An Approximate Solution of Blasius Problem Using Spectral Method

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بسم الله الرحمن الرحيم

IN THE NAME OF ALLAH

THE MOST COMPASSIONATE

THE MOST MERCIFUL

HE IS THE MOST OMNISCIENT.

(AND HE TO WHOM WISDOM IS GRANTED RECEIVETH

INDEED A BENEFIT OVERFLOWING.)

Declaration

I "**Ms. Zunera Shoukat**" hereby solemnly declare that this thesis entitled "**An Approximate Solution of Blasius Problem By Using Spectral Method** ", submitted by me for the partial fulfillment of Master of Science in Mathematics, as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor "**Prof. Dr. Azad Akhter Siddiqui**". No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

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DEDICATED TO

MY BELOVED PARENTS

AND

PROF. DR. AZAD AKHTER SIDDIQUI

Acknowledgment

All praise to Almighty **ALLAH** alone, the most merciful and the most compassionate, the most benevolent and the gracious, whose blessings are abundant and whose favors are unlimited, who gave me the power to do, the right to observe and mind to think and judge. His Holy Prophet **MUHAMMAD** (peace be upon him) the most perfect, who is forever torch of guidance and knowledge for the humanity.

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Abstract

Nearly all of the real world problems are non-linear in nature and they are coded in the language of non-linear differential equations. To find the exact solutions of these problems are usually impossible. So, we direct our attention towards finding the approximate solutions of these equations. This thesis aims at finding the analytical solution of a classical Blasius flat plate problem, non-linear problem, using spectral collocation method. This technique is based on Chebyshev pseduspectral approach that reduced the solution to the solution of a system of algebraic equations. The implementation of this method is carried out in Mathematica and its validity is ensured by comparing it with a built in MATLAB numerical routine called bvp4c. The graphical and tabular representation of the problem is also presented in order to get an insight into the problem.

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CHAPTER 1

Introduction

In the seventeenth century, Gottfried Wilhelm Leibniz $(1646-1716)$ and Isaac Newton $(1647-1716)$ developed the differential equation that was initially performed in the theory of calculus. Three types of differential equations were formulated by Newton in 1671. Newton solved his first differential equation in 1676 and the same year Leibniz denoted a relationship among the two variables X and Y [24]. The term differential equations (DEs) were first introduced by Leibniz. In the field of applied sciences, most of the theoretical and physical phenomena can be expressed as differential equations. By using basic analytical methods for complex shapes, it is not always possible to solve these equations. A number of natural phenomena are represented by equations which involve the rate of change of one variable with respect to the other variable(s) known as "derivative".

An equation containing the derivatives is called as differential equation (DE). In a DE physical quantities are generally represented by function while derivative express rates of change in respective function. Following are some examples of DEs:

$$\frac{dy}{dx} = cosx,$$

$$\frac{d^2y}{dx^2} + x^2 = x,$$

$$\frac{dy}{dx} - lny = 0.$$

The order of a DE is defined as the highest derivative that appears in the equation. e.g.

$$\frac{d^4y}{dx^4} + \frac{dy}{dx} = 0,$$

is a fourth order DE. The degree of differential equation defined as the power of the highest derivative appearing in the equation. e.g.

$$\left(\frac{d^2y}{dx^2}\right)^4 + 4\left(\frac{dy}{dx}\right) - 4x = y,$$

has degree 4.

An ordinary differential equation (ODE) is defined as the equation in which dependent variable contain only one independent variable while on the other hand if it consist of more than one independent variables then it is called partial differential equation (PDE). DEs are also categorized as linear, nonlinear, non-homogeneous and homogeneous.

There are two main types of problems that are categorized by DEs (i.e. boundary value problems and initial value problem) which depend on the type of conditions. In initial value problem conditions are specified at only one point of the domain while in boundary value problem conditions are described at more than one points of the domain. The analytical solutions for every kind of DEs do not exist so we need some numerical technique to solve the problem. To find the solutions of DEs, generally following techniques are useful:

1. Finite Difference Method (FDM).

2. Finite Element Method (FEM).

3. Finite Volume Method (FVM).

4. Spectral Methods.

1.1 Finite Difference Method

The finite difference approach is easiest to understand and applied to solve a broad range of problems that include; time dependent and independent, non-linear and linear problems. This method associates to so called grid-point methods. It was not broadly used to solve engineering problems until the $1940's$. This method can be used to solve problem which contain various kinds of boundary conditions and for a region containing a number of materials.The following method is centered on the calculus of finite differences. It is a straight forward technique in which a PDE is satisfied at a set of interconnected points within the domain, called nodes. The boundary conditions are satisfied at a set of nodes placed on the domain's boundaries. The structure of all nodes is suggested as a mesh or grid. The derivatives in the PDE is approximated using difference approximations derived using Taylor series expansions. To solve the DE using FDM, we follow these steps:

- Formulate the whole domain into small intervals that are called mesh and then label the grid according to created mesh.
- By using backward difference, forward difference quotient or second order central difference quotient approximate the derivative appearing in the DE.
- Obtain an algebraic equation for the first grid points.
- Repeating the above process for all interior grid points leads to a system of algebraic equations in which the values of nodal variables remain unknown. Hereafter, we get number of unknowns equat to the number of equations by putting in the boundary conditions.
- Now to solve this system numerically, different iterative methods like Jacobi method,

Gauss Seidal method etc can be applied.

1.2 Finite Element Method

Finite element method (FEM) is a mathematical technique that is used for finding the approximate solution of differential and integral equation that occurs in various areas of applied sciences. In the previous few eras, Finite Element Method (FEM) has expanded as an essential technique in simulation and modeling of different system of engineering, for instance, transportation, communication, housing and so on. In the advancement of such complex engineering system, engineers deal with a very stringent procedure of simulation, analysis, modeling, visualization, designing, testing, and finally, construction. Application of the FEM has increased rapidly. First time, it was used for solving the stress analysis problems and after that it has been enlarged to numerous other problems such as fluid flow, thermal analysis, heat flow in continuum mechanics.

