Numerical Solution of Burgers' Equation Using Discrete Symmetries

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Dedication

To my beloved family especially to my Parents and Brother, lecturers and all my friends

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Abstract

In this thesis, we meticulously construct an invariant Modified-Crank-Nicolson method that fast convergent to the exact solution of a one-dimensional non-linear heat equation. This innovative construction can be faithfully done by preferentially using discrete symmetry groups. Burgers' equation is reduced to a one-dimensional heat equation by using Hopf-Cole transformation. Moreover, this new transformation function represents the exact solution of Burgers' equation. The innovative invariant numerical scheme is carefully constructed by the composition of continuous and discrete symmetry groups. Furthermore, with this numerical scheme, the convergence and efficiency of the standard Crank-Nicolson method is meaningfully improved for the exact solution of Burgers' equation. The notable performance of this numerical scheme is shown both graphically and in tabular form.

Contents

List of Abbrevations

ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
FTCS	Forward in Time, Central in Space
CNM	Crank-Nicolson Method
M-CNM	Modified-Crank-Nicolson Method

List of Figures

List of Tables

Introduction

Our universe is full of evolving correlated entities like the path of a projectile varies with speed and angle, earth's location varies over time and many more examples like this. These changing entities are known as variables in the language of mathematics and their rate of change in the context of another variable is known as derivative. In mathematics, differential equations are the equations which demonstrate the correspondence amidst these variables and their derivatives. In such way differential equations can be categorized mainly into two types. One is known as ordinary differential equation (ODE), in which the derivative of a dependent variable is taken with respect to one independent variable, while for the second type of differential equation called the partial differential equation (PDE), in which we take the derivative of a dependent variable(s) with respect to more than one independent variables. Both these types of differential equations can be sorted into two forms, one is linear and the other one is non-linear depending upon the degree and the product of dependent variables(s) and its derivatives. For instance, if the degree of a dependent variable(s) and its derivative is one and their product is not present then we call it a linear differential equation. On contrary, a differential equation is said to be non-linear differential equation if any of the above-mentioned cases for linearity gets changed.

Many of the physical phenomenon such as force, momentum, temperature, velocity etc., are usually dependent on several variables and generally deals in PDEs. In the 18^{th} century, scientists like Euler, Lagrange, and Laplace [1] did the introductory work about the PDEs. However, it was during the 19^{th} century that it gained so much popularity chiefly due to the influence of Reimann in certain fields of mathematics [1]. As far as the applications of PDEs are concerned in the field of physics and engineering, Maxwell's equations describe the entire theory of electricity and magnetism [1].

Partial differential equations can be solved either analytically or numerically. Numerical solution is an approximate solution of a PDE when it is impossible to solve it analytically.

In reality, a substantial portion of the physical problems exist in form of non-linear PDEs. In this thesis, Burgers' equation is picked because it is non-linear and admits finite number of continuous symmetries. The focus is to solve the Burgers' equation numerically with a novel approach which is known as Modified-Crank-Nicolson method (M-CNM). We have adopted procedure of transforming the Burgers' equation into diffusion heat equation by means of Hopf-Cole transformation and then approximating the diffusion heat equation. This approximation is carried out by using different finite difference schemes like FTCS, CNM and M-CNM. The numerical scheme M-CNM is obtained by modifying the CNM with the help of discrete symmetries of the Burgers' equation. To check the accuracy, the solutions obtained through these numerical schemes is transformed back to compare with Hopf-Cole transform analytical solution of the Burgers' equation.

1.1 Background of Burgers' Equation

Burgers' equation

$$u_{xx} + 2uu_x = u_t,$$

can be defined as the non-linear model of a Navier-Stokes equation (Rafiq et al., 2011). It is a parabolic equation with the inclusion of viscous term, ν, that is, νu_{xx}. However, for $\nu = 1$, the Burgers' equation turns into an elliptic equation. This equation includes three terms uu_x, u_t, u_x that are convective term, time-dependent term and diffusive term, with $\nu = 1$, respectively.

Burgers' equation was first established by Forsyth [2] in 1906. Yet, it was in 1915 that Bateman [3] derived the Burgers' equation from a physical context and given the steady. Following the discovery of Bateman, in 1940 Burgers presented a more unique solution and significance of the equation. In 1948, Burgers brought in the relationship of the equation in the theory of turbulence (Burgers, 1948). This was the time that the equation has been widely recognized as Burgers' equation due to the vast majority of work done by Burgers in some fields of mathematics (Kutluay et al., 1999). In 1949, Lagerstorm [4] noted a potential of transforming the Burgers' equation into linear heat equation. In 1950, after the establishment of a coordinate transformation Hopf [4] studied the Burgers' equation in the context of gas dynamics. One year later in 1951, based on the suitable initial and boundary conditions, Cole [5] formalized the hypothetical Fourier solution of the Burgers' equation. Another hypothetical solution depended on the test and trail with suitable conditions are obtained by Madsen and Sincovec [6]. Lighthill [7] and Blackstock [8] studied the Burgers' equation in the propagation of one-dimensional acoustic of limited amplitude in 1956 and 1964, respectively. In 1958, Hayes discussed the shock structures in the Navier-Stokes fluid. Without utilizing a few additional conditions Riccati solution was derived from Burgers' equation by Rodin in 1970 [9]. In 1972, Benton and Platzman [10] discovered the thirty-five different solutions in infinite domain for the Burgers' equation. However, in the same year Ames [11] found a way to determine the proper groups by applying the Morgan-Michal method to Burgers' equation. Simultaneously, from 1980-1990 numerous researcher have worked on the Burgers' equation and to exercise the Hopf-Cole transformation in acquiring the analytical solution but it was Shtelen [12] who was able to discover this transformation theoretically.

There has been extensive research in the last few decade aimed at the improvement of the robust computational schemes to deal with the non-linear PDEs found in heat transfer and fluid mechanics. The Burger equation is one of the most popular equation with non-linear propagation effects as well as diffusive effects. As a non-linear PDE, Burgers' equation describes numerous practical problems in engineering which are naturally difficult to solve. It additionally deals in different areas of mathematics. The standard Burgers' equation turns into inviscid Burgers' equation when ν tends to zero, thereby yields a model for non-linear wave propagation. Burgers' equation has been widely used in gas dynamics with its source terms emerged in the theory of aerodynamics. It has great significance in the study of standard problem for numerical methods. Many numerical schemes can be verified through it.

Burgers' equation is mainly used in the field of fluid dynamics and essentially as a model for acoustics, shock theory, cosmology, viscous flow, turbulence, traffic flow, quantum field, heat conduction, mass transport, boundary layer behavior, longitudinal elastic waves in isotropic solids and water wave dispersion. Due to its expansive scope of relevance, it has redirected consideration of a few researchers to its solution. Thus far, the Burgers equation for a small range of arbitrary initial and boundary conditions can be analytically resolved.

For many decades, numerical solution of PDEs has been relevant research subject both in thermal and fluid mechanics. The very first stage is to comprehend the mechanics of the problem, which leads with the help of equations to construct a mathematical model. Such equations in most situations are either ODEs or PDEs. A few suppositions must be made, on the grounds that the real-life problems in engineering are somewhat perplexing to examine. These calculations are then solved by computational methods including the method of finite volume, the method of finite difference and the method of finite element.

Different approaches for mathematical simulation have their own benefits and drawbacks. Finite difference method is the most basic and oldest way of resolving ODEs and PDEs through the discretization process. In order to solve the Burgers' equation by solving the diffusion heat equation explicitly, Bhattacharya [13] was the first to develop the exponential finite difference scheme. Similarly, with the help of uniform implicit difference method Kadalbajoo [14] was the first to solve the time dependent Burgers' equation. Varoglu and Finn [15] presented the numerical solution of Burgers' equation by using finite element method. The transformed Burgers' equation to heat equation by using the Hopf-Cole transformation, and then solving the heat equation with insulated boundary conditions by using explicit and exact-explicit numerical schemes was presented by Kutluay et al., [16]. Based on the least square approach, Nguyen and

Rynen [17] discovered the linear space-time element method. Wani and Thakar discussed a scheme based on Crank-Nicolson method in [18]. A new technique for solving the Burgers' equation by using the method of lines (MOL) and matrix-free modified extended backward difference formula was proposed by Javidi [19]. Cubic spline functions in two spaced variables was used by Jain and Holla [20] in 1978. Malek and Mansi [21] presented the group theoretic approach to solve the Burgers' equation by applying the one-parameter group of transformation to Burgers' equation with suitable initial and boundary conditions. Simultaneously, there are numerous other researchers who contributed to solve the Burgers' equation numerically.

1.2 Symmetry

The difference of linearity holds a special place in differential equations especially in PDEs. Linear PDEs can be solved easily through the numerous methods discussed in the literature like separation of variables, superposition principle, Laplace transform, Fourier transform etc. However, non-linear PDEs are not that easy to be solved analytically. Most of these PDEs appear in the engineering and science, which is why non-linear PDEs are typically much more complicated than linear ones to grasp. Almost every single equation must be analyzed as a single problem. It is a well known fact that methods of symmetry are of great significance when testing differential equations [22]. In recent trends, a symmetry approach is considered to be one of the best methods to solve PDEs.

The solutions of differential equations are based several innovative methods. However, it is to know that most of these methods are drawn from a unified theory of continuous differential equation symmetries. The theory of symmetry methods was first established by a Norwegian mathematician Marius Sophus Lie [23]. Inspired by Galois' theory, he has done most of his work in the field of continuous symmetries, which he used in the study of differential equations and geometry. The algebraic equations like quadratic, cubic and quartic were solved by Evariste Galois in 19^{th} century by using the theoretical approach of groups. In doing so he unified three major branches of mathematics namely Algebra, Analysis and Geometry. On the basis of comparison, Lie introduced his notion, that is, the infinite groups, groups consistently relying on at least one real or complex variable, would most likely be responsible in the treatment of ODEs and PDEs analogous to finite groups requirement of deciding the solvability of finite-degree polynomial equations [22, 24, 25, 26].

The groups that Lie tried were the continuous groups, the differential equation symmetries, which were consistently based on single or multiple real or complex variables. These symmetry groups were later called Lie groups, in which the investigation of symmetries was based on some conditions. Symmetries can be categorized mainly into two types. One is the continuous or Lie point symmetries and the other one is discrete symmetries. Discrete symmetries are defined as the non-continuous symmetries or in other words, those symmetries which lies outside of Lie groups. Some significant applications of discrete symmetries of differential conditions are talked about in [27, 28, 29, 30].

Numerous methods have been established for finding discrete symmetries of a differential equation but the method proposed by Peter E. Hydon is perfect for identifying the discrete symmetries of a differential equation having a finite dimensional Lie algebra of infinitesimal generators of its Lie group of point symmetries [27, 28, 29, 31, 32, 33]. His approach is based on the idea that any point symmetry produces an automorphism for the Lie algebra of the Lie point symmetry generators.

In Chapter 2, we give some basic notions, definitions, theorems, and techniques necessary for finding the continuous or Lie point symmetries of a differential equation. Chapter 3 contains the detailed discussion of discrete symmetries of a differential equation including the Peter E. Hydon's technique for finding the discrete symmetries. In Chapter 4, the comprehensive symmetry analysis of Burgers' equation is carried out. As an immediate application of the discrete symmetries of a Burgers' equation, Chapter 5 includes the original work of the construction of numerical schemes to approximate the exact solution of Burgers' equation. The Hopf-Cole transformation of Burger' equation to diffusion heat equation and the exact solution of the Burgers' equation is also discussed. In Chapter 6, the detailed stability analysis of the newly constructed numerical schemes and the explicit study of solving the Burgers' equation by using these new numerical schemes along with FTCS and CNM is also presented in the form of tables and figures followed by a brief conclusion.

Chapter 2

Lie Point Symmetries of Differential Equations

The motivation behind this chapter is to compactly study some essential notions comprehended with Lie point symmetries of differential equations. Basic definitions and notations are introduced. All the theorems are laid out without proof. A permeable on certain standards which are useful while finding the Lie point symmetries for differential equations is also introduced in this chapter. For details, adequate references are given.

2.1 One-Parameter Lie Group of Point Transformation

For the simplification of an ordinary differential equation by using the suitable change of variables, a point transformation can be define as the transformation of independent and dependent variables, that is x and u respectively maps points (x, u) into points $(\hat{x}, \hat{u})$ [34],

$$\hat{x} = \hat{x}\,(x, u)\,, \qquad \hat{u} = \hat{u}\,(x, u)\,, \tag{2.1}$$

where $\hat{x}$ and $\hat{u}$ are continuous functions. Moreover, in case of symmetry transformation, a point transformation must depend on at least one continuous parameter say ϵ, that is

$$\hat{x} = \hat{x}\,(x, u, \epsilon)\,, \qquad \hat{u} = \hat{u}\,(x, u, \epsilon)\,, \tag{2.2}$$

where $\hat{x}$ and $\hat{u}$ are infinitely differentiable with respect to (x and u).

This section presents the basic definitions required for one-parameter Lie group of point transformations [22].

Definition 2.1.1. *A group $\mathcal{G}$ is said to be r-parameter Lie group, if the group operations*

$$f:\mathcal{G}\times\mathcal{G}\longrightarrow\mathcal{G}\quad f(l,k)=l.k,\quad l,k\in\mathcal{G},$$

and

$$\hat{f}:\mathcal{G}\longrightarrow\mathcal{G}\quad \hat{f}\,(l)=l^{-1},\quad l\in\mathcal{G},$$

acting upon the r-dimensional C^{∞}-manifold are smooth maps between the manifolds.

Definition 2.1.2. *Let M be a C^{∞}-manifold. Then an r-parameter Lie group $\mathcal{G}$ is said to be Lie group of transformation, if there is a smooth map*

$$\psi\colon\mathcal{G}\times M\longrightarrow M,\qquad \psi\,(l,m)=lm,$$

satisfying the following two properties

- $(l_1.l_2)\,m=l_1\,(l_2m)\quad\forall\ l_1,l_2\in\mathcal{G}$ *and* $m\in M$.
- *Let I be the identity element of $\mathcal{G}$ then $Im=m\quad\forall\ m\in M$.*

Now if

$$\hat{\mathbf{b}}=\boldsymbol{\psi}\,(\mathbf{b},\epsilon)\,,\tag{2.3}$$

and

$$\hat{\hat{\mathbf{b}}}=\boldsymbol{\psi}\,(\boldsymbol{\psi}\,(\mathbf{b},\epsilon)\,,\sigma)=\boldsymbol{\psi}\left(\hat{\mathbf{b}},\sigma\right),\tag{2.4}$$

where $\mathbf{b}=(b_1,b_2,...,b_n)$, $\hat{\mathbf{b}}=\left(\hat{b_1},\hat{b_2},...,\hat{b_n}\right)$, $\boldsymbol{\psi}=(\psi_1,\psi_2,...,\psi_n)$ and $\phi(\epsilon,\sigma)$ be the law of composition of parameters $\epsilon,\sigma\in\mathbf{V}$, forms a **one-parameter group of transformations** in the region $\mathcal{D}$ if the following properties hold [35],

- $\mathbf{V}$ forms a group with law of composition ϕ.

- For $\epsilon = \epsilon_0$ corresponding to an identity element, we have $\hat{\mathbf{b}} = \mathbf{b}$ for each $\mathbf{b}$ in the region $\mathcal{D}$.
- For $\hat{\mathbf{b}} \in \mathcal{D}$, the tansformation must be injective in $\mathcal{D}$ for each $\epsilon \in \mathbf{V}$.
- From Eqs. (2.3) and (2.4), we have

$$\hat{\hat{\mathbf{b}}} = \boldsymbol{\psi}\left(\mathbf{b}, \phi\left(\epsilon, \sigma\right)\right), \tag{2.5}$$

  where $\hat{\mathbf{b}}, \hat{\hat{\mathbf{b}}} \in \mathcal{D}$.

Definition 2.1.3. *Let $\mathcal{G}$ be a Lie group and M be the C^∞-manifold with $\phi(\epsilon, \sigma)$ is a composition function. Then a transformation Lie group is said to be one-parameter Lie group of transformation if it satisfies the following conditions*

- *For $\epsilon = 0$ and $-\epsilon$ corresponds to identity and inverse transformation group respectively as ϵ is a continuous parameter with $\epsilon \in \mathbf{V} \subset \mathbb{R}$.*
- *Let $\mathbf{x}$ and $\mathbf{u}$ be any points in the region $\mathcal{D} \subset \mathbb{R}$, then $\hat{\mathbf{x}}$ and $\hat{\mathbf{u}}$ are continuously differentiable w.r.t $\mathbf{x}$ & $\mathbf{u}$ and are analytic in $\epsilon \subset \mathbf{V}$.*
- *The composition function $\phi(\epsilon, \sigma)$ is an analytic function in ϵ and σ, where $\epsilon, \sigma \in \mathbf{V}$.*

2.2 Infinitesimal Transformation and Their Generators

Now we define the infinitesimal transformations and their corresponding generators. Let us consider Eq. (2.3)

$$\hat{\mathbf{b}} = \boldsymbol{\psi}\left(\mathbf{b}, \epsilon\right),$$

then by Taylor expansion at $\epsilon = 0$, we have

$$\hat{\mathbf{b}} = \mathbf{b} + \epsilon \frac{\partial}{\partial \epsilon} \boldsymbol{\psi}\left(\mathbf{b}, \epsilon\right)\Big|_{\epsilon=0} + \frac{\epsilon^2}{2} \frac{\partial^2}{\partial \epsilon^2} \boldsymbol{\psi}\left(\mathbf{b}, \epsilon\right)\Big|_{\epsilon=0} + \mathcal{O}\left(\epsilon^3\right).$$

Consider

$$\frac{\partial}{\partial \epsilon}\boldsymbol{\psi}\left(\mathbf{b},\epsilon\right)\bigg|_{\epsilon=0}=\xi\left(\mathbf{b}\right), \tag{2.6}$$

then the infinitesimal transformation of Lie group is given by

$$\hat{\mathbf{b}}=\mathbf{b}+\epsilon\xi\left(\mathbf{b}\right). \tag{2.7}$$

Equation (2.6) is used in the following **Lie's first fundamental theorem** which provides a technique to re-parametrize a one-parameter group of transformation that is of definitive form.

Theorem 2.2.1. *For Lie group of transformation* (2.3) *to be equivalent to the solution of an initial value problem for the autonomous system of first order ordinary differential equations there exists a parametrization* $\tau(\epsilon)$ *stated by*

$$\frac{\partial\hat{\mathbf{b}}}{\partial\tau}=\xi\left(\mathbf{b}\right), \tag{2.8}$$

with condition $\hat{\mathbf{b}}=\mathbf{b}$ *at* $\tau=0$ [26]. *Particularly,*

$$\tau\left(\epsilon\right)=\int_0^{\epsilon}\lambda\left(\epsilon^{'}\right)d\epsilon^{'}, \tag{2.9}$$

where

$$\lambda\left(\epsilon\right)=\frac{\partial}{\partial h}\phi\left(g,h\right)\bigg|_{(g,h)=\left(\epsilon,\epsilon^{'}\right)},\qquad\lambda(0)=1. \tag{2.10}$$

Now, in the following definition a representation of one-parameter Lie group of transformation will be incorporated in the form of a group generator [26, 22].

Definition 2.2.1. *The infinitesimal generator for one-parameter Lie group of transformation can be defined by the linear differential operator*

$$\mathbf{X}=\boldsymbol{\xi}\left(\mathbf{b}\right).\nabla=\sum_{j=1}^{n}\xi_j\left(\mathbf{b}\right)\frac{\partial}{\partial b^j}, \tag{2.11}$$

where $\boldsymbol{\xi}\left(\mathbf{b}\right)=(\xi_1(\mathbf{b}),\xi_2(\mathbf{b}),...,\xi_n(\mathbf{b}))$ *and* ∇ *is the gradient operator.*

For any differential equation

$$W(x) = W\left(x_1, x_2, ..., x_n\right), \tag{2.12}$$

we can write

$$\mathbf{X}W(x) = \boldsymbol{\xi}(\mathbf{b}).\nabla W(x) = \sum_{j=1}^{n} \xi_j(\mathbf{b}).\frac{\partial W(x)}{\partial b^j}. \tag{2.13}$$

Theorem 2.2.2. *Let* $\mathbf{X}$ *be the linear operator defined by Eq.* (2.13) *and consider Eq.* (2.3) *given by*

$$\hat{\mathbf{b}} = \boldsymbol{\psi}\left(\mathbf{b}, \epsilon\right),$$

then the corresponding generators for the one-parameter Lie group of transformation are

$$\begin{aligned}\hat{\mathbf{b}} = \boldsymbol{\psi}\left(\mathbf{b}, \epsilon\right) &= e^{\epsilon \mathbf{X}}\mathbf{b} = \mathbf{b} + \epsilon \mathbf{X}\mathbf{b} + \frac{\epsilon^2}{2}\mathbf{X}^2\mathbf{b} + \mathcal{O}\left(\epsilon^3\right), \\ &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\mathbf{X}^n\mathbf{b},\end{aligned}$$

and $\mathbf{X}^n = \mathbf{X}\mathbf{X}^{n-1}$ [22, 26].

Moreover, for a one-parameter Lie group of transformation Eq. (2.3) and corresponding infinitesimal generator Eq. (2.13), the generalization [26, 22] of Theorem 2.2.2 for any analytic function $\mathcal{F}$ is given by

$$\mathcal{F}\left(\hat{\mathbf{b}}\right) = \mathcal{F}\left(e^{\epsilon \mathbf{X}}\mathbf{b}\right) = e^{\epsilon \mathbf{X}}\mathcal{F}\left(\mathbf{b}\right).$$

2.3 Prolongation of Lie Group of Point Transformation and Their Generators

The definition of Lie's first fundamental theorem Eq. (2.6) corresponding to one dependent and one independent variable in an ordinary differential equation can be written as

$$\xi(x, u) = \frac{\partial \hat{x}}{\partial \epsilon}\left(x, u, \epsilon\right)\bigg|_{\epsilon=0}, \qquad \eta(x, u) = \frac{\partial \hat{u}}{\partial \epsilon}\left(x, u, \epsilon\right)\bigg|_{\epsilon=0}, \tag{2.14}$$

respectively. Now if we want to apply Eq. (2.2) to an ordinary differential equation [34],

$$W\left(x, u, u', u'', ..., u^{(n)}\right) = 0, \tag{2.15}$$

then first we have to extend the point transformation up to m^{th} order derivative of $u^{(n)}$, $n = 1, 2, ..., m$. Then by recursive relation we have

$$\hat{u}^{(n)} \equiv \frac{D_x \hat{u}^{n-1}}{D_x \hat{x}}, \tag{2.16}$$

with $\hat{u}^{(0)} \equiv \hat{u}$ and D_x is the total derivative w.r.t x given by

$$D_x = \frac{\partial}{\partial x} + u'\frac{\partial}{\partial u} + u''\frac{\partial}{\partial u'} + \cdots .$$

Consequently, we can write

$$\hat{x} = x + \epsilon\xi\left(x, u\right) + \cdots = x + \epsilon \mathbf{X} x + \cdots , \tag{2.17}$$

$$\hat{u} = u + \epsilon\eta(x, u) + \cdots = u + \epsilon \mathbf{X} u + \cdots , \tag{2.18}$$

$$\hat{u'} = u' + \epsilon\eta'(x, u) + \cdots = u' + \epsilon \mathbf{X} u' + \cdots , \tag{2.19}$$

$$\vdots$$

$$\hat{u}^{(m)} = u^{(m)} + \epsilon\eta^{(m)}(x, u) + \cdots = u^{(m)} + \epsilon \mathbf{X} u^{(m)} + \cdots , \tag{2.20}$$

where η, η', η'',$\cdots$, $\eta^{(m)}$ are defined by

$$\eta = \frac{d\hat{u}}{d\epsilon}, \qquad \eta' = \frac{d\hat{u'}}{d\epsilon}, \qquad \eta'' = \frac{d\hat{u''}}{d\epsilon}, \cdots, \eta^{(m)} = \frac{d\hat{u}^m}{d\epsilon}, \quad \text{at} \ \ \epsilon = 0. \tag{2.21}$$

Now, by comparing Eqs. (2.16) and (2.20) implies

$$\hat{u}^{(m)} = u^{(m)} + \epsilon\left(D_x \eta^{m-1} - u^{(m)} D_x \xi\right), \tag{2.22}$$

with $\eta^{(0)} \equiv \eta$.

Moreover, the values of η, η', η'',$\cdots$, $\eta^{(m)}$ can be computed by

$$\eta^{(m)} = D_x \eta^{m-1} - u^{(m)} D_x \xi. \tag{2.23}$$

Similarly, Eqs. (2.17)-(2.20) yields the following prolongation of generator $\mathbf{X}$

$$\mathbf{X}^{(m)} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \eta'\frac{\partial}{\partial u'} + \cdots + \eta^{(m)}\frac{\partial}{\partial u^{(m)}}. \tag{2.24}$$

2.4 Multi-Parameter Lie Group of Point Transformation and Their Infinitesimal Generators

This section deals with the generalization of one-parameter Lie group of point transformations to r-parameter Lie group of point transformations [26, 22]. Let us consider the transformation

$$\hat{\mathbf{b}} = \boldsymbol{\psi}\left(\mathbf{b}, \boldsymbol{\epsilon}\right),$$

where $\hat{\mathbf{b}} = \left(\hat{b_1}, \hat{b_2}, \cdots, \hat{b_n}\right)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ belong to the region $\mathcal{D} \subset \mathbb{R}^{\mathbf{n}}$ with $\boldsymbol{\psi} = (\psi_1, \psi_2, \cdots, \psi_n)$ depending on more than one-parameter say r-parameters ϵ_N, that is $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_r) \in \mathbf{V} \subset \mathbb{R}^{\mathbf{n}}$ satisfying all the properties of a group. The group operation is given by $\boldsymbol{\phi}\left(\boldsymbol{\epsilon}, \boldsymbol{\sigma}\right)$. Then the r-parameter Lie group of transformation is given by

$$\hat{\mathbf{b}} = \boldsymbol{\psi}\left(\mathbf{b}, \boldsymbol{\epsilon}\right) = \prod_{N=1}^{r} \exp\left(\epsilon_N \mathbf{X}_N\right) \mathbf{b}. \tag{2.25}$$

Moreover, the corresponding general infinitesimal transformation [34] for one dependent and one independent variable Eq. (2.3) can be written in the form

$$\mathbf{X}_N = \xi_N(x, u)\frac{\partial}{\partial x} + \eta_N(x, u)\frac{\partial}{\partial u}, \tag{2.26}$$

with

$$\xi_N(x, u) = \left.\frac{\partial \hat{x}}{\partial \epsilon_N}\right|_{\epsilon=0}, \quad \text{and} \quad \eta_N(x, u) = \left.\frac{\partial \hat{u}}{\partial \epsilon_N}\right|_{\epsilon=0}. \tag{2.27}$$

In case of r-parameter group, the vector $\boldsymbol{\xi}\left(\mathbf{b}\right)$ takes the form of a matrix $\xi_{Nj}(\mathbf{x})$, where $\epsilon = 1, 2, \cdots, r$ and $j = 1, 2, \cdots, n$. Then, the associated generator $\mathbf{X}_N$, corresponding to the parameter ϵ_N of the r-parameter Lie group of transformation is defined as

$$\mathbf{X}_N = \sum_{j=1}^{n} \xi_{Nj}(\mathbf{b})\frac{\partial}{\partial b^j}, \quad N = 1, 2, \cdots, r. \tag{2.28}$$

2.5 Lie Algebra of Infinitesimal Generators

We start this section with the definition of an algebraic structure Lie algebra [22].

Definition 2.5.1. *Let $\mathcal{L}$ be the vector space over a field F on which a commutator product $[\,,\,]$ is defined. Then $\mathcal{L}$ is said to be Lie algebra if it satisfies the following properties*

- $[\mathbf{X}_p, \mathbf{X}_q] \in \mathcal{L}$, $\forall\ \mathbf{X}_p, \mathbf{X}_q \in \mathcal{L}$.
- $[\mathbf{X}_p, \mathbf{X}_q] = -[\mathbf{X}_q, \mathbf{X}_p]$, $\forall\ \mathbf{X}_p, \mathbf{X}_q \in \mathcal{L}$.
- $[\mathbf{X}_p, a\mathbf{X}_q + b\mathbf{X}_s] = [\mathbf{X}_p, a\mathbf{X}_q] + [\mathbf{X}_p, b\mathbf{X}_s]$, $\forall\ \mathbf{X}_p, \mathbf{X}_q, \mathbf{X}_s \in \mathcal{L}$ *and for all* $a,b \in F$.
- $[\mathbf{X}_p, [\mathbf{X}_q, \mathbf{X}_s]] + [\mathbf{X}_s, [\mathbf{X}_p, \mathbf{X}_q]] + [\mathbf{X}_q, [\mathbf{X}_s, \mathbf{X}_p]] = 0$, $\forall\ \mathbf{X}_p, \mathbf{X}_q, \mathbf{X}_s \in \mathcal{L}$.

Consequently, from second property it follows that $[\mathbf{X}_p, \mathbf{X}_p] = 0$, which yields the following definition of abelian Lie algebra [34].

Definition 2.5.2. *A Lie algebra $\mathcal{L}$ is said to be abelian if and only if for all $\mathbf{X}_p, \mathbf{X}_q \in \mathcal{L}$, we have*

$$[\mathbf{X}_p, \mathbf{X}_q] = 0.$$

The commutators of two generators $\mathbf{X}_p$ and $\mathbf{X}_q$ is defined by

$$[\mathbf{X}_p, \mathbf{X}_q] = \mathbf{X}_p\mathbf{X}_q - \mathbf{X}_q\mathbf{X}_p. \tag{2.29}$$

Since, Eq. (2.29) satisfies all the properties of a Lie algebra. Therefore, the set of all $\{\mathbf{X}_p\}$, together with the commutator form the Lie algebra under the group. The following two theorems demonstrates the structure of a Lie algebra as a linear combination of r basic generators also known as **Lie's second and third fundamental theorem** [26] respectively.

Theorem 2.5.1. *Let $\mathbf{X}_p$ and $\mathbf{X}_q$ be any two infinitesimal generators of an r-parameter Lie group of point transformation. Then the commutator $[\mathbf{X}_p, \mathbf{X}_q]$ is again an infinitesimal generator*

$$[\mathbf{X}_p, \mathbf{X}_q] = C^k_{pq}\mathbf{X}_k, \tag{2.30}$$

where the coefficients C^k_{pq}, $p,q=1,2,\cdots,r$ are called structure constants.

Theorem 2.5.2. *Consider the structure constants in Eq.* (2.30)*, then the following two properties hold.*

- *The structure constants are antisymmetric in the lower two indices.*

$$C^k_{pq} = -C^k_{qp}.$$

- *Structure constants must satisfy Lie's identity, that is*

$$C^a_{pq}C^d_{ab} + C^a_{qb}C^d_{ap} + C^a_{bp}C^d_{aq} = 0.$$

2.6 Symmetry Condition for an Ordinary Differential Equations

Since, we have defined all the basic mathematical theory. Now we are able to state an essential theorem for finding Lie point symmetries of a differential equation.

Theorem 2.6.1. *An ordinary differential equation*

$$W\left(x, u, u', u'', \cdots, u^{(n)}\right) = 0,$$

admits a group of symmetries with generator $\mathbf{X}$ *if and only if*

$$\mathbf{X}^{(n)}W\big|_{W=0} = 0, \tag{2.31}$$

holds [34]*.*

2.7 Lie Point Symmetries of a Partial Differential Equations

Consider the system of k^{th} order non-linear partial differential equations in P-independent and Q-dependent variables as

$$W_m\left(\mathbf{x}, u, u^{(1)}, u^{(2)}, \cdots, u^{(k)}\right) = 0, \qquad m = 1, 2, 3, \cdots, l, \tag{2.32}$$

where $\mathbf{x} = \left(x^1, x^2, \cdots, x^P\right) \in \mathbf{X} \subset \mathbb{R}^{\mathbf{P}}$ and $\mathbf{u} = \left(u^1, u^2, \cdots, u^Q\right) \in \mathbf{U} \subset \mathbb{R}^{\mathbf{Q}}$ are the corresponding P-independent and Q-dependent variables [34]. Moreover, u^n denotes all the n^{th} order partial derivatives of u w.r.t. x with the corresponding coordinate for $u^{(n)}$ is $\frac{\partial^n u}{(\partial x^{p_1} \partial x^{p_2} \cdots \partial x^{p_n})}$ given by $u^n_{p_1 p_2 \cdots p_j}$, $p = 1, 2, 3, \cdots, P$ for $n = 1, 2, 3, \cdots, k$. For the coordinates $\mathbf{x}, u^1, u^2, \cdots, u^k$, Eq. (2.32) takes the form of an algebraic equation which is a hypersurface in $\left(\mathbf{x}, u, u^1, u^2, \cdots, u^k\right)$-space. Now the point transformation Eq. (2.1) for independent and dependent variables $\hat{x}^p$, $p = 1, 2, 3, \cdots, P$ and $\hat{u}^q$, $q = 1, 2, 3, \cdots, Q$ of the k^{th} order system of partial differential equations [34] is

$$\hat{x}^p = \hat{x}^p\left(x^a, u^b\right), \qquad \hat{u}^q = \hat{u}^q\left(x^a, u^b\right), \tag{2.33}$$

where $a, p = 1, 2, 3, \cdots, P$, and $b, q = 1, 2, 3, \cdots, Q$. Likewise, for any particular parameter say $\epsilon \in \mathbf{V} \subset \mathbb{R}$, Eq. (2.33) takes the form

$$\hat{x}^p = \hat{x}^p\left(x^a, u^b; \epsilon\right), \qquad \hat{u}^q = \hat{u}^q\left(x^a, u^b; \epsilon\right). \tag{2.34}$$

Then the infinitesimal generator of the one-parameter Lie group of point transformations is given by

$$\mathbf{X} = \xi^p\left(x^a, u^b\right)\frac{\partial}{\partial x^p} + \eta^q\left(x^a, u^b\right)\frac{\partial}{\partial u^q}, \tag{2.35}$$

with the corresponding infinitesimal transformation

$$\xi^p \equiv \left.\frac{\partial \hat{x}^p}{\partial \epsilon}\right|_{\epsilon=0}, \qquad \eta^q \equiv \left.\frac{\partial \hat{u}^q}{\partial \epsilon}\right|_{\epsilon=0}. \tag{2.36}$$

Moreover, the extension of an infinitesimal generator Eq. (4.35) for an arbitrary order derivatives [34] is given by

$$\mathbf{X} = \xi^p \frac{\partial}{\partial x^p} + \eta^q \frac{\partial}{\partial u^q} + \eta^q_p \frac{\partial}{\partial u^q_p} + \eta^q_{pr} \frac{\partial}{\partial u^q_{pr}} + \eta^q_{prs} \frac{\partial}{\partial u^q_{prs}} + \cdots, \tag{2.37}$$

where

$$\eta^q_p = \frac{D\eta^q}{Dx^p} - u^q_a \frac{D\xi^a}{Dx^p}, \tag{2.38}$$

$$\eta^q_{pr} = \frac{D\eta^q_p}{Dx^r} - u^q_{pa} \frac{D\xi^a}{Dx^r}, \tag{2.39}$$

with the total derivative $\frac{D}{Dx^p}$ can be define as

$$\frac{D}{Dx^p} = \frac{\partial}{\partial x^p} + u^q_p \frac{\partial}{\partial u^q} + u^q_{pr} \frac{\partial}{\partial u^q_r} + \cdots. \tag{2.40}$$

The following theorem is the symmetry condition for a partial differential equation [34].

Theorem 2.7.1. *Let*

$$\mathbf{X}^{(k)} = \xi^p(x,u)\frac{\partial}{\partial x^p} + \eta(x,u)\frac{\partial}{\partial u} + \eta_p^{(1)}\left(x,u,u^{(1)}\right)\frac{\partial}{\partial u^1} + \cdots$$
$$\cdots + \eta^{(k)}_{p_1,p_2,\cdots,p_j}\left(x,u,u^{(1)},u^{(2)},\cdots,u^{(k)}\right)\frac{\partial}{\partial u^{p_1,p_2,\cdots,p_n}}, \tag{2.41}$$

be the k^{th} order prolonged infinitesimal generator Eq. (4.35) *of the corresponding one-parameter Lie group of transformation*

$$\hat{x} = X\left(x,u;\epsilon\right), \tag{2.42}$$
$$\hat{u} = U\left(x,u;\epsilon\right), \tag{2.43}$$

with

$$\eta^1_p = D_p\eta - \left(D_p\xi_p\right)u_n, \qquad p = 1,2,3,\cdots,P, \tag{2.44}$$
$$\eta^n_{p_1,p_2,\cdots,p_k} = D_{p_k}\eta^{(k-1)}_{p_1,p_2,\cdots,p_{k-1}} - \left(D_{p_k}\xi_n\right)u_{p_1,p_2,\cdots,p_{(k-1)}n}, \tag{2.45}$$

where $p_n = 1,2,3,\cdots,P$ for $n = 1,2,3,\cdots,k$ with $k = 1,2,3,\cdots$. Then a partial differential Eq. (2.32) *admits one-parameter Lie group of transformations Eqs.* (2.41)-(2.43) *if and only if*

$$\mathbf{X}^{(k)}W\left(x,u,u^{(1)},u^{(2)},\cdots,u^{(k)}\right)\big|_{W=0} = 0, \tag{2.46}$$

holds.

