CHAPTER 1

INTRODUCTION

1.1 MOTIVATION

Designing control systems that are capable to deal with system uncertainties, environmental disturbances and perform optimally has gained great attention in past few decades. In order to deal with these design tasks numerous robust control theories have been proposed such as H_∞ theory, H_2 theory and ℓ_1 theory etc. The design objectives of a controller for MIMO system are mostly conflicting and posing the objective in term of single norm is generally not enough. Therefore mixed norm formulation has great interest for control community. It is the motivation that let me to study multi-objective controller synthesis problem for MIMO system.

1.2 CONTROLLER DESIGN GOALS

A control system should be well-designed in order to get desirable performance. Furthermore a well-designed control system will be tolerant of uncertainty in the model or changes that occur in the system, this significant feature of a control system is called robustness [6]. Major tasks for controller are given below:

1.2.1 PERFORMANCE SPECIFICATION

Performance specifications of the system are measures which reflect how well closed loop system is performing. Examples are given below:

- ***Good Regulation***

Disturbances are noises that affect some critical variables of system and there action on system may cause some undesirable performance. The capability of system to abate the effects of disturbances is known as regulation.

- ***Desirable Response***

System variables should respond in desirable way to commands inputs. For example in pitch attitude control of an aircraft, the aircraft should attain the commanded pitch attitude smoothly, should not oscillate etc.

- ***Critical Signals remains in limit***

There are various signal in control loop that are critical, in particular the actuator signals are very important. It is necessary that these signals should be remain in limit and do not touch there saturation level.

1.2.2 ROBUSTNESS SPECIFICATION

Robustness specifications limit the variation in performance of the closed loop system that can be caused by variations in the system to be controlled or differences between the system to be controlled and its model [6]. A robust system should control the perturbations in system due to drift in system components, or in temperature coefficients etc. It is also possible that system is inaccurately modeled or intentionally ignored some frequency modes or nonlinearities.

1.2.3 CONTROL LAW SPECIFICATION

Along with other goals and design specification mentioned above, some time it is also case that, there are some constraints for control law. These control law specifications are often related to the realization of the controller [6]. For examples specification of controller can be order of controller, time invariant or time variant controller and linear on nonlinear controller etc [7].

The control literature is evident of the fact that controller deign goals are conflicting and it is merely impossible to achieve all specification by using single norm optimization problem. In order to achieve these multiple tasks, it is essential to make a multi-objective control problem which includes more than one norm optimization with some time domain constraints.

1.3 SCOPE OF THE WORK

The aim of this work is to study and understanding of multi-objective optimization problem. In this work the optimization problem includes H_∞ and H_2 of some closed-loop transfer function with Pole placement constraints. Multi-objective H_∞/H_2 state feedback control with constraints on pole placement is an approach to synthesize multi-objective controller for MIMO system. In this technique design task such as disturbance rejection, robust stabilization of systems can be expressed by H_∞/H_2 performance and by placing the closed-loop poles in some region of left-Half plane one can achieve desired transient response. This thesis covers synthesizing multi-objective controller for pitch attitude hold autopilot of an aircraft. MATLAB software is used for simulation and verification of performance.

1.4 CHAPTERS ORGANIZATION

This thesis is structured as follows:

Chapter 2 briefly discusses the literature review of the work done in multi-objective controller design.

Chapter 3 is titled as background material, as its name suggests, it contains related material which can be taken as the pre requisites for this thesis.

Chapter 4 discusses the problem setup for multi-objective controller synthesis in detail.

In chapter 5 the different terminologies and concepts related aircraft is briefly discussed. This chapter also includes the simulation results of Pitch attitude hold autopilot.

Finally the chapter 6 concludes the thesis work and contains the future recommendations.

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter we will discuss the literature review of work done in related field. In fact many books and research papers were consulted, details are given in references, but here few research papers discuss briefly.

2.2 BRIEF STUDY UNDERTAKEN

Control systems design is accomplished in order to make the closed-loop system attains some performance indices. The final design is always a trade off among several conflicting control objectives, related with the system physical characteristics. The multi-objective control design can be viewed as the search for a suitable trade off among distinct objectives as, for instance, the control effort minimization versus the achievement of some strict performance specifications [4].

Another research paper "Mixed ℓ_1/H_∞ Control for MIMO systems via Convex Optimization" by M. Snzaier and J. Bu [3] discusses drawback of single norm control. "Clearly, a single norm is usually not enough to capture different, and often conflicting, design specifications, such as simultaneous rejection of disturbances having different characteristics (white noise, bounded energy, persistent); good tracking of classes of inputs; satisfaction of bounds on peak values of some outputs; closed-loop bandwidth, etc" [3].

"Multi-objective control: An overview" a research paper by B. Vroeman and B.D. Jager [2] discusses the various approaches for multi-objective control problem. "Several important controller synthesis problems have been formulated as optimization problems. Notably, LQG or H_2, H_∞ and ℓ_1 control theory have provided us with basic synthesis tools. The underlying premise behind these theories is that all the design objectives can be translated into minimizing a suitably weighted norm of a closed-loop transfer function matrix" [2]. This paper in detail converses the limitation of single norm optimization problem. "The LQG approach proved particularly suited to meet performance specifications while guaranteeing closed-loop stability in the presence of disturbances. Despite of this, LQG control was shown to possess no guaranteed robustness margins if applied in conjunction with an observer or Kalman filter" [2]. H_∞ control is suitable to address the system robust stability in the presence of system uncertainty but it does not directly deal with time domain specifications. This paper highlights many mixed norm optimization problem and there solutions such as ℓ_1/H_∞ and ℓ_1/H_2 using linear programming approach, ℓ_1/H_∞ using Youla parameterization and H_2/H_∞ using Matrix Inequalities (MI) and Algebraic Riccati Equation (ARE).

In [1] it is described that, "H_∞, design deals mostly with frequency-domain aspects and provides little control over the transient behavior and closed-loop pole location. In contrast, satisfactory time response and closed-loop damping can often be achieved by forcing the closed-loop poles into a suitable sub region of the left-half plane".

"General multiobjective control problems are difficult and remain mostly open to this date. By multiobjective control, we refer to synthesis problems with a mix of time- and frequency domain specifications ranging from H_2 and H_∞ performance to regional pole placement, asymptotic tracking or regulation, and settling time or saturation constraints" [5].

It is nice to include all three norms in optimization problem, but two norms (H_2 and H_∞) optimization problem is focused by most approaches. From the literature reviewed, it is clear that above mention norms is not giving guarantee regarding time domain response of closed-loop system therefore it is important to include time domain constraints in optimization problem.

2.3 SUMMARY

In this chapter brief overview of the work done in the field of multi-objective control synthesis for MIMO systems was discussed. It can be concluded that single norm based controller schemes are not sufficient to cover all design requirements. Therefore a mixed norm design technique is required. It is also important for a multi-objective problem setup that it include some time domain constraints in order to get desired time domain response of closed loop system.

CHAPTER 3

BACKGROUND MATERIAL

3.1 INTRODUCTION

This chapter covers the material required as a pre-requisite to understand the multi-objective control problem. First we briefly discuss about signals and system norm, some fundamental concepts of convex set and convex function. This chapter also includes general optimization problem, convex optimization problem, Linear Matrix Inequalities (LMI) and other related topics.

3.2 NORM

Norm is a function which provides a single number which reflects an overall size of vector or a matrix, or of a signal or a system [8]. The properties of norms are articulated in vector space framework.

A norm of *e* (which may be a signal, system vector or matrix,) is a real number denoted $\| e \|$ that satisfy following properties [8]

- Non-negative $\quad \| e \| \geq 0 \;\; \forall\, e \in V$
- Positive $\quad \| e \| = 0 \Leftrightarrow e = 0$
- Homogeneity $\quad \| \alpha . e \| = |\alpha| . \| e \| \;\; \forall\, \alpha \in C$
- Triangular Inequality $\quad \| e_1 + e_2 \| \leq \| e_1 \| + \| e_2 \| \;\; \forall\, e_1, e_2 \in V$

Where V is a vector space defined over complex field C.

3.2.1 SCALAR SIGNALS NORMS

This section discusses few common scalar signal norms.

3.2.1.1 PEAK NORM

If u is any signal then its peak L_∞ norm is defined as its maximum or peak value. The peak norm of a signal is used to specify a firm limit on the absolute value of signal [6]. Mathematically

$$\|u\|_\infty = \sup_{t \geq 0} |u(t)| \tag{3.1}$$

Figure 3.1 shows signal u and its peak norm $\|u\|_\infty$

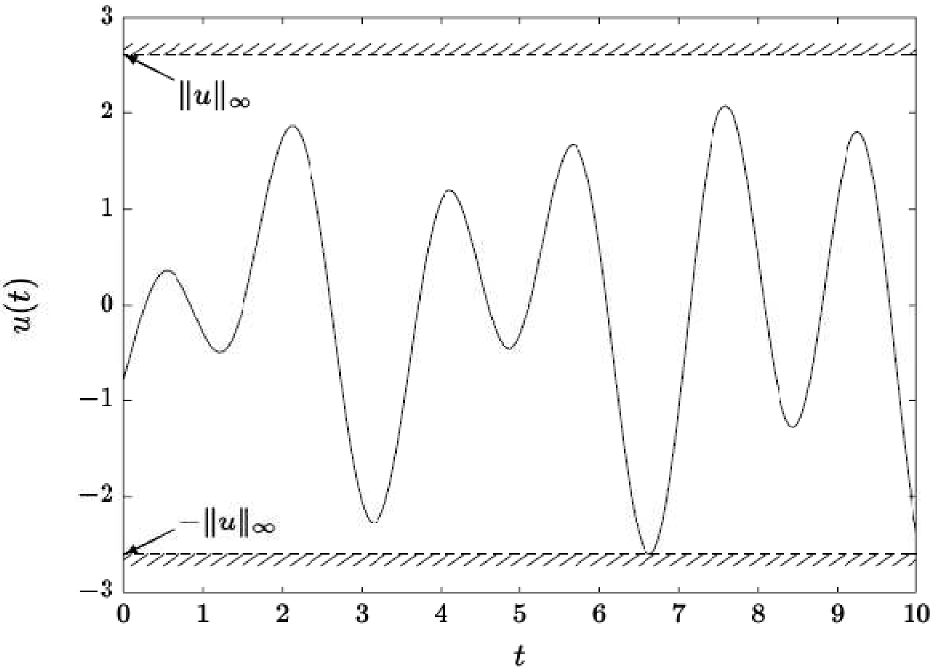

Figure 3.1**:** Signal *u(t)* and its peak norm $\|\boldsymbol{u}\|_\infty$ [6]

3.2.1.2 ROOT MEAN SQUARE NORM

Root mean square (RMS) of a signal is a measure that reflects its eventual average size; it is defined by [6]

$$\|u\|_{rms} \triangleq \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T u(t)^2 \, dt \right)^{\frac{1}{2}} \qquad (3.2)$$

It is essential that the RMS norm is a steady state measure of a signal, any transient in signal doesn’t affect the RMS value of signal. In particular a signal with small RMS value can be very large for some initial time period [6]. Figure 3.2 shows signal *u(t)* and its rms norm $\|u\|_{rms}$

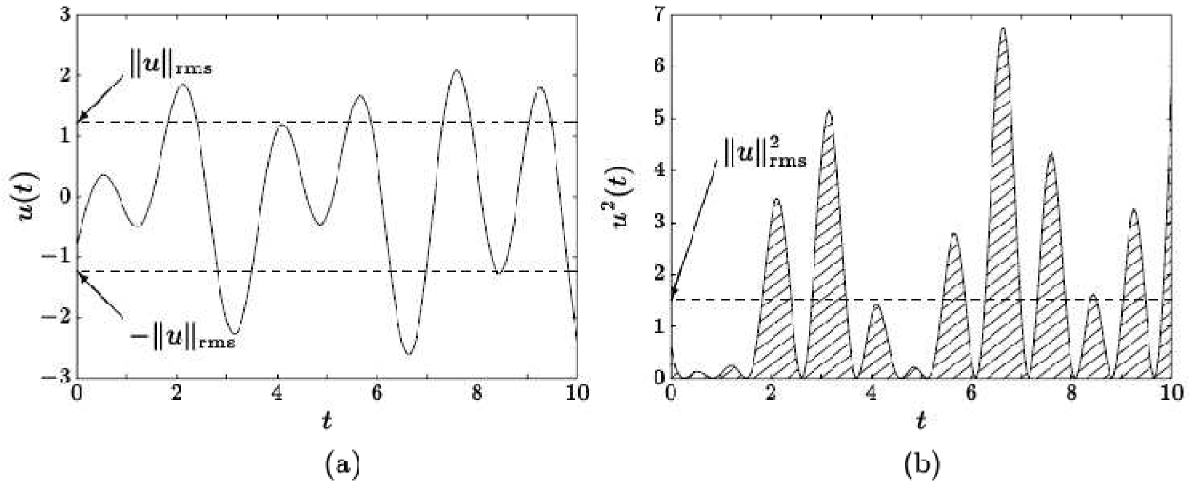

Figure 3.2: (a) A signal u and its RMS value $\|u\|_{rms}$ (b) $\|u\|_{rms}^2$ is the average area under u^2

3.2.1.3 AVERAGE ABSOLUTE NORM

Average absolute norm is less affected by large value of signal it is given as

$$\|u\|_{aa} \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T |u(t)|\,dt \tag{3.3}$$

The average absolute norm $\|u\|_{aa}$ is useful in measuring average fuel or resource use when the fuel or resource consumption is proportional to $|u(t)|$. Figure 3.3 shows the signal *u(t)* and its average absolute norm $\|u\|_{aa}$.

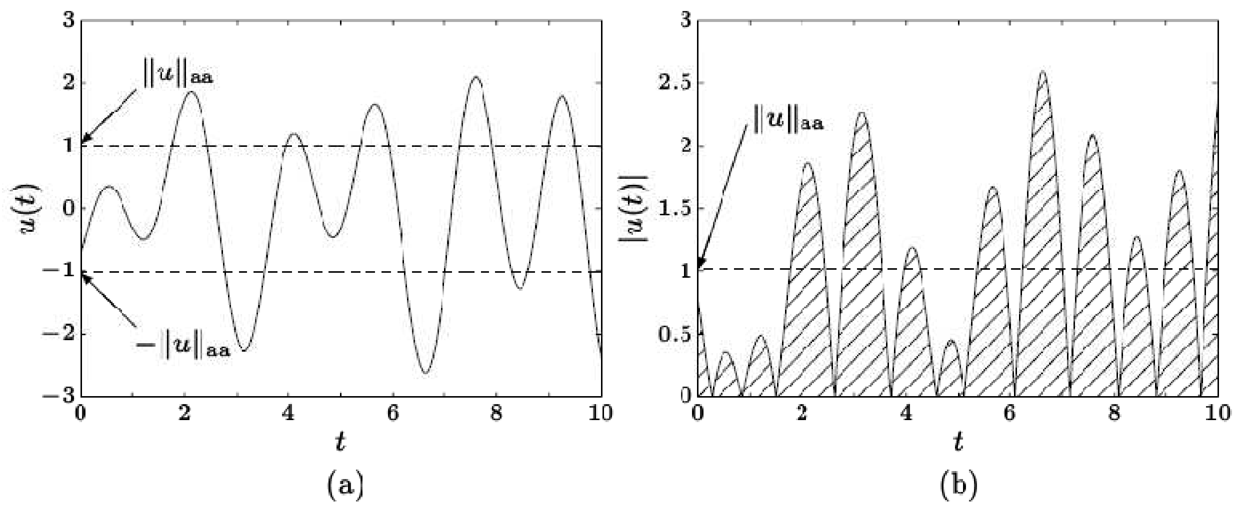

Figure 3.3: (a) A signal u and its RMS value $\|u\|_{aa}$ (b) $\|u\|_{aa}$ is the average area under $|u(t)|$

3.2.1.4 L_2 NORM : SQUARE ROOT TOTAL ENERGY

This norm is the appropriate analog of the RMS norm for decaying signals i.e., signals with finite total energy as opposed to finite steady state power [6]. L_2 norm is defined as

$$\|u\|_2 \triangleq \left(\int_0^\infty u(t)^2 \, dt \right)^{\frac{1}{2}} \tag{3.4}$$

3.2.1.5 L_1 NORM

Just as L_2 norm measures total energy in a signal, while the RMS norm measures its average power, the L_1 norm of a signal can be thought as measure of total resource consumption, while the average absolute norm measures a steady state average resource consumption [6]. It is defined as

$$\|u\|_1 \triangleq \int_0^\infty |u(t)| \, dt \tag{3.5}$$

3.2.2 VECTOR SIGNALS NORMS

Few important norms of vector signals are briefly explained in this section.

