Fundamentals of Airplane Flight Mechanics

David G. Hull

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Flight Mechanics

With 125 Figures and 25 Tables

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Dedicated to

Angelo Miele

who instilled in me his love for flight mechanics.

Preface

Flight mechanics is the application of Newton's laws (F=ma and M=Iα) to the study of vehicle trajectories (performance), stability, and aerodynamic control. There are two basic problems in airplane flight mechanics: (1) given an airplane what are its performance, stability, and control characteristics? and (2) given performance, stability, and control characteristics, what is the airplane? The latter is called airplane sizing and is based on the definition of a standard mission profile. For commercial airplanes including business jets, the mission legs are take-off, climb, cruise, descent, and landing. For a military airplane additional legs are the supersonic dash, fuel for air combat, and specific excess power. This text is concerned with the first problem, but its organization is motivated by the structure of the second problem. Trajectory analysis is used to derive formulas and/or algorithms for computing the distance, time, and fuel along each mission leg. In the sizing process, all airplanes are required to be statically stable. While dynamic stability is not required in the sizing process, the linearized equations of motion are used in the design of automatic flight control systems.

This text is primarily concerned with analytical solutions of airplane flight mechanics problems. Its design is based on the precepts that there is only one semester available for the teaching of airplane flight mechanics and that it is important to cover both trajectory analysis and stability and control in this course. To include the fundamentals of both topics, the text is limited mainly to flight in a vertical plane. This is not very restrictive because, with the exception of turns, the basic trajectory segments of both mission profiles and the stability calculations are in the vertical plane. At the University of Texas at Austin, this course is preceded by courses on low-speed aerodynamics and linear system theory. It is followed by a course on automatic control.

The trajectory analysis portion of this text is patterned after Miele's flight mechanics text in terms of the nomenclature and the equations of motion approach. The aerodynamics prediction algorithms have been taken from an early version of the NASA-developed business jet sizing code called the General Aviation Synthesis Program or GASP. An important part of trajectory analysis is trajectory optimization. Ordinarily, trajectory optimization is a complicated affair involving optimal control theory (calculus of variations) and/or the use of numerical optimization techniques. However, for the standard mission legs, the optimization problems are quite simple in nature. Their solution can be obtained through the use of basic calculus.

The nomenclature of the stability and control part of the text is based on the writings of Roskam. Aerodynamic prediction follows that of the USAF Stability and Control Datcom. It is important to be able to list relatively simple formulas for predicting aerodynamic quantities and to be able to carry out these calculations throughout performance, stability, and control. Hence, it is assumed that the airplanes have straight, tapered, swept wing planforms.

Flight mechanics is a discipline. As such, it has equations of motion, acceptable approximations, and solution techniques for the approximate equations of motion. Once an analytical solution has been obtained, it is important to calculate some numbers to compare the answer with the assumptions used to derive it and to acquaint students with the sizes of the numbers. The Subsonic Business Jet (SBJ) defined in App. A is used for these calculations.

The text is divided into two parts: trajectory analysis and stability and control. To study trajectories, the force equations (F=ma) are uncoupled from the moment equations (M=Iα) by assuming that the airplane is not rotating and that control surface deflections do not change lift and drag. The resulting equations are referred to as the 3DOF model, and their investigation is called trajectory analysis. To study stability and control, both F=ma and M=Iα are needed, and the resulting equations are referred to as the 6DOF model. An overview of airplane flight mechanics is presented in Chap. 1.

Part I: Trajectory Analysis. This part begins in Chap. 2 with the derivation of the 3DOF equations of motion for flight in a vertical plane over a flat earth and their discussion for nonsteady flight and quasi-steady flight. Next in Chap. 3, the atmosphere (standard and exponential) is discussed, and an algorithm is presented for computing lift and drag of a subsonic airplane. The engines are assumed to be given, and the thrust and specific fuel consumption are discussed for a subsonic turbojet and turbofan. Next, the quasi-steady flight problems of cruise and climb are analyzed in Chap. 4 for an arbitrary airplane and in Chap. 5 for an ideal subsonic airplane. In Chap. 6, an algorithm is presented for calculating the aerodynamics of high-lift devices, and the nonsteady flight problems of take-off and landing are discussed. Finally, the nonsteady flight problems of energy climbs, specific excess power, energy-maneuverability, and horizontal turns are studied in Chap. 7.

Part II: Stability and Control. This part of the text contains static stability and control and dynamic stability and control. It is begun in Chap. 8 with the 6DOF model in wind axes. Following the discussion of the equations of motion, formulas are presented for calculating the aerodynamics of a subsonic airplane including the lift, the pitching moment, and the drag. Chap. 9 deals with static stability and control. Trim conditions and static stability are investigated for steady cruise, climb, and descent along with the effects of center of gravity position. A simple control system is analyzed to introduce the concepts of hinge moment, stick force, stick force gradient, and handling qualities. Trim tabs and the effect of free elevator on stability are discussed. Next, trim conditions are determined for a nonsteady pull-up, and lateral-directional stability and control are discussed briefly. In Chap. 10, the 6DOF equations of motion are developed first in regular body axes and second in stability axes for use in the investigation of dynamic stability and control. In Chap. 11, the equations of motion are linearized about a steady reference path, and the stability and response of an airplane to a control or gust input is considered. Finally, the effect of center of gravity position is examined, and dynamic lateral-direction stability and control is discussed descriptively.

There are three appendices. App. A gives the geometric characteristics of a subsonic business jet, and results for aerodynamic calculations are listed, including both static and dynamic stability and control results. In App. B, the relationship between linearized aerodynamics (stability derivatives) and the aerodynamics of Chap. 8 is established. Finally, App. C reviews the elements of linear system theory which are needed for dynamic stability and control studies.

While a number of students has worked on this text, the author is particularly indebted to David E. Salguero. His work on converting GASP into an educational tool called BIZJET has formed the basis of a lot of this text.

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Table of Contents

Chapter 1

Introduction to Airplane Flight Mechanics

Airplane flight mechanics can be divided into five broad areas: trajectory analysis (performance), stability and control, aircraft sizing, simulation, and flight testing. Only the theoretical aspects of trajectory analysis and stability and control are covered in this text. Aircraft sizing and simulation are essentially numerical in nature. Airplane sizing involves an iterative process, and simulation involves the numerical integration of a set of differential equations. They are discussed in this chapter to show how they fit into the overall scheme of things. Flight testing is the experimental part of flight mechanics. It is not discussed here except to say that good theory makes good experiments.

The central theme of this text is the following: Given the three-view drawing with dimensions of a subsonic, jet-powered airplane and the engine data, determine its performance, stability, and control characteristics. To do this, formulas for calculating the aerodynamics are developed.

Most of the material in this text is limited to flight in a vertical plane because the mission profiles for which airplanes are designed are primarily in the vertical plane. This chapter begins with a review of the parts of the airframe and the engines. Then, the derivation of the equations governing the motion of an airplane is discussed. Finally, the major areas of aircraft flight mechanics are described.

1.1 Airframe Anatomy

To begin the introduction, it is useful to review the parts of an airframe and discuss their purposes. Fig. 1.1 is a three-view drawing of a Boeing 727. The body or fuselage of the airplane holds the crew, passengers, and freight. It is carried aloft by the lift provided by the wing and propelled by the thrust produced by jet engines housed in nacelles. This airplane has two body-mounted engines and a body centerline engine whose inlet air comes through an S-duct beginning at the front of the vertical tail. The fuel is carried in tanks located in the wing.

Since a jet transport is designed for efficient high-speed cruise, it is unable to take-off and land from standard-length runways without some configuration change. This is provided partly by leading edge slats and partly by trailing edge flaps. Both devices are used for take-off, with a low trailing edge flap deflection. On landing, a high trailing edge flap deflection is used to increase lift and drag, and brakes, reverse thrust, and speed brakes (spoilers) are used to further reduce landing distance.

A major issue in aircraft design is static stability. An airplane is said to be inherently aerodynamically statically stable if, following a disturbance from a steady flight condition, forces and/or moments develop which tend to reduce the disturbance. Shown in Fig. 1.2 is the body axes system whose origin is at the center of gravity and whose x_b, y_b, and z_b axes are called the roll axis, the pitch axis, and the yaw axis. Static stability about the yaw axis (directional stability) is provided by the vertical stabilizer, whereas the horizontal stabilizer makes the airplane statically stable about the pitch axis (longitudinal stability). Static stability about the roll axis (lateral stability) is provided mainly by wing dihedral which can be seen in the front view in Fig. 1.1.

Also shown in Figs. 1.1 and 1.2 are the control surfaces which are intended to control the rotation rates about the body axes (roll rate P, pitch rate Q, and yaw rate R) by controlling the moments about these axes (roll moment L, pitch moment M, and yaw moment N). The convention for positive moments and rotation rates is to grab an axis with the thumb pointing toward the origin and rotate counterclockwise looking down the axis toward the origin. From the pilot's point of view, a positive moment or rate is roll right, pitch up, and yaw right.

The deflection of a control surface changes the curvature of a wing or tail surface, changes its lift, and changes its moment about the

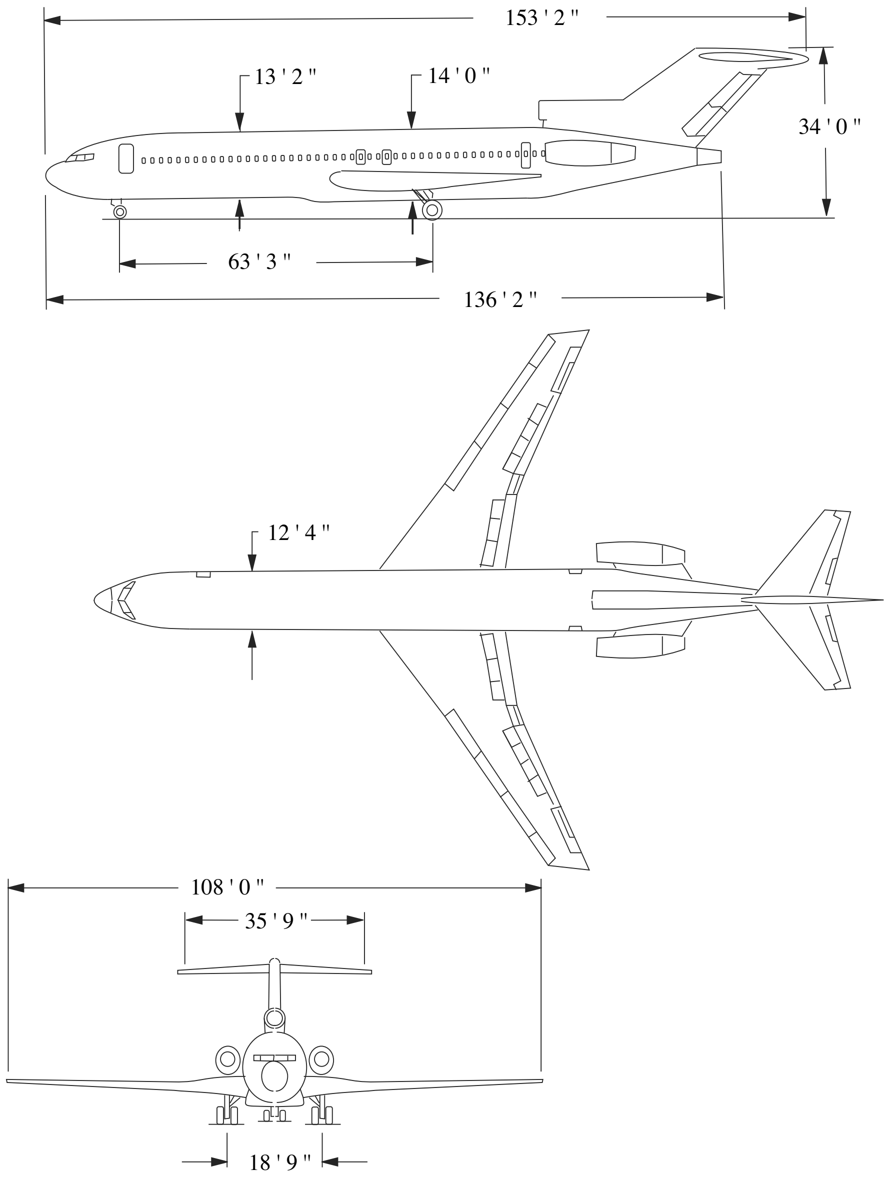

Figure 1.1: Three-View Drawing of a Boeing 727

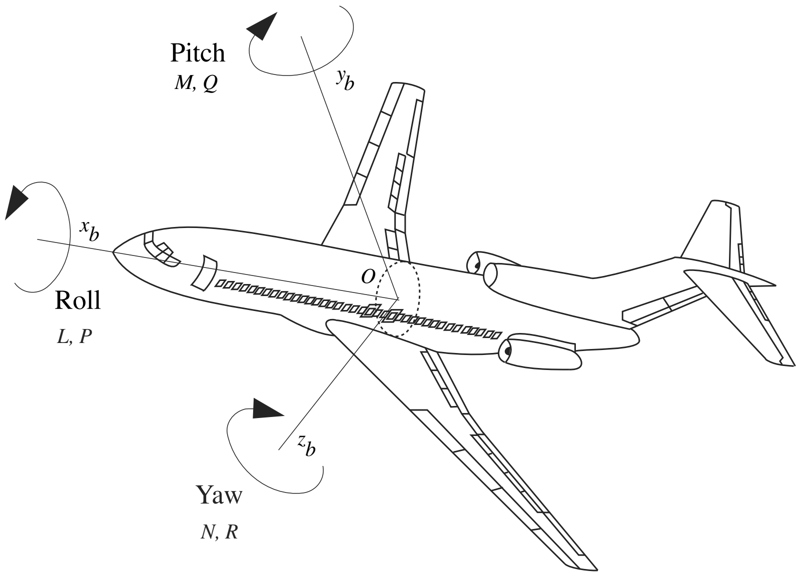

Figure 1.2: Body Axes, Moments, Rates, and Controls

corresponding body axis. Hence, the ailerons (one deflected upward and one deflected downward) control the roll rate; the elevator controls the pitch rate; and the rudder controls the yaw rate. Unlike pitching motion, rolling and yawing motions are not pure. In deflecting the ailerons to roll the airplane, the down-going aileron has more drag than the up-going aileron which causes the airplane to yaw. Similarly, in deflecting the rudder to yaw the airplane, a rolling motion is also produced. Cures for these problems include differentially deflected ailerons and coordinating aileron and rudder deflections. Spoilers are also used to control roll rate by decreasing the lift and increasing the drag on the wing into the turn. Here, a yaw rate is developed into the turn. Spoilers are not used near the ground for rol1 control because the decreased lift causes the airplane to descend.

The F-16 (lightweight fighter) is statically unstable in pitch at subsonic speeds but becomes statically stable at supersonic speeds because of the change in aerodynamics from subsonic to supersonic speeds. The airplane was designed this way to make the horizontal tail as small as possible and, hence, to make the airplane as light as possible. At subsonic speeds, pitch stability is provided by the automatic flight control system. A rate gyro senses a pitch rate, and if the pitch rate is not commanded by the pilot, the elevator is deflected automatically to zero the pitch rate. All of this happens so rapidly (at the speed of electrons) that the pilot is unaware of these rotations.

