The Notion of Closedness in Topological Category of Quantale Valued Closure Spaces

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Dedication

This thesis is dedicated to my respectable Supervisor, Parents and Siblings for their endless devotion, support and encouragement.

Acknowledgement

In the name of **Allah** who is the most gracious and the most merciful. I am grateful to **Allah** for blessing me more than I deserve. I am extremely grateful to my supervisor, **Dr.M.Qasim** for his invaluable advice, continuous support, and patience during my research work. His immense knowledge and experience have encouraged me in all the time of my research.

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Abstract

In this thesis, we consider the topological category of QV-Closure space and explicitly characterize local $\overline{T_0}$ and local T_1 objects in this category. Furthermore, we examine the notion of closedness in $\mathcal{L}$**-Cls**. We consider the topological category of QV-Closure space and explicitly characterize local $\overline{T_0}$ and local T_1 objects in this category. Finally we show that every local T_1 QV-Closure Space is Local $\overline{T_0}$ QV-Closure Space and the notion of closedness coincides with Local $\overline{T_0}$ QV-Closure Space.

Contents

Introduction

Just as groups are multi-faceted algebraic structures, categories are algebraic structures having various complementing natures, such as logical, geometric, combinatorial and computational. MacLane and Eilenberg [2] proposed a category in 1945 that they called natural transformations functors. Later, MacLane and Eilenberg pointed out that the category's objective was to investigate natural transformations, which demanded the usage of categories. The study of category theory facilitates communication between persons working in diverse domains by developing a new language that is cost-effective in terms of new expression. It also adds new meaning to existing problems by introducing new theorems and structures that are independent of one another. Category theory has applications in theoretical Computer Science [3], logics [4], Molecular Biology's DNA and RNA sequences [5], and homological theory [6].

In the year 1971, Horst Herrlich [7] presented a novel sub-branch of mathematics termed as "Categorical Topology". It's a branch of mathematics that straddles general topology and category theory. On a single signal, categorical thoughts, sensations, and consequences are applied to topological settings, assisting with the organization of the massive amount of topological facts.

Classical T_0 separation of topology plays a significant role not solely in mathematics but to get another characterization of locally semi-simple coverings in terms oflight morphisms in algebraic topology [8], however, additionally in computing wherever, this conception corresponds to access the values through observations [9]. Moreover, T_0 axiom is used where topology on Hausdorff space fails to build models [11] , [10] i.e., topological models in lambda calculus and denotational semantics of computer programming language. Furthermore, it is used to classify digital lines in digital topology ([12] and [29]). Due to extensive implementation of the T_0 separation axiom, many mathematicians such as Brummer [14], Marny [15], Hoffmann [16], Harvey [17] and Baran [18] extended this concept to topological category. Also, relationships between various generalizations of T_0 space are analyzed by Weck-Schwarz [19] in year 1991 and Baran [20] in 1995. And in 1991, Baran [18] introduced classical local $\overline{T_0}$ and local T_1 of topology in Categorical Topology in terms of initial and discrete structures. Closure operator has been widely used in calculus ([21] and [22]), algebra ([25], [26], [27]), logic ([23]) and topology ([28], [29]).

In the year 1940, G.Birkhoff [26] noticed that, a complete lattice is a class of all closed sets of closure space. Many authors investigated his work that is, connection among closure spaces and complete lattices and we can find its generalization in [30]. G. Aumann [31] studied in the social sciences on contact relations, while B. Ganter and R. Wille [32] worked in data analysis and knowledge representation on formal contexts, and both employed comparable ideas. Closure operators have been applied in quantum logic and physical systems representation theory [33], [34] in recent years. Considering the huge importance of closure structures, it's been generalized through introducing a few appropriate quantales on it [35]. This motivates us to recollect separation properties of topological category of quantale valued closure spaces.

In this dissertation, the category of Quantale-valued Closure spaces is taken into consideration. Initially, it has been shown that **$\mathfrak{L}$-Cls** is a topological category and its relation to **Cls** is studied and several examples with different quantales are provided. Furthermore, local $\overline{T_0}$ and local T_1 are explicitly characterized for quantale-valued closure space and it is shown that every local T_1 QV-Closure space implies local $\overline{T_0}$ QV-Closure space however, this is not always the case. Moreover, the notion of closedness are examined in topological category of **$\mathfrak{L}$-Cls** and it is examined that this notion of closedness, (i.e., $\{p\}$ is closed) coincides with local $\overline{T_0}$ QV-Closure space. Finally, hereditary and productivity of local $\overline{T_0}$ (resp. local T_1) QV-Closure space are examined. The arrangment of this thesis is done as follows:

In the first chapter, we discuss the basic definitions and concepts of general topology such as topological spaces, continuity, initial topology and lower separation axioms (T_0 and T_1). The prime intension of this chapter is, to define "topological functor". For this, we must know "functors", and for functors we need to explain "categories". In third section we go through the definition of "category" and "functors", their examples and different types. At the end of this chapter we define the topological functor and its examples.

In the second chapter we study quantale its properties and examples. In the very next section of this chapter, closure spaces are mentioned which is a generalization of topological spaces, also we are discussing the continuity and how its initial structure has been defined. In the third section of this chapter, $\mathfrak{L}$ -valued closure space and $\mathfrak{L}$ -valued topological space, their examples, initial and discrete structures are revised.

Categorical definitions of local $\overline{T_0}$ and local T_1 are introduced by Baran [18] which is the extension of the classical definition of local $\overline{T_0}$ and local T_1. In 3rd chapter by using these definitions we characterize local $\overline{T_0}$ and local T_1 quantale valued closure spaces and notion of closedness. Later on, we made an effort to explore the relationship between them and some other properties like hereditary and productivity of local $\overline{T_0}$ and local T_1 quantale valued closure spaces.

In the last chapter of our thesis, we summarizes our thesis with all the results and findings that In **$\mathfrak{L}$-Cls** (the category of Quantale-valued Closure space and continuous maps), Local T_1 implies Local $\overline{T_0}$ = $\{p\}$ is closed.

Chapter 1

Basic Definitions

In this chapter, some prerequisite ideas and concepts are discussed that reader should familiar with. It mainly includes study related to Topological spaces. All the definitions of this section are from [36].

1.1 Topological spaces

Topology is an expanse of mathematics relating to properties that are conserved under incessant distortion including outspreading and meandering, not tearing or jutting. This covers properties such as connectedness, boundaries and continuousness.

Topology was formed as an arena of education, emerging from geometry and set theory through scrutiny of philosophies and impressions, such as, dimensions and transformations. These concepts originated through Leibniz, who in the 17th century [37], perceived Geometria Situs (translating to 'Math of Place') and Analysis Stus (translating to 'Dismantlement of Place'). The expression 'Topology' was derived by Johann Benedict later in the 1800s. Regardless, it was indistinct till the early 20th century, that the likelihood of a topological space was fashioned. However, in the midst of 20th century, topology had emerged as a substantial fragment of mathematics.

In the year 1914, Felix Hausdroff established the term 'Topological Space' and introduced the concept of modern day's terminology, 'Hausdroff Space' [38].

Definition 1.1.1. *Let X be a non empty set and $\tau \subseteq P(X)$, if τ satisfies the following properties then the pair (X, τ) is called a topological space.*

1. X and $\phi \in \tau$.

2. Intersection of the elements of every finite subcollection of τ is in τ i.e., $V_1, V_2, V_3, ..., V_n \in \tau \Rightarrow \bigcap_{k=1}^{n} V_k \in \tau$.

3. If $\{V_i : i \in I\}$ is an indexed family of sets, each of its element belongs to τ, then $\bigcup_{i \in I} V_i \in \tau$.

Example 1.1.2. *Let* $X = \mathbb{R}$, *the topology on* $\mathbb{R}$ *is given by,* $\tau = \{\bigcup_{\alpha \in I} t_\alpha |\ t_\alpha = (u_\alpha, v_\alpha); u_\alpha, v_\alpha \in \mathbb{R}\}$.

Definition 1.1.3. *Every element of* τ *is an open set.*

Example 1.1.4. *The open interval in usual topology is an open set.*

Definition 1.1.5. U *is called a closed set if complement of* U *is open in* τ.

Example 1.1.6. *Closed interval in usual topology is a closed set.*

Definition 1.1.7. *A collection* $\mathfrak{B}$ *of open sets of* X *is a base for the topology* τ *if* $\forall\ p$ *which belongs to* U, $\exists\ G \in \mathfrak{B}$ *with* $p \in G \subset U$.

Example 1.1.8. *The basis of discrete topology is a singleton set.*

Definition 1.1.9. *A class* $\mathfrak{S}$ *of open subsets of* X, *i.e.,* $\mathfrak{S} \subset \tau$, *is a subbase for the topology* τ *on* X *if finite intersection of members of* $\mathfrak{S}$ *form a base for* τ.

Example 1.1.10. *The class* $\mathfrak{S}$ *of all infinite open intervals is a subbase for standard or usual topology* τ_u.

