REAL-TIME MODEL PREDICTIVE CONTROLLER FOR A CLASS OF NONLINEAR SYSTEMS

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"Recite, and your Lord is the most Generous- Who taught by the pen- Taught men that which he knew not." (The Qur'an, Al-'Alaq 30.3-5)

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Ali Hasnain Yousfani

DEDICATION

"To my Parents"

ABSTRACT

The major purpose of this thesis is to design a nonlinear model predictive control (NMPC) scheme which retains the advantages of easily handling the constraints and multi-variables, for which the nonlinear model predictive controller is popular, in real-time setting as well. In the case of fast sampled systems, an additional real-time constraint is imposed on the controller because of an implicit requirement of the optimal control problem to be solved online within a sampling period. Naturally, there will be systems which are complex enough to exhaust any advance hardware in the world or in other cases even if hardware is available the cost of implementation might exceed the designer's budget. The suboptimal solution is the only choice then. Researchers have produced various useful algorithms and real-time schemes to counter the real-time problem but to achieve success, the new modifications made to the problem formulation invalidate the theoretical guarantees provided by the standard NMPC formulations. This is where the work presented in this thesis contributes to the topic. It without adding any additional computation cost recovers the guarantees on stability of the system when it was not constrained in real-time. In practical applications, external disturbances and measurement noises can never be left out hence a computationally fast tube-based approach is used for the guarantee of robust recursive feasibility of the problem. After giving some brief background on the NMPC in chapter 1, the report presents the basic nonlinear model predictive controller design in chapter 2. Chapter 3 surveys the available robust NMPC methods whereas chapter 4 details the adopted robust solution, the tube based approach. Chapter 5 presents the real-time robust NMPC controller scheme. At the end of the chapter, the proposed control scheme is shown to stabilize the system for any arbitrarily small sampling period. For the illustration of the ideas, the mass spring system containing three oscillating masses with nonlinear spring coefficients is stabilized.

Keywords: NMPC, Robust NMPC, Tube-based NMPC, invariant tubes, feasible warm-start, lyapunov decrease constraint, imposed upper bound

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Chapter 1
INTRODUCTION

1.1 Motivation

Model predictive controller (MPC) is a control method which at each sampling instant solves a finite horizon open-loop optimal control problem and computes a control sequence, the first part of which is applied to the system to be controlled. This on-line computation of the optimal solution is the main difference of MPC in comparison to the controllers computed offline. This is why a model predictive controller is preferred where the computation of offline solution is nearly impossible to obtain or not effective enough online.

The ability to handle multivariable systems easily is also one of the reasons that model predictive control has seen its application grow in the industry in past decades. Almost every application in the industry imposes hard constraints such as actuators with the maximum force they can apply or constraints pertaining to the safety limits on pressure, temperature and velocity etc. Efficiency concerns also encourage the designers to operate the system at boundary conditions. Traditionally all these hard constraints on the control action and system states along with shortage of controllers to handle them have seen the designers to resort to ad-hoc methods which in turn makes the system analysis extremely difficult. The ability of MPC to handle these constraints implicitly is a key reason why MPC is considered as one of the few (if not only) suited controllers.

It is normally near to impossible to obtain an accurate mathematical model of a real-world plant, hence a close approximation of the model then is considered for the control problem. Furthermore, the ever present measurement noise and other disturbances in the control loop all add up to make the system parameters uncertain. Controllers which deal with these uncertainties with prior information about the uncertainties (such as bounds and stochastic distributions etc.) are normally referred to as the robust controllers. A number of approaches pertaining to robustness exist in the MPC framework. One such approach is a tube-based model predictive controller which is the adopted solution in the work presented in this thesis.

The natural drawback of the MPC comes with computational concern since an optimal control problem needs to be solved at each sampling instant. This had seen the application of MPC restricted to slower systems only till 1970s. However the advancement in the computation algorithms and processors is enabling the control engineers to use model predictive controller for faster systems as well. Fast sampled systems however still impose a real-time constraint in that an optimal solution needs to be computed at every sampling instant because the proof of theoretical stability and feasibility guarantees depends on the optimality of the solution. For the implementation purposes, almost all the NMPC (Nonlinear Model Predictive controller - model predictive controller considering nonlinear systems) algorithms are resorted to early termination of the optimal control problem, which may cause the systems to be unstable or unfeasible at some point. A number of approaches exist which ensure the stability of the closed-loop system in real-time environment which is the main motivation of this thesis.

The aim of this thesis is to develop a model predictive controller for a class of non-linear systems (class described in the chapters to follow) which ensures stabilization of the origin (i.e. all system states become zero in steady-state condition) in the presence of bounded additive disturbances without having to worry about the time available for the computation of optimal solution. The controller uses a non-linear tube based approach which ensures feasibility of all the input and state constraints under the uncertainties and ensures stabilization through a Lyapunov stability constraint, the feasibility of which is ensured through a warm-start procedure.

The following sections give a basic idea about the non-linear model predictive control and its evolution. Chapter 2 presents the development of the model predictive controller for the system without uncertainties (the nominal system). Chapter 3 presents the various approaches used in the literature for countering the problem of uncertainties in the NMPC design. Chapter 4 presents the basic ideas of the tube-based robust model predictive controller for system with uncertainties. Chapter 5 presents the real-time constraint problem and the solution proposed by this thesis work to ensure the stability and feasibility of all the state and control constraints for any arbitrarily small sampling time.

1.2 Non-Linear Model Predictive Control

Non-Linear Model Predictive controller is a form of control in which the prediction of the system states is made over a finite number of sampling instants (the horizon) and then a sequence of control actions is calculated as to minimize a cost function which penalizes the deviation of system states from the desired trajectory. The control action at every sampling instant is computed by solving the open-loop optimal control problem for a finite horizon using a non-linear mathematical model of the plant with the currently measured state taken as the initial point for the problem which makes it a feedback control policy. The first part of the computed control action is applied to the system to be controlled and at the next sampling instant, measurements are taken again and the whole prediction and optimization procedure is repeated. This is also known as receding horizon control (RHC) which emphasizes the fact that the prediction horizon moves a sampling instant at every step.

The figure 1.1, inspired by (Allgöwer et al. (1999)), shows the basic principle of non-linear model predictive control. At each sampling time *t*, current state is measured. Taking the current measurement as the initial state, future behavior of the system is predicted over a time period T_p and an optimal control input sequence is computed over a time period $T_c \leq T_p$ such that a predetermined cost function, penalizing the deviation of states from reference trajectory and control input from steady-state control, is minimized. If there are no uncertainties in the system and an optimal solution can be computed for the infinite horizon, T_p can be taken as ∞ and the optimal control problem doesn't need to be solved again at any other sampling instant. This is exactly what an LQR (Linear quadratic regulator - a standard optimal technique for linear systems) does. For linear systems with no constraints, obtaining such an optimal solution is possible and that's why an offline computed control policy stabilizes the system. However, it is near to impossible to obtain such a solution for a constrained nonlinear system. Also, in practical applications, the disturbances and uncertainties are ever present entities. Therefore the control input is only applied until the next measurement is available at time $t+\delta$ which makes it a feedback dependent policy whose merits are known in the literature. At new sampling instant, the open-loop optimal control problem is solved again with current measurement as the initial state and again the control is only applied till the next sampling instant.

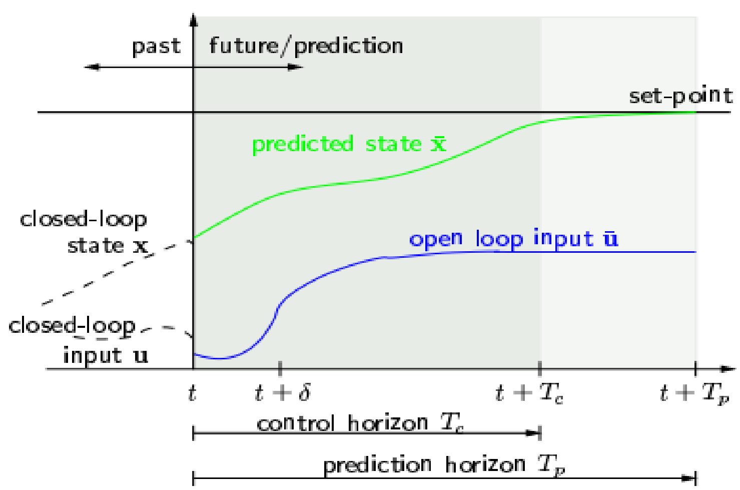

Figure 1.1: Principle of Model Predictive Control

1.3 NMPC- Brief History and Current Developments

Non-linear model predictive control basically stems from the theory of optimal control which aims for a minimal control effort for a certain closed-loop performance. The core of NMPC controllers is the link between two revolutionary contributions to the area of control engineering. One is the Hamilton-Jacobi-Bellman Theory (Dynammic programming) which gives the sufficient conditions for the optimality and on the basis of that a way to obtain an optimal stabilizing feedback control $u = k(x)$. Second is the maximum principle which provides the necessary conditions for optimality and forms the basis for obtaining the algorithms for computing the open loop optimal control $u^0(.;x)$ for a given initial state. The link between the two major schemes is $k(x) = u^0(.;x)$ and was pointed out first in the book by Lee and Markus (1967, p. 423) as "One technique for obtaining a feedback controller synthesis from knowledge of open-loop controllers is to measure the current control process state and then compute very rapidly for the open-loop control function. The first portion of this function is then used during a short time interval, after which a new measurement of the process state is made and a new open-loop control function is computed for this new measurement. The procedure is then repeated". But due to lack of advancement required for computing the solutions online "very rapidly", the proposal was not given much attention to. However, in 1970s, the advent of processors technology and progress in the faster computer algorithms made the model predictive control implementable but still only for slower systems. (Richalet, Rault, Testud and Papon (1976)) were the first to propose its application to the process control industry and it became extremely popular employing Linear models, including impulse and step response models, owing to its implicit ability to handle the constraints and multivariable setting easily. However, in the absence of formal stability guarantees, prediction horizons and cost functions were used as the tuning parameters and for that reason the method was mainly categorized as an ad-hoc solution. Hence they system specific and design guidelines were hard to generalize.

In an article, ((Chen and Shaw (1982)) showed the stability of the closed-loop system with MPC using equality terminal constraints and proved it using Lyapunov function techniques, however, whole of the optimal solution over prediction horizon was applied instead of first part only. In the MPC paradigm as used today, (Keerthi and Gilbert (1988)) show stability of the NMPC using terminal constraints sets for discrete time linear systems and (Mayne and Michalska(1990)) for the continuous time linear systems. The fact that the numerical difficulties make the satisfaction of the equality terminal constraint in non-linear systems nearly impossible, regional terminal constraints along with suitable terminal costs were then proposed and used as alternatives and gained much of the attention from the researchers in 1990s to obtaining NMPC schemes with guaranteed stability. The latter half of the 1990s saw rapid development in the topic owing to the major contributions such as (De Nicolao, Magni and Scattolini (1996)), (Magni and Sepulchre (1997)), (Chen and Allgöwer (1998)). (Mayne, Rawlings, Rao and Scokaert (2000)) in a survey summarize these results which is considered a consensus in the control community over a standard nominal non-linear model predictive control.

An important extension of the nominal NMPC addresses the issues relating to robustness. It is well known that in practical application, it is impossible to avoid the uncertainties in the control loop owing to various factors such as plant-model mismatch, disturbance at the actuation, noise in the measurement and numerical errors etc. Since NMPC is a feedback control law, it exhibits some degree of inherent robustness (Magni and Sepulchre (1997)), (Chen and Shaw (1982)), (Mayne et al. (2000)), (Scokaert et al. (1997)), (Michalska and Mayne (1993)) and (Findeisen et al. (2003c)). But as pointed out by Teel (2004); Grimm et al. (2004), apart from linear systems with convex constraints, it may be very difficult to quantify and its domain indeed includes only very small region of the state-space. This motivates the research in the field of robust NMPC which has seen significant results being published proposing different approaches such as Robust NMPC solving an open-loop min-max problem (Lall and Glover (1994)), (Chen et al.,(1997)), (Blauwkamp and Basar(1999)), H_∞ based NMPC (Magni et al. (2001b), (2001c)), (Chen et al. (1997)), Robust NMPC design using multi objective optimization (Darlington et al.(2000), Rustem(1994)), Robust NMPC via optimizing a feedback controller used in between the sampling times (Kothare et al.(1996)), (Magni et al.(2001b)), tube-based robust NMPC (Mayne and Kerrigan (2007)).

A big limitation which caused delay in the popularity of MPC as compared to the time it was proposed was the implicit requirement that an online optimization problem must be solved online within a given amount of time. Therefore sufficiently fast sampled systems impose a hard real-time constraint on the controllers, the addressing of which is done in topic of fast MPC in the literature. Most fast MPC schemes generally are categorized in two schemes. One is Explicit MPC (A. Bemporad et al. (2002)), (Bemporad et al. (2002)), (Borelli (2003)) in which the controller pre-computes the solutions and then uses them regionally from a look up table on-line. This method however is limited to plants with less complexity (number of dimensions etc.) because of the fact that required storage space (in turn controller cost) increases with exponentially with increase in dimensions. Second approach is to fasten the online optimization (Hansson (2000)), (Kouvaritakis et al. (2002)), (Andersen et al. (2003)), (Ferreau et al. (2008)), (Cannon et al.(2008)), and (Patrinos et al. (2010)). These fast algorithms mainly exploit the special structure of the MPC schemes or sparsity of the different optimization methods. The algorithms while enabling the NMPC to be used for sufficiently fast sampled systems as well, however compromise the theoretical stability and feasibility guarantees associated with the standard MPC problem formulations.

(Zeilinger et al.) propose a real-time MPC formulation for linear systems which guarantees stability and feasibility for an arbitrarily small sampling time claiming no additional computation cost. The work presented in this thesis extends its ideas to the non-linear setting employing tube-based NMPC proposed by (Mayne and Kerrigan (2007)). The central ideas of the proofs are same as in (Zeilinger et al.) while making necessary modifications for the non-linear case. Owing to the vast amount of development in the field, it is impossible to overview all methods or formulations. Hence this thesis will focus on the robust non-linear model predictive controller for only a class of constrained non-linear systems (defined in the following chapter) with guarantees of stability and feasibility in the real-time environment.

Chapter 2
NONLINEAR MODEL PREDICTIVE CONTROLLER

This chapter presents an overview of the basic ideas and implementation algorithm of the model predictive control for non-linear systems, also known as Non-linear Model Predictive Control (NMPC). The literature on nominal NMPC is well-established and due to vastness of the topic, it is not possible to present the evolution of the NMPC formulations over the time in this chapter and thus it only presents the well-known NMPC(with stabilizing terminal constraint and cost) problem formulation for constrained non-linear systems with certain assumptions. For a detailed overview of the topic, the reader is referred to (Allgöwer et al. (2004)), an excellent survey paper by (Mayne et al. (2000)) and the books (Mayne and Rawlingss (2009)), (Grüne and Pannek (2011)) on which most of the notation and content on nominal NMPC (problem formulations and proofs) of this thesis are based.

The NMPC controller developed in this chapter will be referred to as the nominal controller in the rest of this thesis that is it will only account for the system to be controlled without any uncertainty. Nominal NMPC is a standard term in the robust NMPC literature and is a well-established topic. For a detailed overview of the evolution of Stabilizing NMPC schemes and their stability results, interested reader may refer to the survey paper (Mayne et al. (2000)). In this thesis, a model predictive controller similar to (Chen and Allgöwer (1998)), referred by the authors as Quasi-Infinite Horizon Nonlinear Model Predictive Control, is presented. This NMPC scheme uses stabilizing terminal constraints and a suitable cost to ensure stability. NMPC schemes without terminal stabilizing constraints, however, also exist but not yet popular. Interested reader may refer chapter 6 of the book (Grüne and Pannek (2011)) for further information.

In this thesis, only regulation problem is discussed which can be extended to the regulation of constant reference trajectories by a simple change of co-ordinates. Since for time-varying references, obtaining a stabilizing terminal constraint set and cost is not straightforward and possesses considerable complexity, the tracking problem of arbitrary references are not discussed in this work. Hence for the sequel, tracking is assumed to mean tracking of constant piece-wise references. Readers interested in the tracking problem of arbitrary references under NMPC framework may refer to the work of (Bemporad, Casavola and Mosca (1997)), (Bemporad (1998b)) and (Bemporad and Mosca (1998)).

2.1 Non-Linear Model Predictive Controller

NMPC, in its most basic form, is a model based optimizer. That is it uses the mathematical model of a system to predict the future behavior of the system and then finds an optimal control to optimize the future trajectories. Receding horizon control strategy then implies that at each sampling instant new predictions will be made on the basis of new measurements and new optimizing control will be computed. This brings us to the most important ingredient of an NMPC scheme and that is the mathematical model of the system.

In this thesis, as is common in the literature, discrete-time systems of the form

$$x(k+1) = f\big(x(k), u(k)\big), \qquad x(0) = x_0 \tag{2.1}$$

are considered where $x \in \mathbb{R}^n$ *and* $u \in \mathbb{R}^m$ are the system vector and input vector respectively subject to state and input constraints

$$x \in \mathbb{X} \subset \mathbb{R}^n, \qquad u \in \mathbb{U} \subset \mathbb{R}^m \tag{2.2}$$

Following assumptions are made for the system (2.1).

Assumption 2.1

1. $\mathrm{f(x, u)}$ is twice continuously differentiable and $\mathrm{f}(0,0) = 0$ which implies that origin is an equilibrium point of the system with u=0.
2. $\mathbb{X} \subset \mathbb{R}^n$ is convex and closed and $\mathbb{U} \subset \mathbb{R}^m$ is convex and compact. Both are assumed to contain origin in their interior.
3. We assume the uniqueness and existence of the solutions for system (2.1) for any initial condition in $x \in \mathbb{X}$ and $u \in \mathbb{U}$.

Assumption 2.2

All states are measurable.

NMPC schemes in which assumption 2.2 cannot be satisfied employ an observer and are referred as output-feedback NMPC schemes. NMPC with observers is considered as an extension of the standard nominal NMPC controller and is altogether a separate topic because all the stability proofs have to be modified to be valid and hence are not considered in this thesis. Interested reader is referred to the book (Mayne and Rawlings (2009)) for more details on the topic.

Remark 2.3

Satisfaction of Assumption 2.1(1) can always be ensured for the tracking of constant references with a mere change of co-ordinates. Unfamiliar reader may refer to the book (Grüne and Pannek (2011)).

Remark 2.4

The differentiability condition in the assumption 2.1(1) implies at least the existence of a region where the solutions of system (2.1) are unique (Local Lipschitz Condition). However the region is not guaranteed to be larger than or equal to $\mathbb{X}$. An easier case occurs if $\frac{\partial f}{\partial x}$ is bounded in $\mathbb{R}^n$. In that case, the system is globally Lipschitz and assumption 2.1(3) is satisfied easily. For more details on the topic, reader may refer to section 3.1 of the book (Khalil (2002)).

Now that a mathematical model of the system is available, it can be used to iterate the predictions. Since these predictions are done in an embedded controller in a fictitious time, these internal variables must be distinguished from the actual values of the plant. Hence the iterated prediction trajectory is defined by,

$$x_u(0) = x(k), \quad x_u(k+1) = f\big(x_u(k), u(k)\big), \quad k = 0, \dots, N \tag{2.3}$$

where the terms with subscripts represent the predicted values.

The stage cost function *l* (*x(k)*,*u(k)*) which penalizes the deviation of states and inputs from their reference values is defined by

$$l(x(k), u(k)) \triangleq \| x(k) \|_Q^2 + \| u(k) \|_R^2 \tag{2.4}$$

where $\|.\|$ represents the Euclidean norm and $Q > 0$ *and* $R > 0$ are symmetric positive definite weighting matrices and can be used as tuning parameters for improving the performance of the closed-loop system.

