Optimal Conditional Servo-Mechanism Design For a Class of Non-Linear Systems

By

Muhammad Haseeb

NUST201463681MPNEC45014F

Supervisor

Dr. Attaullah Y. Memon

A Thesis submitted to

Pakistan Navy Engineering College (PNEC), Karachi

National University Of Sciences and Technology (NUST), Islamabad

In partial fulfillment of requirements for the degree of

Master of Science (MS) in **Electrical Engineering**

With specialization in Control Systems

July,2018

NATIONAL UNIVERSITY OF SCIENCES AND TECHNOLOGY, PAKISTAN

MASTER'S THESIS

Optimal Conditional Servo-Mechanism Design For a Class of Non-Linear Systems

---

Author:
Muhammad Haseeb

Enrolment No:
NUST201463681MPNEC45014F

Supervisor:
Dr. Attaullah Y. Memon

GEC Members:
Dr. Syed Sajjad Haider Zaidi
Dr. Bilal M. Khan

A thesis submitted in partial fulfillment of the requirements for the degree of Masters of Science in Electrical Engineering with Specialization in Control Systems

Pakistan Navy Engineering College (PNEC), Karachi
Department of Electronics & Power Engineering

July, 2018

Declaration of Authorship

I Muhammad Haseeb, declare that this thesis titled, “Optimal Conditional Servo-Mechanism Design for a Class of Non-Linear Systems” and the work presented in this is my own. I confirm that:

- This work was done wholly or mainly while in the candidature for a Masters of Electrical Engineering with specialization in Control Systems degree at Pakistan Navy Engineering College (PNEC) a constituent unit of National University of Sciences and Technology (NUST), Pakistan.

- Where I have consulted the published work of others, it is clearly attributed.

- Where I have quoted from the work of others, the respective sources are provided. With the exception of such quotations, this thesis is entirely my own work.

- I have acknowledged all the sources of assistance, help and motivation.

Signature of Author: ______________________________

Date: ______________________________

FORM TH – 4

ABSTRACT

OPTIMAL CONDITIONAL SERVO-MECHANISM DESIGN FOR A CLASS OF NONLINEAR SYSTEMS

By

Muhammad Haseeb

In this thesis we present an optimal output feedback based control design to address output regulation problem for a class of minimum phase nonlinear systems. Output regulation problem deals with system's output to track reference signals and reject disturbance signals both generated by an exo-system. Using the proposed scheme, regulation is achieved by including a conditional servo compensator to cater for degraded transient performance that is expected to be encountered in case of addition of a conventional servo compensator. Using closed loop analysis it is shown that under the proposed scheme the systems achieves regulation with steady state regulation error converging to zero with desirable transient performance due to inclusion of the conditional servo compensator. The scheme is applied to various examples to demonstrate the features of the nonlinear optimal control methods introduced by Kokotovic et.al. It is shown that the proposed design not only achieves the optimality but also attains some disc and sector margins which characterize robustness. Realizing the physical scenario where it is not possible to have all the states available for feedback, the design has been modified to output feedback with the help of a High Gain Observer (HGO). The observed states provided by the robust observer are used in the control law which provides similar performance as we get with the state feedback based control design. The simulation results show the efficacy of the proposed control scheme when applied to magnetic suspension system. In addition, analytical stability analysis is provided to show that the proposed controller achieves the design objectives.

ACKNOWLEDGEMENTS

First of all, I would like to thanks Almighty ALLAH for his countless blessings on me, who give me health, strength, ability to accomplish task as well as of the great favour's and blessings that he has bestowed upon his slaves in both spiritual and worldly terms.

I am extremely thankful to my supervisor *"Dr. Attaullah Y. Memon"* for all that have done for me. His endless support, invaluable advice as well as providing me continuous guidance on the research topic, his dedication and commitment throughout the period makes things simpler for me. Without his support I was not able to do this task comfortably.

I am thankful to my Guidance and Evaluation Committee members "*Dr. Sajjad Zaidi*" and "*Dr. Bilal.M.Khan*" for their permission to work as my GEC member and evaluating my thesis work by providing some useful comments.

I am thankful to all of my friends, colleagues that encouraged me especially *Mr. Aatif Mobeen Azhar* who helped me at every step of work, who was there whenever I needed help.

Last but not the least I am really thankful to my parents and other family members for their prayers, love, support & specially encouragement that have made this possible.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

The output regulation is one of the most important problem of the control set theory. The output regulation is actually the designing of a system's controller so that the output of the system asymptotically tracks the reference signal and may reject the disturbance signal. But both signals are generated by a known autonomous system called the exo system which is made by modelling differential equations. An extensive research has been carried out on the output regulation problem for linear, time invariant system like the work of *Davison et. al.* [1] and *Francis et. al.* [2]. By analyzing these papers it is shown that the output regulation problem requires some solvability conditions. i.e. Solution of system of Linear Matrix Equations called *regulator equations* which is equivalent to the characterization of *Hautus et. al.* [3] about the transmission polynomial of the composite system (formed by the actual control system and the exo system) to exhibit certain property and necessary conditions for the existence of any controller that will solve the regulation problem or the tracking problem. An important outcome of these paper is that it reveals that the controller that works for the output regulation problem will consists of two components one is stabilizing and the other is servo compensator. The interconnection of both components is called Internal Model Principle. According to which the servo mechanism is a control for a system to track the reference trajectories while reject the disturbance signals. The reference trajectories and the disturbance signals both are generated by a known system called the exo system.

There are well established method for output regulation problem but fewer have proposed the optimal based solution due to complex nature of problem. This thesis is focused on the design of optimal output feedback controller which is used as a stabilizing compensator that lead to the solution of output regulation problem with the target of increasing the robustness and introducing optimality for a non-linear system which is minimum phase. For that purpose an optimal stabilizing compensator is used which brings the tracking error to a small ring around the origin and from there rendering the error to zero is achieved by incorporating servo compensator with the stabilizing compensator. It's the extension of earlier work of *Attaullah Memon* [7], where the Lyapunov Redesign method is used to design a stabilizing compensator and in [22] where this idea was exploited.

The output regulation problem that is focused in this thesis utilizes the idea of conditional servo compensator an idea that was introduced by Khalil and co researchers [4], [5], [6]. Conditional servo-compensator acts as a classical servo compensator only inside the boundary layer near to zero error manifold. One outstanding feature which is provided by conditional servo is that without affecting the transient performance of the system it also achieves zero steady state tracking error. This idea was first introduced by *Shishagiri et. Al* [4] and [5] where in [4] it considers the design for constant exogeneous signal while [5] incorporates the design with time varying signals. The idea of conditional servo compensator was further dealt by *Attaullah Y Memon* [7] where a Lyapunov

Redesign approach is used for designing the feedback controllers. One distinctive feature of Lyapunov Re design is that it permits to incorporate any given stabilizing controller within the framework and then adding a servo compensator to solve the output regulation problem. Exploiting this flexibility provided by research of *Attaullah Y Memon* [7] and the idea provided in [22] instead of finding the feedback control law that stabilize the system why not seek for the controllers that will likely to provide some additional advantage like providing some optimality with some Robustness, these all could be done in the presence of some parametric uncertainty for example (un modeled fast dynamics, some uncertain parameters that are unknown, static non linearity). In this thesis we will focus on the optimal control methods that were developed by Kokotovic and some other researchers [8], [9], [10] that worked for designing the feedback controllers for the stabilization of system to origin. The well-known methods developed by Kokotovic and some other researchers reveals that in addition to stabilizing the system and reducing the cost, the optimal control methods guarantees stability margins like sector margins and disc margins which describe the robustness properties. There is one major problem that is to be faced in designing an optimal feedback controller is that it requires to solve a famous equation namely Hamilton Jacobean Bellman Equation (HJB) a partial differential equation which is a very complex task to solve if the system is of higher order. Kokotovic and other researchers also developed an Inverse approach [10] which is called an inverse optimal control which is used to solve optimal control problem which exempt the requirement for solving the HJB equation.

However instead of only focusing on the optimality and robustness this thesis will also consider the system whose model have a cascading structure which means that the system which are not in the normal form were considered like the system with internal and external dynamics. Most of the real-world system have cascaded structure because of having some internal and external dynamics present within in the system. So, in that case this thesis will extend the work of *Attaullah Memon* [7], [22] but with some modifications. Firstly, the optimal based control will be used in designing the stabilization compensator for the output regulation problem and its robustness properties will be exploited by changing its parameters. Secondly, it will consider the system with control dynamics that can be expressed as cascaded structure of internal and external part making it challenging to tackle that problem because of fewer number of available system variables that can be used in designing the control component. Finally, realizing the need to implement in the real world scenerio this thesis will consider the practical scenario where only the output is available for feedback instead of all states. In this regard a high gain observer which is called *Extended High Gain Observer* (EHGO) which is based on the idea provided by *Broker et. al.* [11] is used to estimates system states in order to render the system from state feedback to output feedback system.

Chapter 2

PRELIMINARIES

This topic of thesis involves understanding of five different and important concepts like Optimal Control, output regulation, Conditional Servo compensator and extended high gain observer. The understanding of these important concept is important to get idea of about what all this thesis is about. We start our discussion with optimal control theory in section 2.1 and how it can be applied to a nonlinear system. In section 2.2 we discussed about the idea of stability margins. Section 2.3 introduces the output regulation problem for a nonlinear system. Section 2.4 states the introduction of conditional servo compensator and how it overcome the problem of degraded transient performance provided by traditional servo compensator. Finally, section 2.5 includes the extended high gain observer.

2.1 Optimal Control

To stabilize a system feedback is necessary. Optimal control is one of the most important control objective that the designer needs to incorporate while designing any control law for the system. The optimal control is defined as the least amount of input or control input required to accomplish a task or to maintain equilibrium with enhanced transient performance. There are several methods like Ackerman's Pole placement that are used to place the poles at desired eigen values to achieve stabilization. But these methods don't take into account the amount of input available to achieve the task it just specifies the desired poles to seek the gain. Due to these limitations, it is our desire to use optimal control theory of in order to design an effective controller. Optimal controllers are designed as finding a control law such that optimality criteria are achieved by using least control effort and decaying the result to zero in minimum time. For a linear system the LQR method solves the problem of stabilization with optimality.

2.1.1 Linear Quadratic Regulator

The LQR (Linear Quadratic Regulator) is method to choose feedback gains with minimum cost function which results in an optimal control for the system.
Given a system

$$\dot{x} = \mathcal{A}x + \mathcal{B}u \tag{2.1}$$

If it is given that the system is controllable. The state feedback control law is given as

$$U = -Kx \tag{2.2}$$

To design an optimal state feedback control the performance index which is a cost function is defined as

$$J = \frac{1}{2}\int_0^{\infty}(x^TQx + u^TRu)dt \tag{2.3}$$

The main objective is to find a K which minimizes the cost function. This can be done by selecting Q and R as semi definite and semi positive definite respectively.

The feedback gain K can be calculated by using the formula

$$K = R^{-1}\mathcal{B}^T P$$

Where P is calculated by solving Algebraic Ricatti Equation

$$\mathcal{A}^T P + P\mathcal{A} + Q - P\mathcal{B}R^{-1}\mathcal{B}^T P = 0 \tag{2.4}$$

2.1.2 Non-Linear Optimal Control Design

For a nonlinear system the LQR method holds but did not provide global stabilization instead of it provides local region of stabilization. This is due to the fact that the linearization of nonlinear system around the equilibrium point transforms the nonlinear equation into locally Lipchitz function. For nonlinear system, the optimal design tools developed by Kokotovic and some other researchers [10] guarantees robustness and stability margins. The optimality notions described here in this subsection follows from the discussions of *constructive nonlinear control by Rudolph Sepulcher et. al.* [10]. The approach developed by these researchers involves finding a optimal stabilizing feedback control law $\mathcal{u}(\tilde{x})$ for a given nonlinear system.

$$\dot{x} = f(\tilde{x}) + g(\tilde{x})\mathcal{u} \tag{2.5}$$

To find a control law $\mathcal{u}(\tilde{x})$ which achieves the asymptotic stability around the equilibrium point at $x = 0$ and to find a control $\mathcal{u}(\tilde{x})$ that minimizes the objective function which is the cost function given by

$$J = \int_0^\infty (\ell(\tilde{x}) + \mathcal{u}^T \mathcal{R}(\tilde{x})\mathcal{u})dt \tag{2.6}$$

Here in this case the term involved in the cost function are $\ell(\tilde{x})$ and $\mathcal{R}(\tilde{x})$ which is defined as

$$\ell(\tilde{x}) \geq 0, \mathcal{R}(\tilde{x}) > 0 \ \ \forall\ \tilde{x}.$$

The target of optimality is to make the cost function J to its minimum value so when J which is cost function is set to its minimum value it will be termed as optimal value function. The value of J is always finite and is always a function of $J(\tilde{x})$. In this thesis, our objective is to reduce the cost function using the optimal control we will use the optimal value function $J(\tilde{x})$ which is the cost function as a Lyapunov function candidate and is denoted by $V(\tilde{x})$. Thus, when the control law which is $\mathcal{u}(\tilde{x})$ being optimal it will be termed as $\mathcal{u}'(\tilde{x})$ .The relationship between the optimal control given by $\mathcal{u}'(\tilde{x})$ and Lyapunov function which reflects the cost function $V(\tilde{x})$ is given by the subsequent optimality condition.

Theorem 2.1

The feedback control law

$$u'(\tilde{x}) \ = \ -k(\tilde{x}) = -\frac{1}{2}R^{-1}(\tilde{x})\left(L_g V(\tilde{x})\right)^T \tag{2.7}$$

Achieves the asymptotically stability around the equilibrium point at x = 0 if there exists a semi positive definite function $V(x)$ that satisfies the Hamilton Jacobean Bellman Equation.

$$\ell(\tilde{x}) + L_f V(\tilde{x}) - \frac{1}{4}\left(L_g V(\tilde{x})\right) R^{-1}(\tilde{x}) \left(L_g V(\tilde{x})\right)^T = 0\,, \quad V(0) = 0 \tag{2.8}$$

The control law $u'(\tilde{x})$ here in the given scenario is called as an optimal stabilizing control law and the term $V(\tilde{x})$ is termed as an optimal value function. The above control law is used ti minimize cost function given by by (2.2) and guarantees that as time approaches to infinity the state response will be zero $lim_{t\to\infty}\tilde{x}(t) = 0$.

The proof of this can be found in C*onstructive nonlinear control by Rudolph Sepulcher et. al.* [10].

2.1.3 Inverse Optimal Control Design

In designing an optimal stabilization controller, the problem associated in designing an optimal stabilizing control law with the direct approach is to solve Hamilton Jacobian Bellman Equation (HJB) which is a partial differential equation and it becomes difficult to solve if the system consists of more than two or three states or is of higher order. While on the other hand, the proof of theorem 2.1 shows that the robustness is not dependent on the particular choice of the optimal cost functions parameters $\mathcal{R}(\tilde{x})$,$\ell(\tilde{x})$ and both are ≥ 0 . Due to that reason Kokotovic and Freeman [10] then look for the inverse design method in order to provide the system's optimal stabilization solution. The Inverse approach deals with designing of stabilizing feedback control first and it is then proved optimal for a particular cost function which is provided as

$$\mathcal{J} = \int_0^\infty (\ell(\tilde{x}) + u^T \mathcal{R}(\tilde{x})\, u)dt \tag{2.9}$$

Here now this problem now transforms from direct approach to inverse approach since optimal value functions $\mathcal{R}(\tilde{x})$ and $\ell(\tilde{x})$ are not chosen before designing the controller in fact the controller is designed first then it is shown to be optimal for particular cost function. So, for a system the optimal stabilizing control law $u(\tilde{x})$ given in (2.9a) that solves the problem of inverse optimal control if it is expressed in the form provided in (2.10).

$$\dot{x} = f(\tilde{x}) + g(\tilde{x})u \tag{2.9a}$$

The control law u is given by

$$u = -k(\tilde{x}) = -\tfrac{1}{2}\mathcal{R}^{-1}(\tilde{x})\left(L_g V(\tilde{x})\right)^T \tag{2.10}$$

Where, $\mathcal{R}(\tilde{x}) > 0$

A positive semi definite function V $(\tilde{x})$ is known as CLF and with control law $u = -\frac{1}{2}k(\tilde{x})$, we can achieve the negative semi definiteness of $\dot{V}$, such that $\dot{V}$ is given by

$$\dot{V} = L_f V(\tilde{x}) - \tfrac{1}{2} L_g V(\tilde{x})\, k(\tilde{x}) \leq 0 \tag{2.11}$$

Rearranging the equation (2.8) gives us $\ell(\tilde{x})$ as

$$\ell(\tilde{x}) \coloneqq -L_f V(\tilde{x}) + \frac{1}{2} L_g V(\tilde{x}) k(\tilde{x}) \;\; \geq 0 \tag{2.12}$$

The term V $(\tilde{x})$ can be found out by solving Hamilton Jacobian Bellmen equation

$$\ell(\tilde{x}) + L_f V(\tilde{x}) - \frac{1}{4}\left(L_g V(\tilde{x})\right) \mathcal{R}^{-1}(\tilde{x}) \left(L_g V(\tilde{x})\right)^T = 0$$

From the above statements it can be concluded that control law which is of the form given by $u(\tilde{x})$ is known as inverse optimal control law in order to stabilizes system globally if it has the following two characteristics.

i. The global asymptotically stability of the system is achieved when $\tilde{x} = 0$.
ii. The control law defined earlier is of he for given below

$$u = -\frac{1}{2}\mathcal{R}^{-1}(\tilde{x})L_g V(\tilde{x}) \tag{2.13}$$

The term V($\tilde{x}$) which is a control Lyapunov function should be radially unbounded and must be a positive semi definite function that must satisfies the following equation

$$\dot{V} = L_f V(\tilde{x}) - \tfrac{1}{2} L_g V(\tilde{x})\, k(\tilde{x}) \le 0 \tag{2.14}$$

The construction of semi positive definite functions and with appropriate feedback control law through which we can assure the negative semi definiteness of a lyapunov function is the main task of above said design methods. In this thesis, the concept of a “control Lyapunov function” (CLF) provided by Artstein [13] and Sontag [14] will be used for designing a control Lyapunov function. In inverse approach, a CLF (Control Lyapunov function) is required to design first. According to the definition provided in [10] the control Lyapunov function is defined as

Definition 2.1

If there is radially unbounded function V ($\tilde{x}$) which is Positive definite, then the given radially unbounded function which is V ($\tilde{x}$) is known as system’s Control Lyapunov function or (CLF).

$$\dot{x} = f(\tilde{x}) + g(\tilde{x})u$$

If for all $x \neq 0$

$$\begin{aligned} L_g V(\tilde{x}) &= 0 \\ L_f V(\tilde{x}) &< 0 \end{aligned} \tag{2.15}$$

In short, if there is a Lyapunov function for a system such that negative definiteness is achieved then such function is termed as CLF.

The proof of this can be found in C*onstructive nonlinear control by Rodolphe Sepulchre et. al.* [10].

The main advantage of using the Control Lyapunov Function concept in designing an inverse optimal controller is that, once a Control Lyapunov Function is known, with the choice of some explicit expressions we can easily designed an inverse optimal stabilizing control law. An important formula derived by Sontag's is used to design an optimal stabilizing controller by using a CLF is given as

$$u_s(\tilde{x}) = -(c_0 + \frac{\hbar(\tilde{x}) + \sqrt{\hbar^2(\tilde{x}) + (w^T(\tilde{x})w(\tilde{x}))^2}}{(w^T(\tilde{x})w(\tilde{x}))}) \tag{2.16}$$

If $w(\tilde{x}) \neq 0$

$$u_s(\tilde{x}) = 0 \qquad\qquad if\ w(\tilde{x}) = 0$$

Where $\hbar(\tilde{x})$, $w(\tilde{x})$ *is defined as* $L_f V(\tilde{x}) = \hbar(\tilde{x})$ & $(L_g V(\tilde{x}))^T = w(\tilde{x})$

It is shown in [10] the control law of the form (2.16) achieves negative definiteness of $\dot{V}$ for the closed loop system if $V(\tilde{x})$ is Control Lyapunov Function and satisfies the small control property for the class of nonlinear system and the control law itself (2.16) is an optimal stabilizing control for a cost functional given by [10]

$$J = \int_0^\infty (\frac{1}{2} z(\tilde{x}) w^T(\tilde{x}) w(\tilde{x}) + \frac{1}{2z(\tilde{x})} u^T \mathcal{R}(\tilde{x}) u) dt \qquad (2.17)$$

Where

$$z(\tilde{x}) = \begin{cases} c_0 + \frac{a(\tilde{x}) + \sqrt{a^2(\tilde{x}) + \left(b^T(\tilde{x}) b(\tilde{x})\right)^2}}{\left(b^T(\tilde{x}) b(\tilde{x})\right)}, & b_{\tilde{x}} \neq 0 \\ c_0 \qquad , & b_{\tilde{x}} = 0 \end{cases} \qquad (2.18)$$

Apart from achieving optimality one especial feature of control law (2.13) it can provide us sector margin $(\frac{1}{2}, \infty)$ and under some conditions it achieves the disc margins D$(\frac{1}{2})$. The definition of the stability margins is provided in the next section these stability margins guaranteed robustness in the presence of some uncertain parameters called uncertainties which are for example static uncertainties and unmodeled fast dynamics. The definition of all these topics will be covered in next section.

2.2 Stability Margins and Uncertainties

For a system to be stable feedback is necessary. However, how much robust is that nonlinear feedback system is it depends upon the robustness properties of any nonlinear feedback system and is always characterize by the stability margins like sector margins and disc margin. The region around which system is stable is defined by stability margin because they guaranteed that the feedback loop as long as the static nonlinearity $\varphi_i(\cdot)$ remains in the sector (α, β) or disc(α) that is as long as $\alpha y^2 < y\varphi(y) < \beta y^2 \quad \forall\, y \in R$ It will remain stable, as shown in figure.

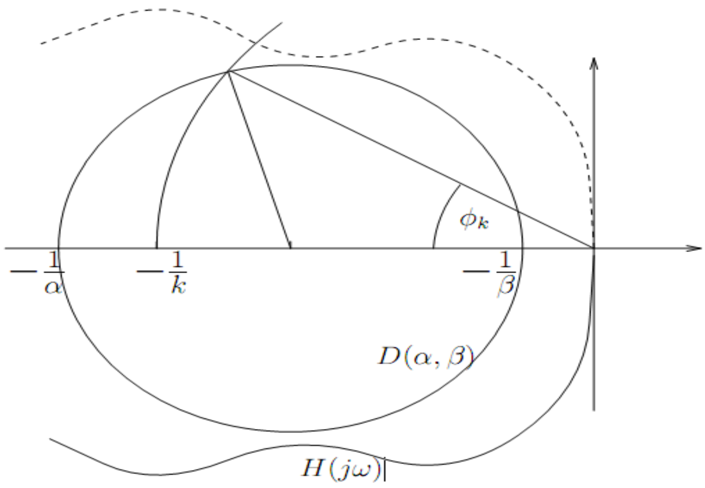

Figure 2.1 Disc Margin D (α, β) for a system [10]

To illustrate this concept let us examine a nonlinear system as shown in fig 2.2. Here H is the nominal nonlinear model of the system whereas Δ is the input uncertainty while u and y are of same dimension. For a nominal system in which no input uncertainty is present the value of Δ is identity I and the overall system is now consists of nominal nonlinear plant H in a feedback loop and the input $u = -k(x)$. In this thesis, the nominal system is denoted by (H, k), where *H* is the nominal system and k is the control while the perturbed system is denoted by (H, k, Δ). Here, the added thing is the perturbation term Δ which is an input uncertainty. One important thing to be note is that here the uncertainty appears to be at input side this lead to an important assumption that the disk margin will guarantees stability only if the uncertainty does not change the relative degree of the system. Hence this restrict the uncertain term Δ to be of zero relative degree. The uncertainties are characterized by two terminologies static and dynamic which do not change the relative degree of the system. The static uncertainty arises from the unknown parameters that are not incorporated while designing the control while the dynamic uncertainties are due to un modeled fast dynamics that characterize robustness properties. The definition for gain margin, sector margin and disc margin are given in [10] described below

Definition 1. (Gain Margin) "The nonlinear system (H, k) is said to have a gain margin (α, β) if the perturbed closed-loop system (H, k, Δ) is globally asymptotically stable for any Δ which is of the form diag $\{\kappa_1, \cdot\cdot, \kappa_m\}$ with constants $\kappa_i \in (\alpha, \beta)$, i = 1, · · ·, m".

Definition 2. (Sector Margin) "The nonlinear system (H, k) is said to have a sector margin
(α, β) if the perturbed closed loop system (H, k, Δ) is globally asymptotically stable for any Δ which is of the form diag $\{\varphi_1(\cdot), \cdot\cdot\cdot, \varphi_m(\cdot)\}$ where $\varphi_i(\cdot)$'s are locally Lipschitz static nonlinearities which belong to the sector (α, β).".

Definition 3. (Disc Margin) "The nonlinear system (H, k) is said to have a disc margin D(α) if the perturbed closed-loop system (H, k, Δ) is globally asymptotically stable for any Δ which is globally asymptotically stable and input feedforward passive with a radially unbounded storage function".

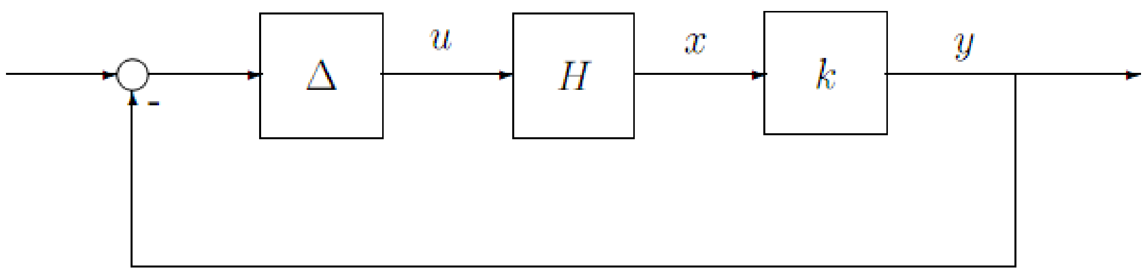

Fig. 2.2: NonLinear Feedback Loop with control u and output y in the presence of input uncertaintyΔ

2.3 Nonlinear Output Regulation Problem

The problem of output regulation is among the fundamental problems of the control theory alternatively known as servomechanism problem. It can be outlined as imposing to track system's output by applying prescribed set of reference signal which is a certain class of constant and/or time varying functions while rejecting certain class of constant and/or time varying disturbances. The objective is that within the family of functions, the controller should provide fixed steady state response. It can be interpreted in another way that error term which is the difference between the reference signal r and system's actual output y should decay to zero with the time approaching infinity over the prescribed set of disturbances. Consider a time-invariant, nonlinear, finite dimensional system described as compact form of (2.1) by the equations,

$$\left.\begin{aligned} \dot{x} &= f(\omega, x, u) \\ e &= h(\omega, x) \end{aligned}\right\} \tag{2.19}$$

with $x \in R^n$ represent the state of the system, $u \in R^m$ denotes the control input, $e \in R^v$ is a vector of regulated outputs that includes errors and other such variables that are required to be regulated to zero. The system is subjected to the set of exogeneous signals $\omega \in R^w$ that include the reference signal which is required to be tracked and to reject the unknown disturbance signal.

