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MASTER'S THESIS

Conditional Servo-Mechanism Designs For a Class of Constrained Non-Linear Systems

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Declaration of Authorship

I Muhammad Aatif Mobeen Azhar, declare that this thesis titled, “Conditional Servo-Mechanism Designs for a Class of Constrained Non-Linear Systems” and the work presented in this is my own. I confirm that:

- This work was done wholly or mainly while in the candidature for a Masters of Electrical Engineering with specialization in Control Systems degree at Pakistan Navy Engineering College (PNEC) a constituent unit of National University of Sciences and Technology (NUST), Pakistan.
- Where I have consulted the published work of others, it is clearly attributed.
- Where I have quoted from the work of others, the respective sources are provided. With the exception of such quotations, this thesis is entirely my own work.
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ABSTRACT

CONDITIONAL SERVO-MECHANISM DESIGNS FOR A CLASS OF CONSTRAINED NONLINEAR SYSTEMS

By

Muhammad Aatif Mobeen Azhar

The problem of servomechanism designs for the class of nonlinear systems exposed to saturation nonlinearities has been studied, with emphasis on improving their transient performance. Two types of systems are considered. First, the systems exhibiting the linear nature but the saturation nonlinearity makes the overall phenomena as nonlinear. For such class of systems, stabilizing compensator is designed based on Composite Nonlinear Feedback (CNF) technique to improve the performance in terms of controlling the damping ratio. Secondly, the systems possessing the nonlinear nature in addition to the saturation constraints are considered. For such systems, robust nonlinear control design technique called passivity based control design is exploited by considering the systems in the normal form. As, normal form representation of system can be realized as cascade connection of two subsystems where the interconnection term plays crucial role. Keeping in view the importance of the interconnection term, stabilizing compensator is designed through the passivity based control design technique that can be augmented with the conditional servo-compensator to achieve the task of output regulation. As, proved already that the conditional servo-compensator is superior than the conventional servo-compensator in terms of transient performance, it has been verified through the simulation examples. Saturation nonlinearities deteriorates the transient performance of the system and sometimes the steady state response as well. It has been figured out that through the saturation level adjustments of the conditional servo-compensator with that of the input control channel, these deteriorating effects of saturation nonlinearities can be minimized or in fact eliminated. Realizing the physical scenario where it is not possible to have all the states available for feedback, the design is extended to the output feedback through a full order observer through the combination of Extended Kalman Filter (EKF) and Extended High Gain Observer (EHGO). This observer provides the robust estimates that are used to replace the states in the control law to recover the performance of the state feedback design. To show the efficacy of the proposed scheme, the technique is applied to the nonlinear benchmark system namely; Translating Oscillator with Rotating Actuator (TORA), using the state feedback passivity based conditional servo-compensator design approach as well as the output feedback control approach that uses a cascaded EKF-EHGO for the state estimation.

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INTRODUCTION

The problem of output regulation is amongst the fundamental problems of the control theory that deals with the controller design to make the output of the system asymptotically track the reference signals and/or reject the disturbance signals. Both of these reference and the disturbance signals are generated by the autonomous system called the *exogeneous system* modelled by the differential equations. An extensive work in this regard has already been done in the past for multivariable, time invariant, finite dimensional, linear systems. e.g. the work of *Davison et. al.* [1] and *Francis et. al.* [2]. These papers established that the output regulation problem requires some solvability conditions. i.e. Solution of system of Linear Matrix Equations called *regulator equations* which is equivalent to the characterization of *Hautus et. al.* [3] about the transmission polynomial of the composite system (formed by the actual control system and the exosystem) to exhibit certain property. The theory concludes with an attractive result that a necessary requirement for the solution of the output regulation problem is that a controller that works in this case can been viewed as the interconnection of two components which are called as *servocompensator* and *stabilizing compensator*. The whole idea is called the *internal model principle* which states that the servo-compensator is a device which is capable enough to generate the reference and the disturbance signals as that of the exo-system whereas the stabilizing compensator works to stabilize the augmented system which is shaped by combining the actual plant and the servo-compensator. Such a general controlling setup is as shown in Fig. 1.1.

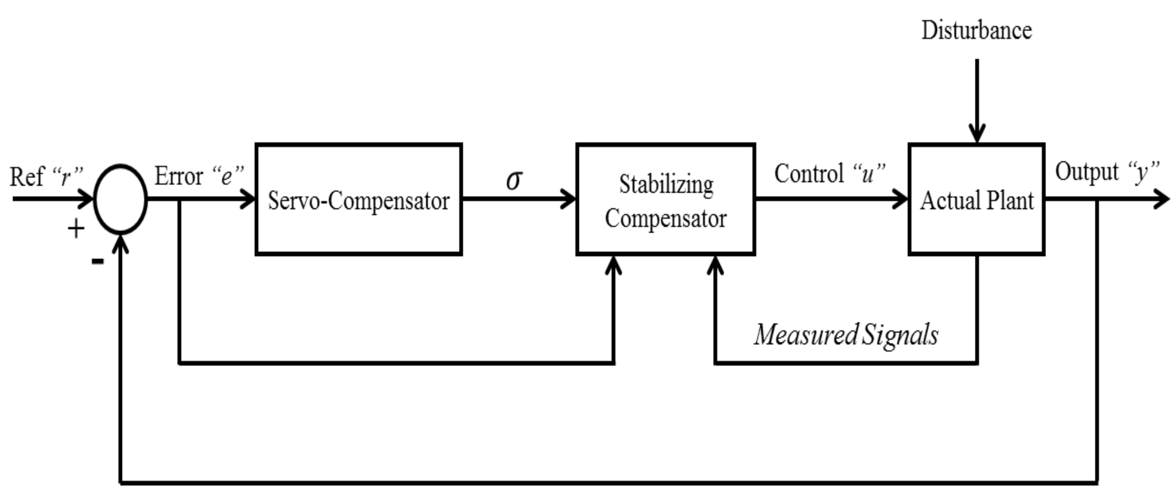

Fig. 1.1: A general control setup for output regulation problem

As a special case, when the reference and disturbance signals are constant, the above-mentioned scheme gets reduced to the classical idea of integral control. One of the pioneering acknowledgements of the internal model principle goes to the work of *Minorsky et. al.* [3] which concludes with an important observation that today's popularly known proportional-integral (PI) controllers performs so nicely that the effect of constant disturbances on the actual system (or plant) almost gets diminished.

This thesis focusses the design of controllers that provides the solution to the output regulation problem for a class of constrained nonlinear systems with emphasis to improve the transient performance. Specifically, we consider the class of constrained or saturated nonlinear systems being minimum-phase that can be transformed into the normal form under compact set of unknown system parameters. For this class of systems, robust continuous feedback control techniques like Energy-Based control, Lyapunov Redesign (i.e. Min-Max control) or Sliding Mode control can be used to force the tracking error ultimately bounded around the origin of closed loop systems by rejecting the bounded disturbances. However, making the tracking error arbitrarily small inside the ball around the origin necessitates the use of high gain feedback in the locality of origin, which leads to the traditional idea of including the servo-compensator with stabilizing compensator as discussed in the previous paragraph.

The output regulation problem being addressed in this thesis utilizes the idea of *conditional servo-compensator* design introduced by the Khalil and co-researchers [4], [5], [6]. Conditional servo-compensator acts as classical servo-compensator only inside the boundary layer near to the zero-error manifold while it is a bounded-input-bounded-state system whose state is guaranteed to be the order of small design parameter. The attractive feature of conditional servo-compensator is that the condition of zero steady state tracking error gets achieved without degraded transient performance. The idea of conditional servo-compensator was initially presented in *Seshagiri et. al.* [4] and [5] in the sliding mode framework, where [4] focusses the conditional integrator design for constant exogenous signals while [5] discusses the general case of time-varying signals. This idea was extended to the Lyapunov Redesign framework by *Attaullah Y. Memon et. al.* [7] for more general feedback controllers. The work of [7] provides the flexibility of starting with any stabilizing state feedback controller and then including the conditional servo-compensator to solve the output regulation problem for certain nonlinear systems. Exploiting this flexibility, *Attaullah Y. Memon et. al.* [8] solves the output regulation problem for the linear system being influenced by the saturation nonlinearities overall making the system nonlinear. These saturation nonlinearities present a ubiquitous issue for all control systems even if we characterize the control system through linear model, the closed loop system nonlinearity stems out from the saturation at the control input. This happens so due to the physical nature of actuators that are used to apply the control effort. These saturation nonlinearities may lead to the stability problems and performance aggravation of the controller because the achievable control objectives are now strongly limited. When the system dynamics are linear and with such nonlinear constraints, the Algebraic Riccatti Equation (ARE) based methods for the solution of output regulation problem necessitates the use of full state feedback. So, in [8] output feedback control technique was developed using full order high gain observer and conditional servo-compensator that achieves robust performance output regulation performance or tracking of reference signals while rejection of disturbance signals both of which are produced by the exo-system.

However, in [8] the system under consideration is assumed to have a control model which is not in the normal form. i.e. Where the system model has, the cascading structure consisting of explicit internal and external dynamics. The contribution of this thesis is twofold which can be considered as the extension of *Attaullah Y. Memon et. al.* [8]. First, it focusses on the systems exhibiting the control dynamics that can be expressed as cascaded structure of internal as well as external dynamics making the control problem a bit challenging due to the limited number of available system variables that can be used in designing the control component. Secondly, this work is concentrated for the systems having nonlinear dynamics at all. i.e. Unlike [8], where the dynamics are linear and saturation nonlinearities make the system nonlinear. This work considers the system dynamics itself nonlinear besides the saturation elements. Starting with the design of stabilizing control based on system's energy functions called *passivity-based control* to stabilize the system states inside the ball around the origin, a conditional servo-compensator idea of [7] is exploited to make the output of the system track the family of reference signals and reject the family of disturbance signals both of which are generated by exo-system. Realizing the practical scenario that only output is available for feedback, a full order high gain observer called *Extended High Gain Observer* (EHGO) based on the idea of *Boker et. al.* [9] is designed to estimate the system states to implement the output feedback version of the designed control consequently.

In section 1.1, we briefly review the background elements that includes the evolution of the output regulation problem for the saturated control systems in the literature and the motivation of my work. This chapter concludes with the section 1.2 that provides the overview of this thesis.

1.1 Background, Literature Review and Motivation

The constraints are ubiquitous in every control system and most often, they appear in the form of rate and magnitude saturation of input and other variables. They can have detrimental effects on the performance of the control system unless elucidated in the control design process of specific controller. Keeping in view the wide recognition of these inherent constraints, several approaches have been developed in the past to cater their damaging effects. Considering the heart of almost all the control systems, these approaches have widely addressed the problem of output regulation for systems having linear dynamics but exposed to saturation nonlinearities. The interest in the subject of saturated control was renewed after the important development by *Sontag et. al.* [10]. The important result from [10] that a linear system subject to input saturation can be globally asymptotically stabilized if the system is asymptotically null controllable with bounded controls in the absence of saturation has attracted the attention of modern researchers to carry out their research activities in the context of constrained systems. A system is said to be asymptotically null controllable with bounded controls if it is stabilizable if it is null controllable and all the open loop poles of the system lies in the closed left half plane. Another crucial result developed in the same paper is that a linear system with input saturation cannot be globally stabilized by using linear feedback control law. Initially the output regulation problem for systems with linear dynamics but subjected to input saturations was studied by *Lin et. al.* [11], [12]. The linear feedback laws based on low-gain technique that solve the problem of semi-global stabilization as well as output regulation were constructed. The short coming of this low-gain design approach is its nature that depicts that a system cannot be operated in its maximum capacity because the moment when the states of the system will be close to the origin, the control input will be far away from its maximum permissible value. *Lin et. al.* [13] worked out to improve the low-gain design and came up with an improved approach called low-and-high gain design technique for the construction of linear control law that solves the semi-global output regulation problem. Today several versions of low-and-high gain control design developed by many other researchers can be found in the literature.

The improvement of the transient performance for the servo-systems utilizing several nonlinear techniques has been studied as early as 1950 [14]. Later, in the late nineties, the new techniques composed of linear and nonlinear techniques were developed to boost the transient performance of the saturated systems. One of such techniques called Composite Nonlinear Feedback (CNF) technique was initially worked out by *Lin et. al.* [15] for the second order linear systems with input saturations. This control consists of two parts. i.e. linear and nonlinear part. The linear part of CNF aims for the low damping ratio when the tracking error is very large resulting the system's output with faster rise time. One the other hand, nonlinear part is designed to increase the damping ratio of the closed loop system at the moment when the tracking error is very much reduced so that the initial overshoot caused by the linear part gets reduced. The extension of CNF for the multivariable systems was provided by *Turner et. al.* [16]. However, both [15] and [16] considered only the state feedback case. *Chen et. al.* [17] developed the measurement feedback version of CNF technique for more general class of systems and verified the design by applying the same to the HDD servo-system. *Lin et. al.* [9] extended the CNF control technique for the solution of output regulation problem in case of nonlinear systems. A similar version of CNF control law was also provided by *Lan et. al.* [19]. Realizing the practical scenario, when all the system states are not available or not measurable, an output feedback or state observer version of CNF was developed by *Lan et. al.* [20]. A similar version of CNF technique can also be found in *Chen et. al.* [21]. Besides the above discussed control techniques of low gain, low-and-high gain and various versions of CNF, several other approaches have been developed for certain specific systems having certain control constraints that includes modern control methodologies like Adaptive Control, Sliding Mode Control (SMC) etc. [47][48][49]. However, all these approaches have been developed for the case of systems having linear dynamics with saturation constraints.

The output regulation problem for nonlinear systems (without saturation) has been addressed in another context that includes the classical idea of Khalil and co-researchers where the robust asymptotic output regulation has been achieved by using the combination of stabilizing compensator and servo-compensator. [22], [23], [24], [25], [26]. In these approaches, a servo-compensator exhibiting the property of generating the reference signals as that of exogeneous system along with their higher order harmonics is utilized with the stabilizing compensator to stabilize the error dynamics of the system arbitrarily small. In the case of constant references and disturbances [24], [25] the tracking error drives the servo-compensator and its inclusion creates an equilibrium point where the tracking error gets zero. On the other hand, for a more general case [22], [23], [26], a system model is identified for the generation of trajectories of the exo-system and its higher order harmonics that are generated by the system nonlinearities. This model is used to design the servo-compensator whose inclusion creates an invariant manifold where the tracking error reduces to zero. To achieve the non-local stabilization of disturbance dependent manifold, robust control techniques are exploited to synthesis the stabilizing compensator. Additionally, this controller designed uses only the error feedback and is worked using the idea of *Esfandiari et. al.* [27] called Separation Principle, where the state feedback version of the controller is designed followed by the saturated high-gain observer that recovers the performance of state feedback. As far now, these techniques have not been used to solve the output regulation problem of nonlinear saturated systems, however, they can be used to solve the same of saturated nonlinear systems as well.

While the above-mentioned designs may solve the output regulation problem in a robust way but in all of them the transient performance is not addressed. In fact, the combination of stabilizing compensator and the servo-compensator achieve the steady-state performance but at a cost of degradation in the transient performance. It happens so due to the facts that the servo-compensator increases the system order and in part it interacts with the control saturation that in case of integral control may lead to the well-known problem of integrator windup. To address the issue of the transient performance degradation, an idea introduced by the *Seshagiri* et. al. [28] called the conditional servo-compensator where the servo-compensator is made to behave conditionally. i.e. It works as traditional servo-compensator only inside the boundary layer while it is a bounded-input-bounded-state system. The idea of making the conventional servo-compensator as conditional one results in far better performance comparatively. The work in [28] is performed in the context of SMC framework. This idea is extended to the lyapunov redesign by *Attaullah Y. Memon et. al.* [7] where there is a flexibility provided that starting with any stabilizing control, a servo-compensator can be added to achieve the tracking of reference signals while rejecting the bounded disturbances. The idea of [7] has been applied to solve the output regulation problem for the systems subject to control constraints in [8] but still the system considered here is such that it possesses linear dynamics.

Summarizing the above discussion about the evolution of the output regulation problem of the saturated control systems, the two immediate conclusions can be drawn. First, all the work is dedicated for such systems which are although exposed to saturation nonlinearities but itself they possess the linear dynamics. Secondly, almost every time the considered control model is in the form where the system dynamics are not explicitly shown. So, the work in this thesis is concentrated to address these two observations. i.e. Firstly the considered system itself possesses nonlinear dynamics besides saturation. Secondly, the system exhibits the cascaded structure where the model depicts the system dynamics explicitly. The significance of looking the system in this way is that most of the physical systems possess similar structure when mathematically modelled. However, viewing system this way presents certain control challenges like stability issues that needs to be carefully addressed in control design process.

1.2 Overview of the Thesis

In Chapter 2, we discuss all the technical terms and the necessary preliminary theories that are involved including saturations, minimum phase systems, non-minimum phase systems, output regulation, conventional as well as conditional servo-compensators, passivity based stabilizing controls and extended high gain observers. In Chapter 3, we put forward the idea of state feedback as well as output feedback output regulation problem of saturated minimum phase nonlinear systems using conventional and conditional servo-compensators. The output feedback version will be designed using full order observer based on the idea of extended high gain observer (EHGO). We also provide the analytical stability analysis and some simulation results verifying the control objectives. In Chapter 4, we extend our work to the non-minimum phase systems and provides with the necessary results that prove the efficacy of our research work. Finally we summarize our results and provide future directions of this research problem in Chapter 5.

Chapter 2

PRELIMINARIES

The title of the thesis involves some important terms like conditional servo-mechanism and constrained class of nonlinear systems. In other words, the output regulation problem will be worked out for the these class of systems. This chapter will discuss some background theories and review some necessary elements required to present our main work in Chapter 3. These elements include the Nonlinear Systems and their Characterizations (Section 2.1), Saturation Nonlinearities and their types (Section 2.2), Stabilizing Controls through Passivity Based Designs (Section 2.3), Nonlinear Output Regulation Problem (Section 2.4), Conditional Sevo-Mechanism Designs (Section 2.5) and Extended High Gain Observer (EHGO) (Section 2.6). We start our discussion by explaining the nonlinear phenomena with the associated systems obeying it and their categorization.

2.1 Nonlinear Systems and their Characterizations

The study of engineering sciences describe that a nonlinear system is a system in which its output is not directly proportional to the input. Nonlinear problems are of interest to the scientists and engineers because there most of the systems in nature exhibit nonlinear phenomena. Their behavior can be understood through their mathematical descriptions. e.g. The flight path of an airplane subjected to certain engine thrust, rudder and elevator angles, particular wind conditions or the behavior of an automobile on cruise control when climbing the hill can be predicted through the mathematical descriptions of their pertinent behavior. The behavior of the processes and the predictions about their responses are normally done through the study of their typical differential or difference equations.

Consider the single-input single-out (SISO) nonlinear system modelled by the following equations,

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x) \tag{2.1}$$

where $x \in \Re^n$ is the state, $u \in \Re$ is the input and $y \in \Re$ is the output. The functions $f(.)$ and $g(.)$ are sufficiently vector fields over the domain $\mathfrak{D} \in \Re^n$ and $h: \mathfrak{D} \to \Re$ is a smooth function which is said to have a relative degree ρ where $1 \leq \rho \leq n$, in a region $\mathfrak{D}_0 \in \mathfrak{D}$, if

$\mathcal{L}_g\mathcal{L}_f^{i-1}h(x) = 0$ for $i = 1, 2, \ldots, \rho - 1$ and $\mathcal{L}_g\mathcal{L}_f^{\rho-1}h(x) \neq 0$ for all $x \in \mathfrak{D}_0$, where $\mathcal{L}_f h(x) = \frac{\partial h}{\partial x} f(x)$ is called the *Lie-Derivative* of function $h(.)$ with respect to x.

Remark 2.1: The smallest integer number ρ such that differentiating the output y i.e. $y^{(\rho)}$ makes the control input u visible in its expression is called the *relative degree*.

The idea of relative degree guarantees that there exists a local change of variables $[z^T \quad \xi^T]^T = T(x),\ z \in \mathfrak{R}^{n-\rho},\ \xi \in \mathfrak{R}^{\rho}$, such that the system (2.1) can be transformed in to the *normal form* as,

$$\left.\begin{aligned} \dot{z} &= \varphi(z, \xi) \\ \dot{\xi} &= A_c\xi + B_c\gamma(x)\big(u - \alpha(x)\big) \\ y &= C_c\xi \end{aligned}\right\} \tag{2.2}$$

where A_c, B_c and C_c are the is the canonical representation of the ρ integrators with $\gamma(x) = \mathcal{L}_g\mathcal{L}_f^{\rho-1}h(x)$ and $\alpha(x) = {-\mathcal{L}_f^{\rho}h(x)}\big/{\mathcal{L}_g\mathcal{L}_f^{\rho-1}h(x)}$. By taking the state feedback control as $u = {(\alpha(x)\gamma(x) + v)}\big/{\gamma(x)}$, the system (2.2) gets reduced to $\dot{\xi} = y^{(\rho)} = v$. This shows that the resulting input-output map is linear that renders that system component z is unobservable through the output. So, the equation $\dot{z} = \varphi(z, \xi)$ is called the internal dynamics of the system whereas, the $\dot{\xi}$ equations represent the systems external dynamics. Setting $\xi = 0$ in the internal dynamics of system (2.2) renders the dynamics as,

$$\dot{z} = \varphi_0(z, 0) \tag{2.3}$$

which is called as the zero dynamics of the system (2.1). When the equilibrium of (2.3) is asymptotically stable, the system is termed as *minimum phase* otherwise it is called *non-minimum phase*.

In the special case of linear time invariant systems, the function $f(x)$ and $g(x)$ in system (2.1) can be chosen as,

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{2.4}$$

where $A \in \mathfrak{R}^{nxn}$, $B \in \mathfrak{R}^{n}$ and $C \in \mathfrak{R}^{1xn}$ with x being the system's state, the control input u and y is the output. The system model (2.4) can equivalently be represented by a transfer function of the form as,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{a_i s^i + a_{i-1}s^{i-1} + \cdots + a_0}{b_j s^j + b_{j-1}s^{j-1} + \cdots + b_0} \tag{2.5}$$

with $i < j$ and $a_i,\ b_j \neq 0$ and the polynomial functions $Y(s)$ and $U(s)$ are the Laplace transformations of the time domain signals $y(t)$ and $u(t)$ respectively. So, for system (2.5), the term *relative degree* can be defined as the difference between the degrees of the numerator and the denominator polynomials (i.e. $\rho = i - j$) or equivalently we have multiple non-unique time domain representations for this system that includes the controllable canonical forms and the observable canonical forms. The controllable canonical form for the system (2.5) can be written as,

$$\left.\begin{aligned} \dot{x} &= A_c\tilde{x} + B_c u \\ y &= C_c\tilde{x} \end{aligned}\right\} \tag{2.6}$$

where the triplet (A_c, B_c, C_c) have the structure as,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = [b_0 \quad b_1 \quad b_2 \quad \ldots \quad b_m]$$

where the coefficients a_i and b_j are the same as they appear in transfer function (2.5). It can be noted that,

$$y^{(k)} = \begin{cases} C_c A_c^k \tilde{x} \quad , & 1 \le k \le i - j - 1 \\ C_c A_c^{i-j}\tilde{x} + C_c A_c^{i-j-1} B_c u, & k = i - j \end{cases}$$

So, we can deduce the same conclusion as that of *Remark 2.1* that *relative degree* is the smallest number, the output y is differentiated so that the control term u appears in the resulting equation. To infer the same deduction about the system classification, here we deduce that if the roots of the numerator of the transfer function (2.5) lie in open left half plane, then the system (2.1) is *minimum phase* otherwise it is *non-minimum phase*.

Remark 2.2 (a): The control system exhibiting mathematical model into normal form, if the internal dynamics have asymptotically stable equilibrium point, the system is called *minimum phase* otherwise it is called *non-minimum phase*.

Remark 2.2 (b): The control system exhibiting certain transfer function of the form (2.5), if all the roots of numerator lie in open left half plane, the system is called *minimum phase* otherwise it is called *non-minimum phase*.

This thesis work is primarily focused for the systems that have nonlinear dynamical behavior and having the stable internal dynamics. Relating these systems with this subsection, we conclude that the selected class of systems is minimum phase nonlinear systems. However, the extension to the non-minimum phase systems will also worked out which will be done by transforming those systems model with appropriate assumptions and suitable transformations into those dynamics that will interpret their control / mathematical model as minimum phase.

2.2 Saturation Nonlinearities and their types

In the context of nonlinear phenomena, one of the devastating factors in the performance limitations of the systems is the saturation. Saturation refers to the certain bounds on the amplitudes and rate of change of the signals. Whenever a constrained control system is talked about, it refers to the actuator, state, input or output constraints. Besides these, the limitations on the sensor packaging, model knowledge and the mobility etc. can also be included into the set of saturations. The control system consists of actuators that are required for the manipulation of the state or output in order to get the desired performance. Physically looking, all the actuators have saturations because only certain amount of force (or equivalent) can be delivered by them. The input level or magnitude constraints are among the most confronted saturation nonlinearities by the actuator since the upper and lower bounds exist naturally for these factors. Sometimes, the rate saturations that are related to the change in signal also come in interaction to the system.

The two examples that depicts the constrained actuators are DC motor and a Valve. DC motor is mostly used for controlling the rotatory motion. This device is has certain bounds on its input current and voltages that may be considered equivalent to the velocity and acceleration in the absence of load. In case of valve, there are the limitations in the level of opening. i.e. Valve can operate between fully opened and fully closed state. However, in the latter case, the rate constrains also exists that are driven by the valve controlling mechanism.

The traditional saturation characteristic block that matches with the input level or actuator magnitude is shown in Fig. 2.1.

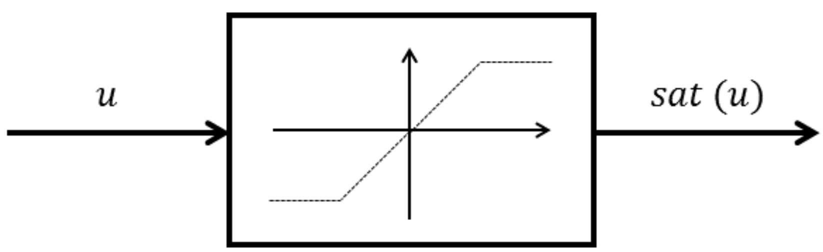

Fig. 2.1: Saturation Characteristic Block

The mathematical description of this saturation function is as follows:

$$sat(u) = \begin{cases} u_{max} \ , & u \geq u_{max} \\ u \ , & u_{min} \leq u \leq u_{max} \\ u_{min} \ , & u \leq u_{min} \end{cases} \tag{2.7}$$

for all $u \in \Re^n$.The values of u_{max} and u_{min} corresponds to the actuator limits that can by chosen wither by measuring the output of the actuator or simply by estimation. The input rate saturation can also be modelled similarly however by the application of difference operator to the saturation. i.e.

$$sat(\Delta u) = \begin{cases} \Delta u_{max} \;, & \Delta u \geq \Delta u_{max} \\ \Delta u \quad , & \Delta u_{min} \leq \Delta u \leq \Delta u_{max} \\ \Delta u_{min} \;\;, & \Delta u \leq \Delta u_{min} \end{cases} \tag{2.8}$$

where $\Delta = 1 - q^{-1}$, Δu_{max} represents the upper and Δu_{min} describes the lower limit of the rate saturations. These saturation constraints can be either hard or soft where hard constraints are related to the scenario where the violation of the limits are not permissible at any cost and soft constraints can violate the specified limits temporarily.

Other types of saturation nonlinearities / constraints include the limitations of the sensor packages, mechanical mobility or steerability, state of the workspace, safety parameters, model knowledge and the calibration factors etc. The states of the system can be determined by the outputs of the sensor and this determination process can be tricky in case of unreliable, noisy and slow sensors. The flexibility in the construction of the control system also presents the limitations which are related to the mechanical or steerability constraints. e.g. the degrees of control freedoms can be restricted in case of wheel robot. The robot can stay stationery on certain point or move forward / backward but side-ward movements cannot be directly performed. The working environment or the state of the system can also add in these constraints. e.g. For a robot to move from one point to another, certain obstacles opposing its motion appears to act. Constraints can also be imposed by the designer or the user to the system. e.g. It may be fruitful to limit the temperature, pressure, voltages, voltages etc. in some cases due to the safety concerns. Saturations can also happen due to the inaccuracy of the system model. i.e. Error and unmodeled dynamics may present strong control challenges. So, summarizing this subsection, it can be concluded that all these saturation constraints cannot be neglected and are necessary to be taken into account while designing the control systems in order to achieve the robustness in the desired performance.

2.3 Stabilizing Controls through Passivity Based Designs

The concepts of passivity provide us with valuable tools for the robust analysis of the nonlinear systems. In 1970s, Willems [29] initially introduced the concepts and systematic developments of passivity. We first establish the passivity concepts and its necessary elements then will discuss how to use passivity to design the robust stabilizing controls for control systems. The passivity notions deliberated here in this subsection follows from the discussions of *constructive nonlinear control by Rodolphe Sepulchre et. al.* [30] and *Nonlinear Control by Hassan Khalil* [31].

To define passivity, starting with the memoryless function $y = h(t, u)$, where u and y are m dimensional vectors and $t \geq 0$. Relating with this memoryless function, consider a simple resistive element having voltage u as input and current y as output. This can be shown as in Fig. 2.2.

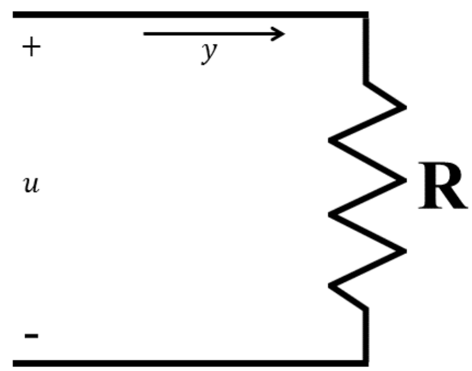

Fig. 2.2: A Passive Resistor

If the power inflow to the circuit is always positive, then this resistor is a *passive* element. Graphically it means that $uy \geq 0$ for all points (u, y) in $u - y$ plane which means that the curve lie in the first or third quadrants. The simplest of such resistor is the one that observes the linear Ohm's law. i.e. $u = Ry$, where R is the resistance of the device. Equivalently, this relation can also be written as $y = Gu$, with G being the conductance of the device that is normally opposite to that of resistance ($G = 1/R$). In case of positive resistance behavior, the $u - y$ curve of the device will be the straight line with slope G and the product $uy = Gu^2$ will always be positive. Fig. 2.3 (a) and (b) shows the nonlinear passive resistors that have characteristic curves in the first and third quadrants whereas Fig. 2.3(c) shows the curve of negative resistance oscillator that is a non-passive resistor.

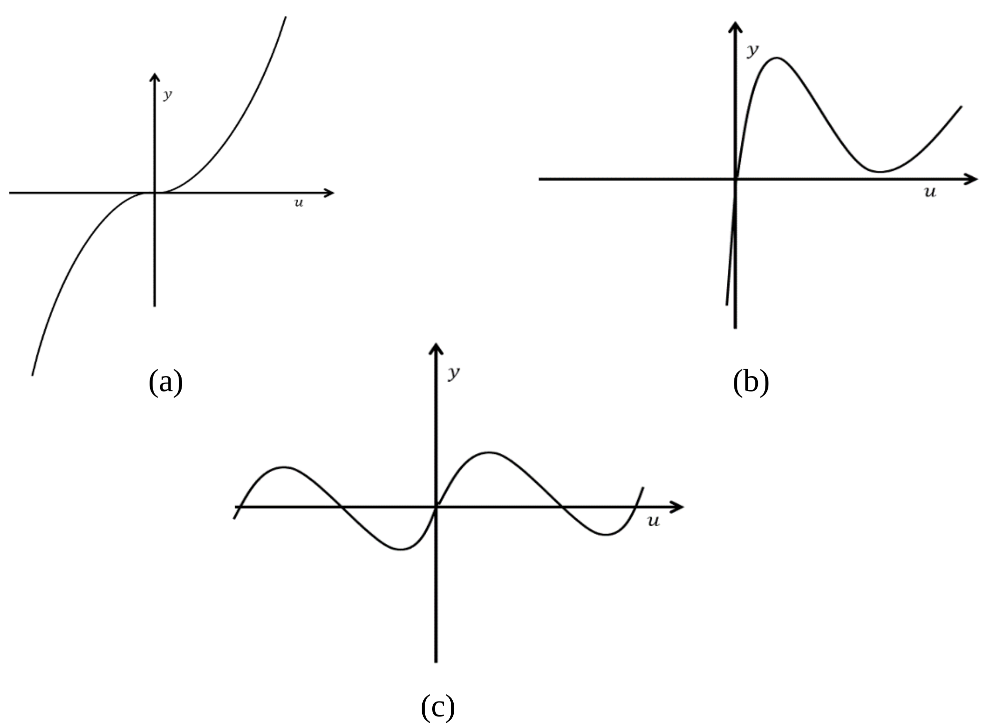

Fig. 2.3 (a) and (b): Characteristics of nonlinear passive resistor. (c): Characteristics of non-passive resistor.

In case of multiport network where u and y are vectors, the power flow to the network can be taken as the inner product. i.e. $u^T y = \sum_{i=1}^{m} u_i y_i = \sum_{i=1}^{m} u_i h_i(t,u)$ and the network is passive when $u^T y \geq 0$ for all values of u. Here, $u^T y$ can be considered as the power inflow to the system. When the function h that satisfy $u^T y = u^T h(t,u) \geq u^T \emptyset(u) > 0$ for all $u \neq 0$, then the function h is called *input strictly passive* since the passivity is strict in the sense that $u^T y = 0$ only when $u = 0$. On the other hand, when $u^T y = u^T h(t,u) \geq y^T \varphi(y)$ for all $y \neq 0$, function h is called *output strictly passive* because the passivity is strict in the sense that $u^T y = 0$ only when $y = 0$. We conclude all these definitions in the following remark.

