Performance Analysis Of Differential Transformation Method

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Dedication

To my family.

Acknowledgments

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Abstract

The aim of this study is to carry out detailed performance analysis of differential transformation method(DTM) for solving linear and non-linear differential equations. DTM is applied to obtain the analytical solution of linear and non-linear ordinary and partial differential equations. Also, the method is later applied to obtain the semi-analytical solution of magneto hydro dynamic (MHD) flow of non-Newtonian fluid between three parallel plates. The results obtained are compared with available exact and numerical solutions. The absolute error is computed and presented graphically. Comparison of the results, confirms that the method is effective, reliable and easy to implement. However, there are many limitations of this method that are highlighted in this study. The analysis of results revealed that DTM performs well only in restricted domain and to achieve strong results in comparison to other methods the number of terms involved in inverse transform should be increased when subintervals increases. Further, the DTM requires extra conditions for solving boundary value partial differential equations. Whereas, a polynomial of higher degree containing unknown variable appears in inverse transform while solving boundary value ordinary differential equations, which increases the computational cost of the method.

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Chapter 1

Introduction

1.1 Literature Review

In general, its not easy to find the exact solution of the nonlinear differential equations so the researchers are always keen to find new methods for obtaining analytical solutions of linear and nonlinear differential equations. Several applications have been developed for finding numerical solutions of differential equations. Some of the famous one's are RK4[1, 2], Euler [1] , DTM etc.
Differential transform method is a semi-analytical method that can effectively solve non-linear differential equation. Initially, DTM was used to solve linear and nonlinear IVPs. In 1986 ZHOU[3] proposed this method for solving linear and nonlinear IVPs in electric circuit analysis. Later Chen and Ho[4] used DTM for solving PDES and obtained the solution in closed series form for linear and nonlinear IVPs. In [5] multi step DTM is proposed to increase the interval of convergence for series solution. It is also used for finding the accurate solution for system of equations. In 2011 Rostam [6] used DTM to solve nonlinear delay differential equation. By using DTM both analytic and exact solutions of the nonlinear systems were obtained and it was shown that DTM is reliable and consume less computational cost. In [7, 11] authors came up with the idea that the results obtained by DTM can be converted into exact answer. This idea is also discussed by [8, 9]. By presenting some examples it was illustrated that the method is reliable, efficient and fast convergent. In [10] authors introduced DTM and MsDTM as effective and reliable tool so that student at undergraduate level can easily

solve linear and nonlinear differential equations. The results of DTM and MsDTM were compared with RK4 to show accuracy and simplicity of the methods. In [12] a reliable algorithm was introduced to calculate differential transform of difficult nonlinearities i.e exponential and logarithmic non linearities. The technique is useful to handle nonlinear terms. In [13, 14] DTM is presented as an approximating technique for solving linear BVPs. The solution of higher order differential i.e 7th, 8th order is developed in these papers. After Chen and Ho [4] much work was done on linear and nonlinear 2D and 3D PDEs. Ayaz [15] proposed some new theorems to solve PDEs of higher dimension and compared there results with decomposition method. The comparison shown that DTM is efficient, fast convergent, easy to implement and consumes less computational cost. In [16] the analytical solution of nonlinear hyperbolic-parabolic PDEs was obtained using DTM. In [17] wave and heat like equations were solved. The conclusion deduced was that DTM is such a powerful technique that gives exact solution. The same 2-D DTM was then used in [18] to solve 2-D nonlinear wave equation and the results were compared with those obtained by Least Square Method. Again the conclusion deduced after comparison was that DTM is fast convergent and more accurate. Also, in [19, 20] nonlinear gas dynamic and Klein-Gordan equations are solved using 2-D DTM. The approximate solutions of these equations are evaluated in the form of series. The obtained results reveal that DTM is easy to implement and fast convergent method. The convergence of DTM is in detail discussed in [33].

In this research, we applied DTM to solve the differential equations involved in MHD flow of third grade non-Newtonian fluids. Non-Newtonian fluids got much popularity due to there various applications in the field of engineering and industry. Non-Newtonian fluids includes ketchup, blood, lubricants, paste etc. There are many different models in literature that describe the behaviour of non-Newtonian fluids between parallel plates. In [34] Adomain decomposition method is successfully applied to study non-Newtonian third grade fluid flow between parallel plates. ADM(Adomain Decomposition Method) results are compared with numerical solution to show the accuracy and reliability of the results. In [35] MSDTM is used to study the non-Newtonian third grade fluid flow between parallel plates. Author obtained the analytical solution of the non-linear differential equations involved and concluded that the method is easily to implement, reliable, takes less computational cost and results obtained by MSDTM are highly accurate.

Magnetohydrodynamics gained much attention and popularity due to its various applications in science and engineering. In [36] author used ADM to obtain analytical solution of non-linear differential equations involved MHD Couette and Poiseuille flow of a third grade fluid and after comparing the obtained results with HPM he concluded that though ADM needs more computational cost but gives more accurate results then HPM.

Here, we studied the MHD flow of third grade non-Newtonian fluids between three parallel plates(first and third are stationary while second is moving).The three cases of plane Couette flow, plane Poiseuille flow and generalized Couette flow are studied. The analytical solution of all the non-linear equations is obtained by DTM in section (4.3). The effect of magnetic field and pressure on velocity are also shown graphically.

1.2 Problem Statement

Non-linear differential equations remains challenging to solve. Many analytical and numerical methods are available in literature to obtain the approximate solutions for such problems. differential transform method (DTM) is a semi-analytical method that provides the analytical and the numerical solution for linear and non-linear differential equations. Keeping in view the effectiveness and ease of application of DTM many researchers have attempted to solve several real life D.Es using this method. However, instead of its significance there are several limitations associated with this method. The aim of this study is to carry out detailed performance analysis of this method to highlight the strength and limitations of the method. In view of the problem statement, the main objective of this study is to carry out the performance analysis of differential transformation method(DTM) that can help researcher to have better understanding of the method.

1.3 Preliminaries

Most of the problems of both engineering and mathematics are expressed in the form of differential equations. The equations involving derivatives (variables that changes due to the other independent variable) are called differential equations. The DEs are further categorized into two types boundary value problems (BVPs) and initial value problems (IVPs). BVPs are the ones in which conditions are defined on boundaries. Where as in IVPs conditions are defined on starting point only.

1.3.1 Ordinary Differential Equations

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an ordinary differential equation (ODE) [27]. Some examples of ordinary differential equations are given below.

Example: The following are examples of linear partial differential equations that commonly arise in problems of maths and physics.

Simple Linear Equation: $y'' + y' = e^x$

Separable Differential Equation: $\frac{dz}{dx} = G(z)$

Non-Linear nth order differential Equation: $y^n = F(x, y, y', ...y^{n-1})$

1.3.2 Partial Differential Equations

A partial differential equation [27] is an equation that contains partial derivatives. In partial differential equations, the unknown function depends on several variables (like u(x,t) where x is distance and t is time) Some examples of the two and three dimensional PDES are given below.

Example : The following are examples of linear partial differential equations that commonly arise in problems of maths and physics.

Linear Heat Equation: $w_{xx} = w_t$

Tricomi equation: $xw_{yy} = w_{xx}$

Laplace Equation: $w_{xx} + w_{yy} = 0$

Example 2: The following are examples of non-linear partial differential equations.

Sine-Gordan Equation: $w_{tt} - w_{xx} + sinw = 0$

Now, here we will discuss some basic definitions of fluid mechanics.

1.3.3 Fluid

A fluid is a substance that has tendency to flow and it takes shape of the containing container.

1.3.4 Incompressible and Compressible Fluid

The fluid whose density doesn't vary with time are incompressible fluids while whose density changes with time are compressible fluids.

1.3.5 Viscosity

The resistance observed by the fluid during it's flow is known as viscosity. Mathematically, it's written as

$$\tau \propto \dot{\gamma}, \tag{1.1}$$

here $\tau = shear\ stress,\ \gamma = rate\ of\ deformation/strain.$

The S.I unit of viscosity is kg/ms.

1.3.6 Newtonian and Non-Newtonian Fluid

The fluids which obeys newton law of viscosity are known as Newtonian fluids. In such fluids shear stress is proportional to rate of deformation. While the fluids whose stress and rate of deformation aren't linearly proportional are called non-Newtonian fluids. Example of such fluids is toothpaste, Gel, lubricants, ketch, oil etc.

1.3.7 Steady and Unsteady Flow

A flow in which the properties of fluid doesn't change with time is called steady flow.

$$\frac{\partial}{\partial t} = 0, \tag{1.2}$$

where as the change in fluid properties with time is called unsteady flow.

$$\frac{\partial}{\partial t} \neq 0. \tag{1.3}$$

1.3.8 Laminar and Turbulent Flow

If the fluid flow is smooth and the path lines of fluid particles doesn't intersect each other then such flow is called laminar flow.

If the fluid flow isn't smooth and the path lines intersect each other then such flow is turbulent flow.

1.3.9 Continuity Equation

Law of conservation of mass states that, 'mass can neither be created nor destroyed.' The mass inflow, outflow and change in mass should be balanced. The mathematical form of this is given as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0, \tag{1.4}$$

here $V = [u(x,y,z), v(x,y,z), w(x,y,z)]$ is velocity field wile $\rho = density$.

1.3.10 Momentum Equation

Newton second law of motion states that

$$\sum F = ma, \tag{1.5}$$

equation (1.5) can also be written as

$$\rho \frac{DV}{Dt} = \sum F, \tag{1.6}$$

whereas $\sum F = F_{surface} + F_{body}$.

The momentum equation for Newtonian fluids is

$$\rho \frac{dV}{dt} = -\nabla P + \nabla.\tau_{nm} + \rho g. \tag{1.7}$$

1.3.11 Magnetohydrodynamics

The study concerned with magnetic field behaviour in electrically conducting fluids is known as Magnetohydrodynamics. The equations representing MHD flow are given in 4.2

1.4 Methods for solving Differential Equation

In general its not possible to find exact solution of all DEs, so the researchers are always keen to find new methods for obtaining analytical solutions of differential equations. Several applications have been developed for finding numerical solutions of differential equations. Some of them are

1. Finite Difference Method

2. RK Method

3. DTM

1.4.1 Finite Difference Method (FDM)

Finite difference method [27] is one of the popular method used to find the numerical solution differential equation. The following steps are used for solving DEs using FDM

1.First the whole domain is divided into small subintervals called mesh. Then label the points in the created mesh.

2. Approximate the derivatives by using forward, backward or central difference method. Now the derivatives will appear as variable with indicies i.e $u_{i,j+1}$.

3. By substituting the values of i, j and then substituting the given boundary conditions algebraic equations are obtained that can be solved using iterative methods like Jacobi Method, Gauss-Seidal etc.

4. By solving the algebraic equation using any iterative method we obtain the solution of unknown variables.

Finite difference method is also known as discretization method.

1.4.2 Runge-Kutta-Method 4 (RK4)

RK4 [27] is an iterative method used to solve 1^{st} order ordinary differential equations. The following steps are used for solving the ODEs

1. Find the step size h if its not given.

2. Using the step size divide the time interval t.

3. Now using the initial condition $y(0) = y_0$ do N iterations of y for different values of time t.

1.4.3 Differential Transformation Method (DTM)

Differential transformation method [4] is a simple and easy technique for solving both mathematical and engineering problems involving complex and high-dimension differential equation. The following steps are followed to find numerical solution of DEs using DTM

1. The differential equation is transformed using DTM.

2. Then the value of K and the given conditions are used to find the unknown polynomials.

3. At last, all polynomials are substituted in inverse transform and the result is obtained in the form of Taylor series. This series can be easy converted into exact solution.

Chapter 2

Solving Ordinary Differential Equations using DTM

Here, first we will discuss about the basic definition, formula, results and theorems of DTM. Furthermore, the implementation of differential transformation method will be presented. Also, the analytical solution of some ordinary differential equations will be obtain using DTM. The analytical solution results will be compared with exact and numerical method to show the efficiency of DTM.

2.1 Differential Transformation Method (DTM)

Definition 2.1.1. *The 1-dimensional transformation [8] of a function w(t) is defined as*

$$W(K) = \frac{1}{K!}\frac{d^K w(x_0)}{dx^K}. \tag{2.1}$$

The inverse of w(x) defined in the form of Taylor series given as

$$w(x) = \sum_{K=0}^{\infty} W(K)(x - x_0)^K. \tag{2.2}$$

Above equation can also be written as

$$w(x) = \sum_{K=0}^{\infty} \frac{d^K w(x_0)}{K!\, dx^K}(x - x_0)^K. \tag{2.3}$$

The following theorems are proved in [4] and here they will be used later in this chapter to solve ODEs

Theorem 2.1.2. *If w(x)=u(x)+v(x) then differential transform of w(x) is*

$$W(K)=U(K)+V(K).$$

Proof: From the definition
W(K)=$[\frac{d^K w(x)}{K!dx^K}]_{x=x_0}$ $=\frac{1}{K!}[\frac{d^K(u(x)+v(x))}{dx^K}]_{x=x_0}$
$=\frac{1}{K!}[\frac{d^K u(x)}{dx^K}]_{x=x_0} + \frac{1}{K!}\frac{d^K v(x)}{dx^K}]_{x=x_0}$ =U(K)+V(K).

Theorem 2.1.3. *If w(x)=az(x) then*

$$W(K)=aZ(K), \text{ where } a=constant.$$

Proof: From the definition
W(K)=$[\frac{d^K w(x)}{K!dx^K}]_{x=x_0}$ =a$[\frac{d^K z(x)}{K!dx^K}]_{x=x_0}$=aZ(K).

Theorem 2.1.4. *If w(x)=$\frac{d^N z(x)}{dx^N}$ then*

$$W(K)=(K+1)(K+2)...(K+N)Z(K+N).$$

Proof: From the definition
W(K)=$[\frac{d^K w(x)}{K!dx^K}]_{x=x_0}$=$\frac{d^K}{K!dx^K}[\frac{d^N z(x)}{dx^N}]_{x=x_0}$=$[\frac{d^{K+N} z(x)}{K!dx^{K+N}}]_{x=x_0}$
$=\frac{1}{K!}(K+N)!\,Z(K+N)$=(K+1)(K+2)....(K+N)Z(K+N).

Theorem 2.1.5. *If w(x)=v(x)w(x) then*

$$W(K)=\sum_{K_1=0}^{K} V(K_1)W(K-K_1).$$

Theorem 2.1.6. *If y(x)=x^n then*

$$Y(K)=\delta(K-n).$$

where

$$\delta(K-n)=\begin{cases}1, & \text{if } K=n;\\ 0, & \text{if } K\neq n.\end{cases}$$

Theorem 2.1.7. *If y(x)=*$e^{\lambda x}$ *then*

$$Y(K)=\frac{(\lambda)^K}{K!}.$$

Theorem 2.1.8. *If y(x)=sin*$(\Omega x+\alpha)$ *then*

$$Y(K)=\frac{(\Omega)^K}{K!}\sin(\frac{K\pi}{2}+\alpha).$$

Theorem 2.1.9. *If y(x)=cos*$(\Omega x+\alpha)$ *then*

$$Y(K)=\frac{(\Omega)^K}{K!}\cos(\frac{K\pi}{2}+\alpha).$$

2.2 Implementation of DTM

DTM is a simple, easy to understand and fast convergent method. Here, in this section its demonstrated that how this method is applied on differential equations.

Consider a nonlinear IVP

$$a(x)y''+b(x)y'+y=f(x,y,y'), \tag{2.4}$$

With initial condition

$$y(0)=p_1, \qquad y'(0)=p_2. \tag{2.5}$$

Where f(x,y,y') is function. Now applying DTM on (2.4) and (2.5)

$$\sum_{K_1=0}^{K}(K-K_1+2)(K-K_1+1)A(K_1)Y(K-K_1+2)$$
$$+\sum_{K_1=0}^{K}(K-K_1+1)B(K_1)Y(K-K_1+1)+Y(K)=F(x,Y,Y') \tag{2.6}$$

$$Y(0)=p_1, \quad Y(1)=p_2. \tag{2.7}$$

Now put value of K in (2.6) to obtain $Y(K)$ and then use this in below given transformation

$$y(x)=\sum_{K=0}^{n}Y(K)x^K, \tag{2.8}$$

to obtain the analytical solution of the given equation.

2.3 Numerical Problems

To check the efficiency and reliability of the method some linear and nonlinear ODEs are tested.

Example 2.3.1. *Consider the linear BVP with boundary conditions*

$$y'' = \frac{w(Lx - x^2)}{2EI}, \tag{2.9}$$

$y(0) = 0, \qquad y(3) = 0,$

here w=15000, L=3 ,I=3/10000, $E = 200 \times 10^9$.

