Solution of Non-linear Ordinary Differential Equation Involving Arbitrary Constants

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MS THESIS WORK

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Dedication

This thesis is dedicated to my beloved parents and friends for their love and support.

Acknowledgement

In the name of ALLAH, the most Gracious, most Merciful.

First and foremost, I thank ALLAH for bestowing me with health, patience, and knowledge to complete this thesis, and without ALLAH's grace, I couldn't have done it. So to ALLAH returns all the praise and gratitude.

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Abstract

In this thesis, we present solution of non-linear ordinary differential equation involving two arbitrary constants. This equation appeared in finding solutions of the Einstein field equations. Depending on different values of the arbitrary constants involved, we have nine possible cases. The analytical solutions are obtained in five cases and for the remaining cases, solutions are obtained numerically.

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Chapter 1

Introduction to Differential Equations

Differential equations were first introduced with the development of the calculus by Newton and Leibniz. An equation that contains derivative of one or more dependent variables with respect to one or more independent variables is referred as differential equation (DE) [1]. Differential equations are used to explain the population growth in species, also for solving the problems in radioactive decay, flow problems, bank interest, cooling and heating processes, orthogonal trajectories, etc. They are also employed to examine the problems involving heat transfer, fluid mechanics and circuit design. In the field of medicines they are used for modeling the spread of disease and cancer growth.

Isaac Newton, [2] described following three types of differential equations:

$$\frac{dz}{dt} = f(t), \tag{1.1}$$

$$\frac{dz}{dt} = f(t, z), \tag{1.2}$$

$$t_1 \frac{\partial z}{\partial t_1} + t_2 \frac{\partial z}{\partial t_2} = z. \tag{1.3}$$

In above equations z is an unknown function of t (or t_1 and t_2) and f is given function. Many well-known scholars have written articles on the development of the subject. In the 1680s, Leibniz, Bernoulli brothers and many others began working on differential equations, after the fluxional equations of Newton during 1670s. The Euler–Lagrange equation was formulated by Euler and Lagrange in 1750. The study of heat flow was published in 1822 by Fourier [3], in which he applied Newton's law of cooling that the heat transfer between two consecutive molecules is proportional to the extremely small temperature difference. In 18th century most generalizations were made in the Leibnizian tradition [4] which was expanded to multi-variable form, which resulted in partial differential equations. Poincaré [5] proposed recurrence theorems, initially in association with that of the three-body problem. New applications have been made for quantum mathematics, dynamic systems and the theory of relativity.

1.1 Classification of Differential Equations

Differential equations are classified in different categories in order to look for the methods to find their solutions. DEs are divided into two types ordinary differential equations and partial differential equations.

Definition 1.1.1. If the dependent variable or variables depend only on a single independent variable then such differential equations are called ordinary differential equations (ODEs).

Ordinary differential equations have played important role in mathematics, computer science and engineering [6]. Furthermore, study of complex processes with ODEs is a developed area and hence there is a rich literature dedicated to their study [7, 8, 9]. ODEs are applied to model biological processes at different stages. For example, kinetics of drugs on a whole body level [10], gene expression [11] and signal processing on cellular level [12].

Definition 1.1.2. If the dependent variable or variables depends on more than one independent variables kthen the resulting equations are called the partial differential equations (PDEs).

Example 1.1.1. (Newton's law)
Force is equal to mass into acceleration and is given as

$$f = ma, \tag{1.4}$$

where m is mass of particle, $a = d^2x/dt^2$ is the particle acceleration and f is force that acts on the particle. Newton's law can be represented by an ordinary differential equation as

$$m\frac{d^2\mathbf{x}(t)}{dt^2} = \mathbf{f}\left(t, \mathbf{x}(t), \frac{d\mathbf{x}(t)}{dt}\right), \tag{1.5}$$

here $\mathbf{x}(t)$ is unknown and is the position of a particle at time t in space. By above equation (1.5), we see that force may depend on time, on the position of the particle in space and on the velocity of the particle.

Example 1.1.2. (The Heat Equation)
For a solid material the temperature T varies over time and in three dimensions of space marked by $\mathbf{x} = (x, y, z)$, according to equation

$$\frac{\partial T(t,\mathbf{x})}{\partial t} = k\left(\frac{\partial^2 T(t,\mathbf{x})}{\partial x^2} + \frac{\partial^2 T(t,\mathbf{x})}{\partial y^2} + \frac{\partial^2 T(t,\mathbf{x})}{\partial z^2}\right), \ k > 0, \tag{1.6}$$

where k is a positive constant that describes material's thermal properties.
The highest derivative appearing in the equation is called order of the differential equation and power of highest derivative is said to be the degree. For example in equation (1.5), the order is 2 and degree is 1.

Definition 1.1.3. A linear differential equation of the form

$$b_n(t)\frac{d^n z}{dt^n} + b_{n-1}(t)\frac{d^{n-1}z}{dt^{n-1}} + \cdots + b_1(t)\frac{dz}{dt} + b_0(t)z = 0, \tag{1.7}$$

is said to be homogeneous, whereas an equation

$$b_n(t)\frac{d^n z}{dt^n} + b_{n-1}(t)\frac{d^{n-1}z}{dt^{n-1}} + \cdots + b_1(t)\frac{dz}{dt} + b_0(t)z = f(t), \tag{1.8}$$

with $f(t) \neq 0$, is said to be non-homogeneous.

Definition 1.1.4. A DE is linear in z if we can write it in the form

$$b_n(t)\frac{d^n z}{dt^n} + b_{n-1}(t)\frac{d^{n-1}z}{dt^{n-1}} + \cdots + b_1(t)\frac{dz}{dt} + b_0(t)z = f(t), \tag{1.9}$$

that is, it satisfies the following conditions
1. Power of dependent variable and all its derivatives is one.

2. The coefficients $b_n, \ldots, b_0$ of $z, z', \ldots, z^{(n)}$ depends only on the independent variable t.

3. No transcendental functions (trigonometric, logarithmic etc.) of z exists.

If any one of these conditions is not satisfied then the DE is said to be non-linear differential equation.

Differential equations are also divided into two types of problems, which depend on the basis of conditions, i.e. initial value problems (IVPs) and boundary value problems (BVPs).

Definition 1.1.5. An initial value problem is one in which all conditions are specified for same value of the independent variable.

Definition 1.1.6. Boundary value problems are those in which conditions are described at more than one values of the independent variable.

An initial value problem may have a unique solution, no solution, or many solutions depending on the initial conditions. The theorem below gives the conditions sufficient for the existence and uniqueness of a first order initial value problem.

Theorem 1.1.1. [1] **a**. If f is continuous on an open rectangle

$$R : \{a < t < b, c < z < d\}, \tag{1.10}$$

that contains (t_0, z_0) then the initial value problem

$$z' = f(t, z), \;\; z(t_0) = z_0, \tag{1.11}$$

has at least one solution on some open sub-interval of (a, b) that contains t_0.

b. If both f and f_z are continuous on R then the initial value problem (1.11) has a unique solution on some open sub-interval of (a, b) that contains t_0.

Example 1.1.3. Consider the initial value problem

$$z' = \frac{t^2 - z^2}{1 + t^2 + z^2}, \;\; z(t_0) = z_0. \tag{1.12}$$

Since

$$f(t,z) = \frac{t^2 - z^2}{1 + t^2 + z^2}, \tag{1.13}$$

and

$$f_z(t,z) = -\frac{2z(1 + 2t^2)}{(1 + t^2 + z^2)^2}, \tag{1.14}$$

are continuous for all (t, z), therefore, by Theorem 1.1.1 the initial value problem (1.12) always has a unique solution.

1.2 Some Analytical Methods for Solving Ordinary Differential Equations

For the 1$^{\text{st}}$ order ordinary differential equations, we have some methods which are further classified on the basis of some criteria i.e. if the variables are separable or not, if the equation is homogeneous or not, or if it is exact or not. If 1$^{\text{st}}$ order ODE is of any one of these types then we have the well-defined process to solve it. Here we discuss some of the methods which we will use later in the thesis.

A differential equation of the type

$$M(t)dt = N(z)dz, \tag{1.15}$$

is called separable equation. The general solution is obtained by direct integration.