1.2 Engine Anatomy

In this section, the various parts of jet engines are discussed. There are two types of jet engines in wide use: the turbojet and the turbofan.

A schematic of a *turbojet* is shown in Fig. 1.3. Air entering the engine passes through the diffuser which slows the air to a desired speed for entering the compressor. The compressor increases the pressure of the air and slows it down more. In the combustion chamber (burner), fuel is added to the air, and the mixture is ignited and burned. Next, the high temperature stream passes through the turbine which extracts enough energy from the stream to run the compressor. Finally, the nozzle increases the speed of the stream before it exits the engine.

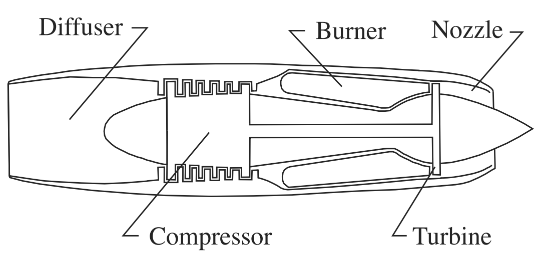

Figure 1.3: Schematic of a Turbojet Engine

The engine cycle is a sequence of assumptions describing how the flow behaves as it passes through the various parts of the engine. Given the engine cycle, it is possible to calculate the thrust (lb) and the fuel flow rate (lb/hr) of the engine. Then, the specific fuel consumption (1/hr) is the ratio of the fuel flow rate to the thrust.

A schematic of a *turbofan* is shown in Fig. 1.4. The turbofan is essentially a turbojet which drives a fan located after the diffuser and before the compressor. The entering air stream is split into a primary part which passes through the turbojet and a secondary part which goes around the turbojet. The split is defined by the bypass ratio, which is the ratio of the air mass flow rate around the turbojet to the air mass flow rate through the turbojet. Usually, the fan is connected to its own turbine by a shaft, and the compressor is connected to its turbine by a hollow shaft which rotates around the fan shaft.

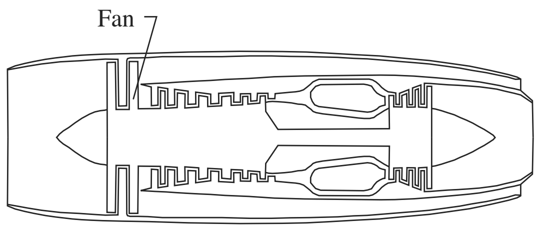

Figure 1.4: Schematic of a Turbofan Engine

1.3 Equations of Motion

In this text, the term flight mechanics refers to the analysis of airplane motion using Newton's laws. While most aircraft structures are flexible to some extent, the airplane is assumed here to be a rigid body. When fuel is being consumed, the airplane is a variable-mass rigid body.

Newton's laws are valid when written relative to an inertial reference frame, that is, a reference frame which is not accelerating or rotating. If the equations of motion are derived relative to an inertial reference frame and if approximations characteristic of airplane motion are introduced into these equations, the resulting equations are those for flight over a nonrotating flat earth. Hence, for airplane motion, the earth is an approximate inertial reference frame, and this model is called the flat earth model. The use of this physical model leads to a small error in most analyses.

A general derivation of the equations of motion involves the use of a material system involving both solid and fluid particles. The end result is a set of equations giving the motion of the solid part of the airplane subject to aerodynamic, propulsive and gravitational forces. To simplify the derivation of the equations of motion, the correct equations for the forces are assumed to be known. Then, the equations describing the motion of the solid part of the airplane are derived.

The airplane is assumed to have a right-left plane of symmetry with the forces acting at the center of gravity and the moments acting about the center of gravity. Actually, the forces acting on an airplane in fight are due to distributed surface forces and body forces. The surface forces come from the air moving over the airplane and through the propulsion system, while the body forces are due to gravitational effects. Any distributed force (see Fig. 1.5) can be replaced by a concentrated force acting along a specific line of action. Then, to have all forces acting through the same point, the concentrated force can be replaced by the same force acting at the point of interest plus a moment about that point to offset the effect of moving the force. The point usually chosen for this purpose is the center of mass, or equivalently for airplanes the center of gravity, because the equations of motion are the simplest.

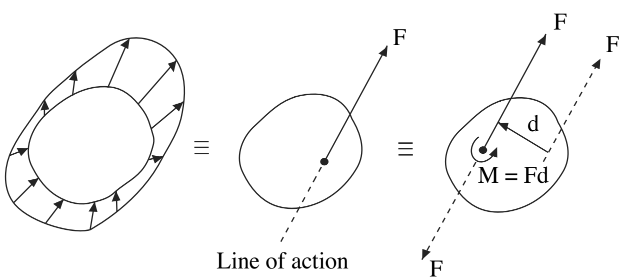

Figure 1.5: Distributed Versus Concentrated Forces

The equations governing the translational and rotational motion of an airplane are the following:

a. Kinematic equations giving the translational position and rotational position relative to the earth reference frame.

b. Dynamic equations relating forces to translational acceleration and moments to rotational acceleration.

c. Equations defining the variable-mass characteristics of the airplane (center of gravity, mass and moments of inertia) versus time.

d. Equations giving the positions of control surfaces and other movable parts of the airplane (landing gear, flaps, wing sweep, etc.) versus time.

These equations are referred to as the six degree of freedom (6DOF) equations of motion. The use of these equations depends on the particular area of flight mechanics being investigated.

1.4 Trajectory Analysis

Most trajectory analysis problems involve small aircraft rotation rates and are studied through the use of the three degree of freedom (3DOF) equations of motion, that is, the translational equations. These equations are uncoupled from the rotational equations by assuming negligible rotation rates and neglecting the effect of control surface deflections on aerodynamic forces. For example, consider an airplane in cruise. To maintain a given speed an elevator deflection is required to make the pitching moment zero. This elevator defection contributes to the lift and the drag of the airplane. By neglecting the contribution of the elevator deflection to the lift and drag (untrimmed aerodynamics), the translational and rotational equations uncouple. Another approach, called trimmed aerodynamics, is to compute the control surface angles required for zero aerodynamic moments and eliminate them from the aerodynamic forces. For example, in cruise the elevator angle for zero aerodynamic pitching moment can be derived and eliminated from the drag and the lift. In this way, the extra aerodynamic force due to control surface deflection can be taken into account.

Trajectory analysis takes one of two forms. First, given an aircraft, find its performance characteristics, that is, maximum speed, ceiling, range, etc. Second, given certain performance characteristics, what is the airplane which produces them. The latter is called aircraft sizing, and the missions used to size commercial and military aircraft are presented here to motivate the discussion of trajectory analysis. The mission or flight profile for sizing a commercial aircraft (including business jets) is shown in Fig. 1.6. It is composed of take-off, climb, cruise, descent, and landing segments, where the descent segment is replaced by an extended cruise because the fuel consumed is approximately the same. In each segment, the distance traveled, the time elapsed, and the fuel consumed must be computed to determine the corresponding quantities for the whole mission. The development of formulas or algorithms for computing these performance quantities is the charge of trajectory analysis. The military mission (Fig. 1.7) adds three performance computations: a constant-altitude acceleration (supersonic dash), constant-altitude turns, and specific excess power (P_S). The low-altitude dash gives the airplane the ability to approach the target within the radar ground clutter, and the speed of the approach gives the airplane the ability to avoid detection until it nears the target. The number of turns is specified to ensure that the airplane has enough fuel for air combat in the neighborhood of the target. Specific excess power is a measure of the ability of the airplane to change its energy, and it is used to ensure that the aircraft being designed has superior maneuver capabilities relative to enemy aircraft protecting the target. Note that, with the exception of the turns, each segment takes place in a plane perpendicular to the surface of the earth (vertical plane). The turns take place in a horizontal plane.

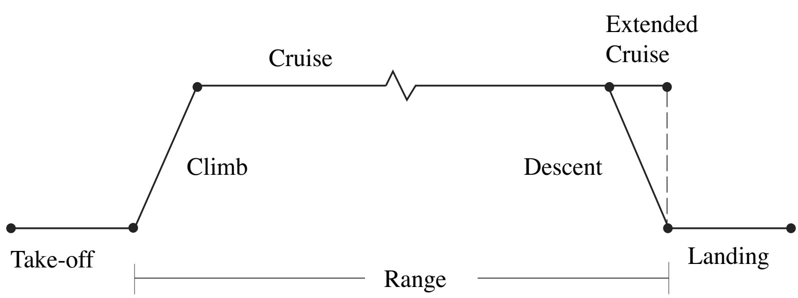

Figure 1.6: Mission for Commercial Aircraft Sizing

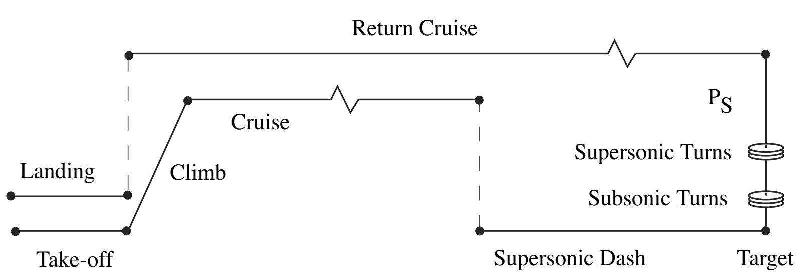

Figure 1.7: Mission for Military Aircraft Sizing

These design missions are the basis for the arrangement of the trajectory analysis portion of this text. In Chap. 2, the equations of motion for flight in a vertical plane over a flat earth are derived, and their solution is discussed. Chap. 3 contains the modeling of the atmosphere, aerodynamics, and propulsion. Both the standard atmosphere and the exponential atmosphere are discussed. An algorithm for predicting the drag polar of a subsonic airplane from a three-view drawing is presented, as is the parabolic drag polar. Engine data is assumed to be available and is presented for a subsonic turbojet and turbofan. Approximate analytical expressions are obtained for thrust and specific fuel consumption.

The mission legs characterized by quasi-steady flight (climb, cruise, and descent) are analyzed in Chap. 4. Algorithms for computing distance, time, and fuel are presented for arbitrary aerodynamics and propulsion, and numerical results are developed for a subsonic business jet. In Chap. 5 approximate analytical results are derived by assuming an ideal subsonic airplane: parabolic drag polar with constant coefficients, thrust independent of velocity, and specific fuel consumption independent of velocity and power setting. The approximate analytical results are intended to be used to check the numerical results and for making quick predictions of performance.

Next, the mission legs characterized by accelerated flight are investigated. Take-off and landing are considered in Chap. 6. Specific excess power, P_S, and constant altitude turns are analyzed in Chap. 7. However, the supersonic dash is not considered because it involves flight through the transonic region.

In general, the airplane is a controllable dynamical system. Hence, the differential equations which govern its motion contain more variables than equations. The extra variables are called control variables. It is possible to solve the equations of motion by specifying the control histories or by specifying some flight condition, say constant altitude and constant velocity, and solving for the controls. On the other hand, because the controls are free to be chosen, it is possible to find the control histories which optimize some index of performance (for example, maximum distance in cruise). Trajectory optimization problems such as these are handled by a mathematical theory known as Calculus of Variations or Optimal Control Theory. While the theory is beyond the scope of this text, many aircraft trajectory optimization problems can be formulated as simple optimization problems whose theory can be derived by simple reasoning.

1.5 Stability and Control

Stability and control studies are concerned with motion of the center of gravity (cg) relative to the ground and motion of the airplane about the cg. Hence, stability and control studies involve the use of the six degree of freedom equations of motion. These studies are divided into two major categories: (a) static stability and control and (b) dynamic stability and control. Because of the nature of the solution process, each of the categories is subdivided into longitudinal motion (pitching motion) and lateral-directional motion (combined rolling and yawing motion). While trajectory analyses are performed in terms of force coefficients with control surface deflections either neglected (untrimmed drag polar) or eliminated (trimmed drag polar), stability and control analyses are in terms of the orientation angles (angle of attack and sideslip angle) and the control surface deflections.

The six degree of freedom model for flight in a vertical plane is presented in Chap. 8. First, the equations of motion are derived in the wind axes system. Second, formulas for calculating subsonic aerodynamics are developed for an airplane with a straight, tapered, swept wing. The aerodynamics associated with lift and pitching moment are shown to be linear in the angle of attack, the elevator angle, the pitch rate, and the angle of attack rate. The aerodynamics associated with drag is shown to be quadratic in angle of attack. Each coefficient in these relationships is a function of Mach number.

Chap. 9 is concerned with static stability and control. Static stability and control for quasi-steady flight is concerned primarily with four topics: trim conditions, static stability, center of gravity effects, and control force and handling qualities. The trim conditions are the orientation angles and control surface deflections required for a particular flight condition. Given a disturbance from a steady flight condition, static stability investigates the tendency of the airplane to reduce the disturbance. This is done by looking at the signs of the forces and moments. Fore and aft limits are imposed on allowable cg locations by maximum allowable control surface deflections and by stability considerations, the aft cg limit being known as the neutral point because it indicates neutral stability. Handling qualities studies are concerned with pilot-related quantities such as control force and how control force changes with flight speed. These quantities are derived from aerodynamic moments about control surface hinge lines. Trim tabs have been introduced to allow the pilot to zero out the control forces associated with a particular flight condition. However, if after trimming the stick force the pilot flies hands-off, the stability characteristics of the airplane are reduced.

To investigate static stability and control for accelerated flight, use is made of a pull-up. Of interest is the elevator angle required to make an n-g turn or pull-up. There is a cg position where the elevator angle per g goes to zero, making the airplane too easy to maneuver. This cg position is called the maneuver point. There is another maneuver point associated with the stick force required to make an n-g pull-up.

While dynamic stability and control studies can be conducted using wind axes, it is the convention to use body axes. Hence, in Chap. 10, the equations of motion are derived in the body axes. The aerodynamics need for body axes is the same as that used in wind axes. A particular set of body axes is called stability axes. The equations of motion are also developed for stability axes.

Dynamic stability and control is concerned with the motion of an airplane following a disturbance such as a wind gust (which changes the speed, the angle of attack and/or the sideslip angle) or a control input. While these studies can and are performed using detailed computer simulations, it is difficult to determine cause and effect. As a consequence, it is desirable to develop an approximate analytical approach. This is done in Chap. 11 by starting with the airplane in a quasi-steady flight condition (given altitude, Mach number, weight, power setting) and introducing a small disturbance. By assuming that the changes in the variables are small, the equations of motion can be linearized about the steady flight condition. This process leads to a system of linear, ordinary differential equations with constant coefficients. As is known from linear system theory, the response of an airplane to a disturbance is the sum of a number of motions called modes. While it is not necessary for each mode to be stable, it is necessary to know for each mode the stability characteristics and response characteristics. A mode can be unstable providing its response characteristics are such that the pilot can easily control the airplane. On the other hand, even if a mode is stable, its response characteristics must be such that the airplane handles well (handling qualities). The design problem is to ensure that an aircraft has desirable stability and response characteristics thoughout the flight envelope and for all allowable cg positions. During this part of the design process, it may no longer be possible to modify the configuration, and automatic control solutions may have to be used.

