# IMPROVED FREQUENCY WEIGHTED MODEL REDUCTION TECHNIQUE

By

AURBA SHAHID KHAN

00000204797

A thesis submitted to the faculty of Information Security Department, Military College of Signals, National University of Sciences and Technology, Islamabad, Pakistan, in partial fulfillment of the requirements for the degree of MS in Systems Engineering

JUNE 2021

# ABSTRACT

In today's era, having a simpler mathematical model is very necessary to analyze and understand a system. "Model Order Reduction" (MOR) is the technique which helps reducing our efforts and computational complexities of the mathematical model which is to be analyzed to understand a system's behavior. It helps us in complex numerical simulation by reducing the order of the system, without bringing a change in its basic properties. MOR is actually the procedure of transforming a higher order complex system into a lower order and non complex system with a reasonable accuracy so we may design the system easily and model and simulate a large complex system doing less effort. Balanced truncation is most commonly used MOR technique that ensures stability and yields an error bound for full frequency range. Sometimes we desire to reduce the system over a specific frequency band, so it motivates the use of frequency weightings in MOR. This thesis focuses on model reduction techniques using frequency weightings. First a full order system is considered and then frequency weighted model order reduction is carried out on that system using proposed techniques, hence yielding a stable reduced order model (ROM). Stability will be guaranteed by having positive/ semidefinite input and output matrices hence assuring the positive semi-definiteness of observability and controllability Gramians. These Gramians will help us formulate a transformation matrix by the help of which we will find new state space realizations of the ROM (internally balanced realizations) which will be stable and gives computable error bounds and approximation error. Existing MOR techniques do preserve stability along with relatively large approximation errors and error bounds but proposed research aim is to develope techniques that yield low approximation error and computable error bound as compared to existing stability preserving techniques.

# DEDICATION

*I dedicate this thesis to my Family, especially my daughter Mirha, Teachers and Friends for their endless love, encouragement and emotional support and most important of all dedicated to myself for the tremoundous will power of not giving up how ever the situations remained tough.*

# ACKNOWLEDGEMENTS

Foremost of all, I am thankful to Allah SWT for showering His blessings upon me and giving me the courage and strength to conclude this thesis. I can not express enough gratitude towards my thesis supervisor Dr Muhammad Imran for his endless support. Without his support I would not have even moved a step forward. I am grateful to him for his understanding behaviour towards my barriers and issues. My supervisors's sincerity and vision has really inspired me. It was a great previlege to work under his guidance.

I would also thank my family, especially my mother and husband, Iftikhar for helping me throughout my tough schedule, supporting me in managing my thesis and one year old daughter. My daughter's smile gave me the constant courage to complete this endaveour. I would also like to mention about Ms. Sammana Batool whose help and guidance made me accomplish the tasks and also my friends Kinza Kazmi and Fatima Tariq whose constant motivation and sweet words made me feel joyous enough in my hard times.

In the end, I would acknowledge myself for the constant faith in my innerself, continously convining myself to accomplish the research.

# TABLE OF CONTENTS

# LIST OF FIGURES

# LIST OF TABLES

# ACRONYMS

| | |
|---|---|
| MOR | Model Order Reduction |
| ROMs | Reduced Order Models |
| BT | Balanced Truncation |
| EVD | Eigen Value Decomposition |
| SVD | Singular Value decomposition |
| HSV | Hankel Singular Values |
| Diag | Diagonal |
| FWMR | Frequency Weighted Model Reduction |
| FLMR | Frequency Limited Model Reduction |

*Chapter 1*

# INTRODUCTION

## 1.1 Overview of Model Order Reduction

While designing a physical system we aim to get some desired performance specifications. For this purpose, it is necessary to understand the dynamical behaviour of that designed system. Understanding this dynamical behaviour might become a difficult task to achieve if the system is complex, hence attaining a simple and reduced mathematical model for a physical system is the need of hour in today's era. In spite of the technological success and progressions, designing of a large system (semiconductor devices, electronic circuits, fluid mechanics, image compression) results in a high-dimensional mathematical model which are difficult to understand. As a result, simulation, computation and cost becomes non efficient and unbearable. Therefore, a simple, non-complex and a ROM is needed instead of complex systems to have easy simulation and analysis. Model Order Reduction (MOR) is the technique which helps in reducing a complex and large-scale model into a lower order model to have a good estimation of the given full order original system, retaining its key properties like stability, passivity and input and output behavior of system. In order to handle the difficulties (caused by having a higher order model system) in a better and more robust way, MOR is the solution. Using MOR technique, we can simulate a large complex system doing less effort by converting it into a lower order and non complex system. MOR proves that it is a well - known simulation tool for a vast area of issues that relate to both research and industrial applications [1] - [22].

As Fig 1.1 explains that when any physical system(s) along with some data is modelled, so a number of ODEs and PDEs are attained. Simulation and control of complex ODEs are highly costly, therefore, doing MOR to have a reduced number of ODEs can be really helpful in simulation and control of the system under consideration.

There are multiple factors which are considered significant in MOR. Approximation error is one of those factors and it is the error between the original system and the reduced system.

MOR also demands to preserve the other factors of system such as stability, passivity and input and output behavior of the system. These factors make MOR techniques computational and time efficient. Error bound gives good approximation for selection of any MOR technique as it gives idea how much error can be accepted for the concerned specific application.

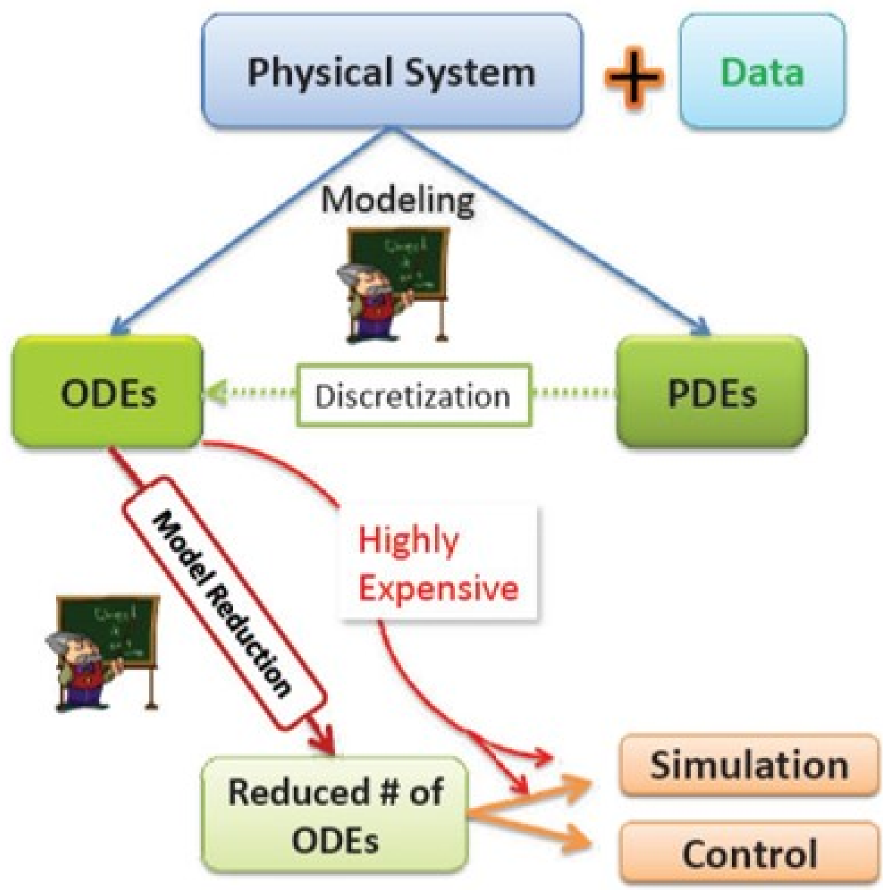


Figure 1.1: Model Order Reduction

### 1.1.1 Balanced Truncation

Balanced truncation (BT) is one of the classes/ methods of MOR (introduced by Moore [1]) which guaranteed a stable ROM along with a $priori$ error bound. It converts a higher model approximation into lower order by eliminating states that have low effect on considered system's response hence making it stable (bounded output for a given bounded input) and giving error bounds. It reduces the given under consideration higher order system to a lower order system. In control theory and its applications, a transfer function has many state space realizations, among them are some really useful realizations called internally balanced realizations. These internally balanced realizations point out the most dominant states of a system. Also, controllability and observability Gramians are equal and diagonal for a balanced realization. Using BT technique, a lower order realization is obtained from a higher order realization and it helps in attaining a lower approximation error over an entire frequency range. Basically, BT eliminates (truncates) the states that have low effect on considered system's response hence making it stable (bounded output for a given bounded input) and giving error bounds hence reducing a higher model model's approximation into a lower order. A lower order system for a full range of frequency is reduced from a full (higher) order system but sometimes it is desired to reduce the system over a specific frequency band, so it motivated the use of frequency weightings. BT is the most consistently and basically used technique of MOR to preserve stability along with other properties of the system and yielding of an approximation error for a responsive error bound in ROMs [2]. While using BT, controllability and observability Gramians are used to get the internally balanced realizations preserving stability by truncating least effective states. BT is actually the application of balanced realization to model reduction theory. Moreover, as mentioned, this method of MOR also gives an error bound formula along with the frequency response error .

Let's consider an original higher order system with transfer function $G(s) = C(sI - A)^{-1}B + D$ where $\{A, B, C, D\}$ represents its $n^{th}$ order minimal realizations with dimensions ($A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$ and $D \in R^{p\times m}$), where $n$ represents order of system and number of inputs and outputs are represented by $m$ and $p$, respectively. $G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r$ of $r^{th}$ order represents the transfer function for the ROM where $r < n$. Since frequency weighted MOR techniques are based on balanced truncation, so a brief summary of BT method is discussed.

Let $P$ (controllability) and $Q$ (observability) Gramians satisfy the following Lyapunov's equations:

$$AP + PA^T + BB^T = 0$$

$$A^TQ + QA + C^TC = 0$$

A transformation matrix $T$ is attained as

$$T^TQT = T^{-1}PT^{-T} = \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

Using this transformation matrix $T$, the original system realization will be transformed, where $\Sigma_1 = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, $\Sigma_2 = diag\{\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_n\}$. Hence obtaining new system realizations as:

$$\overline{A} = T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \overline{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\overline{C} = CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \overline{D} = D$$

ROM is obtained as $G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D$ . Error between the original and ROM is equated by the formula:

$$\| G(s) - G_r(s)\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i$$

Hankel optimal Approximation [2], Pade approximation [36], Krylovtechnique [37] etc are some other approaches which can be used besides BT to do MOR.

**Properties of Balanced Realization**

Some of the properties of Balanced realization (BT) are discussed below:

- Transformation of any realization into a balanced realization is possible only in case of asymptotically stable and minimum original system.
- $\| G(s) - G_r(s) \|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i$ and $\| G(s) - G_{n-1}(s) \|_\infty = 2\sigma_n$ [26]
- It should be noted that while doing BT at very higher frequencies, reduction error (also called as frequency response error) is zero and is non zero at very lower frequencies [36].
- Any transformed realization will be a balanced realization only and only if it is asymptotically stable and minimal [26].
- Input and output normal realizations are the other related relazations. An input normal realization has controllability Gramian as its identity matrix and Observability matrix as its diagonal matrix and vice versa for output normal realizations [39].

**Algorithm of BT**

Algorithm of Balanced Truncation (BT) is explained with the help of Fig 1.2. It has the following steps:

1. A higher order system is considered having realizations $\{A, B, C, D\}$.

2. Performing Singular Value Decompsition (SVD) on the system matrices.

3. Trimming off less dominant Hankel Singular Values (HSV) to obtain equal number of controllabity and observabiliy Gramians.

4. Obtain a ROM using internally balanced realizations.

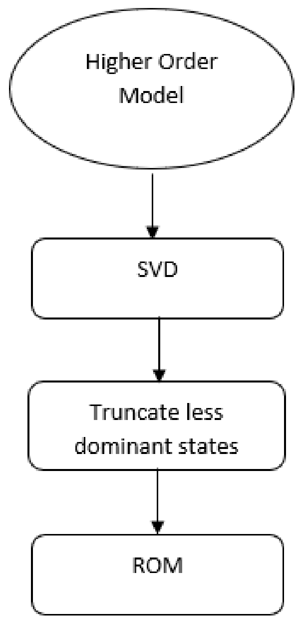


Figure 1.2: Balanced truncation

### 1.2 Frequency Weighted Model Reduction

While doing MOR for all frequencies, a small approximation error between the original system and ROM is desired. However, in some cases approximation error is desired over a specific frequency range instead of full frequencies. In such cases, feedback control systems use Frequency Weighted Model Reduction (FWMR) [3, 21]. This leads towards the FWMR problem which involves frequency weights. A full order stable system having a transfer function $G(s) = C(sI - A)^{-1}B + D$ is desired to be reduced into a lower ROM having transfer function $G_r(s) = C_r(sI - A_r)^{-1}B_r + D_r$. Also, a stable input weighting system and stable output weighting system is given by $V_i(s) = C_v(sI - A_v)^{-1}B_v + D_v$ and $W_o(s) = C_w(sI - A_w)^{-1}B_w + D_w$ where $\{A, B, C, D\}$, $\{A_r, B_r, C_r, D_r\}$ $\{A_v, B_v, C_v, D_v\}$

and $\{A_w, B_w, C_w, D_w\}$ represent $n^{th}$, $r^{th}$, $p^{th}$ and $q^{th}$ order minimal realizations respectively where $r < n$ such that error is achieved as critical as possible.