3.2.2.1 PEAK NORM

Maximum peak of any component of vector signal is known as the Peak or L_∞ norm of vector signal.

$$\|u\|_\infty \triangleq \max_{1 \le i \le n} \|u_i\|_\infty = \sup_{t \ge 0} \max_{1 \le i \le n} |u_i| \tag{3.6}$$

3.2.2.2 RMS NORM

The RMS norm of vector signals is given as

$$\|u\|_{rms} \triangleq \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)^T \, u(t) \, dt \right)^{\frac{1}{2}} \tag{3.7}$$

It is provided that the limit exists.

3.2.2.3 AVERAGE ABSOLUTE NORM

The average absolute norm of a vector signal is defined as

$$\|u\|_{aa} \triangleq \left(\lim_{T \to \infty} \sup \frac{1}{T} \int_0^T \sum_{i=1}^n |u(t)|_i \, dt \right)^{\frac{1}{2}} \tag{3.8}$$

It is measure average total resource consumption of all the components of u [6].

3.2.2.4 L_2 AND L_1 NORM

L_2 and L_1 norm of the vector signals is defined as

$$\|u\|_2 \triangleq \left(\int_0^T \sum_{i=1}^{n} u_i(t)^2 \, dt \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} \|u_i(t)\|_2^2 \right)^{\frac{1}{2}} \quad (3.9)$$

$$\|u\|_1 \triangleq \int_0^\infty \sum_{i=1}^{n} |u_i(t)| \, dt = \sum_{i=1}^{b} \|u_i(t)\|_1 \quad (3.10)$$

3.2.3 SYSTEM NORMS

System norms are measured in terms of its input and output signal norms. This section emphasize on general method of finding different system norms of a LTI system having input *w*, output *z* and transfer function matrix *H*, shown in fig. 3.4.

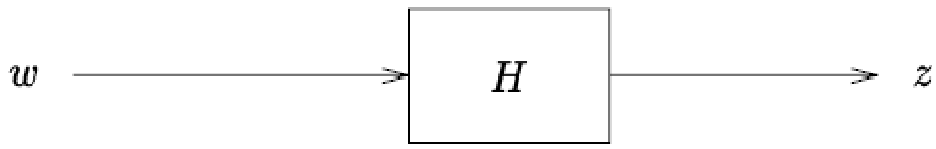

Figure 3.4: A LTI system with output *z* and input *w*

3.2.3.1 H_2 NORM

The H_2 norm of an LTI system is defined as, "the square root of average power (RMS value or 'Power norm') of the response to a white input signal with unit spectral density" [2]. The H_2 norm of a stable system is given as

$$\|H\|_2 \triangleq \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |H(j\omega)|^2 \, d\omega \right)^{\frac{1}{2}} \quad (3.11)$$

Lyapunov equations are used to compute the 2-norm [2]:

$$\|H\|_2^2 = tr[SC^TC] = tr[PBB^T] \quad (3.12)$$

Where *P* is the observability gramian and *S* is the controllability gramian solving

$$B B^T + S A^T + A S = 0$$

$$A^T P + P A + C^T C = 0$$

Another interpretation can be given to H_2 norm, by the Parseval theorem

$$\|H\|_2 = \left(\int_0^\infty h(t)^2\, dt\right)^{\frac{1}{2}} = \|h\|_2 \tag{3.13}$$

the L_2 norm of the impulse response h of the LTI system. Thus H_2 norm of LTI system can be interpreted as L_2 norm of its response to the unit impulse input.

3.2.3.2 ℓ_1 NORM

The ℓ_1 norm of the LTI system is actually its peak gain. It can be defined as the induced norm from L_∞ to L_∞.

$$\|H\|_1 \triangleq \sup_{\|w\|_\infty \neq 0} \frac{\|Hw\|_\infty}{\|w\|_\infty} \tag{3.14}$$

The L_1 norm of impulse response is equal to peak gain of a transfer function [6].

$$\|H\|_1 = \int_0^\infty |h(t)|\, dt = \|h\|_1 \tag{3.15}$$

Stable transfer function has finite peak gain [6].

3.2.3.3 H_∞ NORM

The H_∞ norm of the system is defined as the highest singular value of the system

$$\|H\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}\left(H(j\omega)\right) \tag{3.16}$$

The H_∞ norm of system can be interpreted as the RMS gain of the system or it can also define as induced norm from L_2 to L_2.

$$\|H\|_\infty \triangleq \sup_{\|w\|_2 \neq 0} \frac{\|Hw\|_2}{\|w\|_2} \tag{3.17}$$

For unstable system $\|H\|_\infty = \infty$

3.3 CONVEX SET

A convex set C_o is defined as if the line section between any two points in C_o lies in C_o, i.e., if for any $x_1, x_2 \in C_o$ and any ϕ with $0 \leq \phi \leq 1$, we have

$$\phi x_1 + (1 - \phi)x_2 \in C_o \tag{3.18}$$

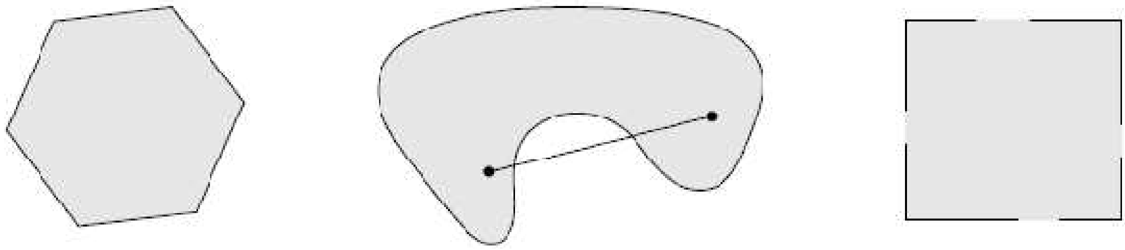

Figure 3.5: Some convex and non-convex sets. *Right* Not Convex *Middle* Not Convex and *Left* Convex [9]

Generally speaking, in a convex set every point in the set can be seen by every other point, along a clear direct path between them, where clear means within the set. Some straightforward convex and non convex sets in $\boldsymbol{R}^2$ are illustrated in figure 3.6. In left the hexagon included the boundary is convex, kidney like shape in the middle is not a convex set because the line segment shown between two dots is not within the set and in right the rectangular shape is not convex because some of its boundary points do not lie in the set.

3.4 CONE

A set C_o is a cone, if for every $x \in C_o$ and $\phi \geq 0$ we have $\phi x \in C_o$. A set C_o is a convex cone if it is convex and a cone, which means that for any $x_1, x_2 \in C_o$ and $\phi_1, \phi_2 \geq 0$, along with

$$\phi_1 x_1 + \phi_2 x_2 \in C_o \tag{3.19}$$

"Points of this form can be described geometrically as forming the two-dimensional pie slice with apex 0 and edges passing through x_1 and x_2" [9].

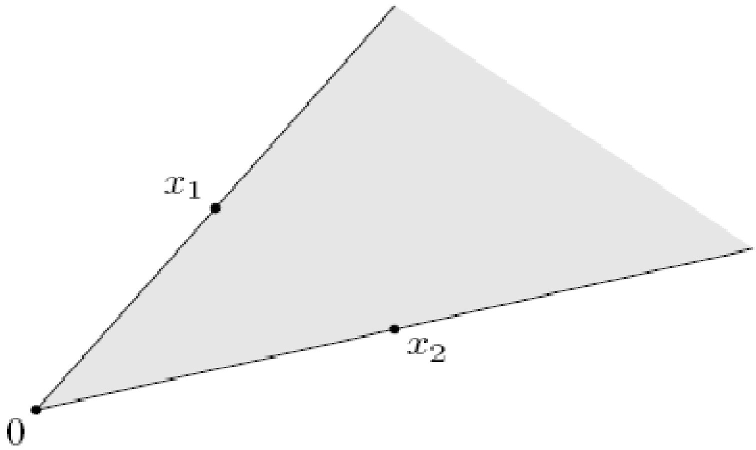

Figure 3.6: All points of the form $\phi_1 x_1 + \phi_2 x_2$ [9]

3.5 GENERALIZED INEQUALITIES

A cone $X \in \boldsymbol{R}^n$ is known a proper cone if it fulfills the conditions given below:

- X is convex.
- X is closed.
- X is solid, i.e. its interior is nonempty.
- X is pointed, i.e. it includes no line

Generalized inequality can be described using a proper cone X, which is a partial ordering on $\boldsymbol{R}^n$ that has number of the characteristics of normal ordering on R.

Partial ordering on $\boldsymbol{R}^n$ given below related with the proper cone X

$$x \preccurlyeq_X y \Leftrightarrow y - x \in X$$

We also write $x \succcurlyeq_K y$ for $x \preccurlyeq_K y$. Similarly, we state an related strict partial ordering by

$$x \prec_X y \Leftrightarrow y - x \in \operatorname{int} X$$

and write $x \succ_X y$ for $y \prec_X x$.

3.6 CONVEX FUNCTION

If domain of function $f : \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ is a convex set and if for all $x,y \in$ domain of f, and ϕ with $0 \leq \phi \leq 1$, then function is called convex function. Mathematically

$$f(\phi x + (1 - \phi)y) \leq \phi f(x) + (1 - \phi)f(y) \tag{3.20}$$

Looking the geometry of this inequality it reflects that the line section between $(x, f(x))$ and $(y, f(y))$, which is the chord from x to y, lies above the graph of f shown in figure 3.7. Strict inequality gives strict convex when $x \neq y$ and $0 < \theta < 1$.

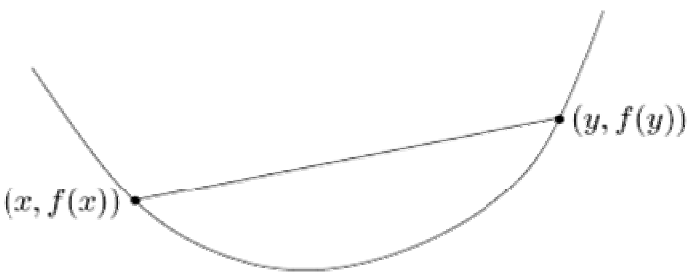

Figure 3.7: Convex function plot. The chord between two points on the graph lies above the plot [9].

3.6.1 EXAMPLES

In this section few examples of convex functions are discussed for further detail refer [9]. It is mentioned earlier that affine and linear functions are convex function few other function are given below;

- **Exponential Functions**. e^{bx} on $\boldsymbol{R}$ is convex, for any $b \in \boldsymbol{R}$
- **Powers of absolute value Functions**. $|x|^q$, for $q \geq 1$, on $\boldsymbol{R}$ is convex
- **Norms Functions**. Every norm if defined on $\boldsymbol{R}^n$ is convex
- **Maximum Functions**. $f(x) = \max\{x_1, \ldots, x_n\}$ on $\boldsymbol{R}^n$ is convex

3.7 OPTIMIZATION PROBLEM

An optimization problem has the form

$$\begin{aligned} &\text{minimize} && f_0(x) \\ &\text{subject to} && f_i(x) \leq b_i,\ i = 1, \ldots, m \\ &&& h_i(x) = 0,\ i = 1, \ldots, p \end{aligned} \tag{3.21}$$

Here the vector $x = (x_1, \ldots, x_n)$ is the optimization variable of the problem, the function f_0 : $\boldsymbol{R}^n \rightarrow \boldsymbol{R}$ is the objective function, the functions f_i : $\boldsymbol{R}^n \rightarrow \boldsymbol{R}$, $i = 1, \ldots, m$, are the (inequality) constraint functions, and the constants $b_1, \ldots, b_m$ are the limits, or bounds, for the constraints. The equations $h_i(x) = 0$ are called the equality constraints, and the functions h_i : $\boldsymbol{R}^n \rightarrow \boldsymbol{R}$ are the equality constraint functions [9].

The optimization domain ($\mathfrak{D}$) is defined as the set of points for which objective and constraints functions are defined. A point $x \in \mathfrak{D}$ which satisfies all constraints functions i.e. equality as well as inequality constraints functions, is called feasible point.

The smallest objective value (x^*) among all vectors that satisfy the constraints is called optimal value, if it has for any y with $f_1(y) \leq b_1, \ldots, f_m(y) \leq b_m$, we have $f_0(y) \geq f_0(x^*)$ [9].

It can be said that x is locally optimal if there is $R > 0$ and x solve the following optimization problem with variable z

$$\begin{aligned} &\text{minimize} && f_0(z) \\ &\text{subject to} && f_i(z) \leq b_i,\ i = 1, \ldots, m \\ &&& h_i(z) = 0,\ i = 1, \ldots, p \end{aligned} \tag{3.22}$$

$$\| z - x \|_2 \leq R$$

this means x minimizes f_0 over nearby points in the feasible set. Optimization problems are classified or categorized on the basis of characteristics of objective and constraints functions.

3.7.1 LINEAR OPTIMIZATION PROBLEM

When the objective and constraint functions are all affine, the problem is called a linear program (LP). A general linear program has the form

$$\begin{aligned} &\text{minimize} \quad C^T x + d \\ &\text{subject to} \quad Gx \preccurlyeq h \\ &\qquad\qquad\quad Ax = b \end{aligned} \qquad (3.23)$$

Where $G \in \boldsymbol{R}^{m\times n}$ and $A \in \boldsymbol{R}^{p\times n}$. Linear programs are, of course, convex optimization problems. It is common to omit the constant d in the objective function, since it does not affect the optimal (or feasible) set [9].

3.7.2 CONVEX OPTIMIZATION PROBLEM

One form of convex optimization problem is given as

$$\begin{aligned} &\text{minimize} && f_0(x) && \\ &\text{subject to} && f_i(x) \preceq 0 && i = 1, 2, \ldots\ldots\ldots\ldots, m \\ & && a_i^T x = b_i && i = 1, 2, \ldots\ldots\ldots\ldots, p \end{aligned} \qquad (3.24)$$

Where $f_0, \ldots, f_m$ are convex functions. Comparing convex optimization problem with the general standard form problem, the convex problem has three additional requirements:

- the objective function must be convex
- the inequality constraint functions must be convex
- the equality constraints functions $h_i(x) = a_i^T x = b_i$ must be affine

It is also important to note that the feasible set of convex optimization problem is convex set because it is the intersection of domain of constraints functions. A primary property of convex optimization problems is that any locally optimal point is also (globally) optimal.

3.7.3 QUADRATIC OPTIMIZATION PROBLEM

The convex optimization problem is called a quadratic program (QP) if the cost function is (convex) quadratic, and the constraint functions are affine. A quadratic program can be expressed in the form

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T Px + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array} \tag{3.25}$$

Where $P \in \boldsymbol{S}_+^n$, $G \in \boldsymbol{R}^{m\times n}$, and $A \in \boldsymbol{R}^{p\times n}$. In a quadratic program, we minimize a convex quadratic function over a polyhedron [9].

