ПО 801

London Mathematical Society Lecture Note Series. 34

Representation Theory of Lie Groups

Proceedings of the SRC/LMS Research Symposium on Representations of Lie Groups, Oxford, 28 June-15 July 1977

M.F. ATIYAH G. LUSZTIG

R. BOTT I.G. MACDONALD

S. HELGASON G.W. MACKEY

D. KAZHDAN W. SCHMID

B. KOSTANT D.J. SIMMS

with the editorial assistance of G.L. LUKE

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE

LONDON NEW YORK NEW ROCHELLE

MELBOURNE SYDNEY

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, N.Y. 10022, USA
296 Beaconsfield Parade, Middle Park, Melbourne 3206, Australia

© Cambridge University Press 1979

ISBN 0 521 22636 8

First published 1979

Printed in Great Britain by
Redwood Burn Ltd., Trowbridge and Esher

Contents

1 · Introduction

M.F. ATIYAH

Lie groups and their representations occupy an important place in mathematics, with applications and repercussions over a wide front. The connections with various aspects of physics are of long-standing, as are the intimate relations with differential equations and differential geometry. More recently the global topology of Lie groups has provided a deep link with questions of number theory. Finally, when viewed as 'non-commutative harmonic analysis' the theory of representations is a branch of linear analysis.

The symposium held in Oxford in July 1977 was designed to provide an introduction to the representation theory of Lie groups on as wide a front as possible. The main lectures, which are reproduced in this volume, should give the reader some indication of the scope and results of the subject. Inevitably there are gaps in various directions, and some areas are treated in greater detail than others. This reflects the particular interests of the participants and is not to be taken as a measure of relevant importance. Broadly speaking the symposium centred on the classical case of real Lie groups and treated only briefly the p-adic and finite fields.

In Part I of these notes we have collected together the introductory material and in Part II, the more advanced lectures.

The symposium was jointly sponsored and financed by the Science Research Council and the London Mathematical Society. The editorial work involved in turning lectures into manuscript was ably supervised by Glenys Luke and I am grateful to her, to the lecturers and to all others involved for their help in producing this volume.

2 · Origins and early history of the theory of unitary group representations

G.W. MACKEY
Harvard University

The theory of group representations was created by Frobenius in 1896 in a more or less deliberate attempt to generalize the theory of characters of finite abelian groups. The latter notion was only formally defined in full generality by Weber in 1881. Weber's definition was an abstraction of one given three years earlier by Dedekind and Dedekind was more or less directly inspired by Gauss' implicit use of characters of order two in his *Disquisitiones Arithmeticae* published in 1801.

To go back a bit further, Lagrange in the early 1770's wrote a two-part memoir making a systematic study of equations of the form

$$Ax^2 + Bxy + Cy^2 = n \quad .$$

Here A, B, C and n are integers and the problem is to find all integer pairs x,y satisfying the equation. Various special cases had been studied by Fermat in the seventeenth century and by Euler in the eighteenth and Lagrange's aim was to construct a systematic general theory. He observed that the transformation

$$x^1 = ax + by \; , \quad y^1 = cx + dy$$

where a, b, c and d are integers with $ad-bc = 1$ carries the equation into an equivalent one having the same values for the 'discriminant' $B^2 - 4AC$ and proved that there can be at most a finite number of inequivalent equations with a given value $D = B^2-4AC$. This number, called the class number, is of key importance in the developed theory. Gauss in the work cited above defined a notion of 'composition' for equivalence classes of forms of a given discriminant (his definition of equivalence was not quite the same as that of Lagrange) and showed in effect that under this composition law the equivalence classes form a group. We say 'in effect' because the concept of 'group' did not then exist. In developing the theory of equations whose class number is greater than one, he used what amounted to characters of order two of the group of equivalence classes and in this connection introduced the word 'character'.

As defined by Weber, a character of a finite abelian group A is a homomorphism $x \to \chi(x)$ of A into the multiplicative group of complex numbers of modulus one. It is evident that the set $\hat{A}$ of all characters of A is itself a finite abelian group under multiplication. Moreover, it is not hard to see that every complex-valued function f on A may be written uniquely as a linear combination of characters

$$f = \sum_{\chi\in\hat{A}} C_\chi . \chi \quad \text{where} \quad C_\chi = \frac{1}{o(A)} \sum_{x\in A} f(x)\, \overline{\chi(x)}$$

and $o(A)$ is the order of A. The analogy with Fourier series expansions is evident and many arguments in nineteenth century number theory may be interpreted as Fourier analysis on finite abelian groups. Dirichlet, in particular, used characters on the multiplicative group of units in the ring of integers mod m and finite Fourier analysis is the key to one step in his celebrated proof that there are an infinite number of primes in any arithmetic progression which can not be extended to contain zero.

The primary impetus to the development of group theory itself was provided by another long memoir of Lagrange published shortly after the one mentioned above. In it he made a penetrating study of the solutions of polynomial equations by radicals. He managed to understand in a unified way the known methods for solving equations of the second, third and fourth degrees and tried (nearly successfully) to understand why the fifth degree equations had proved so intractible. In particular, he saw that the key to the question lay in studying what happened to rational functions of the roots when the roots were permuted amongst themselves. Inspired by this work of Lagrange, Cauchy founded the theory of permutation groups in 1815 and by 1831, Ruffini and Abel had proved the impossibility of solving the general quintic and Galois had worked out his beautiful theory relating solvability to the structure of the 'Galois group' of the equation. It is to Galois that we owe the term group and the concept of normal subgroup. On the other hand, the theorem that the order of a subgroup divides the order of the group is already implicit in Lagrange's paper.

For various reasons, including Galois' premature death at the age of 20, his paper was not published until 1846. At this time Hermite and Kronecker were young men at the beginning of their careers and both became quite active in developing Galois' ideas. However, group theory did not begin to be widely known or to be applied outside of a rather narrow context until around 1870. At that time, three events occurring in the space of as many years, stimulated a considerable expansion in the scope of group theory as well as an increased awareness of the existence and importance of this new branch of mathematics. In 1869 Sophus Lie began to apply the ideas of Galois to differential equations and initiated the systematic study of continuous (actually differentiable) groups. In 1870 C. Jordan published the first book ever to be written on group theory. His *Traités des substitutions et des equations algébriques* contained among other things a clear exposition of Galois theory. Finally, in 1872 Felix Klein announced his celebrated Erlanger program for unifying geometry through group theory and shortly thereafter began a sort of publicity campaign to convince mathematicians of the fruitfulness and wide applicability of the group theoretic point of view.

The parallelism between Fourier analysis on finite commutative groups as indicated above and Fourier analysis as more commonly understood arises of course because the functions $x \to e^{inx}$ are precisely the continuous characters on the compact continuous group obtained from the additive group of the real line by factoring out the discrete subgroup of all integer multiples of 2π. However, the fact that such a connection exists does not seem to have been explicitly noticed until the middle 1920's. The theory of Fourier series and integrals arose in the early nineteenth century to meet the needs of mathematical physics. In the middle of the eighteenth century D. Bernoulli, D'Alembert and Euler succeeded in extending Newton's analysis of particle motion to an analysis of the motion of fluids and deformable solids. More precisely, they found the analogues of Newton's equations of motion. These turned out to be differential equations in which partial derivatives of functions of several variables replaced the ordinary derivatives in Newton's work. Such partial differential equations presented mathematicians with a new and difficult challenge which was by no means met immediately. Progress was slow until Fourier submitted his celebrated memoir on heat conduction to the French academy in 1807. The methods which Fourier used and which are now taught to every mathematics and physics student were quickly seen to apply to many of the partial differential equations arising in physical problems and by the time Fourier's book on heat conduction appeared in 1822, Poisson and Cauchy had been active for years in applying them to a variety of problems. Actually Fourier's expansibility theorem was nearly discovered half a century earlier in connection with studies of the one dimensional wave equation

$$\frac{1}{\nu^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial^2\Psi}{\partial x^2} .$$

However, prejudices of the time made the result implausible to many and in the end the key clues were ignored. Lagrange who developed and systematized the work of Euler, Bernoulli and D'Alembert and incorporated it into his great synthesis of 1787 *Mécanique Analytique* came close to finding Fourier's theorem but he also refused to accept it. In fact he was one of the referees who at first rejected Fourier's memoir of 1807.

A group representation as defined by Frobenius is a homomorphism $x \to L_x$ of a finite group G into the multiplicative group of all $n \times n$ non-singular complex matrices for some $n = 1,2,\ldots$. Its character χ^L is the complex valued function on G defined by $\chi^L(x) = \text{Trace}\ (L_x)$. This definition evidently reduces to that of Dedekind and Weber when $n = 1$. More generally one shows that $\chi^L(x) \equiv \chi^M(x)$ if and only if L and M have the same dimension (or degree) n and there exists a non-singular $n \times n$ matrix W such that $W^{-1}L_xW = M_x$ for all x. One then says that L and M are equivalent. One also shows that for each finite group G there exists a unique finite set $\chi_1, \chi_2, \ldots, \chi_r$ of linearly independent characters on G such that the finite linear combinations $n_1\chi_1 + n_2\chi_2 + \ldots + n_r\chi_r$ (where the n_j are non-negative integers) are precisely the characters of G . Here r is the number of distinct conjugacy classes of G and finding the χ_j (the so-called irreducible characters) can be a highly non-trivial problem.

The immediate stimulus for Frobenius' introduction of group representations and their characters was a problem of Dedekind concerning a little-known concept - the group determinant - which he began to work on in the 1880's. He could solve it in some cases using characters of finite groups and solicited the help of Frobenius in dealing with more general ones. Apparently the problem in group determinants was suggested by the study of the discriminant of an algebraic number field. Frobenius succeeded using his new generalized characters - which he invented expressly for the purpose. The exact story of the relationship between Dedekind's problem and the introduction of higher dimensional characters is complicated and has only recently been elucidated. For further details the reader is referred to three recent articles by Thomas Hawkins in *Archiv for the history of the exact sciences*.

For the next quarter of a century or so the theory of group representations was a branch of pure algebra concerned more or less exclusively with the development of Frobenius' ideas by Frobenius himself, by Burnside and by I. Schur and others. There were striking applications to the structure theory of finite groups (for example, the theorem of Burnside that a group whose order is divisible by only two primes is solvable) but none outside of group theory. However, in the 1920's the situation changed radically. The scope of the theory was enlarged so as to apply to compact Lie groups by work of Hurwitz, Schur, Cartan and Weyl and at the same time important applications were found to number theory and to the new quantum physics.

In 1924 Schur observed that one could apply earlier ideas of Hurwitz on integration over manifolds to define integration of continuous functions defined on compact Lie groups. Using this as a substitute for summing over the group he was able to extend the main ideas of group representation theory from finite groups to compact Lie groups. He also was able to determine all of the irreducible representations of the orthogonal groups. In the next three years Weyl determined the irreducible representations (and their characters) of all the classical compact semisimple Lie groups and in collaboration with F. Peter proved the celebrated Peter-Weyl theorem. This asserts in essence that the matrix coefficients of the irreducible representations of a compact Lie group are plentiful enough so that every continuous function on the group can be uniformly approximated by their linear combinations. It follows that one can obtain an orthonormal basis for the square integrable functions on the groups whose members are such matrix elements. When the group is commutative the basis elements are necessarily complex multiples of characters and the Riesz-Fischer theorem in the theory of

Fourier series is a special case. It was in this work of Weyl that the group theoretical character of classical harmonic analysis was first clearly pointed out. Weyl also pointed out that the classical theory of expansions in spherical harmonics has a group theoretical interpretation and moreover one demanding consideration of higher dimensional representations of a non-commutative group. This observation of Weyl was generalized and further developed by E. Cartan. On the other hand Weyl made heavy use of earlier work of Cartan on the Lie algebras of the classical compact Lie groups in his determination of their representations. Cartan in earlier work had given what amounted to an infinitesimal version of Weyl's results.

The first application of the theory of higher dimensional group representations outside of group theory itself seems to have been made by E. Artin in 1923. Let K be an algebraic number field; that is, a finite extension of the field Q of all rational numbers and let R_K be the ring of all 'algebraic integers' in K. Let H be any subgroup of the group G of all automorphisms of K and let k_H be the subfield of all elements of K which are carried into themselves by all members of H. The so-called zeta function ζ_K of K is defined for all $Re(s) > 1$ by the convergent Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$$

where $\phi(n)$ is the number of ideals I in R_K whose 'norm' $N(I)$ is n. By definition $N(I)$ is the number of elements in the quotient ring R_K/I. As shown by Hecke a few years earlier ζ_K is always continuable to a meromorphic function defined in the whole complex plane and satisfying a simple functional equation relating the values of $\zeta_K(s)$ to those of $\zeta_K(1-s)$. Of course, one can define ζ_{k_H} in the same way and the question arises as to the relationship between these two zeta functions. In the special case when H is commutative it was known from previous work of Takagi that ζ_K factors as a product of ζ_{k_H} and o(H)-1 'L functions'. These L functions are also analytically continuable Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{\Psi(n)}{n^s}$$

which satisfy simple functional equations and have no poles. Artin extended this result in two ways. First he obtained a completely different factorization of ζ_K/ζ_{k_H} valid for all H, commutative or not, and with the factors parameterized by the irreducible characters of H other than the trivial one. The corresponding factorization of ζ_K reflects the decomposition of the regular representation of H in that each factor occurs as many times as the corresponding irreducible representation occurs in the regular representation of H. Secondly when H is commutative he showed that his (conceptually) completely different factorization was the same as that of Takagi. In other words he showed that Takagi's results implied a reinterpretation of the classical L functions in terms of one dimensional characters of H and that the theory of group representations could be used to remove the restriction that H be commutative. Artin's generalized L functions are known to share many properties of the classical ones. However it is still an open question as to whether they are entire.

A rather different application of the theory of group representations to number theory was made by E. Hecke in 1928. Actually this application was made indirectly via the theory of modular forms - a theory having extremely close connections with number theory. Let ℓ and k be positive integers and let Γ_ℓ be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which ad-bc = 1 and a-1, b, c and d-1 are all integer multiples of ℓ. Then a modular form of weight k and level ℓ is an entire function f defined in the upper half of the complex plane which satisfies the identity

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z)$$

for all $\binom{ab}{cd} \in \Gamma_\ell$ as well as certain growth conditions which need not be specified here. One shows that the space of all modular forms of a given weight and level is a finite dimensional vector space and a major problem is to find a basis for this space. It is evident that a modular form of level ℓ is periodic with period ℓ and so of the form

$$\sum_{j=-\infty}^{\infty} c_j e^{2\pi ij/\ell}$$

where the c_j are complex numbers. The imposed growth conditions are such that $c_j = 0$ for $j<0$ and the complex numbers $c_0, c_1, c_2, \ldots$ are called the Fourier coefficients of the form. As functions of j the Fourier coefficients of modular forms have important number theoretical properties.

In the case of forms of level 1 a fairly complete theory was worked out by Klein and his pupil Hurwitz and published in Hurwitz's thesis of 1881. Hecke's paper of 1928 makes the following important observation about forms of higher level. Let $V_{k,\ell}$ denote the vector space of all modular forms of weight k and level ℓ and let $\tilde{V}_{k,\ell}$ be the smallest vector space containing $V_{k,\ell}$ and invariant under the linear operator

$$f(z) \rightarrow f\left(\frac{az+b}{cz+d}\right) (cz+d)^{-2k}$$

for all $\binom{ab}{cd} \in \Gamma_1$. Then $\tilde{V}_{k,\ell}$ is finite dimensional and there is a natural representation of the quotient group Γ_1/Γ_ℓ whose space is $\tilde{V}_{k,\ell}$. The decomposition into multiples of irreducible representations of this representation of the finite group Γ_1/Γ_ℓ carries with it a decomposition of $V_{k,\ell}$ as a direct sum of lower dimensional spaces parameterized by irreducible characters of Γ_1/Γ_ℓ. Thus the problem of finding a basis for $V_{k,\ell}$ is broken down into subproblems and a part of the difficulty revealed to lie in understanding the representation theory of Γ_1/Γ_ℓ. The latter becomes quite difficult when ℓ is divisible by the cube of a prime and was only completely elucidated in the middle 1970's.

During roughly the same period in which Frobenius' theory of group representations was being developed as a new branch of algebra important results were being found in analysis and in physics. These separate and independent advances in algebra, analysis and physics were destined to be further developed and blended together in a very significant way in the next two decades or so. In analysis there was the introduction of the Lebesgue integral in 1902, and at nearly the same time the work of Fredholm on integral equations. The former made possible the beautiful Riesz-Fischer theorem in the theory of Fourier series and through this led the way to the Hilbert space concept. The latter, the work of Fredholm, inspired Hilbert to make his profound study of integral equations - including especially his study of linear operators with continuous spectra. In physics the period from 1900 to 1925 was the period of the so-called 'old quantum theory'. Beginning with Planck's celebrated work on heat radiation in 1900 and proceeding through Einstein's theory of specific heats and the photo electric effect to Bohr's theory of the hydrogen atom in 1913, physicists found numerous ways in which experimental results could be 'explained' or 'deduced' by combining classical mechanics with ad hoc discreteness assumptions.

The synthesis came in 1927. In late 1924 and early 1925 Heisenberg and Schrödinger published mysterious but suggestive papers showing how one could deduce the spectrum of the hydrogen atom without making a priori discreteness assumptions. Their methods were completely different on the surface but were later shown to be mathematically equivalent. They had found the key to the mystery of 'quantization' and within a short time physics was revolutionized by the emergence of the subtle refinement of classical mechanics known as quantum mechanics. Quantum mechanics is a self consistent scheme which reduces to classical mechanics when masses (or energies) are sufficiently large and which leads automatically to the discrete quantum rules of the old quantum theory. Its chief architects other than Heisenberg and Schrödinger were Born, Jordan and Dirac. By a most remarkable coincidence it developed that Hilbert's spectral theorem (generalized to apply to unbounded operators) and the theory of group representations were both to play a central role in the development of quantum mechanics. In 1927 von Neumann showed that the concept of abstract Hilbert space and the spectral theorem for self adjoint operators were just what was needed to make rigorous mathematical sense out of the (sometimes vague and cloudy) formulations of physicists. In the same year Weyl published a paper exploring certain foundational questions via the correspondence between self adjoint operators H and one parameter families of unitary operators $t \to U_t$ set up by the equation $U_t = e^{itH}$. The U_t are all unitary and the correspondence $t \to e^{itH}$ is actually a homomorphism of the additive group of the real line into the group of all unitary operators in the relevant Hilbert space. Thus it is a (usually infinite dimensional) group representation of a non-compact (but locally compact) group. Finally the year 1927 also saw the publication of a paper by Wigner in which he showed how the representation theory of the symmetric group on n elements could be utilized to make rather drastic simplifications in applying 'perturbation theory' to approximate the predictions of quantum mechanics concerning the spectrum of an n electron atom. The possibility of doing this had been suggested to Wigner by von Neumann and the method was further developed and extended to the (compact) rotation group in joint work of the two men. In 1928 the ideas of von Neumann, Wigner and Weyl himself were extended and presented in integrated form in a remarkable book by Weyl entitled *Gruppentheorie and Quantenmechanik*.

In 1930 M.H. Stone published a short note rigorizing some of the considerations of the 1927 paper of Weyl cited above. In particular he proved an analogue of Hilbert's spectral theorem for unitary representations of the additive group of the real line. Combined with von Neumann's extension of the spectral theorem to unbounded self adjoint operators, Stone's theorem implied a natural one-to-one correspondence between all self adjoint operators (bounded or not) and all unitary representations of the additive group of the real line. In this correspondence $t \to U_t$ and H correspond to one another if and only if $U_t = e^{itH}$. Conversely, it is easy to see that in the special case of bounded H the spectral theorem of Hilbert and the later work of Hahn and Hellinger on 'spectral multiplicity theory' imply a complete decomposition and classification theory for all unitary representations of the real line having a bounded H in the above mentioned correspondence. In other words, the work of Hilbert and his students on spectral theory in the first decade or so of the twentieth century was equivalent to studying an important special case of the unitary representation theory of a non-compact commutative group - the additive group of the real line.

In spite of the existence of Stone's theorem, the integration of spectral theory with the theory of representations of finite groups and compact Lie groups did not take place at once. The first direct stimulus toward a more general theory including both was a remarkable paper by A. Haar, published in 1933, showing that one had a natural extension of the Hurwitz integration process which applied not just to compact Lie groups but to all separable, locally compact topological groups. Every such group admits a measure invariant under right (left) translation which is positive on open sets and finite on compact sets. As von Neumann showed a few years later this 'Haar measure' (right or left) is uniquely determined by the group up to a multiplicative constant. Haar observed that using his measure one could extend the Peter-Weyl theorem and the general theory of unitary representations from compact Lie groups to arbitrary separable compact groups.

This stimulus by Haar was followed in 1934 and 1935 by the work of Pontryagin and van Kampen extending the duality between finite commutative groups and their character groups to a similar but deeper theory involving commutative, locally compact topological groups. If G is a commutative, locally compact topological group and $\hat{G}$ is the set of all continuous unitary characters, that is, all continuous homomorphisms of G into the group of all complex numbers of modulus one, then $\hat{G}$ is itself a locally compact, commutative, topological group in a natural way and each x in G defines a member f_x of $\hat{\hat{G}}$ via the formula $f_x(\chi) = \chi(x)$ for all $\chi \in \hat{G}$. The fundamental duality theorem of Pontryagin and van Kampen asserts that the map $x \to f_x$ sets up an isomorphism between G and $\hat{\hat{G}}$ as topological groups. Interesting examples of locally compact, commutative groups which are not Lie groups (and not discrete) are the additive and multiplicative groups of the locally compact field of all p-adic numbers for various primes p. Further examples may be obtained by replacing the p-adic number fields by their finite algebraic extensions and by the field of formal Fourier series with coefficients in a finite field. The 'idèle groups' introduced by Chevalley in 1936 to deal with certain number theoretical questions are infinite subdirect products most of whose factors are multiplication groups of extensions of p-adic number fields and by 1940 Chevalley had shown how to redo the 'class field theory' of Hilbert, Takagi and Artin in a way which exploited Pontryagin duality.

With Haar measure, the Peter-Weyl theorem and Pontryagin -

van Kampen duality at hand, it was possible to construct a fairly complete representation theory of locally compact groups which were either commutative or compact. This theory was presented from rather different points of view in two very influential books, *Topological Groups* by L. Pontryagin was published in Russian in 1938 and in English translation in 1939. A. Weil's *L'intégration dans les groupes topologiques et ses applications* appeared in 1940. Weil emphasized the connections with harmonic analysis much more than Pontryagin. The generalized Fourier transform of Weil led to a complete analysis of the regular representation of an arbitrary locally compact, commutative group. However, the fact that one could generalize Stone's theorem and so analyse an arbitrary representation of such a group was not noticed until later. It was pointed out in 1944 in independent papers of Ambrose, Godement and Naimark.

With compact groups and locally compact commutative groups more or less under control, the obvious next step was to unify the two theories in a single theory which applied to locally compact groups that are neither commutative nor compact. This program began to be developed in a systematic way in 1946 shortly after the end of World War II. However, several significant contributions were made in the preceding decade. In 1939 Wigner published a now celebrated paper analysing the irreducible unitary representations of the inhomogeneous Lorentz group and in 1943 Gelfand and Raikov showed that every locally compact group has 'sufficiently many' irreducible unitary representations. In Wigner's group all but the identity representation turned out to be infinite dimensional and generally speaking it is rather exceptional when a non-compact non-commutative group has only finite dimensional irreducible representations. Although its relevance was not immediately apparent, the work of Stone and von Neumann in 1930 and 1932 on the solutions of the Heisenberg commutation relations is equivalent to determining the irreducible unitary representations of a certain non-commutative, non-compact nilpotent group. Still another important contribution whose relevance became clear only later was the von Neumann - Murray theory of rings of operators published in four instalments between 1936 and 1943.

The systematic theory which developed after 1946 proved to be much richer than its compact and commutative predecessors. The necessity of dealing simultaneously with infinite dimensional irreducibles and with continuous decompositions produced many new difficulties. The resulting literature is now enormous and cannot be briefly summarized. A lengthy summary will be found in the appendix to the author's book *Unitary Group Representations* (University of Chicago Press, 1976). For a much more detailed account of the development from Lagrange to 1946 the reader may consult the author's article 'Harmonic analysis as the exploitation of symmetry' - an historical survey, Rice University Studies (to appear).

3 · Induced representations

GEORGE W. MACKEY

1. Definitions and examples

Let G be a finite group. Then by a representation L of G one means a homomorphism $x \to L_x$ of G into the group of all non-singular linear transformations of some vector space $H(L)$ onto itself. In the theory developed by Frobenius $H(L)$ is a finite dimensional vector space over the complex numbers. The character χ^L of L is the complex valued function $x \to \text{Trace}(L_x)$. Two representations L and M are said to be equivalent, $L \simeq M$, if there exists a bijective linear transformation V from $H(L)$ to $H(M)$ such that $V^{-1}M_xV = L_x$ for all x in G . One shows that L and M are equivalent if and only if $\chi^L(x) \equiv \chi^M(x)$. One defines the direct sum $L \oplus M$ of two representations in an obvious way as the direct sum of these vector spaces and shows that $\chi^{L\oplus U} = \chi^L + \chi^M$. A fundamental theorem asserts that every representation is a direct sum of representations L which are *irreducible* in the sense that $H(L)$ has no proper subspaces which are carried into themselves by all L_x . Thus every character is a sum of characters of irreducible representations, i.e., of 'irreducible characters'. The irreducible characters are *orthogonal* with respect to summation over the group and hence, in particular, are linearly independent. Thus every character is *uniquely* a finite linear combination of irreducible characters with non negative integer coefficients. It follows that to know all characters (or equivalently all equivalence classes of representations) it suffices to know the irreducible characters.

While each group has only a finite number of distinct irreducible characters (or equivalence classes of irreducible representations) actually constructing them can be a problem of considerable difficulty and is in fact one of the main problems of the theory. To deal with the problem Frobenius invented a canonical way of constructing characters of G from characters of its subgroups which often (but by no means always) leads to irreducible characters of G . This construction is called *inducing* and is capable of generalisation to infinite dimensional unitary representations of locally compact groups. We shall be chiefly interested in this generalisation, its properties and its utility in finding and describing irreducible unitary representations. However we shall begin with this simple case treated by Frobenius.

Let χ be a character of the subgroup H of the finite group G . An obvious way to extend χ to G is to define it to be zero for x not in H . Denote the resulting function on G by χ^1 . Since characters are easily seen to be invariant under all inner automorphisms, χ^1 will seldom be a character. We can make it invariant under inner automorphisms by the simple device of summing its transforms by the inner automorphisms. If y_1 and y_2 belong to the same right H coset $\chi^1(y_1xy_2^{-1}) = \chi^1(y_2xy_2^{-1})$ so it makes sense to define

$$\chi^*(x) = \sum_{y\in G/H} \chi^1(yxy^{-1}) \quad .$$

The function χ^* can be shown to be a character of G and is called the character of G induced by χ . The proof that χ^* is a character is most easily given by constructing the representation of which it is the character. Let L be a representation of H whose character is χ and consider the finite dimensional vector space of all functions f from G to the space $\mathcal{H}(L)$ of L which satisfy the identity $f(hx) = L_h(f(x))$ for all $h \in H$ and $x \in G$. This space is invariant under right translation by elements of G and we obtain a representation U^L of G in this space by setting

$$(U^L_x)(f(y)) = f(yx) \quad .$$

A straightforward calculation shows that

$$\chi^{U^L} = \chi^* \quad .$$

The character χ^* cannot be irreducible unless χ is irreducible and need not be irreducible even then. For example if $H = \{e\}$ and $\chi(e) = 1$ then χ^* is the character of the so-called regular representation which contains *every* irreducible representation of G as a constituent. On the other hand, as already remarked χ^* often is irreducible when χ is and the construction provides an important source of irreducible characters. Here are two simple examples.

Example I. Let G be the symmetric group on three letters and let N be the subgroup generated by $a \to b$, $b \to c$, $c \to a$. Then N is normal and G/N is a cyclic group of order 2. Since G has three conjugate classes it can have only one irreducible character in addition to the two one dimensional ones obtained from G/N. To obtain it one has only to form χ^* where χ is either of the two non trivial characters of N.

Example II. Let F be a finite field, let G denote $SL(2,F)$ the group of all 2×2 matrices with determinant 1 and elements in F. Let T be the subgroup of all matrices of the form $\begin{pmatrix} \lambda & 0 \\ a & 1/\lambda \end{pmatrix}$ and notice that the map $\begin{pmatrix} \lambda & 0 \\ a & 1/\lambda \end{pmatrix} \to \lambda$ is a homomorphism of T into the multiplicative group F^* of all non zero members of F. Then for each character χ of F^*, $\begin{pmatrix} \lambda & 0 \\ a & 1/\lambda \end{pmatrix} \to \chi(\lambda)$ is a one dimensional character χ^1 of T and $(\chi^1)^*$ can be shown to be irreducible whenever $\chi^2 \neq 1$; $(\chi_1^1)^*$ and $(\chi_2^1)^*$ are distinct whenever $\chi_1 \neq \chi_2$ and $\chi_1 \neq \chi_2^{-1}$. In this way one obtains about half of the irreducible characters of G.

As already indicated above the chief aim of these lectures is to define and describe the properties of a generalisation of Frobenius' construction in which H is a closed subgroup of a separable locally compact group G and L is a possibly infinite dimensional unitary representation of H. While L will often be finite dimensional the corresponding induced representation U^L will be infinite dimensional except when H has finite index in G. Since characters of infinite dimensional representations only exist in special cases and then must be defined in a round about way it is necessary to avoid characters and deal directly with the representations L and U^L. While it is quite straightforward to generalise the definition of U^L given above we prefer to lead the reader to it by a less direct route in order to emphasize the connection with a natural generalisation of harmonic analysis.

Let G be a separable locally compact group and let S be a space on which G acts as a group of transformations; that is, $[s]x$ is a well defined element of S for each s in S and each x in G and one has $[[s]x]y = [s]xy$ and $[s]e = s$ where e is the identity of G. We say that S is a G space. If S has been provided with a distinguished σ field of subsets (called its Borel sets) in such a way that $(s,x) \to [s]x$ is a Borel function, we say that S is a Borel G space. We remark that G as a topological space has a 'natural' Borel structure - that provided by the σ field generated by its open sets. By a Borel function f from one Borel space to another one means a function f such that $f^{-1}(A)$ is a Borel set wherever A is a Borel set. It turns out that if A_1 and A_2 are Borel subsets of separable complete metric spaces M_1 and M_2 then A_1 and A_2 are isomorphic as Borel spaces if and only if they have the same cardinal number. Moreover the only infinite cardinals that can occur are $\aleph_0$ and C the cardinal of the continuum. Such Borel spaces are called *standard*.

Now let S be a standard Borel G space where G is as above and let μ be a finite measure on S which is *invariant* in the sense that $\mu([E]x) = \mu(E)$ for all x in G and all Borel sets E. Form the Hilbert space $L^2(S,\mu)$ and for each x in G let U_x denote the unitary operator such that $U_x(f) = f([s]x)$. Then it can be shown that $x \to U_x$ is a *unitary representation* of G in the sense that (1) it is a homomorphism of G into the group of all unitary operators in $L^2(S,\mu)$ and (2) $x \to U_x(f)$ is a continuous function from G to $L^2(S,\mu)$ for each f in $L^2(S,\mu)$. Actually (1) is obvious and it is not difficult to prove that $x \to (U_x(f).g)$ is a Borel function for all f and g in $L^2(S,\mu)$. One can show quite generally that (2) is implied by (1) and this property of $(U_x(f).g)$.

There are many instances in mathematics and its applications where one is interested in complex valued functions f defined on a measure space (S,μ) which is invariant under a group G as described above. For example, in considering functions on the real line of period $a > 0$ one may take G to be the additive group of the real line and S to be its quotient by the group of integer multiples of a. Simple translation gives an action of G on S which preserves Lebesgue measure μ. In this particular case the exponential functions

$$y \to e^{\frac{2\pi i n y}{a}} \quad n = 0,\pm1,\pm2,\ \ldots$$

play a special role in that each generates a one dimensional G invariant subspace of $L^2(S,\mu)$ and that there are no other one dimensional G invariant subspaces. Moreover the whole of $L^2(S,\mu)$ is a direct sum of these subspaces and, as everybody knows, the decomposition of an arbitrary function in $L^2(S,\mu)$ as a sum of multiples of the $e^{\frac{2\pi i n y}{a}}$ is a very useful device in attacking many problems. One speaks of the harmonic analysis of f. In more general cases one can attempt to analyse $L^2(S,\mu)$

as a direct sum (or direct integral) of invariant subspaces and hope that the resulting decomposition will be a useful tool in studying various questions about functions f on S. In other words decomposing unitary representations U, defined as above by measure preserving actions of the group G, can be regarded as a natural generalisation of classical harmonic analysis which one may hope will turn out to be as useful.

Given S, μ, G as above let us examine the problem of decomposing the unitary representations U of G. First of all a certain decomposition may result from purely geometric considerations. This happens whenever S admits Borel subsets E such that $\mu(E), \mu(S-E) \neq 0$ and $[E]x = E$ for all x in G. In that case we may restrict μ to E and $S-E$ respectively and obtain the G spaces E and $S-E$ with invariant measures μ_1 and μ_2 and corresponding unitary representations U^1 and U^2. Clearly U is a direct sum of U^1 and U^2. We shall not be interested here in the geometric aspects of the decomposition of U and will accordingly concentrate attention on the case in which no E with the above properties exists. In that case one says that the action of G on (S,μ) is *ergodic* or *metrically transitive*. There is a theorem which permits one to fiber every G space (S,μ) as a sum or integral of ergodic pieces in an essentially unique way.

Suppose then that our G action is ergodic. It is not difficult to show that one has a sharp dichotomy. Either (a) every orbit $[s]G$ is of measure zero or (b) there exists an orbit $[s_0]G$ of positive measure and then its complement is of measure zero. In case (b) one can throw away an irrelevant set of measure zero and obtain a transitive action. In case (a) there is no way of reducing to the transitive case and we say that the action is *properly ergodic*. In principle the transitive and properly ergodic cases are both interesting and deserving of study. However up to now the transitive case has been much more completely investigated. Thus in a study of generalised harmonic analysis it is natural to concentrate attention on the case in which the action of G on S is transitive.

Suppose then that G acts transitively on S and choose $s_0 \in S$. Let H_{s_0} denote the subgroup of all x in G with $[s_0]x = s_0$. It can be proved that H_{s_0} is closed and it is evident that replacing s_0 by another element of S simply changes H_{s_0} to a closed subgroup which is conjugate to it. Moreover the map $x \to [s_0]x$ is a map of G onto S which has the property that x_1 and x_2 have the same image in S if and only if $x_1x_2^{-1} \in H_{s_0}$, that is, if and only if $H_{s_0}x_2 = H_{s_0}x_1$ so that x_1 and x_2 belong to the same right H_{s_0} coset. It can be shown that the map of S onto the coset space G/H_{s_0} just defined is a Borel isomorphism. This S can always be identified with a coset space G/H where H is some closed subgroup of G. Moreover it is not hard to show that the invariant measure in G/H (when it exists) is uniquely determined up to a multiplicative constant by G and H. We thus have a natural unitary representation of G associated with every closed subgroup H of G for which G/H has an invariant measure μ. Functions from G/H may be identified with functions f on G such that $f(hx) = f(x)$ for all h in H and all x in G and when G is finite we see at once that the unitary representation U is precisely the representation U^{I_H} of G induced by the one dimensional identity representation I_H of H. It is now evident how to define U^{I_H} more generally. We take the definition in the finite case and add the condition that $|f(x)|^2$ be integrable with respect to μ when considered as a function on G/H. In particular we see that to a large extent our generalisation of harmonic analysis is just decomposing those induced representations for which the inducing representation is the one dimensional identity.

A slight and more or less obvious generalisation of the construction of U for a G space S turns out to be equivalent in the transitive case to inducing an arbitrary one dimensional unitary representation of H. Suppose indeed that we are given an arbitrary Borel function a from $S \times G$ to the complex numbers of modulus one and for each x in G let $U_x^a(f)(s) = a(s,x)f([s]x)$. Then for each x, U_x^a is clearly a unitary operator. Moreover a simple calculation shows that $x \to U_x^a$ is a unitary representation of G if and only if a satisfies the following identity (in an almost everywhere sense)

$$a(s,x_1x_2) = a(s,x_1)a([s]x_1,x_2) \quad .$$

Such identities are known in homological algebra as one cocycles. Thus for every G space S with invariant measure μ and every one cocycle a we have a well defined unitary representation U^a of G. If b is an arbitrary Borel function from S to the complex numbers of modulus one then $(s,x) \to b(sx)/b(s)$ is a one cocycle and one cocycles of this kind are called coboundaries. It is clear that the product of any two one cocycles is a one cocycle and that with this definition the one cocycles form a commutative group having the coboundaries as a subgroup. On the other hand one verifies easily that U^a and $U^{a'}$ are equivalent representations whenever a/a' is a coboundary, i.e. whenever a and a' are cohomologous. Thus one has a well defined equivalence class of representations associated to each cohomology class of cocycles. Suppose now that the action of G on S is transitive so that S may be identified with G/H_{s_0} for some closed subgroup H_{s_0}. Let a be any cocycle for which the cocycle identity holds everywhere. Then for h_1 and h_2 in H_{s_0} the cocycle identity reduces to

$$a(s_0,h_1h_2) = a(s_0,h_1)a(s_0,h_2)$$

so that $h \to a(s_0,h)$ is a one dimensional character of H_{s_0}. It is not hard to see that this character depends only on the cohomology class of a and that, conversely, cohomologous cocycles lead to the same character. One can show that one can avoid almost everywhere difficulties and that every character arises. Thus there is a natural one to one correspondence between the one dimensional characters of H and cohomology classes of cocycles. Combining this with the mapping $a \to U^a$ one has a natural mapping of one dimensional characters of H into equivalence classes of irreducible unitary representations of G. When G is finite this mapping is easily seen to reduce to the inducing construction. U^a is equivalent to the representation U^χ induced by the character χ of H which defines the cohomology class of a. Since U^a makes sense for infinite G we have a natural way of defining U^χ in this case also. Of course U^χ may also be defined directly by the following obvious modification of the definition when χ is the identity. The space $\mathcal{H}(U^\chi)$ of U^χ is the space of all Borel functions f from G to the complex numbers such that

(a) $f(hx) = \chi(h) f(x)$ for all $h \in H$ and all $x \in G$,

(b) $\int_{G/H} |f(x)|^2 \, d\mu < \infty$.

Here we identify functions which are equal almost everywhere and note that (b) makes sense because $|f(hx)|^2 = |\chi(h)f(x)|^2 = |f(x)|^2$ for all h in H and x in G. If $f \in \mathcal{H}(U^\chi)$ then $(U_x^\chi f)(y) = f(yx)$.

To be led to the definition of U^L for an arbitrary unitary representation L of H it is only necessary to replace $\mathcal{L}^2(S,\mu)$ by $\mathcal{L}^2(S,\mathcal{H}_0,\mu)$ the space of square summable functions on S with values in a second Hilbert space $\mathcal{H}_0$ and replace the scalar function a on $S \times G$ by a suitable function $(s,x) \to A(s,x)$ from $S \times G$ to the bounded linear operators in $\mathcal{H}_0$. We leave details to the reader. The direct definition of U^L of course is the following. The space $\mathcal{H}(U^L)$ of U^L is the space of all Borel functions f from G to $\mathcal{H}(L)$ such that

(a) $f(hx) = L_h(f(x))$ for all $h \in H$ and all $x \in G$

(b) $\int_{G/H} (f(x).f(x))\, d\mu < \infty$.

As before we identify functions which are equal almost everywhere and note that (b) makes sense because $(f(hx).f(h(x)) = (L_h f(x).L_h(f(x)) = (f(x).f(x))$ for all h in H and all x in G on account of the unitariness of the L_h . Finally $U_x^L(f)(y) = f(yx)$.

In all of the above discussion it was assumed that G/H admits an invariant measure. We remark now that this assumption is not really necessary. If G/H does not admit an invariant measure it has at least a measure which is *quasi invariant* in the sense that all of its transforms have the same sets of measure zero that it does. Choose any one of these and copy the above construction. The resulting operators U_x^L will not be unitary but become so when multiplied by appropriate Radon-Nikodym derivatives. One thus obtains a unitary representation whose equivalence class can be shown to be independent of the choice of μ . This U^L is well defined for any unitary representation of any closed subgroup H of G .

In passing from I_H to an arbitrary unitary representation L of H we may seem to have lost the interpretation that decomposing an induced representation is a natural generalisation of harmonic analysis. However this is not the case. It is certainly a natural further generalisation of harmonic analysis to replace complex valued functions by functions having values in a vector space and in particular a Hilbert space. Moreover there is no need for the Hilbert space to be the same at all points. We may have a vector bundle $s \to \mathcal{H}_s$ where each $\mathcal{H}_s$ is a Hilbert space and consider the Hilbert space of square integrable 'cross sections' of this bundle instead of $\mathcal{L}^2(S,\mu)$.

If G acts on the bundle $s \to \mathcal{H}_s$ as a group of 'bundle automorphisms' the corresponding action on the space of square integrable cross sections defines a unitary representation U of G whose decomposition is clearly a natural generalisation of harmonic analysis. We leave details to the reader but it is not difficult to show that the unitary representations of G obtained in this way are precisely the induced representations U^L where L varies over the unitary representations of H and $S = G/H$.

Just as in the case of a finite group G many induced representations are irreducible and inducing is one of the chief devices available for constructing irreducible unitary representations. On the other hand the relationship just described between harmonic analysis and decomposing induced representations tells us that there are many examples of induced representations which are reducible. Indeed these particular reducible representations are precisely those which one is most interested in decomposing into irreducibles. Thus induced representations play a sort of dual role. They are both the representations which one wants to decompose and many of the components into which one decomposes them.

By way of an example let us consider the simply connected double covering $\tilde{\mathcal{E}}$ of the group $\mathcal{E}$ generated by the translations and rotations in physical space. Let N be the commutative normal subgroup of all spatial translations and let K be the double covering of the group of all rotations about some fixed origin. Then K is isomorphic to the group $SU(2)$ of all 2×2 unitary matrices with determinant one and every element of $\tilde{\mathcal{E}}$ is uniquely of the form nk where $n \in N$ and $k \in K$. If M is any irreducible unitary representation of K then $nk \to M_k$ is an irreducible unitary representation of $\tilde{\mathcal{E}}$. Since K is compact these representations are finite dimensional. They were determined by Schur in 1924 and it turns out that there is just one of every positive integer dimension. That of dimension $2j+1$ for

$j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$ is conventionally denoted by the symbol D_j. It follows from a general theorem to be described in section 2 that all other irreducible unitary representations of $\tilde{E}$ are infinite dimensional and are induced by one dimensional representations of the subgroup generated by N and the subgroup K_z of K consisting of members which leave the z axis fixed. Specifically let r be any positive real number and let m be any integer. Then we may identify K_z with the group of all $(e^{i\theta}, \theta \text{ real})$. Given translations $x, y, z, \in N$ and $\theta \in K_z$ let

$$\chi^{r,m}(x,y,z,\theta) = e^{irz} e^{im\theta} \quad .$$

Then $\chi^{r,m}$ is a one dimensional unitary representation of the subgroup NK_z. Moreover the induced representation $U^{\chi^{r,m}}$ is irreducible and every infinite dimensional irreducible unitary representation of E is equivalent to some $U^{\chi^{r,m}}$. Finally $U^{\chi^{r,m}}$ and $U^{\chi^{s,n}}$ are equivalent if and only if $r = s$ and $m = n$.

Now consider the induced representations U^{D_j} where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ so that D_j ranges over the irreducible unitary representations of K. These are of interest in quantum mechanics as $\mathcal{H}(U^{D_j})$ may be identified with the Hilbert space of states of a free particle of positive mass and spin j. They are however far from irreducible and their decomposition is relevant to the physics of the corresponding particle. One finds in fact that

$$U^{D_j} \simeq \int_0^\infty [U^{\chi^{r,-2j}} \oplus U^{\chi^{r,-2j+2}} \oplus \ldots \oplus U^{\chi^{r,2j-2}} \oplus U^{\chi^{r,2j}}] \, dr$$

and each irreducible constituent has a physical interpretation. Indeed to say that the 'state vector' ϕ in $\mathcal{H}(U^j)$ lies in the subspace of $\int_r^{r+\Delta r} U^{\chi^{r,k}} \, dr$ is to say that the particle has momentum between r and $r+\Delta r$ and 'helicity' k. The helicity of a particle is the component of the 'spin angular momentum' along the direction of the momentum vector.

2. The imprimitivity theorem and its applications

The description given in 1. of the irreducible unitary representations of the group $\tilde{E}$ is a special case of a general theorem which describes explicitly or helps to describe explicitly all equivalence classes of irreducible unitary representations of a large and important class of separable locally compact groups. This general theorem is a corollary of the so-called 'imprimitivity theorem' and the latter theorem has among its other corollaries the well-known Stone – von Neumann theorem on the essential uniqueness of the solutions of the Heisenberg commutation relations of quantum mechanics. In this lecture we shall state these theorems and discuss the relationship between them.

Let G be a separable locally compact group and let N be a closed commutative subgroup of G. Suppose that there exists a closed subgroup H such that every element x in G is uniquely of the form nh where n is in N and h is in H. One says then that G is a *semi direct product* of N and H. It is easy to see that G can be reconstructed if we are given simply N, H and the homomorphism $h \to \alpha_h$ of H into the group of all automorphisms of N defined by setting $\alpha_h(n) = hnh^{-1}$. We observe that the group $\tilde{E}$ of section 1 is a semi direct product of a three dimensional real vector group and the group $SU(2)$ and that the symmetric group in three letters is a semi direct product of the cyclic group of order three and the cyclic group of order two.

Let G be such a semi direct product and let $\hat{N}$ denote the character group of N. For each $\chi \in \hat{N}$ and each $h \in H$ let $\alpha_h^*(\chi)$ be the character $n \to \chi(\alpha_h(n))$ and let H_χ be the subgroup of all h in H for which $\alpha_h^*(\chi) = \chi$. It can be shown that H is closed and hence that NH is a closed subgroup of G. Let L be an arbitrary irreducible unitary representation of H_χ . One verifies easily that $nh \to \chi(n)L_h$ is an irreducible unitary representation of the closed subgroup NH_χ . We denote it by χL . The first theorem alluded to above can now be stated as follows.

Theorem A. The induced representation $U^{\chi L}$ is always irreducible and $U^{\chi L_1}$ and $U^{\chi L_2}$ are equivalent if and only if L_1 and L_2 are equivalent. Moreover $U^{\chi_1 L_1}$ and $U^{\chi_2 L_2}$ cannot be equivalent unless χ_1 and χ_2 are in the same H orbit; that is $\alpha_h^*(\chi_1) = \chi_2$ for some h in H . If χ_1 and χ_2 are in the same H orbit then every $U^{\chi_1 L_1}$ is equivalent to $U^{\chi_2 L_2}$ for some L_2 . Finally if there exists a Borel set C which has just one point in common with each H orbit then every irreducible unitary representation of G is equivalent to some $U^{\chi L}$.

This theorem (when C exists) reduces the problem of determining the irreducible unitary representations of G to determining the orbits of H in $\hat{N}$ and to determining the irreducible unitary representations of certain closed subgroups of H . The results of the celebrated 1939 paper of Wigner on the inhomogeneous Lorentz group follows from this theorem since the group in question is a semi direct product of a four dimensional real vector group and the homogeneous Lorentz group. In this case we find that H_χ is either H itself the rotation group in three space, the homogeneous Lorentz group in three dimensional space time or the Euclidean group in the plane. The latter is again a semi direct product in which H is commutative and in which C exists. The homogeneous Lorentz group in three and four space time dimensions present problems which were overcome only in 1947 by works of Bargmann, Gelfand and Naimark. Wigner's paper then does not actually determine all irreducible representations. On the other hand the omitted ones could be argued to be irrelevant to Wigner's physical aims.

The statement of the imprimitivity theorem involves in an essential way the concept of a projection valued measure. Let μ be an arbitrary σ finite measure in a Borel space S and form $L^2(S,\mu)$. For each Borel subset E of S let P_E be defined to be the projection operator which takes f in $L^2(S,\mu)$ into $Q_E f$ where $Q_E(s) = 1$ or 0 according as $s \in E$ or $s \notin E$. It is easy to check that the mapping $E \to P_E$ has the following properties

(1) $P_\emptyset = 0$ and $P_S = I$

(2) $P_{E \cap F} = P_E P_F = P_F P_E$ for all E and F

(3) $P_{E_1 \cup E_2 \cup \ldots} = P_{E_1} + P_{E_2} + \ldots$

wherever $E_1, E_2, \ldots$ are pairwise disjoint Borel sets. In brief $E \to P_E$ is a homomorphism of the σ Boolean algebra of all Borel subsets of S into a Boolean algebra of projection operators in $L^2(S,\mu)$. Guided by this example one makes the following definition. Let S be any Borel space and let $E \to P_E$ be an assignment of a projection operator in a Hilbert space $H(P)$ to each Borel set E in S. Then the assignment $E \to P_E$ is said to be a *projection valued measure* if (1), (2) and (3) above are satisfied. One is usually interested in the special case in which S is standard and $H(P)$ is separable and then it is not difficult to show that *every* projection valued measure on S is 'equivalent' to a 'direct sum' of projection valued measures defined as above by a finite measure in S. Here 'equivalence' and 'direct sum' are defined by obvious analogy with the corresponding concepts for unitary group representations.

Some insight into the nature of projection valued measures may be obtained by considering the rather trivial discrete case in which S is countable and every subset is a Borel set. Then for all E, $P_E = \sum_{s \in E} P_{\{s\}}$ so that P is completely determined by its values at the one point sets. Now it follows from (1), (2) and (3) that the ranges of the $P_{\{s\}}$ are mutually orthogonal closed subspaces whose closed linear span is $\mathcal{H}(P)$. Conversely let $H = \sum_{s \in S} H_s$ be an arbitrary direct sum decomposition of a separable Hilbert space whose components are parameterized by the points of S . For each subset E of S let P_E be the projection on the direct sum of all H_s with $s \in E$. Then P is a projection valued measure on S and $P_{\{s\}}$ is the projection on H_s . In other words given a separable Hilbert space H and a discrete standard Borel space S the projection valued measures P on S with $\mathcal{H}(P) = H$ correspond one-to-one in a natural way to the pairs consisting of a direct sum decomposition of H and a parameterization of the summands by the points of S . Of course some points of S may correspond to 0 .

In the case of non discrete standard Borel spaces one has a similar result using 'direct integrals' of Hilbert spaces instead of direct sums. However for many purposes one may avoid talking about direct integrals of Hilbert spaces and express everything in terms of projection valued measures.

Classically a system of imprimitivity for a representation $x \to M_x$ of a group G is a direct sum decomposition of $\mathcal{H}(M)$ whose components, while not necessarily invariant under the M_x , are at least merely permuted amongst themselves by these M_x . In other words if $H(M) = \sum_{s \in S} H_s$ for some index set S then for all x and s , $M_x(H_s) = H_{s'}$ for some s' depending on s and x . Setting $[s]x^{-1} = s'$ we see that S becomes a G space. If this G space is transitive the system of imprimitivity is said to be transitive. Now suppose that M is unitary and that the H_s are mutually orthogonal. Let $E \to P_E$ be the corresponding projection valued measure so that $P_{\{s\}}$ has H_s as its range. The relationship $M_x(H_s) = H_{[s]x^{-1}}$ translates at once into $M_x P_{\{s\}} M_x^{-1} = P_{\{s\}x^{-1}}$ and equivalently into $M_x P_E M_x^{-1} = P_{[E]x^{-1}}$ for all x in G and all subsets E of S .

This reformation of the classical definition in the unitary case at once suggests a generalisation which permits direct integral decompositions as well as direct sum decompositions. Let U be a unitary representation of the separable locally compact group G. Then a *system of imprimitivity* for U is the pair consisting of a standard Borel G space S and a projection valued measure P defined on S such that $\mathcal{H}(P) = \mathcal{H}(U)$ and $M_x P_E M_x^{-1} = P_{[E]x^{-1}}$ for all Borel sets $E \leq S$ and all x in G. We say that the system of imprimitivity is *based on the* G space S.

A natural example of a system of imprimitivity may be obtained by considering the representation U canonically associated with an invariant measure μ in the standard Borel G space S, $U_x(f)(s) = f([s]x)$. In addition to the operators U_x one has the projection operators P_E where $(P_E(f))(s) = \psi_E(s)f(s)$ for all f in $\mathcal{L}^2(S,\mu)$. The map $E \to P_E$ is a projection valued measure defined on S and a simple calculation shows that $U_X P_E U_x^{-1} = P_{[E]x^{-1}}$ for all x in G and all Borel subsets E of S. Thus P is a system of imprimitivity for U based on S. Exactly the same considerations apply if we introduce a one cocycle a and replace U by U^a as defined in section 1 or more generally if we replace $\mathcal{L}^2(S,\mu)$ by $^2(S,\mathcal{H}_o,\mu)$ for some Hilbert space $\mathcal{H}_o$ and a by an operator valued one cocycle. It follows in particular that there is a canonical system of imprimitivity associated to each induced representation. It may be defined directly and explicitly as follows. Let L be a unitary representation of the closed subgroup H of the separable locally compact group G and let U^L be the unitary representation of G induced by L. Recall that $\mathcal{H}(U^L)$ consists of Borel functions from G to $\mathcal{H}(L)$ satisfying the identity $f(hx) = L_h f(x)$ for all h in H and all x in G as well as a certain integrability condition. For each Borel subset E of the coset space G/H let ψ_E be the function on G such that $\psi_E(x)$ is one or zero according as x is or is not in a right H coset belonging to E. Then $f \to \psi_E f$ is a projection operator P_E^L on $\mathcal{H}(U^L)$. One verifies easily that $E \to P_E^L$ is a projection valued measure defined on G/H and a system of imprimitivity for the induced representation U^L.

The imprimitivity theorem is a sort of converse of the result just described and characterizes induced representations as those admitting a transitive system of imprimitivity. Here is the formal statement.

Theorem B. Let H be a closed subgroup of the separable locally compact group G and let V be a unitary representation of G. Let P be a system of imprimitivity for V based on the coset space G/H. Then there exists a unitary representation L of H and a unitary operator W from $\mathcal{H}(V)$ to $\mathcal{H}(U^L)$ such that

$$WV_xW^{-1} = U_x^L \quad \text{and} \quad WP_EW^{-1} = P_E^L$$

for all x in G and all Borel subsets E of G/H. The equivalence class of the representation L is uniquely determined by the pair V,P and L is irreducible if and only if V and P are irreducible when taken together. More generally there is a canonical isomorphism between the commuting algebra of L on the one hand and the intersection of the commuting algebras of V and P on the other.

In the special case in which H is open as well as closed so that G/H is discrete the proof of the imprimitivity theorem is quite straightforward. For $x \in H$, U_x and $P_{\{H\}}$ commute so that the restriction of U to H has a subrepresentation whose space is the range of $P_{\{H\}}$. This is L and the rest of the proof is mainly a matter of checking and applying definitions. The reader is urged to carry out the details as an exercise. In the general case one cannot define L so directly since it is an 'infinitesimal component' of the restriction of U to H. However this difficulty can be overcome in various ways and half a dozen or so proofs exist in the literature. An early proof and references to others will be found in the author's book cited in the lecture on history.

Both applications of the imprimitivity theorem mentioned at the beginning of this section makes use of the spectral theorem for unitary representations of locally compact commutative groups. Let G be a locally compact commutative group which we take for convenience to be separable. The spectral theorem asserts the existence of a one-to-one correspondence between the unitary representations V of G and the projection valued measures P on $\hat{G}$ such that each V can be obtained from the corresponding P by the following construction. For each pair ϕ,ψ of vectors in $H(P)$ the mapping $E \rightarrow (P_E(\phi).\psi)$ is a complex valued measure and one can form the integral

$$\int \chi(x) \; d(P_\chi(\phi).\psi) \quad .$$

V is then the unique unitary representation of G such that

$$(V_x(\phi).\psi) = \int \chi(x) \; d(P_\chi(\phi).\psi)$$

for all ϕ and ψ in $H(P) = H(V)$.

In the special case in which P is supported by a countable subset of $\hat{G}$ one verifies that V is just the direct sum of the characters in this countable subset, the character χ occurring with multiplicity equal to the dimension of the range of $P_{\{\chi\}}$. In the general case the spectral theorem essentially gives one the direct integral decomposition of V into irreducibles. For most purposes the theorem can be used as it stands without thinking about direct integrals as such.

In the special case in which G is the additive group of the real line the spectral theorem was proved by Stone in 1930.

In that case G is the real line and one has a natural one-to-one correspondence between unitary representations of the real line on the one hand and projection valued measures on the other. Now the spectral theorem of Hilbert (as generalized to unbounded operators by von Neumann) sets up a natural one-to-one correspondence between self adjoint operators H and projection valued measures on the line. Composing these two one-to-one correspondences leads to a third - a natural one-to-one correspondence between self adjoint operators and unitary representations of the real line. This correspondence may also be set up directly and then takes the following simple form. If H is a self adjoint operator then $t \to e^{iHt}$ is the corresponding unitary representation of the real line.

With this background material on self adjoint operators and spectral theory it is easy to explain how the uniqueness of the irreducible solutions of the Heisenberg commutation relations is a corollary of the imprimitivity theorem. Let $Q_1, Q_2 \ldots Q_m$, $P_1, P_2 \ldots P_m$ be two sets of m self adjoint operators. One says that the Heisenberg commutation relations are satisfied if $(P_jQ_j - Q_jP_j) = iI$ for all j and all other pairs commute, i.e., $Q_iQ_j - Q_jQ_i = P_iP_j - P_jP_i = 0$ for all i and j and $P_jQ_i - Q_iP_j = 0$ whenever $i \neq j$. Actually the statement $P_jQ_j - Q_jP_j = iI$ conceals certain ambiguities since the operators are unbounded and hence not everywhere defined. This ambiguity may be avoided by following Hermann Weyl in replacing each Q_j and each P_j by the corresponding unitary representation of the real line. Accordingly one sets $V_t^j = e^{iQ_jt}$ and $U_s^j = e^{iP_js}$. The commutation relations stated above are equivalent (on a formal level at least) to the following commutation relations for the V^j and U^k

$$V_t^jU_s^j = e^{ist}U_s^jV_t^j \quad \text{for all } j, s \text{ and } t ,$$

$$V_t^kV_{t'}^j - V_{t'}^jV_t^k = U_s^kU_{s'}^j - U_{s'}^jU_s^k = 0 \quad \text{for all } k, j, t, t', s \text{ and } s'$$

and

$$V_t^j U_s^k = U_s^k V_t^j \quad \text{for all} \quad t, s, j, k \quad \text{with} \quad j \neq k \ .$$

Now to say that the V^j commute with one another is equivalent to saying that $(t_1, t_2, \ldots, t_m) \to V_{t_1}^1 V_{t_2}^2 \ldots V_{t_m}^m$ is a representation V of the vector group of all n-tuples of real numbers. Similarly to say that the U^k commute with one another is equivalent to saying that $(s_1, s_2, \ldots, s_m) \to U_{s_1}^1 U_{s_2}^2 \ldots U_{s_m}^m$ is a representation U of this same vector group. Setting $V_{t_1 \ldots t_m} = V_{t_1}^1 V_{t_2}^2 \ldots V_{t_m}^m$ and $U_{s_1 s_2 \ldots s_m} = U_{s_1}^1 U_{s_2}^2 \ldots U_{s_m}^m$, the remaining commutation rules can be written quite briefly as follows:

$$V_{t_1 \ldots t_m} U_{s_1 \ldots s_m} = e^{i(s_1 t_1 + \ldots s_m t_m)} U_{s_1 \ldots s_m} V_{t_1 \ldots t_m} \ .$$

Observe next that the most general character of the vector group of all n-tuples of real numbers is $(t_1, t_2, \ldots, t_m) \to e^{i(t_1 s_1 + \ldots t_m s_m)}$ where $(s_1, s_2, \ldots, s_m)$ is an n-tuple of real numbers. This suggests the following generalization of the Weyl form of the Heisenberg commutation relations. One is given a unitary representation V of the separable locally compact group G and a unitary representation U of the dual $\hat{G}$ of G and these representations satisfy the commutation relation $V_x U_\chi = \chi(x) U_\chi V_x$ for all x in G and all χ in $\hat{G}$.

One can ask to what extent this commutation relation determines V and U up to equivalence and so try to generalize the Stone - von Neumann theorem on the uniqueness in the case in which G is an n dimensional real vector group.

This problem may be attacked by using the spectral theorem to replace the representation U of G by the projection valued measure P on $\hat{\hat{G}}\,(=G)$ which defines it. We have $(U_\chi(\phi) . \psi) = \int \chi(x) \; d(P_\chi(\phi) . \psi)$ and an easy calculation (which we omit) shows that $V_x U_\chi = \chi(x) U_\chi V_x$ for all x and χ if and only if $V_x P_E = P_{[E]x^{-1}} V_x$. Thus our original problem involving representations of a commutative group and its dual is replaced by an equivalent problem in which one of the representations U is replaced by a projection valued measure P . In this new problem both V and P are defined on the same group G and the relevant commutation relation $V_x P_E = P_{[E]x^{-1}} V_x$ makes sense whether the group is commutative or not. Indeed it is precisely the relation defining a system of imprimitivity in the special case in which the G space S is G itself the action being right translation. Thus S is a coset space G/H where H is the trivial subgroup consisting of e alone. The imprimitivity theorem applies and tells us that the equivalence classes of irreducible solutions of the commutation relation correspond one-to-one to the equivalence classes of irreducible unitary representations of the group $\{e\}$. Since there is only one of the latter the required uniqueness is an immediate corollary.

Our second application of the imprimitivity theorem (Theorem B) is to the proof of theorem A on the irreducible unitary representations of semi direct products $U Ⓢ H$ with N commutative. If U is an arbitrary unitary representation of $G = N Ⓢ H$ let A and B be the restrictions of U to N and H respectively. Then for all $n \in N$ and $h \in H$ we have $U_{nh} = A_n B_h$. Thus U is uniquely determined by A and B . Conversely let A and B be unitary representations of N and H respectively and define $U_{nh} = A_n B_h$ for all $nh \in G$. Then $nh \to U_{nh}$ will be a unitary representation of G if and only if $U_{(n_1h_1)(n_2h_2)} = U_{n_1h_1} U_{n_2h_2}$ for all n_1, n_2 in N and all h_1, h_2 in H . But $(n_1h_1)(n_2h_2) = n_1h_1(n_2)h_1h_2$ where $h_1(n_2) = h_1n_2h_1^{-1}$ and is in N . Thus the condition is $U_{n_1h_1(n_2)h_1h_2} = U_{n_1h_1} U_{n_2h_2}$. Using the expression for U in terms of A and B this reduces to

$$A_{n_1h_1(n_2)} B_{h_1h_2} = A_{n_1} B_{h_1} A_{n_2} B_{h_2}$$

which simplifies to $A_{h_1(n_2)} B_{h_1} = B_{h_1} A_{n_2}$. Removing now superfluous indices we see that A and B define a unitary representation of G if and only if A and B satisfy the commutation relations

$$B_h A_n = A_{h(n)} B_h$$

for all n in N and all h in H . Since N is commutative the spectral theorem implies that A defines and is defined by a projection valued measure P on $\hat{N}$. Moreover, just as with the uniqueness theorem for solutions of the Heisenberg commutation relations, one verifies that B and A satisfy the indicated commutation relations if and only if B and P satisfy the commutation relations $B_h P_E B_{h^{-1}} = P_{[E]h^{-1}}$ for all h in H and all Borel subsets E of $\hat{N}$. But the latter say precisely that P is a system of imprimitivity for B based in $\hat{N}$. Of course this imprimitivity theorem does not apply at once since H does not act transitively on $\hat{N}$. On the other hand an easy argument shows that whenever U is irreducible the action of H on $\hat{N}$ is ergodic with respect to the invariant measure class whose null sets are the sets E with $P_E = 0$. Moreover when the Borel cross section C of theorem A exists a short further argument shows that this ergodic invariant measure class is supported by an orbit $\mathcal{O}$. The restriction of P to $\mathcal{O}$ is then a transitive system of imprimitivity for B and the imprimitivity theorem allows us to reconstruct P and B and hence U from an irreducible representation of the subgroup H_χ of H leaving fixed some χ in $\mathcal{O}$. The rest of the argument is a matter of calculation.

The reader is urged to work through the attached exercises. If he does he will obtain considerable insight into the Kirillov-Kostant-Auslander theory of nilpotent and solvable Lie groups. The principal features of this theory are already visible in a special case which may be completely analysed with the aid of theorem A.

EXERCISES

Semi direct products and solvable Lie groups

Let V be a finite dimensional real vector space and let A be a linear transformation in V. For each real number t let

$$e^{At} = 1 + At + \frac{At^2}{2} + \dots$$

Then e^{At} is a non singular linear transformation which defines an automorphism of the additive group of V and $t \to e^{At}$ is a homomorphism. Using this homomorphism we may define a semi direct product of the additive groups of V and the real line R

$$(v_1, t_1)(v_2, t_2) = (v_1 + e^{At_1}(v_2), t_1 + t_2) .$$

Call this semi direct product G_A.

Exercise I. Find all possible continuous homomorphisms of the real line into G_A and note that these correspond one-to-one in a natural way to the members of the vector space $V \oplus R$. Then show that the vector space of the Lie algebra L_A of G_A is naturally isomorphic to $V \oplus R$.

Exercise II. Compute the action of the inner automorphisms of G_A on the vector space dual L_A^* of L_A and show that the orbits of this action consist precisely of the following.
(1) All one point sets of the form ℓ, μ where μ is an arbitrary member of R^* and ℓ an arbitrary member of V^* such that $A^*(\ell) = 0$.
(2) All sets of the form $\theta \times R^*$ where θ is an orbit of the action of the dual action of R on V^* which contains

more than one point.

Exercise III. Apply the theory of unitary representations of semi direct products to show that the irreducible unitary representations supplied by that theory (which are exhaustive when the semi direct product has a Borel cross section for the R orbits in V^*) are parametrized by the pairs θ, χ where θ is an R orbit in V^* and χ is a character of a closed subgroup H_θ of R.

Exercise IV. Using the results of II and III show that there is a natural map of the unitary representations of III onto the R orbits in L_A^*. Then show that this map is one-to-one if and only if every H_θ is either $\{0\}$ or R.

Exercise V. Show that every H_θ is either $\{0\}$ or R when and only when A has no pure imaginary eigenvalues.

Exercise VI. Show that G_A is nilpotent if and only if $A^k = 0$ for some k and that there is a Borel cross section for the R orbits in V^* whenever $A^k = 0$ for some k.

It follows from the results of these exercises that the map of Exercise IV sets up a natural one-to-one correspondence between G_A orbits in L_A^* and (equivalence classes of) irreducible unitary representations of G_A whenever G_A is nilpotent and moreover that all irreducibles are induced by *one* dimensional representations of closed subgroups. In 1962 Kirillov proved that this result holds for all simply connected nilpotent Lie groups.

Exercise VII. Show that the exponential map of L_A on G_A is bijective if and only if A has no pure imaginary eigenvalues.

In 1965 Bernat generalized Kirillov's theory as suggested by IV, V and VII. He proved that the *Kirillov* map is bijective for all simply connected solvable Lie groups whose exponential map is bijective.

Exercise VIII. Study the case in which A has pure imaginary eigenvalues so that the map of IV is not one-to-one. Show that for each orbit θ where H_θ is not 0 or R the equivalence classes of irreducible unitary representations of G_A are naturally parametrized by the characters of the fundamental group of θ. Show also that θ is even dimensional and has a natural symplectic structure.

The Auslander-Kostant theory of 1967-1971 is concerned with extending the results of VIII to general solvable Lie groups. In this case one may not be able to obtain all equivalence classes. Indeed if A has two pure imaginary eigenvalues which are not rational multiples of one another, there will be *no* Borel cross section for the G_A orbits in V^*.

3. Some properties of the inducing construction

Let H_1 and H_2 be closed subgroups of the separable locally compact group G and suppose that $H_1 \subset H_2$. Let L be a unitary representation of H_1. Then one can ignore H_2 and simply consider the unitary representation U^L of G induced by L. On the other hand one can ignore G and obtain a unitary representation M of H_2 by regarding H_1 as a subgroup of H_2 and inducing L to H_2. Starting with M one can then form another induced representation U^M of G. How are the two representations U^L and U^M related? A useful theorem called the 'stages theorem' or the 'theorem on inducing in stages' asserts that U^L and U^M are always equivalent. In other words in inducing L from H_1 to G it is possible to 'stop over' at H_2 without affecting the equivalence class of the final result. The proof (which we omit) is both trivial and difficult. For finite groups the reader should be able to construct the proof himself - it is simply a matter of comparing definitions. In the general case one confronts rather exasperating measure theoretical technicalities. However as shown by Blattner these may be reduced by using an alternative definition of U^L which avoids choosing a particular quasi invariant measure in the coset space.

An immediate corollary is the fact that the regular representation of a closed subgroup H of G always induces the regular representation of G. Indeed it is immediate from the definition that the identity representation of the identity subgroup always induces the regular representation. Thus if $H_1 = \{e\}$ and L is the identity then M is the regular representation of H_2 and U^L is the regular representation of G.

One can often obtain valuable information about the structure of an induced representation by choosing the second subgroup H_2 properly and first studying the structure of M. For example let G be the group $SL(2,R)$ of all two by two real matrices with determinant one and let H_1 be the subgroup of all matrices of the form $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. Let I_{H_1} be the one dimensional identity representation of H_1. We may determine the structure of $U^{I_{H_1}}$ by choosing H_2 as the group of all matrices of the form $\begin{pmatrix} \lambda & 0 \\ a & 1/\lambda \end{pmatrix}$. Indeed H_1 is a normal subgroup of H_2 and it follows immediately from the definitions that if M is the representation of H_2 induced by I_{H_1} then M is simply the regular representation of the quotient group H_2/H_1 lifted to be a representation of H_2. Now H_2/H_1 is isomorphic to the commutative group of all diagonal matrices $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ and its regular representation is just the direct integral of all the characters of this commutative group. Thus M is the direct integral of the one dimensional representations of H_2 obtained by lifting the characters χ of the diagonal subgroup. However as in the case of $SL(2,F)$ where F is a finite field the induced representations $U^{\chi'}$ for $\chi^2 \neq I$ are all irreducible and constitute the so-called principal series of irreducible unitary representations of SL(2,R). Now quite generally it is trivial to show that inducing commutes with the taking of direct sums and direct integrals. Hence $U^{I_{H_1}}$ is seen to be a direct integral of all irreducible unitary representations in the principal series. Since one can show further that $U^{\chi_1'}$ and $U^{\chi_2'}$ are equivalent if and only if $\chi_1' = \chi_2'$ or $\chi_1' = \chi_2'^{-1}$, it follows that almost every member of the principal series occurs with multiplicity two.

Let L be a unitary representation of the closed subgroup H of the separable locally compact group G and let $\bar{L}$ denote the contragredient representation. This is defined by the equation $\bar{L}_x = (L_x^*)^{-1}$ where L_x^* denotes the Banach space adjoint of L_x in the dual of $\mathcal{H}(L)$ without taking into account the fact that $\mathcal{H}(L)$ is self dual via a linear anti-automorphism. When L is finite dimensional the character of $\bar{L}$ is the complex conjugate of that of L. It is an immediate consequence of the definitions that $U^{\bar{L}}$ and $\overline{U^L}$ are equivalent.

Now let L and M be unitary representations of the separable locally compact groups G_1 and G_2 respectively. Let $\mathcal{H}(L) \otimes \mathcal{H}(M)$ denote the Hilbert space tensor product of $\mathcal{H}(L)$ and $\mathcal{H}(M)$. Then $(x,y) \to L_x \times M_y$ defines a unitary representation of $G_1 \times G_2$. We call it the (outer) tensor or Kronecker product of L and M and denote it by $L \times M$. As with direct sums, direct integrals and passage to the contragredient, taking tensor products commutes with inducing. More precisely if L and M are unitary representations of the closed subgroups H_1 and H_2 of the separable locally compact groups G_1 and G_2 then $U^{L \times M}$ is equivalent to $U^L \times U^M$. The proof is once again an immediate consequence of the definitions.

A somewhat less obvious result whose proof is still quite easy concerns what happens when an induced representation is restricted to a closed subgroup. Let H_1 and H_2 be closed subgroups of the separable locally compact group G and let L be a unitary representation of H_1. Then the induced representation U^L has a canonically associated transitive system of imprimitivity P^L based on G/H_1. When U^L is restricted to H_2, P^L continues to be a system of imprimitivity but not necessarily a transitive one. The H_2 orbits in G/H_1 are easily seen to correspond one-to-one to the $H_1 : H_2$ double cosets in G and there will normally be more than one. If the orbit space is well behaved - e.g. if there exists a Borel set which meets each orbit just once, one says that H_1 and H_2 are regularly related. Assuming this it is easy to see that U^L restricted to H_2 is a direct integral (direct sum if there are only countably many double cosets) of representations with transitive systems of imprimitivity. By the imprimitivity theorem these are all induced by representations of closed subgroups of H_2 and it is basically a matter of computation to show that the component associated with the double coset $H_1 x H_2$ is induced by the restriction to $H_2 x x^{-1} H_1 x$ of the representation $\xi \to L_{x\xi x^{-1}}$ of $x^{-1} H_1 x$. Thus, when H_1 and H_2 are regularly related, the restriction to H_2 of the induced representation U^L of G is a direct integral over the $H_1 : H_2$ double cosets of explicitly describable induced representations of H_2. This rather elementary result is surprisingly useful. It is known as the 'restriction theorem'.

The restriction theorem has an immediate corollary which is useful in analysing the 'inner' tensor products of induced representations. Let U and V be unitary representations of the same separable locally compact group G. Then $U \times V$ as defined above will be a unitary representation of $G \times G$ which is irreducible whenever U and V are. This representation of $G \times G$ may be restricted to the diagonal subgroup $\tilde{G}$ of $G \times G$ consisting of all x,y with $x = y$. Via the obvious isomorphism between G and $\tilde{G}$ one thus obtains a unitary representation of G which is sometimes called the inner tensor product of U and V and denoted by $U \otimes V$. It will seldom be irreducible even if U and V are irreducible and for each

G, after one has found all equivalence classes of irreducible unitary representations, there remains the problem of decomposing $U \otimes V$ for each pair U and V of irreducibles and finding what its irreducible constituents are. Of course when G is commutative this is just the problem of finding the group structure in $\hat{G}$. When the irreducible representations U and V are induced the theorem we are about to state provides a useful first step in decomposing $U \otimes V$.

Let H_1 and H_2 be closed subgroups of the separable locally compact group G and let L and M be unitary representations of H_1 and H_2 respectively. Then $U^L \otimes U^M$ is obtained from $U^L \times U^M$ by restricting the latter to $\tilde{G} \subseteq G \times G$. On the other hand, as noted above $U^L \times U^M$ is equivalent to $U^{L \times M}$. Thus $U^L \otimes U^M$ is the restriction to $\tilde{G}$ of the representation $U^{L \times M}$ of $G \times G$ induced by the representation $L \times M$ of $H_1 \times H_2$. Evidently the restriction theorem applies - provided that $H_1 \times H_2$ and G are regularly related. It tells us that $U^L \times U^M$ is a direct integral over the $H_1 \times H_2 : \tilde{G}$ double coset of certain explicitly describable induced representations of G. One checks easily that the $H_1 \times H_2 : \tilde{G}$ double cosets in $G \times G$ correspond one-to-one in a natural way to the $H_1 : H_2$ double coset in G and that $H_1 \times H_2$ and $\tilde{G}$ are regularly related if and only if H_1 and H_2 are regularly related. A calculation which we leave to the reader, then shows that the contribution of the double coset $H_1 \times H_2$ to $U^L \otimes U^M$ is the representation of G induced by the inner tensor products pf the restrictions to $xH_1x^{-1} \cap H_2$ of the representations $h \to L_{xhx^{-1}}$ and M of $x^{-1}H_1x$ and H_2 respectively.

In the special case of a finite group or more generally when the subgroups H_1 and H_2 have finite index in G and L and M are finite dimensional the tensor product theorem for induced representations has a simple corollary which allows one to determine the 'intertwining number' of any two induced representations in terms of certain 'intertwining numbers' for the inducing representations.

Whether G is finite or not one defines an *intertwining operator* for two unitary representations U and V of G to be a bounded linear operator T from $\mathcal{H}(U)$ to $\mathcal{H}(V)$ such that $TU_x = V_xT$ for all x in G. The set of all intertwining operators for a given pair U,V is a vector space which we denote by $R(U,V)$. Its dimension is called the *intertwining number* $i(U,V)$ for U and V. Let T be an intertwining operator, let N_T be the null space of T and let R_T be the range of T. One verifies at once that N_T and the closure $\bar{R}_T$ of R_T are closed invariant subspaces of $\mathcal{H}(U)$ and $\mathcal{H}(V)$ respectively. Moreover it is easy to see that T restricted to $N_T^{\perp}$ is an intertwining operator for the subrepresentations of U and V defined by $N_T^{\perp}$ and R_T. It can be shown using the 'polar decomposition theorem' for T that its 'unitary factor' actually sets up an equivalence between these two subrepresentations. It follows that $i(U,V) \neq 0$ if and only if some subrepresentation of U is equivalent to some subrepresentation of V. In particular U is irreducible if and only if $i(U,U) = 1$ and if U and V are both irreducible then $i(U,V) = 1$ or 0 according as U and V are equivalent or inequivalent.

