Depth, Stanley Depth and Regularity of Edge Ideals Associated with Some Graphs

By

Ahtsham ul Haq

Supervised by

Dr. Muhammad Ishaq

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan
2020

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Mr Ahtsham Ul Haq, Regn No. 00000278444 Titled: "Depth, Stanley Depth and Regularity of Edge Ideals Associated with Some Graphs" be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: Dr. Mujeeb ur Rehman Signature: ____________

2. Name: Dr. Muhammad Qasim Signature: ____________

External Examiner: Dr. M. Ahsan Binyamin Signature: ____________

Supervisor's Name: Dr. Muhammad Ishaq Signature: ____________

____
Head of Department

31/12/2020
Date

COUNTERSINGED

Date: 31-12-2020

____
Dean/Principal

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Ahtsham Ul Haq** (Registration No. **00000278444**), of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ______________________

Name of Supervisor: Dr. Muhammad Ishaq

Date: 31/12/2020

Signature (HoD): ______________________

Date: 31/12/2020

Signature (Dean/Principal): ______________________

Date: 31-12-2020

I dedicate this thesis to my loving parents, venerable supervisor, respectable teachers and fellows for their limitless support and encouragement.

Acknowledgement

All praises and glory is to Allah Almighty, the most beneficent and the most merciful, who has created the whole universe. I stand highly grateful and obliged to Him for His showering infinite blessings upon me and bestowing me with the ability and strength to finish this thesis successfully and within time and also blessing me evermore than I actually deserved.

I feel greatly indebted to my venerable supervisor Dr. Muhammad Ishaq, for his never ending support and his honest guidance during my research. Had it not been for his knowledge, expertise and assistance, this work would not have been possible. Moreover, my entire understanding and appreciation of this field are entirely are mainly because of his efforts and positive response to my queries. I would also like to extend my thanks to the Principal, the Head of Department and the entire faculty of the department of mathematics for their kind help during my research work.

My stay at NUST was made highly memorable and enjoyable by my class fellows and friends. I would like to offer special appreciation to Muhammad Usman Rashid, Muhammad Burhan Jafeer and Bakhtawar Shaukat for their invaluable help in completion of both this thesis and also the course work. I would always cherish their suggestions and honest friendship. How can I forget mentioning my classmates Naeem u Din, Naeem ur Rehman, Ahmer Qureshi, Zaheer Abbas, Hifsa Ismail and Mehwish Aslam who where always there when i needed them. Of course, everyone in my class has helped to provide me encouragement along with a friendly environment.

Lastly, I would like to acknowledge the support of my loving family and I would like to extend huge thanks to them for supporting me financially and morally all the way through my studies. Without any doubt, these words cannot fully express how grateful I am for all of the sacrifices that they have made for me. Without their dedicated prayers and support, I would have never been able to reach my destination. I am thankful to all those who have helped me directly or indirectly to complete my research work.

Ahtsham ul Haq

Abstract

This dissertation addresses the algebraic invariants such as depth, Stanley depth, projective dimension and regularity of some graphs. For this purpose, we investigate certain types of monomial ideals of polynomial rings over the fields. Primarily, our interest is to compute Stanley depth and depth of the edge ideals (or equivalently the quotient of the polynomial ring by the monomial ideals) as module over the polynomial ring. Indeed, this work is restricted to those ideals which are generated by square free monomials of degree 2. The geometrical interpretation of such an ideal is the underlying graph. This provides a bridge between algebraic objects and combinatorial objects, that is, the monomial ideals and the graphs. From this correspondence, one can define the algebraic invariants, Stanley depth and depth of graphs. The idea of projective dimension is then explored through Auslander-Buchsbaum formula by using the former concept of depth. Lastly, the regularity of underlying graph is computed by considering minimal free resolutions of modules. In literature, some bounds and exact values of depth and Stanley depth for the edge ideal associated with standard strong product of graphs are given. This research is conducted to address such bounds of edge ideals associated with restricted partial strong product, ladder and cubic circulant graphs. Furthermore, we define a new family of circulant graphs and explore the algebraic concepts of projective dimension and regularity by analyzing their bounds.

Contents

List of Figures

Introduction

Monomial ideals play a crucial role in analyzing the relation between combinatorics and commutative algebra. Generally, the combinatorics' problems are translated as monomial ideals and are solved using methods and techniques used in commutative algebra. The upper bound conjecture proved by Stanley (for simplicial spheres) is marked as the earliest attempt of exploiting this link between aforementioned fields. He used square-free monomial ideals to interlink these two streams (combinatorics and commutative algebra) of mathematics. In the year 1982, Stanley [1] introduced an invariant for finitely generated $\mathbb{Z}^n$-graded modules over the commutative ring, known as Stanley depth. He also provided a conjecture linking the Stanley depth and depth of a module. Later on it was proved by Duval et al. [2] in year 2015 that Stanley's conjecture generally does not hold for P/E type modules, where P is defined as a ring of polynomials with n variables and E is a monomial ideal. Nevertheless, discovery for classes that still fulfill the conjecture is still a challenging phenomena.
In this thesis, some improved lower and upper bounds of Stanley depth and depth are computed for edge ideals associated to the restricted partial strong product of path and cycles. Moreover, it highlights the bounds of regularity, depth and Stanley depth of edge ideal corresponding to the ladder graph and some families of circulant graphs. This thesis consist of six chapters

Chapter 1 includes the fundamental concepts, basic definitions and results from abstract algebra and commutative algebra. It starts with a brief overview of algebra which is followed later by definitions and examples of ring theory. Moreover, it includes types, basic properties, standard operations and primary decompositions of edge ideal. Furthermore, it encompasses some basic of module theory including the generation of modules, exact sequences and graded rings. Toward the end, the chapter enlists simplicial complexes and Stanley-Reisner ideal with some basic examples.

In Chapter 2, a brief introduction of graph theory and fundamental products of graphs (cartesian product, standard strong product, partial and restricted partial strong product) is presented. In addition to that, the chapter also includes some basics of circulants. The chapter concludes with cubic circulants and its previously published results.

Commencing with the introduction of depth and regular sequences, Chapter 3 is comprised of Stanley decomposition, Stanley depth of modules and well known Stanley's conjecture. Additionally, it entails a detailed introduction to compute the Stanley depth for square free monomial ideals. It also includes a concise introduction to Castelnuovo-Mumford regularity of an ideal. Finally, the chapter concludes with some worked out examples of edge ideal and regularity (using graded minimal free resolution of edge ideal).

In Chapter 4, the edge ideal associated to partial strong product of some graphs are introduced and their Stanley depth and depth are computed by using the principle of mathematical induction and depth lemma on short exact sequences.

In Chapter 5, the edge ideal associated to a specific family of circulant graphs are presented and their bounds of regularity are computed by using some well known results.

The edge ideals associated with ladder and cubic circulant graphs are introduced in Chapter 6 and their Stanley depth and depth are determined by making use of principle of mathematical induction and depth lemma on short exact sequences.

Chapter 1

Ring theory and module theory

1.1 Introduction

The branch of mathematics which deals with algebraic structures called *rings* is known as ring theory. For instance, the structure properties of rings and polynomials are explored as rings. Modern ring theory is known to be a very active mathematical discipline that endeavors to study rings in their own right. Ring theory is basically classified into two sections: (1) Commutative ring theory and (2) Non-commutative ring theory. The commutative algebra generally deals with ideas and problems occurring naturally in algebraic number theory and algebraic geometry. Polynomial rings, fields, ring of integers of a number field and coordinate rings of an affine algebraic variety are a few important examples of commutative rings. Likewise, the corresponding theory for non-commutative ring takes examples from non-commutative division ring, representation theory, functional analysis and so on.

The primary concept of a ring first came to surface from an early attempt to prove Fermat's last theorem, which could be traced back to Richard Dedekind [3]. Adolf Fraenkel [4] gave the first axiomatic definition of ring, however, his axioms were more rigorous than those in the most recent definition. After certain attempts from different fields, primarily number theory, the generalized and modern notion of ring (commutative) was established by Emmy Noether and Wolfgang Krull [5].

Recently, the idea of associating a graph with a specific algebraic structure and exploring the interactions between the structure of the algebraic objects and the graph theoretic properties of the graphs connected with them is an absorbing and active area of research. The idea of associating a graph to a commutative ring was initiated by I. Beck in [6].

1.2 Ring theory

In the realm of algebra, the algebraic structures are dealt under the flag of ring theory, which have defined operations of multiplication and addition.

Definition 1.2.1. A non empty set $\mathcal{R}$ alongside the well defined operations " $+$ " and " $\times$ " forms a ring if it fulfills the following axioms:

- $\mathcal{R}$ under " $+$ " is an abelian group.
- Associative law under " $\times$ " holds in $\mathcal{R}$.
- Distributive laws (left and right) holds in $\mathcal{R}$.

If a ring $\mathcal{R}$ is commutative w.r.t multiplication, then it is called a commutative ring. The ring $\mathcal{R}$ is known to have an identity $1 \in \mathcal{R}$ if $\forall\ r \in \mathcal{R}$

$$r \times 1 = 1 \times r = r.$$

Example 1.2.2. Following are a few instances of ring.

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are examples of commutative rings having identity 1.
2. $\mathbb{Z}/n\mathbb{Z}$ with multiplicative identity 1 under multiplication and addition of residue classes, forms a commutative ring.
3. Suppose $\mathcal{R} = \mathbb{R}^3$, then $\mathcal{R}$ is a non commutative ring without unity, where the operation of addition to be the usual addition of vectors and multiplication is the cross product of vectors.

1.2.1 Ring of polynomials

The polynomial ring is a particular type of ring which is formed by a set of polynomials. These polynomials are in one or more than one variable where the coefficients belong to a ring or maybe a field. Polynomial rings are used in several fields of mathematics and the investigation of their properties is among the primary inspirations for the advancement of Commutative Algebra and Ring theory.

Definition 1.2.3. Let $\mathcal{R}$ be a commutative ring having unity, a polynomial in variable x has the form

$$r_0 + r_1 x + \cdots + r_{n-1} x^{n-1} + r_n x^n,$$

with $n \in \mathbb{Z}^+ \cup \{0\}$ and every $r_i \in \mathcal{R}$. The polynomial is of degree n if $r_n \neq 0$. Such a set of polynomials is denoted by

$$\mathcal{R}[x] = \{r_0 + r_1 x + \cdots + r_{n-1} x^{n-1} + r_n x^n : n \in \mathbb{Z}^+ \cup \{0\}, r_i \in \mathcal{R}\}.$$

$\mathcal{R}[x]$ is a commutative ring with unity under polynomial addition and polynomial multiplication and the unity of $\mathcal{R}[x]$ is the unity of coefficient $\mathcal{R}$.

Definition 1.2.4. The polynomial ring in the variables $y_1, y_2, \ldots, y_n$ and coefficients belonging to $\mathcal{R}$ (commutative with identity) is defined inductively

$$\mathcal{R}[y_1, y_2, \ldots, y_n] = \mathcal{R}[y_1, y_2, \ldots, y_{n-1}][y_n].$$

A ring homomorphism is a map from one ring to another that respects the same additive and multiplicative structures.

Definition 1.2.5. Consider two rings R_1 and R_2. A ring homomorphism is a map $\mathcal{H} : R_1 \to R_2$ which satisfies the following axioms for all $r_1, r_2 \in R_1$

- $\mathcal{H}(r_1 + r_2) = \mathcal{H}(r_1) + \mathcal{H}(r_2)$,
- $\mathcal{H}(r_1 r_2) = \mathcal{H}(r_1)\mathcal{H}(r_2)$,

A ring homomorphism which is both injective and surjective is known as ring isomorphism.

1.2.2 Standard operations and properties of ideals

Proposition 1.2.6. *A non-empty subset I of a ring $\mathcal{R}$ is known to be an ideal if and only if $s_1 - s_2 \in I$, $sr \in I$ and $rs \in I$ for all $s_1, s_2, s \in I$ and $r \in \mathcal{R}$.*

Definition 1.2.7. For a proper ideal I, a quotient ring $\mathcal{R}/I$ can be formed, which consists of cosets $r + I$, where $r \in \mathcal{R}$, and the product of cosets is defined as:

$$(r_1 + I)(r_2 + I) = r_1 r_2 + I.$$

Next there are the isomorphism theorems for rings.

Theorem 1.2.8. *(Isomorphism Theorems)*

1. *For a ring homomorphism $\pi : R_1 \to R_2$, $\pi(R_1)$ is isomorphic to $R_1/ker(\pi)$, i.e.,*

$$R_1/ker(\pi) \cong \pi(R_1).$$

2. *Consider the ideals I_1 and I_2 of ring R_1, with $I_1 \subseteq I_2$, then I_2/I_1 is an ideal of R_1/I_1. Also*

$$(R_1/I_1)/(I_2/I_1) \cong R_1/I_2.$$

For ideals I and K of the ring $\mathcal{R}$, the set of sums $a + b$ with $a \in I$, $b \in K$ is not only a subring of $\mathcal{R}$ but also is an ideal in $\mathcal{R}$ (the set is clearly closed under addition and $\alpha(a + b) = \alpha a + \alpha b \in I + K$ since $\alpha a \in I$ and $\alpha b \in K$).

Definition 1.2.9. Assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ be the ideals of ring $\mathcal{R}$. Product of two ideals, say $\mathcal{I}_1$ and $\mathcal{I}_2$, is a set consisting of all possible finite sums of the form $s_1 s_2$, where $s_1 \in \mathcal{I}_1$ and $s_2 \in \mathcal{I}_2$. It is represented by $\mathcal{I}_1\mathcal{I}_2$.

Example 1.2.10. Let $I_1 = 14\mathbb{Z}$ and $I_2 = 21\mathbb{Z}$ in $\mathbb{Z}$. Then $I_1 + I_2$ comprises all integers of the form $14s_1 + 21s_2$ with $s_1, s_2 \in \mathbb{Z}$. For each such type of integer is divisible by 7, so $14\mathbb{Z} + 21\mathbb{Z} \subseteq 7\mathbb{Z}$. On the other hand, $7 = 14(-1) + 21(1)$ shows that $7\mathbb{Z}$ is contained in $14\mathbb{Z} + 21\mathbb{Z}$, hence $14\mathbb{Z} + 21\mathbb{Z} = 7\mathbb{Z}$. In general, $p_1\mathbb{Z} + p_2\mathbb{Z} = d\mathbb{Z}$, whereas $d = (p_1, p_2)$. The product $I_1 I_2$ comprises all possible finite sums of the components of the form $(14s_1)(21s_2)$ where $s_1, s_2 \in \mathbb{Z}$, which clearly gives the ideal $294\mathbb{Z}$.

Definition 1.2.11. For a ring $\mathcal{R}$, principal ideal is an ideal with a single element in its generating set. Finitely generated ideal is an ideal with a finite elements in its generating set.

Definition 1.2.12. A maximal ideal $\mathcal{M}$ in a ring $\mathcal{R}$ is a proper ideal such that there is no proper ideal in between $\mathcal{M}$ and $\mathcal{R}$.
In other words, if $\mathcal{J}$ is an ideal contains $\mathcal{M}$, then either $\mathcal{M} = \mathcal{J}$ or $\mathcal{J} = \mathcal{R}$.

Definition 1.2.13. Local ring is a ring $\mathcal{R}$ with unique maximal ideal.

Example 1.2.14. Ideal generated by $(2) = \{0, 2, 4, 6\}$ is the maximal ideal in $\mathbb{Z}_8$. (2) is also the unique maximal ideal in $\mathbb{Z}_8$. So $\mathbb{Z}_8$ is a local ring.

Definition 1.2.15. A prime ideal $\mathcal{P}$ is a proper ideal of a ring $\mathcal{R}$ such that if for $p_1, p_2 \in \mathcal{R}$, $p_1 p_2 \in \mathcal{P}$, then either $p_1 \in \mathcal{P}$ or $p_2 \in \mathcal{P}$.

Definition 1.2.16. For a ring $\mathcal{R}$, let us suppose two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$. Then their ideal quotient is defined as

$$(\mathcal{I}_1 : \mathcal{I}_2) = \{s \in \mathcal{R} \ : \ s\mathcal{I}_2 \subseteq \mathcal{I}_1\}.$$

Definition 1.2.17. Let $\mathcal{R}$ be a ring and $\mathcal{I}$ is an ideal of $\mathcal{R}$. Then $(0 : \mathcal{I})$ is an ideal known as the annihilator of $\mathcal{I}$ represented as $Ann(\mathcal{I})$ defined as

$$Ann(\mathcal{I}) = \{r \in \mathcal{R} : r\mathcal{I} = 0\}.$$

Definition 1.2.18. An ideal $\mathcal{N}$ of $\mathcal{R}$ is primary ideal if $s_1 s_2 \in \mathcal{N}$, for $s_1, s_2 \in \mathcal{R}$, then either $s_1 \in \mathcal{N}$ or $s_2^k \in \mathcal{N}$ for some $k \geq 1$.

When $\mathcal{N}$ is a primary ideal, $\mathcal{P}$ is a prime ideal and also $\mathcal{P} = \sqrt{\mathcal{N}}$, then $\mathcal{N}$ is called $\mathcal{P}$-primary.

1.2.3 Monomial ideal

Let $S = K[x_1, \ldots, x_n]$ be a ring over field K, monomials forms the natural K-basis for S. Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ where every $b_j \geq 0$. A monomial is any product of the form $x_1^{b_1} \dots x_n^{b_n}$ with $b_j \in \mathbb{Z}_+$. If $w = x_1^{b_1} \dots x_n^{b_n}$ is a monomial, then we write $w = x^{\mathbf{b}}$ with $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}_+^n$, and

$$x^{\mathbf{b_1}} x^{\mathbf{b_2}} = x^{\mathbf{b_1}+\mathbf{b_2}}.$$

An ideal whose generating set only consists of monomials is said to be a monomial ideal. Mon(S) denotes the set of all monomials in S and it forms the basis of S. For any polynomial $f \in S$ and for $b_w \in K$

$$f = \sum_{w \in Mon(S)} b_w w,$$

where support of f is defined as

$$\text{supp}(f) = \{w \in Mon(S) \,:\, b_w \neq 0\}.$$

Proposition 1.2.19. *Consider two monomial ideals I_1 and I_2. Then*

1. *$I_1 \cap I_2$ is a monomial ideal, and $\{lcm(p,q) \;:\; p \in G(I_1)\,,\, q \in G(I_2)\}$ is the generating set of $I_1 \cap I_2$.*
2. *$(I_1 : I_2)$ is a monomial ideal and $(I_1 : I_2) = \bigcap_{q \in G(I_2)} (I_1 : (q))$.*

A monomial $x^{\mathbf{b}}$ is said to be squarefree if $\mathbf{b}$ has components 0 and 1. An ideal with a generating set containing only squarefree monomials is known as squarefree monomial ideal.

1.2.4 Primary decomposition

For an ideal J, primary decomposition is a way of representing J as an intersection $J = \bigcap_{m=1}^{n} N_m$, whereas each N_m is a primary ideal. Let $\{P_m\} = \text{Ass}(N_m)$. If none of the N_m can be omitted in this intersection and $P_r \neq P_s$ for all $r \neq s$ then it is called irredundant primary decomposition.

Example 1.2.20. Let $I = (\, y_1^2 y_3\, ,\, y_4^3\, ,\, y_2^4 y_4^2\, ,\, y_1 y_2 y_3^3\,)$, then

$$\begin{aligned} I &= (y_1^2\, ,\, y_4^3\, ,\, y_2^4 y_4^2\, ,\, y_1 y_2 y_3^3\,) \cap (y_3\, ,\, y_4^3\, ,\, y_2^4 y_4^2\, ,\, y_1 y_2 y_3^3\,) \\ &= (y_1^2\, ,\, y_4^3\, ,\, y_2^4 y_4^2\, ,\, y_1 y_2 y_3^3\,) \cap (y_3\, ,\, y_4^3\, ,\, y_2^4 y_4^2\,) \\ &= (y_1^2\, ,\, y_4^3\, ,\, y_2^4\, ,\, y_1 y_2 y_3^3\,) \cap (y_1^2\, ,\, y_4^3\, ,\, y_4^2\, ,\, y_1 y_2 y_3^3\,) \cap (y_3\, ,\, y_4^3\, ,\, y_2^4\,) \cap (y_3\, ,\, y_4^3\, ,\, y_4^2\,) \\ &= (y_1^2\, ,\, y_4^3\, ,\, y_2^4\, ,\, y_1 y_2 y_3^3\,) \cap (y_1^2\, ,\, y_4^2\, ,\, y_1 y_2 y_3^3\,) \cap (y_3\, ,\, y_4^2\, ,\, y_2^4\,) \cap (y_3\, ,\, y_4^2\,). \end{aligned}$$

In the above example, the obtained primary decomposition is irredundant as $P_r \neq P_s$ for $1 \leq r, s \leq 4$. But generally it does not happen, as in the following example.

Example 1.2.21. Let $I = (\, y_2^4\, ,\, y_3^4\, ,\, y_2^3 y_4^3\, ,\, y_2 y_3 y_4^3\, ,\, y_3^3 y_4^3\,)$, then

$$\begin{aligned} I &= (\, y_2^4\, ,\, y_3^4\, ,\, y_2^3\, ,\, y_2 y_3 y_4^3\, ,\, y_3^3 y_4^3\,) \cap (\, y_2^4\, ,\, y_3^4\, ,\, y_4^3\, ,\, y_2 y_3 y_4^3\, ,\, y_3^3 y_4^3\,) \\ &= (\, y_2^3\, ,\, y_3^4\, ,\, y_2 y_3 y_4^3\, ,\, y_3^3 y_4^3\,) \cap (\, y_2^4\, ,\, y_3^4\, ,\, y_4^3\,) \\ &= (\, y_2^3\, ,\, y_3^4\, ,\, y_2\, ,\, y_3^3 y_4^3\,) \cap (\, y_2^3\, ,\, y_3^4\, ,\, y_3 y_4^3\, ,\, y_3^3 y_4^3\,) \cap (\, y_2^4\, ,\, y_3^4\, ,\, y_4^3\,) \\ &= (\, y_2\, ,\, y_3^4\, ,\, y_3^3 y_4^3\,) \cap (\, y_2^3\, ,\, y_3^4\, ,\, y_3 y_4^3\,) \cap (\, y_2^4\, ,\, y_3^3\, ,\, y_4^3\,) \\ &= (\, y_2\, ,\, y_3^4\, ,\, y_3^3\,) \cap (\, y_2\, ,\, y_3^4\, ,\, y_4^3\,) \cap (\, y_2^3\, ,\, y_3^4\, ,\, y_3\,) \cap (\, y_2^3\, ,\, y_3^4\, ,\, y_4^3\,) \cap (\, y_2^4\, ,\, y_3^3\, ,\, y_4^3\,) \\ &= (\, y_2\, ,\, y_3^3\,) \cap (\, y_2\, ,\, y_3^4\, ,\, y_4^3\,) \cap (\, y_2^3\, ,\, y_3\,) \cap (\, y_2^4\, ,\, y_3^4\, ,\, y_4^3\,) \\ &= (\, y_2\, ,\, y_3^3\,) \cap (\, y_2^3\, ,\, y_3\,) \cap (\, y_2^4\, ,\, y_3^4\, ,\, y_4^3\,). \end{aligned}$$

It is the primary decompostion of I but not irredundant. Here $\mathrm{Ass}(\, y_2\, ,\, y_3^3\,) = \mathrm{Ass}(\, y_2^3\, ,\, y_3\,) = \{(\, y_2\, ,\, y_3\,)\}$. Now for irredundant primary decomposition, take an intersection of $(\, y_2\, ,\, y_3^3\,)$ and $(\, y_2^3\, ,\, y_3\,)$, that is

$$(\, y_2\, ,\, y_3^3\,) \cap (\, y_2^3\, ,\, y_3\,) = (\, y_2^3\, ,\, y_2 y_3\, ,\, y_3^3\,).$$

Hence

$$I = (\, y_2^4\, ,\, y_3^4\, ,\, y_4^3\,) \cap (\, y_2^3\, ,\, y_2 y_3\, ,\, y_3^3\,).$$

Example 1.2.22. Let $U = (\, \varrho_1 \varrho_2\, ,\, \varrho_3 \varrho_5\, ,\, \varrho_2 \varrho_3\, ,\, \varrho_2 \varrho_4\, ,\, \varrho_3 \varrho_4\, ,\, \varrho_1 \varrho_4\,)$ be an ideal of S, then

$$\begin{aligned}
U &= (\varrho_1\varrho_2, \varrho_3\varrho_5, \varrho_2\varrho_4, \varrho_3\varrho_4, \varrho_1\varrho_4) \\
&= (\varrho_1, \varrho_4, \varrho_5) \cap (\varrho_3, \varrho_1\varrho_2, \varrho_2\varrho_4, \varrho_1\varrho_4) \\
&= (\varrho_2, \varrho_4, \varrho_5,) \cap (\varrho_1, \varrho_3, \varrho_2) \cap (\varrho_1\varrho_2, \varrho_3, \varrho_4) \\
&= (\varrho_2, \varrho_4, \varrho_5) \cap (\varrho_2, \varrho_4, \varrho_3) \cap (\varrho_1, \varrho_3, \varrho_4) \cap (\varrho_1, \varrho_2, \varrho_3) \\
&= (\varrho_2, \varrho_4, \varrho_5) \cap (\varrho_1, \varrho_3, \varrho_4) \cap (\varrho_2, \varrho_4, \varrho_3) \cap (\varrho_1, \varrho_2, \varrho_3).
\end{aligned}$$

Since U is square free monomial ideal, so it can be seen that $(\varrho_2, \varrho_4, \varrho_5)$, $(\varrho_1, \varrho_3, \varrho_4)$, $(\varrho_1, \varrho_2, \varrho_3)$ and $(\varrho_2, \varrho_4, \varrho_3)$ are minimal prime ideals of U.

1.3 Module theory

Definition 1.3.1. Consider a commutative ring $\mathcal{R}$, an $\mathcal{R}$-module $\mathcal{M}$ is an commutative group w.r.t addition, along with a scalar multiplication map $\cdot : \mathcal{R} \times \mathcal{M} \to \mathcal{M}$, defined as $\cdot((\alpha, \varrho)) = \alpha\varrho$, which holds the succeeding axioms

1. $\alpha(\varrho_1 + \varrho_2) = \alpha\varrho_1 + \alpha\varrho_2$,
2. $(\alpha_1 + \alpha_2)\varrho = \alpha_1\varrho + \alpha_2\varrho$,
3. $(\alpha_1\alpha_2)\varrho = \alpha_1(\alpha_2\varrho)$,
4. $1\varrho = \varrho$,

$$\forall\ \alpha_1, \alpha_2 \in \mathcal{R} \textit{ and } \varrho_1, \varrho_2 \in \mathcal{M}.$$

Examples 1.3.2. 1. For a commutative group $\mathcal{D}$, let $d \in \mathcal{D}$ and $z \in \mathbb{Z}$, then define $\cdot : \mathbb{Z} \times \mathcal{D} \to \mathcal{D}$, such that

$$\cdot(z, d) = zd = \begin{cases} (-d) + \cdots + (-d)\,, & \text{if } \ z < 0; \\ d + d + \cdots + d\,, & \text{if } \ z > 0; \\ 0\,, & \text{if } \ z = 0. \end{cases}$$

Then $\mathcal{D}$ is a $\mathbb{Z}$-module.

2. The ideals of the ring are also $\mathcal{R}$-modules.

Definition 1.3.3. For a ring $\mathcal{R}$, let us suppose U and V be $\mathcal{R}$-modules. A function $f : U \to V$ is known as $\mathcal{R}$-module homomorphism if

- $f(\varrho_1 + \varrho_2) = f(\varrho_1) + f(\varrho_2)$, for all $\varrho_1, \varrho_2 \in U$.
- $f(r\varrho) = rf(\varrho)$, for all $r \in \mathcal{R}\,,\ \varrho \in U$.

If f is injective and onto then it becomes an $\mathcal{R}$-module isomorphism.

Examples 1.3.4. 1. For a ring R, consider R-module R. Then R-module homomorphism (even from R into itself) needs not to be a ring homomorphism. Consider $R = \mathbb{Z}$, then $\mathbb{Z}$-module homomorphism $x \mapsto 2x$ is not a ring homomorphism.

2. When $R = F[y]$, the ring homomorphism $\phi : h(y) \mapsto h(y^2)$ is not an $F[y]$-module homomorphism.

Definition 1.3.5. Consider a ring $\mathcal{R}$, and a submodule Q of R-module $\mathcal{M}$. Then (additive abelian) quotient group $\mathcal{M}/Q$ becomes an $\mathcal{R}$-module by using scalar multiplication defined as

$$r(m + Q) = rm + Q$$

$\forall\ r \in \mathcal{R}\,,\ m + Q \in \mathcal{M}/Q$.

1.3.1 Generation of modules

For any subset W of R-module M, let

$$RW = \{r_1w_1 + \cdots + r_nw_n\ :\ r_1, \ldots, r_n \in R\,,\ w_1, \ldots, w_n \in W \text{ and } n \in \mathbb{Z}^+\}.$$

If W is a finite set, $\{w_1, \ldots, w_n\}$, then $RW = Rw_1 + Rw_2 + \cdots + wa_n$. Let $M_1 = RW$ for some subset W of M and say M_1 is a submodule of M. W is the generating set for M_1. A submodule M_1 is said to be finitely generated if for $M_1 = RW$, W is a finite subset of M.

Definition 1.3.6. Let $\mathcal{S}$ be an $\mathcal{R}$-module then it is called free on the subset T of $\mathcal{S}$ if for $0 \neq s \in \mathcal{S}$, there are unique non-zero elements $r_1, \ldots, r_k$ of $\mathcal{R}$ and unique $t_1, \ldots, t_k$ in T, such that

$$s = r_1t_1 + \cdots + r_kt_k.$$

Definition 1.3.7. Let $\mathcal{S}$ be a commutative ring, consider a chain of prime ideals in the ring of length m_i

$$Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_{m_i},$$

then dimension of ring $\mathcal{S}$ is defined as

$$\dim \mathcal{S} = \sup\{m_i\}.$$

Suppose $\mathcal{N}$ be an $\mathcal{S}$-module, then Krull dimension of $\mathcal{N}$ is

$$\dim(\mathcal{N}) = \dim(\mathcal{S}/\mathrm{Ann}(\mathcal{N})).$$

For the modules of the type $\mathcal{S}/\mathcal{I}$

$$\dim(\mathcal{S}/\mathcal{I}) = \max\{\dim(\mathcal{S}/J_i)\ :\ J_i \in \mathrm{Ass}(\mathcal{S}/\mathcal{I})\}.$$

1.3.2 Exact sequences

Definition 1.3.8. Let S be a commutative ring, Consider a sequence of S-homomorphisms on S-modules

$$\ldots \longrightarrow \mathcal{U}_{j-1} \xrightarrow{h_j} \mathcal{U}_j \xrightarrow{h_{j+1}} \mathcal{U}_{j+1} \xrightarrow{h_{j+2}} \ldots$$

it is exact at $\mathcal{U}_j$ if $Im(h_j) = ker(h_{j+1})$. The sequence is known to be exact if it is observed to be exact at every $\mathcal{U}_j$. Particularly, $0 \longrightarrow\ V' \xrightarrow{h} \mathcal{U}$ is exact at V' if and only if h is one to one, and $\mathcal{U} \xrightarrow{g} V'' \longrightarrow\ 0$ is exact at V'' if and only if g is onto.

Proposition 1.3.9. *The sequence*

$$0 \longrightarrow\ V' \xrightarrow{h} \mathcal{U} \xrightarrow{g} V'' \longrightarrow\ 0$$

is an exact sequence if and only if h is one to one, g is onto and $Im(h) = ker(g)$.

Remark 1.3.10. *The sequence in Proposition 1.3.9 is called a short exact sequence.*

Proposition 1.3.11. *Let γ be a poset with respect to $\leq$. Then the following are equivalent.*

1. *Any increasing sequence* $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_r \leq \ldots$ *in* γ *is stationary, that is there exist* $r \in \mathbb{N}$ *for which* $\alpha_s = \alpha_r$*, for all* $s \geq r$.

2. *Any* $\emptyset \neq W \subset \gamma$ *possesses a maximal element.*

Let γ be the set of submodules of $\mathcal{N}$ which is ordered w.r.t the relation $\subseteq$ then statement *1* is known as *ascending chain condition* and statement *2* is known as the *maximal condition*.

1.3.3 Graded rings

Consider a commutative semigroup (w.r.t addition) U. A U-graded ring is such type of a ring $\mathcal{R}$ alongside a decomposition

$$\mathcal{R} = \bigoplus_{u \in \mathcal{U}} \mathcal{R}_u \text{ (as a group)},$$

such that $\mathcal{R}_u \mathcal{R}_v \subset \mathcal{R}_{u+v}$ $\forall$ $u, v \in \mathcal{U}$.
Then for $r \in \mathcal{R}$, we can write a unique expression

$$r = \sum_{u \in \mathcal{U}} r_u,$$

where $r_u \in \mathcal{R}_u$ and almost all $r_u = 0$. The element r_u is called the *uth* homogeneous component and if $r = r_u$, then r is homogeneous of degree u. $\mathcal{R}[x]$ and $\mathcal{R}[x, y]$ are $\mathbb{Z}$-graded rings as

- $\mathcal{R}[x] = \mathcal{R} \oplus \mathcal{R}x \oplus \mathcal{R}x^2 \oplus \mathcal{R}x^3 \oplus \mathcal{R}x^4 \oplus \mathcal{R}x^5 \oplus \cdots$.