Furthermore, for a current state '*x*', let the set of admissible control sequences be given by

$$\mathbb{U}_{\mathbb{X}_0}^N(x_0) \triangleq \left\{ \begin{array}{c} u \mid u(k) \in \mathbb{U}, \\ and\ x_u(k+1, x_0) \in \mathbb{X}\ \forall k \in \{0, \dots, N-1\}, \\ x_u(N, x_0) \in \mathbb{X}_0 \end{array} \right\} \tag{2.5}$$

where '*N*' is the prediction horizon and $\mathbb{X}_0 \subseteq \mathbb{X}$ is the terminal constraint set. That is, at each sampling instant, the predicted trajectories are forced to terminate in a neighborhood of the origin which is the key for the stability of the system. Feasibility of the open-loop control problem at time t=0 therefore implies the closed-loop asymptotic stability if the terminal region is an invariant set. One method to obtain such a set will be outlined in the section 2.2.

A set of feasible states, can therefore be defined as

$$\mathbb{X}_N \triangleq \{x_0 \mid \mathbb{U}^N_{\mathbb{X}_0}(x_0) \neq \emptyset\} \tag{2.6}$$

The set of feasible states is usually referred as the region of attraction for the non-linear model predictive controller. The nominal NMPC problem can now be formulated as an algorithm as,

Algorithm 2.5

At each sampling instant t_n, n=0, 1, 2…:

(1) Measure the state $x(k) \in \mathbb{X}$ of the system.
(2) Set $x_0 := x(k)$, solve the optimal control problem

$$\textit{minimize} \qquad J_N\big(x_0, u(.)\big) := \sum_{k=0}^{N-1} l(x_u(k, x_0), u(k)) + F(x_u(N, x_0)) \tag{2.7}$$

$$\textit{such that} \qquad u(.) \in \mathbb{U}^N_{\mathbb{X}_0}(x_0), \tag{2.8}$$

$$\textit{subject to} \qquad x_u(0, x_0) = x_0, \qquad x_u(k+1, x_0) = f\big(x_u(k, x_0), u(k)\big)$$

And denote the obtained optimal control sequence by $u^* \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)$.
(3) Define the NMPC-feedback value

$$\mu_N(x(k)) := u^*(0) \in \mathbb{U} \tag{2.9}$$

and use this control value in the next sampling period.

The corresponding optimal value function is given by

$$V_N(x_0) := \min_{u(.) \in \mathbb{U}^N_{\mathbb{X}_0}(x_0)} J_N\big(x_0, u(.)\big) \tag{2.10}$$

The closed-loop system resulting from algorithm 2.5 with feedback law $\mu_N\big(x(k)\big)$ is then given as,

$$x(k+1) = f\Big(x(k), \mu_N\big(x(k)\big)\Big), \qquad x(0) = x(k) \tag{2.11}$$

Before presenting a method to choose a stabilizing terminal cost and terminal constraint set, few results are stated which will be helpful in outlining the algorithm for choosing appropriate terminal set and cost. Consider the linearized version of the non-linear system obtained by the Jacobian linearization at origin,

$$x(k+1) = Ax(K) + Bu(k) \qquad (2.12)$$

where, $A \coloneqq \frac{\partial f}{\partial x}(0,0)\ and\ B \coloneqq \frac{\partial f}{\partial u}(0,0)$ are the Jacobian matrices.

It is well known that if the pair (A, B) is stabilizable, a state-feedback law $u = Kx$ can always be obtained such that the resulting closed-loop system $x(k+1) = (A+BK)x$ is asymptotically stable. Important results pertaining to this linear state-feedback controlled non-linear closed-loop system are summarized in the following lemma.

LEMMA 2.6 (Chen and Allgöwer (1998))

Suppose the linearized system (2.12) is stabilizable at the origin, then

a) The following slightly modified lyapunov equation

$$(A_K + kI)^T P \ + \ P(A_K + kI) = \ -Q^* \qquad (2.13)$$

yields a solution 'P' which is unique, positive definite and symmetric where $A_K = A + BK$, $Q^* = Q + K^T RK$ and $k \in [0, \infty)$ satisfies

$$k < -\lambda_{max}(A_K) \qquad (2.14)$$

Where λ denotes the eigen value.

b) There exists a neighborhood of the origin which is invariant for this linear state-feedback controller and it stabilizes the system without being constrained inside the region. The neighborhood Ω_α can be specified using an arbitrary constant 'α' as,

$$\Omega_\alpha \coloneqq \{x \in \mathbb{R}^n \mid x^T Px \leq \alpha\} \qquad (2.15)$$

such that

i. $\Omega_\alpha \subset \mathbb{X}$ i.e. the neighborhood lies in the interior of the constraint set.
ii. $Kx \in \mathbb{U},\ \ \forall x \in \Omega_\alpha$, i.e., the linear state-feedback controller stabilizes the non-linear system (2.1) inside the region respecting the control constraints.
iii. The state-feedback controller $u = Kx$ renders Ω_α invariant for the non-linear system (2.1)
iv. For any state lying inside the region, an upper bound for the infinite horizon cost of the non-linear system (2.1) being controlled by $u = Kx$ is given by,

$$J_\infty(x,u) \leq x^T Px,\ \ \forall x \in \Omega_\alpha \qquad (2.16)$$

The proof of this lemma is not stated in this thesis, interested reader may refer to (Chen and Allgöwer (1998)).

2.2 Choosing the Terminal Set and Terminal Cost

The infinite horizon cost obtained for the non-linear system (2.1) controlled by the state-feedback controller $u = Kx$ is given by Eq (2.16). A suitable terminal cost, to be used in algorithm 2.5 is thus chosen to be,

$$F(x_u(N, x_0)) = J_\infty(x_N, u) \le x_u(N, x_0)^T P x_u(N, x_0) \tag{2.17}$$

where the matrix 'P' is obtained by the solution of the lyapunov equation (2.13). Choice of the state feedback gain matrix is only dictated by the requirement that it should stabilize the Jacobian linearization (2.12) of the non-linear system (2.1). However, due to optimality of MPC, a popular choice of the stabilizing gain 'K' is the solution of standard LQR problem.

For choosing a suitable terminal set, the results from the lemma 2.6 can be used to outline a constructive algorithm to obtain the terminal region. Following algorithm outlines the procedure for choosing such a terminal set.

Algorithm 2.7 (Chen and Allgöwer (1998))

1. Find a state-feedback gain matrix 'K' such that $u = Kx$ stabilizes the system (2.12)
2. Find a constant 'k' such that inequality (2.14) is satisfied.
3. Find the unique positive definite symmetric matrix 'P' by solving the lyapunov equation (2.13).
4. Find the largest possible constant 'α_1' such that $u = Kx \in \mathbb{U}, \ \forall x \in \Omega_{\alpha_1}$, where the definition of Ω_α follows from (2.15).
5. Find the largest possible constant $\alpha \in (0, \alpha_1]$ such that the following inequality satisfies in Ω_α,

$$L_\varphi \le \frac{k.\lambda_{min}(P)}{\| P \|} \tag{2.18}$$

where

$$L_\varphi = sup\left\{\frac{\| \varphi(x) \|}{\| x \|} \mid x \in \Omega_\alpha \ and \ x \neq 0\right\} \tag{2.19}$$

and

$$\varphi(x) = f\big(x(k), Kx(k)\big) - A_K x(k) \tag{2.20}$$

Remark 2.8

In this algorithm, the existence of a stabilizing linear feed-back gain is assumed. In the case a linear state-feedback gain is not available, the terminal region is a single point, i.e. 'origin' and the terminal region constraint turns into terminal equality constraint which is shown to stabilize the non-linear system(2.1) in (Mayne and Michalska, (1990) and (Rawlings and Muske, (1993)). However, due to numerical difficulties, considerable longer prediction horizon is required for some systems to ensure feasibility at time t=0. Furthermore, for some systems, it is not possible for all the system states to asymptotically converge to origin. Instead the trajectories stay in a neighborhood around the origin. In those cases, a terminal region is necessary to have. For detailed overview, interested reader may refer to (Chen and Allgöwer (1997a)).

Remark 2.9

The terminal region obtained by algorithm 2.7 is not unique and therefore a largest possible set is searched for. This however depends on many factors including the gain matrix 'K', the corresponding constant 'k' and the strength of the non-linearity of the system $\varphi(x)$. In general, stronger the non-linearity, smaller the terminal region implying lesser feasible states leading to small region of attraction. For linear or less complex non-linear systems, the terminal region might only be affected by the constraints on states and control.

Remark 2.10

For some non-linear systems, algorithm 2.7 may yield a very small region of attraction. In that case, a modification in the algorithm may yield a larger region. In this approach, algorithm is implemented until step 4 and then to find 'α' the following optimization problem is solved instead

$$\max_{x}\{x^T P\varphi(x) - k.x^T Px \;\mid\; x^T Px \leq \alpha\} \qquad (2.21)$$

Once a suitable 'α' is obtained, it may then be used to construct the terminal set from the definition of Ω_α in (2.15). A discussion of obtaining 'α_1' and 'α' can be found in (Michalska and Mayne (1993)).

Remark 2.11

In the case of linear systems, $\varphi(x) = L_\varphi = 0 \quad \forall x \in \mathbb{R}^n$ imply that $k = 0$ satisfies (2.18) and the standard lyapunov equation for linear systems is recovered from (2.13). Also the inequality (2.16) is satisfied with an equality i.e. infinite horizon cost obtained by (2.16) is the same as LQR cost if the stabilizing control is obtained by solving LQR problem.

2.3 Stability of the Non-Linear Model Predictive Controller

It is well known from the history of NMPC that a model predictive scheme with an infinite horizon cost is able to stabilize the system with the closed-loop asymptotic stability guaranteed by the feasibility of constraints then. But the problem is it is impossible to find a solution to the open loop problem of constrained systems with infinite prediction horizon. While practical implementation rules out the scheme with infinite horizon cost, all the theoretical proofs are based on approximating the finite horizon cost with the infinite horizon cost.

In the case of finite horizon, stability is ensured through some explicit additions such as terminal cost and a terminal constraint. In this thesis, non-linear model predictive control similar to (Chen and Allgöwer (1998)) is implemented and therefore only the outline of important relevant results have been presented and for the proofs the reader is referred to the cited work.

The central idea of this NMPC scheme is that at each sampling instant, current measurement of the states of the system is taken on the basis of which open-loop predictions of the nominal system are made. An open loop control problem is then solved to find an optimal control sequence to optimize these open-loop predictions respecting both state and input constraints. To ensure stability, an additional terminal constraint that the system states at the end of prediction horizon terminate in an invariant terminal region is added. In addition to it, infinite horizon cost of the non-linear system if it was being controlled by a linear local state feedback law, is used as terminal cost which gives an upper bound on the optimal value function.

2.3.1 Feasibility and Recursive Feasibility

Since the open-loop optimization problem is solved at each sampling instant, it should also be feasible at all sampling instants. Feasibility at a sampling instant means that there exists at least one (not necessarily optimal) solution to the problem (2.7) subjected to constraints (2.8) under which the objective function (2.4) is bounded. In the following lemma, an important result pertaining to recursive feasibility is stated.

Lemma 2.12

The open loop control problem (2.7) for the system (2.1) subjected to constraints (2.8), with no uncertainties present and all state measurements available at every sampling instant, is feasible at all future sampling instants if it is feasible at t=0.

The proof of this lemma is not stated here, however, the interested reader may refer to (Chen and Allgöwer (1998)).

Remark 2.13

Lemma 2.12 states that once a problem is feasible it stays feasible at all future sampling times if no disturbances are present. This leads to the question how to ensure the feasibility at the start then? The simple answer to the question is to use the prediction horizon as the tuning parameter i.e. increasing the length of the prediction horizon until the problem becomes feasible(Chen and Allgöwer (1998)). This will work because in theory, infinite horizon optimal control problem is proved to stabilize the system as mentioned earlier. Due to computational burden, however, '*N*' is chosen as small as possible.

2.3.2 Asymptotic Stability

On the basis of initial and recursive feasibility, asymptotic stability of the nominal non-linear model predictive controller can be stated.

Theorem 2.14

Suppose that:

1. Assumptions 2.1 are satisfied
2. The Jacobian linearization (2.12) of the non-linear system (2.1) is stabilizable
3. The open-loop optimal control problem is feasible at t=0

Then the closed-loop system (2.11) is stable under the feedback control law (2.9) if no external disturbances are acting on the system. Let $\mathbb{X}_N \in \mathbb{R}^n$ be the set of initial states for which assumption (3) is satisfied then $\mathbb{X}_N$ denotes the region of attraction of the closed-loop system (2.11).

Remark 2.15

The condition (2) in theorem 2.14 is only sufficient not necessary. The fact that the pair (A, B) of the system (2.12) is not stabilizable does not imply that there does not exist a linear feedback controller able to stabilize the non-linear system (2.1). (Chen and Allgöwer (1998)).

Remark 2.16

If the open-loop control problem is feasible at start, it stays feasible for all future sampling instants because of the terminal region constraint. That is true even if the controller is unable to find a global optimal solution in the required specified time and returns a suboptimal solution (Chen and Allgöwer (1998)). That is if the external disturbances and uncertainties are not present, the nominal closed-loop system if once feasible will always converge to the origin. This fact will be useful for the stability of real-time robust NMPC this thesis aims to develop.

Remark 2.17

(Chen and Allgöwer (1997b)) show that if the non-linear system (2.1) is already open-loop asymptotically stable, removal of terminal constraint and terminal cost from the problem formulation does not invalidate the stability of the closed-loop system.

2.4 Example – System of Three Masses

Figure 2.1 (Wang & Boyd (2010)) shows a system of three oscillating masses interconnected by dampers and springs with walls on either side. The control input is the tension exerted by the actuators between the neighboring oscillating masses. The mathematical model of the system is given as,

$$[m]\ddot{x} + [c]\dot{x} + [k]x - [\beta]x^3 = F \quad (2.22)$$

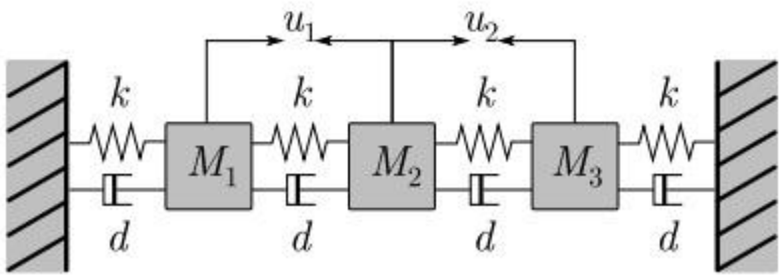

Figure 2.1: System of Three Oscillating Masses

where,

$[m] = \begin{bmatrix} m_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_3 \end{bmatrix}$ represents the masses, $[c] = \begin{bmatrix} c_1 + c_2 & \cdots & -c_2 \\ \vdots & \ddots & \vdots \\ -c_3 & \cdots & c_3 + c_4 \end{bmatrix}$ represents the damping coefficients, $[k] = \begin{bmatrix} k_1 + k_2 & \cdots & -k_2 \\ \vdots & \ddots & \vdots \\ -k_3 & \cdots & k_3 + k_4 \end{bmatrix}$ represents the spring constants and $[\beta] = \begin{bmatrix} \beta_1 + \beta_1 & \cdots & -\beta_2 \\ \vdots & \ddots & \vdots \\ -\beta_3 & \cdots & \beta_3 + \beta_4 \end{bmatrix}$ represents the stiffness coefficient of the springs.

The corresponding linearization of the model at origin is given by,

$$[m]\ddot{x}(t) + [c]\dot{x}(t) + [k]x(t) = F(t) \quad (2.23)$$

The oscillating masses are taken to be $m = 1\ Kg$, the damping constants are $d = 0.1\ N\ s/m$ and the spring constants are taken as $k = 0.9\ N/m$. The stiffness coefficient of the spring is taken as $\beta = 0.5$. The position displacements of the masses are constrained in $\pm 4m$ whereas the velocities of the masses are not constrained at all. The actuation signals are constrained to lie with in $\pm 1N$. The controller is simulated in a matlab code file. The matlab's built-in function *fmincon* is used for the constrained optimization. For the offline computation of terminal sets, the toolbox YALMIP is used.

Figure2.2 shows that the control actions obtained from the solution of nominal optimal control problem asymptotically stabilizes the nominal system to origin.

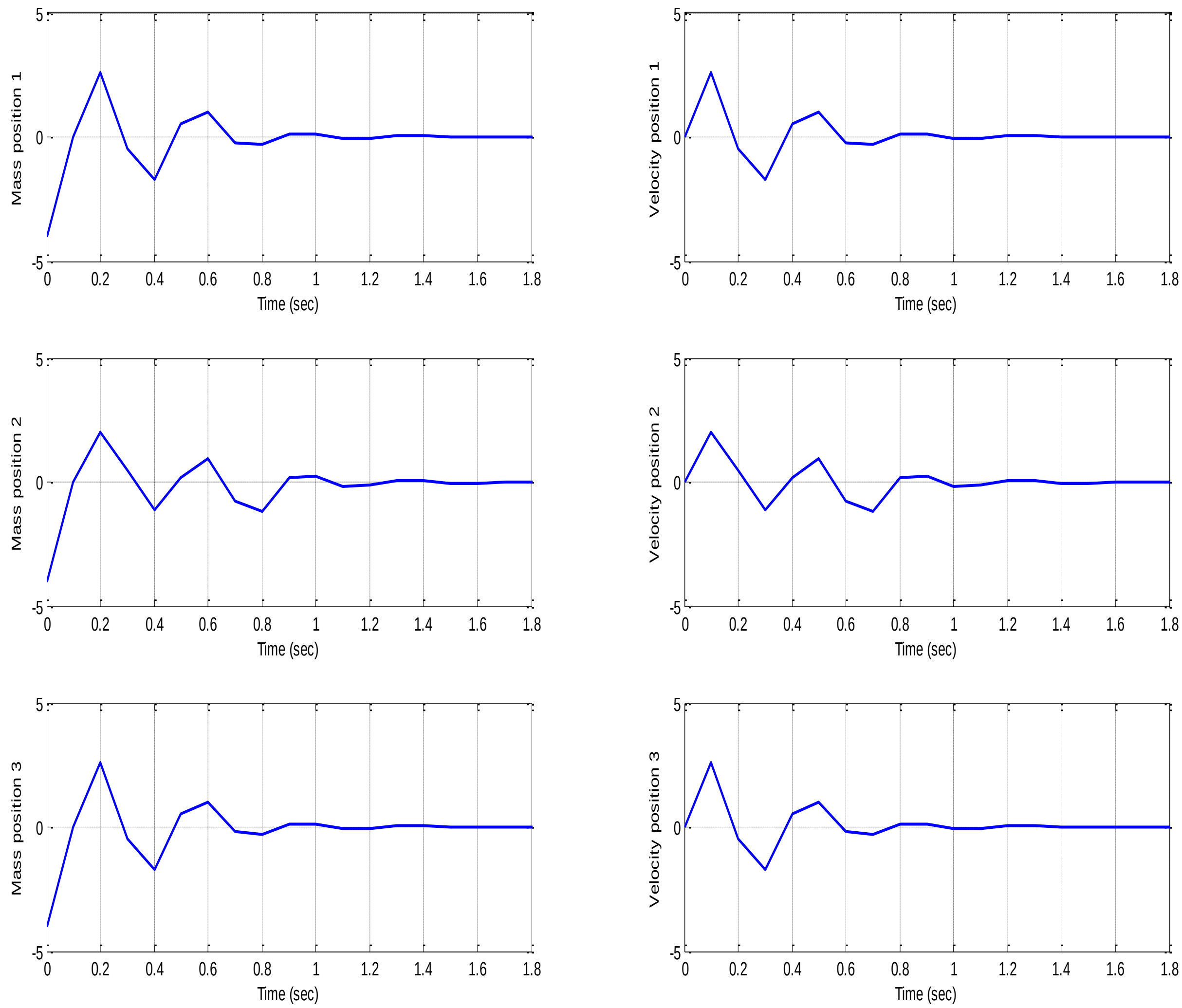

Figure 2.2: Stabilization of the Nominal system to Origin

Chapter 3
ROBUST NONLINEAR MODEL PREDICTIVE CONTROLLER

Nominal Model predictive controller presented in the previous chapter is shown to stabilize the system when there are no uncertainties in the system. It is well known that it is impossible to avoid the uncertainties in a practical application. There are various sources of errors in the system such as model-plant mismatch, measurement noise, numerical errors and external disturbances etc. An important question thus, referred to as the robustness, is what will be the behavior of the closed-loop system under these uncertainties. The researchers have put considerable effort into investigating it but still the robustness analysis of NMPC schemes remains to be the harder task as compared to synthesizing a robust controller because of the constraints and discontinuous nature of the solution of the optimization problem (Bemporad and Morari (1999)). An important tool used for the analysis of the robustness of the robust nonlinear model predictive controllers is input-to-state stability which will be briefly presented in section 3.1.