A differential expression

$$M(t,z)dt + N(t,z)dz, \tag{1.16}$$

is said to be exact in a region R of (t, z) plane, t is along horizontal axis and z is along vertical axis, if it correspondence to total differential of some function $g(t, z)$. A differential equation of the form

$$M(t,z)dt + N(t,z)dz = 0, \tag{1.17}$$

is said to be an exact equation if the expression on the L.H.S is an exact differential.

An equation of the form

$$\frac{dz}{dt} + P(t)z = Q(t)z^n,\ n \in R, \tag{1.18}$$

is referred as the Bernoulli differential equation. For $n = 0, 1$, the equation is linear otherwise equation (1.18) can be reduced to a linear equation.
Consider the 1st order linear differential equation

$$\frac{dz}{dt} + P(t)z = R(t). \tag{1.19}$$

By adding $Q(t)z^2$ on the L.H.S of above equation, we get non-linear differential equation

$$\frac{dz}{dt} + P(t)z + Q(t)z^2 = R(t), \tag{1.20}$$

called the Ricatti equation. In order to reduce to linear equation we use substitution, i.e. $z = z_1 + \frac{1}{u}$, where z_1 is particular solution of equation (1.19) and u is unknown non-zero function of t.
A homogeneous linear differential equation with constant coefficient can be written as an algebraic equation which is called Auxiliary or Characteristic equation of given DE. For higher-order differential equations, we have different techniques to obtain the solution. If it is linear and involves constant coefficients then we have a well-defined process to solve like complementary function. And for solution of non-homogeneous differential equation we have different methods, like, method of undetermined coefficients and variation of parameters method.
In literature, an Euler–Cauchy equation, or Cauchy–Euler equation, or simply Euler's equation is a linear ordinary differential equation with variable coefficients of the type

$$b_0 t^n \frac{d^n z}{dt^n} + b_1 t^{n-1} \frac{d^{n-1} z}{dt^{n-1}} + \cdots + b_{n-1} t \frac{dz}{dt} + b_n z = f(t), \tag{1.21}$$

$b_0, b_1, \ldots, b_{n-1}, b_n$ are real constants. By transformation, $t = e^s$ or $s = lnt$, the equation can be simplified to a linear differential equation with constant coefficients.
In Chapter 2, we discuss some numerical techniques like Euler's method and the Runge-Kutta method for solving ordinary differential equations. We have also discussed the MATLAB built-in functions like $dsolve$, $ode23$ and $ode45$ in this chapter. In Chapter 3, we present solution of non-linear ordinary differential equation that involves arbitrary constants.

Chapter 2

Numerical Methods for Ordinary Differential Equations

It is not always possible to find the analytical solution of a differential equation due to limitation of the available techniques, especially, for non-linear differential equations. In all such situations one generally looks for numerical methods for finding solutions. There are many numerical methods for solving ordinary differential equations with initial conditions. These methods are of two types: single-step methods and multi-step methods. Some well known single-step methods are Euler's method and Runge-Kutta methods. Multi-step methods are Milne's method and Adams-Bashforth method etc. All numerical methods generally vary in accuracy and computation cost.
In this chapter, after discussing some basic concepts, we will discuss Euler's method and its variations and Runge-Kutta method in subsequent sections.

2.1 Error Analysis of Numerical Methods

While using a numerical method we must be aware of various types of errors that may occur. The numerical solutions of ODEs involve following types of errors:

Round-off Error

Round-off error occur by rounding off a number to some finite decimal places. For example, if the irrational number $\pi = 3.141592\dots$ is rounded off to two decimal places as 3.14, then $3.141592\dots - 3.14 = 0.001592\dots$ is the round-off error.

Truncation Error

These errors occur due to the use of approximate formula in calculation or by truncating the infinite series. For example, the Taylor series expansion of e^t is

$$e^t = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!} + \dots. \tag{2.1}$$

If the infinite series (2.1) is used to calculate $r = e^{0.4}$, we get

$$e^{0.4} = 1 + 0.4 + \frac{(0.4)^2}{2!} + \dots + \frac{(0.4)^n}{n!} + \dots. \tag{2.2}$$

It is not possible to add infinite terms numerically, so if we take the first four terms as the approximation, we obtain

$$e^{0.4} \approx 1 + 0.4 + \frac{(0.4)^2}{2!} + \frac{(0.4)^3}{3!} = \overline{r}. \tag{2.3}$$

Then the truncation error $= r - \overline{r} = \frac{(0.4)^4}{4!} + \frac{(0.4)^5}{5!} + \dots.$

Truncation error is divided into two types: The quantity of truncation error that appears in a numerical approximation in one iteration is called local truncation error and the cumulative truncation error caused by many iterations is known as global truncation error.

Absolute Error

If exact value of solution is represented by χ and $\chi^{'}$ is used for approximate value, then the absolute error between two values is given as

$$E_A = |\chi - \chi^{'}|. \tag{2.4}$$

Relative Error

Relative error is given by the absolute error divided by the exact value

$$E_R = \left| \frac{\chi - \chi'}{\chi} \right|. \tag{2.5}$$

2.2 Euler's Method and its Variations

The Euler's method, originated in 1768 by Leonhard Euler, is a single step method and is also known as tangent line method. It is the simplest numerical method to solve the initial value problems.

Consider a differential equation

$$\frac{dz}{dt} = f(t, z), \;\; z(t_0) = z_0. \tag{2.6}$$

We want to find successively $z_1, z_2, \ldots, z_m$, where z_m is the value of z corresponding to $t = t_m$, where $t_m = t_0 + mh$, $m = 1, 2, \ldots$. Take h very small such that the curve is nearly a straight line. The Figure 2.1 shows exact solution and the numerical solutions $z_1, z_2, \ldots, z_m$ obtained by Euler's method and the doted lines represent the errors between the exact and numerical solution.

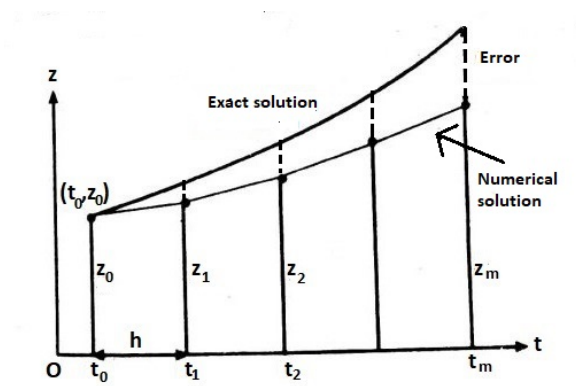

Figure 2.1: Exact solution vs Numerical solutions by Euler's method [13].

Thus in the interval $[t_0, t_1]$ we approximate the curve by tangent at point (t_0, z_0). Equation of tangent line at (t_0, z_0) is

$$z - z_0 = \left. \frac{dz}{dt} \right|_{(t_0, z_0)} (t - t_0), \tag{2.7}$$

$$z = z_0 + (t - t_0) f(t_0, z_0). \tag{2.8}$$

Hence, the value of z corresponding to $t = t_1$ is

$$z_1 \quad = \quad z_0 + (t_1 - t_0) f(t_0, z_0), \tag{2.9}$$

or

$$z_1 = z_0 + h f(t_0, z_0), \tag{2.10}$$

or

$$z_m = z_{m-1} + h f(t_{m-1}, z_{m-1}), \ m = 1, 2, \ldots. \tag{2.11}$$

Euler's method is not very accurate, so in practice, it is seldom used. The draw backs of Euler's method are given as follows:

- Low order of accuracy.
- For maintenance high accuracy, very small h is necessary.
- Computational process is time consuming.

