The Qualitative Theory of Fractional Difference Equations

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(2021)

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A thesis submitted to the National University of Sciences and Technology, Islamabad

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in

Mathematics

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(2021)

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Abstract

Focus of this study is to cover up some gaps and further build up the theory of discrete fractional calculus. This dissertation starts with brief introduction and definitions to discrete fractional calculus. Two new definitions of generalized fractional difference operator are introduced namely Hilfer fractional difference operator and substantial fractional difference operator. A missing property in the literature for delta Laplace transform i.e. delta exponential shift is established. The delta Laplace transform is presented for the newly introduced Hilfer and substantial fractional differences. The double Laplace transform in a delta discrete setting is introduced, and its existence, uniqueness and basic properties are discussed. The delta double Laplace transform is presented for integer and non-integer order partial differences.

Another goal of this study is to establish the existence and UHR stability for various classes of fractional difference equations. Conditions are acquired for RL, Caputo, Hilfer and substantial-type fractional difference equations. Moreover we establish a technique to transforming arbitrary real order delta difference equations with impulses to corresponding summation equations. Existence results are built up for impulsive delta fractional difference equation with nonlocal initial condition and two-point and four-point boundary conditions. The conditions for existence and UHR stability of the solution to multi-point summation boundary value problem are established.

Acknowledgment

'In the name of Allah, the Most Gracious and the Most Merciful'

I thank Allah, the Omniscient for the completion of this Ph.D. dissertation.

I would like to express my gratitude to some personalities who have always been a continuous source of encouragement during the challenges of my whole education career. A special gratitude to my mentor Dr. Mujeeb ur Rehman, he has been an incredible advisor and a counselor for me. I consider it my privilege to have accomplished this thesis under his right guidance. I would also like to thank my respected teachers and Guidance and Examination Committee (GEC) members, Dr. Matloob Anwar, Dr. Asif farooq and Dr. Umer Saeed for their useful discussion, comments and true guidance.

I would like to thank my research colleagues at NUST; Adnan Khaliq, Arjumand Seemab, Rabia Ilyas Butt, Shazia Sadiq, Muhammad Ismail, Zain ul Abideen. I would also like to thank my professional colleagues at UET; Syed Zulqarnain Haider, Dr. Syed Muhammad Abdul Rehman Shah, Ms. Mariam Batool, Dr. Fareeha Zaheer, Dr. Muhammad Jawad, Dr. Zaffer Elahi, Engr. Syed Turab Haider, Dr. Zahid Suleman Butt, Dr. Zaheer Ahmad, Dr. Zahid Rasid. Dr. Muhammad Tariq, Mr. Irfan Ali Naqvi, Dr. Muhammad Sultan, Dr. Azeem Shahzad and to all those who encouraged and facilitated me to see these happy moments.

Words are not enough to express the gratefulness of my parent's and efforts of sibling which made me what I am today.

Syed Sabyel Haider

Dedicated to Hakim-e-waqt

Contents

List of acronym

BC	boundary condition
BVP	boundary value problem
DE	difference equation
DO	difference operator
EO	exponential order
EU	existence and uniqueness
FC	fractional calculus
FD	fractional difference
FDE	fractional difference equation
FDO	fractional difference operator
FPO	fixed point operator
FPT	fixed point theorem
GF	Green's function
IVP	initial value problem
ML	Mittag-Leffler
MPFBVP	multipoint fractional boundary value problem
RL	Riemann-Liouville
UH	Ulam-Hyers
UHR	Ulam-Hyers-Rassias

List of publications

1. Syed Sabyel Haider, Mujeeb Ur Rehman, Ulam-Hyers-Rassias stability and existence of solutions to nonlinear fractional difference equations with multipoint summation boundary condition, Acta Mathematica Scientia, 40B(2), (2020) 589–602. https://doi.org/10.1007/s10473-020-0219-1
2. Syed Sabyel Haider, Mujeeb Ur Rehman, On substantial fractional difference operator. Advances in Difference Equations 2020, 154 (2020). https://doi.org/10.1186/s13662-020-02594-0
3. Syed Sabyel Haider, Mujeeb Ur Rehman, Thabet Abdeljawad, On Hilfer fractional difference operator. Advances in Difference Equations 2020, 122 (2020). https://doi.org/10.1186/s13662-020-02576-2
4. Syed Sabyel Haider, Mujeeb Ur Rehman, Construction of fixed point operators for nonlinear difference equations of non integer order with impulses. Fractional Calculus and Applied Analysis, 23(3) (2020), 886–907 DOI: 10.1515/fca-2020-0045
5. Syed Sabyel Haider, Mujeeb Ur Rehman, Thabet Abdeljawad, A transformation method for delta partial difference equations on time scale. Mathematical Problems in Engineering, Volume 2020, Article ID 3902931, 14 pages, https://doi.org/10.1155/2020/3902931

Chapter 1

Introduction

The mathematical field that deals with operators of an arbitrary real or complex order is called fractional calculus(FC). Some working areas of FC are:

- Hahn h-FC.
- Quantum q-FC.
- (q, h)-FC.
- Stochastic FC.
- Variational FC.
- Fuzzy FC.
- Conformable FC.

Mathematical problems are often solved from recursive set of values by computing the approximation of differential operators or that a discrete analogue can be considered as difference equation(DE). Discrete models often arise for solving continuous models using numerical methods. Computers can work only with discrete data, so continuous equations must be discretized before they can be solved numerically. Forward or delta difference operator(DO) on isolated time scale has been considered for unit graininess. The fractional form of delta difference operator is an important breakthrough owing to non-locality property. On the basis of various motivations, delta fractional difference(FD) operators have been generalized and applied in various contexts.

1.1 History of fractional calculus

The fractional calculus(FC) originated parallel to classical calculus. The FC has lasting and storied past inside the broader mathematical discipline of analysis. Indeed, research into this area was initiated in 1695 in a scientific letter in which L'Hopital inquired Leibniz about the interpretation of a one-half derivative. While precise mathematical investigation of this and related concepts was not realized for almost two centuries. This simple question laid the initial foundation for the area of FC. At first, it seems that questions regarding fractional operators were largely academic being as they were separated effectively from any applicative interest. The "father of an arbitrary order calculus" viewed with respect was Abel (1823). Later, however as the FC matured, it become clear that the FC could be used effectively in a variety of modeling situations. Ostensibly, Leibniz could not possibly have envisioned the very bright and important future for the FC.

Euler (1738), Laplace (1812) and Fourier (1827) are among the many popular mathematicians who gradually make up the theory of FC. Many mathematicians with best mathematical minds are using their own specific documentation and frame work to came up with definitions that encouraged the development of theory of FC. Some other scientists who come up with the middle of the 20th century are Liouville (1832, 1837), Holmgren (1865), Grunwald (1867), Lȩtnikov(1868), Riemann (1876), Laurent (1884), Nȩkrassov (1888), Pincherele (1888), Krųg (1890), Heaviside

(1892), Weyl (1917), Hardy and Littlewood (1926), Marchaud (1927), Davis (1936), Erdelyi (1939), Kober (1940), Zygmund (1945), Riesz (1949), Feller (1952), Levy (1954), and Love (1967). The topic is novel and around 70 years ago, it has become an object of specialized conferences and treatises. In June 1974, Ross arranged the initiatory conference on the topic right after completion of his Ph.D., Oldham and Spanier (1974) put in black and white the first monograph. The texts in book form with title explicitly devoted to FC are Oldham and Spanier (1974), Rubin (1975), Mc Bride (1979), Nishimoto (1991), Samko et al. (1993), Miller and Rǫss (1993), Kiryakova (1994), Pǫdlubny (1999), Kilbs et al. (2006), Diethelm (2010). Furthermore, the treatise by Davis (1936), Gelfand and Shilov (1964), Erdelyi (1965), Caputo (1969), Babenko (1986), Gorenflo and Vassella (1991), Dzherbashian (1993, 1996), Zaslavsky (2005), Magin (2006), Sengul (2010), Holm (2011), Rehman (2011), Asif (2011), Goodrich (2012), Brackins (2014), Saeed (2015), Mctier (2016), Julia (2016), Areeba (2018) and Setniker (2019) contained a comprehensive examination of FC. Despite the fact that few significant fractional derivatives were brought forward, the Riemann-Liouville and Caputo derivatives got special attention of scientists. In the literature, one can discover many definitions of fractional derivatives, for instance, Caputo, Liouville, Hadamard, Erdélyi-Kober, Weyl, Grunwald-Letnikov, Katugampola, $\psi-$Riemann-Liouville, $\psi-$Caputo, substantial, $\psi-$Hilfer definitions and many other. Generally, most of these definitions are not alike with the exception of some those generalize some of previously introduced definitions. The most significant property of fractional derivatives in comparison with ordinary derivative is that the fractional derivatives are not local, this fact allow the execution of memory effects for different phenomenon.

1.2 Background of discrete fractional calculus and difference equations

The origin of calculus of finite differences is found from Brook Taylor (1717), rather it was Jacob Stirling, who found the theory (1730) and introduce the Delta "Δ" symbol for the difference operator, which is common in use currently. Finite differences trace their origins back to Burgi's (1592) and Isaac Newton work. The development on calculus of finite differences in the beginning of nineteenth century by Lacroix and remarkable work of George Boole, Narlund and Stefensen appeared later in nineteenth Century. The work of Thomson (1933) and Jordan (1939) related with classical approach to calculus of finite differences is appreciable. Lagrange and Laplace discussed the solution of partial DE in the area of probability theory and dynamics. Phillips and Wiener (1923) and Courant et al., (1928) considered the partial DE for study. By making use of the Laplace transform, Warschawski (1937) discussed ordinary linear non-homogeneous DE with constant coefficients. Mickens (2001) discussed linear, nonlinear DE and partial DE with applications in different areas especially in: physical, social and biological sciences as well as in economics and warfare.

Practically, each theory of mathematics exhibits its discrete analogue which shape the concept of theory understandable and ease the task of application. For the function having natural number domain the process of integer-order difference can be generalized to non-integer order differences in such a way that it is compatible with classic difference.

Mainly because of vast applications in different areas of FC and due to higher significance level in last twenty years investigation of fractional difference equations(FDE) has been attracting. Scientists from different areas believe that differences and sums of fractional order are appropriate for modeling the physical problems. On contemporary many studies with non-integer-order equations are found, especially in viscoelasticity, control theory, signal processing, rheology, hereditary and solid mechanics, where fractional order derivatives are used.

1.3 Literature survey

In the modern era, focus of mathematician is to construct ambiguity free unified mathematics. The calculus of finite differences is applied to both continuous and discrete functions. Application of FC can be found in [76, 131, 143, 155, 165, 195]. Criteria for being fractional derivative is presented in [143]. Two characteristics properties for not-being fractional operator are showcased by Tarasov [175, 176]. For modeling purpose, choice of fractional verses ordinary operator need not to be suitable always; this fact is demonstrated by Ortigueira [150] for two physical models. The order of the fractional equation is determined that better describes the experimental data, for different types of fractional operators by using the least squares fitting technique [22]. Fractional calculus in the continuous setting is developed in [17, 18, 21, 26, 42–48, 69, 72, 75, 80–82, 94, 95, 121, 128–132, 135, 138, 142, 146, 147, 167, 174, 183, 184, 189–198].

A glimpse of ordinary DE can be found in [15, 50]. The theory of FC for functions of the natural numbers, however, is far less developed. Any significant work was not appeared in this area before 1974 when Diaz and Osler [74] introduced a discrete non-integer order operator. In 1988, Gray and Zhang [105] introduced the type of fractional difference operator(FDO); they developed Leibniz formula, a limited composition and a version of a powerrule for differentiation. However, they dealt exclusively with the nabla (backward) difference operator and hence obtained results distinct from present one, where the delta (forward) difference operator is used exclusively.

A rigorous intrigue in FC of differences was exhibited by Atici and Eloe [29, 30, 32, 35], and they explored characteristics of falling function, a new power law for delta difference operators and the commutativity of sums and differences of arbitrary order [29]. They also presented advance composition formulas for sums and differences of non-integer-order in [30]. For non-integer-order, changes in the domain of function to sums and differences are an important aspect which got special attention by Atici and Eloe [29, 30] for a rigorous and correct dealing of the fractional composition formulas. Anastassiou [23, 24], Bastos et al. [39], Abdeljawad et al. [2, 4, 6, 7, 122], Cheng [62–64], and Sangul [168] made significant contribution in this area. For DE, Bohner and Peterson treated the dynamic equations on time scales in [50] and got surprisingly different results from continuous counterpart. Gronwall's, Hölder's, Jensen's and Opial's inequalities were presented on time scales and some application of inequalities in discrete FC can be found in [33, 82, 90].

Fractional calculus is extended to time scales in [40, 41], and discrete FC in the delta setting is developed in [34–38, 90–92, 96–101, 118, 119, 122]. A variety of results can be found in [4–7, 13, 27, 29–31, 50, 51, 83, 89, 103, 116, 117, 169, 188] which has helped to construct theory of the subject in the topics of discrete FC. The mathematical models of many real world phenomena can be represented by impulsive equations [112, 134, 137, 139, 141, 178] and references therein. Fractional delta DE with impulse have recently took attention, see [185]. The concept of existence of solutions for delta difference system of arbitrary order with boundary condition(BC) has been examined broadly by a lot of researcher, for instance we refer a few of them [16, 58–61, 68, 77, 83, 84, 89–93, 113, 136, 145, 152–154, 159, 161, 169–172, 177]. The majority of researchers considered FDE with an extensive variety of BC, meanwhile among others such as Reunsumrit, Kaewwisetkul and

Sitthiwirattham considered sum BC with two and three points in [126, 162, 169, 170]. However Goodrich [102] considered multi-point sum BC with growth. Coon and Bernstein [49, 66, 67] defined the double Laplace transforms (continuous) and investigated many properties of it. Debnath [71] modified the properties of double Laplace transforms (continuous) and used it to solve functional, integral and partial differential equations. Dhunde and Waghmare [73] discussed convergence, absolute convergence with applications of double Laplace transforms (continuous) to solve Volterra integro-partial differential equation. For applications of triple, quadruple and n-dimensional Laplace transforms (continuous), we refer the readers to [28, 70, 160]. Goodrich and Peterson [103] developed discrete delta Laplace transform analogous to Laplace transform discussed by Bohner and Peterson [50] in the continuous case to solve difference and summation equations with initial data by applying the delta Laplace transform. Delta Laplace transform has also been studied in [29, 50, 116]. However an important shifting property is missing in this setting. Only few simple cases have been addressed by implication of the definition of delta Laplace transform (see Theorem 2.10 and Theorem 2.11) in [103]). Our proposed shifting property is a modest attempt to fill that void. Bohner et al. [52] extend the properties of the Laplace transform to the delta Laplace transform on arbitrary time scales and discussed translation theorems and transforms of periodic functions. Compatible discrete time Laplace transforms with Laplace transforms was introduced in [149]. Savoye [166] highlighted the importance of discrete time problems and relationship of Z transform to Laplace transform on time scale. The qualitative analysis of delay partial DE was given by Zhang in [192].

Hilfer fractional order derivative was introduced in [115]. Furati et al. [86, 87] primarily studied the existence theory of Hilfer fractional differential equations and also explained the type parameter φ as interpolation between the Riemann-Liouville(RL) and the Caputos derivatives. It generate more types of stationary states and gives an extra degree of freedom on the initial conditions.

Some recent studies involving Hilfer fractional derivatives can be found in [65, 163, 173, 180–182]. Majority of the work in discrete FC developed as analogues of continuous FC. Extensive work on Hilfer fractional derivative and on its extensions has been done, i.e., Hilfer-Hadamard [1, 19, 127, 156], K-fractional Hilfer [78], Hilfer-Prabhakar [88], Hilfer-Katugampola [148] and ψ-Hilfer [12] fractional operators. Abdeljawad defined FD with different types of kernel having; discrete power law [2, 3], with discrete exponential and generalized Mittag-Leffler(ML) functions [8, 11], with discrete exponential and ML functions on generalized $h\mathbb{Z}$ time scale [9], and kernel containing product of both power law and exponential function in [10].

The majority of researchers considered fractional substantial and tempered derivative with an extensive variety of applications in physics, see for example [43, 55, 56, 85, 133, 171]. Chen and Deng discussed some useful composition properties of substantial fractional integral and derivative in [57]. It is believed that substantial FC and tempered FC are equivalent concepts. Cao et al. [54] presented the fact that the expression of fractional order tempered integral and derivative is similar to that of fractional order substantial integral and derivative respectively, but they are different in nature. However tempered derivative becomes a special case of substantial derivative for non-negative values of parameter σ. These operators arise from unassociated physical phenomenon. Mathematically, arbitrary order substantial calculus is defined on time and space but the tempered calculus is different from couple of time and space. However, arbitrary order tempered integral and derivative are mostly utilized in truncated exponential power law phenomenon.

1.4 Motivation

Every integer order differential equation can be transformed into DE, but converse is not true generally. Consequently, notice the same for non-integer order and if it does happen, then theory of FDE alone serve the purpose for differential equations as well. Although sizeable literature of FC is available, only small evolutionary research was available in discrete setting till previous decade. In the evolution of discrete FC various unpredicted issues and technical problems arise. A natural question arise: which fractional operator is useful among extensive definitions and what should be the criteria for acceptable operator? Why only Caputo type fractional operators are preferred for obeying nice properties?.

Miller and Ross, proposed a few standards for being fractional order operators, i.e., zero property, compatibility, linearity, and law of exponents. In general, not all fractional operators satisfy the product rule, quotient rule, chain rule, semigroup property and non-singularity property. Fractional derivatives of non-integer orders do not satisfy the Leibniz rule. Fractional derivative that satisfies the Leibniz rule coincides with differentiation of the first order. Because if the linear operator satisfies the Leibniz rule, then the action on the unit is equal to zero. Following operators are considered as integer-order because they violate non-locality property: the conformable, the alternative and M, the local derivative of Kolwankar and Gangal, and the Caputo–Fabrizio with exponential kernels. Derivatives violating non-locality property are of less importance because they can not be considered as fractional derivatives. This means that all results obtained for these operators can be derived by using the integer-order operators. It is dire need to justify a wrong perception that the theory of continuous case trivially holds in the discrete setting.

The gaps found in the theory of non-integer-order operators provide motivation to further work in this area. As compared to ordinary operators, non-integer-order operators may capable of modeling various physical phenomenon. following are the commonly used FDO: Riemann FD, Caputo FD, Baliarsingh generalized FD, proportional FD, tempered FD, Atangana–Baleanu FD, generalized uncertain fractional forward difference, Weyl-Like FD, forward and backward h FDO, q-fractional difference operator, fractional h–differences with exponential kernel, fractional (q, h) difference operator, and FDOs with discrete generalized ML kernel. To overcome the variety of definitions of fractional operators, some general operators need to be introduced. The discrete analogues are practically important and easier to use in real life problems. A brief discussion on the need to study discrete analogue operator is given by Abdeljawad [10]. Discrete version of substantial derivative is a potential candidate to productively describe many physical phenomena. No literature is available for Hilfer FDO and substantial FDO in the delta fractional setting. Also formation of FDO is an important aspect in view of mathematical interest and numerical formulae as well as the applications. This is the motivation behind generalization of the two existing FDO i.e., Riemann-Liouville and Caputo difference operator in Hilfer's sense and also to define substantial difference in delta fractional setting in the (Riemann-Liouville)RL sense. The exact analytical solutions of initial value problem(IVP) for FDE have been studied by delta Laplace transform method. This motivates us to introduce the delta double Laplace transforms with aim to work out problems accompanied by initial conditions for integer and fractional order partial DE.

The mathematical models of many real world phenomena can be represented by impulsive equations. Occurrence of impulse effect shows up as a natural representation of many real world phenomenon and demonstrates incredible potential for breaking down true applications. These impulses are concerned due to small outer influences during establishment process. The length is trivial in comparison with whole length of the subjected phenomena. Therefore it can be assumed that these outer influences are sudden that is they are in the form of impulses. The use of mathematical modeling in the form of impulsive DE is natural. Therefore the study of such sudden effects established potential applications in numerous fields. Fractional delta DE with impulse have recently received attention in [151, 185, 186]. There is no literature related to the existence and uniqueness(EU) theory of solution for fractional delta DE with impulse. Moreover the discrete counterpart of differential equations picked up significance from applications angle of consideration. This inspired us to further discuss the topic under consideration.

The mathematical models of many real world phenomena can be represented by multi-point boundary value problem(BVP). Such models have a large numbers of applications in numerous areas including: modeling and analyzing problems arising in elasticity, from electric power networks, electric railway systems, thermodynamics, telecommunication lines, wave propagation and also in chemistry to analyzing kinetical reaction problems.

The concept of continuity in discrete setting taken from topological spaces. The idea of Ulam-Hyers(UH) type stability is important to both functional and applied problems; especially in biology, economics and numerical analysis. It may have potential application in nonlinear analysis including difference and summation equations. Rassias introduced continuity condition which produced an acceptable stronger results. In the continuous setting, extensive work on

Ulam-Hyers-Rassias(UHR) stability for non-integer-order differential equation has been done. However only a limited work can be found in discrete fractional setting.

1.5 Objective

The theory of discrete FC shows extraordinary potential for breaking down true applications, as classical calculus is still substantially more frequently utilized in such problems. Main objective is to cover up some gaps in the theory of discrete FC. Moreover as the theory is less developed, another purpose of the dissertation is to further build up the theory of discrete FC. Even though there are different approaches to FC, here we deals with the discrete one based on delta differences with isolated time scale. However there are many gaps in the theory of discrete FC, some of the main objectives of dissertation are as follows:

- We will introduce Hilfer and substantial fractional difference operators. We will also explore well-posedness of Hilfer and substantial IVP.
- We will develop technique of the delta double Laplace transforms with aim to work out IVP and BVP for integer and fractional order partial DE. We will also introduced the delta Laplace transforms of Hilfer and substantial fractional difference operators.
- We will examine the well-posedness of fractional order impulsive difference and summation equations on isolated time scale with distinct kind of initial and BC.
- Existence theory of DE for multi-point fractional boundary value problem(MPFBVP) with integral BC on isolated time scale will also be discussed.
- We will develop some basic theory of inequalities on delta calculus. We will use Gronwall's inequality on discrete time scale for qualitative investigation.

In the research undertaken, the fractional order difference operators shall be used to develop basic theory of delta FC. In past, research in this area has been carried out using the fractional differential operators. To understand the significance of new results, we will compare the results to the cases for integer order difference operators.

1.6 Organization of dissertation

In Chapter 2, a few basic but important results from discrete calculus are stated.

In Chapter 3, some new fractional difference operators are introduced. A fractional Hilfer difference operator is introduced in Section 3.1 which interpolate RL and Caputos FD, we also develop some important properties of newly defined operator. A fractional substantial difference operator is introduced in Riemann-Liouville sense. Some composition rules and connection between Riemann-Liouville and substantial differences are developed in Section 3.2.

The motion of Chapter 4 is to introduce the delta double Laplace transform method. The definition, EU, and series representation of the delta double Laplace transform are given in Section 4.1. Some basic properties are derived in Section 4.2. We present the delta double Laplace transform of partial differences in Section 4.3. The delta Laplace transform for fractional Hilfer difference operator is given in Section 4.4. The exponential shift property of delta Laplace transform are proposed in Section 4.5, and an application of exponential shift property, delta Laplace transform for fractional substantial sum and difference operator are presented.

In Chapter 5, fixed point operator(FPO) and Green's function(GF) to different types of FD equations are established. In Section 5.1, we shall present the general method of construction of summation equation related to IVP with the following nonlinear FDE with impulse.

$$\begin{cases} {}^{c}\Delta_a^{\varphi}x(t) + f(\varphi+\rho(t), x(\varphi+\rho(t))) = 0, \quad t \in \mathbb{N}_{a+1-\varphi}, \quad t \neq a+n_j+1-\varphi, \\ \Delta^{j-1}x_k^+ - \Delta^{j-1}x_k = (-1)^{j-1}\Delta^j x_k, \quad t = a+n_j+1-\varphi, \\ x_i = (-1)^i \Delta^i x(a), i = 0, 1, \cdots, r-1, \ \lceil \varphi \rceil = r, \ k \in \mathbb{N}_1^m, \quad j \in \mathbb{N}_0^{k-1}, \end{cases}$$

where ${}^{c}\Delta_a^{\varphi}x(t)$ is the Caputo difference of $x(t)$ for $\varphi > 0$.

In Section 5.2, we apply the general construction to several specific BC. Then the GF for two-point and four-point boundary value problems are derived with some useful properties.

In Section 5.3, first we derive the GF for the following multi-point boundary value problem with summation condition

$$\begin{cases} -\Delta_{\vartheta-2}^{\vartheta}y(t) = g(t), & t \in \mathbb{N}_0^{b+1} \\ y(\vartheta-2) = 0, & y(\vartheta+b+1) + \lambda \sum_{s=\vartheta-1}^{\vartheta+b} y(s) = 0, \end{cases}$$

where $g : \mathbb{N}_0^{b+1} \to \mathbb{R}$, $\vartheta \in (1, 2]$, $b \in \mathbb{N}_0$, and $0 < \lambda \in \mathbb{R}$, and then we derive some useful properties of the GF and construct the FPO for the following nonlinear DE of non-integer order with multi-point summation BC

$$\begin{cases} -\Delta_{\vartheta-2}^{\vartheta}x(t) = h(\rho(t)+\vartheta, x(\rho(t)+\vartheta)), & t \in \mathbb{N}_0^{b+1}, \\ x(\vartheta-2) = p, & x(\vartheta+b+1) + \lambda \sum_{s=\vartheta-1}^{\vartheta+b} x(s) = q, \end{cases}$$

where $h : [\vartheta-2, b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}} \times \mathbb{R} \to \mathbb{R}$, $\vartheta \in (1, 2]$, $b \in \mathbb{N}_0$ and $\lambda > 0$, $p, q \in \mathbb{R}$.

Fixed point operator for the followig Cauchy type problem of Hilfer FD system with $0 < \vartheta < 1$, $0 \leq \varphi \leq 1$, and $\eta = \vartheta + \varphi - \vartheta\varphi$,

$$\begin{cases} \Delta_a^{\vartheta,\varphi}\chi(x) + g(x+\vartheta-1, \chi(x+\vartheta-1)) = 0, \ \text{for } x \in \mathbb{N}_{a+1-\vartheta}, \\ \Delta_a^{-(1-\eta)}\chi(a+1-\eta) = \zeta, \quad \zeta \in \mathbb{R} \end{cases}$$

is obtained in Section 5.4. Fixed point operator for the following substantial FD system with initial condition is obtained in Section 5.5,

$$\begin{cases} {}^{s}\Delta_a^{\varphi}\chi(x) + f(x+\varphi-1, \chi(x+\varphi-1)) = 0, \quad \text{for } x \in \mathbb{N}_a, \\ {}^{s}\Delta^{\varphi-i+1}\chi(x_0 = a+m-\varphi) = \chi_i, \quad i = 0, 1, \cdots, m-1, \end{cases}$$

where $m-1<\varphi\leq m$ with positive integer m.

In Chapter 6, EU of solutions to different types of FDEs are established. Conditions for EU of impulsive DE with IC or BC are obtained by applying the Schaefer's fixed point theorem(FPT) and the contraction mapping in Section 6.1 and 6.2. In Section 6.3, conditions for EU for multi point fractional boundary value problem(MPFBVP) are obtained by applying the Schauder's FPT and Banach FPT. Conditions for EU of solutions for Hilfer FD system and substantial FD system are respectively obtained in Section 6.4 and 6.5 by applying the Brouwer FPT and Banach FPT.

In Chapter 7, UH stability and UHR stability shall be discussed. In Section 7.1, conditions are acquired under which the nonlinear MPFBVP is UH stable, generalized UH stable, UHR stable and generalized UHR stable. In Section 7.2, conditions are acquired under which the nonlinear Hilfer FD system is UH stable, generalized UH stable, UHR stable and generalized UHR stable. Modification and application of discrete Gronwall's inequality in delta setting is also presented. In Section 7.3, conditions are acquired under which the nonlinear substantial FD system is UH stable, generalized UH stable, UHR stable and generalized UHR stable. Examples are presented to demonstrate the applicability of the results.

Finally, in Chapter 8, dissertation is summarize.

Chapter 2

Preliminaries

In this chapter, we present some fundamentals of fractional calculus, special functions and fixed point theory that are helpful in the following chapters.

2.1 Basics of delta fractional calculus

Some basics from discrete FC are given for later use in this sections. The functions we will consider are usually defined on the set $\mathbb{N}_a := \{a, a+1, a+2, \cdots\}$, where $a \in \mathbb{R}$ is fixed. Some times the set $\mathbb{N}_a$ is called isolated time scale. Similarly the set $\mathbb{N}_a^b := \{a, a+1, a+2, \cdots, b\}$ and $[a, b]_{\mathbb{N}_a} := [a, b] \cap \mathbb{N}_a$ [82] for $b = a + k,\ k \in \mathbb{N}_0$. The jump operators $\sigma(t) = t + 1$, and $\rho(t) = t - 1$ defined to be forward and backward respectively, for $t \in \mathbb{N}_a$.

Following concepts are discussed in [103, 116]. The collection of regressive functions is given for $x \in \mathbb{N}_a$ as $\mathscr{R} = \{p_i : 1 + p_i(x) \neq 0\}$. The circle plus sum of $p_1, p_2 \in \mathscr{R}$ is given by $p_1 \oplus p_2 = p_1 + p_2 + p_1 p_2$. The circle minus of $p \in \mathscr{R}$ is given by $\ominus p(x) = \frac{-p(x)}{1+p(x)}$ for $x \in \mathbb{N}_a$. The floor function $\lfloor . \rfloor$ maps a real number to the largest preceding integer. The smallest integer followed by a number is the ceiling $\lceil . \rceil$ function.

Definition 2.1.1. *[50] Assume $f : \mathbb{N}_a \to \mathbb{R}$. The delta definite integral is defined by*

$$\int_b^c f(x)\Delta x = \sum_{x=b}^{c-1} f(x), \quad b, c \in \mathbb{N}_a \text{ and } b \leq c.$$

Note that the value of integral $\int_b^c f(x)\Delta x$, depending on the set $\{b, b+1, \cdots, c-1\}$. Also by the empty sum convention

$$\sum_{x=b}^{b-k} f(x) = 0 \quad \text{whenever } k \in \mathbb{N}_1.$$

The delta indefinite integral is defined by

$$\int_b^{\infty} f(x)\Delta x = \sum_{x=b}^{\infty} f(x).$$

In the next definition, we consider only delta difference with increment 1, and do not bothered with different operators that we will not be using here. One can find the details of Definition 2.1.2 in [106, 124].

Definition 2.1.2. *Assume $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ be a function of two independent variables. Then the partial difference of $\chi(x, y)$ regarding x, taking y as a constant is defined as,*

$$\Delta_x[\chi(x, y)] = \chi(x+1, y) - \chi(x, y).$$

The partial difference of $\chi(x, y)$ regarding y, taking x as a constant is defined as,

$$\Delta_y[\chi(x, y)] = \chi(x, y+1) - \chi(x, y).$$

Partial difference equation is an equation containing partial differences.

Note that $\Delta_{xy} = \Delta_y \Delta_x = \Delta_x \Delta_y = \Delta_{yx}$. Followed by the rule for integer order difference operator $\Delta^n = \Delta\Delta^{n-1}$, we adopt the symbol for partial differences as follows: $\Delta_x^n = \Delta_x \Delta_x^{n-1}$, $\Delta_y^m = \Delta_y \Delta_y^{m-1}$.

The falling function is defined for positive integer n by $x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$.

Definition 2.1.3. *The discrete Taylor monomial based at $s = a$ is given as*

$$h_n(x, a) = \frac{(x-a)^{\underline{n}}}{n!} \quad \text{for} \;\; x \in \mathbb{N}_a.$$

Definition 2.1.4. *The generalized falling function is defined in terms of gamma function by*

$$t^{\underline{\vartheta}} = \frac{\Gamma(\sigma(t))}{\Gamma(\sigma(t) - \vartheta)} \quad t \in \mathbb{N}_a \quad \vartheta \in \mathbb{R},$$

provided that the expression in above equation is justifiable. It is convenient to take $t^{\underline{\vartheta}} = 0$ whenever $t + 1$ is natural number and $t - \vartheta + 1$ is zero or negative integer.

Definition 2.1.5. *The ϑ^{th} order Taylor monomial is defined by $h_\vartheta(t, s) = \frac{(t-s)^{\underline{\vartheta}}}{\Gamma(\vartheta+1)}$ for $t, s \in \mathbb{N}_a$.*

Lemma 2.1.6. *If $t, s \in \mathbb{N}_a$, then $\sum h_\vartheta(t, \sigma(s))\Delta s = -h_{\vartheta+1}(t, s) + C$ as for some constant C and*

$$\sum h_\vartheta(t, a)\Delta t = h_{\vartheta+1}(t, a) + C.$$

Definition 2.1.7. *[103] Assume $p(x) \in \mathscr{R}$ and $x, y \in \mathbb{N}_a$. Then the delta exponential function is given by*

$$e_{p(x)}(x, y) = \begin{cases} \prod_{t=y}^{x-1}[1 + p(t)], & \text{if } x \in \mathbb{N}_y, \\ \prod_{t=x}^{y-1}[1 + p(t)]^{-1}, & \text{if } x \in \mathbb{N}_a^{y-1}. \end{cases}$$

By empty product convention $\prod_{t=y}^{y-1}[h(t)] := 1$ for any function h.

Example 2.1.8. *[103] If $p_1(x) = c$ is a constant such that $c \in \mathscr{R}$ (that is $c \neq -1$), then delta exponential function for constant is given by $e_{p_1}(x, s) = e_c(x, s) = [1 + c]^{x-s}$ for $x \in \mathbb{N}_a$. In particular, for the initial point of the domain of definition $s = a$, we have*

$$e_c(x, a) = [1 + c]^{x-a} \quad \text{for } x \in \mathbb{N}_a.$$

Lemma 2.1.9. [103]*(Fundamental theorem for the difference calculus) Assume* $f : \mathbb{N}_a^b \to \mathbb{R}$ *and* F *is an antidifference of* f *on* $\mathbb{N}_a^{b+1}$. *Then*

$$\sum_{t=a}^{b} f(t) = \sum_{t=a}^{b} \Delta F(t) = F(b+1) - F(a).$$

Lemma 2.1.10. [103] *Assume* $p(x) \in \mathcal{R}$. *Then* $\Delta_x e_{p(x)}(x,y) = p(x) e_{p(x)}(x,y)$.

Lemma 2.1.11. [103]*[Leibniz formula] Assume* $f : \mathbb{N}_{a+\vartheta} \times \mathbb{N}_a \to \mathbb{R}$ *and* $\vartheta > 0$. *Then for* $x \in \mathbb{N}_{a+\vartheta}$,

$$\Delta \sum_{\mathcal{F}=a}^{x-\vartheta} f(x,\mathcal{F}) = \sum_{\mathcal{F}=a}^{x-\vartheta} \Delta_x f(x,\mathcal{F}) + f(x+1, x-\vartheta+1).$$

Lemma 2.1.12. [103] *Assume two functions are defined by* $\psi, \phi : \mathbb{N}_a \to \mathbb{R}$. *Let* $b_1, b_2 \in \mathbb{N}_a$ *such that* $b_1 < b_2$. *Then we have the summation by parts formula,*

$$\sum_{b_1}^{b_2} \psi(\sigma(t)) \Delta\phi(t) \Delta t = \psi(t)\phi(t)\Big|_{b_1}^{b_2+1} - \sum_{b_1}^{b_2} \phi(t) \Delta\psi(t) \Delta t.$$

Lemma 2.1.13. [103] *Assume* $c_1, c_1 \in \mathcal{R}$ *and* $x \in \mathbb{N}_a$. *Then* $e_{c_1}(x,a) e_{c_2}(x,a) = e_{c_1 \oplus c_2}(x,a)$.

Lemma 2.1.14. [103] *Assume* $f, g : \mathbb{N}_a^b \to \mathbb{R}$. *Then for* $x \in \mathbb{N}_a^{b-1}$

$$\Delta[f(x)g(x)] = f(\sigma(x))\Delta g(x) + [\Delta f(x)]g(x).$$

Definition 2.1.15. *[103] Assume* $f : \mathbb{N}_a \to \mathbb{R}$, $\vartheta > 0$. *Then the delta fractional sum of* f *is defined by* $\Delta_a^{-\vartheta} f(x) := \sum_{\mathcal{F}=a}^{x-\vartheta} h_{\vartheta-1}(x, \sigma(\mathcal{F})) f(\mathcal{F})$ *for* $x \in \mathbb{N}_{a+\vartheta}$, *where* $h_\vartheta(t,s) = \frac{(t-s)^{\underline{\vartheta}}}{\Gamma(\vartheta+1)}$ *is the* ϑ^{th} *fractional Taylor monomial based at* s *and* $t^{\underline{\vartheta}}$ *is the generalized falling function.*

Lemma 2.1.16. [103] *Assume* $\varphi \geq 0$ *and* $\vartheta > 0$. *Then* $\Delta_{a+\varphi}^{-\vartheta} (x-a)^{\underline{\varphi}} = \frac{\Gamma(\varphi+1)}{\Gamma(\vartheta+\varphi+1)} (x-a)^{\underline{\vartheta+\varphi}}$ *for* $x \in \mathbb{N}_{a+\vartheta+\varphi}$.

Definition 2.1.17. [31, 144] *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\vartheta > 0$ and $m-1 < \vartheta \leq m$, for $m \in \mathbb{N}_1$. Then the Riemaan-Liouville FD of f at a is defined by*

$$\Delta_a^{\vartheta} f(x) = \Delta^m \Delta_a^{-(m-\vartheta)} f(x) = \sum_{\mathscr{F}=a}^{x+\vartheta} h_{-\vartheta-1}(x, \sigma(\mathscr{F})) f(\mathscr{F}) \quad \text{for } x \in \mathbb{N}_{a+m-\vartheta}.$$

Definition 2.1.18. [2, 4] *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\vartheta > 0$ and $m-1 < \vartheta \leq m$, for $m \in \mathbb{N}_1$. Then the Caputo FD of f at a is defined by*

$${}^c\Delta_a^{\vartheta} f(x) = \Delta_a^{-(m-\vartheta)} \Delta^m f(x) = \sum_{\mathscr{F}=a}^{x-(m-\vartheta)} h_{m-\vartheta-1}(x, \sigma(\mathscr{F})) \Delta^m f(\mathscr{F})$$

for $x \in \mathbb{N}_{a+m-\vartheta}$.

Definition 2.1.19. [50] *Assume $f : \mathbb{N}_a \to \mathbb{R}$. Then the delta Laplace transform of f based at a is defined by*

$$\mathscr{L}_a\{f\}(y) = \int_a^{\infty} e_{\ominus y}(\sigma(x), a) f(x) \Delta x$$

for all complex numbers $y \neq -1$ such that this improper integral converges.

The following concepts are also discussed in [103, 116].

Definition 2.1.20. *[103] A function f is of exponential order(EO) $r_1 > 0$ if there exist a constant $A_1 > 0$ and the following inequality*

$$|f(x)| \leqslant A_1 r_1^x \quad \text{holds for sufficiently large } x \in \mathbb{N}_a.$$

If f is of EO, then $\mathscr{L}_x\{f\}(p)$ converges absolutely for $|p+1| > r_1$, which ensures the existence of the Laplace transform. Even though the converse is not true, we restrict ourself to only EO functions. For $f : \mathbb{N}_a \to \mathbb{R}$, the following are useful expression for the delta Laplace transform of f based at a

$$\mathscr{L}_x\{f\}(p) = \tilde{F}(p) = \int_0^\infty \frac{f(a+j)}{(p+1)^{j+1}} \Delta j$$
$$= \sum_{j=0}^{\infty} \frac{f(a+j)}{(p+1)^{j+1}}$$

for all complex numbers $p \neq -1$ such that this infinite series converges.

Example 2.1.21. *If $c \neq -1$, then for $|p+1| > |c+1|$ we have*

$$\mathscr{L}_x\{e_c(x,a)\}(p) = \frac{1}{p-c}.$$

Lemma 2.1.22. [103] *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of EO $r > 1$ and $\vartheta > 0$. Then*

$$\text{for } |y+1| > r, \text{ we have, } \mathscr{L}_{a+\vartheta}\{\Delta_a^{-\vartheta} f\}(y) = \frac{(y+1)^\vartheta}{y^\vartheta} \tilde{F}_a(y).$$

Lemma 2.1.23. [103] *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ is of EO $r > 0$ and m is positive integer. Then for $|y+1| > r$*

$$\mathscr{L}_a\{\Delta^m f\}(y) = y^m \tilde{F}_a(y) - \sum_{j=0}^{m-1} y^j \Delta^{m-1-j} f(a).$$

Lemma 2.1.24. [103] *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of EO $r \geq 1$ and $m - 1 < \vartheta < m$ with positive integer m. Then for $|y+1| > r$*

$$\mathscr{L}_{a+m-\vartheta}\{\Delta_a^\vartheta f\}(y) = y^\vartheta (y+1)^{m-\vartheta} \tilde{F}_a(y) - \sum_{j=0}^{m-1} y^j \Delta_a^{\vartheta-1-j} f(a+m-\vartheta).$$

Definition 2.1.25. *[103] Assume $f, g : \mathbb{N}_a \to \mathbb{R}$. The convolution product is defined by*

$$(f * g)(x,y) = \sum_{r=a}^{x-1} f(r) g(x - \sigma(r) + a) \quad \text{for} \quad x \in \mathbb{N}_a.$$

Note that by the empty sum convention $(f * g)(a) = 0$.

Lemma 2.1.26. *[103] Assume $f, g : \mathbb{N}_a \to \mathbb{R}$. If both $\mathscr{L}_x f(x)$ and $\mathscr{L}_x g(x)$ exist, then the delta Laplace transform of convolution product can be represented as*

$$\mathscr{L}_x\{(f * g)(x)\} = \mathscr{L}_x\{f(x)\}\mathscr{L}_x\{g(x)\} = \mathscr{L}_x\{(g * f)(x)\}.$$

Lemma 2.1.27. *[103] The following hold for delta Laplace of Taylor monomial of degree $n \geq 0$:*

(i) $\mathscr{L}_x\{h_n(x, a)\}(p) = \frac{1}{p^{n+1}}$ *for* $|p + 1| > 1$,

(ii) $\mathscr{L}_x\{(x - a)^{\underline{n}}\}(p) = \frac{n!}{p^{n+1}}$ *for* $|p + 1| > 1$.