Particularly, for two independent variables (x,t) and one dependent variable u, Eq. (2.41) with $k=2$ can be written as

$$\mathbf{X}^{(2)} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} + \eta_x\left(x,t,u,u_x\right)\frac{\partial}{\partial u_x}$$
$$+ \eta_t\left(x,t,u,u_x,u_t\right)\frac{\partial}{\partial u_t} + \eta_{xx}\left(x,t,u,u_x,u_t,u_{xx}\right)\frac{\partial}{\partial u_{xx}}$$
$$+ \eta_{xt}\left(x,t,u,u_x,u_t,u_{xx},u_{xt}\right)\frac{\partial}{\partial u_{xt}} + \eta_{tt}\left(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}\right)\frac{\partial}{\partial u_{tt}},$$

where Eqs. (2.44) and (2.45) is given by

$$\begin{aligned}\eta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\ &= \eta_x + (\eta_u - \xi_x) u_x - \tau_x u_t - \xi_u (u_x)^2 - \tau_u u_t u_x,\end{aligned} \tag{2.47}$$

$$\begin{aligned}\eta_t &= D_t(\eta) - u_t D_t(\tau) - u_t D_t(\xi), \\ &= \eta_t + (\eta_u - \tau_t) u_t - \xi_t u_x - \tau_u (u_t)^2 - \xi_u u_t u_x,\end{aligned} \tag{2.48}$$

$$\begin{aligned}\eta_{xx} &= D_x(\eta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi), \\ &= \eta_{xx} + (2\eta_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t + (\eta_u - 2\xi_x) u_{xx} - 2\tau_x u_{tx} + (\eta_{uu} - 2\xi_{xu}) (u_x)^2 \\ &\quad - 2\tau_{xu} u_t u_x - \xi_{uu} (u_x)^3 - \tau_{uu} u_t (u_x)^2 - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{tx},\end{aligned} \tag{2.49}$$

$$\begin{aligned}\eta_{xt} &= D_x(\eta_t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\ &= \eta_{tx} + (2\eta_{tu} - \xi_{tx}) u_x + (\eta_{ux} - \tau_{tx}) u_t + (\eta_{uu} - \tau_{tu} - \xi_{ux}) u_t u_x + (\eta_u - \tau_t - \xi_x) u_{tx} \\ &\quad - \tau_{ux} (u_t)^2 - \tau_{uu} u_x (u_t)^2 - \xi_u u_t u_{xx} - \xi_{tu} (u_x)^2 - \xi_t u_{xx} - \tau_x u_{tt} - \xi_{uu} u_t (u_x)^2 \\ &\quad - 2\xi_u u_x u_{tx} - \tau_u u_x u_{xt} - 2\tau_u u_t u_{xt},\end{aligned} \tag{2.50}$$

$$\begin{aligned}\eta_{xx} &= D_t(\eta_t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\ &= \eta_{tt} + (2\eta_{tu} - \tau_{tt}) u_t + (\eta_{uu} - 2\tau_{tu}) (u_t)^2 - \tau_{uu} (u_t)^3 - 3\tau_u u_t u_{tt} - \xi_{tt} u_x - 2\xi_{ut} u_t u_x \\ &\quad - 2\xi_t u_{tx} - \xi_{uu} u_x (u_t)^2 - \xi_u u_x u_{tt} - 2\xi_u u_t u_{xt}.\end{aligned} \tag{2.51}$$

Generally, for one independent and one dependent variables x and u the symmetry condition of Eq. (2.46) gives a non-linear partial differential equation in terms of $(\xi(x,u), \eta(x,u))$, which by comparing the coefficients of powers of derivatives of u generates a system of partial differential equations. Corresponding to each infinitesimal generator a solution of the system can be obtained in terms of ξ and η forming a Lie algebra.

If the obtained symmetries are actual symmetries of a partial differential Eq. (2.32), then it will leave the differential equation invariant under the generated point transformations through the obtained point symmetries [34].

Now we give an example to understand the procedure.

Example 2.7.1. *Consider the Thomas equation*

$$u_{xt} + au_x + bu_t + cu_x u_t = 0, \tag{2.52}$$

where a, b and c are constants such that $a, b > 0$ and $c \neq 0$.

As Eq. (2.52) is of second order, we need to apply the second order prolongation of the infinitesimal generators for partial differential equations, that is

$$\mathbf{X}^{(2)} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \eta_x\frac{\partial}{\partial u_x} + \eta_t\frac{\partial}{\partial u_t} + \eta_{xx}\frac{\partial}{\partial u_{xx}} + \eta_{xt}\frac{\partial}{\partial u_{xt}} + \eta_{tt}\frac{\partial}{\partial u_{tt}},$$

with the coefficients given in Eqs. (2.47)-(2.52). Now in order to apply the Lie point symmetry condition for partial differential equations, let us consider

$$W = u_{xt} + au_x + bu_t + cu_x u_t, \tag{2.53}$$

then by Theorem (2.7.1), we have

$$\mathbf{X}^{(2)}W\big|_{W=0} = 0. \tag{2.54}$$

Substituting the values of $\mathbf{X}^{(2)}$ and W yields

$$\eta_{xt} + a\eta_x + b\eta_t + c\left(\eta_x u_t + \eta_t u_x\right) = 0. \tag{2.55}$$

Using Eqs. (2.47)-(2.49) in Eq. (2.55) and re-arranging the equation with respect to dependence among the derivatives of the equation. Collecting the coefficients of the various monomials in the first and second order derivatives, we get the following system of equations

$$u_x u_{tt} : \quad -\tau_u = 0, \tag{2.56}$$

$$u_t u_{xx} : \quad -\xi_u = 0, \tag{2.57}$$

$$u_{tt} : \quad -\tau_x = 0, \tag{2.58}$$

$$u_{xx} : \quad -\xi_t = 0, \tag{2.59}$$

$$u_x (u_t)^2 : \quad -\tau_{uu} = 0, \tag{2.60}$$

$$(u_x)^2 u_t : \quad -\xi_{uu} = 0, \tag{2.61}$$

$$(u_t)^2 : \quad -\tau_{xu} - c\tau_x + b\tau_u = 0, \tag{2.62}$$

$$(u_x)^2 : \quad -\xi_{tu} - c\xi_t + a\xi_u = 0, \tag{2.63}$$

$$u_x u_t : \quad \eta_{uu} - \xi_{ux} - \tau_{tu} + b\xi_u + a\tau_u + c\eta_u = 0, \tag{2.64}$$

$$u_t : \quad \eta_{xu} - \tau_{xt} + c\eta_x - a\tau_x + b\xi_x = 0, \tag{2.65}$$

$$u_x : \quad \eta_{tu} - \xi_{xt} + c\eta_t - b\xi_t + a\tau_t = 0, \tag{2.66}$$

$$\text{constant} : \quad a\eta_x + b\eta_t + \eta_{xt} = 0. \tag{2.67}$$

From Eqs. (2.56) and (2.58), we obtain

$$\tau = g(t). \tag{2.68}$$

Similarly, Eqs. (2.57) and (4.59) yields

$$\xi = f(x). \tag{2.69}$$

Now from Eq. (2.64) we deduce that

$$\eta = H\,(x,t)\,e^{-cu} + K(x,t). \tag{2.70}$$

Let for our convenience substituting $-\frac{H(x,t)}{c}$ instead of $H(x,t)$, then Eq. (2.70) takes the form

$$\eta = -\frac{H}{c}e^{-cu} + K. \tag{2.71}$$

Using Eqs. (2.68)-(2.71) in Eqs. (2.65) and (4.66), we have

$$cK_x + b\xi_x = 0, \tag{2.72}$$

$$cK_t + a\tau_t = 0. \tag{2.73}$$

From Eqs. (2.72) and (2.73), we get that

$$\xi_x = -\frac{cK_x}{b}, \tag{2.74}$$

$$\tau_t = -\frac{cK_t}{a}, \tag{2.75}$$

and

$$K_{xt} = 0. \tag{2.76}$$

Consequently, we have

$$K = \lambda_1 x + \lambda_2 t + d_1, \tag{2.77}$$

where λ_1, λ_2 and d_1 are the arbitrary constants. Upon the substitution of Eq. (2.77) in Eq. (2.71), we have

$$\eta = -\frac{H}{c}e^{-cu} + \lambda_1 x + \lambda_2 t + d_1 \tag{2.78}$$

Using Eq. (2.78) in Eq. (2.67), we obtain

$$a\lambda_1 + b\lambda_2 = 0. \tag{2.79}$$

So, for $\delta = \frac{\lambda_1}{b}$, yields the coefficients functions ξ, τ and η in the most general form as

$$\xi = -\delta cx + d_3, \tag{2.80}$$

$$\tau = \delta ct + d_2, \tag{2.81}$$

$$\eta = -\frac{H}{c}e^{-cu} + \delta bx - \delta at + d_1, \tag{2.82}$$

where d_1, d_2, d_3 and δ are arbitrary constants, while H is any solution of the Eq. (2.67). The corresponding symmetry generator is given by

$$\mathbf{X} = (-\delta cx + d_3)\frac{\partial}{\partial x} + (\delta ct + d_2)\frac{\partial}{\partial t} + \left(-\frac{H}{c}e^{-cu} + \delta bx - \delta at + d_1\right)\frac{\partial}{\partial u}.$$

Therefore, Thomas equation has a four-dimensional Lie algebra, which is spanned by

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{2.83}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial t}, \tag{2.84}$$

$$\mathbf{X}_3 = \frac{\partial}{\partial u}, \tag{2.85}$$

$$\mathbf{X}_4 = -cx\frac{\partial}{\partial x} + ct\frac{\partial}{\partial t} + (bx - at)\frac{\partial}{\partial u}, \tag{2.86}$$

and the subalgebra of infinite dimension is

$$\mathbf{X}_H = -\frac{H}{c}e^{-cu}\frac{\partial}{\partial u}. \tag{2.87}$$

The corresponding one-parameter Lie group of point transformations are

$$G_1: \quad (x+\epsilon, t, u), \tag{2.88}$$

$$G_2: \quad (x, t+\epsilon, u), \tag{2.89}$$

$$G_3: \quad (x, t, u+\epsilon), \tag{2.90}$$

$$G_4: \quad \left(xr^{-c\epsilon}, te^{c\epsilon}, \frac{b}{c}\left(1-e^{-c\epsilon}\right)x + \frac{a}{c}\left(1-e^{c\epsilon}\right)t + u\right), \tag{2.91}$$

$$G_H: \quad \left(x, t, \frac{1}{c}\log\left[cH\epsilon + e^{cu}\right]\right). \tag{2.92}$$

According to definition each of G_j, $j = 1, 2, 3, 4, H$ is a symmetry group, then let $u = m(x,t)$ is a solution of Eq. (2.52), so are the following functions

$$u_1 = m\left(x-\epsilon, t\right), \tag{2.93}$$

$$u_2 = m\left(x, t-\epsilon\right), \tag{2.94}$$

$$u_3 = m(x,t) + \epsilon, \tag{2.95}$$

$$u_4 = \frac{b}{c}x\left(e^{c\epsilon}-1\right) + \frac{a}{c}t\left(e^{-c\epsilon}-1\right) + m\left(xe^{c\epsilon}, te^{-c\epsilon}\right), \tag{2.96}$$

$$u_H = \frac{1}{c}\log\left[cH\epsilon + e^{cu}\right], \tag{2.97}$$

where ϵ is any real number. The commutator relations among these vector fields is given in Table 2.1.

$[\mathbf{X_i},\mathbf{X_j}]$	$\mathbf{X}_1$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_m}$
$\mathbf{X_1}$	0	0	0	-c$\mathbf{X_1}$+b$\mathbf{X_3}$	$\mathbf{X_{m_x}}$
$\mathbf{X_2}$	0	0	0	c$\mathbf{X_2}$-a$\mathbf{X_3}$	$\mathbf{X_{m_t}}$
$\mathbf{X_3}$	0	0	0	0	$\mathbf{X_{-cm}}$
$\mathbf{X_4}$	c$\mathbf{X_1}$-b$\mathbf{X_3}$	-c$\mathbf{X_2}$+a$\mathbf{X_3}$	0	0	$\mathbf{X}_\mu$
$\mathbf{X_H}$	-$\mathbf{X_{m_x}}$	$\mathbf{X_{m_t}}$	$\mathbf{X_{cm}}$	$\mathbf{X}_\mu$	0

Table 2.1: Commutator table for the Lie algebra $\mathbf{X_i}$ *and* $\mathbf{X_H}$

where

$$\mu = -cxm_x + ctm_t - c\,(bx - at)\,m.$$

Notice that the totality of these symmetries must be a Lie algebra. Therefore, for m to be any solution of the Eq. (2.67), consequently m_x, m_t and $-cxm_x + ctm_t - c\,(bx - at)\,m$ are also the solutions.

Chapter 3

Discrete Symmetries of Differential Equations

This chapter presents details of finding the discrete symmetries of differential equations. Discrete symmetries are defined as the non-continuous point symmetries of a differential equation [27, 28, 29]. Nevertheless, it was never a straightforward way to find the discrete symmetries of a differential equation. Numerous procedures have been produced for finding discrete symmetries of differential equations, yet regularly, either the symmetry condition is too hard to even consider solving, that is, the subsequent system of determining equations is too hard to illuminate or the strategy does not give all the discrete symmetries of the differential equation. This chapter describes the Peter E. Hydon's technique, who was the first to establish an indirect method for finding discrete symmetries of second or higher order differential equations [27, 28, 29, 30, 31, 32, 33] with a property of having a finite dimensional Lie algebra of infinitesimal generators of one-parameter Lie group of point symmetries. The method not only facilitates the impertinent system of determining equations, it also produces all the discrete symmetries of a differential equation in a comprehensive manner.

Our main goal is to understand the framework and find all the discrete symmetries of a differential equation. However, this chapter only describes the method for ODEs whereas PDEs is nearly the equivalent yet fitting extra detail will be given for PDEs where applicable.

3.1 Core Theory

We start this section by recalling some important definitions and theorems [27, 28, 29].

Definition 3.1.1. *A non-continuous point symmetry of a differential equation is called a discrete symmetry.*

Consider an ordinary differential equation

$$u^{(n)} = W\left(x, u, u', u'', \cdots, u^{(n-1)}\right), \tag{3.1}$$

then the representation of one-parameter Lie group of point symmetry of Eq. (3.1) is given by

$$\zeta : (x, u) \longrightarrow (\hat{x}(x, u), \hat{u}(x, u)), \tag{3.2}$$

and the corresponding infinitesimal generator is

$$\mathbf{X} = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}. \tag{3.3}$$

Moreover, the representation of one-parameter Lie group of point symmetry Eq. (3.2) for s-basis elements $\{\mathbf{X}_m\}_{m=1}^{s}$ of the finite dimensional Lie algebra $\mathcal{L}$ of one-parameter Lie group of point symmetry of an ordinary differential Eq. (3.1) is

$$\zeta_m : (x, u) \longrightarrow \left(e^{\epsilon \mathbf{X}_m} x, e^{\epsilon \mathbf{X}_m} u\right). \tag{3.4}$$

The following theorem is iterative form of the generalization of Theorem (1.2.2) for two variables.

Theorem 3.1.1. *Let $\mathcal{F}(x, u)$ be the C^{∞}-function , then for a particular parameter say ϵ, of the Lie group of point symmetries $\zeta(\epsilon)$ with generator Eq.* (3.3)*, an action of a point symmetry is*

$$\mathcal{F}(\hat{x}, \hat{u}) = \mathcal{F}\left(e^{\epsilon \mathbf{X}} x, e^{\epsilon \mathbf{X}} u\right) = e^{\epsilon \mathbf{X}} o \mathcal{F}(x, y) = \zeta \mathcal{F}(x, y). \tag{3.5}$$

The following are the basic theorems [27, 28, 29] for the theory of discrete symmetries.

Theorem 3.1.2. *Let Eq.* (3.2) *be any discrete or continuous point symmetry and* $\mathcal{L}$ *be the Lie algebra of an infinitesimal generator Eq.* (3.3) *of a differential Eq.* (3.1)*. Then for every generator* $\mathbf{X} \in \mathcal{L}$*, we have* $\zeta \mathbf{X} \zeta^{-1} \in \mathcal{L}$ *and for each* ϵ *the corresponding point transformation*

$$\hat{\zeta}_m(\epsilon) = \zeta \zeta_m \zeta^{-1}, \tag{3.6}$$

is also a point symmetry of a differential equation.

Theorem 3.1.3. *Consider* ζ *be any discrete or continuous point symmetry of a differential equation. Then* $\{\zeta \mathbf{X} \zeta^{-1}\}_{m=1}^{s}$ *will be the basis of a Lie algebra* $\mathcal{L}$ *if and only if* $\{\mathbf{X}_m\}_{m=1}^{s}$ *is a basis of* $\mathcal{L}$*.*

Theorem 3.1.4. *Let* $\mathbf{X}_m \longrightarrow \hat{\mathbf{X}}_m$ *be the transformation such that* $\{\mathbf{X}_m\}_{m=1}^{s}$ *and* $\{\hat{\mathbf{X}}_m\}_{m=1}^{s}$ *are the basis of some Lie algebra* $\mathcal{L}$*, then*

$$\{\hat{\mathbf{X}}_m, \hat{\mathbf{X}}_n\} = C_{mn}^{l} \hat{\mathbf{X}}_l, \tag{3.7}$$

if and only if

$$\{\mathbf{X}_m, \mathbf{X}_n\} = C_{mn}^{l} \mathbf{X}_l. \tag{3.8}$$

From Theorem 3.1.2 it concludes that both the basis $\{\mathbf{X}_m\}_{m=1}^{s}$ and $\{\zeta \mathbf{X}_m \zeta^{-1}\}_{m=1}^{s}$ are of the same Lie algebra $\mathcal{L}$. Consequently, each $\mathbf{X}_m$ can be written as a linear combination of $\zeta \mathbf{X}_m \zeta^{-1}$'s, which generalizes the above Theorem 3.1.4 with the help of following lemma [27, 28, 29].

Lemma 3.1.1. *Every discrete or continuous point symmetry of the Lie algebra* $\mathcal{L}$ *of an infinitesimal generator of one-parameter Lie group of point symmetries of Eq.* (3.1) *induces an automorphism. That is, for each* ζ*, there exist a constant* $N \times N$ *nonsingular matrix* $B = b_m^l$ *such that*

$$\mathbf{X}_m = b_m^l \zeta \mathbf{X} \zeta^{-1} = b_m^l \hat{\mathbf{X}}_l, \tag{3.9}$$

preserving all the structure constants.

3.2 Discrete Symmetries Through Peter E. Hydon Technique

This technique was introduced by Peter E. Hydon in 1998. This method mainly consists of two stages. In the first stage, Lemma 3.1.1 has been applied to obtain the corresponding first order partial differential equations, which should satisfy every point symmetry Eq. (3.2) of an ordinary differential Eq. (3.1).

$$\begin{aligned}\mathbf{X}_m\hat{x} &= b^l_m\zeta\mathbf{X}\zeta^{-1}\hat{x}, \quad m=1,2,3,\cdots,s\\ &= b^l_m\zeta\mathbf{X}_l x,\\ &= b^l_m\zeta\xi_l(x,u),\\ &= b^l_m\xi_l\left(\hat{x},\hat{u}\right),\\ &= b^l_m\hat{\xi}_l.\end{aligned} \tag{3.10}$$

Similarly

$$\begin{aligned}\mathbf{X}_m\hat{u} &= b^l_m\zeta\mathbf{X}_l\zeta^{-1}\hat{u}, \quad m=1,2,3,\cdots,s\\ &= b^l_m\zeta\mathbf{X}_l u,\\ &= b^l_m\zeta\eta_l(x,u),\\ &= b^l_m\eta\left(\hat{x},\hat{u}\right),\\ &= b^l_m\hat{\eta}_l.\end{aligned} \tag{3.11}$$

Equations (3.10) and (3.11) together yields a system of partial differential equations. Now, in order to obtain the values of $(\hat{x},\hat{u})$ interms of x, u, b^l_m, the aforementioned system can be solved by method of characteristics equations. Moreover, these obtained values may have some constants and even some unknown functions, whose values will be determined during the second stage. It should be noted that analogous to $b^l_m = \sigma^l_m$ the solution of the preceeding system always admit the trivial symmetry $(\hat{x},\hat{u}) = (x,u)$.

Now in the second stage, we separate non-point symmetry solutions from point symmetry solutions because there may be some solutions which will not satisfy the system of partial differential equations. This process is carried out by applying the symmetry condition on the general solution of the system of partial differential equations.

So, with the help of this technique we can obtain the complete list of all the point symmetries of Eq. (3.1). Since, we know about the continuous symmetries, that is Lie point symmetries. Therefore, other than that every other symmetry is a discrete symmetry. For further discussion, let us write Eq. (3.10) and Eq. (3.11) in a respective marix forms as

$$\begin{bmatrix} \mathbf{X}_1\hat{x} \\ \mathbf{X}_2\hat{x} \\ \mathbf{X}_3\hat{x} \\ \vdots \\ \mathbf{X}_n\hat{x} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 & \cdots & b_1^n \\ b_2^1 & b_2^2 & b_2^3 & \cdots & b_2^n \\ b_3^1 & b_3^2 & b_3^3 & \cdots & b_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & b_n^3 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \\ \vdots \\ \hat{\xi}_n \end{bmatrix}, \tag{3.12}$$

and

$$\begin{bmatrix} \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{u} \\ \mathbf{X}_3\hat{u} \\ \vdots \\ \mathbf{X}_n\hat{u} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 & \cdots & b_1^n \\ b_2^1 & b_2^2 & b_2^3 & \cdots & b_2^n \\ b_3^1 & b_3^2 & b_3^3 & \cdots & b_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & b_n^3 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \\ \hat{\eta}_3 \\ \vdots \\ \hat{\eta}_n \end{bmatrix}. \tag{3.13}$$

By combining Eq. (3.12) and Eq. (3.13), we obtain a system of determining equations,

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{u} \\ \mathbf{X}_3\hat{x} & \mathbf{X}_3\hat{u} \\ \vdots & \vdots \\ \mathbf{X}_n\hat{x} & \mathbf{X}_n\hat{u} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 & \cdots & b_1^n \\ b_2^1 & b_2^2 & b_2^3 & \cdots & b_2^n \\ b_3^1 & b_3^2 & b_3^3 & \cdots & b_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & b_n^3 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 & \hat{\eta}_1 \\ \hat{\xi}_2 & \hat{\eta}_2 \\ \hat{\xi}_3 & \hat{\eta}_3 \\ \vdots & \vdots \\ \hat{\xi}_n & \hat{\eta}_n \end{bmatrix}. \tag{3.14}$$

Moreover, Eq. (3.14) is an un-coupled system of first order partial differential equations. In case of partial differential equation or any other complex ordinary differential equation, this system of determining equations needs not to be linear. In addition to the symmetry condition, if complex valued parameters were permitted to be used then this method also gives the complex discrete symmetries of the given differential equation.

On the off chance that we are to find the discrete symmetries of a partial differential equation rather than an ordinary differential equation, there will be some additional columns for other independent variables. For instance, a partial differential equation with two independent variables (x, t) and one dependent variable u, framework of Eq. (3.14) will take the structure

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{t} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{t} & \mathbf{X}_2\hat{u} \\ \mathbf{X}_3\hat{x} & \mathbf{X}_3\hat{t} & \mathbf{X}_3\hat{u} \\ \vdots & \vdots & \vdots \\ \mathbf{X}_n\hat{x} & \mathbf{X}_n\hat{t} & \mathbf{X}_n\hat{u} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 & \cdots & b_1^n \\ b_2^1 & b_2^2 & b_2^3 & \cdots & b_2^n \\ b_3^1 & b_3^2 & b_3^3 & \cdots & b_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n^1 & b_n^2 & b_n^3 & \cdots & b_n^n \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 & \hat{\tau}_1 & \hat{\eta}_1 \\ \hat{\xi}_2 & \hat{\tau}_2 & \hat{\eta}_2 \\ \hat{\xi}_3 & \hat{\tau}_3 & \hat{\eta}_3 \\ \vdots & \vdots & \vdots \\ \hat{\xi}_n & \hat{\tau}_n & \hat{\eta}_n \end{bmatrix}, \tag{3.15}$$

where

$$\mathbf{X}_m = \xi_m(x, t, u)\frac{\partial}{\partial x} + \tau_m(x, t, u)\frac{\partial}{\partial t} + \eta_m(x, t, u)\frac{\partial}{\partial u}. \tag{3.16}$$

The remainder of the strategy will remain precisely the equivalent.

Let us consider a detailed but simple example to understand the procedure.

Example 3.2.1. *Consider an ordinary differential equation*

$$\frac{d^2u}{dx^2} = f\left(\frac{du}{dx}\right). \tag{3.17}$$

It has two-dimensional abelian Lie algebra of infinitesimal generators of one-parameter Lie group of point symmetries

$$\mathbf{X}_1 = \frac{\partial}{\partial u}, \tag{3.18}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial x}. \tag{3.19}$$

System of determining equation (3.14) for Eq. (3.17) is

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_2\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{u} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 & \hat{\eta}_1 \\ \hat{\xi}_2 & \hat{\eta}_2 \end{bmatrix}, \tag{3.20}$$

$$= \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.21}$$

$$= \begin{bmatrix} b_1^2 & b_1^1 \\ b_2^2 & b_2^1 \end{bmatrix}. \tag{3.22}$$

Solving the system, we have

$$\mathbf{X}_2\hat{x} = b_2^2, \tag{3.23}$$

$$\frac{\partial \hat{x}}{\partial x} = b_2^2, \tag{3.24}$$

$$\hat{x} = b_2^2 x + g(u). \tag{3.25}$$

Now

$$\mathbf{X}_2\hat{x} = b_1^2, \tag{3.26}$$

$$\frac{\partial \hat{x}}{\partial u} = b_1^2, \tag{3.27}$$

$$g(u) = b_1^2 u + c_1. \tag{3.28}$$

So, Eq. (3.25) implies

$$\hat{x} = b_2^2 x + b_1^2 u + c_1. \tag{3.29}$$

Similarly,

$$\hat{u} = b_2^1 x + b_1^1 u + c_2. \tag{3.30}$$

Therefore, the general solution of Eq. (3.22) is

$$(\hat{x}, \hat{u}) = \left(b_2^2 x + b_1^2 u + c_1, b_2^1 x + b_1^1 u + c_2\right). \tag{3.31}$$

By definition Eq. (3.31) is the symmetry condition of Eq. (3.17) if and only if

$$\frac{d^2\hat{u}}{d\hat{x}^2} = f\left(\frac{d\hat{u}}{\hat{x}}\right). \tag{3.32}$$

Now extending the transformation to first and second derivatives

$$\begin{aligned}\frac{d\hat{u}}{d\hat{x}} &= \frac{d\left(b_2^1 x + b_1^1 u + c_2\right)}{d\left(b_2^2 x + b_1^2 u + c_1\right)},\\ &= \frac{b_2^1 + b_1^1 \frac{du}{dx}}{b_2^2 + b_1^2 \frac{du}{dx}}.\end{aligned}$$

Likewise,

$$\begin{aligned}\frac{d^2\hat{u}}{d\hat{x}^2} = \frac{d\hat{u}}{d\hat{x}}\left(\frac{d\hat{u}}{d\hat{x}}\right) &= \frac{d\left(\frac{b_2^1 + b_1^1 \frac{du}{dx}}{b_2^2 + b_1^2 \frac{du}{dx}}\right)}{d\left(b_2^2 x + b_1^2 u + c_1\right)},\\ &= \frac{\left(b_1^1 b_2^2 - b_2^1 b_1^2\right)\frac{d^2u}{dx^2}}{\left(b_2^2 + b_1^2 \frac{du}{dx}\right)^2},\\ &= \frac{\left(b_1^1 b_2^2 - b_2^1 b_1^2\right) f\left(\frac{du}{dx}\right)}{\left(b_2^2 + b_1^2 \frac{du}{dx}\right)^2}.\end{aligned}$$

Thus, the symmetry condition is

$$\frac{\left(b_1^1 b_2^2 - b_2^1 b_1^2\right) f\left(\frac{du}{dx}\right)}{\left(b_2^2 + b_1^2 \frac{du}{dx}\right)^2} = f\left(\frac{b_2^1 + b_1^1 \frac{du}{dx}}{b_2^2 + b_1^2 \frac{du}{dx}}\right). \tag{3.33}$$

The symmetry condition (3.33) is satisfied only if $b_1^1 = b_2^2 = 1$ and $b_2^1 = b_1^2 = 0$. Therefore, the only discrete symmetry of Eq. (3.17) up to equivalence is

$$(\hat{x}, \hat{u}) = (x + c_1, u + c_2). \tag{3.34}$$

3.3 Some Advancements in the Peter E. Hydon Technique

This section talks about certain enhancements in the fundamental startegy which was presented in the last section. If the Lie algebra $\mathcal{L}$ of infinitesimal generators of one-parameter Lie group of point symmetries is abelian, then in such case small enhancements can be made. On the other hand, for a non-abelian Lie algebra $\mathcal{L}$, system of determining equations (3.14) can be substantially simplified in two steps [27, 28, 29].

3.3.1 Canonical Coordinates

Consider Lie algebra to be abelian. Then in such case it is easy to use the canonical coordinates as it ensures that a minimum of one generator within the basis is simplified. This method is particularly effective when dimension of the Lie algebra is one [27]. Consider the canonical coordinates $h(x,u)$ and $k(x,u)$ satisfy

$$\mathbf{X}_1 h = 1, \quad \mathbf{X}_1 k = 0, \tag{3.35}$$

such that

$$\mathbf{X}_1 = \frac{\partial}{\partial h} = \partial_h. \tag{3.36}$$

As a result, the system of determining equations (3.15) for one dimension of Lie algebra embodied in the form

$$\begin{bmatrix} \mathbf{X}_1 \hat{h} & \mathbf{X}_1 \hat{k} \end{bmatrix} = \begin{bmatrix} b_1^1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{3.37}$$

Now Eq. (3.37) implies

$$\begin{aligned} \frac{\partial \hat{h}}{\partial h} &= b_1^1 \neq 0, \qquad \frac{\partial \hat{k}}{\partial h} = 0, \\ \hat{h} &= b_1^1 h + s(k), \quad \hat{k} = t(k), \end{aligned} \tag{3.38}$$

which is the general solution of Eq. (3.37) for some function s and t. The significance of the symmetry condition on this transformation concludes which function among s, t and constant (b_1^1) are passable.

Example 3.3.1. *Consider the Poisson-Boltzman equation* [29]

$$\frac{d^2u}{dx^2} + \frac{r}{t}\frac{du}{dx} + \beta e^u = 0, \; r \neq 0, \; \beta \in \{-1, 1\}. \tag{3.39}$$

It has one-dimensional Lie algebra of point symmetry generators, spanned by

$$\mathbf{X}_1 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial u}. \tag{3.40}$$

In canonical coordinates $h(x,u)$ and $k(x,u)$ by solving the system (3.35), we obtained the corresponding symmetry transformation

$$h = \ln(x), \quad k = u + \ln(x^2). \tag{3.41}$$

Then Eq. (3.39) becomes

$$\frac{d^2k}{dh^2} + (r-1)\left(\frac{dk}{dh} - 2\right) + \beta e^{k(h)} = 0. \tag{3.42}$$

Now according to symmetry condition if Eq. (3.42) holds so must

$$\frac{d^2\hat{k}}{d\hat{h}^2} + (r-1)\left(\frac{d\hat{k}}{d\hat{h}} - 2\right) + \beta e^{\hat{k}(\hat{h})} = 0. \tag{3.43}$$

Since, we know that $\hat{k} = f(k)$. So, for a particular case of $r = 1$, calculating $\frac{d^2\hat{k}}{d\hat{h}^2}$ and applying the symmetry condition, then performing symmetry transformation once more to convert back to (x, u) coordinates, thereby yields the following real set of discrete symmetries of the Poisson-Boltzman equation

$$(\hat{x}, \hat{u}) \in \left\{x^{\mathfrak{y}}, u + 2\ln\left(x^{1-\mathfrak{y}}\mathfrak{y}^{-1}\right)\right\}, \quad \mathfrak{y} \neq 0, \tag{3.44}$$

where $\mathfrak{y}$ is an arbitrary constant.

3.3.2 Non-abelian Lie Algebra

Further taking the discussion of structure of Lie algebra. Consider $\mathcal{L}$ to be a non-abelian Lie algebra. This means that at this point probably some of the equations

$$[\mathbf{X}_m, \mathbf{X}_n] = C^r_{mn}\mathbf{X}_r, \tag{3.45}$$

are non-trivial, which leads to the following theorems [27, 28, 29].

Theorem 3.3.1. *Let $\mathcal{L}$ be a Lie algebra, abelian or non-abelian, and $\mathbf{X}$ be the generator of one-parameter Lie group of point symmetries of a differential equation. Then the commuataor relation*

$$\left[\zeta\mathbf{X}_m\zeta^{-1}, \zeta\mathbf{X}_n\zeta^{-1}\right] = C^r_{mn}\zeta\mathbf{X}_r\zeta^{-1}, \tag{3.46}$$

holds, if and only if

$$[\mathbf{X}_m, \mathbf{X}_n] = C^r_{mn}\mathbf{X}_r. \tag{3.47}$$

Theorem 3.3.2. *Consider* $\zeta \mathbf{X}_m \zeta^{-1}$ *be the generator satisfying the same commutator relation as* $\mathbf{X}_m$. *Then from Eqs.* (3.7)-(3.9), *the structure constants* C^r_{mn} *and the elements of the matrix* $B = \left(b^l_m\right)$ *satisfying the following equations can be written as*

$$C^t_{pq} b^p_m b^q_n = C^r_{mn} b^t_r, \quad \textit{all indices ranges from 1 to } \dim(\mathcal{L}). \tag{3.48}$$

Particularly, let $\dim(\mathcal{L}) = s$, then Eq. (3.48) will have s^3-equations but due to the antisymmetric property of the structure constraints in the lower indices, the number of distinct equations will reduce to $\frac{s^2(s-1)}{2}$. So, it is adequate to confine diligence towards $m < n$.

Thus, the system of determining equations is simplified extensively with the help of these limitations on the elements of the matrix $B = (b^1_1)$. In this way making the system simpler to settle. On the off chance if the number of equations is excessively huge, utilization of some computer algebra is suggested.

Now we give a detailed example to understand the process.

Example 3.3.2. *Consider an ordinary differential equation*

$$\frac{d^4u}{dx^4} = \left(\frac{d^3u}{dx^3}\right)^3 \left(x - \frac{du}{dx}\right). \tag{3.49}$$

It has three dimensional Lie algebra, with the basis

$$\mathbf{X}_1 = \frac{\partial}{\partial u}, \tag{3.50}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \tag{3.51}$$

$$\mathbf{X}_3 = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{3.52}$$

The only non-zero structure constants are

$$C^1_{13} = 2, \quad C^1_{31} = -2, \tag{3.53}$$

$$C^2_{23} = 1, \quad C^2_{32} = -1. \tag{3.54}$$

Now we solve the equations

$$C^t_{pq} b^p_m b^q_n = C^r_{mn} b^t_r, \quad \text{all indices ranges from 1 to 3.} \tag{3.55}$$

As we have already studied that we get the distinct equations if and only if $m < n$. Now in this particular case, $(m, n) = (1, 2), (1, 3), (2, 3)$. Since, the superscript value of 3 in the structure constant is zero. Therefore, we will start with $t = 3$, thereby making it easier to solve.

Consider $t = 3$

$$C^3_{pq} = 0, \quad m, n = 1, 2, 3. \tag{3.56}$$

Consquently, the constraints reduce to linear equations

$$C^r_{mn} b^3_r = 0, \tag{3.57}$$

$$C^1_{mn} b^3_1 + C^2_{mn} b^3_2 + C^3_{mn} b^3_3 = 0. \tag{3.58}$$

For $(m, n) = (1, 2)$, equation is satisfied. Now for $(m, n) = (1, 3)$, we have

$$b^3_1 = 0. \tag{3.59}$$

Likewise, for $(m, n) = (2, 3)$

$$b^3_2 = 0. \tag{3.60}$$

Consider $t = 1$

$$C^1_{pq} = 0, \quad (p, q) \neq (1, 3), (3, 1). \tag{3.61}$$

The constraints reduce to non-linear equations

$$C^1_{13} b^1_m b^3_n + C^1_{31} b^3_m b^1_n = C^r_{mn} b^1_r, \tag{3.62}$$

$$(2) b^1_m b^3_n + (-2) b^3_m b^1_n = C^1_{mn} b^1_1 + C^2_{mn} b^1_2 + C^3_{mn} b^1_3, \tag{3.63}$$

$$2 b^1_m b^3_n - 2 b^3_m b^1_n = C^1_{mn} b^1_1 + C^2_{mn} b^1_2 + C^3_{mn} b^1_3. \tag{3.64}$$

For $(m,n)=(1,2)$, equation is satisfied, whereas for $(m,n)=(1,3)$

$$2b_1^1b_3^3 - 2b_1^3b_3^1 = C_{13}^1b_1^1 + C_{13}^2b_2^1 + C_{13}^3b_3^1, \tag{3.65}$$

$$b_1^1b_3^3 = b_1^1. \tag{3.66}$$

For $(m,n)=(2,3)$

$$2b_2^1b_3^3 - 2b_2^3b_3^1 = C_{23}^1b_1^1 + C_{23}^2b_2^1 + C_{23}^3b_3^1, \tag{3.67}$$

$$2b_2^1b_3^3 = b_2^1. \tag{3.68}$$

Consider $t=2$

$$C_{pq}^2 = 0, \quad (p,q) \neq (2,3),(3,2). \tag{3.69}$$

The constraints reduce to non-linear equations

$$C_{23}^2b_m^2b_n^3 + C_{23}^2b_m^3b_n^2 = C_{mn}^rb_r^2, \tag{3.70}$$

$$b_m^2b_n^3 - b_m^3b_n^2 = C_{mn}^1b_1^2 + C_{mn}^2b_2^2 + C_{mn}^3b_3^2. \tag{3.71}$$

For $(m,n)=(1,2)$, equation is satisfied, and for $(m,n)=(1,3)$, we have

$$b_1^2b_3^3 - b_1^3b_3^2 = C_{13}^1b_1^2 + C_{13}^2b_2^2 + C_{13}^3b_3^2, \tag{3.72}$$

$$b_1^2b_3^3 = 2b_1^2. \tag{3.73}$$

For $(m,n)=(2,3)$

$$b_2^2b_3^3 - b_2^3b_3^2 = C_{23}^1b_1^2 + C_{23}^2b_2^2 + C_{23}^3b_3^2, \tag{3.74}$$

$$b_2^2b_3^3 = b_2^2. \tag{3.75}$$

Thus, we have been able to simplify $B=\left(b_m^l\right)$ as

$$B = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 \\ b_2^1 & b_2^2 & b_2^3 \\ b_3^1 & b_3^2 & b_3^3 \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & 0 \\ b_2^1 & b_2^2 & 0 \\ b_3^1 & b_3^2 & b_3^3 \end{bmatrix}, \tag{3.76}$$

with the following conditions

$$b_1^1 b_3^3 = b_1^1, \tag{3.77}$$

$$2b_2^1 b_3^3 = b_2^1, \tag{3.78}$$

$$b_1^2 b_3^3 = 2b_1^2, \tag{3.79}$$

$$b_2^2 b_3^3 = b_2^2. \tag{3.80}$$

Since, we know that B must be non-singular. Therefore, with $b_1^1 \neq 0$ and $b_2^2 \neq 0$ resulting in $b_2^1 = b_1^2 = 0$ and $b_3^3 = 1$, thus

$$B = \begin{bmatrix} b_1^1 & 0 & 0 \\ 0 & b_2^2 & 0 \\ b_3^1 & b_3^2 & 1 \end{bmatrix}. \tag{3.81}$$

3.3.3 Inequivalent Discrete Symmetries

This section deals with more improvisation of the process and to figure out how to find the inequivalent discrete symmetries [29, 32].