Definition 1.1.11. $A \subseteq X$. *The closure of* A *is the intersection of all closed sets which contain* A.

Example 1.1.12. *Let* $X = \{6, 7, 8, 9\}$ *with topology* $\tau = \{\emptyset, \{6\}, \{7, 8\}, \{6, 7, 8\}, X\}$ *and* C=$\{7, 9\}$ *be a subset of* X.
Open sets:- $\emptyset, \{6\}, \{7, 8\}, \{6, 7, 8\}, X$.
closed sets:- $\emptyset, \{7, 8, 9\}, \{6, 9\}, \{9\}, X$.
Closed sets which contains A*:-* X,$\{7, 8, 9\}$.
$\bar{C} = X \cap \{7, 8, 9\} = \{7, 8, 9\}$.

Definition 1.1.13. *Let X and Y be the nonempty sets and τ and σ be their topologies respectively and $g : (X, \tau) \longrightarrow (Y, \sigma)$ be a map then, g is a continuous map if $\forall\ v \in \sigma$, $g^{-1}(v) \in \tau$.*

Example 1.1.14. *Let τ_l be lower limit topology on $\mathbb{R}$. τ_u be usual topology on it also $g : (\mathbb{R}, \tau_l) \longrightarrow (\mathbb{R}, \tau_u)$ be an identity map , g(y)= y for every y is continous, because inverse image of (c,d) is (c,d) i.e, itself which is in τ_l.*

Example 1.1.15. *Let τ_l be lower limit topology on $\mathbb{R}$ and τ_u be usual topology on it also $g : (\mathbb{R}, \tau_u) \longrightarrow (\mathbb{R}, \tau_l)$ be the identity map , g(y)= y for every y. Then g is not a continous map and the inverse image of $[c, d)$ of τ_l equal to itself which is not in τ_u.*

Definition 1.1.16. *Let X be any set and $(X_i, \tau_i)_{i\in I}$ be the collection of topological spaces and $g_i : X \longrightarrow X_i$ be the mappings then τ_* is an initial topology on X, define as:*

$$\tau_* = \bigcup_{i\in I} \bigcap_{k=1}^{n} \{g_{ik}^{-1}\ (V_{ik})\ ;\ V_{ik}\ \in\ \tau\}.$$

Definition 1.1.17. *$S \subseteq X$. The collection*

$$\tau_S = \{S \cap U \mid U \in \tau\}.$$

is topology on S, called the subspace topology.

Example 1.1.18. *Let X=$\mathbb{R}$, $\tau = \tau_u$ and $\mathbb{Z} \subseteq \mathbb{R}$. Then $\forall\ y \in \mathbb{Z}$*

$$(y - \frac{1}{2}, y + \frac{1}{2}) \cap \mathbb{Z} = \{y\}.$$

That is, $\{y\}$ is the subspace topology on $\mathbb{Z}$ which is induced by τ_u.
Therefore, subspace topology on $\mathbb{Z}$ is a ***discrete topology****.*

Definition 1.1.19. *Let A and B be two topological spaces.The topology with basis $\mathfrak{B}$ is the collection of all sets of the form $W \times X$ is product topology. Where W and X are subsets of A and B respectively.*

Example 1.1.20. *The usual topology τ_u on $\mathbb{R}^2$ is the product topology on $\mathbb{R} \times \mathbb{R}$. Here we have usual topology on $\mathbb{R}$. We know that basis of standard topology on $\mathbb{R}$ is*

$$\mathfrak{B} = \{(s,t)|s,t \in \mathbb{R}, s < t\}.$$

Therefore, basis on $\mathbb{R} \times \mathbb{R}$ is

$$\mathfrak{B}' = \{(s,t) \times (u,v)|s,t,u,v \in \mathbb{R}, s < t, u < v\}.$$

1.2 Separation Axioms

Definition 1.2.1. *(X,τ) is called T_0 or Kolmogrov space if $\forall$ $p,q \in X$ with $p \neq q$, $\exists$ $U \in \tau : p \in U$, $q \notin U$ or $\exists$ $V \in \tau : p \notin V$, $q \in V$.*

Example 1.2.2. *The upper limit topology on $\mathbb{R}$ i.e., τ^l is T_0 space.*

Theorem 1.2.1. *The topological space (Y,σ) is $T_0 \iff \forall\ p, q \in Y$ with $p \neq q$, $\overline{\{p\}} \neq \overline{\{q\}}$.*

Theorem 1.2.2. *All subspaces of T_0 is again T_0.*

Theorem 1.2.3. *Let $\{(Y_i,\sigma_i);\ i \in I\}$ be topological space and $Y = \prod_{i\in I} Y_i$, $\{(Y_i,\sigma_i);\ i \in I\}$ is $T_0 \iff (Y,\sigma_*)$ is T_0 space. Here, σ_* is a product topology on Y.*

Definition 1.2.3. *(X,τ) is called T_1 or accessible space if $\forall$ $p,q \in X$ with $p \neq q$, $\exists$ $U \in \tau : p \in U$, $q \notin U$ and $\exists V \in \tau : p \notin V$, $q \in V$.*

Example 1.2.4. *The co-finite topology & co-countable topology on $\mathbb{R}$ is T_1 space.*

Example 1.2.5. *Standard topology is T_1 space.*

Theorem 1.2.4. *(X,τ) is $T_1 \Leftrightarrow$ every singleton is closed.*

Theorem 1.2.5. *All the subspace of T_1 is again T_1.*

Theorem 1.2.6. *Let $\{(Y_i,\sigma_i);\ i \in I\}$ be topological space and $Y = \prod_{i\in I} Y_i$ if $\{(Y_i,\sigma_i);\ i \in I\}$ is $T_1 \iff (Y,\sigma_*)$ is T_1 space. Here , σ_* is product topology on Y.*

Lemma 1.2.6. *Every T_1 space is also T_0 space but converse is not true.*

1.3 Category Theory

Just as groups are multi-faceted algebraic structures, categories are algebraic structures having various complementing natures, such as logical, geometric, combinatorial and computational. MacLane and Eilenberg [2] proposed a category in 1945 that they called natural transformations functors. Later, MacLane and Eilenberg pointed out that the category's objective was to investigate natural transformations, which demanded the usage of categories. The study of category theory facilitates communication between persons working in diverse domains by developing a new language that is cost-effective in terms of new expression. It also adds new meaning to existing problems by introducing new theorems and structures that are independent of one another. Category theory has applications in theoretical Computer Science [3], logics [4], Molecular Biology's DNA and RNA sequences [5], and homological theory [6].

Definition 1.3.1. *[1] A Category* $\mathbf{C}$ *is a quadruple* $\mathbf{C} = (Obj, hom, id, \circ)$ *which consists of:*

1. $\mathbf{C}$***-objects*** *are members of a class Obj.*

2. *hom is a* $\mathbf{C}$***-morphisms*** *between* $\mathbf{C}$***-objects***.

3. *For every* $\mathbf{C}$*- object Y, A morphism*

$$A \xrightarrow{id_X} A$$

   is the $\mathbf{C}$***-identity***.

4. ***composition law*** *is associated with every* $\mathbf{C}$*-morphism* $A \xrightarrow{g} B$ *and every* $\mathbf{C}$*-morphism* $B \xrightarrow{f} C$*, a* $\mathbf{C}$*-morphism* $A \xrightarrow{f\circ g} C$ *is composition of f and g if,*

   a) ***Associative Property****: Associativity holds in composition. i.e., for each morphisms* $A \xrightarrow{g} B$ *,* $B \xrightarrow{f} C$ *and* $C \xrightarrow{f'} A$ *the following equation holds,*

$$f' \circ (f \circ g) = (f' \circ f) \circ g.$$

*b) **Identity Property**: for each morphism*

$$A \xrightarrow{g} B,$$

then we get,

$$id_B \circ g = g \text{ and } g \circ id_A = g.$$

Example 1.3.2. $\mathbf{C}$= ***Set***: *Objects are class of all sets and* $hom(X, Y) = \{g | g : X \to Y functions\}$, id_X *is an identity map and* $\circ$ *is composition.*

Example 1.3.3. $\mathbf{C}$= ***Top***: *All topological spaces are its objects, continuous maps are its morphisms,* $\circ$ *is composition between them and* $id_{(X,\tau)}$ *is identity morphism on* ***Top***.

Example 1.3.4. $\mathbf{C}$=***POSET***: *Partial ordered sets are its objects, all order that are preserved between partial order sets are its morphisms.*

Example 1.3.5. $\mathbf{C}$=***Grp***, *groups are its objects and morphism is group homomrphism.*

Definition 1.3.6. *([1])*

(i) A category $\mathbf{D}$ *is the subcategory of* $\mathbf{C}$ *if:*

(a) $Obj(\mathbf{D}) \subseteq Obj(\mathbf{C})$,

(b) for every A, $A' \in Obj(\mathbf{C}')$, $hom_{\mathbf{C}'}(A, A') \subseteq hom_{\mathbf{C}}(A, A')$,

(c) for every $\mathbf{C}'$*-object* A*, the* $\mathbf{C}$*-identity on* A *is* $\mathbf{C}'$*-identity on* A,

(d) Composition law of $\mathbf{C}'$ *is the restriction of the composition law of* $\mathbf{C}$ *to the morphisms of* $\mathbf{C}'$.

