Depth and Stanley Depth of Edge Ideals Associated with Corona Product of Certain Trees

By

Naeem ud Din

Supervised by

Dr. Muhammad Ishaq

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan
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Examination Committee Members

1. Name: DR. MUJEEB UR REHMAN Signature:

2. Name: DR. MUHAMMAD QASIM Signature:

External Examiner: DR. ADNAN ASLAM Signature:

Supervisor's Name: DR. MUHAMMAD ISHAQ Signature:

Head of Department

17-11-2020
Date

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Date: 17-11-2020

Dean/Principal

Dedication

I would like to dedicate this thesis to my respectable Supervisor, Teachers, Parents and Siblings for their encouragement and support.

Acknowledgement

All praises for Almighty Supreme Being, the foremost beneficent and also the most merciful, WHO created this whole universe. I am extremely grateful to Almighty Supreme Being for showering multitudinous blessings upon me and giving me the power and strength to finish this thesis with success and blessing me quite I merit. I am deeply grateful to my supervisor Dr. Muhammad Ishaq, for his continuous support and steering throughout my thesis. My understanding and appreciation of the topic square measure entirely thanks to his efforts and positive response to my queries. My studies at NUST are created a lot of unforgettable and that I learned heaps from returning here. Finally, with the deepest feeling, I acknowledge the support of my family, specially to my elder brother for supporting me all the means through my studies. So words cannot specific however grateful I am for all of the sacrifices they need created on behalf of me. I am appreciative to all or any those who directly or indirectly helped me to finish my thesis.

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Abstract

Depth and Stanley depth are the algebraic and geometric invariants, respectively, which have been computed for various classes of edge ideals on graphs. Earlier, for the edge ideal associated with trees and the computed bounds were dependent upon diameter, power and number of connected components. The present thesis is primarily concerned with the exact value and bound of depth and Stanley depth of edge ideals associated with corona product of some trees.

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Introduction

In this thesis, some exact values of depth and Stanley depth are computed for the edge ideals of the corona product of some graphs. This thesis consists of four chapters

Chapter 1 gives the overview definitions and results related to Abstract Algebra and Commutative Algebra. This chapter covers the basics of Rings and Module Theory.

Chapter 2 presents the brief introduction of basic Graph Theory and the fundamental products of graphs.

Chapter 3 reviews the fundamentals of the theory of depth and Stanley depth and Stanley decomposition of modules.

In Chapter 4, edge ideal associated to corona product of some graphs are introduced and their depth and Stanley depth are computed by using induction and Depth Lemma on short exact sequences.

Chapter 1

Preliminaries

This chapter is devoted to the basic definitions of Ring Theory and Module Theory.

1.1 Ring Theory

Definition 1.1.1. A non-empty set R together with the two binary operations addition $''+''$ and multiplication $''\times''$ forms a ring if the following conditions are satisfied.

a. R with respect to the operation of addition forms a commutative group that is, $(R, +)$ is a commutative group.

b. The associative laws hold for R with respect to the operation of multiplication that is, for all t_1, t_2 and $t_3 \in R$

$$t_1 \times (t_2 \times t_3) = (t_1 \times t_2) \times t_3.$$

c. The left and right distributive laws hold, that is for all t_1, t_2 and $t_3 \in R$

   i. left distributive law

$$t_1 \times (t_2 + t_3) = (t_1 \times t_2) + (t_1 \times t_3)$$

   ii. right distributive law

$$(t_1 + t_2) \times t_3 = (t_1 \times t_3) + (t_2 \times t_3).$$

A ring R is said to be commutative ring if it is commutative w.r.t multiplication and if for all $t \in R$ we have $1 \in R$ such that,

$$1 \times t = t \times 1 = t$$

then we say R is a commutative ring with unity.

Throughout this thesis rings will be commutative with unity.

1.1.1 Polynomial ring

Definition 1.1.2. The ring of polynomial denoted by $S = M[z]$, where z is the indeterminate with coefficients from the field M. The sum of the form $a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$, $a_j \in M$ is known as the polynomial of degree n if $a_n \neq 0$. The set of all such polynomials form a ring with respect to component wise addition and multiplication defined by

$$\sum_{i=0}^{k} a_i z^i + \sum_{i=0}^{k} b_i z^i = \sum_{i=0}^{k} (a_i + b_i) z^i$$

$$\sum_{i=0}^{j} (a_i z^i) \times \sum_{i=0}^{k} (b_i z^i) = \sum_{n=0}^{j+k} \sum_{i=0}^{n} (a_i \times b_{n-i}) z^n.$$

$M[z_1, z_2]$ is the ring of polynomials in two indeterminate and also $M[z_1, z_2]{=}M[z_1][z_2]$ where $M[z_1][z_2]$ is the polynomial ring whose coefficients are from $M[z_1]$ and indeterminate is z_2. In general, we have

$$M[z_1, z_2, z_3, ..., z_{n-1}, z_n] = M[z_1, z_2, z_3, ..., z_{n-1}][z_n].$$

Throughout this thesis S represents a polynomial ring in n indeterminate over field M that is, $S := M[z_1, z_2, ..., z_n]$. Let $\mathbb{R}^n_+$ represents the set of all those vectors $b = (b_1, b_2, .., b_n) \in \mathbb{R}^n$ such that each $b_j \geq 0$ and $\mathbb{Z}^+_n = \mathbb{R}^+_n \cap \mathbb{Z}^n$. Any product $z_1^{b_1} \cdots z_n^{b_n}$ is known as a monomial. In general, we can write it as, $v = z^b$ with $b = (b_1, b_2, ..., b_n) \in \mathbb{Z}^+_n$.

1.1.2 Ring homomorphisms

Definition 1.1.3. Let R and $R^{'}$ be two rings

1. The ring homomorphism is a map $\Psi : R \longrightarrow R^{'}$ satisfying the following two axioms:

   a. $\Psi(t_1 + t_2) = \Psi(t_1) + \Psi(t_2)$ for all $t_1, t_2 \in R$.

   b. $\Psi(t_1 \times t_2) = \Psi(t_1) \times \Psi(t_2)$ for all $t_1, t_2 \in R$.

2. The kernal of the ring homomorphism Ψ, denoted by $\ker\Psi$ are those elements of R that maps to additive identity in $R^{'}$.

3. A bijective homomorphism is called an isomorphism.

1.1.3 Ideal of a ring

Definition 1.1.4. The subring V of a ring R is said to be an ideal if it absorbs the multiplication of the elements of R from both side, that is for any $t \in R$, we have $tV \in V$ and $Vt \in V$.

Principal ideal and maximal ideal

An ideal is called principal ideal if it is generated by a single element. An ideal μ of R ($\mu \neq R$) is said be maximal if there exist no proper ideal containing μ. Similarly an ideal ρ is called prime ideal if for any $t_1, t_2 \in R$ such that $t_1 t_2 \in \rho$ implies $t_1 \in \rho$ or $t_2 \in \rho$. In a polynomial ring the ideal is called monomial ideal if it is generated by monomials. Square-free monomial ideal which is generated by square-free monomials. Let $S = M[z_1, z_1, ..., z_6]$ be a ring of polynomial then

$$U = (z_1^2 z_2^4, z_1^3 z_2^3, z_1^5 z_2) \text{ and } U^{'} = (z_1^4 z_2^5, z_1^6 z_2^2)$$

are monomial ideals of S. Also the ideals of the form $I = (z_1)$, $J = (z_1 z_4, z_2 z_5)$ and $K = (z_1, z_2, ..., z_6)$ are the square-free monomial ideals in the ring of polynomial S.

Definition 1.1.5. Let U be an ideal of a ring S. Then S/U forms ring if we define multiplication in S/U as follows

$$(p_1 + U) \times (p_2 + U) = (p_1 \times p_2) + U \quad \text{for all} \quad p_1, p_2 \in S.$$

Nilradical and Jacobson radical

Let S be a ring the nilradical, $N(S)$ be the intersection of all prime ideals of S. The Jacobson radical $J(S)$ is the intersection all maximal ideals of S.

Remark 1.1.6. *As all maximal ideals are prime ideals and therefore, nilradical is a subset of Jacobson radical.*

Definition 1.1.7. A ring having a unique maximal ideal is callde local ring.

Remark 1.1.8. *In the case when a ring is local ring then, Jacobson radical is equal to the maximal ideal.*

1.1.4 Operations on ideals

Addition of ideals

Suppose R is ring. Let V and $V^{'}$ are two ideals of the ring. Then the sum of these ideals is defined as:

$$V + V^{'} = \{q + q^{'} : q \in V, q^{'} \in V^{'}\}$$

this set forms the smallest ideal containing both V and $V^{'}$. The sum can be extended to any arbitrary number of ideals. For the family V_j where $j \in I$ in the sum $\sum_{j \in I} V_j$ the elements are of the form $\sum_{j \in I} y_j$ with only finite many $y_j \neq 0$ where, $y_j \in V_j$.

Multiplication of ideals

The product of two ideals is defined as:

$$VV^{'} = \left\{ \sum_{finite} v_i v_i^{'} : v_i \in V, v_i^{'} \in V^{'} \right\}$$

is an ideal generated by $vv^{'}$ where $v \in V$ and $v^{'} \in V^{'}$. The set $VV^{'}$ is the finite sum of the form $\sum v_i v_i^{'}$. Finite product of ideals can be defined in the similar way. For any $m > 0$, V^m is generated by all the products, $v_1 v_2 ... v_m$ with each $v_i \in V$. We will consider $V^0 = (1)$.

Intersection of ideals

The intersection of any family of ideals again forms an ideal. In $\mathbb{Z}$, for any two principal ideals we have

$$(m\mathbb{Z}) \cap (n\mathbb{Z}) = mn\mathbb{Z}.$$

In general, union of ideals is not an ideal.

Radical of an ideal

Let V be an ideal in a ring R. The radical ideal is defined as

$$\sqrt{V} = \{t \in R : t^m \in V, m > 0\}.$$

In a commutative ring if $\sqrt{V} = V$, then V is called a radical ideal of the ring. All the square-free ideals are radical ideals. The radical ideal V is the intersection of all prime ideals containing V.

Colon ideal

Let V and $V^{'}$ be two ideals of a ring R. Then the quotient ideal is defined as

$$(V : V^{'}) = \{t \in R : tV^{'} \subseteq V\}.$$

$(0 : V)$ is an ideal called the annihilator of V represented as $Ann(V)$ defined as

$$Ann(V) = \{t \in R : tV = 0\}.$$

Definition 1.1.9. Primary ideal V is proper ideal of a ring R given that, if $t_1 t_2 \in V$, for some $t_1, t_2 \in V$, then either $t_1 \in V$ or $t_2^m \in V$ for some $m \geq 1$.

Definition 1.1.10. Let ρ be a prime ideal the height of ρ written as $htt(\rho)$, is defined as

$$htt(\rho) = max\{j : (0) = \rho_0 \subsetneq \rho_1 \subsetneq ... \subsetneq \rho_j = (\rho), \text{where } \rho_i\text{'s are prime ideals}\}.$$

For an ideal V

$$htt(V) = min\{htt(\rho) : V \subset \rho\},$$

where ρ is a prime ideal.

1.1.5 Primary decomposition of ideals

Let R be a ring which is Noetherian and D be a finitely generated $R-$module. [14] A prime ideal $\rho \subset R$ is called associated prime ideal of D, if there exist an element $d \in D$ such that $\rho = Ann(d)$. The set of associated prime ideals of D is represented by by $Ass(D)$.

For an ideal V, primary decomposition is a way of representing V as an intersection $V = \cap_{j=1}^{m} K_j$, where each K_j is primary ideal containing V. Let $\{\rho_j\} = Ass(K_j)$ if neither of the K_j can be omitted in this intersection and $\rho_r \neq \rho_s$ for all $r \neq s$, then it is called irredundant primary decomposition.

Example 1.1.11. Let $U = (\, z_2^4\,,\, z_3^4\,,\, z_2^3 z_4^3\,,\, z_2 z_3 z_4^3\,,\, z_3^3 z_4^3\,)$ be an ideal of S, then

$$\begin{aligned}
U &= (\, z_2^4\,,\, z_3^4\,,\, z_2^3\,,\, z_2 z_3 z_4^3\,,\, z_3^3 z_4^3\,) \cap (\, z_2^4\,,\, z_3^4\,,\, z_4^3\,,\, z_2 z_3 z_4^3\,,\, z_3^3 z_4^3\,) \\
&= (\, z_2^3\,,\, z_3^4\,,\, z_2 z_3 z_4^3\,,\, z_3^3 z_4^3\,) \cap (\, z_2^4\,,\, z_3^4\,,\, z_4^3\,) \\
&= (\, z_2^3\,,\, z_3^4\,,\, z_2\,,\, z_3^3 z_4^3\,) \cap (\, z_2^3\,,\, z_3^4\,,\, z_3 z_4^3\,,\, z_3^3 z_4^3\,) \cap (\, z_2^4\,,\, z_3^4\,,\, z_4^3\,) \\
&= (\, z_2\,,\, z_3^4\,,\, z_3^3 z_4^3\,) \cap (\, z_2^3\,,\, z_3^4\,,\, z_3 z_4^3\,) \cap (\, z_2^4\,,\, z_3^3\,,\, z_4^3\,) \\
&= (\, z_2\,,\, z_3^4\,,\, z_3^3\,) \cap (\, z_2\,,\, z_3^4\,,\, z_4^3\,) \cap (\, z_2^3\,,\, z_3^4\,,\, z_3\,) \cap (\, z_2^3\,,\, z_3^4\,,\, z_4^3\,) \cap (\, z_2^4\,,\, {}_3^3\,,\, z_4^3\,) \\
&= (\, z_2\,,\, x_3^3\,) \cap (\, z_2\,,\, z_3^4\,,\, z_4^3\,) \cap (\, z_2^3\,,\, z_3\,) \cap (\, z_2^4\,,\, z_3^4\,,\, z_4^3\,) \\
&= (\, z_2\,,\, z_3^3\,) \cap (\, z_2^3\,,\, z_3\,) \cap (\, z_2^4\,,\, z_3^4\,,\, z_4^3\,).
\end{aligned}$$

It is the primary decomposition of U but not irredundant. Here, Ass$(\, z_2\,,\, z_3^3\,) =$ Ass$(\, z_2^3\,,\, z_3\,) = \{(\, z_2\,,\, z_3\,)\}$. Now for irredundant primary decomposition, take an intersection of $(\, z_2\,,\, z_3^3\,)$ and $(\, z_2^3\,,\, z_3\,)$, that is

$$(\, z_2\,,\, z_3^3\,) \cap (\, z_2^3\,,\, z_3\,) = (\, z_2^3\,,\, z_2 z_3\,,\, z_3^3\,).$$

Hence

$$U = (\, z_2^4\,,\, z_3^4\,,\, z_4^3\,) \cap (\, z_2^3\,,\, z_2x_3\,,\, z_3^3\,).$$

Example 1.1.12. Let $U = (\, z_1z_2\,,\, z_3z_5\,,\, z_2z_3\,,\, z_2z_4\,,\, z_3z_4\,,\, z_1z_4\,)$ be an ideal of S, then

$$\begin{aligned} U &= (z_1z_2, z_3z_5, z_2z_4, z_3z_4, z_1z_4) \\ &= (z_1, z_4, z_5) \cap (z_3, z_1z_2, z_2z_4, z_1z_4) \\ &= (z_2, z_4, z_5,) \cap (z_1, z_3, z_2) \cap (z_1z_2, z_3, z_4) \\ &= (z_2, z_4, z_5) \cap (z_2, z_4, z_3) \cap (z_1, z_3, z_4) \cap (z_1, z_2, z_3) \\ &= (z_2, z_4, z_5) \cap (z_1, z_3, z_4) \cap (z_2, z_4, z_3) \cap (z_1, z_2, z_3). \end{aligned}$$

Since U is square free monomial ideal so it can be seen that (z_2, z_4, z_5), (z_1, z_3, z_4), (z_1, z_2, z_3) and (z_2, z_4, z_3) are minimal prime ideals of U.

