Solution of Volterra's Population Model by using Spectral Method

Linta Tahir
Regn.#203342

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Supervised by
Prof. Faiz Ahmad

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan

FORM TH-4

National University of Sciences & Technology

MASTER'S THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Ms. Linta Tahir, Regn No. 00000203342 Titled: Solution of Volterra's Population Model by using Spectral Method be accepted in partial fulfillment of the requirements for the award of MS degree.

Examination Committee Members

1. Name: Dr. Muhammad Asif Farooq Signature: _____

2. Name: Dr. Mujeeb ur Rehman Signature: _____

External Examiner: Dr. Qamar Din Signature: _____

Supervisor's Name: Prof. Faiz Ahmad Signature: _____

_____ 16/12/2020
Head of Department Date

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My Beloved Parents

and

My loving Husband

Abstract

In this thesis, we have used spectral method for solving the Volterra's differential equation. We study the basic theory of this method. The spectral method for the solution of non-linear Volterra integro-differential equation is described in details and is applied. The solutions are compared and the results are plotted graphically. We analyze Volterra's population model analytically and numerically by using the spectral method. An analytic approximation was discussed for Volterra's population model and then the use of the Pade approximants gives a better approximation. We discuss the rational Chebyshev polynomial for solving the Volterra's population model numerically. The Laguerre set of polynomials is also used to solve it.

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Chapter 1

Introduction

1.1 Brief History

Many mathematicians and researchers worked on the principle of the spectral method for solving problems numerically that arose in the past few year. Spectral methods are used in scientific computing and applied mathematics for solving numerically certain differential equations. These methods were used firstly in practical flow simulations. The implementation of the spectral method usually learns about "how to approximate a solution of polynomial?."
Now a days, Spectral method is very advanced in many fields and numerious methods have been introduced for solving the problems of differential equations. Here we are focusing on the well known method i-e Chebyshev collocation method. The spectral method

can be used to solve ordinary differential equations, partial differential equations and eigen value problems arising in the differential equations. This method gives the approximate solution of a polynomial. In the field of mathematics, it takes on a global approach. It uses basis functions that are non-zero over the whole domain. The application of spectral methods for non-linear problems arises in fluid dynamics, quantum mechanics, weather predictions, heat conduction and other fields.

The main idea of spectral method is to replace exact derivatives by derivatives of interpolating global polynomial i-e Chebyshev polynomials. Pseudospectral Chebyshev method is very efficient method in which the solution contains Chebyshev collocation points for error minimization. The assumption of approximate solution needs to satisfy the boundary conditions and the equation at the Chebyshev collocation points. This method also has the advantage of dealing with a non-linear differential equations. Parand [17] used Pseudospectral method to solve the non-linear differential equations on semi-infinite intervals. A spectral method is a powerful method that deals with the weighted residual method in which approx-

imate solution can be written in the form of a series and its residual or error goes to zero only in an approximate manner.
We defined the scalar product as:

$$(r, s) = \int_a^b r(x)s(x)w(x)\,dx, \tag{1.1}$$

where $r(x)$ and $s(x)$ are two non-zero vectors of function which is defined on a closed interval $[a, b]$ and $w(x)$ is a weight function. The truncated series expansion of a function $v(x)$ is:

$$v_N(x) = \sum_{i=0}^{N} a_i\phi_i(x), \qquad a \leq x \leq b. \tag{1.2}$$

for some suitable basis function $\phi_i(x)$ and the expansion coefficient a_i must be determined. In this method, the main concern is to find the basis function then to determine the coefficient a_i. There are many choices for choosing the basis function, but here we choose the Chebyshev polynomials as a basis function. Now, we need to introduce the residual R_N that is zero. For example, if the function $v(x)$ is known and its approximate function is $v_N(x)$ then $R_N(x)$ is defined by:

$$R_N(x) = v - v_N(x), \tag{1.3}$$

$v_N(x)$ is an approximate solution then we write the equation as:

$$Lv - f_1 = 0. \tag{1.4}$$

As v is the solution which satisfies the boundary condition depending on the linear operator L and the residual $R_N(x)$ is given by:

$$R_N(x) = Lv_N - f_1. \tag{1.5}$$

In an approximate manner, the residual method R_N of a scalar product should be equal to zero.

$$(R_N, \psi_i) = \int_a^b R_N(x)\psi_i(x)w(x)\,dx = 0, \tag{1.6}$$

where the basis functions $\psi_i(x)$ and the weight $w(x)$ that is associated to determine the basis functions. The weight defines various method for solving the differential equation i-e the spectral or collocation method and the Galerkin type method. The set of collocation points is given to choose the domain D on which residual R should be equal to zero i-e $R_N(x_i) = 0$, where $i = 0, 1, 2, ...N$. This method is known in the literature as the "collocation method".

In this thesis, we discuss about the spectral method. The residual should be zero at various points in spectral method whereas in the Galerkin type method

residual is exactly zero in the mean . In this chapter, we have to apply the Spectral method for solving the ordinary differential equations.

1.2 Chebyshev polynomial

The mathematician, "Pafnuty Lvovich Chebyshev" (1821−1894), was the first who introduced the Chebyshev polynomial on a domain $-1 < x < 1$ as:

$$T_k(x) = \cos(k \cos^{-1}(x)), \qquad k = 0, 1, 2, ... \tag{1.7}$$

The Chebyshev polynomial of first kind $T_k(x)$ can be represented as a linear combination of two Chebyshev polynomials of second kind $U_k(x)$:

$$T_k(x) = \frac{1}{2}(U_k(x) - U_{k-2}(x)). \tag{1.8}$$

where,

$$U_k(x) = \sin(kcos^{-1}(x)), \qquad k = 0, 1, 2, ... \tag{1.9}$$

for approximating a function, the Chebyshev polynomials of first kind T_k are used in spectral method and the fist several Chebyshev polynomials can be defined

as:

$$
\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, ...
\end{aligned}
$$

There exists also a recurrence formula:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x). \quad k \geq 1 \tag{1.10}$$

A finite series of Chebyshev polynomials can be written as:

$$f_N(x) = \sum_{k=0}^{N} a_k T_k(x), \tag{1.11}$$

with a_k is the Chebyshev coefficient and the polynomial approximation of the highest order is thus N. Also it has some basic properties:

$$T_k(-x) = (-1)^k T_k(x). \tag{1.12}$$

$$(T_k, T_j) = \int_{-1}^{1} \frac{T_k(x)T_j(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2} c_k \delta_{kj}, \tag{1.13}$$

with the kronecker symbol δ_{kj} and $c_0 = 2$, $c_k = 1$ for $k \geq 1$. The Chebyshev polynomials defined the

boundary conditions as:

$$T_k(1) = 1,$$
$$T_k(-1) = (-1)^k.$$

1.2.1 Example

Consider the one-dimensional homogenous, linear ordinary differential equation with the boundary conditions.[15]

$$x_{tt} - (t^6 + 3t^2)x = 0, \quad x(-1) = x(1) = 1. \quad (1.14)$$

The exact solution of this problem is:

$$g(t) = \exp([t^4 - 1]/4). \quad (1.15)$$

Apply the spectral method, so to choose the approximate solution that satify the boundary conditions.

$$x(t) = 1 + (1 - t^2)(a_0 + a_1 t + a_2 t^2). \quad (1.16)$$

The residual of this approximation is:

$$R(t; a_0, a_1, a_2) = x_{tt} - (t^6 + 3t^2)x, \quad (1.17)$$

$$R = 2a_2 - 2a_0 - 6a_1 t - 12a_2 t^2 - (t^6 + 3t^2)(1 + (1 - t^2)(a_0 + a_1 t + a_2 t^2)), \quad (1.18)$$

Now to minimize error, we choose a residual zero at a set of points equal in number to the coefficients in $x(t)$. This is called the "collocation and pseudo-spectral method". The above equation becomes:

$$2a_2 - 2a_0 - 6a_1 t - 12a_2 t^2 - (t^6 + 3t^2)(1 + (1 - t^2)(a_0 + a_1 t + a_2 t^2)) = 0, \quad (1.19)$$

Now to choose the collocation points $t_i,\ i = 0, 1, 2, 3, ...n$ to find the $n+1$ linearly independent equations. Consider $t_i = (-0.5, 0, 0.5)$, thus we gives the three linearly independent equations:

$$\begin{aligned} Eq1 =& 2a_2 - 2a_0 - 6a_1(-0.5) - 12a_2(-0.5)^2 - ((-0.5)^6 \\ &+ 3(-0.5)^2)(1 + (1 - (-0.5)^2)(a_0 + a_1(-0.5) \\ &+ a_2(-0.5)^2)), \end{aligned}$$

$$\begin{aligned} Eq2 =& 2a_2 - 2a_0 - 6a_1(0) - 12a_2(0)^2 - ((0)^6 + 3(0)^2) \\ &(1 + (1 - (0)^2)(a_0 + a_1(0) + a_2(0)^2)), \end{aligned}$$

$$\begin{aligned} Eq3 =& 2a_2 - 2a_0 - 6a_1(0.5) - 12a_2(0.5)^2 - ((0.5)^6 \\ &+ 3(0.5)^2)(1 + (1 - (0.5)^2)(a_0 + a_1(0.5) \\ &+ a_2(0.5)^2)), \end{aligned}$$

Table 1.1: Comparison of exact and approximate solutions.

t	$x(t)$	$g(t)$
-1	1.	1.
-0.5	0.806935	0.791065
0	0.794064	0.778801
0.5	0.806935	0.791065
1	1.	1.

The coefficients a_0, a_1, a_2 are determined by solving:

$$Eq1 = Eq2 = Eq3 = 0. \tag{1.20}$$

Hence we get the required values:

$$\begin{aligned} a_0 &= \frac{-784}{3807}, \\ a_1 &= 0, \\ a_2 &= \frac{-784}{3807}. \end{aligned}$$

Putting values in approximate solution, so we would get a required results. The graph of the approximate and exact solution concludes that the approximate solution is quite precise as the exact solution in table [1.1].

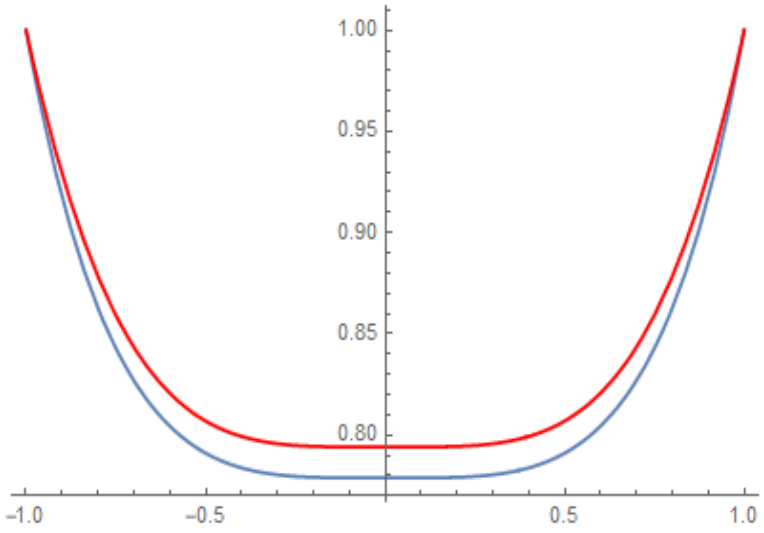

Figure 1.1: This graph indicates the exact and approximate solutions where blue line shows the exact solution and red line shows the approximate solution of the problem.