Error for Euler's Method

Consider

$$z' = \frac{dz}{dt} = f(t, z), \ z(t_0) = z_0. \tag{2.12}$$

Assume z_k is approximate value and $z(t_k)$ is exact value at t_k. The local truncation error at $(k+1)$th step is

$$E_{k+1} = z(t_{k+1}) - z_{k+1} - z(t_k) + z_k. \tag{2.13}$$

Using equation (2.11) with $m = k + 1$ in equation (2.13), we get

$$E_{k+1} = z(t_{k+1}) - z(t_k) - h f(t_k, z_k), \tag{2.14}$$

where $k = 0, 1, 2, \ldots, m-1$.
For $k = 0$

$$E_1 = z(t_1) - z(t_0) - hf(t_0, z_0), \tag{2.15}$$

or

$$E_1 = z(t_0 + h) - z(t_0) - hf(t_0, z_0). \tag{2.16}$$

Using Taylor series, we get

$$E_1 = z(t_0) + hz^{'}(t_0) + \frac{h^2}{2!}z^{''}(c_1) - z(t_0) - hf(t_0, z_0). \tag{2.17}$$

Substituting $z'(t_0) = f(t_0, z_0)$, we get, local truncation error is of order h^2 and is given by

$$|E_1| \quad = \quad \left|\frac{h^2}{2!}z^{''}(c_1)\right|, \tag{2.18}$$

where $c_1 \in (t_0, t_1)$. For $k = 1$

$$|E_2| \quad = \quad \left|\frac{h^2}{2!}z^{''}(c_2)\right|, \tag{2.19}$$

where $c_2 \in (t_1, t_2)$. Similarly for $k + 1 = m$, we have

$$|E_m| = \left|\frac{h^2}{2!}z^{''}(c_m)\right|, \tag{2.20}$$

where $c_m \in (t_{m-1}, t_m)$.
The global truncation error after m steps is

$$|E_1 + E_2 + \cdots + E_m| = \left|\frac{h^2}{2!}z^{''}(c_1) + \frac{h^2}{2!}z^{''}(c_2) + \cdots + \frac{h^2}{2!}z^{''}(c_m)\right|, \tag{2.21}$$

$$|E_1 + E_2 + \cdots + E_m| \leq \left|\frac{h^2}{2!}z^{''}(c_1)\right| + \left|\frac{h^2}{2!}z^{''}(c_2)\right| + \cdots + \left|\frac{h^2}{2!}z^{''}(c_m)\right|. \tag{2.22}$$

Let $|z''(c)|$ is maximum of $|z''(c_1)|, |z''(c_2)|, \ldots, |z''(c_m)|$ then

$$|E_1 + E_2 + \cdots + E_m| \leq \left|\frac{h^2}{2!}z^{''}(c)\right| + \left|\frac{h^2}{2!}z^{''}(c)\right| + \cdots + \left|\frac{h^2}{2!}z^{''}(c)\right|, \tag{2.23}$$

$$|E_1 + E_2 + \cdots + E_m| \leq m \left| \frac{h^2}{2!} z''(c) \right|, \tag{2.24}$$

$$|E_1 + E_2 + \cdots + E_m| \leq \frac{b-a}{2} h |z''(c)|. \tag{2.25}$$

The global truncation error for Euler's method is of order h.
In the subsequent subsections we discuss two variations of Euler's method, namely the improved Euler method and the modified Euler method.

2.2.1 The Improved Euler Method

In this method, geometrically we draw a tangent at (t_0, z_0) as in Euler's method. At t_1, we find the value of z_1^*, by using Euler's method such that $z_1^* = z_0 + hf(t_0, z_0)$. Once we find the value of z_1^* then we have two values (t_0, z_0) and (t_1, z_1^*), then slopes at

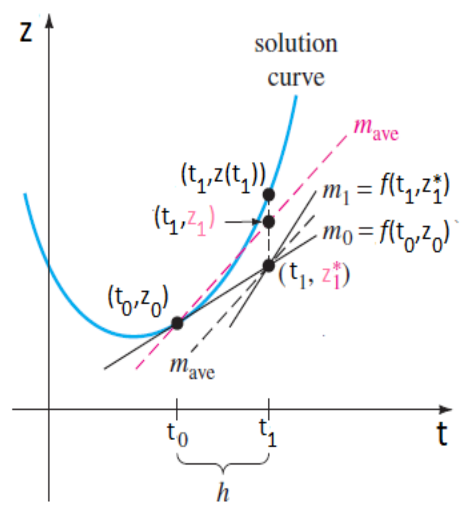

Figure 2.2: Graphical representation of the improved Euler method [1].

(t_0, z_0) and (t_1, z_1^*) are $m_0 = f(t_0, z_0)$ and $m_1 = f(t_1, z_1^*)$, respectively. The average of slopes is given by $\frac{f(t_0,z_0)+f(t_1,z_1^*)}{2} = m_{ave}$, which is the slope of the parallel dashed lines shown in Figure 2.2. In the improved Euler method the value, z_1, at t_1 is obtained by going along the dashed line through (t_0, z_0) with the slope m_{ave}. From Figure 2.2, z_1 is an improvement over z_1^*. Equation of tangent with the average of two slopes at (t_0, z_0) is

$$z - z_0 = (t - t_0)\left(\frac{f(t_0, z_0) + f(t_1, z_1^*)}{2}\right). \tag{2.26}$$

Putting $t = t_1 = t_0 + h$, we get

$$z_1 = z_0 + \frac{h}{2}\left(f(t_0, z_0) + f(t_0 + h, z_0 + hf(t_0, z_0))\right). \tag{2.27}$$

For the second approximate value, z_2, we have

$$z_2 = z_1 + \frac{h}{2}\left(f(t_1, z_1) + f(t_1 + h, z_1 + hf(t_1, z_1))\right). \tag{2.28}$$

In the same way

$$z_m = z_{m-1} + \frac{h}{2}\left(f(t_{m-1}, z_{m-1}) + f(t_{m-1} + h, z_{m-1} + hf(t_{m-1}, z_{m-1}))\right). \tag{2.29}$$

Equation (2.29) is referred as the improved Euler method.

Error for the Improved Euler Method

Using equation (2.29) for $m = k + 1$ in equation (2.13), we have

$$E_{k+1} \quad = \quad z(t_{k+1}) - z(t_k) - \frac{h}{2}\left(f(t_k, z_k) + f(t_k + h, z_k + hf(t_k, z_k))\right). \tag{2.30}$$

Using Taylor's series, we have

$$z(t_{k+1}) = z(t_k) + hz^{'}(t_k) + \frac{h^2}{2!}z^{''}(t_k) + \frac{h^3}{3!}z^{'''}(t_k) + \ldots, \tag{2.31}$$

and

$$\begin{aligned} f(t_k + h, z_k + hf(t_k, z_k)) = f(t_k, z_k) &+ \left(h\frac{\partial}{\partial t} + hf(t_k, z_k)\frac{\partial}{\partial z}\right)f\Big|_{(t_k, z_k)} \\ &+ \frac{1}{2!}\left(h\frac{\partial}{\partial t} + hf(t_k, z_k)\frac{\partial}{\partial z}\right)^2 f\Big|_{(t_k, z_k)} \\ &+ \frac{1}{3!}\left(h\frac{\partial}{\partial t} + hf(t_k, z_k)\frac{\partial}{\partial z}\right)^3 f\Big|_{(t_k, z_k)} + \ldots, \end{aligned} \tag{2.32}$$

or

$$\begin{aligned} f(t_k + h, z_k + hf(t_k, z_k)) = f(t_k, z_k) + hf^{'}(t_k, z_k) + \frac{h^2}{2!}f^{''}(t_k, z_k) \\ + \frac{h^3}{3!}f^{'''}(t_k, z_k) + \dots, \end{aligned} \tag{2.33}$$

where " $'$ " represents the derivative with respect to t. Using equation (2.31) and equation (2.33) in equation (2.30), we get

$$\begin{aligned} E_{k+1} = \left(z(t_k) + hz^{'}(t_k) + \frac{h^2}{2!}z^{''}(t_k) + \frac{h^3}{3!}z^{'''}(t_k) + \dots\right) - z(t_k) \\ - \frac{h}{2}\left(2f(t_k, z_k) + hf^{'}(t_k, z_k) + \frac{h^2}{2!}f^{''}(t_k, z_k) + O(h^3)\right). \end{aligned} \tag{2.34}$$

Substituting $z^{'}(t_k) = f(t_k, z_k)$, $z^{''}(t_k) = f^{'}(t_k, z_k)$ and $z^{'''}(t_k) = f^{''}(t_k, z_k)$ in equation (2.34) and simplifying, we get

$$E_{k+1} = \frac{h^3}{3!}z^{'''}(t_k) - \frac{h^3}{4}z^{'''}(t_k) + \dots, \tag{2.35}$$

or

$$E_{k+1} = \left(\frac{h^3}{3!} - \frac{h^3}{4}\right)z^{'''}(t_k) + \dots. \tag{2.36}$$

The local truncation error is of order h^3 and is given by

$$|E_{k+1}| = \left|\frac{1}{12}h^3 z^{'''}(c_m)\right|, \tag{2.37}$$

where $c_m \in (t_{m-1}, t_m)$.
The global truncation error after m steps is given by

$$|E_1 + E_2 + \dots + E_m| \leq \frac{b-a}{12}|z^{'''}(c)|h^2. \tag{2.38}$$

Equation (2.38) shows the global truncation error for the improved Euler method is of order h^2.