A FWMR approach is being explained using Fig 1.3. If both input and output weights are involved so in this case error is given by $\| \; W(s)(G(s) - G_r(s)V(s)) \; \|_\infty$, whereas if only input weighting is included (shown in Fig 1.4) so error is given by $\| \; (G(s) - G_r(s))V(s) \; \|_\infty$ and if only output weighting is included (shown in Fig 1.5) so error is given by $\| \; (W(s)(G(s) - G_r(s)) \; \|_\infty$.

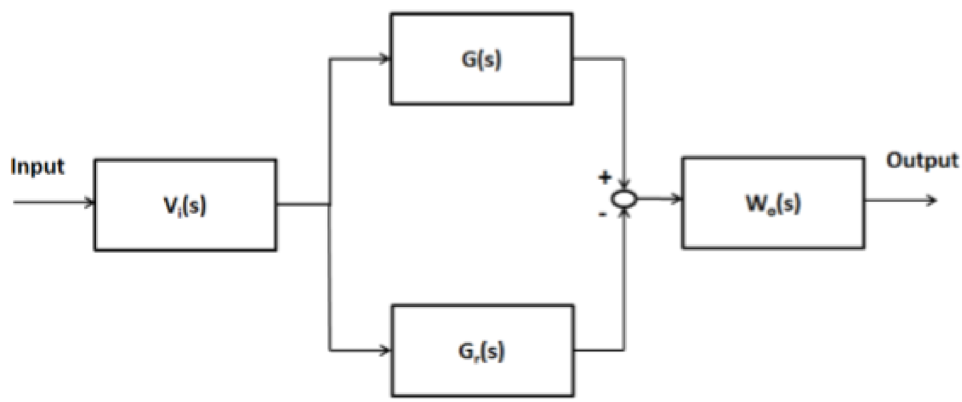


Figure 1.3: Input-Output FWMR error system

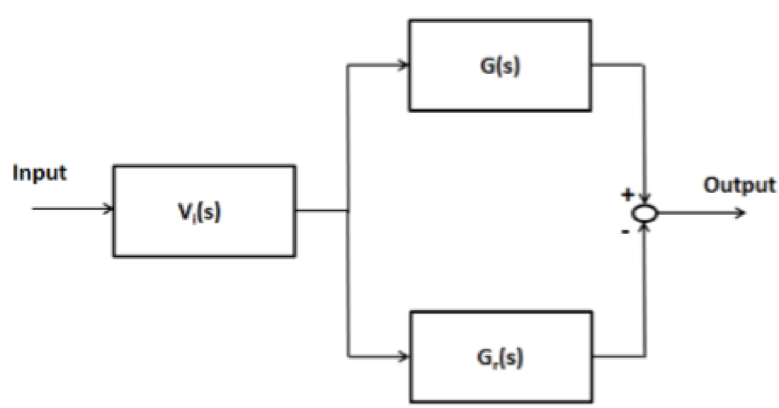


Figure 1.4: Input FWMR error system

A good volume of work has been carried out on this topic by many researchers. Enns [3] had modified FWMR technique using specified input and output model reduction weightings but its drawback was that using double weightings (input and output weightings) method, it may yield systems which are not stable because of the symmetric matrices obtained during procedure may be indefinite [4]. To overcome the instability issue in Enns method, many

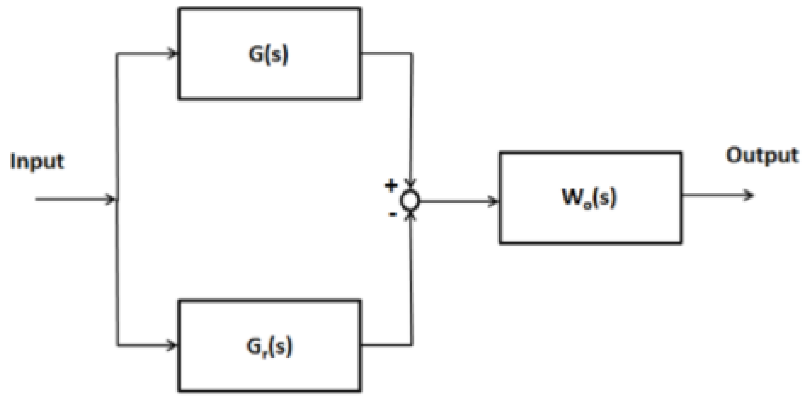


Figure 1.5: Ouput FWMR error system

research works were being carried out [4,5,39,41,30]. Lin and Chiu [4] proposed a technique that yield stable ROM in the presence of double sided weightings, but but its drawback was that their technique is limited to scenarios where only proper weighting functions are included and when forming the augmented system there is absence of pole zero cancellation [35]. Another drawback of this technique is that it gives large error.

Varga and Anderson [39] proposed their technique solving instability issue in Enns [3] method but its limitation was that it only works in those systems where is no pole-zero cancellation and its another drawback is that it is realization dependant i.e different realizations may yield different ROMs. Enns method was modified by Wang et al. [5] giving an advantage of yielding stable ROMs and error bounds even with inclusion of double - sided frequency weightings. Drawback of Wang et al's technique was it yields large approximation errors too because this technique was realization dependent for a same original system, and it used to yield a different model for every different realization [36]. Varga and Anderson again modified Wang et al's technique to have a more improved approximation error. Ghafoor and Sreeram [26] also proposed a technique using partial fraction expansion. Reduced stable models are obtained by either direct truncation or singular perturbation approximation. Drawback of this technique was that it was a parameterized technique. It means using this technique, some specific parameters will influence the responses on which the system will depend [35]. Imran and Ghafoor [30] also did work on MOR. This technique suggests that stability is achieved by taking least negative eigenvalues and subtracting it from all eigenvalues of some input and output related matrices. Drawback of Imran's technique was that it gave large approximation error as its last eigenvalue becomes zero. It should be noted that except Enns technique, all proposed techniques help us to find a stable ROM with large approximation error and error bound.

### 1.3 Frequency Limited Model Reduction

Using FWMR, helps us attain the smallest possible weighted reduction error ie,

$$\| W_o(s)(G(s) - G_r(s))V_i(s)\|_\infty$$

where $V_i(s)$ and $W_o(s)$ are some input weightings and output weightings respectively, for any given system [3]. Results can be manipulated and changed by changing these weights, making usually these input/output weightings fictitious. Frequency limited model reduction (FLMR) is used in cases where the original system $G(s)$ is to be estimated over a specific frequency range. The frequency range is specified by $[\omega_1, \omega_2]$. In absence of any input/output weightings, FLMR is being carried out. In such scenarios, using FWMR can be troublesome because the system designer has to create such weights through which this frequency band is reflected and accommodated, therefore selecting and constructing such weights is an issue [23]. FLMR technique was first proposed by Gawronski and Juang(GJ) [25] where the original system $G(s)$ was approximated without predefining the frequency weights explicitly and without input/output weights using Gramians over a required frequency range $[\omega_1, \omega_2]$. Its drawback was that it may produce an unstable ROM(s) (similar to Enns method [3]) and this technique gave no error bounds as well. Modifications have been made to GJ [25] by Gugercin and Antoulus (GA) [24], Ghafoor and Sreeram (GS) [26], M.Imran and Ghafoor (MIG) [27] and Imran et al. [28] to attain new techniques aiming to get stable ROMs and responsive error bounds by satisfying rank conditions. Another FLMR technique was being proposed by Wang and Zilouchian (WZ) [29] in domain of discrete time system which gave a good approximation error over a full frequency range. Just like Enns [3] technique and GJ [25] technique, this technique doesn't ensure stability phenomenon neither it gives error bound [26] . To resolve the issue in WZ technique, various methods were proposed such as Ghafoor and Sreeram Algorithm 1 ( GSA1) [26] which suggested taking absolute of all eigen values of input output matrices, Ghafoor and Sreeram Algorithm 2 (GSA2) [26] suggesting to ignore the negative eigenvalues of input output matrices, Imran and Ghafoor (IG) [30] suggesting subtracting the least negative value from all the eigenvalues of input output matrices and Hamid et al. [31] methods. .

## 1.4 Problem Summary

All FWMR techniques preserve stability except Enns technique for double sided frequency weighting. However, existing techniques yield large approximation error with loose error bound.

### 1.4.1 Contributions

Three FWMR techniques have been proposed (motivated) from [31] and [32] for continuous time systems which will be providing stable ROMs, lower approximation error and a easily computable error bound.

## 1.5 Thesis Outline

The thesis report compromises of five chapters. An outline is being presented here about brief description of each chapter. Chapter 2 explains the meaning of FWMR in detail, various existing techniques of FWMR and then drawbacks are described in detail. Difference between these existing techniques is explained too using an example. Chapter 3 discusses the three FWMR techniques to have stable ROMs along with a lower approximation error and easily computable error bound. Chapter 4 presents numerical examples and their results prove the effectiveness of proposed techniques. It discusses the achievement of lower approximation error along with easily computable error bound using proposed techniques as compared to existing FWMR techniques and ensuring stability. Chapter 5 discusses the conclusion of thesis and room for future research work.

*Chapter 2*

# FREQUENCY WEIGHTED MODEL REDUCTION: AN OVERVIEW

## 2.1 Introduction

MOR is a very important feature to analyse and design a control system. A controller is designed to run a physical plant which are dynamic systems of a higher order. The controller and plant are usually of comparable orders. Lower order systems and controllers are desirable as compared to higher order systems and controllers because [33, 34, 42].

- Non complex and lower order models are easier to analyse, making their simulation fast.
- They are more reliable.
- They are power and cost efficient.

The main concern of MOR is to reduce complexity of higher order systems, but also to preserve stability, passivity, input and output behaviour of system which are the basic properties of actual system. When MOR is carried out over a specific frequency range so some input and output weights are introduced. The specific frequency range is reflected through these weights. As the approximation error between the original system and ROM is desirable low so when sometimes a lower approximation error is needed over a specific frequency range instead of a full frequency interval, FWMR is used. This scenario is used in feedback control system [3, 21] as described above the problem of a controller and plant. In this chapter, we discuss the existing FWMR techniques in detail.

## 2.2 Existing Techniques

### 2.2.1 Enns Method [3]

Enns was the first one to introduce input and output weightings to the original system.

**Continous time domain**

Consider a continuous LTI time system of $n^{th}$ order having transfer function as :

$$G(s) = C(sI - A)^{-1}B + D$$

where $\{A, B, C, D\ \}$ are the minimal realizations of the original full order system. Also let an input weighting system given by the transfer function be:

$$V_E(s) = C_v(sI - A_v)^{-1}B_v + D_v$$

and output weighting system given by the transfer function be:

$$W_E(s) = C_w(sI - A_w)^{-1}B_w + D_w$$

where $A_v, B_v, C_v, D_v$ are the input weighting minimal realizations and $A_w, B_w, C_w, D_w$ are the output weighting minimal realizations.

Let the augmented systems obtained by using the original, input weighting and output weighting system be:

$$G(s)V_E(s) = C_i(sI - A_i)^{-1}B_i + D_i$$

$$W_E(s)G(s) = C_o(sI - A_o)^{-1}B_o + D_o$$

Let controllability gramian $P_{ie}$ and observability gramian $Q_{oe}$ be

$$P_{ie} = \begin{bmatrix} P_{EN} & P_{12} \\ P_{12}^T & P_V \end{bmatrix}, Q_{oe} = \begin{bmatrix} Q_W & Q_{12}^T \\ Q_{12} & Q_{EN} \end{bmatrix}$$

be a solution to the following Lyapunov equations:

$$A_iP_{ie} + P_{ie}A_i^T + B_iB^T = 0 \tag{2.1}$$

$$A_o^TQ_{oe} + A_oQ_{oe} + C_o^TC_o = 0 \tag{2.2}$$

where,

$$\{A_i, B_i, C_i, D_i\} = \left\{ \begin{bmatrix} A & BC_V \\ 0 & A_V \end{bmatrix}, \begin{bmatrix} BD_V \\ B_V \end{bmatrix}, \begin{bmatrix} C & DC_V \end{bmatrix}, DD_V \right\}$$

$$\{A_o, B_o, C_o, D_o\} = \left\{ \begin{bmatrix} A_W & B_W C \\ 0 & A \end{bmatrix}, \begin{bmatrix} B_W D \\ B \end{bmatrix}, \begin{bmatrix} C_W & D_W C \end{bmatrix}, D_W D \right\}$$

Following equations are yielded by expanding the (1,1) and (2,2) blocks of equation (2.1) and (2.2) respectively.

$$AP_{EN} + P_{E_N}A^T + X = 0 \tag{2.3}$$

$$A^T Q_{EN} + Q_{EN}A + Y = 0 \tag{2.4}$$

where,

$$X = BC_V P_{12}^T + P_{12}C_V^T B^T + BD_V D_V^T B^T \tag{2.5}$$

$$Y = C^T B_W^T Q_{12}^T + Q_{12}B_W C + C^T D_W^T D_W C \tag{2.6}$$

Let transformation matrix obtained $T$ be

$$T^T Q_{EN} T = T^{-1} P_{EN} T^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.7}$$

where $\sigma_i \leq \sigma_i + 1$, $i = 1, 2, \ldots, n-1$, and $\sigma_r > \sigma_r + 1$. Transforming and partitioning the original system, we get

$$\hat{A} = T^{-1}AT = \begin{bmatrix} A_{11} & A12 \\ A_{21} & A_{22} \end{bmatrix}, \hat{B} = T^{-1}B = \begin{bmatrix} B1 \\ B_2 \end{bmatrix} \tag{2.8}$$

$$\hat{C} = CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \hat{D} = D \tag{2.9}$$

Now using these realizations, transfer function for the ROM is obtained and represented by

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D \tag{2.10}$$

**Discrete time domain**

Consider a discrete time system of $n^{th}$ order having transfer function as :

$$H(z) = C(zI - A)^{-1}B + D$$

where $\{A, B, C, D\}$ are the minimal realizations of the original full order system. Also let an input weighting system given by the transfer function be:

$$V_E(z) = C_v(zI - A_v)^{-1}B_v + D_v$$

and output weighting system given by the transfer function be:

$$W_E(z) = C_w(zI - A_w)^{-1}B_w + D_w$$

where $\{A_v, B_v, C_v, D_v\}$ are the input weighting minimal realizations and $\{A_w, B_w, C_w, D_w\}$ are the output weighting minimal realizations.