Suppose now that the unitary representations U and V are finite dimensional. Then the inner tensor product $U \otimes \bar{V}$ has as its vector space the set of all linear transformations T from $\mathcal{H}(U)$ to $\mathcal{H}(V)$. Moreover $(U \otimes \bar{V})_x(T) = U_xTV_x^{-1}$. Hence $U \otimes \bar{V}(T) = T$ if and only if T is an intertwining operator for U and V. Thus $i(U,V)$ is the number of times which $U \otimes \bar{V}$ contains the identity or equivalently $i(U,V) = i(I, U \otimes \bar{V})$. Suppose further that U and V are of the form U^L and U^M respectively where L and M are unitary representations of the closed subgroups H_1 and H_2. [Of course if U and V are finite dimensional then H_1 and H_2 must be of finite index in G and L and M must be finite dimensional.] We conclude that $i(U^L, U^M) = i(I, U^L \otimes \overline{U^M}) = i(I, U^L \otimes U^{\bar{M}})$. Now the tensor product theorem for induced representations implies that $U^L \otimes U^{\bar{M}}$ is a direct sum over the $H_1 : H_2$ double cosets of certain induced representations of G and correspondingly $i(I, U^L \otimes U^{\bar{M}}) = i(U^L, U^M)$ is a sum over the $H_1 : H_2$ double cosets. One computes that the contribution of the double coset $H_1 x H_2$ is just the intertwining number for the restrictions to $xH_1x^{-1} \cap H_2$ of the two representations $h \to L_{xhx^{-1}}$ and $h \to M_h$ of $x^{-1}H_1x$ and H_2 respectively.

This theorem about $i(U^L, U^M)$ has a number of interesting corollaries. For example when $H_2 = G$ there is only one double coset and the statement of the theorem becomes $i(U^L, M) = i(L, M \restriction H_1)$ where $M \restriction H_1$ denotes the restriction of M to H_1. In particular when L and M are both irreducible $i(U^L, M)$ is just the number of times that M appears as an irreducible constituent of U^L and $i(L, M \restriction H_1)$ is just the number of times that L appears as an irreducible constituent of M restricted to H_1. The equality of these two numbers is precisely the celebrated reciprocity theorem of Frobenius. Another corollary is a simple irreducibility criterion for induced representations. Let L be an irreducible unitary representation of H_1. Then U^L is irreducible if and only if $i(U^L, U^L) = 1$. But $i(U^L, U^L)$ is a sum over the $H_1 : H_1$ double coset of certain intertwining operators involving restrictions of L and its conjugates. All contributions are non negative and that of H_1H_1 is 1. Hence U^L is irreducible if and only if all other contributions are zero. Recalling what the contributions are one obtains a useful necessary and sufficient condition for irreducibility.

It is not difficult to go a step further and find an explicit expression for the most general intertwining operator for U^L and U^M. Of course we continue to assume that H_1 and H_2 have finite index in G and that L and M are finite dimensional. Let A be any function from G to the linear operators from $\mathcal{H}(L)$ to $\mathcal{H}(M)$ which satisfies the identity $A(\xi x h) = M_\xi A(x) L_h$ for all $\xi \in H_2$, $h \in H_1$ and $x \in G$. Let f

be any member of $\mathcal{H}(U^L)$ so that f is a function from G to $\mathcal{H}(L)$ which satisfies the identity $f(hx) = L_h f(x)$ for all x in G and all h in H_1. Then $A(xy^{-1})f(y)$ as a function of y for fixed x, is a constant on the right H_1 cosets. Let

$$T_A(f)(x) = \sum_{y \in G/H_1} A(xy^{-1})f(y) \quad .$$

Then $T_A f(x)$ is evidently a member of $\mathcal{H}(U^M)$ and T_A is an intertwining operator for U^L and U^M. It is routine to compute the dimension of the space of all functions A and to verify that it is equal to the expression found above for $i(U^L, U^M)$. It follows then that every member of $R(U^L, U^M)$ is of the indicated form.

Although the above results about intertwining operators for induced representations are valid in a rather limited context they provide a useful heuristic guide in the general case. Note for example that the functions which satisfy the identity $A(\xi x h) = M_\xi A(x) L_h$ are determined throughout a double coset by their value at any one point inside. Moreover given any such function it continues to satisfy the identity if it is altered by being reduced to zero outside of any one double coset. Let us define T_A to be a *double coset intertwining operator* whenever $A(x)$ is zero except for values of x lying in a single double coset. It follows from the above discussion then that every intertwining operator for U^L and U^M is a unique sum of double coset intertwining operators with just one (possibly zero) contribution from each $H_2 : H_1$ double coset.

Now given x_0 in G and given a bounded linear operator A from $\mathcal{H}(L)$ to $\mathcal{H}(M)$ there is at most one function $x \to A_x$ which is zero outside of $H_2 x_0 H_1$ and is defined in $H_2 x_0 H_1$ by $A_{\xi x_0 h} = M_\xi A L_h$ for all $\xi \in H_2$ and all $h \in H_1$. When it exists the corresponding double coset intertwining operator may be written down explicitly. It takes a form which makes sense in the general case provided that certain integrals exist and lead to square summable functions. In other words the double coset intertwining operators which span the space of all intertwining operators when G is a finite group have formal analogues for arbitrary separable locally compact G . The formal analogues may or may not involve convergent integrals and lead to honest intertwining operators. Moreover if these intertwining operators exist they may or may not span the space of all intertwining operators for the induced representations in question. However this may be, analogy with the finite case and the concept of a double coset intertwining operator provides a systematic way of looking for intertwining operators in the general case. Furthermore inspection of the literature shows that many of the known intertwining operators fit into this pattern and may be identified as double coset intertwining operators.

4. Conclusion and connections with the other lectures

In my final lecture I want to talk more systematically about the twin problems of (a) decomposing induced representations and (b) finding the possible irreducible unitary representations of a given G . In doing so I will refer freely to what the other lecturers have told you and try in this way to summarize and tie together the various lecture series which have been and are being presented here.

Let me begin with problem (b). Any general solution must of course be related to the structure of G and in particular, when G has a closed normal subgroup N it is natural to attempt to relate the problem to the corresponding problems for G and G/N . If N is of type I and the natural action of G/N in the set of irreducibles of N has a Borel cross section for the set of orbits, then there is a theory much like that described in section 2 for semidirect products. The problem is reduced to finding the irreducible projective representations of certain subgroups of G/N for certain projective multipliers σ .

A projective representation with multiplier σ is a unitary operator valued function on the group with the usual identity replaced by

$$L_{xy} = \sigma(x,y)L_x.L_y$$

and σ is a Borel function from $G \times G$ to $|z| = 1$ satisfying

$$\sigma(x,yz)\sigma(y,z) = \sigma(xy,z)\sigma(x,y) \quad .$$

Further $x \to L_x(\Psi)$ is a Borel function for each $\Psi \in H(L)$. For each fixed σ there is a theory of σ-representations (projective representations with multiplier σ) parallel to the standard theory. A multiplier σ is said to be trivial if

$$\sigma(x,y) = \frac{\Psi(x,y)}{\Psi(x)\Psi(y)}$$

for some Borel function Ψ from G to $|z| = 1$. It is important to note that if σ is nontrivial, then the difference between ordinary and σ-representations can be quite profound.

For example, let A be a separable, locally compact, commutative group and let $G = A \times A$. Then G is commutative and one can define a projective multiplier σ on G by setting

$$\sigma(x,\chi_1 ; x_2,\chi_2) = \chi_1(x_2) \quad .$$

An easy exercise shows that determining the σ-representations of G is equivalent to finding the most general solution to the generalized Heisenberg commutation relations. Hence there is a unique irreducible σ representation which is infinite dimensional whenever G is infinite. All known multipliers σ for commutative groups are obtained by realizing some quotient of the group in the form $A \times \hat{A}$ and then lifting the multiplier described above. When A is a finite dimensional vector space, then there is a natural symplectic form on $A \times \hat{A}$ given by

$$[(x_1,\ell_1),(x_2,\ell_2)] = \ell_1(x_2) - \ell_2(x_1)$$

and the σ just defined is equivalent to $e^{i[(x_1,\ell_1),(x_2,\ell_2)]}$. Kostant's general 'quantization' procedure associating a Hilbert space to a polarized symplectic manifold is a generalization of the construction of the Hilbert space for the unique $e^{i(\ell_1(x_2)-\ell_2(x_1))}$ representation of $A \times \hat{A}$. Its ubiquitous occurrence in constructing representations of Lie groups is perhaps not so surprising in view of the above remarks.

The so-called Weil representation of $Sp(n)$ can be understood in the same context. Let $A = \mathbb{R}^n$, then every element of $Sp(n)$ defines an automorphism α of $A \times \hat{A}$ which preserves σ. Hence if W is the unique σ-representation of $A \times \hat{A}$ then $(x,\ell) \to W_{\alpha(x,\ell)}$ is equivalent to W and hence there exists V_α such that $V_\alpha W_{x,\ell} V_\alpha^{-1} = W_{\alpha(x,\ell)}$. The operator V_α is uniquely determined up to a multiplicative constant and $\alpha \to V_\alpha$ is a projective representation of $Sp(n)$. This is the Weil representation. Generalizations of it occur in an essential way in the extension of normal subgroup analysis to projective representations (see G.W. Mackey, 'Unitary representations of group extensions', Acta Math., 99, 1958, 265-311).

When the group G is simple, then other methods must be used. We divide the simple, separable, locally compact groups into 4 categories according to whether they are connected or disconnected, compact or non compact. It follows from the work of Gleason, Montgomery and Zippin that every simple, separable locally compact group is a Lie group. A non connected simple group is totally disconnected and a compact, totally disconnected group is finite. Hence the categories are the compact and non compact Lie groups, the finite simple groups and the simple, totally disconnected, non compact groups which further subdivide into the discrete and non discrete.

The compact and non compact simple Lie groups are especially susceptible to a deep analysis because of the availability of differentiation and the fact that many of the properties of the group are reflected in properties of the Lie algebra. In Professor MacDonald's lectures we saw how to use this tool to determine all possible simple Lie groups and to show that they have many common structural features which may be exploited in studying the representation theory. The non compact simple Lie groups can be described purely algebraically in terms of the real field and its algebraic closure. Most of the known examples of non connected simple, locally compact groups are analogues of the simple Lie groups obtained by replacing the real field by a finite field, a countable discrete field or the completion of a countable discrete field with respect to a valuation. In the finite case a concerted attack is being made on the problem of finding all *exceptions* by a small army of mathematicians.

The problem of finding the irreducible unitary representations in the infinite discrete case is hopeless in principle because such groups are not of type I. Of the other four categories the least progress has been made in the totally disconnected non compact case and the solution is complete in the compact case. This latter case was the subject of Professor Bott's lectures. He proved that a compact simple Lie group G has a maximal commutative subgroup T, unique up to conjugacy, whose conjugates exhaust all elements of the group. Let N be the normalizer of T and let $W = N/T$. Then W is a finite group called the Weyl group and the orbits of W in T are precisely the intersections with T of the conjugacy classes in G and so correspond one-to-one to these classes. Professor Bott then went on to prove the celebrated theorem of Hermann Weyl setting up a natural one-to-one correspondence between the orbits of W in the dual T^* of T and the equivalence classes of irreducible unitary representations of G . This correspondence can be described in a number of elegant ways.

For non compact connected Lie groups there is an analogous theory which is much more complicated and which yields only almost all irreducible unitary representations of G . Many important special cases were worked out by Gelfand and his collaborators but the theory in the general case is mainly due to Harish-Chandra and is described in the lectures of Professor Schmid. An important difference lies in the fact that T must in many cases be replaced by a finite number of commutative groups. Another lies in the fact that all irreducibles except the identity are infinite dimensional and that characters are much harder to define. The lecture of Professor Atiyah was concerned with the character problem in this case. The nature of the commutative subgroup which replaces T in the compact case can be appreciated by considerations involving quite general groups G . For each $x \in G$ let C_x denote the conjugate class of x and let H denote the centralizer of x . Then C_x and G/H_x are isomorphic as G spaces. Let us now divide the elements of G into genera by putting x and y with the same genus whenever C_x and C_y are isomorphic as G spaces; that is whenever H_x and H_y are conjugate. Evidently every conjugate class is contained in some genus so that the conjugate classes themselves subdivide into genera. For each genus choose an arbitrary element x and let T_x be the centre of H_x . Let N_x be the normalizer of T_x in H_x and $W_x = N_x/T_x$. Then it can be shown that every element of the genus containing x is conjugate to an element of T_x and the W_x orbits in T_x of the elements of the genus are in one-to-one correspondence with the conjugacy classes in the genus. We say that a genus is degenerate if T_x is conjugate to a proper subgroup of some T_y . If G is a compact simple Lie group then there is only one non degenerate genus and so one commutative subgroup describes all conjugacy classes. For a non compact simple Lie group there are a finite number of genera and the Cartan subgroups are essentially the non degenerate ones. As in Weyl's theory expounded by Professor Bott, Harish-Chandra's theory parameterizes the irreducible unitary representations by W_x orbits in T_x but there are now several families of representations, one for each non degenerate genus of positive measure.

The finite simple groups which have Lie analogues have a representation theory closely analogous to that of the simple Lie groups. However, they share the complication of the non compact theory in that there is always more than one non degenerate genus. Professor Lusztig's lectures summarized the results of this rather new theory.

The Lie analogues based on countably infinite discrete fields are not type I groups and the determination of all these irreducible, unitary representations is not to be expected.

The unitary representation theory of the Lie analogues of the non discrete, totally disconnected simple groups is now under intensive study but the results are far from complete. In studying the centres of the centralizers of elements in $SL(n,F)$, the possible k^{th} order extension fields of F enter in an essential way for all $k \leq n$. The complications that ensue were discussed in Kazhdan's first lecture on group representations and number theory.

Returning tc the relationship between conjugacy classes and irreducible representations, consider the relationship between conjugacy classes and Lie algebras in the case of Lie groups (not necessarily semi-simple). The elements of the Lie algebra L_G of G may be identified with the continuous homomorphisms Ψ of the real line into G and for each $\Psi \in L_G$, there is a well defined element $\Psi(1)$ of G. This mapping is called the exponential mapping exp of L_G into G and satisfies $\exp[\mathrm{Ad}\, x(\phi)] = x \exp\phi\, x^{-1}$ so that exp maps the orbits of the adjoint representation in L_G onto conjugacy classes in G. When exp is bijective, as it is for simple connected nilpotent groups, there is a natural one-to-one correspondence between the orbits in L_G and conjugacy classes. The adjoint action on L_G defines a dual action on L_G^* and because of the duality between conjugacy classes and irreducible unitary representations one may expect some sort of correspondence between co-adjoint orbits in L_G^* and irreducible unitary representations. In the special case of simply connected, nilpotent groups, Kirillov showed in 1962 that there exists a perfect one-to-one correspondence. Later, Bernat showed that when G is solvable and exp bijective then the correspondence is again one-to-one. Kostant and Auslander showed that for any simply connected solvable Lie group of type I, there is a many to one mapping of irreducible unitary representations into co-adjoint orbits. Kostant had earlier made the important observation that the orbits have a natural symplectic structure and Kostant and Auslander exploited this to obtain an explicit description of all the irreducible representations associated with each orbit via Kostant's 'quantization' procedure. This material is described in more detail in Professor Kostant's lectures.

Next let us turn our attention to the decomposition problem itself. The lectures of Professor Kahzdan on integral geometry and those of Professor Helgason on joint eigenspaces of invariant differential operators may be regarded as expositions of two different general methods for dealing with this problem in the special case in which the induced representation to be decomposed is induced by the identity. However both are capable of generalization. Moreover the two methods are by no means unrelated.

Professor Helgason confined himself to the case in which the commuting algebra $R(U^{I_H}, U^{I_H})$ is commutative and G is a connected Lie group. Examples in which this occurs include the case in which H is the maximal compact subgroup of a connected semi-simple Lie group with finite centre and the case in which $H = \tilde{G}$, the diagonal of the direct product $G \times G$. The decomposition in this second case is equivalent to decomposing the regular representation of G since then U^{I_G} can be shown to be equivalent to $\int L^{\lambda} \otimes L^{\bar{\lambda}}\, d\mu(\lambda)$ where $\int L^{\lambda}\, d(\lambda)\, d\mu(\lambda)$ is the decomposition of the regular representation. The commutativity of $R(U^{I_H}, U^{I_H})$ tells us that U^{I_H} is multiplicity free and hence that U^{I_H} has a unique decomposition into irreducible and the associated direct integral decomposition of $\mathcal{H}(U^{I_H})$ also decomposes each $T \in R(U^{I_H}, U^{I_H})$ into constant operators. If the decomposition were into discrete components then each invariant irreducible subspace would be a joint eigenspace for all the operators in $R(U^{I_H}, U^{I_H})$. More generally, there exists a unique, projection valued measure P defined on $\hat{G}$ such that each self-adjoint $T \in R(U^{I_H}, U^{I_H})$ is defined by the projection valued measure $E \to P_{\Psi_T^{-1}(E)}$ where Ψ_T is a Borel function from the support of P to the real line. The functions Ψ_T collectively label the irreducible representations in the decomposition of U^{I_H} by joint eigenvalues of the operators T.

When G is a Lie group so that G/H is a smooth manifold, then we can use invariant differential operators instead of bounded operators in $R(U^{I_H}, U^{I_H})$ and make corresponding statements. When the decomposition is not discrete the eigenspaces will not consist of square integrable functions but functions which can be integrated to form square integrable functions just as the functions $e^{i\lambda x}$ on the real line are integrated in the definition of the Fourier transform.

The fact that one has a natural labelling of the irreducible components of U^{I_H} by joint eigenvalues of invariant differential operators was used by Selberg in his celebrated trace formula to avoid dealing explicitly with infinite dimensional irreducible representations.

The method of integral geometry consists in finding and studying explicit intertwining operators for two induced representations U^{I_H} and U^{I_N} where U^{I_N} has a known structure and one wants information about U^{I_H} . Now as I explained in section 3 on induced representations one can determine explicitly all possible intertwining operators for U^{I_H} and U^{I_N} when G is finite. There is a one dimensional family canonically associated with each $H:N$ double coset and choosing one non zero member of each family gives a basis for the space of all intertwining operators. One can write down explicit formulae for these double coset intertwining operators and these make sense in the general case provided that certain integrals converge. The term 'integral geometry' is used because the formulae reduce to integrating functions defined on G/H over certain orbits of the N action on G/H. These orbits often have simple geometric interpretations. In Professor Kazhdan's second lecture he applied this method to $U^{I_{\tilde{G}}}$ where $\tilde{G}$ is the diagonal of $G \times G$ to obtain the Plancherel formula for G in the special case in which G is the general linear group. As auxiliary group he used $N \times N$ where N is the group of all triangular matrices with ones on the diagram.

In decomposing a unitary representation W into irreducibles $W = \int V^{\lambda} m(\lambda) d\mu(\lambda)$, the measure μ is not uniquely determined. Any other measure on $\hat{G}$ with the same null sets will do. On the other hand, when W is an induced representation, there is an additional condition to impose which often picks out a unique μ. This unique μ, when it exists is called the Plancherel measure and in its strict sense, proving the Plancherel formula means computing the special μ as well as finding the decomposition. We give the appropriate definition when $W = U^L$ and L is a finite dimensional representation of H and H is such that its normalizer $\tilde{H}$ is unimodular and $G/\tilde{H}$ has a finite invariant measure ρ.

Let f be a continuous function on G with compact support and perhaps other restrictions. When G is finite it is easy to compute that

$$\mathrm{Trace}(U_f^L) = \int_{G/\tilde{H}} \left[\int_H f(xhx^{-1}) \chi(h) dh\right] d\rho(x)$$

up to a multiplicative constant. Moreover, if $U^L = \Sigma n_j M^j$ where the M^j are irreducible then

$$\text{Trace}(U_f^L) = \Sigma n_j \text{ Trace}(M_f^j)$$

and so the linear functional

$$f \to \int_{G/\tilde{H}} \int_H f(xhx^{-1}) \chi^L(h) \, dh \, d\rho(x)$$

is a linear combination of linearly independent linear functionals $f \to \text{Trace}(U_f^j)$, the coefficients being proportional to the n_j. Much more generally, the expression

$$\int_{G/\tilde{H}} \int_H f(xhx^{-1}) \chi^L(h) \, dh \, d\rho(x)$$

makes sense and is a linear functional which we try to write in the form $\int \text{Trace}(M_f) d\mu(M)$ where μ is a uniquely determined measure on $\hat{G}$. When this measure exists and is unique, one refers to the formula

$$\int_{G/\tilde{H}} \int_H f(xhx^{-1}) \chi^L(h) \, dh \, d\rho = \int \text{Trace}(M_f) \, d\mu(M)$$

as the Plancherel formula for the induced representation U^L. The existence of μ has so far been proved only in the following special cases:

(1) when $H = <e>$ and $L = I$, then the left hand side becomes $f(e)$ and we have the abstract Plancherel formula of Mauntner and Segal;

(2) when G is commutative, H is arbitrary and $L = I$, then it becomes $\int_H f(h) dh = \int_{H^\perp} \hat{f}(\chi) d\chi$ where $H^\perp$ is the group of all characters which are the identity on H (when G is the real line and H is the group of integers then this is the classical Poisson summation formula);

(3) when H is the maximal compact subgroup of a connected

semi-simple Lie group with finite centre and $L = I$.

The special case in which Γ is a discrete subgroup of a connected semi-simple Lie group G with finite centre and G/Γ has a finite invariant measure is essentially the Selberg trace formula. It may be regarded as a non commutative version of the Poisson summation formula. Both have extensive applications to number theory. The quadratic reciprocity law is one of the earliest general theorems in number theory with some degree of depth and perhaps the simplest proof is due to Dirichlet. Dirichlet's proof is based on a straightforward application of the Poisson summation formula.

REFERENCES

[1] Mackey, G.W. 'The theory of group representations', Mimeographed Lecture Notes, University of Chicago, 1955 (reprinted as part of 10 below).

[2] Mackey, G.W. 'Infinite dimensional group representations', *Bull.Amer.Math.Soc.* 69 (1943), 628-686.

[3] Mackey, G.W. 'Non commutative harmonic analysis with applications to analysis, number theory and physics', Lecture Notes, University of California at Berkeley, 1965.

[4] Auslander, L. and Moore, C. First chapter of 'Unitary representations of solvable Lie groups', *Mem.Amer.Math. Soc.* no.62, 1966.

[5] Mackey, G.W. 'Unitary Group Representations in Physics, Probability and Number Theory', Benjamin/Cummings, Reading, Mass., 1978.

[6] Mackey, G.W. *Induced Representations of Group and Quantum Mechanics*, W.A. Benjamin, New York, Amsterdam, 1968.

[7] Mackey, G.W. 'Infinite dimensional group representations and their applications', C.I.M.E. Lectures at Montecatine Terme, 1970. Edizioni Cremonese, Rome, 1971 (pp.221-330).

[8] Mackey, G.W. 'Induced representations of locally compact groups and applications'. In *Functional Analysis and*

Related Fields, ed. F. Browder, Springer-Verlag, New York, Berlin, 1970.

[9] Lipsman, R.L. 'Group representations', Springer Lecture Notes, no.388, 1974 (166 pp.).

[10] Mackey, G.W. *The Theory of Unitary Group Representations*, University of Chicago Press, London-Chicago, 1976.

4 · The geometry and representation theory of compact Lie groups

RAOUL BOTT (Harvard University)
(Notes by G.L. Luke)

1. On the topology of compact Lie groups

A Lie group G is a C^∞-manifold which is also a group and such that the group laws:

multiplication $m : G \times G \to G$

inverse $i : G \to G$

are C^∞-maps.

Examples

(i) $\mathbb{R}^n$, +; the additive group of n-dimensional Euclidean space,

(ii) R^*, $\cdot$; the multiplicative group of the real line,

(iii) $GL(V)$; the automorphisms of a finite dimensional real vector space V. Choosing a basis for V gives an isomorphism of $GL(V)$ and $GL(n,\mathbb{R})$, the group of $n \times n$ invertible matrices with real coefficients.

If $\mathbb{C}$ denotes the complex numbers and $\mathbb{H}$ the quaternions then $GL(n,\mathbb{C})$ and $GL(n,\mathbb{H})$ are defined similarly.

(iv) compact examples are:

(a) $\mathbb{R}/\mathbb{Z}$, +, the quotient of the additive reals by the integers which is isomorphic to the circle group S^1 of complex numbers of unit norm.

(b) A cartesian product of Lie groups is also a Lie group. A product of circles

$$T^n = S^1 \times \ldots \times S^1 \quad , \quad n \text{ copies}$$

is called a torus.

(c) $SU(n+1)$ = the group of $(n+1)\times(n+1)$ matrices with complex coefficients and satisfying $X\cdot\bar{X}^t = 1$, $\det X = 1$. In particular, if $X \in SU(2)$ then $X = \begin{pmatrix} a & \alpha \\ -\bar{\alpha} & \bar{a} \end{pmatrix}$ when $|a|^2 + |\alpha|^2 = 1$. Hence $SU(2)$ is homeomorphic to S^3, the 3-sphere of unit vectors in $\mathbb{C}^2$.

The fact that the topological spaces S^1 and S^3 can be given the structure of a Lie group suggests the question: 'Which spheres are Lie groups?'. Perhaps the most accessible theorem in this area is the following.

Theorem. *None of the even dimensional spheres, S^{2n}, are Lie groups.*

Proof. Suppose S^{2n} is a Lie group. Consider the multiplication map $m: S^{2n}\times S^{2n} \to S^{2n}$. The restrictions $m: \{e\}\times S^{2n} \to S^{2n}$ and $m: S^{2n}\times\{e\} \to S^{2n}$ both correspond to the identity map.

Let $H^*(S^{2n},\mathbb{R})$ denote the cohomology ring of S^{2n} with real coefficients (the deRham cohomology computed from differential forms). Then $H^0(S^{2n},\mathbb{R}) \cong \mathbb{R}$, $H^{2n}(S^{2n},\mathbb{R}) \cong \mathbb{R}$, $H^m(S^{2n},\mathbb{R}) = 0$ in other degrees.

The multiplication map m induces a ring homomorphism

$$m^* : H^*(S^{2n},\mathbb{R}) \to H^*(S^{2n}\times S^{2n},\mathbb{R})$$

and it is well known that

$$H^*(S^{2n}\times S^{2n},\mathbb{R}) \cong H^*(S^{2n},\mathbb{R}) \otimes H^*(S^{2n},\mathbb{R}) .$$

Using this identification and 1 for the generator in $H^0(S^{2n},\mathbb{R})$, λ for the generator in $H^{2n}(S^{2n},\mathbb{R})$ we see that $m^*(1) = 1\otimes 1$ and $m^*(\lambda)$ must have the form $a(\lambda\otimes 1) + b(1\otimes\lambda)$ for some real numbers a and b. Restricting the multiplication map to $m^* : H^*(S^{2n},\mathbb{R}) \to H^*(S^{2n}\times\{e\},\mathbb{R}) \cong H^*(S^{2n},\mathbb{R})$

where it equals the identity, we see that $a = 1$ and similarly, $b = 1$. Hence $m^*(\lambda) = \lambda\otimes 1 + 1\otimes\lambda$.

However, m^* is also a ring homomorphism and hence

$$m^*(\lambda^2) = m^*(\lambda)^2 = (\lambda\otimes 1 + 1\otimes\lambda)^2$$
$$= \lambda^2\otimes 1 + (\lambda\otimes 1)(1\otimes\lambda) + (1\otimes\lambda)(\lambda\otimes 1) + 1\otimes\lambda^2 \quad .$$

But $\lambda^2 \in H^{4n}(S^{2n},\mathbb{R})$ and hence $\lambda^2 = 0$. Further, as λ may be represented by a form of even degree, we see that

$$(\lambda\otimes 1)(1\otimes\lambda) = (1\otimes\lambda)(\lambda\otimes 1) = \lambda\otimes\lambda$$

and hence $2(\lambda\otimes\lambda) = 0$. Finally, it is clear that $\lambda\otimes\lambda \in H^{2n}(S^{2n},\mathbb{R}) \otimes H^{2n}(S^{2n},\mathbb{R})$ is not zero. Hence no such multiplication exists.

Note: In the case of an odd dimensional sphere, the generator $\lambda \in H^{2n+1}(S^{2n+1},\mathbb{R})$ may be represented by a form of odd degree and so

$$(\lambda\otimes 1)(1\otimes\lambda) = -(1\otimes\lambda)(\lambda\otimes 1)$$

and the argument above fails.

This theorem has the following generalization.

Theorem (H. Hopf). *Let G be a compact connected Lie group, then G has the cohomology (real coefficients) of a product of odd dimensional spheres,*

$$H^*(G,\mathbb{R}) \cong H^*(\prod_\alpha S^{2k_\alpha-1},\mathbb{R}) \quad .$$

The numbers k_α where $2k_\alpha-1$ is the dimension of a component sphere, are called the exponents of G. The cohomology ring of a sphere of dimension $2k_\alpha-1$ is an exterior algebra on one generator $x_{2k_\alpha-1}$ in degree $2k_\alpha-1$ and hence

$$H^*(G;\mathbb{R}) \cong \bigotimes_i E(x_i); i = 2k_\alpha - 1$$

which raises the problem: 'Compute the exponents of G.'

This leads us to the main structure theorem for compact Lie groups.

Lemma. *Let G be a compact Lie group. Then G admits a Riemannian metric which is both left and right invariant.*

For simplicity, we will say that G is primitive if there exists only one bi-invariant Riemannian structure (up to a multiplicative constant).

Theorem. *Let G be a compact, connected, simply-connected primitive Lie group. Then G is one of the following types.*

$A_n \sim SU(n+1)$, $n = 0,1,\ldots$

$B_n \sim \widetilde{SO}(2n+1)$, *the simply connected covering of the special orthogonal group,* $n = 1,2,\ldots$

$C_n \sim Sp(n)$, *the group of* $n\times n$ *matrices with coefficients in* $\mathbb{H}$ *such that* $X\bar{X}^t = 1$, $n = 1,2,\ldots$

$D_n \sim \widetilde{SO}(2n)$, $n = 1,2,\ldots$

These groups are known as the classical groups, and the following as the exceptional groups:

G_2, F_4, E_6, E_7, E_8.

To find the exponents of a group we might proceed by a case-by-case attack. The enumeration of the classical groups suggests using induction. This works perfectly for $SU(n)$ and $Sp(n)$.

The group $SU(n)$ acts naturally on the sphere S^{2n-1} of unit vectors in $\mathbb{C}^n$ and the stability group of a point on the sphere is $SU(n-1)$. Hence he have the fibering

$$SU(n-1) \to SU(n) \to SU(n)/SU(n-1) \simeq S^{2n-1} .$$

Passing to cohomology the twisting disappears,

$$H^*(SU(n),\mathbb{R}) \cong H^*(SU(n-1),\mathbb{R}) \otimes H^*(S^{2n-1},\mathbb{R})$$

and hence, by induction,

$$H^*(SU(n),\mathbb{R}) \cong H^*(S^3,\mathbb{R}) \otimes H^*(S^5,\mathbb{R}) \otimes \ldots \otimes H^*(S^{2n-1},\mathbb{R}) .$$

A similar argument works for $Sp(n)$. The group acts naturally on the sphere S^{4n-1} of unit vectors in quaternionic space $\mathbb{H}^n$ with stability group $Sp(n-1)$ which leads to

$$H^*(Sp(n),\mathbb{R}) \cong H^*(S^3,\mathbb{R}) \otimes H^*(S^7,\mathbb{R}) \otimes \ldots \otimes H^*(S^{4n-1},\mathbb{R}) .$$

It is interesting to note that the Betti numbers for these groups were obtained by R. Brauer in the 1920's by a quite different method.

For the special orthogonal groups, the corresponding fibering is $SO(n-1) \to SO(n) \to S^{n-1}$. The numbers $n-1$ and n are of different parity and the groups $SO(n-1)$ and $SO(n)$ belong to different series. A straightforward induction fails.

We pass now to other methods of computing cohomology. The cohomology of the group-manifold G may be computed via the deRham complex of differential forms, $(\Omega^*(G),d)$ where $\Omega^*(G)$ denotes the exterior algebra of differential forms with real coefficients and d the exterior derivative.

Let $\mathrm{Inv}_{\mathrm{left}}\Omega^*(G)$ or $\Omega^*(\mathfrak{g})$ denote the subcomplex of left invariant forms. Such a form is determined by its value at the identity $e \in G$ and hence $\Omega^*(\mathfrak{g})$ is a finite dimensional space. Let $\mathcal{H}^*(\mathfrak{g})$ denote the subcomplex of bi-invariant differential forms. We will show later that $d|\mathcal{H}^*(\mathfrak{g}) \equiv 0$.

Theorem. *Let* G *be a compact, connected Lie group, then*

$$H^*(G,\mathbb{R}) \cong H^*(\Omega^*(\mathfrak{g}),d) \cong \mathcal{H}^*(\mathfrak{g}) \quad .$$

Note that this isomorphism reduces the problem of computing the cohomology of G to a problem in invariant theory. It was by this method that R. Brauer determined the exponents of the classical groups. The method was also known to Cartan although the actual isomorphism theorem first appeared in a paper of Eilenberg and Chevalley in the 1940's.

Proof of the Theorem. A left invariant differential form is determined by its value at the identity and evaluation yields the isomorphism

$$\mathrm{Inv}_{\mathrm{left}}\Omega^*(G) \cong \Lambda^*(T_e^*(G)) \quad .$$

The exterior algebra $\Lambda^*(T_e^*(G))$ may be interpreted as the multilinear alternating forms on the tangent space $T_e(G)$ of G at the identity. Further, $T_e(G)$ is isomorphic to the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G and an invariant metric gives us an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ which explains our notation $\Omega^*(\mathfrak{g})$.

Consider the inclusion $\Omega^*(\mathfrak{g}) \hookrightarrow \Omega^*(G)$. Averaging the left translations gives a projection

$$\int L_g^* dg : \Omega^*(G) \to \Omega^*(\mathfrak{g})$$

and it follows that both $\Omega^*(\mathfrak{g})$ and $H^*(\Omega^*(\mathfrak{g}))$ are direct summands in $\Omega^*(G)$ and $H^*(\Omega^*(G))$. Note that if w is a closed form then so is dL_g^*w. The connectedness of G implies that e and g may be joined by a path and hence w and $L_g^* w$ are cohomologous. Finally $\int (L_g^* w - w)\,dg$ is an average of coboundaries and hence a coboundary, so w and $\int L_g^* w$ are cohomologous. This shows that

$$H^*(G,\mathbb{R}) \cong H^*(\Omega^*(\mathfrak{g})) \quad .$$

We may use the same argument and right translation restricted to $\Omega^*(\mathfrak{g})$ to show that $\Omega^*(\mathfrak{g})$ and $\mathcal{H}^*(\mathfrak{g})$ have the same cohomology and it remains only to show that $d|\mathcal{H}^*(\mathfrak{g}) \equiv 0$.

The map induced on $T_e^*(G)$ by the inverse map i is simply -1 . Hence the induced map on $\Lambda^p T_e^*(G)$ is $(-1)^p$. Notice also that i^* preserves bi-invariant forms. Hence if w is a bi-invariant p-form then

$$(-1)^{p+1} dw = i^* dw = di^* w = (-1)^p dw$$

and hence $dw = 0$ as required.

2. The adjoint action and its orbits

Throughout this section we will assume that G is compact and connected. We commence by describing our notation.

Let $g \in G$ and $X \in T_e(G)$. Then gX denotes the image of X under the map induced on the tangent space by left translation by g , similarly Xg for right translation and gXg^{-1} for conjugation. We have already defined the adjoint representation of G on $T_e(G)$ by $Ad(g)X = gXg^{-1}$. This is the canonical finite dimensional representation possessed by every Lie group. The derivative of Ad is denoted by

$$ad : T_e(G) \to End(T_e(G))$$

and, in fact, $ad(X)(Y) = [X,Y]$ for $X,Y \in T_e(G)$.

This is easy to see when G is a matrix group. Then

$\mathrm{Ad}(g)(X) = gXg^{-1}$ and $\exp(tX) = 1 + tX + \frac{t^2X^2}{2!} + \ldots$ (matrix multiplication). Hence

$$\begin{aligned} \mathrm{ad}(X)(Y) &= \frac{d}{dt}\,\mathrm{Ad}(tX)(Y)\Big|_{t=0} \\ &= \frac{d}{dt}\,\{e^{tX}Y\,e^{-tX}\}\Big|_{t=0} \\ &= [X,Y]\ , \text{ the coefficient of } t\ . \end{aligned}$$

For example $G = SO(3)$. The tangent space $T_e(G)$ consists of skew symmetric matrices and taking the basis

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

the adjoint representation of $SO(3)$ becomes the usual representation on $\mathbb{R}^3$. The (non-trivial) orbits of the Ad action are the 2-spheres centered at 0 and every sphere cuts a line through 0 in two points. We will generalize this fact later and hence we give it a technical proof.

Fix $X \neq 0$ on a given line and consider the function defined on a sphere by $Y \to \langle X,Y\rangle$ where $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbb{R}^3$. The sphere intersects the line precisely at the maximum and minimum of the function.

Returning to the general compact group G with Lie algebra $T_e(G) = \mathfrak{g}$ and $\langle\ ,\ \rangle$ any Ad-invariant inner product on $\mathfrak{g}$. The infinitesimal version of the invariance $\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle X,Y\rangle$ is computed by putting $g = e^{tZ}$ and differentiating at $t = 0$. This yields

$$\langle [Z,X],Y\rangle + \langle X,[Z,Y]\rangle = 0\ .$$

For $X \in \mathfrak{g}$, let $\mathfrak{g}_X = \{Y \mid [X,Y] = 0\}$, the centralizer of X and $\mathfrak{g}^X = [X,\mathfrak{g}]$, the image of $\mathrm{ad}X$. Then $\mathfrak{g}_X$ is a subalgebra of $\mathfrak{g}$, the Lie algebra of $G_X = \{g \in G \mid Ad(g)X = 0\}$ the stabilizer of X. These spaces fit together in an exact sequence

$$0 \to \mathfrak{g}_X \to \mathfrak{g} \to \mathfrak{g}^X \to 0 .$$

Lemma. *The spaces* $\mathfrak{g}_X$ *and* $\mathfrak{g}^X$ *are perpendicular complements.*

Proof. Let $Y \in (\mathfrak{g}^X)^\perp$. Then $\langle [Y,X],\mathfrak{g}\rangle = \langle Y,[X,\mathfrak{g}]\rangle = 0$ and so $[Y,X] = 0$, $Y \in \mathfrak{g}_X$ and $(\mathfrak{g}^X)^\perp \subseteq \mathfrak{g}_X$. The argument is completed by noting that $\dim(\mathfrak{g}^X)^\perp = \dim \mathfrak{g}_X$.

An element $X \in \mathfrak{g}$ is said to be regular if $\dim \mathfrak{g}_X \le \dim \mathfrak{g}_Y$ for all $Y \in \mathfrak{g}$.

Lemma. *If* X *is regular then* $\mathfrak{g}_X$ *is abelian.*

Proof. Suppose $\mathfrak{g}_X$ is not regular. Then there exist $Y,Z \in \mathfrak{g}_X$ such that $[Y,Z] \neq 0$.

Consider $\mathfrak{g}_{X+tZ}$ for t small. Then $[X+tY,X] = 0$ implies that $ad(X+tY)$ preserves the decomposition $\mathfrak{g} = \mathfrak{g}_X \oplus \mathfrak{g}^X$. Further, since $ad(X)$ acts isomorphically on $\mathfrak{g}^X$, so does $ad(X+tY)$ for t sufficiently small. Hence $\mathfrak{g}^X \subseteq \mathfrak{g}^{X+tY}$ and $\mathfrak{g}_{X+tY} \subseteq \mathfrak{g}_X$ by the previous lemma. However, $Z \notin \mathfrak{g}_{X+tY}$ and hence $\dim_{X+tY} < \dim \mathfrak{g}_X$ which is a contradiction. When X is regular, the abelian Lie algebra $\mathfrak{g}_X$ is called a Cartan subalgebra of $\mathfrak{g}$. It is not hard to show that the Cartan subalgebras are the maximal abelian subalgebras of $\mathfrak{g}$. These generalize the lines through the origin in the case of $SO(3)$.

First Conjugacy Theorem. *Let* G *be a compact connected Lie group with Lie algebra* $\mathfrak{g}$ *and* H *a Cartan subalgebra of* $\mathfrak{g}$. *Then for each* $Y \in \mathfrak{g}$, *the adjoint orbit*

$$O(Y) = Ad(G)(Y)$$

intersects H *in a finite non-empty set of points.*

Proof. Let $X \in \mathfrak{g}$ be such that $H = \mathfrak{g}_X$, $\langle \, , \, \rangle$ be an Ad-invariant inner product on $\mathfrak{g}$ and $f : O(Y) \to \mathbb{R}$ be defined by $f(Z) = \langle Z,X \rangle$. The orbit $O(Y)$ is compact and hence $f\, O(Y)$ achieves its minimum. We can assume $Y \in O(Y)$ is the minimum.

For each $Z \in \mathfrak{g}$, $\frac{d}{dt} f(\mathrm{Ad}(\exp tZ)(Y))\big|_{t=0}$ since $f(Y)$ is the minimum of $f|O(Y)$. Hence $\langle [Z,Y],X \rangle = 0$ for all $Z \in \mathfrak{g}$,

$$\langle [\mathfrak{g},Y],X \rangle = \langle Y,[X,\mathfrak{g}] \rangle = \langle Y,\mathfrak{g}^X \rangle = 0$$

and hence $Y \in \mathfrak{g}_X = H$.

Notice that if $Y \in H$ then $O(Y)$ meets H perpendicularly since $\mathfrak{g}_Y$ and $\mathfrak{g}^Y$ are perpendicular. This implies that $O(Y)$ intersects H in a discrete and hence a finite set of points. Also, the argument actually shows that all the critical points of $f|O(Y)$ belong to $O(Y) \cap H$.

Corollary. *Let* H_1 *and* H_2 *be Cartan subalgebras of* $\mathfrak{g}$. *Then there exists* $g \in G$ *such that* $\mathrm{Ad}(g)H_1 = H_2$.

Proof. Let $H_1 = \mathfrak{g}_{X_1}$ and $H_2 = \mathfrak{g}_{X_2}$. Then there exists $g \in G$ such that $\mathrm{Ad}(g)X_2 \in H_1$. Hence $\mathrm{Ad}(g)H_2 = H_1$.

Let H be a Cartan subalgebra of $\mathfrak{g}$ and $W(H)$ denote the group of automorphisms of H induced by the adjoint automorphism of G. Then $W(H)$ is called the Weyl group of H and is easil seen to be finite.

Proposition. *For each* $Y \in \mathfrak{g}$ *and Cartan subalgebra* H, *the intersection* $O(Y) \cap H$ *is an orbit of the Weyl group.*

Proof. Let $X,Y \in H$ be such that $X = \mathrm{Ad}(g)(Y)$ for some $g \in G$. Then $\mathfrak{g}_X = \mathfrak{g}_{\mathrm{Ad}(g)Y} = \mathrm{Ad}(g)\mathfrak{g}_Y$. But $H \subseteq \mathfrak{g}_Y$ and hence

$Ad(g)H \subseteq \mathfrak{g}_X$. Therefore H and $Ad(g)H$ are both Cartan subalgebras of $\mathfrak{g}_X$.

Let G_X be the stabilizers of X and G_X^o the connected component of the identity. Then $\mathfrak{g}_X$ is the Lie algebra of the compact group G_X^o and there exists $Z \in \mathfrak{g}_X$ such that $Ad(\exp Z)Ad(g)H = H$. But $Ad(\exp Z)(X) = X$ and so

$$X = Ad(\exp Z)(X) = Ad\{(\exp Z) \cdot g\}(Y)$$

and $Ad\{(\exp Z) \cdot g\}$ induces a Weyl group transformation of H .

We illustrate these notions for the group $U(n)$. Then the Lie algebra consists of skew Hermitian matrices which we identify with the Hermitian matrices by multiplication by $\sqrt{-1}$. The adjoint representation is equivalent to the representation of $U(n)$ acting on Hermitian matrices by conjugation.

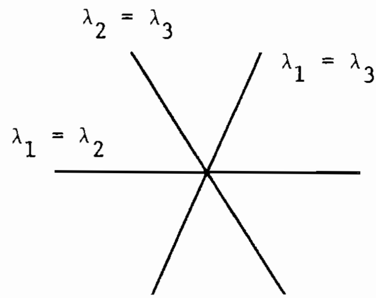
A diagonal Hermitian matrix corresponds to a regular element of $\mathfrak{g}$ if and only if all the diagonal elements are distinct and the corresponding Cartan subalgebra consists of all the diagonal Hermitian matrices. The Weyl group is the set of permutations of the diagonal elements. Finally, the First Conjugation Theorem becomes the well known fact that any Hermitian matrix can be put into diagonal form.

For a general compact Lie group G and Cartan subalgebra $H \subseteq \mathfrak{g}$, the set of singular elements (non-regular) of H is a finite union of hyperplanes. This configuration is called the Infinitesimal Diagram of G .

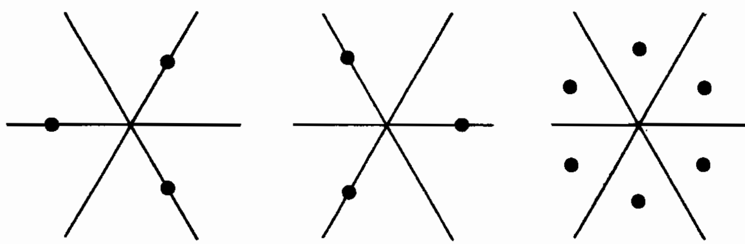
For example, let $G = SU(3)$ and H be the set of real diagonal matrices of trace 0 . Let $Y \in H$ be given by

$$Y = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} , \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0 \ .$$

Then the diagram of $SU(3)$ consists of the 2-plane H and hyperplanes $\lambda_1 = \lambda_2$, $\lambda_2 = \lambda_3$, $\lambda_1 = \lambda_3$.



The Weyl group is generated by reflecting across the hyperplanes in the diagram and the orbits have the following form:



In the general orbit there are 6 critical points, not simply a maximum and minimum.

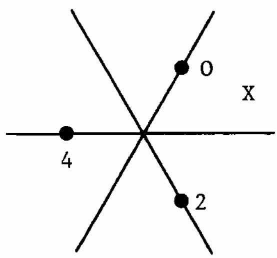
Theorem (Bott). *The critical points of* f *are all non-degenerate and an index of a critical point is equal to twice the number of hyperplanes crossed by a line joining* X *to the critical point.*

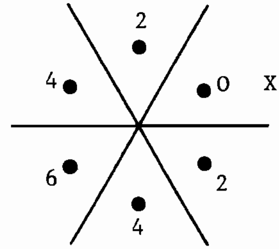
The proof of this theorem appears in [2] and will not be given here. We content ourselves with illustrating the result for SU(3) .

For the general orbit, the critical points have indices as illustrated on the right below and the orbit in g is the flag manifold

U(3)/U(1) × U(1) × U(1) .

For the first degenerate case the orbit is the complex projective space U(3)/U(2)×U(1) and the critical points have indices 0,2,4





Returning to the theorem, we see that as a consequence of general Morse theory, each adjoint orbit is a union of even dimensional cells, one for each critical point. It follows that $H^*(O(Y),\mathbb{Z})$ is torsion-free and the i-th Betti number is the number of critical points of index i . Hence the cohomology of an orbit can be computed from the infinitesimal diagram of the group.

3. The Poincaré series of the loop space of a compact group

The exponential map $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism between a neighbourhood of $0 \in \mathfrak{g}$ and $e \in G$. If $H \subseteq \mathfrak{g}$ is a Cartan subalgebra and $T = \exp H$, the corresponding maximal torus, then $\exp : H \to T$ may be thought of as the universal covering map. As in the case of the algebra H , we define the Weyl group of T to be the group of automorphisms of T induced by conjugation in G .

Theorem. *Each orbit of the conjugation action of* G *intersects the maximal torus* T *precisely in the orbit of the Weyl group.*

Let <u>Ad</u> denote the conjugation action and let G have a bi-invariant Riemannian metric. Then let $P \in T$ be a generic point - not in the image of the singular set of $H = T_e(T)$. By $\Omega(P,O(g))$ we denote the space of smooth paths from P to $O(g)$, the Ad-orbit through g.

We define the energy path u by

$$E(u) = \int |u'(t)|^2 \, dt \quad .$$

Then we have the following fundamental result from differential geometry. The minimum of E on $\Omega(P,O(g))$ exists and is a geodesic from P to $O(g)$ which meets $O(g)$ perpendicularly.

We will prove the above theorem by showing that such a geodesic lies completely in T and hence T and $O(g)$ actually intersect.

Lemma. *Let G be a Lie group which acts on a Riemannian manifold M by isometries. Then a geodesic of M which intersects one of the G-orbits perpendicularly intersects them all perpendicularly.*

Proof. Let $u : [0,1] \to M$ be a geodesic (parametrized by path length) which meets an orbit $O(m)$, $m \in M$ perpendicularly at $u(1)$.

Each $X \in \mathfrak{g}$, the Lie algebra of G, induces a Killing field ξ_X on M and a 1-parameter family of isometries corresponding to e^{sX}. There results a perturbation:

$$u(\ ,\ ) : [0,1] \times (-\varepsilon,\varepsilon) \to M$$

of the geodesic $u = u(\cdot,0)$ by geodesics $u(\cdot,s)$ with the property that $u(1,s)$ meets $O(m)$ perpendicularly.

Consider $g(t) = (\xi_X(u(t)), u'(t))$, the inner product of the Killing field and the velocity vector field along the geodesic. Then if ∇ denotes the Laplace-Beltrami connection operator, we have

$$u'(t)g = (\nabla_{u'(t)}\xi_X, u'(t))$$

$$= (\nabla_{\xi_X} u', u')(u(t)) \quad \text{since} \quad [\xi_X, \frac{d}{dt} u(t,s)] = 0$$

$$= \frac{1}{2}\, \xi_X(u(t))(u',u')$$

$$= 0 \quad \text{since } u \text{ is parametrized by path length.}$$

Hence $g(t)$ is constant along the geodesic u and since it vanishes at $u(1)$ it vanishes everywhere along u .

It follows that u is perpendicular to all the orbits it intersects since ξ_X gives a typical tangent vector to an orbit.

Returning to the theorem, for a general point $P \in T$, $T_P(O(P))$ is the orthogonal complement to $T_P(T)$. Hence a geodesic from P to $O(g)$ which is perpendicular to $O(g)$ is also perpendicular to $O(P)$ and hence lies in T . Therefore $O(g) \cap T \neq \phi$.

The proof that the intersection is an orbit of the Weyl group is similar to that given for the Cartan subalgebra.

We now study the indices of the critical points of the energy function E . In particular, we study the critical points of E on the paths from $P \in T$ to the identity $e \in G$. We know that the critical points are geodesics.

When $G = SU(3)$ and T is the maximal torus of special unitary diagonal matrices with Lie algebra denoted by $\mathfrak{t}$, then the Lie algebra $\mathfrak{g}$ decomposes under the adjoint action restricted to T as

$$\mathfrak{g} = \mathfrak{t} \oplus E_{\alpha_1} \oplus E_{\alpha_2} \oplus E_{\alpha_3}$$

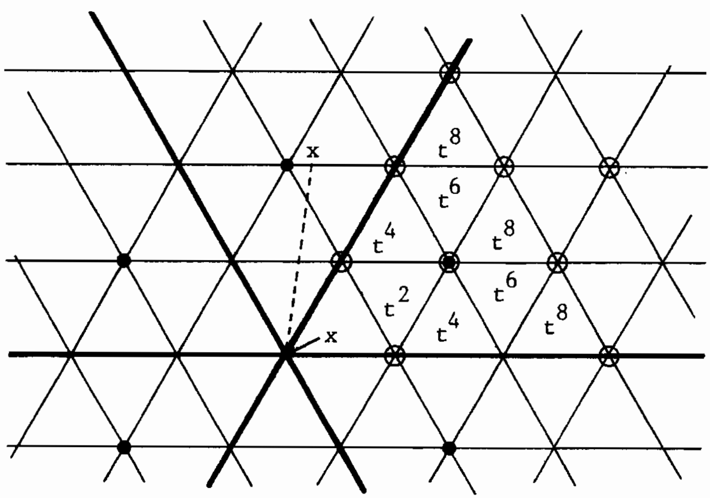
with T acting trivially on t and where $E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_3}$ are distinct two dimensional representations of T

$$\alpha_i : T \to SO(2) \ , \quad i = 1,2,3 \ .$$

Let $U_{\alpha_i} \subseteq T$ be the kernel of α_i . Then the configuration

$$\bigcup_{i=1}^{3} \exp^{-1}(U_{\alpha_i}) \subseteq t$$

is called the Diagram of G . We illustrate this with the Infinitesimal Diagram marked in thicker lines.



The vertices correspond to $\exp^{-1}(Z(G))$ where $Z(G)$ denotes the centre of G , and those distinguished by a dot to $\exp^{-1}(e)$ The geodesics joining a point P to e , lift in the covering to the straight lines joining the points of $\exp^{-1}(P)$ to 0 .

We fix a fundamental Weyl chamber, that is, a component of the infinitesimal diagram. Then each simplex containing a point of $\exp^{-1}(P)$ corresponds, under the Weyl group, to a unique simplex in the fundamental chamber (this fact depends on

$\pi_1(SU(3)) = 0$). Hence, by using the Weyl group, we may move our geodesics into the fundamental chamber so that they are in one-to-one correspondence with the simplices.

As before, the index of the energy function on $\Omega(P,e)$ corresponding to a particular geodesic, is twice the number of lines crossed by the geodesic in the Diagram. This is also illustrated.

Of course, this generalizes to an arbitrary compact, connected, simply connected Lie group G .

We define the Poincaré series of a space X to be

$$P_t(X) = \sum_{i=0}^{\infty} t^i \dim H^i(X, \mathbb{R})$$

and ΩX to be the set of continuous loops in X from a fixed base point. We note that ΩG is homotopy equivalent to $\Omega(P,e)$. Then as a corollary of the previous theorem, since all the indices of E on $\Omega(P,e)$ are even, $H^*(\Omega G, \mathbb{Z})$ has no torsion and

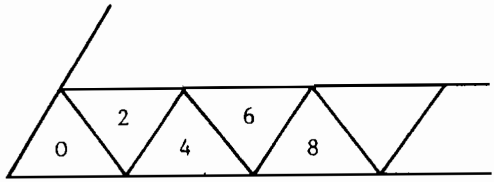
$$P_t(\Omega G) = \sum t^{\lambda(\Delta)}$$

where the sum is taken over the simplices on the fundamental chamber and $\lambda(\Delta)$ is twice the number of hyperplanes crossed by a line joining an interior point of the simplex to 0 .

When G is a simple group, then there is a unique leading simplex in the fundamental chamber with index 2 . Hence $\pi_2(\Omega G) = \mathbb{Z}$ and so $\pi_3(G) = \mathbb{Z}$, $\pi_2(G) = 0$.

These results give us a neat method for computing the Betti numbers from the Diagram. We illustrate it for $G = SU(3)$.

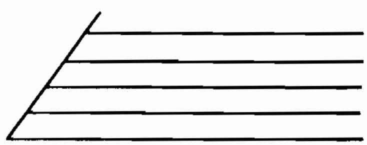
Consider the following strip of the diagram.



In each rhombus the indices are 4 greater than those to the immediate left and hence the contribution of this strip to the Poincaré series is

$$(1+t^2) + t^4(1+t^2) + (t^4)^2(1+t^2) + \ldots = \frac{(1+t^2)}{1-t^4} .$$

Applying the same technique to the series of strips



we see that

$$P_t(\Omega SU(3)) = \frac{1+t^2}{(1-t^4)^2} = \frac{1}{(1-t^2)(1-t^4)} .$$

Recall that $H^*(SU(3),\mathbb{R}) = H^*(S^3),\mathbb{R}) \otimes H^*(S^5,\mathbb{R})$ and so $\frac{1}{1-t^2}$ corresponds to S^3 and $\frac{1}{1-t^4}$ to S^5 .

By comparing this Morse theory approach with the standard constructions in topology one obtains a relation between the exponents - which is easily computable and for E_8 - actually gives all the exponents.

The idea is this:

The general rule for computing $P_t(\Omega G)$ from the corresponding spheres is

$$P_t(\Omega G) = \Pi \frac{1}{1-t^{\ell_i-1}}$$

where ℓ_i is the dimension of a component sphere. The formula holds for G simple, connected and simply connected. Hence we have: If

$$P_t(G) = \Pi(1+t^{2k_i-1}) \text{ , then } P_t(\Omega G) = \Pi \frac{1}{(1-t^{2k_i-2})} .$$

On the other hand using the Diagram we find that

$$P_t(\Omega G) = \frac{Q(t)}{\pi(1-t^{2a_i})}$$

where Q is a polynomial such that $Q(1) = 1$ and $\Sigma a_i \alpha_i$ *is the sum of the positive roots* expressed in terms of the *simple roots* α_i . By comparing the behaviour of these series near $t = 1$, it can be shown that

$$\ell! \Pi d_i \Pi(d_i-1) = \Pi a_i$$

where ℓ is the rank of G and the d_i come from expressing the highest root μ in terms of the simple roots $\alpha_i, \mu = \Sigma d_i \alpha_i$ [2] .

In the case of E_8 , this formula actually gives an expression for $\Pi(k_i-1)$ as a product of ℓ primes

$$\Pi(k_i-1) = 1.7.11.13.17.19.23.29$$

and hence

$$P_t(E_8)$$

$$= (1+t^3)(1+t^{15})(1+t^{23})(1+t^{27})(1+t^{35})(1+t^{39})(1+t^{47})(1+t^{59})$$

4. The Weyl character formula

We commence by reviewing the theory of characters of representations of compact groups.

Let G be a compact group and W a representation of G. Then W is a vector space equipped with a continuous homomorphism from G into the group of automorphisms of W, denoted $x \to W(x)$, $x \in G$. The character χ_W of W is defined by

$$\chi_W(x) = \text{Trace } W(x) , \quad x \in G$$

and characters have the properties:

1) $\chi_{W_1 \oplus W_2} = \chi_{W_1} + \chi_{W_2}$

2) $\chi_{W_1 \otimes W_2} = \chi_{W_1} \cdot \chi_{W_2}$

3) $\int_G \chi_W(x)\,dx = \dim W^G$ where W^G denotes the space of vectors fixed by G. (This property follows from applying the trace homomorphism to the projection $P_W = \int_G W(x)\,dx$ onto W^G.)

4) If W^* denotes the contragredient or dual representation to W, then

$$\chi_{W^*}(x) = \chi(x^{-1}) = \overline{\chi(x)}$$

Next we describe the Schur orthogonality relations. Suppose W and W' are nonequivalent, irreducible representations of G. Then $\text{Hom}(W,W')^G = 0$ by Schur's lemma, that is $(W^* \otimes W')^G$ has zero invariant subspace. Hence $\int_G \overline{\chi_W}\chi_{W'} = 0$. If W and W' are equivalent and irreducible then $\dim \text{Hom}(W,W')^G = 1$ and hence $\int_G \overline{\chi_W}\chi_{W'} = 1$. Therefore, the characters form an orthonormal set of functions on G.

Notice that the characters are class functions (constant on conjugacy classes) and the Peter-Weyl theorem gives us that the characters of irreducible representations form an orthonormal basis of class functions. Hence, W can be expressed as a direct sum of irreducible representations

$$W \approx a_1W_1 \oplus a_2W_2 \oplus \dots \oplus a_nW_n$$

where the $a_i = \dim \mathrm{Hom}(W,W_i)^G = \int_G \overline{\chi_W}\chi_{W_i}$.

Suppose now that G is a compact, connected Lie group. The representation theory reduces to finding the characters of the irreducible representations. We have shown that every conjugacy class in G intersects a fixed maximal torus T and hence a character of a representation is determined by its restriction to T. Of course, each representation W of G restricts to a representation $W|T$ and $\chi_W|T = \chi_{W|T}$. Hence an initial step in calculating the irreducible representations of G is to find the representations and characters of T.

Let $\hat{T}$ denote the set of continuous homomorphisms from T into the circle group. Hence $\hat{T}$ is the set of irreducible representations of T. Then if $\mathbb{Z}[\hat{T}]$ is the ring of formal, finite linear combinations of elements of $\hat{T}$, with integer coefficients, the restriction $W|T$ corresponds, via its character, to an element

$$\sum a_i\lambda_i \in \mathbb{Z}[\hat{T}] \quad , \quad a_i = 0,1,2,\dots,\lambda_i \in \hat{T} \quad .$$

The simplest example occurs when $T = S^1$, the circle group. Then $\hat{S}^1 \cong \mathbb{Z}$. If we let z be a generator of $\hat{S}^1$ so $\hat{S}^1 = \{\dots,z^{-2},z^{-1},1,z,z^2,\dots\}$, then

$$\mathbb{Z}[\hat{S}^1] = \mathbb{Z}[z,z^{-1}] \quad .$$

When $G = SU(2)$, we have a natural representation on $\mathbb{C}^2$. The torus

$$T = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \;\middle|\; |z| = 1 \right\}$$

is maximal in $SU(2)$ and acts naturally on the symmetric powers $S^k(\mathbb{C}^2)$. Thinking of $S^k(\mathbb{C}^2)$ as the space of polynomials in two variables, x and y say, homogeneous of degree k, the element $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ of the torus acts by

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} x^{k-i}y^i = z^{k-2i}x^{k-i}y^i .$$

Hence the character of this representation of T on $S^k(\mathbb{C}^2)$ is

$$\left(z^{-k} + z^{-(k-2)} + \ldots + z^{(k-2)} + z^k\right)$$

and this formula also gives the corresponding element of $\mathbb{Z}[\hat{T}]$ where now z^ℓ denotes the homomorphism

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \to z^\ell .$$

Note that these elements of $\mathbb{Z}[\hat{T}]$ are symmetric under the substitution $z \to z^{-1}$. This follows from the fact that the Weyl group acts on $\mathbb{Z}[\hat{T}]$ fixing those elements which come from restriction of representations of G.

Consider the product

$$(z^{-k} + z^{-(k-2)} + \ldots + z^{k-2} + z^k)(z - z^{-1}) = z^{k+1} - z^{-(k+1)} .$$

Such an expression $z^r - z^{-r}$ is called an elementary alternating expression and the product shows us that the character of $S^k(\mathbb{C}^2)$ has the form

$$\frac{\text{elementary alternating expression}}{\text{smallest elementary alternating expression}} .$$

This is the Weyl Character Formula for SU(2) .

For a general compact connected group G with maximal torus T , let $W(T)$ be the Weyl group and $\lambda \in \hat{T}$. Then

$$\sum_{w \in W(T)} (-1)^w \lambda^w$$

denotes the element of $\mathbb{Z}[\hat{T}]$ given by the sum of λ^w , the image of λ under the action of $w \in W(T)$ on $\hat{T}$ with coefficient $(-1)^w$, the determinant of the action of w on the Lie algebra of T . For an irreducible representation V of G with highest weight λ , the Weyl Character Formula is

$$\chi_V | T = \frac{\Sigma(-1)^w (\lambda+\rho)^w}{\Sigma(-1)^w \rho^w}$$

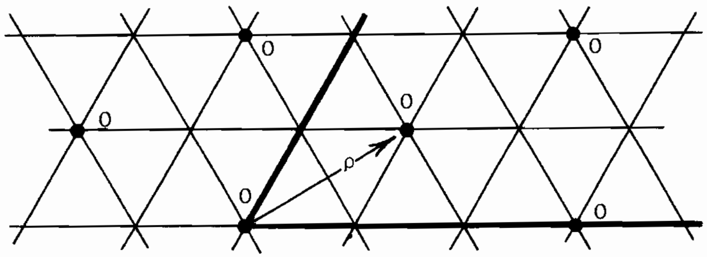
where ρ denotes half the sum of the positive roots. Here we have used the same symbol to denote both a character and its derivative. Also, although $\Sigma(-1)^w \rho^w$ may not actually give a character unless we pass to a covering of G , the quotient certainly makes sense on G . This formula shows how to find the complete set of characters of irreducible representations of G by alternating vertices of the Diagram.

To construct a representation of G from a given $\lambda \in \hat{T}$ in the fundamental chamber, we may proceed as follows. Let L_λ be the line bundle over G/T constructed from the representation λ , $L_\lambda = G \times_\lambda \mathbb{C}$. It can be shown that G/T is a complex manifold and that L_λ is a complex analytic line bundle. The natural representation of G on the holomorphic sections of L_λ is an irreducible representation with highest weight λ . This result is known as the Borel-Weil Theorem.

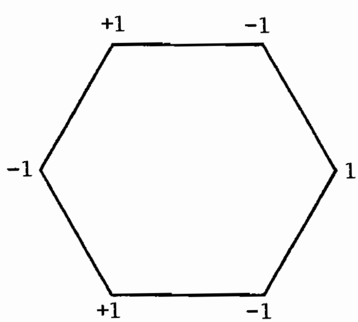
Appendix: We describe the Weyl Character Formula in terms of the Diagram of G (assuming G simple and simply connected).

By using the canonical bilinear form defined on the Lie algebra of T in terms of the roots, the weights may be interpreted as vertices of the Diagram. For a weight λ, we have the following alternating expression $\Sigma(-1)^w \lambda^w$ and the smallest alternating expression comes from alternating the vertex nearest to the origin in the fundamental chamber, namely ρ.

For $G = SU(3)$, the Diagram with ρ is as illustrated



and the formula $\chi_V . \Sigma(-1)^w \rho^w = \Sigma(-1)^w (\lambda+\rho)^w$ may be interpreted geometrically as follows. The hexagon



corresponds to the smallest alternating expression $\Sigma(-1)^w \rho^w$.

In terms of the Diagram, with vertices labelled according to the multiplicity of a weight occurring in the representation V, multiplication of a weight τ with multiplicity k by $\Sigma(-1)^w \rho^w$, corresponds to translating the hexagon on the Diagram to centre at the vertex τ and then replacing the label k at τ

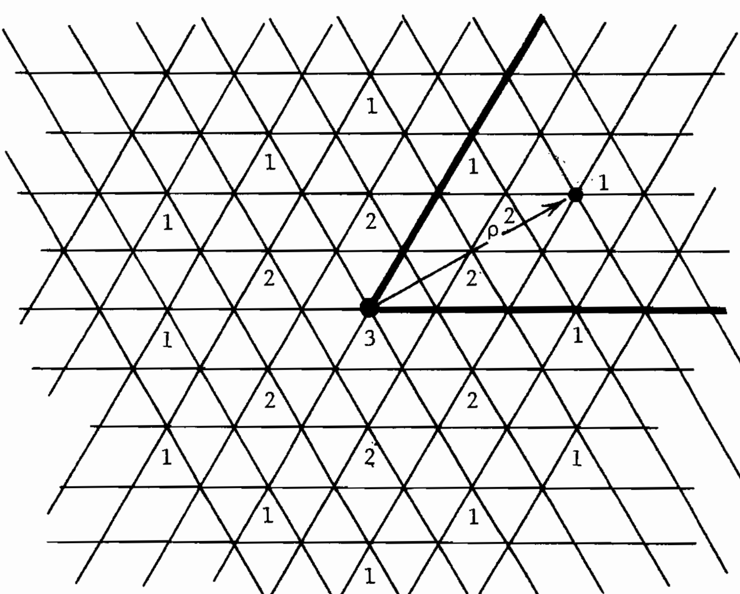
by $\pm k$ at the vertices of the translated hexagon. Finally, on summing all the new labels, we obtain $\Sigma(-1)^w(\lambda+\rho)^w$.

In this way the Hermann-Weyl formula yields the character of $V\lambda$ as the solution of a difference equation; and one can solve it inductively by the following procedure.

1) Start with putting a 1 under some lattice point λ of the fundamental chamber.

2) Now for our hexagon centered anywhere but at $\lambda+\rho$, in the fundamental chamber, the alternating sum of the multiplicities of V must add up to 0 .

3) Use the fact that the multiplicities occur symmetrically relative to the Weyl group. For instance in the last figure I have indicated the result of this procedure when applied to the representation with maximal weight ρ^2 , and I recommend that after cutting out the above hexagon the reader compute the character of the representation with ρ^3 as maximal weight.



References

[1] Bott, R. and Samelson, H. 'Applications of the theory of Morse to symmetric spaces', *American Journal of Mathematics* 80 (1958), 964-1029.

[2] Bott, R., 'An application of the Morse theory to the topology of Lie groups', *Bulletin de la Société Mathématique de France* 84 (1956), 251-282.

[3] Bott, R. and Samelson, H., 'The cohomology ring of G/T', *Proceedings of the National Academy of Sciences* 41 (1955), 490-493.

[4] Serre, J.-P., 'Représentations linéaires et espaces homogènes kähleriens des groupes de Lie compacts', Séminaire Bourbaki, No.100, Paris, 1954.

[5] Stiefel, E., 'Über eine Beziehung zwischen geschlossenen Lie'schen Gruppen und diskontinuierlichen Bewegungsgruppen euklidischer Räume und ihre Anwendung auf die Aufzählung der einfachen Lie'schen Gruppen', *Commentarii Mathematici Helvetici* 14 (1941/2), 350-380.

[6] Stiefel, E., 'Kristellographishe Bestimmung der Characktere der geschlossenen Lieschen Gruppen, *Commentarii Mathematici Helvetici* 17 (1945), 165-200.

5 · Algebraic structure of Lie groups

I.G. MACDONALD
Queen Mary College, London

This survey of the algebraic structure of Lie groups and Lie algebras (mainly semisimple) is a considerably expanded version of the oral lectures at the symposium. It is limited to what is necessary for representation theory, which is another way of saying that very little has been left out. In spite of its length, it contains few proofs or even indications of proofs, nor have I given chapter and verse for each of the multitude of unproved assertions throughout the text. Instead, I have appended references to each section, from which the diligent reader should have no difficulty in tracking down the proofs.

I. Lie Groups and Lie Algebras

1. Vector fields

Let M be a smooth (C^∞) manifold, and for each point $x \in M$ let $T_x(M)$ denote the vector space of *tangent vectors* to M at x. The union of all the $T_x(M)$ is the *tangent bundle* $T(M)$ of M. Locally, if U if a coordinate neighbourhood in M, the restriction of $T(M)$ to U is just $U \times R^n$, where n is the dimension of M. Each smooth map $\phi : M \to N$, where N is another smooth manifold, gives rise to a *tangent map* $T(\phi) : T(M) \to T(N)$, whose restriction $T_x(\phi)$ to the tangent space $T_x(M)$ is a linear mapping of $T_x(M)$ into $T_{\phi(x)}(N)$. In terms of local coordinates in M and N, $T_x(\phi)$ is given by the Jacobian matrix. The familiar rule for differentiating a function of a function now takes the form $T(\phi \circ \psi) = T(\phi) \circ T(\psi)$, so that T is a functor (from smooth manifolds to smooth manifolds).

In particular, if $N = R$, each tangent space $T_y(N)$ may be canonically identified with R. Hence if f is a smooth real-valued function defined on an open neighbourhood of $x \in M$, and ξ is a tangent vector at x, then $T_x(f).\xi$ is a real number, the *directional derivative* of f at x in the direction ξ.

A (smooth) *vector field* on M is a function X which assigns to each $x \in M$ a tangent vector $X(x) \in T_x(M)$, varying smoothly with x; in other words, X is a smooth section of the tangent bundle $T(M)$. X acts on smooth functions as follows:

$$(Xf)(x) = T_x(f).X(x) \quad .$$

In this way X acts as a *derivation* of the R-algebra $C^\infty(M)$ of smooth functions on M; that is to say, X is R-linear and satisfies

$$X(fg) = (Xf).g + f.Xg \tag{1}$$

for $f,g \in C^\infty(M)$; this is just the expression, in the present context, of the rule for differentiating a product of two functions. Conversely, each derivation of $C^\infty(M)$ arises in this way from a unique vector field, and we may therefore *identify* X with the derivation it defines.

Now let X and Y be vector fields (or derivations) on M. Then $X \circ Y : C^\infty(M) \to C^\infty(M)$ is not a derivation, but the Lie bracket

$$[X,Y] = X \circ Y - Y \circ X$$

always is (just check that (1) is satisfied). It follows that the space of vector fields on a manifold M has the structure of a *Lie algebra* over R: it is a (usually infinite-dimensional) vector space over R, equipped with a 'Lie bracket' $[X,Y]$ which is R-bilinear and anticommutative, and in addition satisfies the 'Jacobi identity'

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 .$$

2. The Lie algebra of a Lie group

A *Lie group* G is a smooth manifold which is also a group, the two structures being compatible: that is to say, the mappings $m : G \times G \to G$ and $i : G \to G$ defined by multiplication and inversion $(m(x,y) = xy,\ i(x) = x^{-1})$ are *smooth*.

Examples

1. Any discrete group may be regarded as a Lie group (of dimension 0).
2. The additive group of R^n (or of any finite-dimensional real vector space) is a Lie group. Such a group is called a *vector group*.
3. The *circle group* $T = R/Z$ is a Lie group. The n-*dimensional torus* $T^n = (R/Z)^n$ is a Lie group.
4. The general linear group $GL(n,R)$ of invertible real $n \times n$ matrices is an open submanifold of the space $M(n,R) = R^{n^2}$ of all $n \times n$ matrices, since it is the complement of the hypersurface $\det X = 0$. Hence $GL(n,R)$ is a Lie group, of dimension n^2 . It is not connected but has two components, corresponding to positive and negative determinant. The identity component, consisting of the matrices X with $\det X > 0$, is denoted by $GL^+(n,R)$.

More intrinsically, if V is a real vector space of dimension n , the group $GL(V)$ of invertible linear transformations of V is a Lie group, isomorphic to $GL(n,R)$.
5. Likewise $GL(n,C)$, the group of invertible complex $n \times n$ matrices, is a (complex) Lie group, of complex dimension n^2 . Unlike $GL(n,R)$, it is connected.
6. Let H denote the division ring of quaternions. Then

$GL(n,H)$ is a (real) Lie group of dimension $4n^2$.

For each $x \in G$, let $\lambda_x : G \to G$ denote left translation by x:

$$\lambda_x(y) = xy \quad .$$

Clearly λ_x is a diffeomorphism of G, its inverse being $\lambda_{x^{-1}}$.

Let X be a vector field on G. We say that X is *left-invariant* if X commutes with left translations, i.e. if

$$X \circ \lambda_x = T(\lambda_x) \circ X$$

for all $x \in G$. If we regard X as a derivation, left-invariance is expressed by

$$(Xf) \circ \lambda_x = X(f \circ \lambda_x)$$

for all $f \in C^\infty(G)$ and $x \in G$. It follows immediately that the space of left-invariant vector fields on G is closed under the Lie bracket, and is therefore a *Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$, called the *Lie algebra* of the Lie group G.

Each $X \in \mathfrak{g}$ is determined by its value $X(e) \in T_e(G)$ at the identity element e of G, because

$$X(x) = (X \circ \lambda_x)(e) = T(\lambda_x)X(e) \quad .$$

Conversely, each tangent vector $\xi \in T_e(G)$ determines a left-invariant vector field X_ξ on G by the rule

$$X_\xi(x) = T_e(\lambda_x)\xi \quad .$$

Consequently $\mathfrak{g}$ may be identified with $T_e(G)$, the tangent space to G at the identity element e. In particular it follows that $\dim \mathfrak{g} = \dim G$.

We may also remark here that the tangent bundle $T(G)$ of a Lie group G is *trivial*, i.e. is isomorphic (as a bundle) to $G \times T_e(G)$. Indeed, the mapping $(x,\xi) \to X_\xi(x)$ is an isomorphism of $G \times T_e(G)$ onto $T(G)$.

Now let H be another Lie group and let $\phi : G \to H$ be a smooth homomorphism; let $\mathfrak{g} = T_e(G)$, $\mathfrak{h} = T_e(H)$. The tangent map $T_e(\phi) : \mathfrak{g} \to \mathfrak{h}$ is called the *derived homomorphism* of ϕ and is denoted by ϕ_* . It is a homomorphism of Lie algebras, i.e. we have $\phi_*[X,Y] = [\phi_* X, \phi_* Y]$ for $X,Y \in \mathfrak{g}$. •

Examples

1. If $G = R^n$, then $\mathfrak{g} = R^n$ and $[X,Y] = 0$ for all $X,Y \in \mathfrak{g}$. For each $X \in \mathfrak{g}$, regarded as a derivation of $C^\infty(R^n)$, is of the form $X = \sum_{i=1}^{n} a_i \partial/\partial x_i$, with *constant* coefficients a_i ; any two such derivations clearly commute, because $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$ on smooth functions.
2. Let $G = GL(n,R)$. Define $\alpha : \mathfrak{g} \to M(n,R)$ by

$$\alpha(X)_{ij} = (Xx_{ij})(I_n) \qquad (1 \le i,j \le n)$$

where I_n is the unit matrix (the identity element of G) and $x_{ij} : G \to R$ assigns to each matrix in G its (i,j) element. Then α is an isomorphism of vector spaces and $\alpha[X,Y] = \alpha(X)\alpha(Y) - \alpha(Y)\alpha(X)$. The Lie algebra $\mathfrak{gl}(n,R)$ of $GL(n,R)$ is therefore canonically identified with the Lie algebra of all $n \times n$ matrices. Likewise with C or H in place of R .
3. If V is a real vector space of dimension n (so that $V \cong R^n$), the Lie algebra of $GL(V)$ ($\cong GL(n,R)$) is denoted by $\mathfrak{gl}(V)$. As in Ex.2 we may identify $\mathfrak{gl}(V)$ with the Lie algebra of the ring $\mathrm{End}(V)$ of all linear transformations of V .
4. If G is an abelian Lie group, then $\mathfrak{g}$ is an abelian Lie algebra, i.e. $[X,Y] = 0$ for all $X,Y \in \mathfrak{g}$.

Remark. We could of course have started with *right*-invariant vector fields. However, the inversion $i : x \to x^{-1}$ interchanges right and left, and we get nothing new.

