Properties Of Functions Related To Fejer Type Inequality

HIFSA ISMAIL
Regn No. 00000278476

A thesis submitted in partial fulfillment of the requirements
for the degree of **Master of Science**
in
Mathematics

Supervised by:Dr.Matloob Anwar

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan

2020

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Ms. Hifsa Ismail, Regn No. 00000278476 Titled: Properties of Functions Related to Fejer Type Inequality be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. MUJEEB UR REHMAN Signature: ____________

2. Name: DR. MUHAMMAD QASIM Signature: ____________

External Examiner: DR. NASIR REHMAN Signature: ____________

Supervisor's Name DR. MATLOOB ANWAR Signature: ____________

____
Head of Department

28/09/2020
Date

COUNTERSINGED

Date: 28-09-2020.

____
Dean/Principal

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Hifsa Ismail** (Registration No. **00000278476)**, of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ______________________

Name of Supervisor: Dr. Matloob Anwar

Date: 28/09/2020

Signature (HoD): ______________________

Date: 28/09/2020

Signature (Dean/Principal): ______________________

Date: 28-09-2020

Acknowledgements

Appreciations and admirations belong to one, who created the human being with head and heart, who gave me the potential to complete my work successfully.
Firstly, i would like to show my genuine gratefulness to my supervisor **Dr.Matloob Anwar**, without whose instructions, encouragement and supervision, the research work explored in this dissertation would not be possible.

This research would not have been possible without the research oriented atmosphere to student by rector, NUST.It is his positive attitude that permits such excellent research work from staff and students.
In last, thanks to my parents and classmates for prayers and motivation.

Abstract

The basic aim of this thesis is to consider the Fejer's inequality which is closely related to Hadamard's inequality, when we consider the weight function with it. We try to collect many known theorems from the literature related to Hadamard's inequality and present their different results in terms of Fejer's inequality. Examples are also considered regarding to this inequality. Furthermore, there is an emphasis on Schur-convexity and Schur concavity, playing the role in Fejer's inequality.

Contents

Chapter 1

Introduction

This Chapter covers an introduction, background of the "Hadamard Inequality", "Fejer's Inequality", convex and concave functions.

From the dawn of Newton and Euler ideas regarding mathematical inequalities to the modern day applications; mathematical inequalities have played a vital role in the progression of numerous branches of Mathematics. In the 19th century, there is a considerable role of mathematical inequalities by many Mathematicians and it became a dominant field in research areas . Consequently, generalizing the mathematical inequalities in a more advanced way.

As a result of applications of convex functions, it has been shown that the theory of mathematical inequalities and convex functions are closely linked together.

In Mathematics, inequalities can be defined as the distinction of two qualities relatively reciprocating two different objects. In easy language, the two qualities which are not equal refer as an inequality. With the emergence of Calculus and in the 19th century, the touch of inequalities and its role increasingly became essential. Various areas of science such as physical and engineering sciences have several applications of inequalities.

The concept of convex functions has indeed a dominant role in Mathematics and it keeps much importance and it can be seen in many research articles and books related to this field now-a-days.

The convexity idea is straightforward and characteristic, and can be followed from the Archimedes regarding his famous estimation of π. This idea has immediate and roundabout effects in our regular day to day existence because of its various applications in art, business, medicines, industry etc. An essential part of general theory of convexity refers to the theory of convex function and convex function can be defined as one whose super graph is a convex set.

This theory is considered as an essential part because it contracts much every part of Mathematics, likely out of the blue we experience with this theory in graphical analysis in which we learn the second derivative test in recognizing convexity of a graph. Also in tracing minima and maxima of a function of several variable this their has tremendous part. Also in Mathematical programming, Optimization theory, Engineering, the convexity can be observed.

A great research regarding this field has done by J.L.W.V Jensen. Also in 20th century enormous research was done by Hardy, littlewood and Poyla on publishing first book in inequalities.

In the second half of 20th century a number of generalization of convex function have been made in Mathematics and also in Professional discipline such as engineering and economics.

From the last few decades, the most considerable inequality that has charmed many inequality's experts is "Hermite Hadamard" inequality. 'Jacques Hadamard'(1865-1963) was the first person, whose result has introduced "Hermite Hadamard" inequality in the literature, this result was actually due to 'Charles Hermite'(1882-1901) as pointed out by 'Mitrinouic' and 'Lackovic' in 1985.

So from these facts, most mathematicians refer to it as "Hermite-Hadamard"(or sometimes "Hadamard-Hermite")inequality. The foremost and important subject of Mathematical Analysis is the" Hadamard inequality" with numerous remarkable applications in the field of mathematical sciences.

A number of mathematicians have done their tremendous efforts for its generalization, refinement and for its extension for different classes of functions.
Let $\theta : I \subset R \to R$ be a convex function.Then we have,

$$\theta\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d \theta(x)dx \le \frac{\theta(c)+\theta(d)}{2}, \quad c,d \in I \ with \ c < d. \tag{1.1}$$

This double inequality (1.1) is said to be the first significant result for a convex function with a natural geometric values and different applications. If in the "Hermite-Hadamard" inequality, both inequalities appear in the opposite direction then the case is considered as concave. After its discovery in 1881, there are number of different texts and research papers on it, providing new results, a number of generalization and many more.
Various generalization of "Hermite Hadamard" inequality have been studied such as Jensen's inequality and Fejer's inequality.
Basically the "Fejer's Inequality" is generalization of "Hermite-Hadamard" inequality, when we consider the weight function with the "Hermite-Hadamard" inequality and this weight function is non-negative, integrable and symmetric on $\Psi(x) = \Psi(c+d-x)$ and $\theta : I \subset R \to R$ be a convex function.
It can be written as

$$\theta\left(\frac{c+d}{2}\right) \le \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} \le \frac{\theta(c)+\theta(d)}{2}, \quad c,d \in I \ with \ c < d. \tag{1.2}$$

Likewise in the "Hermite Hadamard" inequality, if both inequalities in (1.2) appear in the opposite direction, then the case is considered as concave.
The integral mean of a convex function connected with the most important inequality are "Hermite-Hadamard" inequalities and its weighted version, known as "Hermite Hadamard Fejer's inequality"

The thesis is divided into the following chapters:

- Chapter 1 covers the background of "Hermite Hadamard" inequality and "Fejer's inequality"

- Chapter 2 covers the basic concepts regarding the convexity, majorization and Schur convexity and Schur concavity.

- Chapter 3 mainly focus on literature review of "Hermite-Hadamard" inequality, and many results are provided here regarding this inequality.

- Chapter 4 is basically the core section of the thesis. In this chapter all results are provided in terms of "Fejers's inequality". This chapter covers results regarding Schur-m convexity and Schur m-concavity as well as with applications of "Fejer's inequality". This chapter highlights our own findings and formulation of results.

- Chapter 5 is the conclusion of thesis.

Chapter 2

Some important definitions and the literature review is presented here.

2.1 Preliminary

In this Chapter, some necessary ideas and concepts are discussed that reader should familiar with. It mainly include some preliminary definitions of convex function and concave functions and their generalization. Also there are some theorems based on these generalizations.

2.1.1 Convex and Concave function

[30] In many areas of mathematics, convex functions keep major importance. While considering the study of optimization problems, convex functions plays a vital role because of many practical applications(designing circuits, modeling, operation research etc). Convex functions also facilitate one for solving inequalities with easy approach.

Definition 2.1.1. *A set $\kappa \subset R^n$ is said to be convex, if for every pairs of points $v_1, v_2 \in \kappa$ and the segment with v_1 and v_2 end points lies entirely inside κ, otherwise called not convex.*

Definition 2.1.2. *Mathematically, A set $\kappa \subset R^n$ is said to be convex, $\forall\ v_1, v_2 \in \kappa$ we have,*

$$\alpha v_1 + (1-\alpha)v_2 \in \kappa, \quad \forall\ \alpha \in [0,1]. \tag{2.1}$$

Example 1. *(1) Empty set and singleton sets are convex.*
(2) R^n is convex.

We want to characterize convex set in terms of convex combination. For this we need to define convex combination. A point $v = \alpha v_1 + (1-\alpha)v_2$ is called convex combination of v_1 and v_2. The set of all convex combination of v_1 and v_2 is called as convex hull written as:

$$Conv[v_1, v_2] = [\alpha_1 v_1 + \alpha_2 v_2; \alpha_1 + \alpha_2 = 1]$$

with

$$\alpha_1 = \frac{v - v_1}{v_2 - v_1} \text{ and } \alpha_2 = \frac{v_2 - v}{v_2 - v_1}, \text{ for } v \in [v_1, v_2] \text{ and } v_1 \neq v_2.$$

Definition 2.1.3. *A convex combination of finitely many point $v_i \in R$ with $i = 1, 2, ..., k$ is the point v of the form,*

$$v = v_1\lambda_1 + v_2\lambda_2 ... v_k\lambda_k \text{ with } v\lambda_1 + \lambda_2, ..., \lambda_k = 1, \ \lambda \geq 0. \tag{2.2}$$

We are also interested in convex function, therefore definition is given as

Definition 2.1.4. *Let $\kappa \subset R^n$ be such that κ is convex. A function $\Phi : \kappa \to R$ is said to be convex, if for all $c_1, c_2 \in \kappa$ and $\lambda \in [0, 1]$, we have*

$$\Phi(\lambda c_1 + (1-\lambda)c_2) \leq \lambda\Phi(c_1) + (1-\lambda)\Phi(c_2). \tag{2.3}$$

Φ is called strictly convex if the inequality is strict for $\lambda \in (0, 1)$ and $c_1 \neq c_2$.
For $\lambda = \frac{1}{2}$

$$\Phi\left(\frac{c_1 + c_2}{2}\right) \leq \frac{\Phi(c_1) + \Phi(c_2)}{2}, \forall c_1, c_2 \in \kappa, \tag{2.4}$$

which is called Jensen's convex function. A function Φ is concave if $-\Phi$ is convex and is strictly concave if $-\Phi$ is strictly convex.
Geometrically the definition of convex function is as:
If the chord connecting any pair of points in its graph rests on or above its points,

then the function Φ is said to be convex. Φ is referred as strictly convex if the chord lies above its graph. A concave function can be defined in similar way but in opposite direction.

A mathematical root of majorization is demonstrated after the effort of Schur on Hadamard determinant inequality. Majorization involves the solution of many Mathematical characterization problems.