- $\mathcal{R}[x, y] = \mathcal{R} \oplus (\mathcal{R}x + \mathcal{R}y) \oplus (\mathcal{R}x^2 + \mathcal{R}xy + \mathcal{R}y^2) \oplus (\mathcal{R}x^3 + \mathcal{R}x^2y + \mathcal{R}xy^2 + \mathcal{R}y^3) \oplus \cdots$.

For a $\mathcal{U}$-graded ring $\mathcal{R}$ and $\mathcal{R}$-module $\mathcal{M}$

$$\mathcal{M} = \bigoplus_{u \in \mathcal{U}} \mathcal{M}_u \text{ (as a group)},$$

with $\mathcal{R}_u \mathcal{M}_v \subset \mathcal{M}_{u+v}$ for all $u, v \in \mathcal{U}$, then $\mathcal{M}$ is said to be a $\mathcal{U}$-graded module. A non zero element of $\mathcal{M}_u$ is called a homogeneous element of degree u.

For a polynomial ring $\mathcal{S}$ defined over the field K, suppose $\mathbf{b} \in \mathbb{Z}^n$, then $h \in \mathcal{S}$ is said to be homogeneous of degree $\mathbf{b}$ when h has the form $\beta x^{\mathbf{b}}$, where $\beta \in K$. Also $\mathcal{S}$ is $\mathbb{Z}^n$-graded with graded components:

$$\mathcal{S}_{\mathbf{b}} = \begin{cases} Kx^{\mathbf{b}}, & \text{if } \mathbf{b} \in \mathbb{Z}^n_+; \\ 0\,, & \text{otherwise.} \end{cases}$$

An $\mathcal{S}$-module $\mathcal{M}$ is $\mathbb{Z}^n$-graded if $\mathcal{M} = \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} \mathcal{M}_{\mathbf{b}}$ and $\mathcal{S}_{\mathbf{b_1}}\mathcal{M}_{\mathbf{b_2}} \subset \mathcal{M}_{\mathbf{b_1}+\mathbf{b_2}}$ for all $\mathbf{b_1}, \mathbf{b_2} \in \mathbb{Z}^n$.

1.4 Simplicial complexes and squarefree monomial ideals

1.4.1 Simplicial complexes

Definition 1.4.1. Let ∇ be the collection of subsets of vertex set $[n] = \{1, \ldots, n\}$, such that if $Z \in \nabla$ called face of ∇ and $Z' \subset Z$, then $Z' \in \nabla$ and ∇ is called simplicial complex on vertex set $[n]$.

Definition 1.4.2. The dimension of the face Z of ∇ is $|Z|-1$ and it is denoted and defined by $dim\nabla = d-1$, where $d = max\{|Z|\colon Z \in \nabla\}$. An edge of simplicial complex is a face of dimension 1 and a vertex of simplicial complex is a 0 dimensional face.

Definition 1.4.3. With respect to inclusion, let us suppose Z be the maximal face of ∇, then Z is known as a facet. Let $Z(\nabla)$ refers to the set of facets of ∇. Obvious $Z(\nabla)$ determines ∇. Whenever $Z(\nabla) = \{Z_1, \ldots, Z_m\}$, then one can write $\nabla = \langle Z_1, \ldots, Z_m\rangle$. If all facets of ∇ have the same cardinality, then ∇ is called pure simplicial complex.

Definition 1.4.4. Let Z be a subset of $[n]$. Z is called a nonface of ∇ if $Z \notin \nabla$. Let $N(\nabla)$ represents the set of minimal nonfaces of ∇.

Example 1.4.5. An example of geometric realization of a simplicial complex is shown in Figure 1.1, which typifies simplicial complex ∇ having dimension 1 and vertex set [5] alongside

$$Z(\nabla) = \{\{2,3\}, \{1,3\}, \{3,4\}, \{1,2\}, \{4,5\}, \{1,5\}\}$$

and

$$N(\nabla) = \{\{1,4\},\{2,4\},\{2,5\},\{3,5\}\}.$$

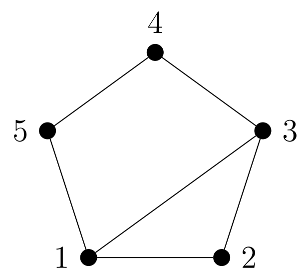


Figure 1.1: Simplicial complex

1.4.2 Facet ideals and Stanley–Reisner ideals

Let $R = K[y_1, \ldots, y_m]$ be a ring of polynomials over the field K and ∇ is a simplicial complex on $[m]$ vertices. For every subset $Z \subset [m]$, we establish

$$y_Z = \prod_{i \in Z} y_i.$$

Definition 1.4.6. An ideal $I(\nabla)$ of R is called facet ideal, if $I(\nabla)$ is generated by the monomials y_Z with $Z \in Z(\nabla)$. Thus, if $Z(\nabla) = \{Z_1, \ldots, Z_p\}$, then $I(\nabla) = (y_{Z_1}, \ldots, y_{Z_p})$.

Definition 1.4.7. An ideal I_∇ of R is called Stanley–Reisner ideal of ∇ if I_∇ is generated by monomials y_Z with $Z \notin Z(\nabla)$. i.e., $I_\nabla = (y_Z : Z \in N(\nabla))$.

Example 1.4.8. Consider ∇ on the vertex set [6], as shown in figure 1.2 with

$$Z(\nabla) = \{\{1,3\},\{1,6\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,6\}\}$$

and

$$N(\nabla) = \{\{1,2\},\{1,4\},\{1,5\},\{2,3\},\{2,6\},\{3,6\},\{4,5\},\{5,6\}\}$$

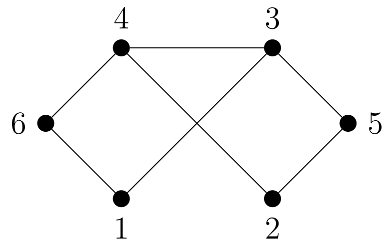


Figure 1.2: Simplicial complex ∇

Hence facet ideal and Stanley–Reisner ideal associated to the simplicial complex ∇ would be

$$I(\nabla) = (y_1y_3, y_1y_6, y_2y_4, y_2y_5, y_3y_4, y_3y_5, y_4y_6)$$

and

$$I_{\nabla} = (y_1y_2, y_1y_4, y_1y_5, y_2y_3, y_2y_6, y_3y_6, y_4y_5, y_5y_6)$$

respectively.

Chapter 2

Preliminaries and basic concepts

2.1 Graph theory: A brief introduction

Finite graphs are the most simple structures in Mathematics. For this particular reason, before any systematic study of graph theory itself, many graph-theoretic problems remained unsolved. Leonhard Euler's 1735 König's bridges Problem [7] is the notorious example of such a problem and the Four-Color Problem which Francis Guthrie originally presented in 1852, as a coloring problem of the map of England's counties. (Although Euler himself sought a simple but ingenious solution for the former, in 1976 Appel and Haken [8] and in 1997 Robertson, Sanders, Seymour and Thomas [9] needed more than 100 years and much advances in graph theory to be resolved in 2 phases.) Such important early experiments include research on polyhedra cycles by Thomas Kirkman and William Hamilton [10], the circuit laws by Gustav Kirchhoff [11], and research by Arthur Cayley and James Sylvester [12] that had ties to theoretical chemistry to the structure of molecules in particular. In 1878, it was Sylvester who suggested the name of "Graph" to the structure he was researching. Graph theory has huge applications in engineering and science especially in chemical engineering, mechanical engineering, architecture, operational research, technology, combinatorics, and computer science.

Over the last ten years, commutative algebraists have been involved in study of the properties of finite simple graphs by employing monomial ideals. Simis, Fróberg, Vasconcelos, and Villarreal are considered to be the pioneers in this field. The departure point for these attempts is to create a monomial ideal by employing the edges of a finite simple graph and it is usually called the edge ideal, and studying the properties of monomial ideal employing the graph properties, and vice versa.

In this chapter basic definition and concept of graph theory are given. This chapter gives a detailed overview of different types of graphs, different representation operations of graph and results which we will use in our last two chapter.

2.2 Basic graph theory

Graph theory consists of the study of graphs, whereas the graphs are the mathematical framework used to model the relation between the objects. The basic fundamental principles of graph theory are presented in this section.

Definition 2.2.1. A graph is a set of points and lines that connects some subset of them (possibly empty). The points are most frequently referred to as graph vertices. Similarly, lines linking the vertexes of a graph are most commonly referred to as graph edges.

Definition 2.2.2. An edge with same end points is known as a loop. The edges with exactly the same set of endpoints are known as multiple edges. A simple graph is a graph with no multiple edges and loops.

Given below is a graph with vertices $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ *and edges* $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$.

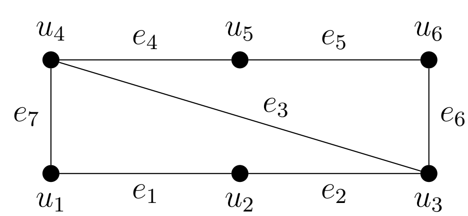


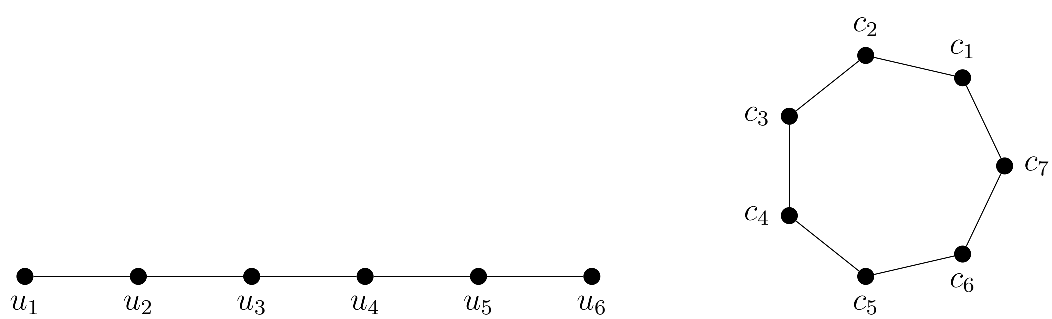
Figure 2.1: Simple Graph

Consider an edge with endpoints u_1, u_2. Then u_1, u_2 are said to be adjacent and they are neighbors of each other. The focus is restricted to only simple graphs in various important applications.

Definition 2.2.3. The total edges incident on vertex v of a graph G is known as degree of v, which is usually represented by $d_G(v)$ or $d(v)$. The set with all the vertices adjacent to v forms the neighbourhood of v, represented by $N_G(v)$.

Definition 2.2.4. The total vertices in $V(G)$ is known as the order of graph G, represented by $n(G)$. While the total edges in $E(G)$ determines the size of graph, written as $e(G)$.

Definition 2.2.5. A graph G is said to be a path if $V(G)$ can be ordered in a way that whenever two vertices are consecutive in the list, there is an edge between them. A graph whose vertex and edge sets have the same cardinality and vertices can be placed around the circle so that whenever two vertices appear consecutive along the circle, an edge lies between them, such a graph is known as a cycle. Deleting one edge from a cycle forms a path. A cycle and path on n vertices are represented by C_n and P_n, respectively.



(a) P_6 (b) C_7

Figure 2.2: Path and Cycle

Definition 2.2.6. A simple graph in which there is an edge between every two vertices is known as a complete graph.

Definition 2.2.7. A subgraph B of a graph A, written as $B \subseteq A$, is a type of graph such that $V(B) \subseteq V(A)$ and $E(B) \subseteq E(A)$ and the endpoints of edges in B are exactly the same as in A.

(a) A (b) B

Figure 2.3: Graph and its Subgraph

Definition 2.2.8. Consider a graph $B = (V(B), E(B))$ is a subgraph of A so that $V(B) \subseteq V(A)$ and $E(B) \subseteq E(A)$. Given a subset $D \subseteq V(A)$, the induced subgraph of A on D is the graph $A_D = (D, E(A_D))$ where $E(A_D) = \{uv \in E(A) | \{u, v\} \subseteq D\}$.

Definition 2.2.9. A^c is called complement of a graph A, is a graph with $(V(A^c), E(A^c))$ where $V(A^c) = V(A)$ and $E(A^c) = \{uv | uv \notin E(A)\}$.

Definition 2.2.10. The neighbours of $u \in V(A)$ are the set$N(u) = \{v \in V(A) | uv \in E(A)\}$. The closed neighbourhood of u is $N[u] = N(u) \cup \{u\}$. The degree of u is $deg(u) = |N(u)|$. If we need to highlight the associated graph, we write $N_A[u]$ or $N_A(u)$.

Proposition 2.2.11. Any graph with k vertices and l edges has at least $k - l$ components.

Definition 2.2.12. If there is an edge between every two vertices of a simple graph then graph is known as complete graph.

Definition 2.2.13. Consider a u, v-path in A. The distance from u to v is the minimum length of u, v-path, written as $d(u, v)$. The path with the maximum length in A

determines the diameter i.e.,

$$diam A = \max_{u,v \in V(A)} d(u, v).$$

Remark 2.2.14. The cycle C_n has diameter $\lfloor \frac{n}{2} \rfloor$ and the path has the diameter $n-1$.

The elimination of a vertex also removes the edges incident on it and the obtained graph is again a graph.

2.3 Some products of graphs

2.3.1 Cartesian product

Definition 2.3.1. Consider two graphs $\mathcal{H}$ and $\mathcal{K}$ with vertex sets $V(\mathcal{H}) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(\mathcal{K}) = \{v_1, v_2, \ldots, v_{n_2}\}$, respectively. The Cartesian product of $\mathcal{H}$ and $\mathcal{K}$ is a graph, with $V(\mathcal{H} \Box \mathcal{K}) = V(\mathcal{H}) \times V(\mathcal{K})$ (the cartesian product of sets), and for $(u_i, v_j), (u_k, v_l) \in V(\mathcal{H} \Box \mathcal{K})$, $(u_i, v_j)(u_k, v_l) \in E(\mathcal{H} \Box \mathcal{K})$, whenever

- $v_j = v_l$ and $u_i u_k \in E(\mathcal{H})$ or
- $v_j v_l \in E(\mathcal{K})$ and $u_i = u_k$

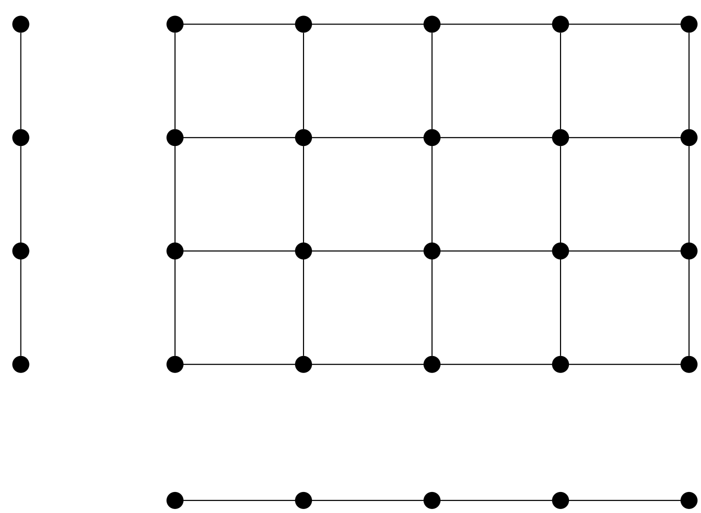


Figure 2.4: Cartesian Product of P_5 and P_4 ($P_5 \Box P_4$)

2.3.2 Partial cartesian product of graphs

Definition 2.3.2. Consider two graphs $\mathcal{H}$ and $\mathcal{K}$ with vertex set $V(\mathcal{H}) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(\mathcal{K}) = \{v_1, v_2, \ldots, v_{n_2}\}$ respectively. If $W \subset V(\mathcal{K})$, then the partial cartesian product of $\mathcal{H}$ and $\mathcal{K}$ w.r.t W is the graph, with $V(\mathcal{H}\Box_W\mathcal{K}) = V(\mathcal{H}) \times V(\mathcal{K})$ (the cartesian product of sets), and for $(u_i, v_j), (u_k, v_l) \in V(\mathcal{H}\Box_W\mathcal{K})$, $(u_i, v_j)(u_k, v_l) \in E(\mathcal{H}\Box_W\mathcal{K})$, whenever

- $v_jv_l \in E(\mathcal{K})$ and $u_i = u_k$ or
- $v_j = v_l \in W$ and $u_iu_k \in E(\mathcal{H})$.

The Partial Cartesian Product of P_5 and P_4 with respect to $\{v_1, v_3\}$ is $P_5\Box_{\{v_1,v_3\}}P_4$.

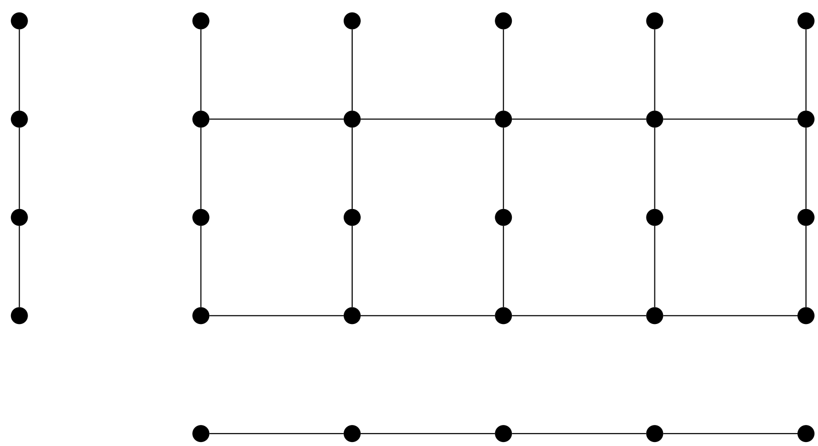


Figure 2.5: The Partial Cartesian Product graph $P_5\Box_{\{v_1,v_3\}}P_4$

2.3.3 Standard strong product of graphs

Definition 2.3.3. Consider two graphs $\mathcal{H}$ and $\mathcal{K}$ with vertex set $V(\mathcal{H}) = \{u_1, u_2, ..u_{n_1}\}$ and $V(\mathcal{K}) = \{v_1, v_2, ..v_{n_2}\}$ respectively. The standard strong product of $\mathcal{H}$ and $\mathcal{K}$ is a graph, with $V(\mathcal{H} \boxtimes \mathcal{K}) = V(\mathcal{H}) \times V(\mathcal{K})$ (the cartesian product of sets), and for $(u_i, v_j), (u_k, v_l) \in V(\mathcal{H} \boxtimes \mathcal{K})$, $(u_i, v_j)(u_k, v_l) \in E(\mathcal{H} \boxtimes \mathcal{K})$, whenever

- $v_jv_l \in E(\mathcal{K})$ and $u_i = u_k$ or
- $v_j = v_l$ and $u_iu_k \in E(\mathcal{H})$ or

- $u_i \in V(\mathcal{H}), , v_j \in V(\mathcal{K})$, $v_j v_l \in E(\mathcal{K})$ and $u_i u_k \in E(\mathcal{H})$ or
- $u_k \in V(\mathcal{H}), v_l \in V(\mathcal{K})$, $v_j v_l \in E(\mathcal{K})$ and $u_i u_k \in E(\mathcal{H})$.

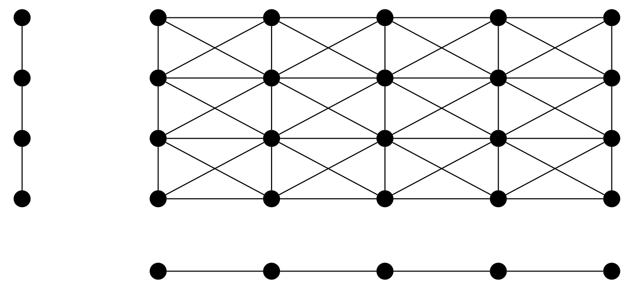


Figure 2.6: Standard Strong Product of P_5 and P_4 ($P_5 \boxtimes P_4$)

2.3.4 Partial strong product of graphs

Definition 2.3.4. Consider two graphs $\mathcal{H}$ and $\mathcal{K}$ with vertex set $V(\mathcal{H}) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(\mathcal{K}) = \{v_1, v_2, \ldots, v_{n_2}\}$ respectively. Now, If $P \subset V(\mathcal{H})$ and $Q \subset V(\mathcal{K})$, then the partial strong product of $\mathcal{H}$ and $\mathcal{K}$ w.r.t P and Q is the graph, with $V(\mathcal{H} \boxtimes_{P,Q} \mathcal{K}) = V(\mathcal{H}) \times V(\mathcal{K})$ (the cartesian product of sets), and for $(u_i, v_j), (u_k, v_l) \in V(\mathcal{H} \boxtimes_{P,Q} \mathcal{K})$, $(u_i, v_j)(u_k, v_l) \in E(\mathcal{H} \boxtimes_{P,Q} \mathcal{K})$, whenever

- $v_j v_l \in E(\mathcal{K})$ and $u_i = u_k$ or
- $v_j = v_l$ and $u_i u_k \in E(\mathcal{H})$ or
- $u_i \in P, v_j \in Q$, $v_j v_l \in E(\mathcal{K})$ and $u_i u_k \in E(\mathcal{H})$ or
- $u_k \in P, v_l \in Q$, $v_j v_l \in E(\mathcal{K})$ and $u_i u_k \in E(\mathcal{H})$.

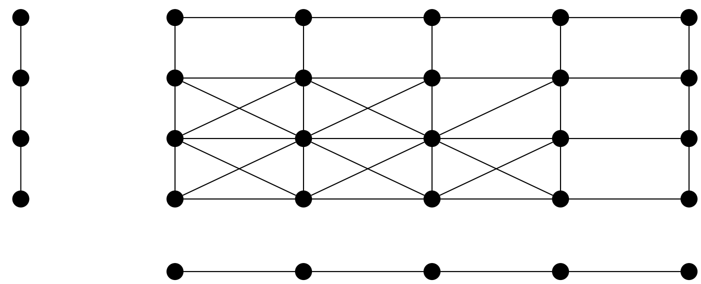


Figure 2.7: The Partial Strong Product Graph $P_5 \boxtimes_{P,Q} P_4$.

2.3.5 Restricted partial strong product of graphs

Definition 2.3.5. Consider two graphs $\mathcal{H}$ and $\mathcal{K}$ with vertex set $V(\mathcal{H}) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(\mathcal{K}) = \{v_1, v_2, \ldots, v_{n_2}\}$ respectively. If $\mathcal{H} \subset V(\mathcal{H})$ and $\mathcal{K} \subset V(\mathcal{K})$, then the restricted partial strong product of $\mathcal{H}$ and $\mathcal{K}$ with respect to $\mathcal{H}$ and $\mathcal{K}$ is the graph, with $V(\mathcal{H}_{\mathcal{H}} \boxtimes_{\mathcal{K}} \mathcal{K}) = V(\mathcal{H}) \times V(\mathcal{K})$ (the cartesian product of sets), and for $(u_i, v_j), (u_k, v_l) \in V(\mathcal{H}_{\mathcal{H}} \boxtimes_{\mathcal{K}} \mathcal{K})$, $(u_i, v_j)(u_k, v_l) \in E(\mathcal{H}_{\mathcal{H}} \boxtimes_{\mathcal{K}} \mathcal{K})$, whenever

- $v_j v_l \in E(\mathcal{K})$ and $u_i = u_k$ or
- $v_j = v_l$ and $u_i u_k \in E(\mathcal{H})$ or
- $u_i \in \mathcal{H}, u_k \notin \mathcal{H}, v_j \in \mathcal{K}, v_l \notin \mathcal{K}$, $v_j v_l \in E(\mathcal{K})$ and $u_i u_k \in E(\mathcal{H})$ or
- $u_i \notin \mathcal{H}, u_k \in \mathcal{H}, v_j \notin \mathcal{K}, v_l \in \mathcal{K}$, $v_j v_l \in E(\mathcal{K})$ and $u_i u_k \in E(\mathcal{H})$.

Example 2.3.6. *The Restricted Partial Strong Product of P_5 and P_4 with respect to $\mathcal{H} = \{u_2, u_4\}$ and $\mathcal{K} = \{v_1, v_3\}$ is $P_{5_{\{u_2,u_4\}}} \boxtimes_{\{v_1,v_3\}} P_4$.*

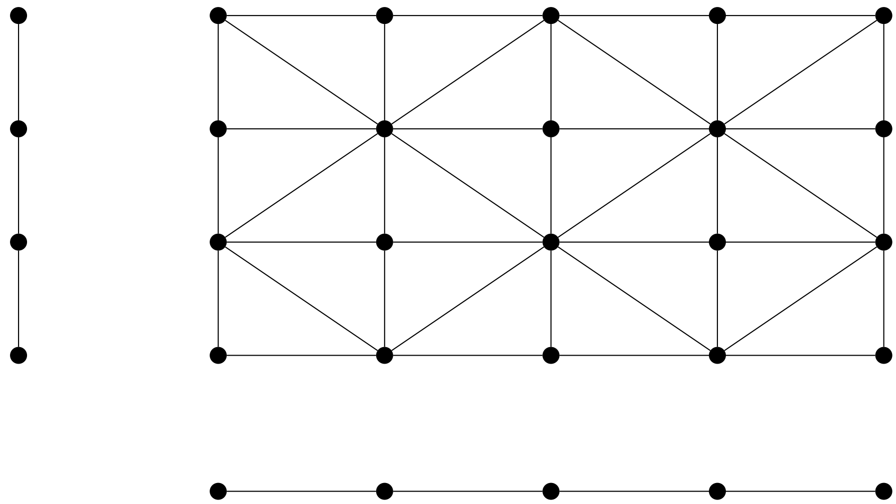


Figure 2.8: Restricted Partial Strong Product of P_5 and P_4 ($P_{5\{u_2,u_4\}} \boxtimes_{\{v_1,v_3\}} P_4$)

Example 2.3.7. *The Restricted Partial Strong Product of P_n and P_2 with respect to $\mathcal{H} = \{x_i : i \in 2\mathbb{Z} \wedge 1 \leq i \leq n\}$ and $\mathcal{K} = \{x_j : j \notin 2\mathbb{Z} \wedge 1 \leq j \leq m\}$ is $P_{n\mathcal{H}} \boxtimes_{\mathcal{K}} P_2$.*

When n is odd

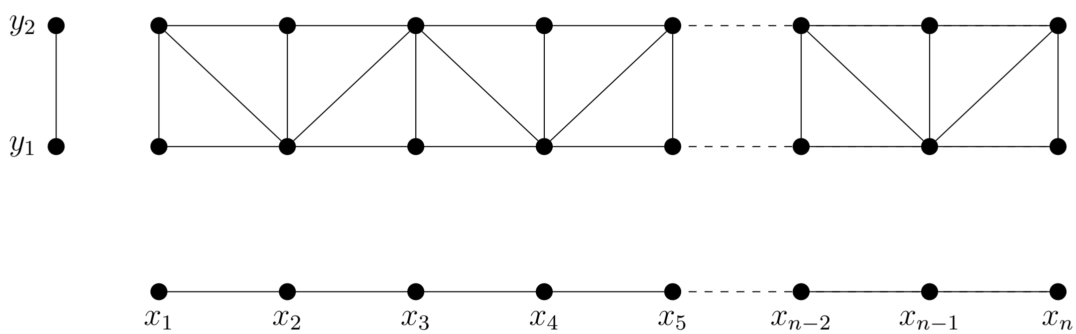


Figure 2.9: Restricted Partial Strong Product of P_n and P_2 ($P_{n\{x_2,x_4,\ldots,x_{n-1}\}} \boxtimes_{\{y_1\}} P_2$)

When n is even

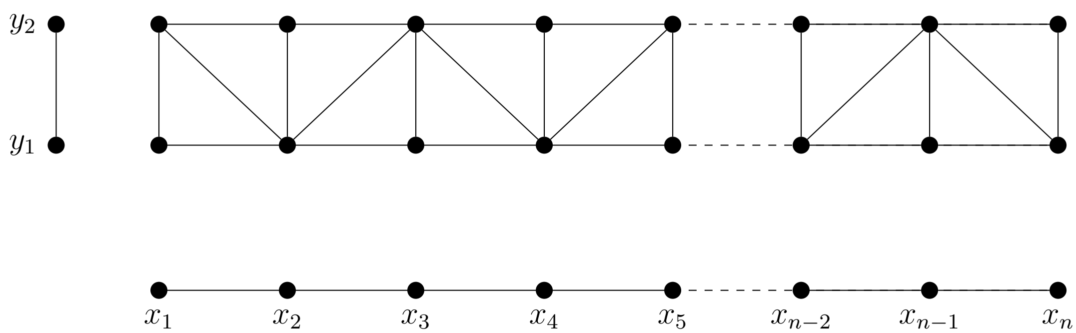


Figure 2.10: Restricted Partial Strong Product of P_n and P_2 ($P_{n\{x_2,x_4,\ldots,x_n\}} \boxtimes_{\{y_1\}} P_2$)

Example 2.3.8. *The Restricted Partial Strong Product of* P_n *and* P_3 *with respect to* $\mathcal{H} = \{x_i : i \in 2\mathbb{Z} \wedge 1 \leq i \leq n\}$ *and* $\mathcal{K} = \{x_j : j \notin 2\mathbb{Z} \wedge 1 \leq j \leq m\}$ *is* $P_{n\mathcal{H}} \boxtimes_{\mathcal{K}} P_3$.

When n is odd

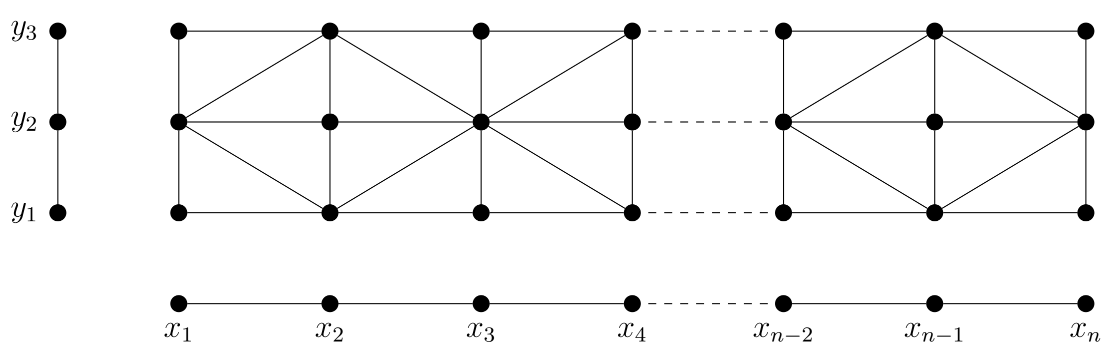


Figure 2.11: Restricted Partial Strong Product of P_n and P_3 ($P_{n\{x_2,x_4,\ldots,x_{n-1}\}} \boxtimes_{\{y_1,y_3\}} P_3$)

When n is even

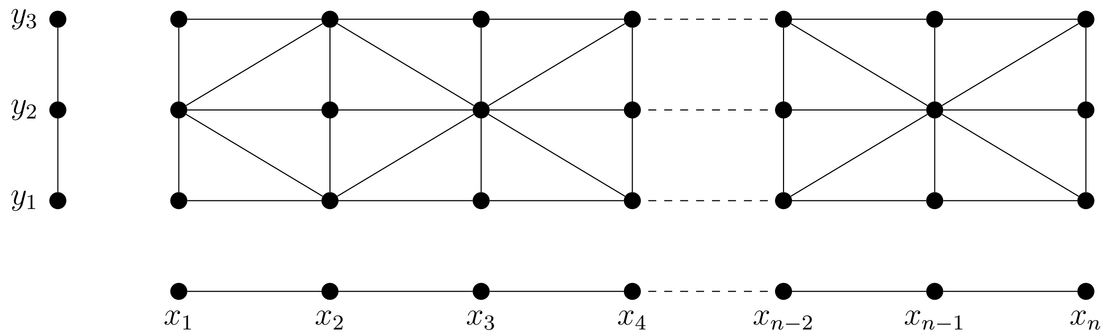


Figure 2.12: Restricted Partial Strong Product of P_n and P_3 ($P_{n\{x_2,x_4,\ldots,x_n\}} \boxtimes_{\{y_1,y_3\}} P_3$)

Example 2.3.9. *The Restricted Partial Strong Product of* C_8 *and* P_2 *with respect to* $\mathcal{H} = \{c_2, c_4, \ldots, c_8\}$ *and* $\mathcal{K} = \{p_1\}$ *is* $C_{8\{c_2,c_4,\ldots,c_8\}} \boxtimes_{\{p_1\}} P_2$.

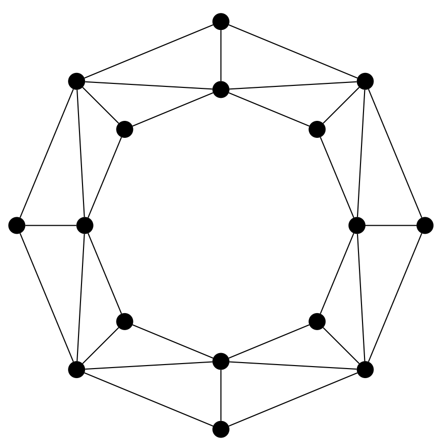


Figure 2.13: $C_{8\{c_2,c_4,\ldots,c_8\}} \boxtimes_{\{p_1\}} P_2$

Example 2.3.10. *The Restricted Partial Strong Product of* C_8 *and* P_3 *with respect to* $\mathcal{H} = \{c_2, c_4, \ldots, c_8\}$ *and* $\mathcal{K} = \{p_1, p_3\}$ *is* $C_{8\{c_2,c_4,\ldots,c_8\}} \boxtimes_{\{p_1,p_3\}} P_3$.