App. A contains the geometric and aerodynamic data used in the text to compute performance, stability and control characteristics of a subsonic business jet called the SBJ throughout the text. App. B gives the relationship between the stability derivatives of Chap. 11 and the aerodynamics of Chap. 8. Finally, App. C contains a review of linear system theory for first-order systems and second-order systems.

1.6 Aircraft Sizing

While aircraft sizing is not covered in this text, it is useful to discuss the process to see where performance and static stability fit into the picture.

Consider the case of sizing a subsonic business jet to have a given range at a given cruise altitude. Furthermore, the aircraft must take-off and land on runways of given length and have a certain maximum rate of climb at the cruise altitude. The first step in the design process is to perform conceptual design. Here, the basic configuration is selected, which essentially means that a three-view drawing of the airplane can be sketched (no dimensions). The next step is to size the engines and the wing so that the mission can be performed. To size an engine, the performance of an actual engine is scaled up or down. See Fig. 1.8 for a flow chart of the sizing process. The end result of the sizing process is a three-view drawing of an airplane with dimensions.

The sizing process is iterative and begins by guessing the take-off gross weight, the engine size (maximum sea level static thrust), and the wing size (wing planform area). Next, the geometry of the airplane is determined by assuming that the center of gravity is located at the wing aerodynamic center, so that the airplane is statically stable. On the first iteration, statistical formulas are used to locate the horizontal and vertical tails. After the first iteration, component weights are available, and statistical formulas are used to place the tails. Once the geometry is known, the aerodynamics (drag polar) is estimated.

The next step is to fly the airplane through the mission. If the take-off distance is too large, the maximum thrust is increased, and the mission is restarted. Once take-off can be accomplished, the maximum rate of climb at the cruise altitude is determined. If it is less than the required value, the maximum thrust is increased, and the mission is restarted. The last constraint is landing distance. If the landing

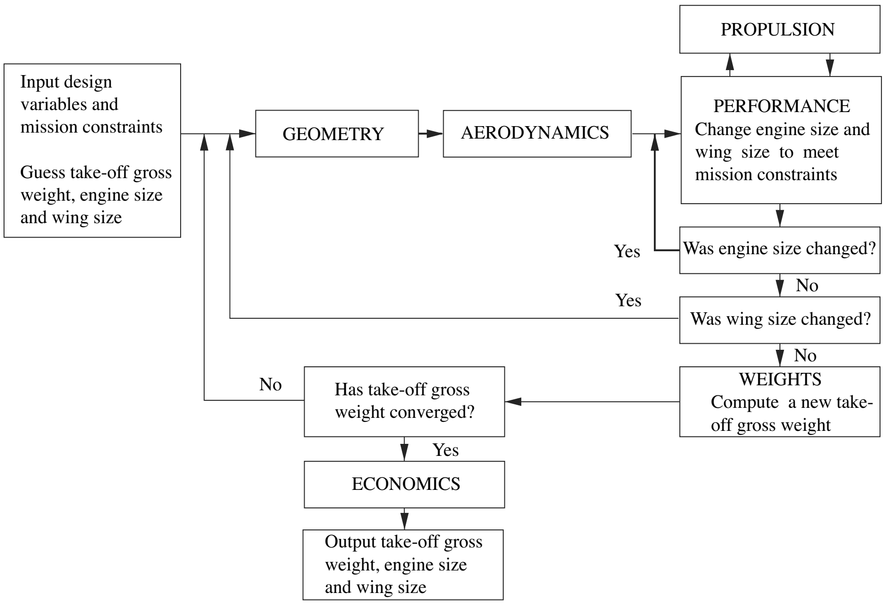

Figure 1.8: Aircraft Sizing Flowchart

distance is too large, the wing planform area is changed, and the mission is restarted. Here, however, the geometry and the aerodynamics must be recomputed.

Once the airplane can be flown through the entire mission, the amount of fuel required is known. Next, the fuel is allocated to wing, tip, and fuselage tanks, and statistical weights formulas are used to estimate the weight of each component and, hence, the take-off gross weight. If the computed take-off gross weight is not close enough to the guessed take-off gross weight, the entire process is repeated with the computed take-off gross weight as the guessed take-off gross weight. Once convergence has been achieved, the flyaway cost and the operating cost can be estimated.

1.7 Simulation

Simulations come in all sizes, but they are essentially computer programs that integrate the equations of motion. They are used to evaluate the flight characteristics of a vehicle. In addition to being run as computer programs, they can be used with working cockpits to allow pilots to evaluate handling qualities.

A major effort of an aerospace company is the creation of a high-fidelity 6DOF simulation for each of the vehicles it is developing. The simulation is modular in nature in that the aerodynamics function or subroutine is maintained by aerodynamicists, and so on.

Some performance problems, such as the spin, have so much interaction between the force and moment equations that they may have to be analyzed with six degree of freedom codes. These codes would essentially be simulations.

Chapter 2

3DOF Equations of Motion

An airplane operates near the surface of the earth which moves about the sun. Suppose that the equations of motion ($F = ma$ and $M = I\alpha$) are derived for an accurate inertial reference frame and that approximations characteristic of airplane flight (altitude and speed) are introduced into these equations. What results is a set of equations which can be obtained by assuming that the earth is flat, nonrotating, and an approximate inertial reference frame, that is, the flat earth model.

The equations of motion are composed of translational (force) equations ($F = ma$) and rotational (moment) equations ($M = I\alpha$) and are called the six degree of freedom (6DOF) equations of motion. For trajectory analysis (performance), the translational equations are uncoupled from the rotational equations by assuming that the airplane rotational rates are small and that control surface deflections do not affect forces. The translational equations are referred to as the three degree of freedom (3DOF) equations of motion.

As discussed in Chap. 1, two important legs of the commercial and military airplane missions are the climb and the cruise which occur in a vertical plane (a plane perpendicular to the surface of the earth). The purpose of this chapter is to derive the 3DOF equations of motion for flight in a vertical plane over a flat earth. First, the physical model is defined; several reference frames are defined; and the angular positions and rates of these frames relative to each other are determined. Then, the kinematic, dynamic, and weight equations are derived and discussed for nonsteady and quasi-steady flight. Next, the equations of motion for flight over a spherical earth are examined to find out how good the flat earth model really is. Finally, motivated by such problems as flight in a headwind, flight in the downwash of a tanker, and flight through a downburst, the equations of motion for flight in a moving atmosphere are derived.

2.1 Assumptions and Coordinate Systems

In deriving the equations of motion for the nonsteady flight of an airplane in a vertical plane over a flat earth, the following physical model is assumed:

a. The earth is flat, nonrotating, and an approximate inertial reference frame. The acceleration of gravity is constant and perpendicular to the surface of the earth. This is known as the *flat earth model.*

b. The atmosphere is at rest relative to the earth, and atmospheric properties are functions of altitude only.

c. The airplane is a conventional jet airplane with fixed engines, an aft tail, and a right-left *plane of symmetry.* It is modeled as a variable-mass particle.

d. The forces acting on an airplane in symmetric flight (no sideslip) are the thrust, the aerodynamic force, and the weight. They act at the center of gravity of the airplane, and the thrust and the aerodynamic force lie in the plane of symmetry.

The derivation of the equations of motion is clarified by defining a number of coordinate systems. For each coordinate system that moves with the airplane, the x and z axes are in the plane of symmetry of the airplane, and the y axis is such that the system is right handed. The x axis is in the direction of motion, while the z axis points earthward if the aircraft is in an upright orientation. Then, the y axis points out the right wing (relative to the pilot). The four coordinate systems used here are the following (see Fig. 2.1):

a. The *ground axes system* Exyz is fixed to the surface of the earth at mean sea level, and the xz plane is the vertical plane. It is an approximate inertial reference frame.

b. The *local horizon axes system* $Ox_hy_hz_h$ moves with the airplane (O is the airplane center of gravity), but its axes remain parallel to the ground axes.

c. The *wind axes system* $Ox_wy_wz_w$ moves with the airplane, and the x_w axis is coincident with the velocity vector.

d. The *body axes system* $Ox_by_bz_b$ is fixed to the airplane.

These coordinate systems and their orientations are the convention in flight mechanics (see, for example, Ref. Mi1).

The coordinate systems for flight in a vertical plane are shown in Fig. 2.1, where the airplane is located at an altitude h above mean sea level. In the figure, $\mathbf{V}$ denotes the velocity of the airplane relative to the

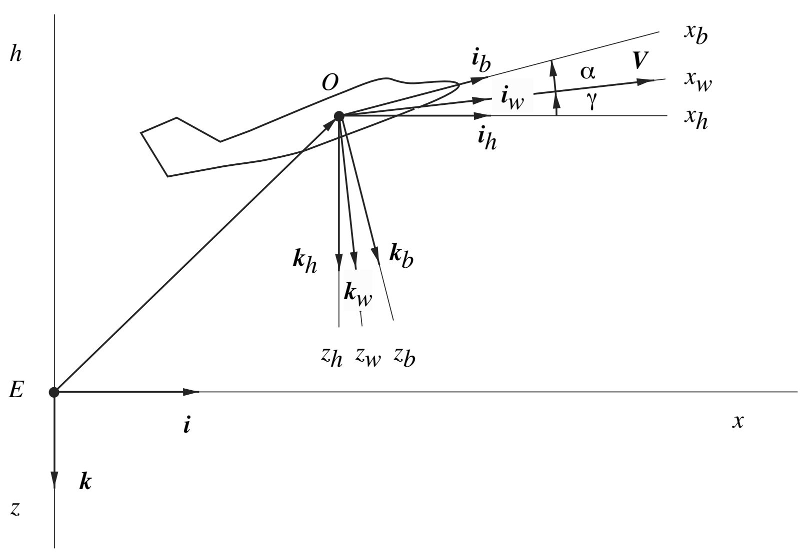

Figure 2.1: Coordinate Systems for Flight in a Vertical Plane

air; however, since the atmosphere is at rest relative to the ground, $\mathbf{V}$ is also the velocity of the airplane relative to the ground. Note that the wind axes are orientated relative to the local horizon axes by the *flight path angle* γ, and the body axes are orientated relative to the wind axes by the *angle of attack* α.

The unit vectors associated with the coordinate directions are denoted by $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ with appropriate subscripts. Since the local horizon axes are always parallel to the ground axes, their unit vectors are equal, that is,

$$\begin{aligned} \mathbf{i}_h &= \mathbf{i} \\ \mathbf{k}_h &= \mathbf{k}\ . \end{aligned} \tag{2.1}$$

Next, the wind axes unit vectors are related to the local horizon unit vectors as

$$\begin{aligned} \mathbf{i}_w &= \cos\gamma\mathbf{i}_h - \sin\gamma\mathbf{k}_h \\ \mathbf{k}_w &= \sin\gamma\mathbf{i}_h + \cos\gamma\mathbf{k}_h\ . \end{aligned} \tag{2.2}$$

Since the unit vectors (2.1) are constant (fixed magnitude and direction), it is seen that

$$\begin{aligned} \tfrac{d\mathbf{i}_h}{dt} &= \tfrac{d\mathbf{i}}{dt} = 0 \\ \tfrac{d\mathbf{k}_h}{dt} &= \tfrac{d\mathbf{k}}{dt} = 0\ . \end{aligned} \tag{2.3}$$

Then, by straight differentiation of Eqs. (2.2), the following relations are obtained:

$$\begin{aligned} \tfrac{d\mathbf{i}_w}{dt} &= -\dot{\gamma}\mathbf{k}_w \\ \tfrac{d\mathbf{k}_w}{dt} &= \ \ \dot{\gamma}\mathbf{i}_w\ . \end{aligned} \tag{2.4}$$

The body axes are used primarily to define the angle of attack and will be discussed later.

2.2 Kinematic Equations

Kinematics is used to derive the differential equations for x and h which locate the airplane center of gravity relative to the origin of the ground axes system (*inertial position*). The definition of velocity relative to the ground (*inertial velocity*) is given by

$$\mathbf{V} = \frac{d\mathbf{EO}}{dt}\ , \tag{2.5}$$

where the derivative is taken holding the ground unit vectors constant. The velocity $\mathbf{V}$ and the position vector $\mathbf{EO}$ must be expressed in the same coordinate system to obtain the corresponding scalar equations. Here, the local horizon system is used where

$$\begin{aligned} \mathbf{V} &= V\mathbf{i}_w = V\cos\gamma\mathbf{i}_h - V\sin\gamma\mathbf{k}_h \\ \mathbf{EO} &= x\mathbf{i} - h\mathbf{k} = x\mathbf{i}_h - h\mathbf{k}_h \ . \end{aligned} \tag{2.6}$$

Since the unit vectors $\mathbf{i}_h$ and $\mathbf{k}_h$ are constant, Eq. (2.5) becomes

$$V\cos\gamma\mathbf{i}_h - V\sin\gamma\mathbf{k}_h = \dot{x}\mathbf{i}_h - \dot{h}\mathbf{k}_h \tag{2.7}$$

and leads to the following scalar equations:

$$\begin{aligned} \dot{x} &= V\cos\gamma \\ \dot{h} &= V\sin\gamma \ . \end{aligned} \tag{2.8}$$

These equations are the *kinematic equations* of motion for flight in a vertical plane.

2.3 Dynamic Equations

Dynamics is used to derive the differential equations for V and γ which define the velocity vector of the airplane center of gravity relative to the ground. Newton's second law states that

$$\mathbf{F} = m\mathbf{a} \tag{2.9}$$

where $\mathbf{F}$ is the resultant external force acting on the airplane, m is the mass of the airplane, and $\mathbf{a}$ is the *inertial acceleration* of the airplane. For the normal operating conditions of airplanes (altitude and speed), a reference frame fixed to the earth is an approximate inertial frame. Hence, $\mathbf{a}$ is approximated by the acceleration of the airplane relative to the ground.

The resultant external force acting on the airplane is given by

$$\mathbf{F} = \mathbf{T} + \mathbf{A} + \mathbf{W} \tag{2.10}$$

where $\mathbf{T}$ is the *thrust*, $\mathbf{A}$ is the *aerodynamic force*, and $\mathbf{W}$ is the *weight*. These concentrated forces are the result of having integrated the distributed forces over the airplane and having moved them to the center of gravity with appropriate moments. Recall that the moments are not needed because the force and moment equations have been uncoupled. By definition, the components of the aerodynamic force parallel and perpendicular to the velocity vector are called the *drag* and the *lift* so that

$$\mathbf{A} = \mathbf{D} + \mathbf{L} \; . \tag{2.11}$$

These forces are shown in Fig. 2.2 where the thrust vector is orientated relative to the velocity vector by the angle ε which is referred to as the *thrust angle of attack*.