3. The exponential map

The usual exponential function e^x is a smooth mapping from $\mathbb{R} = \mathfrak{gl}(1,\mathbb{R})$ onto $GL^+(1,\mathbb{R})$, the multiplicative group of positive real numbers. More generally, if X is any real $n \times n$ matrix, the exponential series $\sum_{n=0}^{\infty} X^n/n!$ converges in the space $M(n,\mathbb{R})$ of $n \times n$ matrices, and its sum $e^X = \exp(X)$ is invertible (with inverse e^{-X}) and has positive determinant (namely $e^{\operatorname{trace} X}$). Hence $X \to e^X$ is a smooth function on $M(n,\mathbb{R}) = \mathfrak{gl}(n,\mathbb{R})$ with values in $GL^+(n,\mathbb{R})$. These examples are particular instances of the *exponential map*, which is defined for any Lie group G, and is a smooth mapping of the Lie algebra $\mathfrak{g}$ into the group G.

The definition runs as follows. A *one-parameter subgroup* of G is a smooth homomorphism $u : \mathbb{R} \to G$. Its derived homomorphism $u_* = T_0(u)$ is a linear mapping of $\mathbb{R}$ into $\mathfrak{g}$, the Lie algebra of G. It is a consequence of the theorem of existence and uniqueness of solutions of linear ordinary differential equations that the mapping $u \to u_*(1)$ is a bijection of the set of one-parameter subgroups of G onto the Lie algebra $\mathfrak{g}$: for each $X \in \mathfrak{g}$ there exists a unique one-parameter subgroup $u_X : \mathbb{R} \to G$ such that $u_{X*}(1) = X$. The exponential map $\exp_G : \mathfrak{g} \to G$ is now defined by

$$\exp_G(X) = u_X(1) \quad .$$

We have $\exp(tX) = u_X(t)$ for all $t \in \mathbb{R}$, so that $\exp(sX)\exp(tX) = \exp((s+t)X)$.

The exponential map is a smooth map whose derivative at $0 \in \mathfrak{g}$ is $1_{\mathfrak{g}}$, the identity mapping of $\mathfrak{g}$. Hence, by the inverse function theorem, $\exp$ is a diffeomorphism of some open neighbourhood of 0 in $\mathfrak{g}$ onto an open neighbourhood of e in G ; that is to say, it provides a chart of G around the identity element. From this it follows that, if G is connected, the image $\exp(\mathfrak{g})$ of the exponential map *generates* G (although in general $\exp : \mathfrak{g} \to G$ is not surjective, except in the cases where G is compact or abelian (and connected)).

For $X, Y \in \mathfrak{g}$ and $t \in R$ we have

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + O(t^2)) \quad , \tag{1}$$

$$[\exp(tX), \exp(tY)] = \exp(t^2[X,Y] + O(t^3)) \tag{2}$$

(where on the left-hand side of (2) the bracket is the commutator $[x,y] = xyx^{-1}y^{-1}$ in G). Thus, under the exponential map, multiplication in G corresponds approximately to addition in $\mathfrak{g}$, and commutator formation in G corresponds approximately to the Lie bracket in $\mathfrak{g}$.

If G is abelian, $\exp_G$ is additive, and therefore a homomorphism of the vector group $\mathfrak{g}$ into G .

If $\phi : G \to H$ is a smooth homomorphism, then we have

$$\phi \circ \exp_G = \exp_H \circ \phi_*$$

(*naturality* of $\exp$).

4. The adjoint representation

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. For each $x \in G$, let $\mathrm{Int}(x) : g \to xgx^{-1}$ be the inner automorphism of G defined by x . $\mathrm{Int}(x)$ is a smooth automorphism of G , and its derived homomorphism is denoted by $\mathrm{Ad}(x)$ or $\mathrm{Ad}_G(x)$:

$$\mathrm{Ad}(x) = \mathrm{Int}(x)_* : \mathfrak{g} \to \mathfrak{g}$$

is an automorphism of the Lie algebra $\mathfrak{g}$, *a fortiori* of the vector space $\mathfrak{g}$. Since $\mathrm{Int}(x) \circ \mathrm{Int}(y) = \mathrm{Int}(xy)$, we have $\mathrm{Ad}(x) \circ \mathrm{Ad}(y) = \mathrm{Ad}(xy)$; also $\mathrm{Ad}(x)$ varies smoothly with x , and therefore

$$\mathrm{Ad} : G \to GL(\mathfrak{g})$$

is a smooth homomorphism of G into the general linear group of $\mathfrak{g}$, called the *adjoint representation* of G .

If G is connected, the kernel of Ad is the centre of G .

The derived homomorphism of Ad is denoted by $\mathrm{ad}_{\mathfrak{g}}$ or ad :

$$\mathrm{Ad}_* = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(\mathfrak{g})$, called the *adjoint representation* of $\mathfrak{g}$. More directly (and without reference to G), $\mathrm{ad}\,\mathfrak{g}$ may be defined by

$$(\mathrm{ad}\, X)Y = [X,Y]$$

for $X,Y \in \mathfrak{g}$. That $[\mathrm{ad}\, X, \mathrm{ad}\, Y] = \mathrm{ad}[X,Y]$ is just a restatement of the Jacobi identity (§1).

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. The *normalizer* $N_G(\mathfrak{h})$ of $\mathfrak{h}$ in G is the group of all $x \in G$ such that $\mathrm{Ad}(x)\mathfrak{h} \subset \mathfrak{h}$, and the *centralizer* $Z_G(\mathfrak{h})$ of $\mathfrak{h}$ in G is the group of all $x \in G$ such that $\mathrm{Ad}(x)|\mathfrak{h} = 1_{\mathfrak{h}}$. Likewise, the *normalizer* $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ is the subalgebra of all $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)\mathfrak{h} \subset \mathfrak{h}$, and the *centralizer* $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$ is the subalgebra of all $X \in \mathfrak{g}$ such that $\mathrm{ad}(X)|\mathfrak{h} = 0$. $N_G(\mathfrak{h})$ and $Z_G(\mathfrak{h})$ are closed subgroups of G and hence (§5) are Lie groups. The Lie algebra of $N_G(\mathfrak{h})$ (resp. $Z_G(\mathfrak{h})$) is $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h})$ (resp. $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{h})$).

Let $\mathfrak{g}$ be a (finite-dimensional) real Lie algebra, and consider the polynomial in t

$$\det(t - \mathrm{ad}_{\mathfrak{g}}(X)) = \sum_{i=0}^{n} d_i(X)t^i \qquad (X \in \mathfrak{g})$$

of degree $n = \dim \mathfrak{g}$. The d_i are polynomial functions in $\mathfrak{g}$. The smallest integer ℓ such that $d_\ell \neq 0$ is called the *rank* of $\mathfrak{g}$, and an element $X \in \mathfrak{g}$ is said to be *regular* if $d_\ell(X) \neq 0$. The set $\mathfrak{g}'$ of regular elements in $\mathfrak{g}$ is therefore the complement of a real algebraic variety in $\mathfrak{g}$, and hence is a dense open subset of $\mathfrak{g}$.

These definitions have global counterparts. Let G be a connected Lie group, and consider the polynomial in t

$$\det(t + 1 - \mathrm{Ad}_G(x)) = \sum_{i=0}^{n} D_i(x) t^i \qquad (x \in G) .$$

of degree $n = \dim G$. The D_i are real analytic functions on G . The least integer ℓ such that $D_\ell \neq 0$ is called the *rank* of G , and an element $x \in G$ is said to be *regular* if $D_\ell(x) \neq 0$. We have $\mathrm{rank}(G) = \mathrm{rank}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G . The set G' of regular elements of G is a dense open subset of G , stable under inner automorphisms, whose complement has measure zero with respect to (left or right) Haar measure on G .

5. Subgroups and subalgebras

By a *Lie subgroup* of a Lie group G we mean a (locally closed) submanifold H of G which is also a subgroup of G . It is almost immediate that H is a closed subgroup of G and a Lie group. The converse of this result is also true, but harder to prove: *every closed subgroup* H *of a Lie group* G *is a submanifold of* G (and therefore a Lie subgroup of G) (E. Cartan's theorem). The Lie algebra $\mathfrak{h}$ of H consists of all $X \in \mathfrak{g}$ such that $\exp(tX) \in H$ for all $t \in \mathbb{R}$.

Examples

1. The *special linear group* $SL(n,\mathbb{R})$, consisting of the real $n \times n$ matrices with determinant 1 , is closed in $GL(n,\mathbb{R})$, hence is a Lie group. Its Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ consists of

the $X \in \mathfrak{gl}(n,R)$ with trace $X = 0$ (because $\det(e^X) = e^{\text{trace } X}$).
Likewise with C in place of R.

2. Let K be any one of R, C or H. Let $\bar{x}$ denote the conjugate of $x \in K$ (so that $\bar{x} = x$ if $K = R$). Let $U(n,K)$ denote the group of all $X \in GL(n,K)$ such that $X\bar{X}^t = 1$. Then $U(n,K)$ is a subgroup of $GL(n,K)$, and is *closed* because it is defined by the polynomial equations

$$\sum_{k=1}^{n} x_{ik}\bar{x}_{jk} = \delta_{ij} \quad ,$$

hence by Cartan's theorem is a Lie group. These equations also imply that $\sum_{i,j} |x_{ij}|^2 = n$, so that $U(n,K)$ is a *bounded* subset of $M(n,K)$, and is therefore *compact*. Hence $U(n,K)$ is a compact Lie group, and its Lie algebra consists of all $X \in M(n,K)$ such that $X + \bar{X}^t = 0$, i.e. such that X is skew-Hermitian (or skew-symmetric, when $K = R$).

(i) When $K = R$, $U(n,K)$ is the *orthogonal group* $O(n)$, which has two components (corresponding to determinant $+1$ and -1). The *special orthogonal group* $SO(n)$, consisting of the orthogonal matrices with determinant $+1$, is a compact connected Lie group. Its Lie algebra $\mathfrak{so}(n)$ consists of the real skew-symmetric $n \times n$ matrices of trace 0.

(ii) When $K = C$, $U(n,K)$ is the *unitary group* $U(n)$, which is connected. The *special unitary group* $SU(n)$, consisting of the unitary matrices $X \in U(n)$ with $\det X = 1$, is a closed subgroup of $U(n)$ and therefore also a Lie group. Its Lie algebra $\mathfrak{su}(n)$ consists of the complex skew-Hermitian $n \times n$ matrices with trace 0.

(iii) When $K = H$, $U(n,K)$ is the *quaternionic unitary group* $Sp(n)$.

If H is a Lie subgroup of G, the Lie algebra $\mathfrak{h}$ of H is a subalgebra of the Lie algebra $\mathfrak{g}$ of G. Conversely, however an arbitrary Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *not* necessarily the Lie algebra of a Lie subgroup of G. What *is* true is that to each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ there exists a connected Lie group H and a smooth injective homomorphism $j : H \to G$ such that j_* is an isomorphism of the Lie algebra of H onto $\mathfrak{h}$; and the pair (H,j) is unique up to isomorphism. The image $j(H)$ is the subgroup of G generated by $\exp_G(\mathfrak{h})$. The connected Lie group H , identified with its image in G , is called the *immersed* subgroup of G corresponding to $\mathfrak{h}$; in general it is not closed in G , and the topology of the Lie group H is not the topology induced from G .

Example. Let G be the torus T^2 , so that $\mathfrak{g} = R^2$; let $\mathfrak{h} = R$, embedded in R^2 by $x \to (x,\theta x)$ where θ is an irrational number. Then $H = R$, and $j(H) \subset G$ is a curve which winds round and round the torus infinitely often, so that $j(H)$ is *dense* in G .

The correspondence between subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ and immersed subgroups H of G has all the properties that one could reasonably expect. The centralizer (resp. normalizer) of H in G is equal to the centralizer (resp. normalizer) of $\mathfrak{h}$ in G . In particular, if G is connected, H is normal in G if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ (i.e. $\mathfrak{N}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}$), and the centre C of G has Lie algebra $\mathfrak{c}$, the centre of $\mathfrak{g}$ (i.e. $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{g})$) . Again, if G is connected, the derived group DG (generated by all commutators $[x,y]$) is an immersed subgroup which corresponds to the derived algebra $\mathfrak{D}\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$, spanned by all brackets $[X,Y]$. It follows that a connected Lie group is solvable if and only if its Lie algebra is solvable (i.e. $\mathfrak{D}^r\mathfrak{g} = 0$ for some $r \geq 1$).

Example. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathrm{Aut}(\mathfrak{g})$ its group of automorphisms. $\mathrm{Aut}(\mathfrak{g})$ is a closed (indeed algebraic) subgroup of $GL(\mathfrak{g})$, hence is a Lie group. Its Lie algebra is the algebra $\mathrm{Der}(\mathfrak{g})$ of *derivations* of $\mathfrak{g}$, a subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

The image $ad(\mathfrak{g})$ of $\mathfrak{g}$ under the adjoint representation (§4) is a subalgebra of $Der(\mathfrak{g})$. To it there corresponds an immersed subgroup $Int(\mathfrak{g})$ of $Aut(\mathfrak{g})$, called the *adjoint group* of $\mathfrak{g}$; it is generated by the automorphisms $\exp(ad X)$, $X \in \mathfrak{g}$. If G is a connected Lie group with $\mathfrak{g}$ as Lie algebra, then $Int(\mathfrak{g})$ is the image of G under the adjoint representation (because $\exp(ad X) = Ad(\exp X)$ by naturality of $\exp$).

If $\mathfrak{g}$ is semisimple, $Int(\mathfrak{g})$ is the identity component of $Aut(\mathfrak{g})$, and $ad(\mathfrak{g}) = Der(\mathfrak{g})$ (i.e., every derivation of $\mathfrak{g}$ is inner).

6. Quotients

Let G be a Lie group, H a *closed* subgroup of G . The quotient set G/H , whose elements are the cosets $xH = \dot{x}$ of H in G , then carries a unique structure of a smooth manifold such that the projection $p : x \to \dot{x}$ of G onto G/H is smooth, and such that a mapping f of G/H into a smooth manifold M is smooth if and only if $f \circ p : G \to M$ is smooth. The tangent space to G/H at the image $\dot{e}$ of e is $\mathfrak{g}/\mathfrak{h} = T_e(G)/T_e(H)$, from which it follows that $\dim(G/H) = \dim G - \dim H$. Moreover, the projection p has a smooth local cross-section defined on an open neighbourhood of $\dot{e}$, from which it follows that locally G looks like the Cartesian product of H with G/H , or more precisely that G is a smooth bundle over G/H with fibre H .

If H is a closed *normal* subgroup of G , then the group structure and the manifold structure on G/H are compatible, i.e. G/H is a Lie group.

Example. If G is a connected Lie group, then $Ad(G) \cong G/Z$ where Z is the centre of G .

7. Homomorphisms and local homomorphisms

Let $\phi : G \to H$ be a smooth homomorphism of Lie groups. The *kernel* N of ϕ is closed in G, hence is a Lie subgroup of G, whose Lie algebra is the kernel of the derived homomorphism $\phi_* : \mathfrak{g} \to \mathfrak{h}$. The *image* $\phi(G)$, on the other hand, need not be a closed subgroup of H but (provided that G is connected) is the *immersed* subgroup of H corresponding to the subalgebra $\phi_*(\mathfrak{g})$ of $\mathfrak{h}$. The immersion is the injective smooth homomorphism $G/N \to H$ induced by ϕ.

For example, the one-parameter subgroups of G (§3) are immersed subgroups.

Let G and H again be Lie groups. A (smooth) *local homomorphism* from G to H is a smooth mapping ϕ of an open neighbourhood U of the identity element in G, with values in H, such that $\phi(xy) = \phi(x)\phi(y)$ whenever x, y and xy all lie in U. If ϕ is also a diffeomorphism of U onto an open neighbourhood of the identity element in H, then ϕ^{-1} is a local homomorphism from H to G, and ϕ is said to be a *local isomorphism* of G with H.

Each local homomorphism ϕ from G to H has a *derived homomorphism* $\phi_* = T_e(\phi) : \mathfrak{g} \to \mathfrak{h}$, which is a homomorphism of Lie algebras; and ϕ is a local isomorphism if and only if ϕ_* is an isomorphism.

Conversely, if $u : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, there exists a local homomorphism ϕ from G to H such that $u = \phi_*$, and moreover ϕ is essentially unique (in the sense that if $u = \phi_{1*} = \phi_{2*}$, then ϕ_1 and ϕ_2 coincide on some open neighbourhood of e in G). It follows that two Lie groups G,H are locally isomorphic if and only if their Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ are isomorphic.

If now G is connected and simply-connected, every local homomorphism from G to H has a unique extension to a (global) smooth homomorphism of G into H (monodromy theorem). Hence the smooth homomorphisms of a connected and simply-connected Lie group G into any Lie group H are in one-one correspondence (via the derived homomorphism) with the Lie algebra homomorphisms of $\mathfrak{g}$ into $\mathfrak{h}$.

8. The universal covering group

Let G be a connected Lie group. Then G has a *universal covering* group $\tilde{G}$, which is a Lie group, characterized up to isomorphism by the following properties: (i) there exists a surjective smooth homomorphism $p : \tilde{G} \to G$ with discrete kernel; (ii) $\tilde{G}$ is connected and simply-connected. The kernel D of p is isomorphic to the fundamental group $\pi_1(G)$, and is a subgroup of the centre of $\tilde{G}$ (because for each $d \in D$ the mapping $x \to xdx^{-1}$ of $\tilde{G}$ into D is continuous, and therefore constant). Hence D , and therefore also $\pi_1(G)$, is *abelian*.

The derived homomorphism $p_* : \mathrm{Lie}(\tilde{G}) \to \mathrm{Lie}(G) = \mathfrak{g}$ is an isomorphism. Hence the connected Lie groups with $\mathfrak{g}$ as Lie algebra are all obtained from $\tilde{G}$ by factoring out a discrete subgroup of the centre of $\tilde{G}$.

Finally, every (finite-dimensional) Lie algebra $\mathfrak{g}$ is the Lie algebra of some connected Lie group G , hence also of its universal covering $\tilde{G}$. In this way is established a one-one correspondence between isomorphism classes of finite-dimensional real Lie algebras and isomorphism classes of connected and simply-connected Lie groups. Thus, for an arbitrary Lie group G , the only information about G that is not captured by its Lie algebra $\mathfrak{g}$ is (i) properties that depend on the different connected components, (ii) properties which depend on different covering groups, i.e. on $\pi_1(G)$.

1. Generalities on Lie algebras

Many of the notions of group theory have counterparts for Lie algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (over any field of characteristic 0). If $\mathfrak{a}$, $\mathfrak{b}$ are vector subspaces of $\mathfrak{g}$, we denote by $[\mathfrak{a},\mathfrak{b}]$ the vector space spanned by all $[X,Y]$ with $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$. A vector subspace $\mathfrak{a}$ of $\mathfrak{g}$ is a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a}$, and an *ideal* in $\mathfrak{g}$ if $[\mathfrak{g},\mathfrak{a}] \subset \mathfrak{a}$: these are the counterparts of the notions of subgroup and normal subgroup, respectively. If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ we can (as in other algebraic contexts) factor it out to form the quotient algebra $\mathfrak{g}/\mathfrak{a}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$, then $[\mathfrak{a},\mathfrak{b}]$ is also an ideal.

The *derived series* of $\mathfrak{g}$ is the decreasing sequence of ideals $(\mathfrak{D}^r\mathfrak{g})_{r\geq 0}$, where $\mathfrak{D}^0\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{D}^{r+1}\mathfrak{g} = [\mathfrak{D}^r\mathfrak{g},\mathfrak{D}^r\mathfrak{g}]$. Just as in group theory, if $\mathfrak{D}^r\mathfrak{g} = 0$ for some r , the Lie algebra $\mathfrak{g}$ is said to be *solvable*.

The *lower central series* of $\mathfrak{g}$ is the decreasing sequence of ideals $(\mathfrak{C}^r\mathfrak{g})_{r\geq 0}$, where $\mathfrak{C}^0\mathfrak{g} = \mathfrak{g}$ and $\mathfrak{C}^{r+1}\mathfrak{g} = [\mathfrak{g},\mathfrak{C}^r\mathfrak{g}]$. The *upper central series* of $\mathfrak{g}$ is the increasing sequence of ideals $(\mathfrak{C}_r\mathfrak{g})_{r\geq 0}$, where $\mathfrak{C}_0\mathfrak{g} = 0$ and $\mathfrak{C}_{r+1}\mathfrak{g}/\mathfrak{C}_r\mathfrak{g}$ is the centre of $\mathfrak{g}/\mathfrak{C}_r\mathfrak{g}$. Just as in group theory, we have $\mathfrak{C}^r\mathfrak{g} = 0$ for large r if and only if $\mathfrak{C}_r\mathfrak{g} = 0$ for large r , and the Lie algebra $\mathfrak{g}$ is then said to be *nilpotent*. An equivalent condition is that $\mathrm{ad}_{\mathfrak{g}}X$ should be nilpotent for all $X \in \mathfrak{g}$.

Every nilpotent Lie algebra is solvable, and a Lie algebra $\mathfrak{g}$ is solvable if and only if its derived algebra $\mathfrak{D}\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ is nilpotent.

The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form $B_{\mathfrak{g}}$ defined by

$$B_{\mathfrak{g}}(X,Y) = \mathrm{trace}(\mathrm{ad}\ X)(\mathrm{ad}\ Y)$$

If $\mathfrak{g}$ is nilpotent, $B_{\mathfrak{g}}$ is identically zero; conversely, if $B_{\mathfrak{g}} = 0$ then $\mathfrak{g}$ is solvable.

Let $\mathfrak{g}$ again be any finite dimensional Lie algebra. Then $\mathfrak{g}$ has a unique maximal solvable ideal (for if $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals, then so is $\mathfrak{a} + \mathfrak{b}$). This ideal $\mathfrak{r}$ is called the *radical* of $\mathfrak{g}$. It is also the orthogonal complement of the derived algebra $\mathfrak{D}\mathfrak{g}$ with respect to the Killing form.

If $\mathfrak{r} = 0$, that is if $\mathfrak{g}$ has no nonzero solvable ideals, then $\mathfrak{g}$ is said to be *semisimple*. An equivalent condition is that $\mathfrak{g}$ is a direct product of *simple* Lie algebras (a Lie algebra is *simple* if it has no nontrivial ideals *and is not abelian*). Yet another equivalent condition is that the Killing form $B_{\mathfrak{g}}$ is nondegenerate.

If $\mathfrak{g}$ is again any finite-dimensional Lie algebra, $\mathfrak{r}$ its radical, then there exists a subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{r} \tag{1}$$

(direct sum). $\mathfrak{l}$ is called a *Levi subalgebra* of $\mathfrak{g}$, and (1) is a *Levi decomposition*. The algebra $\mathfrak{l}$ is semisimple, because it is isomorphic to $\mathfrak{g}/\mathfrak{r}$, which has zero radical. The algebra $\mathfrak{l}$ in (1) is not uniquely determined, but any two are conjugate in $\mathfrak{g}$ under the adjoint group $\mathrm{Int}(\mathfrak{g})$ (I, §5). Also, the Levi subalgebras of $\mathfrak{g}$ are the maximal semisimple subalgebras of $\mathfrak{g}$.

If the radical $\mathfrak{r}$ is the centre $\mathfrak{z}$ of $\mathfrak{g}$, the Lie algebra $\mathfrak{g}$ is said to be *reductive*. An equivalent condition is that the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ should be completely reducible. If $\mathfrak{g}$ is reductive, its derived algebra $\mathfrak{D}\mathfrak{g}$ is semisimple, and $\mathfrak{g}$ is the direct product of $\mathfrak{D}\mathfrak{g}$ and $\mathfrak{z}$. Hence the reductive Lie algebras are just direct products of abelian and semisimple Lie algebras, and we shall therefore concentrate on the latter.

Examples. $\mathfrak{gl}(n,R)$, $\mathfrak{gl}(n,C)$, $\mathfrak{u}(n)$ are reductive but not semisimple. $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$ are semisimple (e.g. by computing the Killing form explicitly).

In particular, if $\mathfrak{g}$ is semisimple, the centre of $\mathfrak{g}$ is zero and hence the adjoint representation embeds $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$, the Lie algebra of $GL(\mathfrak{g})$. From the results of Chap. I, §5 it follows that $\mathfrak{g}$ is isomorphic to the Lie algebra of an immersed subgroup G of $GL(\mathfrak{g})$. Hence every semisimple real Lie algebra is the Lie algebra of some connected Lie group. One can then use Levi's theorem above to show that *every* finite-dimensional Lie algebra over $\mathbb{R}$ is the Lie algebra of a connected Lie group.

If $\mathfrak{g}$ is a real Lie algebra, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} + i\mathfrak{g}$ its complexification, then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}_{\mathbb{C}}$ is semisimple. For the matrix of the Killing form, relative to a basis of $\mathfrak{g}$, is the same for $\mathfrak{g}_{\mathbb{C}}$ as for $\mathfrak{g}$. If $\mathfrak{g}$ is *simple*, then $\mathfrak{g}_{\mathbb{C}}$ is either simple or is the product of two isomorphic simple algebras.

If $\mathfrak{g}$ is a *complex* Lie algebra, let $\mathfrak{g}^{\mathbb{R}}$ denote $\mathfrak{g}$ regarded as a real Lie algebra. If $\mathfrak{g}$ is semisimple (resp. simple) then so is $\mathfrak{g}^{\mathbb{R}}$. We have $(\mathfrak{g}^{\mathbb{R}})_{\mathbb{C}} \cong \mathfrak{g} \times \mathfrak{g}$. We call $\mathfrak{g}^{\mathbb{R}}$ the *realification* of $\mathfrak{g}$.

A subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}^{\mathbb{R}}$ is a *real form* of the complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g} = \mathfrak{g}_o + i\mathfrak{g}_o$. The real simple Lie algebras are either real forms or realifications of complex simple Lie algebras. We shall begin with the structure theory of the complex Lie algebras.

2. Cartan subalgebras

Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. An element $X \in \mathfrak{g}$ is *semisimple* if the linear transformation $\mathrm{ad}\,X : \mathfrak{g} \to \mathfrak{g}$ is semisimple (i.e. diagonalizable over $\mathbb{C}$). A *Cartan subalgebra* of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ consisting of semisimple elements; equivalently, it is the centralizer in $\mathfrak{g}$ of a *regular* element of $\mathfrak{g}$ (Chapter I, §4).

Now let $\mathfrak{g}$ be *complex*. The importance of the Cartan subalgebras for unravelling the structure of $\mathfrak{g}$ lies in the fundamental fact that they are all conjugate under the adjoint group $\mathrm{Int}(\mathfrak{g})$ (Chapter I, §5). (As we shall see later, this is not in general true for real semisimple Lie algebras, and is one of the reasons why their structure theory is more complicated.)

The (complex) dimension of a Cartan subalgebra of $\mathfrak{g}$ is equal to the rank of $\mathfrak{g}$, as defined in I, §4. We shall denote it by ℓ.

Example. If $\mathfrak{g} = \mathfrak{sl}(n,C)$, the diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra. Hence the rank of $\mathfrak{sl}(n,C)$ is $n-1$.

3. Roots

Until further notice, $\mathfrak{g}$ is a complex semisimple Lie algebra. Since all the Cartan subalgebras of $\mathfrak{g}$ are conjugate, there is no harm in choosing one, say $\mathfrak{h}$, once and for all. Since $\mathfrak{h}$ is abelian and the ground field C is algebraically closed, the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$, restricted to $\mathfrak{h}$, splits up as a direct sum of one-dimensional representations. In other words, if $\mathfrak{h}^*$ is the vector space dual of $\mathfrak{h}$, and if for each $\alpha \in \mathfrak{h}^*$ we denote by $\mathfrak{g}^\alpha$ the subspace of all $X \in \mathfrak{g}$ such that $\mathrm{ad}(H).X = \alpha(H)X$ for all $H \in \mathfrak{h}$, then $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}^\alpha$. Two such subspaces $\mathfrak{g}^\alpha, \mathfrak{g}^\beta$ are orthogonal with respect to the Killing form B unless $\alpha+\beta = 0$. Moreover, $\mathfrak{g}^0$ is equal to $\mathfrak{h}$, because $\mathfrak{h}$ is its own centralizer in $\mathfrak{g}$. It follows that $\mathfrak{h}$ is orthogonal to all the $\mathfrak{g}^\alpha$, $\alpha \neq 0$, and therefore the restriction of B to $\mathfrak{h}$ remains nondegenerate.

Since $\mathfrak{g}$ is finite-dimensional, only finitely many of the $\mathfrak{g}^\alpha$ are nonzero. If $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$, then α is said to be a *root* of $\mathfrak{g}$ (relative to the Cartan subalgebra $\mathfrak{h}$) and

$\mathfrak{g}^{\alpha}$ the *root-space* of α . If α is a root, so is $-\alpha$ (otherwise $\mathfrak{g}^{\alpha}$ would be orthogonal to all of $\mathfrak{g}$, contrary to the nondegeneracy of B). For each root α , we have $\dim \mathfrak{g}^{\alpha} = 1$.

If H is a general element of $\mathfrak{h}$, the complex numbers $\alpha(H)$ (α a root) are the nonzero eigenvalues of the linear transformation $\mathrm{ad}_{\mathfrak{g}}H$, i.e. they are the nonzero *roots* of the characteristic equation $\det(\lambda - \mathrm{ad}\, H) = 0$; this is the reason for the terminology.

We denote by R or $R(\mathfrak{g},\mathfrak{h})$ the set of roots: it is a finite subset of $\mathfrak{h}^*$. We have then a direct decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}^{\alpha} \tag{1}$$

called the *root-space* decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$.

The roots span a real subspace V of dimension ℓ in $\mathfrak{h}^*$, so that $\mathfrak{h}^*$ is the complexification of V . We have already observed that the Killing form B remains nondegenerate in restriction to $\mathfrak{h}$, hence defines an isomorphism $\lambda \to H_{\lambda}$ of $\mathfrak{h}^*$ onto $\mathfrak{h}$, and a bilinear form $\langle\lambda,\mu\rangle = B(H_{\lambda},H_{\mu})$ on $\mathfrak{h}^*$. It turns out that the restriction of this to V is real-valued and positive-definite, so that V acquires the structure of a real Euclidean space. Let $\mathfrak{h}_R$ denote the vector space spanned by the H_{α} , $\alpha \in R$; then $\mathfrak{h}$ is the complexification of $\mathfrak{h}_R$ and V is the dual $\mathfrak{h}_R^*$ of $\mathfrak{h}_R$.

In this way we have constructed from $\mathfrak{g}$ a finite set R of nonzero vectors in the Euclidean space V . This set R is called the *root-system* of $\mathfrak{g}$: up to isomorphism, it is independent of the choice of $\mathfrak{h}$, and therefore depends only on $\mathfrak{g}$. It may be thought of as in some sense the 'skeleton' of $\mathfrak{g}$, and it determines $\mathfrak{g}$ up to isomorphism. More precisely, there is the following *isomorphism theorem*: if $\mathfrak{g}'$ is another complex semisimple Lie algebra; $\mathfrak{h}'$ a Cartan subalgebra of $\mathfrak{g}'$; R' the root system of $\mathfrak{g}'$ relative to $\mathfrak{h}'$; and if

$\phi: \mathfrak{h} \to \mathfrak{h}'$ is an isomorphism which induces a bijection of R' onto R, then ϕ can be extended to a Lie algebra isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$.

Example. Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbf{C})$ and let $\mathfrak{h}$ be the Cartan subalgebra consisting of the diagonal matrices with trace 0. Let E_{ij} $(1 \le i,j \le n)$ be the matrix with 1 in the (i,j) place and 0 elsewhere; also let ε_i $(1 \le i \le n)$ be the linear form which maps each diagonal matrix to its ith diagonal element. For each $H \in \mathfrak{h}$ we have

$$[H.E_{ij}] = (\varepsilon_i - \varepsilon_j)(H).E_{ij}$$

so that $\varepsilon_i - \varepsilon_j$ is a root of $(\mathfrak{g},\mathfrak{h})$ whenever $i \ne j$; and since

$$\mathfrak{g} = \mathfrak{h} + \sum_{i \ne j} \mathbf{C}E_{ij}$$

it follows that these are all the roots. The real space V spanned by the roots has dimension $n-1$. For example, the roots $\varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le n-1)$ form a basis of V.

4. Geometry of the root system

For each root $\alpha \in R$ let $w_\alpha : V \to V$ be the reflection in the hyperplane V_α orthogonal to α. Elementary geometry shows that

$$w_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

for $x \in V$, where $\alpha^\vee = 2\alpha/\langle \alpha,\alpha \rangle$ is the *coroot* corresponding to α.

The root system R has the following properties:

(1) $w_\alpha(R) = R$ for each $\alpha \in R$;

(2) $\langle \alpha^\vee, \beta \rangle \in \mathbf{Z}$ for each pair $\alpha, \beta \in R$;

(3) if $\alpha, \beta \in R$ are proportional, then $\beta = \pm\alpha$.

We have no space here for the proofs of these various assertions, which may be found in any text on Lie algebras. Let us however briefly indicate the reason for the integrality property (2). For each pair of roots $\pm\alpha$ one can choose root-vectors $X_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_{\alpha^\vee}$, the image of the coroot $\alpha^\vee$ under the isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ induced by the Killing form. The vector space $\mathfrak{s}_\alpha$ spanned by $X_\alpha, X_{-\alpha}$ and $H_{\alpha^\vee}$ is a Lie subalgebra of $\mathfrak{g}$, and the mapping which takes these three vectors respectively to the matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an isomorphism of $\mathfrak{s}_\alpha$ onto $\mathfrak{sl}(2,\mathbf{C})$. Now a study of the representations of the Lie algebra $\mathfrak{sl}(2,\mathbf{C})$ shows that in any representation ρ the eigenvalues of $\rho\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are *integers*. Since $\langle\alpha^\vee, \beta\rangle = \beta(H_{\alpha^\vee})$ is an eigenvalue of $\mathrm{ad}_{\mathfrak{g}}(H_{\alpha^\vee})$, it follows that $\langle\alpha^\vee, \beta\rangle$ is an integer.

In fact the study of the representations of $\mathfrak{sl}(2,\mathbf{C})$ obtained by restricting $\mathrm{ad}_{\mathfrak{g}}$ to the three-dimensional subalgebras $\mathfrak{s}_\alpha$ is the key to the proofs of the results summarized above.

We can now forget, for the time being, about the Lie algebra $\mathfrak{g}$ and concentrate on the root system R. Abstractly, R can be any finite spanning set of nonzero vectors in a Euclidean space V which satisfies (1),(2),(3) above. The group W generated by the reflections w_α is called the *Weyl group* (of R, or of $\mathfrak{g}$); it acts faithfully as a group of permutations of R, hence is a finite group. Next, the hyperplanes V_α cut up V into congruent open simplicial cones called *chambers*, and a fundamental property of R is that the Weyl group permutes the chambers freely and transitively: that is to say, if we choose a chamber C, then every other chamber is expressible as wC for a unique element $w \in W$. The chamber C is bounded by ℓ (= dim V) hyperplanes $V_{\alpha_i} = V_{-\alpha_i}$ $(1 \le i \le \ell)$. One of each pair of roots $\pm\alpha_i$, say α_i, is such that $\langle\alpha_i, x\rangle > 0$ for all $x \in C$; the resulting set of ℓ roots $\alpha_1, \ldots, \alpha_\ell$ is called a *basis* of R, or a set of *simple* roots. It can also be characterized by the fact that every root α is a linear combination of the simple roots with integer coefficients, either all ≥ 0 or all ≤ 0. The set of bases of R, being in one-one correspondence with the set of chambers, is permuted freely and transitively by the Weyl group W.

Let θ_{ij} be the angle between the simple roots α_i, α_j. Then

$$\cos^2\theta_{ij} = \langle\alpha_i,\alpha_j\rangle^2/\langle\alpha_i,\alpha_i\rangle\langle\alpha_j,\alpha_j\rangle = \frac{1}{4}\langle\alpha_i^\vee,\alpha_j\rangle\langle\alpha_i,\alpha_j^\vee\rangle$$

$= \frac{1}{4} m_{ij}$ say, where $m_{ij} = 0,1,2$ or 3 (since by (2) m_{ij} must be an integer). Since $\theta_{ij} \geq \frac{1}{2}\pi$, the only possible values for the angle θ_{ij} are therefore $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, $\frac{3}{4}\pi$ and $\frac{5}{6}\pi$. If $m_{ij} = 0$, α_i and α_j are orthogonal. If $m_{ij} > 0$, and $\langle\alpha_i,\alpha_i\rangle \geq \langle\alpha_j,\alpha_j\rangle$, then $\langle\alpha_i,\alpha_i\rangle/\langle\alpha_j,\alpha_j\rangle = m_{ij}$.

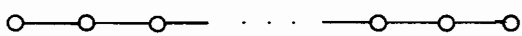
The relative positions of the simple roots may be described by the *Dynkin diagram*; this is a graph whose vertices are in one-one correspondence with the simple roots, the vertices corresponding to α_i and α_j being joined by m_{ij} bonds and (if $m_{ij} > 1$) an arrow-head pointing (like the inequality sign) towards the *shorter* of α_i and α_j.

An equivalent method of describing the relative positions of the simple roots is the *Cartan matrix*, which is the $\ell\times\ell$ matrix of integers whose (i,j) element is $a_{ij} = \langle\alpha_i^\vee,\alpha_j\rangle$. It has 2's down the diagonal, and its off-diagonal elements are ≤ 0.

The Cartan matrix and the Dynkin diagram each determine the other, and either determines R (and hence $\mathfrak{g}$) up to isomorphism

A root system R is said to be *irreducible* if there exists no partition of R into two non-empty subsets R_1, R_2 with each root in R_1 orthogonal to each root in R_2; this is the case if and only if the Lie algebra $\mathfrak{g}$ is *simple*. Since two simple roots are orthogonal if and only if the corresponding vertices of the Dynkin diagram are not directly linked, it is not hard to see that R is irreducible if and only if its Dynkin diagram is *connected*.

Example. If $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ we may take the simple roots to be $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \le i \le n-1)$, in the notation of the Example in §3. The reflection w_α corresponding to the root $\alpha = \varepsilon_i - \varepsilon_j$ interchanges ε_i and ε_j and leaves the remaining ε_k fixed, from which it follows that W is isomorphic to the symmetric group S_n, acting by permuting the ε_i. The Dynkin diagram is a chain



and the Cartan matrix (a_{ij}) has $a_{ij} = 2$ if $i = j$, $a_{ij} = -1$ if $|i-j| = 1$, $a_{ij} = 0$ otherwise.

5. Classification

The classification of the connected Dynkin diagrams is a purely combinatorial undertaking, and leads to the well-known list consisting of the four infinite series A_ℓ $(\ell \ge 1)$, B_ℓ $(\ell \ge 2)$, C_ℓ $(\ell \ge 3)$, D_ℓ $(\ell \ge 4)$ and the five 'exceptional' diagrams E_6, E_7, E_8, F_4, G_2, which will be found in any text on the subject.

Finally, the isomorphism theorem of §3 is complemented by an *existence theorem*, which states that every Dynkin diagram arises from some complex semisimple Lie algebra $\mathfrak{g}$. One constructs $\mathfrak{g}$ by writing down generators and relations, the relations involving only the Cartan integers $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. From all this it follows that the isomorphism classes of complex simple Lie algebras can be labelled by the same symbols $A_\ell, \ldots, G_2$ used above.

Examples. The simple Lie algebra A_ℓ is $\mathfrak{sl}(\ell+1,\mathbb{C})$. The other 'classical' Lie algebras B_ℓ, C_ℓ, D_ℓ may be briefly described as follows. Let E be a complex vector space of finite dimension n, let f be a nondegenerate symmetric or skew-symmetric bilinear form on E, and let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}(E)$ consisting of all $X \in \mathfrak{gl}(E)$ such that $f(Xu,v) + f(u,Xv) = 0$ for all $u,v \in E$. Then $\mathfrak{g} = B_\ell$ if $n = 2\ell+1$ and f is symmetric; $\mathfrak{g} = C_\ell$ if $n = 2\ell$ and f is skew-symmetric; and $\mathfrak{g} = D_\ell$ if $n = 2\ell$ and f is symmetric.

In more concrete terms, B_ℓ is $\mathfrak{so}(2\ell+1,C)$ and D_ℓ is $\mathfrak{s}(2\ell,C)$, where $\mathfrak{so}(n,C) \subset \mathfrak{sl}(n,C)$ consists of the skew-symmetric matrices $(X+X^t = 0)$; and C_ℓ is $\mathfrak{sp}(2n,C) \subset \mathfrak{sl}(2n,C)$, consisting of the matrices X satisfying $XJ + JX^t = 0$, where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

6. Real forms

We shall now take up the structure theory of *real* semisimple Lie algebras. Here the situation is more complicated: it can happen (in fact, as we shall see, it always does) that non-isomorphic real Lie algebras have the same (or isomorphic) complexifications. For example the Lie algebras $\mathfrak{su}(n)$ and $\mathfrak{sl}(n,R)$ both have $\mathfrak{sl}(n,C)$ as their complexification.

If $\mathfrak{g}$ is a complex Lie algebra, a real Lie subalgebra $\mathfrak{g}_o$ of $\mathfrak{g}$ is a *real form* of $\mathfrak{g}$ if $\mathfrak{g}$ is the complexification of $\mathfrak{g}_o$, i.e. if $\mathfrak{g} = \mathfrak{g}_o + i\mathfrak{g}_o$ (direct sum). Such a real form $\mathfrak{g}_o$ determines a mapping $c\colon \mathfrak{g} \to \mathfrak{g}$, namely $Y+iZ \to Y-iZ$ $(Y,Z \in \mathfrak{g}_o)$. This mapping c has the following properties:

(1) c is *semilinear*, i.e. $c(\lambda X+\mu Y) = \bar{\lambda}c(X) + \bar{\mu}c(Y)$ for $X,Y \in \mathfrak{g}$ and $\lambda,\mu \in C$;

(2) c is an *involution*, i.e. $c^2 = 1_\mathfrak{g}$;

(3) $c[X,Y] = [cX,cY]$ for $X,Y \in \mathfrak{g}$.

A bijection $c\colon \mathfrak{g} \to \mathfrak{g}$ with these properties is called a *conjugation* of $\mathfrak{g}$. Conversely, any conjugation c of $\mathfrak{g}$ determines uniquely a real subalgebra $\mathfrak{g}_o = \{X \in \mathfrak{g} : cX = X\}$ such that $\mathfrak{g} = \mathfrak{g}_o + i\mathfrak{g}_o$, that is to say a *real form* of $\mathfrak{g}$. Hence we have a canonical one-one correspondence between conjugations of $\mathfrak{g}$ and real forms of $\mathfrak{g}$.

Now if $\mathfrak{g}_o$ is the Lie algebra of a *compact* Lie group G, the Killing form of $\mathfrak{g}_o$ is negative semi-definite (and negative definite if the centre of $\mathfrak{g}_o$ is zero). For G acts on $\mathfrak{g}$ via the adjoint representation Ad_G; since G is compact, there exists an Ad-invariant positive definite quadratic form Q on $\mathfrak{g}_o$ (take an arbitrary positive definite form, and average it over G). With respect to a Q-orthonormal basis of $\mathfrak{g}_o$, the linear transformation $Ad_G(x)$ for each $x \in G$ is represented by an *orthogonal* matrix, i.e. we have $Ad_G: G \to O(n)$, where $n = \dim \mathfrak{g}_o$. Hence $ad_{\mathfrak{g}_o}: \mathfrak{g}_o \to \mathfrak{so}(n)$ and therefore each $ad\ X$ is represented by a skew-symmetric matrix (Chapter I, §5, Examples). Consequently

$$B(X,X) = \operatorname{trace}(ad\ X)^2 = \sum_{i,j} (ad\ X)_{ij}(ad\ X)_{ji}$$

$$= -\sum_{i,j} (ad\ X)_{ij}^2 \leq 0,$$

and $B(X,X) = 0$ if and only if $ad\ X = 0$, i.e. if and only if X is in the centre of $\mathfrak{g}_o$. For this reason a semisimple real Lie algebra is said to be *compact* if its Killing form is negative definite.

Every complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form, which is unique up to isomorphism. It may be constructed as follows: with the notation of §3, vectors $X_\alpha \in \mathfrak{g}^\alpha$ can be chosen for each root α such that for each pair of roots α,β we have

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \in R, \\ H_\alpha & \text{if } \alpha+\beta = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the constants $N_{\alpha,\beta}$ are *real* and satisfy $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$. From these relations it follows that $B(X_\alpha, X_\beta) = 1$ or 0 according as $\alpha+\beta = 0$ or $\neq 0$. Then the elements iH_α , $X_\alpha - X_{-\alpha}$, $i(X_\alpha + X_{-\alpha})$ span a compact real form of $\mathfrak{g}$.

Another real form of $\mathfrak{g}$ is easily written down, namely the real Lie algebra spanned by the H_α and the X_α . This form is called the *split* (or normal, or anticompact) real form of $\mathfrak{g}$ it is not compact. In a sense to be explained later, these two (the compact and split forms) are at opposite extremes, and in general there will be other real forms as well.

7. Examples: real forms of the classical complex Lie algebras

If $\mathfrak{g} = \mathfrak{sl}(n,C)$, a compact real form of $\mathfrak{g}$ is $\mathfrak{su}(n)$ and a split form is $\mathfrak{sl}(n,R)$. The corresponding conjugations of $\mathfrak{g}$ are respectively $X \to -\bar{X}^t$ and $X \to \bar{X}$.

If n is even, say $n = 2m$, another real form of $\mathfrak{sl}(2m,C)$ is $\mathfrak{sl}(m,H) = \{X \in \mathfrak{gl}(n,H): \mathrm{Re}(\mathrm{trace}\ X) = 0\}$. Any quaternionic matrix may be written as $Y + Zj$, where Y and Z are complex matrices, and we can embed $\mathfrak{sl}(m,H)$ in $\mathfrak{sl}(2m,C)$ by means of the map

$$\eta: Y+Zj \to \begin{pmatrix} Y & Z \\ -\bar{Z} & \bar{Y} \end{pmatrix} ;$$

the image of $\mathfrak{sl}(m,H)$ under η is denoted by $\mathfrak{su}^*(2m)$, and is a real form of $\mathfrak{sl}(2m,C)$. The corresponding conjugation is $X \to J\bar{X}J^{-1}$, where $J = \eta(j)$.

Apart from $\mathfrak{sl}(n,R)$ and $\mathfrak{sl}(m,H)$, the real forms of the classical complex simple Lie algebras $A_\ell, B_\ell, C_\ell, D_\ell$ may all be described uniformly as follows. Let K be any one of R, C, H, and let E be a K-vector space of dimension n (a left vector space if $K = H$). Let $f: E\times E \to K$ be a nondegenerate ε-Hermitian form, where $\varepsilon = \pm 1$ (so that f is K-linear in the first variable, and $f(u,v) = \varepsilon\overline{f(v,u)}$), and let $\mathfrak{g}(E,f) \subset \mathfrak{gl}(E)$ be the subalgebra consisting of all $X \in \mathfrak{gl}(E)$ such that trace $X = 0$ and

$$f(Xu,v) + f(u,Xv) = 0$$

for all $u,v \in E$. Let q denote the Witt index of f (namely the dimension of a maximal totally isotropic subspace of E) and let $p = n-q$, so that $p \geq q \geq 0$ and $p+q = n$. The integers p,q determine f up to isomorphism. The Lie algebras $\mathfrak{g}(E,f)$, for all legitimate choices of K, ε, p and q, together with $\mathfrak{sl}(n,R)$ and $\mathfrak{sl}(n,H)$, exhaust the real forms of the classical complex simple Lie algebras.

(a) Suppose first that $K = R$ and $\varepsilon = +1$. Then f is symmetric, and the algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{so}(p,q)$. It consists of the matrices $X \in \mathfrak{sl}(n,R)$ such that

$$I_{p,q}X + X^tI_{p,q} = 0$$

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, and is a real form of $\mathfrak{so}(n,C)$. [As it stands, $\mathfrak{so}(p,q)$ is not a subalgebra of $\mathfrak{so}(n,C)$, but the isomorphic algebra $J_{p,q}\mathfrak{so}(p,q)J_{p,q}^{-1}$ is, where $J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & iI_q \end{pmatrix}$; the corresponding conjugation of $\mathfrak{so}(n,C)$ is $X \to I_{p,q}\bar{X}I_{p,q}$.] If $q = 0$, then $\mathfrak{so}(p,q) = \mathfrak{so}(n,0) = \mathfrak{so}(n)$ is the compact real form of $\mathfrak{so}(n,C)$. At the other extreme, $q = [\frac{1}{2}n]$ gives the split form.

(b) Next suppose that $K = R$, $\varepsilon = -1$. Then f is skew-symmetric and hence (because f is nondegenerate) n is even, say $n = 2m$, and the index q is equal to m. The algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{sp}(2m,R)$: it consists of the matrices $X \in \mathfrak{sl}(2m,R)$ satisfying $XJ + JX^t = 0$. Hence it is the split real form of $\mathfrak{sp}(2m,C)$, the conjugation being $X \to \bar{X}$.

(c) Now let $K = C$. Here we do not need to distinguish between $\varepsilon = +1$ and $\varepsilon = -1$, because if f is antihermitian then if is hermitian. We may therefore assume that f is *hermitian.* The algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{su}(p,q)$: it consists of the matrices $X \in \mathfrak{sl}(n,C)$ such that

$$(*) \quad I_{p,q}X + \bar{X}^t I_{p,q} = 0$$

and is a real form of $\mathfrak{sl}(n,C)$, the conjugation being $X \to -I_{p,q}\bar{X}^t I_{p,q}$. When $q = 0$, we have $\mathfrak{su}(p,q) = \mathfrak{su}(n,0) = \mathfrak{su}(n)$ the compact real form of $\mathfrak{sl}(n,C)$.

(d) Let $K = H$, $\varepsilon = +1$. Then f is a quaternionic Hermitian form, and the Lie algebra $\mathfrak{g}(E,f)$ is denoted by $\mathfrak{sp}(p,q)$. It consists of the matrices $X \in \mathfrak{gl}(n,H)$ satisfying (*), and under the embedding η of $\mathfrak{gl}(n,H)$ in $\mathfrak{gl}(2n,C)$ it is a real form of $\mathfrak{sp}(2n,C)$. When $q = 0$, we have $\mathfrak{sp}(p,q) = \mathfrak{sp}(n,0) = \mathfrak{sp}(n) = \mathfrak{u}(n,H)$, the Lie algebra of the compact group $Sp(n) = U(n,H)$, which is therefore the compact real form of $\mathfrak{sp}(2n,C)$.

(e) Finally, let $K = H$ and $\varepsilon = -1$. Then f is quaternionic antihermitian, which since f is nondegenerate implies that the index q is $[\frac{1}{2}n]$. The corresponding Lie algebra $\mathfrak{g}(E,f)$ may be taken to consist of the matrices $X \in \mathfrak{sl}(n,H)$ such that $Xj + j\bar{X}^t = 0$; it is denoted by $\mathfrak{sau}(n,H)$. Its image in $\mathfrak{gl}(2n,C)$ under η is a subalgebra $\mathfrak{so}^*(2n)$ of $\mathfrak{so}(2n,C)$, consisting of the $X \in \mathfrak{so}(2n,C)$ such that $XJ + J\bar{X}^t = 0$, and is a real form of $\mathfrak{so}(2n,C)$, the conjugation being $X \to J\bar{X}^t J$.

To summarize, the real forms of A_ℓ $(\ell \geq 1)$ are

$$A_\ell^R = \mathfrak{sl}(\ell+1,R)$$

$$A_\ell^{C,q} = \mathfrak{su}(p,q)$$

(where $p \geq q \geq 0$ and $p+q = \ell+1$, so that $0 \leq q \leq [\frac{1}{2}(\ell+1)]$)

$A_\ell^H = \mathfrak{sl}(m+1,H) \quad (\cong \mathfrak{su}^*(2m))$

(if $\ell = 2m-1$ is odd).

The split form is A_ℓ^R and the compact form is $A_\ell^{C,0} = \mathfrak{su}(\ell+1)$.

The real forms of B_ℓ ($\ell \geq 2$) are

$B_\ell^{R,q} = \mathfrak{so}(p,q)$

(where $p \geq q \geq 0$ and $p+q = 2\ell+1$, so that $0 \leq q \leq \ell$).

The split form is $\mathfrak{so}(\ell+1,\ell)$ $(q=\ell)$ and the compact form is $\mathfrak{so}(2\ell+1)$ $(q=0)$.

The real forms of C_ℓ ($\ell \geq 3$) are

$C_\ell^R = \mathfrak{sp}(2\ell,R)$

$C_\ell^{H,q} = \mathfrak{sp}(p,q)$

(where $p \geq q \geq 0$ and $p+q = \ell$, so that $0 \leq q \leq [\frac{1}{2}\ell]$).

The split form is C_ℓ^R and the compact form is $C_\ell^{H,0} = \mathfrak{sp}(\ell) = \mathfrak{u}(\ell,H)$.

The real forms of D_ℓ ($\ell \geq 4$) are

$D_\ell^{R,q} = \mathfrak{so}(p,q)$

(where $p \geq q \geq 0$ and $p+q = 2\ell$, so that $0 \leq q \leq \ell$)

$D_\ell^H = \mathfrak{s}\alpha\mathfrak{u}(\ell,H) \quad (\cong \mathfrak{so}^*(2\ell))$

The split form is $\mathfrak{so}(\ell,\ell)$ and the compact form is $\mathfrak{so}(2\ell)$. (When $\ell = 4$, we have $D_4^H = D_4^{R,2}$.)

8. The Cartan decomposition

Let us return to the general theory. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{u}$ a compact real form of $\mathfrak{g}$, and

c_u the conjugation (§6) of $\mathfrak{g}$ defined by $\mathfrak{u}$. If c is any conjugation of $\mathfrak{g}$, there exists an automorphism ϕ of $\mathfrak{g}$ such that c_u commutes with $\phi c \phi^{-1}$; hence to find all real forms of $\mathfrak{g}$, up to isomorphism, it is enough to find all conjugations c of $\mathfrak{g}$ which commute with c_u.

If c commutes with c_u, we have $c(\mathfrak{u}) = \mathfrak{u}$ and $c(i\mathfrak{u}) = i\mathfrak{u}$. Let $\mathfrak{k}$ and $i\mathfrak{p}$ be respectively the $+1$ and -1 eigenspaces of c in $\mathfrak{u}$, so that

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$$

(direct sum), and the $+1$ and -1 eigenspaces of c on $i\mathfrak{u}$ are $\mathfrak{p}$ and $i\mathfrak{k}$, so that

$$i\mathfrak{u} = \mathfrak{p} + i\mathfrak{k}$$

Hence if $\mathfrak{g}_o$ is the real form of $\mathfrak{g}$ determined by c, we have

$$(*) \qquad \mathfrak{g}_o = \mathfrak{k} + \mathfrak{p} ;$$

$\mathfrak{k} = \mathfrak{g}_o \cap \mathfrak{u}$ is a subalgebra of $\mathfrak{g}_o$ and $\mathfrak{p} = \mathfrak{g}_o \cap i\mathfrak{u}$ is a vector subspace (*not* a subalgebra) such that

(i) the Killing form $B_{\mathfrak{g}_o}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$;

(ii) the map $cc_u = c_u c = \theta : Y+Z \to Y-Z$ $(Y \in \mathfrak{k},\ Z \in \mathfrak{p})$ is an automorphism of $\mathfrak{g}_o$.

A direct decomposition (*) of $\mathfrak{g}_o$, constructed as above from a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ such that c_u commutes with c, is called a *Cartan decomposition* of the real Lie algebra $\mathfrak{g}_o$, and θ is a *Cartan involution* of $\mathfrak{g}_o$. The Cartan decomposition is determined by the involution θ, since $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ in $\mathfrak{g}_o$. We have $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, and $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to $B_{\mathfrak{g}_o}$.

If θ is *any* involutory automorphism of $\mathfrak{g}_o$, the bilinear form $\langle X,Y\rangle_\theta = -B_{\mathfrak{g}_o}(X,\theta Y)$ is symmetric, and θ is a Cartan involution if and only if $\langle X,X\rangle_\theta$ is positive definite.

The importance of the Cartan decomposition is that it is unique up to conjugacy: if $\mathfrak{g}_o = \mathfrak{k}' + \mathfrak{p}'$ is another Cartan decomposition, there exists $\phi \in \mathrm{Int}(\mathfrak{g}_o)$ such that $\mathfrak{k}' = \phi(\mathfrak{k})$ and $\mathfrak{p}' = \phi(\mathfrak{p})$.

Define the *Cartan signature* (some say *Cartan index*) s of $\mathfrak{g}_o$ to be the signature of the real quadratic form $B_{\mathfrak{g}_o}$, i.e.

$$s = \dim \mathfrak{p} - \dim \mathfrak{k} \ .$$

Then we have

$$-\dim \mathfrak{g} \le s \le \mathrm{rank}\ \mathfrak{g}$$

and $s = -\dim \mathfrak{g} \iff \mathfrak{g}_o$ is compact,

$s = \mathrm{rank}\ \mathfrak{g} \iff \mathfrak{g}_o$ is split.

Examples

1. For the compact real form we have $\mathfrak{p} = 0$, $\mathfrak{k} = \mathfrak{g}_o$, and θ is the identity map. For the split real form of $\mathfrak{g}$, spanned by the H_α and the X_α (§6), $\mathfrak{k}$ is spanned by the $X_\alpha - X_{-\alpha}$ and $\mathfrak{p}$ is spanned by the H_α and the $X_\alpha + X_{-\alpha}$.
2. For the real forms of the classical complex Lie algebras listed in §7, in each case $\theta : X \to -\bar{X}^t$ ($= -X^t$ if X is real) is a Cartan involution. Hence $\mathfrak{k}$ consists of the skew Hermitian matrices in $\mathfrak{g}$, and $\mathfrak{p}$ consists of the Hermitian matrices.

9. The Iwasawa decomposition

From now on the emphasis will be on a fixed real semisimple Lie algebra, which we shall denote by $\mathfrak{g}$ (rather than $\mathfrak{g}_o$);

the complexification of $\mathfrak{g}$, which is a complex semisimple Lie algebra, will be denoted by $\mathfrak{g}_c$ (instead of $\mathfrak{g}$ as heretofore). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a Cartan decomposition of $\mathfrak{g}$, and let θ be the associated Cartan involution, so that θ is the identity on $\mathfrak{k}$, and minus the identity on $\mathfrak{p}$. The bilinear form on $\mathfrak{g}$

$$\langle X,Y\rangle_\theta = -B_{\mathfrak{g}}(X,\theta Y)$$

is symmetric and positive definite (it coincides with $B_{\mathfrak{g}}$ on $\mathfrak{p}$ and with $-B_{\mathfrak{g}}$ on $\mathfrak{k}$), hence endows $\mathfrak{g}$ with the structure of a finite-dimensional real Hilbert space. For any $X \in \mathfrak{g}$, the adjoint of $\operatorname{ad} X$ (with respect to this scalar product) is $-\operatorname{ad} \theta(X)$. Hence, with respect to an orthonormal basis of $\mathfrak{g}$, $\operatorname{ad} X$ is represented by a symmetric matrix if $X \in \mathfrak{p}$, and by a skew-symmetric matrix if $X \in \mathfrak{k}$. It follows that the elements of $\mathfrak{k}$ and $\mathfrak{p}$ are semisimple.

Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subalgebra of the vector space $\mathfrak{p}$, and let $\mathfrak{a}_\mathfrak{p}^*$ be the vector space dual to $\mathfrak{a}_\mathfrak{p}$. For each $\lambda \in \mathfrak{a}_\mathfrak{p}^*$ let

$$\mathfrak{g}^\lambda = \{X \in \mathfrak{g} : [H,X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}_\mathfrak{p}\} .$$

Since $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{a}_\mathfrak{p})$ is a commuting family of self-adjoint linear transformations of $\mathfrak{g}$, it follows that $\mathfrak{g}$ is the orthogonal direct sum of the subspaces $\mathfrak{g}^\lambda$. If $\lambda \neq 0$ and $\mathfrak{g}^\lambda \neq 0$, λ is said to be a *root* of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$, and $\mathfrak{g}^\lambda$ is the *root-space* corresponding to λ. Let $S = S(\mathfrak{g},\mathfrak{a}_\mathfrak{p})$ denote the set of roots. Then we have a *root space decomposition*

$$\mathfrak{g} = \mathfrak{l} + \sum_{\lambda\in S} \mathfrak{g}^\lambda$$

(orthogonal direct sum), in which $\mathfrak{l}$ is the centralizer of $\mathfrak{a}_p$ in $\mathfrak{g}$. The set $S \subset \mathfrak{a}_p^*$ is called the *relative root system* of $\mathfrak{g}$ (with respect to $\mathfrak{a}_p$); up to isomorphism, it depends only on $\mathfrak{g}$. However, there are divergences from the complex case considered in §3. First, S is a root system in $\mathfrak{a}_p^*$, in the sense of §4, but need not be reduced (i.e. need not satisfy condition (3) of §4: this means that it may happen that $\lambda \in S$ and $2\lambda \in S$). Secondly, the dimension m_λ of $\mathfrak{g}^\lambda$ (the *multiplicity* of $\lambda \in S$) may be bigger than 1. Thirdly, $\mathfrak{l}$ is usually bigger than $\mathfrak{a}_p$: in fact we have

$$\mathfrak{l} = \mathfrak{m} + \mathfrak{a}_p$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}_p$ in $\mathfrak{k}$. Finally, the root system S does not of itself determine $\mathfrak{g}$ up to isomorphism; for this purpose we require a more elaborate combinatorial object, which we shall describe in the next section.

Choose a basis of S, and let S^+ be the set of positive roots relative to this basis; and write

$$\mathfrak{n} = \sum_{\lambda \in S^+} \mathfrak{g}^\lambda, \qquad \bar{\mathfrak{n}} = \sum_{\lambda \in S^+} \mathfrak{g}^{-\lambda}.$$

We have $\theta(\mathfrak{g}^\lambda) = \mathfrak{g}^{-\lambda}$ for all $\lambda \in S$ (because θ acts as -1 on $\mathfrak{a}_p$) and therefore $\theta\mathfrak{n} = \bar{\mathfrak{n}}$. $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are nilpotent subalgebras of $\mathfrak{g}$, and we have

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_p + \mathfrak{n}$$

(direct sum); this is the *Iwasawa decomposition* of $\mathfrak{g}$. If $\mathfrak{s} = \mathfrak{a}_p + \mathfrak{n}$, then $\mathfrak{s}$ is a solvable subalgebra of $\mathfrak{g}$.

In Chapter III we shall see that the Iwasawa decomposition has a global counterpart, for any connected semisimple Lie group.

10. The relative root system

We retain the notation of §9. To get more insight into the relative root system S, we shall compare it with the (absolute) root system R of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$, relative to a suitably chosen Cartan subalgebra. For this purpose let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}_\mathfrak{p}$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$; we have $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$, and if we put $\mathfrak{a}_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}$ then

$$\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$$

(direct sum), so that $\mathfrak{a}$ is stable under θ. The dimension of $\mathfrak{a}_\mathfrak{p}$ is called the *relative rank* (or *split rank*) of $\mathfrak{g}$.

The complexification $\mathfrak{h} = \mathfrak{a}_c$ of $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_c$. Hence (§3) we have a root-space decomposition of $\mathfrak{g}_c$ relative to $\mathfrak{h}$:

$$\mathfrak{g}_c = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_c^\alpha$$

where $R = R(\mathfrak{g}_c, \mathfrak{h}) \subset \mathfrak{h}_R^*$ is the root system of $\mathfrak{g}_c$ with respect to $\mathfrak{h}$. Here $\mathfrak{h}_R$ is the real vector space spanned by the H_α, $\alpha \in R$, as in §3, and in fact

$$\mathfrak{h}_R = \mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k} .$$

Moreover, $i\mathfrak{h}_R = \mathfrak{a}_\mathfrak{k} + i\,\mathfrak{a}_\mathfrak{p}$ is a Cartan subalgebra of the compact form $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ of $\mathfrak{g}_c$.

Let $\rho : \mathfrak{h}_R^* \to \mathfrak{a}_\mathfrak{p}^*$ denote restriction to $\mathfrak{a}_\mathfrak{p}$. The kernel of ρ may be identified with $(i\mathfrak{a}_\mathfrak{k})^*$. For each $\alpha \in R$, the restriction $\rho(\alpha)$ of α to $\mathfrak{a}_\mathfrak{p}$ is either 0 or is an element of S. Let R_o be the set of roots $\alpha \in R$ which vanish on $\mathfrak{a}_\mathfrak{p}$, or equivalently such that $H_\alpha \in i\,\mathfrak{a}_\mathfrak{k}$. Then R_o is a root system in $(i\mathfrak{a}_\mathfrak{k})^*$ (except that it may span a proper subspace of $(i\mathfrak{a}_\mathfrak{k})^*$), and is the root system of the complex reductive

Lie algebra $\mathfrak{m}_c$ (where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$) relative to its Cartan subalgebra $(\mathfrak{a}_\mathfrak{k})_c$: we have another root-space decomposition

$$\mathfrak{m}_c = (\mathfrak{a}_\mathfrak{k})_c + \sum_{\alpha \in R_o} \mathfrak{g}_c^\alpha .$$

The projection ρ maps $R-R_o$ onto S, and for each $\lambda \in S$ the multiplicity m_λ is equal to the number of roots $\alpha \in R-R_o$ such that $\rho(\alpha) = \lambda$.

Let c be the conjugation of $\mathfrak{g}_c$ defined by $\mathfrak{g}$, so that $c(X+iY) = X-iY$ for $X,Y \in \mathfrak{g}$. c acts on the root spaces as follows: for each root $\alpha \in R$ define α^σ by

$$\alpha^\sigma(H) = \overline{\alpha(c(H))} \qquad (H \in \mathfrak{h}).$$

Then $c(\mathfrak{g}_c^\alpha) = \mathfrak{g}_c^{\alpha^\sigma}$. The mapping $\alpha \to \alpha^\sigma$ extends by linearity to an involutory isometry of the Euclidean space $\mathfrak{h}_R^*$, under which $R-R_o$ is stable, and R_o is the set of roots $\alpha \in R$ such that $\alpha^\sigma+\alpha = 0$. We have $\alpha^\sigma-\alpha \notin R$ for all $\alpha \in R$.

Abstractly, therefore, we are led to consider pairs (R,σ), where R is a reduced root system in a Euclidean space V, and σ is an involutory isometry of V such that $\sigma(R) = R$. The pair (R,σ) is said to be *normal* if $\alpha \in R \Rightarrow \alpha^\sigma-\alpha \notin R$.

Let $\rho = \frac{1}{2}(1+\sigma)$, so that ρ is the orthogonal projection of V on $V_1 = V^\sigma$ with kernel $V_o = V^{-\sigma}$. Let $R_o = R \cap V_o$ and $S = \rho(R-R_o)$. Then R_o is a reduced root system in V_o (but may span a proper subspace of V_o) and S is a (not necessarily reduced) root system in V_1. We can choose a basis Γ of R such that $\Gamma_o = \Gamma \cap R_o$ is a basis of R_o, and such that $R^+-R_o^+$ is σ-stable, where R^+ and R_o^+ are the sets of positive roots determined by Γ and Γ_o respectively. The involution σ determines an involutory permutation of $\Gamma-\Gamma_o$ as follows: if $\alpha \in \Gamma-\Gamma_o$, there exists a unique $\beta \in \Gamma-\Gamma_o$ such that $\alpha^\sigma \equiv \beta \pmod{Z\Gamma_o}$, and the mapping $\alpha \to \beta$ is a permutation of order 2 . We have $\rho(\alpha) = \rho(\beta) = \frac{1}{2}(\alpha+\beta)$, and $\Delta = \rho(\Gamma-\Gamma_o)$ is a basis of the root system S . Finally, if $W(R)$, $W(R_o)$ and $W(S)$ are the Weyl groups of the root systems R , R_o , S , and if $W(R)^\sigma$ is the centralizer of σ in $W(R)$, then restriction to V_1 defines a homomorphism of $W(R)^\sigma$ onto $W(S)$, the kernel being $W(R_o)$.

Each real semisimple Lie algebra $\mathfrak{g}$ therefore determines a normal pair (R,σ) , which determines $\mathfrak{g}$ up to isomorphism. As described in §4, the reduced root system R may be represented by its Dynkin diagram, the vertices of which represent the elements of the basis Γ of R . The action of σ may be indicated as follows: the vertices of the diagram which represent the elements of Γ_o are coloured black, the remainder white, and two white vertices representing elements $\alpha,\beta \in \Gamma-\Gamma_o$ as above, such that $\rho(\alpha) = \rho(\beta)$, are joined by an arrow $\curvearrowleft\!\curvearrowright$. The resulting diagram is called the *Satake diagram* of $\mathfrak{g}$, and determines $\mathfrak{g}$ up to isomorphism.

Examples

1. If $\mathfrak{g}$ is compact we have $\mathfrak{g} = \mathfrak{k}$, $\mathfrak{p} = 0$, $\mathfrak{a}_\mathfrak{p} = 0$, so that $R_o = R$ and $S = \emptyset$. In this case all the vertices of the Satake diagram of $\mathfrak{g}$ are *black*.
2. At the other extreme, if $\mathfrak{g}$ is split, we may take $\mathfrak{p}$ to be the vector subspace spanned by the H_α and the $X_\alpha+X_{-\alpha}$ (§8, Ex.1), and $\mathfrak{a}_\mathfrak{p}$ to be spanned by the H_α ; thus $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}$ and $\mathfrak{a}_\mathfrak{k} = 0$, so that $R_o = \emptyset$ and σ is the identity. In this case all the vertices of the Satake diagram of $\mathfrak{g}$ are *white*, and it coincides with the Dynkin diagram of R (or $\mathfrak{g}_c$).
3. Let $\mathfrak{g}_1$ be a complex semisimple Lie algebra, $\mathfrak{g} = \mathfrak{g}_1^R$ its realification (§1). Multiplication by i is an endomorphism of $\mathfrak{g}$ satisfying $i^2 = -1$ and $[X,iY] = [iX,Y] = i[X,Y]$ for all $X,Y \in \mathfrak{g}$. Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{k}+i\mathfrak{k}$ is a Cartan decomposition of $\mathfrak{g}$. We may take $\mathfrak{a}_\mathfrak{p} = i\mathfrak{t}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$; then $\mathfrak{a} = \mathfrak{t}+i\mathfrak{t}$

is a Cartan subalgebra of $\mathfrak{g}$, and is the realification of a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}_1$. The root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}_\mathfrak{p}$ is then the same as the root space decomposition of $\mathfrak{g}_1$ with respect to $\mathfrak{h}_1$. Hence if we denote by R_1 the set of roots of $(\mathfrak{g}_1,\mathfrak{h}_1)$, we have $R = R_1 \times R_1$, $R_o = \emptyset$ and $S = R_1$. The Satake diagram of $\mathfrak{g}$ therefore consists of two copies of the Dynkin diagram of R_1 , corresponding vertices in the two copies being joined by arrows.

4. For a concrete example not covered by Exx.1-3, consider $\mathfrak{g} = \mathfrak{sl}(n,H)$ (§7). Then (§8, Ex.2) we may take $\mathfrak{k} = \mathfrak{sp}(n)$ $(= \mathfrak{u}(n,H))$ and $\mathfrak{p}$ to consist of the quaternionic Hermitian matrices with trace 0 . We may then take $\mathfrak{a}_\mathfrak{p}$ to consist of the real diagonal matrices with trace 0 , and $\mathfrak{a}$ to consist of the complex diagonal matrices X with $\mathrm{Re}(\mathrm{trace}\ X) = 0$. If we embed $\mathfrak{sl}(n,H)$ in $\mathfrak{sl}(2n,C)$ as in §7, then $\mathfrak{a}_\mathfrak{c} = \mathfrak{h}$ consists of the diagonal matrices in $\mathfrak{sl}(2n,C)$, and we have $R = \{\varepsilon_i-\varepsilon_j : i \neq j,\ 1 \leq i,j \leq 2n\}$ in the notation of §3, Example. Here $R_o = \{\varepsilon_i-\varepsilon_j : |i-j| = n\}$; hence if we put $\varepsilon_i' = \varepsilon_{i\pm n}$, we may take

$$\Gamma = \{\varepsilon_1'-\varepsilon_1, \varepsilon_1-\varepsilon_2', \varepsilon_2'-\varepsilon_2, \ldots, \varepsilon_{n-1}-\varepsilon_n', \varepsilon_n'-\varepsilon_n\} ,$$

$$\Gamma_o = \{\varepsilon_1'-\varepsilon_1, \varepsilon_2'-\varepsilon_2, \ldots, \varepsilon_n'-\varepsilon_n\}$$

and the Satake diagram is

there being n black vertices and $n-1$ white ones. The relative root system is non-reduced, of type BC_{n-1} .

5. For another example, take $\mathfrak{g} = \mathfrak{su}(p,q)$, consisting of all complex matrices of the form

$$\begin{pmatrix} A & B \\ \bar{B}^t & C \end{pmatrix}$$

where $A \in \mathfrak{u}(p)$, $B \in \mathfrak{u}(q)$, trace A + trace $C = 0$, and B is any $p \times q$ complex matrix. As in §7, we shall assume $p \geq q$.

Let $H(X,Y,Z)$ denote the matrix

$$\begin{pmatrix} X & 0 & Y \\ 0 & Z & 0 \\ Y & 0 & X \end{pmatrix}$$

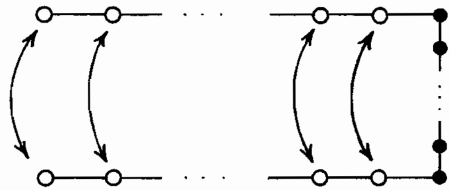
where X,Y,Z are diagonal matrices of sizes $q, p-q, q$ respectively. We may take $\mathfrak{a}_\mathfrak{p}$ to consist of all $H(0,Y,0)$ with Y a real diagonal matrix; $\mathfrak{a}_\mathfrak{t}$ to consist of all $H(iX,0,iZ)$ with X,Z diagonal matrices, and trace $H = 0$. Then $\mathfrak{h} = \mathfrak{a}_c$ consists of all $H(X,Y,Z)$ with X,Y,Z complex diagonal matrices (and trace $H = 0$). Conjugation by the matrix $\begin{pmatrix} \alpha & 0 & \alpha \\ 0 & I & 0 \\ -\alpha & 0 & \alpha \end{pmatrix}$, where $\alpha = \frac{1}{\sqrt{2}} I_q$, transforms $\mathfrak{h}$ into the standard Cartan subalgebra of $\mathfrak{g}_c = \mathfrak{sl}(p+q,C)$. Hence if ξ_j (resp. η_j, resp. ζ_j) is the linear form on $\mathfrak{h}$ whose value at $H(X,Y,Z)$ is the jth diagonal element of X (resp. Y, resp. Z), then the roots of $(\mathfrak{g}_c, \mathfrak{h})$ are the differences between pairs of

$$\xi_1+\eta_1, \ldots, \xi_q+\eta_q, \zeta_1, \ldots, \zeta_{p-q}, \xi_q-\eta_q, \ldots, \xi_1-\eta_1 .$$

We may take the basis Γ to consist of the differences of consecutive pairs of these $p+q$ elements of $\mathfrak{h}^*$. On restriction to $\mathfrak{a}_\mathfrak{p}$, all ξ_j and ζ_k vanish; if $\tilde{\eta}_j$ is the restriction of η_j to $\mathfrak{a}_\mathfrak{p}$, the restrictions to $\mathfrak{a}_\mathfrak{p}$ of the elements of Γ are therefore

$$\tilde{\eta}_1-\tilde{\eta}_2, \ldots, \tilde{\eta}_{q-1}-\tilde{\eta}_q, \tilde{\eta}_q, 0, \ldots, 0, \tilde{\eta}_q, \tilde{\eta}_{q-1}-\tilde{\eta}_q, \ldots, \tilde{\eta}_1-\tilde{\eta}_2 .$$

It follows that the Satake diagram is



there being q pairs of white vertices and $p-q-1$ black vertices. The root system S is of type BC_q .

11. Cartan subalgebras again

Let $\mathfrak{g}$ be a real semisimple Lie algebra. By contrast with the complex case, as we have already observed, it is no longer true in general that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under the adjoint group $\mathrm{Int}(\mathfrak{g})$ (unless $\mathfrak{g}$ is compact); instead, they form a finite number of conjugacy classes. They all have the same dimension (namely rank $\mathfrak{g}$), because their complexifications are Cartan subalgebras of $\mathfrak{g}_c$.

Example. Let $\mathfrak{g} = \mathfrak{sl}(2,R)$ and let $\mathfrak{a}$ (resp. $\mathfrak{b}$) be the subspace of $\mathfrak{g}$ spanned by $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ (resp. $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$). Then $\mathfrak{a}$ and $\mathfrak{b}$ are Cartan subalgebras of $\mathfrak{g}$. They cannot be conjugate in $\mathfrak{g}$, because the subgroup $\exp(\mathfrak{a})$ of $SL(2,R)$ consists of all matrices $\begin{pmatrix}e^{\lambda} & 0\\ 0 & e^{-\lambda}\end{pmatrix}$, hence is isomorphic to the additive group R , and in particular is not compact; whereas $\exp(\mathfrak{b})$ consists of all matrices $\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$, hence is $SO(2)$ and therefore compact.

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, and let θ be the associated Cartan involution. As in §10, let $\mathfrak{a} = \mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a} \cap \mathfrak{p}$ is a maximal abelian subalgebra of the vector space $\mathfrak{p}$.