Definition 3.3.3. *Let ζ and $\hat{\zeta}$ be the two point symmetries of an ordinary differential equation* (3.1)*, then these symmetries are said to be equivalent if there exists $\mathbf{X} \in \mathcal{L}$ such that $\hat{\zeta} = e^{\epsilon \mathbf{X}} \zeta$.*

The system of determining equations (3.14) naturally simplified with the number of reduction of matrices due to the abelian structure of a Lie algebra (i.e. all the structure constants are zero), implying that there are no constraints.

Furthermore, this improvement is not limited to only non-abelian Lie algebra $\mathcal{L}$. Infact, it can be applied to a Lie algebra with zero structure constants, but no considerable simplification of the system is achieved.

On contrary, for the non-abelian case we always try to simplify the matrix $B = \left(b_m^l\right)$ first with the help of non-linear constraints. It is to be noted that for simplification the improvements discussed in previous and preceding sections must be used simultaneously. Let us define some important notions [29, 32] regarding matrices and theorems forming basis for corresponding inequivalent discrete symmetries as

$$(C(n))^r_m = C^r_{mn}, \tag{3.82}$$

and

$$A\left(n,\epsilon\right) = \sum_{j=0}^{\infty} \frac{\epsilon^j}{j!} \left(C(n)\right)^j = e^{\epsilon} C(n). \tag{3.83}$$

Theorem 3.3.3. *Let ζ be the point symmetry and $\mathcal{L}$ be the Lie algebra. Then for a particular parameter say ϵ, the automorphism stimulated by the point symmetry ζ is given by $\zeta = e^{\epsilon \mathbf{X}_m}$ with the corresponding matrix representation*

$$B = A\left(n,\epsilon\right), \tag{3.84}$$

where $\mathbf{X}_m$ is basis element of $\mathcal{L}$.

Theorem 3.3.4. *Let B_1 and B_2 be the corresponding matrix representation of an automorphism stimulated by the point symmetries ζ_1 and ζ_2 respectively. Then the matrix representation of the composition of point symmetries $\zeta_2 \circ \zeta_1$ inducing the automorphism is $B_2 B_1$.*

Theorem 3.3.5. *Let the point symmetries ζ_1 and $\zeta_2 = e^{\epsilon \mathbf{X}} \zeta_1$ stimulating the automorphism with the corresponding matrix representation as B_1 and B_2 respectively. Then for some parameter ϵ_m, $m = 1, 2, 3, \cdots, s$, we have*

$$B_2 = A\left(1,\epsilon_1\right) A\left(2,\epsilon_2\right) \cdots A\left(s,\epsilon_s\right) B_1, \tag{3.85}$$

where s is the dimension of a Lie algebra $\mathcal{L}$.

In order to obtain the inequivalent discrete symmetries, we have to solve the system (3.14) for the inequivalent matrices only. To find the corresponding Lie point symmetries generated by $\mathbf{X}_m$, we need to generate a new matrix say B_2 by multiplying B_1 with each matrix $A(n,\epsilon)$ once, that is, $A(n,\epsilon_s)B_1$ or $B_1 A(n,\epsilon_s)$. Then the obtained matrix B_2 can further be simplified by assigning a particular value to ϵ_m's. This will help in creating zeros in the matrix B_2 resulting in the simplification of determining equations and non-linear constraints. It is to be noted that all the above procedure is applicable if for some ϵ, $C(n)$ is non-zero. On contrary, $A(n, \epsilon)$ is the identity matrix for all ϵ, if for some n, $C(n) = 0$. By solving the determining equations, symmetries obtained in such way stimulates a non-trivial automorphism on a Lie algebra $\mathcal{L}$.

We epitomize the framework with an example [36].

Example 3.3.4. *Consider a first order ODE*

$$\frac{du}{dx} = xu. \tag{3.86}$$

with a two dimensional Lie algebra, spanned by

$$\mathbf{X}_1 = \frac{1}{x}\frac{\partial}{\partial x}, \tag{3.87}$$

$$\mathbf{X}_2 = e^{\frac{x^2}{2}}\frac{\partial}{\partial u}. \tag{3.88}$$

The commutator relation

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_2, \tag{3.89}$$

yields the following non-zero structure constants

$$C^2_{12} = 1, \quad C^2_{21} = -1. \tag{3.90}$$

Now we try to solve the system of non-linear constraints (3.48) for the given values of C^r_{mn}.

$$C^t_{pq} b^p_m b^q_n = C^r_{mn} b^t_r, \quad \text{all indices ranges from 1 to 2.} \tag{3.91}$$

Consider $t = 1$, we have

$$C^1_{pq} = 0, \quad p, q = 1, 2. \tag{3.92}$$

The constraints reduce to

$$C^r_{mn} b^1_r = 0, \tag{3.93}$$

$$C^1_{mn} b^1_1 + C^2_{mn} b^1_2 = 0. \tag{3.94}$$

Now for $(m, n) = (1, 2)$ with $m < n$, Eq. (3.94) takes the form

$$b_2^1 = 0. \tag{3.95}$$

Consider $t = 2$, we have

$$C_{pq}^2 = 0, \qquad (p, q) \neq (1, 2), (2, 1). \tag{3.96}$$

The constraints reduce to

$$b_m^1 b_n^2 - b_m^2 b_n^1 = C_{mn}^1 b_2^2 + C_{mn}^2 b_2^2. \tag{3.97}$$

For $(m, n) = (1, 2)$, Eq. (3.97) can be written as

$$b_2^2 \left(b_1^1 - 1\right) = 0, \tag{3.98}$$

Since, B is non-singular, therefore

$$b_1^1 = 1, \text{ as } b_2^2 \neq 0. \tag{3.99}$$

Substituting all the values of $\left(b_m^l\right)$ into the matrix B, we have

$$B = \begin{bmatrix} b_1^1 & b_1^2 \\ b_2^1 & b_2^2 \end{bmatrix} = \begin{bmatrix} 1 & b_1^2 \\ 0 & b_2^2 \end{bmatrix}. \tag{3.100}$$

Now in order to find the inequivalent matrices using Theorems (3.3.3)-(3.3.5), first we calculate the matrices $C(n)$ and $A\,(n, \epsilon)$.

So,

$$C(1) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.101}$$

and

$$C(2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{3.102}$$

Now

$$A(1,\epsilon) = \exp\left(\epsilon C(1)\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1-\epsilon+\frac{\epsilon^2}{2} \end{bmatrix}, \tag{3.103}$$

Similarly,

$$A(2,\epsilon) = \begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix}. \tag{3.104}$$

Multiplying B with $A(1,\epsilon)$, we have

$$A(1,\epsilon)B = \begin{bmatrix} 1 & b_1^2 \\ 0 & b_2^2\left(1-\epsilon+\frac{\epsilon^2}{2}\right) \end{bmatrix}. \tag{3.105}$$

Let $\epsilon = \epsilon_1 = 1 \pm \iota$, so that $1-\epsilon+\frac{\epsilon^2}{2} = 0$. Therefore, Eq. (3.105) takes the form

$$A(1,\epsilon_1) = \begin{bmatrix} 1 & b_1^2 \\ 0 & 0 \end{bmatrix}. \tag{3.106}$$

Now multiplying $A(2,\epsilon)$

$$A(2,\epsilon)A(1,\epsilon_1)B = \begin{bmatrix} 1 & b_1^2 \\ 0 & 0 \end{bmatrix}. \tag{3.107}$$

Let $B_1 = A(2,\epsilon)A(1,\epsilon_1)B$ is the required inequivalent matrix. To obtain the general solution by utilizing the determining equations (3.14), we have

$$\begin{bmatrix} \mathbf{X}_1\hat{x} & \mathbf{X}_1\hat{u} \\ \mathbf{X}_2\hat{x} & \mathbf{X}_2\hat{u} \end{bmatrix} = \begin{bmatrix} 1 & b_1^2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{x} & 0 \\ 0 & e^{\frac{x^2}{2}} \end{bmatrix}, \tag{3.108}$$

$$= \begin{bmatrix} \frac{1}{x} & b_1^2 e^{\frac{x^2}{2}} \\ 0 & 0 \end{bmatrix}. \tag{3.109}$$

Its general solution is

$$(\hat{x}, \hat{u}) = \left(x + a_1, b_1^2 e^{\frac{x^2}{2}} + a_2\right), \tag{3.110}$$

where a_i are constants.

Upon applying the symmetry condition to Eq. (3.110), we should have

$$\frac{d\hat{u}}{d\hat{x}} = \hat{x}\hat{u}. \tag{3.111}$$

Since,

$$\frac{d\hat{u}}{d\hat{x}} = b_1^2 x e^{\frac{x^2}{2}}. \tag{3.112}$$

So, Eq. (3.111) implies the symmetry condition as

$$b_1^2 x e^{\frac{x^2}{2}} = b_1^2 x e^{\frac{x^2}{2}} + x a_2 + a_1 e^{\frac{x^2}{2}} + a_1 a_2. \tag{3.113}$$

The symmetry condition of Eq. (3.113) is satisfied if $b_1^2 = 1$, and $a_1 = a_2 = 0$. Therefore, up to equivalence there is only one discrete symmetry of Eq. (3.86)

$$(\hat{x}, \hat{u}) = \left(x, e^{\frac{x^2}{2}}\right). \tag{3.114}$$

Chapter 4

Continuous and Discrete Symmetry Analysis of Burgers' Equation

The primary objective of this chapter is to find all the continuous and discrete symmetries of the Burgers' equation. The calculation of all the discrete symmetries of the Burgers' equation is exhaustive, thereby computer algebra is recommended.

4.1 Analysis of Continuous Symmetries

Consider a one-dimensional Burgers' equation

$$u_{xx} + 2uu_x = u_t, \quad 0 < x < 1, \tag{4.1}$$

with initial and boundary conditions

$$u(x, 0) = sin(\pi x), \quad 0 < x < 1, \tag{4.2}$$

$$u(0, t) = u(1, t) = 0, \qquad t > 0. \tag{4.3}$$

Since, Eq. (4.1) is a second order non-linear PDE, so we need to apply the second order prolongation $X^{(2)}$ with the corresponding coefficient Eqs. (1.47)-(1.51). To use the infinitesimal criterion of invariance, let us introduce

$$W = u_{xx} + 2uu_x - u_t, \tag{4.4}$$

then by Theorem (1.7.1), we have

$$X^{(2)}W|_{W=0} \equiv 0, \tag{4.5}$$

which reduces to

$$\eta_{xx} + 2\eta_x u - \eta_t + 2\eta u_x = 0. \tag{4.6}$$

Using the values of η, η_x, η_t and η_{xx} from Eqs. (1.47)-(1.51) in Eq. (4.6), we have

$$\begin{aligned}
&\eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{tx} + (\eta_{uu} - 2\xi_{xu})u_x^2 \\
&- 2\tau_{xu}u_t u_x - \xi_{uu}u_x^3 - \tau_{uu}u_t u_x^2 - 3\xi_u u_x uxx - 2\tau_u u_x u_{tx} + 2(\eta_x + (\eta_u - \xi_x)u_x \\
&- \tau_x u_t - \xi_u u_x^2 - \tau_u u_t u_x)u - (\eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \tau_u u_x^2 - \xi_u u_t u_x) \\
&+ 2\eta u_x = 0.
\end{aligned} \tag{4.7}$$

The comparison of coefficients of u_{tx} yields

$$\tau_{xx} = \tau_{uu} = 0. \tag{4.8}$$

Therefore, Eqn.(4.7) simplified to

$$\begin{aligned}
&\eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x + (\eta_u - 2\xi_x)u_{xx} + (\eta_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 - 3\xi_u u_x u_{xx} \\
&+ 2(\eta_x + (\eta_u - \xi_x)u_x - \xi_u u_x^2)u - (\eta_t + (\eta_u - \tau_t)u_t - \xi_t u_x - \xi_u u_t u_x) - \xi_t u_x \\
&- \xi_u u_t u_x) + 2\eta u_x = 0,
\end{aligned} \tag{4.9}$$

which after some calculus reduces to

$$\begin{aligned}
&\eta_{xx} - \eta_t + 2\eta_x u + (2\eta_{xu} - \xi_{xx} + 2\xi_x u + \xi_t + 2\eta)u_x + (\eta_{uu} - 2\xi_{xu} + 4\xi_u u)u_x^2 \\
&- \xi_{uu}u_x^3 + (\tau_t - 2\xi_u u_x - 2\xi_x)u_t = 0.
\end{aligned} \tag{4.10}$$

By comparing the coefficients of $u_x^{(0)}, u_x^{(1)}, u_x^{(2)}, u_x^{(3)}$ and u_t yields the following system of partial differential equations

$$u_x^{(0)} : \quad \eta_{xx} - \eta_t + 2\eta_x u = 0, \tag{4.11}$$

$$u_x^{(1)} : \quad 2\eta_{xu} - \xi_{xx} + 2\xi_x u + \xi_t + 2\eta = 0, \tag{4.12}$$

$$u_x^{(2)} : \quad \eta_{uu} - 2\xi_{xu} + 4\xi_u u = 0, \tag{4.13}$$

$$u_x^{(3)} : \quad \xi_{uu} = 0, \tag{4.14}$$

$$u_t : \quad \tau_t - 2\xi_u u_x - 2\xi_x = 0. \tag{4.15}$$

After solving the system, one obtains the coefficient functions ξ, τ and η of the form

$$\xi(x,t,u) = \frac{1}{2}(c_1x - 4c_4)t + \frac{1}{2}c_2x + c_5, \tag{4.16}$$

$$\tau(x,t,u) = \frac{1}{2}c_1t^2 + c_2t + c_3, \tag{4.17}$$

$$\eta(x,t,u) = -\frac{1}{4}(x + 2ut)c_1 - \frac{1}{2}c_2u + c_4. \tag{4.18}$$

The corresponding vector field $\mathbf{X}$ is

$$\mathbf{X} = \left(\frac{1}{2}(c_1x - 4c_4)t + \frac{1}{2}c_2x + c_5\right)\frac{\partial}{\partial x} + \left(\frac{1}{2}c_1t^2 + c_2t + c_3\right)\frac{\partial}{\partial t} + \left(-\frac{1}{4}(x + 2ut)c_1 - \frac{1}{2}c_2u + c_4\right)\frac{\partial}{\partial u}. \tag{4.19}$$

Hence, the Lie algebra of infinitesimal symmetry of the Burgers' equation is spanned by the five vector fields, that is

$$\mathbf{X_1} = \frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \frac{1}{4}(x + 2ut)\frac{\partial}{\partial u}, \tag{4.20}$$

$$\mathbf{X_2} = \frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}, \tag{4.21}$$

$$\mathbf{X_3} = \frac{\partial}{\partial t}, \tag{4.22}$$

$$\mathbf{X_4} = -2t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{4.23}$$

$$\mathbf{X_5} = \frac{\partial}{\partial x}. \tag{4.24}$$

The commutation relations between these infinitesimal generators are given in the following table:

$[\mathbf{X_m}, \mathbf{X_n}]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$
$\mathbf{X}_1$	0	$-\mathbf{X_1}$	$-\mathbf{X_2}$	0	$\frac{1}{4}\mathbf{X_4}$
$\mathbf{X}_2$	$\mathbf{X_1}$	0	$-\mathbf{X_3}$	$\frac{1}{2}\mathbf{X_4}$	$-\frac{1}{2}\mathbf{X_5}$
$\mathbf{X}_3$	$\mathbf{X_2}$	$\mathbf{X_3}$	0	$-2\mathbf{X_5}$	0
$\mathbf{X}_4$	0	$-\frac{1}{2}\mathbf{X_4}$	$2\mathbf{X_5}$	0	0
$\mathbf{X}_5$	$-\frac{1}{4}\mathbf{X_4}$	$\frac{1}{2}\mathbf{X_5}$	0	0	0

Table 4.1: Commutator table for the Lie algebra $\mathbf{X_m}$ and $\mathbf{X_n}$

4.2 Analysis of Discrete Symmetries

In this section we find all the discrete symmetries of the Burgers' equation.

4.2.1 Non-zero Structure Constants

The non-zero structure constants C^r_{mn} obtained from commutation relations are

$$C^1_{12} = -1, \qquad C^1_{21} = 1, \qquad C^2_{13} = -1, \qquad C^2_{31} = 1,$$
$$C^3_{23} = -1, \qquad C^3_{32} = 1, \qquad C^4_{15} = \frac{1}{4}, \qquad C^4_{51} = -\frac{1}{4},$$
$$C^4_{24} = \frac{1}{2}, \qquad C^4_{42} = -\frac{1}{2}, \qquad C^5_{25} = -\frac{1}{2}, \qquad C^5_{52} = \frac{1}{2},$$
$$C^5_{34} = -2, \qquad C^5_{43} = 2.$$

4.2.2 Non-linear Constraints

Now in order to simplify the $B = \left(b^l_m\right)$ matrix, we need to substitute the non-zero structure constants in the corresponding non-linear constraints

$$C^s_{pq} b^p_m b^q_n = C^r_{mn} b^s_r, \quad m < n, \;\; m, n, p, q, r, s = 1, 2, 3, 4, 5. \tag{4.25}$$

Consider $s = 1$, we have

$$C^1_{mn} = 0, \qquad (m, n) \neq (1, 2), (2, 1).$$

The constraints reduce to non-linear equations

$$C^1_{12} b^1_m b^2_n + C^1_{21} b^2_m b^1_n = C^r_{mn} b^1_r,$$
$$-b^1_m b^2_n + b^2_m b^1_n = C^1_{mn} b^1_1 + C^2_{mn} b^1_2 + C^3_{mn} b^1_3 + C^4_{mn} b^1_4 + C^5_{mn} b^1_5.$$

For $(m, n) = (1, 2)$

$$-b^1_1 b^2_2 + b^2_1 b^1_2 = C^1_{12} b^1_1 + C^2_{12} b^1_2 + C^3_{12} b^1_3 + C^4_{12} b^1_4 + C^5_{12} b^1_5,$$
$$-b^1_1 b^2_2 + b^2_1 b^1_2 = -b^1_1.$$

For $(m,n) = (1,3)$

$$-b_1^1 b_3^2 + b_1^2 b_3^1 = C_{13}^1 b_1^1 + C_{13}^2 b_2^1 + C_{13}^3 b_3^1 + C_{13}^4 b_4^1 + C_{13}^5 b_5^1,$$
$$-b_1^1 b_3^2 + b_1^2 b_3^1 = -b_2^1.$$

For $(m,n) = (1,4)$

$$-b_1^1 b_4^2 + b_1^2 b_4^1 = C_{14}^1 b_1^1 + C_{14}^2 b_2^1 + C_{14}^3 b_3^1 + C_{14}^4 b_4^1 + C_{14}^5 b_5^1,$$
$$-b_1^1 b_4^2 + b_1^2 b_4^1 = 0.$$

For $(m,n) = (1,5)$

$$-b_1^1 b_5^2 + b_1^2 b_5^1 = C_{15}^1 b_1^1 + C_{15}^2 b_2^1 + C_{15}^3 b_3^1 + C_{15}^4 b_4^1 + C_{15}^5 b_5^1,$$
$$-b_1^1 b_4^2 + b_1^2 b_4^1 = \frac{1}{4} b_4^1.$$

For $(m,n) = (2,3)$

$$-b_2^1 b_3^2 + b_2^2 b_3^1 = C_{23}^1 b_1^1 + C_{23}^2 b_2^1 + C_{23}^3 b_3^1 + C_{23}^4 b_4^1 + C_{23}^5 b_5^1,$$
$$-b_2^1 b_3^2 + b_2^2 b_3^1 = -b_3^1.$$

For $(m,n) = (2,4)$

$$-b_2^1 b_4^2 + b_2^2 b_4^1 = C_{24}^1 b_1^1 + C_{24}^2 b_2^1 + C_{24}^3 b_3^1 + C_{24}^4 b_4^1 + C_{24}^5 b_5^1,$$
$$-b_2^1 b_4^2 + b_2^2 b_4^1 = 0.$$

For $(m,n) = (2,5)$

$$-b_2^1 b_5^2 + b_2^2 b_5^1 = C_{25}^1 b_1^1 + C_{25}^2 b_2^1 + C_{25}^3 b_3^1 + C_{25}^4 b_4^1 + C_{25}^5 b_5^1,$$
$$-b_2^1 b_5^2 + b_2^2 b_5^1 = \frac{1}{2} b_4^1.$$

For $(m,n) = (3,4)$

$$-b_3^1 b_4^2 + b_3^2 b_4^1 = C_{34}^1 b_1^1 + C_{34}^2 b_2^1 + C_{34}^3 b_3^1 + C_{34}^4 b_4^1 + C_{34}^5 b_5^1,$$
$$-b_3^1 b_4^2 + b_3^2 b_4^1 = 0.$$

For $(m,n)=(3,5)$

$$-b_3^1b_5^2+b_3^2b_5^1=C_{35}^1b_1^1+C_{35}^2b_2^1+C_{35}^3b_3^1+C_{35}^4b_4^1+C_{35}^5b_5^1,$$
$$-b_3^1b_5^2+b_3^2b_5^1=0.$$

For $(m,n)=(4,5)$

$$-b_4^1b_5^2+b_4^2b_5^1=C_{45}^1b_1^1+C_{45}^2b_2^1+C_{45}^3b_3^1+C_{45}^4b_4^1+C_{45}^5b_5^1,$$
$$-b_4^1b_5^2+b_4^2b_5^1=0.$$

Thus, for $s=1$, yields the system of non-linear equations as

$$\left.\begin{aligned}
-b_1^1b_2^2+b_1^2b_2^1 &= -b_1^1,\\
-b_1^1b_3^2+b_1^2b_3^1 &= -b_2^1,\\
-b_1^1b_4^2+b_1^2b_4^1 &= 0,\\
-b_1^1b_4^2+b_1^2b_4^1 &= \tfrac{1}{4}b_4^1,\\
-b_2^1b_3^2+b_2^2b_3^1 &= -b_3^1,\\
-b_2^1b_4^2+b_2^2b_4^1 &= 0,\\
-b_2^1b_5^2+b_2^2b_5^1 &= \tfrac{1}{2}b_4^1,\\
-b_3^1b_4^2+b_3^2b_4^1 &= 0,\\
-b_3^1b_5^2+b_3^2b_5^1 &= 0,\\
-b_4^1b_5^2+b_4^2b_5^1 &= 0.
\end{aligned}\right\} \tag{4.26}$$

By solving the system (4.26) of non-linear equations in *Maple*, we get *five* different possibilities to solve it. Due to non-sigularity of the martix B, we can only work with *three* of them. Therefore, for the first case equating $b_2^2=-1$, $b_3^2\neq 0$ and $b_3^1\neq 0$, we obtain the following form of the matrix $B=\left(b_m^l\right)=B_1$

$$B_1=\begin{bmatrix} b_1^1 & b_1^2 & b_1^3 & b_1^4 & b_1^5\\ b_2^1 & b_2^2 & b_2^3 & b_2^4 & b_2^5\\ b_3^1 & b_3^2 & b_3^3 & b_3^4 & b_3^5\\ b_4^1 & b_4^2 & b_4^3 & b_4^4 & b_4^5\\ b_5^1 & b_5^2 & b_5^3 & b_5^4 & b_5^5 \end{bmatrix}=\begin{bmatrix} 0 & 0 & b_1^3 & b_1^4 & b_1^5\\ 0 & -1 & b_2^3 & b_2^4 & b_2^5\\ b_3^1 & b_3^2 & b_3^3 & b_3^4 & b_3^5\\ 0 & 0 & b_4^3 & b_4^4 & b_4^5\\ 0 & 0 & b_5^3 & b_5^4 & b_5^5 \end{bmatrix}. \tag{4.27}$$

Consider $s = 2$, we have

$$C^2_{mn} = 0, \quad (m,n) \neq (1,3),(3,1).$$

The constraints reduce to non-linear equations

$$\begin{aligned} C^2_{13} b^1_m b^3_n + C^2_{31} b^3_m b^1_n &= C^r_{mn} b^2_r, \\ -b^1_m b^3_n + b^3_m b^1_n &= C^1_{mn} b^1_1 + C^2_{mn} b^2_2 + C^3_{mn} b^2_3 + C^4_{mn} b^2_4 + C^5_{mn} b^2_5. \end{aligned}$$

For $(m,n) = (1,3)$

$$\begin{aligned} -b^1_1 b^3_3 + b^3_1 b^1_3 &= C^1_{13} b^2_1 + C^2_{13} b^2_2 + C^3_{13} b^2_3 + C^4_{13} b^2_4 + C^5_{13} b^2_5, \\ -b^1_1 b^3_3 + b^1_3 b^3_1 &= (-1) b^2_2, \\ b^3_1 b^1_3 &= 1, \\ b^3_1 &= \frac{1}{b^1_3}, \ \text{ as } b^1_3 \neq 0. \end{aligned}$$

For $(m,n) = (1,5)$

$$\begin{aligned} -b^1_1 b^3_5 + b^1_3 b^1_5 &= C^1_{15} b^2_1 + C^2_{15} b^2_2 + C^3_{15} b^2_3 + C^4_{15} b^2_4 + C^5_{15} b^2_5, \\ -b^1_1 b^3_5 + b^3_1 b^1_5 &= \frac{1}{4} b^2_4, \\ b^2_4 &= 0. \end{aligned}$$

For $(m,n) = (2,3)$

$$\begin{aligned} -b^1_2 b^3_3 + b^3_2 b^1_3 &= C^1_{23} b^2_1 + C^2_{23} b^2_2 + C^3_{23} b^2_3 + C^4_{23} b^2_4 + C^5_{23} b^2_5, \\ -b^1_2 b^3_3 + b^3_2 b^1_3 &= -b^2_3, \\ b^3_2 b^1_3 &= -b^2_3. \end{aligned}$$

For $(m,n) = (2,4)$

$$\begin{aligned} -b^1_2 b^3_4 + b^3_2 b^1_4 &= C^1_{24} b^2_1 + C^2_{24} b^2_2 + C^3_{24} b^2_3 + C^4_{24} b^2_4 + C^5_{24} b^2_5, \\ -b^1_2 b^3_4 + b^3_2 b^1_4 &= \frac{1}{2} b^2_4, \\ b^2_4 &= 0. \end{aligned}$$

For $(m,n)=(2,5)$

$$\begin{aligned}
-b_2^1 b_5^3 + b_2^3 b_5^1 &= C_{25}^1 b_1^2 + C_{25}^2 b_2^2 + C_{25}^3 b_3^2 + C_{25}^4 b_4^2 + C_{25}^5 b_5^2,\\
-b_2^1 b_5^3 + b_2^3 b_5^1 &= -\frac{1}{2} b_5^2,\\
b_5^2 &= 0.
\end{aligned}$$

For $(m,n)=(3,4)$

$$\begin{aligned}
-b_3^1 b_4^3 + b_3^3 b_4^1 &= C_{34}^1 b_1^2 + C_{34}^2 b_2^2 + C_{34}^3 b_3^2 + C_{34}^4 b_4^2 + C_{34}^5 b_5^2,\\
-b_3^1 b_4^3 &= -2b_5^2,\\
b_3^1 b_4^3 &= 0,\\
b_4^3 &= 0, \ \text{ as } b_3^1 \neq 0.
\end{aligned}$$

For $(m,n)=(3,5)$

$$\begin{aligned}
-b_3^1 b_5^3 + b_3^3 b_5^1 &= C_{35}^1 b_1^2 + C_{35}^2 b_2^2 + C_{35}^3 b_3^2 + C_{35}^4 b_4^2 + C_{35}^5 b_5^2,\\
-b_3^1 b_5^3 + b_3^3 b_1^5 &= 0,\\
-b_3^1 b_5^3 &= 0,\\
b_5^3 &= 0, \ \text{ since } b_3^1 \neq 0.
\end{aligned}$$

However, for $(m,n)=(1,2),(1,4),(4,5)$, the corresponding equations are satisfied.

Thus, Eq. (4.27), the matrix B_1 takes the form

$$B_1 = \begin{bmatrix} 0 & 0 & \frac{1}{b_3^1} & b_1^4 & b_1^5 \\ 0 & -1 & b_2^3 & b_2^4 & b_2^5 \\ b_3^1 & b_3^2 & b_3^3 & b_3^4 & b_3^5 \\ 0 & 0 & 0 & b_4^4 & b_4^5 \\ 0 & 0 & 0 & b_5^4 & b_5^5 \end{bmatrix}. \tag{4.28}$$

Consider $s=3$, we have

$$C_{mn}^3 = 0, \qquad (m,n) \neq (2,3),(3,2).$$

The constraints reduce to non-linear equations

$$\begin{aligned} C^3_{23} b^2_m b^3_n + C^3_{32} b^3_m b^2_n &= C^r_{mn} b^3_r, \\ -b^2_m b^3_n + b^3_m b^2_n &= C^1_{mn} b^3_1 + C^2_{mn} b^3_2 + C^3_{mn} b^3_3 + C^4_{mn} b^3_4 + C^5_{mn} b^3_5. \end{aligned}$$

For $(m,n) = (1,2)$

$$\begin{aligned} -b^2_1 b^3_2 + b^3_1 b^2_2 &= C^1_{12} b^3_1 + C^2_{12} b^3_2 + C^3_{12} b^3_3 + C^4_{12} b^3_4 + C^5_{12} b^3_5, \\ -b^2_1 b^3_2 + b^3_1 b^2_2 &= -b^3_1, \\ b^3_1 &= b^3_1 \neq 0. \end{aligned}$$

For $(m,n) = (1,3)$

$$\begin{aligned} -b^2_1 b^3_3 + b^3_1 b^2_3 &= C^1_{13} b^3_1 + C^2_{13} b^3_2 + C^3_{13} b^3_3 + C^4_{13} b^3_4 + C^5_{13} b^3_5, \\ -b^2_1 b^3_3 + b^3_1 b^2_3 &= -b^3_2, \\ b^3_1 b^2_3 &= -b^3_2. \end{aligned}$$

For $(m,n) = (2,3)$

$$\begin{aligned} -b^2_2 b^3_3 + b^3_2 b^2_3 &= C^1_{23} b^3_1 + C^2_{23} b^3_2 + C^3_{23} b^3_3 + C^4_{23} b^3_4 + C^5_{23} b^3_5, \\ -b^2_2 b^3_3 + b^3_2 b^2_3 &= -b^3_3, \\ b^3_2 b^2_3 &= -b^3_3. \end{aligned}$$

Similarly, for $(m,n) = (i,j)$, $i,j = 1,2,3,4$, with $i < j$, the corresponding equations are satisfied. Consequently, Eq. (4.28), the matrix B_1 can be written as

$$B_1 = \begin{bmatrix} 0 & 0 & \frac{1}{b^1_3} & b^4_1 & b^5_1 \\ 0 & -1 & b^3_2 & b^4_2 & b^5_2 \\ b^1_3 & b^2_3 & b^3_3 & b^4_3 & b^5_3 \\ 0 & 0 & 0 & b^4_4 & b^5_4 \\ 0 & 0 & 0 & b^4_5 & b^5_5 \end{bmatrix}. \tag{4.29}$$

Consider $n = 4$, we have

$$C^4_{mn} = 0, \qquad (m,n) \neq (1,5), (5,1), (2,4), (4,2).$$

The constraints leads to non-linear equations

$$
\begin{aligned}
C^4_{15}b^1_m b^5_n + C^4_{51}b^5_m b^1_n + C^4_{24}b^2_m b^4_n + C^4_{42}b^4_m b^2_n &= C^r_{mn}b^4_r,\\
\frac{1}{4}b^1_m b^5_n - \frac{1}{4}b^5_m b^1_n + \frac{1}{2}b^2_m b^4_n - \frac{1}{2}b^4_m b^2_n &= C^1_{mn}b^4_1 + C^2_{mn}b^4_2 + C^3_{mn}b^4_3 + C^4_{mn}b^4_4 + C^5_{mn}b^4_5.
\end{aligned}
$$

For $(m,n) = (1,2)$

$$
\begin{aligned}
\frac{1}{4}b^1_1 b^5_2 - \frac{1}{4}b^5_1 b^1_2 + \frac{1}{2}b^2_1 b^4_2 - \frac{1}{2}b^4_1 b^2_2 &= C^1_{12}b^4_1 + C^2_{12}b^4_2 + C^3_{12}b^4_3 + C^4_{12}b^4_4 + C^5_{12}b^4_5,\\
-\frac{1}{2}(-1)b^4_1 &= -b^4_1,\\
b^4_1 &= 0.
\end{aligned}
$$

For $(m,n) = (1,3)$

$$
\begin{aligned}
\frac{1}{4}b^1_1 b^5_3 - \frac{1}{4}b^5_1 b^1_3 + \frac{1}{2}b^2_1 b^4_3 - \frac{1}{2}b^4_1 b^2_3 &= C^1_{13}b^4_1 + C^2_{13}b^4_2 + C^3_{13}b^4_3 + C^4_{13}b^4_4 + C^5_{13}b^4_5,\\
-\frac{1}{4}b^5_1 b^1_3 &= -b^4_2,\\
b^4_2 &= \frac{1}{4}b^5_1 b^1_3.
\end{aligned}
$$

For $(m,n) = (1,5)$

$$
\begin{aligned}
\frac{1}{4}b^1_1 b^5_5 - \frac{1}{4}b^5_1 b^1_5 + \frac{1}{2}b^2_1 b^4_5 - \frac{1}{2}b^4_1 b^2_5 &= C^1_{15}b^4_1 + C^2_{15}b^4_2 + C^3_{15}b^4_3 + C^4_{15}b^4_4 + C^5_{15}b^4_5,\\
\frac{1}{4}b^1_1 b^5_5 - \frac{1}{4}b^5_1 b^1_5 + \frac{1}{2}b^2_1 b^4_5 - \frac{1}{2}b^4_1 b^2_5 &= \frac{1}{4}b^4_4,\\
b^4_4 &= 0.
\end{aligned}
$$

For $(m,n) = (2,3)$

$$
\begin{aligned}
\frac{1}{4}b^1_2 b^5_3 - \frac{1}{4}b^5_2 b^1_3 + \frac{1}{2}b^2_2 b^4_3 - \frac{1}{2}b^4_2 b^2_3 &= C^1_{23}b^4_1 + C^2_{23}b^4_2 + C^3_{23}b^4_3 + C^4_{23}b^4_4 + C^5_{23}b^4_5,\\
-\frac{1}{4}b^5_2 b^1_3 - \frac{1}{2}b^4_3 - \frac{1}{2}b^4_2 b^2_3 &= -b^4_3,\\
-\frac{1}{2}\left(\frac{1}{2}b^5_2 b^1_3 + b^4_2 b^2_3\right) &= -\frac{1}{2}b^4_3,\\
\frac{1}{2}b^5_2 b^1_3 + b^4_2 b^2_3 &= b^4_3.
\end{aligned}
$$

For $(m,n)=(2,5)$

$$\begin{aligned}\frac{1}{4}b_2^1b_5^5-\frac{1}{4}b_2^5b_5^1+\frac{1}{2}b_2^2b_5^4-\frac{1}{2}b_2^4b_5^2&=C_{25}^1b_1^4+C_{25}^2b_2^4+C_{25}^3b_3^4+C_{25}^4b_4^4+C_{25}^5b_5^4,\\ -\frac{1}{2}b_5^4&=-\frac{1}{2}b_5^4,\\ b_5^4&=b_5^4\neq 0.\end{aligned}$$

For $(m,n)=(3,4)$

$$\begin{aligned}\frac{1}{4}b_3^1b_4^5-\frac{1}{4}b_3^5b_4^1+\frac{1}{2}b_3^2b_4^4-\frac{1}{2}b_3^4b_4^2&=C_{34}^1b_1^4+C_{34}^2b_2^4+C_{34}^3b_3^4+C_{34}^4b_4^4+C_{34}^5b_5^4,\\ \frac{1}{4}b_3^1b_4^5&=-2b_5^4,\\ -\frac{1}{2}\left(\frac{1}{4}b_3^1b_4^5\right)&=b_5^4.\end{aligned}$$