1.2.2 Example

Consider the non-linear differential equation:

$$y'' + \frac{2xy'}{\sqrt{1 - aly}} = 0, \quad y(0) = 1, y(10) = 0. \quad (1.21)$$

The value of $al = 0.2247$. To solve this problem by Chebyshev polynomials $T_n(x)$ which is defined on a domain [-1,1]. By applying the linear transformation:

$$y = ax + b. \quad (1.22)$$

we have a boundary conditions $y(-1) = 0$ and $y(1) = 10$. Hence, we get the value of shifted Chebyshev

polynomials i-e $T_n(\frac{y}{5} - 1)$.
By applying the spectral method using mathematica, it could easily be determine that the graph obtained from the approximate solution and the given equation is almost same for increasing the value of n. We couldn't determine the exact solution, but the exactness of an approximate solution can be determine by calculating the value with higher n.

1.3 Brief history of Volterra's equation

In 1798, "Thomas Malthus" was the first mathematician who discussed about the concept of population dynamics [1]. He found that, when population increase exponentially and the sources on which population depend increase arithmetically and remain constant.

$$\frac{dP}{dt} = \epsilon p(t), \tag{1.23}$$

Later, in 1838 "Verhulst" established Malthus "principle of population dynamics" both worked together for developing the mathematical concept. "Alfred J.Lotka" [16] works on the theory of population growth in a species then he introduced the "Lotka-Volterra

equations" which is also called the "predator-prey equations". In 1925, the equations was used to examine the predator-prey interactions in his book "Elements of Physical Biology" which shows the great contribution in the analysis of population growth. The logistic equation is called the "Law of population growth" which was introduced by Lotka. Hence, the "Logistic equation" which is defined as:

$$\frac{dP}{dt} = (\epsilon - \lambda p(t))p(t), \tag{1.24}$$

where P is population at time t, ϵ is a constant of proportionality and λ is the carrying capacity.
Later, in 1928, Lotka and Volterra works on the "principle of Volterra Integro-differential equation". It is introduced to examine the population growth in a species. Volterra examine the population growth model then his work resulted to introduced the well-known equation which is named as "Volterra integro-differential equation". It is defined as:

$$\frac{dp}{dt} = a_1 p - b_1 p^2 - c_1 p \int_0^{\tilde{t}} p(x)\, dx, \qquad p(0) = p_0. \tag{1.25}$$

where $a_1 > 0$ is the birth rate coefficient, $b_1 > 0$ is the crowding coefficient, $c_1 > 0$ is the toxicity coeffi-

cient. $p = p(\tilde{t})$ is the population at time $\tilde{t}$ and p_0 is the initial population. The integral term in the right hand side of (1.37) denotes the effect of toxin on the species. Now defining the nondimensional variables:

$$t = \frac{\tilde{t}}{b_1/c_1}, \qquad u = \frac{p}{a_1/b_1}, \tag{1.26}$$

to obtain the equation

$$k\frac{du}{dt} = u - u^2 - u\int_0^t u(x)\,dx, \quad u(0) = u_0. \tag{1.27}$$

where $k = c_1/a_1b_1$ is a nondimensional parameter, which shows a vital role in the behaviour of $u(t)$. Many researchers put their interest to discover the numerical and analytical solution of the Volterra population model by A-M Wazwaz[1], F.M.Scudo[2], R.D.Small[3], and TeBeest[4].

Chapter 2

Solution in series

2.1 Introduction

Many science, engineering applications and mathematical physics/Biology use Volterra equation to solve the problem of population model.
Volterra examined the population growth model when he learned about the hereditary influence. It is defined as "the equation in which both differential and integral operators seems together in the equation is known as the "Volterra integro-differential equations". The nonlinear Volterra equation is:

$$k\frac{du}{dt} = u - u^2 - u\int_0^t u(x)\,dx, \qquad (2.1)$$

Where we choose $u(0) = 0.1$ is the initial condition and the parameter $k = 0.02$, both are positive constants.

To calculate the approximate solution of the Volterra population model, we have to consider the series solution method. This solution agrees with the solution found by the method of Adomian decomposition which give the same solution of the approximate solution. So that, we must need the initial conditions, and this could be seen as $u(x)$ is involving and its derivatives in the above equation.

2.2 Methods for the solution of Volterra population model:

2.2.1 series solution method

The non-linear Volterra integro-differential equation with the value of k=0.02 is given by,

$$\frac{du}{dt} = 50u - 50u^2 - 50u \int_0^t u(x)\, dx, \tag{2.2}$$

Consider the power series solution of the form:

$$u(t) = \sum_{m=0}^{\infty} b_m t^m. \tag{2.3}$$

Now to determine the coefficients b_m, for $m \geq 0$ we have:

$$\sum_{m=1}^{\infty} m b_m t^{m-1} = 50(\sum_{m=0}^{\infty} b_m t^m) - 50(\sum_{m=0}^{\infty} b_m t^m)^2 - 50(\sum_{m=0}^{\infty} b_m t^m) \int_0^t (\sum_{m=0}^{\infty} b_m x^m)\, dx. \quad (2.4)$$

so we get the recurrence relation:

$$(k+1)b_{k+1} = 50b_k - 50 \sum_{m=0}^{k} b_m b_{k-m} - 50 \sum_{m=0}^{k-1} \frac{b_m b_{k-1-m}}{k-m}.$$

putting $u(0) = 0.1$, so we get the required constants:
$b_0 = 0.1, \;\; b_1 = 4.5, \;\; b_2 = 89.75, \;\; b_3 = 847.917, \;\; b_4 = -1893.85$
$b_5 = -176439., \;\; b_6 = -2.3469 \times 10^6 \cdots$
Hence, we find the approximate solution.

$$u(t) = 0.1 + 4.5t + 89.75t^2 + 847.917t^3 - 1893.85t^4 - 17643.t^5 - 2.346 \times 10^6 t^6 - 4.9029 \times 10^6 t^7 + 3.2994 \times 10^8 t^8 + O(t^9).$$

2.2.2 The Decomposition Method

Eq.(2.1) can be written in the decomposition method [7] as:

$$Lu(t) = 50u(t) - 50u^2(t) - 50\int_0^t u(t)u(x)\,dx, \quad (2.5)$$

where $u(0) = 0.1$, the differential and integral operator L is defined by:

$$L = \frac{d}{dt}. \quad (2.6)$$

$$L^{-1}(.) = \int_0^t (.)\,dt. \quad (2.7)$$

As it is an invertible operator so we apply L^{-1} in Eq.(2.5), to get the required equation:

$$u(t) = 0.1 + L^{-1}(50u(t) - 50u^2(t) - 50\int_0^t u(t)u(x)\,dx). \quad (2.8)$$

Now consider the series as:

$$u(t) = \sum_{m=0}^{\infty} u_m(t). \quad (2.9)$$

Now to determine the coefficients of $u_m(t)$, putting the series in Eq.(2.8) will become

$$\sum_{m=0}^{\infty} u_m(t) = 0.1 + L^{-1}(50\sum_{m=0}^{\infty} u_m(t) - 50\sum_{m=0}^{\infty} C_m(t) - 50\int_0^t \sum_{m=0}^{\infty} D_m(x,t)\,dx),$$

where the Adomian polynomials $C_m(t)$ and $D_m(x,t)$ denotes the non linear terms $u^2(t)$ and $u(x)u(t)$. where,

$$u^2(t) = \sum_{m=0}^{\infty} C_m(t), \tag{2.10}$$

and

$$u(x)u(t) = \sum_{m=0}^{\infty} D_m(x,t). \tag{2.11}$$

To introduce the non-linear operator $F(u)$ in the form:

$$F(u) = \sum_{m=0}^{\infty} C_m, \tag{2.12}$$

where C_m are the Adomian polynomial of $u_0, u_1, u_2, ..., u_m$ which is defined by:

$$C_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}[F(\lambda^i U_i)], \quad m = 0, 1, 2, ... \tag{2.13}$$

For $C_m(t)$, we get.

$$\begin{aligned}
C_0(t) &= u_0^2(t),\\
C_1(t) &= 2u_0(t)u_1(t),\\
C_2(t) &= u_1^2(t) + 2u_0(t)u_2(t),\\
C_3(t) &= 2u_1(t)u_2(t) + 2u_0(t)u_3(t), \cdots
\end{aligned}$$

For $D_m(x,t)$,

$$\begin{aligned}
D_0(x,t) &= u_0(x)u_0(t),\\
D_1(x,t) &= u_0(x)u_1(t) + u_1(x)u_0(t),\\
D_2(x,t) &= u_0(x)u_2(t) + u_1(x)u_1(t) + u_2(x)u_0(t), \cdots
\end{aligned}$$

To determine the coefficients of $u(t)$ which are $u_0, u_1, u_2, ...,$
Hence we follow the relation as:

$$u_0(t) = 0.1,$$

$$u_{k+1}(t) = L^{-1}(50u_k(t) - 50C_k(t) - 50\int_0^t D_k(x,t)\,dx),$$

With $u_0(t) = 0.1$, we get the required values:

$$\begin{aligned}
u_0(t) &= 0.1,\\
u_1(t) &= 4.5t - 0.25t^2,\\
u_2(t) &= 90t^2 - 14.5t^3 + 0.41667t^4,\\
u_3(t) &= 862.5t^3 - 449.6t^4 + 31.45t^5 - 0.1667t^6, \cdots
\end{aligned}$$

Putting all values in a series, hence we get the required approximate solution:

$$u(t) = 0.1+4.5t+89.75t^2+847.917t^3-1893.85t^4-17643.t^5$$
$$-2.346\times10^6t^6-4.9029\times10^6t^7+3.2994\times10^8t^8+O(t^9).$$

2.2.3 Example

$$\frac{dU}{dt} = x^2 - (\frac{1}{4})(\frac{dU}{dx})^2, \tag{2.14}$$

where $U(x,0) = 0$. In an operator form Eq.(2.14) will become

$$L_tU = x^2 - \frac{1}{4}(L_xU)^2, \tag{2.15}$$

where

$$L_t = \frac{d}{dt}$$

$$L_x = \frac{d}{dx}$$

and

$$L_t^{-1}(.) = \int_0^t (.)\, dt$$

Apply inverse operator:

$$U(x,t) = x^2t - \frac{1}{4}L_t^{-1}(L_xU)^2, \tag{2.16}$$

Consider the decomposition series,

$$U(x,t) = \sum_{m=0}^{\infty} U_m(x,t), \quad and \;\; U(x,0) = U_0. \;\; (2.17)$$

putting values in Eq.(2.16)

$$\sum_{m=0}^{\infty} U_m(x,t) = x^2 t - \frac{1}{4} L_t^{-1}(L_x \sum_{m=0}^{\infty} U_m(x,t))^2,$$

Here $U_0(x,t) = x^2 t$ and
$U_{k+1}(x,t) = -\frac{1}{4} L_t^{-1}(L_x U_k(x,t))^2$, for $k \geq 0$. Then it follows that:

$$\begin{aligned} U_0 &= x^2 t, \\ U_1 &= -\frac{x^2 t^3}{3}, \\ U_2 &= -\frac{x^2 t^7}{63}, \cdots \end{aligned}$$