1.2 Modules Theory

Let R be a commutative ring, an abelian group D is said to be $R-$module if there exist a map $\star : S \times D \longrightarrow D$ defined as, $\star((t, d)) = td$, for all $t, t_1,\ t_2 \in R$ and for all $d, d_1,\ d_2 \in D$ satisfying the following axioms:

i. $t(d_1 + d_2) = td_1 + td_2$

ii. $(t_1 + t_2)d = t_1d + t_2d$

iii. $(t_1t_2)d = t_1(t_2d)$

iv. $1d = d$.

Module over a field F is called vector space over F.

Examples 1.2.1. 1. For a commutative group L, let $l \in L$ and $z \in \mathbb{Z}$, then define $\star : \mathbb{Z} \times L \to L$, such that

$$\star(z, l) = zl = \begin{cases} (-l) + \cdots + (-l)\,, & \text{if} \quad \text{z} < 0; \\ l + l + \cdots + l\,, & \text{if} \quad \text{z} > 0; \\ 0\,, & \text{if} \quad \text{z} = 0. \end{cases}$$

Then l is a $\mathbb{Z}$-module.

2. For ring R, $R^k = \{(t_1, t_2, \ldots, t_k) : t_i \in R\}$ is an R-module via the scalar multiplication:
$$t(t_1, t_2, \ldots, t_k) = (tt_1, tt_2, \ldots, tt_k).$$
3. A ring R is a module over itself.
4. Ideals of a ring are also R-modules.

1.2.1 R-module homomorphism

Definition 1.2.2. Let D and $D^{'}$ are two R-modules. A map $\gamma : D \to D^{'}$ is an R-module homomorphism if

a. $\gamma(d_1 + d_2) = \gamma(d_1) + \gamma(d_2)$, for all $d_1, d_2 \in D$.

b. $\gamma(td) = t\gamma(d)$, for all $t \in R$, $d \in D$.

If γ is bijective then it becomes an R-module isomorphism.

1.2.2 Graded ring and graded module

For a commutative additive semigroup L. [14] Ring R is said to be L-graded ring if it has a decomposition
$$R = \bigoplus_{l \in L} R_l,$$
such that $R_l R_m \subset R_{l+m}$ for all $l, m \in L$. Then for $t \in R$, we will have a unique representation of the form
$$t = \sum_{l \in L} t_l,$$
where $t_l \in R_l$ and almost all $t_l = 0$ and t_l is known to be the l-th homogeneous component. If $t = t_l$, then t is called homogeneous of degree l. $S = M[y]$ and $S = M[y, z]$ are $\mathbb{Z}$-graded rings as

i. $M[z] = M \oplus Mz \oplus Mz^2 \oplus Mz^3 \oplus Mz^4 \oplus Mz^5 \oplus \cdots$.

ii. $M[y,z] = R \oplus (My + Mz) \oplus (My^2 + Myz + Mz^2) \oplus (My^3 + My^2z + Myz^2 + Mz^3) \oplus \cdots$.

Similarly, for $L-graded$ ring R such that

$$R = \bigoplus_{l \in L} R_l$$

and D be $R-module$. D is called $L-graded$ module if

$$D = \bigoplus_{l \in L} D_l$$

and $R_l D_m \subset D_{l+m}$ for all $l, m \in L$.

Let $d = (d_1, d_2, ..., d_n) \in \mathbb{Z}^n$, then $s \in S$ is known as homogeneous of degree $\mathbf{d}$ if s has the form $\gamma y^{\mathbf{d}}$, where $\gamma \in M$, $y = x_{i_1} x_{i_2} \cdots x_{i_n}$ and $y^d = x_{i_1}^d x_{i_2}^d \cdots x_{i_n}^d$. Also S is $\mathbb{Z}^n$-graded with graded components:

$$S_{\mathbf{d}} = \begin{cases} My^{\mathbf{d}}, & \text{if } \mathbf{d} \in \mathbb{Z}_+^n; \\ 0\,, & \text{otherwise.} \end{cases}$$

An S-module D is called $\mathbb{Z}^n$-graded if $D = \bigoplus_{\mathbf{d} \in \mathbb{Z}^n} D_{\mathbf{d}}$ and $S_{\mathbf{d_1}} D_{\mathbf{d_2}} \subset D_{\mathbf{d_1}+\mathbf{d_2}}$ for all $\mathbf{d_1}, \mathbf{d_2} \in \mathbb{Z}^n$.

1.2.3 Exact sequence

An exact sequence is sequence of objects (Group, Rings, Modules) and morphisms between them. A sequence of R-homomorphisms and R-modules

$$\ldots \longrightarrow L_{j-1} \xrightarrow{l_j} L_j \xrightarrow{l_{j+1}} L_{j+1} \xrightarrow{l_{j+2}} \ldots$$

is said to be exact at L_j if $Im(l_j) = ker(l_{j+1})$. The sequence is exact if it is exact at every L_j. The sequence $0 \longrightarrow F' \xrightarrow{e} E$ is exact at F' if and only if e is injective, and similarly $E \xrightarrow{f} F'' \longrightarrow 0$ is exact at F'' if and only if f is surjective homomorphism. The sequence

$$0 \longrightarrow F' \xrightarrow{e} E \xrightarrow{f} F'' \longrightarrow 0$$

is exact sequence if and only if e is injective, f is surjective and $\mathrm{Im}(e) = \ker(f)$. This exact sequence is called short exact sequence.

Example 1.2.3. 1. Let F and E are R-modules, then

$$0 \longrightarrow C \xrightarrow{e} C \oplus E \xrightarrow{\pi} E \longrightarrow 0$$

is a short exact sequence, where $e(c) = (c, 0)$ and $\pi(c, e) = e$.

2. $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} m\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$ is a short exact equence, where m defines a map $y \mapsto my$, given by multiplication by m and π defines a map $x \mapsto x + m\mathbb{Z}$.

Definition 1.2.4. A relation R over a set M is called partial order, if it is reflexive, transitive and anti-symmetric. A poset is a none-empty set which is ordered by partial order relation. A poset is defined as an ordered pair $P = (M, \leq)$, where M is the ground set of P and $\leq$ is the partial ordered relation of M.

1.2.4 Noetherian ring

Proposition 1.2.5. *Let Σ be a poset with respect to $\leq$. Then the following are equivalent.*

1. *Every increasing sequence $z_1 \leq z_2 \leq \ldots \leq z_n \leq \ldots$ in Σ is stationary, that is there exist $q \in \mathbb{N}$ for which $z_p = z_q$, for all $p \geq q$.*

2. *For all $\emptyset \neq B \subset \Sigma$ has an element which is maximal.*

Definition 1.2.6. Let D be an $R-module$. D is said to be Noetherian if every decreasing chain of $R-modules$ of D is stationary. The ring is said to be Noetherian if it is Noetherian as an $R-module$.

Chapter 2

Graph Theory

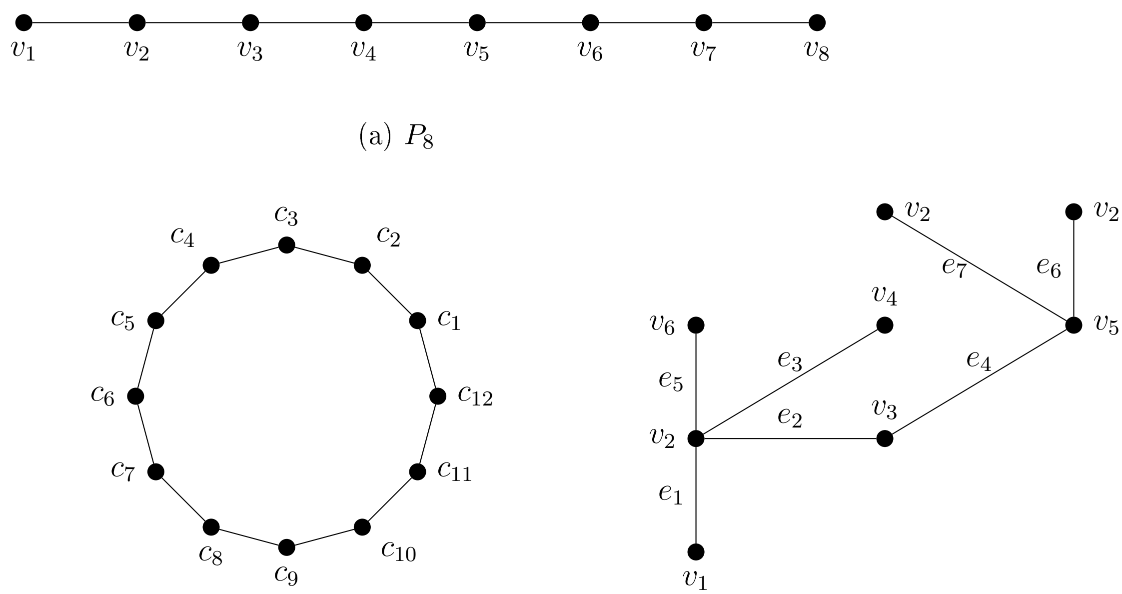
In this chapter we will discuss some basic definitions of Graph Theory and product of graphs. Graph is created set of vertices and which are connected by edges. Graph Theory deals with graphs in which we study the pairwise relationship between objects for different structure modeling.

2.1 Basic definitions

A graph L be a non-empty set of vertices V_L and an edge set E_L and is represented as, $L = (V_L, E_L)$. The vertices $u, v \in V_L$ are known as adjacent in L if these are connected by an edge, which is denoted by uv (or vu). While, two edges $e_1, e_2 \in E_L$ are adjacent if e_1 and e_2 have a common vertex. The order and size of the graph is defined as $\mid V_L \mid$ and $\mid E_L \mid$, respectively. The graph L is finite if it has finite number of vertices and edges, otherwise infinite. If there are two or more edges between two vertices then the edges are known as multiple (parallel) edges. Similarly, if an edge has same starting and end vertex it is said to be a loop.

Path, cycle and tree

A walk is defined as a sequence of alternating vertices and edges. The vertex from where the walk is stared is know as a start vertex while, the vertex at which the walk end is known as the end vertex of the walk. A walk is said to be a closed walk if the start and end vertices are same. A closed walk is also known as a circuit. A trail is defined as a walk with no repeated edges while a path is defined as trail with no repeated vertices. A closed trail is also known as a circuit. Furthermore, a circuit with distinct vertices (except the starting and ending point) is known as a cycle. An acyclic graphhaving no cycle. Moreover, tree is acyclic and connected graph.



(a) P_8

(b) C_{12}

(c) Tree

Figure 2.1: (a) Cycle, (b) Path and (c) Tree.

A graph is known as simple graph if it contains no loops and no multiple edges. A simple graph is called complete if every two arbitrary verteces are adjacent to each others. Since, each vertex is linked with every other vertex therefore, a complete graph has the largest possible size among all the graphs. A complete graph is denoted by K_n, where K_1 is trivially complete. For $u \in V_L$, the neighborhood N_u is defined as $N_u = \{a : au \in E_L\}$. The degree of the vertex u is the number of the edges incident on it or precisely the cardinality of neighborhood N_u. If $\mid N_u \mid= 0$ or $\mid N_u \mid= 1$ then u is called an isolated vertex or pendant vertex respectively. Every edge contributes 1 to the degree of a vertex while loop adds 2 to the degree of the vertex. The degree of L is different from the vertex degree. Graph degree is of two types minimum or maximum degree. The minimum (resp. maximum) degree, denoted by δ(resp. Δ), is the smallest (resp. largest) vertex degree. Therefore, if $v \in V_L$ then

$$\delta \leq deg(v) \leq \Delta.$$

The graph is a regular graph if degree of each vertex of the graph is same.
Throughout this thesis, we take simple and connected graphs.