(ii) $\mathbf{C}'$ *is a full subcategory of* $\mathbf{C}$ *if, for every* $A', A \in Obj(\mathbf{C}')$, $hom_{\mathbf{C}'}(A, A') = hom_{\mathbf{C}}(A, A')$.

Definition 1.3.7. *([1]) Let* $\mathbf{C}$ *be a Category and* $g : X \longrightarrow Y$ *be a morphism between objects of* $\mathbf{C}$*. If* $\forall\ f, h : W \longrightarrow X$ *are morphisms of* $\mathbf{C}$*, if*

$$g \circ f = g \circ h \Rightarrow \ g = f$$

then, g *is **Monomorphism**.*

Example 1.3.8. *Let* $\mathbf{C}$=***Set***, $\forall$ $X, Y \in$ *Obj(**Set**)*. $g : X \longrightarrow Y$ *is a monomorphism* $\Leftrightarrow$ g *is 1 : 1.*

Example 1.3.9. *Let* $\mathbf{C}$=***Top*** , $\forall$ $(X, \tau), (Y, \tau)$ $\in Obj($***Top***$)$, $g : (X, \tau) \longrightarrow$ *(X,* τ'*) is monomorphism* $\Leftrightarrow$ g *is continuous map and 1 : 1.*

Definition 1.3.10. *([1]) Let* $\mathbf{C}$ *be a Category.* $f : X \longrightarrow Y$ *and* $\forall$ g , h : $Y \longrightarrow Z \in$ *Mor(*$\mathbf{C}$*),*

$$g \circ f = h \circ f \Rightarrow g = h.$$

then g *is called* ***Epimorphism***.

Example 1.3.11. *Let* $\mathbf{C}$=***Set***

$$g : X \longrightarrow Y$$

is ***Epimorphism*** $\Leftrightarrow$ g *is Onto*

Example 1.3.12. *Let* $\mathbf{C}$ = ***Top***

$$g : (X, \tau) \longrightarrow (Y, \tau')$$

is ***Epimorphism*** $\Leftrightarrow$ g *is Continuous map and onto.*

Definition 1.3.13. *([1]) Let* $\mathbf{C}$ *be any Category and* $\forall$ E , $F \in$ *Obj(*$\mathbf{C}$*) and* $f : E \longrightarrow F$ $\in$ *hom(*$\mathbf{C}$*) , if* $\exists$ *morphism* $g : F \longrightarrow E$ *Such that,* $f \circ g = 1_E$ *and* $g \circ f = 1_F$.
Then, f *is an* ***Isomorphism*** *of* g.

Example 1.3.14. *In* $\mathbf{C}$=***Top***, $g : (X, \tau) \longrightarrow (Y, \tau')$ *is called isomorphism* $\Longleftrightarrow$ g *is homomorphism.*

Definition 1.3.15. *([1]) If* $g : A \longrightarrow B$ *is monomorphism and epimorphism then ,* g *is called* ***Biomorphism***

Example 1.3.16. *In* ***Top***

$$g : (X, \tau) \rightarrow (y, \tau')$$

is ***Biomorphism*** $\Longleftrightarrow$ *g is bijective and Continous.*

Definition 1.3.17. *([1]) Let* $\mathcal{E}$ *and* $\mathbf{C}$ *be Categories.* $\mathfrak{U} : \mathbf{C} \longrightarrow \mathcal{E}$ *is called a functor if*

(i) $\forall\ X \in Obj(\mathbf{C}) \ \Rightarrow \mathfrak{U}(X) \in Obj(\mathcal{E})$

(ii) $f : X \longrightarrow Y \in hom(\mathbf{C}) \Rightarrow \mathfrak{U}(f) : \mathfrak{U}(X) \longrightarrow \mathfrak{U}(Y) \in hom(\mathcal{E})$.

(iii) $\mathfrak{U}$ *maintains identity morphism; i.e,*

$$\mathfrak{U}(1_A) = 1_{\mathfrak{U}(A)}$$

(iv) $\mathfrak{U}$ *maintains composition; i.e, If* $X \xrightarrow{g} Y \xrightarrow{f} Z \in hom(\mathbf{C})$
then,
$\mathfrak{U}(f \circ g) = \mathfrak{U}(f) \circ \mathfrak{U}(g)$.

Example 1.3.18. *Let* $\mathfrak{U}$ *be functor ,* $\mathfrak{U}$ *:* ***Top*** $\longrightarrow$ ***Set*** *is defined as:* $\mathfrak{U}(X, \tau) = X$ *and* $g : (X, \tau) \longrightarrow (Y, \sigma)$ *is a map,* $\mathfrak{U}(g) = g$ *is a functor.*

Definition 1.3.19. *([1]) Let* $\mathfrak{U} : \mathbf{C} \to \mathcal{E}$ *be a functor, if* $\forall\ A,B \in Obj((\mathbf{C})$ *and* $\forall\ g, f : A \to B \in hom(\mathbf{C})$

$$\mathfrak{U}(g) = \mathfrak{U}(f) \Rightarrow g = f.$$

Then $\mathfrak{U}$ *is called faithfull functor.*

Example 1.3.20. $\mathfrak{U}$ *:* ***Top*** $\to$ ***Set*** *is faithfull functor.*

Definition 1.3.21. *([1]) Let* $\mathfrak{U} : \mathbf{C} \longrightarrow \mathcal{E}$ *be a functor, if* $\forall\ A,\ B \in Obj(\mathbf{C})$ *and* $\forall\ g : \mathfrak{U}(A) \to \mathfrak{U}(B)$ *morphism* $\exists\ f : A \longrightarrow B$ *morphism such that*

$$\mathfrak{U}(g) = f.$$

then $\mathfrak{U}$ *is called a full functor.*

Definition 1.3.22. *([1]) Let* $\mathfrak{U} : \mathbf{C} \longrightarrow \mathcal{E}$ *be a full functor. If* $\forall\ f : A \longrightarrow A$ *in hom(*$\mathbf{C}$*) and* $\mathfrak{U}(f) = 1_A = 1_{\mathfrak{U}(A)}$ *and f is isomorphism.* $\Rightarrow f$ *is identity, then it is called an Amnestic functor.*

Definition 1.3.23. *([1]) If* $\mathfrak{U} : \mathcal{C} \longrightarrow \mathcal{E}$ *is a faithfull and amnestic then* $\mathfrak{U}$ *is called concrete functor.*

Example 1.3.24. $\mathfrak{U}$: ***Top*** $\longrightarrow$ ***Set*** *is amnestic and faithful functor therefore it is also concrete functor but not full functor because in* ***Top*** *morphisms may not preserve.*

Example 1.3.25. $\mathfrak{U}$: ***Grp*** $\longrightarrow$ ***Set*** *is amnestic and faithfull therefore it is concrete functor.*

1.4 Categorical topology

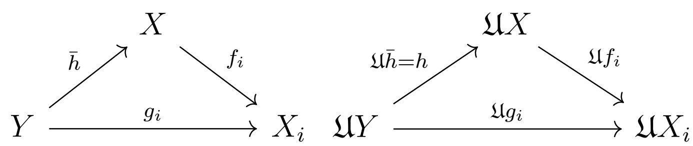
In the year 1971, Horst Herrlich [7] presented a novel sub-branch of mathematics termed as "Categorical Topology". It's a branch of mathematics that straddles general topology and category theory. On a single signal, categorical thoughts, sensations, and consequences are applied to topological settings, assisting with the organization of the massive amount of topological facts.

Definition 1.4.1. *([1]) Let* $\mathbf{C}$ *and* $\mathcal{E}$ *be the two Categories and* $\mathfrak{U} : \mathbf{C} \to \mathcal{E}$ *be a functor, if* $\mathfrak{U}$ *satisfies the followings, then* $\mathfrak{U}$ *is called* ***Topological functor****.*

(i) $\mathfrak{U}$ *is concrete functor. i.e,* $\mathfrak{U}$ *is amnestic and faithfull functor.*

(ii) $\mathfrak{U}$ *consists of fibers i.e;* $\forall\ B \in Obj(\mathcal{E})$; $\mathfrak{U}^{-1}(B)$ *is a set and* $\mathfrak{U}^{-1}(B) = \{X \in obj(\mathbf{C}); \mathfrak{U}(X) = B\}$.