Remark 2.3: For a system $y = h(t,x)$, the following definitions can be stated.

i). It is passive if $[\mathcal{w}(u,y) = u^T y] \geq 0$.
ii). It is lossless if $[\mathcal{w}(u,y) = u^T y] = 0$.
iii). It is input strictly passive if $[\mathcal{w}(u,y) = u^T y] \geq u^T \emptyset(u) > 0,\ \forall\, u \neq 0$.
iv). It is output strictly passive if $[\mathcal{w}(u,y) = u^T y] \geq u^T \varphi(y) > 0,\ \forall\, y \neq 0$.

In all these cases, the inequalities should hold for all (t,u).

Now, defining the same passivity concepts for dynamical system like e.g. system (2.1). Such a system said to be passive if it possesses a positive semidefinite energy function $\mathcal{S}(x)$ and a bilinear supply rate $\mathcal{w}(u,y) = u^T y$ such that the following inequality is satisfied. i.e.

$$\mathcal{S}[x(T)] - \mathcal{S}[x(0)] \leq \int_0^T \mathcal{w}[u(t),y(t)]dt \qquad (2.9)$$

for all u and $T \geq 0$. Expressing in words, it can be concluded from (2.9) that passivity is a property such that the increase in the system's storage $\mathcal{S}$ should not be larger than the amount of the power supplied. The equation (2.9) can be rewritten in the derivative form as,

$$\dot{\mathcal{S}}(x) \leq \mathcal{w}(u,y) \qquad (2.10)$$

Expressing in words, it follows from (2.10) that the property in which the rate of increase of the systems storage is not greater than the supply rate is called passivity. It can be interpreted in another way that; any increase in the system's storage is solely due to the external sources. These definitions of passivity can be motivated by the following RLC circuit example [31]. Consider the network of Fig. 2.4.

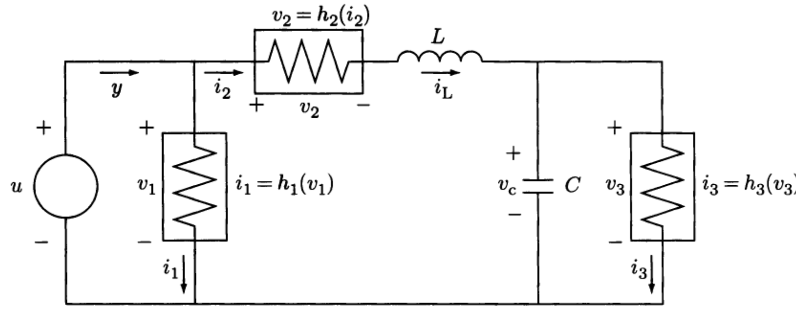

Fig. 2.4: RLC example for defining passivity.

where i indicate the current, v represents the voltage and u denotes the supply source. The network contains the nonlinear resistive elements characterized by $i_1 = h_1(v_1)$, $v_2 = h_2(i_2)$ and $i_3 = h_3(v_3)$. By considering u and y as input and output respectively, the power inflow to the network will be uy. The state space model of this network can be derived by choosing the current through the inductor and voltage through the capacitor as state variables x_1 and x_2 respectively. i.e.

$$L\dot{x}_1 = u - h_2(x_1), \quad C\dot{x}_2 = x_1 - h_3(x_2), \quad y = x_1 + h_1(u)$$

For the energy storage function corresponding to the energy storage elements (capacitor and inductor). i.e. $\mathcal{S}(x) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$, the system can be termed as passive when over the period $[0, T]$, the energy supplied to the network is greater than or equal to the change in energy stored in the network. i.e.

$$\mathcal{S}[x(T)] - \mathcal{S}[x(0)] \leq \int_0^T w[u(t), y(t)]dt$$

$$\mathcal{S}[x(T)] - \mathcal{S}[x(0)] \leq \int_0^T u(t)y(t)dt \tag{2.11}$$

When the inequality (2.11) holds, it means that the difference between the supplied energy and the change in the stored energy should be equal to the amount that is dissipated across the resistors. The inequality (2.11) can be expressed in the instantaneous power form as,

$$\dot{\mathcal{S}}(x) \leq u(t)y(t) \tag{2.12}$$

The inequality (2.12) shows that the power inflow to the network must be greater than or equal to the rate of change of stored energy in the network. Further, investigation of (2.12) across the system trajectories as follows:

$$\begin{aligned}
\dot{\mathcal{S}} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 = x_1[u - h_2(x_1) - x_2] + x_2[x_1 - h_3(x_2)] \\
&= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\
&= x_1u - x_1h_2(x_1) - x_2h_3(x_2) \\
&= x_1u - x_1h_2(x_1) - x_2h_3(x_2) + uh_1(u) - uh_1(u) \\
&= u[x_1 + h_1(u)] - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\
&= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\
uy &= \dot{\mathcal{S}} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2)
\end{aligned} \tag{2.13}$$

Now, when h_1, h_2 and h_3 are passive, then $uy \geq \dot{\mathcal{S}}$. This means that the RLC network overall is passive. The other possibilities are as follows:

- When $h_1 = h_2 = h_3 = 0$, the equation (2.13) reduces to $uy = \dot{\mathcal{S}}$. This mean that the network does not dissipate any energy. So, the system is lossless.
- When h_2 and h_3 are passive, then equation (2.13) gets reduced to $uy \geq \dot{\mathcal{S}} + uh_1(u)$. Now, if $uh_1(u) > 0$ for all $u \neq 0$, then the energy supplied to the network over the period

$[0, T]$ will be equal to the change in the stored energy only when $u(T) = 0$. So, the system is strictly input passive.

- When $h_1 = 0$ and h_3 is passive, then the equation (2.13) will be equivalent to $uy \geq \dot{\mathcal{S}} + yh_2(y)$. Now, if $yh_2(y) > 0$ for all $y \neq 0$, then the energy supplied to the network over the period $[0, T]$ will be equal to the change in stored energy only when $y(T) = 0$. In this case, the system is strictly output passive.
- When $h_1, h_2, h_3 \in [0, \infty]$, then the equation (2.13) will be $uy \geq \dot{\mathcal{S}} + x_1 h_2(x_1) + x_2 h_3(x_2)$ where $x_1 h_2(x_1) + x_2 h_3(x_2)$ is the positive definite function of x. This is the case of strictly state passivity because the energy supplied to the network will be equal to the change in the stored energy only when $x(T) = 0$.

Remark 2.4: The system (2.1) is called a passive one if there exists a storage function $\mathcal{S}(x)$ and supply rate $w(u, y) = u^T y$ such that,

$$w(u, y) = u^T y \geq \dot{\mathcal{S}} = \frac{\partial \mathcal{S}}{\partial x} f(x, u), \quad \forall\ (x, u) \tag{2.14}$$

Moreover, it is:

a. Lossless when $[w(u, y) = u^T y] = \dot{\mathcal{S}}$.
b. Input strictly passive when $[w(u, y) = u^T y] \geq \dot{\mathcal{S}} + u^T \emptyset(u)$ and $u^T \emptyset(u) > 0, \forall\ u \neq 0$.
c. Output strictly passive when $[w(u, y) = u^T y] \geq \dot{\mathcal{S}} + u^T \varphi(y)$ and $u^T \varphi(y) > 0, \forall\ y \neq 0$.
d. State strictly passive when $[w(u, y) = u^T y] \geq \dot{\mathcal{S}} + \vartheta(x)$ with some positive definite function ϑ.

Before employing passivity as control design tool, another important term related to it needs to be defined which is called zero-state observability. Consider a passive system (2.1) with the storage function $\mathcal{S}(x)$ such that the origin of $\dot{x} = f(x)$ is stable. Then (2.1) is called zero-state observable when no solution of $\dot{x} = f(x)$ stay identically in the set $\mathfrak{S} = [y = h(x) = 0]$ other than the zero solution. i.e. $x(t) = 0$. The significance of zero-state observability related to the stability of the origin is concluded in the following remark.

Remark 2.4: Consider the system (2.1), the origin for the case of $\dot{x} = f(x)$ is asymptotically stable when the system itself is:

a. Strictly passive. (Or)
b. Output strictly passive and zero-state observable.

Moreover, when the storage function possesses the property of radially unboundedness, the origin will be globally asymptotically stable.

The statement of the remark 2.4 follows from the Reference [31]. The proof of this result can be found as Reference [31], Proof of Lemma 5.6.

Passivity can be employed to develop robust stabilizing control designs that possess necessary stability margins. i.e. disc or sector margins. Following the developed literature regarding the stabilization designs through passivity (i.e. Reference [30] and [31]), it can be stated that *"the origin of system (2.1) can be globally stabilized by choosing* $u = -\phi(y)$*, where* $\phi(.)$ *is any locally Lipchitz function such that it satisfies* $\phi(0) = 0$ *and* $y^T\phi(y) > 0$ *for all* $y \neq 0$*"*. When a system in non-passive, it can be transformed as passive one and the above statement can also be applied in this case as well. Suppose as a special case that system (2.1) in non-passive and there is a flexibility to choose the output y to make it a passive one. If there exist a storage function $\mathcal{S}(x)$, then to render the passivity from input u to the output y, the output y can be chosen as (2.15) and with this selection if the system gets zero-state observable, the statement discussed above is applicable.

$$y = h(x) = \left[\frac{\partial \mathcal{S}}{\partial x} g(x)\right]^T \tag{2.15}$$

Similarly, when a feedback control law u is used to achieve the passivation. e.g. Following the special assumption that (2.1) is non-passive system and taking $u = \alpha(x) + \beta(x)v$, $y = h(x)$ renders the closed loop system a passive one from input v to output y, then since a feedback control law u is utilized to achieve the passivation, it is termed as *feedback passivation*.

A class of systems that convince the feedback passivation are cascaded systems. The simplest cascade structure formed of two subsystems having the states z and ξ is shown in Fig. 2.5. This structure has important properties. First, that the control input u enters only in ξ subsystem. Secondly, the characteristics of z subsystem are crucial that can be changed by the interconnection that may act as control input or external disturbance.

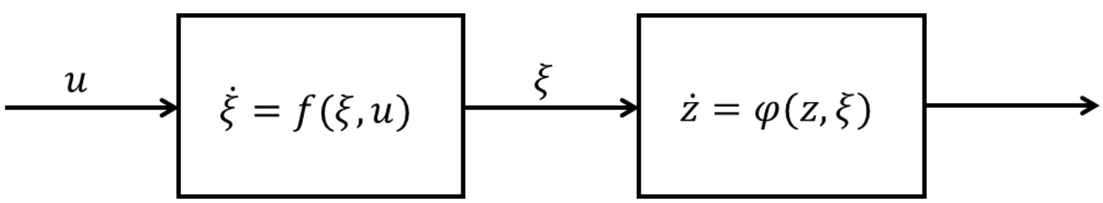

Fig. 2.5: A simple cascade structure

Such a cascaded system can be modelled is very much similar fashion to that of normal form discussed in the subsection 2.1. i.e.

$$\left.\begin{aligned} \dot{z} &= \varphi_a(z) + \Psi(z,y)y \\ \dot{\xi} &= f(z,\xi) + G(\xi)u \\ y &= h(\xi) \end{aligned}\right\} \tag{2.16}$$

where $\varphi_a(0) = 0$, $f(0) = 0$ and $h(0) = 0$. The functions φ_a, Ψ, f and G exhibit the property of locally Lipschitzness with h being a continuous function. It can be seen from (2.16) that last two equations represent a driving subsystem and the first equation is the driven system. With the assumptions that the cascade representation (2.16) is valid globally, there exists a radially unbounded storage function $\mathcal{S}(\xi)$ for the driving system that renders the driving system passive, the subsystem $\dot{z} = \varphi_a(z)$ is stable and with the knowledge of radially unbounded Lyapunov function $\mathcal{W}(z)$ for $\dot{z} = \varphi_a(z)$ such that the following inequality holds. i.e.

$$\frac{\partial W}{\partial z}\varphi_a(z) \leq 0, \qquad \forall z$$

Now, following the reference [31], the desired storage function candidate for the whole system (2.16) can be taken as $\mathfrak{U}(z,\xi) = \mathcal{W}(z) + \mathcal{S}(\xi)$ and its derivative along the system trajectories results as,

$$\dot{\mathfrak{U}} = \frac{\partial \mathcal{W}}{\partial z}\varphi_a(z) + \frac{\partial \mathcal{W}}{\partial z}\Psi(z,y)y + \frac{\partial \mathcal{S}}{\partial \xi}[f(z,\xi) + G(\xi)u]$$
$$\leq \frac{\partial \mathcal{W}}{\partial z}\Psi(z,y)y + y^T u = y^T\left[u + \left(\frac{\partial \mathcal{W}}{\partial z}\Psi(z,y)\right)^T\right]$$

and taking feedback control u as,

$$u = -\left(\frac{\partial \mathcal{W}}{\partial z}\Psi(z,y)\right)^T + v \tag{2.17}$$

results in $\dot{\mathfrak{U}} \leq y^T v$. So, the closed loop system formed of (2.16) and (2.17) renders the system passive from input v to output y with $\mathfrak{U}$ as storage function. To fulfill the condition of zero-state observability, the feedback control law (2.17) can be taken as,

$$u = -\left(\frac{\partial \mathcal{W}}{\partial z}\Psi(z,y)\right)^T - \oint(y). \tag{2.18}$$

where $\oint(0) = 0$ be any locally Lipchitz function such that $y^T\oint(y) > 0$ for all $y \neq 0$. Now taking the same function $\mathfrak{U}$ as Lyapunov candidate for the closed loop system results,

$$\dot{\mathfrak{U}} \leq \frac{\partial \mathcal{W}}{\partial z}\varphi_a(z) - y^T\oint(y) \leq 0$$

$$\dot{\mathfrak{U}} = 0 \quad \Rightarrow \quad z = 0 \text{ and } y = 0 \quad \Rightarrow \quad u = 0$$

Thus, applying invariance principle concludes that the origin of the system (i.e. $z = 0, \xi = 0$) is globally asymptotically stable.

In the context of this thesis work, we will utilize the passivity-based control technique to implement the stabilizing compensator for minimum phase saturated nonlinear systems represented in the cascaded form (2.16).

2.4 Nonlinear Output Regulation Problem

The problem of output regulation is among the fundamental problems of the control theory alternatively known as servomechanism problem. It can be outlined as imposing the system's output to track any prescribed reference signal in a certain class of constant and/or time varying functions while rejecting certain class of constant and/or time varying disturbances. The objective is that within the family of functions, the controller should provide fixed steady state response. It can be interpreted in another way that error term which is the difference between the reference signal r and system's actual output y should decay to zero with the time approaching infinity over the prescribed set of disturbances. Consider a time-invariant, nonlinear, finite dimensional system described as compact form of (2.1) by the equations,

$$\left.\begin{aligned} \dot{x} &= f(\omega, x, u) \\ e &= h(\omega, x) \end{aligned}\right\} \tag{2.19}$$

with $x \in R^n$ represent the state of the system, $u \in R^m$ denotes the control input, $e \in R^v$ is a vector of regulated outputs that includes errors and other such variables that are required to be regulated to zero. The system is subjected to the set of exogeneous signals $\omega \in R^w$ that include the unknown disturbances to be rejected and the references to be tracked.

The exogenous signal ω is supposed to be produced by a neutrally stable system known as exo-system which is designed based on the designer's prescribed knowledge about the reference signals to be tracked and the disturbance signals to be rejected. To solve the problem of output regulation perfectly, the complete knowledge of this signal or the model of the system should be available in real time that is an extremely optimistic scenario and cannot be rendered as practical situation. On the other hand, allowing the case of no knowledge of this signal of system model leads to the error that is ultimately bounded but not zero. Therefore, the generation of these exogeneous signals provides an intermediate solution where ω is allowed to belong to fixed family of time dependent signals enabling the designer to cover major cases of practical significance. A general exo-system which is used in the output regulation problems exhibit the dynamic model described by the following differential equation (2.20) where the initial conditions i.e. $\omega(0)$ are allowed to vary on the prescribed set.

$$\dot{\omega} = \mathfrak{s}(\omega) \tag{2.20}$$

Since the exo-system is neutrally stable system, we can get the model matrix S through linearization at the equilibrium. i.e.

$$S = \left[\frac{\partial \mathfrak{s}}{\partial \omega}\right]_{eq} \tag{2.21}$$

which have all the eigenvalues on the imaginary axis.

Suppose that with the available information from the system, there exist a feedback controller having output u as a function of x and ω which is given by,

$$u = \Phi(x, \omega) \tag{2.22}$$

Thus, the closed loop system formed of (2.19) - (2.22) characterized by the equations (2.23) is said to exhibit the property of *output regulation* if it is possible to design control law (2.22) such that for every exogeneous signal ω (in a prescribed set) and for every initial condition in some neighborhood of the origin, the output error decays to zero as the time tends to infinity.

$$\left.\begin{aligned} \dot{\omega} &= \mathfrak{s}(\omega) = S(\omega) \\ \dot{x} &= f[\omega, x, \Phi(x, \omega)] \end{aligned}\right\} \tag{2.23}$$

Usually, the amount of information available for the system to provide feedback describes the structure of the feedback controller. In case of all the system states x and the states of the exogeneous model ω are available for feedback, a memoryless function similar to the (2.22) can work as desired controller. However, in a more realistic scenario, only the output of the system is available rather than all the state variables. In such case, the error term e which is only measurable quantity imposes to think of dynamic controller of the form,

$$\left.\begin{aligned} \dot{\zeta} &= \Xi(\zeta, e) \\ u &= \theta(\zeta) \end{aligned}\right\} \tag{2.24}$$

with ζ being the internal state of the controller and $\Xi(0) = 0, \theta(0) = 0$ such that the following requirements are met.

- In the absence of exo-system, the origin of the system is asymptotically stable equilibrium point. i.e. if $\omega(t) = 0$, it implies that $x(t) = 0$.
- The error term $e(t)$ converges to zero considering any initial conditions for the system $x(0)$ and the exo-system $\omega(0)$.

Thus, we can say that the output regulated system possesses two responses. i.e. transient response and steady state response. During the transient response, system converges to the steady state response from given initial condition and it exhibits the steady state response for $t \longrightarrow \infty$. The necessary conditions required for the output regulation problem to be solvable are as:

- The system (2.19) must have smooth functions $f(\omega, x, u)$ and $h(\omega, x)$.
- The pair (A, B) is stabilizeable and (C, A) is detectable, where the matrices A, B and C are defined as,

$$A = \left[\frac{\partial f}{\partial x}\right]_{eq}, \qquad B = \left[\frac{\partial f}{\partial u}\right]_{eq}, \qquad C = \left[\frac{\partial h}{\partial x}\right]_{eq}$$

The control action that solve the output regulation problem can be divided into two components. One component is the one that force the system's output to slide on the steady state value / manifold while second component acts to stabilize the system's output on steady state value / manifold. It was shown by *Isidori* [32], that the output regulation problem is solvable if there exists certain continuously differentiable mapping that solve the nonlinear regulator equations. Let $\pi(\omega)$ be the steady state of x and $\sigma(\omega)$ be the steady state of ζ on the zero-error manifold, then the system must satisfy the following set of regulator equations.

$$\left.\begin{aligned} \frac{\partial \pi(\omega)}{\partial \omega}\mathfrak{s}(\omega) &= f[\pi(\omega), \omega, c(\omega)] \\ 0 &= h[\pi(\omega), \omega] \end{aligned}\right\} \tag{2.25}$$

and with the controller satisfying the equations,

$$\left.\begin{aligned} \frac{\partial \sigma(\omega)}{\partial \omega}\mathfrak{s}(\omega) &= \Lambda[\sigma(\omega), 0] \\ c(\omega) &= \vartheta[\sigma(\omega)] \end{aligned}\right\} \tag{2.26}$$

where $c(\omega)$ is the steady state value of control u and it is the polynomial in the component of ω only. Note that the model $\mathfrak{s}(\omega)$ of exo-system is used in the regulator equations (2.25) and (2.26). This suggests that without incorporation of such model, generally the output regulation problem cannot be solved. This fact is known as *internal model principle* and is usually designed as,

$$\left.\begin{aligned} \frac{\partial \sigma(\omega)}{\partial \omega}S\omega &= \varphi\sigma(\omega) \\ c(\omega) &= \Gamma\sigma(\omega) \end{aligned}\right\} \tag{2.27}$$

with $\sigma(\omega)$, φ and Γ represent mappings given by,

$$\sigma(\omega) = \begin{bmatrix} c(\omega) \\ \mathcal{L}_s c(\omega) \\ \mathcal{L}_s^2 c(\omega) \\ \vdots \\ \mathcal{L}_s^{q-1} c(\omega) \end{bmatrix}, \qquad \Gamma = [1 \quad 0 \quad 0 \quad \dots \quad 0]_{1\,\mathrm{x}\,q}, \qquad \varphi = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{q-1} \end{bmatrix}$$

The coefficients $a_0, a_1 \dots, a_{q-1}$ are the real numbers that are obtained by the following equation,

$$\mathcal{L}_s^q c(\omega) = a_0 c(\omega) + a_1 \mathcal{L}_s c(\omega) + a_2 \mathcal{L}_s^2 c(\omega) + \dots + a_{q-1}\mathcal{L}_s^{q-1} c(\omega)$$

such that the characteristics polynomial $\lambda^q - a_{q-1}\lambda^{q-1} - \dots - a_1\lambda - a_0$ has distinct roots on the imaginary axis with pair $\left(\begin{bmatrix} A & 0 \\ GC & \varphi \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}\right)$ being stabilizable where $G = \left[\frac{\partial \Xi}{\partial u}\right]_{eq}$ and $\left([C \quad 0], \begin{bmatrix} A & B\Gamma \\ 0 & \varphi \end{bmatrix}\right)$ being detectable.

Now, with the change of variable $\tilde{x} = x - \pi(\omega)$, the system (2.19) is transformed as,

$$\dot{\tilde{x}} = f(\omega, \tilde{x}, v - c(\omega))$$

where $u = v + c(\omega)$, and component v is designed such that the whole system is stabilized. This component can be designed through robust control techniques like sliding mode control or Lyapunov redesign etc. such that $v = 0$ with $x = \pi(\omega)$ and $\zeta = \sigma(\omega)$ on the zero-error manifold. In practice, the whole controller that solves this regulation problem is the parallel interconnection of the internal model and the stabilizer where,

- The internal model provides $c(\omega)$ the component of u such that $x = \pi(\omega)$ and $\zeta = \sigma(\omega)$.
- The stabilizer provides the steady state component $v = u_{st}(t)$ such that it locally stabilizes the closed loop system and induces the local error convergence towards zero-error manifold. This is as shown in Fig. 2.6.

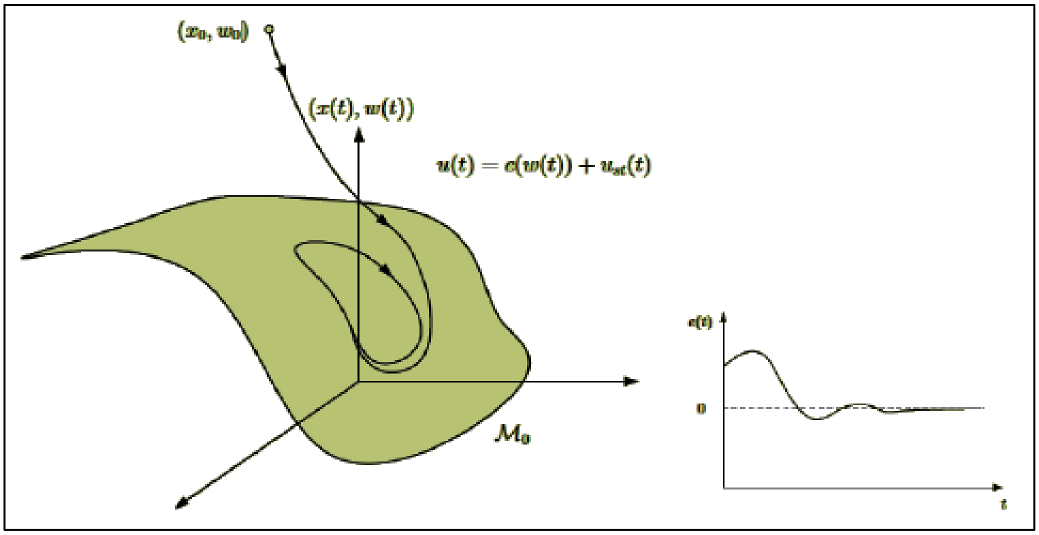

Fig. 2.6: Geometric interpretation of output regulation

In context of this thesis work, we will follow a similar idea for output regulation of the saturated class of minimum phase nonlinear systems.

2.5 Conditional Servo-Mechanism Designs

The output regulation problem discussed so far in the previous section provides two main challenges. First, the design is not robust and only provides the local stability. Secondly, the task of designing the stabilizing controller and the internal model separately can be tricky. Generally, the task of output regulation is accomplished using the idea of servo-mechanism in which a servo-compensator is designed that achieves the output regulation robustly. There are well established methods of designing the servo-compensator in the literature, however, here in this thesis work, the work of *Hassan K. Khalil* [23] and *Attaullah Y. Memon et. al.* [7], [33] is summarized that will lead to our research work in the next chapter. Starting from the conventional servo-compensator design, we will discuss the its drawbacks that leads to the necessity of making the servo-compensator as conditional one and how to achieve do that. The whole discussion follows sequentially from references [23], [7], [33].

Consider the nonlinear system (2.19) in the form,

$$\left.\begin{aligned} \dot{x} &= f(x,\omega) + G(x,\omega)u \\ e &= h(x,\omega) \end{aligned}\right\} \tag{2.28}$$

where the exogeneous signal ω belongs to prescribed set $\mathfrak{Y} \in R^{\omega}$. The functions f, G and h are smooth in the domain $\mathfrak{D} \subset R^n$ and continuous in ω over the set $\mathfrak{Y}$. The error term e represents the vector $[e_1 \quad e_2 \quad e_3 \quad \dots \quad e_n]$, where $e_1 = y - q(\omega)$ with $q(\omega)$ represents the trajectory to be achieved. The solution to the output regulation problem requires the following assumptions.

Assumption 2.3.1: The signal ω and $q(\omega)$ are assumed to be generated by a known neutrally stable exo-system. i.e.

$$\dot{\omega} = S_0\omega \tag{2.29}$$

where S_0 have distinct eigen-values on the imaginary axis and $\omega(t)$ belongs to the compact set $\mathfrak{Y}$.

Assumption 2.3.2: There exists a continuous mapping $x = \pi(\omega)$ with $\pi(0) = 0$ and $u = \chi(\omega)$ on the zero-error manifold which solves the regulator equations given as,

$$\left.\begin{aligned} \frac{\partial\pi(\omega)}{\omega} S_0\omega &= f(\pi,\omega) + G(\pi,\omega)\chi(\omega) \\ 0 &= h(\pi,\omega) \end{aligned}\right\} \tag{2.30a}$$

for all $\omega \in \mathfrak{Y}$.

Assumption 2.3.3: There exist set of real constants $a_0, a_1, \dots, a_{r-1}$, such that the $u = \chi(\omega)$ satisfies the identity given as,

$$\mathcal{L}_s^r\chi(\omega) = a_0\chi(\omega) + a_1\mathcal{L}_s\chi(\omega) + a_2\mathcal{L}_s^2\chi(\omega) + \dots + a_{r-1}\mathcal{L}_s^{r-1}\chi(\omega) \tag{2.30b}$$

for all $\omega \in \mathfrak{Y}$ and the characteristic polynomial $\mathfrak{X}^r - a_{r-1}\mathfrak{X}^{r-1} - \dots - a_1\mathfrak{X} - a_0$ has roots on the imaginary axis. Selecting,

$$S = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{r-1} \end{bmatrix}_{r\,\mathrm{x}\,r}, \quad \mathcal{T}(\omega) = \begin{bmatrix} \chi(\omega) \\ \mathcal{L}_s\chi(\omega) \\ \mathcal{L}_s^2\chi(\omega) \\ \vdots \\ \mathcal{L}_s^{r-1}\chi(\omega) \end{bmatrix}_{r\,\mathrm{x}\,1}, \quad \Gamma = [1 \quad 0 \quad 0 \quad \dots \quad 0]_{1\,\mathrm{x}\,r}$$

It has been shown in [32], that $\chi(\omega)$ can be generated by the internal model,

$$\left.\begin{aligned} \frac{\partial\mathcal{T}}{\partial\omega} S_0 &= S\mathcal{T}(\omega) \\ \chi(\omega) &= \Gamma\mathcal{T}(\omega) \end{aligned}\right\} \tag{2.31}$$

The conventional servo-compensator followed by the above-mentioned assumptions can now be augmented with the system (2.28). i.e.

$$\dot{\varrho} = S\varrho + Je_1 \tag{2.32}$$

where J can be selected as $J = [0 \quad 0 \quad 0 \quad \dots \quad 1]$. We can take the feedback control law for the output regulation problem (2.33) with k is the constant gain and is referred to the maximum permissible control magnitude, $\mathcal{L}_g\mathcal{L}_f^{n-1}h$ is called the high frequency gain. $sign$ as well as sat are the standard signum and saturation functions defined as (2.34) and (2.35).

$$u = -k\ sign\left(\mathcal{L}_g\mathcal{L}_f^{n-1}h\right) sat\left(\frac{s}{\mu}\right) \tag{2.33}$$

$$sign(x) = \begin{cases} 1 & , & x > 0 \\ 0 & , & x = 0 \\ -1 & , & x < 0 \end{cases} \tag{2.34}$$

$$sat(x) = \begin{cases} x & , & |x| \leq 1 \\ sign(x) & , & |x| > 1 \end{cases} \tag{2.35}$$

The sliding surface s can be taken as,

$$s = K_1\varrho + K_2[e_1 \quad e_2 \quad \cdots \quad e_{n-1}]^T + e_n \tag{2.36}$$

and K_1, K_2 matrices can be designed such that,

$$\mathcal{H} = \begin{bmatrix} S & JC_0 \\ -B_0K_1 & A_0 - B_0K_2 \end{bmatrix}$$

is a Hurwitz matrix. The matrices A_0, B_0 and C_0 are the canonical form representations of the system (2.28) when transformed to normal form.

Conventional servo-compensator addresses the challenges posed at the beginning of this section. i.e. It achieves the non-local robust output regulation but with a drawback that the steady state performance usually happens at the expense of transient performance degradation. It happens so because the inclusion of servo-compensator increase the overall system's order and also due to the interaction of the servo-compensator with the control saturation. This issue has been formally addressed under the topic of conditional servo-compensator by *Seshagiri et. al.* [28] as well as *Attaullah Y. Memon et. al.* [7], [33]. The topic of conditional servo-compensator has been discussed in the context of Sliding Mode Framework in [28] and in the context of Lyapunov Redesign Framework in [7], [33]. Conceptually both discussions are similar but the later provide the flexibility of choosing with any stabilizing state feedback controller for the system and then to include servo-compensator to perform the desired task. This flexibility will lead to our work in case of saturated systems. So, owing to the implicational significance, the Lyapunov Redesign Framework based servo-compensator design will be discussed here.

To start in the Lyapunov Redesign framework, transforming the system (2.28) with the change of variable $\tilde{\xi} = x - \pi$, into the form as given by,

$$\dot{\tilde{\xi}} = \tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)[u - \chi(\omega)] \tag{2.37}$$

where $\tilde{f}(\tilde{\xi}, \omega) = f(\tilde{\xi} + \pi, \omega) - f(\pi, \omega) + [G(\tilde{\xi} + \pi, \omega) - G(\pi, \omega)]\chi(\omega)$ and $\tilde{G}(\tilde{\xi}, \omega) = G(\tilde{\xi} + \pi, \omega)$. The system (2.37) possess the form where treating $\chi(\omega)$ as the matched uncertainty, the problem of state feedback regulation can be termed as state feedback stabilization. Assume that there exists a stabilizing state feedback control for the system (2.38) and also a Lyapunov function for the corresponding closed loop system.

$$\dot{\tilde{\xi}} = \tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)u \tag{2.38}$$

Assumption 2.3.4: There exists a locally Lipschitz function $\Delta(\tilde{\xi}, \omega)$, with $\Delta(0, \omega) = 0$ and a continuously differentiable Lyapunov function $\mathcal{V}(\tilde{\xi}, \omega)$ such that $\alpha_1(\|\tilde{\xi}\|) \leq \mathcal{V}(\tilde{\xi}, \omega) \leq \alpha_2(\|\tilde{\xi}\|)$ and,

$$\frac{\partial \mathcal{V}}{\partial \omega} S_o \omega + \frac{\partial \mathcal{V}}{\partial \tilde{\xi}} [\tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)\Delta(\tilde{\xi}, \omega)] \leq -\mathbb{W}(\tilde{\xi}) \tag{2.39}$$

where $\tilde{\xi} \in R^n$, $\omega \in \mathfrak{Y}$, α_1, α_2 are class $\mathcal{K}$ functions and $\mathbb{W}(\tilde{\xi})$ is positive definite continuous function.