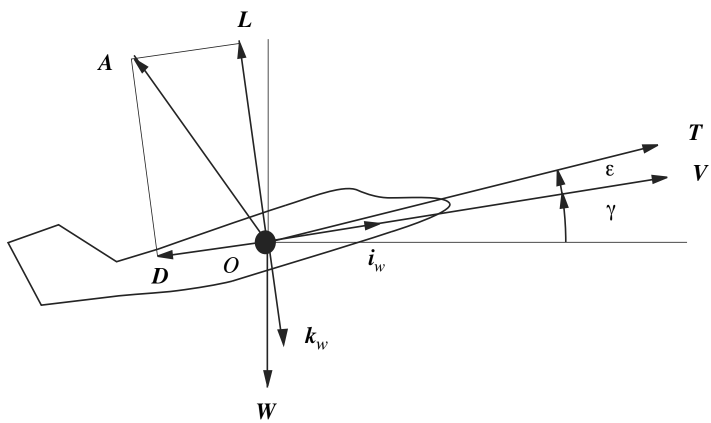

Figure 2.2: Forces Acting on an Airplane in Flight

In order to derive the scalar equations, it is necessary to select a coordinate system. While the local horizon system is used for obtaining the kinematic equations, a more direct derivation of the dynamic equations is possible by using the wind axes system. In this coordinate system, the forces acting on the airplane can be written as

$$\begin{aligned}
\mathbf{T} &= T\cos\varepsilon\mathbf{i}_w - T\sin\varepsilon\mathbf{k}_w \\
\mathbf{D} &= -D\mathbf{i}_w \\
\mathbf{L} &= -L\mathbf{k}_w \\
\mathbf{W} &= -W\sin\gamma\mathbf{i}_w + W\cos\gamma\mathbf{k}_w
\end{aligned} \tag{2.12}$$

so that the resultant external force becomes

$$\mathbf{F} = (T\cos\varepsilon - D - W\sin\gamma)\mathbf{i}_w - (T\sin\varepsilon + L - W\cos\gamma)\mathbf{k}_w \; . \tag{2.13}$$

By definition of acceleration relative to the ground,

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} \tag{2.14}$$

holding the ground axes unit vectors constant. Since the velocity is along the x_w axis, it can be expressed as

$$\mathbf{V} = V\mathbf{i}_w \tag{2.15}$$

where both the velocity magnitude V and the direction of the unit vector $\mathbf{i}_w$ are functions of time. Differentiation leads to

$$\mathbf{a} = \dot{V}\mathbf{i}_w + V\frac{d\mathbf{i}_w}{dt}, \tag{2.16}$$

and in view of Eq. (2.4) the acceleration of the airplane with respect to the ground is given by

$$\mathbf{a} = \dot{V}\mathbf{i}_w - V\dot{\gamma}\mathbf{k}_w. \tag{2.17}$$

By combining Eqs. (2.9), (2.13), and (2.17), the following scalar equations are obtained:

$$\begin{aligned} \dot{V} &= (g/W)(T\cos\varepsilon - D - W\sin\gamma) \\ \dot{\gamma} &= (g/WV)(T\sin\varepsilon + L - W\cos\gamma) \end{aligned} \tag{2.18}$$

where g is the constant acceleration of gravity and where the relation $W = mg$ has been used.

For conventional aircraft, the engines are fixed to the aircraft. This means that the angle between the thrust vector and the x_b axis is constant. Hence, from Fig. 2.3, it is seen that

$$\varepsilon = \alpha + \varepsilon_0 \tag{2.19}$$

where ε_0 is the value of ε when $\alpha = 0$ or the angle which the engine centerline makes with the x_b axis. Then, Eqs. (2.18) can be rewritten as

$$\begin{aligned} \dot{V} &= (g/W)[T\cos(\alpha + \varepsilon_0) - D - W\sin\gamma] \\ \dot{\gamma} &= (g/WV)[T\sin(\alpha + \varepsilon_0) + L - W\cos\gamma] \end{aligned} \tag{2.20}$$

and are the *dynamic equations* for flight in a vertical plane.

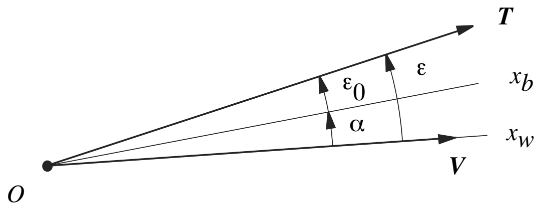

Figure 2.3: Relationship between ε and α

2.4 Weight Equation

By definition of the fuel weight flow rate $\dot{W}_{fuel}$, the rate of change of the weight of the aircraft is given by

$$\dot{W} = -\dot{W}_{fuel} \tag{2.21}$$

Next, the *specific fuel consumption*

$$C = \frac{\dot{W}_{fuel}}{T} \tag{2.22}$$

is introduced because it has some special properties. The *weight equation* becomes

$$\dot{W} = -CT. \tag{2.23}$$

and gives the rate at which the weight of the aircraft is changing in terms of the operating conditions of the propulsion system.

2.5 Discussion of 3DOF Equations

The equations of motion for nonsteady flight in a vertical plane over a flat earth are given by Eqs. (2.8), (2.20), and (2.23), that is,

$$\begin{aligned}
\dot{x} &= V\cos\gamma \\
\dot{h} &= V\sin\gamma \\
\dot{V} &= (g/W)[T\cos(\alpha+\varepsilon_0) - D - W\sin\gamma] \\
\dot{\gamma} &= (g/WV)[T\sin(\alpha+\varepsilon_0) + L - W\cos\gamma] \\
\dot{W} &= -CT
\end{aligned} \tag{2.24}$$

where g and ε_0 are constants. The purpose of this discussion is to examine the system of equations to see if it can be solved. For a fixed geometry airplane in free flight with flaps up and gear up, it is known that drag and lift obey functional relations of the form (see Chap. 3)

$$D = D(h, V, \alpha)\ ,\quad L = L(h, V, \alpha). \tag{2.25}$$

It is also known that thrust and specific fuel consumption satisfy functional relations of the form (see Chap. 3)

$$T = T(h, V, P),\quad C = C(h, V, P)\ . \tag{2.26}$$

In these functional relations, V is the velocity of the airplane relative to the atmosphere. However, since the atmosphere is fixed relative to the earth, V is also the velocity of the airplane relative to the earth. The quantity P is the engine *power setting*. As a consequence, the equations of motion (2.24) involve the following variables:

$$x(t), h(t), V(t), \gamma(t), W(t), P(t), \alpha(t)\ . \tag{2.27}$$

The variables x, h, V, γ and W whose derivatives appear in the equations of motion are called *state variables*. The remaining variables α and P whose derivatives do not appear are called *control variables*.

Actually, the pilot controls the airplane by moving the throttle and the control column. When the pilot moves the throttle, the fuel flow rate to the engine is changed resulting in a change in the rpm of the engine. The power setting of a jet engine is identified with the ratio of the actual rpm to the maximum allowable rpm. Hence, while the pilot actually controls the throttle angle, he can be thought of as controlling the relative rpm of the engine. Similarly, when the pilot pulls back on the control column, the elevator rotates upward, and the lift of the horizontal tail increases in the downward sense. This increment of lift creates an aerodynamic moment about the center of gravity which rotates the airplane nose-up, thereby increasing the airplane angle of attack. Hence, the pilot can be thought of as controlling the angle of attack of the airplane rather than the angle of the control column. In conclusion, for the purpose of computing the trajectory of an airplane, the power setting and the angle of attack are identified as the controls.

The number of *mathematical degrees of freedom* of a system of equations is the number of variables minus the number of equations. The system of equations of motion (2.24) involves seven variables, five equations, and two mathematical degrees of freedom. Hence, the time histories of two variables must be specified before the system can be integrated. This makes sense because there are two independent controls available to the pilot. On the other hand, it is not necessary to specify the control variables, as any two variables or any two relations between existing variables will do. For example, instead of flying at constant power setting and constant angle of attack, it might be necessary to fly at constant altitude and constant velocity. As another example, it might be desired to fly at constant power setting and constant dynamic pressure $\bar{q} = \rho(h)V\ \ /2$. In all, two additional equations involving existing variables must be added to complete the system.

In addition to the extra equations, it is necessary to provide a number of boundary conditions. Since the equations of motion are first-order differential equations, the integration leads to five constants of integration. One way to determine these constants is to specify the values of the state variables at the initial time, which is also prescribed. Then, to obtain a finite trajectory, it is necessary to give one final condition. This integration problem is referred to as an initial-value problem. If some variables are specified at the initial point and some variables are specified at the final point, the integration problem is called a boundary-value problem.

In conclusion, if the control action of the pilot or an equivalent set of relations is prescribed, the trajectory of the aircraft can be found by integrating the equations of motion subject to the boundary conditions. The trajectory is the set of functions $X(t)$, $h(t)$, $V(t)$, $\gamma(t)$, $W(t)$, $P(t)$ and $\alpha(t)$.

In airplane performance, it is often convenient to use lift as a variable rather than the angle of attack. Hence, if the expression for the lift (2.25) is solved for the angle of attack and if the angle of attack is eliminated from the expression for the drag (2.25), it is seen that

$$\alpha = \alpha(h, V, L), \quad D = D(h, V, L). \tag{2.28}$$

If these functional relations are used in the equations of motion (2.24), the lift becomes a control variable in place of the angle attack. It is also possible to write the engine functional relations in the form $P = P(h, V, T)$ and $C = C(h, V, T)$.

Because the system of Eqs. (2.24) has two mathematical degrees of freedom, it is necessary to provide two additional equations relating existing variables before the equations can be solved. With this model, the solution is usually numerical, say by using Runge-Kutta integration. Another approach is to use the degrees of freedom to optimize some performance capability of the airplane. An example is to minimize the time to climb from one altitude to another. The conditions to be satisfied by an optimal trajectory are derived in Ref. Hu. Optimization using this model is numerical in nature.

2.6 Quasi-Steady Flight

Strictly speaking, *quasi-steady flight* is defined by the approximations that the accelerations $\dot{V}$ and $\dot{\gamma}$ are negligible. However, for the performance problems to be analyzed (climb, cruise, descent) additional approximations also hold. They are small flight path inclination, small angle of attack and hence small thrust angle of attack, and small component of the thrust normal to the flight path. The four approximations which define quasi-steady flight are written as follows:

1. $\dot{V}, \dot{\gamma}$ negligible
2. $\gamma \ << 1$ or $\cos\gamma \cong 1, \quad \sin\gamma \cong \gamma$
3. $\varepsilon \ << 1$ or $\cos\varepsilon \cong 1, \quad \sin\varepsilon \cong \varepsilon$
4. $T\varepsilon << W$

For those segments of an airplane mission where the accelerations are negligible, the equations of motion become

$$\begin{aligned} \dot{x} &= V \\ \dot{h} &= V\gamma \\ 0 &= T - D - W\gamma \\ 0 &= L - W \\ \dot{W} &= -CT. \end{aligned} \tag{2.29}$$

Note that if the drag is expressed as $D = D(h, V, L)$ the angle of attack no longer appears in the equations of motion. This means that only the drag is needed to represent the aerodynamics. Note also that the approximations do not change the number of mathematical degrees of freedom; there are still two. However, there are now three states (x, h, W) and four controls (V, γ, P, L).

Because two of the equations of motion are algebraic, they can be solved for two of the controls as

$$L = W \tag{2.30}$$

and

$$\gamma = \frac{T(h, V, P) - D(h, V, W)}{W} \; . \tag{2.31}$$

Then, the differential equations can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= V \\ \frac{dh}{dt} &= V \left[\frac{T(h,V,P) - D(h,V,W)}{W} \right] \\ \frac{dW}{dt} &= -C(h, V, P)T(h, V, P) \end{aligned} \tag{2.32}$$

and still have two mathematical degrees of freedom (V,P).

The time is a good variable of integration for finding numerical solutions of the equations of motion. However, a goal of flight mechanics is to find analytical solutions. Here, the variable of integration depends on the problem. For climbing flight from one altitude to another, the altitude is chosen to be the variable of integration. The states then become x, t, W. To change the variable of integration in the equations of motion, the states are written as

$$x = x(h(t)) \; , \quad t = t(h(t)) \; , \quad W = W(h(t)) \; . \tag{2.33}$$

Next, these expressions are differentiated with respect to the time as

$$\frac{dx}{dt} = \frac{dx}{dh}\frac{dh}{dt}, \quad \frac{dt}{dt} = \frac{dt}{dh}\frac{dh}{dt}, \quad \frac{dW}{dt} = \frac{dW}{dh}\frac{dh}{dt} \tag{2.34}$$

and lead to

$$\frac{dx}{dh} = \frac{dx/dt}{dh/dt}, \quad \frac{dt}{dh} = \frac{1}{dh/dt}, \quad \frac{dW}{dh} = \frac{dW/dt}{dh/dt}. \tag{2.35}$$

Note that the differentials which make up a derivative can be treated as algebraic quantities.

The equations of motion (2.32) with altitude as the variable of integration are given by

$$
\begin{aligned}
\frac{dx}{dh} &= \frac{1}{\frac{T(h,V,P) - D(h,V,W)}{W}} \\
\frac{dt}{dh} &= \frac{1}{V\left[\frac{T(h,V,P) - D(h,V,W)}{W}\right]} \\
\frac{dW}{dh} &= -\frac{C(h,V,P)T(h,V,P)}{V\left[\frac{T(h,V,P) - D(h,V,W)}{W}\right]}\ .
\end{aligned}
\tag{2.36}
$$

From this point on all the variables in these equations are considered to be functions of altitude. There are three states $x(h), t(h), W(h)$ and two controls $P(h), V(h)$ and still two mathematical degrees of freedom.

Eqs. (2.36) lead to optimization problems which can be handled quite easily. For example, suppose that it is desired to minimize the time to climb from one altitude to another. Because the amount of fuel consumed during the climb is around 5% of the initial climb weight, the weight on the right hand side of Eqs. (2.36) can be assumed constant. Also, an efficient way to climb is with maximum continuous power setting (constant power setting). The former assumption is an engineering approximation, while the latter is a statement as to how the airplane is being flown and reduces the number of degrees of freedom to one. With these assumptions, the second of Eqs. (2.36) can be integrated to obtain

$$
t_f - t_0 = \int_{h_0}^{h_f} f(h,V)\ dh, \quad f(h,V) = \frac{1}{V\left[\frac{T(h,V,P) - D(h,V,W)}{W}\right]} \tag{2.37}
$$

because P and W are constant.

The problem of finding the function $V(h)$ which minimizes the time is a problem of the calculus of variations or optimal control theory (see Chap. 8 of Ref. Hu). However, because of the form of Eq. (2.37), it is possible to bypass optimization theory and get the optimal $V(h)$ by simple reasoning.