3.7.4 SECOND ORDER CONE PROGRAMMING

Second order cone programming (SOCP) is closely related to quadratic programming:

$$\begin{array}{lll} \text{minimize} & f^T x & \\ \text{subject to} & \|A_i x + b_i\| \le c^T x + d_i & i = 1, 2, ..., m \\ & Fx = g & \end{array} \tag{3.26}$$

Where optimization variable is $x \in \boldsymbol{R}^n$, $A_i \in R^{n_i x n}$, and $F \in \boldsymbol{R}^{p\times n}$. We call a constraint of the form

$$\|A_i x + b_i\| \le c^T x + d_i$$

Where $A \in \boldsymbol{R}^{k\times n}$, a second-order cone constraint, since it is similar to required the affine function ($Ax + b, c^Tx + d$) to include in the second-order cone in $\boldsymbol{R}^{k+1}$ [9].

3.7.5 VECTOR OPTIMIZATION

The general vector optimization problem can be denoted as

$$\begin{array}{lll} \text{minimize (w.r.t } K) & f_0(x) & \\ \text{subject to} & f_i(x) \le 0 & i = 1, 2, ..., m \\ & h_i(x) = 0 & i = 1, 2, ..., p \end{array} \tag{3.27}$$

Here $x \in \boldsymbol{R}^n$ is the optimization variable, $K \subseteq \boldsymbol{R}^q$ is a proper cone, $f_0 : \boldsymbol{R}^n \to \boldsymbol{R}^q$ is the objective function, $f_i : \boldsymbol{R}^n \to \boldsymbol{R}$ are the inequality constraint functions, and $h_i : \boldsymbol{R}^n \to \boldsymbol{R}$ are the equality constraint functions. The only difference between this problem and the standard optimization problem is that here, the objective function takes values in $\boldsymbol{R}^q$, and the problem specification includes a proper cone *K*, which is used to compare objective values [9].

3.7.5.1 OPTIMAL POINTS AND VALUES

Consider the set of objectives values of feasible points

$$\mathcal{O} = \{f_0(x) | \exists\, x \in \mathcal{D}, f_i(x) \leq 0, i = 1, \ldots, m,\ h_i(x) = 0, i = 1, \ldots, p\} \subseteq \mathrm{R}^q \tag{3.28}$$

This is called the set of achievable objective values. If this set has a minimum element i.e., there is a feasible x such that $f_0(x) \preccurlyeq_K f_0(y)$ for all feasible y, then we say x is optimal for this problem, and refer to $f_0(x)$ as the optimal value of the problem. (When a vector optimization problem has an optimal value, it is unique.) If x^* is an optimal point, then $f_0(x^*)$, the objective at x^*, can be compared to the objective at every other feasible point, and is better than or equal to it. Roughly speaking, x^* is unambiguously a best choice for x, among feasible points [9]. A point x^* is optimal if and only if it is feasible and

$$\mathcal{O} \subseteq f_0(x^*) + K$$

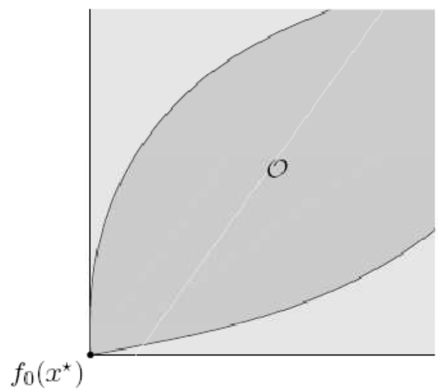

Figure 3.8: The set $\mathcal{O}$ of achievable values for a vector optimization with objective values in R^2, with cone K = R^2_+, is shown shaded. $f_0(x^*)$ is the optimal value of the problem, and x^* is an optimal point [9].

3.7.5.2 PARETO OPTIMAL POINTS AND VALUES

Now consider the case in which the set of achievable objective values does not have a minimum element, so the problem does not have an optimal point or optimal value. In these cases minimal elements of the set of achievable values play an important role. We say that a feasible point x is Pareto optimal (or efficient) if $f_0(x)$ is a minimal element of the set of achievable values $\mathcal{O}$. In this case we say that $f_0(x)$ is a Pareto optimal value for the vector optimization problem. Thus, a point x is Pareto optimal if it is feasible and, for any feasible y, $f_0(y) \preccurlyeq_K f_0(x)$ implies $f_0(y) = f_0(x)$. In other words: any feasible point y that is better than or equal to x (i.e., $f_0(y) \preccurlyeq_K f_0(x)$) has exactly the same objective value as x [9].

A point x is Pareto optimal if and only if it is feasible and

$$(f_0(x) - K) \cap \mathcal{O} = \{f_0(x)\} \tag{3.29}$$

A vector optimization problem can have many Pareto optimal values (and points). The set of Pareto optimal values, denoted P, satisfies

$$P \subseteq \mathcal{O} \cap \mathbf{bd}\, \mathcal{O} \tag{3.30}$$

i.e., every Pareto optimal value is an achievable objective value that l within the boundary of the set of achievable objective values [9].

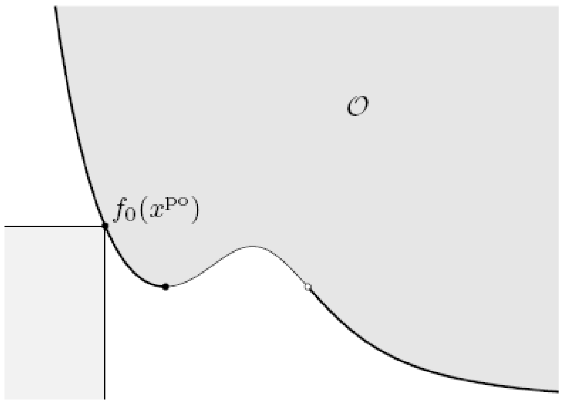

Figure 3.9: The set $\mathcal{O}$ of achievable values for a vector optimization with objective values in R^2, with cone $K = R^2_+$, is shown shaded. Pareto optimal points, whose corresponding values are shown as the darkened curve on the lower left boundary of $\mathcal{O}$. The point labeled $f_0(x^{po})$ is a Pareto optimal value, and x^{po} is a Pareto optimal point [9].

3.8 SCALARIZATION

Scalarization is a standard technique for finding Pareto optimal (or optimal) points for a vector optimization problem, based on the characterization of minimum and minimal points via dual generalized inequalities. Choose any $\lambda \succ_{K^+} 0$, i.e., any vector that is positive in the dual generalized inequality. Now consider the scalar optimization problem

$$\begin{array}{lll} \text{minimize} & \lambda^T f_0(x) & \\ \text{subject to} & f_i(x) \le 0 & i = 1,2,\dots\dots\dots,m \\ & h_i(x) = 0 & i = 1,2,\dots\dots\dots,p \end{array} \quad (3.31)$$

and let x be an optimal point. Then x is Pareto optimal for the vector optimization problem. This follows from the dual inequality characterization of minimal points, and is also easily shown directly. If x were not Pareto optimal, then there is a y that is feasible, satisfies $f_0(y) \preccurlyeq_K f_0(x)$, and $f_0(x) \neq f_0(y)$. Since $f_0(x) - f_0(y) \succcurlyeq_K 0$ and is nonzero, we have $\lambda^T (f_0(x) - f_0(y)) > 0$, i.e., $\lambda^T f_0(x) > \lambda^T f_0(y)$. This contradicts the assumption that x is optimal for the scalar problem [9]. Involve

3.9 MULTICRITERION OPTIMIZATION

A multicriterion or multi-objective optimization problem is a vector optimization problem involves the cone $K = R_+^q$, it is called. The components of f_0, say, $F_1, \dots, F_q$, can be interpreted as q different scalar objectives, each of which we would like to minimize. We refer to F_i as the i^{th} objective of the problem. A multicriterion optimization problem is convex if $f_1, \dots, f_m$ are convex, $h_1, \dots, h_p$ are affine, and the objectives $F_1, \dots, F_q$ are convex.
In a multicriterion problem, an optimal point x^* satisfies

$$F_i(x^*) \le F_i(y) \qquad i = 1,2,\dots\dots\dots,q \quad (3.32)$$

for every feasible y. In other words x^* is simultaneously optimal for each of the scalar problems

$$\begin{array}{lll} \text{minimize} & F_j(x) & j = 1,2,\dots\dots\dots\dots,q \\ \text{subject to} & f_i(x) \le 0 & i = 1,2,\dots\dots\dots\dots,m \\ & h_i(x) = 0 & i = 1,2,\dots\dots\dots\dots,p \end{array} \quad (3.33)$$

When there is an optimal point, we say that the objectives are noncompeting, since no compromises have to be made among the objectives; each objective is as small as it could be made, even if the others were ignored [9].

A Pareto optimal point x^{po} satisfies the following: if y is feasible and $F_i(y) \le F_i(x^{po})$ for $i = \ , ... , q$, then $F_i(x^{po}) = F_i(y)$, $i = 1, \dots, q$. This can be restated as: a point is Pareto optimal if and only if it is feasible and there is no better feasible point. In searching for good points, then, we can clearly limit our search to Pareto optimal points.

3.9.1 SCALARIZING MULTICRITERION OPTIMIZATION

Scalarizing a multicriterion problem by forming the weighted sum objective

$$\lambda^T f_o(x) = \sum_{i=1}^{q} \lambda_i F_i(x) \qquad (3.34)$$

Where $\lambda \succ 0$, we can interpret λ_i as the weight we attach to the i[th] objective. The weight λ_i can be thought of as quantifying our desire to make F_i small (or our objection to having F_i large). In particular, we should take λ_i large if we want F_i to be small; if we care much less about F_i, we can take λ_i small [9].

3.10 LINEAR MATRIX INEQUALITY (LMI)

Constraints of the form given below is called a linear matrix inequality (LMI)

$$A(x) = A_0 + x_1 A_1 + \cdots\cdots\cdots + x_n A_n < 0 \qquad (3.35)$$

Where

- $x = (x_1, \dots\dots, x_n)$ is a vector of optimization variable
- $A_0, \dots\dots, A_n$ are provided symmetric matrices
- <0 represents for "negative definite" [11]

The LMIs are involved in analysis of dynamical system for 100 of years. In year 1890, Lyapunov presented his influential work introducing what we call Lyapunov theory. He proved that the differential equation

$$\frac{dx(t)}{dt} = Ax(t) \qquad (3.36)$$

is stable iff we have a P matrix which is positive-definite

$$A^T P + PA < 0 \qquad (3.37)$$

The requirement $P > 0$, $A^T P + PA < 0$ is what we call a Lyapunov inequality on P, that is a particular form of an LMI. Lyapunov also showed that this first LMI could be unambiguously solved. Indeed, we can pick any $Q = Q^T > 0$ and then solve the linear equation $A^T P + PA = -Q$ for the matrix P, if the system is stable it is guaranteed to be positive-definite [10].

The LMI is a convex constraint on x, therefore the solution set, called as feasible set is a convex subset of R^N. The solution of this LMI is actually a convex optimization problem. The convexity of a LMI is very important characteristics because even though if it does not have analytical solution, it can be solved numerically.

In the majority of applications in control, LMIs don't naturally occur in canonical form but rather in the form given below

$$L(X_1, \dots \dots \dots, X_n) < M(X_1, \dots \dots \dots, X_n) \tag{3.38}$$

Where *M(.)* and *L(.)* are affine functions of variable $X_1, \dots \dots \dots, X_n$. Lyapunov inequality is most straightforward example

$$A^T X + XA < 0 \tag{3.39}$$

Many problems in control and design specifications have LMI formulations [10]. This is especially true for Lyapunov-based analysis and design, but also for optimal LQG control, H_∞ control, covariance control, etc. Further LMIs applications occur in estimation, identification, optimal design, structural design matrix scaling problems, and so on [11].

3.10.1 THREE GENERIC LMI PROBLEMS

There are three basic problem of LMI which are briefly discussed below.

- ***Feasibility Problem***

Finding an answer *x* to LMI system

$$A(x) < 0 \tag{3.40}$$

This is known as feasibility problem.

- ***Linear objective minimization***

It is widely used optimization problem and its play very important role in LMI based design. Minimizing convex cost function under LMI constraints is also convex problem [11]. The linear objective minimization problem is denoted as

$$\begin{aligned} &\text{minimize} \quad && c^T x \\ &\text{subject to} \quad && A(x) < 0 \end{aligned} \tag{3.41}$$

- ***Generalized Eigenvalue minimization***

It is in fact quasi-convex optimization problem but similar techniques can be used for solving this problem.

$$\begin{aligned} &\text{minimize} && \lambda \\ &\text{subject to} && A(x) < \lambda B(x) \\ &&& B(x) > 0 \\ &&& C(x) < 0 \end{aligned} \tag{3.42}$$

3.11 CHAPTER SUMMARY

This chapter briefly discussed background material which is required to understand the concepts of multi-objective control. It described various system and signals norm with mathematical details. This chapter also covered material related to the convex optimization and linear matrix inequalities.

CHAPTER 4

MULTIOBJECTIVE CONTROL SYTHESIS SETUP

4.1 INTRODUCTION

In a practical design problem, one is generally not just confronted with a single objective problem but one has to render a mixture of objectives fulfilled [7]. Mostly the design objectives considered in controller designing are conflicting and addressing all objectives with equal efficiency is quite impossible. Thus, there is an inevitable trade off between design objectives, for example, between the output performance objective and robust stability or between the control efforts and/or regulation and these considerations have led to the study of multi-objective optimization (MO) methods for control systems [7]. H_2/H_∞ with pole constraints optimization is very special case in multi-objective control which is discussed in this chapter.

4.2 PLANT DESCRIPTION

The general control configuration is shown in figure 4.1, where *P* is generalized plant and *K* is generalized controller. The prime control objective is to minimize some norm (ℓ_1, H_2 or H_∞) of transfer function from *w* to *z*.

"Find a controller K, which is based on the information in v, generates a control signal u, which counteracts the influence of w on z, thereby minimizing the closed-loop norms from w to z and w to u [7]."

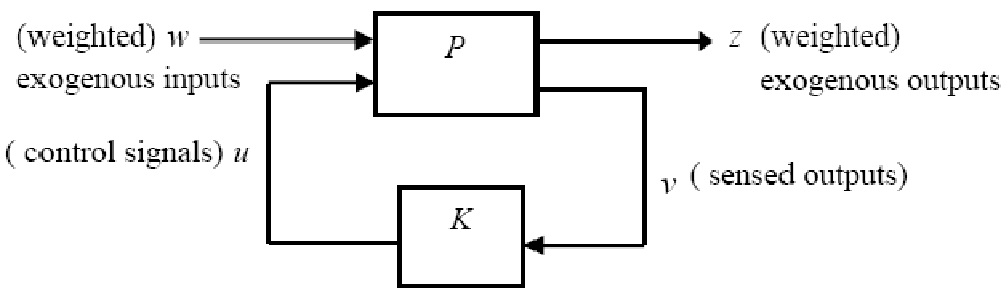

Figure 4.1: General control configuration without model uncertainty

4.2.1 OBTAINING GENERALIZED PLANT

In order to find the generalized plant *P* and controller *K* for any specific case, it is necessary to have block diagram representation and identify the signal *w, u, v* and *z* to drive *P*. One should note that it is an open loop system and remember to break all loops entering and exiting the controller *K*.

Consider one degree of freedom feedback control configuration, conventional block diagram of system is shown in figure 4.2.