2.2.2 The Modified Euler Method

In modified Euler method, we first estimate the value of z at $t_0 + \frac{h}{2}$, using the tangent line at (t_0, z_0) and call this estimated value, z_1^*, which is equal to $z_0 + \frac{h}{2} f(t_0, z_0)$, then

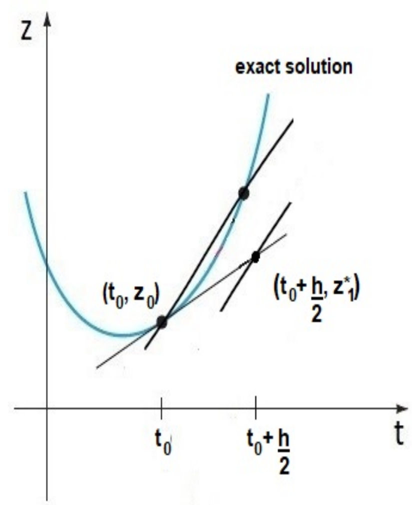

Figure 2.3: Graphical representation of the modified Euler method.

we estimate the value of slope at $t = t_0 + \frac{h}{2}$ by $f(t_0 + \frac{h}{2}, z_1^*)$ then we find the value of z at $t = t_1$ using tangent line that passes from (t_0, z_0) and has slope $f(t_0 + \frac{h}{2}, z_1^*)$ is

$$z_1 = z_0 + hf\left(t_0 + \frac{h}{2}, z_0 + \frac{h}{2} f(t_0, z_0)\right). \tag{2.39}$$

or

$$z_2 = z_1 + hf\left(t_1 + \frac{h}{2}, z_1 + \frac{h}{2} f(t_1, z_1)\right), \tag{2.40}$$

or

$$z_m = z_{m-1} + hf\left(t_{m-1} + \frac{h}{2}, z_{m-1} + \frac{h}{2} f(t_{m-1}, z_{m-1})\right), \quad m = 1, 2, \ldots . \tag{2.41}$$

Equation (2.41) is referred as the modified Euler method. The local truncation error for the modified Euler method is of order h^3 and its global truncation error is of order h^2.

2.3 Runge-Kutta Methods

Runge-Kutta (RK) methods are generalizations of Euler's formula (2.11) in which slope of function f is replaced by a weighted average, u, of slopes over the interval $t_{m-1} \leq t \leq t_m$, i.e.

$$z_m = z_{m-1} + hu, \tag{2.42}$$

where h is the length of the interval. When u is estimated by using slopes at r points then u can be written as

$$u = w_1 k_1 + w_2 k_2 + \cdots + w_r k_r, \tag{2.43}$$

where $w_1, w_2, \ldots, w_r$ are weights of the slopes at various points and $k_1, k_2, \ldots, k_r$ are slopes. Using equation (2.43) in equation (2.42), we get

$$z_m = z_{m-1} + h(w_1 k_1 + w_2 k_2 + \cdots + w_r k_r). \tag{2.44}$$

2^{nd} Order Runge-Kutta Method

The 2^{nd} order RK method has the form

$$z_m = z_{m-1} + h(w_1 k_1 + w_2 k_2), \tag{2.45}$$

where

$$k_1 = f_{m-1}, \tag{2.46}$$

$$k_2 = f(t_{m-1} + ph, z_{m-1} + qhf_{m-1}), \tag{2.47}$$

where p and q are constants. Applying Taylor series on k_2, we have

$$k_2 = f_{m-1} + ph\frac{\partial f_{m-1}}{\partial t} + qhf_{m-1}\frac{\partial f_{m-1}}{\partial z} + O(h^2). \tag{2.48}$$

Ignoring higher order terms in h, we get

$$k_2 = f_{m-1} + ph\frac{\partial f_{m-1}}{\partial t} + qhf_{m-1}\frac{\partial f_{m-1}}{\partial z}. \tag{2.49}$$

Using value of k_1 and k_2 from equations (2.46) and (2.49) in equation (2.45), we get

$$z_m = z_{m-1} + h\Bigg[f_{m-1} + w_2\Bigg(f_{m-1} + ph\frac{\partial f_{m-1}}{\partial t} + qhf_{m-1}\frac{\partial f_{m-1}}{\partial z}\Bigg)\Bigg], \tag{2.50}$$

or

$$z_m = z_{m-1} + hf_{m-1}(w_1 + w_2) + w_2 h^2\Bigg[p\frac{\partial f_{m-1}}{\partial t} + qf_{m-1}\frac{\partial f_{m-1}}{\partial z}\Bigg]. \tag{2.51}$$

The weights w_1, w_2 and constants p, q are to be determined. The Taylor series expansion of z_m about z_{m-1} is given as

$$z_m = z_{m-1} + hz'_{m-1} + \frac{h^2}{2!}z''_{m-1} + O(h^3), \tag{2.52}$$

where

$$z'_{m-1} = f_{m-1}, \tag{2.53}$$

$$z''_{m-1} = \frac{\partial f_{m-1}}{\partial t} + f_{m-1}\frac{\partial f_{m-1}}{\partial z}. \tag{2.54}$$

Using values of z'_{m-1} and z''_{m-1} in equation (2.52) and ignoring higher order terms in h, we get

$$z_m = z_{m-1} + hf_{m-1} + \frac{h^2}{2!}\Bigg[\frac{\partial f_{m-1}}{\partial t} + f_{m-1}\frac{\partial f_{m-1}}{\partial z}\Bigg]. \tag{2.55}$$

Comparing coefficients of powers of h in equations (2.51) and (2.55), we have

$$w_1 + w_2 = 1, \tag{2.56}$$

$$w_2 p = \frac{1}{2}, \tag{2.57}$$

$$w_2 q = \frac{1}{2}. \tag{2.58}$$

Equations (2.56)-(2.58) is a system of three algebraic equations involving fours unknowns. Therefore, it has infinite solutions. Choosing $w_1 = \frac{1}{2}$, we get $w_2 = \frac{1}{2}$ and $p = q = 1$, then equation (2.45) becomes

$$z_m = z_{m-1} + \frac{h}{2}(k_1 + k_2), \tag{2.59}$$

where

$$k_1 = f(t_{m-1}, z_{m-1}), \tag{2.60}$$

$$k_2 = f(t_{m-1} + h, z_{m-1} + hk_1). \tag{2.61}$$

Equation (2.59) is the improved Euler formula. Setting $w_1 = 0$, we get $w_2 = 1$ and $p = q = \frac{1}{2}$, then equation (2.45) becomes

$$z_m = z_{m-1} + hk_2, \tag{2.62}$$

where

$$\begin{aligned} k_1 &= f(t_{m-1}, z_{m-1}), &(2.63)\\ k_2 &= f(t_{m-1} + \frac{h}{2}, z_{m-1} + \frac{hk_1}{2}). &(2.64) \end{aligned}$$

Equation (2.62) is the modified Euler formula. The local truncation error for 2^{nd} order RK method is of order h^3 and global truncation error is of order h^2.