If now $\mathfrak{b}$ is any Cartan subalgebra of $\mathfrak{g}$, there exists a conjugate of $\mathfrak{b}$ which is θ-stable, i.e. such that

$$\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} + \mathfrak{b}_{\mathfrak{p}}$$

where $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{b} \cap \mathfrak{k}$ and $\mathfrak{b}_{\mathfrak{p}} = \mathfrak{b} \cap \mathfrak{p}$. The component $\mathfrak{b}_{\mathfrak{p}}$ is called the *toral part* (or the compact part), and $\mathfrak{b}_{\mathfrak{p}}$ the *vector part* of $\mathfrak{b}$. The vector part $\mathfrak{b}_{\mathfrak{p}}$ is an abelian subalgebra of $\mathfrak{p}$, hence is contained in a maximal abelian subalgebra of $\mathfrak{p}$. By conjugating $\mathfrak{b}$, we may arrange that $\mathfrak{b}_{\mathfrak{p}} \subset \mathfrak{a}_{\mathfrak{p}}$, and then by conjugating again, leaving $\mathfrak{b}_{\mathfrak{p}}$ fixed, we can also arrange that $\mathfrak{b}_{\mathfrak{k}} \supset \mathfrak{a}_{\mathfrak{k}}$. The Cartan subalgebra $\mathfrak{b}$ is said to be *standard* (relative to θ and $\mathfrak{a}$) if these conditions are satisfied.

The classification of Cartan subalgebras up to conjugacy in $\mathfrak{g}$ can now be reduced to a combinatorial problem, as follows. A subset E of the root system R (§10) is said to be *strongly orthogonal* if $\alpha\pm\beta \notin R$ for all pairs $\alpha,\beta \in E$. Now let $\mathfrak{b}$ be a standard Cartan subalgebra of $\mathfrak{g}$. Then there exists a strongly orthogonal set E in R such that the vectors H_{α}, $\alpha \in E$, form a basis of the orthogonal complement of $\mathfrak{b}_{\mathfrak{p}}$ in $\mathfrak{a}_{\mathfrak{p}}$; moreover E is determined by $\mathfrak{b}$ up to conjugacy by the Weyl group W of R. In this way one establishes a one-one correspondence between the conjugacy classes of Cartan subalgebras in $\mathfrak{g}$ and the W-orbits of strongly orthogonal subsets of the set of roots $\alpha \in R$ such that $H_{\alpha} \in \mathfrak{a}_{\mathfrak{p}}$.

The two extreme cases are:

(i) the vector part $\mathfrak{b}_{\mathfrak{p}}$ of $\mathfrak{b}$ is as large as possible: if $\mathfrak{b}$ is standard, this means that $\mathfrak{b}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}$, hence $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{a}_{\mathfrak{k}}$ and consequently $\mathfrak{b} = \mathfrak{a}$. These are the *minimally compact* Cartan subalgebras of $\mathfrak{g}$; they are all conjugate, and they correspond to $E = \emptyset$ above.

(ii) the toral part $\mathfrak{b}_{\mathfrak{k}}$ of $\mathfrak{b}$ is as large as possible: this means that $\mathfrak{b}_{\mathfrak{k}}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{k}$. These are the *maximally compact* (or fundamental) Cartan subalgebras of $\mathfrak{g}$; again they form a single conjugacy class. If $\mathfrak{g}$ itself is compact, they are the only ones.

Examples

1. $\mathfrak{g} = \mathfrak{sl}(n,R)$, $\mathfrak{k} = \mathfrak{su}(n)$, $\mathfrak{p}$ the space of real symmetric $n \times n$ matrices with zero trace; $\mathfrak{a} = \mathfrak{a}_{\mathfrak{p}}$ is the space of diagonal matrices with trace 0 , and $\mathfrak{a}_{\mathfrak{k}} = 0$. Any strongly orthogonal set E is, up to conjugacy by $W = S_n$, of the form

$$E = \{\varepsilon_1-\varepsilon_2, \varepsilon_3-\varepsilon_4, \ldots, \varepsilon_{2k-1}-\varepsilon_{2k}\}$$

where $k \le \frac{1}{2}n$. Hence the number of conjugacy classes of Cartan subalgebras in $\mathfrak{sl}(n,R)$ is $1 + [\frac{1}{2}n]$. For $0 \le k \le [\frac{1}{2}n]$, let $\mathfrak{b}^{(k)}$ denote the set of matrices in $\mathfrak{sl}(n,R)$ which are of the form

$$\mathrm{diag}(X_1, X_2, \ldots, X_k, Y)$$

where $X_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ and Y is a diagonal matrix of size $n-2k$. The $\mathfrak{b}^{(k)}$ are representatives of the classes of Cartan subalgebras.

2. $\mathfrak{g} = \mathfrak{sl}(n,H)$. Here all the Cartan subalgebras are conjugate to the algebra $\mathfrak{a}$ described in §10, Ex.3, since $\mathfrak{a}$ is both maximally and minimally compact.

3. $\mathfrak{g} = \mathfrak{su}(p,q)$ $(p \ge q)$. Here there are $q+1$ conjugacy classes of Cartan subalgebras, representatives of which may be described as follows. With the notation of §10, Ex.4, for each $j = 0,1,\ldots,q$ let $\mathfrak{b}^{(j)}$ consist of all matrices in $\mathfrak{g}$ of the form

$$\begin{pmatrix} iX & 0 & Y \\ 0 & iZ & 0 \\ Y & 0 & iX \end{pmatrix} ,$$

where X,Y,Z are real diagonal matrices of sizes $q-j, q-j, p-q+2j$ respectively. Then $\mathfrak{b}^{(o)}, \ldots, \mathfrak{b}^{(q)}$ are $q+1$ non-conjugate Cartan subalgebras of $\mathfrak{g} = \mathfrak{su}(p,q)$; $\mathfrak{b}^{(o)}$ is minimally compact and $\mathfrak{b}^{(q)}$ is maximally compact.

III. Semisimple Lie groups

1. Semisimple and reductive Lie groups

Let G be a Lie group. The largest connected solvable normal subgroup of G is called the *radical* R of G. It is a closed subgroup of G, and its Lie algebra is the radical (II, §1) of the Lie algebra $\mathfrak{g}$ of G.

A *connected* Lie group G is said to be *semisimple* if $R = \{e\}$, or equivalently if its Lie algebra $\mathfrak{g}$ is semisimple. Every semisimple Lie group G is equal to its derived group $[G,G]$, and the centre of G is discrete. If $\mathfrak{g}$ is *simple*, G is said to be *almost simple*. A connected and simply-connected Lie group G is semisimple if and only if it is a direct product of almost simple groups.

Finally, a connected Lie group is said to be *reductive* if its Lie algebra is reductive. Every compact connected Lie group is reductive.

Examples. $SL(n,\mathbf{R})$, $SU(n)$ are semisimple (indeed almost simple); $GL^{+}(n,\mathbf{R})$, $U(n)$ are reductive but not semisimple.

2. Cartan and Iwasawa decompositions

Let G be a semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be a Cartan decomposition (II, §8) of $\mathfrak{g}$. The immersed subgroup K of G which corresponds to the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ (II, §5) is then *closed* in G, and $\exp_G(\mathfrak{p}) = P$ say is a closed submanifold of G (*not* a subgroup). We have

$$G = K.P$$

and more precisely the mapping $(x,Y) \to x.\exp Y$ is a diffeomorphism of $K \times \mathfrak{p}$ onto G. This is the *Cartan decomposition* of G: it is the global counterpart of the Cartan decomposition of $\mathfrak{g}$. The mapping $\theta : xy \to xy^{-1}$ ($x \in K$, $y \in P$) is an involutory automorphism of G, and K is its group of fixed points.

The group K is compact if and only if the centre of G is finite. In general we have $K = K_o \times V$ where K_o is a maximal compact subgroup of G (necessarily connected), and V is a vector group. It follows that G is diffeomorphic to the product of K_o and a vector group, and therefore $\pi_1(G) = \pi_1(K_o)$.

Next let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_p + \mathfrak{n}$$

be an Iwasawa decomposition (II, §9) of $\mathfrak{g}$, and let A_p and N be the immersed subgroups of G which correspond to the subalgebras $\mathfrak{a}_p$ and $\mathfrak{n}$ of $\mathfrak{g}$. Then A_p and N are closed subgroups of G, and the exponential map $\exp_G$ is an isomorphism of $\mathfrak{a}_p$ onto A_p, and a diffeomorphism of $\mathfrak{n}$ onto N; A_p is a vector group and N is a nilpotent Lie group. We have

$$G = KA_pN$$

and more precisely the mapping $(x,y,z) \to xyz$ is a diffeomorphism of $K \times A_p \times N$ onto G. Finally, A_pN is a closed solvable subgroup of G in which N is normal. This is the *Iwasawa decomposition* of G; it is the global counterpart of the Iwasawa decomposition of $\mathfrak{g}$.

Example. Let $G = SL(n,R)$, $K = SO(n)$. Then we may take A_p to be the group of real diagonal matrices with *positive* elements (and determinant 1), and N to be the group of upper triangular matrices with 1's down the diagonal. The manifold P in the Cartan decomposition consists of the positive definite symmetric matrices with determinant 1 (because if $X \in \mathfrak{p}$ is a symmetric matrix, $\exp(X)$ is symmetric and positive definite). We have $\pi_1(SL(n,R)) = \pi_1(SO(n)) = Z$ if $n = 2$, $Z/2Z$ if $n > 2$.

3. Maximal compact subgroups

Let G be a (connected) semisimple Lie group, K_o a maximal compact subgroup of G. Then $X = G/K_o$ may be given the structure of a complete simply-connected Riemannian manifold with negative curvature. If K_1 is any compact subgroup of G, then K_1 acts on X by left translations as a group of isometries of X, and by a well-known theorem of Riemannian geometry this action has a fixed point $\dot{x} = xK_o \in X$. This means that $yxK_o = xK_o$ for all $y \in K_1$, i.e. that $x^{-1}K_1x \subset K_o$. In particular, if K_1 is a *maximal* compact subgroup of G, we have $x^{-1}K_1x = K_o$, and therefore all maximal compact subgroups of G are conjugate in G.

4. Parabolic subgroups

As in §2 let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_p + \mathfrak{n} \quad , \quad G = KA_pN$$

be Iwasawa decompositions of $\mathfrak{g}$ and G, and assume from now on that *the centre of* G *is finite* (so that K is a maximal compact subgroup of G). Let M and M^* be respectively the centralizer and normalizer of $\mathfrak{a}_p$ (or A_p) in K. Then M^*/M acts on $\mathfrak{a}_p$, hence by transposition also on the dual space $\mathfrak{a}_p^*$, and is isomorphic to the relative Weyl group $W = W(\mathfrak{g},\mathfrak{a}_p)$. The group

$$P_o = MA_pN$$

is a closed subgroup of G (because both M and A_p normalize N) and is the normalizer of N in G. P_o and its conjugates in G are the *minimal parabolic subgroups* of G.

Warning: the group M (and therefore also P_o) *need not be connected.* (However, it has at most finitely many connected components.) In any event, the identity component M^o of M is the subgroup of G corresponding to the Lie algebra $\mathfrak{m}$, the centralizer of $\mathfrak{a}_p$ in $\mathfrak{k}$. Hence the Lie algebra of P_o is

$$\mathfrak{m} + \mathfrak{a}_p + \mathfrak{n} = \mathfrak{l} + \sum_{\lambda>0} \mathfrak{g}^{\lambda}$$

in the notation of II, §9.

Example. Let $G = SL(n,R)$ and let K, A_p, N be the subgroups defined in the Example in §2. Then M consists of the diagonal matrices in which each element is ± 1 (and determinant equal to 1), hence is a finite group of order 2^{n-1}; MA_p is therefore the group of *all* diagonal matrices in G, and $P_o = MA_pN$ the group of upper triangular matrices in G. So P_o has 2^{n-1} connected components, corresponding to the various choices of sign for the diagonal elements.

If $x \in M^*$, the double coset P_oxP_o depends only on the coset xM, that is to say on the image w of x in $M^*/M = W$, so that we may write $P_oxP_o = P_owP_o$ unambiguously. We have then

$$G = \bigcup_{w\in W} P_owP_o \qquad \text{(disjoint union).}$$

This is the *Bruhat decomposition* of G. The group W has a unique element w_1 which transforms each positive root $\lambda \in S^+$ into a negative root, and $P_ow_1P_o$ is a dense open submanifold of G, whose complement has zero Haar measure. All the double cosets $P_o w P_o$ are locally closed submanifolds of G.

The pair of subgroups (P_o, M^*) is a BN-*pair* or *Tits system* in G. Abstractly, a Tits system in a group G consists of a pair of subgroups B, N which together generate G and satisfy certain axioms which we shall not reproduce here. The group $H = B \cap N$ is normal in N, and $W = N/H$ is called the *Weyl group*; it has a distinguished set Δ of involutory generators. For each $x \in N$, the double coset BxB depends only on the image $w = xH$ of x in W, and is denoted by BwB. It is then a consequence of the axioms of a Tits system that G has a Bruhat decomposition

$$G = \bigcup_{w \in W} BwB \qquad \text{(disjoint union).}$$

For each subset E of Δ let W_E be the subgroup of W generated by E. Then $P_E = BW_EB$ is a subgroup of G, and the mapping $E \to P_E$ is an inclusion-preserving bijection of the set of subsets of Δ onto the set of subgroups of G which contain B. Each group P_E is its own normalizer, and no two of them are conjugate in G. In particular, $P_\emptyset = B$ and $P_\Delta = G$. Generalizing the Bruhat decomposition we have

$$G = \bigcup_{w \in W_E \backslash W / W_F} P_E w P_F \qquad \text{(disjoint union)}$$

for any subsets E, F of Δ.

In the present situation, B is the minimal parabolic subgroup P_o, and N is the normalizer M^* of $A_\mathfrak{p}$ in K. The group $H = B \cap N$ is $M^* \cap P_o = M$, and $W = M^*/M$ is the relative Weyl group of G. The subgroups P_E and their conjugates in G are called *parabolic* subgroups. They form $2^{\ell'}$ conjugacy classes, where $\ell' = \text{card}(\Delta) = \dim \mathfrak{a}_\mathfrak{p}$ is the relative rank of $\mathfrak{g}$.

5. The Langlands decomposition of a parabolic subgroup

We retain the notation and assumptions of §3. As in II, §10, let $S = S(\mathfrak{g},\mathfrak{a}_{\mathfrak{p}})$ be the relative root system of $\mathfrak{g}$, and let Δ be the basis of S determined by the Iwasawa decomposition (in which $\mathfrak{n} = \sum_{\lambda\in S^+} \mathfrak{g}^{\lambda}$, where S^+ is the set of positive roots defined by Δ).

Let E be any subset of Δ, $\langle E\rangle$ the subsystem of S generated by E. Let $\mathfrak{g}^E$ be the Lie algebra generated by the root spaces $\mathfrak{g}^{\lambda}$ for $\lambda \in \langle E\rangle$. (If $E = \Delta$, then $\mathfrak{g}^E = \mathfrak{g}$.) Then $\mathfrak{g}^E$ is a semisimple subalgebra of $\mathfrak{g}$. If we put $\mathfrak{k}^E = \mathfrak{k}\cap\mathfrak{g}^E$ and $\mathfrak{p}^E = \mathfrak{p}\cap\mathfrak{g}^E$, then $\mathfrak{g}^E = \mathfrak{k}^E + \mathfrak{p}^E$ is a Cartan decomposition of $\mathfrak{g}^E$, and $\mathfrak{a}_{\mathfrak{p}}^E = \mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{g}^E$ is a maximal abelian subalgebra of the vector space $\mathfrak{p}^E$.

Let $\mathfrak{a}_E$ be the orthogonal complement (with respect to the Killing form $B_{\mathfrak{g}}$) of $\mathfrak{a}_{\mathfrak{p}}^E$ in $\mathfrak{a}_{\mathfrak{p}}$. Equivalently, $\mathfrak{a}_E$ is the intersection of the kernels of the linear forms $\lambda \in E$. As in II, §9, $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a}_E)$ is a commuting set of self-adjoint endomorphisms of $\mathfrak{g}$, and so we have a root-space decomposition

$$\begin{aligned}\mathfrak{g} &= \mathfrak{l}_E + \sum_{\mu\in S_E} \mathfrak{g}^{\mu}\\ &= \mathfrak{l}_E + \mathfrak{n}_E + \bar{\mathfrak{n}}_E\end{aligned}$$

where $\mathfrak{l}_E$ is the centralizer of $\mathfrak{a}_E$ in $\mathfrak{g}$, and $S_E = S - \langle E\rangle$, and $\mathfrak{n}_E = \sum_{\mu\in S_E^+} \mathfrak{g}^{\mu}$, $\bar{\mathfrak{n}}_E = \sum_{\mu\in S_E^+} \mathfrak{g}^{-\mu}$. Now define

$$\mathfrak{m}_E = \mathfrak{l}_E\cap\mathfrak{k} + [\mathfrak{l}_E,\mathfrak{l}_E]\cap\mathfrak{p} \;;$$

then $\mathfrak{m}_E$ is a reductive subalgebra of $\mathfrak{g}$, and we have

$$\mathfrak{l}_E = \mathfrak{m}_E + \mathfrak{a}_E \qquad \text{(direct sum)}.$$

We shall now pass from the Lie algebra to the group G. Let $A_{\mathfrak{p}}^E, A_E, L_E^o, M_E^o, N_E$ respectively be the connected Lie groups immersed in G which correspond to the Lie algebras $\mathfrak{a}_{\mathfrak{p}}^{E}, \mathfrak{a}_{E}, \mathfrak{l}_{E}, \mathfrak{m}_{E}, \mathfrak{n}_{E}$. All these subgroups are closed in G; we have $A_{\mathfrak{p}} = A_{\mathfrak{p}}^{E} A_{E}$ (direct product), and $L_{E}^{o} = M_{E}^{o} A_{E}$.

Let L_E be the centralizer of $\mathfrak{a}_E$ (or A_E) in G, so that L_E^o is the identity component of L_E. Put $M_E(K) = L_E \cap K$, the centralizer of A_E in K, and let $M_E = M_E(K) M_E^o$, so that M_E^o is the identity component of M_E. We have $L_E = M_E A_E$, and the groups L_E, M_E and

$$P_E = M_E A_E N_E = L_E N_E$$

are closed subgroups of G, and P_E contains the minimal parabolic subgroup P_o $(= P_{\emptyset})$. So the groups P_E, for all subsets E of Δ, are the parabolic subgroups of G which contain P_o. The Lie algebra of P_E is $\mathfrak{l}_E + \mathfrak{n}_E = \mathfrak{p}_E$.

Example. Let us illustrate all this by reference to the group $SL(n,R)$. As before (§3, Ex.) we take as minimal parabolic P_o the group of upper triangular matrices with determinant 1, and we take $K = SO(n)$. The relative root system S is the same as the absolute root system R, and is of type A_{n-1}. Each subset E of the basis Δ may be described by a subset $\{m_1,\ldots,m_r\}$ of $\{1,2,\ldots,n-1\}$, where $m_1 < m_2 < \ldots < m_r$, or equivalently by the sequence

$$(n_1,n_2,\ldots,n_{r+1}) = (m_1,m_2-m_1,\ldots,m_r-m_{r-1},n-m_r)$$

of positive integers whose sum is n. Correspondingly we write each matrix $X \in SL(n,R)$ or $\mathfrak{sl}(n,R)$ in block form: $X = (X_{ij})$ where X_{ij} has n_i rows and n_j columns $(1 \le i,j \le r+1)$. Then $\mathfrak{a}_E$ (resp. A_E) consists of all block diagonal matrices ($X_{ij} = 0$ if $i \neq j$) with $X_{ii} = x_i I_{n_i}$ $(1 \le i \le r+1)$ and $\Sigma n_i x_i = 0$ (resp. each $x_i > 0$ and $\Pi x_i^{n_i} = 1$). The centralizer $\mathfrak{l}_E$ of $\mathfrak{a}_E$ consists of all block diagonal matrices X with trace 0, and $\mathfrak{m}_E$ consists of block diagonal matrices X with trace $X_{ii} = 0$ $(1 \le i \le r+1)$, i.e. $\mathfrak{m}_E \cong \prod_i \mathfrak{sl}(n_i, R)$. The corresponding groups are

$$L_E^o = SL(n,R) \cap (\prod_i GL^+(n_i,R))$$

$$L_E = SL(n,R) \cap (\prod_i GL(n_i,R))$$

(so that $(L_E : L_E^o) = 2^r$)

$$M_E(K) = L_E \cap K = SL(n,R) \cap \prod_i O(n_i)$$

$$M_E^o = \prod_i SL(n_i,R)$$

$$M_E = \{\mathrm{diag}(X_1,\dots,X_{r+1}),\ X_i \in GL(n_i,R),\ \det X_i = \pm 1\}$$

$$N_E = \{X = (X_{ij}) : X_{ij} = 0 \text{ if } i > j,\ X_{ii} = I_{n_i}\}$$

$$P_E = \{X \in (X_{ij}) : X_{ij} = 0 \text{ if } i > j,\ \det X = 1\}$$

So P_E consists of the matrices in $SL(n,R)$ which are upper triangular in the block form determined by $(n_1,n_2,\dots,n_{r+1})$.

The subgroup N_E of the parabolic group P_E can be characterized intrinsically: it is the *unipotent radical* of P_E, i.e. the largest connected normal subgroup of P_E whose elements are unipotent, and its Lie algebra $\mathfrak{n}_E$ is the largest ideal of $\mathfrak{p}_E$ whose elements are nilpotent. The subgroup L_E is a *Levi subgroup* of P_E, that is to say a reductive closed subgroup L of P_E such that $P_E = L.N_E$ (semidirect product). Hence if P is any parabolic subgroup of G, N_P its unipotent radical, we have

$$P = L.N_P \qquad \text{(semidirect product)} \qquad (1)$$

where L is a Levi subgroup of P (*Levi decomposition* of P). The Levi component L is not unique, but if L' is another then we have $L' = xLx^{-1}$ for some $x \in N_P$.

The subgroup A_E above can be characterized as the largest connected split abelian subgroup of the centre of the Levi subgroup L_E (an abelian subgroup A of G is *split* if for each $x \in A$, $Ad_G(x)$ is diagonalizable over R). The group M_E can be described as the intersection of the kernels of all continuous homomorphisms $\chi : L_E \to R$. Hence, if P is any parabolic subgroup of G and L a Levi subgroup of P we have $L = MA$, $M \cap A = \{1\}$, and hence by (1)

$$P = MAN_P \tag{2}$$

where $M = \cap \operatorname{Ker}(\chi : L \to R)$ and A is the largest connected split abelian subgroup of the centre of L; moreover the product mapping of $M \times A \times N_P$ onto P is a diffeomorphism. This is the *Langlands decomposition* of P. The group A is called a *split component* of P; it is unique up to conjugation by elements of N_P. The dimension of A is called the *parabolic rank* of P. (Thus the parabolic rank of P_E is $\operatorname{card}(\Delta - E)$.)

Two parabolic subgroups P, P' are said to be *associated* if P and $xP'x^{-1}$ have a common Levi subgroup, for some $x \in G$. Clearly conjugate parabolics are associated, but the converse is false. For example, in $G = SL(n,R)$, the parabolic subgroups $P_E, P_{E'}$ are associated if and only if the sequences (n_i) and (n_i') determined respectively by E and E' are permutations of each other. Hence the number of classes of associated parabolic subgroups in $SL(n,R)$ is equal to the number of partitions of n (whereas the number of conjugacy classes of parabolic subgroups is 2^{n-1}).

6. Cartan subgroups

As before, let G be a connected semisimple Lie group with finite centre, $\mathfrak{g}$ the Lie algebra of G. A *Cartan subgroup* of G is the centralizer B in G of a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$: $B = Z_G(\mathfrak{b})$. It is a closed subgroup of G, but is not necessarily connected. Its identity component B^o is the closed subgroup $\exp_G(\mathfrak{b})$ with Lie algebra $\mathfrak{b}$, and the group of components B/B^o is finite.

Example. Let $G = SL(2,R)$ and let $\mathfrak{a}$, $\mathfrak{b}$ be the Cartan subalgebras of $\mathfrak{g} = \mathfrak{sl}(2,R)$ generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ respectively (II, §11). The corresponding Cartan subgroups of G are A, consisting of all diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ (so that $A \cong R^*$ and therefore has two components) and $B = SO(2)$ (which is connected).

Another warning: Cartan subgroups need not be abelian. (If $\mathfrak{a}$ is the Cartan subalgebra of $\mathfrak{g} = \mathfrak{sl}(3,R)$ consisting of the diagonal matrices, and G is the simply-connected double covering group of $SL(3,R)$, then the centralizer A of $\mathfrak{a}$ in G is not abelian.) However, if G admits a faithful finite-dimensional representation, then all Cartan subgroups of G are abelian; so that in any case the Cartan subgroups of G are abelian modulo the centre of G.

Moreover, if $\mathfrak{b}$ is a maximally compact (or fundamental) Cartan subalgebra of $\mathfrak{g}$, then $B = Z_G(\mathfrak{b})$ is both connected and abelian.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decompositon of $\mathfrak{g}$, let θ be the associated Cartan involution and let $\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} + \mathfrak{b}_{\mathfrak{p}}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$. Let B be the centralizer of $\mathfrak{b}$ in G, let $B_{\mathfrak{p}} = \exp(\mathfrak{b}_{\mathfrak{p}})$, and $B_K = B \cap K$, where K (§2) is the compact subgroup of G with $\mathfrak{k}$ as Lie algebra. $B_{\mathfrak{p}}$ is a vector group (the *vector part* of B) and B_K is a (not necessarily connected) compact group (the *compact part* of B), and we have $B = B_K . B_p$. The identity component B_K^o of B_K is the subgroup of G corresponding to $\mathfrak{b}_{\mathfrak{k}}$.

Examples

1. If $G = SL(n,R)$, $\mathfrak{g} = \mathfrak{sl}(n,R)$, there are up to conjugacy in $\mathfrak{g}$ $[\frac{1}{2}n] + 1$ distinct Cartan subalgebras $\mathfrak{b}_j$ $(0 \leq j \leq [\frac{1}{2}n])$, where the elements of $\mathfrak{b}_j$ are diagonal sums of j 2×2 matrices $\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$ and a diagonal matrix of size $n-2j$ (II, §11, Examples). The centralizer B_j of $\mathfrak{b}_j$ in G consists of the matrices of the same description and determinant equal to 1 . Since the group of nonzero matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is isomorphic to the multiplicative group C^* , it follows that B_j is isomorphic to $(C^*)^j \times (R^*)^{n-2j-1}$ (unless $n = 2j$), hence has 2^{n-2j-1} connected components. If $n = 2j$, B_j is isomorphic to $(C^*)^{j-1} \times T \cong T^k \times R^{k-1}$, where T is the circle group.

2. If $G = SL(n,H)$, $\mathfrak{g} = \mathfrak{sl}(n,H)$, there is up to conjugacy in $\mathfrak{g}$ only one Cartan subalgebra $\mathfrak{a}$, consisting of the complex diagonal matrices $X \in \mathfrak{g}$ such that $\mathrm{Re}(\mathrm{trace}\ X) = 0$ (II, §10, Examples). The corresponding Cartan subgroup A of G consists of the complex diagonal matrices X such that $|\det X| = 1$, hence is isomorphic to $(C^*)^{n-1} \times T \cong T^n \times R^{n-1}$.

3. If $G = SU(p,q)$, the group of linear transformations of C^n with determinant 1 which leave invariant the Hermitian form

$$z_1\bar{z}_1 + \ldots + z_p\bar{z}_p - z_{p+1}\bar{z}_{p+1} - \ldots \quad z_n\bar{z}_n \quad ,$$

then $\mathfrak{g} = \mathfrak{su}(p,q)$ has $q+1$ conjugacy classes of Cartan subalgebras (II, §11, Examples), represented by $\mathfrak{b}^{(j)}$ $(0 \leq j \leq q)$ (loc.cit.). The centralizer $B^{(j)}$ of $\mathfrak{b}^{(j)}$ in G consists of the matrices

$$\begin{pmatrix} e^{iX}\,\mathrm{ch}\,Y & 0 & e^{iX}\,\mathrm{sh}\,Y \\ 0 & e^{iZ} & 0 \\ e^{iX}\,\mathrm{sh}\,Y & 0 & e^{iX}\,\mathrm{ch}\,Y \end{pmatrix}$$

where X, Y, Z are real diagonal matrices of sizes $q-j$, $q-j$ and $p-q+2j$ respectively, and $2 \operatorname{trace} X + \operatorname{trace} Z = 0$; $B^{(j)}$ is connected and abelian, isomorphic to $T^{p+j-1} \times R^{q-j}$.

7. The regular set

Let G be a connected semisimple Lie group with finite centre, $\mathfrak{g}$ its Lie algebra. As in Chapter I, §4, let $\mathfrak{g}'$ (resp. G') denote the set of regular elements of $\mathfrak{g}$ (resp. G) . For any Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, let $\mathfrak{b}' = \mathfrak{g}' \cap \mathfrak{b}$, and define

$$^{\mathfrak{g}}\mathfrak{b}' = \bigcup_{x \in \mathrm{Int}(\mathfrak{g})} x(\mathfrak{b}') = \bigcup_{x \in G} \mathrm{Ad}(x)(\mathfrak{b}') \quad .$$

Likewise, if B is a Cartan subgroup of G , let $B' = G' \cap B$, and define

$$^{G}B' = \bigcup_{x \in G} xB'x^{-1} \quad .$$

Now let $\mathfrak{b}_i$ $(1 \le i \le r)$ be a set of representatives of the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$, and let B_i be the centralizer of $\mathfrak{b}_i$ in G , so that every Cartan subgroup of G is conjugate to exactly one of the B_i . Then we have

$$\mathfrak{g}' = \bigcup_{i=1}^{r} {}^{\mathfrak{g}}\mathfrak{b}_i'$$

and

$$G' = \bigcup_{i=1}^{r} {}^{G}B_i' \quad .$$

Since each $\mathfrak{b}_i'$ (resp. B_i') has only a finite number of connected components, it follows that the number of components of $\mathfrak{g}'$ (resp. G') is finite.

Let again $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$, and let B, B^* be respectively the centralizer and normalizer of $\mathfrak{b}$ in G. The group $B^*/B = W(G,B)$ is finite. It acts on G/B by right multiplication: if $w = nB \in W(G,B)$ and $\dot{x} = xB \in G/B$, then $\dot{x}w = xBnB = xnB$. Also $W(G,B)$ acts on $\mathfrak{b}'$ by the rule $w.H = Ad_G(n).H$. Hence $W(G,B)$ acts on $(G/B) \times \mathfrak{b}'$: $w(\dot{x},H) = (\dot{x}w^{-1}, wH)$. Let

$$\phi : (G/B) \times \mathfrak{b}' \to {}^{\mathfrak{g}}\mathfrak{b}'$$

be the mapping $(\dot{x},H) \to Ad_G(x)H$. Then ϕ is an everywhere regular covering map of degree $|W(G,B)|$.

For the global analogue of this result we must replace the Cartan subgroup B by its *centre* B_o, since B might not be abelian. Define $W(G,B_o) = B^*/B_o$, which is still a finite group. This group acts (on the right) on G/B_o and by conjugation on B'. Let

$$\psi : (G/B_o) \times B' \to {}^{G}B'$$

be the mapping $(\dot{x},b) \to xbx^{-1}$. Then ψ is an everywhere regular covering map of degree $|W(G,B_o)|$.

These results enable integration over $\mathfrak{g}$ (resp. G) to be reduced to integration over $(G/B_i) \times \mathfrak{b}_i$ (resp. $(G/B_{io}) \times B_i$), $i = 1,\ldots,r$, on the lines of Weyl's integration formula for compact Lie groups.

8. Complex Lie groups

In this section we shall briefly review the structure theory of *complex* semisimple Lie groups, which is a much simpler business than the real theory: in particular, the phenomena of disconnectedness (of Cartan subgroups and parabolic subgroups) do not arise in the complex case.

A *complex Lie group* is a complex-analytic manifold G which is a group, the group operations being holomorphic mappings. Semisimplicity, Cartan subgroups, parabolic subgroups etc. are defined exactly as in the real case.

Let G be a complex semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, H the centralizer of $\mathfrak{h}$ in G (the Cartan subgroup of G corresponding to $\mathfrak{h}$). Then $H = \exp_G(\mathfrak{h})$ and is a closed complex-analytic subgroup of G, isomorphic to $(C^*)^\ell$ where $\ell = \dim_C \mathfrak{h}$ is the (complex) rank of $\mathfrak{g}$ (or of G). All Cartan subgroups of G are conjugate to H (because all Cartan subalgebras of $\mathfrak{g}$ are conjugate to $\mathfrak{h}$).

As in Chapter II, §3 let $R \subset \mathfrak{h}_R^*$ be the root system of $(\mathfrak{g},\mathfrak{h})$. We define three lattices L_0, L_1 and L_G in $\mathfrak{h}_R$, as follows. L_1 is the lattice of all $X \in \mathfrak{h}_R$ such that $\alpha(X) \in Z$ for all roots $\alpha \in R$, i.e. L_1 is the dual of the lattice in $\mathfrak{h}_R^*$ spanned by the roots. L_0 is the lattice in $\mathfrak{h}_R$ spanned by the vectors $H_{\alpha^\vee}$ corresponding to the coroots $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$, $\alpha \in R$. Since $\beta(H_{\alpha^\vee}) = \langle\alpha^\vee,\beta\rangle \in Z$ for any two roots $\alpha,\beta \in R$, it follows that L_0 is a sublattice of L_1. Moreover the quotient L_1/L_0 is a *finite* group. Both L_0 and L_1 depend only on the root system R. Finally, the lattice L_G is the kernel of the homomorphism $e : \mathfrak{h} \to H$ defined by $e(X) = \exp_G(2\pi iX)$; e is surjective and therefore induces an isomorphism $\mathfrak{h}/L \cong H$. The lattice L_G lies between L_0 and L_1; also

(1) The homomorphism $e : \mathfrak{h} \to H$ defines an isomorphism of L_1/L_G onto the centre of G.

(2) The canonical mapping $\pi_1(H) \to \pi_1(G)$ is surjective and defines an isomorphism of L_G/L_0 onto $\pi_1(G)$.

It follows that $L_G = L_0$ if and only if G is simply connected, and that $L_G = L_1$ if and only if G is adjoint (i.e. $G = \mathrm{Ad}(G)$). For each lattice L lying between L_0 and L_1 there exists a connected complex semisimple Lie group G with Lie algebra $\mathfrak{g}$ such that $L_G = L$, and G is unique (up to isomorphism).

Example. Let $G = SL(n,C)$, and take H to be the diagonal subgroup of G . Then $\mathfrak{h}_R$ consists of the real diagonal matrices with trace 0 ; L_0 consists of the diagonal matrices with trace 0 and *integer* elements (so that $L_0 \cong Z^{n-1}$) and L_1 consists of the diagonal matrices $\mathrm{diag}(a_1,\ldots,a_n)$ with $\Sigma a_i = 0$ and $a_i - a_j \in Z$ for all i,j . It follows that L_1/L_0 is cyclic of order n ; we have $L_G = L_0$, in agreement with the fact that $SL(n,C)$ is simply-connected. Hence the almost simple connected complex Lie groups with Lie algebra $\mathfrak{sl}(n,C)$ are in one-one correspondence with the subgroups of a cyclic group of order n , i.e. with the divisors of n .

Let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in R} \mathfrak{g}^{\alpha}$$

be the root-space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (II, §3). Let R^+ be the set of positive roots of R relative to a chosen basis, and let

$$\mathfrak{b} = \mathfrak{h} + \sum_{\alpha\in R^+} \mathfrak{g}^{\alpha} \quad .$$

Then $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$, called a *Borel subalgebra*. It is a maximal solvable subalgebra of $\mathfrak{g}$, and every solvable subalgebra of $\mathfrak{g}$ is conjugate to a subalgebra of $\mathfrak{b}$.

Let B be the immersed subgroup of G corresponding to $\mathfrak{b}$. Then B is a closed complex-analytic subgroup of G , called a *Borel subgroup* of G . It is a maximal connected closed solvable subgroup of G , and every connected solvable subgroup of G is conjugate to a subgroup of B.

In the terminology of §4, the Borel subgroups are the minimal parabolic subgroups of G. The parabolic subgroups of G are therefore the subgroups of G which contain a Borel subgroup; they are closed, connected complex-analytic subgroups of G. Moreover, a closed complex-analytic subgroup Q of G is a parabolic subgroup if and only if the homogeneous space G/Q is compact.

Let K be a maximal compact subgroup of G, B a Borel subgroup. Then $T = B \cap K$ is a maximal torus (i.e. Cartan subgroup) of the compact (real) Lie group K. Since $G = KB$ (Iwasawa decomposition) we have

$$G/B = KB/B \cong K/B \cap K = K/T \quad .$$

More generally, if P is a parabolic subgroup of G, then $K_P = P \cap K$ is a subgroup of K containing a maximal torus, and $G/P \cong K/K_P$.

9. Lie groups and algebraic groups

Let $\mathfrak{g}$ be a real semisimple Lie algebra. The adjoint representation $\mathrm{ad}_{\mathfrak{g}}$ maps $\mathfrak{g}$ isomorphically onto a subalgebra of $\mathfrak{gl}(\mathfrak{g}) \cong \mathfrak{gl}(n,\mathbb{R})$ where $n = \dim \mathfrak{g}$. Thus $\mathfrak{g}$ can be realized as a Lie algebra of matrices.

On the other hand, if G is a real semisimple Lie group, it is not necessarily the case that G can be realized as a group of matrices - or, equivalently, that G has a faithful finite-dimensional representation - even if the centre of G is finite.

A semisimple Lie group G is said to be *linear* if G has a faithful finite-dimensional representation, i.e. if there exists a smooth injective homomorphism $i : G \to GL(n,\mathbb{C})$ for some n. It follows then that $i(G)$ is a *closed* subgroup of $GL(n,\mathbb{C})$. An equivalent condition is that G is *algebraic*, i.e. isomorphic to the identity component (with respect to the usual topology) of the group of real points of a semisimple algebraic group defined over R .

Let $\tilde{G}$ and $\tilde{G}_C$ be the simply-connected Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{g}_c$ respectively, and let $\phi : \tilde{G} \to \tilde{G}_C$ be the homomorphism corresponding to the injection $X \to X \otimes 1$ of $\mathfrak{g}$ into $\mathfrak{g}_c$. This homomorphism ϕ is not necessarily injective; it maps the centre of $\tilde{G}$ into the centre of $\tilde{G}_C$, and its kernel D_o is a subgroup of the (discrete) centre of $\tilde{G}$. The group $G_o = \tilde{G}/D_o$ is the 'largest algebraic quotient' of $\tilde{G}$.

If G is any connected Lie group with $\mathfrak{g}$ as Lie algebra, let $N(G)$ denote the intersection of the kernels of the finite-dimensional representations of G. Then (1) there exists a representation of G whose kernel is exactly $N(G)$, and (2) $N(G)$ is the image of $D_o = \ker(\phi)$ under the covering map $\tilde{G} \to G$.

Example. If $G = SL(n,R)$, then $\tilde{G}_C = SL(n,C)$ and $\phi : \tilde{G} \to \tilde{G}_C$ is the composition of the covering map $p : \tilde{G} \to G$ with the embedding $SL(n,R) \to SL(n,C)$. Consequently no proper covering group of $SL(n,R)$ is algebraic.

Finally, if G is compact and semisimple (which means that the centre of G is finite) then $\tilde{G}$ is compact, ϕ is injective, and ϕ maps the centre of $\tilde{G}$ isomorphically onto the centre of $\tilde{G}_C$. Hence every compact semisimple group is algebraic, and

$$\tilde{G}/D \longleftrightarrow \tilde{G}_C/\phi(D)$$

(where D is a subgroup of the (finite) centre of $\tilde{G}$) sets up a one-one correspondence (up to isomorphism) between compact semisimple groups and complex semisimple groups.

References

Chapter I

§1 Chevalley Ch.III, Helgason Ch.I

§2-6 Bourbaki Ch.III, Chevalley Ch.IV, Dieudonné Ch.XIX, Helgason Ch.II

§7 Bourbaki Ch.III, Dieudonné Chs.XVI,XIX

§8 Dieudonné Chs.XVI,XIX

Chapter II

§1 Bourbaki Ch.I, Jacobson Chs.I-III, Serre Chs.I,II

§2 Bourbaki Ch.VII, Humphreys Ch.IV, Jacobson Ch.III, Serre Ch.III

§3 Bourbaki Ch.VIII, Dieudonné Ch.XXI, Helgason Ch.III, Humphreys Chs.II,IV, Jacobson Ch.IV, Serre Ch.VI

§4 Bourbaki Ch.VI, Dieudonné Ch.XXI, Humphreys Ch.III, Jaconson Ch.IV, Serre Ch.V

§5 Bourbaki Ch.VI, Humphreys Ch.V, Jacobson Ch.IV, Serre Chs.V,VI

§6 Dieudonné Ch.XXI, Helgason Ch.III

§7 Helgason Ch.IX, Tits

§8 Hegason Ch.III, Dieudonné Ch.XXI

§9 Dieudonne Ch.XXI, Helgason Ch.VI, Warner Ch.I

§10 Helgason Ch.VI, Warner Ch.I

§11 Warner Ch.I

Chapter III

§1 Bourbaki Ch.III, Dieudonné Ch.XXI, Helgason Ch.II

§2 Dieudonné Ch.XXI, Helgason Ch.VI

§3 Helgason Ch.VI

§4-7 Warner Ch.I

§8 Helgason Ch.VII, Tits, Warner Ch.I

§9 Tits, Warner Ch.II

Bibliography

N.Bourbaki, Groupes de Lie et Algèbres de Lie, Chapters I-VIII (Hermann, 1960-1975)

C.Chevalley, Theory of Lie Groups (Princeton, 1946)

J.Dieudonné, Treatise on Analysis, vols.III,IV,V (Academic Press 1972, 1974, 1977)

S.Helgason, Differential Geometry and Symmetric Spaces (Academic Press, 1962)

J.E.Humphreys, Introduction to Lie Algebras and Representation Theory (Springer, 1972)

N.Jacobson, Lie Algebras (Interscience, 1962)

J.-P.Serre, Algèbres de Lie semisimples complexes (Benjamin, 1966)

J.Tits, Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen (Springer Lecture Notes No.40, 1967)

G.Warner, Harmonic Analysis on semisimple Lie Groups I (Springer, 1972)

6 · Lie groups and physics

D.J. SIMMS
Trinity College, Dublin
(Notes by K.C. Hannabuss)

Introduction

Lie groups play a basic rôle in both classical mechanics and quantum mechanics. In these lectures we shall show how symmetry in quantum mechanics leads to the study of unitary representations of Lie groups, whilst in classical mechanics symmetry leads one to investigate the action of Lie groups on symplectic manifolds. We shall see how fundamental physical quantities such as energy, linear momentum, mass and angular momentum arise naturally in this context. More briefly at the end we shall outline a quantisation procedure which enables one to pass from a classical system to its quantum analogue.

There will not, of course, be time to do more than outline the main ideas, and the bibliography at the end can be used to fill in some of the details. The basic structure of quantum mechanics and the rôle of group theory within it can be found in [5], [6], [8] and [10]. Relativistic symmetry groups are discussed in detail in the last three of these. References [6] and [8] include a discussion of internal symmetry groups. The geometric structure of classical mechanics is discussed in [5], and also in [7] and [9] where the connection with the Lie algebra dual is described. Reference [4] also contains most of the useful material on symplectic geometry which underlies classical mechanics.

1. Quantum mechanics

The conventional mathematical model of quantum mechanics associates with each system a complex Hilbert space H .

From this can be formed the projective space $\hat{H}$, whose elements are the one-dimensional subspaces of H . For each non-zero vector ψ in H we let $\hat{\psi}$ be the one-dimensional subspace (or *ray*) containing ψ . Then $\hat{H} = \{\hat{\psi} \subseteq H ;\ \psi \in H ,\ \psi \neq 0\}$. The *pure states* of the quantum mechanical system are represented by the elements of $\hat{H}$. Since we shall have no need here to discuss general states of a quantum mechanical system we shall often omit the word pure, and simply talk of states. A non-zero vector ψ in H is often called a *wave vector* representing the state. When H is actually modelled on a function space, as is often the case, we call it a *wave function*.

Example. A non-relativistic particle moving in $\mathbb{R}^3$. The relevant Hilbert space is then $H = L^2(\mathbb{R}^3)$.

If ψ is a wave function representing the state $\hat{\psi}$, chosen so that $\|\psi\|^2 = \int_{\mathbb{R}^3} |\psi(x)|^2 d^3x = 1$, then for each Borel set E in $\mathbb{R}^3$ the integral $\int_E |\psi(x)|^2 d^3x$ takes a value in $[0,1]$, which is interpreted as the probability of finding the particle in E .

If ϕ and ψ are non-zero vectors in H then one can form the scalar quantity $|\langle\phi,\psi\rangle|^2/\|\phi\|^2\|\psi\|^2$. By the Cauchy-Schwarz inequality this is a real number lying in the interval $[0,1]$, and it depends only on the rays $\hat{\phi}$ and $\hat{\psi}$ on which ϕ and ψ lie. It is interpreted as the *transition probability* between the two states $\hat{\phi}$ and $\hat{\psi}$, that is, the probability that when the system is known to be in the state $\hat{\phi}$ an observation will find it in the state $\hat{\psi}$. We shall denote this probability by $(\hat{\phi},\hat{\psi})$.

Each *observable* of the system is represented by a self-adjoint operator on H . If $A : H \to H$ is one of these it has a spectral representation of the form

$$A = \int_{\mathbb{R}} \lambda \, P^A(d\lambda)$$

for some projection-valued measure P^A on $\mathbb{R}$. If E is a Borel set in $\mathbb{R}$ then we interpret the quantity $\langle\psi,P^A(E)\psi\rangle/\|\psi\|^2$ in $[0,1]$, which depends only on A, E and $\hat{\psi}$, as the probability that the observable represented by A takes a value in the set E when the system is in a state $\hat{\psi}$.

Example. If we are dealing with a particle in $\mathbb{R}$ so that $H = L^2(\mathbb{R})$, and we take the observable to be the position of the particle then A acts on a vector ψ in its domain by

$$(A\psi)(x) = x\psi(x) \quad .$$

Then $\langle\psi,P^A(E)\psi\rangle/\|\psi\|^2 = \int_E |\psi(x)|^2 dx$, so that the interpretation of this as a probability is consistent with our suggestion in an earlier example.

If H_1 and H_2 are the Hilbert spaces associated with two quantum-mechanical systems then the combined system is represented by $H_1 \otimes H_2$.

Example. Two particles in $\mathbb{R}^3$ are associated with $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3) \cong L^2(\mathbb{R}^6)$. The first three coordinates in $\mathbb{R}^6$ are interpreted as referring to the first particle and the last three coordinates to the second particle. (If the two particles are totally indistinguishable so that the labels 'first' and 'second' have no meaning then the Hilbert space has to be modified.)

2. Relativistic symmetries of quantum-mechanical systems

In the Special Theory of Relativity the laws of physics are held to be the same for all observers in uniform relative motion. (A similar situation pertained in pre-relativistic mechanics where the laws of mechanics were the same in all inertial frames of reference.) This assertion has consequences in quantum mechanics which we shall now investigate.

The Special Theory of Relativity is modelled on a four-dimensional real vector space M (representing space-time), which is equipped with a bilinear form, whose signature is $(+,-,-,-)$. (We shall write $x.y$ for the result of applying the bilinear form to a pair of vectors x and y in M. The variety $\{x \in M : x.x = 0\}$ represents a light-cone in M.) The permissible transformations of M consist of translations by vectors in M together with the linear transformations of M, which preserve the bilinear form. These linear transformations form a group $O(M)$ isomorphic to $O(1,3)$, but we shall concern ourselves only with its connected component $SO_o(M)$. (One can deal with the other components too, but it is simpler and more convenient to omit them. This does not seriously change the nature of the conclusions reached.) The full group of permissible transformations then consists of the natural semi-direct product of M with $SO_o(M)$. This is called the *Poincaré group* or *inhomogeneous Lorentz group*. The action of (a,A) in $M \circledS SO_o(M)$ on x in M takes it to $A.x + a$. This transforms space-time as seen in one frame of reference into space-time as seen in another frame. The multiplication law for (a,A) and (b,B) in $M(s)SO_o(M)$ is $(a,A).(b,B) = (a + A.b, A.B)$.

There is a similar group of permissible transformations in non-relativistic mechanics, called the *Galilean group*. This is isomorphic to $\mathbb{R}^4 \circledS (\mathbb{R}^3 \circledS SO(3))$, where the semi-direct product of $SO(3)$ with $\mathbb{R}^3$ is the natural one, and the action of $\mathbb{R}^3 \circledS SO(3)$ on $\mathbb{R}^4$ defining the other semi-direct product is

$$(v,R) : (x,t) \to (R.x + tv, t)$$

for (v,R) in $\mathbb{R}^3 \circledS SO(3)$, x in $\mathbb{R}^3$ and t in $\mathbb{R}$.

Whenever we have a group of symmetries G of the laws of physics there must be a corresponding action of the group on the states of a quantum-mechanical system. In fact the group element g transforms our old frame of reference into a new one, and corresponding to each state $\hat{\psi}$ will be a new state $T_g(\hat{\psi})$ which viewed from the new frame looks the same as $\hat{\psi}$ did in the old frame. This defines a map T_g from $\hat{H}$ to itself. For consistency we require that $T_{gh}(\hat{\psi}) = T_g(T_h(\hat{\psi}))$ for all g and h in G, and $\hat{\psi}$ in $\hat{H}$.

Since the laws of physics are not supposed to be changed by the action of G, transition probabilities must be preserved. So $(T_g(\hat{\phi}),T_g(\hat{\psi})) = (\hat{\phi},\hat{\psi})$ for all g in G and $\hat{\phi},\hat{\psi}$ in H. It turns out that in cases of interest this forces each T_g to be a genuine projective transformation and indeed that we have the following remarkable theorem of Wigner.

Theorem. If G is a connected group and $g \to T_g$ is a homomorphism from G to the transformations of $\hat{H}$ which satisfy $(T_g(\hat{\phi}),T_g(\hat{\psi})) = (\hat{\phi},\hat{\psi})$ for all $\hat{\phi},\hat{\psi}$ in $\hat{H}$ and g in G, then T_g is 'induced' by a unitary transformation of H. That is, there is a unitary transformation U_g of H (for each g in G) such that

$$\widehat{(U_g\psi)} = T_g(\hat{\psi})$$

for all ψ in H.

Proofs of this result can be found in Wigner [11] and in Bargmann [1]. In these references the group is not assumed to be connected, but the price paid is that anti-unitary operators may be needed. We write $U(H)$ for the unitary operators on H, $U(\hat{H})$ for the induced transformations of $\hat{H}$, and π for the natural projection from $U(H)$ to $U(\hat{H})$.

One can now ask whether the operators U_g can be chosen in such a way that $g \to U_g$ is a homomorphism. Unfortunately this is not usually possible. However, one can form the pull-back group $\pi^* G = \{(g,V) \in G \times U(H) : T_g = \pi(V)\}$, which is an extension of G by the circle group S^1 . Then the homomorphism T automatically lifts to a homomorphism from $\pi^* G$ to $U(H)$. One trouble with $\pi^* G$ is that it depends through π on the Hilbert space used. We would prefer to be able to use an extension depending on G alone, such as its universal simply-connected covering group $\tilde{G}$. Unfortunately there is an obstruction to doing this which lies in $H^2(\mathfrak{g},\mathbb{R})$ (see the lectures of Bott), but if this vanishes it is possible to lift T to a homomorphism from $\tilde{G}$ to $U(H)$. Another advantage of this situation is that the dimension of $\tilde{G}$ is the same as that of G (when G is a Lie group), whereas the dimension of $\pi^* G$ is higher by 1 .

For any semi-simple group and also for the Poincaré group $H^2(\mathfrak{g},\mathbb{R})$ vanishes and we can find a homomorphism from $\tilde{G}$ to $U(H)$. In the case of the Poincaré group $\tilde{G}$ can easily be described. There is a homomorphism from the group $SL(2,\mathbb{C})$ onto $SO_o(M)$ having the two element centre of $SL(2,\mathbb{C})$ as its kernel. This allows us to define an action of $SL(2,\mathbb{C})$ on M (by factoring through $SO_o(M)$), and then to form $M \circledS SL(2,\mathbb{C})$. This is $\tilde{G}$ for G the Poincaré group. Because it is this group which is of most importance in the applications we have in mind we shall henceforth change our terminology and refer to $M \circledS SL(2,\mathbb{C})$ as the Poincaré group. We shall also introduce the notation $E(M)$ for it.

For the Galilean group the situation is rather different. The cohomology group $H^2(\mathfrak{g},\mathbb{R})$ is isomorphic to $\mathbb{R}$ and contains an obstruction which forces us to make a one-dimensional extension of the group.

Returning to the Poincaré group, we know that inducing each homomorphism T from $M \circledS SO_o(M)$ to $U(\hat{H})$ is a homomorphism from $E(M)$ to $U(H)$. We shall now adjust our notation and call this latter homomorphism T as well. So T furnishes us with a unitary representation of $E(M)$ on H. In the lectures of Schmid it is shown that such a unitary representation of a group G gives rise to a representation $\dot{T}$ of its Lie algebra $\mathfrak{g}$ as skew-adjoint operators having a common dense domain H_w (the space of analytic vectors in H). This representation $\dot{T}$ can be uniquely extended to give a representation of the universal enveloping algebra $U\mathfrak{g}$ in the algebra of normal operators defined on H_w. The representations T and $\dot{T}$ are connected by the fact that $T(\exp X) = \exp(\dot{T}(X))$ for each X in $\mathfrak{g}$. For each X in $\mathfrak{g}$ an observable $\frac{1}{i}\dot{T}(X)$ can be defined.

Example. $G = E(M)$, $\mathfrak{g} = e(M)$.

For each m in M we pick $P(m)$ in $\mathfrak{g}$ to be the infinitesimal generator of translations in the direction of m (normalised so that $\exp(P(m))$ is precisely translation through m). The subgroup $SU(2)$ of $SL(2,\mathbb{C})$ fixes a unit vector e_o in M, and the observable $\frac{1}{i}\dot{T}(P(e_o))$ is called the *energy* of the system. The three-dimensional subspace of M orthogonal to e_o (with respect to the bilinear form) is also invariant under $SU(2)$ which acts on it as ordinary rotations. ($SU(2)$ is a double cover of $SO(3)$.) We can pick an orthogonal basis of vectors $\{e_j \in M : j = 1,2,3\}$ for this subspace, each normalised so that $e_j . e_j = -1$. The observables $\frac{1}{i}\dot{T}(P(e_j))$ are called the *linear momenta*.

Any decomposable element α in $\Lambda^2 M$ defines a two dimensional subspace of M consisting of vectors whose exterior product with α is zero. The subgroup of $SL(2,\mathbb{C})$ which fixes this subspace induces a one-parameter group of transformations on it and we write $L(\alpha)$ for its (suitably normalised) infinitesimal generator. In fact the normalisation can be chosen so that the map taking α to $L(\alpha)$ extends linearly to all elements of $\Lambda^2 M$. By convention the generators $L(e_2 \wedge e_3)$, $L(e_3 \wedge e_1)$ and $L(e_1 \wedge e_2)$ are called L_1, L_2 and L_3 respectively. They lie in su(2), and, with respect to suitable coordinates, can be written as

$$L_1 = -\tfrac{1}{2}i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad L_2 = -\tfrac{1}{2}i\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \quad L_3 = -\tfrac{1}{2}i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

They generate about spatial axes in M. The observables $\frac{1}{i}\dot{T}(L_j)$ $j = 1,2,3$, to which they give rise, are called the *angular momenta* of the system.

Just as states change under the action of the relativistic group so do the obserables. Let us suppose that a new frame of reference is related to the old one by the action of g in the group. Then a relativistic transformation which appeared to be h ($\in G$) in the old reference frame looks like ghg^{-1} in the new one. That is, the change of reference frame is represented by acting on G by $\mathrm{Ad}(g)$. As a result of this the change of reference frame affects the infinitesimal generators lying in $\mathfrak{g}$ by the adjoint action of g. If X lies in $\mathfrak{g}$ then the observable $\frac{1}{i}\dot{T}(\mathrm{Ad}_g(X))$ plays the same rôle in the new frame of reference as $\frac{1}{i}\dot{T}(X)$ did in the old frame.

This can be extended to the whole of the universal enveloping algebra $U_{\mathfrak{g}}$. It suggests that elements in the centre of the enveloping algebra $\mathfrak{z}$ will be particularly important, since these will give rise to the same observables in all frames of reference. The elements of $\mathfrak{z}$ are called *Casimir elements*.

Example. $G = E(M)$, $\mathfrak{g} = e(M)$.

For each element m in M we have $P(m)$ in $\mathfrak{g}$. This gives a linear map P in $\mathrm{Hom}(M,\mathfrak{g})$. But using the bilinear form we can identify $\mathrm{Hom}(M,\mathfrak{g})$ with $M \otimes \mathfrak{g}$, and think of P as an element of this tensor product.

Now, quite generally, if we have vector spaces U_1, U_2, V_1 and V_2 and linear maps ϕ_i from U_i to V_i $(i = 1,2)$ we can form the linear map $\phi_1 \otimes \phi_2$ from $U_1 \otimes U_2$ to $V_1 \otimes V_2$. Putting $U_1 = M \otimes M$ which maps under the bilinear form to $V_1 = \mathbb{R}$, and $U_2 = \mathfrak{g} \otimes \mathfrak{g}$ which maps bilinearly under multiplication to $V_2 = U_{\mathfrak{g}}$, we see that there is a linear map from $M \otimes \mathfrak{g} \otimes M \otimes \mathfrak{g} \cong M \otimes M \otimes \mathfrak{g} \otimes \mathfrak{g}$ to $\mathbb{R} \otimes U_{\mathfrak{g}} \cong U_{\mathfrak{g}}$. Identifying P with an element of $M \otimes \mathfrak{g}$ we can map $P \otimes P$ to an element $P.P$ in $U_{\mathfrak{g}}$. Checking the action of G on $P.P$ we find that it lies in $\mathfrak{z}$, the centre of $U_{\mathfrak{g}}$. In coordinate terms

$$P.P = P(e_o)^2 - \sum_{j=1}^{3} P(e_j)^2 \quad .$$

We can apply a somewhat similar analysis to the case of $U_1 = \Lambda^2 M \otimes M$ which maps under exterior multiplication to $\Lambda^3 M$ and then under the Hodge duality map to $V_1 = M$, together with $U_2 = \mathfrak{g} \otimes \mathfrak{g}$ and $V_2 = U_{\mathfrak{g}}$ as before, to justify the definition of an element $W = *(L \wedge P)$ in $M \otimes \mathfrak{g}$. (L can be identified with an element of $\Lambda^2 M \otimes \mathfrak{g}$ and then W is the image of $L \otimes P$ under the composite map $\Lambda^2 M \otimes \mathfrak{g} \otimes M \otimes \mathfrak{g} \cong \Lambda^2 M \otimes M \otimes \mathfrak{g} \otimes \mathfrak{g} \to M \otimes U_{\mathfrak{g}}$.) An element $W.W$ can now be formed in the same way that $P.P$ was.

The element $W.W$ is also central in $U_{\mathfrak{g}}$ and the two elements $Z_1 = P.P$ and $Z_2 = W.W$ generate $\mathfrak{z}$. The two observables $\dot{T}(Z_j)$ $j = 1,2$ are therefore relativistically invariant. (Since both Z_1 and Z_2 are of even order they give rise directly to self-adjoint observables without the need for any factors of i.)

We now turn our attention to the unitary representations of the relativistic symmetry group G on a Hilbert space H.

Usually it will be possible to decompose the space into a direct sum or integral of irreducible subspaces (see the lectures of Mackey). We can think of this as breaking the physical system up into more elementary pieces. We therefore think of the Hilbert space of an irreducible representation as being associated with an *elementary relativistic particle*.

Of course, a composite system containing two elementary particles will not usually be elementary since the tensor product $H_1 \otimes H_2$ will then decompose into a direct integral of irreducible subrepresentations. It is often of physical importance to know precisely what this decomposition is.

This suggests two mathematical problems. The first is to classify all the irreducible unitary representation of the relativistic symmetry groups, in particular of the Poincaré group. (This was essentially done by Wigner in a paper which is reprinted in [3].) The second is to describe in detail the decomposition of tensor products of such representations.

We shall only concern ourselves with the first of these problems, and even there we shall only describe some of the physically most interesting representations.

In his lectures Mackey indicates how the problem of finding all the irreducible unitary representations of semi-direct product groups can often be reduced to a similar problem for certain of their subgroups, and thence solved. He describes this in detail in the case of the Euclidean group $\mathbb{R}^3 \circledS SU(2)$. (More details can be found in [6].) The case of the Poincaré group is similar, and all its irreducible unitary representations can be found by inducing representations of certain subgroups.

The inducing construction provides representations of the group acting on sections of vector bundles. (This is described in the lectures of Mackey but also in those of Bott and of Helgason.)

In general we have in G a closed subgroup K which has a unitary representation L on a Hilbert space V. A vector bundle over $X = G/K$ can be constructed by using the principal bundle $G \to G/K = X$. In fact we first form $G \times V$ and then factor out by the equivalence relation $(gk,v) \sim (g,L(k)v)$ for all k in K, g in G and v in V. This gives a vector bundle G_L over X whose fibre is V. The existence of a quasi-invariant measure μ on X allows us to define the space of square integrable sections $L^2(X,G_L,\mu)$. Since G acts on $G_L = G \times V/\sim$ (by left translation on G), it also acts on these sections, and this gives the induced representation.

One of the subgroups of $G = E(M)$ from which the general procedure tells us that we can induce irreducibles is $K = M \circledS SU(2)$. This gives rise to vector bundles over $X = G/K \approx SL(2,\mathbb{C})/SU(2)$. This homogeneous space can be thought of as the hyperboloid

$$X_m = \{x \in M: \ x.x = m^2, \ x.e_o > 0\} \quad ,$$

for m a positive real number.

As a matter of fact there is a natural vector bundle sitting over X_m, namely the complexified tangent bundle $T_{\mathbb{C}}(X_m)$. If we think of X_m as imbedded in M we can think of $T(X_m)$ as a subspace of $M \times M$ given by $\{(x,v) \in X_m \times M : x.v = 0\}$, and $T_{\mathbb{C}}(M)$ can be obtained from this by letting v lie in the complexification of M (subject to the same constraints). The group element (a,A) in $E(M)$ can be made to act on this bundle by sending (x,v) to $(A.x, \exp(i(a.Ax))A.v)$. The corresponding action of $E(M)$ on L^2-sections of the bundle provides a unitary irreducible representation $U_{(m,1)}$. More complicated representations $U_{(m,s)}$ can be constructed for s in $\frac{1}{2}\mathbb{Z}^+$, by taking the spinor bundle, whose fibre is $\mathbb{C}^2$, and then forming the symmetrised tensor product bundle whose fibre is $\otimes^{2s}_{\text{symm}} \mathbb{C}^2$. In this case the fibres are (2s+1)-dimensional.

As the notation suggests, in the case of $s = 1$ this gives the complex tangent bundle.

The relativistically-invariant observables defined by Z_1 and Z_2 take scalar values in these representations (as they must by Schur's lemma). In fact

$$\dot{U}_{(m,s)}(Z_1) = m^2 1$$

and

$$\dot{U}_{(m,s)}(Z_2) = m^2 s(s+1) 1 \quad .$$

The representations $U_{(m,s)}$ are thought of as describing elementary particles of mass m and spin s. (The parameter m is identified with the mass because m^2 represents the value taken by $P.P$, and in the classical theory a similar relationship links the linear momentum, energy and mass of a particle. The spin is so called because the restriction of $U_{(m,s)}$ to the Euclidean group $\mathbb{R}^3 \circledS SU(2)$ gives a representation in $L^2(\mathbb{R}^3, \mathbb{C}^{2s+1})$ in which s is clearly seen to be the spin of a 'localisable' particle. (See Chapter 3 in [6].)

There is a somewhat similar parametrisation of the irreducible unitary representations of the extension of the Galilean group, but in that case the mass m appears in connection with the centre of the extension.

3. Internal symmetries in quantum mechanics

It became apparent in the 1930's that the newly discovered neutron was remarkably similar to the proton in all its dynamical properties (having the same spin and almost identical mass), and Heisenberg suggested that it might just be another state of the same particle, the nucleon. Ignoring for the moment the small mass difference both particles transformed under the representation $U_{(m,\frac{1}{2})}$ of $E(M)$ (with $m = 939$ MeV). The direct sum of their associated Hilbert spaces could be thought of as $H \otimes V$ where H was the Hilbert space on which $U_{(m,\frac{1}{2})}$ acted and V was a two-dimensional space whose vectors told one whether one was in the proton or neutron state. This was the Hilbert space associated with the nucleon. Unfortunately by its very construction it is not an irreducible representation space for E(M), and so would not seem to define an elementary relativistic system. However, in contrast with the 'external' geometrical symmetries of E(M) one can also produce an 'internal' symmetry group SU(2) acting on V, and then $H \otimes V$ is an irreducible representation space for the direct product group $E(M) \times SU(2)$. Because of its significance in discussing isotopes, and by analogy with the copy of SU(2) contained in E(M) which describes ordinary spin, the new group SU(2) was called the *isospin group*.

Strictly speaking this picture could not be quite accurate because of the slight difference in the masses of the two particles (which was about 0.7% of their total mass). However, confidence in the model was increased by the discovery in the next two decades of other groups of particles similar in their dynamical properties and linked together in other representations of SU(2). For example, three π-mesons transformed with the three-dimensional representation of SU(2).

Early in the 1960's Gell-Mann and Ne'eman realised that if one supposed that the internal symmetry group was SU(3) rather than SU(2) then many more relativistically similar particles could be linked together. In fact 40 known particle states could be assigned to five irreducible 8-dimensional representations. We shall now outline this theory, suppressing all but the internal symmetries.

The group SU(3) is an eight-dimensional compact simple Lie group. Amongst its eight generators are the elements

$$I_1 = -\tfrac{1}{2}i\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad I_2 = -\tfrac{1}{2}i\begin{pmatrix}0 & i & 0\\ -i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \text{ and } I_3 = -\tfrac{1}{2}i\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

which generate the isospin subgroup used in the earlier theory, and

$$Q = \frac{1}{3} i \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The observables associated with the I_j $(j = 1,2,3)$ are called the components of isospin, and that associated with Q is called the electric charge.

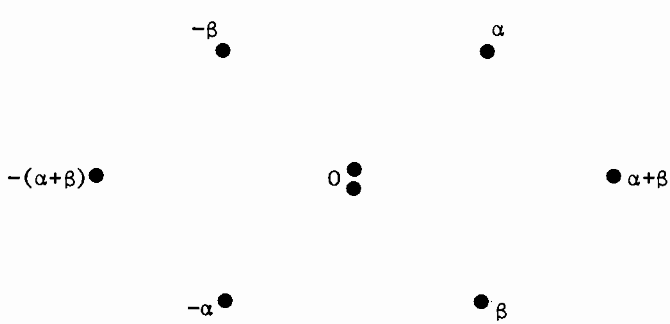
Together Q and I_3 generate a Cartan subalgebra of $su(3)$: $\mathfrak{h} = \{diag(\lambda_1, \lambda_2, \lambda_3) \in su(3) : \lambda_1 + \lambda_2 + \lambda_3 = 0\}$. Following Cartan's method, if one has a representation R of $su(3)$ then one can look for weight vectors ψ_μ in the representation space such that

$$\dot{R}(I_3)\psi_\mu = i\mu(I_3)\psi_\mu$$

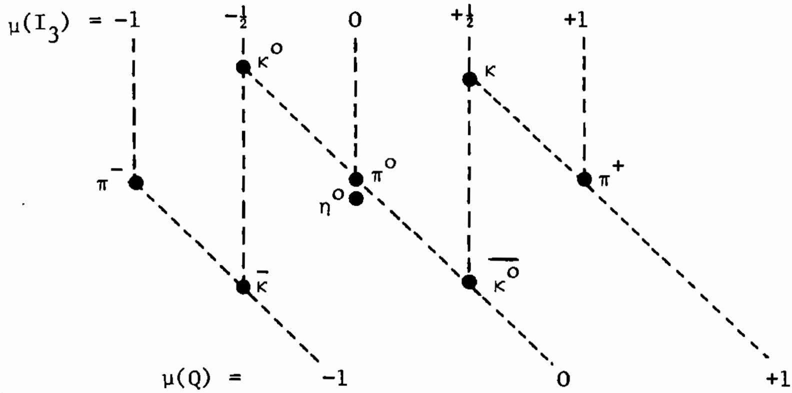
and

$$\dot{R}(Q)\psi_\mu = i\mu(Q)\psi_\mu .$$

The positive roots of $\mathfrak{h}$ in $\mathfrak{g}$ can be taken to be α, β and $\alpha+\beta$, which assign to $diag(\lambda_1, \lambda_2, \lambda_3)$ the values $\lambda_3 - \lambda_2$, $\lambda_1 - \lambda_3$ and $\lambda_1 - \lambda_2$ respectively. The weight diagram for the adjoint representation in which these occur is



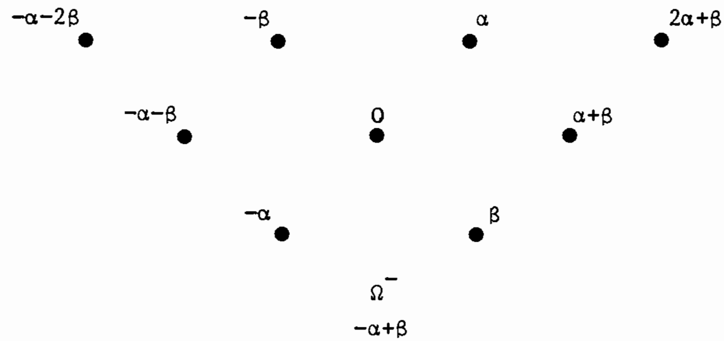
We now give as an example one of the octets of particles whose charges and third component of isospin agree with the values of $\mu(Q)$ and $\mu(I_3)$ in the adjoint representation



All these particles have spin $s = 0$ and mass $m = 628 \pm 135$ MeV . The particles at the same vertical level on this diagram were already linked together by the isospin subgroup.

As we have already mentioned a total of forty known particles could be linked together into such multiplets, and a further nine could be assigned to an irreducible representation whose highest weight is $2\alpha+\beta$. Now this representation is in fact ten-dimensional but at the time no particle was known which could be associated with the weight $-\alpha+\beta$. However, the properties of such a particle could be predicted and a search soon discovered a particle (Ω^-) which fitted the theoretical description exactly. The weight diagram is given below.

One of the reasons why Ω^- could be described so well by the theory was that SU(3) not only linked together particles of approximately the same mass but could also be used to predict the mass differences within a single representation. Such predictions were based on a purely empirical formula but led to the hope that both SU(3) and the Poincaré group might be subgroups of a larger group in which such experimental relationships would find an explanation. Unfortunately a theorem



of O'Raiffeartaigh showed that it was not possible to include both the Poincaré group and SU(3) in a larger Lie group except by having them centralise each other (see [3]). This means that a theoretical understanding of the mass differences is still lacking.

More recently the discovery of a new species of particle has renewed speculation that the internal symmetry group might be yet larger, for example SU(4) . The observable associated with the additional generator needed for a Cartan subgroup has been christened 'charm'.

4. Classical mechanics

In classical mechanics one associated with each system a *configuration space* X which is a C^∞-manifold. The tangent bundle T(X) is called the *velocity phase space*, and points of it represent the *pure states* of the system. (Again we shall have no need to talk of states which are not pure.)

Example. A particle moving on a sphere has configuration space

$$X = S^2 \sim \{x \in \mathbb{R}^3 : |x| = 1\} \quad ,$$

and velocity phase-space

$$T(S^2) \sim \{(x,v) \in S^2 \times \mathbb{R}^3 : x.v = 0\} \quad .$$

The dynamics of the system are derived from a Lagrangian function $L : T(X) \to \mathbb{R}$ which is equal to the kinetic energy minus the potential energy.

Example. A particle in $\mathbb{R}^n$ moving under the influence of a potential V

$$T(X) \sim \mathbb{R}^n \times \mathbb{R}^n = \{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n\} \quad .$$

Here (x,v) represents $\sum_j v^j \frac{\partial}{\partial x^j}$ if we let x^j be the j-th component of x .

Then

$$L(x,v) = \tfrac{1}{2}m|v|^2 - V(x) \quad .$$

Let us take any point (x,v) in $T_x(X)$. Then $T_x(X)$ is naturally isomorphic to the space of tangent vectors to $T(X)$ which are tangential to $T_x(X)$ at (x,v) . Hence the restriction of the differential dL to this subspace defines a linear form on $T_x(X)$. In other words we have a way of mapping $T_x(X)$ to $T_x^*(X)$. If this map defines a diffeomorphism from $T(X)$ to $T^*(X)$ then it is called a Legendre transformation. Such maps enable us to think of the states as being in $T^*(X)$ rather than in $T(X)$. The cotangent bundle is simply called the *phase space* of the system, and the description of classical mechanics to which it gives rise is called the Hamiltonian formulation in order to contrast it with the Lagrangian formulation in $T(X)$. Corresponding to the Lagrangian function on

$T(X)$ is a Hamiltonian function on T^*X . In simple situations this is just the total energy (kinetic plus potential).

Example. A particle in $\mathbb{R}^n$ moving under the influence of a potential V

$$L(x,v) = \tfrac{1}{2}m|v|^2 - V(x)$$

$$dL = \sum \frac{\partial L}{\partial v^j} dv^j + \sum \frac{\partial L}{\partial x^j} dx^j$$

$$= m \sum v^j dv^j - \sum \frac{\partial V}{\partial x^j} dx^j$$

The restriction of dL to vectors tangential to $T_x(X)$ is given by $dx^j = 0$, leaving $m \sum v^j dv^j$, and, on making the identification with an element of $T^*_x(X)$ this becomes $m \sum v^j dv^j$.

If we choose the natural coordinates $\{p_j\}$ in $T^*_x(X)$ (coordinates with respect to the basis dx^j) then the Legendre transformation here is $p_j = mv^j$. In general it is $p_j = \frac{\partial L}{\partial x^j}$.
The Hamiltonian is $H = \frac{1}{2m} |p|^2 + V(x)$.

The phase-space $Y = T^*X$ possesses a natural one-form $\alpha = \sum p_j dq^j$ if we lift coordinates q from X and choose the natural coordinates p_j on T_qX . (This one-form can also be defined in a coordinate-free way.) The two-form $\omega = d\alpha = \sum dp_j \wedge dq^j$ being exact is certainly closed, and it is also non-degenerate. We call a non-degenerate closed two-form a symplectic form, and a C^∞-manifold on which a symplectic form ω is defined is called a *symplectic manifold*. We have thus shown that $T^*(X)$ is a symplectic manifold.

Now on any C^∞-manifold Y the exterior derivative maps $C^\infty(Y)$ into the one-forms $\Omega^1(Y)$. On a symplectic manifold the form ω gives an identification of the tangent and cotangent spaces which enables one to turn the forms into vector fields (and vice versa). Explicitly the vector field ξ on Y comes from the form $\xi \rfloor \omega = \omega(\xi, \cdot)$. Composing this identification with the exterior derivative gives a map from $C^\infty(Y)$ into the vector fields on Y , which takes ϕ in $C^\infty(Y)$ into ξ_ϕ such that $d\phi = \xi_\phi \rfloor \omega$. The vector fields obtained in this way are called Hamiltonian vector fields. We write Ham(Y) for the space of all Hamiltonian vector fields: $\{\xi_\phi : \phi \in C^\infty(Y)\}$.

One has the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{constant functions} & \longrightarrow & C^\infty(Y) & \longrightarrow & \text{Ham}(Y) & \longrightarrow & 0 \\ & & & & & \searrow & \updownarrow \omega & & \\ & & & & & & dC^\infty(Y) \subseteq \Omega^1(Y) & & \end{array}$$

The horizontal sequence becomes an exact sequence of Lie algebras if we turn $C^\infty(Y)$ into a Lie algebra by defining the Poisson bracket of two functions ϕ and ψ to be $[\phi,\psi] = \xi_\phi(\psi)$.

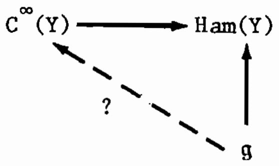
In a classical mechanical system the Hamiltonian H gives rise to a vector field ξ_H on $Y = T^*(X)$. This determines a flow on $T^*(X)$ and (using ω to make the usual identification) also on $T(X)$. Both these flows project onto the same flow on X .

Often we require a mechanical system to be invariant under some relativistic symmetry group G such as the Poincare group (or Galilean group). We then study the action of G on Y , which must preserve ω .

If G is a connected Lie group with Lie algebra $\mathfrak{g}$ then each X in $\mathfrak{g}$ gives a vector field ξ^X on Y defined by

$$(\xi^X\phi)(y) = -\frac{d}{dt}\,\phi(\exp(tX)y)\Big|_{t=0} \quad \forall \; y \in Y ,\ \phi \in C^\infty(Y) \quad .$$

The condition that the action of G preserves ω is equivalent to saying that $\xi^X \rfloor \omega$ is a closed one-form. If $H^1(Y,\mathbb{R}) = 0$, or if $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, then ξ^X must be a Hamiltonian vector field. In this case we have a map from $\mathfrak{g}$ to $\mathrm{Ham}(Y)$. We should like to lift this to a Lie algebra homomorphism from $\mathfrak{g}$ to $C^\infty(Y)$:



This is reminiscent of the situation in quantum mechanics. It is always possible to make an extension $\mathcal{E}$ of $\mathfrak{g}$ by $\mathbb{R}$ by defining

$$\mathcal{E} = \{(X,\phi) \in \mathfrak{g} \times C^\infty(Y) : \xi^X = \xi_\phi\} \quad ,$$

such that $\mathcal{E}$ can be represented in $C^\infty(Y)$. If this extension splits (that is, if there is a homomorphism $\mathfrak{g} \to \mathcal{E}$ such that $\mathfrak{g} \to \mathcal{E} \to \mathfrak{g}$ is the identity), then we can represent $\mathfrak{g}$ itself in $C^\infty(Y)$. Usually, however, there is an obstruction to such a splitting in $H^2(\mathfrak{g},\mathbb{R})$. Fortunately, for the Poincaré group $E(M)$ and for any semi-simple Lie group this obstruction vanishes, and the lifting can always be made. For the Galilean group, however, an extension is necessary, the mass of the particle appearing as the obstruction to the lifting. (See O'Raiffeartaigh in [2].)