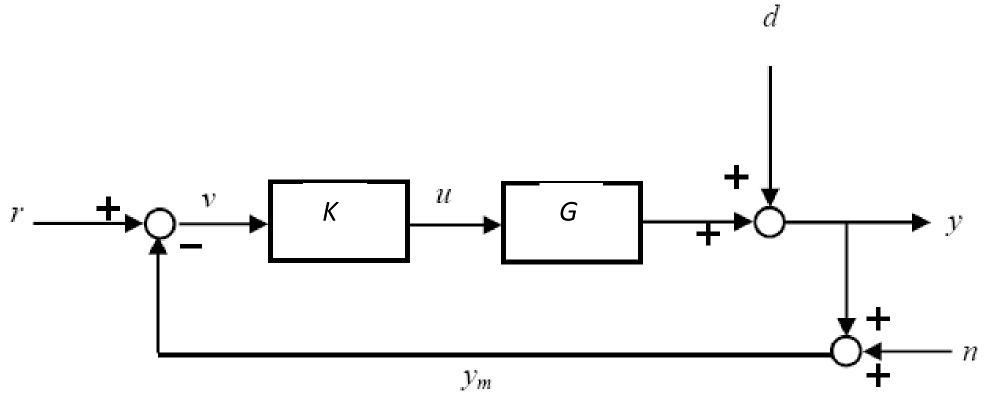

Figure 4.2: One degree of freedom feedback control conventional configuration

First step is to identify the signals of generalized control configuration.

$$w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} d \\ r \\ n \end{bmatrix} \tag{4.1}$$

$$z = e = y - r$$

$$v = r - y_m = r - y - n$$

With this choice of *v*, the controller only has information about the deviation $r - y_m$. Also note that $z = y - r$ which means that performance is specified in terms of the actual output *y* and not in terms of the measured output y_m [8]. The block diagram in Figure 4.3 then yields

$$\begin{aligned} z &= y - r = Gu + d - r = Iw_1 - Iw_2 + 0w_3 + Gu \\ v &= r - y_m = r - y - n = r - Gu - d - n = -Iw_1 + Iw_2 - Iw_3 - Gu \end{aligned} \tag{4.2}$$

And the *P* which represents the transfer function from $[w\ u]^T$ to $[z\ v]^T$ will be

$$P = \begin{bmatrix} I & -I & 0 & G \\ -I & I & -I & -G \end{bmatrix} \tag{4.3}$$

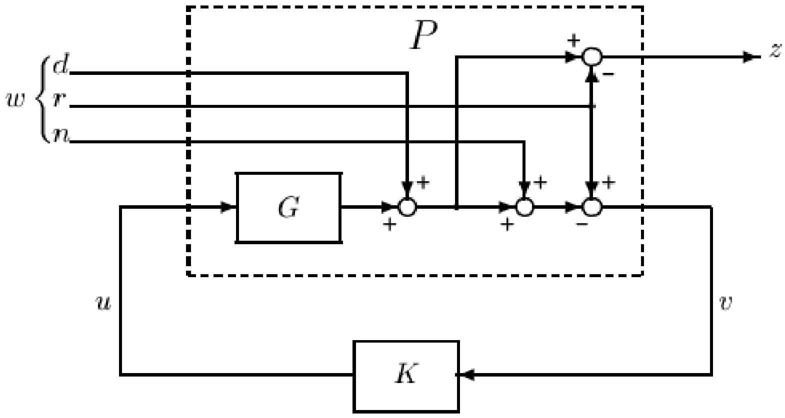

Figure 4.3: Generalized control configuration of figure 4.2

4.2.2 PARTITIONING OF GENERALIZED PLANT *P*

The partition of *P* can be made as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{4.4}$$

Such that its parts are compatible with the signals *w, u, v* and *z* in the generalized control configuration.

$$z = P_{11}w + P_{12}u \tag{4.5}$$

$$v = P_{21}w + P_{22}u \tag{4.6}$$

For one degree of freedom feedback control configuration the P_{11}, P_{12}, P_{21} and P_{22} would be

$$P_{11} = [I \quad -I \quad 0] \tag{4.7}$$

$$P_{12} = [G] \tag{4.8}$$

$$P_{21} = [-I \quad I \quad -I] \tag{4.9}$$

$$P_{22} = [-G] \tag{4.10}$$

4.3 PROBLEM STATEMENT

This thesis work emphasis on H_∞/H_2 design with pole placement constraints using state feedback. The control structure for this problem is shown in figure 4.4.

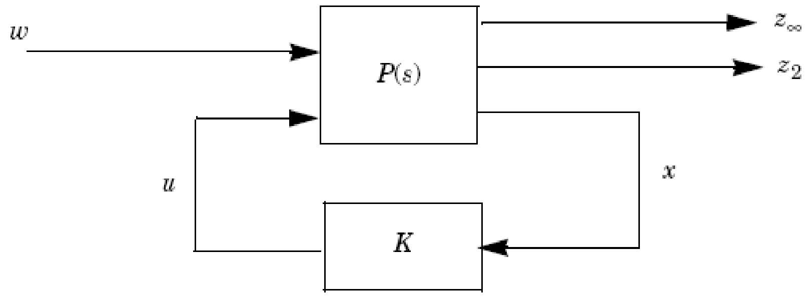

Figure 4.4: State feedback control

P(s) is known Linear Time Invariant (LTI) system and it is assumed that all its states *x* are measurable. Consider the linear time invariant (LTI) system expressed by

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z_\infty = C_\infty x + D_{11} w + D_{12} u \quad (4.11)$$

$$z_2 = C_2 x + D_{21} w + D_{22} u$$

Let $T_2(s)$ and $T_\infty(s)$ are the transfer functions (closed loop) from w to z_2 and w to z_∞ respectively. The aim is to find a feedback law $u=Kx$ that

- Retains the H_∞ norm of $T_\infty(s)$ below given value $\gamma_0 > 0$ maintain
- Retains the H_2 norm of $T_2(s)$ below given value $v_0 > 0$
- Minimization of H_∞/H_2 norms trade off

$$a\|T_\infty(s)\|_\infty + b\|T_2(s)\|_2 \quad (4.12)$$

- Putting the poles of closed-loop system in a given region of the left-half plane.

Where a and b are penalizing factors. There are various practical situations which can be encompassed by this theoretical formulation. For example a problem involving regulation having d disturbance, n white noise and e is the regulation error. By taking

$$w = \begin{bmatrix} d \\ n \end{bmatrix} \tag{4.13}$$

$$z_\infty = e \tag{4.14}$$

$$z_2 = \begin{bmatrix} x \\ u \end{bmatrix} \tag{4.15}$$

Disturbance rejection feature and the LQG feature can be formulated using mixed H_∞/H_2. Placement of closed loop poles in stable left-half plane gives transient responses with well-damping [11].

4.4 POLE PLACEMENT IN LMI REGION

It is a fact that the location of poles has great impact on the transient response of a linear system [13], [1]. The basic condition for the system stability is that, the closed loop poles of the system must lie in the left half of s-plane. Consider a second order system whose poles are λ, given as;

$$\lambda = -\zeta\omega_n \pm j\omega_d \tag{4.16}$$

It can be seen that second order system step response can be completely characterized in terms of natural frequency $\omega_n = |\lambda|$, the damping ratio ζ ,and the damped natural frequency ω_d. By constraining λ to lie in the prescribed region, to obtain a suitable transient response specific limits can be put on these quantities [1]. The different regions in *s*-plane which have great interest are α-stability region $Re(s) < -\alpha$, disks, vertical strips, conics sectors etc as well as any intersection of these regions.

Another appealing region of control purpose is the set $S(\alpha, r, \theta)$ of complex numbers $x + jy$ such that

$$x < -\alpha < 0, \qquad |x + jy| < r, \qquad \tan\theta\, x < -|y| \tag{4.17}$$

As shown in figure 4.5.

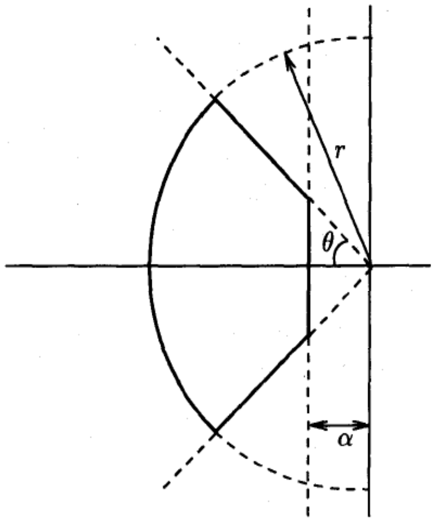

Figure 4.5: Region $S(\alpha, r, \theta)$ [1]

Minimum decay rate α, a maximum undamped natural frequency $\omega_d = r\sin\theta$ and a minimum damping ratio $\zeta = \cos\theta$ is guaranteed if closed loop poles placed in this region**.** This definitely limits the maximum overshoot, the delay time, the frequency of oscillatory modes, the rise time, and the settling time [13].

LMI region concept is very helpful to formulate the objectives of pole placement in LMI terms. Let $\mathfrak{D}$ be a subregion in complex left-half plane [1]. The existence of a symmetric matrix $\alpha = [\alpha_{kl}] \in \boldsymbol{R}^{m \times m}$ and a matrix $\beta = [\beta_{kl}] \in \boldsymbol{R}^{m \times m}$ ensures that $\mathfrak{D}$ the subset of complex plane is LMI region

$$D = \{z \in C : f(z) < 0\} \qquad (4.18)$$

with

$$f_D(z) = \alpha + z\beta + \bar{z}\beta^T = [\alpha_{kl} + \beta_{kl} z + \beta_{kl} \bar{z}]_{1 \leq k, 1 \leq l} \qquad (4.19)$$

The characteristic function f_D takes the value in space of m x m Hermitian matrices and that “< 0” represents negative definiteness [1]. An LMI region that can be characterize by a Linear Matrix Inequality in z and $\bar{z}$, or evenly, a Linear Matrix Inequality in $y = \mathrm{Im}(z)$ and $x = \mathrm{Re}(z)$ which shows, Linear Matrix Inequality section are convex.

Few very important Linear Matrix Inequality section and characteristics functions are given below [11]

- ***Disk Region***

Disk with centre (-*q*, 0) and radius *r*

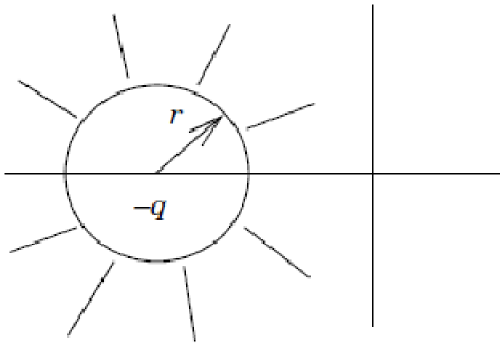

Figure 4.6: LMI Disk Region

The characteristic function of LMI disk region

$$f_D(z) = \begin{bmatrix} -r & \bar{z} + q \\ z + q & -r \end{bmatrix} \tag{4.20}$$

- ***Conic Region***

Conic region with centered at origin and internal angle is θ

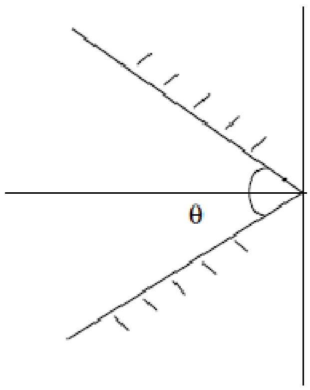

Figure 4.7: Conic LMI region

The characteristic function of conic sector is

$$f_D(z)=\begin{bmatrix} \sin\frac{\theta}{2}\left(z+\bar{z}\right) & -\cos\frac{\theta}{2}\left(z-\bar{z}\right) \\ \cos\frac{\theta}{2}\left(z-\bar{z}\right) & \sin\frac{\theta}{2}\left(z-\bar{z}\right) \end{bmatrix} \tag{4.21}$$

Poles within this sector have damping ratio at least $\cos\frac{\theta}{2}$.

- ***Vertical Strip Region***

The vertical strip region with $h_1 < x < h_2$

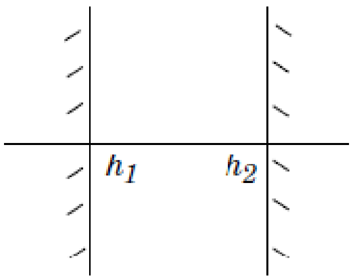

Figure 4.8: Vertical Strip LMI region

The characteristics function is

$$f_D(z)=\begin{bmatrix} 2h_1-\left(z+\bar{z}\right) & 0 \\ 0 & \left(z+\bar{z}\right)-2h_2 \end{bmatrix} \tag{4.22}$$

4.5 LMI FORMULATION

Consider a LTI system P whose state space representation is below

$$\dot{x}=Ax+B_1w+B_2u$$

$$z_\infty=C_\infty x+D_{11}w+D_{12}u \tag{4.23}$$

$$z_2=C_2x+D_{21}w+D_{22}u$$

If control law is defined as u=kx then the state space realization of the closed loop system will be

$$\dot{x} = (A + B_2K)x + B_1w$$

$$z_\infty = (C_\infty + D_{12}K)x + D_{11}w \qquad (4.24)$$

$$z_2 = (C_2 + D_{22}K)x$$

Now we take three design objectives separately, there LMI formulation is given below

4.5.1 H_∞ PERFORMANCE

The closed-loop root mean square gain from w *to* z_∞ does not surpass if and only if there exists a symmetric X_∞ such that [5]

$$\begin{bmatrix} (A+B_2K)X_\infty + X_\infty (A+B_2K)^T & B_1 & X_\infty (C_1 + D_{12}K)^T \\ B_1^T & -I & D_{11}^T \\ (C_1 + D_{12}K)X_\infty & D_{11} & -\gamma^2 I \end{bmatrix} < 0 \qquad (4.25)$$

$$X_\infty > 0$$

4.5.2 H_2 PERFORMANCE

If $T_2(s)$ is transfer function of closed loop from w to z_2 then it H_2 can be taken as

$$\|T_2(s)\|_2^2 = Trace\left((C_2 + D_{22}K)P(C_2 + D_{22}K)^T\right) \qquad (4.26)$$

Solution of Lyapunov equation gives P

$$(A_1 + B_2K)P + P(A_1 + B_2K)^T + B_1B_1^T = 0 \qquad (4.27)$$

Hence the H_2 norm of closed loop $T_2(s)$ doesn't surpass the ν if we have 2 symmetric matrices Q_0 and X_2

$$\begin{bmatrix} (A+B_2K)X_2 + X_2(A+B_2K)^T & B_1 \\ B_1^T & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} Q_0 & (C_2+D_{22}K)X_2 \\ X_2(C_2+D_{22}K)^T & X_2 \end{bmatrix} > 0 \quad (4.28)$$

$$Trace(Q_0) < \nu^2$$

4.5.3 POLE PLACEMENT

The close loop poles lie in the LMI region

$$D = \left\{ z \in C : L + Mz + M^T \bar{z} < 0 \right\} \quad (4.29)$$

Where

$L = L^T = \{\alpha_{ij}\}_{1<i,j<m}$

$M = M^T = \{\beta_{ij}\}_{1<i,j<m}$

Existence of a symmetric matrix X_{pol} satisfying below equation is only condition

$$[\alpha_{ij}X_{pol} + \beta_{ij}(A+B_2K)X_{pol} + \beta_{ij}X_{pol} + \beta_{ij}X_{pol}(A+B_2K)^T]_{1\le i,j\le m} < 0 \quad (4.30)$$

$$X_{pl} > 0$$

Three sets of condition sum up to an non-convex optimization problem involving variables $X_{2,}$ $K, Q_{0,}$, X_∞, and X_{pl} [11]. In order to get LMI structure tractability, only one Lyapunov matrix is required

$$X = X_\infty = X_2 = X_{pl} \quad (4.31)$$

which imposes all discussed objectives. By changing variable $Y=KX$, "this give the suboptimal LMI formulation multi-objective state feedback synthesis problem" [1], [11].