Section 3.1 first presents the description of uncertainty and problem setup used in the formulation then introduces some important definitions which are needed for the analysis of robust NMPC schemes. Section 3.2 presents the results on inherent robustness of the nominal model predictive controller whereas section 3.3 deals with the design of robust nonlinear model predictive controllers. Open-loop robust NMPC with restricted constraints and the min-max robust NMPC (both open and closed loop formulations) are briefly presented. Tube-based model predictive controller, which is the adopted solution for the real-time robust controller, is presented in chapter 4 separately.

3.1 Robustness Problem and Uncertainty Description

let the uncertain system be described by the model of the form

$$x(k+1) = \tilde{f}(x(k), u(k), w(k)), \qquad x(0) = x_0 \tag{3.1}$$

where

$$\tilde{f}(x(k), u(k), w(k)) = f\big(x(k), u(k)\big) + g(x(k), u(k), w(k)) \tag{3.2}$$

In equation (3.2), $f(x,u)$ is the nominal model of the system. $w \in M_W$ is the disturbance which is contained in the compact set $W \subseteq \mathbb{R}^p$ and $g(x,u,w)$ represents the uncertain term and can be modified to model different types of uncertainties, disturbances, measurement noises, modeling errors, aging, actuator nonlinearities etc. Usually in the robust NMPC framework, following assumptions are taken for the uncertain term 'g'

Assumption 3.1

1. $g(x,u,w)$ is lipschitz in all its arguments with the lipschitz constant L_g.
2. $g(.,.,.)$ is bounded from above and the upper bound on its absolute value $|g(.,.,.)|$ is known a priori.

The concept of input-to-state stability (ISS) has proved to be a useful tool for the robustness analysis of the robust NMPC schemes which is defined next.

Definition 3.2 (Input-to-state stability, Magni and Scattolini (2007))

The system

$$x(k+1) = f(x(k), w(k)), \quad x(0) = x_0, \quad with\; w \in M_W \tag{3.3}$$

is said to be ISS in Ω if there exists a KL function β, and a K function γ such that

$$|x(k)| \leq \beta(|x_0, k|) + \gamma(\|w\|), \quad \forall x_0 \in \Omega$$

Definition 3.3 (ISS-Lyapunov function, Magni and Scattolini (2007))

A function $V(.)$ is called an ISS-lyapunov function for the system (3.3) if there exists a set Ω , K functions $\alpha_1, \alpha_2, \alpha_3 \; and \; \sigma$ such that

$$\begin{gathered} V(x) \geq \alpha_1(|x|), \quad \forall x \in \Omega \\ V(x) \leq \alpha_2(|x|), \quad \forall x \in \Omega \\ \Delta V(x,w) = V\big(f(x,w)\big) - V(x) < -\alpha_3(|x|) + \sigma(|w|), \forall x \in \Omega, \forall w \in M_W \end{gathered} \tag{3.4}$$

3.2 Inherent Robustness of the Nominal Nonlinear Model Predictive Controller

Since the model predictive control is a feedback controller by nature, one might expect the existence of a region where the nominal controller will maintain its stability and feasibility properties in the presence of uncertainties, as associated with other feedback controllers. Hence the most obvious strategy would be to ignore the uncertainties and expect the nominal controller to stabilize the perturbed system in an arbitrarily small region of attraction. This approach in the literature is addressed in the name of inherent robustness of the nonlinear model predictive controller. The presence of state constraints and terminal state constraint in the problem formulation of NMPC make it extremely difficult to quantify the robustness of the nominal closed-loop system, hence accounting for the less number of significant results on the topic. There are mainly two approaches which have been used in the literature for the analysis of the robustness of nominal model predicted controllers. One uses the inverse optimality (De Nicolao et al. (1996)) and (Magni and Sepulchre (1997)) and the other uses input-to-state stability property.

3.2.1 Inherent Robustness using Inverse Optimality

It has been proven that for an unconstrained system, the control law obtained from the solution of the infinite horizon problem guarantees robustness under uncertainties for both continuous and discrete-time systems, see for example (Chellaboina et al.(1998)), (Geromel and Cruz (1987)), (S.T.GLAD (1987)) and (Sepulchre et al.(1996)). For continuous time systems, (Magni and Sepulchre (1997)) show that similar robustness properties can also be derived for unconstrained model predictive control using inverse optimality of the control law. For discrete time systems, similar results are shown in (Magni and Scattolini (2007)). Using some strong continuity assumptions on the value function, NMPC is shown to possess robustness properties under and gain perturbations due to external additive disturbances and actuation system nonlinearities in (De Nicolao et al. (1996)).

3.2.2 Inherent Robustness using ISS property

Taking some strong assumptions on continuity of the system and value function, (Jiang and Wang (2001)) present a useful result on the inherent robustness of the nominal nonlinear model predictive controllers using input-to-state stability property. The result can be summarized as,

Theorem 3.4 (Jiang and Wang (2001))

Under the assumptions (2.1) and (3.1), if $V(x,N)$is lipschitz with lipschitz constant L_v, the closed-loop system (3.3) under the control law (2.9) is ISS in $\mathbb{X}_N$ for any perturbation $g(x,u,w)$ such that $|g(x,u,0)| < \frac{\rho}{L_v}\alpha_l(|x|)\, where\ 0 < \rho < 1$ is an arbitrary real number.

3.3 Robust Nonlinear Model Predictive Controllers

The results presented in the previous section for inherent robustness assume that the recursive feasibility (required for the guarantee of stability of Nominal NMPC) is never lost under any perturbation which can only be ensured if the system is unconstrained (i.e., only terminal region and control constraints are applied, no state constraints are present). One of the biggest advantages of NMPC has been its ability to handle constraints easily and hence the use of NMPC in such condition itself is then questionable. (Grimm et al. (2004)) show that nominal predictive controllers for certain constrained nonlinear systems may exhibit no robustness at all under perturbations.

The system of oscillating masses was shown to be stabilized by the nominal NMPC controller when no disturbances were present. Figure 3.1 shows that under uncertainties, the nominal NMPC controller from chapter 2 is no longer able to stabilize the system.

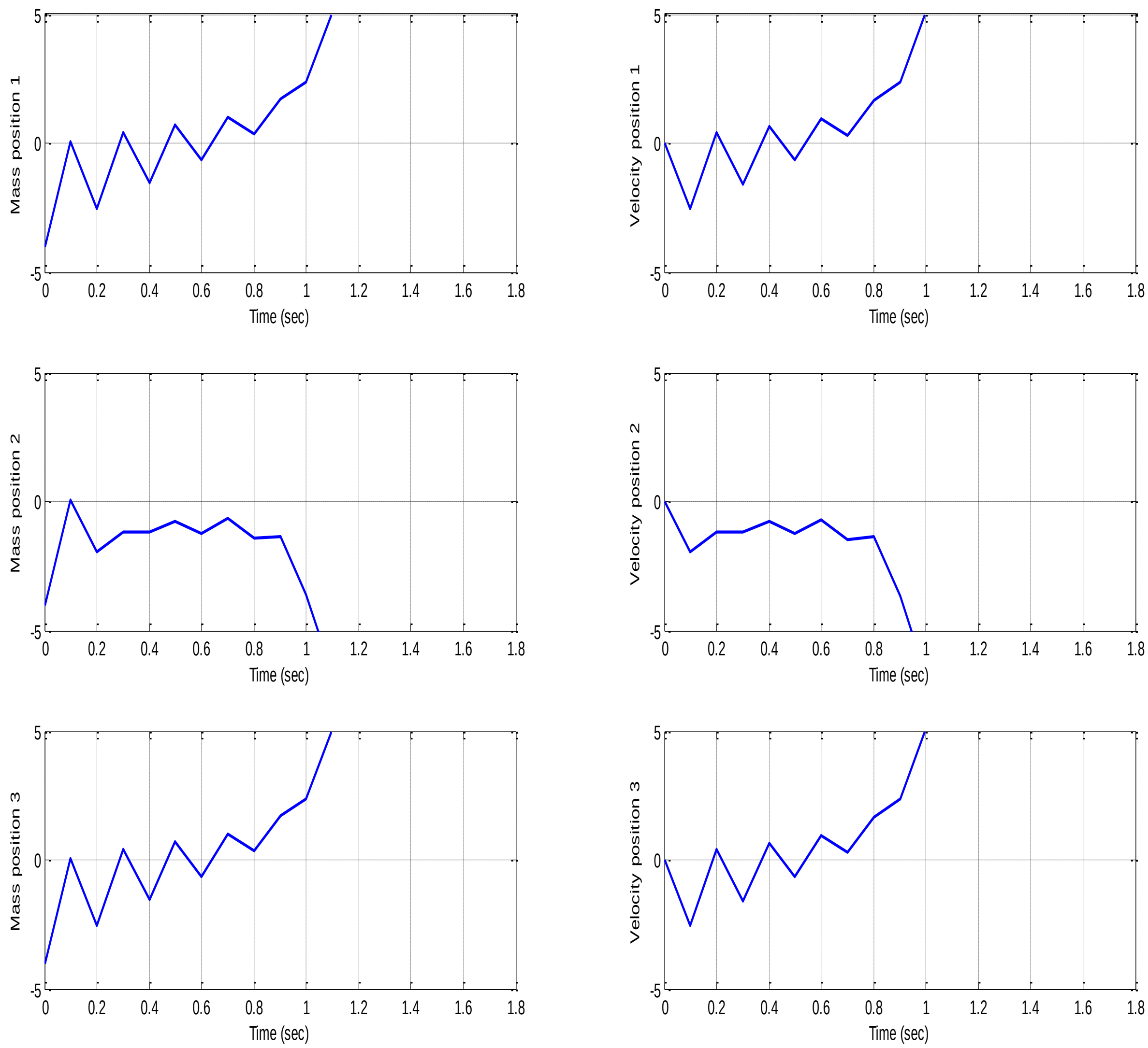

Figure 3.1: Destabilization of states under perturbations

Moreover, even for the stabilizing terminal constraint, the strong continuity assumptions are difficult to prove or satisfy in general leading to the need of robust techniques which consider explicitly countering the effects of disturbances and uncertainties in the system. Several approaches addressing the synthesis of robustly stabilizing model predictive controller have been proposed in the literature, see (Mayne et al. (2000)) and (Magni and Scattolini (2007)) for a survey. The techniques generally fall in following categories.

1. Robust NMPC with tightened constraints employing the tighter control and state constraints. (Limon et al. (2002a)) and (Grimm et al. (2003))

2. Min-Max open-loop and closed-loop robust NMPC formulations employing a modified cost functional. (Chen et al. (1998)), (Magni et al. (2001b)), (Gyurkovics (2002)), (Magni et al. (2003)), (Magni and Scattolini (2005)) and (Limon et al. (2006a))

3. Robust model predictive controller employing tubes of trajectories. (Langson et al. (2004)), (Mayne et al. (2005)) and (Mayne and Kerrigan (2007)).

3.3.1 Robust NMPC with Tightened Constraints

The robust approach presented in this section is the most intuitive way of addressing the robustness problem one could take under the problem formulation i.e., using the constraint handling ability of NMPC to counter the disturbance. The basic idea is to use tighter constraints on both input and states such that states don't leave the feasible set which will ensure stability and feasibility of the problem under perturbations.

Before presenting the problem formulation, an important assumption will have to be made.

Assumption 3.5 (Magni and Scattolini (2007))

The uncertain term in equation (3.1) is bounded by γ, that is $\left|g(.,.,.)\right| \leq \gamma$ for any $x \; and \; u$ satisfying (2.2) and $w \in M_W$.

For tighter state and control constraints, an important definition needs to be introduced.

Definition 3.6 (Pontryagin Difference, Magni and Scattolini (2007))

Let $A, B \subset \mathbb{R}^n$, be two sets, then the Pontryagin difference set is defined as

$$A \sim B = \{x \in \mathbb{R}^n \mid x + y \in A, \forall y \in B\}$$

Consider now the following sets $\widetilde{\mathbb{X}} = X \sim B_\gamma^j \; where \; B_\gamma^j$ is defined as,

$$B_{\gamma}^{j} = \left\{ z \in \mathbb{R}^{n} : \ |z| \leq \frac{L_f^{j} - 1}{L_f - 1}\gamma \right\}$$

The set of admissible control sequences for tighter constraints can be given as,

$$\widetilde{\mathbb{U}}^{N}_{\widetilde{\mathbb{X}}_0}(x_0) \triangleq \left\{ \begin{array}{c} u \mid u(k) \in \mathbb{U}, \\ and\ x_u(k+1, x_0) \in \widetilde{\mathbb{X}}\ \forall k \ \in \ \{0, \dots, N-1\}, \\ x_u(N, x_0) \in \widetilde{\mathbb{X}}_0 \end{array} \right\} \qquad (3.5)$$

Where $\widetilde{\mathbb{X}}\ and\ \widetilde{\mathbb{X}}_0$ are modified according to definition 3.6.

The Nominal Robust Optimal Finite Horizon Control Problem can now be formulated as an algorithm as,

Algorithm 3.7

At each sampling instant t_n, n=0, 1, 2…:

(1) Measure the state $x(k) \in \mathbb{X}$ of the system.
(2) Set $x_0 := x(k)$, solve the optimal control problem

$$minimize \qquad J_N\big(x_0, u(.)\big) := \sum_{k=0}^{N-1} l(x_u(k, x_0), u(k)) + F(x_u(N, x_0)) \qquad (3.6)$$

$$such\ that \qquad u(.) \in \widetilde{\mathbb{U}}^{N}_{\widetilde{\mathbb{X}}_0}(x_0), \qquad (3.7)$$

$$subject\ to \qquad x_u(0, x_0) = x_0, \qquad x_u(k+1, x_0) = f\big(x_u(k, x_0), u(k)\big)$$

And denote the obtained optimal control sequence by $u^* \in \ \widetilde{\mathbb{U}}^{N}_{\widetilde{\mathbb{X}}_0}(x_0)$.

(3) Define the NMPC-feedback value

$$\mu_N(x(k)) := u^*(0) \in \mathbb{U} \qquad (3.8)$$

and use this control value in the next sampling period.

For the stability and feasibility guarantees of the problem defined above, following assumption are be made.

Assumption 3.8 (Magni and Scattolini (2007))

Let $\mu_N(.), F(.)\ and\ \mathbb{X}_0$ be such that

1. Valid region of attraction i.e., $\varphi_f := \{x \in \mathbb{R}^n : F(x) \leq \alpha\} \subseteq \mathbb{X}_N, \varphi_f\ closed, 0 \in \varphi_f$, $\alpha\ is\ a\ \ positive\ constant$
2. Control constraints satisfied in the region of attraction i.e., $\mu_N(x) \in \mathbb{U}$, $\forall x \in \varphi_f$
3. Invariance of the set under NMPC law i.e., $f\big(x, \mu_N(x)\big) \in \varphi_f, \forall x \in \varphi_f$

4. Terminal cost is a lyapunov function i.e., $F(f(x, \mu_N(x)) - F(x) \leq -l(x, \mu_N(x)), \forall x \in \varphi_f$
5. $\alpha v_f(|x|) \leq F(x) \leq \beta v_f(x), \;\; \alpha v_f, \beta v_f$ *are* K *functions*
6. $F(.) is\ lipschitz\ in\ \varphi_f\ with\ a\ lipschitz\ constant\ Lv_f$
7. $\mathbb{X}_0 := \{x \in \mathbb{R}^n : F(x) \leq \alpha_0\}\ is\ such\ that\ \forall x \in \varphi_f, f(x, \mu_N(x)) \in \mathbb{X}_0,$ $\alpha_0\ is\ a\ positive\ constant$

With the above assumptions made, the stability result of the presented scheme can be stated as,

Theorem 3.9 (Limon et al. (2002))

Let $\mathbb{X}_N$ be a set of states in which the optimal problem in algorithm 3.7 is initially feasible, then the closed loop system in equation (3.3) with the feedback control law as in (3.8) is ISS in $\mathbb{X}_N$ if assumptions 3.8 are satisfied with

$$\gamma \leq \frac{\alpha - \alpha_0}{Lv_f L_f^{N-1}}$$

Although the result stated in theorem 3.9 is useful for some systems but is quite conservative in general. The region of attraction using this approach may be very small for some systems and for some it may not even exist. Moreover, continuity of the value function is an important property in order to prove inherent robustness, as shown in (Grimm et al. (2004)), but is firstly very difficult to prove and secondly very conservative to assume. Optimal value function since comprises of a control law obtained from an optimization problem and hence cannot be generally expected to be continuous (Meadows et al. (1995)). Moreover, for system models which inherit discontinuity such as hybrid systems, value function will always be discontinuous. Hence reliance on the inherent robustness results can lead to conservative results under uncertainties. Hence less stringent approaches as described next present the solution to this problem.

3.3.2 Robust NMPC with Min-Max Approaches

The oldest method addressing robust stability problem in the model predictive control frame work (G.Tadmor (1992)) placed it in the H_∞ for linear unconstrained systems. This led to the many other related results for both constrained and unconstrained linear systems, see for example (Scokaert and Mayne (1998)). For the nonlinear systems, (Chen et al. (1997)),(Magni et al. (2001)), (Fontes and Magni (2003)) employ H_∞ based MPC algorithms for the continuous time systems and (Magni et al. (2003)), (Gyurkovics (2002)), (Gyurkovics and Takacs (2003)), (Magni and Scattolini (2005)) and (Mayne (2001)) study discrete-time systems. The basic idea in (Magni et al. (2003)) is to modify the problem such that an H_∞ type cost function is maximized with respect to the disturbance and then the cost function is minimized with respect to the control inputs.

The two main formulations exist namely open-loop min-max NMPC and closed-loop min-max NMPC.

1. Open-loop Min-Max NMPC

As the name suggests, this approach uses the open-loop predictions of the nominal system and maximizes the worst-case disturbance case. For the presentation, following assumption on the disturbance ‘w’ is made.

Assumption 3.10 (Magni and Scattolini (2007))

The disturbance ‘w’ is contained in a compact set W and there exists a K function $\gamma(.)$ such that $|w| \leq \gamma(|(x,u)|)$.

