.

Let us construct a central extension of $L(\mathfrak{g})$. For this we introduce the linear functional $\mathfrak{F}$ on $\mathbb{C}[t, t^{-1}]$ which acts on elements (1′) by formula $\mathfrak{F}(P) = b_1$ where b_1 is the coefficient at t^{-1}. With the help of the functional $\mathfrak{F}$ we define the bilinear form

$$\mathcal{B}(P, Q) = \mathfrak{F}\left(\frac{dP}{dt} Q\right) \tag{5}$$

on $\mathbb{C}[t, t^{-1}]$. It is easy to see that the form $\mathcal{B}$ has the properties

$$\mathcal{B}(P, Q) = -\mathcal{B}(Q, P), \tag{6}$$

$$\mathcal{B}(PQ, R) + \mathcal{B}(QR, P) + \mathcal{B}(RP, Q) = 0. \tag{7}$$

Let B be the Killing bilinear form on the Lie algebra $\mathfrak{g}$ (see Section 1.1.6). It is symmetric and invariant, that is,

$$B(X, Y) = B(Y, X), \quad B([Z, X], Y) = -B(X, [Z, Y]). \tag{8}$$

With the help of the bilinear forms $\mathcal{B}$ and B on $\mathbb{C}[t, t^{-1}]$ and $\mathfrak{g}$ we construct the bilinear form $\tilde{B}$ on $L(\mathfrak{g})$. It acts on elements $P \otimes X$, $P \in \mathbb{C}[t, t^{-1}]$, $X \in \mathfrak{g}$, by the formula

$$\tilde{B}(P \otimes X, Q \otimes Y) = B(X, Y)\mathfrak{F}\left(\frac{dP}{dt} Q\right).$$

By linearity it can be extended onto all elements of the algebra $L(\mathfrak{g})$. In particular,

$$\tilde{B}(t^m \otimes X, t^n \otimes Y) = \delta_{m,-n} B(X, Y).$$

From formulas (6)-(8) the *cocyclicity conditions*

$$\tilde{B}(a, b) = -\tilde{B}(b, a), \quad a, b \in L(\mathfrak{g}), \tag{9}$$

$$\tilde{B}([a,b]_0,c)+\tilde{B}([b,c]_0,a)+\tilde{B}([c,a]_0,b)=0, \quad a,b,c\in L(\mathfrak{g}), \tag{10}$$

of the form $\tilde{B}$ follow. The cocyclicity allows us to extend the Lie algebra $L(\mathfrak{g})$ by the central element c. As a result, we obtain the Lie algebra $\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}c$ with the commutation relations

$$[a+\alpha c, b+\beta c] = [a,b] = [a,b]_0 + \tilde{B}(a,b)c, \quad \alpha,\beta\in\mathbb{C} \tag{11}$$

(we have taken into account that c belongs to the center), where $[\cdot,\cdot]_0$ is the commutator in $L(\mathfrak{g})$ and $\tilde{B}(\cdot,\cdot)$ is the bilinear form constructed above. In particular,

$$[t^m\otimes X+\alpha c, t^n\otimes Y+\beta c] = t^{m+n}\otimes[X,Y] + m\delta_{m,-n}B(X,Y)c.$$

The formulas (9) and (10) ensure that formula (11) really is a commutator (that is, it is anti-symmetric and satisfies the Jacobi identity) and, thus, $\tilde{L}(\mathfrak{g})$ is a Lie algebra (see Section 1.0.1).

In order to construct the affine Lie algebra $\hat{\mathfrak{g}}$ corresponding to the simple Lie algebra $\mathfrak{g}$, we have to extend the algebra $\tilde{L}(\mathfrak{g})$. We adjoint to $\tilde{L}(\mathfrak{g})$ the element $d = t(d/dt)$ and define commutation relations of d with elements of $L(\mathfrak{g})$ and with c by the formulas

$$[d, P\otimes X] = -[P\otimes X, d] = t\frac{dP}{dt}\otimes X, \quad [d,c] = -[c,d] = 0. \tag{12}$$

Thus, *the affine Lie algebra $\hat{\mathfrak{g}}$ is the complex linear space*

$$\hat{\mathfrak{g}} = L(\mathfrak{g})\oplus\mathbb{C}c\oplus\mathbb{C}d \tag{13}$$

with the commutation relations

$$\begin{aligned}
&[(t^m\otimes X)+\alpha c+\mu d, (t^n\otimes Y)+\beta c+\mu' d]\\
&\quad = [(t^m\otimes X)+\alpha c, (t^n\otimes Y)+\beta c] + \mu[d,(t^n\otimes Y)] - \mu'[d,(t^m\otimes X)]\\
&\quad = t^{m+n}\otimes[X,Y] + \mu n(t^n\otimes Y) - \mu' m(t^m\otimes X) + m\delta_{m,-n}B(X,Y)c.
\end{aligned} \tag{14}$$

We recommend to the reader to verify that this bilinear form really is a commutator, that is, it is anti-symmetric and satisfies the Jacobi identity.

The affine Lie algebras $\hat{\mathfrak{g}}$, corresponding to the simple Lie algebras A_ℓ, B_ℓ, C_ℓ, D_ℓ, E_6, E_7, E_8, G_2, F_4, will be denoted by $A_\ell^{(1)}$, $B_\ell^{(1)}$, $C_\ell^{(1)}$, $D_\ell^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $G_2^{(1)}$, $F_4^{(1)}$. They are called *non-twisted affine Lie algebras.* Twisted affine Lie algebras are constructed below.

19.4.2. Roots and root elements of non-twisted affine Lie algebras. If $\mathfrak{h}$ is a Cartan subalgebra of the complex simple Lie algebra $\mathfrak{g}$, then

$$\hat{\mathfrak{h}} = \mathfrak{h}+\mathbb{C}c+\mathbb{C}d \tag{1}$$

is a maximal commutative subalgebra of the affine Lie algebra $\hat{\mathfrak{g}}$. We have defined in Section 15.1.2 the space $\mathfrak{h}'_R$ of linear forms on $\mathfrak{h}$. Let us spread forms λ from $\mathfrak{h}'_R$ (and from all dual space $\mathfrak{h}'$ for $\mathfrak{h}$) onto the whole of the space $\hat{\mathfrak{h}}$ by putting $\lambda(c) = \lambda(d) = 0$. We also introduce the linear form δ on $\hat{\mathfrak{h}}$ by putting

$$\delta(d) = 1; \quad \delta(H) = 0, \quad H \in \mathfrak{h}, \quad \delta(c) = 0. \tag{2}$$

We obtain the space $\tilde{\mathfrak{h}}'_R = \mathfrak{h}'_R \oplus \mathbf{R}\delta$ of linear forms on $\hat{\mathfrak{h}}$. As in the case of semisimple complex Lie algebra, we can decompose the affine Lie algebra $\hat{\mathfrak{g}}$ into a sum of root subspaces

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \sum_{\lambda} \oplus\, \hat{\mathfrak{g}}_\lambda, \tag{3}$$

where $\lambda \in \tilde{\mathfrak{h}}'_R$. Unlike to the case of complex semisimple Lie algebras, the root spaces $\hat{\mathfrak{g}}_\lambda$ can be many-dimensional and a number of these subspaces is infinite. The root decomposition of the affine Lie algebra is of the form

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \sum_{\gamma \in \hat{\Delta}} \hat{\mathfrak{g}}_\gamma, \tag{4}$$

where the set of roots $\hat{\Delta}$ coincides with

$$\hat{\Delta} = \{m\delta \mid m \in \mathbb{Z}\backslash\{0\}\} \cup \{m\delta + \alpha \mid m \in \mathbb{Z}, \alpha \in \Delta\} \tag{5}$$

(Δ is the set of roots for the Lie algebra $\mathfrak{g}$) and

$$\hat{\mathfrak{g}}_{m\delta} = t^m \otimes \mathfrak{h}, \quad \hat{\mathfrak{g}}_{m\delta+\alpha} = t^m \otimes \mathfrak{g}_\alpha. \tag{6}$$

Here $\mathfrak{g}_\alpha$ are the root subspaces of the Lie algebra $\mathfrak{g}$ (see formula (1) of Section 15.1.2). The formula $[\hat{\mathfrak{h}}, \hat{\mathfrak{g}}_\lambda] \subset \hat{\mathfrak{g}}_\lambda$ is verified by a direct evaluation. The comparison of the root subspaces from (6) with basis (4) of Section 19.4.1 shows that the sum in (4) is really direct. It is easy to see from (6) that

$$\dim \hat{\mathfrak{g}}_{m\delta} = \ell, \quad \dim \hat{\mathfrak{g}}_{m\delta+\alpha} = 1,$$

where ℓ is the rank of the simple Lie algebra $\mathfrak{g}$ (the dimension of the Cartan subalgebra $\mathfrak{h}$). The roots $m\delta + \alpha$, $\alpha \in \Delta$, are called *real* and the roots $m\delta$ are called *imaginary*.

The subalgebra

$$\sum_{\substack{s \in \mathbb{Z} \\ s \neq 0}} \hat{\mathfrak{g}}_{s\delta} \oplus \mathbb{C}c$$

of the affine Lie algebra $\hat{\mathfrak{g}}$ is isomorphic to the infinite dimensional Heisenberg algebra with the center $\mathbb{C}c$.

A minimal system of roots Π can be selected in $\hat{\Delta}$ such that every root $\gamma \in \hat{\Delta}$ is represented as a linear combination of the roots from Π with either all positive or all negative integral coefficients. Roots of Π are called *simple.* If a root $\gamma \in \hat{\Delta}$ is a linear combination of the roots of Π with positive (negative) coefficients, then γ is called *positive* (*negative*) *root.*

In order to give simple roots of the affine Lie algebra $\hat{\mathfrak{g}}$ let us note that in the set Δ of roots of the simple Lie algebra $\mathfrak{g}$ there is the root θ such that every root of Δ is obtained from θ by subtraction of simple roots of $\mathfrak{g}$. Using the unit vectors $\mathbf{e}_i$ for representation of roots of simple Lie algebras (see Section 15.1.2), the root θ for the classical simple Lie algebras can be written down as

for the algebra A_ℓ: $\theta = \mathbf{e}_1 - \mathbf{e}_{\ell+1}$,

for the algebra B_ℓ: $\theta = \mathbf{e}_1 + \mathbf{e}_2$,

for the algebra C_ℓ: $\theta = 2\mathbf{e}_1$,

for the algebra D_ℓ: $\theta = \mathbf{e}_1 + \mathbf{e}_2$.

The set Π *of simple roots of the affine Lie algebra* $\hat{\mathfrak{g}}$ *coincides with the roots*

$$\alpha_0 = \delta - \theta, \quad \alpha_1, \alpha_2, \ldots, \alpha_\ell,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ *are simple roots of the Lie algebra* $\mathfrak{g}$. It follows from formula (2) that

$$\alpha_i(d) = 0, \quad i = 1, 2, \ldots, \ell, \quad \alpha_0(d) = 1.$$

Let us construct the system of generating elements for the affine Lie algebra $\hat{\mathfrak{g}}$, which is analogous to the set of generating elements for simple Lie algebras from Theorem 1 of Section 15.1.3. Let ω be the involution of the simple Lie algebra $\mathfrak{g}$ from Section 15.1.3. Let us choose elements E_θ and $F_\theta \equiv E_{-\theta}$ in the root subspaces $\mathfrak{g}_\theta$ and $\mathfrak{g}_{-\theta}$ of the Lie algebra $\mathfrak{g}$ such that $E_{-\theta} = -\omega E_\theta$ and $B(E_\theta, F_\theta) = 2/(\theta, \theta)$. We set

$$[F_\theta, E_\theta] = -H_\theta \tag{6'}$$

and introduce the elements

$$e_0 = t \otimes F_\theta, \quad f_0 = t^{-1} \otimes E_\theta, \quad e_i = 1 \otimes E_i, \quad f_i = 1 \otimes F_i, \quad i = 1, 2, \ldots, \ell. \tag{7}$$

It follows from (6′) that

$$[e_0, f_0] = \frac{2}{(\theta, \theta)} c - 1 \otimes H_\theta \in \hat{\mathfrak{h}}. \tag{8}$$

Let H_i, E_i, F_i, $i = 1, 2 \ldots, \ell$, be the elements of the algebra $\mathfrak{g}$ from Theorem 1 of Section 15.1.3. We take the elements

$$h_i = 1 \otimes H_i, \quad e_i = 1 \otimes E_i, \quad f_i = 1 \otimes F_i, \quad i = 1, 2 \ldots, \ell, \tag{9}$$

and the elements

$$h_0 = \frac{2}{(\theta,\theta)}c - 1 \otimes H_\theta, \quad e_0 = t \otimes F_\theta, \quad f_0 = t^{-1} \otimes E_\theta \tag{10}$$

of the affine Lie algebra $\hat{\mathfrak{g}}$. The matrix

$$A \equiv (a_{ij})_{i,j=0}^{\ell} \equiv (\alpha_j(h_i))_{i,j=0}^{\ell}$$

is called the *Cartan matrix of the affine Lie algebra* $\mathfrak{g}$. The elements (9) and (10) generate the algebra $\tilde{L}(\mathfrak{g})$ and the commutation relations

$$\begin{gathered}[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \\ [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\ (\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0, \quad (\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0, \\ i, j = 0, 1, 2, \dots, \ell\ ,\end{gathered}$$

are valid.

Example 1. The affine Lie algebra $A_\ell^{(1)}$, $\ell > 1$, has the roots

$$\alpha_0 = \delta - (\alpha_1 + \alpha_2 + \ldots + \alpha_\ell), \quad \alpha_1, \alpha_2, \dots, \alpha_\ell. \tag{11}$$

The roots

$$\begin{gathered}k(\alpha_0 + \ldots + \alpha_{i-1}) + (k \pm 1)(\alpha_i + \ldots + \alpha_{j-1}) + k(\alpha_j + \ldots + \alpha_\ell), \\ k = 1, 2, \dots\ , \quad 0 \le i \le j \le \ell,\end{gathered} \tag{12}$$

are positive. The Cartan matrix of $A_\ell^{(1)}$ is of the form

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{13}$$

Example 2. The affine Lie algebra $A_1^{(1)}$ has the simple roots

$$\alpha_0 = \delta - \alpha_1 \quad \text{and} \quad \alpha_1. \tag{14}$$

Positive roots are exhausted by the roots

$$\begin{gathered}(k-1)\delta + \alpha_1 = (k-1)\alpha_0 + k\alpha_1, \\ k\delta - \alpha_1 = k\alpha_0 + (k-1)\alpha_1, \quad k\delta = k\alpha_0 + k\alpha_1.\end{gathered} \tag{15}$$

The Cartan matrix is of the form $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

For every simple root α_i, $i = 0, 1, \dots, \ell$, of the affine Lie algebra $\hat{\mathfrak{g}}$ we define the reflection

$$S_i\lambda = \lambda - \lambda(h_i)\alpha_i \tag{16}$$

of the space of linear forms $\tilde{\mathfrak{h}}'_R = \mathfrak{h}'_R \oplus \mathbf{R}\delta$ where h_i are elements of the subalgebra $\hat{\mathfrak{h}}$ from formulas (9) and (10). If $i = 1, 2, \dots, \ell$, then the reflections (16) coincide with the corresponding elements of the Weyl group W of the Lie algebra $\mathfrak{g}$. Products of reflections (16) generate the transformation group of the space $\hat{\mathfrak{h}}'_R$ which is called the *Weyl group of the affine Lie algebra* $\hat{\mathfrak{g}}$ and is denoted by $\hat{W}$. The group $\hat{W}$ contains the subgroup isomorphic to the group W.

As we shall see for the case of the algebra $A_1^{(1)}$, the Weyl group $\hat{W}$ of the affine Lie algebra $\hat{\mathfrak{g}}$ is infinite (more detailed study of the Weyl group will be given in Section 19.7.2). It is proved (see, for example, [134]) that the Weyl group $\hat{W}$ reflects roots of the algebra $\hat{\mathfrak{g}}$ into roots. In other words, *the root system* $\hat{\Delta}$ *of* $\hat{\mathfrak{g}}$ *is invariant with respect to* $\hat{W}$.

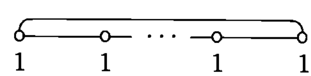
Since the linear form δ from formula (2) has the property $\delta(c) = 0$, then $\delta(h_i) = 0$, $i = 0, 1, \dots, \ell$. Therefore, it follows from (16) that

$$w\delta = \delta \quad \text{for all} \quad w \in \hat{W}.$$

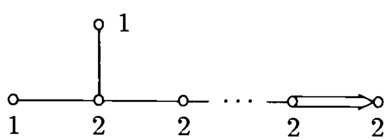
As in the case of semisimple complex Lie algebras, non-twisted Lie algebras can be characterized by the *Dynkin diagrams* which consist of simple roots. Simple roots are graphically represented by cycles. The roots α_i and α_j are connected by $a_{ij}a_{ji}$ lines, where a_{ij} is the corresponding element of the Cartan matrix. If $|a_{ij}| > |a_{ji}|$, then the lines, connecting the roots α_i and α_j, are supplied by arrow pointing towards the root α_i. We arrange the roots $\alpha_1, \alpha_2, \dots, \alpha_\ell$ in the same way as in the case of the simple Lie algebra $\mathfrak{g}$ (see Section 15.1.2) and place the root α_0 on the left. As a result, we obtain the following Dynkin diagrams:

$A_1^{(\ell)}$

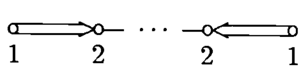
$A_\ell^{(1)}$, $\ell \geq 2$



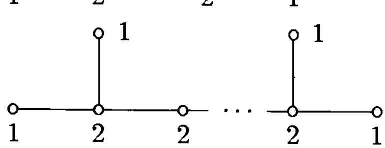
$B_\ell^{(1)}$, $\ell \geq 3$



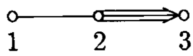
$C_\ell^{(1)}$, $\ell \geq 2$



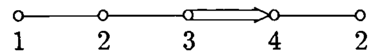
$D_\ell^{(1)}$, $\ell \geq 4$



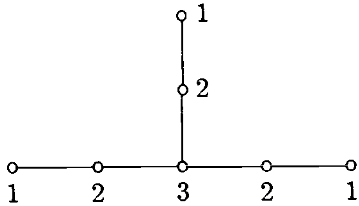
$G_2^{(1)}$



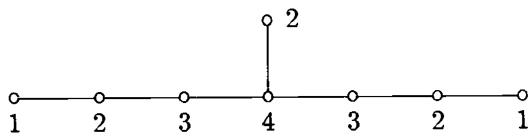
$F_4^{(1)}$



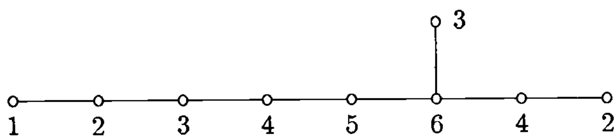
$E_6^{(1)}$



$E_7^{(1)}$



$E_8^{(1)}$



The numbers $a_0, a_1, \dots, a_\ell$ at the simple roots of Dynkin diagrams define the linear form δ, namely,

$$\delta = a_0\alpha_0 + a_1\alpha_1 + \dots + a_\ell\alpha_\ell, \tag{17}$$

and the root θ of the simple Lie algebra $\mathfrak{g}$, namely,

$$\theta = \delta - \alpha_0 = a_1\alpha_1 + \dots + a_\ell\alpha_\ell \tag{18}$$

(we have taken into account that $a_0 = 1$ for all Dynkin diagrams).

19.4.3. The Virasoro algebra. We have introduced in Section 19.4.1 the algebra $\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbf{C}c$ in which the operator $d \equiv d_0 = t(d/dt)$ acts. Let us also define the operators

$$d_s = t^{s+1}\frac{d}{dt}, \quad d_s(c) = 0, \quad s = 0, \pm 1, \pm 2, \dots \tag{1}$$

acting in $\tilde{L}(\mathfrak{g})$. It is evident that the operators d_s are derivations of the Lie algebra $L(\mathfrak{g})$, that is,

$$d_s[a,b]_0 = [d_s a, b]_0 + [a, d_s b]_0, \quad a, b \in L(\mathfrak{g}).$$

Let us show that d_s are derivations of the Lie algebra $\tilde{L}(\mathfrak{g})$. Since c is an element of the center, then it is sufficient to prove that

$$d_s[a + \lambda c, b + \mu c] = [d_s a, b] + [a, d_s b], \quad a, b \in L(\mathfrak{g}). \tag{2}$$

Due to equality (11) of Section 19.4.1 we have

$$d_s[a+\lambda c, b+\mu c] = d_s[a,b]_0 = [d_s a, b]_0 + [a, d_s b]_0. \tag{3}$$

The relation (2) is valid if the right hand sides of formulas (2) and (3) coincide, that is, if

$$\tilde{B}(d_s a, b) + \tilde{B}(a, d_s b) = 0.$$

This equality is proved by direct verification. Thus, the operators d_s are derivations of the algebra $L(\mathbf{g})$.

Let us construct the linear space

$$V = \sum_{s\in Z} \oplus\, \mathbb{C} d_s$$

and supply it with the commutation operation

$$[d_i, d_j]_0 = d_i d_j - d_j d_i.$$

By using formula (1) for the operators d_s we have

$$[d_i, d_j]_0 = (j-i) d_{i+j}, \quad i,j = 0, \pm 1, \pm 2, \dots .$$

These commutation relations turn V into an infinite dimensional Lie algebra.

Let us construct the central extension Vir of the Lie algebra V by adding to V the central element c:

$$\mathrm{Vir} = V \oplus \mathbb{C} c.$$

The commutation relations in Vir are defined by the formula

$$\begin{aligned}[d_i, d_j] &= [d_i, d_j]_0 + \frac{1}{12}(j^3 - j)\delta_{i,-j} c \\ &= (j-i) d_{i+j} + \frac{1}{12}(j^3-j)\delta_{i,-j} c.\end{aligned} \tag{4}$$

The direct evaluations show that the form $\hat{B}(d_i, d_j) = \frac{1}{12}(j^3 - j)\delta_{i,-j}$ satisfies the cocyclicity conditions (9) and (10) of Section 19.4.1. This guarantees that the commutator (4) is antisymmetric and satisfies the Jacobi identity. The Lie algebra Vir is called the *Virasoro algebra*. It is of a great importance for contemporary theoretical physics.

19.4.4. The affine Lie algebra $A_1^{(1)}$. It was shown in Section 19.4.2 that the affine Lie algebra $A_1^{(1)} \equiv \hat{sl}(2, \mathbb{C})$ has two simple roots $\alpha_0 = \delta - \alpha_1$ and α_1. The set of all roots coincides with

$$\hat{\Delta} = \{ j\delta + k\alpha_1 \mid j \in \mathbb{Z}, k \in \mathbb{Z}, (j,k) \neq (0,0) \}.$$

The roots $j\delta$, $j \neq 0$, are imaginary. The positive roots are given in formula (15) of Section 19.4.2. The set of roots of the algebra $A_1^{(1)}$ is shown on Fig. 19.1. The set of positive roots is separated on Fig. 19.1 by the dotted line.

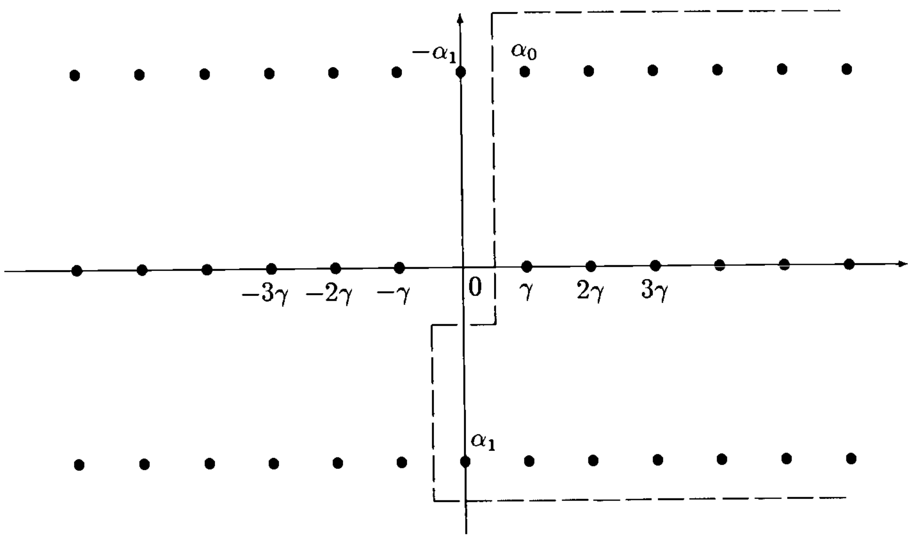


Fig. 19.1.

The generating elements $S_{\alpha_0} \equiv S_0$ and $S_{\alpha_1} = S_1$ of the Weyl group $\hat{W}$ of the algebra $A_1^{(1)}$ act on the simple roots by the formula

$$S_i(\alpha_j) = \alpha_j - \alpha_j(h_i)\alpha_i = \alpha_j - a_{ij}\alpha_i,$$

where a_{ij} are elements of the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We have

$$\left.\begin{aligned} S_0\alpha_0 &= -\alpha_0, \quad & S_0\alpha_1 &= \alpha_1 + 2\alpha_0, \\ S_1\alpha_0 &= \alpha_0 + 2\alpha_1, \quad & S_1\alpha_1 &= -\alpha_1. \end{aligned}\right\} \tag{1}$$

Besides,

$$S_0\delta = S_1\delta = \delta. \tag{2}$$

It follows from (1) that

$$(S_1S_0)\alpha_0 = -S_1\alpha_0 = -\alpha_0 - 2\alpha_1 = \alpha_0 - 2\delta,$$

$$(S_1S_0)^{-1}\alpha_0 = S_0S_1\alpha_0 = \alpha_0 + 2\delta,$$

$$(S_1S_0)\alpha_1 = S_1(\alpha_1 + 2\alpha_0) = 2\alpha_0 + 3\alpha_1 = \alpha_1 + 2\delta,$$
$$(S_1S_0)^{-1}\alpha_1 = S_0S_1\alpha_1 = \alpha_1 - 2\delta.$$

Thus, $(S_1S_0)^j$ does not change the imaginary roots $m\delta$ and shifts the root α_0 by $-(2j)\delta$ and the root α_1 by $(2j)\delta$.

Let $\hat{W}_0$ be the subgroup of the group $\hat{W}$ consisting of the elements $(S_1S_0)^k$, $k = 0, \pm 1, \pm 2, \ldots$. It is clear that $\hat{W}_0$ is a commutative group isomorphic to the additive group of integers. It follows from (1) and (2) that $\hat{W}_0$ is an invariant subgroup of $\hat{W}$ and that

$$\hat{W} = \hat{W}_0 \cup S_0\hat{W}_0 = \hat{W}_0 \cup S_1\hat{W}_0. \tag{3}$$

It is clear that determinant of the transformations S_0 and S_1 is equal to -1. Therefore, determinants of transformations from $\hat{W}_0$ and from $S_0\hat{W}_0$ are correspondingly equal to 1 and -1.

The sets of positive and of negative roots of the algebra $A_1^{(1)}$ will be denoted by $\hat{\Delta}_+$ and $\hat{\Delta}_-$ respectively. If $w \in \hat{W}$, then Φ_w will denote the set

$$\Phi_w = \hat{\Delta}_+ \cap w\hat{\Delta}_- = \{\varphi \in \hat{\Delta}_+ \mid w^{-1}\varphi \in \hat{\Delta}_-\} \tag{4}$$

and $\langle \Phi_w \rangle$ will denote the sum of all roots from Φ_w. It is easy to see that

$$\langle \Phi_{S_i w} \rangle = S_i \langle \Phi_w \rangle + \langle \Phi_{S_i} \rangle, \quad i = 0, 1. \tag{5}$$

The following statement is valid: if $\Phi_w = \Phi_{w'}$ or if $\langle \Phi_w \rangle = \langle \Phi_{w'} \rangle$, then $w = w'$.

Let ρ denote the linear form on $\tilde{h}'_R$ such that $\rho(h_0) = \rho(h_1) = 1$. Then $w\rho = w'\rho$ if and only if $w = w'$. We also have

$$S_i\rho = \rho - \alpha_i, \quad i = 0, 1.$$

We recommend to the reader to prove that

$$\langle \Phi_w \rangle = \rho - w\rho \tag{6}$$

for every $w \in \hat{W}$.

Below we shall need expressions for $\langle \Phi_w \rangle$. Let us show that for $w_m \equiv (S_1S_0)^m$, $m \in \mathbb{Z}$, we have

$$\begin{aligned} \langle \Phi_{w_m} \rangle &= m(2m-1)\alpha_0 + m(2m+1)\alpha_1 \\ &= 2m\alpha_1 + m(2m-1)\delta \end{aligned} \tag{7}$$

and for $S_0 w_m$ we have

$$\begin{aligned} \langle \Phi_{S_0 w_m} \rangle &= (2m^2 + 3m + 1)\alpha_0 + m(2m+1)\alpha_1 \\ &= (2m+1)\alpha_1 + (2m^2 + 3m + 1)\delta. \end{aligned} \tag{8}$$

Remind that

$$w_m\delta = \delta = \alpha_0 + \alpha_1, \quad w_m\alpha_0 = \alpha_0 - 2m\delta, \quad w_m\alpha_1 = \alpha_1 + 2m\delta.$$

It follows from here and from definition of Φ_{w_m} that if $m \geq 0$, then

$$\Phi_{w_m} = \{\alpha_1 + i\delta \mid 0 \leq i \leq 2m - 1\}$$

and if $m < 0$, then

$$\Phi_{w_m} = \{\alpha_0 + i\delta \mid 0 \leq i \leq -2m - 1\}.$$

This leads to formula (7). We have from (5) that

$$\langle \Phi_{S_0 w_m} \rangle = S_0[2m\alpha_1 + m(2m-1)\delta] + \langle \Phi_{S_0} \rangle.$$

Since $\langle \Phi_{S_0} \rangle = \alpha_0 = \delta - \alpha_1$, then we obtain formula (8).

In the same way we derive that

$$\begin{aligned} w_m(\lambda + \rho) - (\lambda + \rho) &= -[2am + (b+1)m(2m-1)]\alpha_0 \\ &\quad - [2am + (b+1)m(2m+1)]\alpha_1, \end{aligned} \tag{9}$$

$$\begin{aligned} S_0 w_m(\lambda + \rho) - (\lambda + \rho) &= -[a(2m+2) + (b+1)(m+1)(2m+1)]\alpha_0 \\ &\quad - [2am + (b+1)m(2m+1)]\alpha_1, \end{aligned} \tag{10}$$

where $\lambda = a\alpha_0 + b\rho$.

19.4.5. Twisted affine Lie algebras. Let $\mathfrak{g}$ be a simple complex Lie algebra and let σ be its automorphism. If $\sigma^m = 1$, then we have the decomposition of $\mathfrak{g}$ into a direct sum of eigenspaces of σ:

$$\mathfrak{g} = \sum_{j=0}^{m-1} \oplus\, \mathfrak{g}_j,$$

where $\mathfrak{g}_j$ corresponds to the eigenvalue $\varepsilon_j = \exp(2\pi i j/m)$. We have

$$[\mathfrak{g}_k, \mathfrak{g}_r] \subset \mathfrak{g}_s \quad \text{where} \quad s \equiv (k+r)(\text{mod } m). \tag{1}$$

Let $L(\mathfrak{g})$ be the algebra (2) of Section 14.4.1 constructed with the help of our Lie algebra $\mathfrak{g}$. We can represent $L(\mathfrak{g})$ as

$$L(\mathfrak{g}) = \sum_{j \in \mathbf{Z}} (t^j \otimes \mathfrak{g}).$$

Due to formula (1) the subspace

$$L(\mathfrak{g},\sigma)=\sum_{j\in\mathbf{Z}}(t^j\otimes\mathfrak{g}_{j(\mathrm{mod}\ m)})$$

has the property

$$[L(\mathfrak{g},\sigma)],L(\mathfrak{g},\sigma)]\subset L(\mathfrak{g},\sigma)$$

and, therefore, is a Lie subalgebra of $L(\mathfrak{g})$. The corresponding *twisted Lie algebra* is defined as

$$\hat{\mathfrak{g}}(\sigma)=L(\mathfrak{g},\sigma)\oplus\mathbf{C}c\oplus\mathbf{C}d,$$

where c and d are such as in Section 19.4.1.

Twisted affine Lie algebras are constructed with the help of automorphisms of simple Lie algebras which correspond to symmetries of Dynkin diagrams. Dynkin diagrams of complex simple Lie algebras are given in Section 15.1.2. It is easy to see that symmetries are admitted by the Dynkin diagrams of the algebras A_ℓ (enumeration of simple roots in opposite order), D_ℓ, E_6, F_4, D_4. The symmetries for the Lie algebras A_ℓ, D_ℓ, E_6, F_4 are of order 2. The Dynkin diagram of the algebra D_4 has also the symmetry of order 3.

The automorphisms of the Lie algebra $\mathfrak{g}$, corresponding to symmetries of the Dynkin diagram, will be denoted by μ. These automorphisms are constructed in the following way. Let $E'_{\alpha_i}\equiv E'_i$, $F'_{\alpha_i}\equiv F'_i$, $H'_i\equiv H'_{\alpha_i}$ be the generating elements of the Lie algebra $\mathfrak{g}$ from Theorem 1 of Section 15.1.3 and let $\bar{\mu}$ be a symmetry of the corresponding Dynkin diagram. Then the automorphism μ of the algebra $\mathfrak{g}$ is given by the formulas

$$\mu(E'_{\alpha_i})=E'_{\bar{\mu}\alpha_i},\quad \mu(F'_{\alpha_i})=F'_{\bar{\mu}\alpha_i},\quad \mu(H'_{\alpha_i})=H'_{\bar{\mu}\alpha_i}.$$

For automorphisms of order 2 and 3 we have the eigenspace decompositions

$$\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1,\quad \mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2. \tag{2}$$

The space $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. Let us construct this subalgebra for every of the Lie algebras $A_{2\ell}$, $A_{2\ell-1}$, $D_{\ell+1}$, D_4. It is sufficient to give generating elements of the subalgebra $\mathfrak{g}_0$.

Algebra $A_{2\ell}$, $\ell\ge 2$. In this case $\bar{\mu}\alpha_i=\alpha_{2\ell-i+1}$. The generating elements H_i, E_i, $i=0,1,\ldots,\ell-1$, of the subalgebra $\mathfrak{g}_0$ have the form

$$H_i=H'_i+H'_{2\ell-i+1},\quad 1\le i\le \ell-1,\quad H_0=2(H'_\ell+H'_{\ell+1}),$$
$$E_i=E'_i+E'_{2\ell-i+1},\quad 1\le i\le \ell-1,\quad E_0=\sqrt{2}(E'_\ell+E'_{\ell+1}).$$

Other generating elements F_i, $i=0,1,\ldots,\ell-1$, are obtained by the equality $F_i=-\omega(E_i)$. We have $\mathfrak{g}_0=B_\ell$. For A_2 we obtain $\mathfrak{g}_0=A_1$.

Algebra $A_{2\ell-1}$. In this case $\bar{\mu}\alpha_i = \alpha_{2\ell-i}$. The generating elements E_i, F_i, H_i, $i = 1, 2, \dots, \ell$, of the subalgebra $\mathfrak{g}_0$ are given by the formulas

$$H_i = H'_i + H'_{2\ell-i}, \quad 1 \le i \le \ell - 1, \quad H_\ell = H'_\ell,$$
$$E_i = E'_i + E'_{2\ell-i}, \quad 1 \le i \le \ell - 1, \quad E_\ell = E'_\ell,$$
$$F_i = -\omega(E_i), \quad 1 \le i \le \ell.$$

By the direct computation we find that $\mathfrak{g}_0 = C_\ell$.

Algebra $D_{\ell+1}$. The symmetry $\bar{\mu}$ of the Dynkin diagram is given by the formulas

$$\bar{\mu}\alpha_i = \alpha_i, \quad i = 1, 2, \dots, \ell - 1; \quad \bar{\mu}\alpha_\ell = \alpha_{\ell+1}, \quad \bar{\mu}\alpha_{\ell+1} = \alpha_\ell.$$

For the generating elements E_i, F_i, H_i, $i = 1, 2, \dots, \ell$, we have

$$H_i = H'_i, \quad 1 \le i \le \ell - 1, \quad H_\ell = H'_\ell + H'_{\ell+1},$$
$$E_i = E'_i, \quad 1 \le i \le \ell - 1, \quad E_\ell = E'_\ell + E'_{\ell+1},$$
$$F_i = -\omega(E_i), \quad i = 1, 2, \dots, \ell.$$

In this case $\mathfrak{g}_0 = B_\ell$.

Algebra D_4. We have

$$\bar{\mu}\alpha_1 = \alpha_4, \quad \bar{\mu}\alpha_2 = \alpha_2, \quad \bar{\mu}\alpha_3 = \alpha_1, \quad \bar{\mu}\alpha_4 = \alpha_3$$

(α_2 is the simple root, connected by lines with all other roots). The generating elements E_1, E_2, F_1, F_2, H_1, H_2 of $\mathfrak{g}_0$ are given by the formulas

$$H_1 = H'_1 + H'_3 + H'_4, \quad H_2 = H'_2,$$
$$E_1 = E'_1 + E'_3 + E'_4, \quad E_2 = E'_2,$$
$$F_i = -\omega(E_i), \quad i = 1, 2.$$

Now we can evaluate that $\mathfrak{g}_0 = G_2$.

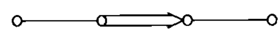
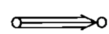
In every case the subalgebra $\mathfrak{g}_0$ is simple. The subspace $\mathfrak{h}_0 = \sum_j \mathbb{C}H_i$ is a Cartan subalgebra of $\mathfrak{g}_0$. Simple roots of $\mathfrak{g}_0$ will be denoted by β_i. They correspond to root elements E_i and F_i considered above.

Since

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_0, \mathfrak{g}_2] \subset \mathfrak{g}_2,$$

then the restriction of the adjoint representation of $\mathfrak{g}$ in $\mathfrak{g}$ onto $\mathfrak{g}_0$ is reducible representation. The representations of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ and in $\mathfrak{g}_2$ will be denoted by T_0 and T_1 respectively. The following statement is valid.

Statement 1. *The representation T_0 is irreducible. It is equivalent to the representation T_1 (if it exists). The highest weight θ_0 of the representation T_0 can be written as $\theta_0 = \sum_i a_i \beta_i$, where a_i are positive integers and β_i are simple roots of the Lie algebra $\mathfrak{g}_0$. The systems of simple roots and the numbers a_i are given by the table*

Lie algebra $\mathfrak{g}$	Lie algebra $\mathfrak{g}_0$	Simple roots β_i and numbers a_i
$A_{2\ell}$, $\ell \geq 2$	B_ℓ	 2 2 2 2
$A_{2\ell-1}$, $\ell \geq 3$	C_ℓ	 1 2 2 1
$D_{\ell+1}$, $\ell \geq 2$	B_ℓ	 1 1 1 1
A_2	A_1	 2
E_6	F_4	 1 2 3 2
D_4	G_2	 1 2

Multiplicities of weights of the representation T_0 do not exceed 1.

We recommend to the reader to prove this statement for every of the algebras $\mathfrak{g}$.

Weights of the representations T_0 and T_1 are obtained by restriction onto $\mathfrak{h}_0$ of the roots of the Lie algebra $\mathfrak{g}$. The subspace $\mathfrak{g}_1$ (the subspace $\mathfrak{g}_2$) of $\mathfrak{g}$ in which the representation T_0 (the representation T_1) of the subalgebra $\mathfrak{g}_0$ is realized can be decomposed into a sum of weight subspaces:

$$\mathfrak{g}_1 = \sum_{\alpha \in \Delta_1} \mathfrak{g}_{1,\alpha} + \mathfrak{g}_{1,0} \quad \left(\mathfrak{g}_2 = \sum_{\alpha \in \Delta_2} \mathfrak{g}_{2,\alpha} + \mathfrak{g}_{2,0} \right), \tag{2'}$$

where $\mathfrak{g}_{1,0}$ ($\mathfrak{g}_{2,0}$) is the subspace corresponding to zero weight. It is easy to show that if $\alpha \in \Delta_i$, $i = 1, 2$, then Δ_i contains the weight $-\alpha$ and

$$[\mathfrak{g}_{i,\alpha}, \mathfrak{g}_{i,-\alpha}] \subset \mathfrak{h}_0.$$

Therefore, if $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, then

$$\dim \mathfrak{h} = \dim \mathfrak{h}_0 + \dim \mathfrak{g}_{1,0} + \dim \mathfrak{g}_{2,0}$$

(the last summand exists only for the algebra D_4, that is, when μ is of order 3). Consequently, if $N = \dim \mathfrak{h} = \operatorname{rank} \mathfrak{g}$ and $\ell = \dim \mathfrak{h}_0 = \operatorname{rank} \mathfrak{g}_0$, then

$$\dim \mathfrak{g}_{1,0} = N - \ell \tag{3}$$

if μ is of order 2, and

$$\dim \mathfrak{g}_{1,0} = \dim \mathfrak{g}_{2,0} = \frac{1}{2}(N - \ell) \tag{4}$$

if μ is of order 3.

Let us introduce invariant bilinear form $(\cdot,\cdot) = cB(\cdot,\cdot)$, $c > 0$, on $\mathfrak{h}$ such that for the element H_α corresponding to the long root α of the algebra $\mathfrak{g}$ we have

$$(H_\alpha, H_\alpha) = 2k,$$

where k is an order of the automorphism μ (k is equal to 2 or 3). Restrict the form $(\cdot,\cdot)$ onto $\mathfrak{h}_0$ and define the mapping ν from $\mathfrak{h}_0$ into the space $\mathfrak{h}_0'$ of linear forms on $\mathfrak{h}_0$ such that

$$(\nu(h))(h_1) = (h, h_1), \quad h_1, h \in \mathfrak{h}_0.$$

This mapping establishes one-to-one correspondence between elements of $\mathfrak{h}_0$ and of $\mathfrak{h}_0'$.

Let us choose an element F_ε ($\varepsilon = \ell$ for the algebra $\mathfrak{g} = A_{2\ell}$ and $\varepsilon = 0$ for other algebras) of $\mathfrak{g}_{1,-\theta_0}$ such that

$$(F_\varepsilon, E_\varepsilon) = 1 \quad \text{where} \quad E_\varepsilon = -\omega(F_\varepsilon).$$

We put

$$[F_\varepsilon, E_\varepsilon] = H_\varepsilon.$$

(It is possible to show that $H_\varepsilon = -\nu^{-1}(\theta_0)$, where θ_0 is the highest weight of the representation T_0.) Now we can construct generating elements of the twisted affine Lie algebra $\hat{\mathfrak{g}}(\mu)$, where $\mathfrak{g}$ is one of the simple Lie algebras $A_{2\ell}$, $A_{2\ell-1}$, $D_{\ell+1}$, D_4.