2.1.2 Majorization

Definition 2.1.5. *If $c = (c_1, c_2, c_3, ..., c_n)$ and $d = (d_1, d_2, d_3, ..., d_n)$ are two nth-ordered real numbers. d is majorized by c(in symbolically $c \prec d$), if*
$\Sigma_{i=1}^{k} c_{[i]} \leq \Sigma_{i=1}^{k} d_{[i]}$, $(k = 1, 2, ..., n-1)$ *and*
$\Sigma_{i=1}^{k} c_{[i]} = \Sigma_{i=1}^{k} d_{[i]}$, $(k = 1, 2, ..., n)$ *where* $c_{[1]} \geq c_{[2]}, ..., \geq c_{[n]}$, $d_{[1]} \geq d_{[2]}, ..., \geq d_{[n]}$ *are readjustment of c and d in a descending order.*

Example 2. *Consider,* $(1, 2, 3) \prec (2, 4, 0)$.
First we will make the readjustment in descending order,
We have
$(3, 2, 1) \prec (4, 2, 0)$.
So, both conditions are satisfied here for majorization.

2.1.3 Schur-covexity and Schur-concavity

It is named after Issai Schur in 1923. Basically if we use the concept of majorization in the definitions of convex and concave functions refers to the result of Schur-convexity and Schur-concavity. In mathematics, Schur-convex is also known as S-function, isotonic function or order preserving function. The definition of Schur-convex can be written as a function which is convex and symmetric. Schur-convexity and Schur-concavity have multiple applications in inequalities, quantum physics, the theory of information and many other research fields. Now a days, in modern analysis Schur-convexity and Schur-concavity have a major role in research fields.

Definition 2.1.6. *[1] If* $c = (c_1, c_2, c_3, ..., c_n)$ *and* $d = (d_1, d_2, d_3, ..., d_n)$ *are two nth-ordered real numbers.*

A real valued function $\Phi : \Upsilon \subset R^n \to R$ *is said to be Schur convex on* Υ *if*

$c \prec d$ *on* $\Upsilon \Rightarrow \Phi(c) \leq \Phi(d)$.

Φ *is schur concave function on* Υ *if and only if* $-\Phi$ *is a Schur convex function.*

Example 3. *Consider,* $(1, 2, 3) \prec (2, 4, 0)$, *and the function we have ,*

$\Phi(x) = max(x)$.

Applying the definition, we have

$max(c) = 3$,

$max(d) = 4$.

So, it implies that $\Phi(c) \leq \Phi(d)$. *It holds for Schur-convex.*

Similarly, for Schur-concave,

$\Phi(x) = min(x)$.

Applying the definition, we have

$min(c) = 1$,

$min(d) = 0$.

So,it implies that $\Phi(c) \geq \Phi(d)$.

2.1.4 Schur-geometrically convex and Schur-geometrically concave

The Schur-geometrically convexity and concavity was put forward by Zhang(2004), and then investigated by Chu et al. (2008), Guan (2007), Sun et al. (2009), and so on. Some authors also use the term "Schur multiplicative convexity" for the Schur- geometrically convexity.

Definition 2.1.7. *[2] If* $c = (c_1, c_2, c_3, ..., c_n)$ *and* $d = (d_1, d_2, d_3, ..., d_n)$ *are two nth ordered real numbers.*

Let $\Upsilon \subset R^n_{++}(c^n, d^n \in R^n_{++},\ c > 0,\ d > 0)$. *A function* $\Phi : \Upsilon \to R_{++}$ *is called Schur geometrically convex if*

$lnc \prec lnd$ on $\Upsilon \Rightarrow \Phi(c) \leq \Phi(d)$.

Φ is Schur geometrically concave if $-\Phi$ is Schur geometrically convex.

In the theory of the Schur geometrically convex function, the below Lemma keeps an important place and considered as basic result.

Lemma 1. *[2] Let $\theta(c) = \theta(c_1, c_2, ...c_n)$ be symmetric and continuous on $\Upsilon \subset R^n_{++}$ and differentiable in Υ^0. Then $\theta : \Upsilon \to R_{++}$ is Schur geometrically convex (Schur geometrically concave) if and only if*

$$(lnc_1 - lnc_2)\left(c_1\frac{\partial\Upsilon}{\partial c_1} - c_2\frac{\partial\Upsilon}{\partial c_2}\right) \geq 0 \ (respectively \leq 0). \tag{2.5}$$

2.1.5 Schur-harmonically convex and Schur-harmonically concave

The notion of Schur-harmonically convex function and Schur-harmonically concave are introduced by Chu (Chu et al. (2011), Chu and Sun (2010), Chu and Lv (2009).

Definition 2.1.8. *[3, 4] Let $\Upsilon \in R^n$.*

(1) *A set Υ is called harmonically convex if $\frac{cd}{\lambda c+(1-\lambda)d} \in \Upsilon$ for every $c, d \in \Upsilon$ and $\lambda \in [0,1]$, where $cd = \Sigma_{i=1}^n c_i d_i$ and $\frac{1}{c} = \left(\frac{1}{c_1}, ..., \frac{1}{c_n}\right)$, $\frac{1}{d} = \left(\frac{1}{d_1}, ..., \frac{1}{d_n}\right)$.*

(2) *A function $\theta : \Upsilon \to R_{++}$ is called Schur harmonically convex on Υ if $\frac{1}{c} \prec \frac{1}{d}$ implies $\theta(c) \leq \theta(d)$. θ is Schur harmonically concave if $-\theta$ is Schur harmonically convex.*

Lemma 2. *[3, 4] Let $\Upsilon \in R^n$ be a symmetric and harmonically convex set with inner points and let $\theta : \Upsilon \to R_{++}$ be a continuously symmetric function which is differentiable in Υ^0, then θ is Schur harmonically convex(Schur harmonically concave) on Υ if and only if*

$$(c_1 - c_2)\left(c_1^2\frac{\partial\theta}{\partial c_1} - c_2^2\frac{\partial\theta}{\partial c_2}\right) \geq 0 \ (repectively \leq 0). \tag{2.6}$$

2.1.6 Schur f-convex and Schur f-concave

Basically the Schur f-convexity and Schur f-concavity are derived from the Schur convexity and Schur concavity. Yang [5] give the Schur f-convexity an schur f-concavity in which can be follows as:

Definition 2.1.9. *[5, 6] Let $\Upsilon \subset R^n$ be a set with non-empty interior and f be a strictly monotone function define on Υ. Let*

$$f(c) = (f(c_1), f(c_2), ...f(c_n)) \text{ and } f(d) = (f(d_1), f(d_2), ...f(d_n)).$$

Then the function $\theta : \Upsilon \rightarrow R$ is said to be Schur f-convex on Υ if $f(c) \prec f(d)$ on Υ implies $\theta(c) \leq \theta(d)$.

θ is said to be Schur f-concave if $-\theta$ is Schur f-convex.

2.1.7 Schur m- power convex and Schur m- power concave

[5] There are variety of inequalities originating from the Schur-convexity and Schur-concavity, same is the case with Schur m-power convex and Schur m-power concave. Yang was the first who introduced the Schur m-power convexity and Schur m-power concavity.

Definition 2.1.10. *[5, 6] Let $\Phi : R_{++} \rightarrow R$ be defined by $\frac{c^m-1}{m}$ if $m \neq 0$ and $\Phi(c) = lnc$ if $m = 0$. Then the function $\theta : \Upsilon \subset R^n \rightarrow R$ is said to be Schur m-power convex on θ if $\Phi(c) \prec \Phi(d)$ on θ implies $\theta(c) \leq \theta(d)$. θ is said to be Schur m-power concave if $-\theta$ is Schur m-power convex.*

Lemma 3. *[5, 6] Let $\chi : \Upsilon \subset R^n_{++} \rightarrow R$ be continuous on Υ and differentiable in Υ^0. Then χ is Schur m-power convex (Schur m-power concave) on Υ if and only if χ is symmetric on Υ and*

$$\frac{c_1^m - c_2^m}{m}\left(c_1^{1-m}\frac{\partial\theta}{\partial c_1} - c_2^{1-m}\frac{\partial\theta}{\partial c_2}\right) \geq 0 \ (respectively \ \leq 0), \ if \ m \neq 0, \tag{2.7}$$

$$(lnc_1 - lnc_2)\left(c_1\frac{\partial\theta}{\partial c_1} - c_2\frac{\partial\theta}{\partial c_2}\right) \geq 0 \ (respectively \leq 0),\ if\ m = 0, \tag{2.8}$$

hold for any $c = (c_1, c_2, ...c_n) \in \Upsilon^0$ *with* $c_1 \neq c_2$*, where* $\Upsilon \subset R$ *is a symmetric set with non-empty interior* Υ^0*.*

Lemma 4. *[7, 8] Then the two discrimination inequalities in Lemma 3 can be written as:*

$$(c_1 - c_2)\left(c_1^{1-m}\frac{\partial\theta}{\partial c_1} - c_2^{1-m}\frac{\partial\theta}{\partial c_2}\right) \geq 0 \ (respectively \leq 0). \tag{2.9}$$

Chapter 3

Hadamard's Inequality

3.0.1 Properties of functions related to Hadamard's type inequality

In this section, we will discuss some important results and some basic properties of "Hadamard's Inequality" and its applications.

Let $\theta : I \subset R \to R$ be a convex function. The famous Hadamard's inequality is given as

$$\theta\left(\frac{c+d}{2}\right) \le \frac{1}{d-c}\int_c^d \theta(x)dx \le \frac{\theta(c)+\theta(d)}{2}. \tag{3.1}$$

In [10], S.S. Dragomir has formed the theorem which is a refinement of the inequality (3.1).

Theorem 3.0.1. *Let $\theta \subset R_{++} \to R$ be a convex function, and*

$$P(t) = \frac{1}{2(d-c)}\int_c^d [\theta(tc+(1-t)x) + (\theta(td+(1-t)x]dx, \ t \in [0,1].$$

Then P is convex on $[0,1]$, and for all $t \in [0,1]$. We have

$$\frac{1}{d-c}\int_c^d \theta(x)dx = P(0) \le P(t) \le P(1) = \frac{\theta(c)+\theta(d)}{2}.$$

Theorem 3.0.2. *Let $\theta \subset R_{++} \to R$ be a continuous function on I. If θ is convex and increasing, and a parameter $m \le 1$ (repectively if θ is convex and decreasing for $m > 1$), then*

$$\frac{1}{d-c}\int_c^d \theta(x)dx \le \frac{c^{1-m}\theta(c)+d^{1-m}\theta(d)}{c^{1-m}+d^{1-m}}. \tag{3.2}$$

If θ is concave and decreasing, and $m \leq 1$ (respectively if θ is concave and increasing, and $m > 1$), then 3.2 is reversed.