Lemma 2.1.28. [117] *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $m - 1 < \vartheta < m$, where m and k are positive integers. Then $[\Delta^k(\Delta_a^{-\vartheta} f)](x) = (\Delta_a^{k-\vartheta} f)(x)$, for $x \in \mathbb{N}_{a+\vartheta}$.*

Lemma 2.1.29. *Suppose that $\vartheta > 0$ and r is a positive integer in such a way that $r-1 < \vartheta \leq r$. For arbitrary real number a, $y(t) = b_1(t-a)^{\underline{\vartheta-1}} + b_2(t-a)^{\underline{\vartheta-2}} \cdots + b_r(t-a)^{\underline{\vartheta-r}}$ for all constants $b_1, b_2, \cdots, b_r$, is a solution of $\Delta^{\vartheta}_{a+\vartheta-r} y(t) = 0$ for $t \in \mathbb{N}_{a+\vartheta-r}$.*

Lemma 2.1.30. *Let r be a positive integer such that $r - 1 < \vartheta \leq r$. If $g : \mathbb{N}_0 \to \mathbb{R}$, then the problem*

$$\begin{aligned} \Delta^{\vartheta}_{\vartheta-r} y(t) =& g(t), \qquad t \in \mathbb{N}_0 \\ y(\vartheta - r - i) =& 0, \qquad 0 \leq i \leq r - 1 \end{aligned}$$

has a solution represented by

$$y(t) = \Delta_0^{-\vartheta} g(t) = \sum_{s=0}^{t-\vartheta} h_{\vartheta-1}(t, \sigma(s))g(s), \qquad t \in \mathbb{N}_{\vartheta-r}.$$

2.2 Special functions

Special functions are particular mathematical functions that play a significant role in the theory of fractional calculus. To proceed further in this work, we first provide necessary information about Euler's gamma function and discrete ML function.

2.2.1 Gamma function

The extension of factorial function for non-integer values is known as gamma function. The gamma function plays a key role in the development of the theory of FC. In 1729, Euler defined the integral $\int_0^1 (\ln(\frac{1}{t}))^{z-1}dt$ which later published in [79]. By slight modification and substituting $t = e^{-\alpha}$, Karl Weierstras introduced new representation which is now known as the gamma function.

Definition 2.1. The gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(z) = \int_0^\infty \alpha^{z-1} e^{-\alpha} d\alpha, \quad z > 0. \tag{2.1}$$

By parts integration of (2.1) yields the fundamental equation

$$\Gamma(z+1) = z\Gamma(z), \quad z > -1, \ z \neq 0. \tag{2.2}$$

The integral defining gamma function is uniformly convergent for all $z \in [a, b]$, where $0 < a \leq b < \infty$, and hence Γ is a continuous function for all $z > 0$.

The domain of gamma function can be further extends to non-negative real number by using reposition of equation (2.2) as follows:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad -1 < z < 0.$$

Since Γ is not defined on $z = 0$, therefore it is not possible to define at -1. Further continues in the same way yields

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)}, \ -2 < z < 0, z \neq -1,$$

and so forth. Therefore $x = 0, -1, -2, \cdots$ are vertical asymptotes of gamma functions.

2.2.2 Mittage-Leffler functions

Gosta Mittag-Leffler (1846-1927) introduced a function as a generalization of exponential function, known as ML function. The one parameter ML function E_ϑ is defined by the power series

$$E_\vartheta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\vartheta + 1)},$$

where $\vartheta > 0$ is a parameter.

The generalization of one parameter ML function was introduced by Wiman in 1905. The two parameter ML function is defined by the power series

$$E_{\vartheta,\varphi}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\vartheta + \varphi)},$$

where $\vartheta, \varphi > 0$ are the parameters.

The discrete ML functions in delta setting are defined by Abdeljawad in [4]

$$E_{\underline{\vartheta,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + (k-1)(\vartheta - 1))^{\underline{k\vartheta}}(z + k(\vartheta - 1))^{\underline{\eta - 1}}}{\Gamma(k\vartheta + \eta)}.$$

Remark 1. If we set $\varphi = 1$ in Example 6.4.1 (hence $\eta = 1$), and take $a = \vartheta - 1$, then it recovers Example 17 in [4]. In fact, the solution of the initial Caputo DE

$${}^{C}\Delta_a^{\vartheta} x(t) = \lambda x(t + \vartheta - 1), \quad x(a) = x_0, \quad \vartheta \in (0, 1],$$

is given by

$$x(t) = x_0 E_{\underline{\vartheta}}(\lambda, t-a) = x_0 \sum_{k=0}^{\infty} \frac{\lambda^k (t-a+k(\vartheta-1))^{\underline{k\vartheta}}}{\Gamma(\vartheta k+1)}.$$

Observe that the case $a = \vartheta - 1$ reduces as (66) in [4]. That is, the formula (66) in [4] represents $E_{\underline{\vartheta}}(\lambda, t-(\vartheta-1))$. Also, one can see that the substitution $\vartheta = 1$ gives the delta discrete Taylor expansion of the delta discrete exponential function.

The observations in Remark 1, suggest the following modified definitions those are different from that appeared in [4].

Definition 2.2.1. *For $\lambda \in \mathbb{R}$, $|\lambda| < 1$ and $\vartheta, \eta, \gamma, z \in \mathbb{C}$ with $Re(\vartheta) > 0$, the discrete Mittag Leffler functions are defined by*

$$\begin{aligned}
E^{\gamma}_{\underline{\vartheta,\eta}}(\lambda, z) &= \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\vartheta-1))^{\underline{\vartheta k+\eta-1}} (\gamma)_k}{\Gamma(\vartheta k+\eta) k!}, \ (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1) \\
E_{\underline{\vartheta,\eta}}(\lambda, z) = E^{1}_{\underline{\vartheta,\eta}}(\lambda, z) &= \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\vartheta-1))^{\underline{\vartheta k+\eta-1}}}{\Gamma(\vartheta k+\eta)}, \\
E_{\underline{\vartheta}}(\lambda z) = E_{\underline{\vartheta,1}}(\lambda, z) &= \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\vartheta-1))^{\underline{\vartheta k}}}{\Gamma(\vartheta k+1)}.
\end{aligned}$$

Using $x^{\underline{\vartheta+\varphi}} = (x-\varphi)^{\underline{\vartheta}}\, x^{\underline{\varphi}}$, note that

$$\begin{aligned}
E^{\gamma}_{\underline{\vartheta,\vartheta}}(\lambda, z) &= \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\vartheta-1))^{\underline{\vartheta k+\vartheta-1}} (\gamma)_k}{\Gamma(\vartheta k+\vartheta) k!} \\
&= \sum_{k=0}^{\infty} \lambda^k \frac{(z+(k-1)(\vartheta-1))^{\underline{k\vartheta}} (z+k(\vartheta-1))^{\underline{\vartheta-1}} (\gamma)_k}{\Gamma(k\vartheta+\vartheta) k!}.
\end{aligned}$$

Definition 2.2.2. *For $\lambda \in \mathbb{R}$, $|\lambda| < 1$ and $\vartheta, \eta, \gamma, z \in \mathbb{C}$ with $Re(\vartheta) > 0$, the discrete ML functions are defined by*

$$\boldsymbol{E}^{\gamma}_{\underline{\vartheta,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\vartheta-1)+\eta-1)^{\underline{\vartheta k+\eta-1}} (\gamma)_k}{\Gamma(\vartheta k+\eta) k!},$$

$$\boldsymbol{E}_{\underline{\vartheta,\eta}}(\lambda, z) = \boldsymbol{E}^{1}_{\underline{\vartheta,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+k(\vartheta-1)+\eta-1)^{\underline{\vartheta k+\eta-1}}}{\Gamma(\vartheta k+\eta)},$$

$$\boldsymbol{E}_{\underline{\vartheta}}(\lambda, z) = \boldsymbol{E}_{\underline{\vartheta,1}}(\lambda, z) = E_{\underline{\vartheta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z + k(\vartheta - 1))^{\underline{\vartheta k}}}{\Gamma(\vartheta k + 1)}.$$

2.3 Ulam-Hyers-Rassias stability

The definitions of Ulam stability for FDE was introduced in [61]. Consider the system

$$\begin{cases} -\Delta_{\vartheta-2}^{\vartheta} x(t) = h(\rho(t) + \vartheta, x(\rho(t) + \vartheta)), & t \in \mathbb{N}_0^{b+1}, \\ x(\vartheta - 2) = p, \qquad x(\vartheta + b + 1) + \lambda \sum_{s=\vartheta-1}^{\vartheta+b} x(s) = q, \end{cases} \tag{2.3}$$

where $h : [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}} \times \mathbb{R} \to \mathbb{R}$, $\vartheta \in (1, 2]$, $b \in \mathbb{N}_0$ and $\lambda > 0$, $p, q \in \mathbb{R}$. Consider the following inequalities:

$$\left| \Delta_{\vartheta-2}^{\vartheta} y(t) + h(\rho(t) + \vartheta, y(\rho(t) + \vartheta)) \right| \leq \epsilon, \qquad t \in [0, b + 1]_{\mathbb{N}_0}, \tag{2.4}$$

$$\left| \Delta_{\vartheta-2}^{\vartheta} y(t) + h(\rho(t) + \vartheta, y(\rho(t) + \vartheta)) \right| \leq \epsilon \eth(\rho(t) + \vartheta), \qquad t \in [0, b + 1]_{\mathbb{N}_0}, \tag{2.5}$$

where $\eth : [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}} \to \mathbb{R}^+$.

Definition 2.3.1. *If there exists a real number $d_h > 0$ such that for each $\epsilon > 0$ and for every solution $y(t) \in K$ of inequality (2.4), then a solution $x(t) \in K$ of system (2.3) satisfying*

$$\left| x(t) - y(t) \right| \leq \epsilon d_h, \qquad t \in [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}}, \tag{2.6}$$

is UH stable. The solution of system (2.3) is generalized UH stable if we substitute the function $\wp_h(\epsilon)$ for the constant ϵd_h in inequality (2.6), where $\wp_h(\epsilon) \in C(R^+, R^+)$ and $\wp_h(0) = 0$.

Definition 2.3.2. *If there exists a real number $d_{h,\eth} > 0$ such that for each $\epsilon > 0$ and for every solution $y(t) \in K$ of inequality (2.5), then a solution $x(t) \in K$ of system (2.3) satisfying*

$$\left| x(t) - y(t) \right| \leq \epsilon \eth(t) d_{h,\eth}, \qquad t \in [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}}, \tag{2.7}$$

is UHR stable with respect to function $\eth$*. The solution of system* (2.3) *is generalized UHR stable if we substitute the function* $\Phi(t)$ *for the function* $\epsilon\eth(t)$ *in inequalities* (2.5) *and* (2.7)*.*

2.4 Gronwall type inequalities

Gronwall's inequality is a bound to the solution of difference and summation equations with different available versions. Also it provides estimation and comparison principle which are helpful for uniqueness and stability of solution for IVP. First it was introduced by Gronwall in 1919 and later by Bellman in 1943. Its discrete analogue was introduced in [33] and generalized in [20, 36]. Here, we present Gronwall's inequality for discrete calculus with the delta DO in modified form. An application of Gronwall's inequality has been given for the stability of solution to fractional order Hilfer DE with different initial conditions. For this purpose, first we develop a Gronwall's inequality for the delta difference operator. Then a simple utilization of Gronwall's inequality leads to stability for Hilfer DE. Choose χ and ϕ such that

$$\chi(x) \leq \chi(a)h_{\eta-1}(x, a+1-\eta) + \Delta^{-\vartheta}_{a+1-\vartheta}\wp(x+\vartheta)\chi(x+\vartheta), \tag{2.8}$$

$$\phi(x) \geq \phi(a)h_{\eta-1}(x, a+1-\eta) + \Delta^{-\vartheta}_{a+1-\vartheta}\wp(x+\vartheta)\phi(x+\vartheta). \tag{2.9}$$

Lemma 2.4.1. *Assume u and w satisfy* (2.8) *and* (2.9)*, respectively. If $\phi(a) \geq \chi(a)$, then $\phi(x) \geq \chi(x)$ for $x \in \mathbb{N}_a$.*

Proof. We give the proof by induction principle. Assume $\phi(\mathscr{F}) - \chi(\mathscr{F}) \geq 0$ holds for $\mathscr{F} =$

$a, a+1, \cdots, x-1$. Then we have

$$\begin{aligned}\phi(x)-\chi(x) \geq & h_{\eta-1}(x, a+1-\eta)(\phi(a)-\chi(a))+\Delta_{a+1-\vartheta}^{-\vartheta} \wp(x+\vartheta) \phi(x+\vartheta) \\ & -\Delta_{a+1-\vartheta}^{-\vartheta} \wp(x+\vartheta) \chi(x+\vartheta) \\ = & h_{\eta-1}(x, a+1-\eta)(\phi(a)-\chi(a)) \\ & +\sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} \frac{(x-\sigma(\mathscr{F}))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} \wp(\mathscr{F}+\vartheta)(\phi(\mathscr{F}+\vartheta)-\chi(\mathscr{F}+\vartheta)),\end{aligned}$$

where the last summation is valid for $x \in \mathbb{N}_{a+\vartheta}$. Now, we shifted the domain of summation to $\mathbb{N}_a$.

$$\begin{aligned}\phi(x)-\chi(x) \geq & h_{\eta-1}(x, a+1-\eta)(\phi(a)-\chi(a)) \\ & +\sum_{\mathscr{F}=a+1}^{x} \frac{(x+\vartheta-\sigma(\mathscr{F}))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} \wp(\mathscr{F})(\phi(\mathscr{F})-\chi(\mathscr{F})).\end{aligned}$$

By given assumption, we have

$$\phi(x)-\chi(x) \geq \wp(x)(\phi(x)-\chi(x)) \quad for\ \mathscr{F}=a, a+1, \cdots, x-1.$$

This implies that $(1-\wp(x))(\phi(x)-\chi(x)) \geq 0$ and for $|\wp(x)|<1$, which is desired result.

□

Following the approach for nabla fractional difference in [36], let $E_v \wp=\Delta_{a+1-\vartheta}^{-\vartheta} \psi(x) \wp(x)$. For constant $\wp$, one can use $E_v \wp$ to express Mittag-Leffler function.

Theorem 2.2. *Assume $\eta=\vartheta+\varphi-\vartheta \varphi$ with $0<\vartheta<1$ and $0 \leq \varphi \leq 1$. The solution of summation equation*

$$\chi(x)=\chi(a) h_{\eta-1}(x, a+1-\eta)+\Delta_{a+1-\vartheta}^{-\vartheta} \psi(x+\vartheta-1) \chi(x+\vartheta-1)$$

is given by

$$\chi(x)=\frac{\chi(a)}{\Gamma(\eta)} \sum_{\ell=0}^{\infty} E_v^{\ell}(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta-1}}.$$

Proof. By method of successive approximation, the following is obtained:

$$\chi_k(x)=\chi_0(x)+\Delta_{a+1-\vartheta}^{-\vartheta} \psi(x+\vartheta-1) \chi_{k-1}(x+\vartheta-1), \quad k=1,2,3, \cdots,$$

where $\chi_0(x) = \chi(a)h_{\eta-1}(x, a+1-\eta)$.

For $k = 1$, we have

$$\begin{aligned}\chi_1(x) =&\chi(a)h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\vartheta}^{-\vartheta}\psi(x+\vartheta-1)\chi_0(x+\vartheta-1)\\ =&\frac{\chi(a)}{\Gamma(\eta)}E_v^0(x+\eta-a-1)^{\underline{\eta-1}} + \frac{\chi(a)}{\Gamma(\eta)}E_v^1(x+\eta-a-1+\vartheta-1)^{\underline{\eta-1}}.\end{aligned}$$

Proceed inductively, we obtained

$$\chi_k(x) =\frac{\chi(a)}{\Gamma(\eta)}\sum_{\ell=0}^{k} E_v^\ell(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta-1}}, \quad k = 1, 2, 3, \cdots$$

and letting $k \to \infty$ then, we have $\chi(x) = \dfrac{\chi(a)}{\Gamma(\eta)}\displaystyle\sum_{\ell=0}^{\infty} E_v^\ell(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta-1}}$.

□

Next, we derived a Gronwall's inequality in delta discrete setting.

Theorem 2.3. *Let $\eta = \vartheta + \varphi - \vartheta\varphi$, with $0 < \vartheta < 1$ and $0 \le \varphi \le 1$. Assume $|\psi(x)| < 1$ for $x \in \mathbb{N}_a$. If χ and ψ are nonnegative real valued functions satisfying*

$$\chi(x) \le\chi(a)h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\vartheta}^{-\vartheta}\psi(x+\vartheta-1)\chi(x+\vartheta-1),$$

then we have

$$\chi(x) \le\frac{\chi(a)}{\Gamma(\eta)}\sum_{\ell=0}^{\infty} E_v^\ell(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta-1}}.$$

Proof. Consider $\phi(x) = \frac{\chi(a)}{\Gamma(\eta)}\sum_{\ell=0}^{\infty} E_v^\ell(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta-1}}$.

The proof of theorem follows from Lemma 2.4.1 and Theorem 7.1. □

For $\eta = 1$, a special case is obtained as follow.

Corollary 2.4.2. *Let $0 < \vartheta < 1$ and $0 \le \varphi \le 1$. Assume $0 < \psi(x) < 1$ for $x \in \mathbb{N}_a$. If χ is nonnegative real-valued function satisfying*

$$\chi(x) \le\chi(a) + \Delta_{a+1-\vartheta}^{-\vartheta}\psi(x+\vartheta-1)\chi(x+\vartheta-1),$$

then we have

$$\chi(x) \le\chi(a)e_v(x, a),$$

where $e_v(x, a)$ is the delta exponential function.

Proof. Following Theorem 7.2 that

$$\chi(x) \leq \chi(a) \sum_{\ell=0}^{\infty} E_v^{\ell}(1).$$

We claimed that $\sum_{\ell=0}^{\infty} E_v^{\ell}(1) = e_v(x,a)$. To justify our claim, we utilized the uniqueness of solution of following IVP, $\Delta\chi(x) = \psi(x)\chi(x),\ \chi(a) = 1$. A unique solution $\chi(x) = e_v(x,a)$ of IVP is given in [103] for regressive function $\psi(x)$. Thus, we have to show that $\sum_{\ell=0}^{\infty} E_v^{\ell}(1)$ satisfies the IVP $\Delta\chi(x) = \psi(x)\chi(x),\ \chi(a) = 1$. Indeed,

$$\begin{aligned}
\Delta \sum_{\ell=0}^{\infty} E_v^{\ell}(1) &= \sum_{\ell=0}^{\infty} \Delta E_v^{\ell}(1) \\
&= \sum_{\ell=1}^{\infty} \Delta E_v(E_v^{\ell-1}(1)) \\
&= \sum_{\ell=1}^{\infty} \Delta \Delta_a^{-1}(\psi(x) E_v^{\ell-1}(1)) = \psi(x) \sum_{\ell=0}^{\infty} E_v^{\ell}(1).
\end{aligned}$$

We have $\sum_{\ell=0}^{\infty} E_v^{\ell}(1)(a) = 1 + \sum_{\ell=1}^{\infty} E_v^{\ell}(1)(a) = 1$, by Definition 2.1.5 and empty sum convention. Then the proof complete. $\square$

Chapter 3

New fractional difference operators

The classical method of obtaining fractional operators relies on iterating an integral to find the integral of the n^{th} order and then exchanging n with any real number. The corresponding derivatives are described after that. We start by introducing a generalized difference operator analogous to Hilfer fractional derivative [115]. To keep the interpolative property of Hilfer fractional difference operators, we carefully choose the starting points of fractional sums. Some important composition properties are developed and utilized to construct fixed point operator for a new class of Hilfer fractional nonlinear difference equation with initial conditions involving Reimann-Liouville fractional sum.

Hilfer's definition in continuous fractional calculus is illustrated as follows: the fractional derivative of order $0 < \vartheta < 1$ and type $0 \leq \varphi \leq 1$ is

$$D_a^{\vartheta,\varphi} f(x) = \left(I_a^{\varphi(1-\vartheta)} \frac{d}{dx} \left(I_a^{(1-\varphi)(1-\vartheta)} f \right) \right)(x).$$

The special cases are Riemann-Liouville fractional derivative for $\varphi = 0$ and the Caputo fractional derivative for $\varphi = 1$. Furati et al. [86, 87] primarily studied the existence theory of Hilfer fractional differential equations and also explained the type parameter φ as interpolation between the RL and the Caputo derivatives. It generates more types of stationary states and gives an extra degree of freedom on the initial condition. Also we develop some important properties of newly defined operator.

3.1 Hilfer fractional difference

Findings of this section appeared in [109]. In this section, we give the general definition of the Hilfer like fractional difference operator. Motivated by the concept of Hilfer fractional derivative [115], and to keeping the interpolative property, we present the following definition. Assume $f : \mathbb{N}_a \to \mathbb{R}$, then the FD of order $m-1 < \vartheta < m$, for $m \in \mathbb{N}_1$ is given by $\Delta_a^{\vartheta,\varphi} f(x) = \Delta_{a+(1-\varphi)(m-\vartheta)}^{-\varphi(m-\vartheta)} \Delta^m \Delta_a^{-(1-\varphi)(m-\vartheta)} f(x)$, for $x \in \mathbb{N}_{a+m-\vartheta}$, where $0 \leq \varphi \leq 1$ is the type of difference operator. Observe that starting point of domain for $\Delta_a^{-(1-\varphi)(m-\vartheta)} f(x)$ is $a+(1-\varphi)(m-\vartheta)$, whereas integer-order differences keep the same domain [116]. The starting point of the last sum is compatible with the starting point for the domain of the function $\Delta^m \Delta_a^{-(1-\varphi)(m-\vartheta)} f(x)$, which is $a+(1-\varphi)(m-\vartheta)$. This allows us the successive composition of operators in above expression and the final domain of $\Delta_a^{\vartheta,\varphi} f(x)$ is $\mathbb{N}_{a+m-\vartheta}$. To get some nice properties, we restrict $0 < \vartheta < 1$ in further analysis.

Definition 3.1.1. *Assume $f : \mathbb{N}_a \to \mathbb{R}$. The fractional difference of order $0 < \vartheta < 1$ and type $0 \leq \varphi \leq 1$ is defined by*

$$\Delta_a^{\vartheta,\varphi} f(x) = \Delta_{a+(1-\varphi)(1-\vartheta)}^{-\varphi(1-\vartheta)} \Delta \Delta_a^{-(1-\varphi)(1-\vartheta)} f(x)$$

for $x \in \mathbb{N}_{a+1-\vartheta}$.

The special cases are Riemann-Liouville fractional difference [31, 144] for $\varphi = 0$ and Caputo fractional difference [2, 4] for $\varphi = 1$.

3.1.1 Properties of Hilfer fractional difference operator

First we develop some composition properties to construct a FPO for a new class of Hilfer nonlinear FDE with initial conditions involving RL fractional sum.

Lemma 3.1.2. *Assume* $f : \mathbb{N}_a \to \mathbb{R}$ *is defined,* $0 < \vartheta < 1$ *and* $0 \leq \varphi \leq 1$*, for* $x \in N_{a+1}$

(i) $\Delta_{a+1-\vartheta}^{-\vartheta}[\Delta_a^{\vartheta,\varphi} f(x)] = \Delta_{a+(1-\varphi)(1-\vartheta)}^{-(\vartheta+\varphi-\vartheta\varphi)} \Delta\Delta_a^{-(1-\varphi)(1-\vartheta)} f(x),$

(ii) $\Delta_{a+1-\vartheta}^{-\vartheta}[\Delta_a^{\vartheta,\varphi} f(x)] = \Delta_{a+(1-\varphi)(1-\vartheta)}^{-(\vartheta+\varphi-\vartheta\varphi)} \Delta_a^{\vartheta+\varphi-\vartheta\varphi} f(x),$

(iii) $\Delta_{a+\vartheta}^{\vartheta,\varphi}[\Delta_a^{-\vartheta} f(x)] = \Delta_{a+(1-\varphi+\vartheta\varphi)}^{-\varphi(1-\vartheta)} \Delta_a^{\varphi(1-\vartheta)} f(x),$

(iv) $\Delta_{a+\vartheta}^{\vartheta,\varphi}[\Delta_a^{-\vartheta} f(x)] = f(x) - \Delta_a^{-(1-\varphi(1-\vartheta))} f(a+1-\varphi(1-\vartheta)) \times h_{\varphi(1-\vartheta)-1}(x, a+1-\varphi(1-\vartheta)).$

Proof. (i) On the left hand side, if we use Definition 3.1.1 and (Theorem 5 [116]), then we obtain

$$\begin{aligned}\Delta_{a+1-\vartheta}^{-\vartheta}[\Delta_a^{\vartheta,\varphi} f(x)] =& \Delta_{a+1-\vartheta}^{-\vartheta}[\Delta_{a+(1-\varphi)(1-\vartheta)}^{-\varphi(1-\vartheta)} \Delta\Delta_a^{-(1-\varphi)(1-\vartheta)} f(x)]\\ =& \Delta_{a+(1-\varphi)(1-\vartheta)}^{-(\vartheta+\varphi-\vartheta\varphi)} \Delta\Delta_a^{-(1-\varphi)(1-\vartheta)} f(x).\end{aligned}$$

(ii) On the left hand side, use (i) and first part of (Lemma 6 [116]),

$$\begin{aligned}\Delta_{a+1-\vartheta}^{-\vartheta}[\Delta_a^{\vartheta,\varphi} f(x)] =& \Delta_{a+(1-\varphi)(1-\vartheta)}^{-(\vartheta+\varphi-\vartheta\varphi)} \Delta\Delta_a^{-(1-\varphi)(1-\vartheta)} f(x)\\ =& \Delta_{a+(1-\varphi)(1-\vartheta)}^{-(\vartheta+\varphi-\vartheta\varphi)} \Delta_a^{\vartheta+\varphi-\vartheta\varphi} f(x).\end{aligned}$$

(iii) Using Definition 3.1.1 and (Theorem 5 [116]), we get

$$\begin{aligned}\Delta_{a+\vartheta}^{\vartheta,\varphi}[\Delta_a^{-\vartheta} f(x)] =& \Delta_{a+\vartheta+(1-\varphi)(1-\vartheta)}^{-\varphi(1-\vartheta)} \Delta\Delta_{a+\vartheta}^{-(1-\varphi)(1-\vartheta)}[\Delta_a^{-\vartheta} f(x)]\\ =& \Delta_{a+(1-\varphi+\vartheta\varphi)}^{-\varphi(1-\vartheta)} \Delta\Delta_a^{-(1-\varphi+\vartheta\varphi)} f(x)\\ =& \Delta_{a+(1-\varphi+\vartheta\varphi)}^{-\varphi(1-\vartheta)} \Delta_a^{\varphi(1-\vartheta)} f(x).\end{aligned}$$

In preceding step, we also used first part of (Lemma 6 [116]).

(iv) Consider the left hand side, use (iii) and second part of (Theorem 8 [116]),

$$\begin{aligned}\Delta_{a+\vartheta}^{\vartheta,\varphi}[\Delta_a^{-\vartheta} f(x)] =&\Delta_{a+(1-\varphi+\vartheta\varphi)}^{-\varphi(1-\vartheta)}\Delta_a^{\varphi(1-\vartheta)} f(x)\\ =&\Delta_{a+1-\varphi(1-\vartheta)}^{-\varphi(1-\vartheta)}\Delta_a^{\varphi(1-\vartheta)} f(x)\\ =&f(x)-\Delta_a^{-(1-\varphi(1-\vartheta))} f(a+1-\varphi(1-\vartheta))\\ &\times h_{\varphi(1-\vartheta)-1}(x,a+1-\varphi(1-\vartheta)).\end{aligned}$$

□

For nonempty set N_a^T, the set of all real-valued bounded functions $B(N_a^T)$ is a normed space with $||f|| = \sup_{x\in\mathbb{N}_a^T}\{f(x)\}$. We consider a weighted space of bounded functions $B_\lambda(N_a^T) := \{f : N_a^T \to \mathbb{R}; |(x-a-\vartheta)^{\underline{\lambda}} f(x)| < M\}$ with $0 \le \lambda < \vartheta$ and $M > 0$. The weighted space of bounded functions is considered for finding left inverse property, but further analysis is not influenced by this space.

Lemma 3.1.3. *Let* $f \in B_\lambda(N_a^T)$ *be given and* $0 < \lambda \le 1$. *Then* $\Delta_a^{-\vartheta} f(a+\vartheta) = 0$ *for* $0 \le \lambda < \vartheta$.

Proof. Since $f \in B_\lambda(N_a^T)$, we have $|(x-a-\vartheta)^{\underline{\lambda}} f(x)| < M$, for some positive integer M and for each $x \in N_a^T$. Therefore it follows

$$\begin{aligned}|\Delta_a^{-\vartheta} f(x)| <&M[\Delta_a^{-\vartheta}(y-a-\vartheta)^{\underline{-\lambda}}](x)\\ \le&M\Gamma(1-\lambda)\frac{(x-a-\vartheta)^{\underline{\vartheta-\lambda}}}{\Gamma(\vartheta-\lambda+1)}.\end{aligned}$$

In the preceding step, we used the fact $\Delta_a^{-\vartheta}(x-a)^{\underline{-\lambda}} = (x-a)^{\underline{\vartheta-\lambda}}\frac{\Gamma(1-\lambda)}{\Gamma(\vartheta-\lambda+1)}$. The desired result is achieved by applying limit process $x \to a+\vartheta$. □

Next, we state the left inverse property.

Lemma 3.1.4. *Assume* $0 < \vartheta < 1$, $0 \le \varphi \le 1$ *and* $\eta = \vartheta + \varphi - \vartheta\varphi$, *then for* $f \in B_{1-\eta}(N_a^T)$,

$$\Delta_{a+\vartheta}^{\vartheta,\varphi}[\Delta_a^{-\vartheta} f(x)] = f(x).$$

Proof. Since $0 \le 1-\eta < 1-\varphi(1-\vartheta)$. Thus Lemma 3.1.3 gives $\Delta_a^{-(1-\varphi+\vartheta\varphi)} f(a+1-\varphi+\vartheta\varphi) = 0$. Hence the result follows from the part (iv) of Lemma 3.1.2. □

3.2 Substantial fractional difference

Findings of this section appeared in [108]. Substantial fractional order integral and derivative were introduced by Chen and Deng [57] in recent form. The definition in RL sense is as follows: Assume that a function f is $(m-1)$-times continuously differentiable on interval (a,∞) and mth order derivatives are integrable on some finite subinterval of $[a,\infty)$, where $m-1<\vartheta<m$ for a positive integer m. Furthermore assume σ to be a constant, then $D_s^{\vartheta}f(x)=D_s^m\left[I_s^{(m-\vartheta)}f(x)\right]$,

where $D_s^m=\left(\dfrac{\partial}{\partial x}+\sigma\right)^m$, and $I_s^{(m-\vartheta)}=\displaystyle\int_{y=a}^{y=x}\frac{(x-y)^{m-\vartheta-1}}{\Gamma(m-\vartheta)}e^{-\sigma(x-y)}f(y)dy.$

Before giving the formal definition of difference operator, we define the product $e_{c_1}(x,a)e_{c_2}(y,a)$ for $x,y\in\mathbb{N}_a$ as a solution of the delta partial difference Cauchy problem

$$c_2\Delta_x\chi(x,y)-c_1\Delta_y\chi(x,y)=0,$$

$$\text{with}\quad \chi(x,a)=e_{c_1}(x,a),\quad \chi(a,y)=e_{c_2}(y,a),$$

where $c_1,c_2\in\mathcal{R}$ for the set $\mathcal{R} of regressive functions$. Note that the product of two exponential functions in continuous calculus enjoy the exponent law, i.e. $e^{c_1x}e^{c_2y}=e^{c_1x+c_2y}$. This is the key motivation behind the product of two delta exponential functions in discrete calculus. Surprisingly, the analogous result does not hold in general for the discrete case. However $e_c(x,0)e_c(y,0)=e_c(x+y,0)$ holds for $x,y\in\mathbb{N}_0$.

Lizama [140] considered abstract fractional difference equations with the kernel of Poisson distribution. To define fractional substantial sum, here we shall use the same kernel in discrete setting, specifically by using the delta exponential and Taylor monomial on discrete time scale as in [10].

Definition 3.2.1. *Assume $f:\mathbb{N}_a\to\mathbb{R}$, $0<\vartheta\in\mathbb{R}$ and a constant $-p\in\mathcal{R}$. Then the fractional* *substantial sum of f of order ϑ is defined by*

${}^s\Delta_a^{-\vartheta} f(x) := \sum_{\mathscr{T}=a}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{T}))e_{-p}(x-\mathscr{T},0)f(\mathscr{T})$, *for* $x \in \mathbb{N}_{a+\vartheta}$.

Definition 3.2.2. *Assume $f : \mathbb{N}_a \to \mathbb{R}$, $m-1 < \vartheta < m$ with positive integer m and a constant $-p \in \mathscr{R}$. Then for $x \in \mathbb{N}_{a+m-\vartheta}$ the fractional substantial difference of f of order ϑ is defined by ${}^s\Delta^{\vartheta} f(x) := {}^s\Delta^m[{}^s\Delta_{a+\vartheta}^{-(m-\vartheta)} f(x)]$, where ${}^s\Delta^m = (\frac{\Delta_x+p}{1-p})^m$, and Δ_x is delta partial difference with respect to x.*

Remark 2. Note that for $p = 0$, substantial sum and difference operators reduce to RL sum (Definition 2.1.15), and RL difference (Definition 2.1.17), respectively.

3.2.1 Properties of substantial fractional sum and difference operator

Lemma 3.2.3. *(Composition of fractional Sums) Assume $f : \mathbb{N}_a \to \mathbb{R}$ and ϑ, φ are positive real numbers. Then for $x \in \mathbb{N}_{a+\vartheta+\varphi}$, we have*

$$[{}^s\Delta_{a+\varphi}^{-\vartheta}({}^s\Delta_a^{-\varphi} f)](x) = ({}^s\Delta_a^{-(\vartheta+\varphi)} f)(x) = [{}^s\Delta_{a+\vartheta}^{-\varphi}({}^s\Delta_a^{-\vartheta} f)](x).$$

Proof. For $x \in \mathbb{N}_{a+\vartheta+\varphi}$, consider the left hand side

$$\begin{aligned}
[{}^s\Delta_{a+\varphi}^{-\vartheta}({}^s\Delta_a^{-\varphi} f)](x) &= \sum_{\mathscr{T}=a+\varphi}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{T}))e_{-p}(x-\mathscr{T},0)({}^s\Delta_a^{-\varphi} f)(\mathscr{T}) \\
&= \sum_{\mathscr{T}=\varphi}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{T}))e_{-p}(x-\mathscr{T},0)\sum_{w=0}^{\mathscr{T}-\vartheta} h_{\varphi-1}(\mathscr{T},\sigma(w)) \\
&\quad \times e_{-p}(\mathscr{T}-w,0)f(w) \\
&= \sum_{\mathscr{T}=\varphi}^{x-\vartheta}\sum_{w=0}^{\mathscr{T}-\vartheta} e_{-p}(x-w,0)\frac{(x-\sigma(\mathscr{T}))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)}\frac{(\mathscr{T}-\sigma(w))^{\underline{\varphi-1}}}{\Gamma(\varphi)}f(w) \\
&= \frac{1}{\Gamma(\vartheta)\Gamma(\varphi)}\sum_{w=0}^{x-(\vartheta+\varphi)} e_{-p}(x-w,0)\sum_{\mathscr{T}=w+\varphi}^{x-\vartheta}(x-\sigma(\mathscr{T}))^{\underline{\vartheta-1}} \\
&\quad \times (\mathscr{T}-\sigma(w))^{\underline{\varphi-1}}f(w).
\end{aligned}$$

Let $\mathscr{F} - \sigma(w) = y$,

$$\begin{aligned}[{}^s\Delta_{a+\varphi}^{-\vartheta}({}^s\Delta_a^{-\varphi} f)](x) =& \frac{1}{\Gamma(\vartheta)\Gamma(\varphi)} \sum_{w=0}^{x-(\vartheta+\varphi)} e_{-p}(x-w,0) \\ & \times \sum_{y=\varphi-1}^{x-\vartheta-w-1} (x-y-w-2)^{\underline{\vartheta-1}} (y)^{\underline{\varphi-1}} f(w)\end{aligned}$$

$$\begin{aligned}[{}^s\Delta_{a+\varphi}^{-\vartheta}({}^s\Delta_a^{-\varphi} f)](x) =& \frac{1}{\Gamma(\varphi)} \sum_{w=0}^{x-(\vartheta+\varphi)} e_{-p}(x-w,0) \\ & \times \Big[\frac{1}{\Gamma(\vartheta)} \sum_{y=\varphi-1}^{x-\vartheta-w-1} (x-w-1-\sigma(y))^{\underline{\vartheta-1}} (y)^{\underline{\varphi-1}}\Big] f(w).\end{aligned}$$

By using Definition 2.1.15, we get

$$[{}^s\Delta_{a+\varphi}^{-\vartheta}({}^s\Delta_a^{-\varphi} f)](x) = \frac{1}{\Gamma(\varphi)} \sum_{w=0}^{x-(\vartheta+\varphi)} e_{-p}(x-w,0) \big[\Delta_{\varphi-1}^{-\vartheta} x^{\underline{\varphi-1}}\big]_{x\to x-w-1} f(w).$$

By Lemma 2.1.16, we have $\Delta_{\varphi-1}^{-\vartheta} x^{\underline{\varphi}} = \frac{\Gamma(\varphi)}{\Gamma(\vartheta+\varphi)} x^{\underline{\vartheta+\varphi-1}}$, which yields the following

$$\begin{aligned}[{}^s\Delta_{a+\varphi}^{-\vartheta}({}^s\Delta_a^{-\varphi} f)](x) =& \sum_{w=0}^{x-(\vartheta+\varphi)} e_{-p}(x-w,0) \big[\frac{1}{\Gamma(\vartheta+\varphi)} (x-w-1)^{\underline{\vartheta+\varphi-1}}\big] f(w) \\ =& \sum_{w=0}^{x-(\vartheta+\varphi)} h_{\vartheta+\varphi-1}(x,\sigma(w)) e_{-p}(x-w,0) f(w) \\ =& ({}^s\Delta_a^{-(\vartheta+\varphi)} f)(x)\end{aligned}$$

for $x \in \mathbb{N}_{a+\vartheta+\varphi}$. We may interchange ϑ and φ to get

$$[{}^s\Delta_{a+\vartheta}^{-\varphi}({}^s\Delta_a^{-\vartheta} f)](x) = ({}^s\Delta_a^{-(\vartheta+\varphi)} f)(x).$$

□

Lemma 3.2.4. *(Left inverse property) Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $\vartheta > 0$ and for positive integer m, $m-1 < \vartheta < m$. Then for $x \in \mathbb{N}_{a+\vartheta}$,*

$$[{}^s\Delta^{\vartheta}({}^s\Delta_a^{-\vartheta} f)](x) = f(x). \tag{3.1}$$

Proof. First, we prove the identity (3.1) for integer m by induction. Consider the base case for

$$m = 1, \qquad {}^{s}\Delta\{{}^{s}\Delta_a^{-1} f(x)\} = \Big(\frac{\Delta_x + p}{1-p}\Big)\Big[\sum_{\mathcal{F}=a}^{x-1} h_0(x, \sigma(\mathcal{F})) e_{-p}(x-\mathcal{F}, 0) f(\mathcal{F})\Big].$$

Since $h_0(x, \sigma(\mathcal{F})) = 1$, we have

$${}^{s}\Delta\{{}^{s}\Delta_a^{-1} f(x)\} = \frac{\Delta_x}{1-p}\Big[\sum_{\mathcal{F}=a}^{x-1} e_{-p}(x-\mathcal{F}, 0) f(\mathcal{F})\Big] + \frac{p}{1-p}\Big[\sum_{\mathcal{F}=a}^{x-1} e_{-p}(x-\mathcal{F}, 0) f(\mathcal{F})\Big].$$

Now, applying Leibniz formula Lemma 2.1.11 on first bracket, we obtain

$$\begin{aligned}
{}^{s}\Delta\{{}^{s}\Delta_a^{-1} f(x)\} =& \frac{1}{1-p}\Big[\sum_{\mathcal{F}=a}^{x-1} \Delta_x e_{-p}(x-\mathcal{F}, 0) f(\mathcal{F}) + e_{-p}(x+1-x, 0) f(x)\Big] \\
&+ \frac{p}{1-p}\Big[\sum_{\mathcal{F}=a}^{x-1} e_{-p}(x-\mathcal{F}, 0) f(\mathcal{F})\Big] \\
=& \frac{1}{1-p}\Big[\sum_{\mathcal{F}=a}^{x-1} -p e_{-p}(x-\mathcal{F}, 0) f(\mathcal{F}) + (1-p)^{1-0} f(x)\Big] \\
&+ \frac{p}{1-p}\Big[\sum_{\mathcal{F}=a}^{x-1} e_{-p}(x-\mathcal{F}, 0) f(\mathcal{F})\Big] = f(x).
\end{aligned}$$

Assume the statement in Equation (3.1) is true for m. For induction step consider

$$\begin{aligned}
{}^{s}\Delta^{m+1}\ {}^{s}\Delta_a^{-(m+1)} f(x) &= {}^{s}\Delta^{m+1}\{\ {}^{s}\Delta_{a+m}^{-1}\ {}^{s}\Delta_a^{-m}\} f(x) \\
&= {}^{s}\Delta^{m}\{{}^{s}\Delta\ {}^{s}\Delta_{a+m}^{-1}\}\ {}^{s}\Delta_a^{-m} f(x) \\
&= {}^{s}\Delta^{m}\ {}^{s}\Delta_a^{-m} f(x) = f(x).
\end{aligned}$$

For positive integer m and $m-1 < \vartheta \leq m$, we have

$${}^{s}\Delta^{\vartheta}[{}^{s}\Delta_a^{-\vartheta} f(x)] = \ {}^{s}\Delta^{m}\{{}^{s}\Delta_{a+\vartheta}^{-(m-\vartheta)}\}[{}^{s}\Delta_a^{-\vartheta} f(x)].$$