We set

$$\hat{\mathfrak{g}}(\mu) = L(\mathfrak{g}, \mu) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where

$$L(\mathfrak{g}, \mu) = \sum_{j \in \mathbf{Z}} (t^j \oplus \mathfrak{g}_{j(\bmod k)})$$

and k is order of the automorphism μ. Let $\hat{\mathfrak{h}}$ be the maximal commutative subalgebra of $\hat{\mathfrak{g}}(\mu)$, that is,

$$\hat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Define the linear form δ on $\hat{\mathfrak{h}}$ such that

$$\delta(d) = 1 \quad \text{and} \quad \delta(h) = 0 \quad \text{if} \quad h \in \mathfrak{h}_0 \oplus \mathbb{C}c.$$

We put

$$e_\varepsilon = t \otimes F_\varepsilon, \quad f_\varepsilon = t^{-1} \oplus E_\varepsilon, \tag{5}$$

$$e_i = 1 \otimes E_i, \quad f_i = 1 \oplus F_i, \tag{6}$$

where $\varepsilon = \ell$ for $\mathfrak{g} = A_{2\ell}$ and $\varepsilon = 0$ for other algebras and $i = 0, 1, \ldots, \ell - 1$ for $A_{2\ell}$ and $i = 1, 2, \ldots, \ell$ for other algebras $\mathfrak{g}$. The elements e_j and f_j, $j = 0, 1, \ldots, \ell$, satisfy the commutation relations

$$[e_i, f_i] = H_i, \quad [e_\varepsilon, f_\varepsilon] = c - \nu^{-1}(\theta_0), \tag{7}$$

where i are such as in formula (6) and θ_0 is the highest weight of the representation T_0. The elements

$$d, e_j, f_j, H_j \quad \text{where} \quad H_\varepsilon = c - \nu^{-1}(\theta_0) \tag{8}$$

generate the affine Lie algebra $\hat{\mathfrak{g}}(\mu)$.

The root decomposition of the algebra $\hat{\mathfrak{g}}(\mu)$ is of the form

$$\hat{\mathfrak{g}}(\mu) = \hat{\mathfrak{h}} + \sum_{\alpha \in \hat{\Delta}} \hat{\mathfrak{g}}(\mu)_\alpha, \tag{9}$$

where the root system $\hat{\Delta}$ coincides with

$$\begin{aligned} \hat{\Delta} = \{s\delta + \gamma \mid s \in \mathbb{Z}, \gamma \in \Delta_r, \quad \text{where} \quad s \equiv r (\text{mod } k), r = 0, 1, \ldots, k-1\} \\ \cup \{s\delta | s \in \mathbb{Z}, s \neq 0\}. \end{aligned} \tag{10}$$

Here Δ_r is the root system Δ_0 of the algebra $\mathfrak{g}_0$ if $r = 0$, and the weight system Δ_r from formula (2) if $r \neq 0$. The root subspaces $\hat{\mathfrak{g}}(\mu)_\alpha$ from (9) are of the form

$$\hat{\mathfrak{g}}(\mu)_{s\delta+\gamma} = t^s \otimes \mathfrak{g}_{s(\text{mod } k),\gamma}, \quad \hat{\mathfrak{g}}(\mu)_{s\delta} = t^s \otimes \mathfrak{g}_{s(\text{mod } k),0}, \tag{11}$$

where $\mathfrak{g}_{r,\gamma}$, $r = 1, 2$, are defined by formula (2), $\mathfrak{g}_{0,0} = \mathfrak{h}_0$, and $\mathfrak{g}_{0,\gamma}$ are the root subspaces of $\mathfrak{g}_0$.

The roots

$$\alpha_\varepsilon = \delta - \theta_0, \quad \alpha_i, \quad i \in \{0, 1, \ldots, \ell\} \backslash \{\varepsilon\}, \tag{12}$$

are simple. Any root of $\hat{\Delta}$ is represented as linear combination of simple roots with non-negative (for positive roots) or with non-positive (for negative roots) integral coefficients.

If $H_0, H_1 \dots, H_\ell$ are elements from formula (8), then the matrix

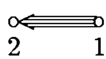
$$A \equiv (a_{ij})_{i,j=0}^{\ell} = (\alpha_j(H_i))_{i,j=1}^{\ell}$$

is called the *Cartan matrix of the affine Lie algebra* $\hat{\mathfrak{g}}(\mu)$.

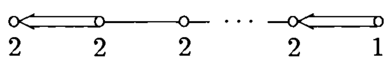
In the same way as in the case of non-twisted affine Lie algebras, the notions of imaginary and real roots and the notion of Weyl group are introduced for twisted affine Lie algebras.

The twisted affine Lie algebras $\hat{\mathfrak{g}}(\mu)$ where $\mathfrak{g}$ is one of the Lie algebras $A_{2\ell}$, $A_{2\ell-1}$, $D_{\ell+1}$, E_6, D_4 is denoted by the same symbol, supplied by the index 2 or 3 depending on an order of the corresponding automorphism μ, that is, by $A_{2\ell}^{(2)}$, $A_{2\ell-1}^{(2)}$, $D_{\ell-1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$, respectively. As in the case of non-twisted affine Lie algebra, we can associate *Dynkin diagram* with every twisted affine Lie algebra. These Dynkin diagrams are of the form

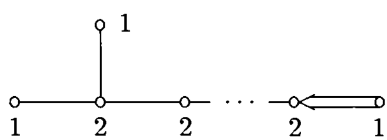
$A_2^{(2)}$



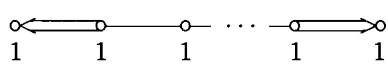
$A_{2\ell}^{(2)}$, $\ell \geq 2$



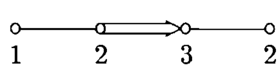
$A_{2\ell-1}^{(2)}$, $\ell \geq 3$



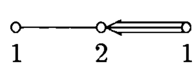
$D_{\ell+1}^{(2)}$, $\ell \geq 2$



$E_6^{(2)}$



$D_4^{(3)}$



where notations are the same as in Section 19.4.2.

19.4.6. The affine Lie algebra $A_2^{(2)}$. The affine Lie algebra $A_2^{(2)}$ is constructed with the help of the Lie algebra $A_2 \simeq \mathfrak{sl}(3, \mathbb{C})$ and its automorphism μ of order 2 (see Section 19.4.5). Using the generators of the subalgebra $\mathfrak{g}_0$, given in Section 19.4.5, we convince ourselves that $\mathfrak{g}_0$ consists of skew-symmetric matrices from $\mathfrak{sl}(3, \mathbb{C})$. In other words, we have $\mathfrak{g}_0 = \mathfrak{so}(3, \mathbb{C})$. It is clear that dim $\mathfrak{g}_0 = 3$.

Since dim $A_2 = 8$, then the decomposition of $\mathfrak{sl}(3,\mathbb{C})$ into a sum of eigenspaces of the automorphism μ has the form

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{g}_0 + \mathfrak{g}_1 \equiv \mathfrak{so}(3,\mathbb{C}) + \mathfrak{g}_1, \quad \dim \mathfrak{g}_1 = 5.$$

Moreover,

$$[\mathfrak{g}_0,\mathfrak{g}_0] \subset \mathfrak{g}_0, \quad [\mathfrak{g}_0,\mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_1,\mathfrak{g}_1] \subset \mathfrak{g}_0.$$

We choose the Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{so}(3,\mathbb{C})$ and the roots $\pm\alpha_1$ of the algebra $\mathfrak{so}(3,\mathbb{C})$ with respect to $\mathfrak{h}_0$, where α_1 is the positive root. Considering the adjoint representation ad of $\mathfrak{sl}(3,\mathbb{C})$ we easily find that the irreducible representation T_0 of the subalgebra $\mathfrak{g}_0$ in the space $\mathfrak{g}_1$ has the weights 0, $\pm\alpha_1$, $\pm 2\alpha_1$ and every of these weights is of multiplicity 1. The highest weight θ_0 is of the form $\theta_0 = 2\alpha_1$ (see the table of Section 19.4.5). Thus,

$$\mathfrak{g}_1 = \mathfrak{g}_{1,0} + \mathfrak{g}_{1,\alpha_1} + \mathfrak{g}_{1,2\alpha_1} + \mathfrak{g}_{1,-\alpha_1} + \mathfrak{g}_{1,-2\alpha_1},$$

where every subspace is one-dimensional.

In the algebra $L(\mathfrak{sl}(3,\mathbb{C})) = \mathbb{C}[t,t^{-1}] \otimes \mathfrak{sl}(3,\mathbb{C})$ we separate the subalgebra

$$L(\mathfrak{sl}(3,\mathbb{C}),\mu) = \sum_{j\in\mathbb{Z}} (t^j \otimes \mathfrak{g}_{j(\mathrm{mod}\ 2)}),$$

where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are such as above. The affine Lie algebra $A_2^{(2)} \equiv \hat{\mathfrak{sl}}(3,\mathbb{C})(\mu)$ coincides with

$$A_2^{(2)} = L(\mathfrak{sl}(3,\mathbb{C}),\mu) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We extend the linear form α_1 onto $\tilde{\mathfrak{h}} \equiv \mathfrak{h}_0 + \mathbb{C}d$ by setting $\alpha_1(d) = 0$. The formulas $\delta|_{\mathfrak{h}_0} = 0$, $\delta(d) = 1$ define the linear form δ. Therefore, the root system $\hat{\Delta}$ of the algebra $A_2^{(2)}$ coincides with

$$\begin{aligned} \hat{\Delta} = \{s\delta + k\alpha_1 \mid s \in \mathbb{Z};\quad & k = 0, \pm 1 \quad \text{if} \quad s \quad \text{is even and} \\ & k = 0, \pm 1, \pm 2 \quad \text{if} \quad s \quad \text{is odd}; \quad (s,k) \neq (0,0)\}. \end{aligned} \tag{1}$$

All roots are of a unit multiplicity.

The roots $\alpha_0 = \delta - 2\alpha_1$ and α_1 are simple, and the roots

$$\left.\begin{aligned} &(2j-1)\delta - 2\alpha_1 \equiv (2j-1)\alpha_0 + (4j-4)\alpha_1, && j \geq 1, \\ &j\delta - \alpha_1 \equiv j\alpha_0 + (2j-1)\alpha_1, && j \geq 1, \\ &j\delta \equiv j\alpha_0 + 2j\alpha_1, && j \geq 1, \\ &(j-1)\delta + \alpha_1 \equiv (j-1)\alpha_0 + (2j-1)\alpha_1, && j \geq 1, \\ &(2j-1)\delta + 2\alpha_1 \equiv (2j-1)\alpha_0 + 4j\alpha_1, && j \geq 1, \end{aligned}\right\} \tag{2}$$

are positive. The root system of the algebra A_2^2 is shown on Fig. 19.2. The positive roots are separated by the dotted line.

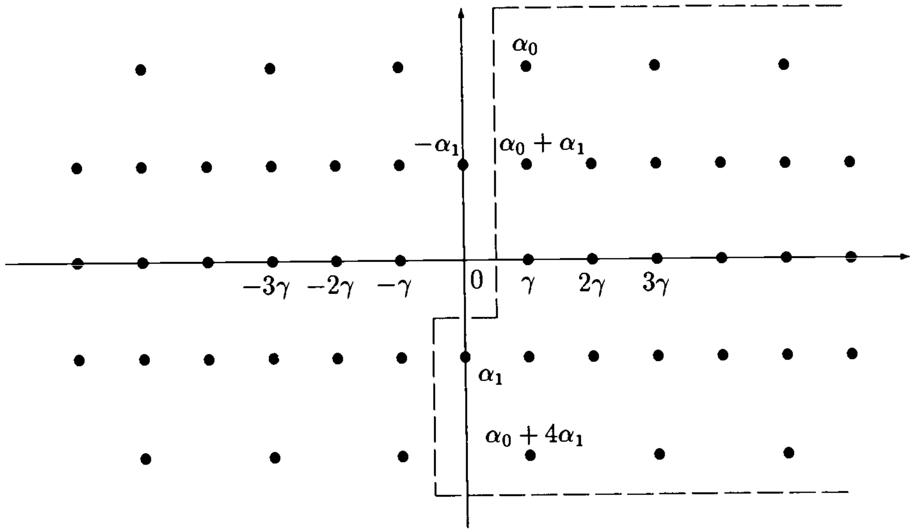


Fig. 19.2.

Direct evaluation shows that the Cartan matrix of the algebra $A_2^{(2)}$ is of the form

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

The Weyl group $\hat{W}$ of $A_2^{(2)}$ is generated by the elements $S_{\alpha_0} \equiv S_0$ and $S_{\alpha_1} \equiv S_1$. Action of S_0 and S_1 onto simple roots are defined with the help of the Cartan matrix. We have

$$\left.\begin{aligned} S_0\alpha_0 &= \alpha_0 - a_{00}\alpha_0 = -\alpha_0, \\ S_0\alpha_1 &= \alpha_1 - a_{01}\alpha_0 = \alpha_1 + \alpha_0, \\ S_1\alpha_0 &= \alpha_0 - a_{10}\alpha_1 = \alpha_0 + 4\alpha_1, \\ S_1\alpha_1 &= \alpha_1 - a_{11}\alpha_1 = -\alpha_1. \end{aligned}\right\} \tag{3}$$

Besides,

$$S_0\delta = S_1\delta = \delta. \tag{4}$$

With the help of formulas (3) we find that

$$\left.\begin{aligned} &(S_1S_0)\alpha_0 = -S_1\alpha_0 = -\alpha_0 - 4\alpha_1 = \alpha_0 - 2\delta, \\ &(S_1S_0)(\alpha_0+\alpha_1) = S_1(-\alpha_0+\alpha_1+\alpha_0) = -\alpha_1 = \alpha_0+\alpha_1-\delta, \\ &(S_1S_0)\alpha_1 = \alpha_1+\delta, \\ &(S_1S_0)(\alpha_0+4\alpha_1) = \alpha_0+4\alpha_1+2\delta. \end{aligned}\right\} \tag{5}$$

Due to formula (4), if we replace α_0, $\alpha_0+\alpha_1$, α_1, $\alpha_0+4\alpha_1$ by $\alpha_0+m\delta$, $\alpha_0+\alpha_1+m\delta$, $\alpha_1+m\delta$, $\alpha_0+4\alpha_1+m\delta$ on the left hand sides of formulas (5), then the right hand sides are also increased by $m\delta$. If we replace S_1S_0 by $(S_1S_0)^{-1}$ in (5), then -2δ, $-\delta$, δ, 2δ on the right hand sides will be replaced correspondingly by 2δ, δ, $-\delta$, -2δ.

Let $\hat{W}_0$ be the commutative subgroup of $\hat{W}$ consisting of the elements $(S_1S_0)^k$; $k = 0, \pm1, \pm2, \dots$. Then $\hat{W}_0$ is an invariant subgroup in $\hat{W}$ consisting of all transformations from $\hat{W}$ with unit determinant. We have

$$\hat{W} = \hat{W}_0 \cup S_0\hat{W}_0 = \hat{W}_0 \cup S_1\hat{W}_0.$$

In the same way as in the case of the affine Lie algebra $A_1^{(1)}$ (see Section 19.4.4), we introduce for A_2^2 the sets of roots Φ_w, $w \in \hat{W}$, and the sums $\langle\Phi_w\rangle$ of the roots from Φ_w. The formulas

$$\langle\Phi_{S_iw}\rangle = S_i\langle\Phi_w\rangle + \langle\Phi_{S_i}\rangle, \quad \langle\Phi_w\rangle = \rho - w\rho$$

are valid, where ρ is the linear form such that $\rho(H_0) = \rho(H_1) = 1$.

Below we shall need the expressions for $\langle\Phi_w\rangle$. For the elements $w_m = (S_1S_0)^m$, $m \in \mathbb{Z}$, we have

$$\begin{aligned} \langle\Phi_{w_m}\rangle &= \frac{m}{2}(3m-1)\alpha_0 + m(3m+2)\alpha_1 \\ &= 3m\alpha_1 + \frac{m}{2}(3m-1)\delta, \end{aligned} \tag{6}$$

$$\begin{aligned} \langle\Phi_{S_0w_m}\rangle &= \frac{1}{2}(3m^2+5m+2)\alpha_0 + (3m^2+2m)\alpha_1 \\ &= -(3m+2)\alpha_1 + \frac{1}{2}(3m^2+5m+2)\delta. \end{aligned} \tag{7}$$

Really, by using formulas (5) it is easy to find that for $m \geq 0$ we have

$$\Phi_{w_m} = \{\alpha_1 + j\delta, (\alpha_0+4\alpha_1) + 2j\delta \mid 0 \leq j \leq m-1\}$$

and for $m < 0$ we obtain

$$\Phi_{w_m} = \{\alpha_0 + 2j\delta, (\alpha_0+\alpha_1) + j\delta \mid 0 \leq j \leq -m-1\}.$$

Therefore, if $m \geq 0$, then

$$\langle \Phi_{w_m} \rangle = \left[m\alpha_1 + \frac{1}{2}m(m-1)\delta \right] + [m(\alpha_0 + 4\alpha_1) + m(m-1)\delta]$$
$$= 3m\alpha_1 + \frac{m}{2}(3m-1)\delta,$$

and if $m < 0$, then

$$\langle \Phi_{w_m} \rangle = \left[-m(\alpha_0 + \alpha_1) + \frac{m}{2}(m+1)\delta \right] + [-m\alpha_0 + m(m+1)\delta]$$
$$= 3m\alpha_0 + \frac{m}{2}(3m-1)\delta.$$

This proves formula (6). For $\langle \Phi_{S_0 w_m} \rangle$ we have

$$\langle \Phi_{S_0 w_m} \rangle = S_0 \langle \Phi_{w_m} \rangle + \langle \Phi_{S_0} \rangle = S_0 \left[3m\alpha_0 + \frac{m}{2}(3m-1)\delta \right] + \langle \Phi_{S_0} \rangle$$
$$= 3m(\alpha_0 + \alpha_1) + \frac{m}{2}(3m-1)\delta + \alpha_0$$
$$= -(3m+2)\alpha_1 + \frac{1}{2}(3m^2 + 5m + 2)\delta.$$

This proves formula (7).

In the same way we derive that

$$w_m(\lambda + \rho) - (\lambda + \rho) = -\left[2am + (c+1)\frac{m}{2}(3m-1) \right] \alpha_0 \\ - [4am + (c+1)m(3m+2)]\alpha_1, \tag{8}$$

$$S_0 w_m(\lambda + \rho) - (\lambda + \rho) = -[(2m+2)a + \frac{1}{2}(c+1)(3m^2 + 5m + 2)]\alpha_0 \\ - [4ma + (c+1)(3m^2 + 2m)]\alpha_1, \tag{9}$$

where $\lambda = a\alpha_0 + c\rho$.

19.4.7. Classification of affine Lie algebras. Let $A = (a_{ij})_{i,j=0}^{\ell}$ be a real $(\ell+1) \times (\ell+1)$ matrix which satisfies the conditions

1) $a_{ii} = 2$, $i = 0, 1, \ldots, \ell$, and a_{ij}, $i \neq j$, are non-positive integers,
2) if $a_{ji} = 0$, then $a_{ij} = 0$.

It is called a *generalized Cartan matrix.* The matrix A is called *indecomposable* if the set $I = \{0, 1, \ldots, \ell\}$ can not be divided into two parts I_1 and I_2 such that $a_{ij} = 0$ for all $i \in I_1$ $j \in I_2$. The matrix A is called *symmetrizable* if there exists non-degenerate diagonal $(\ell+1) \times (\ell+1)$ matrix D such that $D^{-1}A$ is a symmetric matrix.

Let A be an indecomposable symmetrizable generalized $(\ell+1)\times(\ell+1)$ Cartan matrix of rank ℓ. Let $\hat{\mathfrak{h}}$ be a complex vector space of dimension $\ell+2$ and let $\hat{\mathfrak{h}}'$ be the space of complex linear forms on $\hat{\mathfrak{h}}$. We choose linear independent elements

$$\alpha_0, \alpha_1, \ldots, \alpha_\ell \quad \text{and} \quad h_0, h_1, \ldots, h_\ell$$

correspondingly in $\hat{\mathfrak{h}}'$ and in $\hat{\mathfrak{h}}$ such that $\alpha_j(h_i) = a_{ij}$. The forms α_i are called *simple roots*. Now we define elements

$$e_i, f_i, \quad i = 0, 1, 2, \ldots, \ell \ , \tag{1}$$

and for $h \in \hat{\mathfrak{h}}$, e_i, f_i determine the commutation relations

$$\left.\begin{aligned} &[h, h'] = 0, \quad h, h' \in \hat{\mathfrak{h}}, \\ &[e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \quad \text{for} \quad i \neq j, \\ &[h_i, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i, \\ &(\text{ad}\, e_i)^{1-a_{ij}} e_j = 0, \quad (\text{ad}\, f_i)^{1-a_{ij}} f_j = 0. \end{aligned}\right\} \tag{2}$$

The elements $h \in \hat{\mathfrak{h}}$, e_i, f_i generate infinite dimensional Lie algebra (we denote it by $\hat{\mathfrak{g}}(A)$) which is called an *affine Lie algebra.* Affine Lie algebra is a special case of more general infinite dimensional Lie algebras which are called *Kac-Moody algebras.* Kac-Moody algebras are constructed in the same way with the help of indecomposable symmetrized Cartan matrices (without any conditions on a rank).

Theorem 1. *Every affine Lie algebra $\hat{\mathfrak{g}}(A)$ is isomorphic to one of the twisted or non-twisted affine Lie algebras constructed in Sections* 19.4.1 *and* 19.4.5.

The proof of this theorem can be found in [134].

The commutative subalgebra $\hat{\mathfrak{h}}$ of the algebra $\hat{\mathfrak{g}}(A)$ is its Cartan subalgebra. The linear forms $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ are its simple roots. It follows from Theorem 1 that the Dynkin diagrams given in Sections 19.4.1 and 19.4.5 uniquely (up to isomorphism) characterize affine Lie algebras.

The elements $h_0, h_1, \ldots, h_\ell$ together with the element d such that

$$\alpha_i(d) = 0, \quad i = 1, 2, \ldots, \ell, \quad \alpha_0(d) = 1$$

constitute a basis of the Cartan subalgebra $\hat{\mathfrak{h}}$. The central element c of $\hat{\mathfrak{g}}(A)$ is the linear combination of the basis elements $h_0, h_1, \ldots, h_\ell$. In order to give this linear combination we introduce the generalized Cartan matrix A^t which is the transposition of the matrix A. The matrix A^t defines the affine Lie algebra $\hat{\mathfrak{g}}(A^t)$. It is possible to show that the Dynkin diagram of the algebra $\hat{\mathfrak{g}}(A^t)$ is obtained from the Dynkin diagram of $\hat{\mathfrak{g}}(A)$ by changing directions of arrows without changing enumeration of simple roots. A positive integer a_i is attached to every simple root α_i of the Dynkin diagram of the algebra $\hat{\mathfrak{g}}(A)$. Denote numbers a_i corresponding to simple roots of the algebra $\hat{\mathfrak{g}}(A^t)$ by $a_i^\vee$. The integers

$$k = \sum_{i=0}^{\ell} a_i, \quad g = \sum_{i=0}^{\ell} a_i^\vee \tag{3}$$

are called the *Coxeter number* and the *dual Coxeter number*, respectively, of the affine Lie algebra $\hat{\mathfrak{g}}(A)$. The relations

$$\sum_{j=0}^{\ell} a_{ij} a_j = 0, \quad \sum_{i=0}^{\ell} a_{ij} a_i^\vee = 0 \tag{4}$$

are valid, where $(a_{ij}) \equiv A$ is the Cartan matrix of $\hat{\mathfrak{g}}(A)$. In other words, $\mathbf{a} = (a_0, a_1, \dots, a_\ell)$ is the vector such that $A\mathbf{a} = \mathbf{0}$. (Let us remind that the $(\ell+1) \times (\ell+1)$ matrix A is of rank ℓ.) The central element $c \in \hat{\mathfrak{g}}(A)$ is given by the formula

$$c = \sum_{i=0}^{\ell} a_i^\vee h_i. \tag{5}$$

It follows from (4) that $\alpha_i(c) = 0$, $i = 0, 1, \dots, \ell$.

The formulas

$$\left.\begin{aligned} (h_i, h_j) &= a_j (a_j^\vee)^{-1} a_{ij}, \quad i, j = 0, 1, \dots, \ell, \\ (h_i, d) &= 0, \quad i = 1, 2, \dots, \ell, \\ (h_0, d) &= a_0, \quad (d, d) = 0 \end{aligned}\right\} \tag{6}$$

define a symmetric bilinear form on $\hat{\mathfrak{h}}$. We have

$$(c, h_i) = 0, \quad i = 0, 1, \dots, \ell, \quad (c, c) = 0, \quad (c, d) = a_0. \tag{6'}$$

Let $\Lambda_0, \Lambda_1, \dots, \Lambda_\ell$ be elements of the dual space $\hat{\mathfrak{h}}'$ to the Cartan subalgebra $\hat{\mathfrak{h}}$ such that

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(\delta) = 0. \tag{7}$$

They are called fundamental weights of the algebra $\hat{\mathfrak{g}}(A)$. It is easy to see from (5) that

$$\Lambda_i(c) = a_i^\vee. \tag{8}$$

We have introduced the basis $h_0, h_1, \dots, h_\ell, d$ of the Cartan subalgebra $\hat{\mathfrak{h}}$. The elements $\alpha_0, \alpha_1, \dots, \alpha_\ell, \Lambda_0$ constitute a basis of $\hat{\mathfrak{h}}'$.

Let us determine the bilinear form $(\cdot,\cdot)$ on $\hat{\mathfrak{h}}'$. For this we define linear mapping $\nu: \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}'$ which is given by the formula

$$(\nu(h))(h') = (h, h'), \quad h, h' \in \hat{\mathfrak{h}}. \tag{9}$$

It is clear that $\nu(\hat{\mathfrak{h}}) = \hat{\mathfrak{h}}'$, that is, the mapping ν is invertable. For $\alpha, \beta \in \hat{\mathfrak{h}}'$ we set

$$(\alpha, \beta) \equiv (\alpha, \beta)_{\hat{\mathfrak{h}}'} = (\nu^{-1}(\alpha), \nu^{-1}(\beta)) \equiv (\nu^{-1}(\alpha), \nu^{-1}(\beta))_{\hat{\mathfrak{h}}}.$$

For the basis elements $\alpha_0, \alpha_1, \dots, \alpha_\ell, \Lambda_0$ we have

$$\left.\begin{aligned} (\alpha_i, \alpha_j) = a_i^{-1} a_i^{\vee} a_{ij}, \quad i, j = 0, 1, \dots, \ell, \\ (\Lambda_0, \alpha_i) = (\alpha_i, \Lambda_0) = 0, \quad i \neq 0, \\ (\Lambda_0, \alpha_0) = (\alpha_0, \Lambda_0) = a_0^{-1}, \quad (\Lambda_0, \Lambda_0) = 0. \end{aligned}\right\} \tag{10}$$

The form $(\cdot,\cdot)$ on $\hat{\mathfrak{h}}'$ is symmetric.

For the forms δ and θ introduced in Sections 19.4.2 and 19.4.5 we have

$$\delta = \sum_{i=0}^{\ell} a_i \alpha_i, \quad \theta = \delta - a_0 \alpha_0 = \sum_{i=1}^{\ell} a_i \alpha_i.$$

The relations

$$(\delta, \alpha_i) = 0, \quad i = 0, 1, \dots, \ell, \quad (\delta, \delta) = 0, \quad (\delta, \Lambda_0) = 1 \tag{11}$$

are valid. Let us note that $|\alpha_0|^2 \equiv (\alpha_0, \alpha_0) = 2a_0^{-1}$. Therefore,

$$|\theta|^2 = (\theta, \theta) = 2a_0. \tag{11$'$}$$

For the mapping $\nu: \hat{\mathfrak{h}} \to \hat{\mathfrak{h}}'$ we have

$$\nu(c) = \delta, \quad \nu(d) = a_0 \Lambda_0, \quad a_i^{\vee} \nu(h_i) = a_i \alpha_i, \quad i = 0, 1, \dots, \ell. \tag{12}$$

The reflections $S_{\alpha_i} \equiv S_i$, $i = 0, 1, \dots, \ell$, of the space $\hat{\mathfrak{h}}'$ which are given by the formula

$$S_i \alpha = \alpha - \alpha(h_i)\alpha_i, \quad \alpha \in \hat{\mathfrak{h}}', \tag{13}$$

generate the Weyl group $\hat{W}$ of $\hat{\mathfrak{g}}(A)$. We define the action of $\hat{W}$ on $\hat{\mathfrak{h}}$ setting

$$S_i h = h - \alpha_i(h) h_i, \quad h \in \hat{\mathfrak{h}}. \tag{13$'$}$$

Since $\delta(h_i) = 0$, $i = 0, 1, \dots, \ell$, then

$$w\delta = \delta \quad \text{for all} \quad w \in \hat{W}.$$

It is possible to show that the form $(\cdot,\cdot)$ on $\hat{\mathfrak{h}}$ and the corresponding form on $\hat{\mathfrak{h}}'$ are invariant with respect to the Weyl group $\hat{W}$.

Remind that roots of the affine Lie algebra $\hat{\mathfrak{g}}(A)$ are divided into the sets of imaginary and real roots. The set $\hat{\Delta}^{\text{im}}$ of imaginary roots coincides with

$$\hat{\Delta}^{\text{im}} = \{k\delta \mid k \in \mathbb{Z}, k \neq 0\}. \tag{14}$$

For the set $\hat{\Delta}^{\text{re}}$ of real roots of the non-twisted affine Lie algebra we have

$$\hat{\Delta}^{\text{re}} = \{n\delta + \alpha \mid n \in \mathbb{Z}, \alpha \in \Delta\}, \tag{15}$$

where Δ is the set of roots of the corresponding simple complex Lie algebra. The set $\hat{\Delta}^{\text{re}}$ of real roots of the twisted affine Lie algebra $\mathfrak{g}(A)$ can be described in the following way. The roots of the corresponding simple Lie algebra $\mathfrak{g}$ may be of different length. We represent the set Δ of the roots of $\mathfrak{g}$ as $\Delta = \Delta_\ell \cup \Delta_s$ where Δ_ℓ (Δ_s) is the subset of long (short) roots. Then for the set $\hat{\Delta}^{\text{re}}$ of the real roots of the affine Lie algebra $A_2^{(2)}$ we have

$$\begin{aligned} \hat{\Delta}^{\text{re}} = &\left\{\frac{1}{2}[\alpha + (2n-1)\delta] \mid \alpha \in \Delta_\ell, n \in \mathbb{Z}\right\} \\ &\cup \{\alpha + n\delta \mid \alpha \in \Delta_s, n \in \mathbb{Z}\} \cup \{\alpha + 2n\delta \mid \alpha \in \Delta_\ell, n \in \mathbb{Z}\} \end{aligned} \tag{16}$$

and for other twisted affine Lie algebras

$$\hat{\Delta}^{\text{re}} = \{\alpha + n\delta \mid \alpha \in \Delta_s, n \in \mathbb{Z}\} \cup \{\alpha + nk\delta \mid \alpha \in \Delta_\ell, n \in \mathbb{Z}\}, \tag{17}$$

where $k = 3$ for $D_4^{(3)}$ and $k = 2$ for other algebras.

For every root $\alpha \in \hat{\Delta}^{\text{re}}$ we can define the reflection $S_\alpha\colon \hat{\mathfrak{h}}' \to \hat{\mathfrak{h}}'$ which is given by the formula

$$S_\alpha\lambda = \lambda - (\lambda, \alpha^\vee)\alpha, \quad \alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}. \tag{18}$$

It is possible to show that $S_\alpha \in \hat{W}$.

Let us note that for the root $\theta = \delta - a_0\alpha_0$ we have $\theta = a_0\nu(\theta^\vee)$. Since $(\theta,\theta) = 2a_0$, then

$$(\theta^\vee, \theta^\vee) = 2a_0^{-2}.$$

It is clear that $\alpha_0 = a_0^{-1}(\delta - \theta)$. The roots

$$\alpha_0, \alpha_1, \dots, \alpha_\ell$$

are simple. Therefore, the set $\hat{\Delta}_+$ of positive roots of the algebra $\hat{\mathfrak{g}}(A)$ contains only those roots from formula (15) or from formulas (16) and (17) for which $n > 0$

and $\alpha \in \Delta_+$, where Δ_+ is the set of positive roots of the corresponding simple Lie algebra $\mathfrak{g}$.

Let $\hat{\mathfrak{h}}_R$ be the subspace of elements $h \in \hat{\mathfrak{h}}$ for which $\alpha_i(h) \in \mathbf{R}$, $i = 0, 1, \ldots, \ell$. The set

$$C = \{h \in \hat{\mathfrak{h}}_R \mid \alpha_i(h) \geq 0, \quad i = 0, 1, \ldots, \ell\} \tag{19}$$

is called the fundamental chamber in $\hat{\mathfrak{h}}_R$. The set

$$X = \bigcup_{w \in \hat{W}} wC \tag{20}$$

is called the *Tits cone.* The following statements are valid:

1) *For every $h \in X$ the orbit $\hat{W}h$ contains exactly one element of the fundamental chamber.*
2) *X coincides with the set of those elements $h \in \hat{\mathfrak{h}}_R$ for which $\alpha(h) < 0$ only for finite number of roots α from $\hat{\Delta}_+$.*
3) *X is a convex cone in $\hat{\mathfrak{h}}_R$.*
4) *The stabilizer $\hat{W}_h$ for the element $h \in C$ in $\hat{W}$ is generated by some reflections $S_i \equiv S_{\alpha_i}$ corresponding to simple roots.*
5) *If $h \in X$, then h is an interior point in X if and only if the stabilizer $\hat{W}_h$ of h is a finite group.*

Proofs of these statements can be found in [134].

The subalgebra of the algebra $\hat{\mathfrak{g}}(A)$ generated by the elements e_i, $i = 0, 1, \ldots, \ell$ (by the elements f_i, $i = 0, 1, \ldots, \ell$) will be denoted by $\mathfrak{n}_+$ (by $\mathfrak{n}_-$). We have

$$\mathfrak{n}_+ = \sum_{\alpha \in \hat{\Delta}_+} \hat{\mathfrak{g}}(A)_\alpha, \quad \mathfrak{n}_- = \sum_{\alpha \in \hat{\Delta}_+} \hat{\mathfrak{g}}(A)_{-\alpha}. \tag{21}$$

As in the case of semisimple Lie algebras, the decomposition

$$\hat{\mathfrak{g}}(A) = \hat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- \tag{22}$$

is valid.

19.4.8. The universal enveloping algebra. Let

$$\hat{\mathfrak{g}}(A) \equiv \hat{\mathfrak{g}} = \hat{\mathfrak{h}} + \sum_{\alpha \in \hat{\Delta}_+} \hat{\mathfrak{g}}_\alpha + \sum_{\alpha \in \hat{\Delta}_+} \hat{\mathfrak{g}}_{-\alpha}$$

be the decomposition of the affine Lie algebra $\hat{\mathfrak{g}} \equiv \hat{\mathfrak{g}}(A)$ into a sum of root subspaces. It is clear that

$$[\hat{\mathfrak{h}}, \hat{\mathfrak{g}}_\alpha] \subset \hat{\mathfrak{g}}_\alpha, \quad [\hat{\mathfrak{h}}, \hat{\mathfrak{g}}_{-\alpha}] \subset \hat{\mathfrak{g}}_{-\alpha}, \quad [\hat{\mathfrak{g}}_\alpha, \hat{\mathfrak{g}}_\beta] \subset \hat{\mathfrak{g}}_{\alpha+\beta}, \tag{1}$$

where $\hat{\mathfrak{g}}_{\alpha+\beta} \subset \hat{\mathfrak{h}}$ if $\alpha = -\beta$. Let us choose bases in $\hat{\mathfrak{h}}$ and in every of the root subspaces. Union of these bases gives a basis of the algebra $\hat{\mathfrak{g}}$. Let us arrange basis elements in a certain order and enumerate them by the numbers $\ldots, -2, -1, 0, 1, 2, \ldots$. Such a basis will be called *ordered.*

Let $\mathfrak{U}(\hat{\mathfrak{g}})$ be the universal enveloping algebra of the affine Lie algebra $\hat{\mathfrak{g}}$. It can be identified with the multiplicative algebra (with a unit) of polynomials of elements of an ordered basis of $\hat{\mathfrak{g}}$ in which basis elements satisfy the commutation relations, given in $\hat{\mathfrak{g}}$. More detailed description of universal enveloping algebras see in Chapter 1.

Let us fix an ordered basis $\{B_i\}$ of the affine Lie algebra $\hat{\mathfrak{g}}$. The monomials

$$B_{i_1}B_{i_2}\ldots B_{i_k}, \quad 1 \le k < \infty, \quad i_1 \le i_2 \le \ldots \le i_k,$$

of the universal enveloping algebra $\mathfrak{U}(\hat{\mathfrak{g}})$ are called *standard.*

Poincare-Birkhoff-Witt theorem. *The unit and all standard monomials constitute a basis of the algebra* $\mathfrak{U}(\hat{\mathfrak{g}})$.

This theorem is proved in the same way as in the case of semisimple complex Lie algebras (see, for example, [119]).

Let us choose an ordered basis of $\hat{\mathfrak{g}}$ in such way that basis elements from $\mathfrak{n}_- = \sum_{\alpha\in\hat{\Delta}_+} \hat{\mathfrak{g}}_{-\alpha}$ follow at the beginning, then basis elements from $\hat{\mathfrak{h}}$ and, at last, basis elements from $\mathfrak{n}_+ = \sum_{\alpha\in\hat{\Delta}_+} \hat{\mathfrak{g}}_{\alpha}$. If $\mathfrak{U}(\mathfrak{n}_-)$, $\mathfrak{U}(\hat{\mathfrak{h}})$, $\mathfrak{U}(\mathfrak{n}_+)$ are universal enveloping algebras for $\mathfrak{n}_-$, $\hat{\mathfrak{h}}$, $\mathfrak{n}_+$, correspondingly, then it follows from Poincare-Birkhoff-Witt theorem that

$$\mathfrak{U}(\hat{\mathfrak{g}}) = \mathfrak{U}(\mathfrak{n}_-)\mathfrak{U}(\hat{\mathfrak{h}})\mathfrak{U}(\mathfrak{n}_+). \tag{2}$$

It is possible to derive the adjoint action of the subalgebra $\hat{\mathfrak{h}}$ in $\mathfrak{U}(\mathfrak{n}_-)$. Namely, with every $h \in \hat{\mathfrak{h}}$ we associate the operator ad h which acts in $\mathfrak{U}(\mathfrak{n}_-)$ by the formula

$$(\text{ad } h)n = [h, n] = hn - nh, \quad n \in \mathfrak{U}(\mathfrak{n}_-).$$

Direct verification shows that

$$(\text{ad } h)(n_1n_2n_3\ldots n_m) = ((\text{ad } h)n_1)n_2n_3\ldots n_m \\ + n_1((\text{ad } h)n_2)n_3\ldots n_m + \ldots + n_1n_2n_3\ldots((\text{ad } h)n_m).$$

Therefore, a product of weight elements from $\mathfrak{n}_-$ (that is, of elements from weight subspaces) is again a weight element of the operators ad h, $h \in \hat{\mathfrak{h}}$. Moreover, *the algebra* $\mathfrak{U}(\mathfrak{n}_-)$ *decomposes into a direct sum of weight subspaces:*

$$\mathfrak{U}(\mathfrak{n}_-) = \sum_{\lambda} \oplus\, \mathfrak{U}_\lambda(\mathfrak{n}_-), \tag{3}$$

where

$$\mathfrak{U}_\lambda(\mathfrak{n}_-) = \{n \in \mathfrak{U}(\mathfrak{n}_-) \mid (\mathrm{ad}\, h)n = \lambda(h)n\}.$$

It is evident that all linear forms λ are linear combinations of the simple roots $\alpha_0, \alpha_1, \dots, \alpha_\ell$ with non-positive integral coefficients.

Let us find dimensions of the spaces $\mathfrak{U}_\lambda(\mathfrak{n}_-)$. For this we introduce an ordering into the set of positive roots: $\beta_1, \beta_2 \dots$. Then we choose an ordered basis $f^1_{-\beta_i}, f^2_{-\beta_i}, \dots, f^{m_i}_{-\beta_i}$ in every of the root subspaces $\hat{\mathfrak{g}}_{-\beta_i}$. The products

$$(f^1_{-\beta_{j_1}})^{k_1} \dots (f^{m_{j_1}}_{-\beta_{j_1}})^{k_{j_1}} \dots (f^1_{-\beta_{j_r}})^{s_1} \dots (f^{m_{j_r}}_{-\beta_{j_r}})^{s_{j_r}},$$

where $j_1 \le \dots \le j_r$, constitute a basis of the algebra $\mathfrak{U}(\mathfrak{n}_-)$. These basis elements belong to the eigenspaces $\mathfrak{U}_\lambda(\mathfrak{n}_-)$ where

$$\lambda = -(k_1 + \dots + k_{j_1})\beta_{j_1} - \dots - (s_1 + \dots + s_{j_r})\beta_{j_r}.$$

A number of basis elements of $\mathfrak{U}(\mathfrak{n}_-)$ with the weight λ is equal to $K(-\lambda)$ where K is the so called *partition function.* It is non-vanishing only on the set Q_+ of linear forms β which are linear combinations of simple roots $\alpha_0, \alpha_1, \dots, \alpha_\ell$ with non-negative integral coefficients. For $\beta = 0$ we have $K(\beta) = 1$ and for $\beta \in Q_+$ the function $K(\beta)$ is equal to the number of partitions of β into a sum of positive roots, where each root is counted with its multiplicity. Thus,

$$\dim \mathfrak{U}_\lambda(\mathfrak{n}_-) = K(-\lambda). \tag{4}$$

19.5. Representations of Affine Lie Algebras and their Characters

19.5.1. Integrable and weight representations. Let $\mathfrak{H}$ be a complex linear space in which a representation T of the affine Lie algebra $\hat{\mathfrak{g}}(A)$ acts. Suppose that the operators $T(h)$, $h \in \hat{\mathfrak{h}}$, can be simultaneously diagonalized. This means that

$$\mathfrak{H} = \sum_{\lambda \in \hat{\mathfrak{h}}'} \oplus \mathfrak{H}_\lambda, \tag{1}$$

where $\mathfrak{H}_\lambda$ are eigenspaces of all operators $T(h)$, $h \in \hat{\mathfrak{h}}$:

$$\mathfrak{H}_\lambda = \{x \in \mathfrak{H} \mid T(h)x = \lambda(h)x\}.$$

If $\dim \mathfrak{H}_\lambda \neq 0$, then the linear form λ on $\mathfrak{h}$ is called a *weight* of the representation T, and $\mathfrak{H}_\lambda$ is a *weight subspace.* The number $m_\lambda = \dim \mathfrak{H}_\lambda$ is said to be a *multiplicity* of weight λ in the representation T.

If the carrier space $\mathfrak{H}$ of the representation T decomposes into a sum (1) of weight subspaces, then we call T a *weight representation.* Below we consider only such weight representations for which all multiplicities m_λ are finite.

Let $\hat{\mathfrak{g}}(A)$ be the affine Lie algebra generated by the elements e_i, f_i, h_i, $i = 0, 1, \ldots, \ell$, satisfying relations (2) of Section 19.4.7, and by d. Let us fix an integer i, $0 \le i \le \ell$, and constitute the subalgebra $\hat{\mathfrak{g}}_i(A) = \mathbb{C}e_i + \mathbb{C}f_i + \mathbb{C}h_i$. The algebra $\hat{\mathfrak{g}}_i(A)$ is isomorphic to the simple complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Let T be a weight representation of $\hat{\mathfrak{g}}(A)$. This representation is called *integrable* if restriction of it onto each subalgebra $\hat{\mathfrak{g}}_i(A)$ decomposes into a direct sum of finite dimensional irreducible representations of $\hat{\mathfrak{g}}_i(A)$.

