The Mei Symmetries Corresponding to Lagrangian of the Schwarzschild Metric

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In

the loving memory of

my beloved father,

Iftikhar Ahmed Khan

who left too soon but left too much

to get inspired

by

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Kinza Iftikhar

Abstract

This thesis is devoted to find the Mei symmetries corresponding to the Lagrangian of spherically symmetric and static metric. For this purpose, Schwarzschild metric is considered and the criterion for Mei symmetries is analysed. The Lagrangian of the spherically symmetric and static Schwarzschild metric is used to determine the Euler Lagrange equations and the determining equations for the Mei symmetries. Solving the determining equations, four Mei symmetries for the Lagrangian of Schwarzschild metric are obtained. Moreover, the Lie point symmetries and Noether symmetries are reviewed. The obtained Mei symmetries are found to be the subest of these Lie point symmetries. In addition to this, a quick verification of obtained Mei symmetries is also done.

Contents

Chapter 1

Introduction

In this chapter background of differential equations is reviewed. The theory of symmetry groups of ordinary and partial differential equations is discussed with some examples. The criterion of the Lie point symmetries, Noether symmetries and Mei symmetries is presented. The basic concepts, definitions and notations related to these symmetries are presented.

1.1 Brief Background of Differential Equations

The inception of the differential equations is not that straightforward that credits could be only given to one person. It is because although Sir Isaac Newton, after discovering calculus in 1665, wrote his first work in 1671 titled as "The method of fluxion and infinite series" [1] but he did not get it published right away. In 1693, when Gottfried Leibniz gave solution of first differential equation, Newton got his previous work published so it became the official year of inauguration of the differential equations. Therefore, as history tells, these two giants of Mathematics deserve equal credits for the birth of differential equations.

The extension of Leibniz' work was done by Jakob Bernoulli and Johann Bernoulli. In 1695, Jakob Bernoulli came up with a new form of ordinary differential equation

$y' + P(x)y = Q(x)y^n$ called as Bernoulli equation. Then to describe waves, a second order linear partial differential equation i.e. one-dimensional wave equation, was proposed by Jean le Rond d'Alembert in 1746. Within ten years Euler created a three-dimensional version of the wave equation. Leonhard Euler is a household name who contributed much to Mathematics due to his vast number of influential discoveries. The areas of Mathematics covered by Euler include infinitesimal calculus, trigonometry, geometry, number theory and algebra. His famous work carries frequent use of Power series to solve particular cases of differential equations, Euler's identity and Euler's formula. Another feather to his cap was the invention of calculus of variations which takes in his most well-known result, The Euler-Lagrange equation (in collaboration with Joseph Louis Lagrange).

New figures emerged, especially Joseph-Louis Lagrange, Pierre-Simon Laplace and Adrien-Marie Legendre and Joseph Fourier, best known for their concept of Lagrangian multiplier, Laplace's equation and transformation, Legendre polynomials and Legendre transformation, Fourier series respectively. Another renowned Mathematician Friedrich Bessel generalized Bessel functions which were originally introduced by Daniel Bernoulli. In the same era a well-known mathematician Augustin-Louis Cauchy talked about existence and uniqueness of solutions for the first time. The history went on with great names such as Rudolf Lipschitz, Bernhard Riemann, Carl Friedrich Gauss, Emmy Noether and George David Birkhoff carrying out different research for the development of differential equations [2].

Meanwhile, when Evariste Galois formulated basis for group theory while having quest to find out the solutions of polynomial equations, a famous Norwegian mathematician Marius Sophus Lie used groups to find solutions of differential equations afterwards. He suggested that actually the groups of symmetries of the equations are used in standard methods to obtain the solutions. To understand symmetries, first we need to explore transformations and their generators.

1.2 Point Transformations and their Infinitesimal Generators

A **point transformation** is a transformation that transforms a point (x, y) into a new point $(\hat{x}, \hat{y})$

$$\hat{x} = \hat{x}(x, y), \qquad \hat{y} = \hat{y}(x, y), \tag{1.1}$$

where x is independent variable and y is dependent variable. In context of symmetries, point transformations that are dependent on at least one parameter needed to be considered.

1.2.1 One-Parameter Groups of Point Transformations

One-parameter group of point transformations are the transformations that depends on at least one arbitrary parameter $\varepsilon \in \mathbb{R}$

$$\hat{x} = \hat{x}(x, y, \varepsilon), \qquad \hat{y} = \hat{y}(x, y, \varepsilon). \tag{1.2}$$

with the group properties of closure, inverse and identity being satisfied. The identity transformation is obtained by setting $\varepsilon = 0$

$$\hat{x}(x, y, 0) = x, \qquad \hat{y}(x, y, 0) = y. \tag{1.3}$$

The rotations

$$\hat{x} = x\cos\varepsilon - y\sin\varepsilon, \qquad \hat{y} = x\sin\varepsilon + y\cos\varepsilon, \tag{1.4}$$

represent a one-parameter group of point transformations as they depend on only one parameter and also satisfy all the group axioms.

Also one-parameter group of point transformations comprises of scaling such as

$$\hat{x} = e^{\varepsilon}x, \qquad \hat{y} = e^{\varepsilon}y. \tag{1.5}$$

However, translations

$$\hat{x} = x + \varepsilon_1, \qquad \hat{y} = y + \varepsilon_2, \qquad \varepsilon_1, \varepsilon_2 \in \mathbb{R}, \tag{1.6}$$

represent a two-parameter group of point transformations as this set of translations depends on two parameters ε_1 and ε_2. Here one-parameter group of translations in the y direction can be obtained by setting $\varepsilon_1 = 0$ and similarly, one-parameter group of translations in the x direction can be obtained by setting $\varepsilon_2 = 0$.
On the other hand, the reflection

$$\hat{x} = -x, \qquad \hat{y} = -y, \tag{1.7}$$

does not constitute one-parameter group of point transformations but it is still a point transformation [3].
Applying Taylor series about $\varepsilon = 0$ gives infinitesimal representation of point transformation

$$\begin{aligned} \hat{x} &= x + \varepsilon \frac{\partial \hat{x}}{\partial \varepsilon} \mid_{\varepsilon=0} + O(\varepsilon^2), \\ \hat{y} &= y + \varepsilon \frac{\partial \hat{y}}{\partial \varepsilon} \mid_{\varepsilon=0} + O(\varepsilon^2), \end{aligned} \tag{1.8}$$

where the coefficients of infinitesimal transformations are set to be the functions [4]

$$\frac{\partial \hat{x}}{\partial \varepsilon} \mid_{\varepsilon=0} = \xi(x, y), \qquad \frac{\partial \hat{y}}{\partial \varepsilon} \mid_{\varepsilon=0} = \eta(x, y). \tag{1.9}$$

Hence, the infinitesimal generator of transformation is established as

$$\mathbf{X} = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}. \tag{1.10}$$

A group of transformations can be acquired if the infinitesimal generator is known [3].
The generator can be written as

$$\mathbf{X} = (\mathbf{X}x) \frac{\partial}{\partial x} + (\mathbf{X}y) \frac{\partial}{\partial y}. \tag{1.11}$$

This infinitesimal generator can be transformed into new coordinates using transformation law

$$\mathbf{X} = k^{\alpha'} \frac{\partial}{\partial z^{\alpha'}}, \qquad \alpha = 1, ..., N, \tag{1.12}$$

where $z^{\alpha'}$ are new coordinates and $k^{\alpha'} = \frac{\partial z^{\alpha'}}{\partial z^{\alpha}} k^{\alpha}$ are the new components of tangent vector $\mathbf{X}$.

The following example is conferred to grasp this concept clearly.

Example

In view of the given generator

$$\mathbf{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.13}$$

Introducing new set of coordinates

$$r(x, y) = \ln x, \qquad s(x, y) = \frac{x}{y}. \tag{1.14}$$

The infinitesimal generator in the new coordinates is

$$\mathbf{X} = (\mathbf{X}r)\frac{\partial}{\partial r} + (\mathbf{X}s)\frac{\partial}{\partial s}. \tag{1.15}$$

Solving the partial differential equations

$$\mathbf{X}r = \xi(x, y)\frac{\partial r}{\partial x} + \eta(x, y)\frac{\partial r}{\partial y}, \tag{1.16}$$

$$\mathbf{X}r = x\left(\frac{1}{x}\right) + y(0) = 1, \tag{1.17}$$

and

$$\mathbf{X}s = \xi(x, y)\frac{\partial s}{\partial x} + \eta(x, y)\frac{\partial s}{\partial y},$$

$$\mathbf{X}s = \frac{x}{y} + y\left(\frac{-x}{y^2}\right) = 0. \tag{1.18}$$

Putting eq. (1.17) and eq. (1.18) back into eq. (1.15) yields infinitesimal generator in new coordinates r and s

$$\mathbf{X}(r, s) = \frac{\partial}{\partial r}. \tag{1.19}$$

This particular form of generator is known as the normal form or the canonical form of the infinitesimal generators and the coordinates (r, s) are called normal (canonical) coordinates. If the condition $r_x s_y - r_y s_x \neq 0$ is admissible by the canonical coordinates then the transformation is invertible [5].

1.2.2 r-Parameter Groups of Point Transformations

A group of transformations may depend on multiple (more than one) parameters. It means contrary to eq. (1.2) one may write

$$\hat{x} = (x, y, \varepsilon_\alpha), \qquad \hat{y} = (x, y, \varepsilon_\alpha), \tag{1.20}$$

where $\alpha = 1, ..., r$. If all the axioms of groups are satisfied by all these parameters and if they do not depend on each other then these point transformations formulate a r-parameter group (G_r).

An infinitesimal generator can be constructed for each parameter ε_α of r-parameter group of point transformations.

$$\mathbf{X}_\alpha = \xi_\alpha \frac{\partial}{\partial x} + \eta_\alpha \frac{\partial}{\partial y}, \tag{1.21}$$

where the infinitesimals are

$$\xi_\alpha(x, y) = \left.\frac{\partial \hat{x}}{\partial \varepsilon_\alpha}\right|_{\varepsilon_\beta = 0}, \tag{1.22}$$

$$\eta_\alpha(x, y) = \left.\frac{\partial \hat{y}}{\partial \varepsilon_\alpha}\right|_{\varepsilon_\beta = 0}. \tag{1.23}$$

The following example is presented to make it clear how to deal with point transformations that depend on more than one parameters.

Example

Considering the projective transformation of the $x - y$ plane as

$$\hat{x} = \frac{\varepsilon_1 + (1 + \varepsilon_2)x + \varepsilon_3 y}{(1 + \varepsilon_4) + \varepsilon_5 x + \varepsilon_6 y}, \qquad \hat{y} = \frac{\varepsilon_7 + \varepsilon_8 x + (1 + \varepsilon_9)y}{(1 + \varepsilon_4) + \varepsilon_5 x + \varepsilon_6 y}. \tag{1.24}$$

Simplifying eq. (1.24) yields

$$\hat{x} = \varepsilon_1 + x + \varepsilon_2 x + \varepsilon_3 y - \varepsilon_4 x - \varepsilon_5 x^2 - \varepsilon_6 xy + O(\varepsilon^2), \tag{1.25}$$

$$\hat{y} = \varepsilon_7 + \varepsilon_8 x + y + \varepsilon_9 y - \varepsilon_4 y - \varepsilon_5 xy - \varepsilon_6 y^2 + O(\varepsilon^2). \tag{1.26}$$

Application of eq. (1.22) on eq. (1.25) for each parameter $\varepsilon_\alpha(\alpha = 1, ..., 9)$ provides

$$\begin{aligned} &\xi_1 = 1, && \xi_2 = x, && \xi_3 = y, && \xi_4 = -x, && \xi_5 = -x^2, \\ &\xi_6 = -xy, && \xi_7 = 0, && \xi_8 = 0, && \xi_9 = 0. \end{aligned} \tag{1.27}$$

Similarly, application of eq. (1.23) on eq. (1.26) for each parameter $\varepsilon_\alpha(\alpha = 1, ..., 9)$ provides

$$\begin{aligned} &\eta_1 = 0, && \eta_2 = 0, && \eta_3 = 0, && \eta_4 = -y, && \eta_5 = -xy, \\ &\eta_6 = -y^2, && \eta_7 = 1, && \eta_8 = x, && \eta_9 = y. \end{aligned} \tag{1.28}$$

Hence, the generators can be listed as

$$\begin{aligned} &\mathbf{X}_1 = \frac{\partial}{\partial x}, && \mathbf{X}_2 = x\frac{\partial}{\partial x}, && \mathbf{X}_3 = y\frac{\partial}{\partial x}, \\ &\mathbf{X}_4 = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, && \mathbf{X}_5 = -x^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}, && \mathbf{X}_6 = -xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y}, \\ &\mathbf{X}_7 = \frac{\partial}{\partial y}, && \mathbf{X}_8 = x\frac{\partial}{\partial y}, && \mathbf{X}_9 = y\frac{\partial}{\partial y}. \end{aligned} \tag{1.29}$$

However, as $\mathbf{X}_4$ is a linear combination of $\mathbf{X}_2$ and $\mathbf{X}_9$, therefore, there are eight linearly independent generators in total. Hence in this example an 8-parameter group of point transformations is easily investigated.

1.2.3 Prolonged Transformations and their Prolonged Generators

Taking a differential equation into consideration

$$\mathbf{E} = (x, y, y', \dots, y^{(n)}) = 0. \tag{1.30}$$

If we want to apply point transformation eq. (1.2) on it, we must extend or prolong this transformation to its derivatives $y^{(k)}, k = 1, 2, \ldots, n$ as well. Calculating $\hat{y}^{(k)}$ recursively [3] such as

$$\hat{y}^{(k)} = \frac{d\hat{y}}{d\hat{x}^{(k)}} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{\mathbf{D}_x \hat{y}^{(k-1)}}{\mathbf{D}_x \hat{x}}, \tag{1.31}$$

here $\mathbf{D}_x$ is the total derivative w.r.t x.

$$\mathbf{D}_x = \frac{d}{dx} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + y''' \frac{\partial}{\partial y''} + \ldots, \tag{1.32}$$

as eq. (1.8) suggests

$$\begin{aligned} \hat{x} &= x + \varepsilon \xi(x, y) + O(\varepsilon^2), \\ \hat{y} &= y + \varepsilon \eta(x, y) + O(\varepsilon^2), \\ \hat{y}' &= y' + \varepsilon \eta^{(1)}(x, y, y') + O(\varepsilon^2), \\ &\vdots \\ \hat{y}^{(n)} &= y^{(n)} + \varepsilon \eta^{(n)}(x, y, y', \ldots, y^{(n)}) + O(\varepsilon^2), \end{aligned} \tag{1.33}$$

where $\eta^{(1)}, \eta^{(2)}, \ldots, \eta^{(n)}$ are given as

$$\eta^{(1)} = \left.\frac{\partial \hat{y}'}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \eta^{(2)} = \left.\frac{\partial \hat{y}''}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \ldots, \quad \eta^{(n)} = \left.\frac{\partial \hat{y}^{n}}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{1.34}$$

By using eq. (1.33) in eq. (1.31) we get

$$\begin{aligned} \hat{y}' = y' + \varepsilon \eta^{(1)} + O(\varepsilon^2) = \mathbf{D}_x(\hat{y}) &= \frac{d\hat{y}}{d\hat{x}} = \frac{dy + \varepsilon d\eta + \ldots}{dx + \varepsilon d\xi + \ldots}, \\ &= \frac{y' + \varepsilon \left(\frac{d\eta}{dx}\right) + \ldots}{1 + \varepsilon \left(\frac{d\xi}{dx}\right) + \ldots}, \\ &= y' + \varepsilon \left(\frac{d\eta}{dx} - y' \frac{d\xi}{dx}\right) + \ldots. \end{aligned} \tag{1.35}$$

Comparing $\hat{y}'$ from eq. (1.35) with $\hat{y}'$ from eq. (1.33) we get

$$\eta^{(1)} = \frac{d\eta}{dx} - y' \frac{d\xi}{dx} = \mathbf{D}_x \eta - y' \mathbf{D}_x \xi. \tag{1.36}$$

Similarly, for $\hat{y}^{(n)}$ we get

$$\hat{y}^{(n)} = y^{(n)} + \varepsilon \left(\frac{d\eta^{(n-1)}}{dx} - y^n \frac{d\xi}{dx} \right) + \ldots, \tag{1.37}$$

and by comparing $y^{(n)}$ from eq. (1.37) to $y^{(n)}$ from eq. (1.33) we get

$$\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - y^n \frac{d\xi}{dx} = \mathbf{D}_x \eta^{(n-1)} - y^n \mathbf{D}_x \xi. \tag{1.38}$$

Hence we can generalize it to

$$\eta^{(k)} = \frac{d\eta^{(k-1)}}{dx} - y^k \frac{d\xi}{dx} = \mathbf{D}_x \eta^{(k-1)} - y^{(k)} \mathbf{D}_x \xi, \qquad k = 1, \ldots, n \tag{1.39}$$

where $\eta^{(k)}$ is not $k-$th derivative of η rather it is $k-$th prolongation and one can also compute the prolongation of η by putting eq. (1.32) in eq. (1.39) as the first two prolongations are

$$\eta^{(1)} = \eta_{,x} + y'(\eta_{,y} - \xi_{,x}) - y'^2 \xi_{,y}, \tag{1.40}$$

$$\eta^{(2)} = \eta_{,xx} + y'(2\eta_{,xy} - \xi_{,xx}) + y'^2(\eta_{,yy} - 2\xi_{,xy}) - y'^3 \xi_{,yy} + y''(\eta_{,y} - 2\xi_{,x} - 3y'\xi_{,y}). \tag{1.41}$$

where $(,)$ denotes partial derivative w.r.t to the function following it.

Now the prolongation of infinitesimal generator is established such that the infinitesimal transformations are written in the form

$$\begin{aligned} \hat{x} &= x + \varepsilon \mathbf{X} x + O(\varepsilon^2), \\ \hat{y} &= y + \varepsilon \mathbf{X} y + O(\varepsilon^2), \\ \hat{y}' &= y' + \varepsilon \mathbf{X} y' + O(\varepsilon^2), \\ &\vdots \\ \hat{y}^{(n)} &= y^{(n)} + \varepsilon \mathbf{X} y^{(n)} + O(\varepsilon^2). \end{aligned} \tag{1.42}$$

and the prolongation of infinitesimal generator is formulated as

$$\mathbf{X}^{(n)} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} +, \ldots, + \eta^{(n)} \frac{\partial}{\partial y^{(n)}}. \tag{1.43}$$

The below presented example marks the procedure of computing prolongation of generator.

Example Consider the given generator

$$\mathbf{X} = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{1.44}$$

Now we will find its second prolonged generator. To find second prolonged generator we need $\eta^{(1)}$ and $\eta^{(2)}$. From eq. (1.44) we have

$$\xi = xy, \qquad\qquad \eta = y^2, \tag{1.45}$$

and by using eq. (1.39) we can compute $\eta^{(1)}$ and $\eta^{(2)}$ as

$$\eta^{(1)} = \mathbf{D}_x(y^2) - y'\mathbf{D}_x(xy) = yy' - xy'^2, \tag{1.46}$$

$$\eta^{(2)} = \mathbf{D}_x(yy' - xy'^2) - y''\mathbf{D}_x(xy) = -3y'y''x. \tag{1.47}$$

We can also compute $\eta^{(1)}$ directly by using eq. (1.40) as

$$\eta^{(1)} = y'(2y - y) - y'^2x = yy' - y'^2x, \tag{1.48}$$

and also $\eta^{(2)}$ by using eq. (1.41) as

$$\eta^{(2)} = y''(2y - 2y - 3y'x) = -3y'y''x. \tag{1.49}$$

Hence the prolonged generator is found as

$$\mathbf{X}^{(2)} = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (yy' - y'^2x)\frac{\partial}{\partial y'} - 3y'y''x\frac{\partial}{\partial y''}. \tag{1.50}$$

1.3 Lie Point Symmetries of Ordinary Differential Equations

Until now we talked about point transformations. Now we define symmetry group of transformations and Lie point symmetries of ordinary differential equations .

