On Fractional Duhamel's Principle

by

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Dedication

to

My Beloved Parents

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Abstract

In this academic research we generalized the Duhamel's principle and extended this principle for the higher order integer and fractional psi differential equation subject to suitable initial conditions. Furthermore, as application of the generalized Duhamel's principle, some notions like stability, existence and uniqueness of the solutions of the generalized fractional differential equation with initial conditions is investigated. In order to approximate the solutions of the generalized nonlinear fractional differential equation with initial conditions, we introduce a new numerical technique combining the Haar wavelets and Duhamel's principle called Haar-Duhamel's method.

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Chapter 1

Introduction

1.1 Fractional Calculus

The idea of calculus occurs when the concept of derivative arises. The definition of derivative that is used now a days was given by Newton in 1666 [1]. The researchers were developed many physical or geometrical interpretation from the derivative and integration. The integral of a function is the area under the cure [2]. This type of calculus was developed widely over three or four centuries. Now a days, many scientists can understand or describe the physical facts with an ordinary differential equation. If we generalize the ordinary calculus i.e. derivative and integration of integer order to arbitrary non-integer value, we get the fractional calculus i.e. fractional derivative and integration. We defined the fractional calculus as: the branch of mathematics in which we study the properties of fractional derivative and fractional integral is called fractional calculus.

In 1695, Leibniz derived the nth order derivatives formula i.e $\frac{d^n}{dx^n}$. After his publication of the nth derivatives formula, L-Hospital rises a question to Leibniz that if we take n=1/2 what will be the result? This question was the beginning of the fractional calculus. L-Hospital's reply: "An apparent paradox, from which one day very useful results will be drawn." After that time fractional calculus was developed by many mathematicians. Bertram Ross is the first person who done his Ph.D. on the fractional calculus. In 1974, Keith B. Oldam and Jerome Spanier published the monograph and they devoted their publications to the Fractional calculus ([3]). Now a days,many books, journals, articles and conferences held on the fractional calculus and its applications and properties. Many well-known mathematicians Riemann, Liouville, Caputo, Hadamard and Grunwald work in the field of fractional calculus, they also give their own definitions in fractional calculus. There are different definitions in the fractional calculus but Riemann-Liouville integral and derivative are the most famous. Later, a mathematician Caputo gives another definition of fractional derivative to solve the fractional order differential equation Which is the more generalized form of Riemann-Liouville derivative. On the bases of Leibniz's answer studies over 200-300 years and has proved many concepts right.

In this chapter introduction to the theory of fractional calculus, some basic preliminaries and major results already obtained. Also, we will define some special functions. We review literature and research about the fractional derivative and integral with respect to another function and generalized Laplace transform. Its properties and some results are also given. We will give examples to understand the concept of the given result.

In chapter 2, we generalized Duhamel's principle for ψ operator. We will also give its examples to understand the concept of this famous principle.

In chapter 3, the applications of the generalized Duhamel's principle that is the stability and existence of the solutions of the solutions of generalized fractional differential equation (FDE) .

In chapter 4, we review literature and research about wavelet, Haar wavelet, the Haar matrix and the integration matrix. We also approximate function by Haar wavelet and error analysis. Also develop Haar-Duhamel's method to solve the fractional differential equation (FDEs).

1.2 Special Functions

There are many special functions that are very helpful for solving the problems of fractional differential equations. In this section, the definitions of some of the special functions and their properties are discussed.

1.2.1 Gamma Function

Many well-known mathematicians studied the Gamma function. The Gamma function is represented by Γ. It is the generalization of the factorial function (i.e $\Gamma(m+1) = m!$ for $m \in \mathbb{N}$). The gamma function $\Gamma : (0, +\infty) \to \mathbb{R}$ is defined as:

$$\Gamma(w) = \int_0^\infty (s)^{w-1} e^{-s} ds, \; w > 0. \tag{1.1}$$

The integral in equation (1.1) is convergent for $\mathrm{Re}(w) > 0$. There are many properties of the Gamma function, but we list few of them [4].

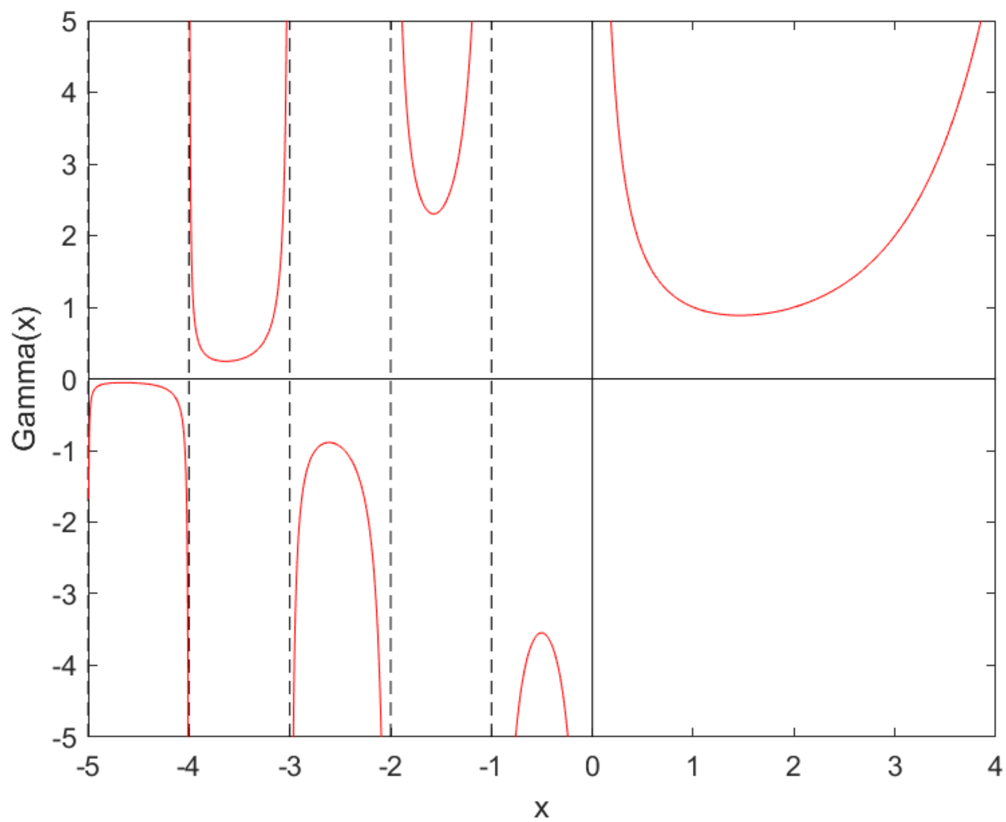


Figure 1.1: The Gamma function for real argument

- **Properties**

$$\Gamma(m+1) = m!$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(w+1) = w\Gamma(w).$$

(a) Duplication formula

$$(2)^{2w-1}\Gamma(w)\Gamma(w+\frac{1}{2}) = \sqrt{\pi}\Gamma(2w).$$

(b) Reflection formula

$$\Gamma(w)\Gamma(1-w) = \frac{\pi}{\sin(\pi w)}.$$

If $w \neq 0$

$$\Gamma(w) = \frac{\Gamma(w+1)}{w} \tag{1.2}$$

Right hand side of the equation (1.2) is defined for $w > 0$. Now, If $w \neq 0$, $w \neq -1$, we have

$$\Gamma(w) = \frac{\Gamma(w+2)}{w(w+1)}. \tag{1.3}$$

The equation (1.3) is valid for $w > -2$, as $w \neq 0, -1$. The process is repeated q-times, we get

$$\Gamma(w) = \frac{\Gamma(w+q)}{(w+q-1)(w+q-2)...(w+1)w}, \; w \neq 0, -1, -2, ...$$

Thus the domain of the Gamma function is $w \in \mathbb{R} \setminus \{0, -1, -2, -3, ...\}$.

1.2.2 Beta Function

The name of the Beta function is used by the Legendr and Whittakar and Waston 1990. It is defined by the following definite integral:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,ds,\ Re(m) > 0,\ Re(n) > 0. \tag{1.4}$$

Sometimes, we replace the Beta function by the Gamma function with a relation that is [4]:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

1.2.3 Mittage-Leffler Function

G.M Mittag-Leffler was a Swedish mathematician who defined and studied the Mittag-Leffler function in 1903 [5]. Generally, it is the parameterized form of the exponential function. It has vast applications in the area of applied sciences, engineering and mathematics.

Definition 1.2.1. *The Mittage-Leffler function of order one is defined as [5]:*

$$E_\gamma(w) = \sum_{q=0}^{\infty} \frac{w^q}{\Gamma(\gamma q + 1)},\ \gamma \in \mathbb{R},\ w \in \mathbb{C}.$$

Later, Agarwal introduced the second order Mittage-Leffler function, which is defined as following:

Definition 1.2.2. *For $\gamma, \eta \in \mathbb{R}$ and $w \in \mathbb{C}$, the Mittage-Leffler function of second order is given as*

$$E_{\gamma,\eta}(w) = \sum_{q=0}^{\infty} \frac{w^q}{\Gamma(\gamma q + \eta)}.$$

If $\eta = 1$, then we get $E_{\gamma,1}(w)$ that is written as $E_\gamma(z)$.

- Some special cases

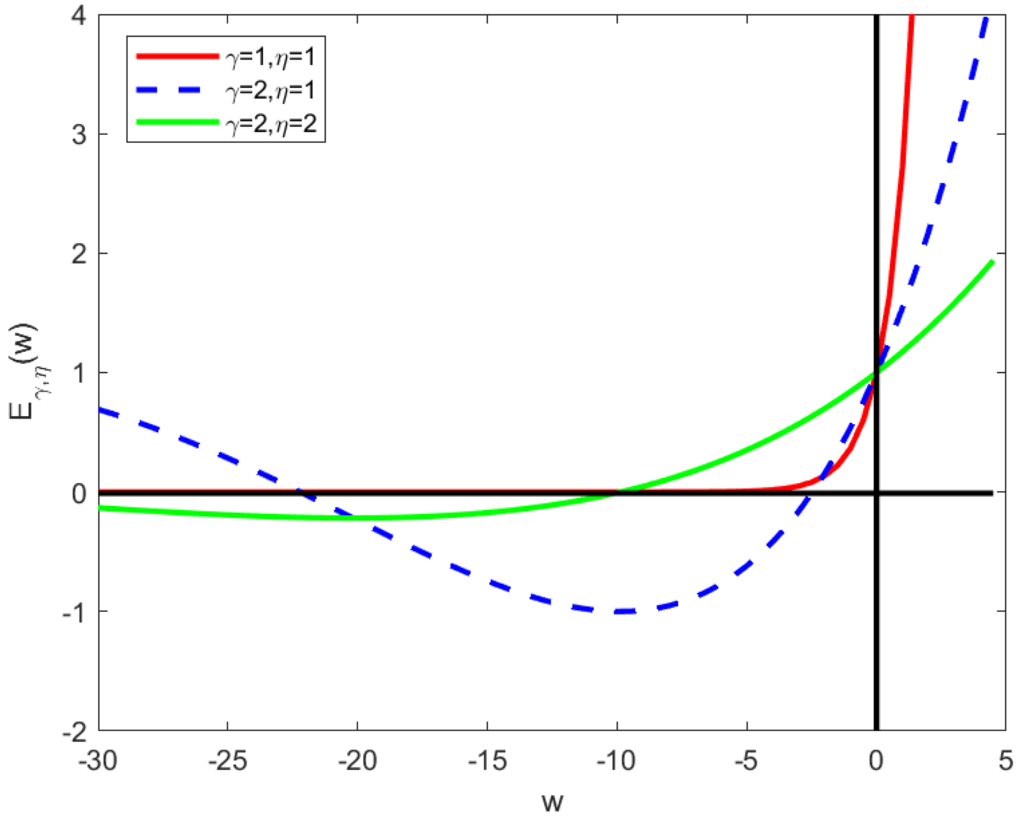


Figure 1.2: The Mittag-leffler function for different values of γ and η.

$$
\begin{aligned}
E_1(w) &= e^w, \\
E_2(w) &= \cosh(\sqrt{w}), \\
E_{1,0}(w) &= we^w, \\
E_{1,2}(w) &= \frac{e^w - 1}{w}, \\
E_{2,2}(w) &= \frac{\sinh(\sqrt{w})}{\sqrt{w}}.
\end{aligned}
$$

Remark 1.2.1. *If $|arg(w)| \in [\mu, \pi]$ and $w \to \infty$ then*

$$E_{\gamma,\eta}(w) = -\sum_{i=2}^{m} \frac{w^{-i}}{\Gamma(\eta - \gamma i)} + O(|w|^{-1-m}).$$

where $m \geq 2$ is any integer.

Theorem 1.2.1. *The following relations hold for* $\gamma > 0,\ \eta > 0$*;*

$$E_{\gamma,\eta}(w) = wE_{\gamma,\gamma+\eta}(w) + \frac{1}{\eta}. \tag{1.5}$$

$$\frac{d^n}{dw^n}\left[w^{\eta-1}E_{\gamma,\eta}(w^{\gamma})\right] = w^{\eta-n-1}E_{\gamma,\eta-n}(w^{\gamma}). \tag{1.6}$$

$$E_{\gamma,\eta}(w) = \eta E_{\gamma,\eta+1}(w) + \gamma w\frac{d}{dw}E_{\gamma,\eta+1}(w). \tag{1.7}$$

1.3 Fractional integral and derivatives

In this section we will discuss some important definitions and results for fractional integral and derivatives.

1.3.1 Riemann-Liouville integral

With the help of Cauchy integral formula, we can define fractional integrals and derivatives. So, first we defined the Cauchy iterative integral formula as:

$$\mathcal{J}_b^m h(x) = \int_b^x \frac{(x-t)^{m-1}}{(m-1)!}h(t)dt, \tag{1.8}$$

where $m \in \mathbb{N}$ and $h \in L_1[b,c],\ b, c \in \mathbb{R}$.
If we use $(m-1)! = \Gamma(m)$ and replace m with any positive real number $\gamma > 0$ in equation (4.1). Then we get the definition of integral fractional.

Definition 1.3.1. *The Riemann-Liouville integral* $\mathcal{J}_b^{\gamma}$ *of fractional order* $\gamma \in \mathbb{R}^+$ *as:*

$$\mathcal{J}_b^{\gamma} h(x) = \int_b^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}h(t)dt, \tag{1.9}$$

where $h \in L_1[b,c]$*, and* $b \le x \le c$*.*

The Riemann-Liouville fractional integral of the function $h(s) = s^{\eta}$ for $\gamma > 0$, and $\eta > -1$, is given as

$$\mathcal{J}_0^{\gamma} s^{\eta} = \frac{1}{\Gamma(\gamma)}\int_0^s (s-t)^{\gamma-1}(s)^{\eta}dt.$$

Substitute $v = \frac{s-t}{s}$ as $t \to 0,\ v \to 1$ and as $t \to s,\ v \to 0$,so, $sdv = -dt$

$$\begin{aligned}
\mathcal{J}_0^{\gamma} s^{\eta} &= \frac{1}{\Gamma(\gamma)} \int_0^1 (sv)^{\gamma-1} (s - sv)^{\eta}\, sdv \\
&= \frac{s^{\gamma+\eta}}{\Gamma(\gamma)} \int_0^1 (v)^{\gamma-1} (1 - v)^{\eta+1-1}\, dv \\
&= \frac{s^{\gamma+\eta}}{\Gamma(\gamma)} \beta(\gamma, \eta + 1).
\end{aligned}$$

Using $\beta(\gamma, \eta + 1) = \frac{\Gamma(\gamma)\Gamma(\eta+1)}{\Gamma(\gamma+\eta+1)}$, we obtained

$$\mathcal{J}_0^{\gamma} s^{\eta} = \frac{\Gamma(\eta + 1)}{\Gamma(\gamma + \eta + 1)} s^{\gamma+\eta}. \tag{1.10}$$

Example 1.3.2. *Consider that $h(s) = s^{\frac{1}{2}}$, then by equation (1.10) the R-L integral of $h(s)$ is*

$$\begin{aligned}
\mathcal{J}_0^{\gamma} s^{\frac{1}{2}} &= \frac{\Gamma(\frac{1}{2} + 1)}{\Gamma(\frac{1}{2} + \gamma + 1)} s^{\frac{1}{2}+\gamma} \\
&= \frac{\sqrt{\pi}}{2\Gamma(\frac{3}{2} + \gamma)} s^{\frac{1}{2}+\gamma}.
\end{aligned} \tag{1.11}$$

Now, we consider some cases i.e.

$$\begin{aligned}
\text{For } \gamma &= \frac{1}{2};\ \mathcal{J}_0^{\frac{1}{2}} s^{\frac{1}{2}} = \frac{\sqrt{\pi} s}{2\Gamma(2)} \approx 0.8862s. \\
\text{For } \gamma &= \frac{3}{2};\ \mathcal{J}_0^{\frac{3}{2}} s^{\frac{1}{2}} = \frac{\sqrt{\pi} s^2}{2\Gamma(3)} \approx 0.4431s^2. \\
\text{For } \gamma &= \frac{5}{2};\ \mathcal{J}_0^{\frac{5}{2}} s^{\frac{1}{2}} = \frac{\sqrt{\pi} s^3}{2\Gamma(4)} \approx 0.1477s^3. \\
\text{For } \gamma &= \frac{7}{2};\ \mathcal{J}_0^{\frac{7}{2}} s^{\frac{1}{2}} = \frac{\sqrt{\pi} s^4}{2\Gamma(5)} \approx 0.0369s^4.
\end{aligned}$$

These integrals are plotted in the Figure1.3.

- **Properties**

(a) Identity operator

If we take $\gamma = 0$, then we obtain the identity operator that is $\mathcal{I}_a^0 h = h$.

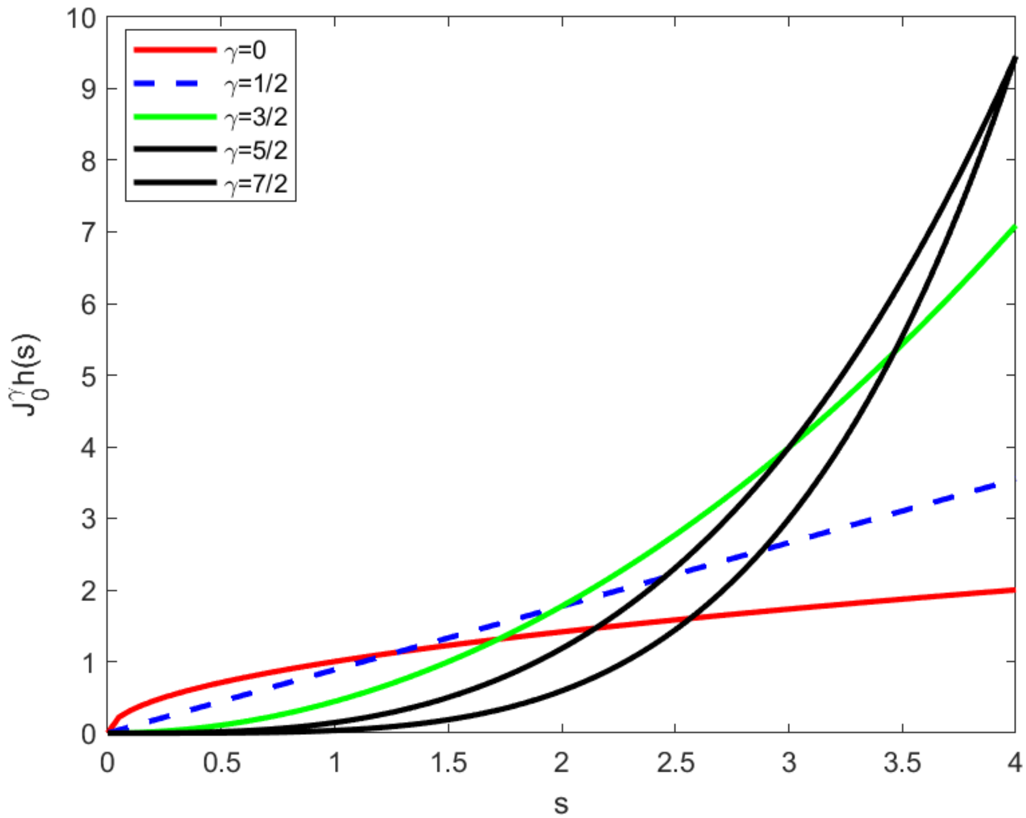


Figure 1.3: The R-L integral of $h(s) = s^{\frac{1}{2}}$ are shown.