The exogenous signal ω is supposed to be produced by a neutrally stable system known as exo-system which is designed based on the designer's prescribed knowledge about the reference signals to be tracked and the disturbance signals to be rejected. To solve the output regulation problem perfectly, it is required to have a complete knowledge of this signal or the model of the system should be available in real time that is an extremely optimistic scenario and cannot be rendered as practical situation. On the other hand, allowing the case of no knowledge of this signal of system model leads to the error that is ultimately bounded but not zero. Therefore, the generation of these exogeneous signals provides an intermediate solution where ω is allowed to belong to fixed family of time dependent signals enabling the designer to cover major cases of practical significance. A general exo-system which is used in the output regulation problems exhibit the dynamic model described by the following differential equation (2.20) where the initial conditions i.e. $\omega(0)$ are allowed to vary on the prescribed set.

$$\dot{\omega} = \mathfrak{s}(\omega) \tag{2.20}$$

Since the exo-system is neutrally stable system, we can get the model matrix S through linearization at the equilibrium. i.e.

$$S = \left[\frac{\partial \mathfrak{s}}{\partial \omega}\right]_{eq} \tag{2.21}$$

which have all the eigenvalues lying on imaginary axis.

Suppose that with the available information from the system, there exist a feedback controller having output u as function of x & ω which is given by,

$$u = \Phi(x, \omega) \tag{2.22}$$

Thus, the closed loop system formed of (2.19) - (2.22) characterized by the equations (2.23) is supposed to exhibit *output regulation* if it is possible to design control law (2.22) such that for every exogeneous signal ω (in a prescribed set) and for every initial condition which lies near the vicinity of origin, so as the time tends to infinity the output error asymptotically decays to zero.

$$\left.\begin{aligned} \dot{\omega} &= \mathfrak{s}(\omega) = S(\omega) \\ \dot{x} &= f[\omega, x, \Phi(x, \omega)] \end{aligned}\right\} \tag{2.23}$$

Usually, the amount of information available for the system to provide feedback describes the structure of the feedback controller. In case of all the system states x and the states of the exogeneous model ω are available for feedback, a memoryless function similar to the (2.22) can work as desired controller. However, in a more realistic scenario, only the output of the system is available rather than all the state variables. In such case, the error term e which is only measurable quantity imposes to think of dynamic controller of the form,

$$\left.\begin{aligned} \dot{\zeta} &= \Xi(\zeta, e) \\ u &= \theta(\zeta) \end{aligned}\right\} \tag{2.24}$$

with ζ being the internal state of the controller and $\Xi(0) = 0$, $\theta(0) = 0$ such that the next requirements described below are met.

- In the absence of exo-system, the origin of the open loop system must have an asymptotically stable equilibrium point. i.e. if $\omega(t) = 0$, it means that $x(t) = 0$.
- The error term $e(t)$ converges to zero considering any initial conditions applied to the system $x(0)$ and the exo-system $\omega(0)$.

Thus, we can say that the output regulated system possesses two responses. i.e. the one is the transient response and the second one implies the steady state response. During the transient response, system converges to the steady state response from given initial condition and it exhibits the steady state response for $t \longrightarrow \infty$. The necessary conditions required for the output regulation problem to be solvable are as:

- The system (2.19) must have smooth functions $f(\omega, x, u)$ and $h(\omega, x)$.
- The pair (A, B) is stabilizeable and (C, A) is detectable, where the matrices A, B and C are defined as,

$$A = \left[\frac{\partial f}{\partial x}\right]_{eq}, \quad B = \left[\frac{\partial f}{\partial u}\right]_{eq}, \quad C = \left[\frac{\partial h}{\partial x}\right]_{eq}$$

The control action that solve the output regulation problem can be divided into two components. One component is the one that force the system's output to slide on the steady state value / manifold while second component acts to stabilize the system's output on steady state value / manifold. It was shown by *Isidori* [15], that the output regulation problem is solvable if there exists certain continuously differentiable mapping that solve the nonlinear regulator equations. Let $\pi(\omega)$ be the steady state of x and $\sigma(\omega)$ be the steady state of ζ on the zero-error manifold, then the system must satisfy the following set of regulator equations.

$$\left.\begin{aligned} \frac{\partial \pi(\omega)}{\partial \omega}\mathfrak{s}(\omega) &= f[\pi(\omega), \omega, c(\omega)] \\ 0 &= h[\pi(\omega), \omega] \end{aligned}\right\} \tag{2.25}$$

and with the controller satisfying the equations,

$$\left.\begin{aligned} \frac{\partial\sigma(\omega)}{\partial\omega}\mathfrak{s}(\omega) &= \Lambda[\sigma(\omega), 0] \\ c(\omega) &= \vartheta[\sigma(\omega)] \end{aligned}\right\} \tag{2.26}$$

where $c(\omega)$ is the steady state value of control u and it is the polynomial in the component of ω only. Note that the model $\mathfrak{s}(\omega)$ of exo-system is used in the regulator equations (2.25) and (2.26). This suggests that without incorporation of such model, generally the output regulation problem cannot be solved. This fact is known as *internal model principle* and is usually designed as,

$$\left.\begin{aligned} \frac{\partial\sigma(\omega)}{\partial\omega}S\omega &= \varphi\sigma(\omega) \\ c(\omega) &= \Gamma\sigma(\omega) \end{aligned}\right\} \tag{2.27}$$

with $\sigma(\omega)$, φ and Γ represent mappings given by,

$$\sigma(\omega) = \begin{bmatrix} c(\omega) \\ \mathcal{L}_s c(\omega) \\ \mathcal{L}_s^2 c(\omega) \\ \vdots \\ \mathcal{L}_s^{q-1} c(\omega) \end{bmatrix}, \quad \Gamma = [1 \quad 0 \quad 0 \quad \ldots \quad 0]_{1\,\mathrm{x}\,q}, \qquad \varphi = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{q-1} \end{bmatrix}$$

The coefficients $a_0, a_1 \ldots, a_{q-1}$ are the real numbers that are obtained by the following equation,

$$\mathcal{L}_s^q c(\omega) = a_0 c(\omega) + a_1 \mathcal{L}_s c(\omega) + a_2 \mathcal{L}_s^2 c(\omega) + \cdots + a_{q-1}\mathcal{L}_s^{q-1} c(\omega)$$

Such that the characteristics polynomial $\lambda^q - a_{q-1}\lambda^{q-1} - \cdots - a_1\lambda - a_0$ has distinct roots on the imaginary axis with pair $\left(\begin{bmatrix} A & 0 \\ GC & \varphi \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}\right)$ being stabilizable where $G = \left[\frac{\partial\Xi}{\partial u}\right]_{eq}$ and $\left([C \quad 0], \begin{bmatrix} A & B\Gamma \\ 0 & \varphi \end{bmatrix}\right)$ being detectable.

Now, transforming the system (2.19) by changing the variables

$$\tilde{x} = x - \pi(\omega),$$
$$\dot{\tilde{x}} = f(\omega, \tilde{x}, v - c(\omega))$$

Where,$u = v + c(\omega)$, and component v is designed such that the whole system is stabilized. This component can be designed through robust and optimal control techniques like sliding mode control, Inverse Optimal Control or Lyapunov redesign etc. such that $v = 0$ with $x = \pi(\omega)$ and $\zeta = \sigma(\omega)$ on the zero-error manifold. In practice, the whole controller that solves this regulation problem is the parallel interconnection of the internal model and the stabilizer where,

- The internal model provides $c(\omega)$ the component of u such that $x = \pi(\omega)$ and $\zeta = \sigma(\omega)$.

- The stabilizer provides the steady state component $v = u_{st}(t)$ such that it locally stabilizes the closed loop system and induces the local error convergence towards zero-error manifold. This is as shown in Fig. 2.6.

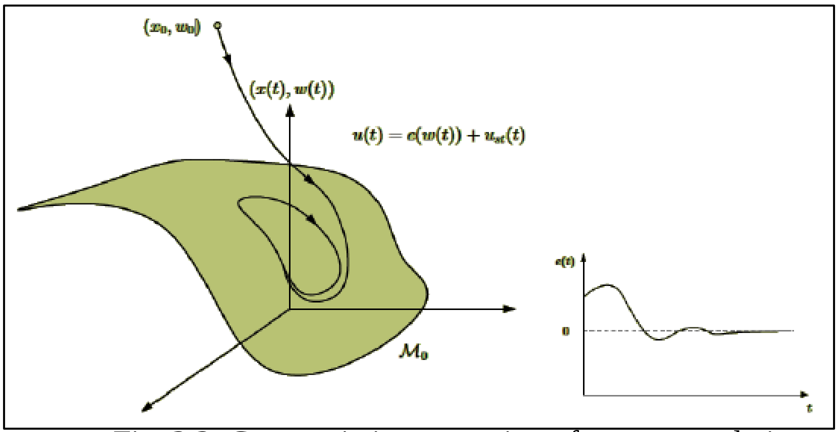

Fig. 2.2: Geometric interpretation of output regulation

In context of this thesis work, we will follow a similar idea for the regulation of the system that must be minimum phase nonlinear.

2.4 Conditional Servo-Mechanism Designs

The output regulation that is discussed so far in the previous section provides two main challenges. First, the design is not robust and only provides the local stability. Secondly, the task of designing the stabilizing controller and the internal model separately can be tricky. Generally, the task of output regulation is accomplished using the idea of servo-mechanism in which a servo-compensator is designed that achieves the output regulation robustly. There are well established methods of designing the servo-compensator in the literature, however, here in this thesis work, the work of *Hassan K. Khalil* [16] and *Attaullah Y. Memon et. al.* [7], [17] is summarized that will lead to our research work in the next chapter. Starting from the conventional servo-compensator design, we will discuss the its drawbacks that leads to the necessity of making the servo-compensator as conditional one and how to achieve do that. The whole discussion follows sequentially from references [16], [7], [17].

Consider the nonlinear system (2.19) in the form,

$$\left.\begin{aligned} \dot{x} &= f(x,\omega) + G(x,\omega)u \\ e &= h(x,\omega) \end{aligned}\right\} \tag{2.28}$$

where the exogeneous signal ω belongs to prescribed set $\mathfrak{Y} \in R^{\omega}$. The functions f, G and h are smooth in the domain $\mathfrak{D} \subset R^n$ and continuous in ω over the set $\mathfrak{Y}$. The error term e represents the vector $[e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_n]$, where $e_1 = y - q(\omega)$ with $q(\omega)$ represents the trajectory to be achieved. The solution to the output regulation problem requires the following assumptions.

Assumption 2.3.1: The signal ω and $q(\omega)$ are assumed to be generated by a known neutrally stable exo-system. i.e.

$$\dot{\omega} = S_0\omega \tag{2.29}$$

where S_0 have distinct eigen-values lies on imaginary (jw) axis where the term $\omega(t)$ is from the given compact set $\mathfrak{Y}$.

Assumption 2.3.2: There exists continuous mapping such that x is defined as $x = \pi(\omega)$ with the term $\pi(0) = 0$ & $u = \chi(\omega)$ on the zero-error manifold which solves the regulator equations given as,

$$\left.\begin{aligned} \frac{\partial \pi(\omega)}{\omega} S_0 \omega &= f(\pi, \omega) + G(\pi, \omega)\chi(\omega) \\ 0 &= h(\pi, \omega) \end{aligned}\right\} \tag{2.30a}$$

for all $\omega \in \mathfrak{Y}$.

Assumption 2.3.3: There exist set of real constants $a_0, a_1, \dots, a_{r-1}$, such that the $u = \chi(\omega)$ satisfies the identity given as,

$$\mathcal{L}_s^r \chi(\omega) = a_0 \chi(\omega) + a_1 \mathcal{L}_s \chi(\omega) + a_2 \mathcal{L}_s^2 \chi(\omega) + \cdots + a_{r-1} \mathcal{L}_s^{r-1} \chi(\omega) \tag{2.30b}$$

for all $\omega \in \mathfrak{Y}$ and the characteristic polynomial $\mathfrak{X}^r - a_{r-1}\mathfrak{X}^{r-1} - \cdots - a_1 \mathfrak{X} - a_0$ has roots on the imaginary axis. Selecting,

$$S = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{r-1} \end{bmatrix}_{r \times r}, \quad \mathcal{T}(\omega) = \begin{bmatrix} \chi(\omega) \\ \mathcal{L}_s \chi(\omega) \\ \mathcal{L}_s^2 \chi(\omega) \\ \vdots \\ \mathcal{L}_s^{r-1} \chi(\omega) \end{bmatrix}_{r \times 1}, \quad \Gamma = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}_{1 \times r}$$

It has been shown in [15] that $\chi(\omega)$ can be generated by the internal model,

$$\left.\begin{aligned} \frac{\partial \mathcal{T}}{\partial \omega} S_0 &= S\mathcal{T}(\omega) \\ \chi(\omega) &= \Gamma \mathcal{T}(\omega) \end{aligned}\right\} \tag{2.31}$$

The conventional servo-compensator followed by the above-mentioned assumptions can now be augmented with the system (2.28). i.e.

$$\dot{\varrho} = S\varrho + Je_1 \tag{2.32}$$

Where, J can be selected as $J = [0 \quad 0 \quad 0 \quad \dots \quad 1]$. We can take the feedback control law for the output regulation problem (2.33) with k is the constant gain and is referred to the maximum permissible control magnitude, $\mathcal{L}_g \mathcal{L}_f^{n-1} h$ is called the high frequency gain. $sign$ as well as sat are the standard signum and saturation functions defined as (2.34) and (2.35).

$$u = -k \, sign\left(\mathcal{L}_g \mathcal{L}_f^{n-1} h\right) sat\left(\frac{s}{\mu}\right) \tag{2.33}$$

$$sign(x) = \begin{cases} 1 & , \; x > 0 \\ 0 & , \; x = 0 \\ -1 & , \; x < 0 \end{cases} \tag{2.34}$$

$$sat(x) = \begin{cases} x & , \; |x| \le 1 \\ sign(x) & , \; |x| > 1 \end{cases} \tag{2.35}$$

The sliding surface s is given below,

$$s = K_1 \varrho + K_2 [e_1 \quad e_2 \quad \dots \quad e_{n-1}]^T + e_n \tag{2.36}$$

However, K_1, K_2 matrices can be designed such that,

$$\mathcal{H} = \begin{bmatrix} S & JC_0 \\ -B_0K_1 & A_0 - B_0K_2 \end{bmatrix}$$

is Hurwitz. The matrices A_0, B_0 and C_0 are the canonical form representations of the system (2.28) when transformed to normal form.

Conventional servo-compensator addresses the challenges posed in the starting of the section. i.e. It achieves the non-local robust output regulation but with a drawback that the steady state performance usually achieved with the major cost of transient performance degradation. It happens so because the inclusion of servo-compensator increases the overall system's order. This issue has been formally addressed under the topic of conditional servo-compensator by *Seshagiri et. al.* [5] as well as *Attaullah Y. Memon et. al.* [7], [17]. The topic of conditional servo-compensator has been discussed in the context of Sliding Mode Framework in [5] and in the context of Lyapunov Redesign Framework in [7], [17]. Conceptually both discussions are similar but the later provide the flexibility of choosing with any given stabilizing state feedback controller for the closed loop system and then to include servo-compensator to perform the desired task. This flexibility will lead to our work if we apply optimal control to the system. So, owing to the implicational significance, the Lyapunov Redesign Framework based servo-compensator design will be discussed here.

To start in the Lyapunov Redesign framework, transforming the system (2.28) by changing the variables as $\tilde{\xi} = x - \pi$, into the form as given by,

$$\dot{\tilde{\xi}} = \tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)[u - \chi(\omega)] \tag{2.37}$$

Where, $\tilde{f}(\tilde{\xi}, \omega) = f(\tilde{\xi} + \pi, \omega) - f(\pi, \omega) + [G(\tilde{\xi} + \pi, \omega) - G(\pi, \omega)]\chi(\omega)$ and $\tilde{G}(\tilde{\xi}, \omega) = G(\tilde{\xi} + \pi, \omega)$. The system (2.37) possess the form where treating $\chi(\omega)$ as the matched uncertainty, the problem of state feedback regulation can be termed as state feedback stabilization. Assume that there exists a stabilizing state feedback control for the system (2.38) and also there is Lyapunov function candidate for the corresponding closed loop system.

$$\dot{\tilde{\xi}} = \tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)u \tag{2.38}$$

Assumption 2.3.4: There exists a locally Lipschitz function $\Delta(\tilde{\xi}, \omega)$, with $\Delta(0, \omega) = 0$ and a continuously differentiable Lyapunov function $\mathcal{V}(\tilde{\xi}, \omega)$ such that $\alpha_1(\|\tilde{\xi}\|) \leq \mathcal{V}(\tilde{\xi}, \omega) \leq \alpha_2(\|\tilde{\xi}\|)$ and,

$$\frac{\partial \mathcal{V}}{\partial \omega} S_o \omega + \frac{\partial \mathcal{V}}{\partial \tilde{\xi}} [\tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)\Delta(\tilde{\xi}, \omega)] \leq -\mathbb{W}(\tilde{\xi}) \tag{2.39}$$

where $\tilde{\xi} \in R^n$, $\omega \in \mathfrak{Y}$, α_1, α_2 are class $\mathcal{K}$ functions and $\mathbb{W}(\tilde{\xi})$ is positive definite continuous function.

Now, writing the system (2.37) as,

$$\dot{\tilde{\xi}} = \tilde{f}(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)\Delta(\tilde{\xi}, \omega) + \tilde{G}(\tilde{\xi}, \omega)u - \tilde{G}(\tilde{\xi}, \omega)[\chi(\omega) + \Delta(\tilde{\xi}, \omega)] \tag{2.40}$$

System (2.40) is in the form where it is required that a saturated high-gain feedback controller is required to deal with the term $\chi(\omega)$. Assuming that $\Omega = \{\mathcal{V}(\tilde{\xi},\omega) < g\}$ is a compact set, $g > 0$ and δ be some function such that,

$$\left\|\chi(\omega) + \Delta(\tilde{\xi},\omega)\right\| \leq \delta(\tilde{\xi}), \qquad \forall\ \tilde{\xi} \in \Omega$$

Assumption 2.3.5: From (2.39), assume that $\left(\frac{\partial \mathcal{V}}{\partial \tilde{\xi}}\right)\tilde{G}(\tilde{\xi},\omega)$ can be stated as,

$$\left(\frac{\partial \mathcal{V}}{\partial \tilde{\xi}}\right)\tilde{G}(\tilde{\xi},\omega) = \mathcal{v}^T(\tilde{\xi})\mathcal{H}(\tilde{\xi},\omega)$$

where $\mathcal{v}(\tilde{\xi})$ is defined as locally Lipchitz known function with given initial condition $\mathcal{v}(0) = 0$, where $\mathcal{H}(\tilde{\xi},\omega)$ is termed as an unknown function. i.e. $\mathcal{H}^T(\tilde{\xi},\omega) + \mathcal{H}(\tilde{\xi},\omega) \geq 2\lambda I_m$, $\left\|\mathcal{H}(\tilde{\xi},\omega)\right\| < \mathbb{k}$, $\mathbb{k} \geq \lambda > 0$ and I_m is the identity matrix.

The version of servo-compensator (2.32) called conditional servo-compensator is then provided by the equation (2.41) and the feedback controller that solves the output regulation problem without degrading the transient performance can be selected as (2.42).

$$\dot{\varrho} = (S - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right) \tag{2.41}$$

$$u = -\mathfrak{a}(\tilde{\xi}) sat\left(\frac{s}{\mu}\right) \tag{2.42}$$

$$\mathfrak{a}(\tilde{\xi}) \geq \frac{\mathbb{k}}{\lambda} + \mathfrak{a}_0, \qquad \mathfrak{a}_0 > 0 \tag{2.43}$$

where $s = \mathcal{v}(\tilde{\xi}) + K_1\varrho$, μ is the boundary layer inside which the servo-action will be performed and matrix K_1 is designed such that $(S - JK_1)$ is Hurwitz.

For the work of this thesis, we will use conditional servo-compensator to perform the task of output regulation for saturated class of minimum phase nonlinear systems due to the superiority of this design discussed in this section that compared to conventional servo-compensator, it provides the robust output regulation without effecting system's transient response.

2.5 Extended High Gain Observer (EHGO)

The control designing process for certain system usually assume that all of its states are available and can be used in the process wherever required. This situation in general not true and in most of the realistic scenarios, we need to use a sensor for each state measurement which is not only costly but also unreliable approach. Secondly, sometimes it is also not possible to measure some of the state even through sensor. That means there is a necessity of some alternate phenomena that may be helpful in such a scenario. To overcome this hindrance, a control engineer uses a technique called the state estimator or state observer in which all the required states of the system are observed / estimated by using only the available information from the system. *Weiwen Wang et. al.* [18] compares some of the widely-used state observers. Observers form the basis of output feedback control design.

If all relevant state variables that are used in the system are observed by the observer, despite the fact that some of state variables can be measured directly or not, it is then known as full-order state observer. When the fewer states out of n state variables are measured using the observer, where n is defined as state vector dimension, the observer is now transformed from to reduced-order state observer or simply we can say that a reduced-order observer.

A state observer is a system that estimates the state variables based on the measurement of the output and control variables. The most commonly used linear observer known as Luenberger Observer can be found frequently in the literature. Consider a linear system as modelled by the equations,

$$\left.\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}u \\ y &= \mathcal{C}x \end{aligned}\right\} \tag{2.44}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are the state space model parameters. The observer is a subsystem that rebuilds the system's state vector it exhibits the mathematical substantially same as that of the original system with a difference that it includes an extra term which is related to the tracking error in order to compensate the inaccuracies in the system matrices $\mathcal{A}$ and B due to the reason that their initial values are not known. So, the mathematical model of the observer for the system (2.44) can be defined as,

$$\dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}u + \mathcal{L}(y - \mathcal{C}\hat{x}) \tag{2.45}$$

It follows from (2.45) that the input to the observer are the output y and the control input u while the matrix $\mathcal{L}$ is the weighing matrix that involves the difference between the measured output y and the estimated output $\mathcal{C}\hat{x}$ is called the observer gain. The observer gain associated term in its model is responsible for improving its performance by continuously correcting the model output. In designing observer, the gain matrix $\mathcal{L}$ is selected such that $(\mathcal{A} - \mathcal{L}\mathcal{C})$ is Hurwitz. i.e. All the eigenvalues lie in left half plane which guarantees the convergence of estimation error to zero. Such a gain matrix can be designed using simple ideas, for example, pole placement.

The main challenge in the Luenberger Observer is that the performance of the observer is highly dependent on the system model accuracy. e.g. the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ for this case. To enhance observer capabilities to deal with the real world issues like uncertainty, noise, disturbance etc., a robust nonlinear observer called High Gain Observer (HGO) was introduced. Representing the nonlinear version of system (2.44) in the normal form as,

$$\left.\begin{aligned} \dot{x}_j &= x_{j+1}, \qquad 1 \le j \le \rho-1 \\ \dot{x}_\rho &= b(x,\theta) + a(x,\theta)u \\ y &= x_1 \end{aligned}\right\} \tag{2.46}$$

where ρ is the relative degree and θ is a vector of unknown disturbances. The High Gain Observer for the system (2.46) can be designed as,

$$\left.\begin{aligned}\dot{\hat{x}}_{j+1} &= \hat{x}_{j+1} + \frac{\hbar_j}{\varepsilon^j}(y - \hat{x}_1), \qquad 1 \le j \le \rho - 1 \\ \dot{\hat{x}}_\rho &= b(\hat{x}) + a(\hat{x})u + \frac{\hbar_\rho}{\varepsilon^\rho}(y - \hat{x}_1)\end{aligned}\right\} \tag{2.47}$$

where ε is the design parameter and normally is chosen as small as possible. The constants $\hbar_1, \hbar_2 \dots \hbar_\rho$ are selected such that the polynomial $\lambda^\rho + \hbar_1\lambda^{\rho-1} + \dots + \hbar_{\rho-1}\lambda + \hbar_\rho$ is Hurwitz. $b(\hat{x})$ and $a(\hat{x})$ are the nominal models of $b(x,\theta)$ and $a(x,\theta)$ respectively. In the cases where the nominal models are not known, they can be ignored and a High Gain Observer can still be designed. However, their inclusion in the observer design yields with high convergence rate which is highly desirable.

In case of nonlinear systems, when the model exhibits the normal form where the internal and external dynamics of the system exists explicitly, a simple High Gain Observer may not work to observe all the states of both the dynamics. In such case, the observer which is normally used is called Extended High Gain Observer (EHGO). This observer estimates the derivatives of the output in addition to an extra signal that is used as virtual output for the auxiliary system. For example, consider the single-input, single-output nonlinear system with well-defined relative degree ρ but consisting of both the state dynamics. i.e.

$$\left.\begin{aligned}\dot{z} &= \vartheta(z,\xi) \\ \dot{\xi}_j &= \xi_{j+1}, \quad 1 \le j \le \rho - 1 \\ \dot{\xi}_\rho &= b(z,\xi) + a(\xi,u) \\ y &= \xi_1\end{aligned}\right\} \tag{2.48}$$

or in compact form as,

$$\left.\begin{aligned}\dot{z} &= \vartheta(z,\xi) \\ \dot{\xi} &= A\xi + B[b(z,\xi) + a(\xi,u)] \\ y &= C\xi\end{aligned}\right\} \tag{2.49}$$

where $z \in \mathcal{R}^{n-\rho}$, $\xi \in \mathcal{R}^\rho$. Extracting the auxiliary system from (2.49) as,

$$\dot{z} = \vartheta(z,\xi), \qquad \sigma = b(z,\xi) \tag{2.50}$$

Any suitable observer called the internal observer can be used to estimate the states of the auxiliary system (2.50) formed of the internal dynamics. For example, *Boker et. al.* [19] used Extended Kalman Filter (EKF) as internal observer. EKF exhibits a similar in structure but differ from technical design as that of Luenberger Observer. For, the auxiliary system (2.50), the EKF takes the form as,

$$\dot{\hat{z}} = \vartheta(\hat{z},\xi) + \mathcal{L}(t)[\sigma - b(\hat{z},\xi)] \tag{2.51}$$

where the observer gain $\mathcal{L}(t)$ can be designed as,

$$\mathcal{L}(t) = \mathcal{P}(t)\mathcal{C}(t)^T R(t)^{-1} \tag{2.52}$$

and $\mathcal{P}(t)$ with $\mathcal{P}(0) = 0$ is the solution of the Riccati Differential Equation,

$$\dot{\mathcal{P}}(t) = \mathcal{A}_1(t)\mathcal{P}(t) + \mathcal{P}(t){\mathcal{A}_1}^T(t) + \mathcal{Q}(t) - \mathcal{P}(t){\mathcal{C}_1}^T(t)\mathcal{R}^{-1}(t)\mathcal{C}_1(t)\mathcal{P}(t) \tag{2.53}$$

The time varying matrices $\mathcal{A}_1(t)$ and $\mathcal{C}_1(t)$ are given by,

$$\mathcal{A}_1(t) = \frac{\partial\vartheta}{\partial z}(\hat{z},\xi), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}(\hat{z},\xi) \tag{2.54}$$

and $\mathcal{Q}(t)$ and $\mathcal{R}(t)$ are symmetric positive definite matrices that which satisfy,

$$0 < r_1 \leq \mathcal{R}(t) \leq r_2 \tag{2.55}$$

$$0 < q_1 I_{n-\rho} \leq \mathcal{Q}(t) \leq q_2 I_{n-\rho} \tag{2.56}$$

Now, the observer that will work for the external dynamics called the external observer will be employed through EHGO. This observer, in addition to the states of the external dynamics, will observe an extra state that has been utilized as the output of the auxiliary system (2.50). Its structure is as follows:

$$\dot{\hat{\xi}} = A\hat{\xi} + B\left[\hat{\sigma} + a\left(\hat{\xi}, u\right)\right] + \mathcal{H}(\varepsilon)\left(y - C\hat{\xi}\right) \tag{2.57}$$

$$\dot{\hat{\sigma}} = \dot{b}\left(\hat{z}, \hat{\xi}, u\right) + \frac{h_{\rho+1}}{\varepsilon^{\rho+1}}\left(y - C\hat{\xi}\right) \tag{2.58}$$

Where,

$$\dot{b}\left(\hat{z}, \hat{\xi}, u\right) = \left({d[b(z,\xi)]}/{dt} \right)\Bigg|_{(\hat{z},\hat{\xi})}, \text{ and} \tag{2.59}$$

$$\frac{d[b(z,\xi)]}{dt} = \frac{db}{dz}\,\vartheta(z,\xi) + \frac{db}{d\xi}\{A\xi + B[b(z,\xi) + a(\xi,u)]\} \tag{2.60}$$

The observer gain matrix $\mathcal{H}(\varepsilon) = \begin{bmatrix} \frac{h_1}{\varepsilon} & \frac{h_2}{\varepsilon^2} & \dots & \frac{h_\rho}{\varepsilon^\rho} \end{bmatrix}^T$ and $h_1, h_2 \dots h_\rho$ are selected such that $\lambda^{\rho+1} + h_1\lambda^\rho + \cdots + h_\rho\lambda + h_{\rho+1}$ is Hurwitz. Moreover, $\varepsilon > 0$ is the small design parameter.