To find the velocity profile $V(h)$ which minimizes the time to climb, it is necessary to minimize the integral (2.37) with respect to the velocity profile $V(h)$. The integral is the area under a curve. The way to minimize the area under the curve is to minimize f with respect to V at each value of h. Hence, the conditions to be applied for finding the minimal velocity profile $V(h)$ are the following:

$$
\left.\frac{\partial f}{\partial V}\right|_{h\ Const} = 0, \quad \left.\frac{\partial^2 f}{\partial V^2}\right|_{h\ Const} > 0. \tag{2.38}
$$

Substitution of the optimal velocity profile $V(h)$ into Eq. (2.37) leads to an integral of the form $\int g(h)dh$. If the aerodynamic and propulsion data are tabular, that is, interpolated tables of numbers, the optimization process must be carried out numerically. On the other hand, if an analytical model is available for the aerodynamics and propulsion, it may be possible to obtain the optimal velocity profile analytically.

Instead of optimizing the climb, it is possible to specify the velocity, say constant velocity. Then, the integration in Eq. (2.37) can be carried out for the time. A question which still remains is what constant velocity should be flown such that the time takes on a best value.

In finding the $V(h)$ which minimizes the time, all possible velocity profiles are in contention for the minimum. In finding the constant velocity which gives the best time, only constant velocity profiles are in contention. The velocity profile which minimizes the time has a lower value of the time than the best constant velocity climb.

2.7 Three-Dimensional Flight

In general, the velocity vector is oriented relative to the body axes by the sideslip angle and the angle of attack. If the velocity vector is in the plane of symmetry of the airplane, the sideslip angle is zero. Such flight is called symmetric. For three-dimensional, symmetric flight over a flat earth, the equations of motion are given by (Ref. Mi1)

$$\begin{aligned}
\dot{x} &= V\cos\gamma\cos\psi \\
\dot{y} &= V\cos\gamma\sin\psi \\
\dot{h} &= V\sin\gamma \\
\dot{V} &= (g/W)[T\cos\varepsilon - D - W\sin\gamma] \\
\dot{\psi} &= (g/WV\cos\gamma)(T\sin\varepsilon + L)\sin\mu \\
\dot{\gamma} &= (g/WV)[(T\sin\varepsilon + L)\cos\mu - W\cos\gamma] \\
\dot{W} &= -CT\ .
\end{aligned} \tag{2.39}$$

In general (see Fig. 2.4), ψ is called *velocity yaw*; γ is called *velocity pitch*; and μ is called *velocity roll*. ψ is also called the *heading angle*, and

μ is called the *bank angle*. These angles are shown in Fig. 2.8 for flight in a horizontal plane.

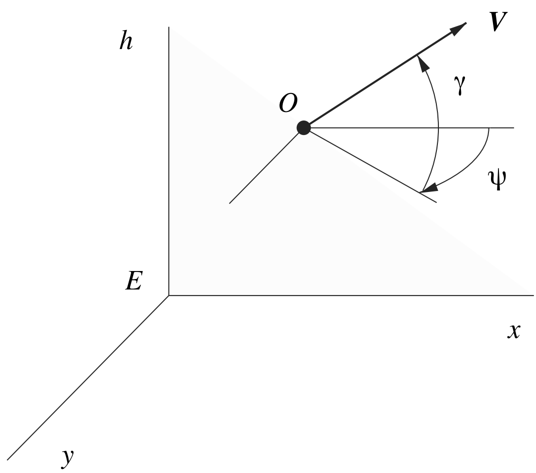

Figure 2.4: Three-Dimensional Flight

Note that if $\psi = 0$, these equations reduce to those for flight in a vertical plane (2.24), or if $\gamma = 0$, they reduce to those for flight in a horizontal plane (see Prob. 2.9).

Note also that the equations for $\dot{h}$ and $\dot{V}$ are the same as those for flight in a vertical plane. This result will be used in Chap. 7 when studying energy-maneuverability.

2.8 Flight over a Spherical Earth

The situation for flight in a vertical plane over a nonrotating spherical earth is shown in Fig. 2.9. Here, r_s is the radius of the surface of the earth, x is a curvilinear coordinate along the surface of the earth, and the angular velocity of the earth is sufficiently small that it can be neglected in the analysis of airplane trajectories. The equations of motion are given by (see Prob. 2.10)

$$\dot{x} = r_s V \cos\gamma/(r_s + h) \tag{2.40}$$

$$\dot{h} = V \sin\gamma \tag{2.41}$$

Note that $D, T,$ and C satisfy the functional relations

$$D = D(h, V, L), \quad T = T(h, P), \quad C = C(h). \tag{3.70}$$

The aerodynamics of the Ideal SBJ (ISBJ, App. A) are given by

$$C_{D_0} = 0.023, \quad K = 0.073. \tag{3.71}$$

The propulsion characteristics of the GE turbojet are given by

$$\text{Troposphere}: a = 1.2, \; b = 0.1 \tag{3.72}$$

$$\text{Stratosphere}: a = 1, \; b = 0, \tag{3.73}$$

with $T_t(P)$ and C_t given in Table 3.7.

The propulsion characteristics of the Garrett turbofan are given by

$$\text{Troposphere}: a = 1, \; b = 0 \tag{3.74}$$

$$\text{Stratosphere}: a = 1, \; b = 0, \tag{3.75}$$

with $T_t(P)$ and C_t given in Table 3.8. It is not possible to just replace the GE turbojet in the ISBJ by the Garrett turbofan. Each turbofan engine weighs about 300 lb more than the turbojet.

Problems

The answers to the problems involving the computation of the SBJ geometry or aerodynamics are given in App. A to help keep you on track. Once you have completed an assignment, you should use the numbers given in App. A instead of those you have calculated.

3.1 Perform the tasks listed below for the wing of the SBJ (App. A). The wing planform extends from the fuselage centerline to the outside of the tip tanks.

Starting from the measured values for the root chord, the tip chord, the span, and the sweep angle of the quarter chord line (App. A) calculate the planform area, the aspect ratio, the taper ratio, and the sweep of the leading edge. Also, calculate the length of the mean aerodynamic chord.

3.2 To demonstrate that Eq. (3.30) reduces to known incompressible thin-wing results, show that, for $M = 0$, ${}_{hc} = 0$, and $= 1$,

$$C_L = \frac{A}{1 + \sqrt{1 + (A=2)^2}}:$$

Then, for large aspect ratio wings $[(A=2)^2 >> 1]$, show that

$$C_L = \frac{2A}{2 + A}:$$

Finally, for a two-dimensional wing (airfoil) for which $A = 1$, show that

$$C_L = 2$$

which is the theoretical thin-airfoil result.

3.3 Prove that the surface area of a right circular cone (excluding the base) is given by

$$A = R\sqrt{R^2 + h^2}$$

where R denotes the base radius and h denotes the height.

3.4 Calculate the wetted area of each component of the SBJ (App. A).

3.5 Assume that the SBJ (App. A) is operating in level ight ($L = W$) at $h = 30{,}000$ ft, $M = 0.7$, and $W = 11{,}000$ lb. The lift coe cient is given by $C_L = 2W= SV^2$.

a. Compute the Mach number for drag divergence.

b. Calculate C_{D_0} and K for this ight condition. In doing this calculation, remember that there are two nacelles and two tip tanks.

3.6 Another form of the parabolic drag polar is given by

$$C_D = C_{D_m}(M) + K_m(M)[C_L \quad C_{L_m}(M)]^2$$

where $C_{D_m}; C_{L_m}$ de ne the minimum drag point. If the Mach number is given, nd the lift coe cient for maximum lift-to-drag ratio and the maximum lift-to-drag ratio for this polar.

Chapter 4

Cruise and Climb of an Arbitrary Airplane

In the mission segments known as cruise and climb, the accelerations of airplanes such as jet transports and business jets are relatively small. Hence, these performance problems can be studied by neglecting the tangential acceleration $\dot{V}$ and the normal acceleration $V\dot{\gamma}$ in the equations of motion. Broadly, this chapter is concerned with methods for obtaining the distance and time in cruise and the distance, time, and fuel in climb for an arbitrary airplane. What is meant by an arbitrary airplane is that cruise and climb performance is discussed in terms of the functional relations $D(h; V; L)$; $T(h; V; P)$, and $C(h; V; P)$. These functional relations can represent a subsonic airplane powered by turbojet engines, a supersonic airplane powered by turbofan engines, and so on. In order to compute the cruise and climb performance of a particular airplane, the functional relations are replaced by computer functions which give the aerodynamic and propulsion characteristics of the airplane.

In Chap. 5, the theory of this chapter is applied to an Ideal SBJ. There, approximate analytical formulas are used for the aerodynamics and propulsion. Analytical solutions are obtained for the distance and time in cruise and the distance, time, and fuel in climb.

The airplane is a controllable dynamical system which means that the equations of motion contain more variables than equations, that is, one or more mathematical degrees of freedom (MDOF). In standard performance problems, the MDOF are reduced to one, and it is associated with the velocity profile flown by the airplane during a particular mission leg. Then, the performance of an airplane can be computed for a particular velocity profile, or trajectory optimization can be used to find the optimal velocity profile. This is done by solving for the distance, time, and fuel in terms of the unknown velocity profile. After selecting distance, time or fuel as the performance index, optimization theory is used to find the corresponding optimal velocity profile. An example is to find the velocity profile which maximizes the distance in cruise from one weight to another, that is, for a given amount of fuel. Besides being important all by themselves, optimal trajectories provide yardsticks by which arbitrary trajectories (for example, constant velocity) can be evaluated.

This chapter begins with a discussion of how flight speeds are represented and what some limitations on flight speed are.

4.1 Special Flight Speeds

In presenting the performance characteristics of an aiplane, several quantities can be used to represent the velocity. The velocity of the airplane relative to the atmosphere, V, is called the *true airspeed*. If σ denotes the ratio of the atmospheric density at the altitude the airplane is operating, ρ, to that at sea level, ρ_s, the *equivalent airspeed* is defined as $V_e = \sqrt{\sigma}\,V$. Note that the equivalent airspeed is proportional to the square root of the dynamic pressure $q = \rho V^2/2$. This airspeed is important for low-speed flight because it can be measured mechanically. The air data system measures the dynamic pressure and displays it to the pilot as *indicated airspeed*. The indicated airspeed is the equivalent airspeed corrupted by measurement and instrument errors. To display the true airspeed to the pilot requires the measurement of static pressure, dynamic pressure, and static temperature. Then, the true airspeed must be calculated. For high-speed flight, it has become conventional to use *Mach number* as an indication of flight speed. High-speed airplanes have a combined airspeed indicator and Mach meter.

In this chapter, the performance of a subsonic business jet (the SBJ of App. A) is computed to illustrate the procedures and results. Because this airplane operates in both low-speed and high-speed regimes, the true airspeed is used to present performance. Note that true airspeed can be roughly converted to Mach number by using a = 1,000 ft/s. Other conversion formulas are useful. If a statute mile has 5,280 ft, 1 mi/hr =

1.4667 ft/s. If a nautical mile has 6,076 ft, 1 kt = 1.6878 ft/s. A kt is one nautical mile per hour.

4.2 Flight Limitations

The lowest speed at which an airplane can maintain steady level flight (constant altitude) is called the stall speed. In steady level flight, the equation of motion normal to the flight path is given by

$$L = W \qquad (4.1)$$

where the component of the thrust has been neglected. Because of the definition of lift coefficient, this equation can be written as

$$\frac{1}{2} C_L \; S_W V^2 = W \qquad (4.2)$$

and says that for a given weight the product $C_L V^2$ is constant. Hence, the lower the speed of the airplane is the higher the lift coefficient must be. Since there is a maximum lift coefficient, there is a minimum speed at which an airplane can be flown in steady level flight. This speed is called the stall speed and is given by

$$V_{stall} = \sqrt{\frac{2W}{\; S_W C_{L_{max}}}} : \qquad (4.3)$$

Some airplanes are speed limited by their structure or control capability. This limit can take the form of a maximum dynamic pressure, so that

$$V_{q_{max}} = \sqrt{\frac{2q_{max}}{\;}} \qquad (4.4)$$

by definition of dynamic pressure. This limit can affect the best climb speed of the airplane as well as the maximum speed.

At high speeds, some airplanes may have a maximum operating Mach number. Hence, by definition of Mach number, the limiting speed is

$$V_{M_{max}} = M_{max} a \qquad (4.5)$$

While this speed decreases with altitude in the troposphere, it is constant in the constant temperature part of the stratosphere.

Note that the stall speed and the maximum dynamic pressure speed are actually constant equivalent airspeeds.

4.3 Trajectory Optimization

An important part of airplane performance is the optimization of trajectories. These problems can be solved by a branch of mathematics called Calculus of Variations, which in recent times is also called Optimal Control Theory. A simple problem which occurs frequently in airplane trajectory optimization is to find the curve $y(x)$ which maximizes the performance index

$$J = \int_{x_0}^{x_f} f(x, y)\, dx \tag{4.6}$$

where x_0 and x_f are the initial and final values of the variable of integration. It is known that the curve $y(x)$ which maximizes the integral (4.6) satisfies the conditions

$$\left.\frac{\partial f}{\partial y}\right|_{x=\text{Const}} = 0, \quad \left.\frac{\partial^2 f}{\partial y^2}\right|_{x=\text{Const}} < 0. \tag{4.7}$$

The first condition gives the curve $y(x)$ which optimizes the integral, and the second condition identifies it as a maximum. For a minimum, < 0 is replaced by > 0. It is possible to verify the nature (maximum or minimum) of the optimal curve by looking at a plot of the function $f(x, y)$ versus y for several values of x.

Because of the simple form of Eq. (4.6), it is easy to verify Eq. (4.7). The integral is the area under the integrand. To maximize the integral, it is necessary to maximize the area. At each value of the variable of integration, the area is maximized by maximizing F with respect to y. The conditions for doing this are Eqs. (4.7).

4.4 Calculations

In this chapter, example calculations are made for the cruise and climb performance of the Subsonic Business Jet (SBJ) in App. A. The airplane is assumed to be operating in a constant gravity standard atmosphere, to have a parabolic drag polar $C_D = C_{D_0}(M) + K(M)C_L^2$, and to be powered by two GE CJ610-6 turbojets. In the calculation of atmospheric properties, drag, thrust, and specific fuel consumption, it is assumed that the altitude h, the velocity V, the weight W, and the power setting P are known. The calculations have been done on MATLAB, and three functions are needed.

The first function contains the standard atmosphere equations of Sec. 3.1. Given h, the function returns the atmospheric properties, particularly density $\rho(h)$ and speed of sound $a(h)$.

The second function computes the drag. With $M = V/a(h)$, Table 3.4 is interpolated for $C_{D_0}(M)$ and $K(M)$. Then, the lift coefficient is obtained from $C_L = 2W/\rho(h)SV^2$, since $L = W$, and the drag coefficient is computed from the parabolic drag polar. The drag is obtained from $D = (1/2)C_D\rho(h)SV^2$.

The third function calculates the thrust and specific fuel consumption of the turbojet. First, the corrected engine speed is determined from Eq. (3.61). Then, the corrected thrust and the corrected specific fuel consumption are obtained from Table 3.5 by a two-dimensional interpolation in terms of M and η. Finally, T and C are computed using the formulas at the beginning of Sec. 3.8.1.