4^{th} Order Runge-Kutta Method

The 4^{th} order Runge-Kutta (RK4) method has the form

$$z_m = z_{m-1} + h(w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4). \tag{2.65}$$

Following the similar procedure as followed for 2^{nd} order RK method, we get $w_1 = w_4 = \frac{1}{6}$ and $w_2 = w_3 = \frac{2}{6}$, therefore equation (2.65) becomes

$$z_m = z_{m-1} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{2.66}$$

where

$$\begin{aligned} k_1 &= f(t_{m-1}, z_{m-1}), &(2.67)\\ k_2 &= f\left(t_{m-1} + \frac{h}{2}, z_{m-1} + \frac{k_1}{2}\right), &(2.68)\\ k_3 &= f\left(t_{m-1} + \frac{h}{2}, z_{m-1} + \frac{k_2}{4}\right), &(2.69)\\ k_4 &= f(t_{m-1} + h, z_{m-1} + k_3). &(2.70) \end{aligned}$$

The local truncation error for RK4 method is of order h^5 and its global truncation error is of order h^4.

2.3.1 RK4 Method for System of First Order IVP

Consider a system of 1st order IVP consisting of two equations given as

$$z_1^{'} = f_1(t, z^1, z^2), \tag{2.71}$$

$$z_2^{'} = f_2(t, z^1, z^2), \tag{2.72}$$

$a \leq t \leq b$, $z_1(a) = \alpha_1$, $z_2(a) = \alpha_2$. RK4 method is given as:
For z^1:

$$z_{m+1}^1 = z_m^1 + \frac{1}{6}(k_1^1 + 2k_2^1 + 2k_3^1 + k_4^1), \tag{2.73}$$

where

$$k_1^1 = hf_1(t_m, z_m^1, z_m^2), \tag{2.74}$$

$$k_2^1 = hf_1\left(t_m + \frac{h}{2}, z_m^1 + \frac{k_1^1}{2}, z_m^2 + \frac{k_1^2}{2}\right), \tag{2.75}$$

$$k_3^1 = hf_1\left(t_m + \frac{h}{2}, z_m^1 + \frac{k_2^1}{2}, z_m^2 + \frac{k_2^2}{2}\right), \tag{2.76}$$

$$k_4^1 = hf_1(t_{m+1}, z_m^1 + k_3^1, z_m^2 + k_3^2). \tag{2.77}$$

For z^2:

$$z_{m+1}^2 = z_m^2 + \frac{1}{6}(k_1^2 + 2k_2^2 + 2k_3^2 + k_4^2), \tag{2.78}$$

where

$$k_1^1 = hf_2(t_m, z_m^1, z_m^2), \tag{2.79}$$

$$k_2^1 = hf_2\left(t_m + \frac{h}{2}, z_m^1 + \frac{k_1^1}{2}, z_m^2 + \frac{k_1^2}{2}\right), \tag{2.80}$$

$$k_3^1 = hf_2\left(t_m + \frac{h}{2}, z_m^1 + \frac{k_2^1}{2}, z_m^2 + \frac{k_2^2}{2}\right), \tag{2.81}$$

$$k_4^1 = hf_2(t_{m+1}, z_m^1 + k_3^1, z_m^2 + k_3^2). \tag{2.82}$$

Example 2.3.1. Consider the following system of initial value problem

$$z_1^{'} = z_1 + 2z_2 = f_1, \tag{2.83}$$

$$z_2^{'} = 4z_1 + 3z_2 = f_2, \tag{2.84}$$

with $z_1(0) = 1$, $z_2(0) = 1$ and $0 \leq t \leq 0.2$, $h = 0.1$.

For first iteration $m = 0$, we get

$k_1^1 = 0.3000$	$k_1^2 = 0.7000$
$k_2^1 = 0.3850$	$k_2^2 = 0.8650$
$k_3^1 = 0.4058$	$k_3^2 = 0.9068$
$k_4^1 = 0.5219$	$k_4^2 = 1.1344$
$z_1^1 = 1.4006$	$z_1^2 = 1.8963$

Table 2.1: Results of RK4 method for $m = 0$.

For second iteration $m = 1$, we get

$k_1^1 = 0.5193$	$k_1^2 = 0.1291$
$k_2^1 = 0.6582$	$k_2^2 = 1.4024$
$k_3^1 = 0.6952$	$k_3^2 = 1.4711$
$k_4^1 = 0.8828$	$k_4^2 = 1.8475$
$z_2^1 = 2.0845$	$z_2^2 = 3.3502$

Table 2.2: Results of RK4 method for $m = 1$.

2.3.2 RK4 Method for Higher Order IVP

Consider the 2^{nd} order IVP given as

$$y^{''} = f(t, y, y^{'}),\ y(a) = \alpha,\ y^{'}(a) = \beta. \tag{2.85}$$

Defining $y^{'} = z$. The above 2^{nd} order equation (2.85) is converted to system of following two 1^{st} order IVP

$$\begin{aligned} y^{'} &= z, & (2.86)\\ z^{'} &= f(t, y, z), & (2.87)\\ y(a) &= \alpha,\ z(a) = \beta. & (2.88) \end{aligned}$$

This is a system of two equations with two unknowns which can be solved by the method describe in section (2.3.1).

Example 2.3.2. Consider the 2^{nd} order initial value problem

$$\begin{aligned} y^{''} - 2y^{'} + 2y &= e^t \cos t, \\ y(0) = 1,\ y^{'}(0) = 2,\ &\text{and } 0 \le t \le 0.2,\ h = 0.1. \end{aligned} \tag{2.89}$$

For first iteration $m = 0$, we get

$k_1^1 = 0.2000$	$k_1^2 = 0.3000$
$k_2^1 = 0.2150$	$k_2^2 = 0.3150$
$k_3^1 = 0.2158$	$k_3^2 = 0.3150$
$k_4^1 = 0.2315$	$k_4^2 = 0.3298$
$z_1^1 = 1.2155$	$z_1^2 = 2.3150$

Table 2.3: Results of RK4 method for $m = 0$.

For second iteration $m = 1$, we get

$k_1^1 = 0.1216$	$k_1^2 = 0.3299$
$k_2^1 = 0.2480$	$k_2^2 = .3556$
$k_3^1 = 0.2493$	$k_3^2 = 0.3455$
$k_4^1 = 0.2661$	$k_4^2 = 0.3588$
$z_2^1 = 1.4459$	$z_2^2 = 2.6635$

Table 2.4: Results of RK4 method for $m = 1$.

Example 2.3.3. In this example, we find numerical solutions of the initial value problem $z^{'} = z - t$, $z(0) = 2$ with step size $h = 0.1$ in interval $0 \le t \le 1$, by using Euler's method and RK4 method and compare the results with the exact solution given by $z(t) = e^t + t + 1$. The results are shown in Tables 2.5 and 2.6.

m	t_m	$z(t_m)$ (exact solution)	z_m (by RK4 method)	z_m (by Euler's method)
1	0.0	2.0000	2.0000	2.0000
2	0.1	2.2052	2.2052	2.2000
3	0.2	2.4214	2.4214	2.4100
4	0.3	2.6499	2.6499	2.6310
5	0.4	2.8918	2.8919	2.8641
6	0.5	3.1487	3.1488	3.1105
7	0.6	3.4221	3.4222	3.3716
8	0.7	3.7138	3.7138	3.6488
9	0.8	4.0255	4.0256	3.9436
10	0.9	4.3596	4.3597	4.2580
11	1.0	4.7183	4.7184	4.5938

Table 2.5: Exact solution and numerical solutions obtained by RK4 and Euler's methods.

m	t_m	$\|E_m\|$ (for RK4 method)	$\|E_m\|$ (for Euler's method)
1	0.0	0.0000	0.0000
2	0.1	0.0000	0.0052
3	0.2	0.0000	0.0114
4	0.3	0.0000	0.0189
5	0.4	0.0001	0.0277
6	0.5	0.0001	0.0382
7	0.6	0.0001	0.0505
8	0.7	0.0000	0.0650
9	0.8	0.0001	0.0819
10	0.9	0.0001	0.1016
11	1.0	0.0001	0.1245

Table 2.6: Absolute error for RK4 and Euler's methods.

Notice that the absolute errors in case of RK4 method are much less than the absolute errors as in Euler's method.