Proof. Let

$$\triangle = \left[c^{1-m} + d^{1-m}\right] \frac{1}{d-c} \int_c^d \theta(x)dx - \left[c^{1-m}\theta(c) + d^{1-m}\theta(d)\right].$$

Since $\theta(x)$ is convex. Then

$$\begin{aligned}
\triangle \leq & [c^{1-m} + d^{1-m}] \left[\frac{\theta(c)+\theta(d)}{2}\right] - [c^{1-m}\theta(c) + d^{1-m}\theta(d)] \\
=& \frac{1}{2}[c^{1-m}\theta(c) + d^{1-m}\theta(c) + c^{1-m}\theta(d) + d^{1-m}\theta(d)] - [c^{1-m}\theta(c) + d^{1-m}\theta(d)] \\
=& \frac{1}{2}[c^{1-m}\theta(c) + d^{1-m}\theta(c) + c^{1-m}\theta(d) + d^{1-m}\theta(d) - 2c^{1-m}\theta(c) - 2d^{1-m}\theta(d)] \\
& \frac{1}{2}\left[c^{1-m}\theta(d) + d^{1-m}\theta(c) - c^{1-m}\theta(c) - d^{1-m}\theta(d)\right] \\
=& \frac{1}{2}\left[c^{1-m}(\theta(d)-\theta(c)) - d^{1-m}(\theta(d)-\theta(c))\right] \\
=& \frac{1}{2}\left[(c^{1-m} - d^{1-m})(\theta(d)-\theta(c)\right] \\
=& -\frac{1}{2(d-c)^2}\left[[(d-c)(\theta(d)-\theta(c))][(d-c)(d^{1-m} - c^{1-m})]\right].
\end{aligned}$$

If $m \leq 1$ and if θ is increasing (respectively $m > 1$ and θ is decreasing), by monotonocity property of function θ and we get $\triangle \leq 0$.

Consider again,

$$\triangle = \left[c^{1-m} + d^{1-m}\right] \frac{1}{d-c} \int_c^d \theta(x)dx - \left[c^{1-m}\theta(c) + d^{1-m}\theta(d)\right]$$

If θ is concave,

$$
\begin{aligned}
\triangle \geq & [c^{1-m}+d^{1-m}]\left[\frac{\theta(c)+\theta(d)}{2}\right]-[c^{1-m}\theta(c)+d^{1-m}\theta(d)] \\
=&\frac{1}{2}[c^{1-m}\theta(c)+d^{1-m}\theta(c)+c^{1-m}\theta(d)+d^{1-m}\theta(d)]-[c^{1-m}\theta(c)+d^{1-m}\theta(d)] \\
=&\frac{1}{2}[c^{1-m}\theta(c)+d^{1-m}\theta(c)+c^{1-m}\theta(d)+d^{1-m}\theta(d)-2c^{1-m}\theta(c)-2d^{1-m}\theta(d)] \\
=&\frac{1}{2}\left[c^{1-m}\theta(d)+d^{1-m}\theta(c)-c^{1-m}\theta(c)-d^{1-m}\theta(d)\right] \\
=&\frac{1}{2}\left[c^{1-m}(\theta(d)-\theta(c))-d^{1-m}(\theta(d)-\theta(c)\right] \\
=&\frac{1}{2}\left[(c^{1-m}-d^{1-m})(\theta(d)-\theta(c)\right] \\
=&-\frac{1}{2(d-c)^2}\left[[(d-c)(\theta(d)-\theta(c))][(d-c)(d^{1-m}-c^{1-m}]\right].
\end{aligned}
$$

if $m \leq 1$ and θ is decreasing (repectively $m > 1$ and θ is increasing). By monotonicity condition $\triangle \geq 0$. □

Theorem 3.0.3. *Let $\theta : I \subset R_{++} \to R$ be a continuous function on I. If θ is convex and increasing, and a parameter $m \leq 1$ (respectively if θ is convex and decreasing, and $m > 1$), then $\forall\ c, d \in I$*

$$
\omega(c,d) = \begin{cases} \frac{1}{d-c}\int_c^d \theta(x)dx, & c \neq d \\ \theta(c), & c = d. \end{cases}
$$

is Schur m-power convex on I^2. If θ is concave and decreasing, and a parameter $m \leq 1$ (respectively if θ is concave and increasing for $m > 1$), then $\omega(c, d)$ is Schur m-power concave on I^2.

Proof. Consider

$$
\omega(c,d) = \frac{1}{d-c}\int_c^d \theta(x)dx \ for\ c \neq d.
$$

By fundamental theorem of Calculus,

$$
\frac{\partial \omega}{\partial c} = \frac{1}{(d-c)^2}\int_c^d \theta(x)dx - \frac{\theta(c)}{d-c},
$$

And

$$\frac{\partial \omega}{\partial d} = -\frac{1}{(d-c)^2}\int_c^d \theta(x)dx + \frac{\theta(d)}{d-c}.$$

By the condition for Schur-m covexity, we have

$$\triangle = (d-c)\left(d^{1-m}\frac{\partial \omega}{\partial d} - c^{1-m}\frac{\partial \omega}{\partial c}\right).$$

It follows that

$$\begin{aligned}
\triangle = &(d-c)\left[d^{1-m}\left(-\frac{1}{(d-c)^2}\int_c^d \theta(x)dx + \frac{\theta(d)}{d-c}\right)) - c^{1-m}\left(\frac{1}{(d-c)^2}\int_c^d \theta(x)dx + \frac{\theta(c)}{d-c}\right)\right] \\
=&(d-c)\left[-\frac{1}{(d-c)^2}\int_c^d \theta(x)dx.d^{1-m} + \frac{\theta(d)}{d-c}.d^{1-m} - \frac{1}{(d-c)^2}\int_c^d \theta(x)dx.c^{1-m} + \frac{\theta(c)}{d-c}.c^{1-m}\right] \\
=&(d-c)\left[-\frac{1}{(d-c)^2}\int_c^d \theta(x)dx\cdot(c^{1-m}+d^{1-m}) + \frac{1}{(d-c)}(c^{1-m}\theta(c)+d^{1-m}\theta(d))\right] \\
=&-\frac{1}{(d-c)}\int_c^d \theta(x)dx\cdot(c^{1-m}+d^{1-m}) + (c^{1-m}\theta(c)+d^{1-m}\theta(d)).
\end{aligned}$$

Applying Theorem 3.0.2, if θ is convex and increasing for $m \leq 1$ (and convex and decreasing for $m > 1$), so

$$\frac{1}{d-c}\int_c^d \theta(x)dx \leq \frac{c^{1-m}\theta(c)+d^{1-m}\theta(d)}{c^{1-m}+d^{1-m}}.$$

$$\begin{aligned}
&= (c^{1-m}+d^{1-m})\cdot\frac{1}{d-c}\int_c^d \theta(x)dx \leq c^{1-m}\theta(c)+d^{1-m}\theta(d) \\
&= -(c^{1-m}+d^{1-m})\cdot\frac{1}{d-c}\int_c^d \theta(x)dx \geq -\left[c^{1-m}\theta(c)+d^{1-m}\theta(d)\right] \\
&= -\frac{1}{(d-c)}\int_c^d \theta(x)dx\cdot(c^{1-m}+d^{1-m}) + (c^{1-m}\theta(c)+d^{1-m}\theta(d)) \geq 0.
\end{aligned}$$

Hence, the condition for Schur-convexity is satisfied.

Similarly, for Schur concavity, we have

$$\frac{1}{d-c}\int_c^d \theta(x)dx \geq \frac{c^{1-m}\theta(c)+d^{1-m}\theta(d)}{c^{1-m}+d^{1-m}}$$

$$
\begin{aligned}
&= (c^{1-m} + d^{1-m}) \cdot \frac{1}{d-c}\int_c^d \theta(x)dx \geq c^{1-m}\theta(c) + d^{1-m}\theta(d) \\
&= -\left(c^{1-m} + d^{1-m}\right) \cdot \frac{1}{d-c}\int_c^d \theta(x)dx \leq -\left[c^{1-m}\theta(c) + d^{1-m}\theta(d)\right] \\
&= -\frac{1}{(d-c)}\int_c^d \theta(x)dx \cdot (c^{1-m} + d^{1-m}) + (c^{1-m}\theta(d) + d^{1-m}\theta(c)) \leq 0.
\end{aligned}
$$

Hence, statement of Theorem 3.0.3 is valid. □

Corollary 3.0.1. *Let $\theta : I \subset R_{++} \to R$ be a continuous function on I. If θ is convex and monotonicity, then $\theta(c, d)$ is Schur-convex. If θ is concave and monotonicity, then $\theta(c, d)$ is Schur-concave.*

Corollary 3.0.2. *Let $\theta : I \subset R_{++} \to R$ be a continuous function on I. If θ is convex and increasing, then $\theta(c, d)$ is Schur geometrically convex and Schur harmonically convex. If θ is concave and decreasing, then $\theta(c, d)$ is Schur-Geometrically concave and Schur-Harmonically concave.*

Theorem 3.0.4. *Let $\theta \subset R_{++} \to R$ be a continuous function,and*

$$P(t) = \frac{1}{2(d-c)}\int_c^d [\theta(tc + (1-t)s) + \theta(td + (1-t)s)]ds, \ t \in [0, 1].$$

For any $t \in [0, 1]$, we define a function of $c, d \in I$ as follows;

$$Q(c, d) = \begin{cases} P(t), \ c \neq d \\ \theta(c), \ c = d \end{cases}$$

(1) for $m \geq 1$ and $\frac{c^{1-m}}{c^{1-m}+d^{1-m}} \leq t \leq 1$, if θ is convex (repectively concave) and decreasing on I, then $P(c, d)$ is Schur m-power convex (respectively Schur m-power concave) on I^2.