Applying DTM on (2.9) *and IVPs results in*

$$(K+1)(K+2)Y(K+2) = \frac{w(L\delta(K-1) - \delta(K-2))}{2EI}, \tag{2.10}$$

$$Y(0) = 0. \tag{2.11}$$

Now by putting K=0,1,2,... in equation (2.10) *and then using equation* (2.11) *in it, we find*

$$Y(2) = 0, \tag{2.12}$$

$$Y(3) = 0.0000625, \tag{2.13}$$

$$Y(4) = -1.0417 \times 10^{-5}, \tag{2.14}$$

$$Y(5) = 0, \tag{2.15}$$

$$Y(6) = 0, \tag{2.16}$$

$$Y(K) = 0 \qquad For \quad K = 0, 2, 6, 7, 8, \cdots \tag{2.17}$$

By using equation (2.2) *we have*

$$y(x) = Y(1)x + 0.0000625x^3 - 1.0417 \times 10^{-5}x^4. \tag{2.18}$$

Here it can be seen that we found all the values except $Y(1)$. *We will find value as shown in [13, 14]*

$$Y(1) = -2.8125 \times 10^{-4}. \tag{2.19}$$

At last, the analytical solution obtained is

$$y(x) = (-2.8125 \times 10^{-4})x + (0.0000625)x^3 - (1.0417 \times 10^{-5})x^4. \quad (2.20)$$

In table 2.1, the results of equation (2.20) *computed at different values of t are presented. The numerical, exact and DTM solution of the problem are shown in figure* (2.1)*. Here it can be seen that DTM gives more accurate value then FDM method . This can also be seen in table 2.2 and figure* (2.2) *that as the time value increase absolute error for DTM also increases whereas FDM method gives less error for higher value of time.*

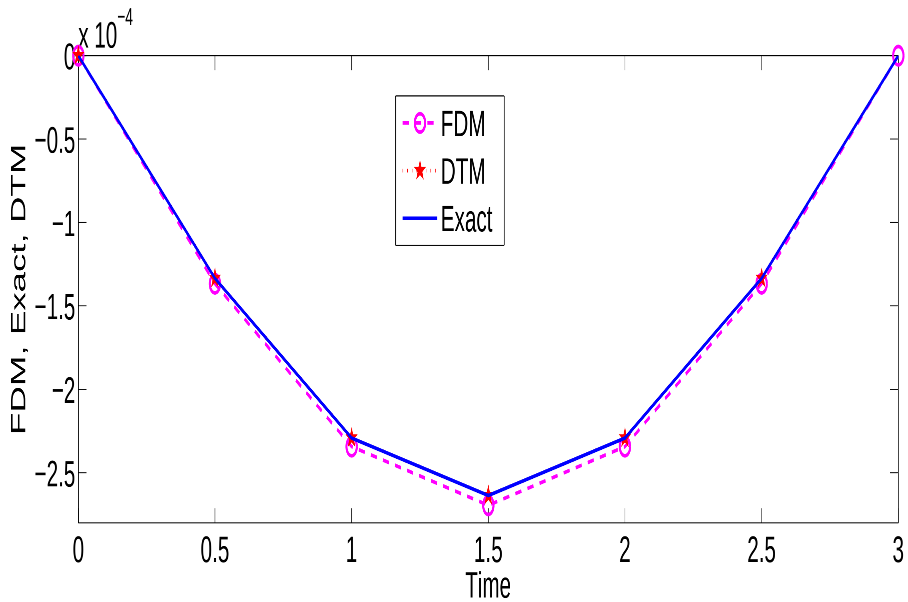

Figure 2.1: FDM, exact and DTM solution of equation 2.9 for h=0.25

Table 2.1: Comparison between DTM, FDM and exact solution

t	Exact	FDM	DTM
0	0	0	0
0.5	-0.00013346	-0.00013672	-0.00013346
1.0	-0.00022917	-0.00023437	-0.00022917
1.5	-0.00026367	-0.00026953	-0.00026367
2.0	-0.00022917	-0.00023438	-0.00022917
2.5	-0.00013346	-0.00013672	-0.00013346
3.0	0	0	$4.33680868994202 \times 10^{-19}$

Table 2.2: Absolute error for FDM and DTM, h=0.5

t	FDM	DTM(K=10)
0	0	0
0.5	3.255×10^{-6}	2.7105×10^{-20}
1.0	5.208×10^{-6}	5.421012×10^{-20}
1.5	5.859×10^{-6}	1.0842×10^{-19}
2.0	5.208×10^{-6}	1.6263×10^{-19}
2.5	$3.255e \times 10^{-6}$	1.8974×10^{-19}
3.0	0	4.33681×10^{-19}

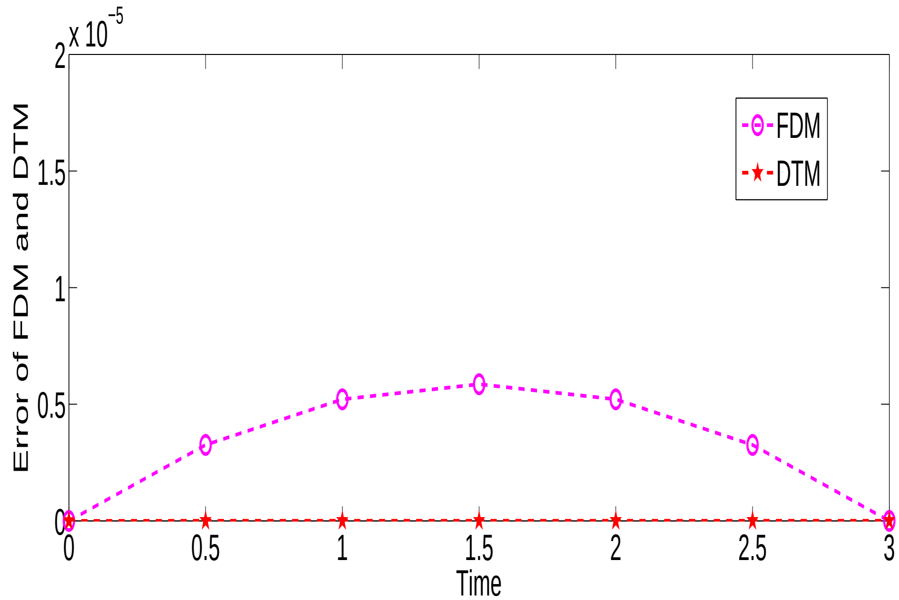

Figure 2.2: Absolute error for FDM and DTM solution of equation 2.9

Example 2.3.2. *Consider the linear equation given with initial condition [27]*

$$y' = y - t^2 + 1 \tag{2.21}$$

$$y(0) = 0.5. \tag{2.22}$$

The exact solution is

$$y(t) = (t+1)^2 - 0.5e^t. \tag{2.23}$$

Applying DTM on (2.21) *and* (2.22), *we obtain*

$$(K+1)Y(K) = Y(K) - \delta(K-2) + \delta(K) \tag{2.24}$$

$$Y(0) = 0.5. \tag{2.25}$$

Now bu putting K=0,1,2... in (2.24)*, we get*

$$Y(1) = \frac{3}{2!},\ Y(2) = 0.75,\ Y(3) = -0.0833,\ Y(4) = -0.0208, ...\ Y(10) = -1.3779e - 06.$$

Putting above values inverse transformation given in (2.2)*, we get*

$$y(t) = 0.5 + \frac{3t}{2!} + 0.75t^2 + \ \cdots \tag{2.26}$$

In table 2.3, the results of equation (2.26) *computed at different values of t are presented. In figure* (2.3) *the absolute error of RK4 and DTM solution is plotted. Here it can be seen that in the time interval [0, 2] RK4 is giving more error then DTM . This can also be seen in table 2.4 in which the absolute error terms are given for different values of t. So it can said that DTM performs better than RK4. This can also be seen in figure* (2.4) *where the analytical, numerical and exact solution w.r.t time are plotted.*

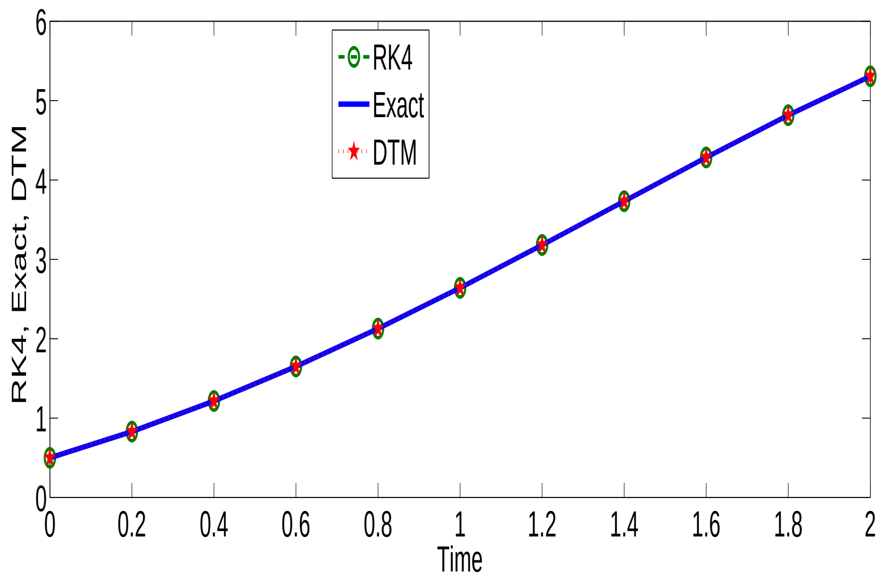

Figure 2.3: Plot of RK4, exact and DTM solution of equation 2.21 for h=0.2

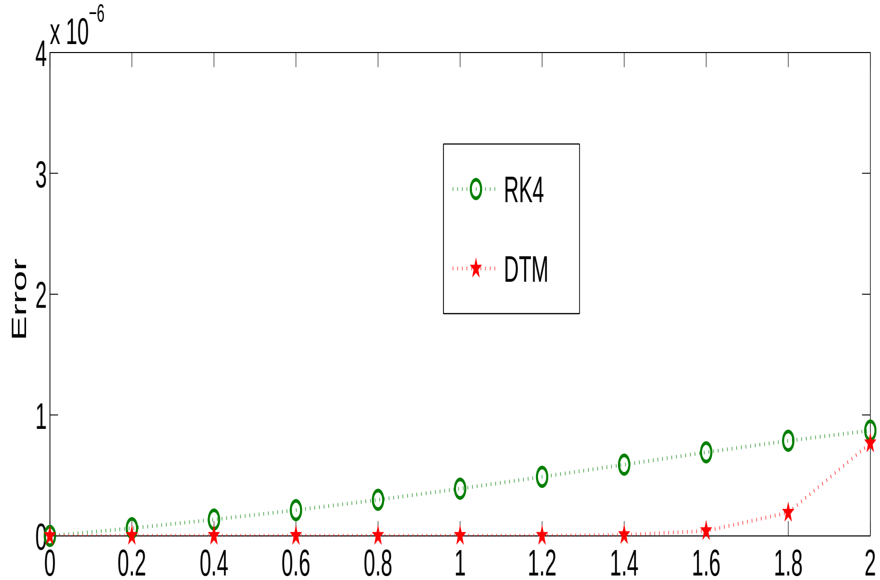

Figure 2.4: Absolute error of RK4 and DTM solution of equation 2.21

Table 2.3: Comparison between DTM, RK4 and exact solution

t	Exact	RK4[28]	DTM(k=12)
0	0.5	0.5	0.5
0.2	0.8293	0.8293	0.8293
0.4	1.2141	1.2141	1.2141
0.6	1.6489	1.6489	1.6489
0.8	2.1272	2.1272	2.1272
1	2.6409	2.6409	2.6409
1.2	3.1799	3.1799	3.1799
1.4	3.7324	3.7324	3.7324
1.6	4.2835	4.2835	4.2835
1.8	4.8152	4.8152	4.8152
2.00	5.305	5.305	5.305

Table 2.4: Absolute error for RK4 and DTM, h=0.2

t	RK4[28]	DTM(K=12)
0	0	0
0.2	6.3484e-08	2.22045e-16
0.4	1.3430e-07	8.8818e-16
0.6	2.1259e-07	1.0925e-13
0.8	2.98178e-07	4.6807e-12
1	3.9047e-07	8.64384e-11
1.2	4.8823e-07	9.3911e-10
1.4	5.8937e-07	7.0756e-09
1.6	6.9058e-07	4.0784e-08
1.8	7.8694e-07	1.91599e-07
2.00	8.7133e-07	7.6605e-07

Nonlinear Case

Example 2.3.3. *Consider the nonlinear equation given with initial condition [27]*

$$y' = -y^2 - 4y - 3 \quad 0 \leq t \leq 1 \tag{2.27}$$

$$y(0) = -2. \tag{2.28}$$

The exact solution of the equation is given below

$$y(t) = -3 + 2(1 + e^{-2t})^{-1}. \tag{2.29}$$

Now by using DTM results on (2.27) *and* (2.28), *we can write*

$$(K+1)Y(K+1) = -\sum_{K1=0}^{K} Y(K_1)Y(K-K_1) - 3\delta(K) - 4Y(K) \quad Where\ K = 0, 1, 2, 3, ... \tag{2.30}$$

with initial condition

$$Y(0) = -2. \tag{2.31}$$

By substituting K=0,1,2,.., in (2.30) *we obtain*

$$Y(1) = 1,\ Y(2) = 0,\ Y(3) = -0.3333,\ Y(4) = 0, ...Y(29) = 2.6148E - 06,\ Y(30) = 0.$$

By using the inverse transformation rule,

$$y(t) = \sum_{K=0}^{30} Y(K)t^K \tag{2.32}$$

$$y(t) = -2 + t - \frac{2t^3}{3!} + \ \cdots \tag{2.33}$$

solution of $y(t)$ *can be obtained. It can also be written as*

$$y(t) = -3 + 2(\frac{1}{2} + \frac{t}{2} - \frac{t^3}{3!} + \ \cdots) = -3 + 2(1 + e^{-2t})^{-1}.$$

This is also the exact solution

This solution computed for different values of t is given in table 2.5 where DTM results are compared with the exact and RK4 results. Here it can be observed that the increase in time increases the solution values. Figure (2.5) *shows that both methods are performing well. But the absolute error of all methods given in table 2.6 and graphical representation given in figure* (2.6) *shows that RK4 gives less error on interval [0.9, 1] than DTM.*

Table 2.5: Comparison between DTM, RK4 and exact solution.

t	Exact	RK4 [28]	DTM(K=30)
0	-2.000	-2.000	-2.000
0.05	-1.950041625042120	-1.950041627648895	-1.950041625042120
0.1	-1.900332005375044	-1.900332010658292	-1.900332005375044
0.15	-1.851114966376682	-1.85111497449505	-1.851114966376682
0.20	-1.802624679775096	-1.802624690968055	-1.802624679775096
0.25	-1.755081337596291	-1.755081352170451	-1.755081337596291
0.3	-1.708687387548409	-1.708687405859141	-1.708687387548409
0.35	-1.663624455663668	-1.663624478093124	-1.663624455663668
0.40	-1.620051037744775	-1.620051064677779	-1.620051037744775
0.45	-1.578100994749992	-1.578101026549401	-1.578100994749992
0.5	-1.537882842739990	-1.537882879723179	-1.537882842739990
0.55	-1.499479788809765	-1.499479831227742	-1.499479788809757
0.60	-1.462950433001965	-1.462950481022370	-1.462950433001842
0.65	-1.428330033914883	-1.428330087609773	-1.428330033913448
0.7	-1.395632222882837	-1.395632282221719	-1.395632222868887
0.75	-1.364851047612713	-1.364851112460892	-1.364851047497127
0.80	-1.335963229732151	-1.335963299854059	-1.335963228898804
0.85	-1.308930530167070	-1.308930605233928	-1.308930524850357
0.9	-1.283702129800975	-1.283702209401957	-1.283702099362572
0.95	-1.260216948725996	-1.260217032381900	–1.260216790510764
1	-1.238405844044235	-1.238405931222698	-1.238405089943087