(b) Linearity

If functions $g(s)$ and $h(s)$ are continuous $\forall\ s \geq 0$. For some $\gamma > 0$ and $\mu \in \mathbb{C}$. Then

$$\mathcal{J}_b^{\gamma}\left(\mu g(s) + h(s)\right) = \mu \mathcal{J}_b^{\gamma} g(s) + \mathcal{J}_b^{\gamma} h(s).$$

(c) Semi group law

If a function $h(s)$ is continuous $\forall\ s \geq 0$.

$$\mathcal{J}_b^{\gamma}\left(\mathcal{J}_b^{\eta} h(s)\right) = \mathcal{J}_b^{\gamma+\eta} h(s) = \mathcal{J}_b^{\eta}\left(\mathcal{J}_b^{\gamma} h(s)\right). \ \forall\ \gamma, \eta \in \mathbb{R}^{+}.$$

Lemma 1.3.1. *If the function $h(s)$ is continuous for all $s \geq 0$ and the integral $\mathcal{J}_b^{n}$*

exists. Then

$$\mathcal{J}_b^n D^n h(s) = h(s) - \sum_{q=0}^{n-1} \frac{(s-b)^q}{q!} D^q h(b).$$

1.3.2 Riemann-Liouville Derivative

After defining the fractional integral, Now we introduce the fractional derivative. There are variety of derivatives definitions. But here we discuss the Riemann-Liouville and Caputo fractional differential operators [6]-[7].

Definition 1.3.3. *The Riemann-Liouville derivative of a function $h \in L_1[b, c]$ of fraction order $\gamma \in \mathbb{R}^+$ is defined as:*

$$D_b^\gamma h(s) = D_b^k \mathcal{J}^{k-\gamma} h(s).$$

If we use the equation (1.9), then we obtained

$$D_b^\gamma h(s) = \frac{1}{\Gamma(k-\gamma)} \frac{d^k}{ds^k} \int_b^s (s-x)^{k-\gamma-1} h(x) dx.$$

where $\gamma = \lceil k \rceil$.

For $\eta > -1$ and $\gamma > 0$, the Riemann-Liouville derivative of the function $h(s) = (s-b)^\eta$ is obtained by using definition (1.3.3), we have

$$\begin{aligned} D_b^\gamma h(s) &= D^n \mathcal{J}_b^{n-\gamma} h(s) \\ &= D^n \mathcal{J}_b^{n-\gamma} (s-b)^\eta . \end{aligned}$$

Substituting equation (1.10), we get

$$D_b^\gamma h(s) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+n+1)} D^n (s-b)^{n-\gamma+\eta} .$$

Applying $D^n s^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} s^{m-n}$, we obtained

$$D_b^\gamma h(s) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\gamma+1)} (s-b)^{\eta-\gamma} . \tag{1.12}$$

Example 1.3.4. *Make use of the equation (1.12), we have*

$$\begin{aligned} D_0^{\frac{1}{2}} s^{\frac{3}{2}} &= \frac{\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{3}{2}-\frac{1}{2}+1)} (s)^{\frac{3}{2}-\frac{1}{2}} \\ &= \frac{3}{4}\sqrt{\pi} s. \end{aligned}$$

- **Properties**

(a) Identity law

If a function $h \in L_1[b,c]$ and $\gamma \in (k-1,k]$, then

$$D_b^{\gamma} \mathcal{J}_b^{\gamma} h = h.$$

(b) Linearity

If the functions $g, h \in L_1[b,c]$, and $D_b^{\gamma} g(s)$ and $D_b^{\gamma} h(s)$ exists for $\gamma \in (k-1,k]\,,\ k \in \mathbb{N},\ \mu \in \mathbb{C}$. Then the Riemann-Liouville derivative is linear i.e.

$$D_b^{\gamma}\left(\mu g(s) + h(s)\right) = \mu D_b^{\gamma} g(s) + D_b^{\gamma} h(s).$$

(c) Semi group property

If a function $h(s)$ is such that the operator $D_b^{\gamma} h(s)$ is exists for all $\gamma \in (k-1,k]\,,\ \ k \in \mathbb{N}$. Then in general

$$D_b^{\gamma}\left(D_b^{\eta} h\right) = D_b^{\gamma+\eta} h \neq D_b^{\eta}\left(D_b^{\gamma} h\right).$$

Hence, the Riemann-Liouville derivative are not commutative i.e.

$$D_b^{\gamma}\left(D_b^{\eta} h\right) \neq D_b^{\eta}\left(D_b^{\gamma} h\right).$$

Lemma 1.3.2. *Suppose that $\mathcal{J}_b^{k-\gamma} h$ is integrable for $\gamma > 0$, $k = \lceil \gamma \rceil$. Then*

$$\mathcal{J}_b^{\gamma} D_b^{\gamma} h(s) = h(s) - \sum_{q=0}^{n-1} \frac{(s-b)^{\gamma-q-1}}{\Gamma(\gamma-q)} \lim_{s \to b} D^{n-q-1} \mathcal{J}_b^{n-\gamma} h(s).$$

1.3.3 The Caputo fractional differentiation operator

The Italian mathematician Caputo reformulated the definition of Riemann-Liouville derivative to give another definition for fractional derivative [8].

Definition 1.3.5. *The Caputo fractional derivative of order $\gamma \in \mathbb{R}^+$ is defined as:*

$$\begin{aligned} {}^{c}D_b^{\gamma} h(t) &= \mathcal{J}_b^{q-\gamma} D^q h(t) \\ &= \frac{1}{\Gamma(q-\gamma)} \int_b^t \frac{\frac{d^q}{d\eta^q} h(\eta)}{(t-\eta)^{\gamma+1-q}} d\eta, \end{aligned} \tag{1.13}$$

where $\eta \in (q-1,q)$, and if $\eta = q \in \mathbb{N}$ then we obtained $\frac{d^q}{dt^q}$.

Lemma 1.3.3. *Consider that $h(t) = (t-b)^{\rho}$ for some $\rho \in \mathbb{R}$. Then*

$$^{c}D_{b}^{\gamma}h(t) = \begin{cases} \frac{\Gamma(\rho+1)}{\Gamma(\rho-\gamma+1)}(t-b)^{\rho-\gamma}; & \gamma \in (n-1,n)\,,\ \rho > n-1 \\ 0; & \gamma \in (n-1,n)\,,\ \rho \leq n-1. \end{cases} \tag{1.14}$$

- **Properties**

(a) Linearity

If the derivatives $^{c}D_{b}^{\gamma}g$ and $^{c}D_{b}^{\gamma}h$ exists. Then Caputo derivatives are linear i.e

$$^{c}D_{b}^{\gamma}\left(kg(s)+h(s)\right) = k^{c}D_{b}^{\gamma}g(s) + {}^{c}D_{b}^{\gamma}h(s),\ \gamma > 0,\ k \in \mathbb{C}.$$

(b) Semi group law

The semi group law does not hold for the Caputo derivative, if the derivatives $^{c}D_{b}^{\gamma}$ and $^{c}D_{b}^{\eta}$ exist then

$$^{c}D_{b}^{\gamma}\left({}^{c}D_{b}^{\eta}h\right) = {}^{c}D_{b}^{\gamma+\eta}h \neq {}^{c}D_{b}^{\eta}\left({}^{c}D_{b}^{\gamma}h\right).$$

Hence, the Caputo fractional derivative are non-commutative.

Example 1.3.6. *Consider that $n-1 < \gamma \leq n$ and $h(t) = t^{\frac{3}{2}}$. Then, by using Lemma (1.3.3), we obtain*

$$^{c}D_{0}^{\gamma}t^{\frac{3}{2}} = \frac{\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{3}{2}-\gamma+1)}t^{\frac{3}{2}-\gamma},$$

$$= \frac{\frac{3}{4}\sqrt{\pi}}{\Gamma(\frac{5}{2}-\gamma)}t^{\frac{3}{2}-\gamma}, \tag{1.15}$$

Now, we consider some cases for fixed values of γ i.e. $\gamma = \frac{2}{3}$, $\gamma = \frac{5}{4}$, $\gamma = \frac{7}{6}$ and $\gamma = \frac{9}{7}$.

$$\text{For } \gamma = \frac{2}{3},\ {}^{c}D_{0}^{\frac{2}{3}}t^{\frac{3}{2}} = \frac{\frac{3}{4}\sqrt{\pi}}{\Gamma(\frac{5}{2}-\frac{2}{3})}t^{\frac{3}{2}-\frac{2}{3}} = 1.2000t^{\frac{5}{6}}.$$

$$\text{For } \gamma = \frac{5}{4},\ {}^{c}D_{0}^{\frac{5}{4}}t^{\frac{3}{2}} = \frac{\frac{3}{4}\sqrt{\pi}}{\Gamma(\frac{5}{2}-\frac{5}{4})}t^{\frac{3}{2}-\frac{5}{4}} = 1.4131t^{\frac{1}{4}}.$$

$$\text{For } \gamma = \frac{7}{6},\ {}^{c}D_{0}^{\frac{7}{6}}t^{\frac{3}{2}} = \frac{\frac{3}{4}\sqrt{\pi}}{\Gamma(\frac{5}{2}-\frac{7}{6})}t^{\frac{3}{2}-\frac{7}{6}} = 1.4886t^{\frac{1}{3}}.$$

$$\text{For } \gamma = \frac{9}{7},\ {}^{c}D_{0}^{\frac{9}{7}}t^{\frac{3}{2}} = \frac{\frac{3}{4}\sqrt{\pi}}{\Gamma(\frac{5}{2}-\frac{9}{7})}t^{\frac{3}{2}-\frac{9}{7}} = 1.4536t^{\frac{13}{18}}.$$

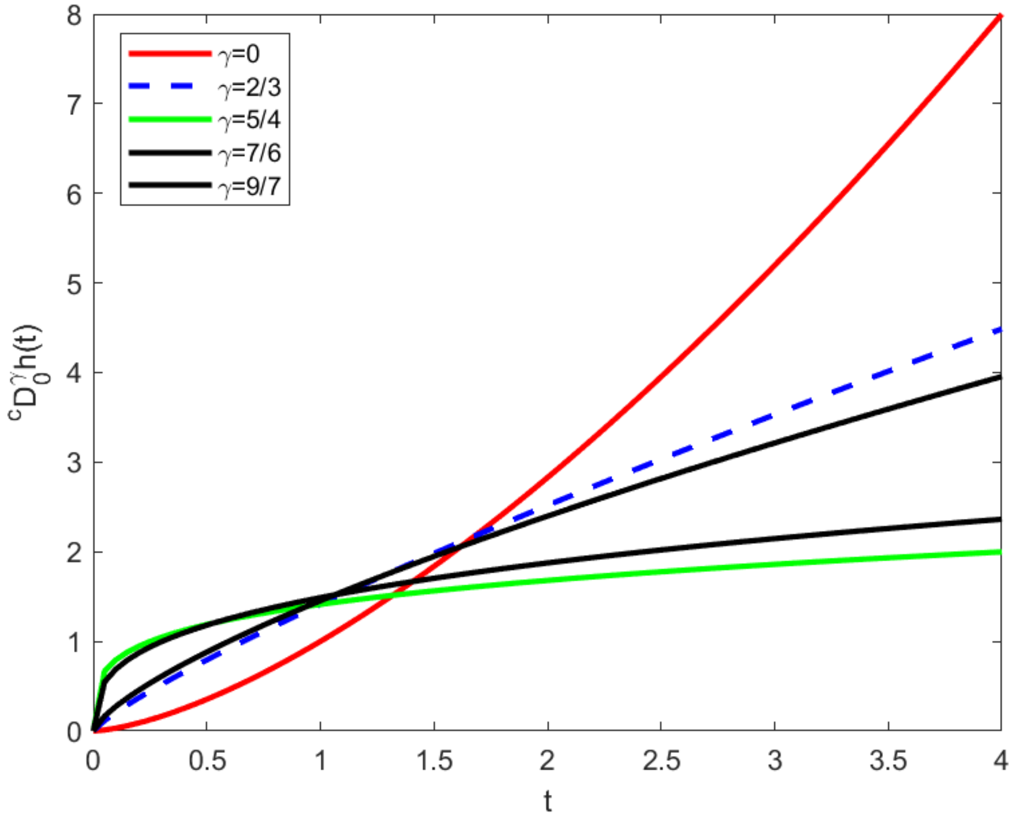


Figure 1.4: The Caputo fractional derivatives of $h(t) = t^{\frac{3}{2}}$

Theorem 1.3.1. *Assume that $h \in L_1[b,c]$ such that the derivatives ${}^cD_b^\gamma h$ and $D_b^\gamma h$ exists, for $\gamma \in \mathbb{R}^+$ and $n = \lceil \gamma \rceil$. Then*

$$^cD_b^\gamma h(s) = D_b^\gamma h(s) - \sum_{q=0}^{n-1} \frac{D^q h(b)}{\Gamma(q-\gamma+1)} (s-b)^{q-\gamma}.$$

Lemma 1.3.4. *If a function $h \in L_1[b,c]$ and $\gamma \geq 0$. Then*

$$^cD_b^\gamma \mathcal{J}_b^\gamma h = h.$$

Lemma 1.3.5. *Let us consider that $\gamma \in \mathbb{R}^+$, $n = \lceil \gamma \rceil$ and $h \in C^n[b,c]$. Then*

$$\mathcal{J}_b^{\gamma c} D_b^\gamma h(s) = h(s) - \sum_{q=0}^{n-1} \frac{D^q h(b)}{q!} (s-b)^q.$$

1.3.4 Hadamard operator

Let $\gamma > 0$, and the function $h(t) \in L^p[b,c]$, then Hadamard integral is defined as [9];

$$^{H}\mathcal{J}_b^{\gamma} h(t) = \frac{1}{\Gamma(\gamma)} \int_b^t \left(\ln(\frac{t}{x}) \right)^{\gamma-1} \frac{h(x)}{x} dx.$$

and if $h \in C^m[b,c]$, then the Hadamard fractional derivative is defined as

$$^{H}\mathcal{D}_b^{\gamma} h(t) = \left(t\frac{d}{dt} \right)^m \mathcal{J}_b^{m-\gamma} h(t),$$

respectively, where $m = \lceil \gamma \rceil + 1 \in \mathbb{N}$.

1.4 The Laplace transform

In this part, we will discuss the definition, properties and the results of the Laplace transform [10]. It is used for solving the initial value problem on the domain $[0, \infty)$. Now a days, Laplace transform is used in the areas of mathematics, physics and engineering etc.

Definition 1.4.1. *The Laplace transform is defined on* $(0, \infty)$ *for the function* $g(p)$ *by*

$$\mathcal{L}\{g(p)\} = g(s) = \int_0^{\infty} e^{-sp} g(p) dp, \; Re(s) > 0.$$

This transform was first discovered by the French mathematician Pierre-simon Laplace.

Definition 1.4.2. *The inverse Laplace transform is defined as:*

$$\mathcal{L}^{-1}\{g(p)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sp} f(s) ds, \; Re(s) > 0.$$

Lemma 1.4.1 (Properties of Laplace transform)**.** *Suppose that the Laplace transform of functions* $g(p)$ *and* $h(p)$ *exists, that is* $G(s)$ *and* $H(s)$ *respectively. Then the following holds [10]:*
(a) The Linear property of the Laplace transform is

$$\mathcal{L}\{\mu g(p) + h(p)\} = \mu G(s) + H(s). \; \mu \in \mathbb{R}.$$

(b) The Laplace transform of convolution of $g(p)$ and $h(p)$ is given as

$$\mathcal{L}\left\{g(p) * h(p)\right\} = G(s)H(s).$$

and the convolution is given as follows:

$$g(p) * h(p) = \int_0^p g(p-\rho)h(\rho)d\rho = \int_0^p g(\eta)h(p-\rho)d\rho.$$

(d) The k-times derivative($k \in \mathbb{N}$) of the Laplace transform is stated as:

$$\mathcal{L}\left\{g^{(k)}(p)\right\} = s^k G(s) - \sum_{q=0}^{n-1} s^{k-q-1} g^{(q)}(0). \tag{1.16}$$

Example 1.4.3. *The Laplace transform of Bessel function of order zero is*

$$\mathcal{L}\left\{J_0(ap)\right\} = \frac{1}{\sqrt{s^2+a^2}}.$$

Lemma 1.4.2 (Laplace transform of the fractional operator)**.** *Let us consider that $p > 0$ and assume the Laplace transform of $g(p)$ exists [10]. Then the Laplace transform of the following operator is given as*

(a) The Laplace transform of Riemann-Liouville fractional derivative of the order $\gamma > 0$ is

$$\mathcal{L}\left\{D_0^\gamma g(p)\right\} = s^\gamma g(s) - \sum_{q=0}^{m-1} s^{m-q-1} D^q \mathcal{J}_0^{n-\gamma} g(p) \mid_{p=0} .$$

(b) If $\mathcal{J}_0^\gamma$ denote the fractional integral, Then its Laplace transform is given by

$$\mathcal{L}\left\{\mathcal{J}_0^\gamma g(p)\right\} = \frac{G(s)}{s^\gamma}.$$

Theorem 1.4.1. *If the $\mathcal{L}\left\{h(t)\right\}$ exists that is $H(s)$ and $p > 0$. Then the Laplace transform of the Caputo fractional derivative is*

$$\mathcal{L}\left\{{}^c D_0^\gamma g(p)\right\} = s^\gamma G(s) - \sum_{q=0}^{m-1} s^{\gamma-q-1} g^{(q)}(0).$$

1.4.1 Application of Laplace transform

Laplace transform are used to solve the initial value problem. we will use classical Laplace transform to solve ordinary differential equations (ODEs) and partial differential equations (PDEs).

Example 1.4.4. *Consider the initial value differential equation*

$$\frac{dy}{dp} + hy(p) = \varphi(p);\ p > 0,\ y(0) = b. \tag{1.17}$$

Applying Laplace transform on the equation (1.17) and using Lemma (1.4.1(d)), we obtain

$$y(s) = \frac{b}{s+h} + \frac{\varphi(s)}{s+h}. \tag{1.18}$$

Applying Laplace inverse on the equation (1.18) and using convolution theorem, we get

$$y(p) = be^{-hp} + \int_0^p \varphi(p-\rho)e^{-h\rho}d\rho.$$

1.5 Fractional integral and derivatives with respect to another Function

There are wide number of definitions of a fractional integrals and derivatives. Recently researchers developed theory of generalized fractional derivatives and integrals which hold a large of fractional operators as special case. Particularly in what follows, we focus on fractional operators of functions w.r.t other functions

Definition 1.5.1. *Suppose that h is an integrable function and $\psi \in C^1[b,c]$ be an increasing function such that $\psi'(s) \neq 0, \forall\ s \in I$, where interval I is $-\infty \leq b < c \leq +\infty$ and let $\gamma > 0$ and $m = \lceil \gamma \rceil$. Then fractional integrals and derivatives with respect to an other function ψ are defined as [11, 12, 13]*

$$\mathcal{J}_{b^+}^{\gamma;\psi} h(s) = \frac{1}{\Gamma(\gamma)} \int_b^s \psi'(x)\left(\psi(s) - \psi(x)\right)^{\gamma-1} h(x)dx,$$

and

$$\mathcal{D}_{b^+}^{\gamma,\psi}h(s) = \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^m \mathcal{J}_b^{m-\gamma}h(s),$$
$$= \frac{1}{\Gamma(m-\gamma)}\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^m \int_b^s \psi'(x)\left(\psi(s)-\psi(x)\right)^{m-\gamma-1}h(x)dx,$$

respectively.

Similarly, right fractional integral and derivative are defined as:

$$\mathcal{J}_{b^-}^{\gamma,\psi}h(s) = \frac{1}{\Gamma(\gamma)}\int_s^b \psi'(x)\left(\psi(x)-\psi(s)\right)^{\gamma-1}h(x)dx,$$

and

$$\begin{aligned} D_{b^-}^{\gamma,\psi}h(s) &= \left(-\frac{1}{\psi'(s)}\frac{d}{ds}\right)^m \mathcal{J}_{b^-}^{m-\gamma,\psi}h(s) \\ &= \frac{1}{\Gamma(m-\gamma)}\left(-\frac{1}{\psi'(s)}\frac{d}{ds}\right)^m \int_s^b \psi'(x)\left(\psi(x)-\psi(s)\right)^{m-\gamma-1}h(x)dx. \end{aligned} \tag{1.19}$$

In equation (1.19) if $\psi(s) = s$, Riemann-Liouville derivative is obtained and for $\psi(s) = \ln(s)$, then we get Hadamard operator is retrieved.
Semi group property:
If $\gamma, \eta > 0$, then fractional integral holds the semi group property

$$\mathcal{J}_b^{\gamma,\psi}\left(\mathcal{J}_b^{\eta,\psi}h(s)\right) = \mathcal{J}_b^{\gamma+\eta,\psi}h(s) = \mathcal{J}_b^{\eta,\psi}\left(\mathcal{J}_b^{\gamma,\psi}h(s)\right).$$

Lemma 1.5.1. *[11] Let γ, $\mu > 0$,*
(1) *If $u(s) = (\psi(s)-\psi(b))^{\mu-1}$. Then*

$$\mathcal{J}_{b^+}^{\gamma,\psi}u(s) = \frac{\Gamma(\mu)}{\Gamma(\gamma+\mu)}\left(\psi(s)-\psi(b)\right)^{\gamma+\mu-1}.$$

(2) *If $v(s) = (\psi(a)-\psi(s))^{\mu-1}$, then*

$$\mathcal{J}_{a^-}^{\gamma,\psi}v(s) = \frac{\Gamma(\mu)}{\Gamma(\gamma+\mu)}\left(\psi(a)-\psi(s)\right)^{\gamma+\mu-1}.$$

1.5.1 Caputo fractional derivative with respect to ψ-function

In this section, we focus on a Caputo fractional derivative with respect to ψ-function. We also present some properties and results of it. Almeida [14] using the concept of the Caputo fractional derivative, introduced a new definition called ψ-Caputo derivative with respect to ψ-function.