The disturbance sequence ‘w’ can be defined as,

$$w = [w(1), w(2), \dots\ w(N-1)]$$

The open-loop min-max problem can now be given as an algorithm as,

Algorithm 3.11

At each sampling instant t_n, n=0, 1, 2…:

(1) Measure the state $x(k) \in \mathbb{X}$ of the system.
(2) Set $x_0 := x(k)$, solve the optimal control problem

$$maximize\ w.r.t\ 'w'\ and\ minimize\ with\ respect\ to\ \text{‘u'} \tag{3.9}$$

$$J_N(x_0, u(.)) := \sum_{k=0}^{N-1} \{l(x_u(k, x_0), u(k)) - l_w(w(k)\} + F(x_u(N, x_0))$$

$$such\ that \qquad u(.) \in \widetilde{\mathbb{U}}^N_{\widetilde{\mathbb{X}}_0}(x_0), \tag{3.10}$$

$$subject\ to \qquad x_u(0, x_0) = x_0, \quad x(k+1) = \tilde{f}(x(k), u(k), w(k))$$

Where l_w is the H_∞ type cost function. Denote the obtained optimal control sequence by $\mu^* \in \widetilde{\mathbb{U}}^N_{\widetilde{\mathbb{X}}_0}(x_0)$.

(3) Define the NMPC-feedback value

$$\mu_N(x(k)) := \mu^*(0) \in \mathbb{U} \tag{3.11}$$

and use this control value in the next sampling period.

The open-loop min-max approach does offer a way to achieve a robust invariant set but it may very conservative in nature due to the intrinsic nature of the receding horizon schemes. Hence a less stringent approach is then to use closed-loop predictions for the problem instead of open-loop predictions.

3.3.2.2 Closed-loop Min-Max NMPC

The drawbacks of open-loop min-max approaches can be overcome by using Closed-loop predictions for the min-max problem. (Scokaert and Mayne (1998)) presented this approach for continuous time and (Magni et al. (2003)) on discrete-time systems. This approach is similar to the open-loop min-max approach i.e., applies similar modifications to the cost function and maximizes the H_∞ type cost function with respect to the disturbance sequence but then minimizes the worst-case cost function with respect to the feedback policies instead of sequence of control actions.

The closed-loop min-max approach can be given as an algorithm as,

Algorithm 3.12

At each sampling instant t_n, n=0, 1, 2…:

(1) Measure the state $x(k) \in \mathbb{X}$ of the system.
(2) Set $x_0 := x(k)$, solve the optimal control problem

$$maximize\ w.r.t\ 'w'\ and\ minimize\ with\ respect\ to\ '\mu' \tag{3.12}$$

$$J_N\big(x_0, \mu(.)\big) := \sum_{k=0}^{N-1} \{l(x_u(k,x_0),u(k)) - l_w(w(k)\} + F(x_u(N,x_0))$$

$$such\ that \qquad \mu(x) \in \widetilde{\mathbb{U}}^N_{\mathbb{X}_0}(x_0), \tag{3.13}$$

$$subject\ to \qquad x_u(0,x_0) = x_0, \qquad x(k+1) = \tilde{f}(x(k),u(k),w(k))$$

Where l_w is the H_∞ type cost function. Denote the obtained optimal control sequence by $\mu^* \in \widetilde{\mathbb{U}}^N_{\mathbb{X}_0}(x_0)$.

(3) Define the NMPC-feedback value

$$\mu_N(x(k)) := \mu^*(0) \in \mathbb{U} \tag{3.14}$$

and use this control value in the next sampling period.

For the stability and feasibility guarantee, following assumption about the disturbance sequence is made.

Assumption 3.13 (Magni and Scattolini (2007))

l_w is such that $\alpha_w(|w|) \le l_w \le \beta_w(|w|)$ where $\alpha_w\ and\ \beta_w$ are K functions.

Following theorem states an important result on the stability of the closed-loop min-max NMPC scheme.

Theorem 3.14 (Magni and Scattolini (2007))

Let $\mathbb{X}_N$ be a set of states in which the optimal problem in algorithm 3.12 is initially feasible and μ^* represents a vector of lipschitz continuous control policies. Then under assumptions 3.8 and 3.13, the closed loop system in equation (3.1) with the feedback control law as in (3.14) is ISS in $\mathbb{X}_N$. In addition, if there exists a $\gamma(.)$ such that

$$\beta_w(\gamma(\left|x, \mu_N(x(k))\right|) - \alpha_l(|x|) \leq -\delta(x)$$

where δ is a K function, the closed-loop system (3.1) and (3.14) is robustly asymptotically stable.

The closed-loop min-max approach although gives advantage on the open-loop approaches but still there are some drawback attached with the scheme. First one is the need for the optimization to be carried out in an infinite dimensional space which can be quite complex for higher dimensional systems. One way to overcome this problem is resorting to finite dimensional parameterization of the control policies in the optimization as proposed in the literature (Mayne (2000)), ((Magni et al. (2003)) and (Fontes and Magni (2003)). Another method to avoid infinite dimensional optimization problem is proposed in (Magni et al. (2003)) by using different prediction (N_p) and control (N_c) horizons. The optimization problem is then solved for only N_c number of policies and for the rest, the auxiliary control law obtained for terminal region can be applied. The method to obtain auxiliary control law presented in chapter 2 is not the only one to obtain such a law, (Magni et al. (2003)) shows how to obtain a non-linear auxiliary control law with similar properties using H_∞ control for the linearized system.

3.3.3 Robust NMPC Employing Tubes

Although with the advent of computer technology, the min-max approaches are becoming more and more feasible, they still demand high computational costs. Also for the higher dimensional and multivariable systems, the complexity of the task is exponentially increased. A promising method, as proposed in (Langson et al. (2004) and (Mayne et al. (2005)) for linear and (Mayne and Kerrigan (2007)) for nonlinear systems, to reduce the computational burden in the robust NMPC problem is the tube-based NMPC controllers. The tube based NMPC scheme is presented in chapter 4 of this thesis.

3.4 Conclusion

This chapter presented a brief overview of the robust NMPC schemes proposed in the literature. The inherent robustness of the nominal NMPC cannot be relied upon as for some systems, the nominal NMPC may exhibit zero robustness and even for other systems, it may lead to very small regions of attraction. Moreover, the strong continuity condition on value function and control law is often difficult to satisfy for many systems.

The most intuitive solution of the robust problem in NMPC framework then is to use the same nominal scheme with just tightened constraints to compensate for the effects of bounded disturbances but this methodology can be guilty of leading to very conservative regions of attraction for many systems. Open-loop min-max approaches overcome the disadvantages of NMPC with restricted constraints but the absence of feedback in the open-loop approaches make the trajectories scattered hence resulting in small regions of attractions. This advantage can be overcome by incorporating feedback in the min-max approaches i.e., use of closed-loop min-max approaches but it can be computationally costly for higher dimensional and faster sampled systems. This disadvantage of computational complexity is overcome by tube-based NMPC scheme which is described in the chapter 4 of this thesis.

Chapter 4
TUBE-BASED ROBUST NONLINEAR MODEL PREDICTIVE CONTROLLER

As mentioned in the previous chapter, open-loop min-max approaches do give sufficient robustness properties under certain assumptions but may give very conservative results for complex systems. Closed-loop min-max approaches overcome the drawbacks of open-loop min-max approaches but the computational complexity attached with them restrict their use to lower dimensional systems because the optimization problem in the case may get impossibly complex for systems of large dimensions. Hence for more complex systems, implementation aspects of the robust problem demand a new solution. One such method is the tube-based nonlinear model predictive controller which provides the robustness guarantees under certain assumptions with very low additional computation required as compared to the nominal case.

Initially proposed for the linear systems (Langson et al. (2004)) and (Mayne et al. (2005)), (Mayne and Kerrigan (2007)) extend this tube-based scheme to nonlinear systems. The basic idea of the approach in both the linear and nonlinear cases stems from the fact that under uncertainties satisfactory control performance is achieved when using feedback as mentioned in chapter 3. The ideal scenario would be to optimize the closed-loop control policies rather open loop control actions. But the optimization problem in this case becomes prohibitively difficult. In the case of linear systems, (Mayne et al. (2005)) proposed to replace the difficult to obtain control policies $\mu = \{\mu_0(.), \mu_1(.), \dots, \mu_{N-1}(.)\}$ in closed-loop min max approaches by simpler control policies $\mu_i(x) = v_i + K(x - z(i))$ where v_i is found by conventional MPC and K is any stabilizing state feedback gain determined offline. x is the uncertain system state whereas $z(i)$ is the prediction obtained from the nominal model. As obvious, the method reduces the computational burden on the robust controller under the model predictive control frame work. The choice of feedback gain K, referred to as the local control law, makes all the possible uncertain trajectories lie in a tube $\{z(0), z(1), \dots\}$ whose center is the nominal trajectory predicted by the conventional MPC. Since the central path is determined by the MPC, the constraint satisfaction is ensured at all times for the nominal case, and the feedback gain then aims to restrict the spread of trajectories of the uncertain system to the tube whose cross section is a constant set ξ. Tightened constraints in the nominal MPC allow for the constraints satisfaction for the uncertain system. Every possible state of the uncertain system, under the effect of feedback gain, then lies in the set $\{z(i)\} \oplus \xi$. The authors prove, under the continuity assumption of value function, the exponential asymptotic stability under bounded additive disturbances.

One would like to do the same with the nonlinear systems as well. But the problem is the local control law found offline (the feedback gain K) in the linear case is difficult to find in the nonlinear case. By conventional NMPC, the central path to be followed by uncertain states can be generated but the local control law that will make the uncertain states converge to the nominal trajectory presents a tough challenge. Furthermore, for the nonlinear systems, a set around nominal trajectories, in which trajectories of the uncertain system have to lie, is difficult to find too.

To overcome these difficulties (Mayne and Kerrigan (2007)) propose using two model predictive controllers. One with the tightened constraints for the generation of a central path, referred to as the nominal controller, and the other to drive the uncertain trajectories towards nominal trajectories, referred to as the ancillary controller. The next section presents the notations and preliminaries. Section 4.2 presents the nominal NMPC whereas section 4.3 presents the ancillary NMPC. The implementation algorithm is presented in section 4.4 and section 4.5 presents the robustness analysis and some important results. A method of choosing suitable tightened constraints is outlined in section 4.6 and the implementation results of the example system are presented in section 4.7. Section 4.8 concludes the chapter.

4.1 Preliminaries

The aim of this controller is that of steering the state of a constrained discrete-time nonlinear system to origin under the presence of bounded additive disturbances in the control loop. The problem is commonly known as the robust regulation of the states to the origin. The dynamics of the system to be controlled is given by a differential equation as,

$$x^{+} = f(x,u) + w, \qquad x(0) = x_0 \tag{4.1}$$

Or equivalently

$$x^{+} = \tilde{f}(x,u,w), \qquad x(0) = x_0 \tag{4.2}$$

where it is assumed that the additive disturbance term w lies in a compact set $\mathbb{W}$ and the origin is contained in its interior. Moreover, the states and control of the system (4.1) are required to satisfy following constraints at all time,

$$x \in \mathbb{X} \subset \mathbb{R}^n, \qquad u \in \mathbb{U} \subset \mathbb{R}^m \tag{4.3}$$

In the sequel, the solution of the system in equation (4.1) is denoted by $\varphi(i, x, \mu, w)$ at time i when the initial condition is x, the control is generated by the policy μ and the disturbance sequence is given as $w = \{w(0), w(1), \dots\}$. Similarly, if the control is generated by a time invariant control law $k(.)$, then the solution of system in equation (4.1) at initial condition x is denoted by $\varphi(i, x, k, w)$.

The nominal system is obtained by leaving the disturbance w out of the system in equation (4.1) hence is given by a differential equation as,

$$z^+ = f(z, v), \qquad z(0) = z_0 \tag{4.4}$$

For the nominal system, a bar is placed on the notations to indicate the nominal parameters. Hence the solution of the nominal system in equation (4.4) is denoted by $\bar{\varphi}(i, z, \mathrm{v})$ where $v \coloneqq \{v(0), v(1), \dots, v(N-1)\}$ is the nominal control sequence obtained by solving the conventional NMPC problem using the nominal model. The deviation between the states of the two system (Nominal (4.4) and Uncertain (4.1)) is given as,

$$e^+ = f(x, u) - f(z, v) + w \tag{4.5}$$

4.2 Nominal Controller – The Central Path

The trajectory of the predicted solutions of the nominal model is used as the central path towards which the ancillary controller will try to steer the states. This nominal trajectory is kept at some distance from the constraints boundary, by using tightened constraints, so that ancillary controller is able to satisfy constraints under the disturbances. The central path is generated using conventional NMPC for the nominal model as described in chapter 2. Following the same ideas, the NMPC scheme is presented again with new notation as suited for the robust setting. The cost function used for the generation of central trajectories is given as,

$$\bar{J}_N(z_0, \mathrm{v}) \coloneqq \sum_{k=0}^{N-1} l(z_v(k, z_0), v(k)) + F(z_u(N, z_0)) \tag{4.6}$$

in which $z_v(k) = \bar{\varphi}(i, z_0, \mathrm{v})$, the solution of the nominal system (4.4) with z_0 as initial state. The nominal stage cost function $l(.)$ is defined as,

$$l(z(k), v(k)) \triangleq \frac{1}{2}\left(\| z(k) \|_Q^2 + \| v(k) \|_R^2\right) \tag{4.7}$$

where $\|.\|$ represents the Euclidean norm and $Q > 0$ *and* $R > 0$ are symmetric positive definite weighting matrices and can be used as tuning parameters for improving the performance of the nominal closed-loop system. The nominal optimal control problem has to satisfy the following tightened constraints,

$$z \in \mathbb{Z} \subset \mathbb{X} \quad and \quad v \in \mathbb{V} \subset \mathbb{U} \tag{4.8}$$

Equation (4.8) indicates constraint tightening. The choice of the tightened constraints in the case of linear systems is quite deterministic as there exists a set around nominal trajectory in which the uncertain states are to be contained. In the nonlinear case it is not easy to bound the deviation $e = x - z$. It is assumed that both these tightened constraints sets are compact. The method of tightening the constraints is not particularly important in this NMPC scheme and they can be tightened by any method available in the literature. Section 4.6 briefly overviews the choice of $\mathbb{Z}\, and\, \mathbb{V}$. The terminal cost $F(.)$ and the terminal constraint set $Z_0 \subseteq \mathbb{X}$ can be chosen from the method as described in the section 2.2 and can be tightened as described in section 4.6.

The set of admissible sequences for a given current nominal state 'z' is compact, bounded because of assumptions on $\mathbb{V}$ and closed because $\bar{\varphi}(i, z, \mathrm{v})$ is continuous. It can be defined as,

$$\mathbb{V}_{Z_0}^N(z_0) \triangleq \left\{ \begin{array}{c} v \mid v(k) \in \mathbb{V}, \\ and\ z_u(k, z_0) \in \mathbb{Z}\ \forall k\ \in \{0, \dots, N-1\}, \\ z_u(N, z_0) \in \mathbb{Z}_0 \end{array} \right\} \tag{4.9}$$

The nominal optimal control problem can now be given as,

Algorithm 4.1

At each sampling instant t_n, n=0, 1, 2…:

(1) Measure the state $z(k) \in \mathbb{Z}$ of the system.
(2) Set $z_0 := z(k)$, solve the optimal control problem

$$minimize \qquad \bar{J}_N(z_0, v(.)) := \sum_{k=0}^{N-1} l(z_u(k, x_0), v(k)) + F(z_u(N, z_0)) \tag{4.10}$$

$$such\ that \qquad v(.) \in \mathbb{V}_{Z_0}^N(z_0), \tag{4.11}$$

$$subject\ to \qquad z_u(0, z_0) = z_0, \qquad z_u(k+1, z_0) = f\big(z_u(k, z_0), v(k)\big)$$

And denote the obtained optimal control sequence by $\mathrm{v}^0 \in \mathbb{V}_{Z_0}^N(z_0)$.

(3) Define the NMPC-feedback value

$$\bar{\mu}_N(z(k)) := \mathrm{v}^0(0) \in \mathbb{V} \tag{4.12}$$

and use this control value in the next sampling period.

The corresponding optimal value function is given by

$$\bar{V}_N(z_0) := \min_{v(.) \in \mathbb{V}_{Z_0}^N(z_0)} J_N(z_0, v(.)) \tag{4.13}$$

The solution of the optimal problem in (4.10) exists (which is assumed to be unique) if the initial state 'z' is feasible because the value function $\bar{V}_N(.)$ is continuous and the set of admissible sequences $\mathbb{V}_{Z_0}^N(z_0)$ is compact. Hence, the set of feasible states, the region of attraction for the closed-loop system, can be defined as,

$$\mathbb{Z}_N \triangleq \{z_0 \mid \mathbb{V}_{Z_0}^N(z_0) \neq \emptyset\} \qquad (4.14)$$

The optimal control sequence obtained from the nominal optimal control problem is given as,

$$\mathrm{v}^0(z) = \{v^0(0;z), v^0(1;z), \dots, v^0(N-1;z)\} \qquad (4.15)$$

And the resulting optimal state trajectory is given as,

$$\mathrm{z}^0(z) = \{z^0(0;z), z^0(1;z), \dots, z^0(N-1;z)\} \qquad (4.16)$$

The nominal closed-loop system resulting from algorithm 4.1 with feedback law $\bar{\mu}_N(z(k)) := v^0(0;z)$ is then given as,

$$z^+ = f(z, \bar{\mu}_N(z(k))), \qquad z(0) = z_0 \qquad (4.17)$$

The nominal closed-loop trajectories, the central path, which are to be used in the ancillary control problem can now be given by,

$$\mathrm{z}^*(z) = \{z^*(0;z), z^*(1;z), \dots, z^*(N-1;z)\} \qquad (4.18)$$

And the associated nominal control sequence is given by,

$$\mathrm{v}^*(z) = \{v^*(0;z), v^*(1;z), \dots, v^*(N-1;z)\} \qquad (4.19)$$

where the sequence (4.18) is generated by the iteration of the closed-loop system given by equation (4.17) such that,

$$z^*(i;z) = \bar{\varphi}(i, z, \bar{\mu}_N(z(k))), and\ v^*(i;z) = \bar{\mu}_N(z^*(i;z))\ \forall i = 0,1, \dots N \qquad (4.20)$$

Moreover, If the region of attraction $\mathbb{Z}_N$ is bounded, and the terminal constraint set and the terminal cost are obtained to satisfy the usual stability axioms as described in section 2.2 then we can state the existence of a lyapunov function. That is, there exits positive constants $c_1 > 0\ and\ \bar{c}_2 > c_1$ such that the optimal value function satisfies the following inequalities,

$$\bar{V}_N^0(z) \geq c_1|z|^2 \qquad (4.21)$$

$$\bar{V}_N^0(z) \leq \bar{c}_2|z|^2 \qquad (4.22)$$

$$\Delta\bar{V}_N^0(z) \leq -c_1|z|^2 \qquad (4.23)$$

for all $z \in \mathbb{Z}_N$, where $\Delta\bar{V}_N^0(z)$ is defined as,

$$\Delta\bar{V}_N^0(z) := \bar{V}_N^0(f(z, \bar{\mu}_N(z))) - \bar{V}_N^0(z) \qquad (4.24)$$

For the closed-loop system in equation (4.17), inequalities (4.20)-(4.22) imply that the closed-loop system is exponentially fast stable with the region of attraction given by $\mathbb{Z}_N$.

4.3 The Ancillary Controller

The basic aim of the ancillary controller is to bound the deviation of the states of the uncertain system $x^+ = f(x, u)$ and the nominal closed-loop system $z^+ = f(z, \bar{\mu}_N(z))$ in the presence of uncertainties. In the case of linear systems, a stabilizing state feedback gain is used for this purpose in addition to the conventional MPC i.e. $u = v + K(x - z)$ where MPC controller achieves the control objectives using tightened constraints and the feedback gain does the disturbance rejection by bounding the deviation between uncertain states and the nominal states in a pre-computed robust invariant set. Reader is referred to section 3.4.3 of the book (Mayne and Rawlings (2009)) for details on the tube-based MPC for the constrained linear systems. In the case of nonlinear systems, the ancillary controller replaces this control law by a nonlinear model predictive controller whose control problem is to minimize a cost function penalizing the deviation between the states of the deterministic system $x^+ = f(x, u)$ and the nominal closed-loop system $z^+ = f(z, \bar{\mu}_N(z))$. That is, we leave out the disturbance term w from the uncertain system and count on the resulting stabilizing controller to restrict the deviation between the states of $x^+ = f(x, u) + w$ and $z^+ = f(z, \bar{\mu}_N(z))$.