So, combining the internal and external observers (2.51), (2.57) and (2.58), the full order observer for (2.49) is characterized by,

$$\left.\begin{aligned} \dot{\hat{\xi}} &= A\hat{\xi} + B\left[\hat{\sigma} + a\left(\hat{\xi}, u\right)\right] + \mathcal{H}(\varepsilon)\left(y - C\hat{\xi}\right) \\ \dot{\hat{\sigma}} &= \dot{b}\left(\hat{z}, \hat{\xi}, u\right) + \frac{h_{\rho+1}}{\varepsilon^{\rho+1}}\left(y - C\hat{\xi}\right) \\ \dot{\hat{z}} &= \vartheta(\hat{z}, \xi) + \mathcal{L}(t)[\sigma - b(\hat{z}, \xi)] \end{aligned}\right\} \tag{2.61}$$

The time varying matrices are now given by,

$$\mathcal{A}_1(t) = \frac{\partial\vartheta}{\partial z}\left(\hat{z}, \hat{\xi}\right), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}\left(\hat{z}, \hat{\xi}\right)$$

For our thesis research work, we will design the output feedback version of the conditional servo-mechanism developed for a class of saturated nonlinear minimum phase systems, utilizing the idea of Extended High Gain Observer (EHGO) discussed in this section.

Chapter 3

OPTIMAL OUTPUT REGULATION PROBLEM FOR NONLINEAR SYSTEM

This chapter will discuss the idea of optimal output regulation problem for a class of nonlinear system using conditional servo mechanism. In the first section 3.1 the problem formulation of output regulation is discussed. The second section 3.2 discusses conventional and conditional servo mechanism designs for these classes of systems (i.e. Class of systems possessing linear dynamics subjected to the control constraints as well as the class of systems exhibiting nonlinear dynamics). The ideas of Optimal based control have been exposed for such designs. In section 3.3, the state feedback developments of previous section have been extended to the output feedback by presenting the appropriate observer designs. Section 3.4 provides the stability analysis of the closed loop system followed by the simulation results presented in the section 3.5 that depicts the efficacy of the developed control designs. Finally, this chapter closes with the section 3.6 that includes the technical discussion of the presented results and the concluding remarks.

In section 3.3, the state feedback developments of previous section have been extended to the output feedback by presenting the appropriate observer designs. Section 3.4 provides the stability analysis of the closed loop system followed by the simulation results presented in the section 3.5 that depicts the efficacy of the developed control designs. Finally, this chapter closes with the section 3.6 that includes the technical discussion of the presented results and the concluding remarks.

3.1 Control Design and Problem Formulation

In this thesis we considered single input single output, nonlinear and minimum phase system. The purpose of this work is to design an optimal based stabilization controler which not only regulates the output of minimum phase system to follow a desired reference signal and reject the disturbance signal both are generated by a known exo system but also provide stability margins that also depicts robustness. Consider a SISO, minimum phase nonlinear system given by

$$\left.\begin{aligned} \dot{\xi} &= f(\xi,\omega) + G(\xi,\omega)\varphi(u) \\ e &= h(\xi,\omega) \end{aligned}\right\} \tag{3.1}$$

Where $\xi \in R^n$the state, e is the regulation error and u is the control input. The overall system is supplied with the reference signal to be tracked and disturbance signal to be rejected both are generated by set of exogeneous input variables ω which belongs to compact set$W \in R^w$. The input uncertainty that is a nonlinear function belongs to a certain range or a sector defined by $[A, B]$, such that it satisfies the inequality given by

$$Au^2 < u\varphi(u) < Bu^2 \qquad \forall\, 0 \leq A \leq B \tag{3.2}$$

The definition for nonlinearities are already defined in chapter 2 in Definition 1,2,&3

Now, with the change of variable $\eta = \xi - \pi(\omega)$, equation 3.1 can be rewritten as

$$\dot{\eta} = \dot{\xi} - \dot{\pi} \tag{3.3}$$

$$\dot{\eta} = f(\eta + \pi, \omega) + G(\eta + \pi, \omega)\varphi(u) - \dot{\pi} \tag{3.4}$$

According to assumption 2.3.4

There exists a continuous mapping $\eta = \pi(\omega)$ with $\pi(0) = 0$ and $u = \chi(\omega)$ on the zero-error manifold which solves the regulator equations given as,

$$\left.\begin{array}{r} \frac{\partial \pi(\omega)}{\omega} S_0 \omega = f(\pi, \omega) + G(\pi, \omega)\chi(\omega) \\ 0 = h(\pi, \omega) \end{array}\right\} \tag{3.5}$$

for all $\omega \in \mathfrak{Y}$.

For the given output regulation problem, the above assumption is mandatory and sufficient enough to solve the regulation problem. It means that there exists zero-error manifold $x = \pi(\omega)$ for (3.1) and $\xi = \pi(\omega)$ for (3.3), with the steady state control as $\phi(\omega)$ on zero-error manifold. This control $\phi(\omega)$ slides the system output on zero-error manifold in the presence of disturbance signals (usually provided from exo-system as a component of exogenous signals besides reference).

Assumption 3.4: There exist real constants $c_0, c_1, \dots, c_{\rho-1}$ such that the steady state control component $\phi(\omega)$ satisfy the following identity

$$\mathcal{L}_s^{\rho}\phi = c_0\phi + c_1\mathcal{L}_s\phi + \cdots + c_{\rho-1}\mathcal{L}_s^{\rho-1}\phi \tag{3.5a}$$

where the polynomial $\Delta^{\rho} - c_{\rho-1}\Delta^{\rho-1} - \cdots - c_1\Delta - c_0$ have all the distinct roots that are located along imaginary axis such that $\mathcal{L}_s\phi = \left({}^{\partial\phi}/_{\partial\omega}\right)\mathcal{S}_0\omega$.

The above assumptions are necessary due to the motivational fact of internal model principle for nonlinear which states that the controller not only generate the prescribed trajectories but can also generate some higher order dynamics. Defining the following matrices,

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{\rho-1} \end{bmatrix}_{\rho x \rho} \quad , \quad \mathcal{T}(\omega) = \begin{bmatrix} \phi(\omega) \\ \mathcal{L}_s\phi(\omega) \\ \mathcal{L}_s^2\phi(\omega) \\ \vdots \\ \mathcal{L}_s^{r-1}\phi(\omega) \end{bmatrix}_{\rho \times 1} \quad , \quad \Gamma = [1 \quad 0 \quad 0 \quad \dots \quad 0]_{1 \times \rho} \tag{3.5b.1}$$

It has been shown in [15], that $\phi(\omega)$ can be generated by the internal model,

$$\left.\begin{aligned} \frac{\partial \mathcal{T}}{\partial \omega}\mathcal{S}_0 &= \mathcal{S}\mathcal{T}(\omega) \\ \phi(\omega) &= \Gamma\mathcal{T}(\omega) \end{aligned}\right\} \tag{3.5b}$$

The internal model (3.10) is only valid when $\phi(\omega)$ have some finite number of harmonics so when $\phi(\omega)$ is the polynomial function of ω. This means that the constants $c_0, c_1, \dots, c_{\rho-1}$ for (3.9) must be known even when $\phi(\omega)$ is uncertain.

Using the equation 3.5. Hence equation 3.4 can be written as

$$\dot{\eta} = f(\eta + \pi, \omega) + G(\eta + \pi, \omega)\varphi(u) - f(\pi, \omega) - G(\pi, \omega)\chi(\omega)$$

$$\dot{\eta} = f(\eta + \pi, \omega) - f(\pi, \omega) + G(\eta + \pi, \omega)\chi(\omega) - G(\pi, \omega)\chi(\omega) - G(\eta + \pi, \omega)\chi(\omega) + G(\eta + \pi, \omega)\varphi(u)$$

It can be rewritten as

$$\dot{\eta} = \tilde{f}(\eta, \omega) + \tilde{G}(\eta, \omega)[\varphi(u) - \chi(\omega)] \tag{3.6}$$

Where,

$$\tilde{f}(\eta, \omega) = f(\eta + \pi, \omega) - f(\pi, \omega) + [G(\eta + \pi, \omega) - G(\pi, \omega)]\chi(\omega) \tag{3.7}$$

$$\tilde{G}(\eta, \omega) = G(\eta + \pi, \omega) \tag{3.8}$$

Here in this case the term $\chi(\omega)$ is the matched uncertainty. Thus this output regulation problem now transformed into stabilization problem this is because the transformation of system into error form lead us to make a stabilization problem. Now, the target is to make regulation error zero and once the regulation error is zero the output asymptotically tracks the reference signal govern by given exo system. The control action that solve the output regulation problem can be divided into two components. One component is the one that force the system's output to slide on the steady state value / manifold while second component acts to stabilize the system's output on steady state value / manifold. To further proceed towards the designing of the control law lets assume the following assumptions hold

Assumption 3.2

There exists a locally Lipschitz function $\Delta(\eta, \omega)$, with $\Delta(0, \omega) = 0$ and a continuously differentiable Lyapunov function $\mathcal{V}(\eta, \omega)$ such that

$$\rho_1(\|\eta\|) \leq \mathcal{V}(\eta, \omega) \leq \rho_2(\|\eta\|) \textit{ and,}$$

$$\left.\begin{array}{c}\frac{\partial v}{\partial \omega} S_o \omega + \frac{\partial v}{\partial \eta}\left[\tilde{G}(\eta,\omega)\varphi\big(\emptyset(\eta)\big)\right] \leq -\mathbb{W}_1(\tilde{x}) \\ \frac{\partial v}{\partial \eta}\tilde{f}(\eta,\omega) \leq -\mathbb{W}_2(\tilde{x}) \\ \mathbb{W}_1(\eta) \leq \mathbb{W}_2(\tilde{\eta})\end{array}\right\} \tag{3.9}$$

where $\eta \in R^n$, $\omega \in \mathfrak{Y}$, ρ_1, ρ_2 are class $\mathcal{K}$ functions and $\mathbb{W}(\eta)$ is positive definite continuous function.

Now, writing the system (3.6) as,

$$\dot{\eta} = \tilde{f}(\eta,\omega) + \tilde{G}(\eta,\omega)\varphi(\emptyset(\eta)) + \tilde{G}(\eta,\omega)\varphi(u) - \tilde{G}(\eta,\omega)\big[\chi(\omega) + \varphi\big(\emptyset(\eta)\big)\big] \tag{3.10}$$

In order to handle uncertainty, in this thesis we use the idea provided by *A. Y. Memon and Khalil* [7] that uses a Lyapunov redesign approach to design the feedback controller to incorporate the uncertainity term

$$[\chi(\omega) + \varphi(\emptyset(\eta))] \tag{3.11}$$

As given in [7],

$$\Omega \;=\; \{sup_{w \epsilon W} V(\eta, w) \leq c_1 \;\subset\; \mathrm{X} \tag{3.12}$$

for some $c_1 > 0$, we defined $\delta(\eta)$ which is continuous function and remained in the certain bound defined in 3.22, its also worth to mention that it does not depends on the non-linearity $\emptyset(\cdot)$ which is sector-bounded, So as given in [7]

$$\| \chi(\mathrm{w}) \;+\; \varphi(\,\emptyset(\eta)) \,\| \leq\; \delta(\eta), \forall \eta \,\in\, \Omega, \forall \mathrm{w} \,\in\, \mathrm{W} \tag{3.13}$$

To make the problem simple take H = 1, such that $(\partial \mathrm{V}/\partial\, \eta)\mathrm{g}(\eta,\mathrm{w})$ as already discussed in chapter 2 can now be written as

$$\left(\frac{\partial \mathrm{V}}{\partial \eta}\right)\mathrm{g}(\eta, \mathrm{w}) \;=\; \upsilon(\eta), \forall \eta \;\in\; \Omega, \forall\, \mathrm{w} \in\; \mathrm{W} \tag{3.14}$$

Conditional servo-compensator is then provided by the equation (3.24) and the feedback controller that solves the output regulation problem without degrading the transient performance can be selected as (3.25).

$$\dot{\varrho} = (S - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right) \tag{3.15}$$

$$u = -\mathfrak{a}(\tilde{\eta}) sat\left(\frac{s}{\mu}\right) \tag{3.16}$$

$$\mathfrak{a}(\tilde{\eta}) \geq \delta(\eta) + \mathfrak{a}_0, \qquad \mathfrak{a}_0 > 0 \tag{3.17}$$

where $s = \upsilon(\eta) + K_1\varrho$, where $\upsilon(\eta)$ is defined as a function which is Locally Lipchitz, with $\upsilon(0) = 0$, μ is defined as the boundary layer inside which the servo-action will be performed and matrix K_1 is designed such that $(S - JK_1)$ is Hurwitz.

3.2 Optimal Nonlinear Feedback Stabilization Controller Design

In this work we took an optimal stabilization feedback controller that acts to stabilize the system robustly and it is assumed that the optimal stabilizing feedback control is available for the system of the form given by

$$\dot{\eta} = \tilde{f}(\eta,\omega) + \tilde{G}(\eta,\omega)u \tag{3.18}$$

Such that when the optimal stabilizing feedback control is applied with some bounded or sector non linearity

$$\dot{\eta} = \tilde{f}(\eta,\omega) + \tilde{G}(\eta,\omega)\emptyset(\eta) \tag{3.19}$$

then the origin of the close loop system will remain stable. The significance of optimal stabilizing control is that it is shown in [10] if the stabilizing controller is optimal with the cost function given by

$$J = \int_0^{\infty}(l(\eta) + u^T R(\eta)u)dt \tag{3.20}$$

Where in this case the term involved in the cost function $l(\eta) \geq 0$ and $R(\eta) > 0$ for all values of $\tilde{x}$. Then according to [10] the optimal stabilization control given by

$$u = -k(\eta) = -\frac{1}{2}R^{-1}(\eta)\left(L_g V(\eta)\right)^T \tag{3.21}$$

Will achieve the sector margin $(\frac{1}{2},\infty)$ and under some conditions it achieves the disk margins D$(\frac{1}{2})$.

Where V(η) is a positive semi definite function called a control Lyapunov function, such that the negative semi definiteness of $\dot{V}$ is achieved with the control u = $-\frac{1}{2}$k(η), that is

$$\dot{V} = L_f V(\eta) - \frac{1}{2}L_g V(\eta) \qquad k(\eta) \leq 0 \tag{3.22}$$

When the function $l(\eta)$ when set to be the right-hand side of equation (2.8)

$$l(\eta) := -L_f V(\eta) + \frac{1}{2}L_g V(\eta) \quad k(\eta) \ \geq 0 \tag{3.23}$$

Then V (η) is a solution of the HJB (Hamilton Jacobian Bellmen) equation

$$l(\eta) + L_f V(\eta) - \frac{1}{4}\left(L_g V(\eta)\right)R^{-1}(\eta)\left(L_g V(\eta)\right)^T = 0 \tag{3.24}$$

But in this thesis, we apply an inverse approach for designing an optimal stabilizing that is an optimal control is designed first then shown to be optimal for particular cost function which is chosen after the control design rather than chosen before the design of controller. As defined in chapter2 this is because the designing of nonlinear optimal controller involves the solution of Hamilton Jacobian Bellman (HJB) equation which is a partial differential equation becomes difficult and infeasible to solve if the system is of higher order. Hence an inverse approach is applied to design a stabilizing controller which is optimal. As discussed in [10] a distinctive feature of this optimal stabilizing control is that it not only stabilizes the system but also provide disc and sector margins which characterize robustness. So, if a Control Lyapunov Function (CLF) *V(x)* is known, an inverse optimal stabilizing control law can be selected from a choice of explicit expressions. The description of inverse optimal control provided by Sepulchure [10] revels that an inverse optimal control stabilization control for the system (3.19) given by

Sontag's formula (2.16) not only stabilizes the system but also provides sector margin $(\frac{1}{2}, \infty)$ and, if $R(\eta)=I$ it also achieves the disk margin $D(\frac{1}{2})$.

Assumption 3.1

There exists a smooth, positive definite, and radially unbounded function V (x) called a control Lyapunov function (CLF) for the system

$$\dot{\eta} = f(\eta) + g(\eta)u \tag{3.25}$$

If for all $\eta \neq 0$

$$\begin{aligned} L_g V(\eta) &= 0 \\ L_f V(\eta) &< 0 \end{aligned} \tag{3.26}$$

By definition, Any Lyapunov function whose time derivative can be rendered negative definite is a CLF. The proof of this can be found in Constructive nonlinear control by Rodolphe Sepulchre et. al. [10]. The importance of CLF based approach is that once a CLF is known an optimal stabilizing control can be designed using Sontag's formula given by (2.16) and in this case the Control Lyapunov Function now becomes an optimal value function. It implies that, once a CLF *V* is known then the optimal feedback control can stabilize the system (3.9) robustly i.e. the system response goes to zero as time approaches to infinity $t \to \infty$ in the presence of sector bound nonlinearity $\emptyset$ that is applied only in the sector [A, B]. The goal of this thesis is to show that we can solve the problem of output regulation with this optimal control and in the presence of sector non-linearity or we can say that the uncertainties that may arise due unmodeled dynamics of the system. This work differs from the work previously done by A. Y. Memon and H. K. Khalil [7] in a sense that in previous work on the system (3.6) the control was totally depend on the control input. But in this work rather than the control linearly depend on input it depends upon the sector bound non linear function $\emptyset$.

3.3 Observer Designs for Output Feedback Version of Servo-Mechanism Designs

This section focuses on the output feedback version of the servo-compensator designs discussed in the previous section. Since, most of the control schemes assume the availability of all the state variables to achieve the desired control objectives. Contrary to this, realizing the physical scenario we cannot measure all the state variables due to the technical or economic reasons which necessitates the mechanism of estimating these variables through a system called state observer or simply observer. It utilizes only the output of the system as input and provides with the estimates of the state variables as output which are used to replace the state variables in the state feedback design making the whole scheme as output feedback. Following from the discussion about the state observers in chapter 2, a nonlinear observer called High Gain Observer (HGO) that recovers the performance of state feedback controller in a robust way, is worked out here to implement the output feedback version of the previously designed control law u. For system (3.1), we will implement the HGO for its transformed model (3.18), based on which the servo-compensator was presented previously whereas for system (3.3), we will implement its extended version (EHGO) for its transformed model (3.18). The need of

EGHO is due to the normal form representation of the system (i.e. internal and external dynamics are shown explicitly). For system (3.23), the HGO can be implemented as,

$$\dot{\hat{\wp}} = \mathbb{A}\hat{\wp} + \mathbb{B}[sat(u) - \Gamma(\omega)] + \mathbb{L}(e - \mathbb{C}_1\hat{\wp}) \tag{3.27}$$

where the observer gain $\mathbb{L}$ is need to be designed. So, the sliding surface s required in control law u and servo-compensator will be as $\hat{s} = \Delta(\hat{\wp}) + K_1\varrho$ where the stabilizing compensator $\Delta(\hat{\wp})$ will be as,

$$\Delta(\hat{\wp}) = -\mathbb{B}^T\mathcal{X}(\varepsilon)\hat{\wp} + \rho(e)\mathbb{B}^T\mathcal{P}\wp \tag{3.28}$$

The HGO observer gain matrix $\mathbb{L}$ can be designed as,

$$\mathbb{L} = \begin{bmatrix}\frac{\alpha_1}{\lambda} & \frac{\alpha_2}{\lambda^2} & \frac{\alpha_3}{\lambda^3} & \cdots & \frac{\alpha_n}{\lambda^n}\end{bmatrix} \tag{3.29}$$

where λ is the small design parameter and the constants α_1, α_2, ..., α_n are selected such that the polynomial $\Omega^n + \alpha_1\Omega^{n-1} + \alpha_2\Omega^{n-2} + \cdots \alpha_n$ is Hurwitz. i.e. all the roots lie in the left half plane.

For our second class of systems, consider the transformed model without matched uncertainty as,

$$\begin{aligned} \dot{z} &= \varphi_a(z) + \Psi(z, y)\hbar(\zeta\ ) \\ \dot{\zeta} &= \mathfrak{f}(z, \zeta) + \mathcal{G}(\zeta)[(u)] \end{aligned}$$

The above system can also be written in the compact form as,

$$\left.\begin{aligned} \dot{z} &= \varphi(z, \zeta) \\ \dot{\zeta} &= \mathcal{M}\zeta + \mathcal{N}[b(z, \zeta) + a(\zeta, u)] \\ y &= \mathcal{O}\zeta \end{aligned}\right\} \tag{3.30}$$

The system (3.30) is in the form where we can present the Extended High Gain Observer (EHGO) design. Extracting the auxiliary system as,

$$\dot{z} = \varphi(z, \zeta), \qquad\qquad \sigma = b(z, \zeta) \tag{3.31}$$

Any suitable observer called the internal observer can be used to estimate the states of the auxiliary system (3.31) formed of the internal dynamics. For example, *Boker et. al*. [9] used Extended Kalman Filter (EKF) as internal observer. For, the auxiliary system (3.31), the EKF design takes the form as,

$$\dot{\hat{z}} = \varphi(\hat{z}, \zeta) + \mathcal{L}(t)[\sigma - b(\hat{z}, \zeta)] \tag{3.32}$$

where the observer gain $\mathcal{L}(t)$ can be designed as,

$$\mathcal{L}(t) = \mathcal{P}(t)\mathcal{C}_1(t)^T R(t)^{-1} \tag{3.32}$$

and $\mathcal{P}(t)$ with $\mathcal{P}(0) = 0$ is the solution of the Riccati Differential Equation,

$$\dot{\mathcal{P}}(t) = \mathcal{A}_1(t)\mathcal{P}(t) + \mathcal{P}(t){\mathcal{A}_1}^T(t) + \mathcal{Q}(t) - \mathcal{P}(t){\mathcal{C}_1}^T(t)\mathcal{R}^{-1}(t)\mathcal{C}_1(t)\mathcal{P}(t) \tag{3.33}$$

The time varying matrices $\mathcal{A}_1(t)$ and $\mathcal{C}_1(t)$ are given by,

$$\mathcal{A}_1(t) = \frac{\partial \varphi}{\partial z}(\hat{z},\zeta), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}(\hat{z},\zeta) \tag{3.34}$$

and $\mathcal{Q}(t)$ and $\mathcal{R}(t)$ are symmetric positive definite matrices that which satisfy,

$$0 < r_1 \leq \mathcal{R}(t) \leq r_2 \tag{3.35}$$

$$0 < q_1 I_{n-\rho} \leq \mathcal{Q}(t) \leq q_2 I_{n-\rho} \tag{3.36}$$

Now, the observer that will work for the external dynamics called the external observer will be employed through EHGO. This observer, in addition to the states of the external dynamics, will observe an extra state that has been utilized as the output of the auxiliary system (3.31).

Its structure is as follows:

$$\dot{\hat{\zeta}} = \mathcal{M}\hat{\zeta} + \mathcal{N}\left[\hat{\sigma} + a(\hat{\zeta},u)\right] + \mathcal{H}(\varepsilon)\left(y - \mathcal{O}\hat{\zeta}\right) \tag{3.37}$$

$$\dot{\hat{\sigma}} = \dot{b}(\hat{z},\hat{\zeta},u) + \frac{\hbar_{\rho+1}}{\varepsilon^{\rho+1}}\left(y - \mathcal{O}\hat{\zeta}\right) \tag{3.38}$$

where,

$$\dot{b}(\hat{z},\hat{\zeta},u) = \left({d[b(z,\zeta)]}/{dt} \right)\Bigg|_{(\hat{z},\hat{\zeta})}, \text{ and} \tag{3.39}$$

$$\frac{d[b(z,\zeta)]}{dt} = \frac{db}{dz}\,\varphi(z,\xi) + \frac{db}{d\zeta}\{\mathcal{M}\zeta + \mathcal{N}[b(z,\zeta) + a(\zeta,u)]\} \tag{3.40}$$

The observer gain matrix $\mathcal{H}(\varepsilon) = \begin{bmatrix} \frac{\hbar_1}{\varepsilon} & \frac{\hbar_2}{\varepsilon^2} & \dots & \frac{\hbar_\rho}{\varepsilon^\rho} \end{bmatrix}^T$ and $\hbar_1, \hbar_2 \dots \hbar_\rho$ are selected such that $\lambda^{\rho+1} + \hbar_1\lambda^\rho + \cdots + \hbar_\rho\lambda + \hbar_{\rho+1}$ is Hurwitz. Moreover, $\varepsilon > 0$ is the small design parameter.

So, combining the internal and external observers (3.32), (3.37) and (3.38), the full order observer for (3.30) is characterized by,

$$\left.\begin{aligned} \dot{\hat{\zeta}} &= \mathcal{M}\hat{\zeta} + \mathcal{N}\left[\hat{\sigma} + a(\hat{\zeta},u)\right] + \mathcal{H}(\varepsilon)\left(y - \mathcal{O}\hat{\xi}\right) \\ \dot{\hat{\sigma}} &= \dot{b}(\hat{z},\hat{\zeta},u) + \frac{\hbar_{\rho+1}}{\varepsilon^{\rho+1}}\left(y - \mathcal{O}\hat{\xi}\right) \\ \dot{\hat{z}} &= \varphi(\hat{z},\zeta) + \mathcal{L}(t)[\sigma - b(\hat{z},\zeta)] \end{aligned}\right\} \tag{3.41}$$

The time varying matrices are now given by,

$$\mathcal{A}_1(t) = \frac{\partial \varphi}{\partial z}(\hat{z},\hat{\zeta}), \qquad \mathcal{C}_1(t) = \frac{\partial b}{\partial z}(\hat{z},\hat{\zeta}) \tag{3.42}$$

So, the sliding surface s required in control law u and servo-compensator will be as $\hat{s} = \Delta(\hat{z},\hat{\zeta}) + K_1\varrho$ where the stabilizing compensator $\Delta(\hat{z},\hat{\zeta})$ will be as,

$$\Delta(\hat{z},\hat{\zeta}) = -\left(\frac{\partial \mathcal{R}}{\partial z}\Psi(\hat{z},y)\right)^T + v \tag{3.43}$$

The component $\mathcal{v}$ will also be designed using the previously discussed techniques but utilizing the state estimates provided by the EHGO.