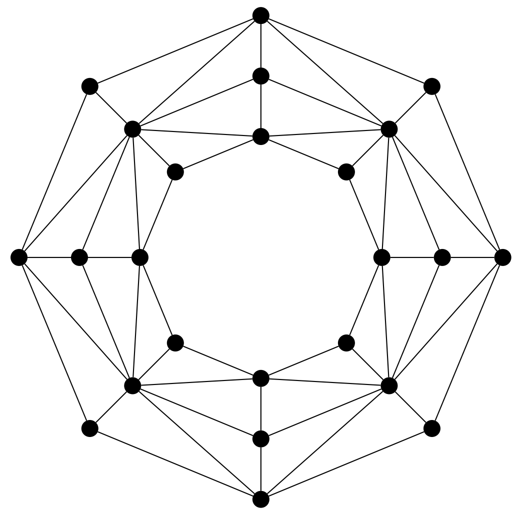


Figure 2.14: $C_{8\{c_2,c_4,\ldots,c_8\}} \boxtimes_{\{p_1,p_3\}} P_3$

2.4 Circulant graphs

Analysis of asymmetries or symmetries of structures give potent outcomes in mathematics. Circulants give a class of symmetric mathematical structures. In the year 1846, Catalan, the renowned mathematician endeavored to introduce properties of circulant graphs and circulant matrices which were investigated by plentiful authors. A magnificent account can be searched in the works of Davis [13].

Circulant graphs are vertex-transitive and regular, and they are a subset of Cayley graphs which is a more general family of graphs. Circulants take different form in a array of graph applications comprising also the theory of designs and error correcting codes and the modeling of data connection networks.

Definition 2.4.1. Suppose $k \geq 2$ and take a subset $P \subset \{1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\}$. Then, the circulant graph $C_k(P)$ with vertex set $\{\alpha_1, \ldots, \alpha_k\}$ is a graph such that $\{\alpha_i, \alpha_j\} \in E(C_k(P)) \Longleftrightarrow |i-j| \in P$ or $k - |i-j| \in P$. For instance, the graph $C_7(1,3)$ is drawn in Figure 2.15.

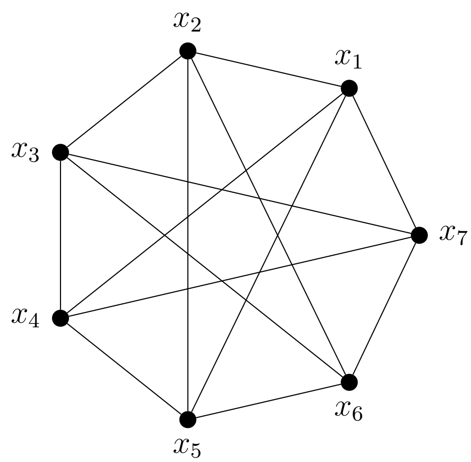


Figure 2.15: $C_7(1,3)$

We generally write C_n for $C_n(1)$. We will usually suppose, without further remark, that the vertices of circulant graph $C_n(S)$ are observed to be the corners of a regular n-gon, labeled anticlockwise.

Examples 2.4.2. 1. $C_n(0) \cong O_n$, totally disconnected graphs on n vertices

2. $C_2(1) \cong P_2$

3. $C_n(1) \cong C_n$

4. $C_n(1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor) \cong K_n$

Theorem 2.4.3. Circulant graph $C_n(R)$ for a set $R = r_1, r_2, ..., r_k$ is connected iff $gcd(n, r_1, r_2, ..., r_k) = 1$.

2.4.1 Cubic circulant graph

A circulant graph is cubic if each vertex has degree three. Consequently, the cubic circulant graphs are of the form $C_{2k}(\alpha, k)$ where $1 \leq \alpha \leq k$.
The structural result by Davis and Domke [14] for cubic circulant graphs is as follows.

Theorem 2.4.4. *[14] Let* $1 \leq \alpha \leq k$ *and* $d = gcd(2k, \alpha)$.

(i) If $\frac{2k}{d}$ is odd, then $C_{2k}(\alpha, k)$ is isomorphic to $\frac{d}{2}$ copies of $C_{\frac{4k}{d}}(2, \frac{2k}{d})$.

(ii) If $\frac{2k}{d}$ is even, then $C_{2k}(\alpha, k)$ is isomorphic to d copies of $C_{\frac{2k}{d}}(1, \frac{k}{d})$.

So, the connected cubic circulant graphs are isomorphic to either $C_{2k}(2, k)$ with odd k (>1) or $C_{2k}(1, k)$ for any $k \geq 2$(for the first circulant, if k is not odd, then this circulant is not connected by using Theorem 2.4.4).

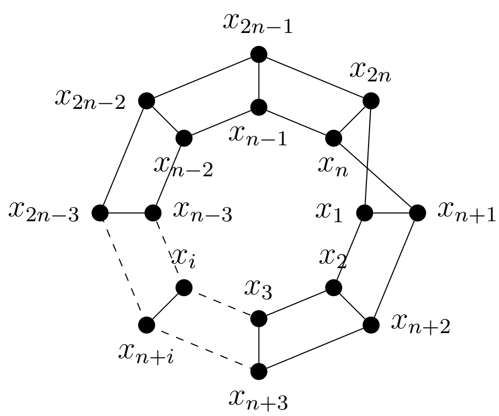


Figure 2.16: From left to right $C_{2n}(1, n)$ and $C_{2n}(2, n)$.

Chapter 3

Depth, Stanley depth and regularity

The present chapter concerns the Stanley depth and depth (named after Richard Stanley [1] in 1982) of $\mathbb{Z}^n$-graded modules over a commutative ring, including the Stanley's conjecture. It summarises the known bounds and values of Stanley depth and depth for monomial ideals of polynomial rings and their quotients. Throughout this chapter, ring R has identity $1 \neq 0$.

3.1 Depth

Definition 3.1.1. Consider an S module N. A zero divisor of a module N is an element $0 \neq s \in S$ such that $sn = 0$, where $0 \neq n \in N$.

Definition 3.1.2. A ring R is called Noetherian if it satisfies the ascending chain condition on its ideals that is given any chain:

$$Z_1 \subset Z_2 \subset \ldots \subset Z_{k+1} \subset \ldots$$

there exists a positive integer n such that

$$Z_n = Z_{n+1} = \ldots.$$

Example 3.1.3. A ring $R = K[x_1, \ldots, x_n]$ with n variables is a Noetherian ring.

Definition 3.1.4. Suppose $\mathcal{W}$ be an $\mathcal{T}$-module. An element t of $\mathcal{T}$ which is non-zero is $\mathcal{W}$ regular if for every $w \in \mathcal{W}$, $tw = 0$ implies $w = 0$.

Definition 3.1.5. A sequence $\alpha = \alpha_1, \ldots, \alpha_n$ of elements of S is said to be N-regular if it satisfies the given axioms:

1. α_k is $N/(\alpha_1, \ldots, \alpha_{k-1})N$ regular for any k;

2. $N \neq (\alpha)N$.

Example 3.1.6. Consider $R = K[x_1, x_2, x_3]$ as a module over itself. As x_1 is regular in $R/(0)R$, x_2 is regular in $R/(x_1)R$, x_3 is regular in $R/(x_1, x_2)R$. x_1, x_2, x_3 is the M-regular sequence in R.

Definition 3.1.7. Consider N, a finitely generated S-module, and let n be unique maximal ideal of local Noetherian ring S. Then, depth of N is common length of all maximal N-sequences in n, represented by depth(N).

Example 3.1.8. Let $Z = (\varrho_1^2, \varrho_2\varrho_3)$ be an ideal of polynomial ring $S = K[\varrho_1, \varrho_2, \varrho_3]$. Then $\mathrm{Ass}(S/Z) = \{(\varrho_1, \varrho_2), (\varrho_1, \varrho_3)\}$. Since the set of zero divisors, say Z is the union of all associated primes, therefore clearly $\varrho_2 - \varrho_3 \notin Z$ and hence a regular element. Thus, $\mathrm{depth}(S/Z) \geq 1$. Also, as $\mathrm{depth}(S/Z) \leq \dim(S/Z) = 1$, hence $\mathrm{depth}(S/Z) = 1$.

Definition 3.1.9. Consider a ring of polynomials S in n variables and let $\mathcal{H}$ be its ideal, then $S/\mathcal{H}$ is Cohen-Macaulay [15] if

$$\dim(S/\mathcal{H}) = \mathrm{depth}(S/\mathcal{H}).$$

Lemma 3.1.10. *(Depth Lemma)[15] Given a short exact sequence* $0 \to \xi_1 \to \xi_2 \to \xi_3 \to 0$ *of* $\mathcal{S}$*-modules where* $\mathcal{S}$ *is a local ring, then*

1. $depth(\xi_2) \geq \min\{depth(\xi_3), depth(\xi_1)\}$.

2. $depth(\xi_3) \geq \min\{depth(\xi_2), depth(\xi_1) + 1\}$.

3. $depth(\xi_1) \geq \min\{depth(\xi_3) - 1, depth(\xi_2)\}$.

3.2 Stanley decomposition and Stanley depth

Definition 3.2.1. Let $P = Z[\alpha_1, \dots, \alpha_n]$ be a ring of polynomials and consider $\mathbb{Z}^n$-graded P-module U. Suppose $u \in U$ and also consider $V \subset \{\alpha_1, \dots, \alpha_n\}$, then $uZ[V]$ represents the Z-subspace of U, whose generating set comprises of elements (homogeneous in degree) of the form ur, where r is a monomial in $Z[V]$. If $uZ[V]$ is a free $Z[V]$-module then it is known as a Stanley space of dimension $|V|$. A Stanley decomposition of U is defined as:

$$\mathcal{D} \,:\, U = \bigoplus_{j=1}^{k} r_j Z[V_j],$$

and

$$\operatorname{sdepth} \mathcal{D} = \min\{\, |V_j|\,,\, j = 1, \dots, k\}.$$

Also,

$$\operatorname{sdepth}_s(U) = \max\{\, \operatorname{sdepth} \mathcal{D} \,:\, \mathcal{D} \text{ is a Stanley decomposition of } U\}.$$

3.2.1 Stanley's conjecture

In 1982, Stanley [1] gave a conjecture about an upper bound for the depth of a $\mathbb{Z}^n$-graded S-modules.

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M).$$

It has been immensely significant as it gave a comparison of two very different invariants of modules. For a ring of polynomials S in n number of variables, Consider $I \subset S$ be the monomial ideal, then for $n \leq 3$, $n = 4$ and $n = 5$ the conjecture for S/I is proved by Apel [16], Anwar [17] and Popescu [18], respectively. Also, when I is an intersection of three monomial prime ideals, or three monomial primary ideals or four monomial prime ideals of S, the conjecture holds for I. But in 2016, Duval et al. [2] proved that Stanley's conjecture is generally false, by giving a counter example for the module of type S/I for which the conjecture does not hold.

3.2.2 Method of computing Stanley depth for squarefree monomial ideals

In 2009, Herzog et al. [19] gave a method of computing the lower bound for Stanley depth of monomial ideals in finite number of steps by using posets. Assume E be a squarefree monomial ideal with $G(E) = (e_1, \ldots, e_m)$. The characteristic poset of E w.r.t $g = (1, \ldots, 1)$, written as $\mathcal{P}_E^{(1,\ldots,1)}$ is defined as

$$\mathcal{P}_E^{(1,\ldots,1)} = \{\gamma \subset [n] \mid \gamma \text{ contains } \operatorname{supp}(e_j) \textit{ for some } j\},$$

where $\operatorname{supp}(e_j) = \{i : x_i | e_j\} \subseteq [n] := \{1, \ldots, n\}$. For each $\rho, \sigma \in \mathcal{P}_E^{(1,\ldots,1)}$ where $\rho \subseteq \sigma$, and

$$[\rho\,,\,\sigma] = \{\gamma \in \mathcal{P}_E^{(1,\ldots,1)} \,:\, \rho \subseteq \gamma \subseteq \sigma\}.$$

Let $\mathcal{P} \,:\, \mathcal{P}_E^{(1,\ldots,1)} = \cup_{j=1}^{k}[\gamma_j\,,\,\eta_j]$ be a partition of $\mathcal{P}_E^{(1,\ldots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\operatorname{supp}(x^{s(j)}) = \gamma_j$, then the Stanley decomposition $\mathcal{D}(\mathcal{P})$ of E is given by

$$\mathcal{D}(\mathcal{P}) \,:\, E = \bigoplus_{j=1}^{r} x^{s(j)} K[\{x_k \,|\, k \in \eta_j\}].$$

Clearly, $\operatorname{sdepth} \mathcal{D}(\mathcal{P}) = \min\{|\eta_1|, \ldots, |\eta_r|\}$ and

$$\operatorname{sdepth}(E) = \max\{\operatorname{sdepth} \mathcal{D}(\mathcal{P}) \mid \mathcal{P} \textit{ is a partition of } \mathcal{P}_E^{(1,\ldots,1)}\}.$$

Example 3.2.2. Consider $I = (\varrho_1\varrho_4, \varrho_1\varrho_2, \varrho_2\varrho_4, \varrho_1\varrho_3) \subset K[\varrho_1, \varrho_2, \varrho_3, \varrho_4]$ be a squarefree monomial ideal and $J = 0$. Set $\sigma_1 = (1, 0, 0, 1)$, $\sigma_2 = (1, 1, 0, 0)$, $\sigma_3 = (0, 1, 0, 1)$ and $\sigma_4 = (1, 0, 1, 0)$. Thus I is generated by $\varrho^{\sigma_1}, \varrho^{\sigma_2}, \varrho^{\sigma_3}, \varrho^{\sigma_4}$ and choose $g = (1, 1, 1, 1)$. The poset $P = P^g_{I/J}$ is given by

$$P = \{(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,0,1), (1,1,1,0), (1,1,0,1), (1,0,1,1),$$
$$(0,1,1,1), (1,1,1,1)\}.$$

Partitions of P are given by

$$\mathcal{P}_1 : \; [(1,1,0,0), (1,1,0,0)] \bigcup [(1,0,1,0), (1,0,1,0)] \bigcup [(0,1,0,1), (0,1,0,1)] \bigcup$$
$$[(1,0,0,1), (1,0,0,1)] \bigcup [(1,1,1,0), (1,1,1,0)] \bigcup [(1,1,0,1), (1,1,0,1)] \bigcup$$
$$[(1,0,1,1), (1,0,1,1)] \bigcup [(0,1,1,1), (0,1,1,1)] \bigcup [(1,1,1,1), (1,1,1,1)].$$

$$\mathcal{P}_2: \ [(1,1,0,0),(1,1,1,0)] \bigcup [(1,0,0,1),(1,1,0,1)] \bigcup [(1,0,1,0),(1,0,1,1)] \bigcup [(0,1,0,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) \ := \ & \varrho_1\varrho_2 K[\varrho_1,\varrho_2] \oplus \varrho_1\varrho_3 K[\varrho_1,\varrho_3] \oplus \varrho_1\varrho_4 K[\varrho_1,\varrho_4] \oplus \varrho_2\varrho_4 K[\varrho_2,\varrho_4] \oplus \\ & \varrho_2\varrho_3\varrho_4 K[\varrho_2,\varrho_3,\varrho_4] \oplus \varrho_1\varrho_2\varrho_4 K[\varrho_1,\varrho_2,\varrho_4] \oplus \varrho_1\varrho_3\varrho_4 K[\varrho_1,\varrho_3,\varrho_4] \oplus \\ & \varrho_1\varrho_2\varrho_3 K[\varrho_1,\varrho_2,\varrho_3] \oplus \varrho_1\varrho_2\varrho_3\varrho_4 K[\varrho_1,\varrho_2,\varrho_3,\varrho_4].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) \ := \ & \varrho_1\varrho_3 K[\varrho_1,\varrho_3,\varrho_4] \oplus \varrho_1\varrho_4 K[\varrho_1,\varrho_2,\varrho_4] \oplus \varrho_1\varrho_2 K[\varrho_1,\varrho_2,\varrho_3] \oplus \varrho_2\varrho_4 K[\varrho_2,\varrho_3,\varrho_4] \oplus \\ & \varrho_1\varrho_2\varrho_3\varrho_4 K[\varrho_1,\varrho_2,\varrho_3,\varrho_4].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(I) \ & \geq \ \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\} \\ & = \ \max\{2,3\} \\ & = \ 3.\end{aligned}$$

Since I is not principal, so $\operatorname{sdepth}(I) = 3$.

Example 3.2.3. Consider $I = (\varrho_1\varrho_4, \varrho_2\varrho_5, \varrho_3\varrho_4\varrho_5) \subset K[\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5]$ and $J = 0$. Set $\sigma_1 = (1,0,0,1,0)$, $\sigma_2 = (0,1,0,0,1)$ and $\sigma_3 = (0,0,1,1,1)$. Thus I is generated by $\varrho^{\sigma_1}, \varrho^{\sigma_2}, \varrho^{\sigma_3}$ and choose $g = (1,1,1,1,1)$. The poset $P = P^g_{I/J}$ is given by

$$\begin{aligned}P = \{ & (1,0,0,1,0),(0,1,0,0,1),(1,1,0,1,0),(1,1,0,0,1),(1,0,1,1,0),(1,0,0,1,1), \\ & (0,1,1,0,1),(0,1,0,1,1),(0,0,1,1,1),(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1), \\ & (1,0,1,1,1),(0,1,1,1,1),(1,1,1,1,1)\}.\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1 :\ & [(1,0,0,1,0),(1,0,0,1,0)] \bigcup [(0,1,0,0,1),(0,1,0,0,1)] \bigcup \\ & [(1,1,0,1,0),(1,1,0,1,0)] \bigcup [(1,0,0,1,1),(1,0,0,1,1)] \bigcup \\ & [(1,1,0,0,1),(1,1,0,0,1)] \bigcup [(1,0,1,1,0),(1,0,1,1,0)] \bigcup \\ & [(0,1,1,0,1),(0,1,1,0,1)] \bigcup [(0,1,0,1,1),(0,1,0,1,1)] \bigcup \\ & [(0,0,1,1,1),(0,0,1,1,1)] \bigcup [(1,1,1,1,0),(1,1,1,1,0)] \bigcup \\ & [(1,1,1,0,1),(1,1,1,0,1)] \bigcup [(1,1,0,1,1),(1,1,0,1,1)] \bigcup \\ & [(1,0,1,1,1),(1,0,1,1,1)] \bigcup [(0,1,1,1,1),(0,1,1,1,1)] \bigcup \\ & [(1,1,1,1,1),(1,1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2 :\ & [(1,0,0,1,0),(1,1,0,1,0)] \bigcup [(0,1,0,0,1),(1,1,0,0,1)] \bigcup \\ & [(1,0,1,1,0),(1,1,1,1,0)] \bigcup [(1,0,0,1,1),(1,1,0,1,1)] \bigcup \\ & [(0,1,1,0,1),(1,1,1,0,1)] \bigcup [(0,1,0,1,1),(0,1,1,1,1)] \bigcup \\ & [(0,0,1,1,1),(1,0,1,1,1)] \bigcup [(1,1,1,1,1),(1,1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_3 :\ & [(1,0,0,1,0),(1,1,1,1,0)] \bigcup [(0,1,0,0,1),(1,1,1,0,1)] \bigcup \\ & [(1,0,0,1,1),(1,1,0,1,1)] \bigcup [(0,1,0,1,1),(0,1,1,1,1)] \bigcup \\ & [(0,0,1,1,1),(1,0,1,1,1)] \bigcup [(1,1,1,1,1),(1,1,1,1,1)].\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) \ :=\ & \varrho_1\varrho_4K[\varrho_1,\varrho_4] \oplus \varrho_2\varrho_5K[\varrho_2,\varrho_5] \oplus \varrho_1\varrho_2\varrho_4K[\varrho_1,\varrho_2,\varrho_4] \oplus \varrho_1\varrho_2\varrho_5K[\varrho_1,\varrho_2,\varrho_5]\oplus \\ & \varrho_1\varrho_3\varrho_4K[\varrho_1,\varrho_3,\varrho_4] \oplus \varrho_1\varrho_4\varrho_5K[\varrho_1\varrho_4\varrho_5] \oplus \varrho_2\varrho_3\varrho_5K[\varrho_2,\varrho_3,\varrho_5]\oplus \\ & \varrho_2\varrho_4\varrho_5K[\varrho_2,\varrho_4,\varrho_5] \oplus \varrho_3\varrho_4\varrho_5K[\varrho_3,\varrho_4,\varrho_5] \oplus \varrho_1\varrho_2\varrho_3\varrho_4K[\varrho_1,\varrho_2,\varrho_3,\varrho_4]\oplus \\ & \varrho_1\varrho_2\varrho_3\varrho_5K[\varrho_1,\varrho_2,\varrho_3,\varrho_5] \oplus \varrho_1\varrho_2\varrho_4\varrho_5K[\varrho_1,\varrho_2,\varrho_4,\varrho_5] \oplus \varrho_1\varrho_3\varrho_4\varrho_5K[\varrho_1,\varrho_3,\varrho_4,\varrho_5]\oplus \\ & \varrho_2\varrho_3\varrho_4\varrho_5K[\varrho_2,\varrho_3,\varrho_4,\varrho_5] \oplus \varrho_1\varrho_2\varrho_3\varrho_4\varrho_5K[\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) \ :=\ & \varrho_1\varrho_4K[\varrho_1,\varrho_2,\varrho_4] \oplus \varrho_2\varrho_5K[\varrho_1,\varrho_2,\varrho_5] \oplus \varrho_1\varrho_3\varrho_4K[\varrho_1,\varrho_2,\varrho_3,\varrho_4]\oplus \\ & \varrho_1\varrho_4\varrho_5K[\varrho_1,\varrho_2,\varrho_4,\varrho_5] \oplus \varrho_2\varrho_3\varrho_5K[\varrho_1,\varrho_2,\varrho_3,\varrho_5] \oplus \varrho_2\varrho_4\varrho_5K[\varrho_2,\varrho_3,\varrho_4,\varrho_5]\oplus \\ & \varrho_3\varrho_4\varrho_5K[\varrho_1,\varrho_3,\varrho_4,\varrho_5] \oplus \varrho_1\varrho_2\varrho_3\varrho_4\varrho_5K[\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_3) &:= \varrho_1\varrho_4 K[\varrho_1,\varrho_2,\varrho_3,\varrho_4] \oplus \varrho_2\varrho_5 K[\varrho_1,\varrho_2,\varrho_3,\varrho_5] \oplus \varrho_1\varrho_4\varrho_5 K[\varrho_1,\varrho_2,\varrho_4,\varrho_5] \oplus \\ &\quad \varrho_2\varrho_4\varrho_5 K[\varrho_2,\varrho_3,\varrho_4,\varrho_5] \oplus \varrho_3\varrho_4\varrho_5 K[\varrho_1,\varrho_3,\varrho_4,\varrho_5] \oplus \varrho_1\varrho_2\varrho_3\varrho_4\varrho_5 K[\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(I) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\ \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\,,\ \operatorname{sdepth}(\mathcal{D}(\mathcal{P}_3))\} \\ &= \max\{2,3,4\} \\ &= 4.\end{aligned}$$

The next example illustrates the method of finding the Stanley depth of S/I.

Example 3.2.4. For $S = K[\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5]$, consider $I = (\varepsilon_1\varepsilon_5, \varepsilon_2\varepsilon_3\varepsilon_4, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_4)$. Then choose $g = (1,1,1,1,1)$ and the poset $P = P^g_{S/I}$ is given by

$$\begin{aligned}P = \{&(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1),\\ &(1,0,1,0,0),(0,1,1,0,0),(0,1,0,1,0),(0,1,0,0,1),(0,0,1,1,0),(0,0,1,0,1),\\ &(0,0,0,1,1),(0,1,1,0,1),(0,1,0,1,1),(0,0,1,1,1)\}.\end{aligned}$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}_1:\ &[(0,0,0,0,0),(0,0,1,1,1)] \bigcup [(1,0,0,0,0),(1,0,0,0,0)] \bigcup \\ &[(0,1,0,0,0),(0,1,0,0,0)] \bigcup [(0,0,1,0,0),(0,0,1,0,0)] \bigcup \\ &[(0,0,0,1,0),(0,0,0,1,0)] \bigcup [(0,0,0,0,1),(0,0,0,0,1)] \bigcup \\ &[(1,0,1,0,0),(1,0,1,0,0)] \bigcup [(0,1,1,0,0),(0,1,1,0,0)] \bigcup \\ &[(0,1,0,1,0),(0,1,0,1,0)] \bigcup [(0,1,0,0,1),(0,1,0,0,1)] \bigcup \\ &[(0,0,1,1,0),(0,0,1,1,0)] \bigcup [(0,0,1,0,1),(0,0,1,0,1)] \bigcup \\ &[(0,0,0,1,1),(0,0,0,1,1)] \bigcup [(0,1,1,0,1),(0,1,1,0,1)] \bigcup \\ &[(0,1,0,1,1),(0,1,0,1,1)].\end{aligned}$$

$$\begin{aligned}\mathcal{P}_2:\ &[(0,0,0,0,0),(1,0,1,0,0)] \bigcup [(0,1,0,0,0),(0,1,1,0,0)] \bigcup \\ &[(0,0,0,1,0),(0,1,0,1,0)] \bigcup [(0,0,0,0,1),(0,1,0,0,1)] \bigcup \\ &[(0,0,0,1,1),(0,1,0,1,1)] \bigcup [(0,0,1,0,1),(0,1,1,0,1)] \bigcup \\ &[(0,0,1,1,0),(0,0,1,1,1)].\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathcal{D}(\mathcal{P}_1) := \; & K[\varepsilon_3,\varepsilon_4,\varepsilon_5]\oplus\varepsilon_1K[\varepsilon_1]\oplus\varepsilon_2K[\varepsilon_2]\oplus\varepsilon_3K[\varepsilon_3]\oplus\varepsilon_4K[\varepsilon_4]\oplus\varepsilon_5K[\varepsilon_5]\oplus\\ & \varepsilon_1\varepsilon_3K[\varepsilon_1,\varepsilon_3]\oplus\varepsilon_2\varepsilon_3K[\varepsilon_2,\varepsilon_3]\oplus\varepsilon_2\varepsilon_4K[\varepsilon_2,\varepsilon_4]\oplus\varepsilon_2\varepsilon_5K[\varepsilon_2,\varepsilon_5]\oplus\\ & \varepsilon_3\varepsilon_4K[\varepsilon_3,\varepsilon_4]\oplus\varepsilon_3\varepsilon_5K[\varepsilon_3,\varepsilon_5]\oplus\varepsilon_4\varepsilon_5K[\varepsilon_4,\varepsilon_5]\oplus\varepsilon_2\varepsilon_3\varepsilon_5K[\varepsilon_2,\varepsilon_3,\varepsilon_5]\oplus\\ & \varepsilon_2\varepsilon_4\varepsilon_5K[\varepsilon_2,\varepsilon_4,\varepsilon_5].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\mathcal{P}_2) := \; & K[\varepsilon_1,\varepsilon_3]\oplus\varepsilon_2K[\varepsilon_2,\varepsilon_3]\oplus\varepsilon_4K[\varepsilon_2,\varepsilon_4]\oplus\varepsilon_5K[\varepsilon_2,\varepsilon_5]\oplus\varepsilon_4\varepsilon_5K[\varepsilon_2,\varepsilon_4,\varepsilon_5]\oplus\\ & \varepsilon_3\varepsilon_5K[\varepsilon_2,\varepsilon_3,\varepsilon_5]\oplus\varepsilon_3\varepsilon_4K[\varepsilon_3,\varepsilon_4,\varepsilon_5].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(S/I) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\,\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\}\\ &= \max\{1,2\}\\ &= 2.\end{aligned}$$

Example 3.2.5. Let $S = M[\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5,\varrho_6]$, consider $U = (\varrho_1\varrho_3,\varrho_2\varrho_5,\varrho_4\varrho_6,\varrho_1\varrho_4\varrho_6)$. Then select $g = (1,1,1,1,1,1)$ and the poset $\rho = \rho^g_{S/U}$ is given by

$$\begin{aligned}\rho = \{ & (0,0,0,0,0,0),(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),(0,0,0,1,0,0),\\ & (0,0,0,0,1,0),(0,0,0,0,0,1),(1,1,0,0,0,0),(1,0,0,1,0,0),(1,0,0,0,1,0),\\ & (1,0,0,0,0,1),(0,1,1,0,0.0),(0,1,0,1,0.0),(0,0,1,1,0,0),(0,0,0,1,1,0),\\ & (0,0,1,0,0,1),(0,0,0,1,1,0),(0,0,0,1,0.1).(0,0,0,0,1,1),(1,1,0,1,0,0),\\ & (1,0,0,1,1,0),(1,0,0,0,1,1),(0,1,1,1,0,0),(0,0,1,1,1,0),(0,0,1,1,0,1),\\ & (0,0,1,0,1,1),(0,0,0,1,1,1),(0,0,1,1,1,1)\}.\end{aligned}$$

The partitions of ρ can be written as

$$\begin{aligned}
\rho_1 :\ & [(0,0,0,0,0,0),(1,0,0,0,0,0)] \bigcup [(0,1,0,0,0,0),(0,1,0,0,0,0)] \bigcup \\
& [(0,0,1,0,0,0),(0,0,1,0,0,0)] \bigcup [(0,0,0,1,0,0),(0,0,0,1,0,0)] \bigcup \\
& [(0,0,0,0,1,0),(0,0,0,0,1,0)] \bigcup [(0,0,0,0,0,1),(0,0,0,0,0,1)] \bigcup \\
& [(1,1,0,0,0,0),(1,1,0,0,0,0)] \bigcup [(1,0,0,1,0,0),(1,0,0,1,0,0)] \bigcup \\
& [(0,1,0,1,0,0),(0,1,0,1,0,0)] \bigcup [(0,0,1,1,0,0),(0,0,1,1,0,0)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,1,0,1),(0,0,0,1,0,1)] \bigcup \\
& [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup [(1,1,0,1,0.0),(1,1,0,1,0,0)] \bigcup \\
& [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup \\
& [(1,1,0,1,0,0),(1,1,0,1,0,0)] \bigcup [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup \\
& [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup [(0,1,1,1,0,0),(0,1,1,1,0,0)] \bigcup \\
& [(0,0,1,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,1,0,1),(0,0,1,1,0,1,)] \bigcup \\
& [(0,0,0,1,1,1),(0,0,0,1,1,1)] \bigcup [(0,0,1,1,1,1),(0,0,1,1,1,1,)].
\end{aligned}$$

$$\begin{aligned}
\rho_2 :\ & [(0,0,0,0,0,0),(1,1,0,1,0,0)] \bigcup [(0,0,1,0,0,0),(0,1,1,1,0,0)] \bigcup \\
& [(0,0,0,0,1,0),(1,0,0,1,1,0)] \bigcup [(0,0,0,0,0,1),(1,0,0,0,1,1)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,0,0,1),(0,0,0,1,0,1)] \bigcup \\
& [(0,0,0,1,0,1),(0,0,0,1,1,1)].
\end{aligned}$$

So the corresponding Stanley decomposition is of the partitions will be

$$\begin{aligned}
\mathcal{D}(\rho_1) \ :=\ & M[\varrho_1] \oplus \varrho_2 M[\varrho_2] \oplus \varrho_3 M[\varrho_3] \oplus \varrho_4 M[\varrho_4] \oplus \varrho_5 M[\varrho_5] \oplus \varrho_6 M[\varrho_6] \oplus \\
& \varrho_1\varrho_2 M[\varrho_1,\varrho_2] \oplus \varrho_1\varrho_4 M[\varrho_1,\varrho_4] \oplus \varrho_2\varrho_4 M[\varrho_2,\varrho_4] \oplus \varrho_3\varrho_4 M[\varrho_3,\varrho_4] \oplus \varrho_4\varrho_5 M[\varrho_4,\varrho_5] \oplus \\
& \varrho_4\varrho_6 M[\varrho_4,\varrho_6] \oplus \varrho_5\varrho_6 M[\varrho_5,\varrho_6] \oplus \varrho_1\varrho_2\varrho_4 M[\varrho_1,\varrho_2,\varrho_4] \oplus \varrho_1\varrho_4\varrho_5 M[\varrho_1,\varrho_4,\varrho_5] \oplus \\
& \varrho_1\varrho_5\varrho_6 M[\varrho_1,\varrho_5,\varrho_6] \oplus \varrho_4\varrho_5 M[\varrho_4,\varrho_5] \oplus \varrho_4\varrho_5 K[\varrho_4,\varrho_5] \oplus \varrho_5\varrho_6 M[\varrho_5,\varrho_6] \oplus \\
& \varrho_1\varrho_2\varrho_4 M[\varrho_1,\varrho_2,\varrho_4] \oplus \varrho_1\varrho_4\varrho_5 M[\varrho_1,\varrho_4,\varrho_5] \oplus \varrho_1\varrho_5\varrho_6 M[\varrho_1,\varrho_5,\varrho_6] \oplus \varrho_2\varrho_3\varrho_4 M[\varrho_2,\varrho_3,\varrho_4] \oplus \\
& \varrho_3\varrho_4\varrho_5 M[\varrho_3,\varrho_4,\varrho_5] \oplus \varrho_3\varrho_4\varrho_6 M[\varrho_3,\varrho_4,\varrho_6] \oplus \varrho_4\varrho_5\varrho_6 M[\varrho_4,\varrho_5,\varrho_6] \oplus \\
& \varrho_3\varrho_4\varrho_5\varrho_6 M[\varrho_3,\varrho_4,\varrho_5,\varrho_6].
\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\rho_2) &:= M[\varrho_1,\varrho_2,\varrho_4]\oplus\varrho_3 M[\varrho_2,\varrho_3,\varrho_4]\oplus\varrho_5 M[\varrho_1,\varrho_4,\varrho_5]\oplus\varrho_6 M[\varrho_1,\varrho_5,\varrho_6]\oplus\\ &\quad \varrho_4\varrho_5 M[\varrho_3,\varrho_4,\varrho_5]\oplus\varrho_3\varrho_6 M[\varrho_4,\varrho_6]\oplus\varrho_4\varrho_6 M[\varrho_4,\varrho_5,\varrho_6].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(S/U) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1)),\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\}\\ &= \max\{1,3\}\\ &= 3.\end{aligned}$$

Some fundamental results on Stanley depth and depth of S-modules are given below.