For $(m,n)=(3,5)$

$$\begin{aligned}\frac{1}{4}b_3^1b_5^5-\frac{1}{4}b_3^5b_5^1+\frac{1}{2}b_3^2b_5^4-\frac{1}{2}b_3^4b_5^2&=C_{35}^1b_1^4+C_{35}^2b_2^4+C_{35}^3b_3^4+C_{35}^4b_4^4+C_{35}^5b_5^4,\\ \frac{1}{4}b_3^1b_5^5-\frac{1}{4}b_3^5b_5^1+\frac{1}{2}b_3^2b_5^4-\frac{1}{2}b_3^4b_5^2&=0,\\ \frac{1}{4}b_3^1b_5^5+\frac{1}{2}b_3^2b_5^4&=0.\end{aligned}$$

Nontheless, the corresponding equations for $(m,n)=(1,4),(2,4),(4,5)$, are satisfied. So, Eq. (4.29), the matrix B_1 is

$$B_1=\begin{bmatrix}0&0&\frac{1}{b_3^1}&0&b_1^5\\ 0&-1&b_2^3&\frac{1}{4}b_1^5b_3^1&b_2^5\\ b_3^1&b_3^2&b_3^3&b_2^5b_3^1+b_2^4b_3^2&b_3^5\\ 0&0&0&0&b_4^5\\ 0&0&0&-\frac{1}{4}\left(\frac{1}{2}b_3^1b_5^4\right)&b_5^5\end{bmatrix}. \tag{4.30}$$

Consider $n=5$, we have

$$C_{mn}^5=0,\qquad (m,n)\neq(2,5),(5,2),(3,4),(4,3).$$

The constraints reduce to non-linear equations

$$
\begin{aligned}
C^5_{25}b^2_m b^5_n + C^5_{52}b^5_m b^2_n + C^5_{34}b^3_m b^4_n + C^5_{43}b^4_m b^3_n &= C^r_{mn}b^5_r,\\
-\frac{1}{2}b^2_m b^5_n + \frac{1}{2}b^5_m b^2_n - 2b^3_m b^4_n + 2b^4_m b^3_n &= C^1_{mn}b^5_1 + C^2_{mn}b^5_2 + C^3_{mn}b^5_3 + C^4_{mn}b^5_4 + C^5_{mn}b^5_5.
\end{aligned}
$$

For $(m,n) = (1,2)$

$$
\begin{aligned}
-\frac{1}{2}b^2_1 b^5_2 + \frac{1}{2}b^5_1 b^2_2 - 2b^3_1 b^4_2 + 2b^4_1 b^3_2 &= C^1_{12}b^5_1 + C^2_{12}b^5_2 + C^3_{12}b^5_3 + C^4_{12}b^5_4 + C^5_{12}b^5_5,\\
-\frac{1}{2}b^5_1 - 2b^3_1 b^4_2 + 2b^4_1 b^3_2 &= -b^5_1,\\
\frac{1}{2}b^5_1 - 2b^3_1 b^4_2 &= 0.
\end{aligned}
$$

For $(m,n) = (1,3)$

$$
\begin{aligned}
-\frac{1}{2}b^2_1 b^5_3 + \frac{1}{2}b^5_1 b^2_3 - 2b^3_1 b^4_3 + 2b^4_1 b^3_3 &= C^1_{13}b^5_1 + C^2_{13}b^5_2 + C^3_{13}b^5_3 + C^4_{13}b^5_4 + C^5_{13}b^5_5\\
\frac{1}{2}b^5_1 b^2_3 - 2b^3_1 b^4_3 + 2b^4_1 b^3_3 &= -b^5_2,\\
-\frac{1}{2}b^5_1 b^2_3 + 2b^3_1 b^4_3 - 2b^4_1 b^3_3 &= b^5_2.
\end{aligned}
$$

For $(m,n) = (1,5)$

$$
\begin{aligned}
-\frac{1}{2}b^2_1 b^5_5 + \frac{1}{2}b^5_1 b^2_5 - 2b^3_1 b^4_5 + 2b^4_1 b^3_5 &= C^1_{15}b^5_1 + C^2_{15}b^5_2 + C^3_{15}b^5_3 + C^4_{15}b^5_4 + C^5_{15}b^5_5,\\
-2b^3_1 b^4_5 &= \frac{1}{4}b^5_4,\\
-2b^3_1\left(-\frac{1}{8}b^1_3 b^5_4\right) &= \frac{1}{4}b^5_4,\\
\frac{1}{4}b^3_1\left(\frac{1}{b^3_1}\right)b^5_4 &= \frac{1}{4}b^5_4,\\
b^5_4 &= b^5_4 \neq 0.
\end{aligned}
$$

For $(m,n) = (2,3)$

$$
\begin{aligned}
-\frac{1}{2}b^2_2 b^5_3 + \frac{1}{2}b^5_2 b^2_3 - 2b^3_2 b^4_3 + 2b^4_2 b^3_3 &= C^1_{23}b^5_1 + C^2_{23}b^5_2 + C^3_{23}b^5_3 + C^4_{23}b^5_4 + C^5_{23}b^5_5,\\
-\frac{1}{2}b^2_2 b^5_3 + \frac{2}{2}b^5_2 b^2_3 - 2b^3_2 b^4_3 + 2b^4_2 b^3_3 &= -b^5_3,\\
\frac{1}{2}b^5_3 + \frac{2}{2}b^5_2 b^2_3 - 2b^3_2 b^4_3 + 2b^4_2 b^3_3 &= -b^5_3.
\end{aligned}
\tag{4.31}
$$

For $(m,n)=(2,4)$

$$
\begin{aligned}
-\frac{1}{2}b_2^2b_4^5+\frac{1}{2}b_2^5b_4^2-2b_2^3b_4^4+2b_2^4b_4^3 &= C_{24}^1b_1^5+C_{24}^2b_2^5+C_{24}^3b_3^5+C_{24}^4b_4^5+C_{24}^5b_5^5,\\
\frac{1}{2}b_4^5 &= \frac{1}{2}b_4^5,\\
b_4^5 &= b_4^5\neq 0.
\end{aligned}
$$

For $(m,n)=(2,5)$

$$
\begin{aligned}
-\frac{1}{2}b_2^2b_5^5+\frac{1}{2}b_2^5b_5^2-2b_2^3b_5^4+2b_2^4b_5^3 &= C_{25}^1b_1^5+C_{25}^2b_2^5+C_{25}^3b_3^5+C_{25}^4b_4^5+C_{25}^5b_5^5,\\
-\frac{1}{2}b_2^2b_5^5+\frac{2}{2}b_2^5b_5^2-2b_2^3b_5^4+2b_2^4b_5^3 &= -\frac{1}{2}b_5^5,\\
b_5^5-2b_2^3b_5^4 &= 0.
\end{aligned}
$$

For $(m,n)=(3,4)$

$$
\begin{aligned}
-\frac{1}{2}b_3^2b_4^5+\frac{1}{2}b_3^5b_4^2-2b_3^3b_4^4+2b_3^4b_4^3 &= C_{34}^1b_1^5+C_{34}^2b_2^5+C_{34}^3b_3^5+C_{34}^4b_4^5+C_{34}^5b_5^5,\\
\frac{1}{2}b_3^2b_4^5 &= 2b_5^5,\\
b_3^2b_4^5 &= b_5^5.
\end{aligned}
$$

For $(m,n)=(3,5)$

$$
\begin{aligned}
-\frac{1}{2}b_3^2b_5^5+\frac{1}{2}b_3^5b_5^2-2b_3^3b_5^4+2b_3^4b_5^3 &= C_{35}^1b_1^5+C_{35}^2b_2^5+C_{35}^3b_3^5+C_{35}^4b_4^5+C_{35}^5b_5^5,\\
-\frac{1}{2}b_3^2b_5^5+\frac{2}{2}b_3^5b_5^2-2b_3^3b_5^4+2b_3^4b_5^3 &= 0,\\
\frac{1}{2}b_3^2b_5^5+2b_3^3b_5^4 &= 0.
\end{aligned}
$$

Likewise, the corresponding equations for $(m,n)=(1,4),(4,5)$, are satisfied. Up to this point, the matrix $B_1=\left(b_m^l\right)$ has been simplified as

$$
B_1=\begin{bmatrix}
0 & 0 & \frac{1}{b_3^1} & 0 & b_1^5\\
0 & -1 & -b_1^3b_3^2 & \frac{1}{4}\frac{b_1^5}{b_3^1} & b_2^5\\
b_3^1 & -b_2^3b_3^1 & -b_2^3b_3^2 & \frac{1}{2}b_2^5b_3^1+b_2^4b_3^2 & b_3^5\\
0 & 0 & 0 & 0 & b_4^5\\
0 & 0 & 0 & -\frac{1}{4}\left(\frac{1}{2}b_3^1b_4^5\right) & b_5^5
\end{bmatrix}, \tag{4.32}
$$

with the following non-linear constraints

$$\begin{aligned}
\frac{1}{4}b_3^1b_5^5+\frac{1}{2}b_3^2b_5^4&=0,\\
\frac{1}{2}b_1^5-2b_1^3b_2^4&=0,\\
-\frac{1}{2}b_1^5b_3^2+2b_1^3b_3^4-2b_1^4b_3^3&=b_2^5,\\
\frac{1}{2}b_3^5+\frac{2}{2}b_2^5b_3^2-2b_2^3b_3^4+2b_2^4b_3^3&=-b_3^5,\\
b_5^5-2b_2^3b_5^4&=0,\\
b_3^2b_4^5&=b_5^5,\\
\frac{1}{2}b_3^2b_5^5+2b_3^3b_5^4&=0.
\end{aligned}$$

Solving the non-linear conditions yields the following simplified form of the matrix B_1 as

$$B_1=\begin{bmatrix}0&0&\frac{1}{b_3^1}&0&b_1^5\\0&-1&-\frac{b_3^2}{b_3^1}&\frac{1}{4}\frac{b_1^5}{b_3^1}&\frac{1}{2}\frac{4b_3^4-b_1^5b_3^2b_3^1}{b_3^1}\\b_3^1&b_3^2&\frac{1}{2}\frac{(b_3^2)^2}{b_3^1}&b_3^4&-\frac{2b_3^4b_3^2}{b_3^1}\\0&0&0&0&b_4^5\\0&0&0&-\frac{1}{8}b_3^1b_4^5&\frac{1}{4}b_2^3b_4^5\end{bmatrix}. \tag{4.33}$$

Similarly, setting $b_2^2=1$, $b_3^1\neq 0$, and $b_3^2\neq 0$, yields B_2

$$B_2=\begin{bmatrix}b_1^1&0&0&b_1^4&0\\b_1^1b_3^2&1&0&-\frac{1}{4}b_1^1b_3^5+\frac{1}{2}b_1^4b_3^2&-\frac{2b_1^4}{b_1^1}\\\frac{1}{2}b_1^1\left(b_3^2\right)^2&b_3^2&\frac{1}{b_1^1}&-\frac{1}{4}b_3^5b_1^1b_3^2&b_3^5\\0&0&0&b_4^4&0\\0&0&0&-\frac{1}{4}b_4^4b_3^2&\frac{b_4^4}{b_1^1}\end{bmatrix}, \tag{4.34}$$

and with $b_2^2 = 1$, $b_3^1 = b_3^2 = 0$, we have B_3 as

$$B_3 = \begin{bmatrix} \frac{1}{2}\frac{(b_1^2)^2}{b_1^3} & b_1^2 & b_1^3 & -\frac{1}{4}\frac{b_1^5 b_1^2}{b_1^3} & b_1^5 \\ 0 & 1 & \frac{2b_1^3}{b_1^2} & -\frac{1}{8}\frac{(b_1^2)^2 b_3^5}{b_1^3} & \frac{1}{2}\frac{2b_1^5+(b_1^2)^2 b_3^5}{b_1^2} \\ 0 & 0 & \frac{2b_1^3}{(b_1^2)^2} & 0 & b_3^5 \\ 0 & 0 & 0 & b_4^4 & -\frac{4b_1^3 b_4^4}{b_1^2} \\ 0 & 0 & 0 & 0 & \frac{2b_1^3 b_4^4}{(b_1^2)^2} \end{bmatrix}. \tag{4.35}$$

4.2.3 Inequivalent Symmetries

Recall Theorems (2.3.3)-(2.3.5), we have

$$C(n)_m^r = C_{mn}^r,$$

and

$$A(n,\epsilon) = \exp(\epsilon C(n)).$$

Calculating $C(n)$, $n = 1, 2, 3, 4, 5$ matrices

$$C(1) = \begin{bmatrix} C_{11}^1 & C_{11}^2 & C_{11}^3 & C_{11}^4 & C_{11}^5 \\ C_{21}^1 & C_{21}^2 & C_{21}^3 & C_{21}^4 & C_{21}^5 \\ C_{31}^1 & C_{31}^2 & C_{31}^3 & C_{31}^4 & C_{31}^5 \\ C_{41}^1 & C_{41}^2 & C_{41}^3 & C_{41}^4 & C_{41}^5 \\ C_{51}^1 & C_{51}^2 & C_{51}^3 & C_{51}^4 & C_{51}^5 \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 \end{bmatrix}. \tag{4.36}$$

Similarly,

$$C(2) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \tag{4.37}$$

$$C(3) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.38}$$

$$C(4) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.39}$$

and

$$C(5) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.40}$$

Now calculating matrices $A(n, \epsilon) = \exp(\epsilon C(n))$,

$$
\begin{aligned}
A(1,\epsilon) &= \exp\left(\epsilon C(1)\right), \\
&= \exp\left(\epsilon\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 \end{bmatrix}\right), \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \epsilon & 1 & 0 & 0 & 0 \\ \frac{\epsilon^2}{2} & \epsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\epsilon}{4} & 1 \end{bmatrix}.
\end{aligned} \tag{4.41}
$$

Likewise,

$$
\begin{aligned}
A(2,\epsilon) &= \exp\left(\epsilon C(2)\right), \\
&= \begin{bmatrix} e^{-\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{\epsilon}{2}} & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{\epsilon}{2}} \end{bmatrix},
\end{aligned} \tag{4.42}
$$

$$
\begin{aligned}
A(3,\epsilon) &= \exp\left(\epsilon C(3)\right), \\
&= \begin{bmatrix} 1 & -\epsilon & \frac{\epsilon^2}{2} & 0 & 0 \\ 0 & 1 & -\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2\epsilon \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned} \tag{4.43}
$$

$$A(4,\epsilon) = \exp\left(\epsilon C(4)\right),$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\epsilon}{2} & 0 \\ 0 & 0 & 1 & 0 & -2\epsilon \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.44}$$

and

$$A(5,\epsilon) = \exp\left(\epsilon C(5)\right),$$
$$= \begin{bmatrix} 1 & 0 & 0 & \frac{\epsilon}{4} & 0 \\ 0 & 1 & 0 & 0 & -\frac{\epsilon}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.45}$$

Now post-multiplying B_1 by $A(2,\epsilon)$

$$B_1A(2,\epsilon) = \begin{bmatrix} 0 & 0 & \frac{1}{b_3^1} & 0 & b_1^5 \\ 0 & -1 & -\frac{b_2^3}{b_1^3} & \frac{1}{4}\frac{b_1^5}{b_3^1} & \frac{1}{2}\frac{4b_3^4-b_1^5b_3^2b_3^1}{b_3^1} \\ b_3^1 & b_3^2 & \frac{1}{2}\frac{(b_3^2)^2}{b_3^1} & b_3^4 & -\frac{2b_3^4b_3^2}{b_3^1} \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & -\frac{1}{8}b_3^1b_5^4 & \frac{1}{4}b_3^2b_4^5 \end{bmatrix} \begin{bmatrix} e^{-\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e^{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{\epsilon}{2}} & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{\epsilon}{2}} \end{bmatrix},$$
$$= \begin{bmatrix} 0 & 0 & \frac{e^{\epsilon}}{b_3^1} & 0 & b_1^5e^{\frac{\epsilon}{2}} \\ 0 & -1 & -\frac{b_3^2e^{\epsilon}}{b_3^1} & \frac{1}{4}\frac{b_1^5e^{-\frac{\epsilon}{2}}}{b_3^1} & \frac{1}{2}\frac{\left(4b_3^4-b_1^5b_3^2b_3^1\right)e^{\frac{\epsilon}{2}}}{b_3^1} \\ b_3^1e^{-\epsilon} & b_3^2 & \frac{1}{2}\frac{(b_3^2)^2e^{\epsilon}}{b_3^1} & b_3^4e^{-\frac{\epsilon}{2}} & -\frac{2b_3^4b_3^2e^{\frac{\epsilon}{2}}}{b_3^1} \\ 0 & 0 & 0 & 0 & b_4^5e^{\frac{\epsilon}{2}} \\ 0 & 0 & 0 & -\frac{1}{8}b_3^1b_4^5e^{-\frac{\epsilon}{2}} & \frac{1}{4}b_3^2b_4^5e^{\frac{\epsilon}{2}} \end{bmatrix}. \tag{4.46}$$

Let $\epsilon = \epsilon_1 = \ln(|b_3^1|)$, then we have

$$b_3^1 : \qquad b_3^1 e^{-\epsilon} = b_3^1 e^{-\ln(|b_3^1|)}, \\ = \frac{b_3^1}{|b_3^1|} = \pm 1. \tag{4.47}$$

$$b_1^3 : \qquad \frac{e^{\epsilon}}{b_3^1} = \frac{e^{\ln(|b_3^1|)^{-1}}}{b_3^1}, \\ = \frac{|b_3^1|}{b_3^1} = \pm 1. \tag{4.48}$$

$$b_2^3 : \qquad \frac{-b_2^3 e^{\epsilon}}{b_3^1} = \frac{-b_2^3 e^{\ln(|b_3^1|)^{-1}}}{b_3^1}, \\ = \frac{-b_2^3(|b_3^1|)}{b_3^1} = \pm\left(-b_2^3\right). \tag{4.49}$$

$$b_3^3 : \qquad \frac{1}{2}\frac{(b_3^2)^2 e^{\epsilon}}{b_3^1} = \frac{1}{2}\frac{(b_3^2)^2 e^{\ln(|b_3^1|)}}{b_3^1}, \\ = \frac{1}{2}\frac{(b_3^2)^2(|b_3^1|)}{b_3^1} = \pm\frac{1}{2}(b_3^2)^2. \tag{4.50}$$

$$b_2^4 : \qquad \frac{1}{4}\frac{b_1^5 e^{-\frac{\epsilon}{2}}}{b_3^1} = \frac{1}{4}\frac{b_1^5 e^{\ln(|b_3^1|)^{-\frac{1}{2}}}}{b_3^1}, \\ = \frac{1}{4}\frac{b_1^5}{\sqrt{|b_3^1|}b_3^1} = \pm\frac{1}{4}b_1^5. \tag{4.51}$$

$$b_3^4 : \qquad b_3^4 e^{-\frac{\epsilon}{2}} = b_3^4 e^{\ln(|b_3^1|)^{-\frac{1}{2}}}, \\ = \frac{b_3^4}{\sqrt{|b_3^1|}} = b_3^4. \tag{4.52}$$

$$b_5^4 : \qquad -\frac{1}{8}b_3^1 b_4^5 e^{-\frac{\epsilon}{2}} = -\frac{1}{8}b_3^1 b_5^4 e^{\ln(|b_3^1|)^{-\frac{1}{2}}}, \\ = -\frac{1}{8}\frac{b_3^1 b_4^5}{\sqrt{|b_3^1|}} = \pm\left(-\frac{1}{8}b_4^5\right). \tag{4.53}$$

$$b_1^5: \qquad b_1^5 e^{\frac{\epsilon}{2}} = b_1^5 e^{\ln\left(|b_3^1|\right)^{\frac{1}{2}}}, \\ = b_1^5 \sqrt{|b_3^1|} = b_1^5. \tag{4.54}$$

$$b_2^5: \qquad \frac{1}{2}\frac{\left(4b_3^4 - b_1^5 b_3^2 b_3^1\right) e^{\frac{\epsilon}{2}}}{b_3^1} = \frac{1}{2}\frac{\left(4b_3^4 - b_1^5 b_3^2 b_3^1\right) e^{\ln\left(|b_3^1|\right)^{\frac{1}{2}}}}{b_3^1}, \\ = \frac{1}{2}\frac{\left(4b_3^4 - b_1^5 b_3^2 b_3^1\right)\sqrt{|b_3^1|}}{b_3^1}, \\ = \pm\frac{1}{2}\left[4b_3^4 - \left(\pm b_1^5 b_3^2\right)\right]. \tag{4.55}$$

$$b_3^5: \qquad -\frac{2b_3^4 b_3^2 e^{\frac{\epsilon}{2}}}{b_3^1} = -\frac{2b_3^4 b_3^2 e^{\ln\left(|b_3^1|\right)^{\frac{1}{2}}}}{b_3^1}, \\ = -\frac{2b_3^4 b_3^2 \sqrt{|b_3^1|}}{b_3^1} = \pm\left(-2b_3^4 b_3^2\right). \tag{4.56}$$

$$b_4^5: \qquad b_4^5 e^{\frac{\epsilon}{2}} = b_4^5 e^{\ln\left(|b_3^1|\right)^{\frac{1}{2}}}, \\ = b_4^5 \sqrt{|b_3^1|} = b_4^5. \tag{4.57}$$

$$b_5^5: \qquad \frac{1}{4} b_2^3 b_4^5 e^{\frac{\epsilon}{2}} = \frac{1}{4} b_2^3 b_4^5 e^{\ln\left(|b_3^1|\right)^{\frac{1}{2}}}, \\ = \frac{1}{4} b_2^3 b_4^5 \sqrt{|b_3^1|} = \frac{1}{4} b_2^3 b_4^5. \tag{4.58}$$

Thus, we have

$$B_1 A(2, \epsilon_1) = \begin{bmatrix} 0 & 0 & \pm 1 & 0 & b_1^5 \\ 0 & -1 & \pm\left(-b_3^2\right) & \pm\frac{1}{4} b_1^5 & \pm\frac{1}{2}\left[4b_3^4 - \left(\pm b_1^5 b_3^2\right)\right] \\ \pm 1 & b_3^2 & \pm\frac{1}{2}\left(b_3^2\right)^2 & b_3^4 & \pm\left(-2b_3^4 b_3^2\right) \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8} b_4^5\right) & \frac{1}{4} b_3^2 b_4^5 \end{bmatrix}. \tag{4.59}$$

Now post-multiplying by $A(3,\epsilon)$

$$B_1A(2,\epsilon_1)A(3,\epsilon) = \begin{bmatrix} 0 & 0 & \pm 1 & 0 & b_1^5 \\ 0 & -1 & \pm\left(-b_3^2\right) & \pm\frac{1}{4}b_1^5 & \pm\frac{1}{2}\left[4b_3^4 - \left(\pm b_1^5 b_3^2\right)\right] \\ \pm 1 & b_3^2 & \pm\frac{1}{2}\left(b_3^2\right)^2 & b_3^4 & \pm\left(-2b_3^4 b_3^2\right) \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & \frac{1}{4}b_3^2 b_4^5 \end{bmatrix} \begin{bmatrix} 1 & -\epsilon & \frac{\epsilon^2}{2} & 0 & 0 \\ 0 & 1 & -\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2\epsilon \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & \pm 1 & 0 & b_1^5 \\ 0 & -1 & \pm\left(\epsilon - b_3^2\right) & \pm\frac{1}{4}b_1^5 & \pm\frac{\epsilon}{2}b_1^5 \pm \frac{1}{2}[4b_3^4 - (\pm b_1^5 b_3^2)] \\ \pm 1 & \pm(-\epsilon) + b_3^2 & \pm\epsilon^2 - \epsilon b_3^2 \pm \frac{1}{2}\left(b_3^2\right)^2 & \pm b_3^4 & 2\epsilon b_3^4 \pm \left(-2b_3^4 b_3^2\right) \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & \pm\left(-\frac{1}{4}b_4^5\epsilon\right) + \frac{1}{4}b_3^2 b_4^5 \end{bmatrix}.$$

Considering $\epsilon = \epsilon_2 = b_3^2$, we have

$$b_3^2: \qquad \pm\left(\epsilon - b_3^2\right) = \pm\left(b_3^2 - b_3^2\right), \\ = 0.$$

$$b_2^3: \qquad \pm(-\epsilon) + b_3^2 = \pm\left(-b_3^2\right) + b_3^2, \\ = 0.$$

$$b_3^3: \qquad \pm\epsilon^2 - \epsilon b_3^2 \pm \frac{1}{2}\left(b_3^2\right)^2 = \pm(b_3^2)^2 - b_3^2 b_3^2 \pm \frac{1}{2}\left(b_3^2\right)^2, \\ = 0.$$

$$b_2^5: \qquad \pm\frac{1}{2}\epsilon b_1^5 \pm \left(4b_3^4 - b_1^5 b_3^2\right) = \pm\frac{1}{2}b_3^2 b_1^5 \pm \left[4b_3^4 - \left(\pm b_1^5 b_3^2\right)\right], \\ = \pm 4b_3^4.$$

$$b_3^5: \qquad \pm 2\epsilon b_3^4 \pm 2b_3^4 b_3^2 = \pm 2b_3^2 b_3^4 \pm 2b_3^4 b_3^2, \\ = 0.$$

$$b_5^5: \qquad \pm\frac{1}{4}\epsilon b_4^5 + \frac{1}{4}b_2^3 b_4^5 = \pm\frac{1}{4}b_3^2 b_4^5 + \frac{1}{4}b_3^2 b_4^5, \\ = 0.$$

So, Eq. (4.59) can be further simplified as

$$B_1 A(2,\epsilon_1)A(3,\epsilon_2) = \begin{bmatrix} 0 & 0 & \pm 1 & 0 & b_1^5 \\ 0 & -1 & 0 & \pm\frac{1}{4}b_1^5 & \pm 4b_3^4 \\ \pm 1 & 0 & 0 & \pm b_3^4 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & 0 \end{bmatrix}. \tag{4.60}$$

Post-multiplying by $A(4,\epsilon)$

$$B_1 A(2,\epsilon_1)A(3,\epsilon_2)A(4,\epsilon) = \begin{bmatrix} 0 & 0 & \pm 1 & 0 & b_1^5 \\ 0 & -1 & 0 & \pm\frac{1}{4}b_1^5 & \pm 4b_3^4 \\ \pm 1 & 0 & 0 & \pm b_3^4 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\epsilon}{2} & 0 \\ 0 & 0 & 1 & 0 & -2\epsilon \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & \pm 1 & 0 & \pm\left(-2\epsilon + b_1^5\right) \\ 0 & -1 & 0 & -\frac{1}{2}\epsilon \pm \frac{1}{4}b_1^5 & \pm 4b_3^4 \\ \pm 1 & 0 & 0 & \pm b_3^4 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & 0 \end{bmatrix}.$$

Let $\epsilon = \epsilon_3 = \frac{1}{2}b_1^5$, then

$$b_1^5 : \qquad \pm\left(-2\epsilon + b_1^5\right) = \pm\left(-b_1^5 + b_1^5\right), \\ = 0.$$

$$b_2^4 : \qquad -\frac{1}{2}\epsilon \pm \frac{1}{4}b_1^5 = -\frac{1}{4}b_1^5 \pm \frac{1}{4}b_1^5, \\ = 0.$$

Therefore, we have

$$B_1A(2,\epsilon_1)A(3,\epsilon_2)A(4,\epsilon_3) = \begin{bmatrix} 0 & 0 & \pm1 & 0 & 0 \\ 0 & -1 & 0 & 0 & \pm4b_3^4 \\ \pm1 & 0 & 0 & \pm b_3^4 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & 0 \end{bmatrix}. \tag{4.61}$$

Now post-multiplying by $A(5,\epsilon)$

$$B_1A(2,\epsilon_1)A(3,\epsilon_2)A(4,\epsilon_3)A(5,\epsilon) = \begin{bmatrix} 0 & 0 & \pm1 & 0 & 0 \\ 0 & -1 & 0 & 0 & \pm4b_3^4 \\ \pm1 & 0 & 0 & \pm b_3^4 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \frac{\epsilon}{4} & 0 \\ 0 & 1 & 0 & 0 & -\frac{\epsilon}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & \pm1 & 0 & 0 \\ 0 & -1 & 0 & 0 & \frac{1}{2}\epsilon \pm 4b_3^4 \\ \pm1 & 0 & 0 & \pm\frac{1}{4}\epsilon \pm b_3^4 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(-\frac{1}{8}b_4^5\right) & 0 \end{bmatrix}.$$

Considering $\epsilon = \epsilon_4 = 4b_3^4$, we have

$$b_3^4 : \qquad \pm\frac{1}{4}\epsilon \pm b_3^4 = \pm\frac{1}{4}\left(4b_3^4\right) \pm b_3^4, \\ = 0, \quad \text{if} \;\; b_3^4 = 0.$$

$$b_3^4 : \qquad \frac{1}{2}\epsilon \pm 4b_3^4 = \frac{1}{2}\left(4b_3^4\right) \pm 4b_3^4, \\ = 0.$$

Finally, we have the most simplified form of the matrix B_1 as

$$B_1A(2,\epsilon_1)A(3,\epsilon_2)A(4,\epsilon_3)A(5,\epsilon_4) = \begin{bmatrix} 0 & 0 & \pm1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \pm1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & \pm\left(\frac{-1}{8}b_4^5\right) & 0 \end{bmatrix}. \tag{4.62}$$

Let

$$B1 = B_1A(2,\epsilon_1)A(3,\epsilon_2)A(4,\epsilon_3)A(5,\epsilon_4),$$

then Eq. (4.62) can be written as

$$B1 = \begin{bmatrix} 0 & 0 & \lambda & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & -\frac{\lambda}{8}b_4^5 & 0 \end{bmatrix}, \tag{4.63}$$

where $\lambda = \pm1$.

Similarly, one can obtain the corresponding $B2$ and $B3$ matrices for the respective aforementioned cases of $\left(b^l_m\right)$ in Eqs. (4.34) and (4.35), that is,

$$B2 = \begin{bmatrix} b^1_1 & 0 & 0 & b^4_1 & 0 \\ b^1_1 b^2_3 & 1 & 0 & \frac{1}{2} b^4_1 b^2_3 & -\frac{2b^4_1}{b^1_1} \\ \frac{1}{2} b^1_1 (b^2_3)^2 & b^2_3 & \frac{1}{b^1_1} & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & -\frac{1}{4}\xi b^2_3 & \frac{\xi}{b^1_1} \end{bmatrix}, \tag{4.64}$$

and

$$B3 = \begin{bmatrix} \frac{1}{2}\frac{(b^2_1)^2}{\sigma} & b^2_1 & \sigma & 0 & 0 \\ 0 & 1 & \frac{2\sigma}{b^2_1} & 0 & 0 \\ 0 & 0 & \frac{2\sigma}{(b^2_1)^2} & 0 & 0 \\ 0 & 0 & 0 & b^4_4 & -\frac{4\sigma b^4_4}{b^2_1} \\ 0 & 0 & 0 & 0 & \frac{2\sigma b^4_4}{(b^2_1)^2} \end{bmatrix}, \tag{4.65}$$

where $\xi = \pm 1$ and $\sigma = \pm 1$.

4.2.4 System of Determining Equations

Now from Eq. (2.15) we know that

$$\mathbf{X_m}\hat{\mathbf{X}}_\mathbf{n} = B\phi_n(\hat{x}, \hat{t}, \hat{u}), \quad m = 1, 2, 3, 4, 5, \quad n = 1, 2, 3, \tag{4.66}$$

where

$$\begin{aligned} \phi_1 &= \xi(\hat{x}, \hat{t}, \hat{u}), \\ \phi_2 &= \tau(\hat{x}, \hat{t}, \hat{u}), \\ \phi_3 &= \eta(\hat{x}, \hat{t}, \hat{u}). \end{aligned}$$

Using Eq. (4.63) in Eq. (4.66) implies

$$\begin{bmatrix} \mathbf{X_1}\hat{x} & \mathbf{X_1}\hat{t} & \mathbf{X_1}\hat{u} \\ \mathbf{X_2}\hat{x} & \mathbf{X_2}\hat{t} & \mathbf{X_2}\hat{u} \\ \mathbf{X_3}\hat{x} & \mathbf{X_3}\hat{t} & \mathbf{X_3}\hat{u} \\ \mathbf{X_4}\hat{x} & \mathbf{X_4}\hat{t} & \mathbf{X_4}\hat{u} \\ \mathbf{X_5}\hat{x} & \mathbf{X_5}\hat{t} & \mathbf{X_5}\hat{u} \end{bmatrix} = \begin{bmatrix} b_1^1 & b_1^2 & b_1^3 & b_1^4 & b_1^5 \\ b_2^1 & b_2^2 & b_2^3 & b_2^4 & b_2^5 \\ b_3^1 & b_3^2 & b_3^3 & b_3^4 & b_3^5 \\ b_4^1 & b_4^2 & b_4^3 & b_4^4 & b_4^5 \\ b_5^1 & b_5^2 & b_5^3 & b_5^4 & b_5^5 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 & \hat{\tau}_1 & \hat{\eta}_1 \\ \hat{\xi}_2 & \hat{\tau}_2 & \hat{\eta}_2 \\ \hat{\xi}_3 & \hat{\tau}_3 & \hat{\eta}_3 \\ \hat{\xi}_4 & \hat{\tau}_4 & \hat{\eta}_4 \\ \hat{\xi}_5 & \hat{\tau}_5 & \hat{\eta}_5 \end{bmatrix}, \tag{4.67}$$

$$= \begin{bmatrix} 0 & 0 & \lambda & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & -\frac{1}{8}b_4^5 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\hat{x}\hat{t} & \frac{1}{2}\hat{t}^2 & -\frac{1}{4}\left(\hat{x}+2\hat{u}\hat{t}\right) \\ \frac{1}{2}\hat{x} & \hat{t} & -\frac{1}{2}\hat{u} \\ 0 & 1 & 0 \\ -2\hat{t} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

So, we have

$$\begin{bmatrix} \mathbf{X_1}\hat{x} & \mathbf{X_1}\hat{t} & \mathbf{X_1}\hat{u} \\ \mathbf{X_2}\hat{x} & \mathbf{X_2}\hat{t} & \mathbf{X_2}\hat{u} \\ \mathbf{X_3}\hat{x} & \mathbf{X_3}\hat{t} & \mathbf{X_3}\hat{u} \\ \mathbf{X_4}\hat{x} & \mathbf{X_4}\hat{t} & \mathbf{X_4}\hat{u} \\ \mathbf{X_5}\hat{x} & \mathbf{X_5}\hat{t} & \mathbf{X_5}\hat{u} \end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 \\ -\frac{1}{2}\hat{x} & -\hat{t} & \frac{1}{2}\hat{u} \\ \frac{1}{2}\lambda\hat{x}\hat{t} & \frac{1}{2}\lambda\hat{t}^2 & \lambda\left(-\frac{1}{4}\hat{x}-\frac{1}{2}\hat{u}\hat{t}\right) \\ b_4^5 & 0 & 0 \\ \frac{1}{4}\lambda b_4^5\hat{t} & 0 & -\frac{1}{8}\lambda b_4^5 \end{bmatrix}. \tag{4.68}$$

By equating the corresponding elements of both matrices, we get

$$\begin{aligned} &\mathbf{X_1}\hat{x} = 0, && \mathbf{X_1}\hat{t} = \lambda, && \mathbf{X_1}\hat{u} = 0, \\ &\mathbf{X_2}\hat{x} = -\frac{1}{2}\hat{x}, && \mathbf{X_2}\hat{t} = -\hat{t}, && \mathbf{X_2}\hat{u} = \frac{1}{2}\hat{u}, \\ &\mathbf{X_3}\hat{x} = \frac{1}{2}\lambda\hat{x}\hat{t}, && \mathbf{X_3}\hat{t} = \frac{1}{2}\lambda\hat{t}^2, && \mathbf{X_3}\hat{u} = \lambda\left(-\frac{1}{4}\hat{x}-\frac{1}{2}\hat{u}\hat{t}\right), \\ &\mathbf{X_4}\hat{x} = b_4^5, && \mathbf{X_4}\hat{t} = 0, && \mathbf{X_4}\hat{u} = 0, \\ &\mathbf{X_5}\hat{x} = \frac{1}{4}\lambda b_4^5\hat{t}, && \mathbf{X_5}\hat{t} = 0, && \mathbf{X_5}\hat{u} = -\frac{1}{8}\lambda b_4^5, \end{aligned}$$

where $b_4^5 \neq 0$ and $\lambda = \pm 1$.

