IMPROVED MODEL ORDER REDUCTION TECHNIQUES WITH ERROR BOUNDS

By

Shabana Bashrat

A thesis submitted to the faculty of Electrical Engineering Department, Military College of Signals, National University of Sciences and Technology, Islamabad, Pakistan, in partial fulfillment of the requirements for the degree of MS in Electrical (Telecommunication) Engineering

August2021

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS Thesis written by **Shabana Bashrat** Registration No. **00000206507**, of **Military College of Signals** has been vetted by undersigned, found complete in all respect as per NUST Statutes/Regulations, is free of plagiarism, errors and mistakes and is accepted as partial, fulfillment for award of MS degree. It is further certified that necessary amendments as pointed out by GEC members of the scholar have been also incorporated in the said thesis.

Signature:_______________________________

Name of Supervisor:______________________

Date: ___________________________________

Signature (HOD):________________________

Date: ________________________________

Signature (Dean/Principal):________________

Date: ________________________________

Declaration

I hereby declare that work carried out in this thesis has not been submitted in support of any degree or professional qualification either at this institution or elsewhere.

Dedication

I dedicate this thesis to my loving parents and wonderful elder brothers for their love, endless support and motivation.

Acknowledgement

My sincere thanks goes to my supervisor Dr. Muhammad Imran for his constant encouragement and dedicated guidance, which enabled me to undertake this research. This dissertation would not have been possible without his continuous support, patience and insightful analysis. Throughout the thesis writing process, his guidance and brain storming motivation remained with me like a beacon. I could not imagine having an enlightened mentor for my MS thesis.

Besides my advisor, I also acknowledge others whose contributions are really significant. To Dr. Adil Masood Siddiqui, Dr. Abdul Ghafoor and Dr. Safia Akram, I am thankful for their encouragement, insightful comments, hard questions and their contributions as members of my committee. Moreover, I would like to express my sincere gratitude to Dr. Abdul Wakeel and Ma'am Sammana Batool for their guidance and suggestions.

My appreciation extends to my lovely friends, Samina Kanwal and Kinza Kazmi whose interest and encouragement enabled me to accomplish this herculean task.

Above ground, I am indebted to my family. Through the many years, my parents have always supported me, encouraged me to explore new directions in life. And finally, my deepest gratitude goes to my lovely brothers and sisters for their motivation, support and patience during study.

Abstract

Model Order Reduction (MOR) is a computational technique to build low order system from high order system by capturing the original properties of the actual system. The demand of MOR is ever higher during the study of dynamic behavior of the complex system. Because the complex high order systems impose difficulties e.g slow computations and expensive storage requirements. So, MOR techniques facilitate to reduce these difficulties in order to provide fast computations and less storage requirements in the process of designing and simulation of large- scale physical systems. MOR capitulates Reduced Order Models (ROMs) which conserve the input output behavior of the original physical system. Construction of stable ROMs with low approximation error between original and compact (reduced) model is the main goal of MOR. MOR can be done in frequency-domain as well as in time-domain. Remarkable research work has been done on various directions of MOR to build ease in the designing, simulation and analysis of the complex dynamic systems. The precedent MOR techniques mostly have limitation of un-stability, large approximation error and lack of a priori error bounds in ROMs. Hence, the aim of this thesis is to construct improved model reduction techniques in order to overcome the existing problems of model reduction techniques in frequency domain. The proposed techniques guarantee the preservation of stability in ROM and low frequency-response approximation error with easily computable error bound as compared to existing MOR techniques. In this thesis, firstly frequency weighted MOR problem is developed, then improved frequency weighted model reduction techniques are proposed for continuous-time systems. A frequency limited

Gramians-based MOR technique is also illustrated with error bounds for continuous-time systems. The applicability of the presented work is demonstrated in the context of some practical numerical examples to show the accuracy and efficacy of the proposed methods.

Contents

1 Introduction 1

1.1 Main Concept ... 1

1.2 Motivation ... 2

1.3 Shortcomings of Existing Literature ... 2

1.4 Problem Formulation ... 4

1.5 Contributions ... 4

1.6 Organization of thesis ... 5

2 Background and Literature Review 6

2.1 The General Idea of MOR ... 6

2.2 Balanced Truncation ... 8

2.3 Frequency Weighted Model Reduction Technique ... 9

2.4 Existing Stability Preserving Frequency Weighted Techniques ... 11

2.4.1 Generalized Lin and Chiu's Technique ... 12

2.4.2 Wang et al's Technique ... 14

2.4.3 Varga and Anderson Technique ... 15

2.4.4 Imran et al's Technique ... 16

2.5 Frequency Limited Model Reduction Technique 17

2.6 Existing Stability Preserving Frequency Limited Techniques............... 18

2.6.1 Gugercin et al's Technique... 18

2.6.2 Ghafoor et al's Technique 19

2.6.3 Imran et al's Technique 20

3 Proposed Techniques22

3.1 New Frequency Weighted MOR Tehniques 23

3.1.1 Error Bound.. 25

3.2 Frequency Limited MOR Techniques 26

3.2.1 Error Bound.. 28

3.3 Algorithms 30

4 Numerical Simulations and Discussion32

4.1 Frequency Weighted MOR Simulations 32

4.2 Frequency Limited MOR Simulations.. 38

5 Conclusion48

5.1 Future Work 48

References50

List of Figures

2.1 MOR Process 7

4.1 $\sigma[G_{200}(s) - G_{45}(s)]$ in $[\omega_1 = 15, \omega_2 = 24]$ 38

4.2 Close-up view of $\sigma[G_{200}(s) - G_{45}(s)]$ 39

4.3 $\sigma[G_{50}(s) - G_{21}(s)]$ in $[\omega_1 = 9, \omega_2 = 20]$ 40

4.4 Close-up view of $\sigma[G_{50}(s) - G_{21}(s)]$ 41

4.5 $\sigma[G_{30}(s) - G_{12}(s)]$ in $[\omega_1 = 9, \omega_2 = 15]$ 42

4.6 Close-up view of $\sigma[G_{30}(s) - G_{12}(s)]$ 42

4.7 $\sigma[G_{20}(s) - G_{11}(s)]$ in $[\omega_1 = 13, \omega_2 = 29]$ 43

4.8 Close-up view of $\sigma[G_{20}(s) - G_{11}(s)]$ 43

4.9 $\sigma[G_{8}(s) - G_{4}(s)]$ in $[\omega_1 = 5, \omega_2 = 12]$ 44

4.10 Close-up view of $\sigma[G_{8}(s) - G_{4}(s)]$ 45

4.11 $\sigma[G_{6}(s) - G_{1}(s)]$ in $[\omega_1 = 13, \omega_2 = 17]$ 45

4.12 Close-up view of $\sigma[G_{6}(s) - G_{1}(s)]$ 46

List of Tables

4.1 Error and Error Bounds Comparison for Example 1 33

4.2 Error and Error Bounds Comparison for Example 2 34

4.3 Error and Error Bounds Comparison for Example 3 35

4.4 Error and Error Bounds Comparison for Example 4 36

4.5 Poles Location of Frequency Limited ROMs 41

4.6 Poles Location of Frequency Limited ROMs 46

Acronyms

MOR	Model Order Reduction
ROMs	Reduced Order Models
RO	Reduced Order
SVD	Singular Value Decomposition
PCA	Principle Component Analysis
HSV	Hankel Singular Values
SISO	Single Input Single Output
LQG	Linear Quadratic Gaussian
CAD	Computer Aided Design
VLSI	Very Large-Scale Integration
LTI	Linear Time-Invariant
PDE	Partial Differential Equation
ODE	Ordinary Differential Equation

Chapter 1

Introduction

This chapter presents a brief introduction to the work accomplished in this thesis. Section1.1explains the main concept of MOR, the main motivation of thesis is explained in Section1.2. Section1.3highlights the shortcomings of the existing literature, the problem formulation is discussed in section1.4. The main contribution of the thesis is elaborated in section1.5. Finally, section1.6gives the idea regarding the organization of the thesis.

1.1 Main Concept

MOR is associated with extraction of a small-scale system that approximates the accurate behavior of a large-scale dynamical system at its predefined input and output parameters. The mathematical approximation techniques for large differential equations (define the physical behavior of the systems) come in the frame to accomplish the process of MOR. Hence, the terms such as "reduced-bases approximation", "retaining of high energy states", "states truncation (having less effect on system response)" "balancing of the Gramians (controllability and observability)" and "order reduction" are associated with the concept of MOR. The theme of MOR, originally has been introduced in mathematics in the context of the differential equations. Later, MOR has been carried over to the control system engineering and other fields such as civil engineering, chemical engineering, process engineering, aerospace engineering, earthquake engineering, mechanical and Very Large-Scale Integration (VLSI) circuits designing [1,2,3].

1.2 Motivation

Simulations or computation science is considered as a reliable tool to identify, analyze and predict the dynamical behavior of the physical system. Computation science has great importance in today's technological world. Computation science is regarded as third discipline other than the classical disciplines of theory and experiment . Computer simulations are now carried out for many chemical and physical processes on routine basis. Computer Aided Design (CAD) and virtual environments have been built in order to provide ease in the designing of new products to make the process faster, more reliable and building less costly prototypes (validate the correct functioning of the designed system before it goes into production). Moreover, the building of a virtual prototype is absolutely cheaper and faster than generating a physical prototype. MOR is a computational technique that speed up the simulations and make computationally expensive in terms of time and memory storage while preserving the original properties of the actual system [4,5,6,7].

1.3 Shortcomings of Existing Literature

Balanced Truncation (BT) is the most common method in MOR techniques to capture low order model from high order model. BT not only ensures stability but also provides error bound for ROMs. The least controllable and observable states are discarded in BT and the most significant observable and controllable states are used to consider low order approximation of original system. Generally, Bt performs the reduction process by using full range of frequency to compute the system's response [8]. However, some applications like filter and controller reduction etc, it is preferred to consider approximation error over a certain frequency range of interest as sometimes, the reduction error is more significant in particular range of frequency. This introduces the concept of frequency weighted MOR [9].

Enns [10] upgraded the BT technique by using frequency weights and this technique preserves the stability for single sided frequency weights (input/output) but it may not yield stable ROMs in case of both sided frequency weights. To handle this issue, many frequency weighted MOR techniques have been presented in literature [11,12,13,14,15,16,17,18,19,20].

Wang et al. [20] proposed a useful technique to achieve the stability of ROMs by ensuring input and output related matrices' positive/semipositive definiteness by using absolute function. This technique is also applicable to controller reduction as the precedent techniques are not applicable to controller reduction due to zero pole cancellation and computes a priori error bound expression. *Varga et al.* [12] established stability by ignoring all negative eigenvalues and retaining only positive eigenvalues.

Later, *Imran et al.* [21] introduced a method to subtract the least negative eigenvalue from all eigenvalues to ensure positive/semipositive definiteness of input and output related matrics to guarantee stability but it leads to large approximation error due to nullification of last eigenvalue.

Gawronski et al. [22] simplified the frequency weighted MOR by considering the approximation in the desired frequency range instead of constructing weights. It is named as frequency limited MOR. In this technique, the controllability and observability Gramians are defined for limited frequency interval. But this technique also capitulates unstable ROMs for the original stable system. Moreover, it does not compute error bound.

To solve the instability issue, *Gugercin et al.* [23] proposed a method to take absolute of negative eigenvalues to ensure the positive or semi-positive definiteness of some input and output related matrices to achieve stability. *Ghafoor et al.* [24] achieved stability by ignoring all negative eigenvalues and retaining only positive eigenvalues. But [23,24] techniques do not affect the all negative eigenvalues equally that lead to large approximation error in some systems. *Imran et al*. [25] guaranteed the stable ROMs in desired frequency range by subtracting the least negative eigenvalue that ensure positive/semipositive definiteness of input and output related matrics. Stability is achieved by existing techniques [23,24,25] but at the cost of large approximation error.

1.4 Problem Formulation

The pioneer frequency weighted and frequency limited interval-Gramians based schemes for continuous-time systems yield unstable ROMs, due to some input/output related matrices that are not conserved to be positive or semi-positive definite. Some existing techniques preserve stability but at the cost of large approximation error and poor error bounds.

1.5 Contributions

The main contribution of this thesis are as follows.

- Proposed MOR techniques are developed for efficient reduction of continuous-time Linear Time-Invariant (LTI) systems.
- Proposed techniques establish stable ROMs with less frequency-response approximation error and computable error bounds as compared to existing techniques in the literature.
- Proposed techniques are applied on frequency weighted and frequency limited

interval-Gramians based MOR techniques.

- Proposed techniques are applied on some practical examples of model reduction to show their efficacy and accuracy.

1.6 Organization of thesis

The rest of the thesis is organized as follows:

- Chapter 2 presents a concise background on the main subjects relevant to this work such as, MOR for LTI continuous-time systems and their modelings, BT, frequency weighted background and its existing literature's shortcomings and frequency limited existing literature's problems.

- Chapter 3 develops the proposed techniques algorithms for frequency weighted and frequency limited interval- Gramians based MOR that guarantee the stability of the ROMs with minimum frequency-response approximation error . Error bounds are also developed for frequency weighted problem involving predefined weights and without predefined weights.

- Chapter 4 presents the numerical simulation and discussion to illustrate the applicability of the proposed solutions.

- Chapter 5 presents the conclusions and future work.

Chapter 2

Background and Literature Review

This chapter illustrates the essential background information on which the presented work is formed: in particular, sections2.1and2.2recall the basic preliminaries regarding MOR procedure. Section2.3represents the basic procedure of model reduction using frequency weights and section2.4recalls some existing stability preserving frequency weighted MOR techniques. Finally, section2.5illustrates the frequency limited MOR method that simplifies the Gramians of the frequency weighted MOR technique to yield near-optimal ROMs and some existing stability preserving frequency limited techniques are discussed in section2.6.