Theorem 3.2.6. *[20, Theorem 1.3] Let $k_1,\dots,k_m$ be some positive integers, then*

$$\operatorname{sdepth}((u_1^{k_1},\dots,u_m^{k_m}))=\operatorname{sdepth}((u_1,\dots,u_m))=\lceil\frac{m}{2}\rceil.$$

In particular, for any $1\leq n\leq m$

$$\operatorname{sdepth}((u_1^{k_1},\dots,u_n^{k_n}))=m-n+\lceil\frac{n}{2}\rceil.$$

Proposition 3.2.7. *[21, Proposition 2.7] For $J\subset\mathcal{M}$ and $\forall\ v\notin J$,*

1. $\operatorname{sdepth}_{\mathcal{M}}(J:v)\geq\operatorname{sdepth}_{\mathcal{M}}(J)$, *[22, Proposition 1.3]*
2. $\operatorname{depth}_{\mathcal{M}}(\mathcal{M}/(J:v))\geq\operatorname{depth}_{\mathcal{M}}(\mathcal{M}/J)$, *[23]*
3. $\operatorname{sdepth}_{\mathcal{M}}(\mathcal{M}/(J:v))\geq\operatorname{sdepth}_{\mathcal{M}}(\mathcal{M}/J)$.

Lemma 3.2.8. *[15, Lemma 3.6] Let I and J be two monomial ideals with $J\subset I$, suppose $\mathcal{M}'=\mathcal{M}[x_{n+1}]$, then*

$$\operatorname{depth}(I\mathcal{M}'/J\mathcal{M}')=\operatorname{depth}(I\mathcal{M}/J\mathcal{M})+1.$$

$$\operatorname{sdepth}(I\mathcal{M}'/J\mathcal{M}')=\operatorname{sdepth}(I\mathcal{M}/J\mathcal{M})+1.$$

Lemma 3.2.9. *[21, Proposition 1.1] Assume that $I\subset\mathcal{M}'=K[x_1,\dots,x_r]$, $J\subset\mathcal{M}''=K[x_{r+1},\dots,x_n]$ be monomial ideals, with $1\leq r\leq n$, then*

$$\operatorname{depth}_{\mathcal{M}}(\mathcal{M}/(I\mathcal{M}+J\mathcal{M}))=\operatorname{depth}_{\mathcal{M}'}(\mathcal{M}'/I)+\operatorname{depth}_{\mathcal{M}''}(\mathcal{M}''/J).$$

Lemma 3.2.10. *[23] Let a short exact sequence* $0 \to \mathcal{U}_1 \to \mathcal{U}_2 \to \mathcal{U}_3 \to 0$ *of* $\mathbb{Z}^n$*-graded* $\mathcal{M}$*-modules. Then*

$$\text{sdepth}(\mathcal{U}_2) \geq \min\{\text{sdepth}(\mathcal{U}_1), \text{sdepth}(\mathcal{U}_3)\}.$$

Lemma 3.2.11 ([24, Lemma 3.6])**.** *Consider a monomial ideal* $I \subset \mathcal{M}$ *and* $\bar{\mathcal{M}} = \mathcal{M}[x_{n+1}, \dots, x_{n+r}]$ *be a ring of polynomials then*

$$\text{depth}(\bar{\mathcal{M}}/I\bar{\mathcal{M}}) = \text{depth}(\mathcal{M}/I\mathcal{M}) + r \quad \textit{and} \quad \text{sdepth}(\bar{\mathcal{M}}/I\bar{\mathcal{M}}) = \text{sdepth}(\mathcal{M}/I\mathcal{M}) + r.$$

Corollary 3.2.12 ([23])**.** *Let* I *be a proper monomial ideal of* $\mathcal{M}$ *and* $u \notin I$*. Then* $\text{depth}(\mathcal{M}/(I : u)) \geq \text{depth}(\mathcal{M}/I)$.

Proposition 3.2.13 ([25, Proposition 2.7])**.** *Let* I *be a proper monomial ideal of* $\mathcal{M}$ *and* $u \notin I$*. Then* $\text{sdepth}(\mathcal{M}/(I : u)) \geq \text{sdepth}(\mathcal{M}/I)$.

3.3 Castelnuovo Mumford regularity of an ideal

Let E be an ideal of homogeneous degree in $S = K[x_1, x_2, ..., x_n]$. Then the minimal graded free resolution of E is given by

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{l,j}(E(G))} \longrightarrow \bigoplus_j S(-j)^{\beta_{l-1,j}(E(G))} \longrightarrow \dots \longrightarrow$$
$$\bigoplus_j S(-j)^{\beta_{0,j}(E(G))} \longrightarrow E(G) \longrightarrow 0$$

Since, $l \leq n$ and $\beta_{i,j}(E)$ is the graded $(i,j)th$ Betti number of $E(G)$. We also let $S(-j)$ denote the shifted polynomial ring in degree j.

Definition 3.3.1. The Castelnuovo Mumford regularity (or regularity) of an ideal E is

$$reg(E) = max\{j - i | \beta_{i,j}(E(G)) \neq 0\}.$$

Example 3.3.2. *Consider the edge ideal* $I(B_2) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1x_3, x_2x_4)$ *associated to the graph* B_2.

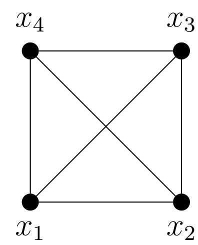


Figure 3.1: (B_2)

The minimal graded free resolution of $I(B_2)$ *is then given by*

$$0 \longrightarrow S^3(-4) \longrightarrow S^8(-3) \longrightarrow S^6(-2) \longrightarrow I(B_2) \longrightarrow 0$$

So, $\beta_{0,2}(I) = 6$, $\beta_{1,3}(I) = 8$, $\beta_{2,4}(I) = 3$
Then

$$reg(I) = max\{2-0, 3-1, 4-2\} = 2.$$

Example 3.3.3. *Consider the edge ideal*

$$I(A_1) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1x_3, x_2x_4, x_2x_5, x_3x_5)$$

associated to the graph A_1.

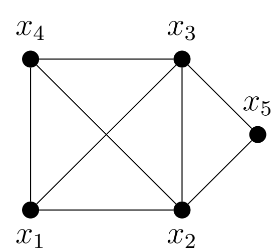


Figure 3.2: (A_1^*)

The minimal graded free resolution of $I(A_1)$ *is then given by*

$$0 \longrightarrow S^2(-5) \longrightarrow S^9(-4) \longrightarrow S^{14}(-3) \longrightarrow S^8(-2) \longrightarrow I(A_1) \longrightarrow 0$$

So, $\beta_{0,2}(I) = 8$, $\beta_{1,3}(I) = 14$, $\beta_{2,4}(I) = 9$, $\beta_{3,5}(I) = 2$
Then

$$reg(I) = max\{2-0, 3-1, 4-2, 5-3\} = 2.$$

Example 3.3.4. *Consider the edge ideal*

$$I(A_2) = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1x_3, x_2x_4, x_2x_5, x_3x_5, x_1x_6, x_4x_6)$$

associated to the graph A_2*.*

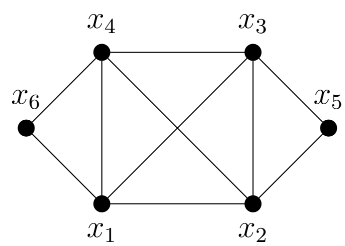


Figure 3.3: (A_2)

The minimal graded free resolution of $I(A_2)$ *is then given by*

$$0 \longrightarrow S^4(-5) \longrightarrow S^{15}(-4) \longrightarrow S^{20}(-3) \longrightarrow S^{10}(-2) \longrightarrow I(A_2) \longrightarrow 0$$

So, $\beta_{0,2}(I) = 10$, $\beta_{1,3}(I) = 20$, $\beta_{2,4}(I) = 15$, $\beta_{3,5}(I) = 4$
Then

$$reg(I) = max\{2-0, 3-1, 4-2, 5-3\} = 2.$$

Chapter 4

Stanley depth and depth for the monomial ideals of restricted partial strong product of path and cycle graphs

Let $V(P_n) := \{x_i : 1 \leq i \leq n\}$ and $V(P_m) := \{x_j : 1 \leq j \leq m\}$.
Let $A := \{x_i : i \in 2\mathbb{Z} \wedge 1 \leq i \leq n\} \subset V(P_n)$ and $B := \{x_j : j \notin 2\mathbb{Z} \wedge 1 \leq j \leq m\} \subset V(P_m)$. For defined A and B, let $P_{nA} \boxtimes_B P_m := P_{n,m}$. Since the graphs $P_{n,m}$ is defined by mn vertices, just for the ease, we name the vertices of $P_{n,m}$ by using m set of variables $\{x_{1j}, x_{2j}, ...x_{nj}\}$ where $1 \leq j \leq m$. For example of $P_{n,m}$ see Fig 4.1.

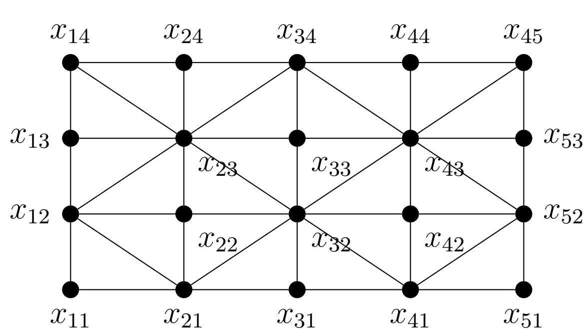


Figure 4.1: $P_{5,4}$

Let $V(C_n) := \{x_i : 1 \leq i \leq n \in 2\mathbb{Z}^+\}$ and $V(P_m) := \{x_j : 1 \leq j \leq m\}$. Let $A := \{x_i : i \in 2\mathbb{Z} \wedge 1 \leq i \leq n\} \subset V(C_n)$ and $B := \{x_j : j \notin 2\mathbb{Z} \wedge 1 \leq j \leq m\} \subset V(P_m)$. For defined A and B Let $C_{n\,A} \boxtimes_B P_m := C_{n,m}$. Since the graphs $C_{n,m}$ is defined on mn vertices, just for the ease, we name the vertices of $C_{n,m}$ by using m set of variables $\{x_{1j}, x_{2j}, ...x_{nj}\}$ where $1 \leq j \leq m$. For example of $C_{n,m}$ see Fig 4.2.

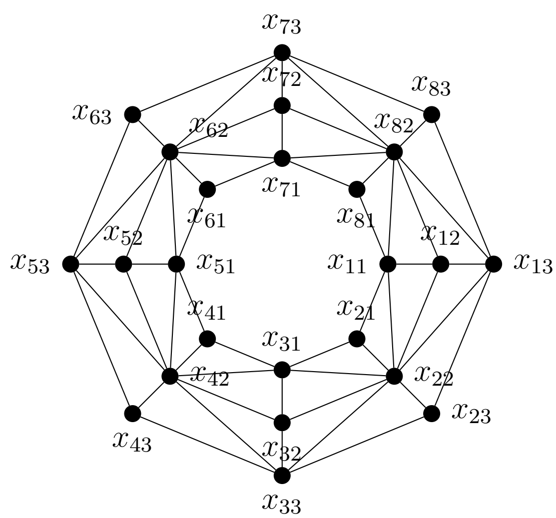


Figure 4.2: $C_{8,3}$

4.1 Bounds for depth of modules associated to $P_{n,m}$.

Let $1 \leq i \leq n$, for convenience, we take $x_i := x_{i1}, y_i := x_{i2}, z_i := x_{i3}$ (see Figure 4.2). We set $S_{n,1} := K[x_1, x_2, \ldots, x_n]$, $S_{n,2} := K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ and $S_{n,2} := K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n]$. Clearly $P_{n,1} \cong P_n$,

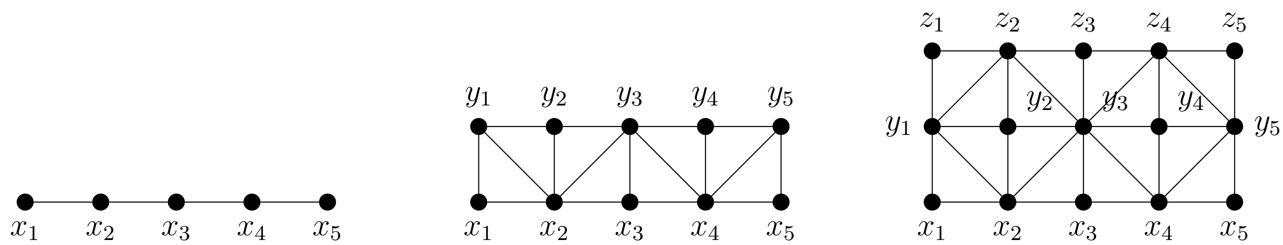


Figure 4.3: From left to right $P_{5,1}, P_{5,2}$ and $P_{5,3}$

The minimal generating sets of monomials for the edge ideals of $P_{n,2}$ and $P_{n,3}$ are given as:
If n is even

$$G(E(P_{n,2})) = \cup_{i=1}^{n-1}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{\cup_{i=1}^{\frac{n-2}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}\}\} \cup \{x_ny_{n-1}, x_ny_n\}.$$

$$G(E(P_{n,3})) = \cup_{i=1}^{n-1}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\} \cup \{\cup_{i=1}^{\frac{n-2}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}\} \cup \{x_ny_{n-1}, x_ny_n, y_{n-1}z_n, y_nz_n\}.$$

If n is odd

$$G(E(P_{n,2})) = \cup_{i=1}^{n-1}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{\cup_{i=1}^{\frac{n-1}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}\}\} \cup \{x_ny_n\}.$$

$$G(E(P_{n,3})) = \cup_{i=1}^{n-1}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\} \cup \{\cup_{i=1}^{\frac{n-1}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}\} \cup \{x_ny_n, y_nz_n\}.$$

Remark 4.1.1. *It is clear that for $n \geq 1$,*
$S_{n,1}/(E(P_{n,1})) \cong S/E(P_n)$. Thus by [26, Lemma 2.8] $\operatorname{depth}(S_{n,1}/E(P_{n,1})) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem 4.1.2. *Let $n \geq 1$, then*

$$\left\lceil \frac{n}{3} \right\rceil \leq \operatorname{depth}(S_{n,2}/E(P_{n,2})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

Proof. When $n = 1$ the result is contained by Remark 4.1.1. Let $n \geq 2$, As $diam(P_{n,2}) = n - 1$, Therefore, using [27]

$$\operatorname{depth}(S_{n,2}/E(P_{n,2})) \geq \left\lceil \frac{n}{3} \right\rceil. \tag{4.1}$$

We prove the reverse of inequality. The required inequality is trivial for $n = 2, 3$. Let $n \geq 4$, We hereby prove the result by the aid of mathematical induction on n. Since $y_1 \notin E(P_{n,2})$ hence by [23]

$$\operatorname{depth}(S_{n,2})/E(P_{n,2})) \leq \operatorname{depth}(S_{n,2})/(E(P_{n,2}) : y_1)). \tag{4.2}$$

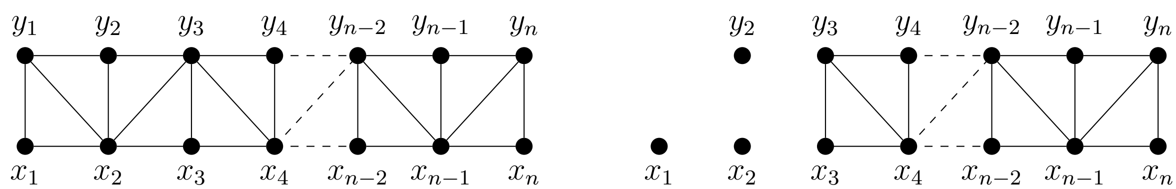


Figure 4.4: From left to right $E(P_{n,2})$ and $(E(P_{n,2}) : y_1)$.

It follows that

$$S_{n,2}/(E(P_{n,2}) : y_1)) \cong S_{n-2,2}/E(P_{n-2,2})[y_1].$$

Therefore, by induction

$$\operatorname{depth}(S_{n,2}/(E(P_{n,2}) : y_1)) \leq \left\lceil \frac{n-2-1}{2} \right\rceil + 1 = \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.3}$$

Now by combining Eqs. (4.1), (4.2) and (4.3), we get

$$\left\lceil \frac{n}{3} \right\rceil \leq \operatorname{depth}(S_{n,2}/E(P_{n,2})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

□

Theorem 4.1.3. *If $n \geq 1$, then*

$$\left\lceil \frac{n}{3} \right\rceil \leq \operatorname{depth}(S_{n,3}/E(P_{n,3})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

Proof. For $n = 1$ the result holds by Remark 4.1.1. Let $n \geq 2$, As $diam(P_{n,3}) = n - 1$, Therefore, using [27]

$$\operatorname{depth}(S_{n,3}/E(P_{n,3})) \geq \left\lceil \frac{n}{3} \right\rceil. \tag{4.4}$$

We prove the reverse of inequality. Above inequality is trivial for $n = 2, 3$. Let $n \geq 4$, We hereby prove the result by the aid of induction on n. Since $y_1 \notin E(P_{n,3})$, by [23]

$$\operatorname{depth}(S_{n,3})/E(P_{n,3})) \leq \operatorname{depth}(S_{n,3})/(E(P_{n,3}) : y_1)). \tag{4.5}$$

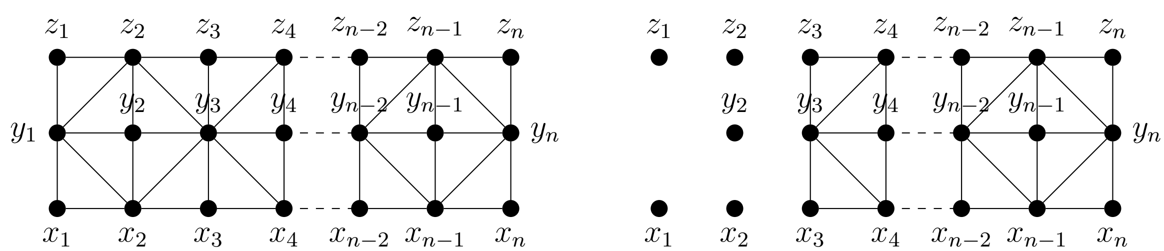


Figure 4.5: From left to right $E(P_{n,3})$ and $(E(P_{n,3}) : y_1)$.

It follows that

$$S_{n,3}/(E(P_{n,3}) : y_1)) \cong S_{n-2,3}/E(P_{n-2,3})[y_1].$$

Therefore, by induction

$$\operatorname{depth}(S_{n,3}/(E(P_{n,3}) : y_1)) \leq \left\lceil \frac{n-2-1}{2} \right\rceil + 1 = \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.6}$$

Now by combining Eqs. (4.4), (4.5) and (4.6), we get

$$\left\lceil \frac{n}{3} \right\rceil \leq \operatorname{depth}(S_{n,3}/E(P_{n,3})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

□

4.2 Bounds for depth of modules associated to some special graphs

In the case of $n \geq 2$, we thus construct a supergraph of $P_{n,2}$ represented by $P^*_{n,2}$ on $V(P^*_{n,2}) := V(P_{n,2}) \cup \{y_{n+1}\}$ and $E(P^*_{n,2}) := E(P_{n,2}) \cup \{y_n y_{n+1}\}$. Also we construct a supergraph of $P^*_{n,2}$ rrepresented by $P^{**}_{n,2}$ by the aid of the vertex set $V(P^{**}_{n,2}) := V(P^*_{n,2}) \cup \{y_{n+2}\}$ and edge set $E(P^{**}_{n,2}) := E(P^*_{n,2}) \cup \{y_1 y_{n+2}\}$ For examples of $P^*_{n,2}$ and $P^{**}_{n,2}$ (see Fig 4.6 and Fig 4.7). Let $S^*_{n,2} := S_{n,2}[y_{n+1}]$ and $S^{**}_{n,2} := S^*_{n,2}[y_{n+2}]$ then we have the following lemmas:

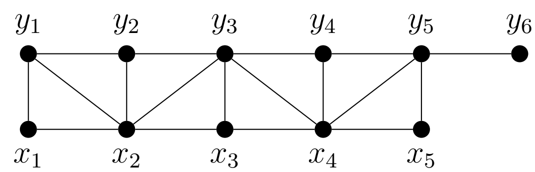


Figure 4.6: $(P^*_{5,2})$

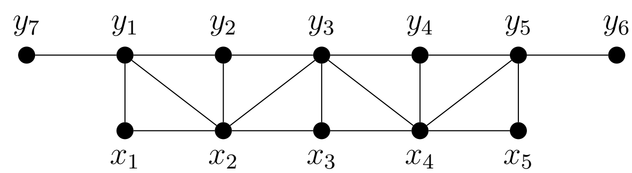


Figure 4.7: $(P^{**}_{5,2})$

Lemma 4.2.1. *Let* $n \geq 2$, *then,*

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \operatorname{depth}(S^*_{n,2}/E(P^*_{n,2})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

Proof. For $n = 1$, it follows from Remark 4.1.1. Let $n \geq 2$, As $diam(P^*_{n,2}) = n$, Thus by [27]

$$\operatorname{depth}(S^*_{n,2}/E(P^*_{n,2})) \geq \left\lceil \frac{n+1}{3} \right\rceil. \tag{4.7}$$

For the reverse inequality. The cases $n = 2, 3$ are trivial. So, let $n \geq 4$, As $y_n \notin E(P^*_{n,2})$, by [23]

$$\operatorname{depth}(S^*_{n,2})/E(P^*_{n,2})) \leq \operatorname{depth}(S^*_{n,2})/(E(P^*_{n,2}) : y_n)). \tag{4.8}$$

It follows that

$$S^*_{n,2}/(E(P^*_{n,2}) : y_n)) \cong S_{n-2,2}/E(P_{n-2,2})[y_n].$$

Therefore, by Theorem 4.1.2

$$\operatorname{depth}(S^*_{n,2}/(E(P^*_{n,2}) : y_n)) \leq \left\lceil \frac{n-2-1}{2} \right\rceil + 1 = \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.9}$$

Now by combining Eqs. (4.7), (4.8) and (4.9), we get

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \operatorname{depth}(S^*_{n,2}/E(P^*_{n,2})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

□

Lemma 4.2.2. *Let $n \geq 2$, then,*

$$\left\lceil \frac{n+2}{3} \right\rceil \leq \operatorname{depth}(S^{**}_{n,2}/E(P^{**}_{n,2})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

Proof. For $n = 1$ it follows from Remark 4.1.1. Let $n \geq 2$, Since $diam(P^{**}_{n,2}) = n + 1$, By [27]

$$\operatorname{depth}(S^{**}_{n,2}/E(P^{**}_{n,2})) \geq \left\lceil \frac{n+2}{3} \right\rceil. \tag{4.10}$$

Now, we prove the reverse of 4.10. The cases $n = 2, 3$ are trivial. Let $n \geq 4$,, Since $y_n \notin E(P^{**}_{n,2})$, by [23]

$$\operatorname{depth}(S^{**}_{n,2})/E(P^{**}_{n,2})) \leq \operatorname{depth}(S^*_{n,2})/(E(P^{**}_{n,2}) : y_n)). \tag{4.11}$$

It follows that

$$S^{**}_{n,2}/(E(P^{**}_{n,2}) : y_n)) \cong S^*_{n-2,2}/E(P^*_{n-2,2})[y_n].$$

Therefore, by Lemma 4.2.5

$$\operatorname{depth}(S^{**}_{n,2}/(E(P^{**}_{n,2}) : y_n)) \leq \left\lceil \frac{n-2-1}{2} \right\rceil + 1 = \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.12}$$

Now by combining Eqs. (4.10), (4.11) and (4.12), we get

$$\left\lceil \frac{n+2}{3} \right\rceil \leq \operatorname{depth}(S^{**}_{n,2}/E(P^{**}_{n,2})) \leq \left\lceil \frac{n-1}{2} \right\rceil.$$

□

For $n \geq 2$, One can construct $P^*_{n,3}$ with $V(P^*_{n,3}) := V(P_{n,3}) \cup \{z_{n+1}\}$ and $E(P^*_{n,3}) := E(P_{n,3}) \cup \{z_n z_{n+1}\}$. Similarly, $P^{**}_{n,3}$ can be constructed with $V(P^{**}_{n,3}) := V(P^*_{n,3}) \cup \{z_{n+2}\}$ and $E(P^{**}_{n,3}) := E(P^*_{n,3}) \cup \{z_1 z_{n+2}\}$. Figures 4.8 and 4.9 represent $P^*_{n,3}$ and $P^{**}_{n,3}$, respectively. Assume $S^*_{n,3} := S_{n,3}[z_{n+1}]$ and $S^{**}_{n,3} := S^*_{n,3}[z_{n+2}]$, we have the following lemmas:

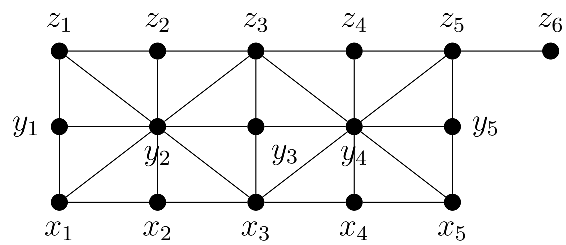


Figure 4.8: ($P^*_{5,3}$)

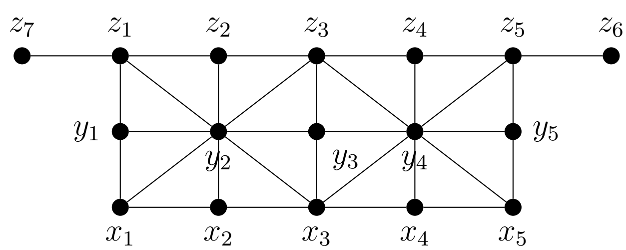


Figure 4.9: ($P^{**}_{5,3}$)

Lemma 4.2.3. *Let $n \geq 2$, then,*

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \operatorname{depth}(S^*_{n,3}/E(P^*_{n,3})) \leq \left\lceil \frac{n}{2} \right\rceil.$$

Proof. For $n = 1$ it follows from Remark 4.1.1. Let $n \geq 2$, Since $diam(P^*_{n,3}) = n$, By [27]

$$\operatorname{depth}(S^*_{n,3}/E(P^*_{n,3})) \geq \left\lceil \frac{n+1}{3} \right\rceil. \tag{4.13}$$

For the reverse of inequality. The cases $n = 2, 3$ are trivial. Let $n \geq 4$,, Since $y_{n-1} \notin E(P^*_{n,3})$, by [23]

$$\operatorname{depth}(S^*_{n,3})/E(P^*_{n,3})) \leq \operatorname{depth}(S^*_{n,3})/(E(P^*_{n,3}) : y_{n-1})). \tag{4.14}$$

It follows that

$$S^*_{n,3}/(E(P^*_{n,3}) : y_{n-1})) \cong S_{n-3,3}/E(P_{n-3,3})[y_{n-1}, z_{n+1}].$$

Therefore, by Theorem 4.1.2

$$\operatorname{depth}(S^*_{n,3}/(E(P^*_{n,3}) : y_{n-1})) \leq \left\lceil \frac{n-3-1}{2} \right\rceil + 2 = \left\lceil \frac{n}{2} \right\rceil. \tag{4.15}$$

Now by combining Eqs. (4.19), (4.20) and (4.21), we get

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \operatorname{depth}(S^*_{n,3}/E(P^*_{n,3})) \leq \left\lceil \frac{n}{2} \right\rceil.$$

□

Lemma 4.2.4. *Let $n \geq 2$, then,*

$$\left\lceil \frac{n+2}{3} \right\rceil \leq \operatorname{depth}(S^{**}_{n,3}/E(P^{**}_{n,3})) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof. For $n = 1$ it follows from Remark 4.1.1. Let $n \geq 2$, Since $diam(P^{**}_{n,3}) = n + 1$, Thus by [27]

$$\operatorname{depth}(S^{**}_{n,3}/E(P^{**}_{n,3})) \geq \left\lceil \frac{n+2}{3} \right\rceil. \tag{4.16}$$

We prove the reverse of inequality. The cases $n = 2, 3$ are trivial. Let $n \geq 4$,, Since $y_{n-1} \notin E(P^{**}_{n,3})$, by [23]

$$\operatorname{depth}(S^{**}_{n,3})/E(P^{**}_{n,3})) \leq \operatorname{depth}(S_{n,3})/(E(P^{**}_{n,3}) : y_{n-1})). \tag{4.17}$$

It follows that

$$S^{**}_{n,3}/(E(P^{**}_{n,3}) : y_{n-1})) \cong S^*_{n-3,3}/E(P^*_{n-3,3})[y_{n-1}, z_{n+1}].$$

Therefore, by Lemma 4.2.5

$$\operatorname{depth}(S^{**}_{n,3}/(E(P^{**}_{n,3}) : y_{n-1})) \leq \left\lceil \frac{n-3}{2} \right\rceil + 2 = \left\lceil \frac{n+1}{2} \right\rceil. \tag{4.18}$$

Now by combining Eqs. (4.22), (4.23) and (4.24), we get

$$\left\lceil \frac{n+2}{3} \right\rceil \leq \operatorname{depth}(S^{**}_{n,3}/E(P^{**}_{n,3})) \leq \left\lceil \frac{n+1}{2} \right\rceil.$$

□

For $n \geq 2$, one can construct $P^{\bullet}_{n,3}$ with $V(P^{\bullet}_{n,3}) := V(P_{n,3}) \cup \{x_{n+1}, z_{n+1}\}$ and $E(P^{\bullet}_{n,3}) := E(P_{n,3}) \cup \{x_n x_{n+1}, z_n z_{n+1}\}$. Similarly, $P^{\bullet\bullet}_{n,3}$ can be constructed by $V(P^{\bullet\bullet}_{n,3}) := V(P^{\bullet}_{n,3}) \cup \{x_{n+2}, z_{n+2}\}$ and $E(P^{\bullet\bullet}_{n,3}) := E(P^{\bullet}_{n,3}) \cup \{x_1 x_{n+2}, z_1 z_{n+2}\}$ (see Fig 4.10 and Fig 4.11). Consider $S^{\bullet}_{n,3} := S_{n,3}[x_{n+1}, z_{n+1}]$ and $S^{\bullet\bullet}_{n,3} := S^{\bullet}_{n,3}[x_{n+2}, z_{n+2}]$, then we have the following lemmas:

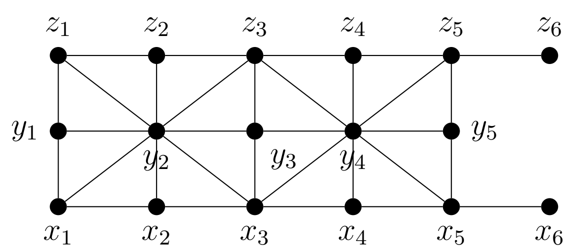


Figure 4.10: $(P^{\bullet}_{5,3})$

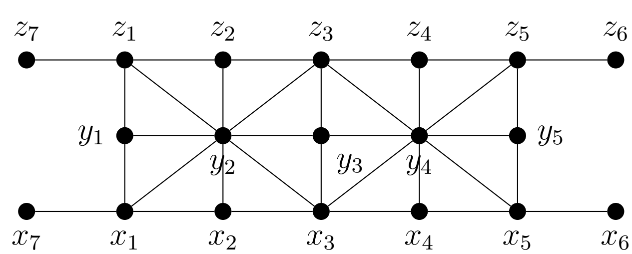


Figure 4.11: $(P^{\bullet\bullet}_{5,3})$

Lemma 4.2.5. *Let $n \geq 2$, then,*

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \operatorname{depth}(S^{\bullet}_{n,3}/E(P^{\bullet}_{n,3})) \leq \left\lceil \frac{n+2}{2} \right\rceil.$$

Proof. For $n = 1$ it follows from Remark 4.1.1. Let $n \geq 2$, Since $diam(P^{\bullet}_{n,3}) = n$, Thus by [27]

$$\operatorname{depth}(S_{n,3}/E(P^{\bullet}_{n,3})) \geq \left\lceil \frac{n+1}{3} \right\rceil. \tag{4.19}$$

For the reverse inequality. The cases $n = 2, 3$ are trivial. Let $n \geq 4$,, Since $y_{n-1} \notin E(P^{\bullet}_{n,3})$, by [23]

$$\operatorname{depth}(S^{\bullet}_{n,3})/E(P^{\bullet}_{n,3})) \leq \operatorname{depth}(S_{n,3})/(E(P^{\bullet}_{n,3}) : y_{n-1})). \tag{4.20}$$

It follows that

$$S^{\bullet}_{n,3}/(E(P^{\bullet}_{n,3}) : y_{n-1})) \cong S_{n-3,3}/E(P_{n-3,3})[y_{n-1}, x_{n+1}, z_{n+1}].$$

