An Introduction to Ordinary Differential Equations
Exercises and Solutions

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1

Radioactive decay and carbon dating

Exercise 1.1 *Radioactive isotopes decay at random, with a fixed probability of decay per unit time. Over a time interval* Δt*, suppose that the probability of any one isotope decaying is* $k\Delta t$*. If there are* N *isotopes, how many will decay on average over a time interval* Δt*? Deduce that*

$$N(t + \Delta t) - N(t) \approx -Nk\Delta t,$$

and hence that $\mathrm{d}N/\mathrm{d}t = -kN$ *is an appropriate model for radioactive decay.*

Over a time interval Δt, $Nk\Delta t$ isotopes will decay. We then have

$$N(t + \Delta t) - N(t) = -Nk\Delta t.$$

Dividing by Δt gives

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = -Nk,$$

and letting $\Delta t \to 0$ we obtain, using the definition of the derivative,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN.$$

Exercise 1.2 *Plutonium 239, virtually non-existent in nature, is one of the radioactive materials used in the production of nuclear weapons, and is a by-product of the generation of power in a nuclear reactor. Its half-life is approximately 24 000 years. What is the value of* k *that should be used in (1.1) for this isotope?*

Since $N(t) = N(s)\mathrm{e}^{-k(t-s)}$, half of the isotopes decay after a time T, where

$$N(s + T) = \tfrac{1}{2}N(s) = N(s)\mathrm{e}^{-kT},$$

i.e. when $\frac{1}{2} = \mathrm{e}^{-kT}$. Thus the half-life $T = \ln 2/k$ (as derived in Section 1.1). If $T = 24000$ then $k = \ln 2/T \approx 2.888 \times 10^{-5}$. □

Exercise 1.3 *In 1947 a large collection of papyrus scrolls, including the oldest known manuscript version of portions of the Old Testament, was found in a cave near the Dead Sea; they have come to be known as the 'Dead Sea Scrolls'. The scroll containing the book of Isaiah was dated in 1994 using the radiocarbon technique*[1]*; it was found to contain between 75% and 77% of the initial level of carbon 14. Between which dates was the scroll written?*

We have

$$N(1994) = pN(s) = N(s)\mathrm{e}^{-k(1994-s)},$$

where $0.75 \leq p \leq 0.77$. Taking logarithms gives

$$\log p = -k(1994 - s),$$

and so

$$s = 1994 + \frac{\log p}{k}.$$

With $k = 1.216 \times 10^{-4}$ this gives (approximately)

$$-372 \leq s \leq -155,$$

dating the scrolls between 372 BC and 155 BC. □

Exercise 1.4 *A large round table hangs on the wall of the castle in Winchester. Many would like to believe that this is the Round Table of King Arthur, who (so legend would have it) was at the height of his powers in about AD 500. If the table dates from this time, what proportion of the original carbon 14 would remain? In 1976 the table was dated using the radiocarbon technique, and 91.6% of the original quantity of carbon 14 was found*[2]*. From when does the table date?*

If the table dates from 500 AD then we would expect

$$N(t) = \mathrm{e}^{-k(t-500)}N(500),$$

and so in 2003 we have

$$N(2003) = \mathrm{e}^{-1503k}N(500).$$

The proportion of ^{14}C isotopes remaining should there be $\mathrm{e}^{-1503k} \approx 83\%$.

[1] A.J. Jull *et al.*, 'Radiocarbon Dating of the Scrolls and Linen Fragments from the Judean Desert', *Radiocarbon* (1995) **37**, 11–19.
[2] M. Biddle, *King Arthur's Round Table* (Boydell Press, 2001).

However, we in fact have 91.6% remaining in 1976. Therefore

$$N(1976) = 0.915N(s) = N(s)e^{-k(1993-s)}.$$

Taking logarithms gives

$$s = 1976 + \frac{\log 0.916}{k} \approx 1255;$$

the table probably dates from during the reign of the English King Edward I, who took the throne in 1270 AD (once the wood was well seasoned) and had a passion for all things Arthurian.

Exercise 1.5 *Radiocarbon dating is an extremely delicate process. Suppose that the percentage of carbon 14 remaining is known to lie in the range* $0.99p$ *to* $1.01p$*. What is the range of possible dates for the sample?*

Suppose that a proportion αp of the original ^{14}C isotopes remain. Then

$$\alpha p N(s) = N(t) = e^{-k(t-s)}N(s),$$

and so

$$\log \alpha + \log p = -k(t-s).$$

It follows that

$$s = t + \frac{\log p}{k} + \frac{\log \alpha}{k}. \tag{S1.1}$$

Denote by S the value of this expression when $\alpha = 1$, i.e. $S = t + (\log p)/k$.

For a proportion $0.99p$ the expression (S1.1) gives

$$s = S - 82.65,$$

while for a proportion $1.01p$ the expression gives

$$s = S + 81.83$$

(both correct to two decimal places). Small errors can give a difference of over 160 years in the estimated date.

2

Integration variables

There are no exercises for this chapter.

3

Classification of differential equations

Exercise 3.1 *Classify the following equations as ordinary or partial, give their order, and state whether they are linear or nonlinear. In each case identify the dependent and independent variables.*

(i) Bessel's equation (ν is a parameter)

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

(ii) Burger's equation (ν is a parameter)

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0,$$

(iii) van der Pol's equation (m, k, a and b are parameters)

$$m\ddot{x} + kx = a\dot{x} - b\dot{x}^3,$$

(iv) $\mathrm{d}y/\mathrm{d}t = t - y^2$,

(v) the wave equation (c is a parameter)

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2},$$

(vi) Newton's law of cooling (k is a parameter and $A(t)$ is a specified function)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - A(t)),$$

(vii) the logistic population model (k is a parameter)

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp(1 - p),$$

(viii) Newton's second law for a particle of mass m moving in a potential $V(x)$,

$$m\ddot{x} = -V'(x),$$

(ix) the coupled equations in (3.9)

$$\begin{aligned} \dot{x} &= x(4 - 2x - y) \\ \dot{y} &= y(9 - 3x - 3y), \end{aligned}$$

and

(x)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbb{A}\mathbf{x},$$

where $\mathbf{x}$ is an n-component vector and $\mathbb{A}$ is an $n \times n$ matrix.

(i) linear 2nd order ODE for $y(x)$;
(ii) nonlinear 2nd order PDE for $u(x, t)$;
(iii) nonlinear 2nd order ODE for $x(t)$;
(iv) nonlinear 1st order ODE for $y(t)$;
(v) linear 2nd order PDE for $y(x, t)$;
(vi) linear 1st order ODE for $T(t)$;
(vii) nonlinear 1st order ODE for $p(t)$;
(viii) 2nd order ODE for $x(t)$, linear if $V'(x) = ax + b$ for some $a, b \in \mathbb{R}$, otherwise nonlinear;
(ix) nonlinear 1st order ODE for the pair $(x(t), y(t))$; and
(x) linear 1st order ODE for the vector $\mathbf{x}(t)$.

4

*Graphical representation of solutions using MATLAB

Exercise 4.1 *Plot the graphs of the following functions:*

(i) $y(t) = \sin 5t \sin 50t$ *for* $0 \le t \le 3$,
(ii) $x(t) = \mathrm{e}^{-t}(\cos 2t + \sin 2t)$ *for* $0 \le t \le 5$,
(iii)

$$T(t) = \int_0^t \mathrm{e}^{-(t-s)} \sin s \, \mathrm{d}s \qquad \text{for} \qquad 0 \le t \le 7,$$

(iv) $x(t) = t \ln t$ *for* $0 \le t \le 5$,
(v) *plot* y *against* x*, where*

$$x(t) = B\mathrm{e}^{-t} + At\mathrm{e}^{-t} \qquad \text{and} \qquad y(t) = A\mathrm{e}^{-t},$$

for A *and* B *taking integer values between* -3 *and* 3.

(i)
```
>> t=linspace(0,3);
>> y=sin(5*t).*sin(50*t);
>> plot(t,y)
```
The result is shown in Figure 4.1.

(ii)
```
>> t=linspace(0,5);
>> x=exp(-t).*(cos(2*t)+sin(2*t));
>> plot(t,x)
```
The result is shown in Figure 4.2.

(iii) Use the short `M`-file `exint.m`:

```
f=inline('exp(-(t-s)).*sin(s)','t','s');

for j=0:100;
     t(j+1)=7*j/100;
     T(j+1)=quad(f,0,t(j+1),[],[],t(j+1));
end
```

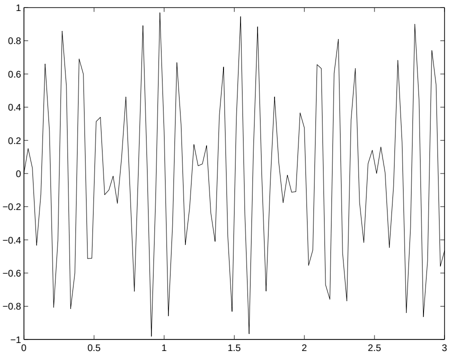

Fig. 4.1. The graph of $y(t) = \sin 5t \sin 50t$ against t.

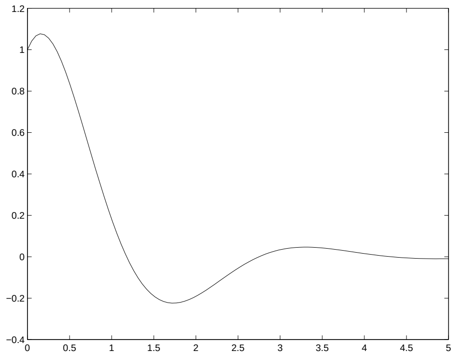

Fig. 4.2. The graph of $x(t) = \mathrm{e}^{-t}(\cos 2t + \sin 2t)$ against t.

```
plot(t,T)
```

The plot is shown in Figure 4.3.

(iv)
```
>> t=linspace(0,5);
>> x=t.*log(t);
>> plot(t,x)
```

The resulting graph is shown in Figure 4.4.

(v) Use the short `M`-file `param.m`:

```
t=linspace(0,5);

hold on

for A=-3:3;
```

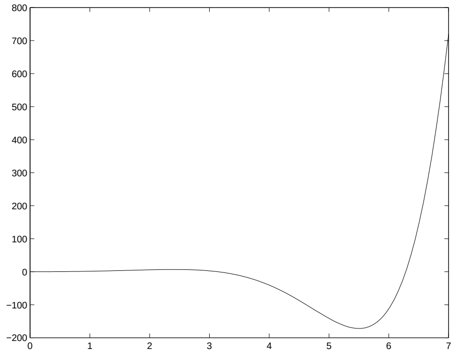

Fig. 4.3. The graph of the integral in Exercise 4.1(iii).

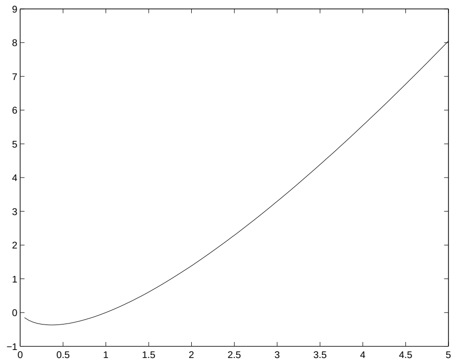

Fig. 4.4. The graph of $x(t) = t \ln t$ against t.

```
    for B=-3:3;
        x=B*exp(-t)+A.*t.*exp(-t);
        y=A*exp(-t);
        plot(x,y)
    end
end

hold off
```

See Figure 4.5.

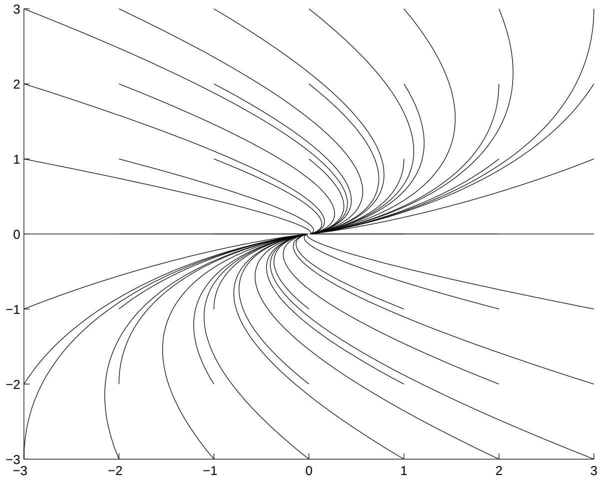

Fig. 4.5. A collection of curves defined parametrically by $x(t) = Be^{-t} + Ate^{-t}$ and $y(t) = Ae^{-t}$.

Exercise 4.2 *Draw contour plots of the following functions:*

(i)

$$F(x, y) = x^2 + y^2 \qquad \text{for} \qquad -2 \le x, y \le 2;$$

(ii)

$$F(x, y) = xy^2 \qquad \text{for} \qquad -1 \le x, y \le 1,$$

with contour lines where $F = \pm 0.1$, ± 0.2, ± 0.4, *and* ± 0.8*;*

(iii)

$$E(x, y) = y^2 - 2\cos x \qquad \text{for} \qquad -4 \le x, y \le 4;$$

(iv)

$$E(x, y) = x - \tfrac{1}{3}x^3 + \tfrac{1}{2}y^2(x^4 - 2x^2 + 2)$$

for $-2 \le x \le 4$ *and* $-2 \le y \le 2$*, showing the contour lines where* $E = 0$, 0.5, 0.8, *1, 2, 3, and 4;*

(v)

$$E(x, y) = y^2 + x^3 - x \qquad \text{for} \qquad -2 \le x, y \le 2.$$

(i)
```
>> [x, y]=meshgrid(-2:.1:2, -2:.1:2);
>> F=x.^2+y.^2;
>> contour(x,y,F)
```
These contours are shown in Figure 4.6.

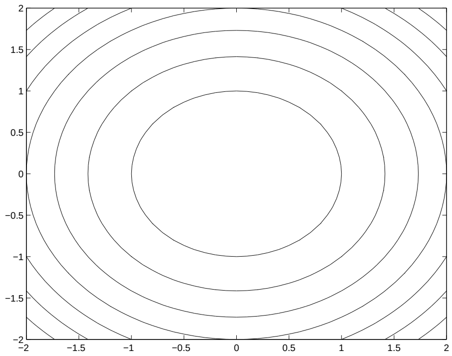

Fig. 4.6. Contours of $x^2 + y^2$

(ii)
```
>> [x, y]=meshgrid(-1:.1:1, -1:.1:1);
>> F=x.*y.^2;
>> contour(x,y,F,[.1 .2 .4 .8 -.1 -.2 -.4 -.8])
```
These contours are shown in Figure 4.7.

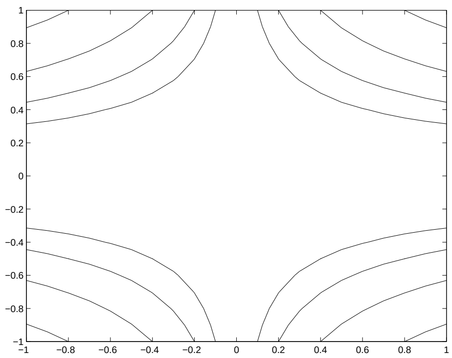

Fig. 4.7. Contours of xy^2

(iii)
```
>> [x, y]=meshgrid(-4:.1:4, -4:.1:4);
>> E=y.^2-2*cos(x);
>> contour(x,y,E)
```
These contours are shown in Figure 4.8.

(iv)
```
>> [x, y]=meshgrid(-2:.1:4, -2:.1:2);
>> E=x-(x.^3)/3+(y.^2/2).*(x.^4-2*x.^2+2);
>> contour(x,y,E,[0 .5 .8 1 2 3 4])
```
These contours are shown in Figure 4.9.

(v)
```
>> [x, y]=meshgrid(-2:.1:2, -2:.1:2);
>> E=y.^2+x.^3-x;
```

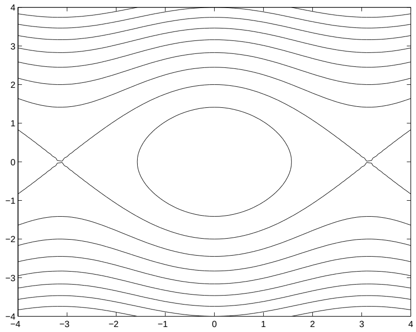

Fig. 4.8. Contours of $y^2 - 2\cos x$

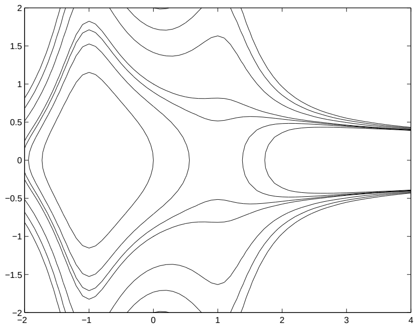

Fig. 4.9. Contours of $x - \frac{1}{3}x^3 + \frac{1}{2}y^2(x^4 - 2x^2 + 2)$

```
>> contour(x,y,E)
```

These contours are shown in Figure 4.10.

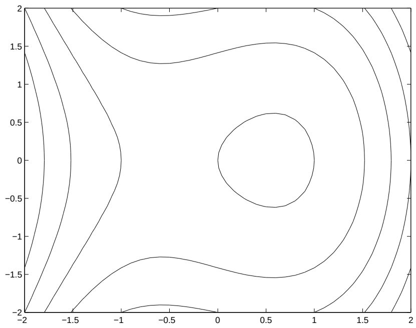

Fig. 4.10. Contours of $y^2 + x^3 - x$

5

'Trivial' differential equations

Exercise 5.1 *Find the general solution of the following differential equations, and in each case find also the particular solution that passes through the origin.*

(i)

$$\frac{d\theta}{dt} = \sin t + \cos t,$$

(ii)

$$\frac{dy}{dx} = \frac{1}{x^2 - 1}$$

(use partial fractions)

(iii)

$$\frac{dU}{dt} = 4t \ln t,$$

(iv)

$$\frac{dz}{dx} = x e^{-2x},$$

and

(v)

$$\frac{dT}{dt} = e^{-t} \sin 2t.$$

(i) Integrating both sides of

$$\frac{d\theta}{dt} = \sin t + \cos t$$

with respect to t we get

$$\theta(t) = -\cos t + \sin t + c.$$

For the solution to pass through the origin we need $\theta(0) = 0$, i.e. $0 = -1 + c$ or $c = 1$, and thus this solution is

$$\theta(t) = 1 - \cos t + \sin t.$$

(ii) We have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x^2 - 1} = \tfrac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right),$$

and so

$$y(x) = \tfrac{1}{2}\left(\log|x-1| - \log|x+1|\right) + c = \tfrac{1}{2}\log\frac{|x-1|}{|x+1|} + c.$$

For this solution to pass through the origin we need

$$0 = y(0) = \tfrac{1}{2}\log\frac{|-1|}{|1|} + c,$$

i.e. $c = 0$, and so we get

$$y(x) = \tfrac{1}{2}\log\frac{|x-1|}{|x+1|}.$$

(iii) Integrating

$$\frac{\mathrm{d}U}{\mathrm{d}t} = 4t\ln t$$

we have

$$U(t) = \int 4t\ln t\,\mathrm{d}t = \frac{1}{2}t^2\ln t - \frac{1}{4}t^2 + c.$$

To ensure that $U(0) = 0$ we need $c = 0$, and so

$$U(t) = \frac{1}{2}t^2\ln t - \frac{1}{4}t^2.$$

(iv) Integrating the right-hand side of

$$\frac{\mathrm{d}z}{\mathrm{d}x} = x\mathrm{e}^{-2x}$$

by parts we have

$$\begin{aligned} z(x) &= \int x\mathrm{e}^{-2x}\,\mathrm{d}x \\ &= -\frac{1}{2}x\mathrm{e}^{-2x} + \frac{1}{2}\int \mathrm{e}^{-2x}\,\mathrm{d}x \\ &= -\frac{1}{2}x\mathrm{e}^{-2x} - \frac{1}{4}\mathrm{e}^{-2x} + c, \end{aligned}$$

and for $z(0) = 0$ we need $0 = -(1/4) + c$, i.e. $c = 1/4$ giving

$$z(x) = \frac{1}{4}(1 - e^{-2x} - 2xe^{-2x}).$$

(v) We need to integrate

$$\frac{dT}{dt} = e^{-t} \sin 2t.$$

The integral of $e^{-t} \sin 2t$ will be a linear combination of $e^{-t} \sin 2t$ and $e^{-t} \cos 2t$. For

$$z(t) = \alpha e^{-t} \sin 2t + \beta e^{-t} \cos 2t$$

we have

$$\begin{aligned} \frac{dz}{dt} &= \alpha(-e^{-t} \sin 2t + 2e^{-t} \cos 2t + \beta(-e^{-t} \cos 2t - 2e^{-t} \sin 2t) \\ &= (-\alpha - 2\beta)e^{-t} \sin 2t + (2\alpha - \beta)e^{-t} \cos 2t, \end{aligned}$$

and so we need

$$-\alpha - 2\beta = 1 \qquad \text{and} \qquad 2\alpha - \beta = 0,$$

i.e. $\alpha = -1/5$ and $\beta = -2/5$. Therefore

$$T(t) = -\frac{e^{-t} \sin 2t + 2e^{-t} \cos 2t}{5} + c.$$

For $T(0) = 0$ we need $0 = -(2/5) + c$, i.e. $c = 2/5$, and so

$$T(t) = \frac{2 - e^{-t}(\sin 2t + 2 \cos 2t)}{5}.$$

Exercise 5.2 *Find the function $f(x)$ defined for $-\pi/2 < x < \pi/2$ whose graph passes through the point $(0, 2)$ and has slope $-\tan x$.*

We want to find a function f that satisfies

$$\frac{df}{dx} = -\tan x \qquad \text{with} \qquad f(0) = 2.$$

So we integrate between the limits that correspond to x values 0 and x,

$$\begin{aligned} f(x) - f(0) &= \int_0^x -\tan \tilde{x} \, d\tilde{x} \\ &= \int_0^x \frac{-\sin \tilde{x}}{\cos \tilde{x}} \, d\tilde{x} \\ &= [\ln |\cos \tilde{x}|]_0^x \\ &= \ln |\cos x|, \end{aligned}$$

and so, since $\cos x > 0$ for $-\pi/2 < x < \pi/2$

$$f(x) = \ln\cos x + 2.$$

Exercise 5.3 *Find the function $g(x)$ defined for $x > -1$ that has slope $\ln(1+x)$ and passes through the origin.*

The required function $g(x)$ satisfies

$$\frac{\mathrm{d}g}{\mathrm{d}x} = \ln(1+x) \qquad \text{with} \qquad g(0) = 0.$$

Integrating both sides of the differential equation between 0 and x gives

$$\begin{aligned} g(x) &= g(0) + \int_0^x \ln(1+\tilde{x})\,\mathrm{d}\tilde{x} \\ &= \Big[(1+\tilde{x})\ln(1+\tilde{x}) - \tilde{x}\Big]_{\tilde{x}=0}^{x} \\ &= (1+x)\ln(1+x) - x, \end{aligned}$$

since $\ln 1 = 0$.

Exercise 5.4 *Find the solutions of the following equations satisfying the given initial conditions:*

(i)

$$\dot{x} = \sec^2 t \qquad \text{with} \qquad x(\pi/4) = 0,$$

(ii)

$$y' = x - \tfrac{1}{3}x^3 \qquad \text{with} \qquad y(-1) = 1,$$

(iii)

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2\sin^2 t \qquad \text{with} \qquad \theta(\pi/4) = \pi/4,$$

(iv)

$$x\frac{\mathrm{d}V}{\mathrm{d}x} = 1 + x^2 \qquad \text{with} \qquad V(1) = 1,$$

and

(v)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)\mathrm{e}^{3t}\right] = \mathrm{e}^{-t} \qquad \text{with} \qquad x(0) = 3,$$

(i) Integrating $\dot{x} = \sec^2 t$ from $\pi/4$ to t gives

$$\begin{aligned} x(t) &= x(\pi/4) + \int_{\pi/4}^{t} \sec^2 \tilde{t}\,\mathrm{d}\tilde{t} \\ &= 0 + \Big[\tan\tilde{t}\Big]_{\tilde{t}=\pi/4}^{t} \\ &= \tan t - \tfrac{1}{2}. \end{aligned}$$

(ii) Integrating $y' = x - \frac{1}{3}x^3$ from -1 to x gives

$$\begin{aligned} y(x) &= y(-1) + \int_{-1}^{x} \tilde{x} - \tfrac{1}{3}\tilde{x}^3\,\mathrm{d}\tilde{x} \\ &= 1 + \left[\frac{\tilde{x}^2}{2} - \frac{\tilde{x}^4}{12}\right]_{\tilde{x}=-1}^{x} \\ &= 1 + \frac{x^2}{2} - \frac{x^4}{12} - \frac{1}{2} + \frac{1}{12} \\ &= \frac{7}{12} + \frac{x^2}{2} - \frac{x^4}{12}. \end{aligned}$$

(iii) Integrating $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2\sin^2 t$ between $\pi/4$ and t we have

$$\begin{aligned} \theta(t) &= \theta(\pi/4) + \int_{\pi/4}^{t} 2\sin^2\tilde{t}\,\mathrm{d}\tilde{t} \\ &= \pi/4 + \int_{\pi/4}^{t} 1 - \cos 2\tilde{t}\,\mathrm{d}\tilde{t} \\ &= \pi/4 + \Big[\tilde{t} - \tfrac{1}{2}\sin 2\tilde{t}\Big]_{\tilde{t}=\pi/4}^{t} \\ &= \pi/4 + t - \tfrac{1}{2}\sin 2t - \pi/4 + \tfrac{1}{2} \\ &= \tfrac{1}{2} + t - \tfrac{1}{2}\sin 2t. \end{aligned}$$

(iv) Dividing $x\frac{\mathrm{d}V}{\mathrm{d}x} = 1 + x^2$ by x and then integrating between 1 and x we obtain

$$\begin{aligned} V(x) &= V(1) + \int_{1}^{x} \frac{1}{\tilde{x}} + \tilde{x}\,\mathrm{d}\tilde{x} \\ &= 1 + \Big[\ln\tilde{x} + \tfrac{1}{2}\tilde{x}^2\Big]_{\tilde{x}=1}^{x} \\ &= 1 + \ln x + \tfrac{1}{2}x^2 - \ln 1 - \tfrac{1}{2} \\ &= \tfrac{1}{2} + \ln x + \tfrac{1}{2}x^2. \end{aligned}$$

(v) Integrating both sides of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)\mathrm{e}^{3t}\right] = \mathrm{e}^{-t}$$

between 0 and t gives

$$\begin{aligned} x(t)\mathrm{e}^{3t} &= x(0) + \int_0^t \mathrm{e}^{-\tilde{t}}\,\mathrm{d}\tilde{t} \\ &= 3 + \left[-\mathrm{e}^{-\tilde{t}}\right]_{\tilde{t}=0}^{t} \\ &= 3 - \mathrm{e}^{-t} + 1 \\ &= 4 - \mathrm{e}^{-t}, \end{aligned}$$

and so

$$x(t) = 4\mathrm{e}^{-3t} - \mathrm{e}^{-4t}.$$

Exercise 5.5 *The Navier-Stokes equations that govern fluid flow were given as an example in Chapter 3 (see equations (3.1) and (3.2)). It is not possible to find explicit solutions of these equations in general. However, in certain cases the equations reduce to something much simpler.*

Suppose that a fluid is flowing down a pipe that has a circular cross-section of radius a. Assuming that the velocity V of the fluid depends only on its distance from the centre of the pipe, the equation satisfied by V is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}V}{\mathrm{d}r}\right) = -P,$$

where P is a positive constant.

Multiply by r and integrate once to show that

$$\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{Pr}{2} + \frac{c}{r}$$

where c is an arbitrary constant. Integrate again to find an expression for the velocity, and then use the facts that (i) the velocity should be finite at all points in the pipe and (ii) that fluids 'stick' to boundaries (which means that $V(a) = 0$) to show that

$$V(r) = \frac{P}{4}(a^2 - r^2),$$

see Figure 5.1. (This is known as Poiseuille flow.)

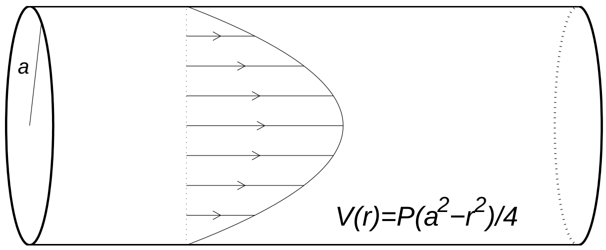

Fig. 5.1. The quadratic velocity profile in a circular pipe.

Multiplying the equation by r we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}V}{\mathrm{d}r}\right) = -Pr,$$

and integrating both sides gives

$$r\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{Pr^2}{2} + c,$$

which implies that

$$\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{Pr}{2} + \frac{c}{r}.$$

Integrating this equation gives

$$V(r) = -\frac{Pr^2}{4} + c\ln r + d.$$

Since $\ln r \to -\infty$ as $r \to 0$, for $V(r)$ to be finite when $r = 0$ we have to take $c = 0$. This then leaves

$$V(r) = -\frac{Pr^2}{4} + d,$$

and to ensure that $V(a) = 0$ we take $d = Pa^2/4$, so that

$$V(r) = \frac{P}{4}(a^2 - r^2)$$

as claimed.

Exercise 5.6 *An apple of mass m falls from a height h above the ground. Neglecting air resistance its velocity satisfies*

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg \qquad v(0) = 0,$$

where $v = \dot{y}$ and y is the height above ground level. Show that the apple hits the ground when

$$t = \sqrt{\frac{2h}{g}}.$$

The velocity at time t is given by

$$v(t) = v(0) - gt = -gt,$$

and its height y above ground level satisfies

$$\dot{y} = v(t) = -gt,$$

and hence

$$y(t) = y(0) - \tfrac{1}{2}gt^2 = h - \tfrac{1}{2}gt^2.$$

It follows that $y(t) = 0$ when $t = \sqrt{2h/g}$ as claimed.

Exercise 5.7 *An artillery shell is fired from a gun, leaving the muzzle with velocity V. If the gun is at an angle θ to the horizontal then the initial horizontal velocity is $V\cos\theta$, and the initial vertical velocity is $V\sin\theta$ (see Figure 5.2). The horizontal velocity remains constant, but the vertical velocity is affected by gravity, and obeys the equation $\dot{v} = -g$. How far does the shell travel before it hits the ground? (Give your answer in terms of V and θ.)*

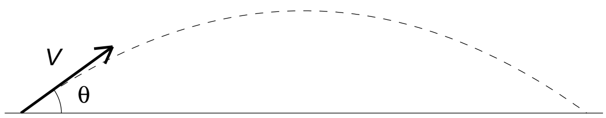

Fig. 5.2. Firing a shell at muzzle velocity V at an angle θ to the horizontal. The shell follows a parabolic path.

The vertical velocity v satisfies

$$\dot{v} = -g \qquad \text{with} \qquad v(0) = V\sin\theta,$$

and so integrating both sides of the differential equation between times 0 and t we obtain

$$v(t) = v(0) - gt = V\sin\theta - gt.$$

The height $y(t)$ of the shell at time t satisfies

$$\dot{y} = v = V\sin\theta - gt \qquad \text{with} \qquad y(0) = 0,$$

and so integrating both sides of this between zero and t we have

$$y(t) = Vt\sin\theta - \tfrac{1}{2}gt^2.$$

The shell strikes the ground at time t_*, where

$$Vt_*\sin\theta - \tfrac{1}{2}gt_*^2 = 0,$$

i.e. when $t^* = (2V\sin\theta)/g$. Since the horizontal velocity is constant and equal to $V\cos\theta$, the shell will have travelled a distance

$$Vt^*\cos\theta = \frac{2V^2\sin\theta\cos\theta}{g} = \frac{V^2\sin 2\theta}{g}.$$

Exercise 5.8 *In Dallas on 22 November 1963, President Kennedy was assassinated; by Lee Harvey Oswald if you do not believe any of the conspiracy theories. Oswald fired a Mannlicher-Carcano rifle from approximately 90 m away. The sight on Oswald's rifle was less than ideal: if the bullet travelled in a straight line after leaving the rifle (at a velocity of roughly 700 m/s) then the sight aimed about 10cm too high at a target 90 m away. How much would the drop in the trajectory due to gravity compensate for this? (The initial vertical velocity v is zero, and satisfies the equation $\dot{v} = -g$, while the horizontal velocity is constant if we neglect air resistance.)*

There is nothing to slow down the horizontal velocity of the bullet if we neglect air resistance: so it takes the bullet 9/70 seconds to travel 90 m. In this time it will have dropped vertically, its height h satisfying

$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} = -g.$$

The solution of this, integrating twice, is

$$h(t) = h(0) - \tfrac{1}{2}gt^2,$$

and so with $t = 9/70$ and $h(0) = 0$ this gives a drop of 0.081 m or 8.1 cm, compensating quite well for the dodgy sight.

Exercise 5.9 *This exercise fills in the gaps in the proof of the Fundamental Theorem of Calculus. Suppose that f is continuous at x, i.e. given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that*

$$|\tilde{x} - x| \le \delta \qquad \Rightarrow \qquad |f(\tilde{x}) - f(x)| \le \epsilon.$$

By writing

$$f(x) = \frac{1}{\delta x}\int_x^{x+\delta x} f(x)\,\mathrm{d}\tilde{x}$$

show that for all δx *with* $|\delta x| \leq \delta(\epsilon)$

$$\left|f(x) - \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}\right| \leq \epsilon,$$

and hence that

$$\lim_{\delta x\to 0}\frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x} = f(x).$$

You will need to use the fact that

$$\left|\int_a^b g(x)\,\mathrm{d}x\right| \leq \int_a^b |g(x)|\,\mathrm{d}x \leq (b-a)\max_{x\in[a,b]}|g(x)|.$$

We have

$$\begin{aligned}\left|f(x) - \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}\right| &= \left|\frac{1}{\delta x}\int_x^{x+\delta x} f(x)\,\mathrm{d}\tilde{x} - \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}\right| \\ &= \frac{1}{\delta x}\left|\int_x^{x+\delta x} f(x) - f(\tilde{x})\,\mathrm{d}\tilde{x}\right| \\ &\leq \frac{1}{\delta x}\int_x^{x+\delta x} |f(x) - f(\tilde{x})|\,\mathrm{d}\tilde{x}.\end{aligned}$$

Then, given any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that if $\delta x < \delta(\epsilon)$ then for every $\tilde{x} \in [x, x+\delta x]$ we have $|f(x) - f(\tilde{x})| \leq \epsilon$, and so

$$\begin{aligned}\left|f(x) - \frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x}\right| &\leq \frac{1}{\delta x}\int_x^{x+\delta x} \epsilon\,\mathrm{d}\tilde{x} \\ &= \frac{1}{\delta x}[\epsilon\,\delta x] \\ &= \epsilon.\end{aligned}$$

Therefore, using the definition of a limit,

$$\lim_{\delta x\to 0}\frac{1}{\delta x}\int_x^{x+\delta x} f(\tilde{x})\,\mathrm{d}\tilde{x} = f(x),$$

as claimed.

6

Existence and uniqueness of solutions

Exercise 6.1 *Which of the following differential equations have unique solutions (at least on some small time interval) for any non-negative initial condition ($x(0) \geq 0$)?*

(i) $\dot{x} = x(1 - x^2)$
(ii) $\dot{x} = x^3$
(iii) $\dot{x} = x^{1/3}$
(iv) $\dot{x} = x^{1/2}(1 + x)^2$
(v) $\dot{x} = (1 + x)^{3/2}$.

In each of these questions we will denote by $f(x)$ the right-hand side of the differential equation. We need to check whether or not f and f' are continuous for $x \geq 0$.

(i) Here $f(x) = x(1 - x^2)$ and $f'(x) = 1 - 3x^2$ are both continuous, so solutions are unique. [In fact for $x(0) \geq 0$ solutions exist for all $t \in \mathbb{R}$, while for $x(0) < 0$ they 'blow up' to $x = -\infty$ in a finite time.]

(ii) For this example $f(x) = x^3$ and $f'(x) = 3x^2$ so solutions are unique. [Solutions blow up in a finite time unless $x(0) = 0$.]

(iii) We have $f(x) = x^{1/3}$ (which is continuous), but $f'(x) = x^{-2/3}/3$, so $f'(x) \to \infty$ as $x \to 0$, and the solution of

$$\dot{x} = x^{1/3} \qquad \text{with} \qquad x(0) = 0$$

is not unique: for any choice of $c \geq 0$, the function

$$x(t) = \begin{cases} 0 & t < c \\ \left(\frac{2(t-c)}{3}\right)^{3/2} & t \geq c \end{cases}$$

solves this equation. [Note that unlike the example $\dot{x} = x^{1/2}$ this equation also makes sense for $x < 0$.]

(iv) We have $f(x) = x^{1/2}(1+x^2)$ which is continuous, and

$$f'(x) = \tfrac{1}{2}x^{-1/2}(1+x^2) + 2x^{3/2}.$$

Near zero $f'(x) \to \infty$, and so the solutions with $x(0) = 0$ are not unique.

(v) The function $f(x) = (1+x)^{3/2}$ is continuous, and $f'(x) = \frac{3}{2}(1+x)^{1/2}$ is also continuous, so solutions are unique. [Solutions blow up in a finite time.]

Exercise 6.2 *The Mean Value Theorem says that if f is differentiable on an interval $[a,b]$ then $f(a) - f(b) = (b-a)f'(c)$ for some $c \in (a,b)$. Suppose that $f(x)$ is differentiable with $|f'(x)| \leq L$ for $a \leq x \leq b$. Use the Mean Value Theorem to show that for $a \leq x, y \leq b$ we have*

$$|f(x) - f(y)| \leq L|x-y|.$$

The result is clearly true if $x = y$. Using the mean value theorem for $x > y$ we have

$$f(x) - f(y) = f'(c)(x-y)$$

for some $c \in (x,y)$. It follows that

$$|f(x) - f(y)| \leq |f'(c)||x-y|,$$

and since $|f'(c)| \leq L$ we have

$$|f(x) - f(y)| \leq L|x-y|. \tag{S6.1}$$

If $y > x$ then $f(y) - f(x) = f'(c)(y-x)$ and on taking the modulus of both sides we once again arrive at (S6.1).

Exercise 6.3 *This Exercise gives a simple proof of the uniqueness of solutions of*

$$\dot{x} = f(x,t) \qquad x(t_0) = x_0, \tag{S6.2}$$

under the assumption that

$$|f(x,t) - f(y,t)| \leq L|x-y|. \tag{S6.3}$$

Suppose that $x(t)$ and $y(t)$ are two solutions of (S6.2). Write down the differential equation satisfied by $z(t) = x(t) - y(t)$, and hence show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2 = 2z[f(x(t),t) - f(y(t),t)].$$

Now use (S6.3) to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2 \leq 2L|z|^2.$$

If $\mathrm{d}Z/\mathrm{d}t \leq cZ$ *it follows that* $Z(t) \leq Z(t_0)\mathrm{e}^{c(t-t_0)}$ *(see Exercise 9.7): use this to deduce that the solution of (S6.2) is unique. Hint: any two solutions of (S6.2) agree when* $t = t_0$*.*

We have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t)$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t).$$

It follows that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(x-y) = \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} = f(x,t) - f(y,t).$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2 = \frac{\mathrm{d}z}{\mathrm{d}t}^2 = 2z\frac{\mathrm{d}z}{\mathrm{d}t} = 2z[f(x,t) - f(y,t)].$$

Using the Lipschitz property (S6.3) we have

$$f(x,t) - f(y,t) \leq |f(x,t) - f(y,t)| \leq L|x-y| = L|z|,$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|^2 \leq 2Lz|z| \leq 2L|z|^2.$$

It follows (using $\mathrm{d}Z/\mathrm{d}t \leq cZ \;\Rightarrow\; Z(t) \leq Z(t_0)\mathrm{e}^{c(t-t_0)}$) that

$$|z(t)|^2 \leq |z(t_0)|^2\mathrm{e}^{2L(t-t_0)}. \tag{S6.4}$$

Since $x(t_0) = y(t_0)$ we have $z(t_0) = 0$, and so (S6.4) becomes $|z(t)|^2 = 0$. It follows that $z(t) = 0$, and so $x(t) = y(t)$, which shows that the two solutions must be identical.

Exercise 6.4 *(T) The proof of existence of solutions is much more involved than the proof of their uniqueness. We will consider here the slightly simpler case*

$$\dot{x} = f(x) \qquad \text{with} \qquad x(0) = x_0, \tag{S6.5}$$

assuming that

$$|f(x) - f(y)| \leq L|x-y|. \tag{S6.6}$$

The first step is to convert the differential equation into an integral equation that is easier to deal with: we integrate both sides of (S6.5) between times 0 *and* t *to give*

$$x(t) = x_0 + \int_0^t f(x(\tilde{t}))\,\mathrm{d}\tilde{t}. \qquad \text{(S6.7)}$$

This integral equation is equivalent to the original differential equation: any solution of (S6.7) will solve (S6.5), and vice versa.

The idea behind the method is to use the right-hand side of (S6.7) as a means of refining any 'guess' of the solution $x_n(t)$ *by replacing it with*

$$x_{n+1}(t) = x_0 + \int_0^t f(x_n(\tilde{t}))\,\mathrm{d}\tilde{t}. \qquad \text{(S6.8)}$$

We start with $x_0(t) = x_0$ *for all* t*, set*

$$x_1(t) = x_0 + \int_0^t f(x_0)\,\mathrm{d}\tilde{t},$$

and continue in this way using (E6.6). The hope is that $x_n(t)$ *will converge to the solution of the differential equation as* $n \to \infty$.

(i) *Use (S6.6) to show that*

$$|x_{n+1}(t) - x_n(t)| \le L \int_0^t |x_n(\tilde{t}) - x_{n-1}(\tilde{t})|\,\mathrm{d}\tilde{t},$$

and deduce that

$$\max_{t\in[0,1/2L]} |x_{n+1}(t) - x_n(t)| \le \frac{1}{2} \max_{t\in[0,1/2L]} |x_n(t) - x_{n-1}(t)|. \qquad \text{(S6.9)}$$

(ii) *Using (S6.9) show that*

$$\max_{t\in[0,1/2L]} |x_{n+1}(t) - x_n(t)| \le \frac{1}{2^{n-1}} \max_{t\in[0,1/2L]} |x_1(t) - x_0(t)|. \qquad \text{(S6.10)}$$

(iii) *By writing*

$$x_n(t) = [x_n(t)-x_{n-1}(t)]+[x_{n-1}(t)-x_{n-2}(t)]+\cdots+[x_1(t)-x_0(t)]+x_0(t)$$

deduce that

$$\max_{t\in[0,1/2L]} |x_n(t) - x_m(t)| \le \frac{1}{2^{N-2}} \max_{t\in[0,1/2L]} |x_1(t) - x_0(t)|$$

for all $n, m \ge N$.

It follows that $x_n(t)$ converges to some function $x_\infty(t)$ as $n \to \infty$, and therefore taking limits in both sides of (E6.6) implies that

$$x_\infty(t) = x_0 + \int_0^t f(x_\infty(\tilde{t}))\,\mathrm{d}\tilde{t}.$$

Thus $x_\infty(t)$ satisfies (S6.7), and so is a solution of the differential equation. The previous Exercise shows that this solution is unique.

(i) We have

$$\begin{aligned} x_{n+1}(t) - x_n(t) &= x_0 + \int_0^t f(x_n(\tilde{t}))\,\mathrm{d}\tilde{t} - x_0 - \int_0^t f(x_{n-1}(\tilde{t}))\,\mathrm{d}\tilde{t} \\ &= \int_0^t f(x_n(\tilde{t})) - f(x_{n-1}(\tilde{t}))\,\mathrm{d}\tilde{t}, \end{aligned}$$

and so

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &= \left| \int_0^t f(x_n(\tilde{t})) - f(x_{n-1}(\tilde{t}))\,\mathrm{d}\tilde{t} \right| \\ &\le \int_0^t |f(x_n(\tilde{t})) - f(x_{n-1}(\tilde{t}))|\,\mathrm{d}\tilde{t} \\ &\le \int_0^t L|x_n(\tilde{t}) - x_{n-1}(\tilde{t})|\,\mathrm{d}\tilde{t}, \\ &= L\int_0^t |x_n(\tilde{t}) - x_{n-1}(\tilde{t})|\,\mathrm{d}\tilde{t}, \end{aligned}$$

using (S6.6).

Since, therefore,

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\le Lt \max_{\tilde{t}\in[0,t]} |x_n(\tilde{t}) - x_{n-1}(\tilde{t})| \\ &\le L\frac{1}{2L} \max_{\tilde{t}\in[0,1/2L]} |x_n(\tilde{t}) - x_{n-1}(\tilde{t})| \end{aligned}$$

for any $t \in [0, 1/2L]$, it follows that

$$\max_{t\in[0,1/2L]} |x_{n+1}(t) - x_n(t)| \le \frac{1}{2} \max_{t\in[0,1/2L]} |x_n(t) - x_{n-1}(t)|, \qquad \text{(S6.11)}$$

as claimed.

(ii) We will write

$$D_n = \max_{t\in[0,1/2L]} |x_n(t) - x_{n-1}(t)|;$$

then (S6.11) reads $D_{n+1} \leq \frac{1}{2} D_n$. It follows easily that

$$D_{n+1} \leq 2^{-(n-1)} D_1$$

which is (S6.10).

(iii) Taking (wlog) $n > m$ we have

$$\begin{aligned} x_n(t) - x_m(t) &= x_n(t) - x_{n-1}(t) + x_{n-1}(t) - x_{n-2}(t) \\ &\quad + \ldots + x_{m+1}(t) - x_m(t), \end{aligned}$$

it follows that

$$\begin{aligned} \max_{t \in [0,1/2L]} |x_n(t) - x_m(t)| &\leq D_n + D_{n-1} + \ldots + D_{m+1} \\ &\leq \left[\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots + \frac{1}{2^{m-1}} \right] D_1 \\ &\leq \frac{D_1}{2^{m-2}}. \end{aligned}$$

7

Scalar autonomous ODEs

Exercise 7.1 *For each of the following differential equations draw the phase diagram, labelling the stationary points as stable or unstable.*

(i) $\dot{x} = -x + 1$
(ii) $\dot{x} = x(2 - x)$
(iii) $\dot{x} = (1 + x)(2 - x)\sin x$
(iv) $\dot{x} = -x(1 - x)(2 - x)$
(v) $\dot{x} = x^2 - x^4$

The Figures all show the phase diagram and the graph of the function $f(x)$ on the right-hand side of the equation.

(i)

$$\dot{x} = -x + 1.$$

There is a single stationary point at $x = 1$, as shown in Figure 7.1.

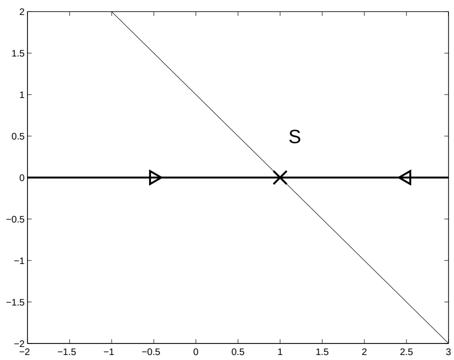

Fig. 7.1. The phase diagram for $\dot{x} = -x + 1$.

(ii)

$$\dot{x} = x(2 - x).$$

There are two stationary points, $x = 0$ and $x = 2$, as shown in Figure 7.2.

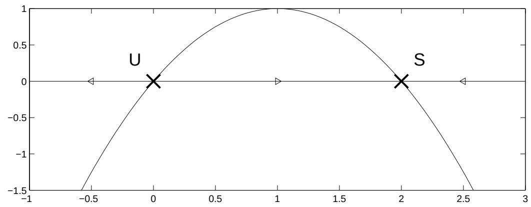

Fig. 7.2. The phase diagram for $\dot{x} = x(2 - x)$.

(iii)

$$\dot{x} = (1 + x)(2 - x) \sin x.$$

There are an infinite number of stationary points, $x = -1$, $x = 2$, and $x = n\pi$ for any integer n, as shown in Figure 7.3.

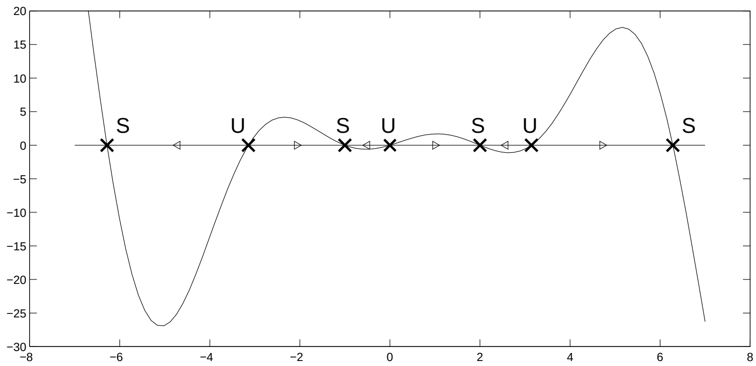

Fig. 7.3. The phase diagram for $\dot{x} = (1 + x)(2 - x) \sin x$ for $-7 \leq x \leq 7$. There are other stationary points at $x = n\pi$ for any integer n.

(iv)

$$\dot{x} = -x(1 - x)(2 - x).$$

There are three stationary points, $x = 0$, $x = 1$, and $x = 2$, as shown in Figure 7.4.

(v)

$$\dot{x} = x^2 - x^4.$$

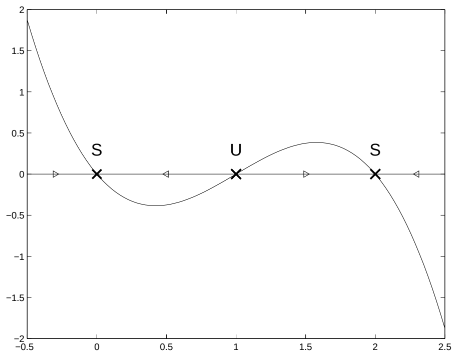

Fig. 7.4. The phase diagram for $\dot{x} = -x(1-x)(2-x)$.

There three are stationary points, at $x = 0$, $x = 1$, and $x = -1$, as shown in Figure 7.5.

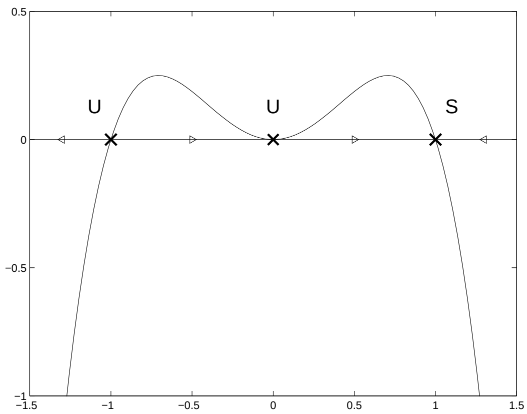

Fig. 7.5. The phase diagram for $\dot{x} = x^2 - x^4$.

Exercise 7.2 *For the equations in Exercise 7.1 determine the stability of the stationary points analytically, by considering the sign of the derivative of the right-hand side.*

(i) When $f(x) = -x + 1$ we have $f'(x) = -1$, and the stationary point at $x = 1$ is stable.

(ii) When $f(x) = x(2-x)$ we have $f'(x) = 2 - 2x$: we have $f'(0) = 2$, so that the stationary point $x = 0$ is unstable; and we have $f'(2) = -2$, and this stationary point is stable.

(iii) For $f(x) = (1+x)(2-x)\sin x$ we have

$$f'(x) = (1-2x)\sin x + (1+x)(2-x)\cos x.$$

So at $x = -1$ we have

$$f'(-1) = 3\sin(-1) \approx -2.52 < 0$$

and this point is stable; at $x = 2$ we have

$$f'(2) = -3\sin 2 \approx -2.73 < 0$$

and this point is also stable.

At $x = n\pi$ we have

$$f'(n\pi) = (1+n\pi)(2-n\pi)(-1)^n;$$

taking $n = 0$ gives $f'(0) = 2$, and this point is unstable. For integer $n \neq 0$,

$$(1+n\pi)(2-n\pi) \leq 0,$$

and so $f'(n\pi) > 0$ (these points are unstable) if n is odd and $f'(n\pi) < 0$ if n is even (and these points are stable).

(iv) We have $f(x) = -x(1-x)(2-x)$, and so

$$f'(x) = -3x^2 + 6x - 2.$$

Therefore

$$f'(0) = -2, \qquad f'(1) = 1, \qquad \text{and} \qquad f'(2) = -2,$$

and so $x = 0$ and $x = 2$ are stable, while $x = 1$ is unstable.

(v) Now we have $f(x) = x^2 - x^4$, and so $f'(x) = 2x - 4x^3$. This gives

$$f'(-1) = 2, \qquad f'(0) = 0, \qquad \text{and} \qquad f'(1) = -2.$$

We can only tell using this method that $x = -1$ is unstable and that $x = 1$ is stable. To find the stability of $x = 0$ we need to work from the phase diagram (it is unstable, or, if you prefer, 'semi-stable', i.e. stable on one side and unstable on the other).

Exercise 7.3 *For all positive values of c find all the stationary points of*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sin x + c,$$

and determine analytically which are stable and unstable. Draw the portion of the phase diagram between $-\pi$ and π. There are three different cases,

$0 \leq c < 1$, $c = 1$, and $c > 1$. You will need to be more careful with the case $c = 1$.

Stationary points occur whenever

$$f(x) \equiv \sin x + c$$

is zero, so whenever $\sin x = -c$. If $0 \leq c < 1$ then there are two solutions in $(-\pi, \pi]$, x_1 between $-\pi$ and $-\pi/2$, and x_2 between $-\pi/2$ and 0. Since $f'(x) = \cos x$ we have

$$f'(x_1) < 0 \qquad \text{and} \qquad f'(x_2) > 0,$$

so that x_1 is stable and x_2 is unstable, see Figure 7.6.

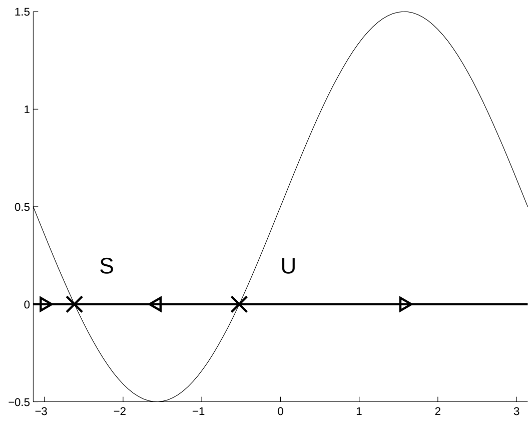

Fig. 7.6. The two stationary points in the interval $[-\pi, \pi]$ for the equation $\dot{x} = \sin x + c$ when $0 \leq c < 1$. This figure has $c = 1/2$.

When $c = 1$ the equation $\sin x = -1$ has only one solution between $-\pi$ and π, which is $x = -\pi/2$. Since $f'(-\pi/2) = \cos(-\pi/2) = 0$ we need to look more closely. We have $f''(x) = -\sin x$, so at $-\pi/2$ the second derivative is positive. So $f(x)$ is a minimum when $x = -\pi/2$. It follows that $f(x)$ itself is positive on both sides on $x = -\pi/2$, so the stationary point is unstable ("stable from the left" but "unstable to the right"), as in Figure 7.7.

Finally when $c > 1$ there are no stationary points, and $f(x) > 0$ for all x. So all trajectories move to the right, as in Figure 7.8.

Exercise 7.4 *A simple model of the spread of an infection in a population is*

$$\begin{aligned} \dot{H} &= -kIH \\ \dot{I} &= kIH, \end{aligned}$$

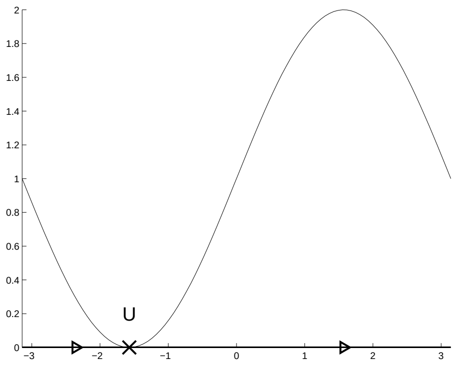

Fig. 7.7. The phase portrait near the single stationary point $x = -\pi/2$ for the equation $\dot{x} = \sin x + 1$.

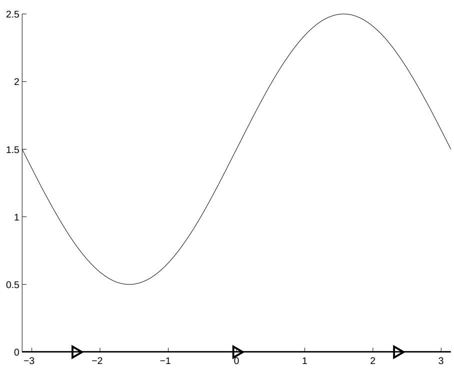

Fig. 7.8. The phase portrait for the equation $\dot{x} = \sin x + c$ when $c > 1$ (this figure has $c = 1.5$). There are no stationary points.

where $H(t)$ is the number of healthy people, $I(t)$ the number of infected people and k the rate of infection. Since $(\mathrm{d}/\mathrm{d}t)(H + I) = 0$, it follows that the size of the population is constant, $H + I = N$, say. Substitute $I = N - H$ in order to obtain a single equation for $H(t)$,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -kH(N - H).$$

Determine the stability of the stationary points for this equation, and draw its phase diagram. Deduce that eventually all the population becomes infected.

We have $\dot{H} = -kIH$ and $I = N - H$. It follows that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -kH(N - H).$$

The stationary points are $H = 0$ (everybody infected) and $H = N$ (nobody infected). If $f(H) = -kH(N-H)$ then $f'(H) = -kN + 2kH$; since $f'(0) = -kN$ and $f'(N) = kN$ it follows that $H = 0$ is stable and $H = N$ is unstable. The phase diagram is shown in Figure 7.9; the whole population is eventually infected.

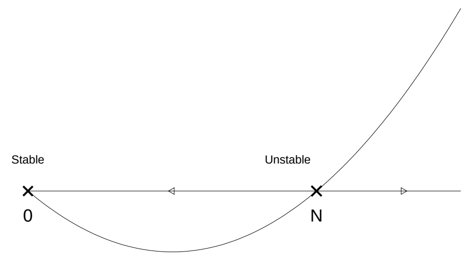

Fig. 7.9. The phase diagram for $\dot{H} = -KH(N-H)$: eventually there are no healthy members of the population left.

Exercise 7.5 *Consider the equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \equiv x^2 - k.$$

Draw the phase diagram for the three cases $k < 0$, $k = 0$ and $k > 0$, labelling the stationary points as stable or unstable in each case. Find the stability of the stationary points using an analytic method when $k > 0$. Show that $f'(0) = 0$ when $k = 0$. Why is this significant?

Draw the bifurcation diagram, with k on the horizontal axis and the fixed points plotted against k, indicating stable fixed points by a solid line and unstable fixed points by a dashed line. (This is known as a saddle node *bifurcation.)*

For $\dot{x} = x^2 - k$ when $k < 0$ there are no stationary points and the particle always moves to the right, as in Figure 7.10.

When $k = 0$ the equation is $\dot{x} = x^2$: there is one stationary point at $x = 0$; we have finite-time blowup for $x(0) > 0$, see Figure 7.11.

However, when $k > 0$ there are two stationary points at $\pm\sqrt{k}$, one stable and one unstable, as shown in Figure 7.12.

We can check the stability and instability of the stationary points for $k > 0$ analytically, since $f'(x) = 2x$: $f'(-\sqrt{k}) = -2\sqrt{k}$ and so this point is stable, while $f'(\sqrt{k}) = 2\sqrt{k}$ and this point is unstable.

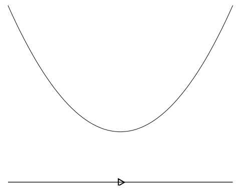

Fig. 7.10. The phase diagram for $\dot{x} = x^2 - k$ when $k < 0$. There are no stationary points.

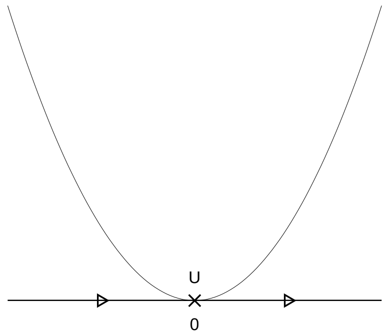

Fig. 7.11. The phase diagram for $\dot{x} = x^2$. There is one stationary point at zero.

When $k = 0$ we have $f'(0) = 0$. This is a necessary condition for a bifurcation, which is precisely what we have here (the so-called saddle node bifurcation). Plotting the stationary points on a graph of (x vs k), with the stable points shown as a solid line and the unstable points as a dashed line we have the bifurcation diagram shown in Figure 7.13 (cf. the pitchfork bifurcation in section 7.6). In this 'saddle-node bifurcation' two stationary points (a stable point and an unstable point) appear 'from nowhere'.

Exercise 7.6 *Draw the phase diagram for the equation*

$$\dot{x} = g(x) = kx - x^2$$

for $k < 0$, $k = 0$, *and* $k > 0$. *Check the stability of the stationary points by considering* $g'(x)$, *and show that the two stationary points exchange stability as* k *passes through zero. Draw the bifurcation diagram for this transcritical bifurcation.*

The stationary points occur where $kx - x^2 = 0$, i.e. at $x = 0$ and $x = k$. Figure 7.14 shows the phase diagrams (along with the graph of f) for $k < 0$, $k = 0$, and $k > 0$.

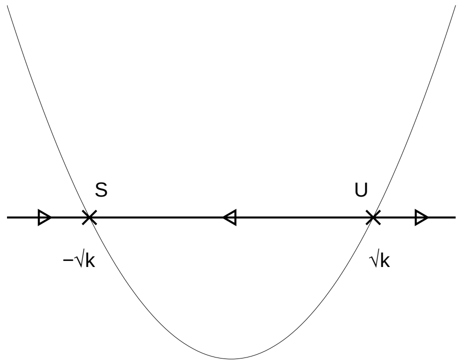

Fig. 7.12. The phase diagram for $\dot{x} = x^2 - k$ when k is positive. There are two stationary points, a stable point at $x = -\sqrt{k}$ and an unstable point at $x = \sqrt{k}$.

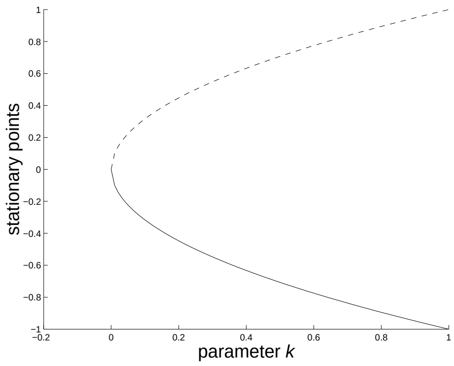

Fig. 7.13. The bifurcation for the 'saddle-node bifurcation' that occurs in the equation $\dot{x} = x^2 - k$.

With $g(x) = kx - x^2$ we have $g'(x) = k - 2x$, and so $g'(0) = k$ and $g'(k) = -k$. It follows that for $k < 0$, the point $x = 0$ is stable and $x = k$ is unstable, while for $k > 0$ this stability is reversed, with $x = 0$ becoming unstable and $x = k$ becoming stable. When $k = 0$ we have only one stationary point, at $x = 0$, and there $g'(0) = 0$; the stability is indeterminate and their is the possibility of a bifurcation. The bifurcation diagram is shown in Figure 7.15.

Exercise 7.7 *One equation can exhibit a number of bifurcations. Find, depending on the value of k, all the stationary points of the equation*

$$\dot{x} = h(x) = -(1 + x)(x^2 - k)$$

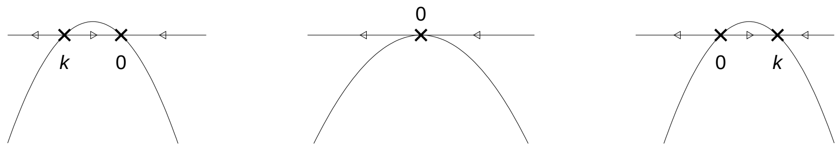

Fig. 7.14. The phase diagram for $\dot{x} = kx - x^2$ for (left to right) $k < 0$, $k = 0$, and $k > 0$.

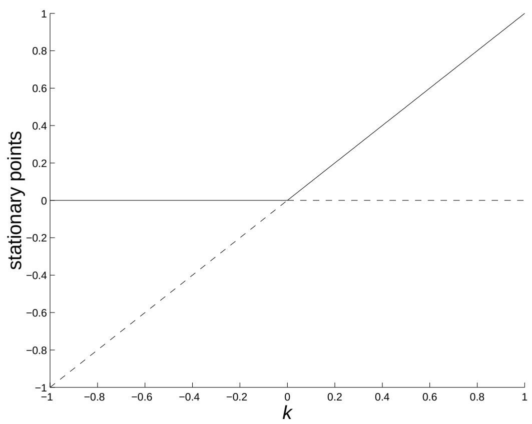

Fig. 7.15. A transcritical bifurcation: stability is exchanged between the origin and $x = k$ as k passes through zero. A solid line indicates a stable stationary point, and a dashed line an unstable stationary point

and by considering $h'(x)$ determine their stability. At which points, and for which values of k, are there possible bifurcations?

Draw representative phase portraits for the five distinct parameter ranges $k < 0$, $k = 0$, $0 < k < 1$, $k = 1$, and $k > 1$, and then draw the bifurcation diagram. Identify the type of the two bifurcations.

The stationary points occur when

$$h(x) = -(1 + x)(x^2 - k) = 0,$$

so at $x = -1$ and $x = \pm\sqrt{k}$. For $k < 0$ there is only one stationary point ($x = -1$), for $k = 0$ there are two ($x = -1$ and $x = 0$), and for $k > 0$ there are three, $x = -1$ and $x = \pm\sqrt{k}$, apart from $k = 1$ when the points at $x = -1$ and $x = -\sqrt{k}$ coincide. Since

$$h'(x) = -3x^2 - 2x + k$$

we have

$$h'(-1) = k - 1 \qquad \text{and} \qquad h'(\pm\sqrt{k}) = -2(k \pm \sqrt{k}).$$

So $x = -1$ is stable for $k < 1$ and unstable for $k > 1$. When $k = 1$ we have $h'(-1) = 0$, and there may be a bifurcation near this point.

When $k = 0$ we have $h'(0) = 0$, meaning that there may be a bifurcation near zero. For $k > 0$ we always have $h'(\sqrt{k}) < 0$ so that this point is always stable. For $k < 1$ we have $h'(-\sqrt{k}) > 0$ implying that $x = -\sqrt{k}$ is unstable, but for $k > 1$ we have $h'(-\sqrt{k}) < 0$ giving stability; for $k = 1$ the derivative takes the indeterminate value $h'(-\sqrt{k}) = 0$, giving the chance of a bifurcation (this again indicates the possibility of a bifurcation near $x = -1$ when $k = 1$, as above).

The phase portraits for $k < 0$, $k = 0$, $0 < k < 1$, $k = 1$, and $k > 1$, are shown in Figure 7.17. The bifurcation diagram is shown in Figure 7.16. The bifurcation at $k = 0$ is a saddle-node bifurcation (two new stationary points appear), and that at $k = 1$ is a transcritical bifurcation (a transfer of stability).

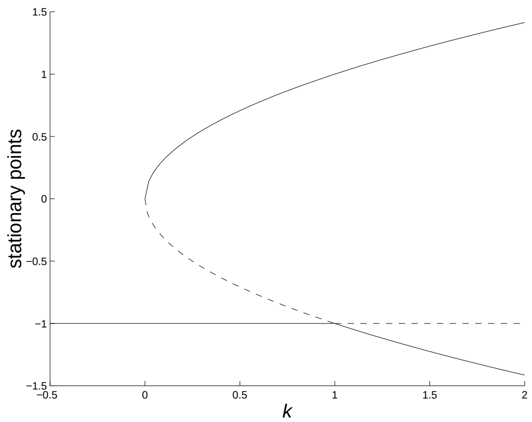

Fig. 7.16. The bifurcation diagram for the equation $\dot{x} = -(1 + x)(x^2 - k)$ [solid/dashed lines indicate stable/unstable stationary points]. There is a saddle-node bifurcation near $x = 0$ as k passes through zero, and a transcritical bifurcation near $x = -1$ and k passes through one.

In the remaining exercises assume that f is a C^1 function, i.e. that both f and $\mathrm{d}f/\mathrm{d}x$ are continuous functions. Note that such an f is smooth enough to guarantee that the equation $\dot{x} = f(x)$ with $x(t_0) = x_0$ has a unique solution. You may also assume that the solutions are defined for all $t \geq 0$.

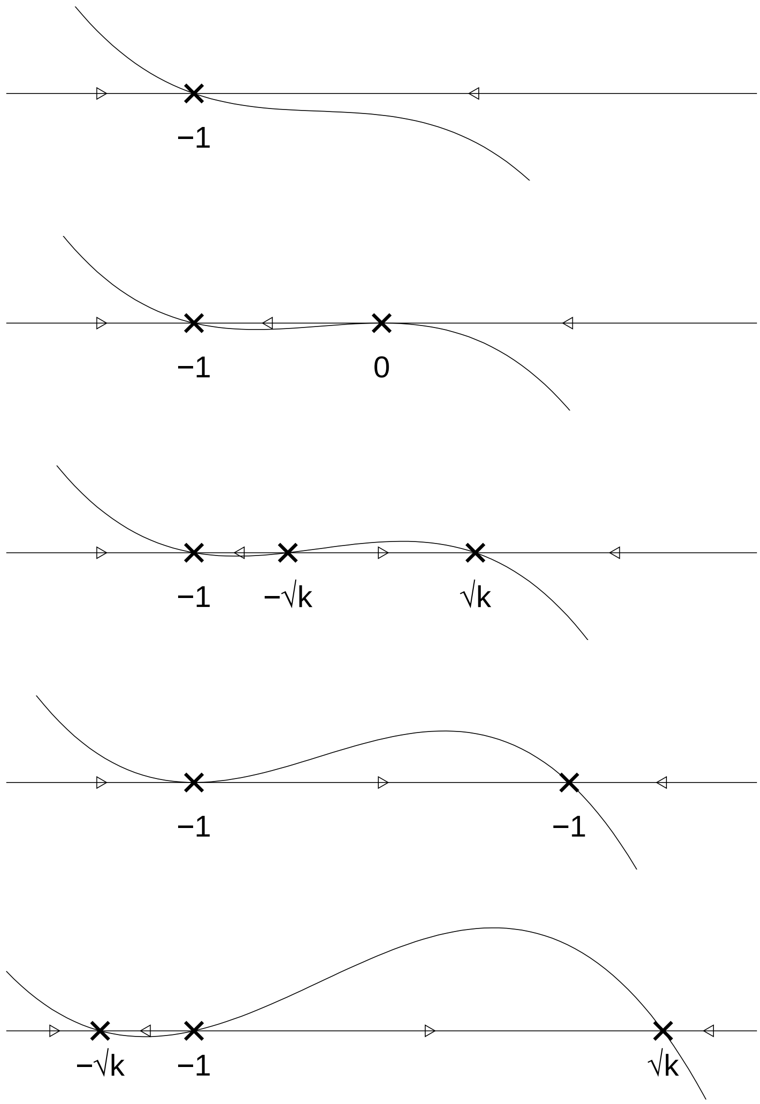

Fig. 7.17. Phase portraits for the equation $\dot{x} = -(1+x)(x^2-k)$ for, from top to bottom: $k<0$, $k=0$, $0<k<1$, $k=1$, and $k>1$.

Exercise 7.8 *Let $x(t)$ be one solution of the differential equation*

$$\dot{x} = f(x).$$

Show that

(i) if $f(x(t^*)) = 0$ for some t^* then $x(t) = x(t^*)$ for all $t \in \mathbb{R}$ (the solution is constant, and $x(t^*)$ is a stationary point); and hence

(ii) if $f(x(t^*)) > 0$ for some t^* then $f(x(t)) > 0$ for all $t \in \mathbb{R}$ (the solution cannot 'reverse direction'). Hint: Use the Intermediate Value Theorem, which states that if g is a continuous function with $g(a) < 0$ and $g(b) > 0$ then there is a point c between a and b with $g(c) = 0$.

Of course, a similar result to (ii) holds if $f(x(t^*)) < 0$ for some t^*.

(i) If $f(x(t^*)) = 0$ then write $x^* = x(t^*)$. Now, since $f(x^*) = 0$, it follows that $x(t) = x^*$ for all $t \in \mathbb{R}$ is a solution of the equation $\dot{x} = f(x)$. Since solutions are unique, there is only one solution with $x(t^*) = x^*$, and so this must be $x(t) = x^*$ for all t. So we must have $x(t) = x(t^*)$ for all t, as claimed. This shows that if a solution is ever at a stationary point, then it is always there.

(ii) First note that since f and x are continuous, $f(x(t))$ is a continuous function of t. We know that $f(x(t^*)) > 0$; now suppose that $f(x(s)) < 0$ for some s. It follows from the intermediate value theorem that there must be some τ between t^* and s with $f(x(\tau)) = 0$. But, then, by part (i), the solution must in fact be constant with $x(t) = x(\tau)$ for all t, a contradiction. So $f(x(t)) > 0$ for all t.

Exercise 7.9 *Show that for autonomous scalar equations, if x^* is attracting then it must also be stable. Hint: use (ii) above.*

If x^* is attracting then there exists some $\epsilon > 0$ such that

$$|x_0 - x^*| < \epsilon \qquad \Rightarrow \qquad |x(t) - x^*| \to 0 \quad \text{as} \quad t \to \infty.$$

Suppose that $x_0 < x^*$ (a similar argument works if $x_0 > x^*$). Then since the solution has to move towards x^*, at some time we must have $x(t) > x_0$, and at some time $t^* > 0$ we must have $f(x(t^*)) > 0$. But then, by part (ii) of the previous question, the solution $x(t)$ must have $f(x(t)) > 0$ for all t, so be increasing for all t. It follows that

$$|x_0 - x^*| < \epsilon \qquad \Rightarrow \qquad |x(t) - x^*| < \epsilon \qquad \text{for all} \qquad t \geq 0,$$

and so x^* is stable.

Exercise 7.10 *Suppose that $x(t)$ is a solution of $\dot{x} = f(x)$ that is moving to the right. Show that either $x(t) \to +\infty$, or $x(t) \to x^*$, where x^* is a stationary point. (Hint: If $x(t)$ does not tend to infinity then it is increasing and bounded above, and so tends to a limit x^*. Show that in this case we must have $f(x^*) = 0$.) A similar result holds if $x(t)$ is moving to the left, with $+\infty$ replaced by $-\infty$.*

Following the hint, either $x(t) \to +\infty$, or $x(t)$ is an increasing function that is bounded above. In the latter case, it follows that $x(t) \to x^*$ for some x^*. Suppose that $f(x^*) \neq 0$. If $f(x^*) < 0$ then $f(x) < 0$ for x sufficiently close to x^*, which contradicts the fact that $x(t)$ must always move right. If $f(x^*) > 0$ then, since f is continuous, for some $\epsilon > 0$ we know that $f(x) > \frac{1}{2}f(x^*)$ for all $x \in (x^* - \epsilon, x^* + \epsilon)$. Since $x(t) \to x^*$, for some T large enough $x(T) > x^* - \epsilon$. But then

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(x(t)) > \tfrac{1}{2}f(x^*) > 0,$$

and integrating this from T to t gives $x(t) > x(T) + \frac{1}{2}(t - T)f(x^*)$. This shows that for t large enough $x(t) > x^*$, contradicting the fact that $x(t) \uparrow x^*$.

Exercise 7.11 *Suppose that $\dot{x} = f(x)$ has a stable stationary point at x_0, with $f(x_0) < 0$. Let g be another C^1 function. Use the following scalar version of the Implicit Function Theorem to show that for ϵ sufficiently small the equation*

$$\dot{x} = f(x) + \epsilon g(x)$$

has a unique stationary point near x_0 which is still stable.
Theorem. *Suppose that $h(x, \epsilon)$, $\partial h/\partial x$, $\partial h/\partial \epsilon$ are all continuous functions of both x and ϵ. Suppose also that $h(x_0, 0) = 0$ and $\partial h/\partial x(x_0, 0) \neq 0$. Then there is an open interval I that contains x_0 such that for each ϵ sufficiently small there is a unique solution $y(\epsilon) \in I$ of*

$$h(y(\epsilon), \epsilon) = 0,$$

and $y(\epsilon)$ depends continuously on ϵ.