(2) for $m < 1$ and $0 \leq t \leq \frac{c^{1-m}}{c^{1-m}+d^{1-m}}$, if θ is concave (respectively convex) and increasing on I, then $Q(c, d)$ is Schur m-power concave (repectively Schur m-power convex) on I^2.

Proof. Consider $Q(c,d)$ for $c \neq d$, we have

$$Q_1(c,d) = \int_c^d \theta(tc + (1-t)s)ds,$$

$$Q_2(c,d) = \int_c^d \theta(td + (1-t)s)ds.$$

Then

$$Q(c,d) = \frac{1}{2(d-c)}\left[Q_1(c,d) + Q_2(c,d)\right] = P(t). \tag{3.3}$$

By using $r = tc + (1-t)v$, then

$$Q_1(c,d) = \frac{1}{1-t}\int_c^{tc+(1-t)d} \theta(r)dr$$

$$= \frac{1}{1-t}\int_0^{tc+(1-t)d} \theta(r)dr - \int_0^c \theta(r)dr.$$

By Fundamental theorem of Calculas,

$$\frac{\partial Q_1(c,d)}{\partial c} = \frac{t}{1-t}\theta(tc + (1-t)d) - \frac{\theta(c)}{1-t} \tag{3.4}$$

$$\frac{\partial Q_1(c,d)}{\partial d} = \theta(tc + (1-t)d). \tag{3.5}$$

$$\frac{\partial Q_2(c,d)}{\partial d} = -\frac{\partial Q_1(c,d)}{\partial c} = -\frac{t}{1-t}\theta(tc + (1-t)d) + \frac{\theta(c)}{1-t}. \tag{3.6}$$

$$\frac{\partial Q_1(c,d)}{\partial d} = -\frac{\partial Q_2(c,d)}{\partial c} = -\theta(tc + (1-t)d). \tag{3.7}$$

Since, $Q_2(c,d) = -Q_1(d,c)$ from (3.4) and (3.7), so

From the condition for Schur-convexity, we have

$$\triangle = (d-c)\left(d^{1-m}\frac{\partial Q(c,d)}{\partial d} - c^{1-m}\frac{\partial Q(c,d)}{\partial c}\right) \tag{3.8}$$

Differentiating equation (3.3),

$$\frac{\partial Q_(c,d)}{\partial d} = -\frac{1}{2(d-c)^2}[Q_1(c,d) + Q_2(c,d)] + \frac{1}{2(d-c)}\left[\frac{\partial Q_1(c,d)}{\partial d} + \frac{\partial Q_2(c,d)}{\partial d}\right]$$

$$\frac{\partial Q_(c,d)}{\partial c} = -\frac{1}{2(d-c)^2}[Q_1(c,d)+Q_2(c,d)] + \frac{1}{2(d-c)}\left[\frac{\partial Q_1(c,d)}{\partial c} + \frac{\partial Q_2(c,d)}{\partial c}\right]$$

So, equation (3.8) follows that,

$$\begin{aligned}\triangle = (d-c)&\left[d^{1-m}\left(-\frac{1}{2(d-c)^2}[Q_1(c,d)+Q_2(c,d)] + \frac{1}{2(d-c)}\left(\frac{\partial Q_1(c,d)}{\partial d} + \frac{\partial Q_2(c,d)}{\partial d}\right)\right)\right] \\ &- \left[c^{1-m}\left(-\frac{1}{2(d-c)^2}[Q_1(c,d)+Q_2(c,d)] + \frac{1}{2(d-c)}\left(\frac{\partial Q_1(c,d)}{\partial c} + \frac{\partial Q_2(c,d)}{\partial c}\right)\right)\right]\end{aligned}$$

$$\begin{aligned}= \frac{1}{2}&\left[\left(d^{1-m}\frac{\partial Q_1(c,d)}{\partial d} - c^{1-m}\frac{\partial Q_1(c,d)}{\partial c}\right) + \left(d^{1-m}\frac{\partial Q_2(c,d)}{\partial d} - c^{1-m}\frac{\partial Q_2(c,d)}{\partial c}\right)\right] \\ &- \frac{Q_1(c,d)+Q_2(c,d)}{2(d-c)}(c^{1-m}+d^{1-m})\end{aligned}$$

$$\begin{aligned}= \frac{1}{2}&\left[\left(d^{1-m} - \frac{c^{1-m}t}{1-t}\right)\theta(tc+(1-t)d) + \left(c^{1-m} - \frac{d^{1-m}t}{1-t}\right)\theta(td+(1-t)c)\right] \\ &+ \frac{1}{2}\left[\frac{d^{1-m}\theta(d)+c^{1-m}\theta(c)}{1-t}\right] \\ &- P(t)(c^{1-m}+d^{1-m})\end{aligned}$$

$$\begin{aligned}= \frac{1}{2}&\left[\left(\frac{(d^{1-m}-(c^{1-m}+d^{1-m})t}{1-t}\right)\theta(tc+(1-t)d)\right] \\ &+ \frac{1}{2}\left[\left(\frac{c^{1-m}-(c^{1-m}+d^{1-m})t}{1-t}\right)\theta(td+(1-t)c)\right] \\ &+ \frac{1}{2}\left[\frac{c^{1-m}\theta(c)+d^{1-m}\theta(d)}{1-t}\right] \\ &- P(t)(c^{1-m}+d^{1-m}).\end{aligned}$$

For $m \geq 1$ and $\frac{c^{1-m}}{c^{1-m}+d^{1-m}} \leq t \leq 1$, then $d^{1-m}-(c^{1-m}+d^{1-m})t \leq c^{1-m}+(c^{1-m}+d^{1-m})t \leq$

0. Since θ is convex and decreasing, thus we get

$$\Delta \geq \frac{1}{2}\left[\left(\frac{(d^{1-m}-(c^{1-m}+d^{1-m})t}{1-t}\right)\theta(tc+(1-t)d)\right]$$
$$+\frac{1}{2}\left[\left(\frac{c^{1-m}-(c^{1-m}+d^{1-m})t}{1-t}\right)\theta(td+(1-t)c)\right]$$
$$+\left[\frac{c^{1-m}\theta(c)+d^{1-m}\theta(d)}{2(1-t)}\right]$$
$$-P(t)(c^{1-m}+d^{1-m})$$

$$=(c^{1-m}\theta(c)+d^{1-m}\theta(d))-P(t)(c^{1-m}+d^{1-m})$$

$$\geq (c^{1-m}\theta(c)+d^{1-m}\theta(d))-\frac{1}{2}(\theta(c)+\theta(d))(c^{1-m}+d^{1-m})$$

$$=\frac{1}{2}(\theta(d)-\theta(c))(d^{1-m}-c^{1-m})\geq 0.$$

If θ is concave and decreasing, thus we get

$$\Delta \leq \frac{1}{2}\left[\left(\frac{(d^{1-m}-(c^{1-m}+d^{1-m})t}{1-t}\right)\theta(tc+(1-t)d)\right]$$
$$+\frac{1}{2}\left[\left(\frac{c^{1-m}-(c^{1-m}+d^{1-m})t}{1-t}\right)\theta(td+(1-t)c)\right]$$
$$+\frac{1}{2}\left[\frac{c^{1-m}\theta(c)+d^{1-m}\theta(d)}{1-t}\right]-P(t)(c^{1-m}+d^{1-m})$$

$$=(u^{1-m}\theta(c)+d^{1-m}\theta(d))-P(t)(c^{1-m}+d^{1-m})$$

$$\leq (c^{1-m}\theta(c)+d^{1-m}\theta(d))-\frac{1}{2}(\theta(c)+\theta(d))(c^{1-m}+d^{1-m})$$

$$=\frac{1}{2}(\theta(d)-\theta(c))(d^{1-m}-c^{1-m})\leq 0.$$

□

Corollary 3.0.3. *For $\frac{1}{2} \leq t \leq 1$, if θ is convex (respectively concave) on I, then $Q(c,d)$ is Schur convex (respectively Schur concave) on I^2; for $0 \leq t \leq \frac{1}{2}$, if θ is concave (respectively convex) on I, then $Q(c,d)$ is Schur concave (repectively Schur convex) on I^2.*

Corollary 3.0.4. *For $\frac{c}{c+d} \leq t \leq 1$, if θ is convex and decreasing on I, then $Q(c,d)$ is Schur geometrically convex on I^2; for $0 \leq t \leq \frac{c}{c+d}$, if θ is concave and increasing on I, then $Q(c,d)$ is Schur geometrically concave on I^2.*

Corollary 3.0.5. *For $\frac{c^2}{c^2+d^2} \leq t \leq 1$, if θ is convex and decreasing on I, then $Q(c,d)$ is Schur harmonically convex on I^2; for $0 \leq t \leq \frac{c^2}{c^2+d^2}$, if θ is concave and increasing on I, then $Q(c,d)$ is Schur harmonically concave on I^2.*

Theorem 3.0.5. *Let θ be a continuous function on $I \subset R$ and let Ψ be a positive continuous weight function on I. Then the function $\forall c, d \in I$*

$$F_\Psi(c,d) = \begin{cases} \frac{1}{\int_c^d \Psi(t)dt} \int_c^d \Psi(x)\theta(x)dx, & c \neq d \\ f(c), & c = d. \end{cases}$$

is Schur-convex (repectively Schur-concave) on I^2 if and only if the inequality

$$\frac{1}{\int_c^d \Psi(t)dt} \int_c^d \Psi(t)\theta(t)dt \leq \frac{\theta(c)\Psi(c) + \theta(d)\Psi(d)}{\Psi(c) + \Psi(d)}.$$

holds(reverses) for all c, d in I.

3.0.2 Applications of Hadamard's inequality

Theorem 3.0.6. *For $c, d \in R_{++}$, and $m \geq 1$. Then*

$$G^2(c,d) \leq \frac{c^m + d^m}{c^{m-1} + d^{m-1}}.P(c,d), \tag{3.9}$$

where $G(c,d) = (cd)^{\frac{1}{2}}$, $P(c,d) = \frac{d-c}{lnc-lnd}$.

Proof:

Let $\theta(x)=\frac{1}{x}$, $x \in (0,\infty)$ *and* $d > c$ *.*

By

$$\frac{1}{d-c}\int_c^d \frac{1}{x}dx = P^{-1}(c,d). \tag{3.10}$$

Since $\theta(x)$ *is convex and decreasing, by corollary 3.0.1, it follows that*

$$\frac{1}{d-c}\int_c^d \frac{1}{x}dx \leq \frac{c^m+d^m}{cd(c^{m-1}+d^{m-1})}. \tag{3.11}$$

and

$$P^{-1}(c,d) \leq \frac{c^m+d^m}{cd(c^{m-1}+d^{m-1})}. \tag{3.12}$$

Thus equation (3.9) is satisfied.