Therefore, by Theorem 4.1.2

$$\operatorname{depth}(S^{\bullet}_{n,3}/(E(P^{\bullet}_{n,3}) : y_{n-1})) \leq \left\lceil \frac{n-3-1}{2} \right\rceil + 3 = \left\lceil \frac{n+2}{2} \right\rceil. \tag{4.21}$$

Now by combining Eqs. (4.19), (4.20) and (4.21), we get

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \operatorname{depth}(S^{\bullet}_{n,3}/E(P^{\bullet}_{n,3})) \leq \left\lceil \frac{n+2}{2} \right\rceil.$$

□

Lemma 4.2.6. *Let $n \geq 2$, then,*

$$\left\lceil \frac{n+2}{3} \right\rceil \leq \operatorname{depth}(S^{\bullet\bullet}_{n,3}/E(P^{\bullet\bullet}_{n,3})) \leq \left\lceil \frac{n+5}{2} \right\rceil.$$

Proof. For $n = 1$ it follows from Remark 4.1.1. Let $n \geq 2$, Since $diam(P^{\bullet\bullet}_{n,3}) = n + 1$, Thus by [27]

$$\operatorname{depth}(S^{\bullet\bullet}_{n,3}/E(P^{\bullet\bullet}_{n,3})) \geq \left\lceil \frac{n+2}{3} \right\rceil. \tag{4.22}$$

For reverse of inequality. The cases $n = 2, 3$ are trivial. Let $n \geq 4$,, Since $y_{n-1} \notin E(P^{\bullet\bullet}_{n,3})$, by [23]

$$\operatorname{depth}(S^{\bullet\bullet}_{n,3})/E(P^{\bullet\bullet}_{n,3})) \leq \operatorname{depth}(S^{\bullet\bullet}_{n,3})/(E(P^{\bullet\bullet}_{n,3}) : y_{n-1})). \tag{4.23}$$

It follows that

$$S^{\bullet\bullet}_{n,3}/(E(P^{\bullet\bullet}_{n,3}) : y_{n-1})) \cong S^{\bullet}_{n-3,3}/E(P^{\bullet}_{n-3,3})[y_{n-1}, x_{n+1}, z_{n+1}].$$

Therefore, by Lemma 4.2.5

$$\operatorname{depth}(S^{\bullet\bullet}_{n,3}/(E(P^{\bullet\bullet}_{n,3}):y_{n-1})) \leq \left\lceil \frac{n-3+2}{2} \right\rceil + 3 = \left\lceil \frac{n+5}{2} \right\rceil. \tag{4.24}$$

Combine Eqs. (4.22), (4.23) and (4.24), we have

$$\left\lceil \frac{n+2}{3} \right\rceil \leq \operatorname{depth}(S^{\bullet\bullet}_{n,3}/E(P^{\bullet\bullet}_{n,3})) \leq \left\lceil \frac{n+5}{2} \right\rceil.$$

□

4.3 Bounds for depth of modules associated to $C_{n,m}$.

Let $1 \leq i \leq n$, for convenience we take $x_i := x_{i1}, y_i := x_{i2}, z_i := x_{i3}$ (see Figure 4.3). We set $S_{n,1} := K[x_1, x_2, \ldots, x_n]$, $S_{n,2} := K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ and $S_{n,2} := K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n]$. Clearly $P_{n,1} \cong P_n$ and $C_{n,1} \cong C_n$,

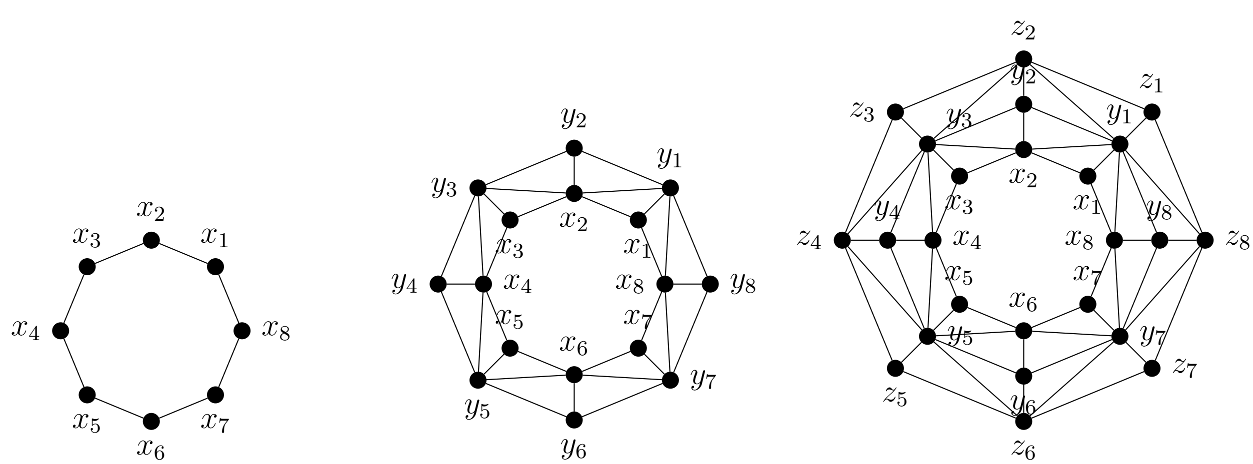


Figure 4.12: From left to right $C_{8,1}, C_{8,2}$ and $C_{8,3}$.

For $C_{n,2}$ and $C_{n,3}$ we define minimal generating set of edge ideals as follow:

$$G(E(C_{n,2})) = G(E(P_{n,2})) \cup \{x_1x_n, y_1y_n, x_ny_1\}.$$

$$G(E(C_{n,3})) = G(E(P_{n,3})) \cup \{x_1x_n, y_1y_n, x_ny_1, y_1z_n, z_1z_n\}.$$

Remark 4.3.1. *Note that for $n \in 2\mathbb{Z}^+$ and $n > 2$, Then $S_{n,1}/(E(C_{n,1})) \cong S/E(C_n)$, and* $\operatorname{depth}(S_{n,1}/E(C_{n,1})) = \left\lceil \frac{n-1}{3} \right\rceil$.

Theorem 4.3.2. *Let n is even, and $n \geq 4$, then*

$$\left\lceil \frac{n-1}{3} \right\rceil \leq \operatorname{depth}(S_{n,2}/E(C_{n,2})) \leq \left\lceil \frac{n-2}{2} \right\rceil.$$

Proof. Let $n \geq 4$, consider following short exact sequence

$$0 \longrightarrow S_{n,2}/(E(C_{n,2}) : x_n) \xrightarrow{.x_n} S_{n,2}/E(C_{n,2}) \longrightarrow S_{n,2}/(E(C_{n,2}), x_n) \longrightarrow 0$$

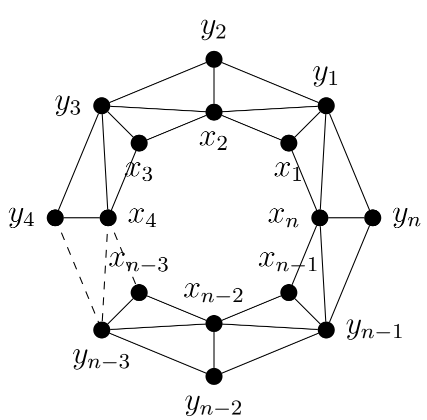
By depth lemma, we have

$$\operatorname{depth}(S_{n,2}/E(C_{n,2})) \geq \min\{\operatorname{depth}(S_{n,2}/(E(C_{n,2}) : x_n)), \operatorname{depth}(S_{n,2}/(E(C_{n,2}), x_n))\}.$$

Since $x_n \notin E(C_{n,2})$, by [23].

$$\operatorname{depth}(S_{n,2}/E(C_{n,2})) \leq \operatorname{depth}(S_{n,2})/(E(C_{n,2}) : x_n)).$$

$$(E(C_{n,2}) : x_n) = (\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, \cup_{i=2}^{\frac{n-4}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}\}$$
$$, x_{n-2}y_{n-3}, x_{n-2}y_{n-2}, x_{n-1}, y_{n-1}, y_n, x_1, y_1).$$



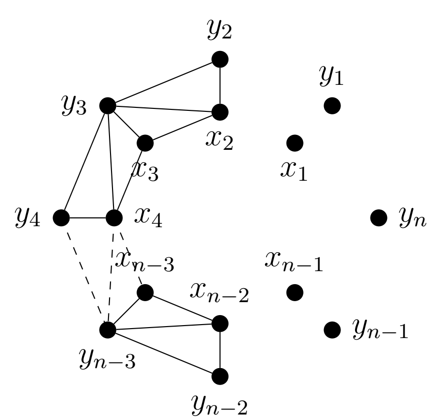


Figure 4.13: From left to right $E(C_{n,2})$ and $(E(C_{n,2}) : x_n)$.

It follows that

$$S_{n,2}/(E(C_{n,2}):x_n) \cong S_{n-3,2}/E(P_{n-3,2})[x_n].$$

Therefore, by Theorem 4.1.2

$$\operatorname{depth}(S_{n,2}/(E(C_{n,2}):x_n)) \leq \left\lceil \frac{n-3-1}{2} \right\rceil + 1 = \left\lceil \frac{n-2}{2} \right\rceil.$$

and

$$\operatorname{depth}(S_{n,2}/E(C_{n,2})) \leq \left\lceil \frac{n-2}{2} \right\rceil. \tag{4.25}$$

For lower bound again consider

$$S_{n,2}/(E(C_{n,2}):x_n) \cong S_{n-3,2}/E(P_{n-3,2})[x_n].$$

Therefore, by Theorem 4.1.2

$$\operatorname{depth}(S_{n,2}/(E(C_{n,2}):x_n)) \geq \left\lceil \frac{n-3}{3} \right\rceil + 1 = \left\lceil \frac{n}{3} \right\rceil.$$

and

$$\operatorname{depth}(S_{n,2}/E(C_{n,2})) \geq \left\lceil \frac{n}{3} \right\rceil. \tag{4.26}$$

Now let,

$$J = (E(C_{n,2}), x_n) = (\cup_{i=1}^{n-2}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, \cup_{i=1}^{\frac{n-2}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}\}, x_{n-1}y_{n-1},$$
$$y_{n-1}y_n, y_ny_1, x_n) = (E(P_{n-1,2}), x_n, y_{n-1}y_n, y_1y_n).$$

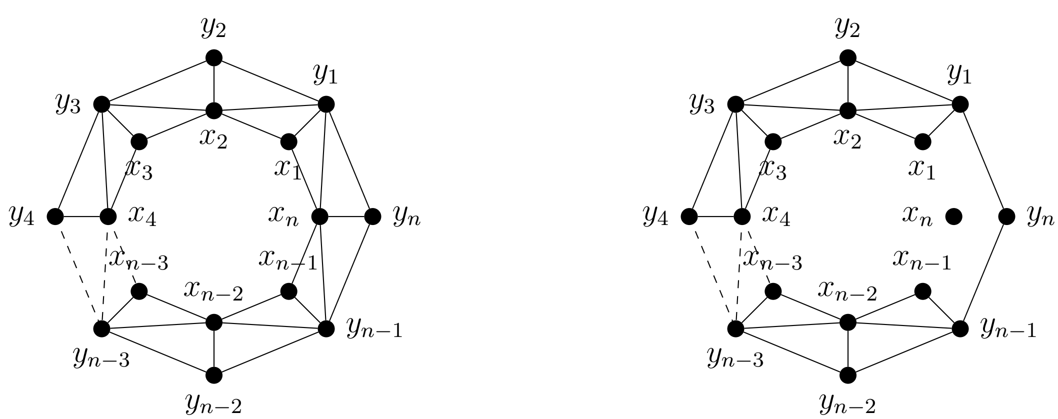


Figure 4.14: From left to right $E(C_{n,2})$ and $(E(C_{n,2}), x_n)$.

Consider

$$0 \longrightarrow S_{n,2}/(J : y_n) \xrightarrow{.y_n} S_{n,2}/J \longrightarrow S_{n,2}/(J, y_n) \longrightarrow 0$$

by applying depth lemma, we get

$$\operatorname{depth}(S_{n,2}/J) \geq \min\{\operatorname{depth}(S_{n,2}/(J : y_n)), \operatorname{depth}(S_{n,2}/(J, y_n))\}.$$

Since $(J, y_n) = (E(P_{n-1,2}), x_n, y_n)$ and $S_{n,2}/(J, y_n)) \cong S_{n-1,2}/E(P_{n-1,2})$.

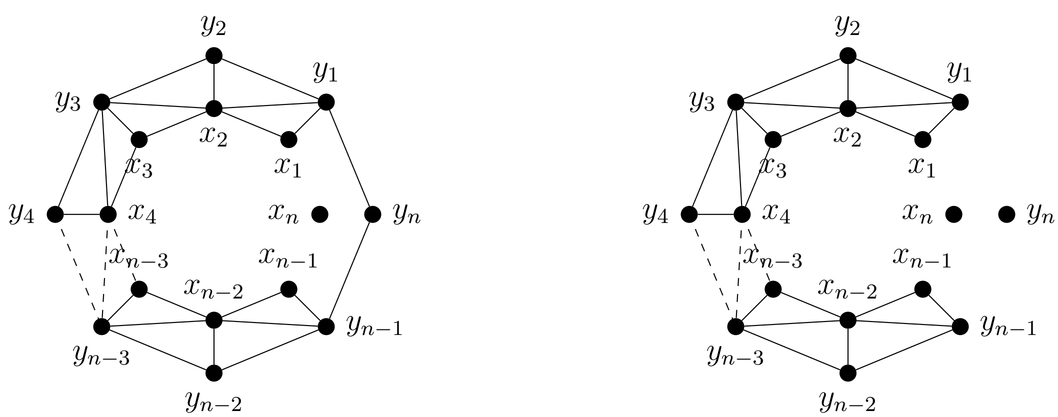


Figure 4.15: From left to right J and (J, y_n).

Therefore, by Theorem 4.1.2

$$\operatorname{depth}(S_{n,2}/(J, y_n)) \geq \left\lceil \frac{n-1}{3} \right\rceil.$$

and

$$(J : y_n) = (\cup_{i=2}^{n-3}\{x_i y_i, x_i x_{i+1}, y_i y_{i+1}\}, \cup_{i=2}^{\frac{n-4}{2}}\{x_{2i} y_{2i-1}, x_{2i} y_{2i+1}\}$$
$$, x_{n-2} y_{n-3}, x_{n-2} y_{n-2}, x_{n-2} x_{n-1}, x_2 x_1, y_{n-1}, x_n, y_1).$$

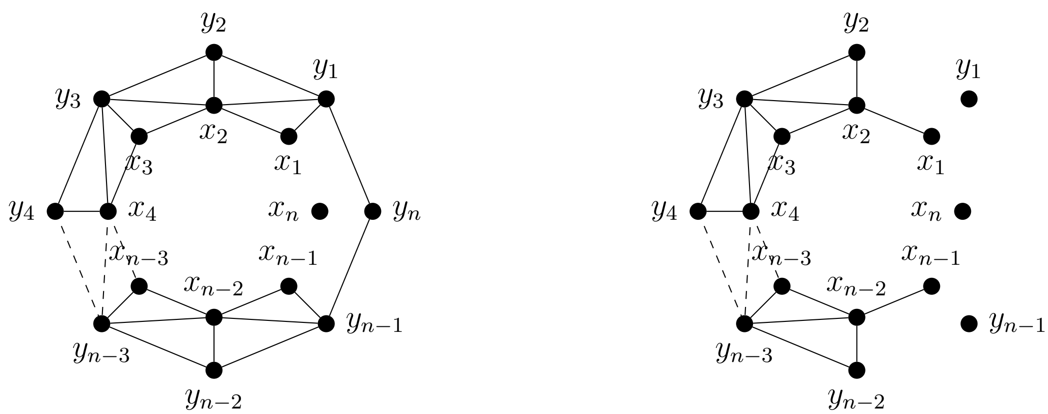


Figure 4.16: From left to right J and $(J : y_n)$.

After renumbering the variables, we can see that

$$S_{n,2}/(J : y_n)) \cong S_{n-3,2}/E(P^{**}_{n-3,2})[y_n].$$

Hence, by Lemma 4.2.2

$$\operatorname{depth}(S_{n,2}/(J : y_n)) \geq \left\lceil \frac{n-3+2}{3} \right\rceil + 1 = \left\lceil \frac{n+2}{3} \right\rceil.$$

By applying depth lemma, we have

$$\operatorname{depth}(S_{n,2}/J) \geq \left\lceil \frac{n-1}{3} \right\rceil.$$

and

$$0 \longrightarrow S_{n,2}/(E(C_{n,2}) : x_n) \xrightarrow{.x_n} S_{n,2}/E(C_{n,2}) \longrightarrow S_{n,2}/J \longrightarrow 0$$

and it follows

$$\operatorname{depth}(S_{n,2}/E(C_{n,2})) \geq \left\lceil \frac{n-1}{3} \right\rceil. \tag{4.27}$$

Combine Eqs. (4.26) and (4.27), we get

$$\left\lceil \frac{n-1}{3} \right\rceil \leq \operatorname{depth}(S_{n,2}/E(C_{n,2})) \leq \left\lceil \frac{n-2}{2} \right\rceil.$$

□

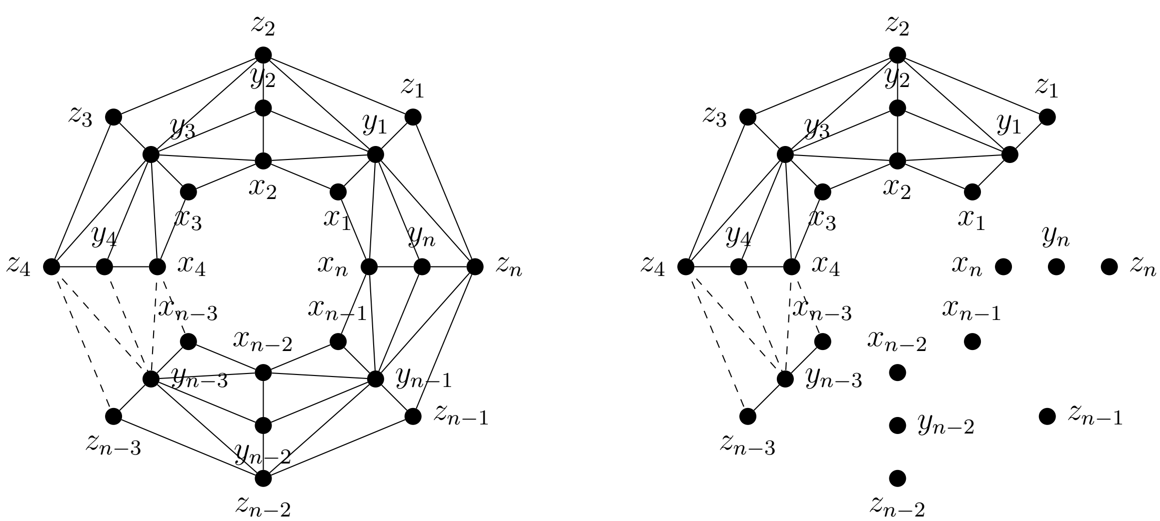
Theorem 4.3.3. *Let n is even, and $n \geq 4$, then*

$$\left\lceil \frac{n-3}{2} \right\rceil \leq \operatorname{depth}(S_{n,3}/E(C_{n,3})) \leq \left\lceil \frac{n-2}{2} \right\rceil.$$

Proof. Let $n \geq 4$, Since $y_{n-1} \notin E(C_{n,3})$, by [23].

$$\operatorname{depth}(S_{n,3})/E(C_{n,3})) \leq \operatorname{depth}(S_{n,3})/(E(C_{n,3}) : y_{n-1}))$$

$$(E(C_{n,3}) : y_{n-1}) = (\cup_{i=1}^{n-4}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=1}^{\frac{n-4}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}, x_{n-3}y_{n-3}, y_{n-3}z_{n-3}, z_{n-2}, z_{n-1}, z_n, y_{n-2}, y_n, x_{n-2}, x_{n-1}, x_n).$$



(a) $(E(C_{n,3}))$ (b) $(E(C_{n,3}) : y_{n-1})$

Figure 4.17

It follows that

$$S_{n,3}/(E(C_{n,3}) : y_{n-1})) \cong S_{n-3,3}/E(P_{n-3,3})[y_{n-1}].$$

Therefore, by Theorem 4.1.2,

$$\operatorname{depth}(S_{n,3}/(E(C_{n,3}) : y_{n-1})) \leq \left\lceil \frac{n-3-1}{2} \right\rceil + 1 = \left\lceil \frac{n-2}{2} \right\rceil.$$

and

$$\operatorname{depth}(S_{n,3}/E(C_{n,3})) \leq \left\lceil \frac{n-2}{2} \right\rceil. \tag{4.28}$$

Let $n \geq 4$, we have

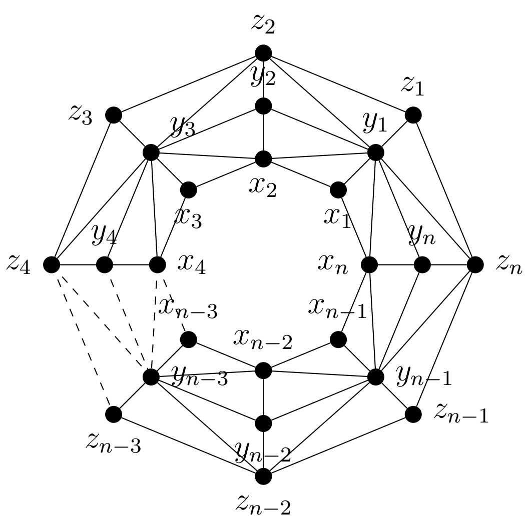
$$0 \longrightarrow S_{n,3}/(E(C_{n,3}) : x_n) \xrightarrow{.x_n} S_{n,3}/E(C_{n,3}) \longrightarrow S_{n,3}/(E(C_{n,3}), x_n) \longrightarrow 0$$

by Depth Lemma

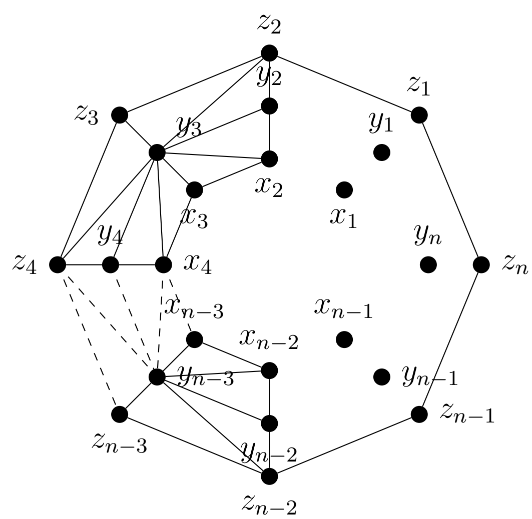
$$\operatorname{depth}(S_{n,3}/E(C_{n,3})) \geq \min\{\operatorname{depth}(S_{n,3}/(E(C_{n,3}) : x_n)), \operatorname{depth}(S_{n,3}/(E(C_{n,3}), x_n))\}.$$

Let

$$\begin{aligned} A = (E(C_{n,3}) : x_n) =& (\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-3}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, \\ & y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, z_{n-2}z_{n-1}, z_{n-1}z_n, z_nz_1, z_1z_2, \\ & y_{n-1}, y_n, y_1, x_{n-1}, x_1). \end{aligned}$$



(a) $(E(C_{n,3}))$



(b) $(E(C_{n,3}) : x_n)$

Figure 4.18

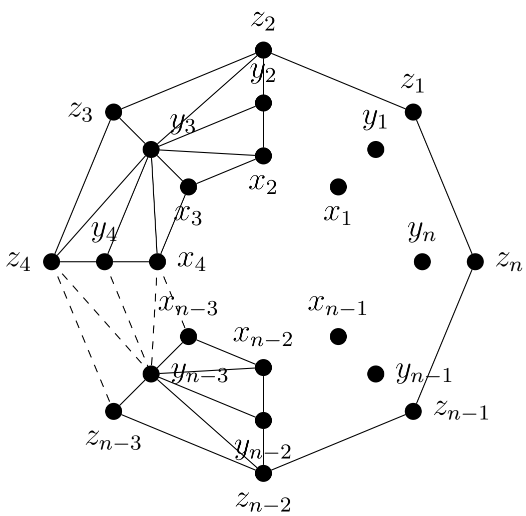
Consider the following exact sequence

$$0 \longrightarrow S'_{n,3}/(A : z_n) \xrightarrow{.z_n} S'_{n,3}/A \longrightarrow S'_{n,3}/(A, z_n) \longrightarrow 0$$

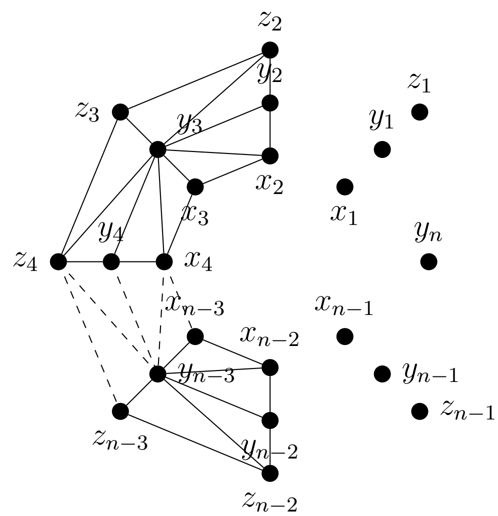
by Depth Lemma

$$\operatorname{depth}(S_{n,3}/A)) \geq \min\{\operatorname{depth}(S_{n,3}/A : z_n))\,,\, \operatorname{depth}(S_{n,3}/A, z_n))\}.$$

$$(A : z_n) = (\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-3}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}$$
$$, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, z_{n-1}, z_1, y_{n-1}, y_n, y_1, x_{n-1}, x_1).$$



(a) (A)



(b) $(A : z_n)$

Figure 4.19: $((E(C_{n,3}) : x_n) : z_n)$.

It follows that

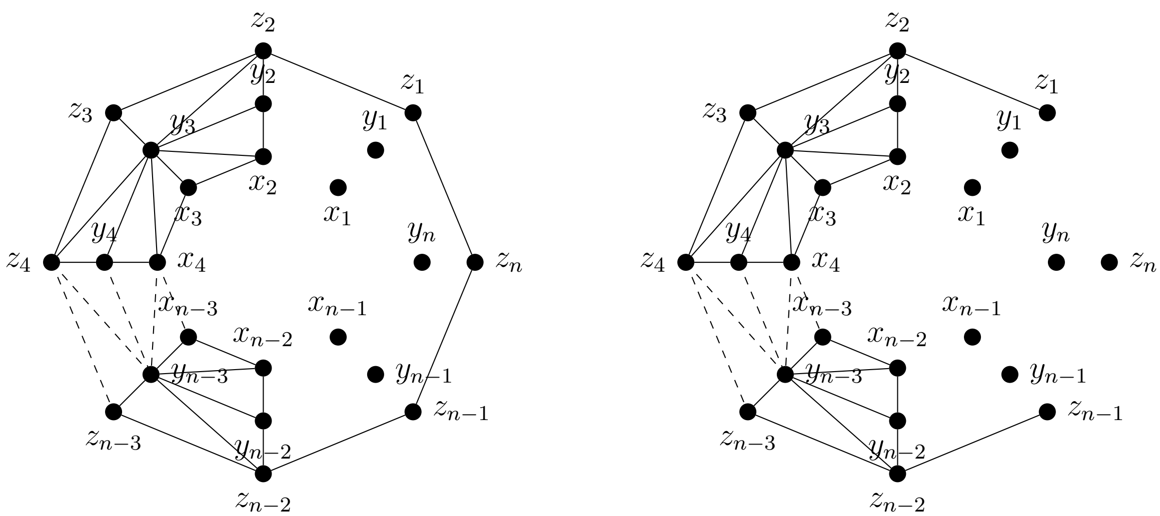
$$S_{n,3}/(A : z_n)) \cong S_{n-3,3}/E(P_{n-3,3})[x_n, z_n].$$

Therefore, by Theorem 4.1.2

$$\operatorname{depth}(S_{n,3}/(A : z_n)) \geq \left\lceil \frac{n-3}{3} \right\rceil + 2 = \left\lceil \frac{n}{3} \right\rceil.$$

Now

$$(A, z_n) = (\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-3}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}$$
$$, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, z_{n-2}z_{n-1}, z_1z_2, z_n, y_{n-1}, y_n, y_1, x_{n-1}, x_1).$$



(a) (A) (b) (A, z_n)

Figure 4.20: $((E(C_{n,3}) : x_n), z_n)$.

We can see that after renumbering the variables,

$$S_{n,3}/(A, z_n)) \cong S_{n-3,3}/E(P^{**}_{n-3,3})[x_n].$$

Therefore, by Lemma 4.2.2,

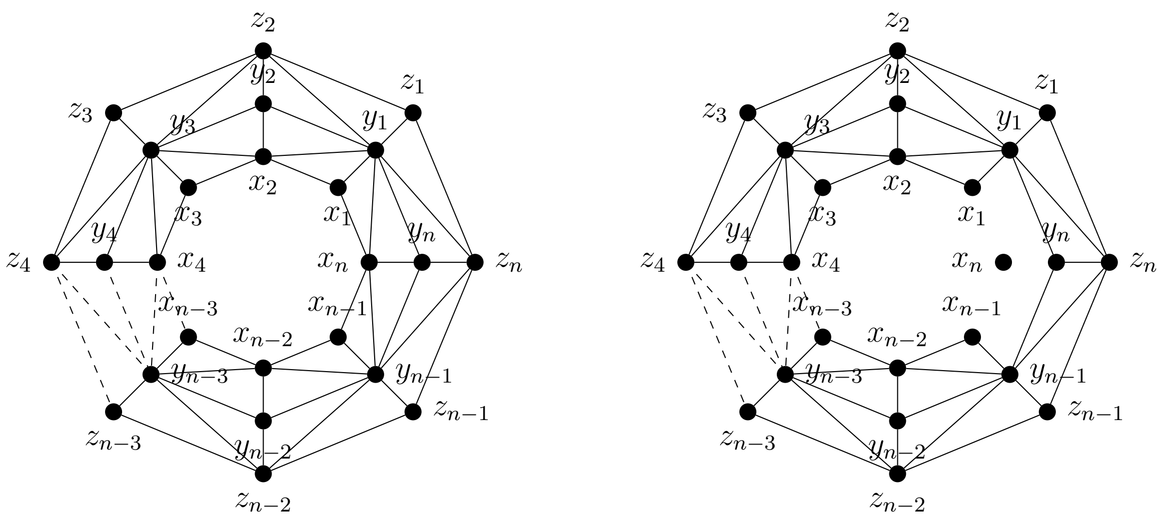
$$\operatorname{depth}(S_{n,3}/(A, z_n)) \geq \left\lceil \frac{n-3}{3} \right\rceil + 1 = \left\lceil \frac{n}{3} \right\rceil.$$

Therefore, by Depth Lemma

$$\operatorname{depth}(S_{n,2}/A) \geq \left\lceil \frac{n}{3} \right\rceil.$$

Now let

$$B = (E(C_{n,3}), x_n) = (\cup_{i=1}^{n-2}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-2}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}, x_{n-1}y_{n-1}, y_{n-1}z_{n-1}, z_{n-1}z_n, z_nz_1, y_{n-1}y_n, y_ny_1, x_n).$$



(a) $(E(C_{n,3}))$ (b) $(E(C_{n,3}), x_n)$

Figure 4.21

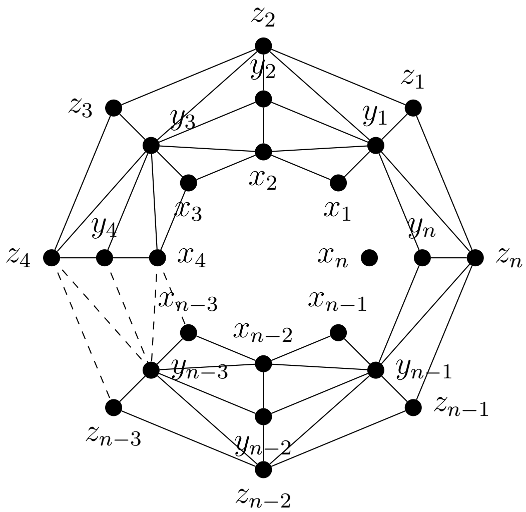
Consider the following exact sequence

$$0 \longrightarrow S_{n,3}/(B : y_n) \xrightarrow{\cdot y_n} S_{n,3}/B \longrightarrow S_{n,3}/(B, y_n) \longrightarrow 0$$

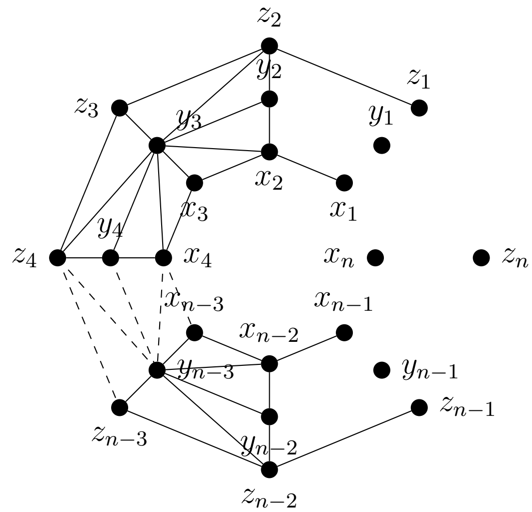
by Depth Lemma

$$\operatorname{depth}(S_{n,3}/B)) \geq \min\{\operatorname{depth}(S_{n,3}/B : y_n)), \operatorname{depth}(S_{n,3}/B, y_n))\}.$$

$$(B : y_n) = (\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-3}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i},$$
$$y_{2i+1}z_{2i}, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, z_{n-2}z_{n-1}, z_n, z_1z_2, y_{n-1}, y_1, x_{n-2}x_{n-1}, x_1x_2, x_n).$$