Definition 1.5.2. *Suppose that $\gamma > 0$, $m \in \mathbb{N}$, ψ is an increasing function such that $\psi'(s) \neq 0$, for all $s \in I$, I is an interval $-\infty \leq b < c \leq +\infty$ and $h, \psi \in C^m([b, c])$. Then left ψ-Caputo derivative is defined as;*

$$\begin{aligned} {}^cD_{b^+}^{\gamma,\psi} h(s) &= \mathcal{J}_{b^+}^{m-\gamma,\psi} \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^m h(s), \\ &= \frac{1}{\Gamma(m-\gamma)} \int_b^s \psi'(x) \left(\psi(s) - \psi(x)\right)^{m-\gamma-1} \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^m h(s)ds. \end{aligned}$$

And the right ψ-Caputo derivative is

$${}^cD_{a^-}^{\gamma,\psi} h(s) = \mathcal{J}_{a^-}^{m-\gamma,\psi} \left(-\frac{1}{\psi'(s)} \frac{d}{ds} \right)^m h(s),$$

where $m = \lceil \gamma \rceil$.

Lemma 1.5.2. *Suppose that a function $h \in C[b, c]$ and $\gamma > 0$, then we have*

$${}^cD_b^{\gamma,\psi} \mathcal{J}_b^{\gamma,\psi} h(t) = h(t).$$

Lemma 1.5.3. *Let $\gamma > 0$ and $\mu > 0$.*
(a) If $u(s) = (\psi(s) - \psi(b))^{\mu-1}$, then

$${}^cD_{b^+}^{\gamma,\psi} u(s) = \frac{\Gamma(\mu)}{\Gamma(\mu-\gamma)} \left(\psi(s) - \psi(b)\right)^{\gamma+\mu-1}.$$

(b) If $v(s) = (\psi(b) - \psi(s))^{\mu-1}$, then

$${}^cD_{b^-}^{\gamma,\psi} v(s) = \frac{\Gamma(\mu)}{\Gamma(\mu-\gamma)} \left(\psi(b) - \psi(s)\right)^{\gamma+\mu-1}.$$

If $h \in C^m[b, c]$ and $\gamma > 0$, then

1.6 The ψ-Laplace Transform

In this section, the definition of the generalized Laplace transform is discussed which has been introduced by the Abdeljawad and Jarad in 2019 $\big($ [15]$\big)$. Which can be used to solve the fractional differential equation involving psi-Riemann-Liouville, and psi-Caputo fractional derivatives. It is also used to solve dynamical systems depending on the fractional operator. This new $\psi-$Laplace transform is the generalization of the classical Laplace transform. Some important properties, results and applications of $\psi-$ Laplace transform are also part of this section.

Definition 1.6.1. *Let $\psi > 0$, be an increasing function such that $\psi(0) = 0$, and h be a real valued function $h : [0, +\infty) \to \mathbb{R}$. Then ψ-Laplace transform is defined by*

$$h(p) = \mathcal{L}_{\psi}\left\{h(s)\right\} = \int_0^{\infty} e^{-p(\psi(s))}\psi'(s)h(s)ds.$$

this integral is valid for all p.

Theorem 1.6.1 (Existence condition of ψ-Laplace transform)**.** *[15] If h is of exponential order-ψ and $h : [0, \infty) \to \mathbb{R}$ is a piecewise continuous function, then its $\psi-$Laplace transform exists for $p > c$.*

Theorem 1.6.2 (Relation between $\psi-$Laplace transform and classical Laplace transform)**.** *Let $h, \psi : [b, \infty) \to \mathbb{R}$ be a real valued continuous function such that $\psi'(s) > 0$ and the $\psi-$Laplace transform of h exists. Then*

$$\mathcal{L}_{\psi}\left\{h(s)\right\}(v) = \mathcal{L}\left\{h\left(\psi^{-1}\left(s + \psi(b)\right)\right)\right\}(v).$$

Lemma 1.6.1 (Linearity property)**.** *If the ψ-Laplace transform of the functions g and h exists on $[a, \infty)$. Then, for constant $v > C$, then $\psi-$Laplace transform is linear i.e.*

$$\mathcal{L}_{\psi}\left\{Cg(s) + h(s)\right\} = C\mathcal{L}_{\psi}\left\{g(s)\right\}(v) + \mathcal{L}_{\psi}\left\{h(s)\right\}(v).$$

Lemma 1.6.2. *If ψ-Laplace transform exists, then the following properties hold*

$$(a)\quad \mathcal{L}_{\psi}\left\{c\right\} = \frac{c}{v},\ \ for\ v > 0,\ \ c\ is\ constant.$$

$$(b)\quad \mathcal{L}_{\psi}\left\{(\psi(s))^n\right\} = \frac{n!}{v^{n+1}}.$$

Theorem 1.6.3 (ψ-Laplace transform of fractional derivatives w.r.t another function)**.** *If* $h(s), \mathcal{J}_0^{m-\gamma,\psi}h(s),\ D^{1,\psi}\mathcal{J}_0^{m-\gamma,\psi}h(s), ..., D^{m-1,\psi}\mathcal{J}_0^{m-\gamma,\psi}h(s)$, *exists, for* $\gamma > 0$, *and* $m = \lceil\gamma\rceil + 1$, *where* $D^{k,\psi} = \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^k$, *are continuous on* $\mathbb{R}^+$ *and of* ψ*-exponential order, while* $D_0^{\gamma,\psi}h(s)$ *is piecewise continuous on* $[0, +\infty)$*. Then*

$$(a)\quad \mathcal{L}_\psi\left\{D_0^{\gamma,\psi}h(s)\right\} = v^n\mathcal{L}_\psi\left\{h(s)\right\} - \sum_{q=0}^{n-1} v^{n-q-1}\left(\mathcal{J}_0^{n-q-\gamma,\psi}h\right)(0).$$

$$(b)\quad \mathcal{L}_\psi\left\{{}^cD_0^{\gamma,\psi}h(s)\right\} = v^n\mathcal{L}_\psi\left\{h(s)\right\} - \sum_{q=0}^{n-1} v^{\gamma-q-1}\left(D^{q,\psi}h\right)(0).$$

Theorem 1.6.4. *Let* $\gamma > 0$ *and* h *be a continuous function over the finite interval* $[0, T]$*, of* ψ*-exponential. Then*

$$\mathcal{L}_\psi\left\{\left(\mathcal{J}_0^{\gamma,\psi}h\right)(s)\right\} = v^{-\gamma}\mathcal{L}_\psi\left\{h(s)\right\}.$$

Lemma 1.6.3. *Suppose that* $\Re(\gamma) > 0$ *and* $|\frac{\mu}{v^\gamma}| < 1$*. Then*

$$\mathcal{L}_\psi\left\{E_n\left(\mu\left(\psi(s)\right)^\gamma\right)\right\} = \frac{v^{\gamma-1}}{v^\gamma - 1}.$$

Theorem 1.6.5 (Convolution theorem)**.** *[15] Suppose* h *and* g *are of exponential order and piecewise continuous on each* $[0, T]$*. Then the convolution of* h *and* g *is defined as:*

$$\left(h *_\psi g\right)(s) = \int_0^{s=\psi^{-1}(\psi(s))} h(\psi^{-1}\left(\psi(s) - \psi(\rho)\right)g(\rho)\psi'(\rho)d\rho.$$

1.6.1 Applications of ψ-Laplace transform

In this subsection, we use ψ-Laplace transform to solve the differential equations and also verified. For the sake of simplicity ${}^c\mathcal{D}_0^{\gamma,\psi}$ will be denoted by ${}^c\mathcal{D}^{\gamma,\psi}$.

Lemma 1.6.4. *Consider the linear homogeneous differential equation*

$${}^c\mathcal{D}^{1,\psi}g(t) - g(t) = 0,\ g(0) \neq 0, \tag{1.20}$$

then the solution of equation (1.20) is $g(t) = g(0)e^{\psi(t)}$*.*

Proof. Applying $\mathcal{L}_\psi$ on both sides of equation (1.20)

$$\mathcal{L}_\psi\left[{}^c\mathcal{D}^{1,\psi}g(t)\right] - \mathcal{L}_\psi\left[g(t)\right] = 0$$

using theorem 1.6.3(a) and initial condition, we obtained

$$vg(v) - \sum_{k=0}^{1} v^{-k}\left(\mathcal{D}^{k,\psi}g\right)(0) - g(v) = 0$$

$$g(0) = \frac{g(0)}{v-1} \tag{1.21}$$

Applying $\mathcal{L}_\psi^{-1}$ on the equation (1.21), we get

$$g(t) = g(0)e^{\psi(t)}$$

□

Lemma 1.6.5. *Consider the non-homogeneous differential equation*

$${}^c\mathcal{D}^{1,\psi}g(t) - g(t) = h(t),\ \ g(0) = 0. \tag{1.22}$$

Then the solution of equation (1.22) is

$$g(t) = \int_0^t e^{(\psi(t)-\psi(\rho))}\psi'(\rho)h(\rho)d\rho.$$

Proof. Applying $\mathcal{L}_\psi$ on both sides of equation (1.22) and using Theorem 1.6.3(a), we obtain

$$vg(v) - \sum_{k=0}^{1} v^{-k}\left(\mathcal{D}^{k,\psi}g\right)(0) - g(v) = h(v).$$

Using condition $g(0) = 0$, and simplifying, we get;

$$g(v) = \frac{h(v)}{v-1}. \tag{1.23}$$

Applying $\mathcal{L}_\psi^{-1}$ on the equation (1.23) and using Convolution theorem (1.6.5), we obtained

$$g(t) = \int_0^t e^{(\psi(t)-\psi(\rho))}\psi'(\rho)h(\rho)d\rho.$$

□

Now we verify that $g(t)$ given by

$$g(t) = \int_0^t e^{(\psi(t)-\psi(\rho))}\psi'(\rho)h(\rho)d\rho. \tag{1.24}$$

Satisfies the initial value problem (1.22).

Applying ${}^c\mathcal{D}^{1,\psi}$ on the equation (1.24), and using Leibniz theorem, we obtained

$$\begin{aligned} {}^c\mathcal{D}^{1,\psi} &= h(t) + \int_0^t e^{(\psi(t)-\psi(\tau))}\psi'(\tau)h(\tau)d\tau \\ {}^c\mathcal{D}^{1,\psi}g(t) - g(t) &= h(t). \end{aligned}$$

Lemma 1.6.6. *Consider the higher order linear differential equation*

$${}^c\mathcal{D}^{n,\psi}g(t) - g(t) = 0. \tag{1.25}$$

$$g(0) = 0, \ \left(\mathcal{D}^{1,\psi}\right)(0) = 0, \ \ldots, \ \left(\mathcal{D}^{m-1,\psi}\right)(0) = 0. \tag{1.26}$$

Then the solution of equations (1.25)-(1.26) is given by

$$g(t) = \left(\mathcal{D}^{m-1,\psi}g\right)(0)\left(\psi(t)\right)^{m-1}E_{n,m}\left(\psi(t)\right)^n$$

Proof. Applying $\mathcal{L}_\psi$ on both sides of equation (1.25) and using Theorem 1.6.3(b), we obtained

$$v^n g(v) - \sum_{k=0}^{m-1} s^{n-k-1}\left(\mathcal{D}^{m-1,\psi}g\right)(0) - g(v) = 0.$$

Using condition (1.26), we get

$$g(v) = \frac{v^{n-m}\left(\mathcal{D}^{m-1,\psi}g\right)(0)}{v^n - 1}. \tag{1.27}$$

Applying $\mathcal{L}_\psi^{-1}$ on equation (1.27),

$$\begin{aligned} g(t) = &\left(\mathcal{D}^{m-1,\psi}g\right)(0)\sum_{k=0}^{\infty}\mathcal{L}_\psi^{-1}\left[\frac{1}{v^{nk+m}}\right] \\ = &\left(\mathcal{D}^{m-1,\psi}g\right)(0)\sum_{k=0}^{\infty}\frac{\left(\psi(t)\right)^{nk+m-1}}{\Gamma(nk+m)} \\ g(t) = &\left(\mathcal{D}^{m-1,\psi}g\right)(0)\left(\psi(t)\right)^{m-1}E_{n,m}\left(\left(\psi(t)\right)^n\right). \end{aligned}$$

□

Lemma 1.6.7. *Consider the higher order non-linear differential equation*

$$ {}^{c}\mathcal{D}^{n,\psi}g(t) - g(t) = h(t), \tag{1.28} $$

$$ g(0) = 0, \ \left(\mathcal{D}^{1,\psi}g\right)(0) = 0, \ ..., \left(\mathcal{D}^{m-1,\psi}g\right)(0) = 0. \tag{1.29} $$

then the solution of the equation (1.28)-(1.29) is given by

$$ g(t) = \int_0^t (\psi(t) - \psi(\rho))^{n-1} E_{n,n}\left((\psi(t) - \psi(\rho))^n\right)\psi'(\rho)h(\rho)d\rho. $$

Proof. Applying $\mathcal{L}_\psi$ on both sides of equation (1.28) and using Theorem 1.6.3(b), we obtain

$$ v^n g(v) - \sum_{k=0}^{m-1} v^{n-k-1}\left(\mathcal{D}^{k,\psi}g\right)(0) - g(v) = h(v). $$

By using equation (1.29), we get

$$ g(v) = \frac{h(v)}{v^n - 1}. \tag{1.30} $$

Applying $\mathcal{L}_\psi^{-1}$ on the equation (1.30) and using convolution theorem (1.6.5), we obtained

$$ g(t) = \int_0^t (\psi(t) - \psi(\rho))^{n-1} E_{n,n}\left((\psi(t) - \psi(\rho))^n\right)\psi'(\rho)h(\tau)d\rho. $$

□

Lemma 1.6.8. *Consider the linear differential equation*

$$ {}^{c}\mathcal{D}^{\gamma,\psi}g(t) - wg(t) = 0, \ g(0) \neq 0; \quad 0 < \gamma \leq 1. \tag{1.31} $$

Then the solution of equation (1.31) is given by

$$ g(t) = g(0)E_\gamma\left(w(\psi(t))^\gamma\right). $$

Proof. Applying $\mathcal{L}_\psi$ on both sides of equation (1.31), we have

$$ \mathcal{L}_\psi\left[{}^{c}\mathcal{D}^{\gamma,\psi}g(t)\right] - w\mathcal{L}_\psi\left[g(t)\right] = 0. $$

Now, using theorem 1.6.3, we obtained

$$v^{\gamma}g(v)-\sum_{q=0}^{m-1}v^{\gamma-q-1}\left(\mathcal{D}^{q,\psi}g\right)(0)-wg(s)=0.$$

Using condition $g(0)\neq 0$, we obtained

$$x(v)=\frac{g(0)v^{\gamma-1}}{v^{\gamma}-w}.$$

After simplification, we get

$$g(v)=g(0)\sum_{q=0}^{\infty}\frac{(w)^{q}}{v^{\gamma q+1}} \tag{1.32}$$

Applying $\mathcal{L}_{\psi}^{-1}$ on both sides of equation (1.32), we obtained

$$g(t)=g(0)E_{\gamma}\left(w(\psi(t))^{\gamma}\right).$$

□

Lemma 1.6.9. *Consider the non-linear differential equation*

$$^{c}\mathcal{D}^{\gamma,\psi}g(t)-wg(t)=0,\ \ n-1<\gamma\leq n. \tag{1.33}$$

$$g(0)=0,\ \left(\mathcal{D}^{1,\psi}g\right)(0)=0,...,\ \left(\mathcal{D}^{n-1,\psi}g\right)(0)\neq 0. \tag{1.34}$$

then solution of the equation (1.33-1.34) is given by

$$g(t)=\left(\mathcal{D}^{m-1,\psi}g\right)(0)\left(\psi(t)\right)^{m-1}E_{\gamma,m}\left(w(\psi(t))^{\gamma}\right).$$

Proof. Applying $\mathcal{L}_{\psi}$ on both sides of equation (1.33), and using theorem1.6.3(b), we obtain

$$v^{\gamma}g(v)-\sum_{q=0}^{m-1}s^{\gamma-q-1}\left(\mathcal{D}^{q,\psi}g\right)(0)-wg(v)=0.$$

Now, using equation (1.34), we get

$$g(v)=\left(\mathcal{D}^{m-1,\psi}g\right)(0)\sum_{q=0}^{\infty}\frac{(w)^{q}}{v^{\gamma q+m}}. \tag{1.35}$$

Applying $\mathcal{L}_{\psi}^{-1}$ on equation (1.35), we obtained

$$g(t)=\left(\mathcal{D}^{m-1,\psi}g\right)(0)\left(\psi(t)\right)^{m-1}E_{\gamma,m}\left(w(\psi(t))^{\gamma}\right).$$

□

Verification:

Now, we verify that the function

$$g(t) = \left(\mathcal{D}^{m-1,\psi} g\right)(0)\left(\psi(t)\right)^{m-1} E_{\gamma,m}\left(w(\psi(t))^{\gamma}\right). \tag{1.36}$$

is solution of the problem (1.33), (1.34).

Applying ${}^{c}\mathcal{D}^{\gamma,\psi}$ on both sides of equation (1.36), and using definition of Mittag-Leffler function, we obtain

$$ {}^{c}\mathcal{D}^{\gamma,\psi} g(t) = \left(\mathcal{D}^{m-1,\psi} g\right)(0) {}^{c}\mathcal{D}^{\gamma,\psi} \sum_{q=0}^{\infty} \frac{(w)^q \left(\psi(t)\right)^{\gamma q+m-1}}{\Gamma(\gamma q+m)}.$$

By using definition (1.5.2)

$$\begin{aligned} {}^{c}\mathcal{D}^{\gamma,\psi} g(t) =& \left(\mathcal{D}^{m-1,\psi} g\right)(0) \sum_{q=1}^{\infty} \left(\frac{1}{\psi'(t)}\right)^{\gamma} \frac{(w)^q}{\Gamma(\gamma q+m)} \frac{\left(\psi(t)\right)^{\gamma q+m-\gamma-1} \left(\psi'(t)\right)^{\gamma} \Gamma(\gamma q+m)}{\Gamma(\gamma q+m-\gamma)} \\ =& w\left(\mathcal{D}^{m-1,\psi} g\right)(0)\left(\psi(t)\right)^{m-1} E_{\gamma,m}\left(w(\psi(t))^{\gamma}\right) \end{aligned}$$

$$ {}^{c}\mathcal{D}^{\gamma,\psi} g(t) = w g(t)$$

$$ {}^{c}\mathcal{D}^{\gamma,\psi} g(t) - w g(t) = 0.$$

Chapter 2

Generalization of the Duhamel's principle

The classical Duhamel's principle introduced by the French mathematician and physicist Jean- Marie Duhamel's in 1830, is well-known. The main aim of this theory is to reduce the Cauchy problem for the given non-homogeneous PDE to the corresponding homogeneous PDE, which is easy to solve.
The classical Duhamel's principle is not directly applicable to the fractional order Cauchy problem because the non-homogeneous fractional order differential equations cannot be reduced directly to the corresponding homogeneous equation. S. Umarov generalized this famous principle for the Cauchy problem of fractional order non-homogeneous generalized differential operators. ([16],[17]).
In this chapter, we generalize Duhamel's principle for the generalized differential operators, including the generalized Caputo fractional differential operator. For simplicity ${}^cD_0^\gamma$ and ${}^cD_0^{\gamma,\psi}$ will be denoted by ${}^cD^\gamma$ and ${}^cD^{\gamma,\psi}$ respectively.

