OUTPUT FEEDBACK STABILIZATION AND REGULATION FOR A CLASS OF UNDER-ACTUATED NON-MINIMUM PHASE BENCHMARK NONLINEAR SYSTEMS

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Dedication

I would like to dedicate this thesis to my parents, my family and my teachers ...

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This thesis is the result of my own work and includes nothing which is the outcome of work done, except where specifically indicated in the text.

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Abstract

In this thesis, the problem of output feedback stabilization and regulation for a class of under-actuated benchmark nonlinear systems is considered. The proposed method utilizes an extended high gain observer (EHGO)-based sliding mode control (SMC) technique to control a class of nonlinear systems which may have unstable zero dynamics. Starting with Lagrangian model of the system and using a suitable coordinate transformation, a generalized normal form representation is derived which decouples the system into its internal and external dynamics.

The internal dynamics is utilized to derive an auxiliary system and the full-order EHGO thus obtained is used for estimation of derivative(s) of the system output that are further used in design of an output feedback control law. It is shown that the proposed output feedback controller stabilizes the system and convergence of estimated states is demonstrated with suitable selection of observer parameters. The proposed stabilizing control scheme is applied to two of the benchmark nonlinear systems, namely Inertia Wheel Pendulum (IWP), and Translational Oscillator with Rotational Actuator (TORA), in order to demonstrate the effectiveness of the technique by simulation.

The technique is extended to further solve for the servomechanism (output regulation) problem for the class of under actuated non-minimum phase nonlinear systems under consideration. Towards this end, the control design is modified to include a 'conditional servo compensator' in order to track reference signals as well as reject disturbances while achieving stabilization and steady-state accuracy with a desired transient performance. The conditional servo compensator is utilized to regulate the controllable states by using SMC while neglect-

ing the internal states initially. The uncontrollable states of the system are estimated using an EHGO and the error signal is then used to synthesize a control input to stabilize the internal states by incorporating it in the sliding surface of the SMC design. The proposed control scheme is then applied to the Translational Oscillator with Rotational Actuator (TORA) system to validate the efficacy of the technique.

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Chapter 1

Introduction

A vast majority of real life control systems, representing a particular subclass of nonlinear systems, are electromechanical systems that cannot be commanded to follow arbitrary trajectories in the entire configuration space. In other words, the control input cannot accelerate the state of these systems in any arbitrary direction. These systems are termed as under-actuated systems and are characterized with fewer number of controls than the number of configuration variables.

Control of under-actuated nonlinear systems is an active research area of modern age due to its direct relevance to real-life physical systems used in military equipments, heavy industries and other commercial equipments. Real-life under-actuated systems, being inherently highly nonlinear, attracted engineers and mathematicians to perform modelling, simulation and rigourous experimental analysis for control of these systems. Advanced control techniques developed for nonlinear systems like optimal control, adaptive control and robust control were utilized and extended for stabilization and control of such under-actuated systems. This led to development of some advanced theories and concepts linked with these type of systems. Examples of under-actuated systems are found in robotics, automobiles, aircrafts, satellites, underwater vehicles, missiles etc, and as benchmark control systems such as Cart Pole system, Translational Oscillator with Rotating Actuator(TORA), Acrobot, Inertia Wheel Pendulum (IWP), Rotary Pendulum (ROTPEN) etc.

In subsequent sections of this chapter, some of the motivating factors that influenced to initiate the research work presented here towards development of control strategies for under-actuated electromechanical systems are presented. Starting with a review of literature and work done by renowned researchers in this area, problem definition and scope of work is precisely described. Later on exact problem targeted in this research work and its proposed solution is identified. Relevant test-bench under-actuated nonlinear systems used for verification and validation of the proposed control scheme are pointed out. Overall objectives achieved in this thesis and major contribution of work towards the subject research area are highlighted. Finally a brief overview of thesis flow is presented.

1.1 Motivation

Control of nonlinear systems is a challenging task as compared to linear systems due mainly to unavailability of systematic procedures and proper design tools. Different design techniques are applied in case of nonlinear systems depending on complexity of the system and its performance requirements. Of particular interest, in this work are the under-actuated nonlinear systems which so commonly arise in the fields of robotics, aerospace systems etc. These systems require some special treatment and a thorough study of their mechanics, kinematics, and dynamics is needed.

For under-actuated systems, a part of the system dynamics is unstable making the system exhibit non-minimum phase properties. Due to lesser number of control or actuators than the configuration space, a subset of the configuration space is unactuated and can only be controlled by indirect coupling movement of actuated variables. Control of these systems becomes complex problem that cannot be solved by classical control techniques since properties like feedback linearizability and passivity are no longer valid. Moreover, many of these systems may possess undetermined relative degree and present structural obstruction for derivation of a straight-forward control law.

Study of under-actuated nonlinear systems and need for developing control algorithms for these systems becomes critical in many practical situations where actuators and motors / drives are heavy or expensive and it is of great advantage to reduce number of these actuators and motors / drives to as minimum level as possible. Some of the reasons of under-actuation in control systems are given as [Fantoni and Lozano, 2002]

- Robotic manipulators where individual links are assumed as rigid becomes under-actuated when non-rigid dynamics is considered.
- Aircrafts, unmanned and space vehicles, satellites etc., having reduced number of motors are built-in under-actuated as weightlessness is of prime importance in these systems.
- Free floating mobile robots having robotic manipulators become under-actuated as a new degree of freedom is added when primary propelling devices are switched off to complete the desired task.
- High order nonlinear systems are forced to artificially become low order under-actuated systems for study and analysis purpose. Examples of such systems are Cart Pole System, Inertia Wheel Pendulum (IWP), TORA system, Ball and Beam system etc.

Dynamical models of under-actuated systems may exhibit complex properties like non-minimum phase zero dynamics, non-holonomic constraints, feed-forward nonlinearities and incapability of feedback linearization. An important and inspiring aspect of research on under-actuated systems is that it is an extension of the research on non-holonomic systems. For the non-holonomic systems, only velocity constraints and kinematic equations of the motion are studied whereas under-actuated system exhibits acceleration constraints and both kinematics and dynamics are required to be considered in the control design process. Thus, the complex structural properties eventually place the study of these systems as an open and challenging problem. Due to these reasons, it is very difficult to present a unified control theory applicable to entire class of under-actuated systems and therefore, most researchers have presented particular design techniques for stabilization and tracking of systems belonging to individual classes of these systems. Some of the researchers have targeted different benchmark under-actuated systems to come up with suitable control laws. Towards that end, a typical class of under-actuated systems is considered in the research work presented in this thesis and dynamical models of test-bench under-actuated systems belonging to it are utilized to develop suitable control algorithms. These benchmark systems include Inertia Wheel Pendulum (IWP) and Translational Oscillator with Rotational Actuator (TORA).

Stability analysis for under-actuated systems is also of major concern, and hence a source of motivation, requiring application of a more severe design constraints for global, semiglobal or local stability of these systems as compared to linear systems. Design techniques like Lyapunov redesign, backstepping, sliding mode control etc are extensively used for control of nonlinear systems. Out of these techniques, sliding mode control has got special attention of researchers due to its robustness to parametric uncertainties and/or external disturbances and also due to its design simplicity. One of the motive of the research presented in this thesis is application of sliding mode algorithm for control of a class of under-actuated systems to account for the high degree of uncertainties and complex internal dynamics. Another important objective targeted in this research is to attain output feedback control of these type of systems as the control problem becomes more challenging when some of the system states are unavailable. This may happen when it is difficult to measure or in some cases it is uneconomical to measure all system states. This necessitates employment of observer based or output feedback based techniques to estimate the unknown states. In brief, the complex nonlinear system structure, severe design constraint requirements for system stability, application of sliding mode framework and incorporation of observer based output feedback control law are major sources of motivation for the research work presented in this thesis.

1.2 Background and Literature Review

In this section, a brief review for *output feedback stabilization* of target class of systems is presented whereas review of research work previously reported on *output feedback regulation* for these type of systems is discussed in Chapter 6. Here, an overview of research done for generalised classification, mathematical model formulation, and output/state feedback stabilization of under-actuated nonlinear systems by renowned researchers is presented.

The unwanted characteristics as mentioned in previous section makes determination of controllability of under-actuated systems typically more difficult than nonlinear systems and previous work done in this area mostly focuses on state feedback. Important work reported under state feedback control is included in Block et al. [2007]; Lopez-Martinez et al. [2010]; Olfati-Saber [2000]; Ortega et al. [2002]; Qaiser [2009]; Spong et al. [2001]. The problem becomes more complex when some of the system states are not available and a control law is deduced from available system states and the output, either under partial state feedback or output feedback control. The requirement of output feedback becomes significant, if actual system or plant is lacking in some of state sensors e.g., velocity sensors. The reason behind this may be economic so as to reduce cost of overall control system or it may be due to the requirement that system should exhibit the ability of eliminating measurement noise in velocity signals. This makes the output stabilization problem of under-actuated systems an open research problem [Xu and Hu, 2013]. In this domain, significant work is reported in Boker and Khalil [2013]; Bou Serhal and Khalil [2012]; Nazrulla and Khalil [2011b]; Xu and Hu [2013].

The above mentioned factors make the control of under-actuated nonlinear systems a challenging problem and therefore, it is under consideration for the past many decades making it an active field of research. A detailed and comprehensive study covering major aspects of the under-actuated systems is done by Reza Olfati-Saber in his thesis [Olfati-Saber, 2000]. An important aspect of this work is that classification of systems is done on the basis of kinetic symmetry properties. Based on these properties, it is demonstrated that complexity of original system dynamics can be reduced and a cascade linear/nonlinaear mathematical form can be obtained after suitable change of control and coordinates. The reduced mathematical form is termed as normal form representation which is characterized by a number of first order ordinary differential equations and can be easily used to analyze the controllability and observability properties of the systems. An important feature of the normal form presented here is that it transforms system dynamics into internal dynamics and external dynamics. Depending on various type of normal form representation and mathematical analysis presented here, it is shown that under-actuated systems can be placed in eight different categories. According to this classification, a number of benchmark under-actuated systems including the Inertia Wheel Pendulum (IWP), Translational Oscillator with Rotating Actuator (TORA) and Acrobot falls together in a similar class, due to identical normal form representation called as strict feedback form. This class of systems is under consideration for detailed mathematical analysis and control law derivation in this thesis.

Towards this end, a passivity-based nonlinear control method to stabilize Inertia Wheel Pendulum is proposed in Spong et al. [2001]. The mathematical formulation presented in Spong et al. [2001], for the development of control law structure that does not require velocity measurement and is non-switching in nature, is further extended in [Ortega et al., 2002]. Here, the stabilization problem of IWP is solved by utilizing a new formulation of passivity based control termed as interconnection and damping assignment for swing up and balance of pendulum. Lately, an important development for a class of under-actuated systems and extension in work of Olfati-Saber [2000] is reported in Qaiser [2009]. In this work, stabilizing algorithms for a class of under-actuated systems in strict feedback normal form are presented and it is shown that some of the shortcomings relevant with previously reported algorithms like energy shaping and damping injection and back-stepping can be avoided by use of Multiple Sliding Surfaces (MSS) Control and Dynamic Surface Control (DSC) techniques. It is further shown that these techniques when applied on typical examples of benchmark systems belonging to this class like IWP, TORA and Acrobot gave improved results for semiglobal asymptotic and local exponential stabilization.

The above mentioned work effectively addresses issues using dynamic feedback control law utilizing entire system states. Some researchers presented solution to stabilization problem of nonlinear under-actuated systems under output feedback using state observer for estimation of system states. High gain observer (HGO) is a type of observer that has been successfully used as a design tool for state estimation and disturbance rejection [Khalil, 2002; Khalil and Praly, 2014]. For the case of minimum phase nonlinear systems, where internal dynamics exhibit stable behaviour, high gain observers have been effectively used to improve system robustness as well as to achieve simplicity in controller design. The technique has been used for a wide class of nonlinear systems and global results have been achieved under global growth condition. In [Esfandiari and Khalil, 1992], it is shown that under lack of global growth conditions observer gain becomes sufficiently high that could destabilize the entire closed loop system. This phenomenon is termed as peaking effect. Due to much faster observer dynamics than closed loop system dynamics, the peaking period is very short and remain very close to the initial values of system variables. This phenomenon gave a simple solution to the problem of peaking effect as proposed in [Esfandiari and Khalil, 1992] where it is suggested that a modified control law based on globally bounded function of state estimates could be devised so that it saturates within period of peaking. After this effective solution of peaking phenomenon, the use of HGO got wide-spread attention. [Teel and Praly, 1995] presented mechanism for semiglobal stabilization of nonlinear systems referred as first nonlinear separation principle. This concept is extended in [Atassi and Khalil, 1999; Khalil and Praly, 2014] by combining fast observer with control saturation so that output feedback control law can recover the trajectories of state feedback controller under sufficiently high value of observer gain.

The HGO based controller design is, however, mostly done for minimum phase systems [Memon and Khalil, 2009] having stable zero dynamics. For estate estimation and disturbance rejection of non-minimum phase systems, an extended version of HGO having one dimension higher than system relative degree have been introduced and effectively utilized. The stabilizing controller for a class of non-minimum phase systems utilizing EHGO is presented by Khalil and coworkers in which concept of an auxiliary system is used that includes unstable internal dynamics. An EHGO based sliding mode controller is proposed in [Nazrulla and Khalil, 2011b] to evolve stabilizing control law for such type of systems. Similarly in Boker and Khalil [2013], a full order observer is designed based on EHGO and Extended Kalman Filter (EKF) to estimate system output as well as internal system states. Both Boker and Khalil [2013]; Nazrulla and Khalil [2011b] demonstrated their results on a test bench under-actuated system namely Translating Oscillator with a Rotating Actuator (TORA). In Bou Serhal and Khalil [2012], application of EHGO for disturbance rejection is demonstrated on a bench mark inverted pendulum system.

In this thesis, work done on state feedback based stabilization control extends the earlier contributions of Olfati-Saber [2000]; Qaiser [2009] while for output feedback based control, it utilizes the work done by Boker and Khalil [2013]; Nazrulla and Khalil [2011b], and a sliding mode control scheme based controller design is proposed. It is demonstrated that closed loop system states and trajectories under output feedback converge to those of state feedback control with suitable choices of observer gains and design parameters. In output feedback, the initial overshoot in states i.e., peaking effect is controlled by applying some constraints on system input. These constraints do not deteriorate the overall closed loop system performance while the system transient behaviour is considerably improved and is comparable with those of state feedback performance. The simulation results on two examples of target class of under-actuated system exhibits the effectiveness of the proposed stabilizing control scheme.

1.3 Contribution

In this thesis, output feedback stabilization/regulation for a class of under-actuated benchmark non-minimum phase system is presented. For the case of output stabilization, the design approach is based on sliding mode framework and an extended high gain observer. The scheme broadly utilizes the earlier contributions of Olfati-Saber [2000] for development of system mathematical model under SFB design and Boker and Khalil [2013]; Nazrulla and Khalil [2011b] for the development of an EHGO to estimate virtual output to an auxiliary system. The contribution of this work, lies in utilization of an important design tool called collocated partial feedback linearization [Olfati-Saber, 2000] and a global coordinate transformation to generate suitable normal form that bifurcates entire system into an internal and external dynamics which can be conveniently utilized for EHGO based controller design [Boker and Khalil, 2013]. Another contribution of this work, is the methodology for derivation of control law. In Olfati-Saber [2000]; Qaiser et al. [2007] design is based on assumption that system states are available to derive a suitable control law whereas in this work an equivalent sliding mode control is used based on the assumption that some of the system states are not available. The whole scheme is based on step by step approach utilizing error dynamics of external system and generating appropriate output to stabilize internal system. The methodology is applied on two examples of target class of systems namely Inertia Wheel Pendulum (IWP) and Translational Oscillator Rotational Actuator (TORA) in Chapter 5.

For stabilization of TORA system, the proposed methodology is applied and an EHGO is implemented along with Extended Kalman Filter (EKF). The controller design approach is an extension to earlier work [Khalid and Memon, 2014]. It broadly utilizes the design approach of Olfati-Saber [2000] to derive system mathematical formulation under state feedback (SFB) design and Boker and Khalil [2013]; Nazrulla and Khalil [2011b] for the development of an EHGO to estimate virtual output to an auxiliary system. The difference in the work presented in this thesis, lies in application of collocated partial feedback linearization [Olfati-Saber, 2000] and a global change of coordinates to derive suitable normal form similar to one used in Boker and Khalil [2013]. Such normal form transforms the whole system dynamics into internal and external systems which can be conveniently used for controller design utilizing system observed states based on EHGO and EKF [Boker and Khalil, 2013]. Another difference in the case of TORA system lies in development of suitable control law. In this work, an observer based sliding mode controller is proposed while in [Olfati-Saber, 2000; Qaiser et al., 2007] only state feedback based design scheme is implemented. Again, the controller design is implemented using a systematic, step by step approach incorporating error output of external system for stabilization of internal system states. The scheme can be implemented on other systems of similar class with appropriate modifications. Thus the proposed scheme allows implementation of control law in a simple and unified way and the overall transient and steady state response is compared with those in [Boker and Khalil, 2013; Nazrulla and Khalil, 2011b]. This thesis contains a detailed account of design and analysis of control techniques proposed for stabilization and output regulation of a class of under-actuated nonlinear systems. Some of the preliminary results presented here have been presented and published. A brief account of the research papers published is given below:

- Khalid, N. and Memon, A. Y. (2016). Output feedback control of a class of under-actuated nonlinear systems using extended high gain observer. Arabian Journal for Science and Engineering, 41(9):3531–3542.

- Khalid, N. and Memon, A. Y. (2014). Output feedback stabilization of an inertia wheel pendulum using sliding mode control. In Control (CONTROL), 2014 UKACC International Conference on, pages 157–162. IEEE.

- Khalid, N. and Memon, A. Y. Output Feedback Control of TORA - A Benchmark Under-actuated Nonlinear System. First International Conference on Latest trends in Electrical Engineering and Computing Technologies (INTELLECT), 2017

1.4 Thesis Organization

The rest of the thesis is organized in six chapters. Chapter 2 presents a brief overview of terms, definitions and basic concepts of under-actuated nonlinear systems. Mathematical formulation and classification of nonlinear system in general and under-actuated systems in particular is discussed in Chapter 3. The model transformation for a class of under-actuated nonlinear systems and its utilization for output feedback stabilization to derive an SMC based control law is given in Chapter 4. Subsequently, in Chapter 5, examples of target class of under-actuated benchmark systems are dealt with proposed control techniques. The output feedback regulation of of under-actuated systems is dealt in Chapter 6. In Section 6.5 of this chapter, TORA system model is taken and a control law is synthesised for analysis and verification of proposed output regulation scheme. Finally, in Chapter 7, a conclusion is drawn and future aspects of the proposed research are highlighted.

Chapter 2

Preliminaries

The objective of this thesis is to solve control problem for a class of under-actuated nonlinear systems that exhibit non-minimum phase behaviour. Before proceeding to main concepts and methodology adopted in this thesis to solve the problem, it is necessary to preview some background theories and preliminary concepts that would be referenced later on in this thesis. In this chapter, some basic concepts including non-minimum phase systems, state observer and extended high gain observers based output feedback control and stability of nonlinear systems are explored. The chapter contains the preliminary material required for the controller development in later chapters.

2.1 Linear Time Invariant Systems

The theory of linear system has been developed much earlier than nonlinear control theory. A lot of research has been done on linear control systems. A linear system essentially follows superposition and homogeneity principles. The superposition principle can be depicted as

$$\left.\begin{array}{rr} \text{if} & g(u_a) = y_a \\ \text{and} & g(u_b) = y_b \\ \text{then} & g(u_a + u_b) = y_a + y_b \end{array}\right\} \tag{2.1}$$

and homogeneity requirement can be defined as

$$\left.\begin{aligned} \text{if} \quad & g(u) = y \\ \text{then} \quad & g(Au) = Ay \end{aligned}\right\} \tag{2.2}$$

where A is any scalar or a constant number. The variables u,u_a,u_b are set of inputs and y,y_a,y_b are corresponding outputs. A linear system that does not change over time is termed as linear time invariant or an LTI system. An LTI system can be represented with state equations as

$$\dot{x} = Ax + Bu \tag{2.3}$$

where x is state vector, A is system matrix, B is input matrix and u is external control input. Depending on configuration of matrix A, an LTI system is characterized by unique and stable equilibrium point independent of initial conditions. The system transient response is composed of natural modes and analytical method could by used to find system general response.

2.2 Linear Non-minimum Phase Systems

An important phenomenon of linear systems is exhibition of minimum / non-minimum phase behaviour. Linear systems can be classified on the basis of location of poles and zeros on complex s plane. An LTI system is termed as minimum phase if the system dynamics and its inverse are stable and causal, While if a system and its inverse show causal but unstable behaviour then such an LTI system is termed as non-minimum phase system. The transfer function $Q(s)$ for a linear system can be represented as

$$Q(s) = \frac{Z(s)}{P(s)} \tag{2.4}$$

Here $Z(s)$, $P(s)$ are polynomial in complex variable s. If roots of polynomial $Z(s)$ known as zeros and/or roots of polynomial $P(s)$ known as poles lie on the left hand side of complex

s plane then system (2.4) is called as minimum phase system. On the other hand, if roots of polynomial $Z(s)$ and/or roots of polynomial $P(s)$ lie on the right hand side of complex s plane then system (2.4) is termed as non-minimum phase system. Although the amplitude response for both minimum phase as well as non-minimum phase systems are identical but the phase response for the non-minimum phase systems is always larger then for systems exhibiting minimum phase behaviour. As an example consider two systems represented by transfer functions $Q_1(s)$, $Q_2(s)$ as

$$Q_1(s) = \frac{1+sT}{1+sT_1} \tag{2.5}$$

$$Q_2(s) = \frac{1-sT}{1+sT_1} \tag{2.6}$$

The distribution of poles and zeros for $Q_1(s)$ and $Q_2(s)$ on the s-plane as shown in Fig. 2.1, clearly shows that the system having transfer function $Q_1(s)$ is minimum phase and the system with transfer function $Q_2(s)$ is non-minimum phase [Schmid, 2005]. The corresponding

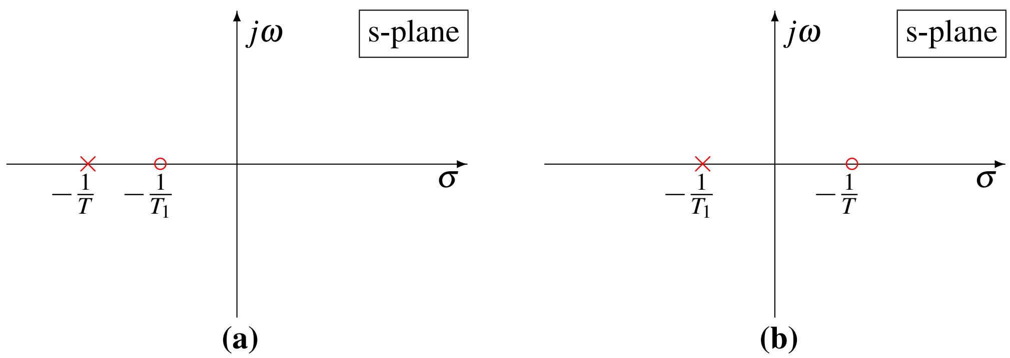

Fig. 2.1 Pole-Zero configuration in s-plane$(a) = Q_1(s)$, $(b) = Q_2(s)$[Schmid, 2005]

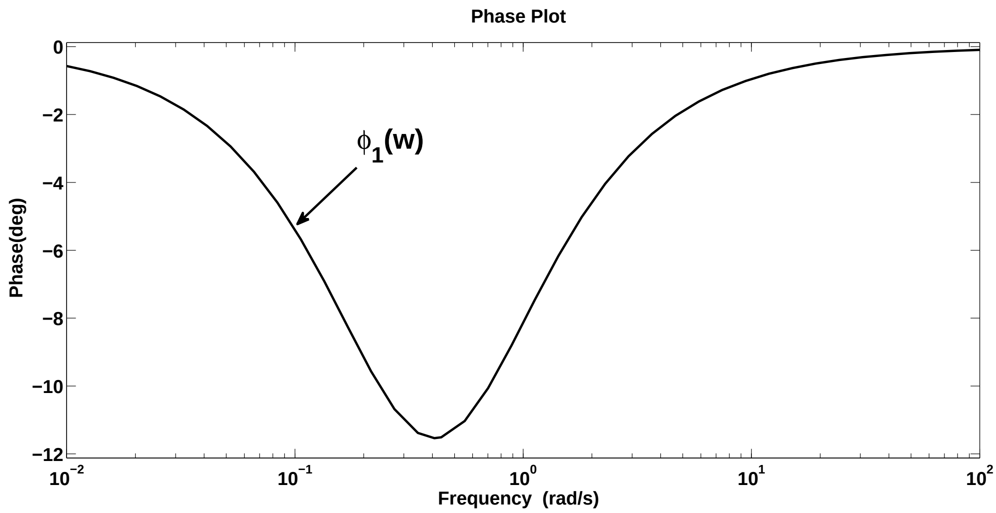

Fig. 2.2 Phase plot for a typical linear minimum phase system

amplitude response for both transfer functions is equal and is given by

$$\left.\begin{aligned} A(w) &= |Q_1(jw)| \\ &= |Q_2(jw)| \\ &= \sqrt{\frac{1+(wT^2)}{1+(wT_1^2)}} \end{aligned}\right\} \tag{2.7}$$