2.1 The General Idea of MOR

Numerous artificial, mechanical and physical procedures can be defined by dynamical systems which can be applied for simulation or control. The modeling of many physical systems can be done using a set of continuous Partial Differential Equations (PDEs) or discrete Ordinary Differential Equations (ODEs). Moreover, the transformation of PDEs is done into a system of linear/non-linear ODEs and approximation of the behavior of continuous system is acquired using discretization. However, the fast developments in large-scale dynamic systems like telecommunications systems, power systems and chemical systems impose complexity in the modeling of these systems which result in large system of ODE equations and make computationally expensive in terms of time and memory storage. MOR techniques are developed and efficiently used to speed up the computation time and save storage requirements for the fairly large-complex system [26,27,28,29]. The idea of MOR has been demonstrated as a valuable tool to obtain

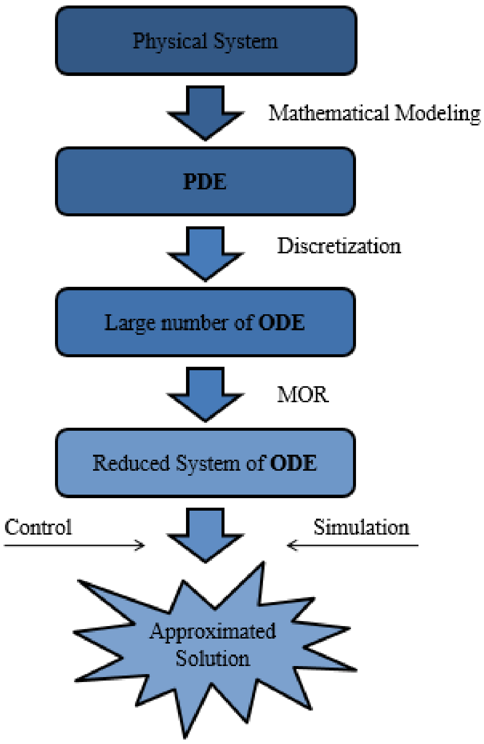

Figure 2.1: MOR Process

efficacy in simulations while guaranteeing desired accuracy. Its applicability to real life problems has made it a poplar tool in many branches of science and engineering. The MOR process is explained pictorially in Figure1.1[1,30]. MOR is a technique to capitulate low order system from high order system by capturing the key properties of the original system. The main goal of MOR is to construct the stable ROMs with less approximation error [31,32,33,34,35].

2.2 Balanced Truncation

Moore [36] proposed the BT method by combining the Principle Component Analysis (PCA) and Singular Value Decomposition (SVD) to cope with the structural instabilities of the dynamic systems. BT is also named as internally balanced realization and it is very significant in control engineering. In balanced realization, the less observable and controllable states are truncated and dominant states that affect the system's response usefully are retained. BT yields stable reduced models with explicit error bounds.

Consider a LTI continuous system in state space

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t)_r = C_r x(t)_r + Du(t) \tag{2.2.1}$$

where { $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ }, n, m and p represent order, number of inputs and number of the outputs of the system respectively. A is the system matrix of the dimension $R^{n\times n}$, B is the input matrix of the dimension $R^{n\times m}$, C is the output matrix of the dimension $R^{p\times n}$ and D is the feedforward matrix of the dimension $R^{p\times m}$. The transfer function of the original system is represented in equation (2.2.1) is

$$G_o(s) = C(sI - A)^{-1}B + D \tag{2.2.2}$$

The controllability and observability Gramians are defined mathmatically as

$$P_C = \int_{-\infty}^{\infty} e^{At}BB^T e^{A^T t} dt \tag{2.2.3}$$

$$Q_O = \int_{-\infty}^{\infty} e^{A^T t}C^T C e^{At} dt \tag{2.2.4}$$

The controllability Gramian $\boldsymbol{P_C}$ and observability Gramian $\boldsymbol{Q_O}$ are the solution of following Lyapunov equations

$$AP_C + P_C A^T + BB^T = 0 \tag{2.2.5}$$

$$A^T Q_O + Q_O A + CC^T = 0 \tag{2.2.6}$$

After calculating the SVD of the matrix, a non-singular transformation matrix T is used to obtain a balanced system from dynamic system by converting observability and controllability Gramians into equal and diagonal matrics [37].

$$T^T Q_O T = T^{-1} P_C T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.2.7}$$

where $\sigma_m \geq \sigma_{m+1},\ m = 1, 2, 3, \ldots, n-1$ and formulate the Hankel Singular Values (HSV) of Σ that are used to measure the robustness of the observable and controllable state [21]. The ROM is obtained by applying the transformation over the original system

$$\hat{A}_t = T^{-1} A_o T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \hat{B}_t = T^{-1} B_o = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$\hat{C}_t = C_o T = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad \hat{D}_t = D_o$$

The transfer function of ROM is obtained as

$$G_r(s) = C_1 (sI - A_{11})^{-1} B_1 + D \tag{2.2.8}$$

2.3 Frequency Weighted Model Reduction Technique

The frequency dependence of the error is critical for the stability of a control system with respect to MOR error, especially in case of feedback controller design wherein the error should be small in crossover frequency ranges and can be larger in case of other frequency ranges. This motivated the *Enns* [10] to use weighted error criterion. Consider the transfer functions of the stable input weight $V_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$ and stable output weight $W_o(s) = C_o(sI - A_o)^{-1}B_o + D_o$ respectively. The augmented systems are given by

$$G_o(s)V_i(s) = C_i(sI - A_i)^{-1}B_i + D_i \qquad (2.3.1)$$

$$W_o(s)G_o(s) = C_o(sI - A_o)^{-1}B_o + D_o \qquad (2.3.2)$$

where

$$\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right] = \left[\begin{array}{cc|c} A & BC_v & BD_v \\ 0 & A_v & B_v \\ \hline C & DC_v & DD_v \end{array}\right]$$

$$\left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right] = \left[\begin{array}{cc|c} A_w & B_wC & B_wD \\ 0 & A & B \\ \hline C_w & D_wC & D_wD \end{array}\right]$$

Let the Gramians

$$P_x = \begin{bmatrix} P_e & P_{12} \\ P_{12}^T & P_v \end{bmatrix}, \quad Q_y = \begin{bmatrix} Q_w & Q_{12}^T \\ Q_{12} & Q_e \end{bmatrix} \qquad (2.3.3)$$

satisfy the following Lyapunov equations:

$$\begin{aligned} A_iP_x + P_xA_i^T + B_iB_i^T &= 0 \qquad (2.3.4) \\ A_o^TQ_y + Q_yA_o + C_o^TC_o &= 0 \qquad (2.3.5) \end{aligned}$$

Expanding the blocks (1,1) and (2,2) respectively, of the Eq (2.3.4) and (2.3.5) yield

$$AP_e + P_eA^T + X_z = 0 \qquad (2.3.6)$$

$$A^TQ_e + Q_eA + Y_z = 0 \qquad (2.3.7)$$

where

$$X_z = BC_iP_{12}^T + P_{12}C_i^TB^T + BD_iD_i^TB^T \qquad (2.3.8)$$

$$Y_z = C^TB_o^TQ_{12}^T + Q_{12}B_oC + C^TD_o^TD_oC \qquad (2.3.9)$$

The contragredient matrix T obtained as

$$T^TQ_eT = T^{-1}P_eT^{-T} = \Sigma \qquad (2.3.10)$$

where Σ formulates the HSV, diagonal elements and arranged in the descending order. By applying transformation and partitioning the original system,the ROMs are obtained by $G_z(s) = C_1(sI - A_{11})^{-1}B_1 + D$.

Remark 1. *Since in Enns [10] technique,* $X_z \geq 0$ *and* $Y_z \geq 0$ *are not always guaranteed, the ROMs may not remain stable in case of double sided frequency weights [20].*

2.4 Existing Stability Preserving Frequency Weighted Techniques

To handle the issue of instability in Enns method [10], a lot of stability preserving techniques have been proposed in literature. In this section we review some well-known frequency weighted stability preserving techniques . Stability is ensured by converting some input output related matrices into positive/semi positive definiteness matrices.

2.4.1 Generalized Lin and Chiu's Technique

Lin and Chiu [11] modified the Enns' method [10] to ensure the stability of the ROMs for strictly proper two sided weights. Let the transformations be applied to input output augmented system realization respectively,

$$T_{Li} = \begin{bmatrix} I & P_{12}P_v^{-1} \\ 0 & I \end{bmatrix}, \quad T_{Lo} = \begin{bmatrix} I & -Q_w^{-1}Q_{12}^T \\ 0 & I \end{bmatrix}$$

The input system of the transformed augmented realization is:

$$\hat{A}_i = T_{Li}^{-1} A_i T_{Li} = \begin{bmatrix} A & X_{12} \\ 0 & A_v \end{bmatrix}, \quad \hat{B}_i = T_{Li}^{-1} B_i = \begin{bmatrix} B_L \\ B_v \end{bmatrix},$$

$$\hat{C}_i = C_i T_{Li} = \begin{bmatrix} C & CP_{12}P_v + DC_v \end{bmatrix}, \qquad \hat{D}_i = D_i = DD_v$$

The output system of the transformed augmented realization is:

$$\hat{A}_o = T_{Lo}^{-1} A_o T_{Lo} = \begin{bmatrix} A_w & Y_{12} \\ 0 & A \end{bmatrix}, \quad \hat{B}_o = T_{Lo}^{-1} B_o = \begin{bmatrix} B \\ B_w D + Q_w^{-1} Q_{12}^T B \end{bmatrix},$$

$$\hat{C}_o = A_o T_{Lo} = \begin{bmatrix} C_L & C_w \end{bmatrix}, \qquad \hat{D}_o = D_o = D_w D$$

where

$$X_{12} = AP_{12}P_v^{-1} + BC_v - P_{12}P_v^{-1}A_v$$

$$B_L = BD_v - P_{12}P_v^{-1}B_v$$

$$Y_{12} = Q_w^{-1}Q_{12}^T A + B_w C - A_w Q_w^{-1} Q_{12}^T$$

$$C_L = D_w C - C_w Q_w^{-1} Q_{12}^T$$

The transformed augmented realization Gramians:

$$\hat{P}_i = T_{Li}^{-1} P_i T_{Li}^{-T} = \begin{bmatrix} P_E - P_{12}P_v^{-1}P_{12}^T & 0 \\ 0 & P_v \end{bmatrix}$$

$$\hat{Q}_o = T_{Lo}^T Q_o T_{Lo} = \begin{bmatrix} Q_w & 0 \\ 0 & Q_E - Q_{12}Q_w^{-1}Q_{12}^T \end{bmatrix}$$

satisfy the following Lyupnov equations:

$$A_i P_i + P_i A_i^T + B_i B_i^T = 0 \quad (2.4.1)$$

$$A_o^T Q_o + Q_o A_o + C_o^T C_o = 0 \quad (2.4.2)$$

Solving the (1,1) of the (2.4.1) and (2,2) block of the (2.4.2) results respectively

$$A P_L + P_L A^T + B_L B_L^T = 0 \quad (2.4.3)$$

$$A^T Q_L + Q_L A + C_L^T C_L = 0 \quad (2.4.4)$$

where $P_L = P_E - P_{12}P_v^{-1}P_{12}^T$ and $Q_L = Q_E - Q_{12}Q_w^{-1}Q_{12}^T$. Diagonalize the weighted gramians P_L and Q_L simultaneously, yield

$$T^T Q_L T = T^{-1} P_L T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \dots, \sigma_n\} \quad (2.4.5)$$

where $\sigma_m \geq \sigma_{m+1}$, $m = 1, 2, 3, \dots, n-1$. The ROMs are acquired by applying the transformation and partitioning the original system. Lin and Chiu [11] produce stable ROMs in case of double-sided weights, assuming no pole-zero cancellation between original system and weights [38].

2.4.2 Wang et al's Technique

Wang et al. [20] proposed a useful technique to achieve the stability of eigenvalues that are calculated from the eigenvalue decomposition of some input and output related matrices by using absolute function. This method is also applicable to controller reduction as the precedent techniques are not applicable to controller reduction due to zero pole cancellation and computes a priori error bound expression.

Let the new controllability P_{wa} and observability Q_{wa} Gramians satisfy the following Lyapunov equations

$$AP_{wa}+P_{wa}A^T+B_{wa}B_{wa}^T = 0 \quad (2.4.6)$$

$$A^TQ_{wa}+Q_{wa}A+C_{wa}^TC_{wa} = 0 \quad (2.4.7)$$

The contragredient matrix T_w obtained as

$$T^TQ_{wa}T=T^{-1}P_{wa}T^{-T}=\Sigma=diag\{\sigma_1,\sigma_2,\ldots,\sigma_n\} \quad (2.4.8)$$

where $\sigma_w \geq \sigma_{w+1}$, $w = 1, 2, 3, \ldots n-1$.

The fictitious input B_{wa} and output C_{wa} matrices are defined as, $B_{wa} = U_{wa}|S_{wa}|^{1/2}$ and $C_{wa} = |R_{wa}|^{1/2}V_{wa}^T$. Since the expressions U_{wa}, S_{wa}, V_{wa} and R_{wa} are computed by orthogonal Eigen Value Decomposition (EVD) of symmetric matrices $X_w=U_{wa}S_{wa}U_{wa}^T$ and $Y_w=V_{wa}R_{wa}V_{wa}^T$ where $S_{wa}=diag(s_1,s_2,\ldots,s_m)\geq 0$ and $R_{wa}=diag(r_1,r_2,\ldots\ldots\ldots,r_t)\geq 0$. The ROMs are computed by partitioning the transformed realization.

Theorem 1. *The following error bound holds, if the rank conditions rank* $[B_{wa} B]$ = *rank* $[B_{wa}]$ *and rank* $\begin{bmatrix} C_{wa} \\ C \end{bmatrix}$ = *rank* $[C_{wa}]$ *are satisfied.*

$$\|W_y(s)(G_o(s)-G_z(s))V_x(s)\|_\infty \leq 2\|W_y(s)L_{wa}\|_\infty\|K_{wa}V_x(s)\|_\infty \sum_{m=n+1}^{n}\sigma_m$$

where

$$L_{wa} = C_{wa} diag(|r_1|^{-1/2}, |r_2|^{-1/2}, \ldots, |r_{mi}|^{-1/2}, 0, \ldots, 0)$$
$$K_{wa} = diag(|s_1|^{-1/2}, |s_2|^{-1/2}, \ldots, |s_{mo}|^{-1/2}, 0, \ldots, 0) U^T B_{wa}$$

$mi = rank[X_z]$ *and* $mo = rank[Y_z]$.