Weight systems of irreducible finite dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ are invariant with respect to the reflection from the Weyl group of $\mathfrak{sl}(2, \mathbb{C})$. In this reason the weight system of every integrable representation T of the affine Lie algebra $\hat{\mathfrak{g}}(A)$ is invariant with respect to the reflections

$$S_i\lambda = \lambda - \lambda(h_i)\alpha_i, \quad i = 0, 1, \ldots, \ell.$$

This means that the weight system of T is invariant with respect to the Weyl group $\hat{W}$ of $\hat{\mathfrak{g}}(A)$. In particular, for multiplicities of weights m_λ we have

$$m_\lambda = m_{w\lambda}, \quad w \in \hat{W}.$$

19.5.2. Verma modules. Irreducible integrable representations of the affine Lie algebra $\hat{\mathfrak{g}}(A)$ can be constructed with the help of so called Verma modules. Verma module is, in fact, a representation of the algebra $\hat{\mathfrak{g}}(A)$ with a highest weight. Representations with highest weight are defined as follows. Let $\mathfrak{U}(\hat{\mathfrak{g}})$ be the universal enveloping algebra of the affine Lie algebra $\hat{\mathfrak{g}}(A)$. A representation T of $\hat{\mathfrak{g}}(A)$ is called a *representation with highest weight* if in the carrier space $\mathfrak{H}$ of T there exists one-dimensional weight subspace $\mathfrak{H}_\Lambda$ with basis element x_Λ such that

$$T(n)x_\Lambda = 0 \quad \text{for all} \quad n \in \mathfrak{n}_+, \tag{1}$$

$$T(h)x_\Lambda = \Lambda(h)x_\Lambda \quad \text{for all} \quad h \in \mathfrak{h}, \tag{2}$$

$$T(\mathfrak{U}(\hat{\mathfrak{g}}))x_\Lambda = \mathfrak{H}_\Lambda, \tag{3}$$

where $T(\mathfrak{U}(\hat{\mathfrak{g}}))x_\Lambda$ denotes the linear space of all vectors $T(a)x_\Lambda$, $a \in \mathfrak{U}(\hat{\mathfrak{g}})$. We say that x_Λ is a *vector of highest weight*. Since $T(1)x_\Lambda = x_\Lambda$ where 1 is the unit in $\mathfrak{U}(\hat{\mathfrak{g}})$ and $\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{n}_-)\mathfrak{U}(\hat{\mathfrak{h}})\mathfrak{U}(\mathfrak{n}_+)$, then

$$\begin{aligned} \mathfrak{H} &= T(\mathfrak{U}(\mathfrak{n}_-)\mathfrak{U}(\hat{\mathfrak{h}})\mathfrak{U}(\mathfrak{n}_+))x_\Lambda \\ &= T(\mathfrak{U}(\mathfrak{n}_-))T(\mathfrak{U}(\hat{\mathfrak{h}}))T(\mathfrak{U}(\mathfrak{n}_+))x_\Lambda \\ &= T(\mathfrak{U}(\mathfrak{n}_-))T(\mathfrak{U}(\hat{\mathfrak{h}}))x_\Lambda = T(\mathfrak{U}(\mathfrak{n}_-))x_\Lambda. \end{aligned} \tag{4}$$

It follows from decomposition (3) of Section 19.4.8 and from definition of the subspaces $\mathfrak{U}_\lambda(\mathfrak{n}_-)$ that

$$\mathfrak{H}_\Lambda = \sum_\lambda T(\mathfrak{U}_\lambda(\mathfrak{n}_-))x_\Lambda \tag{5}$$

and that for $n \in \mathfrak{U}_\lambda(\mathfrak{n}_-)$, $h \in \hat{\mathfrak{h}}$ we have

$$T(h)T(n)x_\Lambda = T(hn)x_\Lambda = \lambda(h)T(n)x_\Lambda - T(n)T(h)x_\Lambda$$
$$= [\Lambda(h) - \lambda(h)]T(n)x_\Lambda .$$

Thus, the following statements are valid:

1) *Representations with highest weight are weight representations.*
2) *Decomposition* (5) *is a direct decomposition into a sum of weight subspaces.*
3) *Weights of a representation* T_Λ *with highest weight* Λ *are of the form* $\Lambda - \sum_{i=0}^{\ell} n_i\alpha_i$, *where* α_i *are simple roots and* n_i *are non-negative integers.*
4) *The weight subspace* $T(\mathfrak{U}_\lambda(\mathfrak{n}_-))x_\Lambda$ *is of the weight* $\Lambda - \lambda$.
5) *dim* $T(\mathfrak{U}_\lambda(\mathfrak{n}_-))x_\Lambda \le K(-\lambda)$, *where* K *is the partition function from Section* 19.4.8.

A representation T with highest weight Λ is called *Verma module* if restriction of T onto $\mathfrak{U}(\mathfrak{n}_-)$ is a freely generated representation of $\mathfrak{U}(\mathfrak{n}_-)$, that is, such that vectors $T(n)x_\Lambda$ and $T(n')x_\Lambda$, $n, n' \in \mathfrak{U}(\mathfrak{n}_-)$, are linearly independent if n and n' are linearly independent in $\mathfrak{U}(\mathfrak{n}_-)$.

The Verma module M_Λ with highest weight Λ can be constructed in the space of the algebra $\mathfrak{U}(\mathfrak{n}_-)$. In this case the unit 1 of $\mathfrak{U}(\mathfrak{n}_-)$ is the highest weight vector and the representation operators are given by formulas

$$M_\Lambda(h)1 = \Lambda(h)1, \quad h \in \hat{\mathfrak{h}},$$
$$M_\Lambda(f_i)1 = f_i, \quad i = 0, 1, \dots, \ell,$$
$$M_\Lambda(e_i)1 = 0, \quad i = 0, 1, \dots, \ell,$$
$$M_\Lambda(h)F_\lambda = (\Lambda - \lambda)(h)F_\Lambda, \quad F_\lambda \in \mathfrak{U}_\lambda(\mathfrak{n}_-),$$
$$M_\Lambda(f_i)F_\lambda = f_iF_\Lambda, \quad F_\lambda \in \mathfrak{U}_\lambda(\mathfrak{n}_-), \quad i = 0, 1, \dots, \ell, \tag{6}$$
$$M_\Lambda(e_i)F_\lambda = e_iF_\Lambda \cdot 1, \quad F_\lambda \in \mathfrak{U}_\lambda(\mathfrak{n}_-), \quad i = 0, 1, \dots, \ell. \tag{7}$$

The right hand side of relation (7) demands an additional explanation. The element e_iF_λ belongs to $\mathfrak{U}(\hat{\mathfrak{g}}) = \mathfrak{U}(\mathfrak{n}_-)\mathfrak{U}(\hat{\mathfrak{h}})\mathfrak{U}(\mathfrak{n}_+)$. We represent it as a sum of products n_-Hn_+ and n'_-H' where $n_-, n'_- \in \mathfrak{U}(\mathfrak{n}_-)$, $H, H' \in \mathfrak{U}(\hat{\mathfrak{h}})$, $n_+ \in \mathfrak{U}(\mathfrak{n}_+)$. Then to every summand there corresponds one of the elements

$$n_-Hn_+ \cdot 1 = 0, \quad n'_-H' \cdot 1 = \Lambda(H')n'_- \cdot 1 = \Lambda(H')n'_- \in \mathfrak{U}(\mathfrak{n}_-).$$

Weights of the Verma module M_Λ with highest weight Λ are of the form $\Lambda - \sum_{i=0}^{\ell} n_i\alpha_i$, where α_i are simple roots and n_i are non-negative integers. It follows from the results of Section 19.4.8 that multiplicity of the weight $\mu = \Lambda - \sum_{i=0}^{\ell} n_i\alpha_i$ in M_Λ (that is, dimension of the corresponding weight subspace) is equal to $K(\mu) = K\left(\Lambda - \sum_i n_i\alpha_i\right)$, where $K(\beta)$ is the partition function.

Linear forms $\lambda \in \hat{\mathfrak{h}}'$ for which $\lambda(h_i)$, $i = 0, 1, \ldots, \ell$, are integers, are called *integral.* The set of all integral linear forms $\lambda \in \hat{\mathfrak{h}}'$ will be denoted by P. A linear form $\lambda \in \hat{\mathfrak{h}}'$ for which the numbers $\lambda(h_i)$, $i = 0, 1, \ldots, \ell$, are non-negative, is called *dominant.* The set of integral dominant forms will be denoted by P_+. By P_{++} will be denoted the subset of those forms λ from P_+ for which $\lambda(h_i) > 0$, $i = 0, 1, \ldots, \ell$ (strictly dominant weights). Remind that ρ denote the linear form on $\hat{\mathfrak{h}}$ such that $\rho(h_i) = 1$, $i = 0, 1, \ldots, \ell$. If $S_i \equiv S_{\alpha_i}$ is the element of the Weyl group $\hat{W}$ corresponding to the simple root α_i, then it is possible to prove that $S_i\rho - \rho = -\alpha_i$.

Let $\Lambda \in P$ be a linear form on $\hat{\mathfrak{h}}$ such that $\Lambda(h_i) \geq 0$ for some fixed i, and let M_Λ be the Verma module with highest weight Λ. Let x_Λ be the highest weight vector of the module M_Λ. By using the relations $[e_i, f_i] = h_i$ and $M_\Lambda(e_i)x_\Lambda = 0$ it is easy to verify that

$$e_i f_i^{\Lambda(h_i)+1} x_\Lambda = 0, \quad e_i f_i^{\Lambda(h_i)} x_\Lambda \neq 0. \tag{8}$$

Since $[e_j, f_i] = 0$ for $i \neq j$, then

$$e_j f_i^{\Lambda(h_i)+1} x_\Lambda = 0, \quad j \neq i. \tag{9}$$

The vector $x_{\Lambda'} = f_i^{\Lambda(h_i)+1} x_\Lambda$ is of the weight

$$\Lambda' = \Lambda - [\Lambda(h_i) + 1]\alpha_i = S_i(\Lambda + \rho) - \rho. \tag{10}$$

Since $\mathfrak{U}(\hat{\mathfrak{g}}) = \mathfrak{U}(\mathfrak{n}_-)\mathfrak{U}(\hat{\mathfrak{h}})\mathfrak{U}(\mathfrak{n}_+)$, then due to formulas (8)-(10) we have

$$T(\mathfrak{U}(\hat{\mathfrak{g}}))x_{\Lambda'} = T(\mathfrak{U}(\mathfrak{n}_-))T(\mathfrak{U}(\hat{\mathfrak{h}}))T(\mathfrak{U}(\mathfrak{n}_+))x_{\Lambda'} = T(\mathfrak{U}(\mathfrak{n}_-))x_{\Lambda'},$$

that is, the vector $x_{\Lambda'}$ generates a subrepresentation of the Verma module M_Λ which is a representation with highest weight and $x_{\Lambda'}$ is its highest weight vector. It is possible to show that this subrepresentation is equivalent to the Verma module $M_{\Lambda'}$.

19.5.3. Integrable representations with highest weight. We have shown that if $\Lambda \in P$ and $\Lambda(h_i) \geq 0$, then the Verma module M_Λ has the subrepresentation (Verma submodule) $M_{\Lambda'}$ where $\Lambda' = S_i(\Lambda + \rho) - \rho$. Let $\Lambda \in P_+$. Then this Verma module contains the subrepresentations M_{λ_i}, $\Lambda_i = S_i(\Lambda + \rho) - \rho$, $i = 0, 1, \ldots, \ell$ (and, possibly, other subrepresentations). Let M be the maximal subrepresentation of the Verma module M_Λ, that is, such that every subrepresentation is contained in M. Let $\mathfrak{H}_\lambda$ be the space in which the Verma module M_Λ acts, and let $\mathfrak{H}$ be the subspace in which the subrepresentation M is realized. The quotient representation M_Λ/M which acts in $\mathfrak{H}_\Lambda/\mathfrak{H}$ is a representation of $\hat{\mathfrak{g}}(A)$ which will be denoted by L_Λ. The following statements are valid:

1) *The representation L_Λ is an irreducible integrable representation of the algebra $\hat{\mathfrak{g}}(A)$ with highest weight Λ.*
2) *Any irreducible integrable representation of $\hat{\mathfrak{g}}(A)$ is equivalent to one of the representations L_Λ, $\Lambda \in P_+$.*
3) *If Λ' is a weight of the representation L_Λ, then for each $w \in \hat{W}$ the linear form $w\Lambda'$ is also a weight of this representation. Moreover, multiplicities of the weights Λ' and $w\Lambda'$ in L_Λ are coinciding.*
4) *If Λ' is a weight of the representation L_Λ, then there exists the element $w \in \hat{W}$ such that the weight $w\Lambda'$ is dominant (that is, belongs to P_+).*
5) *Irreducible representations L_Λ and $L_{\Lambda'}$ are equivalent if and only if $\Lambda = \Lambda'$.*

Proofs of these statements can be found in [134].

19.5.4. Characters of integrable representations. If λ and μ are linear forms from P, then we write $\lambda \geq \mu$ if $\lambda - \mu \in P_+$. Let us introduce formal exponents $e(\lambda)$, $\lambda \in P$, and form the sums

$$E_\Lambda = \sum_{\lambda \in P} c_\lambda e(\lambda), \quad c_\lambda \in \mathbb{C}, \tag{1}$$

where Λ at E_Λ means that $c_\lambda = 0$ if the condition $\lambda \leq \Lambda$ is not fulfilled. We introduce the operation of multiplication $e(\lambda)e(\mu) = e(\lambda + \mu)$ and spread this operation for all sums (1) by linearity. A product of elements of the type (1) is again an element of this type. The multiplication turns the set $\mathcal{E}$ of sums of the type (1) into an commutative associative algebra. The exponents $e(\lambda)$, $\lambda \in P$, are linearly independent elements of $\mathcal{E}$.

Let T_Λ be a weight representation of the affine Lie algebra $\hat{\mathfrak{g}}(A)$ with highest weight Λ, $\Lambda \in P$, and let

$$\mathfrak{H} = \sum_{\lambda \leq \Lambda} \oplus \mathfrak{H}_\lambda$$

be the decomposition of its carrier space into weight subspaces. The element

$$\text{ch } T_\Lambda = \sum_{\lambda \leq \Lambda} (\dim \mathfrak{H}_\lambda) e(\lambda) \tag{2}$$

of the algebra $\mathcal{E}$ is called the *formal character of the representation* T_Λ (the word "formal" will be omitted below).

The results of Section 19.5.2 show that the character of the Verma module M_Λ is of the form

$$\text{ch } M_\Lambda = \sum_{\lambda \le \Lambda} K(\Lambda - \lambda) e(\lambda), \tag{3}$$

where K is the partition function. This character can be represented as

$$\text{ch } M_\Lambda = e(\Lambda) \prod_{\alpha \in \hat{\Delta}_+} (1 + e(-\alpha) + e(-2\alpha) + \dots)^{m(\alpha)}, \tag{4}$$

where $m(\alpha)$ is multiplicity of the root α. Indeed, if we multiply down all multipliers of the right hand side of (4), then we obtain a sum of the type (1) and coefficient at every $e(\lambda)$ coincides with the number of partitions of the weight $\Lambda - \lambda$ into a sum of positive roots, where each root is counted with its multiplicity. This leads to formula (3).

Since formally

$$(1 - e(-\alpha))^{-1} = 1 + e(-\alpha) + e(-2\alpha) + \dots ,$$

then formula (4) can be written down in the form

$$\text{ch } M_\Lambda = e(\Lambda) \prod_{\alpha \in \hat{\Delta}_+} (1 - e(-\alpha))^{-m(\alpha)}. \tag{5}$$

The expression at $e(\Lambda)$ on the right hand side of this formula will be denoted by R^{-1}:

$$R = \prod_{\alpha \in \hat{\Delta}_+} (1 - e(-\alpha))^{m(\alpha)}. \tag{6}$$

Theorem 1. *The character of the irreducible integrable representation L_Λ is given by the formula*

$$ch\ L_\Lambda = \frac{\sum_{w \in \hat{W}} \varepsilon(w) e(w(\Lambda + \rho) - \rho)}{\prod_{\alpha \in \hat{\Delta}_+} (1 - e(-\alpha))^{m(\alpha)}}. \tag{7}$$

This formula means that

$$e(\rho)(ch\ L_\Lambda) R = \sum_{w \in \hat{W}} \varepsilon(w) e(w(\Lambda + \rho)). \tag{8}$$

Here $\varepsilon(w)$ means the sign of determinant of the transformation $w \in \hat{W}$. Each element $w \in \hat{W}$ is a product of reflections $S_i \equiv S_{\alpha_i}$, $i = 0, 1, \dots, \ell$. We have $\varepsilon(w) = +1$ ($\varepsilon(w) = -1$) if w is a product of even (odd) number of such reflections.

Proof of Theorem 1 can be found in [134].

Let us note that L_Λ with $\Lambda = 0$ is the trivial (one-dimensional) representation. Therefore, ch $L_0 = e(0)$. This element is the unit of the algebra $\mathcal{E}$. Setting $\Lambda = 0$ into (8) we obtain the relation

$$R = \prod_{\alpha \in \hat{\Delta}_+} (1 - e(-\alpha))^{m(\alpha)} = \sum_{w \in \hat{W}} \varepsilon(w) e(w\rho - \rho) \tag{9}$$

which is called the *denominator formula.* By using this formula we can represent the character ch L_Λ in the form

$$\text{ch } L_\Lambda = \frac{\sum\limits_{w \in \hat{W}} \varepsilon(w) e(w(\Lambda + \rho))}{\sum\limits_{w \in \hat{W}} \varepsilon(w) e(w\rho)}. \tag{10}$$

We can give sense of a functional equality to formulas (5)-(10). For this we replace the formal exponents $e(\lambda)$ by the exponential functions e^λ on $\hat{\mathfrak{h}}$ setting $e^\lambda(h) = e^{\lambda(h)}$, $h \in \hat{\mathfrak{h}}$. By using character (2) of the representation L_Λ we constitute the series

$$\sum_{\lambda \leq \Lambda} (\dim \mathfrak{H}_\lambda) e^{\lambda(h)}, \quad h \in \hat{\mathfrak{h}}, \tag{11}$$

where $\mathfrak{H}_\lambda$ is the weight subspace of the carrier space of L_Λ. The following statements are valid (proofs of them can be found in the book [134]):

1) *The series* (11) *absolutely converges on the subset*

$$Y = \{h \in \hat{\mathfrak{h}} \mid \text{Re } \delta(h) > 0\} \tag{12}$$

and is a holomorphic function in the interior Int Y *of this domain.*

2) *The series* (11), *as a function on* Int Y, *can be continued analytically to a meromorphic function on the domain*

$$X_c = \{h_1 + ih_2 \mid h_1 \in X, h_2 \in \hat{\mathfrak{h}}_R\}, \tag{13}$$

where X *is the Tits cone* (*see Section* 19.4.7).

3) *The series* $\sum\limits_{w \in \hat{W}} \varepsilon(w) e^{w(\Lambda+\rho)}$ *converges absolutely on* Int X_c *to a meromorphic function and diverges on* $\hat{\mathfrak{h}} \backslash \text{Int } X_c$.

These statements allow us to write down the character of the representation L_Λ as the meromorphic function on Y for which

$$\text{ch } L_\Lambda = \frac{\sum\limits_{w \in \hat{W}} \varepsilon(w) e^{w(\Lambda+\rho)}}{\sum\limits_{w \in \hat{W}} \varepsilon(w) e^{w(\rho)}}. \tag{14}$$

The denominator formula can be written as an equality of functions:

$$\prod_{\alpha\in\hat{\Delta}_+}(1-e^{-\alpha})^{m(\alpha)}=\sum_{w\in\hat{W}}\varepsilon(w)e^{w\rho-\rho}. \tag{15}$$

For every linear combination $\varphi=\sum_{i=0}^{\ell}c_i\alpha_i$, $c_i\in\mathbb{Z}$, we introduce the notation $\kappa_i(\varphi)=c_i$. By putting $u_i=e(-\alpha_i)$, $i=0,1,\dots,\ell$, the denominator formula (9) can be written as equality in the algebra $\mathbb{Z}[u_0,u_1,\dots,u_\ell]$ of formal power series with integral coefficients, namely, as

$$\prod_{\alpha\in\hat{\Delta}_+}\left(1-\prod_{i=0}^{\ell}u_i^{\kappa_i(\alpha)}\right)^{m(\alpha)}=\sum_{w\in\hat{W}}\varepsilon(w)\prod_{i=0}^{\ell}u_i^{\kappa_i(w\rho-\rho)}. \tag{16}$$

The expression (10) for characters in this notations can be represented as

$$\operatorname{ch}L_\Lambda=\frac{\sum_{w\in\hat{W}}\varepsilon(w)\prod_{i=0}^{\ell}u_i^{\kappa_i(\rho-w(\Lambda+\rho))}}{\sum_{w\in\hat{W}}\varepsilon(w)\prod_{i=0}^{\ell}u_i^{\kappa_i(\rho-w\rho)}} \tag{17}$$

(we have multiplied the numerator and the denominator by $e(-\rho)$).

Let $s_0,s_1,\dots,s_\ell$ be fixed positive integers and let q be an indeterminate. With every u^i we associate q^{s_i}, $i=0,1,\dots,\ell$. This gives the homomorphism of the algebra $\mathbb{Z}[u_0,u_1,\dots,u_\ell]$ into $\mathbb{Z}[q]$ which is called *q-specialization of the type* $s_0,s_1,\dots,s_\ell$. The q-specialization of the type $\mathbf{1}=(1,1,\dots,1)$ is called the *principal q-specialization.*

Applying q-specialization of the type $(s_0,s_1,\dots,s_\ell)$ to equality (16), we have the relation

$$\prod_{\alpha\in\hat{\Delta}_+}\left(1-q^{\sum_{i=0}^{\ell}s_i\kappa_i(\alpha)}\right)^{m(\alpha)}=\sum_{w\in\hat{W}}\varepsilon(w)q^{\sum_{i=0}^{\ell}s_i\kappa_i(w\rho-\rho)}, \tag{18}$$

where, remind, $m(\alpha)$ is multiplicity of the root α. Application of this q-specialization to the right hand side of formula (17) leads to the expression

$$\frac{\sum_{w\in\hat{W}}\varepsilon(w)q^{\sum_{i=0}^{\ell}s_i\kappa_i(\rho-w(\Lambda+\rho))}}{\sum_{w\in\hat{W}}\varepsilon(w)q^{\sum_{i=0}^{\ell}s_i\kappa_i(\rho-w\rho)}}. \tag{19}$$

The statements on convergence of the series for characters of representations L_Λ and of the series $\sum_{w\in\hat{W}} \varepsilon(w)e^{w(\Lambda+\rho)}$ show that the series in formulas (18) and (19) converge for certain values of q and $s_0, s_1, \ldots, s_\ell$.

The homomorphism of the algebra $\mathbb{Z}[u_0, u_1, \ldots, u_\ell]$ into $\mathbb{Z}[q]$ for the numbers $1, 1, \ldots, 1$ (principal q-specialization) is denoted by $F_{\mathbf{1}}$. By using formula (18) it is not difficult to show that

$$F_{\mathbf{1}}(e(-\Lambda)\mathrm{ch}\, L_\Lambda) = \prod_{\alpha\in\Delta_+^\vee} \left(\frac{1-q^{(\Lambda+\rho,\alpha)}}{1-q^{(\rho,\alpha)}}\right)^{m(\alpha)}, \tag{20}$$

where $\Delta_+^\vee$ is the positive root system for the affine Lie algebra $\hat{\mathfrak{g}}(A^t)$ (see Section 19.4.7). We recommend to the reader to prove this formula.

19.6. Characters of Representations of the Affine Lie Algebras and Combinatorial Identities

19.6.1. The denominator formula for the algebras $A_1^{(1)}$, $A_2^{(2)}$ and the Jacobi identity. In Section 19.3.4 we have described the Weyl group $\hat{W}$ of the affine Lie algebra $A_1^{(1)}$. It can be represented as $\hat{W} = \hat{W}_0 \cup S_0\hat{W}_0$, where $\hat{W}_0 = \{(S_1S_0)^r \mid r \in \mathbb{Z}\}$ and S_0, S_1 are the reflections generated by the simple roots α_0 and α_1. It is clear that $\varepsilon(w) = 1$ for $w \in \hat{W}_0$ and $\varepsilon(w) = -1$ for $w \in S_0\hat{W}_0$. We have also evaluated in Section 19.3.4 the values of $w\rho - \rho$ for all $w \in \hat{W}$. Using these results in the denominator formula (9) of Section 19.5.4 we obtain that for the algebra $A_1^{(1)}$ this formula can be represented in terms of the roots α_1 and δ as

$$\begin{aligned}&\prod_{n\geq 1}[1-e(-\delta)^n][1-e(-\delta)^{n-1}e(-\alpha_1)][1-e(-\delta)^n e(-\alpha_1)^{-1}]\\ &\quad= \sum_{n\in\mathbf{Z}} e(-\alpha_1)^{2n}e(-\delta)^{n(2n-1)} - \sum_{n\in\mathbf{Z}} e(-\alpha_1)^{2n+1}e(-\delta)^{(n+1)(2n+1)}\end{aligned} \tag{1}$$

and in terms of the roots α_0 and α_1 as

$$\begin{aligned}&\prod_{n\geq 1}[1-e(-\alpha_0)^n e(-\alpha_1)^n][1-e(-\alpha_0)^{n-1}e(-\alpha_1)^n][1-e(-\alpha_0)^n e(-\alpha_1)^{n-1}]\\ &\qquad= \sum_{n\in\mathbf{Z}} e(-\alpha_0)^{n(2n-1)}e(-\alpha_1)^{n(2n+1)}\\ &\qquad\quad - \sum_{n\in\mathbf{Z}} e(-\alpha_0)^{(n+1)(2n+1)}e(-\alpha_1)^{n(2n+1)}.\end{aligned} \tag{2}$$

Putting $e(-\alpha_1) = qt$, $e(-\delta) = q^2$ into (1) we obtain the *Jacobi identity*

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}t)(1-q^{2n-1}t^{-1}) = \sum_{n\in\mathbf{Z}}(-1)^n q^{n^2}t^n. \tag{3}$$

Putting $e(-\alpha_0) = u$ and $e(-\alpha_1) = v$ into (2) and replacing n by $-n$ in the first sum we have

$$\prod_{n=1}^{\infty}(1-u^n v^n)(1-u^n v^{n-1})(1-u^{n-1}v^n) = \sum_{n\in\mathbf{Z}}(-1)^n u^{n(n+1)/2} v^{n(n-1)/2}. \tag{4}$$

This equality can be obtained from (3) with the help of the substitutions $q^2 = uv$, $t^2 = u/v$.

The equality (3) has the following combinatorial sense. *The excess of the number of partitions of the pair (m, m') into an even number of distinct parts of the types $(a, a-1)$, $(b-1, b)$ or (c, c) over those into an odd number of such parts is $(-1)^k$ or 0 according as (m, m') is of the type $(k(k+1)/2, k(k-1)/2)$ or not.*

By using the results of Section 19.3.6 for the affine Lie algebra $A_2^{(2)}$ we obtain in the similar way that the denominator formula for $A_2^{(2)}$ in terms of the roots α_1 and δ is of the form

$$\begin{aligned}\prod_{n\geq 1}&[1-e(-\delta)^n][1-e(-\delta)^{n-1}e(-\alpha_1)][1-e(-\delta)^n e(-\alpha_1)^{-1}]\\ &\times[1-e(-\delta)^{2n-1}e(-\alpha_1)^2][1-e(-\delta)^{2n-1}e(-\alpha_1)^{-2}]\\ =&\sum_{n\in\mathbf{Z}}e(-\alpha_1)^{3n}e(-\delta)^{n(3n-1)/2}-\sum_{n\in\mathbf{Z}}e(-\alpha_1)^{-3n-2}e(-\delta)^{(n+1)(3n+2)/2}\end{aligned} \tag{5}$$

and in terms of the roots α_0 and α_1 is of the form

$$\begin{aligned}\prod_{n\geq 1}&[1-e(-\alpha_0)^{2n-1}e(-\alpha_1)^{4n-4}][1-e(-\alpha_0)^n e(-\alpha_1)^{2n-1}]\\ &\times[1-e(-\alpha_0)^n e(-\alpha_1)^{2n}][1-e(-\alpha_0)^{n-1}e(-\alpha_1)^{2n-1}]\\ &\times[1-e(-\alpha_0)^{2n-1}e(-\alpha_1)^{4n}]\\ &=\sum_{n\in\mathbf{Z}}e(-\alpha_0)^{n(3n-1)/2}e(-\alpha_1)^{n(3n+2)}\\ &-\sum_{n\in\mathbf{Z}}e(-\alpha_0)^{(n+1)(3n+2)/2}e(-\alpha_1)^{n(3n+2)}.\end{aligned} \tag{6}$$

Let us put $e(-\alpha_1) = t^{-1}$, $e(-\delta) = q$ in the formula (5) and replace in the first sum n by $-n$ and in the second sum n by $-n-1$. As a result, we obtain the identity

$$\begin{aligned}\prod_{n=1}^{\infty}&(1-q^n)(1-q^n t)(1-q^{n-1}t^{-1})(1-q^{2n-1}t^2)(1-q^{2n-1}t^{-2})\\ &=\sum_{n\in\mathbf{Z}}q^{(3n^2+n)/2}(t^{3n}-t^{-3n-1}).\end{aligned} \tag{7}$$

Putting $e(-\alpha_0) = u$, $e(-\alpha_1) = v$ in formula (6) and replacing n by $-n$ in the first sum and n by $-n-1$ in the second sum, we receive the identity

$$\prod_{n=1}^{\infty}(1-u^n v^{2n})(1-u^n v^{2n-1})(1-u^{n-1}v^{2n-1})(1-u^{2n-1}v^{4n-4})(1-u^{2n-1}v^{4n})$$
$$= \sum_{n\in\mathbb{Z}} u^{n(3n+1)/2} v^{n(3n-2)} - \sum_{n\in\mathbb{Z}} u^{n(3n+1)/2} v^{(n+1)(3n+1)}. \qquad (8)$$

This identity can be obtained from (7) by setting $t = v^{-1}$ and $q = uv^2$. The identity (8) has the following combinatorial sense. *The excess of the number of partitions of the pair* (m, m') *into an even number of distinct parts of the types* $(a, 2a)$, $(b, 2b-1)$, $(c-1, 2c-1)$, $(2d-1, 4d-4)$ *or* $(2e-1, 4e)$ *over those into an odd number of such parts is* 1 *or* -1 *if* (m, m') *is of the type* $(k(3k+1)/2, k(3k-2))$ *or* $(k(3k+1)/2, (k+1)(3k+1))$, *respectively, and* 0 *otherwise.*

19.6.2. Specialized characters of the algebra $A_1^{(1)}$. Let $\Lambda = a\alpha_0 + c\rho$. Then using expressions (9) and (10) of Section 19.3.4 for $w(\Lambda+\rho)-(\Lambda+\rho)$ and for $S_0 w(\Lambda+\rho)-(\Lambda+\rho)$ we can represent the expression for the numerator of formula (10) of Section 19.5.4, multiplied by $e(-\Lambda-\rho)$, that is, the expression

$$\sum_{w\in\hat{W}} \varepsilon(w) e(w(\Lambda+\rho)-(\Lambda+\rho)) = \frac{\text{ch}\, L_\Lambda}{e(\Lambda)} \sum_{w\in\hat{W}} \varepsilon(w) e(w\rho-\rho),$$

in the form

$$\sum_{n\in\mathbb{Z}} e(-\alpha_0)^{2an+(c+1)n(2n-1)} e(-\alpha_1)^{2an+(c+1)n(2n+1)}$$
$$- \sum_{n\in\mathbb{Z}} e(-\alpha_0)^{a(2n-2)+(c-1)(n+1)(2n+1)} e(-\alpha_1)^{2an+(c+1)n(2n+1)}. \qquad (1)$$

Let us use here the principal q-specialization (1,1), that is, the values $e(-\alpha_0) = e(-\alpha_1) = q$. We shall show that in this case formula (1) takes the form

$$\frac{\text{ch}\, L_\Lambda}{e(\Lambda)} \sum_{w\in\hat{W}} \varepsilon(w) e(w\rho-\rho) \Big|_{e(-\alpha_0)=e(-\alpha_1)=q}$$
$$= \prod_{n=1}^{\infty}(1-q^{(n_0+n_1)n})(1-q^{(n_0+n_1)n-n_0})(1-q^{(n_0+n_1)n-n_1}) \qquad (2)$$
$$= \sum_{w\in\hat{W}} \varepsilon(w) e(w\rho-\rho) \Big|_{e(-\alpha_0)=q^{n_0}, e(-\alpha_1)=q^{n_1}},$$

where $n_0 = (\Lambda+\rho)(h_0)$, $n_1 = (\Lambda+\rho)(h_1)$. Since $\delta(h_0) = \delta(h_1) = 0$, then it is allowed to add δ to Λ. By using this fact we can achieve $\Lambda = a\alpha_0 + c\delta$ where $c \pm 2a \in \mathbb{Z}_+$.

In this case $n_0 = c+2a+1$, $n_1 = c-2a+1$. Setting $e(-\alpha_0) = e(-\alpha_1) = q$ into (1) after some simplifications we receive that

$$\begin{aligned}&\frac{\text{ch } L_\Lambda}{e(\Lambda)} \sum_{w\in\hat{W}} \varepsilon(w)e(w\rho-\rho)\Big|_{e(-\alpha_0)=e(-\alpha_1)=q} \\ &= \sum_{n\in\mathbb{Z}} q^{4(c+1)n^2+4an} - \sum_{n\in\mathbb{Z}} q^{4(c+1)n^2+4(a+c+1)n+2a+c+1}.\end{aligned} \tag{3}$$

Now let us note that if we put $e(-\alpha_0) = q^{m_0}$, $e(-\alpha_1) = q^{m_1}$, $m_0, m_1 \in \mathbb{Z}_+$, into formula (2) of the previous section, then we obtain

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} q^{2(m_0+m_1)n^2+(m_0-m_1)n} - \sum_{n\in\mathbb{Z}} q^{2(m_0+m_1)n^2+(3m_0+m_1)n+m_0} \\ &= \prod_{n=1}^{\infty}(1-q^{(m_0+m_1)n})(1-q^{(m_0+m_1)n-m_0})(1-q^{(m_0+m_1)n-m_1})\end{aligned} \tag{4}$$

(we have replaced n by $-n$ in the first sum). The right hand side of formula (3) is equal to the left hand side of formula (4) if and only if

$$4(c+1) = 2(m_0+m_1), \qquad 4a = m_0 - m_1,$$
$$4(a+c+1) = 3m_0 + m_1, \quad 2a+c+1 = m_0.$$

These relations are valid if and only if

$$m_0 = c+2a+1 \in \mathbb{Z}_+, \quad m_1 = c-2a+1 \in \mathbb{Z}_+.$$

Therefore, the left hand side of formula (3) is represented in the form of the product. This leads to formula (2).

It follows from formula (2) that

$$\begin{aligned}&\frac{\text{ch } L_\Lambda}{e(\Lambda)}\Big|_{e(-\alpha_0)=e(-\alpha_1)=q} \prod_{n=1}^{\infty}(1-q^{2n-1}) \\ &= \prod_{\substack{n=1 \\ n\neq 0,\pm(\Lambda+\rho)(h_0)[\text{mod } (\Lambda-\rho)(h_0+h_1)]}}^{\infty} (1-q^n)^{-1}.\end{aligned} \tag{5}$$

Really, we have from formula (2) of the previous section that

$$\begin{aligned}&\sum_{w\in\hat{W}} \varepsilon(w)e(w\rho-\rho)\Big|_{e(-\alpha_0)=e(-\alpha_1)=q} \\ &= \prod_{n=1}^{\infty}(1-q^{2n-1})\prod_{n=1}^{\infty}(1-q^n).\end{aligned} \tag{6}$$

Dividing both sides of equality (2) by the corresponding parts of this equality we have

$$\left.\frac{\operatorname{ch} L_\Lambda}{e(\Lambda)}\right|_{e(-\alpha_0)=e(-\alpha_1)=q} = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$$
$$\times \prod_{n=1}^{\infty}(1-q^n)^{-1} \prod_{\substack{n=1\\ n\equiv 0,(\Lambda+\rho)(h_0),(\Lambda+\rho)(h_1)[\mathrm{mod}\ (\Lambda+\rho)(h_1+h_0)]}}^{\infty} (1-q^n).$$

After some simplification we receive formula (5).

Let us put $i=(\Lambda+\rho)(h_0)$, $2k+1=(\Lambda+\rho)(h_0+h_1)$ into (5). Then

$$\Lambda(h_0)=i-1, \quad \Lambda(h_1)=2k-1$$

and for $1\le i\le k$ we have

$$\left.\frac{\operatorname{ch} L_\Lambda}{e(\Lambda)}\right|_{e(-\alpha_0)=e(-\alpha_1)=q} \prod_{n=1}^{\infty}(1-q^{2n-1}) = \prod_{\substack{n=1\\ n\not\equiv 0,\pm i(\mathrm{mod}\ 2k+1)}}^{\infty} (1-q^n)^{-1}. \tag{7}$$

For $k=1$ this equality turns into

$$\prod_{n=1}^{\infty}(1-q^{2n-1})^{-1} = \prod_{n=1}^{\infty}(1+q)^n. \tag{8}$$

If

$$\Lambda(h_0)=i-1, \quad \Lambda(h_1)=2k-i-1, \quad 1\le i\le k,$$

then we obtain from (5) that

$$\left.\frac{\operatorname{ch} L_\Lambda}{e(\Lambda)}\right|_{e(-\alpha_0)=e(-\alpha_1)=q} \prod_{n=1}^{\infty}(1-q^{2n-1})$$
$$= \begin{cases} \prod\limits_{\substack{n=1\\ n\not\equiv 0,\pm i(\mathrm{mod}\ 2k)}}^{\infty} (1-q^n)^{-1} \quad \text{if} \quad 1\le i\le k, & (9)\\ \prod\limits_{\substack{n=1\\ n\equiv k(\mathrm{mod}\ 2k)}}^{\infty} (1-q^n) \prod\limits_{\substack{n=1\\ n\not\equiv 0,k(\mathrm{mod}\ 2k)}}^{\infty} (1-q^n)^{-1} \quad \text{if} \quad i=k. & (10)\end{cases}$$

Formulas (7)-(10) lead to the following theorem.

Theorem 1. *Let L_Λ be the irreducible integrable representation of the affine Lie algebra $A_1^{(1)}$ with the highest weight Λ such that $\Lambda(h_0) = k_0$, $\Lambda(h_1) = k_1$, and let*

$$F = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}.$$

Then the principal q-specialization $\chi(\Lambda)$ of the character of the representation L_Λ is given by the formula

$$\chi(\Lambda) = F \cdot \prod_{n=1}^{\infty}(1-q^n)^{-1} \prod_{\substack{n=1 \\ n\equiv 0}}^{\infty}(1-q^n) \prod_{\substack{n=1 \\ n\equiv k_0+1}}^{\infty}(1-q^n) \prod_{\substack{n=1 \\ n\equiv k_1+1}}^{\infty}(1-q^n). \tag{11}$$

This formula can be written in the form

$$\chi(\Lambda) = F \cdot \prod_{\substack{n=1 \\ n\not\equiv 0,\pm(k_0+1)}}^{\infty}(1-q^n)^{-1} \tag{12}$$

if $k_0 \neq k_1$ and in the form

$$\chi(\Lambda) = F \cdot \prod_{\substack{n=1 \\ n\not\equiv 0,k_0+1}}^{\infty}(1-q^n)^{-1} \prod_{\substack{n=1 \\ n\equiv k_0+1}}^{\infty}(1-q^n) \tag{13}$$

if $k_0 = k_1$. In formulas (11)-(13) *all congruences are taken with respect to mod $(k_0 + k_1 + 2)$. The expressions for $\chi(\Lambda)$ are symmetric under permutation of k_0 and k_1.*

Let us write down expressions for $\chi(\Lambda)$ when k_0 and k_1 are small:

$$(k_0, k_1) = (1, 0): \qquad \chi(\Lambda) = F = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}, \tag{14}$$

$$(k_0, k_1) = (1, 1): \qquad \chi(\Lambda) = F \cdot \prod_{n=1}^{\infty}(1+q^{2n-1}), \tag{15}$$

$$(k_0, k_1) = (2, 0): \qquad \chi(\Lambda) = F \cdot \prod_{n=1}^{\infty}(1-q^{4n-2})^{-1}, \tag{16}$$

$$(k_0, k_1) = (1, 2): \qquad \chi(\Lambda) = F \cdot \frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-1})(1-q^{5n-4})}, \tag{17}$$

$$(k_0, k_1) = (0, 3): \qquad \chi(\Lambda) = F \cdot \frac{1}{\prod_{n=1}^{\infty}(1-q^{5n-2})(1-q^{5n-3})}. \tag{18}$$

The expressions at F in the right hand sides of formulas (17) and (18) coincide with the right hand sides of the Rogers-Ramanujan identities (15) and (16) of Section 14.7.1. Derivation of these Rogers-Ramanujan identities on the base of representations of the affine Lie algebra $A_1^{(1)}$ is given by Lepowsky and Wilson [191]. Since their proof is more long, then other derivations of these identities, we do not give it here.

19.6.3. Specialized characters of the algebra $A_2^{(2)}$. Let $\Lambda = a\alpha_0 + c\rho$. By using expressions (8) and (9) of Section 19.3.9 for $w_m(\Lambda+\rho)-(\Lambda+\rho)$ and for $S_0 w_m(\Lambda+\rho)-(\Lambda+\rho)$ the expression for the numerator of formula (10) of Section 19.5.4, multiplied by $e(-\Lambda-\rho)$, that is, the expression

$$\frac{\mathrm{ch}\, L_\Lambda}{e(\Lambda)} \sum_{w\in\hat{W}} \varepsilon(w) e(w\rho-\rho) = \sum_{w\in\hat{W}} \varepsilon(w) e(w(\Lambda+\rho)-(\Lambda+\rho)),$$

can be represented in the form

$$\begin{aligned}&\sum_{n\in Z} e(-\alpha_0)^{2an+(c+1)n(3n-1)/2} e(-\alpha_1)^{4an+(c+1)n(3n+2)}\\ &-\sum_{n\in Z} e(-\alpha_0)^{a(2n+2)+(c+1)(3n^2+5n+2)/2} e(-\alpha_1)^{4an+(c+1)(3n^2+2n)/2}.\end{aligned} \tag{1}$$

Let us set here $e(-\alpha_0) = e(-\alpha_1) = q$ and show that we obtain the formula

$$\begin{aligned}&\frac{\mathrm{ch}\, L_\Lambda}{e(\Lambda)} \sum_{w\in\hat{W}} \varepsilon(w) e(w\rho-\rho)\Big|_{e(-\alpha_0)=e(-\alpha_1)=q}\\ &= \prod_{n=1}^{\infty} (1-q^{2(n_1+2n_0)n-(n_1+4n_0)})(1-q^{2(n_1+2n_0)n-n_1})(1-q^{(n_1+2n_0)n})\\ &\qquad \times (1-q^{(n_1+2n_0)n-(n_1+n_0)})(1-q^{(n_1+2n_0)n-n_0})\\ &\qquad = \sum_{w\in\hat{W}} \varepsilon(w) e(w\rho-\rho)\Big|_{e(-\alpha_0)=q^{n_0}, e(-\alpha_1)=q^{n_1}},\end{aligned} \tag{2}$$

where $n_0 = (\Lambda-\rho)(h_0)$, $n_1 = (\Lambda-\rho)(h_1)$. As in the case of formula (2) of Section 19.6.2, we can consider that $\Lambda = a\alpha_0 + c\rho$ where $c+2a \in \mathbb{Z}_+$, $c-4a\in\mathbb{Z}_+$. Then $n_0 = c+2a+1$, $n_1 = c-4a+1$. Setting $e(-\alpha_0) = e(-\alpha_1) = q$ into (1) we have

$$\begin{aligned}&\frac{\mathrm{ch}\, L_\Lambda}{e(\Lambda)} \sum_{w\in\hat{W}} \varepsilon(w) e(w\rho-\rho)\Big|_{e(-\alpha_0)=e(-\alpha_1)=q}\\ &= \sum_{n\in Z} q^{(6a+3c/2+3/2)n+9(c+1)n^2/2}\\ &- \sum_{n\in Z} q^{(6a+9c/2+9/2)n+c+2a+1+9(c+1)n^2/2}.\end{aligned} \tag{3}$$

Now we put $e(-\alpha_0) = q^{m_1}$, $e(-\alpha_1) = q^{m_0}$, $m_0, m_1 \in \mathbb{Z}_+$, into the formula (6) of Section 19.6.1. As a result, we obtain

$$\begin{aligned}\prod_{n=1}^{\infty} &(1 - q^{2(m_1+2m_0)n-(m_1+4m_0)})(1 - q^{2(m_1+2m_0)n-m_1}) \\ &\times (1 - q^{(m_1+2m_0)n})(1 - q^{(m_1+2m_0)n-(m_1+m_0)})(1 - q^{(m_1+2m_0)n-m_0}) \\ &= \sum_{n\in Z} q^{3(m_0+m_1/2)n^2+(2m_0-m_1/2)n} \\ &- \sum_{n\in Z} q^{3(m_0+m_1/2)n^2+(4m_0-m_1/2)n+m_0}\end{aligned} \tag{4}$$

(we have replaced n by $-(n+1)$ in the second sum). The right hand side of formula (3) is equal to the right hand side of formula (4) if and only if

$$\frac{9}{2}(c+1) = 3\left(m_0 + \frac{m_1}{2}\right), \quad 6a + \frac{3}{2}c + \frac{3}{2} = 2m_0 - \frac{m_1}{2},$$
$$6a + \frac{9}{2}c + \frac{9}{2} = 4m_0 - \frac{m_1}{2}, \quad c + 2a + 1 = m_0.$$

These relations are valid if and only if

$$m_1 = c - 4a + 1 \in \mathbb{Z}_+, \quad m_0 = c + 2a + 1 \in \mathbb{Z}_+.$$

This leads to formula (2).