Hence the following composite system is considered for the design of ancillary controller,

$$x^+ = f(x, u) \tag{4.25}$$
$$z^+ = f(z, \bar{\mu}_N(z)) \tag{4.26}$$

The cost function that is used for penalizing the deviation between the trajectories of the two systems is defined as,

$$J_N(x, z, \mathrm{u}) \triangleq \sum_{i=0}^{N-1} l(x(i) - z^*(i; z), u(i) - v^*(i; z)) \tag{4.27}$$

where, $x(i)$ is the solution of system (4.24), $u(i)$ is minimizing control and $v^*(i; z)\ and\ z^*(i; z)$ are obtained from equations (4.17) and (4.18).

The deterministic nature of the model predictive controller allows one to skip optimization of the nominal system over the whole prediction horizon at each sample time. Hence once the sequences in (4.18) and (4.19) are initialized for the prediction horizon, at every step one nominal MPC optimization is sufficient and can be added to the sequences (4.18) and (4.19) to be used in (4.27). Moreover, for the simplicity, this thesis uses the same definition of the stage cost function $l(.)$ for both nominal (4.7) and ancillary (4.27) cost functions. However, it is not necessary to use the same cost function for both the problems. In fact, the cost function in ancillary controller can be more aggressive than the nominal one if the disturbance rejection is more prior than the control objectives.

Furthermore, for the ancillary control problem, state and control constraints need not be imposed as they are implicitly imposed from the nominal NMPC. Hence the ancillary optimal control problem is to minimize the cost function $J_N(x, z, \mathrm{u})$ in equation (4.27) with respect to u subject to control constraints and only one state constraint and that is the terminal equality constraint $x_u(N, x_0) \in z^*(N; z)$ which ensures the stability of the control problem. The set of admissible sequences for the ancillary control problem can now be defined as,

$$\mathbb{U}_{\mathbb{X}_0}^N(x_0) \triangleq \{u \mid x_u(N, x_0) \in \mathbb{X}_0\} \tag{4.28}$$

where the terminal constraint set $\mathbb{X}_0$ is a single state constraint set, i.e., $\mathbb{X}_0 = \{z^*(N; z)\}$.

The Ancillary optimal control problem can now be defined as,

$$\mathit{minimize} \qquad J_N\big(x, z, u(.)\big) \tag{4.29}$$

$$\mathit{such\ that} \qquad u(.) \in \mathbb{U}_{\mathbb{X}_0}^N(x_0), \tag{4.30}$$

$$\mathit{subject\ to} \qquad x_u(0, x_0) = x_0, \qquad x_u(k+1, x_0) = f(x_u(k, x_0), u(k))$$

The domain of the optimal value function, i.e., the region of attraction of the closed-loop system is dependent on current nominal state 'z' and is given as,

$$\mathbb{X}_N(z) \triangleq \left\{x_0 \mid \mathbb{U}_{\mathbb{X}_0}^N(x_0) \neq \emptyset\right\} \tag{4.31}$$

For each nominal state 'z', the set of feasible uncertain states is bounded. The region of attraction of the composite system is bounded and is defined as,

$$\mathcal{M}_N \coloneqq \{(x, z) \mid z \in \mathbb{Z}_N, \qquad x \in \mathbb{X}_N(z)\} \tag{4.32}$$

That is, the feasibility of the composite system implies that the nominal system can satisfy all the system constraints and the ancillary control can drive the states of the deterministic system to the nominal trajectory. As in the nominal case, if the initial state is feasible, the model predictive scheme ensures feasibility of all the subsequent states.

For any $(x, z) \in \mathcal{M}_N$, the minimizing control sequence obtained by the ancillary control problem is denoted by,

$$\mathrm{u}^0(x, z) = \{u^0(0; x, z), u^0(1; x, z), \dots, u^0(N-1; x, z)\} \tag{4.33}$$

And the resulting optimal state trajectory is given as,

$$\mathrm{x}^0(z) = \{x^0(0; x, z), x^0(1; x, z), \dots, x^0(N-1; x, z)\} \tag{4.34}$$

The optimal control input that is applied to the uncertain system then is given by,

$$\mu_N(x, z) = u^0(0; x, z) \tag{4.35}$$

The resulting closed-loop composite system is then given by,

$$x^+ = f\big(x, \mu_N(x, z)\big) + w \tag{4.36}$$

$$z^+ = f(z, \bar{\mu}_N(z)) \tag{4.37}$$

When $x = z$ then it can be verified that $V_N^0(x, z) = 0, and$

$$u^0(i; x, z) = v^*(i; z) \qquad \forall i = 0,1, \dots, N-1$$

Hence the trajectories of the two systems will essentially be same, i.e., $\mu_N(z, z) = \bar{\mu}_N(z)$. This shows that initially when the trajectories of the two systems will be far apart, the ancillary controller will steer the uncertain states towards the nominal trajectories and as the uncertain trajectories tend to come closer to the nominal trajectories, the ancillary controller tends to produce the same control action as the nominal one. Hence once the trajectories are identical, the controller outputs will be identical too.

4.4 The Implementation Algorithm

For now, assume that these sets have been determined. The implementation algorithm for the robust tube-based NMPC can now be states.

Initialization:

1. $k = 0, x = x(0), z = x$
2. Obtain the sequences $z^*(z)$ *and* $v^*(z)$ by solving the nominal optimal control problem as in algorithm 4.1 over the prediction horizon i.e., for 'N' steps.
3. $u = \bar{\mu}_N(z) = v^*(0; z)$

Step 1: Computation of the Control Action

Solve the ancillary control problem described in (4.29) subject to constraints (4.30) to obtain $u = \mu_N(x, z)$ as defined in (4.35).

Step 2: Apply the Control Action

Apply the control action obtained in step 1 to the uncertain system (4.1).

Step 3: Obtain Measurement

Measure the current state of the uncertain system($x^+ = f(x, u) + w$). And set $x = x^+$.

Step 4: Update predicted variables

1. Obtain $v^* = \bar{\mu}_N(z^*(N))$ and $z^* = f(z^*(N), v^*)$ by solving the nominal optimal control problem once as described in algorithm 4.1.
2. $z = z^*(1)$
3. $z^*(z) = \{z^*(1), \dots, z^*(N), z^*\}$
4. $v^*(z) = \{v^*(1), \dots, v^*(N-1), v^*\}$

Step 5: Update Time

$k = k + 1$

Step 6: Repeat the procedure

Goto Step 1.

4.5 A Tube for the Nonlinear Closed-Loop System

Under some controllability assumptions, the value function for the composite system $V_N^0(x,z)$ possesses the similar properties to that of the nominal case $V_N^0(z)$ except that the bounds are K_∞ functions of the deviation $x-z$ instead of the uncertain states x. That is,

$$\bar{V}_N^0(x,z) \geq c_1|x-z|^2 \tag{4.38}$$
$$\bar{V}_N^0(x,z) \leq \bar{c}_2|x-z|^2 \tag{4.39}$$
$$\Delta\bar{V}_N^0(x,z) \leq -c_1|x-z|^2 \tag{4.40}$$

for all $(x,z) \in \mathcal{M}_N$, where $\Delta\bar{V}_N^0(x,z)$ is defined as,

$$\Delta\bar{V}_N^0(x,z) := \bar{V}_N^0\left(f\left(x, \mu_N(x,z)\right), f(z, \bar{\mu}_N(z))\right) - \bar{V}_N^0(x,z) \tag{4.41}$$

Note that in (4.41), the successor states from the deterministic system are used rather than the uncertain system. From (4.38),(4.39) and (4.40), one can easily obtain,

$$\bar{V}_N^0\left(f\left(x, \mu_N(x,z)\right), f(z, \bar{\mu}_N(z))\right) \leq \gamma\bar{V}_N^0(x,z) \tag{4.42}$$

where $\gamma := 1 - \frac{c_1}{c_2} \in (0,1)$. Hence the origin of the composite deterministic system is exponentially stable with region of attraction given by $\mathcal{M}_N$. The property of the composite lyapunov function stated above is sufficient to imply that the deviation $x-z$ is bounded. This also paves a way to determine the tightened constraint sets which is the topic of section 4.6.

The value function $\bar{V}_N^0(x,z)$ stemming from the ancillary control problem for the composite system (4.25), (4.26) has some continuity and differentiable properties on which the main robust stability result of the tube-based NMPC is based. The properties are summarized as follows.

Proposition 4.2 (Mayne and Kerrigan (2007))

There exists $\forall z \in \mathbb{Z}_N,\ \varepsilon(z) > 0$ such that,

i. The optimal value function of the composite system $x \mapsto V_N^0(x,z)$ is continuously differentiable hence implies that it is lipschitz continuous in the vicinity of the nominal state z i.e., $\{z\} \oplus \varepsilon(z)\mathfrak{B}, where\ \mathfrak{B}$ is a unit ball in $\mathbb{R}^n$.
ii. The function $V_N^0(.)$ is continuous at $(x,0)\ \forall x \in \varepsilon(0)\mathfrak{B}$.

In the case of linear systems, the trajectories of the uncertain system under the robust MPC control law lie in a pre-computed constant set around the nominal trajectory i.e., $x(k) \in \{z(k)\} \oplus S$. In the case of non-linear systems, however, no such constant pre-computed set is available. Instead, the uncertain trajectories lie, for some $d > 0$, in the set $S_d(z(k))$ where the set-valued $S_d(.)$ is defined $\forall z \in \mathbb{Z}_N$ as,

$$S_d(z) \triangleq \{x \mid \bar{V}_N^0(x,z) \leq d\} \quad (4.43)$$

The set $S_d(z)$ is the sublevel set of the optimal value function $x \mapsto V_N^0(x,z)$ and is a neighborhood of z since $S_d(0) = \{z\}$. The set $S_d(z)$, which depends on the value of z, replaces the set $\{z(k)\} \oplus S$ from the case of linear tube based MPC and hence constitutes the tube in which the uncertain trajectories have to lie. Under certain continuity and differentiability assumptions on the optimal value function $V_N^0(x,z)$, following proposition states the result on existence of such a tube.

Proposition 4.3 (Mayne and Kerrigan (2007))

There exists a $d > 0$ such that if the states of the composite system (4.24),(4.25) lie in the composite region of attraction $\mathcal{M}_N$, i.e., if $(x,z) \in \mathcal{M}_N$ and $x \in S_d(z)$ then the successor state of the uncertain system to be controlled satisfies $x^+ \in S_d(z^+)$ i.e.

$$x^+ = f(x, \mu_N(x,z)) + w \in S_d\left(z^+ = f(z, \bar{\mu}_N(z))\right) \quad (4.44)$$

$$\forall w \; satisfying \; |w| \leq \frac{(1-d)\gamma}{k(z)} \quad (4.45)$$

where $k(z)$ is the local lipschitz constant of the optimal value function $x \mapsto V_N^0(x,z)$.

If the disturbance term $w \in \mathbb{W}$ implies $|w| \leq \frac{(1-d)\gamma}{k}$ where k is the upper bound on $k(z)$ then every solution of the uncertain system (4.36) lies in a tube of sets given as,

$$S = \{S_d(z), S_d(z^*(1;z)), \dots\}, \forall w(k) \in \mathbb{W} \;\; \forall k = 0,1,2, \dots \quad (4.46)$$

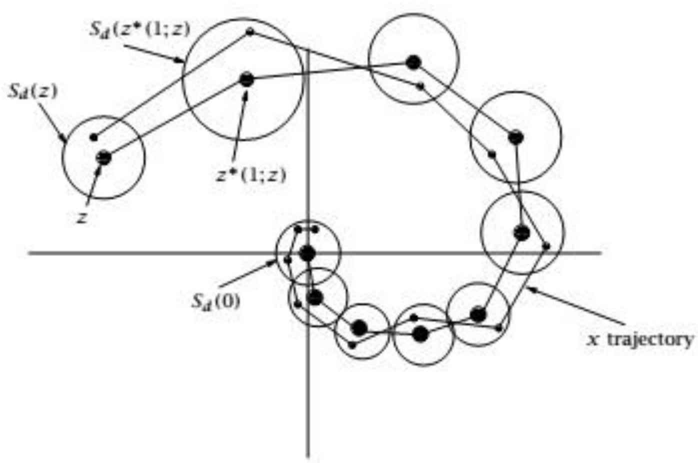

Figure 4.1: Tube for Nonlinear Model Predictive Control

The property of the closed-loop controller stated in proposition 4.3 implies that if initially the problem is feasible, it stays feasible for the rest of time because of the robust invariance of the sets constituting the tube. Figure 4.1 illustrates the result of the proposition and the fact that the sets constituting the tube depend on the value of the nominal closed-loop system states predicted from the system (4.26).

4.6 Choosing the Tightened Constraint sets $\mathbb{Z}$ and $\mathbb{V}$

The main objective of the robust controller is to steer the state trajectories of the uncertain system close to the origin and keep them there while satisfying all the state and control constraints. In the cases where constraints satisfaction is not a priority, the use of NMPC may not be advantageous at all and other options might give useful results. Therefore tightening the constraints sets for the nominal controller is of paramount importance as this allows the margin for ancillary controller for the constraints violation.

Although tightening the constraints is important for the tube-based NMPC scheme presented in the previous sections, the method with which it is achieved is not important at all. The constraints can tightened offline (Grimm et al. 2003) or online (Ma and Braatz (2001)), (Nagy and Braatz (2003)) and (Diehl et al. (2006)).

One simple way to choose the tightened constraints sets is proposed in (Mayne and Kerrigan (2007)) when the constraint sets are polyhedral which often is the case even for the most nonlinear systems. One might choose the tightened constraint sets as $\mathbb{Z} = \alpha\mathbb{X}\ and\ \mathbb{V} = \beta\mathbb{U}\ where\ \alpha, \beta \in (0,1)$ by merely scaling the inequalities. That is, if $\mathbb{X} = \{x \mid Ax \leq a\}$ then $\mathbb{Z} = \{x \mid Ax \leq \alpha a\}$. They can be used as tuning parameters and suitable values of $\alpha\ and\ \beta$ may be chosen by hit and trial method in a Monte Carlo simulation. If in the simulation, the constraints are being violated, the values of α, β may be reduced. Similarly the values of α, β may be increased if the constraints are being too conservative. That is, one might have to choose a trade-off between constraint violation and the degree of robustness associated with each value of α, β.

If the constraint sets are not polyhedral, even then the tightened constraints sets, at least theoretically, can be computed as shown in section 3.6.5 of the book (Mayne and Rawlingss (2009)).

4.7 Example – System of Three Masses

The system of oscillating masses (figure 2.1) was shown to be unstable under Nominal NMPC in chapter 3. Figure 4.2 shows that the control actions obtained from the solution of tube-based robust optimal control problem asymptotically stabilizes the uncertain system in presence of additive disturbances to origin. The oscillating masses are taken to be $m = 1\ Kg$, the damping constants are $d = 0.1\ N\ s/m$ and the spring constants are taken as $k = 0.9\ N/m$. The stiffness coefficient of the spring is taken as $\beta = 0.5$. The position displacements of the masses are constrained in $\pm 4m$ whereas the velocities of the masses are not constrained at all. The actuation signals are constrained to lie with in $\pm 1N$. The disturbance term is bounded as $|w| \leq 0.3$. The controller is simulated in a matlab code file. The matlab's built-in function *fmincon* is used for the constrained optimization. For the offline computation of terminal sets, the toolbox YALMIP is used.

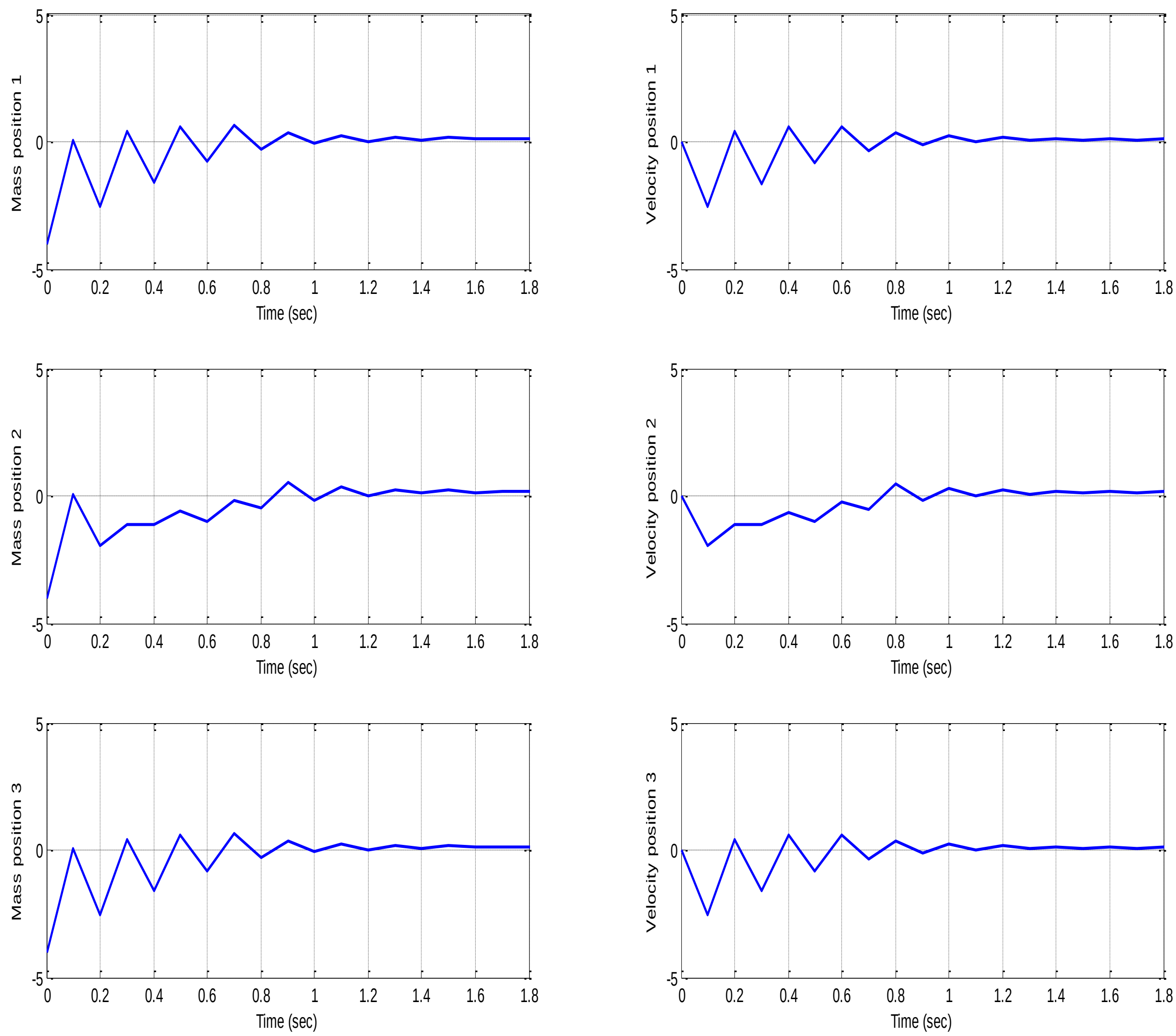

Figure 4.2: Stabilization of the uncertain system to Origin

4.8 Conclusion

This chapter presented a method of robust model predictive control for the nonlinear system with bounded additive disturbances as proposed by (Mayne and Kerrigan (2007). It uses two model predictive controllers at each sampling instant. One, called the nominal controller, uses the nominal model and solves the conventional NMPC problem with tightened constraints on states and inputs. This conventional NMPC generates a central path, say the ideal optimal trajectory. The other, called the ancillary controller, then aims to steer the states of the uncertain system towards that ideal optimal nominal trajectory keeping, in process, the uncertain trajectories in a tube centered on the nominal trajectories. The ancillary controller also uses the same mathematical model(i.e. it doesn't incorporate the uncertain term except that it takes its initial condition from the live measurement of the states of the uncertain system to be controlled and its reference trajectory is generated by a nominal NMPC rather than being origin.