Basic approach behind FEM is to change the initial problem into simple form by dividing the problem domain into numerous sections. Furthermore, we can get a better approximation in FEM by increasing computational efforts. In each sub-domain by using piecewise linear function, a continuous function of an unknown variable is approximated, that known as element made by nodes. The discrete values of the field variable at the nodes are unknowns. To set up equations for the elements, an approximate rule is followed in which elements are connected to one another. To get the mandatory field variable for the whole system this procedure tends to a set of linear equations that can be solved easily. The FEM is preferred over FDM because FEM is comparatively more suitable for problems having complex geometrical domains and writing the codes for FEM are not easy in general.

1.3 Finite Volume Method

The Finite Volume Method (FVM) converts PDEs into a set of abstract algebraic equations over finite volume, which describe the conservation laws on differential volume. Related to finite element method and finite difference method, the first step of the procedure is the formulation of the domain, in which the finite volume method is integrated over each subdomain to transform these equations into algebraic equations. These equations are then solved to calculate the values of the dependent variable for each sub-domain.

Some of terms in FVM in the system of equations are transformed into face fluxes and tested on the finite volume faces. Since the flux input is equal with that of the neighboring range, the FVM is purely conservative. Due to this property, this technique is more preferable technique in computational fluid dynamics (CFD). Another essential feature is that can be constructed on formless polygon meshes in the physical space. Lastly, implementation of types of boundary conditions in FVM in a noninvasive manner is quite easy, as unknown variables are not evaluated at their boundary faces but at their centroid of the volume sub domains.

FVM is relatively appropriate for numerical simulation of various problems involving mass and heat transfer and fluid flow and advancement in FVM have been relatively interlinked with development in CFD. This method is now efficient for dealing with all types of complex physics and their applications.

1.4 Spectral Methods

In the last few eras spectral methods have been expanded quickly because of their extreme assurance. These methods have been used effectively to different numerical simulations in various areas, like engineering, heat conduction, fluid dynamics, mechanics etc. These days, for numerical solution of DEs there are some of the most effective techniques including finite difference method and finite element techniques.

The spectral approach belongs to the class of weighted residual methods (WRMs) in the sense of numerical designs for DEs which are conventionally considered to be a basis of numerical methods such as finite volume, finite element, spectral and boundary element (cf. Finlayson (1972)). WRMs constitute a specific cluster of approximation techniques that minimizes the error in a specific direction and thus leading to certain techniques involving collocation Galerkin, tau formulations and Petrov-Galerkin.

Spectral method can be used to solve ODEs, PDEs and eigenvalues problem efficiently. In spectral method, unknown solution is expended as a global interplant, that is why this method is more advanced as compare to FEM or FDM. The convergence of spectral method is based on regularity of solution. The analytic functions converge more rapidly at the rate of $O(c^N)$ where $0 < c < 1$ see reference [38]. If the functions are smooth then the convergence rate is $O(N^{-m})$, for every m. Even when the functions are not smooth spectral method often works very well, see reference [8]. As compared to finite element methods; spectral methods are computationally more affordable. The main aspect of this method is to hold numerous orthogonal systems of infinitely differentiable global function as trial function.

One of the most important question in spectral method is choosing suitable basis or trial functions having

1. Easy to compute
2. Fast Convergence rate
3. Completeness

The choice of trial functions or basis functions is based on the problem under study. Various trial functions lead to distinct spectral approximation. For example, for periodic problems Fourier series and for non-periodic problem Legendre polynomials and Chebyshev polynomials are used. On the half real line problem, Laguerre polynomial and for whole line problems Hermite polynomials are used.

1.4.1 Various spectral methods

Three most important spectral methods are as follow:

1. Tau Method
2. Collocation Method
3. Galerkin Method

The selection of applying any of these methods depends upon the application.

1.4.2 Tau Method

Tau-spectral method was developed by C. Lanczos in 1938. It is suitable for non-periodic problems with complicated boundary conditions. In this method test function is equal to trial function but the basis functions do not satisfy the boundary conditions. This method minimizes the residual function like the Galerkin method but it differs in that the boundary condition which is also a constraint.

1.4.3 Collocation Method

In practical applications, this kind of spectral approach is used most frequently. Collocation method requires that the given equation satisfies at the nodes points and test function is equal to Dirac delta function at special points called collocation points. Roots of basis functions and Gaussian quadrature points can also be used as collocation points. In this method, if we set the residual function zero at the collocation points then we can find the value of unknown coefficients of the interpolating series. Spectral collocation approach is also known as pseudospectral technique. This method works best for the non-linear problems or with complicated coefficients. For solving DEs, this method was first used by Frazer et al. [17] in 1937. For more detail about this method see ([8], [9], [18] and [20]).

1.4.4 Galerkin Method

This approach is usually credited to Boris Galerkin, but Finlayson [14] and Collatz [37] studied this method in more detailed. The benefits of Galerkin approach are more efficient research and optimal error calculations. There are no specific collocation or mesh points in Galerkin method. In the Galerkin method, test function is equal to trial function, basis functions are smooth functions and they have derivative of all order and satisfy the boundary conditions. The difficulties appear whenever complicated boundary conditions have to be enforced into the trial functions. For more detail about this method see references ([16],[11], [12], [13], [15], [20], [21]).

To see detailed comparison of the three methods mentioned above, see reference [8].

1.5 Difference between spectral method and finite element method

- Spectral methods use basis functions that are defined globally, while in finite element methods basis functions are defined locally.

- Spectral methods are faster converging as compared to finite element method or finite difference methods.
- Spectral methods are more accurate than finite element method.