Finally, we find the solution:

$$U(x,t) = x^2 t - \frac{x^2 t^3}{3} - \frac{x^2 t^7}{63} + ... \qquad (2.18)$$

2.2.4 Converting to non linear ODE

The Volterra's equation convert into a non-linear ODE which is defined as:

$$y(t) = \int_0^t u(x)\,dx. \tag{2.19}$$

This leads to the relations:

$$y^{'}(t) = u(t), \tag{2.20}$$

$$y^{''}(t) = u^{'}(t). \tag{2.21}$$

so we get the required Volterra's equation in which the value of $k = 0.02$:

$$y^{''}(t) = 50y'(t) - 50(y^{'}(t))^2 - 50y(t)y^{'}(t). \tag{2.22}$$

with two initial conditions:

$$y(0) = 0,$$

$$y^{'}(0) = 0.1,$$

Now, Volterra equation can be solve by series solution and the decomposition method. Here we solve the differential equation by Adomian method. Hence the required equation is.

$$L_t y(t) = 50y'(t) - 50(y'(t))^2 - 50y(t)y'(t), \tag{2.23}$$

where L_t and L_t^{-1} is the differential and the integral operator which is defined by:

$$L_t = \frac{d^2}{dt^2},$$

$$L_t^{-1}(.) = \int_0^t \int_0^t (.)\, dt\, dt.$$

Apply L_t^{-1} on both sides of the Eq.(2.23) will become:

$$y(t) = 0.1t + L_t^{-1}(50y'(t) - 50(y'(t))^2 - 50y(t)y'(t)). \tag{2.24}$$

Now consider the decomposition series as:

$$y(t) = \sum_{m=0}^{\infty} y_m(t), \tag{2.25}$$

so that it will be easy to determine the components of $y_m(t)$, Hence equation will become:

$$\sum_{m=0}^{\infty} y_m(t) = 0.1t + L_t^{-1}(50 \sum_{m=0}^{\infty} {y_m}'(t) - 50 \sum_{m=0}^{\infty} \tilde{C}_m(t) - 50 \sum_{m=0}^{\infty} \tilde{D}_m(t)),$$

where $\tilde{C}_m(t)$ and $\tilde{D}_m(t)$ are the Adomian polynomials.

$$\sum_{m=0}^{\infty}(y'(t))^2 = \sum_{m=0}^{\infty}\tilde{C}_m(t) \tag{2.26}$$

$$\sum_{m=0}^{\infty}y(t)y'(t) = \sum_{m=0}^{\infty}\tilde{D}_m(t) \tag{2.27}$$

For finding the polynomials $\tilde{C}_m(t)$,we have:

$$\begin{aligned}
&\tilde{C}_0(t) = (y'_0)^2(t),\\
&\tilde{C}_1(t) = 2y'_0(t)y'_1(t),\\
&\tilde{C}_2(t) = (y_1')^2(t) + 2y_0'(t)y_2'(t),\\
&\tilde{C}_3(t) = 2y_1'(t)y_2'(t) + 2y_0'(t)y_3'(t), \cdots
\end{aligned}$$

For $\tilde{D}_m(t)$,

$$\begin{aligned}
&\tilde{D}_0(t) = y_0(t)y_0'(t),\\
&\tilde{D}_1(t) = y_0'(t)y_1(t) + y_1'(t)y_0(t),\\
&\tilde{D}_2(t) = y_0'(t)y_2(t) + y_1'(t)y_1(t) + y_2'(t)y_0(t), \cdots
\end{aligned}$$

To determine the coefficients of $y_0, y_1, y_2, \cdots$ of y as a function of t, we have:

$$y_0(t) = 0.1t, \tag{2.28}$$

and

$$y_{k+1}(t) = L_t^{-1}\{50{y_k}'(t) - 50\tilde{C}_k(t) - 50\tilde{D}_k(t)\}, \quad k \geq 0. \tag{2.29}$$

With $y_0(t) = 0.1t$, so we get the required values:

$$y_1(t) = 2.25t^2 - 0.0833t^3,$$
$$y_2(t) = 30t^3 - 3.6455t^4 + 0.0833t^5,$$
$$y_3(t) = 215.625t^4 - 78.85t^5 + 5.051t^6 - 0.0842t^7, \cdots$$

As we know,

$$u(t) = y'(t). \tag{2.30}$$

By using Eq.(2.32) we write the above equation as.

$$u(t) = \sum_{m=0}^{\infty} {y_m}'(t) = {y_0}'(t) + {y_1}'(t) + {y_2}'(t) + ..., \tag{2.31}$$

Putting all values in a series, hence the required approximate solution:

$$u(t) = 0.1 + 4.5t + 89.75t^2 + 847.917t^3 - 1893.85t^4 - 17643.t^5$$
$$-2.346 \times 10^6 t^6 - 4.9029 \times 10^6 t^7 + 3.2994 \times 10^8 t^8 + O(t^9).$$

2.3 Pade approximants

It is generally used to determine the rational approximations for functions. we define as:

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)}, \tag{2.32}$$

where,

$$P_N(x) = p_0 + p_1x + p_2x^2 + p_3x^3 + ... + p_Nx^N.$$

$$Q_M(x) = 1 + q_1x + q_2x^2 + q_3x^3 + ... + q_Mx^M.$$

Here $q_0 = 1$. It has $N + M$ degree of the polynomial and $N+M+1$ unknown coefficients to be determined. Consider the approximate solution.

$$u(t) = 0.1+4.5t+89.75t^2+847.917t^3-1893.85t^4-17643.t^5$$
$$-2.346\times10^6t^6-4.9029\times10^6t^7+3.2994\times10^8t^8+O(t^9).$$

Apply Pade approximants in the above equation for finding the mathematical behaviour of the solution $u(t)$. It is defined as "to convert the approximate solution into a rational function". Here $N = M = 4$, so that it has $N+M$ degree of the polynomial. Hence we write as:

$$R_{4,4}(x) = \frac{P_4(x)}{Q_4(x)}, \tag{2.33}$$

$$[\frac{N}{M}] = [\frac{4}{4}] = \frac{p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4}{1 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4}, \quad (2.34)$$

2.4 Analysis

Now to calculate the unknown coefficients and using the approximate solution we have:

$$[\frac{4}{4}] = \frac{0.1 + 2.4647t + 25.961t^2 + 141.28t^3 + 344.t^4}{1.00000 - 20.352t + 277.96t^2 - 1308.t^3 + 4368.t^4}.$$

This equation relates with the Pade approximants. For $u(0) = 0.1$ and $k = 0.02$, the exact value of u_{max} can be obtained from the solution of TeBeest[4]:

$$u_{max} = 1 + \kappa \ln(\frac{\kappa}{1 + \kappa - u_0}), \quad (2.35)$$

putting values we get:

$$u_{max} = 0.923471721 \quad (2.36)$$

and the critical value is:

$$t_{critical} = 0.1118454355 \quad (2.37)$$

Table 2.1: Exact and approximate value of u_{max} for different values of k.

k	Critical t	Approx. u_{max}	Exact u_{max}
0.02	0.1118454355	0.9038380533	0.923471721
0.04	0.2102464437	0.861240177	0.8737199832
0.1	0.4644767322	0.7651130834	0.7697414490
0.2	0.8168581189	0.6579123080	0.6590503816
0.5	1.6267110031	0.4852823482	0.4851902914

2.5 Graphical Behaviour

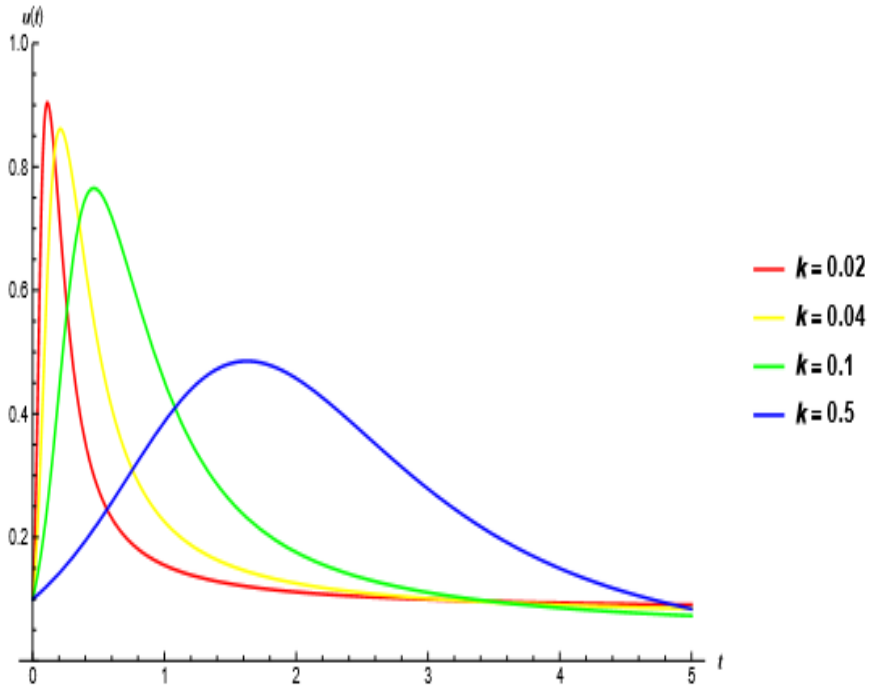

Figure 2.1: Pade's Approximants of $u(t)$ as a function of t.

2.6 Conclusion

The basic idea of this study has been to determine the approximate solution of Volterra population model. In this chapter, we used series solution method [5,6] and the decomposition method [7] in the Volterra population equation.

When we get the approximate solution of $u(t)$ in a series form, then we need to examine the mathematical behaviour of the solution $u(t)$ by using the Pade approximants. It is a very useful method which convert series solution into a rational function to tell us about the further information about $u(t)$. Now, to check the graphical behaviour which shows the Pade approximants of $u(t)$ for $k = 0.02, 0.1, 0.2, 0.5$. In fig. 2.1, the curve that will reach a peak, then moves to the slow exponential decay where $u(t)$ goes to zero as t approaches to infinity.

When k increases, the amplitude decreases and the exponential decay increases in the long run. While when we decrease the value of k, the amplitude of $u(t)$ increases and the exponential decay decreases. Then for different values of k, we determine the approximate and exact value of U_{max}. Further, when we find

more values of $u(t)$ for enhancing the efficiency of series solution and the decomposition method with the Pade approximants [21]. This technique shows better results over series approximation for finding the behaviour of Volterra's population model for small and large value of k.

Chapter 3

Numerical Solution using Rational Chebyshev and Laguerre Polynomials

3.1 Introduction

We shall consider a method to determine the Volterra's population model with the help of spectral method. In the last few years, spectral method for ordinary differential equations in unbounded domains have been used with great success in the field of applied mathematics. Many researchers used spectral method for the unbounded domain in a different way like:

1. Direct approaches: which includes the Laguerre polynomials have been proposed by Maday[8], Funaro[9], Shen[20], Guo and Shen in [10].

2. Indiect approaches: which includes the Rational Chebychev polynomial that converts the unbounded domain into a bounded domain and were investigated by Guo [11,12].

Parand [18] applied a numerical approach for finding the solution of Volterra's equation and it is based on the Rational Chebychev polynomial for solving the unbounded problems. Boyd et al.[14,24] used the collocation method on a semi infinite interval then conducted the result with Rational Chebychev and Laguerre functions.He also introduced [13] some spectral methods on unbounded intervals by using mutually orthogonal systems of Rational functions.