2.4.3 Varga and Anderson Technique

Varga et al. [12] modified the *Wang et al.* [20] technique by reducing the Gramians distance to Enns choice (i.e, the size of P_{wa}-P_e and Q_{wa}-Q_e). The proposed transformation simultaneously diagonalized the controllability and observability Gramians P_{vr}, Q_{vr} respectively,

$$T^T Q_{vr} T = T^{-1} P_{vr} T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.4.9}$$

where $\sigma_m \geq \sigma_{m+1}$, $m = 1, 2, 3, \ldots, n-1$ and Gramians satisfy the following Lyapunov equations

$$AP_{vr} + P_{vr} A^T + B_{vr} B_{vr}^T = 0 \tag{2.4.10}$$
$$A^T Q_{vr} + Q_{vr} A + C_{vr}^T C_{vr} = 0 \tag{2.4.11}$$

The fictitious input B_{vr} and output C_{vr} matrices are defined as, $B_{vr} = U_{vr} S_{vr}^{1/2}$ and $C_{vr} = R_{vr}^{1/2} V_{vr}^T$. The terms U_{vr_1}, S_{vr_1}, V_{vr_1} and R_{vr_1} EVD of symmetric matrices

$$X_e = [U_{vr_1}\, U_{vr_2}] \begin{bmatrix} S_{vr_1} & 0 \\ 0 & S_{vr_2} \end{bmatrix} \begin{bmatrix} U_{vr_1}^T \\ U_{vr_2}^T \end{bmatrix} \tag{2.4.12}$$

$$Y_e = [V_{vr_1} V_{vr_2}] \begin{bmatrix} R_{vr_1} & 0 \\ 0 & R_{vr_2} \end{bmatrix} \begin{bmatrix} V_{Z_1}^T \\ V_{Z_2}^T \end{bmatrix} \qquad (2.4.13)$$

where $\begin{bmatrix} S_{vr_1} & 0 \\ 0 & S_{vr_2} \end{bmatrix} = diag(s_1, s_2, \ldots, s_m)$ and $\begin{bmatrix} R_{vr_1} & 0 \\ 0 & R_{vr_2} \end{bmatrix} = diag(r_1, r_2, \ldots\ldots, r_t), S_{vr_1} \geq,$ $S_{vr_2} < 0$ and $R_{vr_1} \geq 0$, $R_{vr_2} < 0$. The ROMs are computed by partitioning the transformed realization.

2.4.4 Imran et al's Technique

Imran et al. [21] introduced a method to subtract the least negative eigenvalue from all eigenvalues to ensure positive/semipositive definiteness of input and output related matrics to guarantee stability. The proposed transformation simultaneously diagonalized the controllability and observability Gramians P_{im}, Q_{im} respectively,

$$T^T Q_{im} T = T^{-1} P_{im} T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \qquad (2.4.14)$$

where $\sigma_m \geq \sigma_{m+1}$, $m = 1, 2, 3, \ldots, n-1$, controllability and observability Gramians satisfy the following Lyapunov equations

$$AP_{im} + P_{im}A^T + B_{im}B_{im}^T = 0 \qquad (2.4.15)$$

$$A^T Q_{im} + Q_{im}A + C_{im}^T C_{im} = 0 \qquad (2.4.16)$$

The fictitious input an output related matrices B_{im} and C_{im} respectively, are defined as

$$B_{im} = U_{im}(S - s_n I)^{1/2} \qquad for\ s_n < 0 \qquad (2.4.17)$$

$$C_{im} = (R - r_n I)^{1/2} V_{im}^T \qquad for\ r_n < 0 \qquad (2.4.18)$$

The terms U_{im}, S, V_{im} and R are computed by EVD of symmetric matrices $X_e = U_{im} S U^T_{im}$ and $Y_e = V_{im} R V^T_{im}$, where $S = diag(s_1, s_2, \quad, s_m) \geq 0$ and $R = diag(r_1, r_2, \quad, r_n) \geq 0$, $s_1 \geq s_2 \geq s_3 \geq, \ldots s_n$ and $r_1 \geq r_2 \geq r_3 \geq, r_n$. The ROMs are computed by partitioning the transformed realization.

2.5 Frequency Limited Model Reduction Technique

Gawronski et al. [22] presented a method in which the frequency weights are not directly predefined, but approximation is considered in desired frequency interval for linear continuous time systems. Let controllability P_{Gj} and observability Q_{Gj} Gramians are defined for limited frequency interval as $P_{Gj} = P(\omega_2) - P(\omega_1)$ and $Q_{Gj} = Q(\omega_2) - Q(\omega_1)$ respectively. The Gramians are expressed using Parseval's relationship as

$$P_{Gj} = \frac{1}{2\pi} \int_{-\omega_1}^{\omega_2} (j\omega I_2 - A)^{-1} B B^T (-j\omega I_1 - A^T)^{-1} d\omega \quad (2.5.1)$$

$$Q_{Gj} = \frac{1}{2\pi} \int_{-\omega_1}^{\omega_2} (-j\omega I_2 - A^T)^{-1} C^T C (j\omega I_1 - A)^{-1} d\omega \quad (2.5.2)$$

These Gramians are the solution of the following Lyapunov equations

$$A P_{Gj} + P_{Gj} A^T + X_g = 0 \quad (2.5.3)$$

$$A^T Q_{Gj} + Q_{Gj} A + Y_g = 0 \quad (2.5.4)$$

where

$$X_g = (S_o(\omega_2) - S_o(\omega_1)) B B^T + B B^T (S_o^*(\omega_2) - S_o^*(\omega_1)) \quad (2.5.5)$$

$$Y_g = (S_o^*(\omega_2) - S_o^*(\omega_1)) C^T C + C^T C (S_o(\omega_2) - S_o(\omega_1)) \quad (2.5.6)$$

$$S_o(\omega) = \frac{j}{2\pi} \ln((j\omega I + A)(-j\omega I + A)^{-1}) \quad (2.5.7)$$

$$X_g = U \begin{bmatrix} S_{j1} & 0 \\ 0 & S_{j2} \end{bmatrix} U^T, Y_g = V \begin{bmatrix} R_{j1} & 0 \\ 0 & R_{j2} \end{bmatrix} V^T$$

$$S_{j1} = diag(s_1, \ldots.., s_m) \geq 0,\ S_{j2} = diag(s_{m+1}, \ldots, s_n) < 0$$

$$R_{j1} = diag(r_1, \ldots.., r_t) \geq 0, R_{j2} = diag(r_{t+1}, \ldots\ldots, r_n) < 0$$

$m \leq n$ and $t \leq n$ are the number of the positive eigenvalues of X_g and Y_g matrices respectively. $S_o^*(\omega)$ is the conjugate transpose of $S_o(\omega)$. The contragredient matrix T is obtained as

$$T^T Q_j T = T^{-1} P_j T^{-T} = \Sigma \tag{2.5.8}$$

The ROM is obtained by $G_z = C_z(sI - A_z)^{-1} B_z + D$ after applying the transformation and partitioning the original system.

Remark 2. *In Gawronski et al. [22], for a desired frequency range, the symmetric matrices X_g and Y_g are not sometimes positive/semi positive definite that lead to yield unstable ROMs [23].*

2.6 Existing Stability Preserving Frequency Limited Techniques

2.6.1 Gugercin et al's Technique

Inspired by *Wang et al.*[20] frequency weighted MOR technique, *Gujercin et al.* [23] modified the *Gawronski et al.* [22] technique to achieve stability by introducing absolute function. The new controllability P_{ga} and observability Q_{ga} Gramians respectively, acquired as the solution to the following Lyapunov equations

$$AP_{ga}+P_{ga}A^T+B_{ga}B_{ga}^T = 0 \quad (2.6.1)$$
$$A^TQ_{ga}+Q_{ga}A+C_{ga}^TC_{ga} = 0 \quad (2.6.2)$$

The contragredient matrix T obtained as

$$T^TQ_{ga}T = T^{-1}P_{ga}T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \quad (2.6.3)$$

where $\sigma_w \geq \sigma_{w+1}$, $w = 1, 2, 3, \ldots n-1$.

The new fictitious input B_{ga} and output C_{ga} matrices are defined as, $B_{ga} = U_{ga}|S_{ga}|^{1/2}$ and $C_{ga} = |R_{ga}|^{1/2}V_{ga}^T$ respectively. The terms U_{ga}, S_{ga}, V_{ga} and R_{ga} are computed by orthogonal EVD of symmetric matrices $X_g = U_{ga}S_{ga}U_{ga}^T$ and $Y_g = V_{ga}R_{ga}V_{ga}^T$ where $S_{ga} = diag(s_1, s_2, \ldots, s_n) \geq 0$ and $R_{ga} = diag(r_1, r_2, \ldots, r_n) \geq 0$, $|s_1| \geq |s_2| \geq |s_3| \geq, \ldots |s_n|$ and $|r_1| \geq |r_2| \geq |r_3| \geq, \ldots |r_n|$. The ROMs are computed by partitioning the transformed realization.

2.6.2 Ghafoor et al's Technique

Inspired by *Varga et al.*[12] frequency weighted MOR technique, *Ghafoor et al.* [24] modified the *Gawronski et al.* [22] technique respectively, to achieve stability by ignoring negative eigenvalues and retaining only positive values. The new controllability P_{ga} and observability Q_{ga} Gramians respectively, acquired as the solution to the following Lyapunov equations

$$AP_{ga}+P_{ga}A^T+B_{ga}B_{ga}^T = 0 \quad (2.6.4)$$
$$A^TQ_{ga}+Q_{ga}A+C_{ga}^TC_{ga} = 0 \quad (2.6.5)$$

The contragredient matrix T obtained as

$$T^T Q_{ga} T = T^{-1} P_{ga} T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.6.6}$$

where $\sigma_w \geq \sigma_{w+1}$, $w = 1, 2, 3, \ldots n-1$.

2.6.3 Imran et al's Technique

Inspired by *Wang et al.*[20] and *Varga et al.*[12] frequency weighted MOR techniques, *Gujercinet al.* [23] and *Ghafoor et al.* [24] modified the *Gawronski et al.* [22] technique respectively, to handle the issue of unstability. Later, *Imran et al.* [25] applied the frequency weighted MOR technique [21] on frequency limited interval to yield stable ROMs with less approximation error. *Imran et al.* [21] introduced a method to subtract the least negative eigenvalue from all eigenvalues to ensure positive/semipositive definiteness of input and output related matrics to guarantee stability. The proposed transformation simultaneously diagonalized the controllability and observability Gramians P_{im}, Q_{im} respectively,

$$T^T Q_{im} T = T^{-1} P_{im} T^{-T} = \Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \tag{2.6.7}$$

where $\sigma_m \geq \sigma_{m+1}$, $m = 1, 2, 3, \ldots, n-1$, controllability and observability Gramians satisfy the following Lyapunov equations

$$AP_{im} + P_{im}A^T + B_{im}B_{im}^T = 0 \tag{2.6.8}$$

$$A^T Q_{im} + Q_{im}A + C^T_{im}C_{im} = 0 \tag{2.6.9}$$

The fictitious input an output related matrices B_{im} and C_{im} respectively, are defined as

$$B_{im} = U_{im}(S - s_n I)^{1/2} \qquad for\ s_n < 0 \quad (2.6.10)$$

$$C_{im} = (R - r_n I)^{1/2} V^T_{im} \qquad for\ r_n < 0 \quad (2.6.11)$$

The terms U_{im}, S, V_{im} and R are computed by EVD of symmetric matrices $X_e = U_{im} S U^T_{im}$ and $Y_e = V_{im} R V^T_{im}$, where $S = diag(s_1, s_2, \ldots, s_m) \geq 0$ and $R = diag(r_1, r_2, \ldots\ldots, r_n) \geq 0$, $s_1 \geq s_2 \geq s_3 \geq, \ldots s_n$ and $r_1 \geq r_2 \geq r_3 \geq, \ldots\ldots\ldots r_n$. The ROMs are computed by partitioning the transformed realization.

Chapter 3

Proposed Techniques

In this chapter the new proposed methods are being applied on frequency weighted and frequency limited interval Gramians-based model reduction techniques to compute least frequency-response error and easily calculable priori error bound as compared to existing stability conserving frequency weighted (*Wang et al.*[20], *Imran et al*. [21]) and frequency limited (*Gujercin et al*.[23], *Ghafoor et al*.[13], *Imran et al*.[25]) model reduction techniques. These Gramians based schemes are proposed for LTI continuous-time systems.

The pioneer frequency weighted MOR scheme for continuous-time systems proposed by Enns [10] computes lowest frequency-response approximation error but it yields unstable ROMs, due to some input/output related matrices that are not conserved to be positive or semi-positive definite. Same is the case with frequency limited model reduction scheme, the *Gawronski et al.*[22] also capitulates lowest frequency response error but computes unstable ROMs. Whereas, the proposed methods yield stable ROMs with less approximation error by building some variations in X_z and Y_z matrics to ensure the positive/semi-positive definiteness of input/output related matrics respectively, in MOR schemes.

3.1 New Frequency Weighted MOR Tehniques

Let a new controllability P_{Z_k} and observability Q_{Z_k} Gramians respectively, are calculated by solving the following Lyapunov equations:

$$AP_{Z_k} + P_{Z_k}A^T + B_{Z_k}B_{Z_k}^T = 0 \quad (3.1.1)$$

$$A^T Q_{Z_k} + Q_{Z_k}A + C_{Z_k}^T C_{Z_k} = 0 \quad (3.1.2)$$

where $k = 1, 2$. For indefinite symmetric matrices X_Z and Y_Z the new input, output related matrices are defined as B_{Z_k} and C_{Z_k}

$$B_{Z_1} = \begin{cases} U_{Z_1} S_{Z_1}^{1/2} & for\ s_n \geq 0 \\ U_{Z_2}(sin(S_{Z_2}) - S_{Z_2})^{1/2} & for\ s_n < 0 \end{cases} \quad (3.1.3)$$

$$B_{Z_2} = \begin{cases} U_{Z_1} S_{Z_1}^{1/2} & for\ s_n \geq 0 \\ U_{Z_2}((exp(1/S_{Z_2}))^n)^{1/2} & for\ s_n < 0 \end{cases} \quad (3.1.4)$$

and

$$C_{Z_1} = \begin{cases} R_{Z_1}^{1/2} V_{Z_1}^T & for\ r_n \geq 0 \\ (sin(R_{Z_2}) - R_{Z_2})^{1/2} V_{Z_2}^T & for\ r_n < 0 \end{cases} \quad (3.1.5)$$

$$C_{Z_2} = \begin{cases} R_{Z_1}^{1/2} V_{Z_1}^T & for\ r_n \geq 0 \\ ((exp(1/R_{Z_2}))^n)^{1/2} V_{Z_2}^T & for\ r_n < 0 \end{cases} \quad (3.1.6)$$

where n is the order of the system matrix A and the terms U_{Z_1}, U_{Z_2}, S_{Z_1}, S_{Z_2}, V_{Z_1}, V_{Z_2}, R_{Z_1} and R_{Z_2} are attained from following symmetric matrices,

$$X_z = [U\,S\,U^T] = [U_{Z_1}\ U_{Z_2}] \begin{bmatrix} S_{Z_1} & 0 \\ 0 & S_{Z_2} \end{bmatrix} \begin{bmatrix} U_{Z_1}^T \\ U_{Z_2}^T \end{bmatrix} \quad (3.1.7)$$

$$Y_z = [V R V^T] = [V_{Z_1}\ V_{Z_2}] \begin{bmatrix} R_{Z_1} & 0 \\ 0 & R_{Z_2} \end{bmatrix} \begin{bmatrix} V_{Z_1}^T \\ V_{Z_2}^T \end{bmatrix} \quad (3.1.8)$$

where

$$S_{Z_1} = diag(s_1, \ldots\ldots, s_m), S_{Z_2} = diag(s_{m+1}, s_{m+2} \ldots\ldots\ldots, s_n),$$

$$R_{Z_1} = diag(r_1, \ldots\ldots\ldots, r_t), R_{Z_2} = diag(r_{t+1}, r_{t+2} \ldots\ldots\ldots, r_n).$$

Remark 3. *When* $X_g > 0$ *and* $Y_g > 0$, $B_{Z_k} = S_{Z_k} S_{Z_k}^{1/2}$ *and* $C_{Z_k} = R_{Z_k}^{1/2} V_{Z_k}^T$.