Now, the output feedback version of servo-compensator ϱ (3.27) and the control law u (3.28) will be as (3.44) and (3.45) respectively. The simulation results depicting the performance of these output feedback versions will be presented later in the section 3.5 which will prove the efficiency of these output feedback schemes.

$$\dot{\varrho} = (\mathcal{S} - JK_1)\varrho + \mu J sat\left(\frac{\hat{s}}{\mu}\right) \tag{3.63}$$

$$u = -\alpha(\hat{\zeta}) sat\left(\frac{\hat{s}}{\mu}\right) \tag{3.64}$$

3.4 CLOSE LOOP ANALYSIS

This section consists of close loop analysis of entire system. In this section we will show how addition of optimal control will achieve the stability margin in the conditional servo mechanism problem. This section will provide the mathematical analysis of the system to show the robustness provided by the optimal feedback control for designing the stabilizing compensator in order to achieve optimal output regulation. Consider the system defined in [7]

$$\left.\begin{aligned} \dot{\omega} &= \mathcal{S}_0\omega \\ \dot{z} &= \varphi_a(z,\omega) + \Psi(z,y,\omega)\hbar(\zeta,\omega) \\ \dot{\bar{x}} &= \mathfrak{f}(z,\eta,\omega) + \mathcal{G}(\eta,\omega)\chi(\eta,\omega) - \mathcal{G}(\eta,\omega)\varphi\left(sat\left[\alpha(\eta) sat\left(\frac{s}{\mu}\right)\right]\right) - \mathcal{G}(\eta,\omega)[\varphi(\emptyset(\eta)) + \chi(\eta,\omega)] \\ \dot{\varrho} &= (\mathcal{S} - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right) \end{aligned}\right\} \tag{3.27}$$

Suppose for our convenience $A_\varrho \triangleq \mathcal{S} - JK_1$ and $sat\left(\frac{s}{\mu}\right)$ is defined as,

$$sat\left(\frac{s}{\mu}\right) = \begin{cases} \dfrac{s}{\|s\|}\ , & \|s\| \geq \mu \\ \dfrac{s}{\mu}\ , & \|s\| \leq \mu \end{cases} \tag{3.27a}$$

Also, suppose that there exist a continuously differentiable Lyapunov function $\mathcal{V}(\eta,\omega)$ and $\Omega = \{\mathcal{V}(\eta,\omega) \leq c_1\}$ be the compact subset of $\mathbb{X}$ that constitutes the state vector η, with $c_1 > 0$ and assumption (3.2) holds. Now, defining a set $\mathbb{S} = \Omega \times \{\mathcal{V}_0(\varrho) \leq \mu^2 c_2\}$ where c_2 is the positive constant and $\{\mathcal{V}_0(\varrho) \leq \mu^2 c_2\}$ is the compact set such that the initial conditions of the conditional servo-compensator i.e. $\varrho(0)$ belongs to this set [7] and $\mathcal{V}_0(\varrho) = \varrho^T \mathcal{P}_\varrho \varrho$ where $\mathcal{P}_\varrho$ is the solution of ARE as $\mathcal{P}_\varrho A_\varrho + {A_\varrho}^T \mathcal{P}_\varrho = -\mathbb{I}$ (Identity Matrix). The conclusion of the important result that the set $\mathbb{S}$ is positively invariant and each trajectory is $\mathbb{S}$ reaches the positively invariant set $\mathbb{S}_\mu = \{\mathcal{V}(\eta) \leq \rho(\mu)\} \times$

$\{\mathcal{V}_0(\varrho) \leq \mu^2 c_2\}$ in finite time, where $\rho(.)$ is the class $\mathcal{K}$ function is already shown in [7]. To simplify the analysis, assuming that the internal dynamics or driven subsystem is stable. i.e. system is minimum phase and the servo-compensator developed only for the external dynamics or driving subsystem will do the required job of output regulation. So, the internal dynamics can be ignored in analyzing the servo-compensator while they will be included in the analysis of stabilizing compensator.

The following analysis will follow the analysis done in [7] but with some technical differences due to nature of problem under consideration. Here in case as defined earlier we will show that combining both controllers will result in the stability margins provided by the optimal stabilizing control. In order to show that the Lyapunov function V candidate already defined in (3.9) is used. The derivative of Lyapunov function will give

$$\dot{V} = \frac{\partial \mathcal{V}}{\partial \omega} S_o \omega + \frac{\partial \mathcal{V}}{\partial \eta}\left[\tilde{f}(\eta,\omega) + \tilde{G}(\eta,\omega)\varphi(\emptyset(\eta))\right] + \frac{\partial \mathcal{V}}{\partial \eta}\tilde{G}(\eta,\omega)\,\varphi\left(-\alpha(\eta) sat\left(\frac{s}{\mu}\right)\right) - \frac{\partial \mathcal{V}}{\partial \eta}\tilde{G}(\eta,\omega)[\chi(\omega) + \varphi(\emptyset(\eta))] \quad (3.28)$$

According to assumption 3.2

$$\frac{\partial \mathcal{V}}{\partial \omega} S_o \omega + \frac{\partial \mathcal{V}}{\partial \tilde{x}}\left[\tilde{G}(\eta,\omega)\varphi(\emptyset(\eta))\right] \leq -\mathbb{W}_1(\tilde{x})$$

We can write equation (3.28) as

$$\dot{V} = -\mathbb{W}_1(\eta) + \frac{\partial \mathcal{V}}{\partial \eta}\left[\tilde{f}(\eta,\omega)\right] + \frac{\partial \mathcal{V}}{\partial \eta}\tilde{G}(\eta,\omega)\,\varphi\left(-\alpha(\eta) sat\left(\frac{s}{\mu}\right)\right) - \frac{\partial \mathcal{V}}{\partial \eta}\tilde{G}(\eta,\omega)[\chi(\omega) + \varphi(\emptyset(\eta))] \quad (3.29)$$

The Lie derivative defined in Khalil [20] given by

$$\left.\begin{aligned} \frac{\partial \mathcal{V}}{\partial \eta}\tilde{f}(\eta,\omega) &= L_{\tilde{f}}\mathcal{V}(\eta) \\ \frac{\partial \mathcal{V}}{\partial \eta}\tilde{G}(\eta,\omega) &= L_{\tilde{G}}\mathcal{V}(\eta) \end{aligned}\right\} \quad (3.30)$$

Hence equation (3.29) becomes

$$\dot{V} = -\mathbb{W}_1(\eta) + L_{\tilde{f}}\mathcal{V}(\eta) + L_{\tilde{G}}\mathcal{V}(\eta)\varphi\left(-\alpha(\eta) sat\left(\frac{s}{\mu}\right)\right) - \frac{\partial \mathcal{V}}{\partial \eta}\tilde{G}(\eta,\omega)[\chi(\omega) + \varphi(\emptyset(\eta))] \quad (3.31)$$

Assumption 3.3

It is assumed that the trajectories are outside the boundary layer. Outside the boundary layer only stabilizing compensator acts and bring the error to zero, which follows that

$$s \geq \mu \qquad \text{Hence} \qquad sat\left(\frac{s}{\mu}\right) = \frac{s}{\|s\|}$$

Since,

$$s = \vartheta(\eta) + K_1 \varrho$$

Ignoring $K_1\varrho$ because outside the boundary layer the stabilizing compensator is dominant. With $s = \upsilon(\eta)$ which is the stabilizing control and since the control is $\varphi(\emptyset(\eta))$ not $\emptyset(\eta)$. Equation (3.31) becomes

$$\dot{V} = -\mathbb{W}_1(\eta) + L_{\tilde{f}}\mathcal{V}(\eta) - \mathfrak{a}(\eta)L_{\tilde{G}}\mathcal{V}(\eta)\left(\varphi\big(\emptyset(\eta)\big) - \emptyset(\eta)\right) - \frac{\partial \mathcal{V}}{\partial \eta}\tilde{G}(\eta,\omega)[\chi(\omega) + \varphi(\emptyset(\eta))] \quad (3.32)$$

According to equations (3.21), (3.24),(3.13),& (3.14)

$$u = -k(\eta) = -\tfrac{1}{2}R^{-1}(\eta)\left(L_gV(\eta)\right)^T = -\emptyset(\eta)$$

$$l(\eta) + L_fV(\eta) - \frac{1}{4}\left(L_gV(\eta)\right)R^{-1}(\eta)\left(L_gV(\eta)\right)^T = 0$$

$$\left(\frac{\partial \mathrm{V}}{\partial \eta}\right)\mathrm{g}(\eta,\mathrm{w}) = \upsilon(\eta), \forall\eta \in \Omega, \forall \mathrm{w} \in \mathrm{W}$$

Combining above equations along with (3.32) will yield and for simplicity taking $\mathfrak{a}(\eta) = 1$

$$\dot{V} = -\mathbb{W}_1(\eta) - l(\eta) + \frac{1}{4}\left(L_gV(\eta)\right)R^{-1}(\eta)\left(L_gV(\eta)\right)^T + L_{\tilde{G}}\mathcal{V}(\eta)\varphi\big(\emptyset(\eta)\big) - L_{\tilde{G}}\mathcal{V}(\eta)(\frac{1}{2}R^{-1}(\eta)\left(L_gV(\eta)\right)^T - \upsilon(\eta)\,\delta(\eta)$$

$$\dot{V} = -\mathbb{W}_1(\eta) - l(\eta) - \frac{1}{4}\left(L_gV(\eta)\right)R^{-1}(\eta)\left(L_gV(\eta)\right)^T + L_{\tilde{G}}\mathcal{V}(\eta)\varphi\big(\emptyset(\eta)\big) - \upsilon(\eta)\,\delta(\eta)$$

$$\dot{V} = -\mathbb{W}_1(\eta) - l(\eta) + L_{\tilde{G}}\mathcal{V}(\eta)\left(\varphi\big(\emptyset(\eta)\big) - \frac{1}{4}\left(L_gV(\eta)\right)R^{-1}(\eta)\right) - \upsilon(\eta)\,\delta(\eta)$$

$$\dot{V} = -\mathbb{W}_1(\eta) - l(\eta) + L_{\tilde{G}}\mathcal{V}(\eta)\left(\varphi\big(\emptyset(\eta)\big) - \frac{1}{2}\left(\frac{1}{2}L_gV(\eta)R^{-1}(\eta)\right)\right) - \upsilon(\eta)\,\delta(\eta)$$

$$\dot{V} = -\mathbb{W}_1(\eta) - l(\eta) + L_{\tilde{G}}\mathcal{V}(\eta)\left(\varphi\big(k(\eta)\big) - \tfrac{1}{2}k(\eta)\right) - \upsilon(\eta)\,\delta(\eta)$$
(3.33)

The uncertainty in the input which is φ appears in the form R which is diagonal as defined in definition 2.1,2.2 i.e.

$$\mathrm{R} = \mathrm{diag}\ \{\mathrm{r}_i(\eta), \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \mathrm{r}_m(\eta)\}$$

Using (3.21)

$$k(\eta) = -\tfrac{1}{2}R^{-1}(\eta)\left(L_gV(\eta)\right)^T, \qquad \mathrm{l}(\eta) \geq 0 \quad (3.34)$$

It followed from 3.34

$$L_gV(\eta) = -2\big(k(\eta)\big)^TR(\eta) \quad (3.35)$$

Hence equation (3.33) becomes

$$\dot{V} = -\mathbb{W}_1(\eta) - l(\eta) - 2\big(k(\eta)\big)^T R(\eta)\left(\varphi\big(k(\eta)\big) - \tfrac{1}{2}k(\eta)\right) - \upsilon(\eta)\,\delta(\eta) \tag{3.36}$$

Since $R(\tilde{x})$ which is a direct nonlinearity is diagonal or in a Matrix form. So we need to apply summation to sum the equation. Applying summation in equation (3.36) gives

$$\dot{V} = -\mathbb{W}_1(\eta) - l(\eta) - 2\sum_{i=0}^{m}[R_i(\eta)k_i(\eta)]\left(\varphi_i\big(k_i(\eta)\big) - \tfrac{1}{2}k_i(\eta)\right) - \upsilon(\eta)\,\delta(\eta) \tag{3.37}$$

Assumption 3.4

There exists a diagonal matrix R(x)

$$R = diag\,\{r_i(\eta), \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, r_m(\eta)\}$$

Which provides a sector margin $(\frac{1}{2}, \infty)$

Remark 3.4

In this whole analysis the assumption R(η) is diagonal is crucial for the negativity of the term

$$-\big(k(\eta)\big)^T R(\eta)\left(\varphi\big(k(\eta)\big) - \tfrac{1}{2}k(\eta)\right) \tag{3.38}$$

with R non diagonal the negativity of above equation may be violated even with a constant positive definite matrix R and wit the linear gains

$$\varphi_i(\eta) = a_i s \tag{3.39}$$

With this closed loop analysis, it is shown that with the addition of optimal feedback stabilization the system not only provide the optimal control but will also cater the robustness properties of the system by providing the gain and sector margins. It is shown that the optimal based control provides the sector margin of $(\frac{1}{2}, \infty)$.

3.5 Simulation Examples

This section of thesis will demonstrate the above discussed control mechanism when applied to a nonlinear system with the help of simulation. The results of simulation will provide the efficiency of our control design and will show how it can be used to overcome the problem of robustness in a real-world problem. The example will be discussed in detail in the later part of the section. The first simulation in the example will show why there is a need of servo action this is because with the use of servo allow us to design a controller which asymptotically follows the desired reference signal and discard the disturbance signal both are generated by known exo system and reducing tracking error to zero. The first example will depict this result by showing the error asymptotically going to zero which means that the output is tracking a reference signal generated by known exo system. The problem associated with conventional servo is the degraded transient performance it is because the addition of new states will tend to degrade the system's transient response and it is addressed with the help of conditional servo mechanism. In the second part of this example it is shown that how conditional servo will compensate the problem of degraded transient performance. In the next part of the example we applied the inverse optimal stabilizing control to the given system and through various results it is shown that with the addition to stabilize the system the inverse optimal stabilizing controller will provide gain and sector margins which characterize robustness of the system. Thus, it will not only achieve the output regulation but with some robustness. Finally, we will present the results of output feedback version by providing the state estimates through extended high gain observer (EHGO). The results of this example will provide the five-important conclusion that is the key to this thesis. First, in case of no servo action it will show that although the transient performance is good, but the steady state error will not decay to zero. Secondly, the results of servo action depict that by including the conventional servo-compensator the steady state error will be decayed to zero but with degraded transient performance. Third, we will make the servo-compensator as conditional one and for this scenario, the transient performance will get improved along with the zero steady-state error. Next, with the addition of optimal stabilizing controller it will achieve certain sector margin which will provide robustness to the system. Finally, the inclusion of EHGO converts the system from state feedback to output feedback without disturbing the robustness provided by the optimal stabilizing controller.

Example 3.5.1: Consider the system which is given in the form of (3.1) given the following data.

$$\begin{aligned} \dot{z} &= -z + z^2\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\xi_1^3 + z^2 + u \\ y &= \xi_1 \end{aligned} \tag{3.40}$$

Where ξ_2, ξ_1 are the external states while z is the internal state. It is required for the system's output to track the reference signal $\alpha_0 \sin(\omega t)$. So, the system matrix of exo-system will be as,

$$\mathcal{S}_0 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \tag{3.41}$$

Before proceeding to servo design, we establish the stability of the internal dynamics (i.e. z equations). Selecting $\mathcal{R}(z) = \frac{1}{2}z^2$ be the Lyapunov function for zero dynamics. i.e. $\dot{z} = -z$. It follows that $\dot{\mathcal{R}}(z) = -z^2$. i.e. the system is shown to be minimum phase. First, we will consider the case that when there is no servo compensator and only stabilizing compensator is available to do the job. Consider the system given in example 3.5.1. Transforming the system into equation of the form (3.6) with the change of variables variable $\zeta_1 = \xi_1 - \omega_1$, we get

$$\left.\begin{aligned} \dot{z} &= -z + z^2(\zeta_1 + \omega_1) \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= -(\zeta_1 + \omega_1)^3 + z^2 + u + \omega^2\omega_1 \end{aligned}\right\} \tag{3.42}$$

with $\omega = 1$ and $\omega(0) = [0 \quad 1]^T$.

In the first part we will only apply the stabilization control without any servo compensator to show that the error is not going to zero. The stabilization controller that we are applying in simply by using state feedback stabilization method. Since the system is in normal for so the output feedback method guarantees stabilization of the system subject to without any parametric perturbations. The control law that is applied is

$$u = 2\zeta_1 + 3\zeta_2 \tag{3.43}$$

Two cases are considered by changing the initial conditions. Figure 3.1 (a) will show the condition when the initial condition that are taken as $z(0) = 0$, $\zeta_1(0) = 0$ and $\zeta_2(0) = 0$, whereas the figure 3.1 (b) will show the response when the initial condition changed to $z(0) = 0$, $\zeta_1(0) = 1$ and $\zeta_2(0) = 1$. It is clear from both the figures although the stabilizing controller tries to achieve zero regulation error but due to time varying reference signal and the disturbance signal both generated by a known exo system the error did not go to zero. Hence it concludes that there is a need of servo compensator which will work along with the stabilizing compensator.

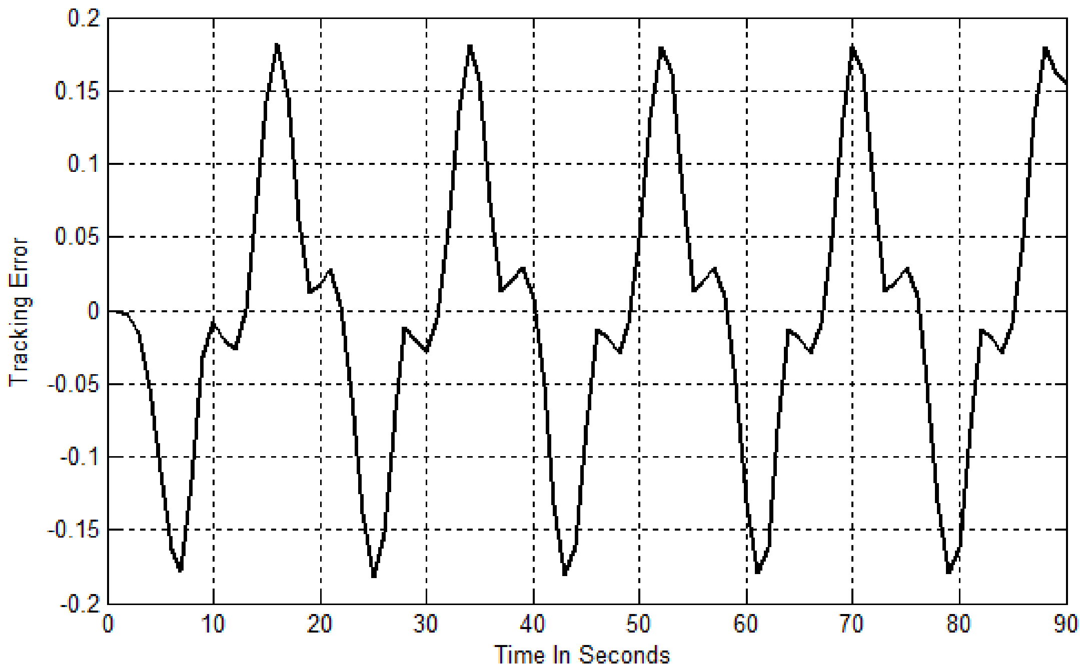

Fig. 3.1(a): Tracking Error in case of No Servo Action [$\zeta_1(0) = 0$].

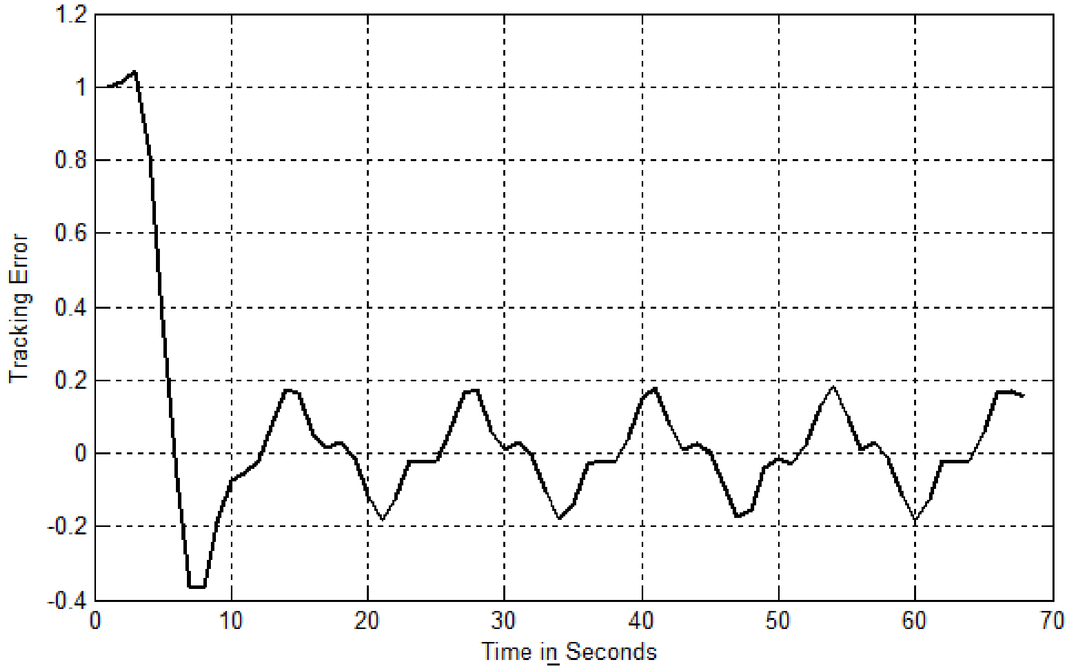

Fig. 3.1(b): Tracking Error in case of No Servo Action [$\zeta_1(0) = 1$].

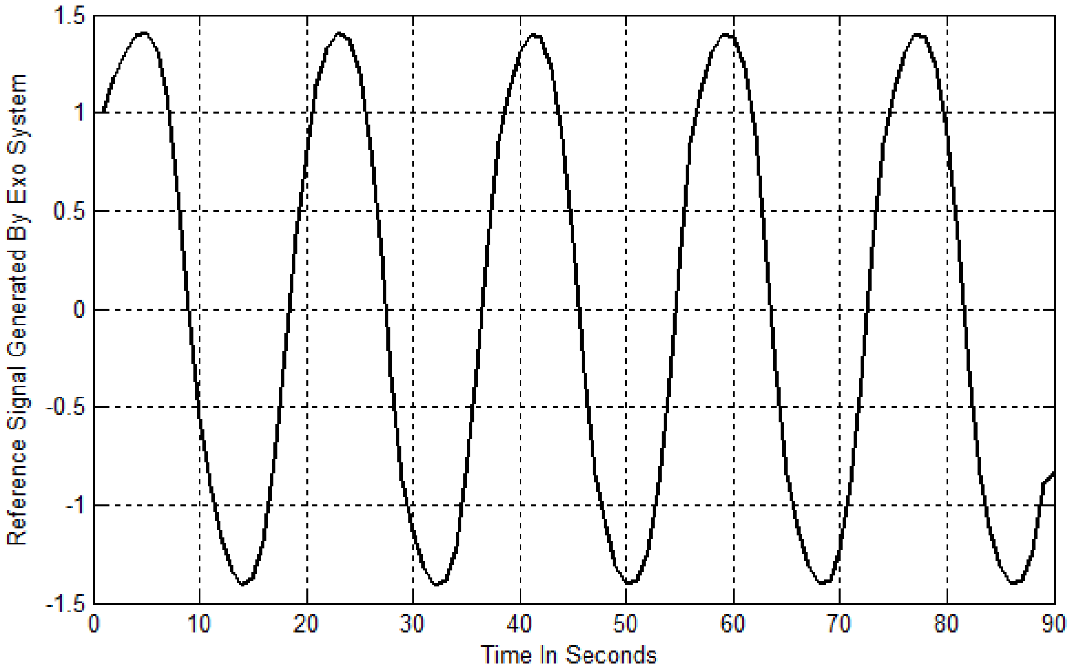

Fig. 3.1c: Reference Signal Generated by known Exo system.

By looking at above figures it can be concluded that there is a need of servo compensator to achieve output regulation. The conventional servo compensator work well in this condition but with a problem of degraded transient performance i.e with the use of conventional servo compensator the regulation error tends to go to zero, but the transient performance is not addressed. A typical conventional servo compensator designed by solving regulator equation provided by (3.5) such that identity (3.5a) holds is given by

$$\mathcal{L}_s^4\phi(\omega) = -9\omega^4\phi(\omega) - 10\omega^2\mathcal{L}_s^2\phi(\omega) \tag{3.44}$$

The S matrix as defined in (3.5b.1) is given by

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{\rho-1} \end{bmatrix}_{\rho x \rho}, \tag{3.45}$$

where $c_0 = -9\omega^4$ and $c_2 = -10\omega^2$.

Hence the $\mathcal{S}$ matrix become

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9\omega^4 & 0 & -10\omega^2 & 0 \end{bmatrix} \tag{3.46}$$

with $\omega = 1$ the $\mathcal{S}$ matrix becomes

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & 0 & -10 & 0 \end{bmatrix} \tag{3.47}$$

and

$$J = [0 \quad 0 \quad 0 \quad 1]^T \tag{3.48}$$

To fulfil the requirement of Huwritzness the K matrix is designed by placing the eigen values in the left half plane. By placing the eigen values of $(\mathcal{S} - JK_1)$ at -0.5, -1, -1.5 and -2 the K_1 matrix becomes

$$K_1 = [-7.5 \quad 6.25 \quad -1.25 \quad 5] \tag{3.49}$$

The conventional servo compensator is designed by using (3.5c) and (3.5d) and the control input u is taken as discussed in conventional compensator as

$$u = -\left(\frac{s}{\mu}\right) \tag{3.50}$$

Where,

$$s = \mathcal{v}(\zeta) + K_1\varrho \tag{3.51}$$

Where $\mathcal{v}(\zeta)$ is the stabilizing compensator that brings the trajectory to a manifold. The stabilizing compensator used in this case is the designed by simple state feedback stabilization technique. Here in this simulation we take the same stabilizing control as taken in the previous example so the s becomes

$$s = 2\zeta_1 + 3\zeta_2 + K_1\varrho \tag{3.52}$$

The initial condition remains the same as used in the previous example. Initial condition that are taken as $z(0) = 0$, $\zeta_1(0) = 1$ and $\zeta_2(0) = 0$.