The difference between two systems $Q_1(s)$,$Q_1(s)$ becomes apparent from corresponding phase response of both systems. As $Q_1(s)$ is minimum phase and $Q_2(s)$ is non-minimum phase, so the difference occurs in their phase response given as follows and shown in Fig. 2.2 and Fig. 2.3 respectively.

$$\left.\begin{aligned} \phi_1(w) &= -\arctan\frac{w(T_1-T)}{1+w^2TT_1} \\ \phi_2(w) &= -\arctan\frac{w(T_1+T)}{1-w^2TT_1} \end{aligned}\right\} \tag{2.8}$$

Non-minimum phase Linear Time Invariant systems exhibit a number of constraints in their general behaviour such as time delayed response, reduction in bandwidth and instability of

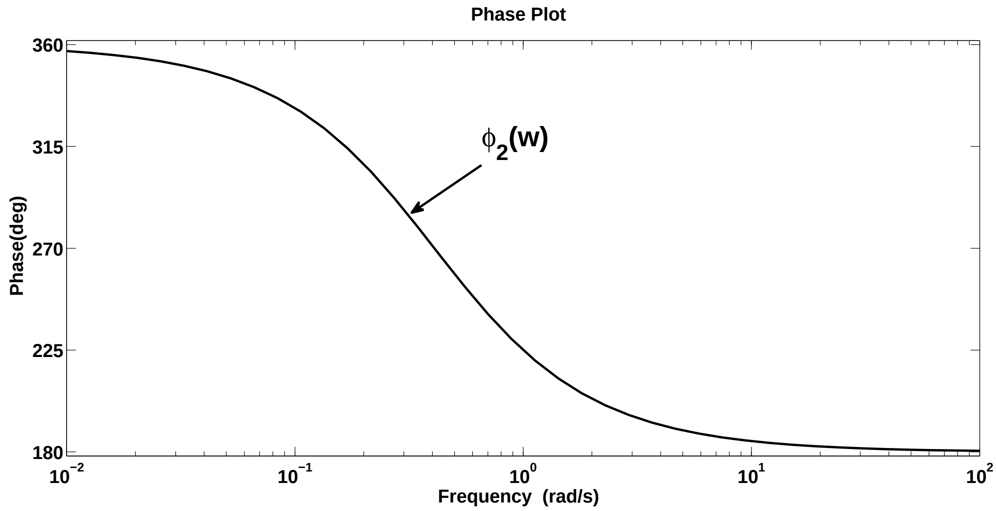

Fig. 2.3 Phase plot of a typical LTI non-minimum phase system

some of internal system states. A critical characteristics of non-minimum phase systems that needs to be addressed is its behaviour under step input. For step input, system initially shows an undesired response with overshoot/undershoot output before it finally converges to desired output. For a typical non-minimum phase LTI system, step response is shown in Fig. 2.4. For these systems number of roots of $Z(s)$ are not equal to number of roots of $P(s)$. This difference of roots is termed as relative degree represented as ρ of system. A general normal form of a system with n number of states and relative degree ρ is given as

$$\left.\begin{aligned} \dot{\eta} &= f(\eta,\xi) \\ \dot{\xi} &= A\xi + B\left(b(\eta,\xi) + a(\eta,\xi)u\right) \\ y &= C\xi \end{aligned}\right\} \tag{2.9}$$

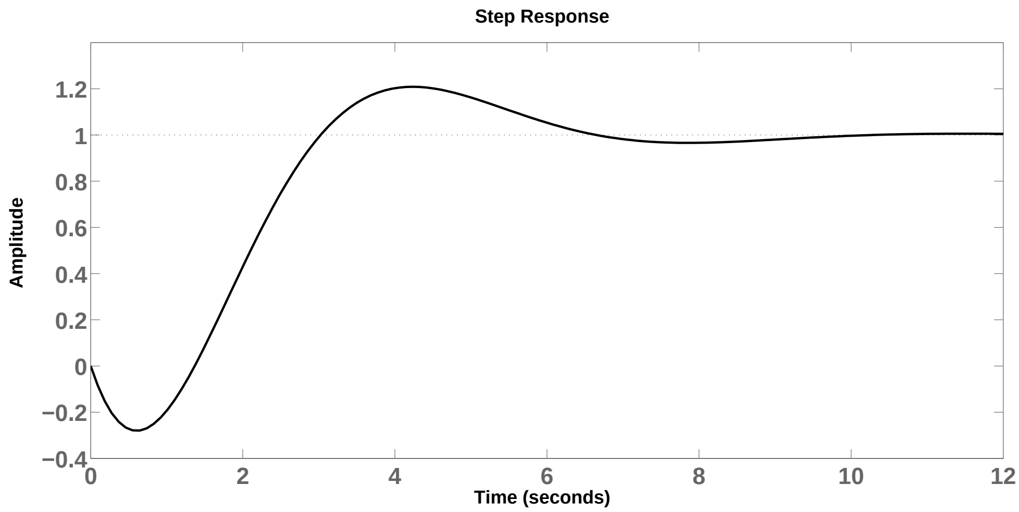

Fig. 2.4 Plot of Step Response of a typical LTI non-minimum phase system

where ξ are the controllable states, η are the internal dynamics (with number of states equal to $n-\rho$) and y is the output. The matrices A,B and C are given as

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \\
B &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{\mathsf{T}} \\
C &= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}
\end{aligned}
$$

with the order of $\rho \times \rho$, $\rho \times 1$ and $1 \times \rho$ respectively.

It is to be noted that controllability / observability of internal dynamics η are important as these properties plays vital role in stabilization or regulation based control designs of these type of systems. For a minimum phase system, if controller makes output convergent to desired trajectories and internal states η are bounded then control problem can be solved and closed loop stability is achieved. However, this is not true for the case of non-minimum phase system as instability of the system is not reflected in the output which is unacceptable. For these systems even if the feedback control law achieves output stabilization or regulation, making system output zero or tracking of reference signal is done perfectly but stability of internal states also needs to be catered for in the overall control law.

2.3 Nonlinear systems

Physical systems, in real world, exhibit nonlinear dynamics and are termed as nonlinear systems. These type of systems can only be approximated to linear systems within a small region of operation. These real life nonlinear systems show different properties than linear systems described in previous section. For nonlinear systems, principle of superposition (2.1) and homogeneity (2.2) does not hold. These systems behave quite irregularly to external control input as compared to linear systems. A number of common nonlinear properties are associated with these systems like finite escape time, limit cycles, bifurcations and chaotic phenomenon etc. Due to these properties, nonlinear systems exhibit much more complex behaviour than linear system. A nonlinear system non-affine in input termed as highly nonlinear system can be represented in generalized form as

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0 \tag{2.10}$$

$$y = h(x(t)) \tag{2.11}$$

Where manifold $x \in \Re^n$ and $u(t) \in \Re^m$ and functions $f : \Re^n \times \Re^m \to \Re^n$ and $h : \Re^n \to \Re$ are continuous smooth vector fields. x_0 is initial condition of x at $t = 0$. A more standard form of nonlinear systems termed as affine in input similar to the form (2.3) is given as

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0 \tag{2.12}$$

$$y = h(x(t)) \tag{2.13}$$

For the nonlinear system (2.10)-(2.13) control theory exists regarding controllability, observability, stabilization, tracking, noise or disturbance decoupling etc., [Olfati-Saber, 2000]. However, these methods are successful in local analysis and under global analysis these methods fail to work. Under global/semiglobal control design and analysis, the dynamics of nonlinear system affine in control (2.12)-(2.13), after change of coordinates, is transformed into a system or subsystem non-affine in control similar to the form in (2.10)-(2.11). A similar situation is faced for the case of under-actuated nonlinear systems and are required to be dealt with special design tools for solution of stabilization and regulation problem.

2.4 Zero Dynamics of Nonlinear Systems

Zero dynamics of a control system is termed as internal system dynamics when system output is limited to zero by the input. For linear systems, poles of zero dynamics are exactly the zeros of the original system. As discussed in Section 2.2, non-minimum phase behaviour of linear system depends on system zeroes and zero dynamics plays vital role in stability of system. For nonlinear systems, where concept of poles and zeroes of system is not directly present, zero dynamics becomes intrinsic property to study system behaviour as relative degree and internal stability of system are largely dependent on it. For the systems having relative degree equal to system order exact feedback linearization is possible. For the case of relative degree less than system order, exact linearization is not possible. However, such systems can be partially linearized under input output feedback linearization. The System 2.12, after partial feedback linearization, takes the form as

$$\dot{\eta} = f(\eta, \xi) \tag{2.14}$$

$$\dot{\xi} = g(\xi, u) \tag{2.15}$$

The above formulation is obtained after nonlinear global coordinate transformation. The dynamics of η does not depend on input u and is termed as internal dynamics as it has no effect on system output. The System (2.14)-(2.15) is composed of an external linear subsystem of dimension equal to system relative degree ρ and an internal subsystem of dimension $(n-\rho)$ respectively. In this way, system stability study becomes equivalent to study of its zero dynamics. For the system in (2.14), it can be represented as

$$\dot{\eta} = q(\eta, 0) \tag{2.16}$$

The system dynamics defined by (2.16) is termed as zero dynamics for nonlinear System (2.12)

2.5 Nonlinear Non-minimum Phase Systems

The non-minimum phase property of nonlinear system is dependent on its zero dynamics. It is an input output property and system output y is said to be non-minimum phase if zero dynamics of (2.14)-(2.15) is unstable. Such a system is termed as non-minimum phase nonlinear system. Control of such system is a challenging task as feedback linearization is not applicable and unstable zero dynamics prevent the use of input output feedback linearization. The most common examples of non-minimum phase system includes inverted pendulum, boiler temperature control, sideways motion control of ships and aircrafts, heave control of submarines and vertical take-off of aircraft etc. The class of test bench under-actuated system dealt in this thesis also fall in this category due to unstable zero dynamics. Examples include Inertia

Wheel Pendulum (IWP), Translational Oscillator Rotational Actuator (TORA) and Acrobot.

2.6 State Observer

In a state feedback control design, it is assumed that all system states are available which is practically often not possible. Measurement of some or all system states is quite difficult, unreliable or some times uneconomical and thus control algorithm often has to utilize system input or output only.

In order to overcome this bottleneck to measure system states different approaches can be utilized. First is to look new procedures that required fever measurements and the other is to adopt more complex dynamical feedback processing algorithms. This second approach is relatively simpler in which an approximation of full state vector on the basis of available measurements is constructed. The simplicity of this approach lies in utilization of earlier devised control algorithms by simply replacing the observed state vectors in place of the actual states. A device used for computing approximation of system states is known as observer. The observer is itself a dynamical system and is an important design tool used in feedback control design for estimation of systems states. It is a dynamical system used for estimations of some or all of system states. Its input values are the values measured from the original system and its state vector generate missing information about the states of the original system. State observer can be termed as a dynamic device that when connected to available system outputs generate the entire states. [Luenberger, 1966]

Observers or state estimators are important building blocks for controller design under output feedback control scheme. If an observer is used for estimation of all system states, regardless of whether some state variables are available for direct measurement or not, it is termed as a full state observer while observer that is used for estimation of some of system states than it is called a reduced-order state observer or, simply, a reduced-order observer.

2.6.1 Linear Observer

For a Linear Time Invariant (LTI) system, state observer measures its control variables depending on its output and control input. A wide range of observers are introduced in literature and are utilized for development of control algorithms. Some of these are explained in [Wang and Gao, 2003], where authors compare some of the widely used state observers. An important type of linear observer is known as Luenberger Observer [Luenberger, 1966].

Consider a linear system, with matrices A,B and C as parameters of its state space model, given as a manifold that solves the regulator equations.

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{2.17}$$

which is a general fully observable, linear, time invariant system where $x(t)$ is n-dimensional state vector, $u(t)$ is input, and $y(t)$ is output of length p. The observer will be another dynamical system that provides the system state vector. The mathematical formulation of observer will substantially be same as that of original system except that of a term depending on the difference between the observed and expected output values. This term is defined as estimation error and it compensate the inaccuracies in the matrices A and B and the lack of initial value knowledge. The mathematical model of the observer is defined as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \tag{2.18}$$

From the observer mathematical model (2.18), it is evident that $y(t)$ and $u(t)$ act as input for observer model and term L called observer gain matrix will effect the dynamical behaviour of the state estimate and state error. By substituting for the output vector, the differential equations for the observer error are given as

$$\left.\begin{aligned}\dot{\hat{e}}(t) &= \dot{\hat{x}}(t) - \dot{x}(t) \\ &= [A - LC]\big(\hat{x}(t) - x(t)\big) \\ &= [A - LC]e(t)\end{aligned}\right\} \tag{2.19}$$

Equations (2.19) shows that observer dynamical behaviour can be adjusted by manipulating the observer gain matrix L. Matrix L act as a weighting matrix for correction of terms involving the difference between the measured output y and the estimated output $C\hat{x}$. This term continuously corrects the model output and is responsible for improving the performance of the observer. The estimation error $e(t)$ exhibits asymptotic convergence, if $(A - LC)$ has all its eigenvalues in the left half s-plane. The selection of gain matrix L using some suitable method, for example, pole placement method is an important part of observer design. It is to be noted that initial values for observer dynamics needs not be identical with initial values of system states as long as the chosen eigenvalues for $(A - LC)$ are in the left half plane. The values of observer gain L can be used to choose location of all eigenvalues of matrix $(A - LC)$ and hence the rate at which observer error goes to zero can be adjusted. A critical part of the design is that it is very much dependent on the accuracy of state space matrices A, B and C.

2.6.2 High Gain Observer

High Gain Observer (HGO) was first introduced, as a continuation of work on state observers, for robust output feedback control of linear systems. Doyle and Stein [Doyle and Stein, 1979] introduced HGO as a design tool to recover system performance achieved under state feedback control. The use of HGO to achieve robustness in design was presented for linear system by a number of scholars in 1980s as in [Petersen and Hollot, 1988]. Saberi and Khalil [Khalil and Saberi, 1987], for the first time, introduced HGO for adaptive stabilization of nonlinear systems. This work opened more avenues towards enhancement of capabilities of HGO in dealing with real world nonlinear system issues which include disturbances, noise and system uncertainties. In this section a brief introduction of HGO for nonlinear system is presented.

The mathematical representation of normal form of nonlinear system is given as

$$\left.\begin{aligned} \dot{x}_i &= x_{i+1} \quad 1 \leq i \leq \rho - 1 \\ \dot{x}_\rho &= b(x,\theta) + a(x,\theta)u \\ y &= x_1 \end{aligned}\right\} \tag{2.20}$$

Here θ is unknown disturbance input to the system and ρ is system relative degree. An HGO for the system in (2.20) can be represented as shown below

$$\left.\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \frac{a_i}{\varepsilon^i}(y - \hat{x}_1) \quad 1 \leq i \leq \rho - 1 \\ \dot{\hat{x}}_\rho &= b(\hat{x}) + a(\hat{x})u + \frac{a_\rho}{\varepsilon^\rho}(y - \hat{x}_1) \end{aligned}\right\} \tag{2.21}$$

Here the design parameter ε is chosen to as small value as possible while $a_1, a_2, \ldots, a_\rho$ are constant parameters selected such that the polynomial $\lambda^\rho + a_1\lambda^{\rho-1} + \ldots + a_{\rho-1}\lambda + a_\rho$ is Hurwitz. Parameters $b(x)$ and $a(x)$ are the nominal model of $b(x,\theta)$ and $a(x,\theta)$ respectively.

The nominal model parameters if available are included in high gain observer design. This makes high gain observer model convergent at a faster rate which is very much desirable for an efficient state observer. However, if nominal model information is not available, high gain observer can still be designed by ignoring nominal model information. The design parameter ε chosen to a very small value could introduce peaking effect in transient phase of high gain observer making the initial condition of scaled estimation error $\frac{x_1(0)-\hat{x}_1(0)}{\varepsilon}$ of the order of $O(1/\varepsilon)$ when $x_1(0) \neq \hat{x}_1(0)$. This may add a term of the form $(\frac{1}{\varepsilon})e^{-\frac{\alpha t}{\varepsilon}}$ for some $\alpha > 0$. The exponential term decays rapidly as time increases under steady state. However, this may lead to unacceptable transient behaviour as well as destabilization of closed loop nonlinear system during transient phase due impulsive behaviour of $(\frac{1}{\varepsilon})e^{-\frac{\alpha t}{\varepsilon}}$ under HGO as ε tends to zero. This unacceptable behaviour of HGO can be avoided by rejecting unrealistic high outputs of state estimates during transient phase by making the control law and nominal functions as globally bounded. The behaviour can be avoided by making control law and/or system states as globally bounded so that system states/control input saturates outside compact sets of interest. This simpler but effective solution to undesired peaking effect of HGO initiated widespread usage of HGO for output feedback regulation of nonlinear systems.

The HGO based observer of states is beneficial for design of nonlinear control systems and concept of separation principle is utilized effectively. Towards that end, observer and controller design can be divided in two phases. In the first phase, control law u is designed under any suitable state feedback control technique. In the next phase, system states are estimated utilizing high gain observer. Finally, the actual states are replaced by estimated states in state feedback design to get an output feedback control law. Such a design is relatively simpler as compared to an alternate attempt of developing an output feedback control law using system output only.

2.6.3 Extended High Gain Observer

The high gain observer provides estimates of systems states utilizing its output only. It is very much effective for fully actuated system or for minimum phase under-actuated system where either there is no zero dynamics or zero dynamics is stable and does not effect stability of overall system dynamics. This is however not true for the case of non-minimum phase under-actuated system where zero dynamics needs to be effectively addressed as it is not stable. An Extended High Gain Observer (EHGO) is used not only to estimate the output and its derivative up to ρ^{th} order where ρ is system relative degree but also provides an estimate of a part of internal dynamics. The estimate of the part of internal dynamics is utilized to make internal dynamics observable when viewed as virtual output of the system. Considering (2.20) as normal form representation of a nonlinear system, than information regarding uncertain term $b(x,\theta)$ and $a(x,\theta)$ can be estimated through the use of EHGO. It is assumed that the sign of high frequency gain $a(x,\theta)$ is known but the uncertain terms $b(x,\theta)$ and $a(x,\theta)$ are unknown. Terms $b(x)$ and $a(x)$ are nominal model of $b(x,\theta)$ and $a(x,\theta)$ respectively. To design an EHGO a virtual output term σ can be defined as

$$\sigma \quad = b(x,\theta) - b(x) + \big(a(x,\theta) - a(x)\big)u \tag{2.22}$$

The system can be represented as an augmented set of equations (2.20) and (2.22) shown as

$$\left.\begin{aligned} &\dot{x}_i = x_{i+1} \quad 1 \le i \le \rho - 1 \\ &\dot{x}_\rho = b(x) + a(x)u + \sigma \\ &\dot{\sigma} = \phi(x,w,u,\dot{u}) \\ &y = x_1 \end{aligned}\right\} \tag{2.23}$$

The extended high gain observer is obtained by considering set of equations (2.23) as an augmented system and an HGO is designed in a similar way as explained in section (2.6.2) for the augmented system. The EHGO for system can be derived as

$$\left.\begin{aligned} &\dot{\hat{x}}_i = \hat{x}_{i+1} + \frac{a_i}{\varepsilon^i}(y - \hat{x}_1) \quad 1 \le i \le \rho - 1 \\ &\dot{\hat{x}}_\rho = b(\hat{x}) + a(\hat{x})u + \hat{\sigma} + \frac{a_\rho}{\varepsilon^\rho}(y - \hat{x}_1) \\ &\dot{\hat{\sigma}} = \frac{a_{\rho+1}}{\varepsilon^{\rho+1}}(y - \hat{x}_1) \end{aligned}\right\} \tag{2.24}$$

where ε is design parameter and it is chosen as small as possible. The value for the constant parameters $a_1, a_2, \ldots, a_{\rho+1}$ are selected such that the polynomial $\lambda^{\rho+1} + a_1\lambda^\rho + \ldots + a_\rho\lambda + a_{\rho+1}$ is Hurwitz. Keeping in view the above mentioned application of an EHGO, it can be used as

- Disturbance estimator
- Sensor of internal dynamics for
    1. Stabilization of non-minimum phase systems

2. Observer for internal dynamics

Consider a class of nonlinear systems having internal dynamics η and external dynamics ξ shown as

$$\left.\begin{aligned} \dot{\eta} &= \phi(\eta,\xi,\theta) \\ \dot{\xi}_i &= \xi_{i+1} \quad 1 \leq i \leq \rho - 1 \\ \dot{\xi}_\rho &= b(\eta,\xi,\theta) + a(\xi,\theta)u \\ y &= \xi_1 \end{aligned}\right\} \tag{2.25}$$

The internal dynamics may be stable or unstable rendering the system to exhibit as minimum phase or non-minimum phase behaviour. For the normal form of system (2.25), having known system structure and parameter θ, it can be assumed that structural information $\phi(\eta,\xi,\theta)$, $b(\eta,\xi,\theta)$ and $a(\xi,\theta)$ are available. The zero dynamics being unobservable but one of its component $\eta_p \in \eta$ where η_p denotes the information of η in $b(\eta,\xi,\theta)$ can be effectively utilized to reconstruct it from the partial information of its component. In this way η_p can be utilized to reconstruct zero dynamics. Keeping this in mind, an auxiliary output σ can be assumed as

$$\sigma = b(\eta,\xi,\theta) \tag{2.26}$$

Term $b(\eta,\xi,\theta)$ can be transformed into η_p as

$$\eta_p = b(\eta,\xi,\theta)v(\xi,\theta) \tag{2.27}$$

Where $v(\xi,\theta)$ is assumed as some known function. Now an extended high gain observer for system represented in set of equations (2.25) is taken as

$$\left.\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \frac{a_i}{\varepsilon^i}(y-\hat{\xi}_1) \quad 1 \le i \le \rho - 1 \\ \dot{\hat{\xi}}_\rho &= a(\hat{\xi},\theta)u + \hat{\sigma} + \frac{a_\rho}{\varepsilon^\rho}(y-\hat{\xi}_1) \\ \dot{\hat{\sigma}} &= \frac{a_{\rho+1}}{\varepsilon^{\rho+1}}(y-\hat{\xi}_1) \end{aligned}\right\} \tag{2.28}$$

Here ε and $a_1, a_2, \ldots, a_{\rho+1}$ are selected as depicted earlier. The term $\hat{\eta}_p$ can obtained as per equation (2.27) as

$$\hat{\eta}_p = \hat{\sigma} v(\hat{\xi},\theta) \tag{2.29}$$

To obtain the observer for internal state $\hat{\eta}_p$ is used as an input and an observer is obtained for zero dynamics as

$$\dot{\hat{\eta}} = \phi(\hat{\eta},\hat{\xi},\theta,\hat{\eta}_p) \tag{2.30}$$

One such observer for internal dynamics will be designed in Chapter 5 belonging to a class of under-actuated non-minimum phase systems namely, Translational Oscillator with Rotating Actuator (TORA) and Inertia Wheel Pendulum (IWP).

2.7 Summary

The preliminary concepts regarding linear and nonlinear systems are given in this chapter. Starting with the general theory of linear time invariant and nonlinear systems, the non-minimum phase property and effects of zero dynamics on system dynamics is discussed. For output feedback control, important design tool for estate estimation termed as high gain observer is explored. Finally, the application of extended high gain observer for estimation of linear as well as nonlinear system states is detailed.

Chapter 3

Under-actuated Nonlinear Systems

This chapter deals with theoretical description and mathematical formulation of under-actuated nonlinear systems. Starting with general description of nonlinear systems, the mathematical formulation for a typical nonlinear mechanical system is presented. Types of nonlinear systems based on nature of actuation leading to under-actuated and fully-actuated systems are presented. A comparison of under-actuated systems with those of fully actuated counterparts is provided to show the complexity of under-actuated systems. Classification of these systems, depending on mathematical formulation is described and finally a class of under-actuated systems having strict feedback normal form termed as Class-1 systems, particularly focused in this thesis, are described. The mathematical formulation and generalized model of Class-1 systems are also presented.

3.1 Mathematical Formulation of Nonlinear Systems

In this section, a generalized dynamical form of a typical under-actuated nonlinear system is formulated. Classical equations of motion for a nonlinear system are employed and distinguishing characteristics of an under-actuated system with that of a fully-actuated system are highlighted. Euler-Lagrangian equations of motion for this class of systems are utilized firstly, to obtain the generalized state space representation of system, and then, using suitable change of control/coordinates a typical normal form representation is derived.

3.1.1 Lagrangian Form

A nonlinear system affine in control $u \in \Re^m$, state vector $x \in \Re^n$ and output $y \in \Re^n$ can be represented as

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0 \tag{3.1}$$

$$y = h(x) \tag{3.2}$$

where $f : \Re^n \times \Re^m \to \Re^n$, $g : \Re^n \times \Re^m \to \Re^n$ and $h : \Re^n \to \Re$ are nonlinear continuous smooth functions. x_0 denotes initial states vector at $t = 0$ and $\dot{x}$ is defined as derivative of states x w.r.t. time (t). The differential equations (3.1-3.2) represent a class of nonlinear physical systems having n degrees of freedom and requires n-dimensional position (configuration) vector $\theta \triangleq [\theta_1, \theta_2, ..., \theta_n] \in Q$ for local representation.