Setting $e(-\alpha_0) = e(-\alpha_1) = q$ into formula (6) of Section 19.6.1 we have

$$\begin{aligned}&\sum_{w\in\hat{W}} \varepsilon(w)e(w\rho - \rho)\Big|_{e(-\alpha_0)=e(-\alpha_1)=q} \\ &\quad = \prod_{n=1}^{\infty}(1 - q^{6n-5})(1 - q^{6n-1}) \cdot \prod_{n=1}^{\infty}(1 - q^n).\end{aligned} \tag{5}$$

Dividing both sides of formula (2) by the corresponding parts of formula (5) we obtain the relation

$$\begin{aligned}\frac{\mathrm{ch}\, L_\Lambda}{e(\Lambda)}\Big|_{e(-\alpha_0)=e(-\alpha_1)=q} &= \prod_{\substack{n=1 \\ n\equiv\pm1 (\mathrm{mod}\ 6)}}^{\infty} (1 - q^n)^{-1} \\ &\times \prod_{n=1}^{\infty}(1 - q^n) \prod_{\substack{n=1 \\ n\equiv 0, n_1+2n_0, \pm n_1 \\ \pm n_0, \pm(n_0+n_1)}}^{\infty} (1 - q^n),\end{aligned} \tag{6}$$

where $n_0 = (\Lambda+\rho)(h_0)$, $n_1 = (\Lambda+\rho)(h_1)$ and in the third product the congruence is taken with respect to mod $(2n_1+4n_0)$. After simple transformations we obtain from here that

$$\frac{\operatorname{ch} L_\Lambda}{e(\Lambda)}\Big|_{e(-\alpha_0)=e(-\alpha_1)=q} = \prod_{\substack{n=1\\ n\equiv\pm1 (\mathrm{mod}\ 6)}}^{\infty} (1-q^n)^{-1}$$
$$\times \prod_{\substack{n=1\\ n\not\equiv 0,(\Lambda+\rho)(2h_0+h_1),\pm(\Lambda+\rho)(h_1),\\ \pm(\Lambda+\rho)(h_0),\pm(\Lambda+\rho)(h_0+h_1)}}^{\infty} (1-q^n)^{-1} \tag{7}$$

if $n_0 \neq n_1$. In the second product the congruence is taken with respect to mod $(\Lambda+\rho)(4h_0+2h_1)$.

19.7. Characters of Representations and Theta Functions

19.7.1. An other form of theta functions. In Section 19.2 we introduced theta functions of two and of many variables. In order to connect theta functions with characters of representations of affine Lie algebras we represent theta functions in another form.

Let us introduce strictly positive definite symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbf{R}^\ell$ and on $\mathbb{C}^\ell$. The group operation on the Heisenberg group $H(\ell,\mathbf{R})$ will be given by the formula

$$(\boldsymbol{\alpha},\boldsymbol{\beta},t)(\boldsymbol{\alpha}',\boldsymbol{\beta}',t') = \left(\boldsymbol{\alpha}+\boldsymbol{\alpha}',\boldsymbol{\beta}+\boldsymbol{\beta}',t+t'+\frac{1}{2}\langle\boldsymbol{\alpha}',\boldsymbol{\beta}\rangle-\frac{1}{2}\langle\boldsymbol{\alpha},\boldsymbol{\beta}'\rangle\right).$$

The action of the group $SL(2,\mathbf{R})$ is defined on $H(\ell,\mathbf{R})$:

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\cdot(\boldsymbol{\alpha},\boldsymbol{\beta},t) = (a\boldsymbol{\alpha}+b\boldsymbol{\beta},c\boldsymbol{\alpha}+d\boldsymbol{\beta},t).$$

We constitute the group G_R which is the semidirect product of the groups $SL(2,\mathbf{R})$ and $H(\ell,\mathbf{R})$:

$$G_R = SL(2,\mathbf{R})\times H(\ell,\mathbf{R}),$$

where for $g\in SL(2,\mathbf{R})$ and for $n\in H(\ell,\mathbf{R})$ we set $gng^{-1} = g\cdot n$. We also can define an action of the group G_R on the space

$$Y = H\times\mathbb{C}^\ell\times\mathbb{C},$$

where H is the upper half-plane of the plane $\mathbb{C}$. Namely, if $(\tau,\mathbf{z},t)\in Y$, that is, $\tau\in H$, $\mathbf{z}\in\mathbb{C}^\ell$, $t\in\mathbb{C}$, then for $A\equiv\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbf{R})$ and for $(\boldsymbol{\alpha},\boldsymbol{\beta},t_0)\in H(\ell,\mathbf{R})$ we set

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}(\tau,\mathbf{z},t) = \left(\frac{a\tau+b}{c\tau+d},\frac{\mathbf{z}}{c\tau+d},t+\frac{c}{2}\frac{\langle\mathbf{z},\mathbf{z}\rangle}{c\tau+d}\right), \tag{1}$$

$$(\boldsymbol{\alpha},\boldsymbol{\beta},t_0)\cdot(\tau,\mathbf{z},t)=\left(\tau,\mathbf{z}+\tau\boldsymbol{\beta}-\boldsymbol{\alpha},t-\langle\boldsymbol{\beta},\mathbf{z}\rangle-\frac{1}{2}\tau\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle+\frac{1}{2}\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle+t_0\right). \quad (2)$$

In particular,

$$(\boldsymbol{\alpha},\mathbf{0},0)\cdot(\tau,\mathbf{z},t)=(\tau,\mathbf{z}-\boldsymbol{\alpha},t), \quad (3)$$

$$(\mathbf{0},\boldsymbol{\beta},0)\cdot(\tau,\mathbf{z},t)=\left(\tau,\mathbf{z}+\tau\boldsymbol{\beta},t-\langle\boldsymbol{\beta},\mathbf{z}\rangle-\frac{1}{2}\tau\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle\right), \quad (4)$$

$$(\mathbf{0},\boldsymbol{\beta},t_0)\cdot(\tau,\mathbf{z},t)=(\tau,\mathbf{z},t+t_0). \quad (5)$$

Let us introduce the lattice L in $\mathbf{R}^\ell$, such that for all $\gamma,\gamma'\in L$ we have $\langle\gamma,\gamma'\rangle\in\mathbb{Z}$. Let L^* be the dual lattice, that is, the set of points $\boldsymbol{\alpha}\in\mathbf{R}^\ell$ such that $\langle\boldsymbol{\alpha},\gamma'\rangle\in\mathbb{Z}$ for all $\gamma'\in L$. It is clear that $L\subset L^*$.

With the help of the lattice L we define the theta function

$$\Theta^L(\tau,\mathbf{z},t)=e^{-2\pi it}\sum_{\gamma\in L}\exp[\pi i\tau\langle\gamma,\gamma\rangle-2\pi i\langle\gamma,\mathbf{z}\rangle]. \quad (6)$$

It is easy to connect this function with the theta function (1) of Section 19.2.6. The series (6) converges absolutely on the set Y to a holomorphic function.

With the help of the function (6) we introduce theta functions of degree 1 with characteristics. Namely, if $\boldsymbol{\mu}\in\mathbb{C}^\ell$, then we set

$$\begin{aligned}\Theta^L_{\boldsymbol{\mu}}(\tau,\mathbf{z},t)&=\Theta^L((\mathbf{0},-\boldsymbol{\mu},0)\cdot(\tau,\mathbf{z},t))\\&=\Theta^L\left(\tau,\mathbf{z}-\tau\boldsymbol{\mu},t+\langle\boldsymbol{\mu},\mathbf{z}\rangle-\frac{1}{2}\tau\langle\boldsymbol{\mu},\boldsymbol{\mu}\rangle\right).\end{aligned} \quad (7)$$

It is easy to verify that

$$\Theta^L_{\boldsymbol{\mu}}((\boldsymbol{\mu}',\mathbf{0},0)\cdot(\tau,\mathbf{z},t))=\exp(2\pi i\langle\boldsymbol{\mu},\boldsymbol{\mu}'\rangle)\Theta^L_{\boldsymbol{\mu}}(\tau,\mathbf{z},t), \quad (8)$$

$$\Theta^L_{\boldsymbol{\mu}}((\mathbf{0},\boldsymbol{\mu}',0)\cdot(\tau,\mathbf{z},t))=\Theta^L_{\boldsymbol{\mu}-\boldsymbol{\mu}'}(\tau,\mathbf{z},t). \quad (9)$$

With the help of the Poisson summation formula we derive that

$$\begin{aligned}\Theta^L_{\boldsymbol{\mu}}\left(-\frac{1}{\tau},\frac{\mathbf{z}}{\tau},t+\frac{\langle\mathbf{z},\mathbf{z}\rangle}{2\tau}\right)&=|L^*/L|^{-1/2}(-i\tau)^{1/2}\\\times\sum_{\boldsymbol{\mu}'\in L^*(\mathrm{mod}\ L)}&\exp(-2\pi i\langle\boldsymbol{\mu},\boldsymbol{\mu}'\rangle)\Theta^L_{\boldsymbol{\mu}'}(\tau,\mathbf{z},t).\end{aligned} \quad (10)$$

For $\boldsymbol{\mu}\in\mathbb{C}^\ell$ and for $m\in\mathbb{Z}_+$ we set

$$\Theta^L_{\boldsymbol{\mu},m}(\tau,\mathbf{z},t)=\exp(-2\pi imt)\sum_{\gamma\in L+\boldsymbol{\mu}/m}\exp[\pi im\tau\langle\gamma,\gamma\rangle)-2\pi im\langle\gamma,\mathbf{z}\rangle]. \quad (11)$$

It is theta function of degree m with characteristic $\boldsymbol{\mu}$. It is clear that $\Theta^L_{\boldsymbol{\mu},m}$ depends on $\boldsymbol{\mu}$ mod mL. It is easy to verify that for $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ from L^* the relations

$$\Theta^L_{\boldsymbol{\mu},m}((m^{-1}\boldsymbol{\mu}',\mathbf{0},0)\cdot(\tau,\mathbf{z},t)) = \exp(2\pi i m^{-1}\langle\boldsymbol{\mu},\boldsymbol{\mu}'\rangle)\Theta^L_{\boldsymbol{\mu},m}(\tau,\mathbf{z},t), \tag{12}$$

$$\Theta^L_{\boldsymbol{\mu},m}((\mathbf{0},m^{-1}\boldsymbol{\mu}',0)\cdot(\tau,\mathbf{z},t)) = \Theta^L_{\boldsymbol{\mu}-\boldsymbol{\mu}',m}(\tau,\mathbf{z},t) \tag{13}$$

are valid. If L' is the lattice L with the new scalar product $\langle\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle' = m\langle\boldsymbol{\gamma},\boldsymbol{\gamma}\rangle$, then

$$\Theta^L_{\boldsymbol{\mu},m}(\tau,\mathbf{z},t) = \Theta^{L'}_{m^{-1}\boldsymbol{\mu}}(\tau,\mathbf{z},mt). \tag{14}$$

The relation (10) leads to the following equality for $\Theta^L_{\boldsymbol{\mu},m}$:

$$\begin{aligned}\Theta^L_{\boldsymbol{\mu},m}\left(-\frac{1}{\tau},\frac{\mathbf{z}}{\tau},t+\frac{\langle\mathbf{z},\mathbf{z}\rangle}{2\tau}\right) &= |L^*/mL|^{-1/2}(-i\tau)^{\ell/2}\\ \times \sum_{\boldsymbol{\mu}'\in L^*(\mathrm{mod}\ mL)} &\exp(-2\pi i m^{-1}\langle\boldsymbol{\mu},\boldsymbol{\mu}'\rangle)\Theta^L_{\boldsymbol{\mu}',m}(\tau,\mathbf{z},t).\end{aligned} \tag{15}$$

Statement 1. *Let* $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$ *and* $\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0 \in \mathbf{R}^\ell$ *be such that*

$$\begin{aligned} &mbd\langle\boldsymbol{\beta},\boldsymbol{\beta}\rangle \equiv 2m\langle\boldsymbol{\alpha}_0,\boldsymbol{\beta}\rangle(\mathrm{mod}\ 2\mathbb{Z}) \quad \text{if} \quad \boldsymbol{\beta}\in L,\\ &mac\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle \equiv 2m\langle\boldsymbol{\alpha},\boldsymbol{\beta}_0\rangle(\mathrm{mod}\ 2\mathbb{Z}) \quad \text{if} \quad c\boldsymbol{\alpha}\in L, \quad m\boldsymbol{\alpha}\in L^*.\end{aligned}$$

Fix $t_0 \in \mathbf{R}$ *and set* $n = (\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0,t_0) \in H(\ell,\mathbf{R})$. *Then there exists* $v(m, A\cdot n) \in \mathbb{C}$ *such that for all* $\boldsymbol{\mu}\in L^*$ *we have*

$$\begin{aligned}\Theta^L_{\boldsymbol{\mu},m}(A\cdot n\cdot(\tau,\mathbf{z},t)) = v(m,A\cdot n) \sum_{\substack{\boldsymbol{\alpha}\in L^*\\ c\boldsymbol{\alpha}(\mathrm{mod}\ mL)}} &\{\exp\pi i[m^{-1}cd\langle\boldsymbol{\alpha},\boldsymbol{\alpha}\rangle\\ &+2m^{-1}bc\langle\boldsymbol{\alpha},\boldsymbol{\mu}\rangle + m^{-1}ab\langle\boldsymbol{\mu},\boldsymbol{\mu}\rangle + 2\langle\boldsymbol{\mu},a\boldsymbol{\alpha}_0+b\boldsymbol{\beta}_0\rangle\\ &+2\langle\boldsymbol{\alpha},c\boldsymbol{\alpha}_0+d\boldsymbol{\beta}_0\rangle]\}\Theta^L_{a\boldsymbol{\mu}+c\boldsymbol{\alpha},m}(\tau,\mathbf{z},t).\end{aligned}$$

Proof of this statement see in [134].

For every $m\in\mathbb{Z}$ we introduce the space $\tilde{\mathfrak{F}}_m$ of holomorphic functions f on $Y = H\times\mathbb{C}^\ell\times\mathbb{C}$ such that

$$\begin{aligned} &f((\boldsymbol{\alpha},\boldsymbol{\beta},t_0)\cdot(\tau,\mathbf{z},t)) = f(\tau,\mathbf{z},t) \quad \text{for} \quad \boldsymbol{\alpha},\boldsymbol{\beta}\in L, \quad t_0+\frac{1}{2}\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle\in\mathbb{Z},\\ &f((\mathbf{0},\mathbf{0},t_0)\cdot(\tau,\mathbf{z},t)) = \exp(-2\pi i m t)f(\tau,\mathbf{z},t) \quad \text{for} \quad t_0\in\mathbf{R}.\end{aligned}$$

It is easy to see that $\tilde{\mathfrak{F}}_0$ is the space of holomorphic functions of $\tau\in H$. It is clear that $\Theta^L_{\boldsymbol{\mu},m}\in\tilde{\mathfrak{F}}_m$ if $\boldsymbol{\mu}\in L^*$.

Statement 2. *Fix $\tau \in H$. If $m \in \mathbb{Z}$, then let $\mathfrak{F}_m^\tau$ be the space of holomorphic functions on $\mathbb{C}^\ell \times \mathbb{C}$, obtained by fixation of this value of τ in functions from $\tilde{\mathfrak{F}}_m$. Then $\mathfrak{F}_0^\tau = \mathbb{C}$ and $\mathfrak{F}_m^\tau = \{0\}$ for $m < 0$. If $m > 0$, then the set of functions*

$$\Theta^L_{\boldsymbol{\mu},m}, \quad \boldsymbol{\mu} \in L^*(\mathrm{mod}\, mL),$$

with our fixed value of τ forms a basis of $\mathfrak{F}_m^\tau$.

Proof of this statement is analogous to that of the corresponding statement of Section 19.2.9. This statement can also be proved by expressing $\Theta^L_{\boldsymbol{\mu},m}$ in terms of the theta functions of Section 19.2.6.

Let $\tilde{\mathfrak{F}} = \sum_{m=0}^{\infty} \oplus \tilde{\mathfrak{F}}_m$. Then $\tilde{\mathfrak{F}}$ is an algebra over the ring $\tilde{\mathfrak{F}}_0$ of holomorphic functions on H. This algebra is called the *ring of theta functions.* It follows from Statement 2 that $\tilde{\mathfrak{F}}$ is the free module over $\tilde{\mathfrak{F}}_0$ with the basis

$$\{\Theta^L_{\boldsymbol{\mu},m} \mid m \in \mathbb{Z}, m > 0, \boldsymbol{\mu} \in L^*(\mathrm{mod}\, mL)\} \cup \{1\}.$$

In this reason $\Theta^L_{\boldsymbol{\mu},m}\Theta^L_{\boldsymbol{\mu}',m'} \in \tilde{\mathfrak{F}}$ and this function can be decomposed into a sum of the theta functions $\Theta^L_{\boldsymbol{\mu}'',m+m'}$. If $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in L^*$ and $m_1, m_2 \in \mathbb{Z}_+$, then we have

$$\Theta^L_{\boldsymbol{\mu}_1,m_1}\Theta^L_{\boldsymbol{\mu}_2,m_2} = \sum_{\boldsymbol{\gamma} \in L(\mathrm{mod}\ (m_1+m_2)L)} d_{\boldsymbol{\gamma}} \Theta^L_{\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+m_1\boldsymbol{\gamma},m_1+m_2}, \tag{16}$$

where

$$d_{\boldsymbol{\gamma}} = \Theta^L_{m_2\boldsymbol{\mu}_1 - m_1\boldsymbol{\mu}_2 + m_1 m_2 \boldsymbol{\gamma}, m_1 m_2(m_1+m_2)}(\tau, \mathbf{0}, 0). \tag{16'}$$

Really, we write $\Theta^L_{\boldsymbol{\mu}_1,m_1}$ and $\Theta^L_{\boldsymbol{\mu}_2,m_2}$ in the form

$$\Theta^L_{\boldsymbol{\mu}_j,m_j}(\tau,\mathbf{z},t) = \exp(-2\pi i m_j t)\sum_{\boldsymbol{\gamma}_j \in L} \exp[\pi i m_j^{-1}\tau\langle m_j\boldsymbol{\gamma}_j + \boldsymbol{\mu}_j, m_j\boldsymbol{\gamma}_j + \boldsymbol{\mu}_j\rangle$$
$$- 2\pi i\langle m_j\boldsymbol{\gamma}_j + \boldsymbol{\mu}_j, \mathbf{z}\rangle],$$

then multiply down these two series and introduce the new summation parameters $\boldsymbol{\gamma} = \boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}' = m_1\boldsymbol{\gamma}_1 + m_2\boldsymbol{\gamma}_2$. Let us write the sums as

$$\sum_{\boldsymbol{\gamma}\in L} \quad \sum_{\boldsymbol{\gamma}' \in m_1\boldsymbol{\gamma} + (m_1+m_2)L}.$$

Then the product $\Theta^L_{\boldsymbol{\mu}_1,m_1}\Theta^L_{\boldsymbol{\mu}_2,m_2}$ can be written as

$$\sum_{\boldsymbol{\gamma}\in L} \Theta^L_{\boldsymbol{\mu}_1+\boldsymbol{\mu}_2+m_1\boldsymbol{\gamma},m_1+m_2}(\tau,\mathbf{z},t)$$
$$\times \exp\left[\pi i\tau\frac{m_1 m_2}{m_1+m_2}\langle\boldsymbol{\gamma} + m_1^{-1}\boldsymbol{\mu}_1 - m_2^{-1}\boldsymbol{\mu}_2, \boldsymbol{\gamma} + m_1^{-1}\boldsymbol{\mu}_1 - m_2^{-1}\boldsymbol{\mu}_2\rangle\right].$$

Extracting here the sum over $\gamma \in L \pmod{(m_1+m_2)L}$ we obtain formula (16).

Above we have introduced the group G_R which acts on the space $Y = H \times \mathbb{C}^\ell \times \mathbb{C}$. Now we extract in the subgroup $H(\ell, \mathbf{R})$ of the group G_R the discrete subgroup

$$H'(\ell, \mathbb{Z}) = \left\{ (\boldsymbol{\alpha}, \boldsymbol{\beta}, t) \in H(\ell, \mathbf{R}) \mid \boldsymbol{\alpha}, \boldsymbol{\beta} \in L, \quad t + \frac{1}{2}\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \in \mathbb{Z} \right\}.$$

The normalizer of the subgroup $H'(\ell, \mathbb{Z})$ in G_R will be denoted by G_Z. It is easy to show that G_Z consists of products $A \cdot n$ of those elements $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $n = (\boldsymbol{\alpha}, \boldsymbol{\beta}, t) \in H(\ell, \mathbf{R})$ for which

$$bd\langle \gamma, \gamma \rangle \equiv 2\langle \boldsymbol{\alpha}, \gamma \rangle \pmod{2\mathbb{Z}}, \quad ac\langle \gamma, \gamma \rangle \equiv 2\langle \boldsymbol{\beta}, \gamma \rangle \pmod{2\mathbb{Z}}$$

for all $\gamma \in L$.

Statement 3. *Let*

$$\mathfrak{F}_m = \sum_{\boldsymbol{\mu} \in L^* \pmod{L}} \oplus \mathbb{C}\Theta^L_{\boldsymbol{\mu}, m}.$$

The group G_Z, acting on $\mathfrak{F}_m$ by the formula

$$R(A \cdot n) f(\tau, \mathbf{z}, t) = f(A \cdot n \cdot (\tau, \mathbf{z}, t))$$

(see formulas (1) *and* (2)*), leaves this space invariant. The matrices of the operators $R(A \cdot n)$ are unitary in the basis $\{\Theta^L_{\boldsymbol{\mu}, m}\}$.*

Proof of this statement can be found in [134].

19.7.2. The lattices M and M'. In order to connect characters of representations of the affine Lie algebra $\hat{\mathfrak{g}}(A)$ with theta functions we introduce lattices in the space $\hat{\mathfrak{h}}'$, dual to the Cartan subalgebra $\hat{\mathfrak{h}}$. For this we have to introduce some additional notations.

Along with the root system $\hat{\Delta}$ of the algebra $\hat{\mathfrak{g}}(A)$ we consider the adjacent root system $\hat{\Delta}^*$. Namely, we put

$$\hat{\Delta} = \hat{\Delta}^* \quad \text{if} \quad a_0 k = 1 \text{ or } 4$$

and assume that multiplicities mult α of roots α in $\hat{\Delta}$ and $\hat{\Delta}^*$ coincide. If $a_0 k \neq 1$ and $a_0 k \neq 4$, then we consider that the set $\hat{\Delta}^{*\mathrm{im}}$ of imaginary roots of the system $\hat{\Delta}^*$ coincides with

$$\hat{\Delta}^{*\mathrm{im}} = \left\{ k^{-1} n\delta \mid n \in \mathbb{Z},\ n \neq 0 \right\}$$

and that multiplicities mult* in $\hat{\Delta}^{*\mathrm{im}}$ are related to multiplicities mult in $\hat{\Delta}^{\mathrm{im}}$ by the formula

$$\operatorname{mult}^* k^{-1} n\delta = \begin{cases} \operatorname{mult} k\delta & \text{if } n \equiv 0 \pmod{k}, \\ \operatorname{mult} k\delta - \operatorname{mult} \delta & \text{if } n \not\equiv 0 \pmod{k}. \end{cases}$$

For the set $\hat{\Delta}^{*\text{re}}$ of real roots we put

$$\begin{aligned}\hat{\Delta}^{*\text{re}} &= \{\alpha + n\delta \mid \alpha \in \Delta^s,\ n \in \mathbb{Z}\} \cup \{k^{-1}(\alpha + n\delta) \mid \alpha \in \Delta^\ell,\ n \in \mathbb{Z}\} \\ &= \{2(\alpha,\alpha)^{-1}(\alpha + n\delta) \mid \alpha \in \Delta,\ n \in \mathbb{Z}\}.\end{aligned}$$

If $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ are simple roots of the system $\hat{\Delta}$, then we set

$$Q = \sum_{i=0}^{\ell} \mathbb{Z}\alpha_i, \quad Q_+ = \sum_{i=0}^{\ell} \mathbb{N}\alpha_i, \quad \mathbb{N} = \mathbb{Z}_+ \cup \{0\}.$$

The set $\hat{\Delta}_+$ of positive roots of $\hat{\Delta}$ coincides with $\hat{\Delta}_+ = \hat{\Delta} \cap Q_+$. For the set $\hat{\Delta}^*_+$ of positive roots of $\hat{\Delta}^*$ we have

$$\hat{\Delta}^*_+ = \hat{\Delta}^* \cap k^{-1} Q_+.$$

If $k^* = a_0^{-1}k$, then $k^* = 1$ if $a_0 k = 1$ or 4, and $k^* = k$ for other cases. The set $\hat{\Delta}^*$ is a root system of some affine Lie algebra $\hat{\mathfrak{g}}(A^*)$. By direct verification we convince that the algebras $\hat{\mathfrak{g}}(A)$ and $\hat{\mathfrak{g}}(A^*)$ are isomorphic if $\hat{\mathfrak{g}}(A)$ differ from $A^{(2)}_{2\ell-1}$ and $D^{(2)}_{\ell+1}$. If $\hat{\mathfrak{g}}(A) = A^{(2)}_{2\ell-1}$ or $D^{(2)}_{\ell+1}$, then $\hat{\mathfrak{g}}(A^*)$ coincides respectively, with $D^{(2)}_{\ell+1}$ and $A^{(2)}_{2\ell-1}$. Every of these correspondences between $\hat{\mathfrak{g}}(A)$ and $\hat{\mathfrak{g}}(A^*)$ leads to the isomorphism Φ between the space $\hat{\mathfrak{h}}'$ of linear forms on $\hat{\mathfrak{h}}$ and the space of linear forms on the Cartan subalgebra $\hat{\mathfrak{h}}^*$ of the affine Lie algebra $\hat{\mathfrak{g}}(A^*)$. This isomorphism is given in such a way that $|\Phi(\alpha)|^2 = k^*|\alpha|^2$. We have

$$\begin{aligned}&\dim \hat{\mathfrak{g}}(A^*)_{\Phi(\alpha)} = \text{mult}^*\,\alpha && \text{if } \alpha \neq 0, \\ &\Phi(\alpha) > 0 && \text{if } \alpha \in \hat{\Delta}^*_+.\end{aligned}$$

By using the isomorphism Φ we introduce the notions Π^* (the system of simple roots $\alpha_0^*, \alpha_1^*, \ldots, \alpha_\ell^*$), δ^*, θ^*, Q^*, $\hat{W}^*$ (the Weyl group), Δ^* (the set of roots of the corresponding simple complex Lie algebra with the simple roots $\alpha_1^*, \ldots, \alpha_\ell^*$), Δ^*_+ (the set of positive roots in Δ^*), Λ_0^* and so on.

In Section 19.4.7 we have introduced the bilinear form $(\cdot,\cdot)$ on $\hat{\mathfrak{h}}'$. For a root $\alpha \in \hat{\Delta}^{\text{re}}$ we set

$$\alpha^\vee = 2\alpha/(\alpha,\alpha) \equiv 2\alpha/|\alpha|^2.$$

For the system Π of simple roots of the algebra $\hat{\mathfrak{g}}(A)$ the lattice

$$Q^\vee = \sum_{\alpha_i \in \Pi} \mathbb{Z}\alpha_i^\vee$$

can be defined. It is not difficult to show that the set $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee$ is a system of simple roots of the affine Lie algebra $\hat{\mathfrak{g}}(A^t)$ defined in Section 19.4.7.

In the space $\hat{\mathfrak{h}}'$ of linear forms on $\hat{\mathfrak{h}}$ we introduce the subspace

$$\bar{\mathfrak{h}}' = \sum_{i=1}^{\ell} \mathbb{C}\alpha_i,$$

where α_i are simple roots of $\hat{\mathfrak{g}}(A)$. Then

$$\hat{\mathfrak{h}}' = \bar{\mathfrak{h}}' \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0.$$

The bilinear form $(\cdot,\cdot)$, determined on $\hat{\mathfrak{h}}'$, is strictly positive definite on $\bar{\mathfrak{h}}'$. If $(\cdot,\cdot)$ is considered on the whole space $\hat{\mathfrak{h}}'$, then its kernel coincides with $\mathbb{C}\delta$.

If $\lambda \in \hat{\mathfrak{h}}'$, then $\bar{\lambda}$ will denote the projection of λ onto $\bar{\mathfrak{h}}'$. We also define the set

$$\bar{Q} = \{\bar{\alpha} \mid \alpha \in Q\}, \quad \bar{Q}^\vee = \{\bar{\alpha} \mid \alpha \in Q^\vee\}.$$

Let us note that

$$\bar{Q}^\vee \subset \bar{Q} \quad \text{if} \quad k = 1 \quad \text{and} \quad \bar{Q} \subset \bar{Q}^\vee \quad \text{if} \quad k \neq 1.$$

For $\alpha \in \hat{\Delta}^{*\mathrm{re}}$ we put $\alpha^\vee = 2\alpha/|\alpha|$ and

$$Q^{*\vee} = \sum_{\alpha \in \Pi^*} \mathbb{Z}\alpha^\vee, \quad \bar{Q}^{*\vee} = \{\bar{\alpha} \mid \alpha \in Q^{*\vee}\}.$$

We have $\bar{Q}^{*\vee} = \bar{Q} \cap \bar{Q}^\vee$. If $k^* \neq 1$, then

$$Q^* = Q^\vee + \mathbb{Z}\delta^*, \quad Q^{*\vee} = Q, \quad \bar{\Delta}^*_+ = \bar{\Delta}^\vee_+.$$

Now we introduce the lattices

$$M = \bar{Q} \cap \bar{Q}^\vee \quad M' = \bar{Q}^\vee = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee. \tag{1}$$

It is easy to show that M is spanned by the elements $\hat{W}\theta^\vee$ with integral coefficients and

$$M = \bar{Q}^\vee \quad \text{if} \quad k = 1, \quad M = \bar{Q} \quad \text{if} \quad k \neq 1. \tag{2}$$

Moreover,

$$M + \mathbb{Z}\delta = Q^{*\vee}, \quad M' + \mathbb{Z}\delta = Q^\vee. \tag{3}$$

For $t \in \mathbb{C}$ we put

$$\hat{\mathfrak{h}}'_t = \{\lambda \in \hat{\mathfrak{h}}' \mid (\lambda, \delta) = t\}, \quad \hat{\mathfrak{h}}'_{t,R} = \hat{\mathfrak{h}}'_t \cap \hat{\mathfrak{h}}'_R,$$

where

$$\hat{\mathfrak{h}}'_R = \{h \in \hat{\mathfrak{h}}' \mid (\alpha_i, h) \in \mathbf{R},\ i = 0, 1, \ldots, \ell\}.$$

It is clear that $\hat{\mathfrak{h}}'_0 = \sum_{i=1}^{\ell} \mathbb{C}\alpha_i$. Since the linear form $(\cdot,\cdot)$ is invariant with respect to $\hat{W}$ and $w\delta = \delta$ for all $w \in \hat{W}$, then the hyperplane $\hat{\mathfrak{h}}'_t$ is invariant with respect to $\hat{W}$.

Let us consider the affine space $\hat{\mathfrak{h}}'_1$ (mod$\mathbb{C}\delta$). It is possible to show that the action of $\hat{W}$ on $\hat{\mathfrak{h}}_0$ is effective. Therefore, the action of $\hat{W}$ on $\hat{\mathfrak{h}}'/\mathbb{C}\delta$ and, consequently, on $\hat{\mathfrak{h}}'_1$ (mod $\mathbb{C}\delta$) is effective. The latter action is of the following geometric sense. Using the projection onto $\bar{\mathfrak{h}}'$, introduced above, we identify $\hat{\mathfrak{h}}'_1$ (mod $\mathbb{C}\delta$) with $\bar{\mathfrak{h}}'$. This define the isomorphism of the Weyl group $\hat{W}$ onto the group $W_{\rm af}$ of affine transformations of $\bar{\mathfrak{h}}'$. This isomorphism will be denoted by af : $\hat{W} \to W_{\rm af}$. It is given by the formula

$$\overline{w\lambda} = ({\rm af}\, w)(\bar{\lambda}), \quad \lambda \in \hat{\mathfrak{h}}'_1.$$

The group $W_{\rm af}$ is called the *affine Weyl group.*

Let W be the subgroup of $\hat{W}$ consisting of elements generated by the reflections $S_{\alpha_i} \equiv S_i$, $i = 1, 2, \ldots, \ell$. It is isomorphic to the Weyl group of the corresponding simple complex Lie algebra $\mathfrak{g}$. For $w \in W$ we have af $w = w$. Moreover,

$$({\rm af}\, S_{\alpha_0})\lambda = S_\theta\lambda + \theta^\vee, \quad \lambda \in \bar{\mathfrak{h}}',$$

that is, af S_{α_0} is the reflection with respect to the hyperplane $\{\lambda \in \bar{\mathfrak{h}}' \mid (\lambda, \theta) = 1\}$.

Since $S_\theta \in W$, then $\hat{W}$ is generated by W and by the element $T_{\theta^\vee} \equiv S_{\alpha_0} S_\theta$. We have

$$({\rm af}\, T_{\theta^\vee})(\lambda) = \lambda + \theta^\vee, \quad \lambda \in \bar{\mathfrak{h}}'.$$

For $\alpha = w\theta^\vee$ where $w \in W$ we set $T_\alpha \equiv w T_{\theta^\vee} w^{-1}$. For this element the relation

$$({\rm af}\, T_\alpha)(\lambda) = \lambda + \alpha$$

is valid. We denote by $\mathcal{T}$ the subgroup of $\hat{W}$, generated by the elements $T_{w\theta^\vee}$, $w \in W$. Then $\mathcal{T} \simeq$ af $\mathcal{T}$ is an commutative subgroup of $\hat{W}$ and there is the isomorphism

$$\hat{W} \simeq W \times \mathcal{T}, \tag{4}$$

where $\mathcal{T}$ is an invariant subgroup. Since lattice M is spanned by the elements $W\theta^\vee$ with integral coefficients, then we have the isomorphism $\alpha \to T_\alpha$ of the commutative group M onto $\mathcal{T}$ which is defined by the formula

$$({\rm af}\, T_\alpha)(\lambda) = \lambda + \alpha.$$

Since the bilinear form $(\cdot,\cdot)$ on $\hat{\mathfrak{h}}'$ is invariant with respect to $\hat{W}$, then $|T_\alpha\lambda|^2 = |\lambda|^2$, $\lambda \in \hat{\mathfrak{h}}'$, $\alpha \in M$. We derive from here and from the definition of T_α that

$$T_\alpha\lambda = \lambda + (\lambda,\delta)\alpha - \left[\frac{1}{2}(\lambda,\delta)|\alpha|^2 + (\alpha,\lambda)\right]\delta \tag{5}$$

for all $\lambda \in \hat{\mathfrak{h}}'$. With the help of this formula one can extend the definition of T_α to arbitrary $\alpha \in \bar{\mathfrak{h}}'$. It is easy to verify that T_α is linear and conserves the bilinear form $(\cdot,\cdot)$. Moreover,

$$T_{\alpha+\beta} = T_\alpha T_\beta, \quad wT_\alpha w^{-1} = T_{w\alpha}, \quad w \in W. \tag{6}$$

In some cases it is convenient to have formula (5) in the other form. In order to obtain this form we note that if $\lambda(c) \neq 0$, then

$$\lambda - \bar{\lambda} = \lambda(c)\Lambda_0 + \frac{1}{2\lambda(c)}(|\lambda|^2 - |\bar{\lambda}|^2)\delta, \tag{7}$$

where $\lambda \in \hat{\mathfrak{h}}'$. This relation allows us to write down the formula (5) in the form

$$T_\alpha\lambda = m\Lambda_0 + \frac{1}{2m}|\lambda|^2\delta + (\bar{\lambda} + m\alpha) - \frac{1}{2m}|\bar{\lambda} + m\alpha|^2\delta, \tag{8}$$

where $m = \lambda(c)$. Let us remark that the expression (λ,δ) from formula (5) can be represented as

$$(\lambda,\delta) = \frac{|\theta|^2}{2a_0}\lambda(h_0) + (\bar{\lambda},\theta).$$

For the Weyl group $\hat{W}^*$ of the affine Lie algebra $\hat{\mathfrak{g}}(A^*)$ we have $\hat{W}^* = W \times \mathcal{T}^*$. The groups $\hat{W}$ and $\hat{W}^*$ differ only in their translation subgroups $\mathcal{T}$ and $\mathcal{T}^*$. The subgroup $\mathcal{T}^*$ consists of the translations T_α, $\alpha \in M'$.

Let us consider the projection mapping π from $\hat{\mathfrak{h}}'_{1,R}$ onto $\bar{\mathfrak{h}}'_R$ and put

$$C_{\rm af} = \{\lambda \in \bar{\mathfrak{h}}'_R \mid (\lambda,\alpha_i) \geq 0,\ i = 1,2,\dots,\ell,\ (\lambda,\theta) \leq 1\}.$$

Identifying $\hat{\mathfrak{h}}'$ and $\hat{\mathfrak{h}}$ with the help of the bilinear form $(\cdot,\cdot)$, we have $\pi^{-1}C_{\rm af} = C \cap \hat{\mathfrak{h}}'_{1,R}$ where

$$C = \{h \in \hat{\mathfrak{h}}_R \mid \alpha_i(h) \geq 0, \quad i = 0,1,\dots,\ell\}.$$

Since af $(w) \circ \pi = \pi \circ w$ for all $w \in \hat{W}$ and C is a fundamental domain for $\hat{W}$ in the Tits cone, then $C_{\rm af}$ is a fundamental domain for $W_{\rm af} \equiv$ af (W) on $\bar{\mathfrak{h}}'_R$. If to say more precisely, then we can formulate the following statement.

Statement 1. *Any point from $\bar{\mathfrak{h}}'_R$ is W-equivalent* mod M *to a unique point from $C_{\rm af}$.*

Now let us give the lattice M for the "classical" affine Lie algebras.

Algebra $A_\ell^{(1)}$. The simple roots of this algebra can be represented in the form

$$\alpha_0 = \delta - \varepsilon_1 + \varepsilon_{\ell+1}, \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \dots, \ell,$$

where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $i, j = 1, 2, \dots, \ell+1$. Then

$$M = \left\{ \sum_{i=1}^{\ell+1} n_i \varepsilon_i \,\middle|\, n_i \in \mathbb{Z}, \sum_{i=1}^{\ell+1} n_i = 0 \right\}.$$

Algebra $B_\ell^{(1)}$. The simple roots are represented as

$$\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2; \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \dots, \ell-1; \quad \alpha_\ell = \varepsilon_\ell,$$

where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. The lattice M is of the form

$$M = \left\{ \sum_{i=1}^{\ell} n_i \varepsilon_i \,\middle|\, n_i \in \mathbb{Z}, \quad \sum_{i=1}^{\ell} n_i \in 2\mathbb{Z} \right\}.$$

Algebra $C_\ell^{(1)}$. The simple roots are

$$\alpha_0 = \delta - 2\varepsilon_1; \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \dots, \ell-1; \quad \alpha_\ell = 2\varepsilon_\ell,$$

where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, and for the lattice M we have

$$M = \left\{ \sum_{i=1}^{\ell} n_i \varepsilon_i \,\middle|\, n_i \in 2\mathbb{Z} \right\}.$$

Algebra $D_\ell^{(1)}$. The simple roots are of the form

$$\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2; \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \dots, \ell-1; \quad \alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell,$$

where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. The lattice M is represented as

$$M = \left\{ \sum_{i=1}^{\ell} n_i \varepsilon_i \,\middle|\, n_i \in \mathbb{Z}, \sum_{i=1}^{\ell} n_i \in 2\mathbb{Z} \right\}.$$

Algebra $A_{2\ell}^{(2)}$. For the simple roots we have

$$\alpha_0 = \frac{1}{2}\delta - \varepsilon_1; \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \dots, \ell; \quad \alpha_\ell = 2\varepsilon_\ell,$$

where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. The lattice M is of the form

$$M = \left\{ \sum_{i=1}^{\ell} n_i \varepsilon_i \;\middle|\; n_i \in \mathbb{Z} \right\}.$$

Algebra $A_{2\ell-1}^{(2)}$. In this case the simple roots are

$$\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2; \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, 2, \dots, \ell - 1; \quad \alpha_\ell = 2\varepsilon_\ell,$$

where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. For the lattice M we have

$$M = \left\{ \sum_{i=1}^{\ell} n_i \varepsilon_i \;\middle|\; n_i \in \mathbb{Z}, \quad \sum_{i=1}^{\ell} n_i \in 2\mathbb{Z} \right\}.$$

Algebra $D_{\ell+1}^{(2)}$. For the simple roots we have

$$\alpha_0 = \delta - \varepsilon_1; \quad \alpha_i = \varepsilon_i - \varepsilon_{i-1}, \quad i = 1, 2, \dots, \ell - 1; \quad \alpha_\ell = \varepsilon_\ell,$$

where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. The lattice M is of the form

$$M = \left\{ \sum_{i=1}^{\ell} n_i \varepsilon_i \;\middle|\; n_i \in \mathbb{Z} \right\}.$$

The lattices M' of these algebras can be determined by the lattices M.

19.7.3. Maximal weights of irreducible integrable representations. Let L_Λ be an irreducible integrable representation of the affine Lie algebra $\hat{\mathfrak{g}}(A)$ with highest weight $\Lambda \in P_+$. The set of all weights of L_Λ will be denoted by $P(\Lambda)$. By $\mathrm{mult}_\Lambda \mu$ we denote the multiplicity of the weight μ in L_Λ.