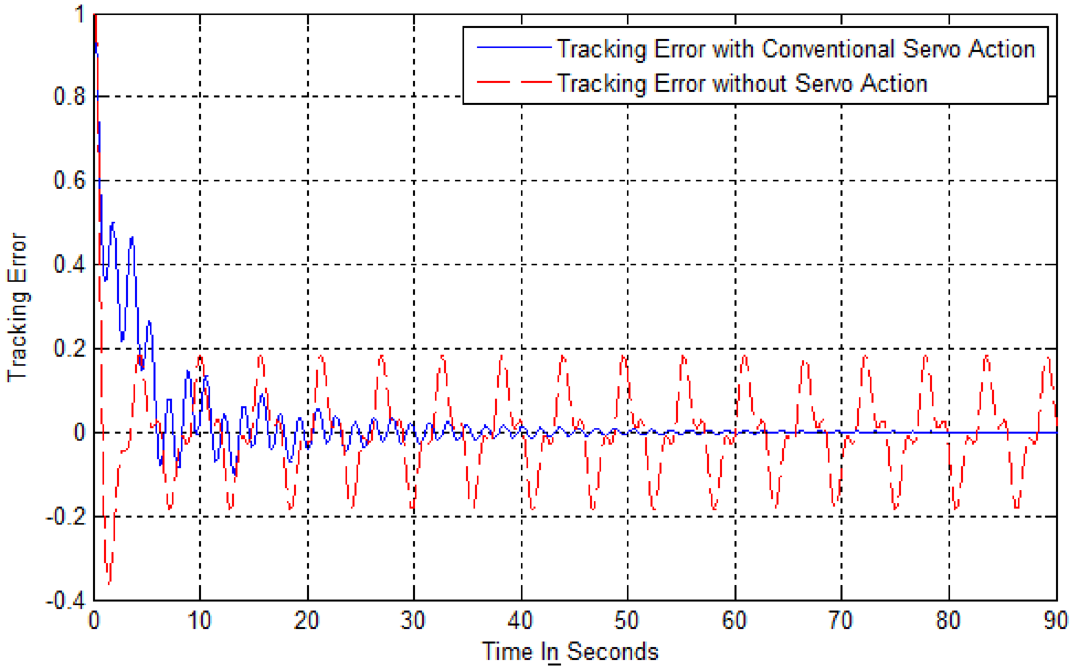

Fig. 3.2: Tracking Error Performance Comparison of Conventional Servo and with No Servo Action.

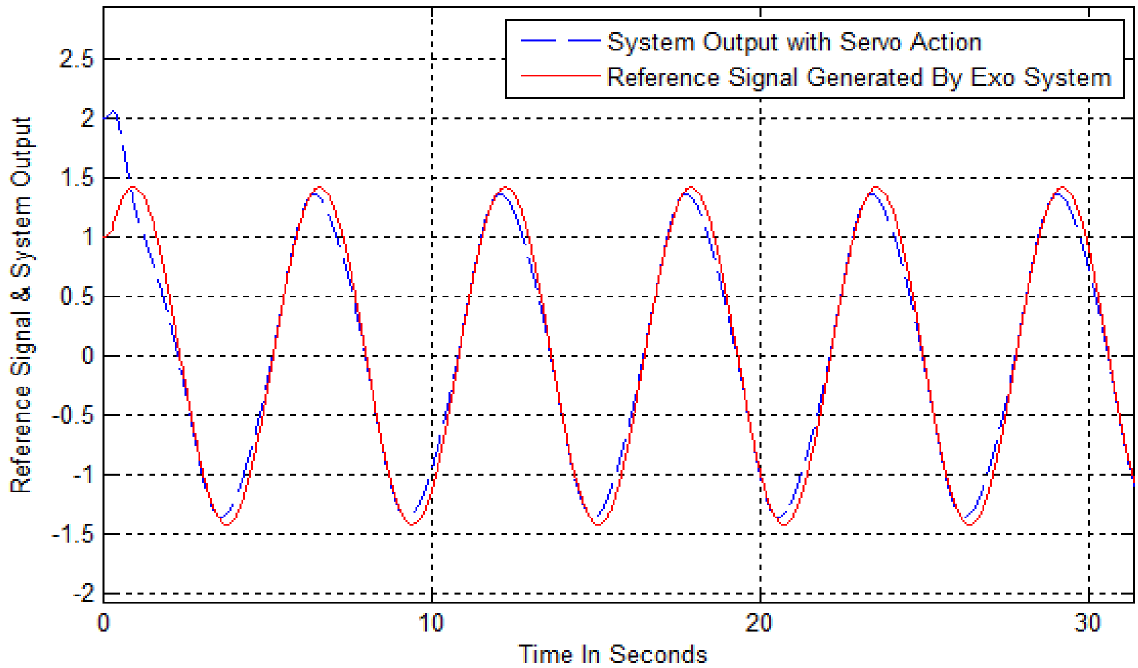

Fig. 3.3a: Conventional Servo. Plot of Reference Signal "ω_1" vs System Output "ξ_1".

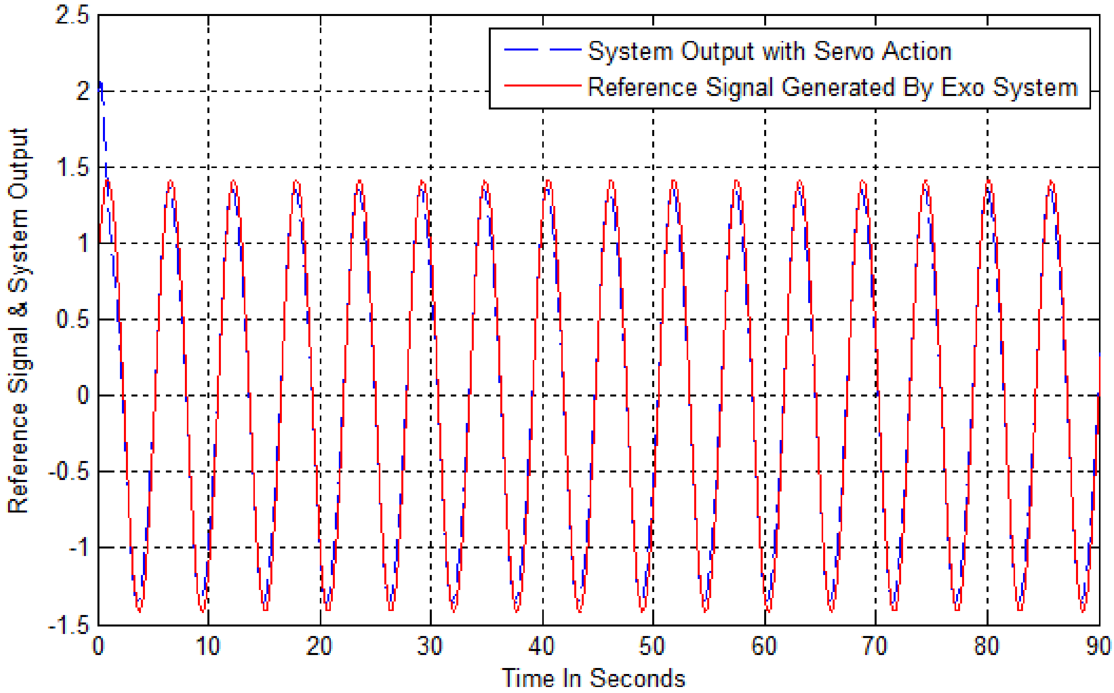

Fig. 3.3b: Conventional Servo. Plot of Reference Signal "ω_1" vs System Output "ξ_1".

Here in this case the figure 3.2 shows the tracking error performance comparison between the two-design when no servo action is applied to the system and when conventional servo is applied to it. It is visible that with the addition of servo compensator will help to achieve zero steady state error but with the expense of degraded transient performance. Although the tracking error goes to zero, but the transient performance is degraded in this case. That's the problem that is faced in case of convention servo compensator. In the next part we will addressed this degraded transient performance with the help of conditional servo compensator.

Fig 3.3a, Fig 3.3b shows the output signal comparison between the reference signal generated by the known exo system and the system output that is generated with applying conventional servo to the system.

To address the problem of transient response a conventional servo is designed as a conditional one provided by (3.25) and (3.26) where a conventional servo is made as a conditional one i.e. that its servo-action is limited to certain region specified by the boundary layer μ. In this way, instead of servo action remain active all the time, is activated only inside that boundary layer while it acts a bounded-input bounded-output system outside this boundary layer. Fig 3.4 shows the comparison between conditional and conventional servo design. It is can be verified that the conditional servo solved the problem of degraded transient performance. In case of conditional servo, the error goes to zero. Similarly, Fig 3.5 shows the output signal and the reference signal. Since the error is going to zero hence the output is tracking the refence signal.

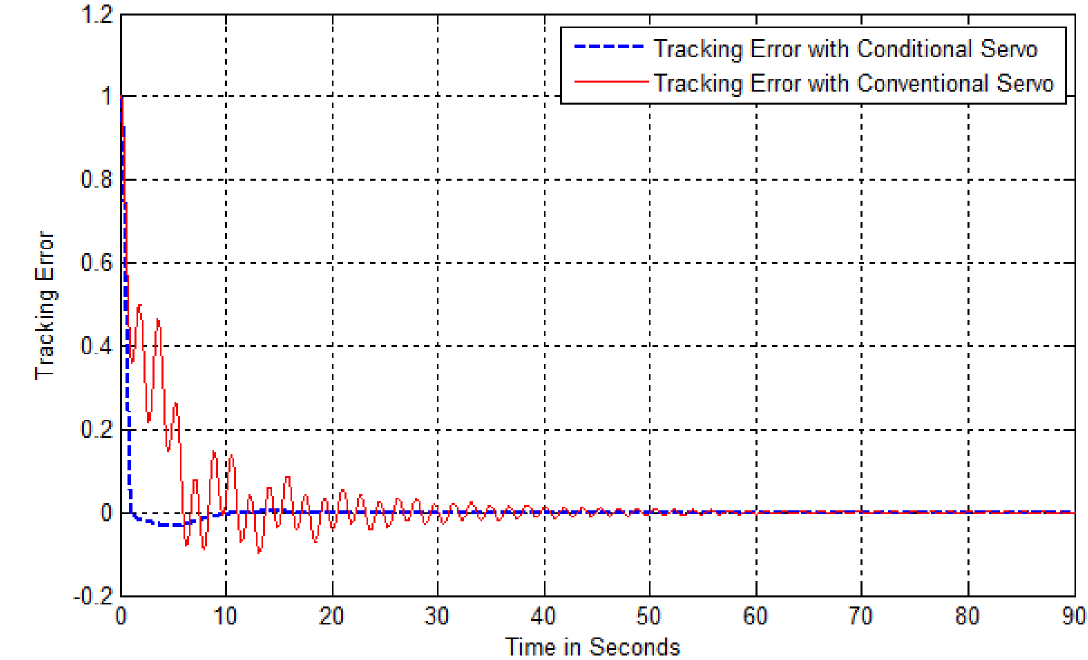

Fig. 3.4: Tracking Error Comparison between Conditional Servo and Conventional Servo

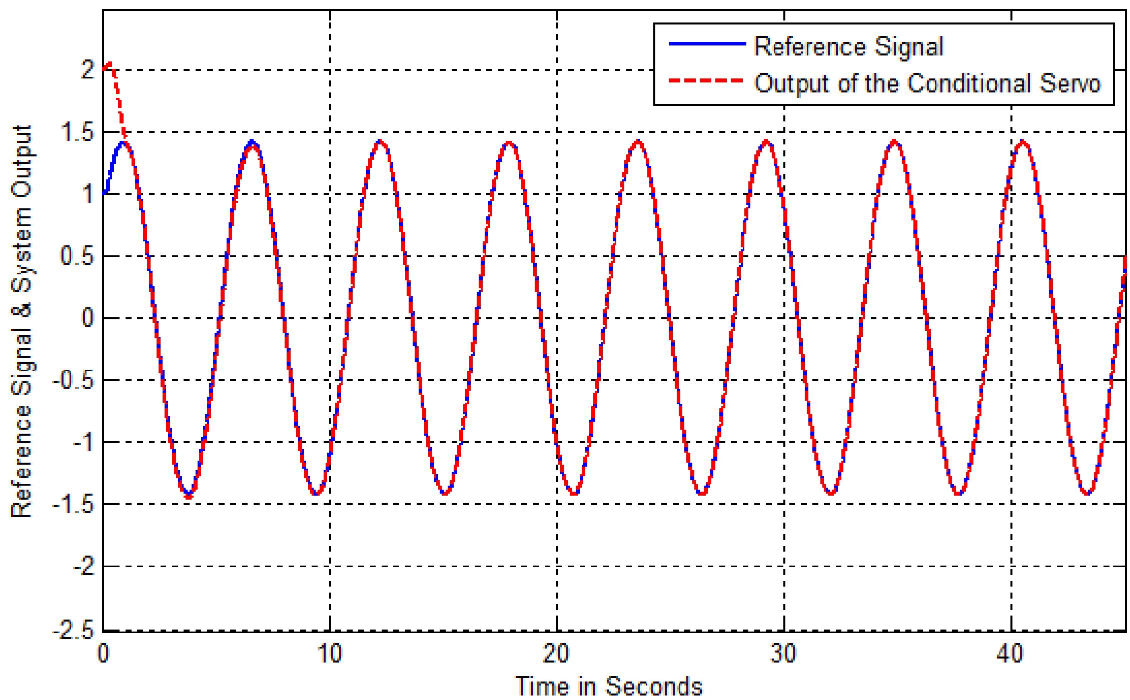

Fig. 3.5: Conditional Servo. Plot of Reference Signal "ω_1" vs System Output "ξ_1".

The above figures prove the superiority provided by the conditional servo design. So far in this example the stabilizing control that was used is simple feedback stabilization which stabilizes the system but does not offer robustness and optimality i.e. more control effort is required to achieve regulation without robustness. The goal of this thesis is to introduce robustness and optimality in the output regulation problem and this can be achieved by the use of non-linear optimal controller. To address this problem the following section considers the same system but this time an optimal based stabilizing control is designed to show the robustness and optimal control provided by nonlinear optimal control. The optimal control methods developed by Kokotovic and some other researchers [10] that not only provide the optimal control but also guarantees sector margins $(\frac{1}{2},\infty)$ which characterize robustness of the system. A major handicap in designing such stabilizing control law is the solution of Hamiltan Jacobian Bellman (HJB) equation which is a partial differential equation and is difficult to solve. In order to solve that problem in this thesis we use an inverse problem i.e. an optimal stabilizing control is designed first and then prove to be optimal for the particular cost function, In this way the problem of solving HJB equation can be tackled. An optimal control designed by Sonntag's [14] provides the sector margin $(\frac{1}{2},\infty)$ and under some condition it also provide the disc margins. The margins indicate the range of non-linearity that can be applied to system with which the system response would not be alter and here in this case the error asymptotically approaches to zero. It will be shown in the subsequent section how inverse optimal control achieve robustness and optimality in terms of applied control. For this we will consider another example to apply optimal stabilizing control.

Example 3.5.2: Consider a system of the form (3.1) with the following data.

$$\begin{aligned} \dot{z} &= -z + z^2\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\xi_1^3 + z^2 - b\xi_2 + \vartheta(u) \\ y &= \xi_1 \end{aligned} \tag{3.53}$$

Where ξ_2, ξ_1 are the external states while z is the internal state and b>0. $\vartheta(u)$ is the control input. In this case a sector bound nonlinearity is added in the control input to investigate the robustness in terms of sector margins provided by the nonlinear optimal based stabilization feedback controller. It is shown earlier in the previous example that the internal state is stable and is response goes to zero as time approaches to infinity. It is required for the system's output to track the reference signal $\alpha_0 \sin(\omega t)$and reject the disturbance signal both are generated by known exo system.

Keeping in view of the performance of the conditional servo an additional extra disturbance signal is applied which is a constant signal from the exo system, the system matrix of exo-system will be as

$$\mathcal{S}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{bmatrix} \tag{3.54}$$

Consider the system given in example 3.5.2. Transforming the system into equation of the form (3.6) with the change of variables variable $\zeta_1 = \xi_1 - \omega_1$, we get

$$\left.\begin{aligned} \dot{z} &= -z + z^2(\zeta_1 + \omega_1) \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= -(\zeta_1 + \omega_1)^3 + z^2 - b\zeta_2 + \vartheta(u) + \omega^2\omega_1 \end{aligned}\right\} \tag{3.55}$$

with $\omega = 1$ and $\omega(0) = [0 \quad 1]^T$. It is already shown in previous example about the stability of internal states by choosing an appropriate Lyapunov function.

The next task is to design an optimal stabilizing controller to stabilize the system. The optimal control methods designed by kokotovic and some other researcher [8] are used for designing optimal stabilization controller but an inverse approach is used just because solving Hamilitian Jacobian Bellman (HJB) equation is a difficult task. The method for finding this control law is discussed in chapter 2.

For designing a control law using an inverse method first a (CLF) which is control Lyapunov function is required. For the given system we define a Lyapunov function candidate which is given by,

$$V(\zeta) = \frac{1}{4}\zeta_1^4 + \frac{1}{2}\zeta^T P\zeta \tag{3.56}$$

Where P matrix m x n is defined as

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \tag{3.57}$$

The Lyapunov function derivative is given by

$$\dot{V} = (1 - p_{22})\zeta_1^3\zeta_2 - p_{22}\zeta_1^4 + (p_{11}\zeta_1 + p_{12}\zeta_2)\zeta_2 - (p_{12}\zeta_1 + p_{22}\zeta_2)b\zeta_2 \tag{3.58}$$

Taking $p_{11} = k \; and \; k > 1, \; p_{12} = p_{21} = p_{22} = 1$, (3.58) becomes

$$\dot{V} = -\zeta_1^4 - (k-1)\zeta_2^2 \tag{3.59}$$

Which is negative definite. According to definition regarding control Lyapunov function [10] any continuously differentiable positive definite function V (ζ), if it has the property defined in (3.60),(3.61) it is said to be control Lyapunov function

$$\frac{\partial V}{\partial \zeta} g(\zeta) = 0 \tag{3.60}$$

For which,

$$\frac{\partial V}{\partial \zeta} f(\zeta) < 0 \tag{3.61}$$

In short if we know any stabilizing control law having a equivalent Lyapunov function V (ζ) then V (ζ) is a control Lyapunov function or (CLF). The negative definiteness in (3.59) satisfying the property of control Lyapunov function. With the use of CLF an optimal stabilizing controller can be find out by using Sontag's formula (2.18) for k>1.

$$\upsilon(\zeta) = -\frac{-\zeta_1^4 - (k-1)\zeta_2^2 + \sqrt{\left(-\zeta_1^4 - (k-1)\zeta_2^2\right)^2 + (\zeta_1 + \zeta_2)^4}}{(\zeta_1 + \zeta_2)} \tag{3.62}$$

The cost function given by (2.17)

$$J = \int_0^\infty (\frac{1}{2} z(\zeta) b^T(\zeta) b(\zeta) + \frac{1}{2z(\zeta)} u^T R(\zeta) u) dt \tag{3.63}$$

Where $b(\zeta) = \zeta_1 + \zeta_2$, $R(\zeta) = I\ and$

$$z(\zeta) = -\frac{-\zeta_1^4 - (k-1)\zeta_2^2 + \sqrt{\left(-\zeta_1^4 - (k-1)\zeta_2^2\right)^2 + (\zeta_1 + \zeta_2)^4}}{(\zeta_1 + \zeta_2)}$$

The servo-compensator for this case will also little bit changed due to the addition of extra disturbance signal. The regulator equations will yield the matrix $\mathcal{S}$ as,

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -9\omega^4 & 0 & -10\omega^2 & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.64}$$

The same procedure is followed as discussed previously to design a servo compensator but this time an optimal based stabilizing control is used to stabilize the system and to achieve the output regulation. The initial conditions are the same as used in the previous example and the disturbance signal is of magnitude 1.

To fulfil the requirement of Hauritzness the K matrix is designed by placing the eigen values in the left half plane. By placing the eigen values of $(\mathcal{S} - JK_1)$ at -0.5, -1, -1.5, -2 and -2.5 the K_1 matrix becomes

$$K_1 = [3.7500 \quad 8.1250 \quad 28.1250 \quad 11.2500 \quad 7.5000] \tag{3.65}$$

The $\mathcal{S} - JK_1$ now becomes

$$\mathcal{S} - JK_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -3.75 & -17.125 & -28.125 & -21.25 & -7.5 \end{bmatrix} \tag{3.66}$$

So, applying optimal based conditional servo using the optimal based control law described in the previous example 3.5.2

$$u = -sat\left(\frac{s}{\mu}\right) \tag{3.67}$$

Where,

$$s = \mathcal{v}(\zeta) + K_1\varrho$$

(3.68)

$$\dot{\varrho} = (S - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right)$$

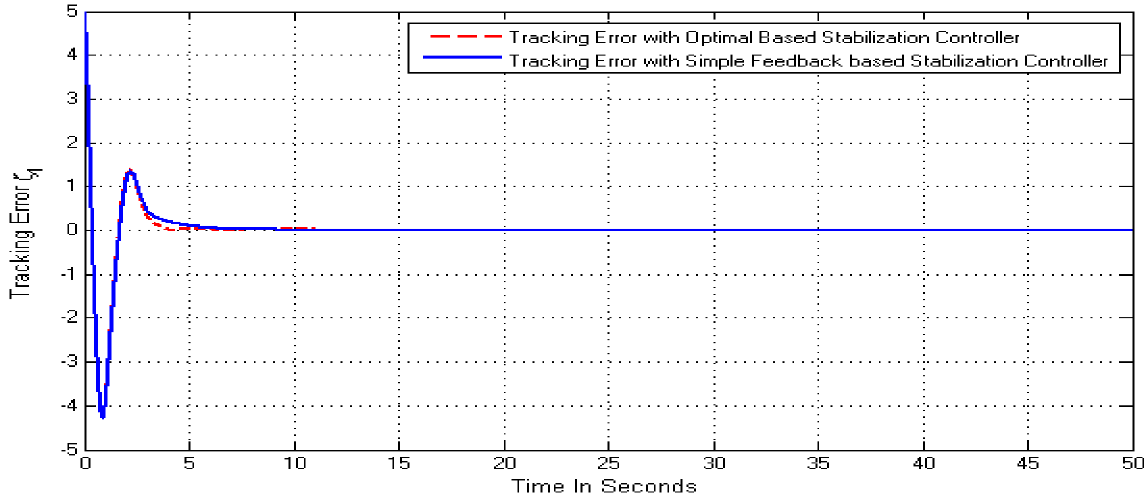

Fig. 3.6: Tracking error Comparison "ξ_1" with Cond. Servo based on Optimal vs Simple Feedback Based Stabilization Controller $\mathcal{v}(\zeta)$.

$\mathcal{v}(\zeta)$ is the optimal based stabilizing compensator that is used to stabilize the system or to bring the trajectories to the zero-error manifold. Fig 3.6 shows the tracking error response comparison of the system when the feedback stabilizing control is based on inverse optimal control technique and simple feedback based stabilization control law is applied. The tracking error is approaching to zero contributing zero steady state error. Here in this case no parametric uncertainty is applied so far.

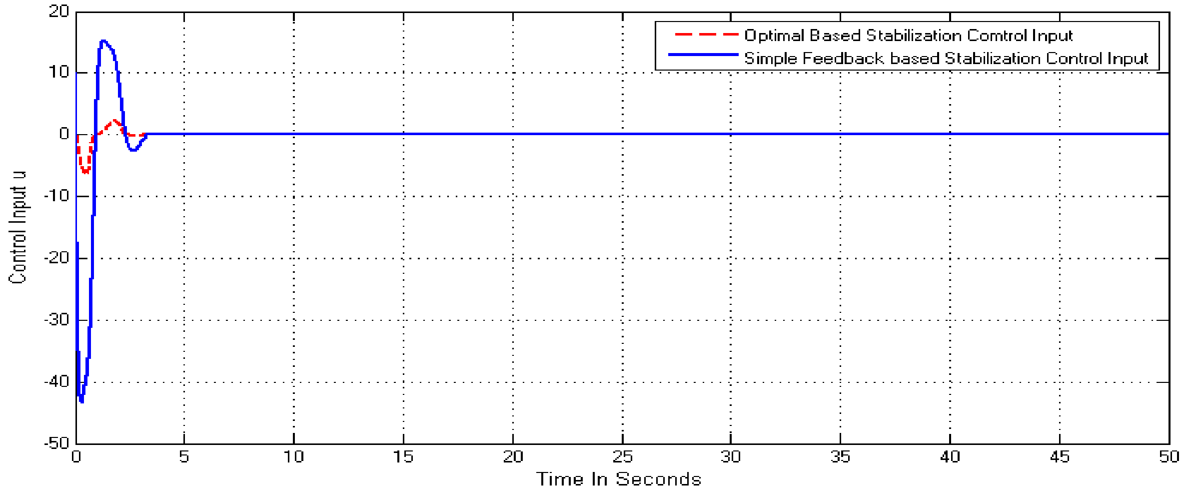

Fig. 3.7: Stabilization Control Input Comparison Between Optimal vs Simple Feedback Controller $\mathcal{v}(\zeta)$.

Fig 3.7 shows the comparison between optimal and simple feedback stabilization control input when applied to the system. It can be seen that optimal based control uses lesser span of control input while that of other uses greater span of input that's one of the advantage that we get with using optimal control law. With the same optimal based stabilization control is applied a parametric uncertainty is added in the control input to check the robustness of the system. Since the optimal based controller provides robustness to the uncertainty that belongs to sector $(\frac{1}{2}, \infty)$ therefore a care is taken in applying that uncertainty i.e. the uncertain term belongs to 1st quadrant . Fig 3.8 shows the graph between the control input u and the perturbed control input $\vartheta(u)$.