Let a contragradient transformation matrix T is derived as

$$T_{Z_k}^T Q_{Z_k} T_{Z_k} = T_{Z_k}^{-1} P_{Z_k} T_{Z_k}^{-T} = diag(\sigma_1, \sigma_2 \cdots \sigma_n) \quad (3.1.9)$$

Where $\sigma_m \geq \sigma_{m+1}$, $m = 1, 2, 3, \ldots\ldots\ldots, n-1$, $\sigma_l > \sigma_{l+1}$. A ROM $\{A_{11}, B_1, C_1, D\}$ is attained by portioning the transformation realization as

$$T_{Z_k}^{-1} A T_{Z_k} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad T_{Z_k}^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$C T_{Z_k} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad D_{Z_k} = D \quad (3.1.10)$$

Remark 4. *Since* $X_z \leq B_{Z_k} B_{Z_k}^T$, $Y_z \leq C_{Z_k}^T C_{Z_k}$, $P_{Z_k} > 0$ *and* $Q_{Z_k} > 0$. *Hence, the realization* (A, B_{Z_k}, C_{Z_k}) *is minimal and stability of ROM is conserved.*

3.1.1 Error Bound

Theorem 2. *The following error bound holds for the proposed frequency weighted MOR techniques,*

$$\|W_o(s)(G_o(s) - G_z(s))V_i(s)\|_\infty \leq 2\|W_o(s)L_{Z_k}\|_\infty \|K_{Z_k}V_i(s)\|_\infty \sum_{m=n+1}^{n} \sigma_m$$

if the following rank conditions $rank[B_{Z_k} \;\; B] = rank[B_{Z_k}]$ *and* $rank \begin{bmatrix} C_{Z_k} \\ C \end{bmatrix} = rank\,[C_{Z_k}]$ *are satisfied.*

where

$$L_{Z_1} = \begin{cases} CV_{Z_1}R_{Z_1}^{-1/2} & for\ r_n \geq 0 \\ CV_{Z_2}(sin(R_{Z_2}) - R_{Z_2})^{-1/2} & for\ r_n < 0 \end{cases} \tag{3.1.11}$$

$$L_{Z_2} = \begin{cases} CV_{Z_1}R_{Z_1}^{-1/2} & for\ r_n \geq 0 \\ CV_{Z_2}((exp(1/R_{Z_2}))^n)^{-1/2} & for\ r_n < 0 \end{cases} \tag{3.1.12}$$

and

$$K_{Z_1} = \begin{cases} S_{Z_1}^{-1/2}U_{Z_1}^T B & for\ s_n \geq 0 \\ (sin(S_{Z_2}) - (S_{Z_2}))^{-1/2}U_{Z_2}^T B & for\ s_n < 0 \end{cases} \tag{3.1.13}$$

$$K_{Z_2} = \begin{cases} S_{Z_1}^{-1/2}U_{Z_1}^T B & for\ s_n \geq 0 \\ ((exp(1/S_{Z_2}))^n)^{-1/2}U_{Z_2}^T B & for\ s_n < 0 \end{cases} \tag{3.1.14}$$

Proof. As the $rank\,[B_{Z_k} \;\; B] = rank[B_{Z_k}]$ and $rank \begin{bmatrix} C_{Z_k} \\ C \end{bmatrix} = rank[C_{Z_k}]$ holds. By substituting $B_1 = B_{Z_1}K_{Z_k}$, $C_1 = L_{Z_k}C_{Z_1}$, $B = B_{Z_k}K_{Z_k}$ and $C = L_{Z_k}C_{Z_k}$ respectively computes

$$
\begin{aligned}
&\|W_o(s)(G(s) - G_z(s))V_i(s)\|_\infty \\
&\quad = \|W_o(s)(C(sI-A)^{-1}B - C_1(sI-A_{11})^{-1}B_1)V_i(s)\|_\infty \\
&\quad = \|W_o(s)(L_{Z_k}C_{Z_k}(sI-A)^{-1}B_{Z_k}K_{Z_k} - L_{Z_k}C_{Z_1}(sI-A_{11})^{-1}B_{Z_1}K_{Z_k})V_i(s)\|_\infty \\
&\quad = \|W_o(s)L_{Z_k}(C_{Z_k}(sI-A)^{-1}B_{Z_k} - C_{Z_1}(sI-A_{11})^{-1}B_{Z_1})K_{Z_k}V_i(s)\|_\infty \\
&\quad \leq \|W_o(s)L_{Z_k}\|_\infty \|(C_{Z_k}(sI-A)^{-1}B_{Z_k} - C_{Z_1}(sI-A_{11})^{-1}B_{Z_1})\|_\infty \|K_{Z_k}V_i(s)\|_\infty
\end{aligned}
$$

If $\{A_{11}, B_{Z_1}, C_{Z_1}, D\}$ is ROM attained by splitting a balanced realization $\{A, B_{Z_k}, C_{Z_k}, D\}$, we have from [20], [21].

$$
\|(C_{Z_k}(sI-A)^{-1}B_{Z_k} - C_{Z_1}(sI-A_{11})^{-1}B_{Z_1})\|_\infty \leq 2 \sum_{m=n+1}^{n} \sigma_m
$$

Therefore,

$$
\|W_o(s)(G_o(s) - G_z(s))V_i(s)\|_\infty \leq 2\|W_o(s)L_{Z_k}\|_\infty \|K_{Z_k}V_i(s)\|_\infty \sum_{m=n+1}^{n} \sigma_m
$$

□

3.2 Frequency Limited MOR Techniques

Let a new controllability P_{Z_k} and observability Q_{Z_k} Gramians respectively, are calculated by solving the following Lyapunov equations:

$$
AP_{Z_k} + P_{Z_k}A^T + B_{Z_k}B_{Z_k}^T = 0 \quad (3.2.1)
$$

$$
A^T Q_{Z_k} + Q_{Z_k}A + C_{Z_k}^T C_{Z_k} = 0 \quad (3.2.2)
$$

where

$$P_{Z_k} = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_2} (j\omega I - A)^{-1} B B^T (-j\omega I - A^T)^{-1} d\omega \tag{3.2.3}$$

$$Q_{Z_k} = \frac{1}{2\pi}\int_{-\omega_1}^{\omega_2} (-j\omega I - A^T)^{-1} C^T C (j\omega I - A)^{-1} d\omega \tag{3.2.4}$$

and $k = 1, 2$. For indefinite symmetric matrices X_Z and Y_Z the new input, output related matrices are defined as B_{Z_k} and C_{Z_k}

$$B_{Z1} = \begin{cases} U_{Z_1} S_{Z_1}^{1/2} & for\ s_n \geq 0 \\ U_{Z_2}(sin(S_{Z_2}) - S_{Z_2})^{1/2} & for\ s_n < 0 \end{cases} \tag{3.2.5}$$

$$B_{Z_2} = \begin{cases} U_{Z_1} S_{Z_1}^{1/2} & for\ s_n \geq 0 \\ U_{Z_2}((exp(1/S_{Z_2}))^n)^{1/2} & for\ s_n < 0 \end{cases} \tag{3.2.6}$$

and

$$C_{Z_1} = \begin{cases} R_{Z_1}^{1/2} V_{Z_1}^T & for\ r_n \geq 0 \\ (sin(R_{Z_2}) - R_{Z_2})^{1/2} V_{Z_2}^T & for\ r_n < 0 \end{cases} \tag{3.2.7}$$

$$C_{Z_2} = \begin{cases} R_{Z_1}^{1/2} V_{Z_1}^T & for\ r_n \geq 0 \\ ((exp(1/R_{Z_2}))^n)^{1/2} V_{Z_2}^T & for\ r_n < 0 \end{cases} \tag{3.2.8}$$

The terms $U_{Z_1}, U_{Z_2}, S_{Z_1}, S_{Z_2}, V_{Z_1}, V_{Z_2}, R_{Z_1}, R_{Z_2}$, are attained from following symmetric matrices,

$$X_z = [U\ S\ U^T] = [U_{Z_1}\ U_{Z_2}] \begin{bmatrix} S_{Z_1} & 0 \\ 0 & S_{Z_2} \end{bmatrix} \begin{bmatrix} U_{Z_1}^T \\ U_{Z_2}^T \end{bmatrix} \tag{3.2.9}$$

$$Y_z = [V\ R\ V^T] = [V_{Z_1}\ V_{Z_2}] \begin{bmatrix} R_{Z_1} & 0 \\ 0 & R_{Z_2} \end{bmatrix} \begin{bmatrix} V_{Z_1}^T \\ V_{Z_2}^T \end{bmatrix} \tag{3.2.10}$$

where

$$S_{Z_1} = diag(s_1, \ldots\ldots, s_m), S_{Z_2} = diag(s_{m+1}, s_{m+2} \ldots\ldots\ldots, s_n),$$

$$R_{Z_1} = diag(r_1, \ldots\ldots\ldots, r_t), R_{Z_2} = diag(r_{t+1}, r_{t+2} \ldots\ldots, r_n).$$

Remark 3: When $X_g > 0$ and $Y_g > 0$, $B_{Z_k} = S_{Z_k} S_{Z_k}^{1/2}$ and $C_{Z_k} = R_{Z_k}^{1/2} V_{Z_k}^T$.

Let a contragradient transformation matrix T is derived as

$$T_{Z_k}^T Q_{Z_k} T_{Z_k} = T_{Z_k}^{-1} P_{Z_k} T_{Z_k}^{-T} = diag(\sigma_1, \sigma_2 \cdots \sigma_n) \tag{3.2.11}$$

Where $\sigma_m \geq \sigma_{m+1}$, $m = 1, 2, 3, \ldots\ldots\ldots, n-1$, $\sigma_l > \sigma_{l+1}$. A ROM $\{A_{11}, B_1, C_1, D\}$ is attained by portioning the transformation realization as

$$T_{Z_k}^{-1} A T_{Z_k} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} T_{Z_k}^{-1} B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$C T_{Z_k} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D \tag{3.2.12}$$

Remark 4: Since $X_z \leq B_{Z_k} B_{Z_k}^T$, $Y_z \leq C_{Z_k}^T C_{Z_k}$, $P_{Z_k} > 0$ and $Q_{Z_k} > 0$. Hence, the realization (A, B_{Z_k}, C_{Z_k}) is minimal and stability of ROM is conserved.

3.2.1 Error Bound

Theorem 3. *The following error bound is hold for the proposed frequency limited MOR techniques, where the weights are not explicitly defined.*

$$\|G_o(s) - G_z(s)\|_\infty \leq 2\|L_{Z_k}\|\|K_{Z_k}\| \sum_{m=n+1}^{n} \sigma_m$$

Proof. If the rank conditions $rank\,[B_{Z_k}\ \ B] = rank\,[B_{Z_k}]$ and $rank \begin{bmatrix} C_{Z_k} \\ C \end{bmatrix} = rank\,[C_{Z_k}]$ are satisfied, substituting $B_1 = B_{Z_1}K_{Z_k}$, $C_1 = L_{Z_k}C_{Z_1}$ and using the equations (3.1.11-3.1.14) respectively, yields

$$
\begin{aligned}
\|(G_o(s) - G_z(s))\|_\infty \\
&= \|(C(sI-A)^{-1}B - C_1(sI-A_{11})^{-1}B_1)\|_\infty \\
&= \|(L_{Z_k}C_{Z_k}(sI-A)^{-1}B_{Z_k}K_{Z_k} - L_{Z_k}C_{Z_1}(sI-A_{11})^{-1}B_{Z_1}K_{Z_k})\|_\infty \\
&= \|L_{Z_k}(C_{Z_k}(sI-A)^{-1}B_{Z_k} - C_{Z_1}(sI-A_{11})^{-1}B_{Z_1})K_{Z_k}\|_\infty \\
&\leq \|L_{Z_k}\|\|(C_{Z_k}(sI-A)^{-1}B_{Z_k} - C_{Z_1}(sI-A_{11})^{-1}B_{Z_1})\|_\infty\|K_{Z_k}\|
\end{aligned}
$$

The RoM $\{A_{11}, B_{Z_1}, C_{Z_1}, D\}$ is attained by splitting a balanced realization $\{A, B_{Z_k}, C_{Z_k}, D\}$, we have from [20], [21].

$$
\|(C_{Z_k}(sI-A)^{-1}B_{Z_k} - C_{Z_1}(sI-A_{11})^{-1}B_{Z_1})\|_\infty \leq 2 \sum_{m=n+1}^{n} \sigma_m
$$

Therefore,

$$
\|G_o(s) - G_z(s)\|_\infty \leq 2\|L_{Z_k}\|\|K_{Z_k}\| \sum_{m=n+1}^{n} \sigma_m
$$

□

Remark 5. *Two choices of $K_{Z_k} \in \{K_{Z_1}; K_{Z_2}\}$ and $L_{Z_k} \in \{L_{Z_1}; L_{Z_2}\}$ form basis to derive error bounds for each proposed technique.*

3.3 Algorithms

The algorithms for proposed work are as follows

Algorithm 1: Generation of New Input and Output Matrices

Input : Negative Diagonal Matrices $S(m)$, $R(t)$
Output: Positive Diagonal Matrices $S(m)$, $R(t)$

```
1  for m ← 1 : n do
2      if S(m) < 0 then
3          S(m) ← (sin(S(m)) − S(m))^(1/2)    // for Technique I
4          S(m) ← (exp(1/S(m)))^(1/2)    // for Technique II
5      else
6          S(m) ← S(m)
7      end
8  end
9  for t ← 1 : n do
10     if R(t) < 0 then
11         R(t) ← (sin(R(t)) − R(t))^(1/2)    // for Technique I
12         R(t) ← (exp(1/R(t)))^(1/2)    // for Technique II
13     else
14         R(t) ← R(t)
15     end
16 end
17 return S(m), R(t)
```

Algorithm 2: Frequency Weighted MOR Algorithm

Input: Original Model(A,B,C,D), Input-Weights(A_x, B_x,C_x,D_x), Output-Weights (A_y,B_y,C_y,D_y)
Output: Reduced Model (A_z, B_z,C_z,D_z)