3.2 Collocation Algorithm

We need to solve the Volterra equation by using the collocation method.
Consider the equation $Lu(x) = f(x)$ (where L is the differential operator) hence, by using the various points we have:

1. Construct the approximate solution.

$$f(t) = \sum_{i=0}^{n} a_i L_i(t), \tag{3.1}$$

2. Insert the series into the given equation.

3. To obtain the residual function.

$$Res(t, a_0, a_1, a_2, ...) = Lu_n(x) - f(x). \tag{3.2}$$

Now we have $N + 1$ unknown and $N + 1$ equations, that can be used to find the constants a_n.

4. Now to choose rooots of order n Rational Chebyshev as n collocation points.

5. Solve this system of equations and to find the constants a_n.

In the last step, the main difficulty is to find the initial approximation by solving a system of non-linear equations. So the best way of finding the initial approximation is to solve a system of equations analytically by using Mathematica or Maple.

As the Pseudospectral method is useful to determine the solution of differential and integral equations. It gives us a highly accurate solutions for differential equations. This method reduces the problem of non-linear equations to the solution of a system of non-linear algebraic equations.
The main goal is to determine the Collocation points and the selection of the basis functions. The basis functions should have three different advantages. It should give us a complete solution, rapid convergence and easy to compute.

3.3 Rational Chebyshev functions

The Rational Chebyshev functions are orthogonal with respect to the weight function $w(x) = 1/(\sqrt{x}(x+1))$ in the interval $[0, +\infty]$ and can be defined as:

$$R_n(x) = T_n(\frac{x-1}{x+1}). \tag{3.3}$$

The RC function satisfy the recurrence formula:

$$R_0(x) = 1, \tag{3.4}$$

$$R_1(x) = \frac{x-1}{x+1}, \tag{3.5}$$

$$R_{n+1}(x) = 2(\frac{x-1}{x+1})R_n(x) - R_{n-1}(x), \ \ where \ n \geq 1. \tag{3.6}$$

The orthogonality relation is defined as:

$$\int_0^\infty R_n(x)R_m(x)w(x)\,dx = \frac{c_m\pi}{2}\delta_{nm}, \tag{3.7}$$

with

$$c_m = \begin{cases} 2 \ \text{for} \ \ m = 0 \\ 1 \ \text{for} \ \ m \neq 1 \end{cases} \tag{3.8}$$

where δ_{nm} is the kronecker function.

3.4 Solving Volterra integro-differential equation by using Rational Chebyshev Polynomial

In this section , we have to apply Rational Chebyshev polynomial in the Volterra's equation with the help of Mathematica. Using Mathematica, we have to check the behaviour of the graph and the mathematical structure of $u(t)$.
By applying the Spectral method in Volterra's equation we have:

1. Consider the approximate solution.

$$u(t) = \sum_{k=0}^{n} a_k R_k(t), \tag{3.9}$$

2. Put the approximate solution into the Volterra integro-differential equation.

$$k\frac{du}{dt} = u - u^2 - u\int_0^t u(x)\,dx, \tag{3.10}$$

for $u(0) = u_0 = 0.1 > 0$ and $k = 0.1$. so the Eq.(3.10) will become:

$$\begin{aligned}\sum_{k=0}^{n} a_k {R_k}'(t) =& 10\sum_{k=0}^{n} a_k R_k(t) - 10(\sum_{k=0}^{n} a_k R_k(t))^2 \\ & - 10\sum_{k=0}^{n} a_k R_k(t)\int_0^t \sum_{k=0}^{n} a_k R_k(x)\,dx,\end{aligned}$$

3. Construct the Residual function:

$$\begin{aligned}Res(t) =& \sum_{k=0}^{n} a_k {R_k}'(t) - 10\sum_{k=0}^{n} a_k R_k(t) + 10(\sum_{k=0}^{n} a_k \\ & R_k(t))^2 + 10\sum_{k=0}^{n} a_k R_k(t)\int_0^t \sum_{k=0}^{n} a_k R_k(x)\,dx\end{aligned}$$

Now we have $n+1$ unknown $\{a_n\}$. To find the unknown coefficients $\{a_n\}$, we need to find $n+1$ equations.

4. Find n collocation points by determine the zeros of $R_{n+1}(t)$.

5. Substitute the collocation points in $Res(t_j, a_0, a_1, a_2, ...$ $0, 1, 2, ..., n-1$ and $u_n(0) = 0.1$ for obtaining a system of $n+1$ equations.

6. Solve this system of equations for obtaining a_n.

Hence, we easily calculate the approximate solution by putting the values of a_n.Then by plotting the graph in mathematica, we would be able to check the behaviour of the graph which tell us about the approximate value of u_{max} and the critical point. For finding the exact value of u_{max}, we need to obtain a formula [4].

$$u_{max} = 1 + \kappa \ln(\frac{\kappa}{1+\kappa-u_0}), \qquad (3.11)$$

Table 3.1: Values of u_{max} for various levels of approximation.

n	Critical t	Approx. u_{max}	Exact u_{max}
2	0.714864	0.561349	0.76974144907
3	0.379906	0.760597	0.76974144907
4	0.318197	0.809141	0.76974144907
6	0.377294	0.776444	0.76974144907
9	0.445509	0.772758	0.76974144907
10	0.418554	0.777775	0.76974144907
12	0.452044	0.778043	0.76974144907

For different values of n, we have a graphs like this:

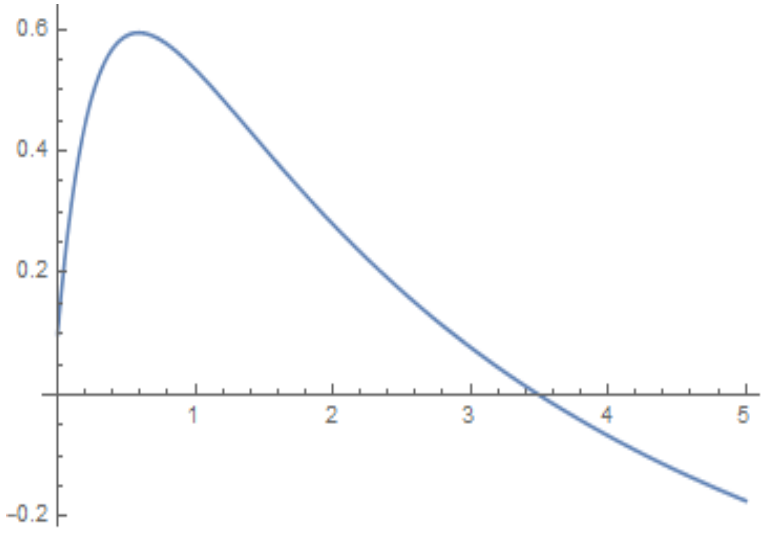

Figure 3.1: Graph of Volterra's Population model for $n = 2$.

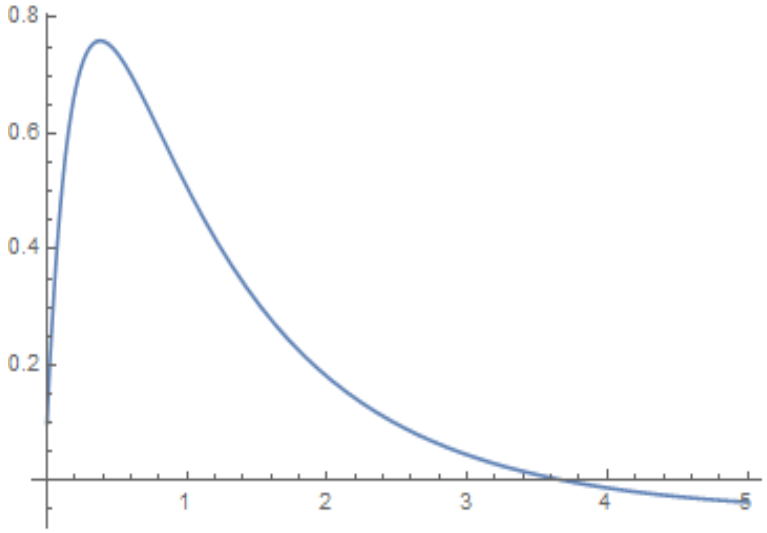

Figure 3.2: Required graph of Volterra's Population model for $n = 3$.

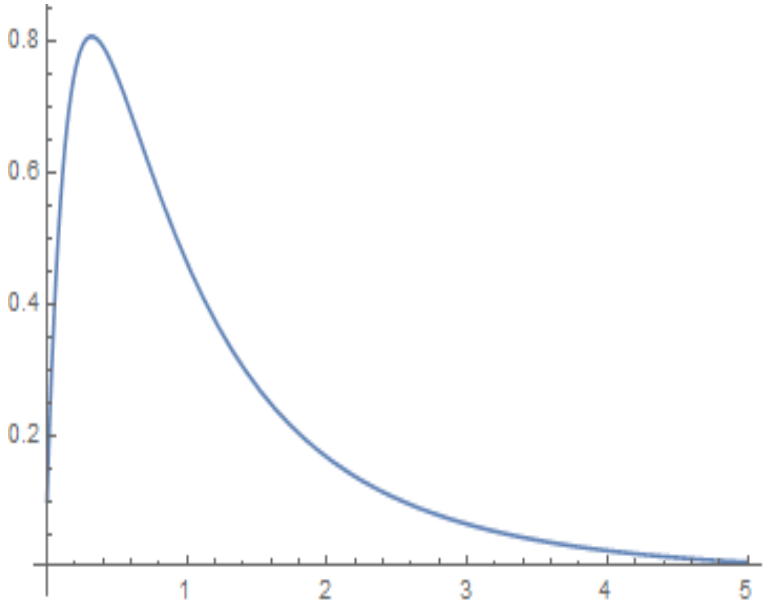

Figure 3.3: Required graph of Volterra's Population model for $n = 4$.

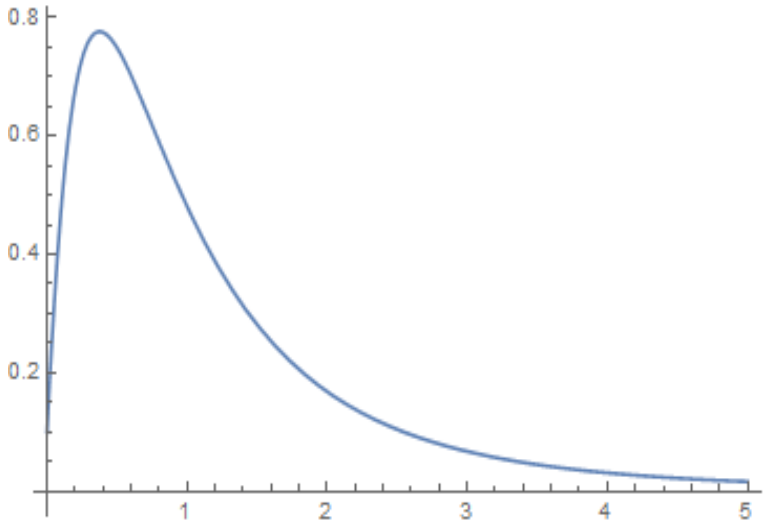

Figure 3.4: Required graph of Volterra's Population model for $n = 6$.