(iii) Existance of initial lift i.e., A source $\mathrm{S} = (X \xrightarrow{f_i} X_i)_{i\in I}$ *is* $\mathfrak{U}-$*initial if for every source* $\tau = (Y \xrightarrow{g_i} X_i)_{i\in I}$ *in* $\mathbf{C}$ *along the same co-domain same as* S *and every* $\mathcal{E}$*-morphism* $\mathfrak{U}Y \xrightarrow{h} \mathfrak{U}X$ *along* $\mathfrak{U}\tau = \mathfrak{U}\mathrm{S} \circ h$, $\exists$ *a single* $\mathbf{C}-$*morphism* $Y \xrightarrow{\bar{h}} X$ *along* $\tau = \mathrm{S} \circ \bar{h}$ *and h=* $\mathfrak{U}\bar{h}$.



Example 1.4.2. $\mathfrak{U}$*:* ***Top*** $\rightarrow$ ***Set*** *is a topological functor, and its initial lift is its initial topology.*

Example 1.4.3. $\mathfrak{U}$*:* ***Grp*** $\rightarrow$ ***Set*** *is not topological functor since initial lift doesnot exist in group. In other words, the subset of a group may not always be its subgroup.*

Chapter 2

Quantale Value Closure Spaces

2.1 Quantale structures

Definition 2.1.1. *([35]) Let $X \neq \phi$, $\leq$ be the relation on X. if it satisfies the followings:*

(i) ***Anti symmetry****:* $\forall\ s, t \in X,\ s \leq t \wedge\ t \leq s \Rightarrow s = t.$

(ii) ***Reflexivity****:* $\forall\ s \in X,\ s \leq s.$

(iii) ***Transitivity****:* $\forall\ s, t, u \in X,\ s \leq t \wedge t \leq u \Rightarrow s \leq u.$

Then, $\leq$ on X is called partial order.
Moreover, if $\leq$ is a partial order on X, then the pair $(X, \leq)$ is called partially ordered set or poset.

Example 2.1.2. $\mathbb{R}$ *is a poset by $\leq$ order.*

Definition 2.1.3. *([39]) All subsets of X, which is poset $(X, \leq)$ have infimum $(\bigwedge)$ and supremum $(\bigvee)$, then $(X, \leq)$ is called a complete lattice. The bottom and top elements are denoted by $\perp$ and $\top$ respectively.*

Example 2.1.4. *All finite posets are complete lattice.*

Definition 2.1.5. *([39]) The qudruple $(\mathfrak{L}, \leq, *, k)$ is an unital quantale, if*
(i) $(\mathfrak{L}, *)$ is a ***monoid structure****, where $*$ is an operation, and k is an identity element.*
(ii) $\forall\ \beta, \alpha_i \in \mathfrak{L}$,

$$(\bigvee_{i \in I} \alpha_i) * \beta = \bigvee_{i \in I} (\alpha_i * \beta).$$

And

$$\beta * (\bigvee_{i\in I} \alpha_i) = \bigvee_{i\in I}(\beta * \alpha_i).$$

Definition 2.1.6. *[39]*

i. The qudruple $(\mathfrak{L}, \leq, *, k)$ *is a commutative quantale, if* $(\mathfrak{L}, *)$ *is a commutative monoid.*

ii. The qudruple $(\mathfrak{L}, \leq, *, k)$ *is an integral quantale if* $\alpha * \top = \top * \alpha$.

Definition 2.1.7. *In quantale, if* $s \in L$ *and* $s \neq \top$*, then* s *is prime element if*

$$a \wedge b \leq s \Rightarrow a \leq s \text{ or } b \leq s. \ \ \forall a, b \in L.$$

Specifically, when $s = \perp$ *then it is called prime bottom element.*

Example 2.1.8. *([40])* $\mathfrak{L} = ([0, \infty], \geq, +, 0)$ *is an* ***Lawvere's quantale*** *with* $+$ *as binary operation and 0 is an identity element.*

Example 2.1.9. $\mathfrak{L} = ([0, 1], \leq, *, k)$ *is an integral and commutative quantale. Where* $\forall$ $a,b \in [0, 1]$, $a * b = \min\{a, b\}$ *and an identity element* $k = 0$.

Example 2.1.10. $\mathfrak{L}$=$([0, 1] \cup \{\perp = -1, \top = \infty\}, \leq, ., 1)$ *is a quantale.*

Example 2.1.11. *([40])* $\mathfrak{L} = \mathit{2} = (\{\perp, \top\}, \leq, \wedge, \perp)$ *is a quantale.*

Example 2.1.12. $\mathfrak{L} = ([0, 1], \leq, *, 1)$ *is a quantale, where* $\forall x, y \in [0, 1]$, $x * y = x.y$.

2.2 Closure Spaces

In the year 1940, G.Birkhoff [26] noticed that, a complete lattice is a class of all closed sets of a closure space. Many authors investigated his work that is, connection among closure spaces and complete lattices and we can find its generalization in [30]. G. Aumann [31] studied in the social sciences on contact relations, while B. Ganter and R. Wille [32] worked in data analysis and knowledge representation on formal contexts, and both employed comparable ideas. Closure operators have been applied in quantum logic and physical systems representation theory [33], [34] in recent years.

Definition 2.2.1. *([41]) Let $X \neq \phi$,and $cl \subseteq P(X)$. If cl satisfies the following, then cl is called closure structure and the pair (X, cl) is called closure space.*

(i) ϕ , $X \in cl$.

(ii) $\forall\ i \in I, \forall\ U_i \in cl \Rightarrow \bigcup_{i \in I} U_i \in cl$.

Definition 2.2.2. *([41]) Let (X, cl),(X', cl') be closure spaces. A function $g : (X, cl) \longrightarrow (X', cl')$ is continuous if $U \in cl' \Rightarrow g^{-1}(U) \in cl$.*

Remark 2.2.1. *[42]*

(i) ***Cls*** *is the category, closure spaces are its objects and continuous maps are its morphisms.*

(ii) ***Top*** *is embedded in* ***Cls*** *as full subcategory.*

Theorem 2.2.1. *([43]) A source $\{g_i : (X, cl) \longrightarrow (X'_i, cl'_i), \forall i \in I\}$ is an initial in* ***Cls*** *iff cl= $\{U \subset X : U = \cup_{i \in I}\ g_i^{-1}(U_i), U_i \in cl_i\}$.*

Example 2.2.3. *Let X=$\{1, 2, 3, 4\}$, cl=$\{\phi, \{1, 2\}, \{2, 3\}, \{1, 3, 2\}, X\}$ is the closure space. Note that cl is not a topology since $\{1, 2\} \cap \{2, 3\}$ is not in cl.*

2.3 Quantale Valued Closure Spaces

In 2017, H.Lai and W. Tholen [35] generalized Closure Spaces by introducing a suitable quantale on it.

Definition 2.3.1. *([35]) A map $c : P(X) \longrightarrow (\mathfrak{L}, \leq, *, k)^X$ is an $\mathfrak{L}-$Valued closure structure on X which satisfies,*

(i) *(Reflexivity)* $\forall\ y \in U \subseteq X$; $k \leqslant (cU)(y)$.

(ii) *(Transitivity)* $\forall\ U, V \subseteq X$, $y \in X$.

$$(\bigwedge_{x \in U} (cU)(x)) * (cV)(y) \leq (cU)(y).$$

Then, (X,c) is an $\mathfrak{L}$-valued closure space.

Definition 2.3.2. *([35]) An $\mathfrak{L}$-valued topological structure on X is a map $c : PX \longrightarrow (\mathfrak{L}, \leq, *, k)^X$ satisfies:*

(i) c is an $\mathfrak{L}$-valued closure structure on X.

(ii) $\forall\ y \in X$ and $\emptyset$, the empty set:

$$(c\phi)(y) = \bot.$$

(iii) $\forall\ y \in X$ and $\forall\ U, V \subseteq X$:

$$c(U \cup V)(y) = (cU)(y) \bigvee (cV)(y).$$

Then, (X,c) is called an $\mathfrak{L}$-valued topological space.

Definition 2.3.3. *[35] A map $g : (X, c) \longrightarrow (X', c')$ is continuous if $(cU)(y) \leq c'(gU)(gy)$, $\forall\ U \subseteq X$ and $y \in X$.*

Remark 2.3.1. *[35]*

*(i) $\mathfrak{L}$-**Cls** (resp. $\mathfrak{L}$-**Top**) is the category with $\mathfrak{L}$-valued closure spaces (resp. $\mathfrak{L}$ -valued topological spaces) as objects and contractive maps as its morphisms.*

*(ii) $\mathfrak{L}$-**Top** is the full subcategory of $\mathfrak{L}$-**Cls**.*

Example 2.3.4. *[35] For terminal quantale **1**, **1-Cls**$\cong$**1-Top** $\cong$ **Set**.*

Example 2.3.5. *[35] Consider $\mathfrak{L}$=($\mathit{2}, \leq, \wedge, \top$), where $\mathit{2}$=$\{\bot < \top\}$. Then **2-Cls** $\cong$ **Cls** and **2-Top** $\cong$ **Top**.*

Example 2.3.6. *[40] If quantale $\mathfrak{L} = (([0, \infty], \geq), +, 0)$ (**Lawvere's quantale**), then $\mathfrak{L}$-**Top** $\cong$ **App**, where **App** is the category of approach spaces and contraction maps [48]. Moreover, we have $\mathfrak{L}$-**Cls** $\cong$ **Cls**$'$, where **Cls**$'$ is the category considered in [49].*

Definition 2.3.7. *[35] Let (X_i, c_i) be $\mathfrak{L}$-valued closure space and $((Y, c) \xrightarrow{g_i} (X_i, c_i))_{i \in I}$, $\forall\ x \in X$ and $A \subseteq X$ then,*

$$(\bar{c}A)(x) = \bigwedge_{i \in I} c_i(g_i A)(g_i x),$$

is an initial structure on X.