Table 2.6: Error term of RK4, DTM(K=30)

t	RK4 [28]	DTM(K=30)
0	0	0
0.05	2.6068e-09	2.2204e-16
0.1	5.2832e-09	0
0.15	8.1184e-09	2.2204e-16
0.2	1.1193e-08	0
0.25	1.4574e-08	2.2204e-16
0.3	1.8311e-08	4.4409e-16
0.35	2.2429e-08	0
0.4	2.6933e-08	2.2204e-16
0.45	3.1799e-08	0
0.5	3.6983e-08	4.4409e-16
0.55	4.2418e-08	7.9936e-15
0.6	4.8020-08	1.2279e-13
0.65	5.3695e-08	1.4351e-12
0.7	5.9339e-08	1.3950e-11
0.75	6.4848e-08	1.1559e-10
0.8	7.0122e-08	8.3335e-10
0.85	7.5067e-08	5.3167e-09
0.9	7.9601e-08	3.0438e-08
0.95	8.3656e-08	1.582e-07
1.00	8.7178e-08	7.5410e-07

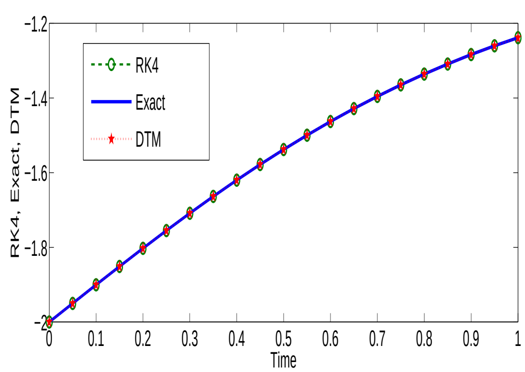

Figure 2.5: Plot of Exact,RK4 and DTM(K=30), h=0.05

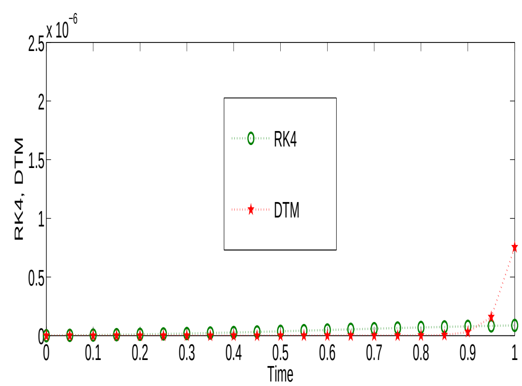

Figure 2.6: Absolute error for RK4 and DTM(K=30), h=0.05

Example 2.3.4. *Consider the nonlinear equation given with initial condition[27]*

$$x' = 1 - x + x^2 e^{2t} \quad 0 \leq t \leq 0.9 \tag{2.34}$$

$$x(0) = 0 \tag{2.35}$$

The exact solution of the equation is given below

$$x(t) = e^{-t} tan(e^t - 1) \tag{2.36}$$

Now by taking differential transform of (2.34), *we can obtain*

$$(K+1)X(K+1) = \delta(K) - X(K) + \sum_{K2=0}^{K} \sum_{K1=0}^{K2} \frac{2^{K1} X(K_2 - K_1 + 1) X(K - K_2 + 1)}{K_1!} \tag{2.37}$$

Taking the differential transform of initial condition given in (2.35), *we have*

$$X(0) = 0 \tag{2.38}$$

Now by substituting value of $K = 0, 1, 2, \cdots$ *in* (2.37) *we get*

$$X(1) = 1, \ X(2) = -\frac{1}{2} \ X(3) = \frac{1}{2!} \ X(4) = \frac{1}{8} \ X(5) = \frac{9}{40} \ X(6) = \frac{163}{6!} \ \cdots$$

By using the inverse transformation rule (2.3), *the following analytical solution is obtained.*

$$x(t) = \sum_{K=0}^{\infty} X(K) t^K \tag{2.39}$$

$$x(t) = t - \frac{t^2}{2!} + \frac{t^3}{2!} + \frac{t^4}{8} + \frac{27t^5}{5!} + \frac{163t^6}{6!} + \ \cdots \tag{2.40}$$

Table 2.7: Comparison between DTM, RK4 and exact solution, h=0.1

t	Exact	RK4[28]	DTM(K=75)
0	0	0	0
0.1	0.0955	0.0955	0.0955
0.2	0.1843	0.1843	0.1843
0.3	0.2703	0.2703	0.2703
0.4	0.3591	0.3591	0.3591
0.5	0.4599	0.4599	0.4599
0.6	0.5907	0.5907	0.5907
0.7	0.7973	0.7973	0.7973
0.8	1.2493	1.2488	1.2492
0.9	3.6413	3.4902	3.5476

Table 2.8: Absolute error for RK4 and DTM

t	RK4[28]	DTM(K=75)
0	0	0
0.1	2.4955e-07	8.3267e-17
0.2	3.4235e-07	0
0.3	1.4908e-07	5.5511e17
0.4	4.4206e-07	1.1102e-16
0.5	1.2644e-06	0
0.6	9.1209e-07	7.771e-16
0.7	1.6848e-06	1.1069e-10
0.8	5.1423e-04	4.1904ee-06
0.9	0.1511	0.0938

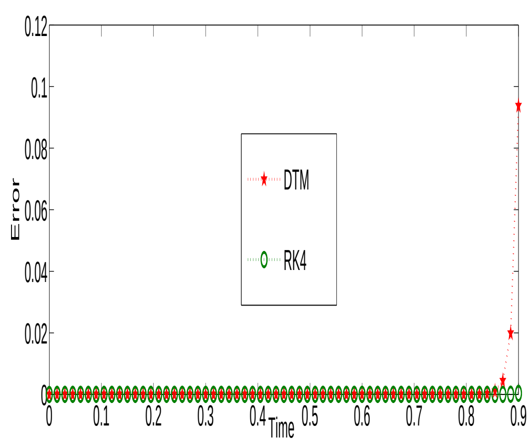

Figure 2.7: Absolute error for RK4 and DTM(K=75). Where as h=0.015

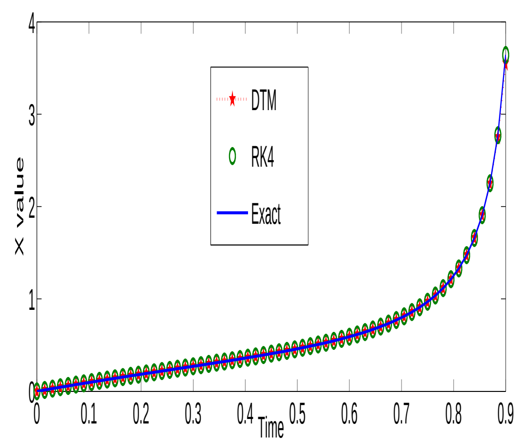

Figure 2.8: Plot of exact,RK4 and DTM for K=75. where as h=0.015

In figure (2.7)*, it can be seen that the absolute error of both methods is below 0.0938. The absolute error of both methods is performing almost similarly on interval [0, 0.855] but the DTM error starts increasing on [0.87, 0.9]. So it can be said that absolute error of DTM increases for higher value of t. This can also be observed in figure* (2.8) *where DTM is giving more value for t=0.9 then RK4. In table 2.7 the numerical comparison is done for h=0.1. Here DTM is performing better than RK4 and this can be observed in table 2.8 where the absolute error of both methods is compared. So for this example it can be said that DTM gives less error for higher value of h.*

2.4 Performance analysis of DTM for ODEs

In this chapter, the semi-analytical solutions of linear and non-linear ODEs are obtained. The results compared with exact and numerical solutions revealed that DTM is a effective, reliable, easy to implement and fast convergent method. The convergence of DTM is discussed in detain in [33]. As the results obtained are in Taylor series form they can be easily converted into exact solution. Unlike RK4, in DTM the unknown terms are found only once, so whenever we have to compute the solution at some value of independent variable we only have to substitute the value in inverse transformation. In RK4 more accurate results are achieved by decreasing the interval difference value that causes the increase in number of iterations due to which accuracy of results increases. While DTM accuracy depends only on number of terms. So in table 2.8 it can be seen that DTM performs well due to large interval difference but in table (2.3) and figure (2.7) RK4 performance is better as the number iterations are increased. So to obtain better results of DTM K should be increased every time when h is decreased. Like other methods DTM also have some limitation. The transforms of all non-linear variables isn't directly defined like e^{-y}, lny, $siny$, $cosy\ etc$ and another technique is used before applying DTM to obtain the analytical solution. In BVPs the conditions are defined on boundary due to which there will exist one or more unknown polynomials that are found by using the boundary condition defined on x_1 in inverse transform and this consumes much computational cost. Another limitation of DTM is that it performs well only in small region specially in case of non-linear problems.

Chapter 3

Solving Partial Differential Equations using DTM

In previous chapter, the analytical solution of ordinary differential equations was obtained using DTM. In this chapter, we will be solving PDEs using DTM. At first, the definition and some important theorems of DTM for solving PDEs will be discussed. Furthermore, the analytical solution of some PDEs will be obtained using DTM. The obtained solution will be compared with exact to show the efficiency and reliability of DTM. The 3-D plots of all results are obtained using Matlab.

3.1 Two-dimensional Differential transformation Method

The 2-dimensional transform[15] of u(x,y) is written as:

$$U(K,H) = \frac{1}{K!\,H!}\frac{\partial^{K+H}u(x_0,y_0)}{\partial x^K \partial y^H}. \tag{3.1}$$

The transformed function is written with upper case where as original one is written with lower case. The inverse transform of U(K,H) is defined by:

$$u(x,y) = \sum_{K=0}^{\infty}\sum_{H=0}^{\infty} U(K,H)x^K y^H. \tag{3.2}$$

From (3.1) and (3.2) its concluded that

$$u(x,y) = \sum_{K=0}^{\infty}\sum_{H=0}^{\infty} \frac{1}{K!\,H!}\frac{d^K u(x_0,y_0)}{\partial x^K \partial y^H} x^K y^H. \tag{3.3}$$

The following theorems have been discussed in [15] and will be used later in this chapter to solve PDEs.

Theorem 3.1.1. *If* $u(x,y) = \frac{\partial^{P+Q} w(x,y)}{\partial x^P \partial y^Q}$*, then*

$U(K,H) = (K+1)(K+2)\ \cdots (K+P)(H+1)(H+2)\ \cdots (H+Q)U(K+P, H+Q).$

Theorem 3.1.2. *If* $u(x,y) = v(x,y)w(x,y)$*, then*

$$U(K,H) = \sum_{P=0}^{K} \sum_{Q=0}^{H} V(P, H-Q)W(K-P, Q).$$

Theorem 3.1.3. *If* $u(x,y) = x^m y^n$*, then*

$$U(K,H) = \delta(K-m, H-n) = \delta(K-m)\delta(H-n)$$

where

$\delta(K-m) = \begin{cases} 1 & \text{if } K = m \\ 0 & \text{if } K \neq m \end{cases}$

Theorem 3.1.4. *If* $u(x,t) = x^n cos(\alpha t + \beta)$ *then*

$$U(K,H) = \frac{(\alpha)^H}{H!}\delta(K-n)cos(\frac{H\pi}{2}).$$

Theorem 3.1.5. $u(x,y) = \frac{\partial v(x,y)}{\partial x}\frac{\partial w(x,y)}{\partial x}$*, then*

$$U(K,H) = \sum_{P=0}^{K} \sum_{Q=0}^{H} (P+1)(K-P+1)V(P+1, H-Q)W(K-P+1, Q).$$

Proof: From definition we have,

$U(0,0) = [\frac{\partial v(x,y)}{\partial x}\frac{\partial w(x,y)}{\partial x}]_{(x_0,y_0)} = V(1,0)W(1,0),$

$U(1,0) = \frac{1}{1!0!}\frac{\partial}{\partial x}[\frac{\partial v(x,y)}{\partial x}\frac{\partial w(x,y)}{\partial x}]_{(x_0,y_0)} = \frac{1}{1!0!}[\frac{\partial^2 v(x,y)}{\partial x^2}\frac{\partial w(x,y)}{\partial x} + \frac{\partial^2 w(x,y)}{\partial x^2}\frac{\partial v(x,y)}{\partial x}]_{(x_0,y_0)}$

$= 2V(2,0)W(1,0) + 2V(1,0)W(2,0),$

$U(2,0) = \frac{1}{2!0!}\frac{\partial^2}{\partial x^2}[\frac{\partial v(x,y)}{\partial x}\frac{\partial w(x,y)}{\partial x}]_{(x_0,y_0)}$

$= 3V(3,0)W(1,0) + 4V(2,0)W(2,0) + 3V(1,0)W(3,0),$

$U(0,1) = V(1,1)V(1,0) + W(1,0)V(1,1),$

$U(1,1) = 2V(2,1)W(1,0) + 2V(2,0)W(1,1) + 2V(1,1)W(2,0) + 2V(1,0)W(2,1),$

In general, we have

$U(K,H) = \sum_{P=0}^{K} \sum_{Q=0}^{H} (P+1)(K-P+1)V(P+1, H-Q)W(K-P+1, Q).$

Theorem 3.1.6. *If u(x,y)=*$\frac{\partial v(x,y)}{\partial y}\frac{\partial w(x,y)}{\partial y}$, *then*

U(K,H)=$\sum_{P=0}^{K}\sum_{Q=0}^{H}(Q+1)(H-Q+1)V(P,H-Q+1)W(K-P,Q+1)$.

Proof: From definition we have,

U(0,0)=$[\frac{\partial v(x,y)}{\partial y}\frac{\partial w(x,y)}{\partial y}]_{(x_0,y_0)}$=V(0,1)W(0,1),

U(1,0)=$\frac{1}{1!0!}\frac{\partial}{\partial x}[\frac{\partial v(x,y)}{\partial y}\frac{\partial w(x,y)}{\partial y}]_{(x_0,y_0)}=\frac{1}{1!0!}[\frac{\partial^2 v(x,y)}{\partial x\partial y}\frac{\partial w(x,y)}{\partial y}+\frac{\partial^2 w(x,y)}{\partial x\partial y}\frac{\partial v(x,y)}{\partial y}]_{(x_0,y_0)}$

=V(1,1)W(0,1)+V(0,1)W(1,1),

U(2,0)=$\frac{1}{2!0!}\frac{\partial^2}{\partial x^2}[\frac{\partial v(x,y)}{\partial y}\frac{\partial w(x,y)}{\partial y}]_{(x_0,y_0)}$

=V(0,1)W(2,1)+V(1,1)W(1,1)+V(2,1)W(0,1),

U(0,1)=2V(0,2)V(0,1)+2W(0,1)V(0,2),

U(1,1)=2V(0,2)V(1,1)+2W(0,1)V(1,2)+2U(1,2)V(0,1)+2U(1,1)V(0,2).

In general, we have

U(K,H)= U(K,H)=$\sum_{P=0}^{K}\sum_{Q=0}^{H}(Q+1)(H-Q+1)V(P,H-Q+1)W(K-P,Q+1)$.

Theorem 3.1.7. *If w(x,y)=*$u(x,y)v(x,y)\omega(x,y)$ *then*

W(K,H)=$\sum_{P=0}^{K}\sum_{Q=0}^{H}\sum_{T=0}^{K-P}\sum_{S=0}^{H-Q}U(P,H-Q-S)V(T,Q)\omega(K-P-T,S)$.

Proof: From definition, we have

W(0,0)=$U(0,0)V(0,0)\omega(0,0)$,

W(1,0)=$\frac{1}{1!0!}\frac{\partial}{\partial x}[u(x,y)v(x,y)\omega(x,y)]_{(x_0,y_0)}$

=$U(0,0)V(0,0)\omega(1,0)+U(0,0)V(1,0)\omega(0,0)+U(1,0)V(0,0)\omega(0,0)$,

W(2,0)=$\frac{1}{2!0!}\frac{\partial^2}{\partial x^2}[u(x,y)v(x,y)\omega(x,y)]_{(x_0,y_0)}$

=$U(0,0)V(0,0)\omega(2,0)+U(0,0)V(1,0)\omega(1,0)+U(0,0)V(2,0)\omega(0,0)+U(1,0)V(0,0)\omega(1,0)+$
$U(1,0)V(1,0)\omega(0,0)+U(2,0)V(0,0)\omega(0,0)$,

W(0,1)=$U(0,1)V(0,0)\omega(0,0)+U(0,0)V(0,1)\omega(0,0)+U(0,0)V(0,0)\omega(0,1)$,

W(0,2)=$U(0,0)V(0,0)\omega(0,2)+U(0,0)V(0,1)\omega(0,1)+U(0,0)V(0,2)\omega(0,0)+U(0,1)V(0,0)\omega(0,1)$
$+U(0,1)V(0,1)\omega(0,0)+U(0,2)V(0,0)\omega(0,0)$.

In general, we have

W(K,H)=$\sum_{P=0}^{K}\sum_{Q=0}^{H}\sum_{T=0}^{K-P}\sum_{S=0}^{H-Q}U(P,H-Q-S)V(T,Q)\omega(K-P-T,S)$.