A point transformation that may or may not depend on any parameter is said to be a **symmetry group of transformations** if it maps a solution to another solution of differential equation and preserves its structure.
For example if we consider the ordinary differential equation

$$\mathbf{E}(x, y, y', \ldots, y^{n}) = 0, \tag{1.51}$$

and apply point transformation from eq. (1.1) to it, if it provides

$$\mathbf{E}(\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n)}) = 0, \tag{1.52}$$

then this point transformation is a symmetry transformation because it did not change the eq. (1.51). More generally, we can say that an nth eq. order differential equation $\mathbf{E}(x, y, y', \ldots, y^{(n)}) = 0$ is invariant under a symmetry transformation $\hat{x} = \hat{x}(x, y), \hat{y} = \hat{y}(x, y), \ldots, \hat{y}^{(n)} = \hat{y}^{(n)}(x, y, y', \ldots, y^{(n)})$ if $\mathbf{E}(\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n)}) = 0$.
If we consider the symmetry group of transformations that is dependent on at least one parameter then this symmetry is known as **Lie point symmetry** named after Norwegian mathematician Sophus Lie.
Differentiating eq. (1.52) yields

$$\left.\frac{\partial \mathbf{E}}{\partial \varepsilon}\right|_{\varepsilon=0} = 0, \tag{1.53}$$

$$= \left.\left(\frac{\partial \mathbf{E}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial \varepsilon} + \frac{\partial \mathbf{E}}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial \varepsilon} + \frac{\partial \mathbf{E}}{\partial \hat{y}'}\frac{\partial \hat{y}'}{\partial \varepsilon} + \ldots, + \frac{\partial \mathbf{E}}{\partial \hat{y}^{(n)}}\frac{\partial \hat{y}^{(n)}}{\partial \varepsilon}\right)\right|_{\varepsilon=0}. \tag{1.54}$$

If we choose $(\frac{\partial \mathbf{E}}{\partial \hat{x}}) |_{\varepsilon=0} = (\frac{\partial \mathbf{E}}{\partial x})$ then by using eq. (1.9) we can write

$$\xi\frac{\partial \mathbf{E}}{\partial x} + \eta\frac{\partial \mathbf{E}}{\partial y} + \eta'\frac{\partial \mathbf{E}}{\partial y'} + \cdots + \eta^{(n)}\frac{\partial \mathbf{E}}{\partial y^{(n)}} = 0, \tag{1.55}$$

which is equivalent to

$$\mathbf{X}\mathbf{E} = 0. \tag{1.56}$$

We now can state the criterion to find Lie point symmetries.

1.3.1 Criterion to Find Lie Point Symmetries

An ordinary differential equation (ODE)

$$\mathbf{E}(x, y, y', y'', \ldots, y^{(n)}) = 0,$$

admits a group of symmetries with generator $\mathbf{X}$ if and only if

$$\mathbf{X}^{(n)}\mathbf{E} \mid_{\varepsilon=0} = 0,$$

holds.

In next example we see how $\mathbf{XE}$ becomes zero considering $\mathbf{E} = 0$.

Example

Suppose we have a linear differential equation

$$y'' + y = 0. \tag{1.57}$$

The generator is given as

$$\mathbf{X} = y\frac{\partial}{\partial y}. \tag{1.58}$$

From eq. (1.58) we get

$$\xi = 0, \qquad \eta = y. \tag{1.59}$$

Now we will find prolonged generator. For this we have to find $\eta^{(1)}$ and $\eta^{(2)}$ using definitions eq. (1.39).

$$\eta^{(1)} = y', \qquad \eta^{(2)} = y''.$$

So the second prolonged generator is obtained as

$$\mathbf{X}^{(2)} = y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''}. \tag{1.60}$$

If we recheck then $\mathbf{X}^{(2)}\mathbf{E} = 0$ as $\mathbf{E} = 0$.

In next example we now elaborate that how this criterion works to find group of symmetries for a second order differential equation.

Example

Consider

$$\mathbf{E}(x,y,y',y''): y''+y=0. \tag{1.61}$$

Now we will find the symmetries of the equation eq. (1.61). The criterion for finding symmetries is

$$\mathbf{X}^{(2)}\mathbf{E}=0. \tag{1.62}$$

Firstly we prolong the generator $\mathbf{X}$ to $\mathbf{X}^{(2)}$ so we get

$$\mathbf{X}^{(2)}=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\eta^{(1)}\frac{\partial}{\partial y'}+\eta^{(2)}\frac{\partial}{\partial y''}. \tag{1.63}$$

Applying the criterion given in eq. (1.62) provides

$$\mathbf{X}^{(2)}E=\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\eta^{(1)}\frac{\partial}{\partial y'}+\eta^{(2)}\frac{\partial}{\partial y''}\right)y''=\eta^{(2)}+\eta=0. \tag{1.64}$$

Using eq. (1.41) we have

$$\eta_{,xx}+y'(2\eta_{,xy}-\xi_{,xx})+y'^2(\eta_{,yy}-2\xi_{,xy})-y'^3\xi_{,yy}+y''(\eta_{,y}-2\xi_{,x}-3y'\xi_{,y})+\eta=0, \tag{1.65}$$

putting $y''=-y$ from eq. (1.61) we get

$$\eta_{,xx}-y\eta_{,y}+2y\xi_{,x}+\eta+y'(2\eta_{,xy}-\xi_{,xx}+3y\xi_{,y})+y'^2(\eta_{,yy}-2\xi_{,xy})-y'^3\xi_{,yy}=0. \tag{1.66}$$

Since ξ,η are the functions of x and y only therefore we can compare coefficients of y' and their powers. Comparing coefficients provides partial differential equations

$$(y'^0): \eta_{,xx}-y\eta_{,y}+2y\xi_{,x}+\eta=0, \tag{1.67}$$

$$(y'^1): 2\eta_{,xy}-\xi_{,xx}+3y\xi_{,y}=0, \tag{1.68}$$

$$(y'^2): \eta_{,yy}-2\xi_{,xy}=0, \tag{1.69}$$

$$(y'^3): \xi_{,yy}=0, \tag{1.70}$$

eq. (1.70) implies

$$\xi = a_1(x)y + a_2(x). \tag{1.71}$$

Finding $\xi_{,xy}$ from eq. (1.71) and putting back in eq. (1.69) gives

$$\eta_{,yy} - 2a_1' = 0, \tag{1.72}$$

eq. (1.72) implies

$$\begin{aligned} \eta_{,y} &= 2a_1'y + a_3(x), \\ \eta &= a_1'y^2 + a_3y + a_4(x). \end{aligned} \tag{1.73}$$

Putting value of $\eta_{,xy}$ and $\xi_{,xx}$ in eq. (1.68) yields

$$\begin{aligned} &4a_1''y + 2a_3' - a_1''y - a_2'' + 3ya_1 = 0, \\ &3a_1''y + 2a_3' - a_2'' + 3ya_1 = 0. \end{aligned} \tag{1.74}$$

Comparing powers of y in eq. (1.74) we get

$$(y^0) : 2a_3'(x) - a_2''(x) = 0. \tag{1.75}$$

$$(y^1) : a_1''(x) + a_1(x) = 0, \tag{1.76}$$

from eq. (1.76) we get

$$a_1 = c_1 \cos x + c_2 \sin x. \tag{1.77}$$

Putting value of a_1' from eq. (1.77) into eq. (1.73) gives

$$\eta = -c_1 \sin xy^2 + c_2 \cos xy^2 + a_3y + a_4. \tag{1.78}$$

Now inserting value of $\eta_{,xx}$ and $\eta_{,y}$ from eq. (1.78) into eq. (1.67) produce

$$(-2c_2 \cos x + 2c_1 \sin x + 2a_1')\, y^2 + (a_3'' + 2a_2')y + a_4'' + a_4 = 0. \tag{1.79}$$

Comparing powers of y in eq. (1.79) yields

$$(y^0): a_4''(x) + a_4(x) = 0 = a_4 = c_5 x + c_6, \tag{1.80}$$

$$(y^1): a_3''(x) + 2a_2'(x) = 0, \tag{1.81}$$

$$(y^2): -2c_2 \cos x + 2c_1 \sin x + 2a_1'(x) = 0,$$

differentiating eq. (1.81) w.r.t x and putting value of a_2'' from eq. (1.75) we get

$$a_3'''(x) + 4a_3'(x) = 0. \tag{1.82}$$

If we consider $a_3'(x) = A_3$, $a_3''(x) = A_3'$, $a_3'''(x) = A_3''$ then eq. (1.82) becomes

$$A_3'' + 4A_3 = 0, \tag{1.83}$$

that gives

$$A_3 = c_5 \cos 2x + c_6 \sin 2x, \tag{1.84}$$

as $A_3 = a_3'(x)$ so by putting into eq. (1.84) and solving we get value of a_3

$$a_3(x) = \frac{1}{2} c_5 \sin 2x - \frac{1}{2} c_6 \cos 2x + c_7. \tag{1.85}$$

Putting value of a_3 into eq. (1.81) we obtain

$$a_2(x) = -\frac{1}{2} c_5 \cos 2x - \frac{1}{2} c_6 \sin 2x + c_8, \tag{1.86}$$

putting value of respective $a_k, k = 1, \ldots, 4$ back in eq. (1.71) and eq. (1.78) produce values of ξ and η

$$\xi(x, y) = (c_1 x + c_2) y + c_3 x^2 + c_7 x + c_8, \tag{1.87}$$

$$\eta(x, y) = c_1 y^2 + (c_3 x + c_4) y + c_5 x + c_6. \tag{1.88}$$

Here $c_k, k = 1, 2, \ldots, 8$ are arbitrary constants.

Hence by using eq. (1.87) and eq. (1.88) the infinitesimal generator of one-parameter.

Lie group of point symmetries of $y'' + y = 0$ is established as

$$\mathbf{X} = [(c_1 \cos x + c_2 \sin x)y - c_5 \cos 2x - c_6 \sin 2x + c_8] \frac{\partial}{\partial x} + \left[(-c_1 \sin x + c_2 \cos x)y^2 + (c_5 \sin 2x - c_6 \cos 2x + c_7)y + c_3 \cos x + c_4 \sin x\right] \frac{\partial}{\partial y}. \tag{1.89}$$

and for each $c_i = 1, c_j = 0$ we get

$$\begin{aligned}
&\mathbf{X}_1 = y \cos x \frac{\partial}{\partial x} - y^2 \sin x \frac{\partial}{\partial y}, && \mathbf{X}_2 = y \sin x \frac{\partial}{\partial x} + y^2 \cos x \frac{\partial}{\partial y}, \\
&\mathbf{X}_3 = \cos x \frac{\partial}{\partial y}, && \mathbf{X}_4 = \sin x \frac{\partial}{\partial y}, \\
&\mathbf{X}_5 = -\cos 2x \frac{\partial}{\partial x} + y \sin 2x \frac{\partial}{\partial y}, && \mathbf{X}_6 = -\sin 2x \frac{\partial}{\partial x} - y \cos 2x \frac{\partial}{\partial y}, \\
&\mathbf{X}_7 = y \frac{\partial}{\partial y}, && \mathbf{X}_8 = \frac{\partial}{\partial x}.
\end{aligned} \tag{1.90}$$

Various systems of differential equations are solved by using Lie point symmetry method [7]-[9].

1.3.2 Symmetry Criterion in Terms of Operator A

In order to define symmetry condition in terms of operator **A** we first have to define some facts about the linear operator **A**.

Suppose we have an ordinary differential equation written as

$$y^{(n)} = w(x, y, y', \ldots, y^{(n-1)}), \tag{1.91}$$

we express the associated partial differential equation as

$$\mathbf{A}f = a^{\alpha} \frac{\partial}{\partial x^{\alpha}} = \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots + w \frac{\partial}{\partial y^{(n-1)}} \right) f = 0, \tag{1.92}$$

just like symmetry generator **X**, the linear operator **A** can be written in its canonical form therefore if we solve eq. (1.92) or transform **A** in its canonical form, we get its solution in both ways. We choose the solutions of eq. (1.92) as ψ^{α} hence

$$\mathbf{A}f = \mathbf{A}\psi^{\alpha} = 0, \tag{1.93}$$

here the first integrals are serving as a link between an ordinary differential equation $y^{(n)} = w$ and partial differential equation $\mathbf{A}f = 0$.

A **first integral** is a non-constant function $\phi(x, y, y', \dots, y^{(n-1)})$ which is locally constant on any solution of eq. (1.91) such that

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + y' \frac{\partial \phi}{\partial y} + y'' \frac{\partial \phi}{\partial y'} + \dots + y^{(n)} \frac{\partial \phi}{\partial y^{(n-1)}} = 0, \tag{1.94}$$

holds if $y^{(n)} = w$ is substituted.

If we observe eq. (1.93), the solutions ψ^α of ordinary differential equation (1.91) are satisfying the same criterion of first integrals given in eq. (1.94) it means every solution of $\mathbf{A}f = 0$ is the first integral ϕ of ODE $y^{(n)} = w(x, y, \dots, y^{(n-1)})$. Moreover, every complete set of n functionally independent solutions ψ^α corresponds to the general solution $y = y(x, \psi_0^\alpha)$ of the ordinary differential equation that can be obtained by eliminating all derivatives of y from the system [3]

$$\psi^\alpha(x, y, \dots, y^{(n-1)}) = \psi_0^\alpha. \tag{1.95}$$

Example

Consider the differential equation

$$y'' + y = 0, \tag{1.96}$$

it implies

$$y'' = -y = w(x, y, y'). \tag{1.97}$$

The corresponding partial differential equation is given as

$$\mathbf{A}f = \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} - y \frac{\partial}{\partial y'} \right) f = 0, \tag{1.98}$$

where $y'' = -y$ from eq. (1.97) is substituted.

We solve it by using method of characteristics and obtained the two solutions such as

$$\psi^1 = y^2 + y'^2, \tag{1.99}$$

$$\psi^2 = x - \arctan\frac{y}{y'}. \tag{1.100}$$

Thus by using eq. (1.99) and eq. (1.100) we write general solution of eq. (1.97) as

$$y = \sqrt{\psi^1}\sin(x - \psi^2). \tag{1.101}$$

Now we can easily see that the linear operator $\mathbf{A}$ and the generator $\mathbf{X}$ are on same lines. If we say that $\mathbf{X}$ is a symmetry of ordinary differential equation (1.91) we must investigate that which criterion must be fulfilled by $\mathbf{X}$ to serve as the symmetry of partial differential equation $\mathbf{A}f = 0$. As symmetry maps solutions to solutions so $\mathbf{X}\psi^\alpha$ is also a solution.

In terms of commutator we consider

$$\begin{aligned}[\mathbf{X}, \mathbf{A}]\psi^\alpha &= (\mathbf{XA} - \mathbf{AX})\psi^\alpha, \\ &= \mathbf{X}(\mathbf{A}\psi^\alpha) - \mathbf{A}(\mathbf{X}\psi^\alpha) = 0.\end{aligned} \tag{1.102}$$

Since $\mathbf{A}\psi^\alpha$ and $[\mathbf{X}, \mathbf{A}]\psi^\alpha$ have the same solutions therefore $\mathbf{A}$ and $[\mathbf{X}, \mathbf{A}]$ are proportional that means

$$[\mathbf{X}, \mathbf{A}] = \lambda(x, y, y', \ldots, y^{(n-1)})\mathbf{A}. \tag{1.103}$$

If we put values of $\mathbf{X}$ and $\mathbf{A}$ and solve the commutator and then after comparing coefficients of ∂_x, ∂_y and ∂'_y we get

$$\left(\frac{\partial}{\partial x}\right) : -\mathbf{A}\xi = \lambda, \tag{1.104}$$

$$\left(\frac{\partial}{\partial y}\right) : \mathbf{X}y' - \mathbf{A}\eta = \lambda y', \tag{1.105}$$

$$\left(\frac{\partial}{\partial y'}\right) : \mathbf{X}y'' - \mathbf{A}\eta^{(1)} = \lambda y''. \tag{1.106}$$

Putting values from eq. (1.104) into eq. (1.105) and eq. (1.106) we get

$$\eta^{(1)} = \mathbf{A}\eta - y'\mathbf{A}\xi = \frac{d\eta}{dx} - y'\frac{d\xi}{dx}, \tag{1.107}$$

$$\eta^{(2)} = \mathbf{A}\eta^{(1)} - y''\mathbf{A}\xi = \frac{d\eta^{(1)}}{dx} - y''\frac{d\xi}{dx}. \tag{1.108}$$

The prolongations of eta have same formulas so both definitions of symmetry are equivalent. Hence the symmetry condition $[\mathbf{X}, \mathbf{A}] = \lambda \mathbf{A}$ in terms of linear operator $\mathbf{A}$ holds.

1.4 Lie Groups and Lie Algebras of Infinitesimal generators

Before going into details of Lie algebra we define Lie group.

Lie group

A group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps, is known to be a Lie group. Lie groups were introduced by a Norwegian mathematician Sophus Lie who formulated the theory of continuous transformation groups in order to model the continuous symmetries.

To every Lie group one can associate a Lie algebra which completely determine the local structure of the Lie group. We define the Lie algebra as

Lie Algebra

Lie algebra $\mathbf{L}$ is a vector space defined on a field $\mathbf{F}$ together with an operation called the Lie bracket satisfying the properties

1. Bilinearity: $[\mathbf{X}, f\mathbf{Y} + g\mathbf{Z}] = [\mathbf{X}, f\mathbf{Y}] + [\mathbf{X}, g\mathbf{Z}], \quad \forall\ \mathbf{X},\mathbf{Y},\mathbf{Z} \in \mathbf{L}$ and f,g $\in \mathbf{F}$.

2. Skew symmetry: $[\mathbf{X},\mathbf{Y}]=$ -$[\mathbf{Y},\mathbf{X}], \quad \forall\ \mathbf{X},\mathbf{Y},\mathbf{Z} \in \mathbf{L}$.

3. Jaccobi identity: $[[\mathbf{X}, \mathbf{Y}], \mathbf{Z}]+[[\mathbf{Z}, \mathbf{X}], \mathbf{Y}]+[[\mathbf{Z}, \mathbf{Y}], \mathbf{X}], \quad \forall\ \mathbf{X},\mathbf{Y},\mathbf{Z} \in \mathbf{L}$.

From property of skew-symmetry we can say $[\mathbf{X}, \mathbf{X}] = 0$ whereas the Lie algebra is called abelian when $[\mathbf{X}, \mathbf{Y}] = 0$.