Definition 2.3.8. *[35] Let X be a non empty set and (X,c) be an* $\mathfrak{L}$*-valued closure space.*

(i) The discrete $\mathfrak{L}$*-valued closure structure on X is given by, if* $\forall y \in X, \forall U \subseteq X$ *then,*

$$(c_{disc}U)(y) = \begin{cases} k, & y \in U \\ \bot, & y \notin U. \end{cases}$$

(ii) The indiscrete $\mathfrak{L}$*-valued closure structure on X is given by* $(c_{ind}U)(y) = \top$.

Chapter 3

Notion Of Closedness In Topological Category Of Quantale Valued Closure Space

3.1 Local $\overline{T_0}$ Quantale Valued Closure Spaces

Definition 3.1.1. *([44]) Let (Y, τ) be a topological space and $p \in Y$. (Y, τ) is called local $\overline{T_0}$ or $\overline{T_0}$ at p (in classical sense) if $\forall\ y \in X$ with $y \neq p$, $\exists\ U \subseteq \tau$ of p not containing y or $\exists\ V \subseteq \tau$ of y not containing p.*

Example 3.1.2. *The topology $\tau = \{\emptyset, \{a\}, X\}$ on X=$\{a, b, c\}$ is not $\overline{T_0}$ but $\overline{T_0}$ at a. Moreover, it is neither at b nor at c.*

Theorem 3.1.3. *([44]) Let (Y, τ) be a topological space. (Y, τ) is $\overline{T_0}$ iff (Y, τ) is $\overline{T_0}$ at p, $\forall\ p \in X$.*

In year 1991, Baran ([18]) introduced classical local $\overline{T_0}$ of topology in Categorical Topology in terms of initial and discrete structures.

Definition 3.1.4. *([18]) Let E be any set and a point $p \in E$. A Wedge product $E \bigvee_p E$ of E at point p is the two disjoint copies of E at point p.*
A point y in $E \bigvee_p E$ is y_1 (resp. y_2) if it is in 1st component (resp. 2nd component). Also E^2 is the cartesian product of E.

Definition 3.1.5. *([18]) Let* $A_p : E\bigvee_p E \longrightarrow E^2$ *be the principal p axis map, defined by*

$$y_i \longmapsto A_p(y_i) = \begin{cases} (y,p), & i = 1 \\ (p,y), & i = 2. \end{cases}$$

Definition 3.1.6. *([18]) Let* $S_p : E\bigvee_p E \longrightarrow E^2$ *be the Skewed p axis map, defined by*

$$y_i \longmapsto S_p(y_i) = \begin{cases} (y,y), & i = 1 \\ (p,y), & i = 2. \end{cases}$$

Definition 3.1.7. *([18]) Suppose* $\nabla_p : E\bigvee_p E \longrightarrow E^2$ *is the fold map at p, defined by*

$$y_i \longmapsto \nabla_p(y_i) = y, \quad \forall\ i = 1, 2.$$

Definition 3.1.8. *([18]) Suppose* (E,τ) *is a top. space.* (E,τ) *is called local* $\overline{T_0}$ *or (*$\overline{T_0}$ *at p) iff the initial topology on* $E\bigvee_p E$ *induced by* $\{E\bigvee_p E \xrightarrow{A_p} E^2$ *and* $E\bigvee_p E \xrightarrow{\nabla_p} \mathfrak{U}D(E)\ \}$ *is disc. top. space.*

Theorem 3.1.1. *([44]) Suppose* (E,τ) *is a top. space.* (E,τ) *is local* $\overline{T_0}$ *in (classical sense) iff* (E,τ) *is local* $\overline{T_0}$*.*

Now, considering categorically, we have following definition given in ([18]).

Definition 3.1.9. *([18]) Suppose* $\mathfrak{U} : \mathbf{C} \longrightarrow \mathbf{Set}$ *is a top. functor* $F \in Obj(\mathbf{C})$ *with* $\mathfrak{U}(F)=E$ *and* $p \in E$*.*
E is local $\overline{T_0} \Longleftrightarrow$ *initial lift of* $\{E\bigvee_p E \xrightarrow{A_p} E^2$ *and* $E\bigvee_p E \xrightarrow{\nabla_p} \mathfrak{U}D(E)\ \}$ *is discrete.*

Theorem 3.1.10. *Let (X,c) be an* $\mathfrak{L}$*-valued closure space and* $p \in X$*. (X,c) is local* $\overline{T_0}$ $\Longleftrightarrow \forall\ x \in X$ *with* $x \neq p$ *,* $\exists\ U \subseteq X$ *with* $x \in U$ *,* $p \notin U$ *and* $\exists\ V \subseteq X$ *with* $p \in V$ *,* $x \notin V$ *such that,*

$$\bot = \bigwedge\{c(U)(p), c(V)(x), k\},$$

where k is an identity element.

Proof: *Let (X,c) be a local* $\overline{T_o}$ *and* $\forall\ x \in X$ *with* $x \neq p$. *Let* $B \subseteq X \bigvee_p X$ *and* $x_1 \in X \bigvee_p X$ *with* $x_1 \notin B$.

Note that,

$$c_{disc}(\nabla_p B)(\nabla_p x_1) = c_{disc}(\nabla_p B)(x) = k,$$

where k is an identity element.

$$k \leq c(proj_1 A_p B)(proj_1 A_p x_1) = c(proj_1 A_p B)(x) = c(V)(x).$$

Since $x \in (proj_1 A_p B)$.

And

$$c(proj_2 A_p B)(proj_2 A_p x_1) = c(proj_2 A_p B)(p) = c(U)(p).$$

Since $x_1 \notin B$ *and (X,c) is* $\overline{T_o}$, *by definition 2.3.7*

$$c(B)(x_1) = \bigwedge\{c(proj_1 A_p B)(proj_1 A_p x_1), c(proj_2 A_p B))(proj_2 A_p x_1), c_{disc}(\nabla_p B)(\nabla_p x_1)\}.$$

$$\perp = \bigwedge\{c(proj_1 A_p B)(x), c(proj_2 A_p B)(p), k\},$$

$$\perp = \bigwedge\{c(proj_1 A_p B)(x), c(proj_2 A_p B)(p), k\},$$

$$\perp = \bigwedge\{c(V)(x), c(U)(p), k\}.$$

Conversely: Let $\overline{c}$ *be an initial structure induced by* $A_p : X \bigvee_p X \longrightarrow (X^2, c^2)$ *and* ∇_p*:* $X \bigvee_p X \longrightarrow (X, c_{disc})$, *where* c^2 *is a product structure on* X^2 *and* $proj_i : X^2 \longrightarrow X$ *, i=1,2 are projection maps, and* c_{disc} *is a discrete structure on X.*

Let $w \in X \bigvee_p X$ *and B be a non empty subset of* $X \bigvee_p X$. $\forall\ x \in X$ *with* $x \neq p$, $\exists\ U \subseteq X$ *with* $x \in U$, $p \notin U$ *and* $\exists\ V \subseteq X$ *with* $p \in V$, $x \notin V$ *such that*

$$\perp = \bigwedge\{c(U)(p), c(V)(x), k\}.$$

Case I: *If* $\nabla_p w = p \in \nabla_p B$ *for some* $p \in X$, *then* $w = p_1 = p_2 \in B$, *it follows from definition 2.3.7,*

$$(\overline{c}B)(w) = k.$$

Case II: *If* $\nabla_p w = p \notin \nabla_p B$, *by definition 2.3.6,*

$$(c_{dis}\nabla_p B)(\nabla_p w) = \bot.$$

And consequently,

$$(\bar{c}B)(w) = \bigwedge\{(c(proj_1 A_p B))(proj_1 A_p w), (c(proj_2 A_p A))(proj_2 A_p w), (c_{dis}(\nabla_p A))(\nabla_p w)\},$$

$$(\bar{c}B)(w) = \bot.$$

Case III: *If* $\nabla_p w = x$ *for some* $x \in X$ *with* $x \neq p$, *it follows that,* $w = x_1$ *or* $w = x_2$.