1.6 Background of Blasius Problem

In applied mathematics, one of the most illustrative and calebrated a non-linear ODE of third order subject to boundary conditions is Blasius problem. In most undergraduate fluid mechanics books, Blasius problem is found that represents laminar viscous flow and steady flow over a semi-infinite plate. A Blasius function denoted by f is a simple solution of third order non-linear ODE i.e. Blasius problem, named after the German fluid dynamicist Paul Richard Heinrich Blasius. He showed that by introducing a stream function and by using similarity transformation with suitable boundary conditions two partial differential equations, namely Navier-Stokes equations can be converted into a single third order ordinary differential equation [28].

Due to absence of second order boundary condition, solution of the Blasius problem is difficult to find analytically. To handle this problem, a wide class of numerical and analytical solution methods was used. To get an approximation of the problem, in 1908, Blasius found power series solution and combined this solution with asymptotic expansions at finite X [10]. In this way numerical solution of Blasius problem can also be adopted. Balsius problem is now more than a century old and it is foundational to study the behavior of fluid. Blasius flow have thin boundary layers similar as the flows past a solid body like ocean currents streaming past an undersea mountain, air rushing past an airplane and even the breath and blood flowing through our bodies.

An adomian decomposition method (ADM) was applied by Wang [41] for approximate solution of classical Blasius problem. Abussita [2] studied the Blasius solution for flow past a flat plate.

For an approximate analytical solution of Blasius problem, Liao [25] applied homotopy analysis method and to find the numerical solution. Abbasbandy [1] used an improved form of ADM. In 1940, Crocco made a transformation which is later used by Wang [9] to tackle Blasius problem from different point of view. In 2009, Parand et al. [29] used Sinc-Collocation method and obtained comparatively accurate solution of the problem. In 2010, an analytical solution of Blasius problem in terms of algorithm of geometric functions was found by Beong in Yun [43]. In 2015, S Ghorbani et al. [19] merged the best approximation theorem and Green's function method for an analytical solution of the Blasius problem.

In fluid mechanics, this problem is recognized as the mother of all boundary layer equations. For large of fluid mechanical situations various but interrelated equations have been developed like Falkner-Skan equation [5]. By using finite diference method, the solution of Falkner-Skan equation is presented by Asaithambi [4]. Lock [26] studied the distribution of velocity for laminar boundary layer equation even during its motion, lower stream was at rest. Later, the flow of two fluids of various densities and viscosities were investigated by Potter [33]. Method of Weyl [42] in [2], was successfully addressed the solution for this model.

CHAPTER 2

Solution of Nonlinear Ordinary Differential Equations

2.1 MATLAB

MATLAB is a software focused primarily upon numerical computing and has wide range of packages and toolboxes in order to enhance its capabilities. Similar to other programming languages, it also contains a huge bank of built in functions for user's convenience. It has a lot of functions for solving ordinary and partial differential equations. There are almost seven differential solvers with each one having its own significance. Out of many, one such solver is bvp4c. The details of the method and how to implement it is given in the subsequent sections.

2.1.1 bvp4c

To solve BVPs in MATLAB, bvp4c is one of the most important built-in tool. It is based on a finite difference method that is applied on the three-stage Lobatto IIIa formula, which is a collocation formula. The polynomial collocation provides a C^1- continuous solution that is valid in fourth order. Mesh selection and error correction are based on the continuous solution residual. BVPs for ODEs in MATLAB using by bvp4c is discussed by Shampine [34].

Let us consider an example to see how to apply MATLAB built-in function bvp4c on a BVP for an ODE.

2.1.2 Example

Consider the following BVP,

$$y^{''} + 2y + 1 = 0,$$

$$y(0) = 0, \qquad y(\pi) = 1.$$

The exact solution of the problem is

$$y(x) = \frac{1}{2} cos\sqrt{2}x + \frac{1}{2}\left[\frac{3 - cos\sqrt{2}\pi}{sin\sqrt{2}\pi}\right] sin\sqrt{2}x - \frac{1}{2}.$$

To find numerical solution we use bvp4c. First we rewrite the problem as a system of two first-order ODEs

$$y_2 = y_1{}^{'},$$

and

$$y_2{}^{'} = y_1{}^{''} = -2y_1 - 1,$$

where $y = y_1$ and $y^{'} = y_2$.

The function f, and the boundary conditions, bc, are coded in MATLAB as the function $twoode$ and $twobc$.

$$function\ dydx = twoode(x, y)$$

$$dydx = [y(2); -2y(1) - 1)];$$

$$function\ res = twobc(ya, yb)$$

$$res = [ya(1); yb(1) - 1];$$

Taking the guess vector as

$$y_1(x) = 1,$$

$$y_2(x) = 0.$$

$$solinit = bvpinit(linspace(0, pi, 140), [1, 0]);$$

Solution of the problem using bvp4c is obtained as

$$sol = bvp4c(@twoode, @twobc, solinit);$$

After evaluating the numerical solution on given domain, we have plotted the result in Fig. (2.1).

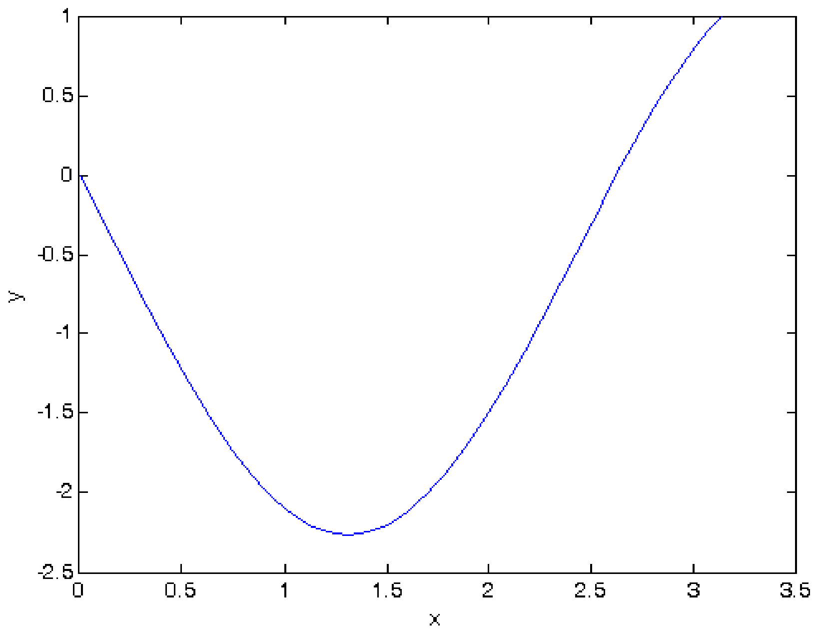


Figure 2.1: Numerical solution of $y(x)$

The graph of the exact solution of the problem is shown in Fig. (2.2).