Finally, using Lemma 3.2.3, we arrive at

$${}^{s}\Delta^{\vartheta}[{}^{s}\Delta_a^{-\vartheta} f(x)] = {}^{s}\Delta^{m}\{{}^{s}\Delta_a^{-m} f(x)\} = f(x).$$

□

Lemma 3.2.5. *(Composition of sum with difference) Assume $f : \mathbb{N}_a \to \mathbb{R}$, $\vartheta > 0$ and $k \in \mathbb{N}_0$. Then for $x \in \mathbb{N}_{a+\vartheta}$*

$$\begin{aligned}[\,{}^{s}\Delta_a^{-\vartheta}(\,{}^{s}\Delta^k f)](x) = &\sum_{j=0}^{k} \binom{k}{j} (-p)^{k-j}\,{}^{s}\Delta^{j-\vartheta} f(x) - e_{-p}(x-a+1,0) \\ &\times \sum_{i=0}^{k-1}\sum_{j=0}^{i-1} \binom{i}{j} (-p)^j h_{\vartheta-j+1}(x,a)\,{}^{s}\Delta^{k-i+1} f(a).\end{aligned} \tag{3.2}$$

Further, $\varphi > 0$ such that for positive integer m, $m-1 < \varphi \leq m$. Then for $x \in \mathbb{N}_{a+m-\varphi+\vartheta}$

$$\begin{aligned}[\,{}^{s}\Delta_{a+m-\varphi}^{-\vartheta}(\,{}^{s}\Delta^{\varphi} f)](x) = &\sum_{j=0}^{m} \binom{m}{j} (-p)^{m-j}\,{}^{s}\Delta^{-(\vartheta-\varphi+m-j)} f(x) \\ &- e_{-p}(x-a+1,0) \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} \binom{i}{j} (-p)^j \\ &\times h_{\vartheta-j+1}(x,a)\,{}^{s}\Delta^{\varphi-i+1} f(a+m-\varphi).\end{aligned} \tag{3.3}$$

Proof. Case I: Suppose $\vartheta \notin \mathbb{N}_1^{k-1}$. First note by Lemma 2.1.14 that

$$\Delta_{\mathscr{F}}\Big[h_{\vartheta-1}(x,\mathscr{F})e_{-p}(x-\mathscr{F}+1,0)\Big] = h_{\vartheta-1}(x,\sigma(\mathscr{F}))\Big[pe_{-p}(x-\mathscr{F},0)\Big] - h_{\vartheta-2}(x,\sigma(\mathscr{F}))\Big[e_{-p}(x-\mathscr{F}+1,0)\Big].$$

Now using Definition 3.2.1 and applying summation by parts formula (Lemma 2.1.12), we have

$$\begin{aligned}{}^{s}\Delta_a^{-\vartheta}\big[\,{}^{s}\Delta^k f(x)\big] = &\sum_{\mathscr{F}=a}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{F}))e_{-p}(x-\sigma(\mathscr{F}),0)\big[\,{}^{s}\Delta^k f(\mathscr{F})\big] \\ = &h_{\vartheta-1}(x,\mathscr{F})e_{-p}(x-\mathscr{F}+1,0)\,{}^{s}\Delta^{k-1} f(\mathscr{F})\Big|_{\mathscr{F}=a}^{x-\vartheta+1} \\ &- \sum_{\mathscr{F}=a}^{x-\vartheta}\Big[ph_{\vartheta-1}(x,\sigma(\mathscr{F}))e_{-p}(x-\mathscr{F},0) \\ &- h_{\vartheta-2}(x,\sigma(\mathscr{F}))e_{-p}(x-\mathscr{F}+1,0)\Big]^{s}\Delta^{k-1} f(\mathscr{F}) \\ = &1.e_{-p}(\vartheta,0)\,{}^{s}\Delta^{k-1} f(x-\vartheta+1) - h_{\vartheta-1}(x,a)e_{-p}(x-a+1,0) \\ &\times\,{}^{s}\Delta^{k-1} f(a) - p\sum_{\mathscr{F}=a}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{F}))e_{-p}(x-\mathscr{F},0)^{s}\Delta^{k-1} f(\mathscr{F}) \\ &+ \sum_{\mathscr{F}=a}^{x-\vartheta} h_{\vartheta-2}(x,\sigma(\mathscr{F}))e_{-p}(x-\mathscr{F}+1,0)^{s}\Delta^{k-1} f(\mathscr{F}).\end{aligned}$$

Combining the first term with last sum, we have

$$\begin{aligned}{}^{s}\Delta_a^{-\vartheta}\big[\,{}^{s}\Delta^{k}f(x)\big] &= {}^{s}\Delta_a^{-(\vartheta-1)}\big[\,{}^{s}\Delta^{k-1}f(x)\big] - p\,{}^{s}\Delta_a^{-\vartheta}\big[\,{}^{s}\Delta^{k-1}f(x)\big]\\ &\quad - h_{\vartheta-1}(x,a)e_{-p}(x-a+1,0)\,{}^{s}\Delta^{k-1}f(a)\\ &=\big(-p\,{}^{s}\Delta_a^{-\vartheta} + {}^{s}\Delta_a^{-(\vartheta-1)}\big)\big[\,{}^{s}\Delta^{k-1}f(x)\big]\\ &\quad - e_{-p}(x-a+1,0)h_{\vartheta-1}(x,a)\,{}^{s}\Delta^{k-1}f(a).\end{aligned}$$

Another application of summation by parts formula

$$\begin{aligned}{}^{s}\Delta_a^{-\vartheta}\big[\,{}^{s}\Delta^{k}f(x)\big] =&\big(p^2\,{}^{s}\Delta_a^{-\vartheta} - 2p\,{}^{s}\Delta_a^{-(\vartheta-1)} + {}^{s}\Delta_a^{-(\vartheta-2)}\big)\big[\,{}^{s}\Delta^{k-2}f(x)\big]\\ &- e_{-p}(x-a+1,0)h_{\vartheta-1}(x,a)\,{}^{s}\Delta^{k-1}f(a)\\ &- e_{-p}(x-a+1,0)\big\{-ph_{\vartheta-1}(x,a)+h_{\vartheta-2}(x,a)\big\}\,{}^{s}\Delta^{k-2}f(a).\end{aligned}$$

Again using summation by parts, we get

$$\begin{aligned}{}^{s}\Delta_a^{-\vartheta}\big[\,{}^{s}\Delta^{k}f(x)\big] =&\big(-p^3\,{}^{s}\Delta_a^{-\vartheta} + 3p^2\,{}^{s}\Delta_a^{-(\vartheta-1)} - 3p\,{}^{s}\Delta_a^{-(\vartheta-2)} + {}^{s}\Delta_a^{-(\vartheta-3)}\big)\\ &\times\big[\,{}^{s}\Delta^{k-3}f(x)\big] - e_{-p}(x-a+1,0)h_{\vartheta-1}(x,a)\,{}^{s}\Delta^{k-1}f(a)\\ &- e_{-p}(x-a+1,0)\big\{-ph_{\vartheta-1}(x,a)+h_{\vartheta-2}(x,a)\big\}\,{}^{s}\Delta^{k-2}f(a)\\ &- e_{-p}(x-a+1,0)\big\{p^2h_{\vartheta-1}(x,a)-2ph_{\vartheta-2}(x,a)+h_{\vartheta-3}(x,a)\big\}\\ &\times\,{}^{s}\Delta^{k-3}f(a).\end{aligned}$$

Further $(k-3)$ times application of summation by parts gives

$$\begin{aligned}[\,{}^{s}\Delta_a^{-\vartheta}(\,{}^{s}\Delta^{k}f)](x) =& \sum_{j=0}^{k}\binom{k}{j}(-p)^{k-j}\,{}^{s}\Delta^{j-\vartheta}f(x) - e_{-p}(x-a+1,0)\\ &\times\sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}h_{\vartheta-j+1}(x,a)\,{}^{s}\Delta^{k-i+1}f(a)\end{aligned}$$

where by assumption $h_{\vartheta-j+1}(x,a)$ is well defined for $\vartheta\notin\mathbb{N}_1^{k-1}$.

Case II: Now Suppose $\vartheta\in\mathbb{N}_1^{k-1}$. Then $k-\vartheta\in\mathbb{N}_1$, we have for $x\in\mathbb{N}_{a+\vartheta}$,

$$\begin{aligned}[\,{}^{s}\Delta_a^{-\vartheta}(\,{}^{s}\Delta^{k}f)](x) =&\{\,{}^{s}\Delta^{k-\vartheta}\,{}^{s}\Delta_{a+\vartheta}^{-(k-\vartheta)}\}\,{}^{s}\Delta_a^{-\vartheta}\,{}^{s}\Delta^{k}f(x)\\ =&\,{}^{s}\Delta^{k-\vartheta}\{\,{}^{s}\Delta_{a+\vartheta}^{-(k-\vartheta)}\,{}^{s}\Delta_a^{-\vartheta}\}\,{}^{s}\Delta^{k}f(x)\\ =&\,{}^{s}\Delta^{k-\vartheta}\big[\,{}^{s}\Delta_a^{-k}\,{}^{s}\Delta^{k}f(x)\big].\end{aligned}$$

By Case I and Equation (3.2), we arrive at

$$\begin{aligned}[\,{}^{s}\Delta_a^{-\vartheta}(\,{}^{s}\Delta^k f)](x) = {}^{s}\Delta^{k-\vartheta}\Big[\sum_{j=0}^{k}\binom{k}{j}(-p)^{k-j}\,{}^{s}\Delta^{j-k}f(x) - e_{-p}(x-a+1,0)\\ \times\sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{k-j+1}(x,a)\,{}^{s}\Delta^{k-i+1}f(a)\Big].\end{aligned}$$

Using Lemma 3.1.2 and Lemma 2.1.16, we get

$$\begin{aligned}[\,{}^{s}\Delta_a^{-\vartheta}(\,{}^{s}\Delta^k f)](x) =&\sum_{j=0}^{k}\binom{k}{j}(-p)^{k-j}\,{}^{s}\Delta^{j-\vartheta}f(x) - e_{-p}(x-a+1,0)\\ &\times\sum_{i=0}^{k-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{\vartheta-j+1}(x,a)\,{}^{s}\Delta^{k-i+1}f(a).\end{aligned}$$

Now consider the Equation (3.3) for $x \in \mathbb{N}_{a+m-\varphi+\vartheta}$, where m is positive integer such that $m-1<\varphi\le m$ and define $g(x) :=\,{}^{s}\Delta_a^{-(m-\varphi)}f(x)$ on $\mathbb{N}_{a+m-\varphi}$, then by Lemma 3.2.4 we have

$${}^{s}\Delta_{a+m-\varphi}^{-\vartheta}(\,{}^{s}\Delta^{\varphi}f)(x) = {}^{s}\Delta_{a+m-\varphi}^{-\vartheta}\,{}^{s}\Delta^{m}g(x).$$

By using Equation (3.2)

$$\begin{aligned}{}^{s}\Delta_{a+m-\varphi}^{-\vartheta}(\,{}^{s}\Delta^{\varphi}f)(x) =&\sum_{j=0}^{m}\binom{m}{j}(-p)^{m-j}\,{}^{s}\Delta^{j-\vartheta}g(x) - e_{-p}(x-a+1,0)\\ &\times\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{\vartheta-j+1}(x,a)\,{}^{s}\Delta^{m-i+1}g(a+m-\varphi)\\ =&\sum_{j=0}^{m}\binom{m}{j}(-p)^{m-j}\,{}^{s}\Delta^{j-\vartheta}\,{}^{s}\Delta_a^{-(m-\varphi)}f(x)\\ &- e_{-p}(x-a+1,0)\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{\vartheta-j+1}(x,a)\\ &\times\,{}^{s}\Delta^{m-i+1}\,{}^{s}\Delta_a^{-(m-\varphi)}f(a+m-\varphi)\\ =&\sum_{j=0}^{m}\binom{m}{j}(-p)^{m-j}\,{}^{s}\Delta^{-(\vartheta-\varphi+m-j)}f(x) - e_{-p}(x-a+1,0)\\ &\times\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j h_{\vartheta-j+1}(x,a)\,{}^{s}\Delta^{\varphi-i+1}f(a+m-\varphi).\end{aligned}$$

□

Lemma 3.2.6. *(Relation between Riemann-Liouville and substantial fractional operators) Assume $f : \mathbb{N}_a \to \mathbb{R}$, $m-1 < \vartheta < m$ with positive integer m and a constant $-p \in \mathcal{R}$. Then*

(i) ${}^s\Delta_a^{-\vartheta} f(x) = e_{-p}(x,0)\Delta_a^{-\vartheta}[e_{-p}(-x,0)f(x)]$ *for $x \in \mathbb{N}_{a+\vartheta}$ where ${}^s\Delta_a^{-\vartheta}$ is substantial fractional sum operator and $\Delta_a^{-\vartheta}$ is Riemann-Liouville fractional sum operator.*

(ii) ${}^s\Delta^{\vartheta} f(x) = e_{-p}(x,0)\Delta_{a+\vartheta}^{\vartheta}[e_{-p}(-x,0)f(x)]$ *for $x \in \mathbb{N}_{a+m-\vartheta}$ where ${}^s\Delta^{\vartheta}$ is substantial fractional difference operator and $\Delta_{a+\vartheta}^{\vartheta}$ is Riemann-Liouville fractional difference operator.*

Proof. (i) Note that $e_{-p}(x-\mathcal{F},0) = e_{-p}(x,0)e_{-p}(-\mathcal{F},0)$. For $x \in \mathbb{N}_{a+\vartheta}$, consider

$$\begin{aligned}{}^s\Delta_a^{-\vartheta} f(x) =& \sum_{\mathcal{F}=a}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathcal{F}))e_{-p}(x-\mathcal{F},0)f(\mathcal{F})\\ =& e_{-p}(x,0)\sum_{\mathcal{F}=a}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathcal{F}))e_{-p}(-\mathcal{F},0)f(\mathcal{F})\end{aligned}$$

By Definition 2.1.15

$${}^s\Delta_a^{-\vartheta} f(x) = e_{-p}(x,0)\Delta_a^{-\vartheta}[e_{-p}(-x,0)f(x)].$$

(ii) For $x \in \mathbb{N}_{a+m-\vartheta}$, consider

$$\begin{aligned}{}^s\Delta^{\vartheta} f(x) =& {}^s\Delta^{m}[\ {}^s\Delta_{a+\vartheta}^{-(m-\vartheta)} f(x)]\\ =& (\frac{\Delta_x + p}{1-p})^m \sum_{\mathcal{F}=a+\varphi}^{x+\vartheta-m} h_{m-\vartheta-1}(x,\sigma(\mathcal{F}))e_{-p}(x-\mathcal{F},0)f(\mathcal{F})\\ =& (\frac{\Delta_x + p}{1-p})^{m-1}\Big[(\frac{\Delta_x + p}{1-p})\sum_{\mathcal{F}=a+\varphi}^{x+\vartheta-m} h_{m-\vartheta-1}(x,\sigma(\mathcal{F}))e_{-p}(x-\mathcal{F},0)f(\mathcal{F})\Big]\end{aligned}$$

$$\begin{aligned}{}^s\Delta^{\vartheta} f(x) =& (\frac{\Delta_x + p}{1-p})^{m-1}\Big[\frac{\Delta_x}{1-p}\Big\{e_{-p}(x,0)\sum_{\mathcal{F}=a+\varphi}^{x+\vartheta-m} h_{m-\vartheta-1}(x,\sigma(\mathcal{F}))e_{-p}(-\mathcal{F},0)f(\mathcal{F})\Big\}\\ &+ \frac{p}{1-p}\ {}^s\Delta_{a+\vartheta}^{-(m-\vartheta)} f(x)\Big].\end{aligned}$$

By using Lemma 2.1.14 and Lemma 2.1.10, we have

$$\begin{aligned}{}^s\Delta^{\vartheta} f(x) =& (\frac{\Delta_x + p}{1-p})^{m-1}\Big[\frac{1}{1-p}\Big\{e_{-p}(\sigma(x),0)\Delta_x^1 \sum_{\mathcal{F}=a+\varphi}^{x+\vartheta-m} h_{m-\vartheta-1}(x,\sigma(\mathcal{F}))\\ &\times e_{-p}(-\mathcal{F},0)f(\mathcal{F}) - p\ \ {}^s\Delta_{a+\vartheta}^{-(m-\vartheta)} f(x)\Big\} + \frac{p}{1-p}\ {}^s\Delta_{a+\vartheta}^{-(m-\vartheta)} f(x)\Big],\end{aligned}$$

using the fact $\frac{e_{-p}(\sigma(x),0)}{1-p} = e_{-p}(x,0)$ and Definition 2.1.15, we obtain

$$^{s}\Delta^{\vartheta} f(x) = (\frac{\Delta_x + p}{1-p})^{m-1}\Big[e_{-p}(x,0)\Delta_x^1 \Delta_{a+\vartheta}^{-(m-\vartheta)}\big\{e_{-p}(-\mathscr{F},0)f(\mathscr{F})\big\}\Big].$$

By Lemma 2.1.28

$$\begin{aligned} ^{s}\Delta^{\vartheta} f(x) &= (\frac{\Delta_x + p}{1-p})^{m-1}\Big[e_{-p}(x,0)\Delta_{a+\vartheta}^{-(m-\vartheta-1)}\big\{e_{-p}(-\mathscr{F},0)f(\mathscr{F})\big\}\Big] \\ &= {}^{s}\Delta^{m-1}\big[\,{}^{s}\Delta_{a+\vartheta}^{-(m-\vartheta-1)} f(x)\big]. \end{aligned}$$

Repetition the same process$m-1$ times, we obtain

$$^{s}\Delta^{\vartheta} f(x) = e_{-p}(x,0)\Delta_{a+\vartheta}^{\vartheta}[e_{-p}(-x,0)f(x)].$$

□

Remark 3. One can find the relation between substantial and Caputo difference by making use of relation between substantial and RL difference Lemma 3.2.6, along with the relation given in [4, Theorem 14] for Caputo and RL difference.

Chapter 4

Delta Laplace and double Laplace transform

In this chapter, we introduced the delta double Laplace transform similar to the one presented by Bernstein [49] in such a way that properties and expressions bear a resemblance to that appearing in Debnath [71] for the continuous calculus. The double convolution product that we introduce in this chapter, resemble with the convolution product defined for delta calculus in [50, 103], but it differs from the one defined by Atici in [29]. We consider the problem with constant coefficients in two independent variables and solve partial difference equations with initial data by applying the delta double Laplace transform. Findings of Sections 4.1 4.2 and 4.3 are appeared in [111].

4.1 The delta double Laplace transforms

In this section, we give abstract definition of the delta double Laplace transform. For convenience, we simplify definition to series representation following the pattern by Goodrich and Peterson [103] for the delta Laplace transform. Also conditions for existence, uniqueness and linearity of the delta double Laplace transform has also been revealed.

Definition 4.1.1. *Assume $f : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$. Then the delta double Laplace transform of f*

based at (a,a) is the successive application of the delta Laplace transform on x and y in any order

$$\begin{aligned}\mathscr{L}_2[f(x,y)](p,q) &= \mathscr{L}_x[\mathscr{L}_y\{f(x,y); y \to q\}; x \to p] \\ &= \mathscr{L}_y[\mathscr{L}_x\{f(x,y); x \to p\}; y \to q] \\ &= \mathscr{L}_y[\tilde{F}(p,y); y \to q] \\ &= \tilde{\tilde{F}}(p,q),\end{aligned}$$

where $\mathscr{L}_x$ and $\mathscr{L}_y$ are the delta Laplace transforms (single) based at a with respect to x and y, respectively and that $\mathscr{L}_2$ is the delta double Laplace transform based at (a,a). The delta double Laplace transform of a function $f(x,y)$ of two variables x and y is defined in p-q plane provided that the following double sum converges

$$\mathscr{L}_2\{f\}(p,q) = \int_a^\infty \int_a^\infty e_{\ominus p}(\sigma(x),a)e_{\ominus q}(\sigma(y),a)f(x,y)\Delta x \Delta y$$

for all complex numbers $p \neq -1$ and $q \neq -1$.

One can easily verify by using Lemma 4.1.2 that $\mathscr{L}_x\mathscr{L}_y = \mathscr{L}_y\mathscr{L}_x$. Later in Theorem 4.2, we will prove that the double infinite series is absolutely convergent. It is well known that absolutely convergent series behave nicely and change in the order of summation $\sum_{k=0}^\infty \sum_{j=0}^\infty$ is allowed. Therefore, we can use $\mathscr{L}_x\mathscr{L}_y = \mathscr{L}_y\mathscr{L}_x$.

Lemma 4.1.2. *Assume $f : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$. Then*

$$\mathscr{L}_2[f(x,y)] = \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}}$$

for all complex numbers $p \neq -1$ and $q \neq -1$ such that the infinite series converges.

Proof. By using the definition of the delta double Laplace transform, we have

$$\mathscr{L}_2\{f\}(p,q) = \int_a^\infty \int_a^\infty e_{\ominus p}(\sigma(x),a)e_{\ominus q}(\sigma(y),a)f(x,y)\Delta x \Delta y.$$

Now by the definition of delta integral from discrete calculus, we get

$$\begin{aligned}\mathscr{L}_2\{f\}(p,q) &= \sum_{y=a}^{\infty}\sum_{x=a}^{\infty} e_{\ominus p}(\sigma(x),a)e_{\ominus q}(\sigma(y),a)f(x,y)\\ &= \sum_{y=a}^{\infty}\sum_{x=a}^{\infty}(1\ominus p)^{\sigma(x)-a}(1\ominus q)^{\sigma(y)-a}f(x,y)\\ &= \sum_{y=a}^{\infty}\sum_{x=a}^{\infty}\frac{f(x,y)}{(p+1)^{x+1-a}(q+1)^{y+1-a}}.\end{aligned}$$

In preceding steps, we used the definition of delta exponential function and the fact that $1\ominus p = \frac{1}{1+p}$ and $1\ominus q = \frac{1}{1+q}$, since p and q are regressive functions. In the following step, we let $x-a=j$ and $y-a=k$ to re-index the sums as follow:

$$\mathscr{L}_2[f(x,y)] = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}}.$$

□

Theorem 4.1. *Assume that the functions $f(x,y) : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$, $g(x) : \mathbb{N}_a \to \mathbb{R}$ and $h(y) : \mathbb{N}_a \to \mathbb{R}$ such that the delta double Laplace transforms exists, then the following holds:*

(i) $\mathscr{L}_2\{g(x)\}(p,q) = \frac{1}{q}\mathscr{L}_x\{g(x)\}(p)$,

(ii) $\mathscr{L}_2\{h(y)\}(p,q) = \frac{1}{p}\mathscr{L}_y\{h(y)\}(q)$,

(iii) $f(x,y) = g(x)h(y), implies \mathscr{L}_2\{f(x,y)\}(p,q) = \mathscr{L}_x\{g(x)\}(p)\mathscr{L}_y\{h(y)\}(q)$.

Proof. Under the assumption stated above and by Lemma 4.1.2:

(i) For $p \neq -1$, $q \neq 0, -1$, we have

$$\begin{aligned}\mathscr{L}_2\{g(x)\}(p,q) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{g(a+j)}{(p+1)^{j+1}(q+1)^{k+1}}\\ &= \sum_{k=0}^{\infty}\frac{1}{(q+1)^{k+1}}\sum_{j=0}^{\infty}\frac{g(a+j)}{(p+1)^{j+1}}\\ &= \frac{1}{q}\mathscr{L}_x\{g(x)\}(p).\end{aligned}$$

(ii) The proof is similar to part (i) for $p \neq 0, -1$, $q \neq -1$.

(iii) For $p \neq -1$, $q \neq -1$, we have

$$\begin{aligned}
\mathscr{L}_2\{f(x,y)\}(p,q) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{g(a+j)h(a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\
&= \sum_{j=0}^{\infty} \frac{g(a+j)}{(p+1)^{j+1}} \sum_{k=0}^{\infty} \frac{h(a+k)}{(q+1)^{k+1}} \\
&= \mathscr{L}_x\{g(x)\}(p)\mathscr{L}_y\{h(y)\}(q).
\end{aligned}$$

□

Example 4.1.3. *(i) If $f(x,y) = 1$ for $x, y \in \mathbb{N}_a$, then $\mathscr{L}_2\{1\} = \frac{1}{pq}$,*

(ii) If $f(x,y) = (x-a)^{\underline{m}}(y-a)^{\underline{n}}$ for $x, y \in \mathbb{N}_a$, then $\mathscr{L}_2\{(x-a)^{\underline{m}}(y-a)^{\underline{n}}\} = \frac{m!n!}{p^{m+1}q^{n+1}}$.

(i) By Lemma 4.1.2

$$\begin{aligned}
\mathscr{L}_2\{1\} &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{(p+1)^{j+1}(q+1)^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{1}{(q+1)^{k+1}} \sum_{j=0}^{\infty} \frac{1}{(p+1)^{j+1}} \\
&= \frac{1}{pq}, \quad \textit{for} \;\; p, q \neq 0, -1.
\end{aligned}$$

(ii) By using Theorem 4.1 part (iii), we get

$$\mathscr{L}_2\{(x-a)^{\underline{m}}(y-a)^{\underline{n}}\} = \mathscr{L}_x\{(x-a)^{\underline{m}}\}\mathscr{L}_y\{(y-a)^{\underline{n}}\}.$$

By using Lemma 2.1.27 on the right hand side of above equation

$$\begin{aligned}
\mathscr{L}_2\{(x-a)^{\underline{m}}(y-a)^{\underline{n}}\} &= \frac{m!}{p^{m+1}} \mathscr{L}_y\{(y-a)^{\underline{n}}\} \\
&= \frac{m!}{p^{m+1}} \frac{n!}{q^{n+1}}.
\end{aligned}$$

If we choose either $m = 0$ or $n = 0$, then as a special case of above equation

$$\begin{aligned}
\mathscr{L}_2\{(y-a)^{\underline{n}}\} &= \frac{n!}{pq^{n+1}}, \quad \textit{for} \;\; p, q \neq 0, -1, \\
\mathscr{L}_2\{(x-a)^{\underline{m}}\} &= \frac{m!}{p^{m+1}q}, \quad \textit{for} \;\; p, q \neq 0, -1.
\end{aligned}$$

Coon and Bernstein [49,66] defined the double Laplace transforms and discussed convergence and existence for continuous case. We present now discrete analogue of the double Laplace transforms.

Definition 4.1.4. *Let $f(x,y) : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$, be a function of EO $r_1, r_2 > 0$ with respect to x and y respectively. If there exists a constant $A > 0$ and $m, n \in \mathbb{N}_0$ such that for each $x \in \mathbb{N}_{a+m}$ and $y \in \mathbb{N}_{a+n}$, then inequality $|f(x,y)| \leqslant Ar_1^x r_2^y$ holds, where $A = max\{A_1, A_2\}$ for $|f(x,a)| \leqslant A_1 r_1^x$ and $|f(a,y)| \leqslant A_2 r_2^y$.*

Theorem 4.2. *If a function $f(x,y) : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ is of EO $r_1, r_2 > 0$, then the delta double Laplace transform $\mathscr{L}_2\{f\}(p,q)$ converges absolutely for p and q provided that$|p+1| > r_1$, and $|q+1| > r_2$.*

Proof. Assume $f(x,y) : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ is of EO $r_1, r_2 > 0$. Then there exists a constant $A > 0$ and $m, n \in \mathbb{N}_0$ such that for each $x \in \mathbb{N}_{a+m}$ and $y \in \mathbb{N}_{a+n}$, $|f(x,y)| \leqslant Ar_1^x r_2^y$. Thus for $|p+1| > r_1$, $|q+1| > r_2$ we consider the following:

$$\begin{aligned}
\sum_{k=n}^{\infty}\sum_{j=m}^{\infty}\Big|\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}}\Big| &\leq \sum_{k=n}^{\infty}\sum_{j=m}^{\infty}\frac{Ar_1^{j+a}r_2^{k+a}}{|p+1|^{j+1}|q+1|^{k+1}} \\
&= \frac{Ar_1^a r_2^a}{|p+1|\ |q+1|}\sum_{k=n}^{\infty}\sum_{j=m}^{\infty}\Big(\frac{r_1}{|p+1|}\Big)^j\Big(\frac{r_2}{|q+1|}\Big)^k \\
&= \frac{Ar_1^a r_2^a}{|p+1|\ |q+1|}\frac{\Big(\frac{r_1}{|p+1|}\Big)^m}{\Big(1-\frac{r_1}{|p+1|}\Big)}\frac{\Big(\frac{r_2}{|q+1|}\Big)^n}{\Big(1-\frac{r_2}{|q+1|}\Big)} \\
&= \frac{Ar_1^{a+m}r_2^{a+n}}{|p+1|^m\ |q+1|^n\Big[(|p+1|-r_1)\ (|q+1|-r_2)\Big]} \\
&< \infty.
\end{aligned}$$

Since $|p+1| > r_1$ and $|q+1| > r_2$, therefore $|p+1| - r_1 > 0$, $|q+1| - r_2 > 0$. Hence the delta double Laplace transform of f converges absolutely. $\square$

Theorem 4.2 ensures the existence of the delta double Laplace transform. In general, the converse does not hold. We should consider functions f of some EO $r > 0$, to ensure the delta double Laplace transform of f does converge somewhere in the complex plane outside the both closed balls of radius r_1 and r_2, centered at -1, that is we can choose $r = max\{r_1, r_2\}$ for $|p+1| > r_1$, $|q+1| > r_2$.

Theorem 4.3. *Suppose $f,\ g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$. If the delta double Laplace transform of $f,\ g$ converges for $|p+1| > r_1$, $|q+1| > r_2$, where $r_1, r_2 > 0$, and let $c_1,\ c_2 \in \mathbb{C}$. Then the delta double Laplace transform of $c_1 f + c_2 g$ converges for $|p+1| > r_1$, $|q+1| > r_2$, and that $\mathscr{L}_2\{c_1 f + c_2 g\}(p,q) = c_1 \mathscr{L}_2\{f\}(p,q) + c_2 \mathscr{L}_2\{g\}(p,q)$ converges for $|p+1| > r_1$, $|q+1| > r_2$.*

Proof. Since $f, g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ and the delta double Laplace transform of f, g converges for $|p+1| > r_1$, $|q+1| > r_2$, where $r_1, r_2 > 0$. We have that for $|p+1| > r_1$, $|q+1| > r_2$,

$$\begin{aligned} c_1 \mathscr{L}_2\{f\}(p,q) + c_2 \mathscr{L}_2\{g\}(p,q) &= c_1 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{f(a+j, a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\ &\quad + c_2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{g(a+j, a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(c_1 f + c_2 g)(a+j, a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\ &= \mathscr{L}_2\{c_1 f + c_2 g\}(p,q). \end{aligned}$$

□

Theorem 4.3 exposed the linearity property of the delta double Laplace transform and Theorem 4.4 revealed the uniqueness.

Theorem 4.4. *Let $f, g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ and $r_1 > 0$, $r_2 > 0$. If $\mathscr{L}_2\{f\}(p,q) = \mathscr{L}_2\{g\}(p,q)$, provided $|p+1| > r_1$, $|q+1| > r_2$, with $p, q \neq 0, -1$, then $f(x,y) = g(x,y)$ for all $x, y \in \mathbb{N}_a$.*

Proof. By hypothesis, we have

$$\mathscr{L}_2\{f\}(p,q) = \mathscr{L}_2\{g\}(p,q)$$

for $|p+1| > r_1$, $|q+1| > r_2$. This implies that

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{g(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}}$$

for $|p+1| > r_1$, $|q+1| > r_2$. Since by Theorem 4.2, the double infinite series is absolute convergent, therefore comparison of both sides of above equation imply that

$$f(a+j,a+k) = g(a+j,a+k), \text{for all } j,k \in \mathbb{N}_0.$$

For each fix j and for all $y \in \mathbb{N}_a$, this implies that

$$f(a+j,y) = g(a+j,y).$$

For each fix k, we get

$$f(x,y) = g(x,y), \text{for all } x,y \in \mathbb{N}_a.$$

□

4.2 Properties of the delta double Laplace transform

In this section, following by Bohner et al. [52] we prove some properties of the delta double Laplace transform. We also define double convolution product of discrete functions following convolution product (single) of discrete functions introduced by Goodrich and Peterson [103]. We present the delta double Laplace transform of double convolution product for later use to solve difference equations.

Theorem 4.5. *Assume that $f : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ and $\mathscr{L}_2[f(x,y)]$ exists. If $\mathscr{L}_2[f(x,y)] = \tilde{\tilde{F}}(p,q)$, then*

$$\mathscr{L}_2[f(x-\alpha, y-\beta)H(x-\alpha, y-\beta)] = e_{\ominus p}(\alpha,0)e_{\ominus q}(\beta,0)\Bigg[\tilde{\tilde{F}}(p,q) - \sum_{s=0}^{c-a-1}\sum_{\mathscr{F}=0}^{c-a-1}\frac{f(a+\mathscr{F}, a+s)}{(p+1)^{\mathscr{F}+1}(q+1)^{s+1}}\Bigg]$$

where $H(x,y)$ is the Heaviside unit step function defined by,

$$H(x-\alpha, y-\beta) = \begin{cases} 0, & \text{if } x-\alpha, y-\beta \in \mathbb{N}_a^{c-1}, \\ 1, & \text{if } x-\alpha, y-\beta \in \mathbb{N}_c. \end{cases}$$

Proof. We have by Lemma 4.1.2,

$$\begin{aligned}\mathscr{L}_2[f(x-\alpha, y-\beta)H(x-\alpha, y-\beta)] &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{f(a-\alpha+j, a-\beta+k)H(a-\alpha+j, a-\beta+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\ &= \sum_{k=\beta+c-a}^{\infty}\sum_{j=\alpha+c-a}^{\infty}\frac{f(a-\alpha+j, a-\beta+k)}{(p+1)^{j+1}(q+1)^{k+1}}.\end{aligned}$$

Re-indexing by $j-\alpha = \mathscr{F}$ and $k-\beta = s$,

$$\begin{aligned}\mathscr{L}_2[f(x-\alpha, y-\beta)H(x-\alpha, y-\beta)] &= \sum_{s=c-a}^{\infty}\sum_{\mathscr{F}=c-a}^{\infty}\frac{f(a+\mathscr{F}, a+s)}{(p+1)^{\alpha+\mathscr{F}+1}(q+1)^{\beta+s+1}} \\ &= \frac{1}{(p+1)^{\alpha}(q+1)^{\beta}}\Bigg[\sum_{s=0}^{\infty}\sum_{\mathscr{F}=0}^{\infty}\frac{f(a+\mathscr{F}, a+s)}{(p+1)^{\mathscr{F}+1}(q+1)^{s+1}} \\ &\quad - \sum_{s=0}^{c-a-1}\sum_{\mathscr{F}=0}^{c-a-1}\frac{f(a+\mathscr{F}, a+s)}{(p+1)^{\mathscr{F}+1}(q+1)^{s+1}}\Bigg] \\ &= e_{\ominus p}(\alpha,0)e_{\ominus q}(\beta,0)\Bigg[\tilde{\tilde{F}}(p,q) \\ &\quad - \sum_{s=0}^{c-a-1}\sum_{\mathscr{F}=0}^{c-a-1}\frac{f(a+\mathscr{F}, a+s)}{(p+1)^{\mathscr{F}+1}(q+1)^{s+1}}\Bigg].\end{aligned}$$

In the last step, we use Lemma 4.1.2 with the fact $e_{\ominus p}(\alpha,0) = \frac{1}{(p+1)^{\alpha}}$ and $e_{\ominus q}(\beta,0) = \frac{1}{(q+1)^{\beta}}$. $\square$

Theorem 4.5 gives a different result from its continuous counterpart stated in [71]. We state the useful shifting Theorem 4.6 for discrete setting.

Theorem 4.6. *Assume that $f : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ and $\mathscr{L}_2[f(x,y)]$ exists. If $\mathscr{L}_2[f(x,y)] = \tilde{\tilde{F}}(p,q)$, then* *(i)* $\mathscr{L}_2[f(x-(c-a), y-(c-a))H(x,y)] = \frac{1}{[(p+1)(q+1)]^{c-a}}\tilde{\tilde{F}}(p,q)$,

(ii) $\mathscr{L}_2[f(x+(c-a), y+(c-a))] = [(p+1)(q+1)]^{c-a}\Bigg[\tilde{\tilde{F}}(p,q)$
$- \sum_{s=0}^{c-a-1}\sum_{\mathscr{J}=0}^{c-a-1} \frac{f(a+\mathscr{J},a+s)}{(p+1)^{\mathscr{J}+1}(q+1)^{s+1}}\Bigg]$,

where $H(x,y)$ is the Heaviside unit step function defined by

$$H(x,y) = \begin{cases} 0, & \text{if } \ x,y \in \mathbb{N}_a^{c-1}, \\ 1, & \text{if } \ x,y \in \mathbb{N}_c. \end{cases}$$

Proof. (i) We have by Lemma 4.1.2, $\mathscr{L}_2[f(x-(c-a), y-(c-a))H(x,y)]$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{f(j+2a-c, k+2a-c)H(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}}$$
$$= \sum_{k=c-a}^{\infty}\sum_{j=c-a}^{\infty} \frac{f(j+2a-c, k+2a-c)}{(p+1)^{j+1}(q+1)^{k+1}}.$$

Re-indexing by $\mathscr{J} = j+a-c$ and $s = k+a-c$,

$$= \sum_{s=0}^{\infty}\sum_{\mathscr{J}=0}^{\infty} \frac{f(a+\mathscr{J}, a+s)}{(p+1)^{\mathscr{J}+c-a+1}(q+1)^{s+c-a+1}}$$
$$= \frac{1}{[(p+1)(q+1)]^{c-a}}\sum_{s=0}^{\infty}\sum_{\mathscr{J}=0}^{\infty} \frac{f(a+\mathscr{J}, a+s)}{(p+1)^{\mathscr{J}+1}(q+1)^{s+1}}$$
$$= \frac{1}{[(p+1)(q+1)]^{c-a}}\tilde{\tilde{F}}(p,q).$$

(ii) By using of Lemma 4.1.2 and re-indexing by $\mathscr{J} = j+c-a$ and $s = k+c-a$,

$$\mathscr{L}_2[f(x+(c-a), y+(c-a))] = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{f(j+c,k+c)}{(p+1)^{j+1}(q+1)^{k+1}}$$

$$= \sum_{s=c-a}^{\infty}\sum_{\mathscr{J}=c-a}^{\infty} \frac{f(a+\mathscr{J}, a+s)}{(p+1)^{\mathscr{J}+a-c+1}(q+1)^{s+a-c+1}}$$
$$= [(p+1)(q+1)]^{c-a}\sum_{s=c-a}^{\infty}\sum_{\mathscr{J}=c-a}^{\infty} \frac{f(a+\mathscr{J}, a+s)}{(p+1)^{\mathscr{J}+1}(q+1)^{s+1}}$$
$$= [(p+1)(q+1)]^{c-a}\left[\tilde{\tilde{F}}(p,q) - \sum_{s=0}^{c-a-1}\sum_{\mathscr{J}=0}^{c-a-1} \frac{f(a+\mathscr{J}, a+s)}{(p+1)^{\mathscr{J}+1}(q+1)^{s+1}}\right].$$

□

Theorem 4.7. *Assume that $f(x,y)$ is periodic with periods $T_1, T_2 \in \mathbb{N}_1$ and $\mathscr{L}_2[f(x,y)]$ exists, then*

$$\mathscr{L}_2[f(x,y)] = \frac{1}{[1-e_{\ominus p}(T_1,0)e_{\ominus q}(T_2,0)]}\sum_{j=0}^{T_1-1}\sum_{k=0}^{T_2-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}}.$$

Proof. Under the assumption, we have by Lemma 4.1.2,

$$\begin{aligned}
\mathscr{L}_2[f(x,y)] &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\
&= \sum_{k=0}^{T_2-1}\sum_{j=0}^{T_1-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} + \sum_{k=T_2}^{\infty}\sum_{j=T_1}^{\infty}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\
&= \sum_{k=0}^{T_2-1}\sum_{j=0}^{T_1-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\
&\quad + \sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\frac{f(a+u+T_1,a+v+T_2)}{(p+1)^{T_1+u+1}(q+1)^{T_2+v+1}}.
\end{aligned}$$

In last step, we used $j = T_1 + u$ and $k = T_2 + v$ to re-index second double summation. In second double summation, periodicity of f implies that

$$\begin{aligned}
\mathscr{L}_2[f(x,y)] &= \sum_{k=0}^{T_2-1}\sum_{j=0}^{T_1-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} + \sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\frac{f(a+u,a+v)}{(p+1)^{T_1+u+1}(q+1)^{T_2+v+1}} \\
&= \sum_{k=0}^{T_2-1}\sum_{j=0}^{T_1-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\
&\quad + \Big[\frac{1}{(p+1)}\Big]^{T_1}\Big[\frac{1}{(q+1)}\Big]^{T_2}\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\frac{f(a+u,a+v)}{(p+1)^{u+1}(q+1)^{v+1}} \\
&= \sum_{k=0}^{T_2-1}\sum_{j=0}^{T_1-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\
&\quad + e_{\ominus p}(T_1,0)e_{\ominus q}(T_2,0)\sum_{v=0}^{\infty}\sum_{u=0}^{\infty}\frac{f(a+u,a+v)}{(p+1)^{u+1}(q+1)^{v+1}} \\
&= \sum_{k=0}^{T_2-1}\sum_{j=0}^{T_1-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} + e_{\ominus p}(T_1,0)e_{\ominus q}(T_2,0)\mathscr{L}_2[f(x,y)] \\
&= \frac{1}{[1-e_{\ominus p}(T_1,0)e_{\ominus q}(T_2,0)]}\sum_{k=0}^{T_2-1}\sum_{j=0}^{T_1-1}\frac{f(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}}.
\end{aligned}$$

□

Definition 4.2.1. *Let $f, g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$. The double convolution product is defined by,*

$$(f**g)(x,y) = \sum_{r=a}^{x-1}\sum_{s=a}^{y-1} f(r,s)g(x-\sigma(r)+a, y-\sigma(s)+a) \;\; for \;\; x,y \in \mathbb{N}_a$$

with empty sums convention $(f**g)(a,a) = 0$.