Now, writing the system (2.37) as,

$$\dot{\tilde{\xi}} = \tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)\Delta(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)u - \tilde{G}(\tilde{\xi}, \omega)[\chi(\omega) + \Delta(\tilde{\xi}, \omega)] \tag{2.40}$$

System (2.40) is in the form where it is required that a saturated high-gain feedback controller is required to deal with the term $\chi(\omega)$. Assuming that $\Omega = \{\mathcal{V}(\tilde{\xi}, \omega) < g\}$ is a compact set, $g > 0$ and δ be some function such that,

$$\|\chi(\omega) + \Delta(\tilde{\xi}, \omega)\| \leq \delta(\tilde{\xi}), \qquad \forall\ \tilde{\xi} \in \Omega$$

Assumption 2.3.5: From (2.39), assume that $\left(\frac{\partial \mathcal{V}}{\partial \tilde{\xi}}\right) \tilde{G}(\tilde{\xi}, \omega)$ can be stated as,

$$\left(\frac{\partial \mathcal{V}}{\partial \tilde{\xi}}\right) \tilde{G}(\tilde{\xi}, \omega) = \mathcal{v}^T(\tilde{\xi})\mathcal{H}(\tilde{\xi}, \omega)$$

where $\mathcal{v}(\tilde{\xi})$ is a locally Lipchitz known function with $\mathcal{v}(0) = 0$ and $\mathcal{H}(\tilde{\xi}, \omega)$ is possibly unknown function. i.e. $\mathcal{H}^T(\tilde{\xi}, \omega) + \mathcal{H}(\tilde{\xi}, \omega) \geq 2\lambda I_m$, $\|\mathcal{H}(\tilde{\xi}, \omega)\| < \Bbbk$, $\Bbbk \geq \lambda > 0$ and I_m is the identity matrix.

The version of servo-compensator (2.32) called conditional servo-compensator is then provided by the equation (2.41) and the feedback controller that solves the output regulation problem without degrading the transient performance can be selected as (2.42).

$$\dot{\varrho} = (S - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right) \tag{2.41}$$

$$u = -\mathfrak{a}(\tilde{\xi}) sat\left(\frac{s}{\mu}\right) \tag{2.42}$$

$$\mathfrak{a}(\tilde{\xi}) \geq \frac{\mathbb{k}}{\lambda} + \mathfrak{a}_0, \qquad \mathfrak{a}_0 > 0 \tag{2.43}$$

where $s = v(\tilde{\xi}) + K_1\varrho$, μ is the boundary layer inside which the servo-action will be performed and matrix K_1 is designed such that $(S - JK_1)$ is Hurwitz.

For the work of this thesis, we will use conditional servo-compensator to perform the task of output regulation for saturated class of minimum phase nonlinear systems due to the superiority of this design discussed in this section that compared to conventional servo-compensator, it provides the robust output regulation without degrading the transient performance of the system.

2.6 Extended High Gain Observer (EHGO)

The control designing process for certain system usually assume that all of its states are available and can be used in the process wherever required. This situation in general not true and in most of the realistic scenarios, we need to use a sensor for each state measurement which is not only costly but also unreliable approach. Secondly, sometimes it is also not possible to measure some of the state even through sensor. That means there is a necessity of some alternate phenomena that may be helpful in such a scenario. To overcome this hindrance, a control engineer uses a technique called the state estimator or state observer in which all the required states of the system are observed / estimated by using only the available information from the system. *Weiwen Wang et. al.* [34] compares some of the widely-used state observers. Observers form the basis of output feedback control design.

If the state observer observes all the state variables of the system, regardless of whether some of them are available for direct measurement or not, it is called a full-order state observer. When the fewer states out of n state variables are measured using the observer, where n is the dimension of the state vector, the observer is called reduced-order state observer or simply reduced-order observer.

A state observer is a system that estimates the state variables based on the measurement of the output and control variables. The most commonly used linear observer known as Luenberger Observer can be found frequently in the literature. Consider a linear system as modelled by the equations,

$$\left.\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}u \\ y &= \mathcal{C}x \end{aligned}\right\} \tag{2.44}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are the state space model parameters. The observer is a subsystem that rebuilds the system's state vector it exhibits the mathematical substantially same as that of the original system with a difference that it includes an extra term which is related to the tracking error in order to compensate the inaccuracies in the system matrices $\mathcal{A}$ and B due to the reason that their initial values are not known. So, the mathematical model of the observer for the system (2.44) can be defined as,

$$\dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}u + \mathcal{L}(y - \mathcal{C}\hat{x}) \tag{2.45}$$

It follows from (2.45) that the input to the observer are the output y and the control input u while the matrix $\mathcal{L}$ is the weighing matrix that involves the difference between the measured output y and the estimated output $\mathcal{C}\hat{x}$ is called the observer gain. The observer gain associated term in its model is responsible for improving its performance by continuously correcting the model output. In designing observer, the gain matrix $\mathcal{L}$ is selected such that $(\mathcal{A} - \mathcal{L}\mathcal{C})$ is Hurwitz. i.e. All the eigenvalues lie in left half plane which guarantees the convergence of estimation error to zero. Such a gain matrix can be designed using simple ideas, for example, pole placement.

The main challenge in the Luenberger Observer is that the performance of the observer is highly dependent on the system model accuracy. e.g. the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ for this case. To enhance observer capabilities to deal with the real world issues like uncertainty, noise, disturbance etc., a robust nonlinear observer called High Gain Observer (HGO) was introduced. Representing the nonlinear version of system (2.44) in the normal form as,

$$\left.\begin{aligned} \dot{x}_j &= x_{j+1}, \qquad 1 \le j \le \rho-1 \\ \dot{x}_\rho &= b(x,\theta) + a(x,\theta)u \\ y &= x_1 \end{aligned}\right\} \tag{2.46}$$

where ρ is the relative degree and θ is a vector of unknown disturbances. The High Gain Observer for the system (2.46) can be designed as,

$$\left.\begin{aligned} \dot{\hat{x}}_{j+1} &= \hat{x}_{j+1} + \frac{h_j}{\varepsilon^j}(y - \hat{x}_1), \qquad 1 \le j \le \rho - 1 \\ \dot{\hat{x}}_\rho &= b(\hat{x}) + a(\hat{x})u + \frac{h_\rho}{\varepsilon^\rho}(y - \hat{x}_1) \end{aligned}\right\} \tag{2.47}$$

where ε is the design parameter and normally is chosen as small as possible. The constants $h_1, h_2 \dots h_\rho$ are selected such that the polynomial $\lambda^\rho + h_1\lambda^{\rho-1} + \cdots + h_{\rho-1}\lambda + h_\rho$ is Hurwitz. $b(\hat{x})$ and $a(\hat{x})$ are the nominal models of $b(x,\theta)$ and $a(x,\theta)$ respectively. In the cases where the nominal models are not known, they can be ignored and a High Gain Observer can still be designed. However, their inclusion in the observer design yields with high convergence rate which is highly desirable.

In case of nonlinear systems, when the model exhibits the normal form where the internal and external dynamics of the system exists explicitly, a simple High Gain Observer may not work to observe all the states of both the dynamics. In such case, the observer which is normally used is called Extended High Gain Observer (EHGO). This observer estimates the derivatives of the output in addition to an extra signal that is used as virtual output for the auxiliary system. For example, consider the single-input, single-output nonlinear system with well-defined relative degree ρ but consisting of both the state dynamics. i.e.

$$\left.\begin{aligned} \dot{z} &= \vartheta(z,\xi) \\ \dot{\xi}_j &= \xi_{j+1}\,, \quad 1 \le j \le \rho - 1 \\ \dot{\xi}_\rho &= b(z,\xi) + a(\xi,u) \\ y &= \xi_1 \end{aligned}\right\} \tag{2.48}$$

or in compact form as,

$$\left.\begin{aligned} \dot{z} &= \vartheta(z,\xi) \\ \dot{\xi} &= A\xi + B[b(z,\xi) + a(\xi,u)] \\ y &= C\xi \end{aligned}\right\} \tag{2.49}$$

where $z \in \mathcal{R}^{n-\rho},\ \xi \in \mathcal{R}^{\rho}$. Extracting the auxiliary system from (2.49) as,

$$\dot{z} = \vartheta(z,\xi), \qquad \sigma = b(z,\xi) \tag{2.50}$$

Any suitable observer called the internal observer can be used to estimate the states of the auxiliary system (2.50) formed of the internal dynamics. For example, *Boker et. al*. [9] used Extended Kalman Filter (EKF) as internal observer. EKF exhibits a similar in structure but differ from technical design as that of Luenberger Observer. For, the auxiliary system (2.50), the EKF takes the form as,

$$\dot{\hat{z}} = \vartheta(\hat{z},\xi) + \mathcal{L}(t)[\sigma - b(\hat{z},\xi)] \tag{2.51}$$

where the observer gain $\mathcal{L}(t)$ can be designed as,

$$\mathcal{L}(t) = \mathcal{P}(t)\mathcal{C}(t)^T R(t)^{-1} \tag{2.52}$$

and $\mathcal{P}(t)$ with $\mathcal{P}(0) = 0$ is the solution of the Riccati Differential Equation,

$$\dot{\mathcal{P}}(t) = \mathcal{A}_1(t)\mathcal{P}(t) + \mathcal{P}(t){\mathcal{A}_1}^T(t) + \mathcal{Q}(t) - \mathcal{P}(t){\mathcal{C}_1}^T(t)\mathcal{R}^{-1}(t)\mathcal{C}_1(t)\mathcal{P}(t) \tag{2.53}$$

The time varying matrices $\mathcal{A}_1(t)$ and $\mathcal{C}_1(t)$ are given by,

$$\mathcal{A}_1(t) = \frac{\partial\vartheta}{\partial z}(\hat{z},\xi), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}(\hat{z},\xi) \tag{2.54}$$

and $\mathcal{Q}(t)$ and $\mathcal{R}(t)$ are symmetric positive definite matrices that which satisfy,

$$0 < r_1 \le \mathcal{R}(t) \le r_2 \tag{2.55}$$

$$0 < q_1 I_{n-\rho} \le \mathcal{Q}(t) \le q_2 I_{n-\rho} \tag{2.56}$$

Now, the observer that will work for the external dynamics called the external observer will be employed through EHGO. This observer, in addition to the states of the external dynamics, will observe an extra state that has been utilized as the output of the auxiliary system (2.50).

Its structure is as follows:

$$\dot{\hat{\xi}} = A\hat{\xi} + B\big[\hat{\sigma} + a\big(\hat{\xi}, u\big)\big] + \mathcal{H}(\varepsilon)\big(y - C\hat{\xi}\big) \qquad (2.57)$$

$$\dot{\hat{\sigma}} = \dot{b}\big(\hat{z}, \hat{\xi}, u\big) + \frac{\hbar_{\rho+1}}{\varepsilon^{\rho+1}}\big(y - C\hat{\xi}\big) \qquad (2.58)$$

where,

$$\dot{b}\big(\hat{z}, \hat{\xi}, u\big) = \left({}^{d[b(z,\xi)]}/_{dt} \right)\Big|_{(\hat{z},\xi)}, \text{ and} \qquad (2.59)$$

$$\frac{d[b(z,\xi)]}{dt} = \frac{db}{dz}\,\vartheta(z,\xi) + \frac{db}{d\xi}\{A\xi + B[b(z,\xi) + a(\xi,u)]\} \qquad (2.60)$$

The observer gain matrix $\mathcal{H}(\varepsilon) = \begin{bmatrix} \frac{\hbar_1}{\varepsilon} & \frac{\hbar_2}{\varepsilon^2} & \dots & \frac{\hbar_\rho}{\varepsilon^\rho} \end{bmatrix}^T$ and $\hbar_1, \hbar_2 \dots \hbar_\rho$ are selected such that $\lambda^{\rho+1} + \hbar_1\lambda^\rho + \cdots + \hbar_\rho\lambda + \hbar_{\rho+1}$ is Hurwitz. Moreover, $\varepsilon > 0$ is the small design parameter.

So, combining the internal and external observers (2.51), (2.57) and (2.58), the full order observer for (2.49) is characterized by,

$$\left.\begin{aligned} \dot{\hat{\xi}} &= A\hat{\xi} + B\big[\hat{\sigma} + a\big(\hat{\xi}, u\big)\big] + \mathcal{H}(\varepsilon)\big(y - C\hat{\xi}\big) \\ \dot{\hat{\sigma}} &= \dot{b}\big(\hat{z}, \hat{\xi}, u\big) + \frac{\hbar_{\rho+1}}{\varepsilon^{\rho+1}}\big(y - C\hat{\xi}\big) \\ \dot{\hat{z}} &= \vartheta(\hat{z}, \xi) + \mathcal{L}(t)[\sigma - b(\hat{z}, \xi)] \end{aligned}\right\} \qquad (2.61)$$

The time varying matrices are now given by,

$$\mathcal{A}_1(t) = \frac{\partial\vartheta}{\partial z}\big(\hat{z}, \hat{\xi}\big), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}\big(\hat{z}, \hat{\xi}\big)$$

For our thesis research work, we will design the output feedback version of the conditional servo-mechanism developed for a class of saturated nonlinear minimum phase systems, utilizing the idea of Extended High Gain Observer (EHGO) discussed in this section.

Chapter 3

OUTPUT REGULATION PROBLEM FOR SATURATED SYSTEMS

In this chapter, we put forward our idea of output regulation for a class of constrained nonlinear systems using conditional servo-mechanism. This chapter is divided into 6 sections. In section 3.1, we provide the system description and the problem formulation with considered classes of saturated systems, section 3.2 discusses the servo-mechanism (conventional and conditional) designs for these classes of systems (i.e. Class of systems possessing linear dynamics subjected to the control constraints as well as the class of systems exhibiting nonlinear dynamics itself along with the saturation nonlinearities). The ideas of Composite Nonlinear Feedback (CNF) and Passivity-Based Control schemes have been exposed for such designs. In section 3.3, the state feedback developments of previous section have been extended to the output feedback by presenting the appropriate observer designs. Section 3.4 provides the stability analysis of the closed loop system followed by the simulation results presented in the section 3.5 that depicts the efficacy of the developed control designs. Finally, this chapter closes with the section 3.6 that includes the technical discussion of the presented results and the concluding remarks.

3.1 System Description and Problem Formulation

In this thesis, we consider two classes of systems. First class of systems is that which possess the linear dynamics and are exposed to input saturations making the system overall nonlinear. In the second class, we consider the systems exhibiting the nonlinear dynamical behavior itself besides being subjected to the nonlinear control constraints. The purpose in both the cases is to have the make the output of the system follow the desired trajectory (and reject a class of disturbance signals) generated by the external autonomous system called exo-system. Starting with the state feedback design in the next section (section 3.2), an output feedback design will also be implemented using the simple High Gain Observer (HGO) for the first case and Extended High Gain Observer (EHGO) later in section 3.3. The systems related to our first class of saturated systems possess the structure of their mathematical model as,

$$\left.\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}sat(u) + \mathcal{E}\omega \\ \dot{\omega} &= \mathcal{S}\omega \\ e &= \mathcal{C}_1 x + \mathcal{F}_1\omega \\ y &= \mathcal{C}_2 x + \mathcal{F}_2\omega \end{aligned}\right\} \quad (3.1)$$

where $x \in \mathbb{R}^q$ represent the system state, u is the control input, $\omega \in \mathbb{R}^q$ represent the exo-system, e represent the error signal and y represent the output of the system. The matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{E}$, $\mathcal{S}$, $\mathcal{C}_1$, $\mathcal{F}_1$, $\mathcal{C}_2$ and $\mathcal{F}_2$ are constant matrices of appropriate dimensions. The function $sat: \mathbb{R} \to \mathbb{R}$ is the actuator saturation which is defined by (2.7) as,

$$sat(u) = sgn(u)min\{u_{max}, |u|\} \quad (3.2)$$

with u_{max} represent the maximum level of saturation of the input. It is assumed that following assumptions hold for the system (3.1). i.e.

A1 The pair $(\mathcal{A}, \mathcal{B})$ is stabilizable with all the eigenvalues of system matrix $\mathcal{A}$ lie in the closed left half plane.

A2 The matrix $\mathcal{S}$ is Anti-Hurwitz. i.e. all the eigenvalues of $\mathcal{S}$ should have nonnegative real parts.

A3 The pair $\left([\mathcal{C}_2 \quad \mathcal{F}_2], \begin{bmatrix} \mathcal{A} & \mathcal{E} \\ 0 & \mathcal{S} \end{bmatrix}\right)$ should be detectable.

On the other hand, the systems related to our second class of saturated systems possess mathematical model with relative degree ρ which under suitable system transformations can be expressed as,

$$\left.\begin{aligned} \dot{z} &= \varphi_a(z, \omega) + \Psi(z, y, \omega)y \\ \dot{\xi} &= f(z, \xi, \omega) + G(\xi, \omega)sat(u) \\ y &= h(\xi, \omega) \end{aligned}\right\} \quad (3.3)$$

where $\varphi_a(0) = 0$, $f(0) = 0$ and $h(0) = 0$. The functions φ_a, Ψ, f and G exhibit the property of locally Lipschitzness with h being a continuous function, $sat(.)$ function is as provided by (3.2) and ω is the exogenous signal provided by exo-system. The z equation of system (3.3) represents the internal dynamics of the system whereas ξ represents the external dynamics. We state the crux of this thesis as the following problems to be addressed.

Problem 3.1.1 (State Feedback Output Regulation): The state feedback version of the output regulation problem is to find a control law,

$$u = \sigma(\eta) \quad (3.4)$$

such that,

i. The closed loop system formed by the interconnection of actual system (3.1) or (3.3) with this control law, is asymptotically stable.

ii. The solution of the closed loop system with the given initial conditions satisfy that $lim_{t\to\infty} e(t) = 0$.

Problem 3.1.2 (Error Feedback Output Regulation): The error feedback version of the output regulation problem is to find the control law,

$$\left.\begin{aligned} \dot{\theta} &= \psi(\theta, \eta) \\ u &= \aleph(\theta) \end{aligned}\right\} \tag{3.5}$$

such that

i. The closed loop system formed by the interconnection of actual system (3.1) or (3.3) with this control law, is asymptotically stable.
ii. The solution of the closed loop system with the given initial conditions satisfy that $lim_{t\to\infty} e(t) = 0$.

Problem 3.1.3 (Output Feedback Output Regulation): The output feedback version of the output feedback regulation problem is to find the observer based control law,

$$\left.\begin{aligned} \dot{\theta} &= \mathfrak{d}(\theta, \hat{\eta}\,) \\ u &= \mathfrak{h}(\theta) \end{aligned}\right\} \tag{3.6}$$

such that

i. The closed loop system formed by the interconnection of actual system (3.1) or (3.3) with this control law, is asymptotically stable.
ii. The solution of the closed loop system with the given initial conditions satisfy that $lim_{t\to\infty} e(t) = 0$.

where in equation (3.4), $\eta = [\omega \quad x]^T$ for system (3.1) and $\eta = [\omega \quad z \quad \xi]^T$ for system (3.3) while in equation (3.5), $\eta = [e]$ for both the systems (3.1) and (3.3). However, the same notations follow in (3.6) for the respective output feedback versions as well.

3.2 Servo-Mechanism Designs for Saturated Systems

This section is focused on the design of conventional servo-compensator for the system (3.1) and (3.3) where former possesses the linear dynamics itself, however it is subjected to the saturation nonlinearities and the later exhibit the nonlinear dynamical behavior itself along with the nonlinearities of the saturation constraints. Starting with the design of conventional servo-compensator of *Hassan K. Khalil* [23], the drawbacks of this design will be discussed that will lead us the design to conditional servo-compensator [28], [7], [33]. Before proceeding with the servo-compensator designs, we make the following necessary assumptions about the system and other parameters.

Assumption 3.1: The system (3.3) is the minimum phase which means that its zero dynamics [i.e. $\dot{z}(z, 0)$] are asymptotically stable. This means that the eigenvalues of the system matrix characterizing the linear zero dynamics lie in the open left half plane or Lyapunov stability in case of nonlinear zero dynamics is satisfied.

Assumption 3.2: Both the systems (3.1) and (3.3) are subjected to the exogeneous signals ω produced by internally stable exo-system modelled as,

$$\dot{\omega} = \mathcal{S}_0 \omega \tag{3.7}$$

where $\mathcal{S}_0$ have distinct eigenvalues on the imaginary axis.

Assumption 3.3: There exist a continuously mapping [$x = \pi(\omega)$ for (3.1), and $\xi = \pi(\omega)$ for (3.3)] and a continuous mapping $\phi(\omega)$ such that the following regulator equations hold.

$$\left.\begin{aligned} \frac{\partial \pi(\omega)}{\partial \omega} \mathcal{S}_0 \omega &= f(z, \pi, \omega) + G(\pi, \omega)\phi(\omega) \\ 0 &= h(\pi, \omega) \end{aligned}\right\} \tag{3.8}$$

for all $\omega \in \mathcal{W}$.

The above assumption states the necessary and sufficient condition for the output regulation problem to be solvable. This means that there exists a zero-error manifold $x = \pi(\omega)$ for (3.1) and $\xi = \pi(\omega)$ for (3.3), with the steady state control as $\phi(\omega)$ on zero-error manifold. This control $\phi(\omega)$ slides the system output on zero-error manifold in the presence of disturbance signals (usually provided from exo-system as a component of exogenous signals besides reference).

Assumption 3.4: There exist real constants $c_0, c_1, \dots, c_{\rho-1}$ such that the steady state control component $\phi(\omega)$ satisfy the following identity

$$\mathcal{L}_s^{\rho} \phi = c_0 \phi + c_1 \mathcal{L}_s \phi + \cdots + c_{\rho-1} \mathcal{L}_s^{\rho-1} \phi \tag{3.9}$$

where the polynomial $\Delta^{\rho} - c_{\rho-1}\Delta^{\rho-1} - \cdots - c_1 \Delta - c_0$ have all the distinct roots on the imaginary axis and $\mathcal{L}_s \phi = \left({}^{\partial \phi}/_{\partial \omega}\right) \mathcal{S}_0 \omega$.

The above assumptions are necessary due to the motivational fact of nonlinear version of internal model principle which states that the controller must be able to generate not only the trajectories characterized by the exo-system but also its higher order harmonics. Defining the following matrices,

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{\rho-1} \end{bmatrix}_{\rho x \rho} , \quad \mathcal{T}(\omega) = \begin{bmatrix} \phi(\omega) \\ \mathcal{L}_s \phi(\omega) \\ \mathcal{L}_s^2 \phi(\omega) \\ \vdots \\ \mathcal{L}_s^{r-1} \phi(\omega) \end{bmatrix}_{\rho \text{ x } 1} , \quad \Gamma = [1 \quad 0 \quad 0 \quad \dots \quad 0]_{1 \text{ x } \rho}$$

It has been shown in [32], that $\phi(\omega)$ can be generated by the internal model,

$$\left.\begin{aligned} \frac{\partial \mathcal{T}}{\partial \omega}\mathcal{S}_0 &= \mathcal{S}\mathcal{T}(\omega) \\ \phi(\omega) &= \Gamma\mathcal{T}(\omega) \end{aligned}\right\} \tag{3.10}$$

The internal model (3.10) is only valid when $\phi(\omega)$ possess finite number of harmonics, which will always be so when $\phi(\omega)$ is the polynomial function of ω. This means that the constants $c_0, c_1, \dots, c_{\rho-1}$ for (3.9) must be known even when $\phi(\omega)$ is uncertain. With these assumptions, the conventional servo-compensator can be designed as (2.32) [23],

$$\dot{\varrho} = \mathcal{S}\varrho + Je_1 \tag{3.11}$$

where $J = [0 \quad 0 \quad 0 \quad \dots \quad 1]$. The state feedback control law u that works for the regulation problem for systems (3.1) and (3.3) based on servo-compensator design can be taken as (2.33). i.e.

$$u = -k\ sign\left(\mathcal{L}_g\mathcal{L}_f^{n-1}h\right) sat\left(\frac{s}{\mu}\right) \tag{3.12}$$

with $sign(.)$ and $sat(.)$ can be chosen as (2.34) and (2.35) respectively and sliding surface s is yet to be defined.

As discussed in chapter 2, the conventional servo-compensator addresses the non-local robust output regulation problem but with a drawback that the steady state performance usually happens at the expense of transient performance degradation. It happens so due to the increase in the overall system's order and also due to the interaction of the servo-compensator with the control saturation. This issue has been formally addressed under the topic of conditional servo-compensator by *Seshagiri et. al.* [28] as well as *Attaullah Y. Memon et. al.* [7], [33]. The topic of conditional servo-compensator has been discussed in the context of Sliding Mode Framework in [28] and in the context of Lyapunov Redesign Framework in [7], [33]. Conceptually both discussions are similar but the later provide the flexibility of choosing with any stabilizing state feedback controller for the system and then to include servo-compensator to perform the desired task. This flexibility lead to our work in case of saturated systems in this thesis. So, owing to the implicational significance, we follow the work of [7] and [33].

Transforming the systems (3.1) and (3.3) through the variable transformation [i.e. $\wp = x - \pi(\omega)$ for (3.1) and $\zeta = \xi - \pi$ for (3.3)] into the forms as,

$$\left.\begin{aligned} \dot{\wp} &= \mathbb{A}\wp + \mathbb{B}[sat(u) - \phi(\omega)] \\ e &= \mathbb{C}_1\wp \end{aligned}\right\} \tag{3.13}$$

and

$$\left.\begin{aligned} \dot{z} &= \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\ \dot{\zeta} &= \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)[sat(u) - \phi(\omega)] \end{aligned}\right\} \tag{3.14}$$

where $\mathfrak{f}(z,\zeta,\omega) = f(z,\zeta+\pi,\omega) - f(z,\pi,\omega) + [G(\zeta+\pi,\omega) - G(\pi,\omega)]$ and $\mathcal{G}(\zeta,\omega) = G(\zeta+\pi,\omega)$. The equation (3.13) and (3.14) represents the systems into the form where viewing $\phi(\omega)$ as the matched uncertainty, the task of output regulation can be achieved by developing the stabilization design for (3.13) and (3.14). Writing the systems (3.13) and (3.14) as,

$$\dot{\wp} = \mathbb{A}\wp + \mathbb{B}[sat(u)] \tag{3.15}$$

and

$$\left.\begin{aligned} \dot{z} &= \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\ \dot{\zeta} &= \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)[sat(u)] \end{aligned}\right\} \tag{3.16}$$

For system (3.16) with assumption (3.1), we know that the internal dynamics (i.e. z equations) are asymptotically stable. This means that the stabilizing controller for external dynamics only (i.e. ζ equations) will stabilize the whole system (3.16). So, ignoring the internal dynamics for some time and if a state feedback controller [parameterized only in the states of external dynamics for system (3.16)] is available that stabilize the system (3.15) and (3.16), we make following necessary assumption.

Assumption 3.5: Suppose there exist a matrix χ for (3.15) and locally Lipschitz function $\chi(\zeta,\omega)$ for (3.16) with $\chi(0,\omega) = 0$, and a continuously differentiable Lyapunov function $\mathcal{V}(\wp,\omega)$ for (3.15) and $\mathcal{V}(\zeta,\omega)$ for (3.16), such that

$$a_1[\|(.)\|] \leq \mathcal{V}(.,\omega) \leq a_2[\|(.)\|] \tag{3.17}$$

$$\frac{\partial\mathcal{V}}{\partial\omega}\mathcal{S}_0\omega + \frac{\partial\mathcal{V}}{\partial\wp}\big[\mathbb{A}\wp + \mathbb{B}[sat(\chi\wp)]\big] \leq -\mathcal{Z}(\wp) \tag{3.18}$$

$$\frac{\partial\mathcal{V}}{\partial\omega}\mathcal{S}_0\omega + \frac{\partial\mathcal{V}}{\partial\zeta}[\mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)[sat(\chi(\zeta,\omega))]\,] \leq -\mathcal{Z}(\zeta) \tag{3.19}$$

where a_1 and a_2 are class $\mathcal{K}$ functions, and $\mathcal{Z}(.)$ is a continuous positive definite function parameterized in $\wp$ and ζ respectively.

The systems (3.13) and (3.14) can be written into the form as,

$$\dot{\wp} = [\mathbb{A} - \mathbb{B}\chi]\wp + \mathbb{B}sat(u) + \mathbb{B}[\chi\wp - \phi(\omega)] \tag{3.20}$$

and

$$\left.\begin{aligned} \dot{z} &= \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\ \dot{\zeta} &= \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega) + \mathcal{G}(\zeta,\omega)sat(u) - \mathcal{G}(\zeta,\omega)[\phi(\omega) + \chi(\zeta,\omega)] \end{aligned}\right\} \tag{3.21}$$

The stabilizing control for system (3.20) and (3.21) can be designed using the Lyapunov Redesign approach with an assumption that [For $\delta(\zeta)$ be continuous function],

$$\mathbb{B}[\chi\wp - \phi(\omega)] \leq 1 - \delta_0, \qquad \delta_0 > 0 \tag{3.22}$$

and

$$[\phi(\omega) + \chi(\zeta,\omega)] \leq \delta(\zeta) \tag{3.23}$$

Now, taking the derivative of $\mathcal{V}(\zeta,\omega)$ along the trajectories of (3.21). i.e.

$$\dot{\mathcal{V}} = \frac{\partial \mathcal{V}}{\partial \omega}\mathcal{S}_0\omega + \frac{\partial \mathcal{V}}{\partial \zeta}[\mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)\,] + \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)sat(u) - \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)[\phi(\omega) + \chi(\zeta,\omega)]$$

$$\leq -\mathcal{Z}(\zeta) + \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)sat(u) - \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)[\phi(\omega) + \chi(\zeta,\omega)] \tag{3.24}$$

A similar expression can also be derived for the system (3.20) as well. From (3.24), it can be seen that due to the matching condition, a stabilizing control can be designed that may cancel the effect of $[\phi(\omega) + \chi(\zeta,\omega)]$. So, before proceeding towards the design of conditional servo-compensator, an important assumption (follows from [7] and [33]) is provided that leads the flexibility of choosing any stabilizing controller, the conditional servo-compensator (will be introduced shortly) can be included to address regulation problem.

Assumption 3.6: Suppose that $\frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)$ can be expressed as,

$$\frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega) = \Delta^T(\zeta)\mathcal{H}(\zeta,\omega) \tag{3.25}$$

where $\Delta(\zeta)$ is a locally Lipschitz known function with $\Delta(0) = 0$ and $\mathcal{H}(\zeta,\omega)$ satisfies the following inequality (If I_m is identity matrix of $m\ x\ m$ dimensions).

$$\mathcal{H}^T(.) + \mathcal{H}(.) \geq 2\lambda I_m, \qquad \|\mathcal{H}(.)\| \leq k; \quad k \geq \lambda > 0 \tag{3.26}$$

A similar result can also be deduced for the system (3.20) as concluded in [8].

With all these assumptions, the conditional servo-compensator [7], [33] also discussed in chapter 2 can be selected as (3.27) and the state-feedback controller for the regulation problem can be chosen as (3.28).

$$\dot{\varrho} = (\mathcal{S} - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right) \tag{3.27}$$

$$u = -\alpha(\zeta) sat\left(\frac{s}{\mu}\right) \tag{3.28}$$

where $\mu > 0$ is the boundary layer that defines the operation region of servo-compensator, $\alpha(\zeta)$ is to be defined later and matrix K_1 is designed such that $(\mathcal{S} - JK_1)$ is a Hurwitz matrix. The conditional servo-mechanism design (3.27)-(3.28) improves the transient performance because its servo-action is activated only inside the boundary layer specified by $\mu > 0$, outside which it acts as bounded-input, bounded-state system. The sliding surface s can be chosen as,

$$s = \Delta(.) + K_1\varrho \tag{3.29}$$

where $\Delta(.)$ is the stabilizing compensator and can be designed through any control scheme that suits best to that specific system. So, starting with any stabilizing control, the conditional servo-compensator (3.27)-(3.28) or conventional servo-compensator (3.11)-(3.12) can be added to achieve the task of non-local robust output regulation. Here in this thesis, for the two classes under consideration, we exploit the two different stabilization schemes. i.e. For (3.1), the stabilization compensator $\Delta(.)$ is designed using Composite Nonlinear Feedback (CNF) technique and for (3.3), passivity-based control scheme is worked out for stabilization design. These two designs are presented in the next two subsections.

3.2.1 Composite Nonlinear Feedback (CNF) Based Stabilization Design

As, the work on conditional servo-compensator design discussed previously offers the flexibility of starting with any stabilizing feedback controller and then including conditional servo-compensator for the output regulation problem. This flexibility enables the designer to apply such design for the saturated systems as well. Owing to this design flexibility and Algebraic Riccati Equation (ARE) based nature of Composite Nonlinear Feedback (CNF) technique, we design the CNF based stabilizing compensator for our first class of considered saturated systems like (3.1) and then including the conditional servo-compensator designed previously achieves the task of output regulation. This CNF design technique initially introduced by *Lin et. al.* [35] yields a control law which is the combination of two components. i.e. linear and nonlinear. Initially, when the output of the system is far away from its steady state value, the linear component works to reduce the damping ratio which thus provides system output with faster rise time. However, on the other hand, when the output of the system gets closer to the steady state value, the nonlinear component works to increase damping ratio so that the overshoot caused by the linear part gets minimized. Many varieties of the CNF designs exist in the literature. e.g. [36], [37], [38]. In this section, we will follow the work of *Weiyao et. al.* [38] to serve our purpose.

Consider the transformed version the system (3.1) where the regulation problem can be treated as stabilization problem, previously shown by (3.13) as,

$$\begin{aligned} \dot{\wp} &= \mathbb{A}\wp + \mathbb{B}[sat(u) - \phi(\omega)] \\ e &= \mathbb{C}_1\wp \end{aligned}$$

Suppose that in the absence of $\phi(\omega)$ as shown in (3.15), the stabilizing control for the system is $\Delta(\wp)$ and it possesses the structure based on CNF technique [38] as,

$$\Delta(\wp) = \vartheta_{Linear} + \vartheta_{Nonlinear} \tag{3.30}$$

where,

$$\vartheta_{Linear} = \mathcal{M}(\varepsilon)\wp, \quad \text{and} \quad \mathcal{M}(\varepsilon) = -\mathbb{B}^T\mathcal{X}(\varepsilon) \tag{3.31}$$

with $\mathcal{X}(\varepsilon)$ is the solution of the Algebraic Riccati Equation (ARE) as follows,

$$\mathbb{A}^T\mathcal{X}(\varepsilon) + \mathcal{X}(\varepsilon)\mathbb{A} - \mathcal{X}(\varepsilon)\mathbb{B}\mathbb{B}^T\mathcal{X}(\varepsilon) + \varepsilon\mathbb{I} = 0 \tag{3.32}$$

having $0 < \varepsilon < 1$ is the design parameter and $\mathbb{I}$ is the identity matrix of appropriate dimensions.