2.1 Duhamel's principle for ODEs and PDEs

In this section, we present the Duhamel's principle for the PDEs and ODEs involving the integer order derivatives. By using Duhamel's principle, we can find the solution of the ODEs and PDEs. This principle allows us to solve non-homogeneous PDE by considering the solution of homogeneous PDE. First, we present Duhamel's principle for ODEs.

2.1.1 Duhamel's principle for ODES

Duhamel's principle states that the solution of homogeneous IVP, can be obtained by the solution of homogeneous IVP [18]. Consider

$$w'(t) + kw(t) = \theta(t),\ t > 0;\ w(0) = 0. \tag{2.1}$$

Let $G(t;\rho)$ be the solution of the homogeneous problem

$$G'(t;\rho) + kG(t;\rho) = 0,\ t \in \mathbb{R}^+;\ G(0;\rho) = \theta(\rho), \tag{2.2}$$

where ρ a new parameters has been introduced. The solution of the problem (2.2) is

$$G(t;\rho) = \theta(\rho)\exp(-kt).$$

The solution of the nonhomogeneous IVP (2.1) is the integral of the solution of the corresponding homogeneous IVP $G(t,\rho)$ (with t replaced by $t-\rho$), with a source is involved as initial condition.

Theorem 2.1.1. *The solution of the non-homogeneous IVP*

$$\frac{d}{dt}w(t) + kw(t) = \theta(t).\ t \in \mathbb{R}^+, \tag{2.3}$$

with initial condition $w(0) = 0$ *is given by*

$$w(t) = \int_0^t G(t-\rho;\rho)d\rho, \tag{2.4}$$

where $G(t;\rho)$ *is solution of the homogeneous problem*

$$\frac{d}{dt}G(t) + kG(t) = 0,\ t > 0; \tag{2.5}$$

satisfying initial condition

$$G(0;\rho) = \theta(\rho),\ \rho \in \mathbb{R}^+.$$

This principle also holds for the second order ODEs with initial conditions. Now we consider the second order ODEs.

Theorem 2.1.2. *The solution of the non-homogeneous problem*

$$\frac{d^2g(t)}{dt^2} + k^2g(t) = h(t), \;\; g(0) = 0, \;\; g'(0) = 0, \tag{2.6}$$

is given by $g(t) = \int_0^t w(t-\rho, \rho)d\rho$*, where* $w(t, \rho)$ *is solution of the problem*

$$\frac{d^2w(t)}{dt^2} + k^2w(t) = 0, \;\; w(0) = 0, \;\; w'(0) = \theta(\rho). \tag{2.7}$$

Proof. Since the solution of the problem (2.6) is

$$g(t) = \int_0^t w(t-\rho;\rho)d\rho. \tag{2.8}$$

Differentiating equation (2.8) and using Leibniz rule, we have

$$\frac{dg(t)}{dt} = w(0) + \int_0^t \frac{dw}{dt}(t-\rho;\rho)d\rho. \tag{2.9}$$

Using $w(0) = 0$ in equation (2.9), we get

$$\frac{dg(t)}{dt} = \int_0^t \frac{dw}{dt}(t-\rho;\rho)d\rho. \tag{2.10}$$

Again differentiating equation (2.10) and using Leibniz rule, we have

$$\frac{d^2g}{dt^2}(t) = w'(0) + \int_0^t \frac{d^2w(t-\rho;\rho)}{dt^2}d\rho. \tag{2.11}$$

Using $w'(0) = \theta(\rho)$, in equation (2.11) and we obtain

$$\frac{d^2g(t)}{dt^2} = \theta(\rho) + \int_0^t \frac{d^2w(t-\rho;\rho)}{dt^2}d\rho. \tag{2.12}$$

Using equation (2.12) in equation (2.6) , we have

$$\frac{d^2g(t)}{dt^2} + k^2g(t) = \theta(\rho) + \int_0^t \frac{d^2w(t-\rho;\rho)}{dt^2} + k^2\int_0^t w(t-\rho;\rho)d\rho,$$

$$= \theta(\rho) + \int_0^t \left[\frac{d^2w(t-\rho;\rho)}{dt^2} + k^2w(t-\rho;\rho)\right]d\rho. \tag{2.13}$$

Using equation (2.7) in (2.13), we obtained

$$\frac{d^2g(t)}{dt^2} + k^2g(t) = \theta(\rho).$$

□

Remark 2.1.1. *The solution of the problem (2.7) is*

$$w(t) = k^{-1}\sin(kt)\theta(t). \tag{2.14}$$

We also know that the solution of the problem (2.6) is

$$g(t) = \int_0^t k^{-1}\sin(k(t-\rho))\theta(\rho)d\rho. \tag{2.15}$$

Now by comparing the solutions (2.14) and (2.15), we see that the solution (2.15) is equivalent to

$$g(t) = \int_0^t w(t-\rho;\rho)d\rho,$$

where $w(t,\rho)$ is the solution of the problem (2.7).

2.1.2 Duhamel's principle for PDEs

In this section we explain Duhamel's principle for the PDEs. This principle allows us to find the solution of a non-homogeneous PDE, in terms of the solution of the homogeneous PDE [19]. We will elaborate this principle for the wave equation.
Since the three-dimensional Euclidean space is denoted by $\mathbb{R}^3$ and a point in $\mathbb{R}^3$ be denoted by $Y = (y_1, y_2, y_3)$. If $W(Y, t, \rho)$ is a solution of the homogeneous wave equation, for each fixed ρ,

$$W_{tt}(Y,t) - k^2\nabla^2 W(Y,t) = 0,\ t > 0,\ Y \in \mathbb{R}^3, \tag{2.16}$$

with conditions

$$W(Y,0,\rho) = 0,\ W_t(Y,0,\rho) = h(Y,\rho). \tag{2.17}$$

Where $h(Y,\rho)$ is a continuous function defined for $Y \in \mathbb{R}^3$. Then the solution of the non-homogeneous wave equation

$$G_{tt}(Y,t) - k^2\nabla^2 G(Y,t) = h(Y,t),\ Y \in \mathbb{R}^3,\ t \in \mathbb{R}^+, \tag{2.18}$$

with initial conditions

$$G(Y,0) = 0,\ G_t(Y,0) = 0,$$

is given by

$$G(Y,t) = \int_0^t W(Y, t-\rho, \rho) d\rho. \tag{2.19}$$

Proof. Differentiating (2.19) and using the Leibniz rule, we have

$$G_t(Y,t) = W(Y,0,t) + \int_0^t W_t(Y, t-\rho; \rho) d\rho.$$

Using equation (2.17), we get

$$G_t(Y,t) = \int_0^t W_t(Y, t-\rho, \rho) d\rho. \tag{2.20}$$

Again differentiating equation(2.20) with respect to t, we obtain

$$G_{tt}(Y,t) = W_t(Y, t-\rho, \rho) + \int_0^t W_{tt}(Y, t-\rho, \rho) d\rho. \tag{2.21}$$

Using equation(2.17) in equation(2.21), we get

$$G_{tt}(Y,t) = h(Y,\rho) + \int_0^t W_{tt}(Y, t-\rho, \rho) d\rho. \tag{2.22}$$

And

$$k^2 \nabla^2 G(Y,t) = \int_0^t k^2 \nabla^2 W d\rho. \tag{2.23}$$

Using equation (2.22) and (2.23) in equation (2.18), we have

$$\begin{aligned} G_{tt}(Y,t) - k^2 \nabla^2 G(Y,t) &= h(Y,\rho) + \int_0^t W_{tt}(Y, t-\rho, \rho) d\rho - \int_0^t k^2 \nabla^2 W d\rho, \\ &= h(Y,\rho) + \int_0^t \left[W_{tt} - k^2 \nabla^2 W \right] d\rho. \end{aligned}$$

By using equation (2.16), we obtained

$$G_{tt}(Y,t) - k^2 \nabla^2 G(Y,t) = h(Y,\rho).$$

□

2.2 Duhamel's principle for fractional differential equation

In this section we establish a fractional Duhamel's principle for the Caputo and Riemann-Liouville type FDEs [20]. The generalization of the fractional Duhamel's principle established in [17] can be directly applied to a nonhomogeneous FDEs reducing them to the corresponding homogeneous equations.

2.2.1 Duhamel's principle for Caputo differential equations

In this subsection we present the fractional Duhamel's principle for the Caputo differential equation.

Theorem 2.2.1. *[21] If $v(y,t;\rho)$ is the solution of homogeneous IVP*

$$^{c}D^{\gamma}v(y,t)-\gamma^{2}\frac{\partial^{2}v(y,t)}{\partial y^{2}}=0,\ \gamma\in(0,1),\ t>\rho,\ y\in\mathbb{R}, \tag{2.24}$$

with initial condition

$$v(y,t)\mid_{t=\rho}={}^{c}D^{1-\gamma}\theta(y,\rho), \tag{2.25}$$

where $\theta(y,t)$ is a differentiable function. Then the solution of inhomogeneous IVP

$$^{c}D^{\gamma}g(y,t)-\gamma^{2}\frac{\partial^{2}g(y,t)}{\partial y^{2}}=\theta(y,t),\ \gamma\in(0,1),\ y\in\mathbb{R},\ t>0, \tag{2.26}$$

satisfying initial conditions $g(y,0)=0,\ y\in\mathbb{R}$, is given by

$$g(y,t)=\int_{0}^{t}v(y,t;\rho)d\rho. \tag{2.27}$$

Proof. Differentiating equation (2.27) with respect to t and using Leibniz rule, we get

$$\frac{\partial}{\partial t}g(y,t)=v(y,t;\rho)\mid_{t=\rho}+\int_{0}^{t}\frac{\partial}{\partial t}v(y,t;\rho)d\rho. \tag{2.28}$$

From equation (2.28) and definition of Caputo derivative, we have

$$^{c}D^{\gamma}g(y,t)-\gamma^{2}\frac{\partial^{2}g(y,t)}{\partial y^{2}}=\mathcal{J}^{1-\gamma}\frac{\partial}{\partial t}g(y,t)-\gamma^{2}\frac{\partial^{2}g(y,t)}{\partial y^{2}},$$

$$= \mathcal{J}^{1-\gamma}\frac{\partial}{\partial t}\int_0^t v(y,t;\rho)d\rho - \int_0^t \gamma^2\frac{\partial^2 v(y,t;\rho)}{\partial y^2}d\rho. \tag{2.29}$$

Applying Leibniz rule on the equation (2.29), we get

$$\begin{aligned} {}^cD^\gamma g(y,t) - \gamma^2\frac{\partial^2 g(y,t)}{\partial y^2} = \mathcal{J}^{1-\gamma}\left[v(y,t;\rho)\mid_{t=\rho} + \int_0^t \frac{\partial}{\partial t}v(y,t;\rho)d\rho\right] \\ - \int_0^t \gamma^2\frac{\partial^2 v(y,t;\rho)}{\partial y^2}d\rho. \end{aligned} \tag{2.30}$$

Using equation (2.25) in equation (2.30), we obtain

$$\begin{aligned} {}^cD^\alpha g(y,t) - \gamma^2\frac{\partial^2 g(y,t)}{\partial y^2} = \mathcal{J}^{1-\gamma}\left({}^cD^{1-\gamma}\theta(y,t)\right) + \int_0^t \Big[\mathcal{J}^{1-\gamma}\frac{\partial}{\partial t}v(y,t;\tau) \\ -\gamma^2\frac{\partial^2 v(y,t;\rho)}{\partial y^2}\Big]d\rho, \end{aligned}$$

$$= \theta(y,t) - \theta(y,0) + \int_0^t \left[{}^cD^\gamma v(y,t;\rho) - \gamma^2\frac{\partial^2 v(y,t;\rho)}{\partial y^2}\right]d\rho. \tag{2.31}$$

Using equation (2.24) in equation (2.31), we obtain

$${}^cD^\gamma g(y,t) - \gamma^2\frac{\partial^2 g(y,t)}{\partial y^2} = \theta(y,t).$$

Further $g(y,0) = 0$. Hence $g(y,t) = \int_0^t v(y,t;\rho)d\rho$ is the solution of the problem (2.26). □

2.2.2 Duhamel's principle for Riemann-Liouville differential equation

In this subsection, we state the Duhamel's principle for fractional differential equation with the Riemann-Liouville derivative.

Consider the non-homogeneous Cauchy problem

$${}^R\mathcal{D}_0^\gamma w(s) + Lw(s) = \theta(s),\ \ s > 0 \tag{2.32}$$

satisfying homogeneous initial condition

$$\mathcal{J}^{1-\gamma}w(0) = 0. \tag{2.33}$$

The fractional Duhamel's principle establishes a relation between the solution of the inhomogeneous Cauchy problem with the homogeneous problem

$$ {}^{R}\mathcal{D}_0^{\gamma} g(s,\rho) + Lg(s,\rho) = 0, \;\; s > \rho \tag{2.34}$$

subject to the inhomogeneous initial condition

$$\mathcal{J}^{1-\gamma} g(s,\rho) \mid_{s=\rho} = \theta(\rho). \tag{2.35}$$

Where $\gamma \in (0,1)$ and $\theta(\rho), \;\rho \geq 0$, is a continuous function.

Theorem 2.2.2. *[20] Suppose that $g(s,\rho)$ is a solution of the homogeneous problem (2.34)-(2.35). Then the solution of the inhomogeneous problem (2.32)-(2.33) is given by Duhamel's integral*

$$w(s) = \int_0^s g(s,\rho)d\rho. \tag{2.36}$$

Proof. Applying ${}^{R}D_0^{\gamma}$ on the equation (2.36), we have

$$ {}^{R}D_0^{\gamma} w(t) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{ds} \int_0^s (s-x)^{-\gamma} \int_0^x g(x,\rho)dxd\rho,$$

$$= \frac{d}{ds} \int_0^s \mathcal{J}^{1-\gamma} g(s,\rho)d\rho. \tag{2.37}$$

From equation (2.37), (2.32) and Leibniz rule, we obtain

$$ {}^{R}D_0^{\gamma} w(s) + Lw(s) = \mathcal{J}^{1-\gamma} g(s,\rho) \mid_{s=\rho} + \int_0^s \frac{d}{ds} \mathcal{J}^{1-\gamma} g(s,\rho)d\rho + L \int_0^s g(s,\rho)d\rho. \tag{2.38}$$

Using equation (2.35) in equation (2.38), we get

$$ {}^{R}D_0^{\gamma} w(s) + Lw(s) = \theta(\rho) + \int_0^s \left[{}^{R}D^{\gamma} g(s,\rho) + Lg(s,\rho) \right] d\rho. \tag{2.39}$$

Using equation (2.34) in equation (2.39), then we obtain

$$ {}^{R}D_0^{\gamma} w(s) + Lw(s) = \theta(\rho).$$

□

2.2.3 Duhamel's principle for psi-differential equation

In this part we established the fractional Duhamel's principle for psi-differential equation.

Let $C = C(x, \frac{\partial}{\partial s}, D^{q,\psi})$ and $D^{\psi,q}=\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^q$ be a linear differential operator whose coefficients not depending on s. Consider the Cauchy problem

$$D^{2,\psi}w(s,z) + Cw(s,z) = \theta(s,z), \;\; s>0, \;\; z \in \mathbb{R}^n \tag{2.40}$$

with initial homogeneous conditions

$$w(0,z) = 0; \;\; D^{1,\psi}w(0,z) = 0. \tag{2.41}$$

If $w(s,\eta,z)$, is a solution of the homogeneous problem

$$D^{2,\psi}G(s,\rho,z) + CG(s,\rho,z) = 0,$$

with initial conditions :

$$G(s,\rho,z)\mid_{s=\rho} = 0, \;\; D^{1,\psi}G(s,\rho,z)\mid_{s=\rho} = \theta(\rho,z).$$

Then solution of the Cauchy problem (2.40) -(2.41) is given by the integral

$$w(s,z) = \int_0^s G(s,\rho,z)\psi'(\rho)d\rho. \tag{2.42}$$

The integral involved in equation(2.42) is the Duhamel's integral for $\psi-$fractional operator.

Lemma 2.2.1. *Assume that g is continuous on $\mathbb{R}^+ \times [0,s]$, and its partial derivatives are jointly continuous in the X-norm, and $\left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n \in L_1(0,s;X)\ \forall \;\; s>0$. If*

$$w(s) = \int_0^s g(s,\rho)\psi'(\rho)d\rho, \tag{2.43}$$

then

$$D^{n,\psi}w(s) = \sum_{l=0}^{n-1} D^{l,\psi}\left[\left(\frac{1}{\psi'(s)}\right)^{n-1-l}\frac{\partial^{n-1-l}g(s,s)}{\partial s^{n-1-l}}\right] + \int_0^s \left(\frac{1}{\psi'(s)}\right)^n \frac{\partial^n g(s,\rho)}{\partial s^n}\psi'(\rho)d\rho. \tag{2.44}$$

Proof. Applying $D^{1,\psi}$ on both sides of the equation (2.43) and using Leibniz rule, we have

$$D^{1,\psi}w(s) = g(s,s) + \int_0^s \frac{1}{\psi'(s)}\frac{\partial g(s,\rho)}{\partial s}\psi'(\rho)d\rho. \tag{2.45}$$

Again applying $D^{1,\psi}$ on the equation (2.45) and using Leibniz rule, we get

$$D^{2,\psi}w(s) = D^{1,\psi}g(s,s) + \frac{1}{\psi'(s)}\frac{\partial g(s,s)}{\partial s} + \int_0^s \left(\frac{1}{\psi'(s)}\right)^2 \frac{\partial^2 g(s,\rho)}{\partial s^2}\psi'(\rho)d\rho.$$

A repeated application of above process for n-times, leads us to

$$D^{n,\psi}w(s) = \sum_{l=0}^{n-1} D^{l,\psi}\left[\left(\frac{1}{\psi'(s)}\right)^{n-1-l}\frac{\partial^{n-1-l}g(s,s)}{\partial s^{n-1-l}}\right] + \int_0^s \left(\frac{1}{\psi'(s)}\right)^n \frac{\partial^n g(s,\rho)}{\partial s^n}\psi'(\rho)d\rho. \tag{2.46}$$

□

2.2.4 Generalization of the Duhamel's principle for psi differential equation

In this subsection we generalize Duhamel's principle for the higher integer order psi-differential equation. Consider the Cauchy problem

$$D^{n-1,\psi}G(s,\rho)\mid_{s=\rho} = \theta(\rho). \tag{2.47}$$

$$D^{q,\psi}(0) = \phi_q(0).\ q = 0, ..., n-1. \tag{2.48}$$

Duhamel's principle establishes a connection between the solutions of the Inhomogeneous Cauchy problem (2.47) with the initial homogeneous condition

$$D^{q,\psi}G(s,\rho)\mid_{s=\rho} = 0\ ,\ \ q = 0, 1, ..., n-2, \tag{2.49}$$

and the Cauchy problem for the corresponding homogeneous equation

$$D^{n,\psi}G(s,\rho) + \sum_{q=0}^{n-1} f_q(A)D^{q,\psi}G(s,\rho) = 0, \tag{2.50}$$

$$D^{q,\psi}G(s,\rho)\mid_{s=\rho}=0\ ,\ q=0,1,...,n-2, \tag{2.51}$$

$$D^{n-1,\psi}G(s,\rho)\mid_{s=\rho}=\theta(\rho). \tag{2.52}$$

Where θ is a continuous function and $G(s,\rho)$ is m times differentiable with respect to s and the partial derivatives $\frac{\partial^i G}{\partial s^i}$ $i\in[0,q-1]$ are jointly continuous in the topology of $\mathrm{EXP}_{A,G}(X)$.

Theorem 2.2.3. *If $G(s,\rho)$ is solution of the problem (2.50)-(2.52). Then a solution of the inhomogeneous Cauchy problem (2.47)-(2.49) is given by*

$$w(s)=\int_0^s G(s,\rho)\psi'(\rho)d\rho. \tag{2.53}$$

Proof. By applying $D^{1,\psi}$ on the above equation (2.53) and using Leibniz theorem, we have

$$D^{1,\psi}w(s)=G(s,s)+\int_0^s \frac{1}{\psi'(s)}\frac{\partial G(s,\rho)}{\partial s}\psi'(\rho)d\rho. \tag{2.54}$$

From equation (2.51) and (2.54) obviously $\left(D^{1,\psi}w\right)(0)=0$.