To use the theorem set

$$h(x;, \epsilon) = f(x) + \epsilon g(x).$$

Then h, $\partial h/\partial x = f'(x) + \epsilon g'(x)$, and $\partial h/\partial \epsilon = g(x)$ are all continuous functions of ϵ and x.

Clearly $h(x^*, 0) = f(x^*) = 0$ at the stationary point x^*, while from our assumption that $f'(x^*) < 0$ we have $\partial h/\partial x(x^*, 0) = f'(x^*) < 0$.

Then in some open interval containing x^* (i.e. $(x^* - \delta, x^* + \delta)$ for some small δ) there is a unique solution y of the equation

$$f(y) + \epsilon g(y) = 0,$$

i.e. a unique stationary point of the equation

$$\dot{x} = f(x) + \epsilon g(x).$$

Since f' and g' are continuous functions of x, and y depends continuously on ϵ, it follows for ϵ sufficiently small that

$$f'(y) + \epsilon g'(y) < 0,$$

and so y is still stable.

8

Separable equations

Exercise 8.1 *Solve the following equations:*

(i) $\dot{x} = t^3(1-x)$ *with* $x(0) = 3$*;*
(ii) $y' = (1+y^2)\tan x$ *with* $y(0) = 1$*;*
(iii) $\dot{x} = t^2 x$ *(general solution);*
(iv) $\dot{x} = -x^2$ *(general solution);*
(v) *for* $\mathrm{d}y/\mathrm{d}t = \mathrm{e}^{-t^2}y^2$ *give the solution in terms of an integral and describe the behaviour of the solution as* $t \to +\infty$*, depending on the initial condition. You may assume that* $\int_0^\infty \mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}/2$*.*

(i) Separate variables in the equation $\mathrm{d}x/\mathrm{d}t = t^3(1-x)$ to give

$$\frac{\mathrm{d}x}{1-x} = t^3\,\mathrm{d}t$$

and integrate between the limits associated with t values 0 and t,

$$\int_3^{x(t)} \frac{\mathrm{d}x}{1-x} = \int_0^t \tilde{t}^3\,\mathrm{d}\tilde{t}$$

to give

$$\Big[-\ln|1-x|\Big]_{x=3}^{x(t)} = \left[\frac{\tilde{t}^4}{4}\right]_{\tilde{t}=0}^{t}.$$

Therefore

$$-\ln|1-x(t)| + \ln 2 = \frac{t^4}{4}.$$

Exponentiating both sides gives

$$\frac{2}{|x(t)-1|} = \mathrm{e}^{t^4/4}$$

and rearranging we have

$$|x(t) - 1| = 2e^{-t^4/4}.$$

Since $x(0) = 3$ for small t we have $y(t) - 1 > 0$, so

$$y(t) = 1 + 2e^{-t^4/4}.$$

Noting that this gives $y(t) - 1 > 0$ for all t this is indeed the required solution.

(ii) Separating the variables in the equation $\mathrm{d}y/\mathrm{d}x = (1 + y^2)\tan x$ we have

$$\frac{\mathrm{d}y}{1 + y^2} = \tan x \,\mathrm{d}x,$$

and so integrating both sides between the limits associated with x values 0 and x we get

$$\int_1^{y(x)} \frac{\mathrm{d}y}{1 + y^2} = \int_0^x \tan \tilde{x} \,\mathrm{d}\tilde{x}$$

which gives

$$\left[\tan^{-1} y\right]_{y=1}^{y(x)} = \left[-\ln\cos\tilde{x}\right]_{\tilde{x}=0}^{x}.$$

Putting in the limits we have

$$\tan^{-1}(y(x)) - \tan^{-1}(1) = -\ln\cos x.$$

Since $\tan^{-1} = \pi/4$ we have

$$y(x) = \tan\left(\frac{\pi}{4} - \ln\cos x\right).$$

(iii) Separating the variables in $\mathrm{d}x/\mathrm{d}t = t^2 x$ we have

$$\frac{\mathrm{d}x}{x} = t^2 \,\mathrm{d}t,$$

and integrating gives

$$\ln|x| = \frac{t^3}{3} + c.$$

Exponentiating both sides we have

$$|x(t)| = Ae^{t^3/3},$$

where $A = e^c$ is positive. Taking $|x(t)| = x(t)$ gives a positive solution, while taking $|x(t)| = -x(t)$ gives a negative solution. Thus the general solution is $x(t) = Ae^{t^3/3}$, allowing any $A \in \mathbb{R}$.

(iv) By separating the variables in $\mathrm{d}x/\mathrm{d}t = -x^2$ we obtain the equation

$$-\frac{\mathrm{d}x}{x^2} = \mathrm{d}t,$$

and integrating both sides we obtain

$$\frac{1}{x} = t + c.$$

It follows that

$$x(t) = \frac{1}{t+c}.$$

For the initial value problem with $x(0) = x_0$, we need $c = 1/x_0$, and so the solution is

$$x(t) = \frac{1}{t + x_0^{-1}}.$$

If $x_0 > 0$ then $x(t) \to 0$ as $t \to \infty$; if $x_0 < 0$ then the solution blows up in a finite time $x(t) \to -\infty$ as $t \to -x_0^{-1}$.

(v) Separate variables in $\mathrm{d}y/\mathrm{d}t = \mathrm{e}^{-t^2} y^2$ to give

$$\frac{\mathrm{d}y}{y^2} = \mathrm{e}^{-t^2}\,\mathrm{d}t,$$

and integrate,

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}y}{y^2} = \int_0^t \mathrm{e}^{-\tilde{t}^2}\,\mathrm{d}\tilde{t}.$$

Although we can integrate the left-hand side, we can find no closed form integral for the right-hand side, so we leave it as it is:

$$\left[-\frac{1}{y}\right]_{y=y_0}^{y(t)} = \int_0^t \mathrm{e}^{-s^2}\,\mathrm{d}s$$

(changing the dummy variable $\tilde{t}$ to an s). So

$$-\frac{1}{y(t)} + \frac{1}{y_0} = \int_0^t \mathrm{e}^{-s^2}\,\mathrm{d}s,$$

which can be rearranged to give

$$y(t) = \frac{1}{y_0^{-1} - \int_0^t \mathrm{e}^{-s^2}\,\mathrm{d}s}.$$

Now, note that

$$\int_0^t \mathrm{e}^{-s^2}\,\mathrm{d}s$$

is increasing in t, and tends to $\sqrt{\pi}/2$ as $t \to \infty$. It follows that if $y_0 < 2/\sqrt{\pi}$ then the solution is always finite, and reaches a limiting value,

$$y(t) \to \frac{1}{y_0^{-1} - (\sqrt{\pi}/2)}.$$

If $y_0 = 2/\sqrt{\pi}$ then as $t \to \infty$, $y(t) \to \infty$, while if $y_0 > 2/\sqrt{\pi}$ the solution blows up in a finite time given by that value of t^* for which

$$\int_0^{t^*} \mathrm{e}^{-s^2}\,\mathrm{d}s = y_0^{-1}.$$

Exercise 8.2 *Solve the linear equation*

$$\dot{x} + px = q$$

by separation of variables.

Separating variables we have

$$\frac{1}{q - px}\,\mathrm{d}x = \mathrm{d}t,$$

and so, integrating

$$-\frac{1}{p}\ln|q - px| = t + c.$$

Multiplying up by $-p$ and exponentiating both sides gives

$$|q - px| = A\mathrm{e}^{-pt}$$

where $A = \mathrm{e}^{-pc}$. Taking either sign for the modulus gives positive or negative values of A, and so we have

$$q - px(t) = A\mathrm{e}^{-pt},$$

and finally

$$x(t) = B\mathrm{e}^{-pt} + \frac{q}{p}$$

(where $B = -A/p$ could be any constant).

Exercise 8.3 *Find the general solution of the equation*

$$xy' = ky$$

that is valid for $x > 0$.

After separating the variables the equation is

$$\frac{\mathrm{d}y}{y} = \frac{k\,dx}{x}.$$

Integrating both sides gives

$$\ln|y| = k\ln x + c,$$

since $x > 0$. Exponentiating this yields

$$|y| = Ax^k$$

with $A = \mathrm{e}^c > 0$. Depending on the sign of $y(1)$ we obtain

$$y(x) = Ax^k$$

for any $A \in \mathbb{R}$ (since $y(x) = 0$ is clearly a solution).

Exercise 8.4 *Find the function $I(t)$ that satisfies*

$$\frac{\mathrm{d}I}{\mathrm{d}t} = p(t)I.$$

(Your answer will involve an integral.)

Separating the variables we obtain

$$\frac{\mathrm{d}I}{I} = p(t)\,\mathrm{d}t.$$

Integrating both sides gives

$$\ln|I| = \int p(t)\,\mathrm{d}t$$

where we have used the notation $\int p(t)\,\mathrm{d}t$ to denote any anti-derivative of p. Thus

$$|I(t)| = \mathrm{e}^{\int p(t)\,\mathrm{d}t},$$

or

$$I(t) = \pm\mathrm{e}^{\int p(t)\,\mathrm{d}t}.$$

Exercise 8.5 *Use the method of separation of variables to show that the general solution of the linear equation*

$$\dot{x} = \lambda x$$

is $x(t) = A\mathrm{e}^{\lambda t}$ for any $A \in \mathbb{R}$.

First notice that $x(t) = 0$ for all t is a solution. Now write

$$\frac{\mathrm{d}x}{x} = \lambda\,\mathrm{d}t,$$

an integrate both sides to give

$$\ln|x| = \lambda t + c.$$

Exponentiating we have

$$|x(t)| = A\mathrm{e}^{\lambda t},$$

where $A = \mathrm{e}^c > 0$. Now, depending on the sign of $x(t)$ we either have $x(t) = A\mathrm{e}^{\lambda t}$ with $A > 0$, or $x(t) = A\mathrm{e}^{\lambda t}$ with $A < 0$. Thus the general solution is

$$x(t) = A\mathrm{e}^{\lambda t}$$

for any $A \in \mathbb{R}$.

Exercise 8.6 *In Exercise 5.6 we showed, neglecting air resistance, that an apple falling from a height h reaches the ground when $t = \sqrt{2h/g}$. If we include air resistance then provided that $v \leq 0$ the equation becomes*

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg + kv^2 \qquad v(0) = 0 \tag{S8.1}$$

with $k > 0$. Show that

$$v(t) = -\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right),$$

and hence that the apple now takes a time

$$t^* = \sqrt{\frac{m}{kg}}\ln\left(\mathrm{e}^{kh/m} - \sqrt{\mathrm{e}^{2kh/m} - 1}\right)$$

to reach the ground. Check that this coincides with the answer with no air resistance ($t^* = \sqrt{2h/g}$) as $k \to 0$. Hint: for small x, $\mathrm{e}^x \approx 1 + x$ and $\ln(1 + x) \approx x$.

Separating variables in (S8.1) we have

$$\frac{\mathrm{d}v}{\kappa v^2 - g} = \mathrm{d}t \qquad v(0) = 0,$$

where we have set $\kappa = k/m$ for notational simplicity. Integrating both sides of the differential equation between limits corresponding to times zero and

t gives

$$\frac{1}{\kappa}\int_0^v \frac{\mathrm{d}\tilde{v}}{\tilde{v}^2 - (g/\kappa)} = \int_0^t \mathrm{d}\tilde{t},$$

which, using the standard integral

$$\int \frac{1}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

yields

$$-\left[\frac{1}{\sqrt{g\kappa}}\tanh^{-1}\left(\frac{\tilde{v}}{\sqrt{g/\kappa}}\right)\right]_{\tilde{v}=0}^{v} = t,$$

i.e.

$$-\frac{1}{\sqrt{g\kappa}}\tanh^{-1}\left(\frac{v}{\sqrt{g/\kappa}}\right) = t,$$

which implies that

$$v(t) = -\sqrt{\frac{g}{\kappa}}\tanh(\sqrt{g\kappa}\,t).$$

Writing $y(t)$ for the height above the ground, we now have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\sqrt{\frac{g}{\kappa}}\tanh(\sqrt{g\kappa}\,t) \qquad \text{with} \qquad y(0) = h.$$

Integrating both sides of the differential equation between times zero and t gives

$$\begin{aligned} y(t) &= h - \sqrt{\frac{g}{\kappa}}\int_0^t \tanh(\sqrt{g\kappa}\,\tilde{t})\,\mathrm{d}\tilde{t} \\ &= h - \sqrt{\frac{g}{\kappa}}\left[\frac{1}{\sqrt{g\kappa}}\ln\cosh(\sqrt{g\kappa}\,\tilde{t})\right]_{\tilde{t}=0}^{t} \\ &= h - \frac{1}{\kappa}\ln\cosh(\sqrt{g\kappa}\,t). \end{aligned}$$

since $\cosh 0 = 1$.

The apple will hit the ground when $y = 0$, so when

$$\ln\cosh(\sqrt{g\kappa}\,t^*) = \kappa h,$$

i.e. when

$$t^* = \frac{1}{\sqrt{g\kappa}}\cosh^{-1}(\mathrm{e}^{\kappa h}).$$

Since

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

it follows that if $\cosh x = y$ then

$$e^{2x} - 2ye^x + 1 = 0,$$

and solving this quadratic for e^x gives

$$e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2} = y \pm \sqrt{y^2 - 1}$$

and so

$$x = \ln(y \pm \sqrt{y^2 - 1}).$$

We want the solution with $t^* > 0$, so we take

$$t^* = \frac{1}{\sqrt{g\kappa}} \ln(e^{\kappa h} + \sqrt{e^{2\kappa h} - 1}),$$

as given in the question.

If k is small then κ is small, so we can use the approximations

$$e^x \approx 1 + x \qquad \text{and} \qquad \ln(1 + x) \approx x$$

(valid for small x) to write

$$\begin{aligned} t^* &\approx \frac{1}{\sqrt{g\kappa}} \ln(1 + \kappa h + \sqrt{2\kappa h}) \\ &\approx \frac{1}{\sqrt{g\kappa}} \ln(1 + \sqrt{2\kappa h}), \\ &\approx \frac{1}{\sqrt{g\kappa}} \sqrt{2\kappa h} \\ &= \sqrt{\frac{2h}{g}}, \end{aligned}$$

where we have used the fact that when x is small, $\sqrt{x}$ is much larger than x.

Exercise 8.7 *Show that for $k \neq 0$ the solution of the differential equation*

$$\frac{dx}{dt} = kx - x^2 \qquad \text{with} \qquad x(0) = x_0$$

is

$$x(t) = \frac{k\, e^{kt} x_0}{x_0(e^{kt} - 1) + k}.$$

Using this explicit solution describe the behaviour of $x(t)$ as $t \to \infty$ for $k < 0$ and $k > 0$. (Note that this is much easier to do using the phase diagram than using the explicit form of the solution.) For $k = 0$ see part (iv) of Exercise 8.1.

Separating variables we have

$$\frac{\mathrm{d}x}{kx - x^2} = \mathrm{d}t.$$

Using partial fractions we have

$$\frac{1}{k}\left[\frac{1}{x} + \frac{1}{k-x}\right]\mathrm{d}x = \mathrm{d}t,$$

and so integrating both sides between the limits corresponds to times zero and t we have

$$\int_{x_0}^{x(t)} \frac{1}{x} + \frac{1}{k-x}\,\mathrm{d}x = \int_0^t k\,\mathrm{d}\tilde{t}.$$

This gives

$$\Big[\ln|x| - \ln|k-x|\Big]_{x=x_0}^{x(t)} = kt,$$

which is

$$\ln|x(t)| - \ln|k - x(t)| - \ln|x_0| + \ln|k - x_0| = kt.$$

Exponentiating both sides gives

$$\frac{|x(t)||k-x_0|}{|x_0||k-x(t)|} = \mathrm{e}^{kt}.$$

From the phase diagram, shown in Figure 8.1 (cf. Figure 7.14 above), we can see that the signs of both $x(t)$ and $k - x(t)$ do not change over time.

Fig. 8.1. The phase diagram for the equation $\dot{x} = kx - x^2$, for $k < 0$ (left) and for $k > 0$ (right).

We can remove the modulus signs and then rearrange to get

$$x(t) = \frac{k\ \mathrm{e}^{kt} x_0}{x_0(\mathrm{e}^{kt} - 1) + k}.$$

When $k > 0$ the term $x_0 e^{kt}$ on the bottom will dominate, and so the solution will tend to k. When $k < 0$ the denominator tends to $k - 1$, and the top tends to zero, so $x(t) \to 0$.

Exercise 8.8 *Show that the solution of the equation*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x(\kappa^2 + x^2)$$

with initial condition $x(0) = x_0$ *is*

$$x(t) = \pm\sqrt{\frac{\kappa^2}{(1 + \kappa^2 x_0^{-2})\mathrm{e}^{2\kappa^2 t} - 1}},$$

where the $\pm$ *is chosen according to the sign of the initial condition. Deduce that* $x(t) \to 0$ *as* $t \to \infty$*. As* t *decreases from zero the solution blows up as* t *approaches a finite value* $t^* < 0$*. When is this 'blow up time'?*

After separating variables we are left with

$$\frac{\mathrm{d}x}{x(\kappa^2 + x^2)} = -\mathrm{d}t.$$

Using the method of partial fractions we can write

$$\frac{1}{x(\kappa^2 + x^2)} = \frac{1}{\kappa^2}\left(\frac{1}{x} - \frac{x}{\kappa^2 + x^2}\right),$$

and so

$$\int_{x_0}^{x(t)} \left(\frac{1}{x} - \frac{x}{\kappa^2 + x^2}\right) \mathrm{d}x = -\int_0^t \kappa^2 \, \mathrm{d}\tilde{t},$$

which gives

$$\left[\ln|x| - \frac{1}{2}\ln(\kappa^2 + x^2)\right]_{x=x_0}^{x(t)} = -\kappa^2 t,$$

or

$$\ln \frac{|x(t)|\sqrt{\kappa^2 + x_0^2}}{|x_0|\sqrt{\kappa^2 + x(t)^2}} = -\kappa^2 t.$$

Now we exponentiate both sides and square to give

$$\frac{x^2(t)(\kappa^2 + x_0^2)}{x_0^2(\kappa^2 + x(t)^2)} = \mathrm{e}^{-2\kappa^2 t},$$

and after a rearrangement we can obtain

$$x(t)^2 = \frac{\kappa^2}{(1 + \kappa^2 x_0^{-2})\mathrm{e}^{2\kappa^2 t} - 1}.$$

Since $\dot{x} < 0$ for $x > 0$, and $\dot{x} > 0$ for $x < 0$, while $\dot{x} = 0$ if $x = 0$, it is clear that the sign of $x(t)$ does not change.

Thus depending on the sign of x_0, we have

$$x(t) = \pm\sqrt{\frac{\kappa^2}{(1 + \kappa^2 x_0^{-2})e^{2\kappa^2 t} - 1}}.$$

As $t \to \infty$ the denominator tends to infinity, and so $x(t) \to 0$. As t decreases from zero the denominator decreases, and the solution will blow up as t approaches t^* where

$$(1 + \kappa^2 x_0^{-2})e^{2\kappa^2 t^*} - 1 = 0,$$

i.e.

$$t^* = -\frac{1}{2\kappa^2}\ln(1 + \kappa^2 x_0^{-2}).$$

Exercise 8.9 *We found the solution of the equation* $\dot{x} = x(\kappa^2 - x^2)$ *in Section 8.6.1,*

$$x(t) = \pm\sqrt{\frac{\kappa^2}{1 + e^{-2\kappa^2 t}(\kappa^2 x_0^{-2} - 1)}}.$$

Show that if $|x_0| > \kappa$ *the solution blows up as* t *decreases towards a finite negative value, and find this critical time.*

The solution will blow up if the expression

$$1 + e^{-2\kappa^2 t}(\kappa^2 x_0^{-2} - 1)$$

becomes zero. If $|x_0| > \kappa$ then $0 < \kappa^2 x_0^{-2} < 1$, and the expression $\kappa^2 x_0^{-2} - 1$ is negative. As t decreases from zero the contribution of the second term becomes larger, and will tends to -1 as $t \to t^*$ with

$$1 + e^{-2\kappa^2 t^*}(\kappa^2 x_0^{-2} - 1) = 0,$$

so

$$t^* = \frac{1}{2\kappa^2}\ln(1 - \kappa^2 x_0^{-2}).$$

[This is negative, since $1 - \kappa^2 x_0^{-2} < 1$.]

Exercise 8.10 *Consider the equation*

$$\dot{x} = x^\alpha \qquad \text{with} \qquad x(0) \geq 0$$

for $\alpha > 0$. *Show that the only value of* α *for which the equation has solutions that are both unique and exist for all time is* $\alpha = 1$. *You should be able to* *find an initial condition for which the solutions are not unique when* $\alpha < 1$ *(cf. (6.3)), and show that solutions with* $x(0) > 0$ *blow up in a finite time if* $\alpha > 1$ *(cf. (6.6)).*

First we solve the equation by separating the variables,

$$\frac{\mathrm{d}x}{x^\alpha} = \mathrm{d}t.$$

Integrating from times zero to t gives

$$\left[\frac{1}{1+\alpha}x^{1+\alpha}\right]_{x_0}^{x(t)} = t,$$

which implies that

$$x(t)^{1-\alpha} - x_0^{1-\alpha} = (1-\alpha)t,$$

and so

$$x(t) = [(1-\alpha)t + x_0^{1-\alpha}]^{1/(1-\alpha)}.$$

If $\alpha < 1$ then (cf. example $\dot{x} = x^{1/2}$ in section 6.1) for any choice of $c \geq 0$ the function

$$x(t) = \begin{cases} 0 & t \leq 0 \\ [(1-\alpha)(t-c) + x_0^{1-\alpha}]^{1/(1-\alpha)} & t \geq c \end{cases}$$

satisfies $\dot{x} = x^\alpha$ with $x(0) = 0$.

If $\alpha > 1$ then for positive initial conditions to solution will blow up in finite time, since $1 - \alpha < 0$: $x(t) \to +\infty$ as $t \to t^*$, where

$$t^* = \frac{x_0^{1-\alpha}}{\alpha - 1}$$

(cf. the example $\dot{x} = x^2$ in section 6.3).

If $\alpha = 1$ then solutions are unique since $f(x) = x$ and $f'(x) = 1$ are both continuous for all $x \in \mathbb{R}$, and the explicit form of the solution $x(t) = x_0\mathrm{e}^t$ shows that the solution exists for all time.

Exercise 8.11 *Assuming that* $f(x)$ *and* $f'(x)$ *are continuous, show that if the solution of*

$$\dot{x} = f(x) \qquad \text{with} \qquad x(0) = x_0$$

blows up to $x = +\infty$ *in finite time then*

$$\int_{x_0}^{\infty} f(x)\,\mathrm{d}x < \infty.$$

The solution of the equation can be found by separating variables:

$$\frac{\mathrm{d}x}{f(x)} = \mathrm{d}t.$$

Integrating between times zero and t gives

$$\int_{x_0}^{x(t)} f(x)\,\mathrm{d}x = t. \tag{S8.2}$$

If there is a solution with $x(0) = x_0$ that blows up to $x = +\infty$ in a finite time, i.e. as $t \to t^*$, then taking limits as $t \to t^*$ on both sides of (S8.2) we get

$$\int_{x_0}^{\infty} f(x)\,\mathrm{d}x = t^* < \infty.$$

9

First order linear equations and the integrating factor

Exercise 9.1 *Use an integrating factor to solve the following differential equations:*

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{x} = x^2$$

(find the general solution and the only solution that is finite when $x = 0$),

(ii)

$$\frac{\mathrm{d}x}{\mathrm{d}t} + tx = 4t$$

(find the solution with $x(0) = 2$),

(iii)

$$\frac{\mathrm{d}z}{\mathrm{d}y} = z\,\tan y + \sin y$$

(find the general solution),

(iv)

$$y' + \mathrm{e}^{-x}y = 1$$

(find the solution when $y(0) = \mathrm{e}$, leaving your answer as an integral),

(v)

$$\dot{x} + x\,\tanh t = 3$$

(find the general solution, and compare it to that for $\dot{x} + x = 3$),

(vi)

$$y' + 2y\cot x = 5$$

(find the solution with $y(\pi/2) = 1$),

(vii)

$$\frac{dx}{dt} + 5x = t$$

(find the general solution),

(viii) with $a > 0$ find the solution of the equation

$$\frac{dx}{dt} + \left[a + \frac{1}{t}\right] x = b$$

for a general initial condition $x(1) = x_0$, and show that $x(t) \to b/a$ as $t \to \infty$ (you would get the same result if you replaced $a + t^{-1}$ by a).

(i) The integrating factor for

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

is

$$I(x) = \exp\left(\int \frac{1}{x}\, dx\right) = \exp \ln x = x,$$

and so we have

$$x\frac{dy}{dx} + y = \frac{d}{dx}(xy) = x^3.$$

Integrating both sides with respect to x gives

$$xy = \frac{x^4}{4} + c$$

and so

$$y(x) = \frac{x^3}{4} + \frac{c}{x}.$$

Note that we only have a solution near $x = 0$ if $y(0) = 0$, and this solution is $y(x) = x^3/4$.

(ii) The integrating factor for

$$\frac{dx}{dt} + tx = 4t$$

is

$$I(t) = \exp\left(\int t\, dt\right) = e^{t^2/2}.$$

Therefore we have

$$e^{t^2/2}\frac{dx}{dt} + e^{t^2/2}tx = \frac{d}{dt}(xe^{t^2/2}) = 4te^{t^2/2}.$$

Integrating both sides between 0 and t gives

$$x(t)\mathrm{e}^{t^2/2} - x(0) = 4\mathrm{e}^{t^2/2} - 4$$

or, since $x(0) = 2$

$$x(t) = 4 - 2\mathrm{e}^{-t^2/2}.$$

(iii) Before looking for the integrating factor we need to rearrange the equation

$$\frac{\mathrm{d}z}{\mathrm{d}y} = z\tan y + \sin y$$

into the standard form

$$\frac{\mathrm{d}z}{\mathrm{d}y} - (\tan y)z = \sin y.$$

Now the integrating factor is

$$I(y) = \exp\left(\int -\tan y\,\mathrm{d}y\right) = \exp(\ln\cos y) = \cos y,$$

and so we have

$$\cos y\frac{\mathrm{d}}{\mathrm{d}y} + z\sin y = \frac{\mathrm{d}}{\mathrm{d}y}(z\cos y) = \sin y\cos y = \tfrac{1}{2}\,\sin 2y.$$

Integrating both sides gives

$$z(y)\cos y = -\frac{\cos 2y}{4} + c,$$

and so

$$z(y) = -\frac{\cos 2y}{4\cos y} + \frac{c}{\cos y}.$$

(iv) The integrating factor for the equation

$$y' + \mathrm{e}^{-x}y = 1$$

is

$$I(x) = \exp\left(\int \mathrm{e}^{-x}\right) = \exp(-\mathrm{e}^{-x}) = \mathrm{e}^{-\mathrm{e}^{-x}}.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\mathrm{e}^{-\mathrm{e}^{-x}}\right) = \mathrm{e}^{-\mathrm{e}^{-x}},$$

and integrating between zero and x we get

$$y(x)\mathrm{e}^{-\mathrm{e}^{-x}} - y(0)\mathrm{e}^{-\mathrm{e}^{0}} = \int_0^x \mathrm{e}^{-\mathrm{e}^{-\tilde{x}}}\,\mathrm{d}\tilde{x},$$

which simplifies, since $y(0) = \mathrm{e}$, to give

$$y(x) = 1 + \mathrm{e}^{\mathrm{e}^{-x}} \int_0^x \mathrm{e}^{-\mathrm{e}^{-\tilde{x}}}\,\mathrm{d}\tilde{x}.$$

(v)

$$\dot{x} + x\tanh t = 3.$$

Here the integrating factor is

$$I(t) = \exp\left(\int \tanh t\,\mathrm{d}t\right) = \exp(\ln\cosh t) = \cosh t.$$

So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x\ \cosh t) = 3\cosh t.$$

Integrating both sides gives

$$x(t)\cosh t = 3\sinh t + c,$$

and so

$$x(t) = 3\tanh t + c\,\mathrm{sech}\,t.$$

As $t \to \infty$,

$$\tanh t = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{\mathrm{e}^t + \mathrm{e}^{-t}} = \frac{1 - \mathrm{e}^{-2t}}{1 + \mathrm{e}^{-2t}} \approx 1 - 2\mathrm{e}^{-2t}$$

and

$$\mathrm{sech}\,t = \frac{2}{\mathrm{e}^t + \mathrm{e}^{-t}} \approx 2\mathrm{e}^{-t}.$$

So as $t \to \infty$ we have $x(t) \approx 3 + 2c\mathrm{e}^{-t}$.

To find the solution of $\dot{x} + x = 3$ we use the integrating factor e^t, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}(x\mathrm{e}^t) = 3\mathrm{e}^t \qquad \Rightarrow x(t)\mathrm{e}^t = 3\mathrm{e}^t + d \qquad \Rightarrow x(t) = 3 + d\mathrm{e}^{-t}.$$

The equations have similar solutions for t large; this is not surprising, since for large t we have $\tanh t \approx 1$ (as shown above).

(vi)

$$y' + 2y\cot x = 5.$$

The integrating factor is

$$I(x) = \exp\left(\int 2\cot x\,\mathrm{d}x\right) = \exp(2\ln|\sin x|) = |\sin x|^2 = \sin^2 x.$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(y\sin^2 x) = 5\sin^2 x.$$

Integrating between $\pi/2$ and x,

$$\Big[y(\tilde{x})\sin^2\tilde{x}\Big]_{\tilde{x}=\pi/2}^{x} = 5\int_{\pi/2}^{x}\sin^2\tilde{x}\,\mathrm{d}\tilde{x}$$

we obtain

$$\begin{aligned} y(x)\sin^2 x - 1 &= 5\int_{\pi/2}^{x}\frac{1-\cos 2\tilde{x}}{2}\,\mathrm{d}\tilde{x} \\ &= 5\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{\tilde{x}=\pi/2}^{x} \\ &= 5\left(\frac{x}{2} - \frac{\sin 2x}{4} - \frac{\pi}{4}\right), \end{aligned}$$

and so

$$y(x) = \frac{1}{\sin^2 x}\left[1 + 5\left(\frac{x}{2} - \frac{\sin 2x}{4} - \frac{\pi}{4}\right)\right].$$

(vii)

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = t.$$

The integrating factor is e^{5t}, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{5t}x(t)) = t\mathrm{e}^{5}t.$$

Integrating both sides gives

$$\begin{aligned} \mathrm{e}^{5t}x(t) &= \int t\mathrm{e}^{5t}\,\mathrm{d}t \\ &= \frac{t\mathrm{e}^{5t}}{5} - \frac{1}{5}\int \mathrm{e}^{5t}\,\mathrm{d}t \\ &= \frac{t\mathrm{e}^{5t}}{5} - \frac{\mathrm{e}^{5t}}{25} + c, \end{aligned}$$

and so

$$x(t) = \frac{t}{5} - \frac{1}{25} + c\mathrm{e}^{-5t}.$$

(viii) The integrating factor for

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \left[a + \frac{1}{t}\right]x = b$$

is

$$\exp\left(\int a + \frac{1}{t}\,\mathrm{d}t\right) = \exp(at + \ln t) = t\mathrm{e}^{at}.$$

Multiplying both sides by this we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t)t\mathrm{e}^{at}) = bt\mathrm{e}^{at}.$$

Integrating between times 1 and t gives

$$x(t)t\mathrm{e}^{at} - x_0\mathrm{e}^{a} = \int_1^t b\tilde{t}\mathrm{e}^{a\tilde{t}}\,\mathrm{d}\tilde{t}.$$

Integrating the right-hand side by parts gives

$$\begin{aligned} x(t)t\mathrm{e}^{at} - x_0\mathrm{e}^{a} &= \left[\frac{b\tilde{t}\mathrm{e}^{a\tilde{t}}}{a}\right]_{\tilde{t}=1}^{t} - \int_1^t \frac{b\mathrm{e}^{a\tilde{t}}}{a}\,\mathrm{d}\tilde{t} \\ &= \frac{bt\mathrm{e}^{at} - b\mathrm{e}^{a}}{a} - \left[\frac{b\mathrm{e}^{a\tilde{t}}}{a^2}\right]_{\tilde{t}=1}^{t} \\ &= b\left(\frac{t\mathrm{e}^{at} - \mathrm{e}^{a}}{a} - \frac{\mathrm{e}^{at} - \mathrm{e}^{a}}{a^2}\right). \end{aligned}$$

Rearranging gives

$$x(t) = \frac{b}{a} + \left(x_0 + \frac{b}{a} + \frac{b}{a^2}\right)\mathrm{e}^{a(1-t)} - \frac{b}{a^2 t},$$

and so $x(t) \to b/a$ as claimed.

Exercise 9.2 *A body is found in a cold room (temperature 5° C) at 3 p.m. and its temperature then is* 19° *C. An hour later its temperature has dropped to* 15° *C. Use Newton's law of cooling to estimate the time of death, assuming that body temperature is* 37° *C.*

Newton's law of cooling is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - A(t)) \qquad \text{or} \qquad \frac{\mathrm{d}T}{\mathrm{d}t} + kT = kA(t).$$

When $A(t) = A$, a constant, we can use the integrating factor e^{kt} and then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{kt}T) = kA\mathrm{e}^{kt}.$$

Integrating between t_0 and t we obtain

$$\mathrm{e}^{kt}T(t) - \mathrm{e}^{kt_0}T(t_0) = A(\mathrm{e}^{kt} - \mathrm{e}^{kt_0}),$$

or

$$T(t) = A + e^{-k(t-t_0)}[T(t_0) - A]. \tag{S9.1}$$

First we find k using $T(3) = 19$, $T(4) = 15$, and $A = 5$:

$$15 = T(4) = 5 + e^{-k}[19 - 5] \qquad \Rightarrow \qquad 10 = 14e^{-k}$$

which gives $k = -\ln(5/7) \approx 0.337$.

If the time of death was t_0 then, using $T(t_0) = 37$, we have

$$19 = T(3) = 5 + e^{-k(3-t_0)}[37 - 5] \qquad \Rightarrow \qquad 14 = 32e^{-k(3-t_0)},$$

so that

$$t_0 = 3 + \frac{\ln(14/32)}{k} \approx 0.543,$$

which fixes the time of death at approximately 12:33 a.m.

Exercise 9.3 *At 7 a.m. in the morning I make my wife a cup of tea using boiling water; after adding some milk it is about* $90°$ *C. When we leave for the station at 7:30 a.m. the tea is still drinkable at about* $45°$ *C. When I get back home at 8 a.m. the neglected tea has cooled to about* $30°$ *C. What is the temperature of our house?*

We use the solution (S9.1) from the previous exercise. We have $T(7) = 90$, $T(7.5) = 45$, and $T(8) = 30$. Therefore

$$45 = A + e^{-k/2}[90 - A] \qquad \text{and} \qquad 30 = A + e^{-k}[90 - A].$$

Since $e^{-k/2} = (45 - A)/(90 - A)$ we have

$$30 = A + \frac{(45 - A)^2}{90 - A},$$

which gives an equation for A,

$$(30 - A)(90 - A) = (45 - A)^2.$$

Solving this equation gives $A = 22.5$, and so the house is 22.5°C.

Exercise 9.4 *Use the integrating factor method to find* $T(t_2)$ *in terms of* $T(t_1)$ *when*

$$\frac{dT}{dt} = -k(T(t) - A(t))$$

and

$$A(t) = \mu + a\cos\omega(t - \phi).$$

Rearrange the equation as

$$\frac{\mathrm{d}T}{\mathrm{d}t} + kT = kA(t).$$

The integrating factor is

$$\exp\left(\int k\,\mathrm{d}t\right) = \mathrm{e}^{kt},$$

and so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(T(t)\mathrm{e}^{kt}) = k\mathrm{e}^{kt}A(t).$$

Integrating both sides between t_1 and t_2 gives

$$T(t_2)\mathrm{e}^{kt_2} - T(t_1)\mathrm{e}^{kt_1} = k\int_{t_1}^{t_2} \mathrm{e}^{kt}A(t)\,\mathrm{d}t.$$

With the choice $A(t) = \mu + a\cos\omega(t-\phi)$ we get

$$T(t_2)\mathrm{e}^{kt_2} - T(t_1)\mathrm{e}^{kt_1} = k\mu\int_{t_1}^{t_2} \mathrm{e}^{kt}\,\mathrm{d}t + ak\int_{t_1}^{t_2} \mathrm{e}^{kt}\cos\omega(t-\phi)\,\mathrm{d}t.$$

An anti-derivative of $\mathrm{e}^{kt}\cos\omega(t-\phi)$ is (cf. example in Section 9.4.2)

$$\frac{k}{k^2+\omega^2}\,\mathrm{e}^{kt}\cos\omega(t-\phi) + \frac{\omega}{k^2+\omega^2}\,\mathrm{e}^{kt}\sin\omega(t-\phi),$$

and so

$$\begin{aligned} T(t_2)\mathrm{e}^{kt_2} - T(t_1)\mathrm{e}^{kt_1} &= \mu[\mathrm{e}^{kt_2} - \mathrm{e}^{kt_1}] \\ &+ak\left[\frac{k}{k^2+\omega^2}\,\mathrm{e}^{kt_2}\cos\omega(t_2-\phi) + \frac{\omega}{k^2+\omega}\,\mathrm{e}^{kt_2}\sin\omega(t_2-\phi)\right] \\ &-ak\left[\frac{k}{k^2+\omega^2}\,\mathrm{e}^{kt_1}\cos\omega(t_1-\phi) + \frac{\omega}{k^2+\omega}\,\mathrm{e}^{kt_1}\sin\omega(t_1-\phi)\right]. \end{aligned}$$

Rearranging we have

$$\begin{aligned} T(t_2) &= \mu + ak\left[\frac{k}{k^2+\omega^2}\,\cos\omega(t_2-\phi) + \frac{\omega}{k^2+\omega}\,\sin\omega(t_2-\phi)\right] \\ &+ \left[T(t_1) - \mu - \frac{ak}{k^2+\omega^2}(k\cos\omega(t_1-\phi) + \omega\sin\omega(t_1-\phi))\right]\mathrm{e}^{-k(t_2-t_1)}. \end{aligned}$$

Exercise 9.5 *A dead body is found outside on a winter's morning at 7 a.m.; its temperature is measured as* $20^\circ C$*. Measured an hour later it has dropped to* $15^\circ C$*. The air temperature* $A(t)$ *fluctuates on a daily cycle about a mean of* $3^\circ C$ *with* $A(t) = 3 - 5\cos\omega(t-2)$*, where* t *is measured in hours with* $t = 0$ *corresponding to midnight, and* $\omega = \pi/12$*.*

(i) *Use the solution from question 9.4 and the temperature observations at 7 a.m. and 8 a.m. to show that*

$$k = -\ln\left\{\frac{12(k^2+\omega^2) - 5k(k\cos 6\omega + \omega\sin 6\omega)}{17(k^2+\omega^2) - 5k(k\cos 5\omega + \omega\sin 5\omega)}\right\}. \tag{S9.2}$$

(ii) *(C) This is a* MATLAB *exercise. Choose an initial guess for k, and then substitute this into the right-hand side of (S9.2) to obtain a new guess. Continue doing this until your 'guess' stabilises. Once this happens you have actually obtained the required solution of (S9.2). Can you see why? [You should find $k \approx 0.3640$.]*

(iii) *If the time of death was t_0, use the fact that body temperature is $37°C$ (so $T(t_0) = 37$) and $T(7) = 20$ to show that*

$$t_0 = 7 + \frac{1}{k}\ln\left\{\frac{17(k^2+\omega^2) - 5k(k\cos 5\omega + \omega\sin 5\omega)}{34(k^2+\omega^2) - 5k(k\cos\omega(t_0-2) + \omega\sin\omega(t_0-2))}\right\}.$$

(iv) *(C) Use* MATLAB *again to refine an initial guess for the time of death as in part (ii). You should find that $t_0 \approx 4.8803$, or 4:53 a.m.*

(i) We have $T(7) = 20$, $T(8) = 15$, $\mu = 3$, $a = 5$, $\phi = 2$, and $\omega = \pi/12$. We need to find t_0 such that $T(t_0) = 37$. We should be able to use the information from 7 a.m. and 8 a.m. to estimate k. Our solution gives (leaving $\omega = \pi/12$ in the equations)

$$\begin{aligned} 15 = 3 + 5k&\left[\frac{k}{k^2+\omega^2}\cos 6\omega + \frac{\omega}{k^2+\omega^2}\sin 6\omega\right] \\ &+ \left[17 - \frac{5k}{k^2+\omega^2}(k\cos 5\omega + \omega\sin 5\omega)\right]\mathrm{e}^{-k}. \end{aligned}$$

No matter how we rearrange this equation we cannot solve it explicitly for k. However, if we rearrange it as

$$\mathrm{e}^{-k} = \frac{12(k^2+\omega^2) - 5k(k\cos 6\omega + \omega\sin 6\omega)}{17(k^2+\omega^2) - 5k(k\cos 5\omega + \omega\sin 5\omega)}$$

we can obtain an equation for k 'in terms of k':

$$k = -\ln\left\{\frac{12(k^2+\omega^2) - 5k(k\cos 6\omega + \omega\sin 6\omega)}{17(k^2+\omega^2) - 5k(k\cos 5\omega + \omega\sin 5\omega)}\right\}. \tag{S9.3}$$

(ii) What we can do with this is to guess k, and then substitute our guess into the right-hand side to obtain a new guess for k, and continue like this until, if we are lucky, our guess settles down to a fixed value which will be the solution we want.

Writing this more mathematically, given an initial guess k_0 we then calculate successive guesses k_n by setting

$$k_{n+1} = -\ln\left\{\frac{12(k_n^2+\omega^2) - 5k_n(k_n\cos 6\omega + \omega\sin 6\omega)}{17(k_n^2+\omega^2) - 5k_n(k_n\cos 5\omega + \omega\sin 5\omega)}\right\}.$$

If $k_{n+1} = k + n = k$ then k must be a solution of (S9.3).

You can do this fairly easily in MATLAB, and the result is $k \approx 0.3640$. [The M-file `findk.m` will do this for you; you can check your answer with the file `temperature.m` which will compute the solution at time t_2 given the temperature at time t_1 and the value of k.]

(iii) To find the time of death will require a similar computer calculation, since once again we have an equation that we cannot solve explicitly. If we set $T(t_0) = 37$ and $T(7) = 20$ then we should be able to find t_0 from (we don't replace k by 0.3640 here to simplify the algebra)

$$\begin{aligned} 20 = 3 + 5k&\left[\frac{k}{k^2+\omega^2}\cos 5\omega + \frac{\omega}{k^2+\omega^2}\sin 5\omega\right] \\ &+\left[34 - \frac{5k}{k^2+\omega^2}(k\cos\omega(t_0-2) + \omega\sin\omega(t_0-2))\right]\mathrm{e}^{-k(7-t_0)} \end{aligned}$$

We can rearrange this as

$$\mathrm{e}^{-k(7-t_0)} = \frac{17(k^2+\omega^2) - 5k(k\cos 5\omega + \omega\sin 5\omega)}{34(k^2+\omega^2) - 5k(k\cos\omega(t_0-2) + \omega\sin\omega(t_0-2))},$$

or

$$t_0 = 7 + \frac{1}{k}\log\left\{\frac{17(k^2+\omega^2) - 5k(k\cos 5\omega + \omega\sin 5\omega)}{34(k^2+\omega^2) - 5k(k\cos\omega(t_0-2) + \omega\sin\omega(t_0-2))}\right\}.$$

(iv) Iterating this (you could use the M-file `findt0.m`) gives $t_0 \approx 4.8803$, which puts the time of death at 4:53 a.m. (Again, you could check that this works with the `temperature.m` file.)

Exercise 9.6 *Show that if y_1 and y_2 are any two solutions of*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0$$

then $y_1(x)/y_2(x)$ is constant. (You do not need to solve the equation!)

Differentiating $y_1(x)/y_2(x)$ with respect to x gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y_1(x)}{y_2(x)}\right] &= \frac{y_2y_1' - y_1y_2'}{y_2^2} \\ &= \frac{y_2[-p(x)y_1] - y_1[-p(x)y_2]}{y_2^2} \end{aligned}$$

$$= \quad 0,$$

and so $y_1(x)/y_2(x)$ is constant.

Exercise 9.7 *Suppose that*

$$\frac{\mathrm{d}x}{\mathrm{d}t} \leq ax$$

(this is known as a differential inequality). Use an appropriate integrating factor to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-at}x\right] \leq 0,$$

and then integrate both sides between appropriate limits to deduce that

$$x(t) \leq x(s)\mathrm{e}^{a(t-s)}$$

for any t and s. Hint: it is a fundamental property of integration that if $f(x) \leq g(x)$ then

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

We can rewrite the equation as

$$\frac{\mathrm{d}x}{\mathrm{d}t} - ax \leq 0,$$

and then use the integrating factor e^{at} to obtain

$$\mathrm{e}^{at}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - ax\right] \leq 0$$

(this is allowed since e^{at} is always positive). The left-hand side of the equation is just $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-at}x)$, so we now have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-at}x\right] \leq 0$$

as required. Integrating both sides between times s and t gives

$$\mathrm{e}^{-at}x(t) - \mathrm{e}^{-as}x(s) \leq 0$$

which on rearrangement yields

$$x(t) \leq \mathrm{e}^{-a(t-s)}x(s).$$

Exercise 9.8 *The function* $\sin\omega t$ *can be written as a combination of complex exponentials,*

$$\sin\omega t = \frac{\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}}{2\mathrm{i}}.$$

Using this form for $\sin\omega t$*, and assuming that the usual rules of integration apply to such complex exponentials, find*

$$\int \mathrm{e}^{kt}\sin\omega t\,\mathrm{d}t.$$

You may also need to use the identity

$$\cos\omega t = \frac{\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}}{2}.$$

See Appendix A for more on these complex exponentials.

We write

$$\begin{aligned}
\int \mathrm{e}^{kt}\sin\omega t\,\mathrm{d}t &= \int \mathrm{e}^{kt}\,\frac{\mathrm{e}^{\mathrm{i}\omega t} - \mathrm{e}^{-\mathrm{i}\omega t}}{2\mathrm{i}}\,\mathrm{d}t \\
&= \frac{1}{2\mathrm{i}}\int \mathrm{e}^{(k+\mathrm{i}\omega)t} - \mathrm{e}^{k-\mathrm{i}\omega)t}\,\mathrm{d}t \\
&= \frac{1}{2\mathrm{i}}\left[\frac{\mathrm{e}^{(k+\mathrm{i}\omega)t}}{k+\mathrm{i}\omega} - \frac{\mathrm{e}^{(k-\mathrm{i}\omega)t}}{k-\mathrm{i}\omega}\right] \\
&= \frac{\mathrm{e}^{kt}}{2\mathrm{i}}\left[\frac{\mathrm{e}^{\mathrm{i}\omega t}}{k+\mathrm{i}\omega} - \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{k-\mathrm{i}\omega}\right] \\
&= \mathrm{e}^{kt}\,\mathrm{Im}\left[(k-\mathrm{i}\omega)(\cos\omega t + \mathrm{i}\sin\omega t)\right] \\
&= \mathrm{e}^{kt}(k\sin\omega t - \omega\cos\omega t)
\end{aligned}$$

since $z - z^* = 2\mathrm{i}\,\mathrm{Im}[z]$.

10

Two 'tricks' for nonlinear equations

Exercise 10.1 *Check that the following equations are exact and hence solve them.*

(i)

$$(2xy - \sec^2 x) + (x^2 + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

(ii)

$$(1 + \mathrm{e}^x y + x\mathrm{e}^x y) + (x\mathrm{e}^x + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

(iii)

$$(x\cos y + \cos x)\frac{\mathrm{d}y}{\mathrm{d}x} + \sin y - y\sin x = 0,$$

and

(iv)

$$\mathrm{e}^x \sin y + y + (\mathrm{e}^x \cos y + x + \mathrm{e}^y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

(i) We have

$$\underbrace{(2xy - \sec^2 x)}_{f(x,y)} + \underbrace{(x^2 + 2y)}_{g(x,y)}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

To check that this equation this exact, i.e. that there is an $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = f(x,y) \qquad \text{and} \qquad \frac{\partial F}{\partial y} = g(x,y)$$

we need to make sure that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

So we check
$$\frac{\partial f}{\partial y} = 2x = \frac{\partial g}{\partial x}.$$
Now to find F we first solve
$$\frac{\partial F}{\partial x} = 2xy - \sec^2 x.$$
Integrating with respect to x we get
$$F(x, y) = x^2 y - \tan x + C(y).$$
In order to find $C(y)$ we partially differentiate $F(x, y)$ with respect to y,
$$\frac{\partial F}{\partial y} = x^2 + \frac{\mathrm{d}C}{\mathrm{d}y} = G(x, y) = x^2 + 2y,$$
and so
$$\frac{\mathrm{d}C}{\mathrm{d}y} = 2y$$
which gives $C(y) = y^2 + c$. So the solution is
$$F(x, y) = x^2 y - \tan x + y^2 = c.$$

(ii) Now for
$$\underbrace{(1 + \mathrm{e}^x y + x\mathrm{e}^x y)}_{f(x,y)} + \underbrace{(x\mathrm{e}^x + 2)}_{g(x,y)} \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
we check
$$\frac{\partial f}{\partial y} = \mathrm{e}^x + x\mathrm{e}^x = \frac{\partial g}{\partial x}$$
and the equation is exact. First we solve
$$\frac{\partial F}{\partial x} = 1 + \mathrm{e}^x y + x\mathrm{e}^x y,$$
integrating both sides with respect to x to give
$$F(x, y) = x + x\mathrm{e}^x y + C(y).$$
So we have
$$\frac{\partial F}{\partial y} = x\mathrm{e}^x + \frac{\mathrm{d}C}{\mathrm{d}y} = x\mathrm{e}^x + 2,$$
so in fact
$$\frac{\mathrm{d}C}{\mathrm{d}y} = 2$$

which gives $C = 2y$. Therefore $F(x, y) = x + x\mathrm{e}^x y + 2y$, and we have

$$x + x\mathrm{e}^x y + 2y = C.$$

This simplifies to give

$$y = \frac{C - x}{2 + x\mathrm{e}^x}.$$

(iii) For

$$\underbrace{(\sin y - y\sin x)}_{f(x,y)} + \underbrace{(x\cos y + \cos x)}_{g(x,y)}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

we check that

$$\frac{\partial f}{\partial y} = \cos y - \sin x = \frac{\partial g}{\partial x},$$

and so the equation is exact.

Now we solve

$$\frac{\partial F}{\partial x} = \sin y - y\sin x,$$

which gives

$$F(x, y) = x\sin y + y\cos x + C(y).$$

In order to find $C(y)$ we differentiate partially with respect to y, and then

$$\frac{\partial F}{\partial y} = x\cos y + \cos x + C'(y) = g(x, y) = x\cos y + \cos x,$$

and so $C'(y) = 0$ and C is a constant. So the solution is

$$x\sin x + y\cos x = c$$

for any constant c.

(iv) Now we have

$$\underbrace{(\mathrm{e}^x\sin y + y)}_{f(x,y)} + \underbrace{(e^x\cos y + x + \mathrm{e}^y)}_{g(x,y)}\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and we check that it is exact by showing that

$$\frac{\partial f}{\partial y} = \mathrm{e}^x\cos y + 1 = \frac{\partial g}{\partial x}.$$

We first solve the equation

$$\frac{\partial F}{\partial x} = \mathrm{e}^x\sin y + y,$$

so that

$$F(x,y) = e^x \sin y + xy + C(y).$$

Now we have

$$e^x \cos y + x + e^y = \frac{\partial F}{\partial y} = e^x \cos y + x + C'(y).$$

and so $C'(y) = e^y$ which implies that $C(y) = e^y + c$. So we obtain the solution

$$e^x \sin y + xy + e^y + c = 0..$$

Exercise 10.2 *Find an integrating factor depending only on x that makes the equation*

$$e^{-y} \sec x + 2 \cot x - e^{-y} \frac{dy}{dx} = 0$$

exact, and hence find its solution. Hint: $\int \operatorname{cosec} x \, dx = \ln |\operatorname{cosec} x - \cot x|$.

To look for an integrating factor $I(x)$ depending only on x we want

$$\underbrace{I(x)(e^{-y} \sec x + 2 \cot x)}_{f(x,y)} + \underbrace{(-I(x)e^{-y})}_{g(x,y)} \frac{dy}{dx} = 0$$

to be exact. This needs

$$\frac{\partial f}{\partial y} = -I(x)e^{-y} \sec x = -I'(x)e^{-y} = \frac{\partial g}{\partial x},$$

i.e.

$$I'(x) = (\sec x)I(x).$$

The solution of this is $I(x) = e^{\int \sec(x)\, dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$. Using this integrating factor the original equation becomes

$$\underbrace{(\sec x + \tan x)(e^{-y} \sec x + 2 \cot x)}_{\tilde{f}(x,y)} + \underbrace{[-(\sec x + \tan x)e^{-y}]}_{\tilde{g}(x,y)} \frac{dy}{dx} = 0.$$

This new equation is exact, since

$$\frac{\partial \tilde{f}}{\partial y} = (\sec x + \tan x)(-e^{-y} \sec x) = -e^{-y}(\sec^2 x + \sec x \tan x) = \frac{\partial \tilde{g}}{\partial x}.$$

So we can now find $F(x,y)$: we will do this by solving

$$\frac{\partial F}{\partial y} = \tilde{g}(x,y) = -(\sec x + \tan x)e^{-y},$$

giving

$$F(x,y) = \mathrm{e}^{-y}(\sec x + \tan x) + C(x).$$

Now we want $\frac{\partial F}{\partial x} = \tilde{f}(x,y)$, so we need

$$\mathrm{e}^{-y}(\sec x \tan x + \sec^2 x) + C'(x) = \mathrm{e}^{-y} \sec x(\sec x + \tan x) + 2\cot x(\sec x + \tan x),$$

which gives

$$C'(x) = 2\,\mathrm{cosec}\, x + 2 \qquad \Rightarrow \qquad 2\ln|\mathrm{cosec}\, x - \cot x| + 2x + c,$$

and so

$$\mathrm{e}^{-y}(\sec x + \tan x) + 2\ln|\mathrm{cosec}\, x - \cot x| + 2x + c = 0.$$

Exercise 10.3 *Show that any equation that can be written in the form*

$$f(x) + g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact, and find its solution in terms of integrals of f and g. Hence find the solutions of

(i)

$$V'(x) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and

(ii)

$$\left(\frac{1}{y} - a\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x} - b = 0,$$

for $x, y > 0$.

Here we have

$$\frac{\partial f}{\partial y} = 0 = \frac{\partial g}{\partial y}.$$

First we can solve

$$\frac{\partial F}{\partial x} = f(x)$$

giving

$$F(x,y) = \int f(x)\,\mathrm{d}x + C(y).$$

We then need

$$\frac{\partial F}{\partial y} = C'(y) = g(y),$$

and so $C(y) = \int g(y)\,\mathrm{d}y + c$, giving the solution

$$\int f(x)\,\mathrm{d}x + \int g(y)\,\mathrm{d}y + c = 0.$$

(i) The solution is $V(x) + y^2 = E$.
(ii) The solution is $\ln y - ay + 2\ln x - bx = c$.

Exercise 10.4 *By substituting $u = y/x$ solve the following homogeneous equations:*

(i)

$$xy + y^2 + x^2 - x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

(the solution is $y = x\tan(\ln|x| + c)$).

(ii)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x^2 + t\sqrt{t^2 + x^2}}{tx}$$

(the solution is $x(t) = \pm t\sqrt{(\ln|t| + c)^2 - 1}$).

(i) Divide through by x^2, then the equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{y^2}{x^2} + 1.$$

This is clearly homogeneous. If we put $u = y/x$ then

$$u' = -\frac{y}{x^2} + \frac{y'}{x},$$

and so

$$u' = -\frac{u}{x} + \frac{u + u^2 + 1}{x},$$

i.e.

$$x\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + u^2.$$

Separating the variables and integrating we have

$$\int \frac{\mathrm{d}u}{1 + u^2}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x,$$

and so

$$\tan^{-1}(u) = \ln|x| + c,$$

or $u = \tan(\ln|x| + c)$. Since $y = xu$ we get

$$y(x) = x\tan(\ln|x| + c).$$

(ii) Rearranging the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x^2 + t\sqrt{t^2 + x^2}}{tx}$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{t} + \sqrt{(t/x)^2 + 1}$$

which is clearly homogeneous. So we substitute $u = x/t$, and then

$$\dot{u} = -\frac{x}{t^2} + \frac{\dot{x}}{t}$$

or

$$t\dot{u} = -u + \dot{x},$$

which gives

$$t\dot{u} = -u + u + \sqrt{u^{-2} + 1} = \sqrt{u^{-2} + 1}.$$

So separating variables we have

$$\frac{\mathrm{d}u}{\sqrt{u^{-2} + 1}} = \frac{1}{t}\,\mathrm{d}t$$

which gives after a rearrangement of the left-hand side

$$\int \frac{u\,\mathrm{d}u}{\sqrt{1 + u^2}} = \int \frac{1}{t}\,\mathrm{d}t.$$

Therefore

$$(1 + u^2)^{1/2} = \ln|t| + c.$$

Squaring both sides we have

$$(1 + u^2) = (\ln|t| + c)^2,$$

and so

$$u = \pm\sqrt{(\ln|t| + c)^2 - 1},$$

which, since $y = tu$, gives the required solution.

Exercise 10.5 *You could solve*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx - x^2.$$

by separating variables (see Exercise 8.7). Instead, substitute $u = x^{-1}$ and show that u satisfies the linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 1 - ku.$$

Solve this equation for $u(t)$, and hence find the solution $x(t)$.

Making the substitution $u = x^{-1}$ we have $\mathrm{d}u/\mathrm{d}t = -x^{-2}\dot{x}$, and so

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{1}{x^2}[kx - x^2] = -\frac{k}{x} + 1 = 1 - ku.$$

This is

$$\frac{\mathrm{d}u}{\mathrm{d}t} + ku = 1$$

which we can solve with the integrating factor e^{kt},

$$\frac{\mathrm{d}}{\mathrm{d}t}(u\mathrm{e}^{kt}) = \mathrm{e}^{kt}.$$

Integrating both sides we get

$$u(t)\mathrm{e}^{kt} = \frac{\mathrm{e}^{kt}}{k} + c,$$

and so

$$u(t) = \frac{1}{k} + c\mathrm{e}^{-kt}.$$

Since $x(t) = u(t)^{-1}$ we have

$$x(t) = \frac{1}{k^{-1} + c\mathrm{e}^{-kt}} = \frac{k}{1 + ck\mathrm{e}^{-kt}}$$

which is the same answer as before (Exercise 8.7) if we set $A = ck$.

Exercise 10.6 *Use an appropriate substitution to solve the equation*

$$\dot{x} = x(\kappa^2 - x^2).$$

You should recover the solution (8.16) found by separating variables.

This is a Bernoulli equation: we can put $u = x^{-2}$ and then

$$\begin{aligned}\dot{u} &= -2x^{-3}\dot{x} \\ &= -2\kappa^2 x^{-2} + 2 \\ &= 2 - 2\kappa^2 u.\end{aligned}$$

Solving the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + 2\kappa^2 u = 2$$

using the integrating factor $\mathrm{e}^{2\kappa^2 t}$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(u\mathrm{e}^{2\kappa^2 t}) = 2\mathrm{e}^{2\kappa^2 t}$$

and so, integrating between times zero and t

$$u(t)\mathrm{e}^{2\kappa^2 t} = u(0) + \frac{\mathrm{e}^{2\kappa^2 t} - 1}{\kappa^2},$$

which gives

$$u(t) = u(0)\mathrm{e}^{-2\kappa^2 t} + \frac{1 - \mathrm{e}^{-2\kappa^2 t}}{\kappa^2}.$$

Since $u(t) = x(t)^{-2}$ we have

$$x(t)^{-2} = x_0^{-2}\mathrm{e}^{-2\kappa^2 t} + \frac{1 - \mathrm{e}^{-2\kappa^2 t}}{\kappa^2}.$$

Rearranging this for $x(t)$ gives

$$x(t) = \pm\sqrt{\frac{\kappa^2}{1 + \mathrm{e}^{-2\kappa^2 t}(\kappa^2 x_0^{-2} - 1)}},$$

the same solution we obtain using the method of separation of variables as (8.16).

11

Second order linear equations: general theory

Exercise 11.1 *By finding the Wronskian of the following pairs of solutions, show that they are linearly independent:*

(i) $x_1(t) = \mathrm{e}^{k_1 t}$ *and* $x_2(t) = \mathrm{e}^{k_2 t}$ *with* $k_1 \neq k_2$*,*
(ii) $x_1(t) = \mathrm{e}^{kt}$ *and* $x_2(t) = t\mathrm{e}^{kt}$*, and*
(iii) $x_1(t) = \mathrm{e}^{\rho t}\sin \omega t$ *and* $x_2(t) = \mathrm{e}^{\rho t}\cos \omega t$*.*

In each case we want to show that $W(x_1, x_2) \neq 0$. We have

(i)

$$W(\mathrm{e}^{k_1 t}, \mathrm{e}^{k_2 t}) = \mathrm{e}^{k_1 t} k_2 \mathrm{e}^{k_2 t} - \mathrm{e}^{k_2 t} k_1 \mathrm{e}^{k_1 t} = (k_2 - k_1)\mathrm{e}^{(k_1 + k_2)t};$$

(ii)

$$W(\mathrm{e}^{kt}, t\mathrm{e}^{kt}) = \mathrm{e}^{kt}(\mathrm{e}^{kt} + kt\mathrm{e}^{kt}) - t\mathrm{e}^{kt} k \mathrm{e}^{kt} = \mathrm{e}^{2kt};$$

and

(iii)

$$\begin{aligned} &W(\mathrm{e}^{\rho t}\sin \omega t, \mathrm{e}^{\rho t}\cos \omega t) \\ &\quad = \mathrm{e}^{\rho t}\sin \omega t(\rho \mathrm{e}^{\rho t}\cos \omega t \\ &\qquad\qquad -\omega \mathrm{e}^{\rho t}\sin \omega t) - \mathrm{e}^{\rho t}\cos \omega t(\rho \mathrm{e}^{\rho t}\sin \omega t + \omega \mathrm{e}^{\rho t}\cos \omega t) \\ &\quad = -\omega \mathrm{e}^{2\rho t}. \end{aligned}$$

In each case $W(x_1, x_2) \neq 0$, and so the two solutions are linearly independent.

Exercise 11.2 *Show that the Wronskian for two solutions* $x_1(t)$ *and* $x_2(t)$ *of the second order differential equation*

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + p_1(t)\frac{\mathrm{d}x}{\mathrm{d}t} + p_2(t)x = 0 \qquad \text{(S11.1)}$$

satisfies

$$\dot{W}(t) = -p_1(t)W(t).$$

(Write $W(t) = x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t)$, differentiate, and use the fact that $x_1(t)$ and $x_2(t)$ satisfy the equation (S11.1).) Deduce that either $W(t) = 0$ for all t, or $W(t) \neq 0$ for all t.

We have $W(t) = x_1\dot{x}_2 - x_2\dot{x}_1$, and so

$$\begin{aligned}\dot{W} &= \dot{x}_1\dot{x}_2 + x_1\ddot{x}_2 - \dot{x}_2\dot{x}_1 - x_2\ddot{x}_1 \\ &= x_1\ddot{x}_2 - x_2\ddot{x}_1 \\ &= x_1[-p_1\dot{x}_2 - p_2x_2] - x_2[-p_1\dot{x}_1 - p_2x_1] \\ &= -p_1[x_1\dot{x}_2 - x_2\dot{x}_1] \\ &= -p_1W.\end{aligned}$$

Solving this linear equation (use an integrating factor) for W gives

$$W(t) = W(0)\exp\left(\int_0^t -p_1(\tilde{t})\,\mathrm{d}\tilde{t}\right).$$

Since e^x is always positive, either $W(t)$ is never zero (if $W(0) \neq 0$) or $W(t)$ is identically zero (if $W(0) = 0$).

Exercise 11.3 *We have seen that if x_1 and x_2 are two solutions of a linear differential equation, then they are linearly independent if and only if their Wronskian is non-zero. The simple example of this question shows that this is not true for general functions that are not the solutions of some differential equation.*

(i) Check carefully that if $f(t) = t^2|t|$ then $\mathrm{d}f/\mathrm{d}t = 3t|t|$ (this is easy when $t \neq 0$; you will have to use the formal definition of the derivative at $t = 0$).

(ii) Let

$$f_1(t) = t^2|t| \qquad \text{and} \qquad f_2(t) = t^3.$$

Show that although these two functions are linearly independent on $\mathbb{R}$, their Wronskian is identically zero.

(i) For $t < 0$, $f(t) = -t^3$, so $\dot{f} = -3t^2 = 3t|t|$; for $t > 0$ similarly $f(t) = t^3$ and $\dot{f} = 3t^2 = 3t|t|$. At $t = 0$,

$$\dot{f}(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{h^2|h|}{h} = \lim_{h\to 0} h|h| = 0.$$

So $\dot{f} = 3t|t|$ as claimed.

(ii) We have $\dot{f}_1(t) = 3t|t|$ from part (i), and $\dot{f}_2(t) = 3t^2$. Therefore

$$W[f_1, f_2](t) = [t^2|t| \times 3t^2] - [t^3 \times 3t|t|] = 3t^4|t| - 3t^4|t| = 0.$$

However, it is clear that the functions are linearly independent on $\mathbb{R}$, since if

$$\alpha t^2|t| + \beta t^3 = 0 \qquad \text{for all} \qquad t \in \mathbb{R}$$

then taking any $t > 0$ implies that $\alpha = -\beta$, while taking any $t < 0$ gives $\alpha = \beta$, and so the only solution is $\alpha = \beta = 0$.

12

Homogeneous linear 2nd order equations with constant coefficients

Exercise 12.1 *Find the general solution of the following differential equations, and then the solution satisfying the specified initial conditions.*

(i) $\ddot{x} - 3\dot{x} + 2x = 0$ *with* $x(0) = 2$ *and* $\dot{x}(0) = 6$*;*
(ii) $y'' - 4y' + 4y = 0$ *with* $y(0) = 0$ *and* $y'(0) = 3$*;*
(iii) $z'' - 4z' + 13z = 0$ *with* $z(0) = 7$ *and* $z'(0) = 42$*;*
(iv) $\ddot{y} + \dot{y} - 6y = 0$ *with* $y(0) = -1$ *and* $\dot{y}(0) = 8$*;*
(v) $\ddot{y} - 4\dot{y} = 0$ *with* $y(0) = 13$ *and* $\dot{y}(0) = 0$*;*
(vi) $\ddot{\theta} + 4\theta = 0$ *with* $\theta(0) = 0$ *and* $\dot{\theta}(0) = 10$*;*
(vii) $\ddot{y} + 2\dot{y} + 10y = 0$ *with* $y(0) = 3$ *and* $\dot{y}(0) = 0$*;*
(viii) $2\ddot{z} + 7\dot{z} - 4z = 0$ *with* $z(0) = 0$ *and* $\dot{z}(0) = 9$*;*
(ix) $\ddot{y} + 2\dot{y} + y = 0$ *with* $y(0) = 0$ *and* $\dot{y}(0) = -1$*;*
(x) $\ddot{x} + 6\dot{x} + 10x = 0$ *with* $x(0) = 3$ *and* $\dot{x}(0) = 1$*;*
(xi) $4\ddot{x} - 20\dot{x} + 21x = 0$ *with* $x(0) = -4$ *and* $\dot{x}(0) = -12$*;*
(xii) $\ddot{y} + \dot{y} - 2y = 0$ *with* $y(0) = 4$ *and* $\dot{y}(0) = -4$*;*
(xiii) $\ddot{y} - 4y = 0$ *with* $y(0) = 10$ *and* $\dot{y}(0) = 0$*;*
(xiv) $y'' + 4y' + 4y = 0$ *with* $y(0) = 27$ *and* $y'(0) = -54$*; and*
(xv) $\ddot{y} + \omega^2 y = 0$ *with* $y(0) = 0$ *and* $\dot{y}(0) = 1$.

(i)

$$\ddot{x} - 3\dot{x} + 2x = 0 \qquad \text{with} \qquad x(0) = 2 \quad \text{and} \quad \dot{x}(0) = 6.$$

Try $x = \mathrm{e}^{kt}$ and obtain the auxiliary equation for k,

$$k^2 - 3k + 2 = 0$$

with solutions $k = 2$ and $k = 4$. The general solution is therefore

$$x(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{4t}.$$

Since $\dot{x}(t) = 2A\mathrm{e}^{2t} + 4B\mathrm{e}^{4t}$ the particular solution with

$$x(0) = A + B = 2 \qquad \text{and} \qquad \dot{x}(0) = 2A + 4B = 6$$

has $A = B = 1$ and is

$$x(t) = \mathrm{e}^{2t} + \mathrm{e}^{4t}.$$

(ii)

$$y'' - 4y' + 4y = 0 \qquad \text{with} \qquad y(0) = 0 \quad \text{and} \quad y'(0) = 3.$$

Try $y(x) = \mathrm{e}^{kx}$ to obtain the auxiliary equation

$$k^2 - 4k + 4 = 0$$

which has $k = 4$ as a repeated root. The general solution is therefore

$$y(x) = A\mathrm{e}^{4x} + Bx\mathrm{e}^{4x}.$$

Since $y'(x) = 4A\mathrm{e}^{4x} + B[\mathrm{e}^{4x} + 4x\mathrm{e}^{4x}]$ the solution with

$$y(0) = A + B = 0 \qquad \text{and} \qquad y'(0) = 4A + B = 3$$

has $A = 1$ and $B = -1$: it is

$$y(x) = (1 - x)\mathrm{e}^{4x}.$$

(iii)

$$z'' - 4z' + 13z = 0 \qquad \text{with} \qquad z(0) = 7 \quad \text{and} \quad z'(0) = 42.$$

Try $z(x) = \mathrm{e}^{kx}$ to obtain

$$k^2 - 4k + 13 = 0,$$

with solutions $k = 4 \pm 6\mathrm{i}$. The general solution is therefore

$$z(x) = \mathrm{e}^{4x}(A\cos 6x + B\sin 6x).$$

We have

$$\begin{aligned} z'(x) &= \mathrm{e}^{4x}(4A\cos 6x + 4B\sin 6x - 6A\sin 6x + 6B\cos 6x) \\ &= \mathrm{e}^{4x}[(4A + 6B)\cos 6x + (4B - 6A)\sin 6x], \end{aligned}$$

and so the solution with

$$z(0) = A + B = 7 \qquad \text{and} \qquad z'(0) = 4A + 6B = 42$$

has $A = 0$ and $B = 7$, and is

$$z(x) = 7\mathrm{e}^{4x}\sin 6x.$$

(iv)

$$\ddot{y} + \dot{y} - 6y = 0 \qquad \text{with} \qquad y(0) = -1 \quad \text{and} \quad \dot{y}(0) = 8.$$

Trying $y(t) = \mathrm{e}^{kt}$ gives the auxiliary equation

$$k^2 + k - 6 = 0$$

with solutions $k = 2$ or $k = -3$. So the general solution is

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-3t}.$$

Since $\dot{y}(t) = 2A\mathrm{e}^{2t} - 3B\mathrm{e}^{-3t}$ the solution with

$$y(0) = A + B = -1 \qquad \text{and} \qquad \dot{y}(0) = 2A - 3B = 8$$

has $A = 1$ and $B = -2$; it is

$$y(t) = \mathrm{e}^{2t} - 2\mathrm{e}^{-3t}.$$

(v)

$$\ddot{y} - 4y = 0 \qquad \text{with} \qquad y(0) = 13 \quad \text{and} \quad \dot{y}(0) = 0.$$

We try $y(t) = \mathrm{e}^{kt}$ then

$$k^2 - 4k = 0$$

with solutions $k = 0$ or $k = 4$. The general solution is

$$y(t) = A + B\mathrm{e}^{4t}.$$

We have $\dot{y}(t) = 4B\mathrm{e}^{4t}$. The initial conditions

$$y(0) = A + B = 13 \qquad \text{and} \qquad \dot{y}(0) = 4B = 0$$

imply that $A = 13$ and $B = 0$, so the solution is

$$y(t) = 13.$$

(vi)

$$\ddot{\theta} + 4\theta = 0 \qquad \text{with} \qquad \theta(0) = 0 \quad \text{and} \quad \dot{\theta}(0) = 10.$$

With $\theta(t) = \mathrm{e}^{kt}$ we have

$$k^2 + 4 = 0,$$

and so $k = \pm 2\mathrm{i}$. The general solution is

$$\theta(t) = A\cos 2t + B\sin 2t.$$

The derivative is given by $\dot{\theta} = -2A\sin 2t + 2B\cos 2t$. In order to satisfy

$$\theta(0) = A = 0 \qquad \text{and} \qquad \dot{\theta}(0) = 2B = 10$$

we must have $A = 0$ and $B = 5$, and the solution is

$$\theta(t) = 5\sin 2t.$$

(vii)

$$\ddot{y} + 2\dot{y} + 10y = 0 \qquad \text{with} \qquad y(0) = 3 \quad \text{and} \quad \dot{y}(0) = 0.$$

Try $y(t) = \mathrm{e}^{kt}$ and then

$$k^2 + 2k + 10 = 0$$

giving the complex conjugate roots $k = -1 \pm 3\mathrm{i}$ and the general solution

$$y(t) = \mathrm{e}^{-t}(A\cos 3t + B\sin 3t).$$

We have

$$\begin{aligned} \dot{y}(t) &= \mathrm{e}^{-t}(-A\cos 3t - B\sin 3t - 3A\sin 3t + 3B\cos 3t) \\ &= \mathrm{e}^{-t}[(3B - A)\cos 3t - (3A + B)\sin 3t]. \end{aligned}$$

The solution with

$$y(0) = A = 3 \qquad \text{and} \qquad \dot{y}(0) = 3B - A = 0$$

has $A = 3$ and $B = 1$, and is

$$y(t) = \mathrm{e}^{-t}(3\cos 3t + \sin 3t).$$

(viii)

$$2\ddot{z} + 7\dot{z} - 4z = 0 \qquad \text{with} \qquad z(0) = 0 \quad \text{and} \quad \dot{z}(0) = 9.$$

When we substitute $z(t) = \mathrm{e}^{kt}$ we obtain the equation

$$2k^2 + 7k - 4 = 0$$

for k. This equation has roots $k = 1/2$ and $k = -4$, so the general solution is

$$z(t) = A\mathrm{e}^{t/2} + B\mathrm{e}^{-4t}.$$

We have $\dot{z}(t) = (A/2)\mathrm{e}^{t/2} - 4B\mathrm{e}^{-4t}$, and so when

$$z(0) = A + B = 0 \qquad \text{and} \qquad \dot{z}(0) = (A/2) - 4B = 9$$

$A = 2$ and $B = -2$; the solution is

$$z(t) = 2(\mathrm{e}^{t/2} - \mathrm{e}^{-4t}).$$

(ix)

$$\ddot{y} + 2\dot{y} + y = 0 \qquad \text{with} \qquad y(0) = 0 \quad \text{and} \quad \dot{y}(0) = -1.$$

With $y(t) = \mathrm{e}^{kt}$ we obtain

$$k^2 + 2k + 1 = 0$$

and so $k = -1$ is a repeated root, giving the general solution

$$y(t) = A\mathrm{e}^{-t} + Bt\mathrm{e}^{-t}.$$

The derivative is $\dot{y}(t) = -A\mathrm{e}^{-t} + B(\mathrm{e}^{-t} - t\mathrm{e}^{-t})$, and so the solution with

$$y(0) = A = 0 \qquad \text{and} \qquad \dot{y}(0) = -A + B = -1$$

has $A = 0$ and $B = -1$: it is

$$y(t) = -t\mathrm{e}^{-t}.$$

(x)

$$\ddot{x} + 6\dot{x} + 10x = 0 \qquad \text{with} \qquad x(0) = 3 \quad \text{and} \quad \dot{x}(0) = 1.$$

We try $x(t) = \mathrm{e}^{kt}$ and then

$$k^2 + 6k + 10 = 0;$$

this implies that $k = -3 \pm \mathrm{i}$, and gives the general solution

$$x(t) = \mathrm{e}^{-3t}(A\cos t + B\sin t).$$

We have

$$\begin{aligned} \dot{x}(t) &= \mathrm{e}^{-3t}(-3A\cos t - 3B\sin t - A\sin t + B\cos t) \\ &= \mathrm{e}^{-3t}[(B - 3A)\cos t - (A + 3B)\sin t]. \end{aligned}$$

The solution with

$$x(0) = A = 3 \qquad \text{and} \qquad \dot{x}(0) = B - 3A = 1$$

has $A = 3$ and $B = 10$:

$$x(t) = \mathrm{e}^{-3t}(3\cos t + 10\sin t).$$

(xi)

$$4\ddot{x} - 20\dot{x} + 21x = 0 \qquad \text{with} \qquad x(0) = -4 \quad \text{and} \quad \dot{x}(0) = -12.$$

Substituting $x(t) = \mathrm{e}^{kt}$ produces the auxiliary equation

$$4k^2 - 20k + 21 = 0$$

with roots $k = 3/2$ and $k = 7/2$. The general solution is

$$x(t) = A\mathrm{e}^{3t/2} + B\mathrm{e}^{7t/2}.$$

We have $\dot{x}(t) = \frac{1}{2}[3A\mathrm{e}^{3t/2} + 7B\mathrm{e}^{7t/2}]$, and so for

$$x(0) = A + B = -4 \qquad \text{and} \qquad \dot{x}(0) = \frac{3A + 7B}{2} = -12$$

we need $A = -1$ and $B = -3$ and the solution is

$$x(t) = -\mathrm{e}^{3t/2} - 3\mathrm{e}^{7t/2}.$$

(xii)

$$\ddot{y} + \dot{y} - 2y = 0 \qquad \text{with} \qquad y(0) = 4 \quad \text{and} \quad \dot{y}(0) = -4.$$

Trying $y(t) = \mathrm{e}^{kt}$ gives the equation for k

$$k^2 - k - 2 = 0$$

with roots $k = 2$ and $k = -1$. The general solution is

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-t}.$$

With $\dot{y}(t) = 2A\mathrm{e}^{2t} - B\mathrm{e}^{-t}$ we need

$$y(0) = A + B = 4 \qquad \text{and} \qquad \dot{y}(0) = 2A - B = -4$$

i.e. $A = 0$ and $B = 4$, and so

$$y(t) = 4\mathrm{e}^{-t}.$$

(xiii)

$$\ddot{y} - 4y = 0 \qquad \text{with} \qquad y(0) = 10 \quad \text{and} \quad \dot{y}(0) = 0.$$

Try $y(t) = \mathrm{e}^{kt}$ and then

$$k^2 - 4 = 0$$

which gives $k = -2$ or $k = 2$ and the general solution

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-2t}.$$

The derivative is given by $\dot{y}(t) = 2A\mathrm{e}^{2t} - 2B\mathrm{e}^{-2t}$, and so for

$$y(0) = A + B = 10 \qquad \text{and} \qquad \dot{y}(0) = 2A - 2B = 0$$

we need $A = B = 5$ and the solution

$$y(t) = 5(\mathrm{e}^{2t} + \mathrm{e}^{-2t}).$$

(xiv)

$$y'' + 4y' + 4y = 0 \qquad \text{with} \qquad y(0) = 27 \quad \text{and} \quad y'(0) = -54.$$

Try $y(x) = \mathrm{e}^{kx}$ and obtain

$$k^2 + 4k + 4 = 0$$

with $k = -2$ a repeated root. The general solution is

$$y(x) = A\mathrm{e}^{-2x} + Bx\mathrm{e}^{-2x}.$$

We have $y'(x) = -2A\mathrm{e}^{-2x} + B(\mathrm{e}^{-2x} - 2x\mathrm{e}^{-2x})$, and so to match the initial conditions we need

$$y(0) = A = 27 \qquad \text{and} \qquad y'(0) = -2A + B = -54,$$

i.e. $A = 27$ and $B = 0$. The solution is

$$y(x) = 27\mathrm{e}^{-2x}.$$

(xv)

$$\ddot{y} + \omega^2 y = 0 \qquad \text{with} \qquad y(0) = 0 \quad \text{and} \quad \dot{y}(0) = 1.$$

Try $y(t) = \mathrm{e}^{kt}$ and the k must satisfy

$$k^2 + \omega^2 = 0,$$

so $k = \pm \mathrm{i}\omega$, and we obtain the general solution

$$y(t) = A\cos\omega t + B\sin\omega t.$$

Since $\dot{y}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$ we need

$$y(0) = A = 0 \qquad \text{and} \qquad \dot{y}(0) = -A\omega + B\omega = 1$$

which gives $A = 0$ and $B = 1/\omega$; the solution we require is

$$y(t) = \frac{\sin\omega t}{\omega}.$$

Exercise 12.2 *If the roots of the auxiliary equation are $k_1 > 0$ and $-k_2 < 0$ then the solution is*

$$x(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{-k_2 t}.$$

For most choices of initial conditions

$$x(0) = x_0 \qquad \dot{x}(0) = y_0$$

we will have $x(t) \to \pm\infty$ as $t \to \infty$. However, there are some special initial conditions for which $x(t) \to 0$ as $t \to \infty$. Find the relationship between x_0 and y_0 that ensures this.