Lemma 4.2.2. *Let $f, g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$. The double convolution product is commutative, namely*

$$(f**g)(x,y) = (g**f)(x,y) \;\; for \;\; x,y \in \mathbb{N}_a.$$

Proof. By Definition 4.2.1 and the change of variables $x - r - 1 + a = u$ and $y - s - 1 + a = v$

$$\begin{aligned}(g**f)(x,y) &= \sum_{r=a}^{x-1}\sum_{s=a}^{y-1} g(r,s)f(x-\sigma(r)+a, y-\sigma(s)+a) \;\; for \;\; x,y \in \mathbb{N}_a, \\ &= \sum_{u=a}^{x-1}\sum_{v=a}^{y-1} g(x-\sigma(u)+a, y-\sigma(v)+a)f(u,v), \\ &= (f**g)(x,y), \;\; \text{for } x,y \in \mathbb{N}_a.\end{aligned}$$

□

Theorem 4.8. *(Convolution theorem) Let $f, g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$. If both $\mathscr{L}_2[f(x,y)]$, and $\mathscr{L}_2[g(x,y)]$ exist, then the delta double Laplace transform of double convolution product is*

$$\mathscr{L}_2\{(f**g)(x,y)\} = \mathscr{L}_2\{f(x,y)\}\mathscr{L}_2\{g(x,y)\}.$$

Proof. Under the assumption of Theorem we have by Lemma 4.1.2, and the fact $(f**g)(a,a) = 0$

$$\begin{aligned}\mathscr{L}_2\{(f**g)(x,y)\} &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(f**g)(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{(f**g)(a+j,a+k)}{(p+1)^{j+1}(q+1)^{k+1}} \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{(p+1)^{j+1}(q+1)^{k+1}} \\ &\quad \times \sum_{r=a}^{a+k-1}\sum_{s=a}^{a+j-1} f(r,s)g\big(a+j-\sigma(r)+a, a+k-\sigma(s)+a\big).\end{aligned}$$

In last step, we used the Definition 4.2.1, next making the change of variables $r \to a + r$ and $s \to a + s$ gives us that

$$\begin{aligned}
\mathscr{L}_2\{(f**g)(x,y)\} &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{k-1}\sum_{s=0}^{j-1} \frac{f(a+r,a+s)g(a+j-\sigma(r),a+k-\sigma(s))}{(p+1)^{j+1}(q+1)^{k+1}} \\
&= \sum_{k=1}^{\infty}\sum_{r=0}^{k-1}\sum_{j=1}^{\infty}\sum_{s=0}^{j-1} \frac{f(a+r,a+s)g(a+j-\sigma(r),a+k-\sigma(s))}{(p+1)^{j+1}(q+1)^{k+1}} \\
&= \sum_{r=0}^{\infty}\sum_{k=1}^{\infty}\sum_{s=0}^{\infty}\sum_{j=1}^{\infty} \frac{f(a+r,a+s)g(a+j-\sigma(r),a+k-\sigma(s))}{(p+1)^{j+1}(q+1)^{k+1}} \\
&= \sum_{r=0}^{\infty}\sum_{\mathscr{F}_2=0}^{\infty}\sum_{s=0}^{\infty}\sum_{\mathscr{F}_1=0}^{\infty} \frac{f(a+r,a+s)g(a+\mathscr{F}_1,a+\mathscr{F}_2)}{(p+1)^{\mathscr{F}_1+r+2}(q+1)^{\mathscr{F}_2+s+2}} \\
&= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{f(a+r,a+s)}{(p+1)^{r+1}(q+1)^{s+1}} \sum_{\mathscr{F}_2=0}^{\infty}\sum_{\mathscr{F}_1=0}^{\infty} \frac{g(a+\mathscr{F}_1,a+\mathscr{F}_2)}{(p+1)^{\mathscr{F}_1+1}(q+1)^{\mathscr{F}_2+1}} \\
&= \mathscr{L}_2\{f(x,y)\}\mathscr{L}_2\{g(x,y)\}.
\end{aligned}$$

In previous steps, we interchange the order of first pairs and second pairs of summation and change variables $j - r - 1 = \mathscr{F}_1$ and $k - s - 1 = \mathscr{F}_2$. □

Corollary 4.2.3. *Let $f, g : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$. If $f(x,y) = \chi_1(x)\psi_1(y)$, and $g(x,y) = \chi_2(x)\psi_2(y)$ and the delta Laplace transforms exists, then*

$$\mathscr{L}_2\{(f**g)(x,y)\} = \mathscr{L}_x\{(\chi_1 * \chi_2)(x)\}\mathscr{L}_y\{(\psi_1 * \psi_2)(y)\},$$

where the product on right and left hand sides are given by Definition 4.2.1 and Definition 2.1.25, respectively.

Proof. By double convolution theorem, we have

$$\mathscr{L}_2\{(f**g)(x,y)\} = \mathscr{L}_2\{f(x,y)\}\mathscr{L}_2\{g(x,y).\}$$

Since $\mathscr{L}_2[f(x,y)] = \mathscr{L}_2[\chi_1(x)\psi_1(y)] = \mathscr{L}_x[\chi_1(x)]\mathscr{L}_y[\psi_1(y)]$, and $\mathscr{L}_2[g(x,y)] = \mathscr{L}_2[\chi_2(x)\psi_2(y)] = \mathscr{L}_x[\chi_2(x)]\mathscr{L}_y[\psi_2(y)]$,

$$\text{Consider} \qquad \begin{aligned}\mathscr{L}_2\{(f**g)(x,y)\} &= \mathscr{L}_x[\chi_1(x)]\mathscr{L}_y[\psi_1(y)]\mathscr{L}_x[\chi_2(x)]\mathscr{L}_y[\psi_2(y)] \\ &= \mathscr{L}_x[\chi_1(x)]\mathscr{L}_x[\chi_2(x)]\mathscr{L}_y[\psi_1(y)]\mathscr{L}_y[\psi_2(y)] \\ &= \mathscr{L}_x\{(\chi_1 * \chi_2)(x)\}\mathscr{L}_y\{(\psi_1 * \psi_2)(y)\}.\end{aligned}$$

Last step followed from single convolution Lemma 2.1.26. □

4.3 Delta double Laplace transform of partial differences

In this section, we examine the action of the delta double Laplace transform on first order partial differences. The results developed for first order partial differences are further used to establish properties of the delta double Laplace transform of generalized order partial difference, similar to that appeared in [25] for fractional order partial derivatives. We usually consider functions $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ of EO $r_1, r_2 > 0$ with respect to x and y, respectively, ensuring that delta and the delta double Laplace transform of $\chi(x,y)$ and its partial differences exist.

Lemma 4.3.1. *Assume $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ such that the delta Laplace transforms exists for constants $p \neq -1$, $q \neq -1$. Then*

$$\mathscr{L}_x\Delta_x[\chi(x,y)] = p\mathscr{L}_x\{\chi(x,y)\} - \chi(a,y), \tag{4.1}$$

$$\mathscr{L}_y\Delta_y[\chi(x,y)] = q\mathscr{L}_y\{\chi(x,y)\} - \chi(x,a), \tag{4.2}$$

$$\mathscr{L}_x\Delta_y[\chi(x,y)] = \Delta_y\mathscr{L}_x\chi(x,y), \tag{4.3}$$

$$\mathscr{L}_y\Delta_x[\chi(x,y)] = \Delta_x\mathscr{L}_y\chi(x,y). \tag{4.4}$$

Proof. By definition of the delta Laplace transform on x,

$$\mathscr{L}_x\Delta_x[\chi(x,y)] = \int_a^\infty e_{\ominus p}(\sigma(x),a)\Delta_x\chi(x,y)\Delta x.$$

Apply summation by parts Theorem 2.1.12 on x, and using the fact

$\Delta_x[e_{\ominus p}(\sigma(x), a)] = \ominus p e_{\ominus p}(x, a)$, we have that

$$\mathscr{L}_x \Delta_x[\chi(x,y)] = e_{\ominus p}(x,a)\chi(x,y)\Big|_{x=a}^{\infty} - \int_a^{\infty} \chi(x,y)[\ominus p e_{\ominus p}(x,a)]\Delta x.$$

Use the the fact $e_{\ominus p}(x,a) = \frac{1}{(p+1)^{x-a}}$ and $\ominus p = \frac{-p}{(p+1)}$,

$$\mathscr{L}_x \Delta_x[\chi(x,y)] = \frac{1}{(p+1)^{x-a}}\chi(x,y)\Big|_{x=a}^{\infty} - \int_a^{\infty} \chi(x,y)\left(\frac{-p}{(p+1)}\right)e_{\ominus p}(x,a)\Delta x.$$

Since $(p+1)e_{\ominus p}(\sigma(x), a) = e_{\ominus p}(x, a)$,

$$\begin{aligned}\mathscr{L}_x \Delta_x[\chi(x,y)] &= [0 - \chi(a,y)] + p\int_a^{\infty} \chi(x,y) e_{\ominus p}(\sigma(x), a)\Delta x \\ &= -\chi(a,y) + p\mathscr{L}_x\{\chi(x,y)\} \\ &= p\mathscr{L}_x\{\chi(x,y)\} - \chi(a,y).\end{aligned}$$

Let $\mathscr{L}_x\chi(x,y) = \tilde{u}(p,y)$. Consider the left hand side of equation (4.3) and use the definition of delta difference

$$\mathscr{L}_x \Delta_y[\chi(x,y)] = \mathscr{L}_x[\chi(x,y+1) - \chi(x,y)].$$

By using linearity property of the delta Laplace transform, we get

$$\begin{aligned}\mathscr{L}_x \Delta_y[\chi(x,y)] &= \mathscr{L}_x\chi(x,y+1) - \mathscr{L}_x\chi(x,y) \\ &= \tilde{u}(p,y+1) - \tilde{u}(p,y). \qquad (4.5)\end{aligned}$$

Now consider the right hand side of equation (4.3) and use $\mathscr{L}_x\chi(x,y) = \tilde{u}(p,y)$

$$\Delta_y \mathscr{L}_x[\chi(x,y)] = \Delta_y \tilde{u}(p,y).$$

By using the definition of delta difference, we get

$$\Delta_y \mathscr{L}_x[\chi(x,y)] = \tilde{u}(p,y+1) - \tilde{u}(p,y). \qquad (4.6)$$

Equality holds in equation (4.3) from equations (4.5) and (4.6). Proof of equations (4.2), (4.4) is similar to those of equations (4.1), (4.3), respectively. □

Theorem 4.9. *Assume* $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ *such that the delta double Laplace transforms exists for constants* $p \neq -1$, $q \neq -1$. *Then*

(i) $\mathscr{L}_2\Delta_x[\chi(x,y)] = p\mathscr{L}_2\{\chi(x,y)\} - \mathscr{L}_y\{\chi(a,y)\}$,

(ii) $\mathscr{L}_2\Delta_y[\chi(x,y)] = q\mathscr{L}_2\{\chi(x,y)\} - \mathscr{L}_x\{\chi(x,a)\}$.

Proof. Since by definition the delta double Laplace transform is the successive application of the delta Laplace transform on x and y in any order, therefore $\mathscr{L}_2 = \mathscr{L}_x\mathscr{L}_y = \mathscr{L}_y\mathscr{L}_x$.

(i) Consider

$$\mathscr{L}_2\Delta_x[\chi(x,y)] = \mathscr{L}_y[\mathscr{L}_x\Delta_x\chi(x,y)].$$

By using equation (4.1) of Lemma 4.3.1 , we get

$$\mathscr{L}_2\Delta_x[\chi(x,y)] = \mathscr{L}_y[p\mathscr{L}_x\{\chi(x,y)\} - \chi(a,y)].$$

Use linearity property of the delta Laplace transform for $\mathscr{L}_y$,

$$\mathscr{L}_2\Delta_x[\chi(x,y)] = p\mathscr{L}_2\{\chi(x,y)\} - \mathscr{L}_y\chi(a,y)].$$

(ii) The proof is similar to that of part (i). □

Note that for constant a, $\Delta_x\{\chi(a,y)\} = \chi(a,y) - \chi(a,y) = 0$. We adopt the following symbols in our result which are non zero in general $\Delta_x\{\chi(a,y)\} = \Delta_x\{\chi(x,y)\}\Big|_{x=a}$, $\Delta_y\{\chi(x,a)\} = \Delta_y\{\chi(x,y)\}\Big|_{y=a}$ that is first we take difference and then evaluate at a.

Lemma 4.3.2. *Assume* $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ *such that delta Laplace transforms exist for constants* $p \neq -1$, $q \neq -1$. *Then* *(i)* $\mathscr{L}_x\Delta_x^2[\chi(x,y)] = p^2\mathscr{L}_x\{\chi(x,y)\} - p\chi(a,y) - \Delta_x\{\chi(a,y)\}$,

(ii) $\mathscr{L}_y\Delta_y^2[\chi(x,y)] = q^2\mathscr{L}_y\{\chi(x,y)\} - q\chi(x,a) - \Delta_y\{\chi(x,a)\}$,

(iii) $\mathscr{L}_x\Delta_{xy}[\chi(x,y)] = p\mathscr{L}_x\Delta_y\{\chi(x,y)\} - \Delta_y\{\chi(a,y)\}$,

(iv) $\mathscr{L}_y\Delta_{xy}[\chi(x,y)] = q\mathscr{L}_y\Delta_x\{\chi(x,y)\} - \Delta_x\{\chi(x,a)\}$.

Theorem 4.10. *Assume $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$, such that delta double Laplace transforms exists for constants $p \neq -1$, $q \neq -1$. Then*

(i) $\mathscr{L}_2\Delta_x^2[\chi(x,y)] = p^2\mathscr{L}_2\{\chi(x,y)\} - p\mathscr{L}_y\{\chi(a,y)\} - \mathscr{L}_y\Delta_x\{\chi(a,y)\}$,

(ii) $\mathscr{L}_2\Delta_y^2[\chi(x,y)] = q^2\mathscr{L}_2\{\chi(x,y)\} - q\mathscr{L}_x\{\chi(x,a)\} - \mathscr{L}_x\Delta_y\{\chi(x,a)\}$,

(iii) $\mathscr{L}_2\Delta_{xy}[\chi(x,y)] = pq\mathscr{L}_2[\chi(x,y)] - q\mathscr{L}_y\{\chi(a,y)\} - p\mathscr{L}_x\{\chi(x,a)\} + \chi(a,a)$.

Proof. (iii) Consider

$$\mathscr{L}_2\Delta_{xy}[\chi(x,y)] = \mathscr{L}_x\mathscr{L}_y\Delta_{xy}[\chi(x,y)].$$

Using Lemma 4.3.2 part (iv), we have

$$\mathscr{L}_2\Delta_{xy}[\chi(x,y)] = \mathscr{L}_x[q\mathscr{L}_y\Delta_x\{\chi(x,y)\} - \Delta_x\{\chi(x,a)\}].$$

By linearity property of delta Laplace transform for $\mathscr{L}_x$ and the fact that $\mathscr{L}_x\mathscr{L}_y = \mathscr{L}_y\mathscr{L}_x$,

$$\mathscr{L}_2\Delta_{xy}[\chi(x,y)] = q\mathscr{L}_y[\mathscr{L}_x\Delta_x\{\chi(x,y)\}] - \mathscr{L}_x[\Delta_x\{\chi(x,a)\}].$$

Equation (4.1) of Lemma 4.3.1 and linearity property of delta Laplace transform for $\mathscr{L}_y$ imply that

$$\mathscr{L}_2\Delta_{xy}[\chi(x,y)] = pq\mathscr{L}_2\{\chi(x,y)\} - q\mathscr{L}_y\chi(a,y) - p\mathscr{L}_x\{\chi(x,a)\} + \chi(a,a).$$

□

Now, we generalized the results for non negative integer m and n.

Lemma 4.3.3. *Assume $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ such that the delta Laplace transforms exists for constants $p \neq -1$, $q \neq -1$. Then*

(i) $\mathscr{L}_x \Delta_x^n[\chi(x,y)] = p^n \mathscr{L}_x\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k} \Delta_x^k \chi(a,y)$,

(ii) $\mathscr{L}_y \Delta_y^m[\chi(x,y)] = q^m \mathscr{L}_y\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j} \Delta_y^j \chi(x,a)$,

(iii) $\mathscr{L}_x \Delta_{xy}^{nm}[\chi(x,y)] = p^n \mathscr{L}_x \Delta_y^m\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k} \Delta_x^k \Delta_y^m \chi(a,y)$,

(iv) $\mathscr{L}_y \Delta_{xy}^{nm}[\chi(x,y)] = q^m \mathscr{L}_y \Delta_x^n\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j} \Delta_x^n \Delta_y^j \chi(x,a)$.

Proof. (i) We prove this part by induction on n. The case for $n = 1$ has been proved in Lemma 4.3.1. Assume that the result is true for $n \geq 1$,

$$\mathscr{L}_x \Delta_x^n[\chi(x,y)] = p^n \mathscr{L}_x\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k} \Delta_x^k \chi(a,y).$$

We will establish result for $n+1$, start with the following

$$\mathscr{L}_x \Delta_x^{n+1}[\chi(x,y)] = \mathscr{L}_x[\Delta_x \Delta_x^n \chi(x,y)].$$

Let $\phi(x,y) = \Delta_x^n[\chi(x,y)]$ we have that

$$\mathscr{L}_x \Delta_x^{n+1}[\chi(x,y)] = \mathscr{L}_x[\Delta_x \phi(x,y)].$$

Again using equation (4.1) of Lemma 4.3.1,

$$\begin{aligned}\mathscr{L}_x \Delta_x^{n+1}[\chi(x,y)] &= p\mathscr{L}_x\{\phi(x,y)\} - \phi(a,y)] \\ &= p\mathscr{L}_x\{\Delta_x^n[\chi(x,y)]\} - \Delta_x^n[\chi(a,y)].\end{aligned}$$

By using assumption for n,

$$\begin{aligned}
\mathscr{L}_x\Delta_x^{n+1}[\chi(x,y)] &= p[p^n\mathscr{L}_x\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k}\Delta_x^k\chi(a,y)] - \Delta_x^n[\chi(a,y)] \\
&= p^{n+1}\mathscr{L}_x\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-k}\Delta_x^k\chi(a,y) - p^{n-n}\Delta_x^n[\chi(a,y)] \\
&= p^{n+1}\mathscr{L}_x\{\chi(x,y)\} - \sum_{k=0}^{n} p^{n-k}\Delta_x^k\chi(a,y).
\end{aligned}$$

The result holds for $n+1$, whenever it holds for n. Hence by induction, result in part (i) holds.

(ii) We prove this part by induction on m, case for $m = 1$ has been proved in Lemma 4.3.1.

Assume that the result is true for $m \geq 1$,

$$\mathscr{L}_y\Delta_y^m[\chi(x,y)] = q^m\mathscr{L}_y\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j}\Delta_y^j\chi(x,a).$$

We will establish result for $m+1$, begin with the following

$$\mathscr{L}_y\Delta_y^{m+1}[\chi(x,y)] = \mathscr{L}_y[\Delta_y\Delta_y^m\chi(x,y)].$$

Let $\phi(x,y) = \Delta_y^m[\chi(x,y)]$, we have that

$$\mathscr{L}_y\Delta_y^{m+1}[\chi(x,y)] = \mathscr{L}_y[\Delta_y\phi(x,y)].$$

Again using equation (4.2) of Lemma 4.3.1,

$$\begin{aligned}
\mathscr{L}_y\Delta_y^{m+1}[\chi(x,y)] &= q\mathscr{L}_y\{\phi(x,y)\} - \phi(x,a)] \\
&= q\mathscr{L}_y\{\Delta_y^m[\chi(x,y)]\} - \Delta_y^m[\chi(x,a)].
\end{aligned}$$

By using assumption for m,

$$\begin{aligned}
\mathscr{L}_y\Delta_x^{m+1}[\chi(x,y)] &= q[q^m\mathscr{L}_x\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j}\Delta_y^j\chi(x,a)] - \Delta_y^m[\chi(x,a)] \\
&= q^{m+1}\mathscr{L}_y\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-j}\Delta_y^j\chi(x,a) - q^{m-m}\Delta_y^m[\chi(x,a)] \\
&= q^{m+1}\mathscr{L}_y\{\chi(x,y)\} - \sum_{j=0}^{m} q^{m-j}\Delta_y^j\chi(x,a).
\end{aligned}$$

The result holds for $m+1$, whenever it holds for m. Hence by induction, result in part (ii) holds.

(iii)

$$\mathscr{L}_x \Delta_{xy}^{nm}[\chi(x,y)] = \mathscr{L}_x \Delta_x^n[\Delta_y^m \chi(x,y)].$$

Let $\psi(x,y) = \Delta_y^m \chi(x,y)$, and use part (i) of the same Lemma,

$$\begin{aligned}
\mathscr{L}_x \Delta_{xy}^{nm}[\chi(x,y)] &= \mathscr{L}_x \Delta_x^n[\psi(x,y)] \\
&= p^n \mathscr{L}_x\{\psi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k} \Delta_x^k \psi(a,y) \\
&= p^n \mathscr{L}_x \Delta_y^m\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k} \Delta_x^k \Delta_y^m \chi(a,y).
\end{aligned}$$

Proof of (iv) is similar to that of part (iii). $\square$

Theorem 4.11. *Assume $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ such that the delta double Laplace transforms exists for constants $p \neq -1$, $q \neq -1$. Then*

(i) $\mathscr{L}_2 \Delta_x^n[\chi(x,y)] = p^n \mathscr{L}_2\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k} \mathscr{L}_y\{\Delta_x^k \chi(a,y)\}$,

(ii) $\mathscr{L}_2 \Delta_y^m[\chi(x,y)] = q^m \mathscr{L}_2\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j} \mathscr{L}_y\{\Delta_y^j \chi(x,a)\}$,

(iii) $\mathscr{L}_2 \Delta_{xy}^{nm}[\chi(x,y)] = p^n q^m \Big[\mathscr{L}_2\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{-1-k} \mathscr{L}_y\{\Delta_x^k \chi(a,y)\}$
$- \sum_{j=0}^{m-1} q^{-1-j} \mathscr{L}_x\{\Delta_y^j \chi(x,a)\} + \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} p^{-1-k} q^{-1-j} \{\Delta_{xy}^{kj} \chi(a,a)\} \Big]$.

Proof. Since by definition, the delta double Laplace transform is the successive application of the delta Laplace transform on x and y in any order, therefore $\mathscr{L}_2 = \mathscr{L}_x \mathscr{L}_y = \mathscr{L}_y \mathscr{L}_x$.

(i) Using Lemma 4.3.3 part (i) and linearity of Laplace, we consider the following

$$\begin{aligned}
\mathscr{L}_2\Delta_x^n[\chi(x,y)] &= \mathscr{L}_y[\mathscr{L}_x\Delta_x^n\chi(x,y)] \\
&= \mathscr{L}_y[p^n\mathscr{L}_x\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k}\Delta_x^k\chi(a,y)] \\
&= p^n\mathscr{L}_y\mathscr{L}_x\{\chi(x,y)\} - \mathscr{L}_y\sum_{k=0}^{n-1} p^{n-1-k}\Delta_x^k\chi(a,y) \\
&= p^n\mathscr{L}_2\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k}\mathscr{L}_y\{\Delta_x^k\chi(a,y)\}.
\end{aligned}$$

(ii) Using Lemma 4.3.3 part (ii) and linearity of Laplace, we consider the following

$$\begin{aligned}
\mathscr{L}_2\Delta_y^m[\chi(x,y)] &= \mathscr{L}_x[\mathscr{L}_y\Delta_y^m\chi(x,y)] \\
&= \mathscr{L}_x[q^m\mathscr{L}_y\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j}\Delta_y^j\chi(x,a)] \\
&= q^m\mathscr{L}_x\mathscr{L}_y\{\chi(x,y)\} - \mathscr{L}_x\sum_{j=0}^{m-1} q^{m-1-j}\Delta_y^j\chi(x,a) \\
&= q^m\mathscr{L}_2\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j}\mathscr{L}_x\{\Delta_y^j\chi(x,a)\}.
\end{aligned}$$

(iii) Using Lemma 4.3.3 part (iii) and linearity of Laplace, we consider the following,

$$\begin{aligned}
\mathscr{L}_2\Delta_{xy}^{nm}[\chi(x,y)] &= \mathscr{L}_y[\mathscr{L}_x\Delta_{xy}^{nm}\chi(x,y)] \\
&= \mathscr{L}_y[p^n\mathscr{L}_x\Delta_y^m\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{n-1-k}\Delta_x^k\Delta_y^m\chi(a,y)] \\
&= p^n[\mathscr{L}_y\mathscr{L}_x\Delta_y^m\{\chi(x,y)\}] - p^n\sum_{k=0}^{n-1} p^{-1-k}\Big[\mathscr{L}_y\Delta_x^k\Delta_y^m\chi(a,y)\Big] \\
&= p^n[\mathscr{L}_2\Delta_y^m\{\chi(x,y)\}] - p^n\sum_{k=0}^{n-1} p^{-1-k}\Big[q^m\mathscr{L}_y\Delta_x^k\{\chi(a,y)\} \\
&\qquad - \sum_{j=0}^{m-1} q^{m-1-j}\Delta_x^k\Delta_y^j\chi(a,a)\Big].
\end{aligned}$$

In previous step, we used Lemma 4.3.3 part (iv). In following step, using Theorem 4.11 part

(ii),

$$\begin{aligned}
\mathscr{L}_2\Delta_{xy}^{nm}[\chi(x,y)] = p^n\Bigg[q^m\mathscr{L}_2\{\chi(x,y)\} - \sum_{j=0}^{m-1} q^{m-1-j}\mathscr{L}_y\{\Delta_y^j\chi(x,a)\}\Bigg] \\
- p^n\sum_{k=0}^{n-1} p^{-1-k}\Bigg[q^m\mathscr{L}_y\Delta_x^k\{\chi(a,y)\} - \sum_{j=0}^{m-1} q^{m-1-j}\Delta_x^k\Delta_y^j\chi(a,a)\Bigg] \\
= p^n q^m\Bigg[\mathscr{L}_2\{\chi(x,y)\} - \sum_{k=0}^{n-1} p^{-1-k}\mathscr{L}_y\{\Delta_x^k\chi(a,y)\} \\
- \sum_{j=0}^{m-1} q^{-1-j}\mathscr{L}_x\{\Delta_y^j\chi(x,a)\} + \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} p^{-1-k}q^{-1-j}\{\Delta_{xy}^{kj}\chi(a,a)\}\Bigg].
\end{aligned}$$

□

Now, we solve partial difference equation.

Example 4.3.4. *(a) Consider the partial difference equation*

$$\Delta_x\chi(x,y) - \Delta_y\chi(x,y) = 0$$

$$\textit{with} \quad \chi(x,a) = (x-a)^{\underline{1}}, \quad \chi(a,y) = (y-a)^{\underline{1}}.$$

Application of the delta Laplace transform to initial conditions by Lemma 2.1.27,

$$\mathscr{L}_x\chi(x,a) = \mathscr{L}_x(x-a)^{\ \underline{1}} = \frac{1}{p^2}, \qquad \mathscr{L}_y\chi(a,y) = \mathscr{L}_y(y-a)^{\ \underline{1}} = \frac{1}{q^2}.$$

Apply the delta double Laplace transform to difference equation and then use linearity property

$$\mathscr{L}_2[\Delta_x\chi(x,y) - \Delta_y\chi(x,y)] = 0,$$

$$\mathscr{L}_2\Delta_x\chi(x,y) - \mathscr{L}_2\Delta_y\chi(x,y) = 0.$$

Using Theorem 4.9,

$$[p\mathscr{L}_2\{\chi(x,y)\} - \mathscr{L}_y\{\chi(a,y)\}] - [q\mathscr{L}_2\{\chi(x,y)\} - \mathscr{L}_x\{\chi(x,a)\}] = 0, \tag{4.7}$$

$$\begin{aligned}
(p-q)\mathscr{L}_2\{\chi(x,y)\} &= \frac{1}{q^2} - \frac{1}{p^2}, \\
\mathscr{L}_2\{\chi(x,y)\} &= \frac{1}{pq^2} + \frac{1}{p^2q}.
\end{aligned}$$

Inverting the delta Laplace transform pairs

$$\chi(x,y) = (x-a)^{\underline{1}} + (y-a)^{\underline{1}}.$$

(b) Consider the same partial difference equation as in (a) with slight different initial conditions

$$\chi(x,a) = (x-a)^{\underline{2}}, \quad and \quad \chi(a,y) = (y-a)^{\underline{2}}.$$

Application of the delta Laplace transform to initial conditions by Lemma 2.1.27,

$$\mathscr{L}_x\chi(x,a) = \mathscr{L}_x(x-a)^{\underline{2}} = \frac{2}{p^3}, \mathscr{L}_y\chi(a,y) = \mathscr{L}_y(y-a)^{\underline{2}} = \frac{2}{q^3}.$$

From equation (4.7)
$$(p-q)\mathscr{L}_2\{\chi(x,y)\} = \frac{2}{q^3} - \frac{2}{p^3},$$
$$\mathscr{L}_2\{\chi(x,y)\} = \frac{2}{pq^3} + \frac{2}{p^2q^2} + \frac{2}{p^3q}.$$

Inverting delta Laplace transform pairs

$$\chi(x,y) = (x-a)^{\underline{2}} + 2(x-a)^{\underline{1}}\ \ (y-a)^{\underline{1}} + (y-a)^{\underline{2}}.$$

Let $f : \mathbb{N}_a \to \mathbb{R}$, then the RL FDO $N-1 < \alpha \leq N$ for $N \in \mathbb{N}_1$ is given by $\Delta_a^{\alpha} f(x) = \Delta^N \Delta_a^{-(N-\alpha)} f(x)$ for $x \in \mathbb{N}_{a+N-\alpha}$. By using the discussion and results from Theorem 2.65 to Theorem 2.70 in [103], we take the starting point of the double Laplace $a+\alpha-N$ and $a+N-\alpha$, respectively for sum and difference operator.

Corollary 4.3.5. *Assume $\chi : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ such that the delta double Laplace transforms exists for constants $p \neq 0, -1$, $q \neq 0, -1$ and denote $\mathscr{L}_2\chi(x,y) = \tilde{\tilde{u}}(p,q)$. Then for $\lceil N-\alpha \rceil = M$ and $\lceil k+\alpha-N \rceil = L$, the delta double Laplace transforms of fractional order operators is given by*

$$(i)\ \mathscr{L}_2[\Delta_a^{-\alpha}\chi(x,y)](p,q) = \frac{(p+1)^{\alpha-N}(q+1)^{\alpha-N}}{p^{\alpha}q^{\alpha}}\tilde{\tilde{u}}(p,q), \ where\ N-1<\alpha<N,$$

$$(ii)\ \mathscr{L}_2[\Delta_x^\alpha \chi(x,y)](p,q) = p^\alpha q^{\alpha-N}(p+1)^{N-\alpha-M}(q+1)^{N-\alpha-M}\tilde{\tilde{u}}(p,q)$$
$$- \sum_{k=0}^{N-1} p^{N-1-k}\Big[q^{k+\alpha-N}(q+1)^{L-(k+\alpha-N)}\tilde{u}(a,q)$$
$$- \sum_{j=0}^{L-1} q^j \Delta_x^{k+\alpha-N-1}\chi(a, a+L-(k+\alpha-N))\Big], \tag{4.8}$$

$$(iii)\ \mathscr{L}_2[\Delta_y^\alpha \chi(x,y)](p,q) = p^{\alpha-N} q^{\alpha}(p+1)^{N-\alpha-M}(q+1)^{N-\alpha-M}\tilde{\tilde{u}}(p,q)$$
$$- \sum_{k=0}^{N-1} q^{N-1-k}\Big[p^{k+\alpha-N}(p+1)^{L-(k+\alpha-N)}\tilde{u}(p,a)$$
$$- \sum_{j=0}^{L-1} p^j \Delta_y^{k+\alpha-N-1}\chi(a+L-(k+\alpha-N), a)\Big].$$

Proof. (i) Proof is an implication of Definition 4.1.1 and Theorem 2.67 in [103].

(ii) Result is obtained by application of Theorem 4.11 part (i) and Theorem 2.70 in [103]. (iii) Result is obtained by application of Theorem 4.11 part (ii) and Theorem 2.70 in [103]. □

Example 4.3.6. *Consider the fractional difference equation for* $0 < \alpha < 1$,

$$\Delta_x^\alpha \chi(x,y) = (y-a)^{\underline{1}} \quad with \quad \chi(a,y) = 0. \tag{4.9}$$

Apply the delta Laplace transforms to initial condition $\mathscr{L}_y\chi(a,y) = \mathscr{L}_y 0 = 0$. *For* $0 < \alpha < 1$, *we have* $N = 1$ *which implies* $k = 0$ *and hence* $\lceil k + \alpha - 1\rceil = L = 0$, *and* $\lceil 1 - \alpha\rceil = M = 1$. *Application of the delta double Laplace transforms on both sides of FDE in* (4.9) *and making use of equation* (4.8) *on left hand side, and on the right hand side we used Example 4.1.3 to obtain,*

$$\frac{p^\alpha q^{\alpha-1}}{(p+1)^\alpha (q+1)^\alpha}\tilde{\tilde{u}}(p,q) - p^0 q^{\alpha-1}(q+1)^{1-\alpha}\tilde{u}(a,q) = \frac{1}{pq^2}.$$

Using $\tilde{u}(a,q) = 0$ *and simplifying the above*

$$\tilde{\tilde{u}}(p,q) = \frac{(p+1)^\alpha (q+1)^\alpha}{p^{\alpha+1} q^{\alpha+1}}.$$

Inverting the delta Laplace transforms pairs by making use of Theorem 4.1 (iii), together with Lemma 2.1.27 (iii),

$$\chi(x,y) = \frac{(x-a)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\frac{(y-a)^{\underline{\alpha}}}{\Gamma(\alpha+1)}.$$

4.4 Delta Laplace transform of Hilfer like fractional difference

Findings of this section appeared in [109]. In this section, we presented the delta Laplace transform for newly defined Hilfer fractional difference operator. Note that, if in Theorem 4.12, we set $\varphi = 0$, then we recovered [103, Theorem 2.70]. Further, if we set $\varphi = 1$, we obtained the delta Laplace transform for the Caputo FD.

Theorem 4.12. *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of EO $r > 1$ with that $\mathscr{L}_a\{f(x)\}(y) = \tilde{F}_a(y)$ and $0 < \vartheta < 1$, $0 \leq \varphi \leq 1$. Then for $|y+1| > r$ we have the delta Laplace transform given as*

$$\begin{aligned}\mathscr{L}_{a+1-\vartheta}\{\Delta_a^{\vartheta,\varphi} f\}(y) =& y^{\vartheta}(y+1)^{1-\vartheta}\tilde{F}_a(y) \\ & - \frac{(y+1)^{\varphi(1-\vartheta)}}{y^{\varphi(1-\vartheta)}}\Delta_a^{-(1-\varphi)(1-\vartheta)} f(a+(1-\varphi)(1-\vartheta)).\end{aligned}$$

Proof: Considering the left hand side and using the Lemmas 2.1.22 and 2.1.23,

$$\begin{aligned}\mathscr{L}_{a+1-\vartheta}\{\Delta_a^{\vartheta,\varphi} f\}(y) =& \mathscr{L}_{a+1-\vartheta}[\Delta_{a+(1-\varphi)(1-\vartheta)}^{-\varphi(1-\vartheta)}\Delta\Delta_a^{-(1-\varphi)(1-\vartheta)} f(x)](y) \\ =& \frac{(y+1)^{\varphi(1-\vartheta)}}{y^{\varphi(1-\vartheta)}}\mathscr{L}_{a+(1-\varphi)(1-\vartheta)}[\Delta\Delta_a^{-(1-\varphi)(1-\vartheta)} f(x)](y) \\ =& \frac{(y+1)^{\varphi(1-\vartheta)}}{y^{\varphi(1-\vartheta)}}\Big[y\mathscr{L}_{a+(1-\varphi)(1-\vartheta)}[\Delta_a^{-(1-\varphi)(1-\vartheta)} f(x)](y) \\ & - \Delta_a^{-(1-\varphi)(1-\vartheta)} f(a+(1-\varphi)(1-\vartheta))\Big] \\ =& \frac{(y+1)^{\varphi(1-\vartheta)}}{y^{\varphi(1-\vartheta)}}\Big[y\frac{(y+1)^{(1-\varphi)(1-\vartheta)}}{y^{(1-\varphi)(1-\vartheta)}}\mathscr{L}_a[f(x)](y) \\ & - \Delta_a^{-(1-\varphi)(1-\vartheta)} f(a+(1-\varphi)(1-\vartheta))\Big] \\ =& y^{\vartheta}(y+1)^{1-\vartheta}\tilde{F}_a(y) \\ & - \frac{(y+1)^{\varphi(1-\vartheta)}}{y^{\varphi(1-\vartheta)}}\Delta_a^{-(1-\varphi)(1-\vartheta)} f(a+(1-\varphi)(1-\vartheta)).\end{aligned}$$

4.5 Delta Laplace transform of substantial fractional difference, exponential shift and double exponential shift property

Some findings of this section appeared in [108]. In this section, we introduce an important shifting property which is missing in the theory of delta Laplace transform. Only few simple cases have been addressed by implication of the definition [103, Theorem 2.10,Theorem 2.11]. Also delta Laplace transform of substantial operators and double exponential shift property of delta Laplace transform have been presented too.

Lemma 4.5.1. *Let $\mathscr{L}\{f(x)\}(y) = \tilde{F}(y)$. Then for $c \in \mathscr{R}$,*

(i) $\mathscr{L}\{e_c(x,a)f(x)\}(y) = \frac{1}{1+c}\tilde{F}(y \ominus c)$, *where* $y \ominus c = \frac{y-c}{1+c}$,

(ii) $\mathscr{L}\{e_c(-x,0)f(x)\}(y) = (1+c)\tilde{F}(y \oplus c)$.

Proof. (i) By Definition 2.1.19 of delta Laplace transform on a,

$$\mathscr{L}_a\{e_c(x,a)f(x)\}(y) = \int_a^\infty e_{\ominus y}(\sigma(x),a)e_c(x,a)f(x)\Delta x.$$

By Example 2.1.8 and by additive inverse property

$$\mathscr{L}_a\{e_c(x,a)f(x)\}(y) = \frac{1}{1+c}\int_a^\infty e_{\ominus y}(\sigma(x),a)e_{\ominus[\ominus c]}(\sigma(x),a)f(x)\Delta x.$$

By using Lemma 2.1.13,

$$\mathscr{L}_a\{e_c(x,a)f(x)\}(y) = \frac{1}{1+c}\int_a^\infty e_{\ominus[y\ominus c]}(\sigma(x),a)f(x)\Delta x.$$

Again by Definition 2.1.19 of delta Laplace transform,

$$\mathscr{L}_a\{e_c(x,a)f(x)\}(y) = \frac{1}{1+c}\tilde{F}(y \ominus c).$$

(ii) By using the fact that $e_c(-x,0) = e_{\ominus c}(x,0) = (1+c)e_{\ominus c}(\sigma(x),0)$, one can prove it on the similar line as in part (i). $\square$

Theorem 4.13. *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of EO $r > 1$ with that $\mathscr{L}_a\{f(x)\}(y) = \tilde{F}_a(y)$ and $\vartheta > 0$. Then for $|y+1| > r$ we have* $\mathscr{L}_{a+\vartheta}\{\,{}^s\Delta_a^{-\vartheta} f\}(y) = \left(\frac{y+1}{y+p}\right)^{\vartheta} \tilde{F}_a(y)$.