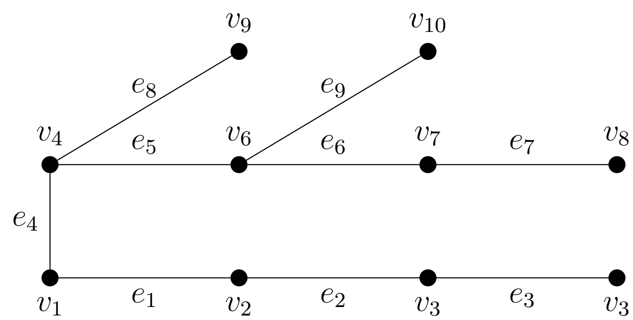


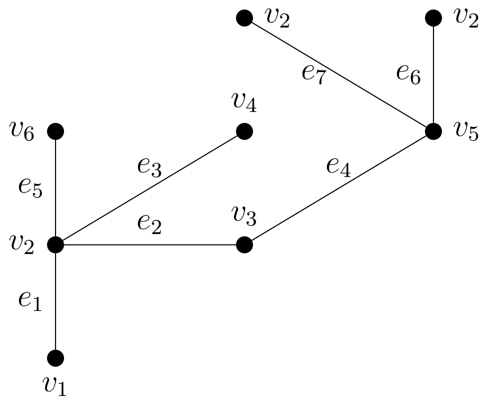
Figure 2.2: Simple connected Graph

Union and intersection of graphs

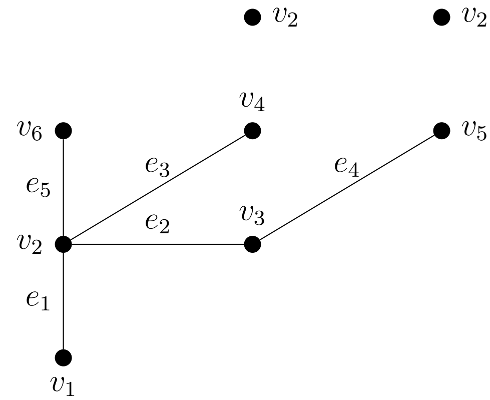
Let L_1 and L_2 are two vertex-disjoint graphs. The union of L_1 and L_2 $L_1 \cup L_2$, is a graph whose vertex and edge set is defined as $V_1 \cup V_2$ and $E_1 \cup E_2$ respectively. Similarly, the intersection, $L_1 \cap L_2$, is a graph with $V_1 \cap V_2$ and $E_1 \cap E_2$ as the vertex and edge sets respectively.

Subgraph and minor of a graph

A graph H is said to be a subgraph of L if $V_H \subseteq V_L$ and $E_H \subseteq E_L$ and is denoted as $H \subseteq L$. Minor of a graph is subgraph $L^{'}$ of a graph L which is formed by deleting the successive edges one by one.



(a) L



(b) $L^{'}$

Figure 2.3: From left to right Tree (L) and minor tree ($L^{'}$).

Definition 2.1.1. A graph is said to be bipartite graph if the vertex V can partitioned into two subsets having no element in common such that each edge has one end in one set the other end in other set. A complete bipartite graph with one partition set has the number of vertex equal to one is called star graph, denoted by S_m.

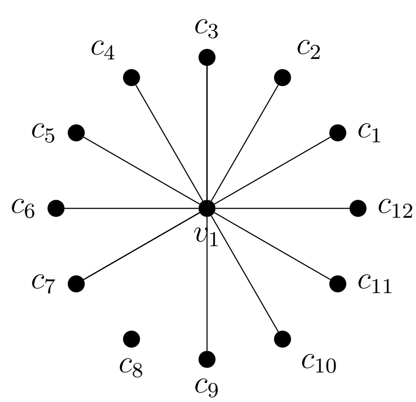


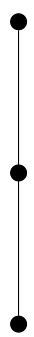
Figure 2.4: S_{13}

2.1.1 Products of graphs

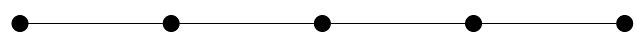
Definition 2.1.2. Let L and $L^{'}$ be two graphs with set of vertices $V(L) = \{v_1, v_2, ...v_n\}$ and $V(L^{'}) = \{v_1^{'}, v_2^{'}, ...v_m^{'}\}$ respectively. The Cartesian product of L and $L^{'}$ is a graph, with set of vertices $V(L \square L^{'}) = V(L) \times V(L^{'})$ (the Cartesian product of sets), for $(v_i, v_i^{'}), (v_j, v_j^{'}) \in V(L \square L^{'})$ and $(v_i v_i^{'})(v_j v_j^{'}) \in E(L \square L^{'})$, whenever

1. $v_i = v_j$ and $v_i^{'} v_j^{'} \in E(L)$ or

2. $v_i^{'} v_j^{'} \in E(L^{'})$ and $v_i^{'} = v_j^{'}$



(a) P_3



(b) P_5

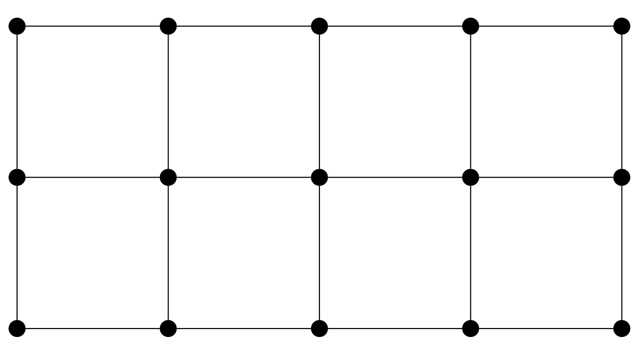


Figure 2.5: Cartesian product of P_3 and P_5 ($P_3 \square P_5$).

Two more examples of Cartesian product of two graphs are shown in the figure 2.6 and 2.7.

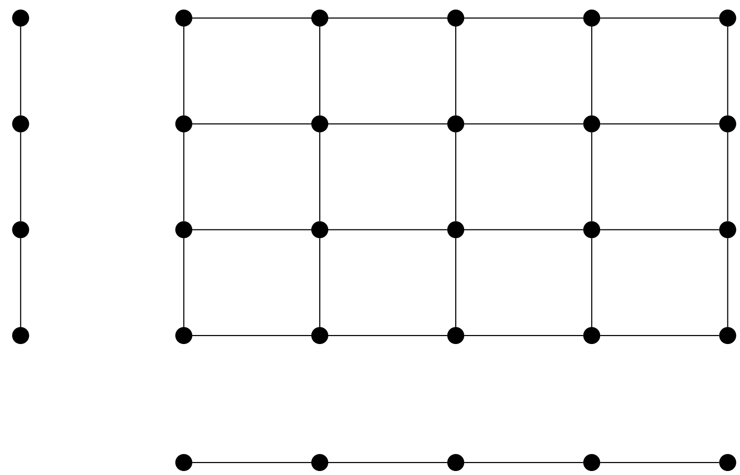


Figure 2.6: Cartesian product of P_5 and P_4 ($P_5 \square P_4$)

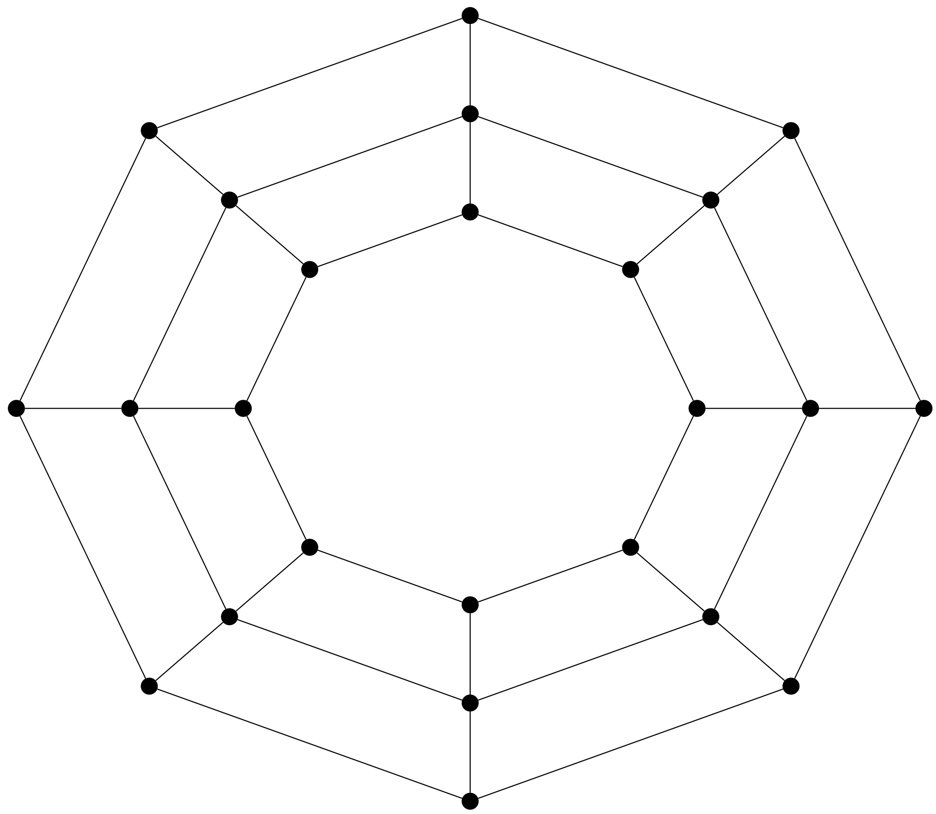


Figure 2.7: Cartesian product of C_8 and P_3 ($C_8 \Box P_3$)

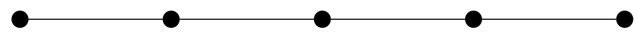
Definition 2.1.3. Let L and $L^{'}$ be two graphs with set of vertices $V(L) = \{v_1, v_2, ...v_n\}$ and $V(L^{'}) = \{v_1^{'}, v_2^{'}, ...v_m^{'}\}$ respectively. The standard strong product of L and $L^{'}$ is the graph, with $V(L \boxtimes L^{'}) = V(L) \times V(L^{'})$ (the Cartesian product of sets), and for $(v_i, v_i^{'}), (v_j, v_j^{'}) \in V(L \boxtimes L^{'})$, $(v_i, v_i^{'})(v_j, v_j^{'}) \in E(L \boxtimes L^{'})$, whenever

a. $v_i^{'} v_j^{'} \in E(L^{'})$ and $v_i = v_j$ or

b. $v_i^{'} = v_j^{'}$ and $v_i v_j \in E(L)$ or

c. $v_i \in V(L), v_i^{'} \in V(L^{'})$, $v_i^{'} v_j^{'} \in E(L^{'})$ and $v_i v_j \in E(L)$ or

d. $v_j \in V(L), v_j^{'} \in V(L^{'})$, $v_i^{'} v_j^{'} \in E(L^{'})$ and $v_i v_j \in E(L)$.



(b) P_5

(a) P_3

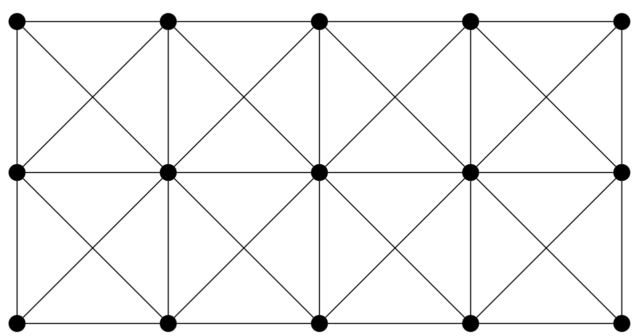


Figure 2.8: Strong product of P_3 and P_5 ($P_3 \boxtimes P_5$)

Chapter 3

Depth and Stanley depth of $\mathbb{Z}^n$ graded modules

In this chapter, we will discuss depth and Stanley depth of $\mathbb{Z}^n$-graded modules over $\mathbb{Z}^n$-graded commutative ring. We will also discuss the well known conjecture of stanley known as Stanley's conjecture. This chapter will summarize the basic results of depth and Stanley depth of some classes of monomial ideals and their quotient rings.

3.1 Regular sequnce and depth

Definition 3.1.1. Let R be a ring and D be a module over R. An element $t \neq 0$ of R is called a zero divisor of module D if $td = 0$ for some non-zero $d \in D$. An element is said to be regular element if it is not a zero divisor.

Definition 3.1.2. A sequence $t = t_1, t_2, ..., t_m$ of element of R is said to be D-regular if it satisfies the following conditions: [5]

i. t_i is $D/(t_1, t_2, ..., t_{i-1})D$ regular for any i;

ii. $D \neq (t)D$.

Definition 3.1.3. Let D be a finitely generated R-module and let μ be the unique maximal ideal of local Noetherian ring R. Then depth of D is the common length of all maximal D-sequences in μ, denoted by depth(D).