Statement 1. *Let L_Λ be such that $\Lambda(c) > 0$, and let $\lambda \in P(\Lambda)$. Let α be a positive imaginary root of $\hat{\mathfrak{g}}(A)$, say, $\alpha = s\delta$, $s > 0$. Then the set I of the integers t, such that $\lambda - t\alpha \in P(\Lambda)$, exhausts all integers from the interval $[-p, \infty)$ for some $p \geq 0$. Moreover, $t \to \mathrm{mult}_\Lambda(\lambda - t\alpha)$ is a non-decreasing function on I.*

Statement 2. *If $\Lambda(c) = m > 0$, then $P(\Lambda) = \hat{W}\{\lambda \in P_+ \mid \lambda \leq \Lambda\}$ where $\lambda \leq \Lambda$ means that $\Lambda - \lambda \in P_+$. Every $\lambda \in P(\Lambda)$ lies in the convex hull of the set $w\Lambda$, $w \in \hat{W}$. If*

$$\lambda - \lambda' \in Q = \sum_{i=0}^{\ell} \mathbb{C}\alpha_i$$

and λ' lies in the convex hull of the set $w\lambda$, $w \in \hat{W}$, then $\mathrm{mult}_\Lambda \lambda' \geq \mathrm{mult}_\Lambda \lambda$. If $\lambda \in P(\Lambda)$, then $|\lambda|^2 \leq |\Lambda|^2$, that is, $P(\Lambda)$ lies inside of the paraboloid:

$$P(\Lambda) \subset \{\lambda \in \hat{\mathfrak{h}}' \mid |\bar{\lambda}|^2 + 2a_0^{-1} m \lambda(d) \leq |\Lambda|^2, \lambda(c) = m\}.$$

Moreover, $|\lambda|^2 = |\Lambda|^2$ *if and only if* $\lambda \in \hat{W}\Lambda$. *Besides,* $|\lambda + \rho|^2 \leq |\Lambda + \rho|^2$, *and* $|\lambda + \rho|^2 = |\Lambda + \rho|^2$ *if and only if* $\lambda = \Lambda$.

These statements are proved in [134].

The linear form $\lambda \in P(\Lambda)$ for which $\lambda + \delta \,\overline{\in}\, P(\Lambda)$ is called *maximal weight of the representation* L_Λ. The set of maximal weights of L_Λ is denoted by $\max(\Lambda)$. Since $w\delta = \delta$, $w \in \hat{W}$, then $\max(\Lambda)$ is invariant with respect to the Weyl group $\hat{W}$. Due to Statement 2 every maximal weight is $\hat{W}$-equivalent to some dominant maximal weight. We derive from Statement 1 that for every weight $\mu \in P(\Lambda)$ there is a unique maximal weight λ and a unique non-negative integer n such that $\mu = \lambda - n\delta$.

The following statement is a consequence of Statements 1 and 2.

Statement 3. *Let* $\lambda \in P_+$ *and* $\Lambda(c) = m > 0$. *Then* $\lambda \to \bar{\lambda}$ *determines one-to-one mapping from* $\max(\Lambda) \cap P_+$ *onto* $mC_{\mathrm{af}} \cap (\bar{\Lambda} + \bar{Q})$. *In particular, the subset of dominant maximal weights of* L_λ *is finite.*

We have defined the sets of weights P, P_+, P_{++}. By using the transition $\lambda \to \bar{\lambda}$ we obtain the sets $\bar{P}$, $\bar{P}_+$, $\bar{P}_{++}$. They are correspondingly the sets of integral, dominant integral and strictly dominant integral weights for the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. With the help of the mapping Φ we also define the sets P^*, P^*_+, P^*_{++}, $\bar{P}^*$, $\bar{P}^*_+$, $\bar{P}^*_{++}$.

Statement 4. *For* $\bar{P}$ *and* $\bar{P}^*$ *we have*

$$\bar{P} = \{\lambda \in \bar{\mathfrak{h}}' \mid (\lambda, \alpha) \in \mathbb{Z} \quad \textit{for all} \quad \alpha \in M'\}, \tag{1}$$

$$\bar{P}^* = \{\lambda \in \bar{\mathfrak{h}}' \mid (\lambda, \alpha) \in \mathbb{Z} \quad \textit{for all} \quad \alpha \in M\}, \tag{2}$$

An element $\mu \in \hat{\mathfrak{h}}'$ *lies in* P *if and only if* $\mu(c) \in \mathbb{Z}$, $\bar{\mu} \in \bar{P}$ *and in* P^* *if and only if* $\mu(c) \in \mathbb{Z}$ *and* $\bar{\mu} \in \bar{P}^*$. *Let* $\gamma \in M$ *and* $\gamma' \in M'$. *Then*

$$(\gamma, \gamma') \in \mathbb{Z}, \quad a_0|\gamma|^2 \in 2\mathbb{Z}, \quad k|\gamma'|^2 \in 2\mathbb{Z}.$$

Proof. The assertions on P, $\bar{P}$, P^* and $\bar{P}^*$ follow from the equalities

$$Q^\vee = M' + \mathbb{Z}\delta, \quad Q^{*\vee} = M + \mathbb{Z}\delta.$$

The last assertion follows from the facts

1) $(M, M') \subset \mathbb{Z}$ (since $M \subset \bar{Q}$ and $M' \subset \bar{Q}^\vee$);
2) $a_0\theta^\vee = \theta \in M'$, $k\tilde{\theta}^\vee = \tilde{\theta} \in M$;
3) M and M' are spanned correspondingly by $W(\theta^\vee)$ and $W(\tilde{\theta}^\vee)$, where $\tilde{\theta}$ is the highest root of the set $\bar{\Delta}_+$.

19.7.4. Characters of representations and theta functions. Let L_Λ be an irreducible integrable representation of $\hat{\mathfrak{g}}(A)$, and let $\max(\Lambda)$ be the set of maximal weights of L_Λ. For multiplicities of weights $\lambda - n\delta \in P(\Lambda)$ where $\lambda \in \max(\Lambda)$ we introduce the generating function

$$b_\lambda^\Lambda = \sum_{n=0}^{\infty} \mathrm{mult}_\Lambda(\lambda - n\delta)e^{-n\delta}. \tag{1}$$

It is considered as a function on $\hat{\mathfrak{h}}$. Since the series (1) is majorized by the series

$$|e^{-\lambda}| \sum_{\mu \in P(\Lambda)} (\mathrm{mult}_\Lambda \mu)|e^\mu|$$

(see Section 19.5.4), then it converges absolutely on

$$Y = \{h \in \hat{\mathfrak{h}} \mid \mathrm{Re}\, \delta(h) > 0\}.$$

For the stabilizer $\hat{W}_\lambda$ of the point $\lambda \in P(\Lambda)$ in $\hat{W}$ we have $\hat{W}_\lambda \cap \mathcal{T} = \{0\}$. It is seen from formula (1) that

$$b_{w\lambda}^\Lambda = b_\lambda^\Lambda \quad \text{for all} \quad w \in \hat{W}.$$

In this reason

$$\begin{aligned} \mathrm{ch}\, L_\Lambda &= \sum_{\mu \in P(\Lambda)} (\mathrm{mult}_\Lambda \mu)e^\mu = \sum_{\lambda \in \max(\Lambda)} e^\lambda b_\lambda^\Lambda \\ &= \sum_{\substack{\lambda \in \max(\Lambda) \\ \lambda (\mathrm{mod}\, \mathcal{T})}} \sum_{T \in \mathcal{T}} e^{T\lambda} b_\lambda^\Lambda. \end{aligned} \tag{2}$$

In order to represent the character ch L_Λ in terms of theta functions we introduce the function

$$\Theta_\lambda = e^{-(|\lambda|^2/2m)\delta} \sum_{T \in \mathcal{T}} e^{T\lambda}, \tag{3}$$

where $\lambda \in \hat{\mathfrak{h}}'$, $\lambda(c) = m > 0$. Taking into account formula (8) of Section 19.7.2 we can write

$$\Theta_\lambda = e^{m\Lambda_0} \sum_{\gamma \in M + m^{-1}\bar{\lambda}} e^{-(1/2)m|\gamma|^2\delta + m\gamma}. \tag{4}$$

It is clear that this series converges absolutely on Y and depends on $\lambda (\mathrm{mod}\, mM + \mathbb{C}\delta)$.

Replacing $\mathcal{T}$ by $\mathcal{T}^*$ in the same way the function Θ'_λ is introduced.

Let us consider coordinates on Y. For $\tau \in H$ (the upper half-plane), $z \in \bar{\mathfrak{h}}'$ and $t \in \mathbb{C}$ we define $h = (\tau, z, t) \in \hat{\mathfrak{h}}$ by requiring that

$$\lambda(h) = -2\pi i(\lambda, \tau\Lambda_0 + z + t\delta) \tag{5}$$

for all $\lambda \in \hat{\mathfrak{h}}'$. This allows us to identify Y with $H \times \bar{\mathfrak{h}}' \times \mathbb{C}$. Identifying $\hat{\mathfrak{h}}'_R$ with $\mathbf{R}^\ell$, the lattice L from Section 19.7.1 with M, and $\langle\cdot,\cdot\rangle$ with $(\cdot,\cdot)$, we can identify functions (4) with functions $\Theta^L_{\mu,m}$ and Θ'_λ with $\Theta^{L'}_{\mu,m}$. Namely[2],

$$\Theta_\lambda = \Theta^M_{\lambda,m}, \quad \Theta'_\lambda = \Theta^{M'}_{\lambda,m}. \tag{6}$$

Let us note that

$$\Theta'_\lambda = \sum_{T \in \mathcal{T}^*(\mathrm{mod}\ \mathcal{T})} \Theta_{T\lambda} = \sum_{\gamma \in M'(\mathrm{mod}\ M)} \Theta^M_{\lambda+m\gamma,m}. \tag{7}$$

We introduce the number

$$s_\Lambda = \frac{|\Lambda+\rho|^2}{2(m+g)} - \frac{|\rho|^2}{2g}, \quad m = \Lambda(c), \tag{8}$$

where $g = \sum_{i=0}^{\ell} a_i^\vee$ is the dual Coxeter number of the algebra $\hat{\mathfrak{g}}(A)$. With the help of s_Λ the number

$$s_\Lambda(\lambda) = s_\Lambda - \frac{|\lambda|^2}{2m}, \quad m = \lambda(c), \tag{9}$$

is introduced. This number is called a *characteristic of the weight* λ. It is rational.

Below it will be convenient to deal with

$$c^\Lambda_\lambda = e^{-s_\Lambda(\lambda)\delta} b^\Lambda_\lambda = e^{-s_\Lambda(\lambda)\delta} \sum_{n=0}^{\infty} \mathrm{mult}_\Lambda(\lambda - n\delta) e^{-n\delta} \tag{10}$$

instead of b^Λ_λ. This formula define c^Λ_λ for $\lambda \in \max(\Lambda)$. If $\lambda \in \hat{\mathfrak{h}}'$ is such that $\lambda - \mu \in \mathbb{C}\delta$ for some $\mu \in \max(\Lambda)$, then we set $c^\Lambda_\lambda = c^\Lambda_\mu$. If the set $(\lambda + \mathbb{C}\delta) \cap \max(\Lambda)$ is empty, then we put $c^\Lambda_\lambda = 0$. Thus, c^Λ_λ is determined for all $\lambda \in \hat{\mathfrak{h}}'$. The function c^Λ_λ is called the *string function* of $\lambda \in \hat{\mathfrak{h}}'$. It is clear that

$$c^\Lambda_{w\lambda} = c^\Lambda_\lambda \quad \text{for all} \quad w \in \hat{W}, \quad \lambda \in \hat{\mathfrak{h}}'.$$

Since $\hat{W} = W \times \mathcal{T}$, then due to formula (5) of section 19.7.2 we have

$$c^\Lambda_{w\lambda+m\gamma+a\delta} = c^\Lambda_\lambda \quad \text{for all} \quad \lambda \in \hat{\mathfrak{h}}', \quad w \in W, \quad \gamma \in M, \quad a \in \mathbb{C}. \tag{11}$$

Using formula (3) for the function Θ_λ we can represent $\mathrm{ch}\, L_\Lambda$ with $\Lambda(c) = m$ in the form

$$\mathrm{ch}\, L_\Lambda = e^{s_\Lambda \delta} \sum_{\substack{\lambda \in P(\mathrm{mod}\ mM + \mathbb{C}\delta) \\ \lambda(c)=m}} c^\Lambda_\lambda \Theta^M_{\lambda,m}. \tag{12}$$

---

[2] We identify $\mathbb{C}^\ell$ with $\bar{\mathfrak{h}}'$ and, therefore, do not use boldface letters for elements from $\mathbb{C}^\ell \sim \bar{\mathfrak{h}}'$.

On the other hand, taking into account the formula $\hat{W} = W \times T$ we obtain from relation (10) of Section 19.5.4 that

$$e^{-s_\Lambda \delta} \text{ch } L_\Lambda = \frac{\sum_{w \in W} \varepsilon(w) \Theta^M_{\overline{w(\Lambda+\rho)}, m}}{\sum_{w \in W} \varepsilon(w) \Theta^M_{\overline{w\rho}, m}}, \tag{13}$$

where the summations are over the Weyl group W of the corresponding simple complex Lie algebra. Comparing formulas (12) and (13) we receive the relation for theta functions

$$\frac{\sum_{w \in W} \varepsilon(w) \Theta^M_{\overline{w(\Lambda+\rho)}, m}}{\sum_{w \in W} \varepsilon(w) \Theta^M_{\overline{w\rho}, m}} = \sum_{\substack{\lambda \in P (\text{mod } mM + \mathbb{C}\delta) \\ \lambda(c) = m}} c^\Lambda_\lambda \Theta^M_{\bar{\lambda}, m}. \tag{14}$$

In conclusion of this section we remark that for multiplicities $\text{mult}_\Lambda(\lambda - n\delta)$ of weights $\lambda - n\delta$ with $\lambda \in \max(\Lambda)$ in irreducible representation L_Λ, where $\Lambda \in P_+$, $\Lambda(c) = m > 0$, the asymptotic relation

$$\text{mult}_\Lambda(\lambda - n\delta) \underset{n \to +\infty}{\sim} \frac{1}{\sqrt{2}} a^{(\ell+1)/4} b n^{-(\ell+3)/4} e^{4\pi(an)^{1/2}} \tag{15}$$

is valid. Here $a = \frac{1}{2} |\bar{\rho}^*|^2 m / g(m+g)$ and

$$b = (\text{vol } \bar{Q}) \left(\frac{g}{m(m+g)} \right)^{1/2} \prod_{\alpha \in \bar{\Delta}^*_+} \frac{\sin \pi \frac{(\alpha, \bar{\Lambda} + \bar{\rho})}{m+g}}{\sin \pi \frac{(\alpha, \bar{\rho})}{g}}.$$

Let us give asymptotic relation (15) for the algebra $A_1^{(1)}$. Let $\Lambda = (m - N)\Lambda_0 + N\Lambda_1 \in P_+$ be highest weight of the positive level m (that is, such that $\Lambda(c) = m$) and let $\lambda = (m-n)\Lambda_0 + n\Lambda_1$. Then $\lambda \in P(\Lambda) + \mathbb{Z}\delta$ if and only if $n \equiv N (\text{mod } 2\mathbb{Z})$. In this case

$$\text{mult}_\Lambda(\lambda - j\delta) \underset{j \to +\infty}{\sim} \frac{\sin \pi \frac{N+1}{m+2}}{2j(m+2)} \exp \left[\pi \left(\frac{2mj}{m+2} \right)^{1/2} \right]. \tag{16}$$

In particular, if $m = 1$, then $\Lambda = \Lambda_0$ and it is possible to show that $\text{mult}_{\Lambda_0}(\Lambda_0 - j\delta) = p(j)$, where p is the classical partition function. Therefore, according to formula (16) we have

$$p(j) \underset{j \to +\infty}{\sim} \frac{1}{4j\sqrt{3}} \exp \left(\pi \sqrt{\frac{2}{3} j} \right). \tag{17}$$

19.7.5. The functions A_λ and A'_λ. Along with $H(\ell, \mathbf{R})$ and G_R the groups $\hat{W}$ and $\hat{W}^*$ act on the space Y. Let us connect these actions. We have

$$T_{\boldsymbol{\alpha}}(\tau, z, t) = (\mathbf{0}, \boldsymbol{\alpha}, 0)(\tau, z, t).$$

Therefore, the groups $\mathcal{T}$ and $\mathcal{T}^*$ can be identified with certain subgroups of the group $H(\ell, \mathbf{R})$. Let us introduce an action of the subgroup W of the group $\hat{W}$ on G_R. Namely, we set

$$w \cdot A = A \quad \text{for} \quad A \in SL(2, \mathbf{R}),$$

$$w(\boldsymbol{\alpha}, \boldsymbol{\beta}, t) = (w\boldsymbol{\alpha}, w\boldsymbol{\beta}, t) \quad \text{for} \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}, t) \in H(\ell, \mathbf{R}).$$

Thus, W turns into a group of automorphisms of G_R and we can define the semidirect product $W \times G_R$ of the groups W and G_R (with the invariant subgroup G_R). The group W acts on Y by the formula

$$w \cdot (\tau, z, t) = (\tau, wz, t).$$

Therefore, an action of $W \times G_R$ on Y can be determined. The formulas

$$R_g f(\tau, z, t) = f(g \cdot (\tau, z, t)), \quad g \in G_R,$$
$$R_w f(\tau, z, t) = f(w \cdot (\tau, z, t)), \quad w \in W,$$

define the action of the group $W \times G_R$ on functions, defined on Y. Let us also note that setting $\det n = 1$ for $n \in H(\ell, \mathbf{R})$ we spread the mapping $\det\colon W \to \pm 1$ to obtain the homomorphism

$$\det\colon W \times H(\ell, \mathbf{R}) \to \pm 1.$$

Now let us define an action of the commutative subgroup $Q^\vee$ on Y. We associate with an element $\alpha \in Q^\vee$ the shift $h \to h + 2\pi i \alpha$, $h \in \hat{\mathfrak{h}}^*$. This action coincides with the action of the subgroup $\{(\alpha, \mathbf{0}, t) \in H(\ell, \mathbf{R}) \mid \alpha \in M', t \in \mathbb{Z}\}$ on Y. Consequently, $Q^\vee$ can be considered as a subgroup of the group $H(\ell, \mathbf{R})$. We separate in the semidirect product $W \times H(\ell, \mathbf{R})$ the subgroups

$$\tilde{\mathcal{T}} = \mathcal{T} \times Q^\vee, \quad \tilde{\mathcal{T}}^* = \mathcal{T}^* \times Q^{*\vee}, \quad \tilde{W} = \hat{W} \times Q^\vee, \quad \tilde{W}^* = \tilde{W}^* \times Q^{*\vee}.$$

They are semidirect products of corresponding groups, where the second multiplier is an invariant subgroup. We have

$$\tilde{\mathcal{T}} = \left\{ (\alpha, \beta, t) \in H(\ell, \mathbf{R}) \,\middle|\, \alpha \in M', \ \beta \in M, \ t + \frac{1}{2}(\alpha, \beta) \in \mathbb{Z} \right\},$$

$$\tilde{\mathcal{T}}^* = \left\{ (\alpha, \beta, t) \in H(\ell, \mathbf{R}) \,\middle|\, \alpha \in M, \ \beta \in M', \ t + \frac{1}{2}(\alpha, \beta) \in \mathbb{Z} \right\},$$

$$\tilde{W} = W \times \tilde{\mathcal{T}}, \quad \tilde{W}^* = W \times \tilde{\mathcal{T}}^*,$$

where $W \times \tilde{\mathcal{T}}$ and $W \times \tilde{\mathcal{T}}^*$ are semidirect products with the invariant subgroups $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{T}}^*$, respectively.

In Section 19.7.1 we have determined the spaces $\tilde{\mathfrak{F}}_m$ and $\mathfrak{F}_m$. The functions $\Theta^M_{\mu,m}$, $\mu \in M'(\mathrm{mod}\, mM)$ form a basis of the space $\mathfrak{F}_m$. The groups $\tilde{W}$ and $\tilde{W}^*$ act on $\tilde{\mathfrak{F}}_m$ and $\mathfrak{F}_m$ and these spaces are invariant with respect to the actions of $\tilde{W}$ and $\tilde{W}^*$. Let us introduce the space

$$\mathfrak{F}^0_m = \{f \in \mathfrak{F}_m \mid R_u f = f \quad \text{for all} \quad u \in \tilde{\mathcal{T}}\}.$$

Then

$$\mathfrak{F}^0_m = \{f \in \mathfrak{F}_m \mid f((\alpha, 0, 0) \cdot (\tau, z, t)) = f(\tau, z, t) \quad \text{for} \quad \alpha \in M'\}.$$

In particular, for $k = 1$, that is, for non-twisted affine Lie algebras, we have $\mathfrak{F}^0_m = \mathfrak{F}_m$.

We also need the spaces

$$\mathfrak{F}^+_m = \{f \in \mathfrak{F}_m \mid R_w f = f \quad \text{for all} \quad w \in \tilde{W}\},$$

$$\mathfrak{F}^-_m = \{f \in \mathfrak{F}_m \mid R_w f = (\det w) f \quad \text{for all} \quad w \in \tilde{W}\}.$$

They can be also defined as

$$\mathfrak{F}^+_m = \{f \in \mathfrak{F}^0_m \mid R_w f = f \quad \text{for all} \quad w \in W\},$$

$$\mathfrak{F}^-_m = \{f \in \mathfrak{F}^0_m \mid R_w f = (\det w) f \quad \text{for all} \quad w \in W\}.$$

Replacing $\tilde{\mathcal{T}}$ and $\tilde{W}$ by $\tilde{\mathcal{T}}^*$ and $\tilde{W}^*$ in an analogous way we introduce the spaces $\mathfrak{F}'^0_m$, $\mathfrak{F}'^+_m$, $\mathfrak{F}'^-_m$. Replacing $\mathfrak{F}_m$ by $\tilde{\mathfrak{F}}_m$ we obtain the spaces $\tilde{\mathfrak{F}}^0_m$, $\tilde{\mathfrak{F}}^\pm_m$, $\tilde{\mathfrak{F}}'^0_m$, $\tilde{\mathfrak{F}}'^\pm_m$.

For $\lambda \in \hat{\mathfrak{h}}'$ such that $\lambda(c) = m > 0$ we define the functions

$$A_\lambda = e^{-(|\lambda|^2/2m)\delta} \sum_{w \in \hat{W}} \varepsilon(w) e^{w\lambda}, \tag{1}$$

$$A'_\lambda = e^{-(|\lambda|^2/2m)\delta} \sum_{w \in \hat{W}^*} \varepsilon(w) e^{w\lambda}, \tag{2}$$

$$S_\lambda = e^{-(|\lambda|^2/2m)\delta} \sum_{\mu \in \hat{W}\lambda} e^{\mu}, \quad S'_\lambda = e^{-(|\lambda|^2/2m)\delta} \sum_{\mu \in \hat{W}^*\lambda} e^{\mu}. \tag{3}$$

Theorem 1. (1) *Let $\lambda, \lambda' \in \hat{\mathfrak{h}}'$ and $\lambda(c) = \lambda'(c) = m > 0$. Then $\Theta_\lambda = \Theta_{\lambda'}$ if and only if $\mathcal{T}\lambda = \mathcal{T}\lambda'$ (mod $\mathbb{C}\delta$), and $S_\lambda = S_{\lambda'}$ if and only if $\hat{W}\lambda = \hat{W}\lambda'$ (mod $\mathbb{C}\delta$). The functions S_λ and $S_{\lambda'}$ are related to the functions Θ_μ and Θ'_μ by the formulas*

$$S_\lambda = |\hat{W}_\lambda|^{-1} \sum_{w \in W} \Theta_{w\lambda}, \quad S'_\lambda = |\hat{W}^*_\lambda|^{-1} \sum_{w \in W} \Theta'_{w\lambda}.$$

(2) $A_\lambda = 0$ *if and only if* $(\lambda, \alpha) = 0$ *for some* $\alpha \in \hat{\Delta}^{\text{re}}$, *and* $A_\lambda = \pm A_{\lambda'}$ *if and only if* $\hat{W}\lambda \equiv \hat{W}\lambda' \pmod{\mathbb{C}\delta}$ *or if* $A_\lambda = A_{\lambda'} = 0$. *The functions* A_λ *and* A'_λ *are connected with functions* Θ_μ *and* Θ'_μ *by the formulas*

$$A_\lambda = \sum_{w \in W} \varepsilon(w)\Theta_{w\lambda} = \sum_{w \in \hat{W} (\text{mod } T)} \varepsilon(w)\Theta_{w\lambda},$$

$$A'_\lambda = \sum_{w \in W} \varepsilon(w)\Theta'_{w\lambda} = \sum_{w \in \hat{W}^* (\text{mod } T^*)} \varepsilon(w)\Theta'_{w\lambda}.$$

(3) *The set*

$$\{\Theta_\lambda \mid \lambda \in P^* \ (\text{mod } T)(\text{mod } \mathbb{C}\delta),\ \lambda(c) = m\} = \{\Theta^M_{\mu,m} \mid \mu \in \bar{P}^* \ (\text{mod } mM)\}$$

is a basis of the space $\mathfrak{F}_m$ (*of the space* $\tilde{\mathfrak{F}}_m$) *over* $\mathbb{C}$ (*over the space* $\tilde{\mathfrak{F}}_0$ *of holomorphic functions of* τ).

(4) *The set*

$$\{\Theta_\lambda \mid \lambda \in P \ (\text{mod } T)(\text{mod } \mathbb{C}\delta),\ \lambda(c) = m\} = \{\Theta^M_{\mu,m} \mid \mu \in \bar{P} \ (\text{mod } mM)\}$$

is a basis of the space $\mathfrak{F}^0_m$ (*of the space* $\tilde{\mathfrak{F}}^0_m$) *over* $\mathbb{C}$ (*over* $\tilde{\mathfrak{F}}_0$).

(5) *The set*

$$\{S_\lambda \mid \lambda \in P_+ \ (\text{mod } \mathbb{C}\delta),\ \lambda(c) = m\}$$

is a basis of the space $\mathfrak{F}^+_m$ (*of the space* $\tilde{\mathfrak{F}}^+_m$) *over* $\mathbb{C}$ (*over* $\tilde{\mathfrak{F}}_0$).

(6) *The set*

$$\{A_\lambda \mid \lambda \in P_{++} \ (\text{mod } \mathbb{C}\delta),\ \lambda(c) = m\}$$

is a basis of the space $\mathfrak{F}^-_m$ (*of the space* $\tilde{\mathfrak{F}}^-_m$) *over* $\mathbb{C}$ (*over* $\tilde{\mathfrak{F}}_0$).

(7) *If* $\Lambda \in P_+$, $\Lambda(c) > 0$, *then*

$$e^{-s_\Lambda \delta}\text{ch}L_\Lambda = \frac{A_{\lambda+\rho}}{A_\rho}.$$

(8) *The set*

$$\{A_{\Lambda+\rho}/A_\rho \mid \Lambda \in P_+ \ (\text{mod } \mathbb{C}\delta),\ \Lambda(c) = m\}$$

is a basis of the space $\tilde{\mathfrak{F}}^+_m$ *over* $\tilde{\mathfrak{F}}_0$. *The space*

$$\tilde{\mathfrak{F}}^- = \sum_{m=0}^{\infty} \tilde{\mathfrak{F}}^-_m$$

is a free $\tilde{\mathfrak{F}}^+$*-module (where* $\tilde{\mathfrak{F}}^+ = \sum_{m=0}^{\infty} \tilde{\mathfrak{F}}_m^+$*) generated by the single element* A_ρ*.*

Proof. The statements (1) and (2) follow from the definitions of the functions A_λ, A'_λ, Θ_λ, Θ'_λ and from properties of the Weyl group $\hat{W}$. The statement (3) follows from Statement 4 of Section 19.7.3 and from the fact that $\mathfrak{F}$ is a free module over $\tilde{\mathfrak{F}}_0$ with the basis $\{\Theta^M_{\mu,m}\}$ (see Section 19.7.1). Now the statement (4) is a consequence of formula (12) of Section 19.7.1 and of Statement 4 of Section 19.7.3.

It is easy to show that P_+ is a fundamental domain for the group $\hat{W}$ acting on $\{\lambda \in P \mid \lambda(c) > 0\} \cup \mathbb{C}\delta$. This assertion and statement (4) of the formulation of our theorem lead to statement (5). Non-vanishing elements from $\{A_\lambda \mid \lambda \in P_+ \pmod{\mathbb{C}\delta},\ \lambda(c) = m\}$ form a basis for the space $\tilde{\mathfrak{F}}_m^-$. This assertion and statement (2) of our theorem lead to statement (6). The statement (7) is another form of formula (14) of Section 19.7.4. It follows from statements (6) and (7) that for each $F \in \tilde{\mathfrak{F}}^-$ the function F/A_ρ is holomorphic. This means that the space $\tilde{\mathfrak{F}}^-$ is a free $\tilde{\mathfrak{F}}^+$-module generated by the element A_ρ. Now statement (8) follows from statement (7) and from the fact that $\lambda \to \lambda + \rho$ is the one-to-one mapping from P_+ onto P_{++}. Theorem is proved.

Let Γ be the subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ such that $c \equiv 0 \pmod{k^*\mathbb{Z}}$, $ac \equiv bc \equiv 0 \pmod{a_0\mathbb{Z}}$ where k^* is the number k taken for the affine Lie algebra $\hat{\mathfrak{g}}(A^*)$ from Section 19.7.2. Let G_0 be the intersection of the normalizer of $\hat{W}$ in $W \times G_R$ with G_R and let $H_0 = H(\ell, \mathbf{R}) \cap G_0$. It is possible to show that

$$H_0 = \{(\alpha, \beta, t) \in H(\ell, \mathbf{R}) \mid (\alpha, \lambda) \in \mathbb{Z},\ (\beta, \mu) \in \mathbb{Z} \quad \text{for} \quad \lambda \in \bar{Q},\ \ \mu \in \bar{Q}^*\}.$$

Let us note that if $k = 1$, then $\Gamma = SL(2,\mathbb{Z})$.

Statement 1. *Let* $m \in \mathbb{Z}_+$*. Then the spaces* $\mathfrak{F}_m$ *and* $\mathfrak{F}_m^0$ *are invariant with respect to the group* G_0*. Elements of the group* G_0 *commute with elements of* W *on the space* $\mathfrak{F}_m^0$*. Matrices corresponding to elements of the group* G_0 *in the basis* $\{\Theta^M_{\mu,m} \mid \mu \in \bar{P}^* \pmod{mM}\}$ *of the space* $\mathfrak{F}_m$ *is unitary. The element* $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *leaves the space* $\mathfrak{F}_m$ *invariant and the matrix of this element in the same basis of* $\mathfrak{F}_m$ *is unitary. Moreover,* S *commutes with elements of* W *and permutes the subspaces* $\mathfrak{F}_m^0$ *and* $\mathfrak{F}_m'^0$*.*

Proof. It is easy to see that G_0 is a subgroup of the group G_Z. This and Statement 3 of Section 19.7.1 show that G_0 leaves the space $\mathfrak{F}_m$ invariant and that elements of G_0 in the basis $\{\Theta^M_{\mu,m}\}$ are unitary. It is clear that for all $g \in G_0$ and $w \in \tilde{W}$ we have $gwg^{-1}w^{-1} \in \tilde{T}$. Therefore, G_0 leaves the space $\mathfrak{F}_m^0$ invariant and elements of G_0 commute with elements from W on this space. This proves the first part of our statement. Analogously, $S \in G_Z$ commutes with elements from W and $S\tilde{T}S^{-1} = S^{-1}\tilde{T}S = \tilde{T}^*$. This and Statement 3 of Section 19.7.1 prove the second part of our statement.

Statement 2. *Let $m \in \mathbb{Z}_+$ and $\mu \in M^* \equiv \bar{P}^* \supset \bar{P}$. The following assertions are valid:*

(1) *Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We choose $\beta \in \bar{P}$ such that $mac|\alpha|^2 \equiv 2(\alpha, \beta)$ $(\mathrm{mod}\ 2\mathbb{Z})$ if $c\alpha \in M$ and $m\alpha \in M^*$. Let $\varepsilon = v(m, B \cdot (0, m^{-1}\beta, 0))$ be the complex number determined in Statement* 1 *of Section* 19.7.1. *Then*

$$\Theta^M_{\mu,m}(B \cdot (\tau, z, t)) = \varepsilon \sum_{\substack{\alpha \in M^* \\ c\alpha (\mathrm{mod}\ mM)}} \exp\{\pi i m^{-1}[(b\mu + d\alpha, a\mu + c\alpha + 2\beta) - (\mu, \alpha)]\}\Theta^M_{a\mu + c\alpha + \beta, m}.$$

(2) *We choose $\alpha \in \mathfrak{h}'$ such that $|\gamma|^2 \equiv 2(\alpha, \gamma)(\mathrm{mod}\ 2\mathbb{Z})$ for all $\gamma \in M$. Then*

$$\Theta^M_{\mu,m}(\tau + 1, z - \alpha, t) = \exp[\pi i(m^{-1}|\mu|^2 + 2(\alpha, \mu))]\Theta^M_{\mu,m}(\tau, z, t).$$

(3) *For the functions $\Theta^M_{\mu,m}$ we have the relation*

$$\Theta^M_{\mu,m}\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right) = |M^*/mM|^{-1/2}(-i\tau)^{\ell/2} \times \sum_{\mu' \in M^* (\mathrm{mod}\ mM)} \exp[-2\pi i m^{-1}(\mu, \mu')]\Theta^M_{\mu',m}(\tau, z, t).$$

(4) *Let $\lambda \in P$, $\lambda(c) = m > 0$. Then*

$$A_\lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right) = |M^*/mM|^{-1/2}(-i\tau)^{\ell/2} \times \sum_{\substack{\lambda' \in P^*_{++} (\mathrm{mod}\ \mathbb{C}\delta) \\ \lambda'(c) = m}} \left[\sum_{w \in W} \varepsilon(w) \exp\left(-2\pi i \frac{(w\bar{\lambda}, \bar{\lambda}')}{m}\right)\right] A'_{\lambda'}(\tau, z, t).$$

Proof of this statement is given in [134].

19.7.6. Expressions for the function A_ρ. Due to the denominator formula

$$\sum_{w \in \hat{W}} \varepsilon(w) e^{w\rho} = e^\rho \prod_{\alpha \in \hat{\Delta}_+} (1 - e^{-\alpha})^{m(\alpha)}$$

the function A_ρ can be expressed in terms of infinite products. These infinite products are contained in the expressions for A_ρ in the same combinations as in the case of the Jacobi theta functions $\theta_1(z,\tau)$ and $\theta_4(z,\tau)$. We shall use the expressions

$$\begin{aligned}\theta_1(z,\tau) &= (-i)\sum_{n=-\infty}^{\infty}(-1)^n e^{i\pi\tau(n+1/2)^2}e^{2\pi i(n+1/2)z}\\ &= (-i)e^{i\pi z}e^{\pi i\tau/4}\prod_{n=0}^{\infty}(1-q^{n+1})(1-e^{2\pi iz}q^{n+1})(1-e^{-2\pi iz}q^n)\\ &= (-i)e^{i\pi z}e^{\pi i\tau/4}(q;q)_\infty(e^{2\pi iz}q;q)_\infty(e^{-2\pi iz};q)_\infty\end{aligned}$$

for the theta function $\theta_1(z,\tau)$ and the expression

$$\begin{aligned}\theta_4(z,\tau) &= \sum_{n=-\infty}^{\infty}(-1)^n e^{i\pi\tau n^2}e^{2\pi inz}\\ &= \prod_{n=0}^{\infty}(1-q^{n+1})(1-e^{2\pi iz}q^{n+1/2})(1-e^{-2\pi iz}q^{n+1/2})\\ &= (q;q)_\infty(q^{1/2}e^{2\pi iz};q)_\infty(q^{1/2}e^{-2\pi iz};q)_\infty\end{aligned}$$

for the theta function $\theta_4(z,\tau)$ where $q=e^{2\pi i\tau}$ (see formulas (1) and (4) of Section 19.2.4). Using these formulas and the formula

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n),\quad q=e^{2\pi i\tau},$$

for Dedekind's η-function we can give the following expressions for the function A_ρ in cases of "classical" affine Lie algebras. For the algebra $A_\ell^{(1)}$

$$A_\rho(\tau,h)=\eta(\tau)^{\ell(1-\ell)/2}\prod_{1\le i<j\le\ell+1}\tilde\theta_1(w_i-w_j,\tau),$$

where $h=\sum_{i=1}^{\ell+1}w_i\varepsilon_i$, $(\varepsilon_i,\varepsilon_j)=\delta_{ij}$, $\tilde\theta_1(z,\tau)=i\theta_1(z,\tau)$. For the algebra $B_\ell^{(1)}$

$$\begin{aligned}A_\rho(\tau,h)=\eta(\tau)^{\ell(1-\ell)}&\prod_{1\le i<j\le\ell}\tilde\theta_1(w_i+w_j,\tau)\tilde\theta_1(w_i-w_j,\tau)\\ \times&\prod_{1\le j\le\ell}\tilde\theta_1(w_j,\tau),\end{aligned}$$

where $h=\sum_{i=1}^{\ell}w_i\varepsilon_i$. For the algebra $C_\ell^{(1)}$

$$\begin{aligned}A_\rho(\tau,h)=\eta(\tau)^{\ell(1-\ell)}&\prod_{1\le i<j\le\ell}\tilde\theta_1(w_i+w_j,\tau)\tilde\theta_1(w_i-w_j,\tau)\\ \times&\prod_{1\le j\le\ell}\tilde\theta_1(2w_j,\tau),\end{aligned}$$

where $w = \sum_{i=1}^{\ell} 2w_i\varepsilon_i$. For the algebra $D_\ell^{(1)}$

$$A_\rho(\tau, h) = \eta(\tau)^{\ell(2-\ell)} \prod_{1\le i<j\le \ell} \tilde{\theta}_1(w_i + w_j, \tau)\tilde{\theta}_1(w_i - w_j, \tau),$$

where $h = \sum_{i=1}^{\ell} w_i\varepsilon_i$. For the algebra $A_{2\ell}^{(2)}$

$$A_\rho(\tau, h) = \eta(\tau)^{\ell(1-\ell)}\eta(2\tau)^{-\ell} \prod_{1\le i<j\le \ell} \tilde{\theta}_1(w_i + w_j, \tau)\tilde{\theta}_1(w_i - w_j, \tau)$$
$$\times \prod_{1\le i\le \ell} \tilde{\theta}_1(2w_i, 2\tau)\theta_4(w_i, \tau),$$

where $h = \sum_{i=1}^{\ell} w_i\varepsilon_i$. For the algebra $A_{2\ell-1}^{(2)}$

$$A_\rho(\tau, h) = \eta(\tau)^{-(\ell-1)^2}\eta(2\tau)^{1-\ell} \prod_{1\le i\le \ell} \tilde{\theta}_1(2w_i, 2\tau)$$
$$\times \prod_{1\le i<j\le \ell} \tilde{\theta}_1(w_i + w_j, \tau)\tilde{\theta}_1(w_i - w_j, \tau),$$

where $h = \sum_{i=1}^{\ell} w_i\varepsilon_i$. For the algebra $D_{\ell+1}^{(2)}$

$$A_\rho(\tau, h) = \eta(\tau)^{1-\ell}\eta(2\tau)^{-(\ell-1)^2} \prod_{1\le i\le \ell} \tilde{\theta}_1(w_i, \tau)$$
$$\times \prod_{1\le i<j\le \ell} \tilde{\theta}_1(w_i + w_j, 2\tau)\tilde{\theta}_1(w_i - w_j, 2\tau),$$

where $w = \sum_{i=1}^{\ell} \frac{1}{2}w_i\varepsilon_i$.

19.7.7. Transformation properties of the function A_ρ. Let us introduce some notations which will be used in formulation of Theorem 1 below. For $\Lambda \in P_+$ such that $\Lambda(c) = m > 0$, we set

$$b(\Lambda) = |\bar{P}/(m+g)M|^{-1/2} \prod_{\alpha\in\Delta_+^*} 2\sin\frac{\pi(\alpha, \bar{\Lambda} + \bar{\rho})}{m+g},$$

where g is the dual Coxeter number of the algebra $\hat{\mathfrak{g}}(A)$. For $\lambda \in \bar{P}_+^*$ we use the Weyl character formula

$$\chi'_\lambda(\exp y) = \frac{\sum_{w\in W} \varepsilon(w)e^{(w(\lambda+\bar{\rho}^*), y)}}{\sum_{w\in W} \varepsilon(w)e^{(w\bar{\rho}^*, y)}}, \quad y \in \bar{\mathfrak{h}}.$$

Below we shall also use the function

$$F(\tau) = e^{2\pi i(|\bar\rho|^2/2g)\tau} \prod_{\alpha\in\hat\Delta_+\setminus\Delta_+} (1 - e^{2\pi i(\alpha,\Lambda_0)\tau})^{m(\alpha)} \tag{1}$$

defined on $H = \{\tau \in \mathbf{C} \mid \operatorname{Im}\tau > 0\}$. Using root systems of affine Lie algebras the function $F(\tau)$ can be expressed in terms of Dedekind's η-function $\eta(\tau)$. Namely, for all non-twisted affine Lie algebras we have

$$F(\tau) = \eta(\tau)^{\ell(h+1)},$$

where $h = \sum_{i=0}^{\ell} a_i$ is the Coxeter number. For twisted affine Lie algebras the function $F(\tau)$ is of the form

$$\begin{aligned}
&\text{for} \quad A_{2\ell}^{(2)}: \quad F(\tau) = \eta\left(\tfrac{\tau}{2}\right)^{2\ell}\eta(\tau)^{\ell(2\ell-3)}\eta(2\tau)^{2\ell},\\
&\text{for} \quad A_{2\ell-1}^{(2)}: \quad F(\tau) = \eta(\tau)^{(\ell-1)(2\ell+1)}\eta(2\tau)^{2\ell+1},\\
&\text{for} \quad D_{\ell+1}^{(2)}: \quad F(\tau) = \eta(\tau)^{2\ell+1}\eta(2\tau)^{(\ell-1)(2\ell+1)},\\
&\text{for} \quad D_4^{(3)}: \quad F(\tau) = \eta(\tau)^7\eta(3\tau)^7.
\end{aligned}$$

Theorem 1. (1) *To every matrix B of the group Γ there corresponds the number $v(B)$ such that*

$$F(B\tau) = v(B)(c\tau + d)^{\ell+2|\Delta_+|}F(\tau),$$
$$A_\rho(B\cdot(\tau,z,t)) = v(B)A_\rho.$$

(2) *For A_ρ we have the expression*

$$\begin{aligned}
A_\rho(\tau,z,t) &= e^{-(|\bar\rho|^2/2g)\delta}e^{\rho}\prod_{\alpha\in\hat\Delta_+}(1-e^{-\alpha})^{m(\alpha)}\\
&= \exp\left[2\pi i\left(\frac{|\bar\rho|^2}{2g}\tau - (\bar\rho,z) - gt\right)\right]\prod_{\alpha\in\hat\Delta_+}\left(1 - e^{2\pi i(\alpha,z+\tau\Lambda_0)}\right)^{m(\alpha)}.
\end{aligned}$$

(3) *The relation*

$$\begin{aligned}
&A_\rho\left(-\frac{1}{\tau},\ \frac{z}{\tau},\ t + \frac{(z,z)}{2\tau}\right)\\
&= |M'/M|^{-1/2}(-i)^{|\Delta_+|}(-i\tau)^{\ell/2}A'_{\rho^*}(\tau,z,t)
\end{aligned}$$

is valid.