The classical equation of motion for this type of systems, is given as

$$\frac{d}{dt}\left[\frac{\partial k}{\partial \dot{\theta}_i}\right] - \left[\frac{\partial k}{\partial \theta_i}\right] = T_i, \quad i = 1, 2, ..., n \tag{3.3}$$

where $\dot{\theta} \triangleq [\dot{\theta}_1, \dot{\theta}_2, ..., \dot{\theta}_n]$ are the generalized velocities and k is total kinetic energy of the system. The term T_i is sum of forces acting on the system denoted by matrix T_i^{ext} and forces obtained from potential energy function V of the system. This can be written as

$$T_i = -\frac{\partial V}{\partial \theta_i}(\theta) + T_i^{ext}, \quad i = 1, 2, ..., n \tag{3.4}$$

Lagrangian ($\mathscr{L}_0$) of this system can be represented as the difference of energies of the system as follows

$$\mathscr{L}_0 = \text{Kinetic Energy - Potential Energy} \tag{3.5}$$

or

$$\mathscr{L}_0 = k(\dot{\theta}, \theta) - V(\theta) \tag{3.6}$$

Using (3.3),(3.4) and (3.6), the Euler Lagrange equations of motion for this system can be derived as

$$\frac{d}{dt}\left[\frac{\partial \mathscr{L}_0}{\partial \dot{\theta}_i}\right] - \left[\frac{\partial \mathscr{L}_0}{\partial \theta_i}\right] = T_i^{ext}, \quad i = 1,2,...,n \tag{3.7}$$

and the total kinetic energy is given as

$$k(\dot{\theta},\theta) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} \tag{3.8}$$

where M is inertia matrix of the system. Using (3.7), and considering T_i^{ext} as control input u, a simplified equation in matrix form for a mechanical system can be derived as follows [Choukchou-Braham et al., 2013]

$$\sum_j m_{kj}(\theta)\ddot{\theta}_j + \sum_{i,j}\Gamma_{ij}^k(\theta)\dot{\theta}_i\dot{\theta}_j + g_k(\theta) = e_k^T F(\theta)u, \quad k = 1,2,...,n \tag{3.9}$$

where e_k is the kth standard basis in $\Re^n$, $g_k(\theta) = \partial_{\theta k}V(\theta)$ and $\Gamma_{ij}^k(\theta)$ the christoffel symbol is defined as follows:

$$\Gamma_{ij}^k(\theta) = \frac{1}{2}\left(\frac{\partial m_{kj}(\theta)}{\partial \theta_i} + \frac{\partial m_{ki}(\theta)}{\partial \theta_j} - \frac{\partial m_{ij}(\theta)}{\partial \theta_k}\right) \tag{3.10}$$

equation (3.9) can be written in matrix notation as

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = F(\theta)u \tag{3.11}$$

where $c_{ij} = \sum_{k=1}^n \Gamma_{kj}^i(\theta)\dot{\theta}_k$ is an element of $C(\theta,\dot{\theta})$. The term $C(\theta,\dot{\theta})\dot{\theta}$ having $\dot{\theta}_i, \dot{\theta}_j$ is a matrix containing centrifugal term for $i = j$ and coriolis-effect terms for $i \neq j$, $F(\theta)$ is $m \times n$ input or control matrix and $G(\theta)$ is the gravity matrix whose elements are given as

$$g_i(\theta) = \frac{\partial V}{\partial \theta_i}(\theta), \quad i \in n$$

The matrix defined by $S_0 = \dot{M}(\theta) - 2C(\theta, \dot{\theta})$ is antisymmetric, such that $\dot{M}(\theta) = C(\theta, \dot{\theta}) + C^T(\theta, \dot{\theta})$. Furthermore, $M(\theta)$ being positive symmetric definite allows to introduce Legendre Transform given as

$$\vartheta = \frac{\partial(\mathscr{L})}{\partial\theta} = M(\theta)\dot{\theta}$$

and equation (3.11) can be further simplified into $2n$ dimensional standard state space representation as

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \vartheta\end{bmatrix} = \begin{bmatrix} M^{-1}(\theta)\vartheta \\ -G(\theta) + C(\theta, M^{-1}\vartheta)M^{-1}(\theta)\vartheta \end{bmatrix} + \begin{bmatrix} 0 \\ F(\theta) \end{bmatrix} u \tag{3.12}$$

Equation (3.12), also called Legendre normal form for a general nonlinear mechanical system, can be represented in state space representation with variables $x_1 = \theta$, $x_2 = \vartheta$ as

$$\frac{d}{dt}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix} M^{-1}(x_1)x_2 \\ -G(x_1) + x_2^T Q(x_1)x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ F(x_1) \end{bmatrix} u \tag{3.13}$$

Equation (3.13) resembles the compact mathematical form affine in control input as given in (3.1).

3.2 Nonlinear System Types based on Actuation Property

The mathematical formulation of nonlinear systems derived in section 3.1 can be used to define two major system types based on actuation properties of generalized configuration vector of entire system space. The number of controls are either equal to or less than dimension of configuration space which leads to define fully actuated or under-actuated nonlinear systems as explained in next sections.

3.2.1 Fully-actuated Systems

A physical system defined by Euler-Lagrange's equations (3.3) is defined as fully actuated if $m = n$ i.e., $F(\theta)$ is invertible. The number of inputs for such systems is equal to the dimension of their configuration space. As a result, these systems are fully feedback linearizable and does not contain any zero dynamics. For these cases a double integrator system is obtained and the control problem is reduced to linear system problem as shown below:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v \end{aligned}\right\} \tag{3.14}$$

$$u = F^{-1}(\theta)\left[M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta)\right] \tag{3.15}$$

3.2.2 Under-actuated Systems

A mechanical system with configuration space $\theta \in Q$ and Lagrangian $\mathscr{L}_0(\dot{\theta},\theta)$ represented by Euler-Lagrange equation (3.7) is said to possess under-actuated dynamics, if rank(m) of $F(\theta) < n = dim(Q)$. Unlike fully-actuated systems, under-actuated systems do not allow exact feedback linearization, due to lesser number of actuators than dimensions of the configuration manifold. As a result of this constraint the generalized system inputs cannot control instantaneous acceleration in every direction. Due to these type of constraints such systems are also termed as non-holonomic mechanical systems [Bloch et al., 1992; Ne_mark and Fufaev, 2004; Reyhanoglu et al., 1999; Wichlund et al., 1995].

A mechanical system is termed as having first order non-holonomic constraints whose Lagrangian system have m ($m < n$) velocity constraints $W^T(\theta)\dot{\theta} = 0$ are non-integrable as detailed in [Choukchou-Braham et al., 2013; Olfati-Saber, 2000]. In other words, no function

$\varphi(t)$ exists such that $\dot{\varphi} = W^T(\theta)\dot{\theta}$. The dynamics of such systems can be written as

$$\left.\begin{aligned} M(\theta)\ddot{\theta} + N(\theta,\dot{\theta}) = W(\theta)\lambda + F(\theta)v \\ W^T(\theta)\dot{\theta} = 0 \end{aligned}\right\} \tag{3.16}$$

Here W is an $m \times n$ matrix, $\lambda \in \Re^m$ is the vector of Lagrange multiplier and the term $W(\theta)\lambda$ is termed as requisite force to maintain constraints. The mechanical systems described in 3.1.1 have been discussed widely by researchers as in [Bloch et al., 1992; Ne_mark and Fufaev, 2004; Olfati-Saber, 2000] and the references therein.

Unlike the systems described in (3.16), having non-holonomic velocity constraints, the lack of actuators in under-actuated systems is often termed as constraints on the acceleration or a second order non-holonomic constraints. The under-actuated test-bench nonlinear systems discussed in this thesis are regarded as non-holonomic systems with second-order acceleration constraints i.e., a constraint that involves second-order time derivatives of the configuration variables. Further details regarding non-holonomic systems with second-order acceleration constraints can be seen in [Fantoni and Lozano, 2002; Reyhanoglu et al., 1999; Wichlund et al., 1995]. Here a brief mathematical description, system classification and some examples of a class of these systems are presented in next sections.

3.2.3 Mathematical Representation for an Under-actuated System

Under-actuated systems with fewer number of controls than the number of configuration variables [Olfati-Saber, 2000] exhibit non-holonomic acceleration constraints and external inputs cannot provide instantaneous accelerations in all possible directions of the configuration space. In this case, some of the system equations are unactuated and system can be divided into two major parts, i.e., an actuated and an unactuated subsystem. In this case exact feedback linearization is not possible because actuation matrix is non-invertible and only partial feedback linearization is possible. For this purpose, the configuration space $\theta \in Q$ can be bifurcated into two sub-systems sometimes called as outer system and an inner system. The equations of motion for an under-actuated nonlinear system with control input v can be expressed as

$$M(\theta)\ddot{\theta}+C(\theta,\dot{\theta})\dot{\theta}+G(\theta)=F(\theta)v \tag{3.17}$$

where $v\in\Re^m$ is the control, $F(\theta)\in\Re^{n\times m}$ is an external non-square control matrix with $m<n$ and having full column rank taking the form as $F(\theta)=[0,I_m]^T$. The n-dimensional configuration vector space can now be divided up as $\theta=(\theta_x,\theta_s)\in\Re^{n-m}\times\Re^m$ where θ_x,θ_s represent unforced and forced components of configuration vector space, respectively. This presence and absence of actuation in some part of configuration manifold determines the shape of the system such that the inertia matrix of these systems is independent of a subset of configuration variables. This subset of variables is termed as external variables and complement of these are called as shape variables. In other words, shape variables are those variables that appear in inertia matrix $M(\theta)$ of a simple under-actuated system. This further implies that for external variables where $\partial M(\theta)/\partial(\theta_x)=0$, the following identity holds

$$\frac{\partial k(\theta,\dot{\theta})}{\partial(\theta_x)}=0 \tag{3.18}$$

which shows kinetic symmetry w.r.t external variables. Thus Lagrangian (3.6) for such systems can now be represented as

$$\mathscr{L}_0=\frac{1}{2}\dot{\theta}^T M(\theta_s)\dot{\theta}-V(\theta) \tag{3.19}$$

and corresponding equation of motion can be decomposed as:

$$\left.\begin{aligned}\frac{d}{dt}\left[\frac{\partial\mathscr{L}_0}{\partial\dot{\theta}_x}\right]-\left[\frac{\partial\mathscr{L}_0}{\partial\theta}\right]&=F_x(\theta)v,\ \ v\in\Re^m\\ \frac{d}{dt}\left[\frac{\partial\mathscr{L}_0}{\partial\dot{\theta}_s}\right]-\left[\frac{\partial\mathscr{L}_0}{\partial\theta}\right]&=F_s(\theta)v,\ \ v\in\Re^m\end{aligned}\right\} \tag{3.20}$$

The Input matrix F in under-actuated systems (3.20) is represented by $F = (F_x, F_s)$ and the rank of F is always less than n. To analyse these type of systems, a number of cases can be defined depending on whether the shape configuration variables are fully actuated, partially actuated or unactuated, and the presence or absence of input coupling due to the input matrix F.

3.3 Classification of Systems

Under-actuated mechanical systems can be classified in a variety of ways depending on system properties, reasons of under-actuation, type of control applied etc. An important aspect on which these types of systems are classified is nature of configuration characteristics [Liu and Yu, 2013]. These characteristics are given as

1. Actuated shape variables
2. Non-interacting inputs
3. Integrable normalized momentums and
4. Some extra conditions requirement

Depending on these characteristics, a number of system classes is inferred based on whether shape configuration vector θ_s is fully actuated, partially actuated or unactuated. Similarly, presence or absence of any input coupling due to the input (force) matrix F_θ and non-integrability of normalized momentums provide a way of classification. For these systems, a generalized methodology can be evolved to reduce it into a cascade normal form. Reza in [Olfati-Saber, 2000] presented eight different classes of under-actuated systems along with three types of normalized forms into which each class can be transformed. These three normal forms for cascade nonlinear systems are given as

1. Strict feedback form

Table 3.1 Classification of Under-actuated Systems

Class	Cascade Normal Form	Examples
I	Strict Feedback	Inertia Wheel Pendulum (IWP), Translational Oscillator with Rotational Actuator (TORA), Acrobot (2 Dimensional, 1 Control)
IIa	Nontriangular Quadratic	Rotating Pendlum, Pendubot, Beam n Ball
IIb	Nontriangular Linear	≈ Flexibie-link Robots
III	Feedforward	Cart-Pole (2 Dimensional, 1 Control)
IVa	Nontriangular Quadratic	Controlled VTOL aircraft (3 Dimensional, 1 Control)
IVb	Nontriangular Quadratic	Controlled VTOL aircraft (3 Dimensional, 1 Control)
V	Strict Feedback	VTOL aircraft (3 Dimensional, 2 Control), ≈Aircraft (6 Dimensional, 4 Control), ≈ Helicopter (6 Dimensional, 4 Control)
VIa	Nontriangular Linear	Three-link Pendulum (3 Dimensional, 2 Control)
VIb	Perturbed Strict Feedback	Three-link Pendulum (3 Dimensional, 2 Control)
VIIa	Nontriangular Linear-quadratic	Three-link Pendulum (3 Dimensional, 2 Control)
VIIb	Perturbed Nontriangular Quadratic	Three-link Pendulum (3 Dimensional, 2 Control)
VIII	Strict Feedforward	3D Cart-Pole System (4 Dimensional, 2 Control)

2. Strict feedforward form

3. Non-triangular linear-quadratic form

Depending on above mentioned properties, different classes of under-actuated systems and their corresponding normal forms are given in Table 3.1. The eight classes of systems shown in Table 3.1 are obtained on the basis of different structural properties of under-actuated systems. The Euler Lagrange equations are given by

$$\begin{bmatrix} m_{xx}(\theta_s) & m_{xs}(\theta_s) \\ m_{sx}(\theta_s) & m_{ss}(\theta_s) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_s \end{bmatrix} + \begin{bmatrix} h_x(\theta,\dot{\theta}) \\ h_s(\theta,\dot{\theta}) \end{bmatrix} = \begin{bmatrix} F_x(\theta)v \\ F_s(\theta)v \end{bmatrix} \tag{3.21}$$

where m_{ij}'s are block matrices obtained after partitioning of inertia matrix $M(\theta)$ and h_x, h_s are position/ velocity components of unforced and forced system dynamics. The parameters like input matrix $F(\theta) = (F_x(\theta), F_s(\theta)), h_x(\theta, \dot{\theta}), h_s(\theta, \dot{\theta})$ corresponds to the nature of actuation or unactuation of shape or external variables, coupling or non-coupling of the inputs, the integrability or non-integrability of the generalized momentum and supplementary condition that are discussed in detail in [Choukchou-Braham et al., 2013; Olfati-Saber, 2000].

3.4 Underactated Systems with Actuated Shape Variables, Integrable Momentums and Non-interacting Inputs

The Class-I systems are identified as systems having actuated shape variables, integrable normalized momentums and decoupled or non-interacting inputs. The input or force matrix $F(\theta)$ can be bifurcated as $F_x(\theta) = 0, F_s(\theta) = I_m$. The mathematical representation of such systems is obtained after system decomposition and partitioning of the inertia matrix $M(\theta)$. It can be represented in generalized form given as

$$\begin{bmatrix} m_{xx}(\theta_s) & m_{xs}(\theta_s) \\ m_{sx}(\theta_s) & m_{ss}(\theta_s) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_s \end{bmatrix} + \begin{bmatrix} h_x(\theta, \dot{\theta}) \\ h_s(\theta, \dot{\theta}) \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix} \tag{3.22}$$

These systems show kinetic symmetry with respect to external variables. In other words $M(\theta) = M(\theta_s)$ i.e., the inertia matrix only depend on shape variables and are independent of external variables. Due to above mentioned properties, the general model of this class of systems is of the following form

$$\left.\begin{aligned} m_{xx}(\theta_s)\ddot{\theta}_x + m_{xs}(\theta_s)\ddot{\theta}_s + \frac{d}{d\theta_s} m_{xx}(\theta_s)\dot{\theta}_x\dot{\theta}_s + \frac{d}{d\theta_s} m_{xs}(\theta_s)\dot{\theta}_s^2 - \frac{\partial V(\theta)}{\partial \theta_x} = 0 \\ m_{sx}(\theta_s)\ddot{\theta}_x + m_{ss}(\theta_s)\ddot{\theta}_s - \frac{1}{2}\frac{d}{d\theta_s} m_{xx}(\theta_s)\dot{\theta}_x^2 + \frac{1}{2}\frac{d}{d\theta_s} m_{ss}(\theta_s)\dot{\theta}_x^2 - \frac{\partial V(\theta)}{\partial \theta_s} = v \end{aligned}\right\} \tag{3.23}$$

A particular feature of this class of systems is that it shows lack of control in its first part and hence exact feedback linearization is not possible using any suitable change of control. However, as shown in [Qaiser, 2009] and the references therein, it is possible to partially linearize the system, so that dynamics of the actuated part is transformed into a linear system using a method called Partial Feedback Linearization. As mentioned earlier, every under-actuated system of this class can be transformed into cascaded strict feedback normal form using a global change of coordinates obtained from Lagrangian of the system. The resulting form contains a cascade of two types of dynamics also called internal and external dynamics. The core internal dynamics is nonlinear while the external system is in linear form. The internal nonlinear system may contain unstable zero dynamics which leads to a non-minimum phase behaviour of the overall system. The stabilization and output feedback control of Class-1 under-actuated test-bench non-minimum phase systems potentially having unstable zero dynamics is the control problem discussed in this thesis.

3.5 Summary

In this chapter, a general overview of under-actuated nonlinear systems is discussed. Starting with Lagrangian form of typical nonlinear systems, mathematical formulation of nonlinear mechanical system possessing kinetic symmetry is presented. Types of nonlinear system based on actuation properties which leads to define and classify under-actuation systems based on systems structural properties is presented and finally typical mathematical representation of a particular class of systems is identified.

Chapter 4

Output Feedback Stabilization of a Class of Under-actuated Systems

In this chapter, a generalized control design procedure for a class of under-actuated nonlinear non-minimum phase systems in strict feedback normal form as identified in Chapter 3 is presented. A step by step procedure for mathematical model transformation of target class of under-actuated systems using collocated partial feedback linearizaion and global change of coordinates to generate normal form representation is discussed. A generalized procedure of observer based controller design is proposed under sliding mode framework. The design is done on the basis of Output Feedback (OFB) control approach while State Feedback (SFB) design is utilized for comparison and analysis of OFB control results. The stability analysis of closed loop system is investigated both under state feedback control and output feedback control scheme. The complete design procedure for the class of system consists of following steps

1. Reduction of nonlinear dynamics into cascade strict feedback form.
2. Control of nonlinear part using Control Lyapunov Function (CLF) approach.
3. Overall controller design under State Feedback Control (SFB) and Output Feedback Control (OFB) utilizing controller obtained in previous step.

4.1 Mathematical Form of a Class of Under-actuated Systems

As mentioned in Section 3.4 core subsystem in cascade strict feedback form of under-actuated systems (3.23) is highly nonlinear and exhibit unstable zero dynamics. Thus exact feedback linearization is not applicable in this case and only Input Output Feedback Linearization is possible.

4.1.1 Collocated Partial Feedback Linearization

Every under-actuated nonlinear system of target class of systems can be transformed into strict feedback normal form using a procedure known as collocated partial feedback linearization. In this process, system is linearized with respect to active degree of freedom while unactuated part of system remains nonlinear. The procedure is performed by a global invertible change of control on the generalized form of system (3.17),(3.22) and is given as

$$v = [F^T(\theta)M^{-1}(\theta)F(\theta)]^{-1}[\tau + F^T(\theta)M^{-1}(\theta)(C(\theta,\dot{\theta}) + G(\theta))] \tag{4.1}$$

The new control input τ becomes $\ddot{\theta}_s$ showing a linear relationship between the dynamics of τ and $\ddot{\theta}_s$. This feedback linearization is called collocated partial feedback linearization where the dynamics of actuated shape variables θ_s are linearized.

4.1.2 Generalized Mathematical Model

The generalized mathematical representation obtained after collocated partial feedback linearization of target class of systems is termed as strict feedback normal form. The change of control (4.1) in Section 4.1.1 can be represented in compact notation as

$$v = \alpha(\theta_x)\tau + \beta(\theta,\dot{\theta}) \tag{4.2}$$

where $\alpha(\theta_x)$ is an $m \times m$ positive definite and symmetric matrix. $\alpha(\theta_x)$ and $\beta(\theta,\dot{\theta})$ are defined as

$$\begin{aligned}\alpha(\theta_x) &= m_{ss}(\theta_x) - m_{sx}(\theta_x)m_{xx}^{-1}(\theta_x)m_{xs}(\theta_x)\\ \beta(\theta,\dot{\theta}) &= h_s(\theta,\dot{\theta}) - m_{sx}(\theta)m_{xx}^{-1}h_x(\theta,\dot{\theta})\end{aligned}$$

The change of control (4.2) transform the entire dynamics into the following form

$$\left.\begin{aligned}\dot{\theta}_{z_1} &= \theta_{z_2}\\ \dot{\theta}_{z_2} &= f_0(\theta_x,\theta_s) + g_0(\theta)\tau\end{aligned}\right\} \tag{4.3}$$

$$\left.\begin{aligned}\dot{\xi}_{x_1} &= \xi_{x_2}\\ \dot{\xi}_{x_2} &= \tau\end{aligned}\right\} \tag{4.4}$$

The dynamics (4.3),(4.4) contains the control input in both internal and external subsystems. This is a source of complexity when controlling under-actuated nonlinear systems. To decouple the two sub-systems, a global change of variables or diffeomorphism is introduced as [Olfati-Saber, 2000]

$$\left.\begin{aligned}\eta_1 &= \theta_x + \gamma(\theta_s)\\ \eta_2 &= m_{xx}(\theta_s)\dot{\theta}_x + m_{xs}(\theta_s)\dot{\theta}_s\end{aligned}\right\} \tag{4.5}$$

The decoupled internal and external sub-systems obtained as a generalized normal form for the class of systems under consideration is obtained as

$$\left.\begin{aligned}\dot{\eta}_1 &= m_{xx}^{-1}(\xi_1)\eta_2 + \gamma(\xi_1)\\ \dot{\eta}_2 &= \kappa(\eta_1,\xi_1)\end{aligned}\right\} \tag{4.6}$$

$$\left.\begin{aligned}\dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= \tau\end{aligned}\right\} \tag{4.7}$$

where

$$\begin{aligned}
\gamma(\xi_1) &= \int_0^{\xi_1} \frac{m_{xs}(\xi_1)}{m_{xx}} ds, \\
\kappa(\eta_1,\xi_1) &= -\frac{\partial V_r(\eta_1,\xi_1)}{\partial \eta_1} \\
V_r(\eta_1,\xi_1) &= V(\eta_1 - \gamma(\xi_1),\xi_1)
\end{aligned}$$

The system dynamical model represented as a cascaded interconnection of linear and nonlinear subsystems (4.6),(4.7) is shown in Fig. 4.1. The systems (4.6) and (4.7) can be further

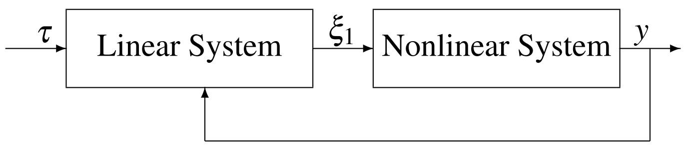

Fig. 4.1 Cascade interconnection of a Class of Under-actuated System

represented in compact notation with original control v as

$$\left.\begin{aligned}
\dot{\eta} &= \phi_0(\eta,\xi) \\
\dot{\xi} &= A\xi + B[\beta(\eta,\xi) + \alpha(\xi,v)] \\
y &= \xi_1
\end{aligned}\right\} \tag{4.8}$$

With $\eta = [\eta_1 \ \eta_2]^T$, ϕ_0=$[m_{xx}^{-1}(\xi_1)\eta_2 + \gamma(\xi_1) \ \kappa(\eta_1,\xi_1)]^T$, and the $m \times m$ matrix A, $m \times 1$ input matrix B represent chain of m integrators. State ξ_1 is taken as the system output y.

4.2 Design of Control Law

The generalized systems model (4.8) is utilized to design an output feedback control law under sliding mode frame work. The states of internal and external sub-systems are estimated utilizing system output under an appropriate observer as shown in the next section.

4.2.1 Observer Design

The required system states are estimated using full order EHGO. Towards that end, the approach of [Boker and Khalil, 2013] is followed and an auxiliary system consisting of internal system η and a virtual output σ is considered as

$$\dot{\eta} = \phi_0(\eta, \xi) \tag{4.9}$$

$$\sigma = \beta(\eta, \xi) \tag{4.10}$$

With above in view, the full order Extended High Gain Observer can be written as

$$\dot{\hat{\xi}} = A\hat{\xi} + B[\hat{\sigma} + \alpha(\hat{\xi}, v)] + H_1[y - \hat{\xi}_1] \tag{4.11}$$

$$\dot{\hat{\sigma}} = \dot{\beta}(\hat{\eta}, \hat{\xi}) + H_2[y - \hat{\xi}_1] \tag{4.12}$$

$$\dot{\hat{\eta}} = \phi_0(\hat{\eta}, \hat{\xi}) + K[\hat{\sigma} - \beta(\hat{\eta}, \hat{\xi})] \tag{4.13}$$

$$y = \xi_1 \tag{4.14}$$

where H_1=$[\alpha_1/\varepsilon \;\; \alpha_2/\varepsilon^2 \; \ldots \; \alpha_m/\varepsilon^m]^T$, H_2=$[\alpha_{m+1}/\varepsilon^{m+1}]$ and K are the observer gains chosen such that the observer dynamics remain faster than those of the overall closed loop system under SFB. The gain parameters $\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1}$ are, therefore, chosen so that roots of the polynomial $p^{m+1} + \alpha_1 p^m + \ldots + \alpha_m p + \alpha_{m+1}$ are in the open left-half plane. The parameter ε is small positive constant and the gain K can be obtained from any suitable online state observer.