Chapter 5
REAL-TIME ROBUST NONLINEAR MODEL PREDICTIVE CONTROLLER

In practical applications, the biggest limitation of an NMPC scheme has been the ability to solve the optimal control problem within a sampling period which in fast sampled systems is difficult to achieve for many systems. Advancement in the field of processors and algorithms has surely helped control engineers to use NMPC in faster system as well. But even if sufficient hardware to solve the problem to optimality is available, it still proves to be costly to be used in certain situations.

This chapter presents the main work of this thesis which is to develop a real-time NMPC scheme which ensures all the constraints satisfaction in a real-time environment with theoretical stability and feasibility guarantees. The organization of the chapter is as follows: Section 5.1 introduces the problem of real-time constraint to be addressed. Section 5.2 presents the real-time robust nonlinear model predictive control scheme and section 5.3 states the stability and feasibility proofs of the closed-loop system. Section 5.4 presents the implementation results for the mass-spring example system.

5.1 Problem Statement

Faster sampled systems impose a hard real-time constraint on the NMPC problem in that the computation time available for the solution of the optimal control problem is limited. Due to this constraint, the computation of optimal solution at each sampling instant is often not feasible. This can result in both loss of stability and feasibility of the closed-loop system. The fast NMPC schemes proposed in the literature push the sampling times used in the NMPC to the ranges of microseconds but the special structure and sparsity of the problem these methods exploit sacrifice the stability and feasibility proofs associated with the standard NMPC schemes. Hence a suboptimal solution which can still guarantee stability and feasibility in the presence of disturbance is the only natural solution. Hence a suboptimal solution which can guarantee stability and feasibility.

A common method which fastens the optimization procedure greatly is the use of warm-start procedure in which the solution of previous step is used as initial point for the optimization in the current step. Algorithm 5.1 describes the real-time NMPC scheme based on a warm-start procedure commonly applied in practice where the initial feasible solution at current states $x(k)$ is taken from the solution computed for the previous measured state $x(k-1)$ as,

$$u_{ws}\big(x(k)\big) = \big\{u_1\big(x(k-1)\big), \dots, u_{N-1}\big(x(k-1)\big), \mu(\bar{\varphi}(N, x(k-1), \boldsymbol{u}(x(k-1)))\big\} \tag{5.1}$$

where μ is given by the local control law obtained for terminal region set.

Algorithm 5.1

Input:

1. Feasible control sequence $\mathbf{u}\big(x(k-1)\big)$
2. Current state measurement $x(k)$
3. Local control law $\mu(x)$

Output:

The $\tau - RT$ control sequence $\mathbf{u}^{\tau}(x(k))$

Algorithm

1. **Warm-start**
   $\mathbf{u}_{ws}\big(x(k)\big) = \big\{u_1\big(x(k-1)\big), \dots, u_{N-1}\big(x(k-1)\big), \mu\big(\bar{\varphi}(N, x(k-1), \mathbf{u}(x(k-1)))\big)\big\}$
2. $\tilde{\mathbf{u}} = \mathbf{u}_{ws}(x(k))$
3. $\boldsymbol{while}\ t < \tau$
   Solve optimal control problem to update $\tilde{\mathbf{u}}$
4. **end while**
5. $\mathbf{u}^{\tau}\big(x(k)\big) = \tilde{\mathbf{u}}$

At every sampling instant, then this warm-start solution is used as the initial point of the optimization and is updated iteratively for a fixed sampling period. At the end of that fixed time, the available solution is used. It is well known, that if the uncertainties were not present in the system, then the scheme presented in algorithm 5.1 stabilizes the closed-loop system even if it is not able to execute any step in the sampling period at any sampling instant. This is because of the terminal region constraint which guarantees recursive feasibility at any sampling instant provided that the problem is feasible at time t=0. Since the warm-start solution (5.1) is renders the problem feasible initially, the nominal cost function can be approximated as the infinite horizon cost which is well known to stabilize the system. See the book (Mayne and Rawlingss (2009)).

Since the open-loop solution of the optimization problem stabilizes the nominal system to origin one would expect the same property of the warm-start when the optimization problem is solved for some fixed time without reaching the optimal solution. That is, at each iteration, one expects the warm-start based optimization procedure to maintain the recursive feasibility of the closed-loop system and ensure that the cost function, which is used as lyapunov function for the closed-loop system, decreases. This, in fact, is not the case with interior point algorithms. Hence the solution provided by the algorithm 5.1 using interior point method for the optimization does not provide the stability or feasibility guarantees. This is due to the fact that the barrier interior point method uses a barrier penalty function as the replacement of the inequality constraints in the cost function. At each iteration then, the augmented cost is minimized rather than actual NMPC cost. Hence if the complete optimal solution is not obtained, there may be possibility that the augmented cost got decreased but the actual MPC cost increased. In that case, the cost function cannot be used as the lyapunov function of the closed-loop system hence the proof associated with the NMPC scheme presented in chapter 2 fails. Furthermore, if the barrier parameter is taken to zero, the solution of the optimization problem with the augmented cost with barrier term only approaches the solution of the original problem in the limit. Hence if the optimization algorithm is then terminated early, it will result in a steady-state offset as it will converge to a steady-state minimizing the augmented cost rather than the actual NMPC cost. Interior point algorithms are, however, required to efficiently solve the optimization problem because of their ability to solve the quadratically constrained Quadratic Programs efficiently as compared to active set methods.

Moreover, as described in chapter 2, the terminal stabilizing constraint ensures that once problem is feasible it will always stay feasible because at the end of the prediction horizon, states have reached an invariant set. If, due to the early termination of the algorithm, a solution satisfying terminal constraint is not found, the initial state at next sampling instant may be infeasible and again the proof associated with the scheme in chapter 2 fails. The uncertain system with additive disturbances was shown to be stabilized to the origin by tube-based NMPC controller in chapter 4. For the solution obtained for figure 4.2, maximum number of iterations in *fmincon* was set to be 100. To illustrate the problem, the maximum number of iterations was reduced to 6 (a random number just to illustrate the point) which corresponds to reducing the time available for the computation of the optimal solution. Figure 5.1 shows that with less number of iterations allowed, the same tube-based nonlinear model predictive controller from chapter 4 no longer stabilizes the uncertain states to the origin.

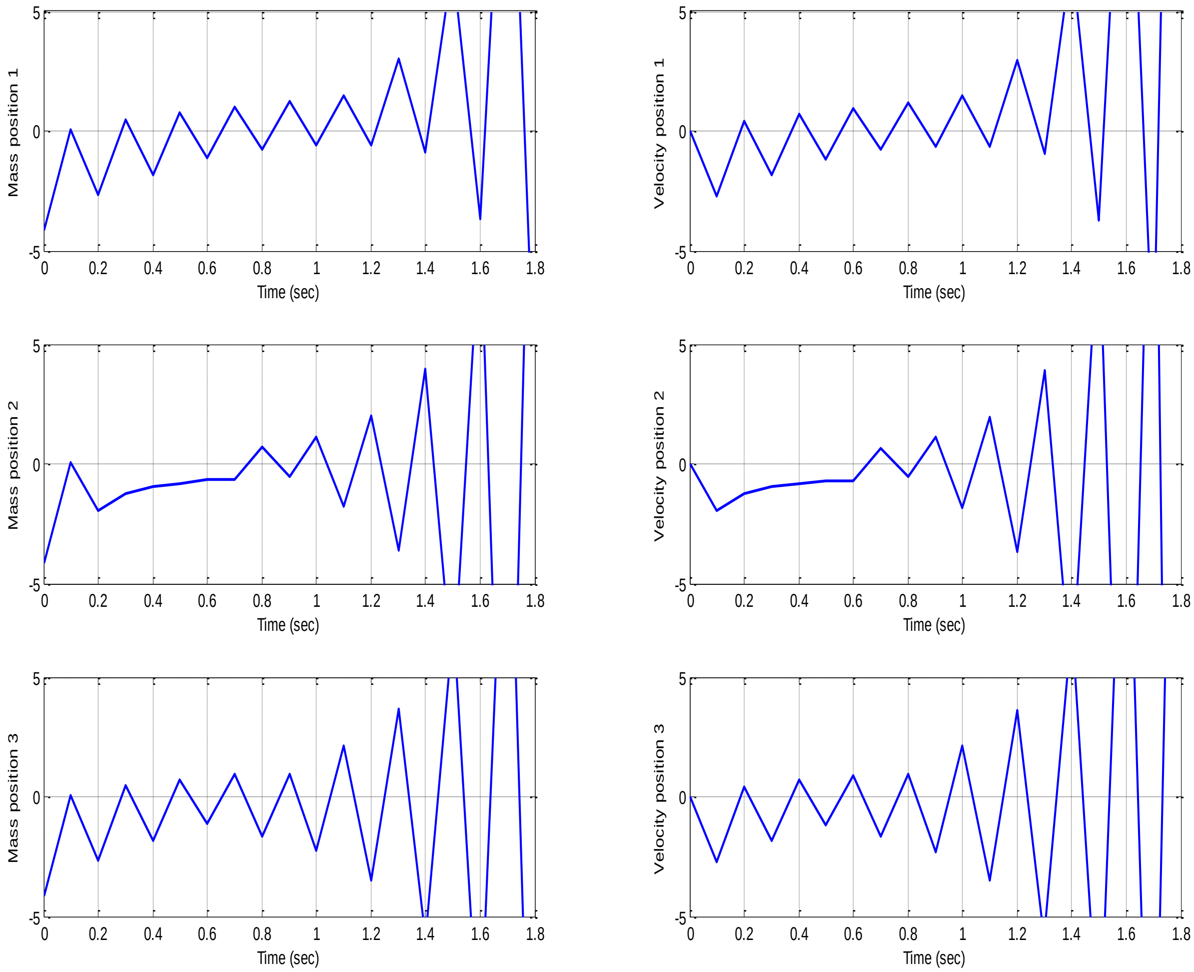

Figure 5.1: Destabilization of the uncertain states under real-time constraint

In addition, it is well known that maintenance of feasibility is of paramount importance in the proof of standard NMPC scheme. In presence of uncertainties, the solution provided by the warm-start procedure in (5.1) doesn't guarantee recursive feasibility at each sampling instant as the warm-start solution is computed from the open-loop trajectories which under disturbances may well differ from the actual one. Hence this motivates the use of robust NMPC scheme in the real-time as described in chapter 4.

In this thesis, a real-time nonlinear model predictive control scheme, satisfying this hard real-time along with system constraints robustly at all times with little additional computation cost, is developed. The stability and recursive feasibility guarantee for any arbitrarily small sampling time then allows one to make a tradeoff between closed-loop performance and real-time constraint. For the feasibility, the warm-start procedure in

(5.1) is used while the recursive feasibility under uncertainties is guaranteed by the tube-based NMPC approach. Input-to-state stability of the closed-loop system is proved using an additional lyapunov constraint in the NMPC scheme and an algorithm bounding the value function.

5.2 The Real-Time Robust Nonlinear Model Predictive Controller

The basic idea of the proposed scheme is to add an additional lyapunov constraint so that the actual NMPC cost is always decreasing in every iteration of the interior point algorithm. Another problem with the NMPC scheme described in chapter 2 is pertaining to the loss of feasibility under uncertainties in real-time environment. Since the recursive feasibility of the NMPC is guaranteed using the terminal constraint. The warm-start procedure presented in algorithm 4.1 then gives a reasonable solution to asymptotically stabilize the nominal system. Under uncertainties however, the warm-start solution (5.1) does not provide any guarantee on the feasibility. Hence Tube-based Robust NMPC scheme, as described in chapter 4, along with the warm start procedure (5.1) is used for the real-time Robust Nonlinear Model Predictive Control. Since it is difficult to prove the asymptotic stability due to uncertainties, the approach using input-to-state stability is used as often is the case in robust NMPC schemes.

Consider the uncertain discrete-time system as given in equation (4.1) subject to constraints (4.3) with associated nominal system given by (4.4). Let the uncertain term w be bounded in a compact set $\mathbb{W}$ and contain origin in its interior. Let the notations presented in chapter 4 be valid in the form as they are described in the chapter. As described in the previous chapter, the tube-based robust NMPC scheme has two parts. One is the nominal NMPC generating the central path and the other is the ancillary NMPC problem steering the uncertain system states towards the central path.

5.2.1 The Real-Time Nominal Controller

It is assumed that the assumptions required for theorem (2.14) are satisfied. The nominal controller is obtained as described in the section 4.2 of the previous chapter. In the initialization procedure of the real-time robust controller, the Nominal NMPC problem is solved first for ‘N’ number of steps and then solved only once at subsequent sampling instants. The output needed from the nominal control problem to be used in the ancillary control problem are the two sequences, the control action sequence obtained by solving the nominal optimal control problem (4.19) and the corresponding closed-loop predicted state trajectory sequence (4.18).

Since the nominal problem which generates the central path to be followed by the uncertain states doesn’t consider any uncertainties in the system and neither any of its variables depends on the actual uncertain plant parameters, it will always stabilize the nominal states to the origin provided that the nominal problem is feasible initially which will be true if initially the system states lie in the region of attraction of the origin as defined in (4.14). The sequences obtained from (4.18) and (4.19) are then used in the cost function of ancillary problem as described in section (4.3) of the previous chapter.

5.2.2 The Real-Time Ancillary Controller

Consider the uncertain system given in (4.1) subject to the state and control constraints given in (4.3). Let the uncertain term w be bounded in a compact set $\mathbb{W}$. Assume that a nominal controller is designed as described in section 4.2 for the tightened constraints tightened as described in section 4.6. The central path to be followed by uncertain trajectories is given by (4.18) and (4.19).

Let the composite system for the ancillary controller be given by (4.25) and (4.26). The cost function penalizing the deviation of the trajectories of the two system is then given by

$$J_N(x, z, \mathbf{u}) \triangleq \sum_{i=0}^{N-1} l(x(i) - z^*(i; z), u(i) - v^*(i; z)) \tag{5.2}$$

The set of admissible sequences for the real-time ancillary controller is then given by,

$$\mathbb{U}_{\mathbb{X}_0}^N(x_0) \triangleq \{u \mid \ x_u(N, x_0) \in \mathbb{X}_0 \ and \tag{5.3}$$

$$J_N(x, z, \mathbf{u}) + F\big(x_{nom} - \ x_u(0, x_0)\big) \leq \Pi_{prev} \} \tag{5.4}$$

where the terms x_{nom} and Π_{prev} are defined in definition 5.3.

The Ancillary optimal control problem $\mathbb{P}_N^\tau$ can now be defined as,

$$minimize \qquad J_N\big(x, z, u(.)\big) + F(x - x_0) \tag{5.5}$$

$$such\ that \qquad u(.) \in \mathbb{U}_{\mathbb{X}_0}^N(x_0)\ and\ x_0 \in S_d(z_0) \tag{5.6}$$

$$subject\ to \qquad x_u(0, x_0) = x_0, \qquad x_u(k+1, x_0) = f(x_u(k, x_0), u(k))$$

The domain of the optimal value function, i.e., the region of attraction of the closed-loop system is dependent on current nominal state 'z' and is given as,

$$\mathbb{X}_N(z) \triangleq \left\{x_0 \mid \mathbb{U}_{\mathbb{X}_0}^N(x_0) \neq \emptyset\right\} \tag{5.7}$$

For each nominal state 'z', the set of feasible uncertain states is bounded. The region of attraction of the composite system is bounded and is defined as,

$$\mathcal{M}_N \coloneqq \{(x, z) \mid z \in \mathbb{Z}_N, \qquad x \in \ \mathbb{X}_N(z)\} \tag{5.8}$$

For any $(x, z) \in \mathcal{M}_N$, the real-time optimal (not necessarily minimizing) control sequence obtained by the real-time ancillary control problem is denoted by,

$$\mathbf{u}^\tau(x, z) = \{u^\tau(0; x, z), u^\tau(1; x, z), \dots, u^\tau(N-1; x, z)\} \tag{5.9}$$

And the resulting optimal state trajectory is given as,

$$\mathbf{x}^{\tau}(x,z) = \{x^{\tau}(0;x,z), x^{\tau}(1;x,z), \dots, x^{\tau}(N-1;x,z), x^{\tau}(N;x,z)\} \tag{5.10}$$

The optimal control input that is applied to the uncertain system then is given by,

$$\mu_N^{\tau}(x,z) = u^{\tau}(0;x,z) \tag{5.11}$$

The resulting closed-loop composite system is then given by,

$$x^{+} = f\big(x, \mu_N^{\tau}(x,z)\big) + w \tag{5.12}$$

$$z^{+} = f(z, \mu_N(z)) \tag{5.13}$$

Definition 5.2

The τ −RT cost under the suboptimal solution, obtained by solving the optimization problem only for a fixed interval τ seconds, is defined as,

$$V^{\tau}(x,z) \triangleq J_N^{\tau}(x,z,\mathbf{u}) + F(x_0 - x_0^{\tau}(x))$$

Definition 5.3

For each $x(k)$, x_{nom} is taken as the state that would have been obtained by ignoring the disturbance, i.e.,

$$x_{nom} = f(x, \mu_N^{\tau}(x,z))$$

And the lyapunov constraint cost as,

$$\Pi_{prev} = V^{\tau}\big(x(k-1), z(k-1)\big) - \frac{1}{2}\varepsilon\|x(k-1) - z(k-1)\|^2$$

The key to the real-time scheme presented in this chapter is to not solve the problem until an optimal solution is found. Instead, the step of optimal control problem is solved iteratively for only a fixed amount of time τ and at the end of the interval, the sub-optimal solution is taken as the solution of the problem. The algorithm 5.2 presents the algorithm to obtain the solution of Real-time Robust ancillary NMPC problem (5.5).

Algorithm 5.2

Input:

1. Feasible control sequence $\mathbf{u}^{\boldsymbol{\tau}}(x(k-1), z(k-1))$
2. The tube center $x_0^{\tau}(x(k-1), z(k-1))$
3. Corresponding state sequence $\mathbf{x}^{\boldsymbol{\tau}}(x(k-1), z(k-1))$
4. Updated central path

$$\mathbf{z}^{*}(z) = \{z^{*}(0;z), z^{*}(1;z), \dots, z^{*}(N-1;z)\}$$

$$\mathbf{v}^{*}(z) = \{v^{*}(0;z), v^{*}(1;z), \dots, v^{*}(N-1;z)\}$$

5. Current state measurement $x(k)$
6. Local control law $\mu(z)$
7. Parameter $\varepsilon_f > 0$

Output:

The $\tau - RT$ control sequence $\mathbf{u}^\tau(x(k), z(k))$ and the tube center $x_0^\tau(x(k), z(k))$

Algorithm

1. Warm-start
   a. $x_0^{ws}(x(k), z(k)) = x_1^\tau(x(k-1), z(k-1))$
   b. $\mathbf{u}^{ws}(x(k), z(k)) = \{u_1^\tau(x(k-1), z(k-1)), \dots, u_{N-1}^\tau(x(k-1), z(k-1)), \mu(x_N^\tau(x(k-1), z(k-1)))\}$
2. $\tilde{\mathbf{u}} = \mathbf{u}^{ws}(x(k))$
3. $\boldsymbol{while}\ t < \tau$
4. Solve optimal control problem iteratively using interior point algorithm to update $\tilde{\mathbf{u}}, \tilde{x}_0$
5. $\boldsymbol{end\ while}$
6. $\boldsymbol{if}\ \|x(k) - z(k)\|_P \leq \varepsilon_f\ \boldsymbol{and}\ J_N(x, z, \mathbf{u}) + F(x(k) - \tilde{x}_0) > F(x(k) - z(k))\ \boldsymbol{then}$
7. $\tilde{x}_0 = z^*(0; z_0)$, $\tilde{\mathbf{u}} = \left\{\mu\left(x_u^\tau(0, x_0) - z^*(0; z_0)\right), \dots, \mu(x_u^\tau(N-1, x_0) - z^*(N-1, z_0))\right\}$
8. $\boldsymbol{end\ if}$
9. $\mathbf{u}^\tau(x(k), z(k)) = \tilde{\mathbf{u}}\ and\ x_0^\tau(x(k), z(k)) = \tilde{x}_0$

Remark 5.3

Note that the real-time ancillary control problem (5.4) and the real-time state and control sequences $\mathbf{x}^\tau(x, z)\ and\ \mathbf{u}^\tau(x, z)$ not only depend on the current state measurement $x(k)$, as is the case in the nominal NMPC presented in chapter 2 or tube-based NMPC presented in chapter 4, but also on the real-time state and control sequences obtained at the previous sampling time $\mathbf{x}^\tau\left(x(k-1), z(k-1)\right)\ and\ \mathbf{u}^\tau\left(x(k-1), z(k-1)\right)$ and the available computation time τ. Since showing the dependence on all mentioned above will make the notation extremely complex, for the simplicity the real-time variables are denoted with the superscript τ.