(a) (B)



(b) $(B : y_n)$

Figure 4.22: $((E(C_{n,3}), x_n) : y_n)$

and

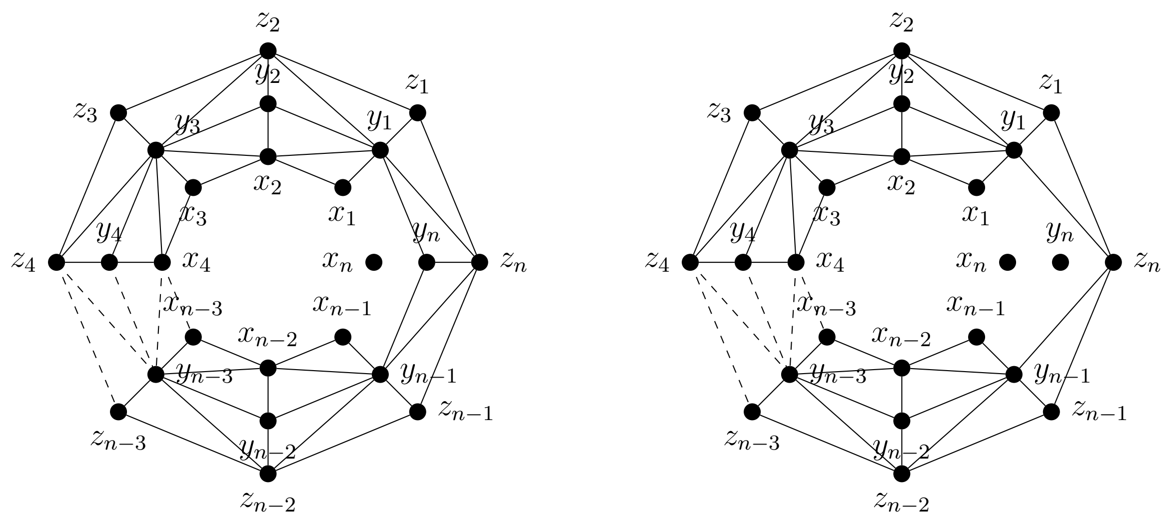
$$S_{n,3}/(B : y_n)) \cong S^{\bullet\bullet}_{n-3,3}/E(P^{\bullet\bullet}_{n-3,3})[y_n].$$

Therefore, by Lemma 4.2.6

$$\operatorname{depth}(S_{n,3}/(B : y_n)) \geq \left\lceil \frac{n-3+2}{3} \right\rceil + 1 = \left\lceil \frac{n+2}{3} \right\rceil.$$

Now

$$C = (B, y_n) = (\cup_{i=1}^{n-2}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-2}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}, x_{n-1}y_{n-1}, y_{n-1}z_{n-1}, z_{n-1}z_n, z_nz_1, y_{n-1}z_n, z_ny_1, x_n, y_n).$$



(a) (B)

(b) (B, y_n)

Figure 4.23: $((E(C_{n,3}), x_n), y_n)$

Consider the following exact sequence,

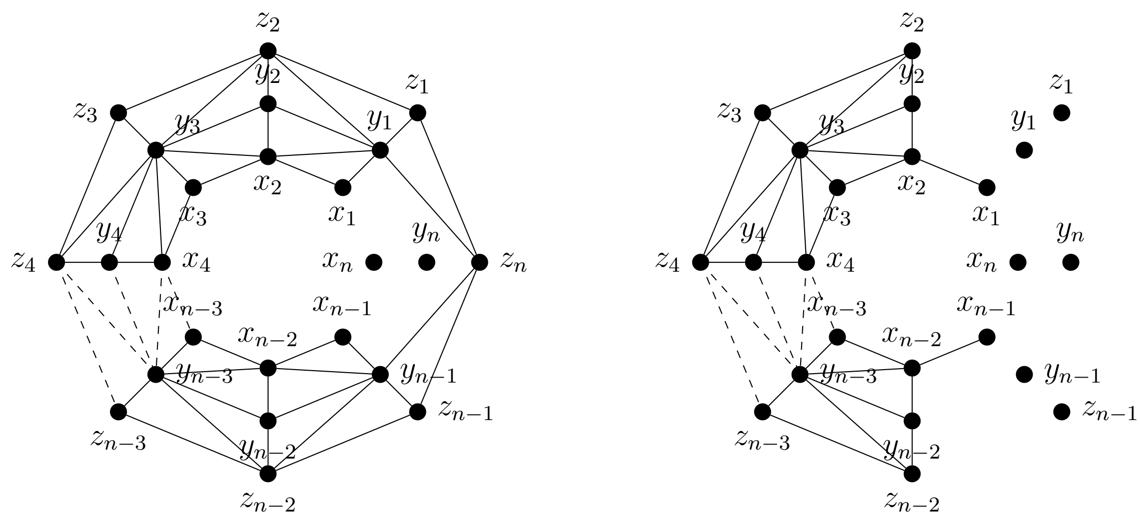
$$0 \longrightarrow S_{n,3}/(C : z_n) \xrightarrow{.z_n} S_{n,3}/C \longrightarrow S_{n,3}/(C, z_n) \longrightarrow 0$$

and by Depth Lemma, we get

$$\text{depth}(S_{n,3}/C)) \geq \min\{\text{depth}(S_{n,3}/C : z_n)) \,, \, \text{depth}(S_{n,3}/C, z_n))\}.$$

and

$$(C : z_n) = (\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-3}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}$$
$$, x_{n-2}y_{n-2}, y_{n-2}z_{n-2}, z_{n-1}, z_1, y_{n-1}, y_n, y_1, x_n, x_{n-1}x_{n-2}, x_1x_2).$$



(a) (C)

(b) $(C : z_n)$

Figure 4.24: $(((E(C_{n,3}), x_n), y_n) : z_n)$.

It follows that

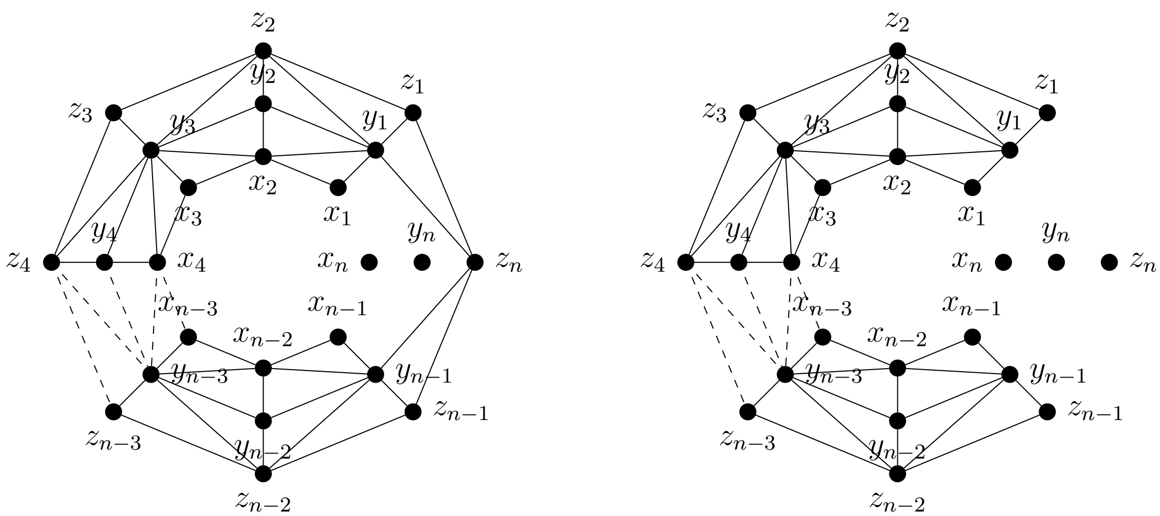
$$S_{n,3}/(C : z_n)) \cong S^{**}_{n-3,3}/E(P^{**}_{n-3,3})[z_n].$$

Therefore, by Lemma 4.2.4,

$$\text{depth}(S_{n,3}/(C : z_n)) \geq \left\lceil \frac{n-3+2}{3} \right\rceil + 1 = \left\lceil \frac{n+2}{3} \right\rceil.$$

Now,

$$(C, z_n) = (\cup_{i=1}^{n-2}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}, y_iz_i, z_iz_{i+1}\}, \cup_{i=2}^{\frac{n-2}{2}}\{x_{2i}y_{2i-1}, x_{2i}y_{2i+1}, y_{2i-1}z_{2i}, y_{2i+1}z_{2i}\}$$
$$, x_{n-1}y_{n-1}, y_{n-1}z_{n-1}, z_n, y_n, x_n).$$



(a) (C) (b) (C, z_n)

Figure 4.25: $(((E(C_{n,3}), x_n), y_n), z_n)$.

$$S_{n,3}/(C, z_n)) \cong S_{n-1,3}/E(P_{n-1,3}).$$

Therefore, by Therorem 4.1.3,

$$\operatorname{depth}(S_{n,3}/(C, z_n)) \geq \left\lceil \frac{n-1}{3} \right\rceil.$$

Therefore, by Depth Lemma,

$$\operatorname{depth}(S_{n,3}/C) \geq \left\lceil \frac{n-1}{3} \right\rceil.$$

Now, we have

$$0 \longrightarrow S_{n,3}/(B : y_n) \xrightarrow{.y_n} S_{n,3}/B \longrightarrow S_{n,3}/(B, y_n) \longrightarrow 0$$

$$\operatorname{depth}(S_{n,3}/B) \geq \left\lceil \frac{n-1}{3} \right\rceil,$$

and

$$0 \longrightarrow S_{n,3}/(A : z_n) \xrightarrow{.z_n} S_{n,3}/A \longrightarrow S_{n,3}/(A, z_n) \longrightarrow 0$$

$$\operatorname{depth}(S_{n,3}/A) \geq \left\lceil \frac{n-3}{3} \right\rceil.$$

Finally,

$$0 \longrightarrow S_{n,3}/A \xrightarrow{.x_n} S_{n,3}/E(C_{n,3}) \longrightarrow S_{n,3}/B \longrightarrow 0$$

By Depth Lemma we get

$$\operatorname{depth}(S_{n,3}/E(C_{n,3})) \geq \left\lceil \frac{n-3}{3} \right\rceil. \tag{4.29}$$

So, combine 4.28 and 4.29, we will get

$$\left\lceil \frac{n-1}{3} \right\rceil \leq \operatorname{depth}(S_{n,3}/E(C_{n,3})) \leq \left\lceil \frac{n}{3} \right\rceil.$$

□

Chapter 5

Projective dimension and regularity of some circulants

5.1 Regularity for edge ideals of $C_{2n}(1, n-1, n)$.

This section deals with the regularity of edge ideal associated with the circulant graph $G = C_{2n}(S)$ with $S = \{1, n-1, n\}$. As an example of $C_{2n}(1, n-1, n)$, the graph $C_{16}(1, 7, 8)$ is drawn in Figure 5.1

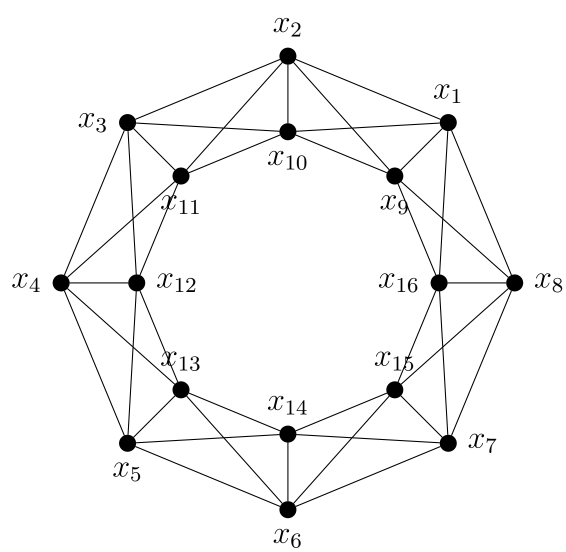


Figure 5.1: $C_{16}(1, 7, 8)$

A convenient approach is use the labeling and representation of the graphs as given in Figure 5.2

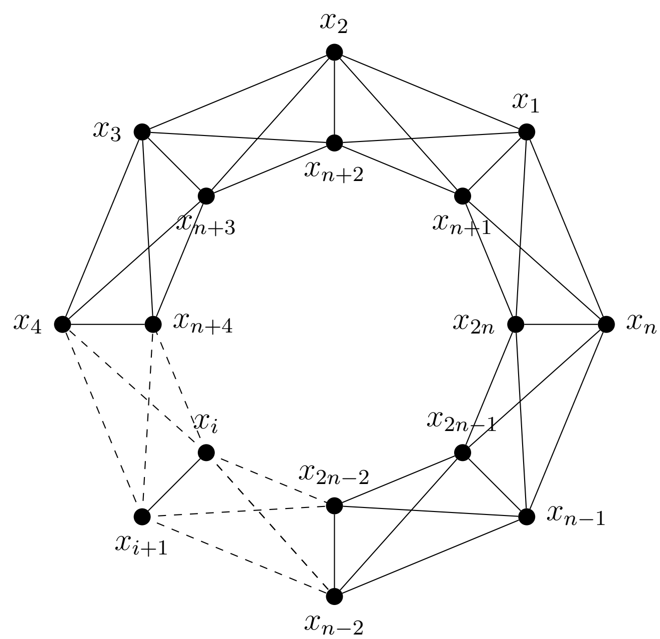


Figure 5.2: $C_{2n}(1, n-1, n)$

One introduces the following families of graphs,

i) The family A_n:

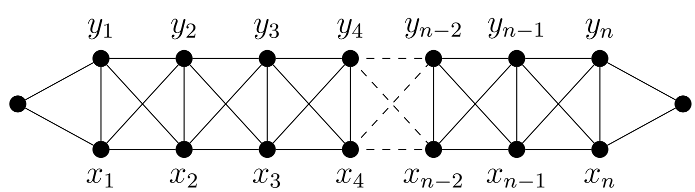


Figure 5.3: A_n

ii) The family B_n:

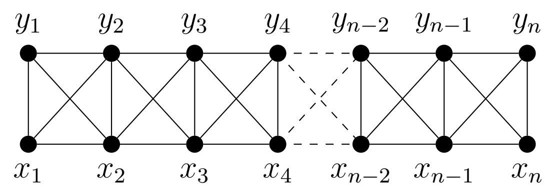


Figure 5.4: B_n

Lemma 5.1.1. *Following the above introduced notations, we have:*

$$reg(I(A_n)) \leq \begin{cases} \frac{n+3}{2} & if\ n\ is\ odd \\ \frac{n+2}{2} & if\ n\ is\ even. \end{cases}$$

Proof. One carries the proof by using principle of mathematical induction on n. By Example 3.3.4, we have $reg(I(A_2)) = 2$. let $n \geq 3$. Graph A_n can be decomposed into A_{n-2} and A_2, i.e.,

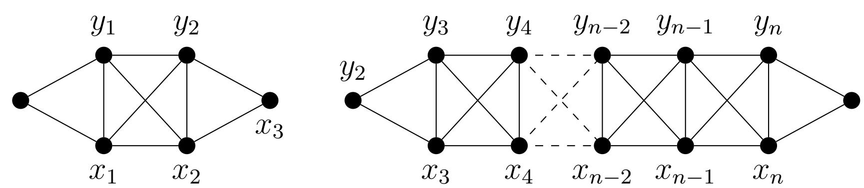


Figure 5.5: A_2 and A_{n-2}

Case I: If n is even, by the principle of mathematical induction and the well known result that $reg(R/I) = reg(I) - 1$, we get:

$$\begin{aligned} reg(R/I(A_n)) &\leq reg(R/I(A_2)) + reg(R/I(A_{n-2})) \\ &\leq 1 + \frac{(n-2)+2}{2} - 1 = \frac{n+2}{2} - 1. \end{aligned}$$

Case II: If n is odd, the proof is similar. $\square$

Lemma 5.1.2. *Following the above introduced notations, we have:*

$$reg(I(B_n)) \leq \begin{cases} \frac{n+3}{2} & if\ n\ is\ odd \\ \frac{n+2}{2} & if\ n\ is\ even. \end{cases}$$

Proof. Suppose that $n \geq 2$. The graph B_n can be decomposed into three subgraphs A, B and C, i.e.,

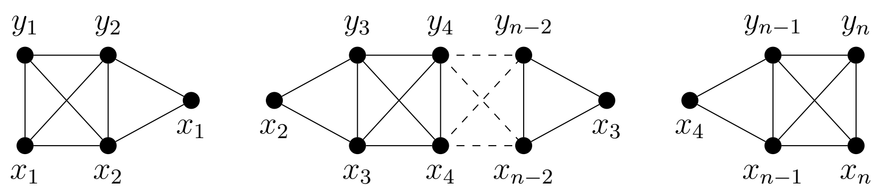


Figure 5.6: Subgraphs of B_n.$(A, B$ and $C)$

Case I: If n is even, By Example 3.3.3 we have $reg(I(A)) = reg(I(C)) = 2$. and by relabeling B we can see that $B \cong A_{n-4}$. By Lemma 5.1.1 and the fact that $reg(R/I) = reg(I) - 1$, we get:

$$\begin{aligned} reg(R/I(B_n)) &\leq reg(R/I(A)) + reg(R/I(B)) + reg(R/I(C)) \\ &\leq 1 + \frac{(n-4)+2}{2} - 1 + 1 = \frac{n+2}{2} - 1. \end{aligned}$$

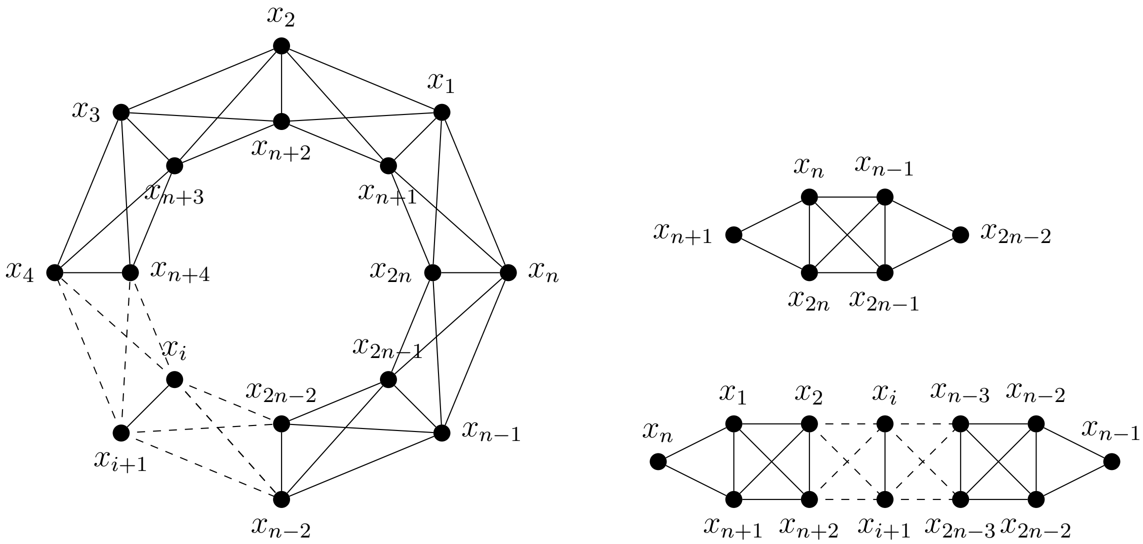
Case II: If n is odd, the proof is similar. □

We now determine bounds on the regularity.

Lemma 5.1.3. *Let $n \geq 4$. If $G = C_{2n}(1, n-1, n)$, then:*

$$reg(I(G)) \leq \begin{cases} \frac{n+3}{2} & if\ n\ is\ odd \\ \frac{n+2}{2} & if\ n\ is\ even. \end{cases}$$

Proof. Suppose that $n \geq 4$. The graph G can be decomposed into the subgraphs A_{n-2} and A_2, i.e.,



(a) $C_{2n}(1, n-1, n)$ (b) A_2 and A_{n-2}

Figure 5.7: G and Subgraphs of G

Case I: If n is even, by Example 3.3.4, we have $reg(I(A_2)) = 2$. By Lemma 5.1.1 and the fact that $reg(R/I) = reg(I) - 1$, we get:

$$\begin{aligned} reg(R/I(G)) &\leq reg(R/I(A_2)) + reg(R/I(A_{n-2})) \\ &\leq 1 + \frac{(n-2)+2}{2} - 1 = \frac{n+2}{2} - 1. \end{aligned}$$

Case I: If n is odd, the proof is analogous. □

Lemma 5.1.4. *Following the above introduced notations, we have:*

$$pdim(I(A_n)) \leq 2n - 1.$$

Proof. The proof is carried out by principle of mathematical induction on n. One can find $pdim(I(A_2)) = 3$ and $pdim(I(A_3)) = 5$ via a direct computation (Macaulay2 [28]). These computations agrees with upper bounds given in problem statement, implies, the basic case holds. Now, assume that $n \geq 4$. The graph A_n can be decomposed A_{n-2} and A_2, i.e.,

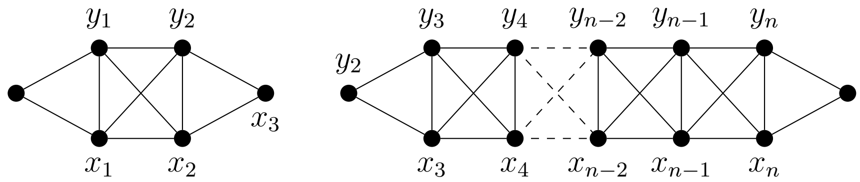


Figure 5.8: A_2 and A_{n-2}

By induction on n, we get:

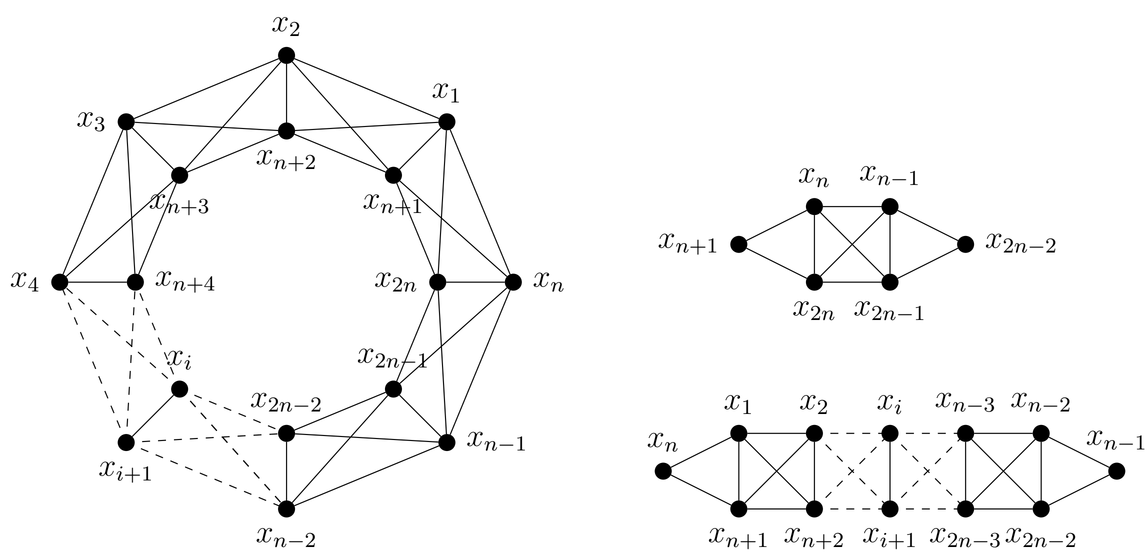
$$\begin{aligned} pdim(I(A_n)) &\leq pdim(I(A_2)) + pdim(I(A_{n-2})) + 1 \\ &\leq 3 + (2(n-2) - 1) + 1 = 2n - 1. \end{aligned}$$

□

Lemma 5.1.5. *Let $n \geq 4$. If $G = C_{2n}(1, n-1, n)$, then:*

$$pdim(I(G)) \leq \begin{cases} 4k - 1 & if\ n = 2k \\ 4k + 1 & if\ n = 2k + 1. \end{cases}$$

Proof. Suppose that $n \geq 4$. The graph G can be decomposed into A_{n-2} and A_2, i.e.,



(a) $C_{2n}(1, n-1, n)$

(b) A_2 and A_{n-2}

Figure 5.9: G and subgraphs of G

Case I: If n is odd, i.e., $n = 2k+1$. By lemma 5.1.4, we get:

$$\begin{aligned} pdim(I(G)) &\leq pdim(I(A_2)) + pdim(I(A_{n-2})) + 1 \\ &\leq 3 + (2(n-2)-1) + 1 = 4k - 1. \end{aligned}$$

Case II: If in is even, i.e., $n = 2k$. By lemma 5.1.4, we get:

$$\begin{aligned} pdim(I(G)) &\leq pdim(I(A_2)) + pdim(I(A_{n-2})) + 1 \\ &\leq 3 + (2(n-2)-1) + 1 = 4k + 1. \end{aligned}$$

□

Chapter 6

Stanley depth and depth of cubic circulant graph

6.1 Stanley depth and depth of module associated with ladder graphs

If $n \geq 2$, then $\mathcal{L}_n := P_2 \square P_n$ is called ladder graph on $2n$ vertices. Examples of $\mathcal{L}_n$ is shown in Figure 6.1. Clearly $|E(\mathcal{L}_n)|= 3n-2$. We label the vertices of the graphs $\mathcal{L}_n$ by using two sets $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$; (see Figure 6.1). Let $S_n := K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be the ring of polynomials in variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ over the field K. Then $I(\mathcal{L}_n)$ is a monomial ideal of S_n with $\mathcal{G}(I(\mathcal{L}_n)) = \cup_{i=1}^{n-1}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_ny_n\}$.

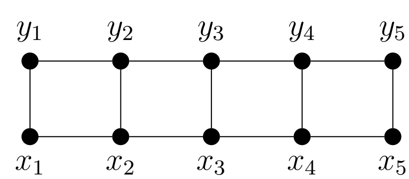


Figure 6.1: $\mathcal{L}_5$

We compute the Stanley depth and depth of $S_n/I(\mathcal{L}_n)$ and $S_n/I(\mathcal{C}\mathcal{L}_n)$. We introduce three super graphs $\mathcal{L}_n^\star$, $\mathcal{L}_n^\diamond$ and $\mathcal{L}_n^\bullet$ of $\mathcal{L}_n$ with vertex sets and edge sets

$V(\mathcal{L}_n^\star) = V(\mathcal{L}_n) \cup \{y_{n+1}\}$, $E(\mathcal{L}_n^\star) = E(\mathcal{L}_n) \cup \{y_n y_{n+1}\}$, $V(\mathcal{L}_n^\diamond) = V(\mathcal{L}_n) \cup \{y_{n+1}, y_{n+2}\}$, $E(\mathcal{L}_n^\diamond) = E(\mathcal{L}_n) \cup \{y_n y_{n+1}, y_1 y_{n+2}\}$, and $V(\mathcal{L}_n^\bullet) = V(\mathcal{L}_n) \cup \{x_{n+1}, y_{n+1}\}$, $E(\mathcal{L}_n^\bullet) = E(\mathcal{L}_n) \cup \{y_n y_{n+1}, x_1 x_{n+1}\}$, respectively. For examples of $\mathcal{L}_n^\star$, $\mathcal{L}_n^\diamond$ and $\mathcal{L}_n^\bullet$; see Figures 6.2, 6.3 and 6.4. It is easy to see that $\mathcal{G}(I(\mathcal{L}_n^\star)) = \mathcal{G}(I(\mathcal{L}_n)) \cup \{y_n y_{n+1}\}$, $\mathcal{G}(I(\mathcal{L}_n^\diamond)) = \mathcal{G}(I(\mathcal{L}_n)) \cup \{y_1 y_{n+2}, y_n y_{n+1}\}$ and $\mathcal{G}(I(\mathcal{L}_n^\bullet)) = \mathcal{G}(I(\mathcal{L}_n)) \cup \{x_1 x_{n+1}, y_n y_{n+1}\}$. For $S_n^\star/I(\mathcal{L}_n^\star)$, $S_n^\diamond/I(\mathcal{L}_n^\diamond)$ and $S_n^\bullet/I(\mathcal{L}_n^\bullet)$, We compute Stanley depth and depth, where $S_n^\star = S_n[y_{n+1}]$, $S_n^\diamond = S_n[y_{n+1}, y_{n+2}]$ $S_n^\bullet = S_n[x_{n+1}, y_{n+1}]$. Then, by using these results, we find Stanley depth and depth of $S_n/I(\mathcal{L}_n)$ and $S_n/I(C_{2n}(a, n))$.

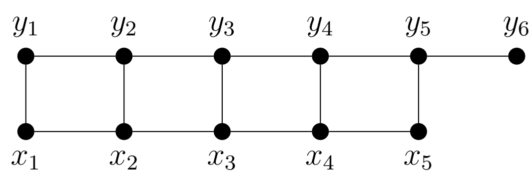


Figure 6.2: $\mathcal{L}_5^\star$

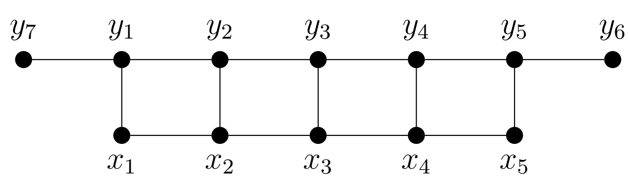


Figure 6.3: $\mathcal{L}_5^\diamond$

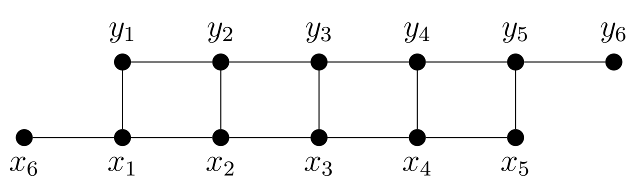


Figure 6.4: $\mathcal{L}_5^\bullet$

As a first result of this section, we compute the precise values of Stanley depth and depth of $S_n^\star/I(\mathcal{L}_n^\star)$ in the following proposition.

Proposition 6.1.1. *For $n \geq 2$, we have* $\operatorname{depth}(S_n^\star/I(\mathcal{L}_n^\star)) = \operatorname{sdepth}(S_n^\star/I(\mathcal{L}_n^\star)) = \lceil \frac{n+1}{2} \rceil$.

Proof. To prove $\operatorname{depth}(S_n^\star/I(\mathcal{L}_n^\star)) = \lceil \frac{n+1}{2} \rceil$ using mathematical induction. If $2 \leq n \leq 3$, then result is obvious. For $n \geq 4$, we have.

$$0 \longrightarrow S_n^\star/(I(\mathcal{L}_n^\star) : y_n) \xrightarrow{\cdot y_n} S_n^\star/I(\mathcal{L}_n^\star) \longrightarrow S_n^\star/(I(\mathcal{L}_n^\star), y_n) \longrightarrow 0. \tag{6.1}$$

$$\begin{aligned}(I(\mathcal{L}_n^\star) : y_n) &= \big(\cup_{i=1}^{n-3} \{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-2}y_{n-2}, x_{n-2}x_{n-1}, y_{n-1}, x_n, y_{n+1}\big) \\ &= (\mathcal{G}(I(\mathcal{L}_{n-2})), x_{n-2}x_{n-1}, y_{n-1}, x_n, y_{n+1}) = (\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1}),\end{aligned}$$

since $S_n^\star/(I(\mathcal{L}_n^\star) : y_n) \cong (S_{n-2}^\star/(I(\mathcal{L}_{n-2}^\star))[y_n]$, by mathematical induction and Lemma 3.2.11,

$$\operatorname{depth}(S_n^\star/(I(\mathcal{L}_n^\star) : y_n) = \lceil \frac{n-2+1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil.$$

Since

$$(I(\mathcal{L}_n^\star), y_n) = \big(\cup_{i=1}^{n-2} \{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-1}y_{n-1}, x_{n-1}x_n, y_n\big) = (\mathcal{G}(I(\mathcal{L}_{n-1}^\star), y_n)),$$

we get $S_n^\star/(I(\mathcal{L}_n^\star), y_n) \cong (S_{n-1}^\star/I(\mathcal{L}_{n-1}^\star))[y_{n+1}]$. Again by Lemma 3.2.11, we have

$$\operatorname{depth}(S_n^\star/(I(\mathcal{L}_n^\star), y_n) = \lceil \frac{n-1+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1.$$

Also $\operatorname{depth}(S_n^\star/(I(\mathcal{L}_n^\star), y_n)) \geq \operatorname{depth}(S_n^\star/(I(\mathcal{L}_n^\star) : y_n))$, so Depth Lemma yields

$$\operatorname{depth}(S_n^\star/I(\mathcal{L}_n^\star)) = \lceil \frac{n+1}{2} \rceil.$$

If $n = 2$, then

$$S_2^\star/I(\mathcal{L}_2^\star) = K[x_1, y_2] \oplus y_1K[y_1, x_2] \oplus x_2K[x_2, y_3] \oplus y_3K[x_1, y_3] \oplus y_1y_3K[y_1, x_2, y_3].$$

If $n = 3$, then

$$\begin{aligned} S_3^\star/I(\mathcal{L}_3^\star) = {} & K[x_1, y_2] \oplus y_1K[y_1, x_2] \oplus x_2K[x_2, y_3] \oplus x_3K[x_1, x_3] \oplus y_3K[x_1, y_3] \\ & \oplus y_4K[x_1, y_4] \oplus y_1x_3K[y_1, x_3, y_4] \oplus y_1y_3K[y_1, x_2, y_3] \oplus y_1y_4K[y_1, x_2, y_4] \\ & \oplus x_2y_3K[x_2, y_3] \oplus x_2y_4K[x_2, y_4] \oplus y_2x_3K[x_1, y_2, x_3] \oplus y_2y_4K[x_1, y_2, y_4] \\ & \oplus x_3y_4K[x_1, y_2, x_3, y_4]. \end{aligned}$$

For $n \geq 4$, by using induction on n, and by Lemma 3.2.10, we obtain

$$\operatorname{sdepth}(S_n^\star/I(\mathcal{L}_n^\star)) \geq \lceil \frac{n+1}{2} \rceil.$$

For the reverse inequality, we again use induction hypothesis on n. The result is obvious for $2 \leq n \leq 3$. If $n \geq 4$, as $y_n \notin I(\mathcal{L}_n^\star)$, therefore, by Proposition 3.2.13

$$\operatorname{sdepth}(S_n^\star/I(\mathcal{L}_n^\star)) \leq \operatorname{sdepth}(S_n^\star/(I(\mathcal{L}_n^\star) : y_n)).$$

Since $S_n^\star/(I(\mathcal{L}_n^\star) : y_n) \cong (S_{n-2}^\star/I(\mathcal{L}_{n-2}^\star))[y_n]$, by mathematical induction on n and Lemma 3.2.11, we have

$$\operatorname{sdepth}((S_{n-2}^\star/I(\mathcal{L}_{n-2}^\star))[y_n]) \leq \lceil \frac{n-2+1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil.$$

□

Remark 6.1.2. *For $k \geq 1$ and $n = 2k$ or $n = 2k+1$ we have that*

$$\operatorname{depth}(S_n^\star/I(\mathcal{L}_n^\star)) = \operatorname{sdepth}(S_n^\star/I(\mathcal{L}_n^\star)) = k+1.$$

With the back-up of Proposition 6.1.1, we have the upcoming results for Stanley depth and depth of $S_n/I(\mathcal{L}_n)$. This result says in particular Stanley's inequality holds for $S_n/I(\mathcal{L}_n)$.