The nonlinear part of (3.30) can be selected as follows:

$$\vartheta_{Nonlinear} = \rho(e)\mathbb{B}^T\mathcal{P}\wp \tag{3.33}$$

where, $\mathcal{P}$ is the positive semidefinite solution of the Lyapunov equation as,

$$[\mathbb{A} + \mathbb{B}\mathcal{M}(\varepsilon)]^T\mathcal{P} + \mathcal{P}[\mathbb{A} + \mathbb{B}\mathcal{M}(\varepsilon)] = -\mathbb{W} \tag{3.34}$$

and $\rho(e)$ is the nonlinear gain that regulates the transient performance of the closed loop system. e.g. By controlling the damping ratio. The only requirement on this nonlinear gain function is $\rho(e) \leq 0$ to preserve the system's stability. So, careful selection of this gain function can lead better performance in terms of quick response time and smaller overshoot. Several versions of this nonlinear gain function can be found in literature, some of which are summarized with selection guidelines by *Weiyao* [39]. According to [39], this gain function should possess the following properties.

→ When the output of the system is far away from its steady state value, the role of $\rho(e)$ should be minimum which results in faster rise time.

→ When the system's output approaches to its final set point, the effect of $\rho(e)$ must be dominated to settle down the output at this set point quickly by increasing the damping ratio and reducing the overshoots caused by the linear control part.

So, the selection of this nonlinear gain $\rho(e)$ is not a unique task subject to the satisfaction of these mentioned properties. One of such satisfying function provided by *Lin et. al.* [35] is as,

$$\rho(e) = -\mathfrak{p}exp^{\mathfrak{g}|e|} \tag{3.35}$$

with $\mathfrak{p} > 0$ and $\mathfrak{g} > 0$ are constant tuning parameters. A modified version of (3.35) can be found [37] provided by *Chen et. al.* i.e.

$$\rho(e) = \frac{\mathfrak{p}}{1 - e^{-1}}\left(exp^{-\left|1-\frac{\Delta e}{e_0}\right|} - exp^{-1}\right) \tag{3.36}$$

where $\Delta e = e - e_0$ with $e_0 = e(0)$ and $\mathfrak{p} > 0$. A similar version provided by *Weiyao et. al.* [39] is as,

$$\rho(e) = -\,\mathfrak{p}\left(exp^{-\mathfrak{g}|e|} - exp^{-e_0}\right) \tag{3.37}$$

At another point in literature, [39] provides another version of this nonlinear gain function given as,

$$\rho(e) = -\mathfrak{p}exp^{\mathfrak{g}\mathfrak{g}_0|e|} \tag{3.38}$$

where,

$$\mathfrak{g}_0 = \begin{cases} \dfrac{1}{|e_0|}, & e \neq 0 \\ 1, & e = 0 \end{cases} \tag{3.39}$$

The nonlinear gain function provided by (3.38) can adapt the variations in the steady state value, hence providing the robustness. However, one can select any of these versions depending upon the application and required performance levels.

3.2.2 Passivity Based Stabilization Design

In this subsection, owing to the flexibility in selecting any suitable stabilizing compensator offered by the conditional servo-mechanism design discussed previously, we consider the design of stabilizing compensator for our second class of considered saturated systems like (3.3). These systems differ from (3.1) in a sense that they exhibit nonlinear behavior itself and can be expressed in terms of internal as well as external dynamics explicitly. Such a system model can be realized as cascaded structure comprising of subsystems (i.e. $\dot{z}$ and $\dot{\xi}$) for both dynamics where the interconnection term play a crucial role in characterizing the system's behavior. When the interconnection term acts as disturbance, its growth as a function of the variable of internal dynamics determines what can be achievable with the feedback. The behavior of such systems can be vividly explained by the idea of passivity-based control which exploits the system's energy functions to render the passivity leading to the robust stabilization designs.

Consider the transformed version the system (3.3) where the regulation problem can be treated as stabilization problem, previously shown by (3.14) as,

$$\dot{z} = \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega)$$
$$\dot{\zeta} = \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)[sat(u) - \phi(\omega)]$$

This system is in the form where the output regulation problem can be termed as state feedback stabilization by treating $\phi(\omega)$ as matched uncertainty. The system can be viewed as a cascade connection with subsystem ($\dot{\zeta}$) as the driving system whereas the subsystem ($\dot{z}$) as driven system. If in the absence of $\phi(\omega)$, the system takes the form as (3.16) and the stabilizing control for the whole cascade is $\Delta(z,\zeta)$ then, to render the passivity concepts as discussed in the chapter 2, we make following assumption about the system.

Assumption 3.7: Suppose that there exists a radially unbounded positive definite storage function $\mathcal{J}(\zeta)$ such that the driving subsystem ($\dot{\zeta}$) is passive in the absence of ω, the origin of $\dot{z} = \varphi_a(z)$ is stable and with the knowledge of radially unbounded function $\mathcal{R}(z)$ for $\dot{z} = \varphi_a(z)$, the following inequality holds.

$$\frac{\partial \mathcal{R}}{\partial z}\varphi_a(z) \leq 0, \qquad \forall\, z$$

Using $\mathcal{U}(z,\zeta) = \mathcal{R}(z) + \mathcal{J}(\zeta)$ as a storage function candidate for the full system (3.14), we get,

$$\dot{\mathcal{U}} = \frac{\partial \mathcal{R}}{\partial z}\varphi_a(z) + \frac{\partial \mathcal{R}}{\partial z}\Psi(z,y)\hbar(\zeta,\omega) + \frac{\partial \mathcal{J}}{\partial \zeta}\left[\mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)[sat(\Delta)]\right]$$

$$\leq \frac{\partial \mathcal{R}}{\partial z}\Psi(z,y)\hbar(\zeta,\omega) + y^T sat(\Delta) \;\; = \;\; y^T\left[sat(\Delta) + \left(\frac{\partial \mathcal{R}}{\partial z}\Psi(z,y)\right)^T\right]$$

and the feedback control,

$$\Delta(z,\zeta) = -\left(\frac{\partial \mathcal{R}}{\partial z}\Psi(z,y)\right)^T + \mathcal{v} \tag{3.40}$$

results in

$$\dot{\mathcal{U}} \leq y^T \mathcal{v} \tag{3.41}$$

where $\mathcal{v} = \lambda(y)$. This follows that the stabilizing compensator can be selected as (3.40) which renders the system with passivity property (3.41) which is similar to the discussion of chapter 2. The component $\mathcal{v}$ in (3.40) plays an important role in recovering the performance results that would have happen in the absence of saturation nonlinearities. The simplest selection of $\mathcal{v}$ can be $-\mathcal{M}\zeta$, where $\mathcal{M} = [m_1 \quad m_2 \quad m_3 \quad \cdots \quad m_q]$ with q being the dimensions of driving system. i.e. ζ. However, such a simple selection may not recover the performance of unsaturated case while designing this component through the robust control schemes may yield better results. Here, in this thesis, we have exploited the idea of low-and-high gain (*Lin et. al.* [40]) as well as optimal control through Sontag's formula (*Sontag* [41]) to design this component. A brief discussion of these two techniques are presented next in the sequel.

The low-and-high gain design technique was presented by *Lin et. al.* [40] to improve the performance of their earlier work of simple low-gain design and its control law takes the form as,

$$\mathcal{v}(\zeta) = \mathcal{v}_{Low}(\zeta) + \mathcal{v}_{High}(\zeta) \tag{3.42}$$

with $\mathcal{v}_{Low}(\zeta) = -\mathcal{B}_0^T\mathcal{X}(\varepsilon)\zeta$, for $\varepsilon \in [0,1]$ and $\mathcal{X}(\varepsilon)$ is the solution of the ARE provided by (3.32) whereas, the high gain component $\mathcal{v}_{High}$ is provided by $\mathcal{v}_{High}(\zeta) = -\varpi\mathcal{B}_0^T\mathcal{X}(\varepsilon)\zeta$ with ϖ being the high gain design parameter chosen as $\varpi \in [1,\infty]$. The matrices $\mathcal{A}_0, \mathcal{B}_0$ to be used in (3.32) characterize the linearized model of driving subsystem ζ.

On the other hand, the optimal control design utilizing the Sontag's formula looks for the Control Lyapunov Function (CLF) $\Upsilon(\zeta)$ and control law takes the form as,

$$\mathcal{v}(\zeta) = \begin{cases} -\left[a + \sqrt{a^2 + b^4}\,\right]\Big/ b & , \quad \text{if } \; b \neq 0 \\ \quad 0 & , \quad \text{if } \; b = 0 \end{cases} \tag{3.43}$$

where $a = \frac{\partial \Upsilon}{\partial \zeta}\mathfrak{f}(z,\zeta)$ and $b = \frac{\partial \Upsilon}{\partial \zeta}\mathcal{G}(\zeta)$. The CLF $\Upsilon(\zeta)$ can be designed by different approaches subject to the condition that it satisfy the small control property [30]. A method to find the CLF is provided by *Sepulchre et. al.* [30] is to select this function as $\Upsilon(\zeta) = \zeta^T\mathcal{X}_0\zeta$, where $\mathcal{X}_0$ is the solution of ARE provided by (3.44) with $\mathcal{A}_0$ and $\mathcal{B}_0$ discussed above.

$$\mathcal{A}_0^T \mathcal{X} + \mathcal{X}\mathcal{A}_0 - \mathcal{X}\mathcal{B}_0\mathcal{B}_0^T \mathcal{X} \; < \; 0 \tag{3.44}$$

The various simulation results and performance comparisons as well as the recovery through the incorporation of this low-and-high gain design or optimal design in the passivity based stabilization control will be discussed later in the section 3.5 that will prove the efficacy of these designs.

3.3 Observer Designs for Output Feedback Version of Servo-Mechanism Designs

This section focusses on the output feedback version of the servo-compensator designs discussed in the previous section. Since, most of the control schemes assume the availability of all the state variables to achieve the desired control objectives. Contrary to this, realizing the physical scenario we cannot measure all the state variables due to the technical or economic reasons which necessitates the mechanism of estimating these variables through a system called state observer or simply observer. It utilizes only the output of the system as input and provides with the estimates of the state variables as output which are used to replace the state variables in the state feedback design making the whole scheme as output feedback. Following from the discussion about the state observers in chapter 2, a nonlinear observer called High Gain Observer (HGO) that recovers the performance of state feedback controller in a robust way, is worked out here to implement the output feedback version of the previously designed control law u. For system (3.1), we will implement the HGO for its transformed model (3.13), based on which the servo-compensator was presented previously whereas for system (3.3), we will implement its extended version (EHGO) [9] for its transformed model (3.14). The need of EGHO is due to the normal form representation of the system (i.e. internal and external dynamics are shown explicitly). For system (3.14), the HGO can be implemented as,

$$\dot{\hat{\wp}} = \mathbb{A}\hat{\wp} + \mathbb{B}[sat(u) - \Gamma(\omega)] + \mathbb{L}(e - \mathbb{C}_1\hat{\wp}) \tag{3.45}$$

where the observer gain $\mathbb{L}$ is need to be designed. So, the sliding surface s required in control law u and servo-compensator will be as $\hat{s} = \Delta(\hat{\wp}) + K_1\varrho$ where the stabilizing compensator $\Delta(\hat{\wp})$ will be as,

$$\Delta(\hat{\wp}) = -\mathbb{B}^T \mathcal{X}(\varepsilon)\hat{\wp} + \rho(e)\mathbb{B}^T \mathcal{P}\wp \tag{3.46}$$

The HGO observer gain matrix $\mathbb{L}$ can be designed as,

$$\mathbb{L} = \begin{bmatrix} \frac{\alpha_1}{\lambda} & \frac{\alpha_2}{\lambda^2} & \frac{\alpha_3}{\lambda^3} & \dots & \frac{\alpha_n}{\lambda^n} \end{bmatrix} \tag{3.47}$$

where λ is the small design parameter and the constants α_1, α_2, ..., α_n are selected such that the polynomial $\Omega^n + \alpha_1\Omega^{n-1} + \alpha_2\Omega^{n-2} + \cdots \; \alpha_n$ is Hurwitz. i.e. all the roots lie in the left half plane. This completes the observer design for our first class of considered saturated systems.

For our second class of considered saturated systems, consider the transformed model without matched uncertainty provided by (3.16) as,

$$\dot{z} = \varphi_a(z) + \Psi(z,y)\hbar(\zeta\ )$$
$$\dot{\zeta} = \mathfrak{f}(z,\zeta) + \mathcal{G}(\zeta)[sat(u)]$$

The above system can also be written in the compact form as,

$$\left.\begin{aligned} \dot{z} &= \varphi(z,\zeta) \\ \dot{\zeta} &= \mathcal{M}\zeta + \mathcal{N}[b(z,\zeta) + a(\zeta,u)] \\ y &= \mathcal{O}\zeta \end{aligned}\right\} \tag{3.48}$$

The system (3.48) is in the form where we can present the Extended High Gain Observer (EHGO) design. Extracting the auxiliary system as,

$$\dot{z} = \varphi(z,\zeta), \qquad\qquad \sigma = b(z,\zeta) \tag{3.49}$$

Any suitable observer called the internal observer can be used to estimate the states of the auxiliary system (3.49) formed of the internal dynamics. For example, *Boker et. al*. [9] used Extended Kalman Filter (EKF) as internal observer. For, the auxiliary system (3.49), the EKF design takes the form as,

$$\dot{\hat{z}} = \varphi(\hat{z},\zeta) + \mathcal{L}(t)[\sigma - b(\hat{z},\zeta)] \tag{3.50}$$

where the observer gain $\mathcal{L}(t)$ can be designed as,

$$\mathcal{L}(t) = \mathcal{P}(t)\mathcal{C}_1(t)^T R(t)^{-1} \tag{3.51}$$

and $\mathcal{P}(t)$ with $\mathcal{P}(0) = 0$ is the solution of the Riccati Differential Equation,

$$\dot{\mathcal{P}}(t) = \mathcal{A}_1(t)\mathcal{P}(t) + \mathcal{P}(t){\mathcal{A}_1}^T(t) + \mathcal{Q}(t) - \mathcal{P}(t){\mathcal{C}_1}^T(t)\mathcal{R}^{-1}(t)\mathcal{C}_1(t)\mathcal{P}(t) \tag{3.52}$$

The time varying matrices $\mathcal{A}_1(t)$ and $\mathcal{C}_1(t)$ are given by,

$$\mathcal{A}_1(t) = \frac{\partial\varphi}{\partial z}(\hat{z},\zeta), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}(\hat{z},\zeta) \tag{3.53}$$

and $\mathcal{Q}(t)$ and $\mathcal{R}(t)$ are symmetric positive definite matrices that which satisfy,

$$0 < \mathscr{r}_1 \leq \mathcal{R}(t) \leq \mathscr{r}_2 \tag{3.54}$$

$$0 < \mathscr{q}_1 I_{n-\rho} \leq \mathcal{Q}(t) \leq \mathscr{q}_2 I_{n-\rho} \tag{3.55}$$

Now, the observer that will work for the external dynamics called the external observer will be employed through EHGO. This observer, in addition to the states of the external dynamics, will observe an extra state that has been utilized as the output of the auxiliary system (3.49).

Its structure is as follows:

$$\dot{\hat{\zeta}} = \mathcal{M}\hat{\zeta} + \mathcal{N}\left[\hat{\sigma} + a\left(\hat{\zeta},u\right)\right] + \mathcal{H}(\varepsilon)\left(y - \mathcal{O}\hat{\zeta}\right) \tag{3.56}$$

$$\hat{\dot{\sigma}} = \dot{b}\left(\hat{z},\hat{\zeta},u\right) + \frac{\hbar_{\rho+1}}{\varepsilon^{\rho+1}}\left(y - \mathcal{O}\hat{\zeta}\right) \tag{3.57}$$

where,

$$\dot{b}\left(\hat{z},\hat{\zeta},u\right) = \left({}^{d[b(z,\zeta)]}/_{dt}\right)\Big|_{(\hat{z},\hat{\zeta})}, \text{ and} \tag{3.58}$$

$$\frac{d[b(z,\zeta)]}{dt} = \frac{db}{dz}\,\varphi(z,\xi) + \frac{db}{d\zeta}\{\mathcal{M}\zeta + \mathcal{N}[b(z,\zeta) + a(\zeta,u)]\} \tag{3.59}$$

The observer gain matrix $\mathcal{H}(\varepsilon) = \begin{bmatrix}\frac{\hbar_1}{\varepsilon} & \frac{\hbar_2}{\varepsilon^2} & \dots & \frac{\hbar_\rho}{\varepsilon^\rho}\end{bmatrix}^T$ and $\hbar_1, \hbar_2 \dots \hbar_\rho$ are selected such that $\lambda^{\rho+1} + \hbar_1\lambda^\rho + \cdots + \hbar_\rho\lambda + \hbar_{\rho+1}$ is Hurwitz. Moreover, $\varepsilon > 0$ is the small design parameter.

So, combining the internal and external observers (3.50), (3.56) and (3.57), the full order observer for (3.48) is characterized by,

$$\left.\begin{aligned}
\hat{\dot{\zeta}} &= \mathcal{M}\hat{\zeta} + \mathcal{N}\left[\hat{\sigma} + a\left(\hat{\zeta},u\right)\right] + \mathcal{H}(\varepsilon)\left(y - \mathcal{O}\hat{\xi}\right)\\
\hat{\dot{\sigma}} &= \dot{b}\left(\hat{z},\hat{\zeta},u\right) + \frac{\hbar_{\rho+1}}{\varepsilon^{\rho+1}}\left(y - \mathcal{O}\hat{\xi}\right)\\
\hat{\dot{z}} &= \varphi(\hat{z},\zeta) + \mathcal{L}(t)[\sigma - b(\hat{z},\zeta)]
\end{aligned}\right\} \tag{3.60}$$

The time varying matrices are now given by,

$$\mathcal{A}_1(t) = \frac{\partial\varphi}{\partial z}\left(\hat{z},\hat{\zeta}\right), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}\left(\hat{z},\hat{\zeta}\right) \tag{3.61}$$

So, the sliding surface s required in control law u and servo-compensator will be as $\hat{s} = \Delta\left(\hat{z},\hat{\zeta}\right) + K_1\varrho$ where the stabilizing compensator $\Delta\left(\hat{z},\hat{\zeta}\right)$ will be as,

$$\Delta(\hat{z},\hat{\zeta}) = -\left(\frac{\partial\mathcal{R}}{\partial z}\Psi(\hat{z},y)\right)^T + \mathcal{v} \tag{3.62}$$

The component $\mathcal{v}$ will also be designed using the previously discussed techniques but utilizing the state estimates provided by the EHGO.

Now, the output feedback version of servo-compensator ϱ (3.27) and the control law u (3.28) will be as (3.63) and (3.64) respectively. The simulation results depicting the performance of these output feedback versions will be presented later in the section 3.5 which will prove the efficiency of these output feedback schemes.

$$\dot{\varrho} = (\mathcal{S} - JK_1)\varrho + \mu J sat\left(\frac{\hat{s}}{\mu}\right) \tag{3.63}$$

$$u = -\alpha(\hat{\zeta})sat\left(\frac{\hat{s}}{\mu}\right) \tag{3.64}$$

3.4 Closed Loop Stability Analysis

This section is focused to prove analytically the stability results about the considered class of systems with the application of conditional servo-mechanism. As, our both considered classes of systems are subjected to the same conditional servo-compensator with a difference lies in the stabilizing compensator, we will initially prove the results by taking the generalized stabilizing compensator for both classes and in the last part of analysis we will discuss the role of each stabilization design explicitly. Owing to the complex nature of nonlinear saturated systems (3.14), we will employ it for most part of the analysis while similar results can also be derived for (3.13) as well. The analytical procedure we will discuss here follows from the work of *Attaullah Y. Memon et. al.* [7], [33] with a technical difference of the control schemes and the considered systems. The crux of is to show that for sufficiently small width of boundary layer μ, the closed loop system formed of (3.7), (3.21), (3.27), (3.28) and (3.29) behaves in such a way that all of its trajectories approaches to the invariant manifold where the error is zero. In the current analysis, we need not to prove the working and the estimation of EHGO because its detailed analysis can be found in the work of *Boker et. al.* [9]. We assume that the state estimated by the observer are correct and the estimation error is zero.

The system in the closed loop form is described by the following equations,

$$\left.\begin{aligned}
&\dot{\omega} = \mathcal{S}_0\omega \\
&\dot{z} = \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\
&\dot{\zeta} = \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega) - \mathcal{G}(\zeta,\omega)sat\left[\alpha(\zeta)sat\left(\frac{s}{\mu}\right)\right] - \mathcal{G}(\zeta,\omega)[\phi(\omega) + \chi(\zeta,\omega)] \\
&\dot{\varrho} = (\mathcal{S} - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right)
\end{aligned}\right\} \quad (3.65)$$

Suppose for our convenience $A_\varrho \triangleq \mathcal{S} - JK_1$ and $sat\left(\frac{s}{\mu}\right)$ is defined as,

$$sat\left(\frac{s}{\mu}\right) = \begin{cases} \dfrac{s}{\|s\|}, & \|s\| \geq \mu \\ \dfrac{s}{\mu}, & \|s\| \leq \mu \end{cases} \quad (3.66)$$

Also, suppose that there exist a continuously differentiable Lyapunov function $\mathcal{V}(\zeta,\omega)$ and $\Omega = \{\mathcal{V}(\zeta,\omega) \leq c_1\}$ be the compact subset of $\mathbb{X}$ that constitutes the state vector ζ, with $c_1 > 0$ and assumption (3.5) holds. Now, defining a set $\mathbb{S} = \Omega \times \{\mathcal{V}_0(\varrho) \leq \mu^2 c_2\}$ where c_2 is the positive constant and $\{\mathcal{V}_0(\varrho) \leq \mu^2 c_2\}$ is the compact set such that the initial conditions of the conditional servo-compensator i.e. $\varrho(0)$ belongs to this set [7] and $\mathcal{V}_0(\varrho) = \varrho^T \mathcal{P}_\varrho \varrho$ where $\mathcal{P}_\varrho$ is the solution of ARE as $\mathcal{P}_\varrho A_\varrho + {A_\varrho}^T \mathcal{P}_\varrho = -\mathbb{I}$ (Identity Matrix). We will first conclude the important result that the set $\mathbb{S}$ is the positively invariant and each trajectory is $\mathbb{S}$ reaches the positively invariant set $\mathbb{S}_\mu = \{\mathcal{V}(\zeta) \leq \rho(\mu)\} \times \{\mathcal{V}_0(\varrho) \leq \mu^2 c_2\}$ in finite time, where $\rho(.)$ is the class $\mathcal{K}$ function. To simplify the analysis, assuming that the internal dynamics or driven subsystem is stable. i.e. system is minimum phase and the servo-compensator developed only for the external dynamics or driving subsystem will do the required job of output regulation. So, the internal dynamics can be ignored in analyzing the servo-compensator while they will be included in the analysis of stabilizing compensator.

So, taking derivative of $\mathcal{V}(\zeta,\omega)$ along the trajectories of the system (3.65) with utilizing the assumption 3.5 and 3.6 results as,

$$\begin{aligned}\dot{\mathcal{V}} &= \frac{\partial \mathcal{V}}{\partial \omega}\mathcal{S}_0\omega + \frac{\partial \mathcal{V}}{\partial \zeta}\left[\mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)\right] - \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)sat\left[\alpha(\zeta)sat\left(\frac{s}{\mu}\right)\right] \\ &\quad - \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)[\phi(\omega) + \chi(\zeta,\omega)] \\ &\le -\mathcal{Z}(\zeta) - (s - K_1\varrho)^T\mathcal{H}(\zeta,\omega)sat\left[\alpha(\zeta)sat\left(\frac{s}{\mu}\right)\right] \\ &\quad - (s - K_1\varrho)^T\mathcal{H}(\zeta,\omega)[\phi(\omega) + \chi(\zeta,\omega)]\end{aligned}$$

Assumption 3.8: Suppose that the function $\alpha(\zeta)$ satisfies,

$$\alpha(\zeta) \ge \frac{k}{\lambda}\delta(\zeta) + \mathfrak{a}_0, \qquad \mathfrak{a}_0 > 0$$

So, inside the set $\mathbb{S}$, $\|\varrho\| \le \mu\sqrt{c_2/\lambda_{min}(\mathcal{P}_\varrho)}$. Using this along with the (3.66) and assumption 3.8, it can be shown that when $\|s\| \ge \mu$, we will have,

$$\begin{aligned}\dot{\mathcal{V}} &\le -\mathcal{Z}(\zeta) - \lambda\alpha(\zeta)\|s\| + k\delta(\zeta)\|s\| + \|\mathcal{K}_1\|\|\varrho\|k[\alpha(\zeta) + \delta(\zeta)] \\ &\le -\mathcal{Z}(\zeta) + \mu\gamma_1 \end{aligned} \qquad (3.67)$$

where $\gamma_1 = max_{\zeta\in\Omega}kk_0[\alpha(\zeta) + \delta(\zeta)]$ and $k_0 = \|\mathcal{K}_1\|\sqrt{c_2/\lambda_{min}(\mathcal{P}_\varrho)}$. On the other hand, we can have the similar results for $\|s\| \le \mu$ as well. i.e.

$$\begin{aligned}\dot{\mathcal{V}} &\le -\mathcal{Z}(\zeta) - \lambda\alpha(\zeta)\frac{\|s\|^2}{\mu} + k\delta(\zeta)\|s\| + \alpha(\zeta)\|\mathcal{K}_1\|\|\varrho\|k\frac{s}{\mu} \\ &\quad + \delta(\zeta)\|\mathcal{K}_1\|\|\varrho\|k \\ &\le -\mathcal{Z}(\zeta) + \mu\gamma_2 \end{aligned} \qquad (3.68)$$

where $\gamma_2 = max_{\zeta\in\Omega}kk_0\left[\alpha(\zeta) + \delta(\zeta)(1 + 1/k_0)\right] \ge \gamma_1$. So, it can be concluded from (3.67) and (3.68) that for sufficiently small value of μ the set $\mathbb{S}$ is positively invariant and all the trajectories starting in the region of set $\mathbb{S}$ will enter in a positively invariant set $\mathbb{S}_\mu = \{\mathcal{V}(\zeta) \le \rho(\mu)\}\times\{\mathcal{V}_0(\varrho) \le \mu^2c_2\}$ in finite time and stays thereafter.

Assumption 3.9: Suppose that $\vartheta(\zeta,\omega) \triangleq \frac{\partial\Delta}{\partial\zeta}\,\mathcal{G}(\zeta,\omega)$ such that it satisfies,

$$\vartheta(\zeta,\omega)+\vartheta(\zeta,\omega)^T \geq 2\lambda_p \mathbb{I}_m, \qquad \|\vartheta(\zeta,\omega)\| \leq \mathbb{k}_p$$

where $\mathbb{k}_p \geq \lambda_p > 0$, for all $\zeta \in \{\mathcal{V}(\zeta) \leq \rho(\mu)\}$ and $\omega \in \mathcal{W}$. Moreover, it follows from assumption 3.8 that $\alpha(0) \geq \frac{\mathbb{K}_p}{\lambda_p}\delta(0) + \mathfrak{a}_0$, $\mathfrak{a}_0 > 0$.

Now, using the quadratic Lyapunov function $\mathcal{V}_1 = \frac{1}{2}s^T s$ and with assumption 3.9, it will be shown next in the sequel that the trajectories reach the boundary layer $\{\|s\| \leq \mu\}$ in finite time. For $(\zeta,\varrho) \in \mathbb{S}_\mu$ and $\|s\| \geq \mu$, we will have,

$$\begin{aligned} s^T\dot{s} \leq& -\alpha(\zeta)\lambda_p\|s\| + \|\vartheta(\zeta,\omega)\|[\phi(\omega)+\chi(\zeta,\omega)]\|s\| \\ &+ \left\|\frac{\partial\Delta}{\partial\zeta}[\mathfrak{f}(z,\zeta,\omega)+\mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)]\right\|\|s\| + \left(\|\varrho\|\|\mathcal{K}_1\|\|A_\varrho\| + \mu\|\mathcal{K}_1\|\|J\|\right)\|s\| \end{aligned} \tag{3.69}$$

Inside the set $\mathbb{S}_\mu$, $\|\varrho\| \leq \mu\sqrt{c_2/\lambda_{min}(\mathcal{P}_\varrho)}$ and the function $\frac{\partial\Delta}{\partial\zeta}[\mathfrak{f}(z,\zeta,\omega)+\mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)]$ is continuous function such that $\frac{\partial\Delta}{\partial\zeta}[\mathfrak{f}(0,0,\omega)+\mathcal{G}(0,\omega)\chi(0,\omega)] = 0$. So, it can be concluded that the norm function of $\left\|\frac{\partial\Delta}{\partial\zeta}[\mathfrak{f}(z,\zeta,\omega)+\mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)]\right\|$ along with the norm function of $\|\varrho\|\|\mathcal{K}_1\|\|A_\varrho\|$ and $\mu\|\mathcal{K}_1\|\|J\|$ can be viewed as a bounded norm by a class $\mathcal{K}$ function. i.e. $\rho_1(\mu)$. So, equation (3.69) can be expressed as,

$$s^T\dot{s} \leq -\alpha(\zeta)\lambda_p\|s\| + \mathbb{K}_p\delta(\zeta)\|s\| + \rho_1(\mu)\|s\|$$

$$\Rightarrow\ \dot{\mathcal{V}}_1 = -\lambda_p\left[\alpha(\zeta) - \frac{\mathbb{K}_p}{\lambda_p}\delta(\zeta) - \frac{\rho_1(\mu)}{\lambda_p}\right]\|s\|$$

$$\leq -\lambda_p\left[\mathfrak{a}_0 - \frac{\rho_1(\mu)}{\lambda_p}\right]\|s\| \tag{3.70}$$

It can be concluded from (3.70) that for appropriately small values of μ, all the trajectories that start inside the set $\mathbb{S}_\mu$ reaches the boundary layer $\{\|s\| \leq \mu\}$ in finite time.

Assumption 3.10: Suppose that there exist positive constants $c_1, c_2, \dots, c_6$ such that (in some neighborhood of ζ),

$$\|\chi(\zeta,\omega)\| \leq c_1\|\Delta(.)\| + c_2\sqrt{\mathcal{Z}(\zeta)}$$

$$\left\|\frac{\partial\Delta}{\partial\zeta}[\mathfrak{f}(z,\zeta,\omega)+\mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)]\right\| \leq c_3\|\Delta(.)\| + c_4\sqrt{\mathcal{Z}(\zeta)}$$

$$\left|\frac{\alpha(\zeta)-\alpha(0)}{\alpha(0)}\right| \leq c_5\|\Delta(.)\| + c_6\sqrt{\mathcal{Z}(\zeta)}$$

Now, with the assumption 3.10, it will be shown that inside the boundary layer there exist a manifold where tracking error is zero such that the closed loop system possesses the trajectories which will approaches to this manifold.

Writing the closed loop system (3.65) inside the boundary layer as,

$$\left.\begin{aligned}
&\dot{\omega} = \mathcal{S}_0\omega \\
&\dot{z} = \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\
&\dot{\zeta} = \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega) - \mathcal{G}(\zeta,\omega)\alpha(\zeta)sat\left(\frac{s}{\mu}\right) - \mathcal{G}(\zeta,\omega)[\phi(\omega) + \chi(\zeta,\omega)] \\
&\dot{\varrho} = \mathcal{S}\varrho + J\Delta
\end{aligned}\right\} \tag{3.71}$$

Referring to the work of *Seshagiri et. al.* [42], a unique matrix (i.e. $\mathfrak{K}$) exists that satisfies,

$$\mathcal{S}\mathfrak{K} = \mathfrak{K}\mathcal{S} \qquad \text{and} \qquad -\mathcal{K}_1\mathfrak{K} = \Gamma$$

Defining the manifold as,

$$\mathfrak{M}_\mu = \{\zeta = 0, \varrho = \bar{\varrho}\}$$

with $\bar{\varrho} = \left({}^{\mu}/_{\alpha(0)}\right)\mathfrak{K}\,\mathcal{T}(\omega)$ and suppose that inside the boundary layer, $\tilde{\varrho} = \varrho - \bar{\varrho}$, $\tilde{s} = \Delta + \mathcal{K}_1\varrho$. With these supposition, inside the boundary layer the closed loop system can be written as,

$$\left.\begin{aligned}
&\dot{\omega} = \mathcal{S}_0\omega \\
&\dot{z} = \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\
&\dot{\zeta} = \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega) - \mathcal{G}(\zeta,\omega)\alpha(\zeta)sat\left(\frac{\tilde{s}}{\mu}\right) \\
&\qquad + \mathcal{G}(\zeta,\omega)\left[\frac{\alpha(\zeta) - \alpha(0)}{\alpha(0)}\right]\phi(\omega) - \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)
\end{aligned}\right\} \tag{3.72a}$$

$$\dot{\tilde{\varrho}} = A_\varrho\tilde{\varrho} + J\tilde{s} = \mathcal{S}\tilde{\varrho} + J\Delta \tag{3.72b}$$

It can be shown that $\mathfrak{M}_\mu$ is the invariant manifold of the system (3.72).