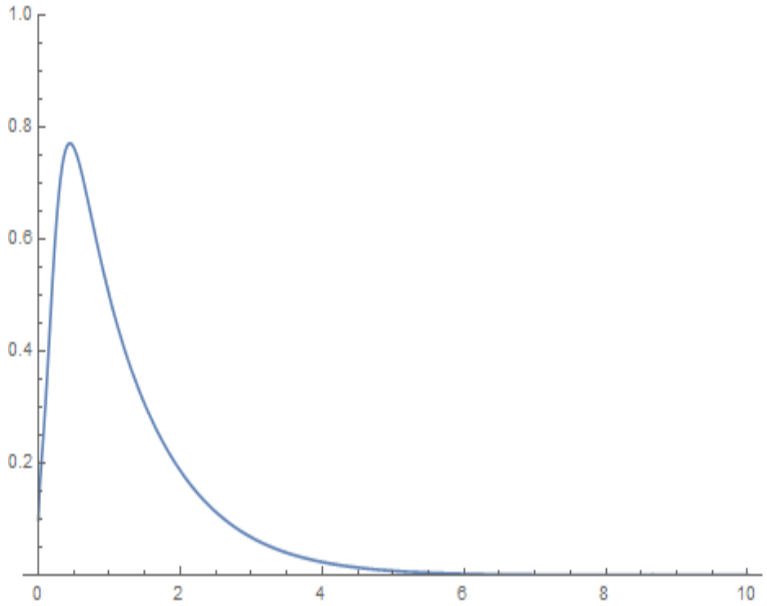

Figure 3.5: Required graph of Volterra's Population model for $n = 9$.

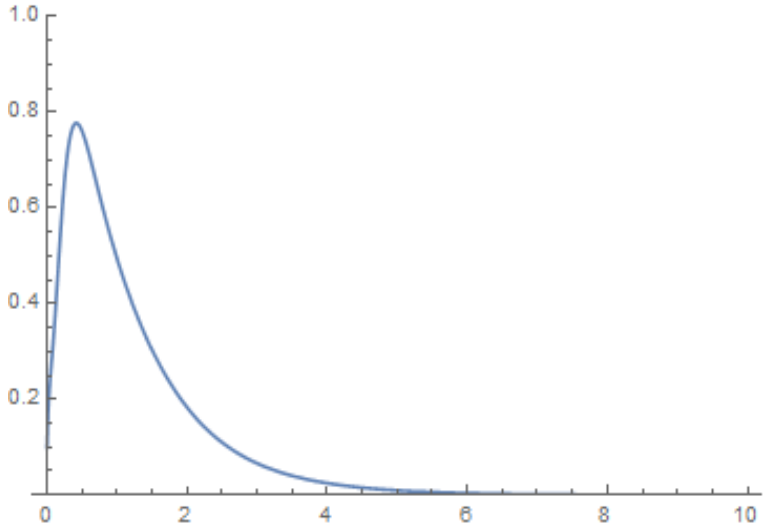

Figure 3.6: Required graph of Volterra's Population model for $n = 10$.

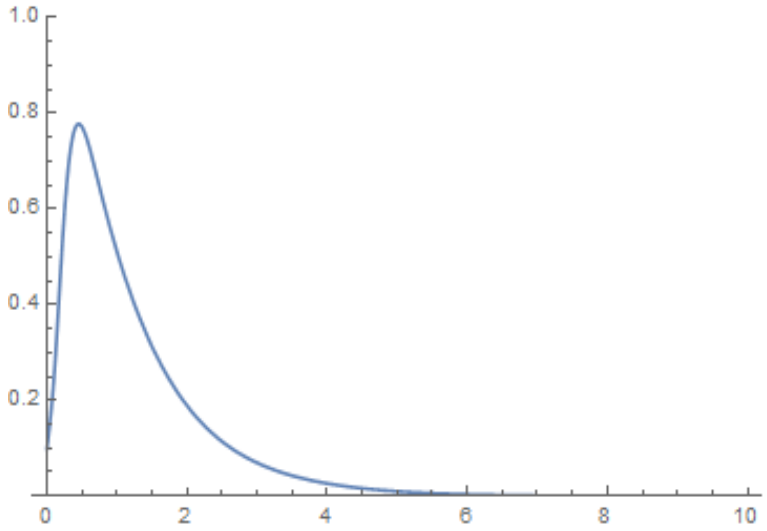

Figure 3.7: Required graph of Volterra's Population model for $n = 12$.

3.5 Laguerre function

We define the Laguerre function, $L(n,t)$, $n = 0, 1, 2, ...$ as follows,

$$L(n,t) = e^{-t/2} L_n(t), \qquad 0 \leq t < \infty \tag{3.12}$$

where $L_n(t)$ is the Laguerre polynomial of order n. The Laguerre function is simply orthogonal on $[0, \infty)$ and has the property that as t tends to infinity, $L(n,t)$ approaches zero for every n. This property is helpful to determine a problem, by spectral method, where the solution asymptotically decays to zero, as in the present case.

3.6 Laguerre polynomial and its properties

The mathematician, Edmond Laguerre (1834-1886), was the first who introduced the second order linear differential equation which is called the Laguerre equation.

$$xy'' + (1-x)y' + ny = 0. \tag{3.13}$$

An obtained solution of the equation:

$$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n! t^k}{(k!)^2 (n-k)!}, \tag{3.14}$$

where $n = 0, 1, 2, ...$ and $0 \leq t < \infty$
Also the orthogonality relation is defined by:

$$\int_0^t e^{-t} L_n(t) L_m(t)\, dt = \begin{cases} 0 \text{ for } m \neq n \\ 1 \text{ for } m = n \end{cases} \tag{3.15}$$

3.7 Collocation Algorithm

To solve the equation $Lu(x) = f(x)$ (where L is the differential operator), we consider the various points:

1. Construct the approximate solution.

$$f(t) = \sum_{i=0}^{n} a_i L(i, t), \tag{3.16}$$

2. Insert the series into the given equation.

3. To obtain the residual function.

$$Res(t, a_0, a_1, a_2, ...) = Lu_n(x) - f(x). \tag{3.17}$$

Now we have $N+1$ unknown and $N+1$ equations, that can be used to find the constants a_n.

4. Now to choose $N+1$ collocation points x_i, $i = 0, 1, 2, ..., N$, then substitute in above step to find the $N+1$ linearly independent equations.

5. Solve this system of equations by Laguerre polynomial and to find the constants a_n.

3.8 Solving Volterra equation by collocation method

In this program , we have to apply Laguerre polynomial in the Volterra Integro-differential equation with the help of Mathematica. Using Mathematica, we have to check the behaviour of the graph and mathematical structure of $u(t)$. The graph reaches the maximum points then followed by the slow exponential decay where $u(t) \to 0 \quad \text{and} \quad t \to \infty$.

Here we consider the Initial condition of $u(0) = 0.1$ and k=0.1. For finding the exact value of u_{max}, we need to obtain a formula that comes from the paper

of TeBeest[4].

$$u_{max} = 1 + \kappa \ln(\frac{\kappa}{1 + \kappa - u_0}), \tag{3.18}$$

Now we can easily calculate the value of critical point and Approximate value of u_{max} that is shown in Table [3.2]. By applying the spectral method in Volterra's equation we have:

1. Consider the approximate solution.

$$u(t) = \sum_{k=0}^{n} a_k L(k,t), \tag{3.19}$$

2. Put the approximate solution into the Volterra integro-differential equation.

$$k\frac{du}{dt} = u - u^2 - u\int_0^t u(x)\,dx, \tag{3.20}$$

for $u(0) = u_0 = 0.1 > 0$ and $k = 0.1$. so the Eq.(3.20) will become:

$$\sum_{k=0}^{n} a_k L'(k,t) = 10\sum_{k=0}^{n} a_k L(k,t) - 10(\sum_{k=0}^{n} a_k L(k,t))^2 - 10\sum_{k=0}^{n} a_k L(k,t)\int_0^t \sum_{k=0}^{n} a_k L(k,x)\,dx,$$

3. Construct the Residual function:

$$Res(t) = \sum_{k=0}^{n} a_k L'(k,t) - 10 \sum_{k=0}^{n} a_k L(k,t) + 10 (\sum_{k=0}^{n} a_k L(k,t))^2 + 10 \sum_{k=0}^{n} a_k L(k,t) \int_0^t \sum_{k=0}^{n} a_k L(k,x)\, dx.$$

Now we have $n+1$ unknowns $\{a_n\}$.To find the unknown coefficients $\{a_n\}$, we need to find $n+1$ equations.

4. Find n collocation points by determine the zeros of $L_{n+1}(t)$.

5. Substitute the collocation points in $Res(t_j, a_0, a_1, a_2, ...$ $0, 1, 2, ..., n-1$ and $u_n(0) = 0.1$ for obtaining a system of $n+1$ equations.

6. Solve this system of equations for obtaining a_n.

Table 3.2: Values of u_{max} for solving IDE.

n	Critical t	Approx. u_{max}	Exact u_{max}
4	0.931667	0.497345	0.76974144907
9	0.596209	0.659436	0.76974144907
11	0.533264	0.695352	0.76974144907
13	0.486188	0.722821	0.76974144907
14	0.466777	0.734172	0.76974144907
22	0.368095	0.788988	0.76974144907
24	0.3533	0.7695944	0.76974144907

For different values of n, we have a graph:

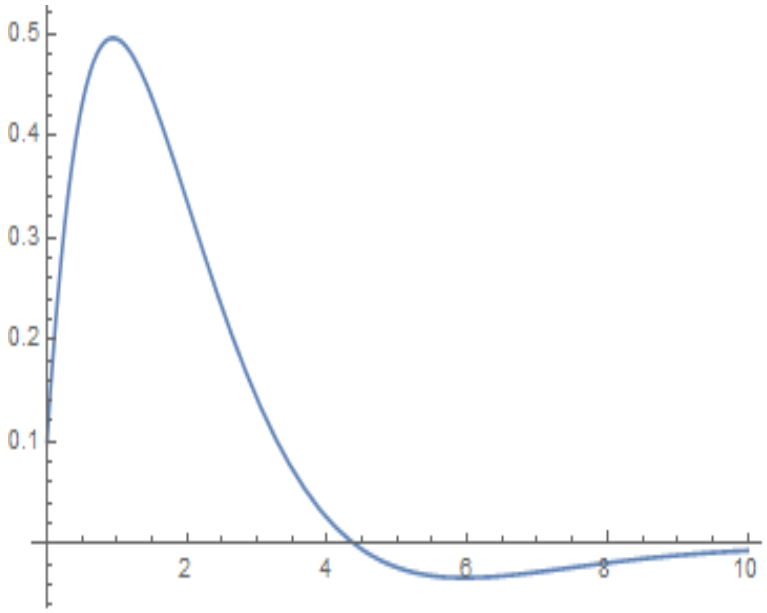

Figure 3.8: Graph obtained for solving the Volterra's integro-differential equation by using Laguerre polynomial for $n = 4$.

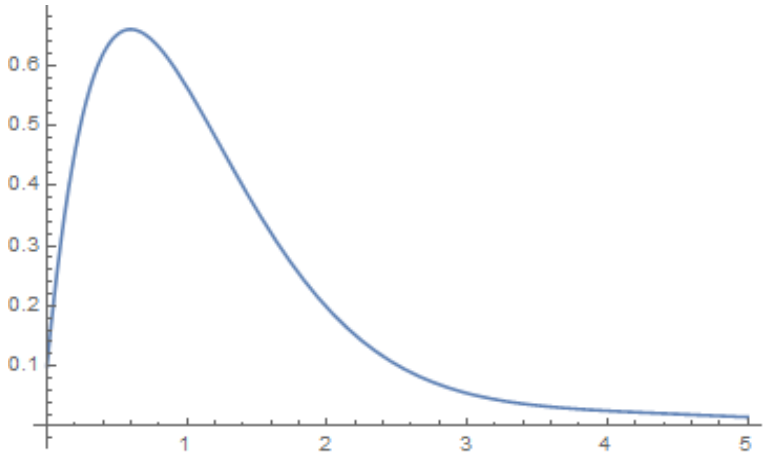

Figure 3.9: Graph obtained for solving the Volterra's integro-differential equation by using Laguerre polynomial for $n = 9$.