Upon substituting the corresponding values of symmetry generators yields the following system of first order non-linear partial differential equations for B_1

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}-\frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{x}=0, \tag{4.69}$$

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}-\frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{t}=\lambda, \tag{4.70}$$

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}-\frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{u}=0, \tag{4.71}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{x}=-\frac{1}{2}\hat{x}, \tag{4.72}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{t}=-\hat{t}, \tag{4.73}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{u}=\frac{1}{2}\hat{u}, \tag{4.74}$$

$$\frac{\partial\hat{x}}{\partial t}=\frac{1}{2}\lambda\hat{x}\hat{t}, \tag{4.75}$$

$$\frac{\partial\hat{t}}{\partial t}=\frac{1}{2}\lambda\hat{t}^2, \tag{4.76}$$

$$\frac{\partial\hat{u}}{\partial t}=\lambda\left(-\frac{1}{4}\hat{x}-\frac{1}{2}\hat{u}\hat{t}\right), \tag{4.77}$$

$$\left(-2t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)\hat{x}=b_4^5, \tag{4.78}$$

$$\left(-2t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)\hat{t}=0, \tag{4.79}$$

$$\left(-2t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)\hat{u}=0, \tag{4.80}$$

$$\frac{\partial\hat{x}}{\partial x}=\frac{1}{4}\lambda b_4^5\hat{t}, \tag{4.81}$$

$$\frac{\partial\hat{t}}{\partial x}=0, \tag{4.82}$$

$$\frac{\partial\hat{u}}{\partial x}=-\frac{1}{8}\lambda b_4^5. \tag{4.83}$$

Likewise, using Eq. (4.64) in Eq. (4.67), we have

$$\begin{bmatrix} \mathbf{X_1}\hat{x} & \mathbf{X_1}\hat{t} & \mathbf{X_1}\hat{u} \\ \mathbf{X_2}\hat{x} & \mathbf{X_2}\hat{t} & \mathbf{X_2}\hat{u} \\ \mathbf{X_3}\hat{x} & \mathbf{X_3}\hat{t} & \mathbf{X_3}\hat{u} \\ \mathbf{X_4}\hat{x} & \mathbf{X_4}\hat{t} & \mathbf{X_4}\hat{u} \\ \mathbf{X_5}\hat{x} & \mathbf{X_5}\hat{t} & \mathbf{X_5}\hat{u} \end{bmatrix} = \begin{bmatrix} b_1^1 & 0 & 0 & b_1^4 & 0 \\ b_1^1 & 1 & 0 & \frac{1}{2}b_1^4 b_3^2 & -\frac{2b_1^4}{b_1^1} \\ \frac{1}{2}b_1^1(b_3^2)^2 & b_3^2 & \frac{1}{b_1^1} & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & -\frac{1}{4}\xi b_3^2 & \frac{\xi}{b_1^1} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\hat{x}\hat{t} & \frac{1}{2}\hat{t}^2 & -\frac{1}{4}\left(\hat{x}+2\hat{u}\hat{t}\right) \\ \frac{1}{2}\hat{x} & \hat{t} & -\frac{1}{2}\hat{u} \\ 0 & 1 & 0 \\ -2\hat{t} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} \frac{1}{2}b_1^1\hat{x}\hat{t} - 2b_1^4\hat{t} & \frac{1}{2}b_1^1\hat{t}^2 & -\frac{1}{4}\left(\hat{x}+2\hat{u}\hat{t}\right)b_1^1 + b_1^4 \\ \frac{1}{2}b_1^1 b_3^2\hat{x}\hat{t} + \frac{1}{2}\hat{x} - b_1^4 b_3^2\hat{t} - \frac{2b_1^4}{b_1^1} & \frac{1}{2}b_1^1 b_2^3\hat{t}^2 + \hat{t} & -\frac{1}{4}\left(\hat{x}+2\hat{u}\hat{t}\right)b_1^1 b_3^2 - \frac{1}{2}\hat{u} + \frac{1}{2}b_1^4 b_3^2 \\ \frac{1}{4}b_1^1(b_3^2)^2\hat{x}\hat{t} + \frac{1}{2}b_3^2\hat{x} & \frac{1}{4}b_1^1(b_3^2)^2\hat{t}^2 + b_3^2\hat{t} + \frac{1}{b_1^1} & -\frac{1}{8}b_1^1(b_3^2)^2\left(\hat{x}+2\hat{u}\hat{t}\right) - \frac{1}{2}b_3^2\hat{u} \\ -2\xi\hat{t} & 0 & \xi \\ \frac{1}{2}\xi b_3^2\hat{t} + \frac{\xi}{b_1^1} & 0 & -\frac{1}{4}\xi b_3^2 \end{bmatrix}.$$

By equating, we get that

$$\mathbf{X_1}\hat{x} = \frac{1}{2}b_1^1\hat{x}\hat{t} - 2b_1^4\hat{t}, \qquad \mathbf{X_1}\hat{t} = \frac{1}{2}b_1^1\hat{t}^2,$$

$$\mathbf{X_1}\hat{u} = -\frac{1}{4}\left(\hat{x}+2\hat{u}\hat{t}\right)b_1^1 + b_1^4,$$

$$\mathbf{X_2}\hat{x} = \frac{1}{2}b_1^1 b_3^2\hat{x}\hat{t} + \frac{1}{2}\hat{x} - b_1^4 b_3^2\hat{t} - \frac{2b_1^4}{b_1^1}, \qquad \mathbf{X_2}\hat{t} = \frac{1}{2}b_1^1 b_2^3\hat{t}^2 + \hat{t},$$

$$\mathbf{X_2}\hat{u} = -\frac{1}{4}\left(\hat{x}+2\hat{u}\hat{t}\right)b_1^1 b_3^2 - \frac{1}{2}\hat{u} + \frac{1}{2}b_1^4 b_3^2,$$

$$\mathbf{X_3}\hat{x} = \frac{1}{4}b_1^1(b_3^2)^2\hat{x}\hat{t} + \frac{1}{2}b_3^2\hat{x}, \qquad \mathbf{X_3}\hat{t} = \frac{1}{4}b_1^1(b_3^2)^2\hat{t}^2 + b_3^2\hat{t} + \frac{1}{b_1^1},$$

$$\mathbf{X_3}\hat{u} = -\frac{1}{8}b_1^1(b_3^2)^2\left(\hat{x}+2\hat{u}\hat{t}\right) - \frac{1}{2}b_3^2\hat{u},$$

$$\mathbf{X_4}\hat{x} = -2\xi\hat{t}, \qquad \mathbf{X_4}\hat{t} = 0,$$

$$\mathbf{X_4}\hat{u} = \xi,$$

$$\mathbf{X_5}\hat{x} = \frac{1}{2}\xi b_3^2\hat{t} + \frac{\xi}{b_1^1}, \qquad \mathbf{X_5}\hat{t} = 0,$$

$$\mathbf{X_5}\hat{u} = -\frac{1}{4}\xi b_3^2,$$

where $b_1^1, b_3^2 \neq 0$ and $\xi = \pm 1$.

Consequently, for $B2$ we have the following system of non-linear partial differential equations

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{x} = \frac{1}{2}b_1^1\hat{x}\hat{t} - 2b_1^4\hat{t}, \tag{4.84}$$

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{t} = \frac{1}{2}b_1^1\hat{t}^2, \tag{4.85}$$

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{u} = -\frac{1}{4}\left(\hat{x} + 2\hat{u}\hat{t}\right)b_1^1 + b_1^4, \tag{4.86}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{x} = \frac{1}{2}b_1^1b_3^2\hat{x}\hat{t} + \frac{1}{2}\hat{x} - b_1^4b_3^2\hat{t} - \frac{2b_1^4}{b_1^1}, \tag{4.87}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{t} = \frac{1}{2}b_1^1b_2^3\hat{t}^2 + \hat{t}, \tag{4.88}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{u} = -\frac{1}{4}\left(\hat{x} + 2\hat{u}\hat{t}\right)b_1^1b_3^2 - \frac{1}{2}\hat{u} + \frac{1}{2}b_1^4b_3^2, \tag{4.89}$$

$$\frac{\partial\hat{x}}{\partial t} = \frac{1}{4}b_1^1(b_3^2)^2\hat{x}\hat{t} + \frac{1}{2}b_3^2\hat{x}, \tag{4.90}$$

$$\frac{\partial\hat{t}}{\partial t} = \frac{1}{4}b_1^1(b_3^2)^2\hat{t}^2 + b_3^2\hat{t} + \frac{1}{b_1^1}, \tag{4.91}$$

$$\frac{\partial\hat{u}}{\partial t} = -\frac{1}{8}b_1^1(b_3^2)^2\left(\hat{x} + 2\hat{u}\hat{t}\right) - \frac{1}{2}b_3^2\hat{u}, \tag{4.92}$$

$$\left(-2t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)\hat{x} = -2\xi\hat{t}, \tag{4.93}$$

$$\left(-2t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)\hat{t} = 0, \tag{4.94}$$

$$\left(-2t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right)\hat{u} = \xi, \tag{4.95}$$

$$\frac{\partial\hat{x}}{\partial x} = \frac{1}{2}\xi b_3^2\hat{t} + \frac{\xi}{b_1^1}, \tag{4.96}$$

$$\frac{\partial\hat{t}}{\partial x} = 0, \tag{4.97}$$

$$\frac{\partial\hat{u}}{\partial x} = -\frac{1}{4}\xi b_3^2. \tag{4.98}$$

Moreover, for $B3$ using Eq. (4.65) in Eq. (4.67) implies

$$\begin{bmatrix} \mathbf{X_1}\hat{x} & \mathbf{X_1}\hat{t} & \mathbf{X_1}\hat{u} \\ \mathbf{X_2}\hat{x} & \mathbf{X_2}\hat{t} & \mathbf{X_2}\hat{u} \\ \mathbf{X_3}\hat{x} & \mathbf{X_3}\hat{t} & \mathbf{X_3}\hat{u} \\ \mathbf{X_4}\hat{x} & \mathbf{X_4}\hat{t} & \mathbf{X_4}\hat{u} \\ \mathbf{X_5}\hat{x} & \mathbf{X_5}\hat{t} & \mathbf{X_5}\hat{u} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\frac{(b_1^2)^2}{\sigma} & b_1^2 & \sigma & 0 & 0 \\ 0 & 1 & \frac{2\sigma}{b_1^2} & 0 & 0 \\ 0 & 0 & \frac{2\sigma}{(b_1^2)^2} & 0 & 0 \\ 0 & 0 & 0 & b_4^4 & -\frac{4\sigma b_4^4}{b_1^2} \\ 0 & 0 & 0 & 0 & \frac{2\sigma b_4^4}{(b_1^2)^2} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\hat{x}\hat{t} & \frac{1}{2}\hat{t}^2 & -\frac{1}{4}\left(\hat{x}+2\hat{u}\hat{t}\right) \\ \frac{1}{2}\hat{x} & \hat{t} & -\frac{1}{2}\hat{u} \\ 0 & 1 & 0 \\ -2\hat{t} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$= \begin{bmatrix} \frac{1}{4}\frac{(b_1^2)^2}{\sigma}\hat{x}\hat{t} + \frac{1}{2}b_1^2\hat{x} & \frac{1}{4}\frac{(b_1^2)^2}{\sigma}\hat{t}^2 + b_1^2\hat{t} + \sigma & -\frac{1}{8}\frac{(b_1^2)^2}{\sigma}\left(\hat{x}+2\hat{u}\hat{t}\right) - \frac{1}{2}b_1^2\hat{u} \\ \frac{1}{2}\hat{x} & \hat{t} + \frac{2\sigma}{b_1^2} & -\frac{1}{2}\hat{u} \\ 0 & \frac{2\sigma}{(b_1^2)^2} & 0 \\ -2b_4^4\hat{t} - \frac{4\sigma b_4^4}{b_1^2} & 0 & b_4^4 \\ \frac{2\sigma b_4^4}{(b_1^2)^2} & 0 & 0 \end{bmatrix}.$$

By equating the corresponding elements yields

$$\begin{aligned} &\mathbf{X_1}\hat{x} = \frac{1}{4}\frac{(b_1^2)^2}{\sigma}\hat{x}\hat{t} + \frac{1}{2}b_1^2\hat{x}, && \mathbf{X_1}\hat{t} = \frac{1}{4}\frac{(b_1^2)^2}{\sigma}\hat{t}^2 + b_1^2\hat{t} + \sigma, && \mathbf{X_1}\hat{u} = -\frac{1}{8}\frac{(b_1^2)^2}{\sigma}\left(\hat{x}+2\hat{u}\hat{t}\right) - \frac{1}{2}b_1^2\hat{u}, \\ &\mathbf{X_2}\hat{x} = \frac{1}{2}\hat{x}, && \mathbf{X_2}\hat{t} = \hat{t} + \frac{2\sigma}{b_1^2}, && \mathbf{X_2}\hat{u} = -\frac{1}{2}\hat{u}, \\ &\mathbf{X_3}\hat{x} = 0, && \mathbf{X_3}\hat{t} = \frac{2\sigma}{(b_1^2)^2}, && \mathbf{X_3}\hat{u} = 0, \\ &\mathbf{X_4}\hat{x} = -2b_4^4\hat{t} - \frac{4\sigma b_4^4}{b_1^2}, && \mathbf{X_4}\hat{t} = 0, && \mathbf{X_4}\hat{u} = b_4^4, \\ &\mathbf{X_5}\hat{x} = \frac{2\sigma b_4^4}{(b_1^2)^2}, && \mathbf{X_5}\hat{t} = 0, && \mathbf{X_5}\hat{u} = 0, \end{aligned}$$

where $b_1^2, b_4^4 \neq 0$ and $\sigma = \pm 1$.

Thus, for $B3$ we have the following system of non-linear partial differential equations

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}-\frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{x}=\frac{1}{4}\frac{(b_1^2)^2}{\sigma}\hat{x}\hat{t}+\frac{1}{2}b_1^2\hat{x}, \tag{4.99}$$

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}-\frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{t}=\frac{1}{4}\frac{(b_1^2)^2}{\sigma}\hat{t}^2+b_1^2\hat{t}+\sigma, \tag{4.100}$$

$$\left(\frac{1}{2}xt\frac{\partial}{\partial x}+\frac{1}{2}t^2\frac{\partial}{\partial t}-\frac{1}{4}(x+2ut)\frac{\partial}{\partial u}\right)\hat{u}=-\frac{1}{8}\frac{(b_1^2)^2}{\sigma}\left(\hat{x}+2\hat{u}\hat{t}\right)-\frac{1}{2}b_1^2\hat{u}, \tag{4.101}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{x}=\frac{1}{2}\hat{x}, \tag{4.102}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{t}=\hat{t}+\frac{2\sigma}{b_1^2}, \tag{4.103}$$

$$\left(\frac{1}{2}x\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}-\frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{u}=-\frac{1}{2}\hat{u}, \tag{4.104}$$

$$\frac{\partial\hat{x}}{\partial t}=0, \tag{4.105}$$

$$\frac{\partial\hat{t}}{\partial t}=\frac{2\sigma}{(b_1^2)^2}, \tag{4.106}$$

$$\frac{\partial\hat{u}}{\partial t}=0, \tag{4.107}$$

$$\left(-2t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)\hat{x}=-2b_4^4\hat{t}-\frac{4\sigma b_4^4}{b_1^2}, \tag{4.108}$$

$$\left(-2t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)\hat{t}=0, \tag{4.109}$$

$$\left(-2t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}\right)\hat{u}=b_4^4, \tag{4.110}$$

$$\frac{\partial\hat{x}}{\partial x}=\frac{2\sigma b_4^4}{(b_1^2)^2}, \tag{4.111}$$

$$\frac{\partial\hat{t}}{\partial x}=0, \tag{4.112}$$

$$\frac{\partial\hat{u}}{\partial x}=0. \tag{4.113}$$

4.2.5 Solution of System of Determining Equations for $B1$

To solve the system of first order non-linear PDEs for $B1$, we consider Eq. (4.82)

$$\begin{aligned}\frac{\partial \hat{t}}{\partial x} &= 0,\\ \hat{t} &= A(t,u). \end{aligned} \tag{4.114}$$

Taking Eq. (4.80) and using Eq. (4.82), we have

$$\begin{aligned}\frac{\partial A(t,u)}{\partial u} &= 0,\\ A(t,u) &= A(t).\end{aligned}$$

Using Eq. (4.114) implies

$$\hat{t} = A(t). \tag{4.115}$$

Now taking Eq. (4.70) and using Eq. (4.82), we have

$$\frac{1}{2}t^2\frac{\partial \hat{t}}{\partial t} - \frac{1}{4}(x+2ut)\frac{\partial \hat{t}}{\partial u} = \lambda. \tag{4.116}$$

Since,

$$\frac{\partial \hat{t}}{\partial \hat{u}} = 0. \tag{4.117}$$

So, Eq. (4.116) can be written as

$$\begin{aligned}\frac{1}{2}t^2\frac{\partial \hat{t}}{\partial t} &= \lambda,\\ \frac{\partial A(t)}{\partial t} &= \frac{2}{t^2}\lambda,\\ A(t) &= -\frac{2}{t}\lambda + c_1,\\ \hat{t} &= -\frac{2}{t}\lambda + c_1.\end{aligned} \tag{4.118}$$

Using Eqs. (4.82) and (4.117) in Eq. (4.73) yields

$$t\frac{\partial \hat{t}}{\partial t} = -\hat{t}.$$

Substituting Eq. (4.118) implies

$$c_1 = 0. \tag{4.119}$$

Therefore, Eq. (4.118) reduces to

$$\hat{t} = -\frac{2}{t}\lambda. \tag{4.120}$$

For $\hat{x}$, we consider Eq. (4.81)

$$\frac{\partial \hat{x}}{\partial x} = \frac{1}{4}\lambda b_4^5 \hat{t}.$$

Subsituting Eq. (4.120)

$$\begin{aligned}
\frac{\partial \hat{x}}{\partial x} &= \frac{1}{4}\lambda b_4^5(-\frac{2}{t}\lambda), \\
\frac{\partial \hat{x}}{\partial x} &= -\frac{\lambda^2}{2t}b_4^5, \\
\hat{x} &= -\frac{\lambda^2}{2t}b_4^5 x + B(t,u).
\end{aligned} \tag{4.121}$$

Taking Eq. (4.78) and substituting Eq. (4.121), we have

$$\begin{aligned}
-2t\left(-\frac{\lambda^2}{2t}b_4^5\right) + \frac{\partial B(t,u)}{\partial u} &= b_4^5, \\
\lambda^2 b_4^5 + \frac{\partial B(t,u)}{\partial u} &= b_4^5, \\
\frac{\partial B(t,u)}{\partial u} &= b_4^5 - \lambda^2 b_4^5, \\
B(t,u) &= b_4^5 u - \lambda^2 b_4^5 u + B(t).
\end{aligned} \tag{4.122}$$

So, Eq. (4.121) implies

$$\hat{x} = -\frac{\lambda^2}{2t}b_4^5 x + b_4^5 u - \lambda^2 b_4^5 u + B(t). \tag{4.123}$$

Now using Eqs. (4.120) and (4.123) in Eq. (4.75)

$$\frac{\lambda^2}{2t^2}b_4^5 x + \frac{dB(t)}{dt} = -\frac{\lambda^2}{t}\hat{x},$$
$$\hat{x} = -\frac{1}{2t}b_4^5 x - \frac{t}{\lambda^2}\frac{dB(t)}{dt}. \tag{4.124}$$

By comparing Eqs. (4.123) and (4.124), we have

$$-\frac{\lambda^2}{2t}b_4^5 x + b_4^5 u - \lambda^2 b_4^5 u + B(t) = -\frac{1}{2t}b_4^5 x - \frac{t}{\lambda^2}\frac{dB(t)}{dt},$$
$$\frac{t}{\lambda^2}\frac{dB(t)}{dt} + B(t) = \frac{\lambda^2}{2t}b_4^5 x - \frac{1}{2t}b_4^5 x + \lambda^2 b_4^5 u - b_4^5 u. \tag{4.125}$$

Upon solving Eq. (4.125) in *Maple* and using Eq. (4.72), yields the following value of $\hat{x}$

$$\hat{x} = -\frac{1}{2t}b_4^5 x. \tag{4.126}$$

Similarly, for $\hat{u}$, consider Eq. (4.83)

$$\frac{\partial \hat{u}}{\partial t} = -\frac{1}{8}\lambda b_4^5,$$
$$\hat{u} = -\frac{1}{8}\lambda b_4^5 x + C(t,u). \tag{4.127}$$

Taking Eq. (4.80) and using Eq. (4.127)

$$-2t\left(-\frac{\lambda}{8}b_4^5\right) + \frac{\partial C(t,u)}{\partial u} = 0,$$
$$\frac{\partial C(t,u)}{\partial u} = -\frac{\lambda}{4}b_4^5 t,$$
$$C(t,u) = -\frac{\lambda}{4}b_4^5 tu + C(t). \tag{4.128}$$

So, Eq. (4.127) implies

$$\hat{u} = -\frac{1}{8}\lambda b_4^5 x - \frac{\lambda}{4}b_4^5 tu + C(t). \tag{4.129}$$

Now taking Eq. (4.77) and using Eq. (4.129)

$$
\begin{aligned}
-\frac{1}{4}\lambda b_4^5 u + \frac{dC(t)}{dt} &= -\frac{\lambda}{4}\left(-\frac{1}{2t}b_4^5 x\right) - \frac{1}{2}\left(-\frac{1}{8}\lambda b_4^5 x - \frac{1}{4}\lambda b_4^5 ut + C(t)\right)\left(\frac{-2\lambda}{t}\right), \\
\frac{dC(t)}{dt} - \frac{\lambda}{t}C(t) &= \frac{1}{8t}\lambda b_4^5 x + \frac{1}{8t}\lambda^2 b_4^5 x + \frac{t}{4}\lambda^2 b_4^5 u + \frac{1}{8t}\lambda b_4^5 x + \frac{1}{4}\lambda b_4^5 u, \\
\frac{dC(t)}{dt} - \frac{\lambda}{t}C(t) &= \frac{\lambda}{8t}b_4^5\left(\lambda+1\right)x + \frac{\lambda}{4}b_4^5\left(\lambda t+1\right)u. \qquad (4.130)
\end{aligned}
$$

Solving the non-homogenous and non-linear Eq. (4.130) in *Maple* and using Eq. (4.74) yields, $C(t) = 0$. Therefore, the value of $\hat{u}$ simplifies to

$$
\hat{u} = -\frac{1}{8}\lambda b_4^5 x - \frac{1}{4}\lambda b_4^5 ut. \qquad (4.131)
$$

Thus, the general solution for $B1$ is

$$
\left(\hat{x}, \hat{t}, \hat{u}\right) = \left(-\frac{1}{2t}b_4^5 x, -\frac{2}{t}\lambda, -\frac{1}{8}\lambda b_4^5 x - \frac{1}{4}\lambda b_4^5 ut\right). \qquad (4.132)
$$

4.2.6 Solution of System of Determining Equations for $B2$

In order to solve the system of non-linear PDEs, we consider Eq. (4.97)

$$
\begin{aligned}
\frac{\partial \hat{t}}{\partial x} &= 0, \\
\hat{t} &= A(t,u). \qquad (4.133)
\end{aligned}
$$

Taking Eq. (4.94) and using Eq. (4.97)

$$
\begin{aligned}
\frac{\partial A(t,u)}{\partial u} &= 0, \qquad (4.134) \\
A(t,u) &= A(t).
\end{aligned}
$$

So, Eq. (4.133) implies

$$
\hat{t} = A(t). \qquad (4.135)
$$

Now taking Eq. (4.85) and using Eq. (4.97)

$$\frac{1}{2}t^2\frac{\partial \hat{t}}{\partial t} - \frac{1}{4}(x+2ut)\frac{\partial \hat{t}}{\partial u} = \frac{1}{2}b_1^1\hat{t}^2. \tag{4.136}$$

Since,

$$\frac{\partial \hat{t}}{\partial u} = 0, \tag{4.137}$$

then Eq. (4.136) can be written as

$$\begin{aligned} \frac{1}{2}t^2\frac{\partial \hat{t}}{\partial t} &= \frac{1}{2}b_1^1\hat{t}^2, \\ \hat{t}^{-2}\frac{\partial \hat{t}}{\partial t} &= b_1^1 t^{-2}, \\ -\frac{1}{\hat{t}} &= -\frac{b_1^1}{t} + c_1, \\ \hat{t} &= \frac{t}{b_1^1 - c_1 t}. \end{aligned} \tag{4.138}$$

Now using Eqs. (4.97) and (4.137)-(4.138) in Eq. (4.88), one can obtain

$$\frac{b_1^1}{(b_1^1 - c_1 t)^2} = \frac{1}{2}b_1^1 b_3^2\left(\frac{t^2}{(b_1^1 - c_1 t)^2}\right) + \frac{t}{b_1^1 - c_1 t}. \tag{4.139}$$

Multiplying $(b_1^1 - c_1 t)^2$ on both sides

$$\begin{aligned} b_1^1 t &= \frac{1}{2}b_1^1 b_3^2 t^2 + t\left(b_1^1 - c_1 t\right), \\ b_1^1 t &= \frac{1}{2}b_1^1 b_3^2 t^2 + b_1^1 t - c_1 t^2, \\ c_1 &= \frac{1}{2}b_1^1 b_3^2. \end{aligned} \tag{4.140}$$

Therefore, Eq. (4.138) implies

$$\begin{aligned}\hat{t} &= \frac{t}{b_1^1 - \left(\frac{1}{2}b_1^1 b_3^2\right)t},\\ &= \frac{t}{\frac{2b_1^1 - b_1^1 b_3^2 t}{2}},\\ &= \frac{2t}{2b_1^1 - b_1^1 b_3^2 t}.\end{aligned}$$

Thus, we have

$$\hat{t} = -\frac{2t}{b_1^1\left(b_3^2 t - 2\right)}. \tag{4.141}$$

Now for $\hat{x}$, we consider Eq. (4.96)

$$\frac{\partial \hat{x}}{\partial x} = \frac{1}{2}\xi b_3^2 \hat{t} + \frac{\xi}{b_1^1}. \tag{4.142}$$

Since,

$$\hat{t} = -\frac{2t}{b_1^1\left(b_3^2 t - 2\right)}.$$

So, Eq. (4.142) is given by

$$\begin{aligned}\frac{\partial \hat{x}}{\partial x} &= \frac{1}{2}\xi b_3^2\left(-\frac{2t}{b_1^1\left(b_3^2 t - 2\right)}\right) + \frac{\xi}{b_1^1},\\ \frac{\partial \hat{x}}{\partial x} &= -\frac{\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)}t + \frac{\xi}{b_1^1},\\ \hat{x} &= -\frac{\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)}tx + \frac{\xi}{b_1^1}x + B(t,u). \qquad (4.143)\end{aligned}$$

Now using the value of $\hat{t}$ and Eq. (4.143) in Eq. (4.93), we have

$$\begin{aligned}-2t\left(-\frac{\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)}t\right) + \frac{\partial B(t,u)}{\partial u} &= -2\xi\left(-\frac{2t}{b_1^1\left(b_3^2 t - 2\right)}\right),\\ \frac{\partial B(t,u)}{\partial u} &= \frac{4\xi}{b_1^1\left(b_3^2 t - 2\right)}t - \frac{2\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)}t^2,\\ B(t,u) &= \frac{4\xi}{b_1^1\left(b_3^2 t - 2\right)}tu - \frac{2\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)}t^2 u + B(t).\end{aligned}$$

Therefore, Eq. (4.143) implies

$$\hat{x} = -\frac{\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} tx + \frac{\xi}{b_1^1} x + \frac{4\xi}{b_1^1\left(b_3^2 t - 2\right)} tu - \frac{2\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} t^2 u + B(t). \tag{4.144}$$

Since,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(-\frac{\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} tx\right) &= \frac{-\xi b_3^2 b_1^1\left(b_3^2 t - 2\right)x + \xi b_1^1 (b_3^2)^2 tx}{\left[b_1^1\left(b_3^2 t - 2\right)\right]^2}, \\
\frac{\partial}{\partial t}\left(\frac{\xi}{b_1^1} x\right) &= 0, \\
\frac{\partial}{\partial t}\left(\frac{4\xi}{b_1^1\left(b_3^2 t - 2\right)} tu\right) &= \frac{4\xi b_1^1\left(b_3^2 t - 2\right)u - 4\xi b_1^1 b_3^2 tu}{\left[b_1^1\left(b_3^2 t - 2\right)\right]^2}, \\
\frac{\partial}{\partial t}\left(-\frac{2\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} t^2 u\right) &= \frac{-4\xi b_3^2 b_1^1\left(b_3^2 t - 2\right)u + 2\xi b_1^1 (b_3^2)^2 t^2 u}{\left[b_1^1\left(b_3^2 t - 2\right)\right]^2}.
\end{aligned}$$

So, Eq. (4.90) takes the form

$$\begin{aligned}
&\frac{-\xi b_3^2 b_1^1\left(b_3^2 t - 2\right)x + \xi b_1^1 (b_3^2)^2 tx}{\left[b_1^1\left(b_3^2 t - 2\right)\right]^2} + \frac{4\xi b_1^1\left(b_3^2 t - 2\right)u - 4\xi b_1^1 b_3^2 tu}{\left[b_1^1\left(b_3^2 t - 2\right)\right]^2} \\
&\quad + \frac{-4\xi b_3^2 b_1^1\left(b_3^2 t - 2\right)u + 2\xi b_1^1 (b_3^2)^2 t^2 u}{\left[b_1^1\left(b_3^2 t - 2\right)\right]^2} + \frac{dB(t)}{dt} = \\
&\frac{1}{4} b_1^1 (b_3^2)^2 \left(-\frac{\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} tx + \frac{\xi}{b_1^1} x + \frac{4\xi}{b_1^1\left(b_3^2 t - 2\right)} tu - \frac{2\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} t^2 u + B(t)\right) \\
&\quad \left(-\frac{2t}{b_1^1\left(b_3^2 t - 2\right)}\right) + \frac{1}{2} b_3^2 \left(-\frac{\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} tx + \frac{\xi}{b_1^1} x + \frac{4\xi}{b_1^1\left(b_3^2 t - 2\right)} tu - \frac{2\xi b_3^2}{b_1^1\left(b_3^2 t - 2\right)} t^2 u + B(t)\right).
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
&\frac{dB(t)}{dt} - \frac{1}{4}b_1^1(b_3^2)^2\left(-\frac{2t}{b_1^1\,(b_3^2t-2)}\right)B(t) - \frac{1}{2}b_3^2B(t) = \\
&\quad \frac{\xi b_3^2 b_1^1\,(b_3^2t-2)x - \xi b_1^1(b_3^2)^2tx}{\left[b_1^1\,(b_3^2t-2)\right]^2} - \frac{4\xi b_1^1\,(b_3^2t-2)u - 4\xi b_1^1 b_3^2tu}{\left[b_1^1\,(b_3^2t-2)\right]^2} \\
&+ \frac{4\xi b_3^2 b_1^1\,(b_3^2t-2)u - 2\xi b_1^1(b_3^2)^2t^2u}{\left[b_1^1\,(b_3^2t-2)\right]^2} + \frac{1}{4}b_1^1(b_3^2)^2\left(-\frac{\xi b_3^2}{b_1^1\,(b_3^2t-2)}tx + \frac{\xi}{b_1^1}x\right)\left(-\frac{2t}{b_1^1\,(b_3^2t-2)}\right) \\
&+ \frac{1}{4}b_1^1(b_3^2)^2\left(\frac{4\xi}{b_1^1\,(b_3^2t-2)}tu - \frac{2\xi b_3^2}{b_1^1\,(b_3^2t-2)}t^2u\right)\left(-\frac{2t}{b_1^1\,(b_3^2t-2)}\right) + \frac{1}{2}b_3^2\left(-\frac{\xi b_3^2}{b_1^1\,(b_3^2t-2)}tx\right) \\
&+ \frac{1}{2}b_3^2\left(\frac{\xi}{b_1^1}x + \frac{4\xi}{b_1^1\,(b_3^2t-2)}tu - \frac{2\xi b_3^2}{b_1^1\,(b_3^2t-2)}t^2u\right). \qquad (4.145)
\end{aligned}
$$

Upon solving the non-homogenous and non-linear Eq. (4.145) in *Maple* and using Eq. (4.84), we get the value of $\hat{x}$

$$\hat{x} = -\frac{2\,(4b_1^4 + \xi x)}{b_1^1\,(b_3^2t-2)}. \qquad (4.146)$$

Now for $\hat{u}$, consider Eq. (4.98)

$$
\begin{aligned}
\frac{\partial \hat{u}}{\partial x} &= -\frac{1}{4}\xi b_3^2, \\
\hat{u} &= -\frac{1}{4}\xi b_3^2 x + B(t,u). \qquad (4.147)
\end{aligned}
$$

Taking Eq. (4.95) and using Eq. (4.147)

$$
\begin{aligned}
\frac{\partial B(t,u)}{\partial u} &= \xi - \frac{1}{2}b_3^2\xi t, \\
B(t,u) &= \xi u - \frac{1}{2}b_3^2\xi tu + B(t). \qquad (4.148)
\end{aligned}
$$

So, Eq. (4.147) can be written as

$$\hat{u} = -\frac{1}{4}\xi b_3^2 x + \xi u - \frac{1}{2}b_3^2\xi tu + B(t). \qquad (4.149)$$

Using Eq. (4.149) in Eq. (4.86) yields

$$\frac{1}{2}xt\left(-\frac{1}{4}b_3^2\xi\right)+\frac{1}{2}t^2\left(-\frac{1}{2}b_3^2\xi u+\frac{dB(t)}{dt}\right)-\frac{1}{4}(x+2ut)\left(\xi-\frac{1}{2}b_3^2\xi t\right)=$$
$$-\frac{1}{4}\left[-\frac{2\left(4b_1^4+\xi x\right)}{b_1^1\left(b_3^2t-2\right)}+2\left(-\frac{1}{4}\xi b_3^2x+\xi u-\frac{1}{2}b_3^2\xi tu+B(t)\right)\left(-\frac{2t}{b_1^1\left(b_3^2t-2\right)}\right)\right]b_1^1$$
$$+b_1^4.$$

Thus, we have

$$\frac{1}{2}t^2\frac{dB(t)}{dt}+\frac{t}{\left(b_3^2t-2\right)}B(t)=\frac{1}{8}b_3^2\xi xt+\frac{1}{4}b_3^2\xi ut^2+\frac{1}{4}(x+2ut)\left(\xi-\frac{1}{2}b_3^2\xi t\right)$$
$$-\frac{1}{4}\left[-\frac{2\left(4b_1^4+\xi x\right)}{b_1^1\left(b_3^2t-2\right)}+2\left(-\frac{1}{4}\xi b_3^2x+\xi u-\frac{1}{2}b_3^2\xi tu\right)\left(-\frac{2t}{b_1^1\left(b_3^2t-2\right)}\right)\right]b_1^1$$
$$+b_1^4. \qquad (4.150)$$

By solving the non-linear and non-homogenous Eq. (4.150) in *Maple* and using Eq. (4.89), one can obtain the value of $\hat{u}$, that is,

$$\hat{u}=-\frac{1}{2}\xi b_3^2tu-b_1^4b_3^2-\frac{1}{4}\xi b_3^2x+\xi u. \qquad (4.151)$$

So, $B2$ has a general solution of the form

$$\left(\hat{x},\hat{t},\hat{u}\right)=\left(-\frac{2\left(4b_1^4+\xi x\right)}{b_1^1\left(b_3^2t-2\right)},-\frac{2t}{b_1^1\left(b_3^2t-2\right)},-\frac{1}{2}\xi b_3^2tu-b_1^4b_3^2-\frac{1}{4}\xi b_3^2x+\xi u\right). \qquad (4.152)$$

4.2.7 Solution of System of Determining Equations for $B3$

Now to solve the system of non-linear partial differential equations for $B3$, we consider Eq. (4.113)

$$\frac{\partial\hat{u}}{\partial x}=0,$$
$$\hat{u}=A(t,u). \qquad (4.153)$$

Taking Eq. (4.107)

$$\frac{\partial \hat{u}}{\partial t} = 0,$$
$$\frac{\partial A(t,u)}{\partial t} = 0,$$
$$A(t,u) = A(u).$$

Using Eq. (4.153) implies

$$\hat{u} = A(u). \tag{4.154}$$

Now taking Eq. (4.110) and using Eq. (4.113)

$$\frac{\partial \hat{u}}{\partial u} = b_4^4,$$
$$\frac{\partial A(u)}{\partial u} = b_4^4,$$
$$A(u) = b_4^4 u + c_1,$$
$$\hat{u} = b_4^4 u + c_1. \tag{4.155}$$

Using Eqs. (4.107) and (4.113) in Eq. (4.104), we have

$$u\frac{\partial \hat{u}}{\partial u} = \hat{u}. \tag{4.156}$$

Since,

$$\frac{\partial \hat{u}}{\partial u} = b_4^4.$$

So, Eq. (4.156) implies

$$c_1 = 0.$$

Therefore, Eq. (4.155) can be written as

$$\hat{u} = b_4^4 u.$$

Similarly, for $\hat{t}$, taking Eq. (4.112)

$$\begin{aligned}\frac{\partial \hat{t}}{\partial x} &= 0,\\ \hat{t} &= B(t,u).\end{aligned} \tag{4.157}$$

Using Eq. (4.112) in Eq. (4.109) implies

$$B(t,u) = B(t).$$

So, Eq. (4.157) is given by

$$\hat{t} = B(t). \tag{4.158}$$

Now substituting Eq. (4.158) in Eq. (4.106)

$$\begin{aligned}\frac{\partial B(t)}{\partial t} &= \frac{2\sigma}{(b_1^2)^2},\\ B(t) &= \frac{2\sigma}{(b_1^2)^2}t + c_2,\\ \hat{t} &= \frac{2\sigma}{(b_1^2)^2}t + c_2.\end{aligned} \tag{4.159}$$

By using Eq. (4.112) in Eq. (4.103), one obtains

$$\left(t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{t} = \hat{t} + \frac{2\sigma}{b_1^2}. \tag{4.160}$$

Since,

$$\frac{\partial \hat{t}}{\partial u} = 0.$$

Hence, Eq. (4.160) yields

$$t\frac{\partial \hat{t}}{\partial t} = \hat{t} + \frac{2\sigma}{b_1^2}. \tag{4.161}$$

By substituting Eq. (4.159) in Eq. (4.161), we have

$$\begin{aligned}\frac{2\sigma}{(b_1^2)^2}t &= \frac{2\sigma}{(b_1^2)^2}t + c_2 + \frac{2\sigma}{b_1^2},\\ c_2 &= -\frac{2\sigma}{b_1^2}.\end{aligned} \tag{4.162}$$

Therefore, Eq. (4.159) implies

$$\hat{t} = \frac{2\sigma}{(b_1^2)^2}t - \frac{2\sigma}{b_1^2}.$$

Finally, for $\hat{x}$, taking Eq. (4.105)

$$\begin{aligned} \frac{\partial \hat{x}}{\partial t} &= 0, \\ \hat{x} &= C(x,u). \end{aligned} \tag{4.163}$$

Using Eq. (4.163) in Eq. (4.111), we have

$$\begin{aligned} \frac{\partial C(x,u)}{\partial x} &= \frac{2\sigma b_4^4}{(b_1^2)^2}, \\ C(x,u) &= \frac{2\sigma b_4^4}{(b_1^2)^2}x + C(u), \\ \hat{x} &= \frac{2\sigma b_4^4}{(b_1^2)^2}x + C(u). \end{aligned} \tag{4.164}$$

Substituting Eq. (4.164) in Eq. (4.108)

$$\begin{aligned} -2t\left(\frac{2\sigma b_4^4}{(b_1^2)^2}\right) + \frac{dC(u)}{du} &= -2b_4^4\hat{t} - \frac{4\sigma b_4^4}{b_1^2}, \\ -\frac{4\sigma b_4^4}{(b_1^2)^2}t + \frac{dC(u)}{du} &= -2b_4^4\left(\frac{2\sigma}{(b_1^2)^2}t - \frac{2\sigma}{b_1^2}\right) - \frac{4\sigma b_4^4}{b_1^2}, \\ -\frac{4\sigma b_4^4}{(b_1^2)^2}t + \frac{dC(u)}{du} &= -\frac{4\sigma b_4^4}{(b_1^2)^2}t + \frac{4\sigma b_4^4}{(b_1^2)^2} - \frac{4\sigma b_4^4}{(b_1^2)^2}, \\ \frac{dC(u)}{du} &= 0, \\ C(u) &= c_3. \end{aligned}$$

So, Eq. (4.164) takes the form

$$\hat{x} = \frac{2\sigma b_4^4}{(b_1^2)^2}x + c_3. \tag{4.165}$$

Now using Eq. (4.105) in Eq. (4.102), we have

$$\left(\frac{1}{2}x\frac{\partial}{\partial x} - \frac{1}{2}u\frac{\partial}{\partial u}\right)\hat{x} = \frac{1}{2}\hat{x}. \tag{4.166}$$

Since,

$$\frac{\partial \hat{x}}{\partial u} = 0.$$

Consequently, by substituting Eq. (4.165) in Eq. (4.166) yields

$$c_3 = 0.$$

Therefore, Eq. (4.165) takes the form

$$\hat{x} = \frac{2\sigma b_4^4}{(b_1^2)^2}x.$$

Hence, for $B3$ the corresponding general solution is

$$(\hat{x}, \hat{t}, \hat{u}) = \left(\frac{2\sigma b_4^4}{(b_1^2)^2}x, \frac{2\sigma}{(b_1^2)^2}t - \frac{2\sigma}{b_1^2}, b_4^4 u\right). \tag{4.167}$$

4.2.8 Analysis of Symmetry Condition

Since, the corresponding general solutions of determining equations are given by

$$B1: \quad (\hat{x}, \hat{t}, \hat{u}) = \left(-\frac{1}{2t}b_4^5 x, -\frac{2\lambda}{t}, -\frac{1}{8}\lambda b_4^5 x - \frac{1}{4}\lambda b_4^5 ut\right), \tag{4.168}$$

$$B2: \quad (\hat{x}, \hat{t}, \hat{u}) = \left(-\frac{2\left(4b_1^4 + \xi x\right)}{b_1^1\left(b_3^2 t - 2\right)}, \frac{-2t}{b_1^1\left(b_3^2 t - 2\right)}, -\frac{1}{2}\xi b_3^2 tu - b_1^4 b_3^2 - \frac{1}{4}\xi b_3^2 x + \xi u\right), \tag{4.169}$$

$$B3: \quad (\hat{x}, \hat{t}, \hat{u}) = \left(\frac{2\sigma b_4^4}{(b_1^2)^2}x, \frac{2\sigma}{(b_1^2)^2}t - \frac{2\sigma}{b_1^2}, b_4^4 u\right) \tag{4.170}$$

Now in the following sections we discuss the corresponding symmetry condition for each general solution of the determininng equation.