Let the augmented systems obtained by using the original, input weighting and output weighting system be:

$$H(z)V_E(z) = C_i(zI - A_i)^{-1}B_i + D_i$$

$$W_E(z)H(z) = C_o(zI - A_o)^{-1}B_o + D_o$$

Let controllability gramian $P_{ie}$ and observability gramian $Q_{ie}$ be

$$P_{ie} = \begin{bmatrix} P_{EN} & P_{12} \\ P_{12}^T & P_V \end{bmatrix}, Q_{oe} = \begin{bmatrix} Q_W & Q_{12}^T \\ Q_{12} & Q_{EN} \end{bmatrix}$$

be a solution to the following Lyapunov equations:

$$A_i P_{ie} A_i^T - P_{ie} + B_i B^T = 0 \tag{2.11}$$

$$A_o^T Q_{oe} A_o - Q_{oe} + C_o^T C_o = 0 \tag{2.12}$$

where,

$$\{A_i, B_i, C_i, D_i\} = \left\{ \begin{bmatrix} A & BC_V \\ 0 & A_V \end{bmatrix}, \begin{bmatrix} BD_V \\ B_V \end{bmatrix}, \begin{bmatrix} C & DC_V \end{bmatrix}, DD_V \right\}$$

$$\{A_o, B_o, C_o, D_o\} = \left\{ \begin{bmatrix} A_W & B_W C \\ 0 & A \end{bmatrix}, \begin{bmatrix} B_W D \\ B \end{bmatrix}, \begin{bmatrix} C_W & D_W C \end{bmatrix}, D_W D \right\}$$

**Remark 1** *There are many applications where input and output realizations may not be minimal.*

Following equations are yielded by expanding the (1,1) and (2,2) blocks of equation (2.11) and (2.12) respectively.

$$AP_{EN}A^T + P_{E_N} + X = 0 \tag{2.13}$$

$$A^T Q_{EN} A + Q_{EN} + Y = 0 \tag{2.14}$$

where,

$$X = AP_{12}C_V^T B^T + BC_V P_{12}^T A^T + BC_V P_V C_V^T B^T + BD_V D_V^T B^T \tag{2.15}$$

$$Y = C^T B_W^T Q_{12}^T A + A^T Q_{12} B_W C + C^T B_W^T Q_{oe} B_W C + C^T D_W^T D_W C \tag{2.16}$$

Let transformation matrix obtained $T_F$ be

$$T_F^T Q_{EN} T_F = T_F^{-1} P_{EN} T_F^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.17}$$

where $\sigma_i \leq \sigma_i + 1$, $i = 1, 2, \ldots, n-1$, and $\sigma_r > \sigma_r + 1$. Transforming and partitioning the original system, we get

$$\hat{A} = T_F^{-1} A T_F = \begin{bmatrix} A_{11} & A12 \\ A_{21} & A_{22} \end{bmatrix}, \hat{B} = T_F^{-1} B = \begin{bmatrix} B1 \\ B_2 \end{bmatrix} \tag{2.18}$$

$$\hat{C} = CT_F = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \hat{D} = D \tag{2.19}$$

Now using these realizations, transfer function for the ROM is obtained and represented by

$$H_r(z) = C_1(zI - A_{11})^{-1} B_1 + D \tag{2.20}$$

*Example 2.1 :* Consider Example 2.3 of [35] having a state space:

$$A = \begin{bmatrix} -4 & -5 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 6 & 2 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

having input weightings as:

$$A_v = \begin{bmatrix} -3 \end{bmatrix}, B_v = \begin{bmatrix} 1 \end{bmatrix}$$

$$C_v = \begin{bmatrix} 1 \end{bmatrix}, D_v = \begin{bmatrix} 0 \end{bmatrix}$$

and output weightings as:

$$A_w = \begin{bmatrix} -4 \end{bmatrix}, B_w = \begin{bmatrix} 1 \end{bmatrix}$$

$$C_w = \begin{bmatrix} 1 \end{bmatrix}, D_w = \begin{bmatrix} 0 \end{bmatrix}$$

Indefinite matrices X and Y are given below:

$$X = \begin{bmatrix} -0.0374 & -0.0064 & -0.0019 \\ -0.0064 & 0.0001 & 0.0001 \\ -0.0021 & 0.0001 & 0.0001 \end{bmatrix}$$

$$y = \begin{bmatrix} -2.0532 & -0.9832 & -0.2432 \\ -0.9832 & 0.3201 & -0.0432 \\ -0.2432 & -0.0433 & -0.0068 \end{bmatrix}$$

Errors using Enns technique are given in the table 2.1

Table 2.1: Errors using Enns [3] technique

| **Weighting** | **Order** | **Enns [3] Error** |
|---|---|---|
| Double | 1 | Unstable |
| | 2 | 0.024873 |
| Input | 1 | 0.3148 |
| | 2 | 0.024873 |
| Output | 1 | 0.32674 |
| | 2 | 0.039382 |

**Remark 2** *While including input weighting only, transformation matrix $T$ is obtained by using the matrices $P_{ie}$ and $Q$ while including output weighting only, transformation matrix $T$ is obtained by using $P$ and $Q_{ie}$*

**Remark 3** *Enns [3] technique has a drawback that ROMs obtained using this technique may become unstable when using double-sided weightings, for cases where matrices$X$ and $Y$ become indefinite.*

### 2.2.2 Wang et al. Method [5]

Issue of stability for double weighting case pointed out in Enns [3] technique was solved making $X$ and $Y$ positive semi definite by using technique of Wang et al.

**Continous time domain**

In this technique, likely to Enns technique, a controllability gramian $P_{WA}$ and observability gramian $Q_{WA}$ is introduced to to give a solution to the following equations:

$$AP_{WA} + P_{WA}A^T + B_{WA}B_{WA}^T = 0 \tag{2.21}$$

$$A^T Q_{WA} + Q_{WA} A + C_{WA}^T C_{WA} = 0 \tag{2.22}$$

Taking a transformation $T$ using above Lyapunov equations as:

$$T^T Q_{WA} T = T^{-1} P_{WA} T^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.23}$$

where $\sigma_i > \sigma_i + 1$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_r + 1$. Doing eigen decomposition of $X$ and $Y$ results in $X = U_{WA} S_{WA} U_{WA}^T$ and $Y = V_{WA} R_{WA} V_{WA}^T$. The matrices $B_{WA}$ and $C_{WA}$ are defined as $B_{WA} = U_{WA} |S_{WA}|^{1/2}$ and $C_{WA} = |R_{WA}|^{1/2} V_{WA}^T$. By transforming and partitioning the original system, ROM is obtained for which the transfer function is given by

$$G_r(s) = C_1 (sI - A_{11})^{-1} B_1 + D_r \tag{2.24}$$

**Discrete time domain**

Campbell et al. [41] proposed a discrete time version for Wang et al. method [5]. A controllability gramian $P_{WA}$ and observability gramian $Q_{WA}$ is introduced to to give a solution to the following equations:

$$A P_{WA} A^T - P_{WA} + B_{WA} B_{WA}^T = 0 \tag{2.25}$$

$$A^T Q_{WA} A - Q_{WA} + C_{WA}^T C_{WA} = 0 \tag{2.26}$$

Taking a transformation $T$ using above Lyapunov equations as:

$$T_F^T Q_{WA} T_F = T^{-1} P_{WA} T_F^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.27}$$

where $\sigma_i > \sigma_i + 1$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_r + 1$. Doing eigen decomposition of $X$ and $Y$ results in $X = U_{WA} S_{WA} U_{WA}^T$ and $Y = V_{WA} R_{WA} V_{WA}^T$. The matrices $B_{WA}$ and $C_{WA}$ are defined as $B_{WA} = U_{WA} |S_{WA}|^{1/2}$ and $C_{WA} = |R_{WA}|^{1/2} V_{WA}^T$. By transforming and partitioning the original system, ROM is obtained for which the transfer function is given by

$$H_r(z) = C_1 (zI - A_{11})^{-1} B_1 + D$$

This technique of Wang et al. suggested to take absolute of the diagonal entries of the matrices $X$ and $Y$ which contained positive as well as negative entries hence transforming negative entries into positive while positive entries remained unchanged leading to make matrices $P_{WA}$ and $Q_{WA}$ positive semidefinite resulting into the stability of the system. Its drawback was that it yields large approximation error.

**Remark 4** *A relationship is established between the input matrix B and new constructed input matrix $B_{WA}$ is established by showing the existance of rank:*

$$rank[B_{WA} \quad B] = rank[B_{WA}]$$

**Error Bound**

*Theorem 1: If these rank conditions*

$$rank[B_{WA} \quad B] = rank[B_{WA}]$$

*and,*

$$\begin{bmatrix} C_{WA} \\ C \end{bmatrix} = rank = [C_{WA}]$$

*are satisfied so the following error bound is held:*

In case of continous time domain:

$$\| W_W(s)(G(s) - G_r(s))V_W(s)\|_\infty \leq 2 \| W_W(s)L\|_\infty \|KV_W(s)\|_\infty \sum_{i=r+1}^{n} \sigma_i \tag{2.28}$$

In case of discrete time domain:

$$\| W_W(z)(H(z) - H_r(z))V_W(z)\|_\infty \leq 2 \| W_W(z)L\|_\infty \|KV_W(z)\|_\infty \sum_{i=r+1}^{n} \sigma_i \tag{2.29}$$

These rank conditions are followed from [5].

### 2.2.3 Varga and Anderson Technique [39]

As the approximation error yielded using technique of Wang et al. [5] was larger so to have a reduced approximation error, Varga and Anderson proposed their technique.

**Continous time domain**

A controllability gramian $P_{VA}$ and observability gramian $Q_{VA}$ is introduced to satisfy the below Lyapunov equations:

$$AP_{VA} + P_{VA}A^T + B_{VA}B_{VA}^T = 0 \tag{2.30}$$

$$A^TQ_{VA} + Q_{VA}A + C_{VA}^TC_{VA} = 0 \tag{2.31}$$

Using above Lyapunov equations, a transformation $T$ is taken as:

$$T^TQ_{VA}T = T^{-1}P_{VA}T^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.32}$$

where $\sigma_i > \sigma_i + 1$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_r + 1$ Doing eigen decomposition of $X$ and $Y$ results in:

$$X = \begin{bmatrix} U_{VA_1} & U_{VA_2} \end{bmatrix} \begin{bmatrix} S_{VA_1} & 0 \\ 0 & S_{VA_2} \end{bmatrix} \begin{bmatrix} U_{VA_1}^T \\ U_{VA_2}^T \end{bmatrix} \tag{2.33}$$

$$Y = \begin{bmatrix} V_{VA_1} & V_{VA_2} \end{bmatrix} \begin{bmatrix} R_{VA_1} & 0 \\ 0 & R_{VA_2} \end{bmatrix} \begin{bmatrix} V_{VA_1}^T \\ V_{VA_2}^T \end{bmatrix} \tag{2.34}$$

The matrices $B_{VA}$ and $C_{VA}$ are defined as $B_{VA} = U_{VA_1}S_{VA_1}^{1/2}$ and $C_{VA} = R_{VA_1}^{1/2}V_{VA_1}^T$. By transforming and partitioning the original system, ROM is obtained for which the transfer function is given by

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D \tag{2.35}$$

**Discrete time domain**

A controllability gramian $P_{VA}$ and observability gramian $Q_{VA}$ is introduced to satisfy the below Lyapunov equations:

$$AP_{VA}A^T - P_{VA} + B_{VA}B_{VA}^T = 0 \tag{2.36}$$

$$A^TQ_{VA}A - Q_{VA} + C_{VA}^TC_{VA} = 0 \tag{2.37}$$

Using above Lyapunov equations, a transformation $T_F$ is taken as:

$$T_F^TQ_{VA}T_F = T_F^{-1}P_{VA}T_F^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.38}$$

where $\sigma_i > \sigma_i + 1$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_r + 1$ Doing eigen decomposition of $X$ and $Y$ results in:

$$X = \begin{bmatrix} U_{VA_1} & U_{VA_2} \end{bmatrix} \begin{bmatrix} S_{VA_1} & 0 \\ 0 & S_{VA_2} \end{bmatrix} \begin{bmatrix} U_{VA_1}^T \\ U_{VA_2}^T \end{bmatrix} \tag{2.39}$$

$$Y = \begin{bmatrix} V_{VA_1} & V_{VA_2} \end{bmatrix} \begin{bmatrix} R_{VA_1} & 0 \\ 0 & R_{VA_2} \end{bmatrix} \begin{bmatrix} V_{VA_1}^T \\ V_{VA_2}^T \end{bmatrix} \tag{2.40}$$

The matrices $B_{VA}$ and $C_{VA}$ are defined as $B_{VA} = U_{VA_1} S_{VA_1}^{1/2}$ and $C_{VA} = R_{VA_1}^{1/2} V_{VA_1}^T$.