Theorem 3.0.7. *For* $a \in (0,\frac{\pi}{2}]$*. Then*

$$\frac{sina}{a} \geq \frac{2}{2a+\pi}\left(1-\frac{2}{2a+\pi}cosa\right)+\frac{2}{\pi}. \tag{3.13}$$

$$\frac{sina}{a} \leq \frac{2}{\pi}+\left(\frac{\pi}{2}-a\right)\left[\frac{sin\left(\frac{a}{2}+\frac{\pi}{4}\right)}{\left(\frac{a}{2}+\frac{\pi}{4}\right)^2}-\frac{cos\left(\frac{a}{2}+\frac{\pi}{4}\right)}{\left(\frac{a}{2}+\frac{\pi}{4}\right)}\right]. \tag{3.14}$$

Proof:

As we have the Hadamard's inequality,

$$\omega(c,d) = \frac{1}{d-c}\int_c^d \theta(x)dx,$$

Consider the functions,

$$\theta(x) = \frac{cosx}{x}-\frac{sinx}{x^2}.$$

Let,

$$\omega(c,d) = \frac{1}{d-c}\int_c^d \left(\frac{cosx}{x}-\frac{sinx}{x^2}\right)dt. \tag{3.15}$$

By Lemma and Corollary 3.0.1, equation (3.15) is Schur-convex on $(0, \frac{\pi}{2}]$*. Since*

$$\left(\frac{a+\frac{\pi}{2}}{2}, \frac{a+\frac{\pi}{2}}{2}\right) \prec \left(\frac{\pi}{2}, a\right) \prec \left(a+\frac{\pi}{2}, 0\right).$$

By definition of Schur-Convexity, then

$$\omega\left(\frac{a+\frac{\pi}{2}}{2}, \frac{a+\frac{\pi}{2}}{2}\right) \leq \omega\left(\frac{\pi}{2}, a\right) \leq \omega\left(a+\frac{\pi}{2}, 0\right).$$

By making substitution from (3.15), we have

$$\frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)} - \frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2} \leq \frac{1}{a-\frac{\pi}{2}} \frac{sinx}{x}|_{\frac{\pi}{2}}^{a} \leq \frac{1}{-\frac{\pi}{2}-a} \frac{sinx}{x}|_{a+\frac{\pi}{2}}^{0}.$$

Firstly, taking these two, we have

$$\frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)} - \frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2} \leq \frac{1}{a-\frac{\pi}{2}} \frac{sinx}{x}|_{\frac{\pi}{2}}^{a},$$

$$(a-\frac{\pi}{2})\left[\frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2} - \frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^3}\right] \leq \frac{sinx}{x}|_{\frac{\pi}{2}}^{a},$$

$$\left(\frac{\pi}{2}-a\right)\left[\frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^3} - \frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2}\right] \leq \left[\frac{sina}{a} - \frac{2}{\pi}\right],$$

$$\left(\frac{\pi}{2}-a\right)\left[\frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^3} - \frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2}\right] + \frac{2}{\pi} \geq \frac{sina}{a}.$$

Hence, we have inequality(3.14).

Now comparing other two, we have

$$\frac{1}{a-\frac{\pi}{2}} \frac{sinx}{x}|_{\frac{\pi}{2}}^{a} \leq \frac{1}{-\frac{\pi}{2}-a} \frac{sinx}{x}|_{a+\frac{\pi}{2}}^{0},$$

$$\frac{sina}{a} - \frac{2}{\pi} \leq \frac{a - \frac{\pi}{2}}{-\frac{\pi}{2} - a}\left[1 - \frac{sin\left(a + \frac{\pi}{2}\right)}{a + \frac{\pi}{2}}\right],$$

$$\frac{sina}{a} \geq \frac{2}{2a + \pi}\left[1 - \frac{2cosa}{2a + \pi}\right] + \frac{2}{\pi}.$$

Thus, the inequality(3.13) obtained.

Chapter 4

Fejer's Inequality

4.0.1 Results and Theorems

Actually "Fejer's inequality" is the generalization of "Hadamard inequality", when we consider the weight function with the "Hadamard inequality".
Our aim is to investigate the different results and applications of "Fejer's inequality" in this Chapter.
We have the following "Hadamard inequality" with the weighted function Ψ *, where "*Ψ*" is non-negative, integrable and is symmetric on* $\Psi(x)=\Psi(c+d-x)$ *and* θ *is a convex function,*

$$\theta\left[\frac{c+d}{2}\right] \leq \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} \leq \frac{\theta(c)+\theta(d)}{2}. \tag{4.1}$$

So, above equation is referred as Fejer's Inequality.

Theorem 4.0.1. *Let* $\theta : [c,d] \rightarrow R$ *be a convex function, and* Ψ *be a non-negative, integrable function on [c,d] and symmetric on* $\Psi(x)=\Psi(c+d-x)$*, then*

$$P(t) = \frac{1}{2}\frac{\int_c^d [\theta(tc+(1-t)x)\Psi(x)+\theta(td+(1-t)x)\Psi(x)]dx}{\int_c^d \Psi(x)dx}, \ t \in [0,1]. \tag{4.2}$$

Then P *is convex on* $[0,1]$*, for all* $t \in [0,1]$ *, we have*

$$\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} = P(0) \leq P(t) \leq P(1) = \frac{\theta(c)+\theta(d)}{2}. \tag{4.3}$$

Theorem 4.0.2. *Let $\theta : I \subset R_{++} \to R$ be a continuous function on I and Ψ be a non-negative integrable, symmetric on $\Psi(x) = \Psi(c+d-x)$ and increasing function. If θ is convex and increasing, and a parameter $m \leq 1$ (respectively if θ is convex and decreasing and $m > 1$), then*

$$\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} \leq \frac{c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)}{c^{1-m}\Psi(c) + d^{1-m}\Psi(d)}. \tag{4.4}$$

and if θ is concave and decreasing and $m \leq 1$ (respectively if θ is concave and increasing, and $m > 1$) then equation (4.4) is reversed.

Proof. Let

$$\triangle = \left[c^{1-m}\Psi(c) + d^{1-m}\Psi(d)\right] \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} - \left[c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)\right].$$

Since θ is convex, by (4.1),

$$\triangle \leq \left[c^{1-m}\Psi(c) + d^{1-m}\Psi(d)\right] \left[\frac{\theta(c)+\theta(d)}{2}\right] - \left[c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)\right]$$

$$= \frac{1}{2}\Big[\left[c^{1-m}\Psi(c) + d^{1-m}\Psi(d)\right]\left[\theta(c)+\theta(d)\right]\Big] - \left[c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)\right]$$

$$= \frac{1}{2}[c^{1-m}\Psi(c)\theta(c) + d^{1-m}\Psi(d)\theta(c) + c^{1-m}\Psi(c)\theta(d) + d^{1-m}\Psi(d)\theta(d)] \\ -\left[c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)\right]$$

$$= \frac{1}{2}[c^{1-m}\Psi(c)\theta(c) + d^{1-m}\Psi(d)\theta(c) + c^{1-m}\Psi(c)\theta(d) + d^{1-m}\Psi(d)\theta(d)] \\ -2c^{1-m}\theta(c)\Psi(c) - 2d^{1-m}\theta(d)\Psi(d)\big]$$

$$= \frac{1}{2}\left[c^{1-m}\Psi(c)\theta(d) + d^{1-m}\theta(c)\Psi(d) - c^{1-m}\theta(c)\Psi(c) - d^{1-m}\theta(d)\Psi(d)\right]$$

$$= \frac{1}{2}\left[c^{1-m}\Psi(c)[\theta(d)-\theta(c)] - d^{1-m}\Psi(d)[\theta(d)-\theta(c)]\right]$$

$$= \frac{1}{2}\left[\left(c^{1-m}\Psi(c) - d^{1-m}\Psi(d)\right)\left(\theta(d) - \theta(c)\right)\right]$$

$$= -\frac{1}{2(d-c)^2}\left[[(d-c)(\theta(d) - \theta(c))][(d-c)(d^{1-m}\Psi(d) - c^{1-m}\Psi(c))]\right].$$

If $m \leq 1$ and if θ is increasing (or $m > 1$ and θ is decreasing), by monotonocity property of function θ and we get $\triangle \leq 0$.

Again consider $\triangle$ for the concavity case,

$$\triangle = (c^{1-m}\Psi(c) + d^{1-m}\Psi(d))\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} - (c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)).$$

If θ is concave,

$$\triangle \geq (c^{1-m}\Psi(c) + d^{1-m}\Psi(d))\left[\frac{\theta(c) + \theta(d)}{2}\right] - (c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d))$$

$$= \frac{1}{2}[c^{1-m}\Psi(c)\theta(c) + d^{1-m}\Psi(d)\theta(c) + c^{1-m}\Psi(c)\theta(d) + d^{1-m}\Psi(d)\theta(d)] - \left[c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)\right]$$

$$\frac{1}{2}[c^{1-m}\Psi(c)\theta(c) + d^{1-m}\Psi(d)\theta(c) + c^{1-m}\Psi(c)\theta(d) + d^{1-m}\Psi(d)\theta(d)] - 2c^{1-m}\theta(c)\Psi(c) - 2d^{1-m}\theta(d)\Psi(d)]$$

$$= \frac{1}{2}\left[c^{1-m}\Psi(c)\theta(d) + d^{1-m}\theta(c)\Psi(d) - c^{1-m}\theta(c)\Psi(c) - d^{1-m}\theta(d)\Psi(d)\right]$$

$$= \frac{1}{2}\left[c^{1-m}\Psi(c)(\theta(d) - \theta(c)) - d^{1-m}\Psi(d)(\theta(d) - \theta(c)\right]$$

$$= \frac{1}{2}\left[(c^{1-m}\Psi(c) - d^{1-m}\Psi(d))(\theta(d) - \theta(c)\right]$$

$$= -\frac{1}{2(d-c)^2}\left[[(d-c)(\theta(d) - \theta(c))][(d-c)(d^{1-m}\Psi(d) - c^{1-m}\Psi(c)]\right].$$

if $m \leq 1$ and θ is decreasing (respectively $m > 1$ and θ is increasing). By monotonicity condition $\triangle \geq 0$. $\square$

Theorem 4.0.3. *Let $\theta \subset R_{++} \to R$ be a continuous function on I and Ψ be a non-negative, integrable, symmetric on $\Psi(x) = \Psi(c+d-x)$ and increasing function. If θ is convex and increasing, and a parameter $m \leq 1$(respectively if θ is convex and decreasing and $m > 1$), then $\forall\ c = d \in I$,*

$$\omega(c,d) = \begin{cases} \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} & , c \neq d \\ \theta(c), c = d. \end{cases}$$

is schur m-power convex on I^2. If θ is concave and decreasing and $m \leq 1$(respectively if θ is concave and increasing $m > 1$), then $\theta(c,d)$ is Schur m-power concave on I^2.