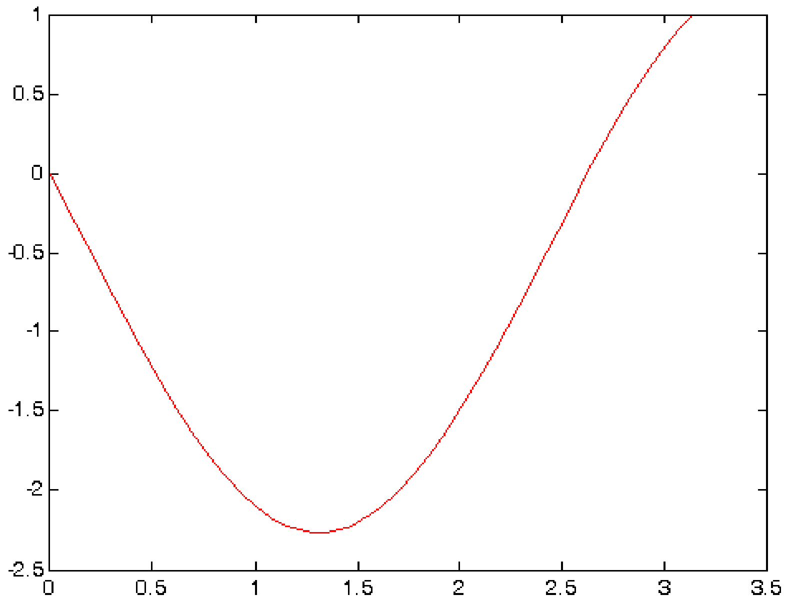


Figure 2.2: Exact solution of $y(x)$

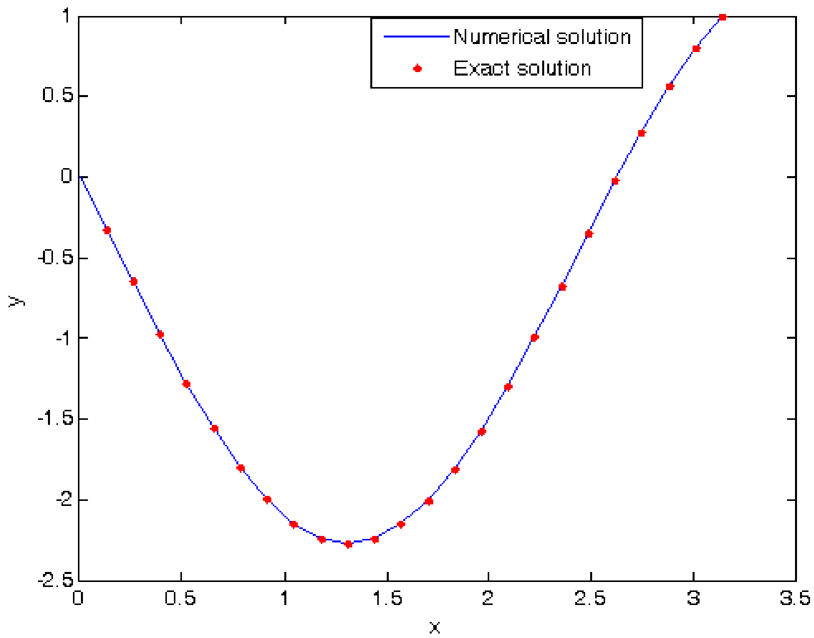


Figure 2.3: Combined graph of exact and numerical solutions

In Fig. (2.3) solid line shows the numerical whereas dotted line shows the exact solution and it is evident from the figure that the exact and numerical solutions are in close agreement.

2.2 Spectral Method

Consider the following non-linear BVP,

$$y^{''} + 2yy^{'} = 0, \qquad y(0) = 0, \qquad y(1) = \frac{1}{2}. \tag{2.1}$$

We find the approximate solution of this problem by using spectral method. For this, let

$$f(x) = y_N(x) = \sum_{i=0}^{N} a_i \psi_i(x), \tag{2.2}$$

be its approximate form, where a_i are unknown coefficients and $\psi_i(x)$ is a set of basis functions. Consider the basis functions $\psi_i(x)$ to be first kind Chebyshev polynomials. Since Chebyshev polynomials are defined on the interval $[-1, 1]$, and our problem is defined over the interval $[0, 1]$. So we define a transformation, which transform our interval from $[0, 1]$ to $[-1, 1]$. For this purpose we are using the transformation

$$T_i(x) = T_i(2x - 1). \tag{2.3}$$

Eq. (2.2) becomes

$$f(x) = y_N(x) = \sum_{i=0}^{N} a_i T_i(x). \tag{2.4}$$

Choose $N = 3$, to have

$$f(x) = y_3(x) = \sum_{i=0}^{3} a_i T_i(x), \tag{2.5}$$

and the boundary conditions lead to

$$y_3(0) = 0,$$

$$y_3(1) = \frac{1}{2}.$$

Unknown coefficients can be found by using roots $x_i's$ of the Chebyshev polynomial as

$$x_1 = 0, x_2 = 0.188255, x_3 = 0.61126, x_4 = 0.950484.$$

By setting the residual to be zero at the points x_2 and x_3 two more equations are obtained. We find the values of unknown co-efficient's a_0, a_1, a_2, a_3 after solving the system of equations as

$$a_0 = 0.70921306, \qquad a_1 = -0.24076791, \qquad a_2 = 0.04078693, \qquad a_3 = -0.00923208.$$

Putting these values in Eq. (2.5) and simplifying we obtain the following expression as an approximate solution.

$$f(x) = 0.9999999999999999 - (0.9740088157453604)x + (0.7694354122570408)x^2 \\ -(0.295422659651168046)x^3,$$

which is shown in Fig. (2.4).