Minimize $\alpha\gamma^2 + \beta\, Trace(Q_0)$ over X, Y, Q_0 and γ^2

subject to

$$\begin{bmatrix} AX + XA^T + B_2Y + Y^T B_2^T & B_1 & XC_1^T + Y^T D_{12}^T \\ B_1^T & -I & D_{11}^T \\ C_1X + D_{12}Y & D_{11} & -\gamma^2 I \end{bmatrix} < 0 \qquad (4.32)$$

$$\begin{bmatrix} Q_0 & C_2X + D_{22}Y \\ XC_2^T + Y^T D_{22}^T & X \end{bmatrix} > 0 \qquad (4.33)$$

$$\left[\alpha_{ij} + \beta_{ij}\left(AX + B_2Y\right)X_{pol} + \beta_{ij}\left(XA^T + Y^T B_2^T\right)^T \right]_{1\le i,j\le m} < 0 \qquad (4.34)$$

$$Trace\left(Q_0\right) < \nu_o^2 \qquad (4.35)$$

$$\gamma^2 < \gamma_o^2 \qquad (4.36)$$

If the optimal solution is denoted by (X^*, Q^*, Y^*, γ^*) the feedback gain would be specified as

$$K^* = Y^*(X^*)^{-1} \qquad (4.37)$$

Guaranteed worst case performance by this gain

$$\left\|T_\infty\right\|_\infty \le \gamma^* \qquad (4.38)$$

$$\left\|T_2\right\|_2 \le \sqrt{Trace\left(Q_0^*\right)} \qquad (4.39)$$

4.6 CHAPTER SUMMARY

This chapter discussed the setup for multi-objective control. It describes in detail about the generalized plant, H_2 and H_∞ performance calculation and there LMI formulation. It also includes the formulation of different LMI region in complex plane. This chapter helps to understand the problem statement and makes ready to use this control scheme for any real system.

CHAPTER 5

MULTI-OBJECTIVE PITCH ATTIUDE HOLD AUTOPILOT

5.1 INTRODUCTION

In previous chapters we have studied the multi-objective problem setup. Now we are ready to implement this control scheme to a real system and analyze its results. In this chapter we will study the mathematical model of an aircraft. This chapter also includes the design H_2/H_∞ multi-objective state feedback controller with poles constraint for Pitch Attitude Hold Autopilot of an aircraft.

5.2 AIRCRAFT SYSTEM MODEL

Model building is a fundamental process [14]. The modeling of an aircraft is an iterative process. A mathematical model based on the laws of physics will suggest what experimental data should be taken, and the model may then undergo considerable refinement in order to fit the data [14]. This section briefly discusses the aircraft system modeling.

5.2.1 REFERENCE AXES

To discuss the aircraft system dynamics, it's necessary to define a system of reference axes or coordinate system. There are three basic systems, each one consisting three mutually perpendicular axes.

The body axes or inertial axes system is rigidly fixed in the airplane and is the system of mutually perpendicular axes passing through the aircraft's center of gravity and whose *x*-axis is parallel to thrust axis, in the direction of the nose of airplane, *y*-axis is positive to right of *x*-axis and *z*-axis is positive downward, perpendicular to the XY plane.

The wind axes system differs from the body axes system in that the *x*-axis is parallel to the relative wind. The *y*-axis is positive to right and *z*-axis is positive downward, perpendicular to XY plane.

In Earth axes system *x*-axis is positive in the direction of north, *y*-axis is positive in the direction of east and *z*-axis is positive towards the center of Earth, perpendicular to the XY-plane.

5.2.2 FORCES AND MOMENTS

Relative motions of aircraft with respect to the air which depend on orientation of aircraft with respect to air flow produce the aerodynamic moments and forces [14].

5.2.2.1 LIFT

The component of the resultant aerodynamic forces on an aeroplane normal to the airplane's velocity vector is called Lift [15]. Lift sustains the weight of the airplane and mostly directed upward. Wing is the major lift producing component of airplane.

$$L = (\rho / 2) V_\infty^2 S C_L$$

$$L = q_\infty S C_L \tag{5.1}$$

Where

L = Lift (Force)
q_∞ = Dynamic pressure
S = Wing Area
C_L = Dimensionless lift coefficient

5.2.2.2 DRAG

The component of the resultant aerodynamic forces on an airplane parallel to the airplane's velocity vector is called Drag. It is a continuing struggle for the practicing aerodynamicist is that of minimizing drag [15].

$$D' = q_\infty S C_d \tag{5.2}$$

Where

D' = Drag (Force)
C_d = Dimensionless drag coefficient

5.2.2.3 MOMENTS

Rotational forces produced by surface pressure and shear stress distribution is called Moment. It is given as

$$M' = q_{\infty} S C C_m \tag{5.3}$$

Where

M' = Moment (Force)

C_m = Dimensionless Moment Coefficient

C = Chord length

5.2.2.4 THRUST

The thrust is defined as the power available to propel the aircraft. It is given as

$$P_A = \eta P \tag{5.4}$$

Where

P = Shaft brake power

η = Propeller efficiency $\eta > 1$

P_A = Thrust

5.2.3 AIRCRAFT MOTION

Aircraft performance is directed by the forces along and perpendicular to the flight path. The translational motion of the airplane is a reaction to these forces [17]. The aircraft motion has six degree of freedom. It has three degrees of freedom motion in translational motion along the body axes as shown in figure 5.1.

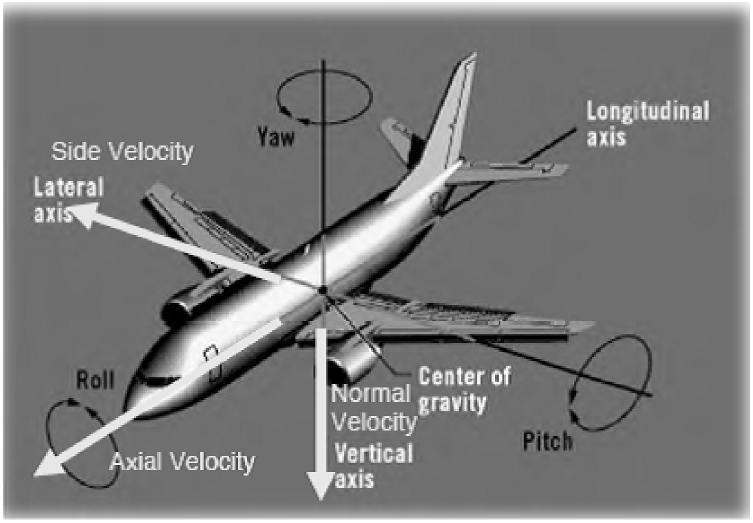

Figure 5.1: Airplane translational degrees of freedom [21]

Aircraft motion has three degrees of freedom in rotation motion as shown in figure 5.2. These rotational angles are called pitch, yaw and roll. These are very important flight dynamics parameters. In flight dynamics, roll, pitch and yaw angles measure not only absolute attitude angles (relative to the North/Horizon) but also changes in attitude angles, relative to the equilibrium orientation of the aircraft. These angles can be defined as

- ***Pitch Angle***

Rotation around body *y*-axis, positive when nose up. It can also define as the angle between body *x*-axis and horizon.

- ***Yaw Angle***

Rotation around body *z*-axis, positive when nose right. It can also define as the angle between body *x*-axis and north.

- ***Roll Angle***

Rotation around body *x*-axis, positive when right wing is down. It can also define as the angle between body *y*-axis and horizon.

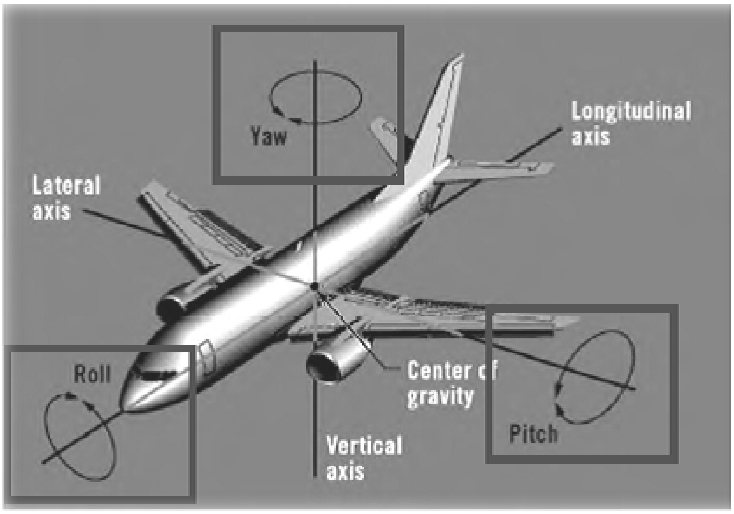

Figure 5.2: Airplane rotational degrees of freedom [21]

5.2.3.1 LONGITUDINAL MOTION

The motion of aircraft around the lateral axis is called longitudinal motion. The primary control surface is elevator for longitudinal motion. Various formulation of Linearized equations of motion are in used [16]. The variables involved in longitudinal motion are shown in figure 5.3.

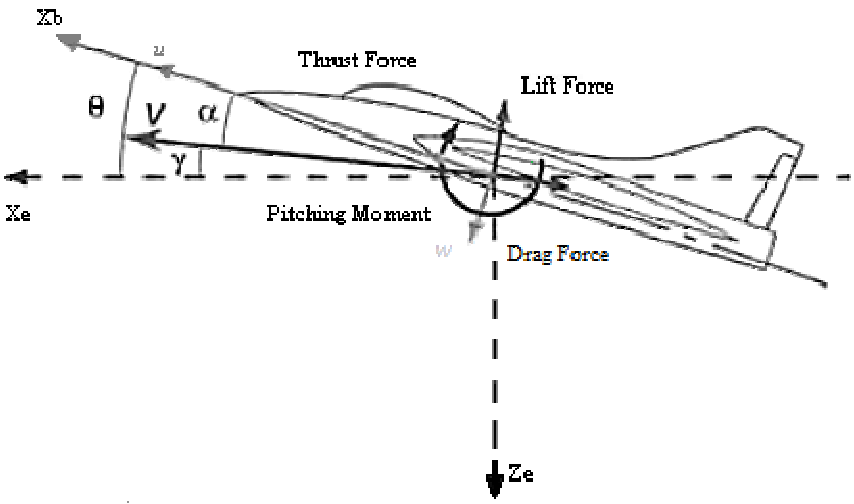

Figure 5.3: Longitudinal Motion of Aircraft [21]

u(t): Axial velocity, along the vehicle centerline

w(t): normal velocity, perpendicular to centerline

V(t): Velocity magnitude, along net direction of flight

$\alpha(t)$: Angle of attack, angle between centerline and direction of flight

$\gamma(t)$: Flight path angle, angle between direction of flight and local horizon

$\theta(t)$: Pitch Angle, angle between centerline and local horizon

The dynamics of the transport aircraft model in a level flight cruise condition at 25,000 ft, 500 ft/s true airspeed and x_{cg} =0.25 $\bar{c}$, are given by [14]

$$\begin{bmatrix} \dot{v}_T \\ \dot{\alpha} \\ \dot{\theta} \\ \dot{q} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -0.0082354 & 18.938 & -32.170 & 0.0 & 5.9022e-05 \\ -0.00025617 & -0.56761 & 0.0 & 1.0 & 2.2633e-06 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 1.3114e-5 & -1.4847 & 0.0 & -0.47599 & -1.4947e-07 \\ 0.0 & -500.00 & 500.00 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} v_T \\ \alpha \\ \theta \\ q \\ h \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.019781 \\ 0 \end{bmatrix} \delta_e \tag{5.5}$$

v_T : Incremental velocity

α : Angle of attack

θ : Pitch angle or attitude

q : Pitch rate

h : Altitude

We will use this aircraft model for designing a pitch attitude hold autopilot in this thesis work.

5.3 AUTOPILOT

An autopilot is an electrical, hydraulic, or mechanical system used to steer a vehicle without human aid. An autopilot can refer particularly to aircraft, missiles, self-steering gear for boats, or auto guidance of spacecraft. Autopilot designing has long history and has many milestones. In current age of technology a modern autopilot is a computer program or software. The computer software receives the aircraft data and controls the flight control system to guide an aircraft.

Most of the flying-qualities specifications do not apply directly to autopilot design [14]. In case of pilot relief modes, the autopilot must be designed to meet specification on steady state error and disturbance rejection, with less emphasis on dynamic response [14]. There are many types of autopilots used in different aircraft. Design of Pitch attitude hold autopilot is discussed in this thesis work.

5.3.1 PITCH ATTITUDE HOLD AUTOPILOT

The pitch attitude hold mode prevents pilots from constantly having to control the pitch attitude. Especially in turbulent air, this can get demanding for the pilot. This autopilot is normally used only when the aircraft is in wing level flight [14]. The controlled variable is $\theta = \gamma + \alpha$, pitch attitude hold tries to keep the current pitch attitude. This system uses the data from the vertical gyroscope as input (feedback), which produce the error signal proportional to the deviation from preset orientation in inertial frame of reference. It then controls the aircraft through the elevators.

5.4 DESIGNING OF MULTI-OBJECTIVE PITCH ATTITUDE HOLD AUTOPILOT

The aircraft model described in sec. 5.2.3.1 is used here for designing a multi-objective autopilot. Minimization of the affect of disturbance w on the pitch attitude θ is a prime purpose of autopilot design. This goal is articulated through subsequent objectives:

- Obtain a fine trade-off between H_2 norm of transfer function from w to $[\theta \ \ u]^T$ and H_∞ norm of on transfer function from w to θ.

- Obtain guaranteed minimum closed-loop damping and decay rate by placing poles of system in the section shown in Figure 5.4

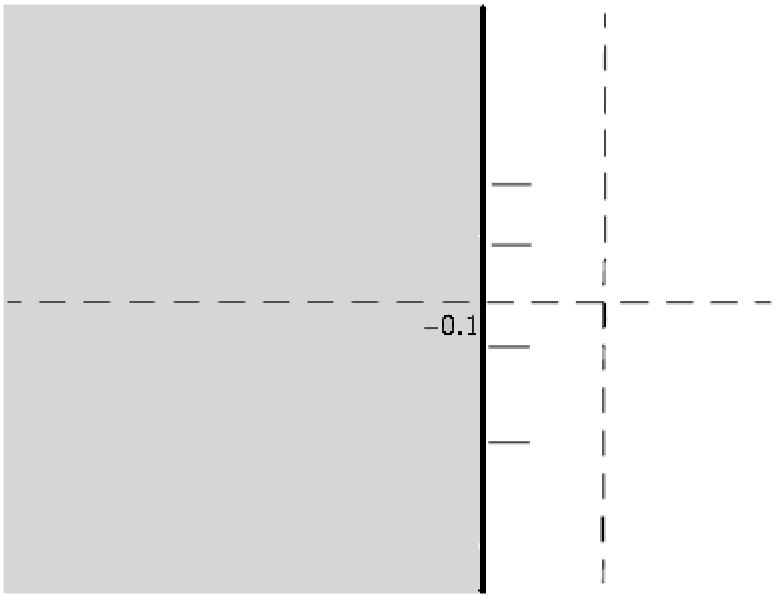

Figure 5.4: Poles Placement region

Where u is the control force and w is disturbance in control force. LMI control toolbox is used to solve, thus the first step is to write down the problem in a proper setup as discussed in sec. 4.3.

$$\dot{x} = Ax + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.019781 \\ 0 \end{bmatrix} u$$

$$z_{\infty} = \theta \qquad (5.6)$$

$$z_2 = \begin{bmatrix} \theta \\ u \end{bmatrix}$$

The matrix A is given in Eq. 5.5 and remaining matrices can be found by comparing the Eq. 5.1 with Eq. 4.11.

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} 0 \end{bmatrix}$$

$$D_{12} = \begin{bmatrix} 0 \end{bmatrix}$$

$$D_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$D_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Optimization problem is solved by using LMI control toolbox (MATLAB code is given in Appendix A). The MATLAB function **msfsyn()** provides the facility of optimizing H_2/H_∞ performance with pole placement constraints. For further detail of **msfsyn()** refer [11].

5.5 SIMULATION RESULTS

We consider four different cases in simulation which accommodates most of the possible combination of this multi-objective optimization problem. In all cases region for pole placement constraints is left-half plane x < -0.1 in order to have minimum decay rate of 0.1 see sec. 4.4. The simulation results include the impulse response of close loop system from w to θ, close loop poles and controller output. Impulse response of any system is described by two characteristics one is peak response and other is settling time. In all these cases we will consider peak response and settling time as performance criteria.