Symmetry Condition for $B1$

For Eq. (4.168) to be symmetry of Eq. (4.1) if and only if, we have

$$\hat{u}_{\hat{t}} = \hat{u}_{\hat{x}\hat{x}} + 2\hat{u}\hat{u}_{\hat{x}},$$

that is,

$$\hat{u}_{\hat{x}\hat{x}} + 2\hat{u}\hat{u}_{\hat{x}} - \hat{u}_{\hat{t}} = 0, \tag{4.171}$$

when

$$u_{xx} + 2uu_x - u_t = 0.$$

Since, we have

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial \hat{t}} &= \frac{\partial\left(-\frac{1}{8}\lambda b_4^5 x - \frac{1}{4}\lambda b_4^5 ut\right)}{\partial\left(-\frac{2\lambda}{t}\right)}, \\
&= \frac{-\frac{1}{8}\lambda b_4^5 \frac{\partial x}{\partial t} - \frac{1}{4}\lambda b_4^5 ut\left(t\frac{\partial u}{\partial t} + u\frac{\partial t}{\partial t}\right)}{\frac{2\lambda}{t^2}}, \\
&= \frac{-\lambda b_4^5 t^2\left(tu_t + u\right)}{8\lambda}.
\end{aligned} \tag{4.172}$$

Now

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial \hat{x}} &= \frac{\partial\left(-\frac{1}{8}\lambda b_4^5 x - \frac{1}{4}\lambda b_4^5 ut\right)}{\partial\left(-\frac{1}{2t} b_4^5 x\right)}, \\
&= \frac{-\frac{1}{8}\lambda b_4^5 \frac{\partial x}{\partial x} - \frac{1}{4}\lambda b_4^5 t\frac{\partial u}{\partial x}}{-\frac{b_4^5}{2t}\frac{\partial x}{\partial x}}, \\
&= \frac{\frac{1}{8}\lambda b_4^5 + \frac{1}{4}\lambda b_4^5 t u_x}{\frac{b_4^5}{2t}\frac{\partial x}{\partial x}}, \\
&= \frac{\frac{1}{8}\lambda + \frac{1}{4}\lambda t u_x}{\frac{1}{2t}\frac{\partial x}{\partial x}}, \\
&= \frac{2t\left(\lambda + 2\lambda t u_x\right)}{8}.
\end{aligned} \tag{4.173}$$

Also,

$$\frac{\partial}{\partial \hat{x}}\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right) = \frac{\partial\left(\frac{\lambda t+2\lambda t^2 u_x}{4}\right)}{\partial\left(-\frac{1}{2t}b_4^5 x\right)},$$
$$= \frac{\frac{1}{4}\left(\lambda \frac{\partial t}{\partial x} + 2\lambda t^2 \frac{\partial u_x}{\partial x}\right)}{-\frac{1}{2t}b_4^5 \frac{\partial x}{\partial x}},$$
$$= \frac{\frac{1}{2}\lambda t^2 u_{xx}}{-\frac{1}{2t}b_4^5}. \tag{4.174}$$

Using Eqs. (4.172)-(4.174) in Eq. (4.171) implies

$$-\frac{\lambda t^3 u_{xx}}{b_4^5} + 2\left(-\frac{1}{8}\lambda b_4^5 x - \frac{1}{4}\lambda b_4^5 ut\right)\left(\frac{\lambda t + 2\lambda t^2 u_x}{4}\right) = \frac{-b_4^5 t^2\left(tu_t + u\right)}{8}.$$

Re-arrangment of the equation leads to

$$-\frac{\lambda t^3 u_{xx}}{b_4^5} + \frac{1}{2}\left(-\frac{1}{8}\lambda^2 b_4^5 xt - \frac{1}{4}\lambda^2 b_4^5 xt^2 u_x - \frac{1}{4}\lambda^2 b_4^5 t^2 u - \frac{1}{2}\lambda^2 b_4^5 t^3 uu_x\right) = \frac{-b_4^5 t^2\left(tu_t + u\right)}{8},$$
$$-\frac{\lambda t^3 u_{xx}}{b_4^5} - -\frac{1}{16}\lambda^2 b_4^5 xt - \frac{1}{8}\lambda^2 b_4^5 xt^2 u_x - \frac{1}{8}\lambda^2 b_4^5 t^2 u - \frac{1}{4}\lambda^2 b_4^5 t^3 uu_x = \frac{-b_4^5 t^2\left(tu_t + u\right)}{8}.$$

Multiplying $-\frac{b_4^5}{\lambda t^3}$ on both sides

$$u_{xx} + \frac{1}{16}\lambda\left(b_4^5\right)^2 \frac{x}{t^2} + \frac{1}{8}\lambda\left(b_4^5\right)^2 \frac{x}{t}u_x + \frac{1}{8}\lambda\left(b_4^5\right)^2 \frac{u}{t} + \frac{1}{4}\lambda\left(b_4^5\right)^2 uu_x = \frac{\left(b_4^5\right)^2 tu_t}{8\lambda t} + \frac{\left(b_4^5\right)^2 u}{8\lambda t}.$$

Upon simplification, one obtains that if we choose $\lambda = 1$ and $b_4^5 = 2\sqrt{2}\gamma$, where $\gamma = \pm 1$, then we have

$$u_{xx} + 2uu_x - u_t = 0.$$

Consequently, for $B1$

$$\hat{x} = -\frac{1}{2}b_4^5 x,$$
$$= -\frac{1}{2t}\left(2\sqrt{2}\gamma\right)x,$$
$$= -\frac{\gamma\sqrt{2}}{t}x. \tag{4.175}$$

Likewise,

$$\hat{t} = -\frac{2}{t}, \tag{4.176}$$

and

$$\hat{u} = -\frac{\gamma\sqrt{2}}{4}\left(x + 2ut\right). \tag{4.177}$$

Therefore, the first discrete symmetry of the Burgers' equation (4.1) up to equivalence is given by

$$\zeta_1 : (\hat{x}, \hat{t}, \hat{u}) \longrightarrow \left(-\frac{\gamma\sqrt{2}}{t}x, -\frac{2}{t}, -\frac{\gamma\sqrt{2}}{4}(x + 2ut)\right). \tag{4.178}$$

Symmetry Condition for $B2$

Now for Eq. (4.169) to be symmetry of Eq. (4.1), we find

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial \hat{t}} &= \frac{\partial\left(-\frac{1}{2}\xi b_3^2 tu - b_1^4 b_3^2 - \frac{1}{4}\xi b_3^2 x + \xi u\right)}{\partial\left(\frac{-2t}{b_1^1\left(b_3^2 t - 2\right)}\right)}, \\
&= \frac{-\frac{1}{2}\xi b_3^2\left(t\frac{\partial u}{\partial t} + u\frac{\partial t}{\partial t}\right) - \frac{1}{4}b_3^2\xi\frac{\partial x}{\partial t} + \xi\frac{\partial u}{\partial t}}{\frac{-2tb_1^1\frac{\partial}{\partial t}\left(b_3^2 t - 2\right) - b_1^1\left(b_3^2 t - 2\right)\frac{\partial}{\partial t}(-2t)}{\left[b_1^1\left(b_3^2 t - 2\right)\right]^2}}, \\
&= \frac{\left(-\frac{1}{2}\xi b_3^2\left(tu_t + u\right) + \xi u_t\right)\left[b_1^1\left(b_3^2 t - 2\right)\right]^2}{4b_1^1}.
\end{aligned} \tag{4.179}$$

Similarly,

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial \hat{x}} &= \frac{\partial\left(-\frac{1}{2}\xi b_3^2 tu - b_1^4 b_3^2 - \frac{1}{4}\xi b_3^2 x + \xi u\right)}{\partial\left(-\frac{2\left(4b_1^4 + \xi x\right)}{b_1^1\left(b_3^2 t - 2\right)}\right)}, \\
&= \frac{-\frac{1}{2}\xi b_3^2 t\frac{\partial u}{\partial x} - \frac{1}{4}b_3^2\xi\frac{\partial x}{\partial x} + \xi\frac{\partial u}{\partial x}}{\frac{-2\xi\frac{\partial x}{\partial x}}{b_1^1\left(b_3^2 t - 2\right)}}, \\
&= \frac{-\frac{1}{2}b_3^2 tu_x - \frac{1}{4}b_3^2 + u_x}{\frac{-2}{b_1^1\left(b_3^2 t - 2\right)}}, \\
&= \frac{b_1^1\left(b_3^2 t - 2\right)\left(-\frac{1}{2}b_3^2 tu_x - \frac{1}{4}b_3^2 + u_x\right)}{-2}.
\end{aligned} \tag{4.180}$$

Now

$$\frac{\partial}{\partial \hat{x}}\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right) = \frac{\frac{1}{4}\partial\left(b_1^1(b_3^2)^2t^2u_x - 2b_1^1b_3^2tu_x + \frac{1}{2}b_1^1(b_3^2)^2t - b_1^1b_3^2 - 2b_1^1b_3^2tu_x + 4b_1^1u_x\right)}{\partial\left(-\frac{2\left(4b_1^4+\xi x\right)}{b_1^1\left(b_3^2t-2\right)}\right)},$$

$$= \frac{\frac{1}{4}\left[b_1^1(b_3^2)^2t^2u_{xx} - 2b_1^1b_3^2tu_{xx} + \frac{1}{2}b_1^1(b_3^2)^2t - b_1^1b_3^2 - 2b_1^1b_3^2tu_{xx} + 4b_1^1u_{xx}\right]}{\frac{-2\xi}{b_1^1\left(b_3^2t-2\right)}},$$

$$= \frac{\frac{1}{4}b_1^1(b_3^2)^2t^2u_{xx} - b_1^1b_3^2tu_{xx} + b_1^1u_{xx}}{\frac{-2\xi}{b_1^1\left(b_3^2t-2\right)}}. \tag{4.181}$$

Substituting Eqs. (4.179)-(4.181) in Eq. (4.171)

$$\frac{\frac{1}{4}b_1^1(b_3^2)^2t^2u_{xx} - b_1^1b_3^2tu_{xx} + b_1^1u_{xx}}{\frac{-2\xi}{b_1^1\left(b_3^2t-2\right)}} + 2\left(-\frac{1}{2}\xi b_3^2tu - b_1^4b_3^2 - \frac{1}{4}\xi b_3^2x + \xi u\right)$$
$$\left(\frac{b_1^1(b_3^2)^2t^2u_x - 2b_1^1b_3^2tu_x + \frac{1}{2}b_1^1(b_3^2)^2t - b_1^1b_3^2 - 2b_1^1b_3^2tu_x + 4b_1^1u_x}{4}\right) =$$
$$\frac{\left(-\frac{1}{2}\xi b_3^2\left(tu_t+u\right) + \xi u_t\right)\left[b_1^1\left(b_3^2t-2\right)\right]^2}{4b_1^1}. \tag{4.182}$$

One can conclude that by setting $b_3^2 = 0$, $b_1^1 = 1$ and $b_1^4 = \mathcal{J}$, where $\mathcal{J}$ is any arbitrary constant, then Eq. (4.182) yields

$$\frac{u_{xx}}{\xi} + 2\xi uu_x = \xi u_t,$$
$$u_{xx} + 2\xi^2uu_x = \xi^2u_t. \tag{4.183}$$

Recall that

$$\xi = \pm 1.$$

So,

$$\xi^2 = 1.$$

Therefore, Eq. (4.183) implies

$$u_{xx} + 2uu_x - u_t = 0.$$

Since, for $B2$

$$\begin{aligned}\hat{x} &= -\frac{2\left(4b_1^4 + \xi x\right)}{b_1^1\left(b_3^2 t - 2\right)},\\ &= \frac{-2\left(4\mathcal{J} + \xi x\right)}{-2},\\ &= 4\mathcal{J} + \xi x.\end{aligned} \tag{4.184}$$

Similarly,

$$\begin{aligned}\hat{t} &= \frac{-2t}{b_1^1\left(b_3^2 t - 2\right)},\\ &= \frac{-2t}{-2},\\ &= t,\end{aligned} \tag{4.185}$$

and

$$\begin{aligned}\hat{u} &= -\frac{1}{2}\xi b_3^2 t u - b_1^4 b_3^2 - \frac{1}{4}\xi b_3^2 x + \xi u,\\ &= \xi u.\end{aligned} \tag{4.186}$$

Hence, the second discrete symmetry of the Burgers' equation (3.1) up to equivalence is given by

$$\zeta_2 : (\hat{x}, \hat{t}, \hat{u}) \longrightarrow \left(4\mathcal{J} + \xi x, t, \xi u\right). \tag{4.187}$$

Symmetry Condition for $B3$

In order to apply the symmetry condition on Eq. (4.170), we find

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{t}} &= \frac{\partial\left(b_4^4 u\right)}{\partial\left(\frac{2\sigma}{(b_1^2)^2}t - \frac{2\sigma}{b_1^2}\right)},\\ &= \frac{b_4^4 \frac{\partial u}{\partial t}}{\frac{2\sigma}{(b_1^2)^2}\frac{\partial t}{\partial t}},\\ &= \frac{b_4^4 u_t}{\frac{2\sigma}{(b_1^2)^2}},\\ &= \frac{(b_1^2)^2 b_4^4 u_t}{2\sigma}.\end{aligned} \tag{4.188}$$

Equivalently,

$$\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{\partial\left(b_4^4 u\right)}{\partial\left(\frac{2\sigma b_4^4}{(b_1^2)^2}x\right)},$$

$$= \frac{b_4^4 \frac{\partial u}{\partial x}}{\frac{2\sigma b_4^4}{(b_1^2)^2}\frac{\partial x}{\partial x}},$$

$$= \frac{(b_1^2)^2 b_4^4}{2\sigma b_4^4}, \tag{4.189}$$

and

$$\frac{\partial}{\partial \hat{x}}\left(\frac{\partial \hat{u}}{\partial \hat{x}}\right) = \frac{\partial\left(\frac{(b_1^2)^2 b_4^4}{2\sigma b_4^4}u_x\right)}{\partial\left(\frac{2\sigma b_4^4}{(b_1^2)^2}x\right)},$$

$$= \frac{\frac{(b_1^2)^2 b_4^4 u_{xx}}{2\sigma b_4^4}}{\frac{2\sigma b_4^4}{(b_1^2)^2}}, \tag{4.190}$$

Now using Eqs. (4.188)-(4.190) in Eq. (4.171) yields

$$u_{xx} + \frac{4\sigma {b_4^4}^2}{(b_1^2)^2}u_x = \frac{2\sigma(b_4^4)^2}{(b_1^2)^2}u_t. \tag{4.191}$$

Choosing $\sigma = 1$, $b_1^2 = \sqrt{2}\mathcal{A}\mathcal{K}$, and $b_4^4 = \mathcal{A}$, where $\mathcal{K} = \pm 1$, and $\mathcal{A}$ is any arbitrary constant. Then we have

$$u_{xx} + \frac{4\sigma \mathcal{A}^2}{(\sqrt{2}\mathcal{A}\mathcal{K})^2}u_x = \frac{2\sigma \mathcal{A}^2}{(\sqrt{2}\mathcal{A}\mathcal{K})^2}u_t,$$

which satisfies the symmetry condition.

Since, for $B3$

$$\hat{x} = \frac{2\sigma b_4^4}{(b_1^2)^2}x,$$

$$= \frac{2(1)\mathcal{A}}{(\sqrt{2}\mathcal{A}\mathcal{K})^2}x,$$

$$= \frac{1}{\mathcal{A}}x. \tag{4.192}$$

Likewise,

$$\begin{aligned}\hat{t} &= \frac{2\sigma}{(b_1^2)^2}t - \frac{2\sigma}{b_1^2},\\ &= \frac{t-\sqrt{2}\mathcal{AK}}{(\sqrt{2}\mathcal{AK})^2},\\ &= \frac{t-\sqrt{2}\mathcal{AK}}{\mathcal{A}^2},\end{aligned} \tag{4.193}$$

with $\mathcal{K}^2 = 1$, and

$$\hat{u} = \mathcal{A}u. \tag{4.194}$$

Therefore, the third discrete symmetry of the Burgers' equation (3.1) up to equivalence is

$$\zeta_3 : (\hat{x}, \hat{t}, \hat{u}) \longrightarrow \left(\frac{1}{\mathcal{A}}x, \frac{t-\sqrt{2}\mathcal{AK}}{\mathcal{A}^2}, \mathcal{A}u\right), \quad \mathcal{A} \neq 0. \tag{4.195}$$

As every discrete symmetry satisfied its respective system of partial differential equations and left the system invariant. Therefore, we conclude that up to equivalence and invariance the following three are the actual discrete symmetries of the Burgers' equation (3.1)

$$\begin{aligned}\zeta_1 &: (\hat{x}, \hat{t}, \hat{u}) \longrightarrow \left(-\frac{\gamma\sqrt{2}}{t}x, -\frac{2}{t}, -\frac{\gamma\sqrt{2}}{4}(x+2ut)\right),\\ \zeta_2 &: (\hat{x}, \hat{t}, \hat{u}) \longrightarrow (4\mathcal{J} + \xi x, t, \xi u),\\ \zeta_3 &: (\hat{x}, \hat{t}, \hat{u}) \longrightarrow \left(\frac{1}{\mathcal{A}}x, \frac{t-\sqrt{2}\mathcal{AK}}{\mathcal{A}^2}, \mathcal{A}u\right),\end{aligned}$$

where $\xi, \mathcal{K} = \pm 1$ and $\mathcal{J}, \mathcal{A}$ are any arbitrary constants.

These results are extremely exhaustive. Therefore, computer algebra is recommended.

Chapter 5

Construction of an Invariant Numerical Scheme for Burgers' Equation using Discrete Symmetry Groups

Crank-Nicolson method was first developed by John Crank and Phyllis Nicolson in the mid-20th century used to approximate diffusion heat equation and other PDEs.

As an immediate application of discrete symmetries, in this chapter an innovative approach of Crank-Nicolson method which is known as Modified-Crank-Nicolson method is introduced to approximate the exact solution of the Burgers' equation. This modification is conducted through the composition of continuous and discrete symmetries and substituting the resultant in the variable u of the CNM to yield an invariant numerical scheme (M-CNM). Furthermore, the linearization framework of the Burgers' equation is carried out by using the Hopf-Cole transformation together with the initial and boundary conditions. In addition to this, the analytical solution of the Burgers' equation by using the Fourier series has been laid out.

The numerical schemes for explicit finite difference schemes (FTCS) and standard Crank-Nicolson method (CNM) for the heat equation are discussed in the appendix.

5.1 Lie Point Symmetry Transformations of Burgers' Equation

In the previous chapter we find Lie point (continuous) symmetries of Burgers' equation. In this section, we find the transformation of Lie point symmetries.

Now taking

$$\mathbf{X_1} = \frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \frac{1}{4}(x+2ut)\frac{\partial}{\partial u}.$$

Here

$$\begin{aligned}
\xi_1(x,t,u) &= \frac{1}{2}xt, \\
\tau_1(x,t,u) &= \frac{1}{2}t^2, \\
\eta_1(x,t,u) &= -\frac{1}{4}x - \frac{1}{2}ut.
\end{aligned}$$

So, by using the definition of Lie point symmetry transformation Eq. (1.8), we have

$$\begin{aligned}
\left.\frac{\partial \hat{x}}{\partial \epsilon}\right|_{\epsilon=0} &= \xi_1(x,t,u), \\
\ln(\hat{x})^2 &= t\epsilon + c_1.
\end{aligned}$$

Using condition, $\hat{x}(0) = x$

$$\begin{aligned}
\ln(\hat{x})^2 - \ln(x)^2 &= t\epsilon, \\
\hat{x} &= xe^{\frac{t\epsilon}{2}}, \\
&= x\left(1 + \frac{t}{2}\epsilon\right).
\end{aligned}$$

Now

$$\begin{aligned}
\left.\frac{\partial \hat{t}}{\partial \epsilon}\right|_{\epsilon=0} &= \tau_1(x,t,u), \\
-\frac{2}{\hat{t}} &= \epsilon + c_2.
\end{aligned}$$

Using condition, $\hat{t}(0) = t$

$$\begin{aligned}
-\frac{2}{\hat{t}} &= \frac{\epsilon t - 2}{t}, \\
\hat{t} &= \frac{2t}{2 - \epsilon t}.
\end{aligned}$$

Likewise,

$$\left.\frac{\partial \hat{u}}{\partial \epsilon}\right|_{\epsilon=0} = \eta_1 (x,t,u),$$
$$\hat{u} = -\frac{1}{4}x\epsilon - \frac{1}{2}ut\epsilon + c_3.$$

Using conditon, $\hat{u}(0) = u$, yields

$$\hat{u} = \frac{1}{4}(4u - x\epsilon - 2ut\epsilon).$$

Thus, the corresponding symmetry transformation G_1 for the symmetry generator $\mathbf{X_1}$ is

$$G_1 : \qquad \left(\hat{x}, \hat{t}, \hat{u}\right) = \left(x + \frac{tx}{2}\epsilon, \frac{2t}{2-\epsilon t}, \frac{1}{4}(4u - x\epsilon - 2ut\epsilon)\right). \tag{5.1}$$

Using the same framework one obtains the Lie point symmetry transformations for the corresponding symmetry generators of the Burgers' equation (3.1) as given in Table 5.1 are

Symmetry Generators	Symmetry Transformations
$\frac{1}{2}xt\frac{\partial}{\partial x} + \frac{1}{2}t^2\frac{\partial}{\partial t} - \frac{1}{4}(x+2ut)\frac{\partial}{\partial u}$	$\left(x + \frac{tx}{2}\epsilon, \frac{2t}{2-\epsilon t}, \frac{1}{4}(4u - x\epsilon - 2ut\epsilon)\right)$
$\frac{1}{2}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u}$	$\left(xe^{\frac{\epsilon}{2}}, te^{\epsilon}, ue^{-\frac{\epsilon}{2}}\right)$
$\frac{\partial}{\partial t}$	$(x, t+\epsilon, u)$
$-2t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$	$(x - 2t\epsilon, t, u+\epsilon)$
$\frac{\partial}{\partial x}$	$(x+\epsilon, t, u)$

Table 5.1: Continuous Symmetry Transformations of Burgers' equation (3.1)

5.2 Invariantization of Crank-Nicolson Method

In this section, we show that Crank-Nicolson method is invariant under the discrete symmetry transformation.

5.2.1 Invariantization of Crank-Nicolson Method using Discrete Symmetry Group ζ_1

As the first discrete symmetry group ζ_1 of the Burgers' equation is

$$\zeta_1 : \quad (\hat{x}, \hat{t}, \hat{u}) \longrightarrow \left(-\frac{\gamma\sqrt{2}}{t} x, -\frac{2}{t}, -\frac{\gamma\sqrt{2}}{4}(x + 2ut) \right). \tag{5.2}$$

Consider that $u_{n,j} = u\left(n\Delta x, j\Delta t\right)$ with $x_n = n\Delta x$ and $t_j = j\Delta t$ be the approximate value of $u(x,t)$ at the mesh points (x_n, t_j), then by using the discrete symmetry group ζ_1 of the Burgers' equation, we have the following transformation

$$u = -\frac{\gamma\sqrt{2}}{4}(x + 2ut). \tag{5.3}$$

So, by substituting Eq. (5.3) into the Crank-Nicolson formulae for approximating the linear parabolic equations, we obtain

$$\begin{aligned}
\alpha &\left[-\frac{\gamma\sqrt{2}}{4}\left(x_{n-1} + 2u_{n-1,j}t_j\right) \right] + 2(1-\alpha)\left[-\frac{\gamma\sqrt{2}}{4}\left(x_n + 2u_{n,j}t_j\right) \right] \\
&+ \alpha\left[-\frac{\gamma\sqrt{2}}{4}\left(x_{n+1} + 2u_{n+1,j}t_j\right) \right] = -\alpha\left[-\frac{\gamma\sqrt{2}}{4}\left(x_{n-1} + 2u_{n-1,j+1}t_{j+1}\right) \right] \\
&+ 2(1+\alpha)\left[-\frac{\gamma\sqrt{2}}{4}\left(x_n + 2u_{n,j+1}t_{j+1}\right) \right] - \alpha\left[-\frac{\gamma\sqrt{2}}{4}\left(x_{n+1} + 2u_{n+1,j+1}t_{j+1}\right) \right],
\end{aligned}$$

which upon simplification reduces to

$$\begin{aligned}
&\alpha\left(x_{n-1} + 2u_{n-1,j}t_j\right) + 2(1-\alpha)\left(x_n + 2u_{n,j}t_j\right) + \alpha\left(x_{n+1} + 2u_{n+1,j}t_j\right) \\
&= -\alpha\left(x_{n-1} + 2u_{n-1,j+1}t_{j+1}\right) + 2(1+\alpha)\left(x_n + 2u_{n,j+1}t_{j+1}\right) - \alpha\left(x_{n+1} + 2u_{n+1,j+1}t_{j+1}\right).
\end{aligned}$$

Now considering

$$J = t_j, \quad \text{and} \quad N = x_n.$$

The transformation Eq. (5.3) takes the form

$$u_{N,J} = -\frac{\gamma\sqrt{2}}{4}\left(N + 2u_{N,J}J\right).$$

Consequently, we have

$$\begin{aligned}&\alpha\left[(N-1)+2u_{n-1,j}J\right]+2(1-\alpha)\left[N+2u_{n,j}J\right]+\alpha\left[(N+1)+2u_{n+1,j}J\right]\\&=-\alpha\left[(N-1)+2u_{n-1,j+1}\left(J+1\right)\right]+2(1+\alpha)\left[N+2u_{n,j+1}\left(J+1\right)\right]\\&\quad-\alpha\left[(N+1)+2u_{n+1,j+1}\left(J+1\right)\right],\end{aligned}$$

which deduces to

$$\alpha u_{N-1,J}+2(1-\alpha)u_{N,J}+\alpha u_{N+1,J}=-\alpha u_{N-1,J+1}+2(1+\alpha)u_{N,J+1}+2u_{N+1,J+1},$$

which is again the same Crank-Nicolson method for the parabolic equations. Hence, the Crank-Nicolson method under the transformation of a discrete symmetry group ζ_1 is invariant. So, with the same procedure one can also show that CNM is invariant under the transformation of the discrete symmetry groups ζ_2 and ζ_3, respectively.

5.3 Construction of an Invariant Numerical Scheme

In this section, we construct an invariant numerical scheme for the Crank-Nicolson method by taking the composition of continuous and discrete symmetry groups converging to the exact solution of the Burgers' equation and giving the most appropriate results as compared to any other finite difference scheme. This construction is purely based on the variable u of these two groups.

It is to be noted that ϵ is a continuous parameter and for the better performance of the numerical scheme, we are opting ϵ to be a very small number.

5.3.1 Construction of an Invariant Numerical Scheme using Discrete Symmetry Group ζ_1

Burgers' equation has a projective symmetry group as

$$\left(\hat{x},\hat{t},\hat{u}\right)=\left(x+\frac{tx}{2}\epsilon,\frac{2t}{2-\epsilon t},\frac{1}{4}\left(4u-x\epsilon-2ut\epsilon\right)\right),\tag{5.4}$$

with

$$\mathcal{S}_1=\frac{1}{4}\left(4u-x\epsilon-2ut\epsilon\right).\tag{5.5}$$

Furthermore, the variable u of the discrete symmetry group ζ_1 is

$$\zeta_1 = -\frac{\sqrt{2}\gamma}{4}(x + 2ut). \tag{5.6}$$

Now the composition of projective group $\mathcal{S}_1$ and discrete symmetry group ζ_1 is

$$\begin{aligned}
\mathcal{S}_1 \circ \zeta_1 &= -\frac{\sqrt{2}\gamma}{4}(x + 2ut), \\
&= -\frac{\sqrt{2}\gamma}{4}\left[x + 2\left(\frac{1}{4}\left(4u - \tilde{x}\epsilon - 2u\tilde{t}\epsilon\right)\right)t\right], \\
&= -\frac{\sqrt{2}\gamma}{4}\left(x + 2ut - \frac{1}{2}\tilde{x}t\epsilon - u\tilde{t}t\epsilon\right),
\end{aligned}$$

where $\tilde{x}$ and $\tilde{t}$ is written for our convienence to differentiate between the continuous and discrete symmetry variables. So, substituting the corresponding $\tilde{x} = -\frac{\sqrt{2}\gamma x}{t}$ and $\tilde{t} = -\frac{2}{t}$ values of the discrete symmetry group ζ_2, we obtain

$$\mathcal{S}_1 \circ \zeta_1 = -\frac{1}{4}\sqrt{2}\gamma\left(2\epsilon + 2t\right)u - \frac{1}{4}\sqrt{2}\gamma\left(\frac{1}{2}\sqrt{2}\gamma x\epsilon + x\right),$$

which is further simplified to obtain the transformation u of the form

$$u = -\frac{1}{4}\sqrt{2}\gamma\left[\left(2\epsilon + 2t\right)u + \left(\frac{\gamma\epsilon}{\sqrt{2}} + 1\right)x\right]. \tag{5.7}$$

Now re-writing the above transformation of variable u as an approximate value of $u(x,t)$ at the grid points (x_n, t_j) as

$$u_{n,j} = -\frac{1}{4}\sqrt{2}\gamma\left[\left(2\epsilon + 2t_j\right)u_{n,j} + \left(\frac{\gamma\epsilon}{\sqrt{2}} + 1\right)x_n\right]. \tag{5.8}$$

So, using the variable u in the Crank-Nicolson method with the above transformation implies

$$\begin{aligned}
&-\frac{\alpha}{4}\sqrt{2}\gamma\left[\left(2\epsilon + 2t_j\right)u_{n-1,j} + \chi_1\right] - \frac{2\sqrt{2}(1-\alpha)\gamma}{4}\left[\left(2\epsilon + 2t_j\right)u_{n,j} + \chi_2\right] \\
&-\frac{\alpha}{4}\sqrt{2}\gamma\left[\left(2\epsilon + 2t_j\right)u_{n+1,j} + \chi_3\right] = \frac{\alpha}{4}\sqrt{2}\gamma\left[\left(2\epsilon + 2t_{j+1}\right)u_{n-1,j+1} + \chi_1\right] \\
&-\frac{2\sqrt{2}(1+\alpha)\gamma}{4}\left[\left(2\epsilon + 2t_{j+1}\right)u_{n,j+1} + \chi_2\right] + \frac{\alpha}{4}\sqrt{2}\gamma\left[\left(2\epsilon + 2t_{j+1}\right)u_{n+1,j+1} + \chi_3\right],
\end{aligned}$$

where

$$\chi_1 = \left(\frac{\gamma\epsilon}{\sqrt{2}} + 1\right) x_{n-1},$$
$$\chi_2 = \left(\frac{\gamma\epsilon}{\sqrt{2}} + 1\right) x_{n},$$
$$\chi_3 = \left(\frac{\gamma\epsilon}{\sqrt{2}} + 1\right) x_{n+1}.$$

which reduces to

$$\alpha\left[\left(2\epsilon+2t_j\right)u_{n-1,j} + \left(\frac{\gamma\epsilon}{\sqrt{2}}+1\right)x_{n-1}\right] + 2(1-\alpha)\left[\left(2\epsilon+2t_j\right)u_{n,j} + \left(\frac{\gamma\epsilon}{\sqrt{2}}+1\right)x_{n}\right]$$
$$+\alpha\left[\left(2\epsilon+2t_j\right)u_{n+1,j} + \left(\frac{\gamma\epsilon}{\sqrt{2}}+1\right)x_{n+1}\right] = -\alpha\left[\left(2\epsilon+2t_{j+1}\right)u_{n-1,j+1} + \left(\frac{\gamma\epsilon}{\sqrt{2}}+1\right)x_{n-1}\right]$$
$$+2(1+\alpha)\left[\left(2\epsilon+2t_{j+1}\right)u_{n,j+1} + \left(\frac{\gamma\epsilon}{\sqrt{2}}+1\right)x_{n}\right] - \alpha\left[\left(2\epsilon+2t_{j+1}\right)u_{n+1,j+1} + \left(\frac{\gamma\epsilon}{\sqrt{2}}+1\right)x_{n+1}\right],$$

which is the acquired Modified-Crank-Nicolson method (M-CNM) corresponding to first discrete symmetry group ζ_1.

5.3.2 Construction of an Invariant Numerical Scheme using Discrete Symmetry Groups ζ_2 and ζ_3

Similarly, the variable u of the discrete symmetry groups ζ_2 and ζ_3 are

$$\zeta_2 = \xi u, \quad \text{and} \quad \zeta_3 = \mathcal{A}u,$$

where the composition of projective and discrete symmetry groups ζ_2 and ζ_3 of variable u as an approximate value of $u(x,t)$ at the grid points (x_n, t_j) are

$$u_{n,j} = \frac{\xi}{2}\left(\left(2 - t_j\epsilon\right)u_{n,j} - \frac{\xi}{2}x_n\epsilon - 2\tau\epsilon\right), \tag{5.9}$$

and

$$u_{n,j} = \left(\frac{\sqrt{2}\mathcal{A}\mathcal{K}\epsilon + 2\mathcal{A}^2 - t_j\epsilon}{2\mathcal{A}}\right)u_{n,j} - \frac{1}{4}x_n\epsilon, \tag{5.10}$$

respectively. Now using the variable u in the Crank-Nicolson method with the transformation Eq. (5.9), we have

$$\frac{\alpha\xi}{2}\left((2-t_j\epsilon)\,u_{n-1,j}-\frac{\xi}{2}x_{n-1}\epsilon-2\tau\epsilon\right)+\frac{2(1-\alpha)\xi}{2}\left((2-t_j\epsilon)\,u_{n,j}-\frac{\xi}{2}x_n\epsilon-2\tau\epsilon\right)$$
$$+\frac{\alpha\xi}{2}\left((2-t_j\epsilon)\,u_{n+1,j}-\frac{\xi}{2}x_{n+1}\epsilon-2\tau\epsilon\right)=-\frac{\alpha\xi}{2}\left((2-t_{j+1}\epsilon)\,u_{n-1,j+1}-\frac{\xi}{2}x_{n-1}\epsilon-2\tau\epsilon\right)$$
$$+\frac{2(1+\alpha)\xi}{2}\left((2-t_{j+1}\epsilon)\,u_{n,j+1}-\frac{\xi}{2}x_n\epsilon-2\tau\epsilon\right)-\frac{\alpha\xi}{2}\left((2-t_{j+1}\epsilon)\,u_{n+1,j+1}-\frac{\xi}{2}x_{n+1}\epsilon-2\tau\epsilon\right),$$

which can be written as

$$\alpha\left((2-t_j\epsilon)\,u_{n-1,j}-\frac{\xi}{2}x_{n-1}\epsilon-2\tau\epsilon\right)+2(1-\alpha)\left((2-t_j\epsilon)\,u_{n,j}-\frac{\xi}{2}x_n\epsilon-2\tau\epsilon\right)$$
$$+\alpha\left((2-t_j\epsilon)\,u_{n+1,j}-\frac{\xi}{2}x_{n+1}\epsilon-2\tau\epsilon\right)=-\alpha\left((2-t_{j+1}\epsilon)\,u_{n-1,j+1}-\frac{\xi}{2}x_{n-1}\epsilon-2\tau\epsilon\right)$$
$$+2(1+\alpha)\left((2-t_{j+1}\epsilon)\,u_{n,j+1}-\frac{\xi}{2}x_n\epsilon-2\tau\epsilon\right)-\alpha\left((2-t_{j+1}\epsilon)\,u_{n+1,j+1}-\frac{\xi}{2}x_{n+1}\epsilon-2\tau\epsilon\right).$$

In similar way, one can obtain the invariant numerical scheme corresponding to Eq. (5.10) as

$$\alpha\left(\mathfrak{X}_1u_{n-1,j}-\frac{1}{4}x_{n-1}\epsilon\right)+2(1-\alpha)\left(\mathfrak{X}_1u_{n,j}-\frac{1}{4}x_n\epsilon\right)+\alpha\left(\mathfrak{X}_1u_{n+1,j}-\frac{1}{4}x_{n+1}\epsilon\right)$$
$$=-\alpha\left(\mathfrak{X}_2u_{n-1,j+1}-\frac{1}{4}x_{n-1}\epsilon\right)+2(1+\alpha)\left(\mathfrak{X}_2u_{n,j+1}-\frac{1}{4}x_n\epsilon\right)-\alpha\left(\mathfrak{X}_2u_{n+1,j+1}-\frac{1}{4}x_{n+1}\epsilon\right),$$

where

$$\mathfrak{X}_1=\frac{\sqrt{2}\mathcal{A}\mathcal{K}\epsilon+2\mathcal{A}^2-t_j\epsilon}{2\mathcal{A}},$$
$$\mathfrak{X}_2=\frac{\sqrt{2}\mathcal{A}\mathcal{K}\epsilon+2\mathcal{A}^2-t_{j+1}\epsilon}{2\mathcal{A}}.$$

The convergence and performance of these invariant numerical schemes will be discussed in the following chapter.