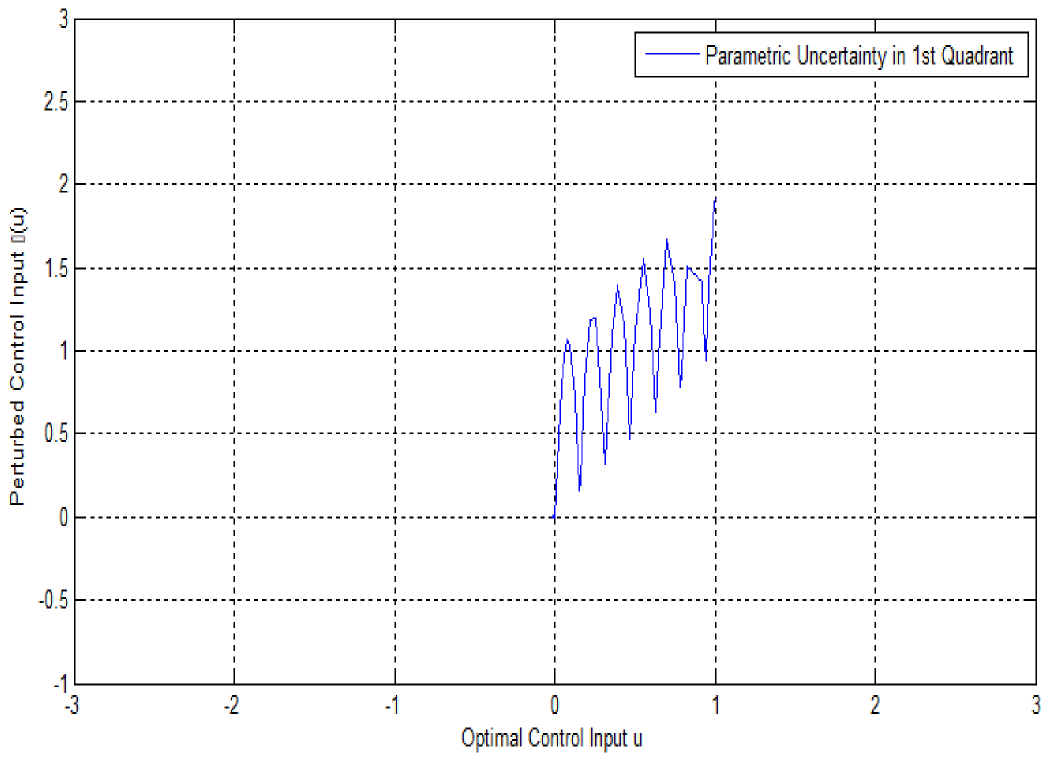

Fig. 3.8: Parametric Uncertainty between control input u and the perturbed control input $\vartheta(u)$

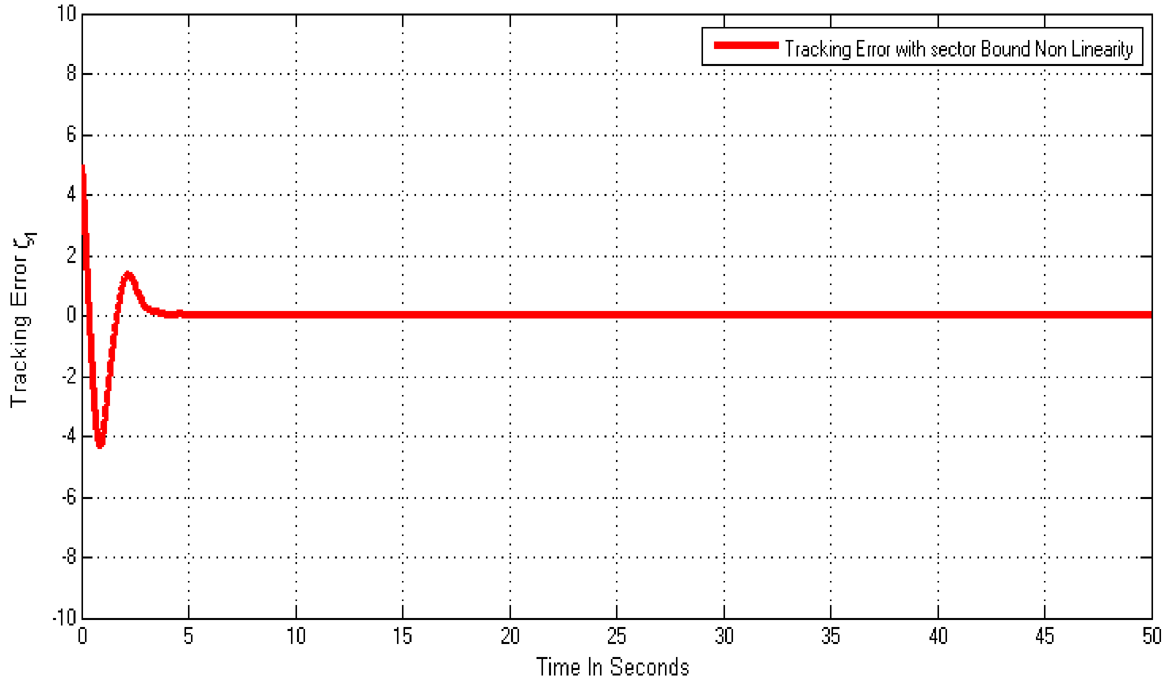

Fig. 3.9 Tracking error "ξ_1" with Cond. Servo based on Optimal Stabilization Controller $v(\zeta)$ with the perturbed control input $\vartheta(v(\zeta))$

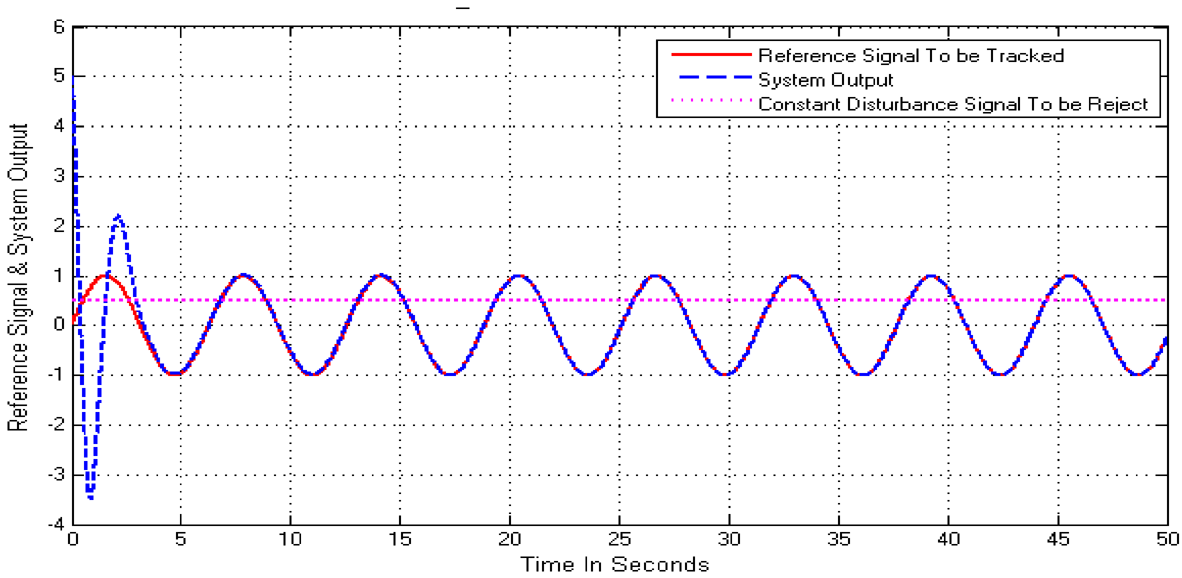

Fig. 3.10: Optimal Conditional Servo. Plot of Reference Signal "ω_1" vs System Output "ξ_1" & disturbance Signal

The above figure 3.9 shows the tracking error "ξ_1" of Cond. Servo based on Optimal Stabilization Controller $\mathscr{v}(\zeta)$ with the perturbed control input $(\mathscr{v}(\zeta))$ and in the presence of disturbance signal. It is clear with the constant disturbance signal provided, the sinusoidal reference signal is applied and the parametric uncertainty is applied the optimal based conditional servo achieve the zero-tracking error with the parametric uncertainty applied in the sector $(\frac{1}{2},\infty)$, which shows the robustness provided by the optimal based stabilization controller. The optimal based stabilization controller also reduces the span of control input since it is designed to provide least control effort to achieve a stabilization task. It can be seen in Fig 3.7 that an optimal based controller uses lesser input signal than other conventional feedback controller. Fig 3.10 shows the reference signal and the output of the system which shows that as the tracking error asymptotically approaches to zero the output follows the reference trajectories produced by exo system and rejecting the disturbance signal of constant magnitude 0.5 produced by the exo system. Fig 3.11 shows the system's response when the same non-linearity is applied to the system whose stabilization controller is other than an optimal based controller. The figure shows that the tracking error is not approaching to zero which depict the non-robustness of the simple feedback stabilization controller and the robustness provided by the optimal based controller. Since the optimal based stabilization controller guarantees sector margins $(\frac{1}{2},\infty)$ and when the same non-linearity is applied to an optimal based controller it handled the uncertainty and achieve the zero-steady state error. This figure also verified the idea provided by Kokotovic and some other researchers [10] which proves that to achieve asymptotic stability an optimal based controller also guarantees sector margins which characterize robustness of the system.

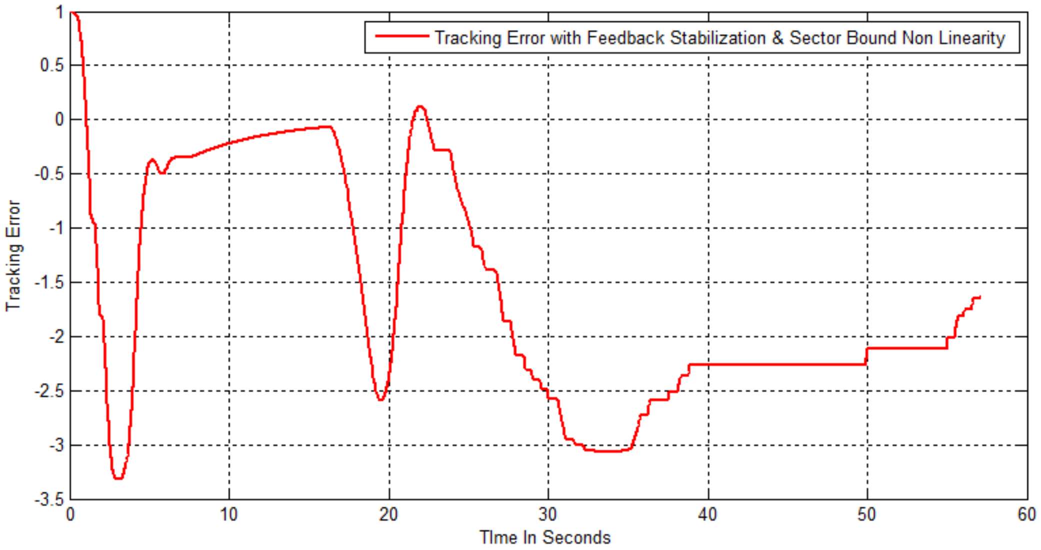

Fig. 3.11 Tracking error "ξ_1" with Cond. Servo based on Feedback Stabilization Controller $\upsilon(\zeta)$ with the perturbed control input $\vartheta(\upsilon(\zeta))$

The previous all designs were based on state feedback based conditional servo where all stated were available for the feedback. The following example will considers a practical case in which a single state which is the output is available for feedback so, the problem now transforms from state feedback-based control to the output feedback-based control. An EHGO is designed to transform the system from state feedback-based control to output feedback-based control. Since an HGO is not useful in this design because of the additional internal state which is to be estimate for that an EHGO is designed. The Extended Kalman Filter based Extended High Gain Observer is designed as discussed previously in section 3.3 for the system given by 3.55. For designing an EHGO first it is required to have an auxiliary system, the auxiliary system in this case is given by

$$\begin{aligned} \dot{z} &= -z + z^2(\zeta_1 + \omega_1) \qquad \sigma = b = z^2 \\ \dot{b} &= 2z \end{aligned} \tag{3.69}$$

The complete EHGO which is based on Extended Kalman Filter EKF is designed by using the steps given in section 3.3 is given by

$$\begin{aligned} \dot{\hat{z}} &= -\hat{z} + \hat{z}^2 y + \mathcal{L}(\hat{\sigma} - \hat{z}^2) \\ \dot{\hat{\zeta}}_1 &= \hat{\zeta}_2 + \frac{\alpha_1}{\varepsilon}(y - \hat{\zeta}_1) \\ \dot{\hat{\zeta}}_2 &= -(\hat{\zeta}_1 + \omega_1)^3 + \hat{\sigma} + u - b\hat{\zeta}_2 + \omega^2\omega_1 + \frac{\alpha_2}{\varepsilon^2}(y - \hat{\zeta}_1) \\ \hat{\sigma} &= \dot{b} + \frac{\alpha_3}{\varepsilon^3}(y - \hat{\zeta}_1) \end{aligned} \tag{3.70}$$

To implement Ricatti Differential equation for a non linear system the matrices $\mathcal{A}_1(t)$ and $\mathcal{C}_1(t)$ is given by

$$\mathcal{A}_1(t) = -1 + 2zy, \qquad \mathcal{C}_1(t) = 2z, \tag{3.71}$$

The gain matrix $\mathcal{L}$ is designed with the solution of Ricatti Differential Equation 3.33. The value of R and Q is chosen to be

$$\mathcal{R} = 10, \qquad \mathcal{Q} = 1$$

The constants α_1, α_2 and α_3 are chosen such that the characteristic equation

$$\Delta^3 + \alpha_1\Delta^2 + \alpha_2\Delta + \alpha_3$$

is Hurwitz. The constants are observer design constants that increase the observer performance. In this example the constants that are used are $\alpha_1 = 7$, $\alpha_2 = 5$ and $\alpha_3 = 1$ and the performance constant $\varepsilon = 0.001$ is chosen. With all these design parameters the output feedback based optimal stabilizing controller is given by

$$z(\hat{\zeta}) = -\frac{-\hat{\zeta}_1^4-(k-1)-\hat{\zeta}_2^2+\sqrt{\left(-\hat{\zeta}_1^4-(k-1)\hat{\zeta}_2^2\right)^2+(\hat{\zeta}_1+\hat{\zeta}_2)^4}}{(\hat{\zeta}_1+\hat{\zeta}_2)} \tag{3.72}$$

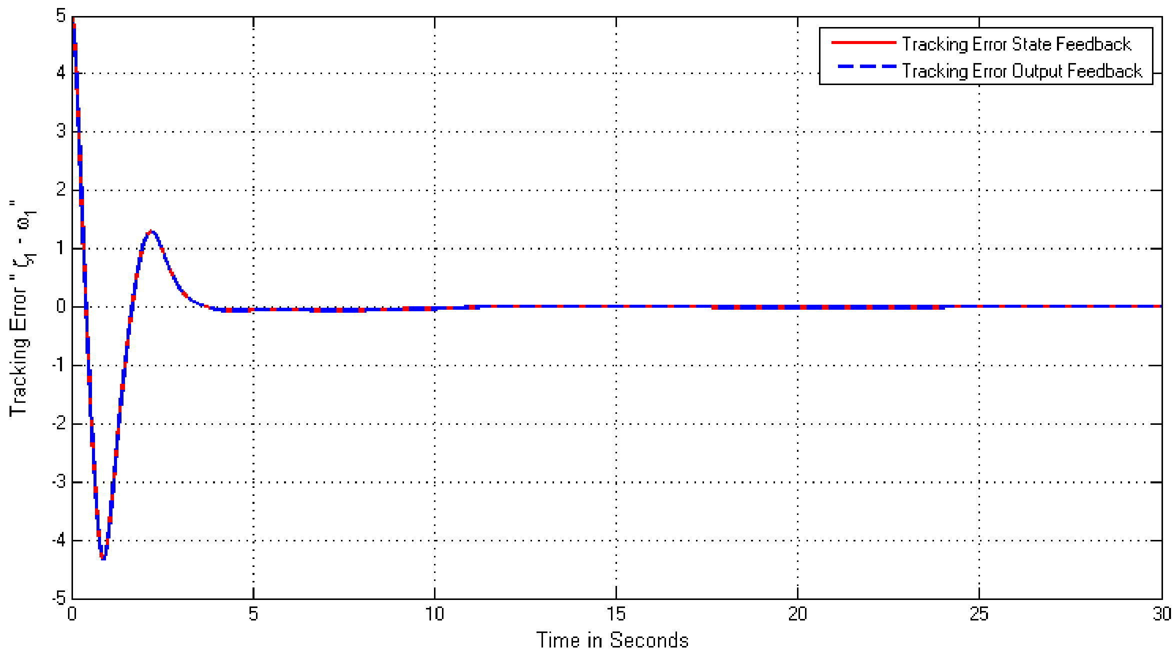

Fig. 3.12: Tracking error "ζ_1" with Cond. Servo based on State Feedback Optimal Stabilization Controller & Output Feedback Optimal Stabilization Controller

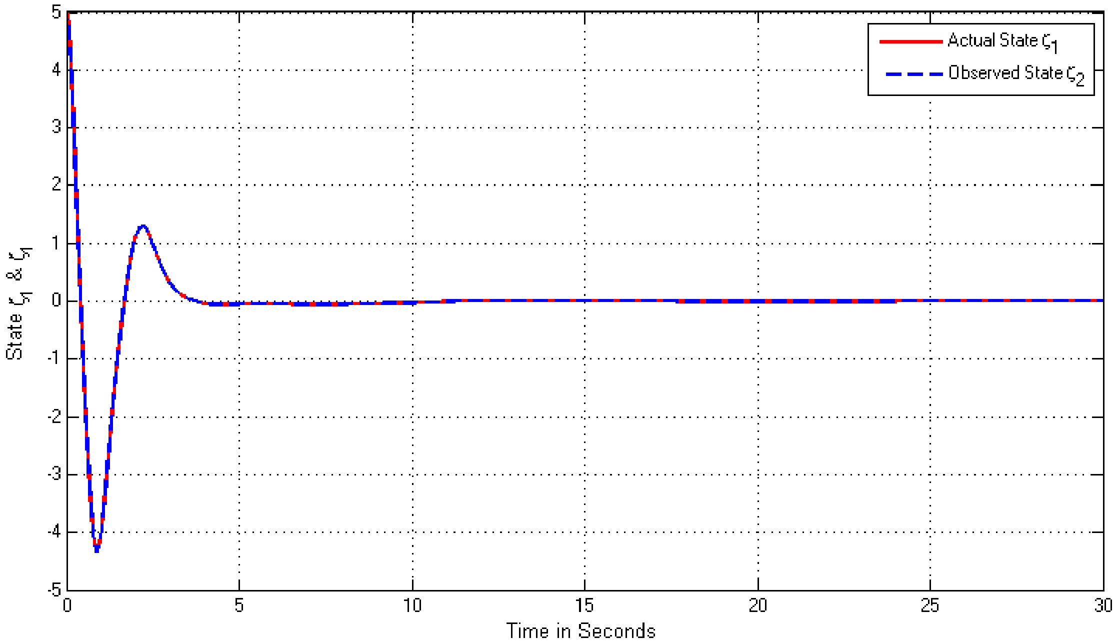

Fig. 3.13 Actual State "ζ_1" and Observed State "$\hat{\zeta}_1$"

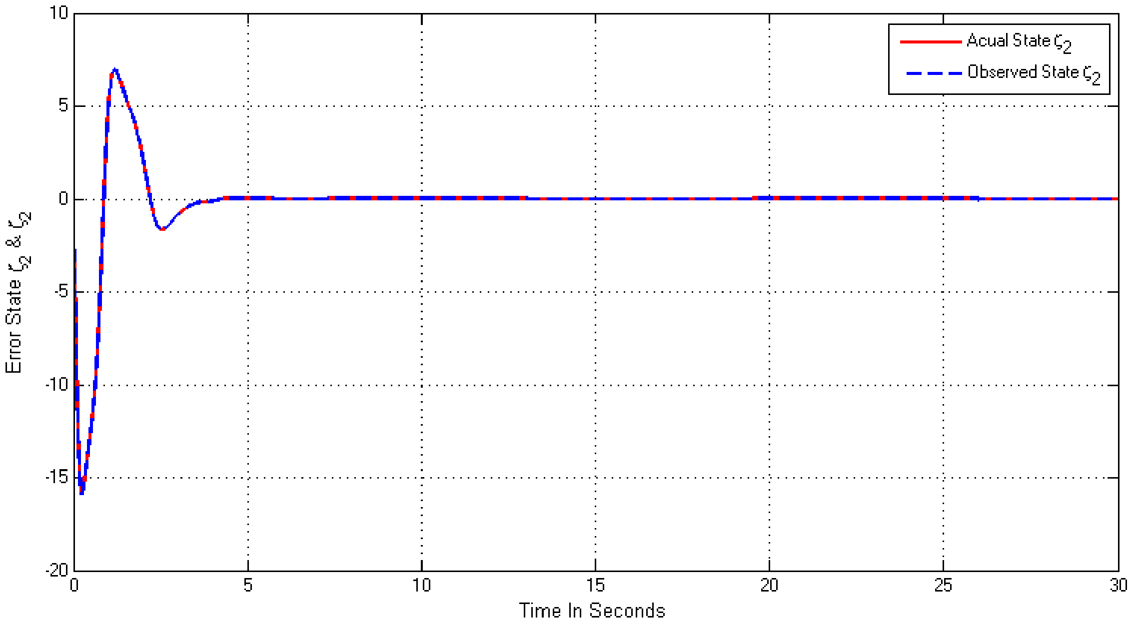

Fig. 3.14 Actual State "ζ_2" and Observed State "$\hat{\zeta}_2$"

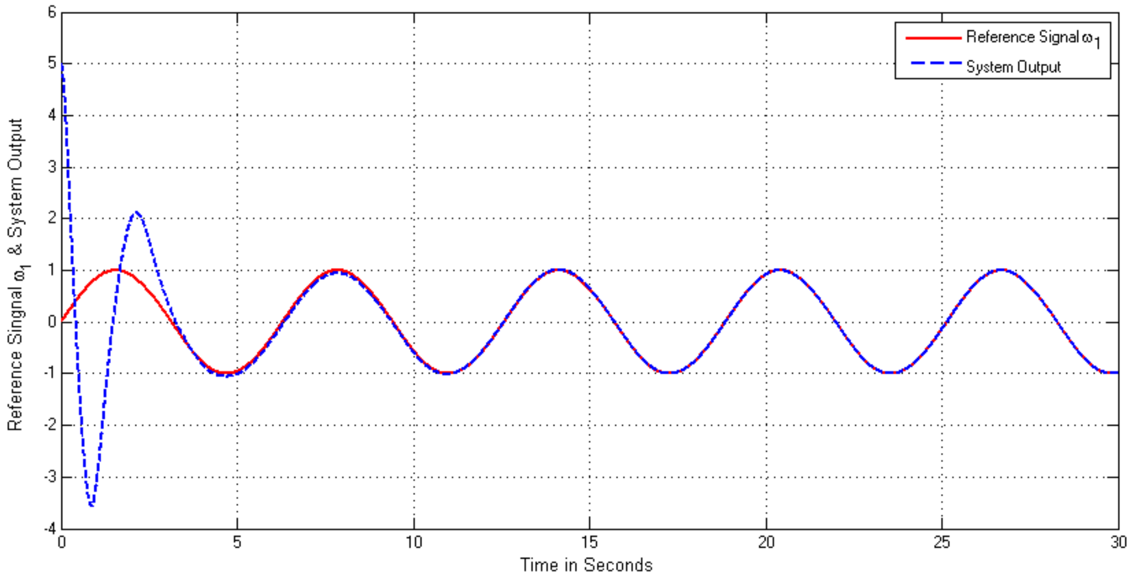

Fig. 3.15 Output Plot between Reference Signal & System Output
$[\zeta_1(0) = 0,\ \zeta_2(0) = 5]$.

Above figures shows the system's response when an optimal based stabilizing control is based on output feedback design instead of state feedback design using EHGO observer i.e. now the states that are used to design the controller is estimated by Extended High Gain Observer. Fig 3.12 shows tracking error "ζ_1" with Cond. Servo based on state feedback Optimal Stabilization Controller & Output Feedback Optimal Stabilization Controller. Similarly, Fig 3.13, 3.14 shows the comparison between actual States and observed state. From the above figures it is evident that by using EHGO the output feedback-based controller recovers the performance provided by state feedback design. Our next task is to investigate the robustness provided by optimal based controller with EHGO.

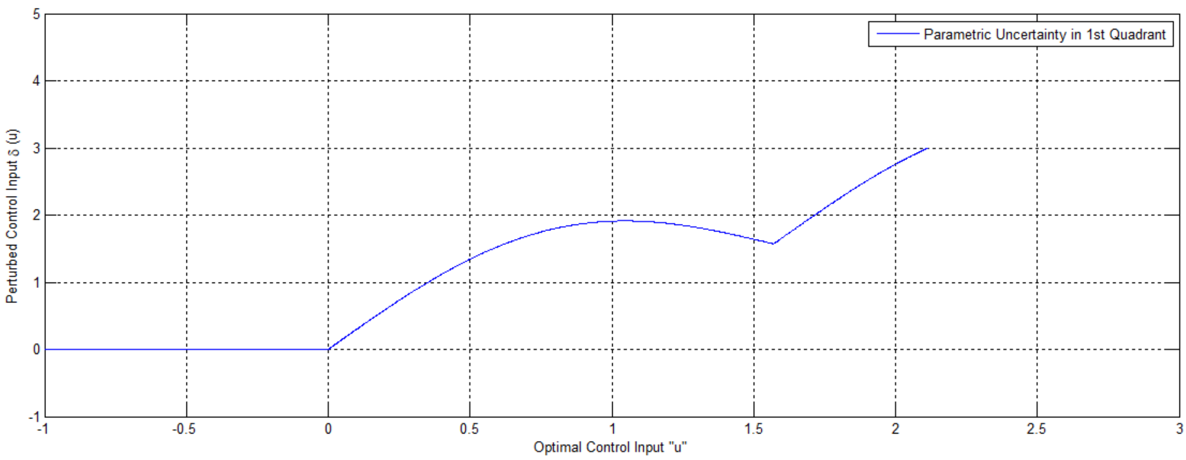

Fig 3.15a: Parametric Uncertainty between control input u and the perturbed control input $\vartheta(u)$

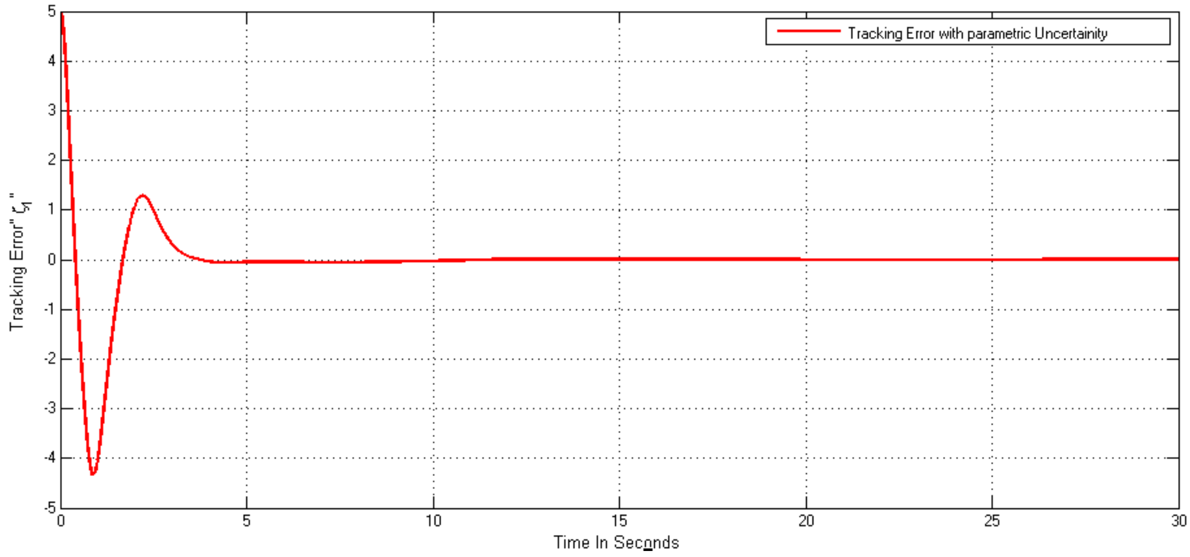

Fig 3.16: Tracking Error with Cond. Servo optimal output feedback-based stabilization controller with perturbed control input $\vartheta(u)$

The above figures show system's response with the system is transformed from state feedback to output feedback and the resultant optimal control is subjected to parametric uncertainty with in sector $(\frac{1}{2}, \infty)$. It is evident from the figures that the robustness provided by optimal stabilization controller holds with EHGO. So, when the control is provided the states observed with EHGO the sector margins holds and the tracking error approaches to zero even with parametric uncertainty.