Theorem 6.1.3. *Let $n \geq 2$. Then* $\operatorname{sdepth}(S_n/I(\mathcal{L}_n)) \geq \operatorname{depth}(S_n/I(\mathcal{L}_n)) = \lceil \frac{n}{2} \rceil$.

Proof. We first show that $\operatorname{depth}(S_n/I(\mathcal{L}_n)) = \lceil \frac{n}{2} \rceil$. If $n = 2$, then $\mathcal{G}(I(\mathcal{L}_n)) = \{x_1y_1, y_1y_2, y_2x_2, x_2x_1\}$, which is a minimal generating set of the edge ideal of C_4. By

[29, Proposition 1.3] it follows $\operatorname{depth}(S_2/I(\mathcal{L}_2)) = 1$. If $3 \le n \le 4$, then one can easily verify the result. For $n \ge 5$,

$$(I(\mathcal{L}_n) : y_n) = \big(\cup_{i=1}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-2}y_{n-2}, x_{n-2}x_{n-1}, x_n, y_{n-1}\big)$$
$$= (\mathcal{G}(I(\mathcal{L}_{n-2})), x_{n-2}x_{n-1}, x_n, y_{n-1}) = (\mathcal{G}(I(\mathcal{L}^\star_{n-2}), x_n, y_{n-1})),$$

so we obtain $S_n/(I(\mathcal{L}_n) : y_n) \cong (S^\star_{n-2}/I(\mathcal{L}^\star_{n-2}))[y_n]$. By Proposition 6.1.1 and Lemma 3.2.11,
$\operatorname{depth}(S_n/(I(\mathcal{L}_n) : y_n) = \lceil \frac{n-2+1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil$. Now consider the following S_n-module isomorphism:

$$(I(\mathcal{L}_n) : y_n)/I(\mathcal{L}_n) \cong y_{n-1}\frac{K[x_1,\dots,x_{n-2},y_1,\dots,y_{n-3}]}{\big(\cup_{i=1}^{n-4}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-3}y_{n-3}, x_{n-3}x_{n-2}\big)}[y_{n-1}]$$
$$\oplus\, x_n\frac{K[x_1,\dots,x_{n-2},y_1,\dots,y_{n-2}]}{\big(\cup_{i=1}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-2}y_{n-2}\big)}[x_n].$$

If $u \in (I(\mathcal{L}_n) : y_n)$ and $u \notin I(\mathcal{L}_n)$. It follows $x_n|u$ or $y_{n-1}|u$. If $x_n|u$ then $u = x_nv_1$ with $v_1 \in S_n$, since $u \notin I(\mathcal{L}_n)$, it follows $v_1 = x_n^\alpha w_1$, with $\alpha \ge 1$ and $w_1 \notin K[x_1,\dots,x_{n-2},y_1,\dots,y_{n-2}]$. Similarly, if $x_n \nmid u$, then $y_{n-1}|u$ and $u = y_{n-1}v_2$ with $v_2 \in S_n$, since $u \notin I(\mathcal{L}_n)$, it follows that $v_2 = y_{n-1}^\beta w_2$ with $\beta \ge 1$ and $w_2 \notin K[x_1,\dots,x_{n-2},y_1,\dots,y_{n-3}]$. Then,

$$\frac{K[x_1,\dots,x_{n-2},y_1,\dots,y_{n-3}]}{\big(\cup_{i=1}^{n-4}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-3}y_{n-3}, x_{n-3}x_{n-2}\big)} \cong S^\star_{n-3}/I(\mathcal{L}^\star_{n-3})$$

and

$$\frac{K[x_1,\dots,x_{n-2},y_1,\dots,y_{n-2}]}{\big(\cup_{i=1}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_{n-2}y_{n-2}\big)} \cong S_{n-2}/I(\mathcal{L}_{n-2}).$$

By Proposition 6.1.1 and Lemma 3.2.11, we have

$$\operatorname{depth}((I(\mathcal{L}_n) : y_n)/I(\mathcal{L}_n)) = \min\{\lceil \frac{n-3+1}{2} \rceil + 1, \lceil \frac{n-2}{2} \rceil + 1\} = \lceil \frac{n}{2} \rceil.$$

Consider

$$0 \longrightarrow (I(\mathcal{L}_n) : y_n)/I(\mathcal{L}_n) \xrightarrow{\cdot y_n} S_n/I(\mathcal{L}_n) \longrightarrow S_n/(I(\mathcal{L}_n) : y_n) \longrightarrow 0. \qquad (6.2)$$

Now we prove that $\operatorname{sdepth}(S_n/I(\mathcal{L}_n)) \ge \lceil \frac{n}{2} \rceil$. For $2 \le n \le 4$, by [24], we have the following Stanley decompositions. If $n = 2$, then

$$S_2/I(\mathcal{L}_2) = K[x_1, y_2] \oplus x_2K[y_1, x_2] \oplus y_1K[y_1].$$

If $n = 3$, then

$$S_3/I(\mathcal{L}_3) = K[x_1, y_2] \oplus y_1K[y_1, x_2] \oplus x_2K[x_2, y_3] \oplus x_3K[x_1, x_3] \oplus y_3K[x_1, y_3]$$
$$\oplus y_1x_3K[y_1, x_3] \oplus y_1y_3K[y_1, x_2, y_3] \oplus y_2x_3K[x_1, y_2, x_3].$$

If $n = 4$, then

$$S_4/I(\mathcal{L}_4) = K[x_1, x_3, y_2] \oplus y_1K[x_2, y_1, y_3] \oplus x_2K[x_2, x_4, y_3] \oplus y_3K[x_1, x_4, y_3]$$
$$\oplus x_4K[x_1, x_4, y_2] \oplus y_4K[x_2, y_1, y_4] \oplus x_1y_4K[x_1, y_2, y_4] \oplus y_1x_3K[y_1, x_3, y_4]$$
$$\oplus y_1x_4K[x_2, x_4, y_1] \oplus y_2y_4K[x_3, y_2, y_4] \oplus x_3y_4K[x_1, x_3, y_4].$$

For $n \geq 5$, by Proposition 6.1.1, induction on n and Lemma 3.2.11 on the exact sequence (6.2), we have the required lower bound. □

Corollary 6.1.4. *Let* $n \geq 2$. *If* $n \equiv 0 \pmod 2$ *then* $\lceil \frac{n}{2} \rceil \leq \operatorname{sdepth}(S_n/I(\mathcal{L}_n)) \leq \lceil \frac{n+1}{2} \rceil$, *otherwise* $\operatorname{sdepth}(S_n/I(\mathcal{L}_n)) = \lceil \frac{n}{2} \rceil$.

Proof. One can easily verify the result for $2 \leq n \leq 3$. If $n \geq 4$, then by Theorem 6.1.3, we only need to show that $\operatorname{sdepth}(S_n/I(\mathcal{L}_n)) \leq \lceil \frac{n+1}{2} \rceil$. As $y_n \notin I(\mathcal{L}_n)$, from Proposition 3.2.13, $\operatorname{sdepth}(S_n/I(\mathcal{L}_n)) \leq \operatorname{sdepth}(S_n/(I(\mathcal{L}_n) : y_n))$. Since

$$S_n/(I(\mathcal{L}_n) : y_n) \cong (S^\star_{n-2}/I(\mathcal{L}^\star_{n-2}))[y_n],$$

by Lemma 3.2.11 and Proposition 6.1.1,

$$\operatorname{sdepth}((S^\star_{n-2}/I(\mathcal{L}^\star_{n-2}))[y_n]) = \lceil \frac{n-2+1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil.$$

and the required result follows. □

Example 6.1.5. *If* $n = 4$, *then we make some calculations for Stanley depth by using CoCoA, (SdepthLib:coc [30]). Calculations show that* $\operatorname{sdepth}(S_4/I(\mathcal{L}_4)) = 3 = \lceil \frac{4+1}{2} \rceil$. *Thus the upper bound in Corollary 6.1.4 is found.*

By using Theorem 6.1.3 and Auslander-Buchsbaum formula [33, Theorem 3.7], we have the exact value of projective dimension of cyclic module $S_n/I(\mathcal{L}_n)$ as follows:

Corollary 6.1.6. *For $n \geq 2$, $pd_{S_n}(S_n/I(\mathcal{L}_n)) = n + \lfloor \frac{n}{2} \rfloor$.*

Theorem 6.1.7. *Let $n \geq 2$. Then* $\operatorname{sdepth}(I(\mathcal{L}_n^\star)) \geq \lceil \frac{n+1}{2} \rceil + 1$.

Proof. The result is clear for $2 \leq n \leq 3$. For $n \geq 4$, as $y_n \notin I(\mathcal{L}_n^\star)$, so we have

$$I(\mathcal{L}_n^\star) = I(\mathcal{L}_n^\star) \cap S' \oplus y_n(I(\mathcal{L}_n^\star) : y_n)S_n^\star,$$

where $S' = K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1}, y_{n+1}]$,
$(I(\mathcal{L}_n^\star) : y_n) = (\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1})S_n^\star$, and $I(\mathcal{L}_n^\star) \cap S' = (\mathcal{G}(I(\mathcal{L}_{n-1}^\star)))S'$. Thus

$$\operatorname{sdepth}(I(\mathcal{L}_n^\star)) \geq \min\Big\{ \operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1})S_n^\star), \operatorname{sdepth}(\mathcal{G}(I(\mathcal{L}_{n-1}^\star))S') \Big\}.$$

By Lemma 3.2.11, we have

$$\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1})S_n^\star) = \operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1})S') + 1,$$

and by [25, Theorem 1.3], it follows that

$$\begin{aligned}\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1})S') \geq \min\Big\{ &\operatorname{sdepth}(I(\mathcal{L}_{n-2}^\star)S_{n-2}^\star) + 3, \\ &\operatorname{sdepth}((y_{n-1}, x_n, y_{n+1})\bar{S}) + \operatorname{sdepth}(S_{n-2}^\star/I(\mathcal{L}_{n-2}^\star))S_{n-2}^\star) \Big\},\end{aligned}$$

where $\bar{S} = K[y_{n-1}, x_n, y_{n+1}]$. The induction hypothesis, [32, Theorem 2.2] and Theorem 6.1.1 yield

$$\begin{aligned}\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1})S') &\geq \min\{\lceil \frac{n-2+1}{2} \rceil + 1 + 3, 2 + \lceil \frac{n-2+1}{2} \rceil\} \\ &= \lceil \frac{n+1}{2} \rceil + 1.\end{aligned}$$

Thus, $\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n, y_{n+1})S_n^\star) > \lceil \frac{n+1}{2} \rceil + 1$. By [31, Lemma 2.11], we have

$$\operatorname{sdepth}(\mathcal{G}(I(\mathcal{L}_{n-1}^\star))S') \geq \operatorname{sdepth}(I(\mathcal{L}_{n-1}^\star)S_{n-1}^\star[y_{n+1}]),$$

by Lemma 3.2.11, we have

$$\operatorname{sdepth}(I(\mathcal{L}_{n-1}^\star)S_{n-1}^\star[y_{n+1}]) \geq \lceil \frac{n-1+1}{2} \rceil + 1 + 1.$$

Therefore,

$$\operatorname{sdepth}(I(\mathcal{L}_n^\star)) \geq \lceil \frac{n+1}{2} \rceil + 1.$$

□

Corollary 6.1.8. *Let* $n \geq 2$. *Then* $\operatorname{sdepth}(I(\mathcal{L}_n^\star)) \geq \operatorname{sdepth}(S_n^\star/I(\mathcal{L}_n^\star)) + 1$.

Theorem 6.1.9. *Let* $n \geq 2$. *Then* $\operatorname{sdepth}(I(\mathcal{L}_n)) \geq \lceil \frac{n}{2} \rceil + 1$.

Proof. It is easy to see that the result holds for $2 \leq n \leq 3$. For $n \geq 4$, since $y_n \notin I(\mathcal{L}_n)$, we have

$$I(\mathcal{L}_n) = I(\mathcal{L}_n) \cap S'' \oplus y_n(I(\mathcal{L}_n) : y_n)S_n,$$

where $S'' = K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1}]$, $(I(\mathcal{L}_n) : y_n) = (\mathcal{G}(I(\mathcal{L}_{n-2}^\star), x_n, y_{n-1}))S_n$, and $I(\mathcal{L}_n) \cap S'' = (\mathcal{G}(I(\mathcal{L}_{n-1}^\star)))S''$. Thus

$$\operatorname{sdepth}(I(\mathcal{L}_n)) \geq \min\Big\{ \operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n)S_n), \operatorname{sdepth}(\mathcal{G}(I(\mathcal{L}_{n-1}^\star))S'')\Big\}.$$

By Lemma 3.2.11,

$$\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n)S_n) = \operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n)S'') + 1,$$

and by [25, Theorem 1.3], it follows

$$\begin{aligned}\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n)S'') \geq \min\Big\{ &\operatorname{sdepth}(I(\mathcal{L}_{n-2}^\star)S_{n-2}^\star) + 2,\\ &\operatorname{sdepth}((y_{n-1}, x_n)\widetilde{S}) + \operatorname{sdepth}(S_{n-2}^\star/I(\mathcal{L}_{n-2}^\star))S_{n-2}^\star)\Big\},\end{aligned}$$

where $\widetilde{S} = K[y_{n-1}, x_n]$. By Theorem 6.1.7, [32, Theorem 2.2] and Theorem 6.1.1, we have

$$\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n)S'') \geq \min\{\lceil \frac{n-2+1}{2} \rceil + 1 + 2, 1 + \lceil \frac{n-2+1}{2} \rceil\} = \lceil \frac{n+1}{2} \rceil.$$

Thus, $\operatorname{sdepth}((\mathcal{G}(I(\mathcal{L}_{n-2}^\star)), y_{n-1}, x_n)S_n) > \lceil \frac{n+1}{2} \rceil$. By [31, Lemma 2.11],

$$\operatorname{sdepth}(\mathcal{G}(I(\mathcal{L}_{n-1}^\star))S'') \geq \operatorname{sdepth}(I(\mathcal{L}_{n-1}^\star)S_{n-1}^\star),$$

by Theorem 6.1.7, we have

$$\operatorname{sdepth}(I(\mathcal{L}_{n-1}^\star)S_{n-1}^\star) \geq \lceil \frac{n-1+1}{2} \rceil + 1.$$

Thus,

$$\operatorname{sdepth}(I(\mathcal{L}_n)) \geq \lceil \frac{n}{2} \rceil + 1.$$

□

Corollary 6.1.10. *Let* $n \geq 2$. *Then* $\text{sdepth}(I(\mathcal{L}_n)) \geq \text{sdepth}(S_n/I(\mathcal{L}_n))$.

Now, we determine Stanley depth and depth of cyclic module $S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})$ by using induction on n and Proposition 6.1.1.

Proposition 6.1.11. *Let* $n \geq 2$. *Then* $\text{sdepth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})), \text{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \geq \lceil \frac{n+1}{2} \rceil$.

Proof. We first show that $\text{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \geq \lceil \frac{n+1}{2} \rceil$ by using induction on n. The result is obvious for $2 \leq n \leq 3$. For $n \geq 4$, consider,

$$0 \longrightarrow S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}) : y_{n+1}) \xrightarrow{\cdot y_{n+1}} S_n^{\diamond}/I(\mathcal{L}_n^{\diamond}) \longrightarrow S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}), y_{n+1}) \longrightarrow 0. \tag{6.3}$$

$$\text{Let} \quad \overline{I} = (I(\mathcal{L}_n^{\diamond}) : y_{n+1}) = (\cup_{i=1}^{n-2} \{x_i y_i, x_i x_{i+1}, y_i y_{i+1}\}, x_{n-1}y_{n-1}, x_{n-1}x_n, y_1 y_{n+2}, y_n)$$
$$= (\mathcal{G}(I(\mathcal{L}_{n-1})), x_{n-1}x_n, y_1 y_{n+2}, y_n),$$

and considering

$$0 \longrightarrow S_n^{\diamond}/(\overline{I} : x_n) \xrightarrow{\cdot x_n} S_n^{\diamond}/\overline{I} \longrightarrow S_n^{\diamond}/(\overline{I}, x_n) \longrightarrow 0. \tag{6.4}$$

Here $(\overline{I}, x_n) = (\mathcal{G}(I(\mathcal{L}_{n-1})), x_n, y_1 y_{n+2}, y_n)$, after a suitable renumbering of the variables, we obtain $(\overline{I}, x_n) = (\mathcal{G}(I(\mathcal{L}_{n-1}^{\star})),\ x_n,\ y_n)$, which further implies that

$$S_n^{\diamond}/(\overline{I}, x_n) \cong (S_{n-1}^{\star}/I(\mathcal{L}_{n-1}^{\star}))[y_{n+1}].$$

By Proposition 6.1.1 and Lemma 3.2.11, it follows that

$$\text{depth}(S_n^{\diamond}/(\overline{I}, x_n)) \geq \lceil \frac{n-1+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1.$$

As

$$(\overline{I} : x_n) = (\cup_{i=1}^{n-3} \{x_i y_i, x_i x_{i+1}, y_i y_{i+1}\}, x_{n-2}y_{n-2}, y_{n-2}y_{n-1}, y_1 y_{n+2}, y_n, x_{n-1})$$
$$= (\mathcal{G}(I(\mathcal{L}_{n-2})), y_{n-2}y_{n-1}, y_1 y_{n+2}, y_n, x_{n-1}),$$

$$(\overline{I} : x_n) = (\mathcal{G}(I(\mathcal{L}_{n-2}^{\diamond})), y_n, x_{n-1})$$

$$S_n^{\diamond}/(\overline{I} : x_n) \cong S_{n-2}^{\diamond}/I(\mathcal{L}_{n-2}^{\diamond})[x_n, y_{n+1}].$$

The induction hypothesis and Lemma 3.2.11 yield

$$\operatorname{depth}(S_n^{\diamond}/(\overline{I}:x_n)) = \lceil\frac{n-2+1}{2}\rceil + 2 = \lceil\frac{n+1}{2}\rceil + 1.$$

Also

$$(I(\mathcal{L}_n^{\diamond}), y_{n+1}) = (\cup_{i=1}^{n-1}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_ny_n, y_1y_{n+2}, y_{n+1})$$
$$= (\mathcal{G}(I(\mathcal{L}_n)), y_1y_{n+2},\, y_{n+1}),$$

after renumbering the variables, $(I(\mathcal{L}_n^{\diamond}), y_{n+1}) = (\mathcal{G}(I(\mathcal{L}_n^{\star})),\, y_{n+1})$, so we have

$$S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}), y_{n+1}) \cong S_n^{\star}/I(\mathcal{L}_n^{\star}).$$

By Proposition 6.1.1, we have $\operatorname{depth}(S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}), y_{n+1}) = \lceil\frac{n+1}{2}\rceil$. Finally, by applying Depth Lemma on (6.5) and (6.4), we conclude that

$$\operatorname{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \geq \lceil\frac{n+1}{2}\rceil.$$

Now, we show that $\operatorname{sdepth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \geq \lceil\frac{n+1}{2}\rceil$. For $2 \leq n \leq 3$, one can easily show that the result holds. For $n \geq 4$, result follows by using mathematical induction on n, Proposition 6.1.1, and Lemma 3.2.10 on the exact sequences (6.5) and (6.4). □

Corollary 6.1.12. *Let $n \geq 2$. If $n \equiv 0 \pmod 2$ then*

$$\lceil\frac{n+1}{2}\rceil \leq \operatorname{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})), \operatorname{sdepth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \leq \lceil\frac{n+1}{2}\rceil + 1,$$

otherwise

$$\lceil\frac{n+1}{2}\rceil \leq \operatorname{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})), \operatorname{sdepth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \leq \lceil\frac{n+2}{2}\rceil.$$

Proof. First, we show the result for depth. It can be easily verified that the result is true for $2 \leq n \leq 5$. Now, if $n \geq 6$, then by Proposition 6.1.11, it is enough to show that If $n \equiv 0 \pmod 2$ then $\operatorname{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \leq \lceil\frac{n+1}{2}\rceil + 1$ otherwise $\operatorname{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \leq \lceil\frac{n+2}{2}\rceil$.
Case 1, If $n \equiv 0 \pmod 2$, Since $x_{n-2}y_n \notin I(\mathcal{L}_n^{\diamond})$, from Corollary 3.2.12 we obtain that $\operatorname{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond})) \leq \operatorname{depth}(S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}) : x_{n-2}y_n))$. As

$$S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}) : x_{n-2}y_n) \cong (S_{n-4}^{\diamond}/I(\mathcal{L}_{n-4}^{\diamond}))[x_{n-2}, y_n],$$

by the principle of mathematical induction and Lemma 3.2.11 we have

$$\operatorname{depth}((S_{n-4}^{\diamond}/I(\mathcal{L}_{n-4}^{\diamond}))[x_{n-2},y_n]) \leq \lceil\frac{n-4+1}{2}\rceil+1+2=\lceil\frac{n+1}{2}\rceil+1.$$

Case 2, If $n\equiv 1\,(\operatorname{mod} 2)$, Since $x_{n-2}y_n\notin I(\mathcal{L}_n^{\diamond})$, from Corollary 3.2.12 we attain that $\operatorname{depth}(S_n^{\diamond}/I(\mathcal{L}_n^{\diamond}))\leq \operatorname{depth}(S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}):x_{n-2}y_n))$. As

$$S_n^{\diamond}/(I(\mathcal{L}_n^{\diamond}):x_{n-2}y_n)\cong (S_{n-4}^{\diamond}/I(\mathcal{L}_{n-4}^{\diamond}))[x_{n-2},y_n],$$

by the principle of mathematical induction and Lemma 3.2.11 we have

$$\operatorname{depth}((S_{n-4}^{\diamond}/I(\mathcal{L}_{n-4}^{\diamond}))[x_{n-2},y_n]) \leq \lceil\frac{n-4+2}{2}\rceil+2=\lceil\frac{n+2}{2}\rceil.$$

Proof for Stanley depth is similar by using Propositions 3.2.13 and 6.1.11. □

Now, we determine the depth and Stanley depth of cyclic module $S_n^{\bullet}/I(\mathcal{L}_n^{\bullet})$ by using induction on n and Proposition 6.1.1.

Proposition 6.1.13. *Let* $n\geq 2$. *Then* $\operatorname{sdepth}(S_n^{\bullet}/I(\mathcal{L}_n^{\bullet})), \operatorname{depth}(S_n^{\bullet}/I(\mathcal{L}_n^{\bullet}))\geq \lceil\frac{n+1}{2}\rceil$.

Proof. We first show that $\operatorname{depth}(S_n^{\bullet}/I(\mathcal{L}_n^{\bullet}))\geq\lceil\frac{n+1}{2}\rceil$. The result is obvious for $2\leq n\leq 3$. For $n\geq 4$,

$$0\longrightarrow S_n^{\bullet}/(I(\mathcal{L}_n^{\bullet}):x_{n+1})\xrightarrow{\cdot x_{n+1}} S_n^{\bullet}/I(\mathcal{L}_n^{\bullet})\longrightarrow S_n^{\bullet}/(I(\mathcal{L}_n^{\bullet}),x_{n+1})\longrightarrow 0. \tag{6.5}$$

Let $(I(\mathcal{L}_n^{\bullet}):x_{n+1})=(\,\cup_{i=2}^{n-1}\{x_iy_i,x_ix_{i+1},y_iy_{i+1}\},x_ny_n,y_ny_{n+1},y_1y_2,x_1)$.

$$(I(\mathcal{L}_n^{\bullet}):x_{n+1})=(\mathcal{G}(I(\mathcal{L}_{n-1}^{\diamond})),\,x_1),$$

which further implies that

$$S_n^{\bullet}/(I(\mathcal{L}_n^{\bullet}):x_{n+1})\cong (S_{n-1}^{\diamond}/I(\mathcal{L}_{n-1}^{\diamond}))[x_{n+1}].$$

By Proposition 6.1.11 and Lemma 3.2.11, it follows that

$$\operatorname{depth}(S_n^{\bullet}/(I(\mathcal{L}_n^{\bullet}):x_{n+1}))\geq\lceil\frac{n-1+1}{2}\rceil+1=\lceil\frac{n}{2}\rceil+1.$$

Also

$$(I(\mathcal{L}_n^\bullet), x_{n+1}) = (\,\cup_{i=1}^{n-1}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\}, x_ny_n, y_ny_{n+1}, x_{n+1})$$
$$= (\mathcal{G}(I(\mathcal{L}_n)), y_ny_{n+1},\, x_{n+1}),$$

after renumbering the variables, $(I(\mathcal{L}_n^\bullet), x_{n+1}) = (\mathcal{G}(I(\mathcal{L}_n^\star)),\, x_{n+1})$, so we have

$$S_n^\bullet/(I(\mathcal{L}_n^\bullet), x_{n+1}) \cong S_n^\star/I(\mathcal{L}_n^\star).$$

Applying Proposition 6.1.1, we have $\operatorname{depth}(S_n^\bullet/(I(\mathcal{L}_n^\bullet), x_{n+1}) = \lceil\frac{n+1}{2}\rceil$. Finally, by using Depth Lemma on (6.5) and (6.4), we conclude that

$$\operatorname{depth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \geq \lceil\frac{n+1}{2}\rceil.$$

Now, we show that $\operatorname{sdepth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \geq \lceil\frac{n+1}{2}\rceil$. For $2 \leq n \leq 3$, one can easily show that the result holds. For $n \geq 4$, result follows by Proposition 6.1.1, and Lemma 3.2.10 on the exact sequence (6.5). □

Corollary 6.1.14. *Let $n \geq 2$. If $n \equiv 0 \pmod 2$ then*

$$\lceil\frac{n+1}{2}\rceil \leq \operatorname{depth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)), \operatorname{sdepth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \leq \lceil\frac{n+1}{2}\rceil + 1,$$

otherwise

$$\lceil\frac{n+1}{2}\rceil \leq \operatorname{depth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)), \operatorname{sdepth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \leq \lceil\frac{n+2}{2}\rceil.$$

Proof. First, we show the result for depth. It can be easily verified that the result is true for $2 \leq n \leq 5$. Now, if $n \geq 6$, then by Proposition 6.1.11, it is enough to show that If $n \equiv 0 \pmod 2$ then $\operatorname{depth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \leq \lceil\frac{n+1}{2}\rceil + 1$ otherwise $\operatorname{depth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \leq \lceil\frac{n+2}{2}\rceil$.
Case 1, If $n \equiv 0 \pmod 2$, Since $x_{n-2}y_n \notin I(\mathcal{L}_n^\bullet)$, from Corollary 3.2.12 we attain that $\operatorname{depth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \leq \operatorname{depth}(S_n^\bullet/(I(\mathcal{L}_n^\bullet) : x_{n-2}y_n))$. As $S_n^\bullet/(I(\mathcal{L}_n^\bullet) : x_{n-2}y_n) \cong (S_{n-4}^\bullet/I(\mathcal{L}_{n-4}^\bullet))[x_{n-2}, y_n]$, by the principle of mathematical induction and Lemma 3.2.11 we have

$$\operatorname{depth}((S_{n-4}^\bullet/I(\mathcal{L}_{n-4}^\bullet))[x_{n-2}, y_n]) \leq \lceil\frac{n-4+1}{2}\rceil + 1 + 2 = \lceil\frac{n+1}{2}\rceil + 1.$$

Case 2, If $n \equiv 1 \pmod 2$, Since $x_{n-2}y_n \notin I(\mathcal{L}_n^\bullet)$, from Corollary 3.2.12 we attain that $\operatorname{depth}(S_n^\bullet/I(\mathcal{L}_n^\bullet)) \leq \operatorname{depth}(S_n^\bullet/(I(\mathcal{L}_n^\bullet) : x_{n-2}y_n))$. As $S_n^\bullet/(I(\mathcal{L}_n^\bullet) : x_{n-2}y_n) \cong (S_{n-4}^\bullet/I(\mathcal{L}_{n-4}^\bullet))[x_{n-2}, y_n]$, by the principle of mathematical induction and Lemma 3.2.11 we have

$$\operatorname{depth}((S_{n-4}^\bullet/I(\mathcal{L}_{n-4}^\bullet))[x_{n-2}, y_n]) \leq \lceil \frac{n-4+2}{2} \rceil + 2 = \lceil \frac{n+2}{2} \rceil.$$

Proof for Stanley depth is similar by using Propositions 3.2.13 and 6.1.11. □

6.2 Cubic Circulant Graphs

By using Corollary 6.1.8, Theorem 6.1.3 and Proposition 6.1.11, we have the following outcomes for $S_{2n}/I(C_{2n}(1,n))$ and $S_{2n}/I(C_{2n}(2,n))$.

Theorem 6.2.1. *If $G = C_{2n}(1,n)$ or $C_{2n}(2,n)$, then for $n \geq 3$ we have that*

$$\operatorname{sdepth}(S_{2n}/I(G)), \operatorname{depth}(S_{2n}/I(G)) \geq \lceil \frac{n-1}{2} \rceil.$$

Proof. Let $G = C_{2n}(1,n)$, If $3 \leq n \leq 4$, one easily see that result holds. For $n \geq 5$,

$$0 \longrightarrow S_{2n}/(I(G) : x_n) \xrightarrow{\cdot x_n} S_{2n}/I(G) \longrightarrow S_{2n}/(I(G), x_n) \longrightarrow 0. \tag{6.6}$$

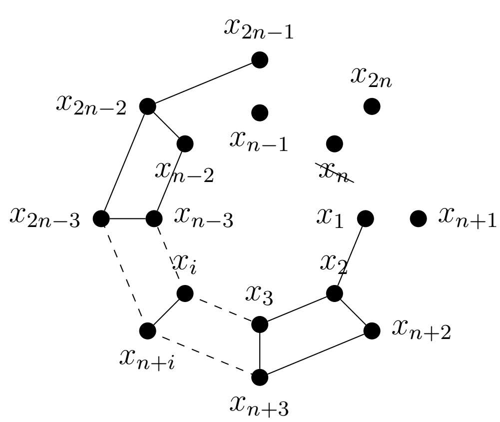


Figure 6.5: $(I(C_{2n}(1,n)) : x_n)$

Since, $(I(G):x_n) = (\mathcal{G}(I(\mathcal{L}^{\bullet}_{n-3})), x_{n-1}, x_{n+1}, x_{2n})$ implies that

$$S_{2n}/I(G) \cong S^{\bullet}_{n-3}/I(\mathcal{L}^{\bullet}_{n-3})[x_n].$$

By Proposition 6.1.13 and Lemma 3.2.11, it follows that $\operatorname{depth}(S_{2n}/(I(G)) \geq \lceil \frac{n-3+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$. Now assume that $J = (I(G), x_n)$

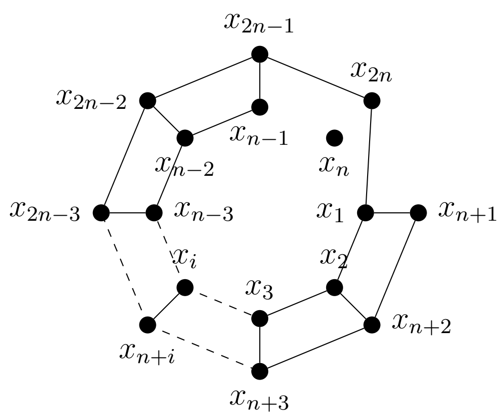


Figure 6.6: $(I(C_{2n}(1,n)), x_n)$

and short exact sequence

$$0 \longrightarrow S_{2n}/(J:x_{2n}) \xrightarrow{\cdot x_{2n}} S_{2n}/J \longrightarrow S_{2n}/(J, x_{2n}) \longrightarrow 0. \qquad (6.7)$$

Since $(J, x_{2n}) = (\mathcal{G}(I(\mathcal{L}_{n-1})), x_n, x_{2n})$, so we have $S_{2n}/(J, x_{2n}) \cong S_{n-1}/I(\mathcal{L}_{n-1})$. By Theorem 6.1.3 and Lemma 3.2.11 we obtain $\operatorname{depth}(S_{2n}/(J, x_{2n}) = \lceil \frac{n-1}{2} \rceil$. Also after renumbering the variables, we have $(J : x_{2n}) = (\mathcal{G}(I(\mathcal{L}^{\bullet}_{n-3})), x_{2n-1}, x_1, x_n)$, since $S_{2n}/(J:x_{2n}) \cong S^{\bullet}_{n-3}/I(\mathcal{L}^{\bullet}_{n-3})[x_{2n}]$. By Proposition 6.1.13 and Lemma 3.2.11,

$$\operatorname{depth}(S_{2n}/(J:x_{2n})) \geq \lceil \frac{n-3+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$$

Finally, by applying Depth Lemma on (6.6) and (6.7), we conclude that $\operatorname{depth}(S_{2n}/I(G)) \geq \lceil \frac{n-1}{2} \rceil$. Now, we show that $\operatorname{sdepth}(S_{2n}/I(G)) \geq \lceil \frac{n-1}{2} \rceil$. If $3 \leq n \leq 4$, then result is obvious. For $n \geq 5$, one can prove the result by applying Lemma 3.2.10 on (6.6) and (6.7).