Proof. Considering the left hand side for $-p \in \mathscr{R}$ and using the Lemma 3.2.6(i),

$$\begin{aligned}
\mathscr{L}_{a+\vartheta}\{\,{}^s\Delta_a^{-\vartheta} f\}(y) =& \mathscr{L}_{a+\vartheta}[e_{-p}(x,0)\Delta_a^{-\vartheta}\{e_{-p}(-x,0)f(x)\}](y) \\
=& \frac{1}{1-p}\Big[\mathscr{L}_{a+\vartheta}\Delta_a^{-\vartheta}\{e_{-p}(-x,0)f(x)\}(y)\Big]_{y\to\frac{y+p}{1-p}} \\
=& \frac{1}{1-p}\Big[\frac{(y+1)^{\vartheta}}{y^{\vartheta}}\mathscr{L}_a\{e_{-p}(-x,0)f(x)\}(y)\Big]_{y\to\frac{y+p}{1-p}}.
\end{aligned}$$

In the preceding steps, we used Lemma 4.5.1(i) and then Lemma 2.1.22. In the following step we applying Lemma 4.5.1(ii),

$$\begin{aligned}
\mathscr{L}_{a+\vartheta}\{\,{}^s\Delta_a^{-\vartheta} f\}(y) =& \frac{1}{1-p}\Big[\frac{(y+1)^{\vartheta}}{y^{\vartheta}}\{(1-p)\tilde{F}_a(y\oplus(-p))\}\Big]_{y\to\frac{y+p}{1-p}} \\
=& \Big[\frac{(y+1)^{\vartheta}}{y^{\vartheta}}\{\tilde{F}_a(y-p-yp)\}\Big]_{y\to\frac{y+p}{1-p}} \\
=& \Big(\frac{y+1}{y+p}\Big)^{\vartheta}\tilde{F}_a(y).
\end{aligned}$$

$\square$

Theorem 4.14. *Assume $f : \mathbb{N}_a \to \mathbb{R}$ is of EO $r \geq 1$ with that $\mathscr{L}_a\{f(x)\}(y) = \tilde{F}_a(y)$ and $m-1 < \vartheta < m$ with positive integer m. Then for $|y+1| > r$,*

$$\begin{aligned}
\mathscr{L}_{a+m-\vartheta}\{\,{}^s\Delta_a^{\vartheta} f\}(y) =& \frac{(y+p)^{\vartheta}}{(1-p)^m}(y+1)^{m-\vartheta}\{\tilde{F}_a(y)\} - \frac{1}{1-p}\sum_{j=0}^{m-1}\Big(\frac{y+p}{1-p}\Big)^j \\
& \times \sum_{\mathscr{T}=a}^{a+m-1-j} h_{-\vartheta-1}(a+m-\vartheta,\sigma(\mathscr{T}))e_{-p}(-\mathscr{T},0)f(\mathscr{T}).
\end{aligned}$$

Proof. Consider the left hand side for $-p \in \mathcal{R}$ and use Lemma 3.2.6 (ii),

$$\begin{aligned}
\mathscr{L}_{a+m-\vartheta}\{\,{}^{s}\Delta_a^{\vartheta} f\}(y) =& \mathscr{L}_{a+m-\vartheta}[e_{-p}(x,0)\Delta_{a+\vartheta}^{\vartheta}\{e_{-p}(-x,0)f(x)\}](y)\\
=& \frac{1}{1-p}\Big[\mathscr{L}_{a+m-\vartheta}\Delta_{a+\vartheta}^{\vartheta}\{e_{-p}(-x,0)f(x)\}(y)\Big]_{y\to\frac{y+p}{1-p}}\\
=& \frac{1}{1-p}\Big[y^{\vartheta}(y+1)^{m-\vartheta}\mathscr{L}_a\{e_{-p}(-x,0)f(x)\}(y)\\
& -\sum_{j=0}^{m-1} y^j\big\{\Delta_a^{\vartheta-1-j}e_{-p}(-x,0)f(x)\big\}_{x\to a+m-\vartheta}\Big]_{y\to\frac{y+p}{1-p}}.
\end{aligned}$$

In the preceding steps, we used Lemma 4.5.1(i) and then Lemma 2.1.24. In the following step we apply Lemma 4.5.1(ii) and Definition 2.1.17,

$$\begin{aligned}
\mathscr{L}_{a+m-\vartheta}\{\,{}^{s}\Delta_a^{\vartheta} f\}(y) =& \frac{1}{1-p}\Big[y^{\vartheta}(y+1)^{m-\vartheta}\{(1-p)\tilde{F}_a(y\oplus(-p))\} - \sum_{j=0}^{m-1} y^j\\
&\Big\{\sum_{\mathscr{T}=a}^{x+\vartheta-1-j} h_{-\vartheta-1}(x,\sigma(\mathscr{T}))e_{-p}(-\mathscr{T},0)f(\mathscr{T})\Big\}_{x\to a+m-\vartheta}\Big]_{y\to\frac{y+p}{1-p}}\\
=& \frac{1}{1-p}\Big[y^{\vartheta}(y+1)^{m-\vartheta}\{(1-p)\tilde{F}_a(y-p-yp)\} - \sum_{j=0}^{m-1} y^j\\
&\times\sum_{\mathscr{T}=a}^{a+m-1-j} h_{-\vartheta-1}(a+m-\vartheta,\sigma(\mathscr{T}))e_{-p}(-\mathscr{T},0)f(\mathscr{T})\Big]_{y\to\frac{y+p}{1-p}}\\
=& \frac{(y+p)^{\vartheta}}{(1-p)^m}(y+1)^{m-\vartheta}\{\tilde{F}_a(y)\} - \frac{1}{1-p}\sum_{j=0}^{m-1}\Big(\frac{y+p}{1-p}\Big)^j\\
&\times\sum_{\mathscr{T}=a}^{a+m-1-j} h_{-\vartheta-1}(a+m-\vartheta,\sigma(\mathscr{T}))e_{-p}(-\mathscr{T},0)f(\mathscr{T}).
\end{aligned}$$

□

The product of two exponential function in continuous calculus $e^{c_1x}e^{c_2y} = e^{c_1x+c_2y}$, motivate the product of two delta exponential function in discrete calculus $e_{c_1}(x,a)e_{c_2}(y,a)$. Surprisingly, analogous result $e_c(x,a)e_c(y,a) = e_c(x+y,a), \quad x,y \in \mathbb{N}_a$, does not holds in general for discrete case, where $c \in \mathcal{R}$. However $e_c(x,a)e_c(y,0) = e_c(x,0)e_c(y,a) = e_c(x+y,a)$ hold but it is not useful for isolated time scale unless $a=0$. Further, more generally $e_c(x,a-b)e_c(y,b) = e_c(x+y,a)$ where $c \in \mathcal{R}$ on all $\mathbb{N}_a, \mathbb{N}_b$ and $\mathbb{N}_{a-b}$. Note that $c \in \mathcal{R}$ on $\mathbb{N}_a$ does not ensure $c \in \mathcal{R}$ on $\mathbb{N}_b$, as a counter example $x-2 \in \mathscr{R}$ on $\mathbb{N}_{\frac{1}{2}}$ but not on $\mathbb{N}_1$. The product $e_{c_1}(x,a)e_{c_2}(y,a) = e_{c_1 \otimes c_2}(x,y,a)$ is defined as unique solution of IVP.

Definition 4.5.2. *Assume $c_1, c_2 \in \mathscr{R}$. Denote the double delta exponential function by $e_{c_1 \otimes c_2}(x,y,a)$ which define to be the unique solution $e_{c_1}(x,a)e_{c_2}(y,a)$ for $x, y \in \mathbb{N}_a$ of the IVP*

$$c_2 \Delta_x \chi(x,y) - c_1 \Delta_y \chi(x,y) = 0, \qquad \chi(a,a) = 1.$$

Note that for derivation of double delta exponential function by application of delta double Laplace transform, we need two initial conditions.

Theorem 4.15. *Let $c_1, c_2 \in \mathscr{R}$. Then the IVP*

$$c_2 \Delta_x \chi(x,y) - c_1 \Delta_y \chi(x,y) = 0,$$

$$\textit{with} \quad \chi(x,a) = e_{c_1}(x,a), \quad \chi(a,y) = e_{c_2}(y,a).$$

has a unique solution $\chi(x,y) = e_{c_1}(x,a)e_{c_2}(y,a)$ for $x, y \in \mathbb{N}_a$.

Proof. Let $\mathscr{L}_2\{\chi(x,y)\}(p,q) = \tilde{\tilde{U}}(p,q)$. Applying delta double Laplace transform on partial difference equation and using linearity property of delta double Laplace transform Theorem 4.3,

$$c_2 \mathscr{L}_2 \Delta_x \chi(x,y) - c_1 \mathscr{L}_2 \Delta_y \chi(x,y) = 0.$$

Using Theorem 4.9,

$$c_2[p\mathscr{L}_2\chi(x,y) - \mathscr{L}_y\chi(a,y)] - c_1[q\mathscr{L}_2\chi(x,y) - \mathscr{L}_x\chi(x,a)] = 0.$$

Take delta Laplace of initial conditions by using Example 2.1.21, $\mathscr{L}_x\chi(x,a) = \frac{1}{p-c_1}$ and

$\mathscr{L}_y \chi(a,y) = \frac{1}{q-c_2}$,

$$[c_2 p - c_1 q]\tilde{\tilde{U}}(p,q) = \frac{c_1}{p-c_1} - \frac{c_2}{q-c_2},$$
$$= \frac{c_2 p - c_1 c_2 - c_1 q + c_1 c_2}{(p-c_1)(q-c_2)},$$
$$\tilde{\tilde{U}}(p,q) = \frac{1}{(p-c_1)(q-c_2)}.$$

Inverting delta Laplace transform pairs

$$\chi(x,y) = e_{c_1}(x,a)e_{c_2}(y,a).$$

□

To find the discrete analogue of [71, Equations 11–15], first we define the following.

Definition 4.5.3. *Let $c_1, c_2 \in \mathcal{R}$. Then delta hyperbolic cosine and sine functions for $x, y \in \mathbb{N}_a$ are defined as follows:*

$$cosh_{c_1 \otimes c_2}(x,y,a) =: \frac{e_{c_1 \otimes c_2}(x,y,a) \ + \ e_{-c_1 \otimes -c_2}(x,y,a)}{2},$$

$$sinh_{c_1 \otimes c_2}(x,y,a) =: \frac{e_{c_1 \otimes c_2}(x,y,a) \ - \ e_{-c_1 \otimes -c_2}(x,y,a)}{2}.$$

Definition 4.5.4. *Let $c_1, c_2 \in \mathcal{R}$. Then delta trigonometric cosine and sine functions for $x, y \in \mathbb{N}_a$ are defined as follows:*

$$cos_{c_1 \otimes c_2}(x,y,a) =: \frac{e_{ic_1 \otimes ic_2}(x,y,a) \ + \ e_{-ic_1 \otimes -ic_2}(x,y,a)}{2},$$

$$sin_{c_1 \otimes c_2}(x,y,a) =: \frac{e_{ic_1 \otimes ic_2}(x,y,a) \ - \ e_{-ic_1 \otimes -ic_2}(x,y,a)}{2i}.$$

Example 4.5.5. *For $p \neq -1, c_1$ and $q \neq -1, c_2$*

(i) $\mathscr{L}_2\{e_{c_1 \otimes c_2}(x,y,a)\} = \frac{1}{(p-c_1)(q-c_2)}$,

(ii) $\mathscr{L}_2\Delta_x\{e_{c_1\otimes c_2}(x,y,a)\} = c_1\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}$,

(iii) $\mathscr{L}_2\Delta_y\{e_{c_1\otimes c_2}(x,y,a)\} = c_2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}$,

(iv) $\mathscr{L}_2\Delta_x^2\{e_{c_1\otimes c_2}(x,y,a)\} = c_1^2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}$,

(v) $\mathscr{L}_2\Delta_y^2\{e_{c_1\otimes c_2}(x,y,a)\} = c_2^2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}$,

(vi) $\mathscr{L}_2\Delta_{xy}\{e_{c_1\otimes c_2}(x,y,a)\} = c_1c_2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}$.

(i) For $p \neq -1, c_1;\ q \neq -1, c_2$. *By using Theorem 4.1 part (iii), we get*

$$\begin{aligned}\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\} &= \mathscr{L}_2[e_{c_1}(x,a)e_{c_2}(y,a)]]\\ &= \mathscr{L}_x[e_{c_1}(x,a)]\mathscr{L}_y[e_{c_2}(y,a)]\end{aligned}$$

By using Example 2.1.21 $\quad = \frac{1}{(p-c_1)(q-c_2)}$.

(ii) By Theorem 4.9 part (i),

$$\begin{aligned}\mathscr{L}_2\Delta_x\{e_{c_1\otimes c_2}(x,y,a)\} &= \frac{p}{(p-c_1)(q-c_2)} - \frac{1}{q-c_2}\\ &= \frac{c_1}{(p-c_1)(q-c_2)}\\ &= c_1\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(iii) By Theorem 4.9 part (ii),

$$\begin{aligned}\mathscr{L}_2\Delta_y\{e_{c_1\otimes c_2}(x,y,a)\} &= \frac{q}{(p-c_1)(q-c_2)} - \frac{1}{p-c_1}\\ &= \frac{c_2}{(p-c_1)(q-c_2)}\\ &= c_2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(iv) First note that $\mathscr{L}_y\Delta_x\{e_{c_1\otimes c_2}(a,y,a)\} = \frac{c_1}{q-c_2}$. *Then by Theorem 4.10 part (i).*

$$\begin{aligned}\mathscr{L}_2\Delta_x^2\{e_{c_1\otimes c_2}(x,y,a)\} &= \frac{p^2}{(p-c_1)(q-c_2)} - \frac{p}{q-c_2} - \frac{c_1}{q-c_2}\\ &= \frac{c_1^2}{(p-c_1)(q-c_2)}\\ &= c_1^2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(v) Observe that $\mathscr{L}_x\Delta_y\{e_{c_1\otimes c_2}(x,a,a)\} = \frac{c_2}{p-c_1}$. *By Theorem 4.10 part (ii), we get*

$$\begin{aligned}
\mathscr{L}_2\Delta_y^2\{e_{c_1\otimes c_2}(x,y,a)\} &= \frac{q^2}{(p-c_1)(q-c_2)} - \frac{q}{p-c_1} - \frac{c_2}{p-c_1} \\
&= \frac{c_2^2}{(p-c_1)(q-c_2)} \\
&= c_2^2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}.
\end{aligned}$$

(vi) By Theorem 4.10 part (iii), we get

$$\begin{aligned}
\mathscr{L}_2\Delta_{xy}\{e_{c_1\otimes c_2}(x,y,a)\} &= \frac{pq}{(p-c_1)(q-c_2)} - \frac{q}{q-c_2} - \frac{p}{p-c_1} + 1 \\
&= \frac{c_1c_2}{(p-c_1)(q-c_2)} \\
&= c_1c_2\mathscr{L}_2\{e_{c_1\otimes c_2}(x,y,a)\}.
\end{aligned}$$

Example 4.5.6. *For* $p \neq -1, \pm c_1$ *and* $q \neq -1, \pm c_2$

(i) $\mathscr{L}_2\{\cosh_{c_1\otimes c_2}(x,y,a)\} = \frac{pq+c_1c_2}{(p^2-c_1^2)(q^2-c_2^2)}$,

(ii) $\mathscr{L}_2\{\sinh_{c_1\otimes c_2}(x,y,a)\} = \frac{c_1q+c_2p}{(p^2-c_1^2)(q^2-c_2^2)}$,

(iii) $\mathscr{L}_2\Delta_x\{\cosh_{c_1\otimes c_2}(x,y,a)\} = c_1\mathscr{L}_2\{\sinh_{c_1\otimes c_2}(x,y,a)\}$,

(iv) $\mathscr{L}_2\Delta_y\{\cosh_{c_1\otimes c_2}(x,y,a)\} = c_2\mathscr{L}_2\{\sinh_{c_1\otimes c_2}(x,y,a)\}$,

(v) $\mathscr{L}_2\Delta_x\{\sinh_{c_1\otimes c_2}(x,y,a)\} = c_1\mathscr{L}_2\{\cosh_{c_1\otimes c_2}(x,y,a)\}$,

(vi) $\mathscr{L}_2\Delta_y\{\sinh_{c_1\otimes c_2}(x,y,a)\} = c_2\mathscr{L}_2\{\cosh_{c_1\otimes c_2}(x,y,a)\}$,

(vii) $\mathscr{L}_2\Delta_x^2\{\cosh_{c_1\otimes c_2}(x,y,a)\} = c_1^2\mathscr{L}_2\{\cosh_{c_1\otimes c_2}(x,y,a)\}$,

(viii) $\mathscr{L}_2\Delta_y^2\{\cosh_{c_1\otimes c_2}(x,y,a)\} = c_2^2\mathscr{L}_2\{\cosh_{c_1\otimes c_2}(x,y,a)\}$,

(ix) $\mathscr{L}_2\Delta_x^2\{\sinh_{c_1\otimes c_2}(x,y,a)\} = c_1^2\mathscr{L}_2\{\sinh_{c_1\otimes c_2}(x,y,a)\}$,

(x) $\mathscr{L}_2\Delta_y^2\{\sinh_{c_1\otimes c_2}(x,y,a)\} = c_2^2\mathscr{L}_2\{\sinh_{c_1\otimes c_2}(x,y,a)\}$,

(xi) $\mathscr{L}_2\Delta_{xy}\{\cosh_{c_1\otimes c_2}(x,y,a)\} = c_1c_2\mathscr{L}_2\{\cosh_{c_1\otimes c_2}(x,y,a)\}$,

(xii) $\mathscr{L}_2\Delta_{xy}\{\sinh_{c_1\otimes c_2}(x,y,a)\} = c_1c_2\mathscr{L}_2\{\sinh_{c_1\otimes c_2}(x,y,a)\}$.

(i) For $p \neq -1, \pm c_1;\ q \neq -1, \pm c_2$,

$$\begin{aligned}
\mathscr{L}_2\{\cosh_{c_1\otimes c_2}(x,y,a)\} &= \mathscr{L}_2\left[\frac{e_{c_1\otimes c_2}(x,y,a)\ +\ e_{-c_1\otimes -c_2}(x,y,a)}{2}\right]\\
&= \left[\frac{\frac{1}{(p-c_1)(q-c_2)} + \frac{1}{(p+c_1)(q+c_2)}}{2}\right]\\
&= \frac{pq+c_1c_2}{(p^2-c_1^2)(q^2-c_2^2)}.
\end{aligned}$$

(ii) For $p \neq -1, \pm c_1;\ q \neq -1, \pm c_2$,

$$\begin{aligned}
\mathscr{L}_2\{\sinh_{c_1\otimes c_2}(x,y,a)\} &= \mathscr{L}_2\left[\frac{e_{c_1\otimes c_2}(x,y,a)\ -\ e_{-c_1\otimes -c_2}(x,y,a)}{2}\right]\\
&= \left[\frac{\frac{1}{(p-c_1)(q-c_2)} - \frac{1}{(p+c_1)(q+c_2)}}{2}\right]\\
&= \frac{c_1q+c_2p}{(p^2-c_1^2)(q^2-c_2^2)}.
\end{aligned}$$

Remaining parts can be obtained by application of Theorem 4.9 and Theorem 4.10.

Example 4.5.7. *For* $p \neq -1, \pm ic_1;\ q \neq -1, \pm ic_2$,

(i) $\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\} = \frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)}$,

(ii) $\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\} = \frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)}$,

(iii) $\mathscr{L}_2\Delta_x\{\cos_{c_1\otimes c_2}(x,y,a)\} = -c_1\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}$,

(iv) $\mathscr{L}_2\Delta_y\{\cos_{c_1\otimes c_2}(x,y,a)\} = -c_2\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}$,

(v) $\mathscr{L}_2\Delta_x\{\sin_{c_1\otimes c_2}(x,y,a)\} = c_1\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}$,

(vi) $\mathscr{L}_2\Delta_y\{\sin_{c_1\otimes c_2}(x,y,a)\} = c_2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}$,

(vii) $\mathscr{L}_2\Delta_x^2\{\cos_{c_1\otimes c_2}(x,y,a)\} = -c_1^2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}$,

(viii) $\mathscr{L}_2\Delta_y^2\{\cos_{c_1\otimes c_2}(x,y,a)\} = -c_2^2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}$,

(ix) $\mathscr{L}_2\Delta_x^2\{\sin_{c_1\otimes c_2}(x,y,a)\} = -c_1^2\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}$,

(x) $\mathscr{L}_2\Delta_y^2\{\sin_{c_1\otimes c_2}(x,y,a)\} = -c_2^2\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}$,

(xi) $\mathscr{L}_2\Delta_{xy}\{\cos_{c_1\otimes c_2}(x,y,a)\} = -c_1c_2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}$,

(xii) $\mathscr{L}_2\Delta_{xy}\{\sin_{c_1\otimes c_2}(x,y,a)\} = -c_1c_2\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}$.

(i) For $p \neq -1, \pm ic_1;\ q \neq -1, \pm ic_2$,

$$\begin{aligned}\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\} &= \mathscr{L}_2\left[\frac{e_{ic_1\otimes ic_2}(x,y,a)\ +\ e_{-ic_1\otimes -ic_2}(x,y,a)}{2}\right]\\ &= \left[\frac{\frac{1}{(p-ic_1)(q-ic_2)} + \frac{1}{(p+ic_1)(q+ic_2)}}{2}\right]\\ &= \frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)}.\end{aligned}$$

(ii) For $p \neq -1, \pm ic_1;\ q \neq -1, \pm ic_2$,

$$\begin{aligned}\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\} &= \mathscr{L}_2\left[\frac{e_{ic_1\otimes ic_2}(x,y,a)\ -\ e_{-ic_1\otimes -ic_2}(x,y,a)}{2i}\right]\\ &= \left[\frac{\frac{1}{(p-ic_1)(q-ic_2)} - \frac{1}{(p+ic_1)(q+ic_2)}}{2i}\right]\\ &= \frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)}.\end{aligned}$$

(iii) For $p \neq -1, \pm ic_1;\ q \neq -1, \pm ic_2$, *and by using Theorem 4.9 part (i),*

$$\begin{aligned}\mathscr{L}_2\Delta_x\{\cos_{c_1\otimes c_2}(x,y,a)\} &= p\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2}\left[\frac{1}{q-ic_2}+\frac{1}{q+ic_2}\right]\\ &= p\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2}\left[\frac{2q}{(q^2+c_2^2)}\right]\\ &= \frac{p^2q-pc_1c_2-p^2q-c_1^2q}{(p^2+c_1^2)(q^2+c_2^2)}\\ &= -c_1\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(iv) For $p \neq -1, \pm ic_1;\ q \neq -1, \pm ic_2$, and by using Theorem 4.9 part (ii),

$$\begin{aligned}
\mathscr{L}_2\Delta_y\{\cos_{c_1\otimes c_2}(x,y,a)\} &= q\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2}\left[\frac{1}{p-ic_1}+\frac{1}{p+ic_1}\right] \\
&= q\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2}\left[\frac{2p}{(p^2+c_1^2)}\right] \\
&= \frac{pq^2-qc_1c_2-pq^2-c_2^2p}{(p^2+c_1^2)(q^2+c_2^2)} \\
&= -c_2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}.
\end{aligned}$$

(v) For $p \neq -1, \pm ic_1;\ q \neq -1, \pm ic_2$, and by using Theorem 4.9 part (i),

$$\begin{aligned}
\mathscr{L}_2\Delta_x\{\sin_{c_1\otimes c_2}(x,y,a)\} &= p\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2i}\left[\frac{1}{q-ic_2}-\frac{1}{q+ic_2}\right] \\
&= p\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2i}\left[\frac{2ic_2}{(q^2+c_2^2)}\right] \\
&= \frac{pqc_1+p^2c_2-c_2p^2-c_1^2c_2}{(p^2+c_1^2)(q^2+c_2^2)} \\
&= c_1\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}.
\end{aligned}$$

(vi) For $p \neq -1, \pm ic_1;\ q \neq -1, \pm ic_2$, and by using Theorem 4.9 part (ii),

$$\begin{aligned}
\mathscr{L}_2\Delta_y\{\sin_{c_1\otimes c_2}(x,y,a)\} &= q\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2i}\left[\frac{1}{p-ic_1}-\frac{1}{p+ic_1}\right] \\
&= q\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)} - \frac{1}{2i}\left[\frac{2ic_1}{(p^2+c_1^2)}\right] \\
&= \frac{c_1q^2+pqc_2-c_1q^2-c_1c_2^2}{(p^2+c_1^2)(q^2+c_2^2)} \\
&= c_2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}.
\end{aligned}$$

(vii) First note that $\mathscr{L}_y\Delta_x\{\cos_{c_1\otimes c_2}(x,y,a)\} = \frac{-c_1c_2}{q^2+c_2^2}$. For $p \neq -1, \pm ic_1$ and $q \neq -1, \pm ic_2$, by using Theorem 4.10 part (i),

$$\begin{aligned}
\mathscr{L}_2\Delta_x^2\{\cos_{c_1\otimes c_2}(x,y,a)\} &= p^2\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} - p\frac{q}{q^2+c_2^2} - \frac{-c_1c_2}{q^2+c_2^2} \\
&= -c_1^2\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} \\
&= -c_1^2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}.
\end{aligned}$$

(viii) First note that $\mathscr{L}_x\Delta_y\{\cos_{c_1\otimes c_2}(x,y,a)\} = \frac{-c_1c_2}{p^2+c_1^2}$. For $p \neq -1, \pm ic_1$ and $q \neq -1, \pm ic_2$, by

using Theorem 4.10 part (ii),

$$\begin{aligned}\mathscr{L}_2\Delta_y^2\{\cos_{c_1\otimes c_2}(x,y,a)\} &= q^2\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} - q\frac{p}{p^2+c_1^2} - \frac{-c_1c_2}{p^2+c_1^2}\\ &= -c_2^2\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)}\\ &= -c_2^2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(ix) First note that $\mathscr{L}_y\Delta_x\{\sin_{c_1\otimes c_2}(x,y,a)\} = \frac{c_1q}{q^2+c_2^2}$. *For* $p \neq -1, \pm ic_1$ *and* $q \neq -1, \pm ic_2$, *by using Theorem 4.10 part (i),*

$$\begin{aligned}\mathscr{L}_2\Delta_x^2\{\sin_{c_1\otimes c_2}(x,y,a)\} &= p^2\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)} - p\frac{c_2}{q^2+c_2^2} - \frac{c_1q}{q^2+c_2^2}\\ &= -c_1^2\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)}\\ &= -c_1^2\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(x) First note that $\mathscr{L}_x\Delta_y\{\sin_{c_1\otimes c_2}(x,y,a)\} = \frac{c_2p}{p^2+c_1^2}$. *For* $p \neq -1, \pm ic_1$ *and* $q \neq -1, \pm ic_2$, *by using Theorem 4.10 part (ii),*

$$\begin{aligned}\mathscr{L}_2\Delta_y^2\{\sin_{c_1\otimes c_2}(x,y,a)\} &= q^2\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)} - q\frac{c_1}{p^2+c_1^2} - \frac{c_2p}{p^2+c_1^2}\\ &= -c_2^2\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)}\\ &= -c_2^2\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(xi) For $p \neq -1, \pm ic_1$ *and* $q \neq -1, \pm ic_2$, *by using Theorem 4.10 part (iii), we get*

$$\begin{aligned}\mathscr{L}_2\Delta_{xy}\{\cos_{c_1\otimes c_2}(x,y,a)\} &= pq\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)} - q\frac{q}{q^2+c_2^2} - p\frac{p}{p^2+c_1^2} + 1\\ &= -c_1c_2\frac{pq-c_1c_2}{(p^2+c_1^2)(q^2+c_2^2)}\\ &= -c_1c_2\mathscr{L}_2\{\cos_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

(xii) For $p \neq -1, \pm ic_1$ *and* $q \neq -1, \pm ic_2$, *by using Theorem 4.10 part (iii), we get*

$$\begin{aligned}\mathscr{L}_2\Delta_{xy}\{\sin_{c_1\otimes c_2}(x,y,a)\} &= pq\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)} - q\frac{c_2}{q^2+c_2^2} - p\frac{c_1}{p^2+c_1^2} + 0\\ &= -c_1c_2\frac{c_1q+c_2p}{(p^2+c_1^2)(q^2+c_2^2)}\\ &= -c_1c_2\mathscr{L}_2\{\sin_{c_1\otimes c_2}(x,y,a)\}.\end{aligned}$$

Now, we states and proves the double shift property.

Theorem 4.16. *Assume that double Laplace of f exists, then for $c_1, c_2 \in \mathcal{R}$,*

$$\mathscr{L}_2\{e_{c_1 \otimes c_2}(x,y,a)f(x,y)\}(p,q) = \frac{1}{(1+c_1)(1+c_2)}\tilde{\tilde{F}}(p \ominus c_1, q \ominus c_2).$$

Proof. By Definition 4.1.1 of delta double Laplace transform,

$$\mathscr{L}_2\{e_{c_1 \otimes c_2}(x,y,a)f(x,y)\} = \int_a^\infty \int_a^\infty e_{\ominus p}(\sigma(x),a)e_{\ominus q}(\sigma(y),a)e_{c_1 \otimes c_2}(x,y,a)f(x,y)\Delta x \Delta y.$$

By Definition 4.5.2,

$$= \int_a^\infty \int_a^\infty e_{\ominus p}(\sigma(x),a)e_{\ominus q}(\sigma(y),a)e_{c_1}(x,a)e_{c_2}(y,a)f(x,y)\Delta x \Delta y.$$

By Example 2.1.8, $e_{c_1}(x,a) = \frac{1}{1+c_1}e_{c_1}(\sigma(x),a)$ and $e_{c_2}(y,a) = \frac{1}{1+c_2}e_{c_2}(\sigma(y),a)$

$$= \frac{1}{(1+c_1)(1+c_2)} \int_a^\infty \int_a^\infty e_{\ominus p}(\sigma(x),a)e_{\ominus q}(\sigma(y),a)e_{c_1}(\sigma(x),a)e_{c_2}(\sigma(y),a)f(x,y)\Delta x \Delta y.$$

By using Lemma 2.1.13, $e_{\ominus p}(\sigma(x),a)e_{\ominus[\ominus c_1]}(\sigma(x),a) = e_{\ominus[p \ominus c_1]}(\sigma(x),a)$ and

$e_{\ominus q}(\sigma(y),a)e_{\ominus[\ominus c_2]}(\sigma(y),a) = e_{\ominus[q \ominus c_2]}(\sigma(y),a)$

$$= \frac{1}{(1+c_1)(1+c_2)} \int_a^\infty \int_a^\infty e_{\ominus[p \ominus c]}(\sigma(x),a)e_{\ominus[q \ominus c_2]}(\sigma(y),a)f(x,y)\Delta x \Delta y.$$

Again by definition of delta double Laplace transform 4.1.1,

$$\mathscr{L}_2\{e_{c_1 \otimes c_2}(x,y,a)f(x,y)\} \quad = \frac{1}{(1+c_1)(1+c_2)}\tilde{\tilde{F}}(p \ominus c_1, q \ominus c_2).$$

$\square$

Chapter 5

Fixed point operators and Green's functions

In order to apply FPT for difference equations, we convert the IVP and BVP to equivalent summation equation. In this chapter, we present an alternative approach to obtain a general method for converting the fractional delta difference equation with impulse to an equivalent summation equations. The basic motive is to exhibit a simple and well established method to construct FPO for impulsive delta difference equations of Caputo type with arbitrary order. We shall demonstrate the applicability of procedure by building the operator for a few particular problems of interest with initial and BC in order to find fixed point. We shall derive Green's function with some of its properties for impulsive delta DE with two and four-point BC and nonlinear FDE with multi-point summation boundary conditions. FPO for Hilfer FDE and substantial FDE shall be obtained too. Findings of Sections 5.1 and 5.2 appeared in [110].

5.1 Fixed point operators for IVP with impulse

Motivated by the work in [158] where authors established a method for transforming differential equations of arbitrary order with impulse to their corresponding integral equations and also considered BVP with nonlinear impulsive FDE for existence of solution. In this section, we use the fundamental theorem to obtain an equivalent summation equation for nonlinear difference equation of non integer order $\varphi > 0$ with impulses:

$$\begin{cases} {}^{c}\Delta_a^{\varphi}x(t) + f(\varphi+\rho(t), x(\varphi+\rho(t))) = 0, \quad t \in \mathbb{N}_{a+1-\varphi}, \quad t \neq a+n_j+1-\varphi, \\ \Delta^{j-1}x_k^{+} - \Delta^{j-1}x_k = (-1)^{j-1}\Delta^{j}x_k, \quad t = a+n_j+1-\varphi, \\ x_i = (-1)^i\Delta^i x(a), i = 0, 1, \cdots, r-1, \ \lceil \varphi \rceil = r, \ k \in \mathbb{N}_1^m, \quad j \in \mathbb{N}_0^{k-1}, \end{cases} \tag{5.1}$$

where ${}^{c}\Delta_a^{\varphi}x(t)$ is the Caputo difference of $x(t)$ for $\varphi > 0$. Choose the numbers $n_k \in \mathbb{N}_0$ in such a way that $0 = n_0 < n_1 < \cdots < n_j < n_{j+1}$, and $a + n_k = \iota_k$ are impulse points. Choose $1 < n_{j+1} - n_j$ to ensure that ι_k and ι_{k+1} are separated by at least two time steps.

Remark 4. We define the notations as $\Delta^{j-1}x_k := \Delta^{j-1}x(t)|_{t=\iota_k}$ and $\Delta^{j-1}x_k^{+} := \Delta^{j-1}x(t)|_{t=\iota_k+c_k}$, where c_k are numbers such that $a + c_k \in \mathbb{N}_a$ for $c_k \in \mathbb{N}_0$ and $c_k \neq n_k$.

The concept of continuity in discrete setting is in the sense of topological spaces. The notation $[a, T]_{\mathbb{N}_a}$ is used to denote the set $[a, T] \cap \mathbb{N}_a$ [107]. Under the choice $n_{j+1} - n_j > 1$, there exist atleast one non-impulsive point between two impulsive points ι_k and ι_{k+1}. The purpose of using the space $PC([a, T]_{\mathbb{N}_a}, \mathbb{R}) = \{x : [a, T]_{\mathbb{N}_a} \to \mathbb{R}, x \in C((\iota_k, \iota_{k+1}]_{\mathbb{N}_a}, \mathbb{R}), \ k \in \mathbb{N}_0^m\}$, is to cover up the continuity of all impulsive and non-impulsive points on $(\iota_k, \iota_{k+1}]_{\mathbb{N}_a}$ through the space $C((\iota_k, \iota_{k+1}]_{\mathbb{N}_a}, \mathbb{R})$ and consequently to cover up the continuity of all impulsive and non-impulsive points on $[a, T]_{\mathbb{N}_a}$. The space $PC([a, T]_{\mathbb{N}_a}, \mathbb{R})$ is a Banach space equipped with norm $||x|| = \sup_{t \in [a,T]_{\mathbb{N}_a}} |x(t)|$. To obtain an impulsive solution of system (5.1), first we establish equivalent summation equation for integer-order difference equation accompanied with impulses and initial conditions

$$\begin{cases} \Delta^r x(t) + g(t) = 0, t \in \mathbb{N}_a, \quad t \neq a+n_j, \quad j \in \mathbb{N}_0^{k-1}, \\ \Delta^{j-1}x_k^{+} - \Delta^{j-1}x_k = (-1)^{j-1}\Delta^{j}x_k, \quad t = a+n_j, \quad k \in \mathbb{N}_1^m, \\ x_i = (-1)^i\Delta^i x(a), i = 0, 1, \cdots, r-1, \text{for a positive integer } r. \end{cases} \tag{5.2}$$

Lemma 5.1.1. *Let $g : \mathbb{N}_a \to \mathbb{R}$ be given and let r be a positive integer. Solution $x(t)$ (5.2), can be explained as if and only if*

$$x(t) = \sum_{i=0}^{r-1} \frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)} x_i + \sum_{i=0}^{r-1} \Big\{ \sum_{a<\iota_k<t} \frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)} \Delta^{i+1} x_k \Big\} - \Delta^{-r} g(t). \tag{5.3}$$

Proof. Assume that $\Delta^r x(t) = -g(t)$. For $s \in (\iota_j, \iota_{j+1}],\ k \in \mathbb{N}_1^m$ and $j \in \mathbb{N}_0^{k-1}$, by using Theorem 2.1.29 and Definition 2.1.15, we have k equations for each value of j:

$$\Delta^{r-1} x(s)|_{s=\iota_{j+1}} - \Delta^{r-1} x(s)|_{s=\iota_j + c_j} = -\sum_{s=\iota_j}^{\iota_{j+1}-1} g(s),\ j \in \mathbb{N}_0^{k-1}.$$

For $s \in (\iota_k, t],$ again we apply Theorem 2.1.29 and Definition 2.1.15

$$\Delta^{r-1} x(s)|_{s=t} - \Delta^{r-1} x(s)|_{s=\iota_k + c_k} = -\sum_{s=\iota_k}^{t-1} g(s).$$

Adding these $k+1$ equations to get

$$\Delta^{r-1} x(t) = \Delta^{r-1} x(a) - \sum_{a}^{t-1} g(s) + \sum_{a<\iota_k<t} (-1)^{r-1} \Delta^r x_k.$$

By following the above procedure, we have

$$\Delta^{r-2} x(t) = \Delta^{r-2} x(a) - \sum_{a}^{t-1} \Delta^{r-1} x(s) + \sum_{a<\iota_k<t} (-1)^{r-2} \Delta^{r-1} x_k.$$

$$\Delta^{r-3} x(t) = \Delta^{r-3} x(a) - \sum_{a}^{t-1} \Delta^{r-2} x(s) + \sum_{a<\iota_k<t} (-1)^{r-3} \Delta^{r-2} x_k.$$

$$\vdots$$

$$\Delta^{2} x(t) = \Delta^{2} x(a) - \sum_{a}^{t-1} \Delta^{3} x(s) + \sum_{a<\iota_k<t} (-1)^{2} \Delta^{3} x_k. \tag{5.4}$$

$$\Delta^{1} x(t) = \Delta^{1} x(a) - \sum_{a}^{t-1} \Delta^{2} x(s) + \sum_{a<\iota_k<t} (-1)^{1} \Delta^{2} x_k. \tag{5.5}$$

$$x(t) = x(a) - \sum_{a}^{t-1} \Delta^{1} x(s) + \sum_{a<\iota_k<t} (-1)^{0} \Delta^{1} x_k. \tag{5.6}$$

Substituting (5.5) into (5.6), we get

$$\begin{aligned} x(t) &= x_0 - \sum_{a}^{t-1}\big[-x_1 - \sum_{a}^{s-1}\Delta^2 x(u) - \sum_{a<\iota_k<s}\Delta^2 x_k\big] + \sum_{a<\iota_k<t}\Delta^1 x_k, \\ &= x_0 + (t-a)x_1 + \sum_{a}^{t-1}\sum_{a}^{s-1}\Delta^2 x(u) + \sum_{a}^{t-1}\sum_{a<\iota_k<s}\Delta^2 x_k + \sum_{a<\iota_k<t}\Delta^1 x_k. \end{aligned} \tag{5.7}$$

Interchange the order of summations and evaluation yields

$$\sum_{a}^{t-1}\sum_{a}^{s-1}\Delta^2 x(u) = \sum_{a}^{t-1}(t-u)\Delta^2 x(u)$$
$$\text{and } \sum_{a}^{t-1}\sum_{a<\iota_k<s}\Delta^2 x_k = \sum_{a<\iota_k<t}(t-\iota_k)\Delta^2 x_k.$$

Thus (5.7) implies that

$$\begin{aligned} x(t) =& x_0 + (t-a)x_1 + \sum_{a}^{t-1}(t-u)\Delta^2 x(u) + \sum_{a<\iota_k<t}(t-\iota_k)\Delta^2 x_k \\ &+ \sum_{a<\iota_k<t}\Delta^1 x_k. \end{aligned} \tag{5.8}$$

Substituting (5.4) into (5.8), we get

$$\begin{aligned} x(t) =& x_0 + (t-a)x_1 + \sum_{a<\iota_k<t}(t-\iota_k)\Delta^2 x_k + \sum_{a<\iota_k<t}\Delta^1 x_k \\ &+ \sum_{u=a}^{t-1}(t-u)\big[x_2 - \sum_{s=a}^{u-1}\Delta^3 x(s) + \sum_{a<\iota_k<u}\Delta^3 x_k\big], \\ =& x_0 + (t-a)x_1 + \sum_{a<\iota_k<t}(t-\iota_k)\Delta^2 x_k + \sum_{a<\iota_k<t}\Delta^1 x_k \\ &- \frac{(t+1-u)^{\underline{2}}}{2}x_2\Big|_{u=a}^{u=t} - \sum_{u=a}^{t-1}\sum_{s=a}^{u-1}(t-u)\Delta^3 x(s) \\ &+ \sum_{u=a}^{t-1}\sum_{a<\iota_k<u}(t-u)\Delta^3 x_k. \end{aligned} \tag{5.9}$$

Interchange in the order of summations and evaluation yields

$$\sum_{u=a}^{t-1}\sum_{s=a}^{u-1}(t-u)\Delta^3 x(s) = \sum_{s=a}^{t-1}\frac{(t+1-s)^{\underline{2}}}{2}\Delta^3 x(s)$$
$$\text{and} \qquad \sum_{u=a}^{t-1}\sum_{a<\iota_k<u}(t-u)\Delta^3 x_k = \sum_{a<\iota_k<t}\frac{(t-\iota_k)^{\underline{2}}}{2}\Delta^3 x_k.$$

Equation (5.9) becomes

$$\begin{aligned}
x(t) =& x_0 + (t-a)x_1 + \frac{(t+1-a)^{\underline{2}}}{2}x_2 - \sum_{s=a}^{t-1}\frac{(t+1-s)^{\underline{2}}}{2}\Delta^3 x(s) \\
&+ \sum_{a<\iota_k<t}\Delta^1 x_k + \sum_{a<\iota_k<t}(t-\iota_k)\Delta^2 x_k + \sum_{a<\iota_k<t}\frac{(t-\iota_k)^{\underline{2}}}{2}\Delta^3 x_k, \\
=& \sum_{i=0}^{2}\frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)}x_i + \sum_{i=0}^{2}\Big\{\sum_{a<\iota_k<t}\frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)}\Delta^{i+1}x_k\Big\} \\
&- \sum_{a}^{t-3} h_{3-1}(t,\sigma(s))\Delta^3 x(s).
\end{aligned}$$

Proceeding inductively, we get

$$\begin{aligned}
x(t) =& \sum_{i=0}^{r-1}\frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)}x_i + \sum_{i=0}^{r-1}\Big\{\sum_{a<\iota_k<t}\frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)}\Delta^{i+1}x_k\Big\} \\
&- \sum_{a}^{t-r} h_{r-1}(t,\sigma(s))\Delta^r x(s), \\
=& \sum_{i=0}^{r-1}\frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)}x_i + \sum_{i=0}^{r-1}\Big\{\sum_{a<\iota_k<t}\frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)}\Delta^{i+1}x_k\Big\} \\
&- \sum_{a}^{t-r} h_{r-1}(t,\sigma(s))g(s), \\
=& \sum_{i=0}^{r-1}\frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)}x_i + \sum_{i=0}^{r-1}\Big\{\sum_{a<\iota_k<t}\frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)}\Delta^{i+1}x_k\Big\} \\
&- \Delta_a^{-r}\Delta^r x(t), \\
=& \sum_{i=0}^{r-1}\frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)}x_i + \sum_{i=0}^{r-1}\Big\{\sum_{a<\iota_k<t}\frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)}\Delta^{i+1}x_k\Big\} \\
&- \Delta_a^{-r}g(t).
\end{aligned}$$

Conversely, one proves that (5.3) satisfies (5.2) by straight forward substitution. □

Now, we extend Lemma 5.1.1 to Caputo type difference operators of fractional order φ.

Consider

$$\begin{cases}
{}^c\Delta_a^{\varphi}x(t) + f(\varphi+\rho(t)) = 0, \quad t \in \mathbb{N}_{a+1-\varphi}, \quad t \neq a+n_j+1-\varphi, \quad j \in \mathbb{N}_0^{k-1}, \\
\Delta^{j-1}x_k^{+} - \Delta^{j-1}x_k = (-1)^{j-1}\Delta^j x_k, \quad t = a+n_j+1-\varphi, \quad k \in \mathbb{N}_1^m, \\
x_i = (-1)^i \Delta^i x(a), i = 0, 1, \cdots, r-1, \text{ where } \lceil\varphi\rceil = r.
\end{cases} \tag{5.10}$$

The proof of the following lemma is obvious therefore it is omitted.