Example. $G = SO(3)$. Since G is semi-simple, the lifting to a map $so(3) \to C^\infty(Y)$ can always be made. If L_j $(j=1,2,3)$ generate rotations about the three axes in $\mathbb{R}^3$ then the corresponding functions ϕ^{L_j} $(j=1,2,3)$ in $C^\infty(Y)$ are called the angular momentum functions.

Observables in classical mechanics are identified with C^∞-functions on Y. They can thus be constructed from the Lie algebra in much the same way as in quantum mechanics. From the Poincaré group generators one obtains energy, linear momentum and angular momentum observables.

Example. Although many groups have to be extended before the homomorphism from g to $Ham(Y)$ can be lifted to $C^\infty(Y)$, every group has some actions on symplectic manifolds which permit a lifting without the need for any extension. We give an example of this due originally to Kirillov.

Any Lie group G acts via the adjoint representation on its Lie algebra g. If g^* is the vector space dual of g then G acts on g^* by the coadjoint action, that is by the transpose of the adjoint action. This action allows us to decompose g^* into orbits under G. Kirillov showed that each of these has a natural symplectic structure. In fact X and W in g give rise to vector fields ξ^X and ξ^W on $\mathcal{O}$ by the general construction given above. At a point f in $\mathcal{O} \subseteq g^*$ one defines $\omega^{\mathcal{O}}$ by

$$\omega_f^{\mathcal{O}}(\xi^X, \xi^W) = \langle f, [X,W] \rangle \quad .$$

It is easy to check that this is well-defined (that it depends only on ξ^X and ξ^W and not on X and W). It can also be shown that $\omega^{\mathcal{O}}$ is a non-degenerate two-form on $\mathcal{O}$, and moreover the Jacobi identity in forces it to be closed. Consequently, $(\mathcal{O}, \omega^{\mathcal{O}})$ is a symplectic manifold.

But also for each X in g one can define $\phi^X \in C^\infty(\mathcal{O})$ by $\phi^X(f) = \langle f,X \rangle$ for all f in $\mathcal{O} \subseteq g^*$. One can check quite easily that this provides a lifting of the map from g to $Ham(\mathcal{O})$.

We shall return to this example shortly.

In quantum mechanics we identified the elementary relativistic systems as those associated with irreducible unitary representations of $E(M)$. For similar reasons we can think of an elementary relativistic system in classical mechanics as one associated with a transitive action of $E(M)$ on a symplectic manifold (Y,ω).

The problem analogous to classifying all the irreducible unitary representations of a group G is thus to find the possible symplectic manifolds on which G acts transitively. (Not all of these have the form $Y = T^*X$.) This problem has been solved completely by Kostant and Souriau ([4] and [9]).

Suppose that G acts transitively on Y and that it is possible to find a homomorphism from $\mathfrak{g}$ to $C^\infty(Y)$. We shall write ϕ^X for the image of X in $\mathfrak{g}$ under this map. Then for each y in Y the map sending X to $\phi^X(y)$ is a linear functional, $\phi^\cdot(y)$. We thus have a map $\phi^\cdot$ from Y to $\mathfrak{g}^*$ which sends y to $\phi^\cdot(y)$. This is called the *moment map*. If we let G act on $\mathfrak{g}^*$ by the coadjoint action of the previous example then the moment map commutes with the action of G. In fact it maps Y onto an orbit of G in $\mathfrak{g}^*$, and even maps the symplectic structure on Y onto the Kirillov symplectic structure on the orbit.

Theorem. If (Y,ω) is a symplectic manifold on which G acts transitively and which permits a lifting of the action to a homomorphism from $\mathfrak{g}$ to $C^\infty(Y)$, then there is a covering map from (Y,ω) to $(\mathcal{O},\omega^{\mathcal{O}})$ for some coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$, which is a G-map.

In those cases where it is necessary to make a central extension $\mathcal{E}$ of $\mathfrak{g}$ before the lifting $\mathfrak{g} \to C^\infty(Y)$ exists the conclusion is that every homogeneous symplectic G-manifold covers a G-orbit in $\mathcal{E}^*$.

As a result of this theorem the elementary relativistic systems in classical mechanics can be found by classifying the possible coadjoint orbits of G in $\mathfrak{g}^*$.

As before we let $\mathfrak{z}$ be the centre of the universal enveloping algebra $U_{\mathfrak{g}}$. The elements of $\mathfrak{z}$ can be identified with G invariant symmetric elements of $U_{\mathfrak{g}}$, and so with G-invariant polynomials on $\mathfrak{g}^*$. The level sets of these polynomials (where they take constant values) pick out G-invariant subsets of $\mathfrak{g}^*$. These must necessarily be unions of G-orbits in $\mathfrak{g}^*$, and in the most favourable case may be single orbits.

Example. To obtain the elementary relativistic systems in classical mechanics we must classify the orbits of the Poincaré group in the dual of its Lie algebra.

We saw earlier that the centre $\mathfrak{z}$ of $U_{\mathfrak{g}}$ is in this case spanned by the two elements $Z_1 = P.P$ and $Z_2 = W.W$ (where $W = *(L \wedge P)$).

We now pick $m > 0$ and $s > 0$ and look for the variety in $e(M)^*$ on which Z_1 and Z_2 take the values m^2 and $m^2 s^2$ respectively. This level set turns out to be an 8-dimensional submanifold of $e(M)^*$ which is the union of two orbits. (On one orbit the polynomial defined by $P.e_o$ is positive and on the other it takes negative values.) The topology of each orbit is that of $\mathbb{R}^6 \times S^2$. If one insists that the orbits be integral orbits (see the lectures of Kostant) then s has to be half-integral just as in quantum theory. These orbits are the nearest one can get to a phase-space of a classical relativistic particle of mass m and spin s .

5. Quantisation

Up till now we have treated classical mechanics and quantum mechanics separately, but there are methods for relating them. Geometric quantisation takes a symplectic manifold (Y,ω) such as the classical mechanical phase space and constructs from it a Hilbert space in the way outlined below.

(i) A line bundle $L \to Y$ is chosen having both a hermitian structure and a connexion ∇ whose curvature form is ω. This is possible if and only if ω defines an integral class in the de Rham cohomology.

(ii) An involutive subbundle F of $T_{\mathbb{C}}(Y)$ is chosen, which is isotropic with respect to ω, and which has half the dimension of Y. Such an F is called a *polarisation*.

(iii) A Hilbert space structure is found for the sections of L which are autoparallel along F (that is $\{s \in \Gamma(L) : \nabla_\xi s = 0 \ \forall \xi \in F\}$). Structures of this kind which are closely related to the hermitian structure on L do exist.

This type of construction is motivated partly by quantum mechanics, partly by the representation theory of compact Lie groups, and partly by the Kirillov method for nilpotent Lie groups.

In representation theory (Y,ω) is an orbit of G in $\mathfrak{g}^*$, and F is chosen to be a G-invariant foliation. Such polarisations can be found by choosing certain Lie subalgebras of the complexification of $\mathfrak{g}$. (See Kostant's discussion of solvable Lie groups.)

Example. From an elementary relativistic classical system of mass m and spin s it is possible in this way to construct the Hilbert space of the corresponding elementary quantum system.

When applied to the phase-space of a general classical mechanical system this procedure is closely related to the quantisation methods used by physicists, and the Schrodinger equation can be derived from it.

Example. $Y = T^*(X)$, $\omega = d\alpha$ is exact. The cotangent spaces are themselves the leaves of a suitable polarisation. Sections autoparallel along them can be represented by their values on the zero section, that is, on X itself. If one chooses a bundle L which is trivial then one obtains the Hilbert space $L^2(X)$, and this gives the usual quantisation.

Bibliography

[1] V.Bargmann: *J. Mathematical Phys*. 5, 1, 1964.

[2] V.Bargmann (ed.) Battelle Rencontre on Group Representations in Mathematics and Physics (especially articles by L.O'Raiffeartaigh and L.Michel). Lecture Notes in Physics No.6. Springer, Berlin, 1970.

[3] F.Dyson (ed.) Symmetry groups in nuclear and particle physics (especially the reprints of the papers of Wigner and of O'Raiffeartaigh). Benjamin, N.Y., 1966.

[4] B.Kostant: Quantization and Unitary Representations, in Lecture Notes in Mathematics, No.170. Springer, Berlin, 1970.

[5] G.W.Mackey: Mathematical Foundations of Quantum Mechanics. Benjamin, N.Y., 1963.

[6] G.W.Mackey: Induced Representations of Groups and Quantum Mechanics. Benjamin, N.Y., 1968.

[7] D.J.Simms and N.M.J.Woodhouse: Lectures on Geometric Quantisation, Lecture Notes in Physics, No.53. Springer, Berlin, 1976.

[8] D.J.Simms: Lie Groups and Quantum Mechanics, Lecture Notes in Mathematics, No.52. Springer, Berlin, 1968.

[9] J.-M.Souriau: Structures des Systèmes Dynamiques. Dunod, Paris, 1970.

[10] V.Varadarajan: Geometry of Quantum Theory, Vol.II. Quantum Theory of Covariant Systems. Van Nostrand-Reinhold, N.Y., 1970.

[11] E.P.Wigner: Group Theory. Academic Press, London and N.Y., 1959.

7 · The Harish-Chandra character

M.F. ATIYAH
Mathematical Institute, Oxford
(Notes by G. Wilson)

In this lecture we shall give an elementary introduction to the idea of the character of an infinite-dimensional representation.

Recall first that if $T: G \to GL(V)$ is a representation of G on a finite-dimensional vector space V, one defines the character of T by

$$\chi_T(g) = \text{trace } T_g , \quad g \in G .$$

From the theory of finite groups one knows that this is an extremely useful object, and one would naturally like to construct something like it in the infinite-dimensional case. To get a clue as to how this might be done, consider the special case of the regular representation, that is, $V = L^2(G)$ and G acts by translation. For G finite it is easy to compute the character of this representation: we have

$$\chi_{reg}(g) = 0 \quad \text{if} \quad g \neq e$$
$$\chi_{reg}(e) = \dim V = |G| .$$

Thus for non-finite G we might expect the character of the regular representation to be zero away from e, and infinite at e, that is, to be the Dirac delta 'function'.

To see how to make sense of this we consider the case of the circle group $G = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. From the theory of Fourier series we know that $L^2(G)$ has an orthonormal basis $\{z^n\}_{n \in Z}$. If $g = e^{i\theta}$ is an element of G, the action of g on the basis is given by $T_g z^n = e^{in\theta}.z^n$, so that the matrix of T_g is an (infinite) diagonal matrix $\mathrm{diag}(e^{in\theta})_{n\in Z}$. Thus trace $T_g = \sum_{-\infty}^{\infty} e^{in\theta}$. Of course this series does not converge in the usual sense; but it does converge 'weakly', that is in the sense of distribution theory. For if $f \in C^{\infty}(G)$, it is well-known that the Fourier series $\sum a_n e^{in\theta}$ of f converges absolutely (where $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\, d\theta$), so that

$$\frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_0^{2\pi} f(\theta)e^{in\theta}\, d\theta = \sum_{-\infty}^{\infty} a_n = f(0) = \delta_o(f) \quad ,$$

δ_o being the Dirac distribution at the identity.

Another way of formulating this is as follows. Recall that for G finite one usually extends a representation T of G to a representation of the group algebra, consisting of formal sums $f = \sum f_i g_i$, $f_i \in \mathbb{C}$, $g_i \in G$: one sets $T_f = \sum f_i T_{g_i}$. If one thinks of f as a function on G, the generalization to the non-finite case is clear: we set $T_f = \int_G f(g) T_g dg$ (here we integrate with values in the Banach space of bounded linear operators on the space of the representation). In the case of the regular representation of $G = S^1$ we have $T_f = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, T_\theta\, d\theta$; thus the matrix of T_f with respect to our basis $\{z^n\}$ is $\mathrm{diag}(a_{-n})$, where a_n are the Fourier coefficients of f. The trace $\sum a_n$ is finite for $f \in C^{\infty}(G)$, so we can define the character to be the distribution $f \to$ trace T_f.

Now consider an arbitrary unitary representation $T : G \to U(\mathcal{H})$, $G = S^1$. Then $\mathcal{H}$ decomposes $\mathcal{H} = \ \mathcal{H}_n$, where $\mathcal{H}_n$ is the subspace on which $e^{i\theta} \in G$ acts as scalar multiplication by $e^{in\theta}$. Suppose each $\mathcal{H}_n$ is finite-dimensional, say $\dim \mathcal{H}_n = M_n$. Then we should like to define the character of T to be the distribution $\chi_T = \sum M_n e^{in\theta}$. To see when this is in fact a distribution, note first that a series $\sum a_n e^{in\theta}$ is the Fourier series of a C^{∞} function if and only if we have $a_n = o(n^k)$ for all integers k (for we must be able to differentiate the series term by term arbitrarily often). It follows easily that the series $\sum M_n e^{in\theta}$ will converge as a distribution if and only if $M_n = O(n^k)$ for some k, that is, if the M_n have (at worst) 'polynomial growth' with respect to n. Whenever this is the case, then, we can define the character of our representation as a distribution on S^1.

Next we indicate how to generalize this to any compact Lie group K. Again consider first the regular representation $L^2(K)$. The Peter-Weyl theorem states that

$$L^2(K) \cong \sum_{\lambda \hat{K}} V_\lambda \otimes V_\lambda^* \quad .$$

Here as usual $\hat{K}$ is the (countable) set of isomorphism classes of irreducible representations of K (they are all finite-dimensional) and V_λ is a fixed representation in the class λ. The above is really a decomposition as $K \times K$-spaces (K acts by both left and right translations on $L^2(K)$, on the left on V_λ and on the right on V_λ^*). If we consider just the left action of K on $L^2(K)$, we get

$$W = L^2(K) = \sum_{\lambda \in \hat{K}} d_\lambda \cdot V_\lambda \quad ,$$

where $d_\lambda = \dim V_\lambda$. The character is given by $\chi_W = \sum d_\lambda \cdot \chi_\lambda$, where χ_λ is the character of V_λ; as for $K = S^1$, this sum converges as a distribution on K to the Dirac distribution δ_e at the identity.

If $T: K \to U(H)$ is now any representation of K, we can decompose it as $H = \sum_{\lambda \in K} H_\lambda$, where H_λ is the subspace of H on which K acts as a direct sum of copies of V_λ. We assume all the H_λ are finite-dimensional, so that we have $H = \sum M_\lambda V_\lambda$, M_λ some non-negative integers. By analogy with the case of S^1 we expect the sum $\sum M_\lambda \chi_\lambda$ to converge to a distribution on K provided the M_λ do not grow too fast in some sense. A convenient yardstick for measuring the growth of M_λ is provided by the Laplacian. Recall that on any compact Riemannian manifold X we have the Laplace operator Δ: it is an elliptic self-adjoint second order differential operator with eigenvalues $0 \le \mu_1 \le \mu_2 \le \ldots$. (On S^1 we have $\Delta = -\frac{d^2}{d\theta^2}$.) We can write any C^∞ function f on X as a convergent sum $f = \sum a_i \Psi_i$, Ψ_i an eigenfunction of Δ with eigenvalue μ_i ; and a formal sum $\sum M_i \Psi_i$ will converge to a distribution on X if and only if $M_i = O(\mu_i^k)$ for some k . In the case when $X = K$ with a bi-invariant metric, Δ is a bi-invariant operator on K , hence (Schur's lemma) Δ acts on each representation V_λ as scalar multiplication by some number $\mu(\lambda) \ge 0$. All the matrix coefficients of V_λ , hence also the character χ_λ , are eigenfunctions of Δ corresponding to the eigenvalue $\mu(\lambda)$. Thus the sum $\sum M_\lambda \chi_\lambda$ will converge to a distribution provided that $M_\lambda = O(\mu(\lambda)^k)$ for some k . When that is the case we define this distribution to be the character of the representation $\sum M_\lambda V_\lambda$.

Remark If K is semi-simple there are other equivalent ways to measure the growth of M_λ . Instead of the eigenvalues $\mu(\lambda)$ we can use the degree $d(\lambda)$ of the irreducible representation V_λ , or the norm $|\lambda|$ of the maximal weight vector.

Finally we consider the case of a non-compact Lie group G . Let $K \subset G$ be a compact subgroup. We have in mind principally the case where G is semi-simple and K is a maximal compact subgroup, e.g. if $G = SL(2,\mathbb{R})$ then $K = SO(2)$. Let $T: G \to U(H)$ be a unitary representation of K , so that H splits up as before: $H = \sum_{\lambda\in K} M_\lambda H_\lambda$. Thus if the M_λ are finite and grow slowly as explained above we have the 'K-character' $\sum M_\lambda \chi_\lambda$. This is a distribution on G . By restriction, it can be viewed as a representation of K , which is not good enough for many purposes. For example when $G = SL(2,\mathbb{R})$, $K = SO(2)$ and its conjugates give the elliptic elements of G , but miss the open set of hyperbolic elements (real eigenvalues). A 'good' character should deal with all conjugacy classes. We wish therefore to define the character as a distribution on G. For this we go back to the second description of the character that we gave for $G = S^1$. Let $f \in C_o^\infty(G)$ (C^∞ with compact support). Then we have the operator $T_f = \int_G T_g f(g)dg$. If we could take the 'trace' of this operator, we could try to define the character to be the distribution $f \to \text{trace } T_f$.

Now the trace can be defined only for a restricted class of operators on H (called 'trace class' operators). Let $\{e_i\}$ be an orthonormal basis for H; then an operator $A: H \to H$ is said to be trace class provided that we have $\sum |\langle Ae_i, e_i\rangle| < \infty$ for all bases $\{e_i\}$ (i.e. the sum of the diagonal elements in the infinite matrix representing A is absolutely convergent). When this is so, the sum $\sum \langle Ae_i, e_i\rangle$ is independent of the choice of basis, and is defined to be the trace of A. Thus we need to know when our operators T_f are trace class. The main results are as follows.

Theorem 1. With the above notation, suppose we have $M_\lambda = O(d(\lambda)^k)$ for some k. Then for every $f \in C_o^\infty(G)$, the operator T_f is trace class, and the map $f \to \text{trace } T_f$ is continuous, that is, it is a distribution on G.

Theorem 2. Let G be connected semi-simple with finite centre, $K \subset G$ a maximal compact subgroup. Then for every irreducible unitary representation of G we have $M_\lambda \leq d(\lambda)$.

Putting the two theorems together we get:

Corollary. If G is connected semi-simple with finite centre, then every irreducible unitary representation of G has a character which is a distribution on G.

It is natural to ask how bad a distribution the character can be. It would be most convenient if it were simply a function on G , as in the finite-dimensional case. (Recall that a locally L^1 function Ψ on G is identified with the distribution $f \to \int_G f(g)\Psi(g)dg$.) For general G this is not so: for example if G is a semi-direct product G = H Ⓢ N with H compact and N abelian, then, as explained in Mackey's lectures, G has irreducible representations that are induced from one-dimensional representations of the normal subgroup N . These representations have distribution characters in the sense explained above; and one can show that (as for finite groups) these characters are supported on N . Thus they can certainly not be any kind of function on G .

In the semi-simple case, however, one has the following theorem of Harish-Chandra.

Theorem. For G connected semi-simple with finite centre, the character of every irreducible unitary representation is a locally L^1 function on G . In fact the function is real analytic except on a 'singular set' of measure zero.

8 · Representations of semi-simple Lie groups

W. SCHMID
Columbia University
(Notes by B.F. Steer)

1. A natural start is to discuss the Peter-Weyl theorem. Let G be a compact topological group with normalized Haar measure dg. Let $\hat{G}$ denote the set of isomorphism classes of irreducible unitary representations. As G is compact each of these is finite dimensional and we choose representatives (π_i, V_i), where V_i is a (finite dimensional) Hilbert space and $\pi_i: G \to \mathrm{Aut}(V_i)$, for each class $i \in \hat{G}$. Let $L^2(G)$ be the space of L^2 functions on G with respect to the measure dg and let $\hat{\oplus}$ denote the Hilbert space direct sum.

Theorem 1.1 (Peter-Weyl) $L^2(G) \cong \hat{\oplus}_{i \in \hat{G}} V_i \otimes V_i^* \cong \hat{\oplus}_{i \in \hat{G}} \mathrm{Hom}(V_i, V_i)$.

This isomorphism respects the action of G on the left (left translation of functions in $L^2(G)$ and action on the left factor V_i) and on the right. It may be made explicit. To do this, define for each $i \in \hat{G}$ a map

$$\pi_i : L^2(G) \to \mathrm{Hom}(V_i, V_i)$$

by the formula

$$\pi_i(f) = \int_G f(g)\pi_i(g)dg \quad , \quad f \in L^2(G)$$

The map $f \to (\pi_i(f))_{i \in \hat{G}}$ is the isomorphism above. It is a Hilbert space isomorphism if we renormalize the natural inner product $\langle \cdot, \cdot \rangle_i$ on $V_i \otimes V_i^*$ by the factor $\dim V_i = \text{degree } i$.

Understanding the representations of G is thus equivalent to understanding $L^2(G)$. It is a very general theorem and does not tell you what the V_i, $i \in \hat{G}$, are. To get more information one must specify G or restrict the class of G to be considered.

If G is a compact Lie group then one can enumerate the irreducible representations by their highest weights. These are lattice points lying in a cone: a closed Weyl chamber which is chosen at the beginning. The Weyl character formula determines the character of the representation in terms of the highest weight. Now the character is, up to a definite constant $(\dim V_i)$, the inverse image of the identity map $I \in \mathrm{Hom}(V_i, V_i)$ under the Peter-Weyl isomorphism. Hence each V_i, and so the class of any irreducible representation, is determined by its character. So one has not just a parametrization of the irreducible representations but a good understanding of their structure. For a compact Lie group the Peter-Weyl theorem thus gives very nearly complete control on $L^2(G)$. It has, too, the following corollary which we have just used.

Corollary 1.2. Any two irreducible representations (of a compact topological group) are equivalent if and only if they have the same character.

Let G now be a *locally compact unimodular* group of *type I*, and let $\hat{G}$ denote the set of isomorphism classes of *irreducible unitary representations*. As before, let (π_i, V_i), $i \in \hat{G}$, be a set of explicit representatives. 'G of type I' is just the hypothesis required to make the Abstract Plancherel Theorem hold. It is significant that a large class of groups - all semi-simple Lie groups, for example - are of type I.

Theorem 1.3 (Abstract Plancherel Theorem). If G is unimodular and of type I then $L^2(G) \cong \int_{\hat{G}} V_i \otimes V_i^* \, d\hat{\mu}(i)$, where:

(a) $\otimes$ denotes the completed tensor product;

(b) $\hat{\mu}$ is a certain positive measure on $\hat{G}$ - the Plancherel measure;

(c) the direct integral of the Hilbert spaces is a Hilbert space of which the elements assign to each $i \in \hat{G}$ an element in $V_i \otimes V_i^*$. This assignment is required to be measurable and square-integrable with respect to $\hat{\mu}$. (One may think of the direct integral as the space of L^2 sections of a vector bundle in the measure category.)

If G is compact one sees that $\hat{\mu}(i)$ = degree i and so $\hat{\mu}$ is performing the renormalization we needed to make the Peter-Weyl isomorphism one of Hilbert spaces.

As in the Peter-Weyl theorem the isomorphism may be made explicit, at least on a dense subset of $L^2(G)$. First notice that $V_i \otimes V_i^*$ is isomorphic to the algebra of Hilbert-Schmidt operators on V_i : we shall write the latter $\mathrm{Hom}_{HS}(V_i)$. Now define $\pi_i(f)$, $i \in \hat{G}$, formally as before: namely

$$\pi_i(f) = \int_G f(g)\pi_i(g)dg \quad .$$

For an arbitrary $f \in L^2(G)$ there is no reason why this integral should make sense. But if $f \in L^1(G)$ then the integral converges for all $i \in \hat{G}$. The analytical assertion above is that if $f \in L^1(G) \cap L^2(G)$ then $\pi_i(f)$ is Hilbert-Schmidt for almost all $i \in \hat{G}$ and $\int_{\hat{G}} \|\pi_i(f)\|_{HS}^2 \, d\hat{\mu} = \|f\|_2^2$. Hence there is a unique continuous extension to $L^2(G)$.

Example 1.4. $G = R$. Then $L^2(R) \cong \int_R V_y \, dy$, where dy is Lebegue measure and V_y is the 1-dimensional representation space spanned by $x \to e^{ixy}$ (a function not in L^2 !)

Remark 1.5. As in the compact case the map is compatible with the group actions on the right and on the left.

Remark 1.6. The Abstract Plancherel Theorem is not as easy to prove as the Peter-Weyl theorem. By itself it does not tell you anything for a particular, concrete, group. For a general locally compact group it is totally inexplicit but it can be very useful indeed for a special class of groups: for example, semi-simple Lie groups.

Henceforth G will be a *connected semi-simple* Lie group with *finite centre*. (The latter condition is probably not essential, but does avoid several unpleasant technical complications.) Harish-Chandra's programme was to determine the various ingredients of the Abstract Plancherel Theorem (e.g. $\hat{G}$, the Plancherel measure $\hat{\mu}$, together with some structural information about the irreducible representations) and to find, for a non-compact semi-simple group, an analogue of the Weyl character formula.

There are two comments to be made. First, in the non-compact case there is not such a tight relation between the irreducible representations and $L^2(G)$ as there is in the compact case. The Plancherel measure $\hat{\mu}$ may miss large subsets of $\hat{G}$ so that to make $\hat{\mu}$ explicit is a very different matter from determining $\hat{G}$ completely. At the moment the Plancherel measure (and so enough of $\hat{G}$ to carry that measure) is, in principle, completely determined; but $\hat{G}$ is not so determined. Here is an example to illustrate these differences. Let $\Gamma \subset G$ be a *cocompact discrete* subgroup. Then

$$L^2(\Gamma \setminus G) \cong \hat{\bigoplus_{i \in \hat{G}}} r_i \pi_i \quad ,$$

where $r_i \in \mathbb{N}$. Such Γ and this decomposition are of interest in a number of arithmetic and geometric questions. (For example, in the case that Γ is a fuchsian group once this decomposition is known so is the spectrum of the Laplacian [26, last chapter].) Now in $L^2(\Gamma \setminus G)$ representations which occur do so discretely so that one cannot leave any given one out. Moreover, as far as we know any given π_i, $i \in \hat{G}$, may occur with non-zero multiplicity in $L^2(\Gamma \setminus G)$ for some Γ. Only partial answers are yet known for the decomposition of $L^2(\Gamma \setminus G)$ whereas, as we have said, the Plancherel measure is, in principle, completely determined.

The second comment is that, although we are really interested in unitary representations, certain constructions will lead us outside the category of unitary representations on Hilbert spaces. We shall need to consider representations on Banach spaces. A representation $\pi: G \to \mathrm{End}(V)$ on a Banach space V is a homomorphism such that the associated map $G \times V \to V$ is continuous. Normally we shall want to consider (topologically) irreducible ones: that is those containing no proper closed invariant subspace. A price has to be paid if one wishes so to extend one's considerations. It is this. Frequently, indeed typically, one can find representations (π_i, V_i), $i = 1,2$, and a continuous G-invariant map $f: V_1 \to V_2$ which has dense image yet is not surjective. This may be interpreted as suggesting that on Banach spaces there are representations which differ essentially only in the choice of topology. We do not wish to distinguish representations which differ only in topology and need a *precise* criterion of when two Banach space representations should be identified. The associated *infinitesimal representation* of the Lie algebra $\mathfrak{g}$ is used to give such a criterion. In finite dimensions a continuous representation is necessarily analytic and hence one obtains a representation of the Lie algebra $\mathfrak{g}$ upon differentiating. That representation determines the original one (of G) and every representation of $\mathfrak{g}$ lifts to a representation of a covering of G, if not to G itself. In infinite dimensions the definition is more intricate.

Definition 1.7. $v \in V$ is an *analytic vector*† for the Banach space representation $\pi: G \to \mathrm{End}(V)$ if and only if the map $G \to V$ defined by $g \to \pi(g).v$ is a real analytic map of G, considered as a real analytic manifold, into V.

Let V_ω denote the space of analytic vectors of V for the representation π. Differentiating we find a representation of $\mathfrak{g}$, the Lie algebra of G, on V_ω.

Theorem 1.8 (Harish-Chandra). If G is a semi-simple Lie group and $\pi: G \to \mathrm{End}(V)$ a Banach space representation then V_ω is dense in V.

This theorem has been extended to arbitrary Lie groups. The idea of the proof is not difficult, but the details are. To see the idea consider the corresponding result for smooth vectors. Let V_s denote the space of smooth vectors. Take a compactly supported smooth function f on G. Define $\pi f \in \mathrm{End}(V)$ by

$$\pi(f) = \int_G f(g)\pi(g)dg \quad ,$$

as earlier. Then $\pi(f)(V) \subset V_s$, for essentially formal reasons. Now take a sequence (f_n), $n \in \mathbb{N}$, of smooth compactly supported functions, approximating the Dirac δ-function at 1. Then $\pi(f_n)(v)$ approximates v. So as to do the same thing for analytic vectors one must approximate the δ-function by analytic functions which die fast at infinity. This can be done using the heat operator. We do need the result for analytic vectors, not just smooth vectors, as we

†Godement in [11] uses 'weakly analytic vectors' which satisfy $g \to \alpha(\pi(g).v)$ analytic for all $\alpha \in V^*$. They are easier to handle.

shall shortly see.

Let $\mathfrak{g}^C = \mathfrak{g} \otimes_R C$ denote the complexified Lie algebra of G.

Definition 1.9. The universal envelopping algebra $\mathfrak{u}(\mathfrak{g}^C)$ of $\mathfrak{g}^C$ is the smallest associative algebra containing $\mathfrak{g}^C$ as a Lie subalgebra.

It may be defined as $T(\mathfrak{g}^C)/\{x\otimes y - y\otimes x - [x,y] ; x,y \in \mathfrak{g}^C\}$, where $T(W)$ denotes the tensor algebra of W.

Given a representation π of G on V one gets one of $\mathfrak{g}$ on V_ω, the analytic vectors, so one of $\mathfrak{g}^C$ and hence one, which also denote by π, of $\mathfrak{u}(\mathfrak{g}^C)$ on V_ω. (We use the same symbol because, for a large class of representations, one determines the other.) We should like to say that (a) $\pi: \mathfrak{u}(\mathfrak{g}^C) \to \mathrm{End}(V_\omega)$ determines $\pi: G \to \mathrm{End}(V)$ uniquely, and that (b) $\pi: \mathfrak{u}(\mathfrak{g}^C) \to \mathrm{End}(V_\omega)$ is irreducible if $\pi: G \to \mathrm{End}(V)$ is. As we shall see (and have noted) the first is true for a large class of representations; but the second is false. The ideas involved in overcoming this difficulty are crucial to the whole development.

Let K be a *maximal compact* subgroup of G. According to Cartan's theorem [see, e.g. 23] any two are conjugate. Moreover, such a K is connected and $N_G K = K$. Let $\hat{K}$ be the set of isomorphism classes of irreducible representations of K. Let $\pi: G \to \mathrm{End}(V)$ be a Banach space representation of G. For each $i \in \hat{K}$, let

$$V_i = \oplus \{W \subset V : W \text{ is K-invariant and K-irreducible of type } i\}$$

$$= \text{K-invariant subspace of } V \text{ of type } i \quad .$$

Definition 1.10. π is *admissible* <=> $\dim V_i < \infty \ \forall\, i \in \hat{K}$.

In future chapters we shall consider only admissible representations. For irreducible unitary representations this is no loss for such are always admissible; we shall note this formally in a moment. Whether it is a definite restriction for irreducible Banach space representations is still unknown. Now let us return for a little to the general situation.

Definition 1.11. $V_\infty = \bigoplus_{i \in \hat{K}} V_i$ is the space of K-finite vectors.

Notice that V_∞ is dense in V by the theorem of Mostow. But not only is this so, $V_\infty \subset V_\omega$. This is a corollary of one proof of theorem 1.8. A map $\mathfrak{g} \otimes_{\mathbb{R}} V_\infty \to V_\omega$ and $\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} V_\infty \to V_\omega$ is thus defined, and it is not hard to see that V_∞ is stable under $\mathfrak{g}^{\mathbb{C}}$; for if U is a finite-dimensional K-invariant subspace so is the image of $\mathfrak{g}^{\mathbb{C}} \otimes_{\mathbb{C}} U$ under the map $x \otimes u \to \pi(x)u$, which is equivariant. So there is attached to $\pi\colon G \to \mathrm{End}(V)$ a representation of $\mathfrak{u}(\mathfrak{g}^{\mathbb{C}})$ on V_∞. This is called the *infinitesimal representation* associated to π, The criterion we were looking for is the following; it gives a precise meaning to the informal idea that two representations differ only in a choice of topology.

Definition 1.12. Two representations (π_i, V_i), $i = 1,2$, are *infinitesimally equivalent* <=> the associated infinitesimal representations are algebraically isomorphic.

The *infinitesimal representation* has the advantage that it is *irreducible if* π *was*. For if $\bar{W} \subset V$ is the closure in V of any $\mathfrak{u}(\mathfrak{g}^{\mathbb{C}})$-invariant subspace of V_∞ then $\bar{W}$ is G-invariant and hence $\bar{W} = V$. But V_∞ is the direct sum of finite-dimensional subspaces and hence $W = V_\infty$. This explains both why we have chosen V_∞ and why we have used V_ω: we need analyticity to lift a subrepresentation of $\mathfrak{g}$ to one of G.

What does all this mean for unitary representations? In the first place there is the following theorem, of which clause (a) was mentioned above.

Theorem 1.13.

(a) (π, V) irreducible unitary representation => π is admissible;

(b) (π, V) irreducible admissible Banach space representation => $\dim V_i \leq (\deg i)^2 \ \forall\, i \in \hat{K}$ (that is, i occurs in $\pi|K$ with multiplicity at most its degree).

The second clause is there because one argument does both. The most elementary is that of Godement [11], but it only works when G admits a faithful finite-dimensional (so not unitary in general!) representation. Using the universal envelopping algebra one shows first that $\dim V_i$ is finite. Then one can complete the proof by using the fact that any such may be represented - up to infinitesimal equivalence - as a subrepresentation of a principal series representation.

Finally, here is a theorem which may be used to find unitary representations for 'small' groups, where '*small*' here means of *low real rank*. (Examples are provided by $SL(2,\mathbb{R})$, $SU(n,1)$, $SO(n,1)$.) To do this we determine certain representations and then look to see which admit an invariant pre-Hilbert space structure. Note that (π, V) is an irreducible unitary representation then V_∞ is a pre-Hilbert space and $\mathfrak{g}$ operates via skew-hermitian operators.

Theorem 1.14.

(a) If (π, V) is an irreducible Banach space representation and a pre-Hilbert space structure exists on V_∞ such that $\mathfrak{g}$ acts via skew-hermitian operators, then that structure is unique up to a scalar multiple. Consequently, infinitesimally equivalent implies unitarily equivalent.

(b) If there is a direct sum of finite-dimensional representations of K admitting an extension of the action of $\mathfrak{k}$ to $\mathfrak{g}$ and a pre-Hilbert space structure such that $\mathfrak{g}$ acts via skew-hermitian operators then there is a unique (up to equivalent) unitary representation of G infinitesimally equivalent to the given one of $\mathfrak{g}$.

2. Here we shall discuss the irreducible unitary representations of $SL(2,\mathbb{R})$ using the method suggested at the end of the last lecture. (It was one of the examples given there of a 'small' group.) $SL(2,\mathbb{R}) \cong SU(1,1)$ and it is often simpler to think in terms of $SU(1,1)$. Let G , then, denote

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha,\beta \in \mathbb{C} \ , \ |\alpha|^2 - |\beta|^2 = 1 \right\} \quad .$$

The complexified Lie algebra $\hat{\mathfrak{g}}^{\mathbb{C}}$ has a basis (over $\mathbb{C}$) thus:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ ,$$

where $\bar{Z} = -Z$ and $\bar{X}_+ = X_-$, the bar denoting conjugation in $\mathfrak{g}^{\mathbb{C}}$. The commutation relations are as follows:

$$[Z,X_+] = 2X_+ \ , \quad [Z,X_-] = -2X_- \ , \quad [X_+,X_-] = Z \quad .$$

Let

$$K = \left\{ k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \ ; \quad \theta \in [0,2\pi) \right\}$$

be a maximal compact subgroup. The vector $iZ = d/d\theta$ is tangent to K and so is an infinitesimal generator. Note, for future use, that $\mathrm{ad}(k_\theta)X_+ = e^{2i\theta}X_+$.

Let (π,V) be an irreducible admissible Banach space representation. As before, let V_∞ denote the K-finite vectors. The universal envelopping algebra $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ operates irreducibly on V_∞ , as we noted just after definition 1.12. $K \cong S^1$ so that $\hat{K} \cong Z$, and a particular isomorphism is $n \to (\rho_n, C)$ where $\rho_n(k_\theta)z = e^{ni\theta}z$, $z \in C$. Let $V(n)$ be the subspace of V of type ρ_n . Then, because iZ is an infinitesimal generator for S^1 ,

$$\pi(Z)v = nv \ , \quad v \in V(n) \quad .$$

Thus, from the commutation formulae, we see that

$$\pi(X_\pm)(V(n)) \subseteq V(n\pm 2) \quad .$$

Hence g^C preserves the parity of n . But (π,V) was supposed irreducible. Thus either $V(n) = 0 \ \forall\, n \equiv 0(2)$ or $V(n) = 0 \ \forall\, n \equiv 1(2)$. Call the first the odd case, the second the even case. In the odd case consider the sequence of odd integers for which $V(n) \neq 0$. If there is one odd integer, say q , missing then the pieces on either side of that integer q are invariant since $\pi(X_\pm)V(n) \subseteq V(n\pm 2)$. So irreducibility implies that if one integer q is missing then either $V(j) = 0 \ \forall\, j \leq q$ or $V(j) = 0 \ \forall\, j \geq q$. There is a similar argument in the even case. The possible configurations for an irreducible representation are thus:

(1) Finite interval : $\exists\, m,n \in Z$, $m < n$ & $m \equiv n(2)$ such that $\begin{cases} V(j) \neq 0 & \text{if } m \leq j \leq n \ , \ j \equiv m(2) \\ V(j) = 0 & \text{otherwise.} \end{cases}$

(2) Half-bounded interval, bounded above : $\exists\, m \in Z$ such that $\begin{cases} V(j) \neq 0 & \text{if } j \geq m \ , \ j \equiv m(2) \\ V(j) = 0 & \text{otherwise.} \end{cases}$

(3) Half-bounded interval, bounded below.

(4) Whole line : $\begin{cases} V(j) \neq 0 & \forall\, j \equiv 1(2) \\ V(j) = 0 & \forall\, j \equiv 0(2) \end{cases}$ or vice versa.

The odd and even cases are similar. Now, for definiteness, we shall restrict ourselves to the odd case. First choose $v_n \in V(n)\backslash 0$ for some n. In configurations 1 & 2 take n maximal, in configuration 3 minimal. Consider first configurations 1 & 2 when n is maximal. Set

$$v_{n-2k} = \pi(X_-)^k v_n \in V(n-2k) \quad .$$

Again from irreducibility we see that $V(j)$ is at most one-dimensional and that $v_{n-2k} = 0$ only in configuration 1 and for $n-2k < m$. Thus

$$\pi(X_+)v_{n-2k} = a_k v_{n-2k+2} \quad , \quad a_k \in \mathbb{C} \quad .$$

Because n was maximal $a_0 = 0$. Using the extension of the representation to $\mathfrak{u}(\mathfrak{g}^{\mathbb{C}})$ and the commutation formulae we see further that

$$\begin{aligned} \pi(X_+)v_{n-2} &= \pi([X_+,X_-])v_n + \pi(X_-,X_+)v_n \\ &= \pi([X_+,X_-])v_n \\ &= \pi(2Z)v_n \\ &= 2nv_n \end{aligned}$$

Hence $a_1 = 2n$ and, inductively, one shows that $a_k = 2kn - 2k(k-1)$. It follows that if $n > 0$ then $\bigoplus_{k=-n}^{n} V(k)$ is an invariant subspace. There is an analogous calculation for configuration 3. Using it we reach the following conclusions with respect to configurations 1, 2 and 3.

(1) *Finite interval*. $m = -n$ and for each $n \in \mathbb{N}$ there exists a unique, irreducible representation σ_n such that $V(j) \neq 0 \iff j \in \{-n,-n+2,\ldots,n-2,n\}$.

(2) *Half-bounded interval, bounded above.* For each $n \in \mathbb{N}$ there exists a unique, irreducible representation π_n^- such that $V^-(j) \neq 0 \iff j = -n, -n-2, \ldots$.

(3) *Half-bounded interval, bounded below.* For each $n \in \mathbb{N}$ there exists a unique, irreducible representation π_n^+ such that $V^+(j) \neq 0 \iff j = n, n+2, \ldots$.

Notice that if $m > 0$ in configuration 2 then the representation is an extension of σ_m by π_{m-1}^-, also $(\pi_n^+)^* = \pi_n^-$.

The configuration 4 is more complicated. Here we do not have $a_o = 0$ so we cannot start from there and determine the numbers a_k inductively. Nonetheless, from any particular a_k we may determine the others inductively in terms of that a_k, just as above. Moreover, the sequence of complex numbers (a_k) $k \in \mathbb{Z}$ determines the representation. For this configuration it is convenient to renormalize the choice of generators and select $v_n \in V(n) \setminus 0$ (for all odd n or all even n as the case may be) such that

$$\pi(Z)v_n = nv_n, \quad \pi(X_+)v_n = \frac{(s+1+n)}{2} v_{n+2}, \quad \pi(X_-)v_n = \frac{(s+1-n)}{2} v_{n-2},$$

for some $s \in \mathbb{C}$. (In the even case if we select $v_o \in V(0)$ and compute a_1 as defined above we find $a_1 = s^2-1$.) Now should s be an integer k we see that $\pi(X_+)v_{-k-1} = 0$ so that $\pi(X_+)V_{(-k-1)} = 0$. Hence if we wish for an irreducible representation we must insist that:

$$\begin{cases} s \text{ not an odd integer in the even case;} \\ s \text{ not an even integer in the odd case.} \end{cases}$$

Subject to this assumption the representations are irreducible. In the irreducible case the map $v_n \to \frac{n+s+1}{n+s-1} v'_n$ gives an isomorphism of $\pi_{s,\text{even}}$ with $\pi_{-s,\text{even}}$ and similarly for s odd. Otherwise they are all distinct. The representations for s and $-s$ are dual. If we call these two families of representations $\pi_{s,odd}$ and $\pi_{s,even}$ we then see that they have the following properties.

(1) $\pi_{s,even}$ irreducible <=> $s \not\equiv 1(2)$, $s \in C$,

$\pi_{s,odd}$ irreducible <=> $s \not\equiv 0(2)$, $s \in C$;

and all these irreducible representations are distinct, except for the isomorphisms $\pi_{s,even} \cong \pi_{-s,even}$, $\pi_{s,odd} \cong \pi_{-s,odd}$.

(2) $\pi_{o,odd}$ is the only semi-simple non-irreducible representation of these families. $\pi_{o,odd} \cong \pi_1^+ \oplus \pi_1^-$.

(3) $\pi_{s,even} \cong \pi^*_{-s,even}$ and similarly for $\pi_{s,odd}$.

(4) $\pi_{2k,odd}$, $k \in N\backslash 0$, has the direct sum of π^+_{2k+1} and π^-_{2k+1} as a subrepresentation, with quotient σ_{2k-1}. The extension is nontrivial and the order is inverted by (3) above) for $\pi_{-2k,odd}$. $\pi_{2k-1,even}$ behaves analogously, with composition factors π^+_{2k}, π^-_{2k} and σ_{2k-2}.

(5) The families $\pi_{s,even}$, $\pi_{s,odd}$; $s \in C$ contain all irreducible admissible representations as quotients or subrepresentations.

It remains to decide which of the infinitesimal representations above correspond to unitary representations of G : to do this we must decide whether we can find an inner product on V such that $\mathfrak{g}^C$ operates by skew-hermitian transformations. If so, then Z must be hermitian and so $V(j) \perp V(k)$ if $k \neq j$ since each is an eigenspace for Z. A little more manipulation using the bases above and one finds that only the following irreducible infinitesimal representations admit such pre-Hilbert structures.†

†For example, in configuration 4 one sees quickly that $s^2-(n+1)^2$ must be negative on computing $\langle\pi(X_-X_+)v_n, v_n\rangle$, using the fact that $\pi(X_-X_+)$ must be negative definite.

$$\left.\begin{array}{ll}
(0) & \text{Trivial representation.} \\
(1) & \pi_n^+ \;\&\; \pi_n^- \;;\; n \in N\setminus 0 . \\
(2) & \pi_{s,even} \;;\; s \in iR \;:\; \pi_{s,even} \cong \pi_{-s,even} \;; \\
 & \pi_{s,odd} \;;\; s \in iR\setminus 0 \;:\; \pi_{s,odd} \cong \pi_{-s,odd} \;. \\
(3) & \pi_{s,even} \;;\; s \in R\setminus 0 \;,\; |s| < 1 \;.
\end{array}\right\} \text{List 2.1}$$

According to Theorem 1.14(b) one may realize all these infinitesimal representations by unitary representations of SL(2,R), but to understand them well we must understand the geometric realizations. References for this are Bargmann [4] and Lang [26], as for the preceding argument. One may also refer to [10]. We shall not discuss the finite non-unitary representations (σ_n), $n \in N\setminus 0$, which all come from the restriction of algebraic representations of SL(2,C).

We shall treat the three classes 1, 2, 3 of list 2.1 one at a time. First we take class 1. Let Δ be the open unit disc in C. The group SU(1,1) operates on it by linear fractional transformations. (In the SL(2,R) picture we should have to take H the upper half-plane. It is more complicated.) Fix an integer $n > 1$ and define a space H_n of holomorphic functions as follows

$$H_n = \{f\colon \Delta \to C \mid f \text{ holomorphic and}$$
$$\int_\Delta (1-|z|^2)^n |f(z)|^2 \frac{dxdy}{(1-|z|^2)^2} < \infty\} \quad .$$

With the obvious inner product,

$$\langle f,g\rangle = \int_\Delta (1-|z|^2)^n\, f(z)\bar{g}(z)\, \frac{dxdy}{(1-|z|^2)^2} \quad ,$$

H_n forms a non-zero Hilbert space. (The form $dxdy/(1-|z|^2)^2$

defines an invariant measure on Δ , unique up to constant.)
On H_n the group acts according to the rule

$$(\pi_n^-(g)f)(z) = (\alpha-\bar{\beta}z)^{-n}f(\frac{\bar{\alpha}z-\beta}{\alpha-\bar{\beta}z}) = (\alpha-\bar{\beta}z)^{-n}f(g^{-1}z)$$

if $g = \begin{pmatrix}\alpha, \beta \\ \bar{\beta}, \bar{\alpha}\end{pmatrix}$ so that $g^{-1} = \begin{pmatrix}\bar{\alpha}, -\beta \\ -\bar{\beta}, \alpha\end{pmatrix}$. Now $z^k \in H_n$, $k \in \mathbb{N}$, and $\pi_n^-(k_\theta)z^k = e^{-(n+2k)i\theta}z^k$; where $k_\theta = \begin{pmatrix}e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}$.
Hence (z^k) , $k \in \mathbb{N}$, are orthogonal vectors in H_n and they clearly form a complete basis. Moreover, they are trivially K-finite vectors and - as we can see directly here - analytic. Thus (π_n^-, H_n) is a realization of the infinitesimal equivalence class π_n^- . As we have noted $\pi_n^+ = (\pi_n^-)^*$ and is realized on H_n by the contragredient representation to (π_n^-, H_n) : $((\pi_n^+ g)f)(z) = (\bar{\alpha}+\beta z)^{-n}f(gz)$. These representations are called the *discrete series* representations; initially because they depended on a discrete parameter but very soon afterwards for a more fundamental reason which we shall see. That reason explains why $\pi_1^{\pm}$ are not regarded as forming part of the discrete series. (These representations can be realized on H_1 = {f:
$H_1 = \{f : \Delta \to \mathbb{C} \mid f$ holomorphic and

$$\lim_{\varepsilon\to 0} \varepsilon \int_\Delta (1-|z|^2)^{1+\varepsilon} |f(z)|^2 \frac{dxdy}{(1-|z|^2)} < \infty\} \quad .)$$

The representations of class 2 of list 2.1 are called the *principal* or *continuous* series. To construct realizations of these we shall think directly in terms of $SL(2,\mathbb{R})$ instead of $SU(1,1)$. The subgroups K,A,N of the Iwasawa decomposition $G = KAN$ are, in the case of $SL(2,\mathbb{R})$, as follows:

$$K = \{k_\theta = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}, \quad 0 \le \theta < 2\pi\} \quad ;$$

$$A = \{a_t = \begin{pmatrix}e^t & 0 \\ 0 & e^{-t}\end{pmatrix}, \quad t \in \mathbb{R}\} \quad ;$$

$$N = \{n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \quad .$$

The principal series consists of those irreducible unitary representations of G induced from (1-dimensional) characters of AN. For $s \in i\mathbb{R}$ consider the character $a_t n_x \to e^{-st}$ on AN and the space

$$H_{s,\text{even}} = \{f: G \to \mathbb{C} : f(-g) = f(g) \ , \ f \mid K \in L^2(K) \ ,$$

$$f(ga_t n_x) = e^{-(s+1)t} f(g)\} \ .$$

H_s may be interpreted as the L^2-sections of the line bundle over $G/AN = K$ whose total space is $G \times_{AN} \mathbb{C}$, where AN acts on $\mathbb{C}$ by the character above $\times e^{-t}$. This explains what the induced action of G should be: $(gf)(h) = f(g^{-1}h)$; $f \in H_s$, $g,h \in G$. An element of H_s is determined by its value on K and we think of the space in terms of the restrictions of the functions to K. The term e^{-t} is the positive square root of the modulus function on AN ; we need it if we are to get a unitary representation with the Hilbert structure that of $L^2(K)$. This being so, one guesses that the K-finite functions are the functions with finite Fourier series. It is, indeed, the case and one may further see that the smooth vectors are the smooth functions and the analytic vectors the analytic functions. Consider the vector $k_\theta \to e^{ni\theta}$, which we shall call $e^{ni\theta}$ for short. It is an eigenfunction for the group K with eigenvalue $k_\phi \to e^{-ni\phi}$. Under the isomorphism $SU(1,1) \cong SL(2,\mathbb{R})$,

$$Z \to -iW = -id/d\theta$$

$$X_\pm \to \frac{1}{2} E_\pm = \frac{1}{2} \{\frac{d}{dt} \mp id/d\theta \mp 2i \frac{d}{dx}\} \quad ;$$

where we have denoted the tangent vectors to K,A,N by

$d/d\theta$, d/dt , d/dx , using the above parametrization of these groups. It is trivial to see that $Ze^{ni\theta} = -ne^{ni\theta}$. To calculate $X_\pm(e^{ni\theta})$, consider H_s as a subspace of functions on G . There, there is an action of G on the left (the one we have) and on the right. The functions have nice invariance properties with respect to AN acting on the right. If r denotes the right action of $\mathfrak{g}^C$ then the two are related by the following equation:

$$(Yf)(g) = (r((\mathrm{ad}g)Y).f)(g) \ : \ Y \in \mathfrak{g}^C \ , \ g \in G \ .$$

In the case when $Y = X_+$ and $g = k_\theta$ (the case which interests us) one has

$$(X_+f)(k_\theta) = (r((\mathrm{ad}\ k_\theta)X_+)f)k_\theta$$

$$= e^{2i\theta}(r(X_+)f)(k_\theta) \quad ,$$

since X_+ is an eigenvector for $\mathrm{ad}(k_\theta)$ with eigenvalue $e^{2i\theta}$. Specializing further to $f = e^{ni\theta}$ and using the equation for X_+ in terms of d/dt , $d/d\theta$, d/dx one finds that

$$X_+(e^{ni\theta}) = \left(\frac{-n+s+1}{2}\right) e^{(n+2)i\theta} \quad ; \text{ and similarly,}$$

$$X_-(e^{ni\theta}) = \left(\frac{n+s+1}{2}\right) e^{(n-2)i\theta} \quad .$$

The representation $H_{s,\mathrm{even}}$ is thus a realization of the infinitesimal equivalence class $\pi_{s,\mathrm{even}}$ in class 2 of the list 2.1.

Let us note here that we could have taken $f \in L^p(K)$ without really modifying the representation. There is an enormous family of possible topologies and we now see some justification for considering infinitesimal equivalence.

It is not so easy to realize the representations of the *complementary series*; class (3) in the above list. We can proceed to define a space $H_{s,even}$ as above, and we get a realization in the appropriate infinitesimal equivalence class. But it is not unitary: the inner product has to be modified.

If (π,V) is a unitary representation we may decompose it as a sum of 1-dimensional eigenspaces of K and so take a basis (v_n), v_n having eigenvalue n. It is natural to consider the matrix coefficients

$$\langle\pi(g)v_n, v_m\rangle$$

of g. We know what happens when $g \in K$, so a convenient decomposition is the Cartan-decomposition $G = KA^+K$, where

$$A^+ = \left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix} : 0 \le t < \infty\right\} .$$

We need, therefore, to understand the action of A^+: essentially to understand the behaviour of the function $t \to \langle\pi(a_t)v_n,v_n\rangle$ for $t > 0$. From the concrete realizations of the infinitesimal equivalence classes one can see that the behaviour is as follows:

Class (1) - the discrete series $\pi_n^{\pm}$, $n \ge 2$, and the mock discrete series $\pi_1^{\pm}$. The function behaves like Ce^{-nt} for some non-zero constant C.

Class (2) - the principal series $\pi_{s,even}, \pi_{s,odd}$: $s \in i\mathbb{R}$ ($s \ne 0$ in odd case). The function behaves like $c_+e^{-(s+1)t} + c_-e^{+(s-1)t}$, except for $\pi_{o,even}$ when it behaves like $c_1e^{-t} + c_2te^{-t}$.

With respect to the Cartan decomposition, Haar measure $dg = \sinh(t)dkdtdk$. Thus we see that the discrete series representations have coefficients in $L^2(G)$. The mock discrete series does not. The coefficients of the principal series are 'almost' in $L^2(G)$. If one had written down the inner product for the complementary series one would find that $\pi_{s,even}$, $-1 \leq s \leq 1$, $s \neq 0$, have coefficients far from $L^2(G)$: the integrals diverge exponentially.

In the Plancherel measure on $\hat{G}$, the mock discrete series and the complementary series form a null set. The discrete series occurs discretely and the principal series continuously. However, if Γ is a discrete cocompact subgroup of G and if the Laplacian has eigenvalues of modulus $\leq \frac{1}{4}$ then complementary series representations do occur in $L^2(\Gamma \backslash G)$.

3. Let G be a connected semi-simple Lie group with finite centre. Let K be a maximal compact subgroup and (π, V) an irreducible unitary representation. For each $i \in \hat{K}$ let $V(i)$ denote the space of K-finite vectors of type i . One finds that $\dim V(i) \leq (\deg i)^2$. This is the crucial property making semi-simple Lie groups of type I. It is the property, too, which implies the existence of a global distributional character for an irreducible unitary representation. (There is a proof of this property for a group admitting a faithful finite-dimensional representation in §2 of [2].) Before discussing this, the *global* or *Harish-Chandra character*, we mention the *infinitesimal character*.

Let $\mathfrak{z}$ denote the centre of $\mathfrak{U}(\mathfrak{g}^C)$. Regard the latter as the algebra of left-invariant differential operators, so that $\mathfrak{z}$ is the algebra of bi-invariant differential operators. The algebraic structure of $\mathfrak{U}(\mathfrak{g}^C)$ is complicated, but $\mathfrak{z}$ *is a polynomial algebra* over C [31]. In the infinitesimal representation associated to (π, V) , $\mathfrak{z}$ commutes with $\mathfrak{U}(\mathfrak{g}^C)$ and preserves $V(i)$, $i \in \hat{K}$. It follows that $\mathfrak{z}$ operates by scalars: that is, there is a homomorphism $\chi_\pi : \mathfrak{z} \to C$ such that $\pi(Z) = \chi_\pi(Z) I$; $\forall Z \in \mathfrak{z}$. This homomorphism is called the *infinitesimal character*. It does *not* determine the representation up to infinitesimal equivalence, as does the global character to which we now turn.

Suppose first that (π,V) is unitary. As in §1 define $\pi f \in \mathrm{End}(V)$, for $f \in C_o^\infty(G)$ - the space of smooth functions of compact support - by $\pi f = \int_G f(g)\pi(g)dg$. There is the following basic proposition of Harish-Chandra.

Proposition 3.1. The operator πf is of trace class. (That is, for any orthonormal basis (e_n), $n \in \mathbb{N}$, $\sum \langle \pi f(e_n), e_n\rangle$ converges absolutely. This number is independent of the basis.)

Consequently we may set $\Theta_\pi f = \text{trace } \pi f$, $f \in C_o^\infty(G)$. If (e_n), $n \in \mathbb{N}$, is an orthonormal basis for V, $f \to \sum_{n=1}^{k} \langle(\pi f)e_n, e_n\rangle$ is a continuous function on $C_o^\infty(G)$ and, for each f, their values tend to $\Theta_\pi f$ as k tends to ∞. But $C_o^\infty(G)$ is an inductive limit of Fréchet spaces and so Θ_π is continuous on $C_o^\infty(G)$ by the Banach-Steinhaus theorem. Hence $\Theta_\pi \in D'(G)$, the distributions on G.

Definition 3.2. $\Theta_\pi \in D'(G)$ is the *Harish-Chandra*, or *global*, *character*.

Θ_π has properties other than those used in defining it. We list some:

(a) Θ_π is invariant under conjugation.

(b) $Z^*\Theta_\pi = \chi_\pi(Z)\Theta_\pi$, $\forall Z \in \mathfrak{z}$; where $X \to X^*$ is the unique anti-isomorphism of $\mathfrak{u}(\mathfrak{g}^C)$ extending $X \to -X$ on $\mathfrak{g}^C$.

(c) If $i \in \hat{K}$ and $p_i : V \to V(i)$ is the unique K-invariant projection to $V(i)$, then

$$\Theta_\pi = \sum_{i\in\hat{K}} \text{trace } F_i \quad ,$$

where the right-hand side converges in the sense of distributions and $F_i : G \to \mathrm{Hom}(V(i),V(i))$ is defined by $F_i(g)(v) = p_i(\pi(g)v) : g \in G,\ v \in V(i)$.

(d) Θ_π determines (π,V) up to infinitesimal equivalence. (This is because Θ_π determines the spherical trace functions, trace $F_i . p_i$ for $i \in \hat{K}$. These determine finite-dimensional representations of certain subalgebras of $L^2(G)$ and a short calculation shows that the 'diagonal matrix coefficients' - with respect to bases whose elements lie in V_i - are also determined. But two irreducible unitary representations with a common non-zero coefficient are infinitesimally equivalent.)

If, now, (π,V) is an irreducible admissible Banach space representation we may use (c) to define Θ_π. Properties (a), (b) and (d) still hold in this case.

A distribution satisfying (a) and (b) is called an *invariant eigendistribution*.

Distributions are hard to manipulate. The following is a deep and difficult theorem of Harish-Chandra. Its proof occupies several of his longer papers.

Theorem 3.3. Every invariant eigendistribution is locally an L^1 function on G. Moreover, that function is analytic on G', the set of regular semi-simple elements of G.

Note that G' is an open dense subset of G and its complement is a real analytic subvariety. Consequently the theorem tells us that if we know the character on G' we know it completely.

In a moment we shall give a sketch of a proof (due to M.F. Atiyah & W. Schmid) [2] of this theorem. But first we shall see what the character is for the concrete representations of $SL(2,R)$ constructed in §2 and indicate the relation of Θ_π with the K-character mentioned by M.F. Atiyah in his lecture.

Assume that π is a unitary representation. If $f \in C^\infty(K)$ then the operator $\pi_K f = \int_K f(k)\pi(k)dk$ is of trace class (because of the bounds on $\dim V(i)$) and the map $f \to \text{trace } \pi_K f$ is a distribution on K . This distribution, which we write τ_π , is called the *K-character*. From the decomposition of V_∞ into invariant subspaces one sees that

$$\tau_\pi = \sum_{i\in\hat{K}} n_i(\pi)\chi_i = \sum_{i\in\hat{K}} \text{trace } F_i|K \quad ,$$

where $n_i(\pi)$ is the multiplicity of i in $V(i)$. The two characters are related in the following way.

Theorem 3.4. $\tau_\pi|K \cap G'$ is (integration against) a real analytic function and $\tau_\pi|K \cap G' = \Theta_\pi|K \cap G'$. (However, τ_π is often highly singular and is not usually locally L^1 on K .)

It can happen that $\tau_\pi \equiv 0$ on $K \cap G'$ but $\tau_\pi \neq 0$, as we shall see.

Now we take the case of $SL(2,\mathbb{R})$ as an example. We shall use the notation of §2 and, in particular, the parametrization of K,A,N . Denote the characters of the principal series representations $\pi_{s,\text{even}}$, $\pi_{s,\text{odd}}$ $(s \in i\mathbb{R})$ by $\Theta_{s,\text{even}}$, $\Theta_{s,\text{odd}}$. The representations are induced from a character on AN , so the character is too and hence will be supported on those conjucacy classes meeting AN : up to a set of measure zero, this is $(\pm A)^G$, the union of the conjugacy classes which meet $\pm A$. Every semi-simple element is conjugate to an element in K or $\pm A$ (but, except for the centre, never to an element in both). So it is sufficient to find $\Theta_{s,\text{even}}$, $\Theta_{s,\text{odd}}$ on $G' \cap K$ and $G' \cap \pm A$

$$\Theta_{s,\text{even}}|K \cap G' = 0 = \Theta_{s,\text{odd}}|K \cap G' \quad ;$$

$$\Theta_{s,even}(a_t) = \frac{e^{-st}+e^{st}}{|e^t-e^{-t}|} = \Theta_{s,odd}(a_t) \quad ;$$

$$\Theta_{s,even}(-a_t) = \frac{e^{-st}+e^{st}}{|e^t-e^{-t}|} = -\Theta_{s,odd}(-a_t) \quad ;$$

where $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ as in §2. In this case $\tau_{s,even}$ and $\tau_{s,odd}$ vanish on $K \cap G'$, because $\Theta_{s,even}$ and Θ_{s-odd} do. But $(\tau_{s,even},1) = 1$, as we can see from the theory of Fourier series.

For the discrete series representations π_n^+, π_n^- (and also for the mock discrete series) we find that on $K \cap G'$ the characters are as follows:

$$\Theta_n^+(k_\theta) = \tau_n^+(k_\theta) = \sum_{r=0}^{\infty} e^{i(n+2r)\theta} = -\frac{e^{i(n-1)\theta}}{e^{i\theta}-e^{-i\theta}} \quad ;$$

$$\Theta_n^-(k_\theta) = \tau_n^-(k_\theta) = \sum_{r=0}^{\infty} e^{i(-n-2r)} = \frac{e^{-i(n-1)\theta}}{e^{i\theta}-e^{-i\theta}}$$

$\theta \neq 0,\pi$; where $k_\theta = \begin{pmatrix} \cos\theta , & \sin\theta \\ -\sin\theta , & \cos\theta \end{pmatrix}$. It is not so easy to see what $\Theta_n^+(a_t)$ should be. Some fortunate accidents tell us. Note that the character can be defined for any reducible representation of finite length so that it makes sense to talk of $\Theta_{n,odd}, \Theta_{n,even}$ for $n \in \mathbb{N}$. If ε = parity of n, it follows from the list of representations in §2 that $\Theta_{n-1,\varepsilon} = \Theta_n^+ + \Theta_n^- + \chi_{\sigma_{n-1}}$, where σ_{n-1} is a finite-dimensional representation. Now $\Theta_{n-1,\varepsilon}$ is the character of the (reducible) principal series representation $\pi_{n-1,\varepsilon}$, and one finds, as above, that

$$\Theta_{n-1,\varepsilon}(a_t) = \frac{e^{-(n-1)t} + e^{+(n-1)t}}{|e^t - e^{-t}|} \quad ;$$

$$\Theta_{n-1,\varepsilon}(-a_t) = (-1)^{\varepsilon}\,\Theta_{n-1,\varepsilon}(a_t) \quad .$$

On $K \cap G'$, of course, $\Theta_{n-1,\varepsilon}$ vanishes. Now σ_{n-1} is the representation of dimension $n-1$ coming from that of $SL(2,\mathbb{C})$. It is unique and a standard form of it is on the space of homogeneous polynomials of degree $n-2$ in two variables (the symmetric product of the fundamental representation of $SL(2,\mathbb{C})$). Its character is easily computed on A:

$$\chi_{\sigma_{n-1}}(a_t) = \frac{e^{(n-1)t} - e^{-(n-1)t}}{e^t - e^{-t}} \quad ;$$

$$\chi_{\sigma_{n-1}}(-a_t) = (-1)^{\varepsilon}\,\chi_{\sigma_{n-1}}(a_t) \quad .$$

We deduce that $\Theta_n^+(a_t) + \Theta_n^-(a_t) = \dfrac{2e^{-(n-1)|t|}}{|e^t - e^{-t}|}$, and $\Theta_n^+(-a_t) + \Theta_n^-(-a_t) = (-1)^{\varepsilon}\, 2e^{-(n-1)|t|}/|e^t - e^{-t}|$. Because a_t is conjugate to a_{-t} (use $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$) and π_n^+ is dual to π_n^- one sees that $\Theta_n^+(a_t) = \Theta_n^+(a_{-t})$ and that $\Theta_n^+(a_t) = \Theta_n^-(a_{-t})$. Consequently

$$\Theta_n^+(a_t) = \frac{e^{-(n-1)|t|}}{|e^t - e^{-t}|} = \Theta_{n,-}(a_t) \quad ,$$

$$\Theta_n^+(-a_t) = \frac{(-1)^{\varepsilon}\, e^{-(n-1)|t|}}{|e^t - e^{-t}|} = \Theta_n^-(-a_t) \quad .$$

As $t \to \infty$, $\Theta_n^+(a_t) \to 0$.

The proof of the regularity theorem (theorem 3.3) in [2] works for any invariant eigendistribution. Here we shall illustrate how the theorem is proved by giving an outline of the proof in the case of $SL(2,\mathbb{R})$. In this instance $SL(2,\mathbb{R})$ gives a good idea of how to proceed in general: the general case is a matter of loading in the structure of $\mathfrak{g}$ to the machinery set up. The proof breaks up into three parts.

(1) $\Theta|G'$ is analytic, where G' is the set of regular semi-simple elements and Θ is an invariant eigendistribution. (We have seen that this is so for the characters of the principal and discrete series.)

(2) If $\Theta|G' = F$ then F is locally L^1 on G. (Note that G' is a dense open subset with complement a real analytic subvariety. Thus $G \setminus G'$ is of measure 0.) So $\Theta = F + S$ where $\operatorname{supp} S \subset G \setminus G'$.

(3) $S = 0$.

Of these three parts the third is the hardest. For use during the proof, let G'' denote the set of regular elements - so $G'' \supset G'$.

The elements of $SL(2,R)$ fall into three types according to their Jordan forms. Let $g \in SL(2,R)$

g is elliptic $\iff$ g is conjugate (in $SL(2,R)$) to

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad 0 < \theta < \pi$$

$\iff |\text{trace } g| < 2$

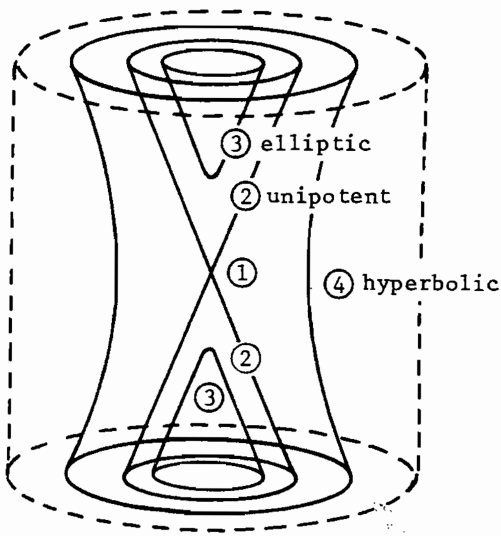
g is hyperbolic $\iff$ g is conjugate to $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\alpha \in R^*$

$\iff |\text{trace } g| > 2$

g is parabolic (or unipotent) $\iff$ g has repeated eigenvalues $+1$ or -1

$\iff |\text{trace } g| = 2$

The trace function t thus distinguishes the three types of elements. It is constant on conjugacy classes and $dt \neq 0$ on G''. Thus for a regular element g, t is a good coordinate to choose when investigating the behaviour of a function or distribution near g ; especially when that function or distribution is invariant (under conjugation). As the distribution Θ is so invariant it is interesting to see what the conjugacy classes of elements look like in $SL(2,\mathbb{R})$. There is a global picture in [2]. The conjugacy classes are the orbits of the conjugation action: the corresponding infinitesimal picture at 1 is the following:



Orbits are (1) the origin, (2) each half of the cone\origin, (3) each sheet of the hyperboloid of two sheets (these are the elliptic elements), (4) each hyperboloid of one sheet.

For a semi-simple Lie group the centre $\mathfrak{z}$ of $\mathfrak{u}(\mathfrak{g}^{C})$ is a polynomial algebra on r generators, where r is the rank. $SL(2,\mathbb{R})$ has rank 1 and so $\mathfrak{z}$ is generated by one element: the *Casimir operator* Ω . This is the operator of order 2 which is represented by the Killing form (identifying $\mathfrak{g}$ with

$\mathfrak{g}^*$) : thus with respect to an orthonormal basis $X_1, \ldots, X_n$ for $\mathfrak{g}^C$, $\Omega = \sum_{i=1}^{n} X_i^2$. Let $g_o \in G''$ and choose, in a neighbourhood, coordinates (t,u,v), where t is the trace function as above and (u,v) are coordinates along the conjugacy classes - so that $\partial/\partial u$, $\partial/\partial v$ are always tangent to these classes. In terms of these coordinates we may write

$$\Omega = D_t + Q \quad ,$$

where D_t involves only differentiation with respect to t ($\partial/\partial u$ and $\partial/\partial v$ are not present) and in Q every term has a $\partial/\partial u$ or $\partial/\partial v$ occurring on the extreme right (and so is the first operator to be applied). Thus Q will annihilate any function constant on conjugacy classes. The characters of the 1, 2 and 3-dimensional irreducible representations of $SL(2,R)$ are 1, t and t^2-1 respectively. For these the Casimir operator takes the values 0, 3 and 8 respectively ($\dim^2-1$). Hence $D_t = (t^2-4)\,\partial^2/\partial t^2 + 3t\,\partial/\partial t$. On the unipotent elements $t^2 = 4$, but away from them $t^2 \neq 4$. So on (the conjugacy classes of elements in) G' the operator is *elliptic*. But any invariant eigendistribution Θ satisfies an equation $D_t\Theta = c\Theta$ for some constant c. So since D_t is (transversally) elliptic on G', there Θ will be a real analytic function, F say. It is now a matter of studying how F behaves as we approach a parabolic point: this is part 2.

Let α, α^{-1} be the eigenvalues of a matrix in $SL(2,R)$. Away from $G \setminus G'$ we may make a coherent choice for α (say $\alpha > 1$ or $\alpha = e^{i\theta}$, $0 < \theta < \pi$) and so we can use α as a coordinate instead of t. With this new coordinate one finds that

$$D_t = D_\alpha = \frac{1}{\Delta}\left(\alpha \frac{\partial}{\partial \alpha}\right)^2 \Delta + c \quad ,$$

where c is a constant and $\Delta = \alpha - \alpha^{-1}$. Thus ΔF is an eigenfunction for $(\alpha \frac{\partial}{\partial\alpha})^2$. Consequently if $c \neq 1$, $F = \frac{1}{\Delta}(c_1\alpha^{\lambda} + c_2\alpha^{-\lambda})$, where $\lambda^2 = c-1$ and c_1, c_2 are constants. (If $c = 1$, there is also a logarithmic term.) The function ΔF is clearly bounded as we approach $g \in G \setminus G'$ since there $\alpha = 1$. To establish that $F \in L^1_{loc}$ it is thus sufficient to show that $1/|\Delta| \in L^1_{loc}$. Direct computation is not easy, even on $SL(2,\mathbb{R})$. (The computation is performed in [2].) Let $S = \{(g,z) \in G \times S^2 \mid gz = z\}$, where S^2 is regarded as the Riemann sphere. The projection $\pi: S \to G$ is a proper map and $g \in G'$, $\# p^{-1}(g) = 2$. It is thus generically a 2-fold covering and would be a 2-fold branched covering but for the points ± 1 where the inverse image is infinite. If one pulls back Haar measure dg one finds that $p^* dg = \Delta\omega$, where ω is a smooth 3-form on $S \subset G \times P^1(C)$. But p is proper, so $\frac{1}{\Delta} \in L^1_{loc}$.

Finally, there is part 3. Let $T = \Theta - F$. It is a distribution with support concentrated on $G \setminus G'$ (double cone). We must show that it is zero. This will be done in two steps: first we shall show that it is zero away from ± 1, then that it is zero at ± 1. Let $g_o \in G \setminus G'$, $g_o \neq \pm 1$. Take coordinates (t,u,v) in a neighbourhood of g_o, where t is the trace as above and is transverse to the conjugacy classes, whilst u,v are coordinates along the conjugacy classes. Since Θ, F are constant along conjugacy classes they are functions of just the one variable t. Hence so is T and it satisfies the equation

$$(D_t - c)T = (D_t - c)F \quad ,$$

since $\Omega\Theta = c\Theta$. The support of T is concentrated on the parabolic points where $t = \pm 2$. For definiteness let us suppose that $t(g_o) = 2$, and let δ denote the Dirac δ-function at 2 . We can write

$$T = \sum_{0\le k\le n} a_k \frac{d^k}{dt^k}\delta \quad .$$

If $T \ne 0$ we may arrange that the 'leading term' $a_n \ne 0$. Recall that $D_t = (t^2-4)\,\partial^2/\partial t^2 + 3t\,\partial/\partial t$ and compute the left-hand side of the equation above.

$$(D_t-c)T = (-4(n+2)+6)a_n \frac{d^{n+1}}{dt^{n+1}}\delta + \text{lower terms} \quad .$$

For no integer n can the 'leading term' vanish because $6-4(n+2)$ is always non-zero. Now consider the right-hand side. F is a locally L^1 function so one can show by an elementary estimate that $(D_t-c)F$ can be at most a multiple of δ . It is now intuitively clear that this equation cannot hold because the left-hand side is at least one degree more singular than the right-hand side. To make this precise define the degree of singularity of a distribution P at 0 to be $\le n$ if $\lim_{\varepsilon\to 0} \varepsilon^n P(\phi_\varepsilon) = 0$ for all smooth ϕ with compact support, where $\phi_\varepsilon(t) = \phi(t/\varepsilon)$. A distribution has degree exactly n (at 0) if it has degree $\le n$ but not degree $\le n-1$. It is easy to see that (1) a locally L^1 function has degree 0 , (2) $\frac{d^k}{dt^k}\delta$ has degree (exactly) $k+1$, and (3) differentiation raises degree by 1 . The above argument is made precise by comparing the degrees of each side.

The proof is completed by making a similar argument at ± 1 to show that the support of T cannot be concentrated there. At these points one has to use distributions in three variables, but the argument is much simpler because here the leading symbol of Ω , being the Killing form, does not vanish.

4. To do representation theory one must have representations. There is the basic method of induction, but even here one must have somewhere to start. For the case of nilpotent Lie groups Kirillov has shown that *all* unitary representations are obtained by inducing from one-dimensional representations. In the case of solvable groups this is no longer so. There are two stages in the construction of a representation: (a) *holomorphic* induction from a one-dimensional character to an intermediate group, (b) *unitary* induction from that subgroup to the whole group. (Of course, for a given representation one step may be vacuous.) By contrast, the case of a semi-simple Lie group is hard; in Kostant's language, there is in general no *positive polarization*. However, we have seen for $SL(2,\mathbb{R})$ that the representations of the *principal series* come from one-dimensional characters by induction. Those of the *discrete series* come from *holomorphic induction*, the stage (a) above. But this is not a good guide to the general situation and we now discuss the analogue of (a) needed in general. There one must use (other) differential operators to select a class of sections. A convenient way to discuss this is in the context of the construction of representations of the *discrete series*.

Let (π,V) be an irreducible unitary representation of G and $\hat{\mu}$ the Plancherel measure on $\hat{G}$.

Definition 4.1. π is *square-integrable* <=> $\hat{\mu}\{\pi\} \neq 0$ <=> π occurs on an invariant subspace of $L^2(G)$.

Definition 4.2. $\hat{G}_d$ = set of isomorphism classes of representations in the *discrete series* = $\{\pi \in \hat{G} : \hat{\mu}\{\pi\} \neq 0\}$.

Theorem 4.3 (Harish-Chandra). $\hat{G}_d \neq \emptyset$ <=> rank G = rank K <=> there exists a compact Cartan subgroup in G.