(4) *Let* $\Lambda \in P_+$ *and* $\Lambda(c) = m > 0$. *Then*

$$\left(\frac{A_{\Lambda+\rho}}{A_\rho}\right)\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\tau}\right)$$
$$= b(\Lambda) \sum_{\substack{\Lambda' \in P_+^* (\mathrm{mod}\, \mathbb{C}\delta) \\ \Lambda'(c)=m}} \chi'_{\bar{\Lambda}'}\left(\exp\left(-2\pi i \frac{\bar{\Lambda}+\bar{\rho}}{m+g}\right)\right)\left(\frac{A'_{\Lambda'+\rho^*}}{A'_{\rho^*}}\right)(\tau, z, t).$$

(5) *For* $\alpha, \beta \in \bar{\mathfrak{h}}'$ *such that* $(\alpha, \gamma) \in \mathbb{Z}$, $(\beta, \mu) \in \mathbb{Z}$ *for all* $\gamma \in \bar{Q}$, $\mu \in \bar{Q}^*$ *we have*

$$A_\rho((\alpha, 0, 0) \cdot (\tau, z, t)) = (-1)^{(\alpha, 2\bar{\rho})} A_\rho(\tau, z, t), \tag{2}$$

$$A_\rho((0, \beta, 0) \cdot (\tau, z, t)) = (-1)^{(\beta, 2\bar{\rho}^*)} A_\rho(\tau, z, t). \tag{2'}$$

Proof. The second assertion follows from the denominator formula and from the definition of the function A_ρ. In order to prove the first assertion let us show that

$$\mathfrak{F}_g^- = \mathbb{C}A_\rho, \quad \mathfrak{F}_g'^- = \mathbb{C}A'_{\rho^*}. \tag{3}$$

Let $g' = \rho^*(c)$. Then according to statement (6) of Theorem 1 of Section 19.7.5 we have $\mathfrak{F}_g^- = \mathbb{C}A_\rho$, $\mathfrak{F}_{g'}'^- = \mathbb{C}A'_{\rho^*}$, $\mathfrak{F}_m^- = \{0\}$ for $m < g$ and $\mathfrak{F}_m'^- = \{0\}$ for $m < g'$. Due to the second part of Statement 1 of Section 19.7.5 we have $\dim \mathfrak{F}_m^- = \dim \mathfrak{F}_m'^-$ for all m. Therefore, $g = g'$ and this proves formulas (3). Due to these formulas and the first part of Statement 1 of Section 19.7.5 we have $A_\rho(B \cdot (\tau, z, t)) = v(B) A_\rho(\tau, z, t)$, $B \in \Gamma$, for some $v(B) \in \mathbb{C}$. The second assertion of our theorem suppose that

$$F(\tau) = \lim_{\substack{z \to 0 \\ z \text{ regular}}} \left[A_\rho(\tau, z, 0) \Big/ \prod_{\alpha \in \Delta_+} (1 - e^{-2\pi i(\alpha, z)}) \right].$$

From here we easily derive the first assertion.

According to assertion (4) of Statement 2 of Section 19.7.5 and to formula (3) we have

$$A_\rho\left(-\frac{1}{\tau}, \frac{z}{\tau}, t + \frac{(z,z)}{2\pi}\right) = c(-i\tau)^{\ell/2} A'_{\rho^*}(\tau, z, t).$$

Due to the denominator formula for finite dimensional representations of simple complex Lie algebras, for c we have the formula

$$c = |M^*/gM|^{-1/2} \sum_{w \in W} \varepsilon(w) \exp[-2\pi i g^{-1}(w\bar{\rho}, \bar{\rho}^*)]$$
$$= |M^*/gM|^{-1/2} \prod_{\alpha \in \Delta_+^*} \left[-2i \sin\left(\pi \frac{(\alpha, \bar{\rho})}{g}\right)\right].$$

If θ is the highest weight for Δ_+^*, then for $\lambda \in P_{++}$ and $\alpha \in \Delta_+^*$ the formula

$$0 < (\alpha, \lambda) \leq (\theta, \lambda) = \lambda(c) - \lambda(h_0) < \lambda(c)$$

is valid. Consequently, $i^{|\Delta_+|}c > 0$. Since in the basis $\{\Theta^M_{\mu,m} \mid \mu \in M^*(\mathrm{mod}\, gM)\}$ of the space $\mathfrak{F}_g$ the element $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is represented by a unitary matrix, then

$$|c|^2 = |\hat{W}/\mathcal{T}|/|\hat{W}^*/\mathcal{T}^*| = |M'/M|^{-1}.$$

Therefore, $c = (-i)^{|\Delta_+|}|M'/M|^{-1/2}$. This proves assertion (3). The assertion (4) follows from assertion (3) and from part (4) of Statement 2 of Section 19.7.5. Proof of formula (2) is simple. Formula (2′) follows from assertion (3) of our theorem and from the equality

$$A_\rho((0,\beta,0)\cdot(\tau,z,t)) = A_\rho(S(\beta,0,0)S^{-1}\cdot(\tau,z,t)).$$

Theorem is proved.

The function A_ρ is used to obtain modular forms (with the help of q-specializations). Let $y, z \in \bar{\mathfrak{h}}'_R$. We set

$$\Delta^{y,z} = \{\alpha \in \hat{\Delta} \mid (\alpha, \Lambda_0 + z) = 0, \quad (\alpha, y) \in \mathbb{Z}\},$$

$$D_{y,z} = \ell + |\Delta^{y,z}|.$$

It is easy to see that $\Delta^{y,z}$ is the set of all roots $\alpha \in \hat{\Delta}$ such that $\exp\{2\pi i(\alpha, \tau(\Lambda_0 + z) - y)\} = 1$ for all $\tau \in H$. The set

$$\Delta_+^{y,z} = \left\{\alpha \in \Delta^{y,z} \mid \bar{\alpha} \in \Delta_+ \quad \text{or} \quad \bar{\alpha} \in \frac{1}{2}\Delta_+\right\}$$

is the system of positive roots of $\Delta^{y,z}$. Let us introduce the function

$$F_{y,z}(\tau) = \left(\frac{A_\rho}{\prod\limits_{\alpha \in \Delta_+^{y,z}} (1 - e^{-\alpha})}\right)\left(\tau, \tau z - y, -\frac{1}{2}(z, \tau z - y)\right).$$

According to the second assertion of Theorem 1 we have

$$F_{y,z}(\tau) = (-1)^d e^{\pi i (y, 2\bar{\rho} - gz)} e^{\pi i g^{-1}|\bar{\rho} - gz|^2 \tau} \times \prod_{\alpha \in \hat{\Delta}_+ \setminus \Delta_+^{y,z}} (1 - e^{2\pi i(\alpha, \tau(\Lambda_0 + z) - y)})^{m(\alpha)},$$

where $d = |\Delta_+^{y,z} \cap (-\hat{\Delta}_+)|$. The function $F_{y,z}$ is holomorphic on H, $F_{y,z}(\tau) \neq 0$ for all $\tau \in H$, and $F_{00}(\tau) = F(\tau)$, where $F(\tau)$ is the function defined before formulation of Theorem 1.

Let us give special cases of the function $F_{y,z}(\tau)$:

$$F_{0,\bar{\rho}^\vee/(h+1)}(\tau) = \eta\left(\frac{\tau}{h+1}\right)^\ell,$$
$$F_{-\bar{\rho}^\vee/(h+1),0}(\tau) = (-i)^{|\Delta_+|}(h+1)^{\ell/2}\eta(\tau(h+1))^\ell.$$

Here h is the Coxeter number of the algebra $\hat{\mathfrak{g}}(A)$.

Replacing A_ρ by A'_{ρ^*} and $\Delta_+^{y,z}$ by $\Delta_+^{*y,z}$ the function $F'_{y,z}(\tau)$ is introduced.

For $\Lambda \in P_+, \Lambda(c) = m > 0$, and for $y, z \in \sum_{i=1}^{\ell} \mathbb{Q}\alpha_i$ (where $\mathbb{Q}$ is the set of rational numbers) we introduce the function

$$\Phi_{y,z}^\Lambda = \left(\frac{A_{\Lambda+\rho}}{A_\rho}\right)\left(\tau, \tau z - y, -\frac{1}{2}(z, \tau z - y)\right).$$

Statement 1. (1) *If* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *and* $c \equiv 0 \pmod{k\mathbb{Z}}$, *then*

$$F_{ay+bz,cy+dz}\left(\frac{a\tau+b}{c\tau+d}\right) = v(B)(c\tau+d)^{D_{y,z}}F_{y,z}(\tau).$$

(2) *Let* $K = 1$ *if* $k = 1$, $K = \prod_{\alpha\in\Omega} \frac{2}{|\alpha|^2}$ *if* $a_0 = 2$ *where*

$$\Omega = \{\alpha \in \Delta_+^{y,z} \mid \bar{\alpha} + (\alpha, y)\delta \in \hat{\Delta}\},$$

and

$$K = |M'/M|^{-1/2} \prod_{\alpha\in\Delta_+^{y,z}} \frac{2}{|\alpha|^2}$$

if $k^* \neq 1$. *Then*

$$F_{y,z}\left(-\frac{1}{\tau}\right) = K(-i)^{\ell+2|\Delta_+|}\tau^{D_{y,z}/2}F'_{z,-y}(\tau).$$

(3) *If* $y_0, z_0 \in \bar{\mathfrak{h}}'$ *are such that* $(y_0, \gamma) \in \mathbb{Z}$, $(\beta, \mu) \in \mathbb{Z}$ *for all* $\gamma \in \bar{Q}$, $\mu \in \bar{Q}^*$, *then we have*

$$F_{y+y_0,z+z_0}(\tau) = (-1)^{(y_0,2\bar{\rho})+(z_0,2\bar{\rho}^*)}$$
$$\times \exp\{\pi i g[(y,z_0) - (y_0,z) + (y_0,z_0)]\}F_{y,z}(\tau).$$

(4) *If* $y, z \in \sum_{i=1}^{\ell} \mathbb{Q}\alpha_i$, *then* $F_{y,z}(\tau)$ *is a holomorphic modular form of weight* $D_{y,z}/2$ *for the group* $\Gamma(N)$ (*see Section* 17.1.3) *where* N *is the least positive integer, divisible by* k, *such that* $(y,\gamma) \in N^{-1}\mathbb{Z}$, $(z,\gamma) \in N^{-1}\mathbb{Z}$ *for all* $\gamma \in \bar{Q}^*$.

(5) *The function* $\Phi^{\Lambda}_{y,z}$ *is a modular form of weight* 0 *with the transformation law*

$$\Phi^{\Lambda}_{y,z}\left(-\frac{1}{\tau}\right) = b(\Lambda) \sum_{\substack{\Lambda' \in P^*_+ (\mathrm{mod}\ \mathbb{C}\delta) \\ \Lambda'(c)=m}} \chi'_{\bar{\Lambda}'}\left[\exp\left(-2\pi i \frac{\bar{\Lambda}+\bar{\rho}}{m+g}\right)\right] \Phi'^{\Lambda'}_{z,-y}(\tau).$$

Proof of this statement can be found in [134].

19.7.8. Polynomial algebras. For a holomorphic function F on $Y = H \times \bar{\mathfrak{h}} \times \mathbb{C} \subset \hat{\mathfrak{h}}$ we define the partial derivatives

$$(\partial_i F)(h) = \lim_{t\to 0} \frac{F(h+th_i) - F(h)}{t}, \quad h \in \hat{\mathfrak{h}}.$$

The Jacobian for the functions $A_{\Lambda_j+\rho}/A_\rho$, $i = 0, 1, \ldots, \ell$, where $\Lambda_0, \Lambda_1, \ldots, \Lambda_\ell$ are the fundamental weights, is of the form

$$J \equiv \det\left(\partial_i \frac{A_{\Lambda_j+\rho}}{A_\rho}\right)^{\ell}_{i,j=0}.$$

Due to Theorem 1 of Section 19.7.5 $J \in \tilde{\mathfrak{F}}^-_g$ and, consequently, $J = b(\tau)A_\rho$ where $b(\tau)$ is a holomorphic function on H. The functions $b(\tau)$ are evaluated (up to multiplicative constants) for the affine Lie algebras $A^{(1)}_\ell$, $B^{(1)}_\ell$, $C^{(1)}_\ell$, $D^{(1)}_\ell$, $G^{(1)}_2$, $A^{(2)}_{2\ell}$, $A^{(2)}_{2\ell-1}$, $D^{(2)}_{\ell+1}$, $D^{(3)}_4$. They are given by the formulas:

for $A^{(1)}_\ell$ and $C^{(1)}_\ell$: $b(\tau) = \eta(\tau)^\ell$,

for $B^{(1)}_\ell$: $b(\tau) = \eta(\tau)^{\ell-3}F_0^{-1}F_\ell^2F_r^3F_{r+1}$ if $\ell = 2r$,

$b(\tau) = \eta(\tau)^{\ell-3}F_0^{-1}F_\ell^2F_rF_{r+1}^3$ if $\ell = 2r+1$,

for $D^{(1)}_\ell$: $b(\tau) = \eta(\tau)^{\ell-4}F_r^3F_{r+1}$ if $\ell = 2r$,

$b(\tau) = \eta(\tau)^{\ell-4}F_r^4$ if $\ell = 2r+1$,

for $G^{(1)}_2$: $b(\tau) = \eta(\tau)F_4$,

for $A^{(2)}_{2\ell}$: $b(\tau) = \eta(\tau)^{\ell-1}F_{\ell+1}$,

for $A^{(2)}_{2\ell-1}$: $b(\tau) = \eta(\tau)^{\ell-2}F_{\ell+1}F_\ell$,

$$\text{for}\quad D^{(2)}_{\ell+1}:\quad b(\tau)=\eta(2\tau)^{\ell-2}F_\ell F_{\ell+1},$$

$$\text{for}\quad D^{(3)}_4:\quad b(\tau)=\eta(\tau)\eta(2\tau)^{-1}\eta(3\tau)^{-1}\eta(4\tau)\eta(6\tau)^2\eta(9\tau)^2 \times\eta(12\tau)^{-1}q^{-49/72}\prod_{\substack{n=1\\ n\equiv 0,\pm 1 (\mathrm{mod}\ 9)}}^{\infty}(1-q^n)^{-1},$$

where

$$F_m=q^{(8M)^{-1}(M-2m)^2}\prod_{\substack{n=1\\ n\equiv 0,\pm m (\mathrm{mod}\ M)}}^{\infty}(1-q^n)$$

and $M=2\ell+1$ for $B^{(1)}_\ell$, $M=\ell$ for $D^{(1)}_\ell$, $M=9$ for $G^{(1)}_2$, $M=2\ell+3$ for $A^{(2)}_{2\ell}$, $M=2\ell+2$ for $A^{(2)}_{2\ell-1}$ and $D^{(2)}_{\ell+1}$, $M=36$ for $D^{(3)}_4$.

For the cases, given above, the function $b(\tau)$ does not vanish on H and the functions $A_{\lambda_i+\rho}/A_\rho$, $0\le i\le\ell$, when restricted onto $Y_{\tau_0}=Y|_{\tau=\tau_0}$, generate the space

$$\tilde{\mathfrak{F}}^{+,\tau_0}\equiv\{f\in\tilde{\mathfrak{F}}^{+}\text{ taken for fixed }\tau=\tau_0\}.$$

Theorem 1. *For any $\tau\in H$ the space $\tilde{\mathfrak{F}}^{+,\tau}$ is a polynomial algebra of $\ell+1$ generators $\Theta_0,\Theta_1,\dots,\Theta_\ell$ where $\Theta_i\in\tilde{\mathfrak{F}}^{+,\tau}_{d_i}$, $d_i=a^\vee_i$. For the affine Lie algebras, enumerated above, the space $\tilde{\mathfrak{F}}^{+}$ is a polynomial algebra over $\tilde{\mathfrak{F}}_0$ (the space of holomorphic functions on H) of generators $A_{\lambda_i+\rho}/A_\rho$, $i=0,1,\dots,\ell$.*

Proof of this theorem can be found in [134].

19.8. The String Function

19.8.1. Properties of the string function. In Section 19.7.4 we introduced the string function

$$c^\Lambda_\lambda=e^{2\pi i s_\Lambda(\lambda)\tau}\sum_{n=0}^{\infty}\mathrm{mult}_\Lambda(\lambda-n\delta)e^{2\pi i n\tau},$$

where $\Lambda\in P_+$, $\Lambda(c)=m>0$, and $\lambda\in\max(\Lambda)$. It is holomorphic on the upper half-plane H. Instead of c^Λ_λ it is more convenient to deal with the function $c(\lambda,\mu,m)$, $\lambda,\mu\in\bar{\mathfrak{h}}'$, $m\in\mathbb{Z}_+$, which is defined with the help of the string function. Namely, for $w\in\hat{W}$, $\Lambda\in P_+$, $\Lambda(c)=m$, $\nu\in\max(\Lambda)$ we set

$$c(\overline{w(\Lambda+\rho)},\bar{\nu},m)=\varepsilon(w)c^\Lambda_\nu,\quad \varepsilon(w)=\det w.$$

For other values of $\lambda,\mu\in\bar{\mathfrak{h}}'$ we put $c(\lambda,\mu,m)=0$. Then $c(\lambda,\mu,m)=0$ unless $\lambda\in\bar{P}$ and $\mu\in\lambda-\bar{\rho}+\bar{Q}\subset\bar{P}$.

Since for $T_\alpha \in \mathcal{T} \subset \hat{W}$ we have $\overline{T_\alpha(\Lambda+\rho)} = \bar{\Lambda}+\bar{\rho}+(m+g)\alpha$ and $\overline{T_\alpha \nu} = \bar{\nu}+m\alpha$ for $\nu \in P(\Lambda)$, then the function $c(\lambda,\mu,m)$ depends only on λ (mod $(m+g)M$) and μ (mod mM). For $w, w' \in W$ we have

$$c(w\lambda, w'\mu, m) = \varepsilon(w)c(\lambda,\mu,m).$$

Using the adjacent affine Lie algebra $\hat{\mathfrak{g}}(A^*)$ from Section 19.7.2 we construct the function $c'(\lambda,\mu,m)$ corresponding to it. It depends on λ (mod $(m+g)M$) and μ (mod mM'). Let us note that $\rho^*(c) = \rho(c) = g$.

Let N be the least positive integer such that $N|\gamma|^2 \in 2\mathbb{Z}$ for all $\gamma \in M^*$. We have $N = 2(\ell+1)$ for $A_\ell^{(1)}$ if ℓ is odd and $N = \ell+1$ if ℓ is even, $N = 8$ for $B_\ell^{(1)}$, $D_\ell^{(1)}$, $A_{2\ell-1}^{(2)}$ if ℓ is odd, $N = 4$ if $\ell \equiv 2 \pmod 4$, and $\ell = 2$ if $\ell \equiv 0 \pmod 4$. We also have $N = 4$ for $C_\ell^{(1)}$, $E_7^{(1)}$, $D_{\ell+1}^{(2)}$, $N = 3$ for $E_6^{(1)}$, $G_2^{(1)}$, $D_4^{(3)}$, $N = 2$ for $F_4^{(1)}$, $A_{2\ell}^{(2)}$, $E_6^{(2)}$, and $N = 1$ for $E_8^{(1)}$.

Theorem 1. (1) *The equality*

$$c\left(-\frac{1}{\tau}\middle|\,\mu,\mu',m\right) = |M^*/(m+g)M'|^{-1/2}|M^*/mM'|^{-1/2}|M'/M|^{1/2}$$
$$\times i^{|\Delta_+|}(-i\tau)^{-\ell/2} \sum_{\substack{\nu\in M^*(\mathrm{mod}\,(m+g)M')\\ \nu'\in M^*(\mathrm{mod}\,mM')}} \exp\left\{2\pi i\left[-\frac{(\mu,\nu)}{m+g}+\frac{(\mu',\nu')}{m}\right]\right\}$$
$$\times c'(\tau \mid \nu,\nu',m)$$

is valid for $c(\lambda,\mu,m) \equiv c(\tau \mid \lambda,\mu,m)$.

(2) *If $a_0 = 1$, then*

$$c(\tau+1 \mid \mu,\mu',m) = \exp\left\{\pi i\left[\frac{|\mu|^2}{m+g} - m^{-1}|\mu'|^2 - g^{-1}|\bar{\rho}|^2\right]\right\}$$
$$\times c(\tau \mid \mu,\mu',m).$$

If $a_0 = 2$, then

$$c(\tau+1 \mid \mu,\mu',m) = \exp\left\{\pi i\left[\frac{|\mu|^2}{m+g} - \frac{|\mu'|^2}{m} - \frac{|\bar{\rho}|^2}{g} - |\mu-\mu'-\bar{\rho}|^2\right]\right\}$$
$$\times c(\tau \mid \mu,\mu',m).$$

(3) *Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We choose $\beta, \beta'' \in \bar{P}$ such that*

$$(m+g)ac|\alpha|^2 \equiv 2(\alpha,\beta)(mod\,2\mathbb{Z}) \quad if \quad c\alpha \in M, \quad (m+g)\alpha \in M^*,$$

$$mcd|\alpha|^2 \equiv 2(\alpha,\beta'')(mod\,2\mathbb{Z}) \quad if \quad c\alpha \in M, \quad m\alpha \in M^*,$$

and set $\beta' = -a\beta''$,

$$\varepsilon = [\exp(-\pi i m^{-1} ab|\beta''|^2)] v(m+g, B\cdot(0,(m+g)^{-1}\beta,0)) \times v(m, B^{-1}\cdot(0,m^{-1}\beta'',0)),$$

where v *is defined by Statement* 1 *of Section* 19.7.1. *Then*

$$F\left(\frac{a\tau+b}{c\tau+d}\right) c\left(\frac{a\tau+b}{c\tau+d}\middle|\mu,\mu',m\right) = \varepsilon(c\tau+d)^{|\Delta_+|}$$

$$\times \sum_{\substack{\alpha\in M^* \\ c\alpha(\mathrm{mod}\,(m+g)M)}} \sum_{\substack{\alpha'\in M^* \\ c\alpha'(\mathrm{mod}\,mM)}} \exp\left\{\frac{\pi i}{m+g}[(a\mu+c\alpha+2\beta, b\mu+d\alpha)-(\mu,\alpha)]\right\}$$

$$\overline{\times \exp\left\{\frac{\pi i}{m}[(a\mu'+c\alpha'+2\beta', b\mu'+d\alpha')-(\mu',\alpha')]\right\}} F(\tau) \times c(\tau \mid a\mu+c\alpha+\beta, a\mu'+c\alpha'+\beta', m)$$

where $F(\tau)$ *is defined by formula* (1) *of Section* 19.7.7.

(4) *If* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(Nm)\cap\Gamma_0(N(m+g))$, *where*

$$\Gamma_0(r) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, c\equiv 0(\mathit{mod}\, r)\right\},$$

and, in addition, $b \equiv 0(\mathit{mod}\, a_0)$, *then*

$$F\left(\frac{a\tau+b}{c\tau+d}\right) c\left(\frac{a\tau+b}{c\tau+d}\middle|\mu,\mu',m\right) = \exp\left[\pi i ab\left(\frac{|\mu|^2}{m+g}-\frac{|\mu'|^2}{m}\right)\right] \varepsilon(c\tau+d)^{|\Delta_+|}F(\tau)c(\tau\mid a\mu, a\mu', m),$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } \ell \text{ is even,} \\ \frac{1}{d}m(m+g) & \text{if } \ell \text{ is odd.} \end{cases}$$

(5) *If* $a_0 = 1$, *then* $F(\tau)c(\tau \mid \mu,\mu',m)$ *is a cusp form of weight* $|\Delta_+|$ *for the discrete group* $\Gamma(Nm)\cap\Gamma(N(m+g))$ *with trivial multiplier system. If* $a_0 = 2$, *then* $F(\tau)c(\tau\mid\mu,\mu',m)$ *is a cusp form of weight* $\frac{1}{2}\ell(2\ell+1)$ *for the same group with the multiplier system* $\left(\frac{c}{d}\right)^\ell$ *where* $\left(\frac{c}{d}\right)$ *is the extended Legendre-Jacobi symbol.*[3]

[3] For relatively prime integers n and m with m odd or with $n\equiv 1(\mathrm{mod}\ 4\mathbb{Z})$ the *extended Legendre-Jacobi symbol* $\left(\frac{n}{m}\right)$, which is equal to $+1$ or -1, is defined as follows. For $n=0$ we put $\left(\frac{0}{\pm 1}\right) = \pm 1$

(6) *The linear span of all string functions for all highest weights Λ from P_+ such that $\Lambda(c) = m$ is invariant with respect to the action*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(\tau) = (c\tau + d)^{\ell/2} f\left(\frac{a\tau + b}{c\tau + d}\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(k^*).$$

Proof of this theorem is given by Kac and Peterson [138].

19.8.2. The matrix of the string functions. Let $m \in \mathbb{Z}_+$ and

$$P_+^{(m)} = \{\lambda \in P_+(\text{mod } \mathbb{C}\delta) \mid \lambda(c) = m\}.$$

According to formula (14) of Section 19.7.4 and Theorem 1 of Section 19.7.5 we can write

$$A_\rho^{-1} A_{\Lambda+\rho} = \sum_{\lambda \in P_+^{(m)}} c_\lambda^\Lambda S_\lambda.$$

Therefore, we may consider $(c_\lambda^\Lambda)_{\Lambda,\lambda \in P_+^{(m)}}$ as the matrix of the $\tilde{\mathfrak{F}}_0$-linear isomorphism from $\tilde{\mathfrak{F}}_{m+g}^-$ onto $\tilde{\mathfrak{F}}_m^+$, given by the formula $F \to A_\rho^{-1} F$.

In order to formulate the theorem on the determinant of the matrix (c_λ^Λ) we introduce the function

$$G(\tau) = e^{2\pi i R\tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})^{\text{mult } (n\delta)}, \tag{1}$$

where $R = |\bar{\rho}|^2/2g(h+1)$ for $k = 1$ and $R = |\bar{\rho}|^2/2g(g+1)$ for $k \neq 1$. This function can be easily evaluated for every of affine Lie algebras. We have

$$G(\tau) = \eta(\tau)^\ell$$

If $n \neq 0$, then we define an automorphism σ_m of the field $\mathbf{F} = \bigcup_N \mathbb{Q}(e^{2\pi i/N})$ (where N runs over all positive integers relatively prime to m) by the formula $\sigma_m(e^{2\pi i/N}) = e^{2\pi i m/N}$. For $\sqrt{n}$ we have

$$\sqrt{n} = e^{-\pi i/4} \sum_{\substack{p \in \mathbb{Z}(\text{mod } 2n\mathbb{Z}) \\ p \equiv n(\text{mod } 2\mathbb{Z})}} e^{\pi i p^2/4n}.$$

Therefore, $\sqrt{n} \in \mathbf{F}$. Now $\left(\frac{n}{m}\right)$ is defined by the formula

$$\sigma_m(\sqrt{n}) = \left(\frac{n}{m}\right)\sqrt{n}.$$

for all non-twisted Lie algebras and for $A_{2\ell}^{(2)}$,

$$\begin{aligned} G(\tau) &= \eta(\tau)^{\ell-1}\eta(2\tau) \quad \text{for } A_{2\ell-1}^{(2)}, \\ G(\tau) &= \eta(\tau)\eta(2\tau)^{\ell-1} \quad \text{for } D_{\ell+1}^{(2)}, \\ G(\tau) &= \eta(\tau)^2\eta(2\tau)^2 \quad \text{for } E_6^{(2)}, \\ G(\tau) &= \eta(\tau)\eta(3\tau) \quad \text{for } D_4^{(3)}. \end{aligned}$$

Theorem 1. *For every $m \in \mathbb{Z}_+$ we have*

$$\det \left(c_\lambda^\Lambda\right)_{\Lambda,\lambda\in P_+^{(m)}} = G^{-|P_+^{(m)}|}. \tag{2}$$

Proof of this theorem can be found in [138].

Let $\left(d_\Lambda^\lambda\right)_{\Lambda,\lambda\in P_+^{(m)}}$ be the inverse matrix for $\left(c_\lambda^\Lambda\right)_{\Lambda,\lambda\in P_+^{(m)}}$. It is the matrix of the isomorphism $F \to A_\rho F$ from $\tilde{\mathfrak{F}}_m^+$ onto $\tilde{\mathfrak{F}}_{m+g}^-$ defined by the formula

$$A_\rho S_\lambda = \sum_{\Lambda\in P_+^{(m)}} d_\Lambda^\lambda A_{\Lambda+\rho}.$$

For $\lambda \in P$, $\lambda(c) = m > 0$ we write

$$A_\rho \Theta_\lambda = \sum_{\substack{\Lambda\in P(\mathrm{mod}\,\mathbb{C}\delta)(\mathrm{mod}\,\mathcal{T}) \\ \Lambda(c)=m+g}} a_\Lambda^\lambda \Theta_\Lambda, \tag{3}$$

where a_Λ^λ is a holomorphic function on H, that is $a_\Lambda^\lambda \in \tilde{\mathcal{F}}_0$. Comparing coefficients at $\exp[-2\pi i(\bar{\Lambda}, z)]$ one has

$$\begin{aligned} a_\Lambda^\lambda &= = \sum_{\substack{T\in\mathcal{T} \\ w\in\hat{W}}} \varepsilon(w) q^{|\overline{T\lambda}|^2/2m+|\overline{w\rho}|^2/2g-|\bar{\Lambda}|^2/2(m+g)} \\ &= \sum_{T\in\mathcal{T}} \varepsilon(\bar{\Lambda} - \overline{T\lambda}) q^{[m(m+g)/2g]|(m+g)^{-1}\bar{\Lambda}-m^{-1}\overline{T\lambda}|^2} \\ &= \sum_{T\in\mathcal{T}} \varepsilon(\overline{T\Lambda} - \bar{\lambda}) q^{[m(m+g)/2g]|(m+g)^{-1}\overline{T\Lambda}-m^{-1}\bar{\lambda}|^2}, \end{aligned}$$

where $\varepsilon(\mu) = \varepsilon(w)$ if $\mu = \overline{w\rho}$ and $\varepsilon(\mu) = 0$ if $\mu \overline{\in} \overline{\hat{W}\rho}$. Therefore, for $\lambda, \Lambda \in P_+^{(m)}$ we have

$$\begin{aligned} d_\Lambda^\lambda &= \sum_{w\in\hat{W}/T\hat{W}_\lambda} a_{\lambda+\rho}^{w\lambda} \\ &= \sum_{\mu\in\hat{W}\lambda} \varepsilon(\bar{\Lambda} + \bar{\lambda} - \bar{\mu}) q^{[m(m+g)/2g]|(\bar{\Lambda}-\bar{\lambda})(m+g)^{-1}-m^{-1}\bar{\mu}|^2}. \end{aligned} \tag{4}$$

Let us fix $\beta \in M^*$. Then it follows from formula (3) that

$$\sum_{\substack{\lambda \in P(\mathrm{mod}\,\mathbb{C}\delta)(\mathrm{mod}\,T) \\ \lambda(c)=m}} e^{2\pi i m^{-1}(\beta,\bar{\lambda})} a^{\lambda}_{\Lambda+\rho}$$

$$= e^{2\pi i m^{-1}(\beta,\bar{\Lambda})} q^{[g(m+g)/2m]|g^{-1}\bar{\rho}-(m+g)^{-1}(\bar{\Lambda}+\bar{\lambda})|^2}$$

$$\times \prod_{\alpha\in\hat{\Delta}_+} \left(1 - e^{2\pi i m^{-1}(\alpha,\beta+\tau(\Lambda+\rho))}\right)^{m(\alpha)}, \tag{5}$$

where $\Lambda \in P$, $\Lambda(c) = m > 0$, $m(\alpha) = \text{mult}\,(\alpha)$, and $q = e^{2\pi i\tau}$. We also have

$$\sum_{\substack{(\Lambda+\rho) \in P(\mathrm{mod}\,\mathbb{C}\delta)(\mathrm{mod}\,T) \\ (\Lambda+\rho)(c)=m+g}} e^{-2\pi i(m+g)^{-1}(\beta,\bar{\Lambda}+\bar{\rho})} a^{\lambda}_{\Lambda+\rho}$$

$$= e^{-2\pi i(m+g)^{-1}(\beta,\bar{\lambda}+\bar{\rho})} q^{[mg/2(m+g)]|g^{-1}\bar{\rho}-m^{-1}\bar{\lambda}|^2}$$

$$\times \prod_{\alpha\in\hat{\Delta}_+} \left(1 - e^{2\pi i(m+g)^{-1}(\alpha,\beta+\tau\lambda)}\right)^{m(\alpha)}, \tag{6}$$

These formulas show that d^{λ}_{Λ} are closely related to q-specialization of the function A_{ρ}. By putting $\beta = 0$ into formula (5) we derive that

$$\sum_{\lambda\in P^{(m)}_+} d^{\lambda}_{\Lambda} = q^{[g(m+g)/2m]|g^{-1}\bar{\rho}-(m+g)^{-1}(\bar{\Lambda}+\bar{\rho})|^2}$$

$$\times \prod_{\alpha\in\hat{\Delta}_+} \left(1 - q^{m^{-1}(\alpha,\Lambda+\rho)}\right)^{m(\alpha)}, \tag{7}$$

where $\Lambda \in P^{(m)}_+$. If max $(\Lambda) = \hat{W}\Lambda$, then

$$d^{\Lambda}_{\Lambda} = \sum_{\lambda\in P^{(m)}_+} d^{\lambda}_{\Lambda}$$

and we have $c^{\Lambda}_{\Lambda} = \left(d^{\Lambda}_{\Lambda}\right)^{-1}$. In the paper [138] it is shown that

$$d^{m\Lambda_0}_{\Lambda} = \varepsilon(a\bar{\rho}) q^{[mg(m+g)/2]|g^{-1}\bar{\rho}+b(m+g)^{-1}(\bar{\Lambda}+\bar{\rho})|^2}$$

$$\times \prod_{\alpha\in\hat{\Delta}_+} \left(1 - q^{m(\alpha,(m+g)\Lambda_0-b(\bar{\Lambda}+\bar{\rho}))}\right)^{m(\alpha)}, \tag{8}$$

where $m \in \mathbb{Z}_+$ and a, b are integers such that $bg + a(m+g) = 1$. This formula is valid either if $k \neq 1$ or if $k = 1$ and all roots $\alpha \in \Delta$ are of the same length, as well as if m and g have no common divisors.

19.8.3. Explicit expressions for the string functions. We give here expressions for the string functions c^Λ_λ for simplest irreducible representations L_Λ. The highest weight Λ is characterized by the numbers $\Lambda(h_0), \Lambda(h_1), \dots, \Lambda(h_\ell)$ and the weight λ by the numbers $\lambda(h_0), \lambda(h_1), \dots, \lambda(h_\ell)$. In this reason c^Λ_λ will be also denoted by $c^{m_0 m_1 \dots m_\ell}_{n_0 n_1 \dots n_\ell}$ where $m_i = \Lambda(h_i)$ and $n_i = \lambda(h_i)$. Remind, that the number $\sum\limits_{i=0}^{\ell} a_i^\vee \Lambda(h_i)$ is called the level of the representation L_Λ. We have

$$\Lambda = a_0^{-1}\Lambda(d)\delta + \sum_{i=0}^{\ell} \Lambda(h_i)\Lambda_i,$$

where Λ_i are the fundamental weights.

We restrict ourselves by those affine Lie algebras $\hat{\mathfrak{g}}(A)$ for which $k \neq 1$ or for which $k = 1$ and Δ consists of roots of the same length (in this case $a_0^\vee = 1$). It is easy to show that for these affine Lie algebras all highest weights of level 1 have the form $\Lambda_j + \mathbf{c}\delta$. But all Λ_j are non-congruent mod $(Q + \mathbb{C}\delta)$. In this case all non-zero string functions c^Λ_λ for $\Lambda \in P_+$ of level 1 are equal to $c(\tau) \equiv c^{\Lambda_0}_{\Lambda_0}(\tau)$ (it is of level 1 since $a_0^\vee = 1$). It follows from Theorem 1 of Section 19.8.2 that

$$c(\tau) = G(\tau)^{-1}.$$

Taking into account the definition of the string function we can write that

$$\sum_{n=0}^{\infty} \mathrm{mult}_\Lambda(\Lambda - n\delta) q^n = \prod_{j=1}^{\infty} \left(1 - q^j\right)^{-m(j\delta)}, \quad q = e^{2\pi i \tau}.$$

Therefore, according to formula (14) of Section 19.7.4 the character ch L_Λ may be represented as

$$e^{-|\Lambda|^2\delta/2} \mathrm{ch}\, L_\Lambda = \frac{\sum\limits_{\gamma \in M + \bar{\Lambda}} e^{\Lambda_0 + \gamma - |\gamma|^2\delta/2}}{\prod\limits_{j=1}^{\infty} (1 - q^{-j\delta})^{m(j\delta)}}$$

if level of the weight Λ is 1. In particular, for $k = 1$ (that is, for the algebras $\hat{\mathfrak{g}}(A) = A_\ell^{(1)}, D_\ell^{(1)}, E_\ell^{(1)}$) we have

$$\mathrm{ch}\, L_{\Lambda_0} = \frac{\sum\limits_{\gamma \in M} e^{\Lambda_0 + \gamma - |\gamma|^2\delta/2}}{\prod\limits_{j=1}^{\infty} (1 - q^{-j\delta})^{\ell}}.$$

This formula can be used for an explicit construction of the representation L_{Λ_0}.

Let now $\mathfrak{g}(A) = B_\ell^{(1)}$, $\ell \geq 3$. In this case $\Lambda \in P_+(\text{mod } \mathbb{C}\delta)$ is of level 1 if Λ coincides with one of the weights Λ_0, Λ_1, Λ_ℓ. Maximal weights of the corresponding representations L_Λ are conjugates (with respect to $\hat{W}$) of either Λ or $\Lambda_1 - \delta$ if $\Lambda = \Lambda_0$ and of Λ_0 if $\Lambda = \Lambda_1$. Consequently, the functions

$$c_{\Lambda_1}^{\Lambda_1} = c_{\Lambda_0}^{\Lambda_0}, \quad c_{\Lambda_0}^{\Lambda_1} = c_{\Lambda_1}^{\Lambda_0}, \quad c_{\Lambda_\ell}^{\Lambda_\ell},$$

exhaust all non-zero string functions c_λ^Λ of level 1. The initial powers of q in the decompositions of c_λ^Λ, that is, the characteristics $s_\Lambda(\lambda)$, are equal to

$$-\frac{2\ell+1}{48} \quad \text{for} \quad c_{\Lambda_1}^{\Lambda_1}, \quad \frac{1}{2} - \frac{2\ell+1}{48} \quad \text{for} \quad c_{\Lambda_0}^{\Lambda_1}, \quad \frac{1-\ell}{24} \quad \text{for} \quad c_{\Lambda_\ell}^{\Lambda_\ell}. \tag{1}$$

We put

$$A(\tau) = \eta(\tau)^{\ell+1}\eta(2\tau)^{-1}c_{\Lambda_\ell}^{\Lambda_\ell}(\tau).$$

Then $A(\tau+1) = A(\tau)$. By using the equality

$$c_{\Lambda_\ell}^{\Lambda_\ell}\left(-\frac{1}{\tau}\right) = (-i\tau)^{-\ell/2}2^{-1/2}\left(c_{\Lambda_1}^{\Lambda_1} - c_{\Lambda_0}^{\Lambda_1}\right)(\tau)$$

from part (1) of Theorem 1 of Section 19.8.1 we have $A\left(-\frac{1}{\tau}\right) = A\left(-\frac{1}{\tau+2}\right)$. Therefore

$$A\left(\frac{a\tau+b}{c\tau+d}\right) = A(\tau)$$

if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is one of the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices generate the group $\Gamma_0(2)$. Consequently,

$$A\left(\frac{a\tau+b}{c\tau+d}\right) = A(\tau) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2).$$

Using formula (1) we verify that $A(\tau)$ is holomorphic at the cusp points $i\infty$ and 0 of the group $\Gamma_0(2)$. Thus, $A(\tau) = \text{const}$. Since $\text{mult}_{\Lambda_\ell}(\Lambda_\ell) = 1$, then const $= 1$. Thus

$$c_{\Lambda_\ell}^{\Lambda_\ell} = \eta(\tau)^{-\ell-1}\eta(2\tau). \tag{2}$$

Replacing τ by $-1/\tau$ in this equality and using the property $\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau)$ of Dedekind's η-function we obtain

$$c_{\Lambda_1}^{\Lambda_1}(\tau) - c_{\Lambda_0}^{\Lambda_1}(\tau) = \eta(\tau)^{-\ell-1}\eta\left(\frac{\tau}{2}\right). \tag{3}$$

Replacing here τ by $\tau+1$ we have

$$c^{\Lambda_1}_{\Lambda_1}(\tau)+c^{\Lambda_1}_{\Lambda_0}(\tau)=\eta\left(\frac{\tau}{2}\right)^{-1}\eta(\tau)^{2-\ell}\eta(2\tau)^{-1}. \tag{4}$$

Formulas (2)–(4) give all string functions of level 1 for the affine Lie algebra $B_\ell^{(1)}$.

Using formula (14) of Section 19.7.4 we can write down formulas for characters of two representations of $B_\ell^{(1)}$:

$$\operatorname{ch} L_{\Lambda_\ell}=\frac{\sum\limits_{\gamma\in\bar{Q}}\exp\left[\Lambda_\ell+\gamma-|\gamma|^2\delta/2-(\gamma,\Lambda_\ell)\delta\right]}{\prod\limits_{j=1}^{\infty}(1-e^{-j\delta})^\ell\left(1-e^{-(2j-1)\delta}\right)},$$

$$\operatorname{ch} L_{\Lambda_0}+\operatorname{ch} L_{\Lambda_1-\delta/2}=\frac{\sum\limits_{\gamma\in\bar{Q}}\exp\left(\Lambda_0+\gamma-|\gamma|^2\delta/2\right)}{\prod\limits_{j=1}^{\infty}\left(1-e^{-j\delta/2}\right)(1-e^{-j\delta})^{\ell-2}\left(1-e^{-2j\delta}\right)}.$$

An analogous evaluations for string functions of the algebra $A_1^{(1)}$, with using the equality $c^{m_1m_2}_{n_1n_2}=c^{m_2m_1}_{n_2n_1}$ as well as the first few terms in the expressions of the string functions and Theorem 1 of Section 19.8.1, leads to explicit expressions for many special cases of c^Λ_λ. We have the formulas

$$c^{10}_{10}=\eta(\tau)^{-1}$$

for level 1,

$$c^{11}_{11}=\eta(\tau)^{-2}\eta(2\tau)$$

$$c^{20}_{20}-c^{20}_{02}=\eta(\tau^{-2})\eta\left(\frac{\tau}{2}\right)$$

for level 2,

$$c^{21}_{21}=\eta(\tau)^{-2}q^{3/10}\prod_{\substack{n=1\\ n\not\equiv\pm1\ (\mathrm{mod}\ 5)}}^{\infty}(1-q^{3n}),$$

$$c^{30}_{12}=\eta(\tau)^{-2}q^{27/40}\prod_{\substack{n=1\\ n\not\equiv\pm2\ (\mathrm{mod}\ 5)}}^{\infty}(1-q^{3n}),$$

$$c^{21}_{21}-c^{21}_{03}=\eta(\tau)^{-2}q^{3/40}\prod_{\substack{n=1\\ n\not\equiv\pm2\ (\mathrm{mod}\ 5)}}^{\infty}(1-q^{n/3})$$

for level 3,

$$c^{40}_{22}=\eta(\tau)^{-2}\eta(6\tau)^{-1}\eta(12\tau)^2,$$

$$c^{40}_{40} - c^{40}_{04} = \eta(2\tau)^{-1},$$
$$c^{31}_{31} - c^{31}_{13} = \eta\left(\frac{\tau}{2}\right)^{-1}$$

for level 4,

$$c^{pq}_{rs} = (-1)^p \eta(\tau)^{-3} \sum_{\substack{m,n=1 \\ m\equiv r \pmod 5 \\ n\equiv(2p+2) \pmod 7 \\ 7m^2+5n^2\equiv 4 \pmod{16}}}^{\infty} \varepsilon(m,n) q^{(7m^2+5n^2)/560}$$

for level 5, where $p+q=r+s=5$, $p\equiv r \pmod 2$ and

$$\varepsilon(m,n) \equiv \exp[2\pi i(m-n+2)/8] \quad \text{if } n \text{ is even},$$
$$\varepsilon(m,n) \equiv \exp[2\pi i(n+1)/4] \quad \text{if } n \text{ is odd},$$

and the formulas

$$c^{33}_{51} = \eta(\tau)^{-3}\eta(2\tau)\eta(3\tau)\eta(6\tau)^{-1}\eta(12\tau),$$
$$c^{51}_{51} + c^{51}_{15} = \eta(\tau)^{-3}\eta(2\tau)\eta(6\tau)^2\eta(12\tau)^{-1},$$
$$c^{51}_{51} - c^{51}_{15} = \eta(\tau)^{-1},$$
$$c^{51}_{51} - 2c^{51}_{33} + c^{51}_{15} = \eta\left(\frac{2\tau}{3}\right)^2 \eta(\tau)^{-3}\eta\left(\frac{4\tau}{3}\right)^{-1}\eta(2\tau),$$
$$c^{33}_{33} - c^{33}_{51} = \eta\left(\frac{\tau}{3}\right)\eta\left(\frac{2\tau}{3}\right)^{-1}\eta(\tau)^{-3}\eta\left(\frac{4\tau}{3}\right)\eta(2\tau)$$

for level 6,

$$c^{52}_{70} = \eta(\tau)^{-3} \sum_{\substack{m,n=0 \\ m\not\equiv n \pmod 2}}^{\infty} e^{2\pi i m/6} q^{(m^2+21n^2)/4},$$

$$c^{52}_{52} = \eta(\tau)^{-3} q^{3/28} \prod_{\substack{n=1 \\ n\equiv 0,\pm 1 \pmod 7}}^{\infty} (1-q^n) \prod_{\substack{n=1 \\ n\not\equiv 1 \pmod 7}} (1-q^{3n})$$

for level 7, and

$$c^{44}_{62} = \eta(\tau)^{-3}\eta(2\tau)\eta(10\tau)$$

for level 8. In the paper [138] other string functions can be found for the algebra $A^{(1)}_1$.