As an example, the linearly independent basic generators listed in eq. (1.90), obtained from general infinitesimal generator of one-parameter Lie group of point symmetries of

$y'' + y = 0$ form a Lie algebra with a commutator defined as

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}, \tag{1.109}$$

and the commutators for eq. (1.90) are found as

$$\begin{aligned}
&[\mathbf{X}_1, \mathbf{X}_2] = 0, && [\mathbf{X}_1, \mathbf{X}_3] = -\mathbf{X}_8 + \mathbf{X}_5, \\
&[\mathbf{X}_1, \mathbf{X}_4] = 3\mathbf{X}_7 + \mathbf{X}_6, && [\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_2, \\
&[\mathbf{X}_1, \mathbf{X}_6] = -\mathbf{X}_1, && [\mathbf{X}_1, \mathbf{X}_7] = -\mathbf{X}_1, \\
&[\mathbf{X}_1, \mathbf{X}_8] = \mathbf{X}_2, && [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_6 - 3\mathbf{X}_7, \\
&[\mathbf{X}_2, \mathbf{X}_4] = -\mathbf{X}_8 - \mathbf{X}_5, && [\mathbf{X}_2, \mathbf{X}_5] = \mathbf{X}_1, \\
&[\mathbf{X}_2, \mathbf{X}_6] = \mathbf{X}_2, && [\mathbf{X}_2, \mathbf{X}_7] = -\mathbf{X}_2, \\
&[\mathbf{X}_2, \mathbf{X}_8] = -\mathbf{X}_1, && [\mathbf{X}_3, \mathbf{X}_4] = 0, \\
&[\mathbf{X}_3, \mathbf{X}_5] = \mathbf{X}_4, && [\mathbf{X}_3, \mathbf{X}_6] = -\mathbf{X}_3, \\
&[\mathbf{X}_3, \mathbf{X}_7] = \mathbf{X}_3, && [\mathbf{X}_3, \mathbf{X}_8] = \mathbf{X}_4, \\
&[\mathbf{X}_4, \mathbf{X}_5] = \mathbf{X}_3, && [\mathbf{X}_4, \mathbf{X}_6] = \mathbf{X}_4, \\
&[\mathbf{X}_4, \mathbf{X}_7] = \mathbf{X}_4, && [\mathbf{X}_4, \mathbf{X}_8] = -\mathbf{X}_3, \\
&[\mathbf{X}_5, \mathbf{X}_6] = 2\mathbf{X}_8, && [\mathbf{X}_5, \mathbf{X}_7] = 0, \\
&[\mathbf{X}_5, \mathbf{X}_8] = 2\mathbf{X}_6, && [\mathbf{X}_6, \mathbf{X}_7] = 0, \\
&[\mathbf{X}_6, \mathbf{X}_8] = -2\mathbf{X}_5, && [\mathbf{X}_7, \mathbf{X}_8] = 0.
\end{aligned} \tag{1.110}$$

The commutator of two symmetry generators again produce a symmetry. Also, commutators of $\mathbf{X}_\alpha$ determine the commutators of the extensions. The commutators of infinitesimal generators of the group of symmetries of heat equation can be found in [13]. Here we must observe that we express the commutators of symmetry generators as a linear combination of basic generators such as $C^\gamma \mathbf{X}_\gamma$. Here C^γ are called structure constants. In general

$$[\mathbf{X}_\alpha, \mathbf{Y}_\beta] = C^\gamma_{\alpha\beta} \mathbf{X}_\gamma. \tag{1.111}$$

In next example we find the structure constants of 3-parameter Lie group of rotations and translations.

Example

The generators of the Lie algebra corresponding to 3 parameters ε_α are

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial y}, \qquad \mathbf{X}_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{1.112}$$

The corresponding Lie algebra is

$$[\mathbf{X}_1, \mathbf{X}_2] = 0, \tag{1.113}$$

$$[\mathbf{X}_1, \mathbf{X}_3] = \frac{\partial}{\partial y} = \mathbf{X}_2 = C_{13}^2\mathbf{X}_2, \tag{1.114}$$

$$[\mathbf{X}_2, \mathbf{X}_3] = -\frac{\partial}{\partial x} = -\mathbf{X}_1 = C_{23}^1\mathbf{X}_1. \tag{1.115}$$

The structure constants are $C_{13}^2 = 1$ and $C_{23}^1 = -1$.

Structure constants do not change under coordinate transformations and due to skew symmetric property of Lie algebra $C_{\alpha\beta}^\gamma = -C_{\beta\alpha}^\gamma$.

1.5 Systems Involving Lagrangians

The classical mechanics mostly comprises of systems of second order differential equations. The notation that is frequently used in classical mechanics is $\dot{q} = dq^\alpha/dt$ where time t is independent variable and generalized coordinates q^α are dependent variables. Using this notation the system of second order differential equations can be written as

$$\ddot{q}^\alpha = w^\alpha(t, q^\beta, \dot{q}^\beta), \qquad \alpha, \beta = 1, \ldots, N, \tag{1.116}$$

which is equivalent to the linear partial differential equation

$$\mathbf{A}f = \left(\frac{\partial}{\partial t} + \dot{q}^\alpha\frac{\partial}{\partial q^\alpha} + w^\alpha(t, q^\beta, \dot{q}^\beta)\frac{\partial}{\partial \dot{q}^\alpha}\right) f = 0. \tag{1.117}$$

As in case of ordinary differential equations the solutions ϕ^α of eq. (1.117) translate into the 2N first integrals of eq. (1.116) [3, 4]. Considering the infinitesimal transformation of time and generalized coordinates

$$\hat{t} = t + \varepsilon\xi(t, q, \dot{q}), \qquad \hat{q}^\alpha = q^\alpha + \varepsilon\eta^\alpha(t, q, \dot{q}), \tag{1.118}$$

the generator and its prolongation in this coordinates can be written as

$$\begin{aligned} \mathbf{X} &= \xi(t, q^\beta)\frac{\partial}{\partial t} + \eta^\alpha(t, q^\beta)\frac{\partial}{\partial q^\alpha}, \\ \mathbf{X}^{[1]} &= \xi(t, q^\beta)\frac{\partial}{\partial t} + \eta^\alpha(t, q^\beta)\frac{\partial}{\partial q^\alpha} + \zeta^\alpha(t, q^\beta, \dot{q^\beta})\frac{\partial}{\partial \dot{q}^\alpha}, \end{aligned} \tag{1.119}$$

where ζ^α is given by

$$\zeta^\alpha = \frac{d\eta^\alpha}{dt} - \dot{q}^\alpha\frac{d\xi}{dt}. \tag{1.120}$$

By recursion, successive prolongations $\mathbf{X}^{[n]}$ of $\mathbf{X}$ can be obtained (one can write $\mathbf{X}$ for the prolongations as well if convenient) and symmetries of eq. (1.116) can be deduced if

$$[\mathbf{X}, \mathbf{A}] = \lambda\mathbf{A}, \tag{1.121}$$

holds.

Once the symmetries are known then the first integrals can be found corresponding to each symmetry using Lagrangian of the system. Whereas the **Lagrangian** L is the difference of kinetic energy T and potential energy V defined as

$$L(t, q^\beta, \dot{q}^\beta) = T - V. \tag{1.122}$$

This correspondence between symmetries and first integrals cannot be established for the number of symmetries less than 2N but it is possible if the system is derivable from an action [3]

$$\mathbf{S} = \int_{t_i}^{t_f} L(t, q^\alpha, \dot{q}^\alpha)dt. \tag{1.123}$$

Let us take a quick overview of Noether symmetries and Noether theorem to establish a relationship between Noether symmetries and the first integrals.

1.5.1 The Noether Symmetries

A **Noether symmetry** is the Lie point transformation under which the action $\mathbf{S}$ is invariant up to a divergence term $g(\varepsilon)$

$$\hat{\mathbf{S}} = \mathbf{S} + g(\varepsilon) = \mathbf{S} + \int_{t_i}^{t_f} \frac{dg(t, q^\alpha, \varepsilon)}{dt} dt. \tag{1.124}$$

The expressions in eq. (1.123) and eq. (1.124) give the same Euler Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0. \tag{1.125}$$

Transformations for which these Euler Lagrange equations are invariant are called invariant transformations. Details can be read in [16].

We consider the prolonged generator given by eq. (1.119) and write eq. (1.123) as

$$\begin{aligned}
\hat{\mathbf{S}} &= \int_{\hat{t}_i}^{\hat{t}_f} \hat{L}(t, \hat{q}^\alpha, \hat{\dot{q}}^\alpha) d\hat{t} + \varepsilon[\xi t_f L - \xi t_i L], \\
&= \int_{t_i}^{t_f} \left[\left(L(t, q^\alpha, \dot{q}^\alpha) + \varepsilon \mathbf{X} L + \dots \right) \left(dt + \varepsilon \frac{d\xi}{dt} dt + \dots \right) \right] + \varepsilon \left[\xi t_f L - \xi t_i L \right], \\
&= \int_{t_i}^{t_f} \left[L dt + \varepsilon \left(\mathbf{X} L + L \frac{d\xi}{dt} \right) dt + \dots \right] + \varepsilon \Big[\xi t_f L - \xi t_i L \Big], \\
&= \int_{t_i}^{t_f} \left[L + \varepsilon \int (\mathbf{X} L + L \mathbf{A} \xi) + \dots \right] dt + \varepsilon \Big[\xi t_f L(t_f, q_f, \dot{q}_f) - \xi t_i L(t_i, q_i, \dot{q}_i) \Big],
\end{aligned} \tag{1.126}$$

where $\frac{d}{dt}$ is replaced by operator $\mathbf{A}$. We can manipulate eq. (1.126) by using

$$G = \varepsilon \Big[\xi t_f L(t_f, q_f, \dot{q}_f) - \xi t_i L(t_i, q_i, \dot{q}_i) \Big] = \int_{t_i}^{t_f} \frac{dg}{dt}, \tag{1.127}$$

so that (1.126) reads

$$\begin{aligned}
\hat{\mathbf{S}} = \mathbf{S} + \varepsilon \int_{t_i}^{t_f} \frac{dg}{dt} dt &= \int_{t_i}^{t_f} L dt + \varepsilon \int_{t_i}^{t_f} \frac{dg}{dt} dt, \\
&= \int_{t_i}^{t_f} L dt + \varepsilon \int_{t_i}^{t_f} (\mathbf{A} g) dt,
\end{aligned} \tag{1.128}$$

comparing first order terms of ε we obtain

$$\mathbf{X} L + L \mathbf{A} \xi = \mathbf{A} g(t, q^\alpha). \tag{1.129}$$

This is the criterion for $\mathbf{X}$ to be a Noether symmetry if there exists a g that satisfies eq. (1.129). The Noether symmetries lead to variational symmetries when $g = 0$. After some algebra eq. (1.129) leads to Noether's theorem [17]-[19].

Noether's Theorem

We may write eq. (1.129) as

$$\xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \dot{\xi} L = \dot{g}. \tag{1.130}$$

here eq. (1.119) replaced $\mathbf{X}$ and $\mathbf{A}$ replaced d/dt. Manipulating eq. (1.130) as [6]

$$\begin{aligned} 0 &= \dot{g} - \xi \frac{\partial L}{\partial t} - \dot{\xi} L - \eta \frac{\partial L}{\partial q} - (\dot{\eta} - \dot{q}\dot{\xi}) \frac{\partial L}{\partial \dot{q}}, \\ &= \frac{d}{dt}(g - \xi L) + \xi \left(\dot{q} \frac{\partial L}{\partial q} + \ddot{q} \frac{\partial L}{\partial \dot{q}} \right) + \dot{\xi} \left(\dot{q} \frac{\partial L}{\partial \dot{q}} \right) - \eta \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \dot{\eta} \frac{\partial L}{\partial \dot{q}}, \\ &= \frac{d}{dt} \left[g - \xi L - (\eta - \xi \dot{q}) \frac{\partial L}{\partial \dot{q}} \right], \end{aligned} \tag{1.131}$$

here we used Euler Lagrange equation (1.125) in place of coefficient of η.

Hence the first integral is obtained as

$$\phi = g - \left[\xi L + (\eta - \xi \dot{q}) \frac{\partial L}{\partial \dot{q}} \right], \tag{1.132}$$

and the statement of Noether's theorem says [17]:

If the Lie derivative $\mathcal{L}$ of a Lagrangian vanishes

$$\mathcal{L}_{\mathbf{X}} L(t, q^\alpha, \dot{q}^\alpha) = \mathbf{X} L(t, q^\alpha, \dot{q}^\alpha) = 0, \tag{1.133}$$

along the generator

$$\mathbf{X} = \xi \frac{\partial}{\partial t} + \eta^\alpha \frac{\partial}{\partial q^\alpha}, \tag{1.134}$$

then $\mathbf{X}$ is the symmetry of the action and to each symmetry there exists a corresponding first integral defined by

$$\phi(t, q, q^\alpha) = g - \left[\xi L + (\eta^\alpha - \dot{q}^\alpha \xi) \frac{\partial L}{\partial \dot{q}^\alpha} \right], \tag{1.135}$$

and g is a function such that

$$\dot{g} = \xi\frac{\partial L}{\partial t} + \eta^{\alpha}\frac{\partial L}{\partial q^{\alpha}} + \zeta^{\alpha}\frac{\partial L}{\partial \dot{q}^{\alpha}} + \dot{\xi}L, \tag{1.136}$$

where $\zeta^{\alpha} = \dot{\eta}^{\alpha} - \dot{q}^{\alpha}\dot{\xi}$. The converse of Noether's theorem is possible but the level of computation is not easy as in that case our system also contain the function g. The computation of first integrals in case of Noether symmetries can be far different from computation of first integrals in case of Lie symmetries.

However, the extension of the Noether theorem for first order Lagrangian presented above, can be formulated in order to solve the systems of higher order. If we consider an nth-order Lagrangian, $L(t, q, \dot{q}, \ldots, q^{(n)})$, that depends on one independent variable and one dependent variable, then we can write the Euler-Lagrange equation as

$$\sum_{\alpha=0}^{n}(-1)^{\alpha}\frac{d^{\alpha}}{dt^{\alpha}}\left(\frac{\partial L}{\partial q^{(\alpha)}}\right), \qquad q^{(\alpha)} = \frac{d^{\alpha}q}{dt^{\alpha}}, \tag{1.137}$$

then $\mathbf{X} = \xi\partial_t + \eta\partial_q$ is the Noether symmetry if there exists a function g such that

$$\dot{g} = \dot{\xi}L + \xi\frac{\partial L}{\partial t} + \sum_{\alpha=0}^{n}(-1)^{\alpha}\zeta^{\alpha}\left(\frac{\partial L}{\partial q^{(\alpha)}}\right), \tag{1.138}$$

where

$$\zeta^{\alpha} = \eta^{(\alpha)} - \sum_{\beta=1}^{\alpha}\binom{\alpha}{\beta}q^{(\alpha+1-\beta)}\xi^{(\beta)}. \tag{1.139}$$

The corresponding first integral can be written as

$$\phi = g - \left[\xi L + \sum_{\alpha=0}^{n-1}\sum_{\beta=0}^{n-1-\alpha}(-1)^{\beta}(\eta - \dot{q}\xi)^{(\alpha)}\frac{d^{\beta}}{dt^{\beta}}\left(\frac{\partial L}{\partial q^{(\alpha+\beta+1)}}\right)\right]. \tag{1.140}$$

Now considering the case of an nth-order Lagrangian with one independent variable and m dependent variables, $L(t, q_{\alpha}, \dot{q}_{\alpha}, \ldots, q_{\alpha}^{(n)})$, $q_{\alpha}^{(n)} = d^{n}q_{\alpha}/dt^{n}$, $\alpha = 1, \ldots, m$, we write the Euler-Lagrange equation as

$$\sum_{\beta=0}^{n}(-1)^{\beta}\frac{d^{\beta}}{dt^{\beta}}\left(\frac{\partial L}{\partial q_{\alpha}^{(\beta)}}\right), \qquad \alpha = 1, \ldots, m. \tag{1.141}$$

then $X=\xi\partial_t+\sum_{\alpha=1}^{m}\eta_\alpha\partial_{q_\alpha}$ is said to be the Noether symmetry if there exists a function g such that

$$\dot{g}=\dot{\xi}L+\xi\frac{\partial L}{\partial t}+\sum_{\alpha=1}^{m}\sum_{\beta=0}^{n}(-1)^{\beta}\zeta_\alpha^\beta\left(\frac{\partial L}{\partial q_\alpha^{(\beta)}}\right), \tag{1.142}$$

where

$$\zeta_\alpha^\beta=\eta_\alpha^{(\beta)}-\sum_{\alpha=1}^{m}\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}q_\alpha^{(\beta+1-\gamma)}\xi^{(\gamma)}. \tag{1.143}$$

The corresponding first integral is given as

$$\phi=g-\left[\xi L+\sum_{\alpha=1}^{m}\sum_{\gamma=0}^{n-1}\sum_{\beta=0}^{n-1-\gamma}(-1)^{\beta}(\eta_\alpha-\dot{q}_\alpha\xi)^{(\gamma)}\frac{d^\beta}{dt^\beta}\left(\frac{\partial L}{\partial q_\alpha^{(\gamma+\beta+1)}}\right)\right]. \tag{1.144}$$

Finally, we mention a notable property [18] that enforces the relationship between a Noether symmetry $\mathbf{X}$ and its corresponding first integral ϕ: it can be proved that ϕ is itself a first order invariant of $\mathbf{X}$, i.e.

$$\mathbf{X}^{[1]}(\phi)=0. \tag{1.145}$$

Following examples elaborate the method of finding Noether symmetries and their corresponding first integrals.

Example

Consider a simple Lagrangian

$$L=\frac{1}{2}y'^2. \tag{1.146}$$

The criterion given in eq. (1.130) now reads

$$(\eta_{,x}-y'\xi_{,x})\,y'+\frac{1}{2}y'^2\xi_{,x}+\frac{1}{2}\xi_{,y}y'^3=g_{,x}+y'g_{,y}. \tag{1.147}$$

Comparing powers of y' yields the determining equations

$$\begin{aligned}
&(y'^0):g_{,x}=0,\\
&(y'^1):\eta_{,x}-g_{,y}=0,\\
&(y'^2):\frac{1}{2}\xi_{,x}-\eta_{,x}=0,\\
&(y'^3):\frac{1}{2}\xi_{,y}=0.
\end{aligned} \tag{1.148}$$

Solving this system of determining equations, we obtain

$$\begin{aligned} \xi &= a_1(x), \\ \eta &= \frac{1}{2}a_1'y + a_2(x), \\ g &= \frac{1}{4}a_1''^2 + a_2'y + a_3(x), \\ 0 &= \frac{1}{4}a_1'''^2 + a_2''y + a_3'. \end{aligned} \tag{1.149}$$

Now putting value of g in $g_{,x}$ and comparing yields values of a_1, a_2, a_3

$$\begin{aligned} a_1 &= c_1 + c_4x + c_5x^2, \\ a_2 &= c_2 + c_3x, \\ a_3 &= c_6. \end{aligned} \tag{1.150}$$

Hence the infinitesimal generator for Noether symmetries corresponding to the Lagrangian given by eq. (1.146) is obtained as

$$\mathbf{X} = \left[c_1 + c_4x + c_5x^2\right]\frac{\partial}{\partial x} + \left[\frac{1}{2}c_4y + c_5xy + c_2 + c_3x\right]\frac{\partial}{\partial y}. \tag{1.151}$$

For each $c_k = 0$ in eq. (1.151) we get five Noether symmetries. a_3 being a constant, is neglected. These symmetries and their corresponding first integrals are

$$\begin{aligned} \mathbf{X}_1 &= \frac{\partial}{\partial x} & \phi_1 &= -\frac{1}{2}y'^2, \\ \mathbf{X}_2 &= \frac{\partial}{\partial y} & \phi_2 &= -y', \\ \mathbf{X}_3 &= x\frac{\partial}{\partial y} & \phi_3 &= y - xy', \\ \mathbf{X}_4 &= x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} & \phi_4 &= -\frac{1}{2}y'(y - xy'), \\ \mathbf{X}_5 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} & \phi_5 &= \frac{1}{2}x^2y'^2 - xyy'. \end{aligned} \tag{1.152}$$

These five Noether symmetries form a Lie algebra

$$\begin{aligned} [\mathbf{X}_1, \mathbf{X}_2] &= 0, & [\mathbf{X}_1, \mathbf{X}_3] &= \mathbf{X}_2, \\ [\mathbf{X}_1, \mathbf{X}_4] &= \mathbf{X}_1, & [\mathbf{X}_1, \mathbf{X}_5] &= 2\mathbf{X}_4, \end{aligned}$$

$$
\begin{aligned}
&[\mathbf{X}_2, \mathbf{X}_3] = 0, && [\mathbf{X}_2, \mathbf{X}_4] = \frac{1}{2}\mathbf{X}_2, \\
&[\mathbf{X}_2, \mathbf{X}_5] = \mathbf{X}_3, && [\mathbf{X}_3, \mathbf{X}_4] = -\frac{1}{2}\mathbf{X}_3, \\
&[\mathbf{X}_3, \mathbf{X}_5] = 0, && [\mathbf{X}_4, \mathbf{X}_5] = \mathbf{X}_5.
\end{aligned} \tag{1.153}
$$

The equation of motion corresponding to this Lagrangian admits three other symmetries but they are not Noether symmetries. Noether symmetries for other systems can be studied in [21]-[24].