Lemma 5.1.2. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and let $r \in \mathbb{N}$ and $r-1 < \varphi \le r$. Solution $x(t)$ of* (5.10), *can explained as if and only if*

$$\begin{aligned} x(t) =& \sum_{i=0}^{r-1} \frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)} x_i + \sum_{i=0}^{r-1} \Big\{ \sum_{a<\iota_k<t} \frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)} \Delta^{i+1} x_k \Big\} \\ & - \Delta_{a+r-\varphi}^{-\varphi} f(\varphi + \rho(t)), \ \textit{for } t \in \mathbb{N}_{a+1}. \end{aligned}$$

Example 5.1.3. *To illustrate the construction in Lemma 5.1.2, consider the example*

$$\begin{cases} {}^c\Delta_0^{\varphi} x(t) + t^{\underline{2-\varphi}} = 0, \quad 1 < \varphi \le 2, \quad t \in \mathbb{N}_{1-\varphi}^{26-\varphi}, \quad t \ne \iota_1, \iota_2, \iota_3, \\ x(\iota_1^+) - x(\iota_1) = 10, \quad \Delta x(\iota_1^+) - \Delta x(\iota_1) = \frac{1}{6}, \ \textit{for } \iota_1 = 7 - \varphi, \\ x(\iota_2^+) - x(\iota_2) = 30, \quad \Delta x(\iota_2^+) - \Delta x(\iota_2) = \frac{1}{6}, \ \textit{for } \iota_2 = 14 - \varphi, \\ x(\iota_3^+) - x(\iota_3) = 50, \quad \Delta x(\iota_3^+) - \Delta x(\iota_3) = \frac{1}{6}, \ \textit{for } \iota_3 = 21 - \varphi, \\ \Delta^i x(0) = 0, \quad i = 0, 1, \end{cases}$$

has solution for $\quad t \in \mathbb{N}_1$,

$$x(t) = \begin{cases} \frac{\Gamma(3-\varphi)}{2} t^{\underline{2}}, & 0 \le t < \iota_1, \\ \frac{\Gamma(3-\varphi)}{2} t^{\underline{2}} + 10 + \frac{1}{6}(t - \iota_1), & \iota_1 \le t < \iota_2, \\ \frac{\Gamma(3-\varphi)}{2} t^{\underline{2}} + 40 + \frac{1}{6}(2t - \iota_1 - \iota_2), & \iota_2 \le t < \iota_3, \\ \frac{\Gamma(3-\varphi)}{2} t^{\underline{2}} + 90 + \frac{1}{6}(3t - \iota_1 - \iota_2 - \iota_3), & \iota_3 \le t \le 25. \end{cases}$$

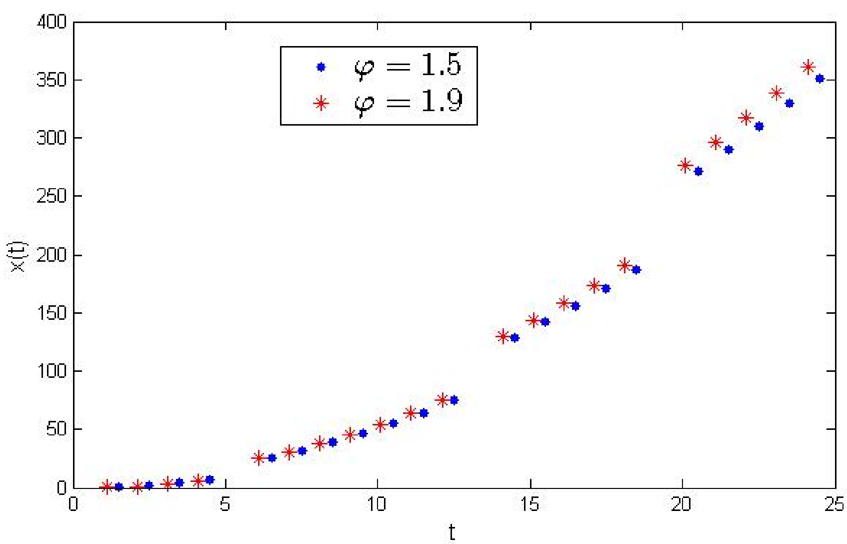

Figure 5.1: The graph of $x(t)$ for $\varphi = 1.5$ and 1.9.

The following corollary conveys that one can apply Lemma 5.1.2 to solve nonlinear impulse difference equation of arbitrary order.

Corollary 5.1.4. *Let $f : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$ be given and let $r \in \mathbb{N}$ and $r-1 < \varphi \le r$. Solution $x(t)$ of* (5.1)*, can explained as if and only if*

$$\begin{aligned} x(t) =& \sum_{i=0}^{r-1} \frac{(t+i-a-1)^{\underline{i}}}{\Gamma(i+1)} x_i + \sum_{i=0}^{r-1} \Big\{ \sum_{a<\iota_k<t} \frac{(t-\iota_k)^{\underline{i}}}{\Gamma(i+1)} \Delta^{i+1} x_k \Big\} \\ & - \Delta_{a+r-\varphi}^{-\varphi} f(\varphi+\rho(t), x(\varphi+\rho(t))). \end{aligned}$$

We apply Corollary 5.1.4 to obtain appropriate FPO for several family of problems for fractional difference equation with impulse. For $t \in \mathbb{N}_{a+1-\varphi}$ consider

$$^{c}\Delta_a^{\varphi} x(t) + f(\varphi+\rho(t), x(\varphi+\rho(t))) = 0, \quad t \neq a+n_j+1-\varphi, \tag{5.11}$$

$$\text{with impulsive conditions,} \quad \Delta^1 x_k = I_k(x(\iota_k)), \tag{5.12}$$

$$\Delta^2 x_k = \bar{I}_k(x(\iota_k)), \quad \text{for} \;\; t = a+n_j+1-\varphi, \text{ and } k \in \mathbb{N}_1^m. \tag{5.13}$$

We use (5.11)–(5.13) under the following set of initial and boundary conditions.

$$\text{For } r-1 < \varphi \le r, \quad (-1)^i \Delta^i x(a) = x_i, \quad i = 0, 1, \cdots, r-1, \tag{5.14}$$

$$\text{for } 1 < \varphi \le 2, \text{the nonlocal conditions, } x(a) = x_0 - h(x), -\Delta x(a) = x_1. \tag{5.15}$$

The first two lemmas are immediate applications of Corollary 5.1.4.

Lemma 5.1.5. *Assume that $f : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are given and $1 < \varphi \le 2$. Then* (5.11)*–*(5.13) *with initial condition* (5.14) *has solution $x(t)$, if and only if*

$$\begin{aligned} x(t) =& x_0 + x_1(t-a) + \sum_{a<\iota_k<t} I_k(x(\iota_k)) + \sum_{a<\iota_k<t} \bar{I}_k(x(\iota_k))(t-\iota_k) \\ & - \Delta_{a+2-\varphi}^{-\varphi} f(\varphi+\rho(t), x(\varphi+\rho(t))). \end{aligned}$$

Lemma 5.1.6. *Let $f : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$ be given and $1 < \varphi \le 2$. Then problem* (5.11)*–*(5.13) *with nonlocal initial condition* (5.15) *has solution $x(t)$, if and only if*

$$\begin{aligned} x(t) =& x_0 - h(x) + x_1(t-a) + \sum_{a<\iota_k<t} I_k(x(\iota_k)) + \sum_{a<\iota_k<t} \bar{I}_k(x(\iota_k))(t-\iota_k) \\ & - \Delta_{a+2-\varphi}^{-\varphi} f(\varphi+\rho(t), x(\varphi+\rho(t))). \end{aligned}$$

5.2 Fixed point operators for impulsive BVP

We use (5.11)–(5.13) under the following sets of BC.

$$\text{for } 0 < \varphi \leq 1, \quad \text{the boundary condition,} \quad c_1 x(a) + c_2 x(T) = \theta \tag{5.16}$$

for $1 < \varphi \leq 2$, the four-point BC,

$$x(a) = \alpha x(\xi),\ x(T) = \beta x(\eta), \quad \xi, \eta \in (a, T)_{\mathbb{N}_a}. \tag{5.17}$$

We state the next two lemmas without proof; we shall provide the related algebraic details in the proof of Lemma 5.2.4.

Lemma 5.2.1. *Let $f : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$ be given and $0 < \varphi \leq 1$ with $c_1 + c_2 \neq 0$. Then problem* (5.11)–(5.12) *with BC* (5.16) *has solution $x(t)$, if and only if*

$$\begin{aligned} x(t) = & \frac{\theta}{c_1 + c_2} + \sum_{a < \iota_k < t} I_k(x(\iota_k)) - \sum_{s=a}^{t-\varphi} h_{\varphi-1}(t, \sigma(s)) f(\varphi + \rho(s), x(\varphi + \rho(s))) \\ & - \frac{c_1}{c_1 + c_2} \Big[\sum_{k=1}^{m} I_k(x(\iota_k)) + \sum_{s=a}^{T-\varphi} h_{\varphi-1}(T, \sigma(s)) f(\varphi + \rho(s), x(\varphi + \rho(s))) \Big]. \end{aligned}$$

For the sake of abbreviation, define

$$p(t) := t(1 - \alpha) + \alpha\xi - a,$$

$$q(t) := t(1 - \beta) + \beta\eta - T,$$

and

$$\delta := (T - \beta\eta + a\beta - a)(1 - \alpha) + (\xi - \xi\beta + a\beta - a)\alpha.$$

We solve the following for $\xi < \eta$, the case for which $\eta < \xi$ can be solved in similar manner.

Lemma 5.2.2. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, $1 < \varphi \leq 2$, and $\delta \neq 0$, for $\alpha, \beta, \xi, \eta \in \mathbb{R}$, $\xi, \eta \in (a, T)_{\mathbb{N}_a}$. Then BVP* (5.11) *with* (5.17) *in absence of impulsive condition, has solution $x(t)$, if and only if*

$$\begin{aligned} x(t) =& \frac{p(t)}{\delta}\Big[\sum_{s=a}^{T-\varphi} h_{\varphi-1}(T, \sigma(s))f(s) - \beta \sum_{s=a}^{\eta-\varphi} h_{\varphi-1}(\eta, \sigma(s))f(s)\Big] \\ &+ \frac{\alpha q(t)}{\delta} \sum_{s=a}^{\xi-\varphi} h_{\varphi-1}(\xi, \sigma(s))f(s) - \sum_{s=a}^{t-\varphi} h_{\varphi-1}(t, \sigma(s))f(s). \end{aligned}$$

Remark 5. One may note that Lemma 5.2.2 is a slight generalization of the problem (1.1) in [91] for three point nonlocal condition.

5.2.1 Green's function for impulsive BVP

Theorem 5.1. *Let $f : \mathbb{N}_a^T \to \mathbb{R}$ be given and $1 < \varphi \leq 2$. Assume $\alpha, \beta, \xi, \eta \in \mathbb{R}$, $\xi, \eta \in (a, T)_{\mathbb{N}_a} with \delta \neq 0$. The solution $x(t)$ of the problem* (5.11) *with boundary conditions* (5.17) *is given by $x(t) = \sum_{s=a}^{T} G(t, s)f(s)$, where Green's function for the boundary value problem is defined by*

$$G(t,s) := \begin{cases} \frac{p(t)}{\delta}\big[h_{\varphi-1}(T, \sigma(s)) - \beta h_{\varphi-1}(\eta, \sigma(s))\big] & \\ +\frac{\alpha q(t)}{\delta} h_{\varphi-1}(\xi, \sigma(s)) - h_{\varphi-1}(t, \sigma(s)), & a \leq s \leq t-\varphi, \\ \frac{p(t)}{\delta}\big[h_{\varphi-1}(T, \sigma(s)) - \beta h_{\varphi-1}(\eta, \sigma(s))\big] & \\ +\frac{\alpha q(t)}{\delta} h_{\varphi-1}(\xi, \sigma(s)), & t-\varphi+1 \leq s \leq \xi-\varphi, \\ \frac{p(t)}{\delta}\big[h_{\varphi-1}(T, \sigma(s)) - \beta h_{\varphi-1}(\eta, \sigma(s))\big] & \\ -h_{\varphi-1}(t, \sigma(s)), & \xi-\varphi+1 \leq s \leq t-\varphi, \\ \frac{p(t)}{\delta}\big[h_{\varphi-1}(T, \sigma(s)) - \beta h_{\varphi-1}(\eta, \sigma(s))\big], & t-\varphi+1 \leq s \leq \eta-\varphi, \\ \frac{p(t)}{\delta} h_{\varphi-1}(T, \sigma(s)) - h_{\varphi-1}(t, \sigma(s)), & \eta-\varphi+1 \leq s \leq t-\varphi, \\ \frac{p(t)}{\delta} h_{\varphi-1}(T, \sigma(s)), & t-\varphi+1 \leq s \leq T-\varphi. \end{cases}$$

Proof. If $a \leq t \leq \xi$, then the solution can be stated from Lemma 5.2.2 as

$$
\begin{aligned}
x(t) =& \frac{-\beta p(t)}{\delta}\Bigg[\sum_{s=a}^{t-\varphi} h_{\varphi-1}(\eta,\sigma(s))f(s) + \sum_{s=t-\varphi+1}^{\xi-\varphi} h_{\varphi-1}(\eta,\sigma(s))f(s) \\
&+ \sum_{s=\xi-\varphi+1}^{\eta-\varphi} h_{\varphi-1}(\eta,\sigma(s))f(s)\Bigg] + \frac{p(t)}{\delta}\Bigg[\sum_{s=a}^{t-\varphi} h_{\varphi-1}(T,\sigma(s))f(s) \\
&+ \sum_{s=t-\varphi+1}^{\xi-\varphi} h_{\varphi-1}(T,\sigma(s))f(s) + \sum_{s=\xi-\varphi+1}^{\eta-\varphi} h_{\varphi-1}(T,\sigma(s))f(s) \\
&+ \sum_{s=\eta-\varphi+1}^{T-\varphi} h_{\varphi-1}(T,\sigma(s))f(s)\Bigg] + \frac{\alpha q(t)}{\delta}\Bigg[\sum_{s=a}^{t-\varphi} h_{\varphi-1}(\xi,\sigma(s))f(s) \\
&+ \sum_{s=t-\varphi+1}^{\xi-\varphi} h_{\varphi-1}(\xi,\sigma(s))f(s)\Bigg] - \sum_{s=a}^{t-\varphi} h_{\varphi-1}(t,\sigma(s))f(s), \\
=& \sum_{s=a}^{t-\varphi}\Big\{ - h_{\varphi-1}(t,\sigma(s)) - \frac{\beta p(t)}{\delta}h_{\varphi-1}(\eta,\sigma(s)) + \frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s)) \\
&+ \frac{\alpha q(t)}{\delta}h_{\varphi-1}(\xi,\sigma(s))\Big\}f(s) + \sum_{s=t-\varphi+1}^{\xi-\varphi}\Big\{\frac{-\beta p(t)}{\delta}h_{\varphi-1}(\eta,\sigma(s)) \\
&+ \frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s)) + \frac{\alpha q(t)}{\delta}h_{\varphi-1}(\xi,\sigma(s))\Big\}f(s), \\
&+ \sum_{s=\xi-\varphi+1}^{\eta-\varphi}\Big\{\frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s)) - \frac{\beta p(t)}{\delta}h_{\varphi-1}(\eta,\sigma(s))\Big\}f(s) \\
&+ \sum_{s=\eta-\varphi+1}^{T-\varphi} \frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s))f(s).
\end{aligned}
$$

Similarly, if $\xi \leq t \leq \eta$, then the solution can be stated as

$$
\begin{aligned}
x(t) =& \sum_{s=a}^{\xi-\varphi}\Big\{\frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s)) + \frac{\alpha q(t)}{\delta}h_{\varphi-1}(\xi,\sigma(s)) - h_{\varphi-1}(t,\sigma(s)) \\
&- \frac{\beta p(t)}{\delta}h_{\varphi-1}(\eta,\sigma(s))\Big\}f(s) + \sum_{s=\xi-\varphi+1}^{t-\varphi}\Big\{\frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s)) - h_{\varphi-1}(t,\sigma(s)) \\
&- \frac{\beta p(t)}{\delta}h_{\varphi-1}(\eta,\sigma(s))\Big\}f(s) + \sum_{s=t-\varphi+1}^{\eta-\varphi}\Big\{\frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s)) \\
&- \frac{\beta p(t)}{\delta}h_{\varphi-1}(\eta,\sigma(s))\Big\}f(s) + \sum_{s=\eta-\varphi+1}^{T-\varphi} \frac{p(t)}{\delta}h_{\varphi-1}(T,\sigma(s))f(s).
\end{aligned}
$$

Furthermore, if $\eta \le t$, then the solution can be stated as

$$\begin{aligned}
x(t) =& \sum_{s=a}^{\xi-\varphi} \Big\{ \frac{p(t)}{\delta} h_{\varphi-1}(T,\sigma(s)) + \frac{\alpha q(t)}{\delta} h_{\varphi-1}(\xi,\sigma(s)) - h_{\varphi-1}(t,\sigma(s)) \\
&- \frac{\beta p(t)}{\delta} h_{\varphi-1}(\eta,\sigma(s)) \Big\} f(s) + \sum_{s=\xi-\varphi+1}^{\eta-\varphi} \Big\{ \frac{p(t)}{\delta} h_{\varphi-1}(T,\sigma(s)) - h_{\varphi-1}(t,\sigma(s)) \\
&- \frac{\beta p(t)}{\delta} h_{\varphi-1}(\eta,\sigma(s)) \Big\} f(s) + \sum_{s=\eta-\varphi+1}^{t-\varphi} \Big\{ \frac{p(t)}{\delta} h_{\varphi-1}(T,\sigma(s)) \\
&- h_{\varphi-1}(t,\sigma(s)) \Big\} f(s) + \sum_{s=t-\varphi+1}^{T-\varphi} \frac{p(t)}{\delta} h_{\varphi-1}(T,\sigma(s)) f(s).
\end{aligned}$$

Finally, define GF as given in the statement. Then it is clear that solution takes the desired form $x(t) = \sum_{s=a}^{T} G(t,s) f(s)$. □

Remark 6. It is observed that the GF given in Theorem 5.1, which is a generalization of GF derived in [177].

The function $H(t,\iota_k)$ can be derived in similar way by the same technique that used in Theorem 5.1:

$$H(t,\iota_k) := \begin{cases}
\frac{p(t)}{\delta}\big[\beta(\eta-\iota_k) - (T-\iota_k)\big] - \frac{\alpha q(t)}{\delta}(\xi-\iota_k) & \\
+(t-\iota_k), & a \le \iota_k \le t, \\
\frac{p(t)}{\delta}\big[\beta(\eta-\iota_k) - (T-\iota_k)\big] - \frac{\alpha q(t)}{\delta}(\xi-\iota_k), & t+1 \le \iota_k \le \xi, \\
\frac{p(t)}{\delta}\big[\beta(\eta-\iota_k) - (T-\iota_k)\big] + (t-\iota_k), & \xi+1 \le \iota_k \le t, \\
\frac{p(t)}{\delta}\big[\beta(\eta-\iota_k) - (T-\iota_k)\big], & t+1 \le \iota_k \le \eta, \\
\frac{-p(t)}{\delta}(T-\iota_k) + (t-\iota_k), & \eta+1 \le \iota_k \le t, \\
\frac{-p(t)}{\delta}(T-\iota_k), & t+1 \le \iota_k \le T.
\end{cases}$$

Corollary 5.2.3. *For $G(t,s)$ and $H(t,\iota_k)$, following equations are hold*

$$\begin{aligned}
\sum_{s=a}^{T} G(t,s) =& \frac{p(t)}{\delta}\Big[h_\varphi(T,a) - \beta h_\varphi(\eta,a)\Big] + \frac{\alpha q(t)}{\delta} h_\varphi(\xi,a) - h_\varphi(t,a), \\
\sum_{\iota_k=a}^{T} H(t,\iota_k) =& \frac{p(t)}{\delta}\Big[\beta h_2(\eta+1,a) - h_2(T+1,a)\Big] + \frac{\alpha q(t)}{\delta} h_2(\xi+1,a) \\
&+ h_2(t,a).
\end{aligned}$$

Lemma 5.2.4. *Let $f : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$ be given, $1 < \varphi \leq 2$, and $\delta \neq 0$. Then problem* (5.11)–(5.13) *with boundary conditions* (5.17) *has solution $x(t)$, if and only if*

$$\begin{aligned} x(t) = &\sum_{a<\iota_k<t} I_k(x(\iota_k)) - \frac{p(t)}{\delta}\Big[\sum_{a<\iota_k<T} I_k(x(\iota_k)) - \beta \sum_{a<\iota_k<\eta} I_k(x(\iota_k))\Big] \\ &- \frac{\alpha q(t)}{\delta}\sum_{a<\iota_k<\xi} I_k(x(\iota_k)) + \sum_{a<\iota_k<T} H(t,\iota_k)\bar{I}_k(x(\iota_k)) \\ &+ \sum_{s=a}^{T} G(t,s) f(\varphi+\rho(s), x(\varphi+\rho(s))). \end{aligned}$$

Proof. Suppose that $x(t)$ solves (5.11). Then by using Corollary 5.1.4, we have

$$\begin{aligned} x(t) =&A + B(t-a) + \sum_{a<\iota_k<t} I_k(x(\iota_k)) + \sum_{a<\iota_k<t} \bar{I}_k(x(\iota_k))(t-\iota_k) \\ &- \sum_{s=a}^{t-\varphi} h_{\varphi-1}(t,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))). \end{aligned} \tag{5.18}$$

Using boundary condition $x(a) = \alpha x(\xi)$ in equation (5.18), we get

$$\begin{aligned} A(1-\alpha) - B\alpha(\xi - a) =&\alpha \sum_{a<\iota_k<\xi} I_k(x(\iota_k)) + \alpha \sum_{a<\iota_k<\xi} \bar{I}_k(x(\iota_k))(\xi-\iota_k) \\ &- \alpha \sum_{s=a}^{\xi-\varphi} h_{\varphi-1}(\xi,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))). \end{aligned} \tag{5.19}$$

Now, using $x(T) = \beta x(\eta)$ in equation(5.18), we get $A(1-\beta) + B(T - \beta\eta - a(1-\beta))$

$$\begin{aligned} =&- \sum_{a<\iota_k<T} \bar{I}_k(x(\iota_k))(T-\iota_k) + \sum_{s=a}^{T-\varphi} h_{\varphi-1}(T,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))) \\ &+ \beta \sum_{a<\iota_k<\eta} \bar{I}_k(x(\iota_k))(\eta-\iota_k) - \beta \sum_{s=a}^{\eta-\varphi} h_{\varphi-1}(\eta,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))) \\ &- \sum_{a<\iota_k<T} I_k(x(\iota_k)) + \beta \sum_{a<\iota_k<\eta} I_k(x(\iota_k)). \end{aligned} \tag{5.20}$$

Solving linear system of equations (5.19) and (5.20) for A and B, we get

$$\begin{aligned} A =&\frac{\alpha(\xi-a)}{\delta}\Big\{ - \sum_{a<\iota_k<T} I_k(x(\iota_k)) + \beta \sum_{a<\iota_k<\eta} \bar{I}_k(x(\iota_k))(\eta-\iota_k) \\ &- \sum_{a<\iota_k<T} \bar{I}_k(x(\iota_k))(T-\iota_k) + \sum_{s=a}^{T-\varphi} h_{\varphi-1}(T,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))) \\ &+ \beta \sum_{a<\iota_k<\eta} I_k(x(\iota_k)) - \beta \sum_{s=a}^{\eta-\varphi} h_{\varphi-1}(\eta,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s)))\Big\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha(T-a-\beta(\eta-a))}{\delta}\Big\{\sum_{a<\iota_k<\xi} I_k(x(\iota_k))+\sum_{a<\iota_k<\xi}\bar{I}_k(x(\iota_k))(\xi-\iota_k)\\
&-\sum_{s=a}^{\xi-\varphi} h_{\varphi-1}(\xi,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\Big\},\\
B=&\frac{(1-\alpha)}{\delta}\Big\{\beta\sum_{a<\iota_k<\eta} I_k(x(\iota_k))-\sum_{a<\iota_k<T} I_k(x(\iota_k))\\
&-\sum_{a<\iota_k<T}\bar{I}_k(x(\iota_k))(T-\iota_k)+\beta\sum_{a<\iota_k<\eta}\bar{I}_k(x(\iota_k))(\eta-\iota_k)\\
&+\sum_{s=a}^{T-\varphi} h_{\varphi-1}(T,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\\
&-\beta\sum_{s=a}^{\eta-\varphi} h_{\varphi-1}(\eta,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\Big\}\\
&-\frac{\alpha(1-\beta)}{\delta}\Big\{\sum_{a<\iota_k<\xi} I_k(x(\iota_k))+\sum_{a<\iota_k<\xi}\bar{I}_k(x(\iota_k))(\xi-\iota_k)\\
&-\sum_{s=a}^{\xi-\varphi} h_{\varphi-1}(\xi,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\Big\}.
\end{aligned}
$$

Substitution of A and B, in equation (5.18) yields

$$
\begin{aligned}
x(t)=&\sum_{a<\iota_k<t}\bar{I}_k(x(\iota_k))(t-\iota_k)-\sum_{s=a}^{t-\varphi} h_{\varphi-1}(t,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\\
&-\frac{p(t)}{\delta}\Big\{\sum_{a<\iota_k<T} I_k(x(\iota_k))+\sum_{a<\iota_k<T}\bar{I}_k(x(\iota_k))(T-\iota_k)\\
&-\beta\sum_{a<\iota_k<\eta} I_k(x(\iota_k))-\beta\sum_{a<\iota_k<\eta}\bar{I}_k(x(\iota_k))(\eta-\iota_k)\\
&-\sum_{s=a}^{T-\varphi} h_{\varphi-1}(T,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\\
&+\beta\sum_{s=a}^{\eta-\varphi} h_{\varphi-1}(\eta,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\Big\}\\
&-\frac{\alpha q(t)}{\delta}\Big\{\sum_{a<\iota_k<\xi} I_k(x(\iota_k))+\sum_{a<\iota_k<\xi}\bar{I}_k(x(\iota_k))(\xi-\iota_k)\\
&-\sum_{s=a}^{\xi-\varphi} h_{\varphi-1}(\xi,\sigma(s))f(\varphi+\rho(s),x(\varphi+\rho(s)))\Big\}+\sum_{a<\iota_k<t} I_k(x(\iota_k))
\end{aligned}
$$

which implies

$$
\begin{aligned}
x(t) = &\sum_{a<\iota_k<t} I_k(x(\iota_k)) - \frac{p(t)}{\delta}\Big\{\sum_{a<\iota_k<T} I_k(x(\iota_k)) - \beta \sum_{a<\iota_k<\eta} I_k(x(\iota_k))\Big\} \\
&- \frac{\alpha q(t)}{\delta}\sum_{a<\iota_k<\xi} I_k(x(\iota_k)) + \sum_{a<\iota_k<t} \bar{I}_k(x(\iota_k))(t-\iota_k) \\
&- \frac{p(t)}{\delta}\Big\{\sum_{a<\iota_k<T} \bar{I}_k(x(\iota_k))(T-\iota_k) - \beta \sum_{a<\iota_k<\eta} \bar{I}_k(x(\iota_k))(\eta-\iota_k)\Big\} \\
&+ \frac{p(t)}{\delta}\Big\{\sum_{s=a}^{T-\varphi} h_{\varphi-1}(T,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))) \\
&- \beta \sum_{s=a}^{\eta-\varphi} h_{\varphi-1}(\eta,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s)))\Big\} \\
&+ \frac{\alpha q(t)}{\delta}\sum_{s=a}^{\xi-\varphi} h_{\varphi-1}(\xi,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))) \\
&- \sum_{s=a}^{t-\varphi} h_{\varphi-1}(t,\sigma(s)) f(\varphi+\rho(s), x(\varphi+\rho(s))) - \frac{\alpha q(t)}{\delta}\sum_{a<\iota_k<\xi} \bar{I}_k(x(\iota_k))(\xi-\iota_k).
\end{aligned}
$$

The desired expression for $x(t)$ follows by Lemma 5.2.2, Theorem 5.1 and the value of $H(t,\iota_k)$. $\square$

5.3 Fixed point operators for MPFBVP

Findings of this section appeared in [107]. In this section, we shall obtain GF and FPO for the given nonlinear difference equation of non-integer order with multi-point summation boundary conditions:

$$
\begin{cases}
-\Delta^{\vartheta}_{\vartheta-2} x(t) = h(\rho(t)+\vartheta, x(\rho(t)+\vartheta)), & t \in \mathbb{N}_0^{b+1}, \\
x(\vartheta-2) = p, \qquad x(\vartheta+b+1) + \lambda \sum_{s=\vartheta-1}^{\vartheta+b} x(s) = q,
\end{cases}
\tag{5.21}
$$

where $h : [\vartheta-2, b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}} \times \mathbb{R} \to \mathbb{R}$, $\vartheta \in (1,2]$, $b \in \mathbb{N}_0$ and $\lambda > 0$, $p, q \in \mathbb{R}$.

5.3.1 Green's function for MPFBVP with summation condition

In this subsection, our purpose is to obtain Green's function for MPFBVP (5.22), further the maximum value of its sum is acquired for later use.

Now, we examine the non-homogeneous MPFBVP with summation conditions:

$$\begin{cases} -\Delta^{\vartheta}_{\vartheta-2} y(t) = g(t), & t \in \mathbb{N}_0^{b+1} \\ y(\vartheta-2) = 0, & y(\vartheta+b+1) + \lambda \sum_{s=\vartheta-1}^{\vartheta+b} y(s) = 0, \end{cases} \tag{5.22}$$

where $g : \mathbb{N}_0^{b+1} \to \mathbb{R}$, $\vartheta \in (1,2]$, $b \in \mathbb{N}_0$ and $0 < \lambda \in \mathbb{R}$.

Theorem 5.2. *The solution y of the MPFBVP (5.22) is given by*

$$y(t) = \sum_{s=0}^{b+1} G(t,s) g(s), \quad t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1},$$

where $G(t,s)$

$$:= \begin{cases} \chi(t,s) = \big[h_{\vartheta-1}(\vartheta+b+1, \sigma(s)) + \lambda h_{\vartheta}(\vartheta+b+1, \sigma(s))\big]\theta t^{\underline{\vartheta-1}}, & 0 \le \sigma(t) - \vartheta \le s \le b+1, \\ \psi(t,s) = \chi(t,s) - h_{\vartheta-1}(t, \sigma(s)), & 0 \le s \le t-\vartheta, \\ 0, \qquad \text{for } (t,s) \in & \{\vartheta-2\} \times [0, b+1]_{\mathbb{N}_0}, \end{cases}$$

with $\theta =: \frac{\vartheta}{(\vartheta+b+1)^{\underline{\vartheta-1}}[\vartheta+\lambda(b+2)]}$ where $h_{\vartheta}(t,s)$ is the ϑ^{th} fractional Taylor monomial given in Definition 2.1.5.

Proof. Suppose y on $\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$ satisfying the equation $-\Delta^{\vartheta}_{\vartheta-2} y(t) = g(t), \quad t \in \mathbb{N}_0^{b+1}$.

By making use of Lemma 2.1.29 and Lemma 2.1.30, we get

$$y(t) = c_1 t^{\underline{\vartheta-1}} + c_2 t^{\underline{\vartheta-2}} - \Delta^{-\vartheta}_{\vartheta-2} g(t).$$

Definition 2.1.15 implies that

$$y(t) = c_1 t^{\underline{\vartheta-1}} + c_2 t^{\underline{\vartheta-2}} - \sum_{s=0}^{t-\vartheta} h_{\vartheta-1}(t, \sigma(s)) g(s). \tag{5.23}$$

Using the first BC $y(\vartheta-2) = 0$, we have

$$c_1 (\vartheta-2)^{\underline{\vartheta-1}} + c_2 (\vartheta-2)^{\underline{\vartheta-2}} - \sum_{s=0}^{-2} h_{\vartheta-1}(\vartheta-2, \sigma(s)) g(s) = 0.$$

First term is zero by Definition 2.1.4, and by empty sum convention, last term also vanishes. Therefore we left with $c_2 = 0$. Let $\sum_{s=\vartheta-1}^{\vartheta+b} y(s) = A$. Now $y(\vartheta+b+1)+\lambda A = 0$ implies that

$$c_1(\vartheta+b+1)^{\underline{\vartheta-1}} - \sum_{s=0}^{b+1} \frac{(\vartheta+b+1-\sigma(s))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(s) + \lambda A = 0.$$

Solving for c_1, we have that

$$c_1 = \frac{-\lambda A}{(\vartheta+b+1)^{\underline{\vartheta-1}}} + \frac{1}{(\vartheta+b+1)^{\underline{\vartheta-1}}\Gamma(\vartheta)} \sum_{s=0}^{b+1} (\vartheta+b-s)^{\underline{\vartheta-1}} g(s).$$

Thus from equation (5.23), we have

$$y(t) = \frac{-\lambda A t^{\underline{\vartheta-1}}}{(\vartheta+b+1)^{\underline{\vartheta-1}}} + \frac{t^{\underline{\vartheta-1}}}{(\vartheta+b+1)^{\underline{\vartheta-1}}\Gamma(\vartheta)} \sum_{s=0}^{b+1} (\vartheta+b-s)^{\underline{\vartheta-1}} g(s) - \sum_{s=0}^{t-\vartheta} \frac{(t-\sigma(s))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(s). \quad (5.24)$$

To evaluate A use y in $A = \displaystyle\sum_{s=\vartheta-1}^{\vartheta+b} y(s)$

$$A = \sum_{s=\vartheta-1}^{\vartheta+b} \left[\frac{-\lambda A s^{\underline{\vartheta-1}}}{(\vartheta+b+1)^{\underline{\vartheta-1}}} + \frac{s^{\underline{\vartheta-1}}}{(\vartheta+b+1)^{\underline{\vartheta-1}}\Gamma(\vartheta)} \sum_{\mathscr{F}=0}^{b+1} (\vartheta+b-\mathscr{F})^{\underline{\vartheta-1}} g(\mathscr{F}) - \sum_{\mathscr{F}=0}^{s-\vartheta} \frac{(s-\sigma(\mathscr{F}))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(\mathscr{F}) \right].$$

$$\begin{aligned} A\left(1 + \frac{\lambda}{(\vartheta+b+1)^{\underline{\vartheta-1}}} \sum_{s=\vartheta-1}^{\vartheta+b} s^{\underline{\vartheta-1}}\right) &= \frac{1}{(\vartheta+b+1)^{\underline{\vartheta-1}}\Gamma(\vartheta)} \sum_{s=\vartheta-1}^{\vartheta+b} s^{\underline{\vartheta-1}} \sum_{\mathscr{F}=0}^{b+1} (\vartheta+b-\mathscr{F})^{\underline{\vartheta-1}} g(\mathscr{F}) \\ &\quad - \sum_{s=\vartheta-1}^{\vartheta+b} \sum_{\mathscr{F}=0}^{s-\vartheta} \frac{(s-\sigma(\mathscr{F}))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(\mathscr{F}). \end{aligned} \quad (5.25)$$

Change the order of summation in last double sum of equation (5.25)

$$\sum_{s=\vartheta-1}^{\vartheta+b} \sum_{\mathscr{F}=0}^{s-\vartheta} \frac{(s-\sigma(\mathscr{F}))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(\mathscr{F}) = \sum_{\mathscr{F}=0}^{b} \sum_{s=\mathscr{F}+\vartheta}^{b+\vartheta} \frac{(s-\sigma(\mathscr{F}))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(\mathscr{F}).$$

Make substitution $s = x + \sigma(\mathscr{F})$ only for inner sum and then use Lemma 2.1.6 and Lemma 2.1.9

$$\begin{aligned} &= \sum_{\mathscr{F}=0}^{b} \sum_{x=\vartheta-1}^{b+\vartheta-\mathscr{F}-1} \frac{x^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(\mathscr{F}) \\ &= \sum_{\mathscr{F}=0}^{b} \frac{x^{\underline{\vartheta}}}{\Gamma(\vartheta+1)} \Bigg|_{\vartheta-1}^{b+\vartheta-\mathscr{F}} g(\mathscr{F}) \end{aligned}$$

$$= \sum_{\mathcal{F}=0}^{b} \frac{(b+\vartheta-\mathcal{F})^{\underline{\vartheta}}}{\Gamma(\vartheta+1)} g(\mathcal{F}). \tag{5.26}$$

Since sums are independent in first double sum of equation (5.25), use Definition 2.1.5, Definition 2.1.1, Lemma 2.1.6 and Lemma 2.1.9 to evaluate the sum

$$\begin{aligned}
\sum_{t=\vartheta-1}^{b+\vartheta} t^{\underline{\vartheta-1}} &= \sum_{t=\vartheta-1}^{b+\vartheta} \Gamma(\vartheta) h_{\vartheta-1}(t,0)\Delta t \\
&= \Gamma(\vartheta) h_{\vartheta}(t,0)\Big|_{\vartheta-1}^{b+\vartheta+1} \\
&= \Gamma(\vartheta) h_{\vartheta}(b+\vartheta+1,0) - \Gamma(\vartheta) h_{\vartheta}(\vartheta-1,0) \\
&= \frac{\Gamma(\vartheta)}{\Gamma(\vartheta+1)} (b+\vartheta+1)^{\underline{\vartheta}} - 0 \\
&= \frac{(b+\vartheta+1)^{\underline{\vartheta}}}{\vartheta}.
\end{aligned} \tag{5.27}$$

Making use of equations (5.26), (5.27) in equation (5.25)

$$A\left(1+\frac{\lambda}{\vartheta}\frac{(b+\vartheta+1)^{\underline{\vartheta}}}{(\vartheta+b+1)^{\underline{\vartheta-1}}}\right) = \frac{(b+\vartheta+1)^{\underline{\vartheta}}}{(\vartheta+b+1)^{\underline{\vartheta-1}}} \sum_{\mathcal{F}=0}^{b+1} \frac{(\vartheta+b-\mathcal{F})^{\underline{\vartheta-1}}}{\Gamma(\vartheta+1)} g(\mathcal{F}) - \sum_{\mathcal{F}=0}^{b} \frac{(b+\vartheta-\mathcal{F})^{\underline{\vartheta}}}{\Gamma(\vartheta+1)} g(\mathcal{F}).$$

Note that $\frac{(b+\vartheta+1)^{\underline{\vartheta}}}{(b+\vartheta+1)^{\underline{\vartheta-1}}} = (b+2)$. Since we choose $0<\lambda$ therefore $\lambda \neq \frac{-\vartheta}{(b+2)}$. $A =$

$$\frac{\vartheta(b+2)}{\Gamma(\vartheta+1)[\vartheta+\lambda(b+2)]} \sum_{\mathcal{F}=0}^{b+1} (\vartheta+b-\mathcal{F})^{\underline{\vartheta-1}} g(\mathcal{F}) - \frac{\vartheta}{\Gamma(\vartheta+1)[\vartheta+\lambda(b+2)]} \sum_{\mathcal{F}=0}^{b} (\vartheta+b-\mathcal{F})^{\underline{\vartheta}} g(\mathcal{F}).$$

Pick value of A in equation (5.24)

$$\begin{aligned}
y(t) =& \frac{-\vartheta\lambda(b+2)t^{\underline{\vartheta-1}}}{\Gamma(\vartheta+1)[\vartheta+\lambda(b+2)](\vartheta+b+1)^{\underline{\vartheta-1}}} \sum_{s=0}^{b+1} (\vartheta+b-s)^{\underline{\vartheta-1}} g(s) \\
&+ \frac{\vartheta\lambda t^{\underline{\vartheta-1}}}{\Gamma(\vartheta+1)[\vartheta+\lambda(b+2)](\vartheta+b+1)^{\underline{\vartheta-1}}} \sum_{s=0}^{b} (\vartheta+b-s)^{\underline{\vartheta}} g(s) \\
&+ \frac{t^{\underline{\vartheta-1}}}{\Gamma(\vartheta)(\vartheta+b+1)^{\underline{\vartheta-1}}} \sum_{s=0}^{b+1} (\vartheta+b-s)^{\underline{\vartheta-1}} g(s) - \sum_{s=0}^{t-\vartheta} \frac{(t-\sigma(s))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(s) \\
=& \frac{t^{\underline{\vartheta-1}}}{\Gamma(\vartheta)(\vartheta+b+1)^{\underline{\vartheta-1}}} \left[1 - \frac{\lambda(b+2)}{\vartheta+\lambda(b+2)}\right] \sum_{s=0}^{b+1} (\vartheta+b-s)^{\underline{\vartheta-1}} g(s) \\
&+ \frac{\lambda t^{\underline{\vartheta-1}}}{\Gamma(\vartheta)[\vartheta+\lambda(b+2)](\vartheta+b+1)^{\underline{\vartheta-1}}} \sum_{s=0}^{b} (\vartheta+b-s)^{\underline{\vartheta}} g(s) - \sum_{s=0}^{t-\vartheta} \frac{(t-\sigma(s))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)} g(s)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\vartheta t^{\underline{\vartheta-1}}}{\Gamma(\vartheta)(\vartheta+b+1)^{\underline{\vartheta-1}}[\vartheta+\lambda(b+2)]}\sum_{s=0}^{b+1}(\vartheta+b-s)^{\underline{\vartheta-1}}g(s)\\
&+\frac{\lambda t^{\underline{\vartheta-1}}}{\Gamma(\vartheta)[\vartheta+\lambda(b+2)](\vartheta+b+1)^{\underline{\vartheta-1}}}\sum_{s=0}^{b}(\vartheta+b-s)^{\underline{\vartheta}}g(s)-\sum_{s=0}^{t-\vartheta}\frac{(t-\sigma(s))^{\underline{\vartheta-1}}}{\Gamma(\vartheta)}g(s).
\end{aligned}
$$

Since $1<\vartheta\leq 2$ and $(\vartheta+b-s)^{\underline{\vartheta}}=0$ for $s=b+1$, therefore for convenience, we can write

$\sum_{s=0}^{b}(\vartheta+b-s)^{\underline{\vartheta}}g(s)=\sum_{s=0}^{b+1}(\vartheta+b-s)^{\underline{\vartheta}}g(s)$.

$$
\begin{aligned}
y(t)=&\theta t^{\underline{\vartheta-1}}\sum_{s=0}^{b+1}\Big[\frac{(\vartheta+b-s)^{\underline{\vartheta-1}}}{\Gamma(\vartheta)}+\frac{\lambda(\vartheta+b-s)^{\underline{\vartheta}}}{\vartheta\Gamma(\vartheta)}\Big]g(s)-\sum_{s=0}^{t-\vartheta}h_{\vartheta-1}(t,\sigma(s))g(s)\\
=&\theta t^{\underline{\vartheta-1}}\sum_{s=0}^{b+1}\big[h_{\vartheta-1}(\vartheta+b+1,\sigma(s))+\lambda h_{\vartheta}(\vartheta+b+1,\sigma(s))\big]g(s)-\sum_{s=0}^{t-\vartheta}h_{\vartheta-1}(t,\sigma(s))g(s).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
y(t)&=\sum_{s=0}^{b+1}\chi(t,s)g(s)-\sum_{s=0}^{t-\vartheta}h_{\vartheta-1}(t,\sigma(s))g(s)\\
&=\sum_{s=0}^{t-\vartheta}[\chi(t,s)-h_{\vartheta-1}(t,\sigma(s))]g(s)+\sum_{s=\sigma(t)-\vartheta}^{b+1}\chi(t,s)g(s)\\
&=\sum_{s=0}^{t-\vartheta}\psi(t,s)g(s)+\sum_{s=\sigma(t)-\vartheta}^{b+1}\chi(t,s)g(s)\\
&=\sum_{s=0}^{b+1}G(t,s)g(s).
\end{aligned}
$$

This completes the proof. $\square$

Next we sum Green's function on $\mathbb{N}_0^{b+1}$ for later use to prove existence of solutions for nonlinear MPFBVP.