3.2 Stanley decomposition and Stanley depth

Definition 3.2.1. Let $S = M[z_1, \ldots, z_n]$ be a polynomial ring, where M is a field and let D be a finitely generated $\mathbb{Z}^n$-graded S-module. [13] Let $d \in D$ be a homogeneous element and consider a subset $K \subset \{z_1, \ldots, z_n\}$, then $dM[K]$ represents the M-subspace of D, whose generating set consists of homogeneous elements of the type dv. This linear M-subspace $dM[K]$ is known as Stanley space whose dimension is $|K|$ if it is a free $M[K]$-module, where $|K|$ is the number of variables in K. The presentation of the M-vector space D as a finite direct sum of Stanley spaces is called Stanley decomposition.

$$\mathcal{D} \,:\, D = \bigoplus_{j=1}^{t} v_j M[K_j],$$

and the Stanley depth of this decomposition $\mathcal{D}$ is

$$\operatorname{sdepth} \mathcal{D} = \min\{\, |K_j| \,,\, j = 1, \ldots, t\}.$$

The Stanley depth of D is

$$\operatorname{sdepth}_S(D) = \max\{\operatorname{sdepth} \mathcal{D}\,\}.$$

3.2.1 Computing Stanley depth for square-free monomial ideals

In 2009, Herzog et al. [16] gave a technique of computing the lower bound for stanley depth of square-free monomial ideals. Let U is a square-free monomial ideal and $G(U) = (u_1, \ldots, u_m)$ be the minimal generating set of U. The characteristic poset of U w.r.t $e = (1, \ldots, 1)$, written as $\rho_U^{(1,\ldots,1)}$ is defined to be

$$\rho_U^{(1,\ldots,1)} = \{l \subset [m] \mid l \text{ contains } \operatorname{supp}(u_i) \; for\ some\ i\},$$

where $\operatorname{supp}(u_i) = \{j \,:\, z_j | u_i\} \subseteq [m] := \{1, \ldots, m\}$. For each $p, q \in \rho_U^{(1,\ldots,1)}$ where $p \subseteq q$, and

$$[p\,,\, q] = \{l \in \rho_U^{(1,\ldots,1)} \,:\, p \subseteq l \subseteq q\}.$$

Let $\rho : \rho_U^{(1,\ldots,1)} = \cup_{i=1}^{k}[l_i\,,\, r_i]$ be a partition of $\rho_I^{(1,\ldots,1)}$, and for every i, suppose $s(i) \in \{0,1\}^m$ is the tuple with $\mathrm{supp}(z^{s(i)}) = l_i$, then the Stanley decomposition $\mathcal{D}(\rho)$ of U is given by

$$\mathcal{D}(\rho) \,:\, I = \bigoplus_{i=1}^{n} z^{s(i)} M[\{z_l \,|\, l \in r_i\}].$$

Clearly, $\mathrm{sdepth}\,\mathcal{D}(\rho) = \min\{|r_1|, \ldots, |r_n|\}$ and

$$\mathrm{sdepth}(U) = \max\{\mathrm{sdepth}\,\mathcal{D}(\rho) \mid \rho \textit{ is a partition of } \rho_U^{(1,\ldots,1)}\}.$$

3.2.2 Stanley's conjecture

In 1982, Stanley [25] presented a conjecture that interrelate two different invariants and gave bound to the depth of a $\mathbb{Z}^n$-graded S-modules

Conjecture 3.2.2. Let D be $\mathbb{Z}^n$-graded S-modules the stanley conjectured that

$$depth(D) \leq sdepth(D).$$

For a polynomial ring S in m indeterminate, let $U \subset S$ is a monomial ideal, then for $m \leq 3$, $m = 4$ and $m = 5$ the conjecture for S/U is proved by Apel [3], Anwar [2] and Popescu [23], respectively. Also, when U is an intersection of three monomial prime ideals, or three monomial primary ideals or four monomial prime ideals of S, the conjecture is true for U. But in general it does not hold, in 2016 Duval et al. [10] proved that Stanley's conjecture is generally false, by giving a counter example for the module of type S/U.

Example 3.2.3. Let $U = (z_1z_4, z_1z_3, z_2z_3, z_3z_4) \subset M[z_1, z_2, z_3, z_4]$ be a square-free monomial ideal and $U' = 0$. Consider $l_1 = (1,0,0,1)$, $l_2 = (1,0,1,0)$, $l_3 = (0,1,1,0)$ and $l_4 = (0,0,1,1)$. Thus U is generated by $z^{l_1}, z^{l_2}, z^{l_3}, z^{l_4}$ and select $g = (1,1,1,1)$. The poset $\rho = \rho^g_{U/U'}$ is given by

$$\rho = \{(1,0,0,1),(1,0,1,0),(0,1,1,0),(0,0,1,1),(1,1,1,0),(1,1,0,1),(1,0,1,1),\\ (0,1,1,1),(1,1,1,1)\}.$$

The Partitions of ρ can be written as

$$\begin{aligned}\rho_1: \ & [(1,0,0,1),(1,0,0,1)] \bigcup [(1,0,1,0),(1,0,1,0)] \bigcup [(0,1,1,0),(0,1,1,0)] \bigcup \\ & [(0,0,1,1),(0,0,1,1)] \bigcup [(1,1,1,0),(1,1,1,0)] \bigcup [(1,1,0,1),(1,1,0,1)] \bigcup \\ & [(1,0,1,1),(1,0,1,1)] \bigcup [(0,1,1,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].\end{aligned}$$

$$\begin{aligned}\rho_2: \ & [(1,0,0,1),(1,1,0,1)] \bigcup [(1,0,1,0),(1,1,1,0)] \bigcup [(0,1,1,0),(0,1,1,1)] \bigcup \\ & [(0,0,1,1),(1,0,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)].\end{aligned}$$

So the corresponding Stanley decomposition of the partitions will be

$$\begin{aligned}\mathcal{D}(\rho_1) \ := \ & z_1z_4M[z_1,z_4] \oplus z_1z_3M[z_1,z_3] \oplus z_2z_3M[z_2,z_3] \oplus z_3z_4M[z_3,z_4] \oplus z_1z_2z_3M[z_1,z_2,z_3] \oplus \\ & z_1z_2z_4M[z_1,z_2,z_4] \oplus z_1z_3z_4M[z_1,z_3,z_4] \oplus z_2z_3z_4M[z_2,z_3,z_4] \oplus \\ & z_1z_2z_3z_4M[z_1,z_2,z_3,z_4].\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\rho_2) \ := \ & z_1z_4M[z_1,z_2,z_4] \oplus z_1z_3M[z_1,z_2,z_3] \oplus z_2z_3M[z_2,z_3,z_4] \oplus z_3z_4M[z_1,z_3,z_4] \oplus \\ & z_1z_2z_3z_4M[z_1,z_2,z_3,z_4].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(U) \ & \geq \ \max\{\operatorname{sdepth}(\mathcal{D}(\rho_1)), \operatorname{sdepth}(\mathcal{D}(\rho_2))\} \\ & = \ \max\{2,3\} \\ & = \ 3.\end{aligned}$$

Because U is not principal, so $\operatorname{sdepth}(U) = 3$.

Example 3.2.4. Let $U = (z_1z_3, z_2z_4, z_1z_4z_5) \subset M[z_1,z_2,z_3,z_4,z_5]$ be a square-free monomial ideal and $U' = 0$. Set $\gamma_1 = (1,0,1,0,0)$, $\gamma_2 = (0,1,0,1,0)$ and $\gamma_3 = (1,0,0,1,1)$ so U is generated by $z^{\gamma_1}, z^{\gamma_2}, z^{\gamma_3}$ and choose $g = (1,1,1,1,1)$. The poset $\rho = \rho^g_{U/U'}$ is written as

$$\begin{aligned}\rho = \{ & (1,0,1,0,0),(0,1,0,1,0),(1,1,1,0,0),(1,1,0,1,0),(1,0,1,1,0),(1,0,1,0,1), \\ & (1,0,0,1,1),(0,1,1,1,0),(0,1,0,1,1),(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1), \\ & (1,0,1,1,1),(0,1,1,1,1),(1,1,1,1,1)\}.\end{aligned}$$

The Partitions of P can be written as

$$\begin{aligned}
\rho_1 :\ & [(1,0,1,0,0),(1,0,1,0,0)] \bigcup [(0,1,0,1,0),(0,1,0,1,0)] \bigcup \\
& [(1,1,1,0,0),(1,1,1,0,0)] \bigcup [(1,1,0,1,0),(1,1,0,1,0)] \bigcup \\
& [(1,0,1,1,0),(1,0,1,1,0)] \bigcup [(1,0,1,0,1),(1,0,1,0,1)] \bigcup \\
& [(1,0,0,1,1),(1,0,0,1,1)] \bigcup [(0,1,1,1,0),(0,1,1,1,0)] \bigcup \\
& [(0,1,0,1,1),(0,1,0,1,1)] \bigcup [(1,1,1,1,0),(1,1,1,1,0)] \bigcup \\
& [(0,1,1,1,1),(0,1,1,1,1)] \bigcup [(1,1,1,0,1),(1,1,1,0,1)] \bigcup \\
& [(1,0,1,1,1),(1,0,1,1,1)] \bigcup [(1,1,0,1,1),(1,1,0,1,1)] \bigcup \\
& [(1,1,1,1,1),(1,1,1,1,1)].
\end{aligned}$$

$$\begin{aligned}
\rho_2 :\ & [(1,0,1,0,0),(1,1,1,1,0)] \bigcup [(0,1,0,1,0),(1,1,0,1,1)] \bigcup \\
& [(1,1,1,0,0),(1,1,1,0,0)] \bigcup [(1,1,0,1,0),(1,1,0,1,0)] \bigcup \\
& [(1,0,1,1,0),(1,0,1,1,0)] \bigcup [(1,0,1,0,1),(1,1,1,0,1)] \bigcup \\
& [(1,0,0,1,1),(1,0,1,1,1)] \bigcup [(1,1,1,1,1),(1,1,1,1,1)].
\end{aligned}$$

$$\begin{aligned}
\rho_3 :\ & [(1,0,1,0,0),(1,1,1,1,0)] \bigcup [(0,1,0,1,0),(1,1,0,1,1)] \bigcup \\
& [(1,0,1,0,1),(1,1,1,0,1)] \bigcup [(1,0,0,1,1),(1,0,1,1,1)] \bigcup \\
& [(0,1,1,1,0),(0,1,1,1,1)] \bigcup [(1,1,1,1,1),(1,1,1,1,1)].
\end{aligned}$$

So the corresponding Stanley decomposition of the partition will be

$$\begin{aligned}
\mathcal{D}(\rho_1) \ :=\ & z_1z_3M[z_1,z_3] \oplus z_2z_4M[z_2,z_4] \oplus z_1z_2z_3M[z_1,z_2,z_3] \oplus z_1z_2z_4M[z_1,z_2,z_4] \oplus \\
& z_1z_3z_4M[z_1,z_3,z_4] \oplus z_1z_3z_5M[z_1z_3z_5] \oplus z_1z_4z_5M[z_1,z_4,z_5] \oplus \\
& z_2z_3z4_5M[z_2,z_3,z_4] \oplus z_2z_4z_5M[z_2,z_4,z_5] \oplus z_1z_2z_3z_4M[z_1,z_2,z_3,z_4] \oplus \\
& z_2z_3z_4z_5M[z_2,z_3,z_4,z_5] \oplus z_1z_2z_3z_5K[z_1,z_2,z_3,z_5] \oplus z_1z_3z_4z_5M[z_1,z_3,z_4,z_5] \oplus \\
& z_1z_2z_4z_5M[z_1,z_2,z_4,z_5] \oplus z_1z_2z_3z_4z_5M[z_1,z_2,z_3,z_4,z_5].
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}(\rho_2) \ :=\ & z_1z_3M[z_1,z_2,z_3,z_4] \oplus z_2z_4M[z_1,z_2,z_4,z_5] \oplus z_1z_2z_3M[z_1,z_2,z_3] \oplus \\
& z_1z_2z_4M[z_1,z_2,z_4] \oplus z_1z_3z_4M[z_1,z_3,z_4] \oplus z_1z_3z_5M[z_1,z_2,z_3,z_5] \oplus \\
& z_1z_4z_5M[z_1,z_3,z_4,z_5] \oplus z_1z_2z_3z_4z_5M[z_1,z_2,z_3,z_4,z_5].
\end{aligned}$$

$$\begin{aligned}\mathcal{D}(\rho_3) \;:=\; & z_1z_3M[z_1,z_2,z_3,z_4] \oplus z_2z_5M[z_2,z_3,z_4,z_5] \oplus z_1z_3z_5M[z_1,z_2,z_3,z_5] \oplus \\ & z_1z_4z_5M[z_1,z_2,z_4,z_5] \oplus z_1z_3z_5M[z_1,z_3,z_4,z_5] \oplus z_1z_2z_3z_4z_5M[z_1,z_2,z_3,z_4,z_5].\end{aligned}$$

now

$$\begin{aligned}\text{sdepth}(U) &\geq \max\{\text{sdepth}(\mathcal{D}(\rho_1))\,,\, \text{sdepth}(\mathcal{D}(\rho_2))\,,\, \text{sdepth}(\mathcal{D}(\rho_3))\} \\ &= \max\{2,3,4\} \\ &= 4.\end{aligned}$$

Because U is not principal, so $\text{sdepth}(U) = 4$.

Now we will illustrate another example for the method of computing the Stanley depth of quotient S/U.

Example 3.2.5. Let $S = M[z_1, z_2, z_3, z_4, z_5, z_6]$, consider $U = (z_1z_3, z_2z_5, z_4z_6, z_1z_4z_6)$. Then select $g = (1,1,1,1,1,1)$ and the poset $\rho = \rho^g_{S/U}$ is given by

$$\begin{aligned}\rho = \{&(0,0,0,0,0,0),(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),(0,0,0,1,0,0),\\ &(0,0,0,0,1,0),(0,0,0,0,0,1),(1,1,0,0,0,0),(1,0,0,1,0,0),(1,0,0,0,1,0),\\ &(1,0,0,0,0,1),(0,1,1,0,0.0),(0,1,0,1,0.0),(0,0,1,1,0,0),(0,0,0,1,1,0),\\ &(0,0,1,0,0,1),(0,0,0,1,1,0),(0,0,0,1,0.1).(0,0,0,0,1,1),(1,1,0,1,0,0),\\ &(1,0,0,1,1,0),(1,0,0,0,1,1),(0,1,1,1,0,0),(0,0,1,1,1,0),(0,0,1,1,0,1),\\ &(0,0,1,0,1,1),(0,0,0,1,1,1),(0,0,1,1,1,1)\}.\end{aligned}$$

The partitions of ρ can be written as

$$\begin{aligned}
\rho_1: \ & [(0,0,0,0,0,0),(1,0,0,0,0,0)] \bigcup [(0,1,0,0,0,0),(0,1,0,0,0,0)] \bigcup \\
& [(0,0,1,0,0,0),(0,0,1,0,0,0)] \bigcup [(0,0,0,1,0,0),(0,0,0,1,0,0)] \bigcup \\
& [(0,0,0,0,1,0),(0,0,0,0,1,0)] \bigcup [(0,0,0,0,0,1),(0,0,0,0,0,1)] \bigcup \\
& [(1,1,0,0,0,0),(1,1,0,0,0,0)] \bigcup [(1,0,0,1,0,0),(1,0,0,1,0,0)] \bigcup \\
& [(0,1,0,1,0,0),(0,1,0,1,0,0)] \bigcup [(0,0,1,1,0,0),(0,0,1,1,0,0)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,1,0,1),(0,0,0,1,0,1)] \bigcup \\
& [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup [(1,1,0,1,0.0),(1,1,0,1,0,0)] \bigcup \\
& [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup \\
& [(1,1,0,1,0,0),(1,1,0,1,0,0)] \bigcup [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup \\
& [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup [(0,1,1,1,0,0),(0,1,1,1,0,0)] \bigcup \\
& [(0,0,1,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,1,0,1),(0,0,1,1,0,1,)] \bigcup \\
& [(0,0,0,1,1,1),(0,0,0,1,1,1)] \bigcup [(0,0,1,1,1,1),(0,0,1,1,1,1,)].
\end{aligned}$$

$$\begin{aligned}
\rho_2: \ & [(0,0,0,0,0,0),(1,1,0,1,0,0)] \bigcup [(0,0,1,0,0,0),(0,1,1,1,0,0)] \bigcup \\
& [(0,0,0,0,1,0),(1,0,0,1,1,0)] \bigcup [(0,0,0,0,0,1),(1,0,0,0,1,1)] \bigcup \\
& [(0,0,0,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,0,0,1),(0,0,0,1,0,1)] \bigcup \\
& [(0,0,0,1,0,1),(0,0,0,1,1,1)].
\end{aligned}$$