4.2.2 Controller Design

For the control law synthesis, the internal sub-system (4.6) is considered first which is unforced and unperturbed complex nonlinear system. It is stabilized using control Lyapunov function (CLF) method [Olfati-Saber, 2000]. Treating $\xi_1 \equiv \omega$ as virtual control input to the system, there exist a state feedback law as

$$\omega(\hat{\eta}_1, \hat{\eta}_2) = \alpha_r(\hat{\eta}_1) - a\rho(c_1\hat{\eta}_1 + c_2\hat{\eta}_2) \tag{4.15}$$

It can be seen that (4.15) can be effectively utilized for stabilization of the internal system (4.9). Here $\alpha_r(\eta_1)$ is a smooth function such that $\kappa(\hat{\eta}_1, \alpha_r(\hat{\eta}_1)) = 0$, $\alpha_r(0) = 0$ and $\rho(c_1\hat{\eta}_1 + c_2\hat{\eta}_2)$ in (4.15) is any scalar sigmoidal function. The external subsystem control law is designed in such a way that its output, acting as virtual input for internal subsystem, generates the desired stabilization function. An equivalent sliding mode control scheme is utilized for generation of requisite stabilization function. The generalized system (4.8) and internal system feedback control law (4.15), satisfies the necessary assumptions needed to design sliding mode controller for external system.

Starting with error dynamics of external configuration vector as

$$\hat{e}_1 = \hat{\xi}_1 - \hat{\xi}_{1d} \tag{4.16}$$

$$\hat{e}_2 = \hat{\xi}_2 - \hat{\xi}_{2d} \tag{4.17}$$

where $\hat{e}_1$ is the error in the generation of stabilization function $\hat{\xi}_{1d} = \omega(\hat{\eta}_1, \hat{\eta}_2)$ by $\hat{\xi}_1$ and $\hat{e}_2$ is the error in the generation of $\hat{\xi}_{2d}$ and $\hat{\xi}_2$. The time-derivatives of the error dynamics equations are

$$\dot{\hat{e}}_1 \triangleq \dot{\hat{\xi}}_1 - \dot{\hat{\xi}}_{1d} = f_1(\hat{\eta}, \hat{\xi}, \hat{\xi}_{1d}, \dot{\hat{\xi}}_{1d}) \tag{4.18}$$

$$\dot{\hat{e}}_2 \triangleq \dot{\hat{\xi}}_2 - \dot{\hat{\xi}}_{2d} = f_2(\hat{\xi}_{2d}, \dot{\hat{\xi}}_{2d}, \tau_{eq}) \tag{4.19}$$

where τ_{eq} is equivalent control law obtained from sliding surface $\hat{s}$ and is used to obtain final control law. The derivatives $\dot{\hat{\xi}}_{1d}, \dot{\hat{\xi}}_{2d}$ can be easily calculated using numerical differentiation with sufficient accuracy. This will, however, require controller to be implemented in discrete form. To avoid this complexity a low pass filter with small positive time constant can be used.

Function f_1 is calculated such that the error dynamics approach to zero and a time varying sliding surface $\hat{s}$ is given as

$$\hat{s} = \lambda \hat{e}_1 + \hat{e}_2 \tag{4.20}$$

The time derivative of sliding surface is used to calculate an equivalent control law τ_{eq}. A sliding mode control law is then evaluated in terms of system states and error dynamics such that the time derivative of sliding surface tends to zero that renders the system asymptotically stable; and is given by

$$\dot{\hat{s}} = f_3(\hat{\eta}, \hat{\xi}, \dot{\hat{\xi}}_1, \dot{\hat{\xi}}_2, \tau_{eq}) \tag{4.21}$$

$$\tau = \tau_{eq} - k(sign(\hat{s})) \tag{4.22}$$

where $k > 0$ is a constant. The control law τ obtained in such way shows high frequency oscillations known as chattering in the sliding phase which are very much undesirable. The chattering effect can be minimized by replacing sign function with a suitable saturation function in (4.22) and the resulting control law with original control v for actual system described in (4.8) takes the form

$$\left.\begin{aligned} \tau &= \tau_{eq} - ksat\left(\frac{\hat{s}}{\mu}\right) \\ v &= \alpha\tau + \beta \end{aligned}\right\} \tag{4.23}$$

where μ is a small positive constant defining boundary layer for sliding surface.

4.3 Stability Analysis

Stability analysis is performed in two steps. First, under state feedback and then under output feedback that uses a full order EHGO. In the case of OFB design, observer dynamics become significant and need to be analyzed to ensure closed loop system stability. In first step, internal nonlinear system and cascaded system stability is analyzed, using suitable analytical tools under the stated assumptions. Lyapunov function method is used to analyze the subsystem represented by (4.13) with virtual control law (4.15) and the steps as in [Olfati-Saber, 2000] are followed. For the state feedback stability analysis, the system (4.11)-(4.14) is considered assuming $(\hat{\eta},\hat{\xi}) = (\eta,\xi)$. As $\hat{\xi}$ act as virtual control input for (4.9), a positive definite function $V_\circ(\hat{\eta}) = \frac{1}{2}\hat{\eta}^2$ is chosen as Lyapunov function candidate and $\dot{V}_\circ(\hat{\eta}) = (\hat{\eta})[\phi_0(\hat{\eta},\hat{\xi})]$ is examined for stability analysis of the subsystem. Assuming $\hat{\xi} = \omega$, the virtual control input; using a sigmoidal function as $\rho(\hat{\eta}_1,\hat{\eta}_2)$ and a smooth function $\alpha(\hat{\eta}_1)$ in (4.15), it can be easily verified that

$$V_\circ(0) = 0; V_\circ(\hat{\eta}) > 0; \dot{V}_\circ(0) = 0; \text{ and } \dot{V}_\circ(\hat{\eta}) \leq 0$$

Using the above conditions and with the help of LaSalle's invariance principle, it can be shown that the origin of the nonlinear subsystem is globally asymptotically stable. The composite system (4.6),(4.7) under control law (4.23) can be represented in terms of internal states and sliding surface as

$$\left.\begin{aligned} \dot{\hat{\eta}}_1 &= m_{xx}^{-1}(\omega(\hat{\eta}_1,\hat{\eta}_2)+\hat{s})\hat{\eta}_2 + \gamma(\omega(\hat{\eta}_1,\hat{\eta}_2)+\hat{s}) \\ \dot{\hat{\eta}}_2 &= \kappa(\hat{\eta}_1,\omega(\hat{\eta}_1,\hat{\eta}_2)+\hat{s}) \end{aligned}\right\} \tag{4.24}$$

$$\left.\begin{aligned} \hat{s} &= \lambda\hat{e}_1 + \hat{e}_2 \\ \dot{\hat{s}} &= f_3(\hat{\eta},\hat{\xi},\dot{\hat{\xi}}_1,\dot{\hat{\xi}}_2,\tau_{eq}) \end{aligned}\right\} \tag{4.25}$$

The internal states and sliding surface formulate complex cascaded system and it will be not be vert easy to find a single Lyapunov function candidate to prove stability of closed loop system. The stability of individual components (4.24) and (4.25) of the overall composite system can however be considered to conclude stability of overall closed loop system. To proceed in this manner, review of necessary assumptions regarding the dynamics of the target class of under-actuated systems is necessary.

The inner, nonlinear subsystem (4.24) with zero input from outer system (4.25) is globally Lipschitz. This assumption may appear as stringent but it holds for all the benchmark systems belonging to the class of under-actuated systems (see for ref. [Qaiser, 2009]) and further meets the assumption for global Lipschitz property as system partial derivatives up to second order are bounded [Qaiser et al., 2007]. A suitable sigmoid function, when chosen as input to internal system, will not alter said property of system (4.24). A zero input from outer system, further renders the inner system as globally asymptotically stable [Olfati-Saber, 2000].

The stability of external, driving system (4.25) can be inferred by considering a quadratic Lyapunov function candidate in terms of $\hat{s}$ as follows

$$V = \frac{1}{2}\hat{s}^2$$

Taking time derivative of V and substituting the values of u and u_{eq}, $\dot{V}$ is obtained as

$$\dot{V} = \hat{s}\dot{\hat{s}} = -\hat{s}ksat\left(\frac{\hat{s}}{\mu}\right)$$

Outside the boundary layer, i.e., When $||\hat{s}|| \geq \mu$

$$\dot{V} = -k\hat{s}\left(\frac{\hat{s}}{||\hat{s}||}\right) = -k||\hat{s}|| < 0$$

Inside the boundary layer, i.e., when $||\hat{s}|| \leq \mu$

$$\dot{V} = -\hat{s}k\left(\frac{\hat{s}}{\mu}\right) = -\left(\frac{k}{\mu}\right)||\hat{s}||^2 < 0$$

which shows that by choosing $k > 0$, the trajectories of the closed loop system starting with arbitrary initial conditions will reach the boundary layer $\hat{s} = \mu$ in finite time and remain inside it thereafter.

The system under output feedback also needs to be addressed as dynamics of EHGO plays significant role in trajectories convergence and system stability. Towards this end, the procedure in [Memon and Khalil, 2009],[Khalil, 2002, Theorem 14.6] is closely followed to show trajectories convergence and boundedness of system states. For the observer based OFB system, scaled estimation error dynamics is formulated and trajectory convergence /boundedness of states is analyzed by appropriate choice of Lyapunov functions.

The scaled estimation error $\tilde{\chi}(\tilde{\chi}_\eta, \tilde{\chi}_{\xi,\sigma})$ for states ($\hat{\eta}$,$\hat{\xi}$,$\hat{\sigma}$) are given as

$$\tilde{\chi}_\eta = \eta - \hat{\eta} \tag{4.26}$$

$$\tilde{\chi}_{\xi_i} = (\xi_i - \hat{\xi}_i)/\varepsilon^{m+1-i}, \ 1 \le i \le m \tag{4.27}$$

$$\tilde{\chi}_\sigma = \beta(\eta, \xi) - \hat{\sigma} \tag{4.28}$$

Let $D(\varepsilon) = diag[\varepsilon^m, \varepsilon^{m-1}, ..., \varepsilon]$, $\tilde{\chi}_{\xi,\sigma} = [\tilde{\chi}_\xi^T \ \tilde{\chi}_\sigma]$ and $D_1(\varepsilon) = diag[D, 1]$, then (4.26)-(4.28) can be written as

$$D(\varepsilon)\tilde{\chi}_\xi = \xi - \hat{\xi} \tag{4.29}$$

$$D_1(\varepsilon)\tilde{\chi}_{\xi,\sigma} = [(\xi - \hat{\xi})^T \ \beta(\eta, \xi) - \hat{\sigma}]^T \tag{4.30}$$

Consider the EHGO based OFB system (4.11)-(4.14), The estimation error dynamics for this system can now be written as

$$\begin{aligned}\dot{\tilde{\chi}}_\eta &= \phi_0(\tilde{\chi}_\eta + \hat{\eta}, D\tilde{\chi}_\xi + \hat{\xi}) - \phi_0(\hat{\eta}, \hat{\xi}) \\ &\quad -H[(\beta(\tilde{\chi}_\eta + \hat{\eta}, D\tilde{\chi}_\xi + \hat{\xi}) - \tilde{\chi}_\sigma) - \beta(\hat{\eta}, \hat{\xi})] \qquad (4.31)\\ &\triangleq g_r(\hat{\eta}, \hat{\xi}, \tilde{\chi}_\eta, D_1\tilde{\chi}_{\xi,\sigma}, t) \qquad (4.32)\\ \varepsilon\dot{\tilde{\chi}}_{\xi,\sigma} &= \Lambda\tilde{\chi}_{\xi,\sigma} + \varepsilon[\bar{B}_1\Delta\dot{\beta} + \bar{B}_2\bar{\Delta}\alpha] \qquad (4.33)\end{aligned}$$

where $\Delta\dot{\beta} = \dot{\beta}(\chi_\eta + \hat{\eta}, \hat{\xi} + D\tilde{\chi}_\xi, v) - \dot{\beta}(\hat{\eta}, \hat{\xi}, v)$,

$\bar{\Delta}\alpha = \Delta\alpha/\varepsilon, \Delta\alpha = \alpha(\hat{\xi} + D\tilde{\chi}_{\xi}, v) - \alpha(\hat{\xi}, v)$ and

$$\Lambda = \begin{bmatrix} -h_1 & 1 & 0 & \cdots & 0 \\ -h_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -h_m & 0 & 0 & \cdots & 1 \\ -h_{m+1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} 0 \\ B \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

It is evident from equations (4.31)-(4.33) that system exhibit two-time scale properties. Using the results [Boker and Khalil, 2013], [Nazrulla and Khalil, 2011b] and [Memon and Khalil, 2009], boundedness and convergence of the trajectories of the closed loop system can be established. Towards that end, compact sets Ω,Σ such that $\hat{\eta}_0 \in \Omega$, $(\hat{\xi}_0, \hat{\sigma}_0) \in \Sigma$ are defined and it is shown that trajectories starting in $\Omega \times \Sigma$ enter an appropriately defined positively invariant set S in finite time. Lyapunov function candidate for $\tilde{\chi}_\eta$ can be defined as $V_1(t, \tilde{\chi}_\eta) = (\tilde{\chi}_\eta)^T P^{-1} \tilde{\chi}_\eta$ satisfying the identity $0 < a_1 I_{n-m} \leq P^{-1} \leq a_2 I_{n-m}$ where P^{-1} is a positive definite matrix and a_1, a_2 are arbitrary constants. Then it can be shown that V_1 satisfies

$$a_1 \|\tilde{\chi}_\eta\|^2 \leq V_1(t, \tilde{\chi}_\eta) \leq a_2 \|\tilde{\chi}_\eta\|^2 \tag{4.34}$$

$$\frac{\partial V_1}{\partial \tilde{\chi}_\eta} g_r(.) \leq -a_3 \|\tilde{\chi}_\eta\|^2, \; \forall \|\tilde{\chi}_\eta\| \leq c_1, \; \forall t \geq t_0 \tag{4.35}$$

where a_3,c_1 are positive constants independent of ε. The inequalities (4.34),(4.35) will ensure that the compact set satisfies $\Omega \subset \Re^{n-m}$. The boundary layer system is obtained from $\tau_b = t/\varepsilon$ and setting $\varepsilon = 0$ for scaled error as $\frac{d\tilde{\chi}_{\xi,\sigma}}{d\tau_b} = \Lambda \tilde{\chi}_{\xi,\sigma}$ and the Lyapunov function candidate $W(\tilde{\chi}_{\xi,\sigma})$ for this subsystem is given as $W(\tilde{\chi}_{\xi,\sigma}) = \tilde{\chi}_{\xi,\sigma}^T P_0 \tilde{\chi}_{\xi,\sigma}$ where P_0 is positive definite solution of Lyapunov equation $P_0 \Lambda + \Lambda^T P_0 = -I$. It can be seen that W satisfies

$$\lambda_{min}(P_0) \|\tilde{\chi}_{\xi,\sigma}\|^2 \leq W(\tilde{\chi}_{\xi,\sigma}) \leq \lambda_{max}(P_0) \|\tilde{\chi}_{\xi,\sigma}\|^2 \tag{4.36}$$

$$\frac{\partial W}{\partial \tilde{\chi}_{\xi,\sigma}} \Lambda \tilde{\chi}_{\xi,\sigma} \leq -\|\tilde{\chi}_{\xi,\sigma}\|^2 \tag{4.37}$$

and the time derivative of V_1,W satisfy

$$\dot{V}_1 \leq -a_1\|\tilde{\chi}_\eta\|^2 + k_2\|\tilde{\chi}_{\xi,\sigma}\| \quad (4.38)$$

$$\dot{W} \leq -\frac{1}{\varepsilon}\|\tilde{\chi}_{\xi,\sigma}\|^2 + k_3\|\tilde{\chi}_{\xi,\sigma}\|\|P_0\| \quad (4.39)$$

Here k_1,k_2 are positive constants independent of ε. Equations (4.38),(4.39) show that trajectories of estimation error dynamics are bounded.

Furthermore, considering the closed loop system comprising of (4.8), (4.21), (4.31), (4.33) and compactly representing it as

$\dot{\chi} = f_r(\chi, \tilde{\chi}_{\xi,\sigma}, \tilde{\chi}_\eta)$, where $\chi = [\eta^T, \xi^T, v]^T$ and $\chi_0 = [\eta_0^T, \xi_0^T, v_0]$

It can be shown as in [Memon and Khalil, 2009] and the reference therein that the trajectories starting in $\Omega \times \Sigma$ remain bounded and converge asymptotically.

The foregoing conclusions can be summarized as the following theorem.

Theorem 4.1 *Under the stated assumptions and suitable values of controller parameters $a, k, C = c_1, c_2, \ldots \in \Re^{m-n}$, observer gains H_1, H_2, K, the closed loop system comprising of composite system (4.8), the full order EHGO based observer (4.11)-(4.13) and the output feedback control law (4.23), there exists $\mu^* > 0$ and $\varepsilon^* > 0$ such that for every $0 < \mu < \mu^*$ and $0 < \varepsilon < \varepsilon^*$, the trajectories of closed loop system are bounded and approach the origin as $t \longrightarrow \infty$.*

4.4 Summary

The controller design for stabilization of target class of under-actuated non-minimum phase nonlinear systen is discussed in this chapter. Utilizing the Lagrangian based generalized nonlinear system model, collocated partial feedback linearization is used for mathematical model transformation. To decouple the two cascaded sub-systems, a suitable global change of coordinates is applied and a generalized mathematical representation in strict feedback normal form is evolved. An EHGO based state estimator is utilized and a generalized output feedback controller for target class of systems is presented. Finally stability of closed loop system is discussed to show boundedness of states and convergence of trajectories.

Chapter 5

Illustrating Examples on Output Feedback Stabilization

In this chapter, two test bench systems of target class of under-actuated systems namely Inertia Wheel Pendulum (IWP) and Translational Oscillator Rotational Actuator (TORA) are selected and design technique presented in Chapter 4 is applied for nonlinear output feedback controller design of these systems. The mathematical formulation for IWP system is taken in Lagrangian form and further transformed into state space and normal form representation. The unknown system states are observed using an extended high gain observer and further utilized in derivation of SMC based control law. The results are analyzed in simulation section where a comparison of results under output feedback with that of state feedback is performed.

The second example of target class of system namely TORA system is chosen in a similar way as that of IWP system to verify the results of proposed output feedback design technique. The design procedure is followed in a similar pattern for TORA system as given in Chapter 4 for generalized systems and results are verified using simulation based analysis. In the simulation section, it is demonstrated that the trajectories of the closed-loop TORA system under the proposed output feedback control technique, approach those of the state feedback control with the choice of proper EHGO parameters. To reduce the peaking effect (initial overshoot in the estimated states), some constraints are applied on the system input. The simulation results included demonstrate the effectiveness of the proposed controller.

5.1 Example 1: Output Feedback Stabilization of IWP

Inertia wheel pendulum, shown in Fig. 5.1, is constructionally simple test-bench nonlinear mechanical system. The system consists of a simple pendulum with a wheel attached to its independent and floating end, whereas the other pendulum end is connected to an unactuated rotating joint. The wheel is driven by a high torque to weight ratio DC motor, that can spin the wheel in both clockwise and anticlockwise directions. When the wheel is rotated at a fairly high speed, angular acceleration is produced and resultant torque is used as an input for controlling the pendulum angle. IWP thus, falls into the category of under-actuated systems, since one end of the pendulum remains unactuated, and IWP is controlled by a single actuator. It is used as a test bed for testing new controller schemes and model identification experiments.

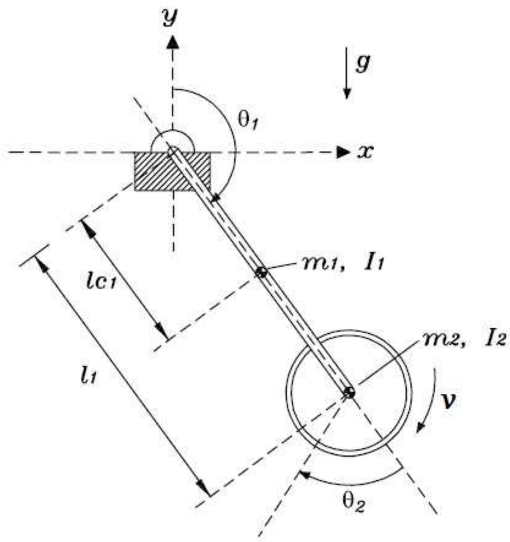

Fig. 5.1 Inertia Wheel Pendulum (IWP)

5.1.1 Mathematical Model of an IWP

To obtain a mathematical model of the IWP, Lagrangian of the system is taken with $\theta = [\theta_1, \theta_2]$ as the configuration variables.

$$L(\theta, \dot{\theta}) = \text{Kinetic Energy} - \text{Potential Energy}$$

$$L(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M \dot{\theta} - V(\theta_1) \tag{5.1}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -m_0 \sin(\theta_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix} \tag{5.2}$$

where

$$\begin{aligned} V(\theta_1) &= m_0 \cos(\theta_1) \\ m_0 &= mg \\ m &= m_1 l_{c1} + m_2 l_1 \\ m_{11} &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + I_2 \\ m_{12} &= m_{21} = m_{22} = I_2 \end{aligned}$$

$$
\begin{aligned}
m_1 &= \text{Pendulum mass } (kg) \\
m_2 &= \text{Wheel mass } (kg) \\
\theta_1 &= \text{Pendulum Angle } (rad) \\
\theta_2 &= \text{Wheel Angle } (rad) \\
v &= \text{Control Input or Torque } (Nm) \\
l_{c1} &= \text{Distance to the centre of the mass } (m) \\
l_1 &= \text{Pendulum length } (m) \\
I_1 &= \text{Pendulum moment of inertia } (kgm^2) \\
I_2 &= \text{Wheel moment of inertia } (kgm^2) \\
g &= \text{Acceleration due to gravity } (m/sec^2), \text{and} \\
M &= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}
\end{aligned}
$$

5.1.2 Control Design

In this section, an output feedback stabilizing controller using extended high gain observer is implemented. Towards that end, it is assumed that the pendulum angle θ_1 can be measured and is available for feedback and a full order observer is implemented using system dynamics in equation (5.2). Starting with system dynamical model, a normal form representation of system model is derived using differential geometrical properties of system and corresponding lie derivatives of system output. The resulting form renders system configuration as an interconnection of internal and external system dynamics. An auxiliary system that utilizes internal dynamics of system is derived and a full order extended high gain observer is then implemented . The estimated state variables are provided to an SMC based nonlinear controller.