Taking the Lyapunov function candidate for the system (3.72) as,

$$\mathcal{V}_2 = \mathcal{V}(z,\zeta) + \frac{\mathfrak{p}}{\mu}\tilde{\varrho}^T\mathcal{P}_\varrho\tilde{\varrho} + \frac{\mathfrak{q}}{2}\tilde{s}^T\tilde{s} \tag{3.73}$$

with $\mathfrak{p}$ and $\mathfrak{q}$ be the positive constants required to choose. Differentiating of this function along the trajectories of (3.72) yields,

$$\dot{\mathcal{V}}_2 = \dot{\mathcal{V}}(z,\zeta,\omega) + \frac{\mathfrak{p}}{\mu}\left[\tilde{\varrho}^T\mathcal{P}_\varrho\dot{\tilde{\varrho}} + \dot{\tilde{\varrho}}^T\mathcal{P}_\varrho\tilde{\varrho}\right] + \mathfrak{q}\tilde{s}^T\dot{\tilde{s}} \tag{3.74}$$

By considering the assumptions 3.5 to 3.10 we explain (3.74) as follows:

The first term on the right-hand side of (3.74) can be written as,

$$\dot{\mathcal{V}} = \frac{\partial \mathcal{V}}{\partial \omega}\mathcal{S}_0\omega + \frac{\partial \mathcal{V}}{\partial \zeta}\left[\mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)\right] + \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)\left[-\alpha(\zeta)sat\left(\frac{\tilde{s}}{\mu}\right) + \left[\frac{\alpha(\zeta)-\alpha(0)}{\alpha(0)}\right]\phi(\omega)\right] - \frac{\partial \mathcal{V}}{\partial \zeta}\mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)$$

$$\leq -\mathcal{Z}(\zeta) - \Delta^T\mathcal{H}(\zeta,\omega)\frac{\alpha(\zeta)}{\mu}(\Delta + \mathcal{K}_1\tilde{\varrho}) + \Delta^T\mathcal{H}(\zeta,\omega)\left[\frac{\alpha(\zeta)-\alpha(0)}{\alpha(0)}\right]\phi(\omega) - \Delta^T\mathcal{H}(\zeta,\omega)\chi(\zeta,\omega)$$

$$\leq -\mathcal{Z}(\zeta) - \mathfrak{a}_0\left({}^{\lambda}/_{\mu}\right)\|\Delta\|^2 + \|\Delta\|k\left({}^{\bar{\alpha}}/_{\mu}\right)\|\mathcal{K}_1\|\|\tilde{\varrho}\| + \|\Delta\|k\left[\frac{\alpha(\zeta)-\alpha(0)}{\alpha(0)}\right]\|\phi(\omega)\| + \|\Delta\|k\|\chi(\zeta,\omega)\|$$

$$\leq -\mathcal{Z}(\zeta) - \left[\mathfrak{a}_0\left({}^{\lambda}/_{\mu}\right) - c_7\right]\|\Delta\|^2 + c_8\|\Delta\|\sqrt{\mathcal{Z}(\zeta)} + \left({}^{c_9}/_{\mu}\right)\|\Delta\|\|\tilde{\varrho}\| \tag{3.75}$$

with $\bar{\alpha}$ being the upper bound on $\alpha(\zeta)$ and c_7 to c_9 are positive constants.

Similarly, the second term on the right-hand side of (3.74) can be described as,

$$\frac{\mathfrak{p}}{\mu}\left[\tilde{\varrho}^T\mathcal{P}_\varrho\dot{\tilde{\varrho}} + \dot{\tilde{\varrho}}^T\mathcal{P}_\varrho\tilde{\varrho}\right] = -\frac{\mathfrak{p}}{\mu}\|\tilde{\varrho}\|^2 + \frac{\mathfrak{p}}{\mu}\left[\tilde{\varrho}^T\mathcal{P}_\varrho J\tilde{s} + \tilde{s}^TJ^T{\mathcal{P}_\varrho}^T\tilde{\varrho}\right]$$

$$\leq -\frac{\mathfrak{p}}{\mu}\|\tilde{\varrho}\|^2 + \frac{2\mathfrak{p}\lambda_{\max}(\mathcal{P}_\varrho)}{\mu}\|\tilde{\varrho}\|\|\tilde{s}\| \tag{3.76}$$

Now, for the third term on the right-hand side of (3.74), we have,

$$\dot{\tilde{s}} = \frac{\partial \Delta}{\partial \zeta}\left[\mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)\right] + \frac{\partial \Delta}{\partial \zeta}\mathcal{G}(\zeta,\omega)\left[-\alpha(\zeta)sat\left(\frac{\tilde{s}}{\mu}\right) + \left[\frac{\alpha(\zeta)-\alpha(0)}{\alpha(0)}\right]\phi(\omega)\right] - \frac{\partial \Delta}{\partial \zeta}\mathcal{G}(\zeta,\omega)\chi(\zeta,\omega) + \mathcal{K}_1(\mathcal{S}\tilde{\varrho} + J\Delta)$$

and

$$\mathfrak{q}\tilde{s}^T\dot{\tilde{s}} = \mathfrak{q}\tilde{s}^T\frac{\partial\Delta}{\partial\zeta}[\mathfrak{f}(z,\zeta,\omega)+\mathcal{G}(\zeta,\omega)\chi(\zeta,\omega)] + \mathfrak{q}\tilde{s}^T\vartheta(\zeta,\omega)\left[-\alpha(\zeta)sat\left(\frac{\tilde{s}}{\mu}\right)+\left[\frac{\alpha(\zeta)-\alpha(0)}{\alpha(0)}\right]\phi(\omega)\right] - \mathfrak{q}\tilde{s}^T\vartheta(\zeta,\omega)\chi(\zeta,\omega) + \mathfrak{q}\tilde{s}^T\vartheta(\zeta,\omega)\mathcal{K}_1(\mathcal{S}\tilde{\varrho}+J\Delta)$$

and

$$\mathfrak{q}\tilde{s}^T\dot{\tilde{s}} \le -\mathfrak{q}\mathfrak{a}_0\left(\frac{\lambda_p}{\mu}\right)\|\tilde{s}\|^2 + \mathfrak{q}c_{10}\|\tilde{s}\|\|\tilde{\varrho}\| + \mathfrak{q}c_{11}\|\tilde{s}\|\|\Delta\| + \mathfrak{q}c_{12}\|\tilde{s}\|\sqrt{\mathcal{Z}(\zeta)} \tag{3.77}$$

with some positive constants c_{10} to c_{12}.

Now, using the values from (3.75), (3.76) and (3.77) into the right-hand side of (3.74). i.e.

$$\dot{\mathcal{V}}_2 \le -\mathcal{Z}(\zeta) - \left[\mathfrak{a}_0\left(\lambda/\mu\right) - c_7\right]\|\Delta\|^2 + c_8\|\Delta\|\sqrt{\mathcal{Z}(\zeta)} + \left(c_9/\mu\right)\|\Delta\|\|\tilde{\varrho}\| - \frac{\mathfrak{p}}{\mu}\|\tilde{\varrho}\|^2 - \mathfrak{q}\mathfrak{a}_0\left(\frac{\lambda_p}{\mu}\right)\|\tilde{s}\|^2 + \mathfrak{q}c_{12}\|\tilde{s}\|\sqrt{\mathcal{Z}(\zeta)} + \left[\frac{2\mathfrak{p}\lambda_{\max}(\mathcal{P}_\varrho)}{\mu} + \mathfrak{q}c_{10}\right]\|\tilde{s}\|\|\tilde{\varrho}\| + \mathfrak{q}c_{11}\|\tilde{s}\|\|\Delta\| \tag{3.78}$$

The equation (3.78) can be written in the quadratic form (i.e. $\Pi = \left[\sqrt{\mathcal{Z}(\zeta)}\|\Delta\|\|\tilde{s}\|\|\tilde{\varrho}\|\right]^T$) as,

$$\dot{\mathcal{V}}_2 \le -\Pi^T\aleph\,\Pi \tag{3.79}$$

where the matrix $\aleph$ is a symmetric matrix given by,

$$\aleph = \begin{bmatrix} 1 & \frac{-c_8}{2} & 0 & \frac{-\mathfrak{q}c_{12}}{2} \\ \frac{-c_8}{2} & \left[\mathfrak{a}_0\left(\lambda/\mu\right)-c_7\right] & -\left(c_9/2\mu\right) & \frac{-\mathfrak{q}c_{11}}{2} \\ 0 & -\left(c_9/2\mu\right) & \frac{\mathfrak{p}}{\mu} & \left(-\frac{\mathfrak{p}\lambda_{\max}(\mathcal{P}_\varrho)}{\mu} - \mathfrak{q}c_{10}\right) \\ \frac{-\mathfrak{q}c_{12}}{2} & \frac{-\mathfrak{q}c_{11}}{2} & \left(-\frac{\mathfrak{p}\lambda_{\max}(\mathcal{P}_\varrho)}{\mu} - \mathfrak{q}c_{10}\right) & -\mathfrak{q}\mathfrak{a}_0\left(\frac{\lambda_p}{\mu}\right) \end{bmatrix}$$

Now, by selecting the appropriate values of positive constants $\mathfrak{p}$ and $\mathfrak{q}$ with sufficiently small value of μ, it is possible to make the principal leading minors of the matrix $\aleph$ as positive definite. This means that in this case, $\dot{\mathcal{V}}_2$ will be negative definite and it can be concluded that the trajectories of the closed loop system inside the boundary layer will approach asymptotically towards the invariant manifold. i.e. $\mathfrak{M}_\mu$. Towards that end, the matrix $\aleph$ can be partitioned as,

$$\aleph = \begin{pmatrix} 1 & -\mathcal{q}_{12}{}^{T} \\ -\mathcal{q}_{12} & \frac{1}{\mu}\mathbb{Q}_{22} + \aleph_{22} \end{pmatrix} \tag{3.80}$$

with

$$\mathcal{q}_{12} = \begin{pmatrix} \frac{c_8}{2} & 0 & \frac{\mathfrak{q}c_{12}}{2} \end{pmatrix}^{T} \tag{3.81}$$

$$\mathbb{Q}_{22} = \begin{bmatrix} \mathfrak{a}_0\lambda & \frac{-c_9}{2} & 0 \\ \frac{-c_9}{2} & \mathfrak{p} & -\mathfrak{p}\lambda_{\max}(\mathcal{P}_\varrho) \\ 0 & -\mathfrak{p}\lambda_{\max}(\mathcal{P}_\varrho) & \mathfrak{q}\mathfrak{a}_0\lambda_p \end{bmatrix} \tag{3.82}$$

and

$$\aleph_{22} = \begin{bmatrix} -c_7 & 0 & \frac{-\mathfrak{q}c_{11}}{2} \\ 0 & 0 & \frac{-\mathfrak{q}c_{10}}{2} \\ \frac{-\mathfrak{q}c_{11}}{2} & \frac{-\mathfrak{q}c_{10}}{2} & 0 \end{bmatrix} \tag{3.83}$$

In (3.82), it can be seen that the positive constants $\mathfrak{p}$ and $\mathfrak{q}$ can be easily chosen to make the principal leading minors of $\mathbb{Q}_{22}$ positive. Initially choose large value of $\mathfrak{p}$ to make the $2\ x\ 2$ minor positive and then choose $\mathfrak{q}$ large enough to make the $3\ x\ 3$ minor positive. Finally, choose sufficiently small value of μ to make the determinant of (3.80) as non-negative which renders that $\dot{\mathcal{V}}_2$ is negative definite. So, the trajectories will approach asymptotically to invariant manifold $\mathfrak{M}_\mu$ as the time approaches infinity. This conclusion is summarized as following remark.

Remark 3.1: With all the assumptions (assumption 3.1 to 3.10) and with initial conditions of both the actual system as well as the exo-system within the prescribed bounded set, there exists μ^* with $\mu \in (0, \mu^*]$, the set $\mathbb{S} = \Omega \times \{\mathcal{V}_0(\varrho) \leq \mu^2 c_2\}$ is a subset of region of attraction with all initial conditions in this set, system states are bounded and the error term approaches zero as time approaches infinity.

Finally, we will show that the stabilizing compensator we have designed using the ideas of passivity achieves the stability properties of the closed loop system. It will also be concluded later in the next section through simulation results that for saturated systems as in our case, the component of the stabilizing compensator we have designed earlier through the low-and-high gain technique recovers the performance of the system within a bounded set of initial conditions that would have happen in the absence of saturation.

Taking the radially unbounded Lyapunov function for the system (3.85) as,

$$\mathcal{U}(z,\zeta) = z^T z + \varkappa(.)\zeta^T \zeta \tag{3.84}$$

with $\varkappa(.)$ be some positive constant. Before proceeding, we make the following assumption.

Assumption 3.11: Suppose that the derivative of the Lyapunov function $\mathcal{U}(z,\zeta)$ satisfies the equality as,

$$\frac{\partial \mathcal{U}}{\partial(z,\zeta)} = \mathfrak{L}(z) + \mathfrak{F}(\zeta)$$

Now, Considering the closed loop form of system (3.14) as,

$$\left.\begin{aligned} \dot{z} &= \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\ \dot{\zeta} &= \mathfrak{f}(z,\zeta,\omega) + \mathcal{G}(\zeta,\omega)\left[sat\left\{-\left(\mathfrak{L}(z)\Psi(z,y)\right)^T + \Delta\right\} - \phi(\omega)\right] \end{aligned}\right\} \tag{3.85}$$

As, the stabilizing compensator is designed by ignoring the matched uncertainty $\phi(\omega)$. Writing the system (3.85) without this term as,

$$\left.\begin{aligned} \dot{z} &= \varphi_a(z) + \Psi(z,y)\hbar(\zeta) \\ \dot{\zeta} &= \mathfrak{f}(z,\zeta) + \mathcal{G}(\zeta)\left[sat\left\{-\left(\mathfrak{L}(z)\Psi(z,y)\right)^T + \Delta\right\}\right] \end{aligned}\right\} \tag{3.86}$$

and taking the derivate of (3.84) along the trajectories of (3.86). i.e.

$$\dot{\mathcal{U}} = \mathfrak{L}(z)\left(\varphi_a(z)\right) + \mathfrak{L}(z)\Psi(z,y)\hbar(\zeta) + \mathfrak{F}(\zeta)\mathfrak{f}(z,\zeta) + \mathcal{G}(\zeta)\left[sat\left\{-\left(\mathfrak{L}(z)\Psi(z,y)\right)^T + \mathcal{v}\right\}\right]$$

Writing $y = \hbar(\zeta)$ and following [31], it can be concluded that the derivative can be expressed as (3.87) which renders that the system is passive. Now the choice of $\mathcal{v}$ will deduce the results of system's stability. Simplest choice of $\mathcal{v}$ can be taken as $\mathcal{v} = \lambda(y) = -\mathcal{M}\zeta$ which can conclude that the origin of the system (3.86) is globally asymptotically stable. However, in case of saturation, the performance may get deteriorated. So, designing this component through a technique like low-and-high gain will enhance the closed loop performance under saturation that will be depicted through simulation results presented in the next section.

$$\dot{\mathcal{U}} \leq y^T \mathcal{v} \tag{3.87}$$

3.5 Simulation Examples

This section presents the simulation results while applying the earlier developed control schemes for the considered class of systems that will prove the efficacy of our designs. The two examples will be discussed in detail where Example 3.5.1 is related to our first class of systems which the system itself possess linear dynamics but the saturation nonlinearities make the overall scenario as nonlinear. For this example, three important conclusions will be deducted through simulation results. First, we will discuss the comparison between the conventional servo-compensator and the conditional servo-compensator which will prove the superiority of later one. Secondly, we will show that our output feedback design recovers the performance of state feedback design robustly which concludes the effectiveness of our observer design and finally, it will be shown that the inclusion of nonlinear component of CNF based stabilizing compensator offers with an additional control over the damping ratio that can be manipulated as per requirement levels. On the other hand, Example 3.5.2 is dedicated for our second class of systems which itself possess the nonlinear behavior besides the saturation nonlinearities. For this example, we will provide five important conclusions will be deducted. First, we will show the results in case of no servo action that will inference that although the transient performance is good but the steady state error will not decay to zero. Secondly, we will provide the results of servo action by including the conventional servo-compensator. In this case, the steady state error will be decayed to zero but with degraded transient performance. At the third spot, we will make the servo-compensator as conditional one but with no saturation. For this scenario, the transient performance will get improved along with the zero steady-state error. Next, for the fourth part of our results, we will add the saturation to the system and will show that the performance of the system will be deteriorated which will be recovered by the low-and-high gain design component of the stabilizing compensator. Finally, we will present the results of output feedback version by providing the state estimates through extended high gain observer (EHGO). At the end of this section, we will show that including the optimal control based stabilizing compensator in the conditional servo results very smooth system's tracking performance.

Example 3.5.1: Consider a system of the form (3.1) with the following data.

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathcal{C}_1 = \mathcal{C}_2 = [1 \quad 0]\,, \qquad \mathcal{E} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{F}_1 = [-1 \quad 0]\,, \qquad \mathcal{F}_2 = [0 \quad 0]$$

$$\mathcal{S} = \begin{bmatrix} 0 & freq \\ -freq & 0 \end{bmatrix}$$

where the output $\mathcal{Y}$ of the system is required to track the reference signal as $(Amp)\sin(freq.\,t)$ with $Amp = 1$, $f = 1$ and $\omega(0) = [0 \quad 1]^T$. Transforming the system into the form as (3.13) through change of variable $\wp_1 = x_1 - \omega_1$, we get

$$\left.\begin{aligned} \dot{\wp}_1 &= \wp_2 \\ \dot{\wp}_2 &= -2\wp_1 + sat(u) - \omega_1 \end{aligned}\right\} \tag{3.87}$$

Selecting the design parameter $\varepsilon = 0.5$ and solving the ARE provided by (3.32) as well as the Lyapunov equation provided by (3.34), the CNF based stabilizing compensator can be designed as,

$$\Delta(\wp) = 0.1213\wp_1 + 0.8618\wp_2 + \rho(e)[-0.5\wp_1 - 1.67\wp_2]$$

and selecting the nonlinear gain function $\rho(e)$ as,

$$\rho(e) = -c_1 e^{-c_2|\wp_1|}$$

with $c_1 = 5$ and $c_2 = 1$. The two designs are implemented. Design I presents the conventional servo-compensator given by,

$$\dot{\varrho} = \mathcal{S}\varrho + J\wp_1$$

while the Design II provide the conditional servo-compensator as,

$$\dot{\varrho} = (\mathcal{S} - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right)$$

with sliding surface $s = \Delta(\wp) + K_1\varrho$ where K_1 is designed by placing the eigenvalues of the matrix $\mathcal{S} - JK_1$ at -0.5 and -1. With the boundary layer defined by $\mu = 0.1$, and the initial conditions $\wp(0) = [3 \quad 0]^T$ the control law as (3.28) is applied to performed the desired regulation task. Figure 3.1(a) shows the comparison of the tracking error of both the designs where it can be shown that although conventional servo-compensator achieve the zero-steady state error but it happens so after very long time. Compared to the servo-compensator on the other hand which achieves the zero steady-state regulation error but in a very earliest time. Secondly, the transient performance is also very much smooth in case of conditional servo-compensator where there is no oscillation while the conventional servo-compensator gives the damped oscillatory response. So, it can be concluded that the overall performance of conditional servo-compensator is far better than the conventional which makes it a superior choice for to achieve the regulation tasks.

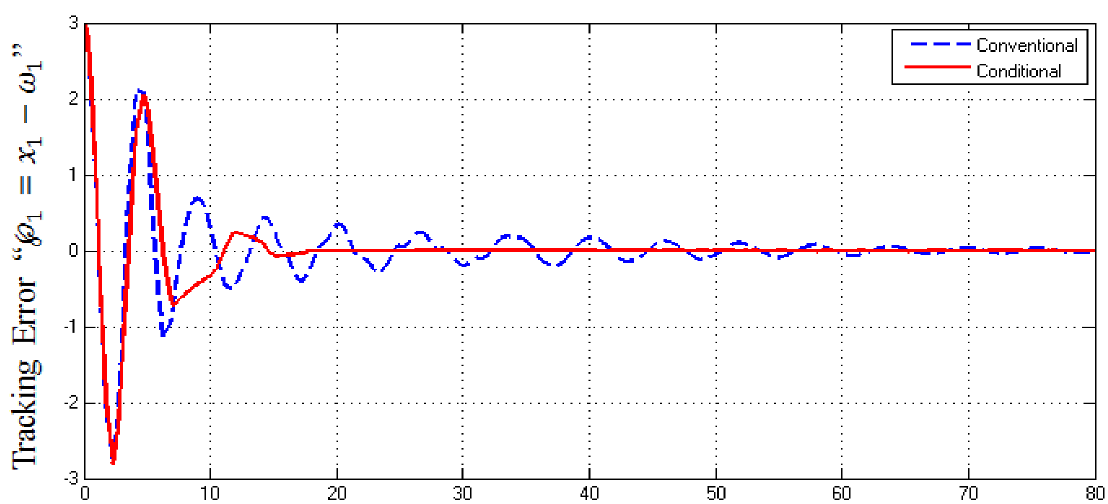

Fig. 3.1(a): Tracking Error Comparison of Conventional & Conditional Servo-Compensator.

Fig. 3.1(b) and (c) shows the plot of reference signal versus the system's output for the conventional servo-compensator and the conditional servo-compensator. The conclusion drawn above can also be verified from these figures as well.

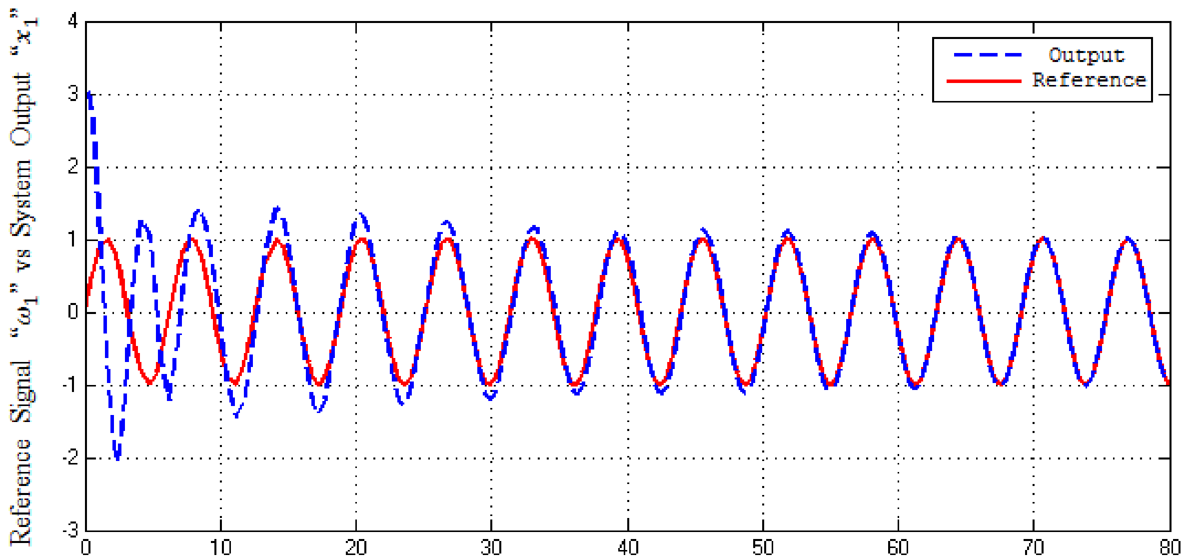

Fig. 3.1 (b). Conventional Servo. Plot of Reference Signal "ω_1" vs System Output "x_1".

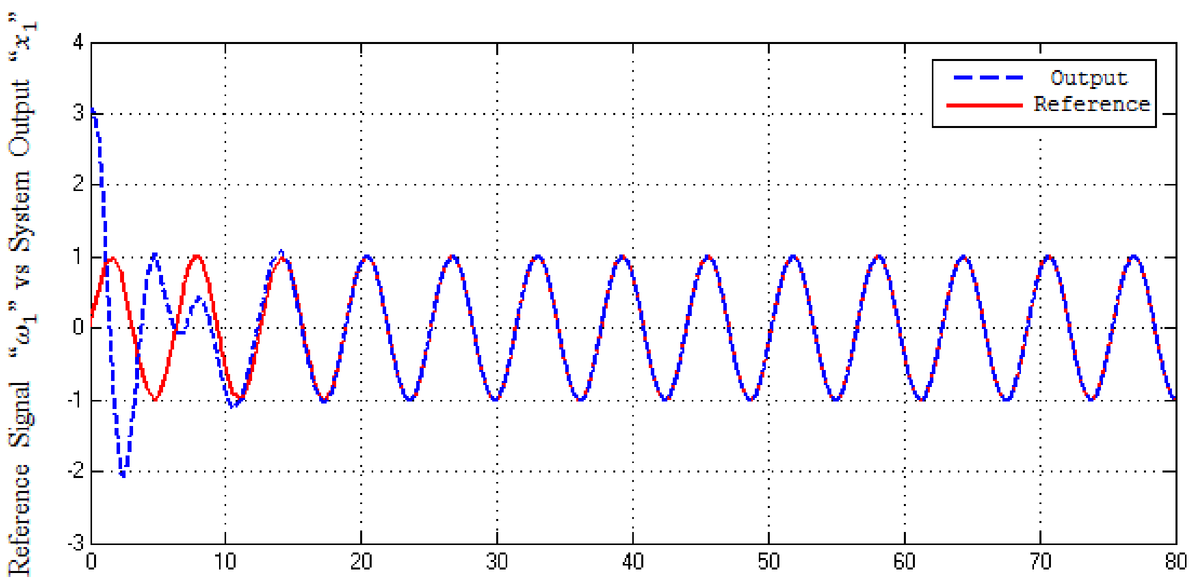

Fig. 3.1 (c). Conditional Servo. Plot of Reference Signal "ω_1" vs System Output "x_1".

Now, for the output feedback version of the design, the High Gain Observer (HGO) is implemented as (3.45) with observer design parameter $\lambda = 0.05$, $\alpha_1 = 7$ and $\alpha_2 = 12$ to make the polynomial $\Omega^n + \alpha_1 \Omega^{n-1} + \alpha_2 \Omega^{n-2} + \cdots \alpha_n$ Hurwitz. The estimated state variables are the used in the sliding surface $\hat{s}$ to observe the performance of the output feedback scenario. Fig. 3.2 shows the simulation plot of tracking error (state feedback vs output feedback) that our observer based design recovers exactly the performance of the state feedback design of conditional servo-compensator in a robust way.

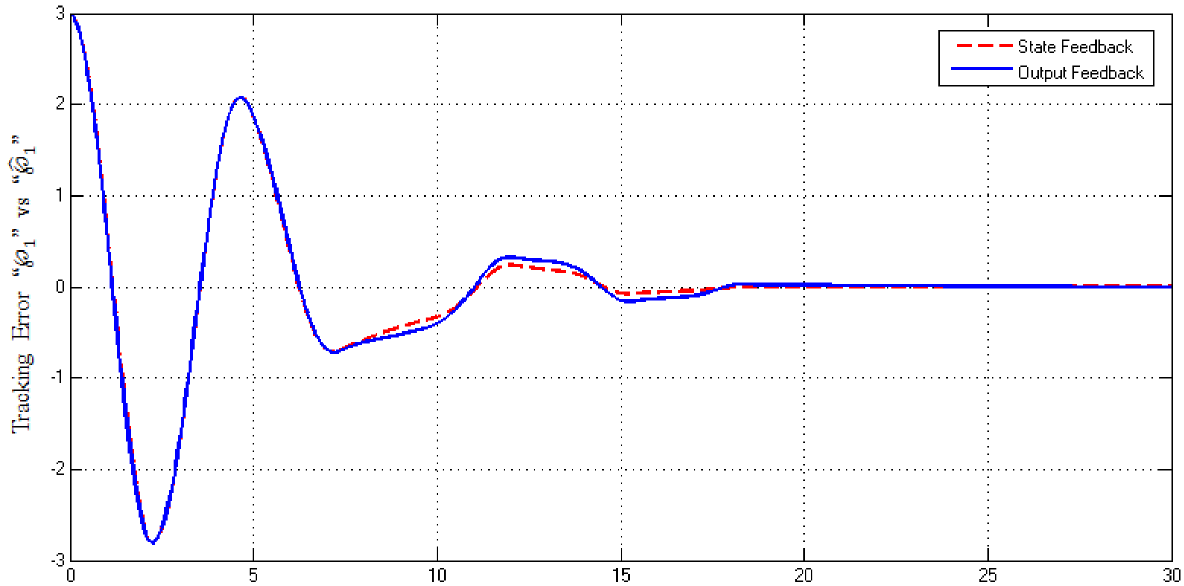

Fig. 3.2: Output Feedback Performance of Tracking Error.

Finally, the important characteristic that a CNF based design provides an additional control over the damping ratio is discussed. The parameter variation (i.e. c_1 and c_2) in the nonlinear gain function $\rho(e)$ may lead to the desired damping ratio whatever a designer wants to be for the specific system under certain scenarios. This is depicted as the simulation results with system's initial condition as $\wp(0) = [0 \quad 0]^T$ and varying these parameters in Fig. 3.3.

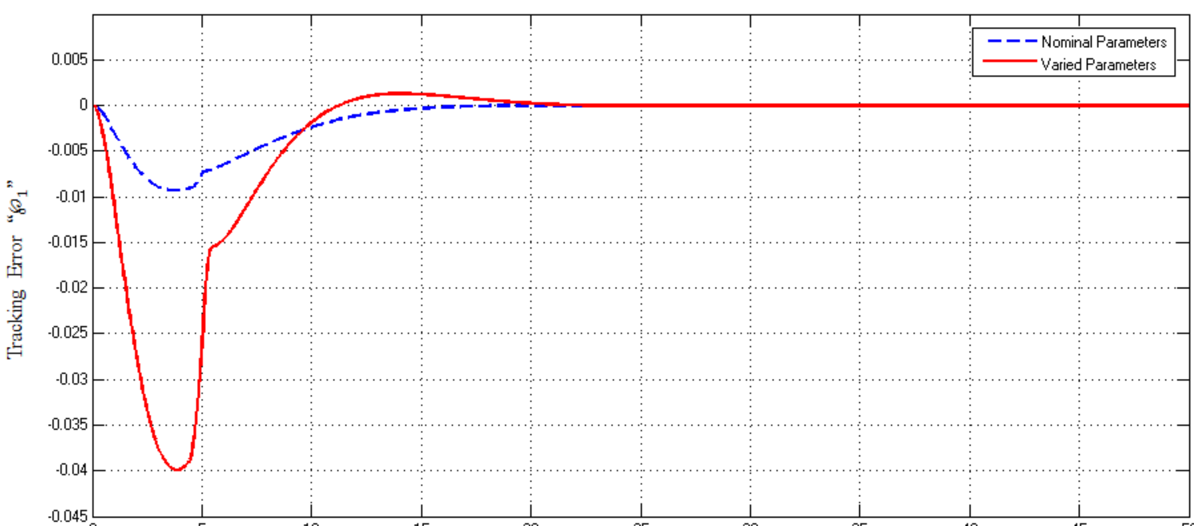

Fig. 3.3: Tracking Error under Nonlinear Gain Function Parameter Variation.

Due to this extra feature, it can be concluded that CNF based conditional servo-compensator can play important role in sophisticated applications where damping ratio play critical role like for example cargo lift control system to address the requirements appropriately.

Example 3.5.2: Consider the system having the structure like (3.3) as follows,

$$\begin{aligned}
\dot{z} &= -z + z^2\xi_1 \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= -\xi_1^3 + z^2 + sat(u) \\
y &= \xi_1
\end{aligned}$$

It is required for the system's output to track the reference signal $\alpha_0 \sin(\omega t)$. So, the system matrix of exo-system will be as,

$$\mathcal{S}_0 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

Before proceeding to servo design, we establish the stability of the internal dynamics (i.e. z equations) and the passivity of the external dynamics. Selecting $\mathcal{R}(z) = \frac{1}{2}z^2$ be the Lyapunov function for zero dynamics. i.e. $\dot{z} = -z$. It follows that $\dot{\mathcal{R}}(z) = -z^2$. i.e. the system is minimum phase. Next, with $\mathcal{J}(\xi) = \frac{1}{4}\xi_1^4 + \frac{1}{2}\xi_2^2$ as the storage function for external dynamics.

$$\begin{aligned}
\dot{\mathcal{J}}(\xi) &= \xi_1^3\xi_2 - \xi_2\xi_1^3 + \xi_2 sat(u) \\
&= \xi_2 sat(u) \quad \leq \quad \xi_1 sat(u)
\end{aligned}$$

It concludes that the system is passive. So, $\mathcal{U}(z,\xi)$ is the storage function candidate for the whole system. Now, transforming the system as (3.14) through change of variable $\zeta_1 = \xi_1 - \omega_1$, we get

$$\left.\begin{aligned}
\dot{z} &= -z + z^2(\zeta_1 + \omega_1) \\
\dot{\zeta}_1 &= \zeta_2 \\
\dot{\zeta}_2 &= -(\zeta_1 + \omega_1)^3 + z^2 + sat(u) + \omega^2\omega_1
\end{aligned}\right\} \tag{3.88}$$

with $\omega = 1$ and $\omega(0) = [0 \quad 1]^T$.

First, we consider the case, when the input is unconstrained and the regulation problem is tried to solve without the servo-compensator design. (i.e. Looking for only stabilizing compensator to do the job). In this case, although the transient performance as the best achievable from the system but the steady state tracking error do not decay to zero. The stabilizing compensator can be designed following the procedure mentioned in the earlier sections as,

$$u = -z^3 - 2\zeta_1 - 3\zeta_2$$

The performance is shown in Fig. 3.4(a) and (b) for two initial conditions. In (a), the initial conditions are taken as $z(0) = 0$, $\zeta_1(0) = 0$ and $\zeta_2(0) = 0$ whereas for the (b) the initial conditions are $z(0) = 0$, $\zeta_1(0) = 2$ and $\zeta_2(0) = 0$. For both the simulation plots, it can be concluded that to achieve the task of output regulation, there is a need of servo-compensator to be included along with the stabilizing compensator.