2.4 MATLAB

MATLAB stands for Matrix Laboratory. MATLAB is a high-level language and interactive computing system for numerical computation and programming. MATLAB is used for developing algorithms, analyzing data, handling graphics and also to create models. There are many built-in functions in MATLAB that are used to solve ordinary differential equations.

dsolve:

dsolve is a built-in function used to find exact solutions of ordinary differential equations. It is applicable for both IVPs and BVPs (of ODEs). The basic syntax is given as:

$$dsolve('eq1',\ 'eq2',\dots,\ 'cond1',\ 'cond2',\dots,\ 'v'),$$

where $''eq1''$ *and* $''eq2''$ stand for ordinary differential equations, $''cond1''$ *and* $''cond2''$ stand for initial or boundary conditions and $''v''$ stands for independent variable. If the independent variable is not specified then MATLAB takes $''t''$ as the independent variable by default. In specifying the equation the letter D denotes differentiation. If z is the dependent variable and t is the independent variable, Dz stands for $\frac{dz}{dt}$. A second derivative is typed as $D2$, third derivative as $D3$, and so on. We write initial conditions $z(a) = A$ as $'z(a) = A'$ and $z'(a) = A$ as $'Dz(a) = A'$. For example, consider the differential equation

$$\frac{dz}{dt} = zt,\ \ z(1) = 1.$$

The *dsolve* syntax is written as:

$$dsolve('Dz = z * t',\ 'z(1) = 1',\ 't').$$

Consider a 2$^{\text{nd}}$ order differential equation

$$z''(t) + 8z'(t) + 2z(t) = cos(t),\ \ z(0) = 0,\ \ z'(0) = 1.$$

The *dsolve* syntax is:

$$dsolve('D2z + 8 * Dz + 2 * z = cos(t)',\ 'z(0) = 0,\ \ Dz(0) = 1',\ 't').$$

dsolve is also used for solving system of ordinary differential equations. The syntax is same as described above. For a system, we write differential equations separated by comas in a single quote. For example, consider system of 1^{st} order differential equations

$$\begin{aligned} t' &= t + 2z - u, \\ z' &= t + u, \\ u' &= 4t - 4z + 5u. \end{aligned} \tag{2.90}$$

The *dsolve* syntax for the above system of equations is:

$$dsolve('Dt = t + 2 * z - u', \ 'Dz = t + u', \ 'Du = 4 * t - 4 * z + 5 * u').$$

*ode*23 and *ode*45:

If *dsolve* fails to find exact solution then one may use built-in functions like *ode*23 and *ode*45 to find numerical solution of initial value problems. *ode*23 is a single step solver. It is coded in MATLAB by combining 2^{nd} and 3^{rd} order Runge- Kutta methods. The built-in function *ode*45 is used for numerical solution of 4^{th} and 5^{th} order Runge-Kutta method. The syntax is given as:

$$[t, z] = solver(odefun, \ tspan, \ z0),$$

where *"[t,z]"* is the output. Here *"t"* and *"z"* are the arrays that represent values of the independent and dependent variables, respectively. *"solver"* stands for MATLAB algorithm like *ode*23 and *ode*45 . *"ode fun"* represent the differential equation, *"tspan"* is the vector defining the beginning and end limits of integration and *"z0"* stands for initial conditions.

For example, consider first order ODE

$$\frac{dz}{dt} = tz^2 + z, \ z(0) = 1, \text{on the interval } t \in [0, 0.5].$$

For this create an m-file as:

$$function \ dzdt = p1(t, z)$$
$$dzdt = tz^2 + z;$$
$$end$$

Then type the code in command window:

$$tspan = [0\ 0.5];$$
$$z0 = 1;$$
$$[t, z] = ode45(@p1, tspan, z0)$$

Finally run the file to obtain the required solution. One can use *plot* command to view the output i.e. $plot(t, z)$. We can also display the results by using the *disp* command i.e. $disp([t, z])$.

For the nth-order ODE: $z^{(n)} = f(t, z, z', \ldots, z^{(n-1)})$, set $z_1 = z, z_2 = z^{'}, \ldots, z_n = z^{(n-1)}$. The result of this substitution will be an equivalent system of n first order ODEs

$$\begin{aligned} z_1^{'} &= z_2, \\ z_2^{'} &= z_3, \\ &\vdots \\ z_n^{'} &= f(t, z_1, z_2, \ldots, z_n). \end{aligned} \tag{2.91}$$

For example, consider a 2nd order differential equation

$$z^{''}(t) + 8z'(t) + 2z(t) = cos(t),\ z(0) = 0,\ z'(0) = 1.$$

Taking $z_1(t) = z(t)$ and $z_2(t) = z'(t)$, we have the system

$$\begin{aligned} z_1^{'}(t) &= z_2(t), \\ z_2^{'}(t) &= -8z_2(t) - 2z_1(t) + cos(t). \end{aligned}$$

Now create an m-file as:

$$\begin{aligned} function\ dzdt &= p2(t, z) \\ dzdt(1) &= z(2); \\ dzdt(2) &= -8 * z_2(t) - 2 * z_1(t) + cos(t); \\ dzdt &= dzdt(1) + dzdt(2); \\ end \end{aligned}$$

Then type the code in command window:

$$tspan = [0\ 5];$$
$$[t, z] = ode45(@p2, tspan, [0\ 1])$$

Finally run the file to obtain the required solution. For solving the system of ordinary differential equations, we follow the same code as we discussed above in example of 2^{nd} order differential equation.

Chapter 3

Solution of Non-linear Ordinary Differential Equation Involving Arbitrary Constants

Differential equations appear as mathematical models of many physical problems. In general relativity solutions of the Einstein field equations, which are essentially a system of non-linear differential equations, lead to models of space-time geometry. An attempt to find spherically symmetric static solutions of the field equations led to the following non-linear ordinary differential equation involving two arbitrary constants.

$$\frac{dy}{dx} = \frac{y}{\sqrt{y^2 + c_1 y + c_2 y^4}}, \; y(0) = 0. \tag{3.1}$$

In this chapter we present solutions of the above initial value problem (3.1). We consider different cases depending on the values of arbitrary constants and they are as follows:

Case	Values of arbitrary constants
Case 1	$c_1 = 0 = c_2$
Case 2	$c_1 = 0, c_2 > 0$
Case 3	$c_1 = 0, c_2 < 0$
Case 4	$c_1 > 0, c_2 = 0$
Case 5	$c_1 < 0, c_2 = 0$
Case 6	$c_1 > 0, c_2 > 0$
Case 7	$c_1 > 0, c_2 < 0$
Case 8	$c_1 < 0, c_2 > 0$
Case 9	$c_1 < 0, c_2 < 0$

Table 3.1: Different cases depending on the values of arbitrary constants.

Case 1: $c_1 = 0,\ c_2 = 0.$

In this case, the IVP (3.1) has following solution

$$y = x. \tag{3.2}$$

Case 2: $c_1 = 0,\ c_2 > 0.$

Taking $c_2 = k^2$ $(k \neq 0)$, we have the IVP (3.1) as

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+k^2y^2}},\ \ y(0) = 0. \tag{3.3}$$

In this case the substitution, $ky = \tan\theta$, leads to the following solution

$$\frac{1}{2k}\left(ky\sqrt{k^2y^2+1} + \ln|\sqrt{k^2y^2+1} + ky|\right) = x. \tag{3.4}$$

Case 3: $c_1 = 0,\ c_2 < 0.$

Taking $c_2 = -k^2$ $(k \neq 0)$, the IVP (3.1) becomes

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-k^2y^2}},\ \ y(0) = 0. \tag{3.5}$$

In this case the substitution, $ky = \sin\theta$, leads to the following solution

$$\frac{1}{2k}\left(\sin^{-1}(ky) + ky\sqrt{1-k^2y^2}\right) = x. \tag{3.6}$$

Case 4: $c_1 > 0$, $c_2 = 0$.

Taking $c_1 = k^2$ $(k \neq 0)$ and $c_2 = 0$, the IVP (3.1) becomes

$$\frac{dy}{dx} = \frac{y}{\sqrt{y^2 + k^2 y}}, \;\; y(0) = 0. \tag{3.7}$$

In this case the substitution

$$y + \frac{1}{2}k^2 = \frac{1}{2}k^2 \sec\theta, \tag{3.8}$$

leads to the solution

$$\sqrt{y^2 + k^2 y} + \frac{k^2}{2} \ln\left|\frac{2y + k^2 + 2\sqrt{y^2 + k^2 y}}{k^2}\right| = x. \tag{3.9}$$

Case 5: $c_1 < 0$, $c_2 = 0$.