By transforming and partitioning the original system, ROM is obtained for which the transfer function is given by

$$H_r(z) = C_1(zI - A_{11})^{-1} B_1 + D \tag{2.41}$$

This technique suggested to replace all the negative entries in matrices X and Y with zero in order to make the matrices X and Y positive semidefinite resulting in stability of the system. Its limitation was that only works in those systems where is no pole-zero cancellation. Likely to the other techniques, the realization $A, B_{VA}, C_{VA}$ is minimal.

**Error Bound**

*Theorem 2: If these rank conditions*

$$rank[B_{VA} \quad B] = rank[B_{VA}]$$

*and,*

$$\begin{bmatrix} C_{VA} \\ C \end{bmatrix} = rank = [C_{VA}]$$

*are satisfied so the following error bound is held:*

In case of continous time domain:

$$\| W_{VA}(s)(G(s) - G_r(s))V_{VA}(s)\|_\infty \leq 2 \| W_{VA}(s)L\|_\infty \|KV_{VA}(s)\|_\infty \sum_{i=r+1}^{n} \sigma_i \tag{2.42}$$

In case of discrete time domain:

$$\| W_{VA}(z)(H(z) - H_r(z))V_{VA}(z)\|_\infty \leq 2 \| W_{VA}(z)L\|_\infty \|KV_{VA}(z)\|_\infty \sum_{i=r+1}^{n} \sigma_i \tag{2.43}$$

### 2.2.4 Imran and Ghafoor Technique [30]

A method was being proposed by Imran and Ghafoor to have a much more reduced approximation error in response to an explicit error bound.

**Continous time domain**

This method proposes controllability gramian $P_{IG}$ and an observability gramian $Q_{IG}$ the solutions to the following equations:

$$AP_{IG} + P_{IG}A^T + B_{IG}B_{IG}^T = 0 \tag{2.44}$$

$$A^TQ_{IG} + Q_{IG}A + C_{IG}^TC_{IG} = 0 \tag{2.45}$$

Taking transformation $T$ as $T^TQ_{IG}T = T^{-1}P_{IG}T^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ where $\sigma_i > \sigma_i + 1$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_r + 1$. Here, eigen decomposition of the matrices $X$ and $Y$ result in $X = U_{IG}S_{IG}UT_{IG}$ and $Y = V_{IG}R_{IG}VT_{IG}$ from which $B_{IG}$ and $C_{IG}$ are defined as :

$$B_{IG} = \begin{cases} U_{IG}(S_{IG} - s_nI)^{1/2} & for\ s_n < 0 \\ U_{IG}S_{IG}^{1/2} & for\ s_n \geq 0 \end{cases} \tag{2.46}$$

$$C_{IG} = \begin{cases} (R_{IG} - r_nI)^{1/2}V_{IG}^T & for\ r_n < 0 \\ R_{IG}^{1/2}V_{IG}^T & for\ r_n \geq 0 \end{cases} \tag{2.47}$$

Transforming and partitioning the original system, ROM is obtained as a transfer function

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D \tag{2.48}$$

**Discrete time domain**

This method proposes controllability gramian $P_{IG}$ and an observability gramian $Q_{IG}$ the solutions to the following equations:

$$AP_{IG}A^T - P_{IG} + B_{IG}B_{IG}^T = 0 \tag{2.49}$$

$$A^TQ_{IG}A - Q_{IG} + C_{IG}^TC_{IG} = 0 \tag{2.50}$$

Taking transformation $T_F$ as $T_F^TQ_{IG}T_F = T_F^{-1}P_{IG}T_F^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ where $\sigma_i > \sigma_i + 1$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_r + 1$. Here, eigen decomposition of the matrices $X$ and $Y$ result in $X = U_{IG}S_{IG}UT_{IG}$ and $Y = V_{IG}R_{IG}VT_{IG}$ from which $B_{IG}$ and $C_{IG}$ are defined as :

$$B_{IG} = \begin{cases} U_{IG}(S_{IG} - s_n I)^{1/2} & for\ s_n < 0 \\ U_{IG} S_{IG}^{1/2} & for\ s_n \geq 0 \end{cases} \tag{2.51}$$

$$C_{IG} = \begin{cases} (R_{IG} - r_n I)^{1/2} V_{IG}^T & for\ r_n < 0 \\ R_{IG}^{1/2} V_{IG}^T & for\ r_n \geq 0 \end{cases} \tag{2.52}$$

Transforming and partitioning the original system, ROM is obtained as a transfer function

$$H_r(z) = C_1(zI - A_{11})^{-1} B_1 + D \tag{2.53}$$

Imran and Ghafoor's technique suggested to subtract all the diagonal entries from the least minimum entry of the diagonal of matrices $X$ and $Y$. This technique gave the advantage that ROM obtained is stable as well as it yields frequency response error bound and improved frequency response error. It also has a similar effect on all the eigenvalues unlike the previous techniques. Its drawback was that sometimes it yields large approximation error.

**Error Bound**

*Theorem 3: Like other previoussly proposed techniques, if these rank conditions*

$$rank[B_{IG} \quad B] = rank[B_{IG}]$$

*and,*

$$\begin{bmatrix} C_{IG} \\ C \end{bmatrix} = rank = [C_{IG}]$$

*are satisfied so the following error bound is held:*

In case of continous time domain:

$$\| W_{IG}(s)(G(s) - G_r(s))V_{IG}(s)\|_\infty \leq 2 \| W_I G(s) L\|_\infty \|K V_I G(s)\|_\infty \sum_{i=r+1}^{n} \sigma_i \tag{2.54}$$

In case of discrete time domain:

$$\| W_{IG}(z)(H(z) - H_r(z))V_{IG}(z)\|_\infty \leq 2 \| W_{IG}(z) L\|_\infty \|K V_{IG}(z)\|_\infty \sum_{i=r+1}^{n} \sigma_i \tag{2.55}$$

### 2.2.5 Advantages/Disadvantages of existing techniques

Table 2.2 interprets the literature survey for all existing proposed techniques.

Table 2.2: Literature survey for all previous techniques

| Techniques | Stability | Approximation Error |
|---|---|---|
| Enns [3] | No | Large |
| Wang et al. [5] | Yes | Large |
| Varga and Anderson [39] | Yes | Large |
| Imran and Ghafoor [30] | Yes | Large |

Also some advantages and disadvantages of all techniques are listed as follows:

***Enns [3]***:

$Advantage$: It only yields stable ROMs in the presence of single sided weighting.

$Disadvantage$: Models obtained by Enns technique may not be stable for two-sided weighting case.

***Wang et al. [5]***:

$Advantages$:

- Stability issue was solved by making X and Y positive semi definite.
- Issue of stability in double weighting case is solved.

$Disadvantage$: It yields large approximation errors.

***Varga and Anderson [39]***:

$Advantages$:

- It reduces approximation error.
- Issue of stability in double weighting case is solved,

$Disadvantage$: It only works in those systems where is no pole-zero cancellation.

***Imran and Ghafoor [30]***

$Advantages$:

- ROM obtained is stable.
- It yields frequency response error bound and improved frequency response error.
- It has a likely effect on all the eigenvalues.

$Disdvantages$:

- Sometimes it yields large approximation error.
- Its last eigenvalue may become zero.

The main drawback of the existing pioneer frequency weighted model reduction technique by Enns was to yield an unstable ROM from the original stable system. Many existing techniques address this limitation and preserve stability in ROMs but at the cost of large approximation error.

*Chapter 3*

# MAIN WORK

## 3.1 Motivation for Proposed Techniques

As all frequency weighted MOR techniques except Enns [3] technique preserve stability for double sided weightings but at cost of large approximation error. Three techniques are proposed (motivated from [31] and [32]) aimed to give a low approximation error along with an easily computable error bound. MOR techniques proposed in [31] and [32] are in the domain of frequency limited whereas this thesis presents their proposed techniques in frequency weighted domain.

## 3.2 Proposed Techniques

Existing frequency weighted MOR techniques except Enns technique preserve stability but give a large approximation error and a lose error bound. Three techniques are proposed (motivated from [31] and [32] aimed to give a low approximation error along with a easily computable error bound. Three techniques are proposed to ensure the stability of ROMs along with a satisfying error bound by making symmetric matrices X and Y positive/semipositive definite. Our first technique is motivated from third technique proposed in [31] while our second and third techniques are motivated from first and second technique proposed in [32], respectively.

### Continous time domain

Let's synthesize controllability and observability Gramians given by $P_{A_i}$ and $Q_{A_i}$, respectively which will satisfy the following Lyapunov equations:

$$AP_{A_i} + P_{A_i}A^T + B_{A_i}B_{A_i}^T = 0 \tag{3.1}$$

$$A^TQ_{A_i} + Q_{A_i}A + C_{A_i}^TC_{A_i} = 0 \tag{3.2}$$

where $i = 1, 2, 3$. For indefinite matrices $X$ and $Y$, the new fictitious matrices $B_{A_i}$ and $C_{A_i}$ are defined as:

$$B_{A_i} = U_{A_i} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & S_2^{1/2} \end{bmatrix} = U_{A_i} S_i^{1/2},$$

$$C_{A_i} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & R_2^{1/2} \end{bmatrix} {V_{A_i}}_i^T = R_i^{1/2} V_i^T$$

for technique 1, i.e. $i = 1$

$$B_{A_1} = U_{A_1} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & ((S_2/\overline{s})^{1/N} - S_2)^{1/2} \end{bmatrix} for\ s_n < 0 \tag{3.3}$$

$$C_{A_1} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & ((R_2/\overline{r})^{1/N} - R_2)^{1/2} \end{bmatrix} {V_{A_1}}^T for\ r_n < 0 \tag{3.4}$$

where $\overline{r} = (r_{q+1} + r_{q+2} + \ldots r_n)/(n-q)$ , $\overline{s} = (s_{p+1} + s_{p+2} + \ldots s_n)/(n-p)$ and $q$ and $p$ are number of positive eigenvalues.

for technique 2, i.e. $i = 2$

$$B_{A_2} = U_{A_2} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & (S_2/s_n)^{1/n} \end{bmatrix}\ for\ s_n < 0 \tag{3.5}$$

$$C_{A_2} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & (R_2/r_n)^{1/2} \end{bmatrix} {V_{A_2}}^T\ for\ r_n < 0 \tag{3.6}$$

for technique 3, i.e. $i = 3$

$$B_{A_3} = U_{A_3} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & (S_2 * s_n)^{1/2} \end{bmatrix}\ for\ s_n < 0 \tag{3.7}$$

$$C_{A_3} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & (R_2 * r_n)^{1/2} \end{bmatrix} {V_{A_3}}^T\ for\ r_n < 0 \tag{3.8}$$

The terms $U_{A_1}$, $U_{A_2}$, $S_1$, $S_2$, $R_1$, $R_2$, $V_1$, $V_2$ are acquired from following symmetric matrices,

$$X = \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} U_{A_1}^T \\ U_{A_2}^T \end{bmatrix} \tag{3.9}$$

$$Y = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} V_{A_1}^T \\ V_{A_2}^T \end{bmatrix} \tag{3.10}$$

where

$$S_1 = diag(s_1, s_2, ...s_q), S_2 = diag(s_{q+1}, s_{q+2}, ...s_n),$$

$$R_1 = diag(r_1, r_2, ...r_p), R_2 = diag(r_{p+1}, r_{p+2}, ...r_n)$$

Similarly, a transformation matrix $T$ is obtained as

$$T^T Q_A T = T^{-1} P_A T^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{3.11}$$

where $\sigma_i > \sigma_{i+1}$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$.