Let us write down some expressions for string functions for the algebra $A^{(1)}_2$:

$$c^{100}_{100} = \eta(\tau)^{-2}$$

for level 1,

$$c_{110}^{110} = \eta(\tau)^{-4}\eta(2\tau)q^{1/20} \prod_{\substack{n=1 \\ n\not\equiv \pm 1 \pmod 5}}^{\infty} (1-q^{2n}),$$

$$c_{200}^{200} - c_{011}^{200} = \eta(\tau)^{-4}\eta\left(\frac{\tau}{2}\right) q^{1/20} \prod_{\substack{n=1 \\ n\not\equiv \pm 1 \pmod 5}}^{\infty} (1-q^{n/2}),$$

$$c_{011}^{200} = \eta(\tau)^{-4}\eta(2\tau)q^{9/20} \prod_{\substack{n=1 \\ n\not\equiv \pm 2 \pmod 5}}^{\infty} (1-q^{2n})$$

for level 2, and

$$c_{111}^{111} = \eta(\tau)^{-6}\eta(2\tau)^3\eta(3\tau)^2\eta(6\tau)^{-1}$$

for level 3. For the affine Lie algebra $A_2^{(2)}$ we have

$$c_{10}^{10} = \eta(\tau)^{-1}$$

for level 1,

$$c_{20}^{20} = \eta(\tau)^{-2}\eta(2\tau)\eta(4\tau)^{-1}q^{1/10} \prod_{\substack{n=1 \\ n\not\equiv \pm 1 \pmod 5}}^{\infty} (1-q^{4n}),$$

$$c_{20}^{01} = \eta(\tau)^{-2}\eta(2\tau)\eta(4\tau)^{-1}q^{9/10} \prod_{\substack{n=1 \\ n\not\equiv \pm 2 \pmod 5}}^{\infty} (1-q^{4n}),$$

$$c_{20}^{20} + c_{01}^{20} = \eta(\tau)^{-2}\eta\left(\frac{\tau}{2}\right)\eta\left(\frac{\tau}{4}\right)^{-1} q^{1/160} \prod_{\substack{n=1 \\ n\not\equiv \pm 2 \pmod 5}}^{\infty} (1-q^{n/4}),$$

$$c_{01}^{01} + c_{20}^{01} = \eta(\tau)^{-2}\eta\left(\frac{\tau}{2}\right)\eta\left(\frac{\tau}{4}\right)^{-1} q^{9/160} \prod_{\substack{n=1 \\ n\not\equiv \pm 2 \pmod 5}}^{\infty} (1-q^{n/4}),$$

$$c_{30}^{11} = 2\eta(\tau)^{-2}\eta(6\tau)^{-1}\eta(12\tau)^2$$

for level 2.

19.8.4. Formulas for the partition function. According to formula (10) of Section 19.5.4 the string function is defined with the help of multiplicities $\mathrm{mult}_\Lambda(\lambda)$ of weights λ in the representation L_Λ. Kac [134] has proved that multiplicity $\mathrm{mult}_\Lambda(\lambda)$ can be represented in terms of the partition function K (see Section 19.4.9):

$$\mathrm{mult}_\Lambda(\lambda) = \sum_{w\in\hat{W}} \varepsilon(w)K(w(\Lambda+\rho)-(\lambda+\rho)). \tag{1}$$

Therefore, for affine Lie algebras of low rank the string function can be evaluated with the help of the partition function. Here we consider the partition function for the affine Lie algebra $A_\ell^{(1)}$.

The Weyl group $\hat{W}$ of the algebra $A_\ell^{(1)}$ is generated by the reflections $S_i \equiv S_{\alpha_i}$, $i = 0, 1, \dots, \ell$, where $\alpha_0, \alpha_1, \dots, \alpha_\ell$ are the simple roots. The space $\hat{\mathfrak{h}}'$, dual to $\hat{\mathfrak{h}}$, is generated by the linear forms $\alpha_0, \alpha_1, \dots, \alpha_\ell, \rho$. We define the element $\sigma \in GL(\hat{\mathfrak{h}}')$ by the formulas

$$\sigma(\alpha_i) = \alpha_{i+1}, \quad i = 0, 1, \dots, \ell - 1, \quad \sigma(\alpha_\ell) = \alpha_0, \quad \sigma(\rho) = \rho.$$

Then $\sigma \hat{W} \sigma^{-1} \subset \hat{W}$ and we can construct the group $W_\sigma = \langle \sigma \rangle \times \hat{W}$ (semidirect product), where $\langle \sigma \rangle$ is the subgroup of $GL(\hat{\mathfrak{h}}')$, generated by σ. The group W_σ leaves the sets $\hat{\Delta}$, $Q = \sum_{i=0}^{\ell} \mathbb{Z}\alpha_i$ and $Q + \rho$ invariant. We define the new action of W_σ on Q by the formula

$$w \cdot \alpha = w(\alpha + \rho) - \rho.$$

This formula allows us to define the corresponding action of W_σ on the set of functions given on Q. In particular, with the help of this action we define the function K' on Q, connected with the partition function K:

$$K' = (1 + S_\ell)(1 - S_{\ell-1}S_\ell) \dots (1 - (-1)^\ell S_1 \dots S_\ell)K.$$

Let us introduce additional notations. If $\alpha \in Q$, then $n_i(\alpha)$, $i = 0, 1, \dots, \ell$, will denote coefficients of the expansion

$$\alpha = n_0(\alpha)\alpha_0 + n_1(\alpha)\alpha_1 + \dots + n_\ell(\alpha)\alpha_\ell,$$

where $\alpha_0, \alpha_1 \dots, \alpha_\ell$ are the simple roots. Remind, that $\sigma_k(q)$ denote the function

$$\sigma_k(q) = \sum_{n=1}^{\infty} \left(\sum_{d|n} d^k \right) q^n,$$

where the second summation is over all divisors of number n.

Statement 1. *There exists polynomial R_ℓ of $\ell + 2$ variables such that for $\beta \in \bar{Q} = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i$ and for q, $|q| < 1$, we have*

$$\varphi(q)^{\ell(\ell+2)} \sum_{n=1}^{\infty} K'(\beta + n\delta) q^n = R_\ell(n_1(\beta), \dots, n_{\ell-1}(\beta), \sigma_1(q), \sigma_3(q), \sigma_5(q)), \quad (2)$$

where $\varphi(q) = \prod_{n=1}^{\infty}(1-q^n) = (q;q)_\infty$. *The polynomials* R_1 *and* R_2 *are of the form*

$$R_1 = 1, \quad R_2 = n_1(\beta) + \frac{3}{2}. \tag{3}$$

If $w = \sigma S_\ell S_{\ell-1} \dots S_1$, *then*

$$\sum_{n \in Z} (-1)^{n\ell} K'(w^n \alpha) = 0, \tag{4}$$

where only a finite number of summands are non-zero.

Proof of this statement can be found in [138].

Let

$$w_i' = S_\ell S_{\ell-1} \dots S_{\ell-i+1}, \quad w_i = \sigma w_i', \quad t_i = w_i^{\ell-i+1}.$$

Direct evaluation shows that

1) If $1 \le i < j \le \ell$, then $w_i' w_j = w_j w_i'$, $t_i w_j = w_j t_i$;
2) $t_i = T_{\bar{\Lambda}_{\ell-i+1}}$, $i = 1, 2, \dots, \ell$, where the element T_γ is defined by the formula (5) of Section 19.7.2;
3) $t_i(\alpha_0) = \alpha_0 + \delta$, $t_i(\alpha_{\ell-i+1}) = \alpha_{\ell-i+1} - \delta$ and $t_i(\alpha_j) = \alpha_j$ for $j \ne 0, \ell - i + 1$.

We put

$$\mathcal{T}_+ = \left\{ t_\ell^{k_\ell} \dots t_1^{k_1} \mid k_1, k_2, \dots, k_\ell \in \mathbb{Z}_+ \cup \{0\} \right\},$$

$$\Phi = \left\{ w_\ell^{k_\ell} \dots w_1^{k_1} \mid 0 \le k_i \le \ell - i \right\}.$$

Elements from $\mathcal{T}_+$ and Φ generate the group W_σ. By putting $\chi(\sigma) = 1$ and $\chi(w) = \det w$ for $w \in \hat{W}$ we obtain a character χ of the group W_σ.

The following theorem is proved in [138].

Theorem 1. *If* $\hat{\mathfrak{g}}(A) = A_\ell^{(1)}$, *then for all* $\alpha \in Q$ *we have*

$$K(\alpha) = \sum_{\substack{t \in \mathcal{T}_+ \\ \varphi \in \Phi}} \chi(t\varphi) K'(t\varphi(\alpha + \rho) - \rho). \tag{5}$$

This theorem and Statement 1 can be used for derivation of explicit formulas for the partition function K for affine Lie algebras of low rank. If $\ell = 1$, then for $A_1^{(1)}$ we have

$$\hat{\Delta}_+ = \{ n_0 \alpha_0 + n_1 \alpha_1 \mid n_0, n_1 \in \mathbb{Z}_+ \cup \{0\}, |n_1 - n_0| \le 1, (n_0, n_1) \ne (0,0) \}.$$

Imaginary roots have the form $n\delta$, $n \in \mathbb{Z}$, where $\delta = \alpha_0 - \alpha_1$. Multiplicities of all roots are equal to 1. In this case $\Phi = \{1\}$, $\mathcal{T}_+ = \{t^k \mid k \geq 0\}$ and

$$t\alpha_0 = \alpha_0 + \delta, \quad t\alpha_1 = \alpha_1 - \delta, \quad t\rho = \rho - \alpha_0, \quad \chi(t) = -1.$$

We also have $R_1 = 1$. Consequently, according to formula (2) $K'(\beta) = p^{(3)}(n_0(\beta))$, where $p^{(r)}(n)$ is defined by the formula

$$\varphi(q)^{-r} = \sum_{n\in\mathbb{Z}} p^{(r)}(n)q^n, \quad |q| < 1.$$

Hence, using formulas (4) and (5) we have the following expressions for the partition function of the algebra $A_1^{(1)}$:

$$K(n_0\alpha_0 + n_1\alpha_1) = \sum_{k=0}^{\infty}(-1)^k p^{(3)}((k+1)n_0 - kn_1 - \frac{1}{2}k(k+1)), \tag{6}$$

$$K(n_0\alpha_0 + n_1\alpha_1) = -\sum_{k=-1}^{-\infty}(-1)^k p^{(3)}((k+1)n_0 - kn_1 - \frac{1}{2}k(k+1)). \tag{7}$$

In the same way it is shown that for the algebra $A_2^{(1)}$ we have

$$\begin{aligned} K(n_0\alpha_0 + n_1\alpha_1 + n_2\alpha_2) &= \sum_{k,\ell=0}^{\infty}(n_1 - n_0 + 2k + \ell + 1) \\ &\times p^{(8)}((k+\ell+1)n_0 - kn_1 - \ell n_2 - \ell(k+\ell+1) - k(k+1)) \\ &- \sum_{k,\ell=0}^{\infty}(n_2 - n_1 + 2k + \ell + 2) \\ &\times p^{(8)}((\ell+1)n_0 + (k+1)n_1 - (k+\ell+1)n_2 - (\ell+1)(k+\ell+1) - k(k+1)). \end{aligned}$$

19.8.5. Hecke modular forms and the string function for $A_1^{(1)}$. Let L be a lattice in $\mathbf{R}^2$ and let B be an indefinite bilinear form on $\mathbf{R}^2$ such that $B(\gamma,\gamma) \in 2\mathbb{Z}\backslash\{0\}$ for all non-zero $\gamma \in L$. We also introduce the lattice

$$L^* = \{\gamma' \in \mathbf{R}^2 \mid B(\gamma,\gamma') \in \mathbb{Z} \text{ for all } \gamma \in L\}.$$

Let G be the connected orthogonal group whose elements leave the form $B(\gamma,\gamma')$ invariant, and let G_0 be the subgroup of G consisting of elements $g \in G$ preserving L and fixing L^*/L. The form $B(\gamma,\gamma)$ can be represented as

$$B(\gamma,\gamma) = \ell_1(\gamma)\ell_2(\gamma), \tag{1}$$

where ℓ_1 and ℓ_2 are real linear forms on $\mathbf{R}^2$. We set

$$\operatorname{sign}\gamma = \operatorname{sign}\ell_1(\gamma) \quad \text{if} \quad \ell_1(\gamma) \neq 0. \tag{2}$$

For each $\boldsymbol{\mu} \in L^*$ we introduce the functions

$$\theta^B_{\boldsymbol{\mu}} = \sum_{\substack{\gamma \in L+\mu \\ B(\gamma,\gamma)>0 \\ \gamma (\mathrm{mod}\, G_0)}} (\operatorname{sign}\gamma) e^{\pi i \tau B(\gamma,\gamma)} \tag{3}$$

on H. They are modular forms of weight 1. Namely, Hecke has proved the following properties of these functions:

1) $\theta^B_{\boldsymbol{\mu}}(\tau+1) = e^{\pi i B(\boldsymbol{\mu},\boldsymbol{\mu})}\theta^B_{\boldsymbol{\mu}}(\tau)$

$$\theta^B_{\boldsymbol{\mu}}\left(-\frac{1}{\tau}\right) = -\frac{\tau}{\sqrt{|L^*/L|}} \sum_{\boldsymbol{\nu}\in L^*(\mathrm{mod}\ L)} e^{2\pi i B(\boldsymbol{\mu},\boldsymbol{\nu})}\theta^B_{\boldsymbol{\nu}}(\tau);$$

2) Let $N \in \mathbb{Z}_+$ be such that $NB(\gamma,\gamma) \in 2\mathbb{Z}$ for all $\gamma \in L^*$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, then there exists $\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$, such that

$$\theta^B_{\boldsymbol{\mu}}\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(c\tau+d)e^{\pi i a b B(\boldsymbol{\mu},\boldsymbol{\mu})}\theta^B_{a\boldsymbol{\mu}}(\tau)$$

for all $\boldsymbol{\mu} \in L^*$.

The functions $\theta^B_{\boldsymbol{\mu}}$ are called *Hecke indefinite modular forms*. Let us show that $\eta(\tau)^3 c^\Lambda_\lambda(\tau)$, where c^Λ_λ is the string function for the algebra $A_1^{(1)}$, coincides with the corresponding function $\theta^B_{\boldsymbol{\mu}}(\tau)$.

If $\hat{\mathfrak{g}}(A) = A_1^{(1)}$, then α_1, δ and ρ form a basis of the space $\hat{\mathfrak{h}}'$. The roots $\alpha \equiv \alpha_1$ and $\alpha_0 = \delta - \alpha_1$ are simple. Fundamental weights are

$$\Lambda_0 = \frac{1}{2}\rho - \frac{1}{4}\alpha_1, \quad \Lambda_1 = \frac{1}{2}\rho + \frac{1}{4}\alpha_1.$$

Restrictions $\bar{\alpha}'$ of the forms $\alpha' \in \hat{\mathfrak{h}}'$ onto the Cartan subalgebra $\mathfrak{h} \subset \hat{\mathfrak{h}}$ of the simple Lie algebra A_1 are of the form

$$\bar{\alpha}_1 = \alpha_1, \quad \bar{\delta} = 0, \quad \bar{\rho} = \frac{1}{2}\alpha_1, \quad \bar{\Lambda}_0 = 0, \quad \bar{\Lambda}_1 = \frac{1}{2}\alpha_1.$$

The lattice M (see Section 19.7.2) coincides with $\mathbb{Z}\alpha_1$. For the normalized symmetric bilinear form $(\cdot\,,\cdot)$ on $\hat{\mathfrak{h}}'$ we have

$$(\alpha_1,\alpha_1) = 2, \quad (\alpha_1,\rho) = 1, \quad (\rho,\rho) = \frac{1}{2}, \quad (\rho,\delta) = 2, \quad (\delta,\delta) = 0, \quad (\delta,\alpha_1) = 0.$$

Let us define the elements s and t of $GL(\hat{\mathfrak{h}}')$ by the formulas

$$s\alpha_1 = -\alpha_1, \quad s\delta = \delta, \quad s\rho = \rho - \alpha_1,$$
$$t\alpha_1 = \alpha_1 - \delta, \quad t\delta = \delta, \quad t\rho = \rho + \alpha_1 - \delta.$$

Then t coincides with element $T_{\alpha_1/2}$ determined by formula (5) of Section 19.7.2. The group W_σ is the semidirect product of the Weyl group $W = \{1, s\}$ of the simple Lie algebra A_1 with the free abelian invariant subgroup generated by the element t. The Weyl group $\hat{W}$ of the algebra $A_1^{(1)}$ is generated by the elements s and t^2.

We put $L = M \oplus M$. If $\mathbf{R}^2$ is identified with $U = \mathbf{R}\alpha_1 \oplus \mathbf{R}\alpha_1$, then the lattice L coincides with $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_1$. We define a quadratic form B on U by the formula

$$B((x,y)) = 2(m+2)x^2 - 2my^2, \quad (x,y) \in \mathbf{R}^2. \tag{4}$$

It is clear that B does not vanish on $L\backslash\{0\}$ and $B((x,y)) \in 2\mathbb{Z}$ for $(x,y) \in \mathbf{R}^2$.

The lattice

$$L^* = \frac{1}{2(m+2)}\mathbb{Z} \oplus \frac{1}{2m}\mathbb{Z}$$

is dual to L with respect to B. Let $SO(B)$ be the group of linear transformations of U conserving the form B, and let $SO_0(B)$ be the connected component of $SO(B)$ containing the identity transformation. Let a be the element of $SO_0(B)$ given by the matrix

$$a = \begin{pmatrix} m+1 & m \\ m+2 & m+1 \end{pmatrix}. \tag{5}$$

Then a acts on $(x,y) \in \mathbf{R}^2$ as

$$a(x,y) = ((m+1)x + my, (m+2)x + (m+1)y).$$

The matrix a generates the subgroup G_0' of $SO_0(B)$ which conserves the lattice L, and the matrix a^2 generates the subgroup G_0 of G_0' fixing the quotient group L^*/L.

If $U^+ = \{u \in U \simeq \mathbf{R}^2 \mid B(u) > 0\}$, then the subset

$$F = \{(x,y) \in \mathbf{R}^2 \mid -|x| < y \le |x|\} \tag{6}$$

is the fundamental domain for G_0' on U^+ and $F \cup aF$ is the fundamental domain for G_0 on U^+.

Now we consider the string function c_λ^Λ. It is given by the formula

$$c_\lambda^\Lambda(\tau) = q^{s_\Lambda(\lambda)} \sum_{j=1}^{\infty} \mathrm{mult}_\Lambda(\lambda - j\delta) q^j. \tag{7}$$

For the affine Lie algebra $A_\ell^{(1)}$ the function $s_\Lambda(\lambda)$ is equal to

$$s_\Lambda(\lambda) = n_0(\Lambda - \lambda) + \frac{|\bar{\Lambda} + \bar{\rho}|^2}{2(\ell+m+1)} - \frac{|\bar{\lambda}|^2}{2m} - \frac{\ell(\ell+2)}{24}, \tag{8}$$

where $m = \Lambda(c) > 0$ and $n_0(\mu)$ is determined by the expansion $\mu = n_0(\mu)\alpha_0 + \ldots + n_\ell(\mu)\alpha_\ell$. Let us show that $n_0(\mu)$ is connected with the quadratic form

$$B((\gamma, \gamma')) = (m + \ell + 1)(\gamma, \gamma) - m(\gamma', \gamma'), \quad (\gamma, \gamma') \in U_\ell,$$

on $U_\ell = \bar{\mathfrak{h}}'_R \oplus \bar{\mathfrak{h}}'_R$. (When $\ell = 1$, then this form is identified with the form (4).) If $(\gamma, \gamma') \in L = M \oplus M$, then $B((\gamma, \gamma')) \in 2\mathbb{Z}$. We have

$$\begin{aligned} n_0(T_\gamma(T_{\gamma'}(\Lambda + \rho) - \lambda) - \rho) = -n_0(\Lambda - \lambda) - \frac{|\bar{\Lambda} + \bar{\rho}|^2}{2(m + \ell + 1)} + \frac{|\bar{\lambda}|^2}{2m} \\ + \frac{1}{2} B\left(\left(\gamma + \gamma' + \frac{\bar{\Lambda} + \bar{\rho}}{m + \ell + 1}, \gamma + \frac{\bar{\lambda}}{m}\right)\right), \end{aligned} \tag{9}$$

where $T_\gamma \in \hat{W}$. In order to prove this equality we write down the left hand side in the form

$$n_0((\Lambda + \rho) - T_{\gamma+\gamma'}(\Lambda + \rho)) - n_0(\lambda - T_\gamma \lambda) - n_0(\Lambda - \lambda)$$

and apply formula (8) of Section 19.7.2. Simple calculations lead to formula (9).

Now we substitute the expression for $\mathrm{mult}_\Lambda(\lambda)$ from formula (1) of Section 19.8.4 into (7) and rearrange summands. We obtain

$$\varphi(q)^{\ell(\ell+2)} c_\lambda^\Lambda(\tau) = q^{s_\Lambda(\lambda)} \sum_{w \in \hat{W}} \varepsilon(w) S(w(\Lambda + \rho) - (\lambda + \rho)), \tag{10}$$

where

$$S(\beta) = \varphi(q)^{\ell(\ell+2)} \sum_{j=0}^{\infty} K(\beta + j\delta) q^j. \tag{11}$$

Lemma 1. *Let $\hat{\mathfrak{g}}(A) = A_1^{(1)}$ and $\beta = n_0\alpha_0 + n_1\alpha_1 \in Q$ where either $n_0 \leq 0$ or $n_1 \leq 0$. Then*

$$S(\beta) = \sum_{k=0}^{\infty} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)}, \tag{12}$$

$$S(\beta) = -\sum_{k=-1}^{-\infty} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)}. \tag{13}$$

Proof. Since $t = T_{\alpha_1/2}$, then from formula (8) of Section 19.7.2 we have

$$n_0(t^k(\beta + \rho) - \rho) = (k + 1)n_0 - kn_1 - k(k + 1)/2.$$

In particular,

$$n_0(t^k(\beta + \rho) - \rho) = n_0(t^{k'}(\beta + \rho) - \rho) \quad \text{if} \quad k + k' = 2n_0 - 2n_1 - 1.$$

Consequently,

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)} = 0.$$

Therefore, formulas (12) and (13) are equivalent. The conditions on n_0 and n_1 show that either $n_0 \le 0$ and $n_0 \le n_1$ or $n_1 \le 0$ and $n_1 \le n_0$. If $n_0 \le 0$ and $n_0 \le n_1$, then

$$(k+1)n_0 - kn_1 - k(k+1)/2 \le 0 \quad \text{for} \quad k \in \mathbb{Z}_+.$$

In this reason we have from formula (6) of Section 19.8.4 that

$$\begin{aligned} S(\beta) &= \varphi(q)^3 \sum_{j=0}^{\infty} K(\beta + j\delta) q^j \\ &= \varphi(q)^3 \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^k p^{(3)} \left((k+1)n_0 - kn_1 - \frac{k(k+1)}{2} + j \right) q^j \\ &= \sum_{k=0}^{\infty} (-1)^k \varphi(q)^3 \sum_{j=0}^{\infty} p^{(3)} \left((k+1)n_0 - kn_1 - \frac{k(k+1)}{2} + j \right) q^j \\ &= \sum_{k=0}^{\infty} (-1)^k q^{-(k+1)n_0 + kn_1 + k(k+1)/2} = \sum_{k=0}^{\infty} (-1)^k q^{-n_0(t^k(\beta+\rho)-\rho)}. \end{aligned}$$

This proves formula (12) for $n_0 \le 0$ and $n_0 \le n_1$. In order to prove this formula for $n_1 \le 0$ and $n_1 \le n_0$, formula (7) of Section 19.8.4 is used instead of formula (6) of the same section. Lemma is proved.

Theorem 1. *Let* $\hat{\mathfrak{g}}(A) = A_1^{(1)}$, $\lambda \in \max(\Lambda)$, $\Lambda \in P_+$, $\Lambda(c) = m > 0$, $\lambda \in P_+$. *Then*

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \theta_{\boldsymbol{\mu}}^B(\tau),$$

where $\boldsymbol{\mu} = ((m+2)^{-1}(\bar\Lambda + \bar\rho), m^{-1}\bar\lambda)$ *and* $\theta_{\boldsymbol{\mu}}^B$ *is the Hecke modular form.*

Proof. According to formula (10) one have

$$\begin{aligned} \eta(\tau)^3 c_\lambda^\Lambda(\tau) = q^{s_\Lambda(\lambda)+1/8} \Bigg\{ &\sum_{n=0}^{\infty} S(t^{2n}(\Lambda+\rho) - (\lambda+\rho)) \\ &+ \sum_{n=-1}^{-\infty} S(t^{2n}(\Lambda+\rho) - (\lambda+\rho)) - \sum_{n=1}^{\infty} S(t^{2n}s(\Lambda+\rho) - (\lambda+\rho)) \\ &- \sum_{n=0}^{-\infty} S(t^{2n}s(\Lambda+\rho) - (\lambda+\rho)), \end{aligned}$$

where $\Lambda \in P_+$, $m = \Lambda(c) > 0$, $\lambda \in P_+$, $\lambda \in \max(\Lambda)$. Let us apply formulas (12) and (13) to the summands of this equality. It is possible since $-\delta + w(\Lambda+\rho) - (\lambda+\rho) \overline{\in} Q_+$,

$w \in \hat{W}$ (see Statement 2 of Section 19.7.3). By using formula (9) we obtain the expression

$$\begin{aligned}\eta(\tau)^3 c_\lambda^\Lambda(\tau) &= \left(\sum_{k,n=0}^{\infty} - \sum_{k,n=-1}^{-\infty}\right)(-1)^k q^{B((\gamma,\gamma'))/2} \\ &\quad - \left(\sum_{k=0}^{\infty}\sum_{n=1}^{\infty} - \sum_{k=-1}^{-\infty}\sum_{n=0}^{-\infty}\right)(-1)^k q^{B((\gamma'',\gamma'))/2},\end{aligned} \tag{14}$$

where

$$\gamma = \left(\frac{k}{2}+n\right)\alpha_1 + \frac{\bar\Lambda+\bar\rho}{m+2}, \quad \gamma'' = \left(\frac{k}{2}+n\right)\alpha_1 - \frac{\bar\Lambda+\bar\rho}{m+2}, \quad \gamma' = \frac{k}{2}\alpha_1 + \frac{\bar\lambda}{m}.$$

This expansion is absolutely convergent.

The quadratic form B has the property $B((\beta,\beta')) = B((-\beta,\beta'))$. Applying this equality to the second summand in (14) we obtain

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \sum_{\substack{k,n\in(1/2)\mathbb{Z} \\ k\equiv n (\mathrm{mod}\ \mathbb{Z}) \\ -k>|n| \text{ or } k\geq|n|}} (-1)^{2k}\operatorname{sign}\left(k+\frac{1}{4}\right) q^{B((\gamma,\gamma'))/2},$$

where $\gamma = k\alpha_1 + (m-2)^{-1}(\bar\Lambda+\bar\rho)$, $\gamma' = n\alpha_1 + m^{-1}\bar\lambda$. We break the sum into two parts (one with integral k and the second with half-integral k) and apply the relation $B((\beta,\beta')) = B((-\beta,\beta'))$ to the second sum. We have

$$\begin{aligned}\eta(\tau)^3 c_\lambda^\Lambda(\tau) &= \sum_{\substack{k,n\in\mathbb{Z} \\ -k>|n| \text{ or } k\geq|n|}} \operatorname{sign}\left(k+\frac{1}{4}\right) q^{B((\gamma,\gamma'))/2} \\ &\quad + \sum_{\substack{k,n\in\mathbb{Z}+1/2 \\ -k\geq|n| \text{ or } k>|n|}} \operatorname{sign}\left(k-\frac{1}{4}\right) q^{B((\gamma'',\gamma'))/2},\end{aligned} \tag{15}$$

where

$$\gamma = k\alpha_1 + \frac{\bar\Lambda+\bar\rho}{m+2}, \quad \gamma'' = k\alpha_1 - \frac{\bar\Lambda+\bar\rho}{m+2}, \quad \gamma' = n\alpha_1 + \frac{\bar\lambda}{m}.$$

We have

$$0 < \mathrm{A} < \frac{1}{2}, \quad 0 \leq \mathrm{B} \leq \frac{1}{2} \quad \text{where} \quad \mathrm{A}\alpha_1 = \frac{\bar\Lambda+\bar\rho}{m+2}, \quad \mathrm{B}\alpha_1 = \frac{\bar\lambda}{m}.$$

Suppose that $\mathrm{A} \geq \mathrm{B}$. We may combine the sums in (15) to obtain

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \sum (\operatorname{sign} x) q^{B((x,y))/2},$$

where the summation is over

$$\left\{(x,y) \in F \mid (x,y) \equiv (\mathrm{A}\,,\mathrm{B}) \text{ or } \left(\frac{1}{2} - \mathrm{A}\,, \frac{1}{2} + \mathrm{B}\right) (\operatorname{mod} \mathbb{Z}^2)\right\}$$

or, equivalently, over

$$\{(x,y) \in F \mid (x,y) \equiv (\mathrm{A}\,,\mathrm{B}) \text{ or } a(\mathrm{A}\,,\mathrm{B}) \,(\operatorname{mod} \mathbb{Z}^2)\}\,.$$

(Here F is the fundamental domain defined by (6).) Since $F \cup aF$ is the fundamental domain for the group G_0, then we obtain

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \theta_{\boldsymbol{\mu}}^B(\tau), \tag{16}$$

where $\boldsymbol{\mu} = \left(\frac{\bar\lambda + \bar\rho}{m+2}, \frac{\bar\lambda}{m}\right)$.

Now we have to remove the restriction $\mathrm{A} \geq \mathrm{B}$. We write

$$\frac{\overline{\sigma\Lambda + \rho}}{m+2} = \mathrm{A}'\alpha_1, \quad \frac{\overline{\sigma\lambda}}{m} = \mathrm{B}'\alpha_1,$$

where $\sigma \in W_\sigma$. Then $\mathrm{A} + \mathrm{A}' = \mathrm{B} + \mathrm{B}' = 1/2$. Therefore, if $\mathrm{A} < \mathrm{B}$, then $\mathrm{A}' > \mathrm{B}'$ and we have

$$\begin{aligned}\eta(\tau)^3 c_\lambda^\Lambda(\tau) &= \eta(\tau)^3 c_{\sigma\lambda}^{\sigma\Lambda}(\tau) = \theta^B_{(\mathrm{A}'\alpha_1, \mathrm{B}'\alpha_1)}(\tau)\\ &= \theta^B_{(\bar\rho - \mathrm{A}\alpha_1, \bar\rho - \mathrm{B}\alpha_1)}(\tau) = \theta^B_{(\bar\rho - \mathrm{A}\alpha_1, -\bar\rho + \mathrm{B}\alpha_1)}(\tau)\\ &= \theta^B_{a(\mathrm{A}\alpha_1, \mathrm{B}\alpha_1)}(\tau) = \theta^B_{(\mathrm{A}\alpha_1, \mathrm{B}\alpha_1)}(\tau).\end{aligned}$$

Thus, formula (15) is also valid for $\mathrm{A} < \mathrm{B}$. Theorem is proved.

19.8.6. Applications of the string functions. Let $m \in \mathbb{Z}_+$. For integral N and n, $N \equiv n (\operatorname{mod} 2)$, we set

$$\tilde\theta_n^N(\tau) = \sum_{(x,y)\in E} (\operatorname{sign} x) q^{(m+2)x^2 - my^2}, \tag{1}$$

where $q = \exp 2\pi i \tau$, $\operatorname{Im} \tau > 0$,

$$E = \left\{(x,y) \in \mathbf{R}^2 \;\middle|\; -|x| < y \leq |x|; (x,y) \text{ or } \left(\frac{1}{2} - x, \frac{1}{2} + y\right) \in \left(\frac{N+1}{2(m+2)}, \frac{n}{2m}\right) + \mathbb{Z}^2\right\}.$$

In fact, $\tilde\theta_n^N(\tau)$ is an alternative form of the Hecke modular form $\theta_{\boldsymbol{\mu}}^B(\tau)$. According to Theorem 1 of Section 19.8.5 for $0 \leq N \leq m$ and for $n \equiv N (\operatorname{mod} 2)$ we have

$$c^{(m-N)\Lambda_0 + N\Lambda_1}_{(m-n)\Lambda_0 + n\Lambda_1}(\tau) = \eta(\tau)^{-3} \tilde\theta_n^N(\tau). \tag{2}$$

Therefore, due to part (4) of Theorem 1 of Section 19.8.1 $\tilde{\theta}_n^N$ is a cups form for the discrete group $\Gamma(4m) \cap \Gamma(4m+8)$ with the trivial multiplicative system.

We derive from formula (16) of Section 19.8.4 that for $0 \leq n, N \leq m$, $n \equiv N \pmod 2$, the expansion

$$\eta(\tau)^{-3}\tilde{\theta}_n^N(\tau) = q^b(1 + b_1 q + b_2 q^2 + \dots)$$

is valid, where

$$b = -\frac{1}{8} + \frac{(N+1)^2}{4(m+1)} - \frac{n^2}{4m} + \max\left(0, \frac{n-N}{2}\right)$$

and

$$1 + b_1 q + b_2 q^2 + \dots = \varphi(q)^{-1}(1 + c_1 q + c_2 q^2 + \dots)$$

(here $\varphi(q) = (q;q)_\infty$ and $c_i \geq 0$). Besides,

$$b_k \underset{k\to\infty}{\sim} \frac{\sin\left(\pi \frac{N+1}{m+2}\right)}{2k(m+2)} \exp\left(\pi\sqrt{k\frac{2m}{m+2}}\right).$$

Comparing formula (2) with the expressions for the string function c_λ^Λ of the algebra $A_1^{(1)}$ for special values of Λ and λ we obtain interesting identities. For example, if $m = 1$, then from the expression for $\tilde{\theta}_0^0(12\tau)$ we have

$$\eta(12\tau)^2 = \sum_{\substack{k,\ell\in\mathbb{Z}\\ k\geq 2|\ell|}} (-1)^{k+\ell} q^{[3(2k+1)^2-(6\ell+1)^2]/2}.$$

If $m = 2$, then from the expression for $\tilde{\theta}_1^1(8\tau)$ we obtain

$$\eta(8\tau)\eta(16\tau) = \sum_{\substack{k,\ell\in\mathbb{Z}\\ k\geq 3|\ell|}} (-1)^k q^{(2k+1)^2-32\ell^2}.$$

If $m = 4$, then with the help of $\tilde{\theta}_3^3(48\tau) - \tilde{\theta}_1^3(48\tau)$ we derive that

$$\eta(24\tau)\eta(96\tau) = \sum_{\substack{k,\ell\in\mathbb{Z}\\ 2k\geq \ell\geq 0}} (-1)^{\ell(\ell+1)/2} q^{8(3k+1)^2-3(2\ell+1)^2}(1 - q^{24(2k+1)}).$$

If $m = 8$, then the expression for $\tilde{\theta}_2^4(2\tau)$ leads to

$$\eta(2\tau)\eta(20\tau) = \sum_{\substack{k,\ell\in\mathbb{Z}\\ 2k\geq \ell\geq 0}} (-1)^k q^{[5(2k+1)^2-(2\ell+1)^2]/4}.$$

For the theta functions

$$\Theta_{nm}(\tau, z) = \sum_{k \in Z + n/2m} q^{mk^2} e^{-2\pi imkz}, \quad q = e^{2\pi i\tau},$$

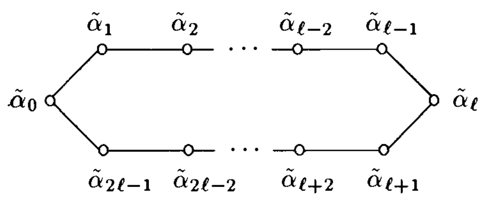
$$A_{nm}(\tau, z) = \Theta_{nm}(\tau, z) - \Theta_{-n,m}(\tau, z) = -2i \sum_{k \in Z + n/2m} q^{mk^2} \sin 2\pi mkz$$

with the help of formula (14) of Section 19.7.4, written down for the algebra $A_1^{(1)}$, and of formula (2) we derive the equality

$$\frac{A_{N+1,m+2}(\tau, z)}{A_{1,2}(\tau, z)} = \eta(\tau)^{-3} \sum_{\substack{0 \le n \le 2m \\ n \equiv N \pmod 2}} \tilde{\theta}_n^N(\tau)\Theta_{nm}(\tau, z).$$

19.9. Reduction of Representations of an Affine Lie Algebra onto a Subalgebra and Hecke Modular Forms

19.9.1. The functions E^{ℓ}_{jk}. Let us arrange 2ℓ simple roots of the affine Lie algebra $A^{(1)}_{2\ell-1}$ as



(1)

This diagram is invariant under the reflection σ with respect to the direct line passing through the roots $\tilde{\alpha}_0$ and $\tilde{\alpha}_\ell$. The automorphism of the algebra $A^{(1)}_{2\ell-1}$ (we denote it by φ) corresponds to this reflection. The set of elements of $A^{(1)}_{2\ell-1}$, which are fixed under φ, forms the subalgebra isomorphic to the affine Lie algebra $C^{(1)}_\ell$. The Dynkin diagram of this subalgebra is obtained from the diagram (1) by projecting the roots onto a horizontal line:

The subalgebra $C^{(1)}_\ell$ can be separated in $A^{(1)}_{2\ell-1}$ with the help of basis elements. The algebra $A^{(1)}_{2\ell-1}$ is generated by the elements

$$\tilde{e}_i, \ \tilde{f}_i, \ \tilde{h}_i, \ i = 0, \ 1, \ \ldots, \ 2\ell - 1, \ d.$$

Then $C_\ell^{(1)}$ is generated by d and by

$$\begin{array}{llll} e_0 = \tilde{e}_0, & e_i = \tilde{e}_i + \tilde{e}_{2\ell-i}, & i = 1,2,\dots,\ell-1, & e_\ell = \tilde{e}_\ell, \\ f_0 = \tilde{f}_0, & f_i = \tilde{f}_i + \tilde{f}_{2\ell-i}, & i = 1,2,\dots,\ell-1, & f_\ell = \tilde{f}_\ell, \\ h_0 = \tilde{h}_0, & h_i = \tilde{h}_i + \tilde{h}_{2\ell-i}, & i = 1,2,\dots,\ell-1, & h_\ell = \tilde{h}_\ell. \end{array}$$

It is clear that the common central element c is of the form

$$c = \sum_{i=0}^{2\ell-1} \tilde{h}_i = \sum_{i=0}^{\ell} h_i.$$

Let $\tilde{\alpha}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_{2\ell-1}$ and $\alpha_0, \alpha_1, \dots, \alpha_\ell$ be the systems of simple roots of $A_{2\ell-1}^{(1)}$ and of $C_\ell^{(1)}$ correspondingly and let

$$\tilde{\mathfrak{h}} = \mathbb{C}d + \sum_{i=0}^{2\ell-1} \mathbb{C}\tilde{h}_i, \quad \hat{\mathfrak{h}} = \mathbb{C}d + \sum_{i=0}^{\ell} \mathbb{C}h_i$$

be the Cartan subalgebras of these algebras. Then

$$\tilde{\alpha}_0 \Big|_{\hat{\mathfrak{h}}} = \alpha_0, \quad \tilde{\alpha}_i \Big|_{\hat{\mathfrak{h}}} = \tilde{\alpha}_{2\ell-i} \Big|_{\hat{\mathfrak{h}}} = \alpha_i, \quad i = 1,2,\dots,\ell-1, \quad \tilde{\alpha}_\ell \Big|_{\hat{\mathfrak{h}}} = \alpha_\ell.$$

Linear form δ will be denoted by $\tilde{\delta}$ for $A_{2\ell-1}^{(1)}$ and by δ for $C_\ell^{(1)}$. We have

$$\tilde{\delta} = \sum_{i=0}^{2\ell-1} \tilde{\alpha}_i, \quad \delta = \alpha_0 + 2\sum_{i=1}^{\ell-1} \alpha_i + \alpha_\ell.$$

Let us note that for $\lambda, \mu \in \hat{\mathfrak{h}}'$ we have $\lambda(h_i) = \mu(h_i)$, $i = 0,1,\dots,\ell$, if and only if $\lambda \equiv \mu (\mathrm{mod}\, \mathbb{C}\delta)$.

For the fundamental weights $\tilde{\Lambda}_i$, $i = 0,1,2,\dots,2\ell-1$, and Λ_i, $i = 0,1,2,\dots,\ell$, of the algebras $A_{2\ell-1}^{(1)}$ and $C_\ell^{(1)}$ the equalities

$$\tilde{\Lambda}_0 \Big|_{\hat{\mathfrak{h}}} = \Lambda_0, \quad \tilde{\Lambda}_i \Big|_{\hat{\mathfrak{h}}} = \tilde{\Lambda}_{2\ell-i} \Big|_{\hat{\mathfrak{h}}} = \Lambda_i, \quad i = 1,2,\dots,\ell-1, \quad \tilde{\Lambda}_\ell \Big|_{\hat{\mathfrak{h}}} = \Lambda_\ell \tag{2}$$

are valid.

Let $L_{\tilde{\Lambda}}$ be an irreducible integrable representation of the algebra $A_{2\ell-1}^{(1)}$ with highest weight $\tilde{\Lambda}$. As it is shown in [138], reduction of $L_{\tilde{\Lambda}}$ onto the subalgebra $C_\ell^{(1)}$ is a completely reducible representation which decomposes into a direct sum of representations of the type L_Λ. We suppose that $\tilde{\Lambda}(d) = 0$.