In conclusion, given $h, V, W,$ and P, it is possible to calculate $\rho(h)$, $a(h)$, $D(h, V, W)$, $T(h, V, P)$, and $C(h, V, P)$

4.5 Flight Envelope

By definition, the flight envelope is the region of the velocity-altitude plane where the airplane can maintain steady level flight. The dynamic equations of motion for steady level flight ($\dot{V} = \dot{\gamma} = \gamma = 0$) are obtained from Eq. (2.29) as

$$\begin{aligned} T(h, V, P) - D(h, V, L) &= 0 \\ L - W &= 0 \end{aligned} \tag{4.8}$$

Then, since $L = W$,

$$T(h, V, P) - D(h, V, W) = 0. \tag{4.9}$$

If the altitude, weight, and power setting are given, this equation can be solved for the velocity.

In Fig. 4.1, the thrust and the drag are sketched versus the velocity. For a subsonic airplane, the drag has a single minimum whose value does not change with the altitude (see for example Fig. 3.3). Also, for a subsonic jet engine, the thrust does not vary greatly with the velocity, but it does change with the altitude, decreasing as the altitude increases. Fig. 4.1 shows that for some altitude, there exist two solutions of Eq. (4.9), a low-speed solution and a high-speed solution. As the altitude decreases (thrust increases), there is some altitude where the low-speed solution is the stall speed and at lower altitudes ceases to exist. As the altitude increases (thrust decreases), there is an altitude where there is only one solution, and above that altitude there are no solutions. The region of the velocity-altitude plane that contains all of the level flight solutions combined with whatever speed restrictions are imposed on the airplane, is called the *flight envelope*.

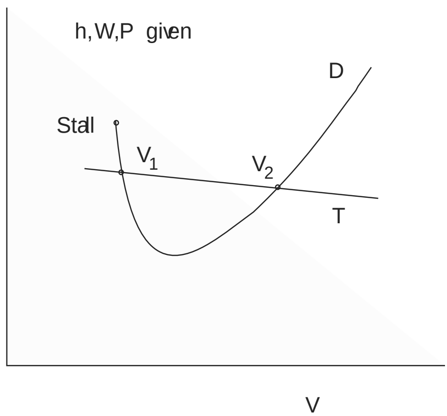

Figure 4.1: Thrust and Drag versus Velocity

The flight envelope has been computed for the SBJ weighing 11,000 lb and operating at maximum continuous thrust ($P = 0.98$). It is shown in Fig. 4.2. The level flight solutions for this weight and power setting are indicated by $T - D = 0$. The stall speed ($C_{L_{max}} = 1.24$), the maximum dynamic pressure ($q_{max} = 300$ lb/ft^2), and the maximum Mach number ($M_{max} = 0.81$) limits are also plotted versus the altitude. The region enclosed by these curves is the flight envelope of the SBJ. The highest altitude at which the airplane can be flown in steady level flight is called the *ceiling* and is around 50,000 ft. Note that the highest speed at which the airplane can be flown is limited by the maximum dynamic pressure or the maximum Mach number.

Next to be discussed are the distance and time during cruise. These quantities have been called the *range* and *endurance*. However, because there is considerable distance and time associated with the climb, the range, for example, could be defined as the sum of the distance in climb and the distance in cruise.

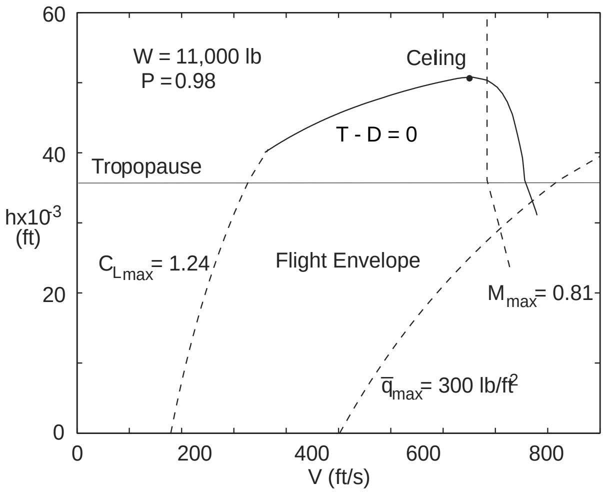

Figure 4.2: Flight Envelope of the SBJ

4.6 Quasi-steady Cruise

Of major importance in the mission profile is the cruise segment because airplanes are designed to carry a given payload a given distance. For a constant altitude cruise, the velocity vector is parallel to the ground, so that the equations of motion for quasi-steady level flight are given by Eqs. (2.29) with $\gamma = 0$, that is,

$$\dot{x} = V \tag{4.10}$$

$$0 = T(h, V, P) - D(h, V, L) \tag{4.11}$$

$$0 = L - W \tag{4.12}$$

$$\dot{W} = -C(h, V, P)T(h, V, P). \tag{4.13}$$

During a cruise, the altitude is constant and is not counted as a variable. Hence, these equations have two states, $x(t)$ and $W(t)$, three controls,

$V(t)$, $P(t)$, and $L(t)$, for a total of five variables. Since there are four equations, this system of equations has one mathematical degree of freedom, which is associated with the velocity profile $V(t)$.

The general procedure followed in studying quasi-steady airplane performance is to solve the equations of motion for each of the variables in terms of the unknown velocity profile. Then, given a velocity profile, the distance and the time for a given fuel can be determined. Since there are an infinite number of velocity profiles, it is desirable to find the one which optimizes some performance index. For cruise, there are two possible performance indices: distance (range) or time (endurance). Hence, the optimization problem is to find the velocity profile which maximizes the distance or the velocity profile which maximizes the time. This process is called trajectory optimization.

4.7 Distance and Time

To compute the distance and the time for a given amount of fuel (W_0 given, W_f given), Eqs. (4.11) and (4.12) require that $L = W$ and that

$$T(h;V;P) - D(h;V;W) = 0 \tag{4.14}$$

This equation can be solved for the power setting as

$$P = P(h;V;W)\,. \tag{4.15}$$

Next, the weight is made the variable of integration, and Eqs. (4.10) and (4.13) are rewritten as

$$\begin{aligned} \frac{dx}{dW} &= -\frac{V}{C(h;V;P)T(h;V;P)} \\ \frac{dt}{dW} &= -\frac{1}{C(h;V;P)T(h;V;P)} \end{aligned} \tag{4.16}$$

where all variables are now considered as functions of W. Then, Eq. (4.15) is used to eliminate the power setting so that

$$\begin{aligned} \frac{dx}{dW} &= -\frac{V}{C(h;V;P(h;V;W))T(h;V;P(h;V;W))} = F(W;V;h) \\ \frac{dt}{dW} &= -\frac{1}{C(h;V;P(h;V;W))T(h;V;P(h;V;W))} = G(W;V;h) \end{aligned} \tag{4.17}$$

where F and G are called the *distance factor* and the *time factor*. Since the altitude is constant and $V = V(W)$, the integration can be performed in principle to obtain $x(W)$ and $t(W)$ as follows:

$$x - x_0 = \int_W^{W_0} F(W; V; h)\,dW \tag{4.18}$$

$$t - t_0 = \int_W^{W_0} G(W; V; h)\,dW. \tag{4.19}$$

Finally, if these equations are evaluated at the final weight, expressions for the cruise distance and cruise time result, that is,

$$x_f - x_0 = \int_{W_f}^{W_0} F(W; V; h)\,dW \tag{4.20}$$

$$t_f - t_0 = \int_{W_f}^{W_0} G(W; V; h)\,dW. \tag{4.21}$$

For each velocity profile $V(W)$, there exists a distance and a time. Once a velocity profile has been selected, the distance and the time for a given amount of fuel can be obtained. In general, the weight interval $W_0 - W_f$ is divided into n subintervals, and the integrals are rewritten as

$$\begin{aligned} x_f - x_0 &= \textstyle\sum_{k=1}^{n} \int_{W_k}^{W_{k+1}} F(W; V; h)\,dW \\ t_f - t_0 &= \textstyle\sum_{k=1}^{n} \int_{W_k}^{W_{k+1}} G(W; V; h)\,dW \end{aligned} \tag{4.22}$$

where

$$W_1 = W_f\,; \quad W_{n+1} = W_0. \tag{4.23}$$

Then, the distance factor and the time factor are assumed to vary linearly with the weight over each subinterval, that is,

$$\begin{aligned} F &= F_k + \tfrac{F_{k+1} - F_k}{W_{k+1} - W_k}(W - W_k) \\ G &= G_k + \tfrac{G_{k+1} - G_k}{W_{k+1} - W_k}(W - W_k) \end{aligned} \tag{4.24}$$

so that Eqs. (4.22) can be integrated analytically to obtain

$$x_f - x_0 = \sum_{k=1}^{n} \frac{1}{2}(F_{k+1} + F_k)(W_{k+1} - W_k) \tag{4.25}$$

$$t_f - t_0 = \sum_{k=1}^{n} \frac{1}{2}(G_{k+1} + G_k)(W_{k+1} - W_k). \tag{4.26}$$

In general, the number of intervals which must be used to get a reasonably accurate solution is small, sometimes just one.

While the distance and time can be computed for different velocity profiles such as constant velocity or constant lift coefficient, it is important for design purposes to find the maximum distance trajectory and the maximum time trajectory.

4.8 Cruise Point Performance for the SBJ

The analysis of the distance factor F and the time factor G is called *point performance* because only points of a trajectory are considered. For a fixed altitude, this can be done by plotting F and G versus velocity for several values of the weight. Regardless of the weight interval $[W_0, W_f]$ that is being used to compute distance and time, F and G can be computed for all values of W at which the airplane might operate. The use of F and G to compute the distance and time for a given velocity profile $V(W)$ and a given weight interval $[W_0, W_f]$ is called *path performance*, because a whole path is being investigated.

Point performance for the SBJ begins with the solution of Eq. (4.14) for the power setting $P(h, V, W)$ using Newton's method. It is shown in Fig. 4.3 for $h = 35{,}000$ ft. Note that the power setting is around 0.90. Then, P is substituted into Eqs. (4.17) to get the distance factor $F(W, V, h)$ and the time factor $G(W, V, h)$. Values of F and G have been computed for many values of the velocity (1 ft/s intervals) and for several values of the weight. These quantities are plotted in Figs. 4.4 and 4.5.

It is observed from Fig. 4.4 that the distance factor has a maximum with respect to the velocity for each value of the weight. This maximum has been found from the data used to compute F. At each weight, the velocity that gives the highest value of F is assumed to represent the maximum. Values of $V(W)$, $F_{max}(W)$, and the corresponding values of $G(W)$ are listed in Table 4.1 and plotted in Figs. 4.6 and 4.7. Note that V and $F_{max}(W)$ are nearly linear in W.

It is observed from Fig. 4.5 that the time factor has a maximum with respect to the velocity for each value of the weight. This maximum has been found from the data used to compute G. At each weight, the velocity that gives the highest value of GF is assumed to represent the

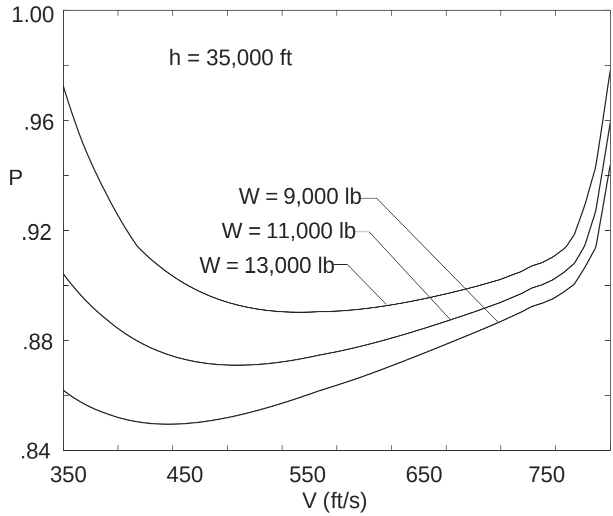

Figure 4.3: Power Setting (SBJ)

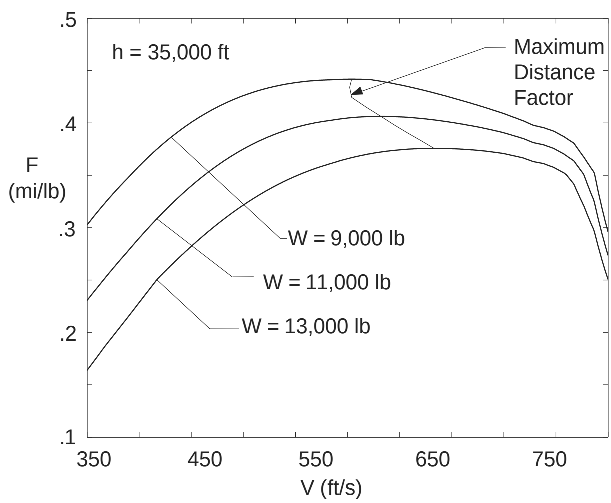

Figure 4.4: Distance Factor (SBJ)

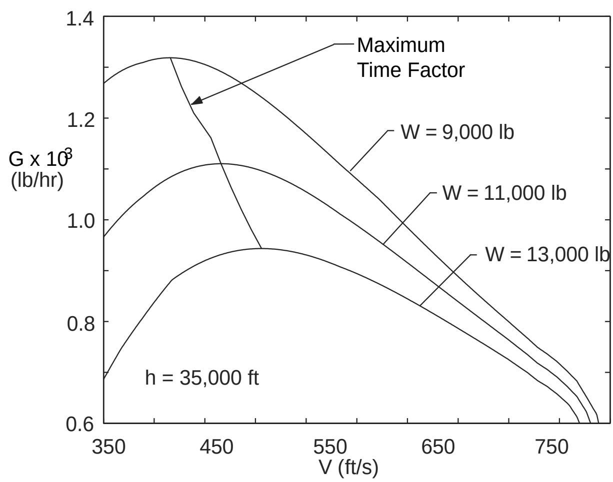

Figure 4.5: Time Factor (SBJ)

maximum. Values of $V(W)$, $G_{max}(W)$, and the corresponding values of $F(W)$ are listed in Table 4.1 and plotted in Figs. 4.6 and 4.7. Note that V and $G_{max}(W)$ are nearly linear in W.

Note that the velocity for maximum distance factor is roughly 35% higher than the velocity for maximum time factor.

4.9 Optimal Cruise Trajectories

At this point, there are two approaches which can be followed. One is to specify a velocity pro le, say for example V = Const, and compute the distance and time for a given W_0 and W_f. The other is to nd the velocity pro le $V(W)$ that optimizes the distance or that optimizes the time. Because distance factor has a maximum, the optimal distance trajectory is a maximum. Similarly, because the time factor has a maximum, the optimal time trajectory is a maximum.. Optimal trajectories are considered rst because they provide a yardstick with which other trajectories can be measured.