Lemma 5.3.1. *The Green's function for MPFBVP* (5.22) *satisfies the identity*

$$
\sum_{s=0}^{b+1}G(t,s)=\frac{\big[\theta\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}}+(\vartheta+b+1)^{\underline{\vartheta+1}}\big\}-(\vartheta+1)(\sigma(t)-\vartheta)\big]}{\Gamma(\vartheta+2)}t^{\underline{\vartheta-1}},\ t\in\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}.
$$

Proof. For $t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$, consider the left hand side

$$\begin{aligned}
\sum_{s=0}^{b+1} G(t,s) &= \sum_{s=0}^{t-\vartheta} \psi(t,s) + \sum_{s=\sigma(t)-\vartheta}^{b+1} \chi(t,s),\\
&= \sum_{s=0}^{t-\vartheta} [\chi(t,s) - h_{\vartheta-1}(t,\sigma(s))] + \sum_{s=\sigma(t)-\vartheta}^{b+1} \chi(t,s),\\
&= \sum_{s=0}^{b+1} \chi(t,s) - \sum_{s=0}^{t-\vartheta} h_{\vartheta-1}(t,\sigma(s)).
\end{aligned}$$

For $t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$, using the expression $\chi(t,s)$ and Lemmas 2.1.6, 2.1.9 we get

$$\begin{aligned}
\sum_{s=0}^{b+1} G(t,s) =& \theta t^{\underline{\vartheta-1}} \Big[\sum_{s=0}^{b+1} [h_{\vartheta-1}(\vartheta+b+1,\sigma(s)) + \sum_{s=0}^{b+1} h_{\vartheta}(\vartheta+b+1,\sigma(s))\Big] + h_{\vartheta}(t,s)\Big|_{s=0}^{\sigma(t)-\vartheta},\\
=& \theta t^{\underline{\vartheta-1}} \left[- h_{\vartheta}(\vartheta+b+1,s)\Big|_{s=0}^{b+2} - h_{\vartheta+1}(\vartheta+b+1,s)\Big|_{s=0}^{b+2} \right] - h_{\vartheta}(t,0),\\
=& \theta t^{\underline{\vartheta-1}} \left[\frac{(\vartheta+b+1)^{\underline{\vartheta}}}{\Gamma(\vartheta+1)} + \frac{(\vartheta+b+1)^{\underline{\vartheta+1}}}{\Gamma(\vartheta+2)} \right] - \frac{t^{\underline{\vartheta}}}{\Gamma(\vartheta+1)},\\
=& \frac{\theta t^{\underline{\vartheta-1}} \big[(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big] - (\vartheta+1) t^{\underline{\vartheta}}}{\Gamma(\vartheta+2)},\\
=& \frac{\big[\theta\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big\} - (\vartheta+1)(\sigma(t)-\vartheta)\big]}{\Gamma(\vartheta+2)} t^{\underline{\vartheta-1}}.
\end{aligned}$$

In last step, we used the fact that $t^{\underline{\vartheta}} = (t-\vartheta+1)t^{\underline{\vartheta-1}}$. ∎

Next lemma gives the maximum value of $\sum_{s=0}^{b+1} G(t,s)$.

Lemma 5.3.2. *The following expression holds for* $\sum_{s=0}^{b+1} G(t,s)$*:*

$$\begin{aligned}
\max_{\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}} \sum_{s=0}^{b+1} G(t,s) =& \frac{1}{\Gamma(\vartheta+2)} \Big[\theta\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big\}\\
& - (\vartheta+1)\Big(\Big\lceil \frac{\theta(\vartheta-1)\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\}}{\vartheta^2+\vartheta} \Big\rceil - 1\Big)\Big]\\
& \times \Big(\vartheta - 2 + \Big\lceil \frac{\theta(\vartheta-1)\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\}}{\vartheta^2+\vartheta} \Big\rceil\Big)^{\underline{\vartheta-1}}\\
=:& \frac{M^*}{\Gamma(\vartheta+2)}.
\end{aligned}$$

Proof. By Lemma 5.3.1, we have

$$\begin{aligned}
\sum_{s=0}^{b+1} G(t,s) =& \frac{\big[\theta\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big\} - (\vartheta+1)(\sigma(t)-\vartheta)\big]}{\Gamma(\vartheta+2)} t^{\underline{\vartheta-1}}\\
=& \frac{F(t)}{\Gamma(\vartheta+2)},
\end{aligned}$$

where $F(t) = \big[\theta\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big\} - (\vartheta+1)(\sigma(t)-\vartheta)\big]t^{\underline{\vartheta-1}}$.

To find critical point of $F(t)$, we consider

$$\begin{aligned}
\Delta F(t) =& -(\vartheta+1)t^{\underline{\vartheta-1}} + (\vartheta-1)t^{\underline{\vartheta-2}}\big[\theta\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big\}\\
&-(\vartheta+1)(t-\vartheta+2)\big],\\
=& -(\vartheta+1)(t-\vartheta+2)t^{\underline{\vartheta-2}} + (\vartheta-1)t^{\underline{\vartheta-2}}\Big[\theta\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big\}\\
&-(\vartheta+1)(t-\vartheta+2)\Big],\\
=& t^{\underline{\vartheta-2}}\Big[-(\vartheta^2+\vartheta)(t-\vartheta+2) + \theta(\vartheta-1)\big\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\big\}\Big].
\end{aligned}$$

In preceding step, we used the fact that $t^{\underline{\vartheta-2}} = \frac{t^{\underline{\vartheta-1}}}{t-\vartheta+2}$ and product rule given in Lemma 2.1.14. The only critical point of $F(t)$ is at $t = \vartheta - 2 + \lceil \frac{\theta(\vartheta-1)\{(\vartheta+1)(\vartheta+b+1)^{\underline{\vartheta}} + (\vartheta+b+1)^{\underline{\vartheta+1}}\}}{\vartheta^2+\vartheta} \rceil$. Note that $F(\vartheta-2) = 0$, however for each fix value of ϑ, λ and b, it is clear from observation that critical point is a point of maximum value. Hence evaluation give the desired result. □

Theorem 5.3. *Assume that $\vartheta \in (1,2]$, $p, q \in \mathbb{R}$ and $f : [0, b+1]_{\mathbb{N}_0} \to \mathbb{R}$. Then solution to the non-homogeneous MPFBVP*

$$\begin{cases} -\Delta^{\vartheta}_{\vartheta-2}x(t) = f(t), & t \in \mathbb{N}_0^{b+1}\\ x(\vartheta-2) = p, & x(\vartheta+b+1) + \lambda\sum_{s=\vartheta-1}^{\vartheta+b} x(s) = q, \end{cases} \tag{5.28}$$

can be represented $\quad x(t) = \phi(t) + \displaystyle\sum_{s=0}^{b+1} G(t,s)f(s), \quad t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1},$

for unique solution $\phi(t)$ of following problem

$$\begin{cases} -\Delta^{\vartheta}_{\vartheta-2}\phi(t) = 0, & t \in \mathbb{N}_0^{b+1}\\ \phi(\vartheta-2) = p, & \phi(\vartheta+b+1) + \lambda\sum_{s=\vartheta-1}^{\vartheta+b} \phi(s) = q. \end{cases} \tag{5.29}$$

Proof. By virtue of Theorem 5.2, suppose that $y(t) = \sum_{s=0}^{b+1} G(t,s)f(s),\ t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$, is a solution of the MPFBVP (5.22) on $\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$. Suppose $x(t)$ and $\phi(t)$ as stated above. Then

$$x(\vartheta-2) = \phi(\vartheta-2) + y(\vartheta-2) = p,$$

$$\text{and} \quad x(\vartheta+b+1)+\lambda\sum_{s=\vartheta-1}^{\vartheta+b}x(s)=y(\vartheta+b+1)+\phi(\vartheta+b+1)+\lambda\sum_{s=\vartheta-1}^{\vartheta+b}\phi(s)=q.$$

$$\text{Finally,} \qquad -\Delta^{\vartheta}_{\vartheta-2}x(t)=-\Delta^{\vartheta}_{\vartheta-2}y(t)-\Delta^{\vartheta}_{\vartheta-2}\phi(t)=f(t) \text{ for } t\in\mathbb{N}_0^{b+1}.$$

□

Theorem 5.3 can be helpful for treating nonlinear system with summation condition as next result reflects.

Corollary 5.3.3. *Assume that $\vartheta\in(1,2]$ and $h:[\vartheta-2,b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}}\times\mathbb{R}\to\mathbb{R}$. Then the solution to system* (5.21) *is given by*

$$x(t)=\phi(t)+\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,x(\rho(s)+\vartheta)),\quad t\in\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}, \tag{5.30}$$

for $\phi(t)$ as given in Theorem 5.3.

5.4 Fixed point operator for Hilfer fractional difference system for IVP

Findings of this section appeared in [109]. To establish the existence theory for Hilfer fractional difference equation with initial conditions:

$$\begin{cases}\Delta_a^{\vartheta,\varphi}\chi(x)+g(x+\vartheta-1,\chi(x+\vartheta-1))=0, \text{ for } x\in\mathbb{N}_{a+1-\vartheta},\\ \Delta_a^{-(1-\eta)}\chi(a+1-\eta)=\zeta, \quad \zeta\in\mathbb{R},\end{cases} \tag{5.31}$$

where $\eta=\vartheta+\varphi-\vartheta\varphi$. We transforms the problem to an equivalent summation equation which in turn defined an appropriate FPO.

Lemma 5.4.1. *Let $g:[a,T]_{\mathbb{N}_a}\times\mathbb{R}\to\mathbb{R}$ be given and $0<\vartheta<1$, $0\le\varphi\le1$. Then χ solves system* (5.31) *if and only if*

$$\chi(x)=\zeta h_{\eta-1}(x,a+1-\eta)-\Delta_{a+1-\vartheta}^{-\vartheta}g(x+\vartheta-1,\chi(x+\vartheta-1)),$$

for all $x \in \mathbb{N}_{a+1}$.

The proof of above lemma is an implication of Lemma 3.1.2 (i) and (ii) and second part of Theorem 8 in [116]. In next chapter, the Brouwer's FPT [61] is utilized for establishing existence conditions. The set Z of all real sequences $u = \{\chi(x)\}_{x=a}^{T}$ with $||u|| = \sup_{x \in \mathbb{N}_a^T} |\chi(x)|$ is a Banach space.

Using Definition 2.1.15 and Lemma 5.4.1, we define an operator $\mathscr{A} : Z \to Z$ by

$$\mathscr{A}\chi(x) = \zeta h_{\eta-1}(x, a+1-\eta) - \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x, \sigma(\mathscr{F}))g(\mathscr{F}+\vartheta-1, \chi(\mathscr{F}+\vartheta-1)). \tag{5.32}$$

The fixed points of $\mathscr{A}$ coincides with the solutions of the problem (5.31).

5.5 Fixed point operator for substantial fractional difference system for IVP

Findings of this section appeared in [108]. In order to apply FPT to establish existence theory for substantial fractional difference equation with initial conditions:

$$\begin{cases} {}^{s}\Delta_a^{\varphi}\chi(x) + f(x+\varphi-1, \chi(x+\varphi-1)) = 0, & \text{for } x \in \mathbb{N}_a, \\ {}^{s}\Delta^{\varphi-i+1}\chi(x_0 = a+m-\varphi) = \chi_i, & i = 0, 1, \cdots, m-1, \end{cases} \tag{5.33}$$

where $m - 1 < \varphi \le m$ with positive integer m. Here we use a type of initial conditions involving non-integer order differences, suggested by Heymans and Podlubny [114]. However these conditions may be converted to whole order conditions by a technique used by Holm [117] in his doctorate dissertation. Physical entity of initial conditions that involves RL derivative has been challenged by few researchers. However Heymans and Podlubny discussed some expositions and provided a physical interpretation for the initial conditions [114, 149]. We convert the problem to an equivalent summation equation to obtain an appropriate fixed point operators.

Lemma 5.5.1. *Lemma Let $f : [a,T]_{\mathbb{N}_a} \times \mathbb{R} \to \mathbb{R}$ be given and $m-1 < \varphi \leq m$ for $m \in \mathbb{N}$. Then u solves system* (5.33) *if and only if*

$$\begin{aligned}\chi(x) =&\frac{e_{-p}(x-a+1,0)h_{\varphi-m+1}(x,a)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}}\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}\chi_i \\ &-\frac{1}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}}\ {}^{s}\Delta_{a+m-\varphi}^{-(\varphi-m+j)}f(x+\varphi-1,\chi(x+\varphi-1)).\end{aligned}$$

The proof of above lemma is an implication of Equation (3.3) of Lemma 3.2.5. The Brouwer's FPT [61] is utilized for establishing the condition for existence, in the next chapter. The set Z of all real sequences $u = \{\chi(x)\}_{x=a}^{T}$ with a norm defined $||u|| = \sup\limits_{x\in\mathbb{N}_a^T} |\chi(x)|$ is a Banach space. Define the operator $\mathscr{A} : Z \to Z$ by

$$\begin{aligned}\mathscr{A}\chi(x) =&\frac{e_{-p}(x-a+1,0)h_{\varphi-m+1}(x,a)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}}\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^{j}\chi_i-\frac{1}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}} \\ &\times\sum_{\mathscr{T}=a+m-\varphi}^{x-\varphi+m-j}h_{\varphi-m+j-1}(x,\sigma(\mathscr{T}))e_{-p}(x-\mathscr{T},0)f(\mathscr{T}+\varphi-1,\chi(\mathscr{T}+\varphi-1)).\end{aligned}$$

Fixed points of $\mathscr{A}$ coincide with the solutions of the problem (5.33).

Chapter 6

Existence and uniqueness of solutions

In this chapter, EU conditions of solutions to IVP and BVP shall acquire for different type of fractional delta difference equations. We deal explicitly with impulsive difference equations for nonlocal initial condition and two and four point boundary conditions, nonlinear FDE with multi point summation BC, Hilfer fractional difference Cauchy system and substantial fractional difference Cauchy system. Findings of sections 6.1 and 6.2 appeared in [110].

6.1 Existence and uniqueness of solutions for IVP with impulse

Following are the assumptions to apply in the contraction principle:

$(\mathcal{C}_1)$ there exists a constant n such that

$$\big|f(t,x)-f(t,\bar{x})\big| \leq n\big|x-\bar{x}\big|, \quad \text{for each } t\in[a,T]_{\mathbb{N}_a} \text{ and all } x,\bar{x}\in\mathbb{R};$$

$(\mathcal{C}_2)$ there exist constants n_1 and n_2 such that

$$(i) \qquad \big|I_k(x)-I_k(\bar{x})\big| \leq n_1\big|x-\bar{x}\big|, \quad \text{for all } x,\bar{x}\in\mathbb{R},\ k\in\mathbb{N}_1^m;$$

$$(ii) \qquad \big|\bar{I}_k(x)-\bar{I}_k(\bar{x})\big| \leq n_2\big|x-\bar{x}\big|, \quad \text{for all } x,\bar{x}\in\mathbb{R},\ k\in\mathbb{N}_1^m;$$

$(\mathcal{C}_3)$ there exists a constant $\mathcal{N}_h$ such that

$$\big|h(x) - h(\bar{x})\big| \leq \mathcal{N}_h\big|x - \bar{x}\big|, \quad \text{for all } x, \bar{x} \in \mathbb{R}.$$

Proof of the following two theorems is similar to that of Theorem 6.4.

Theorem 6.1. *Let $f : [a,T]_{\mathbb{N}_a} \times \mathbb{R} \to \mathbb{R}$ be continuous, and $1 < \varphi \leq 2$. Assume that I_k, $\bar{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous for $k \in \mathbb{N}_1^m$. In addition, assume that* $(\mathcal{C}_1)$, $(\mathcal{C}_2)$ *hold and $\mathcal{N} < \frac{\Gamma(\varphi+1)}{3(T-a)^{\underline{\varphi}}}$, $\mathcal{N}_1 < \frac{1}{3m}$, $\mathcal{N}_2 < \frac{1}{3m}$. Then the problem* (5.11)– (5.13) *with initial condition* (5.14) *has a unique solution.*

Theorem 6.2. *Assume that $f : [a,T]_{\mathbb{N}_a} \times \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are continuous, and $1 < \varphi \leq 2$. Assume that I_k, $\bar{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous for $k \in \mathbb{N}_1^m$. In addition, assume that* $(\mathcal{C}_1) - (\mathcal{C}_3)$ *hold and $\mathcal{N} < \frac{\Gamma(\varphi+1)}{4(T-a)^{\underline{\varphi}}}$, $\mathcal{N}_1 < \frac{1}{4m}$, $\mathcal{N}_2 < \frac{1}{4m}$, $\mathcal{N}_h < \frac{1}{4}$. Then the problem* (5.11)–(5.13) *with initial condition* (5.15) *has a unique solution.*

6.2 Existence and uniqueness of solutions for impulsive BVP

We shall illustrate the usefulness of Lemma 5.2.4 by establishing existence conditions with straight forward use of Schaefer's FPT and accomplish the contraction principle for uniqueness conditions. We define for further use

$$M_1 := \sup_{t\in[a,T]_{\mathbb{N}_a}} \frac{\big|p(t)\big|}{\big|\delta\big|}, \qquad M_2 := \sup_{t\in[a,T]_{\mathbb{N}_a}} \frac{\big|\alpha\big|\big|q(t)\big|}{\big|\delta\big|}.$$

Theorem 6.3. *Let $f : [a,T]_{\mathbb{N}_a} \times \mathbb{R} \to \mathbb{R}$ be continuous, and $1 < \varphi \leq 2$. Assume that I_k, $\bar{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous for $k \in \mathbb{N}_1^m$. Assume $\alpha, \beta, \xi, \eta \in \mathbb{R}$, $\xi, \eta \in (a,T)_{\mathbb{N}_a}$ where$\delta \neq 0$. If* *following conditions hold for positive constants* $L, L_k, \bar{L}_k$ *such that*

$$\big|f(t,x)\big| \leq L,\ \big|I_k(x)\big| \leq L_k,\ \big|\bar{I}_k(x)\big| \leq \bar{L}_k,\quad \textit{for } t \in [a,T]_{\mathbb{N}_a},\ k \in \mathbb{N}_1^m. \tag{6.1}$$

Then the problem (5.11)– (5.13) *with boundary condition* (5.17) *has at least one solution.*

Proof. Define the operator $\mathcal{A} : PC([a,T]_{\mathbb{N}_a}, \mathbb{R}) \to PC([a,T]_{\mathbb{N}_a}, \mathbb{R})$ by

$$\begin{aligned}\mathcal{A}x(t) = &\sum_{a<\iota_k<t} I_k(x(\iota_k)) - \frac{p(t)}{\delta}\Big[\sum_{a<\iota_k<T} I_k(x(\iota_k)) - \beta \sum_{a<\iota_k<\eta} I_k(x(\iota_k))\Big]\\ &- \frac{\alpha q(t)}{\delta}\sum_{a<\iota_k<\xi} I_k(x(\iota_k)) + \sum_{a<\iota_k<T} H(t,\iota_k)\bar{I}_k(x(\iota_k))\\ &+ \sum_{s=a}^{T} G(t,s) f(\varphi+\rho(s), x(\varphi+\rho(s))).\end{aligned}$$

The fixed points of $\mathcal{A}$ coincide with the solutions of the given problem. The continuity of $f,\ I_k$ and $\bar{I}_k$ implies the continuity of operator $\mathcal{A}$. To prove that $\mathcal{A}$ is completely continuous. Let $\Omega_\ell = \{x \in PC([a,T]_{\mathbb{N}_a}, \mathbb{R}) : ||x|| \leq \ell\}$ for some $T < \ell$. Let $k_0 \in \mathbb{N}_1^m$. For $x \in \Omega_\ell$ and $t, \bar{t} \in [a,T]_{\mathbb{N}_a}$, $\iota_{k_0} < t < \bar{t} \leq \iota_{k_0+1}$, by using (6.1), we get

$$\begin{aligned}\big|\mathcal{A}x(t) - \mathcal{A}x(\bar{t})\big| \leq &(\bar{t}-t)\sum_{a<\iota_k<t} Ł_k + \frac{(\bar{t}-t)|\alpha(1-\beta)|}{|\delta|}\Big[\sum_{a<\iota_k<\xi} L_k\\ &+ h_2(\xi,a)\bar{L}_k + h_\varphi(\xi,a)L_k\Big] + \frac{(\bar{t}-t)|(1-\alpha)|}{|\delta|}\Big[\sum_{a<\iota_k<T} L_k\\ &+ h_\varphi(T,a)L_k + h_2(T,a)\bar{L}_k - \beta\Big\{\sum_{a<\iota_k<\eta} L_k + h_2(\eta,a)\bar{L}_k\\ &+ h_\varphi(\eta,a)L_k\Big\}\Big] + \Big|\frac{(\bar{t}-a)^{\underline{2}}}{\Gamma(3)} - \frac{(t-a)^{\underline{2}}}{\Gamma(3)}\Big|\bar{L}_k\\ &+ \Big|\frac{(\bar{t}-a)^{\underline{\varphi}}}{\Gamma(\varphi+1)} - \frac{(t-a)^{\underline{\varphi}}}{\Gamma(\varphi+1)}\Big|L.\end{aligned}$$

The right hand side is independent of x and $\big|\big|\mathcal{A}x(t) - \mathcal{A}x(\bar{t})\big|\big| \to 0$ as $\bar{t} \to t$ for $\iota_{k_0} < t < \bar{t} \leq \iota_{k_0+1}$. By the Arzela-Ascoli theorem we conclude that $\mathcal{A} : PC([a,T]_{\mathbb{N}_a}, \mathbb{R}) \to PC([a,T]_{\mathbb{N}_a}, \mathbb{R})$ is completely continuous. In order to prove $\mathcal{Z} = \{x \in PC([a,T]_{\mathbb{N}_a}, \mathbb{R}) : x = \lambda\mathcal{A}(x), \text{for some } 0 \leq$

$\lambda \leq 1\}$ is bounded. If $x \in \mathscr{Z}$, then $x = \lambda \mathscr{A}(x)$for some $0 \leq \lambda \leq 1$. For any $t \in [a,T]_{\mathbb{N}_a}$, we have

$$\begin{aligned}
\big|x(t)\big| =& \lambda \big|(\mathscr{A})x(t)\big| \\
\leq& \sum_{a<\iota_k<t} \big|I_k(x(\iota_k))\big| + M_1\Big[\sum_{a<\iota_k<T} \big|I_k(x(\iota_k))\big| + |\beta| \sum_{a<\iota_k<\eta} \big|I_k(x(\iota_k))\big|\Big] \\
&+ M_2 \sum_{a<\iota_k<\xi} \big|I_k(x(\iota_k))\big| + \Big|M_1\Big[h_2(T,a) - \beta h_2(\eta,a)\Big] + M_2 h_2(\xi,a) \\
&+ h_2(t,a)\Big|\bar{L}_k + \Big|M_1\Big[h_\varphi(T,a) - \beta h_\varphi(\eta,a)\Big] + M_2 h_\varphi(\xi,a) \\
&+ h_\varphi(t,a)\Big|L \\
\leq& M_1\Big\{\sum_{k=1}^{m} L_k + h_2(T,a)\bar{L}_k + h_\varphi(T,a)L + \big|\beta\big|\big[h_2(\eta,a)\bar{L}_k + h_\varphi(\eta,a)L \\
&+ \sum_{a<\iota_k<\eta} L_k\big]\Big\} + M_2\Big\{\sum_{a<\iota_k<\xi} L_k + h_2(\xi,a)\bar{L}_k + h_\varphi(\xi,a)L\Big\} \\
&+ \sum_{k=1}^{m} L_k + h_2(T,a)\bar{L}_k + h_\varphi(T,a)L := M.
\end{aligned}$$

This implies that the set $\mathscr{Z}$ is bounded. By the virtue of Schaefer's FPT fixed point exist for operator $\mathscr{A}$. Hence given problem has a solution in $PC([a,T]_{\mathbb{N}_a}, \mathbb{R})$. □

Theorem 6.4. *Let $f : [a,T]_{\mathbb{N}_a} \times \mathbb{R} \to \mathbb{R}$ be continuous, and $1 < \varphi \leq 2$. Assume that I_k, $\bar{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous for $k \in \mathbb{N}_1^m$. Assume $\alpha, \beta, \xi, \eta \in \mathbb{R}$, $\xi, \eta \in (a,T)$ where$\delta \neq 0$. In addition, assume that (C_1), (C_2) hold and*

$$\mathcal{n} < \tfrac{\Gamma(\varphi+1)}{3}\Big((1+M_1)(T-a)^{\underline{\varphi}} + \big|\beta\big|M_1(\eta-a)^{\underline{\varphi}} + M_2(\xi-a)^{\underline{\varphi}}\Big)^{-1},$$
$$\mathcal{n}_1 < \tfrac{1}{3}\Big(m + M_1\big(m + \big|\beta\big|(\eta-a)\big) + M_2(\xi-a)\Big)^{-1},$$
$$\mathcal{n}_2 < \tfrac{2}{3}\Big((1+M_1)(T-a)^{\underline{2}} + \big|\beta\big|M_1(\eta-a)^{\underline{2}} + M_2(\xi-a)^{\underline{2}}\Big)^{-1}.$$

Then unique solution to the problem (5.11)–(5.13) *with boundary condition* (5.17) *exists.*

Proof. Let $x, \bar{x} \in \mathbb{R}$. For each $t \in [a,T]_{\mathbb{N}_a}$, we have $\big|\mathscr{A}x(t) - \mathscr{A}\bar{x}(t)\big|$

$$\begin{aligned}
\leq& \sum_{a<\iota_k<t} \big|I_k(x(\iota_k)) - I_k(\bar{x}_k)\big| + \frac{\big|p(t)\big|}{|\delta|}\Big\{\sum_{a<\iota_k<T} \big|I_k(x(\iota_k)) - I_k(\bar{x}_k)\big| \\
&+ \big|\beta\big| \sum_{a<\iota_k<\eta} \big|I_k(x(\iota_k)) - I_k(\bar{x}_k)\big|\Big\} + \frac{\big|\alpha\big|\big|q(t)\big|}{\big|\delta\big|} \sum_{a<\iota_k<\xi} \big|I_k(x(\iota_k)) - I_k(\bar{x}_k)\big|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{a<\iota_k<T} H(t,\iota_k)\big|\bar{I}_k(x(\iota_k))-\bar{I}_k(\bar{x}_k)\big| + \sum_{a<\iota_k<t}\big|\bar{I}_k(x(\iota_k))-\bar{I}_k(\bar{x}_k)\big| \\
&+\sum_{s=a}^{T} G(t,s)\big|f(\varphi+\rho(s),x(\varphi+\rho(s)))-f(\varphi+\rho(s),\bar{x}(\varphi+\rho(s)))\big|, \\
\le&||x-\bar{x}||\Bigg(m\mathcal{n}_1+M_1\Big\{m\mathcal{n}_1+\big|\beta\big|\mathcal{n}_1(\eta-a)\Big\}+M_2\mathcal{n}_1(\xi-a) \\
&+\Big\{M_1\frac{(T-a)^{\underline{2}}}{\Gamma(3)}+\big|\beta\big|M_1\frac{(\eta-a)^{\underline{2}}}{\Gamma(3)}+M_2\frac{(\xi-a)^{\underline{2}}}{\Gamma(3)}+\frac{(T-a)^{\underline{2}}}{\Gamma(3)}\Big\}\mathcal{n}_2 \\
&+\Big\{\frac{(T-a)^{\underline{\varphi}}}{\Gamma(\varphi+1)}+M_1\frac{(T-a)^{\underline{\varphi}}}{\Gamma(\varphi+1)}+\big|\beta\big|M_1\frac{(\eta-a)^{\underline{\varphi}}}{\Gamma(\varphi+1)}+M_2\frac{(\xi-a)^{\underline{\varphi}}}{\Gamma(\varphi+1)}\Big\}\mathcal{n}\Bigg), \\
=&||x-\bar{x}||\Bigg(\Big\{m+M_1\big(m+\big|\beta\big|(\eta-a)\big)+M_2(\xi-a)\Big\}\mathcal{n}_1 \\
&+\Big\{(1+M_1)(T-a)^{\underline{2}}+\big|\beta\big|M_1(\eta-a)^{\underline{2}}+M_2(\xi-a)^{\underline{2}}\Big\}\frac{\mathcal{n}_2}{2} \\
&+\Big\{(1+M_1)(T-a)^{\underline{\varphi}}+\big|\beta\big|M_1(\eta-a)^{\underline{\varphi}}+M_2(\xi-a)^{\underline{\varphi}}\Big\}\frac{\mathcal{n}}{\Gamma(\varphi+1)}\Bigg), \\
<&\Big(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\Big)||x-\bar{x}|| < ||x-\bar{x}||.
\end{aligned}
$$

$\mathscr{A}$ is a contraction mapping. As a consequence of the Banach FPT the given BVP has a unique solution. □

Theorem 6.5. *Let $f:[a,T]_{\mathbb{N}_a}\times\mathbb{R}\to\mathbb{R}$ be continuous, $c_1,c_2,\theta\in\mathbb{R}$, $c_1+c_2\neq 0$ and $0<\varphi\le 1$. Assume that I_k, $\bar{I}_k:\mathbb{R}\to\mathbb{R}$ are continuous for $k\in\mathbb{N}_1^m$. In addition, assume that $(\mathcal{C}_1),(\mathcal{C}_2)$ (i), hold and $\mathcal{n}<\frac{\Gamma(\varphi+1)}{2(T-a)^{\underline{\varphi}}}\Big(1+\frac{|c_1|}{|c_1+c_2|}\Big)^{-1}$, $\mathcal{n}_1<\frac{1}{2m}\Big(1+\frac{|c_1|}{|c_1+c_2|}\Big)^{-1}$. Then the problem* (5.11)–(5.12) *with boundary condition* (5.16) *has a unique solution.*

Finally we give an example to illustrate the usefulness of Theorem 6.3.

Example 6.2.1. *Consider the FDE with impulses and four-point BC*

$$
\begin{cases}
{}^{c}\Delta_0^{\varphi}x(t)+t^{\underline{2}}\ \sin(x)=0, \quad 1<\varphi\le 2,\ t\in\mathbb{N}_{1-\varphi}^{26-\varphi}, \quad t\neq\iota_1,\iota_2,\iota_3, \\
I_k=3, \quad k\in\mathbb{N}_1^3 \\
\bar{I}_k=\frac{1}{5+e^{x^2}}, \\
x(0)=2x(3),\ x(25)=3x(5).
\end{cases}
$$

Since $\delta = -22 \neq 0$, and there exist positive constants $L, L_k, \bar{L}_k$ such that

$$\left|t^2 \ \sin(x)\right| \leq \left|t(t-1)\right| = 600 = L, \ \left|I_k(x))\right| \leq L_k = 3, \ \left|\bar{I}_k(x)\right| \leq \frac{1}{6} = \bar{L}_k,$$

$k \in \mathbb{N}_1^3$. *Hence all the assumptions of Theorem 6.3 are satisfied. Which shows that problem possess at least one solution.*

6.3 Existence and uniqueness of solutions for MPFBVP

Findings of this section appeared in [107]. In this section, we shall study the existence for the nonlinear difference equation of non integer order with multi-point summation boundary conditions. The Schauder's FPT is utilized for existence, and the contraction mapping theorem [191] is utilized for uniqueness of solutions.

Theorem 6.6. *Let $h : [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}} \times \mathbb{R} \to \mathbb{R}$ be continuous function in 2^{nd} variable, $\max_{t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}} \phi(t) \leq M$, where ϕ is the unique solution to the MPFBVP* (5.29). *Let $C = \max\{|h(t,u)| : 0 \leq t \leq b+1, u \in \mathbb{R}, |u| \leq 2M\} > 0$. Then the nonlinear MPFBVP* (5.21) *has a solution provided*

$$M^* < \frac{\Gamma(\vartheta + 2)M}{C}.$$

Proof. Define a norm $||.||$ on space of real valued functions X as $||y|| = \max\{|y(t)| : t \in \mathbb{N}_{\vartheta-2}^{\vartheta+b+1}\}$ so that the pair $(X, ||.||)$ is a Banach space. Thus X is a topological vector space. Define the compact, convex subset $K = \{x \in X : ||x|| \leq 2M\}$ of X. The operator $\mathcal{T} : X \to X$ is given by

$$\mathcal{T}x(t) = \sum_{s=0}^{b+1} G(t,s)h(s, x(\rho(s) + \vartheta)) + \phi(t), \quad t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}.$$

First we shall show that $\mathcal{T}$ is self map. For arbitrary $t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$ and $x \in K$, we have

$$\begin{aligned}
|\mathcal{T}x(t)| =& \Big|\sum_{s=0}^{b+1} G(t,s)h(s,x(\rho(s)+\vartheta)) + \phi(t)\Big| \\
\leq& \sum_{s=0}^{b+1} G(t,s)|h(s,x(\rho(s)+\vartheta))| + |\phi(t)| \\
\leq& C\sum_{s=0}^{b+1} G(t,s) + M \\
\leq& \frac{CM^*}{\Gamma(\vartheta+2)} + M \\
\leq& \frac{C}{\Gamma(\vartheta+2)} \frac{\Gamma(\vartheta+2)M}{C} + M \leq 2M.
\end{aligned}$$

To show $\mathcal{T}$ is continuous on K, let $\varepsilon > 0$ and substitute $\max_{\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}} \sum_{s=0}^{b+1} G(t,s) = L$. Then by Lemma 5.3.2, we have

$$L = \frac{M^*}{\Gamma(\vartheta+2)}.$$

Continuity of h on $\mathbb{R}$, implies its uniform continuity on $[-2M, 2M]$. So there exists $\delta > 0$ for all t, and for all $x, \bar{x} \in [-2M, 2M]$ with $|(t,x) - (t,\bar{x})| < \delta$ we have

$$|h(t,x) - h(t,\bar{x})| < \frac{\varepsilon}{L}.$$

Thus for all $t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$ it follows that

$$|\mathcal{T}x(t) - \mathcal{T}\bar{x}(t)| = \Big|\sum_{s=0}^{b+1} G(t,s)h(s,x(\rho(s)+\vartheta)) - \sum_{s=0}^{b+1} G(t,s)h(s,\bar{x}(\rho(s)+\vartheta))\Big|$$

$$\leq \sum_{s=0}^{b+1} G(t,s)\Big|h(s,x(\rho(s)+\vartheta)) - h(s,\bar{x}(\rho(s)+\vartheta))\Big| < \sum_{s=0}^{b+1} G(t,s)\frac{\varepsilon}{L} \leq \varepsilon.$$

This proves the continuity of operator $\mathcal{T}$ from K into K. Thus the application of Schauder's theorem implies that $\mathcal{T}(x) = x\ for\ some\ x \in K$. This prove the desired result. □

The Contraction mapping theorem [191] is utilized for uniqueness, in the following theorem.

Theorem 6.7. *Assume $h : [\vartheta-2, b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}} \times \mathbb{R} \to \mathbb{R}$ is uniform Lipschitz in 2^{nd} variable, with constant $k > 0$. If $k < \frac{\Gamma(\vartheta+2)}{M^*}$, then the nonlinear fractional BVP (5.21) has unique solution.*

Proof. Define a norm $||.||$ on space of real valued functions X by $||y|| = \max\{|y(t)| : t \in \mathbb{N}_{\vartheta-2}^{\vartheta+b+1}\}$ in such a way that $(X, ||.||)$ is a Banach space. Define operator $\mathcal{T}$ as stated in Theorem 6.6. To show that $\mathcal{T}$ is a contraction map, observe for all $t \in \mathbb{N}_{\vartheta-2}^{b+\vartheta+1}$ and for all $x, \bar{x} \in X$ that

$$\begin{aligned}
||\mathcal{T}x(t) - \mathcal{T}\bar{x}(t)|| &= \max_{t\in\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}} \left| \sum_{s=0}^{b+1} G(t,s)[h(s, x(\rho(s)+\vartheta)) - h(s, \bar{x}(\rho(s)+\vartheta))] \right| \\
&\leq \max_{t\in\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}} \sum_{s=0}^{b+1} G(t,s)k\big|x(\rho(s)+\vartheta) - \bar{x}(\rho(s)+\vartheta)\big| \\
&\leq k\big|\big|x-\bar{x}\big|\big| \max_{t\in\mathbb{N}_{\vartheta-2}^{b+\vartheta+1}} \sum_{s=0}^{b+1} G(t,s) \\
&\leq k\big|\big|x-\bar{x}\big|\big| \frac{M^*}{\Gamma(\vartheta+2)} \leq \alpha\big|\big|x-\bar{x}\big|\big|.
\end{aligned}$$

Thus by given condition $\alpha = \frac{kM^*}{\Gamma(\vartheta+2)} < 1$, consequently $\mathcal{T}$ is a contraction on X. Hence fixed point of $\mathcal{T}$ in X is unique. □

6.4 Existence and uniqueness of solutions for Hilfer fractional difference system

An application of Brouwer's FPT gives us conditions for the existence of solution for a class of Hilfer nonlinear FDE. For uniqueness of solution we applied Banach contraction principle. To solve linear fractional Hilfer difference equation we used successive approximation method and then define the discrete ML function in the delta difference setting. Findings of this section appeared in [109].

Theorem 6.8. *Let $f : [a, T]_{\mathbb{N}_a} \to \mathbb{R}$ be a bounded function in such a way that $|g(x, u)| \leq f(x)|u|$*

for all $u \in Z$ given in Lemma 5.4.1. Then IVP (5.31) has at least one solution on Z, provided

$$L^* \leq \frac{\Gamma(\vartheta+1)}{(T-a-1+\vartheta)^{\underline{\vartheta}}}, \tag{6.2}$$

where $L^* = \sup_{x \in \mathbb{N}_{a+1-\vartheta}^{T}} f(x+\vartheta-1)$.

Proof. For $M > 0$, define the set

$$W = \{u : ||u - \zeta h_{\eta-1}(x, a+1-\eta)|| \leq M, \text{ for } x \in \mathbb{N}_{a+1-\vartheta}^{T}\}.$$

To prove this theorem we just have to show that $\mathscr{A}$ maps W into itself. For $u \in W$, we have

$$\begin{aligned}
&\Big|\mathscr{A}u(x) - \zeta h_{\eta-1}(x, a+1-\eta)\Big| \\
&\qquad \leq f(x+\vartheta-1) \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x, \sigma(\mathscr{F}))|u(\mathscr{F}+\vartheta-1) - 0| \\
&\qquad \leq L^* \sup_{x \in \mathbb{N}_{a+1-\vartheta}^{T}} |\, u(x+\vartheta-1) - 0| \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x, \sigma(\mathscr{F})) \\
&\qquad = L^* ||u - 0|| \Big[\frac{(x-a-1+\vartheta)^{\underline{\vartheta}}}{\Gamma(\vartheta+1)} - 0\Big] \\
&\qquad \leq L^* M \frac{(T-a-1+\vartheta)^{\underline{\vartheta}}}{\Gamma(\vartheta+1)} \leq M.
\end{aligned}$$

We get $||\mathscr{A}u|| \leq M$ which implies that $\mathscr{A}$ is self map. Therefore by Brouwer's FPT $\mathscr{A}$ has at least one fixed point. □

Theorem 6.9. *For $k > 0$ and $u, v \in Z$ assume that $|g(x,u) - g(x,v)| \leq k|u - v|$ for all $x \in [a, T]_{\mathbb{N}_a}$. Then IVP (5.31) has unique solution on Z, provided*

$$k < \frac{\Gamma(\vartheta+1)}{(T-a-1+\vartheta)^{\underline{\vartheta}}}. \tag{6.3}$$

Proof. Let $u, v \in Z$ and $x \in [a, T]_{\mathbb{N}_a}$, we have by assumption

$$\begin{aligned}
\Big|\mathscr{A}u(x) - \mathscr{A}v(x)\Big| \leq & \Big|\sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x, \sigma(\mathscr{F}))\Big| \\
& \times |g(\mathscr{F}+\vartheta-1, u(\mathscr{F}+\vartheta-1)) - g(\mathscr{F}+\vartheta-1, v(\mathscr{F}+\vartheta-1))| \\
\leq & \frac{|0 - (x-a-1+\vartheta)^{\underline{\vartheta}}|}{\Gamma(\vartheta+1)} k|u(\mathscr{F}+\vartheta-1) - v(\mathscr{F}+\vartheta-1)|.
\end{aligned}$$

In the preceding step, we used $\sum_{\mathcal{F}} h_{\varphi-1}(x,\sigma(\mathcal{F})) = -h_{\varphi}(x,\mathcal{F})$ and Lemma 2.1.9. Now taking supremum on both sides we have

$$\sup_{x\in\mathbb{N}_a^T} \Big|\mathcal{A}u(x) - \mathcal{A}v(x)\Big| \leq \frac{k(T-a-1+\vartheta)^{\underline{\vartheta}}}{\Gamma(\vartheta+1)}||u-v||.$$

Using inequality (6.3), we get $||\mathcal{A}u - \mathcal{A}v|| \leq ||u-v||$ which implies $\mathcal{A}$ is contraction. Therefore by Banach FPT $\mathcal{A}$ has unique fixed point. $\square$

To solve the linear Hilfer fractional difference IVP, we use the successive approximation method.