Proof:

Consider

$$\omega(c,d) = \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}.$$

$$\frac{\partial\omega}{\partial c} = \frac{\int_c^d \theta(x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2}\Psi(c) - \frac{\theta(c)\Psi(c)}{\int_c^d \Psi(x)dx}.$$

$$\frac{\partial\omega}{\partial d} = -\frac{\int_c^d \theta(x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2}\Psi(d) + \frac{\theta(d)\Psi(d)}{\int_c^d \Psi(x)dx}.$$

The condition for the Schur-m convexity is

$$\triangle = (d-c)\left(d^{1-m}\frac{\partial\omega}{\partial d} - c^{1-m}\frac{\partial\omega}{\partial c}\right) \geq 0 \tag{4.5}$$

Substituting values in equation (4.5) ,

$$= \left[d^{1-m}\left(-\frac{\int_c^d \theta(x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2}\Psi(d) + \frac{\theta(d)\Psi(d)}{\int_c^d \Psi(x)dx}\right) - c^{1-m}\left(\frac{\int_c^d \theta(x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2}\Psi(c) - \frac{\theta(c)\Psi(c)}{\int_c^d \Psi(x)dx}\right)\right]$$

$$= \frac{1}{\int_c^d \Psi(x)dx}\left[-\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}\Psi(d)d^{1-m} - \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}\Psi(c)c^{1-m}\right] + [d^{1-m}\theta(d)\Psi(d) + c^{1-m}\theta(c)\Psi(c)]$$

$$= \frac{1}{\int_c^d \Psi(x)dx}\left[\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}[-\Psi(c)c^{1-m} - \Psi(d)d^{1-m}]\right] + [d^{1-m}\theta(d)\Psi(d) + c^{1-m}\theta(c)\Psi(c)].$$

By (4.4), we have

If ω is convex and increasing for $m \leq 1$ (respectively convex and decreasing for $m > 1$),

So,

$$\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} \leq \frac{c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)}{c^{1-m}\Psi(c) + d^{1-m}\Psi(d)}.$$

$$= \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}(c^{1-m}\Psi(c) + d^{1-m}\Psi(d)) \leq c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)$$

$$= \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}(-c^{1-m}\Psi(c) - d^{1-m}\Psi(d)) \geq -c^{1-m}\theta(c)\Psi(c) - d^{1-m}\theta(d)\Psi(d)$$

$$= \frac{\int_c^d \theta(x)\Psi(x))dx}{\int_c^d \Psi(x)dx}(-c^{1-m}\Psi(c) - v^{1-m}\Psi(d)) + [c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)] \geq 0.$$

Conditions for the Schur-m convexity is satisfied, and applying Theorem 4.0.2*, we have Theorem* 4.0.3*(valid).*

Similarly, for concavity, we have,

$$\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} \geq \frac{c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)}{c^{1-m}\Psi(c) + d^{1-m}\Psi(d)}.$$

$$= \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}(c^{1-m}\Psi(c) + d^{1-m}\Psi(d)) \geq c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)$$

$$= \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}(-c^{1-m}\Psi(c) - d^{1-m}\Psi(d)) \leq -c^{1-m}\theta(c)\Psi(c) - d^{1-m}\theta(d)\Psi(d)$$

$$\frac{\int_c^d \theta(x)\Psi(x))dx}{\int_c^d \Psi(x)dx}(-c^{1-m}\Psi(c) - d^{1-m}\Psi(d)) + [c^{1-m}\theta(c)\Psi(c) + d^{1-m}\theta(d)\Psi(d)] \leq 0.$$

As the condition for the Schur-m concavity is satisfied, and applying Theorem 4.0.2 *we have Theorem* 4.0.3*(valid).*

Theorem 4.0.4. *Let* $\theta : I \subset R_{++} \to R$ *be a continuous function, and*

$$P(t) = \frac{1}{2}\frac{\int_c^d \left[\theta(tc + (1-t)x)\Psi(x) + \theta(td + (1-t)x)\Psi(x))\right]dx}{\int_c^d \Psi(x)dx}, \ t \in [0,1], \quad (4.6)$$

For any $t \in [0,1]$, *we define a function as follows,* $\forall c, d \in I$,

$$Q(c,d) = \begin{cases} P(t), & c \neq d \\ \theta(c), & c = d. \end{cases}$$

(1) for $m \geq 1$ *if* θ *is convex (respectively concave) and decreasing on* I, *then* $Q(c,d)$ *is Schur m-power convex (respectively Schur m-power concave) on* I^2.

(2) for $m < 1$ *if* θ *is concave (respectively convex) and increasing on* I, *then* $Q(c,d)$ *is Schur m-power concave (respectively Schur m-power convex) on* I^2.

Proof. We will consider the case for $c \neq d$, such that $P(t)$ is given in (4.6), let

$$Q_1(c,d) = \frac{\int_c^d \theta(tc + (1-t)x)\Psi(x)dx}{\int_c^d \Psi(x)dx}.$$

and

$$Q_2(c,d) = \frac{\int_c^d \theta(td + (1-t)x)\Psi(x)dx}{\int_c^d \Psi(x)dx}.$$

Then equation (4.6) becomes,

$$Q(c,d) = P(t) = \frac{1}{2}[Q_1(c,d) + Q_2(c,d)].$$

Now we have the following derivatives,

$$\frac{\partial Q_1}{\partial c} = \frac{\int_c^d \theta(tc+(1-t)x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2} \cdot \Psi(c) - \frac{\Psi(c)\theta(c)}{\int_c^d \Psi(x)dx}, \tag{4.7}$$

$$\frac{\partial Q_1}{\partial c} = \frac{1}{\int_c^d \Psi(x)dx}[\Psi(c)Q_1(c,d) - \theta(c)\Psi(c)].$$

$$\frac{\partial Q_1}{\partial d} = -\frac{\int_c^d \theta(tc+(1-t)x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2} \cdot \Psi(d) + \frac{\theta(tc+(1-t)v)\Psi(d)}{\int_c^d \Psi(x)dx}, \tag{4.8}$$

$$\frac{\partial Q_1}{\partial d} = \frac{1}{\int_c^d \Psi(x)dx}[-\Psi(d)Q_1(c,d) + \theta(tc+(1-t)d)\Psi(d)].$$

$$\frac{\partial Q_2}{\partial c} = \frac{\int_c^d \theta(td+(1-t)x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2} \cdot \Psi(c) - \frac{\Psi(c)\theta(td+(1-t)c)}{\int_c^d \Psi(x)dx}, \tag{4.9}$$

$$\frac{\partial Q_2}{\partial c} = \frac{1}{\int_c^d \Psi(x)dx}[\Psi(c)Q_2(c,d) - \theta(td+(1-t)c)\Psi(c)].$$

$$\frac{\partial Q_2}{\partial d} = -\frac{\int_c^d \theta(td+(1-t)x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2} \cdot \Psi(d) + \frac{\theta(d)\Psi(d)}{\int_c^d \Psi(x)dx}, \tag{4.10}$$

$$\frac{\partial Q_2}{\partial d} = \frac{1}{\int_c^d \Psi(x)dx}[-\Psi(d)Q_2(c,d) + \theta(d)\Psi(d)].$$

Notice that $Q_1(c,d) = -Q_2(c,d)$. From (4.7) to (4.10), we have

$$\frac{\partial Q_2}{\partial d} = -\frac{\partial Q_1}{\partial c} = -\frac{\int_c^d \theta(tc+(1-t)x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2} \cdot \Psi(c) + \frac{\theta(c)\Psi(c)}{\int_c^d \Psi(x)dx}. \tag{4.11}$$

$$\frac{\partial Q_2}{\partial c} = -\frac{\partial Q_1}{\partial d} = \frac{\int_c^d \theta(tc+(1-t)x)\Psi(x)dx}{(\int_c^d \Psi(x)dx)^2} \cdot \Psi(d) - \frac{\theta(tc+(1-t)d)\Psi(d)}{\int_c^d \Psi(x)dx}. \tag{4.12}$$

By the condition for the Schur-convexity as well as for Schur-concavity, we have

$$\Delta = (d-c)\left(d^{1-m}\frac{\partial Q}{\partial d} - c^{1-m}\frac{\partial Q}{\partial c}\right) \geq 0(respectively \leq 0).$$

First we have to prove the case for the Schur-convexity, so

$$\Delta = \frac{1}{2}(d-c)\left(d^{1-m}\left(\frac{\partial Q_1}{\partial d} + \frac{\partial Q_2}{\partial d}\right) - c^{1-m}\left(\frac{\partial Q_1}{\partial c} + \frac{\partial Q_2}{\partial c}\right)\right).$$

$$\Delta = \frac{1}{2}(d-c)\left(d^{1-m}\frac{\partial Q_1}{\partial d} - c^{1-m}\frac{\partial Q_1}{\partial c}\right) + \left(d^{1-m}\frac{\partial Q_2}{\partial d} - c^{1-m}\frac{\partial Q_2}{\partial c}\right).$$

$$\begin{aligned}
&= \frac{d-c}{2\int_d^c \Psi(x)dx}[-\Psi(d)Q_1(c,d)d^{1-m} + \theta(tc+(1-t)d)\Psi(d)d^{1-m} \\
&\quad -\Psi(c)Q_1(c,d)c^{1-m} \\
&\quad +\theta(c)\Psi(c)c^{1-m} - \Psi(d)Q_2(c,d)d^{1-m} + \theta(d)\Psi(d)d^{1-m} - \Psi(c)Q_2(c,d)c^{1-m} \\
&\quad +\theta(td+(1-t)c)\Psi(c)c^{1-m})]
\end{aligned}$$