3.2 Numerical Problems

To check the efficiency and reliability of DTM we will test linear heat, wave , nonlinear Klein-Gordan and diffusion equation.The obtained results are compared with exact solution.

Heat Equation[27]

Example 3.2.1. *Consider the boundary-value problem*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 \le t \le 0.5, \quad 0 \le x \le 1 \tag{3.4}$$

with boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{3.5}$$

and I.C

$$u(x,0) = sin\pi x, \tag{3.6}$$

whose exact solution given as

$$u(x,t) = sin\pi x e^{-\pi^2 t}. \tag{3.7}$$

Taking differential transform of (3.4)*, we get*

$$(K+1)(K+2)U(K+2,H) = (H+1)U(K,H+1). \tag{3.8}$$

And now applying DTM on (3.5) *and* (3.6) *implies*

$$U(0,H) = 0, \qquad U(K,0) = \frac{\pi^K}{K!}\frac{sinK\pi}{2} \qquad (For\ x_0 = 0, t_0 = 0). \tag{3.9}$$

Here in above equation we didn't applied differential transform on $u(1,t) = 0$ *because in this method only* x_0 *or initial value is needed hence we will not be using this condition To find the value of U(K,H), we will substitute the values of K,H in* (3.9) *and then will use it in* (3.8) *to get*

$$U(K,H) = \frac{\pi^{K+2H}}{K!\,H!} sin(K+2H)\frac{\pi}{2}. \tag{3.10}$$

Using the inverse transformation given in equation (3.2), the following solution is obtain

$$u(x,t) = \sum_{K=0}^{\infty} x^K [U(K,0)+U(K,1)t+U(K,2)t^2+U(K,3)t^3+U(K,4)x^4+\cdots] \quad (3.11)$$

$$u(x,t) = x[\pi - \pi^3 t + \frac{\pi^5 t^2}{2!} - \frac{\pi^7 t^3}{3!} + \frac{\pi^9 t^4}{4!} - \cdots] + \frac{x^3}{3!}[-\pi^3 + \frac{\pi^5 t}{1!} - \frac{\pi^7 t^2}{2!} + \frac{\pi^9 t^3}{3!} - \cdots] + \cdots$$
$$= [\pi x - \frac{\pi^3 x^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + \cdots] - \pi^2 t[\pi x - \frac{\pi^3 x^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} \cdots] + \frac{\pi^4 t^2}{2!}[\pi x - \frac{\pi^3 x^3}{3!}$$
$$+ \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} \cdots] - \frac{\pi^6 t^3}{3!}[\pi x - \frac{\pi^3 x^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} \cdots] + \frac{\pi^8 t^4}{4!}[\pi x - \frac{\pi^3 x^3}{3!} + \frac{(\pi x)^5}{5!} \cdots] \cdots$$
$$= [\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} + \cdots][1 - \pi^2 t + \frac{\pi^4 t^2}{2!} - \frac{\pi^6 t^3}{3!} + \cdots] = sin \pi x e^{-\pi^2 t}.$$

This is the exact solution.

Table 3.1: Numerical comparison between Exact, FDM and DTM

x	t	exact	FDM[27]	DTM(K=24=H)
0.2	0	0.5878	0.5878	0.5878
	0.1	0.2191	0.2154	0.2191
	0.2	0.0816	0.0790	0.0816
	0.3	0.0304	0.0289	0.0304
	0.4	0.0113	0.0106	0.0113
	0.5	0.0042	0.0039	0.0042
0.4	0	0.9511	0.9511	0.9511
	0.1	0.3545	0.3486	0.3545
	0.2	0.1321	0.1278	0.1321
	0.3	0.0492	0.0468	0.0492
	0.4	0.0183	0.0172	0.0183
	0.5	0.0068	0.0063	0.0068
0.6	0	0.9511	0.9511	0.9511
	0.1	0.3545	0.3486	0.3545
	0.2	0.1321	0.1278	0.1321
	0.3	0.0492	0.0468	0.0492
	0.4	0.0183	0.0172	0.0183
	0.5	0.0068	0.0063	0.0068
0.8	0	0.5878	0.5878	0.5878
	0.1	0.2191	0.2154	0.2191
	0.2	0.0816	0.0790	0.0816
	0.3	0.0304	0.0289	0.0304
	0.4	0.0113	0.0106	0.0113
	0.5	0.0042	0.0039	0.0042
1	0	1.2246e-16	0	-1.7008e-13
	0.1	4.5643e-17	0	-6.3505e-14
	0.2	1.7012e-17	0	-2.3981e-14
	0.3	6.3403e-18	0	-9.3259e-15
	0.4	2.3631e-18	0	-3.5527e-15
	0.5	8.8075e-19	0	1.33227e-15

Table 3.2: Absolute error of FDM and DTM

x	t	FDM[27]	DTM(K=24=H)
0.2	0	1.4748e-05	0
	0.1	0.0037	1.11022e-16
	0.2	0.0026	8.32667e-17
	0.3	0.0015	2.05842e-14
	0.4	7.4208e-04	2.66614e-11
	0.5	3.2728e-04	6.8301e-09
0.4	0	4.3484e-05	1.1102e-16
	0.1	0.0059	0
	0.2	0.0043	5.5511e-17
	0.3	0.0024	3.3418e-14
	0.4	0.0012	4.3139e-11
	0.5	5.3989e-04	1.10513e-08
0.6	0	4.3484e-05	2.22045e-16
	0.1	0.0059	2.7756e-16
	0.2	0.0043	3.6082e-16
	0.3	0.0024	3.3854e-14
	0.4	0.0012	4.3138e-11
	0.5	5.4716e-04	1.10513e-08
0.8	0	1.4748e-05	9.99201e-16
	0.1	0.0037	7.49401e-16
	0.2	0.0026	9.7145e-17
	0.3	0.0015	2.0581e-14
	0.4	7.4208e-04	2.6662e-11
	0.5	3.2728e-04	6.83006e-09
1	0	1.2246e-16	1.70208e-13
	0.1	1.7012e-17	6.35504e-14
	0.2	4.5644e-17	2.39978e-14
	0.3	6.3404e-18	9.3322e-15
	0.4	2.3631e-18	3.5551e-15
	0.5	8.8075e-19	1.3314e-15

The results of exact, FDM and DTM computed at different values of x and t are tabulated in 3.1 and graphically presented in figure (3.1). *While the absolute error of FDM and DTM is given in 3.2 and graphical presentation is shown in figure* (3.2). *The comparison shows that DTM performs better than FDM and gives less error on interval x=t=[0, 0.8] but give more error when x or t or both are greater than 0.8 .*

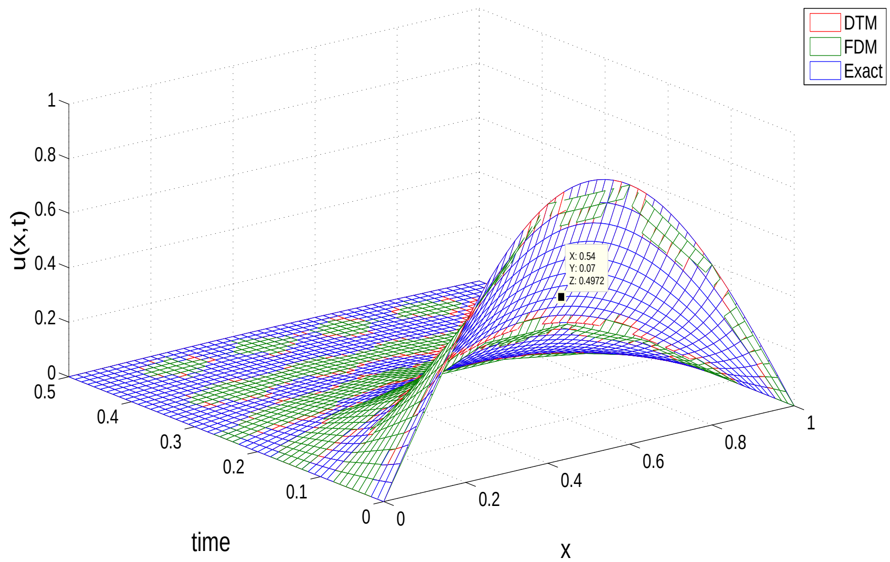

Figure 3.1: Plot of FDM, Exact and DTM solution of heat equation for k=0.02, h=0.01

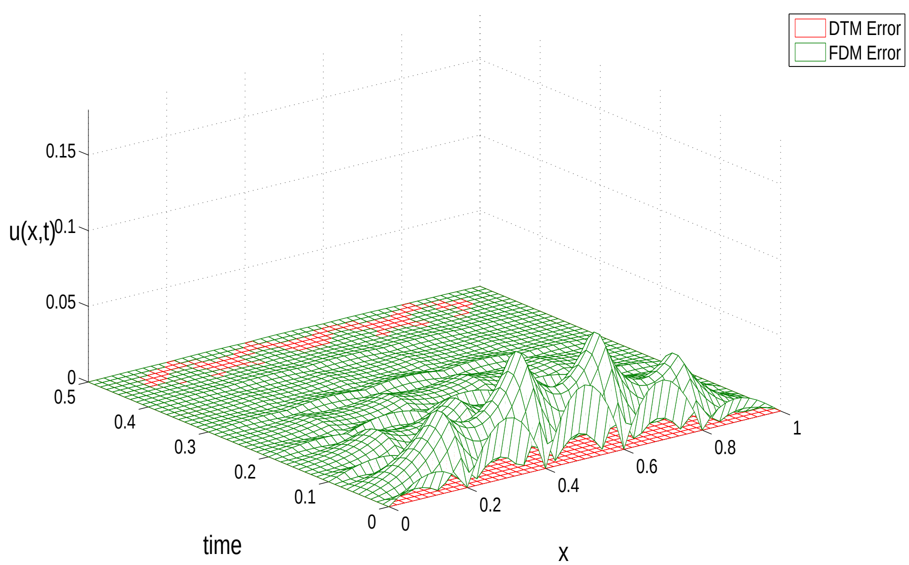

Figure 3.2: Absolute error of FDM and DTM solution for where k=0.02, h=0.01

Wave Equation[27]

Example 3.2.2. *Consider the boundary-value problem*

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 \leq t \leq 1, \quad 0 \leq x \leq 1 \tag{3.12}$$

with boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \tag{3.13}$$

and initial condition

$$u(x,0) = sin\pi x, \qquad \frac{\partial u}{\partial t}|_{t=0} = 0, \tag{3.14}$$

whose exact solution can be expressed as

$$u(x,t) = sin\pi x cos2\pi t. \tag{3.15}$$

Taking differential transform of (3.12), *we can obtain*

$$4(K+1)(K+2)U(K+2,H) = (H+1)(H+2)U(K,H+2). \tag{3.16}$$

By applying differential transform on (3.13) *and* (3.14), *we achieve*

$$U(0,H) = 0, \quad U(K,0) = \frac{\pi^K}{K!}sin\frac{K\pi}{2}, \quad U(K,1) = 0. \tag{3.17}$$

To find the value of U(K,H), we will substitute the values of K,H in (3.17) *and then will use it in* (3.16) *to get*

$$U(K,H) \qquad = \qquad \begin{cases} \frac{2^H(\pi)^{K+H}}{K!H!} sin(K+H)\frac{\pi}{2}, & iff \quad H \neq odd; \\ else \quad 0. \end{cases} \tag{3.18}$$

Using the inverse transform given in equation (3.2), *we obtain*

$$u(x,t) = \sum_{K=0}^{\infty} x^K[U(K,0)+U(K,1)t+U(K,2)t^2+U(K,3)t^3+U(K,4)x^4+\cdots] \tag{3.19}$$

$$\begin{aligned} u(x,t) &= (\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} \cdots) - t^2(2\pi^3 x - \frac{2\pi^5 x^3}{3!} + \frac{2\pi^7 x^5}{5!} - \cdots) + \frac{t^4}{3}(2\pi^5 x - \frac{2\pi^7 x^3}{3!} + \cdots) \cdots \\ &= (\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} \cdots) - \frac{(2\pi t)^2}{2!}(\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} \cdots) + \frac{(2\pi t)^4}{4!}(\pi x - \frac{(\pi x)^3}{3!} + \cdots) \cdots \\ &= (\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \frac{(\pi x)^7}{7!} \cdots)(1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \frac{(2\pi t)^6}{6!} + \frac{(2\pi t)^8}{8!} \cdots \end{aligned}$$

$$u(x,t) = sin\pi x cos 2\pi t. \tag{3.20}$$

This is exact solution.

The results of exact, FDM and DTM are tabulated in 3.3 and graphically presented in figure (3.3)*. While the absolute error of FDM and DTM is given in table 3.4 and graphical presentation is shown in figure* (3.4)*. The comparison shows that for t=[0:0.98] DTM performs better than FDM and gives less error where as when t=1 FDM performance is better than DTM. So it can be said that DTM performs well for small values.*

Table 3.3: Numerical comparison between exact, FDM and DTM

x	t	exact	FDM[27]	DTM(K=27=H)
0.2	0	0.5878	0.5878	0.5878
	0.2	0.1816	0.1903	0.1816
	0.4	-0.4755	-0.4645	-0.4755
	0.6	-0.4755	-0.4912	-0.4755
	0.8	0.1816	0.1464	0.1816
	1	0.5878	0.5860	0.5878
0.4	0	0.9511	0.9511	0.9511
	0.2	0.2939	0.3080	0.2939
	0.4	-0.7694	-0.7516	-0.7694
	0.6	-0.7694	-0.7947	-0.7694
	0.8	0.2939	0.2369	0.2939
	1.0	0.9511	0.9482	0.9511
0.6	0	0.9511	0.9511	0.9511
	0.2	0.2939	0.3080	0.2939
	0.4	-0.7694	-0.7516	-0.7694
	0.6	-0.7694	-0.7947	-0.7694
	0.8	0.2939	0.2369	0.2939
	1.0	0.9511	0.9482	0.9511
0.8	0	0.5878	0.5878	0.5878
	0.2	0.1816	0.1903	0.1816
	0.4	-0.4755	-0.4645	-0.4755
	0.6	-0.4755	-0.4912	-0.4755
	0.8	0.1816	0.1464	0.1816
	1.0	0.5878	0.5860	0.5878

Table 3.4: Absolute error of FDM and DTM

x	t	FDM[27]	DTM(K=27, h=27)
0.2	0	1.47478e-05	0
	0.2	0.0087	5.5511e-17
	0.4	0.0110	1.6653e-16
	0.6	0.0157	2.6201e-14
	0.8	0.0352	8.0940e-11
	1.0	0.0018	4.1185e-08
0.4	0	4.3484e-05	1.11022e-16
	0.2	0.0141	2.2204e-16
	0.4	0.0178	2.2204e-16
	0.6	0.0253	4.22995e-14
	0.8	0.0570	1.3096e-10
	1.0	0.0029	6.6639e-08
0.6	0	4.3484e-05	1.11022e-16
	0.2	0.0141	1.6653e-16
	0.4	0.0178	2.22044e-16
	0.6	0.0253	4.36317e-14
	0.8	0.0570	1.30969e-10
	1.0	0.0029	6.66386e-08
0.8	0	1.4748e-05	1.11022e-16
	0.2	0.0087	3.05311e-16
	0.4	0.0110	4.44089e-16
	0.6	0.0157	2.92543e-14
	0.8	0.0352	8.09539e-11
	1.0	0.0018	4.1185e-08
1	0	1.2246e-16	3.2162e-16
	0.2	3.7844e-17	3.78437e-17
	0.4	9.9076e-17	1.2332e-15
	0.6	9.9076e-17	6.5622e-15
	0.8	3.7844e-17	2.9348e-14
	1.0	1.2246e-16	9.8266e-14