3.6 Discussion and Conclusions

This thesis is a twofold extension of previous work done by *Attaullah Y. Memon et. al.* [7] ,[22] in a sense that an optimal based output regulation is achieved which has two advantages. First it provides an optimal based stabilization control. A second advantage is that it not only provides optimal control but also provides sector margins which characterize robustness. The idea was first introduced in [22] but not applied to some examples. In previous work a Lyapunov redesign based stabilization controller is designed to achieve output regulation but for instance if the system is nonlinear and the requirement is to achieve output regulation when the system subjected to some parametric uncertainty in that case we have designed an optimal controller for output regulation problem which not only solve the problem of robustness when the system is subjected to some parametric uncertainty but also provides an optimal control to achieve output regulation. This work also proves the idea provided by Freeman R.A Kokotovic [8],[9] for the designing of non linear optimal controller that guarantees robustness in a known region. Apart from that it has also been verified on the other hand that conditional servo-compensator is superior to the conventional servo-compensator which in a way provides the acknowledgement to the earlier work on the design conditional servo-compensator [5], [7]. The output feedback version is also implemented through simple High Gain Observer that recovers the performance of state feedback design very robust which proves the efficacy of the designs.

In the second phase the same design tool is applied to the system which itself posses internal dynamics. The system that is considered in such case is in Normal form and minimum phase. Representing the systems in normal form realizes that we can think of as cascade connecting subsystems. The same optimal based control is applied to the cascade structure for designing the stabilizing compensator and along with the conditional servo-compensator, the combination works to solve the servo-mechanism problem very robustly. The same problem is extended to output feedback-based system where Extended High Gain Observer is used. The results of previous section prove the productivity of the design. Hence, it has been shown that to cater the problem of uncertainties which may arise in the system, the optimal based stabilization controller solves the problem robustly.

Chapter 4

OPTIMAL COND. SERVO – MECHANISM PROBLEM FOR MAGNETIC SUSPENSION SYSTEM

In this chapter we consider the example of magnetic suspension system [20] to check the performance of an optimal output feedback based conditional servo compensator and to verify the robustness provided by nonlinear optimal controller. The rest of the chapter is described as follows. In Section 4.1 we discuss the mathematical model of magnetic suspension system along with its transformation into normal form and with change of variables the variable transformation is performed. In Section 4.2 an optimal feedback control is designed which solves the servo mechanism problem for magnetic suspension system and is followed by the output feedback version of the design using HGO in section 4.3. Section 4.4 consists of simulation results that validates our whole design and finally this chapter completed with the concluding remarks provided in section 4.5.

4.1 Magnetic Suspension System Description

A magnetic suspension system [20] shown in fig 4.1 consists of a ball of a magnetic material which is suspended by an electromagnet. The current of electromagnet is controlled by a feedback signal which is obtained from optically sensor that measures ball position. The equation of motion of ball is

$$\mathfrak{m}\ddot{y} = -k\dot{y} + \mathfrak{m}g - \frac{\mathcal{L}_{0} i^2}{2a\left(1+{}^{y}/_{a}\right)^2} + f_d \qquad (4.1)$$

Where $\mathfrak{m}$ is the mass of the ball, k is the viscous friction, a is the positive constant, g is the acceleration due to gravity, $\mathcal{L}_{0}$ is the inductance of electromagnet, f_d is an external disturbance force and $y \geq \mathbf{0}$ is the downward vertical position of the measured from the reference point $y = \mathbf{0}.$ To define equilibrium points let y^* is the nominal equilibrium point, $\hat{\mathfrak{m}}$ is the nominal mass of the ball at equilibrium condition and the corresponding value of current is i^* that is needed to maintain equilibrium condition. The following equation holds in the equilibrium state.

$$\hat{\mathfrak{m}}g = \frac{a\mathcal{L}_{0} i^*}{2(a+y^*)^2} \qquad (4.2)$$

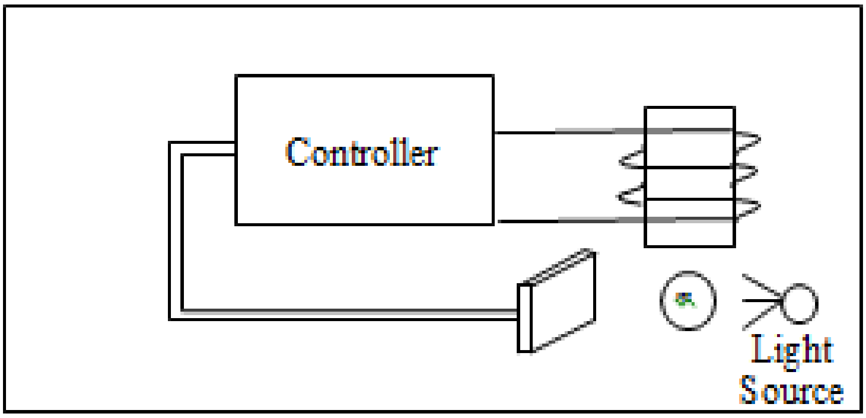

Fig. 4.1: Magnetic Suspension System [20]

To simplify model, the following variables are defined in [21]. Let

$$\xi_1 = \frac{\mathbb{y} - \mathbb{y}^*}{\mathbb{y}^*} \tag{4.3}$$

$$\xi_2 = \frac{\dot{\mathbb{y}}}{\mathbb{y}^* w_n} \tag{4.4}$$

$$u = \frac{i^{*2} - i^2}{i^{*2}} \tag{4.5}$$

$$\tau = w_n t \tag{4.6}$$

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = -\beta\xi_2 + \phi(\xi_1) + \eta(\xi_1)u + \mathfrak{d}(t) \tag{4.7}$$

The functions $\phi(\xi_1), \eta(\xi_1), \beta$ & $\mathfrak{d}$ are defined as

$$\phi(\xi_1) = 1 - \frac{\hat{m}}{m}\left[\frac{(\mathfrak{a} + \mathbb{y}^*)^2}{(\mathfrak{a} + \mathbb{y}^* + \xi_1\mathbb{y}^*)^2}\right] \tag{4.8}$$

$$\eta(\xi_1) = 1 - \phi(\xi_1) \tag{4.9}$$

$$\beta = \frac{k}{m w_n} \tag{4.10}$$

$$\mathfrak{d}(t) = \frac{\mathfrak{f}_d}{m\mathbb{y}^* w_n} \tag{4.11}$$

It is required to balance a ball at a constant reference signal and to reject the known sinusoidal disturbance signal both are generated by known exo system. The constant reference signal to be tracked is denoted by r_0 and the disturbance signal which is to be rejected is

$$d(t) = d_0 \sin(wt)$$

In order to generate these signals, the exo system designed is given by

$$\dot{w} = \begin{bmatrix} 0 & w & 0 \\ -w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (4.12)$$

$$w(0) = \begin{bmatrix} d_0 \\ 0 \\ r_0 \end{bmatrix}$$

The constant reference signal to be tracked $r_0 = w_3$ and the disturbance to be rejected $d_0 = w_1$. The transformation of the system with the change of variables is given by

$$\zeta_1 = \xi_1 - w_3$$
$$\zeta_2 = \xi_2$$
$$\dot{\zeta}_1 = \zeta_2$$
$$\dot{\zeta}_2 = -\beta\zeta_2 + \phi(\zeta_1 + w_3) + \eta(\zeta_1 + w_3)u + \mathfrak{d}(t) \quad (4.13)$$
$$e = \zeta_1$$

The system 4.13 is transformed into a form where the system is converted into state stabilization problem.

4.2 Optimal Conditional Servo Design

In this section of thesis an optimal based conditional servo is designed and the designed control is then applied to a magnetic levitation system discussed above. Th simulation result are shown in the next section which validates our control design for a real-world problem. The results of simulation will provide the efficiency of our control design and will show how it can be used to overcome the problem of robustness in a real-world problem. In order to design an optimal based controller a conditional servo is designed first. A typical conditional servo compensator designed by solving regulator equation provided by (3.5) such that identity (3.5a) holds is given by

$$\mathcal{L}_s^3\phi(\omega) = -\omega^2\mathcal{L}_s\ \phi(\omega) \quad (4.14)$$

The S matrix as defined in (3.5b.1) is given by

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{\rho-1} \end{bmatrix}_{\rho x \rho}, \quad (4.15)$$

where $c_1 = -\omega^2$

Hence the $\mathcal{S}$ matrix become

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{bmatrix} \quad (4.16)$$

with $\omega = 2$ the $\mathcal{S}$ matrix becomes

$$\mathcal{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix} \quad (4.17)$$

and

$$J = [0 \quad 0 \quad 1]^T \tag{4.18}$$

To fulfil the requirement of Huwritzness the K matrix is designed by placing the eigen values in the left half plane. By placing the eigen values of $(\mathcal{S} - JK_1)$ at -0.5, -1 and -1.5 the K_1 matrix becomes

$$K_1 = [0.75 \quad -1.25 \quad 3] \tag{4.19}$$

The conditional servo compensator is designed by using (3.5c) and (3.5d) and the control input u is taken as discussesd in conventional compensator as

$$\dot{\varrho} = (S - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right) \tag{4.20}$$

$$u = -\mathfrak{a}(\tilde{x}) sat\left(\frac{s}{\mu}\right) \tag{4.21}$$

Where,

$$s = \mathscr{v}(\zeta) + K_1\varrho \tag{4.22}$$

Where $\mathscr{v}(\zeta)$ is the stabilizing compensator that brings the trajectory to a manifold. The next task is to designed $\mathscr{v}(\zeta)$ as an optimal one. The next task is to design an optimal stabilizing controller to stabilize the system. The optimal control methods that are designed by kokotovic and some other researchers [8] are used for designing optimal stabilization controller but an inverse approach is used just because it is difficult to solve Hamilitian Jacobian Bellman (HJB) equation. The method for finding this control law is discussed in chapter 2.

For designing a control law using an inverse method first a control Lyapunov function is needed in order to achieve sector margin [8]. For this system a Lyapunov function candidate is given by,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.23}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.24}$$

$$P = \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \tag{4.25}$$

The Recatti inequality which is given by

$$\mathbb{A}^{\mathbb{T}}\mathbb{P} + \mathbb{P}\mathbb{A} - \mathbb{P}\mathbb{B}\mathbb{B}^{\mathbb{T}}\mathbb{P} < \mathbb{O} \tag{4.26}$$

This equation hold only if $c \in (0,1)$. The control Lyapunov function for the nominal non linear system is given by

$$\mathbb{V} = \mathbb{x}^{\mathbb{T}}\mathbb{P}\mathbb{x}$$

For this levitation system the control Lyapunov Function becomes

$$\mathbb{V} = \zeta^{\mathbb{T}}\mathbb{P}\zeta \tag{4.27}$$

The Lyapunov function becomes

$$\mathbb{V}(\zeta) = (\zeta_1 + c\zeta_2)\zeta_1 + (\zeta_2 + c\zeta_1)\zeta_2$$

$$\mathbb{V}(\zeta) = a\zeta_1 + b\zeta_2 \tag{4.28}$$

where $\mathbb{a}=\zeta_1 + c\zeta_2$ & $\mathbb{b}'=\zeta_2 + c\zeta_1$.

The negative definiteness condition (2.15) is satisfied with this condition. With the use of CLF an optimal stabilizing controller can be find out by using Sontag's formula (2.18)

$$z(\zeta) = \begin{Bmatrix} c_0 + \dfrac{a(\zeta) + \sqrt{a^2(\zeta) + \left(b^T(\zeta)b(\zeta)\right)^2}}{\left(b^T(\zeta)b(\zeta)\right)}, & b_\zeta \neq 0 \\ c_0 \qquad , & b_\zeta = 0 \end{Bmatrix} \tag{4.29}$$

Where $a(\zeta) = L_f V(\zeta)$ and $b(\zeta) = (L_g V(\zeta))^T$. The optimal stabilizing control $\mathcal{v}(\zeta)$ is given as

$$\mathcal{v}(\zeta) = -\frac{1}{\eta(\zeta_1 + \mathcal{w}_3)}\left[\frac{2\mathbb{a}\zeta_2}{2\mathbb{b}'} + \left(-\beta\zeta_2 + \phi(\zeta_1 + \mathcal{w}_3)\right) + \sqrt{\frac{2a\zeta_2}{2\mathbb{b}'} + (-\beta\zeta_2 + \phi(\zeta_1 + \mathcal{w}_3))^2 + 4\mathbb{b}'^2(\eta(\zeta_1 + \mathcal{w}_3))^4}\right] \tag{4.30}$$

$$J = \int_0^\infty (\frac{1}{2}z(\zeta)b^T(\zeta)b(\zeta) + \frac{1}{2z(\zeta)}u^T R(\zeta)u)dt \tag{4.31}$$

Where, $b(\zeta) = (L_g V(\zeta))^T$ &

$$z(\zeta) = -\frac{1}{\eta(\zeta_1+\mathcal{w}_3)}\left[\frac{2\mathbb{a}\zeta_2}{2\mathbb{b}'} + \left(-\beta\zeta_2 + \phi(\zeta_1 + \mathcal{w}_3)\right) + \sqrt{\frac{2a\zeta_2}{2\mathbb{b}'} + (-\beta\zeta_2 + \phi(\zeta_1 + \mathcal{w}_3))^2 + 4\mathbb{b}'^2(\eta(\zeta_1 + \mathcal{w}_3))^4}\right] \tag{4.32}$$

The specialty of this control law is that with this control law a sector margin of (1/2,∞) can be achieved. The simulation results provided in the section 4.4 will show how an inverse optimal control provides robustness this also proves the idea provided by kokotovic and some other researchers.

4.3 High Gain Observer Design

The previous all designs were based on state feedback based conditional servo where all states were available for the feedback. In this design a practical case is considered in which only output is available for feedback hence the problem now transforms from state feedback to the output feedback. Here in this case the information of only ζ_1 is provided while the state ζ_2 is missing. For this case $\hat{\zeta}_2$ which is the observed state is found by using non linear high gain observer HGO an idea provided by Khalil.[20]. The observer designed is given by

$\hat{\zeta}_1 = \hat{\zeta}_2 + \frac{g_1(\zeta_1 - \hat{\zeta}_1)}{\epsilon}$

$$\hat{\zeta}_2 = \frac{g_2(\zeta_1 - \hat{\zeta}_1)}{\epsilon^2} \tag{4.33}$$

The values of g_1 and g_2 is chosen such that the roots of the polynomial equation

$$\Delta^2 + g_1\Delta + g_2$$

lies in the left half of the plane, such that the polynomial equation is Hurwitz. The control law is now transformed from state feedback to output feedback. Hence the output feedback version of the optimal stabilizing controller is given by

$$v(\zeta) = -\frac{1}{\eta(\zeta_1 + w_3)}\left[\frac{2a\hat{\zeta}_2}{2b'} + \left(-\beta\hat{\zeta}_2 + \phi(\zeta_1 + w_3)\right) + \sqrt{\frac{2a\hat{\zeta}_2}{2b'} + (-\beta\hat{\zeta}_2 + \phi(\zeta_1 + w_3))^2 + 4b'^2(\eta(\zeta_1 + w_3))^4}\right] \tag{4.34}$$

So the overall control is given as

$$\dot{\varrho} = (S - JK_1)\varrho + \mu J sat\left(\frac{s}{\mu}\right) \tag{4.35}$$

$$u = -\alpha(\tilde{x}) sat\left(\frac{s}{\mu}\right) \tag{4.36}$$

Where,

$$s = v(\zeta) + K_1\varrho \tag{4.37}$$

Now the term $v(\zeta)$ which is the stabilizing compensator that bring the trajectories into zero error manifold is provided in (4.34). Once the trajectories enter into zero error manifold then the regulation takes the control and tracks the reference signal and rejects the difference signal. This is the practical scenario since the state ζ_2 is not directly available for measurement.

4.4 Simulations & Results

This section of thesis will demonstrate the above discussed control mechanism when applied to a nonlinear magnetic suspension system with the help of simulation. The results of simulation will provide the efficiency of our control design and will show how it can be used to overcome the problem of robustness for a magnetic levitation system.
The system described in (4.13) is given by

$$\dot{\zeta}_1 = \zeta_2$$
$$\dot{\zeta}_2 = -\beta\zeta_2 + \phi(\zeta_1 + \mathscr{w}_3) + \eta(\zeta_1 + \mathscr{w}_3)u + \mathfrak{d}(t)$$
$$e = \zeta_1$$

$$\phi(\xi_1 + \mathscr{w}_3) = 1 - \frac{\widehat{\mathfrak{m}}}{\mathfrak{m}}\left[\frac{(\mathfrak{a} + \mathfrak{y}^*)^2}{(\mathfrak{a} + \mathfrak{y}^* + \ (\xi_1 + \mathscr{w}_3)\mathfrak{y}^*)^2}\right]$$

$$\eta(\xi_1 + \mathscr{w}_3) = 1 - \ \phi(\xi_1 + \mathscr{w}_3)$$

$$\beta = \frac{\mathscr{k}}{\mathfrak{m}\mathscr{w}_{\mathfrak{n}}}$$

$$\mathfrak{d}(t) = \frac{\mathfrak{f}_d}{\mathfrak{m}\mathfrak{y}^*\mathscr{w}_{\mathfrak{n}}}$$

Our goal is to track a constant reference signal and to reject the disturbance signal optimally and robustly. The output to be tracked is the position of the ball which is to be attained at a constant position. The following values are used in simulation $\widehat{\mathfrak{m}} = \mathfrak{m}$ =0.2kg, $\mathscr{k}$ =0.001 ,$\mathscr{w}_{\mathfrak{n}}$ =2 rad/sec, $\mathfrak{y}^* = 0.1m$, $\mathfrak{f}_d = 1$, $\mathscr{g}$ =9.81 m/s, $\mathcal{L}_{\mathbb{0}} = 1, \mathfrak{a} = 0.01$. Fig 4.2 shows the system's response when subjected to optimal stabilizing control but the system is not subjected to any disturbance or reference signal. It can be seen that the stabilizing control stabilizes the system and (fig 4.3) shows the comparison between the two controllers one with simple feedback based stabilization controller and other with optimal control that minimizes the cost function.

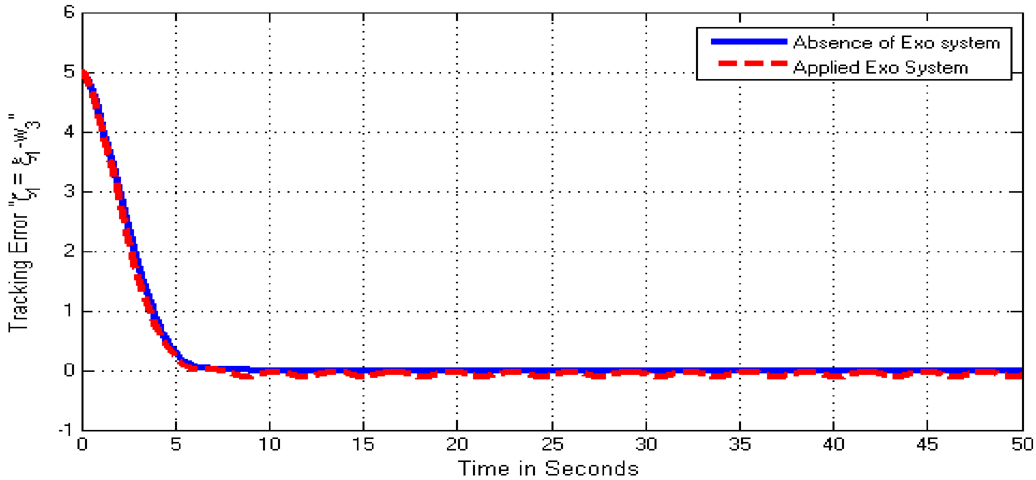

Fig. 4.2: Stabilization Controller Response without the action of exo-system.

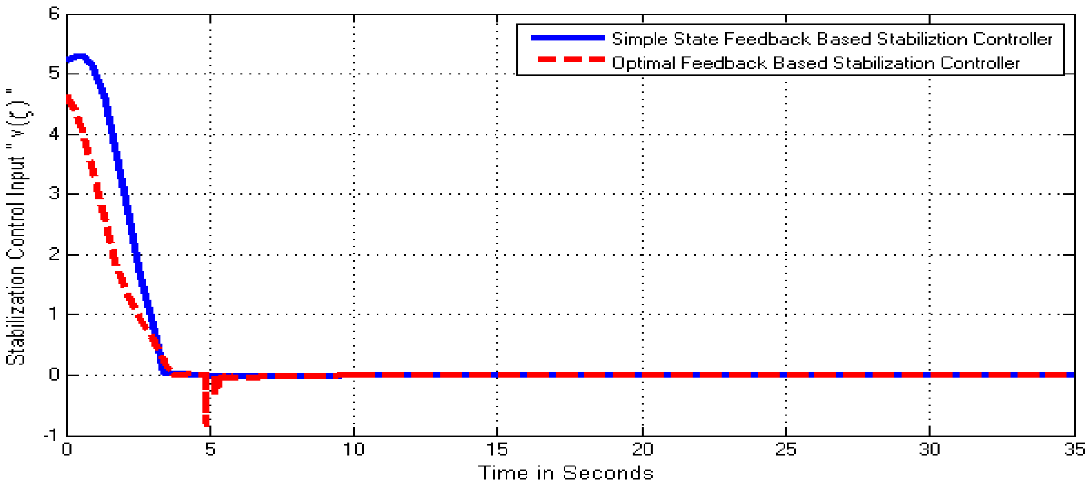

Fig. 4.3: Stabilizing Control Output $v(\zeta)$ Comparison between Optimal and simple Feedback Based Controller which stabilizes the system

Our task is to track the constant reference signal while rejecting the sinusoidal disturbance signal. Fig 4.4 shows the system's response when a constant reference signal and disturbance signal is applied to the system. Clearly, the output is not tracking the reference signal and system's response goes unstable. This clearly indicates that there is a need of servo compensator which brings the error back to zero achieving zero steady state error while rejecting the disturbance signal. On the other hand fig 4.5 shows the tracking error when the optimal based conditional servo is applied to the system and simple feedback based stabilizing controller is applied. It can be seen that the error asymptotically approaches to zero while in fig (4.6) the output which is the position of the ball tracks the constant reference signal in the presence of sinusoidal disturbance signal.

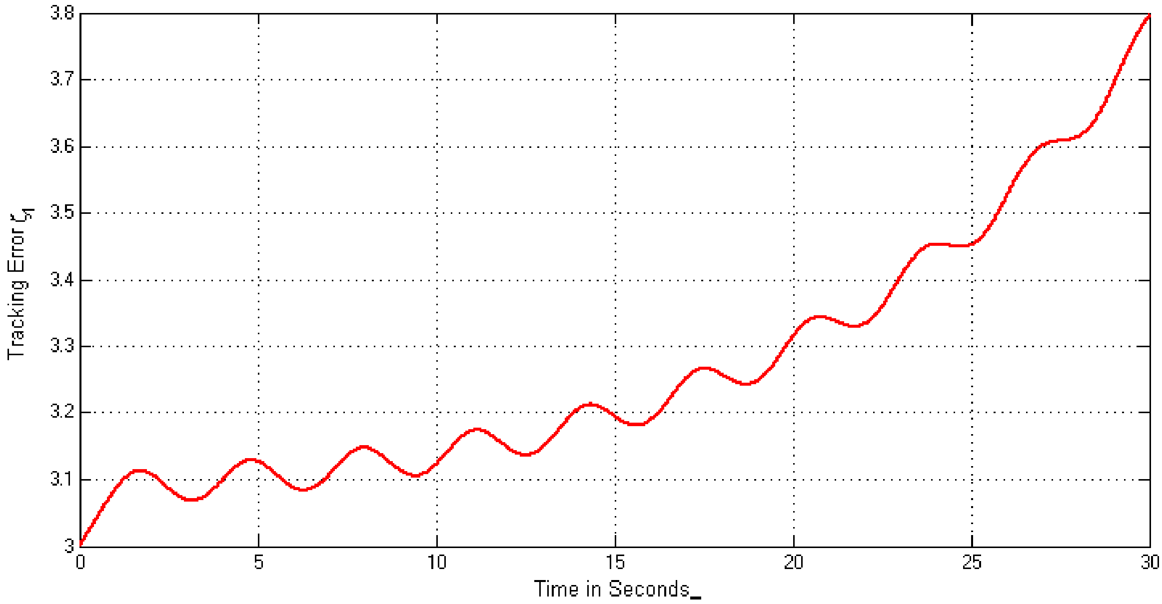

Fig. 4.4: Tracking error without Cond. Servo Compensator

On the other hand fig 4.5 shows the comparison between the tracking error when the optimal based conditional servo is applied to the system and simple feedback based stabilization control is applied . It can be seen that the error asymptotically approaches to zero with enhanced performance in comparison with simple feedback based stabilization controller while in fig (4.6) the output which is the position of the ball tracks the constant reference signal in the presence of sinusoidal disturbance signal. Both are generated by a known exo system. In fig (4.7) the overall control input is plotted. Since the stabilizing compensator is optimal hence the control input can be seen here is optimal control which minimizes the cost function.

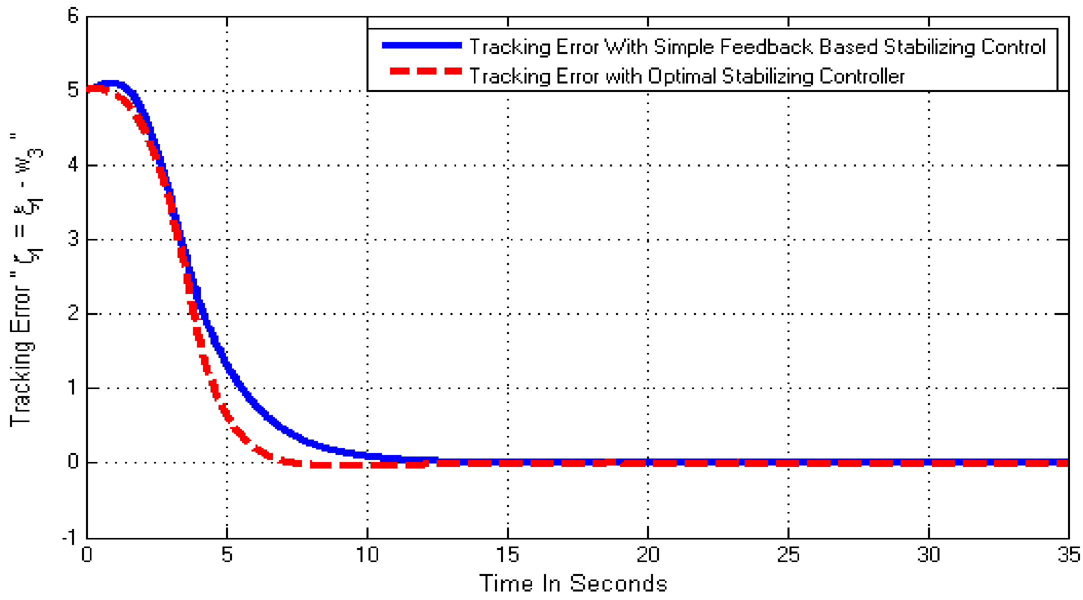

Fig. 4.5: Tracking error Comparison "ζ_1" with Cond. Servo based on Optimal vs Ssimple Feedback Based Stabilization Controller $v(\zeta)$.