[This theorem is proved in [20]. We shall come back to it later.] Thus, examples of groups which have discrete series representations are $SL(2,R)$, $Sp(n,R)$, $SU(p,q)$ and $SO(p,q)$ for $pq \equiv 0(2)$. On the other hand, $SL(n,R)$, $n > 2$, and $SO(p,q)$, pq odd, have no discrete series representations. Nor does any complex semi-simple Lie group.

Suppose that G has a discrete series. First we shall show how to constuct *most* of its discrete series representations. (They will appear as subspaces of a function space and we shall use the Dirac operator to pick out the appropriate subspace.) Then we shall go on (in §5) to discuss the basic results of Harish-Chandra [19,20] on the characters of the discrete series representations. These results, which do not depend upon explicit knowledge of the actual representations, are used to prove that the construction yields, in fact, *all* discrete series representations. (The discrete series representations present some similarities with the representation theory of a compact Lie group; for example, in their parametrization. The Dirac operator for a compact group is discussed in §7 of Bott's paper [7].)

Let G be a semi-simple Lie group such that $\text{rank } G = \text{rank } K$. Let H be a compact Cartan subgroup of G. We may suppose that $H \subset K \subset G$. Now $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\dim \mathfrak{p}$ is even since $\dim G \equiv \text{rank } G \bmod 2$ and $\dim K \equiv \text{rank } K \bmod 2$. Let $\phi: K \to SO(\mathfrak{p})$ be the map induced from the restriction of the adjoint action of G to K. Let us assume that ϕ lifts to $\text{Spin}(\mathfrak{p})$: if not we can take a double covering of K, and hence of G. Because $\dim \mathfrak{p} \equiv 0(2)$, $\text{Spin}(\mathfrak{p})$ has two irreducible (complex half-spin) modules S^+, S^-. The lift of ϕ gives these a K-module structure. For any (finite-dimensional) K-module V there is associated a homogeneous vector bundle $\mathcal{V}$ on G/K. The total space is $G \times_K V$ where K acts on the right on G and on the left on V. There is an obvious action of G on the left on the total space. Now fix an irreducible K-module V and an invariant hermitian structure on V. Consider the vector bundles $\mathcal{V} \otimes \mathcal{S}^+$ and $\mathcal{V} \otimes \mathcal{S}^-$. (These correspond to the K-modules $V \otimes S^+$, $V \otimes S^-$ and all we need is that these, not that V, S^+ and S^- individually, should be K-modules.) There one has a first order homogeneous differential operator (the Dirac operator) which is elliptic and self-adjoint:

$$\tilde{D}_V^+ : \Gamma(\mathcal{V} \otimes \mathcal{S}^+) \to \Gamma(\mathcal{V} \otimes \mathcal{S}^-)$$
$$\tilde{D}_V^- : \Gamma(\mathcal{V} \otimes \mathcal{S}^-) \to \Gamma(\mathcal{V} \otimes \mathcal{S}^+)$$

(The tangent space to G/K may be identified with $G \times_K \mathfrak{p}$, where K acts on $\mathfrak{p}$ by the adjoint action. With this identification the symbol of $\tilde{D}_V$ becomes, when lifted to $G \times \mathfrak{p}$, the Clifford multiplication $\mathfrak{p} \otimes S \overset{m}{\to} S$; $S = S^+ \oplus S^-$. Because $\tilde{D}_V$ is invariant the symbol determines it.) Set $H_V^{\pm} = \ker \tilde{D}_V^{\pm} \cap L^2(\mathcal{V} \otimes \mathcal{S}^{\pm})$. These are Hilbert spaces and, since the operator is homogeneous and the induced action of G is unitary on sections, they are unitary G-spaces. Of course, they might be zero. What we should wish is the following:

(1) For a 'general' V, *one* of the two spaces $H_V^{\pm}$ is irreducible and is a discrete series representation whereas the other is zero.

(2) For 'other' V get $H_V^{\pm} = 0$.

(3) Every discrete series representation may be realized in this way.

These are all true but are proved in stages. First (1) is established when the highest weight of V is 'very non-singular', 4.17. Then (2) is proved when the highest weight is singular and, moreover, one establishes that $H_V^- = 0$ for all V. Finally (1) for other non-singular highest weights is proved using results on characters: this is outlined in §5.

When $G = SL(2,\mathbb{R})$ the operator is the $\bar{\partial}$ operator $= \partial/\partial\bar{z}$ and the process is holomorphic induction. In general, even when G/K has an invariant complex structure (it does in roughly half the cases), the operator is not always $\partial/\partial\bar{z}$ and, although we do indeed get positive polarizations and part of the discrete series is realized by holomorphic induction, not all of it is.

As we have said above, both spaces $H_V^{\pm}$ might be 0. We need a tool to establish the existence of non-trivial sections, s, satisfying $Ds = 0$. That tool is the Atiyah L^2 index theorem. To apply it successfully the following theorem of Borel [5] is crucial.

Theorem 4.4 (Borel). If G is a semi-simple (linear) Lie group $\exists$ $\Gamma \subset G$, a discrete subgroup such that $\Gamma\backslash G$ is compact *and* Γ operates on G/K without fixed point.

The L^2 index theorem is concerned with a smooth (galois) covering, $\tilde{M} \to M$, with group Γ where the base M is a compact Riemannian manifold. One has an elliptic differential operator $D\colon \Gamma(E) \to \Gamma(F)$ between two hermitian vector bundles E and F on M, and consequently an elliptic differential operator $\tilde{D}\colon \Gamma(\tilde{E}) \to \Gamma(\tilde{F})$ between the lifted vector bundles. Set

$$H^+ = \ker \tilde{D} \cap L^2(\tilde{E}) \quad ,$$
$$H^- = \ker \tilde{D}^* \cap L^2(\tilde{F}) \quad .$$

The orthogonal projections from $L^2(\tilde{E})$ to H^+ and from $L^2(\tilde{F})$ to H^- have smooth kernels $K^{\pm}(x,y)$, $x,y \in \tilde{M}$, which are invariant with respect to the diagonal action of Γ on $\tilde{M} \times \tilde{M}$. ($K^{\pm}(x,y)$ may be regarded as extensions of the Bergman kernel.) Let $\Delta \subset \tilde{M}$ be a fundamental domain for the action of Γ.

Definition 4.5. The Γ-dimension of $H^+ = \dim_\Gamma H^+ = \int_\Delta \mathrm{tr}_x K^+(x,x)\tilde{d}x = \int_M k^+(y)dy$, where tr_x is the pointwise trace, dy is the induced measure on M and $\tilde{d}x$ the lift to $\tilde{M}$, k^+ the induced function on M from the Γ-invariant function $\mathrm{tr}_x K^+(x,x)$ on $\tilde{M}$.

When Γ is finite so that $\tilde{M}$ is compact, $\dim H^+ = |\Gamma| \dim \ker D$ and $\dim_\Gamma H^+ = 1/|\Gamma| \dim H^+ = \dim \ker D$. The L^2 index theorem is an extension of this to infinite coverings.

Theorem 4.6 (L^2 index theorem). $\mathrm{Index}_\Gamma \tilde{D} = \dim_\Gamma H^+ - \dim_\Gamma H^- = \mathrm{Index}\ D$.

Combined with the standard index theorem this theorem gives a topological formula for $\mathrm{index}_\Gamma \tilde{D}$. The proof is definitely of a lower order of difficulty than that of the standard index theorem. In our situation, G/K will play the rôle of $\tilde{M}$ and $_\Gamma\backslash G/_K$ that of M, where Γ is a group acting on G/K without fixed point: such exists by Theorem 4.4. The operator will be the Dirac operator $\tilde{D}_V^+: \Gamma(V \otimes S^+) \to \Gamma(V \otimes S^-)$ discussed above. Here $\tilde{D}$ is not just Γ-invariant but G-invariant. This enables us to simplify the computations of the indices in Theorem 4.6. One finds that, with a suitable normalization of Haar measure on G,

$$\mathrm{Index}\ D_V^+ = \dim F_V \ \mathrm{vol}(\Gamma\backslash G) \quad ,$$

where F_V is a certain irreducible K-module depending on V: precisely, if V has highest weight λ then F_V is that with highest weight $\lambda-\rho_n$, where $\rho_n = \frac{1}{2}\Sigma\alpha$, $\alpha \in \Phi_+^n$, the positive non-compact roots. (Should $\lambda-\rho_n$ not be a weight then $\dim F_V$ is the number given by the Weyl character formula evaluated at 1.) When G is a real form of the simply-connected complex group G^C one normalizes in such a way that the dual, M, of G with the corresponding metric has volume 1. The result can not be deduced from Bott's computation [7] by passing to the compact dual M/K of G/K and using Hirzebruch's proportionality principle [24]. For the other index, $\mathrm{Index}_\Gamma \tilde{D}_V^+$, the calculation does not follow quite such well-worn tracks. Let $\hat{\mu}$ denote the Plancherel measure. The

$$L^2(V \otimes S^+) \subset (L^2(G) \otimes V \otimes S^+)^K ,$$

where K acts on the right on $L^2(G)$, and

$$(L^2(G) \otimes V \otimes S^+)^K \cong \int_{\hat{G}} V_i \otimes (V_i^* \otimes V \otimes S^+)^K \, d\hat{\mu}(i) \quad ,$$

according to the Abstract Plancherel Theorem (1.3). The space $(V_i^* \otimes V \otimes S^+)^K$ is finite-dimensional since it corresponds to the K-factor of that representation where K acts trivially, and V_i^* is admissible since unitary. We may gain information about H_V^+ here by the common method of passing to the 'square of $\tilde{D}' = (\tilde{D}_V^+)^* \tilde{D}_V^+ = \tilde{D}_V^- \tilde{D}_V^+$, the spinor Laplacian. Up to addition of a scalar it is the Casimir operator: $\tilde{D}_V^- \tilde{D}_V^+ = -\Omega + c_V$, where c_V is a constant depending on the K-module V. $\mathrm{Ker}\ \tilde{D}_V = \mathrm{Ker}\ \tilde{D}_V^2$ and so

$$H_V^+ = \{f \in L^2 : \Omega f = c_V f\}^{\dagger}$$

Let χ_i denote the infinitesimal character of $i \in \hat{G}$. From the line above one sees that the Dirac operator picks out a subrepresentation of $L^2(V \otimes S^{\pm})$ consisting only of representations i for which $\chi_i(\Omega) = c_V$:

†(c_V same for both $D_V^- D_V^+$ and $D_V^+ D_V^-$: the precise formula is due to Parthasarathy [28].)

$$H_V^+ = \int_{\{i\in G:\chi_i(\Omega)=c_V\}} V_i \otimes (V_i^* \otimes V \otimes S^+)^K \, d\hat{\mu}(i) \quad . \tag{4.7}$$

If one unravels the definition of $\dim_\Gamma$ (using the fact that the operator D is, indeed, G-invariant so that $\dim_G(\ker D)$ has a meaning) one sees that because $\dim_\Gamma = \mathrm{vol}(\Gamma\backslash G)\dim_G$

$$\dim_\Gamma H_V^+ = \mathrm{vol}(\Gamma\backslash G) \int_{\{i\in G:\chi_i(\Omega)=c_V\}} \dim(V_i^* \otimes V \otimes S^+)^K \, d\hat{\mu}(i) \quad .$$

Now one may put together the computations and deduce that

$$\int_{\{i\in G:\chi_i(\Omega)=c_V\}} \{\dim(V_i^* \otimes V \otimes S^+)^K - \dim(V_i^* \otimes V \otimes S^-)^K \, d\hat{\mu}(i)$$

$$= \dim F_V \tag{4.8}$$

From 4.7, 4.8 and knowledge of what F_V is, it is clear that for 'generic' V one of $H_V^\pm$ is non-zero. The next step is to prove that the integrand in 4.8 vanishes outside a finite set. (Thus $H_V^+ - H_V^-$ will involve only discrete series representations.) We shall do this by computing the infinitesimal character of those i for which the integrand is non-zero: we shall show that they all have the *same* infinitesimal character.

Let $H \subset K \subset G$ be a compact Cartan subgroup of G and let Φ be the root system of $(\mathfrak{g}^C, \mathfrak{h}^C)$; let $\Lambda \subset i\mathfrak{h}^*$ be the integral lattice. Say that a root α is compact <=> α root of $(\mathfrak{k}^C, \mathfrak{h}^C)$. Thus $\Phi = \Phi^c \amalg \Phi^n$, where Φ^c is the set of compact roots. The non-compact roots Φ^n describe exactly the action of $\mathfrak{h}$ on $\mathfrak{p}^C$: each $\beta \in \Phi^n$ occurs with multiplicity one there (see [23], [30]). Let $\sigma^\pm$ denote the characters of $S^\pm$. The weights of the half spin representations of $SO(2q)$ are $\frac{1}{2}\sum_{1\le i\le q} \varepsilon_i\mu_i$, $\varepsilon_i = \pm 1$, where $\prod_{1\le i\le q} \varepsilon_i = +1$ in one case, -1 in the other and $\pm\mu_1,\dots,\pm\mu_q$ are the weights of the standard representation of $SO(2q)$. Consequently, for a convenient choice of positive roots, Φ_+ ,

$$(\sigma^+ - \sigma^-) \mid H = \prod_{\beta \in \Phi_+^n} (e^{\beta/2} - e^{-\beta/2}) \quad .$$

Fix such a choice. For each Cartan subgroup, H, there is an isomorphism $\gamma: \mathfrak{z} \stackrel{\cong}{\to} S(\mathfrak{h}^C)^{W_C}$, where W_C denotes the Weyl group of G^C and $S(\mathfrak{h}^C)$ the symmetric algebra on $\mathfrak{h}^C$ [18,31]. Under this isomorphism (interpreting $\gamma(Z)$ as a differential operator on H),

$$Z\Theta \mid H \cap G' = \Delta_H^{-1} \gamma(Z)(\Delta_H \Theta) \mid H \cap G' , \quad Z \in \mathfrak{z} , \tag{4.9}$$

for any function Θ constant along conjugacy classes, where

$$\Delta_H = \prod_{\alpha \in \Phi_+} (e^{\alpha/2} - e^{-\alpha/2}) \quad .$$

(This is the generalization of what we have seen in §3 for $SL(2,\mathbb{R})$ and the Casimir operator.) The key to the computation is that

$$\Delta_H = D_K (\sigma^+ - \sigma^-) \tag{4.10}$$

where D_K is the denominator (for K) in the Weyl character formula. Now, for essentially formal reasons [29], for each $i \in \hat{G}$ $(\sigma^+ - \sigma^-)\tau_i$ is a *finite* linear combination of irreducible characters of K with integer coefficients. Hence, if Θ_i denotes the character of $i \in \hat{G}$ as in §3,

$$\Delta_H \Theta_i \mid H \cap G' = \Delta_H \tau_i \mid H \cap G' = \sum_{\lambda \in F_i} n_\lambda^i e^\lambda \mid H \cap G' , \tag{4.11}$$

where F_i is a finite subset of Λ and $n_\lambda^i \in \mathbb{Z}$. Thus:

$$\chi_i(Z)\Theta_i = Z^*\Theta_i = 1/\Delta_H \sum_{\lambda \in F_i} n_\lambda^i(\gamma(Z^*))(\lambda)e^\lambda \quad ,$$

where $Z \in \mathfrak{z}$, the equations are valid on $H \cap G'$. Consequently, if $n_\lambda^i \neq 0$ for some λ then

$$\chi_i(Z) = (\gamma(Z^*))(\lambda) = \chi_\lambda(Z) \quad ,$$

for all $Z \in \mathfrak{z}$. (χ_λ is the character of the irreducible representation of K with highest weight $\lambda - \rho_c$ †.) Moreover, as the polynomials on $\mathfrak{h}^C$ invariant under W_C separate the W_C orbits in $(\mathfrak{h}^C)^*$ we see that given $n^i_\lambda \neq 0$, then $n^i_\mu \neq 0 \iff \mu = w\lambda$, some $w \in W_C$. This partially computes the infinitesimal character χ_i and proves (because $\Delta_H = D_K(\sigma^+ - \sigma^-)$, as noted) the following proposition.

Proposition 4.13. If, for some highest weight λ , both $(\sigma^+ - \sigma^-)\tau_i$ and $(\sigma^+ - \sigma^-)\tau_j$ contain the character of V_λ with non-zero coefficient then i and j have the same infinitesimal character.

Let $q = \frac{1}{2} \dim \mathfrak{p} = \frac{1}{2} \dim G/K$. Then $(S^+)^* \cong S^+$ or S^- according as q is even or odd. Let χ denote the character of the irreducible K-module V and λ its highest weight. Denote $(V_i \otimes V \otimes S^{\pm})^K$ by $V_i^{\pm}$. Then

$$\dim V_i^+ - \dim V_i^- = (-1)^q\{\text{multiplicity of } \chi \text{ in } (\sigma^+ - \sigma^-)\tau_i\}$$

$$= (-1)^q n^i_{\lambda+\rho_c} \qquad (4.14)$$

From proposition 4.13 we see that $\dim V_i^+ - \dim V_i^- \neq 0$ and $\dim V_j^+ - \dim V_j^- \neq 0 \Rightarrow \chi_i = \chi_j$; as we wished.

The following proposition of Harish-Chandra, combined with the above, now tells us that the discrete series is not empty.

†Recall that if C is a fundamental Weyl chamber for G with respect to H, Φ_+, then the irreducible finite dimensional representations of G may be parametrized by $\lambda \in C \cap \Lambda$ or, alternatively, by their highest weight $\lambda - \rho \in \bar{C} \cap \Lambda$.

Proposition 4.15. For any infinitesimal character χ, the number of irreducible unitary representations which correspond to it is finite.

[This is because the space of invariant eigendistributions with a given infinitesimal character is finite-dimensional, as can be seen quickly from 4.9 if one remembers that there is a finite number of distinct conjugacy classes of Cartan subgroups.]

The discrete series is non-empty because the integral in 4.8 is (in general) non-zero but the integrand is zero outside a finite set by 4.14 and 4.15. Thus $\hat{\mu}(i) \neq 0$ for some $i \in \hat{G}$. We set $\hat{\mu}(i) = d(i)$ and call it the *formal dimension* of i. Although we have shown that $\dim V_i^+ - \dim V_i^- \neq 0 \Rightarrow \chi_i(Z) = (\gamma(Z^*))(\lambda+\rho_c) = \chi_{\lambda+\rho_c}(Z)$, where λ is the highest weight of V, this by no means implies that if $\dim V_i^+ \neq 0 \neq \dim V_j^+$, then $\chi_i = \chi_j$ though we do know that $\chi_i(\Omega) = \chi_j(\Omega)$ where Ω is the Casimir operator. It is not hard to get further. Fix a homomorphism $\chi: \mathfrak{z} \to \mathbf{C}$ and define

$$\Theta(\chi) = \sum_{i \in \hat{G}_d, \chi_i = \chi} d(i)\Theta_i$$

We can compute $\Theta(\chi)$ using 4.11 and letting V vary. (Note that the coefficients in 4.11 satisfy $n_\lambda^i = \operatorname{sign} w \, n_{w\lambda}^i$ for $w \in W$, the Weyl group of K - so $W \subset W_C$.) We deduce that $\Theta(\chi) \mid H \cap G' = 0$ unless χ is determined by a weight λ s.t $\lambda + \rho$ is non-singular, where $\rho = \frac{1}{2} \Sigma \alpha$, $\alpha \in \Phi_+$.

Finally, one fixes an irreducible K-module V with highest weight μ which is 'very non-singular'. That means that μ is well inside a Weyl chamber - precisely, the condition is the following:

$$\langle \mu+\rho_c, \alpha\rangle > \langle B, \alpha\rangle \quad \forall\ \alpha \in \Phi_+ \quad , \tag{4.17}$$

if B is any sum of *distinct* elements of Φ_+^n .

Algebraic arguments now show that (for such V) if $H_V^\pm \neq 0$ then there is a unique $i \in \hat{G}$ such that

1) $\dim(V_i^* \otimes V \otimes S^+)^K = 1$, $(V_i^* \otimes V \otimes S^-)^K = 0$;

2) $(V_j^* \otimes V \otimes S^\pm)^K = 0 \quad \forall\ j \in \hat{G} \setminus \{i\}$.

[Of course, $\Theta_i \mid H \cap G' = (D_K/\Delta_H)\chi_{\mu+\rho_c}$.]

So, for such a V , H_V^+ is a representation in the discrete series. Logically, we get most of them this way and these are parametrized by a subset of W orbits of $\Lambda + \rho$ (supposing G linear). The computations above give us strong information about the K-character of such discrete series representations. Such information can be obtained independently for any discrete series character and is needed to prove exhaustion.

5. We begin with a statement of the main result on discrete series characters. As in Lecture 4, $H \subset K \subset G$ denotes a compact Cartan subgroup, $\Lambda \cong \hat{H} \subset i\mathfrak{h}^*$ is the lattice of weights, $W = N_G H/H = N_K H/H$ is the Weyl group and W^C the Weyl group of $(\mathfrak{g}^C, \mathfrak{h}^C)$. Φ is the root system of $(\mathfrak{g}^C, \mathfrak{h}^C)$ and Φ^c the subset of compact roots; Φ_+ will be a suitable system of positive roots. We shall regard G as a subgroup G^C and shall suppose - for convenience - that G^C is simply-connected. Recall from §3 that if Θ is an invariant eigendistribution and Δ the Weyl denominator then Δ makes sense on each Cartan subgroup, $|\Delta|$ makes sense globally and $|\Delta|\Theta$ is locally bounded.

Theorem 5.1 (Harish-Chandra [20]). For each non-singular $\lambda \in \Lambda$ there exists a unique invariant eigendistribution Θ_λ such that

(a) Θ_λ decays at ∞ (this means that $|\Delta_B||\Theta_\lambda| \to 0$ at ∞ on any Cartan subgroup B);

(b) $\Theta_\lambda \mid H \cap G' = (-1)^q \sum_{w \in W} \mathrm{sign}(w) e^{w\lambda} / \Delta_H$,

where $\Delta_H = \prod_{\alpha \in \Phi_+} (e^{\alpha/2} - e^{-\alpha/2})$.

Moreover, every such Θ_λ is a discrete series character, and conversely.

There are several remarks to be made about this theorem.

5.2 If Θ_λ is bounded at ∞ and satisfies (b) then Θ_λ decays at ∞ if λ is non-singular (i.e. $\langle\lambda,\alpha\rangle \neq 0$ $\forall\, \alpha \in \Phi^c$) . We may see this on $SL(2,R)$. The set of elliptic conjugacy classes is compact so that the assertion only has content for the hyperbolic ones. For λ real, $\lambda \neq 0$ we have on A

$$\Theta_\lambda(t) = \frac{c_1 e^{\lambda t} + c_2 e^{-\lambda t}}{e^t - e^{-t}}$$

If Θ_λ is bounded at ∞ then $c_1 = 0$ or $c_2 = 0$ depending on the sign of λ . So not only is Θ_λ bounded at ∞ , it decays there. For a general group this remark tells us that we may, in the same way, replace 'decay' by 'bounded'.

5.3 If $\pi \in \hat{G}_d$ (i.e. π is in the discrete series) then Θ_π extends continuously from $C_o^\infty(G)$ to the n-th global Sobolev space H_n for some large n . ($f \in H_n \Leftrightarrow f \in L^2$ and $Xf \in L^2$ for all $X \in \mathfrak{u}(\mathfrak{g}^C)$ of degree $\leq n$.) This follows from the proof of the existence of the character plus the fact that if $\pi \in \hat{G}_d$ then the Hilbert-Schmidt norm of $\pi(f)$, $\|\pi(f)\|_{HS}$, is defined for $f \in L^2 \cap L^1$ and bounded by some constant, K_π , times the L^2-norm of f :

$$\|\pi(f)\|_{HS} \leq K_\pi \|f\|_2 \quad \forall\ f \in L^1 \cap L^2 \quad .$$

(The inequality is established quickly from the Abstract Plancherel Theorem.) That Θ_π extends to a global Sobolev space is a slightly stronger statement than Harish-Chandra's result that Θ_π is tempered [20] (Θ_π is tempered <=> $\Theta_\pi \in S$ where S is the analogue on G for the Schwartz space on R^n .)

5.4 If Θ is an invariant eigendistribution and Θ extends continuously to a global Sobolev space then Θ decays at ∞ . (This may be seen by considering Sobolev spaces on Cartan subgroups.)

5.5 If Θ is an invariant eigendistribution and if Θ decays at ∞ then (1) $\Theta \neq 0 \Rightarrow \mathrm{rk}G = \mathrm{rank}\ K$, (2) $\Theta | K \cap G' \neq 0$ and (3) Θ is entirely determined by its value on $K \cap G'$ and hence by its values on $H \cap G'$. (Nonetheless it is far from easy to write down what Θ is on another Cartan subgroup.)

5.6 The discrete series is parametrized by the orbits under W of the non-singular elements in Λ : that is, by certain orbits in the dual of the Lie algebra.

Between them, remarks 5.3-5.5 constitute a proof of theorem 4.3.

In this lecture we shall sketch a proof of Harish-Chandra's theorem (5.1) and, at the same time, prove the exhaustion statement for the discrete series. The argument depends crucially on remarks 5.3-5.5. We state two results contained in those remarks once again.

5.7 (1) Every discrete series character decays at ∞.
(2) If Θ decays at ∞ then Θ is determined by $\Theta \mid H \cap G'$, where Θ is an invariant eigendistribution.

Note that (1) is established without explicit knowledge of the discrete series.

If the highest weight μ of the irreducible K-module V was very non-singular (see 4.17) we had indicated that $H_V^- = 0$, that H_V^+ was an irreducible representation in $\hat{G}_d$ having a character whose restriction to $H \cap G'$ was $\Theta_{\mu+\rho_c}$, where $\rho_c = \frac{1}{2}\Sigma\alpha$, $\alpha \in \Phi_+^c$. This formula is that of clause (b) of theorem 5.1. Because of 5.7(2) the character is thus completely determined and is the $\Theta_{\mu+\rho_c}$ of the theorem. It turns out that the result for the very non-singular case together with the formula for $\Theta(\chi) \mid H \cap G'$ - which can be obtained from 4.11 - implies all of theorem 5.1. ($\Theta(\chi) = \Sigma d(i)\Theta_i$: $i \in \hat{G}_d$ and $\chi_i = \chi$.) This uses a trick due to G.Zuckerman. It is to tensor with finite-dimensional representations of G and to study what happens. It is not hard but it is somewhat surprising that it is successful. One is naturally led outside the category of unitary representations. Indeed, this is a modest example of the principle that one need to consider Banach space representations too. Details are in §8 of [3]. What one proves is that any discrete series representation can be realized as a subrepresentation of one of the form $H_V^+ \otimes W$ where V is an irreducible K-module with very non-singular highest weight μ and W is a *finite-dimensional* representation of G. This is done by a character computation (using 4.11) and by consideration of the possible global characters with a given infinitesimal character χ: that is, we consider

$\Theta(\chi)$ as a sum of irreducible characters. We arrive at the following exhaustion statement.

Proposition 5.8.

(i) If χ is an infinitesimal character (i.e. $\chi: \mathfrak{z} \to \mathbb{C}$) such that $\Theta(\chi) \mid H \cap G' = 0$ then $\nexists$ $i \in \hat{G}_d$ such that $\chi_i = \chi$.

(ii) $\Theta(\chi) = \Sigma\, d(\lambda)\Theta_\lambda$ on $H \cap G'$, where $d(\lambda)$ = Plancherel measure of λ, $\chi_\lambda = \chi$, $\lambda \in \Lambda \cap C$ and C is the fundamental chamber.

The discrete series representations are parametrized in the same way as the finite-dimensional representations of a compact group. For them the Harish-Chandra character formula (5.1) looks like Weyl's character formula. But there is a substantial difference. Weyl's formula determines the character globally because the conjugates of the torus cover the compact group; but the conjugates of H do not cover G so that the Harish-Chandra formula only determines the character in principle. (As we have indicated, the formulae for a general Cartan subgroup are impossibly hard.) There is a formal analogue to the theorem of the highest weight and an analogue (Blattner's conjecture [22]) to the formula [25] for the multiplicity of a weight. In particular, if π_λ is the representation with character Θ_λ (Θ_λ as in 5.1) then the irreducible K-module of highest weight $\lambda + \rho_n - \rho_c$ occurs exactly once in the restriction $\pi_\lambda | K$.

Finally, using powerful results on the explicit form of the Plancherel measure one can prove that if the representation V has highest weight μ such that $\mu + \rho_c$ is non-singular then $H_V^- = 0$ and H_V^+ is irreducible with character $\Theta_{\mu+\rho_c}$. Moreover, if $\mu + \rho_c$ is singular then $H_V^- = 0 = H_V^+$. All discrete series can consequently be realised in this way. [If G is a linear group this last assertion can be proved much more simply, though not the two earlier ones.]

Now we turn briefly to the problem of constructing other series of representations. The discrete series representations are the most difficult and they correspond to a *compact* Cartan subgroup. (There is at most one conjugacy class of such.) To each conjugacy class $\{B\}$ of Cartan subgroup B we have a series of representations - the B series. At one extreme is the *discrete series*; at the other we have the *principal series*, corresponding to a maximally non-compact Cartan subgroup.

As earlier, let us suppose that $G \subset G^C$ and that G^C is simply connected. Let B be a Cartan subgroup of G. Recall that B is not necessarily connected and that B/B^o is perhaps non-abelian. $B^o = A \times T$ where T is compact and A is a vector space. $Z_G A = M.A$ for some M, where M is reductive and $B \cap M$ is a compact Cartan subgroup in M. (M, again, may not always be connected.) Let $\mathfrak{n}$ = Lie algebra spanned by those roots of $(\mathfrak{g}^C, \mathfrak{b}^C)$ whose restriction to $\mathfrak{a}$ is positive. Let N be the corresponding nilpotent subgroup. The group $P = M.A.N$ is a *cuspidal parabolic* subgroup. (Cuspidal means that M - the Levi subgroup of the parabolic group P - has a compact Cartan subgroup, [31].)

Although M may not be semi-simple it is nearly so. There will, however, be some minor problems if it is not connected. These we shall ignore. Let $\tau \in \hat{M}_d$, $e^\nu \in \hat{A}$ and set $\pi_{\tau,\nu} = \mathrm{ind}_P^G(\tau \otimes e^\nu \otimes 1)$; the representation of G induced from the indicated one of P. G/P will not have an invariant measure so we shall have to take the modified induction procedure as for $SL(2,R)$. As we have seen, τ is parametrized by unitary characters of $B \cap M$ modulo the action of the Weyl group. The pair (τ, e^ν) may thus be given in terms of unitary characters of B, and for every character $\sigma \in \hat{B}$ we get a representation π_σ of G such that if $\sigma = (\tau, e^\nu)$ then $\pi_\sigma = \pi_{\tau,\nu}$.

Theorem 5.9.

(i) σ non-singular => π_σ is irreducible and π_σ is independent of the choice of N.

(ii) If σ, σ' are non-singular $\pi_\sigma \approx \pi_{\sigma'}$ $\iff$ $\sigma = w\sigma'$, $w \in W_B = N_G B/B$.

By considering restrictions of the characters to B we see that π_σ is never equivalent to a representation coming from a Cartan subgroup B' not conjugate to B. So to each conjugacy class of Cartan subgroups there is associated a series of irreducible unitary representations parametrized by the non-singular unitary dual of B mod W_B. It is called the *non-degenerate series* corresponding to B.

On the one extreme we have the discrete series if B is compact so that there is no inducing : $P = G$. At the other extreme when A is as large as possible M is compact and we have the unitary principal series. Usually there are others in between, but in the case of $SL(2,\mathbb{R})$ only the two extremes occur: and if G is complex semi-simple only the last extreme occurs. In general (when there may be no compact Cartan subgroup) the *fundamental* series is that corresponding to the (unique) Cartan subgroup with maximal compact part. (So for G semi-simple complex the principal and fundamental series coincide!) In case of each series we can say - if we adopt Kostant's point of view - that they are parametrized by certain orbits in the dual of the Lie algebra.

These constructions give us many unitary representations, but not all. However, they are enough to support the Plancherel measure, as Harish-Chandra has shown. We shall finish with a short discussion of the Plancherel measure. According to the Abstract Plancherel Theorem (1.3)

$$L^2(G) \cong \int_{\hat{G}} V_i \otimes V_i^* d\hat{\mu}(i) \cong \int_{\hat{G}} \mathrm{Hom}(V_i, V_i)_{HS} d\hat{\mu}(i) \quad .$$

As such the measure class only is determined, but once one fixes an explicit isomorphism there is a unique measure.

Suppose $f \in \Omega^{o}_{comp}(G) \equiv C^{\infty}_{o}(G)$ is a smooth function with compact support. Set $f^{*}(g) = \overline{f(g^{-1})}$.

$$f * f^{*}(1) = \|f\|_2^2 = \int_{\hat{G}} \|\pi_i f\|_{HS}^2 \, d\hat{\mu}(i)$$
$$= \int_{\hat{G}} tr(\pi_i f \circ \pi_i f^{*}) d\hat{\mu}(i)$$
$$= \int_{\hat{G}} tr\ \pi_i(f * f^{*}) d\hat{\mu}(i)$$
$$= \int_{\hat{G}} \Theta_i(f * f^{*}) d\hat{\mu}(i)$$

By approximating one sees that for all $f \in C^{\infty}_{o}(G)$,

$$f(1) = \int_{\hat{G}} \Theta_i(f) d\hat{\mu}(i) \tag{5.10}$$

This formula determines $\hat{\mu}$ on $\hat{G}$. In particular, if we have a measurable subset $S \subset \hat{G}$ and a positive measure μ on S such that

$$f(1) = \int_S \Theta_i(f) d\mu(i)$$

for all $f \in C^{\infty}_{o}(G)$ then we know that that measure is the Plancherel measure. From this we learn what to do: ideally we do not need to find all of $\hat{G}$.

Let us consider $SL(2,\mathbb{R})$ by way of example. We have the *principal* series $\pi_{s,even}, \pi_{s,odd}$; $s \in i\mathbb{R}_+$, and the *discrete* series $\pm_n$, $n \in \mathbb{N}\backslash\{0,1\}$. We shall write $\Theta_\cdot$ for the character corresponding to $\pi_\cdot$, as in §3. Let K,A be as there, too (so $K = SO(2)$ and $A \cong \mathbb{R}$) and let $\varepsilon(n)$ denote the sign of n .

For $f \in C^{\infty}_{o}(G)$,

$$2\pi f(1) = \sum_{\substack{n\in Z\\ n\neq 0,\pm 1}} |n-1| \int_{K^G} \Theta_{|n|}^{\varepsilon(n)} (g)g(g)dg \quad ,$$

where K^G = union of conjugacy classes meeting K. One shows this by making into an integral over $K \cong S^1$ and then using Fourier analysis. Now

$$\int_{K^G} = \int_G - \int_{(\pm A)^G} \quad ,$$

so that one can rewrite the right-hand side and obtain:

$$2\pi f(1) = \sum_{\substack{n\in Z\\ n\neq 0,\pm 1}} |n-1| \; \Theta_{|n|}^{\varepsilon(n)} (f) - \int_{(\pm A)^G} \Big(\sum_{\substack{n\in Z\\ n\neq 0,\pm 1}} |n-1| \; \Theta_{|n|}^{\varepsilon(n)} f\Big) \, dg$$

The integral on the right can now be reduced to an integral over $\hat{A}$ of principal series characters. Note that such characters vanish on $K \cap G'$ and hence on K^G. Indeed, this is typical: characters of a B-series will vanish on any more compact Cartan subgroup.

Finally, we remark that recent work of Langlands, Knapp-Zuckerman and Vogan has resulted in a complete classification of the irreducible admissible Banach space representations of a semi-simple Lie group. But one does not know which are unitary.

References

[1] M.F. Atiyah: Elliptic operators, discrete groups and von Neumann algebras, *Astérisque* 32/33 (1976), 43-72.

[2] M.F. Atiyah: Characters of semi-simple Lie groups, Mimeographed lecture notes, University of Oxford.

[3] M.F. Atiyah and W. Schmid: A geometric construction of the discrete series for semi-simple Lie groups, *Inventiones* 42 (1977), 1-62.

[4] V. Bargmann: Irreducible unitary representations of the Lorentz group, *Ann. of Math.* 48 (1947), 568-640.

[5] A. Borel: Compact Clifford-Klein forms of symmetric spaces, *Topology* 2 (1963), 111-122.

[6] A. Borel: Représentations de Groupes localement compacts, Springer lectures notes no.276 (1972).

[7] R. Bott: The index theorem for homogeneous differential operators, 'Differential and Combinatorial Topology', pp.167-186, Princeton University Press, 1965.

[8] J. Dixmier: Représentations intégrables du groupe de De Sitter, *Bull.Soc.Math. France* 89 (1961), 9-41.

[9] T.J. Enright and V.S. Varadarajan: On the infinitesimal characterization of the discrete series, *Ann. of Math.* 102 (1975), 1-16.

[10] I.M. Gelfand, M. Graev and I. Pyateski-Shapiro: Representation Theory and Automorphic Functions, Moscow 1966; translated: W.B. Saunders, 1969.

[11] R. Godement: A theory of spherical functions I, *Trans. A.M.S.* 73 (1952), 496-556.

[12] R. Godement: Théorie des charactères I & II, *Ann. of Math.* (2) 59 (1954), 47-62 and 63-85.

[13] Harish-Chandra: The Plancherel formula for the complex semi-simple Lie groups, *Trans. A.M.S.* 76 (1954), 485-528.

[14] Harish-Chandra: Representations of semi-simple Lie groups I, *Trans. A.M.S.* 75 (1953), 185-243.

[15] Harish-Chandra: Representations of semi-simple Lie groups II & III, *Trans. A.M.S.* 76 (1954), 26-65 and 234-252.

[16] Harish-Chandra: Representations of semi-simple Lie groups IV, *Amer.J.Math.* 77 (1955), 743-777.

[17] Harish-Chandra: Representations of semi-simple Lie groups V & VI, *Amer.J.Math.* 78 (1956), 1-41 and 564-628.

[18] Harish-Chandra: Characters of semi-simple Lie groups, *Trans. A.M.S.* 83 (1956), 98-163.

[19] Harish-Chandra: Discrete series for semi-simple Lie groups I, *Acta Math.* 113 (1965), 241-318.

[20] Harish-Chandra: Discrete series for semi-simple Lie groups II, *Acta Math.* 116 (1966), 1-111.

[21] Harish-Chandra: Harmonic analysis on semi-simple Lie groups, *Bull. A.M.S.* 76 (1970), 529-587.

[22] H. Hecht and W. Schmid: A proof of Blattner's conjecture, *Inventiones* 31 (1975), 129-154.

[23] S. Helgason: Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[24] F. Hirzebruch: Automorphe Formen und der Satz von Riemann-Roch, International Symposium in Algebraic Topology, Mexico 1958.

[25] B. Kostant: A formula for the multiplicity of a weight, *Trans. A.M.S.* 9 (1959), 53-73.

[26] S. Lang: $SL_2(R)$, Addison-Wesley, 1975.

[27] R.P. Langlands: On the classification of irreducible representations of real algebraic groups, mimeographed notes, the Institute for Advanced Study, Princeton, 1973.

[28] R. Parthasarathy: Dirac operators and the discrete series, *Ann. of Math.* 96 (1972), 1-30.

[29] W. Schmid: Some properties of square-integrable representations of semi-simple Lie groups, *Ann. of Math.* 102 (1975), 535-564.

[30] W. Schmid: On the characters of the discrete series, *Inventiones* 30 (1975), 47-144.

[31] G. Warner: Harmonic Analysis on Semi-simple Lie Groups I, Springer Verlag, Heidelberg, 1972.

[32] G. Zuckermann: Tensor products of finite and infinite dimensional representations of semi-simple Lie groups, *Ann. of Math.* 106 (1977), 295-308.

9 · Invariant differential operators and eigenspace representations

SIGURDUR HELGASON

Massachusetts Institute of Technology

1. Introduction

These lectures will focus on the construction and properties of certain representations of Lie groups, which I have called *eigenspace representations*. The construction is rather general and in fact, all representations of Lie groups with which I am familiar can be realized by means of this process. However, there is no particular emphasis on unitary representations.

The construction is based on the concept of an invariant differential operator. Let M be a manifold. A *differential operator* on M is a linear mapping $D: C_c^\infty(M) \to C_c^\infty(M)$ such that for each $p \in M$ there exists a coordinate neighbourhood U with coordinates $(x_1,\ldots,x_m)$ such that

$$(Df)(x) = \sum_{(n)} a_{n_1\ldots n_m}(x)(\partial_1^{n_1}\ldots\partial_m^{n_m} f)(x) \qquad (1)$$

for $f \in C_c^\infty(U)$, $x \in U$ where $\partial_i = \partial/\partial x_i$ and the coefficients $a_{n_1\ldots n_m}$ are C^∞ functions on U.

The following criterion ([18]) is often convenient when one wants to know whether a certain operator is a differential operator.

Theorem 1.1. *A linear mapping* $L: C^\infty(M) \to C^\infty(M)$ *with the property that*

$$\operatorname{supp}(Lf) \subset \operatorname{supp}(f) \qquad f \in C^\infty(M)$$

(supp *denoting support) is a differential operator.*

If L is assumed continuous this is an immediate consequence of Schwartz' theorem about distributions of point support ([8(a)], p.242).

Let D be a differential operator on a manifold M and let ϕ be a diffeomorphism of M onto itself. Put

$$f^{\phi} = f \circ \phi^{-1} \qquad f \in C^{\infty}(M)$$

and define D^{ϕ} by

$$D^{\phi} : f \to (Df^{\phi^{-1}})^{\phi} = (D(f \circ \phi)) \circ \phi^{-1} \quad .$$

Then D^{ϕ} is a differential operator (use Theorem 1.1); it is called *invariant under* ϕ if $D^{\phi} = D$, i.e., if $D(f \circ \phi) = (Df) \circ \phi$ for all f. Note that $(Df)^{\phi} = D^{\phi}f^{\phi}$, justifying the notation.

Examples

(i) A differential operator on $\mathbb{R}^n$ is invariant under all translations if and only if it has constant coefficients (in the standard coordinate system on $\mathbb{R}^n$).

(ii) The d'Alembertian

$$\square = \partial_1^2 - \partial_2^2 - \partial_3^2 - \partial_4^2$$

and the polynomials in it are characterized by their invariance under the Poincaré group.

(iii) The operator

$$E = \frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} + ix\,\frac{\partial}{\partial z}$$

on the Heisenberg group of matrices

$$\sigma = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \qquad x,y,x \in \mathbb{R}$$

is invariant under all left translations $L_{\sigma_0}: \sigma \to \sigma_0\sigma$. To verify this note that under the correspondence $\sigma \to (x,y,z)$ we have $\sigma_0\sigma \to (x_0 + x, y_0 + y, z_0 + z + x_0 y)$ so

$$[E(f \circ L_{\sigma_0}](\sigma) = (\partial_1 f)(\sigma_0\sigma) + i(\partial_2 f)(\sigma_0\sigma) + ix_0(\partial_3 f)(\sigma_0\sigma) + ix(\partial_3 f)(\sigma_0\sigma)$$

and

$$(Ef)(\sigma_0\sigma) = (\partial_1 f)(\sigma_0\sigma) + i(\partial_2 f)(\sigma_0\sigma) + i(x_0+x)(\partial_3 f)(\sigma_0\sigma) .$$

Let us now assume that the manifold M is *homogeneous*, i.e., that it has a separable Lie group G acting transitively on it. Then if $p \in M$ is arbitrary and G_p the subgroup of G fixing p (the *isotropy* group at p) the mapping

$$gG_p \to g\cdot p \tag{2}$$

is a diffeomorphism of G/G_p onto M. This fact is extremely important because the Lie-theoretic information about G/G_p gets transferred to M via (2). On the other hand if H is a closed subgroup of a Lie group G, the space G/H of left cosets is a manifold such that by means of the translations $\tau(g): xH \to gxH$ the group G acts transitively on G/H.

Examples

a) The group $SO(3)$ acts transitively on the two-sphere S^2; the isotropy group at each point can be identified with $SO(2)$ so

$$SO(3)/SO(2) = S^2 \quad . \tag{3}$$

b) Let $\mathbb{M}(2)$ be the group of orientation-preserving isometries of the plane $\mathbb{R}^2$. Then

$$\mathbb{M}(2)/SO(2) = \mathbb{R}^2 \quad . \tag{4}$$

c) Let $SU(1,1)$ denote the group of matrices

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1 \quad .$$

Then $SU(1,1)$ acts transitively on the hyperbolic plane $\mathbb{H}^2$ (the unit disk $|z| < 1$) by

$$g\cdot z = \frac{az+b}{\bar{b}z+\bar{a}}$$

so

$$SU(1,1)/SO(2) = \mathbb{H}^2 \quad . \tag{5}$$

Let D be a differential operator on the manifold G/H (H a closed subgroup of the Lie group G); D is called *invariant* (or G-invariant) if $D^{\tau(g)} = D$ for all $g \in G$. Let $\mathbb{D}(G/H)$ denote the set of G-invariant differential operators on G/H; under addition and composition of operators $\mathbb{D}(G/H)$ is an algebra over the complex numbers. It is easy to verify for Example b) above that $\mathbb{D}(G/H)$ consists of the polynomials in the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$. For Example a) and Example c) the same result holds with L respectively, replaced by the Laplacian

$$L = \frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \sin^{-2}\theta\,\frac{\partial^2}{\partial\phi^2}$$

on S^2 and by the Laplacian

$$L = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{6}$$

on the hyperbolic disk $\mathbb{H}^2$ with the Riemannian metric $ds^2 = (1-x^2-y^2)^{-2}(dx^2+dy^2)$. The reason that for these examples $\mathbb{D}(G/H)$ has just one generator is the high degree of mobility of the spaces: They are *two-point homogeneous* in the sense that for each $r > 0$ the isometry group permutes transitively all pairs of points of distance r . For such spaces we have in fact (cf. [8(a)] and Corollary 2.10 below).

Theorem 1.2. *If M is a two-point homogeneous space, G the Lie group of all isometries, then the only G-invariant differential operators on M are the polynomials in the Laplace-Beltrami operator. The same result holds if M is any isotropic Lorentz manifold.*

Since invariant differential operators form a natural generalization of constant coefficient differential operators there is a host of problems which automatically suggest themselves. In these lectures we will focus attention on the joint eigenfunctions of these operators and on the representations induced on the joint eigenspaces. Let us now explain this in more detail.

Let G be a topological group, V a topological vector space, and $\mathrm{Aut}(V)$ the group of linear homeomorphisms of V onto itself. A *representation* of G on V is a homomorphism $\pi: G \to \mathrm{Aut}(V)$ such that the mapping $(g,v) \to \pi(g)v$ is continuous from $G \times V$ to V .

If H is a closed subgroup of the Lie group G we want to study a particularly interesting class of functions on G/H namely those which are eigenfunctions of each of the operators in $\mathbb{D}(G/H)$. So we fix a homomorphism

$\chi: \mathbb{D}(G/H) \to \mathbb{C}$

and let E_χ denote the corresponding space of joint eigenfunctions, i.e.,

$$E_\chi = \{f \in C^\infty(G/H) \,|\, Df = \chi(D)f \quad \text{for all} \quad D \in \mathbb{D}(G/H)\} \quad .$$

Then E_χ is a closed subspace of the topological vector space $C^\infty(G/H)$, hence it is another topological vector space. We now define a representation T_χ of G on E_χ

$$T_\chi(g)f = f^{\tau(g)} \qquad\qquad g \in G, \; f \in E_\chi \quad .$$

In fact, $f^{\tau(g)} \in E_\chi$ and the definition of the topology of $C^\infty(M)$ ([19]) shows that T_χ has the continuity property required of a representation.

The representations T_χ (introduced in [8(e)]) will be called *eigenspace representations*. These representations of G are a kind of a counterpart to the *induced representations* of G (see e.g., Mackey [15]); the first class of representations is given by the homomorphism $\chi: \mathbb{D}(G/H) \to \mathbb{C}$, the second class by the representations of H.

Problem. For a given coset space G/H determine the joint eigenspaces E_χ and identify the representations T_χ; in particular, for what χ is T_χ irreducible?

This problem is considerably broadened by replacing functions by sections of vector bundles over M and replacing D by a differential operator on the space of sections invariant under the action of G. The Dirac operator becomes an invariant differential operator in this general sense.

We shall now discuss the problem above for some examples. More general cases and indications of proofs will be given later.

Theorem 1.3.

(a) *The eigenspaces of the Laplacian* L *on the sphere* $S^2 = SO(3)/SO(2)$ *are of the form*

$$E_k = \text{span of } \{(\alpha s_1+\beta s_2+\gamma s_3)^k \,|\, \alpha^2+\beta^2+\gamma^2 = 0\} \quad ,$$

k *being a nonnegative integer. Here*

$$(s_1,s_2,s_3) \in S^2 \subset \mathbb{R}^3 \quad \textit{and} \quad \alpha,\beta,\gamma \in \mathbb{C} \quad .$$

(b) *The group* SO(3) *acts irreducibly on each eigenspace* E_k .

This result is well known from the classical theory of spherical harmonics. Next we consider the case $SU(1,1)/SO(2) = \mathbb{H}^2$. The Poisson kernel

$$P(z,b) = \frac{1-|z|^2}{|z-b|^2} \qquad |z| < 1 \; , \quad |b| = 1$$

is of course annihilated by the Laplacian $L = (1-|z|^2)^2 \, (\partial_x^2+\partial_y^2)$. It can also be written

$$P(z,b) = e^{2<z,b>} \quad ,$$

where $<z,b>$ is the non-Euclidean distance (with sign) from 0 to the horocycle through z and b . More generally, if $\nu \in \mathbb{C}$ a direct computation shows that $z \to e^{2\nu<z,b>}$ is an eigenfunction,

$$L_z(e^{2\nu<z,b>}) = 4\nu(\nu-1)e^{2\nu<z,b>} \quad ,$$

and we shall now see that all the eigenfunctions of L are given by superposition of these (cf. [8(d)], p.139, [8(g)]).

Theorem 1.4.

(a) *The eigenfunctions of* L *on* $\mathbb{H}^2$ *are precisely the functions*

$$F(z) = \int_B e^{2\nu<z,b>} dT(b) \quad ,$$

where $\nu \in \mathbb{C}$ *and* T *is an analytic functional on the boundary* B .

(b) *The representation* T_ν *of* $SU(1,1)$ *on the eigenspace*

$$E_\nu = \{u: Lu = 4\nu(\nu-1)u\}$$

is irreducible if and only if ν *is not an integer.*

We recall the concept of analytic functional: Let $\mathfrak{A}(B) \subset C^\infty(B)$ be the subspace of analytic functions on the circumference B: if $f \in \mathfrak{A}(B)$ and $b_0 \in B$, f has a power series expansion near b_0 in terms of the arc parameter. We now describe the topology of $\mathfrak{A}(B)$ and here B is allowed to be any compact analytic Riemannian manifold (cf. [14]) with Laplace-Beltrami operator L_B. For $M > 0$ we put

$$|F|_M = \sup_{k \in Z^+} \frac{1}{(2k)!} M^k \|L_B^k F\| \quad ,$$

where $\| \ \|$ is the L^2 norm on B and put

$$\mathfrak{A}_M(B) = \{F \in C^\infty(B) \mid |F|_M < \infty\} \quad .$$

Then $\mathfrak{A}_M(B)$ is a Banach space, $\mathfrak{A}(B)$ is the union of the spaces $\mathfrak{A}_M(B)$ and is accordingly given the *inductive limit* topology: a convex subset $U \subset \mathfrak{A}(B)$ is a neighbourhood of 0 if $U \cap \mathfrak{A}_M(B)$ is a neighbourhood of 0 in $\mathfrak{A}_M(B)$ for each M. With this topology of $\mathfrak{A}(B)$ the dual space $\mathfrak{A}'(B)$ consists, by definition, of the *analytic functionals*.

While (b) is best understood as a special case of its generalization to symmetric spaces stated later one can see as follows that for certain ν the space E_ν has invariant subspaces. The space $\mathbf{H}^2$ can be viewed as the quadric $-x_1^2 - x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$ with the induced Riemannian structure. The group $SO(2,1)$ then acts on the eigenspaces of the Laplacian L. The 'Lorentzian' Laplacian on $\mathbb{R}^3$ has the form $\square = \partial_r^2 + \frac{2}{r}\partial_r - r^{-2}L$ $(r^2 = -x_1^2 - x_2^2 + x_3^2)$ so if P is a homogeneous polynomial satisfying $\square P = 0$ then the restriction $P|\mathbf{H}^2$ is an eigenfunction of L. In this way we obtain finite-dimensional invariant eigenspaces of L.

Next we consider our problem in the case of the plane $\mathbf{M}(2)/SO(2) = \mathbb{R}^2$ for which $L_0 = \partial_x^2 + \partial_y^2$ generates the invariant differential operators. If $x \in \mathbb{R}^2$ and ω a unit vector, the scalar product (x,ω) is the analog of $\langle z,b\rangle$ for $\mathbf{H}^2$ and $e^{i\lambda(x,\omega)}$ is an eigenfunction of L_0 with eigenvalue $-\lambda^2$. This holds more generally for the function

$$f(x) = \int_{S^1} e^{i\lambda(x,\omega)}\,dT(\omega) \quad ,$$

where T is an analytic functional on the unit circle S^1. But in order to get all the eigenfunctions we need more general functionals.

Given $a,b > 0$ let $E_{a,b}$ denote the space of holomorphic functions f on $\mathbb{C} - (0)$ satisfying

$$\|f\|_{a,b} = \sup_z (|f(z)|e^{-a|z|-b|z|^{-1}}) < \infty \quad . \tag{7}$$

Then $E_{a,b}$ is a Banach space with norm the left hand side of (7) and we give the union $E = \cup_{a,b} E_{a,b}$ the inductive limit topology. Identifying the members of E with their restrictions to S^1 we have a continuous imbedding $E \subset \mathfrak{A}(S^1)$ so the dual E', whose members we call *entire functionals* on S^1, contains $\mathfrak{A}'(B)$. We now have the following solution to our problem for the plane $\mathbb{R}^2$ ([8(g)]).

Theorem 1.5.

(a) *The eigenfunctions of the Laplacian* L_0 *on* $\mathbb{R}^2$ *are precisely the harmonic functions and the functions*

$$f(x) = \int_{S^1} e^{i\lambda(x,\omega)} dT(\omega)$$

where $\lambda \in \mathbb{C} - (0)$ *and* T *is an entire functional on* S^1.

(b) *The action of the group* M(2) *on the eigenspace*

$$\{u\colon L_0 u = -\lambda^2 u\} \qquad (\lambda \in \mathbb{C})$$

is irreducible if and only if $\lambda \neq 0$.

Let us look closer at the exceptional case $\lambda = 0$. Now there is a bigger group acting on the eigenspace. In fact, L_0 is quasi-invariant under the conformal group in the following sense. The group $SL(2,\mathbb{C})$ acts on $\mathbb{R}^2$ (or rather its compactification S^2) by

$$g\colon z \to \frac{az+b}{cz+d} \qquad \text{if} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det g = 1 .$$

We have

$$L_0 = 4\frac{\partial^2}{\partial z \partial \bar{z}} ,$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

so since

$$\left(\frac{\partial}{\partial z}\right)^g = (cz-a)^2\,\frac{\partial}{\partial z} \quad , \quad \left(\frac{\partial}{\partial \bar{z}}\right)^g = (\bar{c}\bar{z}-\bar{a})^2\,\frac{\partial}{\partial \bar{z}}$$

we derive

$$L_0^g = |cz-a|^4\, L_0 \quad .$$

Thus if u is harmonic on $U \subset \mathbb{R}^2$ then $u \circ g$ is harmonic on $(g^{-1}U) \cap \mathbb{R}^2$. But since $SL(2,C)$ does not map $\mathbb{R}^2$ into itself we consider the induced action of $s\ell(2,C)$ (6-dimensional Lie algebra over $\mathbb{R}$) defined as follows. Let $X \in s\ell(2,\mathbb{C})$, $g_t = \exp tX$, $u \in C^\infty(\mathbb{R}^2)$. We define

$$(Xu)(z) = \left(\frac{d}{dt}\, u^{g_t}\right)_{t=0}(z) \quad .$$

Then $L_0u = 0$ implies $L_0Xu = 0$ so we have a representation of $s\ell(2,\mathbb{C})$ (as a real Lie algebra) on the space $\mathfrak{X}$ of harmonic functions on $\mathbb{R}^2$. We have then the following easy complement to Theorem 1.5(b), cf. [8(h)].

Proposition 1.6. *The action of* $s\ell(2,\mathbb{C})$ *on the space* $\mathfrak{X}$ *of harmonic functions on* $\mathbb{R}^2$ *is 'scalar irreducible' i.e., the only continuous operators which commute with the action are the scalar multiples of* I .

We conclude this section with some further cases of eigenspace representations. Let N be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $\lambda \in \mathfrak{n}^*$, the dual of $\mathfrak{n}$ and let $\mathfrak{m} \subset \mathfrak{n}$ be a subalgebra of maximal dimension satisfying $\lambda([\mathfrak{m},\mathfrak{m}]) = 0$. Put $\mathfrak{H} = \mathfrak{m} \cap \text{kernel}(\lambda)$, and let $H \subset N$ be the analytic subgroup corresponding to $\mathfrak{H}$. Using

Kirillov's description of the unitary representations of N, Hole [9] proved the following result.

Theorem 1.7. *The eigenspace representations for* N/H *are irreducible for all nonzero homomorphisms* $\mu: \mathbb{D}(N/H) \to \mathbb{C}$. *The choice of special* μ *and* L^2 *eigenfunctions gives rise to all the irreducible unitary representations of* N.

Next, let G be a simply connected complex semisimple Lie group, N the connected Lie subgroup whose Lie algebra is spanned by the positive root vectors. As we shall prove in §3 the finite-dimensional representations of G are precisely the eigenspace representations of *holomorphic* functions on G/N.

The case of the (ax+b)-group was worked out by Henrik Stetkaer at the conference. He verified that all its unitary representations arise as eigenspace representations if one takes both nontrivial subgroups into account.

2. Invariant differential operators on Lie groups and homogeneous spaces

Given a coset space G/H our aim is now to describe the operators in $\mathbb{D}(G/H)$. First we consider the case when $H = (e)$ and write $\mathbb{D}(G)$ for $\mathbb{D}(G/(e))$, the set of left invariant differential operators on G.

If V is a finite-dimensional vector space over $\mathbb{R}$ the *symmetric algebra* $S(V)$ over V is defined as the algebra of complex-valued polynomial functions on the dual space V^*. If $X_1,\dots,X_n$ is a basis of V, $S(V)$ can be identified with the (commutative) algebra of polynomials

$$\sum_{(k)} a_{k_1 \dots k_n} X_1^{k_1} \dots X_n^{k_n} .$$

Let $\mathfrak{g}$ denote the Lie algebra of G (the tangent space to G at e) and $\exp: \mathfrak{g} \to G$ the exponential mapping which maps a line $\mathbb{R}X$ through 0 in $\mathfrak{g}$ onto a one-parameter subgroup $t \to \exp tX$ of G. If $X \in \mathfrak{g}$ let $\tilde{X}$ denote the vector field on G given by

$$(\tilde{X}f)(g) = X(f \circ L_g) = \left\{\frac{d}{dt} f(g \exp tX)\right\}_{t=0} , \quad f \in C^\infty(G) \qquad (1)$$

where L_g denotes the left translation $x \to gx$ of G onto itself. Then $\tilde{X}$ is a differential operator on G and if $h \in G$

$$(\tilde{X}^{L_h}f)(g) = (\tilde{X}(f \circ L_h))(h^{-1}g) = (\tilde{X}f)(g)$$

so $\tilde{X} \in \mathbb{D}(G)$. Moreover, the bracket on $\mathfrak{g}$ is by definition given by

$$[X,Y]^\sim = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X} \quad , \qquad X,Y \in \mathfrak{g} \quad ,$$

the multiplication on the right hand side being composition of operators.

The following result connecting $S(\mathfrak{g})$ and $\mathbb{D}(G)$ is a modification of results of Harish-Chandra and Schwartz ([6(a), p.192] and [6(c), p.111]).

Theorem 2.1. *Let G be any Lie group with Lie algebra $\mathfrak{g}$. Let $S(\mathfrak{g})$ denote the symmetric algebra over the vector space $\mathfrak{g}$. Then there exists a unique linear bijection*

$$\lambda: S(\mathfrak{g}) \to \mathbb{D}(G)$$

such that $\lambda(X^m) = \tilde{X}^m$ $(X \in \mathfrak{g}, m \in Z^+)$. If $X_1,\dots,X_n$ is any basis of $\mathfrak{g}$ and $P \in S(\mathfrak{g})$ then

$$(\lambda(P)f)(g) = \left\{P(\partial_1,\ldots,\partial_n)f(g\,\exp(t_1X_1+\ldots+t_nX_n)\right\}_{t=0}\ , \qquad (2)$$

where $f \in C^\infty(G)$, $\partial_i = \partial/\partial t_i$ *and* $t = (t_1,\ldots,t_n)$.

Proof. Fix a basis $X_1,\ldots,X_n$ of $\mathfrak{g}$. The the mapping

$$g\,\exp(t_1X_1+\ldots+t_nX_n) \to (t_1,\ldots,t_n)$$

is a coordinate system on a neighbourhood of g in G so formula (2) defines a differential operator $\lambda(P)$ on G . Clearly $\lambda(P)$ is left invariant and by (1), $\lambda(X_i) = \tilde{X}_i$, so by linearity $\lambda(X) = \tilde{X}$ for $X \in \mathfrak{g}$. Also

$$\begin{aligned}(\tilde{X}^2f)(g) = \tilde{X}(\tilde{X}f)(g) &= \left\{\frac{d}{dt}\,(\tilde{X}f)(g\exp tX)\right\}_{t=0}\\ &= \left\{\frac{d}{dt}\left\{\frac{d}{ds}\tilde{X}f(g\exp tX\exp sX)\right\}_{s=0}\right\}_{t=0}\\ &= \left\{\frac{d^2}{dt^2}\ f(g\exp tX)\right\}_{t=0}\end{aligned}$$

which, writing $X = \sum_1^n x_iX_i$, equals

$$\sum x_ix_j(\lambda(X_iX_j)f)(g) = (\lambda(X^2)f)(g) \quad .$$

By the same argument

$$\lambda(X^m) = \tilde{X}^m \qquad\qquad X \in \mathfrak{g},\ m \in Z^+ \quad . \qquad (3)$$

For a fixed $m \in Z^+$, the powers X^m $(X \in \mathfrak{g})$ span the subspace $S^m(\mathfrak{g}) \subset S(\mathfrak{g})$ of homogeneous elements of degree m . In fact, the bilinear form on $S^m(\mathfrak{g}) \times S^m(\mathfrak{g}^*)$ determined by

$$\langle X_1^{m_1}\ldots X_n^{m_n}, x_1^{p_1}\ldots x_n^{p_n}\rangle = \left(\frac{\partial^{m_1+\ldots m_n}}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}}\ x_1^{p_1}\ldots x_n^{p_n}\right)(0)$$

is nondegenerate and the annihilator of all X^m $(X \in \mathfrak{g})$ is 0. Thus (3) shows that although λ is defined by means of a basis, it is actually independent of this basis,

Next we prove that λ is one-to-one. In fact suppose $\lambda(P) = 0$ where $P \neq 0$. With respect to a 'lexicographic ordering' let $aX_1^{m_1}\dots X_n^{m_n}$ be the leading term in P. Let f be a smooth function on a neighbourhood of e in G such that

$$f(\exp(t_1X_1+\dots+t_nX_n)) = t_1^{m_1}\dots t_n^{m_n}$$

for small t. Then $(\lambda(P)f)(e) \neq 0$ contradicting $\lambda(P) = 0$.

Finally, λ maps $S(\mathfrak{g})$ onto $\mathbb{D}(G)$. In fact, if $u \in \mathbb{D}(G)$, there exists a polynomial P such that

$$(uf)(e) = \left\{P(\partial_1,\dots,\partial_n)f(\exp(t_1X_1+\dots+t_nX_n))\right\}_{t=0} .$$

Then by the left invariance of u, $u = \lambda(P)$ so λ is surjective.

Definition. The mapping λ is usually called *symmetrization*.

Exercise. Let $Y_1,\dots,Y_p \in \mathfrak{g}$, not necessarily all different. Then

$$\lambda(Y_1\dots Y_p) = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p} \tilde{Y}_{\sigma(1)}\dots\tilde{Y}_{\sigma(p)} ,$$

where $\mathfrak{S}_p$ is the symmetric group on p letters.

During the proof above we saw that

$$(\tilde{X}^m f)(g) = \left\{\frac{d^m}{dt^m} f(g \exp tX)\right\}_{t=0} .$$

This gives the following version of Taylor's formula.

Corollary 2.2. *If* f *is an analytic function on a neighbourhood of* g *in* G *then*

$$f(g \exp tX) = \sum_{n=0}^{\infty} \frac{1}{n!} (\tilde{X}^n f)(g)$$

for $X \in \mathfrak{g}$ *provided* t *is sufficiently small.*

We now recall some facts concerning the *adjoint representation* Ad of G (or Ad_G) and the *adjoint representation* ad of $\mathfrak{g}$ (or $\mathrm{ad}_{\mathfrak{g}}$). If $g \in G$ the mapping $x \to gxg^{-1}$ is an automorphism of G ; the corresponding automorphism of $\mathfrak{g}$ is denoted $\mathrm{Ad}(g)$. Thus

$$\exp \mathrm{Ad}(g)X = g \exp Xg^{-1} \qquad X \in \mathfrak{g} \ , \ g \in G \ . \tag{4}$$

Then the mapping $g \to \mathrm{Ad}(g)$ is a representation of G on $\mathfrak{g}$. By general theory it induces a representation of $\mathfrak{g}$ on $\mathfrak{g}$, denoted ad . Thus, by definition

$$\mathrm{Ad}(\exp X) = e^{\mathrm{ad}\, X} \qquad X \in \mathfrak{g} \ , \tag{5}$$

where for a linear transformation A , e^A denotes $\sum_0^\infty \frac{1}{n!} A^n$. From (4) and (5) one can deduce

$$\mathrm{ad}\, X(Y) = [X,Y] \qquad X,Y \in \mathfrak{g} \ . \tag{6}$$

These operations can now be extended to differential operators. Let us calculate $(\mathrm{Ad}(g)X)^{\sim}$. Recalling the translations

$$L_g : x \to gx \ , \qquad R_g : x \to xg$$

we have for $f \in C^\infty(G)$,

$$[(\mathrm{Ad}(g)X)^{\sim}f](x) = \left\{\frac{d}{dt}f(x \exp t\,\mathrm{Ad}(g)X)\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}f(xg \exp tXg^{-1})\right\}_{t=0} = \left\{\frac{d}{dt}f^{R_g}(xg \exp tX)\right\}_{t=0}$$

$$= (\tilde{X}f^{R_g})(xg) = (\tilde{X}f^{R_g})^{R_{g^{-1}}}(x)$$

so

$$(\mathrm{Ad}(g)X)^{\sim} = \tilde{X}^{R_{g^{-1}}}$$

Thus we *define* for $D \in \mathbb{D}(G)$

$$\mathrm{Ad}(g)D = D^{R_{g^{-1}}} \qquad (7)$$

Then $\mathrm{Ad}(g)$ is an automorphism of $\mathbb{D}(G)$.

Next we observe that

$$(\mathrm{ad}(X)(Y))^{\sim} = \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}$$

so we *define* for $D \in \mathbb{D}(G)$

$$(\mathrm{ad}\ X)(D) = \tilde{X}D - D\tilde{X} \qquad (8)$$

and then $\mathrm{ad}\ X$ is a derivation of the algebra $\mathbb{D}(G)$. We can also define

$$e^{\mathrm{ad}\ X}(D) = \sum_{0}^{\infty} \frac{1}{n!} (\mathrm{ad}\ X)^n(D) \qquad D \in \mathbb{D}(G) \qquad (9)$$

because $(\mathrm{ad}\ X)^n(D)$ by (8) is a differential operator of order $\leq$ order of D ; thus all the terms in the series (9) lie in a finite-dimensional vector space so there is no convergence problem. Now using Leibnitz' formula for the power of a derivation applied to a product we have for $D_1, D_2 \in \mathbb{D}(G)$,

$$e^{\operatorname{ad} X}(D_1D_2) = \sum_0^\infty \frac{1}{n!}(\operatorname{ad} X)^n(D_1D_2)$$

$$= \sum_0^\infty \frac{1}{n!} \sum_{0\le i,j,i+j=n} \frac{n!}{i!j!}(\operatorname{ad} X)^i(D_1)(\operatorname{ad} X)^j(D_2)$$

so

$$e^{\operatorname{ad} X}(D_1D_2) = e^{\operatorname{ad}X}(D_1)e^{\operatorname{ad} X}(D_2) \quad .$$

Thus $\operatorname{Ad}(\exp X)$ and $e^{\operatorname{ad} X}$ are automorphisms of $\mathbb{D}(G)$; they coincide on $\tilde{g}$, hence on all of $\mathbb{D}(G)$ since by Theorem 2.1, $\tilde{g}$ generates $\mathbb{D}(G)$. Consequently

$$\operatorname{Ad}(\exp X)D = e^{\operatorname{ad} X}(D) \quad , \qquad D \in \mathbb{D}(G) \quad . \tag{10}$$

Lemma 2.3. *Let* $X \in g$, $D \in \mathbb{D}(G)$. *Then* $\tilde{X}D = D\tilde{X}$ *if and only if*

$$D^{R_{\exp tX}} = D \qquad \text{for} \quad t \in \mathbb{R} \quad .$$

In fact we have by (7)-(10)

$$\lim_{t\to 0} \frac{1}{t}(D^{R_{\exp(-tX)}} - D) = \tilde{X}D - D\tilde{X}$$

so the 'if' part is immediate. On the other hand if $\tilde{X}D = D\tilde{X}$ we have $D^{R_{\exp tX}} = D$ by (7) and (10).

Corollary 2.4. *Assume* G *is connected. Let* $Z(G)$ *denote the centre of* $\mathbb{D}(G)$ *and* $I(g) \subset S(g)$ *the set of* $\operatorname{Ad}(G)$ *-invariants. Then*

$$\lambda(I(g)) = Z(G) \quad . \tag{11}$$

Moreover $Z(G)$ *consists of the right invariant differential* *operators in* $\mathbb{D}(G)$.

The last statement is immediate from the lemma (since G is connected). Since

$$\lambda(\mathrm{Ad}(g)P) = \mathrm{Ad}(g)\lambda(P) \qquad\qquad P \in S(\mathfrak{g})$$

statement (11) follows immediately (cf. [4(a)] and [6(a), p.192]).

Now suppose G is a connected Lie group and $H \subset G$ a closed subgroup. Let $\mathfrak{g} \supset \mathfrak{h}$ be their respective Lie algebras and $\mathfrak{m}$ a complementary subspace, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ (direct sum). We now use $\mathfrak{m}$ to introduce coordinates on G/H (cf. [8(a), Ch.II]). Let $(X_1,\ldots,X_r)$ and $(X_{r+1},\ldots,X_n)$ be bases of $\mathfrak{m}$ and $\mathfrak{h}$ respectively, and $\pi\colon G \to G/H$ the natural projection. Then if $g \in G$, the mapping

$$(x_1,\ldots,x_r) \to \pi(g \exp(x_1X_1+\ldots+x_rX_r)) \tag{12}$$

is a diffeomorphism of a neighbourhood of 0 in $\mathfrak{m}$ onto a neighbourhood of $\pi(g)$ in G/H . The inverse of (12) is a local coordinate system near $\pi(g)$, turning G/H into a manifold.

The mapping $\pi\colon G \to G/H$ has a differential $d\pi$ which maps $\mathfrak{g}$ onto the tangent space $(G/H)_0$ to G/H at the origin $0 = \{H\}$. The kernel of $d\pi$ is $\mathfrak{h}$. The translation $\tau(g)\colon xH \to gxH$ satisfies

$$\pi \circ L_g = \tau(g) \circ \pi$$

and since $\pi \circ R_h = \pi$, $\mathrm{Ad}_G(g)X = dR_{g^{-1}} \circ dL_g(X)$, we have for the differentials,

$$d\pi \circ \mathrm{Ad}_G(h)X = d\tau(h)_0 \circ d\pi(X) \qquad\qquad X \in \mathfrak{g} \ .$$

Thus under the isomorphism

$$\mathfrak{g}/\mathfrak{H} \cong (G/H)_0 \tag{13}$$

the linear transformation $Ad_G(h)$ of $\mathfrak{g}/\mathfrak{H}$ corresponds to the linear transformation $d\tau(h)_0$ of $(G/H)_0$.

The coset space G/H is called *reductive* ([16]) if the subspace $\mathfrak{m} \subset \mathfrak{g}$ can be chosen such that

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{H}, \quad Ad_G(h)\,\mathfrak{m} \subset \mathfrak{m} \qquad (h \in H) \quad .$$

If H is compact (or if just $Ad_G(H)$ is compact) then G/H is reductive. In fact $\mathfrak{g}$ will then have a positive definite quadratic form invariant under $Ad_G(H)$ and we can take for $\mathfrak{m}$ the orthogonal complement of $\mathfrak{H}$ in $\mathfrak{g}$. Let

$$\mathbf{D}_H(G) = \{D \in \mathbf{D}(G) \,|\, D^{R_h} = D \quad \text{for all} \quad h \in H\}$$

and if f is a function on G/H we put $\tilde{f} = f \circ \pi$. We have now a Lie-algebraic description of $\mathbf{D}(G/H)$ for a reductive coset space ([8(a), Ch.III]).

Theorem 2.5. *Assume* G/H *reductive. Then the mapping* $u \to D_u$ *where*

$$(D_u f)^{\sim} = u\tilde{f} \qquad f \in C^{\infty}(G/H)$$

is a homomorphism of $\mathbf{D}_H(G)$ *onto* $\mathbf{D}(G/H)$. *The kernel is* $\mathbf{D}_H(G) \cap \mathbf{D}(G)\mathfrak{H}$ *so we have the isomorphism*

$$\mathbf{D}_H(G)/\mathbf{D}_H(G) \cap \mathbf{D}(G)\mathfrak{H} \cong \mathbf{D}(G/H) \quad .$$

Proof. Let $u \in \mathbf{D}_H(G)$ and $f \in C_c^{\infty}(G/H)$. Then $u\tilde{f}$ is right-invariant under H, so of the form $\tilde{f}_1$, $f_1 \in C_c^{\infty}(G/H)$;

D_u is the map $f \to f_1$. It decreases supports so D_u is a differential operator. It is G-invariant because

$$(D_u^{\tau(g)}f)^\sim = ((D_u f^{\tau(g)^{-1}})^{\tau(g)})^\sim = ((D_u f^{\tau(g^{-1})})^\sim)^{L_g}$$
$$= (u(f^{\tau(g^{-1})})^\sim)^{L_g} = u\tilde{f} = (D_u f)^\sim$$

so $D_u^{\tau(g)} = D_u$, whence $D_u \in \mathbb{D}(G/H)$. Also $u \to D_u$ is a homomorphism.

Next we prove that the mapping is surjective. Let $E \in \mathbb{D}(G/H)$. We express E at 0 in terms of $x_1,\dots,x_r$; there exists a polynomial P such that

$$(Ef)(0) = \left[P\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_r}\right)f(\pi(\exp(x_1X_1+\dots+x_rX_r)))\right](0) .$$

By the G-invariance,

$$(Ef)(g\cdot 0) = Ef^{\tau(g^{-1})}(0) =$$
$$\left[P\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_r}\right)f^{\tau(g^{-1})}(\pi(\exp(x_1X_1+\dots+x_rX_r)))\right](0)$$
$$= \left\{P\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_r}\right)\tilde{f}(g\exp(x_1X_1+\dots+x_rX_r))\right\}_{x=0} .$$

In particular take $g = h \in H$. Then

$$(Ef)(0) = \left\{P\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_r}\right)\tilde{f}(\exp \mathrm{Ad}(h)(x_1X_1+\dots+x_rX_r))\right\}_{x_i=0}$$

so we conclude P is Ad(H)-invariant. Put $u = \lambda(P) \in \mathbb{D}(G)$. Then

$$u^{R_{h^{-1}}} = \mathrm{Ad}(h)u = \lambda(\mathrm{Ad}(h)P) = \lambda(P) = u$$

so $u \in \mathbb{D}_H(G)$. Also

$$(u\tilde{f})(g) = (\lambda(P)\tilde{f})(g) =$$

$$\left\{P(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_r})\tilde{f}(g\ \exp(x_1X_1+\ldots+x_rX_r)\right\}_{x_i=0}$$

$$= (Ef)^{\sim}(g) \quad \text{so} \quad D_u = E \quad .$$

Thus our map is surjective.

It remains to prove $D_u = 0 \iff u \in \mathbb{D}_H(G) \cap \mathbb{D}(G)\mathfrak{h}$. For this we insert

Lemma 2.6. $\mathbb{D}(G) = \mathbb{D}(G)\mathfrak{h} \oplus \lambda(S(\mathfrak{m}))$ (direct sum).