The solution is

$$x(t) = A\mathrm{e}^{k_1 t} + B\mathrm{e}^{-k_2 t};$$

this will only tend to zero as $t \to \infty$ if $A = 0$, and then the solution is $x(t) = B\mathrm{e}^{-k_2 t}$ for some B. In this case, since $\dot{x}(t) = -k_2 B\mathrm{e}^{-k_2 t}$, we should have

$$x(0) = B \qquad \text{and} \qquad \dot{x}(0) = -k_2 B.$$

So the solution only tends to zero if $y_0 = -k_2 x_0$.

Exercise 12.3 *Solutions of linear equations with constant coefficients cannot blow up in finite time: it follows that their solutions exist for all $t \in \mathbb{R}$. To see this, we will consider*

$$\ddot{x} + p\dot{x} + qx = 0 \qquad \text{with} \qquad x(0) = x_0 \qquad \text{and} \qquad \dot{x}(0) = y_0$$

for $t \geq 0$ (a similar argument applies for $t \leq 0$). By setting $y = \dot{x}$, we can rewrite this as a coupled pair of first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -py - qx. \end{aligned}$$

Show that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) = (1 - q)xy - py^2,$$

and hence that

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2) \leq (1 + |q| + 2|p|)(x^2 + y^2).$$

Using the result of Exercise 9.7 deduce that for $t \geq 0$

$$x(t)^2 + y(t)^2 \leq (x(0)^2 + y(0)^2)\mathrm{e}^{(1+|q|+2|p|)t},$$

showing that finite-time blowup is impossible. Hint: $xy \leq \frac{1}{2}(x^2+y^2)$*. (The same argument works, essentially unchanged, for*

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0$$

provided that $|p(t)| \leq p$ *and* $|q(t)| \leq q$ *for all* $t \in \mathbb{R}$*.)*

We have

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(x^2+y^2) &= \tfrac{1}{2}(2x\dot{x} + 2y\dot{y}) \\ &= x\dot{x} + y\dot{y} \\ &= xy + y(-py - qx) \\ &= (1-q)xy - py^2. \end{aligned}$$

Now it follows (using the hint) that

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(x^2+y^2) &\leq (1+|q|)\frac{x^2+y^2}{2} + |p|y^2 \\ &\leq \left(\frac{1+|q|}{2} + |p|\right)(x^2+y^2), \end{aligned}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2+y^2) \leq (1+|q|+2|p|)(x^2+y^2).$$

It follows using the result of Exercise 9.7 that

$$x(t)^2 + y(t)^2 \leq (x(0)^2 + y(0)^2)\mathrm{e}^{(1+|q|+2|p|)t},$$

and so $x(t)$ and $\dot{x}(t)$ remain bounded for all $t \geq 0$.

13

Oscillations

Exercise 13.1 *A spring of natural length l and spring constant k is suspended vertically from a fixed point, and a weight of mass m attached. If the system is at rest ($\ddot{x} = \dot{x} = 0$) how far has the spring extended? If the mass is pulled down slightly from this rest position and then released, show that it then oscillates about its equilibrium position with period $2\pi/\omega$, where $\omega^2 = k/m$.*

Denoting by x the extension from the spring's natural length, then the forces on the spring are that due to the weight of the mass, mg, and the restoring force $-kx$ from the spring, so

$$m\ddot{x} = -kx + mg.$$

When the spring is at rest we have $\ddot{x} = 0$ and so $x = mg/k$. If we write $x = y + (mg/k)$ then y satisfies

$$m\ddot{y} = -ky.$$

The solution of this can either be found by trying $y(t) = \mathrm{e}^{\alpha t}$ which gives $m\alpha^2 = -k$ and so $\alpha = \pm\mathrm{i}\sqrt{k/m}$; or preferably you should just recognise the equation as giving rise to simple harmonic motion. Whichever way, the solution is

$$y(t) = A\cos\sqrt{k/m}\,t + B\sin\sqrt{k/m}\,t,$$

and the spring oscillates about its equilibrium position with period $2\pi/\omega$ with $\omega^2 = k/m$ as claimed.

Exercise 13.2 *The acceleration due to gravity in fact depends on the distance R from the centre of the earth: $g = GM/R^2$, where M is the mass of the earth and G Newton's gravitational constant. Show that the period of oscillation of a pendulum will increase as it is taken higher.*

The equation of motion for a pendulum is (cf. (13.5))

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\theta \qquad \text{with} \qquad \omega^2 = \frac{g}{L}.$$

The period of oscillation is $2\pi/\omega$, i.e.

$$2\pi\sqrt{\frac{L}{g}}.$$

Since $g = GM/R^2$ decreases as R increases, it follows that the period will increase as the pendulum moves further from the centre of the earth.

Exercise 13.3 *The earth bulges at the equator: at a latitude θ, the distance to the centre of the earth (measured in kilometres) is approximately*

$$R(\theta) = \sqrt{R_\mathrm{e}^2 \cos^2\theta + R_\mathrm{p}^2 \sin\theta},$$

where $R_\mathrm{e} = 6378$ and $R_\mathrm{p} = 6357$.

I decide to move from Leamington Spa, at a latitude of $52°$, to Seville, which is lies at a latitude of $37°$. My grandfather clock, which keeps perfect time, has a pendulum of length 75 cm. How long would the pendulum need to be to keep perfect time in Seville?

The approximate distance of Leamington from the centre of the earth is 6365 km, while for Seville the Figure is roughly 6370 km. To keep the period of oscillations constant I would need to keep L/g constant. Since g is proportional to R^{-2}, this is the same as keeping LR^2 constant, i.e.

$$L_\mathrm{Leam}R_\mathrm{Leam}^2 = L_\mathrm{Sev}R_\mathrm{Sev}^2,$$

which gives

$$L_\mathrm{Sev} = 75 \times \frac{6365^2}{6370^2}\,\mathrm{cm} = 74.88\,\mathrm{cm}$$

a minimal adjustment.

Exercise 13.4 *The buoyancy force on an object is equal to the weight of water that it displaces: if an object has mass M and displaces a volume V of water then the forces on it are $Mg - Vg$, in units where the density of water is one; see Figure 13.1.*

A bird of mass m is sitting on a cylindrical buoy of density ρ, radius R, and height h, which is floating at rest. How much of the buoy lies below the surface?

The bird flies away. Show that the buoy now bobs up and down, with

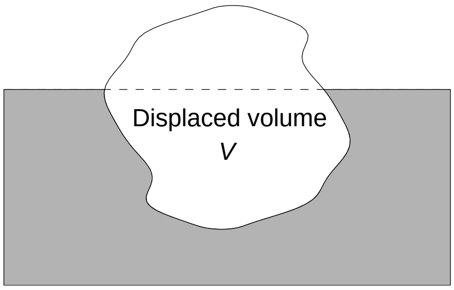

Fig. 13.1. The buoyancy force on an object is equal to the weight of water that it displaces.

the amount below the surface oscillating about ρh with period $2\pi\sqrt{\rho h/g}$ and amplitude $m/\pi R^2$.

Suppose that the buoy is immersed to a depth y. While the bird is there the forces on the buoy are a buoyancy force $\pi R^2 yg$ and gravitational forces due to the mass of the buoy ($M = \rho\pi R^2 h$) and the bird (m). So y obeys the equation

$$(m+M)\ddot{y} = (M+m)g - \pi R^2 yg.$$

At rest y is constant, and so $\ddot{y} = 0$, and so

$$y = \frac{M+m}{\pi R^2}.$$

When the bird leaves the equation of motion becomes

$$M\ddot{y} = Mg - \pi R^2 yg.$$

The new equilibrium depth of immersion is $y = M/\pi R^2 = \rho h$; the initial condition when the bird flies away is

$$y(0) = \frac{M+m}{\pi R^2} = \rho h + \frac{m}{\pi R^2}$$

and $\dot{y}(0) = 0$. The displacement from the equilibrium position, $z = y - \rho h$, satisfies

$$M\ddot{z} = -(\pi R^2 g)z.$$

The general solution of this equation is

$$z(t) = A\cos\omega t + B\sin\omega t,$$

with $\omega = \sqrt{\pi R^2 g/M} = \sqrt{g/\rho h}$. The particular solution satisfying the initial conditions

$$z(0) = m/\pi R^2 \qquad \text{and} \qquad \dot{z}(0) = 0$$

is

$$z(t) = \frac{m}{\pi R^2}\cos\omega t.$$

Therefore the buoy oscillations about $y = \rho h$ with amplitude $m/\pi R^2$ and frequency $2\pi\sqrt{\rho h/g}$, as claimed.

Exercise 13.5 *An open tin can, half full of water, is floating in a canal. The can is 11cm tall, has a diameter of 7.5 cm, and has a mass of 50 . Show that at rest the can is submerged a distance of approximately 6.63 cm below the surface of the canal. If the can is pushed down further it will then perform oscillations about its equilibrium position. Show that the can bobs up and down every 0.21 seconds (a little under five times per second). The acceleration due to gravity is approximately 9.8 m/s^2=980 cm/s^2; the density of water is 1 g/cm^3. You can check your answers in a sink with a baked bean can.*

Denote by y the amount in excess of half the height of the can that is submerged. The forces on the can are then $50g$ downwards and a buoyancy force of $\pi(7.5/2)^2 yg$. So, since $(7.5/2)^2 = 14.0625$,

$$50\ddot{y} = 50g - 14.0625\pi g y,$$

and the equilibrium position is

$$y_{\mathrm{e}} = \frac{50}{14.0625\pi} \approx 1.13,$$

which means that the can is submerged a distance of roughly 6.63 cm below the surface.

If $z = y - y_{\mathrm{e}}$ then z satisfies

$$50\ddot{z} = -14.0625\pi g z,$$

and so the can oscillates with period

$$2\pi/\sqrt{14.0625\pi g/50} \approx 0.21s.$$

[You need to use g in the correct units, 980 cm/s^2.]

Exercise 13.6 *A right circular cone, of height h, radius ρ, and with base radius R, is placed point downward in a lake. Assuming that the apex remains point vertically downwards, show that if the cone is submerged to a depth x then*

$$\ddot{x} = g - \left(\frac{x}{h}\right)^3 \frac{g}{\rho}.$$

(You need not solve this equation.) At equilibrium how far is the cone submerged?

When the cone is submerged to a height x the volume submerged is

$$\frac{1}{3}\pi[(x/h)R]^2 x.$$

Then

$$\frac{1}{3}\pi R^2 h\rho\ddot{x} = \frac{1}{3}\pi R^2 h\rho g - \frac{1}{3}\pi[(x/h)R]^2 xg,$$

which simplifies to give

$$\ddot{x} = g - \left(\frac{x}{h}\right)^3 \frac{g}{\rho}.$$

In equilibrium we must have

$$g = \left(\frac{x}{h}\right)^3 \frac{g}{\rho},$$

and so $x = h\rho^{1/3}$.

Exercise 13.7 *A dashpot is a device designed to add damping to a system, consisting essentially of a plunger in a cylinder of liquid or gas, see Figure 13.2.*

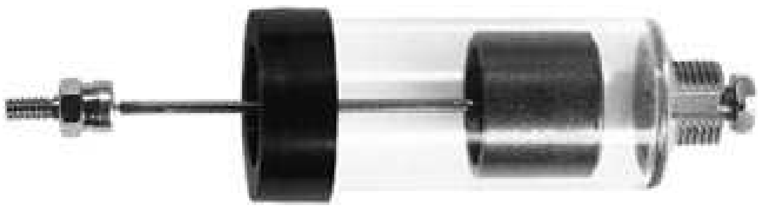

Fig. 13.2. A dashpot. Illustration © 2001 Airpot Corporation. Airpot is a registered trademark of Airpot Corporation.

It produces a resisting force proportional to the velocity, precisely the kind of 'damping' that we used in our model

$$m\ddot{x} + \mu\dot{x} + kx = 0, \tag{S13.1}$$

with μ indicating the 'strength' of the dashpot. Dashpots are used in a variety of applications, for example, cushioning the opening mechanism on a tape recorder, or in car shock absorbers.

A mass-spring-dashpot system consists of a mass attached to a spring and a dashpot, as shown in Figure 13.3. A weight of mass 10 kg is attached to a spring with spring constant 5, and to a dashpot of strength μ. How strong should the dashpot be to ensure that the system is over-damped? What would the period of oscillations be if $\mu = 14$?

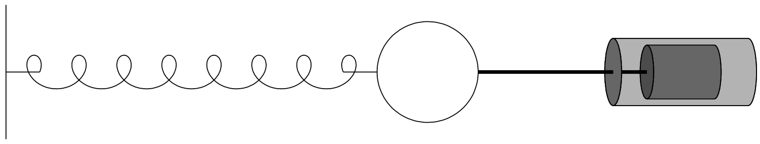

Fig. 13.3. A mass-spring-dashpot system.

Trying $x = e^{\alpha t}$ in the model

$$m\ddot{x} + \mu\dot{x} + kx = 0$$

yields the auxiliary equation for α,

$$m\alpha^2 + \mu\alpha + k = 0,$$

with solutions

$$\alpha = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m}.$$

For the system to be overdamped we require

$$\mu^2 - 4mk > 0,$$

so we need $\mu > 2\sqrt{50} \approx 14.14$. If $\mu = 14$ then the system is under-damped: we would have

$$\alpha = \frac{-14 \pm \sqrt{196 - 200}}{20} = \frac{-14 \pm 2i}{20}$$

giving oscillations that decay like $e^{-0.7t}$ and have period $2\pi/\sqrt{0.1} \approx 19.9$ seconds.

Exercise 13.8 *When first opened, the Millennium Bridge in London (see Figure 13.4) wobbled from side to side as people crossed; you can see this on video at* `www.arup.com/MillenniumBridge`. *Footfalls created small side-to-side movements of the bridge, which were then enhanced by the tendency of* *people to adjust their steps to compensate for the wobbling. With more than a critical number of pedestrians (around 160) the bridge began to wobble violently*[1].

Without any pedestrians, the displacement x of a representative point on the bridge away from its normal position would satisfy

$$M\ddot{x} + k\dot{x} + \lambda x = 0,$$

where

$$M \approx 4 \times 10^5 \,\text{kg}, \quad k \approx 5 \times 10^4 \,\text{kg/s}, \quad \text{and} \quad \lambda \approx 10^7 \,\text{kg/s}^2.$$

Show that the level of damping here is only around 1% *of the critical level.*

Fig. 13.4. The Millennium Bridge in London (courtesy of Arup).

The effective forcing from each pedestrian was found by experiment (which involved varying numbers of people walking across the bridge) to be propor-

[1] A detailed analysis is given in P. Dallard *et al.*, The London Millennium Footbridge, *The Structural Engineer* **79** (2001) 17–33.

tional to $\dot{x}$, with

$$F \approx 300\dot{x}.$$

If there are N pedestrians, the displacement of the bridge satisfies

$$M\ddot{x} + k\dot{x} + \lambda x = 300N\dot{x}.$$

Find the critical number N_0 of pedestrians, such that if there are more than N_0 pedestrians the bridge is no longer damped. Show that if there are 200 pedestrians then there will be oscillations with a frequency of approximately 0.8 Hertz (oscillations per second) and with an amplitude which grows as $e^{t/80}$.

The problem was corrected by adding additional damping, a large part of which was essentially a collection of dashpots, in order to bring the damping up to 20% of the critical level. What would this do to the value of k, and how many people can now walk across the bridge without counteracting all the damping?

Trying $x = e^{\alpha t}$ in the equation gives

$$M\alpha^2 + k\alpha + \lambda = 0,$$

and so

$$\alpha = \frac{-k \pm \sqrt{k^2 - 4M\lambda}}{2M}.$$

The critical level of damping is when $k^2 = 4M\lambda$, i.e. when

$$k = \sqrt{4M\lambda} = 4 \times 10^6.$$

The value of k here, 5×10^4, is 1.25% of the critical level.

The forcing term $300N\dot{x}$ decreases the effect of the damping term $k\dot{x}$. The two terms would balance if

$$300N = 5 \times 10^4$$

which gives $N = 166\frac{2}{3}$. So with 167 pedestrians the bridge is no longer damped.

If there are 200 pedestrians then the equation of motion is

$$M\ddot{x} + \underbrace{(k - 6 \times 10^4)}_{\tilde{k}}\dot{x} + \lambda x = 0.$$

Trying $x(t) = e^{\alpha t}$ yields the auxiliary equation

$$M\alpha^2 + \tilde{k}\alpha + \lambda = 0,$$

with roots

$$\alpha = \frac{-\tilde{k} \pm \sqrt{\tilde{k}^2 - 4M\lambda}}{2M} \approx \frac{1}{80} \pm 5\mathrm{i},$$

and hence solutions

$$x(t) = \mathrm{e}^{t/80}(A\cos 5t + B\sin 5t).$$

The oscillations have frequency $5/2\pi \approx 0.8$Hz, and their amplitude grows like $\mathrm{e}^{t/80}$.

With the damping at 20% of the critical level, i.e. k is now 8×10^5, the damping is now exactly matched if

$$300N = 8 \times 10^5,$$

i.e. $N = 2666\frac{2}{3}$. So it will now take 2667 pedestrians to counteract all the damping, and 2666 should still be able to cross without the wobble setting in.

Exercise 13.9 *In the case of critical damping (see Section 13.3), the general solution of (13.6) is of the form*

$$x(t) = (A + Bt)\mathrm{e}^{-\lambda t/2}.$$

Show that if $\lambda A < 2B$ then $x(t)$ increases initially, reaching its maximum value at

$$t = \frac{2}{\lambda} - \frac{A}{B}.$$

If $x(t) = (A + Bt)\mathrm{e}^{-\lambda t/2}$ then we have

$$\dot{x}(t) = \mathrm{e}^{-\lambda t/2}\left(B - \frac{\lambda}{2}(A + Bt)\right),$$

and so $\dot{x} > 0$ while

$$B - \frac{\lambda}{2}(A + Bt) > 0,$$

i.e. while

$$t < \frac{2}{\lambda} - \frac{A}{B}.$$

14

Inhomogeneous 2nd order linear equations

Exercise 14.1 *Find the general solution to the following differential equations (the homogeneous parts of the equations are all treated in Exercise 12.1) In part (n) also find the one solution that has* $x(0) = n$ *and* $\dot{x}(0) = 0$.

(i) $\ddot{x} - 4x = t^2$,
(ii) $\ddot{x} - 4\dot{x} = t^2$,
(iii) $\ddot{x} + \dot{x} - 2x = 3\mathrm{e}^{-t}$,
(iv) $\ddot{x} + \dot{x} - 2x = \mathrm{e}^{t}$,
(v) $\ddot{x} + 2\dot{x} + x = \mathrm{e}^{-t}$,
(vi) for $\alpha \neq \omega$: $\ddot{x} + \omega^2 x = \sin \alpha t$,
(vii) for $\alpha = \omega$: $\ddot{x} + \omega^2 x = \sin \alpha t$,
(viii) $\ddot{x} + 2\dot{x} + 10x = \mathrm{e}^{-t}$,
(ix) $\ddot{x} + 2\dot{x} + 10x = \mathrm{e}^{-t} \cos 3t$,
(x) $\ddot{x} + 6\dot{x} + 10x = \mathrm{e}^{-3t} \cos t$, *and*
(xi) $\ddot{x} + 4\dot{x} + 4x = \mathrm{e}^{2t}$.

The solutions of the homogeneous equation can all be found in the answers to Exercise 12.1.

(i)

$$\ddot{x} - 4x = t^2.$$

The complementary function is the general solution of the homogeneous equation $\ddot{y} - 4y = 0$, i.e.

$$y(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{-2t}$$

(see part (xiii) of Exercise 12.1).

The function on the right-hand side, t^2, is a second-order polynomial.

Neither t^2 nor any of its derivatives are solutions of the homogeneous equation, so for a particular integral we try a general second order polynomial, $x_{\rm p}(t) = at^2 + bt + c$. Then

$$\dot{x}_{\rm p} = 2at + b \qquad \text{and} \qquad \ddot{x}_{\rm p} = 2a,$$

so we need

$$2a - 4(at^2 + bt + c) = t^2.$$

This requires $a = -1/4$, $b = 0$, and $c = -1/8$. So the particular integral is

$$x_{\rm p}(t) = -\frac{t^2}{4} - \frac{1}{8},$$

and the general solution is

$$x(t) = A{\rm e}^{2t} + B{\rm e}^{-2t} - \frac{t^2}{4} - \frac{1}{8}.$$

If we want $x(0) = 1$ and $\dot{x}(0) = 0$ then, since

$$\dot{x}(t) = 2A{\rm e}^{2t} - 2B{\rm e}^{-2t} - \tfrac{1}{2}t$$

we need

$$1 = x(0) = A + B - \frac{1}{8} \qquad \text{and} \qquad 0 = \dot{x}(0) = 2A - 2B,$$

i.e. $A = B = 9/16$, and so

$$x(t) = 9({\rm e}^{2t} + {\rm e}^{-2t})/16 - \frac{t^2}{4} - \frac{1}{8}.$$

(ii)

$$\ddot{x} - 4\dot{x} = t^2.$$

The complementary function is the general solution of the homogeneous equation $\ddot{y} - 4\dot{y} = 0$, which is

$$y(t) = A + B{\rm e}^{4t}$$

(see part (v) of Exercise 12.1). If we were to try, as before, a second-order polynomial for a particular integral then we would have the problem that one of its derivatives (the second) would be a constant – and this is now a solution of the homogeneous equation. So we have to multiply by an extra factor t, and try $x_{\rm p}(t) = at^3 + bt^2 + ct$. So we have

$$\dot{x}_{\rm p} = 3at^2 + 2bt + c \qquad \text{and} \qquad \ddot{x}_{\rm p} = 6at + 2b.$$

Substituting in we get

$$6at + 2b - 4(3at^2 + 2bt + c) = t^2,$$

requiring $a = -1/12$, $6a - 8b = 0$ so $b = -1/16$, and $2b - 4c = 0$ so that $c = -1/32$, giving

$$x_{\mathrm{p}}(t) = -\frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32}.$$

The general solution is therefore

$$x(t) = A + B\mathrm{e}^{4t} - \frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32}.$$

For the solution with $x(0) = 2$ and $\dot{x}(0) = 0$ we need

$$A + B = 2 \qquad \text{and} \qquad 4B - \frac{1}{32} = 0,$$

so $B = 1/128$ and $A = 255/128$, and

$$x(t) = \frac{1}{128}(255 + \mathrm{e}^{4t}) - \frac{t^3}{12} - \frac{t^2}{16} - \frac{t}{32}.$$

(iii)

$$\ddot{x} + \dot{x} - 2x = 3\mathrm{e}^{-t}.$$

The complementary function is the general solution of the homogeneous equation $\ddot{y} + \dot{y} - 2y = 0$,

$$y(t) = A\mathrm{e}^{t} + B\mathrm{e}^{-2t}$$

(see part (xii) of Exercise 12.1). The function on the right hand side is not a solution of the homogeneous equation, so we can just try $x_{\mathrm{p}}(t) = a\mathrm{e}^{-t}$. We have

$$\dot{x}_{\mathrm{p}} = -a\mathrm{e}^{-t} \qquad \text{and} \qquad \ddot{x}_{\mathrm{p}} = a\mathrm{e}^{-t},$$

so we want

$$a\mathrm{e}^{-t} - a\mathrm{e}^{-t} - 2a\mathrm{e}^{-t} = 3\mathrm{e}^{-t}$$

requiring $a = -3/2$ and giving the particular integral

$$x_{\mathrm{p}}(t) = -\frac{3}{2}\mathrm{e}^{-t}$$

and the general solution

$$x(t) = A\mathrm{e}^{t} + B\mathrm{e}^{-2t} - \frac{3}{2}\mathrm{e}^{-t}.$$

To find the solution with $x(0) = 3$ and $\dot{x}(0) = 0$ we need, since, $\dot{x} = Ae^t - 2Be^{-2t} + (3e^{-t}/2)$,

$$A + B - \frac{3}{2} = 3 \qquad \text{and} \qquad A - 2B + \frac{3}{2} = 0.$$

This gives $A = 5/2$ and $B = 2$, and so

$$x(t) = \frac{5}{2}e^t + 2e^{-2t} - \frac{3}{2}e^{-t}.$$

(iv)

$$\ddot{x} + \dot{x} - 2x = e^t.$$

The complementary function is as in part (iii), but now the right-hand side is a solution of the homogeneous equation, so we have to try $x_p(t) = ate^t$ for a particular integral. For this guess

$$\dot{x}_p = ae^t + ate^t \qquad \text{and} \qquad \ddot{x}_p = 2ae^t + ate^t,$$

and so we want

$$2ae^t + ate^t + ae^t + ate^t - 2ate^t = e^t,$$

requiring $a = 1/3$. So the particular integral is $x_p(t) = te^t/3$ and the general solution is

$$x(t) = Ae^t + Be^{-2t} + \frac{te^t}{3}.$$

For the solution with $x(0) = 4$ and $\dot{x}(0) = 0$ we require, since $\dot{x} = Ae^t + Be^{-2t} + (e^t/3) + (te^t/3)$,

$$A + B = 4 \qquad A - 2B + \frac{1}{3} = 0,$$

and so $A = 23/9$ and $B = 13/9$, giving

$$x(t) = \frac{23e^t + 13e^{-2t} + 3te^t}{9}.$$

(v)

$$\ddot{x} + 2\dot{x} + x = e^{-t}.$$

The complementary function is the general solution of the homogeneous equation $\ddot{y} + 2\dot{y} + y = 0$, which is

$$y(t) = Ae^{-t} + Bte^{-t}$$

(part (ix) of Exercise 12.1). Now observe that the right-hand side e^{-t} is a solution of the homogeneous equation, and so is te^{-t}. Therefore we have to use one power of t more, and try $x_{\rm p}(t) = at^2{\rm e}^{-t}$. For this guess we have

$$\dot{x}_{\rm p} = 2at{\rm e}^{-t} - at^2{\rm e}^{-t} \qquad \text{and} \qquad \ddot{x}_{\rm p} = 2a{\rm e}^{-t} - 4at{\rm e}^{-t} + at^2{\rm e}^{-t}.$$

Substituting in gives

$$2a{\rm e}^{-t} - 4at{\rm e}^{-t} + at^2{\rm e}^{-t} + 2(2at{\rm e}^{-t} - at^2{\rm e}^{-t}) + at^2{\rm e}^{-t} = {\rm e}^{-t};$$

notice that the terms with $t{\rm e}^{-t}$ and $t^2{\rm e}^{-t}$ cancel, leaving just $2a{\rm e}^{-t}$, and so we want $a = 1/2$. So the particular integral is $x_{\rm p}(t) = t^2{\rm e}^{-t}/2$, and the general solution is

$$x(t) = \left(A + Bt + \frac{t^2}{2}\right){\rm e}^{-t}.$$

We have

$$\dot{x}(t) = \left(-A + B + (1 - B)t - \frac{t^2}{2}\right){\rm e}^{-t},$$

and so the solution with $x(0) = 5$ and $\dot{x}(0) = 0$ must have

$$A = 5 \qquad \text{and} \qquad -A + B = 0.$$

This gives $A = B = 5$, and so

$$x(t) = (5 + 5t + \tfrac{1}{2}t^2){\rm e}^{-t}.$$

(vi)

$$\ddot{x} + \omega^2 x = \sin \alpha t.$$

The complementary function is the general solution of the homogeneous problem $\ddot{y} + \omega^2 y = 0$,

$$y(t) = A\cos\omega t + B\sin\omega t$$

(part (xv) of Exercise 12.1). The function on the right-hand side is not a solution of the homogeneous equation, so we would in general try a combination of $\sin\alpha t$ and $\cos\alpha t$. However, we can note here that there is no $\dot{x}$ term on the left-hand side, and it is such a term that would generally make us include a cosine term (since the derivative of $\sin\alpha t$ would give a term in $\cos\alpha t$). So here we just try $x_{\rm p}(t) = a\sin\alpha t$. Since $\ddot{x}_{\rm p} = -a\alpha^2\sin\alpha t$ we need

$$-a\alpha^2\sin\alpha t + \omega^2 a\sin\alpha t = \sin\alpha t,$$

or $a = 1/(\omega^2 - \alpha^2)$. This gives the particular integral

$$x_{\mathrm{p}}(t) = \frac{\sin \alpha t}{\omega^2 - \alpha^2}$$

and the general solution

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{\sin\alpha t}{\omega^2 - \alpha^2}.$$

We have

$$\dot{x}(t) = -A\omega\sin\omega t + B\omega\cos\omega t + \frac{\alpha\cos\alpha t}{\omega^2 - \alpha^2},$$

and so for $x(0) = 6$ and $\dot{x}(0) = 0$ we need

$$A = 6 \qquad \text{and} \qquad B\omega + \frac{\alpha}{\omega^2 - \alpha^2} = 0.$$

This gives $A = 6$, $B = -\alpha/[\omega(\omega^2 - \alpha^2)]$, and so this solution is

$$x(t) = 6\cos\omega t + \frac{\omega\sin\alpha t - \alpha\sin\omega t}{\omega(\omega^2 - \alpha^2)}.$$

(vii) We now have

$$\ddot{x} + \omega^2 x = \sin\omega t.$$

We cannot try $\sin\omega t$ since this is a solution of the homogeneous equation, so we have to try a combination of $t\sin\omega t$ and $t\cos\omega t$, $x_{\mathrm{p}}(t) = at\sin\omega t + bt\cos\omega t$. Then

$$\dot{x}_{\mathrm{p}} = a\sin\omega t + a\omega t\cos\omega t + b\cos\omega t - b\omega t\sin\omega t$$

and

$$\ddot{x}_{\mathrm{p}} = 2a\omega\cos\omega t - a\omega^2 t\sin\omega t - b\omega\sin\omega t - 2b\omega^2 t\cos\omega t.$$

Substituting in we get

$$\begin{aligned} 2a\omega\cos\omega t - a\omega^2 t\sin\omega t - 2b\omega\sin\omega t - b\omega^2 t\cos\omega t \\ + a\omega^2 t\sin\omega t + b\omega^2 t\cos\omega t = \sin\omega t; \end{aligned}$$

the terms with a factor of t cancel, leaving

$$2a\omega\cos\omega t - 2b\omega\sin\omega t = \sin\omega t$$

requiring $a = 0$ and $b = -1/2\omega$. So the particular integral is $x_{\mathrm{p}}(t) = -(t\cos\omega t)/2$ and the general solution is

$$x(t) = A\cos\omega t + B\sin\omega t - \frac{1}{2\omega}t\cos\omega t.$$

The solution with $x(0) = 7$ and $\dot{x}(0) = 0$ must have

$$A = 7 \qquad \text{and} \qquad B\omega - \frac{1}{2} = 0,$$

so $A = 7$, $B = 1/2\omega$, and the solution is

$$x(t) = 7\cos\omega t + \frac{\sin\omega t - t\cos\omega t}{2\omega}.$$

(viii)

$$\ddot{x} + 2\dot{x} + 10x = \mathrm{e}^{-t}.$$

The complementary function is the general solution

$$y(t) = \mathrm{e}^{-t}(A\cos 3t + B\sin 3t).$$

of the homogeneous problem $\ddot{y} + 2\dot{y} + 10y = 0$ (part (vii) of Exercise 12.1). Now to get the correct right-hand side (e^{-t}) we note that although this is involved in the complementary function, it is not a solution of the homogeneous equation, so we can just try $x_{\mathrm{p}}(t) = a\mathrm{e}^{-t}$: we have

$$\dot{x}_{\mathrm{p}} = -a\mathrm{e}^{-t} \qquad \text{and} \qquad \ddot{x}_{\mathrm{p}} = a\mathrm{e}^{-t},$$

so we need

$$a\mathrm{e}^{-t} - 2a\mathrm{e}^{-t} + 10a\mathrm{e}^{-t} = \mathrm{e}^{-t},$$

and this requires $a = 1/9$. The particular integral is $x_{\mathrm{p}}(t) = \mathrm{e}^{-t}/9$, and the general solution is

$$x(t) = \mathrm{e}^{-t}\left(A\cos 3t + B\sin 3t + \frac{1}{9}\right).$$

We have

$$\dot{x}(t) = \mathrm{e}^{-t}\left((-3A - B)\sin 3t + (3B - A)\cos 3t - \frac{1}{9}\right),$$

and so for $x(0) = 8$ and $\dot{x}(0) = 0$ we want

$$A + \frac{1}{9} = 8 \qquad \text{and} \qquad 3B - A - \frac{1}{9} = 0$$

which implies that $A = 71/9$ and $B = 8/3$. Therefore

$$x(t) = \frac{\mathrm{e}^{-t}}{9}(71\cos 3t + 24\sin 3t + 1).$$

(ix)

$$\ddot{x} + 2\dot{x} + 10x = \mathrm{e}^{-t}\cos 3t.$$

The complementary function is as in part (viii), and you can see that now the right-hand side is a solution of the homogeneous equation. So in fact we need to try something like $at\mathrm{e}^{-t}\cos 3t$. But differentiating this will give similar terms with the cosine replaced by a sine, so we have to try

$$x_\mathrm{p}(t) = at\mathrm{e}^{-t}\cos 3t + bt\mathrm{e}^{-t}\sin 3t.$$

So

$$\begin{aligned}\dot{x}_\mathrm{p} &= a\mathrm{e}^{-t}\cos 3t - at\mathrm{e}^{-t}\cos 3t - 3at\mathrm{e}^{-t}\sin 3t + b\mathrm{e}^{-t}\sin 3t \\ &\quad -bt\mathrm{e}^{-t}\sin 3t + 3bt\mathrm{e}^{-t}\cos 3t \\ &= a\mathrm{e}^{-t}\cos 3t + (3b-a)t\mathrm{e}^{-t}\cos 3t - (3a+b)t\mathrm{e}^{-t}\sin 3t \\ &\quad +b\mathrm{e}^{-t}\sin 3t\end{aligned}$$

and

$$\begin{aligned}\ddot{x}_\mathrm{p} &= -a\mathrm{e}^{-t}\cos 3t - 3a\mathrm{e}^{-t}\sin 3t \\ &\quad +(3b-a)[\mathrm{e}^{-t}\cos 3t - t\mathrm{e}^{-t}\cos 3t - 3t\mathrm{e}^{-t}\sin 3t] \\ &\quad -(3a+b)[\mathrm{e}^{-t}\sin 3t - t\mathrm{e}^{-t}\sin 3t + 3t\mathrm{e}^{-t}\cos 3t] \\ &\quad -b\mathrm{e}^{-t}\sin 3t + 3b\mathrm{e}^{-t}\cos 3t \\ &= (6b-2a)\mathrm{e}^{-t}\cos 3t - (8a+6b)t\mathrm{e}^{-t}\cos 3t + (6a-8b)t\mathrm{e}^{-t}\sin 3t \\ &\quad -(6a+2b)\mathrm{e}^{-t}\sin 3t.\end{aligned}$$

Substituting in we get, after careful tidying of terms

$$6b\mathrm{e}^{-t}\cos 3t - 6a\mathrm{e}^{-t}\sin 3t = \mathrm{e}^{-t}\cos 3t,$$

and so we need $a = 0$ and $b = -1/6$. The particular integral is $x_\mathrm{p}(t) = -(t\mathrm{e}^{-t}\cos 3t)/6$, and the general solution is

$$x(t) = \mathrm{e}^{-t}(A\cos 3t + B\sin 3t) - \frac{t\mathrm{e}^{-t}\cos 3t}{6}.$$

With this general solution we have

$$\dot{x}(t) = \mathrm{e}^{-t}\left((2B - A - \tfrac{1}{6})\cos 3t - (3A+B)\sin 3t + \frac{t\cos 3t}{6} - \frac{t\sin 3t}{2}\right),$$

and so to ensure that $x(0) = 9$ and $\dot{x}(0) = 0$ we must have

$$A = 9 \qquad \text{and} \qquad 2B - A - \frac{1}{6} = 0,$$

which gives $B = 55/12$. The solution satisfying these initial conditions in therefore

$$x(t) = \frac{\mathrm{e}^{-t}}{12}(108\cos 3t + 55\sin 3t - 2t\cos 3t).$$

(x)

$$\ddot{x} + 6\dot{x} + 10x = \mathrm{e}^{-3t}\cos t.$$

The complementary function is the general solution of the homogeneous equation $\ddot{y} + 6\dot{y} + 10y = 0$, which is

$$y(t) = \mathrm{e}^{-3t}(A\cos t + B\sin t)$$

(part (x) of Exercise 12.1). So the right-hand side is one solution of the homogeneous equation, and for a particular integral we have to try

$$x_{\mathrm{p}}(t) = Ct\mathrm{e}^{-3t}\cos t + Dt\mathrm{e}^{-3t}\sin t.$$

Therefore

$$\begin{aligned}\dot{x}_{\mathrm{p}}(t) &= C[\mathrm{e}^{-3t}\cos t - 3t\mathrm{e}^{-3t}\cos t - t\mathrm{e}^{-3t}\sin t] \\ &\quad +D[\mathrm{e}^{-3t}\sin t - 3t\mathrm{e}^{-3t}\sin t + t\mathrm{e}^{-3t}\cos t] \\ &= \mathrm{e}^{-3t}\cos t[C + (D - 3C)t] + \mathrm{e}^{-3t}\sin t[D - (C + 3D)t]\end{aligned}$$

and

$$\begin{aligned}\ddot{x}_{\mathrm{p}}(t) &= (D - 3C)\mathrm{e}^{-3t}\cos t + [C + (D - 3C)t][-3\mathrm{e}^{-3t}\cos t - \mathrm{e}^{-3t}\sin t] \\ &\quad -(C + 3D)\mathrm{e}^{-3t}\sin t + [D - (C + 3D)t][-3\mathrm{e}^{-3t}\sin t + \mathrm{e}^{-3t}\cos t] \\ &= \mathrm{e}^{-3t}\cos t[(2D - 6C) + (8C - 6D)t] \\ &\quad +\mathrm{e}^{-3t}\sin t[(6C + 8D)t - (2C + 6D)].\end{aligned}$$

So we have

$$\ddot{x}_{\mathrm{p}} + 6\dot{x}_{\mathrm{p}} + 10x_{\mathrm{p}} = (2D + 10C)\mathrm{e}^{-3t}\cos t - 2C\mathrm{e}^{-3t}\sin t,$$

and need

$$2D + 10C = 1 \qquad \text{and} \qquad 2C = 0,$$

i.e. $D = 1/2$. So the particular integral is $x_{\mathrm{p}}(t) = \frac{1}{2}t\mathrm{e}^{-3t}\sin t$, and the general solution is

$$x(t) = \mathrm{e}^{-3t}(A\cos t + B\sin t + \frac{t}{2}\sin t).$$

We have

$$\dot{x}(t) = \mathrm{e}^{-3t}\left[(B - 3A + \tfrac{1}{2}t)\cos t + \left(3B - A + \tfrac{1}{2} + \frac{3t}{2}\right)\sin t\right],$$

and so to fit the initial conditions we need

$$x(0) = A = 10 \qquad \text{and} \qquad \dot{x}(0) = B - 3A = 0,$$

which gives $A = 10$ and $B = 30$, so the solution is

$$x(t) = \mathrm{e}^{-3t}\left(10\cos t + 30\sin t + \frac{t}{2}\sin t\right).$$

(xi)

$$\ddot{x} + 4\dot{x} + 4\dot{y} = \mathrm{e}^{2t}.$$

The complementary function is the general solution of the homogeneous equation $\ddot{y} + 4\dot{y} + 4y = 0$, and this is

$$y(t) = A\mathrm{e}^{-2t} + Bt\mathrm{e}^{-2t}.$$

The right-hand side is not a solution of the homogeneous equation, so we can try $x_{\mathrm{p}}(t) = C\mathrm{e}^{2t}$. Since

$$\dot{x}_{\mathrm{p}} = 2C\mathrm{e}^{2t} \qquad \text{and} \qquad \ddot{x}_{\mathrm{p}} = 4C\mathrm{e}^{2t}$$

we require

$$4C + 8C + 4C = 1,$$

and so $C = 1/16$, giving the particular integral $x_{\mathrm{p}}(t) = \mathrm{e}^{2t}/16$. The general solution is therefore

$$x(t) = A\mathrm{e}^{-2t} + Bt\mathrm{e}^{-2t} + \frac{\mathrm{e}^{2t}}{16}.$$

The derivative is given by

$$\dot{x}(t) = -2A\mathrm{e}^{-2t} + B\mathrm{e}^{-2t} - 2Bt\mathrm{e}^{-2t} + \frac{\mathrm{e}^{2t}}{8},$$

and so to satisfy the initial conditions we want

$$x(0) = A + \frac{1}{16} = 11 \qquad \text{and} \qquad \dot{x}(0) = -2A + B + \frac{1}{8} = 0,$$

i.e. $A = 175/16$ and $B = 87/4$. So the solution is

$$x(t) = \frac{175\mathrm{e}^{-2t} + 348t\mathrm{e}^{-2t} + 128\mathrm{e}^{2t}}{16}.$$

Exercise 14.2 *Find a particular integral for*

$$\ddot{x} + \dot{x} - 2x = 12\mathrm{e}^{-t} - 6\mathrm{e}^{t}. \tag{S14.1}$$

(You might find parts (iii) and (iv) of the previous exercise useful.)

Since the equation is linear we can take an appropriate linear combination of the particular integrals for

$$\ddot{x} + \dot{x} - 2x = f(t) = 3\mathrm{e}^{-t} \qquad [\text{this was } x_f(t) = -\frac{3}{2}\mathrm{e}^{-t}]$$

and

$$\ddot{x} + \dot{x} - 2x = g(t) = \mathrm{e}^{t} \qquad [\text{this was } x_g(t) = \frac{t\mathrm{e}^{t}}{3}].$$

The right-hand side of (S14.1) is $4f(t) - 6g(t)$, so the particular integral will be

$$x_{\mathrm{p}}(t) = 4x_f(t) - 6x_g(t) = -6\mathrm{e}^{-t} - 2t\mathrm{e}^{3t}.$$

Exercise 14.3 *If you are feeling strong, find a particular integral for*

$$\ddot{x} + 4x = 289t\mathrm{e}^{t}\sin 2t.$$

The complementary function is the solution of the homogeneous equation $\ddot{x} + 4x = 0$, which is

$$x_c(t) = A\cos 2t + B\sin 2t.$$

The right-hand side does not solve the homogeneous equation, and for a particular integral we try

$$x_{\mathrm{p}}(t) = [Ct + D]\mathrm{e}^{t}[E\sin 2t + F\cos 2t].$$

A more convenient form for this is

$$x_{\mathrm{p}}(t) = (\alpha t + \beta)\mathrm{e}^{t}\sin t + (\gamma t + \delta)\mathrm{e}^{t}\cos t.$$

We have

$$\dot{x}_{\mathrm{p}}(t) = [(\alpha+\beta-2\delta) + (\alpha - 2\gamma)t)]\mathrm{e}^{t}\sin 2t + [(2\beta+\gamma+\delta) + (2\alpha+\gamma)t]\mathrm{e}^{t}\cos 2t$$

and

$$\begin{aligned}\ddot{x}_{\mathrm{p}}(t) \quad = \quad & [(2\alpha - 3\beta - 4\gamma - 4\delta) - (3\alpha + 4\gamma)t]\mathrm{e}^{t}\sin 2t \\ & +[(4\alpha + 4\beta + 2\gamma - 3\delta) + (4\alpha - 3\gamma)t]\mathrm{e}^{t}\cos 2t,\end{aligned}$$

and therefore

$$\begin{aligned}\ddot{x}_{\mathrm{p}} + 4x_{\mathrm{p}} \quad = \quad & [(2\alpha + \beta - 4\gamma - 4\delta) + (\alpha - 4\gamma)t]\mathrm{e}^{t}\sin 2t \\ & +[(4\alpha + 4\beta + 2\gamma + \delta) + (4\alpha + \gamma)t]\mathrm{e}^{t}\cos 2t.\end{aligned}$$

To produce $289te^t \sin 2t$ we therefore require

$$\begin{aligned} 2\alpha + \beta - 4\gamma - 4\delta &= 0 \\ \alpha - 4\gamma &= 289 \\ 4\alpha + 4\beta + 2\gamma + \delta &= 0 \\ 4\alpha + \gamma &= 0, \end{aligned}$$

with solution

$$\alpha = 17, \quad \beta = -2, \quad \gamma = -68, \quad \text{and} \quad \delta = 76.$$

[You could find this by writing the simultaneous equations as a matrix equation and using MATLAB's matrix inversion function `inv`.] The particular integral is therefore

$$x_{\mathrm{p}}(t) = (17t - 2)\mathrm{e}^t \sin 2t + (76 - 68t)\mathrm{e}^t \cos 2t.$$

15

Resonance

Exercise 15.1 *For $\alpha \neq \omega$ show that the solution of the equation*

$$\ddot{x} + \omega^2 x = \cos \alpha t \tag{S15.1}$$

with $x(0) = \dot{x}(0) = 0$ is

$$x(t) = \frac{1}{\omega^2 - \alpha^2} (\cos \alpha t - \cos \omega t). \tag{S15.2}$$

We want to find the solution of

$$\ddot{x} + \omega^2 x = \cos \alpha t \qquad \text{with} \qquad x(0) = \dot{x}(0) = 0.$$

The complementary function is the solution of the homogeneous equation

$$\ddot{x} = -\omega^2 x,$$

which is

$$x(t) = A \cos \omega t + B \sin \omega t.$$

To find the particular integral we try $x_p(t) = a \cos \alpha t$ (the term $\dot{x}$ does not occur in the equation, so there is no reason to include the $\sin \alpha t$ term in our guess for a particular integral). Since $\ddot{x}_p = -a\alpha^2 \cos \alpha t$ we want

$$a(-\alpha^2 + \omega^2) \cos \alpha t = \cos \alpha t$$

so the particular integral is

$$x_p(t) = \frac{\cos \alpha t}{\omega^2 - \alpha^2}$$

giving the general solution

$$x(t) = A \cos \omega t + B \sin \omega t + \frac{\cos \alpha t}{\omega^2 - \alpha^2}.$$

To ensure that $x(0) = \dot{x}(0) = 0$ we need

$$A + \frac{1}{\omega^2 - \alpha^2} = 0 \qquad \text{and} \qquad B = 0,$$

so the solution for these initial conditions is

$$x(t) = \frac{\cos \alpha t - \cos \omega t}{\omega^2 - \alpha^2}$$

as given.

Exercise 15.2 *Use the double angle formulae*

$$\cos(\theta \pm \phi) = \cos\theta \cos\phi \mp \sin\theta \sin\phi$$

to find an expression for $\cos x - \cos y$ *as a product of two sine functions, and hence rewrite the solution in (S15.2) as*

$$\frac{2}{\omega^2 - \alpha^2} \sin \frac{(\omega + \alpha)t}{2} \sin \frac{(\omega - \alpha)t}{2}.$$

If α *is close to* ω *then* $|\alpha + \omega|$ *is much larger than* $\omega - \alpha$*; one of the two terms oscillates much faster than the other. A graph of such an expression when* $\omega = 1$ *and* $\alpha = 0.8$ *is shown in Figure 15.1. The periodic variation of the amplitude of the basic oscillation is known as* beating. *You can hear this when, for example, two flutes play slightly out of tune with each other.*

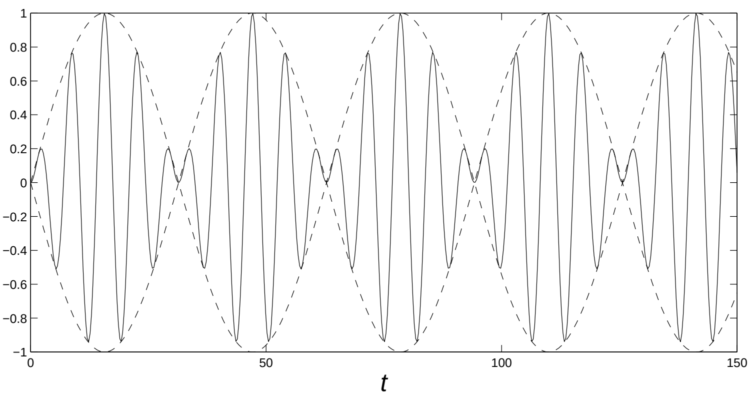

Fig. 15.1. The phenomenon of beats: the graph of $\sin 0.9t \sin 0.1t$ against t (the dashed line shows how the amplitude of the faster oscillation varies like $\sin 0.1t$).

We have

$$\begin{aligned} \cos(\theta + \phi) &= \cos\theta \cos\phi - \sin\theta \sin\phi \\ \cos(\theta - \phi) &= \cos\theta \cos\phi + \sin\theta \sin\phi, \end{aligned}$$

and subtracting these two expressions gives

$$\cos(\theta + \phi) - \cos(\theta - \phi) = -2 \sin\theta \sin\phi.$$

Choosing $\theta + \phi = x$ and $\theta - \phi = y$ implies that

$$\theta = \frac{x+y}{2} \qquad \text{and} \qquad \phi = \frac{x-y}{2},$$

and so

$$\cos x - \cos y = -2 \sin\frac{x+y}{2} \sin\frac{x-y}{2}.$$

We therefore have

$$\frac{1}{\omega^2 - \alpha^2}(\cos\alpha t - \cos\omega t) = \frac{2}{\omega^2 - \alpha^2} \sin\frac{(\omega + \alpha)t}{2} \sin\frac{(\omega - \alpha)t}{2}.$$

Exercise 15.3 *When $\alpha = \omega$ show that the solution of (S15.1) with $x(0) = \dot{x}(0) = 0$ is $x(t) = t \sin\omega t/2\omega$. Recover this solution from that for $\alpha \neq \omega$ by letting $\alpha \to \omega$ in (S15.2) and using L'Hôpital's rule.*

L'Hôpital's rule: if $f(x) \to 0$ as $x \to a$, and $g(x) \to 0$ as $x \to a$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

(provided that f and g have continuous derivatives at $x = a$).

If we try to take the limit as $\alpha \to \omega$ in the solution

$$x(t) = \frac{1}{\omega^2 - \alpha^2}(\cos\alpha t - \cos\omega t)$$

we find that both numerator and denominator tend to zero. However,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(\cos\alpha t - \cos\omega t) = -t \sin\alpha t$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(\omega^2 - \alpha^2) = -2\alpha,$$

so using L'Hôpital's rule the limit is

$$\frac{t \sin\omega t}{2\omega}$$

as given.

Exercise 15.4 *A model for the vibrations of a wine glass is*

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = 0,$$

where λ and ω are constants. Suppose that when struck the glass vibrates at 660 Hz (about the second E above middle C on a piano). Show that

$$\sqrt{4\omega^2 - \lambda^2} = 2640\pi.$$

If it takes about 3 seconds for the sound to die away, and this happens when the original vibrations have reduced to $1/100$ of their initial level, show that

$$\lambda = \frac{2\log 100}{3},$$

and hence that $\lambda = 3.07$ and $\omega = 4.15 \times 10^3$ (both to three significant Figures).

The glass can stand deforming only to $x \approx 1$. A pure tone at 660 Hz is produced at D decibels and aimed at the glass, forcing it at its natural frequency, so that the vibrations are now modelled by

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = \frac{10^{(D/10)-8}}{3}\cos 1320\pi t. \tag{S15.3}$$

How loud should the sound be, i.e. how large should D be, in order to shatter the glass? (Decibels are on a logarithmic scale, hence the exponential on the right-hand side of (S15.3). The strange factor in front of the forcing produces roughly the correct volume level.)

Trying $x(t) = \mathrm{e}^{kt}$ in

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = 0$$

yields the auxiliary equation

$$k^2 + \lambda k + \omega^2 = 0,$$

with solutions

$$k = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\omega^2}}{2}.$$

When $\lambda^2 < 4\omega^2$ we have

$$k = -\frac{\lambda}{2} \pm \frac{\sqrt{4\omega^2 - \lambda^2}}{2}\mathrm{i},$$

and the solution is

$$\mathrm{e}^{-\lambda t/2}(A\cos\Omega t + B\sin\Omega t),$$

with

$$\Omega = \frac{\sqrt{4\omega^2 - \lambda^2}}{2}.$$

The wine glass vibrates at a frequency $\Omega/2\pi$, so we have

$$\frac{\Omega}{2\pi} = \frac{\sqrt{4\omega^2 - \lambda^2}}{4\pi} = 660,$$

or

$$\sqrt{4\omega^2 - \lambda^2} = 2640\pi. \qquad \text{(S15.4)}$$

The vibrations will have reached a hundredth of their initial level when

$$\mathrm{e}^{-\lambda t/2} = \frac{1}{100},$$

and this happens after 3 seconds, so

$$\mathrm{e}^{-3\lambda/2} = \frac{1}{100}.$$

Taking logarithms gives

$$-3\lambda/2 = -\log 100 \qquad \Rightarrow \lambda = \frac{2\log 100}{3}.$$

This implies that $\lambda = 3.07$ and hence, using (S15.4), that $\omega = 4.15 \times 10^3$. We note here for use later that

$$\omega^2 = (1320\pi)^2 + \frac{\lambda^2}{4}.$$

From the treatment in the text, the amplitude of the response to the a forcing $a\cos\alpha t$ is given in (15.6) as

$$\frac{a}{\sqrt{(\omega^2 - \alpha^2)^2 + (\lambda\alpha)^2}}.$$

For

$$\ddot{x} + \lambda\dot{x} + \omega^2 x = \frac{10^{(D/10)-8}}{3}\cos 1320\pi t$$

we have $a = 10^{(D/10)-8}/3$ and $\alpha = 1320\pi$, and so the amplitude of the resulting oscillations in the glass is

$$\frac{10^{(D/10)-8}}{3} \times \frac{1}{\sqrt{(\lambda^2/4) + (1320\lambda\pi)^2}} \approx 2.6182 \times 10^{(D/10)-3}.$$

This amplitude is one when

$$\log_{10} 2.6182 + \frac{D}{10} - 3 = \log_{10} 1 = 0,$$

so when $D \approx 136$.

16

Higher order linear equations with constant coefficients

Exercise 16.1 *Find the general solution of the following equations:*

(i)

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} - 6\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 11\frac{\mathrm{d}x}{\mathrm{d}t} - 6x = \mathrm{e}^{-t},$$

(ii)

$$y''' - 3y' + 2 = \sin x,$$

(iii)

$$\frac{\mathrm{d}^4x}{\mathrm{d}t^4} - 4\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + 8\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 8\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \sin t$$

(if $x = \mathrm{e}^{kt}$ one solution of the corresponding quartic equation is $k = 1 + \mathrm{i}$), and

(iv)

$$\frac{\mathrm{d}^4x}{\mathrm{d}t^4} - 5\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4x = \mathrm{e}^t.$$

(i)

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} - 6\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 11\frac{\mathrm{d}x}{\mathrm{d}t} - 6x = 48\mathrm{e}^{-t}.$$

First we solve the homogeneous equation

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} - 6\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 11\frac{\mathrm{d}x}{\mathrm{d}t} - 6x = 0,$$

by trying $x(t) = \mathrm{e}^{kt}$, which gives the auxiliary cubic

$$k^3 - 6k^2 + 11k - 6 = 0.$$

By inspection one solution is $k = 1$, and so we can factorise

$$k^3 - 6k^2 + 11k - 6 = (k-1)(k^2 - 5k + 6)$$

and obtain the quadratic equation $k^2 - 5k + 6 = 0$ which gives two other roots, $k = 2$ and $k = 3$. So the complementary function is

$$x_c(t) = A\mathrm{e}^t + B\mathrm{e}^{2t} + C\mathrm{e}^{3t}.$$

Since e^{-t} does not solve the homogeneous equation we can try $x_\mathrm{p}(t) = D\mathrm{e}^{-t}$ for a particular integral. Then

$$\dot{x}_\mathrm{p} = \mathrm{d}^3x_\mathrm{p}/\mathrm{d}t^3 = -D\mathrm{e}^{-t} \qquad \text{and} \qquad \ddot{x}_\mathrm{p} = D\mathrm{e}^{-t},$$

so for

$$D[-\mathrm{e}^{-t} - 6\mathrm{e}^{-t} - 11\mathrm{e}^{-t} - 6\mathrm{e}^{-t}] = 48\mathrm{e}^{-t}$$

we must have $D = -2$, and the general solution is

$$x(t) = A\mathrm{e}^t + B\mathrm{e}^{2t} + C\mathrm{e}^{3t} - 2\mathrm{e}^{-t}.$$

(ii)

$$y''' - 3y' + 2y = \sin x.$$

First we find the complementary function by solving the homogeneous equation

$$y''' - 3y' + 2y = 0.$$

Trying $y(x) = \mathrm{e}^{kx}$ gives the cubic equation for k,

$$k^3 - 3k + 2 = 0.$$

Again we can see by guessing that $k = 1$ is a solution. Factorising once more gives

$$k^3 - 3k + 2 = (k-1)(k^2 + k - 2),$$

and so the other roots are the solutions of $k^2 + k - 2 = 0$, which are $k = 1$ and $k = -2$. So we have $k = 1$ a repeated root, and $k = -2$. So the complementary function is

$$y_c(x) = A\mathrm{e}^x + Bx\mathrm{e}^x + C\mathrm{e}^{-2x}.$$

For a particular integral we try $y_\mathrm{p}(x) = D\sin x + E\cos x$, for which

$$y_\mathrm{p}'(x) = D\cos x - E\sin x, \qquad y_\mathrm{p}''(x) = -D\sin x - E\cos x,$$

and

$$y_\mathrm{p}'''(x) = -D\cos x + E\sin x.$$

So we have

$$y_{\mathrm{p}}''' - 3y_{\mathrm{p}}' + 2y = (2D + 4E)\sin x + (2E - 4D)\cos x.$$

To obtain $\sin x$ on the right-hand side we need

$$2D + 4E = 1 \qquad \text{and} \qquad 2E - 4D = 0,$$

so $D = 1/10$ and $E = 1/5$, and so the solution is

$$y(x) = A\mathrm{e}^{x} + Bx\mathrm{e}^{x} + C\mathrm{e}^{-2x} + \frac{\sin x}{10} + \frac{\cos x}{5}.$$

(iii) First we try to solve the homogeneous equation

$$\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} - 4\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + 8\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 8\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0$$

by trying $x = \mathrm{e}^{kt}$. We get a quartic equation for k,

$$k^4 - 4k^3 + 8k^2 - 8k + 4 = 0. \tag{S16.1}$$

If one root is $1 + \mathrm{i}$ then another must be $1 - \mathrm{i}$, and so

$$(k - 1 - \mathrm{i})(k - 1 + \mathrm{i}) = k^2 - 2k + 2$$

must be a factor of (S16.1). So we can divide by this factor using polynomial long division,

$$\begin{array}{lcccccc}
 & & & & k^2 & -2k & +2 \\
 & & -- & -- & -- & -- & -- \\
k^2 - 2k + 2 &) & k^4 & -4k^3 & +8k^2 & -8k & +4 \\
 & & k^4 & -2k^3 & +2k^2 & & \\
 & & -- & -- & -- & -- & -- \\
 & & & -2k^3 & +6k^2 & -8k & +4 \\
 & & & -2k^3 & +4k^2 & -4k & \\
 & & -- & -- & -- & -- & -- \\
 & & & & 2k^2 & -4k & +4 \\
 & & & & 2k^2 & -4k & +4 \\
 & & -- & -- & -- & -- & -- \\
 & & & & & & 0.
\end{array}$$

So in fact the equation factorises into

$$(k^2 - 2k + 2)^2 = 0$$

and the roots are $1 + \mathrm{i}$ twice, and $1 - \mathrm{i}$ twice. So the complementary function is

$$A\mathrm{e}^{t}\sin t + B\mathrm{e}^{t}\cos t + Ct\mathrm{e}^{t}\sin t + Dt\mathrm{e}^{t}\cos t.$$

Now to find the particular integral. Is the right-hand side a solution of the homogeneous equation? Thankfully not, so we can just try a sum of $\sin t$ and $\cos t$, $x(t) = A\sin t + B\cos t$. Then

$$\begin{aligned} \dot{x} &= A\cos t - B\sin t \\ \ddot{x} &= -A\sin t - B\cos t \\ x^{(3)} &= -A\cos t + B\sin t \\ x^{(4)} &= A\sin t + B\cos t. \end{aligned}$$

Substituting in we want

$$A\sin t + B\cos t - 4(-A\cos t + B\sin t) + 8(-A\sin t - B\cos t) \\ -8(A\cos t - B\sin t) + 4(A\sin t + B\cos t) = \sin t.$$

Tidying this up we want

$$(-3A + 4B)\sin t - (4A + 3B)\cos t = \sin t$$

and so $A = -3/25$ and $B = 4/25$. So the particular integral is $(-3\sin t + 4\cos t)/25$ and the general solution is

$$A\mathrm{e}^t \sin t + B\mathrm{e}^t \cos t + Ct\mathrm{e}^t \sin t + Dt\mathrm{e}^t \cos t + \frac{4\cos t - 3\sin t}{25}.$$

(iv) First we solve the homogeneous equation

$$\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} - 5\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 4x = 0$$

by trying $x = \mathrm{e}^{kt}$. Then we want

$$k^4 - 5k^2 + 4 = 0.$$

We can factor this,

$$(k^2 - 4)(k^2 - 1) = 0,$$

and so $k = \pm 2,\ \pm 1$. So the complementary function is

$$A\mathrm{e}^{2t} + B\mathrm{e}^{-2t} + C\mathrm{e}^t + D\mathrm{e}^{-t}.$$

Now to find the particular integral. Sadly, e^t is a solution of the homogeneous equation, so we have to try $x(t) = At\mathrm{e}^t$. Then

$$\dot{x} = A(1+t)\mathrm{e}^t \qquad \ddot{x} = A(2+t)\mathrm{e}^t \qquad \frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = A(3+t)\mathrm{e}^t \qquad \frac{\mathrm{d}^4 x}{\mathrm{d}t^4} = A(4+t)\mathrm{e}^t,$$

and so we want

$$A(4+t)\mathrm{e}^t - 5A(2+t)\mathrm{e}^t + 4At\mathrm{e}^t = \mathrm{e}^t$$

or $-6A = 1$, i.e. $A = -1/6$. So a particular integral is $-te^t/6$ and the general solution is

$$Ae^{2t} + Be^{-2t} + Ce^t + De^{-t} - \frac{te^t}{6}.$$

Exercise 16.2 *The linear independence of three functions f_1, f_2, and f_3 on an interval I depends on the number of solutions of the equation*

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_3(t) = 0 \qquad \text{for all} \qquad t \in I.$$

By differentiating this equation once, and then once more, show that α_1, α_2, and α_3 satisfy the matrix equation

$$\begin{pmatrix} f_1 & f_2 & f_3 \\ \mathrm{d}f_1/\mathrm{d}t & \mathrm{d}f_2/\mathrm{d}t & \mathrm{d}f_3/\mathrm{d}t \\ \mathrm{d}^2 f_1/\mathrm{d}t^2 & \mathrm{d}^2 f_2/\mathrm{d}t^2 & \mathrm{d}^2 f_3/\mathrm{d}t^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Deduce that if $W[f_1, f_2, f_3](t)$, the Wronskian of f_1, f_2, and f_3, defined as

$$W[f_1, f_2, f_3](t) = \begin{vmatrix} f_1 & f_2 & f_3 \\ \mathrm{d}f_1/\mathrm{d}t & \mathrm{d}f_2/\mathrm{d}t & \mathrm{d}f_3/\mathrm{d}t \\ \mathrm{d}^2 f_1/\mathrm{d}t^2 & \mathrm{d}^2 f_2/\mathrm{d}t^2 & \mathrm{d}^2 f_3/\mathrm{d}t^2 \end{vmatrix}$$

is non-zero for any $t \in I$ then f_1, f_2, and f_3 are linearly independent.

We start with

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_2(t) = 0 \qquad \text{for all} \qquad t \in I.$$

Differentiating once we obtain

$$\alpha_1 \dot{f}_1(t) + \alpha_2 \dot{f}_2(t) + \alpha_3 \dot{f}_3(t) = 0 \qquad \text{for all} \qquad t \in I,$$

and differentiating again we have

$$\alpha_1 \ddot{f}_1(t) + \alpha_2 \ddot{f}_2(t) + \alpha_3 \ddot{f}_3(t) = 0 \qquad \text{for all} \qquad t \in I.$$

Combining these three we obtain the matrix equation

$$\begin{pmatrix} f_1(t) & f_2(t) & f_3(t) \\ \mathrm{d}f_1/\mathrm{d}t(t) & \mathrm{d}f_2/\mathrm{d}t(t) & \mathrm{d}f_3/\mathrm{d}t(t) \\ \mathrm{d}^2 f_1/\mathrm{d}t^2(t) & \mathrm{d}^2 f_2/\mathrm{d}t^2(t) & \mathrm{d}^2 f_3/\mathrm{d}t^2(t) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(S16.2)}$$

which should hold for all $t \in I$.

If the determinant of the matrix on the left-hand side is non-zero for any $t \in I$ then we can choose this value of t and multiply by the inverse of the matrix to show that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, which implies that f_1, f_2, and f_3 must be linearly independent on I.

Exercise 16.3 *Show that any three solutions of a third order linear differential equation are linearly independent on an interval I if and only if their Wronskian is non-zero on I.*

If $W(t) \neq 0$ for all $t \in I$ then we know from the previous exercise that the three solutions $x_1(t)$, $x_2(t)$, and $x_3(t)$ are linearly independent.

If $W \equiv 0$ then for some (and in fact for any) $s \in I$ there is a non-trivial solution $(\alpha_1, \alpha_2, \alpha_3)$ of the matrix equation (S16.2) with $t = s$. Now,

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3 x_3(t)$$

is a solution of (S16.3) that satisfies

$$x(s) = \dot{x}(s) = \ddot{x}(s) = 0.$$

But we know one solution of (S16.3) that satisfies these initial conditions, namely $x(t) \equiv 0$. Since the solutions are unique, it follows that we must have

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3 x_3(t) \equiv 0,$$

and so x_1, x_2, and x_3 are linearly dependent on I.

Exercise 16.4 *Suppose that x_1, x_2, and x_3 are three solutions of the third order linear equation*

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + p(t)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + q(t)\frac{\mathrm{d}x}{\mathrm{d}t} + r(t)x = 0, \tag{S16.3}$$

all defined on some interval I. We now show that, just as for two solutions of a second order linear equation,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -p(t)W \tag{S16.4}$$

(see Exercise 11.2). You will need various properties of determinants, which you can prove by longhand (if you wish) in the next exercise.

(i) By differentiating the determinant form of the Wronskian, show that

$$\dot{W} = \begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \mathrm{d}^3x_1/\mathrm{d}t^3 & \mathrm{d}^3x_2/\mathrm{d}t^3 & \mathrm{d}^3x_3/\mathrm{d}t^3 \end{vmatrix}.$$

(You will need parts (i) and (ii) of the next exercise.)

(ii) *Substitute in for* $\mathrm{d}^3x_j/\mathrm{d}t^3$ *using the differential equation (S16.3), and hence show that*

$$\dot{W} = -p(t)\begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{vmatrix},$$

i.e. that (S16.4) holds. (You will need parts (ii) and (iii) of the next exercise.)

(iii) *Solve equation (S16.4) to find an expression for* $W(t)$ *involving an integral, and deduce that either* $W(t) = 0$ *for all* $t \in I$*, or* $W(t) \neq 0$ *for all* $t \in I$*.*

(iv) *Deduce that any three solutions of a third order differential equation are linearly independent if and only if their Wronskian is non-zero.*

(i) Using part (i) from the next exercise, which tells us how to differentiate determinants, we have

$$\frac{\mathrm{d}W}{\mathrm{d}t}(t) = \begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{vmatrix}$$

$$+\begin{vmatrix} x_1 & x_2 & x_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \mathrm{d}^3x_1/\mathrm{d}t^3 & \mathrm{d}^3x_2/\mathrm{d}t^3 & \mathrm{d}^3x_3/\mathrm{d}t^3 \end{vmatrix}.$$

Now, if two rows in a determinant are proportional then the determinant is zero [part (ii) of the next exercise]. In particular this is true if two rows are equal, so the first two terms here vanish and we have just

$$\dot{W}(t) = \begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \mathrm{d}^3x_1/\mathrm{d}t^3 & \mathrm{d}^3x_2/\mathrm{d}t^3 & \mathrm{d}^3x_3/\mathrm{d}t^3 \end{vmatrix}.$$

(ii) From the equation we know that

$$\mathrm{d}^3x_j/\mathrm{d}t^3 = -p(t)\ddot{x}_j - q(t)\dot{x}_j - r(t)x_j,$$

and substituting into the expression for $\dot{W}$ obtained above we have

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ -p\ddot{x}_1 - q\dot{x}_1 - rx_1 & -p\ddot{x}_2 - q\dot{x}_2 - rx_2 & -p\ddot{x}_3 - q\dot{x}_3 - rx_3 \end{vmatrix}.$$

Using part (iii) of the next exercise we can rewrite this as

$$\dot{W}(t) = -p\begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{vmatrix} - q\begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_2 \end{vmatrix} - r\begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ x_1 & x_2 & x_3 \end{vmatrix}.$$

Once again we can use the fact that if two rows are proportional then the determinant is zero [part (ii) of the next exercise] to get rid of the last two terms and end up with just

$$\dot{W}(t) = -p(t)\begin{vmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{vmatrix} = -p(t)W(t),$$

since the determinant is precisely the definition of $W[x_1, x_2, x_3](t)$.

(iii) It follows from the equation

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -p(t)W$$

that

$$W(t) = W(s)\exp\left(-\int_s^t p(\tilde{t})\,\mathrm{d}\tilde{t}\right),$$

and so, since e^z is never zero, either $W(t) = 0$ for all $t \in I$, or $W(t) \neq 0$ for all $t \in I$ (cf. Exercise 11.2).

Exercise 16.5 *For the previous question you will need the following properties of determinants: you should be able to prove them in the* 3×3 *case treated here by simple (if laborious) calculation, using the explicit expression for the determinant of a* 3×3 *matrix*

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = a(sz - ty) - b(rz - tx) + c(ry - sx).$$

(i)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \begin{vmatrix} \dot{a} & \dot{b} & \dot{c} \\ r & s & t \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ \dot{r} & \dot{s} & \dot{t} \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ r & s & t \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

(i.e. differentiate one row at a time; this is essentially the product rule),

(ii) If any two rows are proportional then the determinant is zero: check this for

$$\begin{vmatrix} a & b & c \\ \lambda a & \lambda b & \lambda c \\ x & y & z \end{vmatrix} = 0,$$

(iii) Determinants depend linearly on their rows: show this for the case

$$\begin{vmatrix} a & b & c \\ r & s & t \\ \alpha x_1 + \beta x_2 & \alpha y_1 + \beta y_2 & \alpha z_1 + \beta z_2 \end{vmatrix} = \alpha \begin{vmatrix} a & b & c \\ r & s & t \\ x_1 & y_1 & z_1 \end{vmatrix} + \beta \begin{vmatrix} a & b & c \\ r & s & t \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

(i) One way to show this result would be to use the explicit form of the determinant, differentiate each term using the product rule, and then recombine the results appropriately:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} &= \frac{\mathrm{d}}{\mathrm{d}t}[a(sz-ty)-b(rz-tx)+c(ry-sx)] \\
&= \dot{a}(sz-ty)+a(\dot{s}z-\dot{t}y)+a(s\dot{z}-t\dot{y}) \\
&\quad -\dot{b}(rz-tx)-b(\dot{r}z-\dot{t}x)-b(r\dot{z}-t\dot{x}) \\
&\quad +\dot{c}(ry-sx)+c(\dot{r}y-\dot{s}x)+c(r\dot{y}-s\dot{x}) \\
&= \dot{a}(sz-ty)-\dot{b}(rz-tx)+\dot{c}(ry-sx) \\
&\quad +a(\dot{s}z-\dot{t}y)-b(\dot{r}z-\dot{t}x)+c(\dot{r}y-\dot{s}x) \\
&\quad +a(s\dot{z}-t\dot{y})-b(r\dot{z}-t\dot{x})+c(r\dot{y}-s\dot{x}) \\
&= \begin{vmatrix} \dot{a} & \dot{b} & \dot{c} \\ r & s & t \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ \dot{r} & \dot{s} & \dot{t} \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ r & s & t \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}.
\end{aligned}$$

A quicker way is to note that each term in the expression for the determinant involves one entry from each row. Thus, having differentiated using the product rule, we obtain three terms which we could group together according to which row has been differentiated, and this will give the required expression.