- ***Case 1*: Only pole placement constraints**

In this case we don't have any constraints on the system norms. Pole placement region is left-half plane $x < -0.1$.

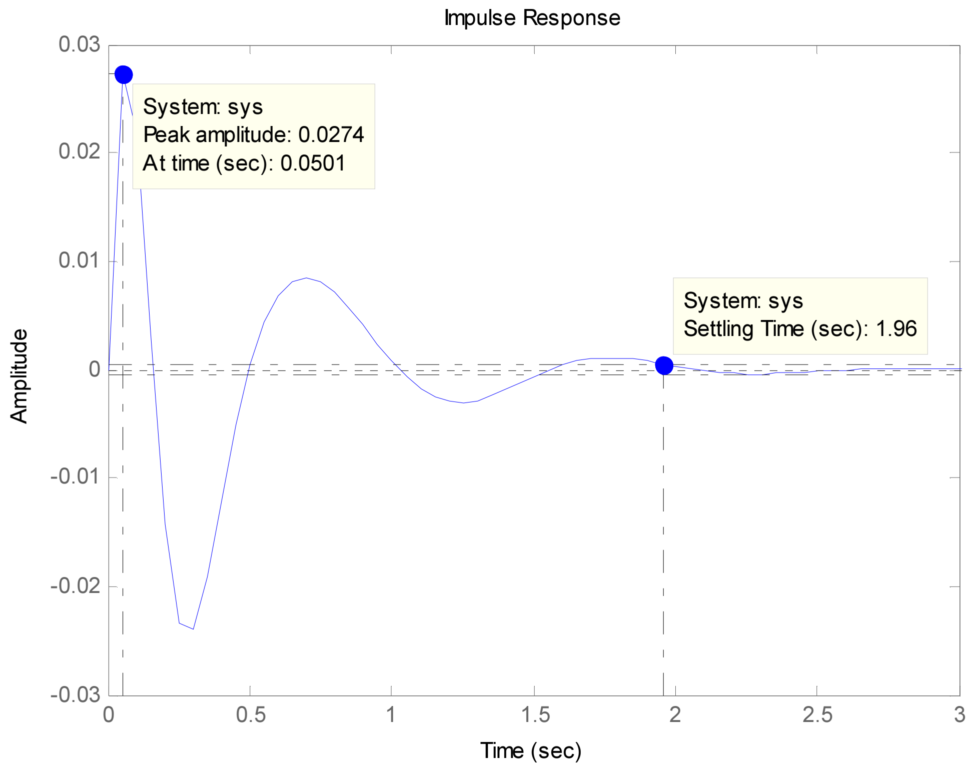

Figure 5.5: Pitch attitude response

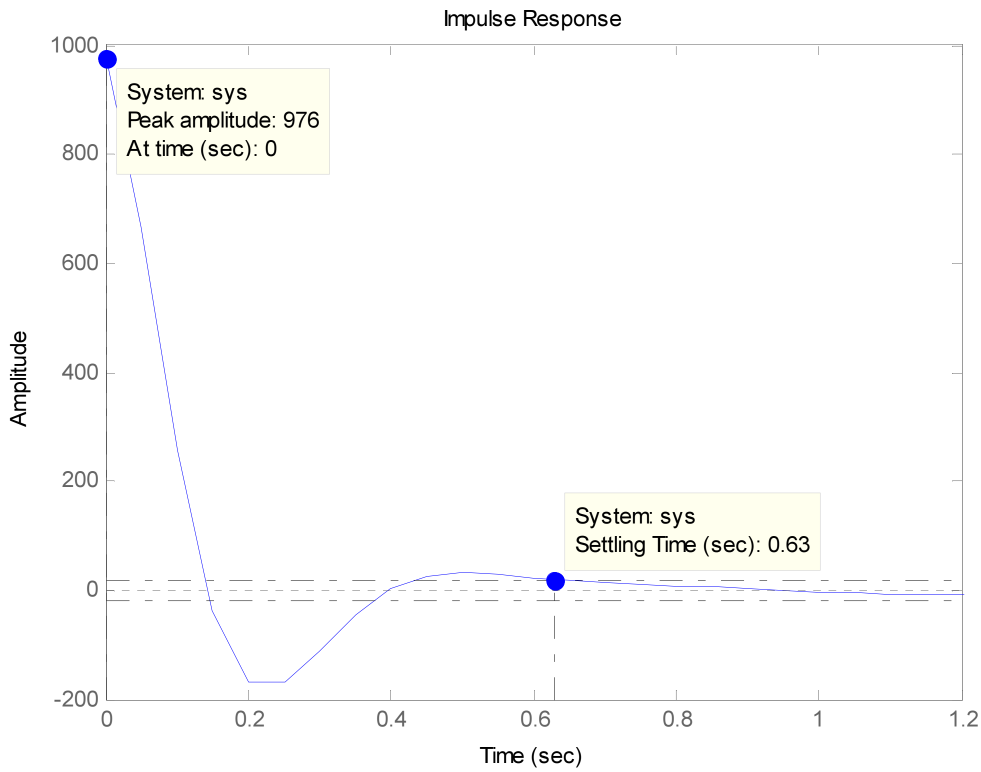

Figure 5.6: Controller output signal u

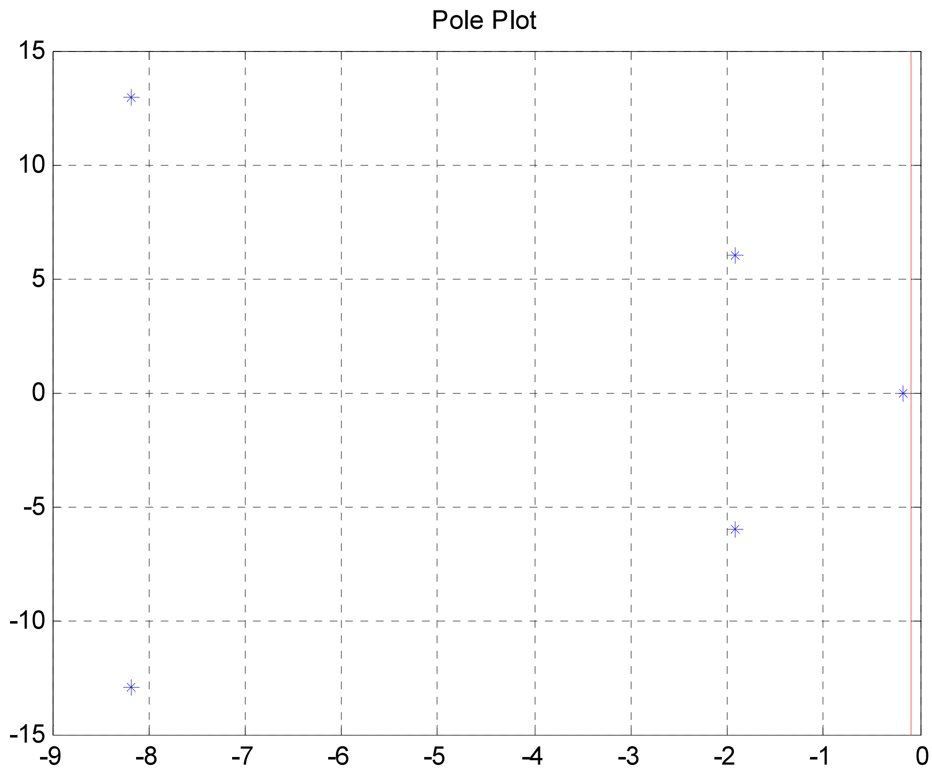

Figure 5.7: Closed loop Poles plot

Results	**Pitch Attitude**	**Controller Output**
Peak Response (rad)	0.0274	976
Settling Time (sec)	1.69	0.63

Table 5.1: Results summary of case 1

- ***Case 2*: *H_∞ Optimization with Pole Placement Constraints***

In this case we minimize the H_∞ norm from w to θ without having any constraints on H_2 norm and keeping the closed loop poles in region $x < -0.1$. In this case it is desired that disturbance influence on the pitch attitude should be minimum no matter what the controller output.

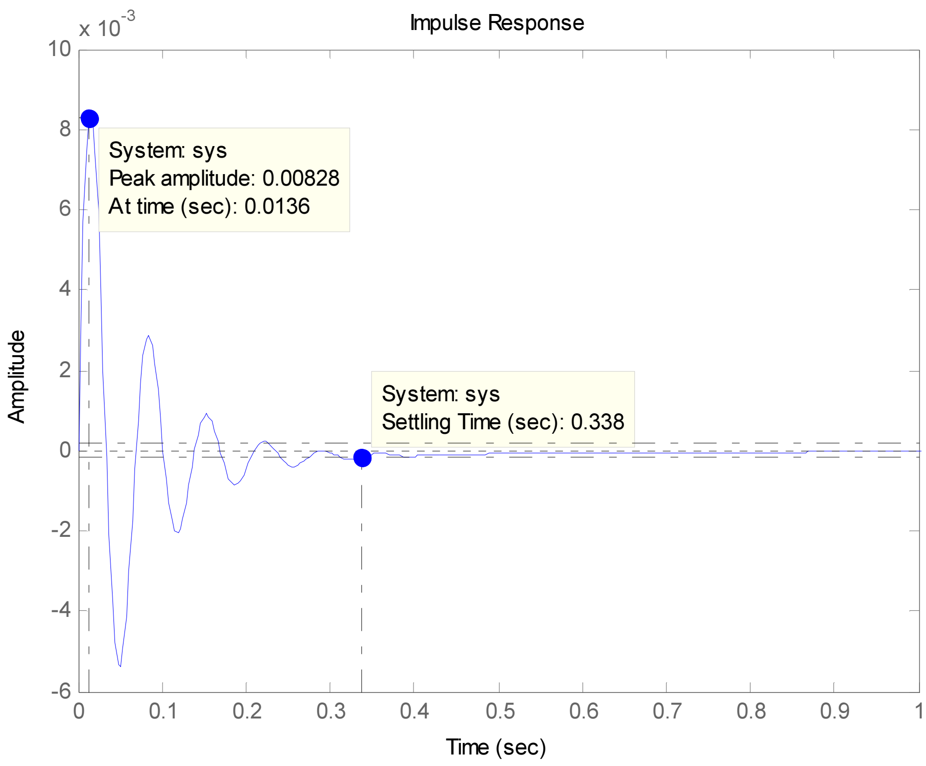

Figure 5.8: Pitch Attitude response

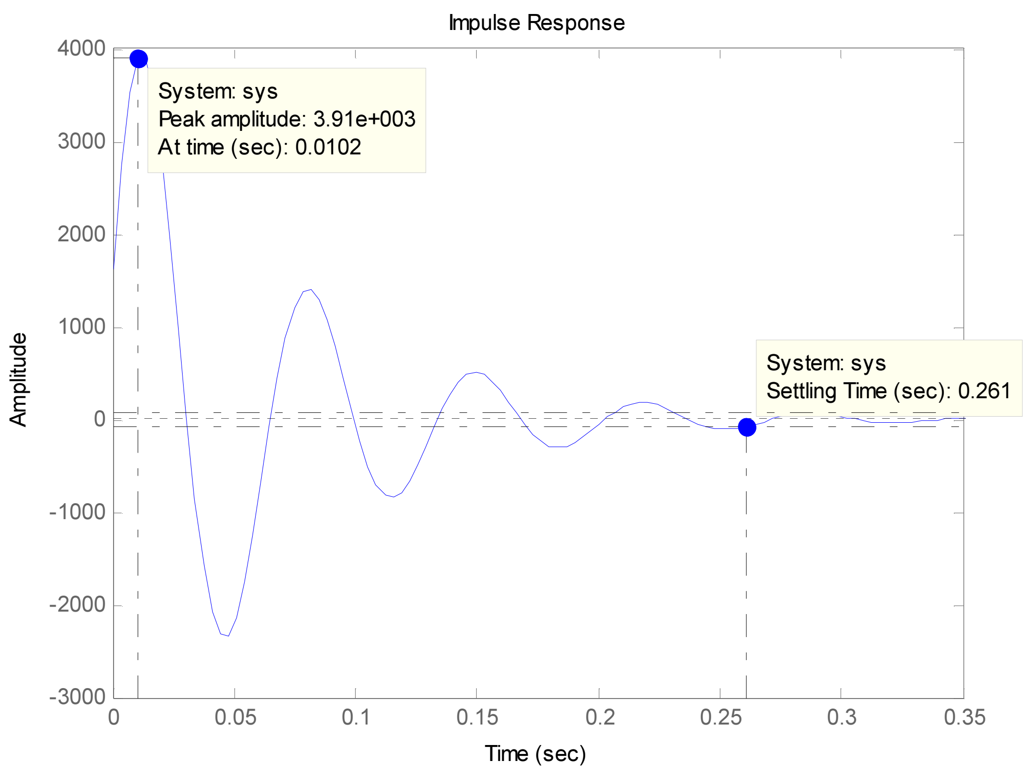

Figure 5.9: Controller output signal u

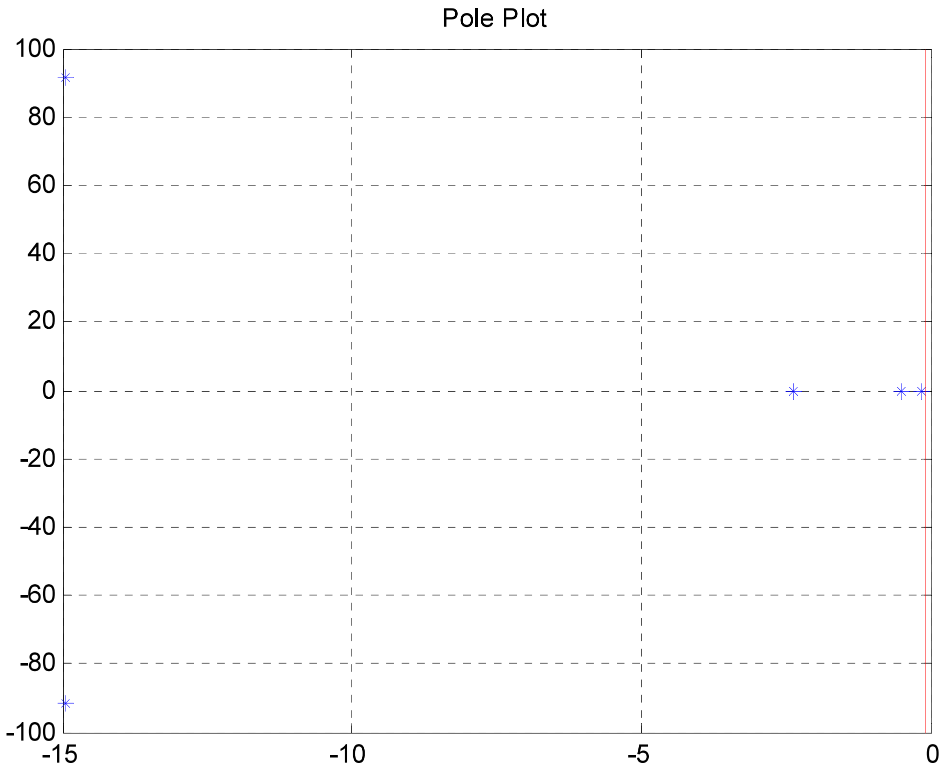

Figure 5.10: Closed loop Poles plot

Results	Pitch Attitude	Controller Output
Peak Response (rad)	0.00828	3910
Settling Time (sec)	0.338	0.261

Table 5.2: Results summary of case 2

- ***Case 3: H_2 Optimization with Poles Placement constraints***

In this case we are minimizing the H_2 norm, without having any constraints on H_∞ norm and keeping the closed loop poles in region $x < -0.1$. By doing this, it is desired to reject the disturbance with minimum controller efforts.