Compute $A_xP_x+P_xA_x^T=-B_xB_x^T$ for controllability Gramian P_x;
Compute $A_y^TQ_y+Q_yA_y=-C_y^TC_y$ for observability Gramian Q_y;
Compute $X_z=-(AP_e+P_eA^T)$ and $Y_z=(A^TQ_e+Q_eA)$ for symmetric matrices;
Decompose X_z and Y_z into $[U_zS_zU_z^T]$ and $[V_zR_zV_z^T]$ using SVD;
Proceed with steps 1-17 in Algorithm 1 to compute new input/output matrices;
Compute $B_{Z_k} \leftarrow U diag(S_1,\ldots S_i\ldots,0\ldots0)$;
Compute $C_{Z_k} \leftarrow diag(R_1,\ldots R_k\ldots,0\ldots0)V^T$;
Compute $AP_{Z_k}+P_{Z_k}A^T=-B_{Z_k}B_{Z_k}^T$ for new controllability Gramian P_{Z_k};
Compute $A^TQ_{Z_k}+Q_{Z_k}A=-C_{Z_k}^TC_{Z_k}$ for new observability Gramian Q_{Z_k};
Compute the contragradient transformation matrix $T_{Z_k}^TQ_{Z_k}T_{Z_k}=T_{Z_k}^{-1}P_{Z_k}T_{Z_k}^{-T}$;
Compute the balanced realization as $\hat{A}_t=T^{-1}A_zT$, $\hat{B}_t=T^{-1}B_z$,$\hat{C}_t=C_zT$;
Select ROM and truncate A, B, C to compute reduced realization A_z, B_z and C_z and it is $D_z=D$;

Algorithm 3: Frequency Limited MOR Algorithm

Input: Original Model(A,B,C,D), Desired Frequency Ranges(ω_1, ω_2)

Output: Reduced Model (A_z, B_z,C_z,D_z)

Compute $S_1(\omega_1) \leftarrow \frac{i}{2\pi} \ln((j\omega_1 I + A)(-j\omega_1 I + A)^{-1})$;

Compute $S_2(\omega_2) \leftarrow \frac{i}{2\pi} \ln((j\omega_2 I + A)(-j\omega_2 I + A)^{-1})$;

Compute $X_g \leftarrow (S_2(\omega_2) - S_1(\omega_1))BB^T + BB^T(S_2^*(\omega_2) - S_1^*(\omega_1))$;

Compute $Y_g \leftarrow (S_2^*(\omega_2) - S_1^*(\omega_1))C^T C + C^T C(S_2(\omega_2) - S_1(\omega_1))$;

Decompose X_g and Y_g into $[U_z S_z U_z^T]$ and $[V_z R_z V_z^T]$ using SVD;

Proceed with steps 1-17 in Algorithm 1 to compute new input/output matrices;

Compute $B_{Z_k} \leftarrow U diag(S_1, \ldots S_i \ldots 0 \ldots 0)$;

Compute $C_{Z_k} \leftarrow diag(R_1, \ldots R_k \ldots, 0 \ldots 0)V^T$;

Compute $AP_{Z_k} + P_{Z_k} A^T = -B_{Z_k} B_{Z_k}^T$ for new controllability Gramian P_{Z_k};

Compute $A^T Q_{Z_k} + Q_{Z_k} A = -C_{Z_k}^T C_{Z_k}$ for new observability Gramian Q_{Z_k};

Compute the contragradient transformation matrix $T_{Z_k}^T Q_{Z_k} T_{Z_k} = T_{Z_k}^{-1} P_{Z_k} T_{Z_k}^{-T}$;

Compute the balanced realization as $\hat{A}_t = T^{-1} A_z T$, $\hat{B}_t = T^{-1} B_z$, $\hat{C}_t = C_z T$;

Select ROM and truncate A, B, C to compute reduced realization A_z, B_z and C_z and it is $D_z = D$;

Chapter 4

Numerical Simulations and Discussion

In this chapter, the proposed techniques are applied on some practical examples of frequency weighted and frequency limited model reduction techniques to show the accuracy and efficacy of the presented work. Examples are taken form literature and comparison of the proposed techniques have been done with existing techniques to illustrate the effectiveness of the proposed techniques.

4.1 Frequency Weighted MOR Simulations

Example 1: Consider a 5^{th} order orignal stable system [39] with stable frequency weights $V_x(s) = \frac{100}{s+100}$ and $W_y(s) = \frac{10}{s+10}$ [40]. The transfer function of the original system is described as

$$G_o(s) = \frac{2s^5 + 4s^4 + 800s^3 + 1200s^2 + 6e^4 s + 4e^4}{s^5 + 22s^4 + 440s^3 + 6600s^2 + 3.8e^4 s + 2.2e^5}$$

The corresponding results of the given system are presented in Table4.1and it can be observed from the results that the proposed techniques capitulate minimum frequency-response approximation error with calculable priori error bound as compared to existing

Table 4.1: Error and Error Bounds Comparison for Example 1

RO	Enns [10]	Wang [20]		Imran [21]		Proposed-I		Proposed-II	
	Error	Error	Bound	Error	Bound	Error	Bound	Error	Bound
1	4.7097	5.6073	52.915	5.6748	30.503	5.2837	513.08	5.2716	141.9
3	3.5208	3.5455	24.413	5.3409	14.137	3.5293	234.52	3.5293	64.829
4	3.5068	7.3238	12.141	11.269	6.8762	6.342	116.75	6.427	32.275

stability preservation techniques [20,21].

Example 2: Consider a 6^{th} order original stable system [21] with following stable input-output frequency weights

$$A_i = \begin{bmatrix} -2.25 & -0 \\ 0 & -0.05 \end{bmatrix}, B_i = \begin{bmatrix} 2 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} C_i = \begin{bmatrix} 1.3 & 0.5 \\ 0.1 & 0.1 \end{bmatrix}, D_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_o = \begin{bmatrix} -4.2 & 0 \\ 0 & -0.025 \end{bmatrix}, B_o = \begin{bmatrix} 1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix} C_o = \begin{bmatrix} 1.3 & 0.5 \\ 0.4 & 0.7 \end{bmatrix}, D_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The results of the presented system are shown in Table4.2and it can be observed from the results that the proposed techniques compute least frequency-response approximation error with computable priori error bound as compared to existing stability conservation techniques.

It can be stated from the Table4.2that the 1^{st} order model obtained using Enns [10] technique is unstable having pole location at $s = 0.0164$ whereas, the proposed techniques, *Imran et al.* [21] and *Wang et al.* [20] have poles with negative real parts that lead to compute stable ROM.

Table 4.2: Error and Error Bounds Comparison for Example 2

RO	Enns [10]	Wang [20]		Imran [21]		Proposed-I		Proposed-II	
	Error	Error	Bound	Error	Bound	Error	Bound	Error	Bound
1	unstable	2462.8	6718.1	2724.4	30109	2362.6	35378	2028.4	93130
2	12.742	136.68	2662.5	123.14	6768	136.46	14152	129.69	41295
3	5.9973	39.9	718.24	225.28	3733.8	36.545	3248.7	16.501	8103
4	0.24849	16.811	287.91	141.42	1745.5	11.034	1317	4.89	3241.6
5	.079136	5.8769	68.684	9.417	415.71	3.7963	302.23	1.7825	765.42

***Example 3**:* Consider a Linear Quadratic Gaussian (LQG) controller for a four-disk system [24]. The system plant to be controlled is represented as LTI, Single-Input-Single-Output (SISO), unstable and non-minimum phase an 8^{th} order system. The transfer function of this plant is described as

$$G_o(s) = \frac{0.006443s^5+0.00232s^4+0.07125s^3+s^2+0.1046s+0.9955}{s^8+0.161s^7+6.004s^6+0.5821s^5+9.983s^4+0.4073s^3+3.982s^2}$$

A full order stable controller $K_o(s)$ using standard LQG technique is defined as

$$K_o(s) = \frac{0.1116s^7 + 0.0224s^6 + 0.6711s^5 + 0.0918s^4 + 1.119s^3 + 0.0902s^2 + 0.4485s + 0.018}{s^8 + 1.313s^7 + 6.853s^6 + 7.359s^5 + 14.09s^4 + 11.43s^3 + 9.177s^2 + 4.49s + 1.377}$$

Table 4.3: Error and Error Bounds Comparison for Example 3

RO	Enns [10]	Wang [20]		Imran [21]		Proposed-I		Proposed-II	
	Error	Error	Bound	Error	Bound	Error	Bound	Error	Bound
1	unstable	4.7523	89.108	3.7822	3198.2	4.6593	2690.1	4.3577	158.01
2	.47908	.1857	63.67	1.3042	2180.1	.59125	1916.9	.39233	111.75
3	.48602	5.2643	40.137	5.5405	1524.3	5.2035	1212	4.4625	72.001
4	.12204	.11569	20.482	1.098	894.81	.11593	616.37	.11816	34.706
5	.12287	1.6913	13.045	2.6299	485.8	1.6325	393.28	.76322	22.092
6	.028233	.02806	6.2782	1.0335	67.92	.028058	190.29	.028112	10.615
7	.028184	.48441	3.1295	3.4767	18.979	.46831	94.854	.21744	5.2982

Let the input-output frequency weights be

$$V_x(s) = (1 + G_o(s)K_o(s))^{-1}, \qquad W_y(s) = (1 + G_o(s)K_o(s))^{-1}G_o(s)$$

The comparison of the approximation error and error bounds is shown in Table4.3. However, it can be noticed from the Table4.3that the 1 st order model build by Enns [10] gives unstable ROM as the pole is located at $s = 0.0003072$ whereas, the *Wang et al.* [20], *Imran et al.*[21] and the proposed techniques produce stable ROMs. Although *Enns* [10] can yield unstable ROMs as shown in Tables4.2and4.3, it computes lowest frequency-response approximation error.

Example 4: Consider a hospital building model sparse system with 48^{th} states [41]. The reduction of the given model is performed at different orders to show the efficacy of the proposed techniques. The model reduction is done using the following stable

Table 4.4: Error and Error Bounds Comparison for Example 4

RO	Enns [10]	Wang [20]		Imran [21]		Proposed-I		Proposed-II	
	Error	Error	Bound	Error	Bound	Error	Bound	Error	Bound
5	.00092531	22.812	218.39	1.4119	13655	.14381	3928	.0637	55.453
7	7.6812*e*¯5	7.8962	134.49	9.5973	12286	5.2571	24096	5.187	34.011
11	.00014655	15.575	60.595	2.3981	9839.9	2.2417	10854	2.234	15.319
21	1.3634*e*¯5	1.0204	4.9859	14.921	5299.3	.63525	890.22	.631	1.2564
22	2.4239*e*¯6	.04578	3.4508	.25526	4926.8	.02871	617.29	.0286	0.8712
23	8.882*e*¯6	.17772	2.443	4.2064	4559.4	.07622	436.84	.0760	.61652
25	3.4602*e*¯6	.21507	1.0174	2.9544	3866.1	.08449	182.12	.0842	.25703
26	2.0156*e*¯7	.00068	.60326	.03153	3550.2	.00040	107.72	.0003	.15202
28	1.1608*e*¯7	.00133	.32491	.01513	2997.3	.00077	58.09	.0007	.08198
31	1.1012*e*¯7	.01733	.16011	25.08	2267.3	.00918	28.517	.0091	.04024
32	4.0385*e*¯8	.00121	.1251	.97626	2044.9	.00064	22.29	.0006	.03145
33	1.7617*e*¯7	.00525	.10024	20.219	1830.8	.00225	17.873	.0022	.02522
34	2.3365*e*¯8	.00113	.07552	.01303	1618.3	0.00060	13.485	.0006	.01903
35	4.158*e*¯8	.00718	.06196	7.7542	1429.9	.00252	11.06	.0025	.01560
36	1.38*e*¯8	.00038	.04876	.3655	1248.7	.00017	8.6824	.0001	.01225
38	2.0139*e*¯8	.00029	.02585	.00495	902.42	.00013	4.6101	.0001	.00650

input/output weights transfer functions respectively,

$$V_i(s) = (0.6)/(s+0.6), \qquad W_o(s) = (s+3.201)/(s+0.0006)$$

Table4.4illustrates the corresponding results of the given model at that reduced orders where the approximation error is significant and it can be noticed from the given model that the stable ROMs build by the proposed techniques capitulate minimum frequency-response approximation error as compared to existing stability conservation techniques with computable error bounds.

4.2 Frequency Limited MOR Simulations

***Example 1**:* Consider a random model sparse system [41]. It is a 200^{th} order system with desired frequency interval [15, 24] rad/s. The comparison of the error function singular values $\sigma[G_{200}(s) - G_{45}(s)]$ is shown in Fig.4.1, where $G_{200}(s)$ is original order stable system and $G_{45}(s)$ is the 45^{th} order ROM that is derived using *Gawronski et al*. [22], *Gugercin et al.* [23], *Ghafoor et al.* [24], *Imran et al*. [25] and proposed techniques. Fig.4.2shows the close-up view of the error plot in the desired frequency range $[\omega_1, \omega_2] = [15, 24]$ rad/s to illustrate the efficacy of the proposed techniques' results. It can be noted that the proposed techniques capitulate comparable approximation error, within the desired frequency range as compared to *Gawronski et al* [22] technique and low frequency response approximation-error as compared to stability preservation techniques [23,24,25].