The real-time Robust NMPC problem (5.5) differs from the nominal NMPC from chapter 2 in two ways,

- ***Robust NMPC:*** The real-time Robust NMPC proposed in this chapter uses the tube-based robust approach instead of the standard NMPC described in chapter 2.

The approach consists of two nonlinear model predictive controllers. One is the nominal controller, designed following the ideas presented in chapter 2, which generates a central path, say the ideal trajectories, if no disturbances or uncertainties were present. The other one is the ancillary controller whose aim is to maintain the states of the uncertain system in the closed vicinity of the central path. The design of nominal controller for tighter constraints allows the ancillary controller some margin to make the uncertain system satisfy the system and control constraints under uncertainties. The tube-based nonlinear model predictive controller is briefly described in chapter 4 of this thesis. For detailed information on the topic, the interested reader is referred to the book (Mayne and Rawlingss (2009)).

There is a slight difference in the cost function used in the real-time robust NMPC and the tube-based NMPC described in chapter 4. That difference is of augmenting the cost function by $F(x - x_0)$ which trades the amount of control effort used for rejecting the disturbances for the amount of control effort used for steering the states of the uncertain system to nominal trajectories. An advantage which the augmented cost brings is that it can be used as an ISS lyapunov function (See theorem 5.1) whereas the cost function proposed in chapter 4 cannot.

- ***Lyapunov Decrease Constraint:*** The constraint (5.4) ensures that at each iteration of the interior point algorithm, the sub-optimal cost, obtained after optimizing for a fixed interval of τ seconds, satisfies condition (4.38-4.40) which is key if the cost function has to provide the ISS lyapunov function for the closed-loop system. That is it explicitly imposes the same guarantee on stability which is lost if the tube-based NMPC based on the approach presented in chapter 4 is applied in real-time. Hence the feasibility of this lyapunov constraint, provided by the warm-start as proved in the lemma 5.1, recovers the stability properties of the tube-based NMPC in real-time. The constraint represents a convex quadratic constraint hence no significant computational cost is added in finding the feasible solution.

Note that the real-time robust NMPC algorithm 5.2 is slightly different than a standard real-time algorithm 5.1. The difference is the proposed algorithm explicitly bounds the lyapunov function from the above.

- ***Upper Bound on Lyapunov Function:*** In the real-time robust NMPC algorithm 5.2, steps 6-8 ensure that in a set containing origin in its interior, the cost function, which has to be employed as ISS lyapunov function of the closed-loop system, is bounded from above by a class $\mathcal{K}_\infty$ function of the states. This bound in the algorithm along with constraint (5.4) ensures that the cost function is an ISS-lyapunov function of the closed-loop system (See theorem 5.1). Although the motivation for including steps 6-8 in the algorithm stems from the theoretical requirement pertaining to the upper bound of the ISS-lyapunov function, one can see an added advantage associated with it in the set of using the optimal cost $\mathcal{E}_f \triangleq \{x \mid \|x - z\|_P \leq \varepsilon_f\}$ obtained from the linearized model for terminal cost. That is, in the real-time environment, when the uncertain states are close to the nominal if the local control law $\mu(z)$, same as the one used in (2.9), provides more optimal solution than the NMPC control law $\mu_N^\tau(x, z)$ from (5.11), the algorithm applies the optimal control action rather than the suboptimal solution provided that all the constraints are satisfied robustly.

5.3 The Stability of the Real-Time Robust Nonlinear Model Predictive Controller

The Nominal NMPC, as described in chapter 2, is shown to stabilize the nominal system (that is if the uncertainties are not present) to origin with the use of a terminal cost and terminal region constraint (Chen and Allgöwer (2000)). Under uncertainties, a combination of an additional NMPC, the ancillary controller, along with this nominal controller, as described in chapter, is shown to stabilize the uncertain system to origin by (Mayne and Kerrigan (2007)). In the real-time environment, however, if the sampling time is not sufficiently long enough, as shown in section 5.1, the suboptimal solution may not stabilize the system. Hence the real-time Robust NMPC scheme, as described in section 5.2, is proposed. In order to investigate the stability of the closed-loop under the proposed scheme, one may note that the two major changes in the scheme in comparison to the nominal scheme as described in chapter 2 recover the stability properties of the nominal scheme. That is, the tube-based scheme under the uncertainties and the lyapunov constraint in the real-time environment render the proposed real-time scheme stabilizing for the closed-loop system (5.12).

In this section, it will be shown that if the lyapunov constraint is recursively feasible for the problem under the proposed real-time control law, the proposed cost function (5.5) is an ISS-lyapunov function for the closed-loop system. Before stating the result, however, the recursive feasibility of the lyapunov constraint needs to be established. Lemma 5.5 shows that the warm-start solution (5.1) guarantees the ancillary controller satisfies the lyapunov constraint at every sampling instant hence establishing the recursive feasibility of the closed-loop system. Before stating the lemma, an important assumption needs to be made.

Assumption 5.4

Initially prior to starting the real-time control of the closed-loop system, i.e., at time k=0, enough computation time and hardware is available to find a feasible solution of the ancillary optimal control problem without using the lyapunov constraint (5.4) to complete the requirement of initial feasible solution for the optimization problem at the start.

Lemma 5.5

Assume that the sub-optimal control sequence $\mathbf{u}^{\tau}(x(k-1), z(k-1))$ and the tube-center $x_0^{\tau}\big(x(k), z(k)\big)$ computed at previous time represent a feasible solution of the ancillary optimal control problem $\mathbb{P}_N^{\tau}(x(k-1))$ defined in (5.5). Then the warm-start presented in algorithm 5.2 is a feasible solution for the problem $\mathbb{P}_N^{\tau}(x(k))$ where $x(k) \in f\big(x, \mu_N(x,z)\big) \oplus \mathbb{W}$ i.e. $x_0^{ws} \in \mathbb{X}_N(z(k))\ and\ \mathbf{u}^{ws} \in \mathbb{U}_{\mathbb{X}_0}^{N}(x_0^{ws})$.

Proof

The real-time optimal control problem $\mathbb{P}_N^{\tau}$ presented in (5.5) is same as the robust optimal control problem presented in (4.28) except that it needs an additional lyapunov constraint to be satisfied. The recursive feasibility without the lyapunov constraint is therefore proved by (Mayne and Kerrigan (2007)). Hence the proof of recursive feasibility only requires that the warm-start solution used in algorithm 5.2 satisfies the lyapunov constraint (5.4).

Since the uncertain term w is not known instantaneously and only an upper bound on it is known, it cannot be taken into account in the optimal controller. Hence the cost obtained from the suboptimal solution of algorithm 5.2 uses the term x_{nom} and is given as,

$$V(x_{nom}, z_0) = J_N\big(x, z, u(.)\big) + F(x - x_0)$$

For the simplicity of the presentation, following slight modification to the notations is made,

$$x_{nom} = f\big(x, \mu_N^{\tau}(x,z)\big) = f_{x,\mu}(x)$$

It can be be easily shown that the term x_{nom} can also be expressed as,

$$x_{nom} = x_1^{\tau}(x^-) + f_{x,\mu}(\Delta x_0^-); \quad where\ \ \Delta x_0^- = x^- - x_0^{\tau}(x^-)$$

Now assume the controller was not able to find the optimal solution and it resorts to the warm-start solution, then the current sample cost which has to satisfy the lyapunov decrease constraint is given as,

$$V(x_{nom}^{ws}, z_0) = J_N\big(x^{ws}, z^{ws}, u^{ws}(.)\big) + F(x - x_0^{ws})$$

The cost at previous time obtained from the τ −RT solution obtained at previous time is given as,

$$V^{\tau}(x^-, z^-) = J_N^{\tau}\big(x^-, z^-, \mu_N^{\tau}(x^-, z^-)\big) + F(x^- - x_0^{\tau}(x^-))$$

Due to invariance of set $S_d(z)$, defined in (4.42),

$$V^{\tau}(x^-, z^-) \geq \|(x_0^{\tau}(x^-) - z_0^-)\|^2 + \|(x^- - z^-) - (x_0^{\tau}(x^-) - z_0^-)\|^2$$

$$V^{\tau}(x^-, z^-) \geq \|(x_0^{\tau}(x^-) - z_0^-)\|^2 + \|\Delta(x_0 - z_0)\|^2$$

Inside the invariant set $S_d(z)\ \forall z \in \mathbb{Z}_0$,

$$V(x_{nom}^{ws}, z_0) \leq F(x_{nom}^{ws}, z_0)$$

$$V(x_{nom}^{ws}, z_0) - V^{\tau}(x^-, z^-) \leq F(x_{nom}^{ws}, z_0) - \|(x_0^{\tau}(x^-) - z_0^-)\|^2 - \|\Delta(x_0 - z_0)\|^2$$

Which implies that,

$$\begin{aligned} V(x_{nom}^{ws}, z_0) - V^{\tau}(x^-, z^-) \\ &\leq F\left(x_1^{\tau}(x^-), z^*(1; z)\right) + F(f_{x,\mu}(\Delta(x_0^- - z_0^-))) - \|(x_0^{\tau}(x^-) - z_0^-)\|^2 \\ &- \|\Delta(x_0 - z_0)\|^2 \end{aligned}$$

Hence,

$$V(x_{nom}^{ws}, z_0) - V^{\tau}(x^-, z^-) \leq -\|(x_0^{\tau}(x^-) - z_0^-)\|^2 - F(\Delta(x_0^- - z_0^-) - f_{x,\mu}(\Delta(x_0^- - z_0^-))$$

Because of the invariance of the set $S_d(z)\ \forall z \in \mathbb{Z}_0$, $\|x\|_Q^2 \geq \|x\|_P^2 + \|x^-\|_P^2$, therefore,

$$V(x_{nom}^{ws}, z_0) - V^{\tau}(x^-, z^-) \leq -\|(x_0^{\tau}(x^-) - z_0^-)\|_Q^2 - \|\ \Delta(x_0^- - z_0^-)\|_Q^2$$

By using the identity of norms, $\frac{1}{2}\|x + y\|_Q^2 \leq \|x\|_Q^2 + \|y\|_Q^2$, we get,

$$V(x_{nom}^{ws}, z_0) - V^{\tau}(x^-, z^-) \leq -\|(x_0^{\tau}(x^-) - z_0^-) + (x^- - z^-) - (x_0^{\tau}(x^-) - z_0^-)\|_Q^2$$

$$V(x_{nom}^{ws}, z_0) - V^{\tau}(x^-, z^-) \leq -\frac{1}{2}\|(x^- - z^-)\|_Q^2$$

Which proves the result.

Hence, given that initially the optimization problem starts from a feasible solution (Assumption 5.4), recursive feasibility is maintained by using this warm-start solution (5.1) in real-time environment and by the tube-based robust approach under uncertainties and disturbances. Due to the lyapunov constraint (5.4), this feasibility implies stability of the closed-loop system. Using Assumption 5.4 and Lemma 5.5, input-to-state stability of the closed-loop system under $\tau -$ RT control law (5.11) obtained from the problem $\mathbb{P}_N^{\tau}$ (5.5) can be proved.

Theorem 5.6

Assume that a nominal NMPC, satisfying all assumptions for theorem 2.14, with terminal cost and terminal constraints set determined as in algorithm 2.7 generates a central trajectory as given in (4.18) and (4.19) satisfying (4.21)-(4.23). The uncertain closed-loop system (5.12) under the $\tau -$ RT control law obtained from algorithm 5.2 is input-to-state stable under the disturbance $w(k) \in \mathbb{W}$ with $\mathbb{X}_N(z)$ from (5.7) representing the region of attraction for the closed-loop system for any $\tau \geq 0$.

Proof

To prove that the real-time cost function associated with the solution obtained from real-time algorithm 5.2 is ISS lyapunov function of the uncertain closed-loop system, inequalities given in (3.4) need to be proved.

(i) **Lyapunov function lower bound** : $V^{\tau}(x,z) \geq \alpha_1|x-z|$

Inside the invariant set $S_d(z)\ \forall z \in \mathbb{Z}_0$, $\|x\|_Q^2 \leq \|x\|_P^2$ because $\|x\|_Q^2 \geq \|x\|_P^2 - \|x^+\|_P^2$, therefore,

$$V^{\tau}(x,z) \geq \|(x_0^{\tau}(x) - z_0)\|_Q^2 + \|(x-z) - (x_0^{\tau}(x) - z_0)\|_Q^2$$

$$V^{\tau}(x,z) \geq \frac{1}{2}\|x-z\|_Q^2 \geq \boldsymbol{\alpha_1|x-z|}$$

(ii) **Lyapunov function upper bound** : $V^{\tau}(x,z) \leq \alpha_2|x-z|$

According to steps 6-8 of the algorithm 5.2, there are two different cases,

case 1: $\|x-z\|_P \leq \varepsilon_f$ and steps 6-8 are not applicable, then

$$V^{\tau}(x,z) \leq F(x,z)$$

$$V^{\tau}(x,z) \leq \|x-z\|_P \leq \boldsymbol{\alpha_2|x-z|}$$

(iii) **Lyapunov decrease condition** : $V^{\tau}\left(f_{x,\mu}(x) + w - z^+\right) - V^{\tau}(x-z) \leq -\alpha_3|x-z| + \sigma|w|$

The current real-time cost obtained from the real-time suboptimal solution $\mathbf{u}^{\tau}\left(x(k), z(k)\right)$ *and* $x_0^{\tau}(x(k), z(k))$ obtained from algorithm 5.2 is given as,

$$V^{\tau}(x,z) = J_N^{\tau}(x,z,\widetilde{\mathbf{u}}) + F(x_0 - \tilde{x}_0)$$

Proof of lemma 5.5 implies that due to the warm-start solution being feasible at all times, the lyapunov decrease constraint is always satisfied, hence

$$V^{\tau}(x,z) \leq V^{\tau}(x^-,z^-) - \frac{1}{2}\epsilon\|x^- - z^-\|_Q^2 + \left|F\left((x-z) - (\tilde{x}_0 - z_0)\right) - F(x_{nom} - \tilde{x}_0)\right|$$

$$V^{\tau}(x,z) \le V^{\tau}(x^-,z^-) - \frac{1}{2}\epsilon\|x^- - z^-\|_Q^2 + F(x-z) - F(\tilde{x}_0 - z_0) - F(x_{nom}) + F(\tilde{x}_0 - z_0)$$

$$V^{\tau}(x,z) - V^{\tau}(x^-,z^-) \le -\frac{1}{2}\epsilon\|x^- - z^-\|_Q^2 + F(x-z) - F(x_{nom})$$

$$V^{\tau}(x,z) - V^{\tau}(x^-,z^-) \le -\frac{1}{2}\epsilon\|x^- - z^-\|_Q^2 + F\big(f_{x,\mu}(x^-) + w - z^-\big) - F\big(f_{x,\mu}(x^-) - z^-\big)$$

$$V^{\tau}(x,z) - V^{\tau}(x^-,z^-) \le -\frac{1}{2}\epsilon\|x^- - z^-\|_Q^2 + F\big(f_{x,\mu}(x^-) - z^-\big) - F\big(f_{x,\mu}(x^-) - z^-\big) + F(w)$$

$$V^{\tau}(x,z) - V^{\tau}(x^-,z^-) \le -\frac{1}{2}\epsilon\|x^- - z^-\|_Q^2 + F(w) \le -\boldsymbol{\frac{1}{2}\epsilon\|x^- - z^-\|_Q^2 + \sigma\|w\|}$$

Which proves the result.

Hence the real-time control action obtained from the combination of the two nonlinear model predictive controllers, one designed for the nominal closed-loop system and the other for making the trajectories of uncertain system to follow the nominal ones, along with the lyapunov constraint for the ancillary robust control problem and the upper bound on value function imposed by the implementation algorithm which executes the optimization only for a fixed available time interval τ is proved to stabilize the uncertain control system for any arbitrarily small sampling time with guaranteed stability.

Remark 5.7

Since the proof of the input-to-state stability is independent of the length of the time interval τ, the result in theorem 5.6 implies the usefulness of the proposed scheme for a broad range of fastly sampled multi-tasking computational platforms.

Remark 5.8

A stabilizing strategy can also be constructed by resorting to the warm-start solution ignoring the lyapunov constraint (5.4) once a feasible warm-start satisfying the constraint (5.4) is available. However, for faster sampled system, it is very likely that the suboptimal solution will not be able to find a solution with in given time which satisfies the lyapunov constraint which will effectively mean that the ancillary optimal control problem runs in open-loop via the warm-start solution. Hence the lyapunov constraint adds no computational cost to the optimization problem and only adds advantage of stability.

Remark 5.9

Due to the guarantee of recursive feasibility provided by the tube-based robust approach, once a problem is feasible and the uncertain states reach in the set $S_d(z)$, lyapunov constraint (5.4) can be left out as the set $S_d(z)$ is robustly invariant (Mayne and Kerrigan(2007)).

Remark 5.10

The use of $\tilde{x}_0 = z^*(0;z_0)$ and , $\tilde{\mathbf{u}} = \{\mu(x_u^\tau(0,x_0) - z^*(0;z_0)), \dots, \mu(x_u^\tau(N-1,x_0) - z^*(N-1,z_0))\}$ in the set $\mathcal{E}_f$ makes the strategy resemble to a dual mode strategy. However the control strategy wont switch once inside this set because the $\mathcal{E}_f$ is not guaranteed to be robustly invariant for the closed-loop system.

Remark 5.11

The upper bound on the value function of the closed-loop system in the neighborhood of the origin is usually assumed in real-time methods in the literature. The proposed scheme however ensures the existence of such a class $\mathcal{K}_\infty$ function of the states through steps 6-8 of the algorithm 5.2 once the nominal states have reached the neighborhood of the origin. Recall that the nominal trajectories are guaranteed (through theorem 2.14) to be reaching the origin within N steps.

Remark 5.12

Only when the robust invariant set $S_d(z)$ is known, the optimization of the first tube-center x_0^τ is possible which adds an additional feedback to the disturbance. However, if the first tube center is fixed, as is the case presented in chapter 4, the optimization can be carried out for the control actions only without invalidating the stability proof.

Remark 5.13

The primary purpose of using a robust approach in this scheme to is to ensure recursive feasibility under the bounded disturbances. The use of tube-based approach however is not necessary but advantageous as it is computationally cheaper than other methods. However any other robust nonlinear model predictive method, such as ones mentioned in chapter 3, providing the property of recursive feasibility could be applied.