(ii) We have

$$\begin{vmatrix} a & b & c \\ \lambda a & \lambda b & \lambda c \\ x & y & z \end{vmatrix} = a(\lambda bz - \lambda cy) - b(\lambda az - \lambda cx) + c(\lambda ay - \lambda bx)$$
$$= \lambda[abz - acy - abz + bcx + acy - bcx]$$
$$= 0.$$

(iii) Expanding the determinant we have

$$\begin{vmatrix} a & b & c \\ r & s & t \\ \alpha x_1 + \beta x_2 & \alpha y_1 + \beta y_2 & \alpha z_1 + \beta z_2 \end{vmatrix}$$
$$= a(s[\alpha z_1 + \beta z_2] - t[\alpha y_1 + \beta y_2]) - b(r[\alpha z_1 + \beta z_2] - t[\alpha x_1 + \beta x_2])$$
$$+c(r[\alpha y_1 + \beta y_2] - s[\alpha x_1 + \beta x_2])$$
$$= \alpha[a(sz_1 - ty_1) - b(rz_1 - tx_1) + c(ry_1 - sx_1)]$$
$$+\beta[a(sz_2 - ty_2) - b(rz_2 - tx_2) + c(ry_2 - sx_2)]$$
$$= \alpha \begin{vmatrix} a & b & c \\ r & s & t \\ x_1 & y_1 & z_1 \end{vmatrix} + \beta \begin{vmatrix} a & b & c \\ r & s & t \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

17

Reduction of order

For further examples of the reduction of order method see also Exercises 18.1(vi), 18.1(vii), and 20.3.

Exercise 17.1 *One solution of the equation*

$$t^2\ddot{y} - (t^2 + 2t)\dot{y} + (t + 2)y = 0$$

is $y(t) = t$. Use the reduction of order method to find a second solution, and hence write down the general solution.

We try $y = tu$, and so

$$\dot{y} = u + t\dot{u} \qquad \text{and} \qquad \ddot{y} = 2\dot{u} + t\ddot{u}.$$

Remembering that when we substitute into the equation the terms in which u has not been differentiated will cancel[1] we have

$$t^2(t\ddot{u} + 2\dot{u}) - (t^2 + 2t)t\dot{u} = 0. \tag{S17.1}$$

Simplifying this we have

$$t^3\ddot{u} - t^3\dot{u} = 0,$$

and so

$$\ddot{u} - \dot{u} = 0.$$

[1] If you don't remember this you will just have more algebra to do:

$$t^2(t\ddot{u} + 2\dot{u}) - (t^2 + 2t)(u + t\dot{u}) + (t + 2)tu = 0,$$

and so

$$t^3\ddot{u} + 2t^2\dot{u} - (t^2 + 2t)u - (t^3 + 2t^2)\dot{u} + (t^2 + 2t)u = 0;$$

the terms involving u now cancel and lead to (S17.1).

Usually we would set $v = \dot{u}$ to obtain a first order equation for v, but in this case we can just try $u(t) = \mathrm{e}^{kt}$ and obtain the auxiliary equation for k,

$$k^2 - k = 0.$$

So $k = 0$ or $k = 1$, and we have

$$u(t) = A + B\mathrm{e}^t.$$

Therefore we have the general solution

$$y(t) = At + Bt\mathrm{e}^t;$$

the 'new solution' is $y(t) = t\mathrm{e}^t$.

Exercise 17.2 *One solution of*

$$(x-1)y'' - xy' + y = 0$$

that is valid for $x > 1$ is $y(x) = \mathrm{e}^x$. Find a second linearly independent solution $z(x)$, and check that the Wronskian of $y(x)$ and $z(x)$ is non-zero for $x > 1$.

If we try $y(x) = \mathrm{e}^x u(x)$ then

$$y'(x) = \mathrm{e}^x u(x) + \mathrm{e}^x u'(x) \qquad \text{and} \qquad y''(x) = \mathrm{e}^x u + 2\mathrm{e}^x u' + \mathrm{e}^x u'',$$

and so, remembering that the terms in which u is not differentiated will cancel, we have

$$(x-1)[2\mathrm{e}^x u' + \mathrm{e}^x u''] - x[\mathrm{e}^x u'] = 0.$$

Cancelling the e^x gives

$$(x-1)u'' + (x-2)u' = 0.$$

Setting $v = u$ we obtain a first order equation for v,

$$v' + \frac{x-2}{x-1}v = 0.$$

We can solve this equation using the integrating factor

$$\begin{aligned} \exp\left(\int \frac{x-2}{x-1}\,\mathrm{d}x\right) &= \exp\left(\int 1 - \frac{1}{x-1}\,\mathrm{d}x\right) \\ &= \exp\left(x - \ln|x-1|\right) \\ &= \frac{\mathrm{e}^x}{x-1}, \end{aligned}$$

since $|x-1| = x-1$ when $x > 1$.

Multiplying both sides by this integrating factor we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{e}^x}{x-1}v(x)\right]=0,$$

and so

$$\frac{\mathrm{e}^x}{x-1}v(x)=A,$$

giving (recall that $v(x)=u'(x)$)

$$u'(x)=A\mathrm{e}^{-x}(x-1).$$

Now we integrate this to give

$$u(x)=-Ax\mathrm{e}^{-x}+B.$$

Since A is an arbitrary constant we can replace A by $-A$ to give

$$y(x)=Ax+B\mathrm{e}^x.$$

In this case, the new solution is just $y(x)=x$.

The Wronskian of $y_1(x)=\mathrm{e}^x$ and $y_2(x)=x$ is

$$W[y_1,y_2](x)=y_1y_2'-y_2y_1'=\mathrm{e}^x-x\mathrm{e}^x=(1-x)\mathrm{e}^x,$$

which is non-zero when $x>1$.

Exercise 17.3 *One solution of*

$$(t\cos t-\sin t)\ddot{x}+\dot{x}t\sin t-x\sin t=0$$

is $x(t)=t$. Find a second linearly independent solution.

With $y=tu$ we have

$$\dot{y}=u+t\dot{u}\qquad\text{and}\qquad\ddot{y}=2\dot{u}+t\ddot{u},$$

and substituting in gives

$$(t\cos t-\sin t)(2\dot{u}+t\ddot{u})+t\sin t(t\dot{u})=0,$$

which after rearrangement is

$$(t^2\cos t-t\sin t)\ddot{u}+(t^2\sin t+2t\cos t-2\sin t)\dot{u}=0.$$

Setting $v=\dot{u}$ we have the first order linear equation

$$(t^2\cos t-t\sin t)\dot{v}+(t^2\sin t+2t\cos t-2\sin t)v=0.$$

The integrating factor for this equation is

$$\begin{aligned}
\exp\left(\int \frac{t^2\sin t + 2t\cos t - 2\sin t}{t^2\cos t - t\sin t}\,\mathrm{d}t\right) &= \exp\left(\int \frac{t\sin t}{t\cos t - \sin t} + \frac{2}{t}\right)\\
&= \exp\left(-\ln(t\cos t - \sin t) + 2\ln t\right)\\
&= \frac{t^2}{t\cos t - \sin t}.
\end{aligned}$$

We therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2 v}{t\cos t - \sin t}\right) = 0,$$

and so

$$\frac{t^2 v}{t\cos t - \sin t} = c,$$

which gives

$$v = \frac{\mathrm{d}u}{\mathrm{d}t} = c\,\frac{t\cos t - \sin t}{t^2}.$$

This implies that

$$u(t) = c\,\frac{\sin t}{t},$$

and so

$$x(t) = tu(t) = c\sin t,$$

and the second linearly independent solution is $x(t) = \sin t$.

Exercise 17.4 *One solution of*

$$(t - t^2)\ddot{x} + (2 - t^2)\dot{x} + (2 - t)x = 0$$

is $x(t) = \mathrm{e}^{-t}$. *Find a second linearly independent solution.*

We put $x(t) = \mathrm{e}^{-t}u(t)$, and so

$$\dot{x} = -\mathrm{e}^{-t}u + \mathrm{e}^{-t}\dot{u} \qquad \text{and} \qquad \ddot{x} = \mathrm{e}^{-t}u - 2\mathrm{e}^{-t}\dot{u} + \mathrm{e}^{-t}\ddot{u}.$$

Substituting into the equation we obtain

$$(t - t^2)[\mathrm{e}^{-t}\ddot{u} - 2\mathrm{e}^{-t}\dot{u}] + (2 - t^2)\mathrm{e}^{-t}\dot{u} = 0.$$

Cancelling the e^{-t} terms and rearranging we have

$$(t - t^2)\ddot{u} + (t^2 - 2t + 2)\dot{u} = 0.$$

Setting $v = \dot{u}$ we obtain a first order equation for v,

$$\dot{v} + \frac{t^2 - 2t + 2}{t - t^2} v = 0.$$

We can rewrite this as

$$\dot{v} + \left(\frac{2}{t} + \frac{1}{1-t} - 1 \right) v = 0,$$

and so the integrating factor is

$$\begin{aligned} \exp\left(\int \frac{2}{t} + \frac{1}{1-t} - 1 \, \mathrm{d}t \right) &= \exp\left(2 \ln t - \ln(1-t) - t\right) \\ &= \frac{t^2}{1-t} \mathrm{e}^{-t}. \end{aligned}$$

The equation for v therefore becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{v t^2 \mathrm{e}^{-t}}{1-t} \right) = 0,$$

and so

$$v = \frac{\mathrm{d}u}{\mathrm{d}t} = c \left[\frac{\mathrm{e}^t}{t^2} - \frac{\mathrm{e}^t}{t} \right],$$

which implies that

$$u(t) = -c \, \frac{\mathrm{e}^t}{t}.$$

This gives a second solution $x(t) = \mathrm{e}^{-t} u(t) = -c/t$; a second linearly independent solution is $x(t) = 1/t$.

Exercise 17.5 *One solution of*

$$y'' - xy' + y = 0$$

is $y = x$. Find a second linearly independent solution in the form of an integral. Expanding the integrand in powers of x using the power series form for e^x,

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and assuming that the resulting expression can be integrated term-by-term show that this second solution can be written as

$$y(x) = A \left[-1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^n (2n-1) n!} \right]$$

(cf. Exercise 20.2(i)).

To find a second solution we put $y(x) = xu(x)$, and so

$$y'(x) = u(x) + xu'(x) \qquad \text{and} \qquad y''(x) = 2u'(x) + xu''(x),$$

and so when we substitute into the original equation we obtain

$$2u' + xu'' - x^2u' = 0$$

which, with $v = u'$, gives

$$v' + \left(\frac{2}{x} - x\right) v = 0.$$

The integrating factor is

$$\exp\left(\int \frac{2}{x} - x \, \mathrm{d}x\right) = \exp(2\ln x - \tfrac{1}{2}x^2) = x^2 \mathrm{e}^{-x^2/2},$$

and so multiplying by this the equation for v becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2\mathrm{e}^{-x^2/2}v(x)\right) = 0,$$

and so

$$u'(x) = v(x) = Ax^{-2}\mathrm{e}^{x^2/2}.$$

Therefore

$$u(x) = A\int x^{-2}\mathrm{e}^{x^2/2}\,\mathrm{d}x$$

and

$$y(x) = Ax\int x^{-2}\mathrm{e}^{x^2/2}\,\mathrm{d}x.$$

Using the power series expansion for e^x,

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we have

$$\begin{aligned} y(x) &= Ax\int x^{-2}\left(\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}\right)\mathrm{d}x \\ &= Ax\int x^{-2} + \sum_{n=1}^{\infty}\frac{x^{2n-2}}{2^n n!}\,\mathrm{d}x \\ &= Ax\left[-x^{-1} + \sum_{n=1}^{\infty}\frac{x^{2n-1}}{2^n(2n-1)n!}\right] \end{aligned}$$

$$= \quad A\left[-1+\sum_{n=1}^{\infty}\frac{x^{2n}}{2^n(2n-1)n!}\right].$$

Exercise 17.6 *One solution of*

$$\tan t\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-3\frac{\mathrm{d}x}{\mathrm{d}t}+(\tan t+3\cot t)x=0$$

is $x(t)=\sin t$. *Find a second linearly independent solution.*

We try $x(t)=u(t)\sin t$ and then

$$\dot{x}=\dot{u}\sin t+u\cos t \qquad \text{and} \qquad \ddot{x}=\ddot{u}\sin t+2\dot{u}\cos t-u\sin t.$$

Substituting into the equation we obtain

$$\tan t[\ddot{u}\sin t+2\dot{u}\cos t]-3[\dot{u}\sin t]=0$$

which is just

$$\ddot{u}\sin t\tan t-\dot{u}\sin t=0.$$

Setting $v=\dot{u}$ gives

$$\dot{v}-v\cot t=0.$$

This can be solved using the integrating factor

$$\exp\left(-\int\frac{\cos t}{\sin t}\,\mathrm{d}t\right)=\exp(-\ln\sin t)=1/\sin t,$$

so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(v/\sin t)=0,$$

which gives the solution

$$v(t)=c\sin t.$$

Therefore $u(t)=c\cos t$ and $x(t)=u(t)\sin t=c\sin t\cos t=c\sin 2t/2$. So we have shown that a second solution is $x(t)=\sin 2t$.

Exercise 17.7 *If we know one solution* $u(t)$ *of the equation*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+p(t)\frac{\mathrm{d}x}{\mathrm{d}t}+q(t)x=0 \tag{S17.2}$$

then the reduction of order method with $x(t)=u(t)y(t)$, *leads to the first order linear equation*

$$u(t)\dot{z}+[2\dot{u}(t)+p(t)u(t)]z=0.$$

for $z = \dot{y}$ (cf. (17.3). Show that

$$z(t) = \frac{A\mathrm{e}^{-\int p(t)\,\mathrm{d}t}}{u(t)^2},$$

and hence find the second linearly independent solution in the form of an integral.

We start with

$$u(t)\dot{z} + [2\dot{u}(t) + p(t)u(t)]z = 0.$$

Dividing by $u(t)$ we obtain

$$\dot{z} + \left[2\frac{\dot{u}}{u} + p(t)\right] z = 0. \qquad \text{(S17.3)}$$

We can solve this equation using the integrating factor

$$\begin{aligned}\exp\left(\int 2\frac{\dot{u}}{u} + p(t)\,\mathrm{d}t\right) &= \exp\left(2\ln u + \int p(t)\,\mathrm{d}t\right)\\ &= u(t)^2 \exp\left(\int p(t)\,\mathrm{d}t\right).\end{aligned}$$

Multiplying both sides of (S17.3) by the integrating factor the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[z(t)u(t)^2 \exp\left(\int p(t)\,\mathrm{d}t\right)\right] = 0,$$

and so

$$z(t)u(t)^2 \exp\left(\int p(t)\,\mathrm{d}t\right) = A,$$

which implies that

$$z(t) = \frac{A\mathrm{e}^{-\int p(t)\,\mathrm{d}t}}{u(t)^2}. \qquad \text{(S17.4)}$$

Since $\dot{y} = z$ and $x(t) = u(t)y(t)$ we have

$$x(t) = u(t)\int \frac{A\mathrm{e}^{-\int p(t)\,\mathrm{d}t}}{u(t)^2}\,\mathrm{d}t.$$

Exercise 17.8 *Suppose that the two solutions of a second order linear differential equation (S17.2) are $u(t)$ and $v(t)$. Use the result of the previous exercise, to show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{v(t)}{u(t)}\right] = \frac{A\mathrm{e}^{-\int p(t)\,\mathrm{d}t}}{u(t)^2},$$

and hence that

$$p(t) = -\frac{u\ddot{v} - v\ddot{u}}{u\dot{v} - v\dot{u}}.$$

Find the function $q(t)$ *such that* $u(t)$ *is a solution of*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{u\ddot{v} - v\ddot{u}}{u\dot{v} - v\dot{u}}\frac{\mathrm{d}x}{\mathrm{d}t} + q(t)x = 0$$

(rearrange the equation for $q(t)$ *and substitute* $x(t) = u(t)$*), and hence show that the second order linear differential equation with solutions* $u(t)$ *and* $v(t)$ *can be written as*

$$(u\dot{v} - v\dot{u})\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - (u\ddot{v} - v\ddot{u})\frac{\mathrm{d}x}{\mathrm{d}t} + (\dot{u}\ddot{v} - \ddot{u}\dot{v})x = 0.$$

This produced Exercises 17.1–17.6 above.

If we know $u(t)$ and use the reduction of order method to find $v(t)$ then we set $x(t) = u(t)y(t)$; if we want to end up with $x(t) = v(t)$ then we must find $y(t) = v(t)/u(t)$. From the previous Exercise, $z(t) = \mathrm{d}y/\mathrm{d}t$ is given by (S17.4), and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{v(t)}{u(t)}\right] = \frac{A\mathrm{e}^{-\int p(t)\,\mathrm{d}t}}{u(t)^2}.$$

Differentiating the left-hand side we have

$$\frac{u\dot{v} - v\dot{u}}{u^2} = \frac{A\mathrm{e}^{-\int p(t)\,\mathrm{d}t}}{u(t)^2},$$

and so

$$A\mathrm{e}^{-\int p(t)\,\mathrm{d}t} = u\dot{v} - v\dot{u}.$$

Differentiating this we obtain

$$-Ap(t)\mathrm{e}^{-\int p(t)\,\mathrm{d}t} = \dot{u}\dot{v} + u\ddot{v} - \dot{v}\dot{u} - v\ddot{u} = u\ddot{v} - v\ddot{u},$$

and so we have

$$p(t) = -\frac{u\ddot{v} - v\ddot{u}}{u\dot{v} - v\dot{u}}.$$

We now know $p(t)$, so the equation satisfied by $u(t)$ and $v(t)$ is of the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{u\ddot{v} - v\ddot{u}}{u\dot{v} - v\dot{u}}\frac{\mathrm{d}x}{\mathrm{d}t} + q(t)x = 0.$$

This implies that

$$q(t) = \frac{1}{x}\left(\frac{u\ddot{v} - v\ddot{u}}{u\dot{v} - v\dot{u}}\,\dot{x} - \ddot{x}\right),$$

and since $u(t)$ is a solution we can put $x(t) = u(t)$ to find

$$q(t) = \frac{1}{u}\left(\frac{u\ddot{v} - v\ddot{u}}{u\dot{v} - v\dot{u}}\,\dot{u} - \ddot{u}\right) = \frac{\dot{u}\ddot{v} - \ddot{u}\dot{v}}{u\dot{v} - v\dot{u}}.$$

The equation is therefore

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{u\ddot{v} - v\ddot{u}}{u\dot{v} - v\dot{u}}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\dot{u}\ddot{v} - \ddot{u}\dot{v}}{u\dot{v} - v\dot{u}}\,x = 0,$$

and multiplying up by $u\dot{v} - v\dot{u}$ yields

$$(u\dot{v} - v\dot{u})\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - (u\ddot{v} - v\ddot{u})\frac{\mathrm{d}x}{\mathrm{d}t} + (\dot{u}\ddot{v} - \ddot{u}\dot{v})x = 0.$$

Exercise 17.9 *Using the result of the previous exercise, find a second order linear differential equation whose solutions are* e^t *and* $\cos t$*. Check that both of these two functions satisfy the resulting equation.*

With $u(t) = \mathrm{e}^t$ and $v(t) = \cos t$ we have

$$u(t) = \dot{u}(t) = \ddot{u}(t) = \mathrm{e}^t$$

and

$$v(t) = \cos t, \qquad \dot{v}(t) = -\sin t, \qquad \text{and} \qquad \ddot{v}(t) = -\cos t.$$

Then the linear equation with $u(t)$ and $v(t)$ as solutions is

$$(-\mathrm{e}^t\sin t - \mathrm{e}^t\cos t)\ddot{x} - (-\mathrm{e}^t\cos t - \mathrm{e}^t\cos t)\dot{x} + (-\mathrm{e}^t\cos t + \mathrm{e}^t\sin t)x = 0.$$

Dividing by e^t we obtain

$$(\sin t + \cos t)\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\cos t\,\frac{\mathrm{d}x}{\mathrm{d}t} + (\cos t - \sin t)x = 0.$$

First we check that $x(t) = \mathrm{e}^t$ is a solution:

$$(\sin t + \cos t)\mathrm{e}^t - 2\mathrm{e}^t\cos t + (\cos t - \sin t)\mathrm{e}^t = 0,$$

as required. For $x(t) = \cos t$ we obtain

$$\begin{aligned}&-(\sin t + \cos t)\cos t + 2\cos t\sin t + (\cos t - \sin t)\cos t\\ &\quad= -\sin t\cos t - \cos^2 t + 2\cos t\sin t + \cos^2 t - \sin t\cos t = 0,\end{aligned}$$

verifying that this is also a solution.

18

The variation of constants formula

Exercise 18.1 *Use the method of variation of constants to find a particular integral for the following equations:*

(i) $y'' - y' - 6y = \mathrm{e}^x$ *(you could use the method of undetermined coefficients for this example, which would be much more sensible);*

(ii) $\ddot{x} - x = t^{-1}$ *(you can leave the answer as an integral);*

(iii) $y'' + 4y = \cot 2x$*. Hint:* $\int \operatorname{cosec} x \, \mathrm{d}x = \ln|\operatorname{cosec} x - \cot x|$*;*

(iv) $t^2\ddot{x} - 2x = t^3$ *(to find the solutions of the homogeneous equation try* $x = t^k$*, see next chapter);*

(v) $\ddot{x} - 4\dot{x} = \tan t$ *(leave your answer as an integral);*

(vi)

$$(\tan^2 x - 1)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\tan^3 x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\sec^4 x = (\tan^2 x - 1)(1 - 2\sin^2 x),$$

one solution of the homogeneous equation is $y(x) = \sec^2 x$*, and the reduction of order method, which is somewhat painful, can be used (if you wish) to show that a second linearly independent solution is* $\tan x$*. You should be able to find a particular integral explicitly for this example.*

(vii)

$$(1 + \sin^2 t)\ddot{x} - (2\tan t + \sin t\cos t)\dot{x} + (1 - 2\tan^2 t)x = f(t),$$

one solution of the homogeneous equation is $\tan t$*, and again the reduction of order method will provide a second solution,* $\cos t$*, after some effort. You should leave your final answer as an integral.*

(i)

$$y'' - y' - 6y = \mathrm{e}^x.$$

To find solutions of the homogeneous equation we try $y(x) = \mathrm{e}^{kx}$ and obtain the auxiliary equation

$$k^2 - k - 6 = 0,$$

so $k = -2$ or $k = 3$. So two linearly independent solutions are e^{-2x} and e^{3x}. So we look for a particular integral of the form

$$y(x) = u(x)\mathrm{e}^{-2x} + v(x)\mathrm{e}^{3x}.$$

Differentiating once we have

$$y'(x) = u'(x)\mathrm{e}^{-2x} - 2\mathrm{e}^{-2x}u(x) + v'(x)\mathrm{e}^{3x} + 3v(x)\mathrm{e}^{3x}.$$

We choose

$$u'(x)\mathrm{e}^{-2x} + v'(x)\mathrm{e}^{3x} = 0,$$

so that

$$y'(x) = -2\mathrm{e}^{-2x}u(x) + 3v(x)\mathrm{e}^{3x}$$

and

$$y''(x) = 4\mathrm{e}^{-2x}u(x) - 2\mathrm{e}^{-2x}u'(x) + 9v(x)\mathrm{e}^{3x} + 3v'(x)\mathrm{e}^{3x} = 0.$$

Substituting for y, y', and y'' into the equation we have

$$\begin{aligned} &4\mathrm{e}^{-2x}u(x) - 2\mathrm{e}^{-2x}u'(x) + 9v(x)\mathrm{e}^{3x} + 3v'(x)\mathrm{e}^{3x} \\ &\quad -[-2\mathrm{e}^{-2x}u(x) + 3v(x)\mathrm{e}^{3x}] - 6[u(x)\mathrm{e}^{-2x} + v(x)\mathrm{e}^{3x}] = \mathrm{e}^{x}, \end{aligned}$$

which simplifies (any terms in which $u(x)$ and $v(x)$ are not differentiated will cancel, since e^{-2x} and e^{3x} are solutions of the homogeneous equation) to give

$$-2\mathrm{e}^{-2x}u'(x) + 3\mathrm{e}^{3x}v'(x) = \mathrm{e}^{x}.$$

So we have two equations for $u'(x)$ and $v'(x)$, which are

$$\begin{aligned} \mathrm{e}^{-2x}u'(x) + \mathrm{e}^{3x}v'(x) &= 0 \\ -2\mathrm{e}^{-2x}u'(x) + 3\mathrm{e}^{3x}v'(x) &= \mathrm{e}^{x}. \end{aligned}$$

The easiest way to solve these for u' and v' is to write them as a matrix equation

$$\begin{pmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{3x} \\ -2\mathrm{e}^{-2x} & 3\mathrm{e}^{3x} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{e}^{x} \end{pmatrix},$$

and multiply both sides by the inverse of the matrix on the left-hand side,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{5\mathrm{e}^x} \begin{pmatrix} 3\mathrm{e}^{3x} & -\mathrm{e}^{3x} \\ 2\mathrm{e}^{-2x} & \mathrm{e}^{-2x} \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{e}^x \end{pmatrix},$$

which gives

$$u'(x) = -\frac{\mathrm{e}^{3x}}{5} \qquad \text{and} \qquad v'(x) = \frac{\mathrm{e}^{-2x}}{5}.$$

Therefore

$$u(x) = -\frac{\mathrm{e}^{3x}}{15} \qquad \text{and} \qquad v'(x) = -\frac{\mathrm{e}^{-2x}}{10}.$$

This gives the particular integral

$$y(x) = u(x)\mathrm{e}^{-2x} + v(x)\mathrm{e}^{3x} = -\frac{1}{15}\mathrm{e}^x - \frac{1}{10}\mathrm{e}^x = -\frac{\mathrm{e}^x}{6}.$$

Of course, it would be much easier to find this by trying $y(x) = A\mathrm{e}^x$ to start with.

(ii)

$$\ddot{x} - x = t^{-1}.$$

Two linearly independent solutions of the homogeneous equation $\ddot{x} = x$ are $x = \mathrm{e}^t$ and $x = \mathrm{e}^{-t}$ (try $x = \mathrm{e}^{kt}$). So for a particular integral we try

$$x(t) = u(t)\mathrm{e}^t + v(t)\mathrm{e}^{-t}.$$

Then

$$\dot{x} = \dot{u}\mathrm{e}^t + u\mathrm{e}^t + \dot{v}\mathrm{e}^{-t} - v\mathrm{e}^{-t}.$$

We introduce the condition

$$\dot{u}\mathrm{e}^t + \dot{v}\mathrm{e}^{-t} = 0$$

so that no second derivatives of u and v occur in $\ddot{x}$:

$$\ddot{x} = \dot{u}\mathrm{e}^t + u\mathrm{e}^t - \dot{v}\mathrm{e}^{-t} + v\mathrm{e}^{-t}.$$

Substituting for $\ddot{x}$ and x into the equation we get, after some cancellations,

$$\dot{u}\mathrm{e}^t - \dot{v}\mathrm{e}^{-t} = \frac{1}{t}.$$

Solving the matrix equation

$$\begin{pmatrix} \mathrm{e}^t & \mathrm{e}^{-t} \\ \mathrm{e}^t & -\mathrm{e}^{-t} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ t^{-1} \end{pmatrix}$$

we obtain

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -\mathrm{e}^{-t} & -\mathrm{e}^{-t} \\ -\mathrm{e}^t & \mathrm{e}^t \end{pmatrix} \begin{pmatrix} 0 \\ t^{-1} \end{pmatrix}$$

and so

$$\dot{u} = \frac{\mathrm{e}^{-t}}{2t} \qquad \text{and} \qquad \dot{v} = -\frac{\mathrm{e}^t}{2t}.$$

Therefore we have

$$x(t) = -\mathrm{e}^t \int \frac{\mathrm{e}^{-t}}{2t}\,\mathrm{d}t + \mathrm{e}^{-t} \int \frac{\mathrm{e}^t}{2t}\,\mathrm{d}t.$$

(iii)

$$y'' + 4y = \cot 2x.$$

Two linearly independent solutions of the homogeneous equation $y''+4y=0$ are $\sin 2x$ and $\cos 2x$. So we look for a particular integral in the form

$$y(x) = u(x)\sin 2x + v(x)\cos 2x.$$

Differentiating gives

$$y' = u'\sin 2x + v'\cos 2x + 2u\cos 2x - 2v\sin 2x.$$

We impose the condition

$$u'\sin 2x + v'\cos 2x = 0$$

so that

$$y' = 2u\cos 2x - 2v\sin 2x$$

and

$$y'' = -4u\cos 2x - 4v\sin 2x + 2u'\cos 2x - 2v'\sin 2x.$$

When we substitute into the equation all the terms in which u and v have not been differentiated will cancel, and we will be left with

$$2u'\cos 2x - 2v'\sin 2x = \cot 2x.$$

Solving the matrix equation

$$\begin{pmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ \cot 2x \end{pmatrix}$$

for u' and v' we obtain

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -2\sin 2x & -\cos 2x \\ -2\cos 2x & \sin 2x \end{pmatrix}\begin{pmatrix} 0 \\ \cot 2x \end{pmatrix},$$

and so

$$u'(x) = \frac{\cos^2 2x}{2\sin 2x} = \frac{1}{2\sin 2x} - \frac{\sin 2x}{2} \qquad \text{and} \qquad v'(x) = -\frac{\cos 2x}{2}.$$

So we have

$$u(x) = \frac{\ln(\operatorname{cosec} 2x - \cot 2x)}{4} + \frac{\cos 2x}{4} \qquad \text{and} \qquad v(x) = -\frac{\sin 2x}{4},$$

which gives the particular integral

$$\begin{aligned} y(x) &= \frac{\ln(\operatorname{cosec} 2x - \cot 2x)\sin 2x}{4} + \frac{\cos 2x \sin 2x}{4} - \frac{\sin 2x \cos 2x}{4} \\ &= \frac{\ln(\operatorname{cosec} 2x - \cot 2x)\sin 2x}{4}. \end{aligned}$$

(iv)

$$t^2\ddot{x} - 2x = t^3.$$

Following the hint, we try $x = t^k$ to find solutions of the homogeneous equation $t^2\ddot{x} - 2x = 0$. This guess gives

$$\dot{x} = kt^{k-1} \qquad \text{and} \qquad \ddot{x} = k(k-1)t^{k-2},$$

and so we want

$$k(k-1)t^k - 2t^k = 0 \qquad \text{i.e.} k^2 - k - 2 = 0,$$

cancelling the factor of t^k. There are two solutions of this quadratic equation, $k = -1$ and $k = 2$, so we obtain two linearly independent solutions $x(t) = t^{-1}$ and $x(t) = t^2$.

We look for a particular integral of the form

$$x(t) = \frac{u(t)}{t} + t^2 v(t).$$

Differentiating we have

$$\dot{x}(t) = \frac{\dot{u}}{t} + t^2\dot{v} - \frac{u}{t^2} + 2tv.$$

After imposing the condition

$$\frac{\dot{u}}{t} + t^2\dot{v} = 0$$

we obtain

$$\dot{x} = -\frac{u}{t^2} + 2tv \qquad \text{and} \qquad \ddot{x} = \frac{2u}{t^3} - \frac{\dot{u}}{t^2} + 2v + 2t\dot{v}.$$

Substituting in, and bearing in mind that all the terms in which u and v have not been differentiated will cancel, we obtain

$$t^2\left[-\frac{\dot{u}}{t^2} + 2t\dot{v}\right] = t^3 \qquad \text{or} \qquad -\dot{u} + 2t^3\dot{v} = t^3.$$

Solving

$$\begin{pmatrix} t^{-1} & t^2 \\ -1 & 2t^3 \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ t^3 \end{pmatrix}$$

by multiplying by the inverse of the matrix on the left-hand side,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{3t^2}\begin{pmatrix} 2t^3 & -t^2 \\ 1 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ t^3 \end{pmatrix},$$

we obtain

$$\dot{u} = -\frac{t^3}{3} \qquad \text{and} \qquad \dot{v} = \frac{1}{3},$$

which implies that

$$u(t) = -\frac{t^4}{12} \qquad \text{and} \qquad v(t) = \frac{t}{3}.$$

This yields the particular integral

$$x(t) = -\frac{t^3}{12} + \frac{t^3}{3} = \frac{t^3}{4}.$$

[Again this would be much quicker with the 'guess' $x(t) = At^3$.]

(v)

$$\ddot{x} - 4x = \tan t.$$

The homogeneous equation has solutions e^{2t} and e^{-2t}, so we try

$$x(t) = \mathrm{e}^{2t}u(t) + \mathrm{e}^{-2t}v(t).$$

Differentiating gives

$$\dot{x} = 2\mathrm{e}^{2t}u(t) - 2\mathrm{e}^{-2t}v(t) + \mathrm{e}^{2t}\dot{u} + \mathrm{e}^{-2t}\dot{v}.$$

We set

$$\mathrm{e}^{2t}\dot{u} + \mathrm{e}^{-2t}\dot{v} = 0$$

and then

$$\dot{x} = 2e^{2t}u - 2e^{-2t}u \qquad \text{and} \qquad \ddot{x} = 4e^{2t}u + 2e^{2t}\dot{u} + 4e^{-2t}v - 2e^{-2t}\dot{v}.$$

Substituting for x and $\ddot{x}$ into the equation gives

$$2e^{2t}\dot{u} - 2e^{-2t}\dot{v} = \tan t.$$

We can now solve the simultaneous equations

$$\begin{pmatrix} e^{2t} & e^{-2t} \\ 2e^{2t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \tan t \end{pmatrix}$$

for $\dot{u}$ and $\dot{v}$,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{-4} \begin{pmatrix} -2e^{-2t} & -e^{-2t} \\ -2e^{2t} & e^{2t} \end{pmatrix} \begin{pmatrix} 0 \\ \tan t \end{pmatrix},$$

giving

$$\dot{u} = \frac{e^{-2t}\tan t}{4} \qquad \text{and} \qquad \dot{v} = -\frac{e^{2t}\tan t}{4}.$$

The best we can do is to write the particular integral in the form

$$x(t) = e^{2t}\int \frac{e^{-2t}\tan t}{4}\,dt - e^{-2t}\int \frac{e^{2t}\tan t}{4}\,dt.$$

For parts (vi) and (vii) we will give here the reduction of order calculations that yield the second solution, but in both cases they are quite unpleasant.

(vi)

$$(\tan^2 x - 1)\frac{d^2y}{dx^2} - 4\tan^3 x\frac{dy}{dx} + 2y\sec^4 x = (\tan^2 x - 1)(1 - 2\sin^2 x).$$

Our first task is to find a second solution of the homogeneous equation

$$(\tan^2 x - 1)\frac{d^2y}{dx^2} - 4\tan^3 x\frac{dy}{dx} + 2y\sec^4 x = 0 \qquad \text{(S18.1)}$$

given that one solution is $\sec^2 x$. To do this we use the reduction of order method, trying $y(x) = u(x)\sec^2 x$. For this we have

$$y'(x) = u'(x)\sec^2 x + 2u(x)\sec^2 x\tan x$$

and

$$y''(x) = u''(x)\sec^2 x + 4u'(x)\sec^2 x\tan x + 2u(x)[2\sec^2 x\tan^2 x + \sec^4 x].$$

When we substitute these expressions into (S18.1) terms in which $u(x)$ is not differentiated will cancel and we obtain

$$(\tan^2 x - 1)(u'' \sec^2 x + 4u' \sec^2 x \tan x) - 4\tan^3 x(u' \sec^2 x) = 0.$$

Putting $v = u'$ we obtain a first order equation for v,

$$v' + v\left[4\tan x - \frac{4\tan^3 x}{\tan^2 x - 1}\right] = 0.$$

To solve this we use an integrating factor

$$\exp\left(\int 4\tan x - \frac{4\tan^3 x}{\tan^2 x - 1}\,\mathrm{d}x\right).$$

One of these integrals is somewhat tricky,

$$I = \int \frac{4\tan^3 x}{\tan^2 x - 1}\,\mathrm{d}x.$$

If we substitute $z = \tan x$ then $\mathrm{d}z = \sec^2 x\,\mathrm{d}x$. Since $\sec^2 x = 1 + \tan^2 x = 1 + z^2$ we have

$$\begin{aligned} I &= \int \frac{4z^3}{(z^2-1)(z^2+1)}\,\mathrm{d}z \\ &= \int \frac{2z}{z^2-1} + \frac{2z}{z^2+1}\,\mathrm{d}z \\ &= \ln(z^2-1) + \ln(z^2+1) \\ &= \ln(z^4-1), \end{aligned}$$

and so

$$I = \ln(\tan^4 x - 1).$$

So we use the integrating factor

$$\exp\left(-4\ln(\cos x) - \ln(\tan^4 x - 1)\right) = \frac{1}{\cos^2 x(\tan^4 x - 1)},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v}{\cos^4 x(\tan^4 x - 1)}\right) = 0,$$

which implies that

$$\begin{aligned} v(x) &= C(\sin^4 x - \cos^4 x) \\ &= C[(1-\cos^2 x)\sin^2 x - (1-\sin^2 x)\cos^2 x] \\ &= C[\sin^2 x - \cos^2 x] \\ &= C\cos 2x, \end{aligned}$$

and so

$$u(x) = (C/2)\sin 2x = C\sin x\cos x.$$

The second solution is therefore

$$y(x) = \sec^2 x \sin x \cos x = \tan x.$$

In order to apply the variation of constants method we now try

$$y(x) = u(x)\sec^2 x + v(x)\tan x.$$

Taking the derivative we have

$$y'(x) = u'(x)\sec^2 x + v'(x)\tan x + 2u(x)\sec^2 x\tan x + v(x)\sec^2 x.$$

We impose the condition

$$u'(x)\sec^2 x + v'(x)\tan x = 0$$

and so obtain

$$y'(x) = 2u(x)\sec^2 x\tan x + v(x)\sec^2 x$$

and

$$\begin{aligned} y''(x) &= 2u'(x)\sec^2 x\tan x + v'(x)\sec^2 x \\ &\quad +[(4\sec^2 x\tan^2 x + \sec^4 x)u(x) + 2v(x)\sec^2 x\tan x]. \end{aligned}$$

Substituting these into the original equation we obtain

$$\begin{aligned} &(\tan^2 x - 1)[2u'(x)\sec^2 x\tan x + v'(x)\sec^2 x] \\ &= (\tan^2 x - 1)(1 - 2\sin^2 x), \end{aligned}$$

and so

$$2u'(x)\sec^2 x\tan x + v'(x)\sec^2 x = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x.$$

Solving the equation

$$\begin{pmatrix} \sec^2 x & \tan x \\ 2\sec^2 x\tan x & \sec^2 x \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ \cos^2 x - \sin^2 x \end{pmatrix}$$

for u' and v' we obtain

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{\sec^2 x(1-\tan^2 x)} \begin{pmatrix} \sec^2 x & -\tan x \\ -2\sec^2 x\tan x & \sec^2 x \end{pmatrix} \begin{pmatrix} 0 \\ \cos^2 x - \sin^2 x \end{pmatrix},$$

noting that the determinant

$$\sec^4 x - 2\sec^2 x\tan^2 x = \sec^2 x(\sec^2 x - \tan^2 x) = \sec^2 x(1 - \tan^2 x).$$

This gives

$$u' = \frac{-\tan x(\cos^2 x - \sin^2 x)}{\sec^2(1-\tan^2 x)} \qquad \text{and} \qquad v' = \frac{\cos^2 x - \sin^2 x}{1-\tan^2 x},$$

which simplify:

$$u' = -\sin x \cos^3 x \qquad \text{and} \qquad v'(x) = \cos^2 x = \frac{1+\cos 2x}{2}.$$

So

$$u(x) = \frac{\cos^4 x}{4} \qquad \text{and} \qquad v(x) = \frac{x}{2} + \frac{\sin 2x}{4}.$$

This gives the particular integral

$$\begin{aligned} y(x) &= \frac{\cos^2 x}{4} + \frac{x\tan x}{2} + \frac{\sin^2 x}{2} \\ &= \frac{1}{4} + \frac{x\tan x}{2} + \frac{\sin^2 x}{4}. \end{aligned}$$

(vii)

$$(1+\sin^2 t)\ddot{x} - (2\tan t + \sin t\cos t)\dot{x} + (1 - 2\tan^2 t)x = f(t).$$

The first task is to find a second linearly independent solution of the homogeneous problem

$$(1+\sin^2 t)\ddot{x} - (2\tan t + \sin t\cos t)\dot{x} + (1 - 2\tan^2 t)x = 0,$$

given that one solution is $\tan t$. We use the reduction of order method, trying $x(t) = y(t)\tan t$, so that

$$\dot{x} = y\sec^2 t + \dot{y}\tan t \qquad \text{and} \qquad \ddot{x} = 2y\sec^2 t\tan t + 2\dot{y}\sec^2 t + \ddot{y}\tan t.$$

Substituting into the equation we obtain

$$(1+\sin^2 t)[2\dot{y}\sec^2 t + \ddot{y}\tan t] - (2\tan t + \sin t\cos t)[\dot{y}\tan t] = 0,$$

or, with $z = \dot{y}$,

$$\dot{z} + z\left(\frac{2\sec^2 t}{\tan t} - \frac{2\tan t + \sin t\cos t}{1+\sin^2 t}\right) = 0.$$

The integrating factor is given by

$$\exp\left(\int \frac{2\sec^2 t}{\tan t} - \frac{2\tan t + \sin t\cos t}{1+\sin^2 t}\,\mathrm{d}t\right).$$

To calculate the integral

$$I = \int \frac{2\tan t + \sin t\cos t}{1+\sin^2 t}\,\mathrm{d}t$$

make the substitution $s = \sin t$. Then $\mathrm{d}s = \cos t\,\mathrm{d}t$, and

$$\begin{aligned} I &= \int \frac{\frac{2s}{1-s^2} + s}{1+s^2}\,\mathrm{d}s = \int \frac{3s - s^3}{1-s^4}\,\mathrm{d}s \\ &= \int \frac{s}{1-s^2} + \frac{2s}{1+s^2}\,\mathrm{d}s \\ &= -\tfrac{1}{2}\ln(1-s^2) + \ln(1+s^2), \end{aligned}$$

and so

$$I = \ln\left(\frac{1+\sin^2 t}{\sqrt{1-\sin^2 t}}\right) = \ln\left(\frac{1+\sin^2 t}{\cos t}\right).$$

So the integrating factor is

$$\exp\left[2\ln(\tan t) - \ln\left(\frac{1+\sin^2 t}{\cos t}\right)\right] = \frac{\sin t \tan t}{1+\sin^2 t}$$

and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{z(t)\sin t \tan t}{1+\sin^2 t}\right] = 0.$$

Therefore

$$\dot{y} = z = C\,\frac{1+\sin^2 t}{\sin t \tan t} = C\left[\frac{\cos t}{\sin^2 t} + \cos t\right],$$

which implies that

$$y(t) = C\left[-\frac{1}{\sin t} + \sin t\right] = -C\,\frac{\cos^2 t}{\sin t},$$

and so the second solution is

$$x(t) = y(t)\tan t = \cos t.$$

To use the variation of constant formula we look for a particular integral of the form

$$x(t) = u(t)\cos t + v(t)\tan t.$$

Differentiating gives

$$\dot{x} = \dot{u}\cos t + \dot{v}\tan t - u\sin t + v\sec^2 t.$$

We set

$$\dot{u}\cos t + \dot{v}\tan t = 0,$$

and then

$$\dot{x} = -u\sin t + v\sec^2 t$$

and

$$\ddot{x} = -\dot{u}\sin t + \dot{v}\sec^2 t - u\cos t + 2v\sec^2 t\tan t.$$

Substituting into the original equation gives

$$(1+\sin^2 t)(-\dot{u}\sin t + \dot{v}\sec^2 t) = f(t).$$

Solving

$$\begin{pmatrix} \cos t & \tan t \\ -\sin t & \sec^2 t \end{pmatrix}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ f(t)/(1+\sin^2 t) \end{pmatrix}$$

for $\dot{u}$ and $\dot{v}$ gives

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{\cos t\sec^2 t - \sin t\tan t}\begin{pmatrix} \sec^2 t & -\tan t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 \\ f(t)/(1+\sin^2 t) \end{pmatrix},$$

which is

$$\dot{u} = -\frac{f(t)\sin t}{(1+\sin^2 t)^2} \qquad \text{and} \qquad \dot{v} = \frac{f(t)\cos^2 t}{(1+\sin^2 t)^2}.$$

A particular integral is

$$x(t) = \tan t\int \frac{f(t)\cos^2 t}{(1+\sin^2 t)^2}\,\mathrm{d}t - \cos t\int \frac{f(t)\sin t}{(1+\sin^2 t)^2}\,\mathrm{d}t.$$

19

Cauchy-Euler equations

Exercise 19.1 *Find the general solution of the following equations, and also the particular solution satisfying the two specified conditions.*

(i) $x^2y'' - 4xy' + 6y = 0$, $y(1) = 0$ *and* $y'(1) = 1$;
(ii) $4x^2y'' + y = 0$, $y(1) = 1$ *and* $y'(1) = 0$;
(iii) $t^2\ddot{x} - 5t\dot{x} + 10x = 0$; $x(1) = 2$ *and* $\dot{x}(1) = 1$;
(iv) $t^2\ddot{x} + t\dot{x} - x = 0$, $x(1) = \dot{x}(1) = 1$;
(v) $x^2z'' + 3xz' + 4z = 0$, $z(1) = 0$ *and* $z'(1) = 5$;
(vi) $x^2y'' - xy' - 3y = 0$, $y(1) = 1$ *and* $y'(1) = -1$;
(vii) $4t^2\ddot{x} + 8t\dot{x} + 5x = 0$, $x(1) = 2$ *and* $\dot{x}(1) = 0$;
(viii) $x^2y'' - 5xy' + 5y = 0$, $y(1) = -2$ *and* $y'(1) = 1$;
(ix) $3x^2z'' + 5xz' - z = 0$, $z(1) = 3$ *and* $z'(1) = -1$; *and*
(x) $t^2\ddot{x} + 3t\dot{x} + 13x = 0$, $x(1) = -1$ *and* $\dot{x}(1) = 2$.

(i)

$$x^2y'' - 4xy' + 6y = 0 \qquad \text{with} \qquad y(1) = 0 \quad \text{and} \quad y'(1) = 1.$$

We try $y = x^k$, and so $y' = kx^{k-1}$ and $y'' = k(k-1)x^{k-2}$. Substituting in gives

$$k(k-1)x^k - 4kx^k + 6x^k.$$

Cancelling x^k we obtain

$$k^2 - 5k + 6 = 0,$$

with roots $k = 2$ and $k = 3$. So the general solution is

$$y(x) = Ax^2 + Bx^3.$$

Since $y'(x) = 2Ax + 3Bx^3$ to fit the initial conditions we want

$$A + B = 0 \qquad \text{and} \qquad 2A + 3B = 1,$$

so $A=-1$ and $B=1$, giving the solution

$$y(x)=x^3-x^2.$$

(ii)

$$4x^2y''+y=0 \qquad \text{with} \qquad y(1)=1 \quad \text{and} \quad y'(1)=0.$$

Trying $y=x^k$ gives the indicial equation

$$4k(k-1)+1=0 \qquad \Rightarrow \qquad 4k^2-4k+1=0$$

which has a repeated root $k=1/2$. So the general solution is

$$y(x)=Ax^{1/2}+Bx^{1/2}\ln x.$$

We have

$$y'(x)=\frac{A}{2x^{1/2}}+\frac{B\ln x}{2x^{1/2}}+Bx^{-1/2},$$

and so to fit the initial conditions we require

$$y(1)=A=1 \qquad \text{and} \qquad y'(1)=\tfrac{1}{2}A+B=0,$$

so $A=1$ and $B=-\frac{1}{2}$, so the solution is

$$y(x)=x^{1/2}+\tfrac{1}{2}x^{1/2}\ln x.$$

(iii)

$$t^2\ddot{x}-5t\dot{x}+10x \qquad \text{with} \qquad x(1)=2 \quad \text{and} \quad \dot{x}(1)=1.$$

With $x(t)=t^k$ we obtain

$$k(k-1)-5k+10=0 \qquad \Rightarrow \qquad k^2-6k+10=0,$$

with roots $k=3\pm\mathrm{i}$. So the general solution is

$$x(t)=t^3[A\cos(\ln t)+B\sin(\ln t)].$$

The derivative is

$$\dot{x}(t)=t^2[(3A+B)\cos(\ln t)+(3B-A)\sin(\ln t)],$$

and so to fit the initial conditions we require

$$x(1)=A=2 \qquad \text{and} \qquad \dot{x}(1)=3A+B=1,$$

i.e. $A=2$ and $B=-5$, so the solution is

$$x(t)=t^3[2\cos(\ln t)-5\sin(\ln t)].$$

(iv)

$$t^2\ddot{x} + t\dot{x} - x = 0 \qquad \text{with} \qquad x(1) = \dot{x}(1) = 1.$$

We try $x(t) = t^k$ and so

$$k(k-1) + k - 1 = 0 \qquad \Rightarrow \qquad k^2 - 1 = 0,$$

which gives $k = -1$ or $k = 1$ and so the general solution is

$$x(t) = \frac{A}{t} + Bt.$$

The derivative is given by $\dot{x} = -At^{-2} + B$, and so to fit the initial conditions we want

$$A + B = 1 \qquad \text{and} \qquad B - A = 1,$$

i.e. $A = 0$ and $B = 1$, giving

$$x(t) = t.$$

(v)

$$x^2z'' + 3xz' + 4z = 0 \qquad \text{with} \qquad z(1) = 0 \quad \text{and} \quad z'(1) = 5.$$

Trying $z = x^k$ we obtain

$$k(k-1) + 3k + 4 = 0 \qquad \Rightarrow \qquad k^2 + 2k + 4 = 0$$

with solutions $k = -1 \pm \sqrt{3}\mathrm{i}$. So the general solution is

$$z(x) = \frac{A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)}{x}.$$

We have

$$z'(x) = \frac{(\sqrt{3}B - A)\cos(\sqrt{3}\ln x) - (B + \sqrt{3}A)\sin(\sqrt{3}\ln x)}{x^2}.$$

To fit the initial conditions we take

$$z(1) = A = 0 \qquad \text{and} \qquad z'(1) = \sqrt{3}B - A = 5,$$

so $A = 0$, $B = 5/\sqrt{3}$, and the solution is

$$z(x) = \frac{5\sin(\sqrt{3}\ln x)}{\sqrt{3}x}.$$

(vi)

$$x^2y'' - xy' - 3y = 0 \qquad \text{with} \qquad y(1) = 1 \quad \text{and} \quad y'(1) = -1.$$

If we try $y = x^k$ we obtain

$$k(k-1) - k - 3 = 0 \qquad \Rightarrow \qquad k^2 - 2k - 3 = 0$$

with roots $k = -1$ and $k = 3$. So the general solution is

$$y(x) = \frac{A}{x} + Bx^3.$$

We have $y'(x) = -Ax^{-2} + 3Bx^2$, so to satisfy the initial conditions we need

$$y(1) = A + B = 1 \qquad \text{and} \qquad y'(1) = -A + 3B = -1,$$

so $B = 0$ and $A = 1$, giving the solution

$$y(x) = \frac{1}{x}.$$

(vii)

$$4t^2\ddot{x} + 8t\dot{x} + 5x = 0 \qquad \text{with} \qquad x(1) = 2 \quad \text{and} \quad \dot{x}(1) = 0.$$

We try $x(t) = t^k$ and then

$$4k(k-1) + 8k + 5 = 0 \qquad \Rightarrow \qquad 4k^2 + 4k + 5 = 0,$$

which has solutions $k = -\frac{1}{2} \pm \mathrm{i}$. The general solution is therefore

$$x(t) = \frac{A\cos(\ln t) + B\sin(\ln t)}{\sqrt{t}}.$$

Differentiating gives

$$\dot{x}(t) = \frac{(B - \frac{1}{2}A)\cos(\ln t) - (A + \frac{1}{2}B)\sin(\ln t)}{t^{3/2}},$$

so in order to satisfy the initial conditions we want

$$x(1) = A = 2 \qquad \text{and} \qquad \dot{x}(1) = B - \tfrac{1}{2}A = 0,$$

so $A = 2$, $B = 1$, and the solution is

$$x(t) = \frac{2\cos(\ln t) + \sin(\ln t)}{\sqrt{t}}.$$

(viii)

$$x^2y'' - 5xy' + 5y = 0 \qquad \text{with} \qquad y(1) = -2 \quad \text{and} \quad y'(1) = 1.$$

Setting $y(x) = x^k$ we obtain

$$k(k-1) - 5k + 5 = 0 \qquad \Rightarrow \qquad k^2 - 6k + 5 = 0$$

which has roots $k = 1$ and $k = 5$ so the general solution is

$$y(x) = Ax + Bx^5.$$

We have $y'(x) = A + 5Bx^4$, and so to obtain the correct values at $x = 1$ we want

$$y(1) = A + B = -2 \qquad \text{and} \qquad y'(1) = A + 5B = 1,$$

which implies that $B = 3/4$ and $A = -11/4$, so the solution is

$$y(x) = \frac{3x^5 - 11x}{4}.$$

(ix)

$$3x^2z'' + 5xz' - z = 0 \qquad \text{with} \qquad z(1) = 3 \quad \text{and} \quad z'(1) = -1.$$

We try $z(x) = x^k$ and then

$$3k(k-1) + 5k - 1 = 0 \qquad \Rightarrow \qquad 3k^2 + 2k - 1 = 0$$

which implies that $k = 1/3$ or $k = -1$. The general solution is therefore

$$z(x) = Ax^{1/3} + \frac{B}{x}.$$

We have $z'(x) = Ax^{-2/3}/3 - Bx^{-2}$, and so to fit the initial conditions we want

$$z(1) = A + B = 3 \qquad \text{and} \qquad z'(1) = \frac{A}{3} - B = -1,$$

so $A = B = 3/2$, and the solution is

$$z(x) = \frac{3}{2}\left(x^{1/3} + \frac{1}{x}\right).$$

(x)

$$t^2\ddot{x} + 3t\dot{x} + 13x = 0 \qquad \text{with} \qquad x(1) = -1 \quad \text{and} \quad \dot{x}(1) = 2.$$

Trying $x(t) = t^k$ we obtain

$$k(k-1) + 3k + 13 = 0 \qquad \Rightarrow \qquad k^2 + 2k + 13 = 0,$$

which has roots $k = -1 \pm 2\sqrt{3}\mathrm{i}$. So the general solution is

$$x(t) = \frac{A\cos(2\sqrt{3}\ln t) + B\sin(2\sqrt{3}\ln t)}{t}.$$

The derivative is given by

$$\dot{x}(t) = \frac{(2\sqrt{3}B - A)\cos(2\sqrt{3}\ln t) - (B + 2\sqrt{3}A)\sin(2\sqrt{3}\ln t)}{t^2},$$

and so to fit the initial conditions we take

$$x(1) = A = -1 \qquad \text{and} \qquad \dot{x}(1) = 2\sqrt{3}B - A = 2,$$

which gives $A = -1$, $B = \sqrt{3}/2$, and the solution

$$x(t) = \frac{\sqrt{3}\sin(2\sqrt{3}\ln t) - 2\cos(2\sqrt{3}\ln t)}{2t}.$$

Exercise 19.2 *If $x = \mathrm{e}^z$ then*

$$\frac{\mathrm{d}}{\mathrm{d}x} = \mathrm{e}^{-z}\frac{\mathrm{d}}{\mathrm{d}z}.$$

Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^{-2z}\left(\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right),$$

and hence that substituting $x = \mathrm{e}^z$ in

$$ax^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0 \tag{S19.1}$$

yields the linear equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (b-a)\frac{\mathrm{d}y}{\mathrm{d}z} + cy = 0. \tag{S19.2}$$

By solving (S19.2) find the solution of (S19.1) when the auxiliary equation

$$ak^2 + (b-a)k + c = 0$$

has

(i) two distinct real roots k_1 and k_2;
(ii) a repeated real root k; and
(iii) a complex conjugate pair of roots $\rho \pm \mathrm{i}\omega$.

When $x = \mathrm{e}^z$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x} = \mathrm{e}^{-z}\frac{\mathrm{d}}{\mathrm{d}z},$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-z}\frac{\mathrm{d}y}{\mathrm{d}z}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-z}\frac{\mathrm{d}y}{\mathrm{d}z}\right] \\ &= \mathrm{e}^{-z}\frac{\mathrm{d}}{\mathrm{d}z}\left[\mathrm{e}^{-z}\frac{\mathrm{d}y}{\mathrm{d}z}\right] \\ &= \mathrm{e}^{-z}\left[-\mathrm{e}^{-z}\frac{\mathrm{d}y}{\mathrm{d}z} + \mathrm{e}^{-z}\frac{\mathrm{d}^2y}{\mathrm{d}z^2}\right] \\ &= \mathrm{e}^{-2z}\left[\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right].\end{aligned}$$

If we substitute $x = \mathrm{e}^z$ in

$$ax^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$$

then we obtain

$$a\mathrm{e}^{2z}\mathrm{e}^{-2z}\left[\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \frac{\mathrm{d}y}{\mathrm{d}z}\right] + b\mathrm{e}^z\mathrm{e}^{-z}\frac{\mathrm{d}y}{\mathrm{d}z} + cy = 0,$$

which is

$$a\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (b-a)\frac{\mathrm{d}y}{\mathrm{d}z} + cy = 0.$$

We try $y = \mathrm{e}^{kz}$ and obtain the auxiliary equation

$$ak^2 + (b-a)k + c = 0.$$

(i) When this equation has two distinct roots k_1 and k_2 we have

$$y = A\mathrm{e}^{k_1z} + B\mathrm{e}^{k_2z} = Ax^{k_1} + Bx^{k_2},$$

since $\mathrm{e}^z = x$;

(ii) when there is a repeated root k we have

$$y = A\mathrm{e}^{kz} + Bz\mathrm{e}^{kz} = Ax^k + Bx^k\ln x,$$

since $z = \ln x$; and finally

(iii) when $k = \rho \pm \mathrm{i}\omega$ we have

$$y = \mathrm{e}^{\rho z}(A\cos\omega z + B\sin\omega z) = x^\rho[A\cos(\omega\ln x) + B\sin(\omega\ln x)].$$

20

Series solutions of second order linear equations

You may find the following two identities useful for these exercises:

$$2 \cdot 4 \cdot 6 \cdots 2n = 2^n n!$$

and

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!}.$$

Exercise 20.1 *Legendre's equation is*

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0. \qquad \text{(S20.1)}$$

If $y(x)$ is given by a power series,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

find the recurrence relation satisfied by the coefficients a_n. Show that if l is a positive integer then there is a solution given by a power series that has only a finite number of terms, i.e. a polynomial. For each value $l = 1$, 2, 3, and 4 find the polynomial solution that has $y(1) = 1$ (these are the 'Legendre polynomials' $P_l(x)$).

We will try to find a solution of Legendre's equation

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0 \qquad \text{(S20.2)}$$

given as a power series in x,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Assuming that we can differentiate term-by-term we have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting in to the original equation we obtain

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0,$$

and so

$$\sum_{n=2}^{\infty} n(n-1)a_n[x^{n-2} - x^n] - 2\sum_{n=1}^{\infty} n a_n x^n + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0.$$

Relabelling the sums so that the powers are all x^n we obtain

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + l(l+1)a_n]x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n = 0.$$

The constant (coefficient of x^0) is

$$2a_2 + l(l+1)a_0 = 0 \qquad \Rightarrow \qquad a_2 = -\frac{l(l+1)}{2}a_0, \tag{S20.3}$$

and the coefficient of x (when $n = 1$) is

$$6a_3 + l(l+1)a_1 - 2a_1 = 0 \qquad \Rightarrow \qquad a_3 = \frac{2 - l(l+1)}{6}a_1. \tag{S20.4}$$

For $n \geq 2$ we have non-zero coefficients of x^n from each term in the series, which we can group together to give

$$(n+1)(n+2)a_{n+2} - n(n+1)a_n + l(l+1)a_n = 0,$$

i.e.

$$\begin{aligned} a_{n+2} &= -\frac{l(l+1) - n(n+1)}{(n+1)(n+2)}\, a_n \\ &= -\frac{(l-n)(l+n+1)}{(n+1)(n+2)}\, a_n. \end{aligned}$$

Now suppose that l is a positive integer. If it is odd then we choose $a_0 = 0$, which implies that $a_2 = 0$ (from (S20.3)), and then the recurrence relation shows that $a_n = 0$ for all even n. With a non-zero choice of a_1, (S20.4) then gives a_3: if $l = 1$ we have $a_3 = 0$, and then the recurrence relation will give $a_n = 0$ for all odd n.

If $l > 1$ then we use the recurrence relation to find a_n for each odd n, until

$$a_{l+2} = -\frac{(l-l)(2l+1)}{(l+1)(l+2)} a_l,$$

which implies that $a_{l+2} = 0$. It then follows that $a_n = 0$ for all $n \geq l+2$. So the power series is the sum of a finite number of terms, i.e. a polynomial. The same argument works if l is an even integer, where we now take $a_0 \neq 0$ and $a_1 = 0$.

If $l = 1$ we have

$$P_1(x) = x.$$

For $l = 2$ we choose $a_1 = 0$ and obtain

$$y(x) = a_0[1 - 3x^2];$$

with $a_0 = -1/2$ we obtain

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

For $l = 3$ we set $a_0 = 0$ and end up with

$$y(x) = a_1 \left[x - \frac{5}{3}x^3\right];$$

to ensure that $P_3(1) = 1$ we choose $a_1 = -3/2$ to give

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Finally, when $l = 4$ we set $a_1 = 0$ to obtain the polynomial solution

$$y(x) = a_0 \left[1 - 10x^2 + \frac{35}{3}x^4\right];$$

we choose $a_0 = 3/8$ so that $y(1) = 1$ and have

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Exercise 20.2 *Find two independent power series solutions of the following equations, and use the ratio test to find their radius of convergence.*

(i) $y'' - xy' + y = 0$ *(cf. Exercise 17.5),*
(ii) $(1 + x^2)y'' + y = 0$,

(iii) $2xy'' + y' - 2y = 0$ *(you should be able to sum the two power series to obtain explicit forms for the two solutions),*

(iv) $y'' - 2xy' + 2ky = 0$*. By finding the recurrence relation for the coefficients in the power series identify those values of* k *for which one solution is a polynomial. Find both solutions when* $k = -2$ *and* $k = 2$*:; in each case you should be able to find a simple expression for one of the two solutions, while the other can be written as a power series whose general term you should be able to find explicitly.*

(i)

$$y'' - xy' + y = 0.$$

The point $x = 0$ is not a singular point, so we can try a simple power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{and} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting these into the equation we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Relabelling the first sum we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

So we have

$$2a_2 + a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)a_{n+2} - (n-1)a_n] x^n = 0.$$

Equating coefficients of x^n on both sides we have

$$a_2 = -\tfrac{1}{2} a_0 \qquad \text{and} \qquad a_{n+2} = \frac{(n-1)a_n}{(n+1)(n+2)}.$$

If we choose $a_0 \neq 0$ and $a_1 = 0$ then we obtain

$$y(x) = a_0 \left[1 - \frac{1}{2} x^2 - \frac{1}{2 \cdot 3 \cdot 4} x^4 - \frac{3}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} x^6 - \dots \right]$$

$$= a_0 \left[1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{(n-1)!2^{n-1}(2n)!} x^{2n} \right]$$
$$= a_0 \left[1 - \sum_{n=1}^{\infty} \frac{1}{n!2^n(2n-1)} x^{2n} \right].$$

The sum converges for all x, since the ratio of consecutive terms is

$$\frac{1}{(n+1)!2^{n+1}(2n+1)} x^{2(n+1)} \Big/ \frac{1}{n!2^n(2n-1)} x^{2n}$$
$$= \frac{2n-1}{(n+1)(2n+1)} x^2$$

which converges to zero as $n \to \infty$.

For a second linearly independent solution we choose $a_0 = 0$ and $a_1 \neq 0$. In this case we obtain $a_3 = 0$ from the recurrence relation, and so $a_n = 0$ for all $n \geq 2$, and the solution is $y(x) = a_1 x$.

(ii)

$$(1 + x^2)y'' + y = 0.$$

Once again $x = 0$ is not a singular point, so we try

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

With

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

the equation becomes

$$\sum_{n=2}^{\infty} n(n-1)a_n[x^{n-2} + x^n] + \sum_{n=0}^{\infty} a_n x^n.$$

So we have

$$\sum_{n=0}^{\infty} [(n+1)(n+2)a_{n+2} + a_n]x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^n = 0$$

or

$$(2a_2 + a_0) + (6a_3 + a_1)x + \sum_{n=2}^{\infty} [(n+1)(n+2)a_{n+2} + (n^2 - n + 1)a_n]x^n.$$

So

$$a_2 = -\tfrac{1}{2}a_0, \qquad a_3 = -\frac{a_1}{6}, \qquad \text{and} \qquad a_{n+2} = -\frac{n^2-n+1}{(n+1)(n+2)}\,a_n.$$

For one solution we take $a_0 \neq 0$ and $a_1 = 0$, to obtain

$$y_1(x) = a_0\left[1 - \frac{x^2}{2} + \frac{3x^4}{4!} - \frac{3\cdot 13x^6}{7!} + \ldots\right]$$

and for the other we choose $a_0 = 0$ and $a_1 \neq 0$,

$$y_2(x) = a_1\left[x - \frac{x^3}{6} + \frac{7x^5}{5!} - \frac{7\cdot 21x^7}{7!} + \ldots\right]$$

The radius of convergence can be found by considering the ratio of successive terms in each series:

$$\frac{a_{n+2}x^{n+2}}{a_nx^n} = \frac{a_{n+2}}{a_n}\,x^2 = -\frac{n^2-n+1}{(n+1)(n+2)}\,x^2 \to x^2$$

as $n \to \infty$. This is less than one provided that $|x| < 1$, and greater than one if $|x| > 1$, so the radius of convergence is 1.

(iii)

$$2xy'' + y' - 2y = 0. \qquad \text{(S20.5)}$$

Written in the standard form this is

$$y'' + \frac{y'}{2x} - \frac{y}{x} = 0,$$

and so $x = 0$ is a regular singular point. We can look for a solution in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\sigma+n}. \qquad \text{(S20.6)}$$

For this we have

$$y'(x) = \sum_{n=0}^{\infty}(n+\sigma)a_nx^{\sigma+n-1}$$

and

$$y''(x) = \sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nx^{\sigma+n-2}.$$

Substituting these expressions into the equation (S20.5) we obtain

$$2\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n x^{\sigma+n-1} + \sum_{n=0}^{\infty}(n+\sigma)a_n x^{\sigma+n-1}$$
$$-2\sum_{n=0}^{\infty} a_n x^{\sigma+n} = 0.$$

If we set the coefficient of the lowest order power in x, $x^{\sigma+n-1}$, to zero, we obtain the indicial equation

$$a_0[2\sigma(\sigma-1)+\sigma] = 0 \qquad \Rightarrow \qquad 2\sigma^2 - \sigma = 0$$

which has solutions $\sigma = 0$ and $\sigma = \frac{1}{2}$. These do not differ by an integer, so we should be able to find two linearly independent series solutions in the form (S20.6)

First we use $\sigma = 0$, for which the equation reads

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

This is the same as

$$2\sum_{n=1}^{\infty} n(n+1)a_{n+1} x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0,$$

and so equating coefficients of powers of x on both sides we have

$$a_1 - 2a_0 = 0 \qquad \text{and} \qquad 2n(n+1)a_{n+1} + (n+1)a_{n+1} - 2a_n = 0$$

where the general relation, valid only for $n \geq 1$, can be simplified to give

$$a_{n+1} = \frac{2}{(2n+1)(n+1)}\, a_n.$$

So, if we choose $a_0 = 1$,

$$\begin{aligned}
y_1(x) &= 1 + 2x + \frac{2^2}{2\cdot 3}\, x^2 + \frac{2^3}{2\cdot 3\cdot 3\cdot 5} x^3 + \frac{2^4}{2\cdot 3\cdot 3\cdot 5\cdot 4\cdot 7}\, x^4 + \dots \\
&= \sum_{n=0}^{\infty} \frac{2^n n! 2^n}{n!(2n)!}\, x^n = \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!}\, x^n \\
&= \sum_{n=0}^{\infty} \frac{(2\sqrt{x})^{2n}}{(2n)!} \\
&= \frac{e^{2\sqrt{x}} + e^{-2\sqrt{x}}}{2}
\end{aligned}$$

$$= \cosh 2\sqrt{x}.$$

This converges for all x since the series for $\cosh x$ converges; or you can check it directly since the ratio of consecutive terms in the power series form of the solution is

$$\frac{a_{n+1}x^{n+1}}{a_n x^n} = \frac{2}{(2n+1)(n+1)}\, x \to 0$$

as $n \to \infty$.

To find a second solution we use $\sigma = \frac{1}{2}$, and then the equation is

$$2\sum_{n=0}^{\infty}(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty}(n+\tfrac{1}{2})a_n x^{n-\frac{1}{2}} - 2\sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} = 0.$$

The coefficient of $x^{-1/2}$ vanishes by the choice of σ, and for $n \geq 1$ the coefficient of $n + \frac{1}{2}$ is

$$2(n+\tfrac{1}{2})(n-\tfrac{1}{2})a_n + (n+\tfrac{1}{2})a_n - 2a_{n-1} = 0.$$

So

$$a_n = \frac{2}{n(2n+1)} a_{n-1}.$$

Now we have (choosing $a_0 = 1$)

$$\begin{aligned}
y_2(x) &= x^{1/2}\left(1 + \frac{2}{3}\,x + \frac{2^2}{3\cdot 2\cdot 5}\,x^2 + \frac{2^3}{3\cdot 2\cdot 5\cdot 3\cdot 7}\,x^3 + \dots\right)\\
&= x^{1/2}\sum_{n=0}^{\infty}\frac{2^n 2^n n!}{n!(2n+1)!}\,x^n\\
&= x^{1/2}\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n+1)!}\,x^n\\
&= \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(2\sqrt{x})^{2n+1}}{(2n+1)!}\\
&= \frac{e^{2\sqrt{x}} - e^{-2\sqrt{x}}}{4}\\
&= \tfrac{1}{2}\sinh 2\sqrt{x},
\end{aligned}$$

and so a second linearly independent solution is $\sinh 2\sqrt{x}$ (and the series converges, as before, for all x).

(iv)

$$y'' - 2xy' + 2ky = 0.$$

Zero is not a singular point, so we try

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

We therefore have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{and} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

and substituting into the original equation we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=1}^{\infty} n a_n x^n + 2k \sum_{n=0}^{\infty} a_n x^n = 0,$$

Relabelling the first sum we have

$$[2a_2 + 2ka_0] + \sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+2} + 2(k-n)a_n] x^n = 0,$$

and so

$$a_2 = -ka_0 \qquad \text{and} \qquad a_{n+2} = \frac{2(n-k)}{(n+1)(n+2)} a_n \quad \text{for} \quad n \geq 1.$$

One solution will be a polynomial if k is a positive integer, since then $a_{k+2} = 0$. Otherwise we have

$$y_1(x) = a_0 \left[1 - kx^2 - \frac{2^2(2-k)}{4!} x^4 - \frac{2^3(2-k)(4-k)}{6!} x^6 - \dots \right]$$

and

$$y_2(x) = a_1 \left[x + \frac{2(1-k)}{3!} x^3 + \frac{2^2(1-k)(3-k)}{5!} x^5 + \dots \right].$$

The ratio of successive terms in these series are

$$\frac{a_{n+2} x^{n+2}}{a_n x^n} = \frac{(n+1)(n+2)(n-k-2)}{n(n-1)(n-k)} x^2 \to x^2$$

as $n \to \infty$, so the radius of convergence is 1.

If $k = -2$ then the two solutions are

$$\begin{aligned} y_1(x) &= a_0 \left[1 + 2x^2 + \frac{2^2}{3} x^4 + \frac{2^3}{3 \cdot 5} x^6 + \dots \right] \\ &= a_0 \sum_{n=0}^{\infty} \frac{2^n 2^n n!}{(2n)!} x^{2n} \end{aligned}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{4^n n!}{(2n)!} x^{2n}$$

and

$$\begin{aligned} y_2(x) &= a_1 \left[x + x^3 + \frac{x^5}{2!} + \frac{x^7}{4!} + \dots \right] \\ &= a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} \\ &= a_1 x \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \\ &= a_1 x \mathrm{e}^{x^2}. \end{aligned}$$

If $k = 2$ then the two solutions are

$$y_1(x) = a_0 \left[1 - 2x^2\right]$$

and

$$y_2(x) = a_1 \left[x - \frac{2}{3!} x^3 - \frac{2^2}{5!} x^5 - \frac{2^3 \cdot 3}{7!} x^7 - \dots \right].$$

In general, for $n \geq 2$ the coefficient of x^{2n+1} is given by

$$\begin{aligned} a_{2n+1} &= -a_1 \frac{2^n \cdot 3 \cdot 5 \cdot (2n-3)}{(2n+1)!} \\ &= -a_1 \frac{2^n (2n-2)!}{2^{n-1}(n-1)!(2n+1)!} \\ &= -2a_1 \frac{(2n-2)!}{(n-1)!(2n+1)!}. \end{aligned}$$

With the convention that $0! = 1$ we can write the solution as

$$y_2(x) = a_1 \left[x - \sum_{n=1}^{\infty} \frac{2(2n-2)!}{(n-1)!(2n+1)!} x^{2n+1} \right].$$

Exercise 20.3 *Find one power series solution of the equation*

$$x(1-x)y'' - 3xy' - y = 0.$$

You should be able to sum this power series to write down the solution explicitly. Now use the reduction of order method to find a second solution.

Zero is a regular singular point, so we try a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\sigma+n}.$$

We therefore have

$$y'(x) = \sum_{n=0}^{\infty}(\sigma+n)a_n x^{\sigma+n-1} \qquad \text{and} \qquad y''(x) = \sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n x^{\sigma+n-2},$$

and substituting into the equation gives

$$\sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n[x^{\sigma+n-1}-x^{\sigma+n}]-3\sum_{n=0}^{\infty}(\sigma+n)a_n x^{\sigma+n}-\sum_{n=0}^{\infty}a_n x^{\sigma+n} = 0.$$

Setting the coefficient of the lowest power of x, $x^{\sigma-1}$ to zero gives the indicial equation for σ,

$$\sigma(\sigma-1) = 0.$$

This gives $\sigma = 0$ or $\sigma = 1$. These roots differ by an integer, so we would anticipate problems.