5.4 Transformation of 1-D Burgers' Equation to 1-D Heat Equation

The newly constructed invariant numerical scheme for the Crank-Nicolson method is only applicable to the linear PDEs. Since, the Burgers' equation is a non-linear PDE. So, for the eligibility of this method to solve the Burgers' equation, we will transform the 1-D Burgers' equation by using the Hopf-Cole transformation [5, 4] to a 1-D diffusion heat equation.

For this, let us consider a transformation

$$u(x,t) = \psi_x(x,t),$$

then we have

$$\begin{aligned} u_x &= \psi_{xx}, \\ u_{xx} &= \psi_{xxx}, \\ u_t &= \psi_{xt}. \end{aligned}$$

Thus, Burgers' Eq. (3.1) can be written as

$$2\psi_x\psi_{xx} + \psi_{xxx} = \psi_{xt}.$$

Integrating w.r.t x

$$\psi_t = 2\left[\psi_x \int \psi_{xx} dx - \int \left(\frac{d}{dx}(\psi_x) \int \psi_x dx\right) dx\right] + \psi_{xx}.$$

After some calculus, we get the following transformation of Burgers' equation

$$\psi_t = \psi_x^2 + \psi_{xx}. \tag{5.11}$$

Now again consider the transformation

$$\psi(x,t) = \ln \phi(x,t),$$

then we obtain

$$\psi_t = \frac{1}{\phi}\phi_t,$$
$$\psi_x = \frac{1}{\phi}\phi_x,$$
$$\psi_{xx} = \frac{\phi\phi_{xx} - \phi_x^2}{[\phi]^2}.$$

Therefore, Eq. (5.11) implies,

$$\frac{1}{\phi}\phi_t = \left(\frac{1}{\phi}\phi_x\right)^2 + \frac{\phi\phi_{xx} - \phi_x^2}{[\phi]^2},$$
$$\phi_t = \frac{1}{\phi}\phi_x^2 + \frac{\phi\phi_{xx} - \phi_x^2}{\phi},$$

transforms to 1-D diffusion heat equation

$$\phi_t = \phi_{xx}. \tag{5.12}$$

Now to transform the boundary conditions, we have the Hopf-Cole transformed exact solution of the Burgers' equation as

$$u = \frac{\phi_x}{\phi}, \tag{5.13}$$

that is,

$$u(0,t) = \frac{\phi_x}{\phi} = 0,$$
$$\phi_x(0,t) = 0. \tag{5.14}$$

and

$$u(1,t) = \frac{\phi_x}{\phi} = 0,$$
$$\phi_x(1,t) = 0. \tag{5.15}$$

Likewise, for the transformation of initial condition, we have

$$u(x,0) = \sin \pi x,$$
$$\frac{\phi_x}{\phi} = \sin \pi x,$$

deduces to

$$\phi(x,0) = \exp\left(\frac{1-\cos\pi x}{\pi}\right). \tag{5.16}$$

Hence, we have transformed the 1-D Burgers' equation

$$u_t = 2uu_x + u_{xx},$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= \sin\pi x, \qquad & 0 < x < 1, \\ u(0,t) &= u(1,t) = 0, \qquad & t > 0. \end{aligned}$$

to the 1-D heat equation

$$\phi_t = \phi_{xx}, \tag{5.17}$$

with non-homogenenous initial condition and insulated homogeneous boundary conditions

$$\phi(x,0) = \exp\left(\frac{1-\cos\pi x}{\pi}\right), \qquad 0 < x < 1, \tag{5.18}$$

$$\phi_x(0,t) = \phi_x(1,t) = 0, \qquad t > 0. \tag{5.19}$$

As the initial temperature is only a function of x and the end points are both insulated. Consequently, this will also model the temperature $u(x,t)$ within the infinite slab in three dimensional space. That is, the temperature will quickly work out as the heat is redistributed with the increasing t. In other words, the original total heat distributes itself uniformly throughout the surface.

5.5 Exact Solution of the Burgers' Equation

Now in this section we will use transformation (5.13) to find the exact Fourier solution to the Burgers' equation. In other words, $\phi(x,t)$ is any solution of the heat equation Eq. (5.17) with the corresponding conditions Eqs. (5.18) and (5.19), then the transformation $\frac{\phi_x(x,t)}{\phi(x,t)}$ is a solution of the Burgers Eq. (2.1) subject to the corresponding conditions Eqs. (2.2) and (2.3).

Consider the transformation (5.13)

$$u = \frac{\phi_x}{\phi}. \tag{5.20}$$

Now using the method of separation of variables. Suppose,

$$\phi(x,t) = X(x)T(t). \tag{5.21}$$

So, Eq. (5.17) implies

$$\frac{\partial^2}{\partial x^2}(XT) = \frac{\partial}{\partial t}(XT), \tag{5.22}$$

we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{dT}{dt}, \tag{5.23}$$

that is,

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{dT}{dt} = -\lambda^2. \tag{5.24}$$

Consequently, Eq. (5.24) can be written as

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2, \tag{5.25}$$

which leads to the second ODE

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0. \tag{5.26}$$

It has a general solution

$$X(x) = d_1\cos(\lambda x) + d_2\sin(\lambda x). \tag{5.27}$$

Since,

$$\frac{\partial X}{\partial x}(x) = -\lambda d_1\sin(\lambda x) + \lambda d_2\cos(\lambda x), \tag{5.28}$$

For $x = 0$, we have $d_2 = 0$, So, Eq. (5.27) implies

$$X(x) = d_1 \cos(\lambda x), \tag{5.29}$$

that is,

$$\frac{\partial X}{\partial x}(x) = -d_1 \sin(\lambda x). \tag{5.30}$$

Now for $x = 1$, we have the infinite sequence of the eigenvalues and corresponding eigenfunctions

$$-d_1 \sin(\lambda) = 0,$$
$$\lambda_n = n\pi, \quad n = 1, 2, 3, \cdots,$$

as $d_1 \neq 0$ and $\lambda \neq 0$.

Therefore, Eq. (5.29) yields

$$X_n(x) = d_1 \cos(n\pi x), \quad n = 1, 2, 3, \cdots. \tag{5.31}$$

Similarly, Eq. (5.24) also implies

$$T_n(t) = c \exp\left(-n^2\pi^2 t\right), \quad n = 1, 2, 3, \cdots. \tag{5.32}$$

Upon substitution of Eq. (5.31) and Eq. (5.32) in Eq. (5.21) we have the product function satisfying the homogenous conditions are

$$\begin{aligned} \phi_n(x,t) &= d_1 \cos(n\pi x) c \exp\left(-n^2\pi^2 t\right), \\ &= d_n \exp\left(-n^2\pi^2 t\right) \cos(n\pi x), \quad n = 1, 2, 3, \cdots, \end{aligned} \tag{5.33}$$

as $d_1 c = \mathcal{Q}_n$.

Now using the principle of superposition Eq. (5.33) takes the form

$$\phi_n(x,t) = \sum_{n=1}^{\infty} \mathcal{Q}_n \exp\left(-n^2\pi^2 t\right) \cos(n\pi x). \tag{5.34}$$

For $t = 0$, Eq. (5.34) can be written as

$$\phi_n(x,0) = \sum_{n=1}^{\infty} \mathcal{Q}_n \cos(n\pi x),$$
$$\exp\left(\frac{1-\cos(\pi x)}{\pi}\right) = \sum_{n=1}^{\infty} \mathcal{Q}_n \cos(n\pi x).$$

Multiplying by $\cos(m\pi x)$ on both sides

$$\exp\left(\frac{1-\cos(\pi x)}{\pi}\right)\cos(m\pi x) = \sum_{n=1}^{\infty} \mathcal{Q}_n \cos(n\pi x)\cos(m\pi x),$$

Integrating w.r.t x

$$\int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\pi}\right)\cos(m\pi x)dx = \int_0^1 \sum_{n=1}^{\infty} \mathcal{Q}_n \cos(n\pi x)\cos(m\pi x)dx,$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} \mathcal{Q}_n \int_0^1 2\cos(n\pi x)\cos(m\pi x)dx,$$

deduces to

$$\int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\phi}\right)\cos(m\pi x)dx = \frac{1}{2}\sum_{n=1}^{\infty} \mathcal{Q}_n \left(\frac{\sin(n+m)\pi x}{n+m} - \frac{\sin(n-m)\pi x}{n-m}\right)\Bigg|_0^1.$$

Now for $m = n$, one obtains

$$\int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\pi}\right)\cos(m\pi x)dx = 0, \quad m = 1, 2, 3, \cdots.$$

Likewise, for $m \neq n$, we have

$$\int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\pi}\right)\cos(m\pi x)dx = \frac{1}{2}\sum_{n=1}^{\infty} \mathcal{Q}_n \left(x - \frac{\sin(2n\pi x)}{2nx}\right)\Bigg|_0^1,$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} \mathcal{Q}_n.$$

Thus,

$$\sum_{n=1}^{\infty} \mathcal{Q}_n = 2\int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\pi}\right)\cos(m\pi x)dx, \quad m = 1, 2, 3, \cdots. \tag{5.35}$$

By using Eq. (5.35) in Eq. (5.34) yields

$$\phi(x,t) = \sum_{n=1}^{\infty} \left(2 \int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\pi} \right) \cos(n\pi x) dx \right) \exp\left(-n^2\pi^2 t\right) \cos(n\pi x).$$

Therefore, the trail solution reduces to the Fourier cosine series to

$$\phi(x,t) = \mathfrak{q}_0 + \sum_{n=1}^{\infty} \mathfrak{q}_n \exp\left(-n^2\pi^2 t\right) \cos(n\pi x), \tag{5.36}$$

with

$$\mathfrak{q}_0 = \int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\pi} \right) dx, \tag{5.37}$$

and

$$\mathfrak{q}_n = 2 \int_0^1 \exp\left(\frac{1-\cos(\pi x)}{\pi} \right) \cos(n\pi x) dx, \qquad n = 1, 2, 3, \cdots, \tag{5.38}$$

where $\mathfrak{q}_0$ and $\{\mathfrak{q}_n\}$ are the coefficients of Fourier cosine series of the initial temperature function.

Hence, the exact Fourier solution to the Burgers' equation using the Hopf-Cole transformation (5.20) is given by

$$u(x,t) = -\pi \frac{\sum_{n=1}^{\infty} \mathfrak{q}_n \exp\left(-n^2\pi^2 t\right) n \sin(n\pi x)}{\mathfrak{q}_0 + \sum_{n=1}^{\infty} \mathfrak{q}_n \exp\left(-n^2\pi^2 t\right) \cos(n\pi x)}, \tag{5.39}$$

where $\mathfrak{q}_0$ and $\mathfrak{q}_n$ are defined by Eqs. (5.37) and (5.38), respectively.

Chapter 6

Stability and Numerical Analysis of an Invariant Numerical Scheme

This chapter deals with the stability and computational analysis of the Burgers' equation. The stability investigation of the newly constructed invariant numerical scheme (M-CNM) in Chapter 5 has been established by means of Von Neumann stability analysis and Lax convergence theorem, which shows that the invariant numerical scheme (M-CNM) corresponding to second discrete symmetry ζ_2 of the Burgers' equation is consistent with the diffusion heat equation, thereby ensuring that the numerical is absolutely convergent to the exact solution of the Burgers' equation. Note that the stability analysis has been done only for the second discrete symmetry ζ_2. However, one can also check the convergence of the remaining numerical schemes following the similar strategy.

The computation results of Burgers' equation are obtained by virtue of FTCS, CNM and M-CNM. For all methods, tables and figures are used to display the results. Moreover, the comparison of all three methods was also discussed to obtain a verdict that which among these three methods has faster convergence rate and error reduction in terms of time and step size, N respectively.

Exact solution of the Burgers' equation, FTCS, CNM and M-CNM are all coded in *MATLAB*.

6.1 Core Theory for Convergence of a Numerical Scheme

In order to discuss the convergence of any numerical scheme, we require three notions to address [37].

6.1.1 Local Truncation Error

Local truncation error is a basic way of providing the comparison between local accuracies of different numerical schemes. It can be define as "the difference between the finite difference approximation at (n, j)th grid point in space and time and its exact differential equation. For instance, an exact solution $\mathcal{U}$ satisfying the partial differential equation say $R(\mathcal{U})$ and a numerical approximation u staisfying the equation $R(u)$ then the local truncation error at the (n, j)th mesh point is $T_{n,j} = R_{n,j}(\mathcal{U})$.

6.1.2 Consistency

The concept of consistency can be regarded as the representation of a partial differential equation by the finite difference approximation. As the grids of space and time are rectified of errors the finite difference equation converges to the original equation, thereby proving the consistency of a finite difference equation with differential equation. Thus, we conclude that the numerical scheme is consistent as the grids of space and time are rectified then the truncation error $T_{n,j} \to 0$.

6.1.3 Stability

Stability of any numerical scheme deals with the propagation of numerical error between the exact solution of the approximating equations to the solution of a differential equation. Any numerical scheme is stable, If the error remains constant or decreases as the approximation in time and space goes on. On the other hand, if error grows with time, the scheme is said to be unstable.

This concludes the convergence of any numerical scheme as $\Delta t \to 0$ and $\Delta x \to 0$ while keeping x_n and t_j constant, the computed solution $u_{n,j}$ of the discretized equation at any point $x_n = n\Delta x$ and $t_j = j\Delta t$ converges to the exact solution $\mathcal{U}_{n,j}$ of the differential equation with the error given as

$$\mathcal{E}_{n,j} = u_{n,j} - \mathcal{U}_{n,j}, \tag{6.1}$$

satisfying the following convergence theorem [37]

$$\lim_{\Delta x, \Delta t \to 0} |\mathcal{E}_{n,j}| \to 0 \text{ at constant } x_n = n\Delta x \text{ and } t_j = j\Delta t. \tag{6.2}$$

Theorem 6.1.1 (Lax Theorem)**.** *If a numerical scheme corresponds to the partial differential equation, then for the convergence, stability is the necessary and sufficient condition subject to an appropriate initial and boundary conditions, that is*

$$\textit{Consistency} + \textit{Stability} \leftrightarrow \textit{Convergence}.$$

6.2 Convergence of an Invaraint Numerical Scheme using Discrete Symmetry Group ζ_2

This section deals with the convergence of a Modified-Crank-Nicolson method obtained by using second discrete symmetry ζ_2. First, we will discuss the stability of concerned numerical scheme. Since, there are numerous techniques to discuss the stability analysis of a finite difference scheme. In this academic thesis, we will discuss one of them, which is the most commonly used method is Von Neumann stability analysis. Consider the following theorems [38, 37] for the main idea of this analysis.

Theorem 6.2.1. *Let $\hat{u} \in \mathcal{L}_2 \in [-\pi, \pi]$ be the function of discrete Fourier transform of $u \in l_2$. Then for $\mathcal{V} \in [-\pi, \pi]$, we have*

$$\hat{u}(\mathcal{V}) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-\iota n \mathcal{V}} u_n, \tag{6.3}$$

and the inverse transformation

$$u_2 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-\iota n \mathcal{V}} \hat{u}(\mathcal{V})\, d\mathcal{V}, \tag{6.4}$$

with the Parselval's relation $\|\hat{u}\|_2 = \|u\|_2$ *as*

$$\|\hat{u}_{n+1}\|_2 \leq \rho\left(\mathcal{V}\right)\|\hat{u}_0\|_2. \tag{6.5}$$

Then in the transformed $\mathcal{L}_2$ *space the finite difference scheme will be stable iff*

$$\rho\left(\mathcal{V}\right) \leqslant 1, \tag{6.6}$$

where $\rho\left(\mathcal{V}\right)$ *is the amplification factor.*

Theorem 6.2.2. *For the operator* $g : l_2 \to \mathcal{L}_2$ *the discrete Fourier transform in* $[-\pi, \pi]$ *is given by*

$$g(u) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{-\iota n\mathcal{V}} u_n. \tag{6.7}$$

Theorem 6.2.3. *Let* $S \pm u = \{\upsilon_j\}$ *be the shift operator with* $\upsilon_j = \upsilon_{j\pm1}$ *for* $j = 0, \pm1, \cdots$. *Then the discrete Fourier transform* (6.7) *takes the form*

$$g\left(S \pm u\right) = e^{\pm\iota\mathcal{V}} g(u), \tag{6.8}$$

$$= e^{\pm\iota\mathcal{V}} \hat{u}\left(\mathcal{V}\right). \tag{6.9}$$

6.2.1 Von Neumann Stability Analysis

In order to apply the Von Neumann stability analysis, consider M-CNM corresponding to second discrete symmetry ζ_2

$$\begin{aligned}
&\alpha\left(\left(2 - t_j\epsilon\right) u_{n-1,j} - \frac{\xi}{2} x_{n-1}\epsilon - 2\tau\epsilon\right) + 2(1-\alpha)\left(\left(2 - t_j\epsilon\right) u_{n,j} - \frac{\xi}{2} x_n\epsilon - 2\tau\epsilon\right)\\
&+ \alpha\left(\left(2 - t_j\epsilon\right) u_{n+1,j} - \frac{\xi}{2} x_{n+1}\epsilon - 2\tau\epsilon\right) = -\alpha\left(\left(2 - t_{j+1}\epsilon\right) u_{n-1,j+1} - \frac{\xi}{2} x_{n-1}\epsilon - 2\tau\epsilon\right)\\
&+ 2(1+\alpha)\left(\left(2 - t_{j+1}\epsilon\right) u_{n,j+1} - \frac{\xi}{2} x_n\epsilon - 2\tau\epsilon\right) - \alpha\left(\left(2 - t_{j+1}\epsilon\right) u_{n+1,j+1} - \frac{\xi}{2} x_{n+1}\epsilon - 2\tau\epsilon\right),
\end{aligned}$$

which can be simplified to

$$(2-t_j\epsilon)\left(\alpha u_{n-1,j}+2\left(1-\alpha\right)u_{n,j}+\alpha u_{n+1,j}\right)+2(1-\alpha)\left(-\frac{\xi}{2}x_n\epsilon-2\tau\epsilon\right)$$
$$-\alpha\left(\frac{\xi}{2}\left(x_{n-1}+x_{n+1}\right)\epsilon+4\tau\epsilon\right)$$
$$=(2-t_{j+1}\epsilon)\left(-\alpha u_{n-1,j+1}+2\left(1-\alpha\right)u_{n,j+1}-\alpha u_{n+1,j+1}\right)+2(1+\alpha)\left(-\frac{\xi}{2}x_n\epsilon-2\tau\epsilon\right)$$
$$-\alpha\left(\frac{\xi}{2}\left(x_{n-1}+x_{n+1}\right)\epsilon+4\tau\epsilon\right). \tag{6.10}$$

Since, we have

$$\begin{aligned} x_{n-1}+x_{n+1} &= (n-1)\Delta x+(n+1)\Delta x, \\ &= 2n\Delta x. \end{aligned}$$

Therefore, Eq. (6.10) implies

$$\begin{aligned} &(2-t_j\epsilon)\left(\alpha u_{n-1,j}+2\left(1-\alpha\right)u_{n,j}+\alpha u_{n+1,j}\right)-n\Delta x\xi\epsilon-4\tau\epsilon \\ &=(2-t_{j+1}\epsilon)\left(-\alpha u_{n-1,j+1}+2\left(1-\alpha\right)u_{n,j+1}-\alpha u_{n+1,j+1}\right)-n\Delta x\xi\epsilon-4\tau\epsilon. \end{aligned}$$

Applying the Von Neumann analysis, we have

$$\begin{aligned} &(2-t_j\epsilon)\left[\alpha e^{\iota\mathcal{V}}\hat{u}_j+2(1-\alpha)u_j+re^{\iota\mathcal{V}}\hat{u}_j\right] \\ &=(2-t_{j+1}\epsilon)\left[-\alpha e^{-\iota\mathcal{V}}\hat{u}_{j+1}+2(1+\alpha)\hat{u}_{j+1}-\alpha e^{\iota\mathcal{V}}\hat{u}_{j+1}\right], \end{aligned}$$

$$\begin{aligned} &(2-t_j\epsilon)\left[\alpha\left(\cos\mathcal{V}-\iota\sin\mathcal{V}\right)+2-2\alpha+\alpha(\cos\mathcal{V}+\iota\sin\mathcal{V})\right]\hat{u}_j \\ &=(2-t_{j+1}\epsilon)\left[-\alpha\left(\cos\mathcal{V}+\iota\sin\mathcal{V}\right)+2+2\alpha-\alpha(\cos\mathcal{V}+\iota\sin\mathcal{V})\right]\hat{u}_{j+1}, \end{aligned}$$

which deduces to

$$\hat{u}_{j+1}=\left(\frac{2-2\alpha+\alpha\cos\mathcal{V}}{2+2\alpha-\alpha\cos\mathcal{V}}\right)\left(\frac{2-t_j\epsilon}{2-t_{j+1}\epsilon}\right)\hat{u}_j. \tag{6.11}$$

After some calculus, Eq. (6.11) takes the form

$$\begin{aligned} \hat{u}_{j+1} &= \left(\frac{1-4\alpha\sin^2\frac{\mathcal{V}}{2}}{1+4\alpha\sin^2\frac{\mathcal{V}}{2}}\right)\left(\frac{2-t_j\epsilon}{2-t_{j+1}\epsilon}\right)\hat{u}_j, \\ &= \rho\left(\mathcal{V}\right)\hat{u}_j. \end{aligned} \tag{6.12}$$

Thus, the amplification factor is

$$\rho(\mathcal{V}) = \left(\frac{1-4\alpha\sin^2\frac{\mathcal{V}}{2}}{1+4\alpha\sin^2\frac{\mathcal{V}}{2}}\right)\left(\frac{2-t_j\epsilon}{2-t_{j+1}\epsilon}\right). \tag{6.13}$$

Since, we know that a solution is stable iff $|\rho(\mathcal{V})| \leqslant 1$. Therefore, Eq. (6.13) implies

$$-1 \leqslant \frac{1-4\alpha\sin^2\frac{\mathcal{V}}{2}}{1+4\alpha\sin^2\frac{\mathcal{V}}{2}} \leqslant 1. \tag{6.14}$$

Hence, this proves that the invariant numerical scheme which is the Modified-Crank-Nicolson method corresponding to second discrete symmetry group ζ_2 is stable for all values of α.

6.2.2 Local Truncation Error

The compact form of an invariant numerical scheme Eq. (4.7) is

$$u_t = \frac{\left(Au_{n,j+1} - \frac{\xi}{2}x_n\epsilon - 2\tau\epsilon\right) - \left(Bu_{n,j} - \frac{\xi}{2}x_n\epsilon - 2\tau\epsilon\right)}{\Delta t}, \tag{6.15}$$

and

$$\begin{aligned} u_{xx} = {} & \frac{\left[(Bu_{n+1,j} - X_{k_3}) - 2\left(Bu_{n,j} - X_{k_2}\right) + \left(Bu_{n-1,j} - X_{k_1}\right)\right]}{2(\Delta x)^2} \\ & + \frac{\left[(Au_{n+1,j+1} - X_{k_3}) - 2\left(Au_{n,j+1} - X_{k_2}\right) + \left(Au_{n+1,j+1} - X_{n+1} - X_{k_1}\right)\right]}{2(\Delta x)^2}, \end{aligned} \tag{6.16}$$

where

$$\begin{aligned} A &= \left(2 - t_{j+1}\epsilon\right), \\ B &= \left(2 - t_j\epsilon\right), \\ X_{k_i} &= \frac{\xi}{2}x_{k_i}\epsilon + 2\tau\epsilon, \quad k_i = n-1, n, n+1 \quad for \quad i = 1, 2, 3. \end{aligned}$$

Now, expanding u by Taylor expansion of two variables

$$
\begin{aligned}
u_{n+1,j} &= u_{n,j} + \Delta x u_x + \frac{(\Delta x)^2}{2} u_{xx} + \frac{(\Delta x)^3}{6} u_{xxx} + \frac{(\Delta x)^4}{24} u_{xxxx} + \cdots, \\
u_{n,j+1} &= u_{n,j} + \Delta t u_t + \frac{(\Delta t)^2}{2} u_{tt} + \frac{(\Delta t)^3}{6} u_{ttt} + \frac{(\Delta t)^4}{24} u_{tttt} + \cdots, \\
u_{n+1,j+1} &= u_{n,j} + \Delta x u_x + \Delta t u_t + \frac{(\Delta x)^2}{2} u_{xx} + \frac{(\Delta t)^2}{2} u_{tt} + \Delta x \Delta k u_{xt} + \frac{(\Delta x)^3}{6} u_{xxx} \\
&\quad + \frac{(\Delta t)^3}{6} u_{ttt} + \frac{(\Delta x)^2 \Delta t}{2} u_{xxt} + \frac{\Delta x (\Delta t)^2}{2} u_{xtt} + \frac{(\Delta x)^4}{24} u_{xxxx} + \frac{(\Delta t)^4}{24} u_{tttt} \\
&\quad + \frac{(\Delta x)^2 (\Delta t)^2}{4} u_{xxtt} + \frac{(\Delta x)^3 \Delta t}{6} u_{xxxt} + \frac{\Delta x (\Delta t)^3}{6} u_{xttt} + \cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
u_{n-1,j+1} &= u_{n,j} - \Delta x u_x + \Delta t u_t + \frac{(\Delta x)^2}{2} u_{xx} + \frac{(\Delta t)^2}{2} u_{tt} - \Delta x \Delta k u_{xt} - \frac{(\Delta x)^3}{6} u_{xxx} \\
&\quad + \frac{(\Delta t)^3}{6} u_{ttt} + \frac{(\Delta x)^2 \Delta t}{2} u_{xxt} - \frac{\Delta x (\Delta t)^2}{2} u_{xtt} + \frac{(\Delta x)^4}{24} u_{xxxx} + \frac{(\Delta t)^4}{24} u_{tttt} \\
&\quad + \frac{(\Delta x)^2 (\Delta t)^2}{4} u_{xxtt} - \frac{(\Delta x)^3 \Delta t}{6} u_{xxxt} - \frac{\Delta x (\Delta t)^3}{6} u_{xttt} + \cdots.
\end{aligned}
$$

Since, the residue of heat equation $u_t = u_{xx}$ is

$$
R_{n,j}(u) = u_t - u_{xx}. \tag{6.17}
$$

Substituting Eqs. (6.15) and (6.16) in Eq. (6.17), we have

$$
\begin{aligned}
R_{n,j}(u) = &\frac{\left(A u_{n,j+1} - \frac{\xi}{2} x_n \epsilon - 2\tau\epsilon\right) - \left(B u_{n,j} - \frac{\xi}{2} x_{n-1} \epsilon - 2\tau\epsilon\right)}{\Delta t} \\
&- \frac{\left[(B u_{n+1,j} - X_{k_3}) - 2\left(B u_{n,j} - X_{k_2}\right) + (B u_{n-1,j} - X_{k_1})\right]}{2(\Delta x)^2} \\
&+ \frac{\left[(A u_{n+1,j+1} - X_{k_3}) - 2\left(A u_{n,j+1} - X_{k_2}\right) + (A u_{n+1,j+1} - X_{n+1} - X_{k_1})\right]}{2(\Delta x)^2},
\end{aligned}
$$

By substituting the corresponding values of $u_{n+1,j}, u_{n,j+1}, u_{n+1,j+1}$ and $u_{n-1,j+1}$, the above equation can be written as

$$
\begin{aligned}
T_{n,j} = &\frac{B}{A}\left(u_t - u_{xx}\right) + \frac{B\Delta t}{2A}\left(u_{tt} - u_{xxt}\right) + \frac{B(\Delta t)^2}{6A} u_{ttt} - \frac{B(\Delta x)^2}{12A} u_{xxxx} \\
&+ \mathcal{O}\left((\Delta t)^3\right) + \mathcal{O}\left((\Delta x)^3\right),
\end{aligned} \tag{6.18}
$$

where the principal part is

$$\frac{B(\Delta t)^2}{6A}u_{ttt} - \frac{B(\Delta x)^2}{12A}u_{xxxx}. \tag{6.19}$$

That is,

$$T_{n,j} = \frac{B(\Delta t)^2}{6A} - \frac{B(\Delta x)^2}{12A} = 0, \tag{6.20}$$

if $\Delta x \to 0$ and $\Delta t \to 0$.

Hence, the invariant numerical scheme (4.7) is consistent with the partial differential equation. Moreover, it is also stable. Therefore, by Lax theorem it implies that the invariant numerical scheme is convergent. Likewise, one can also show that the numerical scheme corresponding to discrete symmetry groups ζ_1 and ζ_3 are neither stable nor consistent with the partial differential equation, thereby does not converge to the exact solution of the Burgers' equation.

6.3 Numerical Analysis

In this section, the computational results of Burgers' equation by using Explicit Finite Difference Method (FTCS), Crank-Nicolson Method (CNM) and Modified-Crank-Nicolson Method (M-CNM) are presented. For all the methods tables and figures are used to show the obtained results.

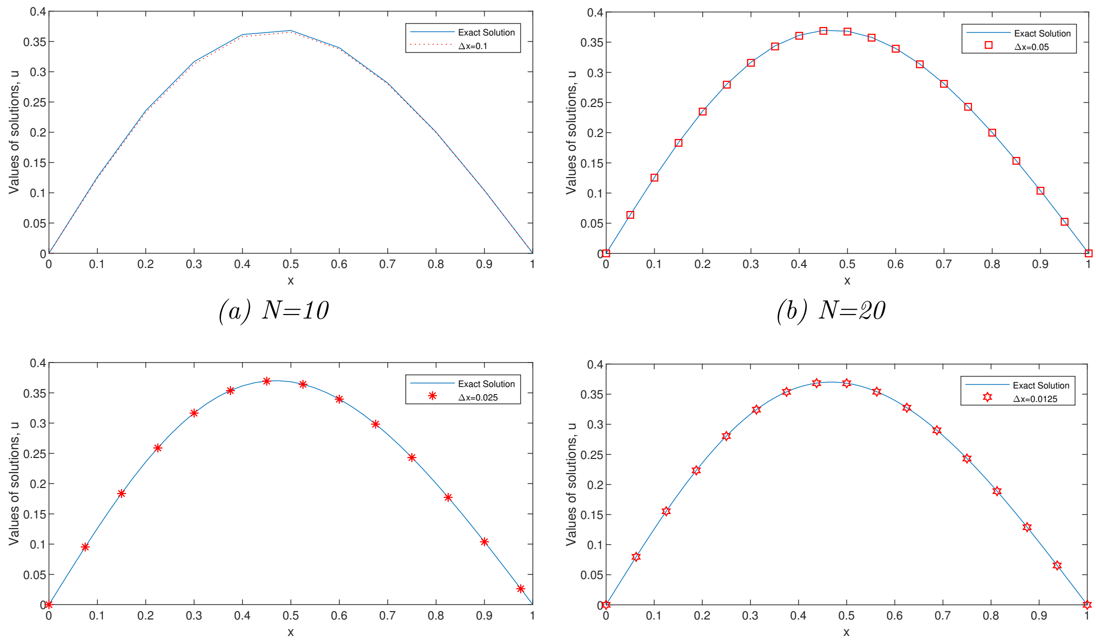
6.3.1 Burgers' Equation Computation using Explicit Finite Difference Method (FTCS)

The numerical computation of Burgers' equation through a Hopf-Cole transformation by using explicit finite difference method (FTCS) are presented in Table 6.1, 6.2, 6.3 and Figure 6.1, 6.2, 6.3 respectively.

x	Numerical Solution				Exact Solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.124116518	0.125486314	0.125829775	0.125915700	0.125949755
0.2	0.232616063	0.234985427	0.235578534	0.235726852	0.235786641
0.3	0.313015514	0.315814130	0.316512951	0.316687598	0.316760166
0.4	0.357912510	0.360596054	0.361264136	0.361430977	0.361503607
0.5	0.365285616	0.367493013	0.368040774	0.368177459	0.368241064
0.6	0.337524500	0.339118106	0.339512244	0.339610515	0.339660562
0.7	0.279949000	0.280965179	0.281215677	0.281278083	0.281313681
0.8	0.199452686	0.200015973	0.200154394	0.200188853	0.200211135
0.9	0.103571779	0.103813384	0.103872608	0.103887342	0.103897943

Table 6.1: Comparison of explicit finite difference (FTCS) solutions with exact solution of Burgers' equation at different step size, N

Table 6.1 reflects the discrete values of exact and explicit finite difference solutions of the Burgers' equation for $t = 0.1$ with $\Delta t = 0.00001$ at various times. Table 6.1 then being illustrated into Figure 6.1 for all the values of a numerical simulation.



(a) N=10 *(b) N=20* *(c) N=40* *(d) N=80*

Figure 6.1: Burgers' equation solutions at different step sizes, N for $t = 0.1$ *with* $\Delta t = 0.00001$ *using explicit finite difference (FTCS)*

From the Figure 6.1, it is clearly observed that the explicit finite difference (FTCS) solutions get nearer to the exact solution as the number of step size, N increases. The numerical values obtained by using the FTCS with different step size appears to have been imbricated and onerous to note from the plots because of the closeness of the computed solutions with the exact solution. However, the sole recognizable difference between the computed solutions and exact solution is when $N = 10$. This additionally demonstrate the less accuracy of computed solutions contrasted with the exact solution.

x	Absolute Error			
	$N = 10$	$N = 20$	$N = 40$	$N = 80$
0.1	0.001833237	0.463441E-03	0.119980E-03	0.34055E-04
0.2	0.003170578	0.801214E-03	0.208108E-03	0.59789E-04
0.3	0.003744652	0.946035E-03	0.247214E-03	0.72567E-04
0.4	0.003591097	0.907553E-03	0.239471E-03	0.72630E-04
0.5	0.002955448	0.748051E-03	0.200290E-03	0.63606E-04
0.6	0.002136062	0.542456E-03	0.148318E-03	0.50047E-04
0.7	0.001364681	0.348502E-03	0.098005E-03	0.35598E-04
0.8	0.000758449	0.195162E-03	0.056741E-03	0.22282E-04
0.9	0.000326164	0.084559E-03	0.025336E-03	0.10601E-04

Table 6.2: Absolute error differences of explicit finite difference (FTCS) solutions with exact solution of Burgers' equation at different step size, N

Table 6.1 demonstrates the absolute error difference of exact solution and the approximation of FTCS of the Burgers' equation presented in Table 6.1. It is clear from the Table 6.2 that error is gradually decreasing and slowly approaching zero as the number of step size, N increases as shown in Figure 6.2.