Using this transformation matrix $T$, new realizations are defined derived and as a result ROM is obtained, given by

$$G_r(s) = C_1(sI - A_{11})^{-1} B_1 + D_{rA} \tag{3.12}$$

**Discrete time domain**

Let's synthesize controllability and observability Gramians given by $P_{A_i}$ and $Q_{A_i}$, respectively which will satisfy the following Lyapunov equations:

$$AP_{A_i} - P_{A_i} A^T + B_{A_i} B_{A_i}^T = 0 \tag{3.13}$$

$$A^T Q_{A_i} - Q_{A_i} A + C_{A_i}^T C_{A_i} = 0 \tag{3.14}$$

where $i = 1, 2, 3$. For indefinite matrices $X$ and $Y$, the new fictitious matrices $B_{A_i}$ and $C_{A_i}$ are defined as:

$$B_{A_i} = U_{A_i} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & S_2^{1/2} \end{bmatrix} = U_{A_i} S_i^{1/2},$$

$$C_{A_i} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & R_2^{1/2} \end{bmatrix} V_{A_i i}^{\ T} = R_i^{1/2} V_i^T$$

for technique 1, i.e. $i = 1$

$$B_{A_1} = U_{A_1} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & ((S_2/\overline{s})^{1/N} - S_2)^{1/2} \end{bmatrix} for\ s_n < 0 \tag{3.15}$$

$$C_{A_1} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & ((R_2/\overline{r})^{1/N} - R_2)^{1/2} \end{bmatrix} {V_{A_1}}^T for\ r_n < 0 \tag{3.16}$$

where $\overline{r} = (r_{q+1} + r_{q+2} + \ldots r_n)/(n-q)$ , $\overline{s} = (s_{p+1} + s_{p+2} + \ldots s_n)/(n-p)$ and $q$ and $p$ are number of positive eigenvalues.

for technique 2, i.e. $i = 2$

$$B_{A_2} = U_{A_2} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & (S_2/s_n)^{1/n} \end{bmatrix} \ for\ s_n < 0 \tag{3.17}$$

$$C_{A_2} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & (R_2/r_n)^{1/2} \end{bmatrix} {V_{A_2}}^T\ for\ r_n < 0 \tag{3.18}$$

for technique 3, i.e. $i = 3$

$$B_{A_3} = U_{A_3} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & (S_2 * s_n)^{1/2} \end{bmatrix} \ for\ s_n < 0 \tag{3.19}$$

$$C_{A_3} = \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & (R_2 * r_n)^{1/2} \end{bmatrix} {V_{A_3}}^T\ for\ r_n < 0 \tag{3.20}$$

The terms $U_{A_1}, U_{A_2}, S_1, S_2, R_1, R_2, V_1, V_2$ are acquired from following symmetric matrices,

$$X = \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} U_{A_1}^T \\ U_{A_2}^T \end{bmatrix} \tag{3.21}$$

$$Y = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} V_{A_1}^T \\ V_{A_2}^T \end{bmatrix} \tag{3.22}$$

where

$$S_1 = diag(s_1, s_2, ...s_q), S_2 = diag(s_{q+1}, s_{q+2}, ...s_n),$$

$$R_1 = diag(r_1, r_2, ...r_p), R_2 = diag(r_{p+1}, r_{p+2}, ...r_n)$$

**Remark 5** *When $X \geq 0$ and $Y \geq 0$, $B_{A_i}$ = $U_{A_i} S_i^{1/2}$ and $C_{A_i}$ = $R_i^{1/2} V_{A_i}^T$.*

Similarly, a transformation matrix $T_F$ is obtained as

$$T_F^T Q_A T_F = T_F^{-1} P_A T_F^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{3.23}$$

where $\sigma_i > \sigma_{i+1}$ , $i = 1, 2, \ldots, n-1$ and $\sigma_r > \sigma_{r+1}$.

Using this transformation matrix $T_F$, new realizations are defined derived and as a result ROM is obtained, given by

$$H_r(z) = C_1(zI - A_{11})^{-1} B_1 + D_{rA} \tag{3.24}$$

**Remark 6** *As $X \leq B_{A_i} B_{A_i}^T$, $Y \leq C_{A_i}^T C_{A_i}$, $B_{A_i} B_{A_i}^T \geq 0$, $C_{A_i}^T C_{A_i} \geq 0$, $P_{A_i} \geq 0$, $Q_{A_i} \geq 0$ the realization ($A$, $B_{A_i}$, $C_{A_i}$) is minimal and stability of ROM is also guaranteed.*

### 3.2.1 Error Bounds

*Theorem 2: If the rank conditions, rank* $[B_{A_i} \quad B]$ *=rank* $\begin{bmatrix} B_{A_i} \end{bmatrix}$ *and rank* $\begin{bmatrix} C_{A_i} \\ C \end{bmatrix}$ *= rank* $[C_{A_i}]$ *are satisfied, so the given error bound holds true:*

$$\|W(s)(G(s) - G_r(s))V(s)\|_\infty \leq 2\ \|L_{A_i}\| \|K_{A_i}\| \sum_{j=r+1}^{n} \sigma_j \tag{3.25}$$

for $i = 1$

$$L_{A_1} = \begin{cases} CV \begin{bmatrix} R_1^{-1/2} & 0 \\ 0 & ((R_2/\overline{r})^{1/N} - R_2)^{-1/2} \end{bmatrix} & for \ \ r_n < 0 \\ CVR^{-1/2} & for \ \ r_n \geq 0 \end{cases} \tag{3.26}$$

$$K_{A_1} = \begin{cases} \begin{bmatrix} S_1^{-1/2} & 0 \\ 0 & ((S_2/\overline{s})^{1/N} - S_2)^{-1/2} \end{bmatrix} U^T B & for \ \ s_n < 0 \\ S^{-1/2} U^T B & for \ \ s_n \geq 0 \end{cases} \tag{3.27}$$

for $i = 2$

$$L_{A_2} = \begin{cases} CV \begin{bmatrix} R_1^{-1/2} & 0 \\ 0 & (R_2/r_n)^{-1/2} \end{bmatrix} & for \ \ r_n < 0 \\ CVR^{-1/2} & for \ \ r_n \geq 0 \end{cases} \tag{3.28}$$

$$K_{A_2} = \begin{cases} \begin{bmatrix} S_1^{1/2} & 0 \\ 0 & (S_2/s_n)^{1/2} \end{bmatrix} U^T B & for \ \ s_n < 0 \\ S^{-1/2} U^T B & for \ \ s_n \geq 0 \end{cases} \tag{3.29}$$

for $i = 3$

$$L_{A_3} = \begin{cases} CV \begin{bmatrix} R_1^{-1/2} & 0 \\ 0 & ((R_2 * r_n)^{-1/2} \end{bmatrix} & for \ \ r_n < 0 \\ CVR^{-1/2} & for \ \ r_n \geq 0 \end{cases} \tag{3.30}$$

$$K_{A_3} = \begin{cases} \begin{bmatrix} S_1^{-1/2} & 0 \\ 0 & (S_2 * s_n)^{-1/2} \end{bmatrix} U^T B & for \;\; s_n < 0 \\ S^{-1/2} U^T B & for \;\; s_n \geq 0 \end{cases} \tag{3.31}$$

*Proof:* By partitioning $B_{A_i} = \begin{bmatrix} B_{A_1} \\ B_{A_2} \end{bmatrix}, C_{A_i} = \begin{bmatrix} C_{A_1} & C_{A_2} \end{bmatrix}$ and then replacing $B_1 = B_{A_1} K_{A_i}, C_1 = L_{A_i} C_{A_1}$ respectively yields

$$\begin{aligned} & \|W(s)(G(s) - G_r(s))V(s)\|_\infty \\ & = \|W(s)(C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1)V(s)\|_\infty \\ & = \|W(s)(L_{A_i}C_{A_i}(sI - A)^{-1}B_{A_i}K_{A_i} - L_{A_i}C_{A_1}(sI - A_{11})^{-1}B_{A_1}K_{A_i})V(s)\|_\infty \\ & = \|WsL_{A_i}(C_{A_i}(sI - A)^{-1}B_{A_i} - C_{A_1}(sI - A_{11})^{-1}B_{A_1})K_{A_i}V(s)\|_\infty \\ & \leq \|W(s)L_{A_i}\|_\infty \|(C_{A_i}(sI - A)^{-1}B_{A_i} - C_{A_1}(sI - A_{11})^{-1}B_{A_1})\|_\infty \|K_{A_i}V(s)\|_\infty \end{aligned}$$

If $\{A_{11}, B_{A_1}, C_{A_1}, D\}$ is ROM achieved by partitioning a balanced realization $\{A, B_{A_i}, C_{A_i}, D\}$, we have [3]

$$\|(C_{A_i}(sI - A)^{-1}B_{A_i} - C_{A_1}(sI - A_{11})^{-1}B_{A_1})\|_\infty \leq 2\sum_{j=r+1}^{n} \sigma_j \tag{3.32}$$

whereas for discrete time domain, the error bound will be:

$$\|W(z)(G(z) - G_r(z))V(z)\|_\infty \leq 2 \; \|L_{A_i}\| \|K_{A_i}\| \sum_{j=r+1}^{n} \sigma_j \tag{3.33}$$

**Remark 7** *Rank conditions listed above are followed from [5] and [26].*

**Remark 8** *Stability of ROM is not guaranteed when X and Y are not positive/semi-positive definite [35].*

### 3.2.2 Algorithm

Given a full order stable system $G(s)$ with input $V(s)$ and output $G(s)$ weights. MOR is done to achieve ROM $G_r(s)$ by using the following steps:

1. Compute $X$ and $Y$ using equations (2.5) and (2.6) for continous time systems and equations (2.15) and (2.16) for discrete time systems, respectively.

2. Compute $B_{A_1}$, $B_{A_2}$and $B_{A_3}$ from equations (3.3),(3.5) and (3.7) for continous time systems and from (3.15), (3.17) and (3.19) for discrete time systems, respectively.

3. Compute $C_{A_1}$, $C_{A_2}$and $C_{A_3}$ from equations (3.4),(3.6) and (3.8) for continous time systems and equations (3.16), (3.18) and (3.20) for discrete time systems, respectively.

4. Solve equations (3.1) and (3.2) (for continous time systems) or equations (3.13) and (3.14) to compute $P_{A_i}$ and $Q_{A_i}$.

5. Obtain a transformation matrix $T$ (for continous time system) and $T_F$ (for disrete time systems) to satisfy equation (3.11) and (3.23), respectively.

6. Compute the minimal realizations to find a ROM as $G_r(s) = C_1(sI - A_{11})^{-1}B_1 + D_{rA}$ for continous time system or $H_r(z) = C_1(zI - A_{11})^{-1}B_1 + D_{rA}$ for discrete time systems.

# Numerical Examples

This section presents some numerical examples in both continous time domain and discrete time domain to show that the proposed MOR techniques give a less approximation error and an easily computable error bound as compared to the previously proposed techniques.

## 4.1 Continous time domain

*Example 1:* Consider a $6^{th}$ order stable full order system along with input and output weightings (example 3.1 of [35]).

Table 4.1, 4.2 and 4.3 show the frequency weighted error and error bound comparison of the proposed techniques with existing techniques $[3, 5, 39, 30]$ respectively for example 1. It can be observed that the proposed techniques give low approximation error with an error bound in the presence of input, output and double weightings respectively.

Table 4.1: Frequency weighted approximation error for double weightings Example 1

| Order | Enns [3] Error | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 14.284 | 58.832 | 16969 | 15.093 | 127.23 | 1200.5 | 86461 | 118.82 | 619.24 | 7.6505 | 885.67 | 38.139 | 803.14 |
| 2 | 3.2793 | 75.302 | 8897.6 | 14.697 | 66.621 | 62.381 | 20035 | 5.5816 | 325.94 | 7.6846 | 463.12 | 38.619 | 424.46 |
| 3 | 2.078 | 20.94 | 1393.2 | 11.927 | 11.303 | 131.47 | 11967 | 3.9657 | 59.052 | 3.0435 | 77.463 | 4.1525 | 57.043 |
| 4 | 0.026015 | 3.814 | 447.84 | 12.28 | 3.2337 | 96.459 | 5779.5 | 0.52279 | 22.2 | 0.43188 | 22.841 | 1.8027 | 21.967 |
| 5 | 0.014759 | 0.88682 | 87.868 | 4.0617 | 0.65376 | 7.4348 | 1298.4 | 0.65099 | 5.4308 | 0.096824 | 4.5404 | 0.34292 | 4.1727 |

Table 4.2: Frequency weighted approximation error for input weightings Example 1

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 14.284 | 1032.4 | 2078.9 | 2019.8 | 185.61 | 1235.1 | 12241 | 182.77 | 3798.4 | 285.39 | 1322.6 | 162.16 | 18424 |
| 2 | 3.2793 | 56.708 | 1045.3 | 41.482 | 93.141 | 63.532 | 2895.4 | 10.798 | 1806.6 | 7.2643 | 663.37 | 11.976 | 8997 |
| 3 | 2.078 | 71.616 | 190.54 | 58.727 | 17.678 | 108.32 | 1565.9 | 12.678 | 432.19 | 10.113 | 125.18 | 14.37 | 1514.5 |
| 4 | 0.026015 | 9.6581 | 61.43 | 4.3143 | 5.1725 | 103.22 | 746.25 | 1.8105 | 162.93 | 0.78976 | 37.099 | 5.9681 | 585.28 |
| 5 | 0.014759 | 3.0357 | 12.085 | 1.7647 | 1.027 | 41.917 | 212.21 | 2.0804 | 40.235 | 0.31772 | 7.3575 | 1.1918 | 112.15 |