So the corresponding Stanley decomposition is of the partitions will be

$$\begin{aligned}
\mathcal{D}(\rho_1) \ := \ & M[z_1] \oplus z_2M[z_2] \oplus z_3M[z_3] \oplus z_4M[z_4] \oplus z_5M[z_5] \oplus z_6M[z_6] \oplus \\
& z_1z_2M[z_1,z_2] \oplus z_1z_4M[z_1,z_4] \oplus z_2z_4M[z_2,z_4] \oplus z_3z_4M[z_3,z_4] \oplus z_4z_5M[z_4,z_5] \oplus \\
& z_4z_6M[z_4,z_6] \oplus z_5z_6M[z_5,z_6] \oplus z_1z_2z_4M[z_1,z_2,z_4] \oplus z_1z_4z_5M[z_1,z_4,z_5] \oplus \\
& z_1z_5z_6M[z_1,z_5,z_6] \oplus z_4z_5M[z_4,z_5] \oplus z_4z_5K[z_4,z_5] \oplus z_5z_6M[z_5,z_6] \oplus z_1z_2z_4M[z_1,z_2,z_4] \oplus \\
& z_1z_4z_5M[z_1,z_4,z_5] \oplus z_1z_5z_6M[z_1,z_5,z_6] \oplus z_2z_3z_4M[z_2,z_3,z_4] \oplus z_3z_4z_5M[z_3,z_4,z_5] \oplus \\
& z_3z_4z_6M[z_3,z_4,z_6] \oplus z_4z_5z_6M[z_4,z_5,z_6] \oplus z_3z_4z_5z_6M[z_3,z_4,z_5,z_6].
\end{aligned}$$

$$\mathcal{D}(\rho_2) := M[z_1,z_2,z_4]\oplus z_3M[z_2,z_3,z_4]\oplus z_5M[z_1,z_4,z_5]\oplus z_6M[z_1,z_5,z_6]\oplus z_4z_5M[z_3,z_4,z_5]\oplus z_3z_6M[z_4,z_6]\oplus z_4z_6M[z_4,z_5,z_6].$$

Then

$$\begin{aligned}\operatorname{sdepth}(S/U) &\geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_1))\,,\,\operatorname{sdepth}(\mathcal{D}(\mathcal{P}_2))\}\\ &= \max\{1,3\}\\ &= 3.\end{aligned}$$

Lemma 3.2.6. *[5, Proposition 1.2.9] (Depth Lemma) Consider the following short exact sequence*

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow H_3 \longrightarrow 0$$

then ***i)*** $\operatorname{depth}(E_1) \geq \min\{\operatorname{depth}(E_2)\,,\,1+\operatorname{depth}(E_3)\}$,
ii) $\operatorname{depth}(E_2) \geq \min\{\operatorname{depth}(E_1)\,,\,\operatorname{depth}(E_3)\}$,
iii) $\operatorname{depth}(E_3) \geq \min\{\operatorname{depth}(E_1)-1\,,\,\operatorname{depth}(E_2)\}$.

Lemma 3.2.7. *[17, Lemma 2.4] Suppose D is a $\mathbb{Z}^m$-graded S-module. let E_1 and E_2 be the submodules of D and consider the short exact sequence of the of the form $0 \to E_1 \to D \to E_2 \to 0$. Then we have*

$$\operatorname{sdepth}(D) \geq \min\{\operatorname{sdepth}(E_1), \operatorname{sdepth}(E_2)\}.$$

Remark 3.2.8. *Let $I \subset S$ be a monomial ideal. Let x be a variable in S and $x \notin I$ then, the short exact sequence*

$$0 \longrightarrow S/(I:x) \xrightarrow{\cdot x} S/I \longrightarrow S/(I,x) \longrightarrow 0,$$

implies that

$$\operatorname{depth}(S/I) \geq \min\{\operatorname{depth}(S/(I:x)), \operatorname{depth}(S/(I,x))\},$$

$$\operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S/(I:x)), \operatorname{sdepth}(S/(I,x))\}.$$

This will be used frequently throughout the thesis.

Proposition 3.2.9. *For a monomial ideal $U \subset S$, consider a monomial s such that $s \in S$ and $s \notin U$ then*

1. $\mathrm{sdepth}_S(U : s) \geq \mathrm{sdepth}_S(U)$, *[23, Proposition 1.3]*

2. $\mathrm{depth}_S(S/(U : s)) \geq \mathrm{depth}_S(S/U)$, *[21, Corollary 1.3].*

Proposition 3.2.10. *[6, Proposition 2.7] For a monomial ideal $U \subset S$, consider a monomial s such that $s \in S$ and $s \notin U$ then*

$$\mathrm{sdepth}_S(S/(U : s)) \geq \mathrm{sdepth}_S(S/U).$$

Theorem 3.2.11. *[22, Theorem 1.1] Suppose that $U \subset S$ be a monomial ideal and $s \in S$ be a monomial regular on S/U, then*

$$\mathrm{sdepth}(S/(U, s)) = \mathrm{sdepth}(S/U) - 1.$$

Lemma 3.2.12. *[16, Lemma 3.6] Let U and $U^{'}$ be two monomial ideals of S and $U^{'} \subset U$, suppose $S' = S[z_{m+1}]$, then*

$$\mathrm{depth}(US'/U^{'}S') = \mathrm{depth}(US/U^{'}S) + 1.$$

$$\mathrm{sdepth}(US'/US') = \mathrm{sdepth}(US/U^{'}S) + 1.$$

Lemma 3.2.13. *[6, Proposition 1.1] Consider $U \subset S' = M[z_1, \ldots, z_m], U^{'} \subset S'' = M[z_{m+1}, \ldots, z_n]$ be monomial ideals, with $1 \leq m \leq n$, then we have*

$$\mathrm{depth}_S(S/(US + U^{'}S)) = \mathrm{depth}_{S'}(S'/U) + \mathrm{depth}_{S''}(S''/U^{'}).$$

Theorem 3.2.14. *[21, Theorem 3.1] Consider $U \subset S' = M[z_1, \ldots, z_m], U^{'} \subset S'' = M[z_{m+1}, \ldots, z_n]$ be monomial ideals, with $1 \leq m \leq n$, then we have*

$$\mathrm{sdepth}_S(S/(US + U^{'}S)) \geq \mathrm{sdepth}_{S'}(S'/U) + \mathrm{sdepth}_{S''}(S''/U^{'}).$$

Lemma 3.2.15. *[15, Lemma 3.1] Let $m \geq 2$ be an integer, and consider $\{M_j : 1 \leq j \leq m\}$ and $\{N_j : 0 \leq j \leq m\}$ be sequence of $\mathbb{Z}^n-$graded $S-$modules and consider the chain of exact sequences of the form*

$$0 \longrightarrow M_1 \longrightarrow N_0 \longrightarrow N_1 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow M_{m-1} \longrightarrow N_{m-2} \longrightarrow N_{m-1} \longrightarrow 0$$

$$0 \longrightarrow M_m \longrightarrow N_{m-1} \longrightarrow N_m \longrightarrow 0$$

such that $\operatorname{depth} M_m \leq \operatorname{depth} N_m$ *and* $\operatorname{depth} M_{j-1} \leq depth M_j$, *for all* $2 \leq j \leq m$ *then* $\operatorname{depth} M_1 = \operatorname{depth} N_0$.

Chapter 4

Depth and Stanley depth of corona product of some trees

4.1 Definition and notations

Definition 4.1.1. Let L and T be two graphs. [29] The corona product of L and T denoted by $L \circ T$, is the graph obtained by taking one copy of L of order n and n copies of T; and the by joining the i-th vertex of L to every vertex in the i-th copy of T.

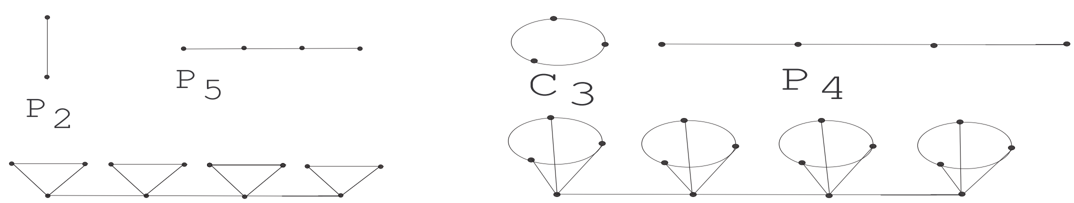


Figure 4.1: From left to right $P_4 \circ P_2$ and $P_4 \circ C_3$.

Definition 4.1.2. A graph with only one vertex is called a trivial graph. We denote the trivial graph by T.

Definition 4.1.3. A graph with one internal vertex and $k-1$ leaves is called a k-star, denoted by S_k.

Definition 4.1.4. Let $z \geq 1$ and $k \geq 2$ be integers and P_z be a path on z vertices $u_1, u_2, \dots, u_z$ that is $E(P_z) = \{u_i u_{i+1} : 1 \leq i \leq z-1\}$ (for $z = 1$, $E(P_z) = \emptyset$). We define a graph on zk vertices by attaching $k-1$ pendant vertices at each u_i. We denote this graph by $P_{z,k}$.

Definition 4.1.5. Let $z \geq 3$ and $k \geq 2$ be integers and C_z be a cycle on z vertices $u_1, u_2, \ldots, u_z$ that is $E(C_z) = \{u_i u_{i+1} : 1 \leq i \leq z-1\} \cup \{u_1 u_z\}$. We define a graph on zk vertices by attaching $k-1$ pendant vertices at each u_i. We denote this graph by $C_{z,k}$.

Figure 4.2: From left to right $P_{3,5}$ and $C_{3,5}$.

Definition 4.1.6. Firecracker is a graph formed by the concatenation of α number of k-stars by linking exactly one leaf from each star. It is denoted by $F_{\alpha,k}$.

Definition 4.1.7. The graph obtained by joining the end vertices of the path joining the leaves of the α stars in $F_{\alpha,k}$. We call this graph circular firecracker and is denoted by $CF_{\alpha,k}$.

Figure 4.3: From left to right $P_{3,5}$ and $C_{3,5}$.

Definition 4.1.8. Let $z \geq 3$ and $k \geq 2$ be integers and P_z be a path on z vertices $u_1, u_2, \ldots, u_z$ that is $E(P_z) = \{u_i u_{i+1} : 1 \leq i \leq z-1\}$ (for $z = 1$, $E(P_z) = \emptyset$). We define a graph by attaching $k_i - 1$ pendants at each u_i when i is odd and no pendants when i is even. We denote this graph by $P_{z;k_1,k_3,\ldots,k_{z-2},k_z}$.

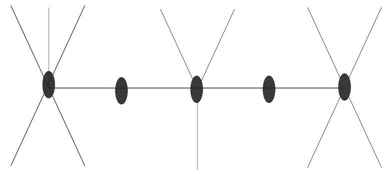


Figure 4.4: $P_{5;5,3,4}$

4.2 Results

Lemma 4.2.1. *Let T be a trivial graph and G be any connected graph on more than one vertices. Consider $X = K[x_1, x_2, ..., x_n]$ be the polynomial ring. Let $U = U(T \circ G)$, then* $\operatorname{depth}(X/U) = 1$.

Figure 4.5: Trivial graphs

Proof. By definition of $T \circ G$ the only vertex x of T has an edge with every vertex of G. Therefore $X/(U : x) \cong K[x]$, and $\operatorname{depth}(X/(U : x)) = 1$. Now $X/(U, x) \cong X_x/U(G)$, where $X_x := X/(x)$. We have $\operatorname{depth}(X/(U, x)) = \operatorname{depth}(X_x/U(G)) \geq 1$, by using Depth Lemma 3.2.6, we have $\operatorname{depth}(X/U) = 1$. $\square$

Proposition 4.2.2. *Let $k \geq 2$, G be any connected graph with $|V(G)| \geq 2$, then*

$$\operatorname{depth}(X/U(S_k \circ G)) = k - 1 + t,$$

where $t = \operatorname{depth}(K[V(G)]/U(G))$.