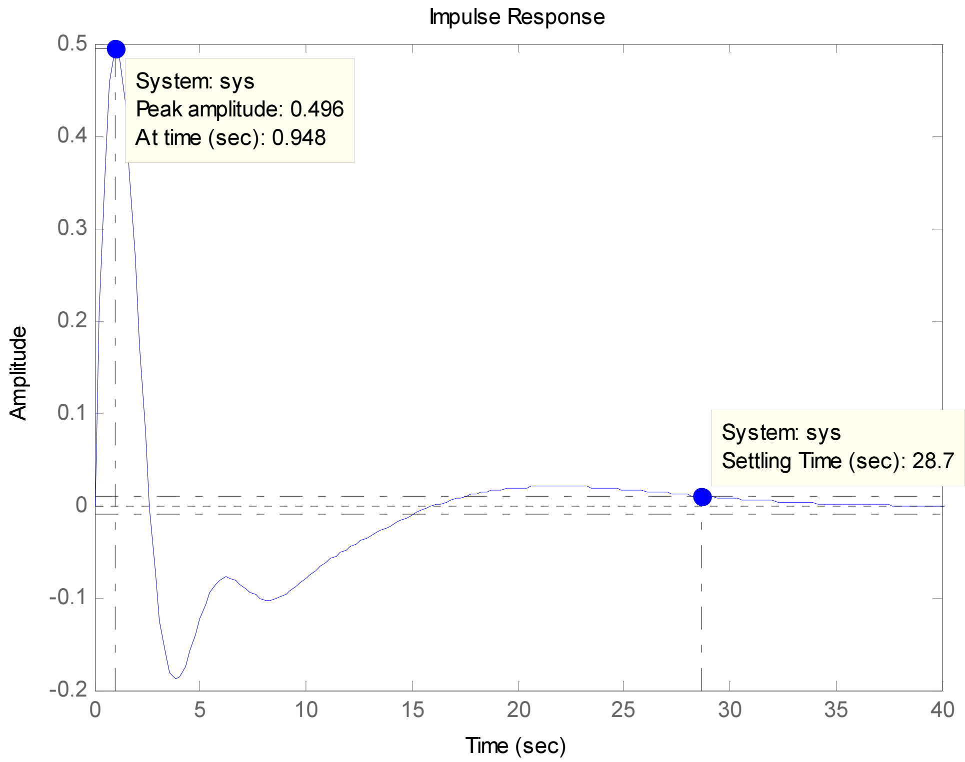

Figure 5.11: Pitch Attitude response

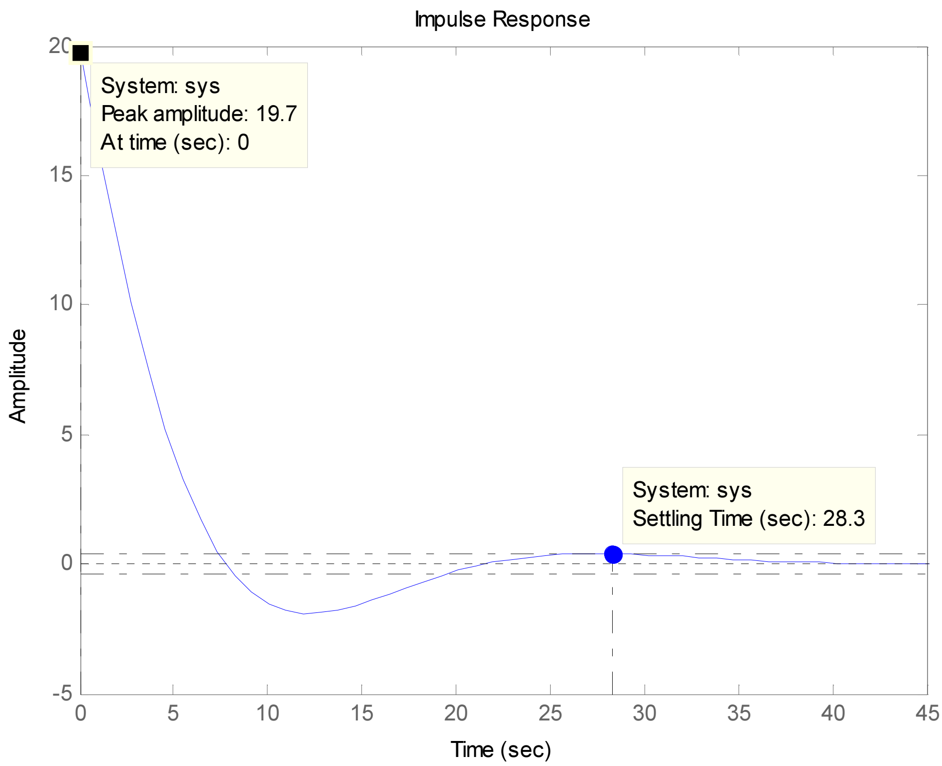

Figure 5.12: Controller output signal u

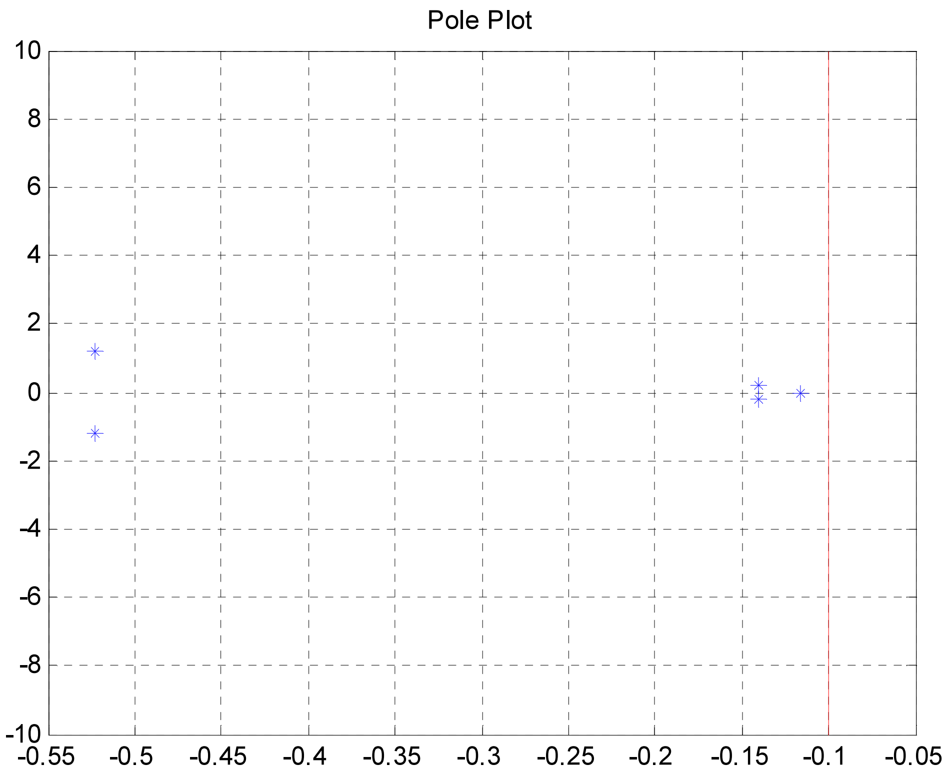

Figure 5.13: Closed loop poles plot

Results	Pitch Attitude	Controller Output
Peak Response (rad)	0.496	19.7
Settling Time (sec)	28.7	28.3

Table 5.3: Results summary of case 3

- ***Case 4: Mix norm Optimization***

This case involves optimization of both H_∞ and H_2 norm simultaneously with weighting factor of 1 and keep the closed loop poles in region $x < -0.1$. If T_2 and T_∞ are the closed-loop transfer function from w to z_2 and w to z_∞ respectively then the cost function will be

$$\alpha \left\| T_\infty \right\|_\infty^2 + \beta \left\| T_2 \right\|_2^2 \qquad (5.7)$$

Where α and β are weighting are penalizing factors, which are equal to 1 in this simulation.

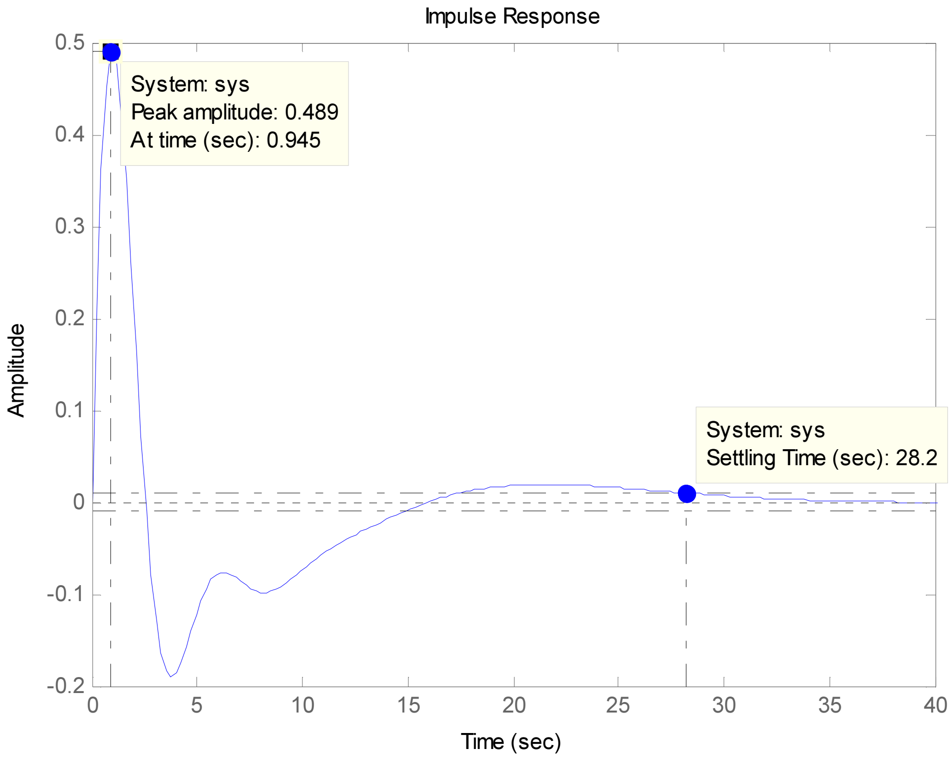

Figure 5.14: Pitch Attitude response

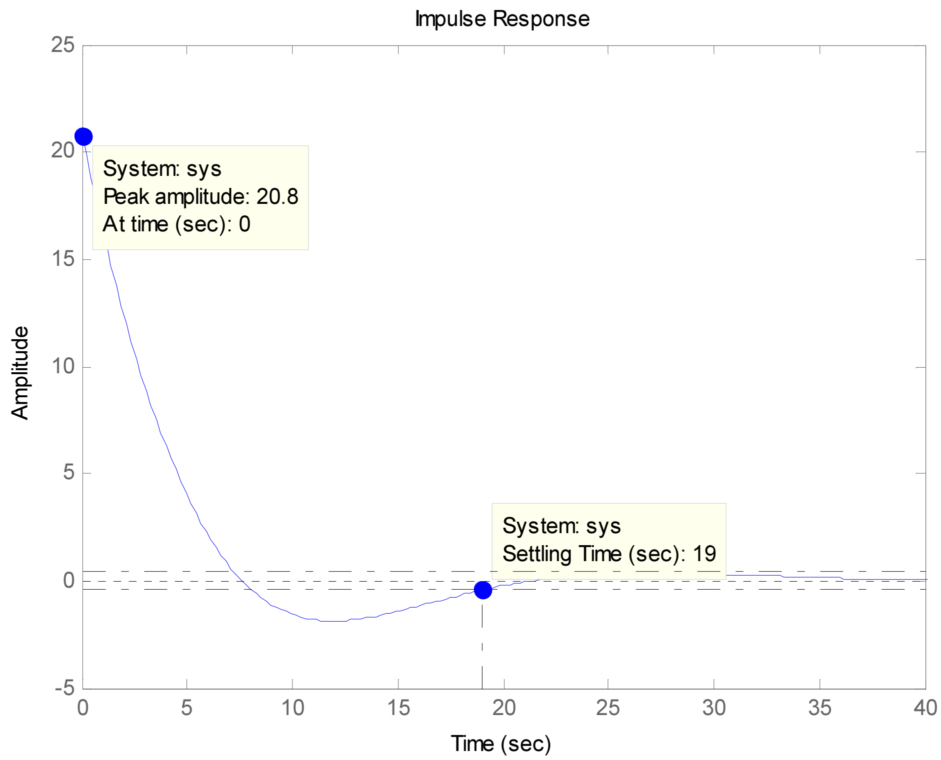

Figure 5.15: Controller output signal u

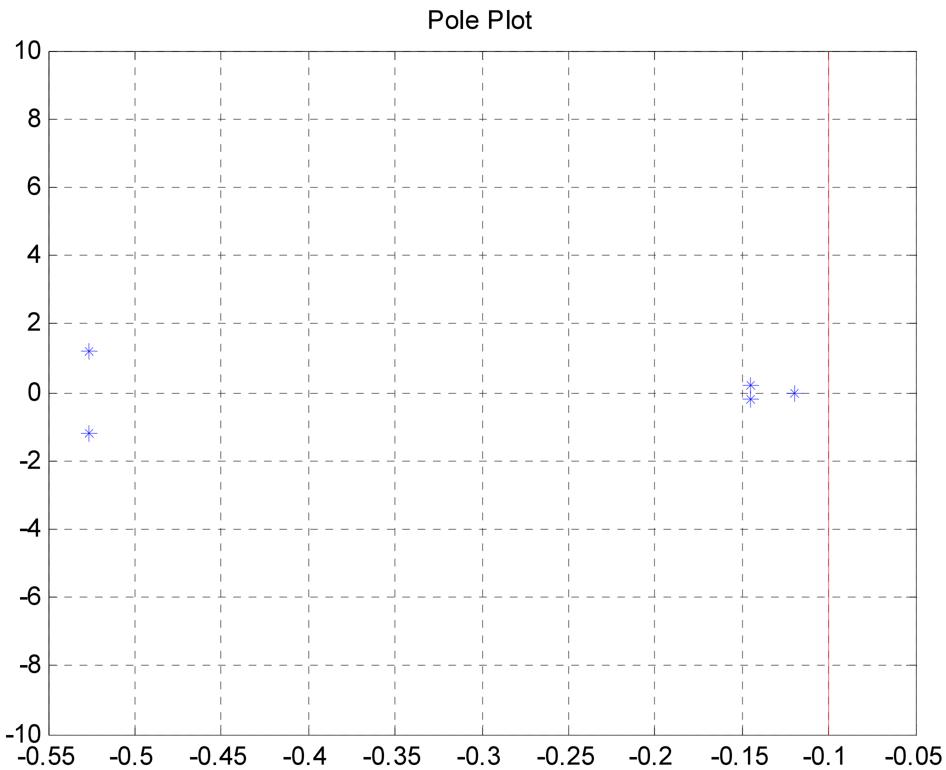

Figure 5.16: Closed loop poles plot

Results	**Pitch Attitude**	**Controller Output**
Peak Response (rad)	0.489	20.8
Settling Time (sec)	28.3	19

Table 5.4: Results summary of case 4

Results comparison of all simulation cases is shown in table 5.1.

		Pitch Attitude	Controller Output
Case 1	**Peak Response (rad)**	0.0274	976
	Settling Time (sec)	1.69	0.63
Case 2	**Peak Response (rad)**	0.00828	3910
	Settling Time (sec)	0.338	0.261
Case 3	**Peak Response (rad)**	0.496	19.7
	Settling Time (sec)	28.7	28.3
Case 4	**Peak Response (rad)**	0.489	20.8
	Settling Time (sec)	28.3	19

Table 5.5: Results summary of all simulation cases

From the above simulation cases we can conclude that, in H_∞ minimization we have good disturbance rejection but on the cost of very large controller output. On the other hand with H_2 minimization we have small controller signal but we don't have good disturbance rejection. Therefore there is trade-off between H_∞ and H_2 performance. We simulate many cases in which we minimize the H_∞ performance with having constraints on H_2 performance. The trade-off curve between H_∞ and H_2 performances is shown in figure 5.16. It is clear from the curve that improvement in H_2 performance cause degradation in H_∞ performance. For a optimum performance we need to compromise in both performances.