Now, by Lemma (2.2.1), we have

$$D^{q,\psi}w(s)=\sum_{l=0}^{q-1}D^{l,\psi}\left[\left(\frac{1}{\psi'(s)}\right)^{q-1-l}\frac{\partial^{q-1-l}G(s,s)}{\partial s^{q-1-l}}\right] + \int_0^s\left(\frac{1}{\psi'(s)}\right)^q\frac{\partial^q G(s,\rho)}{\partial s^q}\psi'(\rho)d\rho. \tag{2.55}$$

Using initial condition (2.51), we obtain

$$D^{q,\psi}w(s)=D^{q-1,\psi}G(s,s)+\int_0^s\left(\frac{1}{\psi'(s)}\right)^q\frac{\partial^q G(s,s)}{\partial s^q}\psi'(\rho)d\rho. \tag{2.56}$$

Now using equation (2.49) in equation (2.56), we have

$$D^{q,\psi}w(s)=\int_0^s\left(\frac{1}{\psi'(s)}\right)^q\frac{\partial^q G(s,s)}{\partial s^q}\psi'(\rho)d\rho. \tag{2.57}$$

Note that $w(s)$ defined in equation (2.53) satisfying the initial condition (2.51). Moreover, by substituting equation (2.57) and (2.56) in equation (2.47), we have

$$D^{n,\psi}w(s)+\sum_{q=0}^{n-1}f_q(A)D^{q,\psi}w(s)=D^{n-1,\psi}G(s,s)+\int_0^s\Big[\left(\frac{1}{\psi'(s)}\right)^n\frac{\partial^n w(s,\rho)}{\partial s^n} \psi'(\rho)\Big]d\rho+\sum_{q=0}^{n-1}f_q(A)\int_0^s\left(\frac{1}{\psi'(s)}\right)^q\frac{\partial^q G(s,\rho)}{\partial s^q}\psi'(\rho)d\rho. \tag{2.58}$$

Using equation (2.52) in equation (2.58), we obtain

$$D^{n,\psi}w(s)+\sum_{q=0}^{n-1}f_q(A)D^{q,\psi}w(s)=\theta(s)+\int_0^s\Big[D^{n,\psi}G(s,\rho)+\sum_{q=0}^{n-1}f_q(A)D^{q,\psi}G(s,\rho)\Big]\psi'(\rho)d\rho. \tag{2.59}$$

Now by using equation (2.50) in equation (2.59), we get

$$D^{n,\psi}w(s)+\sum_{q=0}^{n-1}f_q(A)D^{q,\psi}w(s)=\theta(s).$$

□

Lemma 2.2.2. *If* $v(s)=(\psi(s))^{n-1}E_{\gamma,n}(-\tau(\psi(s))^{\gamma})$ *then*

$$D^{\gamma,\psi}v(s)-\tau v(s)=0,\ \ \gamma\in(n-1,n]$$

where ψ *is increasing differential function.*

Theorem 2.2.4. *If* $w(s)=\mathcal{J}^{k-\gamma,\psi}h(s)$, ψ *is non-decreasing differential function with* $\psi(0)=0$ *and* $\psi'(0)\neq 0$. *where* $\gamma\in(q-1,q]$, $h\in L_1[a,b]$ *and* $a\leq s\leq b$. *Then solution of the Cauchy problem*

$$D^{\gamma,\psi}w(s)+\tau w(s)=f(s)\ ,w(0)=0,\ \left(D^{1,\psi}w\right)(0)=0,...,\left(D^{m-1,\psi}w\right)(0)=0, \tag{2.60}$$

is given by

$$w(s)=\int_0^s g(s,\rho)\psi'(\rho)d\rho. \tag{2.61}$$

Where $g(s,\rho)$ *is a solution of the problem*

$$D^{\gamma,\psi}g(s;\rho)+\tau g(s;\rho)=0,\ \ g(0;\rho)=0,\ \ D^{1,\psi}g(0;\rho)=0,\ \ ...,\ D^{q-1,\psi}g(0;\rho)=\left(\frac{1}{\psi'(0)}\right)^{q-1}h(\gamma). \tag{2.62}$$

Proof. By Lemma 2.2.2, we have

$$g(s,\rho)=(\psi(s))^{q-1}E_{\gamma,q}(-\tau(\psi(s))^{\gamma})h(\rho). \tag{2.63}$$

The function g in equation(2.63) satisfies the equation(2.62). By definition Mittage lefller function definition, we have

$$g(s;\rho) = \sum_{k=0}^{\infty} \frac{(-\tau)^k \left(\psi(s)\right)^{\gamma k+q-1} h(\rho)}{\Gamma(\gamma k+q)}. \tag{2.64}$$

Now applying $D^{m,\psi}$ on the equation (2.64), we have

$$D^{m,\psi} g(s;\rho) = \sum_{k=0}^{\infty} \frac{(-\tau)^k \left(\psi(s)\right)^{\gamma k-m+q-1}}{\Gamma(\gamma k+q-m)} \left(\frac{1}{\psi'(s)}\right)^m h(\rho).$$

Now, it is easily checked

$$g(0;\rho) = 0,\ \ D^{1,\psi} g(0;\rho) = 0,\ \ ,..., D^{q-1,\psi} g(0;\rho) = \left(\frac{1}{\psi'(0)}\right)^{q-1} h(\rho).$$

Now, we have to prove that $w(s)$ defined by equation (2.61) holds equation (2.60). Now equation (2.61) can be written as

$$w(s) = \int_0^s g(s-\rho;\rho)\psi'(\rho)d\rho,$$

$$= \int_0^s (\psi(s)-\psi(\rho))^{q-1} E_{\gamma,q}(-\tau(\psi(s)-\psi(\rho))^{\rho})\psi'(\rho)h(\rho)d\rho,$$

$$= \sum_{m=0}^{\infty}(-\tau)^m \int_0^s \frac{(\psi(s)-\psi(\rho))^{\gamma m+q-1}}{\Gamma(\gamma m+q)}\psi'(\rho)h(\rho)d\rho,$$

$$= \sum_{m=0}^{\infty}(-\tau)^m \mathcal{J}^{\gamma m+q,\psi} h(s). \tag{2.65}$$

Now again applying $D^{\gamma,\psi}$ on the equation (2.65), we have

$$\begin{aligned} D^{\gamma,\psi} w(s) =& \sum_{m=0}^{\infty}(-\tau)^m D^{\gamma,\psi} \mathcal{J}^{\gamma m+q,\psi} h(s) \\ =& \sum_{m=0}^{\infty}(-\tau)^m \mathcal{J}^{\gamma(m-1)+q,\psi} h(s), \\ =& \mathcal{J}^{m-\gamma,\psi} h(s) + \sum_{m=1}^{\infty}(-\tau)^m \mathcal{J}^{\gamma(m-1)+q,\psi} h(s), \\ =& f(s) - \tau \sum_{m=0}^{\infty}(-\tau)^m \mathcal{J}^{\gamma m+q,\psi} h(s). \end{aligned}$$

By using equation (2.65), we get

$$D^{\gamma,\psi}w(s) = f(s) - \tau w(s).$$

□

2.2.5 Generalization of the Duhamel's principle for the Caputo psi-differential operator

In this subsection, we generalize the Duhamel's principle for the Caputo fractional psi differential operator. Consider the operator

$$H^{(\mu,\lambda)}[w](s) = {}^{c}D^{\mu,\psi}w(s) + \int_{0}^{n-1} f(\gamma)^{c}D^{\gamma,\psi}w(s)\lambda\psi'(\gamma)d\gamma, \tag{2.66}$$

where λ represents any arbitrary finite number with sup$\lambda \in [0, n-1]$, $n-1 < \mu < n$ and $\gamma \in (0, n-1)$. The theorem given below presents the solution of the non-homogeneous Cauchy problem involving the Caputo ψ-operator.

Theorem 2.2.5. *The solution of Cauchy problem*

$${}^{c}D^{\mu,\psi}w(s) + \int_{0}^{n-1} f(\gamma)^{c}D^{\gamma,\psi}w(s)\lambda\psi'(\gamma)d\gamma = \theta(s), \;\; s > 0 \tag{2.67}$$

satisfying the homogeneous initial conditions

$$\left({}^{c}D^{q,\psi}w\right)(0) = 0, \;\; q = 0, ..., n-1. \tag{2.68}$$

is given by the Duhamel's principle as;

$$w(s) = \int_{0}^{s} g(s,\rho)\psi'(\rho)d\rho. \tag{2.69}$$

Where $g(s,\rho)$ is the solution of the problem

$${}^{c}D^{\mu,\psi}g(s) + \int_{0}^{n-1} f(\gamma)^{c}D^{\gamma,\psi}g(s)\lambda\psi'(\gamma)d\gamma = 0, \;\; s > \rho \tag{2.70}$$

$${}^{c}D^{q,\psi}g(s,\rho)\mid_{s=\rho+0} = 0, \;\; q = 0, ..., n-2 \tag{2.71}$$

$${}^{c}D^{m-1,\psi}g(s,\rho)\mid_{s=\rho+0} = {}^{c}D^{n-\mu,\psi}\theta(\rho). \tag{2.72}$$

Proof. Since

$$w(s) = \int_0^s g(s,\rho)\psi'(\rho)d\rho.$$

The condition (2.68) holds obviously.

Consider the Cauchy problem (2.67), we have

$$^cD^{\mu,\psi}w(s) + \int_0^{n-1} f(\gamma)^c D^{\gamma,\psi}w(s)\lambda\psi'(\gamma)d\gamma = h(s).\ s > 0 \tag{2.73}$$

By the definition of $^cD^{\mu,\psi}$, we have

$$^cD^{\mu,\psi}w(s) = \frac{1}{\Gamma(q-\gamma)}\int_0^s \psi'(t)(\psi(s)-\psi(t))^{q-\gamma-1}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^q w(t)dt. \tag{2.74}$$

Now, we find

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^q w(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^q \int_0^s g(t,\rho)\psi'(\rho)d\rho. \tag{2.75}$$

By Lemma (2.2.1) and using initial condition (2.68), we get

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^q w(t) = \int_0^s \left(\frac{1}{\psi'(t)}\right)^q \frac{\partial^q g(t,\rho)}{\partial t^q}\psi'(\rho)d\rho,\ q = 0, ..., n-1. \tag{2.76}$$

By substituting equation(2.76) in equation(2.74) and also changing integration order, we get

$$^cD^{\gamma,\psi}w(s) = \int_0^s \frac{1}{\Gamma(q-\gamma)}\int_\rho^s \psi'(t)\left(\psi(s)-\psi(t)\right)^{q-\gamma-1}\left(\frac{1}{\psi'(t)}\right)^q \frac{\partial^q g(t,\rho)}{\partial t^q}\psi'(\rho)dsd\rho. \tag{2.77}$$

The equation (2.77) can be written as

$$^cD^{\gamma,\psi}w(s) = \int_0^s {}^cD_\rho^{\gamma,\psi}g(s,\rho)\psi'(\rho)d\rho, \tag{2.78}$$

$$^cD^{\mu,\psi}w(s) = \frac{1}{\Gamma(n-\mu)}\int_0^s \psi'(t)\left(\psi(s)-\psi(t)\right)^{n-\mu-1}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n w(t)dt. \tag{2.79}$$

Now we calculate,

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n w(t) = \sum_{l=0}^{n-1} D^{l,\psi}\left[\left(\frac{1}{\psi'(t)}\right)^{n-l-1}\frac{\partial^{n-l-1}g(t,t)}{\partial t^{n-l-1}}\right] + \int_0^t \left(\frac{1}{\psi'(t)}\right)^n \frac{\partial^n g(t,\rho)}{\partial t^n}\psi'(\rho)d\rho.$$

By using initial condition (2.72), we obtained

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n w(t) = {}^cD^{n-1,\psi}g(t,t) + \int_0^t \left(\frac{1}{\psi'(t)}\right)^n \frac{\partial^n g(t,\rho)}{\partial t^n}\psi'(\rho)d\rho.$$

This implies that

$$\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n w(t) = {}^cD^{n-\mu,\psi}\theta(t) + \int_0^t \left(\frac{1}{\psi'(t)}\right)^n \frac{\partial^n g(t,\rho)}{\partial t^n}\psi'(\rho)d\rho. \tag{2.80}$$

By substituting (2.80) in equation (2.79), we have

$$\begin{aligned}{}^cD^{\mu,\psi}w(s) = \frac{1}{\Gamma(n-\mu)}\int_0^s \psi'(t)\left(\psi(s)-\psi(t)\right)^{n-\mu-1}\Big[{}^cD^{n-\mu,\psi}\theta(t) \\ + \int_0^t \left(\frac{1}{\psi'(t)}\right)^n \frac{\partial^n g(t,\rho)}{\partial t^n}\psi'(\rho)d\rho\Big]dt.\end{aligned}$$

Again by using definition of ${}^cD^{\mu,\psi}$ and changing integration order, we obtained

$$\begin{aligned}{}^cD^{\mu,\psi}w(s) = \mathcal{J}^{n-\mu,\psi c}D^{n-\mu,\psi}\theta(s) + \int_0^s \frac{1}{\Gamma(n-\mu)} \\ \Big[\int_\rho^s \psi'(t)\left(\psi(s)-\psi(t)\right)^{n-\mu-1}\left(\frac{1}{\psi'(t)}\right)^n \frac{\partial^n g(t,\rho)}{\partial t^n}\psi'(\rho)dt\Big]d\rho, \\ = \theta(s) + \int_0^s {}^cD_\rho^{\mu,\psi}g(s,\rho)\psi'(\rho)d\rho.\end{aligned} \tag{2.81}$$

By using equation (2.78) and (2.81) in equation (2.73)

$$\begin{aligned}&{}^cD^{\mu,\psi}w(s) + \int_0^{n-1} f(\gamma){}^cD^{\gamma,\psi}w(s)\lambda\psi'(\gamma)d\gamma \\ &= \theta(s) + \int_0^s {}^cD_\rho^{\mu,\psi}g(s,\rho)\psi'(\rho) + \int_0^{n-1} f(\gamma)\int_0^s {}^cD_\rho^{\gamma,\psi}g(s,\rho)\psi'(\rho)\lambda\psi'(\gamma)d\rho d\gamma, \\ &= \theta(s) + \int_0^s \left[{}^cD_\rho^{\mu,\psi}g(s,\rho) + \int_0^{n-1} f(\gamma){}^cD_\rho^{\gamma,\psi}g(s,\rho)\lambda\psi'(\gamma)d\gamma\right]\psi'(\rho)d\rho.\end{aligned}$$

By using equation (2.70), we

$${}^cD^{\mu,\psi}w(s) + \int_0^{n-1} f(\gamma){}^cD^{\gamma,\psi}w(s)\lambda\psi'(\gamma)d\gamma = \theta(s).$$

□

2.3 Conclusion

A novel technique named Duhamel's principle has been develop for solving inhomogeneous initial value problems. The method has been applied for solving inhomogeneous FDE. Also, Duhamel's principle has been developed for solving generalized fractional differential equation. Duhamel's technique has been employed to reduce the inhomogeneous IVP to the corresponding homogeneous IVP.

Chapter 3

Applications of the Duhamel principle

The concept of Duhamel's principle is very useful because it will help us in finding the solution of the inhomogeneous equation using corresponding homogeneous equation. This principle has a wide rang of applications in the field of applied mathematics, engineering, and physics. The interest in this principle is caused by many applications to problems of mechanics, geometry, applied physics, and other applied fields (see, e.g.,[22, 23, 24, 25].

Furthermore, in literature [26, 27, 28, 29], IBVPs for both fractional ordinary differential equations and partial fractional differential equations are studied. An enormous the results of fractional calculus like stability, existence, uniqueness, etc. of the solution have been obtained for the fractional differential equations (FDEs) ([22, 24, 30, 31]). Seemab. A in [32] was established the existence result for fractional non-linear partial differential equations (PDEs) containing fractional Caputo derivative of order $1 < \gamma < 2$.

In this chapter, we shall discuss the applications of the generalized Duhamel's principle for the psi-differential operator. In first section of this chapter we will present stability analysis of the FDEs. Second section is about the existence of this principle.

3.1 Stability of solution for differential equation

In this section, we will discuss the stability of the solutions of the system of differential equations. Particularly we will discuss the behavior of the solutions of the system of the form

$$\frac{dx}{dt} = U\left(x(t), y(t)\right), \ \frac{dy}{dt} = V\left(x(t), y(t)\right), \tag{3.1}$$

where x and y are unknown scalar functions, and the first partial derivatives of U and V are continuous in region a D of the $xy-$plane. Such system is called autonomous, because U and V do not depend on t. We will require a number of definitions, for more discussion.

Definition 3.1.1. *[33] A critical point (c, d) of (3.1) is said to be an isolated critical point if $\exists$ a circle*

$$\sqrt{(x(t)-c)^2 + (y(t)-d)^2} = R, \ R > 0.$$

containing no other critical point inside it.

Example 3.1.2. *Consider the system*

$$\frac{dx}{dt} = x - y, \ \frac{dy}{dt} = x + y. \tag{3.2}$$

The critical point $(0, 0)$ of (3.2) is isolated because it is only critical point of (3.2). Now, we introduce the idea of the stability of the solution $x(t) = c, \ y(t) = d, \ t \in (-\infty, +\infty)$, of (3.1) or the stability of a critical point.

Definition 3.1.3. *[33] If (c, d) is an isolated critical point of the system (3.1), then (c, d) is said to be stable if any given $\varepsilon > 0$, $\exists\ \delta > 0$ such that, whenever the solution (x, y) satisfies*

$$\sqrt{[x(0)-c]^2 + [y(0)-d]^2} < \delta,$$

the solution for $t \geq 0$ exists and satisfies

$$\sqrt{[x(t)-c]^2 + [y(t)-d]^2} < \varepsilon.$$

The critical point (c, d) is said to be asymptotically stable if it is stable and in addition $\exists\ \delta_0 > 0$ such that

$$\lim_{t\to\infty} x(t) = c,\ \lim_{t\to\infty} y(t) = d,$$

whenever

$$\sqrt{[x(0) - c]^2 + [y(0) - d]^2} < \delta_0.$$

An isolated critical point, that is not stable, is said to be unstable.

Example 3.1.4. *Consider the system*

$$\frac{dx}{dt} = -x,\ \frac{dy}{dt} = -2y. \tag{3.3}$$

The critical point of (3.3) is $(0, 0)$ only. Let $x(0) = c,\ y(0) = d$. The solution of (3.3) is $x(t) = c\exp(-t),\ y(t) = d\exp(-2t)$. By using definition (3.1.3), we have

$$\sqrt{c^2 + d^2} < \delta.$$

Again applying definition (3.1.3), we obtained

$$\sqrt{(x(t))^2 + (y(t))^2} = \sqrt{(c\exp(-t))^2 + (d\exp(-2t))^2} \le \sqrt{c^2 + d^2} \le \delta.$$

We choose $\varepsilon = \delta$. So, by definition (3.1.3) critical point $(0, 0)$ is stable. Also

$$\lim_{t\to\infty} c\exp(-t) = 0,\ \lim_{t\to\infty} d\exp(-2t) = 0.$$

Hence $(0, 0)$ is asymptotically stable.

3.2 Stability analysis of FDEs

The stability of the solutions of the problems play an important role in the field of PDEs. Since the fractional derivatives have weakly singular kernels, therefore the stability of FDEs is more complex than that of the ODE. The author in [34] discussed the stability of the linear FDEs with Caputo fractional derivative of order $0 < \gamma <$

1. Recently Qian et al. [35] discussed the analysis on stability of linear FDEs with Riemann-Liouville fractional derivative of order $0 < \gamma < 1$.
In this section, we introduce the stability of the solutions of the generalized linear and nonlinear FDEs with Caputo fractional psi-differential operator.

Definition 3.2.1. *The Mittag-Leffler function of two parameter γ and ρ is defined as:*

$$E_{\gamma,\rho}(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\gamma i + \rho)}$$

If $\gamma = 1$, then this will become $E_{\gamma,1}(w)$ that is also written as $E_{\gamma}(w)$ and this is the Mittag-Leffler function of one parameter.

Remark 3.2.1. *[35] If $|arg(w)| \in [\mu, \pi]$ and $w \to \infty$*
then

$$E_{\gamma,\rho}(w) = -\sum_{i=2}^{m} \frac{w^{-i}}{\Gamma(\rho - \gamma i)} + O(|w|^{-1-m})$$

where $m \geq 2$ is any integer.

Lemma 3.2.1 (Gronwall Inequality)**.** *[35] Suppose that the functions w, ν are continuous in $[s_0, s_1]$. If*

$$\nu(s) \leq \mu + \int_{s_0}^{s} [w(\rho)\nu(\rho) + r]\, d\rho,$$

then

$$\nu(s) \leq (\mu + r(s_1 - s_0)) \exp\left(\int_{s_0}^{s} w(\rho) d\rho\right), \ s \in [s_0, s_1].$$

where $\nu(s) \geq 0$, $\mu \geq 0$ and $r \geq 0$.