5.1.3 Normal Form Representation

IWP system model (5.2) is transformed in terms of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ into a set of equations given as

$$\left.\begin{aligned} \ddot{\theta}_1 &= -\frac{m_{12}}{m_{11}m_{22}-m_{12}m_{21}}v + \frac{m_{22}m_0}{m_{11}m_{22}-m_{12}m_{21}}\sin(\theta_1) \\ \ddot{\theta}_2 &= \frac{m_{11}}{m_{11}m_{22}-m_{12}m_{21}}v - \frac{m_{21}m_0}{m_{11}m_{22}-m_{12}m_{21}}\sin(\theta_1) \\ y &= \theta_1 \end{aligned}\right\} \tag{5.3}$$

Where pendulum angle θ_1 is taken as system output y. State space representation of plant model is derived assuming system states as follows

$$x_1 = \theta_1, \dot{x}_1 = \dot{\theta}_1 = x_2$$

$$x_3 = \theta_2, \dot{x}_3 = \dot{\theta}_2 = x_4$$

The state space representation of IWP System model and its output can be represented as following set of equations

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{m_{12}}{m_{11}m_{22}-m_{12}m_{21}}v + \frac{m_0m_{22}}{m_{11}m_{22}-m_{12}m_{21}}\sin(x_1) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\frac{m_{11}}{m_{12}m_{21}-m_{22}m_{11}}v + \frac{m_0m_{21}}{m_{12}m_{21}-m_{11}m_{22}}\sin(x_1) \\ y &= x_1 \end{aligned}\right\} \tag{5.4}$$

For the output y, it can be easily seen that it is an under-actuated system and its relative degree ρ is given as

$$\rho = 2 < n \tag{5.5}$$

The degree of under-actuation depicts that overall system can be further bifurcated into internal and external dynamics (η, ξ) given as

$$\eta = (\eta_1, \eta_2)$$
$$\xi = (\xi_1, \xi_2)$$

The dynamics of (η, ξ) is determined following the procedure in [Khalil, 2002] and a change of coordinates is applied as follows

$$\left.\begin{aligned} \eta_1 &= -m_{11}x_1 - m_{12}x_3 \\ \eta_2 &= -m_{11}x_2 - m_{12}x_4 \\ \xi_1 &= x_1 \\ \xi_2 &= x_2 \end{aligned}\right\} \tag{5.6}$$

The change of coordinates mentioned in (5.6) gives normal form representation of IWP model. The internal dynamics is obtained as

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= -m_0 \sin(\xi_1) \end{aligned}\right\} \tag{5.7}$$

Corresponding external dynamics and output are given as

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \frac{m_0 m_{22}}{m_{11}m_{22} - m_{12}m_{21}} \sin(\xi_1) - \frac{m_{12}}{m_{11}m_{22} - m_{12}m_{21}} v \end{aligned}\right\} \tag{5.8}$$

$$y = \xi_1 \tag{5.9}$$

5.1.4 EHGO based Full Order Observer

Normal form of an IWP as in equations (5.7),(5.8) and (5.9) can be represented into compact matrix notation similar to generalized form (4.8) as follows

$$\left.\begin{aligned} \dot{\eta} &= A_1\eta + \phi_0(\xi_1) \\ \dot{\xi} &= A\xi + B[\beta(\eta,\xi) + \alpha(\xi,v)] \\ y &= \xi_1 \end{aligned}\right\} \tag{5.10}$$

Here for IWP system

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \phi_0 &= \begin{bmatrix} 0 \\ -m_0 sin(\xi_1) \end{bmatrix} \\ A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \beta &= \frac{m_0 m_{22}}{m_{11}m_{22} - m_{12}m_{21}} \sin(\xi_1) \\ \alpha(\xi,v) &= -\frac{m_{12}}{m_{11}m_{22} - m_{12}m_{21}} v \end{aligned}$$

Auxiliary system having virtual output σ is given as below

$$\dot{\eta} = A_1\eta + \phi_0(\xi) \tag{5.11}$$

$$\sigma = \beta(\eta,\xi) \tag{5.12}$$

Using virtual output σ and internal and external dynamics of IWP, the full order EHGO is derived as

$$\begin{aligned}\dot{\hat{\xi}} &= A\hat{\xi} + B[\hat{\sigma} + \alpha(\hat{\xi}, v)] + h_1[y - \hat{\xi}_1] \quad (5.13)\\ \dot{\hat{\sigma}} &= \dot{\beta}(\hat{\eta}, \hat{\xi}) + h_2[\hat{\sigma} - \hat{\xi}_1]\\ \dot{\hat{\eta}} &= A_1\hat{\eta} + \phi_0(\hat{\xi}) + h_3[\hat{\sigma} - \beta] \quad (5.14)\\ y &= \xi_1\end{aligned}$$

5.1.5 OFB Controller Design

The estimated system states obtained from EHGO block in Section 5.1.4 are utilized in proposed SMC based controller design and an output feedback control law is derived to achieve stabilization of the IWP system. Towards that end, the system (5.2) is transformed into normal form by using collocated partial feedback linearization and global coordinate transformation for change of control and input decoupling [Olfati-Saber, 2000]. In particular, using equation (5.2), the applied torque can be represented as

$$v = \alpha u + \beta \quad (5.15)$$

where

$$\left.\begin{aligned}\alpha &= m_{22} - \left(\frac{m_{12}m_{21}}{m_{11}}\right)\\ \beta &= \frac{m_{21}}{m_{11}}\{m_0 \sin(\theta_1)\}\\ u &= \ddot{\theta}_2\end{aligned}\right\} \quad (5.16)$$

To further decouple the system input, following global change of variables [Qaiser, 2009] is introduced

$$\left.\begin{aligned} z_1 &= m_{11}\dot{\theta}_1 + m_{12}\dot{\theta}_2 \\ z_2 &= \theta_1 \\ z_3 &= \dot{\theta}_2 \end{aligned}\right\} \tag{5.17}$$

Using observed states from EHGO block (5.13),(5.14) and coordinate transformation in (5.6), the estimated system states $(\hat{\eta}, \hat{\xi})$ in terms of $(\hat{z}_1, \hat{z}_2, \hat{z}_3)$ are given as

$$\left.\begin{aligned} \hat{z}_1 &= -\hat{\eta}_2 \\ \hat{z}_2 &= \hat{\xi}_1 \\ \hat{z}_3 &= -\frac{1}{m_{22}}\hat{\eta}_2 - \frac{m_{11}}{m_{12}}\hat{\xi}_2 \end{aligned}\right\} \tag{5.18}$$

The advantage of the above change of coordinates is that it yields the system equations in the following strict-feedback form

$$\dot{\hat{z}}_1 = m_0 \sin(\hat{z}_2) \tag{5.19}$$

$$\dot{\hat{z}}_2 = \frac{(\hat{z}_1 - m_{12}\hat{z}_3)}{m_{11}} \tag{5.20}$$

$$\dot{\hat{z}}_3 = u \tag{5.21}$$

The resulting system (5.19)-(5.21) can now be represented as a cascade connection of two subsystems; namely: a linear subsystem (5.19)-(5.20) and a nonlinear subsystem (5.21), as shown in Fig. 5.2.

5.1.6 Internal System Control Law

To devise a control law for internal system, Control Lyapunov Function method is adopted to make the subsystem represented by (5.19) globally asymptotically stable. By choosing z_2 as

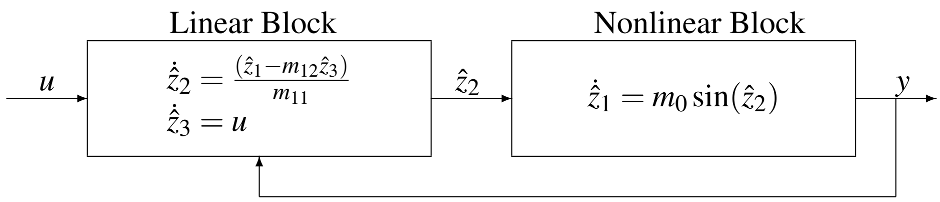

Fig. 5.2 Cascade Interconnection of Inertia Wheel Pendulum (IWP)

an input for the internal system and any positive definite function $V_\circ(z_1)$ as Lyapunov function candidate, the time derivative of $V_\circ(z_1)$ is taken to find an appropriate control law as shown below [Olfati-Saber, 2000]

$$V_\circ(\hat{z}_1) = \frac{1}{2}\hat{z}_1^2 \tag{5.22}$$

$$\dot{V}_\circ(\hat{z}_1) = m_0 \hat{z}_1 \sin(\hat{z}_2)$$

$$\dot{V}_\circ(\hat{z}_1) = \hat{z}_1 m_0 \sin(\alpha(\hat{z}_1)) \tag{5.23}$$

where

$$\alpha(\hat{z}_1) = \hat{z}_2$$

From (5.22) and (5.23), it can be seen that a sigmoidal function $\alpha(\hat{z}_1)$ will make $\dot{V}_\circ(\hat{z}_1)$ as negative definite and hence choosing

$$\alpha(\hat{z}_1) = -a \arctan(c_1 \hat{z}_1) \tag{5.24}$$

with $0 < a \leq \pi/2$ and $c_1 > 0$ will render origin of the nonlinear system (5.19) as globally asymptotically stable.

5.1.7 Composite System Control Law

By using sliding mode control for subsystem (5.20)-(5.21) the control law for entire system is derived. Taking error surfaces for z_2 and z_3 as

$$\hat{e}_1 = \hat{z}_2 - \hat{z}_{2d} \tag{5.25}$$

$$\hat{e}_2 = \hat{z}_3 - \hat{z}_{3d} \tag{5.26}$$

where $\hat{e}_1$ is the estimated error in the generation of stabilization function $\hat{z}_{2d} = \alpha(\hat{z}_1)$ by $\hat{z}_2$ and $\hat{e}_2$ is the estimated error in the generation of $\hat{z}_{3d}$ by $\hat{z}_3$.

The time-derivatives of the equations (5.25)-(5.26) are

$$\dot{\hat{e}}_1 = \dot{\hat{z}}_2 - \dot{\hat{z}}_{2d} \tag{5.27}$$

$$\dot{\hat{e}}_2 = \dot{\hat{z}}_3 - \dot{\hat{z}}_{3d} \tag{5.28}$$

Substituting $\dot{\hat{z}}_2$ and $\dot{\hat{z}}_3$ from (5.20) and (5.21) in (5.27),(5.28) gives

$$\dot{\hat{e}}_1 = \frac{1}{m_{11}}\hat{z}_1 - \frac{m_{12}}{m_{11}}\hat{z}_3 - \dot{\hat{z}}_{2d} \tag{5.29}$$

$$\dot{\hat{e}}_2 = u - \dot{\hat{z}}_{3d} \tag{5.30}$$

In order to make $\hat{e}_1$ zero, using (5.29), $\bar{z}_3$ is defined as

$$\bar{z}_3 = \frac{m_{11}}{m_{12}}\left(k_1\hat{e}_1 + \frac{1}{m_{11}}\hat{z}_1 - \dot{\hat{z}}_{2d}\right)$$

where $k_1 > 0$. The derivatives $\dot{\hat{z}}_{2d}$ and $\dot{\hat{z}}_{3d}$ can be calculated using numerical differentiation with sufficient accuracy. However, this will require the controller to be implemented as a discrete-time system. To avoid this complexity, a low pass filter with small positive time constant v_{3d} can be used [Qaiser, 2009]. The outputs $\hat{z}_{3d}$ and $\dot{\hat{z}}_{3d}$ obtained from the low pass filter can be used to generate the desired control signal as

$$v_{3d}\dot{\hat{z}}_{3d} + \hat{z}_{3d} = \bar{z}_3$$

with initial conditions

$$\hat{z}_{3d}(0) = \bar{z}_3(0) = 0$$

Next, a time varying surface $s(t)$ in space $\Re^2$ by a scalar equation $s(e;t) = 0$ is defined as

$$\begin{aligned}
\hat{s} &= \lambda \hat{e}_1 + \hat{e}_2 \\
\dot{\hat{s}} &= \lambda \left(\frac{1}{m_{11}} \hat{z}_1 - \frac{m_{12}}{m_{11}} \hat{z}_3 - \dot{\hat{z}}_{2d} \right) + \hat{u}_{eq} - \dot{\hat{z}}_{3d} \\
\hat{u}_{eq} &= \dot{\hat{z}}_{3d} - \lambda \left(\frac{1}{m_{11}} \hat{z}_1 - \frac{m_{12}}{m_{11}} \hat{z}_3 - \dot{\hat{z}}_{2d} \right) \\
u &= \hat{u}_{eq} - ksign(\hat{s})
\end{aligned} \tag{5.31}$$

where k is a strictly positive constant. The control law u obtained shows high frequency oscillations known as chattering in sliding phase which can be removed using saturation function as in [Memon and Khalil, 2010]. The resulting control law u under output feedback is given as follows

$$u = \hat{u}_{eq} - ksat\left(\frac{\hat{s}}{\mu}\right) \tag{5.32}$$

and finally input to IWP system v is given as

$$v = \alpha u + \beta \tag{5.33}$$

as the desired control law.

5.1.8 SFB Controller Design

The OFB controller shown in section (5.1.5) utilize estimated system states under EHGO in section (5.1.4). In order to compare performance of controller under OFB with corresponding SFB based design, a state feedback controller design utilizing original system states is considered as in previous work of Khalid and Memon [2014]. Starting with reduced order strict feedback form for IWP system (5.3) as shown below

$$\left.\begin{aligned} \dot{z}_1 &= m_0 \sin(z_2) \\ \dot{z}_2 &= \frac{(z_1 - m_{12} z_3)}{m_{11}} \\ \dot{z}_3 &= u \end{aligned}\right\} \tag{5.34}$$

The corresponding error dynamics and SFB control law is given below.

$$\begin{aligned} \dot{e}_1 &= \dot{z}_2 - \dot{z}_{2d} \\ \dot{e}_2 &= \dot{z}_3 - \dot{z}_{3d} \\ s &= \lambda e_1 + e_2 \\ u_{eq} &= \dot{z}_{3d} - \lambda \left(\frac{1}{m_{11}} z_1 - \frac{m_{12}}{m_{11}} z_3 - \dot{z}_{2d} \right) \\ u &= u_{eq} - ksat\left(\frac{s}{\mu}\right) && (5.35) \\ v &= \alpha u + \beta && (5.36) \end{aligned}$$

The control law obtained in (5.35),(5.36) is used to compare system response under SFB control.

5.1.9 Simulation Results

In this section, simulation results for the proposed control scheme to stabilize the IWP system are provided. To provide for an appropriate way of performance comparison with the results

Table 5.1 IWP System Parameters

Parameter	Value	Unit
l_1	0.124	m
l_{c1}	0.063	m
m_1	0.020	kg
m_2	0.300	kg
I_1	$47e^{-6}$	kgm^2
I_2	$32e^{-6}$	kgm^2
m_{11}	$4.83e^{-3}$	kgm^2
m_{12}, m_{21}, m_{22}	$32e^{-6}$	kgm^2
g	9.81	m/sec^2

provided in Olfati-Saber [2000], Spong et al. [2001] and Qaiser [2009], same plant / model specifications are used. At first simulation results for the state feedback controller (5.35),(5.36) as in Khalid and Memon [2014] are shown and then results for the output feedback controller (5.32),(5.33) are elaborated. The IWP system parameters chosen for this purpose are given in Table 5.1 and the controller parameters used in the SFB/OFB design are shown in Table 5.2.

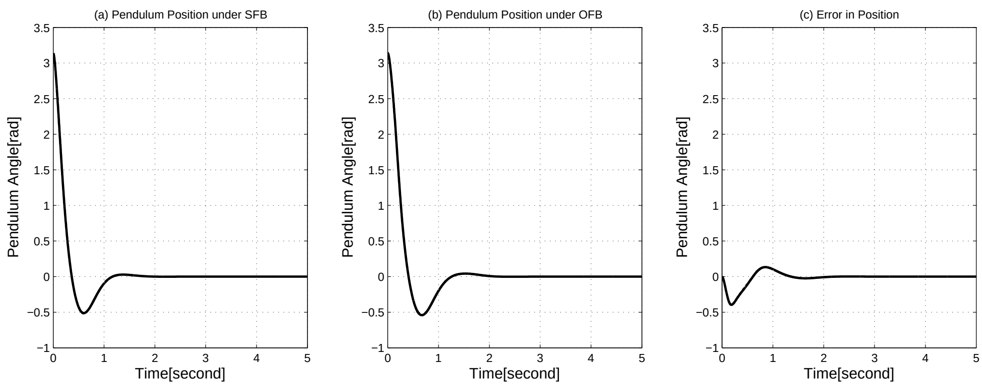

Fig. 5.3 Pendulum Angle(θ_1) (a) Under SFB (b) Under OFB (c) Convergence Error

Figures 5.3 - 5.6 show the simulation results for various states/control input of the IWP system under state feedback control, output feedback control and the corresponding convergence errors respectively. These results show convergence to the equilibrium of the pendulum angle, pendulum velocity and wheel velocity. These states converge to equilibrium point in fairly small interval of time. As wheel angle θ_2 does not play important role in IWP dynamics, it is

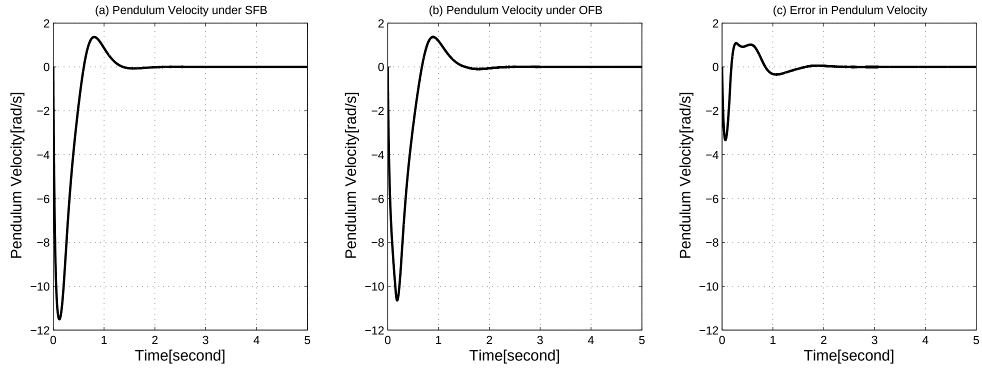

Fig. 5.4 Pendulum Velocity($\dot{\theta}_1$) (a) Under SFB (b) Under OFB (c) Convergence Error

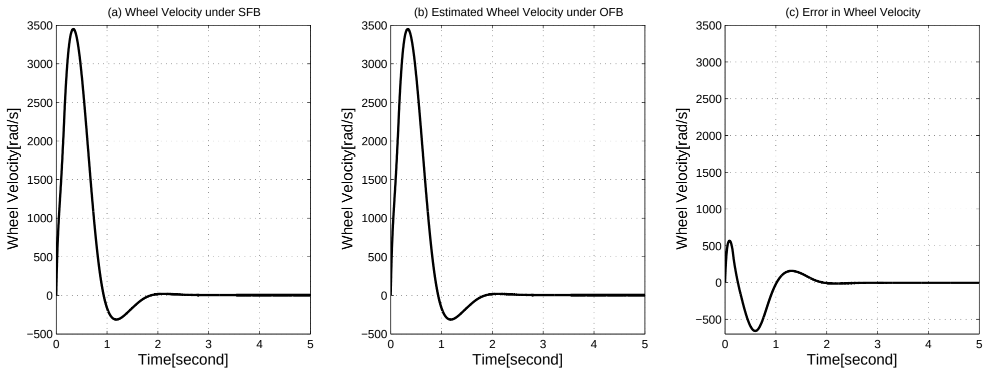

Fig. 5.5 Wheel Velocity($\dot{\theta}_2$) (a) Under SFB (b) Under OFB (c) Convergence Error

not shown as state variable here. Figures 5.3a - 5.6a show simulation results for the state feedback controller (5.35),(5.36). Figures 5.3b - 5.6b show simulation results for output feedback controller which uses an extended high gain observer (5.32),(5.33). Figures 5.3c - 5.6c show the corresponding convergence error between actual and estimated values of different system states/control input. Figure 5.7 shows phase portrait plot under SFB and OFB control. The convergence of system estimated states using EHGO based control law and under state feedback (SFB) based controller is investigated. The results show that estimated states converge to actual states arbitrarily fast. However small delay is seen in the initial phase. As mentioned earlier, in output feedback design only pendulum angle is used for estimation of all the system states. Initial values of these system states are not precisely known. Furthermore, in order to avoid peaking phenomenon, appropriate saturation function is used in design of control law.

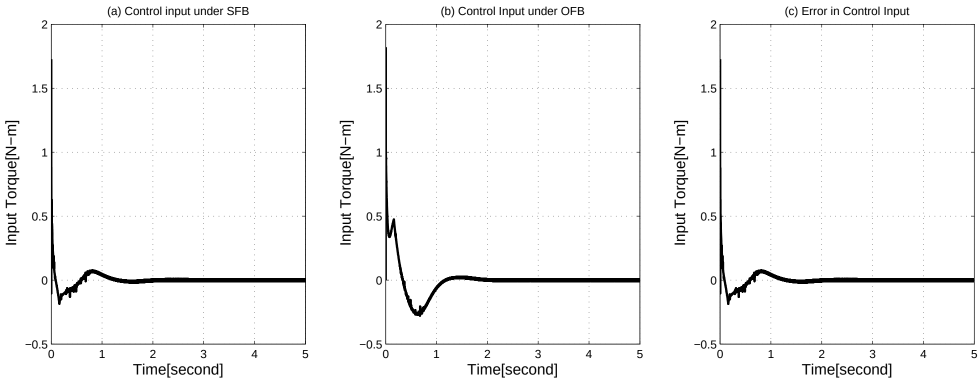

Fig. 5.6 Control Input(v) (a) Under SFB (b) Under OFB (c) Convergence Error

Table 5.2 Controller Parameters for IWP System

Parameter	Value
a	$\pi/2$
c_1	9
k_1	4
λ	10
μ	0.01
k	100
v_{3d}	10

All these factors result in a slight degradation in the system transient response. Therefore resulting behavior of states under EHGO based output feedback controller is expected to be slightly delayed as compared with actual states.

5.2 Example 2: Output Feedback Stabilization of TORA

Translational Oscillator with Rotational Actuator (TORA) also known as Oscillating Eccentric Rotor (OER) is a widely studied benchmark nonlinear system. It belongs to a class of nonlinear under-actuated systems and is considered as a simplified version of dual-spin aircraft. The interaction between rotation and translation in the TORA system is similar to the interaction between spin and nutation in a dual-spin spacecraft. The system consists of an un-actuated translational oscillation cart and an actuated eccentric rotor attached to the cart. The

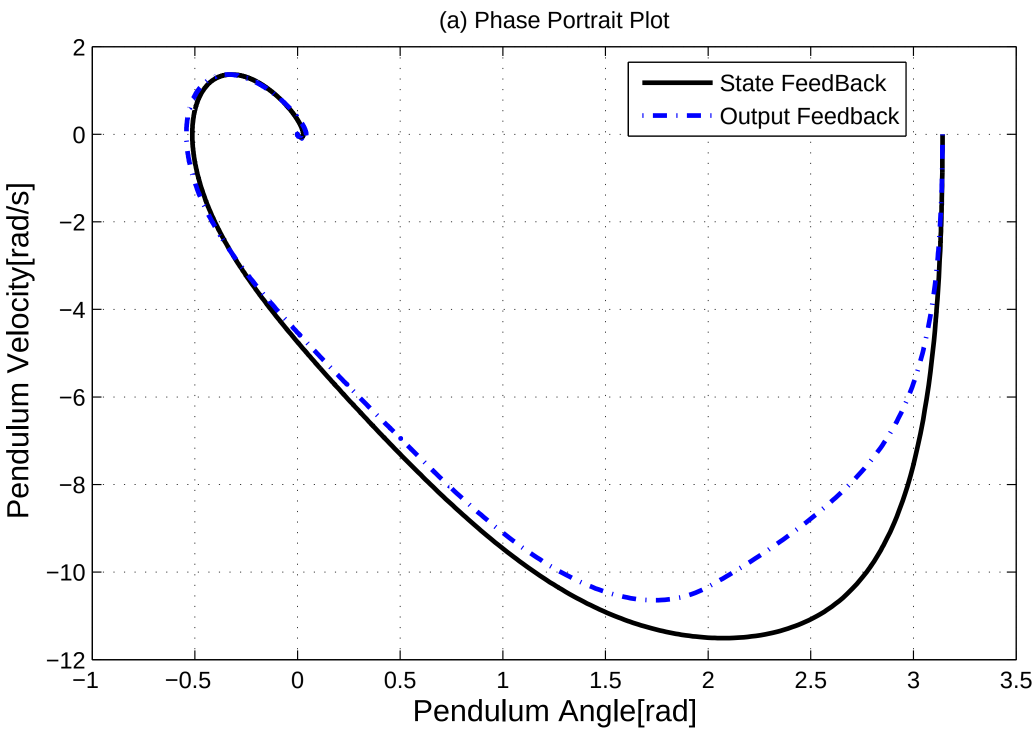

Fig. 5.7 Phase Potrait Plot(SFB vs OFB)

stabilization problem of TORA system includes application of control input torque acting on the rotor to stabilize both the translational position of the unactuated oscillating cart and the rotational position of the actuated eccentric rotor.

5.2.1 Background

The complexity of TORA system dynamics and its non-minmum phase nature due to unstable zero dynamics, makes stabilization of TORA system a challenging problem. Prominent work reported for stabilization of TORA system under state feedback control (SFB) includes [Lee and Chang, 2012; Olfati-Saber, 2000; Qaiser et al., 2007], while for output feedback control (OFB) the notable work done is reported in [Boker and Khalil, 2013; Nazrulla and Khalil, 2011b; She et al., 2012]. Reza Olfati-Saber has addressed and solved stabilization problem of TORA system using state feedback with a physically meaningful Lyapunov function [Olfati-

Saber, 2000]. Furthermore, the author shows classification of under-actuated systems into eight different types depending on kinetic symmetric properties of these systems. Accordingly, TORA system is classified as Class-I under-actuated system due to strict feedback normal form representation of the system obtained after suitable coordinate transformation and collocated partial feedback linearization. In [Qaiser et al., 2007], stabilization of TORA system is achieved via Multiple Sliding Surface (MSS) and Dynamic Surface Control (DSC) with simpler control law and better response time. In [Lee and Chang, 2012], Lee and Chang, experimentally implemented an adaptive backstepping control scheme for TORA system.

The dynamic state feedback control of TORA system mentioned above efficiently solves stabilization problem of the system. To extend this result to the output feedback, a suitable observer is required. A High Gain Observer (HGO), as introduced by Khalil [Khalil, 2002], is an effective nonlinear tool that is commonly used for observer based control design of nonlinear systems. The effectiveness of output feedback control of TORA system increases when system is lacking velocity sensor(s) or when measurement noise in velocity sensor needs to be eliminated. HGO based output feedback controller offers a very good solution for nonlinear systems in terms of robustness and design simplicity. This technique has received wide-spread attention, especially after establishment of concept of separation principle [Atassi and Khalil, 1999; Khalil and Praly, 2014], and presentation of a simpler but effective solution to peaking phenomenon [Esfandiari and Khalil, 1992]. The HGO based controller design is, however, mostly done for minimum phase nonlinear systems [Memon and Khalil, 2009] having stable zero dynamics while for under-actuated systems like TORA having unstable zero dynamics, the extended high gain observers(EHGO) have been successfully implemented. Nazrulla and Khalil [Nazrulla and Khalil, 2011b] utilize Sliding Mode Control (SMC) and EHGO for stabilization of TORA system while Boker and Khalil [Boker and Khalil, 2013] establish a full order observer comprising of an EHGO and an Extended Kalman Filter (EKF) for the estimation of output and internal system states respectively. Similarly in [She et al., 2012], global stabilization of under-actuated mechanical systems including TORA is presented. The key feature of [She et al., 2012] is that the system dynamics are decoupled into internal and external dynamics, and an observer based control law is synthesized requiring only the state variables of position (and not velocity).