Taking $c_1 = -k^2$ $(k \neq 0)$ and $c_2 = 0$, the IVP (3.1) becomes

$$\frac{dy}{dx} = \frac{y}{\sqrt{y^2 - k^2 y}}, \;\; y(0) = 0. \tag{3.10}$$

Taking the substitution

$$y - \frac{1}{2}k^2 = \frac{1}{2}k^2 \sec\theta, \tag{3.11}$$

we get the following solution

$$\sqrt{y^2 - k^2 y} - \frac{k^2}{2} \ln\left|\frac{2y - k^2 + 2\sqrt{y^2 - k^2 y}}{k^2}\right| = x. \tag{3.12}$$

Case 6: $c_1 > 0$, $c_2 > 0$.

In this case, taking $c_1 = 1$ and $c_2 = 1$, the IVP (3.1) becomes

$$\frac{dy}{dx} = \frac{y}{\sqrt{y^2 + y + y^4}}, \;\; y(0) = 0. \tag{3.13}$$

For $y \in [-0.682, 0)$ we have complex values of $\sqrt{y^2 + y + y^4}$, therefore, solution does not exist for this range. The solution lies in $(-\infty, -0.682) \cup (0, \infty)$. Obtaining analytical solution of IVP (3.13) is not that trivial. So, we obtain numerical solution using MATLAB built-in functions $ode23$, $ode45$ and by RK4 method and the results are shown in Table 3.2. The graphs of the solution for $y \in (-\infty, -0.682)$ and $(0, \infty)$ are shown in Figures 3.1 and 3.2 respectively.

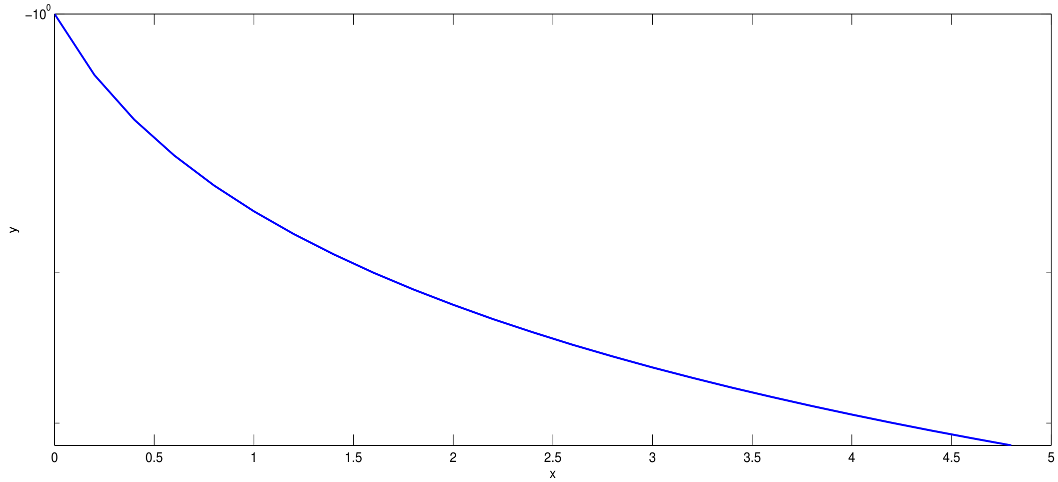

Figure 3.1: Numerical solution of the IVP for case 6, when $y \in (-\infty, -0.682)$.

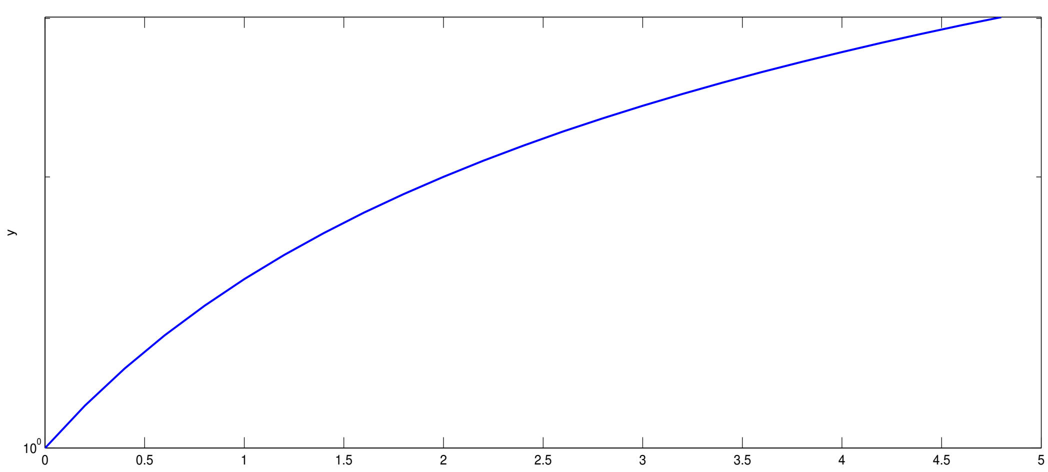

Figure 3.2: Numerical solution of the IVP for case 6, when $y \in (0, \infty)$.

x_m	y_m (by $ode23$)	y_m (by $ode45$)	y_m (by RK4 method)
0	-1.0000	-1.0000	-1.0000
0.0502	–	-1.0485	–
0.0800	-1.0758	–	–
0.1005	–	-1.0941	–
0.1507	–	-1.1374	–
0.2000	–	–	-1.1780
0.2010	–	-1.1787	–
0.2442	-1.2130	–	–
0.2760	–	-1.2374	–
0.3510	–	-1.2932	–
0.4000	–	–	-1.3283
0.4260	–	-1.3464	–
0.4751	-1.3802	–	–
0.5010	–	-1.3975	–
0.5760	–	-1.4468	–
0.6000	–	–	-1.4622
0.6510	–	-1.4944	–
0.7260	–	-1.5405	–
0.7751	-1.5701	–	–
0.8000	–	–	-1.5848
0.8010	–	-1.5853	–
0.8760	–	-1.6289	–
0.9510	–	-1.6714	–
1.0000	–	–	-1.6986
1.5510	–	-1.9798	–
3.0000	-2.5834	-2.5833	-2.5833

Table 3.2: Results of IVP (3.13) by using $ode23$, $ode45$ and RK4 method.

Notice that $ode23$ involves least and $ode45$ most number of intermediate points to obtained result at the boundary point, which differ slightly in the fourth decimal place. As $ode45$ is fifth and $ode23$ is third order method, therefore, ode45 is more accurate. The solution obtained by RK4 matches with the one obtained by $ode45$ with lesser number of intermediate points involved, therefore, we have chosen RK4 for our calculations in the remaining cases.

Case 7: $c_1 > 0$, $c_2 < 0$.

Taking $c_1 = 1$ and $c_2 = -1$, the IVP (3.1) is given as

$$\frac{dy}{dx} = \frac{y}{\sqrt{y^2 + y - y^4}}, \ y(0) = 0. \tag{3.14}$$

For $y \in (-\infty, 0) \cup [1.325, \infty)$ we have complex values of $\sqrt{y^2 + y - y^4}$. Therefore, the solution lies in $(0, 1.324]$ only. The numerical solution obtained using RK4 method is shown in Figure 3.3.

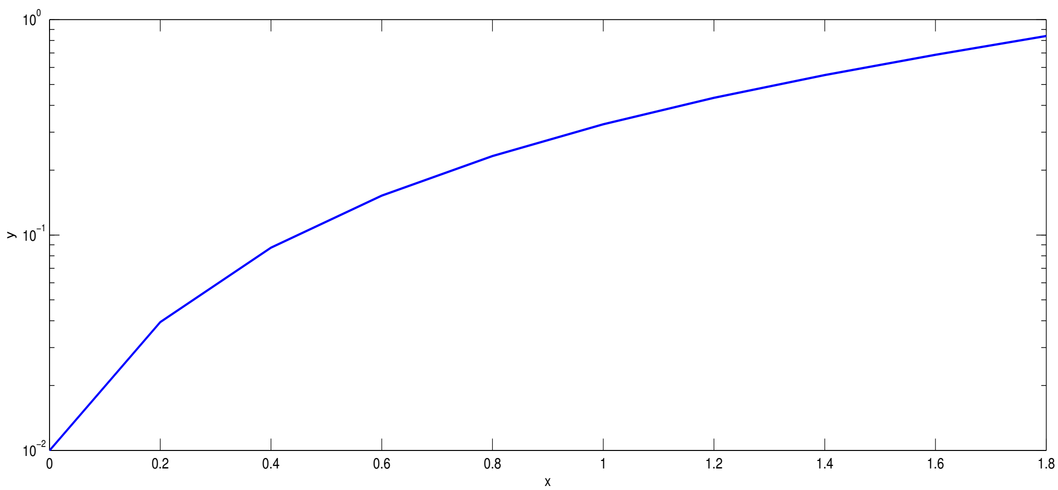

Figure 3.3: Numerical solution of the IVP for case 7, when $y \in (0, 1.324]$.