We use the larger choice, $\sigma = 1$, in order to find one series solution of this form, and put

$$y(x) = \sum_{n=1}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2},$$

and then we have

$$\sum_{n=2}^{\infty} n(n-1)a_n[x^{n-1}-x^n] - 3\sum_{n=1}^{\infty} n a_n x^n - \sum_{n=1}^{\infty} a_n x^n = 0.$$

With some relabelling and rearrangement this becomes

$$-\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty}[n(n+1)a_{n+1} - 3na_n]x^n - \sum_{n=1}^{\infty} a_n x^n = 0.$$

From the coefficient of x we have

$$2a_2 - 4a_1 = 0 \qquad \Rightarrow \qquad a_2 = 2a_1$$

and from the coefficient of x^n for $n \geq 2$ we obtain

$$a_{n+1} = \frac{n+1}{n}\, a_n.$$

Therefore, taking $a_1 = 1$, one solution is

$$\begin{aligned} y(x) &= x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ldots \\ &= x[1 + 2x + 3x^2 + 4x^3 + \ldots] \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}[1 + x + x^2 + x^3 + x^4 + \ldots] \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

To use the reduction of order method we try

$$y(x) = \frac{xu(x)}{(1-x)^2},$$

and so

$$y'(x) = \frac{xu'}{(1-x)^2} + \frac{u}{(1-x)^2} + \frac{2xu}{(1-x)^3}$$

and

$$y''(x) = \frac{xu''}{(1-x)^2} + \frac{2u'}{(1-x)^2} + \frac{4xu'}{(1-x)^3} + \frac{4u}{(1-x)^3} + \frac{6xu}{(1-x)^4}.$$

When we substitute these expressions into the original equation – remembering that the terms in which u has not been differentiated will (thankfully) cancel – we obtain

$$x(1-x)\left[\frac{xu''}{(1-x)^2} + \frac{2u'}{(1-x)^2} + \frac{4xu'}{(1-x)^3}\right] - \frac{3x^2u'}{(1-x)^2} = 0.$$

Multiplying up by $(1-x)^2$ we obtain

$$x(1-x)\left[xu'' + 2u' + \frac{4xu'}{1-x}\right] - 3x^2u' = 0.$$

Setting $v = u'$ and rearranging gives

$$v' + v\left[\frac{2}{x} + \frac{1}{1-x}\right] = 0.$$

The integrating factor for this equation is

$$\exp\left(\int \frac{2}{x} + \frac{1}{1-x}\,\mathrm{d}x\right) = \exp(2\ln x - \ln(1-x)) = \frac{x^2}{1-x},$$

and so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{x^2v}{1-x}\right] = 0.$$

Therefore

$$v(x) = u'(x) = A\frac{1-x}{x^2} = A\left[\frac{1}{x^2} - \frac{1}{x}\right],$$

This gives

$$u(x) = A\left[-\frac{1}{x} - \ln x\right]$$

and so the second solution is

$$y(x) = \frac{1}{(1-x)^2} + \frac{x\ln x}{(1-x)^2}.$$

Exercise 20.4 *Find one series solution of the 'modified Bessel equation'*

$$x^2y'' + xy' - x^2y = 0.$$

Zero is a regular singular point, so we try a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\sigma+n},$$

and then

$$y'(x) = \sum_{n=0}^{\infty}(\sigma+n)a_n x^{\sigma+n-1} \qquad \text{and} \qquad y''(x) = \sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n x^{\sigma+n-2}.$$

Substituting these into the equation we have

$$\sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n x^{\sigma+n} + \sum_{n=0}^{\infty}(\sigma+n)a_n x^{\sigma+n} - \sum_{n=0}^{\infty} a_n x^{\sigma+n+2} = 0.$$

The coefficient of x^σ is

$$\sigma(\sigma-1) + \sigma = \sigma^2,$$

so we have a repeated root $\sigma = 0$. We can therefore try a standard power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Relabelling the final sum we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

From the coefficient of x we obtain $a_1 = 0$, and from the coefficient of x^n with $n \geq 2$ we have

$$a_{n+2} = \frac{a_n}{n^2}.$$

So, taking $a_0 = 1$, one solution is

$$\begin{aligned} y(x) &= 1 + x^2 + \frac{x^4}{2^2} + \frac{x^6}{2^2 4^2} + \frac{x^8}{2^2 4^2 6^2} + \dots \\ &= 1 + x^2 \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2}. \end{aligned}$$

Exercise 20.5 *Find a series solution for Bessel's equation of order one,*

$$x^2 y'' + x y' + (x^2 - 1) y = 0. \tag{S20.7}$$

You should obtain

$$y(x) = cx \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!\, n!};$$

with the choice $c = 1/2$ this gives the standard form of the Bessel function $J_1(x)$,

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!\, n!} \left(\frac{x}{2}\right)^{2n+1}.$$

From the analysis in the text (Section 20.4) we look for a solution $\sum_{n=0}^{\infty} a_n x^{\sigma+n}$ with $\sigma = 1$:

$$y(x) = \sum_{n=1}^{\infty} a_n x^n.$$

With this choice for $y(x)$ we have

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{and} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into the equation we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n [x^{n+2} - x^n] = 0.$$

Relabelling we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty}(n-1)a_n x^n + \sum_{n=3}^{\infty} a_{n-2}x^n = 0.$$

The coefficient of x vanishes; the coefficient of x^2 is $3a_2$, and so $a_2 = 0$, and the recurrence relation, valid for $n \geq 3$,

$$a_n = -\frac{a_{n-2}}{(n+1)(n-1)},$$

shows that $a_n = 0$ for all even n. With $a_1 = c$ we therefore obtain the solution

$$\begin{aligned} y(x) &= c\left[x - \frac{x^3}{2\cdot 4} + \frac{x^5}{2\cdot 4\cdot 4\cdot 6} - \frac{x^7}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8} + \dots\right] \\ &= cx\left[1 - \frac{x^2}{2\cdot 4} + \frac{x^4}{2\cdot 4\cdot 4\cdot 6} - \frac{x^6}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8} + \dots\right] \\ &= 2cx\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n}n!(n+1)!}; \end{aligned}$$

with the choice $c = 1/4$ we obtain

$$J_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!n!}\left(\frac{x}{2}\right)^{2n+1}.$$

Exercise 20.6 *In order to find a second solution of (S20.7), substitute*

$$y(x) = J_1(x)\ln x + \frac{1}{x}\left[\sum_{n=0}^{\infty} b_n x^n\right],$$

where $J_1(x)$ is the series solution from the previous question, to show that

$$b_1 + b_0 x + \sum_{n=2}^{\infty}[(n^2-1)b_{n+1} + b_{n-1}]x^n = -2\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)}{(k+1)!\,k!}\left(\frac{x}{2}\right)^{2k+1}.$$

Hence show that $b_0 = -1$, $b_1 = 0$, and that b_n obeys the recurrence relation

$$(n^2-1)b_{n+1} + b_{n-1} = 0$$

if n is even and, for $k = 1, 2, 3, \dots$,

$$[(2k+1)^2 - 1]b_{2(k+1)} + b_{2k} = -\frac{(-1)^k(2k+1)}{2^{2k}(k+1)!\,k!}. \qquad \text{(S20.8)}$$

Deduce that $b_j = 0$ for all odd values of j.

Denoting by H_n the sum

$$H_n = \sum_{j=1}^{n} \frac{1}{j},$$

verify that

$$b_{2k} = \frac{(-1)^k (H_k + H_{k-1})}{2^{2k} k!\,(k-1)!}$$

solves (S20.8) and hence write down a second solution of (S20.7).

To find a second solution of

$$x^2 y'' + xy' + (x^2 - 1)y = 0 \tag{S20.9}$$

we try

$$y(x) = J_1(x) \ln x + \frac{b_0}{x} + \sum n = 0^\infty b_{n+1} x^n,$$

and so

$$y'(x) = J_1'(x) \ln x + \frac{J_1(x)}{x} - \frac{b_0}{x^2} + \sum_{n=1}^{\infty} n b_{n+1} x^{n-1}$$

and

$$y''(x) = J_1''(x) \ln x + \frac{2J_1'(x)}{x} - \frac{J_1(x)}{x^2} + \frac{2b_0}{x^3} + \sum_{n=2}^{\infty} n(n-1) b_{n+1} x^{n-2}.$$

Substituting these expressions into the equation (S20.9) gives

$$x_2 J_1'' \ln x + 2xJ_1' - J_1 + \frac{2b_0}{x} + \sum_{n=2}^{\infty} n(n-1) b_{n+1} x^n + xJ_1' \ln x + J_1 - \frac{b_0}{x}$$
$$+ \sum_{n=1}^{\infty} n b_{n+1} x^n + (x^2 - 1) \left[J_1 \ln x + \frac{b_0}{x} + \sum_{n=0}^{\infty} b_{n+1} x^n \right] = 0,$$

which after cancellations becomes

$$2xJ_1'(x) + \sum_{n=2}^{\infty} n(n-1) b_{n+1} x^n + \sum_{n=1}^{\infty} n b_{n+1} x^n + b_0 x$$
$$+ \sum_{n=0}^{\infty} (b_{n+1} x^{n+2} - b_{n+1} x^n) = 0.$$

Using the series solution for $J_1(x)$ we obtain

$$-2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(n+1)!n!} \left(\frac{x}{2}\right)^{2n+1} = \sum_{n=0}^{\infty} (n^2 - 1) b_{n+1} x^n + \sum_{n=2}^{\infty} b_{n-1} x^n + b_0 x.$$

The coefficients of x^0 and x give

$$b_1 = 0 \qquad \text{and} \qquad b_0 = -1.$$

For even $n \geq 2$ we have

$$(n^2 - 1)b_{n+1} + b_{n-1} = 0,$$

which, given that $b_1 = 0$, shows that all odd coefficients are zero.

For odd $n = 2k+1$, the coefficient of x^{2k+1} gives

$$[(2k+1)^2 - 1]b_{2(k+1)} + b_{2(k-1)} = \frac{(-1)^k(2k+1)}{(k+1)!k!2^{2k}}.$$

We write

$$H_n = \sum_{j=1}^{n} \frac{1}{j},$$

and want to check that

$$b_{2k} = \frac{(-1)^k(H_k + H_{k-1})}{2^{2k}k!(k-1)!}$$

satisfies this equation. We have

$$\begin{aligned}
[(2k+1)^2 - 1]&\left[\frac{(-1)^{k+1}(H_{k+1} + H_k)}{2^{2k+2}(k+1)!k!}\right] + \frac{(-1)^k(H_k + H_{k-1})}{2^{2k}k!(k-1)!} \\
&= \frac{-(-1)^k}{2^{2k}k!(k-1)!}\left[\frac{((2k+1)^2 - 1)(H_{k+1} + H_k)}{4(k+1)k} - (H_k + H_{k-1})\right] \\
&= \frac{-(-1)^k}{2^{2k}k!(k-1)!}\left[(H_{k+1} + H_k) - (H_k + H_{k-1})\right] \\
&= \frac{-(-1)^k}{2^{2k}k!(k-1)!}\left[\frac{1}{k+1} + \frac{1}{k}\right] \\
&= \frac{-(-1)^k}{2^{2k}k!(k-1)!}\,\frac{2k+1}{k(k+1)} \\
&= \frac{(-1)^k(2k+1)}{(k+1)!k!2^{2k}},
\end{aligned}$$

as required.

The second solution to Bessel's equation of order one is therefore

$$y(x) = \ln x \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!}\left(\frac{x}{2}\right)^{2n+1} + \frac{1}{x}\left[\sum_{n=0}^{\infty} \frac{(-1)^n(H_n + H_{n-1})}{(n+1)!n!2^{2n}}\,x^{2n}\right].$$

Exercise 20.7 *Show that when n is a positive integer one solution of Bessel's equation*

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-n^2)y=0$$

can be written as the power series

$$J_n(x)=\sum_{j=0}^{\infty}(-1)^j\frac{1}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}. \tag{S20.10}$$

Bessel's equation of order n is

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-n^2)y=0.$$

Since zero is a regular singular point we try a solution of the form

$$y(x)=\sum_{j=0}^{\infty}a_jx^{\sigma+j},$$

and then from the treatment in section 20.4 the indicial equation implies that $\sigma=\pm n$. We take $\sigma=n$, the larger of the two roots, and so try for a solution in the form

$$y(x)=\sum_{j=n}^{\infty}a_jx^j.$$

We therefore have

$$y'(x)=\sum_{j=n}^{\infty}ja_jx^{j-1} \qquad \text{and} \qquad y''(x)=\sum_{j=n}^{\infty}j(j-1)a_jx^{j-2}.$$

Substituting these two expressions into the equation we obtain

$$\sum_{j=n}^{\infty}j(j-1)a_jx^j+\sum_{j=n}^{\infty}ja_jx^j+\sum_{j=n}^{\infty}a_j[x^{j+2}-n^2x^j]=0.$$

The coefficients of x^n and x^{n+1} give

$$n(n-1)a_n+na_n-n^2a_n=0 \quad \text{and} \quad n(n+1)a_{n+1}+(n+1)a_{n+1}-n^2a_{n+1}=0,$$

so a_n is arbitrary and $a_{n+1}=0$. The coefficient of x^j for $j\geq n+2$ gives

$$j(j-1)a_j+ja_j+a_{j-2}-n^2a_j=0,$$

which provides the recurrence relation

$$a_j=-\frac{a_{j-2}}{j^2-n^2}=-\frac{a_{j-2}}{(j+n)(j-n)}.$$

So we have

$$\begin{aligned}
J_n(x) &= a_n\left[x^n - \frac{1}{(2n+2)2}\,x^{n+2} + \frac{1}{(2n+2)(2n+4)2\cdot 4}\,x^{n+4}\right. \\
&\qquad \left. - \frac{1}{(2n+2)(2n+4)(2n+6)2\cdot 4\cdot 6}\,x^{n+6} + \ldots\right] \\
&= a_n x^n \sum_{j=0}^{\infty}(-1)^j \frac{n!}{2^j(n+j)!2^j j!}\,x^{2j} \\
&= 2^n a_n n! \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}.
\end{aligned}$$

Choosing $a_n = 1/(2^n n!)$ we obtain the standard form of $J_n(x)$,

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}.$$

Exercise 20.8 *The gamma function generalises the factorial function to values that are not integers. For any real number z we define*

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Integrate by parts in order to show that for a positive integer n

$$\Gamma(n+1) = n\Gamma(n).$$

Since $\Gamma(1) = 1$, deduce that $\Gamma(n+1) = n!$. (Using the gamma function in place of one of the factorials in the power series (S20.10) gives

$$J_\nu(x) = \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!\,\Gamma(\nu+j+1)}\left(\frac{x}{2}\right)^{\nu+2j},$$

and this formula now applies for any real number ν. This is where the strange normalisation of J_ν for non-integer ν comes from (see comments after Example 20.3).)

We have

$$\Gamma(z) = \int_0^{\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Integrating by parts we obtain

$$\Gamma(n+1) = \left[-t^n\mathrm{e}^{-t}\right]_0^{\infty} + \int_0^{\infty} nt^{n-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

$$\begin{aligned} &= n\int_0^\infty t^{n-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &= n\Gamma(n), \end{aligned}$$

and we have

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t = 1,$$

so

$$\Gamma(n+1) = n!$$

Exercise 20.9 *(C) Write a short program to generate the coefficients in the power series expansion of $J_\nu(x)$ for any value of ν using the recurrence relation (20.17). Investigate how many terms of the expansion you need to take in order to approximate the solution well on a fixed interval ($0 \le x \le 10$, say). (You might like to look at the* `M`*-file* `besselseries.m`*, which produced the Bessel function figures in this chapter.)*

Below is the listing of the file `besselseries.m`, which calculates the coefficients in the power series expansion, and then plots various approximations given by a finite number of terms, along with the 'exact' values generated by MATLAB's `besselj` function.

```
%% Bessel function of order nu from its series expansion

%% Sum up to power x^(2n)

nu=input('nu = '); N=input('N= ');

a(1)=-1/(4*(1+nu));

%% a(i) is coefficient of x^(2i)

for i=2:N;

   a(i)=-a(i-1)/(4*i*(i+nu));

end

x=linspace(0,10,1000);

y=1+0.*x; hold on
```

```
for k=1:N;
   y=y+a(k)*x.^(2*k);
   ay=y.*x.^nu;
   if (k/2)~=floor(k/2) & k>1;
        plot(x,ay);
   end
end

%% Change the normalisation

z=besselj(nu,x)*2^nu*gamma(1+nu);

plot(x,z,'linewidth',2)

ylim([1.5*min(z),1.5*max(z)])
```

Exercise 20.10 *(T) The Bessel functions might seem exotic, but they arise very naturally in problems that have radial symmetry. For example, the vibrations of a circular drum satisfy*

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \tag{S20.11}$$

where $u(r,\theta,t)$ is the displacement of the circular skin of the drum at a point expressed in polar coordinates. In the method of separation of variables we look for a solution of the form

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t),$$

and try this guess in the equation. Substitute this in to (S20.11) and show that

$$\frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{1}{rR}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}. \tag{S20.12}$$

The left-hand side of this equation is a function of t alone, and the right-hand side a function of r and θ, so in order to be always equal they must both be constants. Choosing

$$\frac{1}{T}\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -k^2$$

(there are good physical reasons for choosing this constant to be negative) show that we can rearrange (S20.12) to give

$$-\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = \frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + r^2k^2. \tag{S20.13}$$

Now the left-hand side is a function of θ alone, while the right-hand side is a function of r alone: so both sides must be equal to a constant. Now we choose

$$-\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = \nu^2$$

(again there are good physical reasons why this constant should be positive); show that in this case (S20.13) can be rearranged to give

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + (r^2k^2 - \nu^2)R = 0.$$

Finally substitute $x = rk$ to show that R satisfies Bessel's equation of order ν,

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + x\frac{\mathrm{d}R}{\mathrm{d}x} + (x^2 - \nu^2)R = 0.$$

Substituting $u = R(r)\Theta(\theta)T(t)$ into

$$\frac{\partial^2 u}{\partial^2 t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2},$$

we obtain

$$R\Theta\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = \frac{\Theta T}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{RT}{r^2}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}.$$

Dividing by $R\Theta T$ gives

$$\frac{1}{T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = \frac{1}{rR}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}.$$

The left-hand side is a function of t alone, and the right-hand side depends only on r and θ. For these always to be equal they must equal a constant, which we choose as $-k^2$. In this case we have

$$\frac{1}{rR}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{1}{r^2\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} = -k^2.$$

Multiplying through by r^2 and rearranging yields

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + k^2r^2 = -\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2}.$$

In this equation the left-hand side depends only on r and the right-hand side depends only on θ: so both sides must be equal to a constant, which we choose to be ν^2. This gives

$$\frac{r}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + k^2r^2 = \nu^2,$$

and if we differentiate on the left we have

$$\frac{r^2}{R}\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}r} + k^2r^2 = \nu^2.$$

Multiplying up by R and rearranging we obtain

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + (k^2r^2 - \nu^2)R = 0.$$

Finally we set $x = kr$; then

$$r\frac{\mathrm{d}}{\mathrm{d}r} = x\frac{\mathrm{d}}{\mathrm{d}x},$$

and so the equation becomes

$$x^2\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + x\frac{\mathrm{d}R}{\mathrm{d}x} + (x^2 - \nu^2)R = 0,$$

which is Bessel's equation of order ν.

21

Euler's method

Exercise 21.1 *Apply Euler's method to the general linear equation $\dot{x} = \lambda x$. Find the approximation x_n, and using (21.6) show that as $h \to 0$ the numerical solution converges to the true solution.*

We have

$$x_{n+1} = x_n + h\lambda x_n = (1 + h\lambda)x_n,$$

and so

$$x_n = (1 + h\lambda)^n x_0.$$

The approximation this provides to $x(t)$ is given by x_n, where $nh = t$, so is

$$x(t) \approx x_n = (1 + \frac{\lambda t}{n})^n x_0,$$

where we have replaced h by t/n in this expression. We now use the limiting behaviour

$$\left(1 + \frac{t}{n}\right)^n \to \mathrm{e}^t$$

to deduce that as $h \to 0$, i.e. as $n \to \infty$, we have

$$\left(1 + \frac{\lambda t}{n}\right)^n x_0 \to \mathrm{e}^{\lambda t} x_0,$$

which is the exact solution.

Exercise 21.2 *There are variants of the Euler method that have the advantage of better stability properties, but have the disadvantage of no longer being explicit schemes. For example, the* backwards Euler *method is*

$$x_{n+1} = x_n + hf(x_{n+1}, t_{n+1}),$$

which has to be solved at each stage to find x_{n+1} in terms of x_n. Apply this method to the linear equation $\dot{x} = x$, and show that once again the method converges to the true solution $x(t) = \mathrm{e}^t$ as $t \to \infty$.

If we use the backwards Euler method for $\dot{x} = x$ then we have

$$x_{n+1} = x_n + hx_{n+1}.$$

In this simple case we can solve easily to find an expression for x_{n+1} in terms of x_n, namely

$$x_{n+1} = \frac{x_n}{1-h}.$$

This has solution

$$x_n = \frac{x_0}{(1-h)^n}.$$

The approximation this gives for $x(t)$ when $t = nh$ is

$$x(t) \approx x_0 \left(1 - \frac{t}{n}\right)^{-n},$$

and as $h \to 0$, i.e. as $n \to \infty$, this converges to

$$\frac{x_0}{\mathrm{e}^{-t}} = x_0 \mathrm{e}^t,$$

the exact solution.

Exercise 21.3 *Another variant of the standard Euler method is the trapezoidal Euler method. If $x(t)$ is the solution of $\dot{x} = f(x,t)$ then we have*

$$x(t+h) = x(t) + \int_t^{t+h} f(x(s), s)\,\mathrm{d}s.$$

Use the trapezium rule to approximate the integral to derive this scheme,

$$x_{n+1} = x_n + h[\tfrac{1}{2} f(x_n, t_n) + \tfrac{1}{2} f(x_{n+1}, t_{n+1})].$$

Starting with

$$x(t+h) = x(t) + \int_t^{t+h} f(x(s), s)\,\mathrm{d}s,$$

we can approximate the integral using the trapezium rule,

$$\int_t^{t+h} f(x(s), s)\,\mathrm{d}s \approx \frac{h}{2}[f(x(t), t) + f(x(t+h), t+h)],$$

see Figure 21.1.

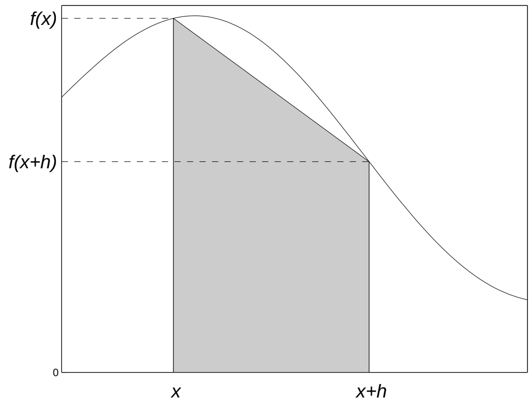

Fig. 21.1. The trapezium rule: the integral between x and $x+h$ is approximated by the shaded area.

With $x_n = x(nh)$ this gives the numerical approximation

$$x_{n+1} = x_n + h[\tfrac{1}{2}f(x_n, t_n) + \tfrac{1}{2}f(x_{n+1}, t_{n+1})].$$

Exercise 21.4 *(T) Since* $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$, *if we calculate the derivative of* e^x *at* $x = 0$ *as a limit it follows that*

$$\lim_{h\to 0} \frac{\mathrm{e}^h - 1}{h} = 1.$$

By rearranging this (note that $1 = \lim_{h\to 0} 1$*) show that*

$$\mathrm{e} = \lim_{h\to 0}(1+h)^{1/h},$$

and hence that

$$\mathrm{e}^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^{1/n}.$$

[Hint: if

$$\lim_{h\to 0} f(h) = \lim_{h\to 0} g(h) = y,$$

and $\kappa(x)$ *is continuous at* $x = y$*, then*

$$\lim_{h\to 0} \kappa[f(h)] = \lim_{h\to 0} \kappa[g(h)].$$

You will need to use this once for each step.]

Using the hint we take $f(h) = (\mathrm{e}^h - 1)/h$, $g(h) = 1$, and

$$\kappa(x) = (xf(x) + 1)^{1/x}.$$

Since κ is continuous at $x = 1$ the limit is preserved, and so we have

$$\mathrm{e} = \lim_{h \to 0} (1+h)^{1/h}.$$

Since the functions $x \mapsto \mathrm{e}^x$ and $h \mapsto (1+h)^{1/h}$ are also continuous, we have

$$\mathrm{e}^x = \lim_{h \to 0} (1+h)^{x/h}.$$

If we put $n = x/h$ then we will have $n \to \infty$ and $h = x/n$, which gives

$$\mathrm{e}^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{1/n},$$

as required.

Exercise 21.5 *(T) In this question we suppose that f satisfies the Lipschitz condition*

$$|f(x) - f(y)| \leq L|x - y| \tag{S21.1}$$

and consider the Euler θ-method for approximating the solutions of $\dot{x} = f(x)$,

$$x_{n+1} = x_n + h[(1-\theta)f(x_n) + \theta f(x_{n+1})].$$

For $\theta = 0$ this is the standard Euler method; for $\theta = \frac{1}{2}$ this is the trapezoidal Euler method; and for $\theta = 1$ this is the 'backwards Euler' method. Since x_{n+1} is not given explicitly as a function of x_{n+1}, we need a way of reliable way of calculating it numerically.

(i) The first thing we must check is that there is a unique solution for x_{n+1}. Suppose that

$$\begin{aligned} y &= x_n + h[(1-\theta)f(x_n) + \theta f(y)] \qquad \text{and} \\ z &= x_n + h[(1-\theta)f(x_n) + \theta f(z)], \end{aligned}$$

i.e. that both y and z satisfy the equation. By subtracting these two equations show that

$$y - z = \theta h[f(y) - f(z)],$$

and hence deduce that

$$|y - z| \leq hL\theta|y - z|,$$

and therefore that $y = z$ provided that $h < 1/L\theta$.

(ii) Suppose therefore that $h < 1/L\theta$. *Given an initial guess* y_0 *for* x_{n+1}, *we can refine this guess successively by setting*

$$y_{j+1} = x_n + h[(1-\theta)f(x_n) + \theta f(y_j)]; \qquad \text{(S21.2)}$$

if $y_{j+1} = y_j = y$ *then*

$$y = x_n + h[(1-\theta)f(x_n) + \theta f(y)],$$

and so y *would be the required value for* x_{n+1}. *Show that*

$$|y_{j+1} - y_j| \le hL\theta|y_j - y_{j-1}|, \qquad \text{(S21.3)}$$

and hence that successive values of y_j *are closer together. Thus, for large* j, *we would expect that* $y_{j+1} \approx y_j$ *and that* y_j *is a good approximation to* x_{n+1}.

(iii) Still assuming that $h < 1/L\theta$, *use (S21.3) show that*

$$|y_{j+1} - y_j| \le (hL\theta)^j|y_1 - y_0|,$$

and hence that

$$|y_j - y_k| \le \frac{(hL\theta)^J}{1 - hL\theta}|y_1 - y_0|$$

for any $j, k \ge J$.

It follows that $\{y_j\}$ *is a Cauchy sequence, and so converges to a limit* y. *Taking limits as* $j \to \infty$ *on both sides of (S21.2) we get*

$$y = x + \tfrac{1}{2}h[f(x_n) + f(y)],$$

and thus $x_{n+1} = y$.

(i) We have

$$\begin{aligned} y &= x_n + h[(1-\theta)f(x_n) + \theta f(y)] \qquad \text{and} \\ z &= x_n + h[(1-\theta)f(x_n) + \theta f(z)], \end{aligned}$$

and subtracting these two equations we obtain

$$y - z = \theta h[f(y) - f(z)].$$

Using the Lipschitz condition from (S21.1),

$$|f(y) - f(z)| \le L|y - z|,$$

this implies that

$$|y - z| = \theta h|f(y) - f(z)| \le hL\theta|y - z|.$$

Now if $y \neq z$ and $hL\theta < 1$, i.e. if $h < 1/L\theta$, we have

$$|y - z| < |y - z|,$$

which is impossible. So for $h < 1/L\theta$ we have $y = z$ and the solution is unique.

(ii) We have

$$y_{j+1} = x_n + h[(1-\theta)f(x_n) + \theta f(y_j)]$$

and

$$y_j = x_n + h[(1-\theta)f(x_n) + \theta f(y_{j-1})],$$

and so

$$y_{j+1} - y_j = h\theta[f(y_j) - f(y_{j-1})].$$

Using the Lipschitz property of f we can obtain

$$|y_{j+1} - y_j| \leq hL\theta|y_j - y_{j+1}|. \tag{S21.4}$$

(iii) If we write $\delta_j = |y_{j+1} - y_j|$ then (S21.4) is

$$\delta_j \leq hL\theta\, \delta_{j-1},$$

and so

$$\delta_j \leq (hL\theta)^{j-1}\delta_1,$$

i.e.

$$|y_{j+1} - y_j| \leq (hL\theta)^j |y_1 - y_0|.$$

Assuming wlog that $j \geq k \geq J$, it follows that

$$\begin{aligned}
|y_j - y_k| &= |y_j - y_{j-1} + y_{j-1} - y_{j-2} + y_{j-2} - \ldots + y_{k+1} - y_k| \\
&\leq |y_j - y_{j-1}| + |y_{j-1} - y_{j-2}| + \ldots + |y_{k+1} - y_k| \\
&\leq \sum_{n=J}^{\infty} |y_{n+1} - y_n| \\
&\leq \left[\sum_{n=J}^{\infty} (hL\theta)^n\right] |y_1 - y_0| \\
&\leq \frac{(hL\theta)^J}{1 - hL\theta} |y_1 - y_0|.
\end{aligned}$$

Exercise 21.6 *(C) For a number of values of t and h compare the exact solution of $\dot{x} = x$ with the solution from Euler's method, and verify the error estimate in (21.7).*

The exact solution is $x(t) = x(0)\mathrm{e}^t$, while the solution of Euler's method is $x_n = x_0(1+h)^{t/h}$. Comparing the solutions for $x(0) = 1$ at time $t = 1$ (the exact solution is $x(1) = \mathrm{e}$) for $h = 2^{-n}$ we have

$$\frac{x(1) - x_{1/h}}{h} = \frac{\mathrm{e} - (1 + 2^{-n})^n}{2^{-n}},$$

and so, when $h = 2^{-n}$, we have

n	$[x_{1/h} - x(1)]/h$
1	0.9366
2	1.1075
3	1.2200
4	1.2857
5	1.3213
6	1.3400
7	1.3495
8	1.3543
9	1.3567
10	1.3579
11	1.3585
12	1.3588
13	1.3590
14	1.3591
15	1.3591

showing that for h small enough we have

$$x_{1/h} - x(1) = 1.3591h$$

and the Euler scheme is indeed of order h. These numbers were produced by the simple MATLAB code

```
for i=1:20;
  x(i)=(1+2^(-i))^(2^i);
  d(i)=(exp(1)-x(i))/(2^(-i))
end
```

Exercise 21.7 *(C) Implement the backwards Euler scheme of Exercise 21.2 numerically, and apply it to the equation* $\dot{x} = x(1-x)$ *to find the solution when* $x(0) = \frac{1}{2}$ *for* $0 \le t \le 8$. *In order to find* x_{n+1} *given* x_n *you can use the approach of Exercise 21.5, and iterate*

$$g_{k+1} = x_n + hg_k(1 - g_k)$$

to give a succession of 'guesses' g_k for x_{n+1} until g_k appears to stabilise (e.g. until $|g_{k+1}-g_k| < h^3$). You will need to choose h carefully to ensure that your sequence of guesses converges. (Can you work out, using the theoretical results of Exercise 21.5, what value of h should suffice?) The M*ATLAB* `M`*-file* `backeuler.m`*, implementing this scheme, can be downloaded from the web.*

The `M`-file `backeuler.m` is as follows:

```
%% backwards Euler method

T=8;        %% final time

h=0.5;       %% timestep

%% MATLAB does not allow an index 0
%% on a vector, so x_n is x(n+1) here

t(1)=0;       %% initial time
x(1)=0.5;     %% initial condition

for n=1:T/h;

t(n+1)=n*h;

%% Use iterative method to find x(n+1) given x(n)

gn=x(n); g=gn+2*h^3;

while abs(gn-g)>h^3;

%% method is O(h) so approximate x(n+1) to within O(h^3)

g=gn;
gn=x(n)+h*g*(1-g);

end

x(n+1)=gn;

end
```

```
plot(t,x)
```

The resulting solution is shown in figure 21.2.

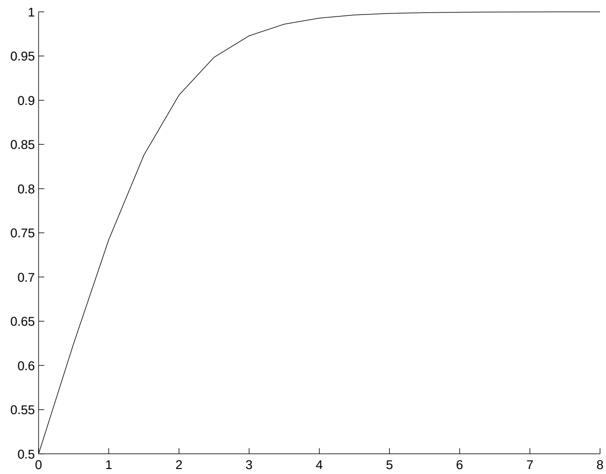

Fig. 21.2. The solution of the equation $\dot{x} = x(1-x)$ with $x(0) = \frac{1}{2}$, calculated using the backwards Euler scheme with $h = 0.5$.

We have $f(x) = x(1-x)$, and so $f'(x) = 1-2x$. If the numerical solution is consistent with the true solution then $x(t) \leq 1$ for all t, and so we should have $f'(x(t)) \leq 1$, and therefore (using the result of Exercise 6.2), we should have $L \leq 1$. Since $\theta = 1$ for the backwards Euler scheme, we expect that our iteration for x_{n+1} will work provided that $h < 1$. The numerics above are calculated with $h = 0.5$.

Exercise 21.8 *(C) Write a* MATLAB *program to implement the Runge-Kutta method introduced at the end of the chapter. Apply this method to $\dot{x} = t - x^2$ when $h = 0.5$, and compare this to the solution obtained using Euler's method with the same timestep. You can download the* MATLAB `M`*-file* `rungekutta.m` *from the web if you wish.*

The method is implemented in the `M`-file `rungekutta.m`:

```
%% Runge-Kutta scheme

T=12;
```

```
h=0.5;

t(1)=0;      %% initial time
x(1)=0;      %% initial condition

for n=1:T/h;

   t(n+1)=n*h;

   f1=t(n)-x(n)^2;
   f2=(t(n)+(h/2))-(x(n)+(h*f1/2))^2;
   f3=(t(n)+(h/2))-(x(n)+(h*f2/2))^2;
   f4=(t(n)+h)-(x(n)+h*f3)^2;
   x(n+1)=x(n) + h * (f1+(2*f2)+(2*f3)+f4)/6;

end

[t x]        %% display values

plot(t,x)
```

This is shown (as a solid line) along with the Euler solution with the same timestep (dashed line) in figure 21.3.

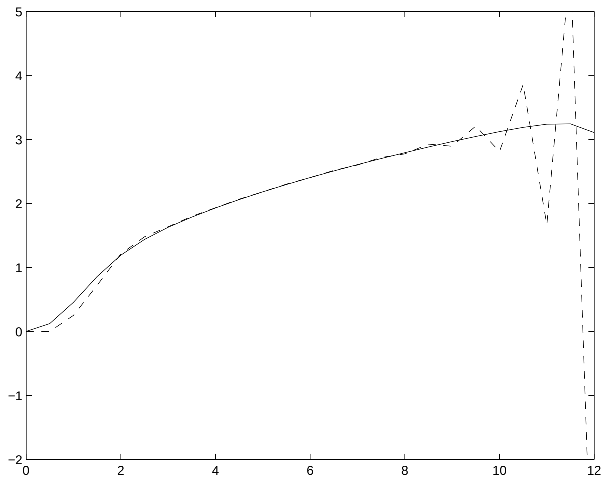

Fig. 21.3. The solution of $\dot{x} = t - x^2$ calculated using the fourth order Runge-Kutta scheme (solid line) and the Euler method (dashed line), both with $h = \frac{1}{2}$.

22

Difference equations

Exercise 22.1 *Find the solutions of the following difference equations satisfying the given initial conditions.*

(i) $x_{n+2} - 4x_{n+1} + 3x_n = 0$ *with* $x_0 = 0$ *and* $x_1 = 1$*;*
(ii) $2x_{n+1} - 3x_n - 2x_{n-1} = 0$ *with* $x_1 = x_2 = 1$*;*
(iii) $x_{n+2} = 2x_{n+1} - 2x_n$ *with* $x_0 = 1$ *and* $x_1 = 2$*;*
(iv) $x_{n+2} + 6x_{n+1} + 9x_n = 0$ *with* $x_0 = 1$ *and* $x_1 = 6$*;*
(v) $2x_n = 3x_{n-1} - x_{n-2}$ *with* $x_0 = 3$ *and* $x_1 = 2$*; and*
(vi) $x_{n+2} - 2x_{n+1} + 5x_n = 0$ *with* $x_0 = \sqrt{5}$ *and* $x_1 = 5\cos\tan^{-1} 2$.

(i)

$$x_{n+2} - 4x_{n+1} + 3x_n \qquad \text{with} \qquad x_0 = 0 \qquad \text{and} \qquad x_1 = 1.$$

We try $x_n = k^n$ and obtain the auxiliary equation for k,

$$k^2 - 4k + 3 = 0.$$

This has solutions $k = 1$ and $k = 3$, and so the general solution is

$$x_n = A + B3^n.$$

For the initial conditions we need

$$x_0 = A + B = 0 \qquad \text{and} \qquad x_1 = A + 3B = 1,$$

i.e. $A = -1/2$ and $B = 1/2$, and so the solution is

$$x_n = \frac{3^n - 1}{2}.$$

(ii)

$$2x_{n+1} - 3x_n - 2x_{n-1} = 0 \qquad \text{with} \qquad x_1 = x_2 = 1.$$

With $x_n = x^k$ we obtain

$$2k^2 - 3k - 2 = 0,$$

which has solutions $k = 2$ and $k = -\frac{1}{2}$. The general solution is therefore

$$x_n = A2^n + B(-\tfrac{1}{2})^n.$$

In order to satisfy the initial conditions we must have

$$x_1 = 2A - \tfrac{1}{2}B = 1 \qquad \text{and} \qquad x_2 = 4A + \tfrac{B}{4} = 1.$$

So we have $A = 3/10$, $B = -4/5$, and the solution is

$$x_n = \frac{3 \times 2^n - 8(-\frac{1}{2})^n}{10}.$$

(iii)

$$x_{n+2} = 2x_{n+1} - 2x_n \qquad \text{with} \qquad x_0 = 1 \quad \text{and} \quad x_1 = 2.$$

Setting $x_n = n^k$ we obtain the auxiliary equation

$$k^2 = 2k - 1,$$

and so $k = 1$ twice. The general solution is therefore

$$x_n = A + Bn.$$

The initial conditions require

$$x_0 = A = 1 \qquad \text{and} \qquad x_1 = A + B = 2,$$

and so $A = B = 1$ and the solution is

$$x_n = n + 1.$$

(iv)

$$x_{n+2} + 6x_{n+1} + 9x_n = 0 \qquad \text{with} \qquad x_0 = 1 \quad \text{and} \quad x_1 = 6.$$

We set $x_n = n^k$, and so

$$k^2 + 6k + 9 = 0.$$

This has a repeated root $k = -3$, so the general solution is

$$x_n = A(-3)^n + Bn(-3)^n.$$

The initial conditions are

$$x_0 = A = 1 \qquad \text{and} \qquad x_1 = A - 3A - 3B = 6,$$

and so $A = 1$, $B = -1$, and the solution is

$$x_n = (-3)^n(1+n).$$

(v)

$$2x_n = 3x_{n-1} - x_{n-2} \qquad \text{with} \qquad x_0 = 3 \quad \text{and} \quad x_1 = 2.$$

With $x_n = k^n$ we obtain

$$2k^2 = 3k - 1,$$

and so $k = 1$ or $k = \frac{1}{2}$. The general solution is therefore

$$x_n = A + B2^{-n}.$$

In order to satisfy

$$x_0 = A + B = 3 \qquad \text{and} \qquad x_1 = A + \tfrac{1}{2}B = 2$$

we need $A = 1$ and $B = 2$, giving the solution

$$x_n = 1 + 2^{1-n}.$$

(vi)

$$x_{n+2} - 2x_{n+1} + 5x_n = 0 \qquad \text{with} \qquad x_0 = \sqrt{5} \quad \text{and} \quad x_1 = 5\cos\tan^{-1} 2.$$

Trying $x_n = n^k$ we obtain

$$k^2 - 2k + 5 = 0,$$

which has a complex conjugate pair of roots, $k = 1 \pm 2\mathrm{i} = \sqrt{5}\mathrm{e}^{\mathrm{i}\tan^{-1}(2)}$. The general solution of the equation is therefore

$$x_n = 5^{n/2}(A\cos n\tan^{-1}(2) + B\sin n\tan^{-1}(2)).$$

In order to fit the initial conditions we need

$$x_0 = A = \sqrt{5} \qquad \text{and} \qquad 5(A\cos\tan^{-1}(2) + B\sin\tan^{-1}(2)),$$

and so $A = \sqrt{5}$, $B = 0$, and the solution is

$$x_n = 5^{(1+n)/2}\cos n\tan^{-1}(2).$$

Exercise 22.2 *Show that if the auxiliary equation*

$$ak^2 + bk + c = 0$$

has a repeated root $k = \lambda$ then the difference equation

$$ax_{n+2} + bx_{n+1} + cx_n = 0$$

can be rewritten in the form

$$x_{n+2} - 2\lambda x_{n+1} + \lambda^2 x_n = 0.$$

Since the roots of the equation $ak^2+bk+c = 0$ are $k = (-b\pm\sqrt{b^2 - 4ac})/2a$, if the equation has a repeated real root then $b^2 = 4ac$ and the root is $k = \lambda = -b/2a$. If we divide the original difference equation

$$ax_{n+2} + bx_{n+1} + cx_n = 0$$

through by a we obtain

$$x_{n+2} + \frac{b}{a}x_{n+1} + \frac{c}{a}x_n = 0.$$

Since $c = b^2/4a$ we can rewrite this as

$$x_{n+2} + \frac{b}{a}x_{n+1} + \frac{b^2}{4a^2}x_n = 0,$$

which is just

$$x_{n+2} - 2\lambda x_{n+1} + \lambda^2 x_n = 0,$$

as claimed.

Exercise 22.3 *The "golden ratio" is the ratio (greater than one) of the sides of a rectangle with the following property: remove a square whose sides are the length of the shorter side of the rectangle, and the remaining rectangle is similar to the original one (its sides are in the same ratio), see Figure 22.1. This ratio was used by the Greeks in constructing the Parthenon (among many other monuments), and has been a favourite tool of artists ever since.*

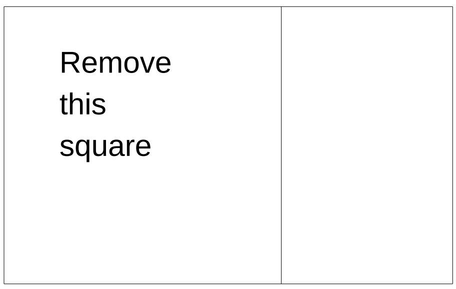

Fig. 22.1. The golden rectangle.

Suppose that $\{x_n\}$ *is a sequence of numbers satisfying the recurrence relation*

$$x_{n+2} = x_{n+1} + x_n.$$

Show that if all the elements of the sequence are integers then the ratio of consecutive terms, x_{n+1}/x_n*, converges to the golden ration. Show that the same result is true if all the terms in the sequence have the same sign. (In particular this is true for the Fibonacci numbers, which have* $x_0 = 0$ *and* $x_1 = 1$*.)*

First we have to find an expression for the golden ratio (the ratio of the longer side of the rectangle to the shorter). If the original sides are x and rx, then the sides of the smaller rectangle are $(r-1)x$ and x; the ratio between them is $1/(r-1)$, and this should be r. So

$$\frac{1}{r-1} = r$$

giving

$$r^2 - r - 1 = 0.$$

The solutions of this are

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

We want the positive square root (since $r > 0$), giving $r = (1+\sqrt{5})/2$.

Now we find an expression for x_n. We try $x_n = k^n$ in

$$x_{n+1} = x_{n+1} + x_n \tag{S22.1}$$

and find that

$$k^2 = k + 1.$$

So

$$k = \frac{1 \pm \sqrt{5}}{2},$$

and the general solution of (S22.1) is

$$x_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The ratio x_{n+1}/x_n is given by

$$\frac{x_{n+1}}{x_n} = \frac{A\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + B\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n}.$$

If $A \neq 0$ then as $n \to \infty$ the second terms on both the top and the bottom of this fraction tend to zero (since $(1-\sqrt{5})/2$ is smaller than 1), and so we would expect the limiting value to be

$$\frac{1+\sqrt{5}}{2},$$

precisely the golden ration.

To be more careful, we have

$$\frac{x_{n+1}}{x_n} = \left(\frac{1+\sqrt{5}}{2}\right) \frac{1}{1+(B/A)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n} + \frac{B\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n},$$

and this tends to $(1+\sqrt{5})/2$ since

$$\frac{1-\sqrt{5}}{1+\sqrt{5}} < 1$$

(for the first term) and $(1-\sqrt{5})/2 < 1$ (for the second).

If $A = 0$ the above argument does not work, and then

$$x_{n+1} = \frac{1-\sqrt{5}}{2}\, x_n,$$

and it is not possible for all terms in the sequence to be integers. Similarly, since $(1-\sqrt{5})/2 < 0$, the signs of the terms in the sequence must alternate. So it is not possible for all terms in the sequence to have the same sign either.

Exercise 22.4 *Find the general solution of the following difference equations, and then find the solution that satisfies the specified initial conditions:*

(i) $x_{n+2} - 4x_n = 27n^2$, *with* $x_0 = 1$ *and* $x_1 = 3$;
(ii) $x_{n+1} - 4x_n + 3x_{n-1} = 36n^2$, *with* $x_0 = 12$ *and* $x_1 = 0$;
(iii) $x_{n+1} - 4x_n + 3x_{n-1} = 2^n$, *with* $x_0 = -4$ *and* $x_1 = -6$;
(iv) $x_{n+1} - 4x_n + 3x_{n-1} = 3^n$, *with* $x_0 = 2$ *and* $x_1 = 13/2$;
(v) $x_{n+2} - 2x_{n+1} + x_n = 1$, *with* $x_0 = 3$ *and* $x_1 = 6$;
(vi) $x_{n+2} + x_n = 2^n$, *with* $x_0 = x_1 = 0$;
(vii) $x_{n+2} + x_{n+1} + x_n = c$ *(the general solution is enough here).*

(i)

$$x_{n+2} - 4x_n = 27n^2 \qquad \text{with} \qquad x_0 = 1 \quad \text{and} \quad x_1 = 3.$$

First solve the homogeneous problem

$$x_{n+2} - 4x_n = 0$$

by trying $x_n = k^n$. Then k must satisfy

$$k^2 - 4 = 0,$$

so that $k = 2$ or -2, giving the complementary function $A2^n + B(-2)^n$. For the particular solution we try a polynomial of the same order, $x_n = an^2 + bn + c$, so that

$$x_{n+2} = a(n+2)^2 + b(n+2) + c = an^2 + (2a+b)n + (4a+2b+c),$$

and we need

$$\begin{aligned} &an^2 + (2a+b)n + (4a+2b+c) - 4(an^2+bn+c) \\ &\quad = -3an^2 + (2a-3b)n + (4a+2b-3c) = 27n^2. \end{aligned}$$

So we take $a = -9$, $b = -6$, and $c = -16$, so that the general solution is

$$A2^n + B(-2)^n - 9n^2 - 6n - 16.$$

To fit the initial conditions we want

$$x_0 = A + B - 16 = 1 \qquad \text{and} \qquad x_1 = 2A - 2B - 9 - 6 - 16 = 3,$$

i.e. $A = 17$ and $B = 0$, so that

$$x_n = 17(2^n) - 9n^2 - 6n - 16.$$

(ii)

$$x_{n+1} - 4x_n + 3x_{n-1} = 36n^2 \qquad \text{with} \qquad x_0 = 12 \qquad \text{and} \qquad x_1 = 0.$$

To solve the homogeneous problem

$$x_{n+1} - 4x_n + 3x_{n-1} = 0$$

we try $x_n = k^n$, and then

$$k^2 - 4k + 3 = 0,$$

giving $k = 1$ or $k = 3$. So the complementary function is $A + B3^n$. In this case a derivative of the right-hand side (n^2) is a solution of the homogeneous equation ($\mathrm{d}^2/\mathrm{d}n^2(n^2) = 2$), and so we have to try a polynomial of one order higher, $x_n = an^3 + bn^2 + cn$ (we don't need the constant term, since that will just give zero). Then

$$\begin{aligned} x_{n-1} &= a(n-1)^3 + b(n-1)^2 + c(n-1) \\ &= a(n^3 - 3n^2 + 3n - 1) + b(n^2 - 2n + 1) + c(n-1) \\ &= an^3 + (b-3a)n^3 + (3a-2b+c)n + (b-a-c) \end{aligned}$$

and

$$\begin{aligned} x_{n+1} &= a(n+1)^3 + b(n+1)^2 + c(n+1) \\ &= a(n^3+3n^2+3n+1) + b(n^2+2n+1) + c(n+1) \\ &= an^3 + (3a+b)n^2 + (3a+2b+c)n + (a+b+c). \end{aligned}$$

Substituting into the difference equation gives

$$\begin{aligned} &an^3 + (3a+b)n^2 + (3a+2b+c)n + (a+b+c) \\ &\quad -4[an^3+bn^2+cn] \\ &\quad +3[an^3+(b-3a)n^3+(3a-2b+c)n+(b-a-c)] = 36n^2 \end{aligned}$$

or tidying up

$$-6an^2 + (12a-4b)n + (-2a+4b-6c) = 36n^2.$$

So we need $a=-6$, $b=-18$, and $c=-10$. So the general solution is

$$A + B3^n - 6n^3 - 18n^2 - 10n.$$

To fit the initial conditions we need

$$x_0 = A+B = 12 \qquad \text{and} \qquad A+3B-6-18-10 = 0,$$

which implies that $A=1$, $B=11$, and the solution is

$$x_n = 11(3^n) - 6n^3 - 18n^2 - 10n + 1.$$

(iii)

$$x_{n+1} - 4x_n + 3x_{n-1} = 2^n \qquad \text{with} \qquad x_0 = -4 \quad \text{and} \quad x_1 = -6.$$

We have already solved the homogeneous equation, and note that 2^n is not a solution. So we can just try $x_n = a2^n$, and then we need

$$a2^{n+1} - 4a2^n + 3a2^{n-1} = 2^n,$$

or

$$4a - 8a + 3a = 4,$$

i.e. we want $a=-4$, so the general solution is

$$A + B3^n - 2^{n+2}.$$

The initial conditions require

$$x_0 = A+B-4 = -4 \qquad \text{and} \qquad A+3B-8 = -6,$$

which implies that $A = -1$, $B = 1$, and the solution is

$$x_n = 3^n - 2^{n+2} - 1.$$

(iv)

$$x_{n+1} - 4x_n + 3x_{n-1} = 3^n \quad \text{with} \quad x_0 = 2 \quad \text{and} \quad x_1 = 13/2.$$

Now, 3^n is a solution of the homogeneous equation. So we have to try $x_n = an3^n$. So

$$x_{n-1} = a(n-1)3^{n-1} \quad \text{and} \quad x_{n+1} = a(n+1)3^{n+1},$$

and we want

$$an3^{n+1} + a3^{n+1} - 4an3^n + 3an3^{n-1} - 3a3^{n-1} = 3^n,$$

which tidies up to give

$$a3^{n+1} - a3^n = 3^n,$$

or $2a = 1$, i.e. $a = 1/2$. So the general solution is

$$x_n = A + B3^n + \frac{n3^n}{2}.$$

The initial conditions,

$$x_0 = A + B = 2 \quad \text{and} \quad x_1 = A + 3B + \frac{3}{2} = \frac{13}{2}$$

imply that $A = \frac{1}{2}$ and $B = \frac{3}{2}$, and therefore

$$x_n = \frac{1 + 3^{n+1} + n3^n}{2}.$$

(v)

$$x_{n+2} - 2x_{n+1} + x_n = 1 \quad \text{with} \quad x_0 = 3 \quad \text{and} \quad x_1 = 6.$$

To solve the homogeneous equation

$$x_{n+2} - 2x_{n+1} + x_n = 0$$

we try $x_n = k^n$. Then

$$k^2 - 2k + 1 = 0$$

which gives $k = 1$ twice. So the complementary function is

$$A(1^n) + Bn(1^n) = A + Bn.$$

To try to find a particular solution we can't try $x_n = c$, since this would be a solution of the homogeneous equation; nor can we try $x_n = cn$, since this also solves the homogeneous equation. So we have to try $x_n = cn^2$. Substituting this in we get

$$c[n^2 + 4n + 4 - 2(n^2 + 2n + 1) + n^2] = 2c = 1,$$

so we want $c = 1/2$ and the general solution is

$$A + Bn + \frac{n^2}{2}.$$

The initial conditions require us to choose A and B such that

$$x_0 = A = 3 \qquad \text{and} \qquad x_1 = A + B + \frac{1}{2} = 6,$$

i.e. $A = 3$ and $B = 5/2$: the solution is

$$x_n = \frac{6 + 5n + n^2}{2}.$$

(vi)

$$x_{n+2} + x_n = 2^n \qquad \text{with} \qquad x_0 = x_1 = 0.$$

In order to solve the homogeneous equation

$$x_{n+2} + x_n = 0,$$

we put $x_n = k^n$ and find that k must satisfy $k^2 = -1$. The roots are $\pm\mathrm{i} = \mathrm{e}^{\pm\mathrm{i}\pi/2}$, and so the solution is

$$x_n = A\sin(n\pi/2) + B\cos(n\pi/2).$$

To find the particular solution note that 2^n is not a solution of the homogeneous equation, so we can just try $x_n = a2^n$, and need

$$a2^{n+2} + a2^n = 2^n,$$

or $5a = 1$. So the general solution is

$$A\sin(n\pi/2) + B\cos(n\pi/2) + \frac{2^n}{5}.$$

To fit the initial conditions we take

$$x_0 = B = 0 \qquad \text{and} \qquad x_1 = A + \frac{2}{5} = 0,$$

so

$$x_n = \frac{2^n - 2\sin(n\pi/2)}{5}.$$

(vii)

$$x_{n+2} + x_{n+1} + x_n = c.$$

To solve

$$x_{n+2} + x_{n+1} + x_n = 0$$

we put $x_n = k^n$ and get

$$k^2 + k + 1 = 0.$$

The solutions of this are

$$k = \frac{-1 \pm \sqrt{1-4}}{2} \qquad k = -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}.$$

Rewriting this in modulus and argument form gives

$$k = \mathrm{e}^{\pm \mathrm{i}2\pi/3},$$

and so

$$x_n = A\cos 2n\pi/3 + B\sin 2n\pi/3.$$

We now try $x_n = C$, and then $3C = c$, which implies that $C = c/3$, giving the solution

$$x_n = \frac{c}{3} + A\cos 2n\pi/3 + B\sin 2n\pi/3.$$

Exercise 22.5 *Find the solution of the difference equation*

$$x_{n+1} = x_n(1 + x_n),$$

with $x_1 = 1$. *Now show that if* $x_1 = c$ *then*

$$x_n = c\,\frac{\Gamma(c+n)}{\Gamma(c+1)},$$

where the Γ *function, which was defined in Exercise 20.8, satisfies* $\Gamma(x+1) = x\Gamma(x)$.

When $x_1 = 1$ we have

$$x_1 = 1, \qquad x_2 = 2 \cdot 1, \qquad x_3 = 3 \cdot 2 \cdot 1, \qquad \ldots$$

and so $x_n = n!$. To check this using induction, we know that $x_1 = 1 = 1$. Now suppose that $x_k = k!$; then $x_{k+1} = (k+1)k! = (k+1)!$. It follows that $x_n = n!$ for all n, as claimed.

If $x_1 = c$ then

$$x_2 = c(c+1), \qquad x_3 = c(c+1)(c+2), \qquad x_4 = c(c+1)(c+2)(c+3), \qquad \ldots.$$

It follows that $x_n = c\Gamma(c+n)/\Gamma(c+1)$, which is simple to check by induction, since $x_1 = c$, and if $x_k = c\Gamma(c+k)/\Gamma(c+1)$,

$$x_{k+1} = \frac{c(c+k)\Gamma(c+k)}{\Gamma(c+1)} = \frac{c\Gamma(c+k+1)}{\Gamma(c+1)} = x_{k+1},$$

as required.

23

Nonlinear first order difference equations

Exercise 23.1 *Show that there is an orbit of period 3 containing the point $x = 1$ for the difference equation*

$$x_{n+1} = \frac{14}{3}x_n^2 - \frac{13}{2}x_n + \frac{7}{3}.$$

We have

$$\begin{aligned} f(1) &= \frac{14}{3} - \frac{13}{2} + \frac{7}{3} \\ &= \frac{1}{2}, \\ f^2(1) &= f(\tfrac{1}{2}) = \frac{14}{3}\frac{1}{4} - \frac{13}{2}\frac{1}{2} + \frac{7}{3} \\ &= \frac{1}{4}, \\ f^3(1) &= f(1/4) = \frac{14}{3}\frac{1}{16} - \frac{13}{2}\frac{1}{4} + \frac{7}{3} \\ &= 1, \end{aligned}$$

and so there is a periodic orbit of period 3, consisting of the points 1, $\frac{1}{2}$, and 1/4.

Exercise 23.2 *Suppose that the differential equation $\dot{x} = f(x)$ has a stationary point x^* where $f'(x^*) < 0$. We saw that the point x^* is a stable fixed point for*

$$x_{n+1} = x_n + hf(x_n).$$

provided that $h < 1/|f'(x^*)|$. Assuming that x_0 is sufficiently close to x^*, show that if $h > 1/(2|f'(x^*)|)$ then x_n is alternately greater than and less than x^*, while if $h < 1/(2|f'(x^*)|)$ the orbit x_n approaches x^* monotonically.

Writing $x_n = x^* + \delta_n$ we have

$$\begin{aligned} x^* + \delta_{n+1} &= x^* + \delta_n + hf(x^* + \delta_n) \\ &= x^* + \delta_n + hf(x^*) + hf'(x^*)\delta_n + \dots, \end{aligned}$$

and so, since $f(x^*) = 0$, if we drop the higher order terms (which are very small when δ_n is small), we obtain

$$\delta_{n+1} = (1 + hf'(x^*))\delta_n.$$

If $1/(2|f'(x^*)|) < h < 1/|f'(x^*)|$ then, since $f'(x^*) < 0$, we have

$$-1 < 1 + hf'(x^*) < 0,$$

and so the sign of δ_n switches between successive values of n; while if $h < 1/(2|f'(x^*|)$ we have $0 < 1 + hf'(x^*) < 1$, and the sign of δ_n doesn't change.

Exercise 23.3 *In this question we consider the trapezoidal Euler method*

$$x_{n+1} = x_n + \tfrac{1}{2}h[f(x_n) + f(x_{n+1})].$$

Show that $x_{n+1} = x_n = x^*$ *if and only if* $f(x^*) = 0$, *i.e. that the fixed points of the numerical scheme coincide with the stationary points of the differential equation* $\dot{x} = f(x)$.

Using the chain rule show that

$$\frac{\mathrm{d}x_{n+1}}{\mathrm{d}x_n} = \frac{1 + \frac{1}{2}hf'(x_n)}{1 - \frac{1}{2}hf'(x_{n+1})},$$

and hence that a fixed point x^* *is stable if* $f'(x^*) < 0$ *and unstable if* $f'(x^*) > 0$, *i.e. that whatever the timestep the stability coincides with that of the corresponding stationary point in the differential equation.*

For the trapezoidal Euler method

$$x_{n+1} = x_n + \tfrac{1}{2}h[f(x_n) + f(x_{n+1})],$$

we have a fixed point $x_n = x_{n+1} = x^*$ if

$$x^* = x^* + \tfrac{1}{2}h[f(x^*) + f(x^*)],$$

i.e. if $f(x^*) = 0$. So the fixed points of the numerical scheme are the same as those of the differential equation $\dot{x} = f(x)$.

Applying the chain rule we differentiate both sides with respect to x_n and obtain

$$\frac{\mathrm{d}x_{n+1}}{\mathrm{d}x_n} = 1 + \tfrac{1}{2}h\left[f'(x_n) + f'(x_{n+1})\frac{\mathrm{d}x_{n+1}}{\mathrm{d}x_n}\right],$$

Rearranging this we have

$$\frac{\mathrm{d}x_{n+1}}{\mathrm{d}x_n} = \frac{1+\frac{1}{2}hf'(x_n)}{1-\frac{1}{2}hf'(x_{n+1})}.$$

The fixed point is stable provided that at $x_n = x^*$ this derivative has modulus less than 1. Since x^* is a fixed point we also have $x_{n+1} = x^*$, and so the fixed point is stable if

$$\left|\frac{1+\frac{1}{2}hf'(x^*)}{1-\frac{1}{2}hf'(x^*)}\right| < 1,$$

which is the same as

$$|1+\tfrac{1}{2}hf'(^*)| < |1-\tfrac{1}{2}hf'(x^*)|.$$

Squaring both sides we obtain

$$1+hf'(x^*)+\frac{1}{4}h^2f'(x^*)^2 < 1-hf'(x^*)+\frac{1}{4}h^2f'(x^*),$$

i.e. if $f'(x^*) < 0$. In a similar way we obtain instability of x^* whenever $f'(x^*) > 0$. So the stability of the fixed points coincides with that of the stationary points of the original differential equation whatever the value of h.

Exercise 23.4 *(T) It follows from the definition of the derivative that*

$$f(x^*+h) = f(x^*) + f'(x^*)h + o(h),$$

where $o(h)$ indicates that the remainder terms satisfy

$$\frac{o(h)}{h} \to 0 \qquad \text{as} \qquad h \to 0.$$

In particular, given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|o(h)| \le \epsilon h$$

for all $|h| \le \delta$. Use this to show rigorously that a fixed point x^* of $x_{n+1} = f(x_n)$ is stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$. (Recall that a fixed point x^* is stable if given an $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x_0 - x^*| < \delta$ we have $|f^n(x_0) - x^*| < \epsilon$ for all $n = 0, 1, \ldots$. In fact you should be able to show that when $|f'(x^*)| < 1$ the fixed point is attracting.)

If we write $x_n = x^* + \delta_n$ then we have

$$\begin{aligned} x^* + \delta_{n+1} &= x_{n+1} = f(x_n) = f(x^* + \delta_n) \\ &= f(x^*) + f'(x^*)\delta_n + o(\delta_n) \\ &= x^* + f'(x^*)\delta_n + o(\delta_n), \end{aligned}$$

and so

$$\delta_{n+1} = f'(x^*)\delta_n + o(\delta_n).$$

Now choose $\epsilon = \frac{1}{2}|f'(x^*) - 1|$, and then δ^* small enough that $|o(h)| \leq \epsilon h$ for all h with $|h| \leq \delta^*$; this means that if $|\delta_n| \leq \delta^*$ then

$$\delta_{n+1} = [f'(x^*) + g(\delta_n)]\delta_n, \tag{S23.1}$$

where $|g(\delta_n)| < \epsilon$ for $|\delta_n| < \delta^*$.

By the way we have chosen ϵ, for all $|\delta_n| < \delta^*$ we have

$$|f'(x^*) + g(\delta_n)| < 1 \qquad \text{if} \qquad |f'(x^*)| < 1,$$

and

$$|f'(x^*) + g(\delta_n)| > 1 \qquad \text{if} \qquad |f'(x^*)| > 1.$$

It follows by iterating (S23.1) that if $|f'(x^*)| < 1$ then $\delta_n \to 0$ as $n \to \infty$, and so x^* is stable; while if $|f'(x^*)| > 1$ then $|\delta_n| \to \infty$ as $n \to \infty$, so that x_n leaves any small neighbourhood of x^*, showing that x^* is unstable.

Exercise 23.5 *(T) Suppose that f has a periodic orbit of order k, consisting of the points $\{x_1, \ldots, x_k\}$. Show that each of the points on the orbit is a fixed point for the map $g(x) = f^k(x)$.*

A periodic orbit is said to be stable if each point on the orbit is a stable fixed point of f^k. Show that a periodic orbit $\{x_1, x_2\}$ of period 2 is stable provided that

$$|f'(x_1)f'(x_2)| < 1,$$

and that a periodic orbit $\{x_1, \ldots, x_k\}$ of period k is stable provided that

$$|f'(x_1)f'(x_2)\cdots f'(x_{k-1})f'(x_k)| < 1.$$

Note in particular that if one point on the orbit is a stable fixed point of f^k then so are all the others.

If $f(x)$ has a periodic orbit of period k, $\{x_1, \ldots, x_k\}$, then

$$x_j = f^j(x_1) \qquad \text{and} \qquad f^k(x_1) = x_1.$$

Applying f^j to both sides of $f^k(x_1) = x_1$ shows that

$$f^k(x_j) = f^{k+j}(x_1) = f^{j+k}(x_1) = f^j(x_1) = x_j,$$

so all the points on the orbit are fixed points of f^k.

Each element of a periodic orbit of period 2, $\{x_1, x_2\}$, is a fixed point of the map $g = f^2$. By definition this orbit is stable if $|g'(x_j)| < 1$. Now,

$$g(x) = f(f(x)),$$

and so using the chain rule

$$g'(x) = f'(f(x))f'(x).$$

It follows that x_1 lies on a stable periodic orbit if

$$|g'(x_1)| = |f'(f(x_1))f'(x_1)| = |f'(x_2)f'(x_1)| < 1;$$

we obtain the same condition starting with x_2, since $f(x_2) = x_1$.

If $g = f^k$ then we have

$$g(x) = \underbrace{f(f(\cdots f(f(}_{k \text{ times}}x))\cdots)),$$

and so

$$g'(x) = f'(f^{k-1}(x))f'(f^{k-2}(x))\cdots f'(x).$$

A periodic orbit is stable if $|g'(x_j)| < 1$, i.e. if

$$\begin{aligned} |g'(x_1)| &= |f'(f^k(x_j))f'(f^{k-1}(x_j))\cdots f'(x_j)| \\ &= |f'(x_{j-1})f'(x_{j-2})\cdots f'(x_1)f'(x_k)\cdots f'(x_j)| \\ &= |f'(x_1)\cdots f'(x_k)| < 1. \end{aligned}$$

Exercise 23.6 *(T) Consider the iterated map*

$$y_{n+1} = f(y_n) = ry_n + y_n^2$$

for $r \le 0$. Find the two fixed points, and show that the fixed point at $y = 0$ is stable for $-1 < r \le 0$ and unstable for $r < -1$.

Show that if y lies on an orbit of period 2 then

$$y^2 + (r+1)y + (r+1) = 0,$$

and deduce that there is a period 2 orbit if $r < -1$. [Hint: we must have $f^2(y) = y$, and you can factorise the resulting equation since $f(0) = 0$ and $f(1-r) = 1-r$.]

If y_1 and y_2 are the points on this orbit, show that

$$f'(y_1)f'(y_2) = 4 + 2r - r^2,$$

and hence that this orbit is stable for $1 - \sqrt{6} < r < -1$.

There are two fixed points, i.e. values of y for which $y = f(y) = ry + y^2$. One is $y = 0$, and the other is $y = 1 - r$.

The stability of the fixed point at $y = 0$ is determined by the modulus of $f'(0)$. Since $f'(0) = r$, this fixed point is stable if $-1 < r \leq 0$ and unstable for $r < -1$.

If y lies on an orbit of period 2 then $f^2(y) = y$, so

$$r[ry + y^2] + [ry + y^2]^2 = y.$$

This implies that

$$r^2y + ry^2 + r^2y^2 + 2ry^3 + y^4 = y,$$

and so, cancelling a factor of y (this would give the fixed point at $y = 0$) we have

$$y^3 + 2ry^2 + (r + r^2)y + (r^2 - 1) = 0.$$

Since $y = 1 - r$ is also a solution, a factor of this equation will be $y - 1 + r$:

$$(y - 1 + r)(y^2 + (r + 1)y + (r + 1)) = 0.$$

The initial factor gives the fixed point at $1 - r$, so there will be a periodic orbit of period 2 if

$$y^2 + (r + 1)y + (r + 1) = 0 \qquad \text{(S23.2)}$$

has a solution; since the solution is

$$y = \frac{-(r + 1) \pm \sqrt{(r + 1)^2 - 4(r + 1)}}{2},$$

for this solution to be real we require

$$(r + 1)^2 - 4(r + 1) > 0 \qquad \Rightarrow \qquad r^2 - 2r - 3 > 0,$$

which implies that $r < -1$.

If y_1 and y_2 are the points on this orbit then we have

$$f'(y_1)f'(y_2) = (r + 2y_1)(r + 2y_2) = r^2 + 2r(y_1 + y_2) + 4y_1y_2.$$

Since y_1 and y_2 are the solutions of (S23.2), it follows that

$$y_1 + y_2 = -(r + 1) \qquad \text{and} \qquad y_1y_2 = (r + 1).$$

[You can see this if you expand $(y - y_1)(y - y_2) = 0$ and compare with (S23.2).] So therefore

$$f'(y_1)f'(y_2) = r^2 - 2r(r+1) + 4(r+1) = -r^2 + 2r + 4.$$

Using the result of Exercise 23.5, this orbit is stable provided that

$$|-r^2 + 2r + 4| < 1.$$

The maximum value of the expression in the modulus signs is when $r = -1$, and is 1; the expression has the value -1 when

$$-r^2 + 2r + 4 = -1,$$

i.e. when $r^2 - 2r - 5 = 0$,

$$r = \frac{2 \pm \sqrt{4+20}}{2} = 1 \pm \sqrt{6}.$$

So this periodic orbit is stable while $1 - \sqrt{6} < r < -1$.

Exercise 23.7 *(C) Apply Euler's method with timestep h to the equation $\dot{x} = x(k - x)$ (cf. Exercise 7.6). Investigate how the stability of the fixed points depends on k and h. Now implement this Euler scheme numerically and verify your results (e.g. compare the cases $k = 1$ and $k = 3$ with timestep $h = 1$). (You could adapt the* MATLAB *M-file* `euler.m`*, which is available on the web.)*

Euler's method applied to this equation gives rise to the map

$$x_{n+1} = g(x_n) := x_n + hx_n(k - x_n).$$

The fixed points of this map are the same as the stationary points of the differential equation $\dot{x} = x(k - x)$, namely zero and k.

To investigate the stability of the fixed points we consider

$$g'(x) = 1 + hk - 2hx$$

at the two fixed points. We have

$$g'(0) = 1 + hk \qquad \text{and} \qquad g'(k) = 1 - hk.$$

We saw in Exercise 7.6 that for $k < 0$ the origin is stable and $x = k$ is unstable, while for $k > 0$ the origin is unstable and $x = k$ is stable.

For Euler's method, the origin will be stable if

$$|g'(0)| = |1 + hk| < 1,$$

i.e. if $-2 < hk < 0$, and unstable otherwise. The fixed point at $x = k$ will be unstable if $|1-hk| > 1$, so for any choice of $k < 0$ (h is always positive), and also if $hk > 2$. Thus for a fixed h the fixed points have the correct stability properties for $|k| < 2/h$, while to ensure the correct stability properties for a fixed k we need to take $h < 2/|k|$.

The output shown graphically in Figure 23.1 ($k = 1$ and $h = 1$ with the correct stability properties) and Figure 23.2 ($k = 3$ and $h = 1$ with strange stability properties) was produced by the two M-files `tccobweb.m` and `tcgraph.m`.

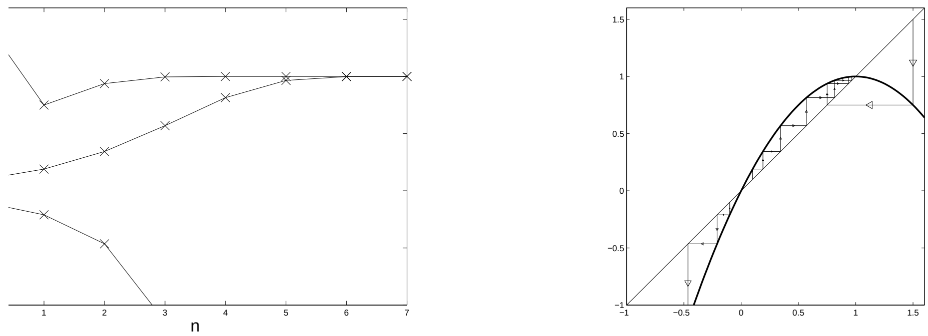

Fig. 23.1. Successive values of x_n (on the left) and the cobweb diagram (on the right) for the map $x_{n+1} = x_n + x_n(1 - x_n)$. The fixed points have the same stability properties as those of the corresponding differential equation $\dot{x} = x(1-x)$.

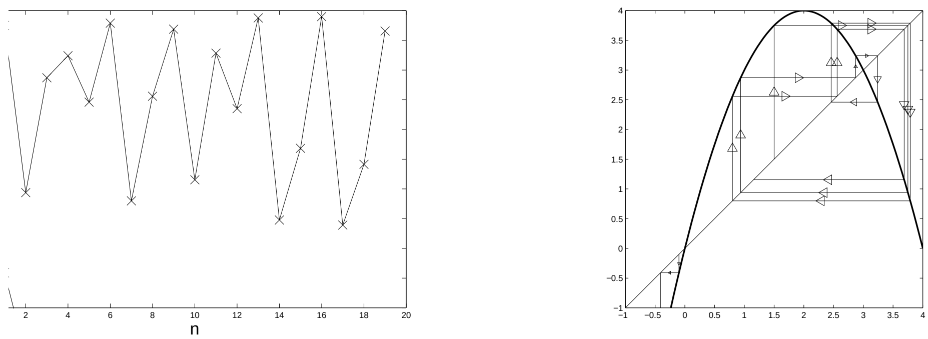

Fig. 23.2. Successive values of x_n (on the left) and the cobweb diagram (on the right) for the map $x_{n+1} = x_n + x_n(3 - x_n)$. Although zero is still unstable, the fixed point at $x = 3$ is also unstable, in contrast to the behaviour of the differential equation $\dot{x} = x(3 - x)$.

24

The logistic map

Exercise 24.1 *Consider the iterated map*

$$x_{n+1} = rx_n(1 - x_n^2).$$

Show that for $0 < r < 3\sqrt{3}/2$ *if* $0 \le x_n \le 1$ *then* $0 \le x_{n+1} \le 1$. *Show that if* $r < 1$ *then the only fixed point in* $[0, 1]$ *is zero, and that this is stable.*

When $r > 1$ *there is another fixed point in* $[0, 1]$. *Find the value of this fixed point (as a function of* r*). For which values of* r *is it stable, and for which values is it unstable?*

What would you expect to happen when $r > 2$*?*

Clearly $x_{n+1} \ge 0$. The thing to check is the maximum value of $f(x) = rx(1 - x^2)$ for $x \in [0, 1]$. This occurs when $f'(x) = 0$, and

$$f'(x) = r(1 - 3x^2).$$

So the maximum is always at $x = 1/\sqrt{3}$, and is $2r/3\sqrt{3}$. So this is no greater than 1 provided that $r \le 3\sqrt{3}/2$ as claimed.

The fixed points are given by $x = f(x)$, so are solutions of

$$x = rx(1 - x^2).$$

One of them is $x = 0$, leaving $1 = r - rx^2$, or $x^2 = (r - 1)/r$. When $r < 1$ there is only one fixed point at $x = 0$, since the quadratic equation has no real root.

The stability of this fixed point can be found by considering $f'(x)$ at $x = 0$. We have $f'(0) = r$, so for $r < 1$ the fixed point will be stable (while for $r > 1$ it will be unstable).

When $r > 1$ there is another fixed point in $[0, 1]$, namely $\sqrt{1 - (1/r)}$. If we look at

$$f'(1 - (1/r)) = 3 - 2r$$

we can see that this point is stable when $1 < r < 2$ and unstable once $r > 2$.

For $r > 2$ we would expect an orbit of period 2, then period 4, then 8... the "period doubling cascade".

Exercise 24.2 *(C) Use the* M*-files* `logistic.m` *(which draws cobweb diagrams),* `xnvsn.m` *(which plots successive values of x_n vs n), and* `bifurcation.m` *(which draws the bifurcation diagram for a given range of r) to investigate the dynamics of the logistic map. Modify the programs to investigate the dynamics of the map in the previous exercise.*

The figures below show some of the behaviour of the map

$$x_{n+1} = rx_n(1 - x_n^2)$$

which was the subject of the previous exercise. There we saw that for $r < 1$ the origin is stable, as shown in Figure 24.1.