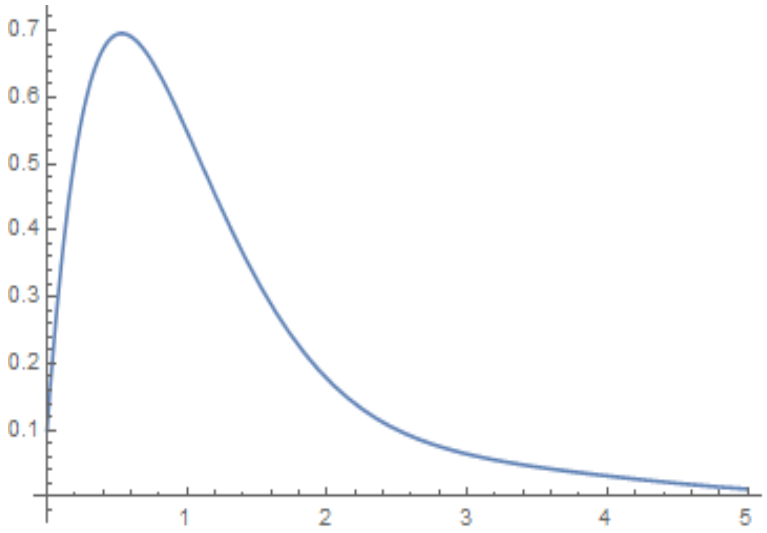

Figure 3.10: Graph obtained for solving the Volterra's integro-differential equation by using laguerre polynomial for $n = 11$.

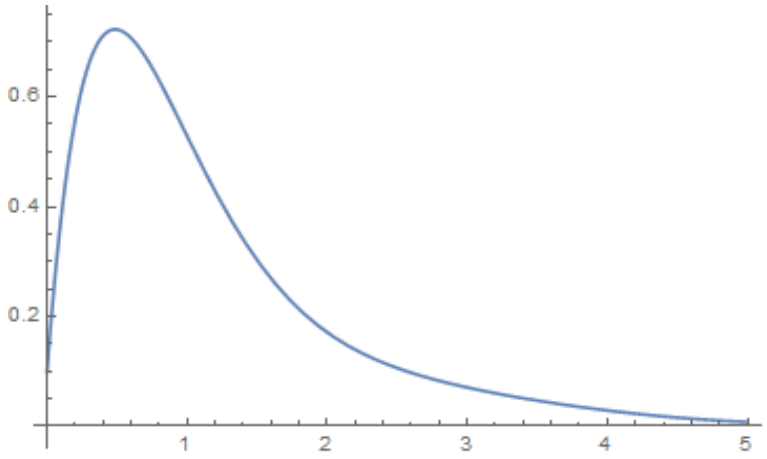

Figure 3.11: Graph obtained for solving the Volterra's integro-differential equation by using laguerre polynomial for $n = 13$.

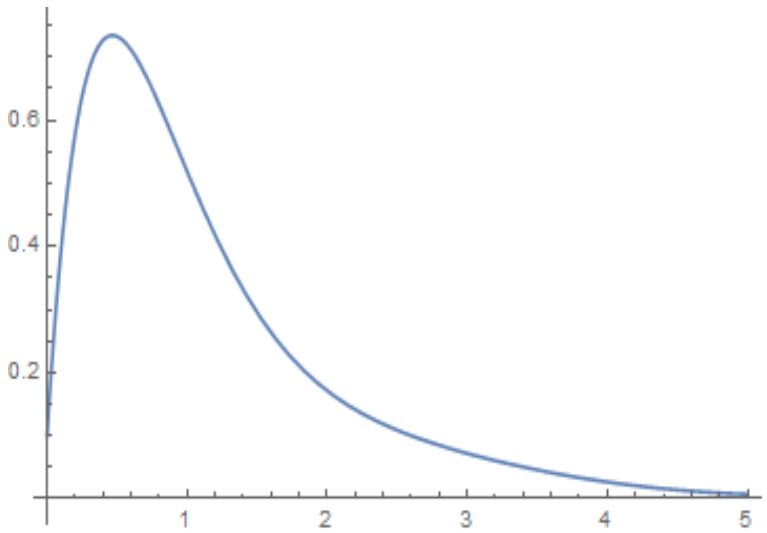

Figure 3.12: Graph obtained for solving the Volterra's integro-differential equation by using Laguerre polynomial for $n = 14$.

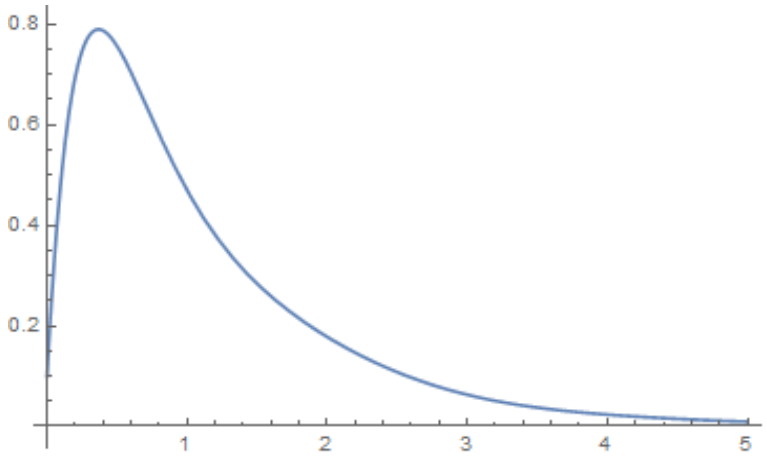

Figure 3.13: Graph obtained for solving the Volterra's integro-differential equation by using Laguerre polynomial for $n = 22$.

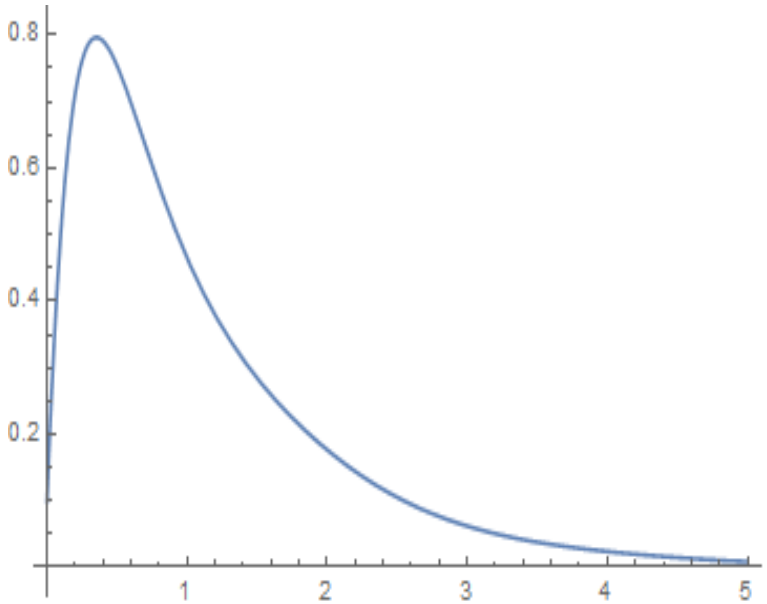

Figure 3.14: Graph obtained for solving the Volterra's integro-differential equation by using Laguerre polynomial for $n = 24$.

3.9 Conclusion

The results of the present work examine the mathematical structure of $u(t)$. We compare the numerical results of all the methods that was easily calculated by using Mathematica. The main goal of this study has been to construct an approximation for the solution of nonlinear Volterra's integro-differential equation in a semi-infinite interval. In this chapter, we solve the problems on semi-infinite domain without any domain truncation and shifting the problem to a finite domain by using collocation method [15]. By applying the Rational Chebyshev and Laguerre polynomial function into the Volterra integro-differential equation, the graph obtained from the results shows that both of the presented approaches have good reliability and efficiency.

We also conclude that the approximate value and the exact value of u_{max} for $k = 0.1$ is approximately same for two or three decimal places by using these method. When the value of k is small, the population is relatively insensitive to toxins and when k is large, the population of this type is sensitive to toxins. Hence, the order of these collocation approaches can be in-

creased by finding the further values of $u(t)$.

Chapter 4

Analysis and Solution of the Problem

4.1 Volterra's Population Equation

The growth of a species within a closed system is modeled by the following Volterra's equation

$$\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(s)ds, \quad u(0) = u_0, \qquad (4.1)$$

where $\kappa = c/(ab)$ is a dimensionless parameter with a, b, c respectively represent the birth rate, the intraspace competition coefficient and the toxicity coefficient. Each of these parameters is positive. Also t is a scaled time and $u(t)$ is a population variable (4.1).

It is easy to see that if, $u_0 > 0$, then $u(t) > 0$ for $t > 0$, because otherwise, at a point where $u(t)$ changes sign, the derivative would be negative while the right hand side of (4.1) would vanish at this point.

Small [3] used time scaling to find the following analytical solution valid for $\kappa << 1, u_0 < 1$,.

$$u_s(t) = e^{-t} - 1 + \frac{1}{1 + (1/u_0 - 1)e^{-t/\kappa}} \tag{4.2}$$

Small's solution, for $\kappa = 0.07, u_0 = 0.2$, is presented graphically in Fig.4.1. Although it was derived for $\kappa << 1$, the figure is an elegant representation of the qualitative features of the population function, $u(t)$, even for larger κ. We define a *global accuracy index* of an approximate solution $u_a(t)$ as

$$\alpha_a = \frac{\int_0^\infty u_a(t)dt}{\int_0^\infty u(t)dt}. \tag{4.3}$$

We shall find later, [see Eq. (4.14)], that when $\kappa << 1$, $\int_0^\infty u(t)dt = \kappa + 1$. This index for Small's solution (4.2), shown in Fig.4.1, is found to be 0.8293, which indicates a global error of a little more than 17 percent. However for $\kappa = 0.01, u_0 = 0.2$, this error reduces to less than 3 percent.

Wazwaz [1] used Adomian Decomposition Method to find a solution in the form of a power series. However, the series converges within a finite interval and a function which vanishes at infinity cannot be adequately represented, on the whole domain, by a trun-

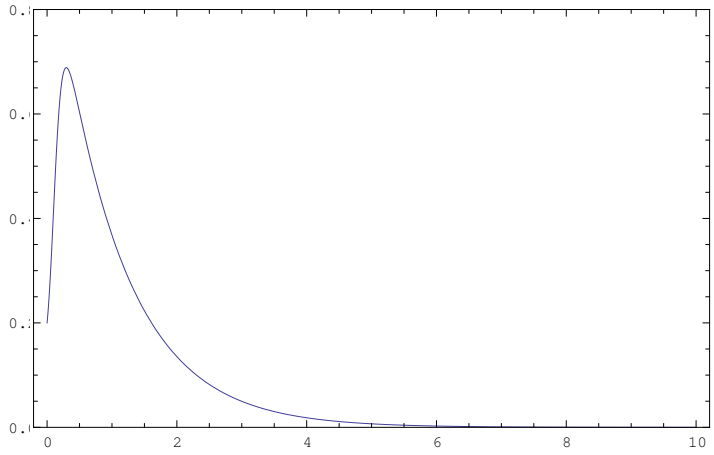

Figure 4.1: Small's solution for $\kappa = 0.07, u_0 = 0.2$

cated power series. A useful technique to overcome this problem is to represent the solution in the form of a rational function. Wazwaz [1] followed this path by resorting to Padé approximants. Parand et al. [21] employed sinc as well as *rational* Legendre functions in their solutions by the spectral methods. These solutions successfully mimic the qualitative features but suffer from lack of precision, especially for large times. In the present work, we make use of a result of TeBeest [4], to precisely determine the integral,

$$I = \int_0^\infty u(s)ds, \tag{4.4}$$

which provides us with an exact parameter to be used as a yardstick to measure accuracy of an approximate

solution. It also leads to an accurate estimate of a decay constant for large time. The solution is obtained in a few simple steps and describes the population function $u(t)$ on the entire domain. For large times, the population is found to decay exponentially.