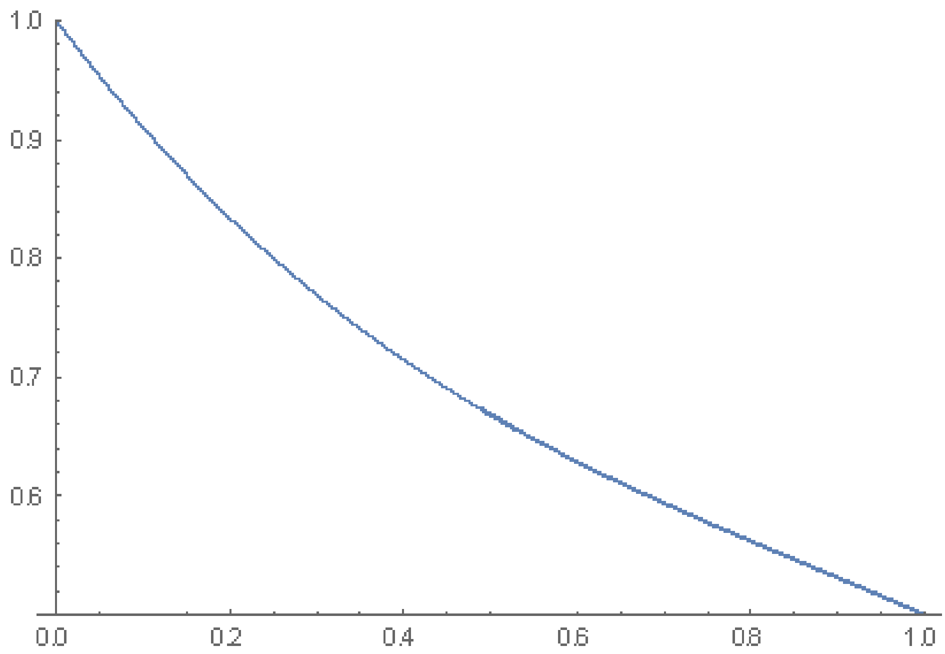


Figure 2.4: Approximation of $f(x)$, for $N = 3$ using Chebyshev Spectral Method

By increasing N, we actually increase the number of terms of our series solution, which in turn causes the error to decrease. So, we have the freedom to choose such a value for N, which satisfies our tolerance criterion as well as computational need.

2.3 Blasius Problem

Let us consider a uniform flow over a flat semi-infinite plate. In this situation, equations of flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial y^2}, \tag{2.7}$$

where u and v are the x and y components of the velocities. Boundary conditions are

$$u(x,0) = 0, \qquad v(x,0) = 0, \tag{2.8}$$

$$y \to \infty \quad \Rightarrow \quad u(x,y) \to U, \tag{2.9}$$

where U is the constant speed of fluid outside the boundary layer. Let $\psi(x,y)$ is some stream function, then

$$u = \frac{\partial \psi}{\partial y}, \tag{2.10}$$

and

$$v = -\frac{\partial \psi}{\partial x}. \tag{2.11}$$

Taking partial derivative of Eq. (2.10) w.r.t. "x" and Eq. (2.11) w.r.t. "y", we have

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y}, \tag{2.12}$$

$$\frac{\partial v}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y}, \tag{2.13}$$

using Eqs. (2.12) and (2.13) in Eqs. (2.6) and (2.7) respectively, we get

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0,$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = v\frac{\partial^3 \psi}{\partial y^3}. \tag{2.14}$$

To transform Eq. (2.14) using the following similarity transformations,

$$\eta = a\frac{y}{\sqrt{x}}, \tag{2.15}$$

$$\psi(x,y) = b\sqrt{x}f(\eta), \tag{2.16}$$

where a and b are constants chosen, such that $f(\eta)$ is dimensionless, as

$$a = \sqrt{\frac{U}{\nu}}, b = \sqrt{\nu U}, \tag{2.17}$$

where ν is kinematic viscosity of the fluid and η is dimensionless similarity variable. After differentiation of Eq. (2.16), we get

$$\frac{\partial\psi}{\partial x} = \frac{1}{2}\sqrt{\nu U}\frac{f(\eta)}{\sqrt{x}} - \frac{U}{2}\frac{y}{x}f^{'}(\eta), \tag{2.18}$$

and now differentiate Eq. (2.16), to have

$$\frac{\partial\psi}{\partial y} = Uf^{'}(\eta). \tag{2.19}$$

Differentiate Eq. (2.19) with respect to x and y, we get

$$\frac{\partial^2\psi}{\partial x\partial y} = -\frac{U}{2x}\eta f^{''}(\eta), \tag{2.20}$$

$$\frac{\partial^2\psi}{\partial y^2} = Uf^{''}(\eta)\frac{a}{\sqrt{x}}. \tag{2.21}$$

By taking derivative of Eq. (2.21), we get

$$\frac{\partial^3\psi}{\partial y^3} = \frac{U^2}{\nu x} f^{'''}(\eta). \tag{2.22}$$