a. *If* $w = x_1 = x_2 \in B$, *then* $\nabla_p w \in \nabla_p B$ *and* $proj_i A_p w \in proj_i A_p B$ *for i=1,2 , by definition 2.3.7*

$$(\bar{c}B)(w) = \bigwedge\{(c(proj_i A_p B))(proj_i A_p w), (c_{dis}(\nabla_p A))(\nabla_p w)\},$$

$$(\bar{c}B)(w) = k.$$

b. *If* $w = x_1, x_2 \notin B$, *then* $\nabla_p w \notin \nabla_p B$, *by definition 2.3.7*

$$(\bar{c}B)(w) = \bot.$$

c. *If* $w = x_1 \notin B$ *but* $x_2 \in B$ *,by def 2.3.8*

$$(c_{dis}\nabla_p B)(\nabla_p w) = k.$$

And

$$c(proj_1 A_p B))(proj_1 A_p w) = c(proj_1 A_p B)(p),$$

$$c(proj_2 A_p B))(proj_2 A_p w) = c(proj_2 A_p B)(x).$$

It follows that,

$$(\bar{c}B)(w) = \bigwedge\{c(proj_i A_p B)(proj_i A_p w), c_{dis}(\nabla_p A)(\nabla_p w)\},$$

$$(\bar{c}B)(w) = \bot.$$

Therefore , $\forall\ w \in X \bigvee_p X$ *and* $B \subseteq X \bigvee_p X$, *we have,*

$$(\overline{c}B)(w) = \begin{cases} k, & w \in B \\ \perp, & w \notin B. \end{cases}$$

By definition 2.3.8, $\overline{c}$ *is an* $\mathfrak{L}$*-valued discrete structure on* $X \bigvee_p X$*, and by definition of local* $\overline{T_o}$*.*
Thus, (X, c) *is local* $\overline{T_o}$*.*

Theorem 3.1.11. *Let (X,c) be an* $\mathfrak{L}-$*valued closure space and* $p \in X$*, where* $\mathfrak{L}$ *is an integral quantale and* $\mathfrak{L}$ *has a prime bottom element. (X,c) is local* $\overline{T_o} \iff \forall\ x \in X$ *with* $x \neq p$ *,* $\exists\ U \subseteq X$ *with* $x \in U$ *,* $p \notin U$ *and* $\exists\ V \subseteq X$ *with* $p \in V$ *,* $x \notin V$ *such that*

$$c(U)(p) = \perp \ \ or \ \ c(V)(x) = \perp.$$

where $\perp$ *is a bottom element.*

Proof. : It follows from theorem 3.1.10 and definition of *integral quantale* and *prime bottom element.*

□

3.2 Local T_1 Quantale Valued Closure Spaces

Definition 3.2.1. *([44]) Suppose* (Y, τ) *is a topological space and* $p \in Y$*.* (Y, τ) *is called local* T_1 *or* T_1 *at p (in classical sense) iff* $\forall\ y \in Y$ *with* $y \neq p$*,* $\exists\ U \subseteq \tau$ *of p not containing y and* $\exists\ V \subseteq \tau$ *of y not containing p.*

Example 3.2.2. *The topology* $\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, X\}$ *on X={1,2,3,4} is not* T_1 *space but* T_1 *at* 1*.*

Theorem 3.2.3. *([44]) Let* (Y, τ) *be a topological space.* (Y, τ) *is* T_1 *iff* (Y, τ) *is* T_1 *at p,* $\forall\ p \in Y$*.*

In 1991, Baran [18] introduced classical local T_1 of topology in Categorical Topology in terms of initial and discrete structures.

Definition 3.2.4. *([18]) Suppose* (E,τ) *is a top. space.* (E,τ) *is called local* T_1 *or (*T_1 *at p) iff the initial top. on* $E\bigvee_p E$ *induced by {*$E\bigvee_p E \xrightarrow{S_p} E^2$ *and* $E\bigvee_p E \xrightarrow{\nabla_p} \mathfrak{U}D(E)$ *} is disc. top. space.*

Theorem 3.2.5. *([44]) Let* (Y,τ) *be a topological space.* (Y,τ) *is local* T_1 *in (classical sense) iff* (Y,τ) *is local* T_1*.*

Now, considering categorically, we have following def given in [18].

Definition 3.2.6. *([18]) Suppose* $\mathfrak{U}:\mathbf{C}\longrightarrow\mathbf{Set}$ *is a top. functor* $F\in$ *Obj(*$\mathbf{C}$*) with* $\mathfrak{U}$*(F)=E and p*$\in$ *E.*
E is local $T_1 \Longleftrightarrow$ *initial lift of {*$E\bigvee_p E \xrightarrow{S_p} E^2$ *and* $E\bigvee_p E \xrightarrow{\nabla_p} \mathfrak{U}D(E)$ *} is discrete.*

Theorem 3.2.7. *Let* (X,c) *be an* $\mathfrak{L}$*-valued closure space and* $p\in X$*.* (X,c) *is local* T_1 $\Longleftrightarrow \forall\ x\in X$ *with* $x\neq p$*,* $\exists\ U\subseteq X$ *with* $x\in U$ *,* $p\notin U$ *and* $\exists V\subseteq X$ *with* $p\in V$ *,* $x\notin V$ *such that*

$$c(U)(p)\wedge k=\bot=c(V)(x)\wedge k.$$

Proof. Suppose, (X,c) is local T_1 and $\forall\ x\in X$ with $x\neq p$. Let $B\subseteq X\bigvee_p$ X and $x_1\in$ X $\bigvee_p$ X with $x_1\notin$ B.
Note that

$$c_{disc}(\nabla_p B)(\nabla_p x_1)=c_{disc}(\nabla_p B)(x)=k.$$

Where k is an identity element.

$$k\leq c(proj_1S_pB)(proj_1S_px_1)=c(proj_1S_pB)(x)=c(V)(x).$$

Since x $\in proj_1S_pB$.
And

$$,c(proj_2S_pB)(proj_2S_px_1)=c(proj_2S_pB)(x),$$

since $x_1\notin$ B and (X,c) is local T_1, by definition 2.3.7,

$$c(B)(x_1)=\bigwedge\{c(proj_1S_pB)(proj_1S_px_1),c(proj_2S_pB)(proj_2S_px_1),c_{disc}(\nabla_pB)(\nabla_px_1)\},$$

$$\bot=\bigwedge\{c(proj_1S_pB)(x),(c(proj_2S_pB)(x),k\},$$

$$\perp = \bigwedge\{c(V)(x), k\},$$

and consequently,

$$c(V)(x) \wedge k = \perp.$$

Similarly, let B $\subseteq$ X $\bigvee_p$ X and $x_2 \in$ X $\bigvee_p$ X with $x_2 \notin$ B,then we have

$$\perp = \bigwedge\{c(U)(p), k\},$$

and consequently,

$$c(U)(p) \wedge k = \perp.$$

Conversely: Let $\bar{c}$ be an initial structure induced by $S_p : X \bigvee_p X \longrightarrow (X^2, c^2)$ and ∇_p: $X \bigvee_p X \longrightarrow$ (X,c_{disc}), where c^2 is a product structure on X^2 and $proj_i : X^2 \longrightarrow$ X , i=1,2 are projection maps and c_{disc} is a discrete structure on X.

Let w $\in X \bigvee_p X$ and B be the non empty subset of $X \bigvee_p X$ and $\forall$ x $\in$ X with x $\neq$ p $\exists$ U $\subseteq$ X with x $\in$ U , p $\notin$ U and $\exists$ V $\subseteq$ X with p $\in$ V , x $\notin$ V such that,

$$c(U)(p) \wedge k = \perp = c(V)(x) \wedge k.$$

Case I: If $\nabla_p w =$ p $\in \nabla_p$B then, w$= p_1 = p_2 \in$ B,

it follows from definition 2.3.8,

$$(\bar{c}B)(w) = k.$$

Case II: If ∇_pw=p $\notin \nabla_p B$,by definition 2.3.8

$$(c_{dis}\nabla_p B)(\nabla_p w) = \perp.$$

And consequently,

$$(\bar{c}B)(w) = \bigwedge\{c(proj_1 S_p B)(proj_1 S_p w), c(proj_2 S_p B)(proj_2 S_p w), c_{dis}(\nabla_p A)(\nabla_p w)\},$$

$$(\bar{c}B)(w) = \perp.$$

Case III: If $\nabla_p w = x$ for some x $\in$ X with x $\neq$ p, it follows that, $w = x_1$ or w$=x_2$

a. If $w = x_i \in$ B for $i = 1, 2$, then $\nabla_p w \in \nabla_p$B and $proj_i S_p w \in proj_i S_p B$ forx (X, c) , by definition 2.3.8,

$$(\bar{c}B)(w) = \bigwedge\{c(proj_i S_p B)(proj_i S_p w), c_{dis}(\nabla_p A)(\nabla_p w)\},$$

$$(\bar{c}B)(w) = k.$$

b. If w= $x_i \notin$ B for $i = 1, 2$, then $\nabla_p w \notin \nabla_p$B , by definition 2.3.8