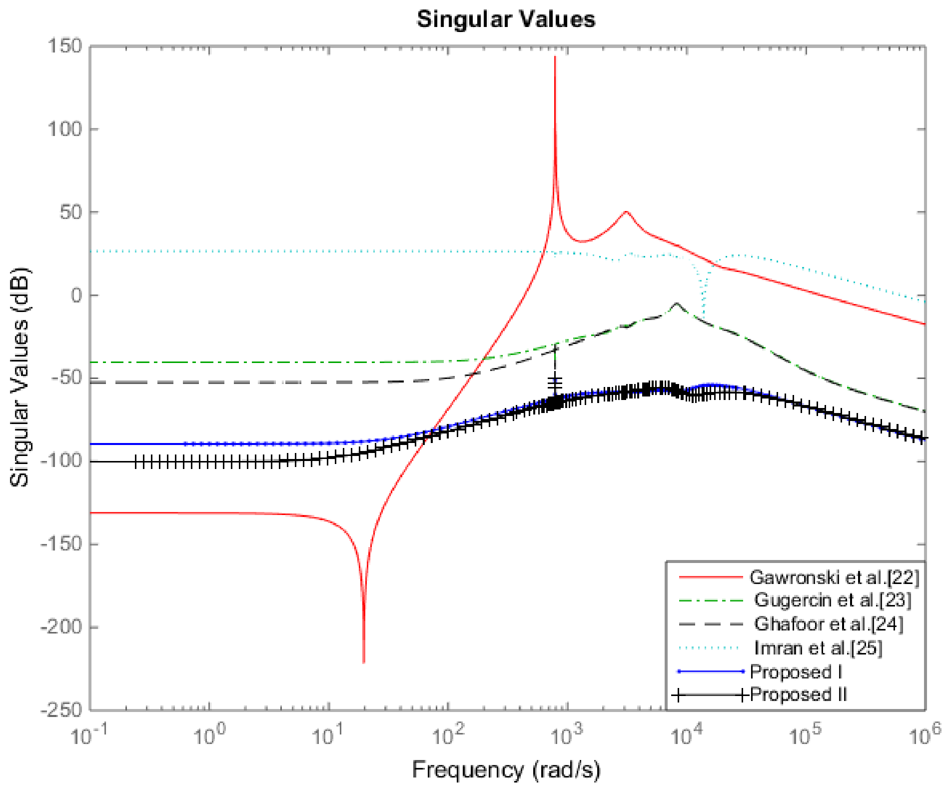

Figure 4.1: $\sigma[G_{200}(s) - G_{45}(s)]$ in $[\omega_1 = 15,\ \omega_2 = 24]$

***Example 2**:* Consider a 50^{th} order stable high pass Chebyshev type-2 filter with stop-

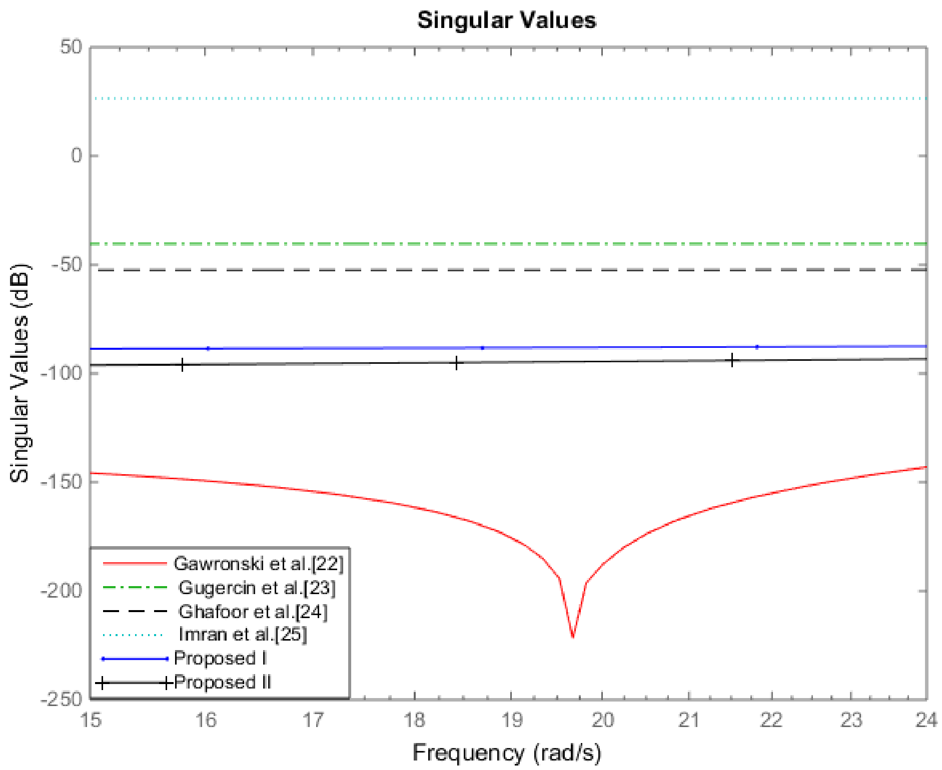

Figure 4.2: Close-up view of $\sigma[G_{200}(s) - G_{45}(s)]$

band edge frequency 38.5 Hz and stopband ripple 32.02 dB [42]. The comparison of the error function singular values $\sigma[G_{50}(s) - G_{21}(s)]$ is shown in Fig.4.3, within the desired frequency range $[\omega_1, \omega_2] = [9, 20]$ rad/s, where $G_{50}(s)$ is original order system and $G_{21}(s)$ is 21^{th} order ROM that is constructed using *Gawronski et al.* [22], *Gugercin et al.* [23], *Imran et al.* [25] and proposed techniques. Fig.4.4shows the close-up view of the error plot in the desired frequency range $[\omega_1, \omega_2] = [9, 20]$ rad/s to elaborate the efficacy of the proposed techniques . Fig.4.4illustrates the computing of lowest frequency-response approximation error, within the desired frequency range as compared to stability preservation techniques including *Gugercin et al.* [23], *Ghafoor et al.* [24] and *Imran et al.*[25].

Example 3: Consider a 30^{th} order high pass stable Chebyshev type-2 filter with stop band edge frequency 27.5 Hz and stop band ripple of 13.02 dB [42]. The comparison of the error function singular values $\sigma[G_{30}(s) - G_{12}(s)]$ is presented in Fig.4.5within

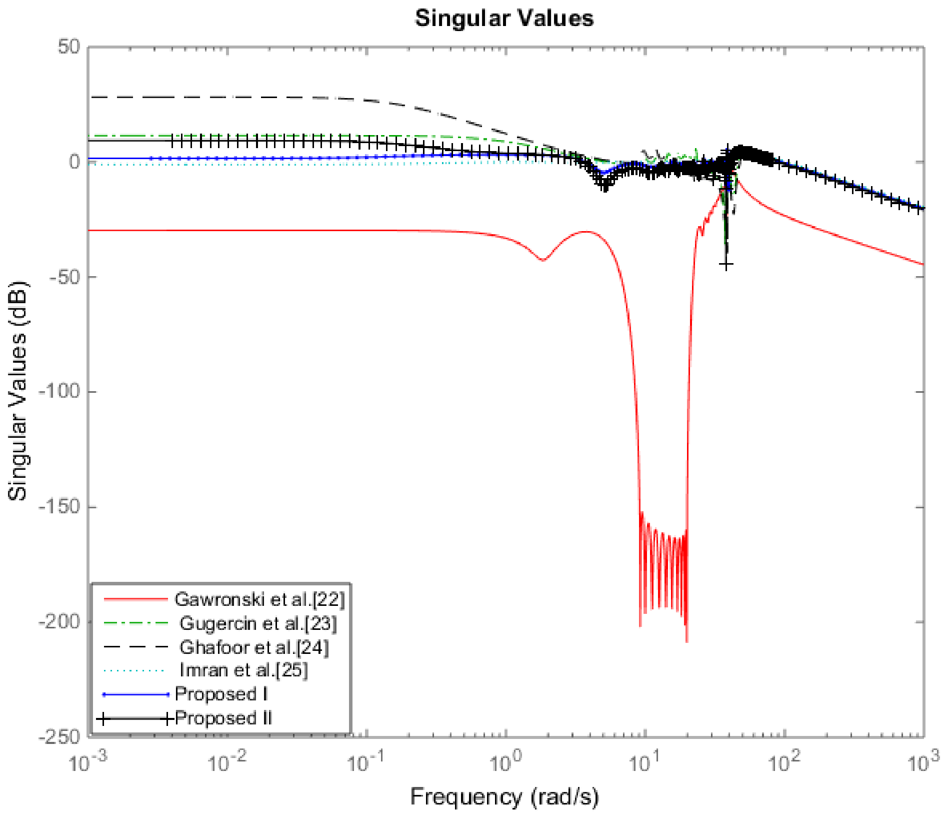

Figure 4.3: $\sigma[G_{50}(s) - G_{21}(s)]$ in $[\omega_1 = 9,\ \omega_2 = 20]$

desired frequency range $[\omega_1, \omega_2] = [9, 15]$ rad/s, where $G_{30}(s)$ is the original order stable system and $G_{12}(s)$ is 12^{th} order ROM that is capitulated using *Gawronski et al*. [22], *Gugercin et al.* [23], *Imran et al*. [25] and proposed techniques. Fig.4.6illustrates the close-up view of the error plot in the desired frequency range $[\omega_1, \omega_2] = [9, 15]$ rad/s to show the efficacy of the proposed techniques' results. Fig.4.6demonstrates the computing of lowest frequency-response approximation error, within the desired frequency range as compared to stability preservation techniques including *Gugercin et al.* [23], Ghafoor et al. [24] and *Imran et al.* [25].

Example 4: Consider a 20^{th} order type-2 Chebyshev high pass stable filter with stop-band edge frequency 35.5 Hz and stopband ripple of 13.02 dB [42]. The comparison of the error function singular values $\sigma[G_{20}(s) - G_{11}(s)]$ is presented in Fig.4.7within desired frequency range $[\omega_1, \omega_2] = [13, 29]$ rad/s, where $G_{20}(s)$ is the original order sta-

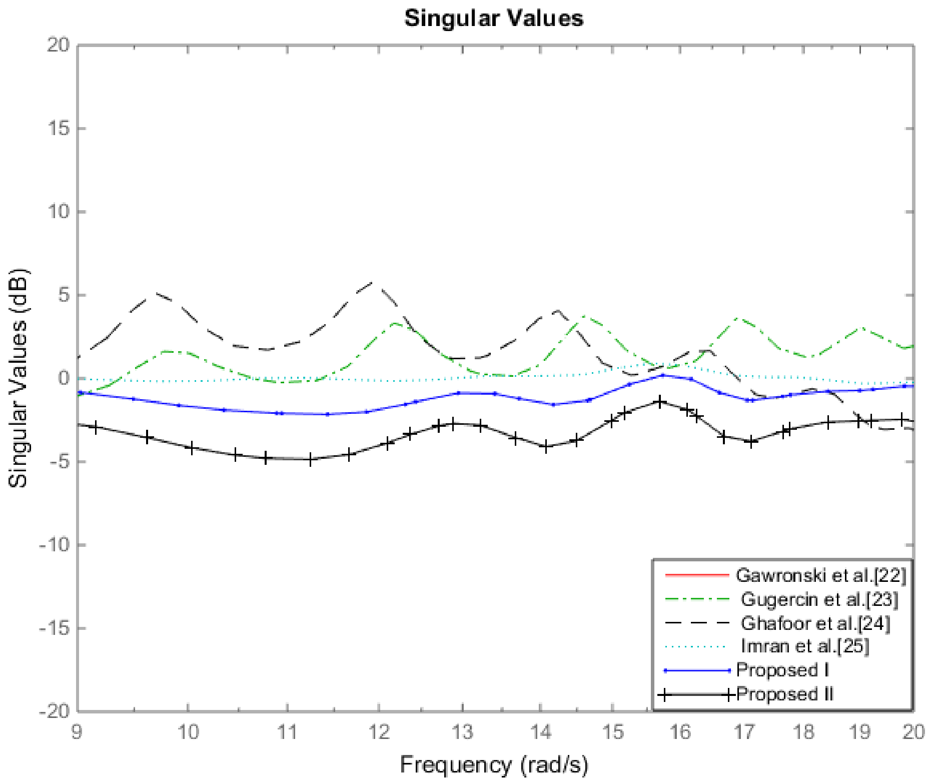

Figure 4.4: Close-up view of $\sigma[G_{50}(s) - G_{21}(s)]$

Table 4.5: Poles Location of Frequency Limited ROMs

Techniques	Poles of Example 5	
Gawronski et al. [22]	-2.7147 ±8.9013i	-11.6622, 8.9836
Gugercin et al. [23]	-0.9625 ±16.6018i	-1.3339 ±9.5342i
Ghafoor et al. [24]	-0.2614 ±15.1758i	-1.1222 ± 9.6005i
Imran et al. [25]	-0.6544±17.7506i	-1.5397 ± 12.7585i
Proposed-I	-2.5532 ±17.0912i	-2.4541 ± 12.1899i
Proposed-II	-2.7070 ± 17.0382i	-2.6278 ± 12.0768i

ble system and $G_{11}(s)$ is 11^{th} order ROM that is derived using *Gawronski et al.* [22], *Gugercin et al.* [23], *Ghafoor et al* [24], *Imran et al.* [25] and proposed techniques. Fig.4.8illustrates the close-up view of the error plot in the desired frequency range $[\omega_1, \omega_2] = [13, 29]$ rad/sec to show the efficacy of the proposed techniques' results.