Substituting above derivatives from Eqs. (2.18)-(2.22) in Eq. (2.14), we get

$$-\frac{U}{2x}\frac{ay}{\sqrt{x}} f^{'}(\eta) f^{''}(\eta) + \frac{ay}{\sqrt{x}}\frac{U}{2x} f^{'}(\eta) f^{''}(\eta) - \frac{a\sqrt{\nu U}}{2x} f(\eta) f^{''}(\eta) = \frac{U}{x} f'''(\eta),$$

or

$$f^{'''}(\eta) + \frac{1}{2} f(\eta) f^{''}(\eta) = 0. \tag{2.23}$$

Eq. (2.23) is called the Blasius equation. Also, transform boundary conditions Eqs. (2.8) and (2.9) are

$$\eta(x,0) = 0,$$

$$u(x,0) = U f^{'}(\eta(x,0)),$$

$$0 = U f^{'}(0) \Rightarrow f^{'}(0) = 0. \tag{2.24}$$

Now

$$v(x,0) = -\frac{1}{2}\sqrt{\nu U}\frac{f(\eta)}{\sqrt{x}} + \frac{U}{2}\frac{y}{x} f^{'}(\eta),$$

$$0 = -\frac{1}{2}\sqrt{\nu U}\frac{f(0)}{\sqrt{x}} + \frac{U}{2}\frac{y}{x} f^{'}(0),$$

which implies

$$f(0) = 0. \tag{2.25}$$

Similarly $y \to \infty \Rightarrow \eta \to \infty$, which gives

$$f^{'}(\infty) = 1, \tag{2.26}$$

as $\eta \to \infty$. By using similarity transformation method and with the help of stream function, PDE of required function are transformed into third order non-linear ODE.

CHAPTER 3

Approximate Solution of the Blasius Problem using Chebyshev Spectral Method

3.1 Solution of Blasius problem

In 1912 Topfer was the first who provided a numerical solution to the Blasius problem [40]. To solve this problem, he used similarity reduction and symmetry principles with Range-Kutta (RK) method for integrating the ODE. R. Cortell [10] presented a Range-Kutta algorithm for high order IVP to obtain numerical solution of the Blasius flat-plate problem. In the solution of the Blasius problem $\alpha = f''(0)$, plays a crucial role. Howarth [23] has developed a solution of the Blasius problem which is considered as a highly accurate numerical solution and obtained $\alpha = f''(0) = 0.332057$. Abbasbandy [1] used ADM method to obtain $\alpha = f''(0) = 0.333729$ with 0.383% error, also Tajvidi et al. [39] calculated $\alpha = f''(0) = 0.333329$ with 0.009% error.

The standard form of this problem is given in Eqs. (2.23)-(2.26). So here first we solve the problem with the bvp4c function in MATLAB and is shown in Fig. (3.1).

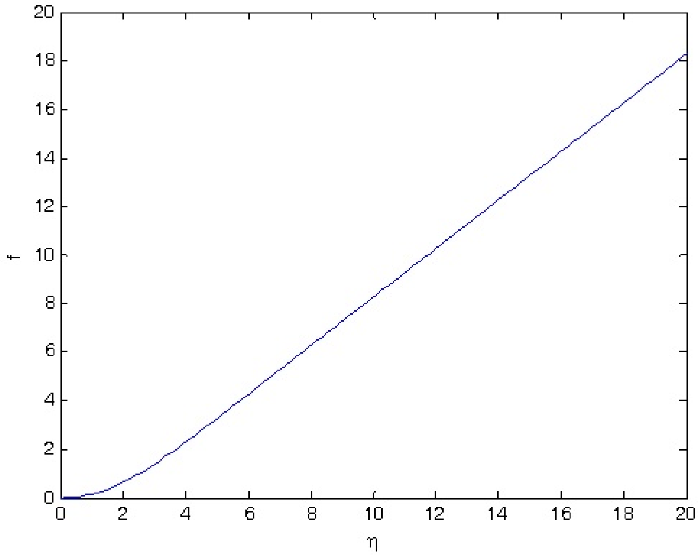


Figure 3.1: Solution of Blasius flat-plate problem using bvp4c

3.1.1 Solving problems in semi-infinite domain

Spectral methods have been successfully applied and problems were defined in unbounded domain. There are different methods in which collocation method is based on the nodes of Gauss formulas related to semi-infinite intervals [22]. In above approach computations involve orthogonal polynomials, like as Laguerre polynomials. However, in semi-infinite and finite domain there is a deficiency in numerical techniques for the solution of PDEs. To solve problems in semi-infinite domain, numerous spectral methods are used.

1. First technique to solve the problem on unbounded domain is by using Laguerre polynomials see [27], [35] and [36].

2. The second technique is to reformulate the original problem of semi-infinite domain to singular in the limited domain by variable change, and then to estimate the resulting unique problem using Jacobi polynomial.

3. Rational orthogonal functions are based on the third approach. By mapping Chebyshev

polynomials as called rational Chebyshev functions in the semi-infinite interval, Boyd [6] developed a new spectral basis.

4. Fourth technique is called domain truncation in which semi-infinite domain is replace with $[0, L]$ interval, by selecting L sufficiently large [6].

Writers [32], [30] and [31] applied a spectral method that based on a rational Tau method for the solution of non-linear ODEs. They used the Tau method with the sum of operational derivative matrices, Legendre and logical Chebyshev in order to reduce the non-linear ODEs solution to the algebraic equations solution.

Boyd, et al. [7] used collocation techniques at an unbounded interval and contrasted the rational Chebyshev, Laguerre and Fourier sine. Guo, et al. [22] define a new set of rational Legendre functions for the Korteweg-de Vries equation.

To get unknown coefficients in spectral methods there are some properties which are very helpful. Proofs are maybe found in [3].

Theorem: The ith degree polynomial ψ_i of an orthogonal set has i real distinct zeros, all of which lie in the interval (a,b).

Theorem: The polynomial of an orthogonal set satisfy a recurrence relation of the form

$$x\psi_i(x) = A_i\psi_i(x) + B_i\psi_i(x) + C_i\psi_i(x), i \geq 1,$$

where $A_i,\ B_i$ and C_i are constants that may depend on i.