Proof. If $k = 2$. let e be a variable corresponding to a leaf in S_2, we have

$$X/(U : e) \cong K[V(G)]/U(G) \otimes_K K[e],$$

by using 3.2.12, we have

$$\operatorname{depth}(X/(U : e)) = t + 1.$$

It is easy to see that $X/(U,e) \cong K[V(T\circ G)]/U(T\circ G)\otimes_K K[V(G)]/U(G))$. Hence by using 4.2.1, we have

$$\operatorname{depth}(X/(U,e)) = 1+t = \operatorname{depth}(X/(U:e)).$$

Thus by Depth Lemma we have $\operatorname{depth}(X/U) = 1+t$, holds for $k=2$.
Consider $k\geq 3$, the proof is done by induction on k. Let e be a variable corresponding to a leaf in S_k, we have

$$X/(U:e) \cong \bigotimes_{j=1}^{k-2}{}_K K[V(T)\circ G]/U(T\circ G)\otimes_K K[V(G)]/U(G)\otimes_K K[e]$$

$$\operatorname{depth}(X/(U:e)) = \sum_{j=1}^{k-2}\operatorname{depth}(K[V(T\circ G)]/U(T\circ G)) + \operatorname{depth}(K[V(G)]/U(G)) + \operatorname{depth} K[e],$$

by 3.2.12, we have

$$\operatorname{depth}(X/(U:e)) = k-2+t+1 = k-1+t.$$

It is easy to see that $X/(U,e) \cong K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)\otimes_K (K[V(G)]/U(G))$, hence by using Induction on k and we have

$$\operatorname{depth}(X/(U,e)) = (k-2+t)+t = k+2t-2 \geq k-1+t = \operatorname{depth}(X/(U:e)).$$

Thus by Depth Lemma we have $\operatorname{depth}(X/U) = k-1+t$, this complete the proof. □

Theorem 4.2.3. *Let $z\geq 1$ and $k\geq 2$ be integers and G be a connected graph with $|V(G)|\geq 2$. Then $\operatorname{depth}(X/U(P_{z,k}\circ G)) = z(k-1+t)$, where $t = \operatorname{depth}((K[V(G)])/U(G))$.*

Proof. If $z=1$ then proof follows from Proposition 4.2.2. Assume that $z=2$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_2 in $P_{z,k}$. Let $U = U(P_{z,k}\circ G)$, for $0\leq i\leq k-2$, $U_i := (U_i, e_{i+1})$, where $U_0 = U$. Consider the chain of short exact sequences of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X/(U_0:e_1) & \longrightarrow & X/(U_0) & \longrightarrow & X/(U_0,e_1) & \longrightarrow & 0\\
0 & \longrightarrow & X/(U_1:e_2) & \longrightarrow & X/(U_1) & \longrightarrow & X/(U_1,e_2) & \longrightarrow & 0\\
 & & & & \vdots & & & & \\
0 & \longrightarrow & X/(U_{k-2}:e_{k-1}) & \longrightarrow & X/(U_{k-2}) & \longrightarrow & X/(U_{k-2},e_{k-1}) & \longrightarrow & 0\\
0 & \longrightarrow & X/(U_{k-1}:u_1) & \longrightarrow & X/(U_{k-1}) & \longrightarrow & X/(U_{k-1},u_1) & \longrightarrow & 0
\end{array}$$

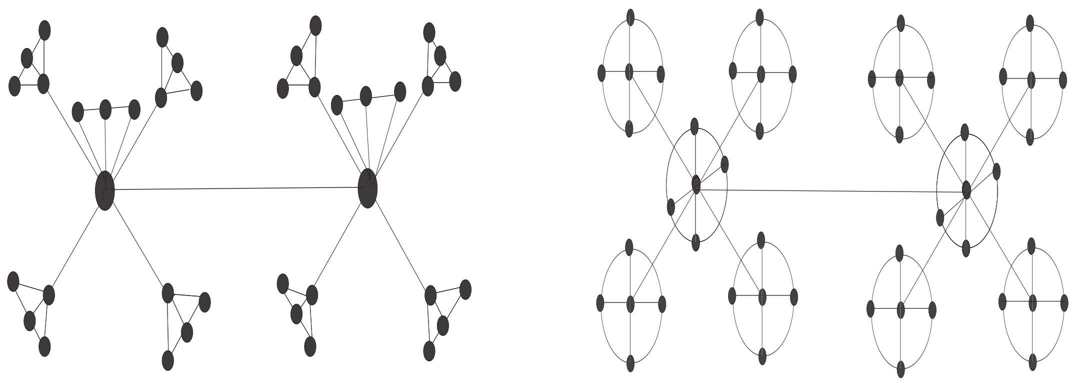


Figure 4.6: From left to right $P_{2,5} \circ P_3$ and $P_{2,5} \circ C_4$.

$$X/(U_i : e_{i+1}) \cong K[V(S_k \circ G)]/U(S_k \circ G) \bigotimes_{j=1}^{k-2-i}{}_K K[V(T \circ G)]/U(T \circ G)$$
$$\bigotimes_{j=1}^{i+1}{}_K K[V(G)]/U(G) \otimes_K K[e_{i+1}],$$

$$\operatorname{depth}(X/(U_i : e_{i+1})) = \operatorname{depth}(K[V(S_k \circ G)]/U(S_k \circ G)) + \textstyle\sum_{j=1}^{k-2-i} \operatorname{depth}(K[V(T \circ G)]/U(T \circ G))$$
$$+ \sum_{j=1}^{i+1} \operatorname{depth}(K[V(G)]/U(G)) + 1,$$

hence by Lemma 4.2.1 and Proposition 4.2.2, we get

$$\begin{aligned} \operatorname{depth} X/(U_i : e_{i+1}) &= k - 1 + t + \textstyle\sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} t + 1 \\ &= k + t + k - 2 - i + (i+1)t = 2(k - 1 + t) + U(t - 1). \quad (4.1) \end{aligned}$$

$$X/(U_{k-1} : u_1) \cong \bigotimes_{j=1}^{k-1}{}_K K[V(T \circ G)]/U(T \circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/U(G) \otimes_K K[u_1]$$

by Proposition 4.2.2, we get

$$\operatorname{depth}(X/(U_{k-1} : u_1)) = k + kt = 2(k - 1 + t) + (k - 2)(t - 1). \quad (4.2)$$

And

$$X/(U_{k-1}, u_1) \cong K[V(S_k \circ G)]/U(S_k \circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/U(G)$$

$$\operatorname{depth} X/(U_{k-1}, u_1) = \operatorname{depth} K[V(S_k \circ G)]/U(S_k \circ G) + \sum_{j=1}^{k} \operatorname{depth} K[V(G)]/U(G)$$

by Proposition 4.2.2, we get

$$\operatorname{depth}(X/(U_{k-1},u_1)) = k-1+t+kt = 2(k-1+t)+(k-1)(t-1). \tag{4.3}$$

Hence by Lemma 3.2.15, we get

$$\operatorname{depth}(X/U(P_{2,k}\circ G)) = 2(k-1+t)$$

hold for $z=2$. Now consider $z \geq 2$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_z in $P_{z,k}$. Let $U = U(P_{z,k}\circ G)$, for $0 \leq i \leq k-2$, $U_i := (U_i, e_{i+1})$ where $U_0 = U$. Consider the chain of short exact sequences of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X/(U_0 : e_1) & \longrightarrow & X/(U_0) & \longrightarrow & X/(U_0, e_1) & \longrightarrow & 0 \\
0 & \longrightarrow & X/(U_1 : e_2) & \longrightarrow & X/(U_1) & \longrightarrow & X/(U_1, e_2) & \longrightarrow & 0 \\
 & & & & \vdots & & & & \\
0 & \longrightarrow & X/(U_{k-2} : e_{k-1}) & \longrightarrow & X/(U_{k-2}) & \longrightarrow & X/(U_{k-2}, e_{k-1}) & \longrightarrow & 0 \\
0 & \longrightarrow & X/(U_{k-1} : u_1) & \longrightarrow & X/(U_{k-1}) & \longrightarrow & X/(U_{k-1}, u_1) & \longrightarrow & 0
\end{array}$$

we have,

$$X/(U_i : e_{i+1}) \cong K[V(P_{z-1,k}\circ G)]/U(P_{z-1,k}\circ G) \bigotimes_{j=1}^{k-2-i}{}_K K[V(T\circ G)]/U(T\circ G)$$
$$\bigotimes_{j=1}^{i+1}{}_K K[V(G)]/U(G) \otimes_K K[e_{i+1}].$$
$$\operatorname{depth}(X/(U_i : e_{i+1})) = \operatorname{depth}(K[V(P_{z-1,k}\circ G)]/U(P_{z-1,k}\circ G)) +$$
$$\sum_{j=1}^{k-2-i} \operatorname{depth}(K[V(T\circ G)]/U(T\circ G)) + \sum_{j=1}^{i+1} \operatorname{depth}(K[V(G)]/U(G)) + 1$$

hence by Lemma 4.2.1, Proposition 4.2.2 and induction on z we get,

$$\operatorname{depth} X/(U_i : e_{i+1}) = (z-1)(k-1+t) + \textstyle\sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} t + 1 =$$
$$(z-1)(k-1+t) + k-2-i + (i+1)t + 1 = z(k-1+t) + U(t-1). \tag{4.4}$$

$$X/(U_{k-1} : u_1) \cong K[V(P_{z-2,k}\circ G)]/U(P_{z-2,k}\circ G) \bigotimes_{j=1}^{k-1}{}_K K[V(T\circ G)]/U(T\circ G)$$
$$\bigotimes_{j=1}^{k}{}_K K[V(G)]/U(G) \otimes_K K[u_1]$$

by Proposition 4.2.2, we get

$$\operatorname{depth}(X/(U_{k-1} : u_1)) = z(k-1+t) + (k-2)(t-1). \tag{4.5}$$

And

$$X/(U_{k-1}, u_1) \cong K[V(P_{z-1,k} \circ G)]/U(P_z - 1, k \circ G) \bigotimes_{j=1}^{k}{}_K K[V(G)]/U(G)$$

$$\operatorname{depth} X/(U_{k-1}, u_1) = \operatorname{depth} K[V(P_{z-1,k} \circ G)]/U(P_{z-1,k} \circ G) + \sum_{j=1}^{k} \operatorname{depth} K[V(G)]/U(G),$$

by Proposition 4.2.2 and induction on z we get

$$\operatorname{depth}(X/(U_{k-1}, u_1)) = (z-1)(k-1+t) + kt = z(k-1+t) + (k-1)(t-1). \tag{4.6}$$

Hence by Lemma 3.2.15, we get

$$\operatorname{depth}(S/U(P_{z,k} \circ G)) = z(k-1+t).$$

hold for $z \geq 4$. □

Theorem 4.2.4. *Let $z \geq 3$ and $k \geq 2$ be integers and G be a connected graph with $|V(G)| \geq 2$. Then $\operatorname{depth}(X/U(C_{z,k} \circ G)) = z(k-1+t)$, where $t = \operatorname{depth}((K[V(G)])/U(G))$.*

Proof. Assume that $z = 3$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_3 in $C_{z,k}$. Let $U = U(C_{z,k} \circ G)$, for $0 \leq i \leq k-2$, $U_i := (U_i, e_{i+1})$ where $U_0 = U$. Consider the chain of short exact sequences of the form

$$\begin{array}{ccccccccc}
0 \longrightarrow & X/(U_0 : e_1) & \longrightarrow & X/(U_0) & \longrightarrow & X/(U_0, e_1) & \longrightarrow 0 \\
0 \longrightarrow & X/(U_1 : e_2) & \longrightarrow & X/(U_1) & \longrightarrow & X/(U_1, e_2) & \longrightarrow 0 \\
 & & & \vdots & & & \\
0 \longrightarrow & X/(U_{k-2} : e_{k-1}) & \longrightarrow & X/(U_{k-2}) & \longrightarrow & X/(U_{k-2}, e_{k-1}) & \longrightarrow 0 \\
0 \longrightarrow & X/(U_{k-1} : u_1) & \longrightarrow & X/(U_{k-1}) & \longrightarrow & X/(U_{k-1}, u_1) & \longrightarrow 0
\end{array}$$

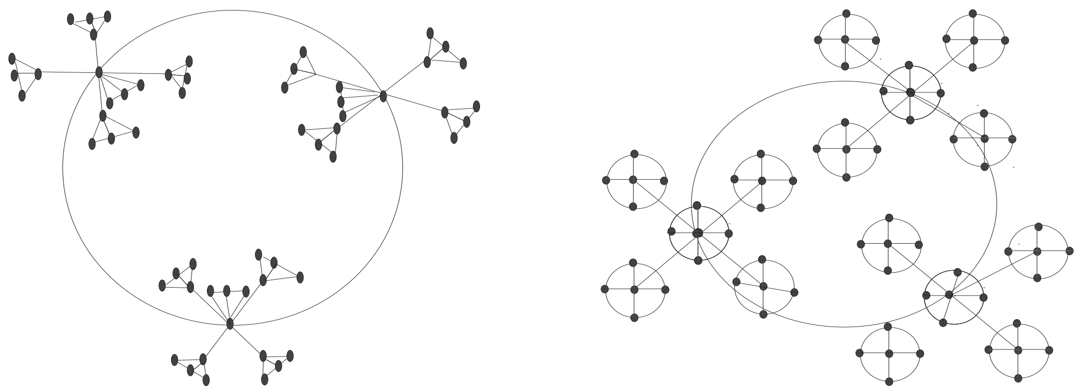


Figure 4.7: From left to right $P_{2,5} \circ P_3$ and $P_{2,5} \circ C_4$.

we have,

$$X/(U_i : e_{i+1}) \cong K[V(P_{2,k}\circ G)]/U(P_{2,k}\circ G)\bigotimes_{j=1}^{k-2-i}{}_K K[V(T\circ G)]/U(T\circ G)\bigotimes_{j=1}^{i+1}{}_K K[V(G)]/U(G)\underset{K}{\otimes_K} K[e_{i+1}]$$

$$\operatorname{depth}(X/(U_i : e_{i+1})) = \operatorname{depth}(K[V(P_{k,2}\circ G)]/U(P_{k,2}\circ G)) + \textstyle\sum_{j=1}^{k-2-i} \operatorname{depth}(K[V(T\circ G)]/U(T\circ G))$$
$$+\sum_{j=1}^{i+1} \operatorname{depth}(K[V(G)]/U(G)) + 1$$

hence by Lemma 4.2.1, Proposition 4.2.2 and 4.2.3, we get

$$\operatorname{depth} X/(U_i : e_{i+1}) = 2(k-1+t) + \textstyle\sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} +1 =$$
$$2(k-1+t) + k - 2 - i + it + t + 1 = 3(k-1+t) + U(t-1). \quad (4.7)$$

$$X/(U_{k-1} : u_1) \cong \bigotimes_{j=1}^{k-1}{}_K K[V(T\circ G)]/U(T\circ G)\bigotimes_{j=1}^{k-1}{}_K K[V(T\circ G)]/U(T\circ G)\bigotimes_{j=1}^{k+1}{}_K K[V(G)]/U(G)\otimes_K K[u_1]$$

by Lemma 4.2.1, we get

$$\operatorname{depth}(X/(U_{k-1} : u_1)) = 3(k-1+t) + (k-2)(t-1). \quad (4.8)$$

And

$$X/(U_{k-1}, u_1) \cong K[V(P_{2,k} \circ G)]/U(P_{2,k} \circ G)\bigotimes_{j=1}^{k}{}_K K[V(G)]/U(G).$$

$$\operatorname{depth} X/(U_{k-1}, u_1) = \operatorname{depth} K[V(P_{2,k} \circ G)]/U(P_{2,k} \circ G) + \sum_{j=1}^{k} \operatorname{depth} K[V(G)]/U(G).$$

by Theorem 4.2.3 we get

$$\operatorname{depth}(X/(U_{k-1},u_1)) = 2(k-1+t)+kt = 3(k-1+t)+(k-1)(t-1). \qquad (4.9)$$

Hence by Lemma 3.2.15, we get

$$\operatorname{depth}(X/U(C_{3,k}\circ G)) = 3(k-1+t)$$

holds for $z=3$.