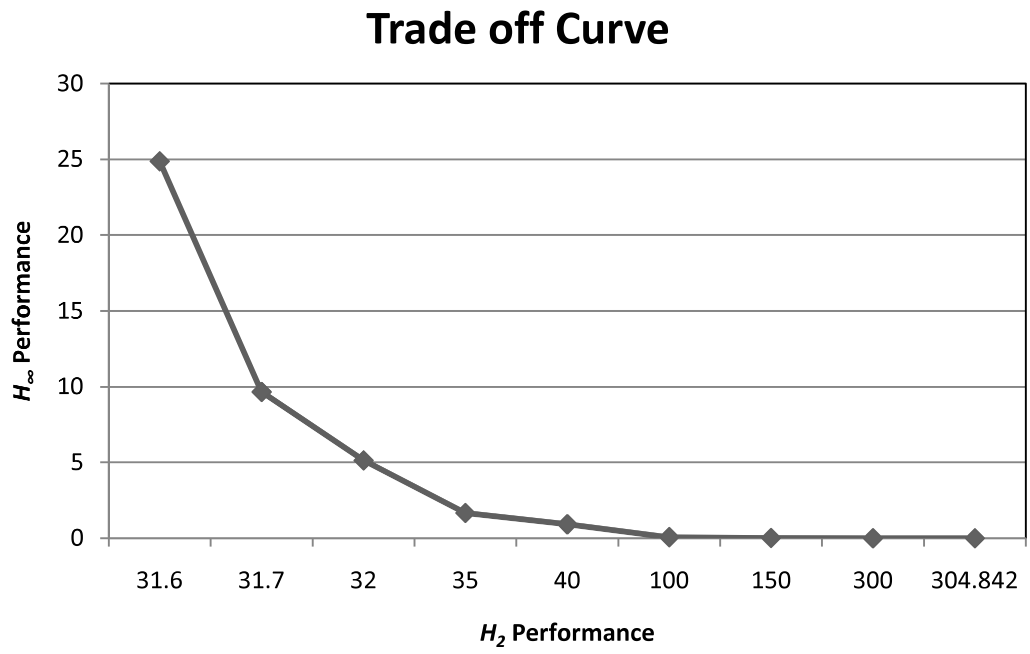

Figure 5.17: Trade off curve between H_∞ and H_2 performance

5.6 SIMULINK MODEL

The simulation of pitch attitude autopilot is also done using MATLAB Simulink tool. The feedback gain matrix K is the input for this model which it takes from workspace after executing the MATLAB code (given in appendix A). Simulink model is shown in figures below.

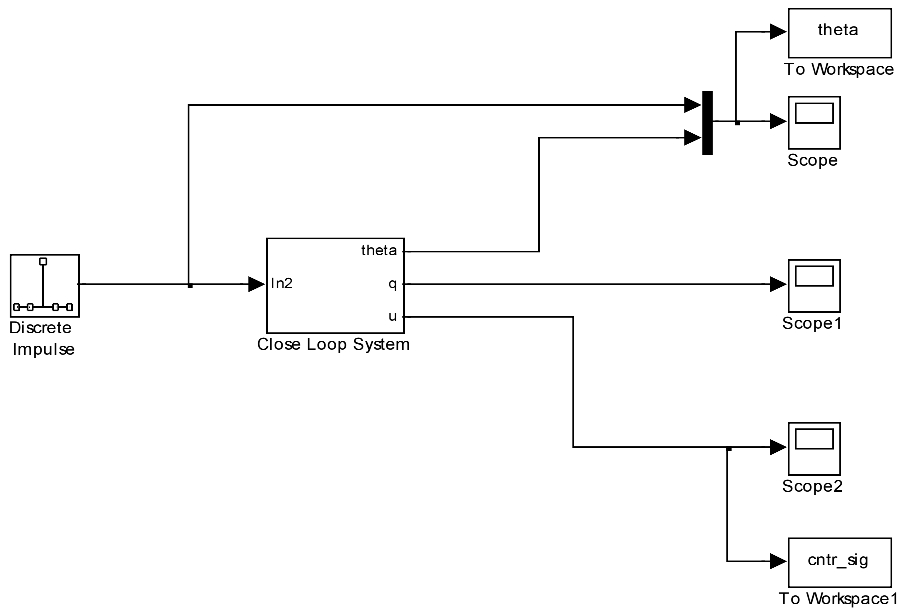

Figure 5.18: Simulink Model of Pitch Attitude Hold Autopilot

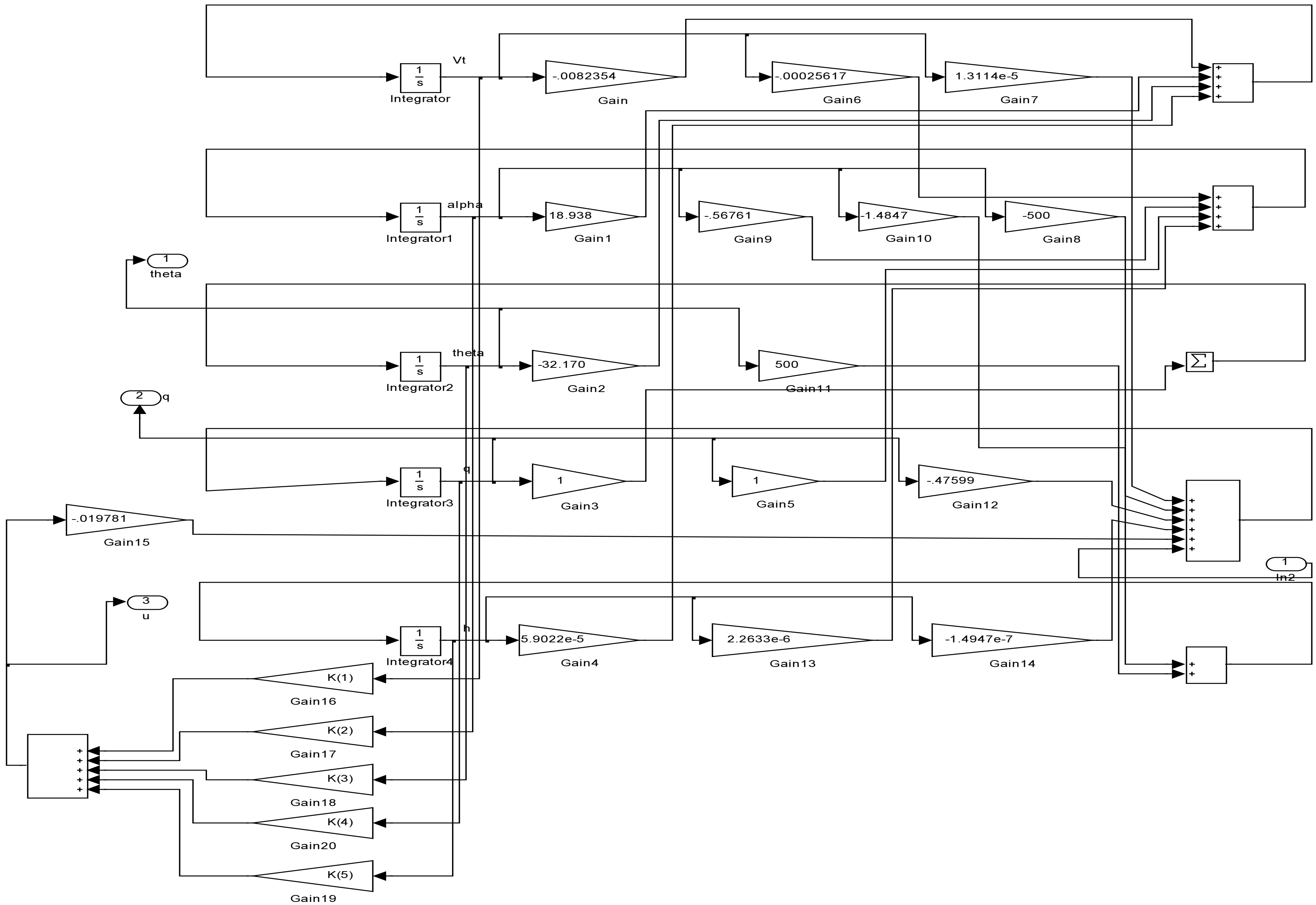

Figure 5.19: Simulink model of Closed-loop system

Simulation results of *Case 4* (mix norm optimization with Pole placement constraints) is shown in figures below,

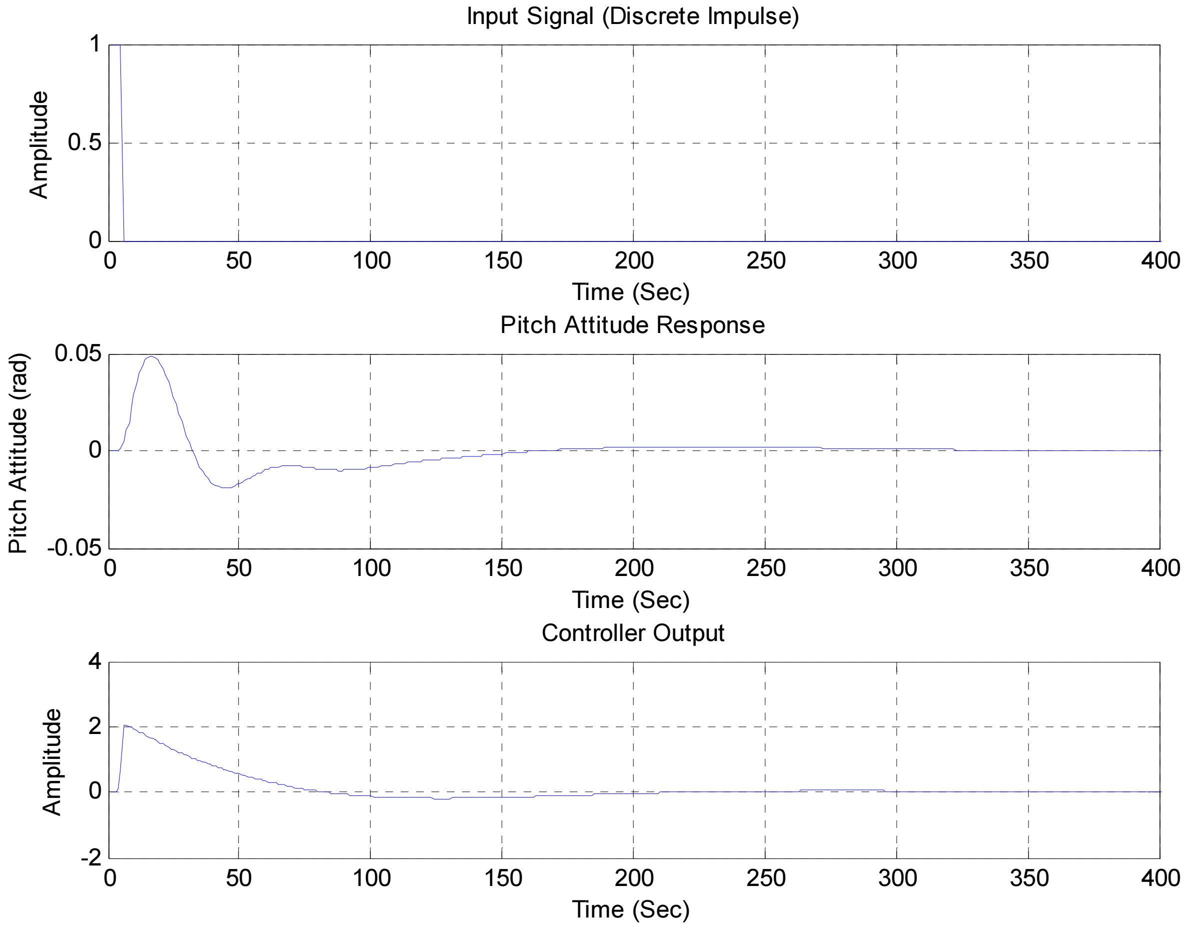

Figure 5.20: Results plot of Simulink Model

5.7 SUMMARY

In this chapter briefly discussed the various terminologies and concept related to the aircraft, such as forces and moments of an aircraft, aircraft motion, linear model for longitudinal motion etc. It also discussed the designing of pitch attitude hold autopilot. The simulation results of various cases are also presented in this chapter. Finally the autopilot is designed and analyzed by using MATLAB Simulink tool.

CHAPTER 6

CONCLUSION AND FUTURE RECOMMENDATION

6.1 CONCLUSION

In this thesis work H_∞/H_2 multi-objective state feedback controller with Pole placement constraints for Pitch attitude hold of an aircraft is designed. Results are obtained by using MATLAB simulation. Various cases were simulated, including only pole placement constraints, H_∞ optimization with pole placement constraints, H_2 optimization with pole placements constraints and H_2/H_∞ mix norm optimization with pole placement constraints. Minimizing the H_∞ norm of transfer function from w to θ, it is tried to diminish the affect of disturbance w on pitch attitude θ. By minimizing the H_2 norm, it is attempted to minimize the controller efforts.

From the simulation results it is very much clear that, In case of H_∞ optimization with pole placement constraints the Pitch attitude hold autopilot rejects the disturbance very quickly but for achieving this quick transient behavior controller puts extremely high effort, in contrary H_2 optimization with pole placement constraints rejects the disturbance very slowly but uses minimum controller efforts. Therefore there is trade-off between H_2 and H_∞ performance.

6.2 FUTURE WORK

Multi-objective controller synthesis is very vast field and it is still open for researchers and engineers. The prime objective of this thesis work was to study and understand multi-objective optimization problem, which is done up to great extent. This thesis provides fundamentals of multi-objective optimization and also discussed case study problem. Therefore there are many directions in which one can continue his research, such as

- Inclusion of ℓ_1 norm in optimization problem with H_2 and H_∞
- In order to control the time domain response of the system, use directly Time Domain Constrains (TDC) instead of pole placement constrains.
- Blending the modern control techniques such as Sliding Mode Control (SMC) or Fuzzy Logic Control (FLC) with these design techniques.

- Mixed objective problem is highly complex and nonlinear constraints optimization problem. Most of the solution techniques involve convex optimization. But it is extremely open research area to use evolutionary optimization techniques for finding the solution of nonlinear constraints problem.

Appendix A

MATLAB Codes

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% MATLAB Program
% State Feedback Controller for Aircraft Pitch Attitude Hold
% Auther: Muhammad Bilal
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

close all
clear all
clc
echo off


disp('                      Muhammad Bilal Nirban');
disp('                      *********************');
disp('  ');
disp('  ');
disp('            STATE-FEEDBACK CONTROL OF AIRCRAFT PITCH ATTITUDE HOLD');


echo on
pause  % Strike any key to continue...
clc

a=[-0.0082354   18.938   -32.17   0         5.9022e-5
   -0.00025617  -0.56761  0.0     1         2.633e-6
   0.0           0.0      0.0     1         0
   1.3114e-5    -1.4847   0.0    -0.47599   -1.4947e-7
   0.0          -500      500     0.0       0.0]

b=[0 0 0 1 0; 0 0 0 -0.01978 0]'

c1 = [0 0 1 0 0]
c2 = [0 0 1 0 0;0 0 0 0 0]
c=[c1;c2];
d11= 0;d12=0; d21 =[0 0]';d22=[0 1]';
d=[d11 d12;d21 d22];

P = ltisys(a,b,c,d)

region = lmireg

[gopt hopt K Pcl] = msfsyn(P,[2 1],[0 0 0 0],region)

figure
splot(ssub(Pcl,1,1),'im');
figure
plot(spol(ssub(Pcl,1,1)),'*')
hold on
y=[-10:1:10]; plot(-0.1*ones(size(y)),y,'r');
title('Pole Plot')
figure
splot(ssub(Pcl,1,3),'im'
```

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