3.1.2 Approximation of Blasius problem

As we studied in Chapter 1 that spectral methods are suitable for non-linear problems with complicated coefficients. Since Blasius problem is a non-linear ODE on a semi-infinite interval and we cannot solve it exactly, so we use Spectral method on the Blasius flat-plate problem to obtain its approximate solution. Here we are using its standard form

$$f^{'''}(\eta) + \frac{1}{2}f(\eta)f^{''}(\eta) = 0, \tag{3.1}$$

$$f(0) = 0, \quad f^{'}(0) = 0, \quad f^{'}(\infty) = 1. \tag{3.2}$$

First, we write f in its approximate form as

$$f_N(\eta) = \sum_{j=0}^{N} a_j \phi_j(\eta). \tag{3.3}$$

It is an approximate solution of the Blasius flat-plate problem, where a_j are unknown coefficients and $\phi_j(\eta)$ is a set of trial functions. The residual function $R_N(\eta)$ of Eq. (3.1) can be written as

$$R_N(\eta) = f_N^{'''}(\eta) + \frac{1}{2}f_N(\eta)f_N^{''}(\eta). \tag{3.4}$$

We need to solve system of $(N+1)$ equations to find the value of $(N+1)$ unknown coefficients in Eq. (3.3). By applying boundary conditions on Eq. (3.3), we find the subsequent equations

$$\sum_{j=0}^{N} {a_j}^N \phi_j(0) = 0, \tag{3.5}$$

$$\sum_{j=0}^{N} {a_j}^N {\phi_j}^{'}(0) = 0, \tag{3.6}$$

$$\sum_{j=0}^{N} {a_j}^N {\phi_j}^{'}(\infty) = 1. \tag{3.7}$$

Now we set residual to zero at N from $(N+1)$ nodes $\eta_j,\ j = 1, 2, 3, ..., N+1$. By using the Gauss-Radau formula, we can find these nodes. In this method, we use zeroes of polynomial in Eq. (3.4) as nodes.

$$R_N(\eta_j) = 0, j = 1, 2, 3, ..., N. \tag{3.8}$$

3.1.3 Chebyshev Spectral Method

To find a solution of this problem by spectral method, we use trial function $\phi_j(\eta)$ as Chebyshev polynomials in Eq. (3.3). First kind of Chebyshev polynomials $T_n(x)$, are solution of the Chebyshev differential equation

$$(1-x^2)T_n{}''(x) - xT_n{}''(x) + n^2T_n(x) = 0, \tag{3.9}$$

where

$$T_n(x) = \sum_{k=0}^{\frac{n}{2}} \frac{n!x^{n-2k}(x^2-1)^k}{(2k)!(n-2k)!}, n \geq 0.$$

Since our problem is defined over the semi-infinite interval $[0, \infty)$ and Chebyshev polynomials are defined on $[-1, 1]$. To solve this problem using Chebyshev spectral method first we use the approach domain truncation which changes our interval from semi-infinite interval to $[0, a]$, and then we define a linear transformation which transforms our interval from $[0, a]$ to $[-1, 1]$. For this purpose, we use small transformation

$$T_n(\eta) = T_n\left(\frac{\eta - 10}{10}\right). \tag{3.10}$$

In the next step we use this transformed Chebyshev polynomials $T_j(\eta)$ in Eq. (3.3) as a basis function. Thus we obtain

$$f_N(\eta) = \sum_{j=0}^{N} a_j T_j(\eta). \tag{3.11}$$

The residual function $R_N(\eta)$ given in Eq. (3.4) can be minimized easily if we increase the number of terms N.

In the next step we find the unknown coefficients a_i using the N zeros of transformed Chebyshev polynomials $T_j(\eta)$. We can also use zeros of $T_j(\eta) + T_{j+1}(\eta)$, which are also called Gauss-Radau nodes. To find these nodes we use Mathematica.

For $N = 5$, we get the following expression as approximate solution of the Blasius problem

$$f_5(\eta) = 0 + 1.110223024625156 * 10^{-16}\eta + 0.166025\eta^2 - 0.01287136551922731\eta^3$$
$$+0.000490128255821227\eta^4 - 0.00000734933195400 8114\eta^5,$$

that is shown in Fig. (3.2)

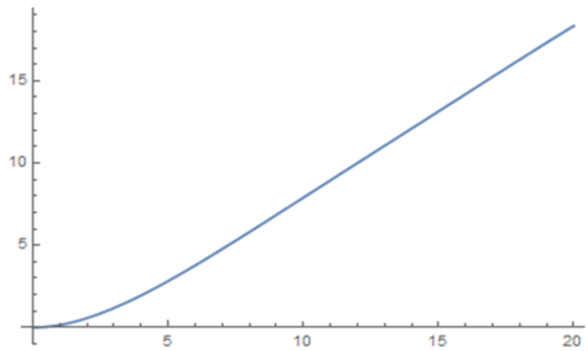


Figure 3.2: Approximate solution of the Blasius Problem

By using the following method, here are some approximate solutions of this problem for