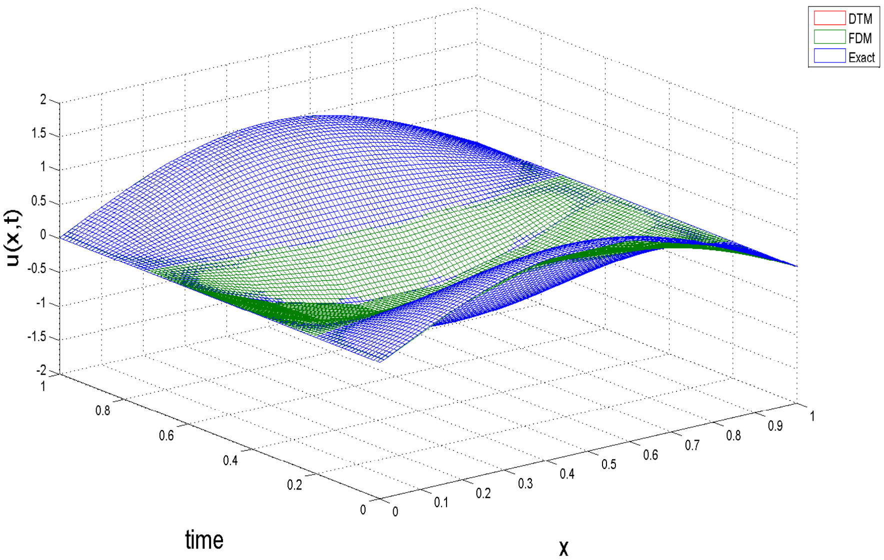

Figure 3.3: Plot of FDM, Exact and DTM solution of wave equation for 0<x<1 and 0<t<1 where k=0.01, h=0.01

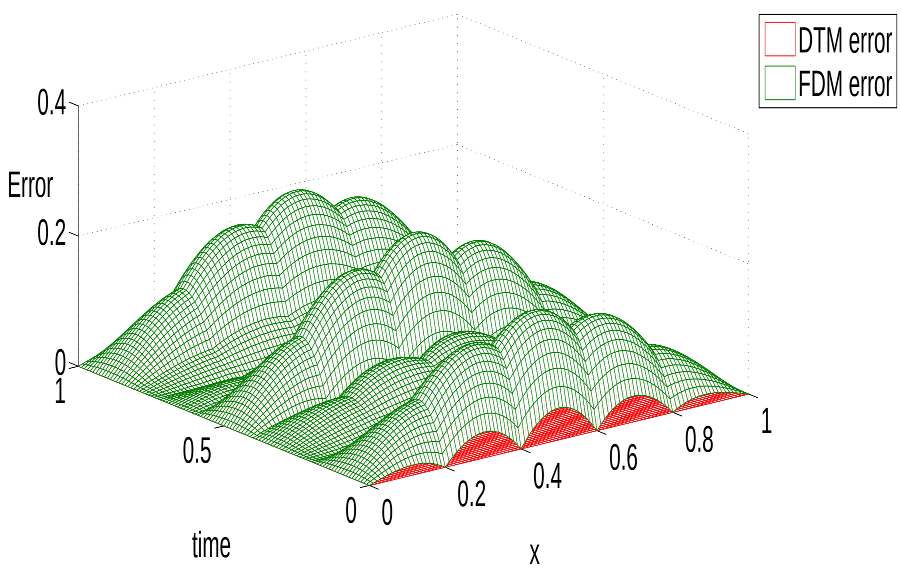

Figure 3.4: Absolute error of FDM and DTM solution for 0<x<1 and 0<t<1 where k=0.01, h=0.01

Nonlinear Case

Example 3.2.3. *Consider the nonlinear Klein Gordan equation[25]*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^2 = -xcost + x^2cos^2t, \qquad -1 \leq t \leq 1, \quad -1 \leq x \leq 1 \tag{3.21}$$

with I.C

$$u(x,0) = x, \qquad \frac{\partial u(x,t)}{\partial t}|_{t=0} = 0, \tag{3.22}$$

whose exact solution is

$$u(x,t) = xcost. \tag{3.23}$$

Taking differential transform of (3.21) *and* (3.22)*, implies*

$$\begin{aligned}&(H+1)(H+2)U(K,H+2) - (K+1)(K+2)U(K+2,H) + \\ &\sum_{P=0}^{K}\sum_{Q=0}^{H} U(P,H-Q)U(K-P,Q) = -\sum_{P=0}^{K}\sum_{Q=0}^{H}\delta(K-P)\delta(P-1,Q)\frac{cos(H-Q)\frac{\pi}{2}}{(H-Q)!} \\ &+\sum_{P=0}^{K=0}\sum_{Q=0}^{H}\delta(P-2,Q)\sum_{T=0}^{K-P}\sum_{S=0}^{H-Q}\frac{\delta(T)\delta(K-P-T)}{S!\,(H-Q-S)!}cos\frac{\pi S}{2}cos(H-Q-S)\frac{\pi}{2}.\end{aligned} \tag{3.24}$$

$$U(K,0) = \delta(K-1), \qquad U(K,1) = 0 \tag{3.25}$$

Putting values of K and H in (3.24) *then using* (3.25) *in it, we obtain*

$$U(K,H) = \begin{cases} \frac{\delta(K-1)}{H!} sin(K+H)\frac{\pi}{2} & if\ K = 1 \\ 0 & otherwise \end{cases}. \tag{3.26}$$

Using (3.26) *in the inverse transform given in* (3.2)*, we get*

$$u(x,t) = x(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \ \cdots) \tag{3.27}$$

$$u(x,t) = xcost. \tag{3.28}$$

This is same as exact solution.

The results of exact and DTM computed at different values of x and t are tabulated in 3.5 and graphically presented in figure (3.5)*. While the absolute error of DTM is given in 3.6 and graphical presentation is shown in figure* (3.6)*. The comparison shows that DTM performs effectively and gives negligible error.*

Table 3.5: Comparison between Exact and DTM

t	x	exact	DTM(K=1, H=50)
0.01	-1	-1	-1
	-0.8	-0.8	-0.8
	-0.6	-0.6	-0.6
	-0.4	-0.4	-0.4
	-0.2	-0.2	-0.2
	0	0	0
	0.2	0.2	0.2
	0.4	0.4	0.4
	0.6	0.6	0.6
	0.8	0.8	0.8
	1	1	1
0.1	-1	-0.995	-0.995
	-0.8	-0.7960	-0.7960
	-0.6	-0.5970	-0.5970
	-0.4	-0.3980	-0.3980
	-0.2	-0.1990	-0.1990
	0	0	0
	0.2	0.1990	0.1990
	0.4	0.3980	0.3980
	0.6	0.5970	0.5970
	0.8	0.7960	0.7960
	1	0.995	0.995
0.9	-1	-0.6216	-0.6216
	-0.8	-0.4973	-0.4973
	-0.6	-0.3730	-0.3730
	-0.4	-0.2486	-0.2486
	-0.2	-0.1243	-0.1243
	0	0	0
	0.2	0.1243	0.1243
	0.4	0.2486	0.2486
	0.6	0.3730	0.3730
	0.8	0.4973	0.4973
	1	0.6216	0.6216

Table 3.6: Absolute error value of DTM

t	x	DTM(K=1, H=50)
0.01	-1	0
	-0.8	1.1102e-16
	-0.6	0
	-0.4	5.5511e-17
	-0.2	2.7756e-17
	0	0
	0.2	2.7756e-17
	0.4	5.5511e-17
	0.6	0
	0.8	1.1102e-16
	1	0
0.1	-1	0
	-0.8	0
	-0.6	1.1102e-16
	-0.4	1.1102e-16
	-0.2	5.5511e-17
	0	0
	0.2	5.5511e-17
	0.4	1.1102e-16
	0.6	1.1102e-16
	0.8	0
	1	0
0.9	-1	0
	-0.8	0
	-0.6	5.5511e-17
	-0.4	5.5511e-17
	-0.2	2.7756e-17
	0	0
	0.2	2.7756e-17
	0.4	5.5511e-17
	0.6	5.5511e-17
	0.8	0
	1	0

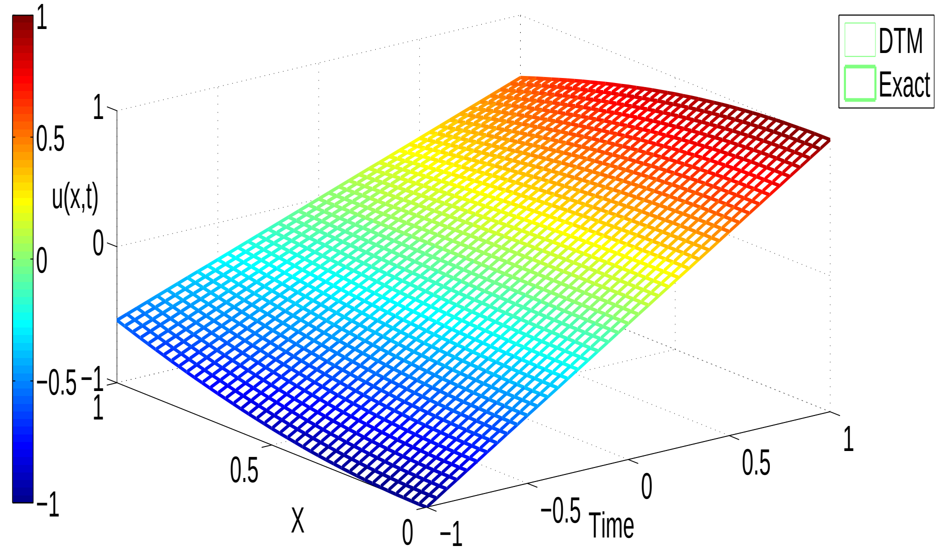

Figure 3.5: Plot of Exact and DTM solution of Klein equation for -1<x<1 and 0<t<1 where k=0.02, h=0.01

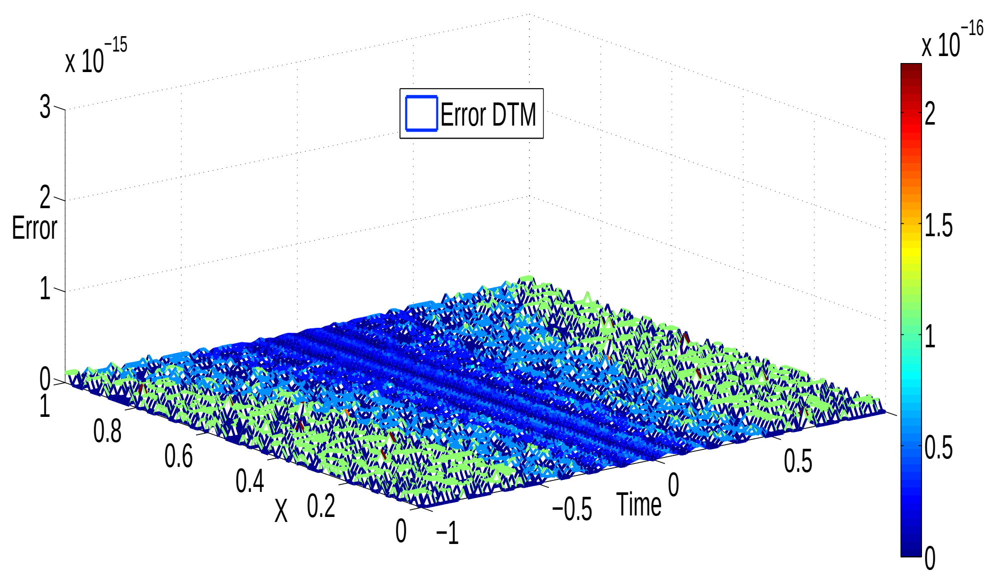

Figure 3.6: Absolute error of DTM solution of Klein equation for -1<x<1 and 0<t<1 where k=0.02, h=0.01

Example 3.2.4. *Consider the nonlinear diffusion equation[26]*

$$u_t = (u^2 u_x)_x, \qquad 0 \leq t \leq 1, \quad 0 \leq x \leq 1. \tag{3.29}$$

with I.C

$$u(x,0) = \frac{x+a}{2c}. \tag{3.30}$$

whose exact solution is

$$u(x,t) = \frac{x+a}{2\sqrt{c^2-t}}. \tag{3.31}$$

Taking differential transform of (3.29) *and* (3.30), *implies*

$$U(K,0) = \frac{\delta(K-1)+\delta(K)}{2c}, \quad wherec = a = 1 \tag{3.32}$$

$$(H+1)U(K,H+1) = \sum_{P=0}^{K}\sum_{T=0}^{K-P}\sum_{Q=0}^{H}\sum_{S=0}^{H-Q} U(P,H-Q-S)U(T,Q)U(K-P-T+2,S)$$
$$+2\sum_{P=0}^{K}\sum_{T=0}^{K-P}\sum_{Q=0}^{H}\sum_{S=0}^{H-Q}(T+1)(K-P-T+1)U(P,H-Q-S)U(T+1,Q)U(K-P-T+1,S). \tag{3.33}$$

Putting values of K and H in (3.33) *and then by using* (3.34) *in* (3.33), *we get*

$$U(K,H) = \begin{cases} \frac{1.3.5...(2H-1)}{2^{H+1}H!c^{2H+1}} & if\ K=0,1, H=1,2,3,... \\ 0 \quad otherwise \end{cases}. \tag{3.34}$$

Using (3.34) *in the inverse transformation rule given in* (3.2), *we get*

$$u(x,t) = \frac{(x+a)}{2}\left(\frac{1}{c} + \frac{t}{2c^3} + \frac{3t^2}{2!\,4} + \frac{15t^3}{3!\,8}\ \cdots\right). \tag{3.35}$$

After simplification (3.35) *becomes,*

$$u(x,t) = \frac{(x+a)}{2\sqrt{c^2-t}}. \tag{3.36}$$

This is same as exact solution.

The results of exact and DTM computed at different values of x and t are tabulated in 3.7 and graphically presented in figure (3.7). *While the absolute error of DTM is given* *in 3.8 and graphical presentation is shown in figure* (3.8). *The comparison shows that DTM performs well and gives negligible error for t=[0.01, 0.1] and x=[0.1, 0.5, 0.9] but for t=0.9 an increase in error value can be seen. So it can be concluded that DTM gives more error for higher values.*

Table 3.7: Comparison between Exact and DTM

t	x	exact	DTM(K=1,2, H=20)
0.01	0.1	0.5528	0.5528
	0.5	0.7538	0.7538
	0.9	0.9548	0.9548
0.1	0.1	0.5798	0.5798
	0.5	0.7906	0.7906
	0.9	1.0014	1.0014
0.9	0.1	1.7393	1.6758
	0.5	2.3717	2.2851
	0.9	3.0042	2.8945

Table 3.8: Absolute error for DTM

t	x	DTM(K=1, H=20)
0.01	0.1	1.1102e-16
	0.5	1.1102e-16
	0.9	0
0.1	0.1	1.1102e-16
	0.5	1.1102e-16
	0.9	2.22045e-16
0.9	0.1	0.0636
	0.5	0.0866
	0.9	0.10968

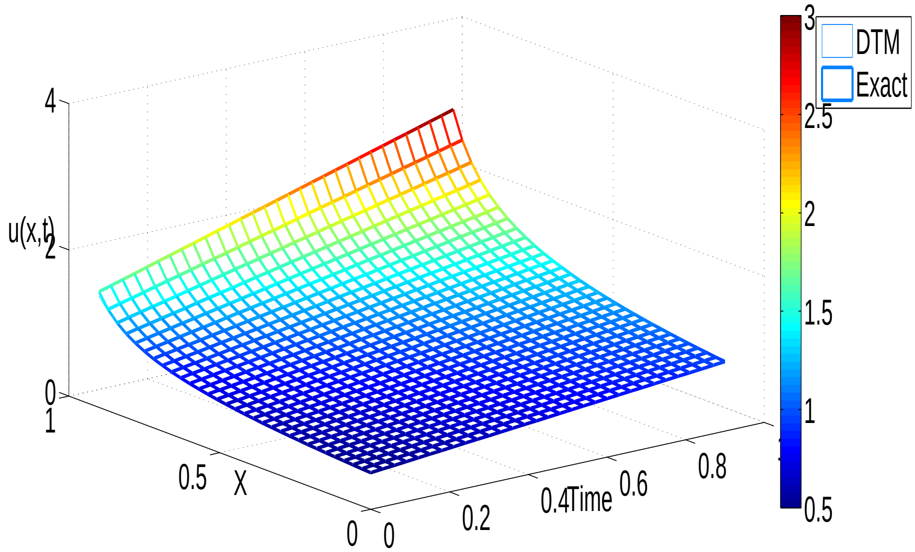

Figure 3.7: Plot of Exact and DTM solution of wave equation for $0<x<1$ and $0<t<1$ where k=0.1, h=0.1

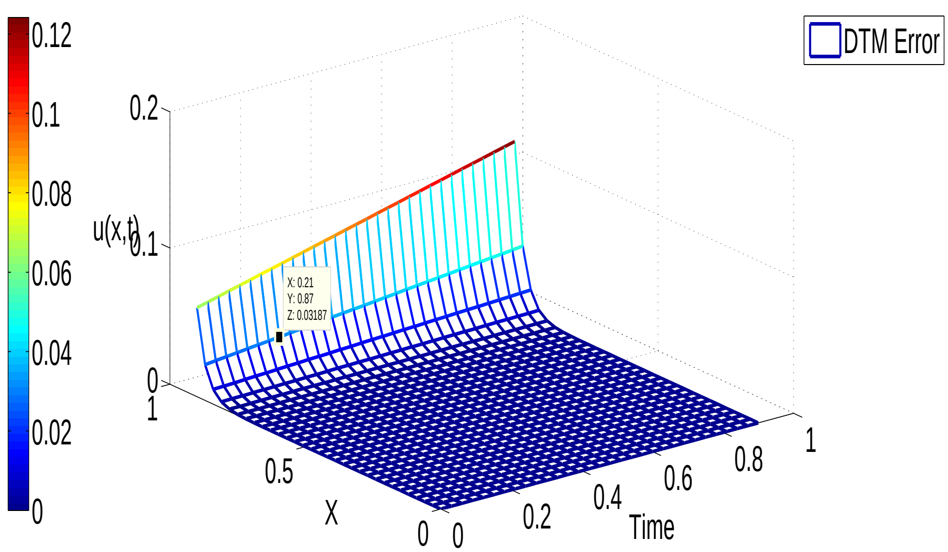

Figure 3.8: Absolute error of DTM solution for $0<x<1$ and $0<t<1$ where k=0.1, h=0.1

3.3 Performance analysis of DTM for PDEs

In this chapter, the semi-analytical solutions of linear and non-linear PDEs are obtained. The results compared with exact and numerical solutions revealed reliability, ease of implementation and fast convergence of the method. Unlike FDM, the accuracy of DTM results depends only on K. The increase in K causes the decrease in error due to which more accurate results are achieved. Here, the comparison of DTM with FDM also revealed that if exact solution includes a trigonometric function then FDM performs better on boundary conditions. This can be seen in table 3.2 and 3.6 that only at t=1 FDM performs better than DTM this is because DTM gives approximate solution.