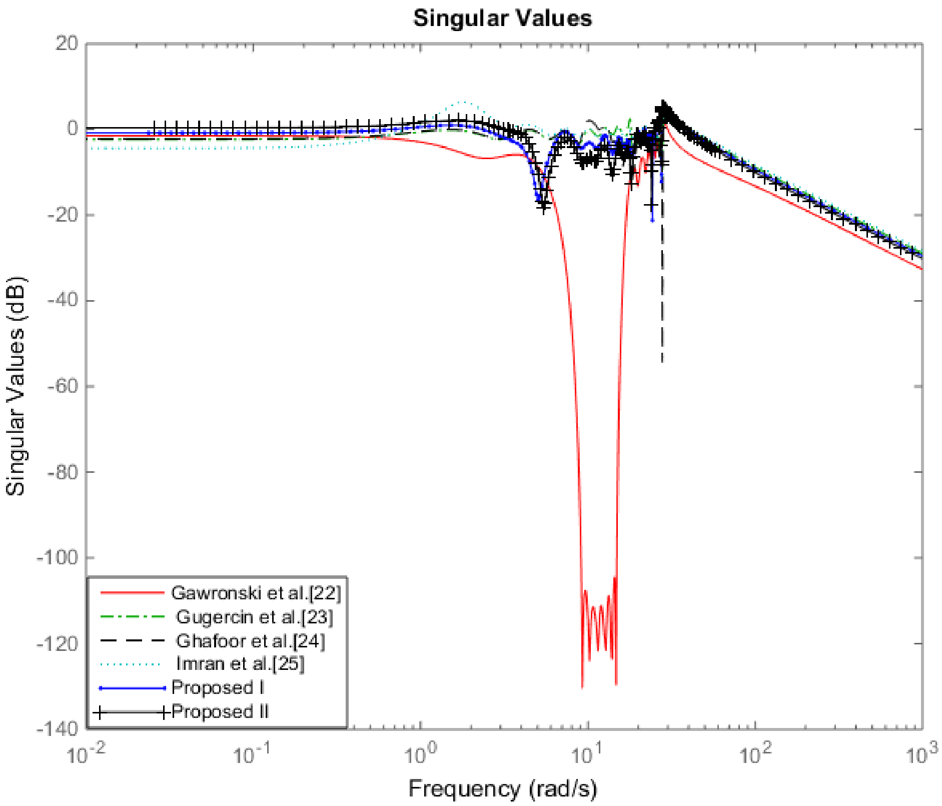

Figure 4.5: $\sigma[G_{30}(s) - G_{12}(s)]$ in $[\omega_1 = 9,\ \omega_2 = 15]$

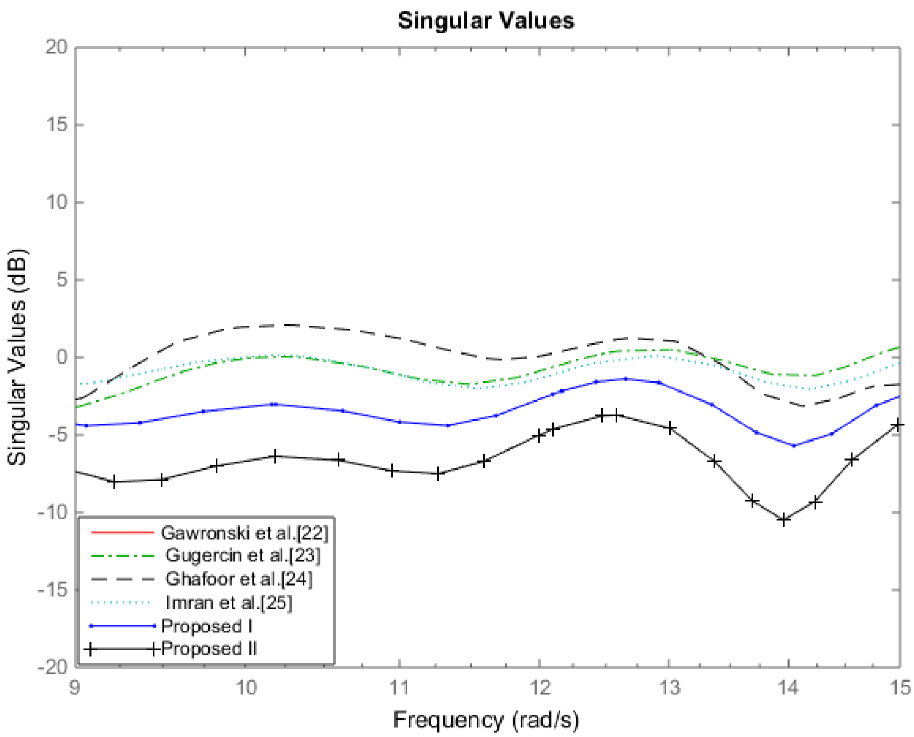

Figure 4.8: Close-up view of $\sigma[G_{20}(s) - G_{11}(s)]$

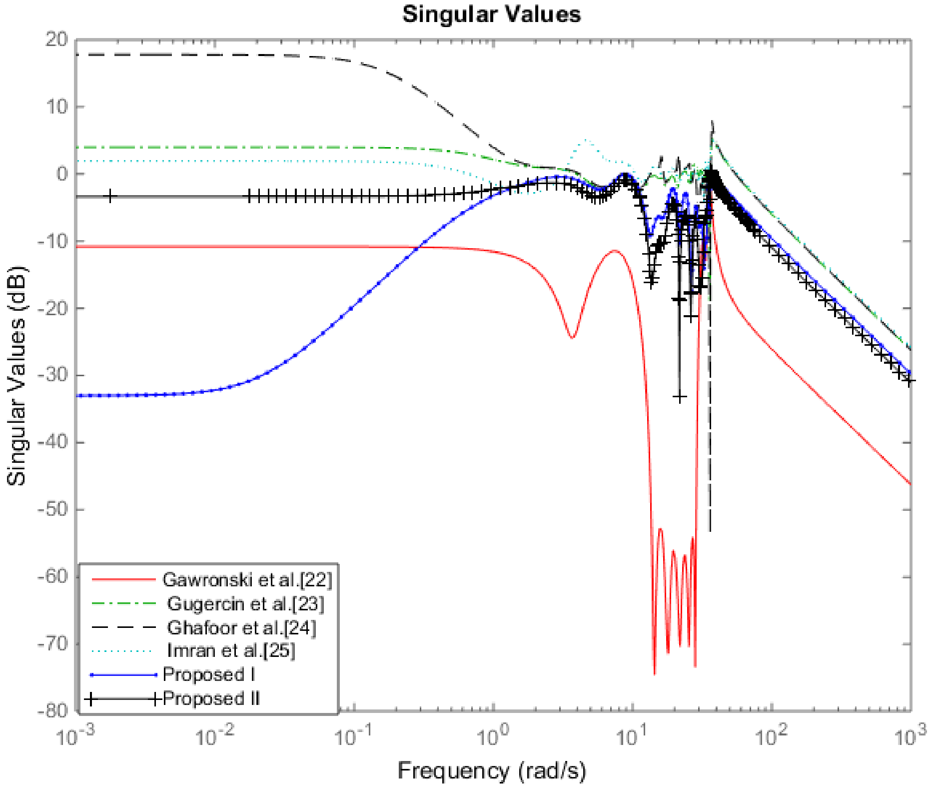

Figure 4.7: $\sigma[G_{20}(s) - G_{11}(s)]$ in $[\omega_1 = 13,\ \omega_2 = 29]$

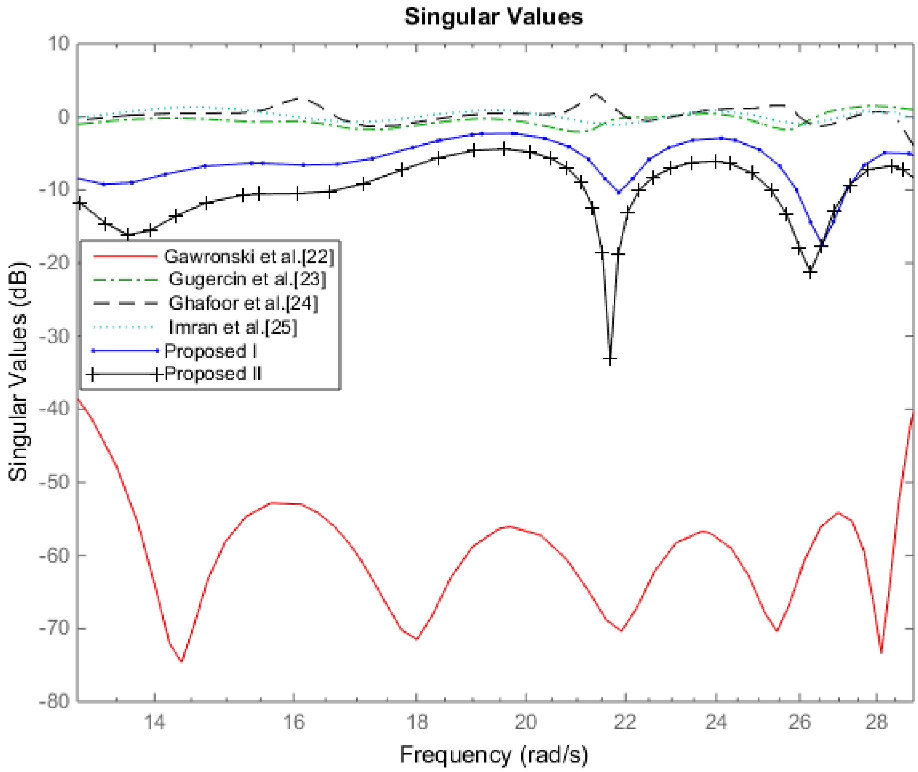

Figure 4.8: Close-up view of $\sigma[G_{20}(s) - G_{11}(s)]$

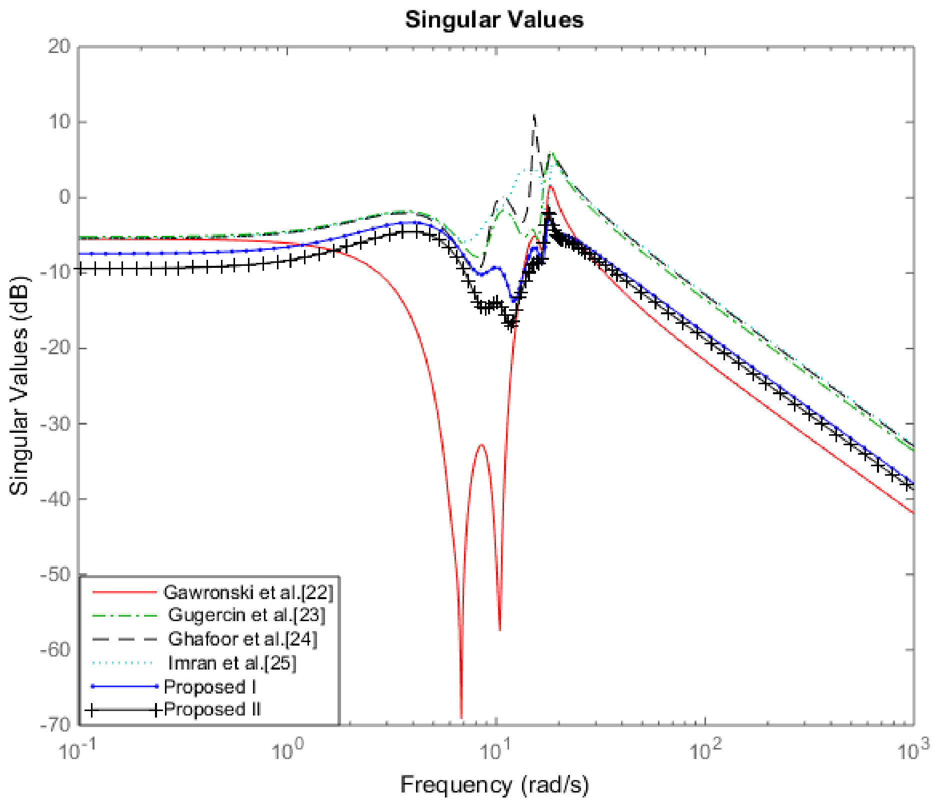

Figure 4.9: $\sigma[G_8(s) - G_4(s)]$ in $[\omega_1 = 5,\ \omega_2 = 12]$

Example 5: Consider an 8^{th} order high pass stable Chebyshev type-2 filter with stop-band edge frequency 17.5 Hz and stopband ripple of 11.2 dB [42]. The comparison of the error function singular values $\sigma[G_8(s) - G_4(s)]$ is presented in Fig.4.9within desired frequency range $[\omega_1, \omega_2] = [5, 12]$ rad/s, where $G_8(s)$ is original order stable system and $G_4(s)$ is 4^{th} order ROM is derived using *Gawronski et al*. [22], *Ghafoor et al*. [24], *Gugercin et al*. [23], *Imran et al*. [25] and proposed techniques. The close-up view of the error plot in the desired frequency range $[\omega_1, \omega_2] = [5, 12]$ rad/s is shown in the Fig.4.8. It can be observed that the proposed techniques yield lowest approximation error within the desired frequency range as compared to stability preservation techniques [23] ,[25], [24]. The poles locations of the 4^{th} order ROM is presented in Table4.5. It can be seen from the Table4.5that the 4^{th} order model is derived using *Gawronski et al*. [22] technique is unstable having pole location at $s = 8.9836$ whereas, the proposed techniques, *Imran et al*. [25] and *Gugercin et al*. [23] and *Ghafoor et al*.