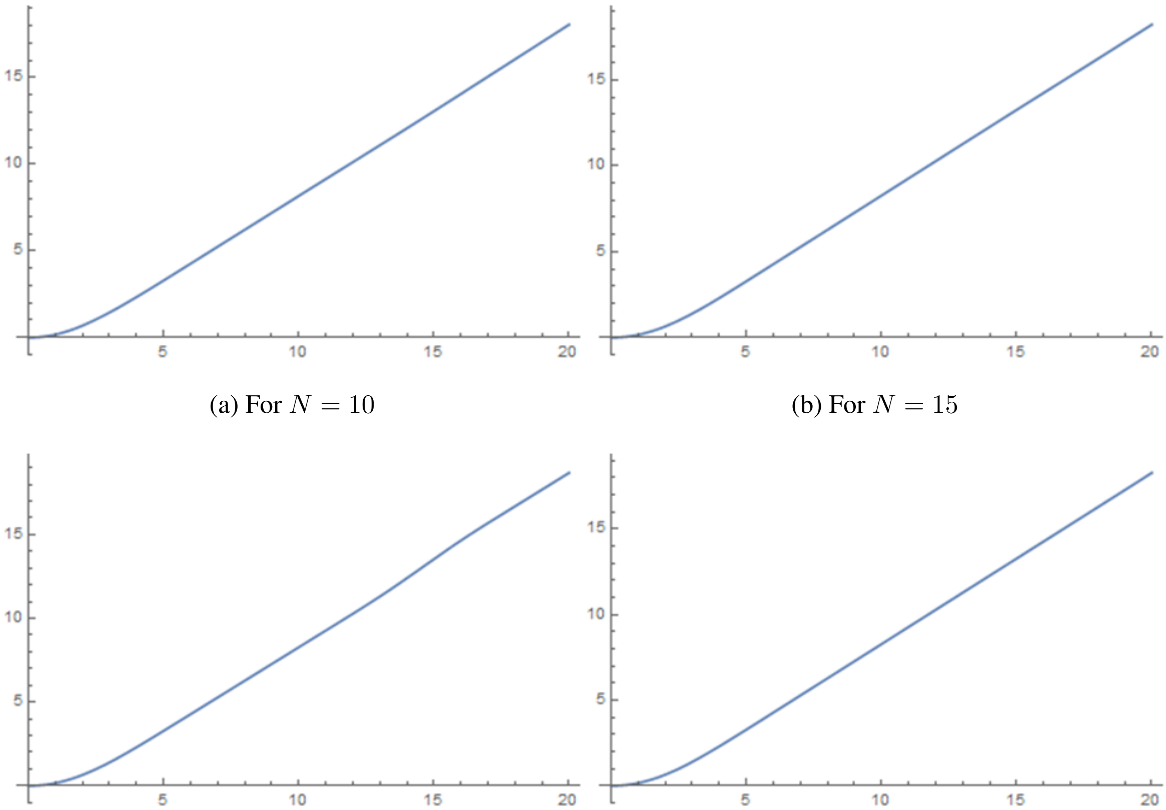
$N = 10$ and 15.

$$f_{10}(\eta) = -3.5527136788005 * 10^{-15} + 1.110223024625156 * 10^{-16}\eta$$
$$+0.16602499999999998\eta^2 + 0.02248356428539726\eta^3$$
$$-0.013755143474056852\eta^4 + 0.0025394160109805408\eta^5$$
$$-0.00025897875142205494\eta^6 + 0.000016014108458038347\eta^7$$
$$-5.976317766601015 * 10^{-7}\eta^8 + 1.23977606557046 * 10^{-8}\eta^9$$
$$-1.098852966268973 * 10^{-10}\eta^{10},$$

$$f_{15}(\eta) = 1.458535678030333 * 10^{-16} - 3.313815235807477 * 10^{-17}\eta$$
$$+0.1660249999999996\eta^2 + 0.00021709678567897805\eta^3$$
$$-0.0012053172264257644\eta^4 + 0.001221553416182771\eta^5$$
$$-0.0010785875037519796\eta^6 + 0.00037617151908165956\eta^7$$
$$-0.00007312200624057126\eta^8 + 0.000009076612541672305\eta^9$$
$$-7.612920082640843 * 10^{-7}\eta^{10} + 4.391259250999106 * 10^{-8}\eta^{11}$$
$$-1.723625258282282 * 10^{-9}\eta^{12} + 4.409833684404594 * 10^{-11}\eta^{13}$$
$$-6.6432559128151677 * 10^{-13}\eta^{14} + 4.475636356067715 * 10^{-15}\eta^{15}.$$

$f_N(\eta)$ for different values of N are shown in Fig. (3.3).

3.2 Graphs of $f_N(\eta)$ for different values of N



(a) For $N = 10$

(b) For $N = 15$

(c) For $N = 20$

(d) For $N = 25$

Figure 3.3: Graphs of $f_N(\eta)$ for different values of N

Table 3.1: Approximate solution $f_N(\eta)$ evaluated at different points by adding more terms N.

η	$N = 10$	$N = 15$	$N = 20$	$N = 25$
0.5	0.043532	0.041482	0.041492	0.041492
1.0	0.177049	0.165491	0.165568	0.165568
2.0	0.690476	0.64999	0.650014	0.650011
3.0	1.44674	1.39657	1.39677	1.39678
4.0	2.33996	2.30494	2.30572	2.30572
5.0	3.29403	3.28052	3.28321	3.28322
6.0	4.26770	4.26931	4.27951	4.27955
7.0	5.24469	5.25482	5.27926	5.27915

Table 3.2: Approximation of $f_N(\eta)$ for Chebyshev spectral method and bvp4c.

η	Chebyshev Spectral Method	bvp4c
0	0	0
1.0072	0.1679	0.1680
2.0144	0.6591	0.6591
3.0216	1.4150	1.4151
4.0288	2.3332	2.3333
5.0360	3.3189	3.3190
6.0432	4.3227	4.3227
7.0504	5.3295	5.3296
8.0576	6.3367	6.3368
9.0647	7.3438	7.3440
10.0719	8.3511	8.3512
11.0791	9.3582	9.3584
12.0863	10.3654	10.3655
13.0935	11.3725	11.3727
14.1007	12.3797	12.3799
15.1079	13.3869	13.3871
16.1151	14.3911	14.3943
17.1223	15.4013	15.4015
18.1295	16.4088	16.4089

CHAPTER 4

Conclusion

In this thesis, spectral method has been employed to find an approximate solution of the nonlinear classical Blasius flat plate problem. To achieve this goal, the pseudospectral method with Chebyshev polynomials as a basis functions were used. As the Chebyshev polynomials are defined on the interval $[-1, 1]$, this interval can be transformed according to our problem. To get unknown coefficients, roots of these polynomials are used. Also, we find that spectral method solution can be made as accurate as desired by adding more terms.

On contrary, bvp4c was also used to find out the required problem's numerical solution. A comparison of the results is shown in Table (3.2), that indicates a close agreement between the two results.

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