5.2.2 Mathematical Formulation

The Translational Oscillator Rotational Actuator (TORA) system is shown in Fig. 5.8. It comprises of a translational oscillating platform that is controlled by a rotational eccentric mass or rotor. The platform is connected via an ideal spring and may be subjected to some disturbance force. An angular displacement in rotor causes a linear displacement in horizontal platform. Platform position and rotor angle of rotation are taken as configuration variables. The angular movement of rotor is used to control linear displacement of horizontal platform. The Lagrangian model (difference of kinetic energy and potential energy) of TORA system is

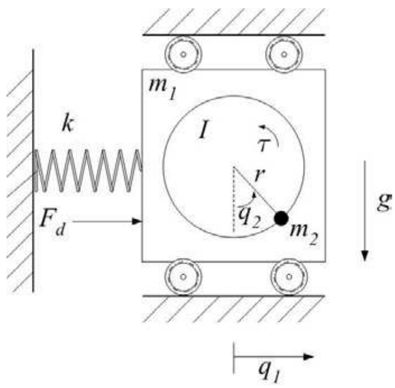

Fig. 5.8 Translational Oscillator with Rotational Actuator (TORA)

given as

$$
\begin{aligned}
L(q,\dot{q}) &= \text{Kinetic Energy} - \text{Potential Energy} \\
L(q,\dot{q}) &= \frac{1}{2}\dot{q}^T M\dot{q} - V(q_1,q_2)
\end{aligned}
$$

or

$$L(q,\dot{q}) = \frac{1}{2}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}\begin{bmatrix}m_1+m_2 & m_2 r\cos(q_2)\\ m_2 r\cos(q_2) & m_2 r^2+I\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} - V(q_1,q_2) \tag{5.37}$$

where

$$\begin{aligned}
m_1 &= \text{Translational oscillating platform or cart mass}(kg)\\
m_2 &= \text{Rotational eccentric mass or rotor mass}(kg)\\
r &= \text{Rotor radius}(m)\\
q_1 &= \text{Platform linear displacement}(m)\\
q_2 &= \text{Rotor position or angle}(rad)\\
\tau &= \text{Control input or torque } (Nm)\\
I &= \text{Rotor moment of inertia } (kgm^2)\\
g &= \text{Acceleration due to gravity } (m/sec^2), \text{and}\\
k &= \text{Spring constant}(N/m)\\
M &= \text{Inertia matrix of system}
\end{aligned}$$

The Platform linear displacement q_2 appears in inertia matrix M of system, it is called shape variable while q_1 is called external variable. Furthermore, q_2 is actuated and hence TORA falls within Class-1 of under-actuated system [Olfati-Saber, 2000]. The Euler Lagrange equations of motion for TORA are given as

$$\begin{aligned}
(m_1+m_2)\ddot{q}_1 + m_2 r\cos(q_2)\ddot{q}_2 - m_2 r\sin(q_2)\dot{q}_2^2 + kq_1 &= 0\\
m_2 r\cos(q_2)\ddot{q}_1 + (m_2 r^2+I)\ddot{q}_2 + m_2 g r\sin(q_2) &= \tau
\end{aligned}$$

The unactuated part of TORA system dynamics disallows exact feedback linearization and only partial feedback linearization is possible. Collocated partial feedback linearization [Olfati-Saber, 2000], an important design tool in this regard, is used with invertible change of control as follows

$$\tau = \alpha(q_2)u + \beta \tag{5.38}$$

where

$$\begin{aligned} \alpha(q_2) &= \left[(m_2r^2 + I) - \frac{(m_2r\cos(q_2))^2}{m_1+m_2}\right] \\ \beta &= \left[\frac{m_2^2r^2\sin(q_2)\cos(q_2)}{m_1+m_2}\dot{q}_2^2 - \frac{m_2rkq_1^2\cos(q_2)}{m_1+m_2} + m_2gr\sin(q_2)\right] \\ u &= \ddot{q}_2 \end{aligned} \tag{5.39}$$

In order to decouple the control input, a global change of coordinates is used

$$\left.\begin{aligned} \eta_1 &= q_1 + \frac{m_2r\sin(q_2)}{m_1+m_2} \\ \eta_2 &= (m_1+m_2)\dot{q}_1 + m_2r\cos(q_2) \\ \xi_1 &= q_2 \\ \xi_2 &= \dot{q}_2 \end{aligned}\right\} \tag{5.40}$$

Strict feedback normal form representation is obtained which can be represented as a cascade of two subsystems with original control input τ. The first subsystem is

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \frac{k}{m_1+m_2}\left(\frac{m_2r\sin(\xi_1)}{m_1+m_2} - \eta_1\right) \end{aligned}\right\} \tag{5.41}$$

and the second subsystem

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \frac{1}{\delta(\xi_1)}\Big[(m_1+m_2)\tau - m_2 r\cos(\xi_1)\Big\{m_2 r\xi_2^2\sin(\xi_1) - k \\ &\qquad \left(\eta_1 - \frac{m_2 r\sin(\xi_1)}{m_1+m_2}\right)\Big\}\Big] \end{aligned}\right\} \tag{5.42}$$

where

$$\begin{aligned} \delta(\xi_1) &= (I+m_2r^2)(m_1+m_2) - m_2^2 r^2\cos^2(\xi_1) \\ &\geq (I+m_2r)^2 m_1 + m_2 I \\ &> 0 \end{aligned}$$

The normal form of system (5.41)-(5.42) bifurcates the entire system into an internal system $\eta = (\eta_1, \eta_2)$ and an external system $\xi = (\xi_1, \xi_2)$ as shown in Fig. 5.9, such that, output ξ_1 of external system acts as an input for internal system. For the internal dynamics (5.41) and

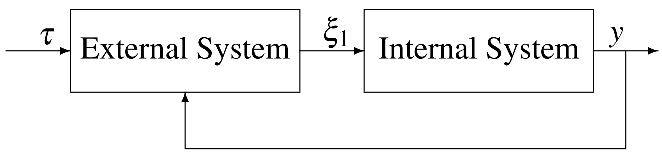

Fig. 5.9 Cascade Interconnection of TORA

taking ξ_1 as output y, the zero dynamics is obtained by setting $y = \xi_1 = 0$ then $\dot{\xi}_1 = 0 \Rightarrow \xi_2 = 0 \Rightarrow \dot{\xi}_2 = 0$ and zero dynamics of TORA system is given as

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= -\frac{k}{m_1+m_2}\eta_1 \end{aligned}\right\} \tag{5.43}$$

Eigen values, for zero dynamics (5.43), can be calculated as $\left(\pm j\sqrt{\frac{k}{m_1+m_2}}\right)$, which shows that TORA is a non-minimum phase system.

5.2.3 Control Design

As for the case of IWP system control design for the TORA system is carried out in two steps. First, a SFB control is defined using SMC which is then extended to an OFB design by employing an EHGO/EKF [Boker and Khalil, 2013; Nazrulla and Khalil, 2011b] in the closed loop, as given in the following subsections.

State feedback Control Law

In this section, a sliding mode controller is implemented utilizing full states of external system for stabilization of entire system. The internal system being unforced nonlinear system, is stabilized using Control Lyapunov Function (CLF) [Olfati-Saber, 2000]. Utilizing ξ_1 as virtual control input for the internal system, at first, it is assumed that

$$\alpha(\eta_2) = \xi_{1d} = -a\tan^{-1}(c_2\eta_2) \tag{5.44}$$

where a,c_2 are any positive constants. The error dynamics are

$$e_1 = \xi_1 - \xi_{1d} \tag{5.45}$$

$$e_2 = \xi_2 - \xi_{2d} \tag{5.46}$$

$$e_1 = \xi_1 + a\tan^{-1}(c_2\eta_2)$$

$$e_2 = \xi_2 - \xi_{2d}$$

where e_1 is the error in the generation of stabilization function $\alpha(\eta_2) = \xi_{1d}$ by ξ_1 and e_2 is the error in the generation of ξ_{2d} by ξ_2. The time derivative of error surfaces (5.45)-(5.46) are

$$\dot{e}_1 = \dot{\xi}_1 - \dot{\xi}_{1d} \tag{5.47}$$

$$\dot{e}_2 = \dot{\xi}_2 - \dot{\xi}_{2d} \tag{5.48}$$

Substituting $\dot{\xi}_1$ and $\dot{\xi}_2$ from (5.42) in (5.47),(5.48) gives

$$\dot{e}_1 = \xi_2 - \dot{\xi}_{1d} \tag{5.49}$$

$$\begin{aligned} \dot{e}_2 = & \frac{1}{\delta(\xi_1)}\Bigg[(m_1+m_2)\tau - m_2 r\cos(\xi_1)\Bigg\{m_2 r\xi_2^2\sin(\xi_1) - \\ & k\left(\eta_1 - \frac{m_2 r\sin(\xi_1)}{m_1+m1}\right)\Bigg\}\Bigg] - \dot{\xi}_{2d} \end{aligned} \tag{5.50}$$

In order to make e_1 zero, using (5.49), $\overline{\xi}_2$ is defined as

$$\overline{\xi}_2 = -k_1 e_1 + \dot{\xi}_{1d} \tag{5.51}$$

where $k_1 > 0$. The derivatives $\dot{\xi}_{1d}$ and $\dot{\xi}_{2d}$ can be evaluated numerically using any fast processor. However, this will require the controller to be implemented in discrete-time domain. To avoid this complexity, a low pass filter with small positive time constant v_{3d} can be used in a similar way as for the case of IWP [Qaiser et al., 2007]. The outputs ξ_{2d} and $\dot{\xi}_{2d}$ obtained from the low pass filter can be used to generate the desired control signal as

$$v_{3d}\dot{\xi}_{2d} + \xi_{2d} = \overline{\xi}_2$$

with initial conditions

$$\xi_{2d}(0) = \overline{\xi}_2(0) = 0$$

Sliding surface $s(t)$ in space $\Re^2$ for error dynamics is defined as

$$s = \lambda e_1 + e_2$$

Time derivative of sliding surface is

$$\begin{aligned}
\dot{s} &= \lambda \dot{e}_1 + \dot{e}_2 \\
\dot{s} &= \lambda(\xi_2 - \dot{\xi}_{1d}) + \frac{1}{\delta(\xi_1)}\Bigg[(m_1+m_2)\tau - m_2 r\cos(\xi_1)\Bigg\{m_2 r\xi_2^2\sin(\xi_1) - \\
&\quad k\Bigg(\eta_1 - \frac{m_2 r\sin(\xi_1)}{m_1+m1}\Bigg)\Bigg\}\Bigg] - \dot{\xi}_{2d} \qquad (5.52)
\end{aligned}$$

for $\dot{s} = 0$, τ_{eq} is given as

$$\begin{aligned}
0 &= \lambda(\xi_2 - \dot{\xi}_{1d}) + \frac{1}{\delta(\xi_1)}\Bigg[(m_1+m_2)\tau_{eq} - m_2 r\cos(\xi_1)\Bigg\{m_2 r\xi_2^2\sin(\xi_1) - \\
&\quad k\Bigg(\eta_1 - \frac{m_2 r\sin(\xi_1)}{m_1+m_1}\Bigg)\Bigg\}\Bigg] - \dot{\xi}_{2d} \qquad (5.53) \\
\tau_{eq} &= \frac{m_2 r\cos(\xi_1)}{m_1+m_2}\Bigg\{m_2 r\xi_2^2\sin(\xi_1) - k\Bigg(\eta_1 - \frac{m_2 r\sin(\xi_1)}{m_1+m_2}\Bigg)\Bigg\} + \frac{\delta(\xi_1)}{m_1+m_2}\dot{\xi}_{2d} \\
&\quad - \frac{\lambda\delta(\xi_1)}{m_1+m_2}(\xi_2 - \dot{\xi}_{1d}) \qquad (5.54)
\end{aligned}$$

and corresponding control input under sliding mode framework for the system is

$$\tau = \tau_{eq} - k_2 sign(s) \qquad (5.55)$$

where k_2 is a strictly positive constant. The control law τ obtained shows high frequency oscillations known as chattering in sliding phase which can be removed using saturation function. The desired control law is

$$\tau = \tau_{eq} - k_2\big(sat\big(\frac{s}{\mu}\big)\big) \qquad (5.56)$$

Output Feedback Control Law

In output feedback control, only system output is utilized in feedback design and other unknown states are estimated using a suitable state observer. After estimation of system states, a control law based on sliding mode technique is derived utilizing observed states instead of actual system states. As mentioned earlier and also in [Khalid and Memon, 2014], high gain observer exhibit somewhat sluggish response to state estimation for nonlinear systems like TORA. An EHGO/EKF is utilized for estimation of unknown states under the assumption that only output of system is available as feedback. The scheme closely follows the approach of [Boker and Khalil, 2013; Nazrulla and Khalil, 2011b]. For output feedback control, normal form representation assuming ξ_1 as output is given as

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\left\{\begin{bmatrix}\frac{km_2 r\cos(\xi_1)}{\delta(\xi_1)} & 0\end{bmatrix}\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\end{bmatrix} + a(\xi,u)\right\} \tag{5.57}$$

$$\begin{bmatrix}\dot{\eta}_1\\ \dot{\eta}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m_1+m_2} & 0\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{km_2 r\sin(\xi_1)}{(m_1+m_2)^2}\end{bmatrix} \tag{5.58}$$

$$y = \xi_1 \tag{5.59}$$

where

$$a(\xi,u) = \left[\frac{(m_1+m_2)u}{\delta(\xi_1)} - \frac{m_2^2 r^2 \xi_2^2 \sin(\xi_1)\cos(\xi_1)}{\delta(\xi_1)} - \frac{km_2^2 r^2 \sin(\xi_1)\cos(\xi_1)}{\delta(\xi_1)(m_1+m_2)}\right]$$

Now an auxiliary system $\sigma(\xi,\eta)$ is considered from external dynamics (ξ_1,ξ_2) containing internal dynamics η as well, to design an EHGO for external system.

$$\sigma(\xi,\eta) = \begin{bmatrix}\frac{km_2 r\cos(\xi_1)}{\delta(y)} & 0\end{bmatrix}\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix} \tag{5.60}$$

The resulting full order observer for external system is given as

$$\begin{bmatrix}\dot{\hat{\xi}}_1\\ \dot{\hat{\xi}}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\xi}_1\\ \hat{\xi}_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\left[\hat{\sigma} + a(\hat{\xi},u)\right] + \begin{bmatrix}\frac{\alpha_1}{\varepsilon}\\ \frac{\alpha_2}{\varepsilon^2}\end{bmatrix}\left[y - \hat{\xi}_1\right] \tag{5.61}$$

and from auxiliary problem, estimate of virtual output σ is given as

$$\dot{\hat{\sigma}} = \hat{\sigma} + \frac{\alpha_3}{\varepsilon^3}\left[y - \hat{\xi}_1\right] \tag{5.62}$$

and an Extended Kalman Filter (EKF) based observer [Boker and Khalil, 2013] for internal system is

$$\begin{bmatrix} \dot{\hat{\eta}}_1 \\ \dot{\hat{\eta}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m_1+m_2} & 0 \end{bmatrix} \begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{km_2 r\cos(y)}{(m_1+m_2)^2} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \left[\hat{\sigma} - \frac{km_2 r\sin(y)}{\delta(y)}\hat{\eta}_1\right] \tag{5.63}$$

The gain matrix L(t) is obtained as $L(t) = P(t)C_1^T(t)R^{-1}(t)$ while P(t) is obtained through the solution of Algebraic Ricatti Equation.

To complete the output feedback controller design, the SFB controller is extended by using estimated states obtained from internal observer (5.63) and from external system observer (5.61). Towards that end, it is assumed that only rotor angular position $q_2 = \xi_1$ can be measured and is available for feedback. The state feedback controller (5.56) is then implemented as output feedback controller with $\hat{\tau}$ as control input to the TORA system

$$\hat{\tau} = \hat{\tau}_{eq} - k_2\left(sat\left(\frac{\hat{s}}{\mu}\right)\right) \tag{5.64}$$

$$\hat{\tau} = \hat{\tau}_{eq} - k_2 sign(\hat{s}) \tag{5.65}$$

$$\begin{aligned} \hat{\tau}_{eq} = & \left[\frac{m_2 r\cos(y)}{m_1+m_2}\left\{m_2 r\hat{\xi}_2^2 \sin(y) - k\left(\hat{\eta}_1 - \frac{m_2 r\sin(y)}{m_1+m_2}\right)\right\} + \frac{\delta(y)}{m_1+m_2}\dot{\hat{\xi}}_{2d}\right. \\ & \left. - \frac{\lambda\delta(y)}{m_1+m_2}(\hat{\xi}_2 - \dot{\hat{\xi}}_{1d})\right] \end{aligned} \tag{5.66}$$

$$\hat{s} = \lambda\hat{e}_1 + \hat{e}_2 \tag{5.67}$$

The design parameter ε and arbitrary constants $\alpha_1, \alpha_2, \alpha_3$ in observer (5.61),(5.62) are carefully set to ensure asymptotic convergence of the estimation errors to zero arbitrarily fast. Peaking effect is minimized by saturating the estimated states during initial period. Separation principle of paper [Atassi and Khalil, 1999] is utilized and it is shown that output feedback

Table 5.3 TORA System Parameters

Parameter	Value	Unit
k	186.3	N/m
m_1	1.3608	kg
m_2	0.096	kg
r	0.0592	m
I	0.0002175	kgm^2
g	9.81	m/sec^2

Table 5.4 Controller Parameters for TORA System

Parameter	Value
a	$\pi/2$
c_2	20
k_1	10
k_2	0.17
λ	10
μ	0.05
ε	0.01
α_1	3
α_2	3
α_3	1
v_{3d}	10

controller can recover the performance of state feedback controller. These results are verified in next section.

5.2.4 Simulation Results

In this section, verification of control design for TORA system using results obtained by numerical simulations is done. For fair performance comparison, same physical parameters are chosen for the TORA system as in [Nazrulla and Khalil, 2011b] and are shown in Table 5.3.The selected controller parameters are given in Table 5.4.

Figures 5.10 - 5.15 show the simulation results for various states of the TORA system under state feedback control, output feedback control and the corresponding convergence er-

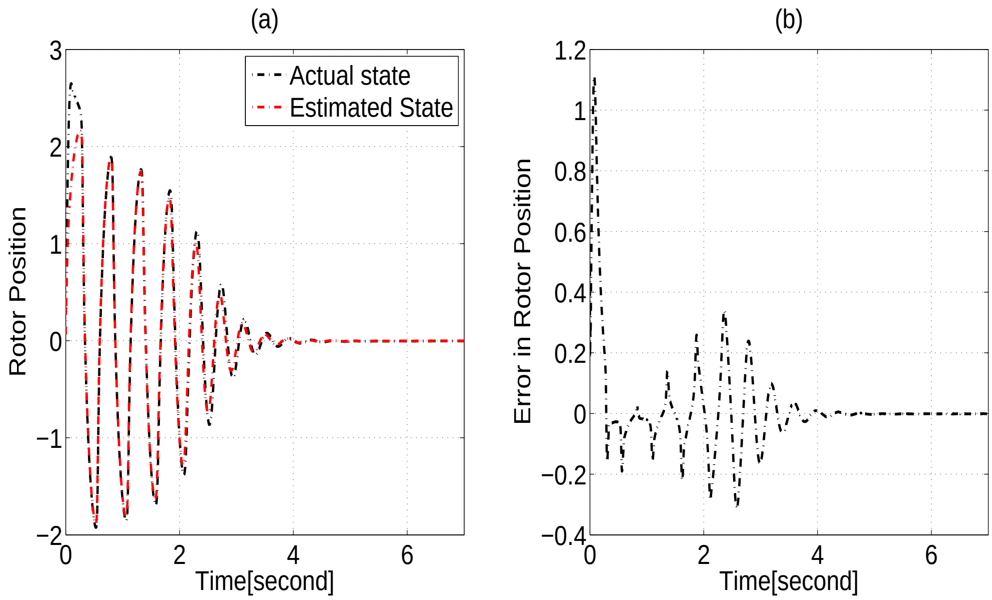

Fig. 5.10 Rotor Angle(ξ_1) (a) State Feedback vs Output Feedback (b) Convergence Error in Rotor Position

rors. These results show convergence to the equilibrium of rotor angle/velocity and cart angle/velocity. These states converge to equilibrium point in fairly small interval of time.

Figures 5.10a - 5.15a show comparison of simulation results for state feedback controller (5.55),(5.56) and for output feedback controller (5.64),(5.65), which uses an extended high gain observer (5.61),(5.63). Figures 5.10b - 5.15b show the corresponding convergence error between actual and estimated values of different internal and external system states. The behaviour of system observed states using EHGO with that of state feedback based control law is examined. It is evident that observed states converge to actual states arbitrarily fast. However small delay is seen in the transient phase. As stated earlier, in output feedback design only rotor angle of TORA system is used for estimation of system states. Initial values of these system states are not accurately known. Furthermore, to prevent states from peaking, saturation functions are implemented in the design. All these factors result in a slight deterioration in the system transient response.

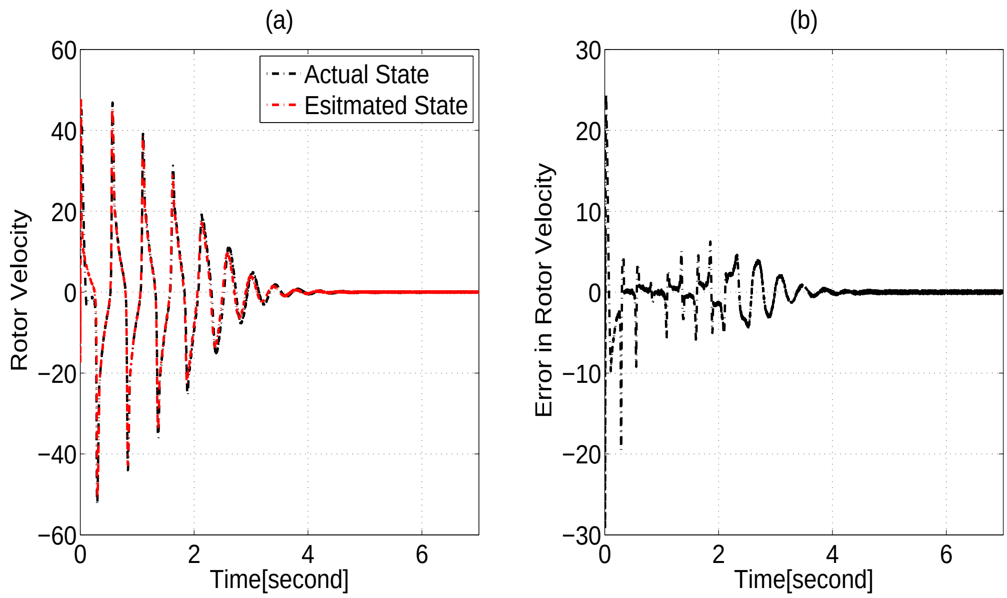

Fig. 5.11 Rotor Velocity(ξ_2) (a) State Feedback vs Output Feedback (b) Convergence Error in Rotor Velocity

5.3 Summary

Two important examples of target class of under-actuated nonlinear system are chosen in this chapter. Inertia wheel pendulum is taken as first example and proposed design technique is applied on it. Starting with Lagrangian model of the system, a normal form representation of the system is derived under suitable coordinate transformation and collocated partial feedback linearization. A sliding mode controller is synthesized, first under state feedback and then, under output feedback control design. In output feedback design, extended high gain observer is used to estimate derivatives of system output as well as to estimate the virtual output of an auxiliary system. To complete the output feedback controller design, the estimated system states so obtained are fed to stabilizing controller in a similar way as in state feedback design. The outcome of design indicate that the system response under OFB controller converges to that of the SFB based design. It is also evident that both observed and actual states converge arbitrarily fast with proper selection of the design parameters.

In second example, a stabilizing controller for Translational Oscillator with Rotational

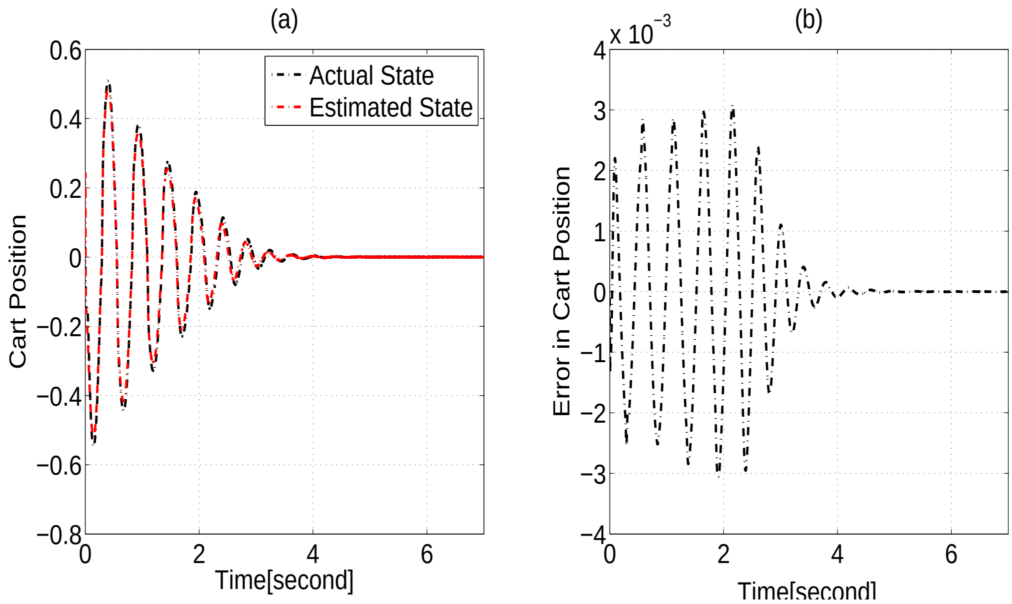

Fig. 5.12 Cart Position(xc) (a) State Feedback vs Output Feedback (b) Convergence Error in Cart Position

Actuator (TORA), a benchmark non-minimum phase system is presented. The external and internal states observed under EHGO and EKF respectively are utilized in control law based on sliding mode control technique, in a similar way as for IWP system . The comparison of results under output feedback with that of state feedback design validate the effectiveness of technique and it is shown that system states and output converge to steady state value without any significant degradation of transient performance. The analysis indicates that the proposed output feedback control technique may be extended to other under-actuated systems of similar class.