4.2 Padé Approximation

Let $f : \mathbb{R} \to \mathbb{R}, f \in C^{(r+s)}$, and R, S be respectively polynomials of degree r and s. If

$$f(x) - \frac{R(x)}{S(x)} = O(x^{r+s+1}), \tag{4.5}$$

then the rational function $R(x)/S(x)$ is said to be a Padé Approximant of order r, s for $f(x)$. We denote it by $P_{[r,s]}f(x)$. Polynomials $R(x)$ and $S(x)$ are found by equating coefficients of powers of $x^k, k = 0, 1, .., r + s$, to zero in $S(x)f(x) - R(x)$ which leads to a system of $r + s$ linear equations in as many unknowns.

Example Let $f(x) = \ln(1 + x)$. The function has a Maclaurin series representation

$$\ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \quad \frac{x^k}{k}, \quad -1 < x \leq 1. \tag{4.6}$$

As an illustration, we shall seek $P_{[2,2]} \ln(1+x)$. Assume

$$\ln(1+x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2} + O(x^5), \qquad (4.7)$$

which leads to

$$(x - x^2/2 + x^3/3 - x^4/4)(1 + b_1 x + b^2 x^2) - (a_0 + a_1 x + a_2 x^2) = 0 \qquad (4.8)$$

Equating coefficient of each power $x^k,\ k = 0, 1, .., 4$ in (4.8), we find the following system for the parameters, a_0, a_1, a_2, b_1, b_2,

$$\begin{aligned} 0 &= a_0, \\ 1 &= a_1, \\ b_1 - \frac{1}{2} &= a_2, \\ b_2 - \frac{b_1}{2} - \frac{1}{3} &= 0, \\ -\frac{b_2}{2} + \frac{b_1}{3} - \frac{1}{4} &= 0. \end{aligned}$$

Solving the system, staring from the lower end, we find,

$$a_0 = 0, a_1 = 1, a_2 = \tfrac{1}{2}, b_1 = 1, b_2 = \tfrac{1}{6}.$$

Thus

$$P_{[2,2]} \ln(1+x) = \frac{x + \frac{x^2}{2}}{1 + x + \frac{x^2}{6}}. \qquad (4.9)$$

It is remarkable that, more often than not, a Padé approximant is nearer the truth compared with the polynomial used for its derivation. For example, the series in (4.6) diverges for $x = 2$, but from (4.9), we find

$$P_{[2,2]} \ln(1+x) \mid_{x=2} = 1.091,$$

which differs from $\ln 3 = 1.099$ by less than one percent.

4.3 Qualitative Analysis of the Problem

Let $u_0 < 1$. It is clear that the population $u(t)$ will initially increase. It will do so until $t = t_c$, to a level less than unity, where the derivative vanishes. It will have a local maximum at t_c beyond which it starts to decrease. A question arises as to whether it might have a local minimum at a point, $t_1 > t_c$, and start rising again. The answer to this point is in the negative because in that case,

$$u'(t_1) = 0, \quad u''(t_1) = -(1/\kappa)u^2(t_1) < 0,$$

therefore a local minimum cannot occur at t_1. Hence $u(t)$ decreases for all $t > t_c$. On the other hand if $u_0 \geq 1$ then $u(t)$ starts to decrease immediately and, because of the above observation, continues to do so subsequently.
Since $u(t)$ is a decreasing function of time and is bounded below by zero,

$$\lim_{t\to\infty} u(t) = l \geq 0.$$

However l must vanish, otherwise the integral, $\int_0^\infty u(t)dt$, will diverge and the limiting process in the Volterra equation (4.1) gives

$$\begin{aligned} \kappa \lim_{t\to\infty} \frac{du}{dt} &= = l - l^2 - l.\infty, \\ &= -\infty, \end{aligned}$$

which contradicts the fact that $u(t) > 0$ for $t \geq 0$.
Divide the Volterra equation (4.1) with u and consider the limit when $t \to \infty$. We have

$$\kappa \lim_{t\to\infty} \frac{1}{u}\frac{du}{dt} = 1 - I.$$

where I was defined in (4.,4). Since $u(t) > 0$ and $du/dt < 0$, the limit on the left side will be less than or equal to zero. Hence

$$I \geq 1. \tag{4.10}$$

TeBeest [4], defined the *accumulated toxicity variable,*

$$y(t) = \int_0^t u(s)ds.$$

The chain rule, applied to $u(t)$ leads to

$$\begin{aligned} \frac{du}{dt} &= \frac{du}{dy}\ \frac{dy}{dt}, \\ &= u\frac{du}{dy}. \end{aligned} \tag{4.11}$$

Volterra's problem, in the new independent variable, y, reduces to the *linear* differential equation,

$$\kappa\frac{du}{dy} = 1 - u - y, \qquad u(0) = u_0,$$

to which TeBeest found the solution

$$u(y) = (1 + \kappa - y) - (1 + \kappa - u_0)e^{-y/\kappa}, \tag{4.12}$$

and he noted that for, $u_0 < 1$, the population reaches its maximum value of u_{max} at y_c where

$$\begin{aligned} u_{max} &= 1 + \kappa \ln(\frac{\kappa}{1 + \kappa - u_0}), \\ y_c &= \kappa \ln(\frac{1 + \kappa - u_0}{\kappa}). \end{aligned}$$

Since du/dy and du/dt vanish together [see (4.11)], maximum value of u as a function of t will also equal

u_{max}. The curious fact $y_c = 1 - u_{max}$ is explained by noting that when $u(t) = u_{max}$, Eq. (4.1) leads to $0 = 1 - u_{max} - y_c$.

Since $y(t)$ is an increasing function of t, there are only two possibilities as $t \to \infty$, namely,

1. y diverges to ∞,

2. y approaches a finite limit.

The first option must be excluded because, if this was to happen, the left side of Eq.(4.12) would be finite while the right side approaches $-\infty$. Let

$$\lim_{t\to\infty} y(t) = I(u_0, \kappa),$$

a finite positive number. We shall omit the arguments, κ and u_0 and just write I to denote the limiting accumulated toxicity. An important feature of Eq.(4.12) is that as t becomes large, y approaches I and

$$\lim_{t\to\infty} u(y(t)) = 0.$$

Thus for given u_0, κ, I can be precisely determined by solving the equation,

$$1 + \kappa - I = (1 + \kappa - u_0)e^{-I/\kappa}, \tag{4.13}$$

which follows from (4.12) in the limit $t \to \infty$.

It follows from (4.13) that,

1. if $u_0 < 1 + \kappa, \quad I < 1 + \kappa,$
2. if $u_0 = 1 + \kappa, \quad I = 1 + \kappa,$
3. if $u_0 > 1 + \kappa, \quad I > 1 + \kappa,$

and if $\kappa << 1$, then

$$I = \kappa + 1. \tag{4.14}$$

Combining (4.10) and (4.14), we have $1 \leq I \leq 1 + \kappa$. It is remarkable that, in case of $u_0 = 1 + \kappa$, the function, $u(t) = (1 + \kappa)e^{-t}$, is the *exact* solution of the problem.

4.3.1 Behavior for Large Time

We can write

$$\begin{aligned} \int_0^t u(s)ds &= \int_0^\infty u(t)dt - \int_t^\infty u(s)ds, \\ &= I - \int_t^\infty u(s)ds. \end{aligned} \tag{4.15}$$

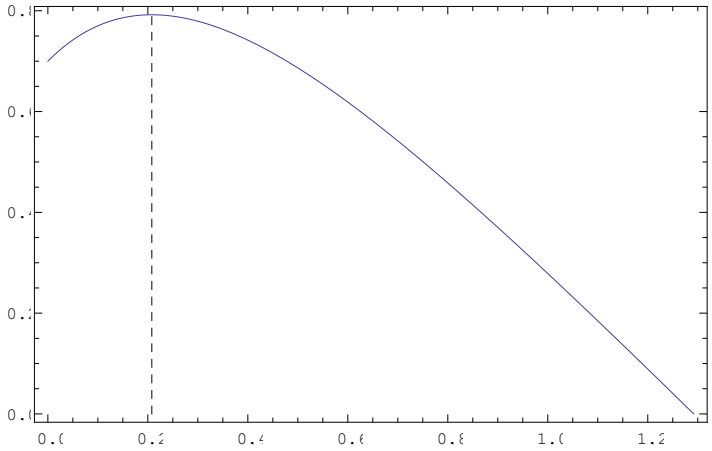

Figure 4.2: Population u as a function of the accumulated toxicity y. The dashed line indicates position of u_{max}.

Substitution of (4.15) in (4.1) leads to

$$\kappa \frac{du}{dt} = u - Iu - \{u^2 + u \int_t^\infty u(s)ds\}.$$

For large t, terms enclosed by braces can be dropped and we conclude that, asymptotically, the solution behaves as

$$u(t) = Ae^{-bt} \tag{4.16}$$

where A is a constant and $b = (I-1)/\kappa$.

4.4 Solution of the Problem

As an illustration, we shall solve the problem, in some detail, for the pair $\kappa = 0.3, u_0 = 0.7$. Later

we shall also briefly describe solution for the case $\kappa = 0.4, u_0 = 1.3$. The calculations are completed in five steps.

1. Solve Eq.(4.13) for I. Newton-Raphson method quickly produces the result, $I = 1.29191$, hence $b = (I-1)/\kappa = 0.973035$.

2. Assume a solution, valid for short times, in the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Substitute in Eq.(4.1) and determine the coefficients recursively. We find, $a_0 = u_0, a_1 = (a_0 - a_0^2)/\kappa$ and $a_n, n \geq 2$ are given by the following recurrence relation,

$$\kappa a_{n+1} = a_n - \sum_{p=0}^{n} a_p a_{n-p} - \sum_{p=0}^{n-1} \frac{a_p a_{n-p-1}}{p+1}, \; n = 1, 2, ...$$

For the example at hand, the first seven coefficients are listed below.