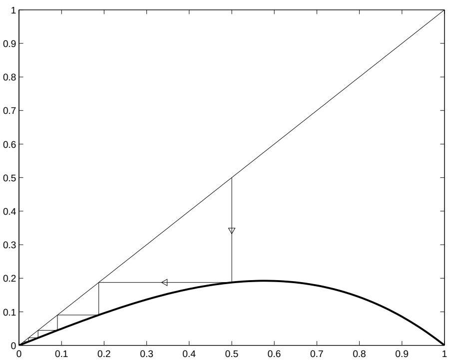

Fig. 24.1. The cobweb diagram for $r = 0.5$. The origin is the only fixed point, and it is stable.

For $r > 1$ the origin loses stability, and there is a new fixed point which is stable while $r < 2$. This shown in Figure 24.2.

When $r > 2$ the non-zero fixed point loses stability, and a stable periodic orbit of period 2 appears (see Figure 24.3).

As predicted above, the period-doubling sequence then sets in: Figure 24.4 shows an orbit of period 4, and Figure 24.5 has an orbit of period 8.

Beyond a critical value of r the orbits appear to be chaotic, as shown in Figure 24.6 for $r = 2.5$.

Figure 24.7 shows the bifurcation diagram for this equation for all allowable values of r ($0 \leq r \leq 3\sqrt{3}/2$), while Figure 24.8 shows the 'interesting' region where $r > 1.9$.

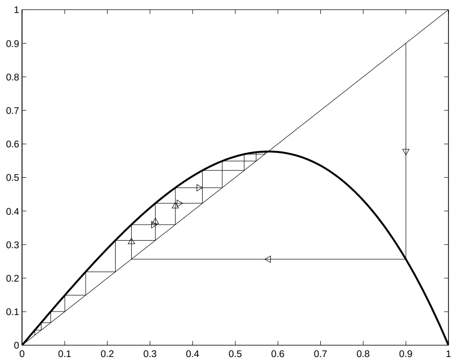

Fig. 24.2. The cobweb diagram for $r = 1.5$. The new non-zero fixed point is now stable, and the origin is unstable.

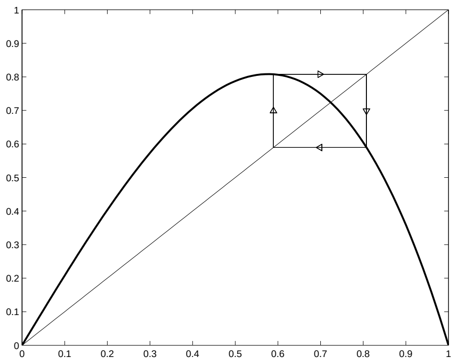

Fig. 24.3. The cobweb diagram when $r = 2.1$: there is a stable periodic orbit of period 2.

Having the bifurcation diagram allows us to find other parameter values with interesting behaviour. For example, there is a stable periodic orbit of period three for $r = 2.46$, as shown in Figure 24.9.

Exercise 24.3 *(C & T) The* `M`*-file* `f2.m` *plots the graph of* $f(x)$ *and of* $f^2(x)$ *in the left-hand Figure, and the graph of* f^2 *restricted to the little box in a blown-up version on the right (see Figure 24.14). By looking at a succession of pictures as* r *increases from 0 to 4, observe that the rescaled version of* f^2 *behaves in the same way that* f *does as* r *increases. This can be made precise, and explains the period doubling cascade. Since the fixed point of* f *becomes unstable and gives rise to a period 2 orbit, the same thing happens to* f^2*: its fixed point (a period 2 orbit for* f*) becomes unstable and gives rise to a period 2 orbit (a period 4 orbit for* f*). Since whatever happens*

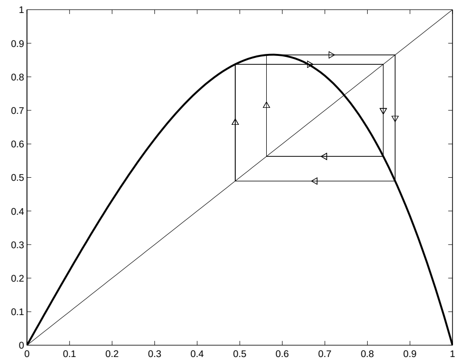

Fig. 24.4. A stable periodic orbit of period 4 when $r = 2.25$.

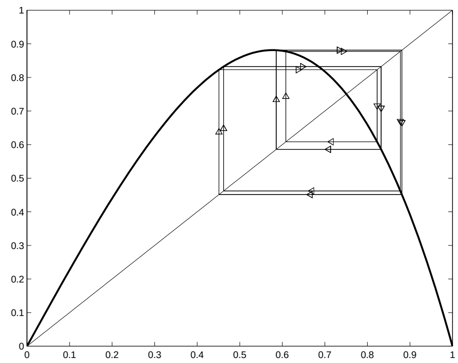

Fig. 24.5. A stable periodic orbit of period 8 when $r = 2.29$.

to f happens to f^2, whatever happens to f^2 happens to $(f^2)^2 = f^4$: its fixed point (a period 4 orbit for f) will become unstable and give rise to a period 2 orbit (a period 8 orbit for f). Similar reasoning holds for each orbit of period 2^k, showing that it becomes unstable and produces an orbit of period 2^{k+1}. The map formed by restricting f^2 to the little box, and then rescaling to the interval $[0, 1]$, is known as the renormalisation *of f. You can investigate the dynamics of the renormalised map as r changes using the* `M`*-file* `renormalised.m`.

We know that for $r < 3$ there are no periodic orbits, and the non-zero fixed point is stable. So f^2 should only have fixed point at the origin and at the non-zero fixed point. This is shown in Figure 24.10 when $r = 2.5$. The process of creating a new map from the old one by looking at the little box and then blowing it up is called ‘renormalisation’. The `M`-file `renormalised.m` enables you to investigate the dynamics of this map as r

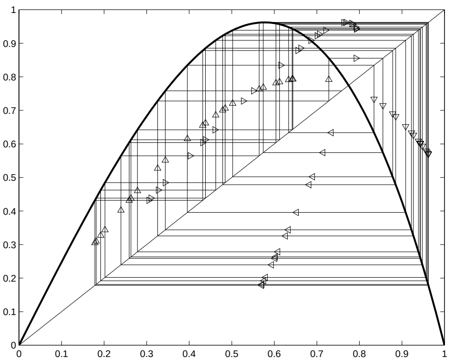

Fig. 24.6. A chaotic orbit when $r = 2.5$.

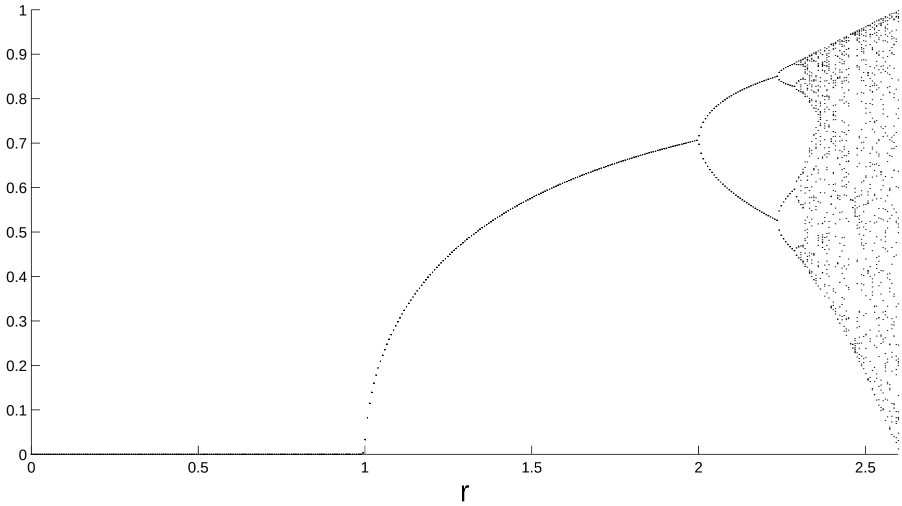

Fig. 24.7. The bifurcation diagram for the map $x_{n+1} = rx_n(1 - x_n^2)$. Thirty points on the orbit (after an initial 500 iterations) are plotted on the vertical axis for a range of values of r.

is varied (the sequence of pictures will match that in the text for the logistic map itself, and the sequence above for the cubic version of Exercise 24.1, so we will not repeat the whole set here).

When $r > 3$, however, a periodic orbits of period 2 appears. This situation is shown in Figure 24.12, and the cobweb diagram for the renormalised map in Figure 24.13.

When the period 2 orbit becomes unstable for f, this means that fixed point of f^2 will become unstable and give rise to an orbit of period 2, as illustrated in Figure 24.14.

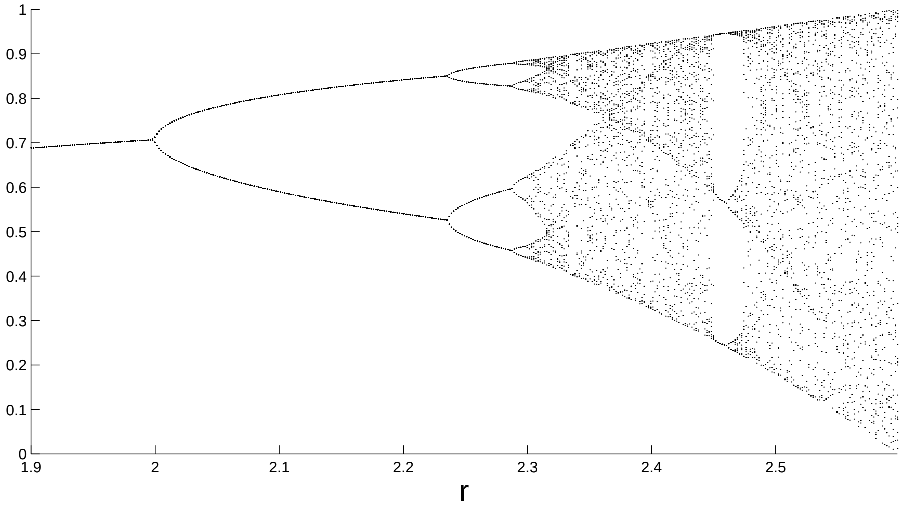

Fig. 24.8. The more 'interesting' part of Figure 24.7 showing greater detail for $1.9 \leq r \leq 3\sqrt{3}/2$.

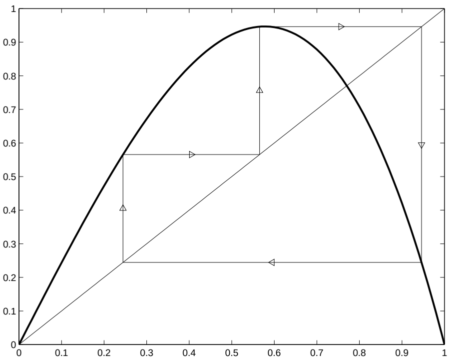

Fig. 24.9. A stable orbit of period 3 when $r = 2.46$.

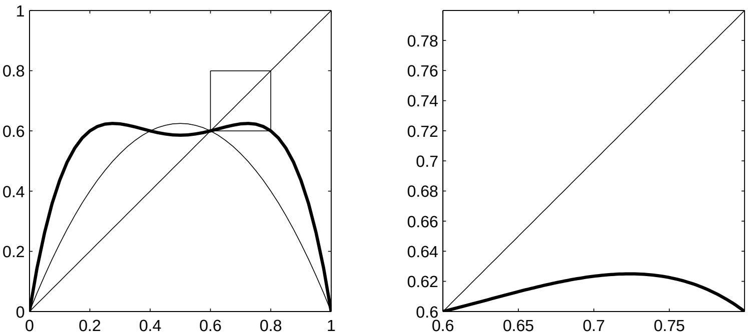

Fig. 24.10. For $r < 3$ (in the figure $r = 2.5$) there are no periodic orbits, and f^2 has no fixed points 'of its own', merely inheriting those of f. On the left is the graph of f and f^2 (in bold), while on the right the 'renormalised' map only has the uninteresting fixed point at the origin, as shown in Figure **??**.

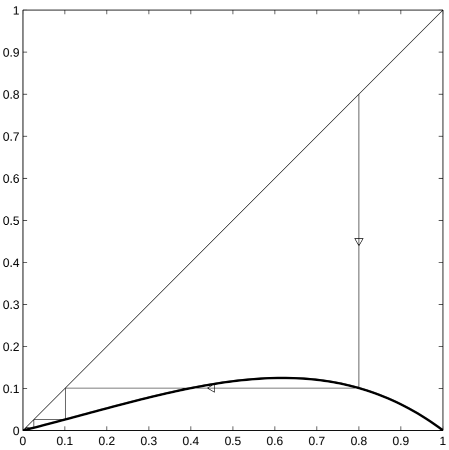

Fig. 24.11. The renormalised map when $r = 2.5$ has only a fixed point at zero.

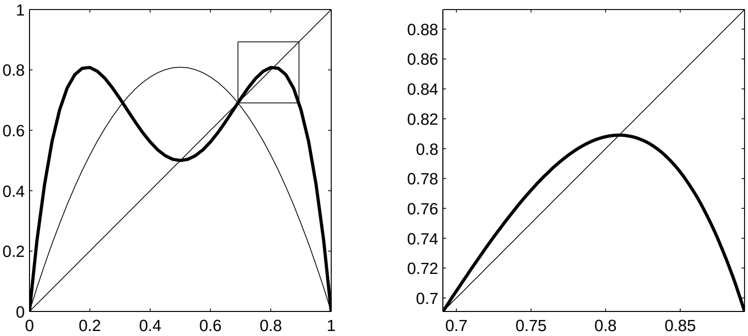

Fig. 24.12. On the left is the graph of f and f^2 (in bold) for $r = 1 + \sqrt{5}$. On the right is a magnified version of the box in the left-hand Figure, showing that f^2 has a new fixed point.

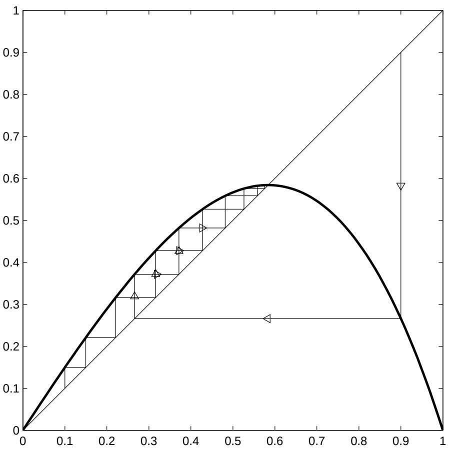

Fig. 24.13. The phase diagram for the renormalised map when $r = 1 + \sqrt{5}$; this now has a non-zero fixed point, corresponding to a period 2 orbit.

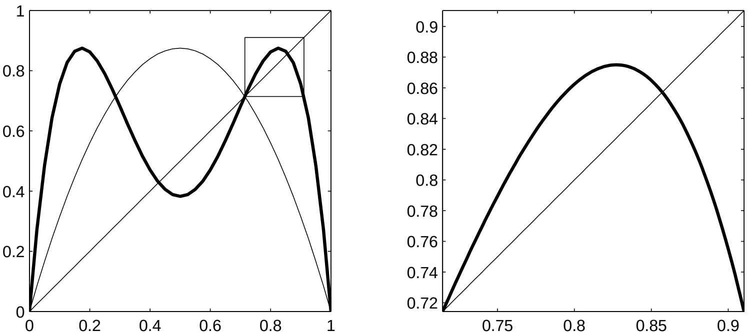

Fig. 24.14. For $r = 3.5$ there is in fact an orbit of period 4. The non-zero fixed point of the 'renormalised' map in the right-hand picture is in fact now unstable, as can be seen in the cobweb diagram of Figure 24.15.

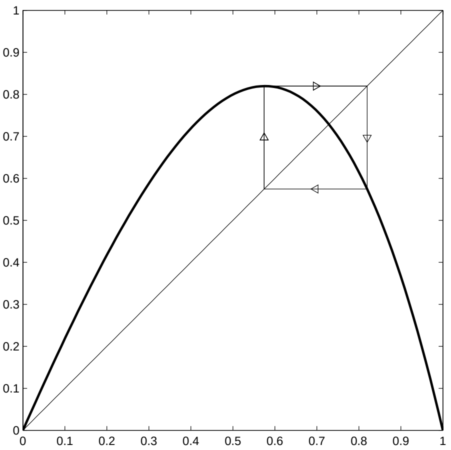

Fig. 24.15. The cobweb diagram of the renormalised map when $r = 3.5$: this map has a periodic orbit of period 2, corresponding to an orbit of period 4 for f.

25

Vector first order equations and higher order equations

Exercise 25.1 *By choosing an appropriate collection of new variables $x_1, \dots x_n$ rewrite the nth order differential equation*

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} = f\left(\frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}, \dots, \frac{\mathrm{d}x}{\mathrm{d}t}, x, t\right)$$

as a set of n coupled linear first order equations. Find the conditions on the function $f(x_n, \dots, x_1, t)$ for the original differential equation to have a unique solution.

We set $x_1 = x$, $x_2 = \dot{x}$, and in general $x_j = \mathrm{d}^{j-1}x/\mathrm{d}t^{j-1}$. Then we can write

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \vdots &= \vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(x_{n-1}, \dots, x_2, x_1, t). \end{aligned}$$

With $\mathbf{x} = (x_1, \dots, x_n)$ and

$$\mathbf{f}(\mathbf{x}) = (x_2, x_3, \dots, f(x_{n-1}, \dots, x_1, t)),$$

we can rewrite the original nth order equation as

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}).$$

The equation has a unique solution provided that $\mathbf{f}$ and its partial derivatives are continuous functions of $\mathbf{x}$ and t. Since the first $n-1$ components of $\mathbf{f}$ clearly satisfy these conditions, we only require that f and $\partial f/\partial x_j$ for each $j = 1, \dots, n$ are continuous functions of $\mathbf{x} = (x_1, \dots, x_n)$ and t.

Exercise 25.2 *Suppose that* $\mathbf{f}$ *is a Lipschitz function of* $\mathbf{x}$*, i.e. that for some* $L > 0$

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|.$$

Use an argument similar to that of Exercise 6.3 to show that if $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are two solutions of*

$$\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{f}(\mathbf{x}) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0 \qquad \text{(S25.1)}$$

and $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{y}(t)$ *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{z}|^2 \leq 2L|\mathbf{z}|^2,$$

and hence that the solution of (S25.1) is unique. [You might find the Cauchy-Schwarz inequality $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ *useful.]*

Suppose that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ satisfy

$$\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{f}(\mathbf{x}) \qquad \text{with} \qquad \mathbf{x}(0) = \mathbf{x}_0$$

and

$$\mathrm{d}\mathbf{y}/\mathrm{d}t = \mathbf{f}(\mathbf{y}) \qquad \text{with} \qquad \mathbf{y}(0) = \mathbf{x}_0.$$

Then, if $\mathbf{z} = \mathbf{x} - \mathbf{y}$ we have

$$\mathrm{d}\mathbf{z}/\mathrm{d}t = \mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) \qquad \text{and} \qquad \mathbf{z}(0) = 0.$$

Taking the inner product of the equation with $\mathbf{z}$ we have

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{z}|^2 = \mathbf{z} \cdot [\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})],$$

and so

$$\begin{aligned} \tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{z}|^2 &\leq |\mathbf{x} \cdot [\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})]| \leq |\mathbf{z}||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \\ &\leq L|\mathbf{z}|^2, \end{aligned}$$

using the Lipschitz condition on $\mathbf{f}$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|.$$

It follows that $\mathrm{d}|\mathbf{z}|^2/\mathrm{d}t \leq 2L|\mathbf{z}|^2$, and so, using the result of Exercise 9.7, we must have

$$|\mathbf{z}(t)|^2 \leq \mathrm{e}^{2Lt}|\mathbf{z}(0)|^2.$$

However, $\mathbf{z}(0) = 0$, from which it follows that $\mathbf{z}(t) = 0$ for all t, and so $\mathbf{x}(t) = \mathbf{y}(t)$ for all t: the solutions are unique.

26

Explicit solutions of coupled linear systems

Exercise 26.1 *Find the general solutions of the following differential equations by converting them into a single second-order equation. Also find the solution that satisfies the given initial conditions.*

(i)

$$\begin{aligned}\dot{x} &= 4x - y \\ \dot{y} &= 2x + y + t^2,\end{aligned} \qquad x(0) = 0 \quad \text{and} \quad y(0) = 1;$$

(ii)

$$\begin{aligned}\dot{x} &= x - 4y + \cos 2t \\ \dot{y} &= x + y,\end{aligned} \qquad x(0) = 1 \quad \text{and} \quad y(0) = 1;$$

(iii)

$$\begin{aligned}\dot{x} &= 2x + 2y \\ \dot{y} &= 6x + 3y + \mathrm{e}^t,\end{aligned} \qquad x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1;$$

(iv)

$$\begin{aligned}\dot{x} &= 5x - 4y + \mathrm{e}^{3t} \\ \dot{y} &= x + y,\end{aligned} \qquad x(0) = 1 \quad \text{and} \quad y(0) = -1;$$

(v)

$$\begin{aligned}\dot{x} &= 2x + 5y \\ \dot{y} &= -2x + \cos 3t,\end{aligned} \qquad x(0) = 2 \quad \text{and} \quad y(0) = -1;$$

(vi)

$$\begin{aligned}\dot{x} &= x + y + \mathrm{e}^{-t} \\ \dot{y} &= 4x - 2y + \mathrm{e}^{2t},\end{aligned} \qquad y(0) = -1 \quad \text{and} \quad x(0) = 1;$$

and

(vii)

$$\begin{aligned}\dot{x} &= 8x + 14y \\ \dot{y} &= 7x + y,\end{aligned} \qquad x(0) = y(0) = 1.$$

(i)

$$\begin{aligned}\dot{x} &= 4x - y \\ \dot{y} &= 2x + y + t^2,\end{aligned} \qquad \text{with} \qquad x(0) = 0 \quad \text{and} \quad y(0) = 1.$$

From the first equation we have $y = 4x - \dot{x}$, and so $\dot{y} = 4\dot{x} - \ddot{x}$. Substituting these expressions into the equation for y we obtain

$$4\dot{x} - \ddot{x} = 2x + [4x - \dot{x}] + t^2,$$

and so

$$\ddot{x} - 5\dot{x} + 6x = -t^2.$$

We can find the solution of the homogeneous equation

$$\ddot{z} - 5\dot{z} + 6z = 0$$

by trying $z = \mathrm{e}^{kt}$, and then $k^2 - 5k + 6 = 0$. This gives $k = 2$ or $k = 3$, and so the complementary function is $z(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{3t}$. To find the particular integral we try

$$x_{\mathrm{p}}(t) = Ct^2 + Dt + E,$$

for which

$$\dot{x}_{\mathrm{p}}(t) = 2Ct + D \qquad \text{and} \qquad \ddot{x}_{\mathrm{p}}(t) = 2C.$$

We therefore want

$$2C - 5[2Ct + D] + 6[Ct^2 + Dt + E] = -t^2,$$

which implies that

$$6C = -1, \qquad -10C + 6D = 0, \qquad \text{and} \qquad 2C - 5D + 6E = 0,$$

i.e. $C = -1/6$, $D = -5/18$, and $E = -19/18$. We therefore have

$$x(t) = A\mathrm{e}^{2t} + B\mathrm{e}^{3t} - \frac{3t^2 + 5t + 19}{18}.$$

Since $y = 4x - \dot{x}$ we have

$$\begin{aligned} y(t) &= 4A\mathrm{e}^{2t} + 4B\mathrm{e}^{3t} - \frac{4(3t^2 + 5t + 19)}{18} - 2A\mathrm{e}^{2t} - 3B\mathrm{e}^{3t} + \frac{6t5}{18} \\ &= 2A\mathrm{e}^{2t} + B\mathrm{e}^{3t} - \frac{12t^2 + 14t + 71}{18}. \end{aligned}$$

Now in order to satisfy the initial conditions we require

$$x(0) = A + B - \frac{19}{18} = 0 \qquad \text{and} \qquad y(0) = 2A + B - \frac{71}{18} = 1.$$

This gives $A = 6$, $B = -89/18$, and the solution

$$\begin{aligned} x(t) &= 6\mathrm{e}^{2t} - \frac{89\mathrm{e}^{3t} + 3t^2 + 5t + 19}{18} \\ y(t) &= 12\mathrm{e}^{2t} - \frac{89\mathrm{e}^{3t} - 12t^2 - 14t - 71}{18}. \end{aligned}$$

(ii)

$$\begin{aligned} \dot{x} &= x - 4y + \cos 2t \\ \dot{y} &= x + y \end{aligned} \qquad \text{with} \qquad x(0) = 1 \quad \text{and} \quad y(0) = 1.$$

From the second equation we have $x = \dot{y} - y$, and so $\dot{x} = \ddot{y} - \dot{y}$. Substituting these into the equation for $\dot{x}$ we obtain

$$\ddot{y} - \dot{y} = \dot{y} - y - 4y + \cos 2t,$$

or

$$\ddot{y} - 2\dot{y} + 5y = \cos 2t.$$

In order to solve this we first find the solution of the homogeneous equation

$$\ddot{z} - 2\dot{z} + 5z = 0$$

by trying $z(t) = \mathrm{e}^{kt}$. This gives the auxiliary equation for k,

$$k^2 - 2k + 5 = 0,$$

with solutions $k = 1 \pm 2\mathrm{i}$. Therefore

$$z(t) = \mathrm{e}^t(A\cos 2t + B\sin 2t).$$

For a particular integral we try $y_{\mathrm{p}}(t) = C\cos 2t + D\sin 2t$, and then we have

$$\dot{y}_{\mathrm{p}}(t) = -2C\sin 2t + 2D\cos 2t \qquad \text{and} \qquad \ddot{y}_{\mathrm{p}}(t) = -4C\cos 2t - 4D\sin 2t.$$

Substituting these into the equation we require

$$\begin{aligned} &[-4C\cos 2t - 4D\sin 2t] - 2[-2C\sin 2t + 2D\cos 2t] \\ &\qquad +5[C\cos 2t + D\sin 2t] \\ &= \cos 2t[C - 4D] + \sin 2t[D + 4C] \\ &= \cos 2t, \end{aligned}$$

i.e. we need

$$C - 4D = 1 \qquad \text{and} \qquad D + 4C = 0,$$

which implies that $C = 1/17$ and $D = -4/17$, and so

$$y(t) = \mathrm{e}^t(A\cos 2t + B\sin 2t) + \frac{\cos 2t - 4\sin 2t}{17}.$$

Now we can recover $x(t)$ using $x = \dot{y} - y$, giving

$$x(t) = \mathrm{e}^t[2B\cos 2t - 2A\sin 2t] - \frac{7\cos 2t + 6\sin 2t}{17}.$$

In order to satisfy the initial conditions we now need

$$x(0) = 2B - \frac{7}{17} = 1 \qquad \text{and} \qquad y(0) = A + \frac{1}{17} = 1,$$

which gives $A = 16/17$, $B = 12/17$, and the solution

$$\begin{aligned} x(t) &= \frac{\mathrm{e}^t(24\cos 2t - 32\sin 2t) - 7\cos 2t + 6\sin 2t}{17} \\ y(t) &= \frac{\mathrm{e}^t(16\cos 2t + 12\sin 2t) + \cos 2t - 4\sin 2t}{17}. \end{aligned}$$

(iii)

$$\begin{aligned} \dot{x} &= 2x + 2y \\ \dot{y} &= 6x + 3y + \mathrm{e}^t \end{aligned} \qquad \text{with} \qquad x(0) = 0 \quad \text{and} \quad \dot{x}(0) = 1.$$

From the first equation we have $y = \frac{1}{2}\dot{x} - x$, $\dot{y} = \frac{1}{2}\ddot{x} - \dot{x}$, and so

$$\tfrac{1}{2}\ddot{x} - \dot{x} = 6x + 3[\tfrac{1}{2}\dot{x} - x] + \mathrm{e}^t,$$

which is

$$\ddot{x} - 5\dot{x} - 6x = 2\mathrm{e}^t.$$

First we solve the homogeneous problem $\ddot{z} - 5\dot{z} - 6z = 0$ by setting $z = \mathrm{e}^{kt}$; we obtain two roots $k = -1$ and $k = 6$ as the solutions of $k^2 - 5k - 6 = 0$, giving the complementary function

$$z(t) = A\mathrm{e}^{-t} + B\mathrm{e}^{6t}.$$

To find a particular integral we try $x_{\mathrm{p}}(t) = C\mathrm{e}^t$, and so we require $-10C = 2$, or $C = -1/5$. So the solution for $x(t)$ is

$$x(t) = A\mathrm{e}^{-t} + B\mathrm{e}^{6t} - \frac{\mathrm{e}^t}{5}.$$

Using $y = \frac{1}{2}\dot{x} - x$ we therefore have

$$\begin{aligned} y(t) &= \tfrac{1}{2}\left[-A\mathrm{e}^{-t} + 6B\mathrm{e}^{6t} - \frac{\mathrm{e}^t}{5}\right] - \left[A\mathrm{e}^{-t} + B\mathrm{e}^{6t} - \frac{\mathrm{e}^t}{5}\right] \\ &= -\frac{3A\mathrm{e}^{-t}}{2} + 2B\mathrm{e}^{6t} + \frac{\mathrm{e}^t}{10}. \end{aligned}$$

Now we satisfy the initial conditions, by taking

$$x(0) = A + B - \frac{1}{5} = 0 \qquad \text{and} \qquad \dot{x}(0) = -A + 6B - \frac{1}{5} = 1,$$

i.e. $B = 1/5$ and $A = 0$, yielding the solution

$$\begin{aligned} x(t) &= \frac{\mathrm{e}^{6t} - \mathrm{e}^t}{5} \\ y(t) &= \frac{4\mathrm{e}^{6t} + \mathrm{e}^t}{10}. \end{aligned}$$

(iv)

$$\begin{aligned} \dot{x} &= 5x - 4y + \mathrm{e}^{3t} \\ \dot{y} &= x + y \end{aligned} \qquad \text{with} \qquad x(0) = 1 \quad \text{and} \quad y(0) = -1.$$

From the second equation we have $x = \dot{y} - y$ and $\dot{x} = \ddot{y} - \dot{y}$; substituting these into the first equation we obtain

$$\ddot{y} - \dot{y} = 5[\dot{y} - y] - 4y + \mathrm{e}^{3t},$$

which simplifies to give

$$\ddot{y} - 6\dot{y} + 9y = \mathrm{e}^{3t}.$$

The solution of the homogeneous equation

$$\ddot{z} - 6\dot{z} + 9z = 0$$

can be found by setting $z = \mathrm{e}^{kt}$. From this we obtain the auxiliary equation $k^2 - 6k + 9 = 0$, which has a repeated root $k = 3$. The complementary function is therefore

$$z(t) = A\mathrm{e}^{3t} + Bt\mathrm{e}^{3t}.$$

In order to find a particular integral we have to try $y_{\mathrm{p}}(t) = Ct^2\mathrm{e}^{3t}$. For this guess we have

$$\dot{y}_{\mathrm{p}}(t) = C(3t^2 + 2t)\mathrm{e}^{3t} \qquad \text{and} \qquad \ddot{y}_{\mathrm{p}}(t) = C(9t^2 + 12t + 2)\mathrm{e}^{3t},$$

and so we need

$$C[(9t^2 + 12t + 2) - 6(3t^2 + 2t) + 9t^2] = 1,$$

i.e. $C = \frac{1}{2}$. It follows that

$$y(t) = (A + Bt + \tfrac{1}{2}t^2)\mathrm{e}^{3t},$$

and so, using $x = \dot{y} - y$, we also have

$$x(t) = \mathrm{e}^{3t}[(2A + B) + (2B + 1)t + t^2].$$

Choosing A and B to satisfy the initial conditions we require

$$x(0) = 2A + B = 1 \qquad \text{and} \qquad y(0) = A = -1,$$

i.e. $A = -1$, $B = 3$, and the solution is

$$\begin{aligned} x(t) &= \mathrm{e}^{3t}(1 + 7t + t^2) \\ y(t) &= \mathrm{e}^{3t}(-1 + 3t + \tfrac{1}{2}t^2). \end{aligned}$$

(v)

$$\begin{aligned} \dot{x} &= 2x + 5y \\ \dot{y} &= -2x + \cos 3t \end{aligned} \qquad \text{with} \qquad x(0) = 2 \quad \text{and} \quad y(0) = -1.$$

From first equation we have $y = (\dot{x} - 2x)/5$, and so

$$\frac{\ddot{x} - 2\dot{x}}{5} = -2x + \cos 3t,$$

or more neatly

$$\ddot{x} - 2\dot{x} + 10x = 5\cos 3t.$$

Solving the homogeneous equation $\ddot{z} - 2\dot{z} + 10z = 0$ using the substitution $z(t) = \mathrm{e}^{kt}$ yields the auxiliary equation

$$k^2 - 2k + 10 = 0,$$

which has roots $k = 1 \pm 3\mathrm{i}$, and so the complementary function is

$$z(t) = \mathrm{e}^{t}(A\cos 3t + B\sin 3t).$$

For the particular integral we try $x_{\mathrm{p}}(t) = C\cos 3t + D\sin 3t$, so

$$\dot{x}_{\mathrm{p}}(t) = -3C\sin 3t + 3D\cos 3t \qquad \text{and} \qquad \ddot{x}_{\mathrm{p}}(t) = -9C\cos 3t - 9D\sin 3t.$$

Substituting these into the equation gives

$$\begin{aligned} &(-9C\cos 3t - 9D\sin 3t) - 2(-3C\sin 3t + 3D\cos 3t) \\ &\quad +10(C\cos 3t + D\sin 3t) \\ &(C - 6D)\cos 3t + (D + 6C)\sin 3t = 5\cos 3t, \end{aligned}$$

and so

$$C - 6D = 5 \qquad \text{and} \qquad D + 6C = 0.$$

This implies that $C = 5/37$, $D = -30/37$, and so

$$x(t) = \mathrm{e}^t(A\cos 3t + B\sin 3t) + \frac{5\cos 3t - 30\sin 3t}{37}.$$

Now, using $y = (\dot{x} - 2x)/5$, we have

$$\begin{aligned} y(t) &= \frac{1}{5}\Bigg(\mathrm{e}^t([3A+B]\cos 3t + [3B-A]\sin 3t) \\ &\qquad -\frac{30\sin 3t + 90\cos 3t}{37} - 2\mathrm{e}^t(A\cos 3t + B\sin 3t) \\ &\qquad -\frac{10\cos 3t - 60\sin 3t}{37}\Bigg) \\ &= \frac{1}{5}\Bigg(\mathrm{e}^t([A+B]\cos 3t + [B-A]\sin 3t) \\ &\qquad +\frac{30\sin 3t - 100\cos 3t}{37}\Bigg). \end{aligned}$$

Now for the initial conditions we want

$$x(0) = A + \frac{5}{37} = 2 \qquad \text{and} \qquad y(0) = \frac{1}{5}\left(A + B - \frac{100}{37}\right) = -1,$$

which implies that $A = 69/37$, $B = -154/37$, and the solution is

$$\begin{aligned} x(t) &= \frac{\mathrm{e}^t(69\cos 3t - 154\sin 3t) + 5\cos 3t - 30\sin 3t}{37} \\ y(t) &= \frac{\mathrm{e}^t(-17\cos 3t - (223/5)\sin 3t) + 6\sin 3t - 20\cos 3t}{37}. \end{aligned}$$

(vi)

$$\begin{aligned} \dot{x} &= x + y + \mathrm{e}^{-t} \\ \dot{y} &= 4x - 2y + \mathrm{e}^{2t} \end{aligned} \qquad \text{with} \qquad x(0) = 1 \qquad \text{and} \qquad y(0) = -1.$$

From the first equation we have $y = \dot{x} - x - \mathrm{e}^{-t}$, so

$$\dot{y} = \ddot{x} - \dot{x} + \mathrm{e}^{-t},$$

and substituting these into the equation for $\dot{y}$ we obtain

$$\ddot{x} - \dot{x} + \mathrm{e}^{-t} = 4x - 2[\dot{x} - x - \mathrm{e}^{-t}] + \mathrm{e}^{2t},$$

which is just

$$\ddot{x} + \dot{x} - 6x = \mathrm{e}^{-t} + \mathrm{e}^{2t}.$$

In order to solve the homogeneous equation

$$\ddot{z} + \dot{z} - 6z = 0$$

we put $z = e^{kt}$ and obtain the auxiliary equation $k^2 + k - 6 = 0$, with solutions $k = 2$ and $k = -3$, so that the complementary function is

$$z(t) = Ae^{2t} + Be^{-3t}.$$

For a particular integral we can try $x_p(t) = Ce^{-t} + Dte^{2t}$. This gives

$$\dot{x}_p(t) = -Ce^{-t} + D(1+2t)e^{2t} \qquad \text{and} \qquad \ddot{x}_p(t) = Ce^{-t} + D(4+4t)e^{2t},$$

and so we want

$$C[1 - 1 - 6]e^{-t} + D[4 + 4t + 1 + 2t - 6t] = e^{-t} + e^{2t},$$

which implies that $C = -1/6$, $D = 1/5$, and

$$x(t) = Ae^{2t} + Be^{-3t} - \frac{1}{6}e^{-t} + \frac{1}{5}te^{2t}.$$

We can now obtain $y(t)$ using $y = \dot{x} - x - e^{-t}$:

$$y(t) = Ae^{2t} - 2Be^{-3t} - \frac{2}{3}e^{-t} + \frac{1}{5}(t + 1)e^{2t}.$$

For the initial conditions we require

$$x(0) = A + B - \frac{1}{6} = 1 \qquad \text{and} \qquad y(0) = A - 2B - \frac{2}{3} + \frac{1}{5} = -1.$$

The solutions of these equations for A and B are $A = 18/30$ and $B = 17/30$, and so

$$\begin{aligned} x(t) &= \frac{18e^{2t} + 17e^{-3t} - 5e^{-t} + 6te^{5t}}{30} \\ y(t) &= \frac{24e^{2t} - 34e^{-3t} - 20e^{-t} + 6te^{2t}}{30}. \end{aligned}$$

(vii)

$$\begin{aligned} \dot{x} &= 8x + 14y \\ \dot{y} &= 7x + y \end{aligned} \qquad \text{with} \qquad x(0) = y(0) = 1.$$

We have $y = (\dot{x} - 8x)/14$, and so

$$\ddot{x} - 8\dot{x} = 98x + \dot{x} - 8x,$$

which simplifies to give

$$\ddot{x} - 9\dot{x} - 90x = 0.$$

Trying $x(t) = \mathrm{e}^{kt}$ we obtain the auxiliary equation $k^2 - 9k - 90 = 0$, which has roots $k = -6$ and $k = 15$. The solution for $x(t)$ is therefore

$$x(t) + A\mathrm{e}^{-6t} + B\mathrm{e}^{15t},$$

while $y(t) = (\dot{x} - 8x)/14$, and so

$$y(t) = -A\mathrm{e}^{-6t} - \tfrac{1}{2}B\mathrm{e}^{15t}.$$

In order to satisfy the initial conditions we need

$$A + B = 1 \qquad \text{and} \qquad -A - \tfrac{1}{2}B = 1,$$

so $A = -3$, $B = 4$, and the solution is

$$\begin{aligned} x(t) &= 4\mathrm{e}^{15t} - 3\mathrm{e}^{-6t} \\ y(t) &= 3\mathrm{e}^{-6t} - 2\mathrm{e}^{15t}. \end{aligned}$$

27

Eigenvalues and eigenvectors

Exercise 27.1 *Find the eigenvectors and eigenvalues of the following matrices:*

(i)

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix},$$

(ii)

$$\begin{pmatrix} 2 & 2 \\ 0 & -4 \end{pmatrix},$$

(iii)

$$\begin{pmatrix} 7 & -2 \\ 26 & -1 \end{pmatrix},$$

(iv)

$$\begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix},$$

(v)

$$\begin{pmatrix} 7 & 1 \\ -4 & 11 \end{pmatrix},$$

(vi)

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix},$$

(vii)

$$\begin{pmatrix} 6 & 0 \\ 0 & -13 \end{pmatrix},$$

(viii)

$$\begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix},$$

(ix)

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix},$$

(x)

$$\begin{pmatrix} -7 & 6 \\ 12 & -1 \end{pmatrix}.$$

(i) With

$$\mathbb{A} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

we find the eigenvalues λ by setting $|\mathbb{A} - \lambda\mathbb{I}| = 0$, so

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = 0.$$

The solutions are this quadratic equation are the eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = 2$. To find the eigenvector corresponding to $\lambda_1 = -1$ we solve $(\mathbb{A} - \lambda_1\mathbb{I})\mathbf{v} = \mathbf{0}$,

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so $v_1 + v_2 = 0$, i.e. $v_2 = -v_1$. A representative choice of eigenvector is therefore

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For the eigenvector corresponding to $\lambda_2 = 2$ we consider

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

which implies that $v_1 = 2v_2$, so the eigenvector is

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(ii) The matrix

$$\mathbb{A} = \begin{pmatrix} 2 & 2 \\ 0 & -4 \end{pmatrix}$$

is one of those that we can recognise using Example 27.3 as having eigenvalues that we can simply read off: they are the diagonal elements $\lambda_1 = 2$ and $\lambda_2 = -4$. The eigenvector corresponding to $\lambda_1 = 2$ is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For the eigenvector corresponding to $\lambda_2 = -4$ we have to consider $(\mathbb{A} - \lambda_2\mathbb{I})\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 6 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

which gives $6v_1 + 2v_2 = 0$, so $v_2 = -3v_1$, and a representative eigenvector is

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(iii) With

$$\mathbb{A} = \begin{pmatrix} 7 & -2 \\ 26 & -1 \end{pmatrix}$$

we have

$$|\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 7-\lambda & -2 \\ 26 & -1-\lambda \end{vmatrix} = (7-\lambda)(-1-\lambda) + 52 = 0.$$

This simplifies to

$$\lambda^2 - 6\lambda + 45 = 0,$$

with roots

$$\lambda = \frac{6 \pm \sqrt{36 - 180}}{2} = 3 \pm 6\mathrm{i}.$$

Since the eigenvalues form a complex conjugate pair, so do the corresponding eigenvectors; so we only have to perform the calculations to find one of them. We will do this for the eigenvalue $\lambda_+ = 3 + 6\mathrm{i}$. The eigenvector $\mathbf{v}_+$ satisfies $(\mathbb{A} - \lambda_+\mathbb{I})\mathbf{v}_+ = \mathbf{0}$, so we have

$$\begin{pmatrix} 4-6\mathrm{i} & -2 \\ 26 & -4-6\mathrm{i} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

which means that we need $(4-6\mathrm{i})v_1 - 2v_2 = 0$, so $v_2 = (2-3\mathrm{i})v_1$, yielding the eigenvector

$$\mathbf{v}_+ = \begin{pmatrix} 1 \\ 2-3\mathrm{i} \end{pmatrix}.$$

The eigenvector corresponding to $\lambda_- = 3-6\mathrm{i}$ is the complex conjugate of this,

$$\mathbf{v}_= \begin{pmatrix} 1 \\ 2+3\mathrm{i} \end{pmatrix}.$$

(iv) We now have

$$\mathbb{A} = \begin{pmatrix} 9 & 2 \\ 2 & 6 \end{pmatrix},$$

and so the eigenvalues are the solutions of

$$|\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 9-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = (9-\lambda)(6-\lambda) - 4 = 0.$$

This gives the quadratic equation

$$\lambda^2 - 15\lambda + 50 = 0,$$

with solutions $\lambda_1 = 5$ and $\lambda_2 = 10$. To find the eigenvalue corresponding to $\lambda_1 = 5$ we consider

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

which implies that $2v_1 + v_2 = 0$, so $v_2 = -2v_1$ and the eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

For the eigenvector corresponding to $\lambda_2 = 10$ we want

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so we require $-v_1 + 2v_2 = 0$, i.e. $v_1 = 2v_2$ and the eigenvector is

$$\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(v)

$$\mathbb{A} = \begin{pmatrix} 7 & 1 \\ -4 & 11 \end{pmatrix}.$$

We have

$$|\mathbb{A}-\lambda\mathbb{I}| = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 11-\lambda \end{vmatrix} = (7-\lambda)(11-\lambda)+4 = 0.$$

The quadratic equation

$$\lambda^2 - 18\lambda + 81 = 0$$

has a repeated root $\lambda = 9$, so there is only one eigenvalue. Consequently we only expect to find one eigenvector, given by

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}.$$

This implies that $-2v_1 + v_2 = 0$, so $v_2 = 2v_1$ and the eigenvector is

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(vi)

$$\mathbb{A} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

The eigenvalues are the solutions of

$$|\mathbb{A}-\lambda\mathbb{I}| = \begin{vmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 9 = \lambda^2 - 4\lambda + 13 = 0,$$

so are

$$\lambda_\pm = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3\mathrm{i}.$$

The eigenvector corresponding to $\lambda_+ = 2 + 3\mathrm{i}$ is given by

$$\begin{pmatrix} -3\mathrm{i} & -3 \\ 3 & -3\mathrm{i} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so we must have $-3\mathrm{i}v_1 - 3v_2 = 0$, i.e. $v_2 = -\mathrm{i}v_1$ and the eigenvectors are

$$\mathbf{v}_\pm = \begin{pmatrix} 1 \\ \mp\mathrm{i} \end{pmatrix}.$$

(vii)

$$\mathbb{A} = \begin{pmatrix} 6 & 0 \\ 0 & -13 \end{pmatrix}.$$

This is a diagonal matrix, as in Example 27.2. The eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -13$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(viii)

$$\mathbb{A} = \begin{pmatrix} 4 & -2 \\ 1 & 2 \end{pmatrix}.$$

To find the eigenvalues consider

$$|\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 4-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) + 2 = \lambda^2 - 6\lambda + 10 = 0,$$

which has solutions

$$\lambda_\pm = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm \mathrm{i}.$$

The eigenvector corresponding to $\lambda_+ = 3 + \mathrm{i}$ is given by

$$\begin{pmatrix} 1-\mathrm{i} & -2 \\ 1 & -1-\mathrm{i} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so we require $v_1 = (1+\mathrm{i})v_2$, and one choice for the eigenvector is

$$\mathbf{v}_+ = \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix}.$$

The eigenvector corresponding to $\lambda_- = 3 - \mathrm{i}$ is just the complex conjugate of this,

$$\mathbf{v}_- = \begin{pmatrix} 1-\mathrm{i} \\ 1 \end{pmatrix}.$$

(ix)

$$\mathbb{A} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

The eigenvalues satisfy

$$|\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = 0,$$

and so there is just one eigenvalue $\lambda = 2$. It follows that there is just one eigenvector given by

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

i.e. $v_1 - v_2 = 0$ and so the eigenvector is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(x) Finally

$$\mathbb{A} = \begin{pmatrix} -7 & 6 \\ 12 & -1 \end{pmatrix}$$

has eigenvalues given by the solutions of the characteristic equation

$$\begin{vmatrix} -7-\lambda & 6 \\ 12 & -1-\lambda \end{vmatrix} = (-7-\lambda)(-1-\lambda) - 72 = \lambda^2 + 8\lambda - 65 = 0,$$

i.e. $\lambda_1 = -13$ or $\lambda_2 = 5$. The eigenvector corresponding to $\lambda_1 = -13$ can be found from

$$\begin{pmatrix} 6 & 6 \\ 12 & 12 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so has $v_1 + v_2 = 0$, i.e. is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvector for $\lambda_2 = 5$ satisfies

$$\begin{pmatrix} -12 & 6 \\ 12 & -6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so has $2v_1 = v_2$; it is

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

28

Distinct real eigenvalues

Exercise 28.1 *Write down the general solution and draw the phase portrait for the equation* $\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}$*, when the eigenvalues and eigenvectors of* $\mathbb{A}$ *are as follows. You should take particular care with stable (or unstable) nodes to ensure that the trajectories approach (or move away from) the origin tangent to the correct eigenvector.*

(i) $\lambda_1 = 1$, $\mathbf{v}_1 = (1,\ 1)$ *and* $\lambda_2 = 2$, $\mathbf{v}_2 = (1,\ -1)$*;*
(ii) $\lambda_1 = 1$, $\mathbf{v}_1 = (1,\ 0)$ *and* $\lambda_2 = -2$, $\mathbf{v}_2 = (1,\ 1)$*;*
(iii) $\lambda_1 = -2$, $\mathbf{v}_1 = (1,\ 2)$ *and* $\lambda_2 = -3$, $\mathbf{v}_2 = (2,\ -3)$*;*
(iv) $\lambda_1 = 3$, $\mathbf{v}_1 = (2,\ 3)$ *and* $\lambda_2 = -5$, $\mathbf{v}_2 = (0,\ 1)$*;*
(v) $\lambda_1 = 3$, $\mathbf{v}_1 = (1,\ 2)$ *and* $\lambda_2 = 1$, $\mathbf{v}_2 = (1,\ -3)$*;*
(vi) $\lambda_1 = 2$, $\mathbf{v}_1 = (0,\ 1)$ *and* $\lambda_2 = -3$, $\mathbf{v}_2 = (1,\ 5)$*;*
(vii) $\lambda_1 = 1$, $\mathbf{v}_1 = (1,\ 1)$ *and* $\lambda_2 = 2$, $\mathbf{v}_2 = (2,\ 1)$*; and*
(viii) $\lambda_1 = -3$, $\mathbf{v}_1 = (1,\ 3)$ *and* $\lambda_2 = -1$, $\mathbf{v}_2 = (-3,\ 2)$*.*

(i)

$$\lambda_1 = 1 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 2 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = A\mathrm{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The origin is an unstable node, and solutions move away tangent to $\mathbf{v}_1$ (since $|\lambda_1| < |\lambda_2|$).

(ii)

$$\lambda_1 = 1 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -2 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

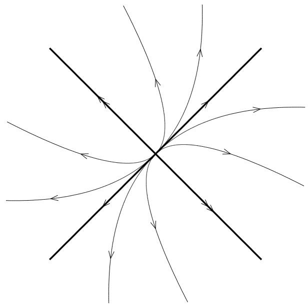

Fig. 28.1. The phase portrait for Exercise 28.1(i).

The general solution is

$$\mathbf{x}(t) = Ae^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and the origin is a saddle point.

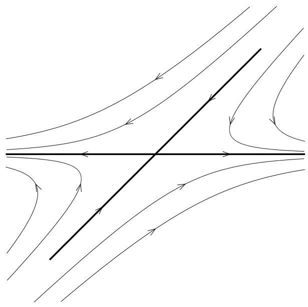

Fig. 28.2. The phase portrait for Exercise 28.1(ii).

(iii)

$$\lambda_1 = -2 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -3 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = Ae^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{-3t}\begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Trajectories approach to origin, a stable node, tangent to $\mathbf{v}_1$ (since $|\lambda_1| < |\lambda_2|$).

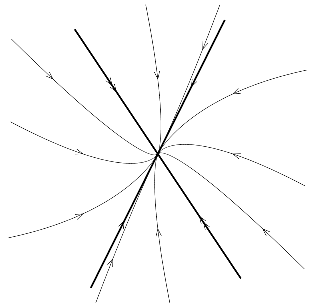

Fig. 28.3. The phase portrait for Exercise 28.1(iii).

(iv)

$$\lambda_1 = 3 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -5 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The origin is a saddle point, and the general solution is

$$\mathbf{x}(t) = A\mathrm{e}^{3t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + B\mathrm{e}^{-5t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

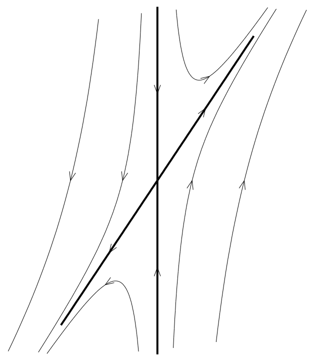

Fig. 28.4. The phase portrait for Exercise 28.1(iv).

(v)

$$\lambda_1 = 3 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 1 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = A\mathrm{e}^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + B\mathrm{e}^{t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Trajectories move away from the origin, which is an unstable node, tangent to $\mathbf{v}_2$ (since $|\lambda_2| < |\lambda_1|$).

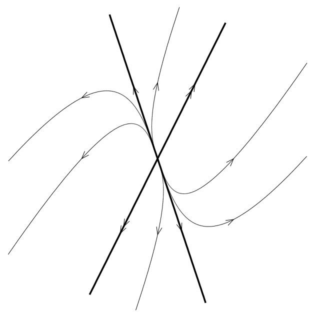

Fig. 28.5. The phase portrait for Exercise 28.1(v).

(vi)

$$\lambda_1 = 2 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -3 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = A\mathrm{e}^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + B\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

and the origin is a saddle.

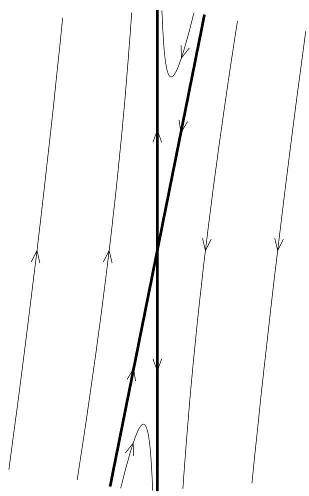

Fig. 28.6. The phase portrait for Exercise 28.1(vi).

(vii)

$$\lambda_1 = 1 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 2 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = A\mathrm{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The origin is an unstable node, and trajectories move away tangent to $\mathbf{v}_1$, since $|\lambda_1| < |\lambda_2|$.

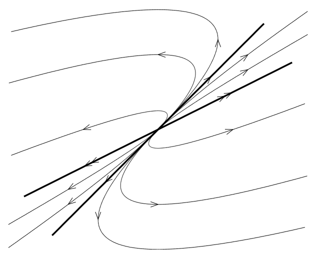

Fig. 28.7. The phase portrait for Exercise 28.1(vii).

(viii)

$$\lambda_1 = -3 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -1 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = A\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + B\mathrm{e}^{-t}\begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

The origin is a stable node, approached tangent to $\mathbf{v}_2$ since $|\lambda_2| < |\lambda_1|$.

Exercise 28.2 *For the following equations find the eigenvalues and eigenvectors of the matrix on the right-hand side, and hence find the coordinate transformation that will decouple the equations. Show that this transformation has the desired effect. (You can also write down the general solution and draw the phase portrait for the equation if you wish.)*

(i)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 8 & 14 \\ 7 & 1 \end{pmatrix}\mathbf{x}$$

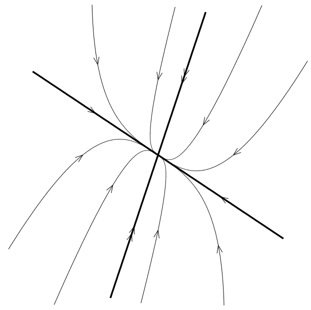

Fig. 28.8. The phase portrait for Exercise 28.1(viii).

(ii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 2 & 0 \\ -5 & -3 \end{pmatrix} \mathbf{x}$$

(iii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 11 & -2 \\ 3 & 4 \end{pmatrix} \mathbf{x}$$

and

(iv)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 1 & 20 \\ 40 & -19 \end{pmatrix} \mathbf{x}.$$

Throughout these solutions we use $\mathbb{A}$ to denote the matrix appearing on the right-hand side of the differential equation, and $\tilde{\mathbf{x}} = (\tilde{x},\ \tilde{y})$.

(i) The eigenvalues of the matrix

$$\begin{pmatrix} 8 & 14 \\ 7 & 1 \end{pmatrix}$$

are the solutions of

$$\begin{vmatrix} 8-\lambda & 14 \\ 7 & 1-\lambda \end{vmatrix} = (8-\lambda)(1-\lambda) - 98 = \lambda^2 - 9\lambda - 90 = 0,$$

and so are $\lambda_1 = -6$ and $\lambda_2 = 15$. The corresponding eigenvectors are: for $\lambda - 1 = -6$, given by

$$\begin{pmatrix} 14 & 14 \\ 7 & 7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so is $\mathbf{v}_1 = (1,\ -1)$; for $\lambda_2 = 15$ given by

$$\begin{pmatrix} -7 & 14 \\ 7 & -14 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so is $\mathbf{v}_2 = (2,\ 1)$.

If we write

$$\mathbf{x} = \tilde{x} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \tilde{y} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \tilde{\mathbf{x}}$$

then we have

$$\tilde{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \mathbf{x},$$

and so

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 8 & 14 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \tilde{\mathbf{x}} \\ &= \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -6 & 30 \\ 6 & 15 \end{pmatrix} \tilde{\mathbf{x}} \\ &= \begin{pmatrix} -6 & 0 \\ 0 & 15 \end{pmatrix} \tilde{\mathbf{x}}. \end{aligned}$$

(ii) The eigenvalues of the matrix

$$\begin{pmatrix} 2 & 0 \\ -5 & -3 \end{pmatrix}$$

are $\lambda_1 = -3$ and $\lambda_2 = 2$ (see Example 27.3). The eigenvector corresponding to $\lambda_1 = -3$ is $\mathbf{v}_1 = (0,\ 1)$, while that corresponding to $\lambda_2 = 2$ can be found by considering

$$\begin{pmatrix} 0 & 0 \\ -5 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so is $\mathbf{v}_2 = (1,\ -1)$.

If we set

$$\mathbf{x} = \tilde{x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \tilde{y} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \tilde{\mathbf{x}}$$

we have

$$\tilde{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \mathbf{x} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\tilde{\mathbf{x}} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -3 & -2 \end{pmatrix}\tilde{\mathbf{x}} \\ &= \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\tilde{\mathbf{x}}.\end{aligned}$$

(iii) The eigenvalues of the matrix

$$\begin{pmatrix} 11 & -2 \\ 3 & 4 \end{pmatrix}$$

are the solutions of the characteristic equation,

$$\begin{vmatrix} 11-\lambda & -2 \\ 3 & 4-\lambda \end{vmatrix} = (11-\lambda)(4-\lambda)+6 = \lambda^2 - 15\lambda + 50 = 0,$$

and so are $\lambda_1 = 5$ and $\lambda_2 = 10$. The eigenvector corresponding to λ_1 is given by

$$\begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so is $\mathbf{v}_1 = (1,\ 3)$; while that corresponding to λ_2 can be found from

$$\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so is $\mathbf{v}_2 = (2,\ 1)$.

We now set

$$\mathbf{x} = \tilde{x}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \tilde{y}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\tilde{\mathbf{x}},$$

and so

$$\tilde{\mathbf{x}} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}^{-1}\mathbf{x} = -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}\mathbf{x}.$$

We therefore have

$$\begin{aligned}\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 11 & -2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\tilde{\mathbf{x}} \\ &= -\frac{1}{5}\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 & 20 \\ 15 & 10 \end{pmatrix}\tilde{\mathbf{x}}\end{aligned}$$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \tilde{\mathbf{x}}.$$

(iv) To find the eigenvalues of

$$\begin{pmatrix} 1 & 20 \\ 40 & -19 \end{pmatrix}$$

we consider

$$\begin{vmatrix} 1-\lambda & 20 \\ 40 & -19-\lambda \end{vmatrix} = (1-\lambda)(-19-\lambda) - 800 = \lambda^2 + 18\lambda - 819 = 0,$$

which has solutions $\lambda_1 = 21$ and $\lambda_2 = -39$. The eigenvector corresponding to $\lambda_1 = 21$ is given by

$$\begin{pmatrix} -20 & 20 \\ 40 & -40 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so is $\mathbf{v}_1 = (1,\ 1)$; while that corresponding to $\lambda_2 = -39$ is given by

$$\begin{pmatrix} 40 & 20 \\ 40 & 20 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

i.e. it is $\mathbf{v}_2 = (1,\ -2)$.

We now set

$$\mathbf{x} = \tilde{x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tilde{y} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \tilde{\mathbf{x}},$$

which implies that

$$\tilde{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \mathbf{x} = -\frac{1}{3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{x}.$$

Thus

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 20 \\ 40 & -19 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \tilde{\mathbf{x}} \\ &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 21 & -39 \\ 21 & 78 \end{pmatrix} \tilde{\mathbf{x}} \\ &= \begin{pmatrix} 21 & 0 \\ 0 & -39 \end{pmatrix} \tilde{\mathbf{x}}. \end{aligned}$$

Exercise 28.3 *(C) Given a matrix*

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the `M`*-file* `lportrait.m` *will draw the phase portrait for the linear equation* $\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}$*. The program draws the trajectory forwards and backwards from a given initial condition, placing an arrow there indicating the direction the solution moves as* t *increases. Draw the phase portraits for the equations in the previous exercise using this program.*

All the phase portraits above in the solutions for Exercise 28.1 were produced using `lportait.m`.

Exercise 28.4 *(T) Using the chain rule, if* $y = y(x(t))$ *then*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t},$$

from which it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Therefore if

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_1 x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \lambda_2 y \tag{S28.1}$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\lambda_2 y}{\lambda_1 x}.$$

Solve this to find the equation of the curves traced out by trajectories of (S28.1).

We can separate variables in the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\lambda_2 y}{\lambda_1 x}$$

to give

$$\frac{\mathrm{d}y}{\lambda_2 y} = \frac{\mathrm{d}x}{\lambda_1 x}.$$

Integrating both sides we obtain

$$\lambda_2 \ln|y| = \lambda_1 \ln|x| + c,$$

or taking exponentials,

$$|y|^{\lambda_2} = A|x|^{\lambda_1} \qquad \text{with} \qquad A > 0,$$

or

$$|y| = C|x|^{\lambda_1/\lambda_2}$$

(where $C = A^{1/\lambda_2}$ is another arbitrary, but positive, constant). We obtain all possible solutions by removing the modulus signs:

$$y = Cx^{\lambda_1/\lambda_2},$$

cf. (28.13).

Exercise 28.5 *(T) We have seen in this chapter that if* $\mathbb{A}$ *has distinct real eigenvalues* λ_1 *and* λ_2*, with corresponding eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*, then*

$$\mathbb{P}^{-1}\mathbb{A}\mathbb{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\mathbb{P} = [\mathbf{v}_1 \ \ \mathbf{v}_2]$*. It follows, conversely, that the matrix with these eigenvalues and eigenvector is*

$$\mathbb{A} = \mathbb{P}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbb{P}^{-1}.$$

(This is how the `M`*-file* `makematrix.m` *constructs matrices with specified eigenvalues and eigenvectors.) Find the matrices whose eigenvalues and eigenvectors are as follows:*

(i) $\lambda_1 = 3$, $\mathbf{v}_1 = (1,\ 2)$ *and* $\lambda_2 = 6$, $\mathbf{v}_2 = (1,\ -1)$*;*
(ii) $\lambda_1 = 3$, $\mathbf{v}_1 = (1,\ 0)$ *and* $\lambda_2 = -1$, $\mathbf{v}_2 = (2,\ 1)$*; and*
(iii) $\lambda_1 = 5$, $\mathbf{v}_1 = (1,\ 1)$ *and* $\lambda_2 = 1$, $\mathbf{v}_2 = (1 - 1,\ )$*.*

(You could now check your phase portraits for Exercise 28.1, using the `M`*-file* `makematrix.m` *to find the matrix with the specified eigenvalues and eigenvectors, and then* `lportrait.m` *to draw the phase portraits.)*

(i) The matrix with eigenvalues and eigenvectors $\lambda_1 = 3$, $\mathbf{v}_1 = (1,\ 2)$ and $\lambda_2 = 6$, $\mathbf{v}_2 = (1,\ -1)$ is given by

$$\begin{aligned}
\mathbb{A} &= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \times -\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 5 & -1 \\ -2 & 4 \end{pmatrix}.
\end{aligned}$$

(ii) The matrix with eigenvalues and eigenvectors $\lambda_1 = 3$, $\mathbf{v}_1 = (1,\ 0)$ and $\lambda_2 = -1$, $\mathbf{v}_2 = (2,\ 1)$ is given by

$$\begin{aligned} \mathbb{A} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -6 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -8 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

(iii) The matrix with eigenvalues and eigenvectors $\lambda_1 = 5$, $\mathbf{v}_1 = (1,\ 1)$ and $\lambda_2 = 1$, $\mathbf{v}_2 = (1 - 1,\ )$ is given by

$$\begin{aligned} \mathbb{A} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \times -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -5 & -5 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}. \end{aligned}$$

Exercise 28.6 *Suppose that $\mathbb{A}$ has two eigenvalues, $\lambda_1 = 0$ with eigenvector $\mathbf{v}_1$ and $\lambda_2 \neq 0$ with eigenvector $\mathbf{v}_2$.*

(i) Write down the general solution of the equation $\dot{\mathbf{x}} = \mathbb{A}\mathbf{x}$.

(ii) After changing to a coordinate system referred to the eigenvectors the equation will become

$$\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tilde{\mathbf{x}},$$

i.e

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = \lambda\tilde{y}.$$

By solving these equations draw the phase portrait in the $(\tilde{x}, \tilde{y})$ system, and hence sketch the phase portrait for the original coordinates.

(iii) Draw the phase portrait for the equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \mathbf{x}.$$

(i) The general solution is

$$\mathbf{x}(t) = A\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + B\mathrm{e}^{\lambda_2 t}\mathbf{v}_2 = A\mathbf{v}_1 + B\mathrm{e}^{\lambda_2 t}\mathbf{v}_2.$$

(ii) When

$$\mathrm{d}\tilde{x}/\mathrm{d}t = 0 \qquad \text{and} \qquad \mathrm{d}\tilde{y}/\mathrm{d}t = \lambda\tilde{y}$$

the solutions are $\tilde{x}(t) = \tilde{x}(0)$ and $\tilde{y}(t) = \mathrm{e}^{\lambda t}\tilde{y}(0)$. So $\tilde{x}$ is constant, and $\tilde{y}$ either increases exponentially (if $\lambda > 0$) or tends to zero exponentially (if $\lambda < 0$). The phase portrait for the case $\lambda > 0$ is shown to the left in Figure 28.9, while the phase portrait in the original coordinates in shown to the right.

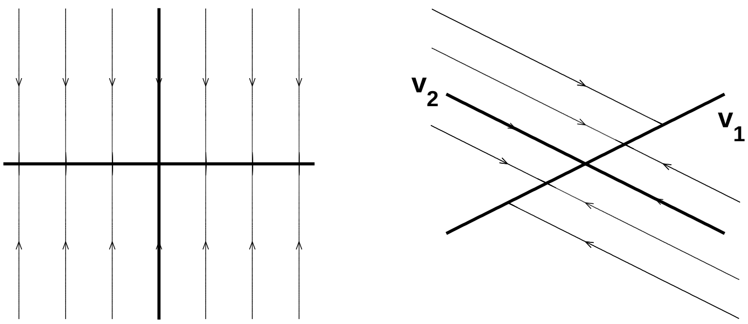

Fig. 28.9. Phase portraits when one real eigenvalue is zero: canonical coordinates on the left, original coordinates on the right.

(iii) The matrix occurring in the equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \mathbf{x} \tag{S28.2}$$

has eigenvalues given by the solutions of the characteristic equation

$$\begin{vmatrix} -2-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = (-2-\lambda)(-1-\lambda) - 2 = \lambda^2 + 3\lambda = 0,$$

so $\lambda_1 = 0$ or $\lambda_2 = -3$. The corresponding eigenvectors are $\mathbf{v}_1 = (1,\ 1)$ and $\mathbf{v}_2 = (-2,\ 1)$, and the phase portrait is shown in Figure 28.10.

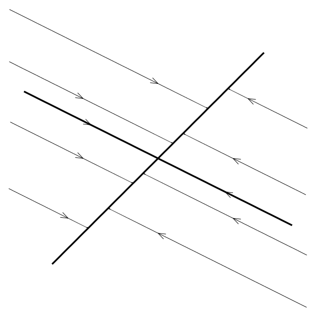

Fig. 28.10. The phase portrait for equation (S28.2).

29

Complex eigenvalues

Exercise 29.1 *Draw the phase portrait for the equation* $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbb{A}\mathbf{x}$*, when the eigenvalues* ($\lambda_\pm$) *and eigenvectors* ($\eta_\pm$) *of* $\mathbb{A}$ *are as follows. Also given is the sign of* $\dot{x}$ *when* $x = 0$ *and* $y > 0$*.*

(i) $\lambda_\pm = 1 \pm 3\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (1,\ 2 \mp \mathrm{i})$, $\dot{x} < 0$;
(ii) $\lambda_\pm = \pm 3\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (1 \pm 2\mathrm{i},\ 1 \mp 3\mathrm{i})$, $\dot{x} < 0$;
(iii) $\lambda_\pm = -2 \pm \mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (1 \mp \mathrm{i},\ 3 \pm \mathrm{i})$, $\dot{x} > 0$;
(iv) $\lambda_\pm = -1 \pm \mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (1,\ \pm\mathrm{i})$, $\dot{x} > 0$;
(v) $\lambda_\pm = 2 \pm 2\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (\pm 3\mathrm{i},\ 5 \mp 4\mathrm{i})$, $\dot{x} < 0$;
(vi) $\lambda_\pm = 5 \pm 3\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (2 \pm 5\mathrm{i},\ \mp\mathrm{i})$, $\dot{x} < 0$;
(vii) $\lambda_\pm = \pm 7\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (1 \pm \mathrm{i},\ -1 \pm 2\mathrm{i})$, $\dot{x} > 0$; *and*
(viii) $\lambda_\pm = -13 \pm 17\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (\pm 6\mathrm{i} - 8,\ 4 \mp 5\mathrm{i})$, $\dot{x} > 0$.

(i)

$$\lambda_\pm = 1 \pm 3\mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (1,\ 2 \mp \mathrm{i}) \quad \dot{x} < 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

Since the real part of λ is positive the origin is an unstable spiral; since $\dot{x} < 0$ when $x = 0$ and $y > 0$ the orbits rotate anti-clockwise around the origin, and the phase portrait is as in Figure 29.1.

(ii)

$$\lambda_\pm = \pm 3\mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (1 \pm 2\mathrm{i},\ 1 \mp 3\mathrm{i}) \quad \dot{x} < 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

The eigenvalues have zero real part, so the origin is a centre. Since $\dot{x} < 0$ when $x = 0$ and $y > 0$ the orbits rotate anti-clockwise around the origin, and the phase portrait is as in Figure 29.2.

(iii)

$$\lambda_\pm = -2 \pm \mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (1 \mp \mathrm{i},\ 3 \pm \mathrm{i}) \quad \dot{x} > 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

Since the real part of the eigenvalues is negative the origin is a stable

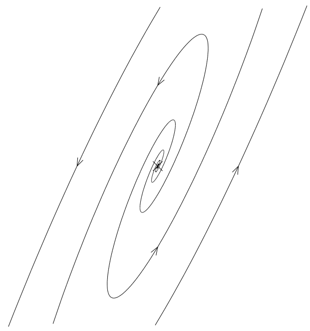

Fig. 29.1. Phase portrait for Exercise 29.1(i).

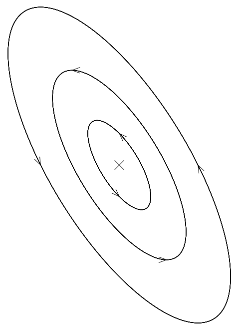

Fig. 29.2. Phase portrait for Exercise 29.1(ii).

spiral; with $\dot{x} > 0$ when $x = 0$ and $y > 0$ the solutions circle the origin is a clockwise direction, and the phase portrait is shown in Figure 29.3.

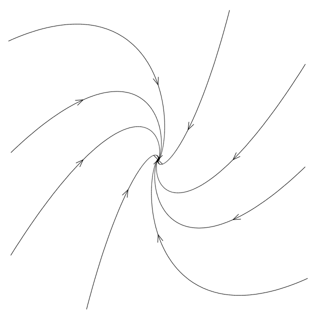

Fig. 29.3. Phase portrait for Exercise 29.1(iii).

(iv)

$$\lambda_\pm = -1 \pm \mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (1,\ \pm\mathrm{i}) \quad \dot{x} > 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

Since the real part of the eigenvalues is negative the origin is a stable spiral; since $\dot{x} > 0$ when $x = 0$ and $y > 0$ the solutions move clockwise around the origin, and the phase portrait is shown in Figure 29.4.

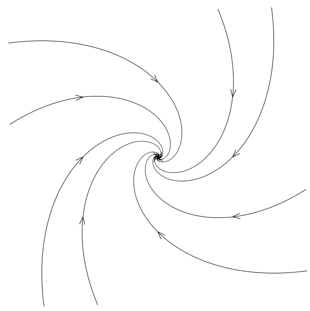

Fig. 29.4. Phase portrait for Exercise 29.1(iv).

(v)

$$\lambda_\pm = 2 \pm 2\mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (\pm 3\mathrm{i},\ 5 \mp 4\mathrm{i}) \quad \dot{x} < 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

Since the real part of the eigenvalues is positive the origin is an unstable spiral; since $\dot{x} < 0$ when $x = 0$ and $y > 0$ orbits circle the origin anti-clockwise, and the phase portrait is as in Figure 29.5.

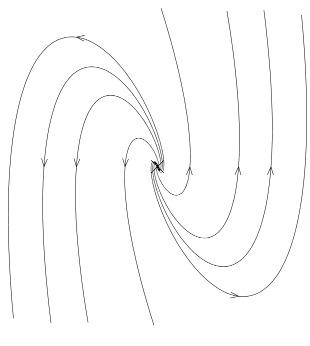

Fig. 29.5. Phase portrait for Exercise 29.1(v).

(vi)

$$\lambda_\pm = 5 \pm 3\mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (2 \pm 5\mathrm{i},\ \mp\mathrm{i}) \quad \dot{x} < 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

Since the real part of the eigenvalues is positive the origin is an unstable spiral; since $\dot{x} < 0$ when $x = 0$ and $y > 0$ the orbits spiral around the origin anti-clockwise, and the phase portrait is shown in Figure 29.6.

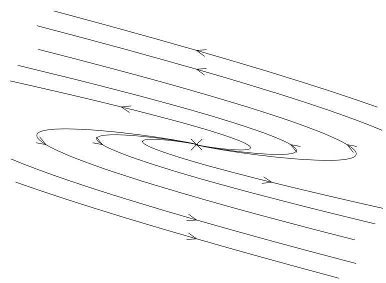

Fig. 29.6. Phase portrait for Exercise 29.1(vi).

(vii)

$$\lambda_\pm = \pm 7\mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (1\pm\mathrm{i},\ -1\pm2\mathrm{i}) \quad \dot{x} > 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

The real part of the eigenvalues is zero, so the origin is a centre. Orbits circle to origin clockwise, since $\dot{x} > 0$ when $x = 0$ and $y > 0$. The phase portrait is shown in Figure 29.7.

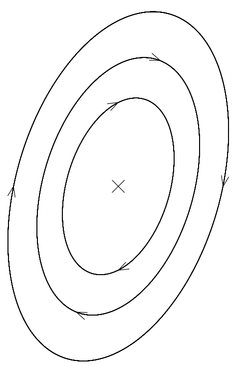

Fig. 29.7. Phase portrait for Exercise 29.1(vii).

(viii)

$$\lambda_\pm = -13\pm17\mathrm{i} \quad \text{with} \quad \boldsymbol{\eta}_\pm = (\pm6\mathrm{i}-8,\ 4\mp5\mathrm{i}) \quad \dot{x} > 0 \quad \text{when} \quad x = 0 \text{ and } y > 0.$$

The real part of the eigenvalues is negative, so the origin is a stable spiral. Since $\dot{x} > 0$ when $x = 0$ and $y > 0$, orbits circle the origin clockwise, as shown in Figure 29.8.

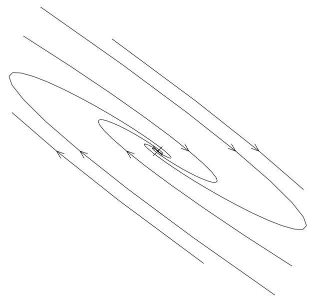

Fig. 29.8. Phase portrait for Exercise 29.1(viii).

Exercise 29.2 *Write down the general solution of the equation* $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbb{A}\mathbf{x}$ *when the eigenvalues* ($\lambda_\pm$) *and eigenvectors* ($\boldsymbol{\eta}_\pm$) *of* $\mathbb{A}$ *are those in the previous exercise.*

Throughout this exercise, the arbitrary constant C is the complex number $\alpha + \mathrm{i}\beta$.