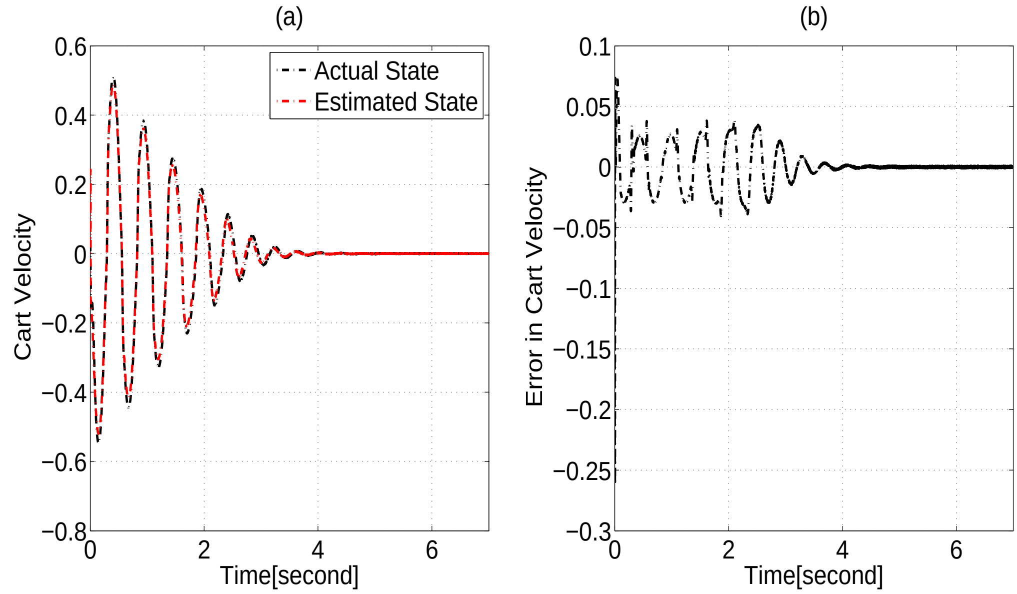

Fig. 5.13 Cart Velocity($\dot{x}c$) (a) State Feedback vs Output Feedback (b) Convergence Error in Cart velocity

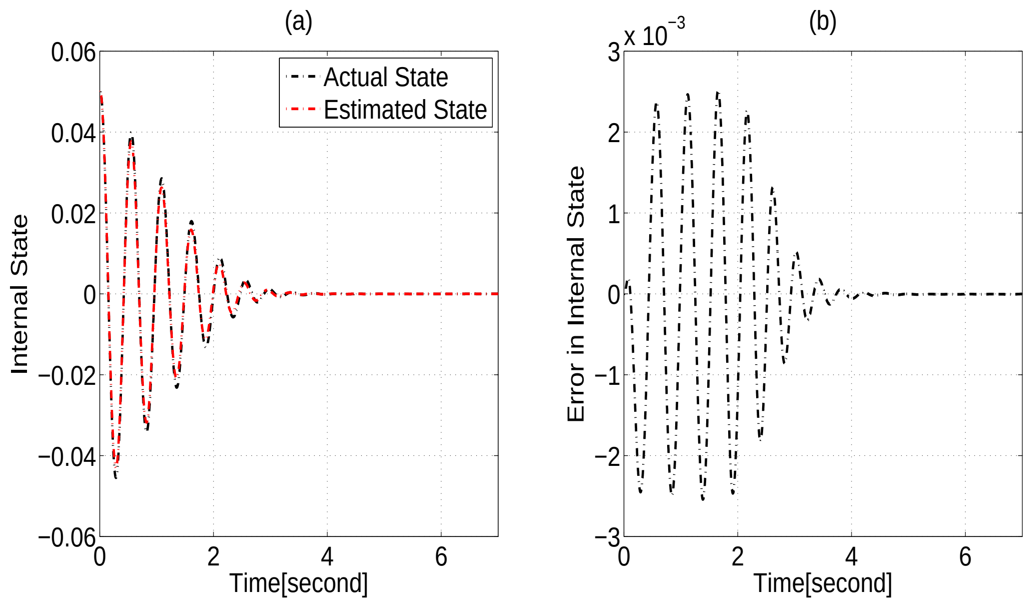

Fig. 5.14 Internal State(η_1) (a) State Feedback vs Output Feedback (b) Convergence Error in internal State(η_1)

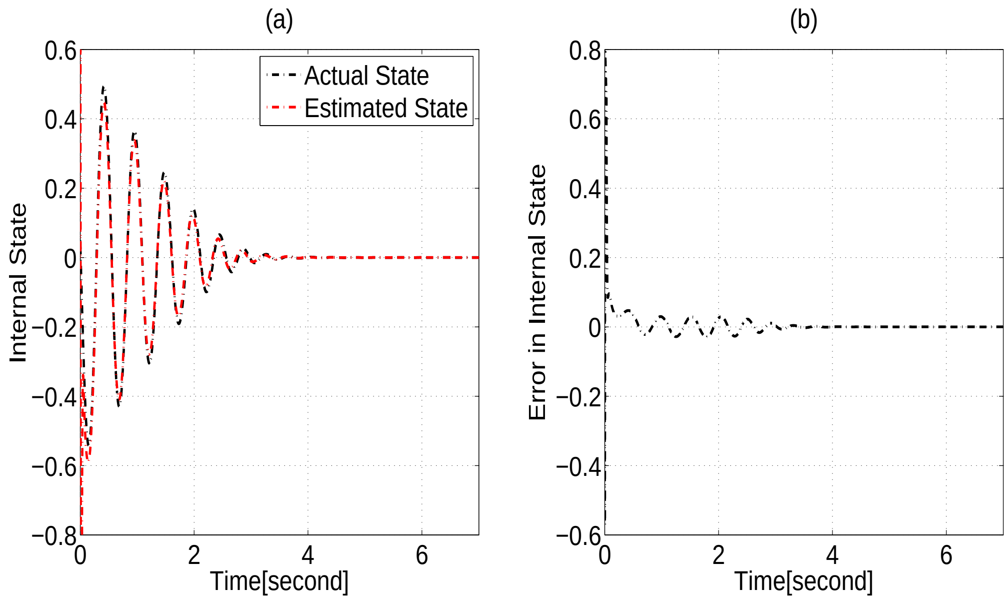

Fig. 5.15 Internal State(η_2) (a) State Feedback vs Output Feedback (b) Convergence Error in internal State(η_2)

Chapter 6

Output Regulation of a Class of Under-actuated Systems

Output feedback stabilization of target class of under-actuated nonlinear systems considered so far in this thesis is further extended, in this chapter, towards a more important problem of output regulation. The chapter starts with a general introduction and description of work done in past by renowned researchers for output regulation problem of linear as well as nonlinear systems. Application of servo-compensation based output feedback regulation on generalized model of target class of non-minimum phase under-actuated systems is presented. A generalized methodology is then given to design a suitable control law based on conditional servo compensator scheme.

In the next part of this chapter, the generalized output feedback servo compensator algorithm as described in first part, is applied on a benchmark under-actuated nonlinear system called Translational Oscillator with a Rotating Actuator (TORA). The system model used here is same as considered for stabilization problem in Chapter 5. In section 5.2 detailed system description, mathematical model and its importance as test bench system is discussed. For output regulation of TORA, the strict feedback normal form obtained in Chapter 5 for stabilization purpose is reutilized. The design is performed in two stages; first for external dynamics using conditional servo compensator and second for unstable internal dynamics using extended high gain observer. The results of two stages are integrated to derive a composite control to regulate complete system. Finally the simulation results are presented to demonstrate the effectiveness of the control scheme. The concepts and results are concluded in the last section.

6.1 Introduction

The output regulation problem, also termed as servomechanism problem, addresses design of a control law for a system to track a predefined reference input and reject a class of disturbance signals in an uncertain environment while maintaining closed-loop stability. The reference signals to be tracked and disturbance signals to be rejected are generated by an autonomous system called exo-system. The general mathematical formulation of output regulation is applicable to many real world control problems like cruise control of land and airborne vehicles, landing and take-off of aircrafts, satellite movement in space, speed control of electric motors etc.

Although work was started in this direction long time ago, yet, precise problem formulation in modern state space domain was reported not earlier than 1970. Like most of the research reported in control engineering, the output regulation problem was also studied for a class of linear systems. The linear output regulation problem was solved collectively by many researchers, including [Desoer and Wang, 1980; Francis and Wonham, 1976; Huang, 1995; Johnson, 1971; Lin et al., 1996; Saberi et al., 2003], to name just a few. The research on output regulation of linear systems resulted in an important concept of internal model principle of control theory and it was observed that this problem is related with solution of a set of two linear matrix equations termed as Regulator Equations. The study in this direction for nonlinear system was started almost at same time when work on output regulation problem for linear systems was on the peak. Initially Francis and Wonham considered nonlinear regulation problem for a special case of constant exogenous signals. For next many years, a number of researchers started studying the nonlinear output regulation problem. The key work reported during this period was by Isidori and Byrnes [1990]. In this paper, it was shown that neutral stability assumption on the exo-system, and in the presence of some other stabilizability/detectability conditions, it is possible to solve nonlinear output regulation problem provided certain algebraic equations and partial differential equations, termed as nonlinear regulator equations, are solvable. In Huang and Rugh [1992a] another important solution to local output regulation problem was presented. Afterwards, a large number of publications followed taking into account various aspects of the problem and its solutions. In [Byrnes et al., 1997; Huang and Lin, 1994; Huang and Rugh, 1992b; Wang et al., 2000] approximate results were suggested when exact solution of regulator equations was difficult to generate. Similarly results for structurally stable systems where uncertainty in system parameters could be ignored are presented in [Byrnes et al., 1997; Isidori, 1995]. Output regulation for systems having uncertainty in parameters within a predefined compact set was studied by a number of authors including [Byrnes et al., 1997; Isidori and Tarn, 1995; Khalil, 1994, 2000]. An adaptive internal model allowing uncertain exo-system was considered in [Serrani et al., 2001] that solved semiglobal output regulation problem. The output regulation problem is extensively discussed in Byrnes and Isidori [2000]; Byrnes et al. [1997]; Huang [2004]; Isidori et al. [2003] where a detailed list of references is available.

The results presented until year 2000 focused on solution of local or semiglobal stability problem where initial conditions stay only in a small predefined compact set. For a more wider domain, focusing on global perspective along-with a degree of robustness, such work was extended to global version of output regulation. The number of results presented in this area are however few and were presented not before year 2000. Papers including [Byrnes and Isidori, 2003; Chen and Huang, 2003, 2004; Ding, 2001; Marconi and Serrani, 2002; Serrani and Isidori, 2000] targeted in this domain. Furthermore, work reported on output regulation, either in local or global domain, targeted minimum phase nonlinear systems having stable or no zero dynamics. In [Khalil, 2000; Mahmoud and Khalil, 1997] servomechanism problem is addressed for minimum phase nonlinear system. In [Mahmoud and Khalil, 1997] a robust controller is designed under state feedback to ensure boundedness of all states as well as convergence of tracking error and showed that system performance can be recovered under output feedback using high gain observer. In [Khalil, 2000], it is shown that robustness in design can be achieved under output regulation for a minimum phase system by utilizing some knowledge of the nonlinearities of the system while the plant parameters and uncertainties are not used in the design calculations. Later on Khalil and group of co-researchers [Seshagiri and Khalil, 2005a,b; Singh and Khalil, 2005] introduced concept of conditional servo compensator for output regulation of minimum phase systems that improved steady state regulation error without degrading transient behaviour of the system.

While a lot of work was done on stabilization and regulation of minimum phase nonlinear systems, little work is reported in this direction for the case of non-minimum phase systems. The complexity of non-minimum phase nonlinear systems having unstable zero dynamics cannot be removed by straight forward application of feedback linearization and hence like, output feedback stabilization problem, the output regulation problem also becomes a challenging task. This area is, therefore, an active research area in the field of nonlinear control systems. For the case of stabilization problem, some work is reported in literature that is already described in Chapter 4. For the case of output regulation of non-minimum phase nonlinear systems, again, not too many publications are reported in literature and some researchers presented the solution to this problem in this area are [Celani, 2011; Chiang and Isidori, 2015; Huang and Guoqiang, 2004; Nazrulla and Khalil, 2011a; Su, 2016]. In [Huang and Guoqiang, 2004] and [Celani, 2011], a benchmark nonlinear system TORA is investigated to robustly reject disturbances based on output regulation method that uses only measurements of the output. In paper [Su, 2016], practical implementation for solution of local output regulation problem for TORA benchmark system is demonstrated.

The work on output regulation presented in this thesis is an extension to the previous work of output feedback stabilization for a class of under-actuated nonlinear benchmark system

[Khalid and Memon, 2014] and utilizes earlier work on output regulation / servo mechanism for minimum phase system presented in Seshagiri and Khalil [2005b]. The generalized procedure developed for evolution of normal form representation after collocated partial feedback linearization and global change of coordinates applied earlier for stabilizing control as detailed in subsection 4.1.2 is equally applicable for output regulation. Towards this end, the technique for output regulation presented in this work broadly utilizes the work presented in Isidori [2000]; Nazrulla and Khalil [2011a,b]; Wazir [2015] to solve the regulation problem of target class of under-actuated benchmark nonlinear systems and the problem is addressed by formulating reduced auxiliary system so that the controllable states act as control input for zero dynamics to make it stabilizable and an extended high gain observer is implemented to estimate virtual output as well as system controllable states and actual output. In previously reported work the internal states are stabilized to steady state value by regulator equation that requires solution of partial differential equations and complex mathematical calculations that is quite tedious. Another difference in work for output regulation presented here is in servo compensator design and a conditional servo compensator is used due to its advantages to conventional one as mentioned in Seshagiri and Khalil [2005b] and the steady state values of internal dynamics are driven arbitrarily fast to corresponding equilibrium points.

6.2 Nonlinear Output Feedback Regulation

Output feedback regulation is termed as deriving the output of a system to asymptotically track any prescribed reference trajectories in a certain class of functions of time while rejecting any disturbances in a certain class of disturbances. One of the possible approach to solve this problem is via internal models [Isidori et al., 2003]. If the trajectory which is required to be tracked or the disturbance which is required to be rejected belongs to a set of all possible trajectories generated by some fixed linear exo-system and such an internal model of a system is incorporated in a controller, then the controller can make asymptotic decay of the tracking error to zero for every possible trajectories present in the system and does it robustly with respect to parameter uncertainties. Regulation via such internal model based control is very effective for highly nonlinear systems as this reshapes the nonlinear dynamics of the closed loop system to induce an invariant manifold in the presence of uncertainties. The procedure depicted here to solve nonlinear servomechanism problem closely follows the procedure as in [Serrani, 2005].

A time-invariant, nonlinear, finite dimensional system having state vector $x \in \Re^n$, control input $u \in \Re^m$, external disturbance inputs $w \in \Re^r$, measured output $y \in \Re^p$ and regulated output or tracking error $e \in \Re^q$ for solution to output regulation problem can be described as

$$\dot{x} = f(x,u,w), \quad x(0) = x_0 \tag{6.1}$$

$$e = h(x,w) \tag{6.2}$$

$$y = k(x,w) \tag{6.3}$$

where $f(x,u,w)$, $h(x,w)$ and $k(x,w)$ are nonlinear continuous smooth functions of their arguments and that $f(0,0,0) = 0$, $h(0,0) = 0$, $k(0,0) = 0$. The disturbance input w consists of a set of exogenous input variables which includes references to be tracked and unknown disturbances to be rejected. x_0 denotes initial states vector at $t = 0$ and $\dot{x}$ is defined as derivative of states x w.r.t. time t. The regulated output e includes tracking errors and any other variables that needs to be regulated.

The neutrally stable system known as exo-system generates an exogenous signal denoted as w. It is an autonomous system that models the class of disturbance and reference signals taken into consideration by the designer. Ideally a complete knowledge or model of reference signals to be tracked and disturbance signals to be rejected by the closed loop control system should be available in real time. This, however, would not be a practical and realizable solution. On the other hand, with no prior knowledge of exo-system may lead to a design of closed loop control system exhibiting an ultimate bounded error e that will not necessarily be zero. To implement an intermediate solution that is both practicable and realistic, an exo-system is chosen that contains a set of reference signals to be tracked and disturbance signals to be rejected. In this way, w is only allowed to belong to a fixed family of functions of time enabling the designer to cover major cases of practical cases. A general exo-system used in output regulation problem is generated by nonlinear autonomous system

$$\dot{w} = s(w) \tag{6.4}$$

The function s is assumed as smooth function with $s(0) = 0$. For simplicity and in almost all known design methods. the case of autonomous linear time-invariant exo-system is considered given as

$$\dot{w} = Sw \tag{6.5}$$

where the corresponding initial conditions $w(0)$ is allowed to vary on a prescribed set. The exo-system being neutrally stable have all eigenvalues on imaginary axis. The matrix S can be expressed as skew-symmetric matrix so that all the trajectories of exo-system (6.5) are bounded in forward time and does not decay to zero as $t \to \infty$. A dynamic feedback controller can be defined as

$$\left.\begin{aligned} \dot{\xi} &= \phi(\xi, y) \\ u &= \theta(\xi, y) \end{aligned}\right\} \tag{6.6}$$

where $\phi(0,0) = 0$, $\theta(0,0) = 0$ such that for all initial conditions in the neighbourhood of origin, all the state variables of the closed-loop system are bounded and $\lim_{t\to\infty} e(t) = 0$. For the system (6.1-6.3) and controller (6.6), a set of partial derivatives are defined as

$$A = \left[\frac{\partial f}{\partial x}\right]_0 B = \left[\frac{\partial f}{\partial u}\right]_0 C = \left[\frac{\partial h}{\partial x}\right]_0$$

$$F = \left[\frac{\partial \phi}{\partial \xi}\right]_0 G = \left[\frac{\partial \phi}{\partial u}\right]_0 H = \left[\frac{\partial \theta}{\partial \phi}\right]_0$$

To solve output regulation problem, the origin of system needs to be asymptotically stable equilibrium when exo-system is not present. Furthermore the pair(A,B) and the pair$(C.A)$ should be detectable. The closed loop system can be given as

$$\left.\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= f(x, \theta(\xi, k(x,w)), w) \\ \dot{\xi} &= \phi(\xi, k(x,w)) \\ e &= h(x,w) \end{aligned}\right\} \tag{6.7}$$

and if $\pi(w)$ is steady state of x and $\sigma(w)$ is steady state value of ξ on a zero error manifold, as shown in Fig. 6.1, such that the manifold

$$\mathscr{M}_0 = \{(x, \xi, w) : x = \pi(w), \xi = \sigma(w)\}$$

is to be invariant than control action can be divided as the one that drives the system to steady state and the other that stabilizes it at steady state. For error dynamics $e = 0$, such that $0 = h(\pi(w), w)$, then the system must satisfy a set of equations termed as regulator equations and are given as

$$\left.\begin{aligned} \frac{\partial \pi(w)}{\partial w} s(w) &= f(\pi(w), w, c(w)) \\ 0 &= h(\pi(w), w) \end{aligned}\right\} \tag{6.8}$$

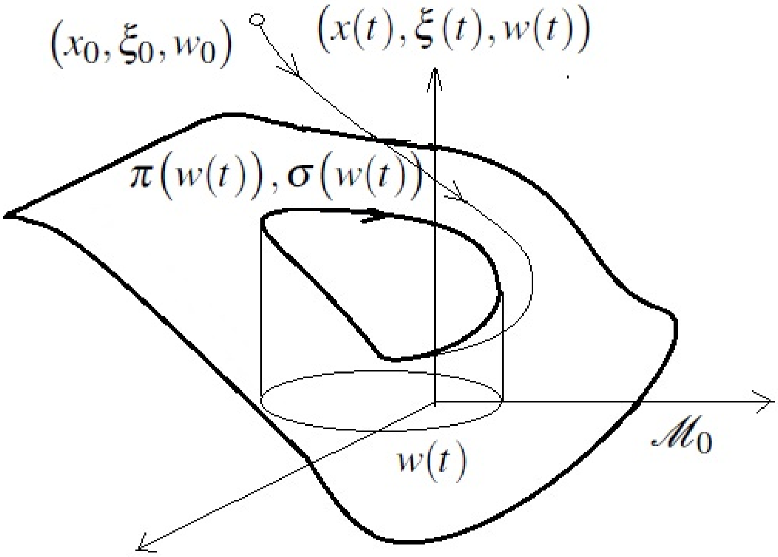

Fig. 6.1 Pictorial Representation of Zero Error Manifold [Serrani, 2005]

and the set of controller equations are

$$\left.\begin{aligned} \frac{\partial \sigma(w)}{\partial w} s(w) &= \phi(\sigma(w), 0) \\ c(w) &= \theta(\sigma(w)) \end{aligned}\right\} \tag{6.9}$$

Here polynomial $c(w)$ is the steady state value of control input u having only w. It is to be noted that problem of output regulation is solved using model of exo-system $s(w)$. This is generally known as internal model principle and is usually designed as

$$\left.\begin{aligned} \frac{\partial \sigma(w)}{\partial w} sw &= \Phi(\sigma(w), 0) \\ c(w) &= \Gamma(\sigma(w)) \end{aligned}\right\} \tag{6.10}$$

where $\sigma(w)$ represent the mapping

$$\sigma(w) = \begin{bmatrix} c(w) & L_s c(w) & L_(s)^2 c(w) & \cdots & L_s^{q-1} c(w) \end{bmatrix}^{\mathsf{T}}$$

where

$$
\begin{aligned}
L_s c(w) &= \frac{\partial c(w)}{\partial w} s(w) = \frac{\partial c(w)}{\partial w} S w \\
\Gamma &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{1*q} \\
\Phi &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{q-1} \end{bmatrix}
\end{aligned}
$$

The last row in matrix Φ contains coefficients that are obtained from

$$L_s^q c(w) = a_0 c(w) + a_1 L_s c(w) + a_2 L_(s)^2 c(w) + \cdots + a_{q-1} L_s^{q-1} c(w)$$

where the characteristic polynomial $\lambda^q - a_{q-1}\lambda^{q-1} - \ldots - a_1\lambda - a_0$ has distinct roots on the imaginary axis and the pair $\begin{bmatrix} A & 0 \\ GC & \Phi \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}$ is stabilizable and the pair $\begin{bmatrix} C & 0 \end{bmatrix}, \begin{bmatrix} A & B\Gamma \\ 0 & \Phi \end{bmatrix}$ is detectable. By using a change of variables

$$\left.\begin{aligned} \tilde{x} &= x - \pi(w)) \\ \tilde{\xi}' &= \xi' - \sigma'(w) \end{aligned}\right\} \tag{6.11}$$

the stabilizer takes the form

$$\left.\begin{aligned} \dot{\xi}'' &= l(\xi'', e) \\ u'' &= \Gamma(\xi'', e) \end{aligned}\right\} \tag{6.12}$$

In this way, output regulator, as shown in Fig. 6.2 may be called as a parallel interconnection of an internal model based servo compensator (6.10) and a stabilizing compensator shown in (6.12). For stabilizing compensator, local stabilization can be achieved by linearization and there exists set of matrices L,M and N for which a linear stabilizer compensator can be derived. In this case the overall controller can be chosen of the form

$$\left.\begin{aligned} \dot{\xi}' &= \Phi\xi' + \theta_1 e \\ \dot{\xi}'' &= L\xi'' + My \\ u &= \Gamma\xi' + N\xi'' \end{aligned}\right\} \tag{6.13}$$

The regulation of system is thus achieved locally as the internal model provides $c(w)$ component of u such that $x = \pi(w)$ and $\xi = \sigma(w)$, while the stabilizer provides stabilizing component of u that induces local exponential convergence towards the zero error manifold $\mathscr{M}_0$.

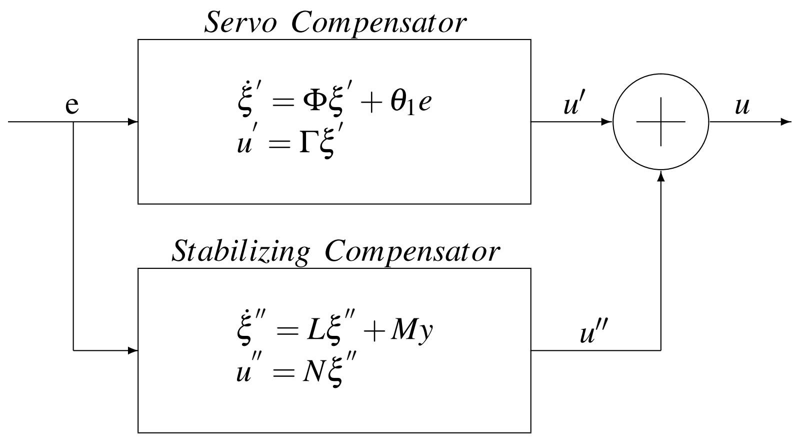

Fig. 6.2 Structure of Regulator

6.3 Nonlinear Servomechanism

The output regulation method used to regulate tracking error as discussed in section 6.2 is associated with two main disadvantages. Firstly it does not cater for robustness issue as only local stability is addressed and secondly separate controllers are required to be designed for stabilization and internal model based compensators. Another technique termed as servo compensator is reported in literature for output regulation based control of nonlinear systems. The servo compensator based output feedback regulation is an extension of regulator based method as discussed in section 6.2 and is a well defined technique reported in literature for regulation of minimum phase nonlinear systems as for example in Mahmoud and Khalil [1997]; Seshagiri and Khalil [2005b]. In these reported works, system normal form representation is utilized to control external dynamics using conventional/conditional servo compensators along-with suitable observer for output feedback regulation of complete system. In this regard, the work reported in Seshagiri and Khalil [2005b] is extended in this thesis for output feedback regulation of target class of under-actuated benchmark nonlinear systems. It utilizes mathematical representation in strict feedback normal form derived earlier to solve stabilization problem while an extended high gain observer is used for state estimation of zero dynamics. Finally, the conditional servo compensator is implemented for regulation of external dynamics so that trajectory tracking and boundedness of systems states is achieved and system exhibits a degree of robustness under unknown disturbances.