$a_0 = 0.7, a_1 = 0.7, a_2 = -1.28333, a_3 = -0.790741, a_4 = 2.55478, a_5 = -0.0808745, a_6 = -4.36321$

3. Assume a solution, valid for all $t > 0$, in the form

$$u(t) = e^{-bt} \sum_{n=0}^{\infty} c_n t^n, \tag{4.17}$$

Evaluate $c_n, n = 0, 1, 2, ..,$ by equating the above series with the one in the last step and comparing powers of t. Thus

$$\begin{aligned} c_0 + c_1 t + c_2 t^2 + ... &= e^{bt}(a_0 + a_1 t + a_2 t^2 + ...) \\ &= (1 + bt + b^2/2t^2 + ...)(a_0 + a_1 t + \\ &= a_0 + (a_0 b + a_1)t + (a_0 b^2/2 + a_1 b \end{aligned}$$

Hence $c_0 = a_0, c_1 = a_0 b + a_1, ...,$ and in general

$$c_n = \textstyle\sum_{p=0}^{n} a_p b^{n-p}/(n-p)!$$

For our example, first seven coefficients are as follows,

$c_0 = 0.7, c_1 = 1.38112, c_2 = -0.27083, c_3 = -1.60061, c_4 = 1.31146, c_5 = 1.86487, c_6 = -3.39591.$

4. In the final step, we truncate the series, $\sum_{n=0}^{\infty} c_n t^n$, say, after $2k$ terms. To find an expression which is valid for all t *and* which for large t behaves

like Ae^{-bt}, all we need is to replace the polynomial $\sum_{n=0}^{2k} c_n t^n$ by its equivalent Padé approximant $P[k,k]$. The desired approximate solution representing the solution for $0 \leq t < \infty$ is $u_k(t) = P[k,k]e^{-bt}$.

Let $P_{[k,k]}f(t)$ denote the Padé $[k,k]$ approximant for the function f. Also let $u_k(t)$ denote the $k-th$ approximation to the population function and let,

$$g(t) = \sum_{i=0}^{2k} c_i t^i,$$

then the approximate solution can be expressed, in a compact form, as

$$u_k(t) = e^{-bt} P_{[k,k]}(e^{bt} g(t)). \tag{4.18}$$

It is convenient to use the Mathematica command PadeApproximant to efficiently find the part $P_{[k,k]}(e^{bt}g(t))$ on the right side of (4.18).

4.5 Padé Approximants

Using the theory described in the last section, we find $u_4(t)$ in the following form

$$u_4(t) = e^{-bt}\frac{0.7 + 2.29833t + 3.27123t^2 + 2.40193t^3 + 0.773}{1.0 + 1.31029t + 2.47484t^2 + 1.34192t^3 + 0.536}$$

Fig.4.3 depicts the curve representing the above solution together with, $u_{10}(t) = e^{-bt}\frac{R_{10}(t)}{S_{10}(t)}$. The polynomials R, S are found to be

$R_{10}(t) = 0.7 + 3.90012t + 11.5718t^2 + 22.9791t^3 +$
$33.0982t^4 + 35.6889t^5 + 28.9452t^6 + 17.3468t^7 +$
$7.32891t^8 + 1.96809t^9 + 0.255022t^{10}$
$S_{10}(t) = 1.0 + 3.59857t + 9.818t^2 + 17.1349t^3 +$
$23.629t^4 + 24.0363t^5 + 19.1182t^6 + 11.1119t^7 +$
$4.67529t^8 + 1.23225t^9 + 0.166902t^{10}$.

Curves representing the two solutions are seen to overlap. This is an indication of the robustness of the scheme for finding the approximate solution. A measure of accuracy of an approximation, the global accuracy index, α_k was defined as,

$$\alpha_k = \frac{I_k}{I}, \quad k = 3, 4, ..,$$

Table 4.1: Accuracy Index of Approximate Solutions
$\kappa = 0.3, u_0 = 0.7,\ I = 1.29191055$

k	I_k	α_k
3	1.30843	1.01279
4	1.28904	0.99778
6	1.29158	0.99974
8	1.29182	0.99993
10	1.29186	0.999960

where I_k and I respectively denote the total accumulated toxicity associated with $u_k(t)$ and the exact solution $u(t)$ i.e.

$$I_k = \int_0^\infty u_k(t)dt, \qquad I = \int_0^\infty u(t)dt.$$

In Table 4.1, values of this index for some representative approximate solutions are listed. It is remarkable that even for, $u_3(t)$, this index differs from unity by a little more than one percent.

In Table 4.2, values of $u_4(t)$, and $u_{10}(t)$ are tabulated. It was noted above that, for large t, the solution should behave like Ae^{-bt}. Considering $t = 15$ to be sufficiently large, we find $u_{10}(15) = 7.11495 \times (10)^{-7}$, which leads to $A = 1.55213$. Values of the expression Ae^{-bt} are also listed in the Table. It appears that even for $t = 2$ the exponential approximation suffers from an error of about 4 parts in 200.

Table 4.2: Comparison of approximate solutions

$\kappa = 0.3, u_0 = 0.7,\ A = 1.55213, b = 0.973035$

t	$u_4(t)$	$u_{10}(t)$	Ae^{-bt}
0	0.7	0.7	1.55213
2	0.217294	0.218161	0.221697
4	0.0313713	0.0318674	0.031666
6	0.0044455	0.0045577	0.004523
8	0.0006299	0.0006499	0.000646
10	0.00008940	0.00009266	0.00009228
12	0.0000127	0.0000132	0.00001318
14	1.80713×10^{-6}	1.8839×10^{-6}	1.8826×10^{-6}
16	2.5726×10^{-7}	2.6872×10^{-7}	2.68899×10^{-7}

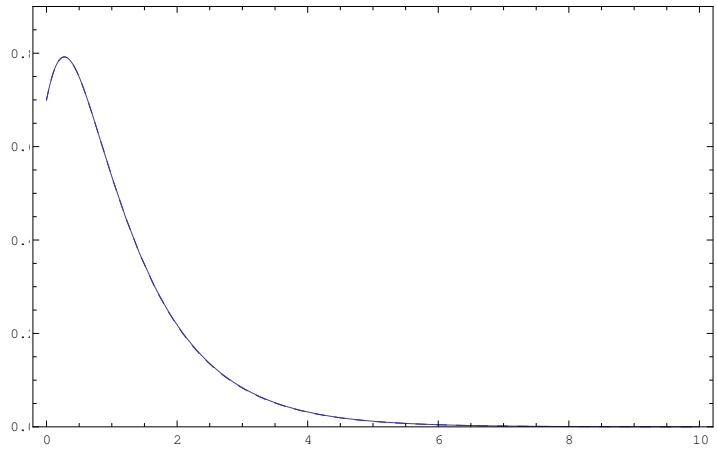

Figure 4.3: Approximate solutions $u_4(t)$ (dashed) and $u_{10}(t)$ (solid).

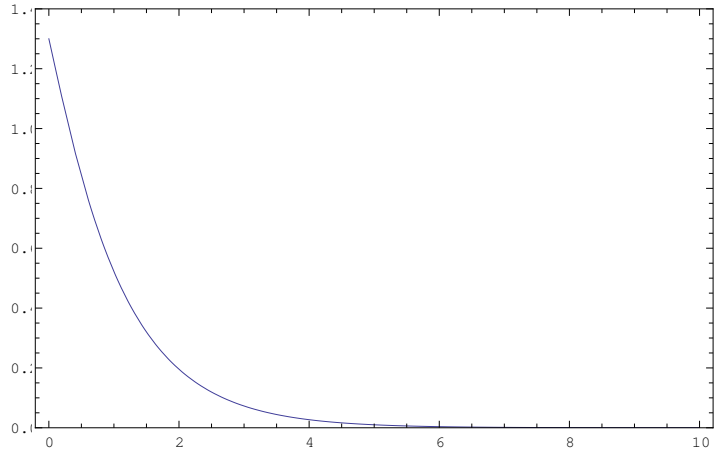

Figure 4.4: With $u_0 = 1.3, \kappa = 0.4$, the approximate solution $u_4(t)$.

4.6 Exponentially Decaying Solution

Now we consider an example with $\kappa = 0.4, u_0 = 1.3$. In this case, total accumulated toxicity, $I = \int_0^\infty u(t)dt$, found by solving Eq. (4.13) is found to be 1.3969572, which leads to the decay constant $b = 0.99239$. An approximate solution $u_4(t)$ is found in the following form

$$u_4(t) = e^{-bt}\frac{1.3 + 3.97751t + 4.76402t^2 + 2.63096t^3 + 0.559}{1.0 + 2.81723t + 3.35862t^2 + 1.82588t^3 + 0.393} \tag{4.19}$$

This solution is presented in Fig.4.4. The global accuracy index for the above solution is found to be 1.00001, which amounts to it being exact for most purposes. Using $u_4(12)$ as a reference for *large* t, we

Table 4.3: Comparison of approximate solutions

$\kappa = 0.4, u_0 = 1.3,\ A = 1.42803, b = 0.99239$

t	$u_4(t)$	Ae^{-bt}
0	1.3	1.42803
1	0.522096	0.529354
2	0.195599	0.196226
4	0.0269797	0.026634
6	0.003708	0.003705
8	0.0005094	0.0005091
10	0.00006998	0.00006996
12	9.61275×10^{-6}	9.61275×10^{-6}
14	1.3205×10^{-6}	1.3209×10^{-6}
16	1.8141×10^{-7}	1.8150×10^{-7}

find $A = 1.42803$. In Table 4.3, values of $u_4(t),\ Ae^{-bt}$ are listed. It is remarkable that for $t \geq 2$, the error committed by the expression Ae^{-bt} is less than 7 parts in 2000.

4.7 Comments on a Recent Paper

In a recent paper [22], the authors have solved the Volterra's population problem by the spectral method. They have used the modified Bessel functions of the first kind as the basis and the zeros of the rational Chebyshev polynomials as collocation points. They have claimed to have obtained results possessing high

accuracy. We shall show that their claims are false since, by using their technique, it is *impossible* to obtain any result with reasonable accuracy.

1. In their solution of the problem, by an application of the spectral method, Parand et al. [22] assume the population function $u(t)$ of the form

$$u(t) = \sum_{k=0}^{n} a_k I_k(x), \tag{4.20}$$

where $I_k(x)$ denotes the modified Bessel function of the first kind of order k. The function $I_k(x)$ has the property,

$$\lim_{x\to\infty} I_k(x) = \infty, \quad k > 0.$$

It is obvious, that the assumed solution of the problem will approach ∞ or $-\infty$ depending on the sign of the last term, a_n, in the summation (7). This violates the expected behavior of $u(t)$ which must decay to zero as t tends to infinity. Strangely enough, the authors, in Fig.4.1, of their paper [3] claim that their solution does follow the expected theoretical behavior. They also make apparently false claim, in Table 4.1, that their

curves attain the maximum value almost exactly at the respective points as predicted by the theory.

2. We have discussed in Section 2, above that the integral of the population function $I = \int_0^\infty u(t)dt$, is finite and for given values of κ and u_0, can be calculated precisely by using Eq.(4.6). Since the solution of Parand et al. [22] fails to have the correct asymptotic behavior, its integral will diverge and the solution will lack one of the essential features predicted by theory. This makes their claims of accuracy, *even for large* t, extremely doubtful.

3. For the spectral method to yield good results, the choice of a suitable basis and collocation points must be made so as to minimize the error of approximation. If the solution is assumed to be of the form, $u(t) = \sum_{k=0}^{n} \varphi_k(t)$, where $\{\varphi_k(t)\}_{k=0}^{\infty}$ is a suitable basis, then the error will be minimized only if the collocation points are chosen to the zeros of the function, $\varphi_{n+1}(t)$. Since the authors failed to follow this path, their results are not likely to possess the accuracy depicted by the Tables in [4.3].

4.8 Conclusion

In view of the above remarks, it is evident that claims made by Parand et al. in [22] are not supported mathematically by the algorithm presented by them. We suggest that they should consider a voluntary redaction of their paper.

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