In case of PDEs also, DTM have some limitations. Just like ODEs, in PDEs also the transformation of only limited variables are defined. There are many variables which are combination of more than two functions and there transformation aren't defined due to which its very challenging to solve them. So it's needed that the transformation of more complex variables should be defined. In PDEs extra conditions are needed for obtaining the solution of BVPs by DTM where as the solution of BVPs with limited conditions cannot be obtained as the number of unknown variables will be greater than the conditions.

Chapter 4

Study of MHD flow of third grade non-Newtonian fluid between three parallel plates using DTM

So far, we have computed the analytical solution of ODEs and PDEs using DTM. Now here, we will study the MHD flow of third grade fluid between three parallel plates. We applied DTM to compute the analytical solution of the equations involved in the model. We also compared the solution with ADM to show reliability and efficiency of the method. The effect of magnetic field and pressure gradient on the velocity are also graphically analyzed.

4.1 Problem Formulation and Governing Equations

Consider a uni-directional fully developed third grade (non-Newtonian) fluid flow between three infinitely parallel plates in the presence of transverse magnetic field. The governing equations of an incompressible and isothermal fluid flow are:

$$\nabla.V = 0, \tag{4.1}$$

$$\rho(\frac{\partial V}{\partial t} + (V.\nabla)V) = \rho f + \nabla.T + J \times B, \tag{4.2}$$

here, J is electric current density, whereas $B = B_0 + b$ is the total magnetic field, B_0 and b are imposed magnetic field and induced magnetic fields respectively. By using

Ohm's law, J is is defined as

$$J = \sigma(E + V \times B), \tag{4.3}$$

here σ is electric conductivity. Now by taking the cross product of equation (4.2) with B, we obtain

$$J \times B = -\sigma B^2 V. \tag{4.4}$$

Using equation (4.3) in equation (4.1) implies

$$\rho(\frac{\partial V}{\partial t} + (V.\nabla)V) = \rho f + \nabla.T - \sigma B^2 V, \tag{4.5}$$

here the Cauchy stress tensor for third grade fluid is,

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2(A_1 A_2 + A_2 A_1) + \beta_3(trA_2)A_1, \tag{4.6}$$

where as

$$A_1 = (\nabla V)^T + \nabla V, \quad A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}\nabla V + (\nabla V)^T A_{n-1}, \; n = 2, 3, 4 \; \cdots \tag{4.7}$$

In this problem it is assumed that for uni-dimensional flow velocity field will be

$$V = (v(y), 0, 0). \tag{4.8}$$

By using equation (4.8) and steady state condition in equation (4.2), we obtain

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 v}{\partial^2 y} + 6(\beta_2 + \beta_3)\frac{\partial^2 v}{\partial^2 y}(\frac{\partial v}{\partial y})^2 - \sigma B_0^2 v = 0, \tag{4.9}$$

$$-\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}(2\alpha_1 + \alpha_2)(\frac{\partial v}{\partial y})^2 = 0, \tag{4.10}$$

here $\sigma = electric\ conductivity,\ B_0 = applied\ magneticfield,\ \frac{dp}{dx} = pressure\ gradient$

$$\frac{\partial p}{\partial z} = 0. \tag{4.11}$$

Now introducing generalized pressure [36]

$$\hat{p} = -p(x, y) + \frac{\partial}{\partial y}(2\alpha_1 + \alpha_2)(\frac{\partial v}{\partial y})^2, \tag{4.12}$$

and by using it in equation (4.10), we have

$$\frac{\partial \hat{p}}{\partial y} = 0, \tag{4.13}$$

and this shows that $\hat{p} = \hat{p}(x)$. Using this in equation (4.9), implies

$$-\frac{\partial \hat{p}}{\partial x} + \mu \frac{\partial^2 v}{\partial^2 y} + 6\beta \frac{\partial^2 v}{\partial^2 y}(\frac{\partial v}{\partial y})^2 - \sigma B_0^2 v = 0, \tag{4.14}$$

here $\beta = \beta_3 + \beta_2 = non\ Newtonian\ parameter$, $m = \sigma B_0^2/\mu = magnetic\ parameter$

4.1.1 Plane Couette Flow

Consider the MHD flow of a non-Newtonian third grade fluid between three parallel plates as shown in figure (4.1).

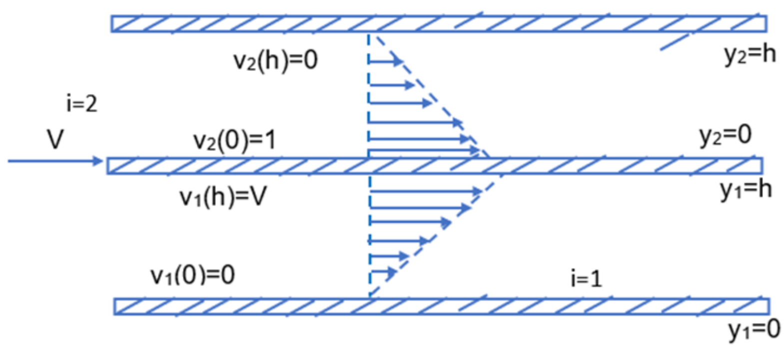

Figure 4.1: Schematic digram of Couette flow

The first and last plate are stationary while the middle one is moving with constant velocity V. The pressure gradient is zero while the properties of fluid vary only along y-axis. So in absence of pressure gradient the equation becomes

$$v_i'' + \frac{6\beta}{\mu}(v_i')^2 v_i'' - m^2 v_i = 0, \quad i = 1, 2 \tag{4.15}$$

with the B.Cs:

$$v_1(0) = 0, \qquad v_1(h) = V, \tag{4.16}$$

$$v_2(0) = 1, \qquad v_2(h) = 0. \tag{4.17}$$

Here, i=1 represent the fluid flow between first and second plate while i=2 represent fluid flow between second and third fluid. By introducing following non-dimensional parameters

$$v^* = \frac{v}{V}, \quad y^* = \frac{y}{h}, \quad \beta^* = \frac{\beta}{\mu h^2/V^2}, \quad m^{*2} = \frac{\sigma {B_0}^2}{\mu/h^2}, \tag{4.18}$$

and then by dropping the '*', the equation (4.15) becomes

$$v_i'' + 6\beta v_i'^2 v_i'' - m^2 v_i = 0, \quad i = 1, 2 \tag{4.19}$$

with boundary conditions

$$v_1(0) = 0, \qquad v_1(1) = 1, \tag{4.20}$$

$$v_2(0) = 1, \qquad v_2(1) = 0, \tag{4.21}$$

4.1.2 Plane Poiseuille Flow.

Now consider the steady laminar pressure driven flow of the third grade fluid between the three stationary parallel plates as shown in figure (4.2).

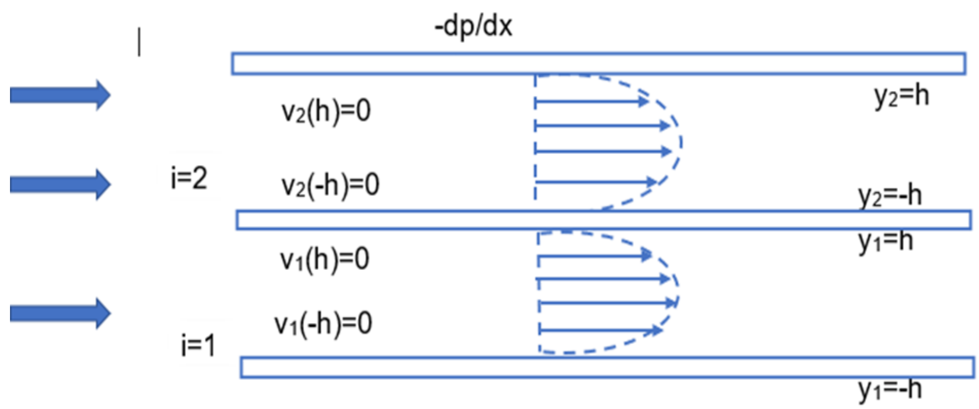

Figure 4.2: Schematic digram of Poiseuille flow

Let the separation between plates(1, 2 and 2, 3) be 2h and plates are at y=-h and y=h. Thus, the equation in presence of constant pressure gradient and magnetic field becomes

$$v_i'' + 6\frac{\beta}{\mu} v_i'^2 v_i'' - m^2 v_i = \frac{d\hat{p}}{\mu dx}, \quad i = 1, 2 \tag{4.22}$$

with the boundary conditions:

$$v_1(-h) = 0, \qquad v_1(h) = 0, \tag{4.23}$$

$$v_2(-h) = 0, \qquad v_2(h) = 0. \tag{4.24}$$

Introducing the non-dimensional parameters

$$v^* = \frac{v}{V}, \quad y^* = \frac{y}{h}, \quad \beta^* = \frac{\beta V^2}{\mu h^2}, \quad m^{*2} = \frac{\sigma {B_0}^2}{\mu/h^2}, \quad p^* = \frac{\hat{p}}{\mu V/h}, \tag{4.25}$$

after dropping '*', equation (4.22) becomes

$$v_i'' + 6\beta v_i'^2 v_i'' - m^2 v_i = \frac{dp}{dx}, \quad i = 1, 2 \tag{4.26}$$

$$v_1(-1) = 0, \qquad v_1(1) = 0, \tag{4.27}$$

$$v_2(-1) = 0, \qquad v_2(1) = 0, \tag{4.28}$$

4.1.3 Generalized Couette Flow

Here, in this case we assume the motion of third grade non-Newtonian fluid flow produced due to the movement of the middle plate and the constant pressure gradient as shown in figure (4.3).

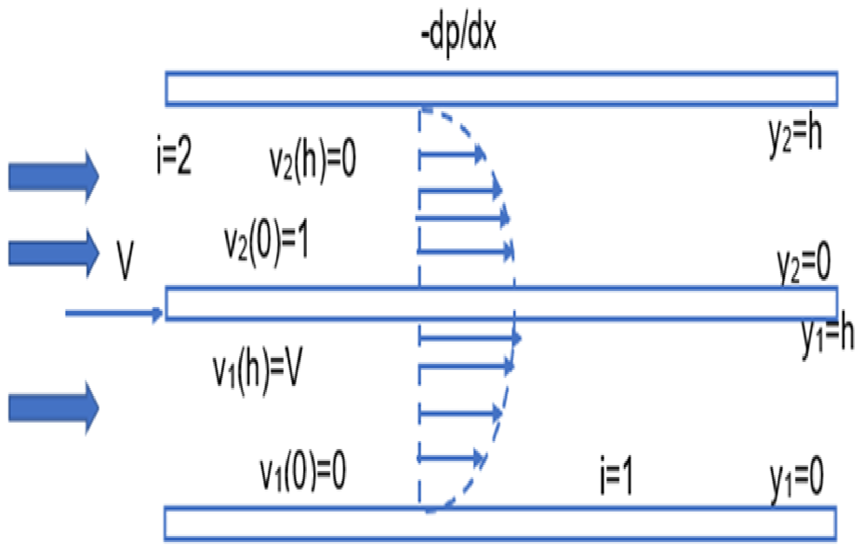

Figure 4.3: Schematic digram of generalized Couette flow

The boundary conditions mentioned in subsection (4.1.1) will be used here and the governing equation will be

$$v_i'' + 6\frac{\beta}{\mu}v_i'^2 v_i'' - m^2 v_i = \frac{d\hat{p}}{\mu dx}, \quad i = 1, 2. \tag{4.29}$$

Using the non-dimensional parameters given in (4.1.1), equation (4.29) becomes

$$v_i'' + 6\beta v_i'^2 v_i'' - m^2 v_i = \frac{dp}{dx}, \quad i = 1, 2. \tag{4.30}$$

with boundary conditions

$$v_1(0) = 0, \qquad v_1(1) = 1, \tag{4.31}$$

$$v_2(0) = 1, \qquad v_2(1) = 0. \tag{4.32}$$

All the equations given in subsections 4.1.1, 4.1.2 and 4.1.3 will be solved using DTM.

4.2 Solution of problems by DTM

Here, the equations given in 4.1.1-4.1.3 are solved using DTM.

4.2.1 Plane Couette Flow

Applying DTM on equation (4.19) and corresponding boundary conditions (4.20) and (4.21), we get

$$(K+1)(K+2)V_i(K+2)+6\beta\sum_{K_2=0}^{K}\sum_{K_1=0}^{K_2}(K_1+1)(K_1+2)(K_2-K_1+1)(K-K_2+1)V_i(K+2)$$
$$V_i(K_2-K_1+1)V_i(K-K_2+1) - m^2V_i(K) = 0, \ K = 1, 2, \cdots \tag{4.33}$$

$$V_1(0) = 0, \qquad V_1(1) = a, \tag{4.34}$$

$$V_2(0) = 1, \qquad V_2(1) = b. \tag{4.35}$$

Now by using value of K and the boundary conditions (4.34) and (4.35) in equation (4.33), the unknown variables of form $V_1(K)$ and $V_2(K)$ are computed. The variables are substituted in (2.2), to get analytical solution. Values of a and b are computed as shown in [13]. The obtained analytical solution is tabulated in table 4.1

y	ADM[36]	DTM(K=7)
0	0	0
0.1	0.099988449889147	0.0999995875009634
0.20	0.199977599796053	0.199999200001973
0.3	0.299968149736175	0.299998862503055
0.40	0.399960799720373	0.399998600004188
0.5	0.499956249752604	0.499998437505290
0.60	0.599955199827626	0.599998400006187
0.7	0.699958349928699	0.699998512506598
0.80	0.799966400025280	0.799998800006110
0.9	0.899980050070727	0.899999287504158
1	1	1

Table 4.1: Comparison of results obtained for plane Couette flow when i=1, m=0.01 and β=0.5.

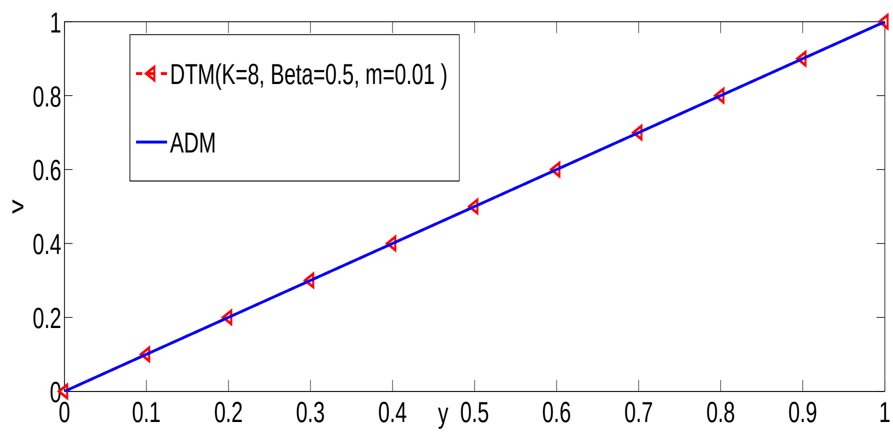

Figure 4.4: Comparison of results obtained for plane Couette flow for i=1, $\beta = 0.5$ and m=0.01.