Consider

$$\begin{aligned}
&= \frac{1}{2}[-\Psi(d)Q_1(c,d)d^{1-m} + \theta(tc+(1-t)d)\Psi(d)d^{1-m} - \Psi(c)Q_1(c,d)c^{1-m} + \theta(c)\Psi(c)c^{1-m} \\
&\quad -\Psi(d)Q_2(c,d)d^{1-m} + \theta(d)\Psi(d)d^{1-m} - \Psi(c)Q_2(c,d)c^{1-m} + \theta(td+(1-t)c)\Psi(c)c^{1-m})]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[\Psi(d)d^{1-m}\theta(tc+(1-t)d) + \Psi(c)c^{1-m}\theta(td+(1-t)c) + \Psi(c)\theta(c)c^{1-m} + \Psi(d)\theta(d)d^{1-m}\right] \\
&\quad -\Psi(d)d^{1-m}\frac{[Q_1+Q_2]}{2} - \Psi(c)c^{1-m}\frac{[Q_1+Q_2]}{2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[\Psi(d)d^{1-m}\theta(tc+(1-t)d) + \Psi(c)c^{1-m}\theta(td+(1-t)c) + \Psi(c)\theta(c)c^{1-m} + \Psi(d)\theta(d)d^{1-m}\right] \\
&\quad -[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}]\frac{Q_1+Q_2}{2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[\Psi(d)d^{1-m}\theta(tc+(1-t)d) + \Psi(c)c^{1-m}\theta(td+(1-t)c) + \Psi(c)\theta(c)c^{1-m} + \Psi(d)\theta(d)d^{1-m}\right] \\
&\quad -P(t)[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}].
\end{aligned}$$

If θ is convex and decreasing for $m \geq 1$, then we have

$$\Delta \geq \frac{1}{2}[t\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(d)d^{1-m} - t\theta(d)\Psi(d)d^{1-m} + t\theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} - t\theta(c)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}] - P(t)[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}]$$

Here, we are using $P(t) \leq P(1)$ from equation (4.3),

$$= \frac{1}{2}[\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}] - \frac{\theta(c)+\theta(d)}{2}[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}]$$

$$\geq \frac{1}{2}[\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}] - \theta(c)\Psi(d)d^{1-m} - \theta(d)\Psi(c)c^{1-m} - \theta(c)\Psi(c)c^{1-m} - \theta(d)\Psi(d)d^{1-m}] \geq 0.$$

So, the condition for the Schur-convexity is satisfied for $m \geq 1$. Now to prove the case for the Schur-concavity,

If θ is concave and decreasing, then we have

$$\Delta \leq \frac{1}{2}\Big[t\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(d)d^{1-m} - t\theta(d)\Psi(d)d^{1-m} + t\theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} - t\theta(c)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}\Big] - P(t)\big[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}\big].$$

Note that, $P(t) \geq P(0)$, by equation (4.3) and using Theorem 4.0.2,

$$= \frac{1}{2}[2\theta(d)\Psi(d)d^{1-m} + 2\theta(c)\Psi(c)c^{1-m}] - \frac{\int_c^d \theta(x)\Psi(x)dx}{\Psi(x)dx}[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}]$$

$$= \frac{1}{2}[2\theta(d)\Psi(d)d^{1-m} + 2\theta(c)\Psi(c)c^{1-m}] - \theta(d)\Psi(d)d^{1-m} - \theta(c)\Psi(c)c^{1-m} \leq 0.$$

So, the case for the Schur-concavity is proved for $m \geq 1$,

Case (2) is similar for $m < 1$.

If θ is concave and increasing for $m < 1$, then we have,

$$\Delta \leq \frac{1}{2}[t\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(d)d^{1-m} - t\theta(d)\Psi(d)d^{1-m} + t\theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} - t\theta(c)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}] - P(t)[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}] \tag{4.13}$$

Note that $P(t) \leq P(1)$, by equation (4.3),

$$= \frac{1}{2}[\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}] \\ -\frac{\theta(c)+\theta(d)}{2}[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}] \tag{4.14}$$

$$\leq \frac{1}{2}[\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}] \\ -\theta(c)\Psi(d)d^{1-m} - \theta(d)\Psi(c)c^{1-m} - \theta(c)\Psi(c)c^{1-m} - \theta(d)\Psi(d)d^{1-m}] \leq 0.$$

The condition for the Schur-concavity is satisfied for $m < 1$. Now, if θ is convex and increasing, then we have

$$\Delta \geq \frac{1}{2}[t\theta(c)\Psi(d)d^{1-m} + \theta(d)\Psi(d)d^{1-m} - t\theta(d)\Psi(d)d^{1-m} + t\theta(d)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} \\ -t\theta(c)\Psi(c)c^{1-m} + \theta(c)\Psi(c)c^{1-m} + \theta(d)\Psi(d)d^{1-m}] - P(t)[\Psi(d)d^{1-m} + \Psi(c)c^{1-m}] \tag{4.15}$$

Note that, $P(t) > P(0)$, by equation (4.3) and using Theorem 4.0.3,

$$= \frac{1}{2}[2\theta(d)\Psi(d)d^{1-m} + 2\theta(c)\Psi(c)c^{1-m}] - \theta(d)\Psi(d)d^{1-m} - \theta(c)\Psi(c)c^{1-m} \geq 0. \tag{4.16}$$

The condition for the Schur-convexity is satisfied for $m < 1$. □

Now we will consider possibility of further generalization in the companion mappings.
Let $\alpha : [0,1] \to [0,1]$ *be a monotonic non-decreasing continuous function on [0,1]. Let* $G_\alpha : [0,1] \to R$ *be a function is defined by*

$$G_\alpha(t) = \frac{1}{2}\frac{\int_c^d [\theta(\alpha(t)c + (1-\alpha(t)x)\Psi(x) + \theta(\alpha(t)d + (1-\alpha(t))x)\Psi(x)]dx}{\int_c^d \Psi(x)dx}. \tag{4.17}$$

Lemma 5. *Suppose that θ is convex (respectively concave), then $G_\alpha(t)$ is convex (respectively concave) if α is a linear function.*
Proof: It follows that from equation (4.17), if θ is convex, also α is linear,then we have the composition $\theta \circ \alpha$ is linear.

Lemma 6. *Let* $\theta(x)=\frac{cosx}{x}-\frac{sinx}{x^2}$, $t\epsilon(0,\Pi]$. *Then* $\theta(x)$ *is convex and decreasing on*$(0,\frac{\Pi}{2}]$, *and* $\theta(x)$ *is convex and increasing on* $[\frac{\Pi}{2},\pi]$.

Proof: Since,

$$\theta^1(x)=\frac{2sinx-2xcosx-x^2sinx}{x^3}.$$

$$\theta^2(x)=\frac{-x^3cost+3x^2sinx+6xcosx-6sinx}{x^4}.$$

Let

$$g(x)=2sinx-2xcosx-x^2sinx \text{ and } g(0)=0.$$

Then

$$g^1(x)=-x^2cost\leq 0.$$

And $g(x)\leq g(0)$, *and* $\theta^1(x)\leq 0$. *Thus,* $\theta(x)$ *is decreasing on* $(0,\frac{\Pi}{2}]$.
Let $h(x)=-x^3cosx+3x^2sinx+6xcosx-6sinx$ *and* $h(0)=0$. *Then* $h^1(x)=x^3sinx\geq 0$ *and* $h(x)\leq h(0)=0$. *Further* $f^2(x)\geq 0$. *So* $\theta(x)$ *is convex and increasing on* $(0,\frac{\Pi}{2}]$
Similarly, we get that $\theta(x)$ *is convex and increasing on* $(0,\frac{\Pi}{2}]$.

4.0.2 Applications of Fejer's Inequality

There are numerous applications of "Fejer's inequality" which have received attention in back years. Relating to the integral mean of a convex function are the most famous inequalities ***"Hadamard inequality"*** *and* ***"Fejer's inequality"****. Some applications of "Fejer's inequality" are as follows:*

Theorem 4.0.5. *For* $c,d\in R_{++}$,*and* $m\geq 2$.*Then*

$$G^2(c,d)\leq\frac{c^m+d^m}{c^{m-2}+d^{m-2}}.P(c,d). \tag{4.18}$$

where $G(c,d) = cd,\ P(c,d) = \frac{d^2-c^2}{2(lnc-lnd)}$

Proof:

Let $\theta(x)=\frac{1}{x^2},\ \ \Psi(x)=x,\ x\in(0,\infty)$ *and* $d>c$ *.*

Consider the fejer's inequality and putting values,

$$\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} = \frac{\int_c^d \frac{1}{x}dx}{\int_c^d xdx} = \frac{2(ln(c)-ln(d))}{d^2-c^2} = P^{-1}(c,d). \tag{4.19}$$

By definition of convex function, since $\theta(x)$ *is convex and decreasing. By Theorem 4.0.2, it follows that*

$$\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx} \leq \frac{c^{1-m}\theta(c)\Psi(c)+d^{1-m}\theta(d)\Psi(d)}{c^{1-m}\Psi(c)+d^{1-m}\Psi(d)}. \tag{4.20}$$

After simplifying,we have

$$\leq \frac{c^m+d^m}{c^2d^2(c^{m-2}+d^{m-2})}. \tag{4.21}$$

If θ *is convex and monotonicity, then* θ *is Schur-convex. As from the 4.0.3, we have the following equation for Schur-convexity,*

$$\frac{1}{\int_c^d \Psi(x)dx}\left[\frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}[-\Psi(c)c^{1-m}-\Psi(d)d^{1-m}]+[d^{1-m}\theta(d)\Psi(d)+c^{1-m}\theta(c)\Psi(c)]\right]\geq 0, \tag{4.22}$$

Then

$$\frac{\int_c^d \frac{1}{x}dx}{\int_c^d xdx} \leq \frac{c^m+d^m}{c^2d^2[c^{m-2}+d^{m-2}]},$$

Thus equation(4.18) is satisfied.