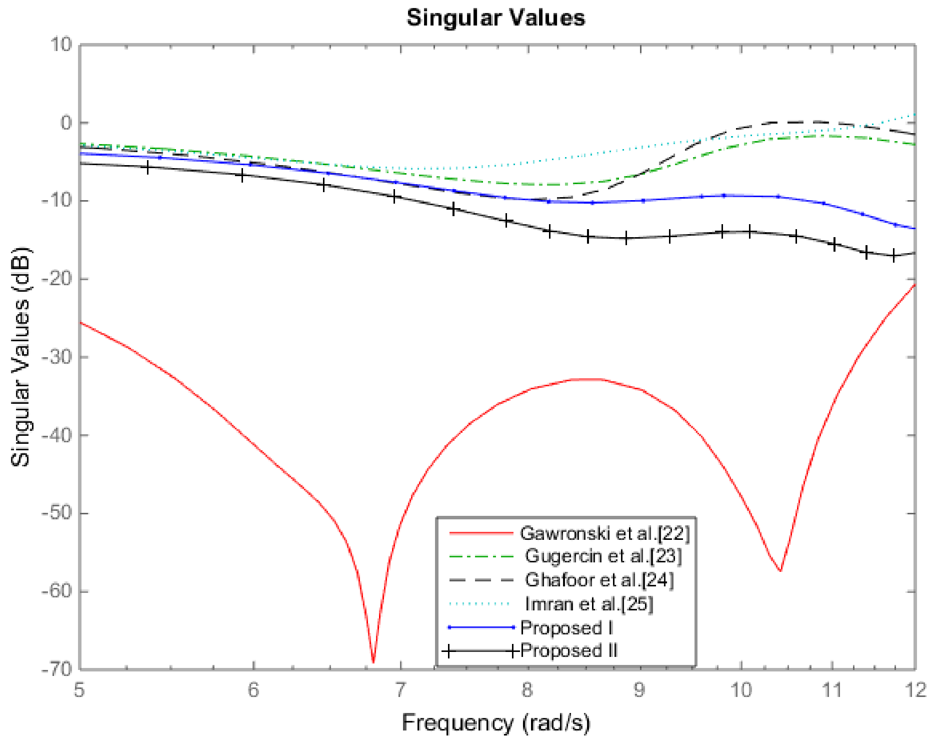

Figure 4.10: Close-up view of $\sigma[G_8(s) - G_4(s)]$

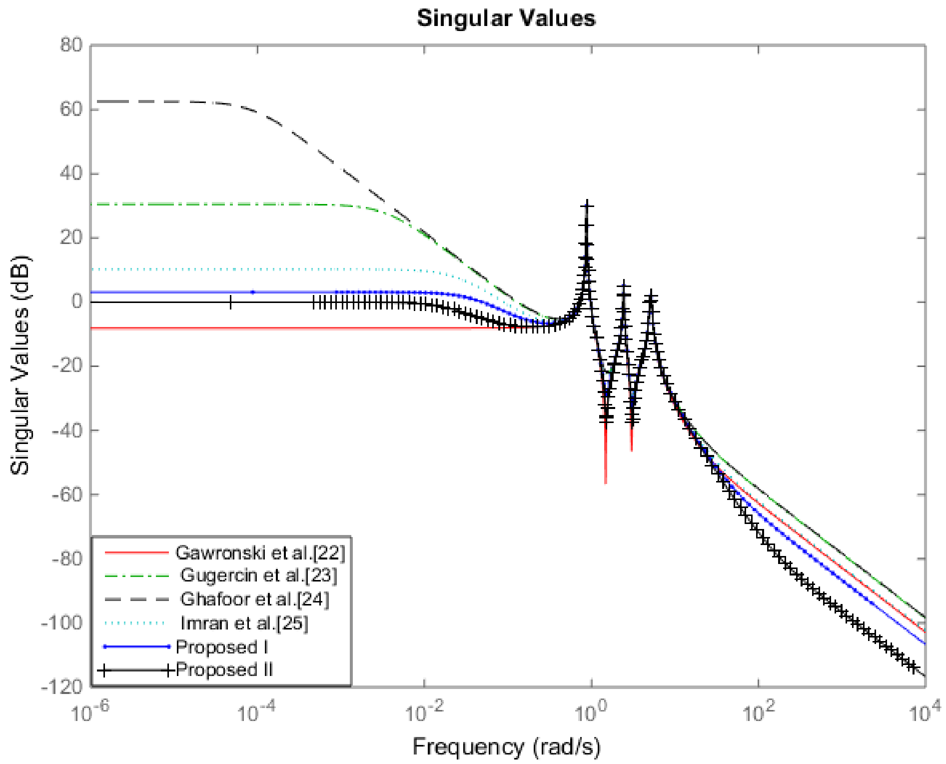

Figure 4.11: $\sigma[G_6(s) - G_1(s)]$ in $[\omega_1 = 13,\ \omega_2 = 17]$

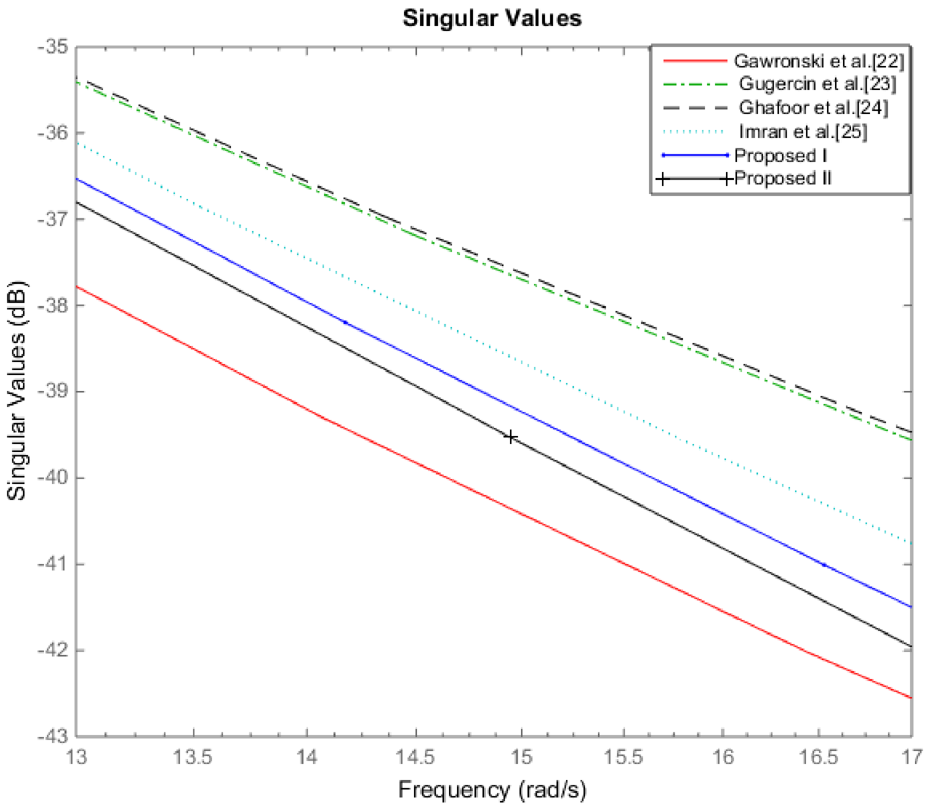

Figure 4.12: Close-up view of $\sigma[G_6(s) - G_1(s)]$

Table 4.6: Poles Location of Frequency Limited ROMs

Techniques	ROMs of Example 6	Poles
Gawronski et al. [22]	$\frac{-0.07208}{s-6.458}$	6.4576
Gugercin et al. [23]	$\frac{-0.1195}{s+0.003525}$	-0.0035
Ghafoor et al. [24]	$\frac{-0.1222}{s+0.00009114}$	-0.000091139
Imran et al. [25]	$\frac{0.07594}{s+0.02644}$	-0.0264
Proposed-I	$\frac{0.04647}{s+0.04405}$	-0.0441
Proposed-II	$\frac{0.01463}{s+0.02374}$	-0.0237

[24] produce stable ROMs.

***Example 6**:* Consider a 6^{th} order stable three mass mechanical system [43]. The comparison of the error function singular values $\sigma[G_6(s) - G_1(s)]$ is presented in Fig.4.11 within desired frequency range $[\omega_1, \omega_2] = [13, 17]$ rad/s, where $G_6(s)$ is the original order system and $G_1(s)$ is 1^{st} order ROM is computed using W.Gawronski *et al*. [22], *Gugercin et al*. [23], *Imran et al*. [25] and proposed techniques. The close-up view of the error plot in the desired frequency range $[\omega_1, \omega_2] = [13, 17]$ rad/s is illustrated in the Fig.4.12. The poles locations of the 1^{st} ROM with reduced transfer function is presented in4.6.It can be noticed from the Table4.6that the 1^{st} order model derived using *Gawronski et al*. [22] technique is unstable having pole location at $s = 6.4576$ whereas, the proposed techniques capitulates stable ROM with lowest frequency-response approximation error. Although *Gawronski et al.* [22] may compute unstable ROMs as shown in Tables4.5and4.6, it yields lowest frequency-response approximation error.

Chapter 5

Conclusion

In this thesis, two new frequency weighted and frequency limited-interval Gramians-based MOR are proposed for the LTI continuous-time systems. The proposed methods yield stable reduced systems with least frequency-response approximation error and computable error bounds to overcome the problems of the model reduction techniques. The proposed techniques are also illustrated with help of numerical simulations and examples are taken from the literature. The simulation results demonstrate that the proposed methods are fruitful and comparable with some other existing model reduction techniques.

5.1 Future Work

While developing this thesis, some recommendations are indicated for future work, which are listed below:

- In this thesis, the proposed methods have considered the model reduction of Single-Input-Single-Output (SISO). It is interesting to see the results for Multi-Input-Multi-Output (MIMO) systems.
- In this thesis the proposed MOR techniques have been applied on stable system.

In future the proposed methods can be extended for unstable systems.

- In this thesis the proposed techniques use BT to process the frequency weighted and frequency limited-interval Gramians based model reduction process. In future work, instead of using BT the proposed methods can be used for different model reduction techniques like Pade approximation,Hankel norm and Krylov etc.

- The proposed methods have been considered for LTI continuous-time system. In future work, we have planned to see the results for discrete-time systems, non-linear and time-variant systems.

Bibliography

[1] S. B. Nouri, "Advanced model-order reduction techniques for large scale dynamical systems," Ph.D. dissertation, Carleton University, 2014.

[2] E. Beltrami, *Mathematics for dynamic modeling*. Academic press, 2014.

[3] L. Perko, *Differential equations and dynamical systems*. Springer Science & Business Media, 2013, vol. 7.

[4] G. De Luca, G. Antonini, and P. Benner, "A parallel, adaptive multi-point model order reduction algorithm," in *2013 IEEE 22nd Conference on Electrical Performance of Electronic Packaging and Systems*. IEEE, 2013, pp. 115–118.

[5] W. H. Schilders, H. A. Van der Vorst, and J. Rommes, *Model order reduction: theory, research aspects and applications*. Springer, 2008, vol. 13.

[6] A. C. Antoulas, *Approximation of large-scale dynamical systems*. SIAM, 2005.

[7] V. Mehrmann and T. Stykel, "Balanced truncation model reduction for large-scale systems in descriptor form," in *Dimension Reduction of Large-Scale Systems*. Springer, 2005, pp. 83–115.

[8] M. Imran, M. J. Awan, and A. Ghafoor, "Model reduction techniques with error bounds," in *2017 13th IEEE Conference on Automation Science and Engineering (CASE)*, 2017, pp. 1510–1515.

[9]H. I. Toor, M. Imran, A. Ghafoor, D. Kumar, V. Sreeram, and A. Rauf, “Frequency limited model reduction techniques for discrete-time systems,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 67, no. 2, pp. 345–349, 2020.

[10]D. F. Enns, “Model reduction with balanced realizations: An error bound and a frequency weighted generalization,” in *The 23rd IEEE conference on decision and control*. IEEE, 1984, pp. 127–132.

[11]C.-A. Lin and T.-Y. Chiu, “Model reduction via frequency weighted balanced realization,” in *1990 American Control Conference*. IEEE, 1990, pp. 2069–2070.

[12]A. Varga and B. D. Anderson, “Accuracy-enhancing methods for balancing-related frequency-weighted model and controller reduction,” *Automatica*, vol. 39, no. 5, pp. 919–927, 2003.

[13]A. Ghafoor and V. Sreeram, “Partial-fraction expansion based frequency weighted model reduction technique with error bounds,” *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1942–1948, 2007.

[14]V. Sreeram, B. Anderson, and A. Madievski, “New results on frequency weighted balanced reduction technique,” in *Proceedings of 1995 American Control Conference-ACC’95*, vol. 6. IEEE, 1995, pp. 4004–4009.

[15]U. M. Al-Saggaf and G. F. Franklin, “On model reduction,” in *1986 25th IEEE Conference on Decision and Control*. IEEE, 1986, pp. 1064–1069.

[16]V. Sreeram and S. Sahlan, “Improved results on frequency-weighted balanced truncation and error bounds,” *International Journal of Robust and Nonlinear Control*, vol. 22, no. 11, pp. 1195–1211, 2012.

[17]S. Sahlan and V. Sreeram, “New results on partial fraction expansion based frequency weighted balanced truncation,” in *2009 American Control Conference*. IEEE, 2009, pp. 5695–5700.

[18]G. A. Latham and B. D. Anderson, “Frequency-weighted optimal hankel-norm approximation of stable transfer functions,” *Systems & Control Letters*, vol. 5, no. 4, pp. 229–236, 1985.

[19]Y. Hung and K. Glover, “Optimal hankel-norm approximation of stable systems with first-order stable weighting functions,” *Systems & control letters*, vol. 7, no. 3, pp. 165–172, 1986.

[20]G. Wang, V. Sreeram, and W. Liu, “A new frequency-weighted balanced truncation method and an error bound,” *IEEE Transactions on Automatic Control*, vol. 44, no. 9, pp. 1734–1737, 1999.

[21]M. Imran, A. Ghafoor, and V. Sreeram, “A frequency weighted model order reduction technique and error bounds.”

[22]W. Gawronski and J.-N. Juang, “Model reduction in limited time and frequency intervals,” *International Journal of Systems Science*, vol. 21, no. 2, pp. 349–376, 1990.

[23]S. Gugercin and A. C. Antoulas, “A survey of model reduction by balanced truncation and some new results,” *International Journal of Control*, vol. 77, no. 8, pp. 748–766, 2004.

[24]A. Ghafoor and V. Sreeram, “A survey/review of frequency-weighted balanced model reduction techniques,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 130, no. 6, 2008.

[25]M. Imran and A. Ghafoor, “A frequency limited interval gramians-based model reduction technique with error bounds,” *Circuits, Systems, and Signal Processing*, vol. 34, no. 11, pp. 3505–3519, 2015.

[26]A. Daraghmeh, "Model order reduction of linear control systems: comparison of balance truncation and singular perturbation approximation with application to optimal control," Ph.D. dissertation, 2016.

[27]K. Salah, "A novel model order reduction technique based on artificial intelligence," *Microelectronics journal*, vol. 65, pp. 58–71, 2017.

[28]A. De Gaetano and O. Arino, "Mathematical modelling of the intravenous glucose tolerance test," *Journal of mathematical biology*, vol. 40, no. 2, pp. 136–168, 2000.

[29]J. C. Willems and J. W. Polderman, *Introduction to mathematical systems theory: a behavioral approach*. Springer Science & Business Media, 1997, vol. 26.

[30]M. Imran, "Development of model order reduction techniques," Ph.D. dissertation, National University of Science & Technology, Pakistan, 2014.

[31]A. K. Prajapati and R. Prasad, "A new model order reduction method for the design of compensator by using moment matching algorithm," *Transactions of the Institute of Measurement and Control*, vol. 42, no. 3, pp. 472–484, 2020.

[32]Z.-Y. Qiu, Y.-L. Jiang, and J.-W. Yuan, "Interpolatory model order reduction method for second order systems," *Asian Journal of Control*, vol. 20, no. 1, pp. 312–322, 2018.

[33]S. Haider, A. Ghafoor, M. Imran, and F. M. Malik, "Frequency interval gramians based structure preserving model order reduction for second order systems," *Asian Journal of Control*, vol. 20, no. 2, pp. 790–801, 2018.

[34]K. S. Haider, A. Ghafoor, M. Imran, and F. M. Malik, "Model reduction of large scale descriptor systems using time limited gramians," *Asian Journal of Control*, vol. 19, no. 3, pp. 1217–1227, 2017.

[35]D. Kumar, A. Jazlan, and V. Sreeram, “Model reduction based on limited-time interval impulse response gramians,” *Asian Journal of Control*, vol. 23, no. 1, pp. 572–581, 2021.

[36]M. Morari and E. Zafiriou, *Robust process control*. Morari, 1989.

[37]A. K. Prajapati and R. Prasad, “Model order reduction by using the balanced truncation and factor division methods,” *IETE Journal of Research*, vol. 65, no. 6, pp. 827–842, 2019.

[38]A. Ghafoor, *Frequency-weighted model reduction and error bounds*. University of Western Australia, 2007.

[39]A. Ghafoor and M. Imran, “Passivity preserving frequency weighted model order reduction technique,” *Circuits, Systems, and Signal Processing*, vol. 36, no. 11, pp. 4388–4400, 2017.

[40]W. M. W. Muda, V. Sreeram, and H. H.-C. Lu, “Passivity-preserving frequency weighted model order reduction techniques for general large-scale rlc sytems,” in *2010 11th International Conference on Control Automation Robotics & Vision*. IEEE, 2010, pp. 1310–1315.

[41]Y. Chahlaoui and P. Van Dooren, “A collection of benchmark examples for model reduction of linear time invariant dynamical systems.” 2002.

[42]H. I. Toor, M. Imran, A. Ghafoor, M. Zeeshan, and A. Rauf, “Improved frequency limited model reduction,” in *2018 IEEE Conference on Systems, Process and Control (ICSPC)*. IEEE, 2018, pp. 17–22.

[43]M. Imran and A. Ghafoor, “Frequency limited model reduction techniques with error bounds,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 65, no. 1, pp. 86–90, 2017.