6.3.1 Conventional Servo Compensator

The Euler's Lagrange equation described in Chapter 3, equation (3.21) transformed into strict feedback normal form (4.8) with error dynamics e can be represented as

$$\dot{\eta} = \phi_0(\eta, \xi, \theta) \tag{6.14}$$

$$\dot{\xi}_i = \xi_{i+1}, \quad 1 \leq i \leq \rho - 1 \tag{6.15}$$

$$\dot{\xi}_\rho = b(\eta, \xi, \theta) + a(\eta, \xi, \theta)u \tag{6.16}$$

$$e_1 = \xi_1 - q(w, \theta) \tag{6.17}$$

$$e = h(\xi, w, \theta) \tag{6.18}$$

Here the high frequency component $a(.) \neq 0$, $b(0,0,\theta) = 0$, $\eta \in D_\eta \subset \Re^{\eta-\rho}$, $\xi \in D_\xi \subset \Re^\rho$ and $\theta \in \Theta \subset \Re^\rho$ is a vector of unknown constant parameters. The error dynamics $e =$

$[e_1\, e_2\, e_3\, \cdots\, e_\rho]$ shows error output and $q(w,\theta)$ is the reference trajectory to be tracked. Control u is considered here as a single input and zero dynamics or internal states are assumed having roots with negative real eigenvalues making it a minimum phase system. For sake of simplicity the minimum phase system is considered here. Later on non-minimum phase systems of target class of under-actuated systems with unstable zero dynamics will be considered for application of servo compensator algorithm and will be applied on an example of target class of systems.

To solve the output regulation problem a concept of exo-system and regulator equations are assumed as discussed in section 6.2. The disturbance in θ and $q(w,\theta)$ are assumed to be generated by a neutrally stable exo-system

$$\dot{w} = S_0 w \tag{6.19}$$

A continuous mapping $\xi = \pi(w)$ and $u = \chi(w)$ exists on zero error manifold that solves the regulator equations

$$\left.\begin{aligned} \frac{\partial \pi_i(w)}{\partial w} S_0 w &= \pi_{i+1}(w), \quad 1 \leq i \leq \rho - 1 \\ \frac{\partial \pi_\rho(w)}{\partial w} S_0 w &= b(0,\pi(w),w) + a(0,\pi(w),w)\chi(w) \\ 0 &= h(\pi(w),w) \end{aligned}\right\} \tag{6.20}$$

for all $w \in W$. The mapping $u = \chi(w)$ satisfies the identity

$$L_s^m \chi = c_0\chi + c_1 L_s \chi + c_2 L_s^2 \chi + \ldots + c_{m-1} L_s^{m-1} \chi$$

for all $w \in W$ and $c_0, c_1, \cdots, c_{m-1}$ are set of real constants having values so as characteristic polynomial $\lambda^m - c_{m-1}\lambda^{m-1} - \cdots - c_1\lambda^1 - c_0$ has distinct zeros on the imaginary axis. Now an internal model can be utilized to find $\chi(w)$ as

$$\left.\begin{aligned}\frac{\partial\tau(w)}{\partial w}S_0 w = S\tau(w)\\ \chi(w) = \Gamma(\tau(w))\end{aligned}\right\} \tag{6.21}$$

where

$$\Gamma = \begin{bmatrix}1 & 0 & 0 & ... & 0\end{bmatrix}_{1*q}$$

$$S = \begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -c_0 & -c_1 & -c_2 & \cdots & -c_{q-1}\end{bmatrix}$$

$$\tau(w) = \begin{bmatrix}\chi(w) & L_s c(w) & L_s^2 c(w) & \cdots & L_s^{m-1}c(w)\end{bmatrix}^{\mathsf{T}}$$

Under the above mentioned assumptions, the conventional servo compensator based control law can be designed as

$$u = -ksign\left(L_g L_f^{\rho-1}h\right)sat(s/\mu) \tag{6.22}$$

Here k is a positive constant gain, μ is a design parameter and *sign* and *sat* are functions defined as

$$sign(x) = \begin{cases}+1 & x > 0\\ 0 & x = 0\\ -1 & x < 0\end{cases}$$

$$sat(x) = \begin{cases} x & |x| \leq 1 \\ sign(x) & |x| > 1 \end{cases}$$

$L_g L_f^{\rho-1} h$ is high frequency gain and s is sliding surface given as

$$s = K_1\phi + K_2 \begin{bmatrix} e_1 & e_2 & \cdots & e_{\rho-1} \end{bmatrix}^\intercal + e_\rho \tag{6.23}$$

and the conventional servo compensator is augmented with

$$\dot{\phi} = S\phi + Je_1 \tag{6.24}$$

where $K_2 = \begin{bmatrix} k_1 & k_2 & k_{\rho-1} \end{bmatrix}$. The matrices K_1 and K_2 are chosen such that closed loop system remain Hurwitz.

6.3.2 Conditional Servo Compensator

The conventional servo compensator discussed in previous section show better performance as stability achieved is non-local and closed loop system is robust to disturbances. The improved steady state performance often happens at the cost of degraded transient response as the overall system order is increased due to addition of servo compensator and its interaction with control saturation. The problem is addressed by number of researchers and conditional servo compensator is introduced to solve it [Seshagiri and Khalil, 2005b].

The core design of conditional servo compensator is similar to the conventional counterpart as the control input 'u' and the sliding surface 's' remains the same as in (6.22) and (6.23) respectively. It differs in implementation of augmented servo compensator 'ϕ' and how the value of matrices K_1 and K_2 are selected. In conditional servo compensator equation (6.24) is modified as

$$\dot{\phi} = (S - JK_1)\phi + \mu J sat\left(\frac{s}{\mu}\right) \tag{6.25}$$

Here $\mu > 0$ is the width of the boundary layer and K_1 is chosen to make $S - JK_1$ as Hurwitz. While $K_2 = \begin{bmatrix} k_1 & k_2 & \cdots k_{\rho-1} \end{bmatrix}$ is selected such that the polynomial $\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1$ is hurwitz.

6.4 Control Law under Output Feedback Regulation

The design of control law under output feedback regulation is based on conditional servo compensator based regulating controller and an internal dynamics stabilizing control law. The servo compensator based regulation mechanism utilizes design procedure as detailed in (6.3.2). The observe part of design consist of an EHGO for state estimation of external dynamics as discussed in section (2.6.3) in a similar way for stabilization of TORA as presented in section (5.2.3). For stabilization of internal dynamics, the design technique utilizes similar approach as for the case of stabilization problem. Towards that end, an auxiliary system is formulated for stabilization of internal dynamics utilizing a suitable control law as auxiliary control input to be generated by external dynamics. The scheme is implemented in the dynamics of sliding surface variable "s" by augmentation of this auxiliary control input so that overall regulation purpose is achieved. The overall design algorithm is applied on output feedback regulation of TORA system presented in next section.

6.5 Illustrating Example: Output Feedback Regulation of TORA System

In this section, the mathematical formulation for output feedback regulation of TORA system is presented. The design broadly utilizes the methodology suggested in Nazrulla and Khalil [2011a] and Wazir [2015].

6.5.1 Mathematical Formulation

The simplified diagrammatical representation of TORA is shown in Fig. 5.8, its nonlinear mathematical form is shown in section 5.2.2. The normal form representation of TORA derived after change of variables is, for convenience, again shown here in the form of two subsystems and an error output. The first subsystem is

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \frac{k}{m_1+m_2}\left(\frac{m_2 r\sin(\xi_1)}{m_1+m_2} - \eta_1\right) \end{aligned}\right\} \tag{6.26}$$

the second subsystem and the output is given as

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \frac{1}{\delta(\xi_1)}\left[(m_1+m_2)\tau - m_2 r\cos(\xi_1)\left\{m_2 r\xi_2^2\sin(\xi_1) - k\right.\right. \\ &\qquad \left.\left.\left(\eta_1 - \frac{m_2 r\sin(\xi_1)}{m_1+m_2}\right)\right\}\right] \\ y &= \xi 1 \end{aligned}\right\} \tag{6.27}$$

$$\begin{aligned} \text{where} \quad \delta(\xi_1) &= (I+m_2r^2)(m_1+m_2) - m_2^2 r^2\cos^2(\xi_1) \\ &\geq (I+m_2r)^2 m_1 + m_2 I \\ &> 0 \end{aligned}$$

and y denotes the available output. The system has relative degree $\rho = 2$ while internal dynamics is unstable. For output feedback regulation, a controller is designed so that the output $y = \xi_1$ follows the trajectory defined by a linear exo-system (6.28) while the internal dynamics remain stable and stabilized to zero. The linear exo-system is taken as

$$\dot{w} = S_0 w \tag{6.28}$$

where $w = [w_1 \quad w_2]^\mathsf{T}, \quad S_0 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$ and ω is a constant parameter.

6.5.2 Conditional Servo Compensator Design

The servo compensator is designed by replacing output y in (6.27) with error dynamics e_1 given as

$$e_1 = \xi_1 - w_1 \tag{6.29}$$

The external dynamics, taken as reduced system, where (6.29) is taken as output is defined as

$$\left.\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \frac{m_1 + m_2}{\delta(\xi_1)} u \\ y &= \xi_1 - w_1 \end{aligned}\right\} \tag{6.30}$$

Taking $\quad \Gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$ the control input u takes the form as

$$u = -Ksat\left(\frac{s}{\mu}\right) \tag{6.31}$$

Sliding surface s for external dynamics under conditional servo compensator design contains error dynamics and its derivative obtained using high gain observer. Additionally it contains a control input for internal dynamics that will be discussed in next section.

$$s = K_1\phi + k_2 e_1 + e_2 + additional\, control\, for\, \eta \tag{6.32}$$

and the constant K is

$$K = \beta \frac{\delta(\xi_1)}{m_1 + m_2} \tag{6.33}$$

where k_2 is any positive constant and μ and k_1 are design parameters. K_1 is selected such that $S - JK_1$ is Hurwitz. While ϕ is generated by

$$\dot{\phi} = (S - JK_1)\phi + \mu J sat\left(\frac{s}{\mu}\right) \tag{6.34}$$

6.5.3 Stabilization Controller for Internal Dynamics

In the second phase of controller design, the stabilization controller for zero dynamics is formulated by considering an auxiliary system given as

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \frac{k}{m_1 + m_2}\left[\frac{m_2 r \sin(\xi_1)}{m_1 + m_2} - \eta_1\right] \\ \dot{e}_1 &= u_a \\ \sigma &= \frac{m_2 r \cos(\xi_1)}{\delta(\xi_1)}\left[k \times \left(\eta_1 - \frac{m_2 r \sin(\xi_1)}{m_1 + m_2}\right) - m_2 r \xi_2^2 \sin(\xi_1)\right] \end{aligned}\right\} \tag{6.35}$$

where the auxiliary output σ is estimated from an extended high gain observer as

$$\left.\begin{aligned} \dot{\hat{\xi}}_1 &= \hat{\xi}_2 + \frac{a_1}{\varepsilon}(\xi_1 - \hat{\xi}_1) \\ \dot{\hat{\xi}}_2 &= \hat{\sigma} + \hat{a}(\xi)u + \frac{a_2}{\varepsilon^2}(\xi_1 - \hat{\xi}_1) \\ \hat{\sigma} &= \frac{a_3}{\varepsilon^3}(\xi_1 - \hat{\xi}_1) \end{aligned}\right\} \tag{6.36}$$

where ε is a design parameter and the constant parameters a_1,a_2 and a_3 are selected such that the polynomial $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ have negative real roots. The design for stabilization of internal dynamics closely follows to the one given in previous section and in Nazrulla and Khalil [2011a]. An auxiliary output $\hat{y}_a$ is defined by replacing ξ_2 with u_a for auxiliary system

(6.35) given as

$$\hat{y}_a = \frac{\delta(\xi_1)\sigma}{m_2 r\cos(\xi_1)} + \frac{m_2 r(u_a)^2 sin(\xi_1)}{k} + \frac{m_2 r\sin(\xi_1)}{m_1+m_2} \tag{6.37}$$

such that

$$\hat{y}_a \cong \eta_1 \tag{6.38}$$

Assuming $u_a = \frac{1}{\varepsilon_a}(v_a - e_1)$ where v_a is the control input to be designed and ε_a is a positive constant. Now the auxiliary system can be expressed as

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \frac{k}{m_1+m_2}\left[\frac{m_2 r}{m_1+m_2}\sin(e_1) - \eta_1\right] \\ \varepsilon_a \dot{e}_1 &= v_a - e_1 \\ \hat{y}_a &= \eta_1 \end{aligned}\right\} \tag{6.39}$$

For small value of ε_a the above system act like a singularly perturbed system as it exhibits two time scale properties. For the case of $\varepsilon_a = 0 \Rightarrow v_a = e_1$ results in $sin(e_1) = sin(v_a) \triangleq w_a$ and (6.39) can be written as a linear system given as

$$\left.\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= \frac{k}{m_1+m_2}\left[\frac{m_2 r}{m_1+m_2} w_a - \eta_1\right] \\ \hat{y}_a &= \eta_1 \end{aligned}\right\} \tag{6.40}$$

The auxiliary system (6.40) is observable and control problem reduces to design of control law w_a. A state feedback control law $w_a = \kappa_1\eta_1 + \kappa_2\eta_2$ can be derived. A full order observer based design can be done as

$$\left.\begin{aligned} \dot{\hat{\eta}}_1 &= \hat{\eta}_2 + \tilde{a}_1(\hat{y}_a - \hat{\eta}_1) \\ \dot{\hat{\eta}}_2 &= \tilde{a}_1(\hat{y}_a - \hat{\eta}_1) - \frac{k\hat{\eta}_1}{m_1 + m_2} + \frac{km_2 r}{(m_1 + m_2)^2}\hat{w}_a \end{aligned}\right\} \tag{6.41}$$

Here $\hat{w}_a$ is a control input with estimated states $\hat{\eta}_1$, $\hat{\eta}_2$ and $\tilde{a}_1$, $\tilde{a}_2$ are selected so that the polynomial $\lambda^2 + \tilde{a}_1\lambda + \tilde{a}_2$ is Hurwitz. The control input is taken as

$$\left.\begin{aligned} u_a &= N(\hat{\eta}, e_1) \\ N(\hat{\eta}, e_1) &= \frac{1}{\varepsilon_a}\left[sin^{-1}\{sat(\kappa_1\hat{\eta}_1 + \kappa_2\hat{\eta}_2) - e_1\}\right] \end{aligned}\right\} \tag{6.42}$$

6.5.4 Composite Control Law

The control law for entire cascaded system is derived using the control law derived for internal system in section 6.5.3. The sliding surface (6.32) having estimated error dynamics in control law (6.31) can be modified as

$$\hat{s} = K_1\phi + k_2\hat{e}_1 + \hat{e}_2 - u_a \tag{6.43}$$

where error states $\hat{e}_1, \hat{e}_2$ are estimated from high gain observer

$$\left.\begin{aligned} \dot{\hat{e}}_1 &= \hat{e}_2 + \frac{g_1}{\varepsilon_0}(e_1 - \hat{e}_1) \\ \dot{\hat{e}}_2 &= \frac{g_2}{\varepsilon_0^2}(e_1 - \hat{e}_1) \end{aligned}\right\} \tag{6.44}$$

states ϕ are estimated from

$$\dot{\phi} = (S - JK_1)\phi + \mu J sat\left(\frac{\hat{s}}{\mu}\right) \tag{6.45}$$

and $\hat{y}_a$ is obtained as

$$\hat{y}_a = \frac{\delta(\hat{\xi}_1)\hat{\sigma}}{m_2 r \cos(\hat{\xi}_1)} + \frac{m_2 r (u_a)^2 sin(\hat{\xi}_1)}{k} + \frac{m_2 r \sin(\hat{\xi}_1)}{m_1 + m_2} \tag{6.46}$$

Finally the control signal u is taken as

$$u = -Ksat\left(\frac{\hat{s}}{\mu}\right) \tag{6.47}$$

6.5.5 Simulation based Analysis

Table 6.1 TORA System Parameters and Initial Conditions

Parameter	Value	Unit
k	186.3	N/m
m_1	1.3608	kg
m_2	0.096	kg
r	0.0592	m
I	0.0002175	kgm^2
g	9.81	m/sec^2

Table 6.2 Controller Parameters

Parameter	Value
β	500
μ	0.01
k_2	5
κ_1	-1
κ_2	-8
ε	0.01
ε_0	0.001
ε_a	0.1

In this section, performance of TORA system is analyzed for verification and validation. For simulation purpose, the system parameters/gains and controller parameters are set with corresponding initial conditions. The same system parameters are chosen here for TORA

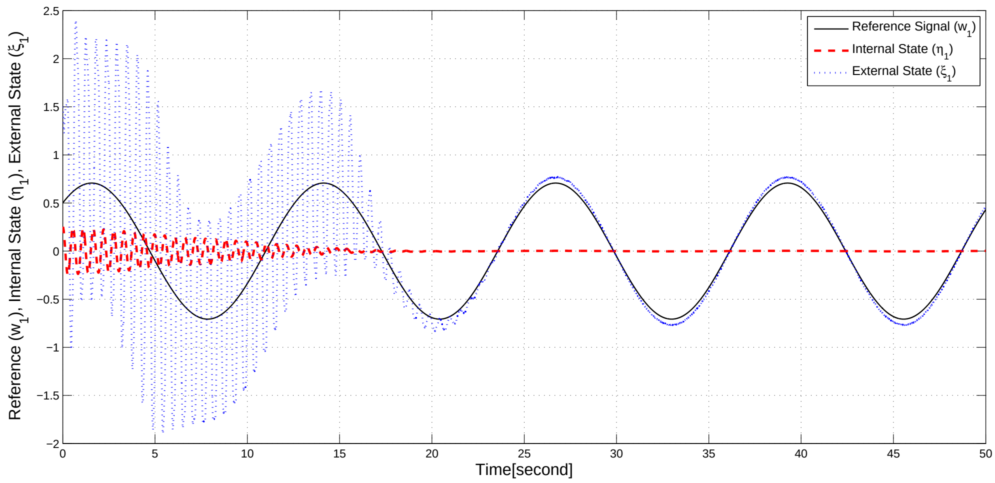

Fig. 6.3 Reference signal w_1, Internal state η_1, Output $y = \xi 1$

system as in Chapter 5 and, for convenience, are shown in Table 6.1. The controller parameters are given in Table 6.2. The parameter for exo-system is chosen as $\Omega = 0.5$ with initial value $w(0) = [0.5 \quad 0.5]^{\mathsf{T}}$. The poles for extended high gain observer are set at $s = -11, -12, -13$. The poles for linear high gain observer are set at $s = -5, -6$. The poles for observer of internal states are set at $s = -2, -3$. For conditional servo compensator, the matrices are given as

$$J = [0 \quad 0 \quad 1]^{\mathsf{T}}, \quad S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \text{ and } K_1 = [125 \quad 80 \quad 20].$$

The overall system performance and convergence of system states is shown in Fig. (6.3). The plot shows that after some transient phase zero dynamics state η_1 is stabilized to equilibrium point and remains bounded thereafter. Similarly the external state ξ_1 converge to equilibrium, remains bounded and track reference input signal. In this way, the regulated output $y = \xi_1$ moves along-with desired reference trajectory after some initial transient phase.

6.6 Summary

In this chapter, the regulation problem for nonlinear systems is discussed. Starting with general description and background work presented by renowned researchers in this domain, the classical method of nonlinear output regulation is described. The detailed algorithm of conventional servo compensator for output feedback regulation and its advantages with respect to other control scheme is provided. Finally conditional servo compensator based technique is elaborated and its effectiveness to improve system steady state and transient response is presented.

In second part of this chapter, the output feedback regulation of a test bench nonlinear system TORA is discussed and the generalized procedure elaborated in first part for servo compensator based output regulation is applied on the TORA system. Utilizing the strict feedback normal form of TORA system, the system is classified as an auxiliary system having unstable internal dynamics and an external linear system. A conditional servo compensator is developed for reduced system while an EHGO is used for auxiliary system. A composite control law is finally evolved and is utilized for simulation based analysis of TORA system.

Chapter 7

Conclusions

Non-minimum phase nonlinear systems exhibit complex dynamics requiring more control efforts than linear systems. Under-actuated systems are generally non-minimum phase nonlinear systems where some part of system dynamics is unactuated. Real life mechanical and electromechanical systems are mostly under-actuated or are prone to be under-actuated upon occurrence of fault. Control of these type of systems is, therefore, an active and important research area.

This thesis focuses on the problem of output feedback stabilization and regulation for a class of under-actuated non-minimum phase nonlinear systems with emphasis on presenting a simple and step by step controller design methodology applicable on target class of systems. The controller design is unique, in non-hybrid form and easy to implement as compared to some of previously reported designs. The state feedback design is incorporated in Sliding Mode Control (SMC) framework and an Extended High Gain Observer (EHGO) based output feedback scheme is utilized for state estimation. The SMC and EHGO based control law inherently incorporate robustness in design and provide faster convergence of states. An important feature of methodology is derivation of suitable normal form representation for target class of under-actuated systems that is effectively utilized for solution of output feedback control problem. The normal form bifurcates generalized system into internal and external dynamics. The internal dynamics being highly nonlinear is stabilized using control Lyapunov

Function method and the stabilization is achieved using error dynamics from external linear system under SMC framework.

For output regulation, the concept of servomechanism is incorporated for reference tracking and disturbance rejection in the presence of time varying exogenous signals that are generated by a known exo-system. A conditional servo compensator being used for output regulation improves transient performance without any significant degradation of steady state response. Stability of closed loop system is ensured and system performance is verified by numerical simulations. Analytical results are provided for an arbitrarily large set of initial conditions showing effectiveness of proposed methodology.

The rest of the chapter summarizes the thesis, provides its sequential flow and outlines directions of future work. Future aspects relevant to proposed work that may be taken by researchers are highlighted and new dimensions of research in this domain are identified.

7.1 Summary of Thesis

The thesis is divided into seven chapters.

First chapter sheds light on inspiring factors that acted as driving force for the preparation of this thesis and highlight a brief about the state of the art. The work done by renowned researchers in the field of under-actuated systems is summarized. Finally the contribution of work towards stabilizing controller design is presented.

Second chapter provides basic definitions and concepts of linear as well as nonlinear systems. The concept of zero dynamics and non-minimum phase property of systems is described in detail. Usage of state observer for output feedback control is provided in last section where high gain observer and extended high gain observer are elaborated in detail.

The third chapter is about under-actuated nonlinear systems. It provides Lagrangian based mathematical formulation of nonlinear systems. Types of nonlinear systems based on actuation property and mathematical representations of under-actuated systems is described in this chapter. A broader classification of nonlinear under-actuated systems is provided and a typical class of systems targeted in this thesis for stabilization and regulation purpose is identified.

In chapter four, the design methodology for stabilization of target class of systems is proposed. A generalized methodology is formulated and application of an important design tool in this case is elaborated. Subsequently in this chapter, observer based generalised controller design is formulated and stability of both controller and observer part of design is explored.

In chapter five, two examples of target class of under-actuated systems are selected and proposed design methodology is applied on these systems. Starting with system mathematical formulation, design algorithm is applied for development of observer based output feedback control law. Finally, simulation based results are utilized for verification of design paradigm.

In chapter six, the proposed design technique is extended for output regulation of target class of under-actuated nonlinear system. Starting with general introduction of servo mechanism problem, work done by renowned scholars in this area is highlighted. The servo compensator based design approach for target class of systems is presented and step by step design procedure is shown. Further in this chapter, example of TORA system is taken and the design methodology is applied for output feedback regulation of TORA system.

7.2 Future Work

In this thesis, we have considered the problem of stabilization and output regulation for a class of non-minimum phase nonlinear systems. The proposed control methodology offers a systematic design procedure and presents a flexibility in selection of various design parameters as per the requirements of the system under consideration. An interesting direction of future work would be to extend the class of systems for which the proposed scheme is applicable. Towards this end, it would be interesting to understand whether the proposed framework allows the flexibility of extending the class of systems considered in this thesis, or relax the assumptions we have made. Of particular interest, in this regard, are the under-actuated systems mentioned in Table 3.1

An interesting, yet challenging, direction of future research would be investigating the output regulation problem for the under-actuated nonlinear systems with control constraints or saturation. The presence of saturation in the input channel imposes strong limitations to the achievable control objectives such as transient performance etc. It will be interesting to see if the proposed control scheme can be modified to achieve the desired level of robustness and performance for these systems.

Last but not the least, further research work needs be done in understanding how to tune the controller as well as observer parameters in order to achieve specific control objectives, as well as to identify possible limitations on the achievable performance.

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