Now consider that $z \geq 4$. Let $e_1, e_2, \ldots, e_{k-1}$ be leaves attached to u_z in $C_{z,k}$. Let $U = U(C_{z,k}\circ G)$, for $0 \leq i \leq k-2$, $U_i := (U_i, e_{i+1})$ where $U_0 = U$. Consider the chain of short exact sequences of the form

$$\begin{array}{ccccccc}
0 \longrightarrow & X/(U_0 : e_1) \longrightarrow & X/(U_0) & \longrightarrow & X/(U_0, e_1) \longrightarrow & 0 \\
0 \longrightarrow & X/(U_1 : e_2) \longrightarrow & X/(U_1) & \longrightarrow & X/(U_1, e_2) \longrightarrow & 0 \\
 & & \vdots & & & \\
0 \longrightarrow & X/(U_{k-2} : e_{k-1}) \longrightarrow & X/(U_{k-2}) & \longrightarrow & X/(U_{k-2}, e_{k-1}) \longrightarrow & 0 \\
0 \longrightarrow & X/(U_{k-1} : u_1) \longrightarrow & X/(U_{k-1}) & \longrightarrow & X/(U_{k-1}, u_1) \longrightarrow & 0
\end{array}$$

$$X/(U_i : e_{i+1}) \cong K[V(P_{z-1,k}\circ G)]/U(P_{z-1,k}\circ G) \bigotimes_{j=1}^{k-2-i}{}_K K[V(T\circ G)]/U(T\circ G) \bigotimes_{j=1}^{i+1}{}_K K[V(G)]/U(G) \otimes_K K[e_i$$

$$\operatorname{depth}(X/(U_i : e_{i+1})) = \operatorname{depth}(K[V(P_{z-1,k}\circ G)]/U(P_{z-1,k}\circ G)) + \textstyle\sum_{j=1}^{k-2-i} \operatorname{depth}(K[V(T\circ G)]/U(T\circ G))$$
$$+ \sum_{j=1}^{i+1} \operatorname{depth}(K[V(G)]/U(G)) + 1$$

Hence by Lemma 4.2.1, Proposition 4.2.2 and 4.2.3, we get

$$\begin{aligned}
\operatorname{depth} X/(U_i : e_{i+1}) &= (z-1)(k-1+t) + \textstyle\sum_{j=1}^{k-2-i} 1 + \sum_{j=1}^{i+1} t + 1 \\
&= (z-1)(k-1+t) + k - 2 - i + it + t + 1 = z(k-1+t) + U(t-1). \qquad (4.10)
\end{aligned}$$

$$X/(U_{k-1} : u_1) \cong K[V(P_{z-3,k}\circ G)]/U(P_{z-3,k}\circ G) \bigotimes_{j=1}^{k-1}{}_K K[V(T\circ G)]/U(T\circ G) \bigotimes_{j=1}^{k-1}{}_K K[V(T)\circ G]/U(T\circ G)$$
$$\bigotimes_{j=1}^{k+1}{}_K K[V(G)]/U(G) \otimes_K K[u_1]$$

by Proposition 4.2.2 and Theorem 4.2.3, we get

$$\operatorname{depth}(X/(U_{k-1}:u_1)) = z(k-1+t)+(k-2)(t-1). \tag{4.11}$$

And

$$X/(U_{k-1},u_1) \cong K[V(P_{z-1,k}\circ G)]/U(P_{z-1,k}\circ G)\bigotimes_{j=1}^{k}{}_K K[V(G)]/U(G)$$

$$\operatorname{depth} X/(U_{k-1},u_1) = \operatorname{depth} K[V(P_{z-1,k}\circ G)]/U(P_{z-1,k}\circ G)+\sum_{j=1}^{k}\operatorname{depth} K[V(G)]/U(G)$$

by Lemma 4.2.2 and Theorem 4.2.3, we get

$$\operatorname{depth}(X/(U_{k-1},u_1)) = (z-1)(k-1+t)+kt = z(k-1+t)+(k-1)(t-1). \tag{4.12}$$

Hence by Lemma 3.2.15, we get

$$\operatorname{depth}(X/U(C_{z,k}\circ G)) = z(k-1+t)$$

hold for $z \geq 4$. □

Theorem 4.2.5. *Let $\alpha \geq 2$ and $k \geq 2$ be integers and G be a connected graph with $|V(G)| \geq 2$. Then*

$$\operatorname{depth} S/U(F_{\alpha,k}\circ G) = \begin{cases} \alpha(k-1+t)+\frac{\alpha}{2}(t-1) & \alpha = even \\ \alpha(k-1+t)+\frac{\alpha-1}{2}(t-1) & \alpha = odd \end{cases}$$

Proof. Consider $\alpha = 2$. Let $\{e_1.e_2,.....,e_{k-1}\}$ be leaves attached to u_2 in $F(2,k)$. Let $U = U(F_{2,k}\circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow X/(U:e_1) \longrightarrow X/U \longrightarrow X/(U,e_1) \longrightarrow 0$$

where e_1 is leave of second star that is attached to the previous star.

$$X/(U:e_1) \cong K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)\bigotimes_{j=1}^{k-2}{}_K K[V(T\circ G)]/U(T\circ G)$$
$$\bigotimes_{j=1}^{2}{}_K K[V(G)]/U(G)\otimes_K K[e_1]$$

$$\operatorname{depth}(X/(U:e_1)) = \operatorname{depth}(K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G))+\sum_{j=1}^{k-2}\operatorname{depth}(K[V(T\circ G)]/U(T\circ G))$$
$$+2\operatorname{depth}(K[V(G)]/U(G))+1,$$

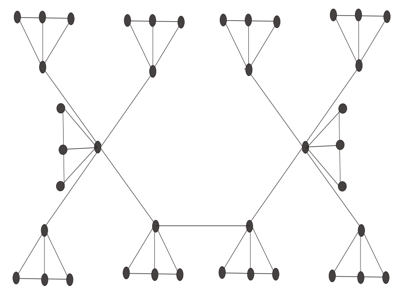


Figure 4.8: $F_{3,5} \circ P_3$

hence by Lemma 4.2.1 and Proposition 4.2.2, we get

$$\operatorname{depth} X/(U : e_1) = k-2+t+\textstyle\sum_{j=1}^{k-2} 1 + 2t + 1 =$$
$$k-2+t+k-2+2t+1 = 2(k-1+t)+(t-1). \quad (4.13)$$

$$X/(U, e_1) \cong K[V(S_k \circ G)]/U(S_k \circ G) \otimes_K K[V(S_{k-1} \circ G)]/U(S_{k-1}) \otimes_K K[V(G)]/U(G)$$

.

$$\begin{aligned} \operatorname{depth} X/(U, e_1) &= (k-1+t)+(k-2+t)+t \\ \operatorname{depth} X/(U, e_1) &= 2(k-1+t)+(t-1). \end{aligned} \quad (4.14)$$

So by using 3.2.6, we have

$$\operatorname{depth} X/U(F_{2,k} \circ G) = 2(k-1+t)+(t-1).$$

Now consider $\alpha = 3$. Let $\{e_1.e_2, \ldots.., e_{k-1}\}$ be leaves attached to u_3 in $F(3,k)$. Let $U = U(F_{3,k} \circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow X/(U : e_1) \longrightarrow X/U \longrightarrow X/(U, e_1) \longrightarrow 0$$

where e_1 is leave of third star that is attached to the previous star. We have

$$X/(U : e_1) \cong K[V(S_k \circ G)]/U(S_k \circ G) \otimes_K K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G)$$
$$\bigotimes_{j=1}^{k-2}{}_K K[V(T \circ G)]/U(T \circ G) \bigotimes_{j=1}^{2}{}_K K[V(G)]/U(G) \otimes_K K[e_1]$$

$$\operatorname{depth}(X/(U : e_1)) = \operatorname{depth}(K[V(S_k \circ G)]/U(S_k \circ G)) + \operatorname{depth}(K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G))$$
$$+ \sum_{j=1}^{k-2} \operatorname{depth}(K[V(T \circ G)]/U(T \circ G)) + 2 \operatorname{depth}(K[V(G)]/U(G)) + 1$$

hence by Lemma 4.2.1 and Proposition 4.2.2, we get

$$\operatorname{depth} X/(U:e_1) = k-1+t+k-2+t+\textstyle\sum_{j=1}^{k-2} 1+2t+1 =$$
$$k-1+t+k-2+t+k-2+2t+1 = 3(k-1+t)+(t-1). \quad (4.15)$$

$$X/(U,e_1) \cong K[V(F_{2,k}\circ G)]/U(F_{2,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)\otimes_K K[V(G)]/U(G)$$

$$\begin{aligned}\operatorname{depth} X/(U,e_1) &= 2(k-1+t)+(t-1)+(k-2+t)+t \\ \operatorname{depth} X/(U,e_1) &= 3(k-1+t)+2(t-1). \qquad (4.16)\end{aligned}$$

So by using 3.2.6, we have

$$\operatorname{depth} X/U(F_{3,k}\circ G) = 3(k-1+t)+(t-1)$$

holds for $\alpha = 3$. Consider α is even so $\alpha-1$ and $\alpha-2$ are odd and even respectively. Let $\{e_1.e_2, \ldots.., e_{k-1}\}$ be leaves attached to u_α in $F(\alpha, k)$. Let $U = U(F_{\alpha,k}\circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow X/(U:e_1) \longrightarrow X/U \longrightarrow X/(U,e_1) \longrightarrow 0$$

where e_1 is leave of last star that is attached to the previous star. We have

$$X/(U:e_1) \cong K[V(F_{\alpha-2,k}\circ G)]/U(F_{\alpha-2,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)$$
$$\bigotimes_{j=1}^{k-2}{}_K K[V(T\circ G)]/U(T\circ G)\bigotimes_{j=1}^{2}{}_K K[V(G)]/U(G)\otimes_K K[e_1]$$
$$\operatorname{depth}(X/(U:e_1)) = \operatorname{depth} K[V(F_{\alpha-2,k}\circ G)]/U(F_{\alpha-2,k}\circ G)+\operatorname{depth}(K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G))$$
$$+\sum_{j=1}^{k-2}\operatorname{depth}(K[V(T\circ G)]/U(T\circ G))+2\operatorname{depth}(K[V(G)]/U(G))+1$$

hence by Lemma 4.2.1, Proposition 4.2.2 and induction on α, we get

$$\operatorname{depth} X/(U:e_1) = (\alpha-2)(k-1+t)+\tfrac{\alpha-2}{2}(t-1)+(k-2+t)+\textstyle\sum_{j=1}^{k-2} 1+2t+1$$
$$= \alpha(k-1+t)+\frac{\alpha}{2}(t-1). \quad (4.17)$$

$$X/(U,e_1) \cong K[V(F_{\alpha-1,k}\circ G)]/U(F_{\alpha-1,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)\otimes_K K[V(G)]/U(G)$$

$$\begin{aligned}\operatorname{depth} X/(U, e_1) &= (\alpha-1)(k-1+t)+\frac{\alpha-2}{2}(t-1)+(k-2+t)+t \\ \operatorname{depth} X/(U : e_1) &= \alpha(k-1+t)+\frac{\alpha}{2}(t-1). \qquad (4.18)\end{aligned}$$

So by using 3.2.6, we have

$$\operatorname{depth} X/(F_{\alpha,k} \circ G) = \alpha(k-1+t)+\frac{\alpha}{2}(t-1)$$

holds for α is even. Consider α is odd so $\alpha-1$ and $\alpha-2$ are even and odd respectively. Let $\{e_1.e_2, \ldots.., e_{k-1}\}$ be leaves attached to u_α in $F(\alpha, k)$. Let $U = U(F_{\alpha,k} \circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow X/(U : e_1) \longrightarrow X/U \longrightarrow X/(U, e_1) \longrightarrow 0$$

where e_1 is leave of last star that is attached to the previous star. We have

$$\begin{aligned}X/(U : e_1) \cong{} & K[V(F_{\alpha-2,k} \circ G)]/U(F_{\alpha-2,k} \circ G) \otimes_K K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G) \\ & \bigotimes_{j=1}^{k-2}{}_K K[V(T \circ G)]/U(T \circ G) \bigotimes_{j=1}^{2}{}_K K[V(G)]/U(G) \otimes_K K[e_1]\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(X/(U : e_1)) ={} & \operatorname{depth} K[V(F_{\alpha-2,k} \circ G)]/U(F_{\alpha-2,k} \circ G) + \operatorname{depth}(S_{k-1}/U(S_{k-1} \circ G)) \\ & + \sum_{j=1}^{k-2} \operatorname{depth}(K[V(T \circ G)]/U(T \circ G)) + 2 \operatorname{depth}(K[V(G)]/U(G)) + 1\end{aligned}$$

hence by Lemma 4.2.1, Proposition 4.2.2 and induction on α, we get

$$\begin{aligned}\operatorname{depth} X/(U : e_1) &= (\alpha-2)(k-1+t)+\tfrac{\alpha-3}{2}(t-1)+(k-2+t)+\textstyle\sum_{j=1}^{k-2} 1+2t+1 \\ &= \alpha(k-1+t)+\frac{\alpha-1}{2}(t-1).\end{aligned}$$

$$X/(U, e_1) \cong K[V(F_{\alpha-1,k} \circ G)]/U(F_{\alpha-1,k} \circ G) \otimes_K S K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G) \otimes_K K[V(G)]/U(G)$$

$$\begin{aligned}\operatorname{depth} X/(U, e_1) &= (\alpha-1)(k-1+t)+\frac{\alpha-1}{2}(t-1)+(k-2+t)+t \\ \operatorname{depth} X/(U, e_1) &= \alpha(k-1+t)+\frac{\alpha+1}{2}(t-1). \qquad (4.19)\end{aligned}$$