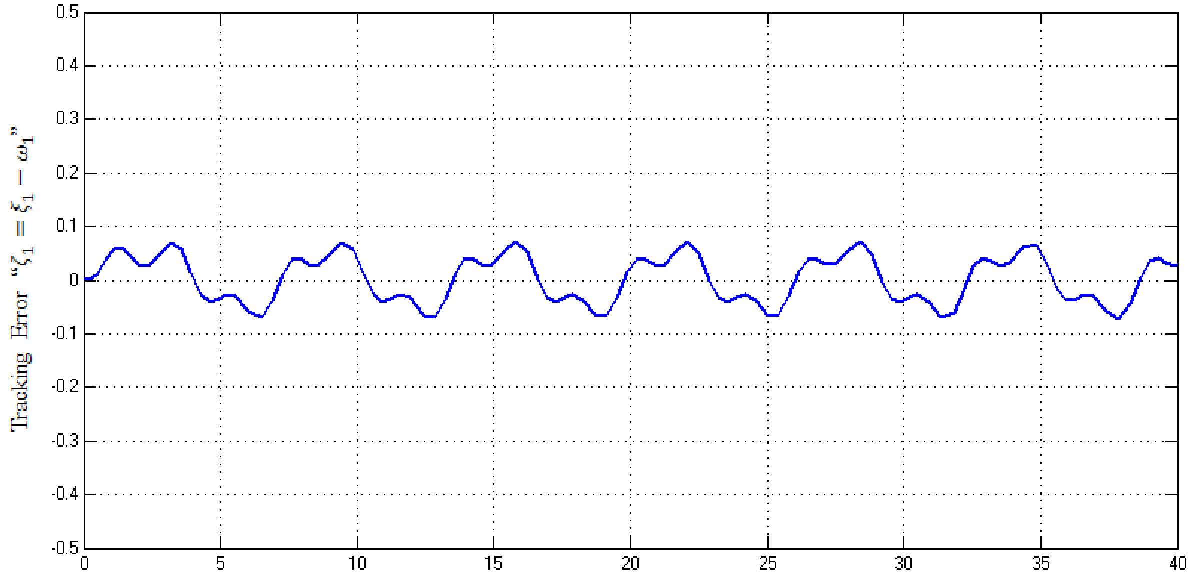

Fig. 3.4(a): Tracking Error in case of No Servo Action [$\zeta_1(0) = 0$].

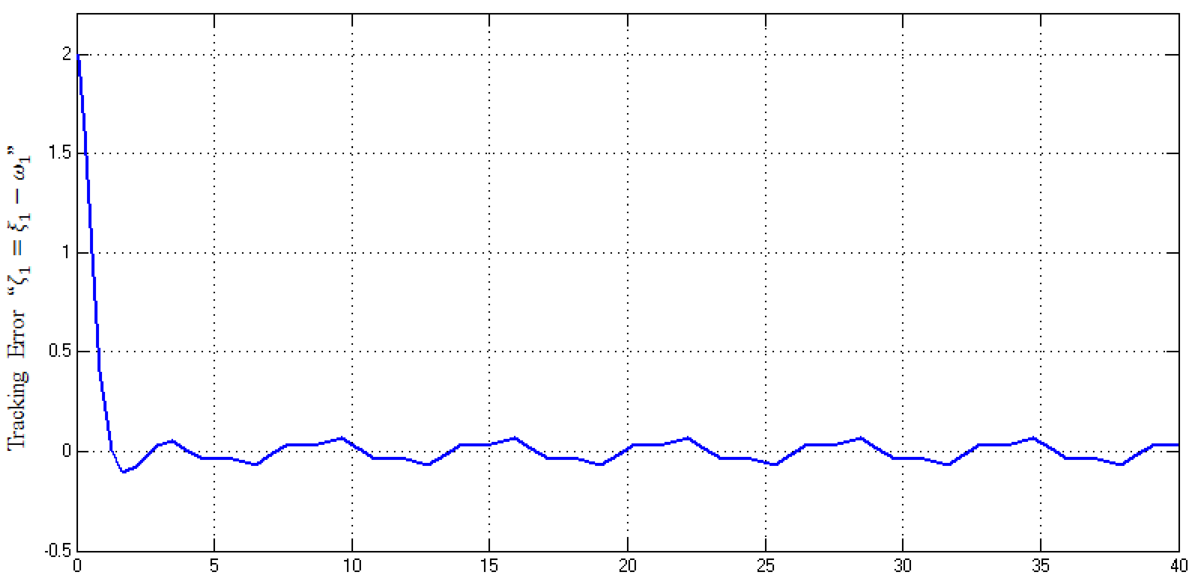

Fig. 3.4(b): Tracking Error in case of No Servo Action [$\zeta_1(0) = 2$].

It can be concluded that to achieve the task of robust output regulation, there is a need of servo-compensator to be augmented with stabilizing compensator. This combination works perfect to achieve the zero steady-state regulation error but with degraded transient performance. Solving the regulator equations provided by (3.8) such that the identity (3.9) holds as follows,

$$\mathcal{L}_s^4\phi(\omega) = -9\omega^4\phi(\omega) - 10\omega^2\mathcal{L}_s^2\phi(\omega)$$

where $c_0 = -9\omega^4$ and $c_1 = -10\omega^2$. So, the matrix $\mathcal{S}$ can be designed as,

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9\omega^4 & 0 & -10\omega^2 & 0 \end{bmatrix}$$

with $\omega = 1$ and

$$J = [0 \quad 0 \quad 0 \quad 1]^T$$

The matrix K_1 is designed by placing the eigenvalues of $(\mathcal{S} - JK_1)$ at -0.5, -1, -1.5 and -2 to fulfill its conditions of Hurwitzness. i.e.

$$K_1 = [-7.5 \quad 6.25 \quad -1.25 \quad 5]$$

So, applying the conventional servo-compensator design of (3.11) and taking the control law u as,

$$u = -\left(\frac{s}{\mu}\right)$$

with $s = z^3 + 2\zeta_1 + 3\zeta_2 + K_1\varrho$. Taking the initial conditions as $z(0) = 0, \zeta_1(0) = 2, \zeta_2(0) = 0$ and $\varrho(0) = [0 \quad 0 \quad 0 \quad 0]^T$ with $\mu = 0.1$, Fig. 3.5 presents the tracking performance of this conventional servo-mechanism design compared with earlier plot of no servo-action. This time the steady-state error is reduced to zero, but the transient performance is degraded to very much extent. Fig. 3.6 shows the simulation plot of reference signal vs the system's output in case of conventional servo-compensator.

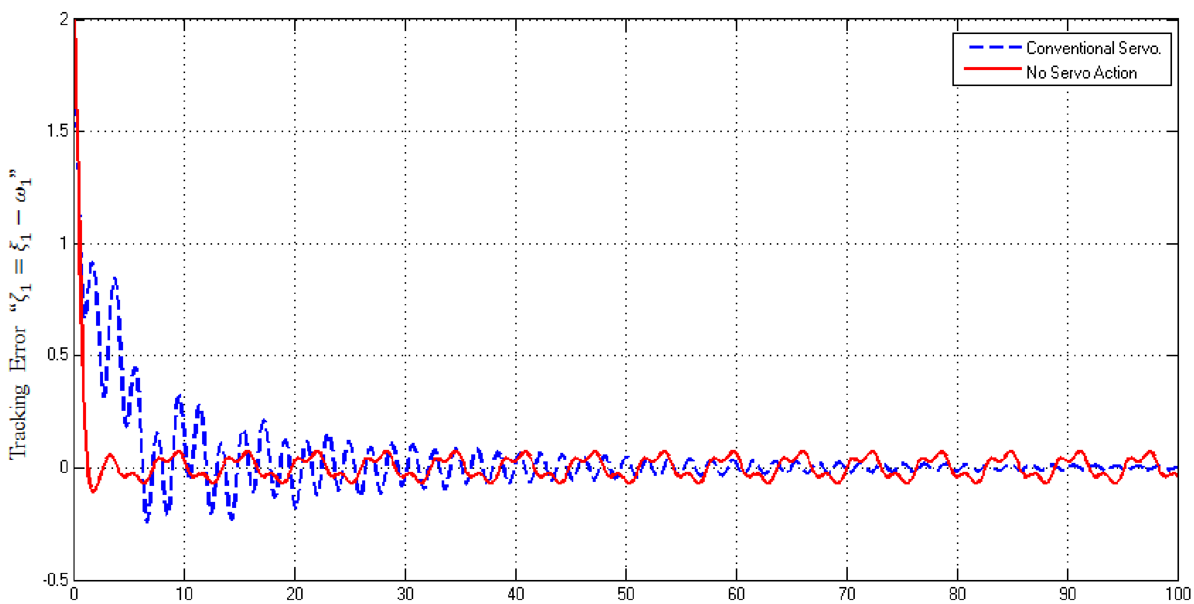

Fig. 3.5: Tracking Error Performance Comparisons of Conventional Servo. with No Servo. Action

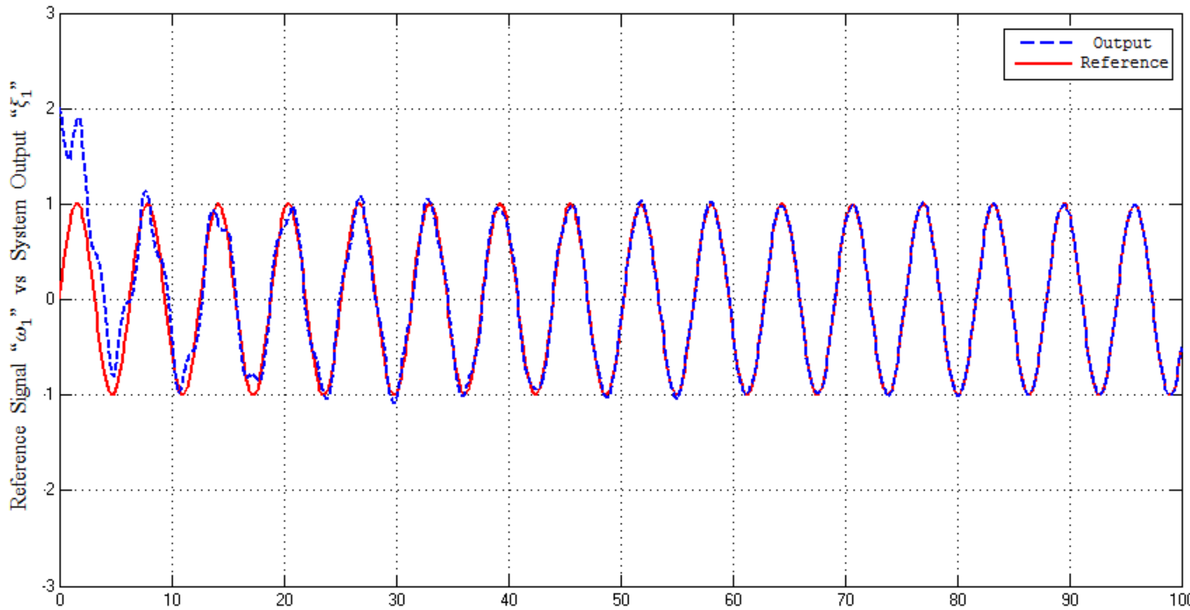

Fig. 3.6: Simulation plot of Reference Signal vs System's Output (Conventional Servo.).

To address the transient performance, a conditional servo-compensator design as provided by (3.27) – (3.29) where the conventional servo-compensator is made as conditional one which means that its servo-action is limited to certain region specified by the boundary layer μ. In this way, instead of servo action remain active all the time, is activated only inside that boundary layer while it acts a bounded-input bounded-output system outside this boundary layer. Fig. 3.7 shows the tracking performance results for the conditional servo-compensator along with the conventional servo-compensator where it can be verified that the transient performance is improved (almost same as it was with no servo action) with zero-steady state error. Fig. 3.8 shows the simulation plot for reference vs system output for conditional servo.

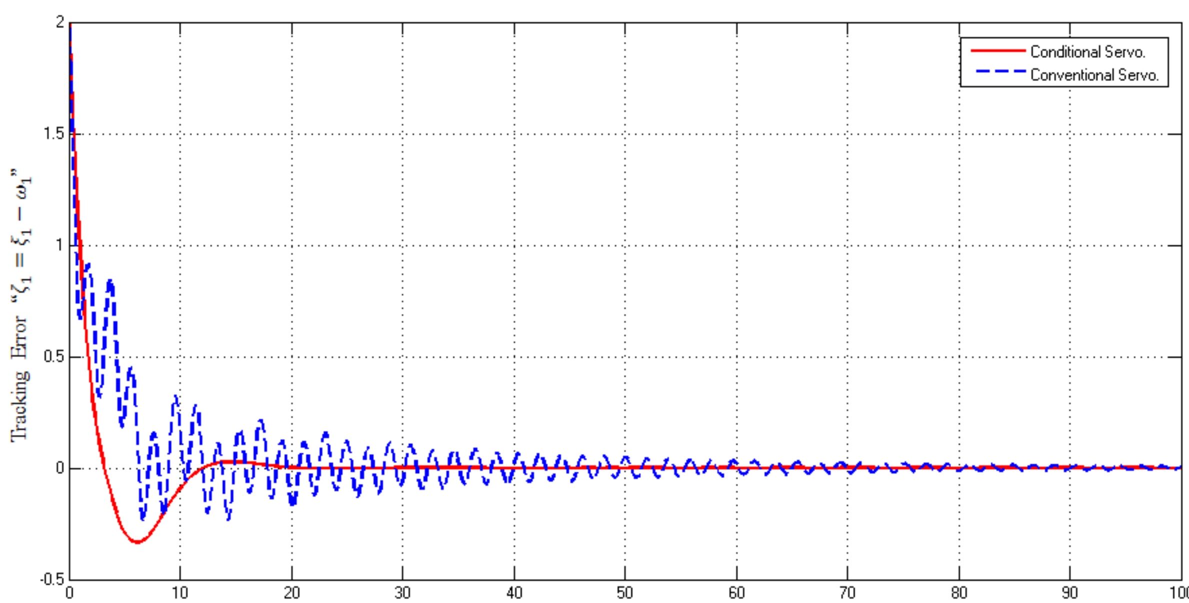

Fig. 3.7: Tracking Error Comparison of Conventional & Conditional Servo-Compensator.

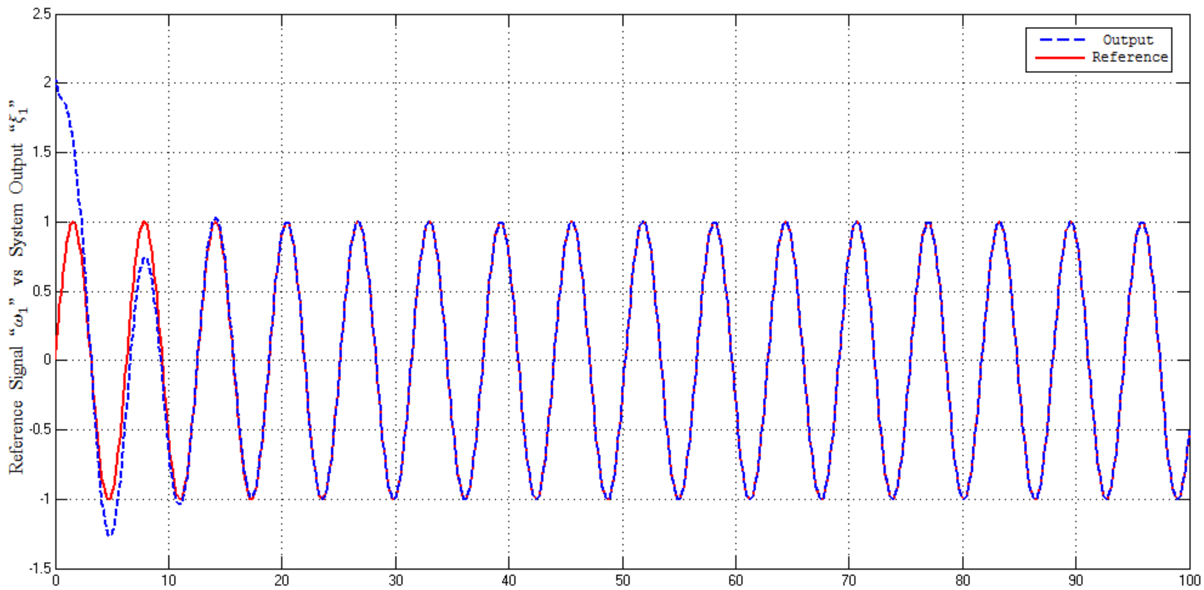

Fig. 3.8: Conditional Servo. Plot of Reference Signal "ω_1" vs System Output "ξ_1".

Keeping in view the robust performance of the conditional servo-compensator, an extra signal in the form of constant disturbance is imposed on the system. i.e. in addition to the reference signal from exo-system, it also provides this disturbance signal. By doing so, the system matrix for the exo-system will be little modified as follows,

$$\mathcal{S}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{bmatrix}$$

The servo-compensator for this case will also little bit changed due to the addition of extra disturbance signal. The regulator equations will yield the matrix $\mathcal{S}$ as,

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -9\omega^4 & 0 & -10\omega^2 & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Following the same procedure as discussed before, the conditional servo-compensator is designed and it is applied along with the stabilizing compensator to the system to achieve the task of output regulation. With the same initial conditions for the system and a constant disturbance of magnitude 1, the tracking performance of the system is shown in Fig. 3.9 where it can be shown that the conditional servo-compensator achieves the task of output regulation robustly. Fig. 3.10 presents the reference vs output simulation plot. These results also votes for the superiority of conditional servo-compensator over the conventional servo-compensator due to the fact that the servo-action is limited to the certain specified region of operation.

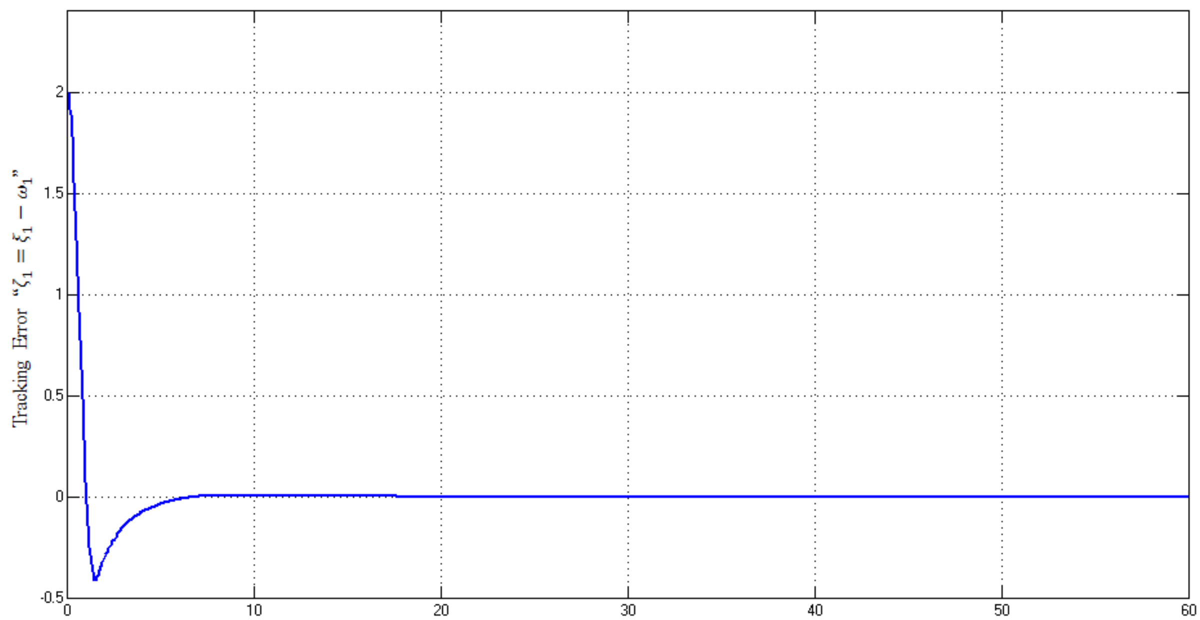

Fig. 3.9: Tracking Performance of Conditional Servo. with Reference as well as Disturbance.

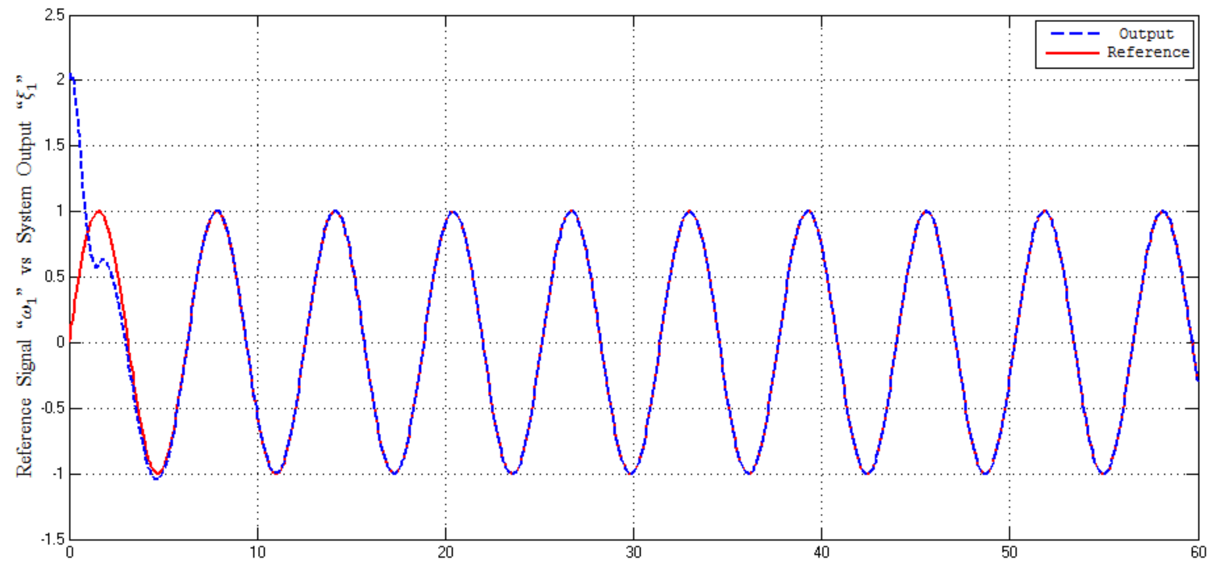

Fig. 3.10: Simulation Plot – Reference vs Output – In the Presence of Constant Disturbance.

Now, once the superiority of the conditional servo-mechanism has been proved, the regulation problem is made tedious by introducing the saturation nonlinearity in the system. This is done in a way that the constraints have been imposed on the control input channel which means that now, the signal flow capacity of the input channel is capped. By doing so, the limited part of the control signal of designed in the earlier case will act to achieve the task of output regulation. So, the performance of the system is deteriorated and the steady state performance is delayed with the saturation nonlinearities. Actually, in this scenario the control signal hit the saturation nonlinearities two time. First, the saturation of the conditional servo-compensator and second the saturation of the input channel. What happens in this situation is that when the saturation level of the control channel gets equal to that of the conditional servo-compensator, the performance of the system with input saturation is exactly the same as it is in the absence of this input channel saturation. On the other hand, when the saturation level of the input channel is less than that of the conditional servo-compensator, the transient as well as the steady-state response of the system gets deteriorated. (i.e. Transients are distorted and decay to zero very late in the time compared to that of the unsaturated input channel). Fig. 3.11(a) and (b) depicts this behavior with different initial conditions and the saturation levels.

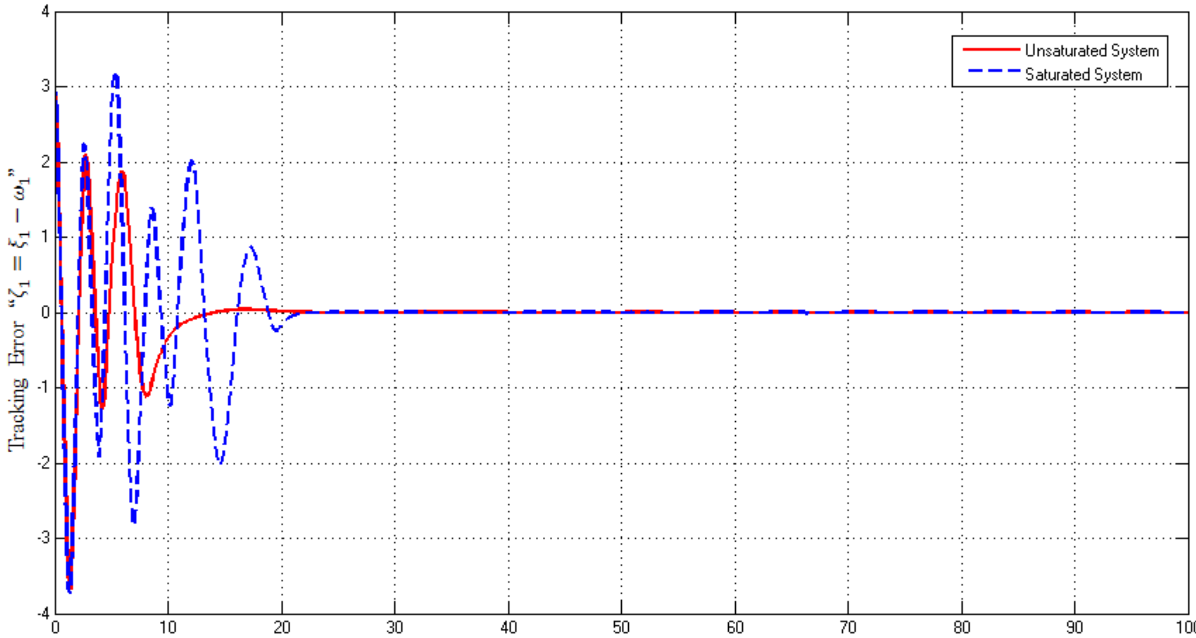

Fig. 3.11(a): Saturated and Unsaturated System Tracking Error Response [$\zeta_1(0) = 3$].

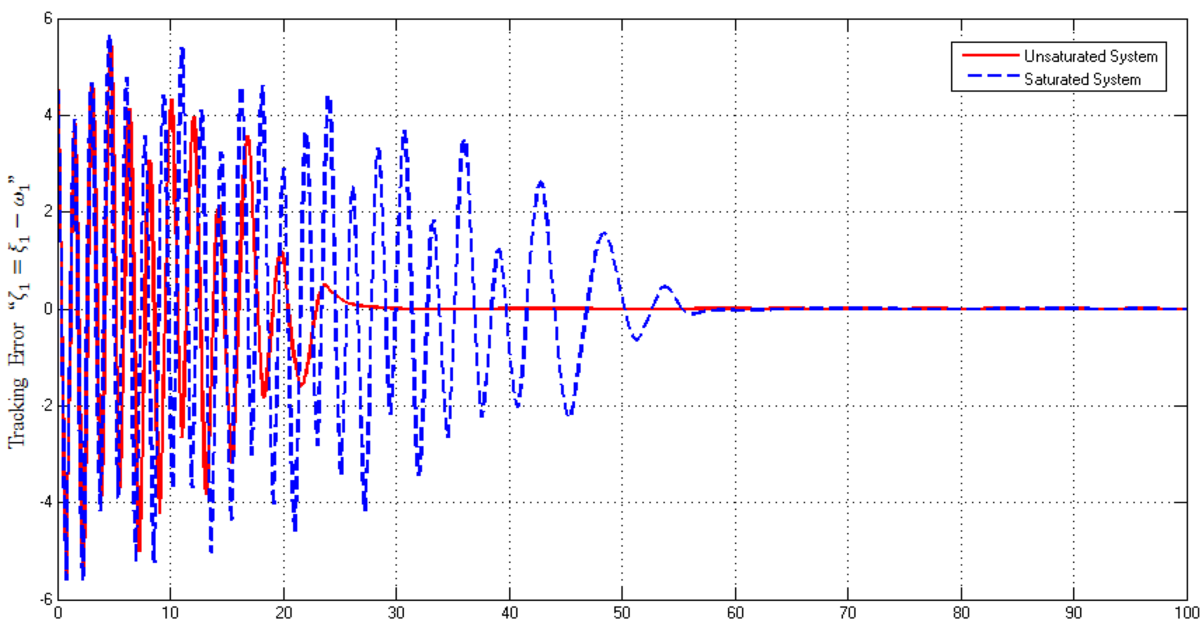

Fig. 3.11(b): Saturated and Unsaturated System Tracking Error Response [$\zeta_1(0) = 5$].

The above figures shows that the saturation effects the system performance by deteriorating the transient as well as the steady-state performance of the system. As, the design of servo-compensator depends upon the designer whereas the constraints on the input channel is the natural characteristics of the channel which cannot be changed. When the saturation limit of the servo-compensator (i.e. $sat\left(\frac{s}{\mu}\right)$) becomes equal to that of the input channel, these deteriorated effects can be controlled. This is shown as in Fig. 3.12(a) and (b) for the same initial conditions as that of Fig. 3.11(a) and (b).

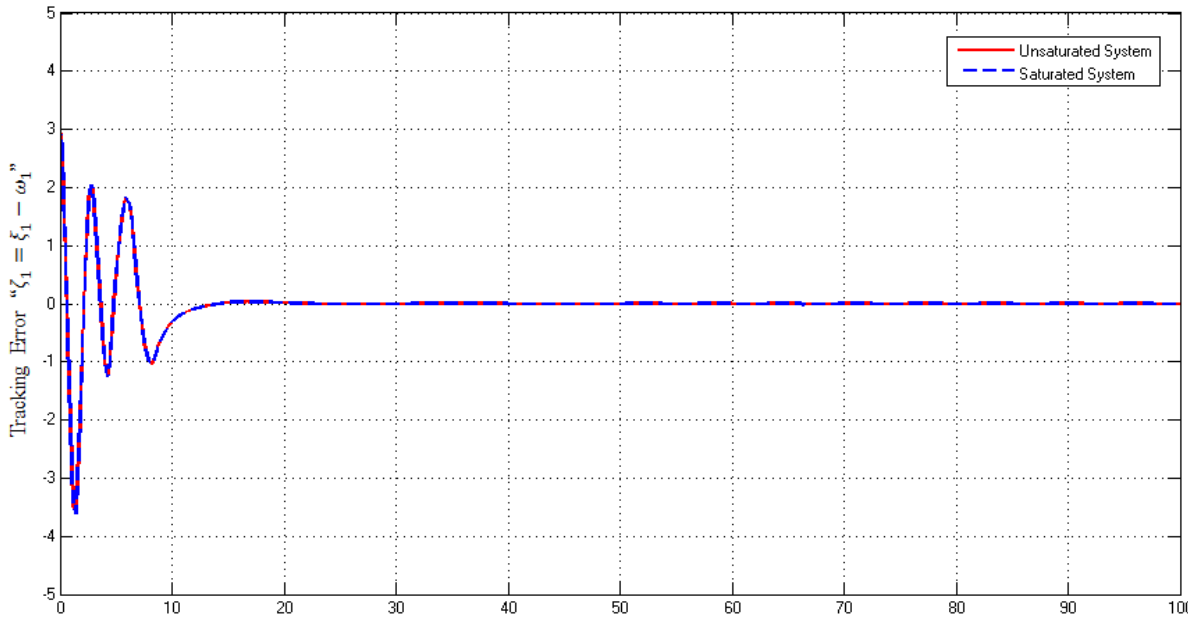

Fig. 3.12(a): Exterminated Effects of Saturation - Tracking Error Response [$\zeta_1(0) = 3$].

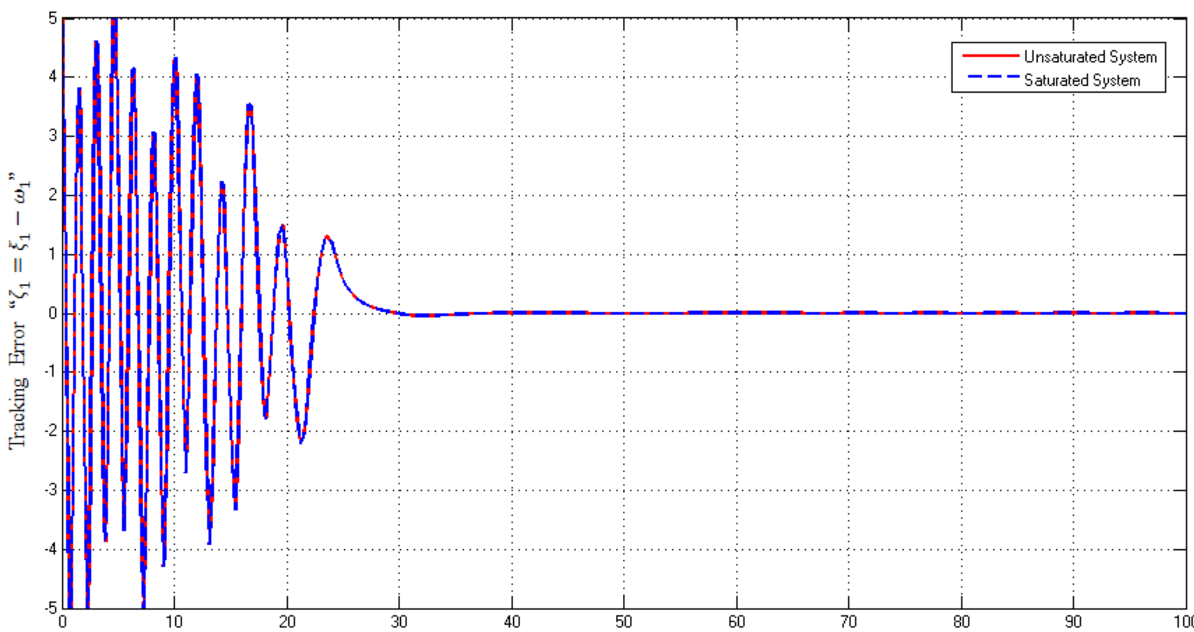

Fig. 3.12(b): Exterminated Effects of Saturation - Tracking Error Response [$\zeta_1(0) = 5$].