Example

Now consider a higher order Lagrangian

$$
L = \frac{1}{2}y''^2. \tag{1.154}
$$

The E-L equation for the Lagrangian is

$$
y^{(iv)} = 0. \tag{1.155}
$$

Applying Noether criterion on the above Lagrangian produce

$$
\begin{aligned}
&\Big[\eta_{,xx} + 2y'\eta_{,xy} + y'^2\eta_{,yy} + y''\eta_{,y} - 2y''\left(\xi_{,x} + y'\xi_{,y}\right) \\
&\quad - y'\left(\xi_{,xx} + 2y'\xi_{,xy} + y'^2\xi_{,yy} + y''\xi_{,y}\right)\Big]y'' + \frac{1}{2}\Big[\xi_{,x} + y'\xi_{,y}\Big]y''^2 \\
&\quad = g_{,x} + y'g_{,y} + y''g_{,y'}.
\end{aligned} \tag{1.156}
$$

Next by comparing the coefficients of powers of y' we obtain the system of determining equations

$$
\begin{aligned}
(y'^0) &: g_{,x} = 0. \\
(y') &: g_{,y} = 0, \\
(y'') &: \eta_{,xx} + 2y'\eta_{,xy} + y'^2\eta_{,yy} - y'\xi_{,xx} - g_{,y'} = 0, \\
(y''^2) &: \eta_{,y} - \frac{3}{2}\xi_{,x} = 0, \\
(y'y''^2) &: \xi_{,y} = 0.
\end{aligned}
$$

These determining equations yield

$$\begin{aligned}
\xi &= a_1(x),\\
\eta &= \frac{3}{2}a_1' y + a_2(x),\\
g &= a_1'' y'^2 + \left(\frac{3}{2}a_1''' y + a_2''\right) y' + a_3(x,y),\\
0 &= y' g_{,y} + g_{,x},
\end{aligned} \tag{1.157}$$

from these equations a_1, a_2, a_3 can be deduced

$$\begin{aligned}
a_1 &= c_1 + c_6 x + c_7 x^2,\\
a_2 &= c_4 + c_5 x + c_6 x^2 + c_7 x^3,\\
a_3 &= -a_2''' y + a_4,\\
a_4 &= c_8.
\end{aligned} \tag{1.158}$$

a_4 is a constant so we neglect it and the infinitesimal generator is obtained as

$$\mathbf{X} = \left[c_1 + c_6 x + c_7 x^2\right]\frac{\partial}{\partial x} + \left[\frac{3}{2}c_6 y + 3c_7 xy + c_2 + c_3 x + c_4 x^2 + c_5 x^3\right]\frac{\partial}{\partial y}, \tag{1.159}$$

and for each c_k in eq. (1.159) we get seven Noether symmetries. These symmetries and their corresponding first integrals are

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial x} & \phi_1 &= -y'y''' + \frac{1}{2}y''^2,\\
\mathbf{X}_2 &= \frac{\partial}{\partial y} & \phi_2 &= y''',\\
\mathbf{X}_3 &= x\frac{\partial}{\partial y} & \phi_3 &= xy''' - y'',\\
\mathbf{X}_4 &= x^2\frac{\partial}{\partial y} & \phi_4 &= x^2 y''' - 2xy'' + 2xy',\\
\mathbf{X}_5 &= x^3\frac{\partial}{\partial y} & \phi_5 &= x^3 y'''^2 y'' + 6xy' - 6y,\\
\mathbf{X}_6 &= x\frac{\partial}{\partial x} + \frac{3}{2}y\frac{\partial}{\partial y} & \phi_6 &= -xy'y + \frac{1}{2}xy''^2 - \frac{1}{2}y'y'' + \frac{3}{2}yy''',\\
\mathbf{X}_7 &= x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}, & \phi_7 &= x(3y - xy')y''' - (3y - xy' - \frac{1}{2}x^2 y'')y'' + 2y'^2.
\end{aligned} \tag{1.160}$$

and the Lie algebra of these symmetry generators is found as

$$
\begin{aligned}
&[\mathbf{X}_1, \mathbf{X}_2] = 0, && [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \\
&[\mathbf{X}_1, \mathbf{X}_4] = 2\mathbf{X}_3, && [\mathbf{X}_1, \mathbf{X}_5] = 3\mathbf{X}_4, \\
&[\mathbf{X}_1, \mathbf{X}_6] = \mathbf{X}_1, && [\mathbf{X}_1, \mathbf{X}_7] = 2\mathbf{X}_6, \\
&[\mathbf{X}_2, \mathbf{X}_3] = 0, && [\mathbf{X}_2, \mathbf{X}_4] = 0, \\
&[\mathbf{X}_2, \mathbf{X}_5] = 0, && [\mathbf{X}_2, \mathbf{X}_6] = \frac{3}{2}\mathbf{X}_2, \\
&[\mathbf{X}_2, \mathbf{X}_7] = 3\mathbf{X}_3, && [\mathbf{X}_3, \mathbf{X}_4] = 0, \\
&[\mathbf{X}_3, \mathbf{X}_5] = 0, && [\mathbf{X}_3, \mathbf{X}_6] = \frac{1}{2}\mathbf{X}_3, \\
&[\mathbf{X}_3, \mathbf{X}_7] = 2\mathbf{X}_4, && [\mathbf{X}_4, \mathbf{X}_5] = 0, \\
&[\mathbf{X}_4, \mathbf{X}_6] = -\frac{1}{2}\mathbf{X}_4, && [\mathbf{X}_4, \mathbf{X}_7] = \mathbf{X}_5, \\
&[\mathbf{X}_5, \mathbf{X}_6] = -\frac{3}{2}\mathbf{X}_5, && [\mathbf{X}_5, \mathbf{X}_7] = 0, \\
&[\mathbf{X}_6, \mathbf{X}_7] = \mathbf{X}_7.
\end{aligned}
\tag{1.161}
$$

If we find Lie point symmetries for the same Lagrangian we get $\mathbf{X}_8 = y\frac{\partial}{\partial y}$ in addition to the seven Noether symmetries presented in eq. (1.160). This shows the difference of Lie point symmetries and Noether point symmetries.

If we put the the required values in eq. (1.140) we get the expression for first integrals

$$\phi = g - \frac{1}{2}\xi y''^2 + (\eta - y'\xi)y''' - (\eta' - y''\xi - y'\xi')y''. \tag{1.162}$$

Hence we can find function g for each Noether symmetry which are

$$
\begin{aligned}
g_1 &= 0, \\
g_2 &= 0, \\
g_3 &= 0, \\
g_4 &= 2xy', \\
g_5 &= 6xy' - 6y,
\end{aligned}
\tag{1.163}
$$

$$g_6 = 0,$$

$$g_7 = 2y'^2.$$

At the end of this subsection, we present the Noether symmetries of a Lagrangian for the system of higher dimension [19].

Example

Consider the Lagrangian of higher order

$$L(t,x,y,\dot{x},\dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2). \tag{1.164}$$

The Euler-Lagrange equations given by eq. (1.141) for the said Lagrangian are

$$\ddot{x} = 0, \qquad \ddot{y} = 0. \tag{1.165}$$

The Noether criterion given by eq. (1.142) for Lagrangian written in eq. (1.164) gives

$$\begin{aligned} &\left[\eta^1_{,t} + \dot{x}\eta^1_{,x} + \dot{y}\eta^1_{,y} - \dot{x}\left(\xi_{,t} + \dot{x}\xi_{,x} + \dot{y}\xi_{,y}\right)\right]\dot{x} + \left[\eta^2_{,t} + \dot{x}\eta^2_{,x} + \dot{y}\eta^2_{,y}\right.\\ &\qquad \left. - \dot{y}\left(\xi_{,t} + \dot{x}\xi_{,x} + \dot{y}\xi_{,y}\right)\right]\dot{y} = g_{,t} + \dot{x}g_{,x} + \dot{y}g_{,y}. \end{aligned} \tag{1.166}$$

Comparing powers of $\dot{x}$ and $\dot{y}$ we get system of partial differential equations

$$\begin{aligned} (constant) &: g_{,t} = 0,\\ (\dot{x}) &: \eta^1_{,t} - g_{,x} = 0,\\ (\dot{y}) &: \eta^2_{,t} - g_{,y} = 0,\\ (\dot{x}^2) &: \eta^1_{,x} - \xi_{,t} = 0,\\ (\dot{y}^2) &: \eta^2_{,y} - \xi_{,t} = 0,\\ (\dot{x}^3) &: -\xi_{,x} = 0,\\ (\dot{y}^3) &: -\xi_{,y} = 0,\\ (\dot{x}\dot{y}) &: \eta^1_{,y} + \eta^2_{,x} = 0,\\ (\dot{x}^2\dot{y}) &: -\xi_{,y} = 0,\\ (\dot{x}\dot{y}^2) &: -\xi_{,x} = 0, \end{aligned} \tag{1.167}$$

From these determining equations we obtain

$$\begin{aligned}
\xi &= a_1(t), \\
\eta^1 &= \dot{a_1}x + a_5 y + a_6(t), \\
\eta^2 &= -a_5 x + \dot{a_1} y + a_4(t), \\
g &= \frac{1}{2}\ddot{a_1}x^2 + \dot{a_5}xy + \dot{a_6}x + a_7(y,t), \\
0 &= \frac{1}{2}\dddot{a_1}x^2 + \ddot{a_6}x + \frac{1}{2}\dddot{a_1}y^2 + \ddot{a_4}y + a_8(t).
\end{aligned} \tag{1.168}$$

and by solving eq. (1.168) we deduce

$$\begin{aligned}
a_1 &= c_1 + c_2 t + c_3 t^2, \\
a_4 &= c_7 + c_8 t, \\
a_5 &= c_4, \\
a_6 &= c_5 + c_6 t.
\end{aligned} \tag{1.169}$$

Hence, we write general infinitesimal generator as

$$\begin{aligned}
\Big[c_1 + c_2 t + c_3 t^2\Big]\frac{\partial}{\partial t} + \Big[c_2 x + 2c_3 t x + c_4 y + c_5 + c_6 t\Big]\frac{\partial}{\partial x} \\
+ \Big[-c_4 x + c_2 y + 2c_3 t y + c_7 + c_8 t\Big]\frac{\partial}{\partial y}.
\end{aligned} \tag{1.170}$$

From eq. (1.170), corresponding to each c_k, we found eight Noether symmetries for the Lagrangian given by eq. (1.164)

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial t}, & \mathbf{X}_2 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\
\mathbf{X}_3 &= t^2\frac{\partial}{\partial t} + 2t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), & \mathbf{X}_4 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\
\mathbf{X}_5 &= \frac{\partial}{\partial x}, & \mathbf{X}_6 &= t\frac{\partial}{\partial x}, \\
\mathbf{X}_7 &= \frac{\partial}{\partial y}, & \mathbf{X}_8 &= t\frac{\partial}{\partial y}.
\end{aligned} \tag{1.171}$$

Here, first three symmetry generators form a Lie algebra

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \qquad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \qquad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3. \tag{1.172}$$

and the Lie algebra obtained from last four symmetry generators is

$$\begin{aligned} &[\mathbf{X}_5, \mathbf{X}_6] = 0, && [\mathbf{X}_5, \mathbf{X}_7] = 0, \\ &[\mathbf{X}_5, \mathbf{X}_8] = 0, && [\mathbf{X}_6, \mathbf{X}_7] = 0, \\ &[\mathbf{X}_6, \mathbf{X}_8] = 0, && [\mathbf{X}_7, \mathbf{X}_8] = 0. \end{aligned} \tag{1.173}$$

1.5.2 Relation Between Lie and Noether Symmetries

Now we have gained enough knowledge about Lie point symmetries and Noether point symmetries that we can summarize how they are related. We quickly recall the definitions of Lie point symmetries and Noether symmetries. A Lie point symmetry of an ordinary differential equation (ODE) is a point transformation in the space of variables which preserves the set of solutions of the ODE. In another words Lie point symmetry is an invariance of the differential equations of motion under the point transformations however a Noether point symmetry can be defined as an invariance of action integral under the infinitesimal transformation of time t and generalized coordinates q^α. From all the previous discussion, we know that Noether symmetries are the subset of Lie symmetries and hence after finding the Lie symmetries for the corresponding Euler Lagrange equation, one can check if these specific Lie symmetries satisfy the Noether symmetry criterion given by eq. (1.129) or not, and if they do so then they are also Noether symmetries .

Now we show how one can construct a Lagrangian to find Noether symmetries by utilising the given symmetry and its corresponding first integral. To do this we use the relationship of a Lagrangian with the symmetry and its corresponding first integral. According to Noether, to every symmetry we can associate a first integral given by eq. (1.135). This equation relates first integral ϕ, symmetry $\mathbf{X}$, boundary term g and Lagrangian L [20]. If we consider the equation of motion

$$\ddot{q} = 0. \tag{1.174}$$

We can find the Lie symmetries for eq. (1.174) and also their first integrals using the method for Lie point symmetry. The symmetries and first integrals are

$$
\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial t}, & \phi_1 &= \dot{q} \\
\mathbf{X}_2 &= \frac{\partial}{\partial q}, & \phi_2 &= \dot{q} \\
\mathbf{X}_3 &= t\frac{\partial}{\partial t}, & \phi_3 &= t\dot{q} - q \\
\mathbf{X}_4 &= t\frac{\partial}{\partial q}, & \phi_4 &= t\dot{q} - q \\
\mathbf{X}_5 &= t^2\frac{\partial}{\partial t} + tq\frac{\partial}{\partial q}, & \phi_5 &= t\dot{q} - q \\
\mathbf{X}_6 &= q\frac{\partial}{\partial t}, & \phi_6 &= \frac{t\dot{q} - q}{\dot{q}} \\
\mathbf{X}_7 &= q\frac{\partial}{\partial q}, & \phi_7 &= \frac{t\dot{q} - q}{\dot{q}} \\
\mathbf{X}_8 &= tq\frac{\partial}{\partial t} + q^2\frac{\partial}{\partial q}, & \phi_8 &= \frac{t\dot{q} - q}{\dot{q}}.
\end{aligned}
\tag{1.175}
$$

The Lie algebras corresponding to the symmetry generators given in eq. (1.175) are

$$
\begin{aligned}
[\mathbf{X}_1, \mathbf{X}_2] &= 0, & [\mathbf{X}_1, \mathbf{X}_3] &= \mathbf{X}_1, \\
[\mathbf{X}_1, \mathbf{X}_4] &= \mathbf{X}_2, & [\mathbf{X}_1, \mathbf{X}_5] &= 2\mathbf{X}_3 + \mathbf{X}_7, \\
[\mathbf{X}_1, \mathbf{X}_6] &= 0, & [\mathbf{X}_1, \mathbf{X}_7] &= 0, \\
[\mathbf{X}_1, \mathbf{X}_8] &= \mathbf{X}_6, & [\mathbf{X}_2, \mathbf{X}_3] &= 0, \\
[\mathbf{X}_2, \mathbf{X}_4] &= 0, & [\mathbf{X}_2, \mathbf{X}_5] &= \mathbf{X}_4, \\
[\mathbf{X}_2, \mathbf{X}_6] &= \mathbf{X}_1, & [\mathbf{X}_2, \mathbf{X}_7] &= \mathbf{X}_2, \\
[\mathbf{X}_2, \mathbf{X}_8] &= \mathbf{X}_3 + 2\mathbf{X}_7, & [\mathbf{X}_3, \mathbf{X}_4] &= \mathbf{X}_4, \\
[\mathbf{X}_3, \mathbf{X}_5] &= \mathbf{X}_5, & [\mathbf{X}_3, \mathbf{X}_6] &= -\mathbf{X}_6, \\
[\mathbf{X}_3, \mathbf{X}_7] &= 0, & [\mathbf{X}_3, \mathbf{X}_8] &= 0, \\
[\mathbf{X}_4, \mathbf{X}_5] &= 0, & [\mathbf{X}_4, \mathbf{X}_6] &= \mathbf{X}_3 - \mathbf{X}_7, \\
[\mathbf{X}_4, \mathbf{X}_7] &= \mathbf{X}_4, & [\mathbf{X}_4, \mathbf{X}_8] &= \mathbf{X}_5,
\end{aligned}
\tag{1.176}
$$

$$[\mathbf{X}_5, \mathbf{X}_6] = -\mathbf{X}_8, \qquad [\mathbf{X}_5, \mathbf{X}_7] = 0,$$
$$[\mathbf{X}_5, \mathbf{X}_8] = 0, \qquad [\mathbf{X}_6, \mathbf{X}_7] = -\mathbf{X}_6,$$
$$[\mathbf{X}_6, \mathbf{X}_8] = 0, \qquad [\mathbf{X}_7, \mathbf{X}_8] = \mathbf{X}_8.$$

If we consider the Lagrangian for the equation (1.174)

$$L = \frac{1}{2}\dot{q}^2, \tag{1.177}$$

then we can find Noether symmetries for this Lagrangian. The Noether symmetries and the corresponding first integrals for this Lagrangian are

$$\begin{aligned}
\mathbf{N}_1 &= \frac{\partial}{\partial t}, & \phi_{\mathbf{N}_1} &= -\frac{1}{2}\dot{q}^2 \\
\mathbf{N}_2 &= \frac{\partial}{\partial q}, & \phi_{\mathbf{N}_2} &= -\dot{q} \\
\mathbf{N}_3 &= t\frac{\partial}{\partial q}, & \phi_{\mathbf{N}_3} &= q - t\dot{q} \\
\mathbf{N}_4 &= t^2\frac{\partial}{\partial t} + tq\frac{\partial}{\partial q}, & \phi_{\mathbf{N}_4} &= (q - t\dot{q})^2 \\
\mathbf{N}_5 &= tq\frac{\partial}{\partial t} + q^2\frac{\partial}{\partial q}, & \phi_{\mathbf{N}_5} &= \dot{q}(q - t\dot{q}).
\end{aligned} \tag{1.178}$$

The coefficients of $\mathbf{X}_1$ and $\mathbf{X}_2$ are the two solutions of eq. (1.174) so they are called solution symmetries. One can quickly observe that the three Noether symmetries as well as their corresponding integrals are not present in contrast of Lie point symmetries $\mathbf{X}_6$, $\mathbf{X}_7$, $\mathbf{X}_8$. These are actually the Lie point symmetries which are not Noether symmetries (non Noetherian symmetries).