Now let $G = C_{2n}(2, n)$, If $3 \leq n \leq 4$, then result obviously holds. For $n \geq 5$,

$$0 \longrightarrow S_{2n}/(I(G) : x_1) \xrightarrow{\cdot x_1} S_{2n}/I(G) \longrightarrow S_{2n}/(I(G), x_1) \longrightarrow 0. \qquad (6.8)$$

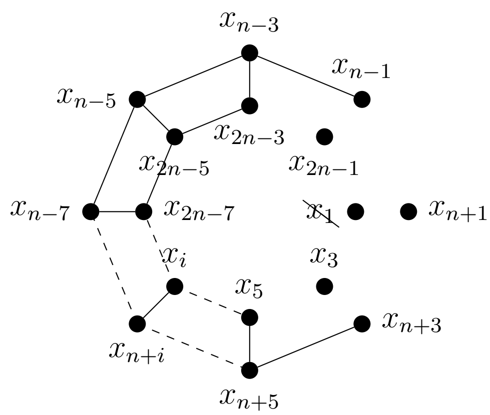


Figure 6.7: $(I(C_{2n}(2, n)) : x_1)$

after renumbering the variables, we have $(I(G) : x_1) = (\mathcal{G}(I(\mathcal{L}^{\diamond}_{n-3})), x_{2n-1}, x_{n+1}, x_3)$, that follows that $S_{2n}/I(G) \cong S^{\diamond}_{n-3}/I(\mathcal{L}^{\diamond}_{n-3})[x_1]$. By Proposition 6.1.11 and Lemma 3.2.11, it follows that $\text{depth}(S_{2n}/(I(G)) \geq \lceil \frac{n-3+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$. Now assume that $J = (I(G), x_1)$ and

$$0 \longrightarrow S_{2n}/(J : x_{n+1}) \xrightarrow{\cdot x_{n+1}} S_{2n}/J \longrightarrow S_{2n}/(J, x_{n+1}) \longrightarrow 0. \qquad (6.9)$$

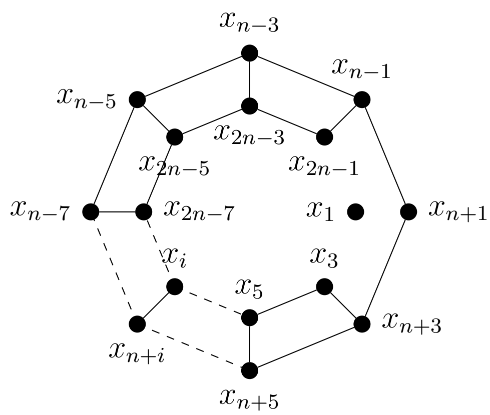


Figure 6.8: $(I(C_{2n}(2,n)), x_1)$

Since $(J, x_{n+1}) = (\mathcal{G}(I(\mathcal{L}_{n-1})), x_1, x_{n+1})$, so we have $S_{2n}/(J, x_{n+1}) \cong S_{n-1}/I(\mathcal{L}_{n-1})$. By Theorem 6.1.3 and Lemma 3.2.11, we obtain depth$(S_{2n}/(J, x_{n+1}) = \lceil \frac{n-1}{2} \rceil$. Also, we have $(J : x_{n+1}) = (\mathcal{G}(I(\mathcal{L}^{\diamond}_{n-3})), x_{n-1}, x_{n+3}, x_1)$, so $S_{2n}/(J : x_{n+1}) \cong S^{\diamond}_{n-3}/I(\mathcal{L}^{\diamond}_{n-3})[x_{n+1}]$. By Proposition 6.1.11 and Lemma 3.2.11,

$$\operatorname{depth}(S_{2n}/(J : x_{n+1})) \geq \lceil \frac{n-3+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil.$$

Finally, by applying Depth Lemma on (6.6) and (6.7), we conclude that depth$(S_{2n}/I(G)) \geq \lceil \frac{n-1}{2} \rceil$. Now, we show that sdepth$(S_{2n}/I(G)) \geq \lceil \frac{n-1}{2} \rceil$. If $3 \leq n \leq 4$, then one can easily see that result holds. For $n \geq 5$, the required result follows by using Lemma 3.2.10 on (6.8) and (6.9). □

Corollary 6.2.2. *If $G = C_{2n}(1,n)$ or $C_{2n}(2,n)$, and $n \geq 3$, If $n \equiv 0 \pmod 2$ then*

$$\lceil \frac{n-1}{2} \rceil \leq \operatorname{sdepth}(S_{2n}/I(G)), \operatorname{depth}(S_{2n}/I(G)) \leq \lceil \frac{n+1}{2} \rceil,$$

otherwise

$$\lceil \frac{n-1}{2} \rceil \leq \operatorname{sdepth}(S_{2n}/I(G)), \operatorname{depth}(S_{2n}/I(G)) \leq \lceil \frac{n}{2} \rceil + 1.$$

Proof. First, we prove result for depth; this can be easily verified if $3 \leq n \leq 4$. For $n \geq 5$, by Theorem 6.2.1, we just need to show that If $n \equiv 0 \pmod 2$ then

$\operatorname{depth}(S_{2n}/I(G)) \leq \lceil \frac{n}{2} \rceil + 1$ otherwise $\operatorname{depth}(S_{2n}/I(G)) \leq \lceil \frac{n+1}{2} \rceil$

Case 1, If $n \equiv 0 \,(\operatorname{mod} 2)$. Let $G = C_{2n}(1, n)$, Since $x_n \notin I(G)$, thus $\operatorname{depth}(S_{2n}/I(G)) \leq \operatorname{depth}(S_{2n}/(I(G) : x_n))$ by Corollary 3.2.12. As $S_{2n}/(I(G) : x_n) \cong S^{\bullet}_{n-3}/I(\mathcal{L}^{\bullet}_{n-3})[x_n]$.
As $n \equiv 0 \,(\operatorname{mod} 2)$ then $(n-3) \equiv 1 \,(\operatorname{mod} 2)$. Therefore, by Corollary 6.1.14, and Lemma 3.2.11,

$$\operatorname{depth}(S^{\bullet}_{n-3}/I(\mathcal{L}^{\bullet}_{n-3})[x_n] \leq \lceil \frac{n-3+2}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil.$$

Now let $G = C_{2n}(2, n)$, Since $x_1 \notin I(G)$, thus $\operatorname{depth}(S_{2n}/I(G)) \leq \operatorname{depth}(S_{2n}/(I(G) : x_1))$ by Corollary 3.2.12. As $S_{2n}/(I(G) : x_1) \cong S^{\diamond}_{n-3}/I(\mathcal{L}^{\diamond}_{n-3})[x_1]$.
As $n \equiv 0 \,(\operatorname{mod} 2)$ then $(n-3) \equiv 1 \,(\operatorname{mod} 2)$. Therefore, by Corollary 6.1.12, and Lemma 3.2.11,

$$\operatorname{depth}(S^{\diamond}_{n-3}/I(\mathcal{L}^{\diamond}_{n-3})[x_1]) \leq \lceil \frac{n-3+2}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil.$$

Case 2, If $n \equiv 1 \,(\operatorname{mod} 2)$. Let $G = C_{2n}(1, n)$, Since $x_n \notin I(G)$, thus $\operatorname{depth}(S_{2n}/I(G)) \leq \operatorname{depth}(S_{2n}/(I(G) : x_n))$ by Corollary 3.2.12. As $S_{2n}/(I(G) : x_n) \cong S^{\bullet}_{n-3}/I(\mathcal{L}^{\bullet}_{n-3})[x_n]$.
As $n \equiv 1 \,(\operatorname{mod} 2)$ then $(n-3) \equiv 0 \,(\operatorname{mod} 2)$. Therefore, by Corollary 6.1.14, and Lemma 3.2.11,

$$\operatorname{depth}(S^{\bullet}_{n-3}/I(\mathcal{L}^{\bullet}_{n-3})[x_n] \leq \lceil \frac{n-3+1}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1.$$

Now let $G = C_{2n}(2, n)$, Since $x_1 \notin I(G)$, thus $\operatorname{depth}(S_{2n}/I(G)) \leq \operatorname{depth}(S_{2n}/(I(G) : x_1))$ by Corollary 3.2.12. As $S_{2n}/(I(G) : x_1) \cong S^{\diamond}_{n-3}/I(\mathcal{L}^{\diamond}_{n-3})[x_1]$.
As $n \equiv 1 \,(\operatorname{mod} 2)$ then $(n-3) \equiv 0 \,(\operatorname{mod} 2)$. Therefore, by Corollary 6.1.12, and Lemma 3.2.11,

$$\operatorname{depth}(S^{\diamond}_{n-3}/I(\mathcal{L}^{\diamond}_{n-3})[x_1]) \leq \lceil \frac{n-3+1}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1.$$

It remains to show the result for Stanley depth, for this it is similar as in the case of depth by using Proposition 3.2.13 instead of Corollary 3.2.12. □

Label the graphs $C_{2n}(1, n)$ and $C_{2n}(2, n)$ by using two sets of variables $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ (see Figure 6.9). Let $S_n := K[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ be the

ring of polynomials in variables $x_1, x_2, \ldots,$
$x_n, y_1, y_2, \ldots, y_n$ over the field K. Then $I(I(C_{2n}(1,n))$ and $I(I(C_{2n}(2,n))$ are monomial ideals of S_n with $\mathcal{G}(I(C_{2n}(1,n))) = \mathcal{G}(I(\mathcal{L}_n)) \cup \{x_1y_n, y_1x_n\}$ and $\mathcal{G}(I(C_{2n}(2,n))) = \mathcal{G}(I(\mathcal{L}_n)) \cup \{x_1x_n, y_1y_n\}$.

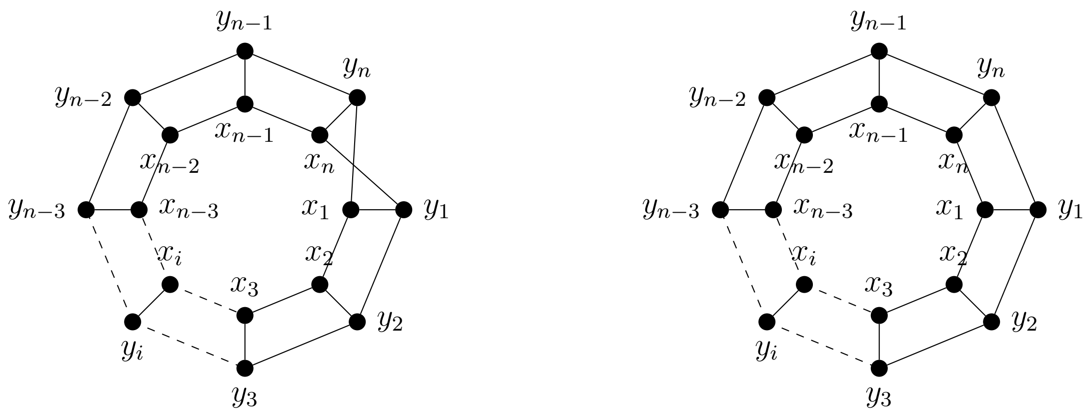


Figure 6.9: From left to right $C_{2n}(1,n)$ and $C_{2n}(2,n)$.

Proposition 6.2.3. *Let* $n \geq 3$. *Then* $\operatorname{sdepth}(I(C_{2n}(1,n))/I(\mathcal{L}_n)) \geq \lceil \frac{n+1}{2} \rceil$.

Proof. For $3 \leq n \leq 5$, by using [24], there exist Stanley decompositions.
If $n = 3$, then The poset P is given by

$$P = \{(1,0,0,0,0,1), (0,0,1,1,0,0)\}.$$

Partitions of P are given by

$$\mathcal{P}: \ [(1,0,0,0,0,1), (1,0,0,0,0,1)] \bigcup [(0,0,1,1,0,0), (0,0,1,1,0,0)].$$

and the corresponding Stanley decomposition is

$$I(C_6(1,3))/I(\mathcal{L}_3) = x_1y_3K[x_1, y_3] \oplus x_3y_1K[x_3, y_1].$$

If $n = 4$, then

$$I(C_8(1,4))/I(\mathcal{L}_4) = x_1y_4K[x_1, x_3, y_2, y_4] \oplus x_4y_1K[x_2, x_4, y_1, y_3].$$

If $n=5$, then

$$I(C_{10}(1,5))/I(\mathcal{L}_5) = x_1y_5K[x_1,x_4,y_3,y_5] \oplus x_1x_3y_5K[x_1,x_3,y_2,y_5] \oplus x_1y_2y_5K[x_1,x_4,y_2,y_5]$$
$$\oplus x_5y_1K[x_3,x_5,y_1,y_4] \oplus x_5y_1y_3K[x_2,x_5,y_1,y_3] \oplus x_2x_5y_1K[x_2,x_5,y_1,y_4].$$

For $n \geq 6$, we have

$$\begin{aligned}
&I(C_{2n}(1,n))/I(\mathcal{L}_n)\\
&\cong x_1y_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},x_{n-1},y_2]}{(x_3y_3,x_3x_4,y_3y_4,\dots,x_{n-3}y_{n-3},x_{n-3}x_{n-2},y_{n-3}y_{n-2},x_{n-2}y_{n-2},x_{n-2}x_{n-1},y_2y_3)}[x_1,y_n]\\
&\oplus y_1x_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},y_{n-1},x_2]}{(x_3y_3,x_3x_4,y_3y_4,\dots,x_{n-3}y_{n-3},x_{n-3}x_{n-2},y_{n-3}y_{n-2},x_{n-2}y_{n-2},y_{n-2}y_{n-1},x_2x_3)}[y_1,x_n]\\
&= x_1y_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},x_{n-1},y_2]}{(I(\mathcal{L}_{n-4}),y_2y_3,x_{n-2}x_{n-1})}[x_1,y_n]\\
&\oplus y_1x_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},y_{n-1},x_2]}{(I(\mathcal{L}_{n-4}),x_2x_3,y_{n-2}y_{n-1})}[y_1,x_n].
\end{aligned}$$

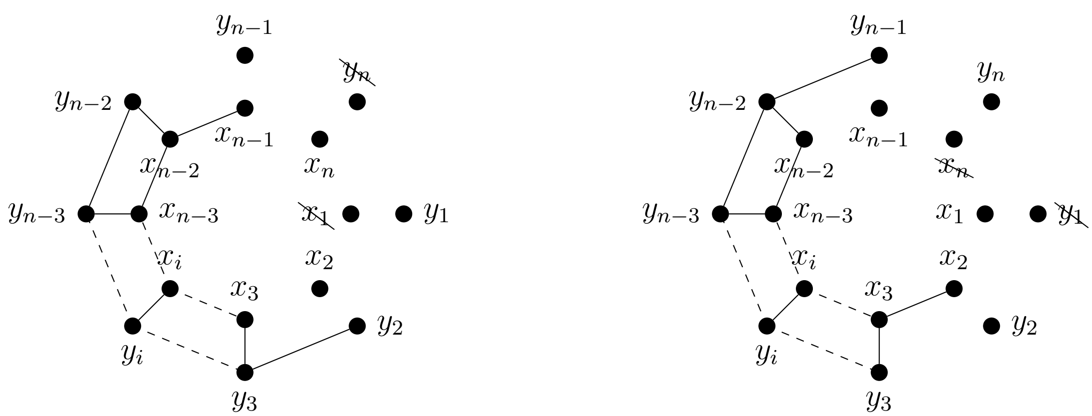


Figure 6.10: $I(C_{2n}(1,n))/I(\mathcal{L}_n)$.

If $u \in I(C_{2n}(1,n))$ such that $u \notin I(\mathcal{L}_n)$. It follows that $(x_1y_n)|u$ or $(y_1x_n)|u$. If $(x_1y_n)|u$ then $u = x_1^{\gamma_1}y_n^{\delta_1}v_1$, $v_1 \in K[x_3,y_3,\dots,x_{n-2},y_{n-2},x_{n-1},y_2]$, since $v_1 \notin I(\mathcal{L}_n)$, it follows that $v_1 \notin (x_3y_3,x_3x_4,y_3y_4,\dots,x_{n-3}y_{n-3},x_{n-3}x_{n-2},y_{n-3}y_{n-2},x_{n-2}y_{n-2},x_{n-2}x_{n-1},y_2y_3)$. Now if $(y_1x_n)|u$ then $u = y_1^{\gamma_2}x_n^{\delta_2}v_2$, $v_2 \in K[x_3,y_3,\dots,x_{n-2},y_{n-2},y_{n-1},x_2]$, since $v_2 \notin I(\mathcal{L}_n)$, it follows that

$v_2 \notin (x_3y_3, x_3x_4, y_3y_4, \ldots, x_{n-3}y_{n-3}, x_{n-3}x_{n-2}, y_{n-3}y_{n-2}, x_{n-2}y_{n-2}, y_{n-2}y_{n-1}, x_2x_3)$. Clearly we can see that

$$\frac{K[x_3, y_3, \ldots, x_{n-2}, y_{n-2}, x_{n-1}, y_2]}{(I(\mathcal{L}_{n-4}), y_2y_3, x_{n-2}x_{n-1})} \cong S^{\bullet}_{n-4}/I(\mathcal{L}^{\bullet}_{n-4})$$

and

$$\frac{K[x_3, y_3, \ldots, x_{n-2}, y_{n-2}, y_{n-1}, x_2]}{(I(\mathcal{L}_{n-4}), x_2x_3, y_{n-2}y_{n-1})} \cong S^{\bullet}_{n-4}/I(\mathcal{L}^{\bullet}_{n-4}).$$

By Proposition 6.1.13 and Lemma 3.2.11, we have

$$\text{sdepth}(I(C_{2n}(1,n))/I(\mathcal{L}_n)) \geq \min\{\lceil\frac{n-4+1}{2}\rceil + 2, \lceil\frac{n-4+1}{2}\rceil + 2\} = \lceil\frac{n+1}{2}\rceil.$$

□

Proposition 6.2.4. *For* $n \geq 3$, *Then* $\text{sdepth}(I(C_{2n}(2,n))/I(\mathcal{L}_n)) \geq \lceil\frac{n+1}{2}\rceil$.

Proof. For $3 \leq n \leq 5$, by using [24] we have

If $n = 3$, then The poset P is given by

$$P = \{(1,0,1,0,0,0), (0,0,0,1,0,1), (1,0,1,0,1,0), (0,1,0,1,0,1)\}.$$

Partitions of P are given by

$$\begin{aligned}\mathcal{P}: \ &[(1,0,1,0,0,0), (1,0,1,0,0,0)] \bigcup [(0,0,0,1,0,1), (0,0,0,1,0,1)] \bigcup \\ &[(1,0,1,0,1,0), (1,0,1,0,1,0)] \bigcup [(0,1,0,1,0,1), (0,1,0,1,0,1)]\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}I(C_6(2,3))/I(\mathcal{L}_3) = x_1x_3K[x_1, x_3] \oplus y_1y_3K[y_1, y_3] \oplus x_1x_3y_2K[x_1, x_3, y_2] \\ \oplus x_2y_1y_3K[x_2, y_1, y_3].\end{aligned}$$

If $n = 4$, then

$$\begin{aligned}I(C_8(2,4))/I(\mathcal{L}_4) = x_1x_4K[x_1, x_4, y_2] \oplus y_1y_4K[x_2, y_1, y_4] \oplus x_1y_3x_4K[x_1, y_3, x_4] \\ \oplus y_1x_3y_4K[y_1, x_3, y_4].\end{aligned}$$

If $n = 5$, then

$$I(C_{10}(2,5))/I(\mathcal{L}_5) = x_1x_5K[x_1,x_5,y_2] \oplus y_1y_5K[x_2,y_1,y_5] \oplus x_1x_3x_5K[x_1,x_3,x_5]$$
$$\oplus x_1y_3x_5K[x_1,y_3,x_5] \oplus x_1y_4x_5K[x_1,y_4,x_5] \oplus y_1x_3y_5K[y_1,x_3,y_5] \oplus y_1y_3y_5K[y_1,y_3,y_5]$$
$$\oplus x_2y_1x_4y_3y_5K[x_2,y_1,x_4,y_3,y_5] \oplus x_1x_3x_5y_2K[x_1,x_3,x_5,y_2] \oplus x_1x_5y_2y_4K[x_1,x_5,y_2,y_4]$$
$$\oplus x_2y_1y_3y_5K[x_2,y_1,y_3,y_5] \oplus x_1y_2x_3y_4x_5K[x_1,y_2,x_3,y_4,x_5] \oplus x_2y_1x_4y_5K[x_2,y_1,x_4,y_5]$$
$$\oplus y_1y_3x_4y_5K[y_1,y_3,x_4,y_5] \oplus y_1x_3x_5y_4K[y_1,x_3,x_5,y_4] \oplus y_1x_4y_5K[y_1,x_4,y_5].$$

For $n \geq 6$, we have,

$$I(C_{2n}(2,n))/I(\mathcal{L}_n)$$
$$\cong x_1x_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},y_{n-1},y_2]}{(x_3y_3,x_3x_4,y_3y_4,\dots,x_{n-3}y_{n-3},x_{n-3}x_{n-2},y_{n-3}y_{n-2},x_{n-2}y_{n-2},y_{n-2}y_{n-1},y_2y_3)}[x_1,x_n]$$
$$\oplus y_1y_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},x_{n-1},x_2]}{(x_3y_3,x_3x_4,y_3y_4,\dots,x_{n-3}y_{n-3},x_{n-3}x_{n-2},y_{n-3}y_{n-2},x_{n-2}y_{n-2},x_{n-2}x_{n-1},x_2x_3)}[y_1,y_n]$$
$$= x_1x_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},y_{n-1},y_2]}{(I(\mathcal{L}_{n-4}),y_2y_3,y_{n-2}y_{n-1})}[x_1,x_n]$$
$$\oplus\, y_1y_n\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},x_{n-1},x_2]}{(I(\mathcal{L}_{n-4}),x_2x_3,x_{n-2}x_{n-1})}[y_1,y_n].$$

If $u \in I(C_{2n}(2,n))$ such that $u \notin I(\mathcal{L}_n)$. It follows that $(x_1x_n)|u$ or $(y_1y_n)|u$. If $(x_1x_n)|u$ then $u = x_1^{\gamma_1}x_n^{\delta_1}v_1$, $v_1 \in K[x_3,y_3,\dots,x_{n-2},y_{n-2},y_{n-1},y_2]$, since $v_1 \notin I(\mathcal{L}_n)$, it follows that $v_1 \notin (x_3y_3,x_3x_4,y_3y_4,\dots,x_{n-3}y_{n-3},x_{n-3}x_{n-2},y_{n-3}y_{n-2},x_{n-2}y_{n-2},y_{n-2}y_{n-1},y_2y_3)$. Now if $(y_1y_n)|u$ then $u = y_1^{\gamma_2}y_n^{\delta_2}v_2$, $v_2 \in K[x_3,y_3,\dots,x_{n-2},y_{n-2},x_{n-1},x_2]$, since $v_2 \notin I(\mathcal{L}_n)$, it follows that
$v_2 \notin (x_3y_3,x_3x_4,y_3y_4,\dots,x_{n-3}y_{n-3},x_{n-3}x_{n-2},y_{n-3}y_{n-2},x_{n-2}y_{n-2},x_{n-2}x_{n-1},x_2x_3)$. Clearly we can see that

$$\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},y_{n-1},y_2]}{(I(\mathcal{L}_{n-4}),y_2y_3,y_{n-2}y_{n-1})} \cong S^{\diamond}_{n-4}/I(\mathcal{L}^{\diamond}_{n-4})$$

and

$$\frac{K[x_3,y_3,\dots,x_{n-2},y_{n-2},x_{n-1},x_2]}{(I(\mathcal{L}_{n-4}),x_2x_3,x_{n-2}x_{n-1})} \cong S^{\diamond}_{n-4}/I(\mathcal{L}^{\diamond}_{n-4}).$$

By Proposition 6.1.11 and Lemma 3.2.11, we have

$$\operatorname{sdepth}(I(C_{2n}(2,n))/I(\mathcal{L}_n)) \geq \min\{\lceil\frac{n-4+1}{2}\rceil+2,\lceil\frac{n-4+1}{2}\rceil+2\} = \lceil\frac{n+1}{2}\rceil.$$

□

Theorem 6.2.5. *If $G = C_{2n}(1,n)$ or $C_{2n}(2,n)$ and $n \geq 3$. Then,* $\operatorname{sdepth}(I(G)) \geq \lceil \frac{n+1}{2} \rceil$.

Proof. Consider

$$0 \longrightarrow I(\mathcal{L}_n) \longrightarrow I(G) \longrightarrow I(G)/I(\mathcal{L}_n) \longrightarrow 0,$$

then by Lemma 3.2.10,

$$\operatorname{sdepth}(I(G)) \geq \min\{\operatorname{sdepth}(I(\mathcal{L}_n)), \operatorname{sdepth}(I(G)/I(\mathcal{L}_n))\}.$$

By Theorem 6.1.9, it follows that

$$\operatorname{sdepth}(I(\mathcal{L}_n)) \geq \lceil \frac{n}{2} \rceil + 1,$$

and by Proposition 6.2.3 and 6.2.4, we have

$$\operatorname{sdepth}(I(G)/I(\mathcal{L}_n)) \geq \lceil \frac{n+1}{2} \rceil = \lceil \frac{n-1}{2} \rceil + 1.$$

□

Corollary 6.2.6. *If $G = C_{2n}(1,n)$ or $C_{2n}(2,n)$ and $n \geq 3$. Then*

$$\operatorname{sdepth}(I(G)) \geq \operatorname{sdepth}(S_n/I(G)).$$

Proof. Let $n \geq 3$, by Theorem 6.2.5 and corollary 6.2.2 we have,
If $n \equiv 0 \pmod 2$

$$\operatorname{sdepth}(I(G)) \geq \lceil \frac{n+1}{2} \rceil \geq \operatorname{sdepth}(S_n/I(G)).$$

□

By considering Theorem 6.2.1, Corollary 6.2.2 and Auslander-Buchsbaum formula [33, Theorem 3.7], we have,

Corollary 6.2.7. *If $G = C_{2n}(1,n)$ or $C_{2n}(2,n)$, For $n \geq 3$,*

$$n + \lfloor \frac{n}{2} \rfloor - 1 \leq pd_{S_n}(S_n/I(G)) \leq n + 1 + \lfloor \frac{n-1}{2} \rfloor.$$

Summary

- In this thesis, the existing lower bounds and values for the Stanley depth and depth of modules are discussed.
- Our newly computed bounds are presented for Stanley depth and of edge ideals associated with different families of graphs.
- The detailed procedure is given to compute the bounds and values for Stanley depth and depth of the edge ideals corresponding to ladder graph and restricted partial strong product of the graphs.
- The lower and upper bounds of depth, Stanley depth and projective dimension are presented for the cubic circulant graphs.
- The upper bound of regularity and projective dimension for some families of circulant graph are also given.

Bibliography

[1] Stanley, R. P. (1982). Linear Diophantine equations and local cohomology. *Inventiones mathematicae, 68*(2), 175-193.

[2] Duval, A. M., Goeckner, B., Klivans, C. J., Martin, J. L. (2016). A non-partitionable Cohen–Macaulay simplicial complex. *Advances in Mathematics, 299*, 381-395.

[3] Edwards, Harold M., (1977). Fermat's Last Theorem. A Genetic Introduction to Algebraic Number Theory, New York: Springer.

[4] Fraenkel, A. (1922). Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre. Math. Ann. 86, 230-237. Mathematische Annalen. 86. 230-237. 10.1007/BF01457986.

[5] The development of Ring Theory (http://www-gap.dcs.st-and.ac.uk/ history/HistTopics/Ring_theory.html).

[6] Anderson, Dan, Naseer, M,(1993). Beck's Coloring of a Commutative Ring. Journal of Algebra. **159**. 500-154.

[7] Sachs, H., Steibitz, M., and Wilson, R.J. (1988) "An Historical Note: Euler's Königsberg Letters." in Journal of Graph Theory 12 1: 133-39.

[8] Appel, K.; Haken, W. (1977) Every planar map is four colorable. Part I: Discharging. Illinois J. Math. 21, no. 3, 429–490. doi:10.1215/ijm/1256049011. https://projecteuclid.org/euclid.ijm/1256049011.

[9] Robertson, Sanders, N., Seymour, D., Graham, P., (1996). A New Proof Of The Four-Colour Theorem. Electron. Res. Announc. Amer. Math. Soc.. 2. 10.1090/S1079-6762-96-00003-0.

[10] Mckelvie, C.L.. (1975). William Hamilton Maxwell. 22. 10.1093/nq/22.10.450-a.

[11] Warburg, E.. (1925). Zur Erinnerung an Gustav Kirchhoff. The Science of Nature. 13. 205-212. 10.1007/BF01558883.

[12] Cayley, A., Halsted, G. (1895). ARTHUR CAYLEY.. Science (New York, N.Y.). 1. 450-1. 10.1126/science.1.17.450.

[13] P. J. Davis, (1979) Circulant Matrices, Wiley, New York.

[14] Davis, G.J.; Domke, G.S. (2002). 3-Circulant Graphs. J. Combin. Math. Combin. Comput. **40** 133–142.

[15] Bruns, W., Herzog, H. J. (1998). Cohen-macaulay rings. *Cambridge university press*.

[16] Apel, J. (2003). On a conjecture of RP Stanley; part II—quotients modulo monomial ideals. *Journal of Algebraic Combinatorics, 17*(1), 57-74.

[17] Anwar, I., Popescu, D. (2007). Stanley conjecture in small embedding dimension. *Journal of Algebra, 318*(2), 1027-1031.

[18] Popescu, A. (2010). Special stanley decompositions. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 363-372.

[19] Herzog, J., Vladoiu, M., Zheng, X. (2009). How to compute the Stanley depth of a monomial ideal. *Journal of Algebra, 322*(9), 3151-3169.

[20] Cimpoeaş, M. (2008). Stanley depth of complete intersection monomial ideals. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 205-211.

[21] Cimpoeas, M. (2012). Several inequalities regarding Stanley depth. *Romanian Journal of Math. and Computer Science, 2*(1), 28-40.

[22] Popescu, D. (2009). An inequality between depth and Stanley depth. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie,* 377-382.

[23] Rauf, A. (2010).Depth and Stanley depth of multigraded modules. *Communications in Algebra, 38*(2), 773-784.

[24] J. Herzog, M. Vladoiu, X. Zheng, *How to compute the Stanley depth of a monomial ideal*, *J. Algebra*, **322**(9) (2009), 3151–3169.

[25] M. Cimpoeas, *Several inequalities regarding Stanley depth*, *Romanian Journal of Mathematics and Computer Science*, **2** (2012), 28–40.

[26] Morey, S. (2010). Depths of powers of the edge ideal of a tree. *Communications in Algebra*, 38(11), 4042-4055.

[27] Fouli, L., Morey, S. (2015). A lower bound for depths of powers of edge ideals. *Journal of Algebraic Combinatorics, 42*(3), 829-848.

[28] Grayson, D. R., Stillman, M. E. Macaulay 2, a software system for research in algebraic geometry. http://www.math.uiuc.edu/Macaulay2/ Accessed September 20, 2010.

[29] M. Cimpoeas, (2013) *Stanley depth of squarefree Veronese ideals*, *An. St. Univ. Ovidius Constanta*, 21(3), 67–71.

[30] G. Rinaldo, (2008) *An algorithm to compute the Stanley depth of monomial ideals*, *Le Matematiche*, **LXIII**(ii), 243–256.

[31] M. Ishaq, (2012) *Upper bounds for the Stanley depth*, *Comm. Algebra*, 40(1), 87–97.

[32] C. Biro, D. M. Howard, M. T. Keller, W. T. Trotter,S. J. Young, (2010) *Interval partitions and Stanley depth*, *J. Combin. Theory Ser. A*, **117**, 475–482.

[33] M. Auslander, D. Buchsbaum, (1957) *Homological dimension in local rings*, *Trans Amer Math Soc*, **85**, 390–405.