Now, to implement the output feedback version of this above discussed design, the Extended Kalman Filter (EKF) based Extended High Gain Observer (EHGO) is designed as discussed in the section 3.3. For the system (3.88), the auxiliary system will be,

$$\dot{z} = -z + z^2(\zeta_1 + \omega_1), \quad \sigma = b = z^2$$

So, following the steps of section 3.3, the whole (EKF – EHGO) design will be as,

$$\begin{aligned}
\dot{\hat{z}} &= -\hat{z} + \hat{z}^2 y + \mathcal{L}(\hat{\sigma} - \hat{z}^2) \\
\dot{\hat{\zeta}}_1 &= \hat{\zeta}_2 + \frac{\alpha_1}{\varepsilon}(y - \hat{\zeta}_1) \\
\dot{\hat{\zeta}}_2 &= -\left(\hat{\zeta}_1 + \omega_1\right)^3 + \hat{\sigma} + sat(u) + \omega^2\omega_1 + \frac{\alpha_2}{\varepsilon^2}(y - \hat{\zeta}_1) \\
\dot{\hat{\sigma}} &= \dot{b} + \frac{\alpha_3}{\varepsilon^3}\left(y - \hat{\zeta}_1\right)
\end{aligned}$$

where $\dot{b} = 2z$ and,

$$\mathcal{A}_1(t) = -1 + 2zy, \qquad \mathcal{C}_1(t) = 2z, \qquad \mathcal{R} = 10, \qquad \mathcal{Q} = 1$$

with the choice of $\alpha_1 = 7$, $\alpha_2 = 5$ and $\alpha_3 = 1$ such that $\Delta^3 + \alpha_1\Delta^2 + \alpha_2\Delta + \alpha_3$ is Hurwitz polynomial whereas the gain matrix $\mathcal{L}$ is designed with the solution of Riccati Differential Equation (3.52) and appropriately as (3.51). So, the output feedback based sliding surface of conditional servo-compensator is $s = \hat{z}^3 + 2\hat{\zeta}_1 - 3\hat{\zeta}_2 + K_1\varrho$.

The simulation results are provided in the Fig. 3.13 that shows that the output feedback design recovers the performance of state feedback design robustly. Fig. 3.13(a) shows the tracking error response in the output feedback design, Fig. 3.13(b) shows the plot of output vs reference signal, Fig 3.13(c) and (d) shows the state estimation performance of the EHGO.

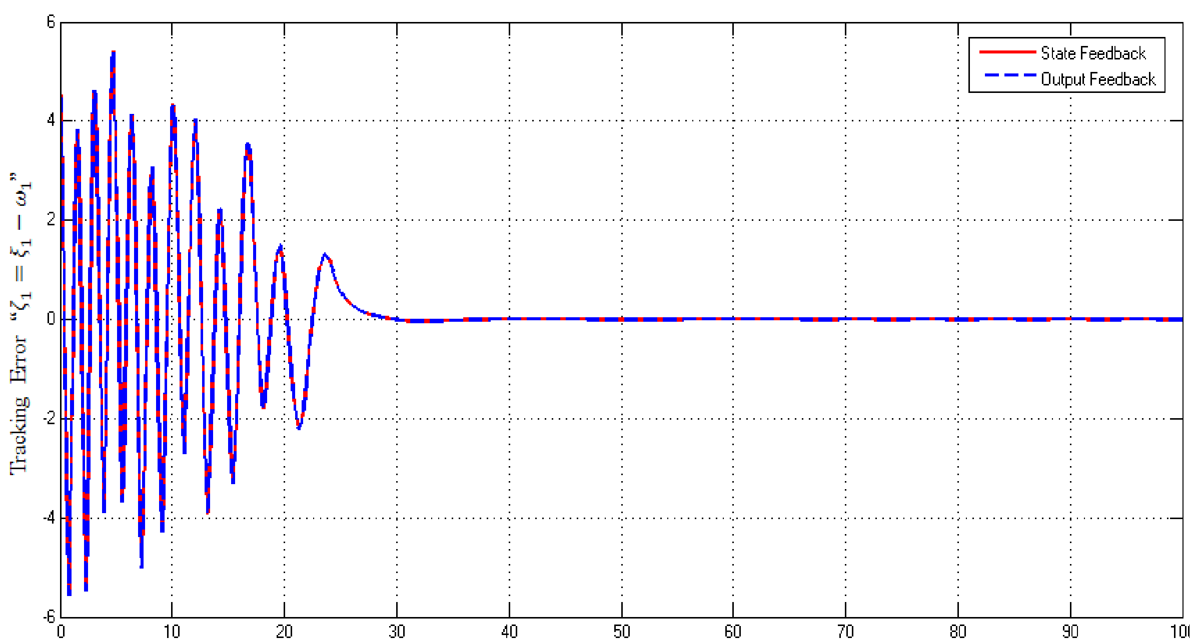

Fig. 3.13(a). Tracking Error Plot – State Feedback vs Output Feedback [$\zeta_1(0) = \hat{\zeta}_1(0) = 5$].

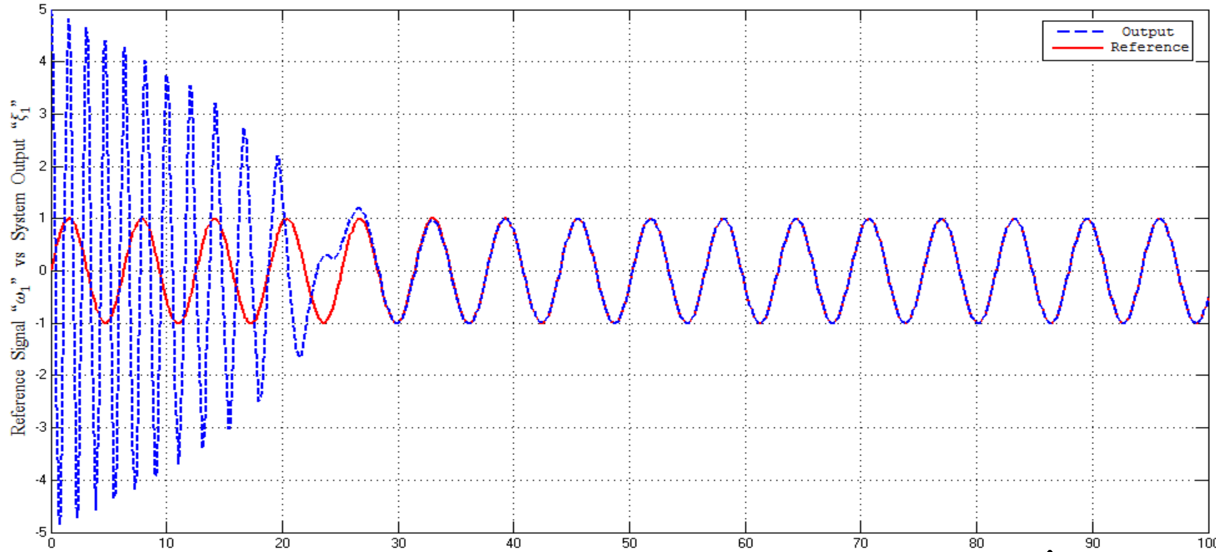

Fig. 3.13(b): Output Feedback Plot – Reference vs System Output [$\zeta_1(0) = \hat{\zeta}_1(0) = 5$].

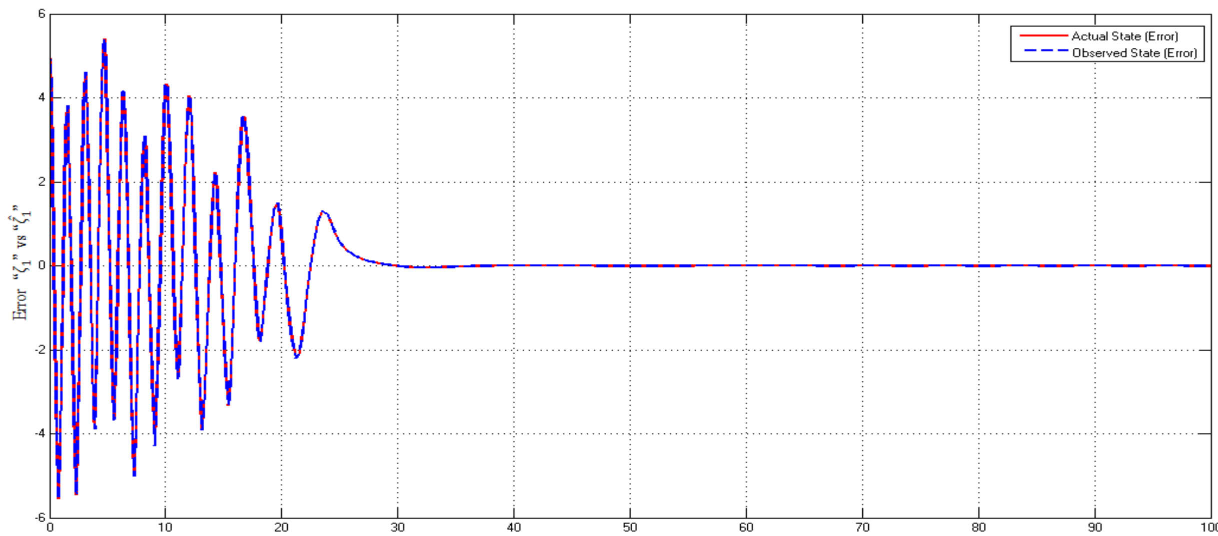

Fig. 3.13(c): EHGO Error Term Estimation ($\zeta_1$vs $\hat{\zeta}_1$).

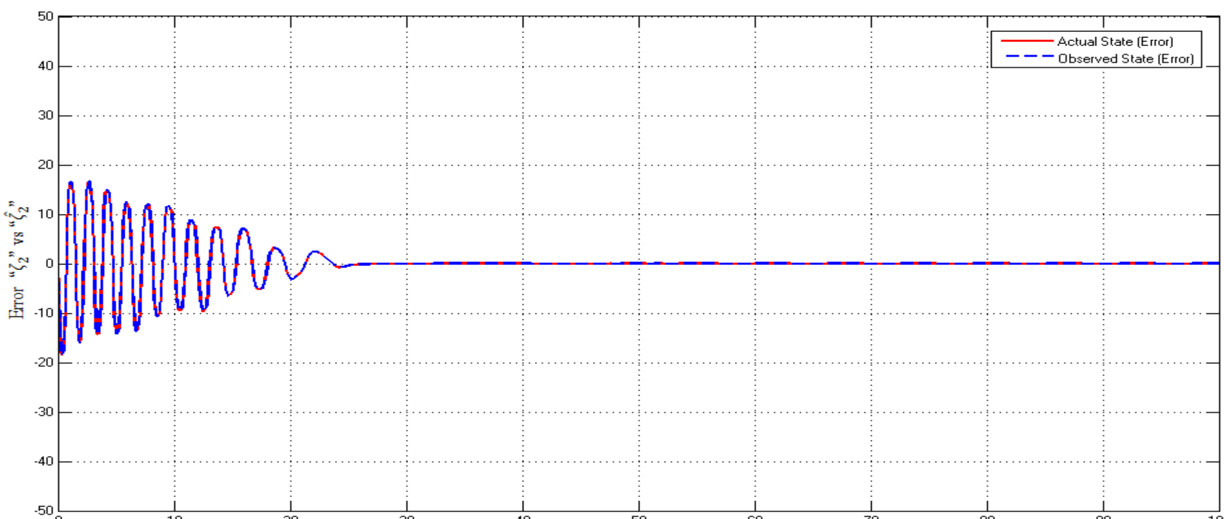

Fig. 3.13(d): EHGO Error Term Estimation ($\zeta_2$vs $\hat{\zeta}_2$).

3.6 Discussion and Conclusions

The work presented in this chapter provides twofold extensions to the work of *Attaullah Y. Memon et. al.* [8] which is focused for the servo-mechanism problem of nonlinear saturated systems. First, for the case of systems possessing the linear dynamics itself but the saturation nonlinearity of the input control channel makes the overall system as nonlinear, we have designed the stabilizing compensator of the control law (composed of stabilizing compensator and the conditional servo-compensator) utilizing the robust technique of Composite Nonlinear Feedback (CNF) design approach where the stabilizer consists of two parts i.e. linear and nonlinear through which an additional control is provided to the designer to adjust the desired damping ratio by just varying the parameters of nonlinear gain function. It has also been verified on the other hand that conditional servo-compensator is superior to the conventional servo-compensator which in a way provides the acknowledgement to the earlier work on the design conditional servo-compensator [7], [28], [33]. The output feedback version is also implemented through simple High Gain Observer that recovers the performance of state feedback design very robust which proves the efficacy of the designs.

Secondly, the same problem is extended for the systems which itself possess nonlinear dynamics along with the constraints on the input channel. In such case, the systems are considered in the normal form and also being minimum phase. Representing the systems in normal form realizes that we can think of as cascade connecting subsystems. For such cascade connections, the nonlinear design tool known as “Passivity Based Control” is utilized for designing the stabilizing compensator and along with the conditional servo-compensator, the combination works to solve the servo-mechanism problem very robustly. Output feedback version is implemented using the idea of Extended High Gain observer [9], where the results of previous section shows the performance of these designs. It has been shown that, to cope the deteriorated effects of the saturation nonlinearities, the designer must equate the operating limits of the conditional servo-compensator with those of the input channel. By doing so, the results showed that the performance is recovered that would have happen in the absence of these saturation nonlinearities.

Chapter 4

SERVO – MECHANISM PROBLEM FOR TRANSLATING OSCILLATOR WITH ROTATING ACTUATOR (TORA) SYSTEMS

In this chapter, we solve the output regulation or servo–mechanism problem for the nonlinear benchmark system of Translating Oscillator with Rotating Actuator generally known as TORA systems. The TORA problem was originally conceived as a simplified version of dual-spin spacecraft. The interaction between the rotation and the translation in the oscillating eccentric rotor is analogous to the interaction between spin and nutation in a dual-spin spacecraft [43]. Actually, the TORA system is a non-minimum phase system but through suitable variable transformations and necessary assumptions, we can derive the minimum phase realization of this system which is capable to be worked out through the approaches developed in the previous chapter. This chapter is divided into the following sections. Section 4.1 discusses the mathematical model of TORA along with the transformation of this model into the normal form. This normal form is converted into the minimum phase realization of the system model through some variable transformations and necessary assumptions. In section 4.2, we design the state feedback control law for resulting minimum phase model that solves the servo-mechanism for TORA systems followed by the output feedback version of the design using the EHGO in section 4.3. The simulation results are provided in the section 4.4 that validates our whole design approach and finally this chapter closes with the section 4.5 that provides some concluding remarks.

4.1 TORA System Description and Problem Formulation

The Fig. 4.1 presents the simplified form of TORA system. The equation of motion of motion of this system are as given by [44]. i.e.

$$\left.\begin{aligned} (M_c + m_r)\ddot{x} + m_r e\left(\ddot{\theta} cos\theta - \dot{\theta}^2 sin\theta\right) + kx = 0 \\ m_r e \ddot{x} cos\theta + \left(m_r e^2 + \ddot{\theta}\right) = N \end{aligned}\right\} \tag{4.1}$$

where M_c is the mass of cart fixed to a wall by a linear spring possessing stiffness k, with an unbalanced mass m_r possessing moment of inertia I about the center of mass where the center is located at the distance e from the axis of rotation.

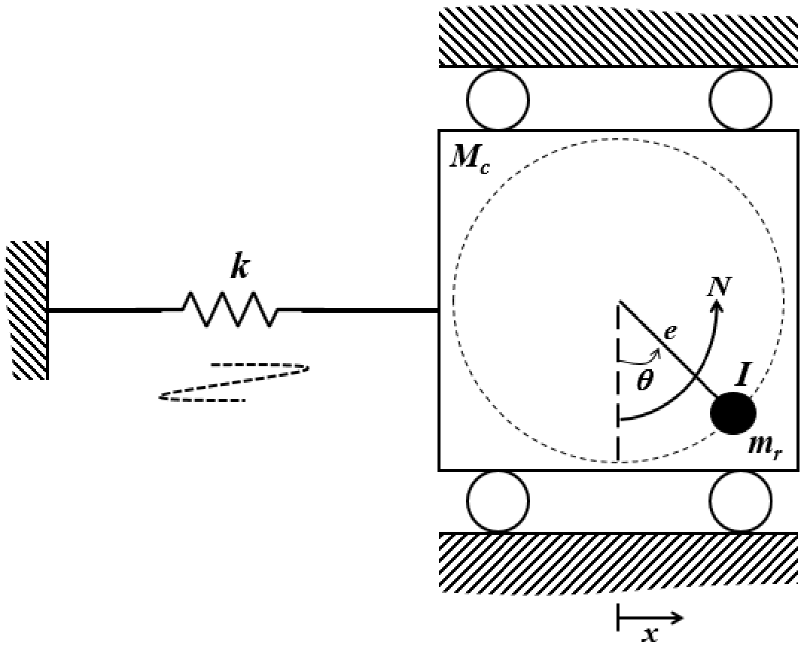

Fig. 4.1: Translating Oscillator with Rotating Actuator (TORA).

The terms $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ are the displacement, velocity and the acceleration of the translational platform respectively whereas correspondingly $\theta(t)$, $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ are the angular position, velocity and the acceleration of the rotating disk respectively. N represents the control torque applied to the rotational disk. After suitable normalizations, as suggested in [45], the following equations can represent the TORA model. i.e.

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-x_1 + \epsilon x_4^2 sinx_3}{1 - \epsilon^2 cos^2 x_3} - \frac{\epsilon cosx_3}{1 - \epsilon^2 cos^2 x_3} u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{\epsilon cosx_3 (x_1 - \epsilon x_4^2 sinx_3)}{1 - \epsilon^2 cos^2 x_3} + \frac{1}{1 - \epsilon^2 cos^2 x_3} u \end{aligned}\right\} \quad (4.2)$$

with $0 < \epsilon < 1$ and x_1 shows the normalized position of the translational platform from the equilibrium point, $x_3 = \theta$ represents angle of eccentric rotation of the rotational platform or disk and u is the normalized control torque applied to the rotational disk. We see that the system (4.2) is not in the normal form. So, to apply our approach of last chapter, we first must transform the model into the normal form which is possible through the following change of variables,

$$\left.\begin{aligned} z_1 &= x_1 + \epsilon sin x_3 \\ z_2 &= x_2 + \epsilon x_4 cos x_3 \\ \mathcal{X}_1 &= x_3 \\ \mathcal{X}_2 &= x_4 \end{aligned}\right\} \tag{4.3}$$

and

$$v = \frac{1}{1 - \epsilon^2 cos^2 \mathcal{X}_1}\{\epsilon cos\mathcal{X}_1[z_1 - (1 + \mathcal{X}_2^2)\epsilon sin\mathcal{X}_1] + u\} \tag{4.4}$$

The model (4.2) can be rewritten in the cascaded form as,

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \epsilon sin\mathcal{X}_1 \\ \dot{\mathcal{X}}_1 &= \mathcal{X}_2 \\ \dot{\mathcal{X}}_2 &= v \\ y &= \mathcal{X}_1 \end{aligned}\right\} \tag{4.5}$$

where v is the auxiliary control variable and y is the available output. The system is in the normal form possessing relative degree $\rho = 2$. The states z_1 and z_2 denotes the internal dynamics of the system whereas $\mathcal{X}_1$ and $\mathcal{X}_2$ are the states of external dynamics. By setting $\mathcal{X}_1 = 0$ in the internal dynamics, they result into the zero dynamics as,

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 \end{aligned}\right\} \tag{4.6}$$

which possess the eigenvalues at $\pm j$ which means that it is a weakly minimum phase system having only stable zero dynamics instead of asymptotically stable. First, we have to convert this system into the minimum phase with asymptotically stable zero dynamics before applying the servo-mechanism approach of previous chapter to achieve the regulation task. Towards that end, viewing $\mathcal{X}_1$ as the control input to the internal dynamics of (4.5) and seek a desired control signal which will stabilize these dynamics. Suppose $\mathcal{X}_{1d}$ is the desired control carriable, and let it be a sigmoidal membership function as ([30], [46]),

$$\mathcal{X}_{1d} = -arctan(\vartheta z_2) \tag{4.7}$$

with ϑ being a positive constant. Choosing a Lyapunov function as,

$$\mathcal{V}_z = {}^1\!/_2\,(z_1)^2 + {}^1\!/_2\,(z_2)^2 \tag{4.8}$$

Now, taking the derivative of $\mathcal{V}_z$ along the trajectories of the zero dynamics and substituting the value of $\mathcal{X}_{1d}$ results as,

$$\begin{aligned} \dot{\mathcal{V}}_z &= z_1 z_2 - z_1 z_2 + z_2 \epsilon \sin(\mathcal{X}_1) \\ \dot{\mathcal{V}}_z &= z_2 \epsilon \sin\big(\mathcal{X}_1 - arctan(\vartheta z_2)\big) = -z_2 \epsilon \sin\big(arctan(\vartheta z_2)\big) \\ \dot{\mathcal{V}}_z &\leq 0 \end{aligned} \tag{4.9}$$

This shows that the internal dynamics are now stable in the sense of Lyapunov. Following the LaSalle's Invariance Principle [31], it can be verified that the states of the closed loop system comprising internal dynamics will asymptotically converge to the equilibrium point which proves the asymptotic stability.

Now, considering that $\mathcal{X}_1(t)$ is not a real control variable and defining the deviation from its desired value as,

$$\xi_1 = \mathcal{X}_1 - \mathcal{X}_{1d} = \mathcal{X}_1 + arctan(\vartheta z_2)$$

and

$$\dot{\xi}_1 = \mathcal{X}_2 - \dot{\mathcal{X}}_{1d} = \xi_2$$
$$\dot{\xi}_2 = \dot{\mathcal{X}}_2 - \ddot{\mathcal{X}}_{1d} = v - \ddot{\mathcal{X}}_{1d}$$

where,

$$\dot{\mathcal{X}}_{1d} = \vartheta \frac{z_1 - \epsilon sin(\mathcal{X}_1)}{1 + (\vartheta z_2)^2}$$

$$\ddot{\mathcal{X}}_{1d} = \vartheta \frac{z_2 - \epsilon \mathcal{X}_2 cos(\mathcal{X}_1)}{1 + (\vartheta z_2)^2} + \vartheta^3 \frac{2z_2[z_1 - \epsilon sin(\mathcal{X}_1)]^2}{[1 + (\vartheta z_2)^2]^2}$$

So, the dynamic model (4.5) can be written as,

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \epsilon sin(\xi_1 + \mathcal{X}_{1d}) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v - \ddot{\mathcal{X}}_{1d} \\ y &= \xi_1 \end{aligned}\right\} \qquad (4.10)$$

We need to design a controller so that the output $y = \xi_1$ tracks the trajectory defined by the exo-system while the internal dynamics remain stable and stabilized to zero. Taking the exo-system as,

$$\dot{\omega} = S_0 \omega \qquad (4.11)$$

where,

$$\omega = [\omega_1 \quad \omega_2]^T$$

and

$$S_0 = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}$$

With Ω is a constant parameter.

4.2 State Feedback Servo-Mechanism Design

Once realizing the minimum phase model of TORA system in the previous section, we are now ready to apply our approach of passivity based conditional servo-mechanism design developed in the previous chapter. The system (4.10) can also be written as,

$$\left.\begin{aligned} \dot{z} &= f_a(z) + F(z,y)y \\ \dot{\xi} &= f(z,\xi) + G(\xi)v \\ y &= h(\xi) \end{aligned}\right\} \tag{4.12}$$

where $f_a(z) = \begin{bmatrix} z_2 \\ -z_1 + \epsilon\, sin(\mathcal{X}_{1d}) \end{bmatrix}$, $F(z,y)y = \begin{bmatrix} 0 \\ \epsilon\varphi(z_2,\xi_1) \end{bmatrix}$, $f(z,\xi) = \begin{bmatrix} \xi_2 \\ -\ddot{\mathcal{X}}_{1d} \end{bmatrix}$, $G(\xi) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $h(\xi) = \xi_1$ and $\varphi(z_2,\xi_1) = si\,n(\xi_1 + \mathcal{X}_{1d}) - sin(\mathcal{X}_{1d})$. Now, selecting the Lyapunov function as (4.8) for the zero dynamics yields,

$$\dot{\mathcal{V}}_z = \epsilon z_2 si\,n(\mathcal{X}_{1d}) = -\epsilon z_2 si\,n\big(arctan(\vartheta z_2)\big) \le 0$$

and selecting the storage function for the external dynamics as,

$$\mathcal{V}_\xi = \frac{1}{2}(\xi_1 - \xi_2)^2 + \xi_1\xi_2 \tag{4.13}$$

and with the assumption that $\left\|\ddot{\mathcal{X}}_{1d}\right\| < 1$, we have

$$\dot{\mathcal{V}}_\xi = \xi_1\xi_2 + \xi_2 v \quad \le \xi_1 v \tag{4.14}$$

So, the storage function candidate for the whole system (4.12) can be taken as,

$$\mathcal{V} = \mathcal{V}_z + \mathcal{V}_\xi \tag{4.15}$$

and differentiating along the trajectories results as,

$$\dot{\mathcal{V}} = \frac{\partial\mathcal{V}_z}{\partial z} f_a(z) + \frac{\partial\mathcal{V}_z}{\partial z} F(z,y)y + \frac{\partial\mathcal{V}_\xi}{\partial\xi}[f(z,\xi) + G(\xi)v]$$

$$\le \frac{\partial\mathcal{V}_z}{\partial z} F(z,y)y + y^T v \quad = y^T\left[v + \left(\frac{\partial\mathcal{V}_z}{\partial z}F(z,y)\right)^T\right]$$

So, the state feedback control can be taken as,

$$v = -\left(\frac{\partial\mathcal{V}_z}{\partial z}F(z,y)\right)^T + w \tag{4.16}$$

which results in the passivity property as $\dot{\mathcal{V}} \le y^T w$.

So, in the coordinates of system (4.10), the state feedback control law v can be derived as (4.17) which acts as stabilizing compensator $\Psi(z,\xi)$ in our case. i.e.

$$\Psi(z,\xi) = z_2[sin(\xi_1 + \mathcal{X}_{1d}) - sin(\mathcal{X}_{1d})] + w \tag{4.17}$$

where the simplest choice of w is $w = k_1\xi_1 + k_2\xi_2$.

Now, to design conditional servo-compensator for TORA system, we augment (4.10) with an exo-system (4.11) where we have an error signal e_1 represented as,

$$e_1 = \xi_1 - \omega_1 \tag{4.18}$$

As, the internal dynamics of the system (4.10) are stable (minimum phase), wo in the conditional servo-compensator design process, they can be ignored. Also, assuming that $\cos(\theta) = \theta$, for $\|\theta\| \cong 1$ to simplify our calculations. So, the reduced model of (4.10) for the conditional servo-compensator design process can be taken as,

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \epsilon\xi_1\xi_2 + v \\ e_1 &= \xi_1 - \omega_1 \end{aligned}\right\} \tag{4.19}$$

Solving the nonlinear regulator equations,

$$\left.\begin{aligned} \frac{\partial\pi}{\partial\omega} S_0\omega &= f[\pi(\omega), c(\omega), \omega] \\ 0 &= h[\pi(\omega), \omega] \end{aligned}\right\} \tag{4.20}$$

yields,

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4\Omega^4 & 0 & -5\Omega^2 & 0 \end{bmatrix}$$

and along with,

$$J = [0 \quad 0 \quad 0 \quad 1]^T$$

The conditional servo-compensator can be designed as,

$$\dot{\varrho} = (S - JK_1)\varrho + \mu sat\left(\frac{s}{\mu}\right) \tag{4.21}$$

where the choice of K_1 is to made such that $S - JK_1$ is Hurwitz. Converting the system (4.10) in the error form through the variable transformation (4.17). The sliding surface s can be chosen as $s = \Psi(z,\xi) + K_1\varrho$ and the control law that will solve the servo-mechanism problem can be taken as,

$$u = -ksat\left(\frac{s}{\mu}\right) \tag{4.22}$$

4.3 Output Feedback Servo-Mechanism Design

This section is focused on extending the state feedback design of previous section towards the output feedback design through the Extended High Gain Observer idea discussed in the chapter 3. The EKF will be implemented to estimate the states of internal dynamics. Towards that end, the auxiliary system comprising of internal dynamics only for the system (4.12) will be as,

$$\dot{z} = f_a(z) + F(z, y)y \tag{4.23}$$

Taking the Lyapunov function $\mathcal{V}_z$ that prove its minimum phase condition. Suppose that for (4.23), the virtual output can be defined as,

$$\sigma = b = \frac{\partial \mathcal{V}_z}{\partial z}\big(F(z, 0)\big) \tag{4.24}$$

So, the EKF based internal observer can be designed as,

$$\dot{\hat{z}} = f_a(\hat{z}) + F(\hat{z}, y)y + \mathcal{L}\big(\sigma - \hat{b}\big) \tag{4.25}$$

where the gain matrix $\mathcal{L}(t)$ is as,

$$\mathcal{L}(t) = \mathcal{P}(t)C_1(t)R^{-1}(t) \tag{4.26}$$

with matrix $\mathcal{P}(t)$ is the solution of the RDE provided by (3.52).

Now, the EHGO for the states of external dynamics will be as,

$$\dot{\hat{\xi}} = f\big(\hat{z}, \hat{\xi}\big) + G\big(\hat{\xi}\big)v + \mathcal{H}(\varepsilon)\left(y - h\big(\hat{\xi}\big)\right) \tag{4.27a}$$

$$\dot{\hat{\sigma}} = \dot{b} + \left({}^{\alpha_{\rho+1}}/_{\varepsilon^{\rho+1}}\right)\left(y - h\big(\hat{\xi}\big)\right) \tag{4.27b}$$

with ρ being the relative degree of the system (4.12) and $\mathcal{H}(\varepsilon) = \begin{bmatrix}\frac{\alpha_1}{\varepsilon} & \frac{\alpha_2}{\varepsilon^2} & \dots & \frac{\alpha_\rho}{\varepsilon^\rho}\end{bmatrix}^T$ where $\alpha_1 \ldots \alpha_\rho$ are chosen as in chapter 3 and $\dot{b}$ is defined as,

$$\dot{b} = \frac{db}{dt}_{|(\hat{z},\hat{\xi})} \tag{4.28}$$

So, combining the observers (4.25) and (4.27), the overall observer design will be as,

$$\left.\begin{aligned}
\dot{\hat{z}} &= f_a(\hat{z}) + F(\hat{z}, y)y + \mathcal{L}\big(\hat{\sigma} - \hat{b}\big)\\
\dot{\hat{\xi}} &= f\big(\hat{z}, \hat{\xi}\big) + G\big(\hat{\xi}\big)v + \mathcal{H}(\varepsilon)\left(y - h\big(\hat{\xi}\big)\right)\\
\dot{\hat{\sigma}} &= \dot{b} + \left({}^{\alpha_{\rho+1}}/_{\varepsilon^{\rho+1}}\right)\left(y - h\big(\hat{\xi}\big)\right)
\end{aligned}\right\} \tag{4.29}$$

So, using the sliding surface $\hat{s} = \Psi(\hat{z}, \hat{\xi}) + K_1 \varrho$ in equation (4.21) and (4.22), the resulting control law will be the output feedback version of the design which recovers the performance of the state feedback design robustly.

4.4 Simulation Results

This section is focused on the simulation results for the TORA systems. Writing the system (4.12) in the error form through variable transformation $e_1 = \xi_1 - \omega_1$ yields,

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -z_1 + \epsilon sin(e_1 + \omega_1 + \mathcal{X}_{1d}) \\ \dot{e}_1 &= e_2 \\ \dot{e}_2 &= v - \ddot{\mathcal{X}}_{1d} + \omega_1 \end{aligned}\right\} \tag{4.30}$$

$$\mathcal{X}_{1d} = -arctan(\vartheta z_2), \qquad \dot{\mathcal{X}}_{1d} = \vartheta \frac{z_1 - \epsilon sin(\mathcal{X}_1)}{1 + (\vartheta z_2)^2},$$

$$\ddot{\mathcal{X}}_{1d} = \vartheta \frac{z_2 - \epsilon \mathcal{X}_2 cos(\mathcal{X}_1)}{1 + (\vartheta z_2)^2} + \vartheta^3 \frac{2z_2[z_1 - \epsilon sin(\mathcal{X}_1)]^2}{[1 + (\vartheta z_2)^2]^2}$$

with $\Omega = 1$ for the exo-system (4.11) also $\mathcal{X}_1 = \xi_1 + \mathcal{X}_{1d} = e_1 + \omega_1 + \mathcal{X}_{1d}$ and $\mathcal{X}_2 = \xi_2 + \dot{\mathcal{X}}_{1d} = e_2 + \omega_2 + \dot{\mathcal{X}}_{1d}$. The stabilizing compensator (4.17) in the error coordinates will be as,

$$\Psi(z, e) = z_2[sin(e_1 + \omega_1 + \mathcal{X}_{1d}) - sin(\mathcal{X}_{1d})] + k_1 e_1 + k_2 e_2 \tag{4.31}$$

Selecting the eccentricity value as $\epsilon = 0.5$, $\vartheta = 1.5$, $k_1 = 2$ and $k_2 = 3$, Fig. 4.2 shows the stabilizing response of the system in the absence of exo-system. i.e. $y = e_1 = \xi_1$ with $\xi_1(0) = 5$. It can be verified that the stabilizing compensator (4.31) robustly stabilizes the TORA system (4.30).