$$c_{dis}(\nabla_p B)(\nabla_p w) = c_{dis}(\nabla_p B)(x) = \perp,$$

and consequently,

$$(\bar{c}B)(w) = \perp.$$

c. If w = $x_1 \notin$ B but $x_2 \in$ B , by definition 2.3.8

$$c_{dis}(\nabla_p B)(\nabla_p w) = c_{dis}(\nabla_p B)(x) = k.$$

And

$$c(proj_1 S_p B)(proj_1 S_p w) = c(proj_1 S_p B)(x) = c(V)(x),$$

$$c(proj_2 S_p B)(proj_2 S_p w) = c(proj_2 S_p B)(x) = c(U)(p).$$

By definition 2.3.7

$$(\bar{c}B)(w) = \bigwedge\{c(proj_i S_p B)(proj_i S_p w), c_{dis}(\nabla_p B)(\nabla_p w)\},$$

$$(\bar{c}B)(w) = \bigwedge\{c(V)(x), k\} = \perp.$$

Also,

$$c(V)(x) \wedge k = \perp.$$

Similar to above, if w $=x_2 \notin$ B but $x_1 \in$ B, then we have

$$c(U)(p) \wedge k = \perp.$$

Therefore , $\forall$ w $\in$ X $\bigvee_p$ X and B $\subseteq$ X $\bigvee_p$ X, we have,

$$(\overline{c}B)(w) = \begin{cases} k, & w \in B \\ \perp, & w \notin B. \end{cases}$$

by definition 2.3.8, $\overline{c}$ is an $\mathfrak{L}$-valued discrete structure on $X \bigvee_p X$ and by definition of local T_1.

Thus, (X,c) is local T_1. □

Theorem 3.2.8. *Let (X,c) be an $\mathfrak{L}-$valued closure space and $p \in X$, where $\mathfrak{L}$ is an integral quantale. (X,c) is local T_1 $\Longleftrightarrow$ $\forall\ x \in X$ with $x \neq p$, $\exists\ U \subseteq X$ with $x \in U$, $p \notin U$ and $\exists\ V \subseteq X$ with $p \in V$, $x \notin V$ such that,*

$$c(U)(p) = \perp = c(V)(x),$$

where $\perp$ is a bottom element.

Proof. It follows from theorem 3.2.7 and definition of *integral quantale.* □

3.3 Notion Of Closedness in Quantale Valued Closure Spaces

Definition 3.3.1. *([46]) Let (Y, τ) be topological space and $p \in Y$. $\{p\}$ is closed set (in classical sense) iff $\{p\}^c$ is open set. i.e. $\{p\}^c \in \tau$.*

In 1991, Baran [18] introduced classical notion of closedness of general topology in Categorical Topology in terms of initial and discrete structures. Therefore, infinite wedge product has been introduced for this purpose.

Definition 3.3.2. *([46]) Let E be any set and a point $p \in E$. The infinite Wedge product $\bigvee_p^\infty E$ of E at point p is, the infinite disjoint copies of E at point p.*

Definition 3.3.3. *([46]) Let $A_p^\infty : \bigvee_p^\infty E \longrightarrow E^\infty$ be the Infinite principal p axis map, defined by*

$$y_i \longmapsto A_p^\infty(y_i) = (p, p, ..., y, ...)\ , \qquad \forall\ i \in I.$$

Definition 3.3.4. *([46]) Let* $\nabla_p^\infty : \bigvee_p^\infty E \longrightarrow E^\infty$ *be an Infinite fold map at p, defined by*

$$y_i \longmapsto \nabla_p^\infty(y_i) = y, \qquad \forall\ i \in I.$$

Definition 3.3.5. *([46]) Let* (E, τ) *be a topological space and* $p \in E$. $\{p\}$ *is called closed set* $\Longleftrightarrow$ *the initial top. on* $\bigvee_p^\infty E$ *induced by* $\{\bigvee_p^\infty E \xrightarrow{A_p^\infty} E^\infty$ *and* $\bigvee_p^\infty E \xrightarrow{\nabla_p^\infty} (E, \tau_{disc})\}$ *is discrete top.*

Theorem 3.3.6. *([45]) Let* (E, τ) *be topological space and* $p \in E$. $\{p\}$ *is a closed set (in classical sense)* $\Longleftrightarrow$ $\{p\}$ *is closed set.*

Now considering categorical counterpart, we have following definition of closed objects given in [46].

Definition 3.3.7. *([46]) Suppose* $\mathfrak{U} : \mathbf{C} \longrightarrow \boldsymbol{Set}$ *is a top. functor,* $F \in Obj(\mathbf{C})$ *with* $\mathfrak{U}(F) = E$ *and* $p \in E$.
$\{p\}$ *is closed* $\Longleftrightarrow$ *the initial lift of* $\{\bigvee_p^\infty E \xrightarrow{A_p^\infty} E^\infty$ *and* $\bigvee_p^\infty E \xrightarrow{\nabla_p^\infty} \mathfrak{U}D(E) = E\}$ *is discrete.*

Theorem 3.3.8. *Let* (X, c) *be an* $\mathfrak{L}$*-valued closure space.* $\{p\}$ *is closed* $\Longleftrightarrow \forall\ x \in X$ *with* $x \neq p$, $\exists\ U \subseteq X$ *with* $x \in U$, $p \notin U$ *and* $\exists\ V \subseteq X$ *with* $p \in V$, $x \notin V$ *such that,*

$$\bot = \bigwedge\{c(U)(p), c(V)(x), k\}$$

Proof. Let (X, c) be an $\mathfrak{L}$-valued closure space, $p \in X$, suppose $\{p\}$ is closed ,so for all x in X with $x \neq p$. let $\{x_j\} \subseteq$ B $\subseteq \bigvee_p^\infty$X and $w = x_i \ \in \bigvee_p^\infty X$.
Note that,

$$(c_{dis}\nabla_p^\infty B)(\nabla_p^\infty x_1) = k.$$

Since, $x \in \nabla_p^\infty B$.

$$c(proj_i A_p^\infty B)(proj_i A_p^\infty w) = c(proj_i A_p^\infty B)(x)$$
$$c(proj_j A_p^\infty B)(proj_j A_p^\infty w) = c(proj_j A_p^\infty B)(p)$$
$$c(proj_k A_p^\infty B)(proj_k A_p^\infty w) = c(proj_k A_p^\infty B)(p).$$

Also,

$$k \leq c(proj_k A_p^\infty B)(proj_k A_p^\infty w) = c(proj_k A_p^\infty B)(p) = c(U)(p),$$

as $p \in proj_k A_p^\infty B$

Since w $= x_i \notin B$ and $\{p\}$ is closed, by definition 2.3.7,

$$\begin{aligned}(cB)(w) = &\bigwedge\{c_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w), c(proj_i A_p^\infty B)(proj_i A_p^\infty w), c(proj_j A_p^\infty B)(proj_j A_p^\infty w),\\ &c(proj_k A_p^\infty B)(proj_k A_p^\infty w)\},\\ \perp = &\bigwedge\{k, c(U)(p), c(V)(x)\}.\end{aligned}$$

Conversely: Let $\bar{c}$ be an initial structure on wedge $\bigvee_p^\infty X$ induced by $A_p^\infty : \bigvee_p^\infty X \longrightarrow (X^\infty, c_*)$ and ∇_p^∞: $\bigvee_p^\infty X \longrightarrow$ (X,c_{disc}), where c_* is a product $\mathfrak{L}-$ closure structure induced by $proj_k : X^\infty \longrightarrow X \ \forall(k \in I)$ projection map and c_{disc} is the discrete $\mathfrak{L}$-closure structure.

Suppose, w $\in \bigvee_p^\infty X$ and B $\subseteq \bigvee_p^\infty X$.

Case I: If $\nabla_p^\infty w$=p $\in \nabla_p^\infty B$ for some p $\in$ X , $w = p_1 = p \in \bigvee_p^\infty X$.

It follows that,

$$(\bar{c}B)(w) = k,$$

where k is an identity element.