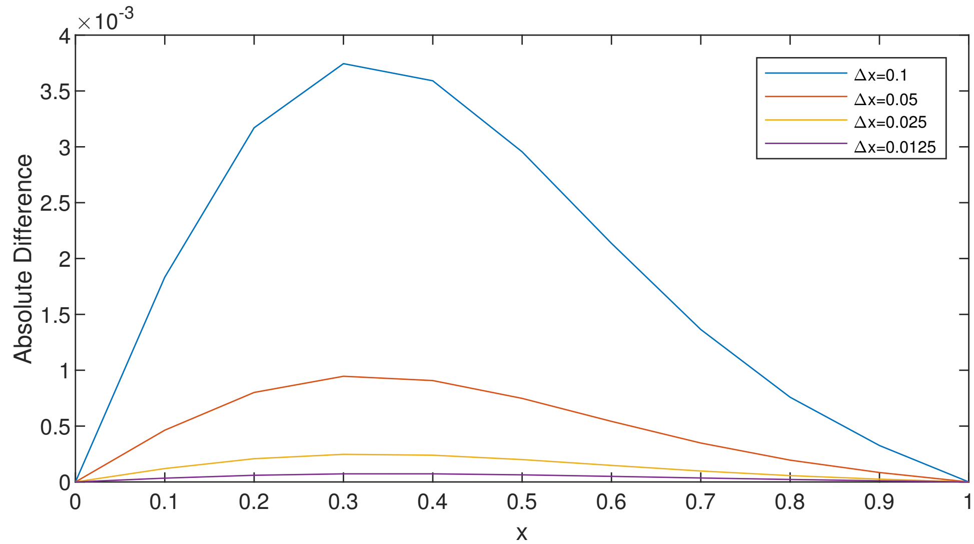


Figure 6.2: Relative error difference of explicit finite difference (FTCS) solutions and exact solution of Burgers' equation for $t = 0.1$ with $\Delta t = 0.00001$ at different step size

x	Numerical Solution				
	$t = 0.1$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	0.125915700	0.044186236	0.005923086	0.000818905	0.113683E-03
0.2	0.235726852	0.083517712	0.011256632	0.001557463	0.216234E-03
0.3	0.316687598	0.113835409	0.015472577	0.002143264	0.297613E-03
0.4	0.361430977	0.132203343	0.018158317	0.002518966	0.349854E-03
0.5	0.368177459	0.137168587	0.019057027	0.002647908	0.367845E-03
0.6	0.339610515	0.128752859	0.018090429	0.002517655	0.349829E-03
0.7	0.281278083	0.108249805	0.015362731	0.002141142	0.297572E-03
0.8	0.200188853	0.077928886	0.011146785	0.001555341	0.216193E-03
0.9	0.103887342	0.040730539	0.005855196	0.000817594	0.113658E-03

Table 6.3: Explicit finite difference (FTCS) solutions of Burgers' equation for different time, t with space step size of N=80

Table 6.3 displays the discrete computation of Burgers' equation solutions at different time, $t = 0.1$, $t = 0.2$, $t = 0.4$, $t = 0.6$ and $t = 0.8$ with the time step size of $\Delta t = 0.00001$. All the obtained values of the numerical scheme are shown in the figure.

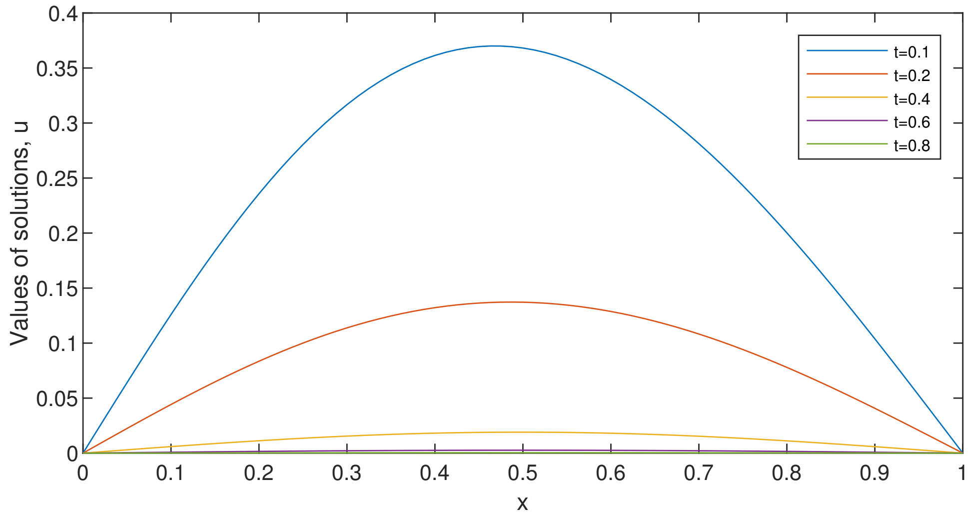


Figure 6.3: Explicit finite difference (FTCS) solutions of Burgers' equation for different time, t

Figure 6.3 indicates that even though, the value of the Burgers' equation solution decreases as the time expands, the shape is yet held as a curve shape.

It has been evidently noted that the numerical solutions obtained through explicit finite difference method (FTCS) are passably encompass agreement with the exact solution. This additionally means that the computed results obtained through explicit difference approximation (FTCS) have high precision if apply the higher number of step size with condition in the scope of time step fulfill the Von Neumann stability, which can be obtained as $\Delta t \leqslant \frac{1}{2}(\Delta x)^2$ [37]. Hence, this method can be used to approximate the Burgers' equation solution.

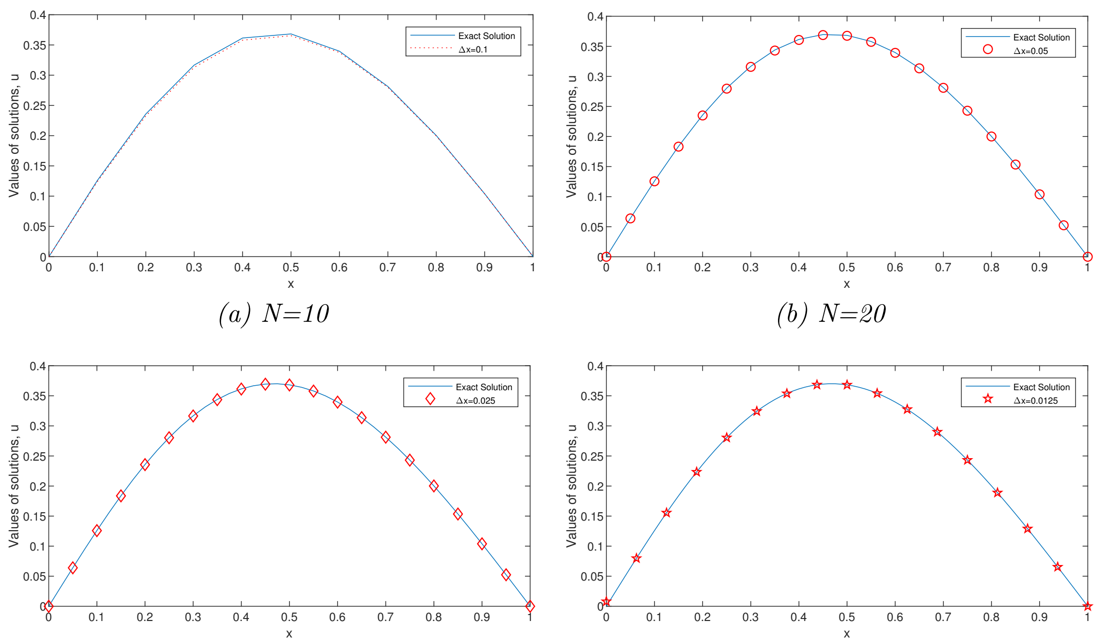
6.3.2 Burgers' Equation Computation using Crank-Nicolson Method (CNM)

In this section, the computation of Hopf-Cole transformed Burgers' equation by using Crank-Nicolson method are presented in Table 6.4, 6.5, 6.6 and Figure 6.4, 6.5, 6.6 respectively.

x	Numerical Solution				Exact Solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.124121709	0.125491666	0.125835169	0.125921105	0.125949755
0.2	0.232626020	0.234995674	0.235588855	0.235737192	0.235786641
0.3	0.313029387	0.315828364	0.316527277	0.316701947	0.316760166
0.4	0.357929046	0.360612961	0.361281137	0.361448001	0.361503607
0.5	0.365303242	0.367510969	0.368058813	0.368195518	0.368241064
0.6	0.337541466	0.339135332	0.339529536	0.339627823	0.339660562
0.7	0.279963572	0.280979934	0.281230476	0.281292894	0.281313681
0.8	0.199463347	0.200026745	0.200165193	0.200199659	0.200211135
0.9	0.103577408	0.103819064	0.103878300	0.103893037	0.103897943

Table 6.4: Comparison of Crank-Nicolson method (CNM) solutions with the exact solution of Burgers' equation at different step size, N

Table 6.4 shows the Crank-Nicolson method (CNM) method solutions and exact solution of the Burgers' equation for time, $t = 0.1$ with time step size of $\Delta t = 0.00001$. The following Figure 6.4 is based on Table 6.4 for all the points of x.



(a) N=10 *(b) N=20* *(c) N=40* *(d) N=80*

Figure 6.4: Burgers' equation solutions at different step size, N for $t = 0.1$ with $\Delta t = 0.00001$ using Crank-Nicolson method (CNM)

Figure 6.4 has a similar shape to Figure 6.1. This is due to the results obtained by using Crank-Nicolson method (CNM) are inadequately close to exact solution of the Burgers' equation and consequently, reducing the transparency among the curves particularly for $N = 10$. Nevertheless, this also suggest that the obtained solutions for $N = 10$ are much less accurate as compared to different step size.

x	Absolute Error			
	$N = 10$	$N = 20$	$N = 40$	$N = 80$
0.1	0.001828046	0.458089E-03	0.114586E-03	0.28650E-04
0.2	0.003160621	0.790968E-03	0.197787E-03	0.49449E-04
0.3	0.003730778	0.931801E-03	0.232888E-03	0.58218E-04
0.4	0.003574561	0.890646E-03	0.222470E-03	0.55606E-04
0.5	0.002937822	0.730095E-03	0.182251E-03	0.45546E-04
0.6	0.002119096	0.525230E-03	0.131026E-03	0.32739E-04
0.7	0.001350109	0.333748E-03	0.083205E-03	0.20787E-04
0.8	0.000747787	0.184389E-03	0.045942E-03	0.11476E-04
0.9	0.000320535	0.078879E-03	0.019644E-03	0.04906E-04

Table 6.5: Absolute error differences of Crank-Nicolson method (CNM) solutions with the exact solution of Burgers' equation at different step size, N

Table 6.5 displays the absolute error difference between Crank-Nicolson method (CNM) solutions and the exact solution of Burgers' equation for time, $t = 0.1$ with a time step size of $\Delta t = 0.00001$. One can see that the error is decreasing significantly and quickly approaching zero with each increment of a step size. Therefore, one can deduce that the veracity of a computed solutions pivots upon the step size. The higher the step size, the more minute the value of absolute difference between the computed solutions and exact solution as shown in the Figure 6.5.

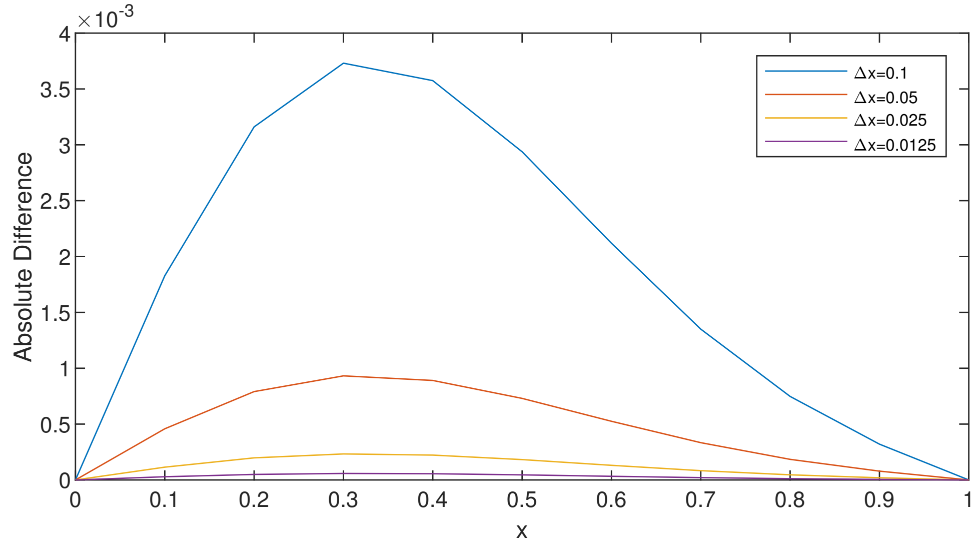


Figure 6.5: Relative error difference of Crank-Nicolson method solutions and exact solution of Burgers' equation for $t = 0.1$ with $\Delta t = 0.00001$ at different step size, N

x	Numerical Solution				
	$t = 0.1$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	0.125921105	0.044190673	0.005924246	0.000819144	0.113727E-03
0.2	0.235737192	0.083526060	0.011258836	0.001557918	0.216319E-03
0.3	0.316701947	0.113846705	0.015475602	0.002143891	0.297729E-03
0.4	0.361448001	0.132216346	0.018161861	0.002519702	0.349990E-03
0.5	0.368195518	0.137181948	0.019060739	0.002648682	0.367988E-03
0.6	0.339627823	0.128765283	0.018093946	0.002518390	0.349965E-03
0.7	0.281292894	0.108260163	0.015365713	0.002141768	0.297688E-03
0.8	0.200199659	0.077936295	0.011148946	0.001555796	0.216278E-03
0.9	0.103893037	0.040734395	0.005856330	0.000817832	0.113702E-03

Table 6.6: Crank-Nicolson method (CNM) solutions of Burgers' equation for different time, t with space step size of N=80

Table 6.6 exhibits the simulacrum results of Crank-Nicolson method (CNM) vacillates in time notably $t = 0.1$, $t = 0.2$, $t = 0.4$, $t = 0.6$ and $t = 0.8$. It is to be noted that the obtained solutions of the Burgers' equation are shrinking and gradually approaching zero with time as follows in Figure 6.6.

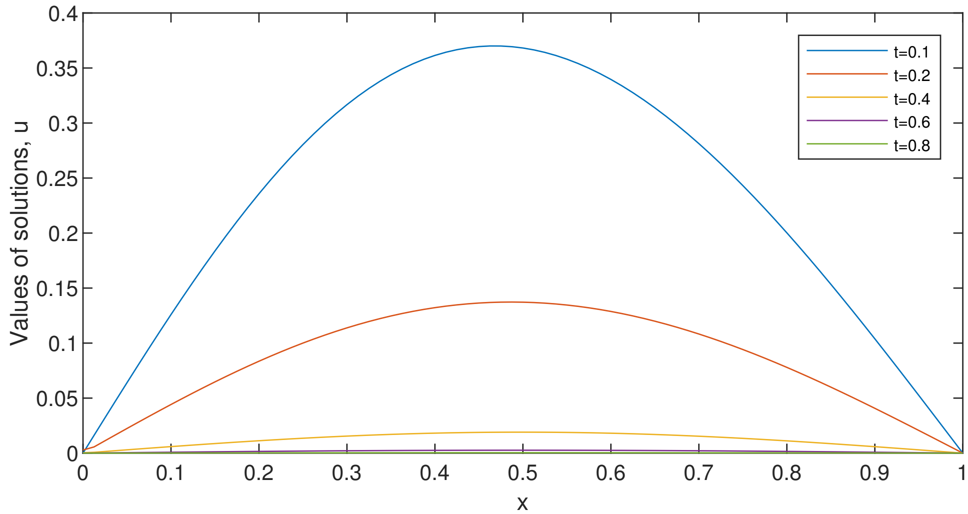


Figure 6.6: Crank-Nicolson method (CNM) solutions of Burgers' equation for different time, t

The Crank-Nicolson method (CNM) is known to be efficient in using to approximate Burgers' equation solutions according to the numbers and graphs scrutinized for this method. The explanation is that the simulated solutions are reminiscent of the exact solution.

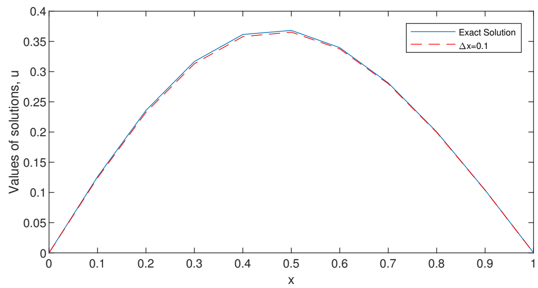
6.3.3 Burgers' Equation Computation using Modified-Crank-Nicolson Method (M-CNM)

This section deals with the numerical computation of Burgers' equation by means of Hopf-Cole transformation using Modified-Crank-Nicolson method (M-CNM) through second discrete symmetry ζ_2 which are presented in Table 6.7, 6.8, 6.9 and Figure 6.7, 6.8, 6.9 respectively.

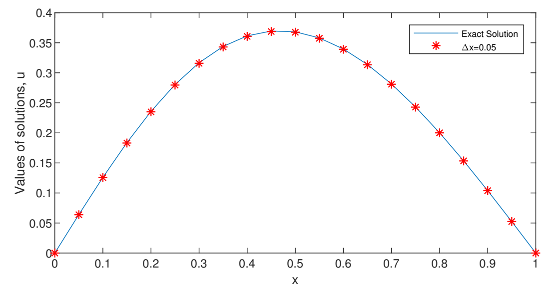
x	Numerical Solution				Exact Solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.124125305	0.125499708	0.125845588	0.125932744	0.125949755
0.2	0.232642366	0.235015589	0.235610615	0.235759881	0.235786641
0.3	0.313055072	0.315856751	0.316557007	0.316732333	0.316760166
0.4	0.357960001	0.360645852	0.361314942	0.361482232	0.361503607
0.5	0.365335452	0.367544519	0.368092950	0.368229900	0.368241064
0.6	0.337571430	0.339166241	0.339560806	0.339659209	0.339660562
0.7	0.279988479	0.281005581	0.281256351	0.281318797	0.281313681
0.8	0.199481036	0.200045127	0.200183739	0.200218176	0.200211135
0.9	0.103586233	0.103828653	0.103888035	0.103902704	0.103897943

Table 6.7: Comparison of Modified-Crank-Nicolson method (M-CNM) solutions with the exact solution of Burgers' equation at different step size, N

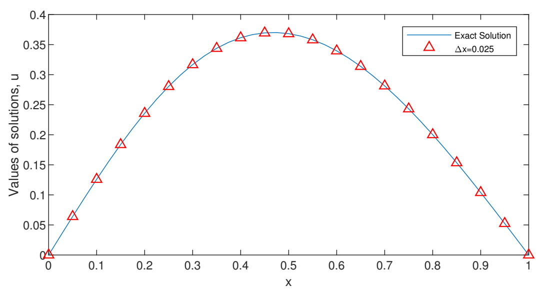
Table 6.7 displays the discrete results obtained by using Modified-Crank-Nicolson method (M-CNM) with $\epsilon = 10^{-7}$, $\tau = 1$, $\xi = 1$ for time, $t = 0.1$ with the time step, $\Delta t = 0.00001$. The main reason for choosing such values for ϵ, τ and ξ is that these are the ideal values for this method to be highly accurate with an adherent convergence rate. Table 6.7 then being illustrated into Figure 6.7 for all values of x. Note that the value of ϵ must be smaller as this method fails for any $\epsilon < 10^{-7}$.



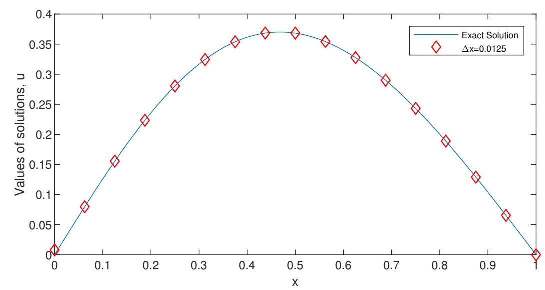
(a) N=10



(b) N=20



(c) N=40



(d) N=80

Figure 6.7: Burgers' equation solutions at different step size, N for $t = 0.1$ with $\Delta t = 0.00001$ using Modified-Crank-Nicolson method (M-CNM)

It is prominent from the Figure 6.7 that the numerical results are conceivably in good congruity with the exact solution of the Burgers' equation. As one can see that the visibility of the curves between the computed and exact solution for $N = 10$ has been slowly diminished. However, the solutions obtained by using M-CNM with different step size expounds to have been completely cohered and the graphs are convolutedly distinguishable.

x	Absolute Error			
	$N = 10$	$N = 20$	$N = 40$	$N = 80$
0.1	0.001824450	0.450047E-03	0.104167E-03	0.17011E-04
0.2	0.003144275	0.771052E-03	0.176026E-03	0.26760E-04
0.3	0.003705094	0.903415E-03	0.203159E-03	0.27833E-04
0.4	0.003543606	0.857755E-03	0.188665E-03	0.21375E-04
0.5	0.002905612	0.696545E-03	0.148114E-03	0.11164E-04
0.6	0.002089132	0.494321E-03	0.099756E-03	0.01353E-04
0.7	0.001325202	0.308100E-03	0.057330E-03	0.05116E-04
0.8	0.000730099	0.166008E-03	0.027396E-03	0.07041E-04
0.9	0.000311710	0.069290E-03	0.009908E-03	0.04761E-04

Table 6.8: Absolute error differences of Modified-Crank-Nicolson method (M-CNM) solutions with the exact solution of Burgers' equation at different step size, N

Table 6.8 can be inquired from Table 6.7 which shows that the absolute error difference between the computed solutions using Modified-Crank-Nicolson method (M-CNM) and exact solution for time, $t = 0.1$ with a step size of time, $\Delta t = 0.00001$ has been truncated conspicuously as adorned into Figure 6.8.

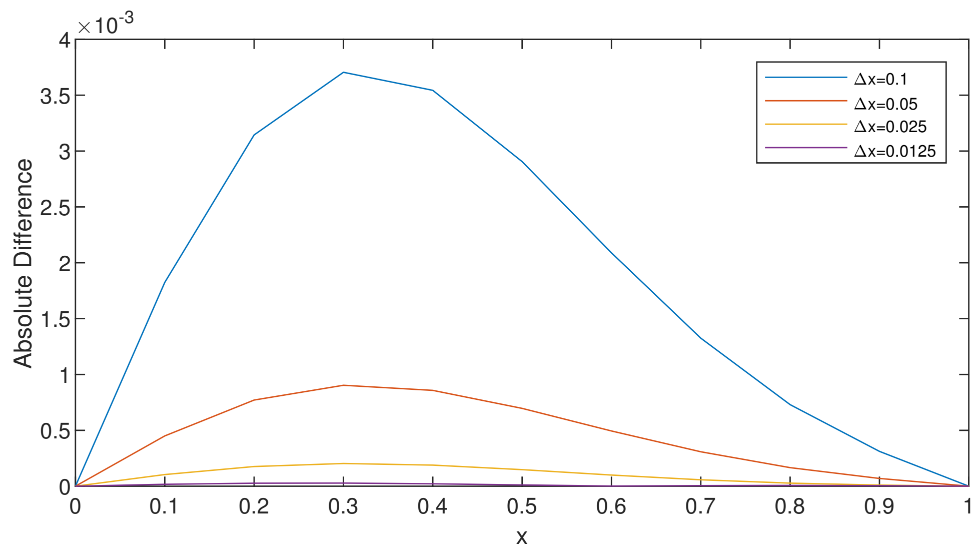


Figure 6.8: Relative error difference of Modified-Crank-Nicolson method (M-CNM) solutions and exact solution of Burgers' equation for $t = 0.1$ with $\Delta t = 0.00001$ at different step size, N

x	Numerical Solution				
	$t = 0.1$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	0.125932744	0.044199367	0.005927927	0.000821367	0.115635E-03
0.2	0.235759881	0.083543494	0.011266837	0.001563145	0.220946E-03
0.3	0.316732333	0.113870667	0.015486850	0.002151325	0.304337E-03
0.4	0.361482232	0.132244163	0.018175152	0.002528520	0.357836E-03
0.5	0.368229900	0.137210789	0.019074788	0.002658041	0.376325E-03
0.6	0.339659209	0.128792410	0.018107447	0.002527443	0.358044E-03
0.7	0.281318797	0.108283119	0.015377393	0.002149672	0.304764E-03
0.8	0.200218176	0.077952997	0.011157617	0.001561727	0.221606E-03
0.9	0.103902704	0.040743183	0.005860932	0.000820992	0.116544E-03

Table 6.9: Modified-Crank-Nicolson method (M-CNM) solutions of Burgers' equation for different time, t with space step size of N=80

Table 6.9 exhibits the obtained solutions for Burgers' equation by using Modified-Crank-Nicolson method (M-CNM) with $\epsilon = 10^{-7}$, $\tau = 1$, $\xi = 1$ at different times with a step size of $\Delta t = 0.00001$. It is clearly observed that the numerical solutions reflect the accurate physical behavior of a problem with more sophisticated convergence rate as follows in the Figure 6.9.

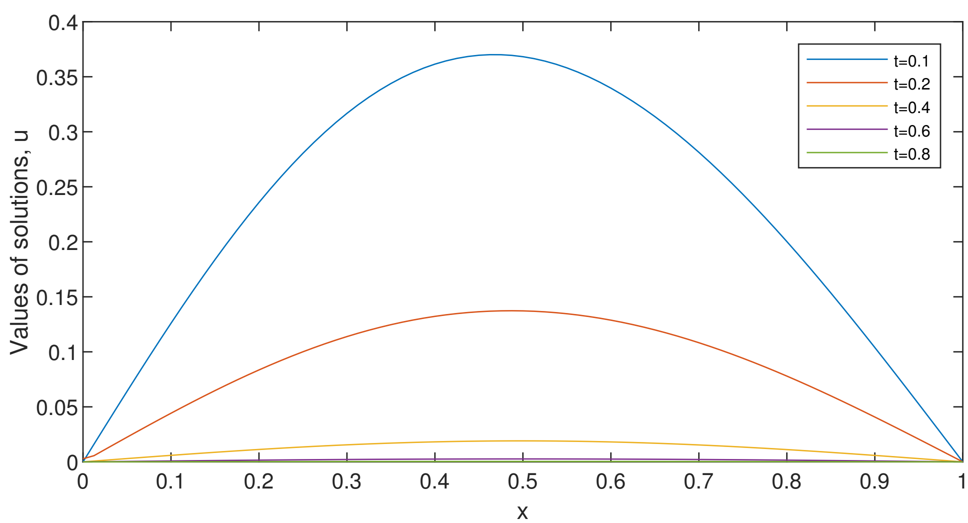


Figure 6.9: Modified-Crank-Nicolson method (M-CNM) solutions of Burgers' equation for different time, t

In the Figure 6.9, one can see that the corresponding curve for $N = 80$, which was initially held as an arch shaped in the aforementioned methods appears to have been flattened, thereby elucidating the precision of the convergence rate of Modified-Crank-Nicolson method (M-CNM).

However, the solutions of Modified-Crank-Nicolson method (M-CNM) obtained through first and third discrete symmetries of the Burgers' equation are scattered everywhere and diverging completely from the exact solution of the Burgers' equation as shown in the Figure 6.10.

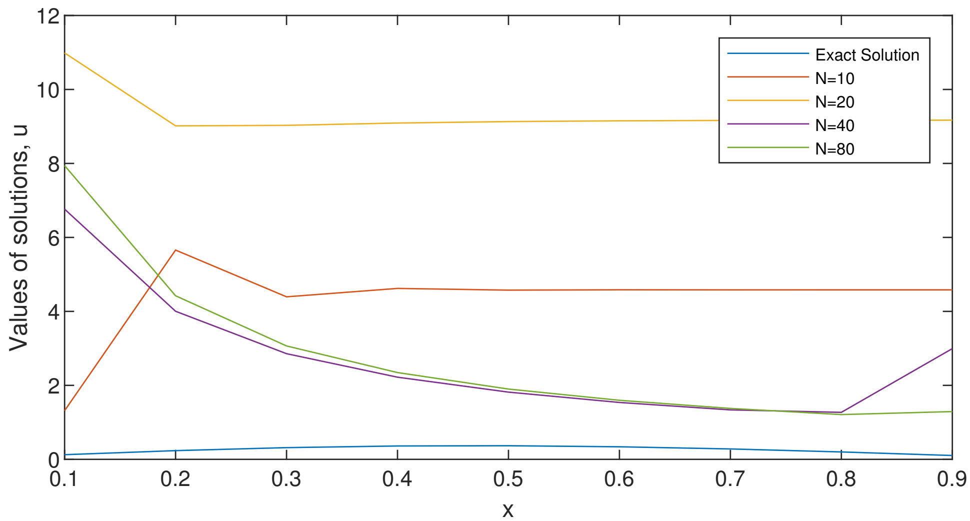


Figure 6.10: Modified-Crank-Nicolson method (M-CNM) solutions corresponding to first and second discrete symmetries with exact solution of the Burgers' equation for $t = 0.1$ *with* $\Delta t = 0.00001$ *at different step size,* N

This also proves a notion that though any differential equation will yield many discrete symmetries, but this does not guarantee that all those discrete symmetries will approximate the exact solution of that differential equation. However, one thing is clear that in all the obtained discrete symmetries, at least one discrete symmetry will approximate the exact solution of the corresponding differential equation.

6.3.4 Comparison of Transformed Burgers' Equation For FTCS, CNM and M-CNM

In this thesis, Burgers' equation has been transformed to diffusion heat equation by using the Hopf-Cole transformation as appeared in Chapter 4. Here, FTCS, CNM and M-CNM were picked to fathom the transformed Burgers' equation. The computed results then can be transformed back to yield a solution of Burgers' equation. To determine the precision and convergence rate of the methods used, a comparison is presented in Table 6.10, 6.11 with the corresponding absolute error difference in Table 6.12, 6.13.

x	Numerical Solution						Exact Solution
	FTCS	CNM	FTCS	CNM	FTCS	CNM	
	$N = 20$		$N = 40$		$N = 80$		
0.1	0.125486314	0.125491666	0.125829775	0.125835169	0.125915700	0.125921105	0.125949755
0.2	0.234985427	0.234995674	0.235578534	0.235588855	0.235726852	0.235737192	0.235786641
0.3	0.315814130	0.315828364	0.316512951	0.316527277	0.316687598	0.316701947	0.316760166
0.4	0.360596054	0.360612961	0.361264136	0.361281137	0.361430977	0.361448001	0.361503607
0.5	0.367493013	0.367510969	0.368040774	0.368058813	0.368177459	0.368195518	0.368241064
0.6	0.339118106	0.339135332	0.339512244	0.339529536	0.339610515	0.339627823	0.339660562
0.7	0.280965179	0.280979934	0.281215677	0.281230476	0.281278083	0.281292894	0.281313681
0.8	0.200015973	0.200026745	0.200154394	0.200165193	0.200188853	0.200199659	0.200211135
0.9	0.103813384	0.103819064	0.103872608	0.103878300	0.103887342	0.103893037	0.103897943

Table 6.10: Comparison of FTCS and CNM solutions with the exact solution of the Burgers' Equation at different step size, N

Table 6.10 shows the computation simulation of the Burgers' equation by using both FTCS and CN methods with the increments of step size, N at time, $t = 0.1$ with a time step size of, $\Delta t = 0.00001$. One can clearly observe that the convergence rate of CNM is faster than FTCS as the step size, N increases.

x	Absolute Error					
	FTCS	CNM	FTCS	CNM	FTCS	CNM
	$N = 20$		$N = 40$		$N = 80$	
0.1	0.463441E-03	0.458089E-03	0.119980E-03	0.114586E-03	0.34055E-04	0.28650E-04
0.2	0.801214E-03	0.790968E-03	0.208108E-03	0.197787E-03	0.59789E-04	0.49449E-04
0.3	0.946035E-03	0.931801E-03	0.247214E-03	0.232888E-03	0.72567E-04	0.58218E-04
0.4	0.907553E-03	0.890646E-03	0.239471E-03	0.222470E-03	0.72630E-04	0.55606E-04
0.5	0.748051E-03	0.730095E-03	0.200290E-03	0.182251E-03	0.63606E-04	0.45546E-04
0.6	0.542456E-03	0.525230E-03	0.148318E-03	0.131026E-03	0.50047E-04	0.32739E-04
0.7	0.348502E-03	0.333748E-03	0.098005E-03	0.083205E-03	0.35598E-04	0.20787E-04
0.8	0.195162E-03	0.184389E-03	0.056741E-03	0.045942E-03	0.22282E-04	0.11476E-04
0.9	0.084559E-03	0.078879E-03	0.025336E-03	0.019644E-03	0.10601E-04	0.04906E-04

Table 6.11: Comparison of Absolute error difference of FTCS and CNM solutions with the exact solution of the Burgers' Equation at different step size, N

Table 6.11 reflects the absolute error difference of both FTCS and CNM computed solutions with the exact solution based on Table 6.10. As it is clearly evident from Table 6.11 that for $N = 20$, the error difference of CNM is smaller in contrast to FTCS. However, for $N = 40$, and $N = 80$, the error difference has been auxiliary truncated to a lesser degree due to the increase in step size, N.

x	Numerical Solution						Exact Solution
	CNM	M-CNM	CNM	M-CNM	CNM	M-CNM	
	$N = 20$		$N = 40$		$N = 80$		
0.1	0.125491666	0.125499708	0.125835169	0.125845588	0.125921105	0.125932744	0.125949755
0.2	0.234995674	0.235015589	0.235588855	0.235610615	0.235737192	0.235759881	0.235786641
0.3	0.315828364	0.315856751	0.316527277	0.316557007	0.316701947	0.316732333	0.316760166
0.4	0.360612961	0.360645852	0.361281137	0.361314942	0.361448001	0.361482232	0.361503607
0.5	0.367510969	0.367544519	0.368058813	0.368092950	0.368195518	0.368229900	0.368241064
0.6	0.339135332	0.339166241	0.339529536	0.339560806	0.339627823	0.339659209	0.339660562
0.7	0.280979934	0.281005581	0.281230476	0.281256351	0.281292894	0.281318797	0.281313681
0.8	0.200026745	0.200045127	0.200165193	0.200183739	0.200199659	0.200218176	0.200211135
0.9	0.103819064	0.103828653	0.103878300	0.103888035	0.103893037	0.103902704	0.103897943

Table 6.12: Comparison of CNM and M-CNM solutions with the exact solution of the Burgers' Equation at different step size, N

Table 6.12 promulgates the CNM and M-CNM solutions with the exact solution of the Burgers' equation for time, $t = 0.1$ with step size of time, $\Delta t = 0.00001$ at different step size, N. It can be seen that the computational simulations of M-CNM is eclipsing CNM as the grids are refined.

x	Absolute Error					
	CNM	M-CNM	CNM	M-CNM	CNM	M-CNM
	$N = 20$		$N = 40$		$N = 80$	
0.1	0.458089E-03	0.450047E-03	0.114586E-03	0.104167E-03	0.28650E-04	0.17011E-04
0.2	0.790968E-03	0.771052E-03	0.197787E-03	0.176026E-03	0.49449E-04	0.26760E-04
0.3	0.931801E-03	0.903415E-03	0.232888E-03	0.203159E-03	0.58218E-04	0.27833E-04
0.4	0.890646E-03	0.857755E-03	0.222470E-03	0.188665E-03	0.55606E-04	0.21375E-04
0.5	0.730095E-03	0.696545E-03	0.182251E-03	0.148114E-03	0.45546E-04	0.11164E-04
0.6	0.525230E-03	0.494321E-03	0.131026E-03	0.099756E-03	0.32739E-04	0.01353E-04
0.7	0.333748E-03	0.308100E-03	0.083205E-03	0.057330E-03	0.20787E-04	0.05116E-04
0.8	0.184389E-03	0.166008E-03	0.045942E-03	0.027396E-03	0.11476E-04	0.07041E-04
0.9	0.078879E-03	0.069290E-03	0.019644E-03	0.009908E-03	0.04906E-04	0.04761E-04

Table 6.13: Comparison of Absolute error difference of CNM and M-CNM solutions with the exact solution of the Burgers' Equation at different step size, N

Table 6.13 is based on Table 6.12 which displays the error difference of CNM and M-CNM solutions with the exact Burgers' equation. It is clearly observed that both numerical methods are reasonably in good agreement with the exact solution as the error has been reduced significantly with each increment of a step size, N. Moreover, M-CNM behaves more refined and swiftly approaching zero as compared to CNM.

6.4 Conclusion

The main objective of this comparison is to parade the precision of all three simulations in *MATLAB*. All three numerical schemes demonstrate that the more precise the solutions are, when certain constraints apply, as the number of step size, N increases. For simple and efficient confirmation of the accuracy of the used numerical schemes the absolute difference between exact and numerical solutions has been determined, in which all three numerical schemes demonstrate a promising result. In addition to the precision, graphic layout for all the solutions are analyzed as time is increased, in which the result is achieved that the solution decreases as time increases. All three numerical schemes focus on various approaches for solving the Burgers' equation. Logically, M-CNM obtained by using second discrete symmetry group ζ_2 yields more accurate solutions compare to FTCS and CNM as seen in the error difference Tables 6.12-6.13. The explanation is that the truncation error of M-CNM is of second order with some constants in terms of time derivative, whereas FTCS and CNM has first and second order truncation error in terms of time derivative respectively, thereby confirming the convergence of the numerical scheme M-CNM discussed earlier in Section 6.2. In short, all three numerical methods are applicable to approximate the solution of Burgers' equation, however due to high accuracy, M-CNM can therefore be considered to be competitive with the other two methods and worth recommendation. Simultaneously, the results of this analysis showed that the trajectory of the computer simulations is on the correct course.

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Appendix

This appendix discusses the numerical schemes for FTCS and CNM.

Explicit Finite Difference Scheme (FTCS)

The finite difference scheme for the heat equation is given by

$$u_{n,j+1} = \alpha u_{n,j} + \alpha \left(u_{n+1,j} - 2u_{n,j} + u_{n-1,j} \right),$$

where $\alpha = \frac{\Delta t}{(\Delta x)^2}$ and for the mesh points (x_n, t_j), Δx and Δt are the space and time step size respectively. This scheme is conditionally convergent with a bound $0 < \alpha < \frac{1}{2}$ on α and a truncation error of $\mathcal{O}(\Delta t)$ [37].

Crank-Nicolson Method (CNM)

The Crank-Nicolson scheme for diffusion heat equation is

$$2u_{n,j} + \alpha \left(u_{n+1,j} - 2u_{n,j} + u_{n-1,j} \right) = 2u_{n,j+1} + \alpha \left(-u_{n+1,j+1} + 2u_{n,j+1} - u_{n-1,j+1} \right).$$

This numerical scheme is implicit and unconditionally stable [37] with a great significance for the time-accurate solutions. It has a truncation error of $\mathcal{O}(\Delta t^2)$.