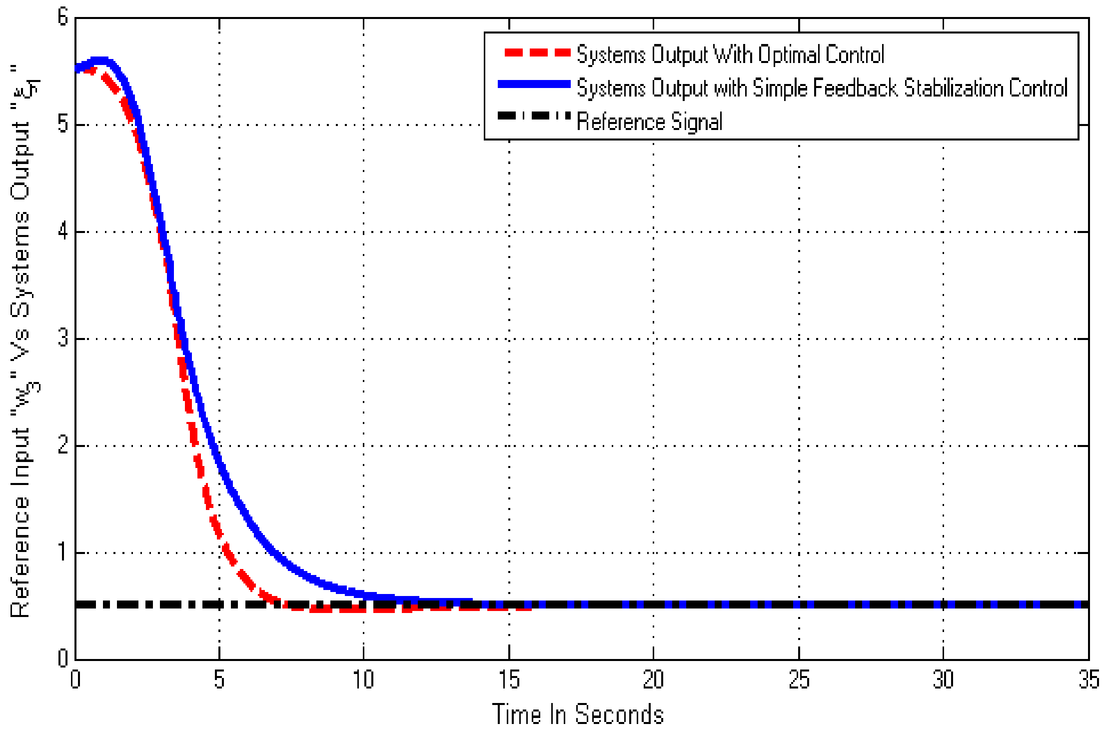

Fig. 4.6: Conditional Servo. Plot of System Output "ξ_1".

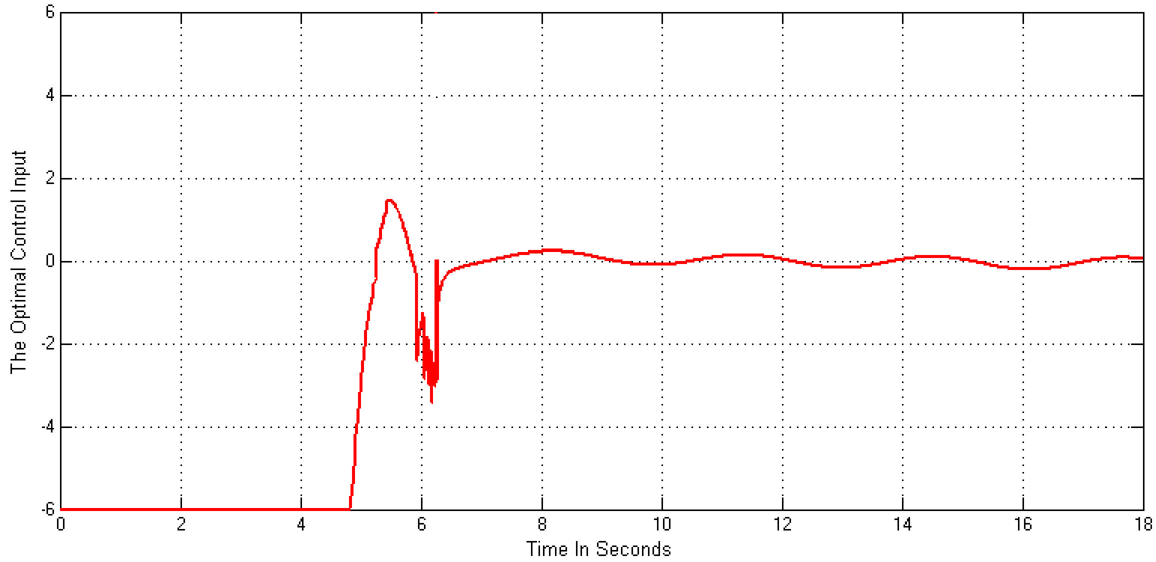

Fig. 4.7: Optimal Output regulation Control Input u.

The optimal control methods that are developed by Kokotovic and some other researchers guarantees stability margins under certain conditions. Hence our next task is to check the robustness of the system. The robustness is checked by applying parametric uncertainty which lies in the sector $(1/2,\infty)$. Fig 4.8 shows the uncertainty that is applied to the magnetic suspension system to check the robustness provided by inverse optimal controller. Fig 4.9 shows that with the perturbed control input that lies in the sector $(1/2,\infty)$ the optimal stabilization controller sustains the uncertainty and the magnetic levitation system achieves zero steady state error and the output shown in figure 4.10 asymptotically tracks the reference signal. The results of it satisfies that the proposed control law achieves optimality and robustness in the presence of parametric uncertainty.

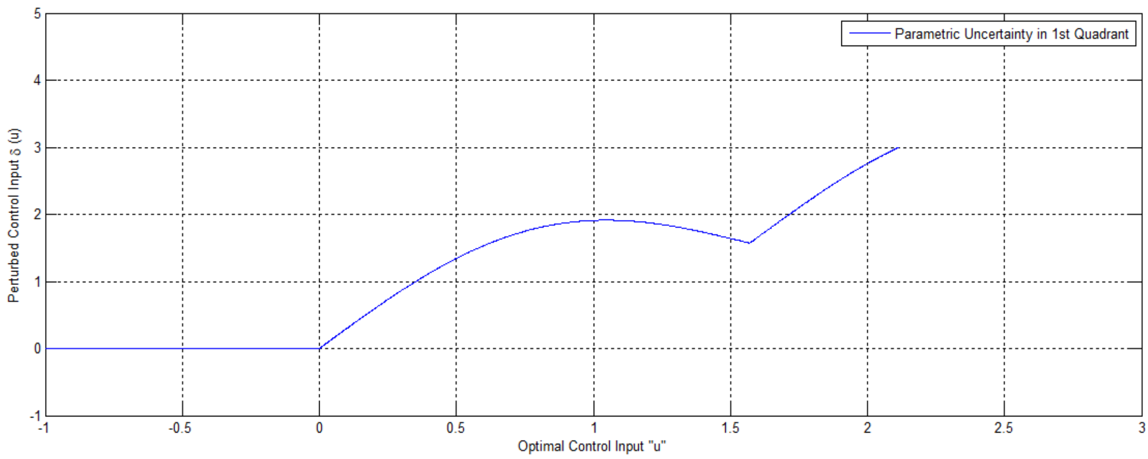

Fig 4.8: Parametric Uncertainty between control input u and the perturbed control input $\vartheta(u)$

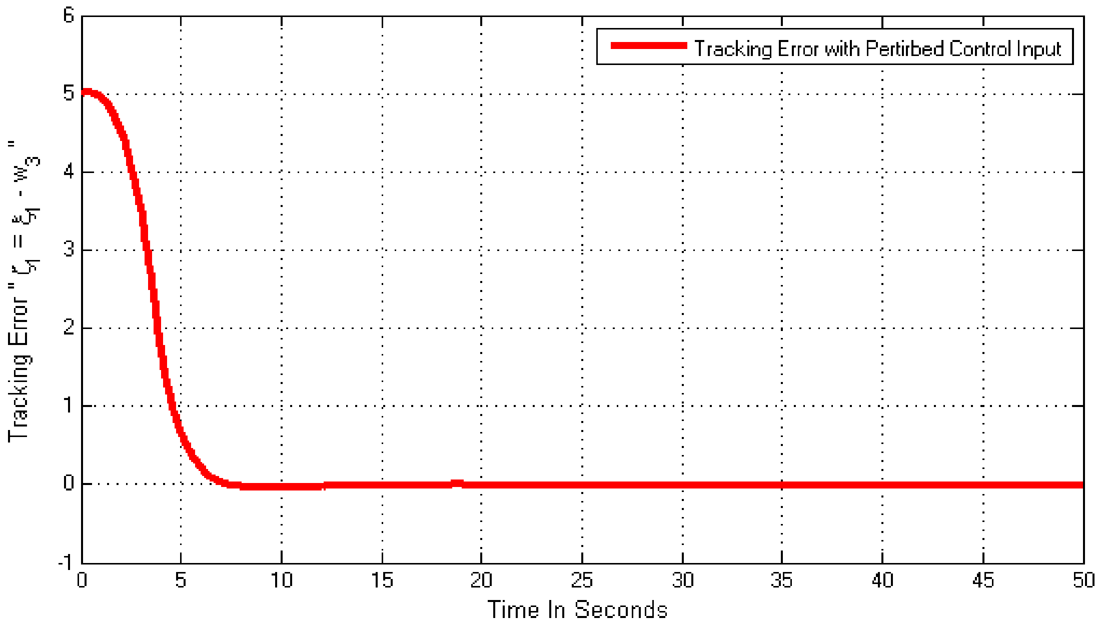

Fig 4.9: Tracking Error with Cond. Servo optimal output feedback-based stabilization controller with perturbed control input $\vartheta(u)$

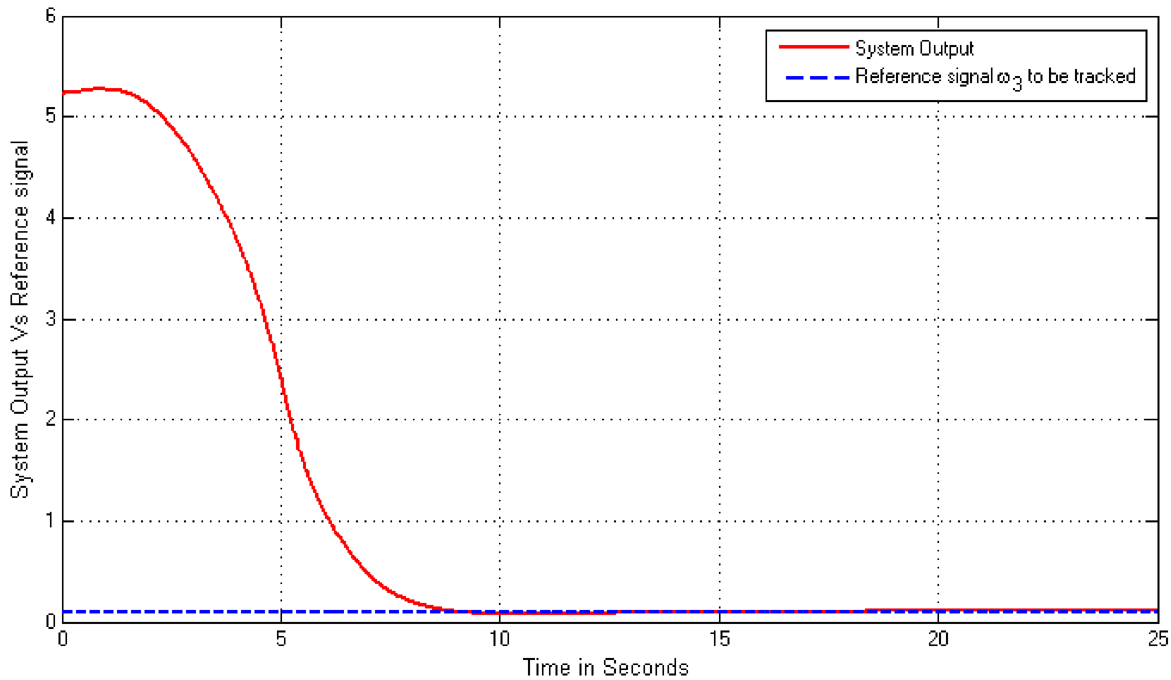

Fig. 4.10: Output Plot between Reference Signal & System Output

The previous all designs were based on state feedback based conditional servo where all stated were available for the feedback. In the next part we will consider a practical case in which only output is available for feedback hence the problem now transforms from state feedback to the output feedback. Here in this case the information of only ζ_1 is provided while the state ζ_2 is missing. For this case $\hat{\zeta}_2$ which is the observed state is found by using non linear high gain observer HGO an idea provided by Khalil.[20]. The observer designed is given in section 4.3.

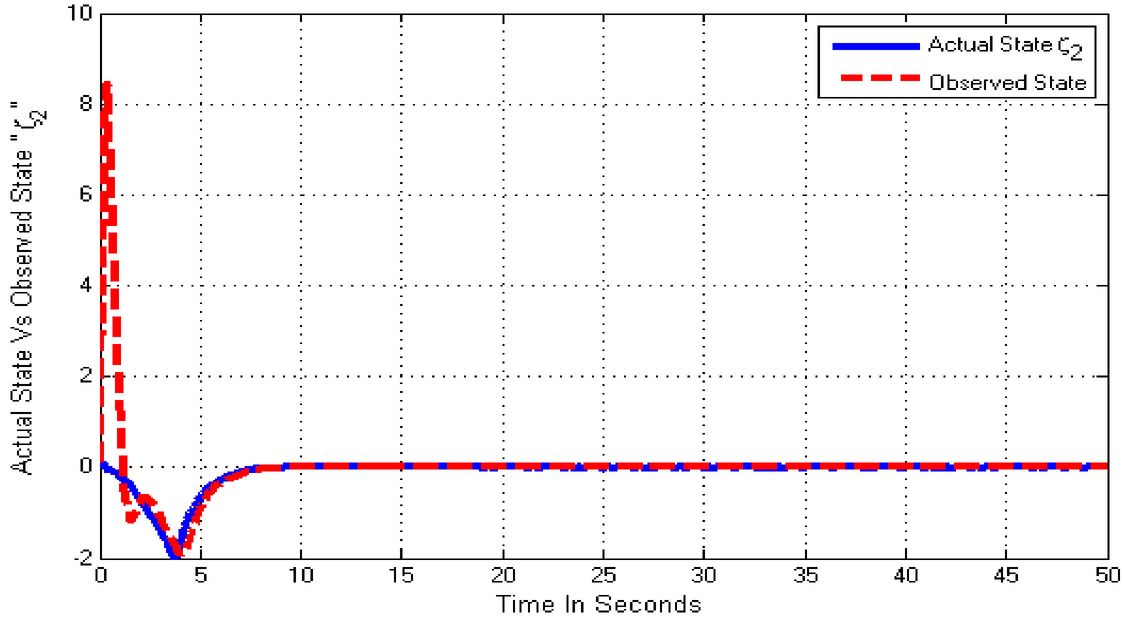

Fig. 4.11 Actual State "ζ_2" and Observed State "$\hat{\zeta}_2$"

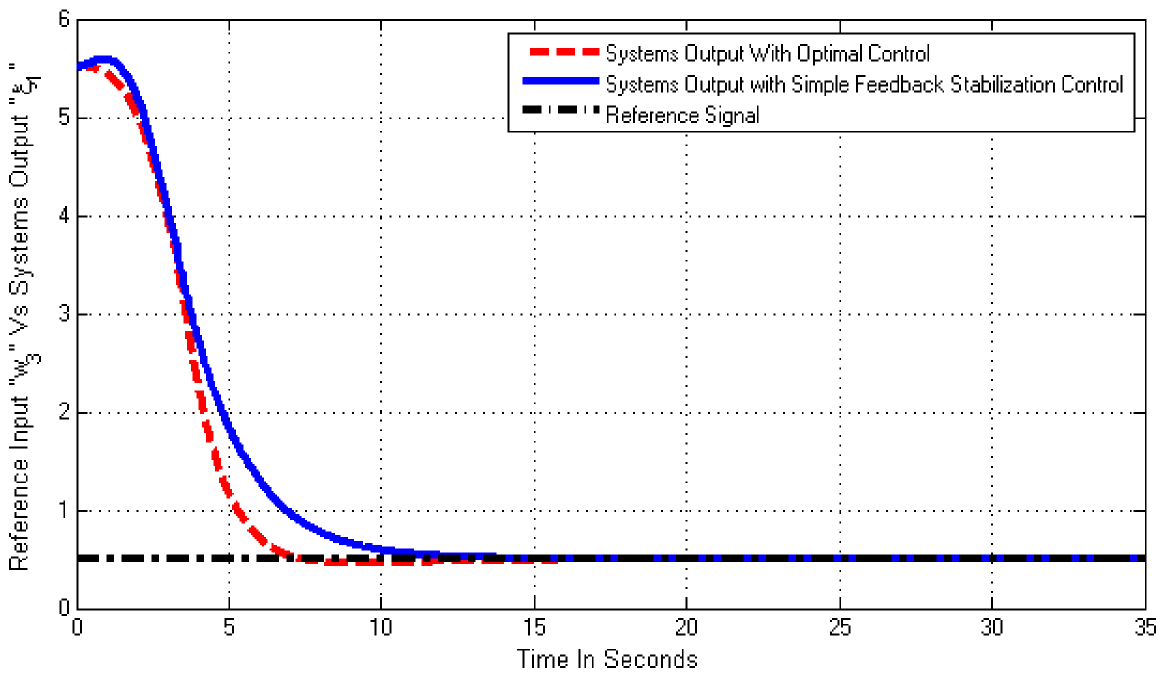

Fig. 4.12: Output Plot between Reference Signal & System Output

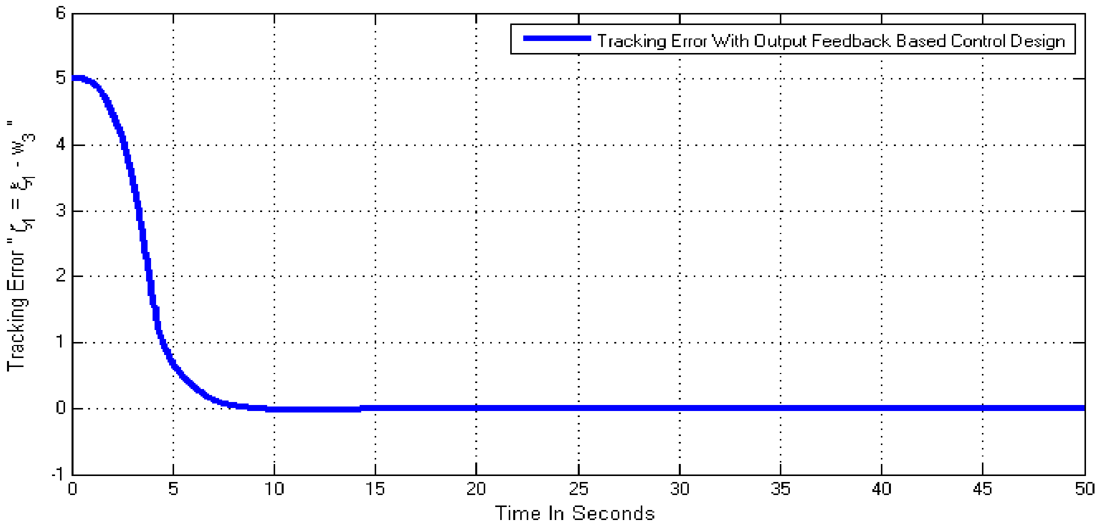

Fig. 4.13: Tracking Error "ζ_1" based on output feedback based optimal Output regulation

Above figures shows system's response when the optimal based stabilizing control is based on output feedback design instead of state feedback design using HGO observer i.e. now the states that are used to design the controller is estimated by High Gain Observer. Fig 4.13 shows tracking error "ζ_1" with Cond. Servo based on state feedback Optimal Stabilization Controller & Output Feedback Optimal Stabilization Controller. Similarly, Fig 4.11 shows the comparison between actual State "ζ_2" and observed state $\hat{\zeta}_2$. The output of the system asymptotically tracks the reference signal as shown in fig 4.12. From the above figures it is clear that with the use of HGO the performance provided by state feedback design is same as we get with output feedback-based controller by using HGO. By the choice of design parameters it recovers the performance as we get with output feedback based controller.

The conditional servo control based on inverse optimal stabilization controller which is discussed in the previous chapter is applied to magnetic suspension system. Through suitable variable transformation the magnetic suspension system is converted to a form where conditional servo can be applied. The optimal based stabilization control an idea provided by Freeman R.A Kokotovic [8], [9] for the designing of nonlinear optimal controller that guarantees robustness in a known region is used for stabilizing compensator. The whole control law i.e. stabilizing + servo compensator is applied to a magnetic levitation system in order to achieve regulation. The simulation results verify that the optimal based conditional servo control law for magnetic levitation system not only achieve regulation but also provides optimality with respect to cost function and the robustness in terms of sector margin.

4.5 Discussion & Conclusions

In thesis we focus on the solution of the servo-mechanism problem for a certain class of nonlinear system with the principle objective to show optimality and robustness. In our approach we utilized the idea of conditional servo-compensator provided by *Attaullah Memon* [7], where the Lyapunov Redesign method is used to design a stabilizing compensator. The optimal based stabilizing compensator is designed using an approach developed by kokotovic and other researchers [8], [9] and [10] and is then integrated with conditional servo based design.. Finally realizing a real world scenario where all states are not available for feedback the output feedback based design is implemented with the use of HGO and EHGO [11]. This work is the extension of earlier work of *Attaullah Memon* [22].

We start our discussion with the introduction in chapter 1 followed by some preliminaries in chapter 2 that are necessary to illustrate the concept involved in the understanding of whole work. The problem of optimal conditional servo was formulated in chapter 3 in which we considered a nonlinear system when it is desired to have regulation optimally and robustly. We have provided an extension to the reference work [7], [22] in a sense that the stabilizing compensator of the conditional servo-compensator design is designed using the technique of Inverse Optimal Control and applied to a nonlinear system. With this we are able to achieve optimal output regulation but also some sector margins that are used to characterize system's robustness. . The output feedback version of such design is implemented by the simple Extended High Gain Observer (EHGO). The simulation results provided at the end of that chapter proves the efficacy of the design.

In chapter 4, we have applied our approach developed in the chapter 3 to the practical nonlinear system known as magnetic levitation system. Through suitable variable transformation the magnetic suspension system is converted to a form where conditional servo can be applied.

Once, the system is transformed into the normal form, we have shown that our approach of optimal based conditional servo-compensator can be applied which yields robust results. Both the state feedback designs as well as the output feedback designs using EHGO are implemented and the results are shown at the end of Chapter 4.

Our observation during the whole design process is that the optimal based control design technique requires to solve a famous equation namely Hamilton Jacobean Bellman Equation (HJB) a partial differential equation which is a very complex task to solve if the system is of higher order. An Inverse approach developed by Kokotovic and other researcher is applied which is called an inverse optimal control which is used to solve optimal control problem which exempt the requirement for solving the HJB equation. In inverse approach an optimal stabilizing control is designed first and that control is then shown to be optimal for particular cost function. The desire of particular cost function is achieved by designing the control law first and checking the cost so by changing a control lyapunov function we were able to achieve the cost function.

This research can further be extended by incorporating some new control design techniques like sub optimal control through which the cost can further be minimized. Furthermore, a promising direction would be to extend this approach towards the non-minimum phase systems subjected to the control constraints, defining their operation region and increasing their region of attraction to making the design work globally.

Bibliography

[1]. E. J. Davison, *"The robust control of servo-mechanism problem for linear time invariant multivariable systems"*, Published in IEEE Transactions on Automatic Control, Vol. 21, pp. 25-34, 1976.

[2]. B. A. Francis and W. M. Wonham, *"The internal model principle of control theory"*, Published in Automatica, Vol. 12, pp. 457-465, 1976.

[3]. N. Minorsky, *"Directional stability of automatically steered bodies"*, Published in American Society Journal of Naval Engg., Vol. 34, pp. 280-309, 1922.

[4]. S. Seshagiri and H. K. Khalil, *"Robust output feedback regulation of minimum phase nonlinear systems using conditional integrators"*, Published in Automatica, Vol. 41, Issue 1, pp. 43-54, 2003.

[5]. S. Seshagiri and H. K. Khalil, *"Robust output feedback regulation of minimum phase nonlinear systems using conditional servo-compensators"*, Published in International Journal of Robust and Nonlinear Control, Vol. 15, pp. 83-102, 2005.

[6]. A. Singh and H. K. Khalil, *"Regulation of nonlinear systems using conditional integrators"*, Published in International Journal of Robust and Nonlinear Control, Vol. 15, pp. 339-362, 2005.

[7]. A. Y. Memon and H. K. Khalil, *"Lyapunov redesign approach to output regulation of nonlinear systems using conditional servo-compensators"*, Published in the proceedings of American Control Conference, Washington USA, June 11-13, pp. 395-400, 2008.

[8]. Freeman R.A Kokotovic .P..”*Optimality of Robust Non Linear Feedback Controls"* Volume-34, Issue 4, pp. 1365-1391, 1996.

[9]. Freeman R.A Kokotovic .P..”*Robust Control of Non Linear Systems",* Birkhauser, Boston, 1996.

[10]. Sepulchre. R. Jankovic,M., and Kokotovic,P.., *"Constructive Non Linear Control"* Springer, London, 1997.

[11]. A. M. Boker and H. K. Khalil, *"Nonlinear observers comprising high-gain observers and extended Kalman filters"*, Published in Automatica, Vol. 49, pp. 3583-3590, 2013.

[12]. B.D.O. Anderson, J.B. Moore, *"Optimal Control: Linear Quadratic Methods"* Prentice-Hall, Englewood Cliffs, NJ, 1990.

[13]. Z. Artstein, “Stabilization with relaxed controls,” Nonlinear Analysis, vol 7, pp. 1163-1173,1983

[14]. E.D. Sontag, "*A Lyapunov like characterization of asymptotic controllability*" SIAM J. on Control and Optimization, vol. 21, pp. 462-471,1983

[15]. A. Isidori, *"Nonlinear Systems"*, Third Edition, Book Published by Springer-Verlag, New York, 1995.

[16]. H. K. Khalil, *"On the design of robust servomechanisms for minimum phase nonlinear systems"*, Published in International Journal of Robust and Nonlinear Control, Vol. 10, pp. 339-361, 2000.

[17]. A. Y. Memon and H. K. Khalil, *"Output regulation of nonlinear systems using conditional servocompensators"*, Published in Automatica, Vol. 6, pp. 1119-1128, 2010.

[18]. W. Wang and Z. Ghao, *"A comparison study of advanced state observer design techniques"*, Published in the proceedings of American Control Conference, Vol. 6, pp. 4754-4759, 2003.

[19]. A. M. Boker and H. K. Khalil, *"Nonlinear observers comprising high-gain observers and extended Kalman filters"*, Published in Automatica, Vol. 49, pp. 3583-3590, 2013.

[20]. Khalil H. K., "*Nonlinear Control Systems"*, 3rd edition, Prentice-Hall: Upper Saddle River, New Jersey, 2002.

[21]. Attaullah Y. Memon "*Regulation of Nonlinear Systems Using Conditional Servocompensators*" Thesis Michigan State University.

[22]. Attaullah Y. Memon *"Optimal Output Regulation of Minimum Phase Nonlinear Systems"* UKACC International Conference on Control 2012 Cardiff, UK, 3-5 September 2012.