If we solve eq. (1.136) for $\mathbf{X}_6$, by putting all required values, assuming $g = 0$, we get partial differential equation

$$q\frac{\partial L}{\partial \dot{q}} = -\frac{t\dot{q} - q}{\dot{q}}, \tag{1.179}$$

solving this and using eq. (1.174) we deduce a Lagrangian corresponding to $\mathbf{X}_6$

$$L = \ln\frac{\dot{q}}{q} - \frac{\dot{q}t}{q} + s(t). \tag{1.180}$$

Similarly for $\mathbf{X}_7$ eq. (1.136) becomes

$$qL + (-q\dot{q})\frac{\partial L}{\partial \dot{q}} = -\frac{t\dot{q} - q}{\dot{q}}, \tag{1.181}$$

solving this we get the Lagrangian corresponding to $\mathbf{X}_7$

$$L = \frac{(t\dot{q} - q)^2}{2\dot{q}q^2} + \dot{q}s(q). \tag{1.182}$$

For $\mathbf{X}_8$ eq. (1.136) reads

$$qtL + (q^2 - qt)\frac{\partial L}{\partial \dot{q}} = -\frac{t\dot{q} - q}{q}, \tag{1.183}$$

and follows from which the Lagrangian for $\mathbf{X}_8$

$$L = (t\dot{q} - q)\left[\frac{1}{q^2}\ln(t\dot{q} - q) - \ln(\dot{q}) + \frac{s(q)}{t^2}\right], \tag{1.184}$$

where s is an arbitrary function. Hence, we can say that these three symmetries $(\mathbf{X}_6, \mathbf{X}_7, \mathbf{X}_8)$ are the Noether symmetries for these particular Lagrangians given by equations (1.180), (1.182), (1.184). It means if we consider appropriate Lagrangian the symmetries can fulfil the criterion for Noether symmetries establishing the fact that all the symmetries of a given ODE for one specifically given Lagrangian need not to be Noether symmetries. But by using eq. (1.136) for the obtained Lie symmetry and its corresponding first integral one can obviously find an appropriate Lagrangian for which the symmetry is a Noether symmetry. Another symmetry of interest is the Mei symmetry.

1.5.3 The Mei symmetries

In 2000, F.X. Mei proposed a new kind of symmetry called as Mei symmetry or the form invariance [25]. Mei symmetry can be defined as the form invariance of differential equations of motion when the transformed functions replace dynamical functions (such as Lagrangian, Birkhoffian, Hamiltonian etc.) under the infinitesimal transformations.

Also by Noether's theorem we know that symmetries lead to the first integrals. This is also true for Mei symmetries as they also provide first integrals known as Mei conserved quantities. The Lie symmetry method and the Noether symmetry method have grown so much over time and been used in handling various problems. On the other hand, so much work and research on the Mei symmetries is still undone and they are still on their way to be applied on various problems. Our main purpose is to find Mei symmetries as well as Noether symmetries for a particular Lagrangian later presented in Section 2.

First we should built the definition and criterion of Mei symmetries to be able to find them.

Suppose we have a Lagrangian

$$L = L(t, q^\alpha, \dot{q}^\alpha). \tag{1.185}$$

Consider the one-parameter group of infinitesimal transformations

$$\begin{aligned} \hat{t} &= t + \varepsilon\xi(t, q^\beta), \\ \hat{q}^\alpha &= q^\alpha + \varepsilon\eta^\alpha(t, q^\beta), \end{aligned} \tag{1.186}$$

where α,$\beta = 1, \ldots, n$ and $\varepsilon \in R$. The corresponding infinitesimal generator is

$$\mathbf{X} = \xi\frac{\partial}{\partial t} + \eta^\alpha\frac{\partial}{\partial q^\alpha}. \tag{1.187}$$

The Lagrangian from eq. (1.185) under the transformations given by eq. (1.186) becomes

$$\begin{aligned} \hat{L} &= L(\hat{t}, \hat{q}^\alpha, \dot{\hat{q}}^\alpha), \\ &= L\left(t + \varepsilon\xi, q^\alpha + \varepsilon\eta^\alpha, \frac{\dot{q}^\alpha + \varepsilon\dot{\eta}^\alpha}{1 + \varepsilon\dot{\xi}}\right). \end{aligned} \tag{1.188}$$

Taylor series expansion of eq. (1.188) about $\varepsilon = 0$ gives

$$\hat{L} = L(t, q^\alpha, \dot{q}^\alpha) + \varepsilon\mathbf{X}^{(1)}(L) + O(\varepsilon^2), \tag{1.189}$$

where

$$\mathbf{X}^{(1)} = \xi\frac{\partial}{\partial t} + \eta^\alpha\frac{\partial}{\partial q^\alpha} + (\dot{\eta}^\alpha - \dot{\xi}\dot{q}^\alpha)\frac{\partial}{\partial \dot{q}^\alpha}, \tag{1.190}$$

is the first prolongation of infinitesimal generator $\mathbf{X}$.

Writing Euler Lagrange equation as

$$\mathbf{E}_\alpha(L) = 0, \tag{1.191}$$

where $\mathbf{E}_\alpha$ is the Euler operator

$$\mathbf{E}_\alpha = \frac{d}{dt}\frac{\partial}{\partial \dot{q}^\alpha} - \frac{\partial}{\partial q^\alpha}. \tag{1.192}$$

If the eq. (1.191) remains the same when the new Lagrangian $\hat{L}$ from eq. (1.189) is substituted in place of Lagrangian L, i.e.

$$\mathbf{E}_\alpha(\hat{L}) = 0, \tag{1.193}$$

then this invariance is known as the Mei symmetry of Euler Lagrange equation. Hence we can present the criterion to find Mei symmetries as [26]-[28]

Criterion

If the infinitesimals ξ and η satisfy

$$\mathbf{E}_\alpha[\mathbf{X}^{(1)}(L)] = 0, \qquad \alpha = 1, \ldots, n, \tag{1.194}$$

then the corresponding invariance is the Mei symmetry for the Lagrangian in eq. (1.185).

Before we use this criterion to find Mei symmetries we should seek the relation between Mei symmetries and Noether symmetries as it is of great significance in finding Mei conserved quantities and Noether conserved quantities.

1.5.4 Relation Between Noether and Mei Symmetries

Firstly we present an important theorem [26].

Theorem

If the infinitesimals ξ and η^α of the Mei symmetry corresponding to the Lagrangian given by eq. (1.185) and the boundary function $g(t,q^\alpha,\dot{q}^\alpha)$ admits the structural equation

$$\mathbf{X}^{(1)}(L)\dot{\xi}+\mathbf{X}^{(1)}\Big(\mathbf{X}^{(1)}(L)\Big)+z(t)\frac{\partial\mathbf{X}^{(1)}(L)}{\partial q^\alpha}\dot{q}^\alpha\dot{\xi}+\dot{g}=0, \tag{1.195}$$

then the Mei symmetry can lead to new conserved quantity

$$\phi_1=\frac{\partial\mathbf{X}^{(1)}(L)}{\partial\dot{q}}\eta^\alpha+\left(\mathbf{X}^{(1)}(L)-\frac{\partial\mathbf{X}^{(1)}(L)}{\partial\dot{q}}\dot{q}-z(t)\frac{\partial\mathbf{X}^{(1)}(L)}{\partial t}\right)\xi+g. \tag{1.196}$$

This theorem help us build a relation between Noether and Mei symmetries. If we consider integral functional $\mathbf{S}$(q)

$$\mathbf{S}(q)=\int_{t_1}^{t_2}\mathbf{X}^{(1)}(L)\left(L(t,q^\alpha(t),\dot{q}^\alpha(t))\right)dt, \tag{1.197}$$

admitting boundary conditions $q^\alpha(t)\mid_{t=a}=q^\alpha(a)$ and $q^\alpha(t)\mid_{t=b}=q^\alpha(b)$ where $\alpha=1,\ldots,n$.

Euler Lagrange equations for eq. (1.197) can be deduced that has the same form as eq. (1.194). Also we know that Noether symmetry refers to invariance of action integral so if

$$\hat{\mathbf{S}}(\hat{q})=\mathbf{S}(q), \tag{1.198}$$

remains true under infinitesimal transformations given by eq. (1.186) then the invariance is known as Noether symmetry. For ξ and η there exists a boundary function $g(t,q^\alpha,\dot{q}^\alpha)$ such that

$$\begin{aligned}\frac{\partial\mathbf{X}^{(1)}(L)}{\partial t}\xi+\frac{\partial\mathbf{X}^{(1)}(L)}{\partial q^\alpha}\eta^\alpha+\frac{\partial\mathbf{X}^{(1)}(L)}{\partial\dot{q}^\alpha}(\dot{\eta}^\alpha-\dot{q}^\alpha\dot{\xi})\\ +\mathbf{X}^{(1)}L\dot{\xi}=-\dot{g}.\end{aligned} \tag{1.199}$$

We obtain same equation as eq. (1.195) and it is known as Noether identity for problem given by eq. (1.197). From this we can deduce Noether first integral or Noether conserved quantity which is same as eq. (1.196). For detailed discussion one may refer to [29].

Chapter 2

The Mei Symmetries for the Lagrangian of Schwarzschild Metric

Before finding the Mei symmetries for Schwarzschild metric we should have some knowledge about Schwarzchild metric.

2.1 The Schwarzschild Metric

In 1916, a German physicist Karl Schwarzschild gave the fist exact solution of Einstein field equations of general relativity which is known as Schwarzschild metric or Schwarzchild solution. It actually depicts the gravitational field outside a spherical mass, provided that the angular momentum of the mass, the electric charge of the mass and the universal cosmological constant are all zero.
In the Schwarzschild coordinates (t, r, θ, ϕ), with the signature convention $(-, +, +, +)$, Schwarzschild metric has the form

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.0)$$

where r_s is the Schwarzschild radius of the massive body defined as $r_s = \frac{2GM}{c^2}$, G is gravitational constant, c is the speed of light, t is the time coordinate, r is the radial coordinate, θ is the colatitude of a point on 2-sphere, ϕ is the longitude of a point on

2-sphere.

The Schwarzschild solution is considered to be the most general static and spherically symmetric solution of the Einstein field equations. Being static metric corresponds to the time independence of the metric and the invariance of line element under the transformation of time like coordinates, say $x^0 \to -x^0$. While spherically symmetric means metric has no preferable angular direction say, $dx^\alpha \to -dx^\alpha$ is possible without changing the form of metric, where x^α are spatial coordinates. One may study [30] to have some in-depth knowledge about Schwarzschild solution.

2.2 Review of the Noether Symmetries for the Lagrangian of Schwarzschild Metric

Ibrar Hussain, Fazal M. Mahomed and Asghar Qadir. [31]:

Writing the Lagrangian for Schwarzschild metric by

$$\mathbf{L} = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2. \tag{2.1}$$

The criterion for Noether symmetries is given as

$$\mathbf{X}^{(1)}(L) + (\mathbf{A}\xi)L = \mathbf{A}g, \tag{2.2}$$

where $\mathbf{X}^1 = \xi\partial s + \eta^\alpha\partial q^\alpha + \left(\dot{\eta}^\alpha - \dot{q}^\alpha\dot{\xi}\right)\partial\dot{q}^\alpha$ is the first extended generator, $\mathbf{A} = \partial s + \dot{q}^\alpha\partial q^\alpha$ is a linear operator and L is the Lagrangian.

By putting the required values in eq. (2.2) we get

$$\eta^2\left[-\frac{2m\dot{t}^2}{r^2} - 2\left(1 - \frac{2m}{r}\right)^{-2}\frac{m\dot{r}^2}{r^2} + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\right] + 2\eta^3r^2\sin\theta\cos\theta\dot{\phi}^2$$
$$+\left[\dot{\eta}^1 - \dot{t}\dot{\xi}\right]\left[-2\left(1 - \frac{2m}{r}\right)\dot{t}\right] + \left[\dot{\eta}^2 - \dot{r}\dot{\xi}\right]\left[2\left(1 - \frac{2m}{r}\right)^{-1}\dot{r}\right] + \left[\dot{\eta}^3 - \dot{\theta}\dot{\xi}\right]\left[2r^2\dot{\theta}\right]$$
$$+\left[\dot{\eta}^4 - \dot{\phi}\dot{\xi}\right]\left[2r^2\sin^2\theta\dot{\phi}\right] + \xi_{,s}\left[-\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2\right.$$
$$\left. + r^2\sin^2\theta\dot{\phi}^2\right] + \dot{t}\xi_{,t}\left[-\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right]$$

$$
\begin{aligned}
&+\dot{r}\xi_{,r}\left[-\left(1-\frac{2m}{r}\right)\dot{t}^2+\left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right]\\
&+\dot{\theta}\xi_{,\theta}\left[-\left(1-\frac{2m}{r}\right)\dot{t}^2+\left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right]\\
&+\dot{\phi}\xi_{,\phi}\left[-\left(1-\frac{2m}{r}\right)\dot{t}^2+\left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right]\\
&=g_{,s}+\dot{t}g_{,t}+\dot{r}g_{,r}+\dot{\theta}g_{,\theta}+\dot{\phi}g_{,\phi}.
\end{aligned}
\tag{2.3}
$$

After solving eq. (2.3) and comparing powers of $\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}$ following determining equations are obtained

$$(constant) : g_{,s} = 0, \tag{2.4a}$$

$$(\dot{t}) : 2\left(1-\frac{2m}{r}\right)\eta^1_{,s}+g_{,t}=0, \tag{2.4b}$$

$$(\dot{r}) : 2\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,s}-g_{,r}=0, \tag{2.4c}$$

$$(\dot{\theta}) : 2r^2\eta^3_{,s}-g_{,\theta}=0, \tag{2.4d}$$

$$(\dot{\phi}) : 2r^2\sin^2\theta\eta^4_{,s}-g_{,\phi}=0, \tag{2.4e}$$

$$(\dot{t}^2) : 2\left(1-\frac{2m}{r}\right)\eta^1_{,t}+\frac{2m}{r^2}\eta^2-\left(1-\frac{2m}{r}\right)\xi_{,s}=0, \tag{2.4f}$$

$$(\dot{r}^2) : 2\left(1-\frac{2m}{r}\right)^{-1}\frac{m}{r^2}\eta^2-2\eta^2_{,r}+\xi_{,s}=0, \tag{2.4g}$$

$$(\dot{\theta}^2) : 2\eta^2+2r\eta^3_{,\theta}-r\xi_{,s}=0, \tag{2.4h}$$

$$(\dot{\phi}^2) : 2\eta^2+2r\cot\theta\eta^3+2r\eta^4_{,\phi}-r\xi_{,s}=0, \tag{2.4i}$$

$$(\dot{t}\dot{r}) : \left(1-\frac{2m}{r}\right)^2\eta^1_{,r}-\eta^2_{,t}=0, \tag{2.4j}$$

$$(\dot{t}\dot{\theta}) : \left(1-\frac{2m}{r}\right)\eta^1_{,\theta}-r^2\eta^3_{,\theta}=0, \tag{2.4k}$$

$$(\dot{t}\dot{\phi}) : \left(1-\frac{2m}{r}\right)\eta^1_{,\phi}-r^2\sin^2\theta\eta^4_{,t}=0, \tag{2.4l}$$

$$(\dot{r}\dot{\theta}) : \left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,\theta}+r^2\eta^3_{,r}=0, \tag{2.4m}$$

$$(\dot{r}\dot{\phi}) : \left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,\phi}+r^2\sin^2\theta\eta^4_{,r}=0, \tag{2.4n}$$

$$(\dot{\theta}\dot{\phi}) : \eta^3_{,\phi} + \sin^2\theta \eta^4_{,\theta} = 0, \tag{2.4o}$$

$$(\dot{t}^3) : \xi_{,t} = 0, \tag{2.4p}$$

$$(\dot{r}^3) : \xi_{,r} = 0, \tag{2.4q}$$

$$(\dot{\theta}^3) : \xi_{,\theta} = 0, \tag{2.4r}$$

$$(\dot{\phi}^3) : \xi_{,\phi} = 0, \tag{2.4s}$$

Now we solve this system of partial differential equations to get values of $\xi, \eta^1, \eta^2, \eta^3, \eta^4$. From the equations (2.4p)-(2.4s) we can easily deduce that ξ is a function of s only i.e

$$\xi = \xi(s). \tag{2.5}$$

Differentiating eq. (2.4b) w.r.t s and using eq. (2.4a) yields

$$\eta^1_{,ss} = 0. \tag{2.6}$$

Differentiating eq. (2.4c) w.r.t s and using eq. (2.4a) yields

$$\eta^2_{,ss} = 0. \tag{2.7}$$

Differentiating eq. (2.4d) w.r.t s and using eq. (2.4a) yields

$$\eta^3_{,ss} = 0. \tag{2.8}$$

Differentiating eq. (2.4e) w.r.t s and using eq. (2.4a) yields

$$\eta^4_{,ss} = 0. \tag{2.9}$$

Now by taking derivative of eq. (2.4f) two times w.r.t s we get

$$\xi_{,sss} = 0, \tag{2.10}$$

from which, by using eq. (2.5), we can deduce

$$\xi = c_1 s^2 + c_2 s + c_3, \tag{2.11}$$

where c_k are some arbitrary constants.

Now differentiating eq. (2.4b) w.r.t r gives

$$g_{,tr} = 2\left(1-\frac{2m}{r}\right)\eta^1_{,sr} + \frac{4m}{r^2}\eta^1_{,s}, \tag{2.12}$$

and differentiating eq. (2.4c) w.r.t t gives

$$g_{,tr} = 2\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,st}. \tag{2.13}$$

By comparing eq. (2.12) and eq. (2.13) we obtain

$$2\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,st} + 2\left(1-\frac{2m}{r}\right)\eta^1_{,sr} + \frac{4m}{r^2}\eta^1_{,s} = 0. \tag{2.14}$$

By differentiating eq. (2.4j) w.r.t s and manipulating we get

$$2\left(1-\frac{2m}{r}\right)\eta^1_{,sr} = 2\left(1-\frac{2m}{r}\right)^{,-1}\eta^2_{,st} = 0. \tag{2.15}$$

Putting eq. (2.15) in eq. (2.14) produce

$$\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,st} + \frac{m}{r^2}\eta^1_{,s} = 0. \tag{2.16}$$

If we differentiate eq. (2.4b) w.r.t θ, it reads

$$g_{,t\theta} = -2\left(1-\frac{2m}{r}\right)\eta^1_{,s\theta}. \tag{2.17}$$

and if we differentiate eq. (2.4d) w.r.t t, we get

$$g_{,t\theta} = 2r^2\eta^3_{,st}. \tag{2.18}$$

By comparing eq. (2.17) and eq. (2.18) we obtain

$$\left(1-\frac{2m}{r}\right)\eta^1_{,s\theta} + r^2\eta^3_{,st} = 0. \tag{2.19}$$

If we differentiate eq. (2.4k) w.r.t s, it becomes

$$\left(1-\frac{2m}{r}\right)\eta^1_{,s\theta} = r^2\eta^3_{,st}. \tag{2.20}$$

Substituting value of $\eta^3_{,st}$ from eq. (2.20) into eq. (2.19) gives

$$\eta^1_{,s\theta} = 0. \tag{2.21}$$

Now differentiating eq. (2.4b) w.r.t ϕ produce

$$g_{,t\phi} = -\left(1 - \frac{2m}{r}\right)\eta^1_{,s\phi}. \tag{2.22}$$

and differentiating eq. (2.4e) w.r.t t produce

$$g_{,t\phi} = r^2 \sin^2\theta \eta^4_{,st}. \tag{2.23}$$

Equating eq. (2.22) and eq. (2.23) leads to

$$\left(1 - \frac{2m}{r}\right)\eta^1_{,s\phi} + r^2\sin^2\theta\eta^4_{,st} = 0. \tag{2.24}$$

Also by differentiating eq. (2.4l) w.r.t s we get

$$r^2\sin^2\theta\eta^4_{,st} = \left(1 - \frac{2m}{r}\right)\eta^1_{,s\phi}. \tag{2.25}$$

Substituting eq. (2.25) into eq. (2.24) yields

$$\eta^1_{,s\phi} = 0. \tag{2.26}$$

Differentiating eq. (2.4c) w.r.t θ gives

$$g_{,r\theta} = 2\left(1 - \frac{2m}{r}\right)^{-1}\eta^2_{,s\theta}. \tag{2.27}$$