(i) When $\lambda_\pm = 1 \pm 3\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1,\ 2 \mp \mathrm{i})$ the general solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= C\mathrm{e}^{(1+3\mathrm{i})t}\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix} + C^*\mathrm{e}^{(1-3\mathrm{i})t}\begin{pmatrix} 1 \\ 2+\mathrm{i} \end{pmatrix} \\
&= 2\,\mathrm{Re}\left[C\mathrm{e}^{(1+3\mathrm{i})t}\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix}\right] \\
&= 2\mathrm{e}^t\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos 3t + \mathrm{i}\sin 3t)\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix}\right] \\
&= 2\mathrm{e}^t\mathrm{Re}\left[\Big((\alpha\cos 3t - \beta\sin 3t) + \mathrm{i}(\beta\cos 3t + \alpha\sin 3t)\Big)\begin{pmatrix} 1 \\ 2-\mathrm{i} \end{pmatrix}\right] \\
&= 2\mathrm{e}^t\begin{pmatrix} \alpha\cos 3t - \beta\sin 3t \\ +2(\alpha\cos 3t - \beta\sin 3t) + (\beta\cos 3t + \alpha\sin 3t) \end{pmatrix} \\
&= \mathrm{e}^t\begin{pmatrix} A\cos 3t + B\sin 3t \\ (2A-B)\cos 3t + (A+2B)\sin 3t \end{pmatrix},
\end{aligned}$$

where $A = 2\alpha$ and $B = -2\beta$.

(ii) When $\lambda_\pm = \pm 3i$ with $\boldsymbol{\eta}_\pm = (1 \pm 2\mathrm{i},\ 1 \mp 3\mathrm{i})$ the general solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= C\mathrm{e}^{3\mathrm{i}t}\begin{pmatrix} 1+2\mathrm{i} \\ 1-3\mathrm{i} \end{pmatrix} + C^*\mathrm{e}^{-3\mathrm{i}t}\begin{pmatrix} 1-2\mathrm{i} \\ 1+3\mathrm{i} \end{pmatrix} \\
&= 2\,\mathrm{Re}\left[C\mathrm{e}^{3\mathrm{i}t}\begin{pmatrix} 1+2\mathrm{i} \\ 1-3\mathrm{i} \end{pmatrix}\right] \\
&= 2\,\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos 3t + \mathrm{i}\sin 3t)\begin{pmatrix} 1+2\mathrm{i} \\ 1-3\mathrm{i} \end{pmatrix}\right] \\
&= 2\,\mathrm{Re}\left[\Big((\alpha\cos 3t - \beta\sin 3t) + \mathrm{i}(\alpha\sin 3t + \beta\cos 3t)\Big)\begin{pmatrix} 1+\mathrm{i} \\ 1-3\mathrm{i} \end{pmatrix}\right] \\
&= 2\begin{pmatrix} \alpha\cos 3t - \beta\sin 3t - \alpha\sin 3t - \beta\cos 3t \\ \alpha\cos 3t - \beta\sin 3t + 3\alpha\sin 3t + 3\beta\cos 3t \end{pmatrix} \\
&= \begin{pmatrix} (A-B)\cos 3t - (A+B)\sin 3t \\ (A+3B)\cos 3t + (3A-B)\sin 3t \end{pmatrix},
\end{aligned}$$

where $A = 2\alpha$ and $B = 2\beta$.

(iii) When $\lambda_\pm = -2 \pm \mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1 \mp \mathrm{i},\ 3 \pm \mathrm{i})$ the explicit solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= C\mathrm{e}^{(-2+\mathrm{i})t}\begin{pmatrix} 1-\mathrm{i} \\ 3+\mathrm{i} \end{pmatrix} + C^*\mathrm{e}^{(-2-\mathrm{i})t}\begin{pmatrix} 1+\mathrm{i} \\ 3-\mathrm{i} \end{pmatrix} \\
&= 2\,\mathrm{Re}\left[C\mathrm{e}^{(-2+\mathrm{i})t}\begin{pmatrix} 1-\mathrm{i} \\ 3+\mathrm{i} \end{pmatrix}\right] \\
&= 2\mathrm{e}^{-2t}\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos t + \mathrm{i}\sin t)\begin{pmatrix} 1-\mathrm{i} \\ 3+\mathrm{i} \end{pmatrix}\right] \\
&= 2\mathrm{e}^{-2t}\mathrm{Re}\left[\Big((\alpha\cos t - \beta\sin t) + \mathrm{i}(\alpha\sin t + \beta\cos t)\Big)\begin{pmatrix} 1-\mathrm{i} \\ 3+\mathrm{i} \end{pmatrix}\right] \\
&= 2\mathrm{e}^{-2t}\begin{pmatrix} \alpha\cos t - \beta\sin t + \alpha\sin t + \beta\cos t \\ 3\alpha\cos t - 3\beta\sin t - \alpha\sin t - \beta\cos t \end{pmatrix} \\
&= \mathrm{e}^{-2t}\begin{pmatrix} (A+B)\cos t + (A-B)\sin t \\ (3A-B)\cos t - (A+3B)\sin t \end{pmatrix},
\end{aligned}$$

with $A = 2\alpha$ and $B = 2\beta$.

(iv) When $\lambda_\pm = -1 \pm \mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1,\ \pm\mathrm{i})$ the explicit solution is given by

$$\mathbf{x}(t) = C\mathrm{e}^{(-1+\mathrm{i})t}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + C^*\mathrm{e}^{(-1-\mathrm{i})t}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}\left[C\mathrm{e}^{(-1+\mathrm{i})t}\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{-t}\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos t+\mathrm{i}\sin t)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{-t}\mathrm{Re}\left[\Big((\alpha\cos t-\beta\sin t)+\mathrm{i}(\alpha\cos t+\beta\cos t)\Big)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\right]\\
&= \mathrm{e}^{-t}\begin{pmatrix}A\cos t-B\sin t\\ -A\cos t-B\sin t\end{pmatrix},
\end{aligned}
$$

with $A=2\alpha$ and $B=2\beta$.

(v) When $\lambda_\pm = 2\pm 2\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (\pm 3\mathrm{i},\ 5\mp 4\mathrm{i})$ the explicit solution is given by

$$
\begin{aligned}
\mathbf{x}(t) &= C\mathrm{e}^{(2+2\mathrm{i})t}\begin{pmatrix}3\mathrm{i}\\ 5-4\mathrm{i}\end{pmatrix}+C^*\mathrm{e}^{(2-2\mathrm{i})t}\begin{pmatrix}-3\mathrm{i}\\ 5+4\mathrm{i}\end{pmatrix}\\
&= 2\,\mathrm{Re}\left[C\mathrm{e}^{(2+2\mathrm{i})t}\begin{pmatrix}3\mathrm{i}\\ 5-4\mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{2t}\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos 2t+\mathrm{i}\sin 2t)\begin{pmatrix}3\mathrm{i}\\ 5-4\mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{2t}\mathrm{Re}\left[\Big((\alpha\cos 2t-\beta\sin 2t)+\mathrm{i}(\alpha\sin 2t+\beta\cos 2t)\Big)\begin{pmatrix}3\mathrm{i}\\ 5-4\mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{2t}\begin{pmatrix}-3\alpha\sin 2t-3\beta\cos 2t\\ 5\alpha\cos 2t-5\beta\sin 2t+4\alpha\sin 2t+4\beta\cos 2t\end{pmatrix}\\
&= \mathrm{e}^{2t}\begin{pmatrix}3A\sin 2t+3B\cos 2t\\ (5B-4A)\sin 2t-(5A+4B)\cos 2t\end{pmatrix},
\end{aligned}
$$

where $A=-2\alpha$ and $B=-2\beta$.

(vi) When $\lambda_\pm = 5\pm 3\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (2\pm 5\mathrm{i},\ \mp\mathrm{i})$ the explicit solution is

$$
\begin{aligned}
\mathbf{x}(t) &= C\mathrm{e}^{(5+3\mathrm{i})t}\begin{pmatrix}2+5\mathrm{i}\\ -\mathrm{i}\end{pmatrix}+C^*\mathrm{e}^{(5-3\mathrm{i})t}\begin{pmatrix}2-5\mathrm{i}\\ \mathrm{i}\end{pmatrix}\\
&= 2\,\mathrm{Re}\left[C\mathrm{e}^{(5+3\mathrm{i})t}\begin{pmatrix}2+5\mathrm{i}\\ -\mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{5t}\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos 3t+\mathrm{i}\sin 3t)\begin{pmatrix}2+5\mathrm{i}\\ -\mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{5t}\mathrm{Re}\left[\Big((\alpha\cos 3t-\beta\sin 3t)+\mathrm{i}(\beta\cos 3t+\alpha\sin 3t)\Big)\begin{pmatrix}2+5\mathrm{i}\\ -\mathrm{i}\end{pmatrix}\right]\\
&= 2\mathrm{e}^{5t}\begin{pmatrix}2\alpha\cos 3t-2\beta\sin 3t-5\beta\cos 3t-5\alpha\sin 3t\\ \beta\cos 3t+\alpha\sin 3t\end{pmatrix}
\end{aligned}
$$

$$= \quad e^{5t}\begin{pmatrix} (2A-5B)\cos 3t-(5A+2B)\sin 3t \\ B\cos 3t + A\sin 3t \end{pmatrix},$$

where $A = 2\alpha$ and $B = 2\beta$.

(vii) When $\lambda_\pm = \pm 7\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (1 \pm \mathrm{i},\ -1 \pm 2\mathrm{i})$ the explicit solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= Ce^{7\mathrm{i}t}\begin{pmatrix} 1+\mathrm{i} \\ -1+2\mathrm{i} \end{pmatrix} + C^*e^{-7\mathrm{i}t}\begin{pmatrix} 1-\mathrm{i} \\ -1-2\mathrm{i} \end{pmatrix} \\
&= 2\,\mathrm{Re}\left[Ce^{7\mathrm{i}t}\begin{pmatrix} 1+\mathrm{i} \\ -1+2\mathrm{i} \end{pmatrix}\right] \\
&= 2\,\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos 7t + \mathrm{i}\sin 7t)\begin{pmatrix} 1+\mathrm{i} \\ -1+2\mathrm{i} \end{pmatrix}\right] \\
&= 2\,\mathrm{Re}\left[\Big((\alpha\cos 7t - \beta\sin 7t) + \mathrm{i}(\alpha\sin 7t + \beta\cos 7t)\Big)\begin{pmatrix} 1+\mathrm{i} \\ -1+2\mathrm{i} \end{pmatrix}\right] \\
&= \begin{pmatrix} A\cos 7t - B\sin 7t - A\sin 7t - B\cos 7t \\ -A\cos 7t + B\sin 7t - 2A\sin 7t - 2B\cos 7t \end{pmatrix} \\
&= \begin{pmatrix} (A-B)\cos 7t - (A+B)\sin 7t \\ (B-2A)\sin 7t - (A+2B)\cos 7t \end{pmatrix},
\end{aligned}$$

where $A = 2\alpha$ and $B = 2\beta$.

(viii) When $\lambda_\pm = -13 \pm 17\mathrm{i}$ with $\boldsymbol{\eta}_\pm = (\pm 6\mathrm{i} - 8,\ 4 \mp 5\mathrm{i})$ the explicit solution is given by

$$\begin{aligned}
\mathbf{x}(t) &= Ce^{(-13+17\mathrm{i})t}\begin{pmatrix} 6\mathrm{i}-8 \\ 4-5\mathrm{i} \end{pmatrix} + C^*e^{(-13-17\mathrm{i})t}\begin{pmatrix} -6\mathrm{i}-8 \\ 4+5\mathrm{i} \end{pmatrix} \\
&= 2\,\mathrm{Re}\left[Ce^{(-13+17\mathrm{i})t}\begin{pmatrix} 6\mathrm{i}-8 \\ 4-5\mathrm{i} \end{pmatrix}\right] \\
&= 2e^{-13t}\mathrm{Re}\left[(\alpha+\mathrm{i}\beta)(\cos 17t + \mathrm{i}\sin 17t)\begin{pmatrix} 6\mathrm{i}-8 \\ 4-5\mathrm{i} \end{pmatrix}\right] \\
&= 2e^{-13t}\mathrm{Re}\left[\Big((\alpha\cos 17t - \beta\sin 17t) + \mathrm{i}(\beta\cos 17t + \alpha\sin 17t)\Big)\begin{pmatrix} 6\mathrm{i}-8 \\ 4-5\mathrm{i} \end{pmatrix}\right] \\
&= e^{-13t}\begin{pmatrix} -6B\cos 17t - 6A\sin 17t - 8A\cos 17t + 8B\sin 17t \\ 4A\cos 17t - 4B\sin 17t + 5B\cos 17t + 5A\sin 17t \end{pmatrix} \\
&= e^{-13t}\begin{pmatrix} (8B-6A)\sin 17t - (8A+6B)\cos 17t \\ (4A+5B)\cos 17t + (5A-4B)\sin 17t \end{pmatrix}.
\end{aligned}$$

Exercise 29.3 *For the following equations find the eigenvalues and eigenvectors of the matrix on the right-hand side, and hence find the coordinate* *transformation that will put the equations into their standard simple (canonical) form. Show that this transformation has the desired effect.*

(i)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x};$$

(ii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -2 & 3 \\ -6 & 4 \end{pmatrix} \mathbf{x};$$

(iii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -11 & -2 \\ 13 & -9 \end{pmatrix} \mathbf{x};$$

and

(iv)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 7 & -5 \\ 10 & -3 \end{pmatrix} \mathbf{x}.$$

Throughout these solutions we use $\mathbb{A}$ to denote the matrix appearing on the right-hand side of the differential equation, and $\tilde{\mathbf{x}} = (\tilde{x},\ \tilde{y})$.

(i) The eigenvalues of

$$\mathbb{A} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

are the solutions of

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = -\lambda(-1-\lambda) + 1 = \lambda^2 + \lambda + 1 = 0,$$

which are

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}.$$

The eigenvector corresponding to $\lambda_+ = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ is given by

$$\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} & -1 \\ 1 & \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so $v_2 = (\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i})v_1$, and a representative eigenvector is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 - \sqrt{3}\mathrm{i} \end{pmatrix}.$$

So we set

$$\mathbf{x} = \tilde{x}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \tilde{y}\begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -\sqrt{3} \end{pmatrix}\tilde{x},$$

which implies that

$$\tilde{\mathbf{x}} = \begin{pmatrix} 2 & 0 \\ 1 & -\sqrt{3} \end{pmatrix}^{-1}\mathbf{x} = -\frac{1}{2\sqrt{3}}\begin{pmatrix} -\sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}\mathbf{x}.$$

Therefore we have

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= -\frac{1}{2\sqrt{3}}\begin{pmatrix} -\sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -\sqrt{3} \end{pmatrix}\tilde{\mathbf{x}} \\ &= -\frac{1}{2\sqrt{3}}\begin{pmatrix} -\sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & \sqrt{3} \\ 1 & \sqrt{3} \end{pmatrix}\tilde{\mathbf{x}} \\ &= -\frac{1}{2\sqrt{3}}\begin{pmatrix} \sqrt{3} & -3 \\ 3 & \sqrt{3} \end{pmatrix}\tilde{\mathbf{x}} \\ &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\tilde{\mathbf{x}}. \end{aligned}$$

(ii) The matrix

$$\begin{pmatrix} -2 & 3 \\ -6 & 4 \end{pmatrix}$$

has eigenvalues given by the roots of the characteristic equation

$$\begin{vmatrix} -2-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} = (-2-\lambda)(4-\lambda) + 18 = \lambda^2 - 2\lambda + 10 = 0,$$

i.e.

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3\mathrm{i}.$$

The eigenvector corresponding to $\lambda_+ = 1 + 3\mathrm{i}$ is given by

$$\begin{pmatrix} -3-3\mathrm{i} & 3 \\ -6 & 3-3\mathrm{i} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so $-(1+\mathrm{i})v_1 + v_2 = 0$, and a representative eigenvector is

$$\begin{pmatrix} 1 \\ 1+\mathrm{i} \end{pmatrix}.$$

If we set

$$\mathbf{x} = \tilde{x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tilde{y}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\tilde{\mathbf{x}}$$

then

$$\tilde{\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \mathbf{x} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mathbf{x},$$

and we have

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ -6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tilde{\mathbf{x}} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} \tilde{\mathbf{x}} \\ &= \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \tilde{\mathbf{x}}. \end{aligned}$$

(iii) To find the eigenvalues of the matrix

$$\begin{pmatrix} -11 & -2 \\ 13 & -9 \end{pmatrix}$$

we consider

$$\begin{vmatrix} -11-\lambda & -2 \\ 13 & -9-\lambda \end{vmatrix} = (-11-\lambda)(-9-\lambda)+26 = \lambda^2+20\lambda+125 = 0,$$

and obtain

$$\lambda = \frac{-20 \pm \sqrt{400-500}}{2} = -10 \pm 5\mathrm{i}.$$

The eigenvector corresponding to the eigenvalue $-10+5\mathrm{i}$ can be determined from

$$\begin{pmatrix} -1-5\mathrm{i} & -2 \\ 13 & 1-5\mathrm{i} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

i.e. $(-1-5\mathrm{i})v_1 - 2v_2 = 0$. A representative eigenvector is therefore

$$\begin{pmatrix} -2 \\ 1+5\mathrm{i} \end{pmatrix}.$$

We set

$$\mathbf{x} = \tilde{x} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \tilde{y} \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & 5 \end{pmatrix} \tilde{\mathbf{x}},$$

and then

$$\tilde{\mathbf{x}} = \begin{pmatrix} -2 & 0 \\ 1 & 5 \end{pmatrix}^{-1} \mathbf{x} = -\frac{1}{10} \begin{pmatrix} 5 & 0 \\ -1 & -2 \end{pmatrix} \mathbf{x}.$$

Therefore

$$\begin{aligned}\frac{d\tilde{\mathbf{x}}}{dt} &= -\frac{1}{10}\begin{pmatrix} 5 & 0 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} -11 & -2 \\ 13 & -9 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 1 & 5 \end{pmatrix}\tilde{\mathbf{x}} \\ &= -\frac{1}{10}\begin{pmatrix} 5 & 0 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 20 & -10 \\ -35 & -45 \end{pmatrix}\tilde{\mathbf{x}} \\ &= \begin{pmatrix} 10 & 5 \\ -5 & 10 \end{pmatrix}\tilde{\mathbf{x}}.\end{aligned}$$

(iv) The eigenvalues of

$$\begin{pmatrix} 7 & -5 \\ 10 & -3 \end{pmatrix}$$

are given by the solutions of

$$\begin{vmatrix} 7-\lambda & -5 \\ 10 & -3-\lambda \end{vmatrix} = (7-\lambda)(-3-\lambda) + 50 = \lambda^2 - 4\lambda + 29 = 0,$$

i.e.

$$\lambda = \frac{4 \pm \sqrt{16 - 116}}{2} = 2 \pm 5\mathrm{i}.$$

The eigenvector corresponding to $2 + 5\mathrm{i}$ is determined by

$$\begin{pmatrix} 5-5\mathrm{i} & -5 \\ 10 & -5-5\mathrm{i} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

and so is

$$\begin{pmatrix} 1 \\ 1-\mathrm{i} \end{pmatrix}.$$

Now if we set

$$\mathbf{x} = \tilde{x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tilde{y}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\tilde{\mathbf{x}},$$

which gives

$$\tilde{\mathbf{x}} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}^{-1}\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\mathbf{x},$$

we obtain

$$\begin{aligned}\frac{d\tilde{\mathbf{x}}}{dt} &= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 7 & -5 \\ 10 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\tilde{\mathbf{x}} \\ &= \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 7 & 3 \end{pmatrix}\tilde{\mathbf{x}}\end{aligned}$$

$$= \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix} \tilde{\mathbf{x}}.$$

Exercise 29.4 *(T) In the previous chapter we used the result that the eigenvectors corresponding to distinct eigenvalues are linearly independent. Use this result to show that the real and imaginary parts of complex eigenvectors are linearly independent.*

Denote the eigenvalues by $\lambda_\pm$, and the corresponding eigenvectors by $\boldsymbol{\eta}_\pm = \mathbf{v}_1 \pm \mathrm{i}\mathbf{v}_2$. Since $\lambda_+ \neq \lambda_-$, the two eigenvectors $\boldsymbol{\eta}_+$ and $\boldsymbol{\eta}_-$ are linearly independent. Now suppose that

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \mathbf{0}. \tag{S29.1}$$

Since $2\mathbf{v}_1 = \boldsymbol{\eta}_+ + \boldsymbol{\eta}_-$ and $2\mathbf{v}_2 = -\mathrm{i}(\boldsymbol{\eta}_+ - \boldsymbol{\eta}_-)$, equation (S29.1) is the same as

$$\alpha(\boldsymbol{\eta}_+ + \boldsymbol{\eta}_-) - \beta\mathrm{i}(\boldsymbol{\eta}_+ - \boldsymbol{\eta}_-) = \mathbf{0},$$

or

$$(\alpha - 2\mathrm{i}\beta)\boldsymbol{\eta}_+ + (\alpha + 2\mathrm{i}\beta)\boldsymbol{\eta}_- = \mathbf{0}.$$

Since $\boldsymbol{\eta}_+$ and $\boldsymbol{\eta}_-$ are linearly independent it follows that

$$\alpha - 2\mathrm{i}\beta = 0 \qquad \text{and} \qquad \alpha + 2\mathrm{i}\beta = 0,$$

from which we obtain $\alpha = \beta = 0$, and so $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Exercise 29.5 *(T) Following the same line of reasoning as in Exercise 28.5, show how to construct a matrix with a complex conjugate pair of eigenvalues* $\lambda_\pm = \rho \pm \mathrm{i}\omega$ *and corresponding eigenvectors* $\boldsymbol{\eta}_\pm = \mathbf{v}_1 \pm \mathrm{i}\mathbf{v}_2$*. Hence find the matrices with the following eigenvalues and eigenvectors:*

(i) $\lambda_\pm = 3 \pm 3\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (2 \pm \mathrm{i},\ 1 \mp \mathrm{i})$*;*
(ii) $\lambda_\pm = \pm 3\mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (\pm\mathrm{i},\ 3 \pm 2\mathrm{i})$*; and*
(iii) $\lambda_\pm = -2 \pm \mathrm{i}$ *with* $\boldsymbol{\eta}_\pm = (1 \pm \mathrm{i},\ 1 \mp \mathrm{i})$*.*

(The `M`*-file* `makematrix.m` *will do this for you. You could use this to check that the signs of* $\dot{x}$ *given in Exercise 29.1 are correct by finding the appropriate matrix* $\mathbb{A}$ *and then looking at* $\dot{x}$ *when* $x = 0$ *and* $y > 0$*.)*

If $\mathbb{A}$ has complex conjugate eigenvalues $\lambda_\pm = \rho \pm \mathrm{i}\omega$ with corresponding eigenvectors $\boldsymbol{\eta}_\pm = \mathbf{v}_1 \pm \mathrm{i}\mathbf{v}_2$, we have seen that if we define

$$\mathbb{P} = [\mathbf{v}_1 \ \ \mathbf{v}_2]$$

then

$$\mathbb{P}^{-1}\mathbb{A}\mathbb{P} = \begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix},$$

so it follows that the matrix $\mathbb{A}$ with these specified eigenvalues and eigenvectors can be written in the form

$$\mathbb{A} = \mathbb{P}\begin{pmatrix} \rho & \omega \\ -\omega & \rho \end{pmatrix}\mathbb{P}^{-1}.$$

(i) The matrix with eigenvalues $\lambda_\pm = 3 \pm 3\mathrm{i}$ and eigenvectors $\boldsymbol{\eta}_\pm = (2 \pm \mathrm{i},\ 1 \mp \mathrm{i})$ is

$$\begin{aligned}\mathbb{A} &= \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ -3 & 3 \end{pmatrix} \times -\frac{1}{3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} \\ &= -\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -5 \\ 2 & 2 \end{pmatrix}.\end{aligned}$$

(ii) The matrix with eigenvalues $\lambda_\pm = \pm 3\mathrm{i}$ and eigenvectors $\boldsymbol{\eta}_\pm = (\pm\mathrm{i},\ 3 \pm 2\mathrm{i})$ is

$$\begin{aligned}\mathbb{A} &= \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \times -\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -3 & 0 \end{pmatrix} \\ &= -\frac{1}{3}\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -9 & 0 \\ -6 & 3 \end{pmatrix} \\ &= -\frac{1}{3}\begin{pmatrix} -6 & 3 \\ -39 & 6 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 13 & -2 \end{pmatrix}.\end{aligned}$$

(iii) The matrix with eigenvalues $\lambda_\pm = -2 \pm \mathrm{i}$ and eigenvectors $\boldsymbol{\eta}_\pm = (1 \pm \mathrm{i},\ 1 \mp \mathrm{i})$ is

$$\begin{aligned}\mathbb{A} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \times -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix} 4 & 2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}.\end{aligned}$$

30

Yet more phase portraits: a repeated real eigenvalue

Exercise 30.1 *Find the eigenvalue and eigenvector of the matrices occurring in the following equations, and hence draw the phase portrait. Find also the coordinate transformation that will put the equation into canonical form and show that this works. Write down the general solution.*

(i)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x};$$

(ii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -6 & 2 \\ -2 & -2 \end{pmatrix} \mathbf{x};$$

(iii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -3 & -1 \\ 1 & -5 \end{pmatrix} \mathbf{x};$$

(iv)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} \mathbf{x};$$

and

(v)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$$

Throughout these solutions we use $\mathbb{A}$ to denote the matrix appearing on the right-hand side of the differential equation, and $\tilde{\mathbf{x}} = (\tilde{x},\ \tilde{y})$.

(i)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

The eigenvalue is the solution of

$$\begin{vmatrix} 5-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 4 = \lambda^2 - 6\lambda + 9 = 0,$$

so is $\lambda = 3$. The corresponding eigenvector, determined by

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0}$$

is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

When $x = 0$ and $y > 0$, $\dot{x} = -4y < 0$; $\dot{x} = 0$ on the line $5x = 4y$, and the phase portrait is as shown in Figure 30.1.

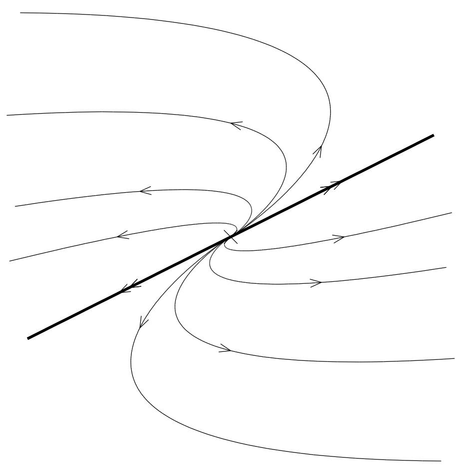

Fig. 30.1. Phase portrait for the equation in Exercise 30.1(i).

We now choose a second vector that is linearly independent from $\mathbf{v}$, e.g. here we choose the vector

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

that is orthogonal to $\mathbf{v}$, but any choice would do. We then set

$$\mathbf{v}_1 = (\mathbb{A} - \lambda\mathbb{I})\mathbf{v}_2 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}.$$

Now we make the change of coordinates

$$\mathbf{x} = \tilde{x}\begin{pmatrix} 10 \\ 5 \end{pmatrix} + \tilde{y}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ 5 & -2 \end{pmatrix}\tilde{\mathbf{x}}, \tag{S30.1}$$

which implies that

$$\tilde{\mathbf{x}} = \begin{pmatrix} 10 & 1 \\ 5 & -2 \end{pmatrix}^{-1}\mathbf{x} = \frac{1}{-25}\begin{pmatrix} -2 & -1 \\ -5 & 10 \end{pmatrix}\mathbf{x}.$$

Therefore

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= \frac{1}{-25}\begin{pmatrix} -2 & -1 \\ -5 & 10 \end{pmatrix}\begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 10 & 1 \\ 5 & -2 \end{pmatrix}\tilde{\mathbf{x}} \\
&= -\frac{1}{25}\begin{pmatrix} -2 & -1 \\ -5 & 10 \end{pmatrix}\begin{pmatrix} 30 & 13 \\ 15 & -1 \end{pmatrix}\tilde{\mathbf{x}} \\
&= -\frac{1}{25}\begin{pmatrix} -75 & -25 \\ 0 & -75 \end{pmatrix}\tilde{\mathbf{x}} \\
&= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\tilde{\mathbf{x}}.
\end{aligned}$$

The explicit solution is

$$\begin{aligned}
\mathbf{x}(t) &= [At\mathrm{e}^{3t} + B\mathrm{e}^{3t}]\begin{pmatrix} 10 \\ 5 \end{pmatrix} + A\mathrm{e}^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{(S30.2)} \\
&= \begin{pmatrix} 10At\mathrm{e}^{3t} + (A+10B)\mathrm{e}^{3t} \\ 5At\mathrm{e}^{3t} + (5B-2A)\mathrm{e}^{3t} \end{pmatrix}.
\end{aligned}$$

[Rather than try to remember the form of the explicit solution,

$$\mathbf{x}(t) = [At\mathrm{e}^{\lambda t} + B\mathrm{e}^{\lambda t}]\mathbf{v}_1 + A\mathrm{e}^{\lambda t}\mathbf{v}_2, \tag{S30.3}$$

it is perhaps safer to derive it each time from the decoupled system. In this case we have

$$\begin{aligned}
\mathrm{d}\tilde{x}/\mathrm{d}t &= 3\tilde{x} + \tilde{y} \\
\mathrm{d}\tilde{y}/\mathrm{d}t &= 3\tilde{y},
\end{aligned}$$

And so clearly $\tilde{y}(t) = A\mathrm{e}^{3t}$. Using this in the equation for $\mathrm{d}\tilde{x}/\mathrm{d}t$ we obtain

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = 3\tilde{x} + A\mathrm{e}^{3t}.$$

Using the integrating factor e^{-3t} this is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{x}\mathrm{e}^{-3t}\right] = A,$$

and so

$$\tilde{x}(t)\mathrm{e}^{-3t} = At + B \qquad \Rightarrow \qquad \tilde{x}(t) = [At + B]\mathrm{e}^{3t}.$$

Now using (S30.1) we obtain (S30.2). In the remaining solutions we will use (S30.3) directly.]

(ii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -6 & 2 \\ -2 & -2 \end{pmatrix}\mathbf{x}.$$

The eigenvalue of the matrix $\mathbb{A}$ is the solution of

$$\begin{vmatrix} -6-\lambda & 2 \\ -2 & -2-\lambda \end{vmatrix} = (-6-\lambda)(-2-\lambda) + 4 = \lambda^2 + 8\lambda + 16 = 0,$$

i.e. $\lambda = -4$. The corresponding eigenvector is determined by

$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so is

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The derivative $\dot{x} = 0$ on the line $6x = 2y$, and when $x = 0$ and $y > 0$ this derivative is positive. The phase portrait is shown in Figure 30.2.

In order to transform the equation into its canonical form we choose another direction $\mathbf{v}_2$. Again, we will make the canonical choice of a vector perpendicular to $\mathbf{v}$, taking

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We now set

$$\mathbf{v}_1 = (\mathbb{A} + 4\mathbb{I})\mathbf{v}_2 = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix},$$

and make the change of coordinates

$$\mathbf{x} = \tilde{x}\begin{pmatrix} -4 \\ -4 \end{pmatrix} + \tilde{y}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ -4 & -1 \end{pmatrix}\tilde{\mathbf{x}}.$$

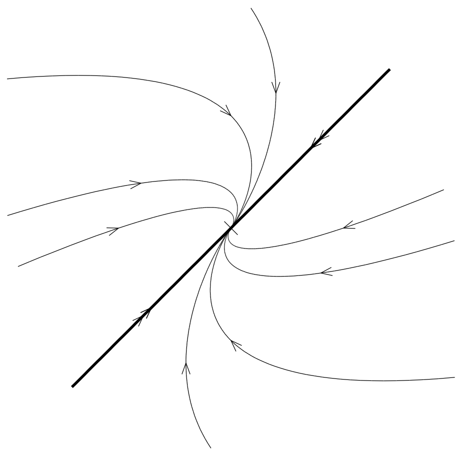

Fig. 30.2. Phase portrait for the equation of Exercise 30.1(ii).

It follows that

$$\tilde{\mathbf{x}} = \begin{pmatrix} -4 & 1 \\ -4 & -1 \end{pmatrix}^{-1} \mathbf{x} = \frac{1}{8}\begin{pmatrix} -1 & -1 \\ 4 & -4 \end{pmatrix}\mathbf{x},$$

and the differential equation satisfied by $\tilde{\mathbf{x}}$ is

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= \frac{1}{8}\begin{pmatrix} -1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} -6 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} -4 & 1 \\ -4 & -1 \end{pmatrix}\tilde{\mathbf{x}} \\ &= \frac{1}{8}\begin{pmatrix} -1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 16 & -8 \\ 16 & 0 \end{pmatrix}\tilde{\mathbf{x}} \\ &= \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}\tilde{\mathbf{x}}. \end{aligned}$$

The explicit solution is

$$\begin{aligned} \mathbf{x}(t) &= [At\mathrm{e}^{-4t} + B\mathrm{e}^{-4t}]\begin{pmatrix} -4 \\ -4 \end{pmatrix} + A\mathrm{e}^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} (A-4B)\mathrm{e}^{-4t} - 4At\mathrm{e}^{-4t} \\ -(A+4B)\mathrm{e}^{-4t} - 4At\mathrm{e}^{-4t} \end{pmatrix}. \end{aligned}$$

(iii)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -3 & -1 \\ 1 & -5 \end{pmatrix}\mathbf{x}.$$

The eigenvalue is the repeated real root of

$$\begin{vmatrix} -3-\lambda & -1 \\ 1 & -5-\lambda \end{vmatrix} = (-3-\lambda)(-5-\lambda)+1 = \lambda^2+8\lambda+16 = 0,$$

i.e. $\lambda = -4$ again. This time the eigenvector is (by inspection) $\mathbf{v} = (1,\ 1)$.

We have $\dot{x} = 0$ on the line $y = -3x$, and when $x = 0$ and $y > 0$, $\dot{x} < 0$, so the phase portrait is as in Figure 30.3.

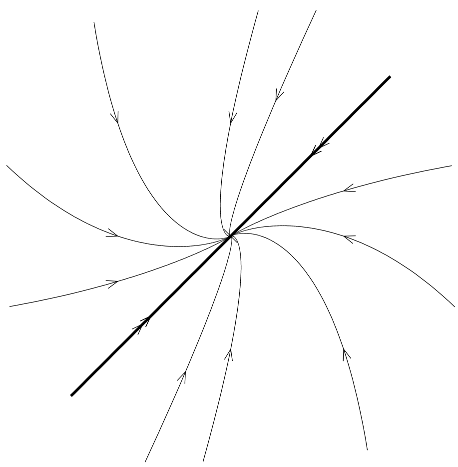

Fig. 30.3. The phase portrait for the equation of Exercise 30.1(iii).

The axes for our new coordinate system we choose to be

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

and set

$$\mathbf{x} = \tilde{x}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \tilde{y}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\tilde{\mathbf{x}},$$

which implies that

$$\tilde{\mathbf{x}} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}^{-1}\mathbf{x} = -\frac{1}{4}\begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix}\mathbf{x}.$$

The derivative of $\tilde{\mathbf{x}}$ satisfies the differential equation

$$\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = -\frac{1}{4}\begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\tilde{\mathbf{x}}$$

$$= -\frac{1}{4}\begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} -8 & -2 \\ -8 & 6 \end{pmatrix}\tilde{\mathbf{x}}$$
$$= \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}\tilde{\mathbf{x}}.$$

The explicit solution is given by

$$\begin{aligned}\mathbf{x}(t) &= [At\mathrm{e}^{-4t} + B\mathrm{e}^{-4t}]\begin{pmatrix} 2 \\ 2 \end{pmatrix} + A\mathrm{e}^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} (A+2B)\mathrm{e}^{-4t} + 2At\mathrm{e}^{-4t} \\ (2B-A)\mathrm{e}^{-4t} + 2At\mathrm{e}^{-4t} \end{pmatrix}.\end{aligned}$$

(iv)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}\mathbf{x}.$$

Here the repeated eigenvalue is $\lambda = 13$. The matrix is a multiple of the identity, so every vector is an eigenvector. Since the eigenvalue is positive, the origin is an unstable star, and the phase portrait is shown in Figure 30.4.

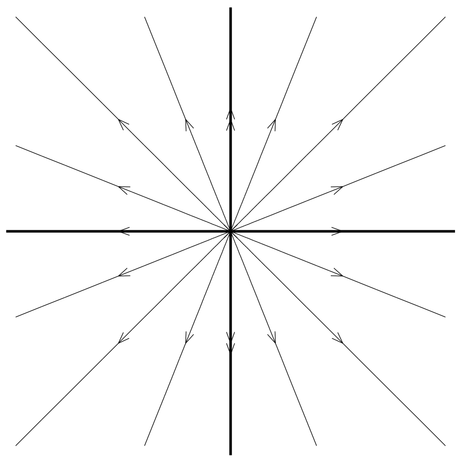

Fig. 30.4. The phase portrait for the equation in Exercise 30.1(iv).

The equation is already in its 'canonical form', and the explicit solution is simply

$$\mathbf{x}(t) = \mathbf{v}\mathrm{e}^{13t}$$

for any vector $\mathbf{v}$.

(v)

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$$

The eigenvalue is the solution of

$$\begin{vmatrix} 7-\lambda & -4 \\ 1 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 4 = \lambda^2 - 10\lambda + 25 = 0,$$

i.e. $\lambda = 5$. The eigenvector can be determined from

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

so is

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We have $\dot{x} = 0$ when $7x = 4y$, and for $x = 0$ and $y > 0$ the derivative $\dot{x} < 0$; the phase portrait is shown in Figure 30.5.

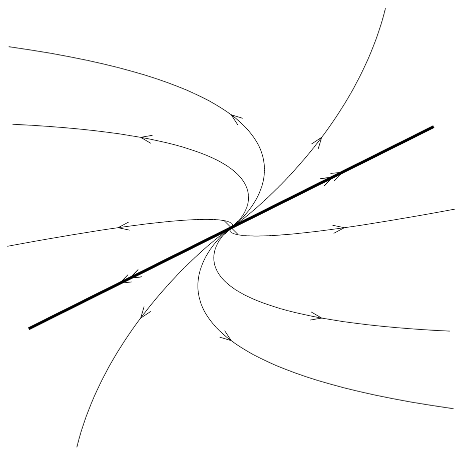

Fig. 30.5. The phase portrait for the equation in Exercise 30.1(v).

To transform the equation into its canonical form we choose

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_1 = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix},$$

set

$$\mathbf{x} = \tilde{x} \begin{pmatrix} 10 \\ 5 \end{pmatrix} + \tilde{y} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ 5 & -2 \end{pmatrix} \tilde{\mathbf{x}}.$$

It follows that

$$\tilde{\mathbf{x}} = \begin{pmatrix} 10 & 1 \\ 5 & -2 \end{pmatrix}^{-1} \mathbf{x} = -\frac{1}{25} \begin{pmatrix} -2 & -1 \\ -5 & 10 \end{pmatrix} \mathbf{x}$$

and so

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= -\frac{1}{25} \begin{pmatrix} -2 & -1 \\ -5 & 10 \end{pmatrix} \begin{pmatrix} 7 & -4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 10 & 1 \\ 5 & -2 \end{pmatrix} \tilde{\mathbf{x}} \\ &= -\frac{1}{25} \begin{pmatrix} -2 & -1 \\ -5 & 10 \end{pmatrix} \begin{pmatrix} 50 & 15 \\ 25 & -5 \end{pmatrix} \tilde{\mathbf{x}} \\ &= -\frac{1}{25} \begin{pmatrix} -125 & -25 \\ 0 & -125 \end{pmatrix} \tilde{\mathbf{x}} \\ &= \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} \tilde{\mathbf{x}}. \end{aligned}$$

The explicit solution is given by

$$\begin{aligned} \mathbf{x}(t) &= [At\mathrm{e}^{5t} + B\mathrm{e}^{5t}] \begin{pmatrix} 10 \\ 5 \end{pmatrix} + A\mathrm{e}^{5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} (A+10B)\mathrm{e}^{5t} + 10At\mathrm{e}^{5t} \\ (5B-2A)\mathrm{e}^{5t} + 5At\mathrm{e}^{5t} \end{pmatrix}. \end{aligned}$$

Exercise 30.2 *(T) The characteristic equation for a* 2×2 *matrix*

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $|\mathbb{A} - k\mathbb{I}| = 0$*, i.e.*

$$k^2 - (a+d)k + (ad - bc) = 0.$$

By explicit calculation show that $\mathbb{A}$ *satisfies its own characteristic equation, i.e. that*

$$\mathbb{A}^2 - (a+d)\mathbb{A} + (ad-bc)\mathbb{I} = \mathbb{O},$$

where $\mathbb{O}$ *is the* 2×2 *matrix of zeros. This is a particular case of the Cayley-Hamilton Theorem.*

We have

$$\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad \mathbb{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix}.$$

Substituting these into the expression

$$\mathbb{A}^2 - (a+d)\mathbb{A} + (ad-bc)\mathbb{I}$$

we obtain

$$\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & (a+d)b \\ (a+d)c & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so we do indeed have

$$\mathbb{A}^2 - (a+d)\mathbb{A} + (ad-bc)\mathbb{I} = \mathbb{O},$$

and $\mathbb{A}$ satisfies its own characteristic equation.

Exercise 30.3 *(T) If $\mathbb{A}$ has a repeated eigenvalue λ with eigenvector $\mathbf{v}$ then its characteristic equation can be written*

$$(k-\lambda)^2 = 0.$$

Use the Cayley-Hamilton Theorem from the previous exercise to deduce that

$$(\mathbb{A}-\lambda\mathbb{I})^2 = \mathbb{O},$$

and hence that $(\mathbb{A}-\lambda\mathbb{I})\mathbf{x}$ is an eigenvector of $\mathbb{A}$ for any choice of non-zero vector $\mathbf{x} \neq \mathbf{v}$.

Using $(\mathbb{A}-\lambda\mathbb{I})^2 = \mathbb{O}$ it follows in particular that

$$(\mathbb{A}-\lambda\mathbb{I})^2\mathbf{x} = \mathbb{O}.$$

Setting $\tilde{\mathbf{v}} = (\mathbb{A}-\lambda\mathbb{I})\mathbf{x}$ this equation can be rewritten as

$$(\mathbb{A}-\lambda\mathbb{I})\tilde{\mathbf{v}} = \mathbb{O},$$

or, upon rearrangement,

$$\mathbb{A}\tilde{\mathbf{v}} = \lambda\tilde{\mathbf{v}},$$

showing that $\tilde{\mathbf{v}}$ is an eigenvector of $\mathbb{A}$ with eigenvalue λ. It follows, since there is only one 'eigendirection', that $(\mathbb{A}-\lambda\mathbb{I})\mathbf{x} = \tilde{\mathbf{v}}$ is in the same direction as $\mathbf{v}$.

Exercise 30.4 *(T) By following the ideas of Exercise 28.5, show how to construct a matrix with a single eigenvalue λ and corresponding eigenvector $\mathbf{v}$. (There will be many such matrices.) Find two matrices with eigenvalue -1 and eigenvector $(1,\ 1)$.*

Given a matrix $\mathbb{A}$ with eigenvector $\mathbf{v}$ and eigenvalue λ, we have seen that for an arbitrary choice of vector $\mathbf{v}_2$ in a different direction to $\mathbf{v}$, if we set $\mathbf{v}_1 = (\mathbb{A} - \lambda\mathbb{I})\mathbf{v}_2$ this is in the same direction as $\mathbf{v}$, and then if we define

$$\mathbb{P} = [\mathbf{v}_1 \ \ \mathbf{v}_2] \tag{S30.4}$$

we have

$$\mathbb{P}^{-1}\mathbb{A}\mathbb{P} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{S30.5}$$

To find a matrix $\mathbb{A}$ with specified eigenvalue λ and eigenvector $\mathbf{v}$ we can choose $\mathbf{v}_1$ to be $\mathbf{v}$, and $\mathbf{v}_2$ to be arbitrary. The matrix $\mathbb{A}$ that we obtain from (S30.5),

$$\mathbb{A} = \mathbb{P}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\mathbb{P}^{-1}$$

with $\mathbb{P}$ defined as in (S30.4), will have the required eigenvalue and eigenvector (and will be consistent, in that $\mathbf{v}_1 = (\mathbb{A} - \lambda\mathbb{I})\mathbf{v}_2$).

So, to construct a matrix with a repeated eigenvalue -1 and eigenvector $\mathbf{v}_1 = (1,\ 1)$ we first have to choose another direction. If we choose $\mathbf{v}_2 = (1,\ -1)$ (the orthogonal direction) then

$$\mathbb{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \mathbb{P}^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},$$

and we can take

$$\begin{aligned} \mathbb{A} &= -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 & -1/2 \\ 1/2 & -3/2 \end{pmatrix}. \end{aligned}$$

For another matrix with the same eigenvalue and eigenvector, we make a different choice for $\mathbf{v}_2$, for example $\mathbf{v}_2 = (1,\ 0)$. Then we have

$$\mathbb{P} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \mathbb{P}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

We now obtain the matrix

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}.$$

Exercise 30.5 *(T) Suppose that the matrix $\mathbb{A}$ has zero as a repeated eigenvalue, with eigenvector $\mathbf{v}$. Then we can change to coordinates referred to $\mathbf{v}_2$ and $\mathbf{v}_1 = \mathbb{A}\mathbf{v}_2$, where $\mathbf{v}_2$ is any vector in a different direction to $\mathbf{v}$, so that $\tilde{\mathbf{x}} = \tilde{x}\mathbf{v}_1 + \tilde{y}\mathbf{v}_2$. The equation becomes*

$$\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tilde{\mathbf{x}},$$

and so

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} = \tilde{y} \qquad \text{and} \qquad \tilde{y} = 0.$$

(i) Solve the equations for $\tilde{x}(t)$ and $\tilde{y}(t)$, and hence write down the general solution for $\mathbf{x}(t)$.

(ii) Draw the phase diagram in the $(\tilde{x}, \tilde{y})$ plane, and hence in the (x, y) plane.

(iii) Draw the phase diagram for the equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}.$$

(i) We have $\tilde{y}(t) = A$, which gives $\mathrm{d}\tilde{x}/\mathrm{d}t = A$, and hence $\tilde{x}(t) = At + B$. Since $\mathbf{x} = \tilde{x}\mathbf{v}_1 + \tilde{y}\mathbf{v}_2$ the general solution is

$$\mathbf{x}(t) = [At + B]\mathbf{v}_1 + A\mathbf{v}_2.$$

(ii) The phase portrait in the $(\tilde{x}, \tilde{y})$ plane is shown in the left-hand picture of Figure 30.6, and the phase portrait in the (x, y) plane in the right-hand picture.

(iii) The matrix in the equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{x}$$

has repeated eigenvalue zero corresponding to the eigenvector $(1,\ 1)$. The phase portrait is shown in Figure 30.7.

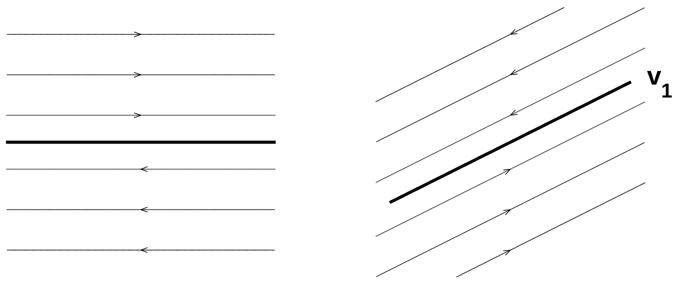

Fig. 30.6. Phase portraits for a repeated real eigenvalue zero with eigenvector $\mathbf{v}_1$, in the transformed plane (left) and the original coordinates (right). In both cases the bold line consists entirely of stationary points.

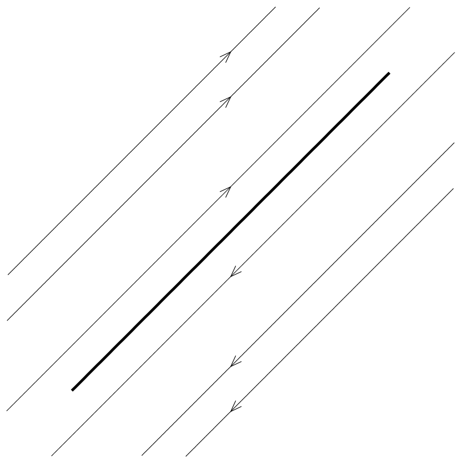

Fig. 30.7. The phase portrait for the equation in Exercise 30.5(iii). All points on the bold line are stationary.

31

Summary of phase portraits for linear equations

Exercise 31.1 *Draw the phase portrait for the equation* $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbb{A}\mathbf{x}$ *when the eigenvalues and eigenvectors of* $\mathbb{A}$ *are the following:*

(i) $\lambda_1 = 3$ *with* $\mathbf{v}_1 = (1,\ 1)$ *and* $\lambda_2 = -2$ *with* $\mathbf{v}_2 = (1,\ -2)$*;*

(ii) complex conjugate eigenvalues $\lambda_\pm = -1 \pm 3\mathrm{i}$*, with* $\dot{x} < 0$ *when* $x = 0$ *and* $y > 0$*;*

(iii) a single eigenvalue $\lambda = 13$ *with eigenvector* $(3,\ 2)$*, and* $\dot{x} > 0$ *when* $x = 0$ *and* $y > 0$*;*

(iv) $\lambda_1 = -2$ *with* $\mathbf{v}_1 = (2,\ 1)$ *and* $\lambda_2 = -3$ *with* $\mathbf{v}_2 = (1,\ -1)$*;*

(v) a single eigenvalue $\lambda = -3$ *with eigenvector* $(1,\ -1)$*, and* $\dot{x} > 0$ *when* $x = 0$ *and* $y > 0$*;*

(vi) $\lambda = \pm 2\mathrm{i}$*, where* $\dot{y} < 0$ *when* $y = 0$ *and* $x > 0$*;*

(vii) $\lambda_1 = 1$ *with* $\mathbf{v}_1 = (3,\ 2)$ *and* $\lambda_2 = 5$ *with* $\mathbf{v}_2 = (1,\ -4)$*;*

(viii) $\lambda = 5 \pm \mathrm{i}$*, and* $\dot{y} > 0$ *when* $y = 0$ *and* $x > 0$*; and*

(ix) a single eigenvalue $\lambda = -7$*, with the matrix* $\mathbb{A}$ *a multiple of the identity.*

(i) When $\lambda_1 = 3$ with $\mathbf{v}_1 = (1,\ 1)$ and $\lambda_2 = -2$ with $\mathbf{v}_2 = (1,\ -2)$, the two eigenvalues are both real, but have opposite sign. The origin is therefore a saddle point, and the phase portrait is shown in Figure 31.1.

(ii) When there are complex conjugate eigenvalues $\lambda = -1 \pm 3\mathrm{i}$, with $\dot{x} < 0$ when $x = 0$ and $y > 0$, the origin is a stable spiral (the real part of $\lambda_\pm$ is negative), with orbits circling anti-clockwise, as shown in Figure 31.2.

(iii) When there is a single eigenvalue $\lambda = 13$ with eigenvector $(3,\ 2)$, and $\dot{x} > 0$ when $x = 0$ and $y > 0$, the origin is an unstable improper node. The phase portrait is shown in Figure 31.3.

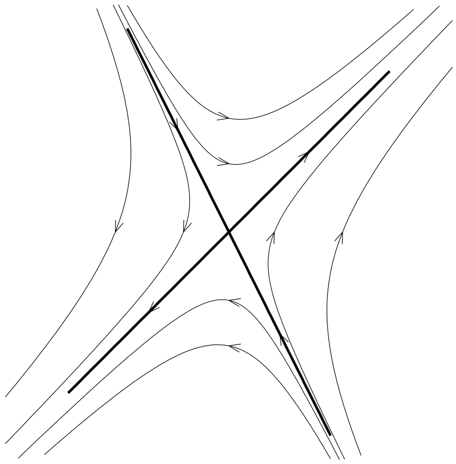

Fig. 31.1. The phase portrait for Exercise 31.1(i).

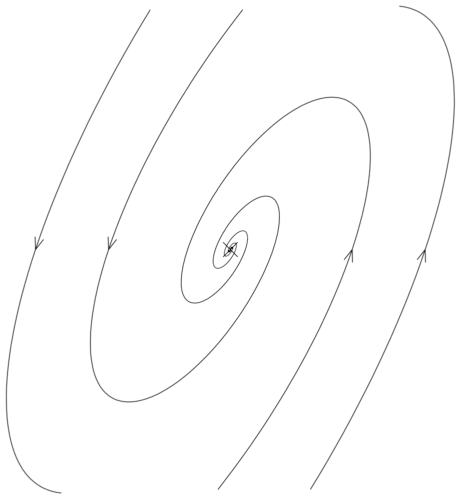

Fig. 31.2. The phase portrait for Exercise 31.1(ii).

(iv) When $\lambda_1 = -2$ with $\mathbf{v}_1 = (2,\ 1)$ and $\lambda_2 = -3$ with $\mathbf{v}_2 = (1,\ -1)$ both eigenvalues are real and negative, so the origin is a stable node, and trajectories approach tangent to $\mathbf{v}_1$, as shown in Figure 31.4.

(v) When there is a single eigenvalue $\lambda = -3$ with eigenvector $(1,\ -1)$, and $\dot{x} > 0$ when $x = 0$ and $y > 0$, the origin is a stable improper node. The phase portrait is shown in Figure 31.5.

(vi) When $\lambda = \pm 2\mathrm{i}$, with $\dot{y} < 0$ when $y = 0$ and $x > 0$, the origin is a

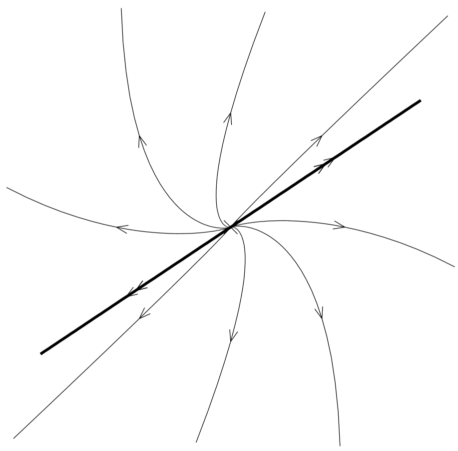

Fig. 31.3. The phase portrait for Exercise 31.1(iii).

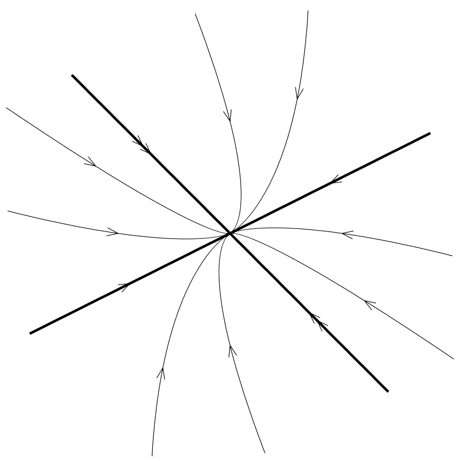

Fig. 31.4. The phase portrait for Exercise 31.1(iv).

centre, and trajectories spiral clockwise around it, as shown in Figure 31.6.

(vii) When $\lambda_1 = 1$ with $\mathbf{v}_1 = (3,\ 2)$ and $\lambda_2 = 5$ with $\mathbf{v}_2 = (1,\ -4)$ the origin is an unstable node (both eigenvalues are real and positive). Trajectories move away from the origin tangent to $\mathbf{v}_1$: the phase portrait is shown in Figure 31.7.

(viii) When $\lambda = 5 \pm \mathrm{i}$, and $\dot{y} > 0$ when $y = 0$ and $x > 0$, the origin is a centre. Orbits circle it anti-clockwise, as shown in Figure 31.8.

(ix) Finally, if there is a single eigenvalue $\lambda = -7$ and the matrix $\mathbb{A}$ a

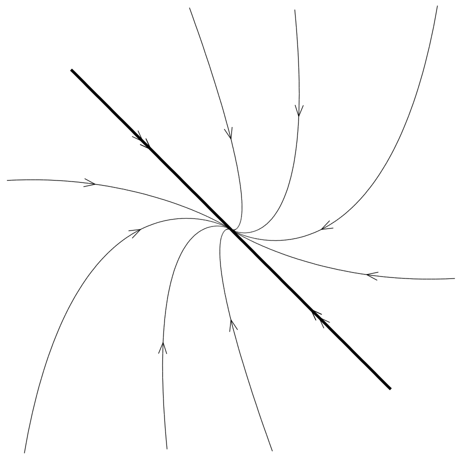

Fig. 31.5. The phase portrait for Exercise 31.1(v).

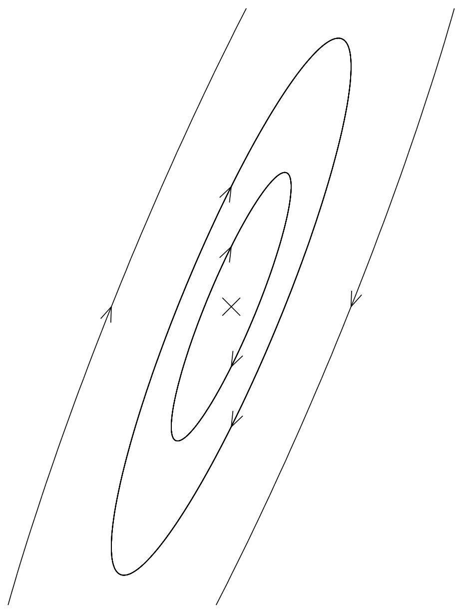

Fig. 31.6. The phase portrait for Exercise 31.1(vi).

multiple of the identity the origin is a stable star. The phase portrait is shown in Figure 31.9.

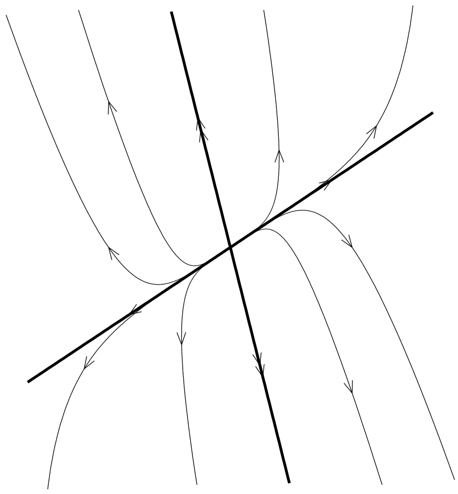

Fig. 31.7. The phase portrait for Exercise 31.1(vii).

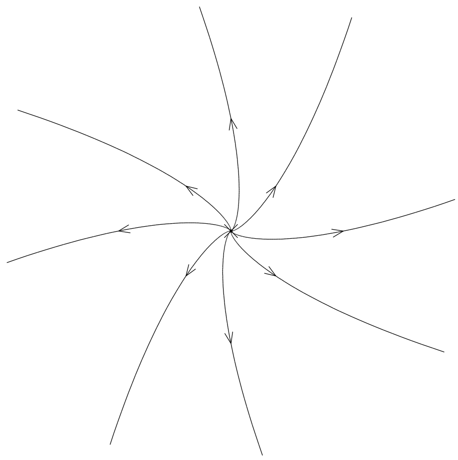

Fig. 31.8. The phase portrait for Exercise 31.1(viii).

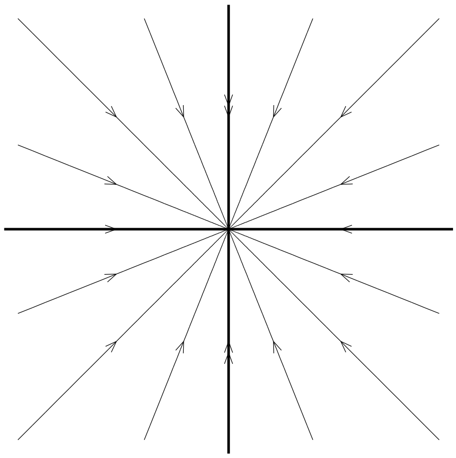

Fig. 31.9. The phase portrait for Exercise 31.1(ix).

32

Coupled nonlinear equations

In the solutions for this and all subsequent chapters, the eigenvalues and eigenvectors have been calculated using MATLAB.

Exercise 32.1 *The Hartman-Grobman theorem guarantees that the phase portrait for a nonlinear equation looks like the linearised phase portrait sufficiently close to a stationary point* provided that the eigenvalues have non-zero real part. *In particular, the linearised system may not give a qualitatively correct picture when the linearised equation produces a centre, as this example demonstrates. First show that the origin is a centre for the linearised version of the equation*

$$\begin{aligned} \dot{x} &= -y + \lambda x(x^2 + y^2) \\ \dot{y} &= x + \lambda y(x^2 + y^2). \end{aligned}$$

Now write down the equation satisfied by r, where

$$r^2 = x^2 + y^2,$$

and hence show that the stability of the origin depends on the sign of λ. Draw the phase portrait for $\lambda < 0$.

The linearisation of the equations

$$\begin{aligned} \dot{x} &= -y + \lambda x(x^2 + y^2) \\ \dot{y} &= x + \lambda y(x^2 + y^2). \end{aligned}$$

near the origin is just

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

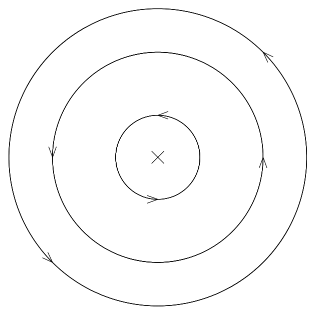

Fig. 32.1. The phase portrait for the linearised equation near the origin.

The matrix has eigenvalues $\pm\mathrm{i}$, so the linearisation yields a centre, see Figure 32.1.

However, if we set $x = r\cos\theta$ and $y = r\sin\theta$ then we have (see the discussion after (29.8))

$$\begin{aligned}\dot\theta &= \frac{x\dot y - y\dot x}{r^2}\\ &= \frac{x^2+\lambda xy(x^2+y^2)+y^2-\lambda xy(x^2+y^2)}{r^2}\\ &= 1,\end{aligned}$$

so that $\theta(t) = \theta_0 + t$, and, since $r^2 = x^2 + y^2$ then we obtain

$$\begin{aligned}2r\dot r &= 2x\dot x + 2y\dot y\\ &= 2\lambda x^2(x^2+y^2) + 2\lambda y^2(x^2+y^2)\\ &= 2\lambda r^4,\end{aligned}$$

and so we have

$$\dot r = \lambda r^3.$$

We can solve this equation explicitly, since we can separate the variables to give

$$\frac{\mathrm{d}r}{r^3} = \lambda\,\mathrm{d}t.$$

Integrating both sides between limits corresponding to times 0 and t we obtain

$$\left[-\frac{1}{2r^2}\right]_{r=r_0}^{r(t)} = \lambda t,$$

and so

$$r(t) = \frac{1}{\sqrt{r_0^{-2} - 2\lambda t}}.$$

Therefore if $\lambda > 0$ trajectories move away from the origin, while if $\lambda < 0$ trajectories move slowly towards the origin, see Figure 32.2 for the case $\lambda = -1$.

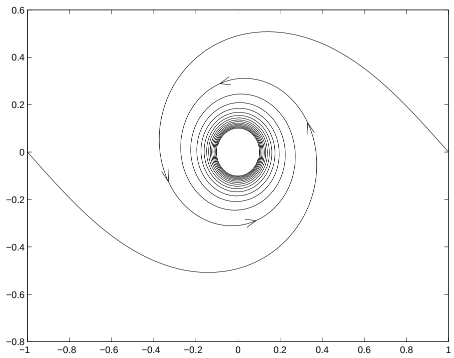

Fig. 32.2. The phase portrait for the nonlinear equation for $\lambda = -1$: trajectories spiral slowly in towards the origin on curves like those shown here.

33

Ecological models

Pictures of the local phase portraits (the linearised phase diagrams) near all the stationary points are given in the solutions of Exercise 33.1, but not for the subsequent exercises.

Exercise 33.1 *For each of the following models of two species, describe first the type of situation being modelled, then find the stationary points, determine their stability type and draw the phase portrait for $x, y \geq 0$. Finally, say what the phase portrait means for the two species.*

(i)

$$\begin{aligned} \dot{x} &= x(2 - x - y) \\ \dot{y} &= y(2 - 2y - 2x) \end{aligned}$$

(ii)

$$\begin{aligned} \dot{x} &= x(2 - x - y) \\ \dot{y} &= y(2 - 2y - x/4) \end{aligned}$$

(iii)

$$\begin{aligned} \dot{x} &= x(2 - x - 3y) \\ \dot{y} &= y(2 - 2y - 3x) \end{aligned}$$

(iv)

$$\begin{aligned} \dot{x} &= x(1 - 2y) \\ \dot{y} &= y(-2 + 3x), \end{aligned}$$

find also the equations of the curves along which the solutions move,

(v)

$$\begin{aligned}\dot{x} &= x(4 - x/2 - 3y)\\ \dot{y} &= y(-2 + x)\end{aligned}$$

(vi)

$$\begin{aligned}\dot{x} &= x(10 - x - 3y)\\ \dot{y} &= y(1 + x - 10y)\end{aligned}$$

(vii)

$$\begin{aligned}\dot{x} &= x(3 - x - y)\\ \dot{y} &= y(-2 + x).\end{aligned}$$

(You could use the MATLAB *program* `lotkaplane.m` *to help draw some of these phase portraits. It asks for the parameters that occur in the general form of the equations*

$$\begin{aligned}\dot{x} &= x(A + ax + by)\\ \dot{y} &= y(B + cx + dy),\end{aligned}$$

and then draws the trajectory forwards and backwards through specified initial conditions.)

To find out what type of situation is being modelled you can consider first the behaviour of one species without the other, and then the interaction terms.

(i)

$$\begin{aligned}\dot{x} &= x(2 - x - y)\\ \dot{y} &= y(2 - 2y - 2x).\end{aligned}$$

On its own x evolves according to $\dot{x} = x(2 - x)$, a normal population equation. So x can survive on the natural resources available. Similarly, on its own y satisfies $\dot{y}(2 - 2y)$, so y is also living on natural resources. The interaction terms means that y disadvantages x and vice versa, so we have a competition situation.

The stationary points are at

$$(0,0), \qquad (0,1), \qquad (2,0).$$

(There is no fourth stationary point with $x, y \geq 0$.)

The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 2 - 4y - 2x \end{pmatrix}.$$

At the origin we get

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

so that both eigenvalues are equal to 2 and this point is an unstable star, see Figure 33.1.

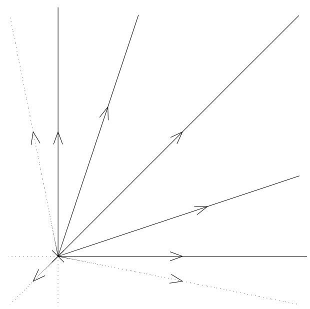

Fig. 33.1. The phase portrait near $(0, 0)$.

At $(0, 1)$ we get

$$J(0, 1) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}.$$

The eigenvalues are 1 and -2, so that this point is a saddle. The $\lambda = 1$ eigenvector is given by

$$\begin{pmatrix} 0 & 0 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0},$$

and so is $(3,\ -2)$; the eigenvector for $\lambda = -2$ is just along the y axis, $(0,\ 1)$. The linearised phase portrait near this stationary point is shown in Figure 33.2.

At $(2, 0)$ we get

$$J(2, 0) = \begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix},$$

so the eigenvalues are both -2 and we have an stable improper node. The one eigenvector is along the x axis, $(1,\ 0)$. The local phase portrait is shown in Figure 33.3.

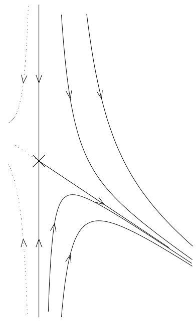

Fig. 33.2. The phase portrait near $(0, 1)$.

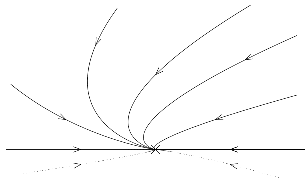

Fig. 33.3. The phase portrait near $(2, 0)$.

The global phase portrait, incorporating these three local pictures, is as shown in Figure 33.4.

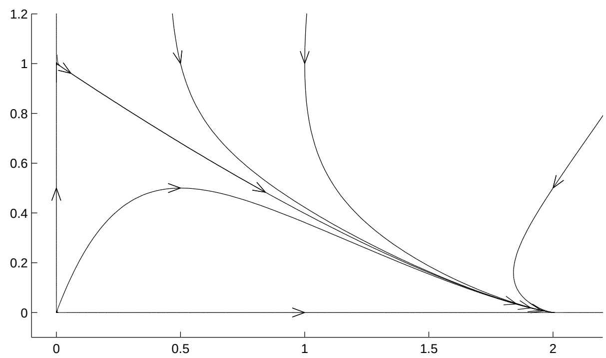

Fig. 33.4. The phase portrait for Exercise 33.1(i).

Species x is stronger, and always drives species y to extinction.

(ii)

$$\dot{x} = x(2 - x - y)$$

$$\dot{y} \;=\; y(2 - 2y - x/4)$$

The stationary points occur when

$$x(2 - x - y) = 0 \qquad \text{and} \qquad y(2 - 2y - x/4) = 0,$$

so we have

$$(0,0), \qquad (0,1), \qquad (2,0), \qquad \text{and} \qquad (8/7, 6/7).$$

The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 2 - 2x - y & -2x \\ -y/4 & 2 - 4y - x/4 \end{pmatrix}.$$

At the origin

$$J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and so the origin is an unstable star, and the phase portrait is as in Figure 33.1.

At the stationary point on the x axis, $(2,0)$, we have

$$J(2,0) = \begin{pmatrix} -2 & -4 \\ 0 & 3/2 \end{pmatrix}.$$

The eigenvalues are -2 with corresponding eigenvector $(1,\ 0)$, and $3/2$ with eigenvector $(1,\ -7/8)$, and so this stationary point is a saddle, as in Figure 33.5.

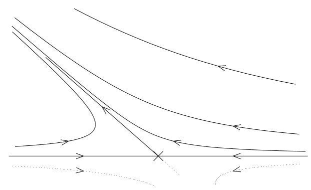

Fig. 33.5. The phase portrait near $(2,0)$.

At the stationary point on the y axis, $(0,1)$, we have

$$J(0,1) = \begin{pmatrix} 1 & 0 \\ -1/4 & -2 \end{pmatrix},$$

and so the eigenvalues here are -2 with eigenvector $(0,\ 1)$, and 1 with eigenvector $(1,\ -1/12)$; the phase portrait of the linearised equation near this point is shown in Figure 33.6.

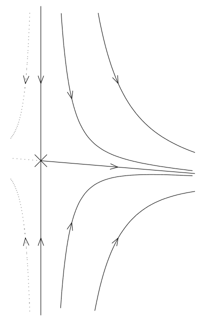

Fig. 33.6. The linearised phase portrait near $(0, 1)$.

Finally at the interior stationary point $(8/7, 6/7)$ we have

$$J(8/7, 6/7) = \begin{pmatrix} -8/7 & -12/7 \\ -3/14 & -12/7 \end{pmatrix},$$

and so the eigenvalues are given by

$$\begin{aligned} \begin{vmatrix} -8/7-\lambda & -12/7 \\ -3/14 & -12/7-\lambda \end{vmatrix} &= \left(-\frac{8}{7}-\lambda\right)\left(-\frac{12}{7}-\lambda\right)-\frac{18}{49} \\ &= \lambda^2+\frac{20}{7}\lambda+\frac{78}{49}=0, \end{aligned}$$

which gives

$$\lambda = \frac{-20 \pm \sqrt{400-312}}{14} = \frac{-10 \pm \sqrt{22}}{7},$$

The eigenvector corresponding to $\lambda_{\pm} = (-10 \pm \sqrt{22})/7$ is given by

$$\begin{pmatrix} 2/7 \mp \sqrt{22}/7 & -12/7 \\ -3/14 & -2/7 \mp \sqrt{22}/7 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{0},$$

i.e.

$$\mathbf{v}_{\pm} = \begin{pmatrix} 12 \\ 2 \pm \sqrt{22} \end{pmatrix}.$$

This point is a stable node; the local phase portrait is shown in Figure 33.7.

The global phase portrait is shown in Figure 33.8.

(iii)

$$\begin{aligned} \dot{x} &= x(2-x-3y) \\ \dot{y} &= y(2-2y-3x). \end{aligned}$$

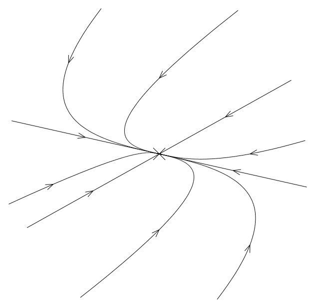

Fig. 33.7. The local phase portrait near $(8/7, 6/7)$.

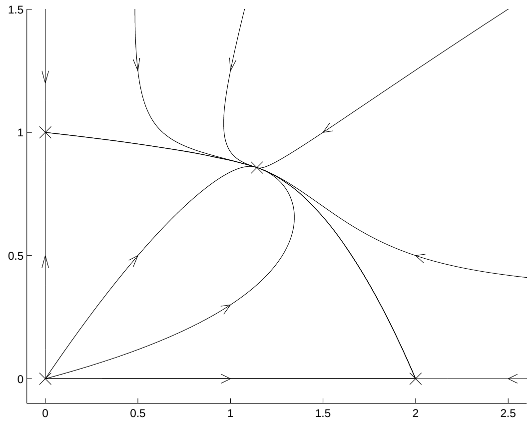

Fig. 33.8. The phase portrait for Exercise 33.1(ii).

A competition case again. Here the stationary points are $(0, 0)$, $(0, 1)$, $(2, 0)$ and a fourth given by finding the solution of

$$2 - x - 3y = 0 \qquad \text{with} \qquad 2 - 2y - 3x = 0.$$

Solving these gives $x = 2/7$ and $y = 4/7$. So we have four stationary points

$$(0, 0), \qquad (0, 1), \qquad (2, 0), \qquad \text{and} \qquad (2/7, 4/7).$$

The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 2 - 2x - 3y & -3x \\ -2y & 2 - 4y - 2x \end{pmatrix}.$$

At the origin this is

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

so that the origin is an unstable star again (see Figure 33.1).

At $(0,1)$ we have

$$\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix},$$

so the eigenvalues are -2 (with eigenvector along the y axis) and -1 with eigenvector given by

$$\begin{pmatrix} 0 & 0 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0},$$

i.e. $(3,\ -2)$. This point is a stable node, and trajectories approach tangent to the $(3,\ -2)$ direction, and shown in Figure 33.9.

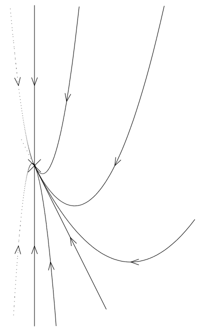

Fig. 33.9. The local phase portrait $(0,1)$.

At $(2,0)$ we have

$$\begin{pmatrix} -2 & -6 \\ 0 & -4 \end{pmatrix},$$

so the eigenvalues are -2 (along the x axis) and -4, with eigenvector given by

$$\begin{pmatrix} -6 & -6 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0},$$

i.e. $(1,-1)^T$. This is another stable node, approached by trajectories tangent to the x axis, as in Figure 33.10.

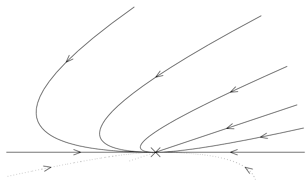

Fig. 33.10. The phase portrait near $(2, 0)$.

Finally, at the interior stationary point $(2/7, 4/7)$ we have

$$J(2/7, 4/7) = \begin{pmatrix} -2/7 & -6/7 \\ -8/7 & -8/7 \end{pmatrix} = -\frac{2}{7} \begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}.$$

The eigenvalues will be $-2/7$ times the eigenvalues of

$$\begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}$$

and will have the same eigenvectors. The eigenvalues of this simpler matrix are given by the solutions of

$$\begin{vmatrix} 1-\lambda & 3 \\ 4 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 8 = 0.$$

These are real and of opposite sign, so this point is a saddle, as shown in Figure 33.11. (Ideally we should calculate the directions of the eigenvectors, but this would be too messy.)

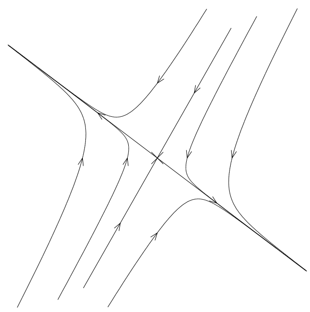

Fig. 33.11. The phase portrait near the interior stationary point $(2/7, 4/7)$.

The full phase portrait is shown in Figure 33.12.

The stable manifold of the saddle point separates the phase space into two regions: in one species x always “wins”, while in the other it is species y that comes out on top.