Case 8: $c_1 < 0$, $c_2 > 0$.

Taking $c_1 = -1$ and $c_2 = 1$, the IVP (3.1) is given as

$$\frac{dy}{dx} = \frac{y}{\sqrt{y^2 - y + y^4}}, \ y(0) = 0. \tag{3.15}$$

For $y \in (0, 0.682]$ we have complex values of $\sqrt{y^2 - y + y^4}$. Therefore, the solution lies in $(-\infty, 0) \cup (0.682, \infty)$. The graphs of the solution for $y \in (-\infty, 0)$ and $(0.682, \infty)$ are shown in Figures 3.4 and 3.5 respectively.

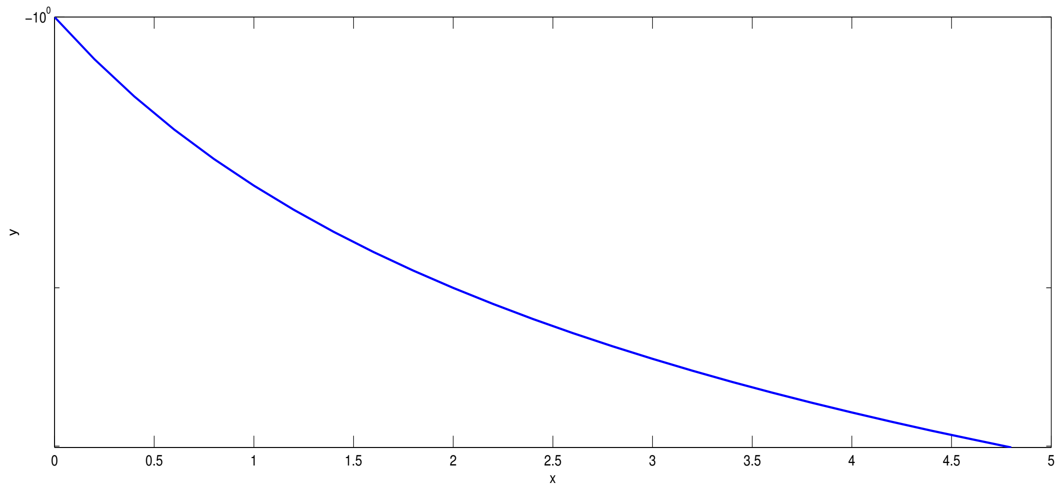

Figure 3.4: Numerical solution of the IVP for case 8, when $y \in (-\infty, 0)$.

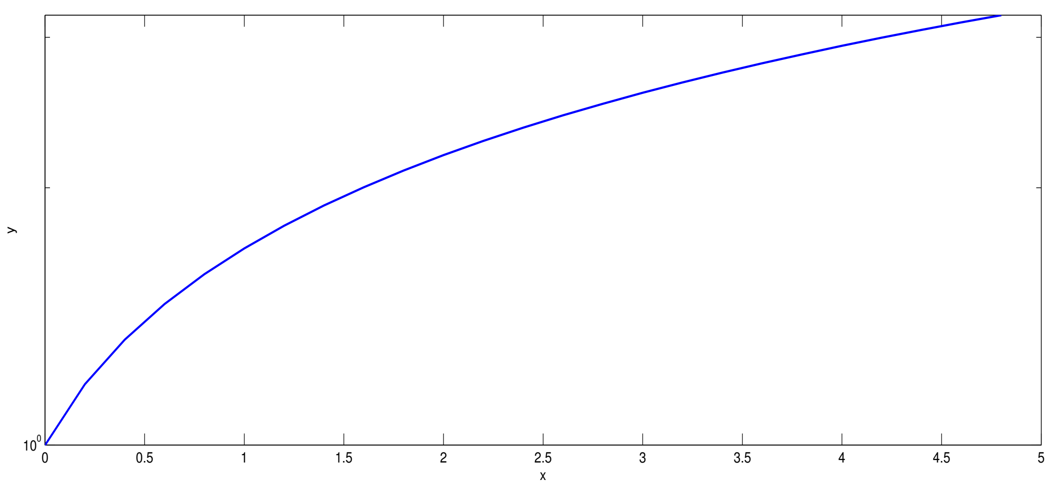

Figure 3.5: Numerical solution of the IVP for case 8, when $y \in (0.682, \infty)$.

Case 9: $c_1 < 0$, $c_2 < 0$.

Taking $c_1 = -1$ and $c_2 = -1$, the IVP (3.1) is given as

$$\frac{dy}{dx} = \frac{y}{\sqrt{y^2 - y - y^4}}, \ y(0) = 0. \tag{3.16}$$

For $y \in (-\infty, -1.324) \cup (0, \infty)$ we have complex values of $\sqrt{y^2 - y - y^4}$. Therefore, the solution lies in $[-1.324, 0)$ only. The numerical solution obtained using RK4 method is shown in Figure 3.6.

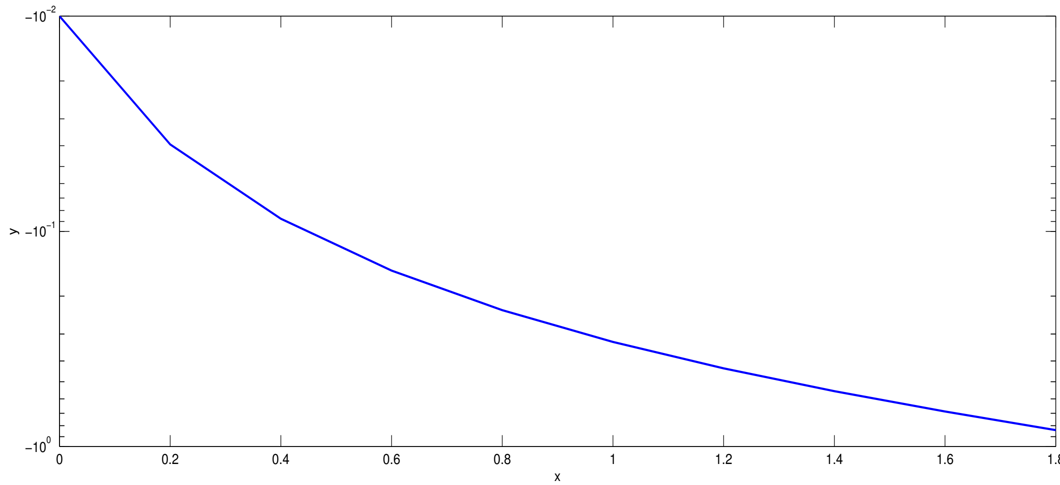

Figure 3.6: Numerical solution of the IVP for case 9, when $y \in [-1.324, 0)$.

Chapter 4

Conclusion

In this thesis, solution of non-linear ordinary differential equation of first order is discussed for different values of the constants involved.
In Chapter 1, we have given a brief introduction of ordinary differential equations and also discussed some analytical methods for solving ODEs. In Chapter 2, some numerical methods like Euler's method & it's variations and Runge-Kutta methods for solving IVPs of ordinary differential equations are discussed. A brief review of MATLAB built-in functions is also given. In Chapter 3, we investigate the solution of 1^{st} order non-linear ordinary differential equation involving two arbitrary constants. We have nine cases based on different values of the parameters. For five cases exact analytical solutions are obtained. For remaining four cases we obtain numerical solutions. The ranges of validity of the solutions are also discussed.

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