Table 4.3: Frequency weighted approximation error for output weightings Example 1

| Order | Enns[3] | Wang et al.[5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **Error** | **Error** | **Bound** | **Error** | **Bound** | **Error** | **Bound** | **Error** | **Bound** | **Error** | **Bound** | **Error** | **Bound** |
| 1 | 24.592 | 1038.8 | 1503.6 | 785.68 | 803.98 | 842.4 | 1358.8 | 192.27 | 2842.6 | 162.79 | 2770.8 | 184.67 | 2700.9 |
| 2 | 12.054 | 50.501 | 399.2 | 49.385 | 229.64 | 49.315 | 325.37 | 8.4858 | 727.13 | 8.3763 | 749.52 | 8.4593 | 701.47 |
| 3 | 7.657 | 160.68 | 224 | 48.769 | 139.21 | 151.68 | 180.04 | 27.314 | 389.92 | 26.467 | 429.79 | 27.106 | 383.32 |
| 4 | 1.5121 | 41.302 | 92.909 | 27.304 | 49.274 | 50.915 | 78.551 | 8.657 | 175.34 | 6.1575 | 171.02 | 7.8819 | 166.88 |
| 5 | 0.42969 | 12.188 | 22.945 | 11.01 | 12.141 | 11.262 | 19.896 | 2.1169 | 43.503 | 1.9863 | 42.198 | 2.0814 | 41.303 |

*Example 2:* Consider a $6^{th}$ order stable full order system having a state space:

$$A = \begin{bmatrix} -5 & 1 & 0 & 0 & 0 & 0 \\ -89 & 0 & 1 & 0 & 0 & 0 \\ -199 & 0 & 0 & 1 & 0 & 0 \\ -290 & 0 & 0 & 0 & 1 & 0 \\ -500 & 0 & 0 & 0 & 0 & 1 \\ -76 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0.6 \\ 0.7 & 0.3 \\ 2 & 0.1 \\ 1 & 0.2 \\ 0.5 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

having input and output weightings as:

$$A_v = \begin{bmatrix} -4.25 & 0.1 \\ 0 & -0.03 \end{bmatrix}, B_v = \begin{bmatrix} 1 & 10.5 \\ 0.2 & 10.3 \end{bmatrix}$$

$$C_v = \begin{bmatrix} 0.3 & 0.5 \\ 0.1 & 0.1 \end{bmatrix}, D_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_w = \begin{bmatrix} -4 & 0 \\ 0 & -0.25 \end{bmatrix}, B_w = \begin{bmatrix} 1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}$$

$$C_w = \begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.7 \end{bmatrix}, D_w = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Table 4.4, 4.5 and 4.6 show the frequency weighted error and error bound comparison of the proposed techniques with existing techniques [3, 5, 39, 30] respectively for example 2. It can be observed that the proposed techniques give low approximation error with an error bound in the presence of input, output and double weightings respectively.

Table 4.4: Frequency weighted approximation error for double weightings Example 2

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 169.61 | 739.55 | 98431 | 317.77 | 1185.4 | 799.13 | 93671 | 133.25 | 2536.8 | 108.42 | 6595.6 | 185.29 | 2431.4 |
| 2 | 60.139 | 180.28 | 53733 | 27.579 | 646.34 | 181.37 | 42524 | 30.053 | 1374.2 | 30.055 | 3602.6 | 30.019 | 1308.5 |
| 3 | 206.72 | 688.16 | 32683 | 264.27 | 394.53 | 637.55 | 24161 | 116.82 | 834.22 | 110.78 | 2201.7 | 127.28 | 772.61 |
| 4 | 91.807 | 281.94 | 16281 | 88.234 | 196.38 | 289.75 | 9966 | 47.194 | 414.97 | 46.236 | 1099.1 | 49.451 | 363.33 |
| 5 | 10.627 | 33.589 | 1940.7 | 6.9364 | 25.272 | 32.56 | 2479.6 | 5.6004 | 48.187 | 5.4632 | 137.81 | 5.7668 | 40.481 |

Table 4.5: Frequency weighted approximation error for input weightings Example 2

| Order | Enns[3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 169.61 | 524.16 | 7540.8 | 524.21 | 848.95 | 523.85 | 8692.3 | 87.357 | 13902 | 87.366 | 12842 | 87.354 | 15870 |
| 2 | 60.139 | 214.73 | 4061.3 | 214.73 | 452.67 | 215.61 | 4259.7 | 35.788 | 7510.8 | 35.788 | 6868.8 | 35.786 | 8690 |
| 3 | 206.72 | 378.32 | 2357.5 | 366.95 | 258.77 | 345.74 | 2317.4 | 63.747 | 4381.5 | 61.718 | 3945.8 | 66.894 | 5163.4 |
| 4 | 91.807 | 180.83 | 1055.5 | 179.74 | 110.65 | 181.51 | 820.13 | 30.206 | 1990.1 | 30.011 | 1712.5 | 30.501 | 2469.8 |
| 5 | 10.627 | 55.946 | 147.32 | 53.741 | 17.248 | 104.23 | 237.45 | 9.4176 | 268.62 | 9.0888 | 257.91 | 9.6544 | 284.99 |

Table 4.6: Frequency weighted approximation error for output weightings Example 2

| Order | Enns[3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 106.89 | 863.82 | 2192.9 | 766.45 | 1400.8 | 891.96 | 1830.2 | 152.63 | 925.1 | 131.07 | 2610.4 | 184.64 | 1146.6 |
| 2 | 30.067 | 180.26 | 1097.2 | 180.31 | 704.64 | 180.83 | 846.16 | 30.049 | 458.23 | 30.05 | 1311.7 | 30.024 | 565.7 |
| 3 | 69.513 | 465.05 | 585.84 | 446.3 | 382.14 | 468.86 | 454.34 | 77.738 | 243.31 | 75.063 | 709.05 | 83.794 | 290.57 |
| 4 | 19.112 | 119.04 | 193.93 | 117.38 | 134.45 | 120.64 | 154.54 | 19.955 | 78.728 | 19.624 | 246.42 | 20.374 | 80.574 |
| 5 | 3.6465 | 22.58 | 43.433 | 22.326 | 29.061 | 24.307 | 34.573 | 3.7955 | 17.792 | 3.7307 | 53.636 | 3.8607 | 20.281 |

*Example 3:* Consider a $5^{th}$ order stable full order system having a state space:

$$A = \begin{bmatrix} -20 & -10 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 & 0 \\ 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 10 & 0 & -10 \\ 0 & 0 & 0 & 10 & -2 \end{bmatrix}, B = \begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 \end{bmatrix}$$

having input and output weightings as:

$$A_v = \begin{bmatrix} -2.1 \end{bmatrix}, B_v = \begin{bmatrix} 1 \end{bmatrix}$$
$$C_v = \begin{bmatrix} 7.9 \end{bmatrix}, D_v = \begin{bmatrix} 1 \end{bmatrix}$$

$$A_w = \begin{bmatrix} -2.1 \end{bmatrix}, B_w = \begin{bmatrix} 1 \end{bmatrix}$$
$$C_w = \begin{bmatrix} 7.9 \end{bmatrix}, D_w = \begin{bmatrix} 1 \end{bmatrix}$$

Table 4.7 and 4.8 show the frequency weighted error and error bound comparison of the proposed techniques with existing techniques [3, 5, 39, 30] respectively for example 3. It can be observed that the proposed techniques give low approximation error with an error bound in the presence of input and output weightings respectively.

Table 4.7: Frequency weighted approximation error for input weightings Example 3

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 7.4393 | 12.666 | 194.91 | 12.313 | 23.889 | 21.635 | 209.25 | 2.5335 | 34315 | 2.4635 | 29483 | 7.1066 | 372410 |
| 2 | 21.022 | 50.542 | 135.28 | 113.64 | 16.77 | 42.714 | 138.72 | 10.097 | 23815 | 22.297 | 20697 | 7.9857 | 223610 |
| 3 | 3.5998 | 7.7533 | 83.774 | 7.6923 | 10.077 | 12.155 | 79.9 | 1.5508 | 14751 | 1.5385 | 12438 | 24.177 | 121360 |
| 4 | 3.5505 | 37.673 | 41.511 | 36.486 | 5.0067 | 57.552 | 35.077 | 7.5372 | 7309.2 | 7.2951 | 6179.6 | 7.8862 | 19748 |

Table 4.8: Frequency weighted approximation error for output weightings Example 3

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 2.254 | 12.666 | 194.91 | 12.313 | 119.44 | 21.635 | 209.25 | 3.0772 | 37.388 | 2.5181 | 42.789 | 2.8912 | 37.443 |
| 2 | 2.043 | 50.542 | 135.28 | 113.64 | 83.852 | 42.714 | 138.72 | 9.1569 | 25.674 | 11.317 | 29.854 | 9.2852 | 25.768 |
| 3 | 1.5333 | 7.7533 | 83.774 | 7.6923 | 50.387 | 12.155 | 79.9 | 1.5783 | 16.293 | 1.5443 | 18.251 | 1.5699 | 16.275 |
| 4 | 1.5328 | 37.673 | 41.511 | 36.486 | 25.033 | 57.552 | 35.077 | 7.8899 | 8.0385 | 7.3693 | 9.055 | 7.8201 | 8.0396 |

*Example 4:* Consider the state space of $8^{th}$ full order stable controller of (Example 6 of [26]) with the following input and output weightings given as:

$$A_v = \begin{bmatrix} -3 \end{bmatrix}, B_v = \begin{bmatrix} 1 \end{bmatrix}$$

$$C_v = \begin{bmatrix} 100 \end{bmatrix}, D_v = \begin{bmatrix} 0 \end{bmatrix}$$

$$A_w = \begin{bmatrix} -10 \end{bmatrix}, B_w = \begin{bmatrix} 1 \end{bmatrix}$$

$$C_w = \begin{bmatrix} 200 \end{bmatrix}, D_w = \begin{bmatrix} 0 \end{bmatrix}$$

Table 4.9 and 4.10 show the frequency weighted error and error bound comparison of the proposed techniques with existing techniques [3, 5, 39, 30] respectively for example 4. It can be observed that the proposed techniques give low approximation error with an error bound in the presence of input and output weightings respectively.

Table 4.9: Frequency weighted approximation error for input weightings Example 4

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 150.19 | 30.259 | 175.48 | 29.435 | 15.209 | 32.19 | 133.99 | 3.7886 | 2511.4 | 3.6801 | 2336.7 | 4.4554 | 9928.9 |
| 2 | 120.44 | 24.398 | 129.73 | 24.217 | 11.237 | 26.709 | 99.437 | 3.0503 | 1856.7 | 3.0276 | 1726.4 | 3.3327 | 7392.5 |
| 3 | 121.09 | 36.852 | 90.865 | 160.16 | 7.8478 | 26.463 | 67.298 | 4.5759 | 1300.5 | 16.758 | 1205.8 | 3.5572 | 5308 |
| 4 | 50.598 | 10.315 | 52.89 | 10.351 | 4.521 | 10.405 | 37.757 | 1.2893 | 757.22 | 1.2937 | 694.79 | 1.2195 | 3629.4 |
| 5 | 51.31 | 12.793 | 34.511 | 11.571 | 2.9424 | 11.846 | 24.422 | 1.6016 | 494.13 | 1.4516 | 452.23 | 2.568 | 2349.8 |
| 6 | 23.665 | 4.8054 | 17.031 | 4.8084 | 1.4434 | 4.8186 | 11.846 | 0.60066 | 243.89 | 0.60104 | 221.86 | 0.59414 | 1314.6 |
| 7 | 23.227 | 5.5145 | 8.4571 | 5.116 | 0.71722 | 4.755 | 5.899 | 0.69061 | 121.1 | 0.64054 | 110.24 | 1.6467 | 641.63 |

Table 4.10: Frequency weighted approximation error for output weightings Example 4

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 2.2173 | 17.771 | 97.591 | 17.755 | 68.824 | 24.549 | 215.39 | 2.5592 | 57912 | 2.4803 | 13.747 | 2.2193 | 743.92 |
| 2 | 1.8479 | 14.802 | 72.6 | 14.793 | 51.196 | 18.141 | 157.35 | 1.9726 | 43454 | 1.9492 | 10.297 | 1.8491 | 553.38 |
| 3 | 1.7519 | 14.034 | 51.35 | 14.024 | 36.191 | 24.473 | 118.09 | 2.0286 | 31662 | 1.9852 | 7.476 | 1.7531 | 391.2 |
| 4 | 0.86739 | 6.9336 | 30.508 | 6.9364 | 21.48 | 14.858 | 81.636 | 0.80088 | 20725 | 0.82364 | 4.7893 | 0.86705 | 232.18 |
| 5 | 0.86858 | 6.9476 | 20.134 | 6.9481 | 14.172 | 14.723 | 54.651 | 1.213 | 13683 | 1.1328 | 3.1678 | 0.86851 | 153.19 |
| 6 | 0.41577 | 3.3257 | 9.9649 | 3.3259 | 7.0081 | 14.033 | 29.295 | 0.41387 | 7292.2 | 0.41427 | 1.662 | 0.41574 | 75.753 |
| 7 | 0.40781 | 3.2639 | 4.93 | 3.2632 | 3.4671 | 14.587 | 14.335 | 0.69318 | 3588 | 0.61865 | 0.819 | 0.4079 | 37.477 |

*Example 5:* Consider the state space of a $48^{th}$ full order stable "Building Model" system of [40] with the following input and output weightings given by:

$$A_v = \begin{bmatrix}-5\end{bmatrix}, B_v = \begin{bmatrix}1\end{bmatrix}$$

$$C_v = \begin{bmatrix}190\end{bmatrix}, D_v = \begin{bmatrix}0\end{bmatrix}$$

$$A_w = \begin{bmatrix}-4\end{bmatrix}, B_w = \begin{bmatrix}1\end{bmatrix}$$

$$C_w = \begin{bmatrix}1000\end{bmatrix}, D_w = \begin{bmatrix}0\end{bmatrix}$$

Table 4.11 show the frequency weighted error and error bound comparison of the proposed techniques with existing techniques $[3, 5, 39, 30]$ respectively for example 5. It can be observed that the proposed techniques give low approximation error with an error bound in the presence of output weightings.