Proof. Given $P \in S(\mathfrak{g})$, I claim $\exists\, Q \in S(\mathfrak{m})$ with $\lambda(P-Q) \in \mathbb{D}(G)\mathfrak{h}$. This is clear if P has degree 1 . We assume claim true for $P \in S(\mathfrak{g})$ of degree $< d$. We must prove it holds for P of degree d . We may assume $P = X_1^{e_1} \ldots X_n^{e_n}$ in terms of the bases $X_1,\ldots,X_r$ of $\mathfrak{m}$, $X_{r+1},\ldots,X_n$ of $\mathfrak{h}$. If $e_{r+1}+\ldots+e_n = 0$ we can take $Q = P$. If $e_{r+1}+\ldots+e_n > 0$, $\lambda(P)$ is a linear combination $\tilde{X}_{\alpha_1} \ldots \tilde{X}_{\alpha_d}$, where $X_{\alpha_i} \in \mathfrak{h}$ for some i . Put $\mathbb{D}^d(G) = \lambda(\sum_{e \le d} S^e(\mathfrak{g}))$. Then

$$\tilde{X}_{\alpha_1} \ldots \tilde{X}_{\alpha_d} - \tilde{X}_{\alpha_1} \ldots \tilde{X}_{\alpha_{i-1}} \tilde{X}_{\alpha_{i+1}} \ldots \tilde{X}_{\alpha_d} \tilde{X}_{\alpha_i} \in \mathbb{D}^{d-1}(G)$$

so

$$\lambda(P) - D \in \mathbb{D}(G)\mathfrak{h} \qquad \text{for some} \qquad D \in \mathbb{D}^{d-1}(G) \quad .$$

By induction hypothesis, there exists a $Q \in S(\mathfrak{m})$ such that

$$\lambda(Q) - D \in \mathbb{D}(G)\mathfrak{h} \quad ,$$

whence $\lambda(P-Q) \in \mathbb{D}(G)\mathfrak{h}$. This gives the decomposition.

Next we prove the directness. Let $P \in S(\mathfrak{m})$, $P \neq 0$. Then there exists a function $f^*(x_1,\ldots,x_r)$ such that

$$\left(P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}\right) f^*\right)(0) \neq 0 \quad .$$

Choose $f \in C^\infty(G/H)$ such that

$$f(\pi(\exp(x_1X_1+\ldots+x_rX_r))) = f^*(x_1,\ldots,x_r)$$

for x_i sufficiently small. Then

$$\lambda(P)(f \circ \pi)(e) \neq 0$$

so $\lambda(P) \notin \mathbb{D}(G)\mathfrak{H}$.

Since both summands are stable under $Ad_G(H)$ we deduce

Corollary 2.7. *Let* $I(\mathfrak{m})$ *denote the set of* $Ad_G(H)$*-invariants in* $S(\mathfrak{m})$. *Then*

$$\mathbb{D}_H(G) = (\mathbb{D}_H(G) \cap \mathbb{D}(G)\mathfrak{H}) \oplus \lambda(I(\mathfrak{m})) \quad .$$

We can now finish the proof of Theorem 2.5. Let $u \in \mathbb{D}_H(G)$ such that $D_u = 0$. Let

$$u = u_1 + u_2$$

as in Corollary 2.7. Then $D_{u_1} = 0$ so $D_{u_2} = 0$. But $u_2 = \lambda(P_2)$, $P_2 \in I(\mathfrak{m})$. We claim $u_2 = 0$. If not, then as we saw above there exists $f \in C^\infty(G/H)$ with $u_2\tilde{f} \neq 0$ so $D_{u_2} \neq 0$, a contradiction. Thus $u_2 = 0$ so $u \in \mathbb{D}_H(G) \cap \mathbb{D}(G)\mathfrak{H}$.

Theorem 2.8. *Let* G/H *be a reductive homogeneous space. The mapping* $Q \to D_{\lambda(Q)}$ *is a linear bijection of* $I(\mathfrak{m})$ *onto* $\mathbb{D}(G/H)$. *Explicitly, if* $Q \in I(\mathfrak{m})$ *then*

$$(D_{\lambda(Q)}f)(g\cdot 0) = \left[Q\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}\right)\tilde{f}(g \exp(x_1X_1+\ldots+x_rX_r)) \right](0)$$

While the mapping $Q \to D_{\lambda(Q)}$ is not in general multiplicative (even when $\mathbb{D}(G/H)$ is commutative) we have

$$D_{\lambda(P_1 P_2)} = D_{\lambda(P_1)} D_{\lambda(P_2)} + D_{\lambda(Q)}$$

where $Q \in I(\mathfrak{m})$ has degree $< \deg P_1 + \deg P_2$.

By induction we obtain,

Corollary 2.9. *If* $I(\mathfrak{m})$ *has a finite system of generators* $P_1, \ldots, P_\ell$ *and we put* $D_i = D_{\lambda(P_i)}$ *then each* $D \in \mathbb{D}(G/H)$ *can be written*

$$D = \sum_{(n)} a_{n_1 \ldots n_\ell} D_1^{n_1} \ldots D_\ell^{n_\ell} .$$

Corollary 2.10. *Let* M *be a two-point homogeneous space.* $M = G/K$ *where* $G = I(M)$. *Then* $\mathbb{D}(G/K)$ *consists of the polynomials in the Laplace-Beltrami operator.*

In fact, since $\mathrm{Ad}_G(K)$ acts transitively on the unit sphere in the tangent space $(G/K)_0$ it is clear that $I(\mathfrak{m})$ is generated by $X_1^2 + \ldots + X_r^2$, if (X_i) is an orthonormal basis of $\mathfrak{m}$.

A particularly interesting class of homogeneous spaces are the *symmetric coset spaces* G/H where G is a connected Lie group with an involutive automorphism σ whose fixed point set is the group H. If in addition H is compact we call G/H *Riemannian symmetric coset space* (because it has a G-invariant Riemannian metric).

The spaces (3), (4), (5) in §1 are examples of these coset spaces; for (5) the involution σ is given by

$$\sigma: \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \to \begin{pmatrix} a & -b \\ -\bar{b} & \bar{a} \end{pmatrix} .$$

Theorem 2.11. *Let* G/H *be a symmetric coset space,* G *semisimple and* H *connected. Then* $\mathbb{D}(G/H)$ *is commutative.*

For the proof let us first assume that H is compact. Let $C_c(G)$ denote the space of continuous functions on G of compact support; this space is an algebra under convolution:

$$(f * g)(x) = \int_G f(xy^{-1})g(y)dy = \int_G f(y)g(y^{-1}x)dy \quad .$$

Here we used the bi-invariance of the Haar measure dy on G. Since σ is involutive it preserves dy so the map $f \to f^\sigma$ is an automorphism of $C_c(G)$. By the symmetry of G/H, each $g \in G$ can be written $g = hp$ where $h \in H$ and $\sigma(p) = p^{-1}$.

Let $C^\#(G)$ denote the set of functions in $C_c^\infty(G)$ which are bi-invariant under H. As noticed by Gelfand [4(b)], $C^\#(G)$ is commutative under convolution. In fact, if $f,g \in C_c(G)$ and we put $\check{f}(g) = f(g^{-1})$ then (* denoting convolution)

$$\check{f} * \check{g} = (g * f)^\vee \quad .$$

Since $(f * g)^\sigma = f^\sigma * g^\sigma$ and since $f^\sigma = \check{f}$ for $f \in C^\#(G)$ we obtain $f * g = g * f$ for $C^\#(G)$.

Now let $D,E \in \mathbb{D}_H(G)$ and $f,g \in C^\#(G)$. Then since D is left invariant, $D(f * g) = f * Dg = Dg * f$ so

$$DE(f * g) = D(Eg * f) = Eg * Df = Df * Eg$$

$$ED(f * g) = ED(g * f) = Eg * Df \quad .$$

Thus E and D commute on the subspace $C^\# * C^\#$ of $C^\#(G)$, hence, by density, on all of $C^\#(G)$.

Now if $F \in C_c^\infty(G)$ is right invariant under H we consider the function

$$F^*(x,y) = \int_H F(yhx)dh \quad ,$$

where dh is normalized Haar measure on H. (Then (subscript 1 denoting differentiation with respect to the first argument), we have since F^* is bi-invariant under H in the first variable

$$(D_1F^*)(x,y) = \int_H (DF)(yhx)\,dh = (DF)^*(x,y) \quad ,$$

so

$$(DF)(y) = (D_1F^*)(e,y) \quad .$$

Hence

$$(DEF)(y) = (D_1(EF)^*)(e,y) = (D_1E_1F^*)(e,y) = (E_1D_1F^*)(e,y)$$
$$= (EDF)(y) \quad ,$$

so by Theorem 2.5, $\mathbb{D}(G/H)$ is commutative.

Now we drop the assumption that H is compact. Let

$$\mathfrak{g} = \mathfrak{H} + \mathfrak{g} \tag{14}$$

be the decomposition of $\mathfrak{g}$ into the eigenspaces of $d\sigma$ for the eigenvalue $+1$ and -1, respectively. Then $\mathfrak{H}$ is the Lie algebra of H. By a standard result on semisimple Lie algebras (cf. [2, p.100], [8(c), p.29]), there exists a Cartan decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \quad , \tag{15}$$

which is compatible with (14) in the sense that the corresponding *Cartan involution* of $\mathfrak{g}$

$$\theta: T + X \to T - X \qquad\qquad T \in \mathfrak{k} \ , \ X \in \mathfrak{p}$$

commutes with σ. Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$,

$\mathfrak{h}_c$ and $\mathfrak{q}_c$ the subspaces of $\mathfrak{g}_c$ generated by $\mathfrak{h}$ and $\mathfrak{q}$. Put $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. Then $\mathfrak{u}$ is a compact real form of $\mathfrak{g}_c$ and since $\sigma\theta = \theta\sigma$ we have

$$\mathfrak{u} = \mathfrak{u} \cap \mathfrak{h}_c + \mathfrak{u} \cap \mathfrak{q}_c \quad ,$$

$$\mathfrak{h}_c = \mathfrak{u} \cap \mathfrak{h}_c + i(\mathfrak{u} \cap \mathfrak{h}_c) \quad .$$

The subspace

$$\mathfrak{s} = \mathfrak{u} \cap \mathfrak{h}_c + i(\mathfrak{u} \cap \mathfrak{q}_c) \tag{16}$$

is a real form of $\mathfrak{g}_c$ and (16) is a Cartan decomposition of $\mathfrak{s}$. Let $S = \mathrm{Int}(\mathfrak{s})$ be the adjoint group of $\mathfrak{s}$, put $\mathfrak{h}^* = \mathfrak{u} \cap \mathfrak{h}_c$, and let H^* be the analytic subgroup of S with Lie algebra $\mathfrak{h}^*$. Then H^* is compact and S/H^* is a Riemannian symmetric coset space. By the first part of the proof, $\mathbb{D}(S/H^*)$ is commutative so by Theorem 2.5,

$$\mathbb{D}_{H^*}(S)/\mathbb{D}_{H^*}(S) \cap \mathbb{D}(S)\mathfrak{h}^* \qquad \text{is commutative.} \tag{17}$$

Passing to the complexifications, (17) means that

$$Z_{\mathfrak{h}_c}(\mathfrak{g}_c)/Z_{\mathfrak{h}_c}(\mathfrak{g}_c) \cap U(\mathfrak{g}_c)\mathfrak{h}_c \qquad \text{is commutative.} \tag{18}$$

Here $U(\mathfrak{g}_c)$ denotes the universal enveloping algebra of $\mathfrak{g}_c$ and $Z_{\mathfrak{h}_c}(\mathfrak{g}_c)$ denotes the centralizer of $\mathfrak{h}_c$ in $U(\mathfrak{g}_c)$. But just as (17) is equivalent to (18), property (18) is equivalent to

$$\mathbb{D}_H(G)/\mathbb{D}_H(G) \cap \mathbb{D}(G)\mathfrak{h} \qquad \text{is commutative}$$

which by Theorem 2.5 proves $\mathbb{D}(G/H)$ commutative.

Remark. Theorem 2.11, for H compact, is stated in a different form in Harish-Chandra [6(b), Lemma 1] and is proved in Selberg [20]. The result is indicated in [4(b)], but the 'proof' as described in [1, p.340] is insufficient because it is based on the statement that $Z(G)$ gets mapped *onto* $\mathbb{D}(G/H)$ by applying the operators in $Z(G)$ to functions on G which are right invariant under H (cf. Theorem 2.5). As shown in [8(b)] this 'onto' property fails when G is a suitable real form of the exceptional groups E_6, E_7 or E_8, but it happens to be true whenever G is a real or complex classical group.

The present proof of Theorem 2.11, valid for H compact or not was written in 1960 following a discussion with Harish-Chandra. In [13], Lichnérowicz proves the same result under the assumption that G/H has an invariant volume element.

3. Finite-dimensional representations

We shall now sketch the basic theory of finite-dimensional representations of a simply connected complex semisimple Lie group G and show how these representations arise as eigenspace representations (Theorem 3.5). Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G and $\mathfrak{g} = \mathfrak{h} + \sum_\alpha \mathfrak{g}^\alpha$ the corresponding root space decomposition. For each root α let $H_\alpha \in \mathfrak{h}$ be chosen such that $\langle H, H_\alpha \rangle = \alpha(H)$ $(H \in \mathfrak{h})$ where $\langle , \rangle$ denotes the Killing form of $\mathfrak{g}$. Let $W = W(\mathfrak{g}, \mathfrak{h})$ denote the Weyl group. Let π be a finite-dimensional representation of $\mathfrak{g}$ on V. An element λ in the dual $\mathfrak{h}^*$ is called a *weight* of π is there exists a $v \neq 0$ in V such that $\pi(H)v = \lambda(H)v$ $(H \in \mathfrak{h})$. The following facts (i)-(iv) are classical results of É.Cartan.

(i) *Each* π *has a weight and* W *permutes the weights of* π.

(ii) *$\lambda \in \mathfrak{h}^*$ is a weight of some π if and only if*

$$2\,\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in Z \qquad \textit{for each root } \alpha\,. \tag{1}$$

In particular the weights are real on the space $\mathfrak{h}_{\mathbb{R}} = \sum_\alpha \mathbb{R}H_\alpha$.

An element $\lambda \in \mathfrak{h}^*$ is called *integral* if (1) holds. Assuming the dual $\mathfrak{h}^*_{\mathbb{R}}$ ordered, λ is called *dominant integral* if

$$2\,\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in Z^+ \qquad \text{for each root } \alpha > 0\,. \tag{2}$$

Given λ integral exactly one transform $s\lambda$ ($s \in W$) is dominant integral. Hence we have

(iii) *Let π be an irreducible finite-dimensional representation of $\mathfrak{g}$, λ its highest weight. Then λ is dominant integral.*

(iv) *Assume $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ dominant integral. Then there exists an irreducible finite-dimensional representation π (unique up to equivalence) with highest weight λ .*

In addition to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ consider the subalgebras

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}^\alpha\,, \qquad \bar{\mathfrak{n}} = \sum_{\alpha<0} \mathfrak{g}^\alpha\,, \qquad \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$$

and let H , N , $\bar{N}$ and B denote the corresponding analytic subgroups. We can choose $X_\alpha \neq 0$ in $\mathfrak{g}^\alpha$ such that the space

$$\mathfrak{u} = \sum_\alpha \mathbb{R}(iH_\alpha) + \sum_\alpha \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_\alpha \mathbb{R}(i(X_\alpha + X_{-\alpha}))$$

is a compact real form of $\mathfrak{g}$. Let $T \subset U$ be the (compact) analytic subgroups of G corresponding to $\sum_\alpha \mathbb{R}(iH_\alpha) = \mathfrak{t} \subset \mathfrak{U}$. In the coset space G/B consider the U-orbit of the origin $U\cdot 0 = U/U \cap B$. Since $U \cap B$ has Lie algebra $\mathfrak{t}$ we have $\dim_{\mathbb{R}} U\cdot 0 = \dim_{\mathbb{R}} G/B$ so we conclude: 1) G/B is compact; a) U/T has a complex structure. From the Bruhat decomposition we know that

$$\bar{N}HN \text{ is open and dense in } G. \tag{3}$$

Consider now a holomorphic homomorphism $\omega: H \to \mathbb{C}^*$. Extending ω to B by the condition $\omega(hn) = \omega(h)$ $(h \in H, n \in N)$ we consider the vector space

$$V_\omega = \{F \text{ holomorphic on } G\colon F(gb) \equiv \omega(b)F(g)\} .$$

The homomorphism $\omega: B \to \mathbb{C}^*$ defines a complex line bundle over G/B in which V_ω is the space of holomorphic sections. It is well known that $\dim V_\omega < \infty$; this is for example contained in [3] and the following simple proof was communicated to me by J.-P.Serre. (Another proof based on Liouville's theorem is given in [8(d), p.148].) The vector space V_ω becomes a Banach space when topologized by uniform convergence on the compact set G/B. Using the fact that a uniformly bounded sequence of holomorphic functions has a subsequence converging uniformly on compacts, V_ω is easily seen to be locally compact. Thus V_ω is a locally compact Banach space, hence finite-dimensional. We now prove the Borel-Weil theorem ([21]) that V_ω is irreducible.

Lemma 3.1. *The representation* π_ω *of* B *on* V_ω *is irreducible.*

Proof. By the semisimplicity of G, $V_\omega = \oplus_i V_i$ where G acts irreducibly on V_i. Let $F \in V_i$ be a lowest weight

1) Cf. S.Bochner, Tensor fields with finite basis, Ann. of Math. 53 (1951), 400-411.

vector. Then $F(\bar{n}g) \equiv F(g)$. Thus $F(\bar{n}hn) = F(h) = \omega(h)F(e)$ so by (3) $\mathbb{C}F$ is the same for all i . This proves the lemma.

Let π be any irreducible finite-dimensional representation of $\mathfrak{g}$ (and G) on V , and v_0 a highest weight vector. Let $\omega : H \to \mathbb{C}^*$ be the homomorphism defined by $\pi(h)v_0 = \omega(h)v_0$ $(h \in H)$. Then if v^* belongs to the dual V^* the function $F(g) = \langle \pi(g)v_0, v^* \rangle$ belongs to V_ω and the mapping $v^* \to F$ sets up an equivalence between π' (the contragredient of π) and π_ω .

Now we need a description of $\mathbb{D}(G/N)$; since G/N is not in general reductive, Theorem 2.8 does not apply. Nevertheless, we begin by determining the invariants of $\mathrm{Ad}_G(N)$ in the symmetric algebra $S(\mathfrak{g}/\mathfrak{n})$. Equivalently, if $\sigma : \mathfrak{g} \to \bar{\mathfrak{n}} + \mathfrak{h}$ is the projection corresponding to the direct decomposition $\mathfrak{g} = (\bar{\mathfrak{n}} + \mathfrak{h}) + \mathfrak{n}$ we determine the set $I(\bar{\mathfrak{n}} + \mathfrak{h})$ of invariants of the group $\sigma \circ \mathrm{Ad}_G(N)$ in $S(\bar{\mathfrak{n}} + \mathfrak{h})$.

Lemma 3.2. $I(\bar{\mathfrak{n}} + \mathfrak{h}) = S(\mathfrak{h})$.

Proof. The inclusion $S(\mathfrak{h}) \subset I(\bar{\mathfrak{n}} + \mathfrak{h})$ is trivial since $\sigma \circ \mathrm{Ad}_G(n) = I$ on $\mathfrak{h}$. For the converse let $\alpha_1 > \alpha_2 > \ldots > \alpha_p > 0$ be the positive roots and put $\beta_i = -\alpha_i$. Select a basis $E_{\beta_i}, F_{\beta_i} = JE_{\beta_i}$ of $\mathfrak{g}^{\beta_i}$, J denoting the complex structure. For $X \in \mathfrak{n}$ let $d(X)$ denote the derivation of $S(\bar{\mathfrak{n}} + \mathfrak{h})$ extending the endomorphism $\sigma \circ \mathrm{ad}_{\mathfrak{n}}(X)$ of $\bar{\mathfrak{n}} + \mathfrak{h}$. Then $P \in I(\bar{\mathfrak{n}} + \mathfrak{h})$ if and only if $d(X)P = 0$ $(X \in \mathfrak{n})$. Writing

$$P = \sum_{(m,n)} r_{(m,n)} E_{\beta_1}^{m_1} F_{\beta_1}^{n_1} \ldots E_{\beta_p}^{m_p} F_{\beta_p}^{n_p} , \qquad r_{(m,n)} \in S(\mathfrak{h})$$

we use the relations $d(X_{\alpha_1})P = 0$, $d(X_{\alpha_1})(\mathfrak{h} + \sum_{i>1} \mathfrak{g}^{\beta_i}) = 0$ to conclude that E_{β_1}, F_{β_1} do not occur in P . Repeating this with $X_{\alpha_2}, \ldots, X_{\alpha_p}$ the lemma follows.

Now given $U \in S(\mathfrak{H})$ define D_U by

$$(D_U f)(gN) = \left\{U_h(f(ghN))\right\}_{h=e} , \qquad f \in C^\infty(G/N) . \qquad (4)$$

This is well-defined since $nNh^{-1} \subset N$ for each $h \in H$. In terms of the fibration of G/N over G/B with general fiber $\{ghN: h \in H\}$, D_U is a differential operator on the bundle given by a constant coefficient operator on each fiber. It is clear that $D_U \in \mathbb{D}(G/N)$.

Lemma 3.3. *The mapping* $U \to D_U$ *is an isomorphism of* $S(\mathfrak{H})$ *onto* $\mathbb{D}(G/N)$.

It remains to prove that the mapping is surjective. Let $D \in \mathbb{D}(G/N)$. Let $X_1,\dots,X_r,\dots,X_q$ be a basis of $\mathfrak{H} + \bar{\mathfrak{n}}$ with the r first elements in $\mathfrak{H}$, the rest in $\bar{\mathfrak{n}}$. We express D at the origin 0 in G/N in the coordinate system $\exp(\sum_1^q t_i X_i)N \to (t_1,\dots,t_q)$. Then

$$(Df)(0) = \left\{P(\partial_1,\dots,\partial_q) f(\exp(\sum_1^q t_i X_i)N)\right\}_{t_i=0} ,$$

where P is a polynomial. We decompose P into monomials and let P^* denote the sum of the terms of highest total degree. For $n \in N$ we write the mapping $\tau(n): gN \to ngN$ in coordinates

$$\tau(n)\exp(\sum_i t_i X_i)N = \exp(\sum_i s_i X_i)N$$

and if $F(t_1,\dots,t_q) = f(\exp(\sum_i t_i X_i)N)$ the invariance $D(f \circ \tau(n))(0) = (Df)(0)$ implies

$$\left\{P(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_q})(F(t_1,\dots,t_q) - F(s_1,\dots,s_q))\right\}_{t_i=0} = 0 . \quad (5)$$

Although the coordinate change $(t_1,\dots,t_q) \to (s_1,\dots,s_q)$ is not linear, the *highest order* term $P^*(\partial_1,\ldots,\partial_q)$ in $P(\partial_1,\ldots,\partial_q)$ is transformed by the *linear* transformation $d\tau(n)_0$. Since F is arbitrary, (5) implies that P^* is invariant under the action of N , so by Lemma 3.2, $P^* \in S(\mathfrak{H})$. We can then define D_{P^*} by (5). Then $D - D_{P^*}$ has lower order than D at 0 , hence everywhere, by the invariance. By induction we get $D = D_U$ for some $U \in S(\mathfrak{H})$ so Lemma 3.3 is proved.

Corollary 3.4. *The holomorphic joint eigenfunctions of the operators in* $\mathbb{D}(G/N)$ *are the holomorphic functions* f *satisfying*

$$f(ghN) \equiv f(gN)\omega(h)$$

where $\omega\colon H \to \mathbb{C}^*$ *is a holomorphic homomorphism.*

The functions $g \to f(gN)$ $(g \in G)$ are therefore precisely the members of V_ω so Corollary 3.4 and Lemma 3.1 give the following result ([8(d)], p.145).

Theorem 3.5. *For each homomorphism* $\mu\colon \mathbb{D}(G/N) \to \mathbb{C}$ *let* $\mathfrak{X}_\mu$ *be the space of holomorphic functions* f *on* G/N *satisfying*

$$Df = \mu(D)f \qquad\qquad D \in \mathbb{D}(G/N)$$

and π_μ *the representation of* G *on* $\mathfrak{X}_\mu$. *Then* π_μ *is finite-dimensional and irreducible and all such representations of* G *arise from a suitable* μ .

If we drop the holomorphy condition in Theorem 3.5 the eigenspace representations for G/N constitute the principal series for G (except that we are not in Hilbert spaces so do not concern ourselves with the problem of unitarity).

4. The Euclidean case

Now we consider $\mathbb{R}^n$ as a coset space $\tilde{M}(n)/O(n)$ where $\tilde{M}(n)$ denotes the group of all isometries of $\mathbb{R}^n$. For $\lambda \in \mathbb{C}$ we consider the eigenspace $\mathfrak{E}_\lambda = \{f \in C^\infty(\mathbb{R}^n) : Lf = -\lambda^2 f\}$, L being the Laplacian.

Theorem 4.1. [8(g)] *The action of* $\tilde{M}(n)$ *on* $\mathfrak{E}_\lambda$ *is irreducible if and only if* $\lambda \neq 0$.

The following lemma reduces the proof to a problem for the compact group $O(n)$.

Lemma 4.2. *Let* $\lambda \in \mathbb{C}$. *Then there exists a sphere* $S_\lambda : |x| = R_\lambda$ *such that if* $f \in C^\infty(\mathbb{R}^n)$ *satisfies*

$$Lf = -\lambda^2 f \quad , \qquad\qquad f \equiv 0 \quad \text{on} \quad S_\lambda$$

then

$$f \equiv 0 \quad \text{on} \quad \mathbb{R}^n \quad .$$

Proof. Let B_λ denote the ball $|x| < R_\lambda$ and define the norm $\| \ \|$ on $C^\infty(B_\lambda)$ by

$$\|u\|^2 = \int_{B_\lambda} (|u(x)|^2 + \langle \text{grad } u, \text{grad } \bar{u} \rangle) dx \quad .$$

If $f \in C^\infty(\mathbb{R}^n)$ vanishes identically on the boundary S_λ of B_λ and if $\varepsilon > 0$ there exists a $\phi \in C_c^\infty(B_\lambda)$ such that $\|f-\phi\| < \varepsilon$. To see this, let for $\delta > 0$, α_δ be an even function in $C^\infty(\mathbb{R})$ satisfying

1) $\alpha_\delta \equiv 1$ on $[0, R_\lambda - 4\delta]$;
2) $\alpha_\delta \equiv 0$ on $[R_\lambda - \delta, R_\lambda)$;

3) $|\alpha'_\delta| \le \delta^{-1}$;

4) $|\alpha_\delta| \le 1$.

Let $\alpha \in C^\infty(B_\lambda)$ be defined by $\alpha(x) = \alpha_\delta(|x|)$. Then if δ is small enough, $\phi = \alpha f$ has the desired property. In fact, expressing the gradient in polar coordinates $(\theta_1,\ldots,\theta_n)$, where $\theta_n = r$, the relation

$$\frac{\partial}{\partial\theta_i}(\alpha f - f) = (\alpha - 1)\frac{\partial f}{\partial\theta_i} + \frac{\partial\alpha}{\partial\theta_i} f$$

shows that we just have to prove

$$\int_{B_\lambda} \left|\frac{\partial\alpha}{\partial r} f\right|^2 dx \to 0 \qquad \text{as} \qquad \delta \to 0 \quad . \tag{1}$$

But

$$\int_{B_\lambda} \left|\frac{\partial\alpha}{\partial r} f\right|^2 dx \le \delta^{-2} \int_{R_\lambda-4\delta}^{R_\lambda-\delta} dr \int_{(\theta)} |f|^2 r^{n-1} d\theta_1 \ldots d\theta_{n-1}$$

and

$$|f(\theta_1,\ldots,\theta_{n-1},r)|^2 = \left|\int_r^{R_\lambda} \frac{\partial f}{\partial p}\, dp\right|^2 \le \int_r^{R_\lambda} dp \int_r^{R_\lambda} \left|\frac{\partial f}{\partial p}\right|^2 dp \; .$$

Hence

$$|f(\theta_1,\ldots,\theta_{n-1},r)|^2 \le 4\delta \int_{R_\lambda-4\delta}^{R_\lambda} \left|\frac{\partial f}{\partial p}\right|^2 dp \;, \quad R_\lambda - 4\delta \le r \le R_\lambda \quad ,$$

so if $m = \max\left|\frac{\partial f}{\partial p}\right|^2$ on B_λ we obtain

$$\int_{B_\lambda} \left|\frac{\partial\alpha}{\partial r} f\right|^2 dx \le 16m\, \mathrm{vol}(\Sigma_\delta) \quad ,$$

where Σ_δ is the shell $R_\lambda - 4\delta \le |x| \le R_\lambda - \delta$. This proves (1).

If $\phi,\psi \in C_c^\infty(B_\lambda)$ we have

$$\int_{B_\lambda} \phi L\psi\, dx = -\int_{B_\lambda} \langle \mathrm{grad}\ \phi, \mathrm{grad}\ \psi \rangle\, dx$$

so by approximation as above

$$\int_{B_\lambda} fL\psi \, dx = - \int_{B_\lambda} \langle \text{grad } f, \text{grad } \psi \rangle \, dx \quad ,$$

if $f \equiv 0$ on S_λ . If, in addition, $Lf = -\lambda^2 f$ then by the symmetry of L ,

$$\lambda^2 \int_{B_\lambda} f\psi \, dx = \int_{B_\lambda} \langle \text{grad } f, \text{grad } \psi \rangle \, dx$$

so approximating $\bar{f}$ by ψ we deduce

$$\lambda^2 \int_{B_\lambda} |f|^2 dx = \int_{B_\lambda} \langle \text{grad } f, \text{grad } \bar{f} \rangle dx \quad . \tag{2}$$

On the other hand, if $\phi \in C_c^\infty(B_\lambda)$ and we write

$$\phi(x) = \int_{-R_\lambda}^{x_j} \frac{\partial \phi}{\partial x_j} (x_1, \ldots, t, \ldots, x_n) dt$$

we get by Schwarz's inequality and integration over B_λ , the Poincaré inequality,

$$\int_{B_\lambda} |\phi|^2 dx \le 4R_\lambda^2 \int_{B_\lambda} \langle \text{grad } \phi, \text{grad } \bar{\phi} \rangle \, dx \quad .$$

Again we get by approximation in $\| \ \|$,

$$\int_{B_\lambda} |f|^2 dx \le 4R_\lambda^2 \int_{B_\lambda} \langle \text{grad } f, \text{grad } \bar{f} \rangle \, dx \quad . \tag{3}$$

Thus if $|\lambda|^2 4R_\lambda^2 < 1$, (2) and (3) imply $f \equiv 0$ on B_λ , hence of $\mathbb{R}^n$ by analyticity. Thus the lemma holds for every such R_λ .

Remark. If $\lambda = 0$, the proof shows that R_λ can be chosen arbitrarily (as is also clear from the maximum principle). But the function $\sin(\lambda^{\frac{1}{2}} x)$ on $\mathbb{R}$ shows that in general R_λ will depend on λ .

Lemma 4.3. *Let* $\lambda \neq 0$. *Then the map* $F \to f$ *of* $L^2(S^{n-1})$ *into* $\mathfrak{E}_\lambda$ *given by*

$$f(x) = \int_{S^{n-1}} e^{i\lambda(x,\omega)} F(\omega)d\omega \qquad (4)$$

is one-to-one.

Obvious by differentiation.

If a group G acts on a space X a function f on X is called G-*finite* if the translates $\{f^g | g \in G\}$ span a finite-dimensional space V_f . If, in addition, $V_f = \oplus_i V_i$ where each V_i is G-invariant and irreducible and the representation of G on each V_i equivalent to a fixed representation δ , the function f is said to be G-*finite of type* δ .

Lemma 4.4. *Let* $\lambda \neq 0$. *The* O(n)-*finite eigenfunctions in* $\mathfrak{E}_\lambda$ *are precisely*

$$f(x) = \int_{S^{n-1}} e^{i\lambda(x,\omega)} F(\omega)d\omega \quad ,$$

where F *is an* O(n)-*finite function on* S^{n-1} .

Proof. Let S_λ be as in Lemma 4.2 and let δ be an irreducible representation of $O(n)$. Let $C_\delta^\infty(S_\lambda)$, $C_\delta^\infty(S^{n-1})$, $\mathfrak{E}_{\lambda,\delta}$ be the space of O(n)-finite functions of type δ in the spaces $C^\infty(S_\lambda)$, $C^\infty(S^{n-1})$, $\mathfrak{E}_\lambda$, respectively. Consider the maps

$$f \in \mathfrak{E}_{\lambda,\delta} \to (f|S_\lambda) \in C_\delta^\infty(S_\lambda) \quad ,$$

$$F \in C_\delta^\infty(S^{n-1}) \to f \in \mathfrak{E}_{\lambda,\delta} \qquad \text{(cf. (4))}.$$

By Lemmas 4.2-4.3 these maps are one-to-one so

$$\dim C_\delta^\infty(S^{n-1}) \le \dim \mathfrak{E}_{\lambda,\delta} \le \dim C_\delta^\infty(S_\lambda) \quad .$$

But then the equality signs must hold so the lemma follows since $\oplus_\delta \mathfrak{E}_{\lambda,\delta}$ constitutes all the $O(n)$-finite functions in $\mathfrak{E}_\lambda$.

For $\lambda \neq 0$ let $\mathfrak{X}_\lambda$ denote the space of functions f in (4); giving f the norm $\|f\| = \|F\|_2$ (as we can by Lemma 4.3) turns $\mathfrak{X}_\lambda$ into a Hilbert space.

Corollary 4.5. *Let* $\lambda \neq 0$. *Then* $\mathfrak{X}_\lambda$ *is dense in* $\mathfrak{E}_\lambda$.

. Let $f \in \mathfrak{E}_\lambda$; by expanding the function $k \to f(k \cdot x)$ on $O(n)$ into $O(n)$-finite functions by the Peter-Weyl theory (cf. also [6(e), §3]) we get a series $f = \sum_n f_n$ which converges in the topology of $\mathfrak{E}_\lambda$ and where, by Lemma 4.4, each f_n lies in $\mathfrak{X}_\lambda$.

We can now prove Theorem 4.1. An approximation argument like the one used for Corollary 4.5 shows that it suffices to prove $\widetilde{M}(n)$ acts irreducibly on $\mathfrak{X}_\lambda$. Let $V \subset \mathfrak{X}_\lambda$ be a closed invariant subspace, $V \neq 0$. Then there exists an $f \in V$, $f(0) = 1$. Writing f in the form (4) we see that the average $\phi(x) = \int_{O(n)} f(k \cdot x)\,dk$ belongs to V and is given by

$$\phi(x) = \int_{S^{n-1}} e^{i\lambda(x,\omega)}\,d\omega \quad .$$

But since V is translation-invariant it will then for each $t \in \mathbb{R}^n$ contain the function

$$x \to \int_{S^{n-1}} e^{i\lambda(x,\omega)}\, e^{i\lambda(t,\omega)}\,d\omega \quad .$$

But then Lemma 4.3 shows that the annihilator of V in $\mathfrak{X}_\lambda$ is 0 so $V = \mathfrak{X}_\lambda$, whence the irreducibility.

5. The case of a noncompact symmetric space

We shall now discuss Theorem 1.4 and its generalization to a symmetric space G/K where G is a connected semisimple Lie group with finite centre and K a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be their respective Lie algebras so $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to the Killing form $\langle , \rangle$ of $\mathfrak{g}$. For a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ consider the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha \quad ,$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \qquad \text{for} \quad H \in \mathfrak{a}\}$$

and $\Sigma = \{\alpha\}$ is the set of restricted roots. The dimension $m_\alpha = \dim \mathfrak{g}_\alpha$ is called the multiplicity of α. It is known (by classification) that the triple $(\mathfrak{a}, \Sigma, m)$ characterizes $\mathfrak{g}$ and G/K. Thus any problem about G/K should, in principle, be answerable in terms of these data.

Let M and M', respectively, denote the centralizer and normalizer of $\mathfrak{a}$ in K and W the (finite) factor group M'/M, the Weyl group. It can be viewed as a group of linear transformations of $\mathfrak{a}$. Let $I(\mathfrak{a}) = I(\mathfrak{a}^*)$ be the set of W-invariant polynomials.

Example. Let I_r denote the unit $r \times r$ matrix and consider the symmetric matrix

$$J_{p,q} = \begin{pmatrix} 0 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & -I_{q-p} \end{pmatrix} .$$

The corresponding quadratic form has signature (p,q). The identity component of the corresponding orthogonal group is $SO_0(p,q)$. The corresponding Lie algebra is

$$\{X \mid {}^tX J_{p,q} + J_{p,q} X = 0\} \quad ,$$

where ${}^tX$ denotes the transpose of the $(p+q) \times (p+q)$ matrix X. The mapping $X \to J_{p,q} X J_{p,q}$ is a Cartan involution of this Lie algebra, the Killing form being $B(X,Y) = (p+q-2)\mathrm{Tr}(XY)$.

Now let us consider the case $p = 2$, $q = 4$, $G = SO_0(2,4)$ and let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the Cartan decomposition given by the involution $X \to J_{2,4} X J_{2,4}$. Now for the maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ we can take

$$\mathfrak{a} = \left\{ H = \begin{bmatrix} h_1 & & & & & \\ & h_2 & & & & \\ & & -h_1 & & & \\ & & & -h_2 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix} : h_1, h_2 \in \mathbb{R} \right\} \quad .$$

If $f_1, f_2 \in \mathfrak{a}^*$ are defined by $f_1(H) = h_1$, $f_2(H) = h_2$ the set Σ of roots is found to be

$$\Sigma = \{\pm f_1, \pm f_2, \pm(f_1 \pm f_2)\}$$

with multiplicities

$$m_{\pm f_1} = 2 \; , \quad m_{\pm f_2} = 2 \; , \quad m_{\pm(f_1 \pm f_2)} = 1 \quad .$$

We can take for Σ^+ the set $f_1, f_2, f_1 \pm f_2$. Thus $\dim \mathfrak{n} = 6$.

The Weyl group W is here found to be generated by the maps $f_i \to \epsilon_i f_{\sigma(i)}$ where σ is a permutation of $\{1,2\}$ and $\epsilon_i = \pm 1$ and the invariants are given by

$$I(A) = I(\mathfrak{a}) = \mathbb{C}[f_1^2+f_2^2, f_1^2 f_2^2] \quad .$$

Returning now to the general case we shall construct some special joint eigenfunctions of the operators in $\mathbb{D}(G/K)$. For an ordering of the restricted roots, let $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$, and $N \subset G$ the corresponding analytic subgroup. Then we have the Iwasawa decompositions $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$, $G = NAK$ so each $g \in G$ is uniquely written $g = n \exp A(g)k(g)$ $(n \in N, A(g) \in \mathfrak{a}, k(g) \in K)$. Then we put $X = G/K$, $B = K/M$ and define the map $A: X \times B \to \mathfrak{a}$ by $A(gK,kM) = A(k^{-1}g)$. Let $\rho = \frac{1}{2} \sum_{\alpha>0} m_\alpha \alpha$. We need the following well known lemma.

Lemma 5.1.
(i) If $\lambda \in \mathfrak{a}_c^*$, $b \in B$, $e^{(i\lambda+\rho)(A(x,b))}$ *is a joint eigenfunction of* $\mathbb{D}(G/K)$,

$$D_x(e^{(i\lambda+\rho)(A(x,b))}) = P_D(\lambda) e^{(i\lambda+\rho)(A(x,b))}, \quad D \in \mathbb{D}(G/K). \quad (1)$$

(ii) The mapping $D \to P_D$ *is an isomorphism of* $\mathbb{D}(G/K)$ *onto* $I(\mathfrak{a}^*)$.

Since the ring $S(\mathfrak{a}^*)$ can be shown to be integral over $I(\mathfrak{a}^*)$ one deduces the following consequence of (ii) above.

Corollary 5.2. *Each homomorphism* $\mu: \mathbb{D}(G/K) \to \mathbb{C}$ *has the form* $D \to P_D(\lambda)$ *for some* $\lambda \in \mathfrak{a}_c^*$.

Consequently, each joint eigenspace of $\mathbb{D}(G/K)$ has the form

$$\mathfrak{E}_\lambda = \{f \in C^\infty(X) : Df = P_D(\lambda) f \quad \text{for} \quad D \in \mathbb{D}(G/K)\}$$

for some $\lambda \in \mathfrak{a}_c^*$. Let T_λ denote the representation of G on $\mathfrak{E}_\lambda$ given by $(T_\lambda(g)f)(x) = f(g^{-1}\cdot x)$. We would now like to have a criterion for the irreducibility of T_λ in terms of the triple $(\mathfrak{a},\Sigma,m)$. We shall now state such a result and then explain its relationship to the theory of spherical functions and to the spherical principal series.

Let $\alpha_0 = \alpha/\langle\alpha,\alpha\rangle$ for $\alpha \in \Sigma$, let Σ_0 denote the set of indivisible roots and put

$$\Gamma_X(\lambda) = \prod_{\alpha\in\Sigma_0} \Gamma(\tfrac{1}{2}(\tfrac{1}{2}m_\alpha + 1 + \langle i\lambda,\alpha_0\rangle))\Gamma(\tfrac{1}{2}(\tfrac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda,\alpha_0\rangle)) .$$

Theorem 5.3. *Let* $\lambda \in \mathfrak{a}_c^*$ *and let* T_λ *be the eigenspace representation of* G *on* $\mathfrak{E}_\lambda$. *Then*

$$T_\lambda \text{ is irreducible} \iff \frac{1}{\Gamma_X(\lambda)} \neq 0 .$$

Remark 1. Consider the spherical function on X,

$$\phi_\lambda(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}\, db$$

where db is the K-invariant normalized measure on B. Up to a constant multiple this is the unique K-invariant function in $\mathfrak{E}_\lambda$ and on the positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ it has the asymptotic behaviour

$$\phi_\lambda(\exp H\cdot 0) \sim \sum_{s\in W} c(s\lambda)e^{(is\lambda-\rho)(H)} \qquad H \in \mathfrak{a}^+ \tag{2}$$

(cf. [6(d)]) where Harish-Chandra's c-function satisfies ([6(d)], [5])

$$c(\lambda)c(-\lambda) = c_0 \frac{\prod_{\alpha\in\Sigma_0} \Gamma(<i\lambda,\alpha_0>)}{\Gamma_X(\lambda)} , \qquad c_0 = \text{const.}$$

Now let $\{\phi_\lambda\}$ denote the closed subspace of $\mathfrak{E}_\lambda$ generated by all the G-translates of ϕ_λ. Then G acts irreducibly on $\{\phi_\lambda\}$ because any closed invariant subspace V will contain a non-zero K-invariant, so $\phi_\lambda \in V$. Thus the irreducibility question for T_λ amounts to the problem of deciding for which $\lambda \in \mathfrak{a}_c^*$ we have $\{\phi_\lambda\} = \mathfrak{E}_\lambda$. It is therefore natural that the answer turns out to be related to the c-function. Note however that in the ratio representing $c(\lambda)$ there is extensive cancellation.

Remark 2. It is of interest to compare the eigenspace representations T_λ $(\lambda \in \mathfrak{a}_c^*)$ of G/K to the spherical principal series. This is by definition the family of representations τ_λ $(\lambda \in \mathfrak{a}_c^*)$ of G induced by the character $man \to e^{i\lambda(\log a)}$ of MAN. It is known (Parthasarathy, Ranga-Rao and Varadarajan [17] for complex G and Kostant [11] for real G) for which $\lambda \in \mathfrak{a}_c^*$, τ_λ is irreducible. Their method is purely algebraic (working with $U(\mathfrak{g})$ instead of G) and the points λ of non-irreducibility appear as the zeros of a certain family of polynomials. By examining the poles of the Γ-function one verifies that these λ are precisely the ones in Theorem 5.3. Thus we have the following consequence.

Corollary 5.4. *Let* $\lambda \in \mathfrak{a}_c^*$. *Then* T_λ *is irreducible if and only if* τ_λ *is irreducible.*

One might ask whether this could be proved directly by proving *a priori* that T_λ and τ_λ are equivalent in some weak sense. This however is out of the question because, while $T_\lambda = T_{s\lambda}$ for every $s \in W$, τ_λ is not in general equivalent to $\tau_{s\lambda}$. On the other hand, the representations

τ_λ $(\lambda \in \mathfrak{a}_c^*)$ can be viewed as the eigenspace representations for the coset space G/MN ([8(d)], Ch.III).

The proof of Theorem 5.3 is given in [8(f)], [8(d), II]; it involves determining the range of the Fourier transform

$$f(x) \to \tilde{f}(\lambda,b) = \int_X e^{(i\lambda+\rho)(A(x,b))} f(x)dx , \quad f \in C_c^\infty(X)$$

and the determination of those $\lambda \in \mathfrak{a}_c^*$ for which the Poisson transform

$$F(b) \to \int_B e^{(i\lambda+\rho)(A(x,b))} F(b)db \qquad F \in C^\infty(B)$$

is one-to-one. For the latter problem we use results of Kostant and Rallis on K-harmonic polynomials on $\mathfrak{p}$ ([11], [12]). The proof gives the following partial solution to the joint eigenfunction problem [8(d), p.207].

Theorem 5.5. *The K-finite joint eigenfunctions of* $\mathbb{D}(G/K)$ *are precisely the Poisson integrals*

$$u(x) = \int_B e^{(i\lambda+\rho)(A(x,b))} F(b)db \quad ,$$

where F *is a K-finite continuous function on* B .

Since each $f \in \mathfrak{E}_\lambda$ has a convergent expansion $f = \sum_{\delta\in\hat{K}} f_\delta$ ($\hat{K}$ = set of equivalence classes of irreducible representations of K) where f_δ is K-finite function in $\mathfrak{E}_\lambda$ of type δ , it follows that f is a superposition

$$f(x) = \int_B e^{(i\lambda+\rho)(A(x,b))} dT(b) \quad , \qquad (3)$$

where T is a 'functional' on B (possibly depending on λ and f). In a remarkable joint work [10], Kashiwara, Kowata, Minemura, Okamota, Oshima, and Tanaka proved that the T

which arise in (3) above are precisely the analytic functionals on B as the special case in Theorem 1.4 and some further evidence had led me to hope.

6. The tangent space to a symmetric space

Following the notation of §5 let X_o denote the tangent space to the symmetric space $X = G/K$ at the fixed point o of K. We consider X_o as the homogeneous space G_o/K where G_o is the group of affine transformations of X_o generated by the translations and the natural action of K on X_o. Our problem is to determine the joint eigenfunctions of the differential operator algebra $\mathbb{D}(G_o/K)$ and to answer the question of irreducibility of the corresponding eigenspace representations.

The algebra $\mathbb{D}(G_o/K)$ is clearly canonically isomorphic to the algebra $I(\mathfrak{p})$ of K-invariants in the symmetric algebra $S(\mathfrak{p})$. But $I(\mathfrak{p})$ is isomorphic to $I(\mathfrak{a})$ via Chevalley's restriction theorem. Since $S(\mathfrak{a})$ is integral over $I(\mathfrak{a})$ each homomorphism of $I(\mathfrak{a})$ into C is given by evaluation at a point $\lambda \in \mathfrak{a}_c^*$. Identifying $\mathfrak{p}$ and X_o let for $p \in I(\mathfrak{p})$ $\partial(p)$ denote the corresponding member of $\mathbb{D}(G_o/K)$. If $\lambda \in {}_c^*$ let $\mathfrak{E}_\lambda(\mathfrak{p})$ denote the joint eigenspace

$$\mathfrak{E}_\lambda(\mathfrak{p}) = \{f \in C^\infty(\mathfrak{p}) \mid \partial(p)f = p(i\lambda)f \quad \text{for all} \quad p \in I(\mathfrak{p})\} ,$$

with the topology induced by that of $C^\infty(\mathfrak{p})$. Each joint eigenspace of the operators $\partial(I(\mathfrak{p})) = D(G_o/K)$ is of this form for a suitable λ and $\mathfrak{E}_{s\lambda}(\mathfrak{p}) = \mathfrak{E}_\lambda(\mathfrak{p})$ for each $s \in W$. For each $\lambda \in \mathfrak{a}_c^*$ let $A_\lambda \in \mathfrak{a}_c$ be determined by $\langle A_\lambda, H\rangle = \lambda(H)$ for all $H \in \mathfrak{a}$. If F is a continuous function on K/M then its 'Poisson transform'

$$f(X) = \int_{K/M} e^{i\langle k.A_\lambda, X\rangle} F(kM)\,dk_M$$

belongs to $\mathfrak{E}_\lambda(\mathfrak{p})$. Here dk_M denotes the normalized K-invariant measure on K/M. We recall that $\lambda \in \mathfrak{a}_c^*$ is said to be *regular* if $\alpha(A_\lambda) \neq 0$ for all $\alpha \in \Sigma$. We have now the following analog to Theorem 5.5, proved in [8(i)].

Theorem 6.1. *Assume* $\lambda \in \mathfrak{a}_c^*$ *regular. Then the* K-*finite elements in* $\mathfrak{E}_\lambda(\mathfrak{p})$ *are the functions*

$$f(X) = \int_{K/M} e^{i<k.A_\lambda, X>} F(kM)\,dk_M$$

where F *is a* K-*finite continuous function on* K/M.

In contrast to Theorem 5.5 the regularity assumption can of course not be dropped in Theorem 6.1. But then we have the following substitute.

Theorem 6.2. *Let* $\lambda \in \mathfrak{a}_c^*$ *and let* $f \in \mathfrak{E}_\lambda(\mathfrak{p})$ *be* K-*finite. Then* f *is a finite sum*

$$f(X) = \sum_j q_j(X) \int_{K/M} e^{i<k.A_\lambda, X>} F_j(kM)\,dk_M$$

where each q_j *is a polynomial in* $\mathfrak{E}_o(\mathfrak{p})$ *and each* F_j *is a* K-*finite continuous function on* K/M.

For the eigenspace representations we have the following result, proved in 8(i).

Theorem 6.3. *Let* $\lambda \in \mathfrak{a}_c^*$ *and let* T_λ *denote the natural representation of* G_o *on the joint eigenspace* $\mathfrak{E}_\lambda(\mathfrak{p})$. *Then* T_λ *is irreducible if and only if* λ *is regular.*

7. The compact case

Theorem 7.1. *Let* K *be a closed subgroup of a compact Lie group* U . *Then each eigenspace representation* T_χ *on* U/K *is irreducible.*

Proof. The space U/K has a U-invariant Riemannian structure and the corresponding Laplace-Beltrami operator belongs to $\mathbb{D}(U/K)$. Since the eigenspaces of the Laplacian on a compact Riemannian manifold are finite-dimensional it is clear that $\dim E_\chi < \infty$ for each χ . Let $E_\chi = \sum_i E_i$ be the direct decomposition into U-irreducible subspaces. Each E_i contains a function which is $\neq 0$ at the point $0 = \{K\}$, so by integration over K each E_i contains a K-invariant function f_i , $f_i(0) = 1$. Now if $D \in \mathbb{D}(U)$ the operator

$$D_0 = \int_U D^{R(k)}\, dk$$

belongs to $\mathbb{D}_K(U)$ and the function $\tilde{f}_i(u) = f_i(u\cdot 0)$, being bi-invariant under K_τ satisfies

$$(D\tilde{f}_i)(e) = (D_0\tilde{f}_i)(e) \quad .$$

But $D_0\tilde{f}_i = c\tilde{f}_i$ $(c \in \mathbb{C})$ where c is independent of i . Hence

$$D(\tilde{f}_i-\tilde{f}_j)(e) = 0$$

for all i,j and all $D \in \mathbb{D}(U)$ so by the analyticity and Corollary 2.2, $f_i \equiv f_j$, proving the irreducibility of T_χ .

For the case when U/K is symmetric these eigenfunctions and eigenspace representations can be described much more explicitly. For this we assume U simply connected and assume U/K to be the compact dual of the symmetric space G/K in §5.

This means that $\mathfrak{u}$ is a compact real form of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$ and that if G_c is the simply connected Lie group with Lie algebra $\mathfrak{g}_c$ then U and G are analytic subgroups corresponding to $\mathfrak{u}$ and $\mathfrak{g}$, respectively. The mapping $A: G \to \mathfrak{a}$ given by the Iwasawa decomposition $g = n \exp A(g)k$ can be extended uniquely to a holomorphic map $A: G_c^0 \to \mathfrak{a}_c$ of a neighbourhood G_c^0 of e in G_c into the complexification $\mathfrak{a}_c$ of $\mathfrak{a}$ as follows (Stanton [23], Sherman [22(a),(b)]). For the complexified Iwasawa decomposition $\mathfrak{g}_c = \mathfrak{n}_c + \mathfrak{a}_c + \mathfrak{k}_c$ the mapping

$$(X,H,T) \to \exp X \exp H \exp T$$

is a holomorphic diffeomorphism of a neighbourhood of $(0,0,0)$ in $\mathfrak{n}_c \times \mathfrak{a}_c \times \mathfrak{k}_c$ onto a neighbourhood G_c^0 of e in G_c. The mapping $\exp X \exp H \exp T \to H$ is then the desired holomorphic mapping $A: G_c^0 \to \mathfrak{a}_c$. Let U_0 be an open ball around e in U contained in G_c^0 and B_0 the corresponding ball around the origin in U/K. Let $\mathfrak{a}^*$ denote the dual of $\mathfrak{a}$ and let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{H} = \mathfrak{H} \cap \mathfrak{k} + \mathfrak{a}$. We have now the following result ([8(h),(d), p.79]).

Theorem 7.2. *The restrictions to* B_0 *of the joint eigenfunctions of* $\mathbb{D}(U/K)$ *are precisely the functions*

$$f(uK) = \int_{K/M} e^{-\mu(A(k^{-1}uk))} F(kM)\,dk_M \qquad (u \in U_0)$$

where dk_M *is the invariant measure,* $F \in C^\infty(K/M)$ *and* $\mu \in \mathfrak{a}^*$ *satisfies*

$$\frac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in Z^+ \qquad \textit{for all restricted roots } \alpha > 0 . \tag{2}$$

The linear functions μ *on* $\mathfrak{H}$ *satisfying* (2) *and* $\mu(\mathfrak{H}\cap\mathfrak{k}) = 0$

are precisely the highest weights of the eigenspace representations. (If s^* *is the maximal Weyl group element* $-s^*\mu$ *is the highest weight of the representation of* U *on the space of functions* (1), $F \in C^{\infty}(K/M)$.)

Bibliography

[1] F.A. Berezin, I.M. Gelfand, M.I. Graev, and M.A. Naimark, 'Representations of groups', *Uspehi Math. Nauk.* 11 (1956), 13-40, *Amer.Math.Soc.Transl.* 16 (1960), 325-353.

[2] M. Berger, 'Les espaces symétriques non compacts', *Ann. Sci. École Norm. Sup.* 74 (1957), 85-177.

[3] H. Cartan and J.-P. Serre, 'Un théorème de finitude concernant les variétés analytiques compactes', *C.R. Acad.Sci. Paris* 237 (1953), 128-130.

[4] I.M. Gelfand, (a) 'The center of an infinitesimal group algebra', *Mat.Sb.* 26 (1950), 103-112; (b) 'Spherical functions on symmetric spaces', *Dokl.Akad.Nauk. SSSR* 70 (1950), 5-8.

[5] S.G. Gindikin and F.I. Karpelevič, 'Plancherel measure of Riemannian symmetric spaces of non-positive curvature', *Dokl.Akad.Nauk. SSSR* 145 (1962), 252-255.

[6] Harish-Chandra, (a) 'Representations of semisimple Lie groups' I, *Trans.Amer.Math.Soc.* 75 (1953), 185-243, VI *Amer.J.Math.* 78 (1956), 564-628; (b) 'On the Plancherel formula for the right invariant functions on a semisimple Lie group', *Proc.Nat.Acad.Sci. USA* 40 (1954), 200-204; (c) 'The characters of semisimple Lie groups', *Trans.Amer.Math.Soc.* 83 (1956), 98-163; (d) 'Spherical functions on a semisimple Lie group I', *Amer.J.Math.* 80 (1958), 241-310; (e) 'Discrete series for semisimple Lie groups II', *Acta Math.* 116 (1966), 1-111.

[7] M. Hashizume, A. Kowata, K. Minemura and K. Okamoto,

'An integral representation of an eigenfunction of the Laplacian on the Euclidean space', *Hiroshima Math.J.* 2 (1972), 535-545.

[8] S. Helgason, (a) 'Differential operators on homogeneous spaces', *Acta Math.* 102 (1959), 239-299; (b) 'Fundamental solutions of invariant differential operators on symmetric spaces', *Amer.J.Math.* 86 (1964), 565-601; (c) 'Lie groups and symmetric spaces', *Battelle Rencontres*, 1-71, Benjamin, New York, 1968; (d) 'A duality for symmetric spaces with applications to group representations', I, *Advan.Math.* 5 (1970), 1-154, II, *Advan.Math.* 22 (1976), 187-219; (e) 'Group representations and symmetric spaces', *Actes, Congr.Intern.Math. Nice* 1970, Vol.2, 313-319; (f) 'The surjectivity of invariant differential operators on symmetric spaces I', *Ann. of Math.* 98 (1973), 451-479; (g) 'Eigenspaces of the Laplacian; integral representations and irreducibility', *J.Funct. Anal.* 17 (1974), 328-353; Proc. AMS Symposium in Differential Geometry, Stanford, 1973; (h) 'Some results on eigenfunctions on symmetric spaces and eigenspace representations', *Math.Scand.* 41 (1977), 79-89; (i) 'A duality for symmetric spaces with applications to group representations III. Tangent space analysis', *Advan.Math.* (to appear).

[9] A. Hole, 'Representations of nilpotent Lie groups on eigenspaces' (to appear).

[10] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, 'Eigenfunctions of invariant differential operators on a symmetric space', *Ann. of Math.* 107 (1978), 1-39.

[11] B. Kostant, 'On the existence and irreducibility of certain series of representations in "Lie Groups and their

Representations"' (I.M. Gelfand, Ed.), pp.231-329, Halstead Press, New York, 1975.
Math. 93 (1971), 753-809.

[13] A. Lichnèrowicz, 'Opérateurs différentiels invariants sur un espace symétrique', *C.R.Acad.Sci. Paris* 257 (1963), p.3548.

[14] J.L. Lions and E. Magenes, 'Problèmes aux limites non homogènes (VII)', *Ann.Math.Pura Appl.* 4, 63 (1963), 201-224.

[15] G.W. Mackey, 'Induced representations of locally compact groups I', *Ann. of Math.* 55 (1952), 101-139.

[16] K. Nomizu, 'Invariant affine connections on homogeneous spaces', *Amer.J.Math.* 76 (1954), 33-65.

[17] K.R. Parthasarathy, R. Ranga Rao and V.S. Varadarajan, 'Representations of complex semisimple Lie groups and Lie algebras', *Ann.Math.* 85 (1967), 383-429.

[18] J. Peetre, 'Rectification à l'article "Une charactérisation abstraite des opérateurs différentiels"', *Math. Scand.* 8 (1960), 116-120.

[19] L. Schwartz, 'Théorie des Distributions', 2nd ed., Hermann, Paris 1966.

[20] A. Selberg, 'Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series', *J.Ind.Math.Soc.* 20 (1956), 47-87.

[21] J.-P. Serre, 'Représentations linéaires et èspaces homogènes Kähleriens des groupes de Lie compactes', *Sém. Bourbaki* 1954.

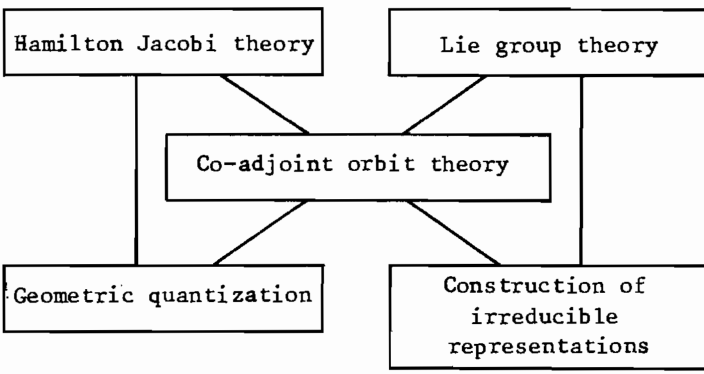
[22] T.O. Sherman, (a) 'Fourier analysis on the sphere', *Trans.Amer.Math.Soc.* 209 (1975), 1-31; (b) 'Fourier analysis on compact symmetric spaces (Local theory)', preprint.

[23] R.J. Stanton, 'On mean convergence of Fourier series on compact Lie groups', *Trans.Amer.Math.Soc.* 218 (1976), 61-87.

10 · Quantization and representation theory

BERTRAM KOSTANT
(Massachusetts Institute of Technology)

The following diagram represents the main topics discussed in these lectures, the lines representing the connections emphasized.



For the most part we will deal with the top of the picture, discussing the complete integrability of the Toda lattice, a finite analogue of the Korteweg - de Vries equation, and some finite generalizations. The quantization of the system leads us to the Whittaker model (see [5]) and the representation theory of solvable groups.

1. Hamilton Jacobi theory

Consider phase space $\mathbb{R}^{2n}$ with canonical co-ordinate functions $p_1,\ldots,p_n,q_1,\ldots,q_n$ and a Hamiltonian $H \in C^\infty(\mathbb{R}^n)$. The problem is to solve

$$\frac{\partial H}{\partial p_i} = \frac{dq_i}{dt} \quad , \quad \frac{\partial H}{\partial q_i} = -\frac{dp_i}{dt} \quad , \quad i = 1,\ldots,n \quad ,$$

a system describing the evolution in time of a mechanical system with total energy H (the p_i's are the momentum variables and the q_j's the position variables). For example, for a particle on the line with total energy

$$H = \frac{p^2}{2m} + f(q) \quad ,$$

Hamilton's equations are the following equations of Newtonian mechanics

$$\frac{p}{m} = \frac{dq}{dt} \quad , \quad f'(q) = -\frac{dp}{dt} \quad .$$

The usual technique for dealing with the equation is to introduce a symplectic structure (a closed, non-singular 2-form ω). The canonical choice in this case is

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i \quad .$$

In general, a symplectic form ω on a manifold X defines a mapping from smooth functions to vector fields, assigning to a function ϕ the vector field ξ_ϕ satisfying

$$\eta(\phi) = \omega(\xi_\phi, \eta)$$

for all vector fields η . We then define a Lie algebra structure on functions by putting

$$[\phi,\psi] = \xi_\phi(\psi) \quad , \quad \phi,\psi \in C^\infty(X) \quad .$$

The bracket operation is called the Poisson bracket.

In this notation the problem of solving Hamilton's equations becomes that of integrating the vector field ξ_H .

A function G such that $[G,H] = 0$ is called a conserved quantity. For example, if H is a function of the differences $q_i - q_j$ then $M = \Sigma p_i$, the total momentum, is a conserved quantity. A set of functions which Poisson commute with each other is called a system in involution and there are at most n such functions on $\mathbb{R}^{2n}$ with differentials which are linearly independent at each point.

Suppose we have a system $H_1, \ldots, H_n$ $(H_1 = H)$ of conserved quantities in involution with differentials which are everywhere linearly independent. These functions define a foliation by the requirement that the functions are constant on the leaves and these leaves are Lagrangian submanifolds of $\mathbb{R}^{2n}$ (the tangent spaces are maximally isotropic with respect to ω). Such a foliation F by Lagrangian leaves is called a real polarization. Each leaf inherits a flat affine connection given by taking the vector fields ξ_{H_i} to be parallel.

Let $C_F(\mathbb{R}^{2n})$ be the set of functions constant along the leaves and

$$C_F^1(\mathbb{R}^{2n}) = \{f \in C^\infty(\mathbb{R}^{2n}) \mid [f, C_F(\mathbb{R}^{2n})] \subseteq C_F(\mathbb{R}^{2n})\} \quad .$$

A function $f \in C_F^1(\mathbb{R}^{2n})$ is easily seen to be linear along the leaves with respect to the flat connection. More properly this is done locally obtaining a sheaf of such functions.

If we have been able to find a system $H_1, \ldots, H_n$ as above, we then seek $G_1, \ldots, G_n \in C_F^1$, such that $H_1, \ldots, H_n, G_1, \ldots, G_n$ is a co-ordinate system. The flow of ξ_H takes the form

$$H_i(t) = \text{constant} \ , \quad G_i(t) = a_i + b_i t \quad , \qquad i = 1, \ldots, n \quad .$$

Finally, expressing the original p's and q's in terms of the new co-ordinates

$$p_i = P_i(H_1, \ldots, H_n, G_1, \ldots, G_n) \ , \ \ q_i = Q_j(H_1, \ldots, H_n, G_1, \ldots, G_n)$$

gives an explicit solution for the position and momentum of the system.

2. Symplectic manifolds, G-actions and co-adjoint orbits

Let X be a smooth manifold and ω a closed, non-singular 2-form on X. Then ω is called a symplectic structure. Darboux's theorem states that there exist local co-ordinates $p_1,\ldots,p_n,q_1,\ldots,q_n$ such that in that co-ordinate patch $\omega = \Sigma dp_i \wedge dq_i$. Hence locally (X,ω) is like the phase space example described in the previous section.

An important family of symplectic manifolds comes from the co-tangent spaces of manifolds. Let $X^{2n} = T^*(M^n)$, then X^{2n} has a natural symplectic structure. In terms of local co-ordinates $u_1,\ldots,u_n$ on open $U \subseteq M^n$ and the natural projection $\pi: T^*(M^n) \to M^n$, put $q_i = u_i \circ \pi$. Each point $b \in T^*(U)$ may be expressed in the form $b = \Sigma p_i(b) du_i$. The functions $p_1,\ldots,p_n,q_1,\ldots,q_n$ are canonical coordinates on $T^*(U)$. Finally, let $\omega = \Sigma dp_i \wedge dq_i$. It is easy to check that ω is independent of the original coordinates and hence is globally well-defined. Notice that the co-tangent bundle X has a natural polarization F by the fibres of the bundle and that the $q_i \in C_F$ and the $p_i \in C_F^1$.

Another family of examples comes from group actions. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}' = \mathrm{Hom}(\mathfrak{g},\mathbb{R})$. Then G acts on $\mathfrak{g}'$ by the co-adjoint action. The space $\mathfrak{g}'$ decomposes into G orbits,

$$\mathfrak{g}' = \bigcup_{f \in \mathfrak{g}} G.f$$

and each orbit has a canonical symplectic structure. Let $f \in \mathfrak{g}'$ and $x,y \in \mathfrak{g}$. Then x and y define vector fields ξ^x and ξ^y on the orbit $O = G.f$. The symplectic form ω_f is given by

$$\omega_f(\xi^y,\xi^x) = \langle f,[x,y]\rangle$$

To illustrate this, consider the $ax+b$ group,

$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \neq 0\} \quad .$$

Then $\dim \mathfrak{g} = 2$ and generators $x,y \in \mathfrak{g}$ may be chosen such that $[x,y] = y$. Interpreting x and y as coordinate functions on $\mathfrak{g}'$ and $\{f \in \mathfrak{g}' \mid \langle f,y\rangle > 0\}$ as the upper half plane in $\mathfrak{g}'$, the co-adjoint orbits are the upper half plane, the lower half plane and the single points along the real axis $y = 0$. The symplectic form on the upper half plane orbit coincides with the usual hyperbolic volume form $\omega = dx \wedge \frac{dy}{y^2}$.

Another simple example is given by $G = SO(3)$. In this case the non-trivial co-adjoint orbits are 2-spheres and the symplectic form on any one of these is given by the vector triple product. If $x \in \mathfrak{g}'$, $x \neq 0$, Y and Z are tangent to the orbit at x then

$$\omega_x(Z,Y) = (Y \times Z).x \quad .$$

In the previous section we saw how a symplectic structure on X induces a Poisson bracket on $C^\infty(X)$ and a homomorphism of Lie algebras from $C^\infty(X)$ to vector fields. A vector field in the image $A(X)$ of this map is called a Hamiltonian vector field. Hence we have an exact sequence of Lie algebras

$$0 \to \mathbb{R} \to C^\infty(X) \to A(X) \to 0$$

where $C^\infty(X)$ occurs as a central extension of the Lie algebra of Hamiltonian vector fields. In general, this sequence is non-split. For example, consider phase space $\mathbb{R}^{2n}$ with co-ordinates $p_1,\ldots,p_n,q_1,\ldots,q_n$. Then $\xi_{p_i} = -\partial/\partial q_i$ and $\xi_{q_i} = \partial/\partial p_i$. The functions $p_1,\ldots,p_n,q_1,\ldots,q_n,1$ generate the Heisenberg algebra

$$[q_i,p_i] = \delta_{ij} \ , \quad [1,p_i] = 0 \ , \quad [1,q_j] = 0 \quad .$$

The image in $A(\mathbb{R}^{2n})$ is abelian and hence it is clear that no splitting exists.

Suppose now that we have a Lie group G acting on X by transformations which preserve the symplectic structure. Then the Lie algebra $\mathfrak{g}$ of G induces locally Hamiltonian vector fields on X. In many cases these vector fields will automatically be Hamiltonian; for example, if $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ or if X is a co-adjoint orbit. Then we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(X) & \longrightarrow & A(X) & \longrightarrow & 0 \\ & & & & & \nwarrow & \uparrow & & \\ & & & & & & \mathfrak{g} & & \end{array}$$

If this map from $\mathfrak{g}$ to $A(X)$ lifts to a homomorphism of Lie algebras from $\mathfrak{g}$ to $C^\infty(X)$, then we say the G-action is Hamiltonian. For example, if $\mathfrak{g}$ is semi-simple then the lifting exists $(H^2(\mathfrak{g},\mathbb{R}) = 0)$ and it is unique $(H^1(\mathfrak{g},\mathbb{R}) = 0)$.

Co-adjoint orbits are Hamiltonian. Let $f \in \mathfrak{g}'$, $O = G.f$ and $x \in \mathfrak{g}$. Then x defines a function ϕ^x on O by $\phi^x(f) = \langle f,x\rangle$ and the map $x \to \phi^x$ is a homomorphism of Lie algebras,

$$[\phi^x,\phi^y] = \phi^{[x,y]} \quad \text{and} \quad \xi_{\phi^x} = \xi^x \quad .$$

In general, the mapping from $\mathfrak{g}$ to $C^\infty(X)$ induces a mapping from X to $\mathfrak{g}'$ known as the moment mapping, by assigning to $p \in X$, $f_p \in \mathfrak{g}'$ given by $f_p(x) = \phi^x(p)$. If X is a general orbit of G (not necessarily co-adjoint) then the image of the moment mapping is a co-adjoint orbit.

Theorem. Let G be a simply connected Lie group and (X,ω) a symplectic manifold with a Hamiltonian G-action which is transitive. Then the moment mapping is a covering map from X to a co-adjoint orbit.

3. The Poisson algebra

Let A be an associative, commutative algebra. A Poisson structure on A is a bilinear form

$$[\ ,\ ]: A \times A \to A$$

with respect to which A is a Lie algebra and such that

$$[ab,c] = a[b,c] + [a,c]b \quad \text{for all} \quad a,b,c \in A \quad .$$

For example, the Poisson bracket on smooth functions on a symplectic manifold satisfies these properties.

The smooth functions on $\mathfrak{g}'$, the dual space to the Lie algebra $\mathfrak{g}$ of G, possesses a natural Poisson structure even though it is not in general a symplectic manifold. The Poisson structure on $C^\infty(T^*(G))$ restricts to the left invariant functions which are determined by their values on $\mathfrak{g}' = T^*_e(G)$. This correspondence induces the Poisson structure on $C^\infty(\mathfrak{g}')$.

Let $x_1,\ldots,x_n$ be a basis for $\mathfrak{g}$ and $z_1,\ldots,z_n$ the dual basis for $\mathfrak{g}'$. Let $\partial(z_i) = \partial/\partial x_i$ be differentiation in the direction z_i. Then if $\phi,\psi \in C^\infty(\mathfrak{g})$, $f \in \mathfrak{g}'$,

$$[\phi,\psi]\,(f) = \sum_{i,j} \partial(z_i)(\phi)(f).\partial(z_i)(\psi)(f).\langle f,[x_i,x_j]\rangle$$

where $[x_i,x_j]$ is the Lie bracket on $\mathfrak{g}$. Defining

$$\sigma(\phi): \mathfrak{g}' \to \mathfrak{g}$$

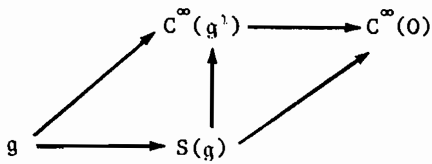
by

$$\sigma(\phi)(f) = \sum_i \partial(z_i)(\phi)(f)x_i$$

the formula becomes

$$[\phi,\psi](f) = \langle f,[\sigma(\phi)(f),\sigma(\psi)(f)]\rangle \quad .$$

This Poisson structure on $C^\infty(\mathfrak{g}')$ is universal in the sense that if $O \subseteq \mathfrak{g}'$ is an orbit of the co-adjoint action, then the restriction from $C^\infty(\mathfrak{g}')$ to $C^\infty(O)$ is a homomorphism of Poisson algebras. The space $S(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}'$ is preserved by this Poisson bracket which coincides with the bracket induced by the Lie bracket on $\mathfrak{g}$ (see later). Putting these homomorphisms together we obtain the commutative diagram:



4. The Toda lattice

The Toda lattice is a completely integrable classical mechanical system consisting of n particles on a line, each with mass 1 and subject to a system of springs which behave exponentially. The Hamiltonian energy function is

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

The problem of determining the behaviour of this system was solved by J. Moser and in his solution the p_i's and e^{q_i}'s appear as rational functions of the G's and H's , the relationship coming from the theory of continued fractions. He was also able to determine the scattering of the system showing that the i-th particle described a path asymptotic to lines of the form $c_i t + d_i$ and $c_i' t + d_i'$ near ∞ and $-\infty$, respectively.

Recall that in terms of the natural basis λ_i of weights the simple roots of $SL(n,\mathbb{R})$ are $\lambda_1-\lambda_2,\lambda_2-\lambda_3,\ldots,\lambda_{n-1}-\lambda_n$; it is no coincidence that the potential in the system above contains a similar expression. In fact, every simple Lie algebra has a corresponding completely integrable system. For example, for the algebra B_3 the potential is

$$e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3}$$

so that there are three particles where the third particle is attracted to a fixed mass as well as to the second particle. In all cases we can give explicitly the formula for $q_i(t)$ in terms of the fundamental finite dimensional representations of the corresponding group. In fact, $q_i(t)$ is a linear combination of terms which are logs of a finite sum of exponentials arising from the weights of the fundamental representations. The highest and lowest weights dictate the asymptotic behaviour as $t \to \pm\infty$. In the example just cited, $q_1(t)$ is the log of 7 exponentials corresponding to the 7 dimensional representation of B_3. To compute $q_2(t)$ and $q_3(t)$ we need also the 21 and the 8 dimensional representations of B_3.

The following idea is due to P. Lax. Convert phase space $\mathbb{R}^{2n}$ into a space of matrices on which the Hamiltonian becomes the Killing form,

$$H = \tfrac{1}{2}\,\mathrm{tr}\,\eta^2 \quad .$$

Then the system of conserved quantities in involution, the H's , become functions of eigenvalues and the flows on the corresponding leaves, isospectral flows. It turns out that the G's are characters of representations of a certain torus.

First we reduce the problem. The total momentum, $M = \Sigma p_i$, is an invariant of motion (a conserved quantity) and the linear space $M^{-1}(0)$, a $2n-1$ dimensional submanifold of $\mathbb{R}^{2n}$ with tangent vector field

$$\xi_M = -\sum \frac{\partial}{\partial q_i} \quad .$$

This field is isotropic for the symplectic form

$$\omega = \sum dp_i \wedge dq_i \quad .$$

Let X^{2n-2} denote the space of integral curves of ξ_M on $M^{-1}(0)$. Then ω determines a symplectic structure ω_X on X^{2n-2}. The functions p_i, $q_i - q_{i+1}$ and H are still well defined on X^{2n-2} and hence we can consider the problem on this collapsed space.

For Lax's matrix formulation, consider the set of traceless, symmetric Jacobi matrices

$$\begin{pmatrix} b_1 & \sqrt{a_1} & & & \\ \sqrt{a_1} & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & \sqrt{a_{n-1}} \\ & & & \sqrt{a_{n-1}} & b_n \end{pmatrix}$$

$a_1, \ldots, a_{n-1} > 0$, $\Sigma b_i = 0$, zeros elsewhere. This set of matrices is a $2n-2$ dimensional manifold $\tilde{J}$. Let

$$\tilde{\omega}_J = \sum_{j=1}^{n-1} \left(\sum_{i=1}^{j} db_i \right) \wedge \frac{da_j}{a_j} \quad .$$

Then $(\tilde{J}, \omega_{\tilde{J}})$ is a symplectic manifold.

The correspondences between the two systems are:

$$X^{2n-2} \longleftrightarrow \tilde{J}$$

$$\omega_X \longleftrightarrow \omega_{\tilde{J}}$$

$$e^{q_i - q_{i+1}} \longleftrightarrow a_i$$

$$p_i \longleftrightarrow b_i$$

$$H \longleftrightarrow \tfrac{1}{2} \operatorname{tr} \eta^2 \quad .$$

The conserved quantities $H_1,\ldots,H_n$ are obtained from the eigenvalues by

$$\det(\lambda I-\eta) = \lambda^n + \sum_{i=1}^{n-1} H_i(\eta)\lambda^{n-1-i} \quad .$$

If the continued fraction

$$\cfrac{1}{\lambda - b_n + \cfrac{a_{n-1}}{\lambda - b_{n-1} + \cfrac{a_{n-2}}{\lambda \quad \cfrac{}{a_1}}}}$$

is expressed in the form $\Sigma\, r_i^2/\lambda-\lambda_i$ then the poles λ_i are constants of the flow and the r_i's are (essentially) linear. Further the a_i's and b_j's are rational functions of the r_i's and λ_j's .

Our role here began when we observed that ω_J is in fact the symplectic structure of a certain orbit of a group of lower triangular matrices and that the continued fractions expansion corresponds to the Bruhat decomposition of certain elements.

Before we proceed to the generalization of this situation, let us observe that a Jacobi matrix

$$\begin{pmatrix} b_1 & \sqrt{a_1} & & & \\ \sqrt{a_1} & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & \sqrt{a_{n-1}} \\ & & & \sqrt{a_{n-1}} & b_n \end{pmatrix}$$

is conjugate (via a diagonal matrix) to

$$\begin{pmatrix} b_1 & a_1 & & & \\ 1 & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ & & & 1 & b_n \end{pmatrix}$$

Let J denote the set of Jacobi matrices of this form. Conjugation induces a symplectic isomorphism

$$(\tilde{J},\omega_{\tilde{J}}) \cong (J,\omega_J)$$

where ω_J is given by the same expression as $\omega_{\tilde{J}}$.