Differentiating eq. (2.4d) w.r.t r gives

$$g_{,r\theta} = 4r\eta^3_{,s} + 2r^2\eta^3_{,sr}. \tag{2.28}$$

Equating eq. (2.27) and eq. (2.28) produce

$$\left(1 - \frac{2m}{r}\right)^{-1}\eta^2_{,s\theta} - r^2\eta^3_{,sr} - 2r\eta^3_{,s} = 0. \tag{2.29}$$

If we differentiate eq. (2.4m) w.r.t s, we get

$$r^2\eta^3_{,sr} = -\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,s\theta}. \tag{2.30}$$

Putting eq. (2.30) into eq. (2.29), we obtain

$$\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,s\theta} - r\eta^3_{,s} = 0, \tag{2.31}$$

The derivative of eq. (2.4c) w.r.t ϕ is

$$g_{,r\phi} = 2\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,s\phi}. \tag{2.32}$$

and derivative of eq. (2.4e) w.r.t r is

$$g_{r\phi} = 4r\sin^2\theta\eta^4_{,s} + 2r^2\sin^2\theta\eta^4_{,sr}. \tag{2.33}$$

Comparing both equations given by eq. (2.32) and eq. (2.33) gives

$$\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,s\phi} - 2r\sin^2\theta\eta^4_{,s} - r^2\sin^2\theta\eta^4_{,sr} = 0. \tag{2.34}$$

By taking derivative of eq. (2.4n) w.r.t s, it becomes

$$\left(1-\frac{2m}{r}\right)\eta^2_{,s\phi} = -r^2\sin^2\theta\eta^4_{,sr}. \tag{2.35}$$

Putting eq. (2.35) into eq. (2.34) leads to

$$\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,s\phi} - r\sin^2\theta\eta^4_{,s} = 0. \tag{2.36}$$

Now differentiating eq. (2.4f) w.r.t s produce

$$\frac{2m}{r^2}\eta^2_{,s} + 2\left(1-\frac{2m}{r}\right)\eta^1_{,st} - \left(1-\frac{2m}{r}\right)\xi_{,ss} = 0. \tag{2.37}$$

Differentiating eq. (2.37) w.r.t θ and using eq. (2.4r) eq. (2.21) yields

$$\eta^2_{,s\theta} = 0. \tag{2.38}$$

Similarly differentiating eq. (2.37) w.r.t ϕ and using eq. (2.4s) and eq. (2.26) yields

$$\eta^2_{,s\phi} = 0. \tag{2.39}$$

Substituting eq. (2.38) in eq. (2.31) gives

$$\eta^3_{,s} = 0, \tag{2.40}$$

and substituting eq. (2.39) in eq. (2.36) gives

$$\eta^4_{,s} = 0. \tag{2.41}$$

If we differentiate eq. (2.4h) w.r.t s and utilise eq. (2.40), it becomes

$$2\eta^2_{,s} - r\xi_{,ss} = 0, \tag{2.42}$$

and differentiating eq. (2.42) w.r.t t and utilising eq. (2.4p), we get

$$\eta^2_{,st} = 0. \tag{2.43}$$

If we put eq. (2.43) in eq. (2.16) it becomes

$$\eta^1_{,s} = 0. \tag{2.44}$$

Put value of $\eta^2_{,s}$ from eq. (2.42) and $\eta^1_{,s}$ from eq. (2.44) in eq. (2.37) to get

$$\xi_{,ss} = 0, \tag{2.45}$$

which means c_1 in eq. (2.11) is zero and hence (2.11) becomes

$$\xi = c_2 s + c_3. \tag{2.46}$$

By means of eq. (2.45), eq. (2.42) gives

$$\eta^2_{,s} = 0. \tag{2.47}$$

By putting equations (2.44), (2.47) (2.40) and (2.41), in equations (2.4b), (2.4c), (2.4d) and (2.4e), respectively, we obtain

$$\begin{aligned} g_{,t} &= 0, & g_{,r} &= 0, \\ g_{,\theta} &= 0, & g_{,\phi} &= 0, \end{aligned} \tag{2.48}$$

and by considering eq. (2.4a) we can say g is only a constant so we write

$$g = c_4. \tag{2.49}$$

We differentiate eq. (2.4j) w.r.t r to get

$$4\left(1-\frac{2m}{r}\right)\frac{m}{r^2}\eta^1_{,r} + \left(1-\frac{2m}{r}\right)^2 \eta^1_{,rr} - \eta^2_{,tr} = 0, \tag{2.50}$$

and differentiate eq. (2.4g) w.r.t t use eq. (2.4p) to get

$$\eta^2_{,tr} = \left(1-\frac{2m}{r}\right)^{-1}\frac{m}{r^2}\eta^2_{,t}. \tag{2.51}$$

Putting eq. (2.51) in (2.50) and manipulating gives

$$4\left(1-\frac{2m}{r}\right)^2 \eta^1_{,r} + \frac{r^2}{m}\left(1-\frac{2m}{r}\right)^3 \eta^1_{,rr} - \eta^2_{,t} = 0. \tag{2.52}$$

Comparing eq. (2.4j) and (2.52) to produce

$$\left(1-\frac{2m}{r}\right)\eta^1_{,rr} + \frac{3m}{r^2}\eta^1_{,r} = 0. \tag{2.53}$$

Now we differentiate eq. (2.4k) w.r.t r

$$\left(1-\frac{2m}{r}\right)\eta^1_{,r\theta} + \frac{2m}{r^2}\eta^1_{,\theta} - 2r\eta^3_{,t} - r^2\eta^3_{,tr} = 0, \tag{2.54}$$

also derivative of eq. (2.4j) w.r.t θ is

$$\left(1-\frac{2m}{r}\right)^2 \eta^1_{,r\theta} - \eta^2_{,t\theta}. \tag{2.55}$$

By differentiating eq. (2.4m) w.r.t t

$$\eta^2_{,t\theta} = -\left(1-\frac{2m}{r}\right) r^2\eta^3_{,tr}. \tag{2.56}$$

and putting back in eq. (2.55) yields

$$r^2\eta^3_{,tr} = -\left(1-\frac{2m}{r}\right)\eta^1_{,r\theta}. \tag{2.57}$$

Using eq. (2.57) and comparing with eq. (2.4k) produce

$$\left(1-\frac{2m}{r}\right)\eta^1_{,r\theta} - \frac{1}{r}\left(1-\frac{3m}{r}\right)\eta^1_{,\theta} = 0, \tag{2.58}$$

Now taking derivative of eq. (2.4l) w.r.t r gives

$$\left(1-\frac{2m}{r}\right)\eta^1_{,r\phi} + \frac{2m}{r^2}\eta^1_{,\phi} - 2r\sin^2\theta\eta^4_{,t} - r^2\sin^2\theta\eta^4_{,tr} = 0. \tag{2.59}$$

and the derivative of eq. (2.4j) w.r.t ϕ is

$$\left(1-\frac{2m}{r}\right)^2\eta^1_{,r\phi} - \eta^2_{,t\phi} = 0. \tag{2.60}$$

By differentiating eq. (2.4n) w.r.t t

$$\eta^2_{,t\phi} = -\left(1-\frac{2m}{r}\right)r^2\sin^2\theta\eta^4_{,tr}. \tag{2.61}$$

and putting back into eq. (2.60) gives

$$r^2\sin^2\theta\eta^4_{,tr} = -\left(1-\frac{2m}{r}\right)\eta^1_{,r\phi}. \tag{2.62}$$

Using eq. (2.62) and equating with eq. (2.4l) produce

$$\left(1-\frac{2m}{r}\right)\eta^1_{,r\phi} - \frac{1}{r}\left(1-\frac{3m}{r}\right)\eta^1_{,\phi} = 0. \tag{2.63}$$

We differentiate eq. (2.58) w.r.t r and get

$$\left(1-\frac{2m}{r}\right)\eta^1_{,rr\theta} + \frac{2m}{r^2}\eta^1_{,r\theta} + \frac{1}{r^2}\eta^1_{,\theta} - \frac{6m}{r^3}\eta^1_{,\theta} - \frac{1}{r}\left(1-\frac{3m}{r}\right)\eta^1_{,r\theta} = 0. \tag{2.64}$$

Differentiating eq. (2.53) w.r.t θ and putting value of $\eta^1_{,rr\theta}$ into eq. (2.64) gives

$$\left(1-\frac{2m}{r}\right)\eta^1_{,r\theta} - \frac{1}{r}\left(1-\frac{6m}{r}\right)\eta^1_{,\theta} = 0. \tag{2.65}$$

Comparing eq. (2.65) with eq. (2.58) yields

$$\eta^1_{,\theta} = 0. \tag{2.66}$$

Similarly taking derivative of eq. (2.63) w.r.t r gives

$$\left(1 - \frac{2m}{r}\right)\eta^1_{,rr\phi} + \frac{2m}{r^2}\eta^1_{,r\phi} + \frac{1}{r^2}\eta^1_{,\phi} - \frac{6m}{r^3}\eta^1_{,\phi} - \frac{1}{r}\left(1 - \frac{3m}{r}\right)\eta^1_{,r\phi} = 0. \tag{2.67}$$

Differentiating eq. (2.53) w.r.t ϕ and putting values of $\eta^1_{,rr\phi}$ into (2.67) gives

$$\left(1 - \frac{2m}{r}\right)\eta^1_{,r\phi} - \frac{1}{r}\left(1 - \frac{6m}{r}\right)\eta^1_{,\phi} = 0. \tag{2.68}$$

Comparing eq. (2.68) with eq. (2.63) produce

$$\eta^1_{,\phi} = 0. \tag{2.69}$$

Substituting eq. (2.66) into eq. (2.4k) leads to

$$\eta^3_{,t} = 0. \tag{2.70}$$

and substituting eq. (2.69) into eq. (2.4l) gives

$$\eta^4_{,t} = 0. \tag{2.71}$$

By taking derivative of eq. (2.4h) w.r.t t it becomes

$$2\eta^2_{,t} + 2r\eta^3_{,t\theta} - r\xi_{,st} = 0. \tag{2.72}$$

Now using eq. (2.70) and eq. (2.4p) in (2.72) results into

$$\eta^2_{,t} = 0. \tag{2.73}$$

If we put (2.73) into (2.4j), it reads into

$$\eta^1_{,r} = 0. \tag{2.74}$$

Differentiating (2.4f) w.r.t θ and using (2.66) we obtain

$$\eta^2_{,\theta} = 0, \tag{2.75}$$

and differentiating (2.4f) w.r.t ϕ and using (2.69) yields

$$\eta^2_{,\phi} = 0. \tag{2.76}$$

Now substitution of (2.75) into eq. (2.4m) reads into

$$\eta^3_{,r} = 0, \tag{2.77}$$

and substitution of (2.76) into eq. (2.4n) reads into

$$\eta^4_{,r} = 0. \tag{2.78}$$

By taking derivative of eq. (2.4h) w.r.t r and making use of (2.77), we write

$$\eta^2_{,r} = 0. \tag{2.79}$$

The Equations given by eq. (2.73), eq. (2.75), eq. (2.76) and eq. (2.79) result into

$$\eta^2 = c_5, \tag{2.80}$$

where c_5 is an arbitrary constant.

Making use of eq. (2.46), eq. (2.79) and eq. (2.80) in eq. (2.4g) produce

$$\left(1 - \frac{2m}{r}\right) c_2 = -\frac{2m}{r^2} c_5, \tag{2.81}$$

from which one can easily deduce

$$c_2 = 0, \tag{2.82}$$

$$c_5 = 0. \tag{2.83}$$

By using eq. (2.82) in (2.46) we have

$$\xi_{,s} = 0, \tag{2.84}$$

that means

$$\xi = c_3. \tag{2.85}$$

Also by using (2.83) in eq. (2.83) we get

$$\eta^2 = 0. \tag{2.86}$$

By making use of eq. (2.84) and eq. (2.86) in eq. (2.4f) we deduce

$$\eta^1_{,t} = 0. \tag{2.87}$$

Considering eq. (2.66), eq. (2.69), eq. (2.74) and eq. (2.87), we can write

$$\eta^1 = c_6. \tag{2.88}$$

Substituting eq. (2.84) and eq. (2.86) in eq. (2.4h) results into

$$\eta^3_{,\theta} = 0. \tag{2.89}$$

Using eq. (2.84) and eq. (2.86) in eq. (2.4i) gives

$$\cot\theta\eta^3 + \eta^4_{,\phi} = 0. \tag{2.90}$$

Now differentiating eq. (2.4o) w.r.t θ and using eq. (2.89) we get

$$2\sin\theta\cos\theta\eta 43_{,\theta} + sin^2\theta\eta^4_{,\theta\theta} = 0. \tag{2.91}$$

By manipulation eq. (2.91) becomes

$$2\cot\theta\eta^4_{,\theta} + \eta^4_{,\theta\theta} = 0. \tag{2.92}$$

By solving eq. (2.92) gives

$$\eta^4 = c_7 \cos\phi \cot\theta - c_8 \sin\phi \cot\theta + c_9. \tag{2.93}$$

Derivative of eq. (2.93) w.r.t ϕ is

$$- c_7 \sin\phi \cot\theta - c_8 \cos\phi \cot\theta, \tag{2.94}$$

which upon substituting in eq. (2.90) produce

$$\eta^3 = c_7 \sin\phi + c_8 \cos\phi. \tag{2.95}$$

where c_7, c_8 and c_9 are also arbitrary constants.

Hence we have determined the values for infinitesimals ξ, η^1, η^2, η^3, η^4 and the boundary function g as

$$\begin{aligned}
&\xi = C_1, \qquad \eta^1 = C_2, \qquad \eta^2 = 0,\\
&\eta^3 = C_3 \sin\phi + C_4 \cos\phi,\\
&\eta^4 = C_3 \cos\phi \cot\theta - C_4 \sin\phi \cot\theta + C_5,\\
&g = C_6,
\end{aligned} \tag{2.96}$$

where $(C_1, C_2, C_3, C_4, C_5, C_6) = (c_3, c_6, c_7, c_8, c_9, c_4)$ are some arbitrary constants. Using the infinitesimals found in eq. (2.96) we can write the generator for Noether symmetries as

$$\mathbf{X}^{(1)} = C_1 \frac{\partial}{\partial s} + C_2 \frac{\partial}{\partial t} + (C_3 \sin\phi + C_4 \cos\phi) \frac{\partial}{\partial \theta} + (C_3 \cos\phi \cot\theta - C_4 \sin\phi \cot\theta + C_5) \frac{\partial}{\partial \phi}, \tag{2.97}$$

and for $C_k = 0$, five Noether symmetries are obtained

$$\begin{aligned}
&\mathbf{X}_1 = \frac{\partial}{\partial s}, && \mathbf{X}_2 = \frac{\partial}{\partial t},\\
&\mathbf{X}_3 = \sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi}, && \mathbf{X}_4 = \cos\phi \frac{\partial}{\partial \theta} - \sin\phi \cot\theta \frac{\partial}{\partial \phi},\\
&\mathbf{X}_5 = \frac{\partial}{\partial \phi}.
\end{aligned} \tag{2.98}$$

Here, we can observe that the infinitesimals ξ, η along with the boundary function g satisfy the equation (1.199). This equation is obviously same as the structural equation (1.195) on the basis of which the relationship of Noether symmetries and Mei symmetries is established in section (1.5.4). These symmetries form the Lie algebra

$$\begin{aligned} &[\mathbf{X}_1, \mathbf{X}_2] = 0, && [\mathbf{X}_1, \mathbf{X}_3] = 0, \\ &[\mathbf{X}_1, \mathbf{X}_4] = 0, && [\mathbf{X}_1, \mathbf{X}_5] = 0, \\ &[\mathbf{X}_2, \mathbf{X}_3] = 0, && [\mathbf{X}_2, \mathbf{X}_4] = 0, \\ &[\mathbf{X}_2, \mathbf{X}_5] = 0, && [\mathbf{X}_3, \mathbf{X}_4] = \mathbf{X}_5, \\ &[\mathbf{X}_3, \mathbf{X}_5] = -\mathbf{X}_4 && [\mathbf{X}_4, \mathbf{X}_5] = \mathbf{X}_3. \end{aligned} \tag{2.99}$$

2.3 Evaluation of the Mei Symmetries

First, we find the Euler Lagrange equations for the system. These equations are

$$\ddot{t} = -2\left(1-\frac{2m}{r}\right)^{-1}\frac{m\dot{t}\dot{r}}{r^2}, \tag{2.100}$$

$$\ddot{r} = \left(1-\frac{2m}{r}\right)^{-1}\frac{m\dot{r}^2}{r^2} - \left(1-\frac{2m}{r}\right)\frac{m\dot{t}^2}{r^2} + \left(1-\frac{2m}{r}\right)r\dot{\theta}^2 + \left(1-\frac{2m}{r}\right)r\sin^2\theta\dot{\phi}^2, \tag{2.101}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \sin\theta\cos\theta\dot{\phi}^2, \tag{2.102}$$

$$\ddot{\phi} = -\frac{2\dot{r}\dot{\phi}}{r} - 2\cot\theta\dot{\theta}\dot{\phi}. \tag{2.103}$$

According to [31], the Lie point symmetries of these Euler Lagrange equations (geodesic equations) are

$$\begin{aligned} &\mathbf{X}_1 = \frac{\partial}{\partial s}, && \mathbf{X}_2 = \frac{\partial}{\partial t}, \\ &\mathbf{X}_3 = \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}, && \mathbf{X}_4 = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}, \\ &\mathbf{X}_5 = \frac{\partial}{\partial\phi}, && \mathbf{X}_6 = s\frac{\partial}{\partial s}. \end{aligned} \tag{2.104}$$

There is one extra Lie symmetry obtained in addition to the Noether symmetries given in the last section. Next we find the Mei symmetries for the same Lagrangian given in eq. (2.1) to see how they come up in comparison to the Noether and Lie symmetries.