Table 4.11: Frequency weighted approximation error for output weightings Example 5

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 0.80396 | 38.582 | 293.4 | 38.585 | 152.75 | 71.601 | 540.89 | 0.8038 | 6.1124 | 0.80386 | 1030.7 | 6.9061 | 945.84 |
| 2 | 0.29072 | 13.952 | 234.33 | 13.952 | 120.5 | 23.27 | 463.73 | 0.29067 | 4.8819 | 0.29067 | 813.14 | 0.29071 | 766.71 |
| 3 | 0.29067 | 30.462 | 187.82 | 17.824 | 96.635 | 55.411 | 405.4 | 0.63462 | 3.9129 | 0.37133 | 652.07 | 0.94138 | 613.65 |
| 4 | 0.1868 | 8.1667 | 141.39 | 8.321 | 72.905 | 16.569 | 347.73 | 0.17014 | 2.9456 | 0.17335 | 491.94 | 0.16531 | 460.98 |
| 5 | 0.17673 | 17.35 | 124.33 | 14.968 | 64.229 | 16.859 | 312.96 | 0.36146 | 2.5902 | 0.31183 | 433.4 | 0.67683 | 404.14 |
| 6 | 0.065389 | 8.2774 | 107.45 | 8.358 | 55.633 | 8.5023 | 280.54 | 0.17245 | 2.2385 | 0.17412 | 375.4 | 0.16758 | 347.64 |
| 7 | 0.065048 | 13.425 | 91.762 | 12.703 | 47.2 | 19.227 | 253.03 | 0.27969 | 1.9117 | 0.26465 | 318.49 | 0.31102 | 299.73 |
| 8 | 0.12923 | 3.1894 | 76.866 | 3.2123 | 39.168 | 7.0566 | 225.73 | 0.063609 | 1.1827 | 0.066922 | 264.3 | 0.065612 | 254.13 |
| 9 | 0.06454 | 8.7956 | 66.704 | 7.8661 | 33.971 | 8.9103 | 202.38 | 0.18324 | 1.3897 | 0.16388 | 229.23 | 0.21287 | 220.64 |
| 10 | 0.059884 | 3.0532 | 56.772 | 3.0664 | 28.902 | 5.9131 | 179.84 | 0.063609 | 1.1827 | 0.063884 | 195.03 | 0.063052 | 187.84 |
| 11 | 0.059702 | 5.4947 | 50.227 | 5.167 | 25.611 | 6.9375 | 159.47 | 0.11447 | 1.0464 | 0.10765 | 172.82 | 0.12359 | 165.86 |
| 12 | 0.022686 | 3.0292 | 43.801 | 3.0474 | 22.379 | 2.401 | 139.47 | 0.063109 | 0.91252 | 0.063488 | 151.01 | 0.062606 | 144.29 |
| 13 | 0.022542 | 5.1277 | 37.723 | 4.9123 | 19.178 | 5.5694 | 124.07 | 0.10683 | 0.7859 | 0.10234 | 129.41 | 0.11283 | 124.94 |
| 14 | 0.017153 | 2.2806 | 31.907 | 1.6 | 16.089 | 1.8822 | 109.21 | 0.047513 | 0.66474 | 0.033334 | 108.57 | 0.071215 | 106.64 |
| 15 | 0.01713 | 4.6487 | 26.587 | 4.3253 | 13.405 | 7.4813 | 95.171 | 0.096849 | 0.5539 | 0.090111 | 90.457 | 0.10735 | 88.897 |

*Example 6:* Consider a $5^{th}$ order stable full order system having a state space:

$$A = \begin{bmatrix} -10 & -50 & -70 & =60 & -40 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1240 & 460 & 20 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

having input and output weightings (cited from example 2.3 and example 2.4, respetively of [35]) as:

$$A_v = \begin{bmatrix} -3 \end{bmatrix}, B_v = \begin{bmatrix} 1 \end{bmatrix}$$

$$C_v = \begin{bmatrix} 1 \end{bmatrix}, D_v = \begin{bmatrix} 0 \end{bmatrix}$$

$$A_w = \begin{bmatrix} -0.6250 & -0.2500 \\ 1 & 0 \end{bmatrix}, B_w = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$C_w = \begin{bmatrix} 0 & 0.1250 \end{bmatrix}, D_w = \begin{bmatrix} 0 \end{bmatrix}$$

Table 4.12 and 4.13 show the frequency weighted error and error bound comparison of the proposed techniques with existing techniques [3, 5, 39, 30] respectively for example 6. It can be observed that the proposed techniques give low approximation error with an error bound in the presence of input and output weightings respectively.

Table 4.12: Frequency weighted approximation error for input weightings Example 6

| Order | Enns [3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 5.1621 | 90.602 | 1262.8 | 90.578 | 137.53 | 91.61 | 1644.8 | 18.127 | 129.28 | 18.127 | 129.25 | 18.116 | 1.1762 |
| 2 | 0.7263 | 10.36 | 188.26 | 9.9006 | 20.889 | 7.8586 | 223.46 | 2.2057 | 18.585 | 2.2057 | 18.581 | 1.9802 | 0.17865 |
| 3 | 0.30882 | 10.019 | 57.258 | 9.4274 | 6.7115 | 9.5869 | 44.944 | 2.1399 | 5.0271 | 2.1399 | 5.026 | 1.8855 | 0.057399 |
| 4 | 0.0058825 | 1.3996 | 7.0112 | 1.2333 | 0.79614 | 1.0387 | 6.0133 | 0.35967 | 0.61372 | 0.35967 | 0.61358 | 0.24667 | 0.0068089 |

Table 4.13: Frequency weighted approximation error for output weightings Example 6

| Order | Enns[3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 8.0899 | 136.87 | 258.62 | 124.02 | 145.85 | 151.17 | 587.13 | 27.375 | 66.573 | 24.805 | 9998.7 | 28.78 | 9770.6 |
| 2 | 1.0709 | 6.1922 | 50.108 | 6.0747 | 28.796 | 36.121 | 362.81 | 1.2384 | 12.899 | 1.2149 | 1974.1 | 1.3293 | 1389.9 |
| 3 | 0.73068 | 11.357 | 21.113 | 6.4754 | 10.745 | 38.364 | 174.08 | 2.2713 | 5.4349 | 1.2951 | 736.63 | 6.2242 | 403.25 |
| 4 | 0.018686 | 1.1005 | 2.6765 | 0.81838 | 1.3619 | 21.292 | 33.107 | 0.22009 | 0.68896 | 0.16368 | 93.367 | 2.7359 | 85.209 |

*Example 7:* Consider example 2 of [28] that uses an elliptic band-pass $6^{th}$ order filter which passes frequencies between [5,15] rad/s, and with 0.5 dB of ripple in the pass-band, and 30 dB of attenuation in the stop band along with input weights considering an elliptic band-pass $2^{nd}$ order filter which passes frequencies between [3,15] rad/s, and with 0.1 dB of ripple in the pass-band, and 240 dB of attenuation in the stop band and output weights considering elliptic band-pass $6^{th}$ order filter which passes frequencies between [3,7] rad/s,bandwith 0.1 dB of ripple in the pass-band, and 30 dB of attenuation in the stop band.

Table 4.14 and 4.15 show the frequency weighted error and error bound comparison of the proposed techniques with existing techniques [3, 5, 39, 30] respectively for example 7. It can be observed that the proposed techniques give low approximation error with an error bound in the presence of input and output weightings respectively.

Table 4.14: Frequency weighted approximation error for input weightings Example 7

| Order | Enns[3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 1.8943 | 5.9942 | 34.26 | 5.9942 | 4.0302 | 6.0255 | 24.49 | 1.0003 | 373.49 | 0.99903 | 372.83 | 0.99903 | 371.19 |
| 2 | 1.4192 | 4.7983 | 20.16 | 4.7896 | 2.3706 | 4.7947 | 14.491 | 0.80375 | 220.04 | 0.80231 | 219.55 | 0.79879 | 218.37 |
| 3 | 0.84964 | 4.7981 | 12.388 | 4.7893 | 1.4559 | 4.7892 | 8.874 | 0.80424 | 135.41 | 0.80247 | 135.04 | 0.79874 | 134.14 |
| 4 | 0.21113 | 1.6735 | 4.7021 | 1.6672 | 0.55246 | 1.6792 | 3.3943 | 0.28182 | 51.433 | 0.28079 | 51.281 | 0.27825 | 50.905 |
| 5 | 0.080413 | 1.6735 | 2.3311 | 1.6672 | 0.27354 | 1.6787 | 1.6618 | 0.28182 | 25.584 | 0.28079 | 25.478 | 0.27825 | 25.217 |

Table 4.15: Frequency weighted approximation error for output weightings Example 7

| Order | Enns[3] | Wang et al. [5] | | Varga Anderson [39] | | Imran Ghafoor [30] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 0.94939 | 6.2188 | 31.604 | 5.9045 | 19.305 | 5.7282 | 26.739 | 1.3073 | 7.6561 | 1.3004 | 6.7909 | 0.98685 | 11.222 |
| 2 | 0.72459 | 3.5617 | 18.784 | 3.5934 | 10.832 | 4.3155 | 16.132 | 1.1178 | 4.5049 | 1.0864 | 4.0881 | 0.58781 | 6.3327 |
| 3 | 0.43967 | 3.5419 | 10.846 | 2.893 | 6.0569 | 3.2512 | 9.9036 | 1.0747 | 2.7393 | 1.0553 | 2.4882 | 0.48918 | 3.5508 |
| 4 | 0.10954 | 1.2638 | 4.451 | 1.0881 | 2.6102 | 1.3004 | 4.9275 | 0.4919 | 1.2124 | 0.42422 | 1.0695 | 0.18436 | 1.5246 |
| 5 | 0.040464 | 1.2694 | 2.002 | 1.0669 | 1.0871 | 1.4883 | 1.9522 | 0.4535 | 0.54261 | 0.42617 | 0.4895 | 0.18133 | 0.63924 |

## 4.2 Discrete time domain

*Example 1:* Consider an elliptic band-pass $6^{th}$ order filter which passes frequencies between [0.2,0.4] rad/s, and with 0.2 dB of ripple in the passband, and 20 dB of attenuation in the stop band along with input weights given by:

$$A_v = \begin{bmatrix} -1.1619 & -0.6959 & -0.1378 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_v = \begin{bmatrix} 2.8081 & 2.2444 & 0.8325 \end{bmatrix}, D_v = \begin{bmatrix} 1 \end{bmatrix}$$

and output weights given by:

$$A_w = \begin{bmatrix} -1.1619 & -0.6959 & -0.1378 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_w = \begin{bmatrix} 1.8081 & 2.2444 & 0.8325 \end{bmatrix}, D_w = \begin{bmatrix} 1 \end{bmatrix}$$

Table 4.16 and 4.17 presents the frequency weighted error and error bound comparison (in case of discrete time systems) of the proposed techniques with existing techniques [3, 41, 39, 21] respectively for example 1. It can be observed that the proposed techniques yield stable ROMs and give low approximation error with an error bound in the presence of input and output weightings respectively for Enns [3], Campbell et al.'s [41], Varga and Anderson's [39] and Imran and Ghafoor [21]. .