5. The generalized Toda lattice

Let $\mathfrak{g}$ be a real, split, semi-simple Lie algebra, $\mathfrak{H}$ a Cartan subalgebra and

$$\mathfrak{b} = \mathfrak{H} + \sum_{\phi>0} \mathbb{R}e_\phi$$

$$\bar{\mathfrak{b}} = \mathfrak{H} + \sum_{\phi>0} \mathbb{R}e_{-\phi}$$

Borel subalgebras where ϕ is a root and e_ϕ a corresponding root vector. Let B and $\bar{B}$ be the corresponding subgroups of G where G is the subgroup of the complexified adjoint group which stabilizes the real Lie algebra $\mathfrak{g}$. Let θ be the Cartan involution associated with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and define $x^* = -\theta x$, $x \in \mathfrak{g}$. Then the bilinear form

$$P(x,y) = Q(x,y^*) \quad ,$$

where Q is the Killing form, is positive definite. We may assume that the choices are made so that $\bar{\mathfrak{b}}^* = \mathfrak{b}$. Further, if $a \in \bar{B}$, $x \in \mathfrak{b}$, then the co-adjoint action of $\bar{B}$ on $\mathfrak{b}$ coincides with the orthogonal projection of the adjoint action

$$\text{Co-ad}(a)(x) = \text{Proj}_{\mathfrak{b}} \text{Ad}(a)(x) \quad .$$

Let $x \in \mathfrak{g}$, then x is said to be regular if

$$\mathfrak{g}^x = \{y \in \mathfrak{g} \mid [x,y] = 0\}$$

is such that

$$\dim \mathfrak{g}^x = \text{rank } \mathfrak{g} = \ell \quad .$$

A regular nilpotent element is called principal nilpotent.

Let $\alpha_1,\ldots,\alpha_\ell$ be simple, positive roots. Then

$$e = \sum_{i=1}^{\ell} e_{\alpha_i}$$

is a principal nilpotent element. If $\mathfrak{g} = s\ell(n,\mathbb{R})$ then these choices may be made so that

$$e = \begin{pmatrix} 0 & 1 & \\ & & 1 \\ & & 0 \end{pmatrix} \quad , \quad \text{zeros elsewhere.}$$

Let $0 = \bar{B}.e$ under the action described above. We will show that $\dim 0 = 2\ell$. Each element of the form $\Sigma\, a_i\, e_{\alpha_i}$, $a_i > 0$ may be obtained from e by conjugation by an element of the Cartan subgroup $H \subseteq \bar{B}$. Let $\bar{N} = [\bar{B},\bar{B}]$, the group corresponding to the span of the negative root vectors

$$\text{Ad}(\exp \textstyle\sum c_i\, e_{-\alpha_i})(e) = e^{\text{ad}(\sum c_i\, e_{-\alpha_i})}(e)$$

$$= e + [\textstyle\sum c_i e_{-\alpha_i}, e] + \ldots$$

$$= e + \sum_{i,j} c_i[e_{-\alpha_i}, e_{\alpha_j}] + \dots$$

The elements $[e_{-\alpha_i}, e_{\alpha_i}]$, $i = 1,\dots,\ell$ span $\mathfrak{H}$ and it follows that

$$O = \mathfrak{H} + \sum \mathbb{R}^+ e_{\alpha_i} \quad .$$

The orbit $O = \bar{B}.e$ carries a natural symplectic structure. We illustrate this for $g = s\ell(n,\mathbb{R})$. Let $\varepsilon_1,\dots,\varepsilon_\ell$ be the basis of $\mathfrak{H}$ dual to the simple roots $\alpha_1,\dots,\alpha_\ell$. Then the coefficients a_i of the matrix

$$\begin{pmatrix} b_1 & a_1 & & & \\ & b_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & b_{n-1} & a_{n-1} \\ & & & & b_n \end{pmatrix}$$

are the functions $\phi^{e_{-\alpha_i}}$ (recall $\phi^{e_{-\alpha_i}}(x) = Q(e_{-\alpha_i},x))$ and hence

$$\phi^{\varepsilon_j} = \sum_{i=1}^{j} b_j \quad .$$

The functions $\phi^{e_{-\alpha_i}}$, ϕ^{ε_j} give coordinates on O . Further,

$$[\phi^{\varepsilon_i}, \phi^{e_{-\alpha_j}}] = \phi^{[\varepsilon_i, e_{-\alpha_j}]} = \delta_{ij}\, \phi^{e_{-\alpha_j}}$$

$$[\phi^{e_{-\alpha_i}}, \phi^{e_{-\alpha_j}}] = 0$$

$$[\phi^{\varepsilon_i}, \phi^{\varepsilon_j}] = 0 \quad .$$

In general, if (X,ω) is a symplectic manifold with local coordinate functions $u_1,\dots,u_n$ and $\omega = \Sigma\, b_{ij}\, du_i \wedge du_j$, then $[u_i,u_j] = a_{ij}$ where (a_{ij}) is one half the inverse matrix to (b_{ij}) . Hence the symplectic form on $O = \bar{B}.e$ is

$$\omega = \sum_{j=1}^{n-1} \left(\sum_{i=1}^{j} db_i \right) \wedge \frac{da_j}{a_j} .$$

6. The Lie algebra theory of the generalized Toda lattice

Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$, then by a theorem of Chevalley

$$S(\mathfrak{g})^G = \mathbb{R}[I_1,\dots,I_\ell] ,$$

that is, the algebra $S(\mathfrak{g})^G$ of symmetric invariants is a polynomial algebra with homogeneous generators $I_1,\dots,I_\ell$ called the primitive elements. We may take I_1 to be the Killing form.

We have already described how the subspace $S(\mathfrak{g})$ of $C^\infty(\mathfrak{g}')$ inherits a Poisson structure by regarding its elements as left invariant functions on $T^*(G)$. It is not difficult to check that if $\phi \in S(\mathfrak{g})^G$ then ϕ Poisson commutes with $S(\mathfrak{g})$.

The conserved quantities in involution (the H's) on the phase space $\bar{B}.e$, of the generalized Toda lattice will be constructed using the primitive elements $I_1,\dots,I_\ell$. Of course, these functions are constant on G-orbits but we will be concerned with their behaviour on $\bar{B}.e$ translated by another principal nilpotent element f . We construct as algebra isomorphism

$$S(\mathfrak{g})^G \cong S(\bar{\mathfrak{b}})_f$$

where $S(\bar{\mathfrak{b}})_f \subseteq S(\bar{\mathfrak{b}})$ is defined by

$$S(\bar{\mathfrak{b}})_f = \mathbb{R}[I_1^f,\dots,I_\ell^f]$$

where

$$I_j^f(z) = I_j(f+z) \quad , \quad f = \sum_{i=1}^{\ell} e_{-\alpha_i} \quad , \quad z \in \mathfrak{b}$$

(regarding $S(\bar{\mathfrak{b}})$ as polynomials on $\mathfrak{b}$).

Lemma. The algebra $S(\bar{\mathfrak{b}})_f$ is Poisson commutative.

Proof. Let I , $\hat{I} \in S(\mathfrak{g})^G$, $z \in \mathfrak{b}$. Then

$$[I^f,\hat{I}^f](z) = Q(z,[\sigma(I^f)(z),\sigma(\hat{I}^f)(z)] \quad .$$

We introduce the splitting

$$\sigma(I)(x) = \sigma(I)_{\bar{\mathfrak{b}}}(x) + \sigma(I)_{\mathfrak{n}}(x)$$

where $\sigma(I)_{\bar{\mathfrak{b}}}(x) \in \bar{\mathfrak{b}}$ and $\sigma(I)_{\mathfrak{n}}(x) \in \mathfrak{n}$. Then

$$\sigma(I^f)(z) = \sigma(I)_{\bar{\mathfrak{b}}}(f+z)$$

and hence

$$\begin{aligned} [I^f,\hat{I}^f](z) &= Q(z,[\sigma(I^f)(z),\sigma(\hat{I}^f)(z)]) \\ &= Q(z,[\sigma(I)_{\bar{\mathfrak{b}}}(f+z),\sigma(\hat{I})_{\bar{\mathfrak{b}}}(f+z)]) \\ &= Q(z+f,[\sigma(I)_{\bar{\mathfrak{b}}}(f+z),\sigma(\hat{I})_{\bar{\mathfrak{b}}}(f+z)]) \end{aligned}$$

since f is Q-orthogonal to $[\bar{\mathfrak{b}},\bar{\mathfrak{b}}]$. But $z+f$ is regular and hence $\mathfrak{g}^{z+f}$ is spanned by

$$\{\sigma(I)(z+f) \mid I \in S(\mathfrak{g})^G\} \quad .$$

Hence

$$[I^f,\hat{I}^f](z) = -Q(z+f,[\sigma(I)_{\mathfrak{n}}(z+f),\sigma(\hat{I})_{\bar{\mathfrak{b}}}(z+f)])$$

$$= Q(z+f,[\sigma(I)_{\mathfrak{n}}(z+f),\sigma(\hat{I})_{\mathfrak{n}}(z+f)])$$

using the fact that Q is invariant, hence

$$Q([a,b],c) = Q(a,[b,c])$$

and also that

$$[\sigma(I)(z+f),z+f] = 0 \quad .$$

Finally, $z+f \perp [\mathfrak{n},\mathfrak{n}]$ and so $[I^f,\hat{I}^f] \equiv 0$.

Let $J = f+O$ where $O = \bar{B}.e$ and $f = \sum_{i=1}^{\ell} e_{-\alpha_i}$ so that for the case of $SL(n,\mathbb{R})$, J is the space of Jacobi matrices defined in §4. Also the translation by f carries the symplectic structure on O to a symplectic structure ω_J on J which for the case of $SL(n,\mathbb{R})$ agrees with the symplectic structure on J defined in §4.

Let $\tau\colon \mathfrak{g} \to \mathbb{R}^{\ell}$ be given by

$$\tau(x) = (I_1(x),\ldots,I_{\ell}(x)) \quad .$$

Assume that the e_{α_i} are chosen so that $Q(e_{-\alpha_i},e_{\alpha_i}) > 0$. Now for any ξ in the image $\tau(\mathfrak{H}^*)$ let $J(\xi) = \tau^{-1}(\xi) \cap J$. It may be shown that the primitive elements $I_1,\ldots,I_{\ell}$ still have linearly independent differentials when restricted to J . It follows that $J(\xi)$ is a submanifold of dimension ℓ and

$$J = \bigcup_{\xi} J(\xi) \qquad (*)$$

is the foliation corresponding to the polarization of J defined by $I_i|J$, $i = 1,\ldots,\ell$. Explicitly if $y = f+z \in J$ and $T_y(J(\xi))$ is the tangent space to $J(\xi)$ at y then

$$T_y(J(\xi)) = [n_y, y]$$

where n_y is the Q-orthocomplement in $\mathfrak{n}$ of $[y,[\bar{\mathfrak{n}},\mathfrak{n}]]$. The latter is a subspace of codimension ℓ in $\bar{\mathfrak{n}}$.

Now for any $z_o \in \mathfrak{H}^*$ let $z = f + z_o$ so that z is a regular element in $\bar{\mathfrak{b}}$. Then z and z_o are conjugate by an element $\bar{m} \in \bar{B}$. Let G^{z_o} and G^z be their respective centralizers in G so that G^{z_o} is the Cartan subgroup corresponding to $\mathfrak{H}$. The groups G^{z_o} and G^z are not connected. Let G_*^z be the unique component of G^z which is $\bar{m}$ conjugate to that component in G^{z_o} containing the set of elements $g_o \in G^{z_o}$ such that $g_o^{\alpha_i} < 0$ for $i = 1,\ldots,\ell$. Then for any $g \in G_*^z$ there exists unique elements $\bar{n} = \bar{n}(g) \in \bar{N} = [\bar{B},\bar{B}]$, $n = n(g) \in N = [B,B]$ and $h = h(g) \in H$ where H is the connected Lie subgroup corresponding to $\mathfrak{H}$, so that

$$a(\aleph)^{-1} g = \bar{n} h n$$

where $\aleph$ is the unique element of the Weyl group W such that $\phi < 0$ for $\phi > 0$ and $a(\aleph) \in G$ is the unique element in the normalizer of $\mathfrak{H}$ such that $a(\aleph) e_{\alpha_i} = e_{\aleph\alpha_i}$.

Now for any $\xi \in \tau(\mathfrak{H}^*)$, $J(\xi)$ has a flat affine connection as the $J(\xi)$ are leaves of a polarization. On the other hand G_*^z has a natural flat affine connection since G^z is an abelian Lie group.

Theorem. Let $z_o \in \mathfrak{H}^*$ and put $\xi = \tau(z_o)$. Then for any $g \in G_*^z$, $\mathrm{Ad}\, n(g) z \in J(\xi)$. Moreover, the map $g \to \mathrm{Ad}\, n(g) z$ defines an isomorphism

$$\beta\colon G_*^z \to J(\xi)$$

of flat affinely connected manifolds.

Note that G_*^z is complete with respect to the flat affine connection. This means the polarization defined by the $J(\xi)$ satisfies the Pukansky condition.

Corollary. For any $I \in S(\mathfrak{g})^G$ let ξ_I be the Hamiltonian vector field on J corresponding to the function $I|J \in C^\infty(J)$. Then ξ_I can be integrated for all values of the time parameter, that is, it generates a global one parameter group.

We are of course interested in the case where $I = I_1$ is the Hamiltonian of our generalized Toda lattice. Given the initial condition $z = f + z_o$ in the notation of the theorem and $g \in G_*^z$ we have the following result.

Theorem. The trajectory of ξ_I through $\beta(g)$ is $\beta(g \exp tz)$ for $t \in \mathbb{R}$.

To be more explicit we first write

$$a(\aleph)^{-1} g \exp tz = \bar{n}(t)h(t)n(t) \quad .$$

Then

$$\beta(g \exp tz) = \mathrm{Ad}\, n(t) z$$

$$= f + x(t) + \sum_{i=1}^{\ell} s_i(t) e_{\alpha_i}$$

where $x(t) \in \mathfrak{H}$ and $s_i(t) > 0$. Thus for the case of the usual Toda lattice

$$s_i(t) = e^{q_i(t) - q_{i+1}(t)} \quad .$$

In general, $s_i(t) = h(t)^{-\alpha_i}$, so the problem of determining the trajectory is the same as computing $h(t)^{-\alpha_i}$. However, if λ_i, $i = 1, \ldots, \ell$ are the highest weights of the fundamental representations σ_i of $\mathfrak{g}$ it suffices to compute $h(t)^{\lambda_i}$ since the $h(t)^{-\alpha_i}$ are related to the $h(t)^{\lambda_i}$ by the Cartan matrix. If V_{λ_i} is the module of σ_i and v_{λ_i} and $v_{\gamma\lambda_i}$ are respectively the highest and the lowest weight vectors of V_{λ_i}, then (replacing the adjoint group by the simply connected group)

$$h(t)^{\lambda_i} = (\sigma_i(g \exp tz)v_{\lambda_i}, v_{\gamma\lambda_i})$$

where the inner product $(\, , \,)$ is invariant under the compact form $\mathfrak{k} + i\mathfrak{p}$. However, writing

$$g \exp tz = \bar{m}(g_o \exp tz_o)\bar{m}^{-1}$$

we have, as a sum of exponentials,

$$h(t)^{\lambda_i} = \sum_{\nu} c_\nu \, g_o^\nu \, e^{t<\nu, z_o>}$$

where ν runs over a weight basis of V_{λ_i} and the constants c_ν are expressed in terms of the operator $\sigma_i(\bar{m})$. Since this may be given explicitly we have a formula for $h(t)^{\lambda_i}$ and hence $h(t)^{-\alpha_i}$. Furthermore, we may choose z_o to be in the fundamental chamber so that the asymptotic line for $t \to +\infty$ is given by

$$t<\lambda_i, z_o> + \log c_{\lambda_i}$$

and the asymptotic line for $t \to -\infty$ by

$$t<\aleph\lambda_i, z_o> + \log c_{\aleph\lambda_i} \quad .$$

Moser's results on the asymptotic behaviour follow easily from these facts when $\mathfrak{g}$ is the Lie algebra of $SL(n,\mathbb{R})$.

7. Quantization of the Toda lattice

We have constructed ℓ commuting elements $I_1^f,\ldots,I_\ell^f$ in $S(\bar{\mathfrak{b}})$ and now seek the quantum analogue of this picture. The orbit $O = \bar{B}.e$ determines a unique representation π_O of $\bar{B}$. The quantum analogue will consist of constructing ℓ elements $\tilde{I}_1^k,\ldots,\tilde{I}_\ell^k$ in $U(\bar{\mathfrak{b}})$ and then constructing the simultaneous spectral resolution of $\pi_O(\tilde{I}_1^k),\ldots,\pi_O(\tilde{I}_\ell^k)$.

For example, if $\mathfrak{g} = B_3$ then

$$I_1^f = \tfrac{1}{2}(p_1^2+p_2^2+p_3^2) + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3}$$

and the quantum analogue involves finding the spectral resolution of the operator

$$-\left(\frac{d}{dq_1}\right)^2 - \left(\frac{d}{dq_2}\right)^2 - \left(\frac{d}{dq_3}\right)^2 + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3} \quad .$$

The splitting $\mathfrak{g} = \mathfrak{k} + \bar{\mathfrak{b}}$ induces

$$U(\mathfrak{g}) = U(\mathfrak{g})\,\mathfrak{k} + U(\bar{\mathfrak{b}})$$

which in turn defines a mapping from $U(\mathfrak{g})$ to $U(\bar{\mathfrak{b}})$ which restricted to the centre of $U(\mathfrak{g})$ becomes a homomorphism of algebras. The centre of $U(\mathfrak{g})$ is in 1-1 correspondence with $S(\mathfrak{g})^G$ and we let $\tilde{I}_j^k$ be the image in $U(\bar{\mathfrak{b}})$ of the primitive element $I_j \in S(\mathfrak{g})^G$.

On the other hand the direct sum

$$S(\mathfrak{g}) = S(\bar{\mathfrak{b}}) \oplus \mathfrak{k}\,S(\mathfrak{g})$$

defines a projection map from $S(\mathfrak{g})$ to $S(\bar{\mathfrak{b}})$. Let $I_j^k \in S(\bar{\mathfrak{b}})$ be the image of I_j. First we consider the sense in which $\pi_O(\tilde{I}_j^k)$ is the quantization of $I_j^k \mid O$.

The representation π_O is defined as follows (a more complete review of the theory is given in the next section). The isotropy subgroup of $\bar{B}$ at e is $[\bar{N},\bar{N}]$ and the triple

$$[\bar{\mathfrak{n}},\bar{\mathfrak{n}}] \subseteq \bar{\mathfrak{n}} \subseteq \bar{\mathfrak{b}}$$

defines a polarization at e. Let $x \in \bar{\mathfrak{n}}$. Then

$$x = \sum_{\phi>0} d_\phi e_{-\phi}$$

and the character of $\bar{N}$ corresponding to e is

$$\bar{\chi}(\exp x) = e^{2\pi i \Sigma d_i c_i}$$

where $c_i = Q(e_{\alpha_i}, e_{-\alpha_i})$. The representation π_0 is simply $\mathrm{Ind}\ \bar{\chi}$.

Since $\bar{B} = H\bar{N}$, the representation π_0 defines a representation

$$\pi_0 : U(\bar{\mathfrak{b}}) \to \mathrm{End}\ C^\infty(H)$$

of $U(\bar{\mathfrak{b}})$ as differential operators on H. However, we can regard O as the cotangent bundle of H. Using the standard filtration of $U(\bar{\mathfrak{b}})$ we see that $I_i^k \mid O$ is the symbol of $\pi_0(\tilde{I}_1^k)$. Thus $\pi_0(\tilde{I}_i^k)$ (modulo an appropriate power of $\sqrt{-1}$) is a quantization of $I_i^k \mid O$. We note that for $s\ell(n,\mathbb{R})$, $\pi_0(I_1^k)$ may be expressed in terms of the Bessel's operator.

Next we describe the simultaneous eigenfunction expansion of the $\pi_0(\tilde{I}_i^k)$ using eigenfunctions which decay exponentially at ∞. However, before describing the $\pi_0(\tilde{I}_j^k)$ let us recall some elementary facts of the harmonic analysis theory of the representation theory of groups. Let $C^\infty(G)$ and $C_o^\infty(G)$ be the spaces of smooth functions of arbitrary and compact support, respectively, on a Lie group G, with their usual topologies. Let $\mathrm{Dist}_o(G)$ and $\mathrm{Dist}(G)$ be their respective continuous duals. There are natural embeddings of both G

and $U(\mathfrak{g})$ in $\mathrm{Dist}_o(G)$.

Let π be a representation of G on a Hilbert space H . Then let $H^\infty \le H$ be the space of smooth vectors. The space H^∞ with the locally convex topology induced by the semi-norms $\| \ \|_u$, $u \in U(g)$

$$\|v\|_u = \|uv\| \quad , \quad v \in H^\infty$$

is complete. Let $H^{-\infty}$ be its continuous dual. Then $U(g)$ also acts on $H^{-\infty}$. In fact, there is a natural action of $\mathrm{Dist}_o(G)$ on both H^∞ and $H^{-\infty}$. Note that if $v \in H^{-\infty}$ and $\phi \in C_o^\infty(G)$ then $\pi(\phi)v \in H^\infty$. Let $v, v' \in H^{-\infty}$ and define

$$d_{v,v'}(\phi) = \langle v, \pi(\phi)v' \rangle \quad .$$

The distribution $d_{v,v'}$ on G is a function of either v or v' and belongs to $H^{-\infty}$.

The element $v' \in H^{-\infty}$ is called a Whittaker vector (with respect to χ) if $\pi(a)v' = \chi(a)v'$ for all $a \in \bar{N}$.

In [5] we have proved that if π is any member of the principal series, there exists (up to scalars) a unique Whittaker vector $v' \in H^{-\infty}$. Now $\mathfrak{H}_{\mathbb{C}}'$ parametrizes the spherical principal series. For each $\lambda \in \mathfrak{H}_{\mathbb{C}}'$ let π_λ denote the corresponding principal series representation on H_λ . Let $v' \in H_\lambda^{-\infty}$ be a Whittaker vector and let $v \in H_\lambda^\infty$ be a spherical vector. Then if $a \in H$, the function $d_\lambda(a) = d_{v,v'}(a)$ is an eigenvector for the operators $\pi_0(\tilde{I}_i^k)$. This is clear using the Iwasawa decomposition $G = KH\bar{N}$ and the fact that $\pi_\lambda(\tilde{I}_i)$ is a scalar $c_i(\lambda)$. Then

$$\pi_0(\tilde{I}_i^k)d_\lambda = c_i(\lambda)d_\lambda \quad .$$

For $SL(2,\mathbb{R})$, d_λ can be expressed in terms of the

Whittaker function $W_{0,s(\lambda)}$ or Bessel functions $J_{n(\lambda)}$.

Let $\chi: \mathfrak{n} \to i\mathbb{R}$ so that $\chi(x) = iQ(t,x)$. Then χ extends to a homomorphism $\chi: U(\mathfrak{n}) \to \mathbb{C}$. Let $U_\chi(\mathfrak{n})$ be the kernel of this extension so that

$$U(\mathfrak{g}) = U(\bar{\mathfrak{b}}) \oplus \mathfrak{u}\, U_\eta(\mathfrak{n}) \quad .$$

For any $\tilde{I} \in U(\mathfrak{g})$, let $\tilde{I}^f$ be its image in $U(\bar{\mathfrak{b}})$ with respect to this decomposition. If $W(\bar{\mathfrak{b}}) \subseteq U(\bar{\mathfrak{b}})$ is the image of the centre Z of $U(\mathfrak{g})$ then $W(\bar{\mathfrak{b}}) = \mathbb{C}[\tilde{I}_1^f, \ldots, \tilde{I}_\ell^f]$ is a polynomial algebra. There exists a filtration (the x_o-filtration in $U(\bar{\mathfrak{b}})$, see [5]) such that modulo the appropriate power of $\sqrt{-1}$, $I_j^f \in S(\bar{\mathfrak{b}})$ is the symbol of $\tilde{I}_j^f$. This carries over to the orbit 0 and hence, modulo the appropriate power of $\sqrt{-1}$, using this filtration, $\pi_0(\tilde{I}_j^f)$ is a quantization of $I_j^f | 0$.

Remark. If $\tau: 0 \to 0$ is given by

$$\tau(x + \sum_{i=0}^{\ell} r_i e_{\alpha_i}) = x + \sum_{i=1}^{\ell} r_i^2 e_{\alpha_i} \quad ,$$

then it can be shown that

$$I_j^k = I_j^f \circ \tau \quad .$$

The simultaneous eigenfunctions for the $\pi_0(\tilde{I}_j^f)$ are found in a way similar to the definition above except that v is replaced by a Whittaker vector with respect to χ. Using an ellipticity argument it can be shown that the corresponding distribution is actually a function on H.

8. Representation theory of solvable groups

If a group G is of Type I then an arbitrary unitary representation may be expressed 'uniquely' as the direct integral of irreducible representations. The general solvable group is not of this type as is shown by the following counter example of Mauntner. This group is the semi-direct product $\mathbb{R} \times_s (\mathbb{C} \times \mathbb{C})$ with $s\colon \mathbb{R} \to \mathrm{Aut}(\mathbb{C} \times \mathbb{C})$ given by

$$s(t)(z_1, z_2) = (e^{it} z_1, e^{i\lambda t} z_2) \quad , \quad \lambda \text{ irrational.}$$

Let G be a connected, simply connected solvable group. Then the property of being of Type I may be characterized in terms of the co-adjoint orbit structure.

Let $f \in G.f$. Then if G^f denotes the isotropy group, $O = G/G^f$ and in general, G^f need not be connected. Hence the orbit may possess interesting geometry. On O we have the symplectic 2-form ω_f given by

$$\omega_f(Y,X) = \langle f,[X,Y]\rangle \ , \quad X,Y \in \mathfrak{g} \ ,$$

which defines a cohomology class

$$[\omega_f] \in H^2(O,\mathbb{R})$$

independent of the representative $f \in O$. The orbit is said to be integral if $[\omega_f]$ lies in the image $H^2(O,\mathbb{Z}) \to H^2(O,\mathbb{R})$.

The isotropy algebra $\mathfrak{g}^f$, the Lie algebra of G^f , is characterized by

$$\langle f,[\mathfrak{g}^f,\mathfrak{g}]\rangle = 0 \quad .$$

Hence $\langle f,[\mathfrak{g}^f,\mathfrak{g}^f]\rangle = 0$ and the map

$$2\pi i f\colon \mathfrak{g}^f \to i\mathbb{R}$$

is a homomorphism of Lie algebras. The orbit is integral iff

$2\pi i f$ induces a homomorphism

$$\chi_f \colon G_o^g \to S^1$$

of Lie groups.

Theorem (Auslander and Kostant). Let G be a connected, simply connected solvable Lie group. Then G is of Type I iff every co-adjoint orbit is

(1) the intersection of an open and a closed set,

(2) $[\omega_f] = 0$.

All algebraic solvable groups satisfy these conditions.

Let $\hat{G}$ denote the set of irreducible, unitary representations of G . These representations may be constructed from the orbits but the theory differs from that of nilpotent groups (developed by Kirillov) in that there may be more than one representation coming from a single orbit. We describe (without proofs) the precise parametrization. The main point is that for a given $f \in \mathfrak{g}'$, G^f need not be connected and hence there may be more than one character of G^f with differential $2\pi i f$.

Let $\widehat{G^f}$ denote the set of all characters χ on G^f such that

$$d(\chi) \mid \mathfrak{g}^f = 2\pi i f \quad .$$

Then for each $\gamma \in (G^f/G_o^f)^\wedge$, $\gamma \cdot \chi \in \widehat{G^f}$ (G_o^f is the identity component of G^f). In fact, $(G^f/G_o^f)^\wedge$ acts simply, transitively on $\widehat{G^f}$.

In what follows we describe how to assign to each $\chi \in \widehat{G^f}$ a representation π_χ of G . Then defining $\widehat{G(O)} = \{\pi_\chi \mid \chi \in \widehat{G^f}\}$ we have

$$\hat{G} = \bigcup_{\text{all orbits}} \widehat{G(0)} .$$

There is a special class of solvable groups, the exponential solvable groups, where the representation theory is simpler and the parametrization of $\hat{G}$ given by the co-adjoint orbits. This theory was developed by Bernat. A Lie group G is said to be exponential if the exponential map

$$\exp: \mathfrak{g} \to G$$

is bijective. We describe the construction briefly.

Let the dimension of an orbit $0 = G.f$ be $2k$. Then a real polarization at f is a subalgebra $\mathfrak{H}$ such that

$$\mathfrak{g}^f \subset \mathfrak{H} \subset \mathfrak{g} ,$$

$\dim \mathfrak{g} - \dim \mathfrak{H} = \dim \mathfrak{H} - \dim \mathfrak{g}^f = k$, and $\langle f,[\mathfrak{H},\mathfrak{H}]\rangle = 0$. Let H_o be the corresponding subgroup of G and $H = G^f.H_o$. Then χ_f extends to a character of H , $\eta_f: H \to S^1$. In the case of a nilpotent group we simply take

$$\pi_\chi = \operatorname{Ind}_H^G \eta_f .$$

However, in the solvable case, this representation is not, in general, irreducible unless the polarization satisfies the following extra condition.

Let $\mathfrak{H}^\perp \subseteq \mathfrak{g}'$ be the space of functionals vanishing on $\mathfrak{H}$. Then $H.f$ is contained in $f + \mathfrak{H}^\perp$ as an open subset.

The Pukansky condition. The Pukansky condition is that

$$H.f = f + \mathfrak{H}^\perp .$$

If $\mathfrak{H}$ satisfies this condition then $\operatorname{Ind}_H^G \eta_f$ is an irreducible representation of G . In geometric terms, we have a fibration $G/G^f \to G/H$ with fibre G^f/H . The fibres are leaves of a Lagrangian foliation and the Pukansky condition requires that the leaves be complete.

Although a real polarization $\mathfrak{H}$ at f satisfying the Pukansky condition is not unique, all of them lead to the same representation.

For a general solvable group, it is necessary to introduce the notion of a complex polarization. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. A complex polarization at f is a complex vector space $\mathfrak{H}$,

$$\mathfrak{g}^f_{\mathbb{C}} \subset \mathfrak{H} \subset \mathfrak{g}_{\mathbb{C}}$$

such that

1) $\langle f,[\mathfrak{H},\mathfrak{H}]\rangle = 0$

2) $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim_{\mathbb{C}} \mathfrak{H} = \dim_{\mathbb{C}} \mathfrak{H} - \dim_{\mathbb{C}} \mathfrak{g}^f_{\mathbb{C}}$

3) $\mathfrak{H}$ is stable under G^f

4) $\mathfrak{H} + \bar{\mathfrak{H}}$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{d} = \mathfrak{H} \cap \mathfrak{g}$ and $\mathfrak{e} = (\mathfrak{H} + \bar{\mathfrak{H}}) \cap \mathfrak{g}$. Then

$$\mathfrak{d}_{\mathbb{C}} = \mathfrak{H} \cap \bar{\mathfrak{H}} \quad \text{and} \quad \mathfrak{e}_{\mathbb{C}} = \mathfrak{H} + \bar{\mathfrak{H}} \quad .$$

Let D_o and E_o be the corresponding subgroups of G , $D = G^f.D_i$ and $E = G^f.E_o$. In this case the Pukansky condition is that $E.f$ is closed.

There is a natural symplectic form B_f on $\mathfrak{e}/\mathfrak{d}$: let $x,y \in \mathfrak{e}$ and $\hat{x},\hat{y}$ be the images of x and y in the quotient $\mathfrak{e}/\mathfrak{d}$, then

$$B_f(\hat{x},\hat{y}) = \langle f,[x,y]\rangle \quad .$$

On the complexification,

$$(\mathfrak{e}/\mathfrak{d})_{\mathbb{C}} = (\mathfrak{H}/\mathfrak{d})_{\mathbb{C}} \oplus (\bar{\mathfrak{H}}/\mathfrak{d})_{\mathbb{C}}$$

there is a natural j operator,

$$j|(\mathfrak{H}/\mathfrak{d})_{\mathbb{C}} = -i \quad \text{and} \quad j|(\mathfrak{H}/\mathfrak{d})_{\mathbb{C}} = i$$

which makes E/D into a Kahler manifold.

A polarization $\mathfrak{H}$ is said to be positive if the symmetric form $S_f(\hat{x},\hat{y}) = B_f(j\hat{x},\hat{y})$ is positive on $\mathfrak{e}/\mathfrak{d}$. To construct a representation from a positive polarization we proceed as follows. A character χ_f on G^f has a unique extension η_f to D, a character with derivative $2\pi i f$. Then the representation $\mathrm{Ind}(\chi_f,\mathfrak{H})$ is a certain subrepresentation of $\mathrm{Ind}{\uparrow}_D^G(\eta_f)$.

The space E/D has a complex structure in which $(\mathfrak{H}/\mathfrak{d})_{\mathbb{C}}$ is the space of anti-holomorphic tangent vectors and an invariant measure corresponding to S_f. Let V be the space of smooth functions ϕ on E such that $\phi(xa) = \eta_f(a^{-1})\phi(x)$, $x \in E$, $a \in D$, $|\phi|$ is square integrable on E/D and such that $.z = 2\ i\langle f,z\rangle$ for all $z \in (\mathfrak{H}/\mathfrak{d})_{\mathbb{C}}$ where $\phi.x(y) = \frac{d}{dt}\phi(y\exp(-tx))\big|_{t=0}$ for $x \in \mathfrak{e}$. Then E has a unitary representation π_V on the Hilbert space V. Let $\mathrm{Ind}(\chi_f,\mathfrak{H} = \mathrm{Ind}{\uparrow}_E^G(\pi_V)$.

Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$ and let $\mathfrak{H}$ be a polarisation at $f \in \mathfrak{g}'$. Then $\mathfrak{H}$ is said to be admissible if $\mathfrak{H} \cap \mathfrak{n}_{\mathbb{C}}$ is a polarization at $f' = f|\mathfrak{n}$. Notice that G preserves $\mathfrak{n}$ and if $G^{f'} = \{x \in G | x.f' = f'\}$ then $G^{f'} \supseteq G^f$. The polarization $\mathfrak{H}$ is said to be strongly admissible if, in addition, $G^{f'}$ preserves $\mathfrak{H} \cap \mathfrak{n}_{\mathbb{C}}$.

Theorem (Auslander and Kostant). Let G be a connected, simply connected solvable group and $f \in \mathfrak{g}'$. Then there exists a strongly admissible polarization $\mathfrak{H}$ and f. The representation $\mathrm{Ind}(\chi_f,\mathfrak{H})$ is irreducible and independent of $\mathfrak{H}$.

Hence we have described a mapping

$$G^f \to \hat{G}: \chi_f \to \mathrm{Ind}(\chi_f, \ )$$

which is injective and gives

$$\hat{G} = \bigcup_{\substack{\text{co-adjoint} \\ \text{orbits } 0}} G(0)$$

when G is of Type I.

References

[1] R. Abraham. Foundations of Mechanics, Benjamin.

[2] L. Auslander and B. Kostant. 'Polarization and unitary representations of solvable Lie groups', *Inventiones Math.* 14 (1971), 255-354.

[3] P. Cartier. 'Vecteurs différentiables dans les représentations unitaires des groupes de Lie', Séminaire Bourbaki, Exposé No.454, 1974/5.

[4] B. Kostant. 'Quantization and unitary representations', Lectures in Modern Analysis and Applications III, Lecture Notes in Mathematics, No.170, Springer-Verlag.

[5] B. Kostant. 'Whittaker vectors and representation theory', M.I.T. preprint.

[6] J. Moser, 'Finitely many mass points on the line under the influence of an exponential potential - an integrable system', Dynamical Systems, Theory and Applications, Lecture Notes in Physics, No.38, Springer-Verlag.

[7] M. Vergne, 'Représentations unitaires des groupes de Lie resoluble', Séminaire Bourbaki, Exposé No.447, 1973/4.

[8] G. Warner, 'Harmonic Analysis on Semi-simple Groups I', Springer-Verlag, 1972.

11 · Integral geometry and representation theory

D. KAZHDAN
(Harvard University)
(Notes by G. Lusztig)

1. An important question in representation theory is that of decomposing induced representations. Let G be a Lie group and let H be a closed subgroup; assume that $X = H\backslash G$ has a G-invariant measure so that we have a unitary representation on $L^2(X)$. We shall explain how it decomposes in the special case where $G = SL(2,\mathbb{R})$, H = group of upper triangular unipotent matrices in G. In this case, X can be identified with $\mathbb{R}^2-(0,0)$ with the usual action of $SL(2,\mathbb{R})$, which leaves invariant the Lebesgue measure. For each $\lambda \in \mathbb{R}^*$ we have an operator $\rho_\lambda: L^2(X) \to L^2(X)$ given by $(\rho_\lambda f)(x) = f(\lambda x)$, $x \in X$. It commutes with the action of G. The operators ρ_λ give rise to a decomposition of $L^2(X)$ compatible with the G-action, into a direct integral

$$L^2(X) = \int_{\pi\in\hat{\mathbb{R}}^*} L_\pi \, d\pi \quad ,$$

where $d\pi$ is a Haar measure on the set $\hat{\mathbb{R}}^*$ of unitary characters of $\mathbb{R}^*$ and, for $\pi \in \hat{\mathbb{R}}^*$, L_π is the Hilbert space of all functions f on X which are square integrable on any circle in $\mathbb{R}^2$ with centre at the origin and satisfy $\rho_\lambda(f) = \pi(\lambda)f$, for all $\lambda \in \mathbb{R}^*$. This gives the required decomposition of $L^2(X)$, indeed the G-modules L_π are irreducible except when $\pi^2 = 1$. Note also that the G-modules $L_\pi, L_{\pi'}$ are isomorphic if and only if $\pi' = \pi^{\pm 1}$; we can express this fact by saying that $L^2(X)$ has a 'double spectrum'.

Now let f be a smooth complex value function with compact support on G. Following Gelfand, we associate to f the function $\tilde{f}: G \times G \to \mathbb{C}$ defined by

$$\tilde{f}(g_1,g_2) = \int_{t\in\mathbb{R}} f(g_1^{-1} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g_2)\, dt \quad .$$

It is clear that $\tilde{f}$ will satisfy

$$\tilde{f}(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g_1 , \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} g_2) = \tilde{f}(g_1,g_2)$$

for all $a \in \mathbb{R}^*$, $b \in \mathbb{R}$, $c \in \mathbb{R}$. Thus, $\tilde{f}$ may be regarded as a function on $X \times X$, invariant under all transformations of $X \times X$ of the form $(x,x') \to (ax,ax')$, $a \in \mathbb{R}^*$. Using the decomposition of $L^2(X)$ described above, we see that $\tilde{f}$ can be regarded as an element of

$$L' = \int_{\pi\in\mathbb{R}^*} L_\pi \otimes L_{\pi^{-1}}\, d\pi \quad .$$

The map $f \to \tilde{f}$ extends to a map of $G\times G$-modules $\psi: L^2(G) \to L'$. ψ is not surjective since L' has a 'double spectrum' while $L^2(G)$ has a simple spectrum (as a $G\times G$-module). Roughly speaking, the image of ψ is half of L'. All this remains valid if in the previous discussion $\mathbb{R}$ is replaced by $\mathbb{C}$, so that $SL(2,\mathbb{R})$ is replaced by $SL(2,\mathbb{C})$. On the other hand, the description of the kernel of ψ depends very much on the ground field: in the complex case, $\ker \psi = 0$, while in the real case, $\ker \psi$ is a direct sum of countably many irreducible representations (the discrete series). This is intimately related with questions in integral geometry, as we shall now explain. We may regard $SL(2)$ as a quadric $ad - bc = 1$ in the four-dimensional space with coordinates (a,b,c,d). The subsets of $SL(2)$ of the form $g_1^{-1}Hg_2$ are precisely the straight lines on our quadric. The map $f \to \tilde{f}$ considered above associates to a function f defined on the quadric, the function $\tilde{f}$ on the set of lines on the quadric obtained by integrating f along each of these lines. The question whether $\ker \psi = 0$ can then be reformulated as follows: can the function f be reconstructed from the function $\tilde{f}$, i.e. from the knowledge of the integrals of f on each line on the quadric? This is a typical question of integral geometry.

2. Recently in [3] a general theorem was proved which throws some light on this question of integral geometry. Before stating this result, we consider a very simple variant of it, for finite sets. Consider a diagram of finite sets

$$X \xleftarrow{\ \pi\ } Z \xrightarrow{\ \rho\ } Y$$

such that $\pi \times \rho: Z \to X \times Y$ identifies Z with a subset of $X \times Y$. Assume that there exist two constants $c \neq c'$ such that, if $G_x = \rho(\pi^{-1}x)$, then $|G_x| = c$ for all $x \in X$ and $|G_x \cap G_{x'}| = c'$ for all $x \neq x'$ in X . Define $R: \mathbb{C}(x) \to \mathbb{C}(Y)$ by

$$(Rf)(y) = \sum_{\substack{x \in X \\ (x,y) \in Z}} f(x) \quad , \qquad (f \in \mathbb{C}(X))$$

and $R^t: \mathbb{C}(Y) \to \mathbb{C}(X)$ by

$$(R^t f')(x) = \sum_{\substack{y \in Y \\ (x,y) \in Z}} f'(y) \quad , \qquad (f' \in \mathbb{C}(Y)) \quad .$$

One checks immediately that, if δ_{x_0} is the characteristic function of $x_0 \in X$, then $(R^t \circ R)(\delta_{x_0})(x) = |G_x \cap G_{x_0}|$; it follows that $\ker R = 0$. (This applies, for example, in the case where X is an affine space of dimension ≥ 2 over a finite field, Y is the set of affine lines in this affine space and Z is the incidence relation between points and lines.)

Let us now drop the assumption that X,Y,Z are finite, we assume only that X,Y,Z are smooth manifolds, π,ρ are C^∞ fibrations and π is proper. Assume also that $\pi\times\rho: Z\to X\times Y$ identifies Z with a submanifold of $X\times Y$ and that we are given a smooth measure dz on Z. We define the "Radon transform" R from smooth functions f on X with compact support to smooth measures on Y with compact support by the formula

$$\int_{Y_0} R(f) = \int_{\rho^{-1}(Y_0)} \pi^* f.dz$$

for any open set $Y_0\subset Y$. The transpose R^t of R associates to a smooth function f' on Y a smooth measure on X_1 by the formula

$$\int_{X_0} R^*(f') = \int_{\pi^{-1}(X_0)} \rho^* f'.dx$$

for any relatively compact open set $X_0\subset X$. If we assume that on X and Y we are given some fixed, nowhere vanishing, smooth measures, we may identify smooth functions and smooth measures on X and Y, and we may consider the composition $R^t\circ R$ as a map from smooth functions with compact support on X to smooth functions on X.

We shall also make the following two assumptions.

(a) For each $x\in X$, consider the compact submanifold $G_x = \rho(\pi^{-1}(x))$ of Y. Then if $x\neq x'$, G_x and $G_{x'}$ intersect transversally in Y.

(b) Let $N_z\subset T^*(X\times Y)$ be the conormal bundle of Z in $X\times Y$ (= annihilator of tangent bundle of Z). Then $N_z - 0\subset (T^*(X)-0)\times(T^*(Y)-0)$ and ρ induces an immersion $N_z - 0\to T^*(Y)-0$.

It is proved in [3] that under these assumptions, $R^t\circ T$ is an elliptic pseudo-differential operator on X.

It follows that a function f in the kernel of R is automatically C^∞ and, if it has sufficiently small support, it is actually zero; moreover, if X is compact, $\ker R$ is finite dimensional.

3. Let E be an n-dimensional affine space over k and let H_k be the set of all k-dimensional subspaces of E $(k < n-1)$. The assumptions of (a),(b) of [3] are satisfied if we take $X = E$, $Y = H_k$ and Z = the incidence relation between points and k-planes in E. This example has been investigated by Gelfand, Graev and Shapiro in [2]. Before explaining this result we shall need some notation.

If W is a complex vector space, we have a canonical decomposition

$$W \otimes_{\mathbb{R}} \mathbb{C} = W' \oplus W''$$

defined as follows: consider the map $I: W \otimes_{\mathbb{R}} \mathbb{C} \to W \otimes_{\mathbb{R}} \mathbb{C}$ defined by $I(w \otimes \lambda) = (iw) \otimes \lambda$ and define W' to be the i-eigenspace of I and W'' to be the $(-i)$-eigenspace of I. Let Λ be the line bundle over H_k whose fibre at $x \in H_k$ is $\Lambda^k(L') \otimes \Lambda^k(L'')$ where L is the vector space of translations of the affine space x. (The correspondence $x \to L$ gives a map $p: H_k \to G_k$ where G_k is the set of all k-dimensional subspaces of the vector space V of translations of E.) Then the Radon transform (integration over x) can be regarded as a linear map

$$R: \mathcal{S}(E) \to \mathcal{S}(H_k, \Lambda)$$

where $\mathcal{S}$ denotes the space of Schwartz functions on E or sections of Λ. Let Ω^{kk} be the bundle of differential forms of type (k,k) on G_k and let $\pi^*\Omega^{kk}$ be its pull-back under the natural projection $\pi: G_k \times E \to G_k$. We define a linear map

$$D: C^\infty(H_k,\Lambda) \to C^\infty(G_k \times E, \pi^*\Omega^{kk})$$

as follows. Let $f \in C^\infty(H_k,\Lambda)$, $L \in G_k$, $e \in E$ be given. Let f_L be the restriction of f to the (n-k)-dimensional affine space $p^{-1}(L)$ (with space of translations V/L) and let $\bar{e}$ be the point in this affine space defined by e. Let $\nu_1',\ldots,\nu_k'$ be vectors in (V/L) and let $\nu_1'',\ldots,\nu_k''$ be vectors in $(V/L)''$. The expression

$$\frac{\partial^{2k} f_L}{\partial\nu_1'\ldots\partial\nu_k'.\partial\nu_1''\ldots\partial\nu_k''}(\bar{e}) \in \Lambda^k(L') \otimes \Lambda^k(L'')$$

is multilinear in ν_i',ν_i'' and symmetric in $\nu_1',\ldots,\nu_k'$ and in $\nu_1'',\ldots,\nu_k''$, hence it defines a linear map

$$S^k((V/L)') \otimes S^k((V/L)'') \to \Lambda^k(L') \otimes \Lambda^k(L'')$$

(where S^k is the k-th symmetric power) hence a linear map

$$\Lambda^k((L')^*) \otimes S^k((V/L)') \otimes \Lambda^k(L'') \otimes S^k((V/L)'') \to \mathbb{C} \quad .$$

Now, given two vector spaces V_1,V_2, there is a canonical surjective map

$$\Lambda^k(V_1 \otimes V_2) \to \Lambda^k(V_1) \otimes S^k(V_2) \quad ;$$

taking $(V_1,V_2) = ((L')^*,(V/L)')$ or $((L'')^*,(V/L)'')$ and composing with the previous map, we get a linear map

$$\Lambda^k((L'^* \otimes (V/L)') \otimes \Lambda^k((L'')^* \otimes (V/L)'') \to \mathbb{C} \quad .$$

If we identify the tangent space $T_{G_k}(L)$ to G_k at L with $L^* \times (V/L)$, we get a linear map

$$\Lambda^k(T_{G_k}(L)') \otimes \Lambda^k(T_{G_k}(L)'') \to \mathbb{C} \quad .$$

We define $(Df)_{L,e}$ to be this linear map. This completes the definition of D . Note that D is a differential operator of order $2k$.

The first main result of [2] is that a section $f \in \mathcal{S}(H_k,\Lambda)$ is in the image of R if and only if Df is a closed form on G_k (depending on a parameter in E). The second main result of [2] is that R is injective. The idea of the proof is as follows. Let γ be a 2k-dimensional homology class of G_k . Given $f \in \mathcal{S}(E)$, we may integrate $D(R(f_1))$ (regarded as a differential form on G_k , depending on a parameter in E) over a cycle representing γ . The result is a function $s_\gamma(f_1)$ on E which does not depend on the choice of the cycle in the homology class γ . Using the fact that all our constructions are equivariant with respect to the group of affine transformations of E , we see that $s_\gamma(f_1)$ must in fact be of the form $c_\gamma . f_1$, where c_γ is a constant independent of f_1 . To prove that $\ker R = 0$, it is then enough to find γ such that $c_\gamma \neq 0$. The cycle in G_k consisting of all k-planes contained in a fixed (k+1)-plane will have this property.

(In the real case, a result analogous to the first main result of [2] holds; however, in this case, R is not in general injective since no suitable cycles in the real Grassmannian can be found.)

The above result on the injectivity of R is not in itself sufficient for the application to representation theory given in no.1. One needs an inversion formula i.e. a way to recover $f \in \mathcal{S}(E)$ when only the restriction of $R(f)$ to a certain n-dimensional manifold of H_k is known. The problem of characterizing these n-dimensional submanifolds of H_k for which such an inversion formula exists is a very interesting one. The reader is referred to [1],[3] for a study of this problem in the case $k = 1$.

References

[1] I.M. Gelfand, M.I. Graev. 'Complexes of straight lines in the space C^n', *Funkt.Analiz.* 2, No.3 (1968), 39-52.

[2] I.M. Gelfand, M.I. Graev, Z.Ya.Shapiro. 'Integral Geometry on k-dimensional spaces', *Funkt.Analiz.* 1, No.1 (1967), 15-31.

[3] V. Guillemin. 'On some results of Gelfand in integral geometry', Symposium on Global Analysis, Durham, July 1976.

12 · On the reflection representation of a finite Chevalley group

G. LUSZTIG
(University of Warwick)

One of the most natural representations of a finite Chevalley group is the permutation representation on the set of Borel subgroups. Although the degrees of the irreducible components of this representation are known (except in type E_8), very little is known about their characters, say, on regular semi-simple elements. In his thesis [9], Kilmoyer constructed a remarkable irreducible component of this permutation representation (called the reflection representation). For $PGL_n(F_q)$, it is just the representation of degree $q^{n-1}+q^{n-2}+\ldots+q$ on the functions on the projective space whose sum of values is zero; more generally, for a simple group of type A, D or E this representation has degree $q^{e_1}+q^{e_2}+\ldots+q^{e_\ell}$ where e_i are the exponents of the Weyl group. One of the main results of this paper is a formula for the character of the reflection representation, for a group of A, D or E on any regular semisimple element. For type A this is very easy to prove; however, in the general case, we have to use some l-adic cohomology machinery.

In the first part of this paper, we shall define some virtual representations of the Weyl group of a complex semisimple group; in the second part these virtual representations are used to show that the character of any irreducible component of the permutation representation of a finite Chevalley group, considered above, is given, on the set of regular semisimple elements by a universal formula, independent of the finite ground field. This result is then used in the third part to study the reflection representation.

1. Let G be a simple adjoint algebraic group over $\mathbb{C}$. We denote by G' the set of regular semisimple elements in G and by X the set of Borel subgroups in G. The set of orbits of G on $X \times X$ (pairs of Borel subgroups) will be denoted $\underline{\underline{W}}$ and called the Weyl group. (It has a natural group structure which can be described, for instance, as follows. Let T be a maximal torus in G. Any Borel subgroup B containing T gives rise to an isomorphism of the Weyl group $W(T)$ of T with $\underline{\underline{W}}$ by the formula $w \to$ orbit of $(B, \dot{w}B\dot{w}^{-1})$, where $\dot{w}$ is a representative for w in the normalizer of T. If we change B, this isomorphism is changed into its composition with an inner automorphism of $\underline{\underline{W}}$. We shall often identify $\underline{\underline{W}}$ and $W(T)$ as indicated and use the notation $\dot{w}$ without further explanation.)

The orbit corresponding to $w \in \underline{\underline{W}}$ will be denoted $\mathcal{O}_w$. For each $s \in G'$, let $\Gamma_s \subset X \times X$ be defined as the graph of the map $\mathrm{ad}(s): X \to X$; in other words, $\Gamma_s = \{(B, sBs^{-1}) \mid B \in X\}$. We have the following.

Lemma 1.1. *For each* $w \in \underline{\underline{W}}$, Γ_s *is transversal to* $\mathcal{O}_w$ *in* $X \times X$.

Let $y = (B, sBs^{-1}) \in \Gamma_s \cap \mathcal{O}_w$. We denote by $\underline{b}, \underline{g}$ the Lie algebras of B, G. The tangent space $T_y(X \times X)$ is just $(\underline{g}/\underline{b}) \times (\underline{g}/\mathrm{ad}(s)\underline{b})$. The tangent space $T_y(\Gamma_s)$ is $\{(v,v') \in (\underline{g}/\underline{b}) \times (\underline{g}/\mathrm{ad}(s)\underline{b}) \mid v' = \mathrm{ad}(s)v\}$. The tangent space $T_y(\mathcal{O}_w)$ is $\{(\phi_1(v), \phi_2(v)) \in (\underline{g}/\underline{b}) \times (\underline{g}/\mathrm{ad}(s)\underline{b}) \mid v \in \underline{g}/(\underline{b} \cap \mathrm{ad}(s)\underline{b})\}$, where ϕ_1, ϕ_2 are the natural projections of $\underline{g}/(\underline{b} \cap \mathrm{ad}(s)\underline{b})$ onto $\underline{g}/\underline{b}$, $\underline{g}/\mathrm{ad}(s)\underline{b}$ respectively. To prove that $T_y(X \times X) = T_y(\Gamma_s) + T_y(\mathcal{O}_w)$ amounts to proving the following statement: given $(v,v') \in \underline{g} \times \underline{g}$, there exists $v_1 \in \underline{g}$, $v_2 \in \underline{g}$ such that $v_1 + v_2 \equiv v (\mathrm{mod}\ \underline{b})$, $\mathrm{ad}(s)v_1 + v_2 \equiv v' (\mathrm{mod}\ \mathrm{ad}(s)\underline{b})$. This is the same as proving that the image of the map $1 - \mathrm{ad}(s^{-1})$: $\underline{g} \to \underline{g}$ spans, together with $\underline{b}$, the whole of $\underline{g}$.

Let $\underline{t} \subset \underline{g}$ be the Lie algebra of the unique maximal torus

T in G containing s. Then $(1-\mathrm{ad}(s^{-1}))\underline{g}$ is just the sum I of all root spaces in $\underline{g}$ with respect to $\underline{t}$. Let $x \in \underline{g}$ be a vector in $\underline{g}$ orthogonal (with respect to the Killing form) to I and to $\underline{b}$. It follows that, on the one hand, $x \in \underline{t}$ (hence it is semisimple) and, on the other hand, x is in the orthogonal of $\underline{b}$ (hence it is nilpotent). Thus, $x = 0$ and the Lemma is proved.

Remark. This argument uses the non-degeneracy of the Killing form, hence it would not apply in a characteristic p situation when the Killing form may be degenerate. The following argument (for which I am indebted to Roger Richardson) can be used in that situation. Let Z be the set of all Borel subalgebras $\underline{b}$ of $\underline{g}$ such that $I + \underline{b}$ has minimum dimension. Clearly, Z is a closed subvariety of the variety of all Borel subalgebras and Z is stable under conjugation by elements in T. By a well known fixed point theorem of Borel, there exists $\underline{b}_0 \in Z$ fixed by all elements of T, i.e. such that $\underline{b}_0 \supset \underline{t}$. But for such $\underline{b}_0$, it is clear that $I + \underline{b}_0 = \underline{g}$. For arbitrary $\underline{b}$, we have $\dim(I+\underline{b}) \geq \dim(I+\underline{b}_0)$ hence $I + \underline{b} = \underline{g}$, as required.

Given $w \in \underline{\underline{W}}$ and $s \in G'$ we define $Y_{s,w} = \{B \in X \mid (B, sBs^{-1}) \in \theta_w\}$. This can be regarded as the intersection of Γ_s and θ_w and therefore, by Lemma 1.1, it is a non-singular variety of pure dimension $\ell(w)$, where ℓ is the length function on $\underline{\underline{W}}$. Let T be a maximal torus in G. One can show that, when s varies in $G' \cap T$, the cohomology groups $H^i_c(Y_{s,w}, \mathbb{C})$ form a locally constant (flat) vector bundle $\mathcal{H}^i$ over $G' \cap T$. According to Deligne [4],[5], this vector bundle has a canonical finite filtration consisting of flat sub-bundles $\ldots \subset \mathcal{W}^i_{n-1} \subset \mathcal{W}^i_n \subset \ldots$ (the weight filtration). Now the Weyl group $W(T)$ of T acts naturally on the vector bundle $\mathcal{W}^i$ (compatibly with the weight filtration and with the flat structure;

it induces the usual action of $W(T)$ on $G' \cap T$. We have

Proposition 1.2. *The successive quotients $\mathcal{W}^i_n/\mathcal{W}^i_{n-1}$ of the weight filtration are trivial flat vector bundles over* $G' \cap T$.

The Weyl group $W(T)$ acts naturally on the space of constant global sections of $\mathcal{W}^i_n/\mathcal{W}^i_{n-1}$; this representation will be denoted $\rho^i_{n,w}$. (Using the isomorphism $W(T) \cong \underline{\underline{W}}$, which is well defined up to an inner automorphism of $\underline{\underline{W}}$, we see that $\rho^i_{n,w}$ can also be regarded as a representation of $\underline{\underline{W}}$). We put $\rho_{n,w} = \sum_i (-1)^i \rho^i_{n,w}$ and $\rho_w = \Sigma \rho_{n,w}$ (a virtual representation of $W(T)$ or $\underline{\underline{W}}$).

To prove the Proposition we note that the flat vector bundles $\mathcal{W}^i_n/\mathcal{W}^i_{n-1}$ have the property of being completely reducible (cf. Deligne [5]), i.e. any flat sub-bundle admits a flat complement. It is therefore enough to show that H^i has a finite filtration by flat sub-bundles whose successive quotients are trivial. Choose a Borel subgroup B_o containing T; let U_o be its unipotent radical and let U_o^- be the unipotent radical. For $s \in G' \cap T$, we define a partition of $Y_{s,w}$ into locally closed subvarieties $Y^{w'}_{s,w}$ ($w' \in \underline{\underline{W}}$) where $Y^{w'}_{s,w} = \{B \in Y_{s,w} \mid (B_o, B) \in \theta_{w'}\}$. It will then be enough to show that, when s varies in $G' \cap T$, $H^i_c(Y^{w'}_{w,s}, \mathbb{C})$ form trivial flat bundles over $G' \cap T$. We will actually show that the bundle E over $G' \cap T$ whose fibre at $s \in G' \cap T$ is $Y^{w'}_{s,w}$, is a trivial bundle. Let $\dot{w}, \dot{w}'$ be representatives for w, w' in the normalizer of T (using the isomorphism $W(T) \to \underline{\underline{W}}$ given by B_o). Any $B \in Y^{w'}_{s,w}$ can be written uniquely in the form $B = \dot{w}' u B_o u^{-1} \dot{w}'^{-1}$, where $u \in U_o^- \cap \dot{w}'^{-1} U_o \dot{w}'$ satisfies $u^{-1} \dot{w}'^{-1} s \dot{w}' u \in B_o \dot{w} B_o$. Let $s_{w'} = \dot{w}'^{-1} s \dot{w}'$. As $s_{w'}$ is regular, the map $u \to u^{-1} s_{w'} u s_{w'}^{-1} = \tilde{u}$ of $U_o^- \cap \dot{w}'^{-1} U_o \dot{w}'$ into itself is an automorphism of varieties. This defines an isomorphism of $Y^{w'}_{s,w}$ onto $\{\tilde{u} \in U_o^- \cap \dot{w}'^{-1} U_o \dot{w}' \cap B_o \dot{w} B_o\}$. Thus the bundle E is trivial and the Proposition is proved.

From now on, G will denote a simple adjoint algebraic group over k, an algebraic closure of the field F_p with p elements. We shall use the same notations $X, \underline{\underline{W}}, \theta_w, G', T$, $Y_{s,w}$, ... as in the complex case. All the results proved in the complex case remain valid in the present case (we now use l-adic cohomology instead of ordinary cohomology). The representations $\rho^i_{n,w}$ of $\underline{\underline{W}}$ in this new situation will be the same as the representations constructed in the complex case, for a group of the same type as G; note that the Weyl groups of these two groups can be identified. (This follows from known results on the behaviour of cohomology under reduction mod p of a scheme over Z.) We now assume that we are given an F_q-rational structure on G (F_q a finite sub-field of k), with corresponding Frobenius map $F:G \to G$ such that G is split over F_q, i.e. F acts trivially on $\underline{\underline{W}}$. Let T be an F-stable maximal torus. There is a unique element $w_1 \in W(T)$ such that $F(t) = \dot{w}_1 t^q \dot{w}_1^{-1}$ for all $t \in T$.

F induces maps $F:Y_{s,w} \to Y_{F(s),w}$ ($s \in G' \cap T$) hence an endomorphism $F^*:\mathcal{H}^i \to \mathcal{H}^i$ (compatible with the flat structure and with the weight filtration). Hence it defines endomorphisms F^* of the trivial bundles $\mathcal{W}^i_n/\mathcal{W}^i_{n-1}$ and of their spaces of constant sections.

Proposition 1.3. *All eigenvalues of the endomorphism* $w_1^{-1}F^*$ *of the space of constant sections of* $\mathcal{W}^i_{2n}/\mathcal{W}^i_{2n-1}$ *are equal to* q^n. *We have* $\mathcal{W}^i_{2n-1}/\mathcal{W}^i_{2n-2} = 0$.

It is easy to see that the truth of this statement for some F-stable maximal torus in G implies its truth for any other F-stable maximal torus. Therefore, we may assume that T is contained in an F-stable Borel subgroup B_o; in which case we have $w_1 = 1$. The eigenvalues of F^* on the space of constant sections of $\mathcal{W}^i_n/\mathcal{W}^i_{n-1}$ are, in any case, known to have complex absolute value $q^{n/2}$.

It will be enough to show that in this case, there exists a finite filtration of $\mathcal{H}^i$ by flat sub-bundles stable under F^* and such that its successive quotients are trivial, with F^* having on their spaces of constant section only eigenvalues which are integral powers of q . As in the proof of Proposition 1.2, we see that it is enough to prove that the eigenvalues of F^* on the cohomology of the variety

$$U_o^- \cap \dot{w}'^{-1} U_o \dot{w}' \cap B_o \dot{w} B_o$$

are integral powers of q , for any $w' \in W(T)$. (U_o, U_o^- are defined as in the proof of Proposition 1.2.) This can be shown by decomposing this variety into locally closed pieces each of which is an iterated fibration over a point with fibres k or k^* .

Corollary 1.4. *Let* $s \in T^F \cap G'$ *and let* $w_1 \in W(T)$ *be defined as above. Then*

$$|Y^F_{s,w}| = \sum_{n=0}^{\ell(w)} \mathrm{Tr}(w_1, \rho_{2n,w}) q^n .$$

This follows from the previous Proposition and the trace formula for Frobenius.

Remark. If $w, w' \in \underline{\underline{W}}$ are conjugate, we have $\rho_w = \rho_{w'}$. [Indeed, we can assume that $w' = rwr$, $\ell(r) = 1$, $\ell(w') \geq \ell(w)$. We can check directly as in [6, Proof of Thm. 1.6] that $|Y^F_{s,w'}| = |Y^F_{s,w}|$ if $\ell(w') = \ell(w)$ and, if $\ell(w') > \ell(w')$, $|Y^F_{s,w'}| = q|Y^F_{s,w}| + (q-1)|Y^F_{s,rw}|$. We then apply the Corollary and set $q = 1$.] One can show that, when G is of type A_ℓ , we have $\rho_w = \sum_{\chi \in W} \mathrm{Tr}(w,\chi)\chi$. This is, however, not true in the general case.

2. Let $\mathfrak{F}$ be the vector space of all complex valued functions on X^F ; this is a G^F-module in a natural way (the permutation representation on the set of F_q-rational Borel subgroups). The endomorphism algebra of this G^F-module can be described (following [2]) as follows. It has a basis consisting of endomorphisms T_w (one for each $w \in \underline{\underline{W}}$) , where

$$(T_w f)(x) = \sum_{\substack{y \in X^F \\ (x,y) \in \mathcal{O}_w^F}} f(y) \ , \quad (f \in \mathcal{F} \ , \ x \in X^F) \quad ;$$

the multiplication in this algebra is given by

$$\begin{aligned} &T_w T_{w'} = T_{ww'} \quad &&\text{if } \ w,w' \in W \ , \ \ell(ww') = \ell(w) + \ell(w') \\ &(T_w+1)(T_w-q) = 0 \quad &&\text{if } \ w \in W \ , \ \ell(w) = 1 \quad . \end{aligned} \tag{*}$$

Let A_q denote the abstract algebra over $\mathbb{C}$ with basis $(T_w)_{w\in\underline{\underline{W}}}$ and multiplication defined by (*); here q can be any complex number. It is known (see [7]) that this algebra is semisimple for all $q \in \mathbb{R}_{>0}$. If $\check{A}_q$ denotes the set of isomorphism classes of irreducible representations of A_q , then the $\check{A}_q$ are the fibres of a finite unramified covering of the open subset of $\mathbb{C}$ consisting of those q for which A_q is semisimple. This covering must be trivial over the set $\mathbb{R}_{>0}$. In particular there is a natural bijection $\check{A}_q \leftrightarrow \check{A}_1$. Note that A_1 is just the group algebra of W . For each irreducible representation $\chi \in \check{\underline{\underline{W}}}$ and each $q \in \mathbb{R}_{>0}$, we denote by χ_q the corresponding element in $\check{A}_q$.

Returning to our G^F-module $\mathcal{F}$, we decompose it as a $G^F \times A_q$-module:

$$\mathfrak{F} \cong \bigoplus_{\chi \in \check{\underline{\underline{W}}}} R_\chi \otimes \chi_q$$

where R_χ is an irreducible G^F-module. We shall be interested in the character of the G^F-modules R_χ on a regular semisimple element $s \in G^F$. Let T be the unique maximal torus in G containing s and let $w_1 \in W(T)$ be such that $F(t) = \dot{w}_1 t^q \dot{w}_1^{-1}$ for all $t \in T$. We also denote by w_1 the corresponding element of $\underline{W}$; it is well defined only up to conjugacy.

Proposition 2.1. $\mathrm{Tr}(s,R_\chi) = |\underline{\underline{W}}|^{-1} \sum_{w\in\underline{\underline{W}}} \mathrm{Tr}(w_1,\rho_w)\mathrm{Tr}(w,\chi)$.

(In particular this is independent of the ground field F_q.)

We compute $\mathrm{Tr}(sT_w,\mathcal{F})$ in two different ways. On the one hand it is equal to

$$\{B\in X^F \mid (B,sBs^{-1}) \in \theta_w\} = \{B\in X \mid (B,sBs^{-1}) \in \theta_w ,\ FB = B\}$$
$$= |Y^F_{s,w}| \quad .$$

On the other hand, it is equal to

$$\sum_{\chi\in\underline{\underline{W}}^\vee} \mathrm{Tr}(s,R_\chi)\mathrm{Tr}(T_w,\chi_q) \quad .$$

Thus we have, for all $w \in \underline{\underline{W}}$:

$$|Y^F_{s,w}| = \sum_{\chi\in\underline{\underline{W}}^\vee} \mathrm{Tr}(s,R_\chi)\mathrm{Tr}(T_w,\chi_q) \quad .$$

We now use the orthogonality formula [3]:

$$\sum_{w\in\underline{\underline{W}}} q^{-\ell(w)}\mathrm{Tr}(T_w,\chi_q)\mathrm{Tr}(T_{w^{-1}},\chi'_q) = \begin{cases} 0 & \text{if } \chi\neq\chi' \ (\chi,\chi'\in\underline{\underline{W}}) \\ P(q)\dfrac{\dim \chi}{\dim R} & \text{if } \chi=\chi' \ (\chi\in\underline{\underline{W}}) \end{cases}$$

where

$$P(q) = \sum_{w\in\underline{\underline{W}}} q^{\ell(w)}$$

and deduce that

$$\mathrm{Tr}(s,R_\chi) = \sum_{w\in\underline{\underline{W}}} |Y^F_{s,w}|\,\mathrm{Tr}(T_{w^{-1}},\chi_q)\,q^{-\ell(w)}\,\frac{\dim R_\chi}{\dim \chi}\cdot\frac{1}{P(q)} \quad .$$

We now substitute for $|Y^F_{s,w}|$ the value given by Corollary 1.4:

$$\mathrm{Tr}(s,R_\chi) = \sum_{w\in\underline{\underline{W}}} \Big(\sum_n \mathrm{Tr}(w_1,\rho_{2n,w})q^n\Big)\mathrm{Tr}(T_{w^{-1}},\chi_q)\,q^{-\ell(w)}\,\frac{\dim R_\chi}{\dim \chi}\cdot\frac{1}{P(q)} \quad .$$

The left hand side of this equality is an integer less than $|\underline{\underline{W}}|$ in absolute value (cf. [$\underline{6}$, 7.9, 6.8]). The right hand side of this equality can be regarded as a rational function in the complex variable $\sqrt{q}$ (indeed, by a theorem of Benson and Curtis [1], $\dim R_\chi$ can be regarded as a polynomial in q, and $\mathrm{Tr}(T_{w^{-1}},\chi_q)$ can be regarded as a polynomial in $\sqrt{q}$). This rational function takes only finitely many values on an infinite subset of C (corresponding to q a power of a prime) hence it must be a constant. Its value for $q = 1$ is $|\underline{\underline{W}}|^{-1}\sum_{w\,\underline{\underline{W}}}\mathrm{Tr}(w_1,\rho_w)\mathrm{Tr}(w,\chi)$.
(Indeed, for $q = 1$, $\mathrm{Tr}(T_{w^{-1}},\chi_q)$ becomes $\mathrm{Tr}(w,\chi) = \mathrm{Tr}(w^{-1},\chi)$, $\dim R_\chi$ becomes $\dim\chi$ and $P(q)$ becomes $|\underline{\underline{W}}|$.) This ends the proof.

3. From now on, we assume that all roots of G have the same length, i.e. that G is of type A_ℓ $(\ell \geq 1)$, D_ℓ $(\ell \geq 4)$ or E_ℓ ($\ell = 6$, 7 or 8). Let χ be the standard ℓ-dimensional representation of $\underline{\underline{W}}$. The corresponding G^F-module R_χ will be called the reflection representation of G^F. It has been observed by Kilmoyer that R_χ is the unique irreducible representation of G^F such that for any F-stable parabolic subgroup $P \subset G$, we have $\langle R_\chi,\mathrm{Ind}_{P^F}^{G^F}(1)\rangle_{G^F} = \ell - \ell'$, where ℓ' is the semisimple rank of the Levi part of P .

Let $s \in G^F$ be a regular semisimple element, and let

$w_1 \in \underline{\underline{W}}$ the corresponding element of $\underline{\underline{W}}$ (defined up to conjugacy). We have

Theorem 3.1. $\mathrm{Tr}(s,R\ ) = \mathrm{Tr}(w_1,\chi)$.

This is well known and easy to prove when G is of type A_ℓ . From now on, we shall assume that G is of type D_ℓ $(\ell \geq 4)$ or E_ℓ $(\ell = 6, 7$ or $8)$. Moreover, we see from Proposition 2.1 that the truth of 3.1 for a particular q implies its truth for arbitrary q . We may, therefore, assume that the characteristic p of F_q is so large that the results of Kazhdan [8] and Springer [10] on Green functions are valid. Let $u \in G^F$ be a subregular unipotent element (it follows from results of Dynkin that u is well defined up to G^F-conjugacy and that its centralizer in G^F has order $q^{\ell+2}$). Let $T \subset G$ be an F-stable maximal torus, let w be the corresponding element of $\underline{\underline{W}}$ (defined up to conjugacy) and let $\theta : T^F \to \mathbb{C}^*$ be a homomorphism. Consider the virtual G^F-module $R_T^G(\theta)$ defined in [6],[8].

As a consequence of the results of Kazhdan and Springer (*loc.cit.*), we have $\mathrm{Tr}(u,R_T^G(\theta)) = 1 + q\mathrm{Tr}(w,\chi)$.

We now prove the following

Lemma 3.2. We have

$$\sum_{\substack{(T,\theta) \\ \mathrm{mod}\ G^F}} \mathrm{Tr}(u,R_T^G(\theta)) \frac{R_T^G(\theta)}{\langle R_T^G(\theta), R_T^G(\theta)\rangle_{G^F}} = \sum_{\rho \in (G^F)^\vee} \mathrm{Tr}(u,\rho)\rho \quad .$$

We denote by f the character of the left hand side and by f' the character of the right hand side. To show that $f = f'$, it is enough to prove that $\langle f,f'\rangle_{G^F} = \langle f,f\rangle_{G^F} = \langle f',f'\rangle_{G^F}$.

First note that $\langle f',f'\rangle$ is equal to the order of the centralizer of u in G^F , hence it is $q^{\ell+2}$.

On the other hand, we have

$$\langle f,f\rangle = \langle f,f'\rangle = \sum_{\substack{(T,\theta) \\ \mathrm{mod}\ G^F}} \frac{\mathrm{Tr}(u,R_T^G(\theta))^2}{\langle R_T^G(\theta),R_T^G(\theta)\rangle_{G^F}}$$

$$= |\underline{\underline{W}}|^{-1} \sum_{w\in\underline{\underline{W}}} (1+q\mathrm{Tr}(w,\chi))^2 \det(q-w,\chi) \quad .$$

A computation with characters of $\underline{\underline{W}}$ (using the fact that 2 is not an exponent of $\underline{\underline{W}}$) shows that the last expression equals $q^{\ell+2}$, as required.

Let us now use the disjointness theorem 6.3 of [6]. If we denote $R_w = R_T^G(1)$ (where $w \in \underline{\underline{W}}$ corresponds to T) we deduce from Lemma 3.2 that

$$|W|^{-1} \sum_{w\in\underline{\underline{W}}} (1+q\mathrm{Tr}(w,\chi))R_w = \sum_{\substack{\rho\in(G^F)^\vee \\ \text{unipotent}}} \mathrm{Tr}(u,\rho)\rho \quad .$$

According to [6, 7.14] the identity representation of G^F can be expressed as

$$1 = |\underline{\underline{W}}|^{-1} \sum_{w\in\underline{\underline{W}}} R_w \quad .$$

Subtracting this from the previous identity gives

$$\sum_{\substack{\rho\in(G^F)^\vee \\ \text{unipotent} \\ \rho\neq 1}} \mathrm{Tr}(u,\rho)\rho = q|\underline{\underline{W}}|^{-1} \sum_{w\in\underline{\underline{W}}} \mathrm{Tr}(w,\chi)R_w \quad .$$

Let us assume that $q = p$. It is known that the character of any ρ in the sum at a regular unipotent element is zero; on the other hand this is clearly congruent mod p to its character at u . Thus $q^{-1}\mathrm{Tr}(u,\rho)$ is an integer for all ρ in the sum. It follows that

$$|\underline{\underline{W}}|^{-1} \sum_{w\in\underline{\underline{W}}} \mathrm{Tr}(w,\chi)R_w$$

is a virtual representation of G^F. It is clearly irreducible, of degree

$$|\underline{\underline{W}}|^{-1} \sum_{w \in \underline{\underline{W}}} Tr(w,\chi)\det(1-wq,\chi)^{-1}(q^{e_1+1}-1)\ldots(q^{e_\ell+1}-1) = q^{e_1}+\ldots+q^{e_\ell}$$

where $e_1,\ldots,e_\ell$ are the exponents of $\underline{\underline{W}}$. It is the unique unipotent representation $\neq 1$ whose character is non-zero at u (its value at u equals q). Its inner products with $\mathrm{Ind}_{P^F}^{G^F}(1)$ are easily seen to be the same as for R_χ; therefore it must coincide with R_χ. We have proved that when $q = p$ is sufficiently large, we have

$$R_\chi = |\underline{\underline{W}}|^{-1} \sum_{w \in \underline{\underline{W}}} Tr(w,\chi)R_w \quad .$$

If s is as in the statement of Theorem 3.1, we have (cf. [6, 7.9])

$$Tr(s,R_\chi) = \langle R_\chi, R_{w_1} \rangle = Tr(w_1,\chi) \quad .$$

Thus Theorem 3.1 is proved when $q = p$ is sufficiently large and hence also for arbitrary q.

References

[1] C.T. Benson and C.W. Curtis. 'On the degrees and rationality of certain characters of finite Chevalley groups', *Trans.Amer.Math.Soc.* 165 (1972), 251-273 and 202 (1975).

[2] N. Bourbaki. 'Groupes et algèbres de Lie', Ch.IV, Hermann, Paris.

[3] C.W. Curtis and T.V. Fossum. 'On centralizer rings and characters of representations of finite groups', *Math. Z.* 107 (1968), 402-406.

[4] P. Deligne. 'Poids dans la cohomologie des variétés algébriques', Proc. of I.C.M. Vancouver 1974, vol.1, 79-85.

[5] P. Deligne. 'Théorie de Hodge II', *I.H.E.S. Publ.Math.* 40 (1972), 5-58.

[6] P. Deligne and G. Lusztig. 'Representations of reductive groups over finite fields', *Ann. of Math.* 103 (1976), 103-161.

[7] J.A. Green. 'On the Steinberg characters of finite Chevalley groups', *Math.Z.* 117 (1970), 272-288.

[8] D. Kazhdan. 'Proof of Springer's hypothesis', to appear.

[9] R. Kilmoyer. 'Some irreducible complex representations of a finite group with a BN-pair', Ph.D. dissertation, M.I.T. 1969.

[10] T.A. Springer. 'Trigonometric sums, Green functions of finite groups and representations of Weyl groups', *Inventiones math.* 36 (1976), 173-207.

Index