Case II: If $\nabla_p^\infty w = p \notin \nabla_p^\infty B$, then

$$c_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w) = \perp.$$

Since, c_{disc} is the discrete $\mathfrak{L}$-closure structure, and consequently,

$$(\bar{c}B)(w) = \perp.$$

Case III: Let $\nabla_p^\infty w = x$ for some $x \in X$, then we have $w = x_i$, $(\forall i \in I)$.

a. If $w = x_i \in$ B, then $\nabla_p^\infty w \in \nabla_p^\infty$B and $proj_i A_p^\infty w \in proj_i A_p^\infty B$, it follows that

$$(\bar{c}B)(w) = k.$$

b. If $w = x_i \notin$ B, then $\nabla_p^\infty w \notin \nabla_p^\infty$B and consequently,

$$c_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w) = \bot,$$

and

$$(\bar{c}B)(w) = \bot.$$

c. If $w = x_i \notin$ B but $x_j \in$ B with i $\neq$ j. For i $\neq$ k $\neq$ j, by definition 2.3.8.

$$c_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w) = c_{dis}(\nabla_p^\infty B)(x) = k.$$

Since, x $\in \nabla_p^\infty B$.

$$c(proj_i A_p^\infty B)(proj_i A_p^\infty w) = c(proj_i A_p^\infty B)(x),$$
$$c(proj_j A_p^\infty B)(proj_j A_p^\infty w) = c(proj_j A_p^\infty B)(p),$$

And

$$c(proj_k A_p^\infty B)(proj_k A_p^\infty w) = c(proj_k A_p^\infty B)(p).$$

Since, $p \in proj_k A_p^\infty B$ and by definition 2.3.4, then we get

$$k \leq c(proj_k A_p^\infty B)(p).$$

It follows from definition 2.3.7,

$$\begin{aligned}(\bar{c}B)(w) = &\bigwedge\{c_{dis}(\nabla_p^\infty B)(\nabla_p^\infty w), c(proj_i A_p^\infty B)(proj_i A_p^\infty w), c(proj_j A_p^\infty B)(proj_j A_p^\infty w),\\ &c(proj_k A_p^\infty B)(proj_k A_p^\infty w)\},\\ = &\bigwedge\{k, c(U)(p), c(V)(x)\}.\end{aligned}$$

where $U = proj_i A_p^\infty B$ and $V = proj_j A_p^\infty B$.
By our assumption,

$$\bot = \bigwedge\{k, c(U)(p), c(V)(x)\}.$$

And consequently,

$$(\bar{c}B)(w) = \bot.$$

Similar to above, if $w = x_j \notin$ B but $x_i \in$ B with $i \neq j$. For $i \neq k \neq j$, it follows that

$$(\bar{c}B)(w) = \bot.$$

Then $w \in \bigvee_p^{\infty} X$ and all non-empty subset B of $\bigvee_p^{\infty} X$, we have

$$(\overline{c}B)(w) = \begin{cases} k, & w \in B \\ \perp, & w \notin B. \end{cases}$$

by definition 2.3.8, $\overline{c}$ is the discrete $\mathfrak{L}$-closure structure and by definition 3.3.7 $\{p\}$ is closed. □

3.4 Relationship among local $\overline{T_0}$, local T_1 and notion of closedness in category $\mathfrak{L}$-Cls

Lemma 3.4.1. *Every local T_1 QV-closure space is local $\bar{T_0}$ QV-closure space but converse is not true.*

Proof. It follows from Theorems 3.1.10 and 3.2.7. □

Example 3.4.2. *Let $X = \{a, b, c\}$ and $P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Consider a quantale $\mathfrak{L} = (([0, 1], \leq), ., 1)$, here $[0, 1]$ is real a unit interval with $\leq$ as partial order, "." the product i.e., the quantale operation and 1 is an identity element. Let $c : P(X) \longrightarrow \mathfrak{L}^X$ be a map defined by, $\forall x \in X$, and $\forall\ \phi \neq U \subset X$. $c(U)(x) = 1$ if $x \in U$ and $c(\{b\})(c) = c(\{a, b\})(c) = c(\{c\})(b) = c(\{a, c\})(b) = \frac{1}{2}$, $c(\{b\})(a) = c(\{c\})(a) = c(\{b, c\})(a) = 0$.*
Clearly, (X, c) be a QV-closure space. Note that, it is local $\overline{T_0}$ at a but not local T_1 at a.

Theorem 3.4.1. *Let (X, c) be $\mathfrak{L}$-valued closure space then following are equivalent,*

(i) (X,c) is local $\overline{T_0}$.

(ii) Every singleton set is closed.

Proof. It follows from theorems 3.1.10 and 3.3.8. □

3.5 Epireflective Properties of Local $\overline{T_0}$ and Local T_1 Quantale valued Closure Spaces

The Category L$\overline{T_0}$ $\mathfrak{L}-$**Cls** resp (LT_1 $\mathfrak{L}-$**Cls**) whose objects are Local $\overline{T_0}$ (resp. Local T_1) Quantale-Valued Closure Spaces and Morphisms are Continuous maps is the full subcategory of $\mathfrak{L}$-**Cls** and isomorphic closed.

Theorem 3.5.1. *Let (X, c) be $\mathfrak{L}$-valued closure space . $A \subset X$ and $p \in X$. If (X, c) is local $\bar{T}_0$ (resp. local T_1), then (A, c_A) is also local $\bar{T}_0$ (resp. local T_1).*

Proof. Let $f : A \hookrightarrow X$ be the inclusion map defined by $f(x) = x$ and c_A be the initial lift of $f : A \hookrightarrow (X, c)$. Suppose $U \subset A$ with $x \in U$, $p \notin U$ and $\exists\ V \subset A$ with $p \in V$, $x \notin V$. Since, c_A is an initial lift of f.

$$c_A(U)(p) = c(f(U))(f(p)) = c(U)(p),$$

and

$$c_A(V)(p) = c(f(V))(f(x)) = c(V)(x).$$

Since, (X, c) is local T_1 (resp. local $\bar{T}_0$) . Then by theorem 3.1.10

$$c(U)(p) = \perp.$$

or (resp. and)

$$c(V)(x) = \perp.$$

It follows that,

$$c_A(U)(p) = \perp,$$

or (resp. and)

$$c_A(V)(x) = \perp.$$

Hence (A, c_A) is local $\bar{T}_0$ (resp. local T_1). □

Theorem 3.5.2. *Let (X_i, c_i) $\forall i \in I$ be an $\mathfrak{L}$-valued closure spaces. $X = \prod_{i \in I} X_i$ and $p = (p_1, p_2, p_3, ...) \in X$. Where $p_i \in X_i$, $\forall i \in I$. If $(X_i, c_i) \forall i \in I$ is local $\overline{T_0}$ (resp. local T_1), then the product space, (X, c) is also local $\overline{T_0}$ (resp. local T_1).*

Proof. Let $\{(X_i, c_i); \forall i \in I\}$ be local $\bar{T}_0(resp.$ local $T_1)$ $\mathfrak{L}$-valued closure spaces and $X = (x_1, x_2, ..., x_n, ...) \in$ X.
Since, $\exists\ j \in I$ such that $x_j \neq p_j$.
By assumption (X_i, c_i) is local $\bar{T}_0$ (resp. local T_1), $\exists\ U_j \subset X_j$ with $x_j \in U_j$ and $\exists\ U_j \subset X_j$ with $p_j \in V_j$ and $x_j \notin V_j$.

$$c_j(U_j)(p_j) = \perp,$$

or (resp. and)

$$c_j(V_j)(x_j) = \perp.$$

If

$$c_j(U_j)(p_j) = \perp.$$

By definition of initial lift.

$$c_*(U)(p) = \bigwedge_{i \in I} \{c_i(proj_i U)(proj_i p)\}.$$

$$\begin{aligned} c_*(U)(p) &= \bigwedge \{c_1(U_1)(p_1), c_2(U_2)(p_2), ...\} \\ &= \perp. \end{aligned}$$

Similarly, if $c_*(V)(x) = \perp$. Then the product $\mathfrak{L}$-valued closure spaces is local $\overline{T_0}$ (resp. local T_1). □

Chapter 4

Conclusion

In this dissertation, the category of Quantale-valued Closure spaces is taken into consideration. Initially, it has been shown that **$\mathfrak{L}$-Cls** is a topological category and its relation to the category **Cls** of classical Closure spaces and continuous maps is studied and several examples with different quantales are provided. Furthermore, local $\overline{T_0}$ and local T_1 are explicitly characterized for quantale-valued closure space and it is shown that every local T_1 QV-Closure space implies local $\overline{T_0}$ QV-Closure space but converse is not true in general. In addition, the notion of closedness are examined in topological category of **$\mathfrak{L}$-Cls** and it is examined that this notion of closedness, (i.e., $\{p\}$ is closed) coincides with local $\overline{T_0}$ QV-Closure space. Finally, hereditary and productivity of local $\overline{T_0}$ (resp. local T_1) QV-Closure space are examined.

Comparing our results with other topological categories, we have followings.

(i) In **CHY** (Category of Cauchy spaces) [50], Local $\overline{T_0}$ = Local T_1 = $\{p\}$ is closed [51].

(ii) In **Born** (the category of Bornological spaces and bounded maps) [7] , all objects are Local $\overline{T_0}$, Local T_1 and every set $\{p\}$ is closed [52].

(iii) In **Top** [45], Local T_1 = $\{p\}$ is closed implies Local $\overline{T_0}$ [45].

(iv) In **$\mathfrak{L}$-GS** (the category of Quantale-valued Gauge spaces) [53], Local T_1 implies Local $\overline{T_0}$ = $\{p\}$ is closed [47].

In our thesis, we found the following results.

In **$\mathfrak{L}$-Cls** (the category of Quantale-valued Closure space and continuous maps), by Theorems 3.1.10, 3.2.7 and 3.3.8, Local T_1 implies Local $\overline{T_0}$ = Every singleton is closed.

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