Table 4.16: Frequency weighted approximation error for input weightings Example 1

| Order | Enns[3] | Campbell et al.'s [41] | | Varga Anderson [39] | | Imran Ghafoor [21] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 1.8943 | 5.9942 | 34.26 | 5.9942 | 4.0302 | 6.0255 | 24.49 | 1.0003 | 373.49 | 0.99903 | 372.83 | 0.99903 | 371.19 |
| 2 | 1.4192 | 4.7983 | 20.16 | 4.7896 | 2.3706 | 4.7947 | 14.491 | 0.80375 | 220.04 | 0.80231 | 219.55 | 0.79879 | 218.37 |
| 3 | 0.84964 | 4.7981 | 12.388 | 4.7893 | 1.4559 | 4.7892 | 8.874 | 0.80424 | 135.41 | 0.80247 | 135.04 | 0.79874 | 134.14 |
| 4 | 0.21113 | 1.6735 | 4.7021 | 1.6672 | 0.55246 | 1.6792 | 3.3943 | 0.28182 | 51.433 | 0.28079 | 51.281 | 0.27825 | 50.905 |
| 5 | 0.080413 | 1.6735 | 2.3311 | 1.6672 | 0.27354 | 1.6787 | 1.6618 | 0.28182 | 25.584 | 0.28079 | 25.478 | 0.27825 | 25.217 |

Table 4.17: Frequency weighted approximation error for output weightings Example 1

| Order | Enns[3] | Campbell et al.'s [41] | | Varga Anderson[39] | | Imran Ghafoor[21] | | Proposed technique 1 | | Proposed technique 2 | | Proposed technique 3 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Error | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound | Error | Bound |
| 1 | 0.94939 | 6.2188 | 31.604 | 5.9045 | 19.305 | 5.7282 | 26.739 | 1.3073 | 7.6561 | 1.3004 | 6.7909 | 0.98685 | 11.222 |
| 2 | 0.72459 | 3.5617 | 18.784 | 3.5934 | 10.832 | 4.3155 | 16.132 | 1.1178 | 4.5049 | 1.0864 | 4.0881 | 0.58781 | 6.3327 |
| 3 | 0.43967 | 3.5419 | 10.846 | 2.893 | 6.0569 | 3.2512 | 9.9036 | 1.0747 | 2.7393 | 1.0553 | 2.4882 | 0.48918 | 3.5508 |
| 4 | 0.10954 | 1.2638 | 4.451 | 1.0881 | 2.6102 | 1.3004 | 4.9275 | 0.4919 | 1.2124 | 0.42422 | 1.0695 | 0.18436 | 1.5246 |
| 5 | 0.040464 | 1.2694 | 2.002 | 1.0669 | 1.0871 | 1.4883 | 1.9522 | 0.4535 | 0.54261 | 0.42617 | 0.4895 | 0.18133 | 0.63924 |

### 4.3 Conclusion

All the results shown in tables interpret that errors and their responsive error bounds achieved from the proposed techniques are comparitevely improved than the existing techniques.

*Chapter 5*

# CONCLUSIONS AND FUTURE WORK

## 5.1 Overview of Thesis

Three FWMR tehniques which are motivated from [31] and [32] are presented in this thesis for continous and discrete time LTI systems.

Chapter 2 discusses the phenomena and motivation of FWMR and explains in detail about FWMR. In this chapter previously proposed techniques using FWMR are being explained in detail and their drawbacks are mentioned as well.

In chapter 3, three FWMR techniques are proposed for LTI continuous and discrete time systems. Stability of a system is assured using proposed techniques, also they provide a lower approximation error as compared to existing stability preserving techniques. These techniques proved to be a solution for instability issue in Enns [3] in case of double weightings. ROMs obtained using proposed tehniques are stable along with an easily computable error bound.

Examples are presented to demonstrate that the proposed techniques give a lower and better approximation of the original system as compared to the existing techniques.

This thesis presented three new frequency weighted MOR techniques for continuous and discrete time systems. After simulation of examples it is being shown that the proposed techniques yield lower approximation error (and an easily computable error bound) as compared to existing techniques. Proposed techniques preserved stability as well. Although Enns [3] technique yields comparatively least approximation error as compared to existing and newly proposed techniques but it yields unstable ROM in case of double sided weightings.

## 5.2 Future Work

The following topics can be used as a future research to enhance the results from this thesis:

- As stability is not guaranteed in Enns techniques [31] for double sided weightings. Due to indefinite matrices X and Y. It will be interesting to see and validate conditions which will help to achieve stability of ROM in case of double sided weightings.

- FWMR techniques proposed in this thesis (Chapter 2) and techniques proposed in [3, 5, 39, 30] are realisation dependent. Research can be carried out to find and label the realisation of original system that may yield lower approximation error and tighter error bound.

- Different FWMR techniques have different formulas to make fictitious input and output matrices. It remains an open problem that which formula among all formulas yield lowest error.

- The techniques that are proposed in this thesis are applied only to stable linear time invariant systems and to check whether the results are valid for the case of non-linear and time varying systems would be interesting.

# BIBLIOGRAPHY


[1] B. C. Moore, “Principal component analysis in linear systems: controllability, observability, and model reduction”, *IEEE Transaction on Automatic Control*, vol. 26, no. 1, pp. 17-32, 1981.

[2] K. Glover. (1984): All optimal hankel-norm approximation of linear multivariable systems and their $L_\infty$ - error bounds, *International Journal of Control*, 39: 1115-95.

[3] D. F. Enns, “Model reduction with balanced realizations: an error bound and a frequency weighted generalization”, *Proceedings of Conference on Decision and Control*, Las Vagas, pp. 127-132, 1984.

[4] Lin, C.-A., And Chiu, T.-Y., 1992, “Model Reduction Via Frequency Weighted Balanced Realization” Control Theory Adv. Technol. 0911-0704, 8, Pp. 341-451.

[5] G. Wang, V. Sreeram, and W. Q. Liu, “A new frequency-weighted balanced truncation method and an error bound,” IEEE Trans. Autom. Control, vol. 44, no. 9, pp. 1734–1737, Sep. 1999.

[6] P. P. Vuillemin, C. P. Vassal, and D. Alazard, “A frequency-limited $H_2$ model approximation method with application to a medium-scale flexible aircraft”, *Advances in Aerospace Guidance, Navigation and Control,* Springer Berlin Heidelberg, pp. 569-583, 2013.

[7] Y. Shamash, “Stable reduced-order models using Pad-type approximations”, *IEEE Transactions on Automatic Control,* vol. 19, pp. 615-619, 1974.

[8] Y. Liu, and B. D. O. Anderson, “Singular perturbation approximation of balanced systems”, *International Journal of Control,* vol. 50, no. 4, pp. 1379-1405, 1989.

[9] Z.Z. Qi, Y. L. Jiang, and Z. H. Xiao,“Model order reduction based on general orthogonal polynomials in the time domain for coupled systems”, *Journal of the Franklin Institute,* vol. 351, no. 6, pp. 3200-3214, 2014.

[10] T. Stykel, “Gramian-based model reduction for descriptor systems”, *Mathematics of Control Signals and Systems,* vol. 16, no. 4, pp. 297-319, 2004.

[11] V. Mehrmann, and T. Stykel, “Balanced truncation model reduction for large-scale systems in descriptor form”, *Dimension Reduction of Large-Scale Systems,* Springer Berlin Heidelberg, pp. 83-115, 2005.

[12] P. Aghaee, A. Zilouchian, S. Ravesh, and A. Zadegan, “Principal of frequency-domain balanced structure in linear systems and model reduction”, *Computer Electronics Engineering,* vol. 29, no. 3, pp. 463–477, 2003.

[13] M. Tahavori, and H. R. Shaker, “Model reduction via time-interval balanced stochastic truncation for linear time invariant systems”,*International Journal of Systems Science,* vol. 44, no. 3, pp. 493-501, 2013.

[14] P. Benner, S. Gugercin, and K. Willcox, “A survey of model reduction methods for parametric systems”, MPIMD/13-14, Max Planck Institute Magdeburg Preprint, 2013.

[15] P. Benner, T. Breite, and T. Damm. (2010): Krylov subspace methods for model order reduction of bilinear discrete-time control systems, *Proceedings in Applied Mathematcs and Mechanics*, 1: 601-02.

[16] A. Davoudi, J. Jatskevich, P. L. Chapman, and A. Bidram, “Multi-resolution modeling of power electronics circuits using model-order reduction techniques”, *IEEE Transaction on Circuits and Systems I: Regular Papers*, vol. 60, no. 3, pp. 810-823, 2013.

[17] X. Du and G. H Yang, “$H_\infty$ model reduction of linear continuous-time systems over finite-frequency interval”, *IET Control Theory and Applications,* vol. 4, no. 3, pp. 499-508, 2010.

[18] W. Yang, L. Zhang, P. Shi and Y. Zhu, “Model reduction for a class of nonstationary Markov jump linear systems”, *Journal of the Franklin Institute,* vol. 349, no. 7, pp. 2445-2460, 2012 .

[19] M. B. Ha, C. Batlle and E. Fossas, “A new estimation of the lower error bound in balanced truncation method”, *Automatica,* vol. 50, no. 8, pp.2196-2198, 2014.

[20] H. R. Shaker and M. Tahavori, “Time interval model reduction of Bilinear systems”, *International Journal of Control,* vol. 87, no. 8, pp. 1487-1495, 2014.

[21] M. Imran, A. Ghafoor, V. Sreeram, Frequency Weighted Model Order Reduction Technique And Error Bounds For Discrete Time Systems. Math. Probl. Eng. 2014, 1–8 (2014).

[22] Rodriguez Sanchez, Raul & Buchschmid, Martin & Müller, Gerhand. (2017). Model order reduction in design of parameterized structures under different load configurations. Procedia Engineering. 199. 378-383. 10.1016/j.proeng.2017.09.049.

[23] U. M. Al Saggaf, G. F. Franklin.(1987): An error bound for discrete reduced oreder model of a linear multivariable system, *IEEE Transaction on Automatic Control,* 32: 815-19

[24] S. Gugercin, A. C. Antoulas, “A survey of model reduction by balanced truncation and some new results”, *International Journal of Control*, vol. 77, no. 8, pp. 748-766, 2004.

[25] W. Gawronski and J.-N. Juang, “Model reduction in limited time and frequency intervals”, *International Journal of Systems and Science*, vol. 21, no. 2, pp. 349-37, 1990.

[26] A. Ghafoor, V. Sreeram, “A survey/review of frequency-weighted balanced model reduction techniques”, *ASME Journal of Journal of Dynamic Systems, Measurement and Control*, vol. 130, no. 6, 2008.

[27] M. Imran, and Abdul Ghafoor, “A frequency limited interval Gramians-based model reduction technique with error bounds”, *Circuit Systems and Signal Processing*, vol. 34, no. 11, pp. 3505-3519, 2015.

[28] Imran, Muhammad, and Abdul Ghafoor. ”Frequency limited model reduction techniques with error bounds.” *IEEE Transactions on Circuits and Systems II: Express Briefs* 65.1: 86-90 (2017).

[29] D. Wang, and A. Zilouchian. (2000): ”Model reduction of discrete linear syetem via frequency domain balanced realization”, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 47: 830-37.

[30] M. Imran, A. Ghafoor, and V. Sreeram, “A frequency weighted model order reduction technique and error bounds”, *Automatica*, vol. 50, no. 12, pp. 3304-3309, 2014.

[31] H. I. Toor, M. Imran, A. Ghafoor, D. Kumar, V. Sreeram, and A. Rauf, “Frequency limited model reduction techniques for discretetime systems,” *IEEE Transactions on Circuits and Systems II: Express Briefs,* vol. 67, no. 2, pp. 345–349, 2020.

[32] M. Imran, M. J. Awan, and A. Ghafoor, “Model reduction techniques with error bounds,” in *2017 13th IEEE Conference on Automation Science and Engineering (CASE),* pp. 1510–1515, 2017.

[33] Obnita G. and B. D. O. Anderson, “Model Reduction for Control System Design”, *Springer.* 2001.

[34] K. Zhou, J. C. Doyle and K. Glover, “Robust and optimal control”, *Prentice Hall,* 1996.

[35] Ghafoor, A. (2007). Frequency-Weighted Model Reduction and Error Bounds. (Ph.D.dissertion) University of Western Austrlia.

[36] Z. Bai, and R. W. Freund, “A partial Pade-via-Lanczos method for reduced-order modeling”, *Linear Algebra Applications,* vol. 332, no. 1, pp. 139-164, 2001.

[37] S. Gugercin, “An iterative SVD-Krylov based method for model reduction of largescale dynamical systems”, Linear Algebra Applications, vol. 428, no. 8, pp. 1964- 1986, 2008.

[38] Kabamba, P. T., “Balanced Forms: Canonicity and Parameterizations,” *IEEE Transactions on Automatic Control,* Vol. AC-30, no. 11(113), pp. 1106–1109, 1985

[39] A. Varga and B. D. O. Anderson, “Accuracy enhancing methods for the frequency-weighted balancing related model reduction,” in *Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No.01CH37228),* vol. 4, pp. 3659–3664 vol.4, 2001.

[40] http://slicot.org/20-site/126-benchmark-examples-for-modelreduction.

[41] K. Campbell, V. Sreeram and G. Wang, “A frequency weighted discrete system balanced truncation method and an error bound”, *Proceedings of American Control Conference,* Chicago, pp. 2403-2404, 2000.

[42] B. D. O. Anderson and Y. Liu, ”Controller reduction: concepts and approaches,” *IEEE Transactions on Automatic Control,* vol. 34, no. 8, pp. 802-812, Aug. 1989, doi: 10.1109/9.29422.