Statement 1. *There exist power series $E^\ell_{\tilde{\Lambda}\Lambda}$ in one variable $q = e^{-\delta}$ such that*

$$ch\, L_{\tilde{\Lambda}}|_{\hat{h}} = \sum_{\substack{\Lambda \in P_+ \\ \Lambda(c) = \tilde{\Lambda}(c),\ \Lambda(d)=0}} E^\ell_{\tilde{\Lambda}\Lambda}\, ch\, L_\Lambda. \tag{3}$$

For these series we have

$$E^{\ell}_{\tilde{\Lambda}\Lambda}(q) = 0 \quad \text{if} \quad \tilde{\Lambda}\Big|_{\hat{h}} \not\equiv \Lambda \left(\bmod Q = \sum_{i=0}^{\ell} \mathbb{Z}\alpha_i\right), \tag{4}$$

$$E^{\ell}_{\tilde{\Lambda}'\Lambda'}(q) = q^{[\tilde{\Lambda}(h)-\Lambda(h)]/2} E^{\ell}_{\tilde{\Lambda}\Lambda}(q), \quad h = \sum_{j=0}^{\ell} j h_j, \tag{5}$$

where $\tilde{\Lambda}'(d) = \Lambda'(d) = 0$ *and*

$$\tilde{\Lambda}'(\tilde{h}_i) = \begin{cases} \tilde{\Lambda}(\tilde{h}_{\ell-i}) & \text{for } i = 0, 1, \ldots, \ell, \\ \tilde{\Lambda}(\tilde{h}_{3\ell-i}) & \text{for } i = \ell+1, \ell+2, \ldots, 2\ell-1, \end{cases}$$
$$\Lambda'(h_i) = \Lambda(h_{\ell-i}), \quad i = 0, 1, \ldots, \ell.$$

Proof. Let V be the space of vectors $\mathbf{v}$ from the carrier space of the representation $L_{\tilde{\Lambda}}$ for which $L_{\tilde{\Lambda}}(e_i)\mathbf{v} = 0$, $i = 0, 1, \ldots, \ell$, and let $V = \sum_{\lambda \in I} \oplus V_\lambda$ be the decomposition of V into a sum of weight subspaces. Then $\lambda \in P_+$, where P_+ is the set of dominant integral weights for the subalgebra $C_\ell^{(1)}$. Using the definition of highest weight representations it is easy to show that we have

$$L_{\tilde{\Lambda}}\Big|_{C_\ell^{(1)}} = \sum_{\lambda \in I} \oplus\, (\dim\, V_\lambda) L_\lambda,$$

where, in fact, $(\dim\, V_\lambda)$ indicates multiplicity of the representation L_λ of $C_\ell^{(1)}$ in the decomposition.

If $\lambda \in I$, then let $\Lambda \in P_+$ be such that $\Lambda(h_i) = \lambda(h_i)$, $i = 0, 1, \ldots, \ell$, and $\Lambda(d) = 0$. Then

$$\lambda = \tilde{\Lambda}\Big|_{\hat{\mathfrak{h}}} - \sum_{i=0}^{\ell} n_i \alpha_i = \Lambda - n\delta,$$

where $n_i \in \mathbb{Z}$, $n_i \geq 0$. This means that

$$\tilde{\Lambda}\Big|_{\hat{\mathfrak{h}}} \equiv \Lambda \,(\bmod Q), \quad \tilde{\Lambda}(c) = \lambda(c) = \Lambda(c), \quad n = n_0 \geq 0.$$

Setting $E^{\ell}_{\tilde{\Lambda}\Lambda}(q) = \sum_{n=0}^{\infty} (\dim\, V_{\Lambda - n\delta}) q^n$ we obtain properties (3) and (4). In order to prove property (5) we define the linear transformations $\tilde{A}$ and A of the spaces

$$\sum_{i=0}^{2\ell-1} \mathbb{C}\tilde{\alpha}_i, \qquad \sum_{i=0}^{\ell} \mathbb{C}\tilde{\alpha}_i,$$

respectively, by setting

$$\tilde{A}(\tilde{\alpha}_i) = \tilde{\alpha}_{\ell-i}, \quad i = 0,1,\ldots,\ell, \quad \tilde{A}(\tilde{\alpha}_i) = \tilde{\alpha}_{3\ell-i}, \quad i = \ell+1, \ell+2, \ldots, 2\ell-1,$$
$$A(\alpha_i) = \alpha_{\ell-i}, \quad i = 0,1,\ldots,\ell.$$

Then the symmetry of the Dynkin diagrams guarantees fulfillment of the relations

$$\tilde{A}\left(e^{-\tilde{\Lambda}}\mathrm{ch}\, L_{\tilde{\Lambda}}\right) = e^{-\tilde{\Lambda}'}\mathrm{ch}\, L_{\tilde{\Lambda}'},$$
$$A\left(e^{-\Lambda}\mathrm{ch}\, L_{\Lambda}\right) = e^{-\Lambda'}\mathrm{ch}\, L_{\Lambda'}.$$

Substituting these expressions into (3) one obtains the relation

$$A\left(\exp\left(-\tilde{\Lambda}\Big|_{\hat{\mathfrak{h}}} + \Lambda\right) E^{\ell}_{\tilde{\Lambda}\Lambda}(q)\right) = \exp\left(-\tilde{\Lambda}'\Big|_{\hat{\mathfrak{h}}} + \Lambda'\right) E^{\ell}_{\tilde{\Lambda}'\Lambda'}(q).$$

This relation and the equalities

$$A(\delta) = \delta, \quad A(\Lambda_j - \Lambda_0) - (\Lambda_{\ell-j} - \Lambda_0) = \frac{1}{2}j\delta - (\Lambda_j - \Lambda_0)$$

lead to property (5). Lemma is proved.

We shall study the series $E^{\ell}_{\tilde{\Lambda}\Lambda}(q)$ when $\tilde{\Lambda}$ coincides with one of the fundamental weights $\tilde{\Lambda}_j$, $j = 0,1,\ldots,\ell$. In this case Λ is a fundamental weight for $C^{(1)}_{\ell}$. The series $E^{\ell}_{\tilde{\Lambda}_j\Lambda_k}$, $j,k = 0,1,\ldots,\ell$, will be denoted by E^{ℓ}_{jk}. Properties (4) and (5) mean that

$$E^{\ell}_{jk}(q) = 0 \quad \text{if} \quad j \not\equiv k \pmod 2,$$
$$E^{\ell}_{\ell-j,\ell-k}(q) = q^{(j-k)/2}E^{\ell}_{jk}(q).$$

It follows from here that

$$q^{(j-k)/2}E^{\ell}_{jk}(q) = \begin{cases} O(1) & \text{if } j \le k, \\ O(q^{(j-k)/2}) & \text{if } j \ge k. \end{cases} \tag{6}$$

Below we also use the notation

$$e^{\ell}_{jk}(\tau) = \exp[2\pi i(\tilde{r}_j - r_k)\tau]E^{\ell}_{jk}(e^{2\pi i\tau}), \tag{7}$$

where

$$\tilde{r}_j = \frac{j(2\ell-j)}{4\ell} - \frac{2\ell-1}{24}, \quad r_k = \frac{k(2\ell-k+2)}{4(\ell+2)} - \frac{\ell(2\ell+1)}{24(\ell+2)}. \tag{8}$$

19.9.2. The matrix (e^{ℓ}_{jk}). The matrix $(e^{\ell}_{jk})^{\ell}_{j,k=0}$ will be denoted by E_{ℓ}.

Theorem 1. *The matrix E_ℓ obeys the transformation lows*

$$E_\ell(\tau+1) = \tilde{\mathcal{D}}_\ell E_\ell(\tau)\mathcal{D}_\ell^{-1}, \quad E_\ell\left(-\frac{1}{\tau}\right) = \tilde{S}_\ell E_\ell(\tau) S_\ell, \tag{1}$$

where $\tilde{\mathcal{D}}_\ell$ and $\mathcal{D}_\ell$ are the diagonal constant matrices:

$$(\tilde{\mathcal{D}}_\ell)_{jj'} = \delta_{jj'}\exp(2\pi i\tilde{r}_j), \quad (\mathcal{D}_\ell)_{kk'} = \delta_{kk'}\exp(2\pi i r_k)$$

($\tilde{r}_j$, r_k are given by formula (8) *of Section* 19.9.1), *the matrices $\tilde{S}_\ell$ and S_ℓ are the constant matrices with entries*

$$(\tilde{S}_\ell)_{jj'} = \begin{cases} 2(2\ell)^{-1/2}\cos\frac{jj'\pi}{\ell} & \text{for } j' = 1,2,\dots,\ell-1, \\ (2\ell)^{-1/2}\cos\frac{jj'\pi}{\ell} & \text{for } j' = 0,\ell, \end{cases}$$

$$(S_\ell)_{kk'} = \sqrt{\frac{2}{\ell+2}}\sin\frac{\pi(k+1)(k'+1)}{\ell+2}.$$

If E_ℓ^{even} and E_ℓ^{odd} are respectively the matrices $(e^\ell_{jk})_{j,k\ \text{even}}$ and $(e^\ell_{jk})_{j,k\ \text{odd}}$, then

$$\det E_\ell^{\text{even}} = \det E_\ell^{\text{odd}} = 1$$

for odd ℓ and

$$\det E_\ell^{\text{even}} = \eta(\tau)\eta(2\tau)^{-1}, \quad \det E_\ell^{\text{odd}} = \eta(\tau)^{-1}\eta(2\tau)$$

for even ℓ.

Proof of this theorem is analogous to that of Theorem 1 of Section 19.8.2 and we omit it.

19.9.3. Evaluation of E^ℓ_{jk}. In order to evaluate the functions E^ℓ_{jk} we need the principal specialization F_1 of the functions $e^{-\tilde{\Lambda}_j}\,\text{ch}\,L_{\tilde{\Lambda}_j}$ and $e^{-\Lambda_j}\,\text{ch}\,L_{\Lambda_j}$. Using the results of Sections 19.4.2 and 19.5.4 we find that

$$e^{-\tilde{\Lambda}_j}\,\text{ch}\,L_{\tilde{\Lambda}_j}\Big|_{e^{-\tilde{\alpha}_0}=\dots=e^{-\tilde{\alpha}_{2\ell-1}}=x} = \varphi(x^{2\ell})/\varphi(x), \qquad j = 0,1,2,\dots,2\ell-1, \tag{1}$$

$$e^{-\Lambda_j}\,\text{ch}\,L_{\Lambda_j}\Big|_{e^{-\alpha_0}=\dots=e^{-\alpha_\ell}=x} = F_{2\ell+4,2j+2}(x)/\varphi(x), \qquad j = 0,1,2,\dots,\ell, \tag{2}$$

where $\varphi(x) = \prod_{n=1}^{\infty}(1-x^n) = (x,x)_\infty$ and

$$F_{N,r}(x) = \prod_{n=0}^{\infty}(1-x^{Nn+r})(1-x^{Nn+N-r})(1-x^{Nn+N})$$
$$= (x^r;x^N)_\infty(x^{N-r};x^N)_\infty(x^N;x^N)_\infty.$$

Since $q = e^{-\delta} = x^{2\ell}$, then the specialization F_1 for relation (3) of Section 19.9.1 leads to an identity among power series in $q^{1/2\ell}$.

By using the Jacobi identity

$$\prod_{n=0}^{\infty}(1-x^{2n+2})(1-zx^{2n+1})(1-z^{-1}x^{2n+1}) = \sum_{n\in\mathbb{Z}}(-1)^n z^n x^{n^2}$$

(see Section 19.6.1) we can write a series expansion for $F_{N,r}$. Collecting like fractional powers we obtain

$$\begin{aligned} x^{j^2/2}F_{2\ell+4,2j+2}(x) &= x^{j^2/2}\sum_{n\in\mathbb{Z}}(-1)^n x^{(\ell+2)n^2+(2j-\ell)n} \\ &= \sum_{\substack{k\equiv j (\mathrm{mod}\ 2)\\ k (\mathrm{mod}\ 2\ell)}} x^{k^2/2}P^{\ell}_{jk}(q), \end{aligned} \tag{3}$$

where

$$P^{\ell}_{jk}(q) = \sum_{n\in\mathbb{Z}}(-1)^{n\ell+(j-k)/2}q^{A(\ell,j,k,n)}, \tag{4}$$

$$A(\ell,j,k,n) = \frac{1}{2}\ell(\ell+2)n^2 - \frac{1}{2}[\ell(j+1)-(\ell+2)k]n + \frac{1}{8}(j-k)(j-k+2).$$

Introducing the function

$$P^{\infty}_{jk}(q) = (-1)^{(j-k)/2}q^{(j-k)(j-k+2)/8}$$

we can represent relation (4) as

$$\begin{aligned} P^{\ell}_{jk}(q) &= q^{(j+1)^2/4(\ell+2)}\sum_{j'\equiv j(\mathrm{mod}\ 2\ell+4)} q^{-(j'+1)^2/4(\ell+2)}P^{\infty}_{j'k}(q) \\ &= q^{-k^2/2\ell}\sum_{k'\equiv k(\mathrm{mod}\ 2)} q^{k'^2/4\ell}P^{\infty}_{j'k}(q). \end{aligned} \tag{4'}$$

From here we get the symmetry relations

$$P^{\ell}_{-j-2,-k}(q) = -P^{\ell}_{jk}(q), \quad P^{\ell}_{\ell+1,k}(q) = P^{\ell}_{-\ell-3,k}(q), \quad P^{\ell}_{j,\ell}(q) = P^{\ell}_{j,-\ell}(q)$$

and periodicity of $q^{-(j+1)^2/4(\ell+2)}P^\ell_{jk}(q)$ (of $q^{k^2/4\ell}P^\ell_{jk}(q)$) with respect to j (with respect to k) with period $2(\ell+2)$ (with period 2ℓ).

We set

$$\tilde{P}^\ell_{jk}(q) = \begin{cases} P^\ell_{jk}(q) - P^\ell_{j,-k}(q) & \text{if } j \equiv k (\text{mod } 2),\ 1 \le k \le \ell - 1, \\ P^\ell_{jk}(q) & \text{if } j \equiv k (\text{mod } 2),\ k = 0,\ \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that ℓ is a prime integer. Equating like fractional powers in the principal specialization of relation (3) of Section 19.9.1 we obtain the equality

$$\sum_{j=0}^{\ell} q^{(i-j)/2} E^\ell_{ij}(q) \tilde{P}^\ell_{jk}(q) = \varphi(q)\delta_{ik}.$$

Thus, the matrix $(q^{(i-j)/2}E^\ell_{ij}(q))$ is inverse to the matrix $(\varphi(q)^{-1}\tilde{P}^\ell_{ij}(q))$. Let us show that this assertion is valid for all integers ℓ, $\ell \ge 2$. For this we use the series

$$\theta^N_r(\tau) = \sum_{n\in\mathbb{Z}} (-1)^{nN} e^{\pi i N\tau(n+r/2N)^2} \tag{5}$$

with $r, N \in \mathbb{Z}$, $r \equiv N(\text{mod } 2)$. They obey the transformation low

$$\theta^N_r\left(-\frac{1}{\tau}\right) = \sqrt{\frac{-i\tau}{N}} \sum_{\substack{s\equiv N(\text{mod } 2)\\ s(\text{mod } 2N)}} e^{\pi i r s/2N}\theta^N_s(\tau). \tag{6}$$

The functions (4) are related to these series as

$$P^\ell_{jk}(q) = q^{-c_{jk}}(-1)^{(j-k)/2}\theta^{\ell(\ell+2)}_{\ell(j+1)-k(\ell+2)}(\tau), \tag{7}$$

where $e^{2\pi i\tau} = q$ and

$$c_{jk} = -\frac{(j+1)^2}{4(\ell+2)} + \frac{k^2}{4\ell} + \frac{1}{8} = r_j - \tilde{r}_k - \frac{1}{2}(j-k) + \frac{1}{24}.$$

We put

$$p^\ell_{jk}(\tau) = q^{c_{jk}} P^\ell_{jk}(q), \quad \tilde{p}^\ell_{jk}(\tau) = q^{c_{jk}}\tilde{P}^\ell_{jk}(q)$$

and introduce the matrix $P_\ell(\tau) = (\eta(\tau)^{-1}\tilde{p}^\ell_{jk}(\tau))$ where $\eta(\tau) = q^{1/24}\varphi(\tau)$ is Dedekind's η-function. Let us show that

$$E_\ell(\tau)P_\ell(\tau) = 1. \tag{8}$$

The matrix $E_\ell(\tau)$ obeys the transformation relations (1) of Section 19.9.2. The matrix $P_\ell(\tau)$ satisfies similar relations

$$P_\ell(\tau+1) = \mathcal{D}_\ell P_\ell(\tau)\tilde{\mathcal{D}}_\ell^{-1}, \quad P_\ell\left(-\frac{1}{\tau}\right) = S_\ell P_\ell(\tau)\tilde{S}_\ell. \tag{9}$$

Really, the first relation is evident. In order to prove the second relation we use formulas (6) and (7). Since $(\ell/2, (\ell+2)/2) = 1$ for even ℓ and $(\ell, \ell+1) = 1$ for odd ℓ, then $\nu = \ell(j+1) - (\ell+2)k$ for

$$0 \le j+1 \le \ell+1, \quad -(\ell-1) \le k \le \ell, \quad j \equiv k (\mathrm{mod}\ 2)$$

or for

$$-(\ell+1) \le j \le 0, \quad -\ell \le k \le \ell-1, \quad j \equiv k (\mathrm{mod}\ 2)$$

runs over the set

$$\{\nu \in \mathbb{Z}/2N\mathbb{Z} \mid \nu \equiv N (\mathrm{mod}\ 2)\}, \quad N = \ell(\ell+2).$$

Therefore,

$$\begin{aligned}\sqrt{\frac{\ell(\ell+2)}{-i\tau}}\, p^\ell_{jk}\left(-\frac{1}{\tau}\right) &= i \sum_{\substack{0 \le j'+1 \le \ell+1 \\ -(\ell-1) \le k' \le \ell \\ j' \equiv k' (\mathrm{mod}\ 2)}} \exp\left[\pi i \left(\frac{kk'}{\ell} - \frac{(j+1)(j'+1)}{\ell+2}\right)\right] p^\ell_{j'k'}(\tau) \\ &= i \sum_{\substack{-\ell-1 \le j'+1 \le 0 \\ -\ell \le k' \le \ell-1 \\ j' \equiv k' (\mathrm{mod}\ 2)}} \exp\left[\pi i \left(\frac{kk'}{\ell} - \frac{(j+1)(j'+1)}{\ell+2}\right)\right] p^\ell_{j'k'}(\tau).\end{aligned}$$

This together with the symmetry properties of $P^\ell_{jk}(q)$ lead to the second relation in (9).

Formulas (1) of Section 19.9.2 and (9) show that elements of the matrix $E_\ell(\tau)P_\ell(\tau)$ are invariant under the action of the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ onto τ. Therefore, they are invariant with respect to the group $SL(2,\mathbb{Z})$. In order to show that $E_\ell(\tau)P_\ell(\tau) = \mathrm{const}$ it is sufficient to prove the estimate

$$(E_\ell(\tau)P_\ell(\tau))_{jk} - \delta_{jk} = O(1) \quad \text{when} \quad \tau \to i\infty.$$

This estimate is obtained with the help of formula (6) of Section 19.9.1 if we take into account the fact that only the terms with $n = 0$ or $n = 1$ in formula (4) have possible contributions to the lowest order. This proves equation (8).

Now we evaluate the inverse matrix to $P_\ell(\tau)$. We have to solve the system of linear equations

$$\sum_{k=0}^{\ell} \tilde{P}^\ell_{jk}(q) V^{(i)}_k = \delta_{ji}, \quad 0 \le i, j \le \ell, \tag{10}$$

with $V_k^{(i)} = \varphi(q)^{-1} q^{(k-i)/2} E_{ki}^{\ell}(q)$. Let us extend the range of suffixes of $V_k^{(i)}$ by requiring that for $0 \le k \le \ell - 1$ we have $V_{-k}^{(i)} = V_k^{(i)}$. Then the system (10) can be written as

$$\Sigma \equiv \sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv j \pmod 2}} P_{jk}^{\ell}(q) V_k^{(i)} = \delta_{ji}, \quad 0 \le i, j \le \ell. \tag{11}$$

Using the symmetry relations for P_{jk}^{ℓ}, $-\ell - 3 \le j \le -1$, we may represent the sum Σ as

$$\Sigma = - \sum_{\substack{-\ell \le k \le \ell-1 \\ k \equiv j \pmod 2}} P_{-j-2,-k}^{\ell}(q) V_k^{(i)}$$
$$= - \sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv j \pmod 2}} P_{-j-2,-k}^{\ell}(q) V_k^{(i)} = -\delta_{-j-2,i}.$$

In particular, for $j = -1$ we have

$$\sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv -1 \pmod 2}} P_{-1,k}^{\ell}(q) V_k^{(i)} = - \sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv -1 \pmod 2}} P_{-1,-k}^{\ell}(q) V_k^{(i)} = 0.$$

We analogously obtain

$$\sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv \ell+1 \pmod 2}} P_{\ell+1,k}^{\ell}(q) V_k^{(i)} = \sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv -\ell-3 \pmod 2}} P_{-\ell-3,k}^{\ell}(q) V_k^{(i)}$$
$$= - \sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv \ell+1 \pmod 2}} P_{\ell+1,k}^{\ell}(q) V_k^{(i)} = 0.$$

Thus, we get the equality

$$\sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv j \pmod 2}} P_{jk}^{\ell}(q) V_k^{(i)} = \delta_{ji} - \delta_{-j-2,i}, \quad 0 \le i \le \ell, \quad -\ell - 2 \le j \le \ell + 1.$$

By virtue of the quasi-periodicity of P_{jk}^{ℓ} in j, it can be extended to all values of j as

$$\sum_{\substack{-\ell+1 \le k \le \ell \\ k \equiv j \pmod 2}} q^{-(j+1)^2/4(\ell+2)} P_{jk}^{\ell}(q) V_k^{(i)}$$
$$= q^{-(i+1)^2/4(\ell+2)} \sum_{i' \equiv i \pmod{2\ell+4}} (\delta_{ji'} - \delta_{j,-i'-2}),$$

where $0 \le i \le \ell$, $j \in \mathbb{Z}$. Substituting here expression (4′) for P_{jk}^{ℓ} we derive that

$$\sum_{\substack{k \in \mathbb{Z} \\ k \equiv j \pmod 2}} P_{jk}^{\infty}(q) V_k^{(i)} = q^a \sum_{i' \equiv i \pmod{2\ell+4}} (\delta_{ji'} - \delta_{j,-i'-2}), \tag{12}$$

where $a = [(j+1)^2 - (i+1)^2]/4(\ell+2)$ and $V_k^{(i)}$ are determined for all $k \in \mathbb{Z}$ with the help of the equality

$$q^{-k^2/4\ell} V_k^{(i)} = q^{-k'^2/4\ell} V_{k'}^{(i)}, \quad k' \equiv k (\mathrm{mod}\, 2\ell).$$

In (12) the coefficients $P_{jk}^{\infty}(q)$ depend only on the difference $j - k$. Hence, the system of equations (12) can be easily solved by taking the Fourier transforms. We obtain

$$\sum_{k\in\mathbb{Z}} z^k V_k^{(i)} = \left[\prod_{n=0}^{\infty}(1-q^{n+1})(1-z^2q^{n+1})(1-z^{-2}q^n)\right]^{-1} \times \sum_{\nu\in\mathbb{Z}} q^{(\ell+2)\nu^2+(i+1)\nu}\left(z^{2(\ell+2)\nu+i} - z^{-2(\ell+2)\nu-i-2}\right).$$

Calculation with the help of residues gives $\varphi(q)^3 V_k^{(i)}$ as a double series. Taking into account the relation between $V_k^{(i)}$ and E_{jk}^{ℓ} we have

$$\begin{aligned}
&\varphi(q)^2 q^{(j-k)/2} E_{jk}^{\ell}(q) = \varphi(q)^2 E_{\ell-j,\ell-k}^{\ell}(q) \\
&= \left(\sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty}\right)(-1)^n q^{n(n+1)/2+n(k-j)/2+m(k+1)+m(\ell+2)(m+n)} \\
&+ \left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} - \sum_{m=-1}^{-\infty}\sum_{n=0}^{-\infty}\right)(-1)^n q^{n(n+1)/2+n(k+j)/2+m(k+1)+m(\ell+2)(m+n)},
\end{aligned}$$

where $j \equiv k \pmod 2$ and $0 \le j \le k \le \ell$. Replacing $m + n/2$ by m and separating the sum over n into even and odd parts, we can rewrite this formula in the form

$$\begin{aligned}
&\eta(\tau)^2 e_{jk}^{\ell}(\tau) = \eta(\tau)^2 e_{\ell-j,\ell-k}^{\ell}(\tau) \\
&= \left(\sum_{m\ge n\ge 0} + \sum_{\substack{m\ge 0>n \\ m+n\ge 0}} - \sum_{0>n>m} - \sum_{\substack{n\ge 0>m \\ m+n<0}}\right) q^{(\ell+2)\left[m+\frac{k+1}{2(\ell+2)}\right]^2 - \ell\left(n+\frac{j}{2\ell}\right)^2} \\
&+ \left(\sum_{0>n>m} + \sum_{\substack{n\ge 0\ge m \\ m+n<-1}} - \sum_{m\ge n\ge 0} - \sum_{\substack{m\ge 0>n \\ m+n>-1}}\right) q^{(\ell+2)\left[m+\frac{1}{2}+\frac{k+1}{2(\ell+2)}\right]^2 - \ell\left(n+\frac{1}{2}+\frac{j}{2\ell}\right)^2},
\end{aligned} \tag{13}$$

where $j \equiv k \pmod 2$ and $0 \le j \le k \le \ell$.

Comparing the right hand side of (13) with the formula (3) of Section 19.8.5 for Hecke modular form we conclude that

$$\eta(\tau)^2 e_{jk}^{\ell}(\tau) = \theta_{\boldsymbol{\mu}}^{B}(\tau), \quad \boldsymbol{\mu} = \left(\frac{k+1}{2(\ell+1)}, \frac{j}{2\ell}\right), \tag{14}$$

where $B((x,y))$ is the same as in Section 19.8.5, that is, the function $e^{\ell}_{jk}(\tau)$ and the string function $c^{\Lambda}_{\lambda}(\tau)$ are expressed in terms of the same Hecke modular forms.

Let

$$K(q)=\prod_{n=0}^{\infty}\frac{1}{(1-q^{8n+1})(1-q^{8n+7})}, \tag{15}$$

$$L(q)=\prod_{n=0}^{\infty}\frac{1}{(1-q^{8n+3})(1-q^{8n+5})}, \tag{16}$$

$$G(q)=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{17}$$

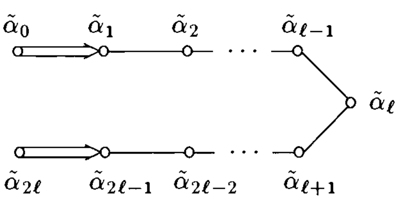
$$H(q)=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{18}$$

Using expression (13) for $e^{\ell}_{jk}(q)$ and formulas (2) and (3) of Section 19.9.2 at $\ell=2$ and $\ell=3$ M. Jimbo and T. Miwa have proved [128] that

$$K(q)^2+qL(q)^2=\frac{\varphi(q^2)^6}{\varphi(q)^3\varphi(q^4)\varphi(q^8)^2}, \tag{19}$$

$$G(q)G(q^9)+q^2H(q)H(q^9)=\frac{\varphi(q^3)^2}{\varphi(q)\varphi(q^9)}. \tag{20}$$

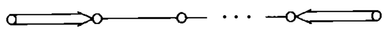
19.9.4. Reduction $C^{(1)}_{2\ell}\supset C^{(1)}_{\ell}$. We arrange the simple roots of the affine Lie algebra $C^{(1)}_{2\ell}$ in the form



(1)

and consider the automorphism φ of $C^{(1)}_{2\ell}$ corresponding to the symmetry of this Dynkin diagram under the reflection σ with respect to the direct horizontal line passing through the root $\tilde\alpha_\ell$. We have $\sigma\tilde\alpha_i=\tilde\alpha_{2\ell-i}$, $i=0,1,\dots,2\ell$. This automorphism is constructed in the same way as in the case of the algebra $A^{(1)}_{2\ell-1}$. Elements of the subalgebra $C^{(1)}_{\ell}$ is invariant under automorphism φ. The Dynkin diagram of

$C_\ell^{(1)}$ is obtained from the diagram (1) by projecting it onto a horizontal line:



Let $J^\ell_{jk}(q)$, $q = e^{-\delta}$, $0 \le j,k \le \ell$, be the coefficients of the expansion of the characters ch $L_{\tilde\Lambda_0}$, ch $L_{\tilde\Lambda_1}, \dots,$ ch $L_{\tilde\Lambda_\ell}$ of the fundamental highest weigh representations of the algebra $C_{2\ell}^{(1)}$ in the characters ch L_{Λ_0}, ch $L_{\Lambda_1}, \dots,$ ch L_{Λ_ℓ} of the fundamental highest weigh representations of $C_\ell^{(1)}$. These coefficients are expressed in terms of Hecke modular form $\theta^B_{\boldsymbol\mu}(\tau)$ for which

$$B(\gamma,\gamma) = 2(\ell+2)x^2 - 8(\ell+1)y^2, \quad \gamma = (x,y). \tag{2}$$

The lattice L is of the form $L = \mathbb{Z}^2$ and the dual lattice is $L^* = \frac{1}{2\ell+4}\mathbb{Z} \oplus \frac{1}{8\ell+8}\mathbb{Z}$. The matrix

$$a = \begin{pmatrix} 2\ell+3 & 4\ell+4 \\ \ell+2 & 2\ell+3 \end{pmatrix}$$

conserves the form (2). The subgroup G_0, introduced in Section 14.8.5, coincides with $G_0 = \{a^{2n} \mid n \in \mathbb{Z}\}$. The group G_0 conserves elements of the quotient group L^*/L.

We have

$$\eta(2\tau)\eta(\tau)q^b J^\ell_{jk}(q) = \theta^B_{\boldsymbol\mu}(\tau), \quad q = e^{2\pi i\tau}, \tag{3}$$

where B is given by the formula (2) and

$$b = \frac{j-k}{2} - \frac{(j+1)^2}{4(\ell+1)} + \frac{(k+1)^2}{4(\ell+2)} + \frac{1}{8}, \quad \boldsymbol\mu = \left(\frac{k+1}{2(\ell+2)} + \frac{1}{2}, \frac{j+1}{4(\ell+1)} + \frac{1}{4}\right).$$

For determinants of matrices (J^ℓ_{jk}) we get

$$\det\,(J^\ell_{jk})_{j,k\ \text{even}} = \varphi(q)^{-\ell/2}\varphi(q^2)^{\ell/2}, \tag{4}$$

$$\det\,(J^\ell_{jk})_{j,k\ \text{odd}} = \varphi(q)^{-\ell/2}\varphi(q^2)^{-\ell/2} \tag{5}$$

if ℓ is even, and

$$\det\,(J^\ell_{jk})_{j,k\ \text{even}} = \varphi(q)^{-(\ell-1)/2}\varphi(q^2)^{(\ell-3)/2}\varphi(q^4), \tag{6}$$

$$\det\,(J^\ell_{jk})_{j,k\ \text{odd}} = \varphi(q)^{-(\ell+1)/2}\varphi(q^2)^{-(\ell-3)/2}\varphi(q^4)^{-1} \tag{7}$$

if ℓ is odd. Proof of formulas (3)-(7) see in [128].

Let us introduce the renotation

$$\theta^\ell_{mn}(\tau) = \theta^B_{\boldsymbol\mu}(\tau), \quad \boldsymbol\mu = \left(\frac{m}{2(\ell+2)}, \frac{n}{8(\ell+1)}\right),$$

for the form θ^B_μ from formula (3) and define the matrices

$$T_1(\tau) = (\theta^\ell_{mn})_{\substack{m=1,3,\dots,2[\ell/2]+1\\ n=0,4,\dots,4[\ell/2]}},$$

$$T_2(\tau) = (\theta^\ell_{mn})_{\substack{m=2,4,\dots,2[(\ell+1)/2]\\ n=2,6,\dots,4[(\ell+1)/2]-2}},$$

where $[r]$ is an integral part of the number r.

M Jimbo and T. Miwa [128] have proved that

$$\det T_1(\tau) = \begin{cases} \eta(\tau)\eta(2\tau)^{\ell+1} & \text{if } \ell \text{ is even,} \\ \eta(2\tau)^{\ell+2}\eta(4\tau)^{-1} & \text{if } \ell \text{ is odd,} \end{cases}$$

$$\det T_2(\tau) = \begin{cases} \eta(2\ell)^{\ell} & \text{if } \ell \text{ is even,} \\ \eta(\tau)\eta(2\tau)^{\ell-1}\eta(4\tau) & \text{if } \ell \text{ is odd.} \end{cases}$$

For $\ell = 2$ along with determinants it is possible to evaluate the matrix elements J^2_{jk}. They are expressed in terms of Dedekind's η-functions:

$$\theta^2_{1,0}(\tau) + \theta^2_{3,0}(\tau) = \eta(\tau)^2\,\eta\left(\frac{3\tau}{2}\right)^2 \Big/ \eta\left(\frac{\tau}{2}\right)\eta(3\tau),$$

$$\theta^2_{1,0}(\tau) - \theta^2_{3,0}(\tau) = \eta\left(\frac{\tau}{2}\right)\eta(2\tau)\eta(3\tau)^5 \Big/ \eta(\tau)\eta\left(\frac{3\tau}{2}\right)^2\eta(6\tau)^2,$$

$$\theta^2_{1,4}(\tau) + \theta^2_{3,4}(\tau) = \eta(\tau)\eta(3\tau)^2 \Big/ \eta\left(\frac{3\tau}{2}\right),$$

$$-\theta^2_{1,4}(\tau) + \theta^2_{3,4}(\tau) = \eta(\tau)\,\eta\left(\frac{3\tau}{2}\right)\eta(6\tau) \Big/ \eta(3\tau),$$

$$\theta^2_{2,2}(\tau) = \eta(2\tau)^2,$$

$$\theta^2_{1,1}(\tau) + \theta^2_{1,7}(\tau) = \eta(\tau)\eta(3\tau)^2 \big/ \eta(6\tau),$$

$$\theta^2_{1,1}(\tau) - \theta^2_{1,7}(\tau) = \eta(\tau)\eta(4\tau)^5 \big/ \eta(2\tau)^2\eta(8\tau)^2,$$

$$\theta^2_{1,9}(\tau) = \eta(\tau)^2\eta(6\tau)^2 \big/ \eta(2\tau)\eta(3\tau),$$

$$\theta^2_{1,5}(\tau) = \eta(\tau)\eta(8\tau)^2 \big/ \eta(4\tau).$$

Using these relations and formula (9) for $\ell = 2$ the following equality for η-function is derived:

$$\eta(\tau)^2\eta(4\tau)\eta(6\tau)^9 + \eta(2\tau)^3\eta(3\tau)^6\eta(12\tau)^3$$
$$= 2\eta(\tau)\eta(2\tau)\eta(3\tau)^3\eta(4\tau)^3\eta(6\tau)^2\eta(12\tau)^2.$$

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Bibliography Notes

Below we separately indicate references results of which were included into our book (they are listed after the letter A) and references which were used under preparation of the text (they are listed after the letter B). References are given by two numbers. The first number indicates the corresponding volume, where the reference is taken from, and the second one indicates the number of reference in this volume.

References related to Chapters 1–3 are given in footnote on p. 595 of the first volume. In this reason our notes are begun from Chapter 4.

Chapter 4:

A: [1.49].
B: [2.29].

Chapter 5:

A: [1.30], [1.49].
B: [1.59], [2.29], [2.253], [2.254].

Chapter 6:

A: [1.3], [1.14], [1.49], [2.206], [2.217], [2.396].
B: [1.35], [1.45], [1.48], [1.55], [2.34], [2.323], [2.429].

Chapter 7:

A: [1.27], [1.49], [2.8], [2.28], [2.32], [2.48], [2.146], [2.179], [2.400], [2.401], [2.412], [2.413], [2.418].
B: [1.41], [1.42], [2.13], [2.169], [2.220], [2.221], [2.222], [2.232], [2.242], [2.243], [2.275], [2.282], [2.304], [2.322].

Chapter 8:

A: [1.6], [1.7], [1.49], [2.31], [2.33], [2.38], [2.176], [2.205], [2.250], [2.256], [2.279], [2.411], [2.430].
B: [1.34], [1.35], [1.48], [1.57], [1.59], [2.15], [2.26], [2.39], [2.58], [2.118], [2.173], [2.175], [2.266], [2.348], [2.368], [2.369–371], [2.391], [2.422], [2.461].

Chapter 9:

A: [1.49], [2.9], [2.96], [2.198], [2.265], [2.295–297], [2.317], [2.384–386], [2.392], [2.398], [2.399].
B: [1.31], [2.1], [2.16], [2.17], [2.23], [2.24], [2.43], [2.44], [2.46], [2.51], [2.62], [2.72], [2.75], [2.76-79], [2.108], [2.132], [2.158], [2.186], [2.219], [2.253], [2.262], [2.293], [2.311], [2.312], [2.326], [2.349], [2.441].

Chapter 10:

A: [1.13], [1.49], [2.45], [2.48], [2.98], [2.106], [2.107], [2.127–129], [2.183], [2.203], [2.230], [2.231], [2.270–272], [2.315], [2.333], [2.342], [2.366], [2.397], [2.402], [2.403], [2.410], [2.415], [2.419].

B: [2.62], [2.99], [2.105], [2.150], [2.157], [2.174], [2.180], [2.181], [2.227], [2.234], [2.235], [2.238], [2.259], [2.267-269], [2.272-274], [2.280], [2.284], [2.287], [2.289 -292], [2.298], [2.308], [2.314], [2.324], [2.345], [2.347], [2.363], [2.375], [2.420].

Chapter 11:

A: [1.24], [2.96], [2.116], [2.117], [2.199–202], [2.204], [2.300], [2.301], [2.321], [2.337], [2.377], [2.408], [2.409], [2.417].

B: [1.31], [2.3], [2.10], [2.11], [2.22], [2.71], [2.108], [2.115], [2.122], [2.147], [2.149], [2.163], [2.164], [2.171], [2.172], [2.263], [3.307], [2.344], [2.438], [3.111],[3.162], [3.165], [3.166].

Chapter 12:

A: [1.49], [1.56], [2.19], [2.89], [2.119], [2.137], [2.141], [2.142], [2.211], [2.248], [2.302], [2.330], [2.404].

B: [2.29], [2.92], [2.94], [2.102], [2.110], [2.111], [2.135], [2.136], [2.145], [2.160], [2.161], [2.168], [2.178], [2.249], [2.278], [2.328], [2.329], [2.335], [2.378],[2.436], [2.437], [2.146], [3.174], [3.253], [3.254].

Chapter 13:

A: [2.6], [2.7], [2.27], [2.59], [2.84–86], [2.354], [2.356], [2.359], [2.453], [2.454], [2.456], [2.463–466], [3.193], [3.270].

B: [2.53], [2.65–67], [2.69], [2.87], [2.90], [2.91], [2.103], [2.166], [2.225], [2.239], [2.247], [2.255], [2.327], [2.353], [2.355], [2.358], [2.360], [2.361], [2.364],[2.446], [2.449], [2.450], [2.452], [2.460], [2.467], [2.468], [3.7], [3.8], [3.38], [3.39], [3.108], [3.184], [3.195], [3.283], [3.303].

Chapter 14:

A: [2.6], [2.7], [2.12], [2.23], [2.24], [2.121], [2.187], [2.388], [2.447], [2.448], [2.455], [2.462], [2.466], [3.2], [3.3], [3.7], [3.25], [3.53], [3.69–71], [3.86–89], [3.95–98], [3.139], [3.140], [3.142–144], [3.155], [3.171–173], [3.176], [3.205], [3.207], [3.231–235], [3.255], [3.258], [3.277], [3.284], [3.300], [3.316].

B: [2.120], [2.261], [2.387], [2.457–459], [3.5], [3.6], [3.10], [3.15], [3.20], [3.28–31], [3.35–37], [3.44–46], [3.54–57], [3.59], [3.64–66], [3.75], [3.81], [3.94], [3.100], [3.117], [3.118], [3.124–126], [3.145], [3.164], [3.170], [3.175], [3.177], [3.178], [3.181], [3.183], [3.198], [3.203], [3.204], [3.206], [3.208], [3.217], [3.218], [3.220–222], [3.236–238], [3.251], [3.252], [3.256], [3.257], [3.259–261], [3.278], [3.282], [3.294–296], [3.298], [3.299], [3.301], [3.313], [3.315], [3.317], [3.18].

Chapter 15:

A: [1.21], [1.51], [2.129], [2.165], [2.195], [3.13], [3.119], [3.148],[3.149], [3.242], [3.272], [3.273], [3.302].

B: [1.25], [1.50], [1.52], [2.73], [2.126], [2.151], [2.155], [2.162], [2.228], [3.4], [3.27], [3.34], [3.40], [3.106], [3.107].

Chapter 16:

A: [1.8], [1.22], [1.51], [1.53], [2.37], [2.185], [2.191], [2.195], [2.226], [3.76], [3.82], [3.99], [3.241], [3.304].

B: [1.33], [1.58], [1.59], [2.68], [2.70], [2.74], [2.100], [2.139], [2.148], [2.151–153], [2.155], [2.192–194], [2.212–215], [2.218], [2.229], [2.244], [2.257], [2.258], [2.288], [2.305], [2.312], [2.331], [2.339–341], [2.428], [2.431], [2.432], [3.11], [3.42], [3.43], [3.49], [3.80], [3.90], [3.147], [3.153], [3.154], [3.167], [3.168], [3.180], [3.211–213].

Chapter 17:

A: [2.35], [2.124], [2.125], [2.240], [2.309], [2.376], [3.393], [3.1], [3.14], [3.18], [3.19], [3.33], [3.47], [3.77], [3.78], [3.85], [3.101], [3.112–114], [3.120–123], [3.197], [3.227], [3.228], [3.250], [3.285], [3.288–290], [3.305–308].

B: [2.42], [2.60], [2.61], [2.80], [2.81], [2.83], [2.123], [2.143], [2.144], [2.151], [2.152], [2.154], [2.159], [2.167], [2.207–209], [2.416], [3.17], [3.41], [3.60], [3.61], [3.74], [3.83], [3.84], [3.92], [3.93], [3.102], [3.103], [3.110], [3.150], [3.201], [3.202], [3.228], [3.229], [3.271], [3.276], [3.280], [3.291–293], [3.314].

Chapter 18:

A: [1.24], [2.5], [2.57], [2.130], [2.133], [2.134], [2.188], [2.189], [2.318], [2.319], [2.394], [2.405], [2.406], [2.407], [3.67], [3.72], [3.156], [3.158], [3.262–267].

B: [2.4], [2.58], [2.101], [2.132], [2.156], [2.157], [2.190], [2.236], [2.237], [2.241], [2.285], [2.303], [2.365], [2.379], [2.380], [2.382], [2.383], [2.407], [2.435], [3.21–24], [3.26], [3.116], [3.141], [3.159], [3.160], [3.196], [3.215], [3.216], [3.247], [3.248], [3.287].

Chapter 19:

A: [3.9], [3.52], [3.62], [3.115], [3.128], [3.129], [3.132], [3.134–136], [3.138],[3.185], [3.188], [3.189], [3.192], [3.226], [3.230], [3.240], [3.274], [3.275].

B: [2.223], [2.439], [3.48], [3.50], [3.51], [3.63], [3.68], [3.73], [3.109], [3.127], [3.130], [3.131], [3.133], [3.137], [3.150], [3.151], [3.169], [3.179], [3.183],[3.186], [3.187], [3.190], [3.191], [3.194], [3.199], [3.200], [3.219], [3.223–225], [3.249], [3.259], [3.268], [3.269], [3.280], [3.309–312].

Subject Index