Considering the criterion for the Mei symmetries as

$$\mathbf{E}_\alpha[\mathbf{X}^{(1)}(L)] = 0. \tag{2.105}$$

Here L is the Lagrangian, whereas $\mathbf{E}_\alpha = \frac{d}{ds}\frac{\partial}{\partial \dot{q}^\alpha} - \frac{\partial}{\partial q^\alpha}$ is the Euler operator and $\mathbf{X}^{(1)} = \xi\frac{\partial}{\partial s} + \eta^\alpha\frac{\partial}{\partial q^\alpha} + (\dot{\eta}^\alpha - \dot{q}^\alpha\dot{\xi})\frac{\partial}{\partial \dot{q}^\alpha}$ is the first extended infinitesimal generator. Applying first extended generator on the Lagrangian given in eq. (2.1) gives

$$\begin{aligned}\mathbf{X}^{(1)}(L) = \eta^2&\left[-\frac{2m\dot{t}^2}{r^2} - 2\left(1-\frac{2m}{r}\right)^{-2}\frac{m\dot{r}^2}{r^2} + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\right] + 2\eta^3 r^2\sin\theta\cos\theta\dot{\phi}^2\\ &+\left[\dot{\eta}^1 - \dot{t}\dot{\xi}\right]\left[-2\left(1-\frac{2m}{r}\right)\dot{t}\right] + \left[\dot{\eta}^2 - \dot{r}\dot{\xi}\right]\left[2\left(1-\frac{2m}{r}\right)^{-1}\dot{r}\right]\\ &+\left[\dot{\eta}^3 - \dot{\theta}\dot{\xi}\right]\left[2r^2\dot{\theta}\right] + \left[\dot{\eta}^4 - \dot{\phi}\dot{\xi}\right]\left[2r^2\sin^2\theta\dot{\phi}\right].\end{aligned} \tag{2.106}$$

For $q^1 = t$ eq. (2.105) yields

$$\left[\frac{d}{ds}\frac{\partial}{\partial \dot{t}} - \frac{\partial}{\partial t}\right]\left[\mathbf{X}^{(1)}(\mathbf{L})\right] = 0. \tag{2.107}$$

Using eq. (2.106) in eq. (2.107), solving it further and after cancellation of some alike terms the coefficients of $\dot{t}$, $\dot{r}$,$\dot{\theta}$, $\dot{\phi}$ and their powers are compared to get system of determining equations as follows

$$(constant): \quad \eta^1_{,ss} = 0, \tag{2.108a}$$

$$(\dot{t}): \quad -\frac{m}{r^2}\eta^2_{,s} - \left(1-\frac{2m}{r}\right)\eta^1_{,st} + \left(1-\frac{2m}{r}\right)\xi_{,ss} = 0, \tag{2.108b}$$

$$(\dot{r}): \quad \eta^1_{,sr} = 0, \tag{2.108c}$$

$$(\dot{\theta}): \quad \eta^1_{,s\theta} = 0, \tag{2.108d}$$

$$(\dot{\phi}): \quad \eta^1_{,s\phi} = 0, \tag{2.108e}$$

$$(\dot{t}^2): \quad -\frac{2m}{r^2}\eta^2_{,t} - \left(1 - \frac{2m}{r}\right)\eta^1_{,tt} + \left(1 - \frac{2m}{r}\right)^2 \frac{m}{r^2}\eta^1_{,r} + 4\left(1 - \frac{2m}{r}\right)\xi_{,st} = 0, \tag{2.108f}$$

$$(\dot{r}^2): \quad \left(1 - \frac{2m}{r}\right)\eta^1_{,rr} + \frac{3m}{r^2}\eta^1_{,r} = 0, \tag{2.108g}$$

$$(\dot{\theta}^2): \quad \eta^1_{,\theta\theta} + \left(1 - \frac{2m}{r}\right) r\eta^1_{,r} = 0, \tag{2.108h}$$

$$(\dot{\phi}^2): \quad \eta^1_{,\phi\phi} + \left(1 - \frac{2m}{r}\right) r\sin^2\theta\eta^1_{,r} + \sin\theta\cos\theta\eta^1_{,\theta} = 0, \tag{2.108i}$$

$$(\dot{t}\dot{r}): \quad \frac{2m}{r^3}\eta^2 + 2\left(1 - \frac{2m}{r}\right)^{-1}\frac{m^2}{r^4}\eta^2 - \frac{m}{r^2}\eta^2_{,r} - \frac{m}{r^2}\eta^1_{,s} - \left(1 - \frac{2m}{r}\right)\eta^1_{,tr} + 2\left(1 - \frac{2m}{r}\right)\xi_{,sr} = 0, \tag{2.108j}$$

$$(\dot{t}\dot{\theta}): \quad -\frac{m}{r^2}\eta^2_{,\theta} - \left(1 - \frac{2m}{r}\right)\eta^1_{,t\theta} + 2\left(1 - \frac{2m}{r}\right)\xi_{,s\theta} = 0, \tag{2.108k}$$

$$(\dot{t}\dot{\phi}): \quad -\frac{m}{r^2}\eta^2_{,\phi} - \left(1 - \frac{2m}{r}\right)\eta^1_{,t\phi} + 2\left(1 - \frac{2m}{r}\right)\xi_{,s\phi} = 0, \tag{2.108l}$$

$$(\dot{r}\dot{\theta}): \quad -\left(1 - \frac{2m}{r}\right)\eta^1_{,r\theta} + \frac{1}{r}\left(1 - \frac{3m}{r}\right)\eta^1_{,theta} = 0, \tag{2.108m}$$

$$(\dot{r}\dot{\phi}): \quad -\left(1 - \frac{2m}{r}\right)\eta^1_{,r\phi} + \frac{1}{r}\left(1 - \frac{3m}{r}\right)\eta^1_{,\phi} = 0, \tag{2.108n}$$

$$(\dot{\theta}\dot{\phi}): \quad -\eta^1_{,\theta\phi} + \cot\theta\eta^1_{,\phi} = 0, \tag{2.108o}$$

$$(\dot{t}^3): \quad \xi_{,tt} - \left(1 - \frac{2m}{r}\right)\frac{m}{r^2}\xi_{,r} = 0, \tag{2.108p}$$

$$(\dot{t}^2\dot{r}): \quad \left(1 - \frac{2m}{r}\right)\xi_{,tr} - \frac{m}{r^2}\xi_{,t} = 0, \tag{2.108q}$$

$$(\dot{t}^2\dot{\theta}): \quad \xi_{,t\theta} = 0, \tag{2.108r}$$

$$(\dot{t}^2\dot{\phi}): \quad \xi_{,t\phi} = 0, \tag{2.108s}$$

$$(\dot{t}\dot{r}^2): \quad \left(1 - \frac{2m}{r}\right)\xi_{,rr} + \frac{m}{r^2}\xi_{,r} = 0, \tag{2.108t}$$

$$(\dot{t}\dot{\theta}^2): \quad \xi_{,\theta\theta} + \left(1 - \frac{2m}{r}\right) r\xi_{,r} = 0, \tag{2.108u}$$

$$(\dot{t}\dot{\phi}^2): \quad \xi_{,\phi\phi} + \left(1 - \frac{2m}{r}\right) r\sin^2\theta\xi_{,r} + \sin\theta\cos\theta\xi_{,\theta} = 0, \tag{2.108v}$$

$$(\dot{t}\dot{r}\dot{\theta}): \quad \xi_{,r\theta} - \frac{1}{r}\xi_{,\theta} = 0, \tag{2.108w}$$

$$(\dot{t}\dot{r}\dot{\phi}): \quad \xi_{,r\phi} - \frac{1}{r}\xi_{,\phi} = 0, \tag{2.108x}$$

$$(\dot{t}\dot{\theta}\dot{\phi}): \quad \xi_{,\theta\phi} - \cot\theta\xi_{,\phi} = 0, \tag{2.108y}$$

where (,) represents partial derivative. Now, putting $q^2 = r$ in eq. (2.105) yields

$$\left[\frac{d}{ds}\frac{\partial}{\partial \dot{r}} - \frac{\partial}{\partial r}\right]\left[\mathbf{X}^{(1)}(\mathbf{L})\right] = 0. \tag{2.109}$$

Again using eq. (2.106) in eq. (2.109), further simplification along with the coefficients of ($\dot{t}$, $\dot{r}$, $\dot{\theta}$, $\dot{\phi}$) and their powers' comparison yields some exactly similar determining equations as the previously obtained subequations (2.108p)-(2.108y) and the remaining ones are listed as

$$(constant): \quad \eta^2_{,ss} = 0, \tag{2.110a}$$

$$(\dot{t}): \quad \left(1 - \frac{2m}{r}\right)^{-1}\eta^2_{,st} + \frac{m}{r^2}\eta^1_{,s} = 0, \tag{2.110b}$$

$$(\dot{r}): \quad -\left(1 - \frac{2m}{r}\right)^{-1}\frac{m}{r^2}\eta^2_{,s} + \eta^2_{,sr} - \xi_{,ss} = 0, \tag{2.110c}$$

$$(\dot{\theta}): \quad \left(1 - \frac{2m}{r}\right)^{-1}\eta^2_{,s\theta} - r\eta^3_{,s} = 0, \tag{2.110d}$$

$$(\dot{\phi}): \quad \left(1 - \frac{2m}{r}\right)^{-1}\eta^2_{,s\phi} - r\sin^2\theta\eta^4_{,s} = 0, \tag{2.110e}$$

$$\begin{aligned}(\dot{t}^2): \quad & 2\left(1 - \frac{2m}{r}\right)^{-1}\frac{m^2}{r^4}\eta^2 + \left(1 - \frac{2m}{r}\right)^{-1}\eta^2_{,tt} - \frac{m}{r^2}\eta^2_{,r} - 2\frac{m}{r^3}\eta^2 \\ & + 2\frac{m}{r^2}\eta^1_{,t} = 0,\end{aligned} \tag{2.110f}$$

$$\begin{aligned}(\dot{r}^2): \quad & 2\left(1 - \frac{2m}{r}\right)^{-2}\frac{m^2}{r^4}\eta^2 + 2\left(1 - \frac{2m}{r}\right)^{-1}\frac{m}{r^3}\eta^2 \\ & - \left(1 - \frac{2m}{r}\right)\frac{m}{r^2}\eta^2_{,r} + \eta^2_{,rr} - 4\xi_{,sr} = 0,\end{aligned} \tag{2.110g}$$

$$\begin{aligned}(\dot{\theta}^2): \quad & -2\left(1 - \frac{2m}{r}\right)^{-1}\frac{m}{r}\eta^2 + \left(1 - \frac{2m}{r}\right)\eta^2_{,\theta\theta} + r\eta^2_{,r} - 2r\eta^3_{,\theta} \\ & - \eta^2 = 0,\end{aligned} \tag{2.110h}$$

$$(\dot{\phi}^2): \quad -2\left(1-\frac{2m}{r}\right)^{-1}\frac{m}{r}\sin^2\theta\eta^2+\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,\phi\phi}+r\sin^2\theta\eta^2_{,r}$$
$$+\left(1-\frac{2m}{r}\right)^{-1}\sin\theta\cos\theta\eta^2_{,\theta}-\sin^2\theta\eta^2-2r\sin\theta\cos\theta\eta^3$$
$$-2r\sin^2\theta\eta^4_{,\phi}=0, \tag{2.110i}$$
$$(\dot{r}\dot{t}): \quad -2\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\eta^2_{,t}+\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,tr}-2\left(1-\frac{2m}{r}\right)^{-1}\xi_{,st}$$
$$+\frac{m}{r^2}\eta^1_{,r}=0, \tag{2.110j}$$
$$(\dot{r}\dot{\theta}): \quad -\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\eta^2_{,\theta}+\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,r\theta}-\frac{1}{r}\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,\theta}$$
$$-2\left(1-\frac{2m}{r}\right)^{-1}\xi_{,s\theta}-r\eta^3_{,r}=0, \tag{2.110k}$$
$$(\dot{r}\dot{\phi}): \quad -\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\eta^2_{,\phi}+\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,r\phi}-\frac{1}{r}\left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,\phi}$$
$$-2\left(1-\frac{2m}{r}\right)^{-1}\xi_{,s\phi}-r\sin^2\theta\eta^4_{,r}=0, \tag{2.110l}$$
$$(\dot{t}\dot{\theta}): \quad \left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,t\theta}+\frac{m}{r^2}\eta^1_{,\theta}-r\eta^3_{,t}=0, \tag{2.110m}$$
$$(\dot{t}\dot{\phi}): \quad \left(1-\frac{2m}{r}\right)^{-1}\eta^2_{,t\phi}+\frac{m}{r^2}\eta^1_{,\phi}-r\sin^2\theta\eta^4_{,t}=0, \tag{2.110n}$$
$$(\dot{\theta}\dot{\phi}): \quad \left(1-\frac{2m}{r}\right)^{-1}\eta^2_{\theta\phi}-\left(1-\frac{2m}{r}\right)^{-1}\cot\theta\eta^2_{,\phi}-r\eta^3_{,\phi}-r\sin^2\theta\eta^4_{,\theta}=0. \tag{2.110o}$$

Next putting $q^3=\theta$ in eq. (2.105), it becomes

$$\left(\frac{d}{ds}\frac{\partial}{\partial\dot{\theta}}-\frac{\partial}{\partial\theta}\right)\left(\mathbf{X}^{(1)}(\mathbf{L})\right)=0. \tag{2.111}$$

Now by using eq. (2.106) in eq. (2.111), it is further simplified and the comparison of coefficients of ($\dot{t}$, $\dot{r}$, $\dot{\theta}$, $\dot{\phi}$) and their powers is done that again provides some similar equations ((2.108p)-(2.108y)) as written above. The left out equations are

$$(constant): \quad \eta^3_{,ss}=0, \tag{2.112a}$$
$$(\dot{t}): \quad \eta^3_{,st}=0, \tag{2.112b}$$

$$(\dot{r}): \quad \eta^3_{,s} + r\eta^3_{,sr} = 0, \tag{2.112c}$$

$$(\dot{\theta}): \quad \eta^2_{,s} + r\eta^3_{,s\theta} - r\xi_{,ss} = 0, \tag{2.112d}$$

$$(\dot{\phi}): \quad \eta^3_{,s\phi} - \sin\theta\cos\theta\eta^4_{,s} = 0, \tag{2.112e}$$

$$(\dot{t}^2): \quad r^2\eta^3_{,tt} - \left(1 - \frac{2m}{r}\right) m\eta^3_{,r} = 0, \tag{2.112f}$$

$$(\dot{r}^2): \quad r^2\eta^3_{,rr} + \left[2r + \left(1 - \frac{2m}{r}\right)^{-1} m\right] \eta^3_{,r} = 0, \tag{2.112g}$$

$$(\dot{\theta}^2): \quad 2\eta^2\theta + r\eta^3_{,\theta\theta} + \left(1 - \frac{2m}{r}\right) r^2\eta^3_{,r} - 4r\xi_{,s\theta} = 0, \tag{2.112h}$$

$$\begin{aligned}(\dot{\phi}^2): \quad & \eta^3_{,\phi\phi} + \left(1 - \frac{2m}{r}\right) r\sin^2\theta\eta^3_{,r} + \sin\theta\cos\theta\eta^3_{,\theta} + \sin^2\theta\eta^3 - \cos^2\theta\eta^3 \\ & - 2\sin\theta\cos\theta\eta^4_{,\phi} = 0,\end{aligned} \tag{2.112i}$$

$$(\dot{\theta}\dot{t}): \quad \eta^2_{,t} + r\eta^3_{,t\theta} - 2r\xi_{,st} = 0, \tag{2.112j}$$

$$(\dot{\theta}\dot{r}): \quad -\eta^2 + r\eta^2_{,r} + r^2\eta^3_{,r\theta} - 2r^2\xi_{,sr} = 0, \tag{2.112k}$$

$$(\dot{\theta}\dot{\phi}): \quad \eta^2_{,\phi} + r\eta^3_{,\theta\phi} - r\cot\theta\eta^3_{,\phi} - 2r\xi_{,s\phi} - r\sin\theta\cos\theta\eta^4_{,\theta} = 0, \tag{2.112l}$$

$$(\dot{t}\dot{r}): \quad r^2\eta^3_{,tr} + \left[r - \left(1 - \frac{2m}{r}\right)^{-1} m\right] \eta^3_{,t} = 0, \tag{2.112m}$$

$$(\dot{t}\dot{\phi}): \quad \eta^3_{,t\phi} - \sin\theta\cos\theta\eta^4_{,t} = 0, \tag{2.112n}$$

$$(\dot{r}\dot{\phi}): \quad \eta^3_{,r\phi} - \sin\theta\cos\theta\eta^4_{,r} = 0. \tag{2.112o}$$

For last variable $q^4 = \phi$, eq. (2.6) yields

$$\left[\frac{d}{ds}\frac{\partial}{\partial\dot{\phi}} - \frac{\partial}{\partial\phi}\right]\left[\mathbf{X}^{(1)}(\mathbf{L})\right] = 0. \tag{2.113}$$

By using eq. (2.106) in eq. (2.113), it is simplified and then equating to zero the coefficients of $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ and their powers produce some similar equations ((2.108p) – (2.108y)). Remaining equations are

$$(constant): \quad \eta^4_{,ss} = 0, \tag{2.114a}$$

$$(\dot{t}): \quad \eta^4_{,st} = 0, \tag{2.114b}$$

$$(\dot{r}): \quad \eta^4_{,s} + r\eta^4_{,sr} = 0, \tag{2.114c}$$

$$(\dot{\theta}): \quad \sin\theta\cos\theta\eta^4_{,s} + \sin^2\theta\eta^4_{,s\theta} = 0, \tag{2.114d}$$

$$(\dot{\phi}): \quad \eta^2_{,s} + r\cot\theta\eta^3_{,s} + r\eta^4_{,s\phi} - r\xi_{,ss} = 0, \tag{2.114e}$$

$$(\dot{t}^2): \quad r^2\eta^4_{,tt} - \left(1 - \frac{2m}{r}\right) m\eta^4_{,r} = 0, \tag{2.114f}$$

$$(\dot{r}^2): \quad r^2\eta_{,rr} + \left[2r + \left(1 - \frac{2m}{r}^{-1}\right) m\right] \eta^4_{,r} = 0, \tag{2.114g}$$

$$(\dot{\theta}^2): \quad 2\cot\theta\eta^4_{,\theta} + \eta^4_{,\theta\theta} + \left(1 - \frac{2m}{r}\right) r\eta^4_{,r} = 0, \tag{2.114h}$$

$$(\dot{\phi}^2): \quad 2\eta^2_{,\phi} + 2r\cot\theta\eta^3_{,\phi} + r\eta^4_{,\phi\phi} + r\sin\theta\cos\theta\eta^4_{,\theta} + \left(1 - \frac{2m}{r}\right) r^2\sin^2\theta\eta^4_{,r} - 4r\xi_{,s\phi} = 0, \tag{2.114i}$$

$$(\dot{\phi}\dot{t}): \quad \eta^2_{,t} + r\cot\theta\eta^3_{,t} + r\eta^4_{,t\phi} - 2r\xi_{,st} = 0, \tag{2.114j}$$

$$(\dot{\phi}\dot{r}): \quad -\eta^2 + r\eta^2_{,r} + r^2\cot\theta\eta^3_{,r} + r^2\eta^4_{,r\phi} - 2r^2\xi_{,sr} = 0, \tag{2.114k}$$

$$(\dot{\phi}\dot{\theta}): \quad \eta^2_{,\theta} - r\cot^2\theta\eta^3 - r\eta^3 + r\cot\theta\eta^3_{,\theta} + r\eta^4_{,\theta\phi} - 2r\xi_{,s\theta} = 0, \tag{2.114l}$$

$$(\dot{t}\dot{r}): \quad r^2\eta^4_{,tr} + \left[r - \left(1 - \frac{2m}{r}\right)^{-1} m\right] \eta^4_{,t} = 0, \tag{2.114m}$$

$$(\dot{t}\dot{\theta}): \quad \cot\theta\eta^4_{,t} + \eta^4_{,t\theta} = 0, \tag{2.114n}$$

$$(\dot{r}\dot{\theta}): \quad \cot\theta\eta^4_{,r} + \eta^4_{,r\theta} = 0. \tag{2.114o}$$

Now we solve the above system of partial differential equations to find values of ξ, η^1, η^2, η^3 and η^4.

Differentiating eq. (2.108v) w.r.t t and making use of eq. (2.108r) and eq. (2.108s) we get

$$\xi_{,tr} = 0. \tag{2.115}$$

Using eq. (2.115) in eq. (2.108q) we get

$$\xi_{,t} = 0. \tag{2.116}$$

From eq. (2.116) $\xi_{,tt} = 0$ which makes eq. (2.108p)

$$\xi_{,r} = 0, \tag{2.117}$$

using eq. (2.117) in eq. (2.108w) and eq. (2.108x) we get

$$\xi_{,\theta} = 0, \qquad \xi_{,\phi} = 0. \tag{2.118}$$

From eq. (2.116), eq. (2.117) and eq. (2.118) we now know that ξ is a function of s only i.e.

$$\xi = \xi(s). \tag{2.119}$$

If we differentiate eq. (2.108b) w.r.t s and make use of eq. (2.108a) and eq. (2.110a) we get

$$\xi_{,sss} = 0, \tag{2.120}$$

by utilising eq. (2.119), eq. (2.120) can be solved to get value of ξ

$$\xi = c_1 s^2 + c_2 s + c_3, \tag{2.121}$$

where c_1, c_2, c_3 are arbitrary constants.