Table 4.1 SBJ Optimal Cruise Point Performance

h = 35,000 ft

	Maximum Distance			Maximum Time		
W lb	V ft/s	F_{max} mi/lb	G hr/lb	V ft/s	G_{max} hr/lb	F mi/lb
9,000	604	.442	1.073E-3	416	1.386E-3	.374
9,500	602	.434	1.058E-3	427	1.262E-3	.367
10,000	605	.424	1.031E-3	439	1.210E-3	.362
10,500	618	.415	.985E-3	456	1.161E-3	.361
11,000	631	.406	.943E-3	466	1.110E-3	.353
11,500	645	.398	.905E-3	477	1.064E-3	.345
12,000	658	.390	.870E-3	487	1.021E-3	.338
12,500	671	.383	.837E-3	496	.981E-3	.332
13,000	683	.376	.807E-3	506	.944E-3	.326

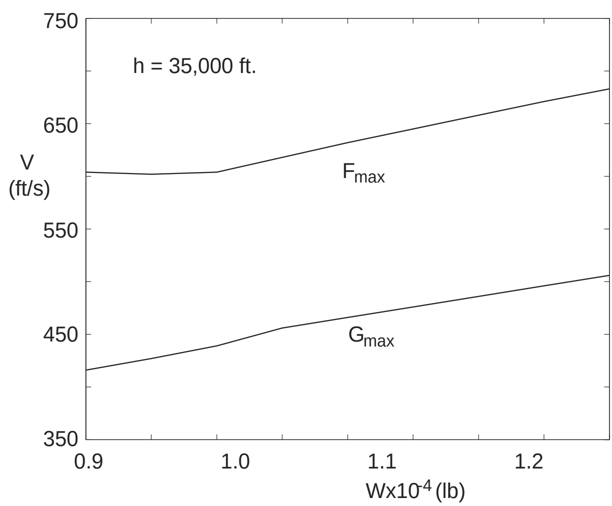

Figure 4.6: Optimal Velocity Pro les (SBJ)

4.9.1 Maximum distance cruise

The velocity pro le $V(W)$ for maximizing the distance (4.20) is obtained by maximizing the distance factor with respect to the velocity for each

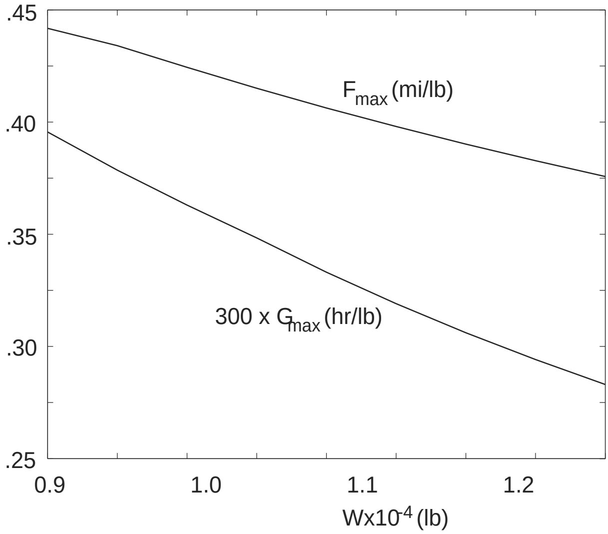

Figure 4.7: Maximum Distance and Time Factors (SBJ)

value of the weight (see Sec. 4.3). This velocity profile is then used to compute the maximum distance factor which is used to compute the maximum distance and to compute the time factor which is used to compute the time along the maximum distance trajectory.

This process has been carried out for the SBJ at h = 35,000 ft with W_0 = 12,000 lb and W_f = 10,000 lb ($W_0 - W_f$ = 2,000 lb of fuel). The values of F_{max} and G shown in Table 4.1 are used to compute the maximum distance and the corresponding time from Eqs. (4.25) and (4.26). As an example, to compute the maximum distance and the corresponding time for h = 35,000 ft, W_0 = 12,000 lb, and W_f =10,000 lb (2000 lb of fuel) using 5,000 lb weight intervals (n = 4), the values to be used in Eqs. (4.25) and (4.26) are listed in Table 4.2. Then, Eq. (4.25) gives the maximum distance of 813 mi (see Table 4.3). Similarly, the time along the maximum distance trajectory is obtained from Eq. (4.26) as 1.90 hr. This computation has also been made for one interval (n = 1), and the results agree well with those of n = 4. This happens because F_{max} (Fig. 4.5) and G are nearly linear in W.

Table 4.2 Maximum Distance Cruise

$h = 35{,}000$ ft, $W_0 = 12{,}000$ lb, $W_f = 10{,}000$ lb

k	W_k (lb)	$F_{max;k}$ (mi/lb)	G_k (hr/lb)
1	10,000	.424	.001031
2	10,500	.415	.000985
3	11,000	.406	.000943
4	11,500	.398	.000905
5	12,000	.390	.000870

Table 4.3: SBJ Optimal Cruise Path Performance

$h = 35{,}000$ ft, $W_0 = 12{,}000$ lb, $W_f = 10{,}000$ lb

		4 Intervals	1 Interval
Maximum Distance	Distance (mi)	813	814
	Time (hr)	1.90	1.90
Maximum Time	Distance (mi)	704	700
	Time (hr)	2.20	2.23

To actually fly the maximum distance velocity profile, it is necessary to know the weight as a function of time. If it is not available, the optimal path can only be approximated. Other velocity profiles are possible: constant lift coefficient, constant velocity, constant power setting, etc. The importance of the optimal profile is that the usefulness of the other profiles can be evaluated. For example, if a particular velocity profile is easy to fly and it gives a distance within a few percent of the maximum distance, it could be used instead. It can be shown that the maximum distance is almost independent of the velocity profile $V(W)$.

4.9.2 Maximum time cruise

The velocity profile $V(W)$ for maximizing the time (4.21) is obtained by maximizing the time factor with respect to the velocity for each value of the weight (see Sec. 4.3). This velocity profile is then used to compute the maximum time factor which is used to compute the maximum time and to compute the distance factor which is used to compute the distance along the maximum time trajectory.

This process has been carried out for the SBJ with h = 35,000 ft, W_0 =12,000 lb, and W_f = 10,000 lb ($W_0 - W_f$ = 2,000 lb of fuel). The optimal velocity profile $V(W)$, the maximum time factor $G_{max}(W)$, and the distance factor $F(W)$ are shown in Table 4.1. Then, the maximum time and the corresponding distance are obtained from Eqs. (4.25) and (4.26). For the maximum time of the SBJ, the maximum time has been found by using four intervals to be 2.20 hr and the distance is 704 mi (Table 4.3). These results have also been obtained using only one interval (700 mi and 2.23 hr). The agreement between the results using one interval and the results obtained by using four intervals is very good.

4.10 Constant Velocity Cruise

In this section an example of arbitrarily specifying the velocity profile is presented. The maximum distance path requires that the velocity change as the weight changes (Table 4.1). Since there is no weight meter on an airplane, the pilot cannot fly this trajectory very well. At constant altitude, the indicated airspeed is proportional to the airspeed. Hence, the pilot can fly a constant velocity trajectory fairly well even though the controls must be adjusted to maintain constant velocity.

To obtain the distance for a particular velocity, the values of the distance factor F for that velocity for several values of the weight are used with Eq. (4.25). Similarly, the time along a constant velocity path is obtained by using the values of the time factor G for that velocity for several values of the weight and by using Eq. (4.26).

This process has been carried out for the SBJ operating at h = 35,000 ft from W_0 = 12,000 lb to W_f = 10,000 lb (2,000 lb of fuel). The results are shown in Table 4.4 for several values of the velocity used for the cruise. With regard to the distance, note that there is a cruise velocity for which the distance has a best value. This speed can be calculated by computer or by curve fitting a parabola to the three points containing the best distance. The curve fit leads to V = 634 ft/s and $x_f - x_0$ = 812 mi. This value of the distance is almost the same as the maximum distance $x_f \quad x_0 = 813$ mi. Hence, the airplane can be flown at constant velocity and not lose much distance relative to the maximum. It is emphasized that this conclusion could not have been reached without having the maximum distance path. As an aside, it is probably true that the airplane can be flown with any velocity profile (for example, constant power setting) and get close to the maximum distance.

A similar analysis with similar results can be carried out for the time.

Note that the term maximum distance is applied to the case where all possible velocity profiles are in contention for the maximum. On the other hand the term best distance is used for the case where the class of paths in contention for the maximum is restricted, that is, constant velocity paths. The maximum distance should be better than or at most equal to the best distance.

Table 4.4 Constant Velocity Cruise

$h = 35{,}000$ ft, $W_0 = 12{,}000$ lb, $W_f = 10{,}000$ lb

V (ft/s)	Distance (mi)	Time (hr)	V (ft/s)	Distance (mi)	Time (hr)
350	461	1.93	600	809	1.98
400	582	2.13	650	812	1.83
450	680	2.22	700	801	1.68
500	750	2.20	750	781	1.53
550	791	2.11	800	749	1.37

4.11 Quasi-steady Climb

The equations of motion for quasi-steady climbing flight are given by Eqs. (2.29), that is,

$$\dot{x} = V \tag{4.27}$$

$$\dot{h} = V \tag{4.28}$$

$$0 = T(h;V;P) \quad D(h;V;L) \quad W \tag{4.29}$$

$$0 = L \quad W \tag{4.30}$$

$$\dot{W} = \quad C(h;V;P)T(h;V;P) \tag{4.31}$$

This system of ve equations has seven variables, $x(t); h(t); W(t); V(t)$, (t), $P(t)$, and $L(t)$. Hence, it has two mathematical degrees of freedom. Since it is easy to solve for L and , the degrees of freedom are associated with V and P. Experience shows that it is best to climb at maximum continuous thrust so the power setting is held constant leaving one degree of freedom, the velocity.

During the climb, an aircraft consumes around 5% of its weight in fuel. Hence, it is possible to assume that the weight of the aircraft is constant on the right-hand sides of the equations of motion. Then, the integration of the weight equation gives an estimate of the fuel consumed during the climb.

Since $L = W$, Eq. (4.28) can be solved for the ight path inclination or climb angle as

$$= \frac{T(h;V;P) \quad D(h;V;W)}{W} : \tag{4.32}$$

Two other important quantities are the rate of climb

$$\dot{h} = V \quad = \left[\frac{T(h;V;P) \quad D(h;V;W)}{W} \right] V \tag{4.33}$$

and the fuel factor

$$H = \frac{dh}{dW} = \frac{\dot{h}}{\dot{W}} = \frac{\frac{[T(h;V;P) \quad D(h;V;W)]V}{W}}{C(h;V;P)T(h;V;P)} : \tag{4.34}$$

In order to be able to solve for the distance, the time, and the fuel in climbing from one altitude to another, the altitude is made the variable of integration. The di erential equations of motion become the following:

$$\begin{aligned} \frac{dx}{dh} &= \frac{1}{(h;V;P;W)} \\ \frac{dt}{dh} &= \frac{1}{\dot{h}(h;V;P;W)} \\ \frac{dW}{dh} &= \frac{1}{H(h;V;P;W)}; \end{aligned} \tag{4.35}$$

where all variables are now functions of h. Since P and W are constant and $V = V(h)$, the integration can be performed to obtain $x(h)$, $t(h)$, and $W(h)$:

$$x - x_0 = \int_{h_0}^{h} \frac{1}{\gamma(h, V, P, W)}\, dh \tag{4.36}$$

$$t - t_0 = \int_{h_0}^{h} \frac{1}{\dot{h}(h, V, P, W)}\, dh \tag{4.37}$$

$$W_0 - W = \int_{h_0}^{h} \frac{1}{H(h, V, P, W)}\, dh\,. \tag{4.38}$$

Finally, if these equations are evaluated at the final altitude, the following expressions result for the distance, the time, and the fuel :

$$x_f - x_0 = \int_{h_0}^{h_f} \frac{1}{\gamma(h, V, P, W)}\, dh \tag{4.39}$$

$$t_f - t_0 = \int_{h_0}^{h_f} \frac{1}{\dot{h}(h, V, P, W)}\, dh \tag{4.40}$$

$$W_0 - W_f = \int_{h_0}^{h_f} \frac{1}{H(h, V, P, W)}\, dh. \tag{4.41}$$

Hence, there are three possible optimal trajectories: minimum distance, minimum time, or minimum fuel.

Once the velocity profile is known, the distance, the time, and the fuel can be obtained by approximate integration. Here, the altitude interval $[h_0, h_f]$ is divided into n subintervals, that is,

$$\begin{aligned} x_f - x_0 &= \textstyle\sum_{k=1}^{n} \int_{h_k}^{h_{k+1}} dh/\gamma \\ t_f - t_0 &= \textstyle\sum_{k=1}^{n} \int_{h_k}^{h_{k+1}} dh/\dot{h} \\ W_0 - W_f &= \textstyle\sum_{k=1}^{n} \int_{h_k}^{h_{k+1}} dh/H \end{aligned} \tag{4.42}$$

where $h_1 = h_0$ and $h_{n+1} = h_f$. Next, it is assumed that the climb angle, the rate of climb, and the fuel factor vary linearly with the altitude over each altitude interval as follows:

$$\begin{aligned} \gamma &= \gamma_k + (\Delta\gamma/\Delta h)(h - h_k) \\ \dot{h} &= \dot{h}_k + (\Delta\dot{h}/\Delta h)(h - h_k) \\ H &= H_k + (\Delta H/\Delta h)(h - h_k) \end{aligned} \tag{4.43}$$

where

$$(\ ) = (\ )_{k+1} \quad (\ )_k : \qquad (4.44)$$

If Eqs. (4.43) are substituted into Eqs. (4.42) and the integrations are performed, the following approximate expressions are obtained for the distance, the time, and the fuel consumed during the climb:

$$\begin{aligned} x_f \quad x_0 &= \textstyle\sum_{k=1}^{n} (\ h/\ \ ) \ln(\ _{k+1}/\ _k) \\ t_f \quad t_0 &= \textstyle\sum_{k=1}^{n} (\ h/\ \dot{h}) \ln(\dot{h}_{k+1}/\dot{h}_k) \qquad (4.45) \\ W_0 \quad W_f &= \textstyle\sum_{k=1}^{n} (\ h/\ H) \ln(H_{k+1}/H_k) : \end{aligned}$$

4.12 Climb Point Performance for the SBJ

Climb point performance involves the study of the ight path angle , the rate of climb $\dot{h}$ and the fuel factor H as de ned in Eqs. (4.32) through (4.34). Values of these quantities have been computed for many values of the velocity (1 ft/s intervals) and several values of the altitude for W =11,000 lb and P = 0.98.

The ight path angle is presented in Fig. 4.8. Note that the maximum occurs at sea level and is around 22 deg. As the altitude increases the maximum reduces to zero at the ceiling. At each altitude, the maximum is determined by nding the velocity (computed at 1 ft/s intervals) that gives the highest value of . This value of the velocity is used to compute the rate of climb and the fuel factor. Then $V(h)$, $_{max}(h)$, $\dot{h}(h)$, and $H(h)$ are listed for several values of h in Table 4.5 and plotted in Figs. 4.11 and 4.12.