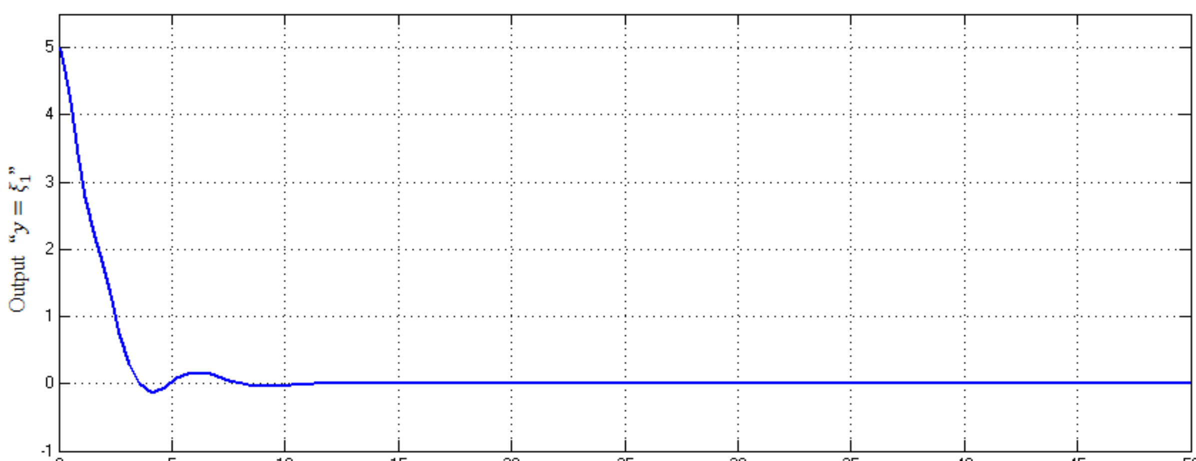

Fig. 4.2: Stabilization Response of the system in the absence of exo-system.

Now exposing the system with the exo-system which constitutes the reference signal to be tracked, the error plot is shown in the Fig. 4.3. It can be verified that although the transient performance is best but the steady state error is not zero. Fig. 4.4 shows the output versus the reference signal response which shows that the output is not perfectly tracking the reference signal.

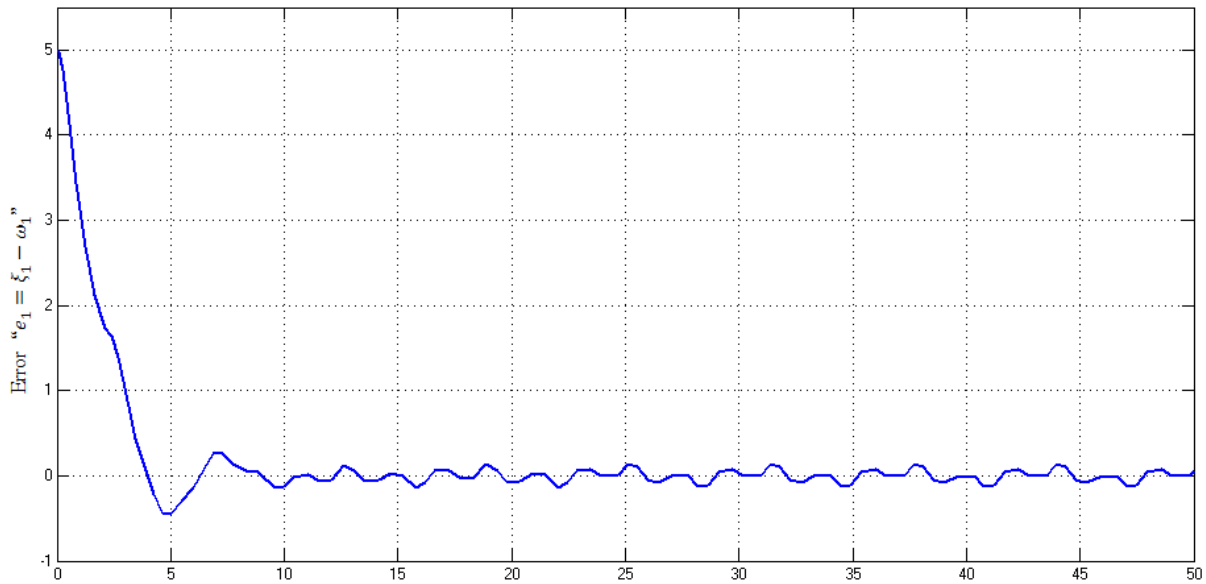

Fig. 4.3: Tracking Error Plot with No Servo Action.

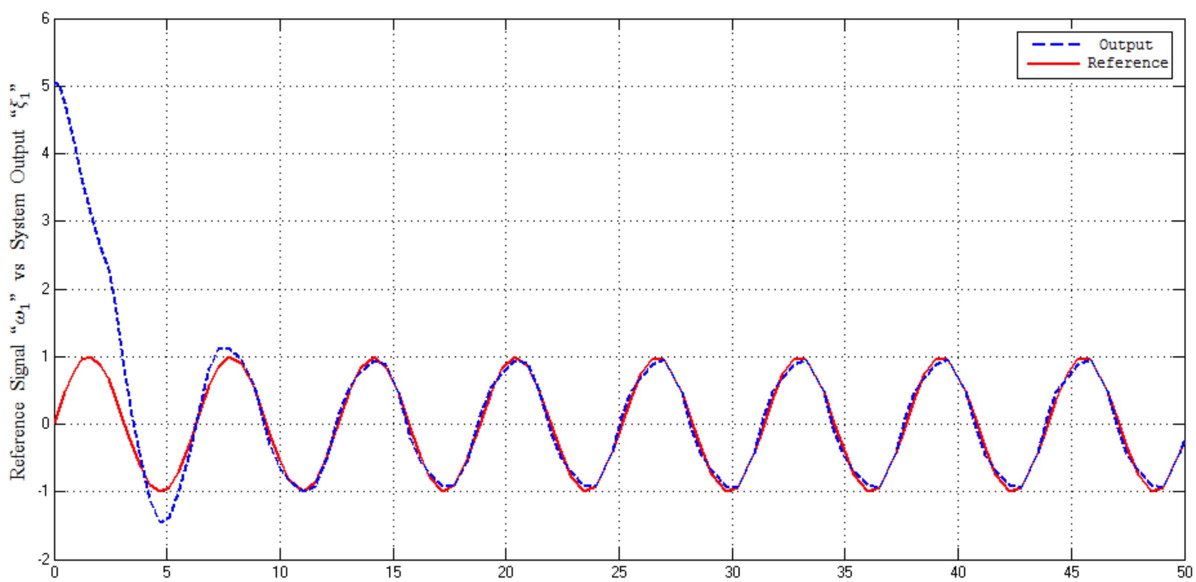

Fig. 4.4: Plot of Reference Signal "ω_1" vs System's Output "ξ_1" (No Servo Action).

Now, to force the steady state error term to zero without deteriorating the transient performance of the system, a conditional servo-compensator (4.21) is augmented with the stabilizing compensator. The simulation results are shown in Fig. 4.5 and Fig. 4.6 which shows that the conditional servo-compensator achieves the zero-steady state error without degrading the transient performance of the system.

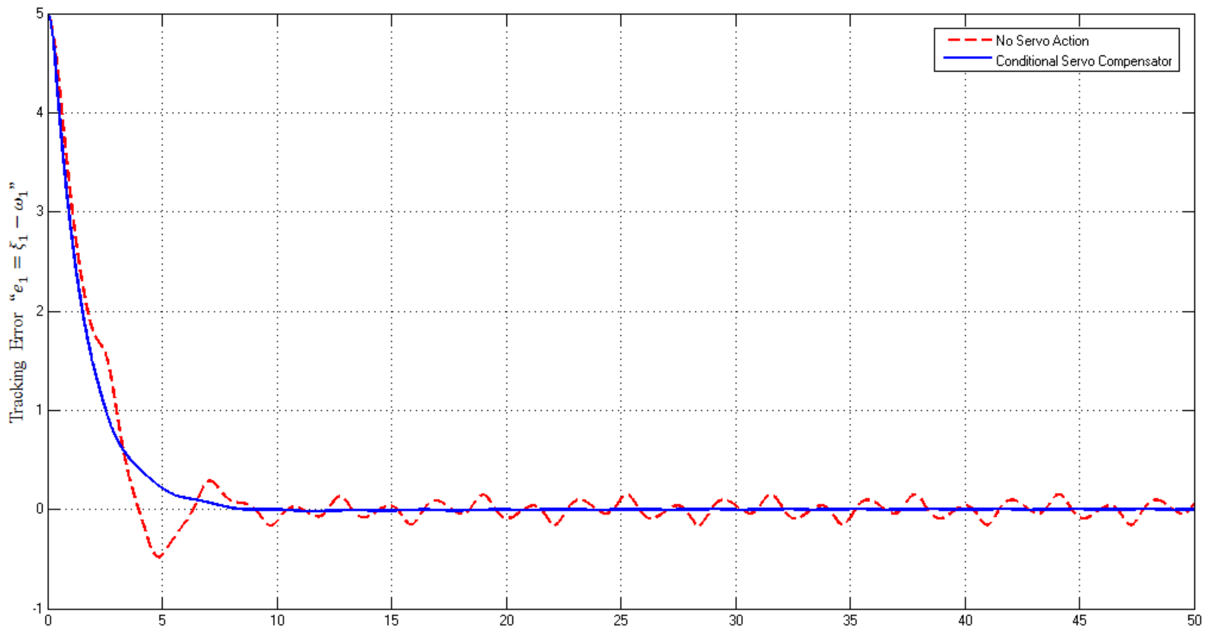

Fig. 4.5: Tracking Error Response – No Servo Action vs Conditional Servo Action.

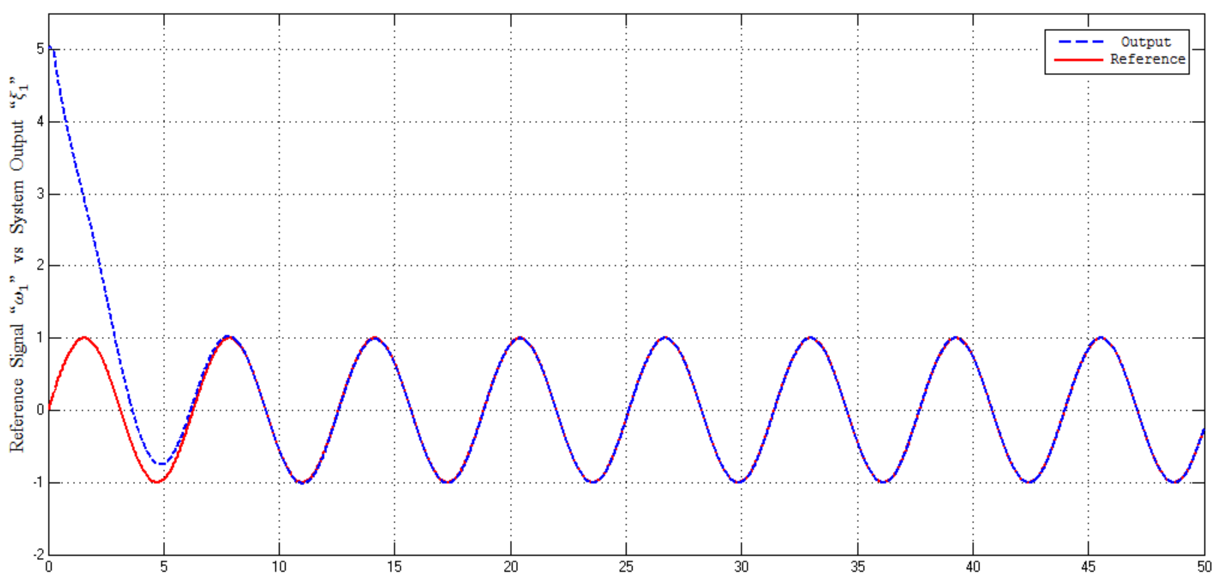

Fig. 4.6: Tracking Error Plot (Conditional Servo – Compensator).

Now, introducing the saturation constraints / nonlinearities in the input channel. We see that the performance of the system gets deteriorated both in the transient performance as well as in the steady-state performance in the form of delayed results. This effect is as shown in the Fig. 4.6. To cope with these saturation effects, the same technique is used as that of chapter 3. i.e. the saturation limit of the conditional servo-compensator may be chosen less than or equal to that of the saturation level of the input channel. By doing so, the effects of saturation are minimized. In fact, the exact performance is achieved as it should be in the absence of the saturation. These results are depicted in the Fig. 4.7 and 4.8.

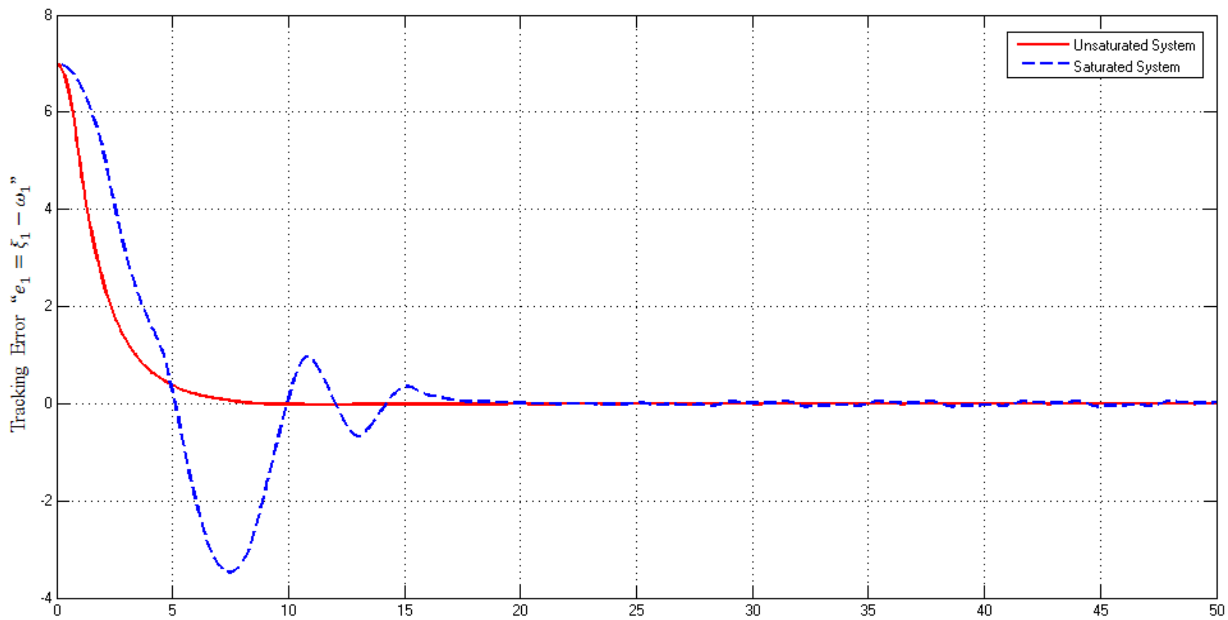

Fig. 4.7: Tracking Error Response under Saturation Nonlinearities $e_1(0) = 7$.

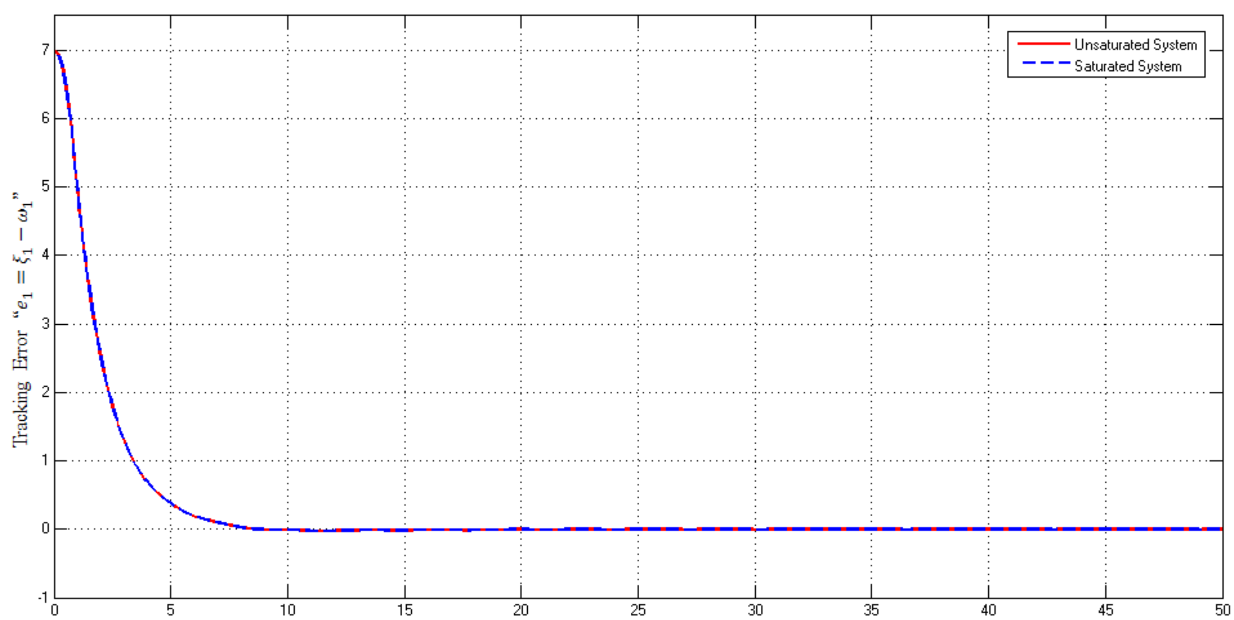

Fig. 4.8: Minimized Effect of Saturation (Tracking Error Response at $e_1(0) = 7$).

Now for output feedback response of the system, utilizing the observer design provided by (4.29) where $\sigma = z_2\, sin(\mathcal{X}_{1d})$ and,

$$\mathcal{A}_1(t) = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon cos(\hat{e}_1 + \omega_1 + \mathcal{X}_{1d}) \end{bmatrix}, \quad \mathcal{C}_1(t) = [0 \quad z_2 \cos(\mathcal{X}_{1d}) + \sin(\mathcal{X}_{1d})]$$

So, the observer (4.29) can be written as,

$$\begin{aligned}
\dot{\hat{z}}_1 &= \hat{z}_2 + L_1\left[\hat{\sigma} - \hat{z}_2\, sin\left(\hat{\mathcal{X}}_{1d}\right)\right] \\
\dot{\hat{z}}_2 &= -\hat{z}_1 + \epsilon sin(\hat{e}_1 + \omega_1 + \mathcal{X}_{1d}) + L_2\left[\hat{\sigma} - \hat{z}_2\, sin\left(\hat{\mathcal{X}}_{1d}\right)\right] \\
\dot{\hat{e}}_1 &= \hat{e}_2 + \frac{\alpha_1}{\varepsilon}(y - \hat{e}_1) \\
\dot{\hat{e}}_2 &= v - \ddot{\hat{\mathcal{X}}}_{1d} + \frac{\alpha_2}{\varepsilon^2}(y - \hat{e}_1) \\
\dot{\hat{\sigma}} &= \hat{z}_2\, cos\left(\hat{\mathcal{X}}_{1d}\right).\frac{-1}{1 + (\vartheta z_2)^2} + sin\left(\hat{\mathcal{X}}_{1d}\right) + \frac{\alpha_3}{\varepsilon^3}(y - \hat{e}_1)
\end{aligned}$$

where $\mathcal{L}(t) = [L_1 \quad L_2]^T$ is designed using (4.26). With $\varepsilon = 0.05$, $\alpha_1 = 5$, $\alpha_2 = 3$ and $\alpha_3 = 1$, the output feedback tracking error simulation results are shown in Fig. 4.9 which shows that the output feedback control law recovers the performance of state feedback control law robustly. Fig. 4.10 shows the tracking performance of system's output versus the reference signal which also proves the efficacy of the observer design. Fig. 4.11 and Fig. 4.12 depicts the estimation performance of the EKF-EHGO observer.

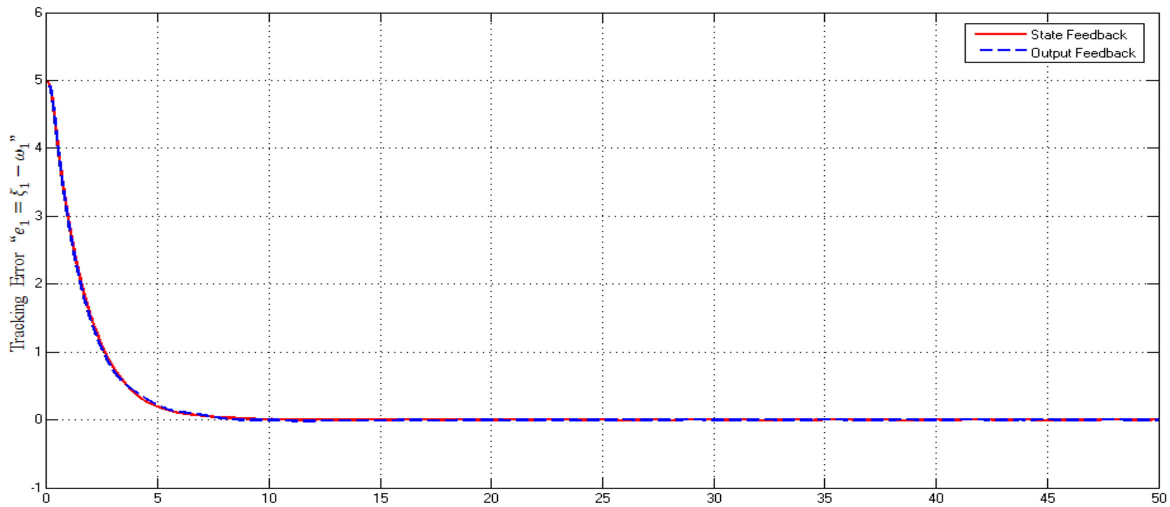

Fig. 4.9: Tracking Error Response (State Feedback vs Output Feedback).

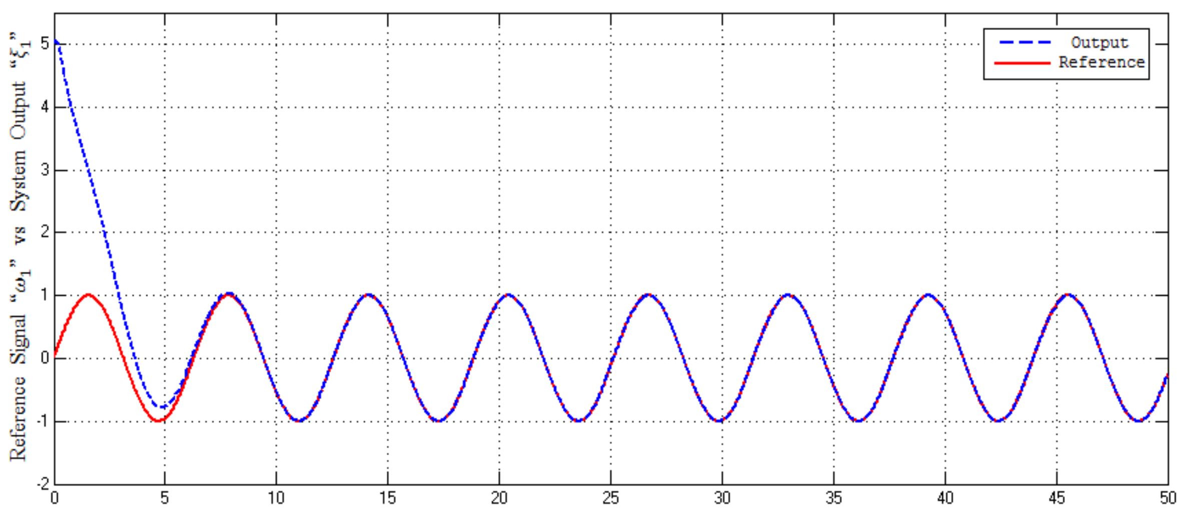

Fig. 4.10: Plot of Reference Signal ω_1 vs Systems Output ξ_1 under OFB design.

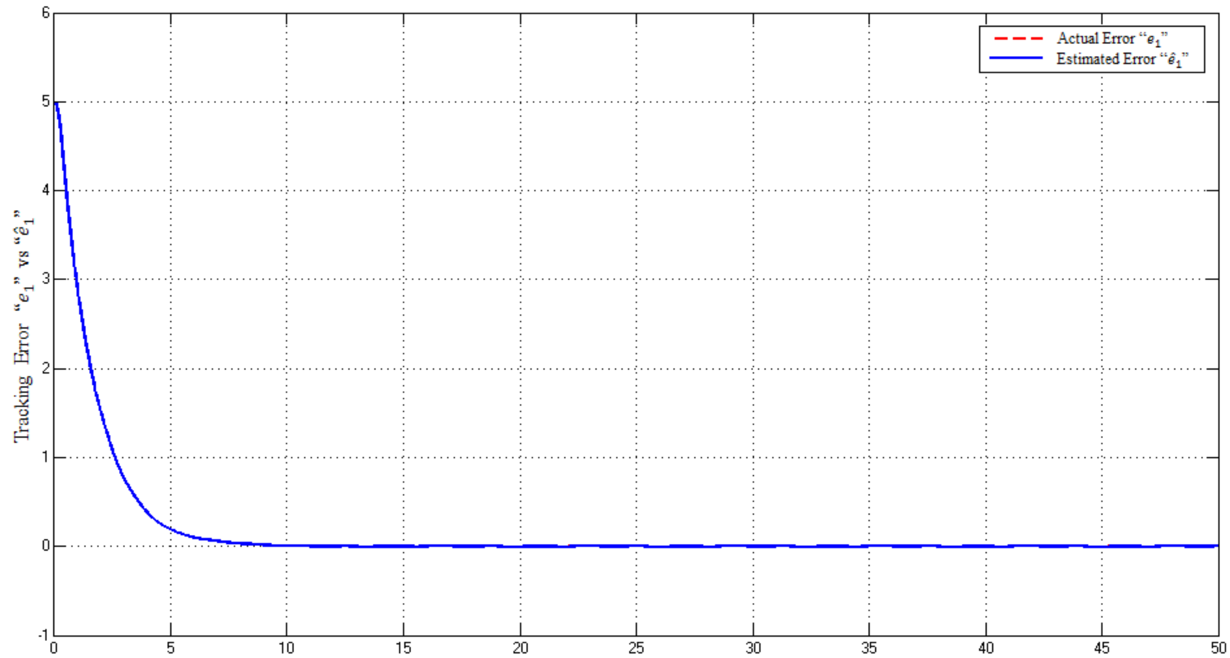

Fig. 4.11: Actual Error Signal "e_1" vs Observed Error Signal "$\hat{e}_1$".

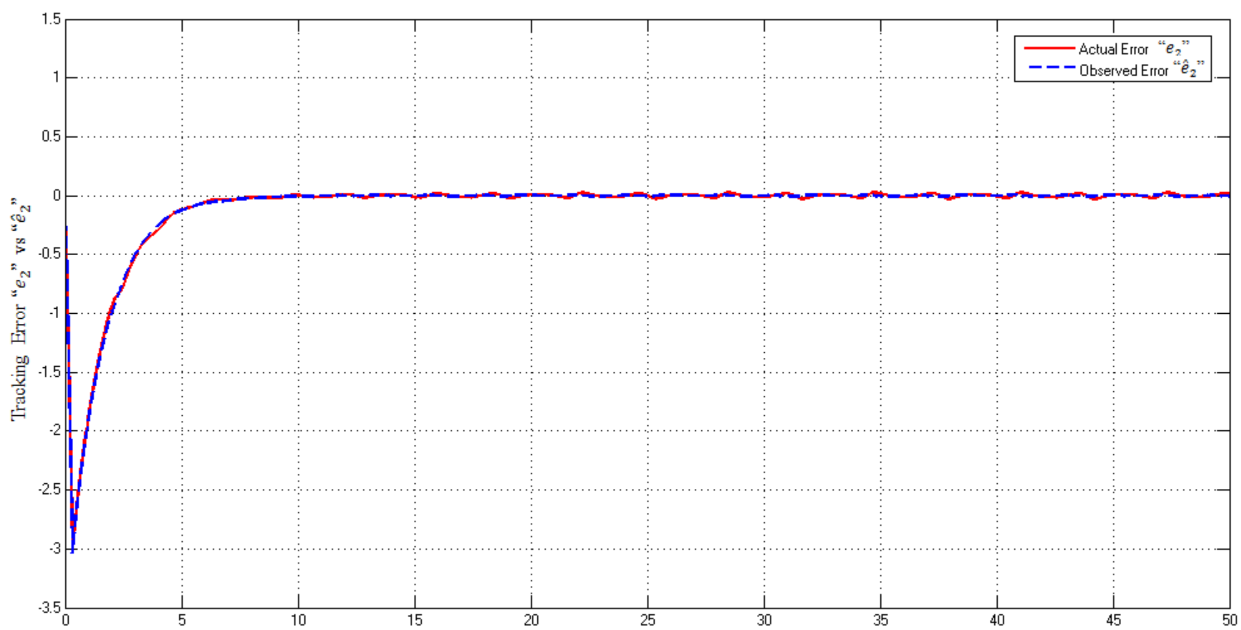

Fig. 4.12: Actual Error Signal "e_2" vs Observed Error Signal "$\hat{e}_2$".

So, these results prove the efficiency of the approach of last chapter which is passivity based conditional servo-mechanism. Both the state feedback design as well as the output feedback design provides the robust results whereas the effects of saturation nonlinearities can also be minimized by a simple adjustment of saturation levels. Towards the observer design, as discussed by the *Boker et. al.* [9], the internal observer can also be other than the EKF worked here.

4.5 Discussion and Conclusions

The conditional servo-mechanism design approach of previous chapter which is based on the passivity based control design technique have been applied to the Translating Oscillator with Rotating Actuator (TORA) systems successfully. The mathematical model of TORA systems is weakly minimum phase that do not satisfy the Lyapunov stability properties. We have shown that it is possible through suitable variable transformations, to convert that mathematical model into the form which satisfy the Lyapunov stability properties. e.g. condition of being minimum phase. Such model transformation for TORA systems has also been used in [41] and [46]. Once, the system is transformed into the minimum phase realization, all the techniques which are applicable specifically for minimum phase systems can also be applied for such realized models which has been proved by the simulation results discussed in the previous section.

Chapter 5

CONCLUSIONS

This thesis work concentrates on the solution of servo-mechanism problem for the class of nonlinear systems with a focus on the saturation nonlinearities. We have utilized the conditional servo-compensator based approach designed by *Attaullah Y. Memon et. al.* [7], [33]. The output feedback design is implemented using the Extended Kalman Filter based Extended High Gain Observer (EKF-EHGO) approach developed by the *Boker et. al.* [9]. This work is actually the extension of the work of *Attaullah Y. Memon et. al.* [8].

We started our discussion by introducing the topic and some literature review in chapter 1 followed by the preliminary discussions provided in chapter 2. We formulated the problem in chapter 3 where we have considered two types of systems exposed to the saturation constraints. First, the systems which itself possess the linear behavior but the saturation nonlinearity makes the overall scenario as nonlinear. Secondly, we have considered the systems that itself possess nonlinear nature in addition to the saturation constraints in the input channel. For the first type of systems, we have provided an extension to the reference work [8] in a sense that the stabilizing compensator of the conditional servo-compensator design is designed using the technique of Composite Nonlinear Feedback (CNF) where by simply tuning the parameters of the nonlinear gain function, an extra control feature is provided. i.e. the damping ratio can be adjusted as required by the designer. The output feedback version of such design is implemented by the simple High Gain Observer (HGO). The simulation results provided at the end of that chapter proves the efficacy of the design. On the other hand, for the second type of systems, we have considered that when the system exhibiting the nonlinear behavior is represented in the normal form (explicitly in internal as well as external dynamics), it can be considered as the cascade connection of two subsystems. Such a realization leads us to work out on the concepts of passivity to design the stabilizing compensator because in such cases the interconnection term connecting the internal and the external dynamics play important role and can be robustly manipulated using the concepts of passivity. Starting with the Lyapunov stability of the internal dynamics, we have proved the passivity of the external dynamics and the Lyapunov function that realize the Lyapunov stability results, is utilized manipulate the interconnection term in the control design process where the closed loop system results in the passive design. The necessary assumptions and the mathematical analysis have been provided which concludes the closed loop stability results analytically. The output feedback version of this design is implemented using the idea of *Boker et. al.* [9] where a full order observer known as Extended Kalman Filter based Extended High Gain Observer (EKF-EHGO) is implemented to estimate the system states of both dynamics. The simulation results provided in the simulation section of chapter 3 proves the efficacy of the observer as well as the overall design mechanism as well.

In chapter 4, we have applied our approach developed in the chapter 3 to the practical nonlinear benchmark system known as Translating Oscillator with Rotating Actuator (TORA) system. This system is weakly minimum phase whose mathematical model do not satisfy the Lyapunov stability properties. We have shown that it is possible to transform such model through suitable variable transformations, into such form where we can prove the Lyapunov stability and minimum phase conditions. Once, the system is transformed into the minimum phase form, we have shown that our approach of passivity based conditional servo-compensator can be applied which yields robust results. Both the state feedback designs as well as the output feedback designs using EKF-EHGO are implemented and the results are shown at the end of Chapter 4.

Our major observation is that the performance degradation that occurs due to the saturation of the control channel can be improved or in fact fully recovered by just simple adjustments. As, in the design of conditional servo-compensator, it is the choice of the designer to choose the saturation level of compensator. It has been observed that as long as the saturation level of the conditional servo-compensator is chosen less than or equal to that of the saturation level of control input channel, the performance is fully recovered compared to what it should be in the absence of saturations. However, in practical the saturation is classical nature of every system and cannot be isolated so this level adjustment works perfectly in this scenario. Through these adjustment, the control magnitude does not get saturated and it works to achieve the desired control task.

Finally, we present some extensions of this research work which can be worked out in future. As, the combination of stabilizing compensator + servo-compensator works perfect to achieve the task of output regulation for the control systems. However, the optimal designs of stabilizing compensators which focus on the control cost can be worked out in future to make the design more economic and feasible. Secondly, the stability margins concentrating the parameter perturbations and the region of operation of stabilizing + servo-compensator combinations can be identified in future to make the design very robust which can lead towards the universal control designs. A promising direction would be to extend this approach towards the nonminimum phase systems subjected to the control constraints, defining their operation region and increasing their region of attraction to making the design work globally.

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