Definition 3.2.2. *[35] Consider the fractional differential system*

$$D^{\gamma,\psi} w(s) - Bw(s) = 0. \tag{3.4}$$

with initial condition $D^{\gamma-1,\psi} w(s) \mid_{s=0} = w_0 = (w_{10}, w_{20}, ..., w_{n0})^T$,
where $w(s) = [w_1(s), w_2(s), ..., w_n(s)]^T$, $0 \leq \gamma \leq 1$ and $B \in \mathbb{R}^{n \times n}$. The system (3.4) is said to be
(1) Stable iff for any w_0, there exists $\epsilon > 0$, such that $\|w(s)\| < \epsilon$, for $s \geq 0$;
(2) Asymptotically stable iff $\lim_{s\to\infty} \|w(s)\| = 0$.

The stability of the fractional differential system containing Riemann-Liouville derivative of order $0 < \gamma < 1$ was investigated by Qian et al.[35]. Motivated by this paper, we generalized the results of stability for generalized fractional differential equation containing Caputo derivative.

3.2.1 Stability analysis of linear generalized Caputo differential system

Consider the linear FDEs involving Caputo-psi differential operator as

$$ {}^{c}D^{\gamma,\psi}w(s) - Bw(s) = 0, \ \gamma \in (0,1) \tag{3.5} $$

where ψ is an increasing function, $w(s) = [w_1(s), ..., w_n(s)]^T \in \mathbb{R}^n, \ B = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, and the initial condition

$$ {}^{c}D^{n-1,\psi}w(s) \mid_{s=0} = w_0 = [w_{10}, ..., w_{n0}]^T . \tag{3.6} $$

Theorem 3.2.1. *If all the eigenvalues of B satisfy*

$$ \left| \arg\left(\mu(B)\right) \right| > \frac{\gamma\pi}{2}. \tag{3.7} $$

Then the solution (3.5) is asymptotically stable.

Proof. The solution of (3.5) is given by

$$ w(s) = w_0 \left(\psi(s)\right)^{n-1} E_{\gamma,n}\left(B\left(\psi(s)\right)^{\gamma}\right). \tag{3.8} $$

Suppose that B matrix is similar to a diagonal matrix, i.e $\exists$ an invertible matrix H such that

$$ \mu = H^{-1}BH = \mathrm{dia}\left(\mu_1, ..., \mu_n\right). $$

Then,

$$ \begin{aligned} E_{\gamma,n}\left(B\left(\psi(s)\right)^{\gamma}\right) &= HE_{\gamma,n}\left(\mu\left(\psi(s)\right)^{\gamma}\right)H^{-1} \\ &= H\mathrm{diag}\left[E_{\gamma,n}\left(\mu_1\left(\psi(s)\right)^{\gamma}\right), ..., E_{\gamma,n}\left(\mu_n\left(\psi(s)\right)^{\gamma}\right)\right]H^{-1}. \end{aligned} $$

By Remark (3.2.1), we have

$$E_{\gamma,n}\left(\mu_i\left(\psi(s)\right)^{\gamma}\right) = -\sum_{q=2}^{p}\frac{\left(\mu_i\left(\psi(s)\right)^{\gamma}\right)^{-q}}{\Gamma(n-\gamma q)} + O(|\mu_i\left(\psi(s)\right)^{\gamma}|^{-1-p},\ 1\leq i\leq n,$$
$$= -\sum_{q=2}^{p}\frac{\left(\mu_i\right)^{-q}\left(\psi(s)\right)^{-\gamma q}}{\Gamma(n-\gamma q)} + O\left(|\mu_i|^{-1-p}\left(\psi(s)\right)^{-\gamma-\gamma p}\right)\to 0,\quad s\to\infty.$$

Thus

$$\left\|E_{\gamma,n}\left(\mu\left(\psi(s)\right)^{\gamma}\right)\right\| = \left\|\mathrm{diag}\left[E_{\gamma,n}\left(\mu_1\left(\psi(s)\right)^{\gamma}\right),...,E_{\gamma,n}\left(\mu_n\left(\psi(s)\right)^{\gamma}\right)\right]\right\|\to 0.$$

So, the result holds.

Now, we consider that B matrix is similar to a Jordan canonical form i.e $\exists$ an invertible matrix H such that

$$J = H^{-1}BH = \mathrm{diag}\left(J_1,...,J_l\right),$$

$J_k,\ 1\leq k\leq l$, has the following form

$$\begin{pmatrix}\mu_k & 1 & & \\ & \mu_k & \ddots & \\ & & \ddots & 1\\ & & & \mu_k\end{pmatrix}.$$

$$E_{\gamma,n}\left(B\left(\psi(s)\right)^{\gamma}\right) = H\mathrm{diag}\left[E_{\gamma,n}\left(J_1\left(\psi(s)\right)^{\gamma}\right),...,E_{\gamma,n}\left(J_l\left(\psi(s)\right)^{\gamma}\right)\right]H^{-1}$$
$$= \sum_{q=0}^{\infty}\frac{\left(J_k\left(\psi(s)\right)^{\gamma}\right)^{q}}{\Gamma(\gamma q+n)},$$
$$= \frac{\left(\psi(s)\right)^{\gamma q}}{\Gamma(\gamma q+n)}\begin{pmatrix}\mu_k^{q} & {}^{q}C_1\mu_k^{q-1} & \dots & {}^{q}C_{n_k-1}\mu_k^{q-n_k+1}\\ & \mu_k^{q} & \ddots & \\ & & \ddots & {}^{q}C_1\mu_k^{q-1}\\ & & & \mu_k^{q}\end{pmatrix},$$
$$= \begin{pmatrix}E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right) & \frac{1}{1!}\frac{\partial}{\partial\mu_i}E_{\gamma,n}\left(\mu_i\left(\psi(s)\right)^{\gamma}\right) & \dots & \frac{1}{(n_i-1)!}\left(\frac{\partial}{\partial\mu_k}\right)^{n_i-1}E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right)\\ & E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right) & \ddots & \\ & & \ddots & \frac{1}{1!}\frac{\partial}{\partial\mu_k}E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right)\\ & & & E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right)\end{pmatrix}.$$

By some calculations and using Remark (3.2.1). If $|\arg(\mu_k(B))| > \frac{\gamma\pi}{2},\ 1 \le k \le l$ and $s \to \infty$, then we have
$\left|E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right)\right| \to 0$ and $\left|\frac{1}{j!}\left(\frac{\partial}{\partial\mu_k}\right)^j E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right)\right| \to 0,\ 0 \le j \le n_k - 1,\ 1 \le k \le l$.
These can be seen from the following

$$E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right) = -\sum_{q=2}^{r}\frac{(\mu_k)^{-q}\left(\psi(s)\right)^{-\gamma q}}{\Gamma(n-\gamma q)} + O\left(|\mu_k|^{-1-r}\left(\psi(s)\right)^{-\gamma-\gamma r}\right).$$

This implies that $\left|E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right)\right| \to 0$, as $t \to \infty$, and

$$\begin{aligned}\frac{1}{j!}\left(\frac{\partial}{\partial\mu_k}\right)^j E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right) &= \frac{1}{j!}\left(\frac{\partial}{\partial\mu_k}\right)^j \Bigg\{-\sum_{q=2}^{r}\frac{(\mu_k)^{-q}\left(\psi(s)\right)^{-\gamma q}}{\Gamma(n-\gamma q)} \\ &\quad + O\left(|\mu_k|^{-1-r}\left(\psi(s)\right)^{-\gamma-\gamma r}\right)\Bigg\}, \\ &= -\sum_{q=2}^{r}\frac{(-1)^j(q+j-1)\ldots(q+1)q\mu_k^{-q-j}\left(\psi(s)\right)^{-\gamma q}}{j!\Gamma(n-\gamma q)} \\ &\quad + O\left(|\mu_k|^{-1-r}\left(\psi(s)\right)^{-\gamma-\gamma r}\right).\end{aligned}$$

This shows that $\left|\frac{1}{j!}\left(\frac{\partial}{\partial\mu_k}\right)^j E_{\gamma,n}\left(\mu_k\left(\psi(s)\right)^{\gamma}\right)\right| \to 0,\ 1 \le j \le n_k - 1$, as $s \to \infty$. It now follows that

$$\|w(s)\| = \|w_0\left(\psi(s)\right)^{n-1} E_{\gamma,n}\left(B\left(\psi(s)\right)^{\gamma}\right)\| \to 0,$$

as $s \to +\infty$ for non-zero initial value w_0. □

3.2.2 Stability analysis of the generalized perturbed fractional system

Consider the non-linear FDEs

$$^{c}\mathcal{D}^{\gamma,\psi}w(s) - Bw(s) = g(s)w(s);\quad (0 < \gamma < 1), \tag{3.9}$$

where the matrix $B = (b_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, and vector $w(s) = [w_1(s), w_2(s), ..., w_n(s)]^T \in \mathbb{R}^{n\times n}$. $g(s)$ is a matrix of order $n \times n$ which depends upon s.

satisfying non-homogeneous initial condition

$$ {}^{c}\mathcal{D}^{n-1,\psi} w(s) \mid_{s=0} = w_0. \tag{3.10} $$

Theorem 3.2.2. *Suppose that $\|g(s)\|$ is bounded, that is for some $N > 0$, $\|g(s)\| \leq N$ and all the eigenvalues of B satisfy*

$$ |\, arg\,(\mu(B))\,| > \frac{\gamma\pi}{2}. \tag{3.11} $$

Then the solution of system (3.9)-(3.10) is asymptotically stable.

Proof. By the Duhamel's principle the solution of the system (3.9) with condition (3.10) is

$$ \begin{aligned} w(s = {} & w_0\,(\psi(\rho))^{\gamma-1}\,E_{\gamma,\gamma}\,(B\,(\psi(\rho))^{\gamma}) \\ & + \int_0^s (\psi(s-\rho))^{\gamma-1}\,E_{\gamma,\gamma}\,(B\,(\psi(s-\rho))^{\gamma})\,\psi'(\rho)g(\rho)w(\rho)d\rho. \end{aligned} \tag{3.12} $$

By applying norm on both sides of equation (3.12) and also by triangle property of norm, we obtained

$$ \begin{aligned} \|w(s)\| \leq {} & \left\|w_0\,(\psi(\rho))^{\gamma-1}\,E_{\gamma,\gamma}\,(B\,(\psi(\rho))^{\gamma})\right\| \\ & + \int_0^s (\psi(s-\rho))^{\gamma-1}\,\|E_{\gamma,\gamma}\,(B\,(\psi(s-\rho))^{\gamma})\|\,\|\psi'(\rho)\|\,\|g(\rho)\|\,\|w(\rho)\|\,d\rho. \end{aligned} $$

By using Lemma (3.2.1), we have

$$ \begin{aligned} \|w(s)\| \leq {} & \left\|w_0\,(\psi(\rho))^{\gamma-1}\,E_{\gamma,\gamma}\,(B\,(\psi(\rho))^{\gamma})\right\| \\ & \exp\left\{\int_0^s (\psi(s-\rho))^{\gamma-1}\,\|E_{\gamma,\gamma}\,(B\,(\psi(s-\rho))^{\gamma})\|\,\|f(\rho)\|\,\psi'(\rho)d\rho\right\}, \end{aligned} $$

$$ = \left\|w_0\,(\psi(\rho))^{\gamma-1}\,E_{\gamma,\gamma}\,(B\,(\psi(\rho))^{\gamma})\right\| \exp\left\{\int_0^s (\psi(\rho))^{\gamma-1}\,\|E_{\gamma,\gamma}\,(B\,(\psi(\rho))^{\rho})\|\,\|g(s-\rho)\|\,\psi'(\rho)d\rho\right\}. $$

Since $g(s)$ is bounded. So, we have

$$ \|w(s)\| \leq \left\|w_0\,(\psi(\rho))^{\gamma-1}\,E_{\gamma,\gamma}\,(B\,(\psi(\rho))^{\gamma})\right\| \exp\left\{\int_0^s N\left\|(\psi(\rho))^{\gamma-1}\,E_{\gamma,\gamma}\,(B\,(\psi(\rho))^{\gamma})\right\|\psi'(\rho)d\rho\right\}. $$

First, we consider that B matrix is similar to a diagonal matrix. i.e. $\exists$ an invertible matrix H, such that

$$\int_0^s \left\|(\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(B\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\right\| d\rho = \int_0^s \Bigg\| H \text{diag}\Bigg[(\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_1\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho), (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_2\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho), \ldots .., (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_m\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\Bigg] H^{-1}\Bigg\| d\rho. \tag{3.13}$$

Now, we will prove that $\exists$ a constant $K > 0$, such that

$$\int_0^s \left| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_k\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\right| d\rho \leq K, \; 1 \leq k \leq m.$$

For $s > s_0 > 0$, we have

$$\int_0^s \left| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_k\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\right| d\rho = \int_0^{s_0} \left| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_k\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\right| d\rho + \int_{s_0}^s \left| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_k\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\right| d\rho.$$

By Remark (3.2.1), we obtained

$$\int_0^s \left| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_k\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\right| d\rho \leq \int_0^{s_0} (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(|\mu_k|\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho) d\rho + \int_{s_0}^s \Bigg| -\sum_{q=2}^{r} \frac{(\mu_k)^{-q}\left(\psi(\rho)\right)^{-\gamma q+\gamma-1}}{\Gamma(\gamma-\gamma q)}\psi'(\rho) + O\left(|\mu_k|^{-1-r}\left(\psi(\rho)\right)^{-\gamma q-1}\right)\psi'(\rho)\Bigg| d\rho,$$

$$\leq \int_0^{s_0} \sum_{q=0}^{\infty} \frac{|\mu_k|^{q}\left(\psi(\rho)\right)^{\gamma q+\gamma-1}}{\Gamma(\gamma q+\gamma)}\left(\psi'(\rho)\right) d\rho + \sum_{q=2}^{p} \frac{|\mu_k|^{-q}}{|\Gamma(\gamma-\gamma q)|}\int_{s_0}^s \left(\psi(\rho)\right)^{\gamma-\gamma q-1}\psi'(\rho) d\rho + O\left(|\mu_k|^{-1-r}\left(\psi(s)\right)^{-\gamma r}\right). \tag{3.14}$$

By integrating the equation (3.14), we have

$$\int_0^s \left| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(\mu_k\left(\psi(\rho)\right)^{\gamma}\right)\psi'(\rho)\right| d\rho = \sum_{q=0}^{\infty} \frac{|\mu_k|^{q}\left(\psi(s_0)\right)^{\gamma q+\gamma}}{\Gamma(\gamma q+\gamma+1)} + \sum_{q=2}^{r} \frac{|\mu_k|^{-q}\left(\psi(s)\right)^{\gamma-\gamma q}}{|\Gamma(\gamma-\gamma q+1)|} - \sum_{k=2}^{r} \frac{|\mu_k|^{-q}\left(\psi(s_0)\right)^{\gamma-\gamma q}}{|\Gamma(\gamma-\gamma q+1)|} + O\left(|\mu_k|^{1-r}\left(\psi(s)\right)^{-\gamma r}\right),$$

$$\to (\psi(s_0))^{\gamma} E_{\gamma,\gamma+1}\left(|\mu_k| (\psi(s_0))^{\gamma}\right) + \sum_{q=2}^{r} \frac{|\mu_k|^{-q} (\psi(s_0))^{\gamma-\gamma q}}{|\Gamma(\gamma-\gamma q+1)|} \le N, \text{ as } s \to +\infty.$$

thus equation (3.13) becomes

$$\int_0^s \left\| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma}\left(B (\psi(\rho))^{\rho}\right) \psi'(\rho) \right\| d\rho \le C \; for \; s \ge 0.$$

Now, we suppose that matrix B is similar to a Jordan form. Since $s > s_0 > 0$, we have

$$\begin{aligned} &\int_0^s \left| (\psi(\rho))^{\gamma-1} \frac{1}{j!} \left(\frac{\partial}{\partial \mu_k}\right)^j E_{\gamma,\gamma}\left(\mu_k (\psi(\rho))^{\gamma}\right) \psi'(\rho) \right| d\rho \\ &= \int_0^{s_0} \left| (\psi(\rho))^{\gamma-1} \frac{1}{j!} \left(\frac{\partial}{\partial \mu_k}\right)^j E_{\gamma,\gamma}\left(\mu_k (\psi(\rho))^{\gamma}\right) \psi'(\rho) \right| d\rho \\ &+ \int_{s_0}^s \left| (\psi(\rho))^{\gamma-1} \frac{1}{j!} \left(\frac{\partial}{\partial \mu_k}\right)^j E_{\gamma,\gamma}\left(\mu_k (\psi(\rho))^{\gamma}\right) \psi'(\rho) \right| d\rho. \end{aligned} \tag{3.15}$$

By Mittag-Leflier definition and using Remark (3.2.1) in the equation (3.15), we get

$$\begin{aligned} &\int_0^s \left| (\psi(\rho))^{\gamma-1} \frac{1}{j!} \left(\frac{\partial}{\partial \mu_k}\right)^j E_{\gamma,\gamma}\left(\mu_k (\psi(\rho))^{\gamma}\right) \psi'(\rho) \right| d\rho \\ &\le \int_0^{s_0} \sum_{q=0}^{\infty} \frac{q(q-1)...(q-j+1)|\mu_k|^{q-j}}{j!\Gamma(\gamma q+\gamma)} (\psi(\rho))^{\gamma q+\gamma-1} \psi'(\rho) d\rho \\ &+ \int_{s_0}^s \left| (\psi(\rho))^{\gamma-1} \frac{1}{j!} \left(\frac{\partial}{\partial \mu_k}\right)^j \left\{ -\sum_{q=2}^{r} \frac{(\mu_k)^{-q} (\psi(\rho))^{-\gamma q}}{\Gamma(\gamma-\gamma q)} + O\left(|\mu_k|^{-1-r} (\psi(\rho))^{-\gamma-\gamma r}\right) \right\} \psi'(\rho) \right| d\rho, \\ &= \sum_{q=0}^{\infty} \frac{q(q-1)...(q-j+1)|\mu_k|^{q-j}}{j!\Gamma(\gamma q+\gamma)} \int_0^{s_0} (\psi(\rho))^{\gamma q+\gamma-1} \psi'(\rho) d\rho + \\ &\int_{s_0}^s \left| \left\{ -\sum_{q=2}^{r} \frac{(-1)^j (q+j-1)! \, (\mu_k)^{-q-j}}{j!(q-1)!\Gamma(\gamma-\gamma q)} (\psi(\rho))^{\gamma-\gamma q-1} \psi'(\rho) \right.\right. \\ &\left.\left. + O\left(|\mu_k|^{-1-r} (\psi(\rho))^{-\gamma-\gamma r} \psi'(\rho)\right) \right\} \right| d\rho. \end{aligned} \tag{3.16}$$

Integrating equation (3.16) and also use property of absolute value

$$\int_0^s \Big| (\psi(\rho))^{\gamma-1} \frac{1}{j!} \left(\frac{\partial}{\partial \mu_k} \right)^j E_{\gamma,\gamma} \left(\mu_k \left(\psi(\rho) \right)^\gamma \right) \psi'(\rho) \Big| d\rho$$
$$\leq \sum_{q=0}^{\infty} \frac{q(q-1)...(q-j+1)|\mu_i|^{q-j}}{j!\Gamma(\gamma q+\gamma+1)} (\psi(s_0))^{\gamma q+\gamma} + \sum_{q=2}^{r} \frac{(q+j-1)!|\mu_k|^{-q-j}}{j!(q-1)!|\Gamma(\gamma-\gamma q+1)|} (\psi(s))^{\gamma-\gamma q}$$
$$- \sum_{q=2}^{r} \frac{(q+j-1)!|\lambda_k|^{-q-j}}{j!(q-1)!|\Gamma(\gamma-\gamma q+1)|} (\psi(s_0))^{\gamma-\gamma q} + O\left(|\mu_k|^{-1-r} (\psi(s))^{-\gamma r} \right)$$
$$\rightarrow (\psi(s_0))^\gamma \frac{1}{j!} \left(\frac{\partial}{\partial |\mu_k|} \right)^j E_{\gamma,\gamma} \left(|\mu_k| \left(\psi(s_0) \right)^\gamma \right) + \sum_{q=2}^{r} \frac{(q+j-1)!|\mu_k|^{-q-j} (\psi(s_0))^{\gamma-\gamma q}}{j!(q-1)!\Gamma(\gamma-\gamma q+1)} \leq D,$$

as $s \rightarrow +\infty$.
This shows that $\exp\left\{ M \int_0^s \left\| (\psi(\rho))^{\gamma-1} E_{\gamma,\gamma} \left(B \left(\psi(\rho) \right)^\gamma \right) \psi'(\rho) \right\| d\rho \right\}$ is bounded.
We also note that $\left\| w_0 \left(\psi(\rho) \right)^{\gamma-1} E_{\gamma,\gamma} \left(B \left(\psi(s) \right)^\gamma \right) \right\| \rightarrow 0$ as $s \rightarrow +\infty$.
Finally, we have $\lim\limits_{s\rightarrow\infty} w(s) = 0$.
So, the solution (3.12) is asymptotically stable. □

3.3 Existence and uniqueness of solutions for generalized FDEs

In this part, we will discuss the existence and uniqueness of the solutions of the generalized differential equation. Consider the operator $\triangle(s,\rho)$, which is defined as

$$\triangle(s,y) = s^\rho + \int_0^{n-1} f(\gamma) s^\gamma d\gamma, \tag{3.17}$$

where $n-1 < \rho \leq n$, $f(\gamma)$ is continuous for all $\gamma > 0$, and analytic in $y \in G \subset \mathbb{C}$.