"Camille Jordan" [9] introduced the Jordan's inequality for $a\in(0,\frac{\Pi}{2}]$*.*

$$\frac{2}{\pi} \leq \frac{sinx}{x} \leq 1, \tag{4.23}$$

with equality holds iff $x=\frac{\pi}{2}$*. By Corollary 3.0.1 and 3.0.2, further inequalities and new refinements for Jordan's inequality are obtained, as follows*

Theorem 4.0.6. *For $a \in (0, \frac{\pi}{2}]$. Then,*

$$\frac{sina}{a} \geq \frac{a^2 - \frac{\pi^2}{4}}{-\left(\frac{\pi}{2}+a\right)^2}\left[1 - \frac{2}{2a+\pi}\right] + \frac{2}{\pi}, \tag{4.24}$$

$$\frac{sina}{a} \leq \frac{2}{\pi} + \left(\frac{\pi^2}{8} - \frac{a^2}{2}\right)\left[\frac{sin\left(\frac{a}{2}+\frac{\pi}{4}\right)}{\left(\frac{a}{2}+\frac{\pi}{4}\right)} - \frac{cos\left(\frac{a}{2}+\frac{\pi}{4}\right)}{\left(\frac{a}{2}+\frac{\pi}{4}\right)^2}\right]. \tag{4.25}$$

Proof:

As we have the "Fejer's inequality",

$$\omega(c,d) = \frac{\int_c^d \theta(x)\Psi(x)dx}{\int_c^d \Psi(x)dx}.$$

Let

$$\theta(x) = \frac{cosx}{x^2} - \frac{sinx}{x^3},$$
$$and\ \Psi(x) = x.$$

and it follows that

$$\omega(c,d) = \frac{2}{d^2 - c^2}\int_c^d \left(\frac{cosx}{x} - \frac{sinx}{x^2}\right)dx. \tag{4.26}$$

By Lemma and Corollary 3.0.1, equation (4.26) is Schur-convex on $(0, \frac{\pi}{2}]$. Since

$$\left(\frac{a+\frac{\pi}{2}}{2}, \frac{a+\frac{\pi}{2}}{2}\right) \prec \left(\frac{\pi}{2}, a\right) \prec \left(a+\frac{\pi}{2}, 0\right).$$

By definition of Schur-Convexity, then

$$\omega\left(\frac{a+\frac{\pi}{2}}{2}, \frac{a+\frac{\pi}{2}}{2}\right) \leq \omega\left(\frac{\pi}{2}, a\right) \leq \omega\left(a+\frac{\pi}{2}, 0\right).$$

By making substitution, we have

$$\frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)} - \frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2} \leq \frac{2}{a^2 - \frac{\pi^2}{4}}\frac{sinx}{x}|_{\frac{\pi}{2}}^{a} \leq \frac{2}{-\left(\frac{\pi}{2}+a\right)^2}\frac{sinx}{x}|_{a+\frac{\pi}{2}}^{0},$$

Firstly, taking these two, we have

$$\frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)} - \frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2} \leq \frac{2}{a^2-\frac{\pi^2}{4}}\frac{sinx}{x}|_{\frac{\pi}{2}}^{a},$$

$$\frac{a^2-\frac{\pi^2}{4}}{2}\left[\frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)} - \frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2}\right] \leq \frac{sinx}{x}|_{\frac{\pi}{2}}^{a},$$

$$\left(\frac{\pi^2}{8}-\frac{a^2}{2}\right)\left[\frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^3} - \frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2}\right] \leq \left[\frac{sina}{a}-\frac{2}{\pi}\right],$$

$$\left(\frac{\pi^2}{8}-\frac{a^2}{2}\right)\left[\frac{sin\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)^2} - \frac{cos\left(\frac{a+\frac{\pi}{2}}{2}\right)}{\left(\frac{a+\frac{\pi}{2}}{2}\right)}\right] + \frac{2}{\pi} \leq \frac{sina}{a}.$$

so from above, we have inequality(4.25).

Now comparing other two, we have

$$\frac{2}{a^2-\frac{\pi^2}{4}}\frac{sinx}{x}|_{\frac{\pi}{2}}^{a} \leq \frac{2}{-\left(\frac{\pi}{2}+a\right)^2}\frac{sinx}{x}|_{a+\frac{\pi}{2}}^{0},$$

$$\frac{sina}{a}-\frac{2}{\pi} \leq \frac{a^2-\frac{\pi^2}{4}}{-\left(\frac{\pi}{2}+a\right)^2}\left[1-\frac{sin\left(a+\frac{\pi}{2}\right)}{a+\frac{\pi}{2}}\right],$$

$$\frac{sina}{a} \leq \frac{a^2-\frac{\pi^2}{4}}{-\left(\frac{\pi}{2}+a\right)^2}\left[1-\frac{2cosa}{2a+\pi}\right]+\frac{2}{\pi}.$$

Equation (4.24) is satisfied.

Chapter 5

Conclusion

Every Fejer's type inequality is the Hadamard's type inequality when we consider the weight function is equal to one. We must say that every Fejer's type inequality comes in category of Hadamard's type inequality.
Concept of Fejer's type inequality is much more easy to understand in Hadamard's type inequality. With the help of literature we came to know about many results in terms of Hadamard's type inequality. Authors provided many directions to resolve the problems regarding Hadamard's type inequality. Therefore using these results of Hadamard's type inequality, many results regarding Fejer's type inequality are formed, presented in our thesis.
In our thesis, we have some concepts that revolves around Schur-convexity and Schur-concavity. However, Schur Geometrically convexity and Schur Geometrically concavity are different direction to study. So, this is an interesting direction which we prepare for future work. Also,some problems arise while studying the Schur-Convexity and Schur-Concavity. So, that should also be studied in future.

Bibliography

[1] *Marshall AW, Olkin I, Arnold BC. Inequalities: theory of majorization and its applications. New York: Academic press; 1979 Dec.*

[2] *X. M. ZHANG, Geometrically-Convex Functions, Anhui Univ. Press, Hefei, 2004(in Chinese)*

[3] *Yuming C, Tianchuan S. The Schur harmonic convexity for a class of symmetric functions. Acta Mathematica Scientia. 2010 Sep 1;30(5):1501-6.*

[4] *Wu Y, Qi F. Schur-harmonic convexity for differences of some means. Analysis (Munich). 2012 Nov 1;32:263-70.*

[5] *Yang ZH. Schur power convexity of Stolarsky means. PUBLICATIONES MATHEMATICAE-DEBRECEN. 2012 Feb 1;80(1-2):43-66.*

[6] *Yang ZH. Schur power convexity of the Daróczy means. Math. Inequal. Appl. 2013 Jul 1;16(3):751-62.*

[7] *Wang W, Yang SG. Schur m-power convexity of generalized Hamy symmetric function. Journal of Mathematical Inequalities. 2014 Sep 1.*

[8] *Wang W, Yang S. Schur-Power Convexity of a Class of Multiplicatively Convex Functions and Applications. InAbstract and Applied Analysis 2014 Jan 1 (Vol. 2014). Hindawi.*

[9] *Mitrinovic DS, Pecaric J, Fink AM. Classical and new inequalities in analysis. Springer Science and Business Media; 2013 Apr 17.*

[10] *Dragomir SS. Further properties of some mappings associated with Hermite-Hadamard inequalities. Tamkang Journal of Mathematics. 2003 Mar 31;34(1):45-*

58.

[11] Yuming C, Tianchuan S. The Schur harmonic convexity for a class of symmetric functions. Acta Mathematica Scientia. 2010 Sep 1;30(5):1501-6.

[12] V. CHULJAK, A remark on Schur-convexity of the mean of a convex function, J. Math. Ineqal., 9(4)(2015), 1133-1142.

[13] Dragomir SS, Pearce CE. Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000. Online: http://www. staff. vu. edu. au/RGMIA/monographs/hermite hadamard. html. 2004.

[14] Elezović N, Pečarić J. A note on Schur-convex functions. The Rocky Mountain Journal of Mathematics. 2000 Oct 1:853-6.

[15] Guan K, Guan R. Some properties of a generalized Hamy symmetric function and its applications. Journal of mathematical analysis and applications. 2011 Apr 15;376(2):494-505.

[16] Klén R, Visuri M, Vuorinen M. On Jordan type inequalities for hyperbolic functions. Journal of Inequalities and Applications. 2010 Dec;2010:1-4.

[17] Manfrino RB, Ortega JA, Delgado RV. Inequalities: a mathematical olympiad approach. Springer Science and Business Media; 2009 Sep 18.

[18] Merkle M. Conditions for convexity of a derivative and some applications to the Gamma function. Aequationes Mathematicae. 1998 May 1;55(3):273-80.

[19] M.MERKLE, Representaion of the error terms in Jensen's and some related inequalities with applications, J. Math. Analysis Appl.,231(1991), 76-90.

[20] Niculescu CP. The Hermite–Hadamard inequality for log-convex functions. Nonlinear Analysis: Theory, Methods and Applications. 2012 Jan 1;75(2):662-9.

[21] Özdemir ME. A theorem on mappings with bounded derivatives with applications to quadrature rules and means. Applied mathematics and computation. 2003 Jun 20;138(2-3):425-34.

[22] *Peajcariaac JE, Tong YL. Convex functions, partial orderings, and statistical applications. Academic Press; 1992 Jun 3.*

[23] *Qi F, Niu DW, Guo BN. Refinements, generalizations, and applications of Jordan's inequality and related problems. Journal of Inequalities and Applications. 2009 Dec 1;2009(1):271923.*

[24] *W.T. SULAIMAN, Schur Convexity and Hadamard's Inequality, Global Journal of Sci. frontier Research, (2011), 11,1.*

[25] *Tseng KL, Hwang SR, Dragomir SS. New Hermite–Hadamard-type inequalities for convex functions (I). Applied Mathematics Letters. 2012 Jun 1;25(6):1005-9.*

[26] *Wu S, Debnath L. A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality. Applied Mathematics Letters. 2006 Dec 1;19(12):1378-84.*

[27] *Yang ZH. Schur power convexity of Gini means. Bull. Korean Math. Soc. 2013 Mar 1;50(2):485-98.*

[28] *Yang ZH. New sharp Jordan type inequalities and their applications. arXiv preprint arXiv:1206.5502. 2012 Jun 24.*

[29] *Gilanyi A, Pales Z. On convex functions of higher order. Mathematical Inequalities and Applications. 2008 Apr;11(2):271-82.*

[30] *Niculescu C, Persson LE. Convex functions and their applications. New York: Springer; 2006.*