4.2.2 Plane Poiseuille Flow

Applying DTM on equation (4.26) and corresponding boundary conditions (4.27) and (4.28), we get

$$(K+1)(K+2)V_i(K+2)+6\beta\sum_{K_2=0}^{K}\sum_{K_1=0}^{K_2}(K_1+1)(K_1+2)(K_2-K_1+1)(K-K_2+1)$$
$$V_i(K+2)V_i(K_2-K_1+1)V_i(K-K_2+1)-m^2V_i(K)=\delta(K)\frac{dp}{dx}. \quad (4.36)$$

Here i=1, 2 and $K=1,2\ \cdots$

$$V_1(0)=0, \qquad V_1(1)=A, \quad (4.37)$$

$$V_2(0)=0, \qquad V_2(1)=B. \quad (4.38)$$

Now by using the value of K and boundary conditions (4.37) and (4.38) in equation (4.36), we will obtain the unknown variables. By substituting the variables in (2.2), the analytical solution of $v_1(y)$ is obtained. Boundary conditions given for $v_1(y)$ and $v_2(y)$ are the same so the analytical solution will also be the same. Value of A that is equal to B is computed as shown in [13]. The comparison between results of ADM and DTM is tabulated in table 4.2.

y	**ADM**[36]	**DTM(K=7)**
0	0.0488769220014881	0.0488453909445173
0.1	0.0483794607669509	0.0483404127868781
0.20	0.0468882053910040	0.0468401983122849
0.3	0.0444065145285811	0.0443483935117025
0.40	0.0409399000045561	0.0408712593999515
0.5	0.0364959043077160	0.0364177162270056
0.60	0.0310839418703092	0.0309993525659633
0.7	0.0247151169211607	0.0246303958245323
0.80	0.0174020343540525	0.0173276407268707
0.9	0.009158623706779	0.00911033231262392
1	0	0

Table 4.2: Comparison of results obtained for plane Poiseuille when i=1,2, m=0.01, $\frac{dp}{dx}=-0.1$ and β=2.

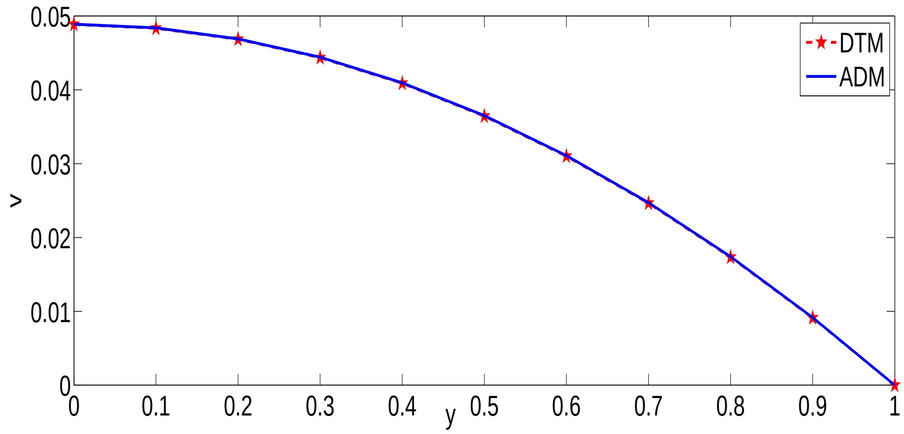

Figure 4.5: Comparison of results obtained for Poiseuille flow for i=1, $\beta = 2$, $\frac{dp}{dx} = -0.1$ and m=0.1.

4.2.3 Generalized Couette Flow

Applying DTM on equation (4.30) and corresponding boundary conditions (4.31) and (4.32), we get

$$(K+1)(K+2)V_i(K+2) + 6\beta \sum_{K_2=0}^{K} \sum_{K_1=0}^{K_2} (K_1+1)(K_1+2)(K_2-K_1+1)(K-K_2+1)$$
$$V_i(K+2)V_i(K_2-K_1+1)V_i(K-K_2+1) - m^2 V_i(K) = \delta(K)\frac{dp}{dx}. \quad (4.39)$$

Here i=1, 2 and $K = 1, 2 \cdots$

$$V_1(0) = 0, \qquad V_1(1) = A, \quad (4.40)$$

$$V_2(0) = 1, \qquad V_2(1) = B. \quad (4.41)$$

By using K and boundary conditions given in equation (4.40) and (4.41) in (4.39), we will find the unknown variables. Substitution of the variables in (2.2), gives the analytical solution of $v_1(y)$ and $v_2(y)$. Values of A and B are computed by method shown in [13]. The comparison between DTM and ADM results is given in table 4.3.

y	ADM[36]	DTM(K=5)
0	0	0
0.1	0.123139106940344	0.123137043936472
0.20	0.241214850497521	0.241216389399580
0.3	0.354196813899843	0.354205055300158
0.40	0.462054936238774	0.462070852788963
0.5	0.564759659521834	0.564782454114820
0.60	0.662282068615257	0.662309461482768
0.7	0.754594024076398	0.754622475912209
0.80	0.841668287875887	0.841693166095052
0.9	0.923478642009538	0.923494337253860
1	1	1

Table 4.3: Comparison of results obtained for Generalized Couette flow when i=1, m=0.01 and β=0.5.

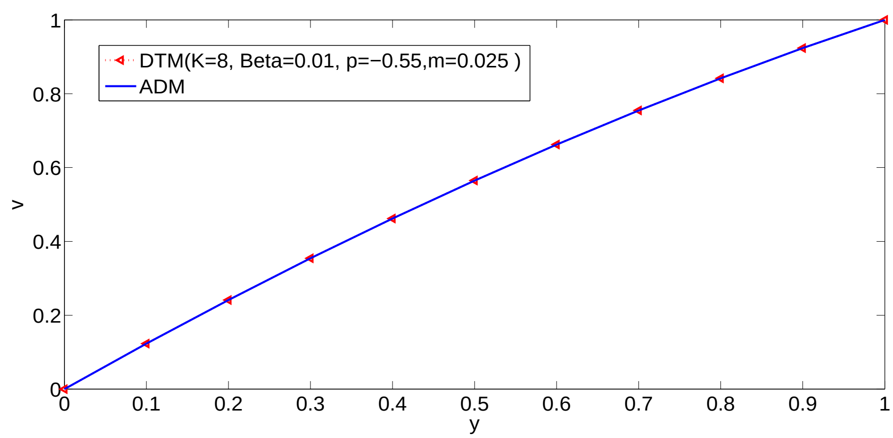

Figure 4.6: Comparison of results obtained for Generalized Couette flow for i=1, $\beta = 0.01$, $\frac{dp}{dx} = -0.55$ and m=0.025.

4.3 Results and Discussion

Here, we studied the MHD flow of third grade non-Newtonian fluid. DTM is used to obtain the analytical solution of differential equations involved. Three cases Couette flow, Poiseuille flow and generalized Couette flow are discussed. The comparison between DTM and ADM[36] solutions of all three cases is graphically represented in figures (4.4), (4.5) and (4.6) respectively. The results obtained using DTM shows good agreement with solution of ADM for restricted values of all parameters. Further, the effect of the dimensionless parameters 'm and dp/dx' on fluid velocity are examined. Figures (4.7), (4.8), (4.11), (4.15) and (4.16) shows the effect of magnetic field on the fluid velocity. It can be seen that the increase in magnetic field causes decrease in velocity. This is because the applied magnetic field produces a drag force due to which decrease in velocity is caused. The effect of pressure gradient on velocity of Poiseuille and generalized Couette flow is graphically presented in figure (4.12), (4.17) and (4.18). From the results it can be observed that velocity increases with the increase in pressure gradient.

Further, the wall shear stress is computed for all three cases and the effect of magnetic field and pressure on it are graphically presented. Here, its observed that the magnetic field is directly proportional to shear stress near moving wall while inversely proportional to shear stress at static wall, as shown in figures (4.9), (4.10) , (4.13),(4.19), (4.20). Whereas, the increase in shear stress near stationary wall is noted due to increase in pressure gradient, as shown in (4.14), (4.21) and (4.22). Based on the above analysis it can be conclude that the values of shear stress is directly proportional to magnetic and inversely proportional to pressure gradient in case of moving wall. While in case of static wall magnetic field is inversely and pressure gradient is directly proportional to shear stress.

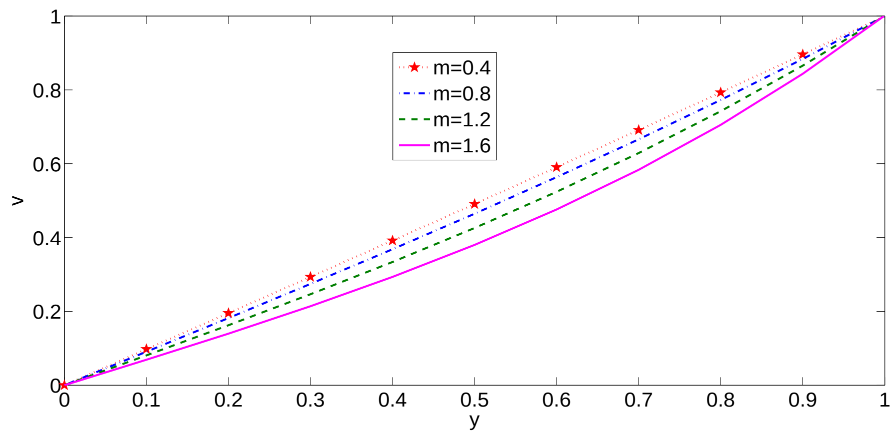

Figure 4.7: Plot of the $v_1(y)$ for $\beta = 0.01$ while m vary for plane Couette flow.

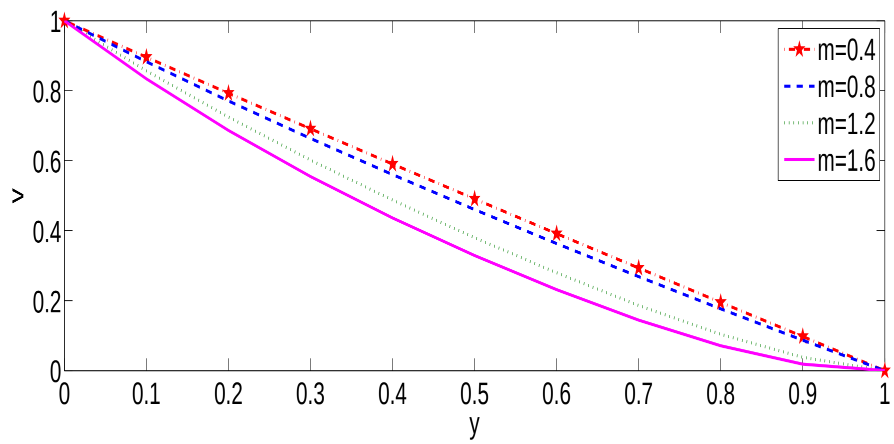

Figure 4.8: Plot of the $v_2(y)$ for $\beta = 0.01$ while m vary for plane Couette flow.

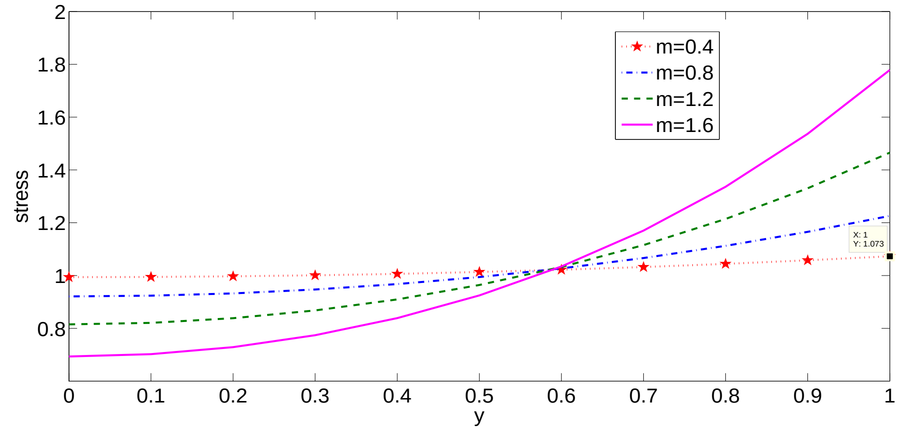

Figure 4.9: Plot of the values of stress for i=1, $\beta = 0.01$ while m vary for plane Couette flow.

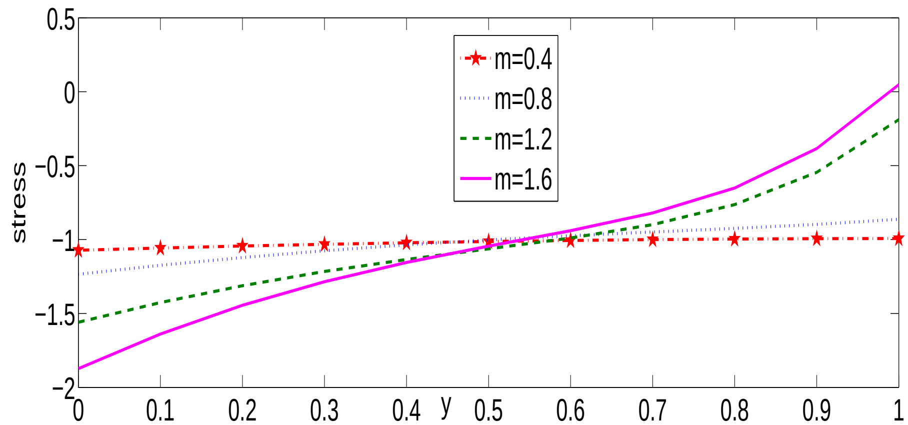

Figure 4.10: Plot of the values of stress for i=2, $\beta = 0.01$ while m vary for plane Couette flow.

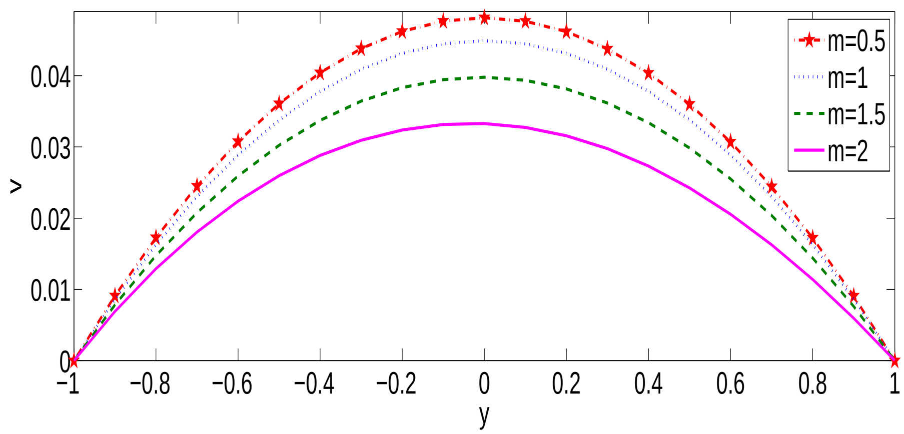

Figure 4.11: Plot of plane Poiseuille flow velocity $v_1(y) = v_2(y)$ for $\beta = 1.2$, $\frac{dp}{dx} = -0.1$ and K=6 while m vary.

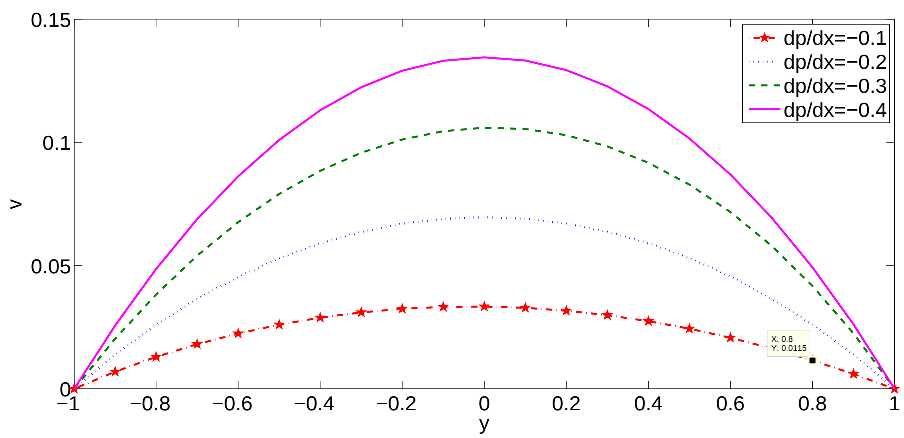

Figure 4.12: Plot of plane Poiseuille flow velocity $v_1(y) = v_2(y)$ for $\beta = 1$, m=1 and K=6 while $\frac{dp}{dx}$ vary.