So by using 3.2.6, we have

$$\operatorname{depth} X/(F_{\alpha,k} \circ G) = \alpha(k-1+t)+\frac{\alpha-1}{2}(t-1)$$

holds for α is an odd. □

Theorem 4.2.6. *Let $\alpha \geq 3$ and $k \geq 2$ be integers and G be a connected graph with $|V(G)| \geq 2$. Then*

$$\operatorname{depth} X/U(CF_{\alpha,k} \circ G) = \begin{cases} \alpha(k-1+t) + \frac{\alpha}{2}(t-1) & \alpha = even \\ \alpha(k-1+t) + \frac{\alpha+1}{2}(t-1) & \alpha = odd \end{cases}$$

Proof. Consider $\alpha = 3$. Let $\{e_1.e_2, ..., e_{k-1}\}$ be leaves attached to u_3 in $CF(3,k)$. Let $U = U(CF_{3,k} \circ G)$. Consider the short exact sequence of the form

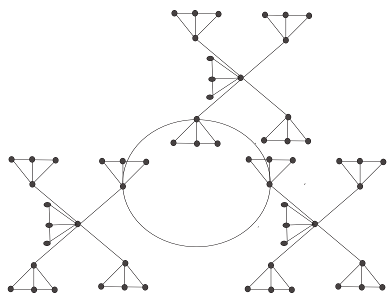


Figure 4.9: $CF_{3,5} \circ P_3$

$$0 \longrightarrow X/(U : e_1) \longrightarrow X/U \longrightarrow X/(U, e_1) \longrightarrow 0$$

where e_1 is leave of third star that is attached to the previous star and first star. We have

$$X/(U : e_1) \cong \bigotimes_{j=1}^{2}{}_K K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G)$$
$$\bigotimes_{j=1}^{k-2}{}_K K[(T \circ G)]/U(T \circ G) \bigotimes_{j=1}^{3}{}_K K[V(G)]/U(G) \otimes_K K[e_1]$$

$$\operatorname{depth}(X/(U : e_1)) = 2 \operatorname{depth}(K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G)) +$$
$$\sum_{j=1}^{k-2} \operatorname{depth}(K[V(T \circ G)]/U(T \circ G)) + 3 \operatorname{depth}(K[V(G)]/U(G)) + 1$$

hence by Lemma 4.2.1 and Proposition 4.2.2, we get

$$\operatorname{depth} X/(U : e_1) = 2(k-2+t) + \sum_{j=1}^{k-2} 1 + 3t + 1 = 3(k-1+t) + 2(t-1). \quad (4.20)$$

$$X/(U,e_1) \cong K[V(F_{2,k}\circ G)]/U(F_{2,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)\otimes_K K[V(G)]/U(G)$$

$$\begin{aligned} \operatorname{depth} X/(U,e_1) &= 2(k-1+t)+(t-1)+(k-2+t)+t \\ \operatorname{depth} X/(U:e_1) &= 3(k-1+t)+2(t-1). \end{aligned} \tag{4.21}$$

So by using 3.2.6, we have

$$\operatorname{depth} X/U(CF_{3,k}\circ G) = 3(k-1+t)+2(t-1).$$

Now consider $\alpha = 4$. Let $\{e_1.e_2, ..., e_{k-1}\}$ be leaves attached to u_4 in $CF(4,k)$. Let $U = U(CF_{4,k}\circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow X/(U:e_1) \longrightarrow X/U \longrightarrow X/(U,e_1) \longrightarrow 0$$

where e_1 is leave of third star that is attached to the previous star and first star. We have

$$\begin{aligned} X/(U:e_1) \cong\ & K[V(S_k\circ G)]/U(S_k\circ G)\bigotimes_{j=1}^{2}{}_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G) \\ & \bigotimes_{j=1}^{k-2}{}_K K[V(T\circ G)]/U(T\circ G)\bigotimes_{j=1}^{3}{}_K K[V(G)]/U(G)\otimes_K K[e_1] \end{aligned}$$

$$\begin{aligned} \operatorname{depth}(X/(U:e_1)) =\ & \operatorname{depth}(K[V(S_k\circ G)]/U(S_k\circ G)) + 2\operatorname{depth}(K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)) \\ & + \sum_{j=1}^{k-2}\operatorname{depth}(K[V(T\circ G)]/U(T\circ G)) + 3\operatorname{depth}(K[V(G)]/U(G)) + 1 \end{aligned}$$

hence by Lemma 4.2.1 and Proposition 4.2.2, we get

$$\operatorname{depth} X/(U:e_1) = k-1+t+2(k-2+t)+\textstyle\sum_{j=1}^{k-2} 1+3t+1 = 4(k-1+t)+2(t-1). \tag{4.22}$$

$$X/(U,e_1) \cong K[V(F_{3,k}\circ G)]/U(F_{3,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)\otimes_K K[V(G)]/U(G)$$

$$\begin{aligned} \operatorname{depth} X/(U,e_1) &= 3(k-1+t)+(t-1)+(k-2+t)+t \\ \operatorname{depth} X/(U,e_1) &= 4(k-1+t)+2(t-1). \end{aligned} \tag{4.23}$$

So by using 3.2.6, we have

$$\operatorname{depth} X/(F_{4,k}\circ G) = 4(k-1+t)+2(t-1)$$

holds for $\alpha = 4$. Consider α is odd so $\alpha - 1$ and $\alpha - 3$ are even. Let $\{e_1.e_2, \ldots.., e_{k-1}\}$ be leaves attached to u_α in $CF(\alpha, k)$. Let $U = U(CF_{\alpha,k} \circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow X/(U : e_1) \longrightarrow X/U \longrightarrow X/(U, e_1) \longrightarrow 0$$

where e_1 is leave of last star that is attached to the previous star and first star. We have

$$X/(U : e_1) \cong K[V(F_{\alpha-3,k} \circ G)]/U(F_{\alpha-3,k} \circ G) \bigotimes_{j=1}^{2}{}_K K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G)$$
$$\bigotimes_{j=1}^{k-2}{}_K K[V(T \circ G)]/U(T \circ G) \bigotimes_{j=1}^{3}{}_K K[V(G)]/U(G) \otimes_K K[e_1]$$

$$\operatorname{depth}(X/(U : e_1)) = \operatorname{depth} K[V(F_{\alpha-3,k} \circ G)]/U(F_{\alpha-3,k} \circ G) + 2 \operatorname{depth}(K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G))$$
$$+ \sum_{j=1}^{k-2} \operatorname{depth}(K[V(T \circ G)]/U(T \circ G)) + 3 \operatorname{depth}(K[V(G)]/U(G)) + 1$$

hence by Lemma 4.2.1, Proposition 4.2.2 and Theorem 4.2.5, we get

$$\begin{aligned} \operatorname{depth} X/(U : e_1) &= (\alpha - 3)(k - 1 + t) + \tfrac{\alpha-3}{2}(t - 1) + 2(k - 2 + t) + \textstyle\sum_{j=1}^{k-2} 1 + 3t + 1 \\ &= \alpha(k - 1 + t) + \frac{\alpha + 1}{2}(t - 1). \end{aligned} \quad (4.24)$$

$$X/(U, e_1) \cong K[V(F_{\alpha-1,k} \circ G)]/U(F_{\alpha-1,k} \circ G) \otimes_K K[V(S_{k-1} \circ G)]/U(S_{k-1} \circ G) \otimes_K K[V(G)]/U(G)$$

$$\begin{aligned} \operatorname{depth} X/(U, e_1) &= (\alpha - 1)(k - 1 + t) + \frac{\alpha - 1}{2}(t - 1) + (k - 2 + t) + t \\ \operatorname{depth} X/(U, e_1) &= \alpha(k - 1 + t) + \frac{\alpha + 1}{2}(t - 1). \end{aligned} \quad (4.25)$$

So by using 3.2.6, we have

$$\operatorname{depth} X/U(CF_{\alpha,k} \circ G) = \alpha(k - 1 + t) + \frac{\alpha + 1}{2}(t - 1)$$

holds for α is odd. Consider α is even so $\alpha - 1$ and $\alpha - 3$ are odds. Let $\{e_1.e_2, \ldots.., e_{k-1}\}$ be leaves attached to u_α in $F(\alpha, k)$. Let $U = U(F_{\alpha,k} \circ G)$. Consider the short exact sequence of the form

$$0 \longrightarrow X/(U : e_1) \longrightarrow X/U \longrightarrow X/(U, e_1) \longrightarrow 0$$

where e_1 is leave of last star that is attached to the previous star and first star. We have

$$X/(U:e_1) \cong K[V(F_{\alpha-3,k}\circ G)]/U(F_{\alpha-3,k}\circ G)\bigotimes_{j=1}^{2}{}_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)$$
$$\bigotimes_{j=1}^{k-2}{}_K K[V(T\circ G)]/U(T\circ G)\bigotimes_{j=1}^{3}{}_K K[V(G)]/U(G)\otimes_K K[e_1]$$

$$\operatorname{depth}(X/(U:e_1)) = \operatorname{depth} K[V(F_{\alpha-3,k}\circ G)]/U(F_{\alpha-3,k}\circ G)+2\operatorname{depth}(K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G))$$
$$+\sum_{j=1}^{k-2}\operatorname{depth}(K[V(T\circ G)]/U(T\circ G))+3\operatorname{depth}(K[V(G)]/U(G))+1$$

hence by Lemma 4.2.1, Proposition 4.2.2 and induction on α, we get

$$\operatorname{depth} X/(U:e_1) = (\alpha-3)(k-1+t)+\tfrac{\alpha-4}{2}(t-1)+2(k-2+t)+\textstyle\sum_{j=1}^{k-2}1+3t+1$$
$$= \alpha(k-1+t)+\frac{\alpha}{2}(t-1). \quad (4.26)$$

$$X/(U,e_1) \cong K[V(F_{\alpha-1,k}\circ G)]/U(F_{\alpha-1,k}\circ G)\otimes_K K[V(S_{k-1}\circ G)]/U(S_{k-1}\circ G)\otimes_K K[V(G)]/U(G)$$

$$\operatorname{depth} X/(U,e_1) = (\alpha-1)(k-1+t)+\frac{\alpha-2}{2}(t-1)+(k-2+t)+t$$
$$\operatorname{depth} X/(U,e_1) = \alpha(k-1+t)+\frac{\alpha}{2}(t-1). \quad (4.27)$$

So by using 3.2.6, we have

$$\operatorname{depth} X/U(CF_{\alpha,k}\circ G) = \alpha(k-1+t)+\frac{\alpha}{2}(t-1)$$

holds for α is an even. □

Proposition 4.2.7. *[1] For a star graph $\mathcal{S}_k$, let $U=U(\mathcal{S}_k)$, then*

$$\operatorname{depth}(S/I) = \operatorname{sdepth}(X/U) = 1,$$

and

$$\operatorname{depth}(X/U^t), \operatorname{sdepth}(X/U^t) \geq 1.$$

Theorem 4.2.8. *Let $z \geq 3$ and $k \geq 2$ be integers and consider $U = U(P_{z;k_1,k_3,\dots,k_z})$. Then* $\operatorname{depth}(X/U) = \frac{z+1}{2}$.

Proof. The proof is done by induction on z. Now consider $z = 3$, we have $(I : u_3) = (x : x \in N(u_3)) + U(S_{k_1})$ and $X/(U : u_3) \cong S^{'}[u_3]/U(S_{k_1})$, thus by Lemma 3.2.12 and 4.2.7,

$$\operatorname{depth} X/(U : u_3) = \operatorname{depth} S^{'}/(S_{k_1}) + 1 = 2$$

clearly $(I, u_3) = (U(S_{k_1+1}), u_3)$ and $X/(U, u_3) \cong S^{''}[N(u_3)]/U(S_{k_1+1})$ so by 4.2.7,

$$\begin{aligned} \operatorname{depth} X/(U, u_3) &= \operatorname{depth} S^{''}/(U(S_{k_1+1})) + |N(u_3)| \\ \operatorname{depth} X/(U, u_3) &= 1 + k_3 - 1 = k_3 \end{aligned}$$

hence by using 3.2.6, we have

$$\operatorname{depth} S/U(P_{z;k_1,k_3}) = 2,$$

holds for $z = 3$. Assume that $z \geq 4$, consider a short exact sequence of the form

$$0 \longrightarrow X/(U : u_z) \longrightarrow X/U \longrightarrow X/(U, u_z) \longrightarrow 0,$$

then $(U : u_z) = (x : x \in N(u_z) + U(P_{z-2;k_1,\dots,k_{z-2}})$. Since $X/(U : u_z) \cong S_{z-2}[u_z]/U(P_{z-2;k_1,\dots,k_{z-2}})$ so by induction on z, we get

$$\begin{aligned} \operatorname{depth} X/(U : u_z) &= \operatorname{depth} S_{z-2}/U(P_{z-2;k_1,\dots,k_{z-2}}) + 1 \\ \operatorname{depth} X/(U : u_z) &= \frac{z-2+1}{2} + 1 \\ \operatorname{depth} X/(U : u_z) &= \frac{z+1}{2}. \end{aligned}$$

And $(U, u_z) = U(P_{z-2;k_1,\dots,k_{z-2}+1,u_z})$. Since $X/(U, u_z) \cong S_{z-1}[N(u_z)]/U(P_{z-2;k_1,\dots,k_{z-1}+1,u_z})$ by using induction on z, we get

$$\begin{aligned} \operatorname{depth} X/(U, u_z) &= \operatorname{depth} S_{z-2}/U(P_{z-1;k_1,\dots,k_{z-2}+1}) + k_z \\ \operatorname{depth} X/(U, u_z) &= \frac{z-2+1}{2} + k_z \\ &= \frac{z+1}{2} + k_z - 1. \end{aligned}$$

Hence by 3.2.6 we have $\operatorname{depth} X/U = \frac{z+1}{2}$ □

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