Next from eq. (2.108a) we can write η^1 as

$$\eta^1 = a_1(t, r, \theta, \phi)s + a_2(t, r, \theta, \phi). \tag{2.122}$$

where a_1, a_2 are some arbitrary functions of mentioned arguments, but from equations (2.108c), (2.108d) and (2.108e) we realize that a_1 must be the function of t only. Hence eq. (2.122) becomes

$$\eta^1 = a_1(t)s + a_2(t, r, \theta, \phi), \tag{2.123}$$

Now differentiating eq. (2.112d) w.r.t t and using eq. (2.112b) and eq. (2.116) we yield

$$\eta^2_{,st} = 0, \tag{2.124}$$

utilising eq. (2.124) in eq. (2.110b) it produces

$$\eta^1_{,s} = 0, \tag{2.125}$$

which means $a_1(t)$ must be equal to zero and hence η^1 from eq. (2.123) translates into

$$\eta^1 = a_2(t, r, \theta, \phi). \tag{2.126}$$

From eq. (2.110a) η^2 can be written as

$$\eta^2 = a_3(t, r, \theta, \phi)s + a_4(t, r, \theta, \phi). \tag{2.127}$$

Now we differentiate eq. (2.108b) w.r.t r and use eq. (2.125) and eq. (2.117) to get

$$\eta^2_{,sr} = 0, \tag{2.128}$$

and after putting value of $\eta^2_{,sr}$ in eq. (2.110c) and using $\xi_{,ss} = c_1$, we solve it to get value of $\eta^2_{,s}$

$$\eta^2_{,s} = -\frac{2c_1}{m}(r^2 - 2mr). \tag{2.129}$$

If we differentiate eq. (2.108b) w.r.t θ and use eq. (2.105) and eq. (2.118), we obtain

$$\eta^2_{,s\theta} = 0, \tag{2.130}$$

and putting it in eq. (2.110d) yields

$$\eta^3_{,s} = 0, \tag{2.131}$$

putting eq. (2.131) in eq. (2.112d) and utilising $\xi_{,ss} = c_1$, produce

$$\eta^2_{,s} = 2rc_1, \tag{2.132}$$

equating eq. (2.129) and eq. (2.132) generates

$$c_1 = 0, \tag{2.133}$$

which means ξ in eq. (2.121) now becomes

$$\xi = c_2 s + c_3. \tag{2.134}$$

Also if $c_1 = 0$ then

$$\eta^2_{,s} = 0, \tag{2.135}$$

which means a_3 must be equal to zero and hence η^2 in eq. (2.127) becomes

$$\eta^2 = a_4(t, r, \theta.\phi). \tag{2.136}$$

Solving eq. (2.112a) gives

$$\eta^3 = a_5(t, r, \theta, \phi)s + a_6(t, r, \theta, \phi), \tag{2.137}$$

but as $\eta^3_{,s} = 0$ then a_5 must equals zero and therefore eq. (2.137) translates into

$$\eta^3 = a_6(t, r, \theta, \phi). \tag{2.138}$$

Eq. (2.114a) can be solved to get value of η^4 as

$$\eta^4 = a_7(t, r, \theta, \phi)s + a_8(t, r, \theta, \phi), \tag{2.139}$$

if we differentiate eq. (2.108b) w.r.t. ϕ and use eq. (2.105) and eq. (2.118) to get

$$\eta^2_{,s\phi} = 0, \tag{2.140}$$

which upon putting in eq. (2.110e) gives

$$\eta^4_{,s} = 0, \tag{2.141}$$

that means a_4 in η^4 must be zero, so eq. (2.139) now reads

$$\eta^4 = a_8(t, r, \theta, \phi). \tag{2.142}$$

If we differentiate eq. (2.108m) w.r.t. r we get

$$\left(1-\frac{2m}{r}\right)\eta^1_{,rr\theta}+\frac{2m}{r^2}\eta^1_{,r\theta}+\frac{1}{r^2}\eta^1_{,\theta}-\frac{6m}{r^3}\eta^1_{,\theta}-\frac{1}{r}\left(1-\frac{3m}{r}\right)\eta^1_{,r\theta}=0, \tag{2.143}$$

also differentiating eq. (2.108g) w.r.t. θ yields

$$\left(1-\frac{2m}{r}\right)\eta^1_{,rr\theta}=-\frac{3m}{r^2}\eta^1_{,rr\theta}, \tag{2.144}$$

putting eq. (2.144) back into eq. (2.143) and solving gives

$$\left(1-\frac{2m}{r}\right)\eta^1_{,r\theta}-\frac{1}{r}\left(1-\frac{6m}{r}\right)\eta^1_{,\theta}=0, \tag{2.145}$$

equating eq. (2.145) and eq. (2.108m) produce

$$\eta^1_{,\theta}=0. \tag{2.146}$$

Next, if we differentiate eq. (2.108n) w.r.t r we obtain

$$\left(1-\frac{2m}{r}\right)\eta^1_{,rr\phi}+\frac{2m}{r^2}\eta^1_{,r\phi}+\frac{1}{r^2}\eta^1_{,\phi}-\frac{6m}{r^3}\eta^1_{,\phi}-\frac{1}{r}\left(1-\frac{3m}{r}\right)\eta^1_{,r\phi}=0, \tag{2.147}$$

also differentiating eq. (2.108g) w.r.t. ϕ yields

$$\left(1-\frac{2m}{r}\right)\eta^1_{,rr\phi}=-\frac{3m}{r^2}\eta^1_{,rr\phi}, \tag{2.148}$$

putting eq. (2.148) back into eq. (2.147) and solving gives

$$\left(1-\frac{2m}{r}\right)\eta^1_{,r\phi}-\frac{1}{r}\left(1-\frac{6m}{r}\right)\eta^1_{,\phi}=0, \tag{2.149}$$

equating eq. (2.149) and eq. (2.108n) produce

$$\eta^1_{,\phi}=0. \tag{2.150}$$

using eq. (2.146) into eq. (2.108h) we get

$$\eta^1_{,r}=0. \tag{2.151}$$

With the help of equations (2.146),(2.151) and (2.151) we can say a_2 is function of t only, then eq. (2.126) becomes

$$\eta^1 = a_2(t). \tag{2.152}$$

By using eq. (2.118) and eq. (2.146) in eq. (2.108k) we get

$$\eta^2_{,\theta} = 0. \tag{2.153}$$

Now using eq. (2.118) and eq. (2.150) in eq. (2.108l) gives

$$\eta^2_{,\phi} = 0. \tag{2.154}$$

Putting eq. (2.146) and eq. (2.153) in eq. (2.110m) yields

$$\eta^3_{,t} = 0. \tag{2.155}$$

Putting eq. (2.150) and eq. (2.154) in eq. (2.110n) yields

$$\eta^4_{,t} = 0. \tag{2.156}$$

Making use of eq. (2.118) and eq. (2.153) in eq. (2.110k), we get

$$\eta^3_{,r} = 0. \tag{2.157}$$

Making use of eq. (2.118) and eq. (2.154) in eq. (2.110l), we get

$$\eta^4_{,r} = 0. \tag{2.158}$$

(2.112j) can be solved by utilising eq. (2.116) and eq. (2.155) to get

$$\eta^2_{,t} = 0. \tag{2.159}$$

By realising equations (2.153),(2.154) and (2.159), $\eta^2(a_4)$ must be function of r only therefore eq. (2.136) becomes

$$\eta^2 = a_4(r). \tag{2.160}$$

Now if we use eq. (2.117) and eq. (2.157) in eq. (2.112k), we get

$$-\eta^2 + r\eta^2_{,r} = 0, \tag{2.161}$$

as η^2 is a function of r only so by substituting $a_4 = r^m$, eq. (2.161) can be solved to get

$$\eta^2 = c_4 r, \tag{2.162}$$

where c_4 is an arbitrary constant.
By putting eq. (2.117) and the value of η^2 from eq. (2.162) in (2.110g) we obtain

$$c_4 = 0, \tag{2.163}$$

which means eq. (2.162) gives

$$\eta^2 = 0, \tag{2.164}$$

substituting value of η^2 from eq. (2.164) into eq. (2.110f) we get

$$\eta^1_{,t} = 0. \tag{2.165}$$

The equations (2.146), (2.150), (2.151) and (2.165) suggest η^1 is just a constant i.e.

$$\eta^1 = c_5. \tag{2.166}$$

substituting eq. (2.164) into eq. (2.110h) gives

$$\eta^3_{,\theta} = 0. \tag{2.167}$$

Using value of η^2 in eq. (2.110i) gives

$$\eta^3 = -\tan\theta\eta^4_{,\phi}, \tag{2.168}$$

If we take derivative of eq. (2.168), we obtain

$$\eta^3_{,\phi} = -\tan\theta\eta^4_{,\phi\phi}, \tag{2.169}$$

using eq. (2.164) and eq. (2.169) in eq. (2.110o) to get

$$\eta^4_{,\phi\phi} + \sin\theta\cos\theta\eta^4_{,\theta}. \tag{2.170}$$

If we put equations (2.118), (2.154), (2.158) and (2.170) in eq. (2.114i), we are left with

$$\eta^3_{,\phi} = 0, \tag{2.171}$$

As η^3 is zero w.r.t all the variables therefore it must be equal to a constant i.e.

$$\eta^3 = c_6. \tag{2.172}$$

Using eq. (2.172) in eq. (2.110o) gives

$$\eta^4_{,\theta} = 0, \tag{2.173}$$

also by putting eq. (2.172) in eq. (2.168) we get

$$\eta^4_{,\phi} = -c_6\cot\theta, \tag{2.174}$$

if we put eq. (2.172) and eq. (2.175) in eq. (2.112i), it produces

$$c_6 = 0, \tag{2.175}$$

by means of which eq. (2.172) becomes

$$\eta^3 = 0, \tag{2.176}$$

and eq. (2.174) also becomes

$$\eta^4_{,\phi} = 0, \tag{2.177}$$

keeping the equations (2.156),(2.158),(2.173) and (2.177) in view we can iterate

$$\eta^4 = c_7. \tag{2.178}$$

So we have found all the required infinitesimals and if we assume $(c_2, c_3, c_5, c_7) = (C_1, C_2, C_3, C_4)$ then we can write

$$\begin{aligned} &\xi = C_1 s + C_2, && \eta^1 = C_3, && \\ &\eta^2 = 0, && \eta^3 = 0, && \eta^4 = C_4. \end{aligned} \tag{2.179}$$

Hence the generator can be written as

$$\mathbf{X}^{(1)} = (C_1 s + C_2)\frac{\partial}{\partial s} + C_3\frac{\partial}{\partial t} + C_4\frac{\partial}{\partial \phi}, \tag{2.180}$$

For $C_k = 0$ we get four symmetries

$$\begin{aligned} &\mathbf{X}_1 = s\frac{\partial}{\partial s}, && \mathbf{X}_2 = \frac{\partial}{\partial s}, \\ &\mathbf{X}_3 = \frac{\partial}{\partial t}, && \mathbf{X}_4 = \frac{\partial}{\partial \phi}. \end{aligned} \tag{2.181}$$

These four symmetries are the required Mei symmetries.

We see that out of these four Mei symmetries, three symmetries $\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$ are same as three Noether symmetries $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_5$ particularly found in eq. (2.98) corresponding to the Lagrangian given by eq. (2.1). These three symmetries actually satisfy the equation (1.199) but $\mathbf{X}_1$ does not satisfy this equation (1.199) therefore, it is not a Noether symmetry. And similarly, the Noether symmetries $\mathbf{X}_3, \mathbf{X}_4$ do not satisfy the equation (1.194) and hence they are not Mei symmetries. This observation marks the difference between both symmetries.

One can also observe that the four Mei symmetries obtained for Lagrangian given in eq. (2.1) are the subset of the Lie point symmetries as obtained in eq. (2.104) for the system of equations of motion given by eq. (2.100)-(2.103). The obtained Mei symmetries satisfy the Lie algebra

$$\begin{aligned} &[\mathbf{X}_1, \mathbf{X}_2] = -\mathbf{X}_2, && [\mathbf{X}_1, \mathbf{X}_3] = 0, \\ &[\mathbf{X}_1, \mathbf{X}_4] = 0, && [\mathbf{X}_2, \mathbf{X}_3] = 0, \\ &[\mathbf{X}_2, \mathbf{X}_4] = 0, && [\mathbf{X}_3, \mathbf{X}_4] = 0. \end{aligned} \tag{2.182}$$

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2.4 Verification of the Mei Symmetries

One may verify if the obtained symmetries fulfil the criterion of Mei symmetries given by eq. (2.105). By making use of obtained values of infinitesimals given by eq. (2.179) we write $\mathbf{X}^{(1)}(\mathbf{L})$ as

$$\mathbf{X}^{(1)}(\mathbf{L}) = 2C_1\left(1-\frac{2m}{r}\right)\dot{t}^2 - 2C_1\left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - 2C_1 r^2\dot{\theta}^2 - 2C_1 r^2\sin^2\theta\dot{\phi}^2. \quad (2.183)$$

We apply the Euler operator for each dependent variable one by one as required by criterion given in eq. (2.105).

For $q^1 = t$ criterion given by eq. (2.105) gives

$$\left(\frac{d}{ds}\frac{\partial}{\partial \dot{t}} - \frac{\partial}{\partial t}\right)(\mathbf{X}^{(1)}(\mathbf{L})) = 0, \quad (2.184)$$

using eq. (2.183), the left hand side of eq. (2.184) gives

$$\frac{d}{ds}\left(4\left(1-\frac{2m}{r}\right)C_1\dot{t}\right) = \frac{8m}{r^2}C_1\dot{t}\dot{r} - \frac{8m}{r^2}C_1\dot{t}\dot{r} = 0, \quad (2.185)$$

that means criterion holds true for $q^1 = t$.

For $q^2 = r$ criterion given by eq. (2.105) produce

$$\left(\frac{d}{ds}\frac{\partial}{\partial \dot{r}} - \frac{\partial}{\partial r}\right)(\mathbf{X}^{(1)}(\mathbf{L})) = 0. \quad (2.186)$$

On putting eq. (2.183) in eq. (2.186) we get

$$\frac{d}{ds}\left(-4C_1\left(1-\frac{2m}{r}\right)^{-1}\dot{r}\right) - \left(4C_1\frac{m}{r^2}\dot{t}^2 + 4C_1\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2 - 4C_1 r\dot{\theta}^2 - 4C_1 r\sin^2\theta\dot{\phi}^2\right)$$

$$= 8C_1\dot{r}^2\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2} - 4C_1\left(1-\frac{2m}{r}\right)^{-1}\ddot{r} - 4C_1\frac{m}{r^2}\dot{t}^2 - 4C_1\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2 + 4C_1 r\dot{\theta}^2 + 4C_1 r\sin^2\theta\dot{\phi}^2, \quad (2.187)$$

putting value of $\ddot{r}$ from eq. (2.101)

$$= 8C_1\dot{r}^2\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2} - 4C_1\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2 + 4C_1\frac{m}{r^2}\dot{t}^2 - 4C_1 r\dot{\theta}^2 - 4C_1 r\sin^2\theta\dot{\phi}^2$$
$$- 4C_1\frac{m}{r^2}\dot{t}^2 - 4C_1\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2 + 4C_1 r\dot{\theta}^2 + 4C_1 r\sin^2\theta\dot{\phi}^2 = 0. \tag{2.188}$$

It goes to zero and hence criterion holds for $q^2 = r$.

And for $q^3 = \theta$ criterion becomes

$$\left(\frac{d}{ds}\frac{\partial}{\partial\dot{\theta}} - \frac{\partial}{\partial\theta}\right)(\mathbf{X}^{(1)}(\mathbf{L})) = 0, \tag{2.189}$$

solving left hand side, we get

$$\begin{aligned} &\frac{d}{ds}\left(-4r^2C_1\dot{\theta}\right) - \left(-4r^2\sin\theta\cos\theta C_1\dot{\phi}^2\right) \\ &= -8C_1 r\dot{r}\dot{\theta} + 8C_1 r\dot{r}\dot{\theta} - 4C_1 r^2\sin\theta\cos\theta\dot{\phi}^2 + 4C_1 r^2\sin\theta\cos\theta\dot{\phi}^2 = 0, \end{aligned} \tag{2.190}$$

hence it also holds for $q^3 = \theta$.

Also for $q^4 = \phi$

$$\left(\frac{d}{ds}\frac{\partial}{\partial\dot{\phi}} - \frac{\partial}{\partial\phi}\right)(\mathbf{X}^{(1)}(\mathbf{L})) = 0, \tag{2.191}$$

the left hand side gives

$$\begin{aligned} &\frac{d}{ds}\left(-4r^2\sin^2\theta C_1\dot{\phi}\right) \\ &= -8C_1 r\sin^2\theta\dot{r}\dot{\phi} - 8C_1 r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 8C_1 r\sin^2\theta\dot{r}\dot{\phi} + 8C_1 r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} = 0, \end{aligned} \tag{2.192}$$

which tells criterion in eq. (2.105) holds true for $q^4 = \phi$ as well. Hence eq. (2.181) presents four symmetries for the Lagrangian in eq. (2.1) of Schwarzschild metric.

Chapter 3

Summary

In this thesis, from the extensive history of differential equations, the major developments with time are briefly revisited. The solutions of these differential equations always served as a gateway to further exploration. To acquire and comprehend these solutions of the differential equations, symmetry methods has become a very powerful and extra ordinary technique in the recent times. This work commence with the investigation of the symmetries including Lie Point symmetries, Noether symmetries and Mei symmetries. The study naturally includes the Ordinary differential equations. The definition of symmetry groups of point transformations and their infinitesimal generators, is presented. The method to prolong infinitesimal generators is discussed. The criterion of Lie point symmetries is studied and applied to some well known differential equations of first and higher orders as well. The evaluation of Lie algebras and Lie brackets of the basic symmetry generators is performed. After defining the Lagrangian, Noether symmetries and Mei symmetries are defined along with their respective criterion. The theorems (without proofs) related to first integrals or conserved quantities of Noether symmetries and Mei symmetries are quoted and used in examples in order to find them. With the help of already established facts, the relationship of Lie symmetry with Noether symmetry and relationship of Noether symmetry with Mei symmetry is accomplished.

Chapter 2 primarily focuses on finding the Mei symmetries for the Lagrangian of spherically symmetric and static metric. This chapter begins with the review of Noether symmetries for the Lagrangian of spherically symmetric and static metric from the paper [31]. In this review, the Schwarzschild solution being the most important solution of Einstein field equations is considered. After revisiting the criterion to find Noether symmetries, the Lagrangian for Schwarzschild metric is presented. Firstly, considering the criterion for Noether symmetries, the linear operator **A** along with the prolonged infinitesimal generator is used to establish the system of determining equations, which are then solved to find the unknown infinitesimals $(\xi, \eta^1, \eta^2, \eta^3, \eta^4)$ and the boundary function g. One out of these five infinitesimals appeared to be zero while others four, depending upon five arbitrary constants, lead to five Noether symmetries with the boundary function found to be a constant. Secondly, the Lie point symmetries obtained in the paper are presented. After this the main task to find Mei symmetries corresponding to the Lagrangian of Schwarzschild metric is executed. To find the Mei symmetries, the Euler Lagrange equations are compiled one by one for the four Schwarzschild coordinates (t, r, θ, ϕ). Taking the criterion of Mei symmetries into account, the infinitesimal generator is prolonged and the system of determining equations for all the dependent variables is obtained. This system is then solved independently to evaluate the values of the five infinitesimals $(\xi, \eta^1, \eta^2, \eta^3, \eta^4)$. Two out of which are found to be zero and the remaining three depend on four arbitrary constants, corresponding to which we found four Mei symmetries. We observed that the three of the Mei symmetries are just the same as three of the Noether symmetries for the same Lagrangian. No explicit relationship between these two symmetries could be found. However, the obtained Mei symmetries are found to be the subset of Lie point symmetries. In the end, the verification of the obtained Mei symmetries is also done in order to endorse the criterion.

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