Lemma 3.3.1. *Consider the operator $A_q(s,w)$, which is defined as*

$$G_q(s,y) = B_{\rho-q-1}(s,y) + \int_q^{n-1} f(\gamma) B_{\gamma-q-1}(s,y) d\gamma, \; q = 0,...,n-1. \tag{3.18}$$

where $B_\rho(s,y) = \mathcal{L}_\psi^{-1}[\frac{v^\rho}{h(v,y)}](s)$,$y \in G \subset C$, and $\mathcal{L}_\psi^{-1}$ represents the inverse ψ- Laplace transform. Then $G_q(s,C)\nu_q$ solves the Cauchy problem

$${}^cD^{\rho,\psi} w(s) + \int_0^n f(\gamma) {}^cD^{\gamma,\psi} w(s) d\gamma = 0, \tag{3.19}$$

$$({}^cD^{p,\psi}w)(0) = \sigma_{p,q}\nu_p \ , \ p = 0, ..., n-1. \tag{3.20}$$

where $\sigma_{p,q}$ shows the Kronecker delta

$$\sigma_{p,q} = \begin{cases} 1 & \textit{if } p = q \\ 0 & \textit{if } p \neq q. \end{cases}$$

Proof. By applying $\mathcal{L}_\psi$ on equation (3.19), we obtain

$$\mathcal{L}_\psi \left[{}^cD^{\rho,\psi}w(s)\right] + \sum_{q=1}^{n-1} \int_{(q-1,q)} f(\gamma)\mathcal{L}_\psi \left[{}^cD^{\gamma,\psi}w(s)\right] d\gamma = 0.$$

By using the property of the ψ-Laplace transform
$v^\rho \bar{w}(v) - \sum_{i=0}^{n-1} v^{\rho-i-1}({}^cD^{i,\psi}w)(0)+$

$$\sum_{q=1}^{n-1} \int_{q-1}^{q} f(\gamma) \left[v^\gamma \bar{w}(v) - \sum_{p=0}^{q-1} v^{\gamma-p-1}({}^cD^{p,\nu}w)(0) \right] d\gamma = 0.$$

By simplifying the above equation, we have
$s^\rho \bar{w}(v) + \int_0^{n-1} f(\gamma)v^\gamma \bar{w}(v) d\gamma =$

$$\sum_{i=0}^{n-1} v^{\rho-i-1}({}^cD^{i,\psi}w)(0) + \int_0^{n-1} \sum_{p=0}^{q-1} v^{\gamma-p-1}({}^cD^{p,\psi}w)(0) f(\gamma) d\gamma.$$

By virtue of equation (3.20), we have

$$\bar{w}(v) \left[v^\rho + \int_0^{n-1} f(\gamma)v^\gamma d\gamma \right] = v^{\rho-q-1}\nu_q + \int_q^{n-1} v^{\gamma-q-1}\nu_q d\gamma.$$

Now using equation (3.18), we get

$$\bar{w}(v)\triangle(v, C) = \nu_q \left(v^{\rho-q-1} + \int_q^{n-1} f(\gamma)v^{\gamma-q-1}\nu_q d\gamma \right).$$

So, finally, we obtained

$$G_q(s, y) = v^{\rho-q-1} + \int_q^{n-1} v^{\gamma-q-1} f(\gamma) d\gamma.$$

So, the solution is given by $w_q = G_q(s, C)\nu_q, \ \ q = 0, ..., n-1.$ □

Corollary 3.3.1. *Let the collection of solution operators be represented by* $G_q(s, y)$, $q = 0, ..., n-1$, *which is defined in Lemma (3.3.1). Then the Cauchy problem*

$$ {}^cD^{\mu,\psi}w(s)+\int_0^m f(\gamma)^cD^{\gamma,\psi}u(s)\lambda d\gamma = 0,\ ({}^cD^{p,\psi}w)(0) = \sigma_{p,q}\nu_p\ ,\ p = 0, ..., m-1. \quad (3.21) $$

has a solution

$$ w(s) = \sum_{q=0}^{n-1} G_q(s, C)\nu_q. \quad (3.22) $$

Theorem 3.3.1. *Supose that* $g(s)$ *is continuous on* $s \in [0, T]$, *and* ν_q *be a continuous for all* $q = 0, ..., m-1$. *Then the Cauchy problem (3.21) has a unique solution*

$$ v(s) = \sum_{q=0}^{m-1} G_q(s, C)\nu_q + \int_0^s G_{n-1}(s-\rho, C)g(\rho)\psi'(\rho)d\rho. \quad (3.23) $$

Proof. We separate the Cauchy problem (3.21) in to two problems

$$ {}^cD^{\mu,\psi}v(s) + \int_0^m f(\gamma)^cD^{\gamma,\psi}v(s)\lambda d\gamma = 0, \quad (3.24) $$

$$ {}^cD^{q,\psi}v(0) = \nu_q,\ q = 0, ..., m-1. \quad (3.25) $$

And

$$ {}^cD^{\mu,\psi}w(s) + \int_0^m f(\gamma)^cD^{\gamma,\psi}w(s)\lambda d\gamma = h(s),\ s > 0, \quad (3.26) $$

$$ {}^cD^{q,\psi}[w](0) = 0,\ \ q = 0, ..., m-1. \quad (3.27) $$

By using corollary (3.3.1) the unique solution to the Cauchy problem (3.24)-(3.25) is given by

$$ v(s) = \sum_{q=0}^{m-1} G_q(s, C)\nu_q. \quad (3.28) $$

For the solution of Cauchy problem (3.26)-(3.27), follows from the fractional Duhamel's principle, it enough to solve the Cauchy problem for the the homogeneous equation:

$$ {}^cD^{\mu,\psi}v(s) + \int_0^m f(\gamma, C)^cD^{\gamma,\psi}v(s)\lambda d\gamma = 0, \quad (3.29) $$

$$ {}^cD^{\psi,q}w(s, \rho)\mid_{s=\rho} = 0,\ q = 0, ..., m-2. \quad (3.30) $$

$$ {}^cD^{\psi,m-1}w(s, \rho)\mid_{s=\rho} = h(\rho). \quad (3.31) $$

Again using the result of corollary (3.3.1) to obtain the solution of this problem ,(also note that $G_q(s,\rho,z) = G_q(s-\rho,z),\ q = 0,...,m-1$, is given by the

$$w(s,\rho) = S_{m-1}(s-\rho,C)h(\rho). \tag{3.32}$$

Thus equation (3.23) obtained by the Duhamel's integral of $w(s,\rho)$ and equation (3.28). The uniqueness of a solution also follows from the obtained representation (3.23).

Chapter 4

Haar-Duhamel methods for fractional differential equation

Exact solutions of many FDEs are unknown. Therefore, different numerical techniques have been applied for providing approximate solutions. So, many numerical techniques i.e. the Adomian decomposition method (ADM) [36], the homotopy perturbation method (HPM) [37], wavelet methods [38, 39] etc. have been used for approximating the solution of FDEs. There are different types of wavelet but Haar wavelet is the orthonormal simplest of them [40]. Lepik in 2007-8 [41, 42] solved differential equations by using Haar wavelet algorithm. Hariharan [40] in 2009, found the approximate solution of Fisher's equation using Haar wavelet method. In the same way, Kannan [43] and Hariharan solved Fitzhugh-Nagumo equation. Berwal [44] in 2013 solved Telegraph equation using Haar wavelet technique. The good characteristics of this technique is to convert a fractional differential equation into an algebraic equation and possibility to integrate a rectangular function analytically arbitrary time. The disadvantage of this technique is their discontinuity. In this chapter we will discuss wavelet, Haar wavelet and their properties. Also, we present operational matrix of fractional integration by Haar wavelets technique. We also used Haar wavelet with Duhamel's principle to develop a method for solving fractional differential equations.

4.1 Wavelet

The Haar wavelet was first presented in the thesis of the A.Haar (1909). The concept of the wavelet is not a new. The concept of wavelet originates from different field involving engineerings, Physics and applied mathematics. It has many different origins in the history of the mathematics. It has been used in the numerical analysis and signal processing. Now a days, it has been commonly used in the field of numerical solution of the initial and boundary value problem.

Wavelets are defined as orthonormal system of functions with a compact support obtained with the assistance of dilation and translation. Its basis is formed by a particular functions defined on the finite interval using dilation and translation. If the properties of orthonormality are not necessary, then large class of functions are also called "wavelets". There are many form of wavelets e.g Haar wavelets [45, 46], Legender wavelets [47], Battle-Lemarie [48].

Wavelets are constructed from the specific transformation i.e. compression and translation of a single valued function called the mother wavelet which is given as:

$$G_{c,d}(s) = |c|^{-\frac{1}{2}} G\left(\frac{s-d}{c}\right). \tag{4.1}$$

If $c > 1$ in equation (4.1), then wavelet has larger support in time domain and having lower frequencies. On the other hand, if we take $c < 1$, then wavelet has smaller support in time domain and having higher frequencies become compressed form of mother wavelet.

The parameters are discretized as $c = c_0^{-q},\ d = md_0c_0^{-q}$. Then we obtained the class of discrete wavelets

$$G_{q,m}(s) = |c_0|^{\frac{1}{2}} G\left(c_0^q s - md_0\right),$$

where $G_{q,m}(s)$ form a basis for $L^2(\mathbb{R})$. These wavelets became orthonormal basis, if $c_0 = 2$, and $d_0 = 1$.

An orthonormal wavelets form called the Haar wavelet, which has been used by many researchers. Mathematically, Haar wavelets family carries with its rectangular functions. Haar wavelets has many applications in the field of engineering and science. It is also used to determine the eigenfunctions corresponding to the eigenvalues[49],[50].

4.2 Haar wavelet

Note that the Haar wavelet are the simplest orthonormal of the wavelet. It was first introduced by Alfred Haar in 1910. They are the piecewise function defined on the real line which have $-1, 0, 1$ values only. Commonly, Haar wavelets are defined for $0 \leq s < 1$, but in general case $s \in [c, d]$. we will split the $[c, d]$ interval into $2T$ subintervals of equal width, $\Delta s = \frac{d-c}{2T}$, The set of orthogonal Haar wavelet at $[c, d]$ interval is defined.

$$G_0(s) = \begin{cases} 1, & s \in [c, d], \\ 0, & \text{otherwise} \end{cases}$$

and

$$G_i(s) = \begin{cases} 1; & s \in [\eta_1(i), \eta_2(i)) \\ -1; & s \in [\eta_1(i), \eta_2(i)) \\ 0; & \text{otherwise} \end{cases} \tag{4.2}$$

where

$$\begin{aligned} \eta_1(i) &= c + (d - c)\,\frac{q}{r}, \\ \eta_2(i) &= c + (d - c)\,\frac{2q + 1}{r}, \\ \eta_3(i) &= c + (d - c)\,\frac{q + 1}{r}. \end{aligned} \tag{4.3}$$

We defined $T = 2^J$, $r = 2^j$, where $j = 0, 1, 2, ..., J$ and $q = 0, 1, 2, .., r - 1$. The j and q parameters involved here have a definite significance. The quantity j represents the dilation parameter or level of wavelet, because the wavelet turns small or support decreases by increasing j, and J represents the level of maximal resolution for the Haar wavelet. We can deduced from the following subsistence or width of the i-th wavelet equation as

$$\eta_3(i) - \eta_1(i) = \frac{d - c}{r} = \frac{d - c}{2^j}. \tag{4.4}$$

The q parameter represents the translation, because q denotes the position of the wavelet on the x-axis; by giving values of q from 0 to $r - 1$, the starting point of the

i-th wavelet $\eta_1(i)$ translated from $x = c$ to $x = \frac{c+(r-1)d}{r}$. The relationship between i, q and r is $i = r + q + 1$. The above equation(4.2) is true for $i \geq 3$. For $i = 2$, the corresponding scaling function is given as

$$G_2(s) = \begin{cases} 1; & s \in \left[c, \frac{c+d}{2}\right) \\ -1; & s \in \left[\frac{c+d}{2}, d\right) \\ 0; & \text{otherwise.} \end{cases} \tag{4.5}$$

Since the Haar wavelet functions are orthogonal, then the equation(4.2) become

$$\int_c^d G_i(s)G_l(s)ds = \begin{cases} 1; & \text{for } i = l; \\ -1; & \text{for } i \neq l; \\ 0; & \text{otherwise} \end{cases} \tag{4.6}$$

Example 4.2.1. *Suppose that $c = 0,\ d = 1$, the wavelet number is $i = 2$, if $J = 2$, then $j = 0, 1, 2$, and now, we consider $j = 0,\ q = o$, and $r = 1$, the equation (4.5) will become*

$$G_2(s) = \begin{cases} 1, & 0 \leq s < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq s < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the wavelet number is $i = 3$, for $q = o,\ j = 1$ and $r = 2$. so, equation (4.5) will become

$$G_3(s) = \begin{cases} 1, & 0 \leq s < \frac{1}{4}, \\ -1, & \frac{1}{4} \leq s < \frac{1}{2}, \\ 0, & elsewhere. \end{cases}$$

- **Properties**

1 Any arbitrary function can be written as a linear combination of $G_0(s),\ G_0(2s), ...,$ $G_0(2^q s), ...$ and their translation functions.

2 Any arbitrary function can be written as a linear combination of constant function $G_1(s),\ G_1(2s), ..., G_1(2^q s), ...$ and their translation.

4.3 Approximation of functions by Haar wavelet

Let w be the function defined on $[0, 1]$. It can be written in the form of Haar wavelet as

$$w(s) = \sum_{i} K_i G_i(s). \tag{4.7}$$

where G_i are the basis functions and co-efficient of Haar wavelet represented by K_i. We decompose the equation (4.7) into $2T$-terms as

$$\tilde{w}(s) = \sum_{i=0}^{2T} K_i G_i(s). \tag{4.8}$$

The corresponding approximation function of the equation (4.8) can be written as

$$\tilde{w}(s_l) = \sum_{i=0}^{2T} K_i G_i(s_l). \tag{4.9}$$

The matrix form of equation (4.9) is

$$W^T = KG, \tag{4.10}$$

where K can be found as

$$K = W^T G,$$

where W^T and G are row vector of dimension $2T$, and G is Haar matrix. There are many ways to calculate the error function of the wavelet estimations, but we define one of these error function as:

$$\Delta = \int_c^d \left[w(s) - \tilde{w}(s)\right]^2 ds.$$

Haar wavelet are related to the groups of piecewise functions. If the function is differentiable, then constant function convergence rate function is $O(\frac{1}{T^2})$.

4.4 Fractional integral by Haar wavelet

The Haar wavelets Algorithm are helpful for computing the numerical solution of integral and differential equations due to its simplicity. We compute the integrals by Haar wavelet function. If we integrating Haar function γ-times, we have

$$\mathcal{J}_a^{\gamma} G_i(s) = P^{\gamma,i}(s) = \int_a^s \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} G_i(t)dt. \tag{4.11}$$

$$i = 1, 2, 3, ..., 2T.$$

Generally, these integrals can be calculated by using equation (4.11).

$$P^{\gamma,i}(s) = \begin{cases} \int_{\eta_1(s)}^{s} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt, & \eta_1(s) \le s < \eta_2(s), \\ \int_{\eta_1(s)}^{s} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt, & \eta_2(s) \le s < \eta_3(s), \\ \int_{\eta_1(s)}^{s} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt, & s > \eta_3(s), \end{cases}$$

and

$$P^{\gamma,i}(s) = \begin{cases} \int_{\eta_1(s)}^{s} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt, & \eta_1(s) \le s < \eta_2(s), \\ \int_{\eta_1(s)}^{\eta_2(s)} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt - \int_{\eta_2(s)}^{s} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt, & \eta_2(s) \le s < \eta_3(s), \\ \int_{\eta_1(s)}^{\eta_2(s)} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt - \int_{\eta_2(s)}^{\eta_3(s)} \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)} dt, & s > \eta_3(s). \end{cases}$$

So, finally we get the integrals of the Haar wavelet function of order γ as

$$P^{\gamma,i}(s) = \begin{cases} \frac{(s-\eta_1(s))^{\gamma}}{\Gamma(\gamma+1)}, & \eta_1(s) \le s < \eta_2(s), \\ \frac{(s-\eta_1(s))^{\gamma}}{\Gamma(\gamma+1)} - 2\frac{(s-\eta_2(s))^{\gamma}}{\Gamma(\gamma+1)}, & \eta_2(s) \le s < \eta_3(s), \\ \frac{(s-\eta_1(s))^{\gamma}}{\Gamma(\gamma+1)} - 2\frac{(s-\eta_2(s))^{\gamma}}{\Gamma(\gamma+1)} + \frac{(s-\eta_3(s))^{\gamma}}{\Gamma(\gamma+1)}, & s > \eta_3(s), \end{cases} \tag{4.12}$$

The equation (4.12) is valid for $i > 1$. If we take $i = 1$, we have $\eta_1(1) = a,\ \eta_2(1) = \eta_3(1) = b$ and

$$P^{\gamma,1}(s) = \frac{1}{\Gamma(\gamma+1)} (s-a)^{\gamma}. \tag{4.13}$$

4.4.1 Haar Matrix

First of all, we define the grid point as

$$\tilde{s}_l = c + l\frac{d-c}{4^J},\ l = 0, 1, 2, 3, ..., 4^J. \tag{4.14}$$

where J is the maximal level of resolution. We use collocation points as

$$s_l = \frac{1}{2}\left(\tilde{s}_{l-1} - \tilde{s}_l\right), \ l = 1, 2, 3, ..., 4^J. \tag{4.15}$$

and replacing s by s_l in equation (4.2),(4.12) and equation (4.13). It gives the results of these sets in the form of a matrix. For this we present the Haar matrices $G, P^1, ..., P^u$ of order $2T \times 2T$. The entries of Haar matrices will be $G(i,l) = G_i(l), \ P^u(i,l) = P^{ui}(s_l)$, where $u = 1, 2, 3, 4, ...$

For example suppose that $c = 0, \ d = 1$ and $J = 1$. so, the Haar matrix will be a 4×4 matrix. From the equation (4.14), the grid points for $l = 0, 1, 2, 3, 4, 5$ are $\tilde{s}_0 = 0, \ \tilde{s}_1 = \frac{1}{4}, \ \tilde{s}_2 = \frac{2}{4}, \ \tilde{s}_3 = \frac{3}{4}$, and $\tilde{s}_4 = 1$. To find collocation points using these grid points in equation (4.15), we obtained $s_1 = \frac{1}{8}, \ s_2 = \frac{3}{8}, \ s_3 = \frac{5}{8}$ and $s_4 = \frac{7}{8}$. The entries of Haar matrix G can be calculated by using equation (4.2).

Now for $G_1(s)$ at the points $s_1, \ s_2, \ s_3, \ s_4 \in [\eta_1(1), \eta_2(1))$, the first row of G matrix contains all entries 1. Similarly, for $G_2(s), \ G_3(s)$ and $G_4(s)$. So, G matrix is

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

4.4.2 Integration Matrix for Haar wavelet

Now we develop the integration matrix (or operational matrix) for the Haar wavelets. These matrices have been mostly used to solve the FDEs.