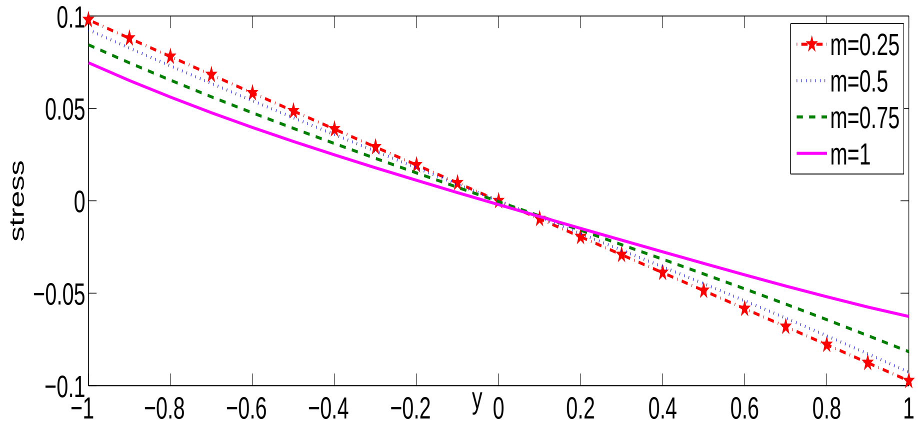

Figure 4.13: Plot of plane Poiseuille flow stress for $\beta = 1.2$, $\frac{dp}{dx} = -0.1$ and K=6 while m vary.

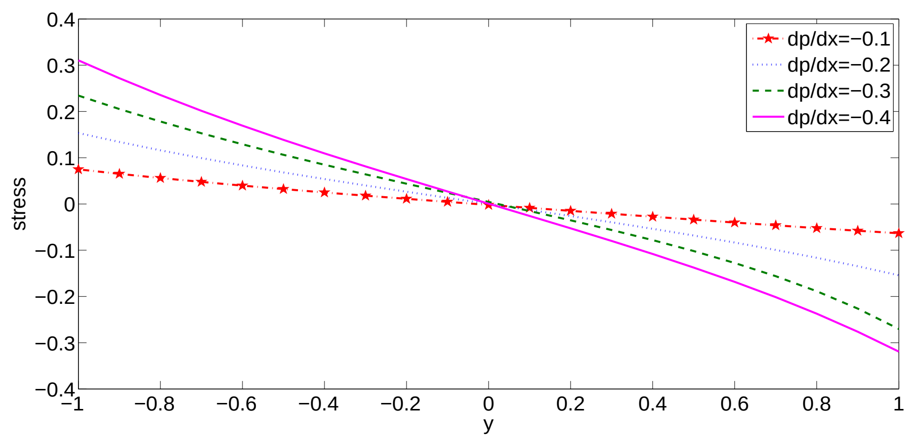

Figure 4.14: Plot of stress for plane Poiseuille flow when $\beta = 1$, m=1 and K=6 while $\frac{dp}{dx}$ vary.

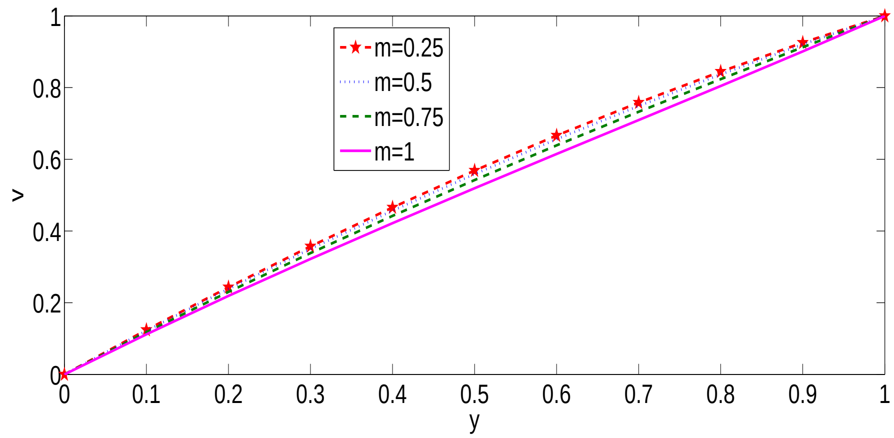

Figure 4.15: Plot of generalized Couette flow velocity for i=1, $\beta = 0.05$, $\frac{dp}{dx} = -0.75$ and K=7 while m vary.

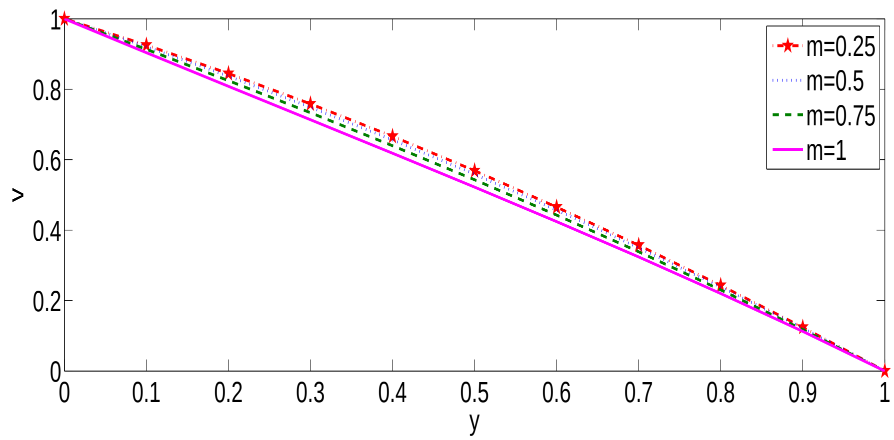

Figure 4.16: Plot of generalized Couette flow velocity for i=2, $\beta = 0.05$, $\frac{dp}{dx} = -0.75$ and K=7 while m vary.

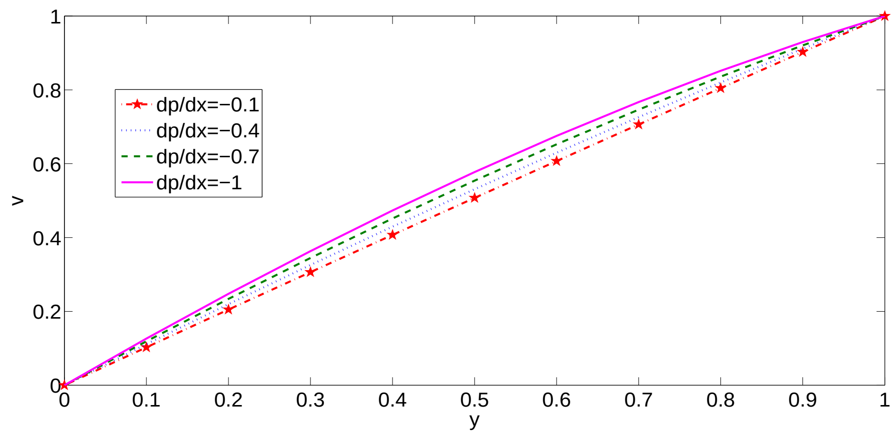

Figure 4.17: Plot of generalized Couette flow velocity for i=1, $\beta = 0.1$, m=0.1 and K=7 while $\frac{dp}{dx}$ vary.

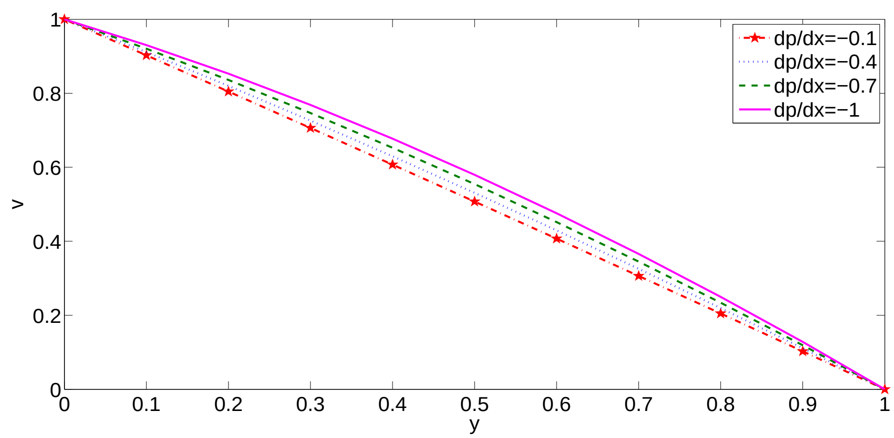

Figure 4.18: Plot of generalized Couette flow velocity for i=2,$\beta = 0.1$, m=0.1 and K=7 while $\frac{dp}{dx}$ vary.

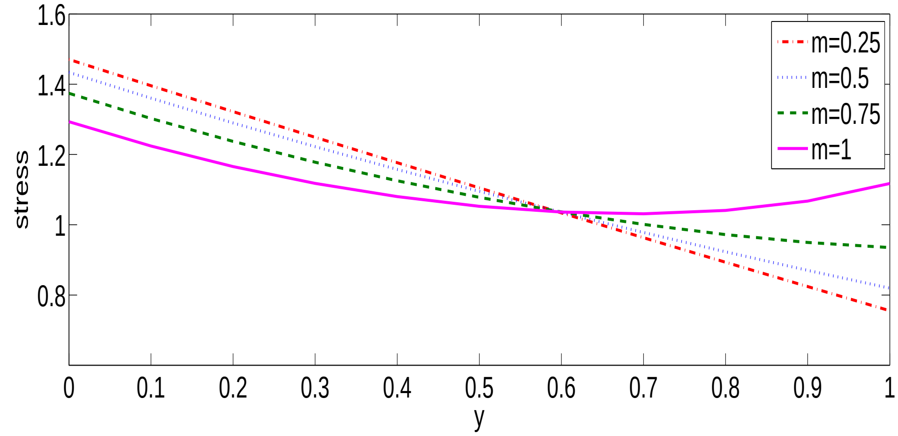

Figure 4.19: Plot of generalized Couette flow stress for i=1, $\beta = 0.05$, $\frac{dp}{dx} = -0.75$ and K=7 while m vary.

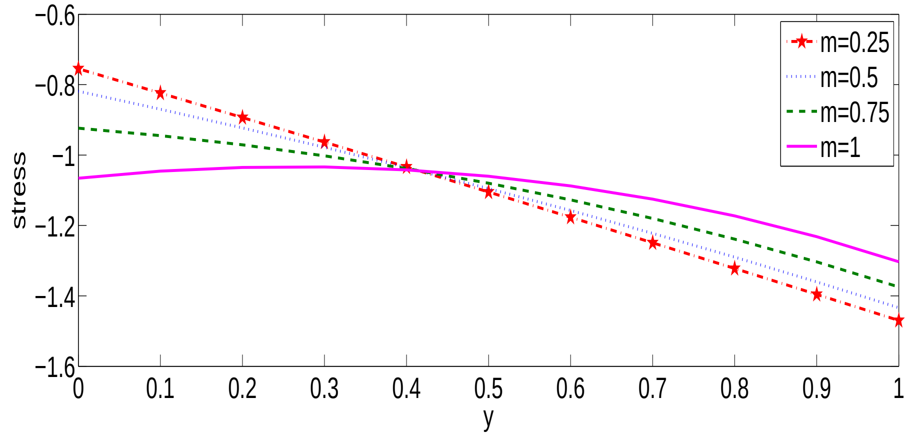

Figure 4.20: Plot of generalized Couette flow stress for i=2, $\beta = 1.2$, $\frac{dp}{dx} = -0.1$ and K=7 while m vary.

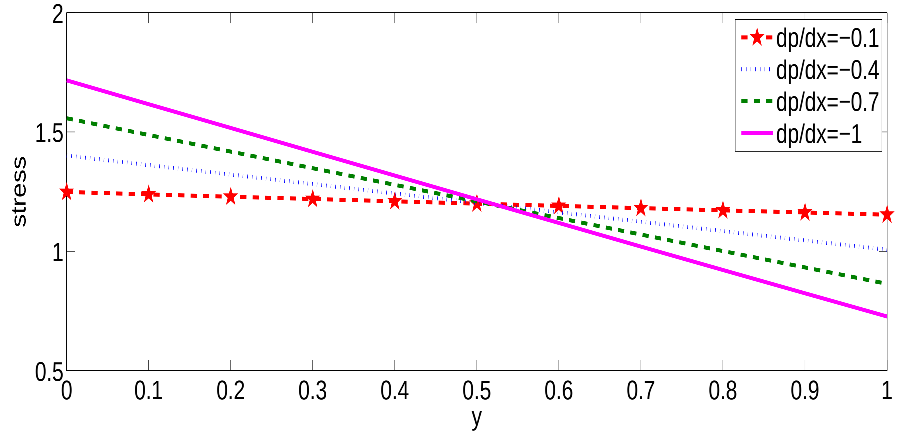

Figure 4.21: Plot of generalized Couette flow stress for i=1, $\beta = 0.1$, m=0.1 and K=7 while $\frac{dp}{dx}$ vary.

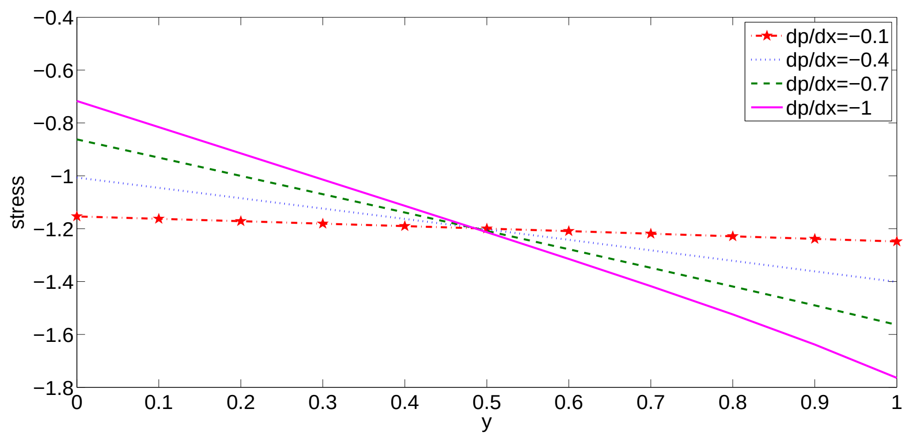

Figure 4.22: Plot of generalized Couette flow stress for i=2, $\beta = 1$, m=1 and K=7 while $\frac{dp}{dx}$ vary.

4.4 Performance analysis of DTM for non-linear ODEs

In this chapter, the semi-analytical solutions of non-linear ODEs involved in MHD flow of non-Newtonian fluids are obtained. The results are compared with numerical solution to show the effectiveness and reliability of method. Here, DTM results are in good agreement with ADM results, for small values of m, dp/dx and β. Here, while solving the BVPs an unknown variable appears in inverse transform whose value was obtained using Matlab, many times we got more than one real roots and to achieve desired result all real roots were checked by substituting them one by one in inverse transform . But this consumes much computational cost and to resolve this problem it's really important to find that which one root should be used to get good results.

Chapter 5

Conclusions

In this study, at first the analytical solution and absolute error of both ODEs and PDEs are obtained using DTM. The results obtained are compared with exact and numerical solution. The comparison revealed the easy implementation and reliability of the method. After doing the analysis of obtained results, we deduce that DTM performs well for large interval difference of independent variable. So to achieve more accurate results number of terms should be increased when interval difference is decreased. Whereas in case of PDEs DTM doesn't performs well at t=1 if the exact solution involve a trigonometric function because DTM gives approximate solution. It is also observed that, when the BVPs are solved by DTM an unknown expression is followed till the end and much computational cost is consumed to compute unknown variable using boundary condition especially when more than one real root is obtained. It's difficult to know which root will give best result. However, for PDEs we don't have to find the unknown variable as DTM solve only those PDEs that have extra conditions and all of there unknown variables are found using those conditions. The analysis of all the results also revealed that DTM performs well only in restricted domain. Another limitation of DTM is that transformations are defined for only specific variables due to which it's not easy to solve those whose direct transforms aren't defined. Thus, to increase the efficiency of method much more research work is needed to address all these limitations.

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Appendix