The rate of climb is shown in Fig. 4.9. Note that the maximum $\dot{h}$ occurs at sea level and is around 150 ft/s (9,000 ft/min). As the altitude increases the maximum $\dot{h}$ reduces to zero at the ceiling. At each altitude, the maximum $\dot{h}$ is determined by nding the velocity that gives the highest value of $\dot{h}$ which is computed at 1 ft/s intervals. This value of the velocity is used to compute the ight path angle and the fuel factor. Then, $V(h)$, (h), $\dot{h}_{max}(h)$, and $H(h)$ are listed for several values of h in Table 4.6 and plotted in Figs. 4.11 and 4.12.

Because it is not possible to y at the airplane ceiling, several other ceilings have been de ned. The *service ceiling* is the altitude at which the maximum rate of climb is 100 ft/min. The *cruise ceiling* is

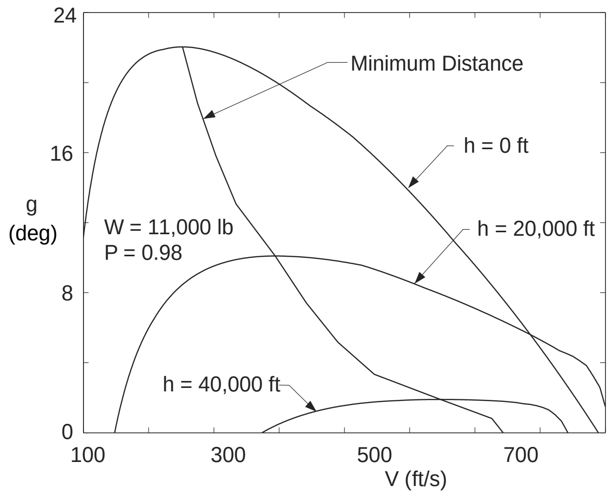

Figure 4.8: Climb Angle (SBJ)

Table 4.5 SBJ Optimal Climb Point Performance:

W=11,000 lb, P=.98

	Maximum			
h ft	V ft/s	max deg	h ft/s	H ft/lb
0	252	22.0	96.9	63.4
5,000	275	18.8	90.3	66.6
10,000	303	15.8	83.7	70.0
15,000	334	13.1	76.2	72.7
20,000	394	10.1	69.4	79.0
25,000	442	7.40	57.1	79.5
30,000	490	5.17	44.2	75.7
35,000	546	3.34	31.8	66.7
40,000	647	1.90	21.5	52.6
45,000	726	0.82	10.4	29.7

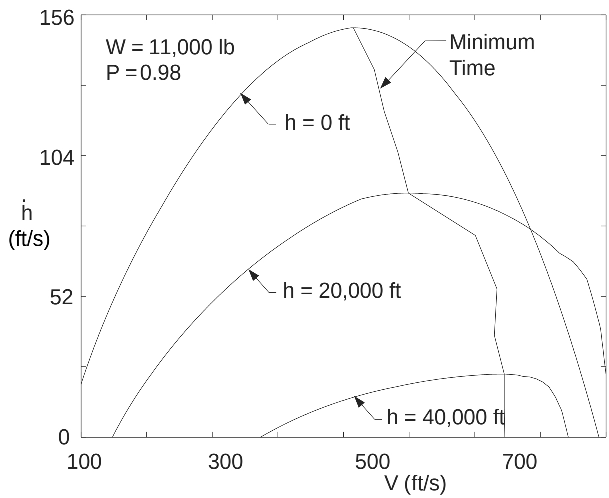

Figure 4.9: Rate of Climb (SBJ)

Table 4.6 SBJ Optimal Climb Point Performance: $\dot{h}$

W=11,000 lb, P=.98

	Maximum $\dot{h}$			
h ft	V ft/s	deg	$\dot{h}_{max}$ ft/s	H ft/lb
0	515	16.8	151.	96.8
5,000	547	14.2	136.	96.5
10,000	562	12.3	120.	96.0
15,000	583	10.4	105.	94.7
20,000	599	8.63	90.2	92.4
25,000	701	6.09	74.5	84.9
30,000	734	4.27	54.7	75.9
35,000	730	2.95	37.6	66.5
40,000	745	1.79	23.3	51.6
45,000	745	0.81	10.9	20.8

the altitude where the maximum rate of climb is 300 ft/min. Thecombat ceiling is the altitude where the maximum rate of climb is 500 ft/min.

The fuel factor is presented in Fig. 4.10. Note that the maximum H occurs at sea level and is around 100 ft/lb. As the altitude increases the maximum H reduces to zero at the ceiling. At each altitude, the maximum H is determined by nding the velocity (computed at 1 ft/s intervals) that gives the highest value of . This value of the velocity is used to compute the rate of climb and the fuel factor. Then, $V(h)$, (h), $\dot{h}(h)$, and $H_{max}(h)$ are listed for several values of h in Table 4.7 and plotted in Figs. 4.11 and 4.12.

4.13 Optimal Climb Trajectories

There are three possible performance indices for computing optimal climb trajectories: distance, time, or fuel. Because , $\dot{h}$, and H each have a maximum, the distance, time and fuel trajectories each have a minimum.

4.13.1 Minimum distance climb

The velocity pro le for minimizing the distance (4.39) is obtained by maximizing the climb angle with respect to the velocity at each value of the altitude (Sec. 4.3). This velocity pro le is used to compute the maximum climb angle, the rate of climb, and the fuel factor which are used to compute the minimum distance and the time and fuel along the minimum distance trajectory. This process has been carried out for the SBJ climbing from sea level to h = 35,000 ft with W = 11,000 lb and P = 0.98. The optimal velocity pro le $V(h)$, the maximum climb angle (h), the rate of climb $\dot{h}(h)$ and the fuel factor $H(h)$ shown in Table 4.5 are used to compute the minimum distance and the time and fuel along the minimum distance trajectory from Eqs. (4.45). These quantities have been found by using four intervals to be 42.2 mi, 9.22 min, and 484. lb respectively (Table 4.8). Note that the fuel is around 4% of the climb weight, justifying the approximation of weight constant on the right-hand sides of the climb equations of motion.

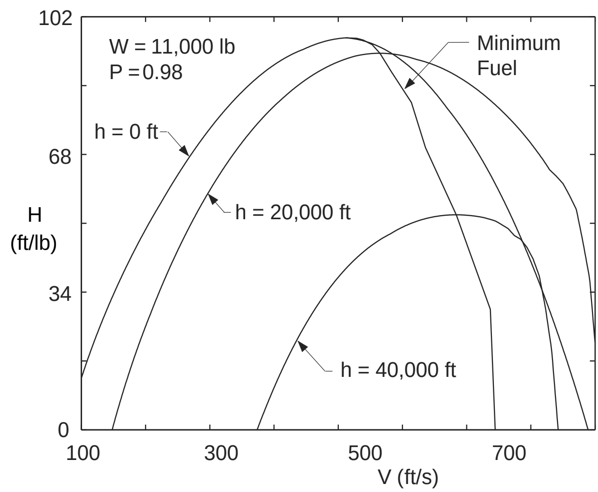

Figure 4.10: Fuel Factor (SBJ)

Table 4.7 SBJ Optimal Climb Point Performance:H

W=11,000 lb, P=.98

	Maximum H			
h ft	V ft/s	 deg	$\dot{h}$ ft/s	H_{max} ft/lb
0	512	16.9	151.	96.8.
5,000	529	14.7	136.	96.7.
10,000	539	12.8	120.	96.3
15,000	553	10.9	105.	95.1
20,000	565	9.10	89.7	93.0
25,000	583	6.89	70.1	88.4
30,000	614	4.86	52.1	80.8
35,000	636	3.24	35.9	69.7
40,000	684	1.89	22.5	53.1
45,000	737	0.82	10.5	29.8

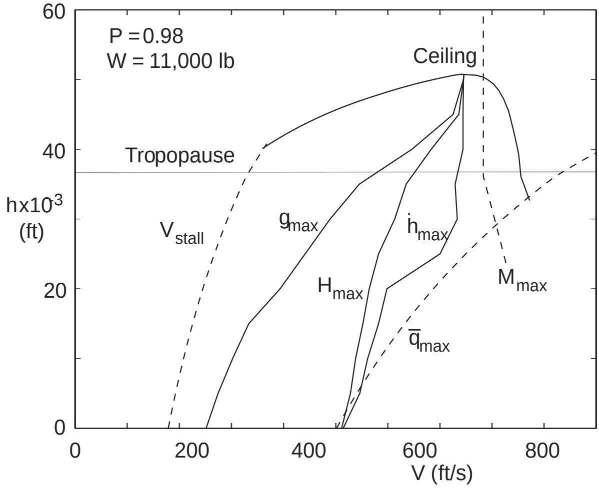

Figure 4.11: Optimal Velocity Pro les (SBJ)

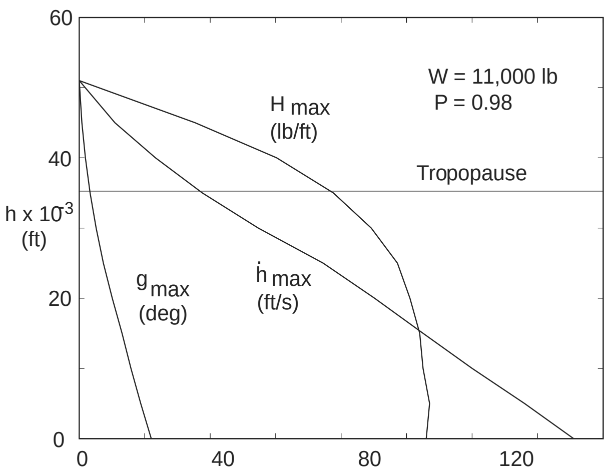

Figure 4.12: Maximum Climb Angle and Rate of Climb (SBJ)

These results have also been obtained using only one interval. The agreement between n = 1 and n = 4 is very good in and h because they are nearly linear in h. The error in H is around 10%.

4.13.2 Minimum time climb

The velocity pro le for minimizing the time (4.40) is obtained by maximizing the rate of climb with respect to the velocity at each value of the altitude (Sec. 4.3). This velocity pro le is used to compute the climb angle, the maximum rate of climb, and the fuel factor which are used to compute the minimum time and the distance and fuel along the minimum time trajectory.

This process has been carried out for the SBJ climbing from sea level to h = 35,000 ft with W = 11,000 lb and P = 0.98. The optimal velocity pro le V(h), the climb angle (h), the maximum rate of climb h(h) and the fuel factor H(h) shown in Table 4.6 are used to compute the distance, the time, and the fuel along the minimum time trajectory from Eqs. (4.45). These quantities have been found by using four intervals to be 51.4 mi, 6.97 min, and 399 lb respectively (Table 4.8). Note that the fuel is less than 4% of the climb weight, justifying the approximation of weight constant on the right-hand sides of the climb equations of motion.

These results have also been obtained using only one interval. The agreement between n = 1 and n = 4 is very good in and h because they are nearly linear in h. The error in H is less than 10%.

4.13.3 Minimum fuel climb

The velocity pro le for minimizing the distance (4.41) is obtained by maximizing the fuel factor with respect to the velocity at each value of the altitude (Sec. 4.3). This velocity pro le is used to compute the climb angle, the rate of climb, and the fuel factor which are used to compute the minimum fuel and the distance and time along the minimum fuel trajectory.

This process has been carried out for the SBJ climbing from sea level to h = 35,000 ft with W = 11,000 lb and P = 0.98. The optimal velocity pro le V(h), the climb angle (h), the rate of climb h(h) and the maximum fuel factor H(h) shown in Table 4.7 are used to compute the distance, the time, and the fuel along the minimum fuel trajectory from Eqs. (4.45). These quantities have been found by using four intervals to be 47.2 mi, 7.17 min, and 390 lb respectively (Table 4.8). Note that the fuel is less than 4% of the climb weight, justifying the approximation of weight constant on the right-hand sides of the climb equations of motion.

These results have also been obtained using only one interval. The agreement between n = 1 and n = 4 is very good in and h because they are nearly linear in h. The error in H is less than 10%.

Table 4.8 SBJ Optimal Climb Path Performance

$W = 11{,}000$ lb, $P = .98$, $h_0 = 0$ ft, $h_f = 35{,}000$ ft

		7 Intervals	1 Interval
Minimum Distance	Distance (mi)	42.2	38.4
	Time (min)	9.22	10.0
	Fuel (lb)	484.	538.
Minimum Time	Distance (mi)	51.4	47.7
	Time (min)	6.97	7.15
	Fuel (lb)	399.	433
Minimum Fuel	Distance (mi)	47.2	45.8
	Time (min)	7.17	7.28
	Fuel (lb)	390.	424.

4.14 Constant Equivalent Airspeed Climb

An example of selecting an arbitrary velocity pro le is to assume that the airplane is own at constant equivalent airspeed. Note that in Fig. 4.11 the optimal velocity pro les for the SBJ are functions of the altitude and may be di cult to y. On the other hand, the pilot has an instrument for equivalent (indicated) airspeed, so it is possible to y a constant equivalent airspeed trajectory. Here, the velocity pro le is given by

$$V(h) = \frac{V_e}{\sqrt{(h)}} \qquad (4.46)$$

4.15 Descending Flight

In the mission pro le, the descent segment is replaced by extending the cruise segment and, hence, is not very important. However, descending ight is just climbing ight for the case where the thrust is less than the drag. Here, the descent angle becomes negative so the descent angle is de ned as = ; the rate of climb becomes negative so the rate of descent is de ned as $\dot{z} = \dot{h}$; and the fuel factor is de ned as $dh=dW$ since both dh and dW are negative. The calculation of the minimum descent angle, the minimum descent rate, and the minimum fuel is the same as that for climbing ight, as is the calculation of the distance, time, and fuel.

Problems

All of the numbers in this chapter have been computed for the SBJ in App. A. Make similar computations for the airplane of Fig. 4.13 which is the SBJ of App. A with a lengthened fuselage to accommodate more passengers and with two Garrett TFE 731-2 turbofan engines. The take-o gross weight is 17,000 lb which includes 800 lb of reserve fuel and 6,200 lb of climb/cruise fuel.

1. Create functions that calculate the atmospheric properties, the drag, and the thrust and SFC. To calculate the drag, you need $C_{D_0}(M)$ and $K(M)$.

2. Calculate the ight envelope.

3. Calculate the maximum distance trajectory in cruise.

4. Calculate the minimum time trajectory in climb.

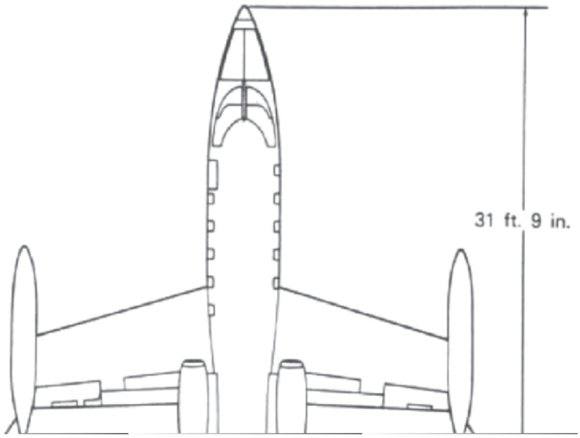

Figure 4.13: Turbofan Business Jet