5.4 Example – System of Three Masses

The system of oscillating masses (figure 2.1) was shown to be unstable when subjected to the real-time constraint in figure 5.1. Figure 5.2 shows that with the application of the real-time NMPC proposed in section 5.2, the guarantee of stability and recursive feasibility of the closed-loop system under uncertainties is recovered as provided by the tube-based controller from chapter 4. The oscillating masses are taken to be $m = 1\ Kg$, the damping constants are $d = 0.1\ N\ s/m$ and the spring constants are taken as $k = 0.9\ N/m$. The stiffness coefficient of the spring is taken as $\beta = 0.5$. The position displacements of the masses are constrained in $\pm 4m$ whereas the velocities of the masses are not constrained at all. The actuation signals are constrained to lie with in $\pm 1N$. The disturbance term is bounded as $|w| \leq 0.3$. The controller is simulated in a matlab code file. The matlab's built-in function *fmincon* is used for the constrained optimization. The maximum number of iterations in the *fmincon* is taken to be 6 corresponding to insufficient computation time as described in section 5.1. For the offline computation of terminal sets for the nominal controller, the toolbox YALMIP is used.

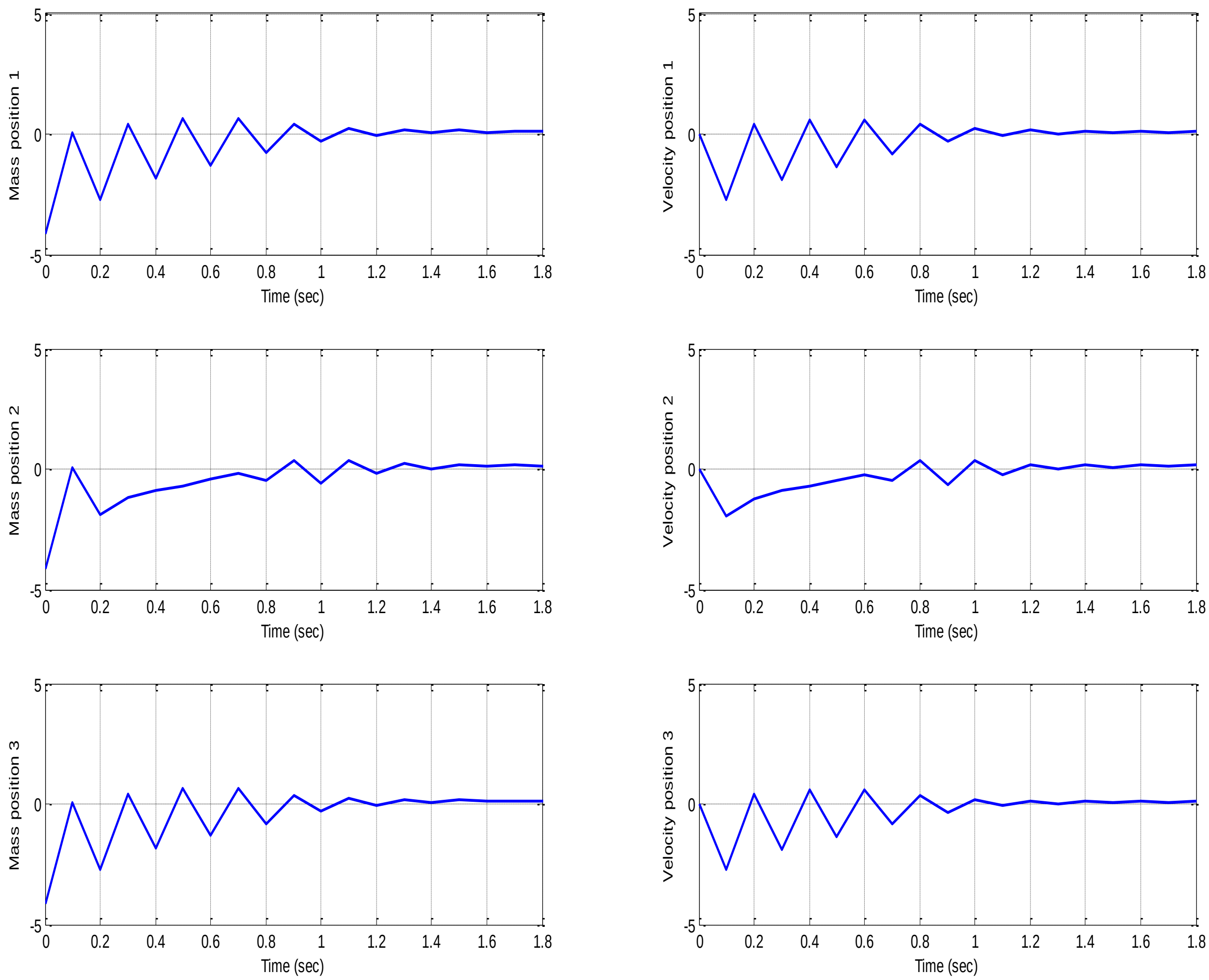

Figure 5.2: Stabilization of the uncertain system to Origin for any arbitrarily small sampling time

5.5 Conclusion

This chapter presented a nonlinear model predictive controller scheme which retains the stability properties of the robust NMPC controllers in real-time environment. The nominal controller stabilizes the system through terminal constraint and cost whereas the ancillary controller maintains the stability and ensures recursive feasibility under bounded additive disturbances. The stability in the presence of real-time constraint is imposed explicitly by adding a lyapunov decrease constraint in the ancillary control problem. A warm-start solution is used for both the control problems. In the case of nominal controller, it reduces the computation time for obtaining the optimal solution whereas in the ancillary controller, its major purpose is to ensure that the lyapunov decrease constraint can be satisfied at all sampling times. Lemma 5.5 shows that if the real-time control system starts initially from a feasible point, it stays feasible for the subsequent sampling instants. Theorem 5.6 proves that algorithm 5.2 asymptotically stabilizes the uncertain system for any arbitrarily small sampling time. The implementation results are shown for a system of three oscillating masses to illustrate the point.

Chapter 6
FUTURE WORK DIRECTIONS

The work presented in this thesis can be extended in various directions such as,

1. **Real-time NMPC for tracking problem**

The real-time NMPC scheme is proposed for regulation problem. That is, it stabilizes the system to the origin or to any other constant equilibrium point with a change of coordinates. A difficult but interesting prospect is to design a scheme which guarantees real-time stability for the problem of tracking piecewise constant or time-varying reference trajectories. Following difficulties arise in the tracking problem

a. The terminal cost obtained for the nominal controller plays an important role in the proof of theorem 5.6 as it provides the upper bound for ISS lyapunov function. However in the time varying case, the method described in section 2.2 fails to provide a stabilizing terminal cost. This is due to the fact that in the time-varying case, the linearization and the subsequent LQR problem do not lead to an easily solvable riccati equation such as (2.13).

b. The feasibility of the lyapunov decrease constraint which ensures stability of the control problem is guaranteed by a warm-start solution such as (5.1). In the tracking case, piecewise constant or time-varying, a warm-start solution is not readily available as the reference trajectory is varying all the time hence the best solution which guarantees the lyapunov decrease constraint is satisfiable at all times is difficult to find. All is not lost however. A method to obtain a warm-start for tracking is presented in the book (Grüne and Pannek (2011)). A useful result can be obtained using this warm-start for the tracking problem however the proof might be rigorous and involving.

2. **Output Feedback NMPC**

In all the chapters of thesis, it is assumed that all states are measured. In many applications it is not feasible to be able to measure all system states or is not economical even if it is feasible. In the absence of state measurement, observers are used. NMPC with observers is usually referred to as the Output feedback NMPCs and various output feedback NMPCs are proposed in the literature. An interesting result could be obtained if the stability provided by an output feedback NMPC can be guaranteed with real-time constraint.

3. Real-Time NMPC for Unstable Systems Using Soft Constraints

In the real-time scheme presented in chapter 5, it is assumed that initially all the time in the world is available for the initialization of a feasible solution. This might not be the case in a hybrid system or any other situation for that matter. In that case, a good strategy could be to relax some of the state constraints which only represent the designer's requirements not critical to stability. Such constraints in the literature are referred to as the soft constraints. Critical bound state constraints and constraints on control input are always taken as hard constraints. Soft constraints have been used in the literature and a useful result can be obtained if the real-time scheme presented in this chapter can be combined with soft constraints strategy to obtain robust stabilization of the unstable systems in real-time with guaranteed stability.

List of Figures

REFERENCES

Frank Allgöwer, Rolf Findeisen, and Zoltan K. Nagy. (2004). *Nonlinear Model Predictive Control: From Theory to Application*, *Journal of the Chinese Institute of Chemical Engineers. 35(3), 299-315*

E. B. Lee and L. Markus. *Foundations of Optimal Control Theory*. Wiley, New York, 1967. 2

J. Richalet, A. Rault, J. Testud, and J. Papon. Model predictive heuristic control: Applications to industrial processes. *Automatica*, 14(5):413–428, Sep 1978.

Chen, C. C., & Shaw, L. (1982). On receding horizon feedback control. Automatica, 18, 349}352.

Keerthi, S.S., Gilbert, E.G.: Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: stability and moving-horizon approximations. J. Optim.Theory Appl. **57**(2), 265–293 (1988)

Mayne, D. Q., & Michalska, H. (1990). Receding horizon control of non-linear systems. IEEE Transactions on Automatic Control, 35(5), 814}824.

G. de Nicolao, L. Magni, and R. Scattolini. On the robustness of receding-horizon control with terminal constraints. *IEEE Transactions on Automatic Control*, 41(3):451–453, Mar 1996. 4, 21

Magni, L., & Sepulchre, R. (1997). Stability margins of nonlinear receding-horizon control via inverse optimality. Systems *&* Control Letters, 32, 241}245.

Chen, H., & Allgower, F. (1998b). A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. Automatica, 14(10), 1205}1217.

D. Mayne, J. B. Rawlings, C. Rao, and P. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, Jun 2000.

Scokaert, P. O. M., Rawlings, J. B., & Meadows, E. S. (1997). Discretetime stability with perturbations: Applications to model predictive control. Automatica, 33(3), 463}470.

Michalska, H., & Mayne, D. Q. (1993). Robust receding horizon control of constrained nonlinear systems. IEEE Transactions on Automatic Control, 38, 1623}1632.

R. Findeisen, L. Imsland, F. Allgower and B. A.Foss. State and output feedback nonlinear model predictive control:An overview. European Journal of Control, vol. 9, no. 2-3, pages 190206, 2003.

A. R. Teel. *Discrete Time Receding Horizon Optimal Control: Is the Stability Robust?*, volume 301/2004 of *Lecture Notes in Control and Information Sciences*, pages 3–27. Springer Verlag, Berlin, 2004.

G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel. Examples when nonlinear model predictive control is nonrobust. *Automatica*, 40(10):1729–1738, Oct 2004.

S. Lall and K. Glover. *A game theoretic approach to moving horizon control*, pages 131–144. Advances in Model-Based Predictive Control. Oxford University Press, Oxford, U.K., 1994.

H. Chen, C. Scherer, and F. Allgöwer. A Game Theoretic Approach to Nonlinear Robust Receding Horizon Control of Constrained Systems. In *Proceedings of the American Control Conference*, volume 5, pages 3073–3077, Jun 1997.

Blauwkamp, R. and T. Basar, "A Receding-Horizon Approach to Robust Output Feedback Control for Nonlinear Systems," *Proc. 38th IEEE Conf. Decision Contr*., p. 4879, San Diego, U.S.A. (1999).

Magni, L., G. de Nicolao, R. Scattolini, and F. Allgöwer, "Robust Receding Horizon Control for Nonlinear Discrete-Time Systems," *Proceedings 15th IFAC World Congress*, Barcelona, Spain (2001b).

Magni, L., H. Nijmeijer, and A. van der Schaft, "A RecedingHorizon Approach to the Nonlinear H∞ Control Problem," *Automatica*, **37**(5), 429 (2001c).

Darlington, J., C. C. Pantelides, B. Rustem, and B. A. Tanyi, "Decreasing the Sensitivity of Open-Loop Optimal Solutions in Decision Making under Uncertainty," *Eur. J. Oper. Res*., **121**, 343 (2000).

Rustem, B., "Stochastic and Robust Control of Nonlinear Economic Systems," *Eur. J. Oper. Res.*, **73**, 304 (1994).

M. V. Kothare, V. Balakrishnan, and M. Morari. Robust Constrained Model Predictive Control using Linear Matrix Inequalities. *Automatica*, 32(10):1361–1379, Oct 1996.

D. Q. Mayne and E. C. Kerrigan. Tube-based robust nonlinear model predictive control. In *Proceedings of the 7th IFAC Symposium - NOLCOS2007*, Preatoria, South Africa, Aug 2007.

Melanie N. Zeilinger, Davide M. Raimondo, Alexander Domahidi, Manfred Morari, Colin N. Jones. On Real Time Robust Model Predictive Control. *Automatica*, 50(3):683–694, March 2014.

A. Bemporad, F. Borrelli, and M. Morari, *Model Predictive Control Based on Linear Programming - The Explicit Solution*, IEEE Transactions on Automatic Control **47** (2002), no. 12, 1974–1985.

A. Bemporad, M. Morari, V. Dua, and E. Pistikopoulos. The Explicit Linear Quadratic Regulator for Constrained Systems. *Automatica*, 38(1):3–20, Jan 2002.

F. Borrelli, A. Bemporad, and M. Morari, *A Geometric Algorithm for Multi-Parametric Linear Programming*, Journal of Optimization Theory and Applications **118** (2003), no. 3, 515–540.

A. Hansson, *A primal-dual interior-point method for robust optimal control of linear discrete-time systems*, IEEE Transactions on Automatic Control **45** (2000), no. 9, 1639–1655.

B. Kouvaritakis, M. Cannon, and J. A. Rossiter, *Who needs QP for linear MPC anyway?*, Automatica **38** (2002), 879–884.

E. D. Andersen, C. Roos, and T. Terlaky, *On implementing a primaldual interior–point method for conic quadratic optimization*, Mathematical Programming **95** (2003), 249–277.

H. J. Ferreau, H. G. Bock, and M. Diehl, *An online active set strategy to overcome the limitations of explicit* MPC, International Journal of Robust and Nonlinear Control **18** (2008), 816–830.

M. Cannon, W. Liao, and B. Kouvaritakis, *Efficient mpc optimization using pontryagin's minimum principle*, Int. J. of Robust and Nonlinear Control **18** (2008), 831–844.

P. Patrinos, P. Sopasakis, and H. Sarimveis, *A global piecewise smooth newton method for fast large-scale model predictive control*, Tech. Report TR2010-02, National Technical University of Athens, 2010.

A. Bemporad, A. Casavola, and E. Mosca. Nonlinear Control of Constrained Linear Systems via Predictive Reference Management. *IEEE Transactions on Automatic Control*, 42(3):340–349, 1997.

A. Bemporad. Reference Governor for Constrained Nonlinear Systems. *IEEE Transactions on Automatic Control*, 43(3):415–419, Mar 1998b.

A. Bemporad and E. Mosca. Fulfilling Hard Constraints in Uncertain Linear Systems by Reference Managing. *Automatica*, 34(4):451–461, 1998.

H. K. Khalil, *Nonlinear systems*. Prentice hall Upper Saddle River, 2002, vol. 3.

Rawlings, J. B., & Muske, K. R. (1993). Stability of constrained receding horizon control. IEEE Transactions on Automatic Control, AC-38(10), 1512}1516.

Chen, H. and F. Allgower (1997a). Nonlinear model predictive control schemes with guaranteed stability. In *NATO AdI'ilnced Study Institute* all *NOlllinear Model Based Process Contl'ol*, ed. R. Berber, Kluwer Academic Publishers. Dordrecht. The Netherlands.

Chen, H. and F. Allgower (1997b). A quasi-infinite horizon nonlinear predictive control scheme for stable systems: Application to a CSTR.In *ProC'. IPAC Int. Symp. Adv Conltrol*
of Chemical Processes. ADCHEM, Banff, pp. 471-476.

Y. Wang and S. Boyd, *fast_mpc: code for fast model predictive control*, http://www.stanford.edu/~boyd/fast_mpc, Sept 2008.

A. Bemporad and M. Morari. *Robust Model Predictive Control: A Survey*, volume 245/1999 of *Lecture Notes in Control and Information Sciences*, pages 207–226. Springer Verlag, Berlin, 1999.

L. Magni and R. Scattolini. Robustness and Robust Design of MPC for Nonlinear Discrete-Time Systems. In R.Findeisen et al. (Eds.): Assessment and Future Directions, volume LNCIS 358, pages 239254. Springer-Verlag Berlin Heidelberg, 2007.

V. S. Chellaboina and W. M. Haddad. Stability margins of discrete-time nonlinear-nonquadratic optimal regulators. In *IEEE CDC*, pages 1786–1791, 1998.

J. C. Geromel and J. J. Da Cruz. On the robustness of optimal regulators for nonlinear discrete-time systems. *IEEE Trans. on Automatic Control*, AC-32:703–710, 1987.

S. T. Glad. Robustness of nonlinear state feedback- a survey. *Automatica*, 23:425–435, 1987.

R. Sepulchre, M. Jankovic, and P. V. Kokotovic. *Constructive Nonlinear Control*. Springer-Verlag, 1996.

Z.-P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.

D. Limon, T. Alamo, and E. F. Camacho. Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties. In *IEEE CDC*, pages 4619–4624, 2002.

G. Grimm, M. J. Messina, S. E. Tuna and A. R. Teel. Nominally Robust Model Predictive Control with State Constraints. In 42nd IEEE CDC, pages 1413-1418, 2003.

H. Chen, C. W. Scherer and F. Allgöwer. A robust model predictive control scheme for constrained linear systems. In Proc. DYCOPS 5, Corfu, Greece, 1998.

Eva Gyurkovics. Receding Horizon $H\infty$ Control for Nonlinear Discrete-Time Systems. IEE Proceedings - Control Theory and Applications, vol. 149, pages 540-546, 2002.

L. Magni, G. De Nicolao, R. Scattolini and F. Allgöwer. Robust Model Predictive Control of Nonlinear Discrete-Time Systems. International Journal of Robust and Nonlinear Control, vol. 13, pages 229-246, 2003.

L. Magni and R. Scattolini. Control Design for Nonlinear Systems: Trading Robustness and Performance with the Model Predictive Control Approach. IEE Proceedings - Control Theory & Application, pages 333-339, 2005.

D. Limon, T. Alamo, F. Salas and E. F. Camacho. Input to State stability of min-max MPC controllers for nonlinear systems with bounded uncertainties. Automatica, vol. 42, pages 797- 803, 2006.

W. Langson, I. Chryssochoos, S. V. Rakovi´ c, and D. Mayne. Robust model predictive control using tubes. *Automatica*, 40(1):125–133, Jan 2004.

D. Q. Mayne, M. M. Seron, and S. V. Rakovi´ c. Robust Model Predictive Control of Constrained Linear Systems with Bounded Disturbances. *Automatica*, 41(2):219–224, Feb 2005.

E. S. Meadows, M. A. Henson, J. W. Eaton, and J. B. Rawlings. Receding horizon control and discontinuous state feedback stabilization. *Int. J. Control*, 62:1217–1229, 1995.

G. Tadmor. Receding horizon revisited: An easy way to robustly stabilize an LTV system. *Systems & Control Letters*, 18:285–294, 1992.

F. A. C. C. Fontes and L. Magni. Min-max model predictive control of nonlinear systems using discontinuous feedbacks. *IEEE Trans. on Automatic Control*, 48:1750–1755, 2003.

D. Q. Mayne. Control of constrained dynamic systems. *European Journal of Control*, 7:87–99, 2001.