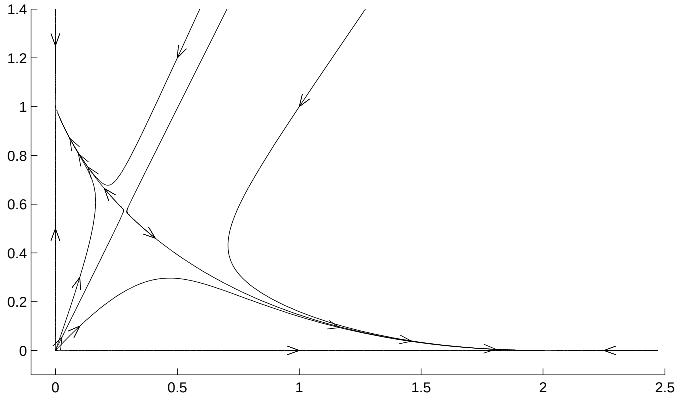

Fig. 33.12. The phase portrait for Exercise 33.1(iii).

(iv)

$$\begin{aligned}\dot{x} &= x(1-2y)\\ \dot{y} &= y(-2+3x).\end{aligned}$$

Left alone species x satisfies $\dot{x} = x$ and reproduces without bound, happily feeding on the natural resources. But species y alone has $\dot{y} = -2y$ and dies out. The interaction shows that this is a predator-prey situation (with x the prey), since presence of y disadvantages x, but presence of x is good for y.

There are only two stationary points, $(0,0)$ and $(2/3,1/2)$. The Jacobian is

$$\begin{pmatrix} 1-2y & -2x \\ 3y & -2+3x \end{pmatrix}.$$

At the origin this is

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

with eigenvalues 1 (along the x axis) and -2 (along the y axis) so this is a saddle (see Figure 33.13).

At the interior stationary point we have

$$\begin{pmatrix} 0 & -4/3 \\ 3/2 & 0 \end{pmatrix},$$

so that the eigenvalues are $\pm\sqrt{2}\mathrm{i}$, and this point is a linear centre, as shown in Figure 33.14.

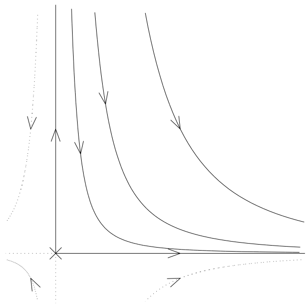

Fig. 33.13. The phase portrait near $(0, 0)$.

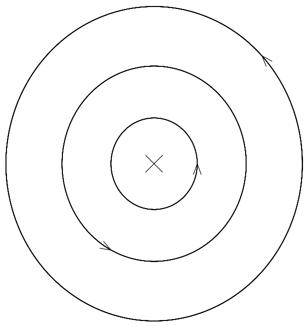

Fig. 33.14. The interior stationary point $(2/3, 1/2)$ is a linearised centre.

In fact it is surrounded by closed orbits: we can find the equation of these. Dividing $\dot{y}$ by $\dot{x}$ we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(-2+3x)}{x(1-2y)},$$

and this equation can be separated to give

$$\int \frac{1}{y} - 2\,\mathrm{d}y = \int -\frac{2}{x} + 3\,\mathrm{d}x.$$

Integrating both sides we get

$$\ln y - 2y = -2\ln x + 3x + c,$$

and so

$$y\mathrm{e}^{-2y} = Kx^{-2}\mathrm{e}^{3x}.$$

The phase portrait is shown in Figure 33.15.

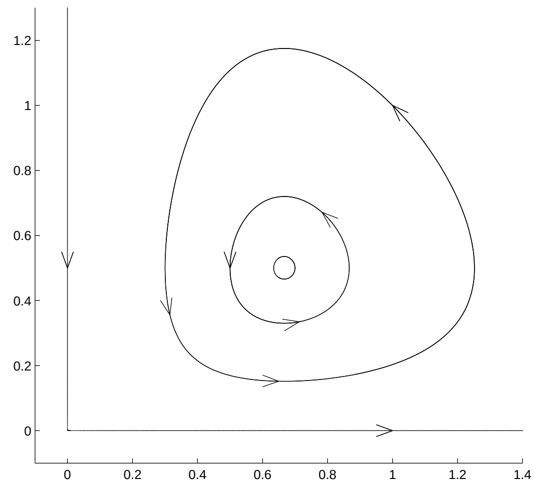

Fig. 33.15. The phase portrait for Exercise 33.1(iv). The populations cycle repeatedly.

(v)

$$\begin{aligned}\dot{x} &= x(4 - x/2 - 3y)\\ \dot{y} &= y(-2 + x).\end{aligned}$$

On its own x obeys $\dot{x} = x(4 - x/2)$, the logistic population equation, so settles down to a steady population of $x = 8$. On its own once again y will die out ($\dot{y} = -2y$), and the interaction terms show that x is the prey and y the predator once again.

There are now three stationary points,

$$(0,0), \qquad (8,0), \qquad \text{and} \qquad (2,1).$$

The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 4 - x - 3y & -3x \\ y & -2 + x \end{pmatrix}.$$

So at the origin

$$J(0,0) = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

and this gives a saddle (with the x axis unstable and the y axis stable); see Figure 33.16.

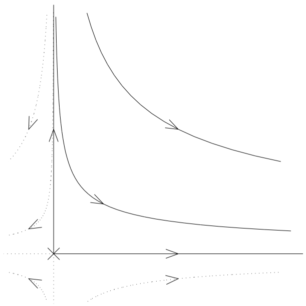

Fig. 33.16. The phase portrait near the origin.

At $(8,0)$ we get

$$J(8,0) = \begin{pmatrix} -4 & -24 \\ 0 & 6 \end{pmatrix}$$

so the eigenvalues are -4 (along the x axis) and 6: the eigenvalue in the unstable direction can be found from

$$\begin{pmatrix} -10 & -24 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

and so is $(12,-5)^T$. This point is a saddle, as shown in Figure 33.17.

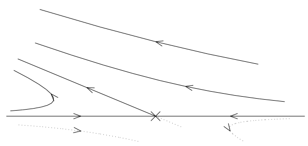

Fig. 33.17. The local phase portrait at $(8,0)$.

At the interior stationary point we have

$$J(2,1) = \begin{pmatrix} -1 & -6 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are the solutions of

$$\begin{vmatrix} -1-\lambda & -6 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + \lambda + 6 = 0$$

and so are

$$\lambda = -1 \pm \sqrt{1-24}2 :$$

a complex conjugate pair with negative real part, so this point is a stable focus, as shown in Figure 33.18.

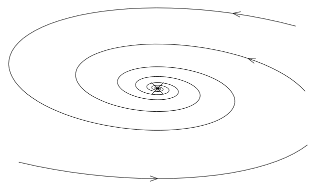

Fig. 33.18. The phase portrait of the linearised equation near $(2, 1)$.

The global phase portrait is shown in Figure 33.19.

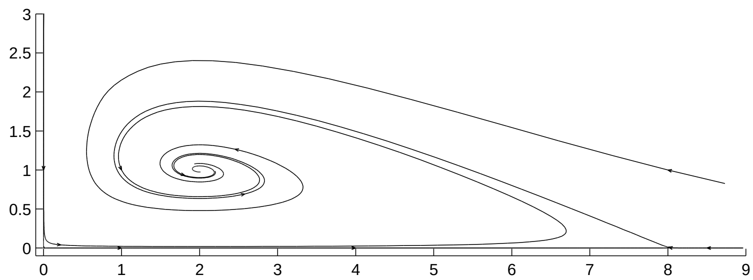

Fig. 33.19. The phase portrait for Exercise 33.1(v).

Provided that there is a mixture of the species, they eventually settle down to the coexistent steady state $(2, 1)$.

(vi)

$$\begin{aligned}\dot{x} &= x(10 - x - 3y) \\ \dot{y} &= y(1 + x - 10y).\end{aligned}$$

On its own x satisfies $\dot{x} = x(10 - x)$: reproducing quickly and settling to a population of 10. On its own y satisfies $\dot{y} = y(1 - 10y)$, so reproduces more slowly, and has a much smaller limiting population of $1/10$. The interaction terms show that the presence of y disadvantages x, while the presence of x advantages y. This could be a predator-prey situation, or perhaps a host-parasite model.

The stationary points are at

$$(0, 0), \qquad (10, 0), \qquad (0, 1/10), \qquad \text{and} \qquad (97/13, 11/13).$$

The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 10 - 2x - 3y & -3x \\ y & 1 + x - 20y \end{pmatrix}.$$

At the origin we get

$$\begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix},$$

so this is an unstable node as in Figure 33.20.

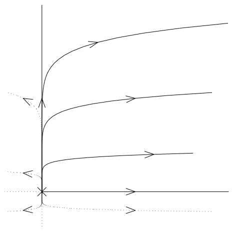

Fig. 33.20. The phase portrait near the origin.

At $(0, 1/10)$ we have

$$\begin{pmatrix} 9.7 & 0 \\ 0.1 & -1 \end{pmatrix}:$$

the point is a saddle, with a eigenvalue -1 in the direction of the y axis, and eigenvalue 9.7 corresponding to the eigenvector $(1, -9.6)^T$. See Figure 33.21.

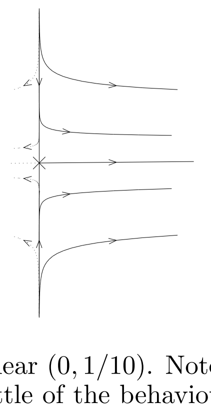

Fig. 33.21. The phase portrait near $(0, 1/10)$. Note that this point is very close to the origin, so that we will see little of the behaviour shown in Figure 33.20 in our final phase diagram.

At $(10,0)$ we have

$$\begin{pmatrix} -10 & -30 \\ 0 & 11 \end{pmatrix},$$

another saddle, with eigenvalues -10 along the x axis and eigenvalue 11 in the direction $(30,-21)^T$; see Figure 33.22.

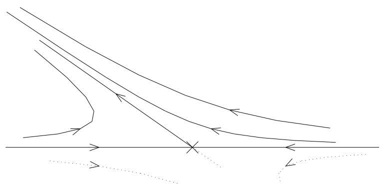

Fig. 33.22. The linearised phase portrait at $(10,0)$.

At the interior stationary point

$$J(97/13,11/13) = \begin{pmatrix} -97/13 & -291/13 \\ 11/13 & -110/13 \end{pmatrix}.$$

A messy calculation (or a quick use of MATLAB) shows that the eigenvalues here are a complex conjugate pair with negative real part, so this is a stable focus, as you can see in Figure 33.23.

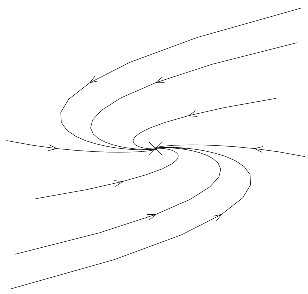

Fig. 33.23. The stable focus $(97/13,11/13)$.

The global phase portrait is shown in Figure 33.24.

(vii) The model

$$\begin{aligned} \dot{x} &= x(3-x-y) \\ \dot{y} &= y(-2+x) \end{aligned}$$

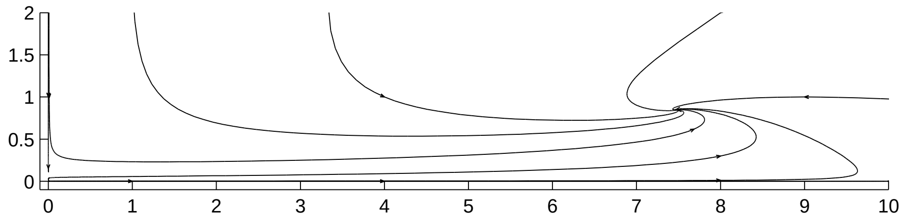

Fig. 33.24. The phase portrait for Exercise 33.1(vi). The population always settles down to a coexistent state.

has stationary points at

$$(0,0), \qquad (2,1), \qquad \text{and} \qquad (3,0).$$

The matrix of partial derivatives is given by

$$\mathrm{D}\mathbf{f}(x,y) = \begin{pmatrix} 3-2x-y & -x \\ y & -2+x \end{pmatrix}.$$

At the origin we have

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\boldsymbol{\xi},$$

and the eigenvalues and eigenvectors of the matrix are

$$\lambda_1 = 3 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -2 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is a saddle point; see Figure 33.25.

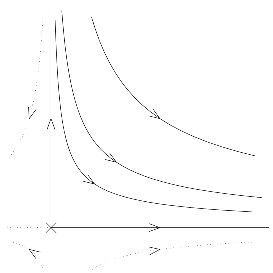

Fig. 33.25. The origin is a saddle point.

At $(2,1)$ we have

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} -2 & -2 \\ 1 & 0 \end{pmatrix}\boldsymbol{\xi}.$$

The eigenvalues are $-1 \pm \mathrm{i}$; this point is a stable spiral, as shown in Figure 33.26.

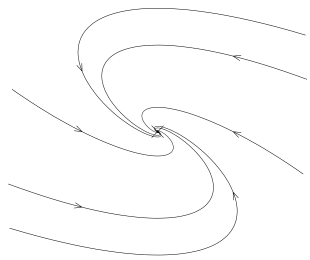

Fig. 33.26. The interior stationary point $(2,1)$ is a stable spiral.

At $(3,0)$ we have

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} -3 & -3 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi}.$$

The matrix has eigenvalues and eigenvectors

$$\lambda_1 = -3 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 1 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

This is a saddle point.

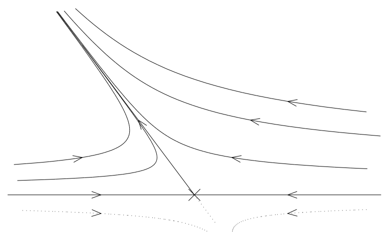

Fig. 33.27. A second saddle point at $(3,0)$.

The global phase portrait is shown in Figure 33.28.

Exercise 33.2 *The situation in which two species cooperate, so that the presence of one enhances the environment for the other, can be modelled by a coupled pair of equations of the form*

$$\begin{aligned} \dot{x} &= x(A - ax + by) \\ \dot{y} &= y(B + cx - dy), \end{aligned}$$

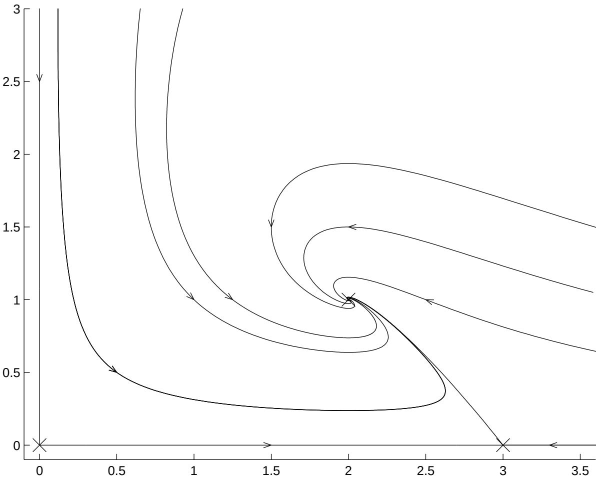

Fig. 33.28. The phase portrait for Exercise 33.1(vii).

where all the parameters are positive. Draw the phase portraits for the following cooperative equations:

(i)

$$\begin{aligned}\dot{x} &= x(1-x+y)\\ \dot{y} &= y(1+x-2y).\end{aligned}$$

and

(ii)

$$\begin{aligned}\dot{x} &= x(2-x+y)\\ \dot{y} &= y(4+2x-y).\end{aligned}$$

(i) The equations

$$\begin{aligned}\dot{x} &= x(1-x+y)\\ \dot{y} &= y(1+x-2y)\end{aligned}$$

have the stationary points

$$(0,0), \qquad (0,\tfrac{1}{2}), \qquad (1,0), \qquad \text{and} \qquad (3,2).$$

The matrix of partial derivatives $\mathbf{Df}$ is

$$\mathrm{D}\mathbf{f}(x,y) = \begin{pmatrix} 1-2x+y & x \\ y & 1+x-4y \end{pmatrix}.$$

The linearisation near $(0,0)$ is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\xi} :$$

this points is an unstable star.

Near $(0, \frac{1}{2})$ we have

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 3/2 & 0 \\ 1/2 & -1 \end{pmatrix} \boldsymbol{\xi},$$

so the eigenvalues here are -1 with eigenvector $(0,\ 1)$ and $3/2$ with eigenvector $(5,\ 1)$. So this point is a saddle.

Near $(1,0)$ the linearisation is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \boldsymbol{\xi}.$$

One eigenvalue is -1 with eigenvector $(1,\ 0)$, and the other is 2 with eigenvector $(1,\ 3)$. This point is another saddle.

Near the interior stationary point $(3,2)$ the linearisation is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} -3 & 3 \\ 2 & -4 \end{pmatrix} \boldsymbol{\xi}.$$

The eigenvalues here are $\lambda_1 = -1$ with eigenvector $(3,\ 2)$ and $\lambda_2 = -6$ with eigenvector $(1,\ -1)$. In particular, both of these eigenvalues are negative, so the interior stationary point is a stable node.

The phase portrait is shown in Figure 33.29: the species settle to the coexistent state $(8,6)$.

(ii) The equations

$$\begin{aligned} \dot{x} &= x(2 - x + y) \\ \dot{y} &= y(4 + 2x - y) \end{aligned}$$

have only three non-negative stationary points, at

$$(0,0), \qquad (2,0), \qquad \text{and} \qquad (0,4).$$

The matrix of partial derivatives is

$$\mathrm{D}\mathbf{f}(x,y) = \begin{pmatrix} 2 - 2x + y & x \\ 2y & 4 + 2x - 2y \end{pmatrix},$$

and so the linearisation at the origin is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \boldsymbol{\xi};$$

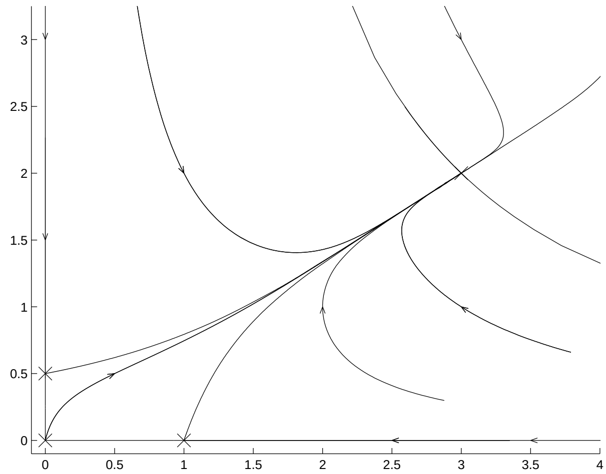

Fig. 33.29. The phase portrait for the cooperative system of Exercise 33.2(i), settling to a coexistent state with bounded numbers of both species.

this is an unstable node, with the eigenvectors directed along the axes.

At the stationary point $(2, 0)$ on the x axis the linearisation is

$$\frac{d\boldsymbol{\xi}}{dt} = \begin{pmatrix} -2 & 2 \\ 0 & 6 \end{pmatrix} \boldsymbol{\xi}.$$

The eigenvalues and eigenvectors are

$$\lambda_1 = -2 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 6 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

and this is a saddle point.

At the stationary point on the y axis, $(0, 4)$, we obtain the linearised equation

$$\frac{d\boldsymbol{\xi}}{dt} = \begin{pmatrix} 6 & 0 \\ 8 & -4 \end{pmatrix} \boldsymbol{\xi}.$$

Here the eigenvalues and eigenvectors are

$$\lambda_1 = -4 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 6 \qquad \text{with} \qquad \mathbf{v}_2 = \begin{pmatrix} 5 \\ 4 \end{pmatrix},$$

another saddle.

The phase portrait is shown in Figure 33.30: the number of both species increases to $+\infty$ as $t \to \infty$.

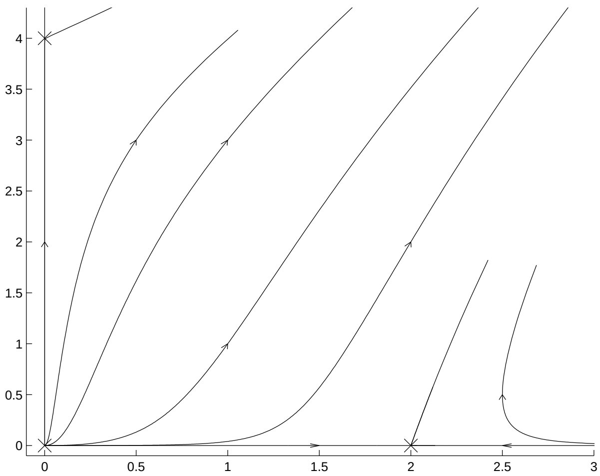

Fig. 33.30. For the cooperative system of Exercise 33.2(ii) both populations increase forever.

Exercise 33.3 *(T) Consider the general model of two competing species,*

$$\begin{aligned}\dot{x} &= x(A - ax - by)\\ \dot{y} &= y(B - cx - dy),\end{aligned}$$

where all the parameters are positive. Assuming that the intercepts of the nullclines (lines on which $\dot{x} = 0$ and $\dot{y} = 0$) with the x and y axes do not coincide, by considering the relative positions of these intercepts show that there are four distinct possibilities for the behaviour of solutions, and find the parameter ranges over which they occur. Check that your results are consistent with what you found for the competitive examples in Exercise 33.1.

The nullclines for $\dot{x} = 0$ occur when $x = 0$ and when $A - ax - by = 0$; the second of these two lines intercepts the y axis at $y = A/b$ and the x axis at $x = A/a$. The nullclines for $\dot{y} = 0$ occur when $y = 0$ and when $B - cx - dy = 0$; the second of these two lines intercepts the y axis at $y = B/d$ and the x axis at $x = B/c$. The relative positions of these intercepts determine the asymptotic behaviour of the model. We can therefore distinguish four cases:

[Y] When $A/b < B/d$ and $A/a < B/c$. The nullclines do not intersect, except on the axes, yielding only three stationary points with $x, y \geq 0$, as shown in Figure 33.31. The trajectories are attracted to the stationary point $(0, B/d)$ on the y axis.

[X/Y] When $A/b < B/d$ and $A/a > B/c$ the nullclines intersect in the interior of

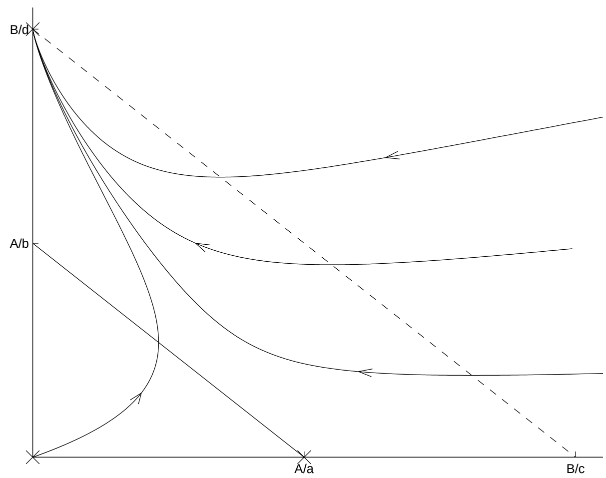

Fig. 33.31. The nullclines when $A/b < B/d$ and $A/a < B/c$ ($\dot{x} = 0$ on solid line, $\dot{y} = 0$ on dashed line). Trajectories are attracted to $(0, B/d)$ (case [Y]).

the quadrant, and we have four stationary points in total, see Figure 33.32. In this case there are two possible attracting stationary points, one on the x axis and one on the y axis. The nullclines give little indication of the rôle of the stable manifold of the interior stationary points as the 'separatrix' that divides the two regions leading to these different outcomes.

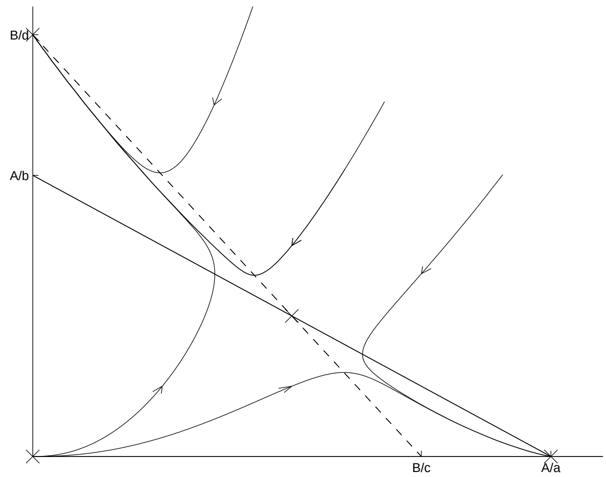

Fig. 33.32. The nullclines when $A/b < B/d$ and $A/a > B/c$ ($\dot{x} = 0$ on solid line, $\dot{y} = 0$ on dashed line). In this case there are two possible attracting points, those on either axis (case [X/Y]).

[X] When $A/b > B/d$ and $A/a > B/c$ the nullclines do not intersect and we have only three stationary points, see Figure 33.33. This time it is the point on the x axis, $(A/a, 0)$, which attracts all the trajectories.

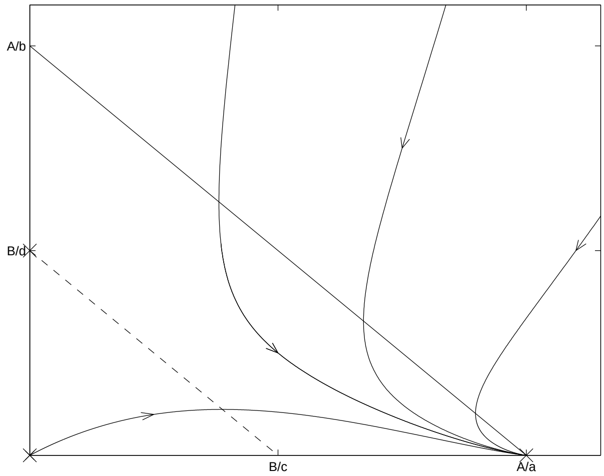

Fig. 33.33. The nullclines when $A/b > B/d$ and $A/a > B/c$ ($\dot{x} = 0$ on solid line, $\dot{y} = 0$ on dashed line). The stationary point $(A/a, 0)$ attracts all trajectories (case [X]).

[XY] When $A/b > B/d$ and $B/c > A/a$ there is once again an interior stationary point, as shown in Figure 33.34, and now all trajectories are attracted to this coexistent state.

The following parts of Exercise 33.1 are competitive:

(i)

$$\begin{aligned} \dot{x} &= x(2 - x - y) \\ \dot{y} &= y(2 - 2y - 2x). \end{aligned}$$

Here $A = B = 2$, $a = b = 1$, and $c = d = 2$: we have

$$A/b = 2 > 1 = B/d \qquad \text{and} \qquad A/a = 2 > 1 = B/c,$$

so we are in case [X], and all trajectories are attracted to the stationary point on the x axis.

(ii)

$$\begin{aligned} \dot{x} &= x(2 - x - y) \\ \dot{y} &= y(2 - 2y - x/4). \end{aligned}$$

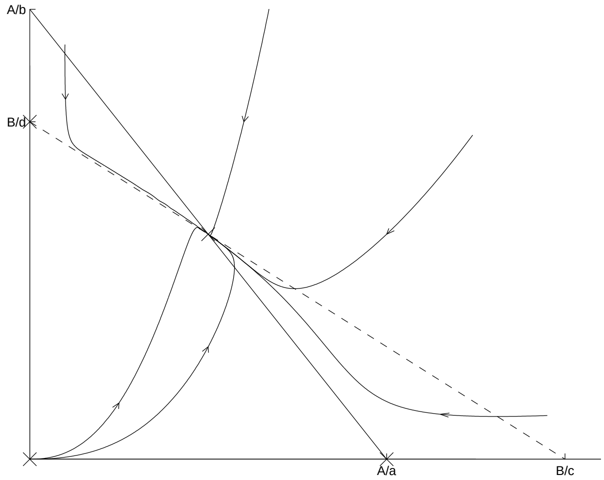

Fig. 33.34. The nullclines when $A/b > B/d$ and $B/c > A/a$ ($\dot{x} = 0$ on solid line, $\dot{y} = 0$ on dashed line). There is an attracting coexistent state (case [XY]).

Now $A = B = 2$, $a = b = 1$, $c = 1/4$ and $d = 2$. We have

$$A/b = 2 > 1 = B/d \qquad \text{and} \qquad A/a = 2 < 8 = B/c,$$

so we are in case [XY] and the interior (coexistent) stationary point is attracting.

(iii)

$$\begin{aligned} \dot{x} &= x(2 - x - 3y) \\ \dot{y} &= y(2 - 2y - 3x). \end{aligned}$$

Now $A = B = 2$, $a = 1$, $b = c = 3$, and $d = 2$. So

$$A/b = 2/3 < B/d = 1 \qquad \text{and} \qquad A/a = 1 > 2/3 = B/c;$$

we are in case [X/Y] and could end up at the stationary point on either axis depending on the initial condition.

34

Newtonian dynamics

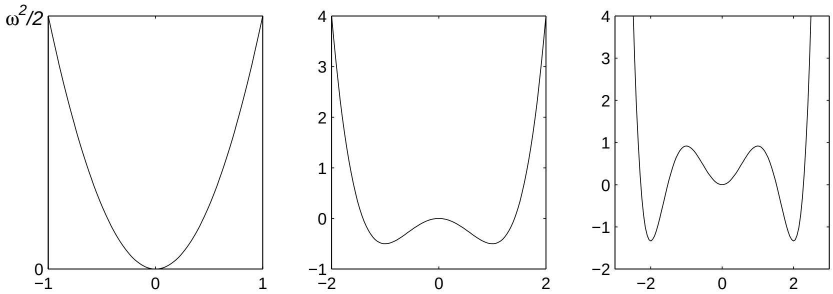

Fig. 34.1. From left to right, the potentials from parts (i), (ii), and (iii) of Exercise 34.1, plotted against x.

Exercise 34.1 *For the following choices of potential functions $V(x)$ write down the total energy for a particle of unit mass, and assuming that this is conserved write down a coupled system for x and $y = \dot{x}$. Draw the phase portrait and interpret the dynamics.*

(i) $V(x) = \frac{1}{2}\omega^2 x^2$;
(ii) $V(x) = \frac{1}{2}x^4 - x^2$; *and*
(iii) $V(x) = \frac{1}{6}x^6 - \frac{5}{4}x^4 + 2x^2$.

(Pictures of these potentials are shown in Figure 34.1.)

(i) When $V(x) = \frac{1}{2}\omega^2 x^2$ the energy is

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega^2 x^2.$$

Differentiating this gives

$$0 = \frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}\ddot{x} + \omega^2 x\dot{x},$$

yielding the differential equation

$$\ddot{x} = -\omega^2 x.$$

This equation gives rise to simple harmonic motion, and has the general solution

$$A\cos\omega t + B\sin\omega t.$$

Here we look at this from the phase plane point of view.

If we put $y = \dot{x}$ then we get

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega^2 x.\end{aligned}$$

There is only one stationary point, at the origin, and there the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\pm i\omega$, and since the equation is linear we really do have a centre. Indeed, it is easy to see that the trajectories move on ellipses, since

$$E = \tfrac{1}{2}\omega^2 x^2 + \tfrac{1}{2}y^2$$

is constant. This should bring home the fact that oscillations correspond to closed orbits in the phase plane.

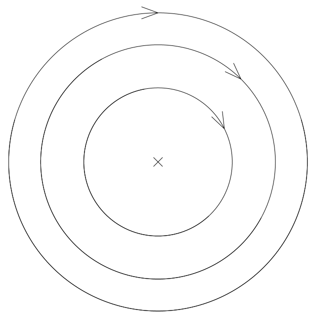

Fig. 34.2. The phase portrait for a particle in the potential $V(x) = \frac{1}{2}\omega^2 x^2$.

(ii) For the potential $V(x) = \frac{1}{2}x^4 - x^2$ the total energy is

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^4 - x^2.$$

Differentiating we have

$$\dot{x}\ddot{x} + 2x^3\dot{x} - 2x\dot{x} = 0,$$

and so dividing by $\dot{x}$ we obtain

$$\ddot{x} + 2x^3 - 2x = 0.$$

Setting $y = \dot{x}$ we obtain the coupled system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -2x^3 + 2x. \end{aligned}$$

The stationary points of this equation are at $(\pm 1, 0)$ and $(0,0)$. The matrix of partial derivatives is

$$\mathrm{D}\mathbf{f}(x,y) = \begin{pmatrix} 0 & 1 \\ -6x^2 + 2 & 0 \end{pmatrix},$$

and so

$$\mathrm{D}\mathbf{f}(\pm 1, 0) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix},$$

which shows that $(\pm 1, 0)$ are centres (eigenvalues $\pm 2\mathrm{i}$); and at $(0,0)$ we have

$$\mathrm{D}\mathbf{f}(0,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}:$$

the eigenvalues are $\pm\sqrt{2}$ with eigenvectors $(1,\ \pm\sqrt{2})$, and this is a saddle point. The phase portrait is shown in Figure 34.3.

(iii) When the potential is

$$V(x) = \frac{1}{6}x^6 - \frac{5}{4}x^4 + 2x^2$$

the total energy is given by

$$E = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{6}x^6 - \tfrac{5}{4}x^4 + 2x^2.$$

Since E is constant, if we differentiate we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 = \dot{x}\ddot{x} + x^5\dot{x} - 5x^3\dot{x} + 4x\dot{x}.$$

Dividing by $\dot{x}$ we obtain the second order equation

$$\ddot{x} + x^5 - 5x^3 + 4x = 0,$$

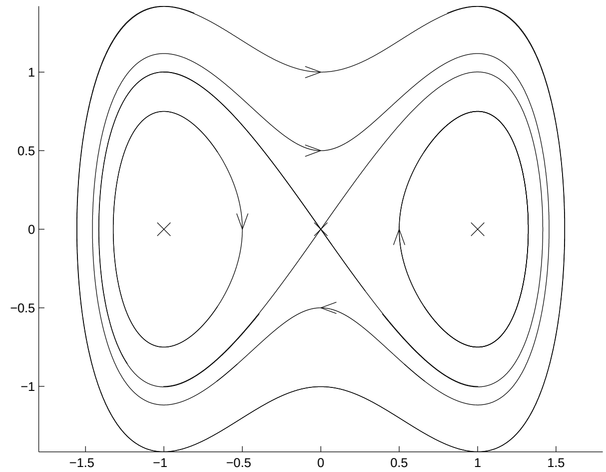

Fig. 34.3. The phase portrait for a particle moving in the potential $V(x) = \frac{1}{4}x^4 - x^2$.

and on setting $y = \dot{x}$ this can be rewritten as the coupled equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^5 + 5x^3 - 4x. \end{aligned}$$

The stationary points for this system occur at

$$(-2,0), \quad (-1,0), \quad (0,0), \quad (1,0), \quad \text{and} \quad (2,0).$$

The matrix of partial derivatives is

$$\mathbf{Df} = \begin{pmatrix} 0 & 1 \\ -5x^4 + 15x^2 - 4 & 0 \end{pmatrix}.$$

The linearisations at the two points $(\pm 2, 0)$ are the same:

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -24 & 0 \end{pmatrix} \boldsymbol{\xi},$$

so that this point is a centre (the eigenvalues are $\pm 2\sqrt{6}\mathrm{i}$).

The linearisations at the two points $(\pm 1, 0)$ are also the same:

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ 6 & 0 \end{pmatrix} \boldsymbol{\xi},$$

with eigenvalues $\pm\sqrt{6}$ and eigenvectors $(1, \ \pm\sqrt{6})$. These points are both saddles, while at the origin the linearisation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \boldsymbol{\xi}$$

shows that this point is another centre.

Since the energy is constant all the 'linearised centres' here are indeed centres for the nonlinear equation: the phase portrait is shown in Figure 34.4.

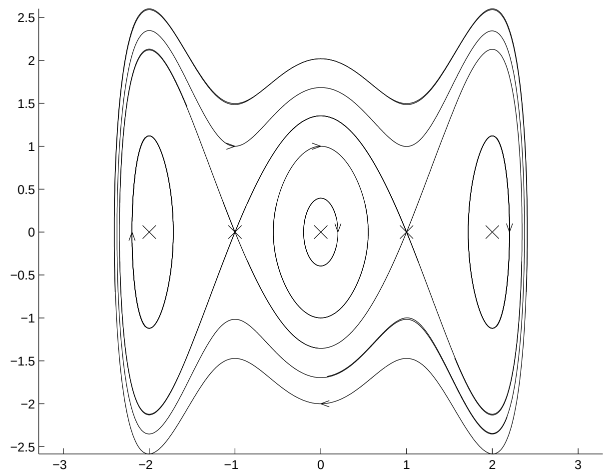

Fig. 34.4. The phase portrait for a particle moving in the potential $V(x) = \frac{1}{6}x^6 - \frac{5}{4}x^4 + 2x^2$.

Exercise 34.2 *For the functions $V(x)$ in parts (i) and (ii) of exercise 34.1 write down the kinetic energy of a particle of unit mass moving on a wire whose height as a function of x is $V(x)$. Taking $g = 1$ write down the total energy, and hence derive the second order equation satisfied by x. Write down a coupled system for x and $y = \dot{x}$, and draw the phase portrait.*

(i) The coordinates of a particle of unit mass moving on a wire whose height at position x is $V(x) = \frac{1}{2}\omega^2 x^2$ are

$$\mathbf{x} = (x, \tfrac{1}{2}\omega^2 x^2);$$

its velocity is therefore

$$\dot{\mathbf{x}} = (\dot{x}, \omega^2 x\dot{x})$$

and so

$$|\dot{\mathbf{x}}|^2 = \dot{x}^2(1 + \omega^4 x^2).$$

The total energy of the particle (taking $g = 1$) is therefore

$$E = \tfrac{1}{2}\dot{x}^2(1 + \omega^4 x^2) + \tfrac{1}{2}\omega^2 x^2.$$

Since E is constant if we differentiate with respect to t we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 = \dot{x}\ddot{x}(1+\omega^4x^2) + \omega^4x\dot{x}^3 + \omega^2x\dot{x}.$$

Dividing by $\dot{x}$ we obtain the second order equation

$$(1+\omega^4x^2)\ddot{x} + \omega^4x\dot{x}^2 + \omega^2x = 0.$$

With $y = \dot{x}$ we can write this as the coupled system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\frac{\omega^2x(1+\omega^2y^2)}{1+\omega^4x^2}.\end{aligned}$$

Although this looks complicated, we know that the stationary points will be the same as for the much simpler problem in part (i) of the previous exercise. Indeed, it is easy to see that if $y = 0$ (which we need to ensure that $\dot{x} = 0$) then the only stationary point must have $x = 0$ also.

The matrix of partial derivatives here is very messy,

$$\mathrm{D}\mathbf{f}(x,y) = \begin{pmatrix} 0 & 1 \\ -\frac{\omega^2(1+\omega^2y)(1-\omega^4x^2)}{(1+\omega^4x^2)^2} & \frac{\omega^4x}{1+\omega^4x^2}\end{pmatrix},$$

but at the origin it reduces to the much more manageable

$$\mathrm{D}\mathbf{f}(0,0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0\end{pmatrix},$$

showing that the origin is a centre for the linearised equation (the eigenvalues are $\pm\mathrm{i}\omega$). This will still be a centre for the nonlinear equation due to the conservation of the energy E.

The phase portrait (equivalent to a plot of curves of constant E) is shown in Figure 34.5.

(ii) The position of the particle is given by

$$\mathbf{x} = (x, \tfrac{1}{2}x^4 - x^2),$$

and so its velocity is

$$\dot{\mathbf{x}} = (\dot{x}, \dot{x}(2x^3 - x)).$$

The total energy is therefore

$$\begin{aligned}E &= \tfrac{1}{2}|\dot{\mathbf{x}}|^2 + [\tfrac{1}{2}x^4 - x^2]\\ &= \tfrac{1}{2}\dot{x}^2(1+(2x^3-x)^2) + \tfrac{1}{2}x^4 - x^2,\end{aligned}$$

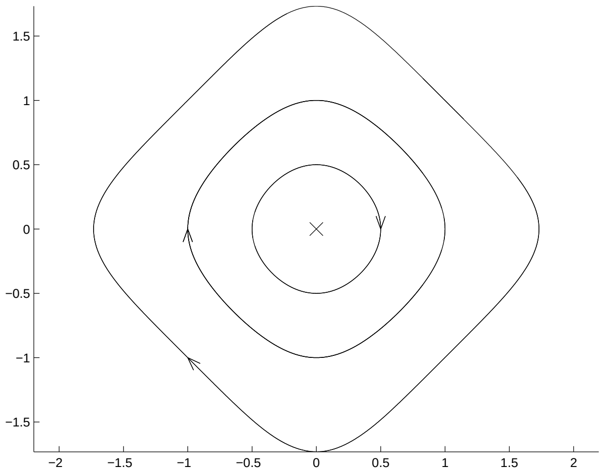

Fig. 34.5. The phase portrait for a particle rolling on a wire in the shape $\frac{1}{2}x^2$ ($\omega = 1$).

and so differentiating we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0 = \dot{x}\ddot{x}(1 + (2x^3 - x)^2) + \dot{x}^3(2x^3 - x)(6x^2 - 1) + 2x^3\dot{x} - 2x\dot{x}.$$

Dividing by $\dot{x}$ we obtain

$$(1 + (2x^3 - x)^2)\ddot{x} = -\dot{x}^2(2x^3 - x)(6x^2 - 1) - 2x^3 + 2x,$$

and on setting $y = \dot{x}$ we have the coupled system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \frac{-(2x^3 - x)(6x^2 - 1)y^2 - 2x^3 + 2x}{1 + (2x^3 - x)^2}. \end{aligned}$$

The stationary points occur when $y = 0$ (for $\dot{x} = 0$) and then $\dot{y} = 0$ requires $x = \pm 1$ or $x = 0$.

In order to find the stability of the stationary points it is much easier to use the general analysis of Section 34.2 than to write down the matrix of partial derivatives of the right-hand side of the governing equation (although we did this for the very simple example in part (i) of this Exercise above). Indeed, the general analysis shows that the stability of the fixed points is the same as for the simple equation $\ddot{x} = -V'(x)$. We analysed this in part (ii) of the previous Exercise: $(\pm 1, 0)$ are centres, and $(0, 0)$ is a saddle. The qualitative behaviour is the same, although the details of the phase portrait (the curves of which E is constant) and somewhat different, see Figure 34.6.

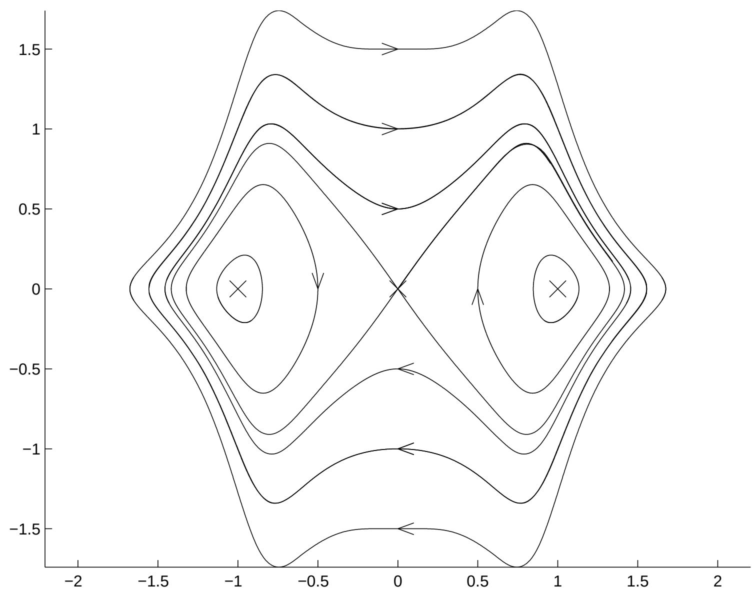

Fig. 34.6. The phase portrait for a particle moving on a wire in the shape $\frac{1}{2}x^4 - x^2$.

Exercise 34.3 *Write down the equation of motion for a particle of unit mass moving in each of the potentials in Exercise 34.1, when there is an additional damping force* $-\dot{x}$ *(in part (i) take* $\omega = 1$*). Draw the phase portrait for each case.*

(i) With $\omega = 1$ and a damping force $-\dot{x}$ the equations are now

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x - y.\end{aligned}$$

The origin is still the only stationary point; the equation is linear already,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{x},$$

and the matrix has eigenvalues $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$, so the origin is a stable spiral. The phase portrait is shown in Figure 34.7.

(ii) The equations are

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -2x^3 + 2x - y,\end{aligned}$$

with stationary points $(\pm 1, 0)$ and $(0, 0)$.

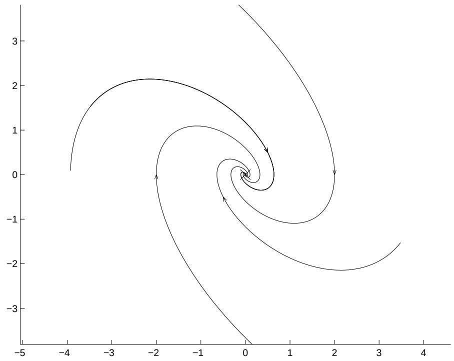

Fig. 34.7. The phase portrait for a particle moving in the potential $\frac{1}{2}x^2$ with damping $-\dot{x}$.

The matrix of partial derivatives is

$$\mathrm{D}\mathbf{f}(x,y) = \begin{pmatrix} 0 & 1 \\ -6x^2+2 & -1 \end{pmatrix}.$$

At $(\pm 1, 0)$ the linearisation is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix} \boldsymbol{\xi},$$

with eigenvalues given by the solutions of

$$\begin{vmatrix} -\lambda & 1 \\ -4 & -1-\lambda \end{vmatrix} = -\lambda(-1-\lambda) + 4 = \lambda^2 + \lambda + 4 = 0,$$

i.e.

$$\lambda = \frac{-1 \pm \sqrt{1-16}}{2} = -\tfrac{1}{2} \pm \tfrac{\sqrt{15}}{2}\mathrm{i},$$

so these two points are stable spirals.

At the origin the linearisation is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \boldsymbol{\xi},$$

with eigenvalues and eigenvectors

$$\lambda_1 = 1 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -2 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

so this point is a saddle.

The phase portrait is shown in Figure 34.8.

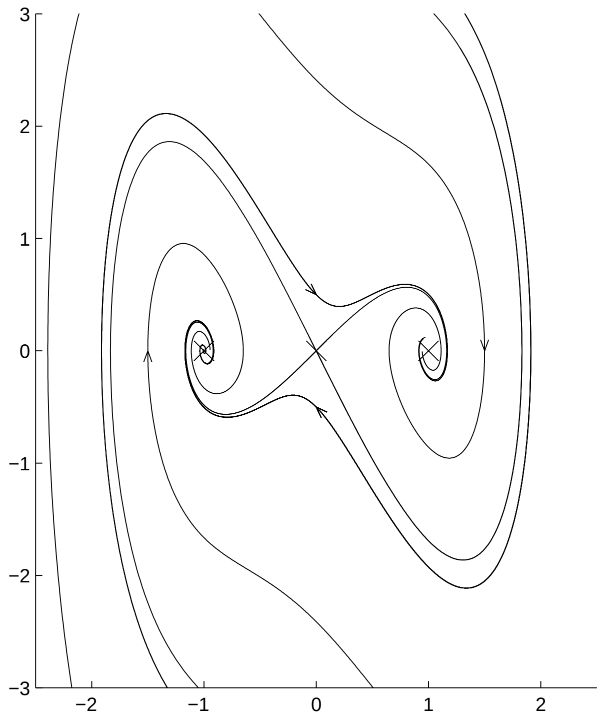

Fig. 34.8. The phase portrait for a particle moving the potential $\frac{1}{2}x^4 - x^2$ with damping $-\dot{x}$.

(iii) The equations of motion are now

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^5 + 5x^3 - 4x - y, \end{aligned}$$

with stationary points at $(\pm 2, 0)$, $(\pm 1, 0)$, and $(0, 0)$. The matrix $\mathrm{D}\mathbf{f}$ of partial derivatives is given by

$$\mathrm{D}\mathbf{f}(x, y) = \begin{pmatrix} 0 & 1 \\ -5x^4 + 15x^2 - 4 & -1 \end{pmatrix}.$$

The linearisation at $(\pm 2, 0)$ is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -24 & -1 \end{pmatrix} \boldsymbol{\xi},$$

and the eigenvalues are $-\frac{1}{2} \pm \frac{\sqrt{95}}{2}\mathrm{i}$, so these two points are stable spirals, as is the origin where the linearisation is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix} \boldsymbol{\xi}$$

and the relevant eigenvalues are $-\frac{1}{2} \pm \frac{\sqrt{15}}{2}\mathrm{i}$.

Near the other two stationary points, $(\pm 1, 0)$, the linearisation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix} \boldsymbol{\xi}$$

yields eigenvalues and eigenvectors

$$\lambda_1 = 2 \quad \text{with} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -3 \quad \text{with} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix},$$

so this point is a saddle.

The phase portrait is shown in Figure 34.9.

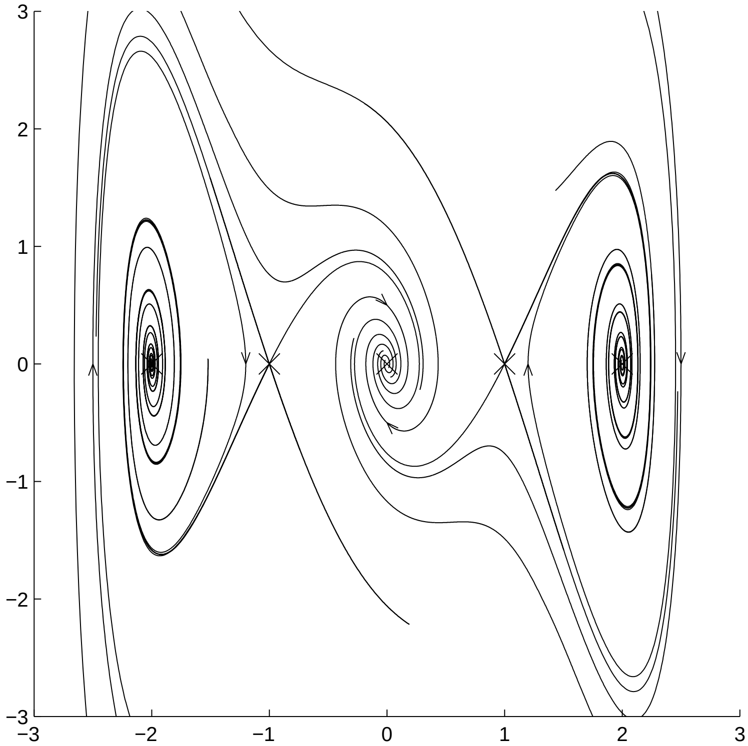

Fig. 34.9. The phase portrait for a particle moving in the potential $\frac{1}{6}x^6 - \frac{5}{4}x^4 + 2x^2$ with damping $-\dot{x}$.

Exercise 34.4 *(T) A particle of unit mass moves on a wire whose height as a function of x is $V(x)$, and is subject to an additional damping force $-k\dot{x}$. Write down the equation of motion, and show that the behaviour of this system is qualitatively the same as that of*

$$\ddot{x} = -V'(x) - k\dot{x}.$$

The equation of motion for the particle on a wire was derived as equation (34.6), namely

$$\ddot{x} = -V'(x)\left[\frac{1 + V''(x)\dot{x}^2}{1 + V'(x)^2}\right].$$

With the addition of a damping force $-k\dot{x}$ the equation becomes

$$\ddot{x} = -V'(x)\left[\frac{1 + V''(x)\dot{x}^2}{1 + V'(x)^2}\right] - k\dot{x}. \tag{S34.1}$$

We want to compare the qualitative behaviour of this equation with that of

$$\ddot{x} = -V'(x) - k\dot{x}.$$

The general analysis of dissipative systems in Section 34.3 shows that the stationary points occur at turning points of V: at maxima the stationary point is a saddle, while at minima it is a stable spiral. We want to recover the same results for the more complicated model (S34.1)

First we write the equations as a coupled system: setting $y = \dot{x}$ we obtain

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -V'(x)\left[\frac{1 + V''(x)y^2}{1 + V'(x)^2}\right] - ky.\end{aligned}$$

At a stationary point we must have $\dot{x} = 0$, which implies that $y = 0$; then have $\dot{x} = 0$ we require

$$-V'(x)\left[\frac{1}{1 + V'(x)^2}\right] = 0,$$

and since the term in square brackets is always strictly greater than zero, stationary points only occur when $V'(x) = 0$. So our two systems have the same stationary points.

Now we look at the matrix of partial derivatives. This is the same horrific expression as we obtained in Section 34.2 before, with the addition of a $-k$ term in the lower right entry:

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 & 1\\ -V''\left[\frac{1+V''y^2}{1+V'^2}\right] - V'\left[\frac{(1+V'^2)V'''y^2 - 2V'V''(1+V''y^2)}{(1+V'^2)^2}\right] & \frac{-2yV'V''}{1+V'^2} - k\end{pmatrix}.$$

As before, we obtain a significant simplification of this at any stationary point $(x^*, 0)$, since we have both $y = 0$ and $V'(x^*) = 0$:

$$\mathrm{D}\mathbf{f}(x^*, 0) = \begin{pmatrix} 0 & 1 \\ -V''(x^*) & -k \end{pmatrix}.$$

This gives rise to the same linearisation near the stationary point $(x^*, 0)$ as we obtained in equation (34.10) for the simple system

$$\ddot{x} = -V'(x) - k\dot{x}. \tag{S34.2}$$

It follows that not only does the 'damped particle on a wire' equation have the same stationary points as (S34.2), but that all the stationary points have the same stability properties. It follows that the two systems have the same qualitative behaviour.

Exercise 34.5 *(C) Investigate the dynamics of the equations in exercise 34.1 both with and without damping, using the* `M`*-file* `newtonplane.m`*. The program asks for the level of damping k, and then a succession of initial conditions. The equation is specified in the file* `newtonde.m`*, currently set up for the example $V(x) = x - \frac{1}{3}x^3$ in the main text. By changing this file you should be able to consider all the examples in Exercise 34.1, and also the equivalent problems for a ball rolling on a wire.*

The phase portraits in the solutions for this chapter were all drawn using the `M`-file `newtonplane.m`. It can also be useful to plot the contours of constant E using the MATLAB's `contour` command.

35

The 'real' pendulum

Exercise 35.1 *Draw the phase portrait for the damped pendulum equations in (35.5) when $k = 2$ and when $k = 3$.*

Here we use the analysis from Section 35.2, inserting the correct value of k. In both cases the points at $(\pm\pi, 0)$ are still saddles; it is the stability type of the stable equilibrium position $(0, 0)$ that depends on K.

- When $k = 2$ the matrix on the right-hand side of the linearised equation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \boldsymbol{\xi}$$

has a repeated real eigenvalue -1. So the origin is a stable improper node. The single eigenvector is $(1,\ -1)$. The phase portrait is shown in Figure 35.1.

- When $k = 3$ we have the linearisation

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix} \boldsymbol{\xi},$$

and the relevant eigenvalues and eigenvectors are

$$\lambda_1 = \frac{-3 + \sqrt{5}}{2} \qquad \text{with} \qquad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 - \sqrt{5} \end{pmatrix}$$

and

$$\lambda_2 = \frac{-3 - \sqrt{5}}{2} \qquad \text{with} \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 + \sqrt{5} \end{pmatrix}.$$

Both these eigenvalues are negative, and so this point is a stable node. Trajectories approach tangent to the eigenvector $\mathbf{v}_1$, and the phase portrait is shown in Figure 35.2.

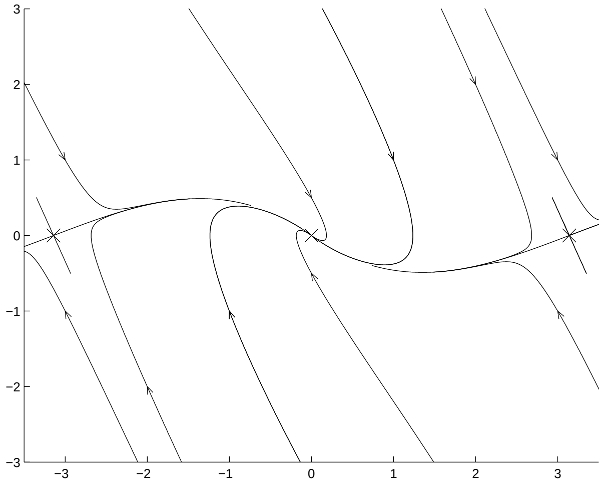

Fig. 35.1. The phase portrait for the damped pendulum when $k = 2$.

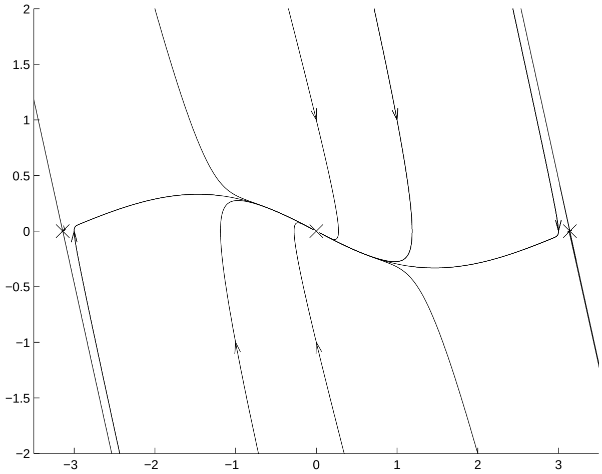

Fig. 35.2. The phase portrait for the damped pendulum when $k = 3$.

Exercise 35.2 *Consider the equation for a pendulum with a quadratic damping term*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x - ky|y|. \end{aligned}$$

Show that if $E = \frac{1}{2}y^2 - \cos x$ *then*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -ky^2|y|. \tag{S35.1}$$

Show that the point $(0,0)$ *is a centre for the linearised equation, but using (S35.1) deduce that for the nonlinear equation it behaves like a stable spiral, and hence draw the phase diagram. (Remember that 'linearised centres' do not have to be centres for the nonlinear equation.)*

We consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x - ky|y|. \end{aligned}$$

Defining $E = \frac{1}{2}y^2 - \cos x$, the total energy, we have

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= y\dot{y} + \sin x\dot{x} \\ &= -y\sin x - ky^2|y| + y\sin x \\ &= -ky^2|y|. \end{aligned}$$

It follows that this energy decreases while $y \neq 0$, i.e. while the particle is moving.

The matrix of partial derivatives is

$$\mathrm{D}\mathbf{f}(x,y) = \begin{pmatrix} 0 & 1 \\ -\cos x & -2k|y| \end{pmatrix},$$

so at the origin the linearisation is

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\boldsymbol{\xi},$$

giving a centre. So solutions will certainly circle around the origin.

However, near the origin we have

$$E \approx \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 - 1 + \ldots$$

(using the series expansion $\cos x \approx 1 - \frac{1}{2}x^2$, valid for small x). Since E decreases it follows that the distance from the origin, $\sqrt{x^2+y^2}$, decreases, and so the origin behaves like a stable spiral. The phase portrait is shown in Figure 35.3.

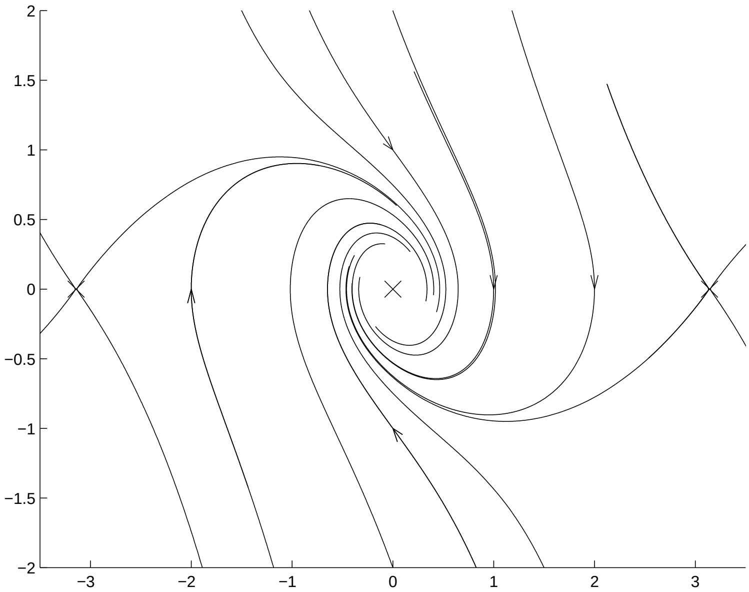

Fig. 35.3. The phase portrait for the nonlinearly damped pendulum.

36

Periodic orbits

Exercise 36.1 *Use Dulac's criterion to show that periodic orbits in the equations*

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -ky - V'(x)\end{aligned}$$

are only possible if $k = 0$.

If we write

$$\mathbf{f}(x, y) = (f(x, y),\ g(x, y)) = (y,\ -ky - V'(x))$$

then we have

$$\nabla \cdot \mathbf{f} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -k.$$

Therefore we can apply Dulac's criterion with $h = 1$, and deduce that there can be no periodic orbits in any region if $k \neq 0$.

Exercise 36.2 *Consider the coupled system*[1]

$$\begin{aligned}\dot{x} &= y + \tfrac{1}{4}x(1 - 2r^2) \\ \dot{y} &= -x + \tfrac{1}{2}y(1 - r^2),\end{aligned}$$

where $r^2 = x^2 + y^2$. *First, show that the system has only one stationary point which lies at the origin. Now, by finding the equation satisfied by* r, *show that trajectories enter (and do not leave) the region* D, *where*

$$D = \{(x, y) :\ \tfrac{1}{2} \leq r^2 \leq 1\}.$$

[1] This example is taken from P.A. Glendinning, *Stability, instability, and chaos* (Cambridge University Press, 1994)

Use the Poincaré-Bendixson Theorem to deduce that the system has a periodic orbit lying within D.

The stationary points occur when $\dot{x} = \dot{y} = 0$, i.e. when

$$y + \frac{x}{4}(1 - 2r^2) = 0 \qquad \text{and} \qquad x = \tfrac{1}{2}y(1 - r^2).$$

Substituting for x in the first equation we obtain

$$y\left[1 + \frac{(1 - r^2)(1 - 2r^2)}{8}\right] = 0.$$

If $y = 0$ then it follows from the equation for $\dot{y}$ that $x = 0$; so we now consider the term in the square brackets. For this to be zero we require

$$(1 - r^2)(1 - 2r^2) + 8 = 0 \qquad \Rightarrow \qquad 2r^4 - 3r^2 + 9 = 0.$$

This has a real solution for r^2 if the discriminant $(b^2 - 4ac)$ is positive. But $b^2 - 4ac = 9 - 72 < 0$, so the equation has no real solution. Thus there is only one stationary point, which is at the origin.

We can find the equation for $\dot{r}$ simply. From the identity $r^2 = x^2 + y^2$ we obtain

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= xy + \frac{1}{4}x^2(1 - 2r^2) - xy + \frac{y^2}{2}(1 - r^2) \\ &= \frac{r^2}{4} + \frac{y^2}{4} - \frac{r^4}{2} \\ &= \frac{r^2}{4}(1 + \sin^2\theta) - \frac{r^4}{2}, \end{aligned}$$

since in polar coordinates $y = r\sin\theta$.

It follows that if $r^2 < 1/2$ then $\dot{r} > 0$, while if $r^2 > 1$ we must have $\dot{r} < 0$. Thus on trajectories r increases if $r^2 < 1/2$, and decreases if $r > 1$. Therefore trajectories will enter, and cannot leave the region D. Since D contains no stationary points (the only such point is the origin), the Poincaré-Bendixson Theorem can be applied to deduce the existence of a periodic orbit within D.

37

The Lorenz equations

There are no exercises for this chapter.

38

What next?

This sequence of exercises treats the problem of the vibrating string using the method of separation of variables. This produces a simple boundary value problem, and serves to introduce the idea of Fourier series.

The equation for the vibrations of a string stretched between $x = 0$ *and* $x = 1$ *and attached at both endpoints is*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{S38.1}$$

with $u(x,t)$ *representing the height of the string at position* x *at time* t*. Since the string is fixed at the endpoints, we should have* $u(0,t) = u(1,t) = 0$ *for all* t*. See Figure 38.1.*

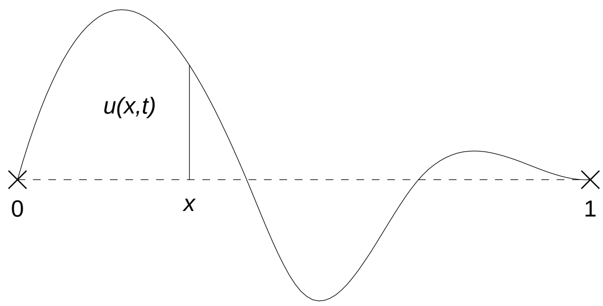

Fig. 38.1. A vibrating string, fixed at the endpoints $x = 0$ and $x = 1$.

Exercise 38.1 *Show that the principle of superposition is valid: if two functions* $u_1(x,t)$ *and* $u_2(x,t)$ *satisfy the equation and the boundary conditions,* *then $u(x,t) = \alpha u_1(x,t) + \beta u_2(x,t)$ also satisfies both the equation and the boundary conditions.*

Suppose that

$$\frac{\partial^2 u_1}{\partial t^2} = c^2 \frac{\partial^2 u_1}{\partial x^2} \qquad \text{with} \qquad u_1(0,t) = u_1(1,t) = 0$$

and

$$\frac{\partial^2 u_2}{\partial t^2} = c^2 \frac{\partial^2 u_2}{\partial x^2} \qquad \text{with} \qquad u_2(0,t) = u_2(1,t) = 0.$$

Then if $u(x,t) = \alpha u_1(x,t) + \beta u_2(x,t)$ we have

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \alpha \frac{\partial^2 u_1}{\partial t^2} + \beta \frac{\partial^2 u_2}{\partial t^2} \\ &= \alpha c^2 \frac{\partial^2 u_1}{\partial x^2} + \beta c^2 \frac{\partial^2 u_2}{\partial x^2} \\ &= c^2 \frac{\partial^2 u}{\partial x^2}, \end{aligned}$$

while clearly

$$u(0,t) = \alpha u_1(0,t) + \beta u_2(0,t)$$

and similarly for $u(1,t)$. So $u(x,t)$ is also a solution of the equation satisfying the boundary conditions.

Exercise 38.2 *Show that if we guess that a solution has the form $u(x,t) = X(x)T(t)$ then $X(x)$ and $T(t)$ must satisfy*

$$\frac{1}{c^2 T} \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{1}{X} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}.$$

Now we try $u(x,t) = X(x)T(t)$ in the equation. This yields

$$X \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = c^2 T \frac{\mathrm{d}^2 X}{\mathrm{d}x^2},$$

or equivalently

$$\frac{1}{c^2 T} \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{1}{X} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}.$$

□

Since the left-hand side is a function of t alone, and the right-hand side is a function of x alone, the only way that they can be equal is if they are both constant. If we choose this constant to be $-\lambda$ then we obtain two equations:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -\lambda c^2 T \qquad \text{(S38.2)}$$

and the boundary value problem

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\lambda X \qquad \text{with} \qquad X(0) = X(1) = 0. \qquad \text{(S38.3)}$$

Exercise 38.3 *Show that if $\lambda \le 0$ then the only solution of (S38.3) is $X(x) = 0$ for all $x \in [0,1]$. (You can find the general solution using the methods you have learned in this book, and then choose the constants in order to satisfy the boundary conditions.)*

In order to find the general solution of the second order differential equation

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\lambda X$$

we try $X(x) = \mathrm{e}^{kx}$, and obtain the auxiliary equation $k^2 = -\lambda$. Since $\lambda < 0$ this gives two real roots, $\pm k = \pm\sqrt{|\lambda|}$, so the general solution is

$$X(x) = A\mathrm{e}^{kx} + B\mathrm{e}^{-kx}.$$

The boundary conditions require

$$X(0) = A + B = 0 \qquad \text{and} \qquad X(1) = A\mathrm{e}^{k} + B\mathrm{e}^{-k} = 0,$$

which implies that $A = B = 0$, and so $X(x) = 0$.

Exercise 38.4 *Show that if $\lambda > 0$ then we only have $X(x) \neq 0$ if we choose $\lambda = n^2\pi^2$ for some integer n, and then*

$$X(x) = AX_n(x), \qquad \text{where} \qquad X_n(x) = \sin n\pi x$$

and A is an arbitrary constant.

Now the general solution of the second order equation $X'' = -\lambda X$ is

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

In order to satisfy the boundary condition $X(0) = 0$ we must have $A = 0$; then to satisfy the boundary condition at $x = 1$ we need

$$B\sin\sqrt{\lambda} = 0.$$

Choosing $B = 0$ just gives $X(x) = 0$; but we can also satisfy this boundary condition by choosing $\sqrt{\lambda} = n\pi$ for some integer n. So if we take $\lambda = n^2\pi^2$ we have the solution

$$X(x) = AX_n(x) \qquad \text{with} \qquad X_n(x) = \sin n\pi x.$$

□

The values $\lambda_n = n^2\pi^2$*, and the corresponding solutions* $X_n(x)$*, are known as the eigenvalues and eigenfunctions for the problem*

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\lambda X \qquad \text{with} \qquad X(0) = X(1) = 0. \tag{S38.4}$$

Exercise 38.5 *By requiring the solution of (S38.4) to be non-zero we have restricted the possible values of* λ *to the eigenvalues* $\lambda_n = n^2\pi^2$*. Find the solution of (S38.2) when* $\lambda = \lambda_n$*, and hence show that once solution of (S38.1) is*

$$u(x,t) = (A\sin n\pi ct + B\cos n\pi ct)\sin n\pi x. \tag{S38.5}$$

Use the principle of superposition to show that

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \sin n\pi ct + B_n \cos n\pi ct)\sin n\pi x \tag{S38.6}$$

solves (S38.1) for any choice of coefficients A_n *and* B_n*.*

When $\lambda = n^2\pi^2$, the equation (S38.2) for $T(t)$ becomes

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -n^2\pi^2 c^2 T.$$

This has the general solution

$$T(t) = A\sin n\pi ct + B\cos n\pi ct,$$

and since we originally put $u(x,t) = X(x)T(t)$ this gives us the solution

$$u(x,t) = (A\sin n\pi ct + B\cos n\pi ct)\sin n\pi x.$$

The principle of superposition means that we can add up linear combinations of individual solutions. Assuming that this works for an infinite number of solutions, we obtain a solution in the form of a series,

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \sin n\pi ct + B_n \cos n\pi ct)\sin n\pi x. \tag{S38.7}$$

Exercise 38.6 *Assuming that any solution of (S38.1) can be written in the form (S38.6), the problem becomes to determine the coefficients* A_n *and*

B_n. Show that if the initial position and velocity of the string, $u(x,0)$ and $\partial u/\partial t(x,0)$, are given then A_n and B_n must satisfy

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin n\pi x \qquad \text{(S38.8)}$$

and

$$\frac{\partial u}{\partial t}(x,0)=\sum_{n=1}^{\infty}n\pi c\,A_n\sin n\pi x.$$

If the solution is given in the form (S38.7) then we have

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin n\pi x,$$

and since, assuming that we can differentiate the series term-by-term, we have

$$\frac{\partial u}{\partial t}(x,t)=\sum_{n=1}^{\infty}(n\pi cA_n\cos n\pi ct-n\pi cB_n\sin n\pi ct)\sin n\pi x,$$

it follows that

$$\frac{\partial u}{\partial t}(x,0)=\sum_{n=1}^{\infty}n\pi cA_n\sin n\pi x.$$

An expansion of a function $f(x)$ as a sum of sine functions,

$$f(x)=\sum_{n=1}^{\infty}c_n\sin n\pi x \qquad \text{(S38.9)}$$

is known as a Fourier series expansion of f. It is one of the wonders of mathematics that any reasonably smooth function f that has $f(0)=f(1)=0$ can be expanded in such a series. (If we also include cosine functions then we can remove the restrictions at the endpoints.) Finding the coefficients c_n is also relatively straightforward, at least in principle.

Exercise 38.7 *Check that*

$$\int_0^1\sin n\pi x\,\sin m\pi x\,\mathrm{d}x=\begin{cases}0 & n\neq m\\ \frac{1}{2} & n=m.\end{cases}$$

('The functions $\sin n\pi x$ and $\sin m\pi x$ are orthogonal on $[0,1]$'.)

If $n\neq m$ then, using the double angle formula

$$2\sin a\sin b=\cos(a-b)-\cos(a+b)$$

we have

$$\begin{aligned}\int_0^1 \sin n\pi x \sin m\pi x \, \mathrm{d}x &= \tfrac{1}{2}\textstyle\int_0^1 \cos(n-m)\pi x - \cos(n+m)\pi x \, \mathrm{d}x \\ &= \tfrac{1}{2}\left[\tfrac{1}{(n-m)\pi}\sin(n-m)\pi x - \tfrac{1}{(n+m)\pi}\sin(n+m)\pi x\right]_0^1 \\ &= 0,\end{aligned}$$

while if $n = m$ we have

$$2\sin^2 a = 1 - \cos 2a,$$

and so

$$\begin{aligned}\int_0^1 \sin^2 n\pi x \, \mathrm{d}x &= \tfrac{1}{2}\textstyle\int_0^1 1 - \cos 2n\pi x \, \mathrm{d}x \\ &= \tfrac{1}{2}.\end{aligned}$$

Exercise 38.8 *Multiply both sides of (S38.9) by* $\sin m\pi x$ *and, assuming that it is possible to integrate the series term-by-term, show that the coefficient* c_m *is given by*

$$c_m = 2\int_0^1 f(x)\,\sin m\pi x \, \mathrm{d}x.$$

If we multiply both sides of (S38.9) by $\sin m\pi x$ and integrate we obtain

$$\int_0^1 f(x)\sin m\pi x \, \mathrm{d}x = \int \sin m\pi x \sum_{n=1}^{\infty} c_n \sin n\pi x \, \mathrm{d}x.$$

Assuming that we can change the order of the sum and integral on the right-hand side we have

$$\begin{aligned}\int_0^1 f(x)\sin m\pi x \, \mathrm{d}x &= \sum_{n=1}^{\infty}\left(\int_0^1 \sin m\pi x \, c_n \sin n\pi x \, \mathrm{d}x\right) \\ &= \tfrac{1}{2}c_m,\end{aligned}$$

since all the integrals vanish except when $n = m$. Therefore the coefficient c_m is given by the integral

$$c_m = 2\int_0^1 f(x)\sin m\pi x \, \mathrm{d}x.$$

Sturm-Liouville theory treats the more general eigenvalue problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = \lambda w(x)y \qquad \text{with} \qquad y(a) = y(b) = 0$$

(in equation (S38.3) we had $p(x) = w(x) = 1$, $q(x) = 0$, $a = 0$, and $b = 1$). There are, again, an infinite set of eigenvalues λ_n for which there is a corresponding non-zero eigenfunction $y_n(x)$. The eigenfunctions are 'orthonormal on $[a, b]$ with respect to the weight function $w(x)$',

$$\int_a^b y_n(x)y_m(x)w(x)\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases} \tag{S38.10}$$

Furthermore, any function $f(x)$ satisfying the boundary conditions can be expanded as a generalised Fourier series using the eigenfunctions $y_n(x)$,

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x). \tag{S38.11}$$

Exercise 38.9 *Using the orthonormality relation in (S38.10) show that the coefficients c_m in (S38.11) are given by*

$$c_m = \int_a^b f(x)y_m(x)w(x)\,\mathrm{d}x.$$

Now we assume that

$$\int_a^b y_n(x)y_m(x)w(x)\,\mathrm{d}x = \begin{cases} 0 & n \neq m \\ 1 & n = m. \end{cases}$$

If we try to write

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

then to find the coefficients we multiply both sides by $y_m(x)w(x)$ and integrate between 0 and 1. Feeling free to change the order of summation and integral we obtain

$$\begin{aligned} \int_0^1 f(x)y_m(x)w(x)\,\mathrm{d}x &= \int_0^1 y_m(x)w(x)\sum_{n=1}^{\infty} c_n y_n(x)\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} c_n \left(\int_0^1 y_m y_n(x)w(x)\,\mathrm{d}x\right) \\ &= c_m, \end{aligned}$$

and so

$$c_m = \int_0^1 f(x)y_m(x)w(x)\,\mathrm{d}x.$$