Now we find the entries of P^1 by using equation (4.2), (4.12) and (4.13). For calculating $P^{11}(s_l)$ at the points $s_1, \ s_2, \ s_3, \ s_4 \in [\eta_1(1), \eta_2(1))$, and we use $P^{\gamma,i}(s) = \frac{1}{\Gamma(\gamma+1)}\left[s - \eta_1(i)\right]^{\gamma}$, for $s \in [\eta_1(i), \eta_2(i)]$. For $\gamma = 1$, the entries of first row of P^1 is $0.125, \ 0.375, \ 0.625, \ 0.875$. Similarly, other entries of P^1 and P^2 operational matrices are

$$P^1 = \frac{1}{8}\begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ P^2 = \frac{1}{128}\begin{bmatrix} 1 & 9 & 25 & 49 \\ 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 \\ 0 & 0 & 1 & 7 \end{bmatrix},$$

$$P^{1.5} = \begin{bmatrix} 0.03324519 & 0.17274707 & 0.37169252 & 0.61570954 \\ 0.03324519 & 0.17274707 & 0.30520214 & 0.27021539 \\ 0.03324519 & 0.10625669 & 0.05944356 & 0.04507156 \\ 0 & 0 & 0.03324519 & 0.10625669 \end{bmatrix}.$$

4.4.3 Riemann-Liouville integral by Haar wavelet

Now we consider Riemann-Liouville integration of a function $w(s)$ defined on $[0, 1]$ by Haar wavelet. Suppose that function $g(s)$ is integrable. Then function $w(s)$ can be approximated as

$$w(s) = \sum_i K_i G_i(s). \tag{4.16}$$

Now taking finite terms of the series in (4.16)

$$w(s) = \sum_{i=0}^{n-1} K_i G_i(s). \tag{4.17}$$

After substituting collocation point into equation(4.17),we have

$$W(s) = K_n G_n(s). \tag{4.18}$$

Applying Riemann-Liouville integration on the equation(4.18), we have

$$\mathcal{J}_a^{\gamma} W(s) = K_n \mathcal{J}_a^{\gamma} G_n(s) = K_n P_{n\times n}^{\gamma} G_n(s). \tag{4.19}$$

We can compute $\mathcal{J}_a^{\gamma} W(s) = K_n P_{n\times n}^{\gamma} G_n(s)$ by using equation (4.12) and (4.13).

Example 4.4.1. *Consider that*

$$w(s) = \cos(ws), \ \ s \in [0, 1] \ \ \textit{and} \ \gamma \in (1, 2]. \tag{4.20}$$

First of all, we calculate the exact R-L integral of the equation (4.20), as

$$\mathcal{J}_0^{\gamma} w(s) = \mathcal{J}_0^{\gamma} \left\{\cos(ws)\right\}. \tag{4.21}$$

By using Taylor series of $\cos(\gamma s)$ in equation (4.21), we have

$$\mathcal{J}_0^{\gamma} w(s) = \sum_{q=0}^{\infty} \frac{(-1)^q (w)^{2q}}{\Gamma(2q+1)} \mathcal{J}_0^{\gamma} s^{2q}. \tag{4.22}$$

Now using $\mathcal{J}_0^\gamma s^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\gamma+1)} s^{\gamma+\eta}$ and by definition of Mittag-Leffler function in equation(4.22)

$$\mathcal{J}_0^\gamma w(s) = s^\gamma E_{2,\gamma+1}\left(-(ws)^2\right). \tag{4.23}$$

The above equation (4.23) gives the exact R-L integral of functions in equation (4.20). The approximate and exact R-L integral by Haar wavelet are plotted in Fig4.1(a) and also for $J = 5$ and distinct values of γ, the absolute error between approximate and exact are shown.

Example 4.4.2. *Consider*

$$w(s) = \exp(ws),\ \ s \in [0,1] \ \textit{and}\ \gamma \in (1,2]. \tag{4.24}$$

By using Taylor series of $\exp(ws)$, we have

$$w(s) = \sum_{q=0}^{\infty} \frac{(ws)^q}{\Gamma(q+1)}. \tag{4.25}$$

Applying R-L integral on the equation(4.25), we have

$$\mathcal{J}_0^\gamma w(s) = \sum_{q=0}^{\infty} \frac{(w)^q}{\Gamma(q+1)} \mathcal{J}_0^\gamma s^q. \tag{4.26}$$

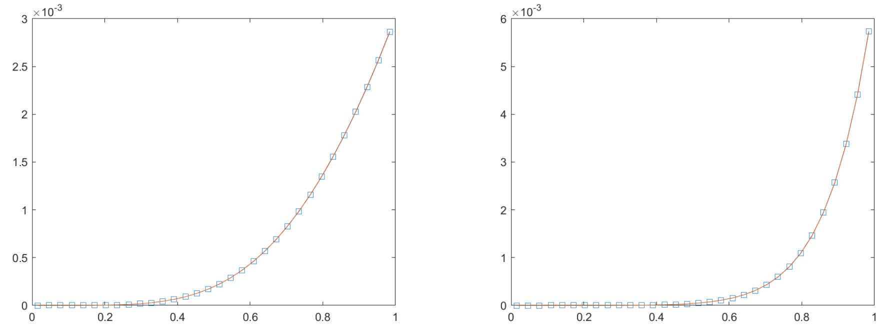
By using $\mathcal{J}_0^\gamma s^q = \frac{\Gamma(q+1)}{\Gamma(q+\gamma+1)} s^{q+\gamma}$ in equation (4.26)

$$\mathcal{J}_0^\gamma w(s) = \sum_{q=0}^{\infty} \frac{(w)^q}{\Gamma(q+\gamma+1)} s^{q+\gamma}. \tag{4.27}$$

By Mittag-Leffler function

$$\mathcal{J}_0^\gamma w(s) = s^\gamma E_{1,\gamma+1}(ws). \tag{4.28}$$

The above equation (4.28) is exact R-L integral of equation (4.24). The exact and approximate R-L integral and their absolute error is plotted in the Fig4.1(b) for distinct values of γ and $J = 5$.



(a) $J = 5,\ \gamma = 5,\ w = 7.$ (b) $J = 5,\ w = 7,\ \gamma = 6.$

Figure 4.1: The exact R-L integral of $w(s)$ and by Haar wavelet R-L integral are plotted

4.5 Error Analysis

In this part, we discuss an inequality [51] in the form of upper bound, that shows the Haar wavelet convergence.

Theorem 4.5.1. *If a function $w(s)$ is differentiable and its first order derivative is bounded i.e $|w'(s)| < T$, $T > 0$, $\forall s \in (a, b)$, the approximation of $w(s)$ is represented by $w_r(s)$. Then*

$$\|w(s) - w_r(s)\| = O\left(\frac{1}{r}\right).$$

Proof. Let $w(s)$ be defined on $[a, b)$ as

$$w(s) = \sum_{n=0}^{\infty} K_n G_n(s), \tag{4.29}$$

where $K_n = \langle w(s), G_n(s) \rangle$. We consider r-th terms of equation (4.29), which is represented by $w_r(s)$ and is the approximation of $w(s)$, that is

$$w(s) \simeq w_r(s) = \sum_{n=0}^{r-1} K_n G_n(s), \tag{4.30}$$

where $r = 2^{q+1}$, $q = 0, 1, 2, ...$, the equation (4.30) become

$$w(s) - w_r(s) = \sum_{n=o}^{\infty} K_n G_n(s) - \sum_{n=0}^{r-1} K_n G_n(s) \tag{4.31}$$

$$= \sum_{n=r}^{\infty} K_n G_n(s).$$

Applying norm on equation (4.31), we obtained

$$\|w(s) - w_r(s)\|^2 = \int_a^b (w(t) - w_r(s))^2 \, ds$$

$$= \int_a^b \left(\sum_{n=r}^{\infty} K_n G_n(s) \right)^2 ds$$

$$= \sum_{n=r}^{\infty} \sum_{n=r'}^{\infty} K_n K_{n'} \int_a^b G_n(s) G_{n'}(s) ds. \tag{4.32}$$

By using orthogonality property on equation (4.32), we get

$$\|w(s) - w_r(s)\|^2 = \sum_{n=r}^{\infty} K_n^2 = \sum_{n=2^q+1}^{\infty} K_n^2, \tag{4.33}$$

where $K_n = \int_a^b w(s) G_n(s) ds = \langle w(s), G_n(s) \rangle$. Since

$$G_n(s) = \begin{cases} 1, & a + 2^{-j}q \leq s \leq a + \left(\frac{2q+1}{2}\right) 2^{-j}, \\ -1, & a + \left(\frac{2q+1}{2}\right) 2^{-j} \leq s \leq a + (q+1)2^{-j}, \\ 0, & otherwise. \end{cases}$$

By substituting, we got

$$K_n = 2^{\frac{j}{2}} \left\{ \int_{a+2^{-j}q}^{a+\left(\frac{2q+1}{2}\right)} w(s) ds - \int_{a+\left(\frac{2q+1}{2}\right)}^{a+2^{-j}(q+1)} w(s) ds \right\}.$$

By mean value theorem for integral, $\exists \;\; \gamma, \lambda$ such that

$$\gamma \in \left[a + 2^{-j}q, a + (\frac{2q+1}{2})2^{-j} \right),$$

$$\lambda \in \left[a + (\frac{2q+1}{2})2^{-j}, a + 2^{-j}(q+1) \right).$$

So,

$$K_n = 2^{\frac{j}{2}} \left\{ \left(a + (\frac{2q+1}{2})2^{-j} - (a + 2^{-j}q) \right) w(\gamma) \right.$$

$$\left. - \left(a + 2^{-j}(q+1) - \left(a + (\frac{2q+1}{2})2^{-j} \right) \right) w(\lambda) \right\},$$

$$K_n = 2^{\frac{-j}{2}-1}\left(w(\gamma) - w(\lambda)\right).$$

Again by mean value theorem, $\exists\ \gamma < \rho < \lambda$ such that

$$\begin{aligned} K_n &= 2^{\frac{-j}{2}-1}\left(\lambda - \gamma\right) w'(\rho) \\ &\leq 2^{\frac{-j}{2}-1} 2^{-j} T, \ \because\ |w'(\rho)| < T \\ &= 2^{\frac{-3j}{2}-1} T. \end{aligned}$$

Therefore the equation (4.33) become

$$\begin{aligned} \|w(s) - w_r(s)\|^2 = & \sum_{n=2^{q+1}}^{\infty} K_n^2 = \sum_{j=q+1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} K_n^2\right) \\ \leq & \sum_{j=q+1}^{\infty} \left(\sum_{n=2^j}^{2^{j+1}-1} \left(2^{\frac{-3j}{2}-1} T\right)^2\right) \\ = & \sum_{j=q+1}^{\infty} \left(2^{-3j-2} T^2 \sum_{n=2^j}^{2^{j+1}-1}\right) \\ = & \sum_{j=q+1}^{\infty} \left(2^{-3j-2} T^2 \left(2^{j+1} - 1 - 2^j + 1\right)\right) \end{aligned}$$

$$\|w(s) - w_r(s)\|^2 = T^2 \sum_{j=q+1}^{\infty} 2^{-2j-2}. \tag{4.34}$$

Applying geometric sum formula in the equation (4.34), we have

$$\begin{aligned} \|w(s) - w_r(s)\|^2 = & \frac{2^{-2q-2}}{3} T^2 \\ = & \frac{(r)^{-2}}{3} T^2, \ \because\ r = 2^{q+1} \\ \|w(s) - w_r(s)\|^2 = & O\left(\frac{1}{r}\right). \end{aligned}$$

If the numerical value of T is given, then we can find the exact value of error bound $\frac{2^{-2q-2}}{3}T^2$ for the equation (4.31). Now to find the value of T, we have suppose that $w(s)$ and its first order derivative continuous and differentiable on $[a, b]$, also $w' \in [a, b]$. $w'(s)$ can be given as

$$w'(s) \simeq \sum_{n=0}^{r-1} K_n G_n(s), \tag{4.35}$$

where $K_n = \langle w'(s), G(s)\rangle$. The matrix form of equation (4.35) can be written as

$$w'(s) \simeq K^T G, \tag{4.36}$$

where $K_n = [K_0, K_1, ..., K_{r-1}]^T$, and $G = [G_0, G_1, ..., G_{r-1}]$. Integration of equation (4.35), leads to

$$w(s) - w(a) \simeq \sum_{n=0}^{r-1} K_n \int_a^s G_n(t)dt. \tag{4.37}$$

We defined the points s_j as,

$$s_j = \frac{j - 0.001}{r}, \quad j = 1, 2, 3, ..., r.$$

By using s_j in equation (4.37), we get

$$w(s_j) - w(a) \simeq \sum_{n=0}^{r-1} K_n \int_a^{s_j} G_n(s)ds. \tag{4.38}$$

We can write equation (4.38) in matrix form as

$$W - W(a) = KF. \tag{4.39}$$

The above equation (4.39) is system of linear equation. Where $K = [K_0, K_1, ..., K_{r-1}]^T$, $W(a) = [w(a), w(a), ..., w(a)]$ and $W = [w(s_1), w(s_2), ..., w(s_r)]^T$ and $F = \left[\int_a^s G_n(s)ds\right]_{0 \le n < r-1}$, Now the vector K can be determined by solving the system of linear equation (4.38) and using vector K in equation (4.35), $w'(s)$ can be calculated for each $s \in [a, b]$. Suppose that $s_i \in [a, b]$ and $w'(s_i)$ can be calculated for each $i = 1, 2, 3, ..., l$, where l is equidistant, then approximation of T may be considered as $\epsilon + max|w'(s_i)|_{i \le \le l}$. □

4.6 Haar-Duhamel's method for solving FDEs

We developed a new method to obtain the solutions of the non-linear FDEs numerically, called Haar-Duhamel's method. In general, this method requires to use operational matrix for FDEs. Interestingly, accuracy is not compromised, rather enhanced by using Haar-Duhamel's technique for solving FDEs subject to the initial conditions.

Example 4.6.1. *Consider the FDEs of order $\gamma \in (n-1, n]$*

$${}^cD_0^{\gamma,\psi} v(s) + \mu v(s) = w(s), \tag{4.40}$$

satisfying the conditions

$$v(0)=0,\ v'(0)=0,...,v^{n-1}(s)\mid_{s=0}=0. \tag{4.41}$$

We can check that $v(s)=s^{\gamma+1}$ is solution of the problem (4.40)-(4.41) with $w(s)=\Gamma(\gamma+2)s+\lambda s^{\gamma+1}$. By Duhamel's principle the solution of the problem (4.40)-(4.41) is given as

$$v(s)=\int_0^s (s-\rho)^{\gamma-1}E_{\gamma,\gamma}\left(-\mu\left(s-\rho\right)^{\gamma}\right)\varphi(\rho)d\rho. \tag{4.42}$$

where

$$w(\rho)=\mathcal{J}_0^{2-\gamma}\varphi(\rho). \tag{4.43}$$

We find the numerical solution of the equation (4.40) with condition (4.41) by Haar wavelet that is presented in section (4.3). By Haar wavelet, we can approximate $\varphi(s)$ as

$$\varphi(\rho)=KG(\rho). \tag{4.44}$$

Now by using equation (4.44) in equation (4.43), we have

$$w(\rho)=K\mathcal{J}_0^{2-\gamma}G(\rho). \tag{4.45}$$

After putting collocation points, the above equation (4.45) become

$$W(\rho)=KP^{2-\gamma}G(\rho). \tag{4.46}$$

The equation (4.46) represents a system of linear equation. Where G is Haar matrix and P is operational matrix and K is unknown matrix, which is determined by any Algebraic method using MATLAB program.

Now φ can be determined by using K in equation (4.44). By substituting $\varphi(\rho)$ in equation (4.42), we got

$$\begin{aligned}v(s)=&\int_0^s (s-\rho)^{\gamma-1}E_{\gamma,\gamma}\left(-\mu\left(s-\rho\right)^{\gamma}\right)KG(\rho)d\rho,\\ =&K\int_0^s (s-\rho)^{\gamma-1}E_{\gamma,\gamma}\left(-\mu\left(s-\rho\right)^{\gamma}\right)G(\rho)d\rho,\\ =&KE_n^{\gamma}G.\end{aligned}$$

The above linear system is calculated with the help of MATLAB.

In Table4.1, we give the absolute error for constant value of $\mu = 1,\ n = 2,\ 1 < \gamma \leq 2$ and for distinct values of resolution level J, as J increases, the absolute error decreases.

J	Absolute Error
5	2.07395×10^{-2}
6	1.81276×10^{-2}
7	1.57557×10^{-2}
8	1.37121×10^{-2}

Table 4.1: This table shows the absolute error for distinct values of J.

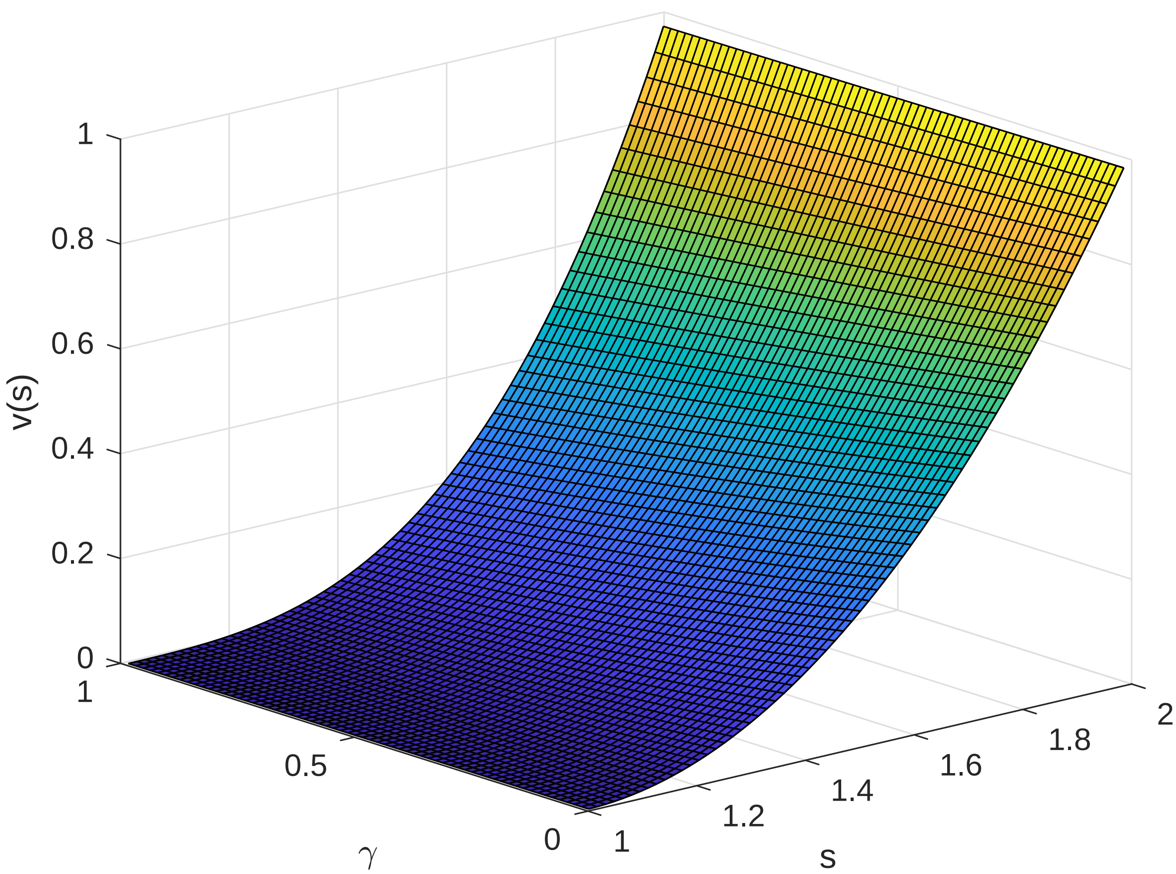


Figure 4.2: Approximate and exact solutions by Haar-Duhamel's for $J = 7$.

□

Chapter 5

Summary

In first chapter we have given a brief introduction and some basic definitions of fractional calculus. Basic notions like Gamma and Mittag-Lefler functions are introduced. Also, we have defined fractional integral and derivatives with respect to another function. We have discussed the classical and generalized Laplace transform and their important properties, results and applications.

In second chapter, we provided a method for finding the solutions of a generalized non-linear fractional differential equations with initial conditions known as Duhamel's principle. A detailed discussion of this method for ordinary differential equations and partial differential equations is carried out. We propose this principle for fractional differential equations subject to initial conditions. We also generalized Duhamel's principle for psi-differential equation and extend this principle for the higher integer order psi-differential equation. We also developed this principle for the fractional higher order psi-differential equation.

In chapter 3 we have presented the applications of the generalized Duhamel's principle. We discussed the stability of the solutions of generalized linear and non-linear FDEs involving Caputo psi-differential equation. Also, we develop the existence and uniqueness of the solutions of the generalized FDEs.

Haar Wavelets which are the primary tool to develop the numerical methods are discussed in detail for the ordinary fractional differential equations and partial fractional differential equations subject to the initial conditions. In chapter 4, we developed a numerical technique for finding the approximate solution of the generalized non-linear FDEs with initial conditions, called Haar-Duhamel's method. Furthermore, to check the accuracy and effectiveness of the proposed method, the results of essential numerical applications are documented in a graphical as well as tabular form.

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Appendix