Example 6.4.1. *Let $\eta = \vartheta + \varphi - \vartheta\varphi$, with $0 < \vartheta < 1$ and $0 \leq \varphi \leq 1$. Consider the IVP for linear Hilfer fractional difference equation,*

$$\begin{cases} \Delta_a^{\vartheta,\varphi}\chi(x) - \lambda\chi(x+\vartheta-1) = 0, \\ \Delta_a^{-(1-\eta)}\chi(a+1-\eta) = \zeta, \quad \zeta \in \mathbb{R}. \end{cases} \tag{6.4}$$

The solution of (6.4) is given by

$$\chi(x) = \zeta h_{\eta-1}(x, a+1-\eta) + \lambda\Delta_{a+1-\vartheta}^{-\vartheta}\chi(x+\vartheta-1).$$

The Definition 2.1.15 and successive approximation yields the following

$$\chi_k(x) = \chi_0(x) + \lambda \sum_{\mathcal{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathcal{F}))\chi_{k-1}(\mathcal{F}+\vartheta-1), \tag{6.5}$$

for $k = 1, 2, 3, \cdots$, where $\chi_0(x) = \zeta h_{\eta-1}(x, a+1-\eta)$.

Initially for $k = 1$ and by Lemma 2.1.16

$$\chi_1(x) = \zeta h_{\eta-1}(x, a+1-\eta) + \lambda\zeta h_{\eta-1+\vartheta}(x+\vartheta-1, a+1-\eta).$$

Similarly for $k = 2$

$$\begin{aligned}\chi_2(x) =&\zeta\Big[h_{\eta-1}(x, a+1-\eta) + \lambda h_{\eta-1+\vartheta}(x+\vartheta-1, a+1-\eta) + \lambda^2 h_{\eta-1+2\vartheta}(x \\ &+ 2(\vartheta-1), a+1-\eta)\Big] \\ =&\zeta\Big[\lambda^0 \frac{(x+\eta-a-1)^{\underline{0.\vartheta+\eta-1}}}{\Gamma(\eta)} + \lambda^1 \frac{(x+\eta-a-1+(\vartheta-1))^{\underline{1.\vartheta+\eta-1}}}{\Gamma(\vartheta+\eta)} \\ &+ \lambda^2 \frac{(x+\eta-a-1+2(\vartheta-1))^{\underline{2.\vartheta+\eta-1}}}{\Gamma(2\vartheta+\eta)}\Big].\end{aligned}$$

Proceeding inductively and let $k \to \infty$

$$\chi(x) =\zeta\Big[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+k(\vartheta-1))^{\underline{k\vartheta+\eta-1}}}{\Gamma(k\vartheta+\eta)}\Big].$$

Now, we use property $x^{\underline{\vartheta+\varphi}} = (x-\varphi)^{\underline{\vartheta}}\ x^{\underline{\varphi}}$ in the following step,

$$\chi(x) =\zeta\Big[\sum_{k=0}^{\infty} \lambda^k \frac{(x+\eta-a-1+(k-1)(\vartheta-1))^{\underline{k\vartheta}}(x+\eta-a-1+k(\vartheta-1))^{\underline{\eta-1}}}{\Gamma(k\vartheta+\eta)}\Big].$$

Now, from the discrete form (6.5), we have numerical formula

$$\chi(a+n) =\chi(a) + \frac{\lambda}{\Gamma(\vartheta)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\vartheta)}{\Gamma(n-j+1)} \chi(a+j-1), \tag{6.6}$$

with $\chi(a) = \zeta \frac{\Gamma(n+\eta)}{\Gamma(\eta)\Gamma(n+1)}$. From (6.6), we can have

$$\chi(n) =\zeta \frac{\Gamma(n+\eta)}{\Gamma(\eta)\Gamma(n+1)} + \frac{\lambda}{\Gamma(\vartheta)} \sum_{j=1}^{n} \frac{\Gamma(n-j+\vartheta)}{\Gamma(n-j+1)} \chi(j-1).$$

For different values of φ numerical solutions for $\vartheta = 0.8$ and $\vartheta = 0.5$ are shown in Fig. 1 and Fig. 2 respectively. Fig. 1 and Fig. 2 show the interpolative behavior of Hilfer difference operator between the RL [188] and the Caputo difference operator [187].

Definition 6.4.2. *For $\lambda \in \mathbb{R}$ and $\vartheta, \eta, z \in \mathbb{C}$ with $Re(\vartheta) > 0$, the discrete ML functions are defined by*

$$E_{\underline{\vartheta,\eta}}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+(k-1)(\vartheta-1))^{\underline{k\vartheta}}(z+k(\vartheta-1))^{\underline{\eta-1}}}{\Gamma(k\vartheta+\eta)}.$$

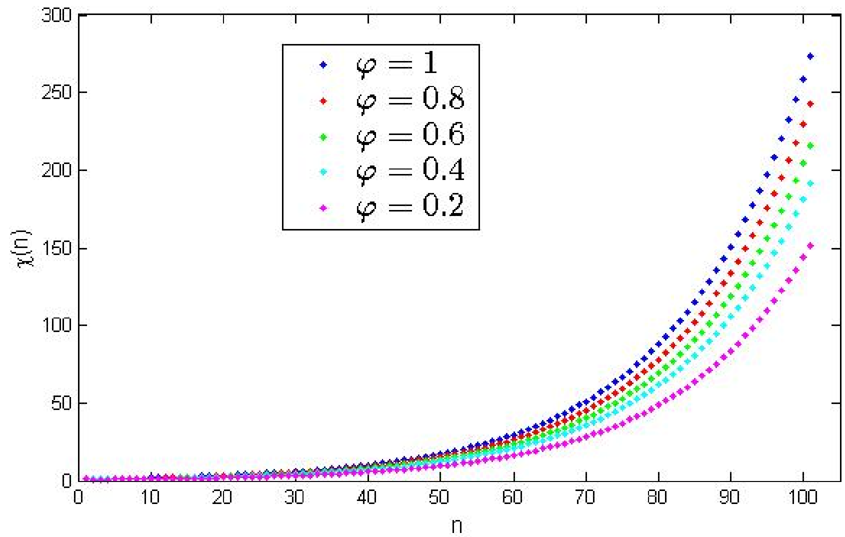

Figure 6.1: Solutions for $\lambda = 0.1$, $\vartheta = 0.8$ and different values of φ.

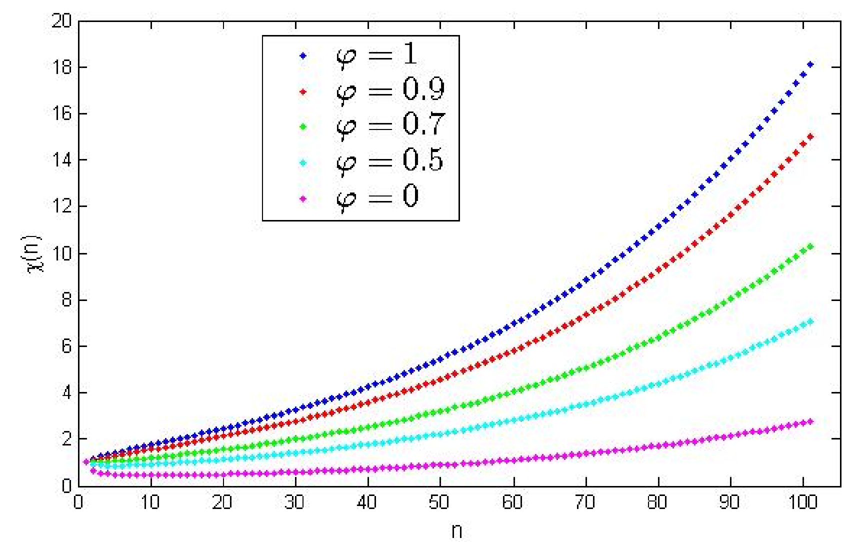

Figure 6.2: Solutions for $\lambda = 0.1$, $\vartheta = 0.5$ and different values of φ.

Note that $E_{\underline{\vartheta,\eta}}(\lambda, z)$ are same discrete Mittag-Leffler functions appeared in [4].

Next we solve the non-homogeneous Hilfer fractional difference IVP.

Example 6.4.3. *Let $\eta = \vartheta + \varphi - \vartheta\varphi$, with $0 < \vartheta < 1$ and $0 \leq \varphi \leq 1$. Consider Hilfer non-homogeneous fractional difference equation,*

$$\begin{cases} \Delta_a^{\vartheta,\varphi}\chi(x) - \lambda\chi(x+\vartheta-1) = f(x), \\ \Delta_a^{-(1-\eta)}\chi(a+1-\eta) = \zeta, \quad \zeta \in \mathbb{R}. \end{cases} \tag{6.7}$$

The solution of (6.7) *is given by*

$$\chi(x) =\zeta h_{\eta-1}(x, a+1-\eta) + \lambda\Delta^{-\vartheta}_{a+1-\vartheta}\chi(x+\vartheta-1) + \Delta^{-\vartheta}_{a+1-\vartheta}f(x).$$

The Definition 2.1.15 and successive approximation yields

$$\chi_k(x) =\chi_0(x) + \lambda\sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x, \sigma(\mathscr{F}))\chi_{k-1}(\mathscr{F}+\vartheta-1) + \Delta^{-\vartheta}_{a+1-\vartheta}f(x),$$

for $k = 1, 2, 3, \cdots$, *where* $\chi_0(x) = \zeta h_{\eta-1}(x, a+1-\eta)$.

Initially for $k = 1$ *and by Lemma 2.1.16*

$$\chi_1(x) =\zeta h_{\eta-1}(x, a+1-\eta) + \lambda\zeta h_{\eta-1+\vartheta}(x+\vartheta-1, a+1-\eta) + \Delta^{-\vartheta}_{a+1-\vartheta}f(x).$$

Similarly for $k = 2$

$$\begin{aligned}\chi_2(x) =&\zeta\Big[h_{\eta-1}(x, a+1-\eta) + \lambda h_{\eta-1+\vartheta}(x+\vartheta-1, a+1-\eta) + \lambda^2 h_{\eta-1+2\vartheta}(x\\ &+2(\vartheta-1), a+1-\eta)\Big] + \lambda\Delta^{-2\vartheta}_{a+1-\vartheta}f(x+\vartheta-1) + \Delta^{-\vartheta}_{a+1-\vartheta}f(x)\\ =&\zeta\Big[\lambda^0\frac{(x+\eta-a-1)^{\underline{0.\vartheta+\eta-1}}}{\Gamma(\eta)} + \lambda^1\frac{(x+\eta-a-1+(\vartheta-1))^{\underline{1.\vartheta+\eta-1}}}{\Gamma(\vartheta+\eta)}\\ &+\lambda^2\frac{(x+\eta-a-1+2(\vartheta-1))^{\underline{2.\vartheta+\eta-1}}}{\Gamma(2\vartheta+\eta)}\Big] + \lambda\Delta^{-2\vartheta}_{a+1-\vartheta}f(x+\vartheta-1)\\ &+\Delta^{-\vartheta}_{a+1-\vartheta}f(x).\end{aligned}$$

Proceeding inductively and let $k \to \infty$

$$\begin{aligned}\chi(x) =&\zeta\Big[\sum_{k=0}^{\infty}\lambda^k\frac{(x+\eta-a-1+k(\vartheta-1))^{\underline{k\vartheta+\eta-1}}}{\Gamma(k\vartheta+\eta)}\Big]\\ &+\sum_{k=1}^{\infty}\lambda^{k-1}\Delta^{-k\vartheta}_{a+1-\vartheta}f(x+(k-1)(\vartheta-1))\\ =&\zeta\Big[\sum_{k=0}^{\infty}\lambda^k\frac{(x+\eta-a-1+k(\vartheta-1))^{\underline{k\vartheta+\eta-1}}}{\Gamma(k\vartheta+\eta)}\Big]\\ &+\sum_{k=1}^{\infty}\lambda^{k-1}\sum_{\mathscr{F}=a+1-\vartheta}^{x-k\vartheta}h_{k\vartheta-1}(x, \sigma(\mathscr{F}+(k-1)(\vartheta-1)))f(\mathscr{F})\end{aligned}$$

$$
\begin{aligned}
\chi(x) =&\zeta\Big[\sum_{k=0}^{\infty}\lambda^k\frac{(x+\eta-a-1+k(\vartheta-1))^{\underline{k\vartheta+\eta-1}}}{\Gamma(k\vartheta+\eta)}\Big]\\
&+\sum_{k=0}^{\infty}\lambda^k\sum_{\mathcal{F}=a+1-\vartheta}^{x-k\vartheta-\vartheta}\frac{(x-\sigma(\mathcal{F})+k(\vartheta-1))^{\underline{k\vartheta+\vartheta-1}}}{\Gamma(k\vartheta+\vartheta)}f(\mathcal{F})\\
=&\zeta\Big[\sum_{k=0}^{\infty}\lambda^k\frac{(x+\eta-a-1+k(\vartheta-1))^{\underline{k\vartheta+\eta-1}}}{\Gamma(k\vartheta+\eta)}\Big]\\
&+\sum_{\mathcal{F}=a+1-\vartheta}^{x-\vartheta}\sum_{k=0}^{\infty}\lambda^k\frac{(x-\sigma(\mathcal{F})+k(\vartheta-1))^{\underline{k\vartheta+\vartheta-1}}}{\Gamma(k\vartheta+\vartheta)}f(\mathcal{F}).
\end{aligned}
$$

In preceding step, we have interchanged summation of second expression. Now, we use property $x^{\underline{\vartheta+\varphi}}=(x-\varphi)^{\underline{\vartheta}}\,x^{\underline{\varphi}}$ *in the following step,*

$$
\begin{aligned}
\chi(x) =&\zeta\Big[\sum_{k=0}^{\infty}\lambda^k\frac{(x+\eta-a-1+(k-1)(\vartheta-1))^{\underline{k\vartheta}}(x+\eta-a-1+k(\vartheta-1))^{\underline{\eta-1}}}{\Gamma(k\vartheta+\eta)}\Big]\\
&+\sum_{\mathcal{F}=a+1-\vartheta}^{x-\vartheta}\sum_{k=0}^{\infty}\lambda^k\frac{(x-\sigma(\mathcal{F})+(k-1)(\vartheta-1))^{\underline{k\vartheta}}(x-\sigma(\mathcal{F})+k(\vartheta-1))^{\underline{\vartheta-1}}}{\Gamma(k\vartheta+\vartheta)}f(\mathcal{F}).
\end{aligned}
$$

Using Definition 6.4.2, we have

$$
\chi(x)=\zeta E_{\underline{\vartheta,\eta}}(\lambda,x+\eta-a-1)+\sum_{\mathcal{F}=a+1-\vartheta}^{x-\vartheta}\Big[E_{\underline{\vartheta,\vartheta}}(\lambda,x-\sigma(\mathcal{F}))\Big]f(\mathcal{F}).
$$

Note that above is the generalization of Caputo fractional difference IVP [4], one can prevail it for $\varphi=1$.

6.5 Existence and uniqueness of solutions for substantial fractional difference system

The Brouwer's FPT is utilized for existence, and the Banach FPT is utilized for uniqueness of solutions for a class of substantial nonlinear FDE. Findings of this section appeared in [108].

Theorem 6.10. *Let* $g:[a,T]_{\mathbb{N}_a}\to\mathbb{R}$ *be a bounded function in such a way that* $|f(x,u)|\le g(x)|u|$ *for all* $u\in Z$*. Then IVP* (5.33) *has at least a solution on* Z*, provided that*

$$
L^*\le(1-p)^m, \tag{6.8}
$$

where $L^* = \sup_{x\in\mathbb{N}_a^T} \sum_{\mathcal{J}=a+m-\varphi}^{x-\varphi+m-j} h_{\varphi-m+j-1}(x,\sigma(\mathcal{J}))e_{-p}(x-\mathcal{J},0)g(\mathcal{J}+\varphi-1).$

Proof. For $M>0$, define the set

$$B = \Big\{u(x) : ||u - \frac{e_{-p}(x-a+1,0)h_{\varphi-m+1}(x,a)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}} \sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j u_i|| \le M\Big\}.$$

To prove theorem we just show that $\mathcal{A}$ maps B into B. For $u \in B$, we have

$$\Big|\mathcal{A}u(x) - \tfrac{e_{-p}(x-a+1,0)h_{\varphi-m+1}(x,a)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}} \sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j u_i\Big|$$

$$\le \frac{\sum_{\mathcal{J}=a+m-\varphi}^{x-\varphi+m-j} h_{\varphi-m+j-1}(x,\sigma(\mathcal{J}))e_{-p}(x-\mathcal{J},0)g(\mathcal{J}+\varphi-1)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}}|u-0|.$$

Since $\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell} = (1-p)^m$. Take supremum on both sides we have

$$\sup_{x\in\mathbb{N}_a^T}\Big|\mathcal{A}u(x) - \frac{e_{-p}(x-a+1,0)h_{\varphi-m+1}(x,a)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}} \sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j u_i\Big| \le \frac{L^*M}{(1-p)^m}.$$

By using inequality (6.8), we get $||\mathcal{A}u|| \le M$ which implies that $\mathcal{A}$ is self map and therefore by Brouwer fixed point theorem it has at least a fixed point. □

Theorem 6.11. *Under the assumption* (H_1) $|f(x,u)-f(x,v)| \le k|u-v|$, *for* $k>0$ *and for all* $u,v \in Z$ *and* $x \in [a,T]_{\mathbb{N}_a}$. *The IVP* (5.33) *has a unique solution on* Z, *provided*

$$k < \frac{|1-p|^{2m-\varphi-j}}{|h_{\varphi-m+j}(T,a-\varphi+m)|}. \tag{6.9}$$

Proof: Let $u,v \in Z$ and $x \in [a,T]_{\mathbb{N}_a}$. Then, we have

$$\begin{aligned}\Big|\mathcal{A}u(x) - \mathcal{A}v(x)\Big| \le& \Big|\frac{\sum_{\mathcal{J}=a+m-\varphi}^{x-\varphi+m-j} h_{\varphi-m+j-1}(x,\sigma(\mathcal{J}))e_{-p}(x-\mathcal{J},0)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}}\Big| \\ &\times |f(\mathcal{J}+\varphi-1,u(\mathcal{J}+\varphi-1)) - f(\mathcal{J}+\varphi-1,v(\mathcal{J}+\varphi-1))| \\ \le& \frac{|h_{\varphi-m+j}(x,a-\varphi+m)|}{|1-p|^{2m-\varphi-j}}k|u(\mathcal{J}+\varphi-1)-v(\mathcal{J}+\varphi-1)|.\end{aligned}$$

In the preceding step, we use condition (H_1), $\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell} = (1-p)^m$, Lemma 2.1.9, $\sum_{\mathscr{F}} h_{\varphi-1}(x,\sigma(\mathscr{F})) = -h_{\varphi}(x,\mathscr{F})$ and the inequality

$$\Big|\sum_{\mathscr{F}=a+m-\varphi}^{x-\varphi+m-j} h_{\varphi-m+j-1}(x,\sigma(\mathscr{F}))e_{-p}(x-\mathscr{F},0)\Big| < \frac{\Big|\sum_{\mathscr{F}=a+m-\varphi}^{x-\varphi+m-j} h_{\varphi-m+j-1}(x,\sigma(\mathscr{F}))\Big|}{|1-p|^{m-\varphi-j}}.$$

Now taking supremum on both sides

$$\sup_{x\in\mathbb{N}_a^T}\Big|\mathscr{A}u(x)-\mathscr{A}v(x)\Big| \leq \frac{|h_{\varphi-m+j}(T,a-\varphi+m)|}{|1-p|^{2m-\varphi-j}}k||u-v||.$$

By using inequality (6.9), we get $||\mathscr{A}u-\mathscr{A}v|| \leq ||u-v||$ which implies $\mathscr{A}$ is contraction. Therefore by Banach FPT it has unique fixed point.

Chapter 7

Stability analysis

In this chapter, four kinds of Ulam stability to initial and boundary value problems is discussed. Conditions shall acquire for different type of fractional delta difference problems namely, nonlinear FDE with multi-point summation BC, Hilfer fractional difference with initial condition and substantial fractional difference with initial condition. Also continuous dependence of solution to fractional order Hilfer difference equation with on initial conditions will be discussed through modified Gronwall's inequality.

The definition of Ulam stability for fractional difference equations is introduced in [61]. Consider the system (5.31) and the following inequalities:

$$\left|\Delta_a^{\vartheta,\varphi} v(y) + g(y+\vartheta-1, v(y+\vartheta-1))\right| \leq \epsilon, \qquad y \in [a,T]_{\mathbb{N}_a}, \tag{7.1}$$

$$\left|\Delta_a^{\vartheta,\varphi} v(y) + g(y+\vartheta-1, v(y+\vartheta-1))\right| \leq \epsilon\eth(\rho(y)+\varphi), \qquad y \in [a,T]_{\mathbb{N}_a}, \tag{7.2}$$

where $\eth : [a,T]_{\mathbb{N}_a} \to \mathbb{R}^+$.

Definition 7.0.1. [109] *A solution $u \in Z$ of system (5.31) is UH stable if there exists a real number $d_f > 0$ such that for each $\epsilon > 0$ and for every solution $v \in Z$ of inequality (7.1), if* *it satisfies*

$$\Big|\Big|v-u\Big|\Big| \leq \epsilon d_f. \tag{7.3}$$

A solution of system (5.31) *is generalized UH stable if we substitute the function* $\wp_f(\epsilon)$ *for the constant* ϵd_f *in inequality* (7.3)*, where* $\wp_f(\epsilon) \in C(R^+, R^+)$ *and* $\wp_f(0) = 0$.

Definition 7.0.2. [109] *A solution* $u \in Z$ *of system* (5.31) *is UHR stable with respect to function* $\eth$ *if there exists a real number* $d_{f,\eth} > 0$ *such that for each* $\epsilon > 0$ *and for every solution* $v \in Z$ *of inequality* (7.2) *if it satisfies*

$$\Big|\Big|v-u\Big|\Big| \leq \epsilon \eth(y) d_{f,\eth}, \qquad y \in [a,T]_{\mathbb{N}_a}. \tag{7.4}$$

The solution of system (5.31) *is generalized UHR stable if we substitute the function* $\Phi(y)$ *for the function* $\epsilon\eth(y)$ *in inequalities* (7.2) *and* (7.4).

Now consider the system (5.33) and the following inequalities:

$$\Big|\ {}^{s}\Delta_a^{\varphi} v(x) + f(\rho(x)+\varphi, v(\rho(x)+\varphi))\Big| \leq \epsilon, \qquad x \in [a,T]_{\mathbb{N}_a}, \tag{7.5}$$

$$\Big|\ {}^{s}\Delta_a^{\varphi} v(x) + f(\rho(x)+\varphi, v(\rho(x)+\varphi))\Big| \leq \epsilon\eth(\rho(x)+\varphi), \qquad x \in [a,T]_{\mathbb{N}_a}, \tag{7.6}$$

where $\eth : [a,T]_{\mathbb{N}_a} \to \mathbb{R}^+$.

Definition 7.0.3. *[108] If there exists a real number* $d_f > 0$ *such that for each* $\epsilon > 0$ *and for every solution* $v \in Z$ *in inequality* (7.5) *the solution* $u \in Z$ *of system* (5.33) *is UH stable if it satisfies*

$$\Big|\Big|v-u\Big|\Big| \leq \epsilon d_f. \tag{7.7}$$

The solution of system (5.33) *is generalized UH stable if we substitute the function* $\wp_f(\epsilon)$ *for the constant* ϵd_f *in inequality* (7.7)*, where* $\wp_f(\epsilon) \in C(R^+, R^+)$ *and* $\wp_f(0) = 0$.

Definition 7.0.4. *[108] If there exists a real number $d_{f,\eth} > 0$ such that for each $\epsilon > 0$ and for every solution $v \in Z$ of inequality (7.6) then, a solution $u \in Z$ of system (5.33) is UHR stable with respect to function $\eth$ if it satisfies*

$$\big|\big|v - u\big|\big| \leq \epsilon \eth(x) d_{f,\eth}, \qquad x \in [a, T]_{\mathbb{N}_a}. \tag{7.8}$$

The solution of system (5.33) is generalized UHR stable if we substitute the function $\Phi(x)$ for the function $\epsilon\eth(x)$ in inequalities (7.6) and (7.8).

7.1 Stability of solutions for MPFBVP

In this section, we shall discuss UHR stability for the nonlinear difference equation of non integer order with multi-point summation boundary conditions. Findings of this section appeared in [107].

Theorem 7.1. *Assume for constant $L_h > 0$, $\big|h(t, x(t)) - h(t, y(t))\big| \leq L_h \Big|x(t) - y(t)\Big|$ holds, for each $t \in [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}}$ and $x(t), y(t) \in K$. Let $x(t) \in K$ be a solution of system (5.21) and $y(t) \in K$ be a solution of inequality (2.4) and M^* is given in Lemma 5.3.2. Then for $M^* L_h < 1$ then the nonlinear MPFBVP (5.21) is UH stable and consequently, generalized UH stable.*

Proof. By Corollary 5.1.4 the solution $x(t)$ of nonlinear MPFBVP (5.21) is given by equation (5.30). From inequality (2.4) for $t \in [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}}$, it follows that

$$\Big|y(t) - \big(\phi(t) - \sum_{s=0}^{b+1} G(t, s) h(\rho(s) + \vartheta, y(\rho(s) + \vartheta)\big)\Big| \leq \epsilon. \tag{7.9}$$

By making use of equation (5.30) and inequality (7.9) together for $t \in [\vartheta - 2, b + \vartheta + 1]_{\mathbb{N}_{\vartheta-2}}$, we have

$$\begin{aligned}
\Big|y(t)-x(t)\Big| =&\Big|y(t)-\Big(\phi(t)-\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,x(\rho(s)+\vartheta))\Big)\Big|\\
\leq&\Big|y(t)-\Big(\phi(t)-\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,y(\rho(s)+\vartheta))\Big)\Big|\\
&+\Big|\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,x(\rho(s)+\vartheta))-\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,y(\rho(s)+\vartheta))\Big|\\
\leq&\ \ \epsilon+M^*L_h\Big|y(t)-x(t)\Big|.
\end{aligned}$$

In preceding step, we use the assumption on h and Lemma 5.3.2. Simplification yields the following

$$\Big|y(t)-x(t)\Big|\leq\frac{\epsilon}{1-M^*L_h}=\epsilon d_h,\quad \text{with}\quad d_h=\frac{1}{1-M^*L_h}.$$

Therefore solution of the MPFBVP (5.21) is UH stable. Further by using $\wp_h(\epsilon)=\epsilon d_h$, $\wp_h(0)=0$, implies that solution of system (5.21) is generalized UH stable. $\square$

Theorem 7.2. *Assume for constant $L_h>0$, $\Big|h(t,x(t))-h(t,y(t))\Big|\leq L_h\Big|x(t)-y(t)\Big|$ holds, for each $t\in[\vartheta-2,b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}}$ and $x(t),y(t)\in K$. Let $x(t)\in K$ be a solution of system* (5.21) *and $y(t)\in K$ be a solution of inequality* (2.5) *where M^* is given in Lemma 5.3.2. Then for $M^*L_h<1$, the nonlinear MPFBVP* (5.21) *is UHR stable with respect to function $\eth:[\vartheta-2,b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}}\to\mathbb{R}^+$ and consequently, generalized UHR stable.*

Proof. By Corollary 5.1.4 the solution $x(t)$ of nonlinear MPFBVP (5.21) is given by equation (5.30). From inequality (2.5), for $t\in[\vartheta-2,b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}}$, it follows that

$$\Big|y(t)-\big(\phi(t)-\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,y(\rho(s)+\vartheta))\big)\Big|\leq\epsilon\eth(\rho(t)+\vartheta). \tag{7.10}$$

By making use of equation (5.30) and inequality (7.10) together for $t\in[\vartheta-2,b+\vartheta+1]_{\mathbb{N}_{\vartheta-2}}$, we have

$$
\begin{aligned}
\Big|y(t)-x(t)\Big| =& \Big|y(t)-\Big(\phi(t)-\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,x(\rho(s)+\vartheta))\Big)\Big|\\
\leq& \Big|y(t)-\Big(\phi(t)-\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,y(\rho(s)+\vartheta))\Big)\Big|\\
&+\Big|\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,x(\rho(s)+\vartheta))-\sum_{s=0}^{b+1}G(t,s)h(\rho(s)+\vartheta,y(\rho(s)+\vartheta))\Big|\\
\leq&\ \epsilon\eth(\rho(t)+\vartheta)+M^*L_h\Big|y(t)-x(t)\Big|.
\end{aligned}
$$

In preceding step, we use the assumption on h and Lemma 5.3.2. Simplification yields the following

$$
\Big|y(t)-x(t)\Big| \leq \frac{\epsilon\eth(\rho(t)+\vartheta)}{1-M^*L_h}=\epsilon d_{h,\eth}, \quad \text{with} \quad d_{h,\eth}=\frac{\eth(\rho(t)+\vartheta)}{1-M^*L_h}.
$$

Therefore solution of the MPFBVP (5.21) is UHR stable. Further $\Phi(\rho(t)+\vartheta)=\epsilon\eth(\rho(t)+\vartheta)$, implies that solution of system (5.21) is generalized UHR stable. □

Finally, to illustrate the usefulness of Theorem 7.1, we present the following example.

Example 7.1.1. *Consider the delta DE with summation condition*

$$
\begin{cases}
-\Delta_{-0.8}^{1.2}x(t)=\zeta(t+0.2)^{\underline{3}}x(t+0.2), & t\in[0,14]_{\mathbb{N}_0}\\
x(-0.8)=p, \qquad x(15.2)+10\sum_{s=0.2}^{14.2}\ x(s)=q,
\end{cases}
$$

we have $L_h=221.76\zeta$, *for* $t\in[-0.8,15.2]_{\mathbb{N}_{-0.8}}$, *since* $b=13$, $\vartheta=1.2$ *and* $\lambda=10$, *therefore* $M^*=2.12$. *Then for* $\frac{1}{\zeta}<470.1$, *the solution to given problem with inequalities*

$$
\Big|\Delta_{-0.8}^{1.2}y(t)+h(t+0.2,y(t+0.2))\Big|\leq\epsilon \qquad t\in[0,14]_{\mathbb{N}_0},
$$

$$
\Big|\Delta_{-0.8}^{1.2}y(t)+h(t+0.2,y(t+0.2))\Big|\leq\epsilon\eth(t+0.2) \qquad t\in[0,14]_{\mathbb{N}_0},
$$

is respectively UH stable and UHR stable with respect to function $\eth:[-0.8,15.2]_{\mathbb{N}_{-0.8}}\to\mathbb{R}^+$.

7.2 Stability of solutions for Hilfer fractional difference system

In this section, conditions are acquired for four type of Ulam stability. An application of modified Gronwall's inequality has been given for the stability of solution to fractional order Hilfer difference equation with initial conditions. Findings of this section appeared in [109].

Theorem 7.3. *For $k > 0$ assume that $|g(x,u) - g(x,v)| \leq k|u - v|$, for all $x \in [a,T]_{\mathbb{N}_a}$. Let $u \in Z$ be a solution of system* (5.31) *and $v \in Z$ be a solution of inequality* (7.1)*. Then for k in inequality* (6.3)*, the nonlinear IVP* (5.31) *is UH stable and consequently, generalized UH stable.*

Proof. For simplicity the solution of IVP (5.31) can be rewritten as by using Equation (6.1)

$$u(x) = \phi(x) - \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{F}))g(\mathscr{F}+\vartheta-1, u(\mathscr{F}+\vartheta-1)), \tag{7.11}$$

where $\phi(x) = \zeta h_{\eta-1}(x, a+1-\eta)$. Now, for $[a,T]_{\mathbb{N}_a}$ it follows from inequality (7.1) that

$$\left| v(x) - \Big(\phi(x) - \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{F}))g(\mathscr{F}+\vartheta-1, v(\mathscr{F}+\vartheta-1)) \Big) \right| \leq \epsilon. \tag{7.12}$$

For $[a,T]_{\mathbb{N}_a}$, making use of equation (7.11) and inequality (7.12) together for $[a,T]_{\mathbb{N}_a}$, we have

$$\begin{aligned}
\Big| v(x) - u(x) \Big| =& \left| v(x) - \Big(\phi(x) - \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{F}))g(\mathscr{F}+\vartheta-1, u(\mathscr{F}+\vartheta-1)) \Big) \right| \\
\leq& \left| v(x) - \Big(\phi(x) - \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{F}))g(\mathscr{F}+\vartheta-1, v(\mathscr{F}+\vartheta-1)) \Big) \right| \\
&+ \Big| \sum_{\mathscr{F}=a+1-\vartheta}^{x-\vartheta} h_{\vartheta-1}(x,\sigma(\mathscr{F})) \Big| \\
&\times \Big| g(\mathscr{F}+\vartheta-1, v(\mathscr{F}+\vartheta-1)) - g(\mathscr{F}+\vartheta-1, u(\mathscr{F}+\vartheta-1)) \Big| \\
\leq& \ \epsilon + |0 - h_{\vartheta}(x, a+1-\vartheta)| k \Big| v(\mathscr{F}+\varphi-1) - u(\mathscr{F}+\varphi-1) \Big|.
\end{aligned}$$

In preceding step, we used assumption and the same argument used in Theorem 6.9. Now, taking supremum on both sides and simplifying, we have

$$||v-u|| \leq \frac{\epsilon}{1-h_{\vartheta}(T,a+1-\vartheta)k} = \epsilon d_f, \quad \text{with} \;\; d_f = \frac{1}{1-h_{\vartheta}(T,a+1-\vartheta)k}.$$

Therefore by inequality (6.3), (5.31) is UH stable. Further by using $\wp_f(\epsilon) = \epsilon d_f$, $\wp_f(0) = 0$, which implies that (5.31) is generalized UH stable. □

Theorem 7.4. *For $k > 0$ assume that $|g(x,u) - g(x,v)| \leq k|u-v|$, for all $x \in [a,T]_{\mathbb{N}_a}$. Let $u \in Z$ be a solution of system* (5.31) *and $v \in Z$ be a solution of inequality* (7.2)*. Then for k in inequality* (6.3)*, the nonlinear IVP* (5.31) *is UHR stable with respect to function $\eth : [a,T]_{\mathbb{N}_a} \to \mathbb{R}^+$ and consequently, generalized UHR stable.*

To illustrate the usefulness of Theorem 7.3, we present the following example.

Example 7.2.1. *Consider the fractional Hilfer difference equation with initial conditions involving Reimann-Liouville fractional sum*

$$\begin{cases} -\Delta_{0.3}^{0.7,0.5} u(x) = (x-0.3)u(x-0.3), & x \in [0.3, 9.3]_{\mathbb{N}_{0.3}} \\ \Delta_{0.3}^{-(0.15)} u(0.45) = \zeta. \end{cases}$$

Here $a = 0.3$, $T = 9.3$, $\vartheta = 0.7$ and $\varphi = 0.5$. Therefore $\eta = 0.85$. Thus for $k < 0.1974$, the solution to the given problem with inequalities

$$\left|\Delta_{0.3}^{0.7,0.5} v(x) + (x-0.3)v(x-0.3)\right| \leq \epsilon \qquad x \in [0.3, 9.3]_{\mathbb{N}_{0.3}},$$

$$\left|\Delta_{0.3}^{0.7,0.5} v(x) + (x-0.3)v(x-0.3)\right| \leq \epsilon\eth(x-0.3) \qquad x \in [0.3, 9.3]_{\mathbb{N}_{0.3}},$$

is UH stable and UHR stable with respect to function $\eth : [0.3, 9.3]_{\mathbb{N}_{0.3}} \to \mathbb{R}^+$.

7.2.1 Application of modified Gronwall's inequality in delta difference setting

A simple utilization of Gronwall's inequality leads to stability for Hilfer difference equation. Let $\eta = \vartheta + \varphi - \vartheta\varphi$, then for $0 < \vartheta < 1$ and $0 \leq \varphi \leq 1$, we have $0 < \eta \leq 1$. Following result illustrates the application of Gronwall's inequality, for the system

$$\begin{cases} \Delta_a^{\vartheta,\varphi} v(x) + g(x+\vartheta-1, v(x+\vartheta-1)) = 0, \text{for } x \in \mathbb{N}_{a+1-\vartheta}, \\ \Delta_a^{-(1-\eta)} v(a+1-\eta) = \xi, \quad \xi \in \mathbb{R}. \end{cases} \tag{7.13}$$

Theorem 7.5. *Assume Lipschitz condition $|g(x,u) - g(x,v)| \leq k|u-v|$ holds for function g. Then the solution to Hilfer fractional difference system is stable.*

Proof. Let $u \in Z$ be a solution of system (5.31) and $v \in Z$ be a solution of system (7.13). Then the corresponding summation equations are

$$u(x) = \zeta h_{\eta-1}(x, a+1-\eta) - \Delta_{a+1-\vartheta}^{-\vartheta} g(x+\vartheta-1, u(x+\vartheta-1)),$$

$$v(x) = \xi h_{\eta-1}(x, a+1-\eta) - \Delta_{a+1-\vartheta}^{-\vartheta} g(x+\vartheta-1, v(x+\vartheta-1)).$$

For the absolute value of the difference, we have $|u(x) - v(x)|$

$$\begin{aligned} \leq & |\zeta - \xi||h_{\eta-1}(x, a+1-\eta)| \\ & + |\Delta_{a+1-\vartheta}^{-\vartheta}(g(x+\vartheta-1, u(x+\vartheta-1)) - g(x+\vartheta-1, v(x+\vartheta-1)))| \\ \leq & |\zeta - \xi| h_{\eta-1}(x, a+1-\eta) + \Delta_{a+1-\vartheta}^{-\vartheta} k|u(x+\vartheta-1) - v(x+\vartheta-1)|. \end{aligned}$$

Then it follows from the Theorem 2.3 that

$$|u(x) - v(x)| \leq \frac{|\zeta - \xi|}{\Gamma(\eta)} \sum_{\ell=0}^{\infty} E_K^{\ell}(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta-1}}.$$

By using Lemma 2.1.16, we get $E_K^{\ell}(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta-1}} = \frac{K^{\ell}\Gamma(\eta)}{\Gamma(\eta+\vartheta\ell)}(x+\eta-a-1+\ell(\vartheta-1))^{\underline{\eta+\vartheta\ell-1}}$.

To shape in the form of discrete Mittag-Leffler function, we use property $x^{\underline{\vartheta+\varphi}} = (x-\varphi)^{\underline{\vartheta}}\, x^{\underline{\varphi}}$,

$$\begin{aligned}
|u(x)-v(x)| \leq & |\zeta-\xi| \sum_{\ell=0}^{\infty} \frac{K^{\ell}}{\Gamma(\eta+\vartheta\ell)} (x+\eta-a-1+(k-1)(\vartheta-1))^{\underline{k\vartheta}} \\
& \times (x+\eta-a-1+k(\vartheta-1))^{\underline{\eta-1}} \\
= & |\zeta-\xi| E_{\underline{\vartheta,\eta}}(K, x+\eta-a-1),
\end{aligned}$$

where $E_{\underline{\vartheta,\eta}}(\lambda, x)$ is discrete Mittag-Leffler functions defined in [4]. Replace system (7.13) with

$$\begin{cases} \Delta_a^{\vartheta,\varphi} v(x) + g(x+\vartheta-1, v(x+\vartheta-1)) = 0, \\ \Delta_a^{-(1-\eta)} v(a+1-\eta) = \zeta_n, \end{cases} \tag{7.14}$$

for $x \in \mathbb{N}_{a+1-\vartheta}$ and $\zeta_n \to \zeta$. The solutions are denoted by v_n. Now we have

$$|u(x)-v_n(x)| \leq |\zeta-\zeta_n| E_{\underline{\vartheta,\eta}}(k, x+\eta-a-1).$$

This leads to $|u(x)-v_n(x)| \to 0$, when $\zeta_n \to \zeta$ for $n \to \infty$. This complete the proof. □

7.3 Stability of solutions for substantial fractional difference system

In this section, conditions will be acquire for four type of Ulam stability to fractional order substantial difference equation with initial conditions. Findings of this section appeared in [108].

Theorem 7.6. *Assume condition (H_1) holds. Let $u \in Z$ be a solution of system* (5.33) *and $v \in Z$ be a solution of inequality* (7.5). *Then for k in inequality* (6.9), *the nonlinear IVP* (5.33) *is UH stable and consequently, generalized UH stable.*

Proof. By Lemma 5.5.1, for simplicity we can rewrite the solution of IVP (5.33) as

$$u(x) = \phi(x) - \sum_{\mathscr{T}=a+m-\varphi}^{x-\varphi+m-j} \frac{h_{\varphi-m+j-1}(x, \sigma(\mathscr{T})) e_{-p}(x-\mathscr{T}, 0)}{(1-p)^m} f(\mathscr{T}+\varphi-1, u(\mathscr{T}+\varphi-1)), \tag{7.15}$$

where $\phi(x) = \frac{e_{-p}(x-a+1,0)h_{\varphi-m+1}(x,a)}{\sum_{\ell=0}^{m}\binom{m}{\ell}(-p)^{m-\ell}} \sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\binom{i}{j}(-p)^j u_i$. From inequality (7.5) for $[a,T]_{\mathbb{N}_a}$, it follows that

$$\Big|v(x) - \Big(\phi(x) - \sum_{\mathcal{F}=a+m-\varphi}^{x-\varphi+m-j} \frac{h_{\varphi-m+j-1}(x,\sigma(\mathcal{F}))e_{-p}(x-\mathcal{F},0)}{(1-p)^m} \times f(\mathcal{F}+\varphi-1, v(\mathcal{F}+\varphi-1))\Big)\Big| \leq \epsilon. \tag{7.16}$$

Making use of equation (7.15) and inequality (7.16) together for $[a,T]_{\mathbb{N}_a}$, we have

$$\begin{aligned}\Big|v(x) - u(x)\Big| =& \Big|v(x) - \Big(\phi(x) - \sum_{\mathcal{F}=a+m-\varphi}^{x-\varphi+m-j} \frac{h_{\varphi-m+j-1}(x,\sigma(\mathcal{F}))e_{-p}(x-\mathcal{F},0)}{(1-p)^m} \\ & \times f(\mathcal{F}+\varphi-1, u(\mathcal{F}+\varphi-1))\Big)\Big| \\ \leq& \Big|v(x) - \Big(\phi(x) - \sum_{\mathcal{F}=a+m-\varphi}^{x-\varphi+m-j} \frac{h_{\varphi-m+j-1}(x,\sigma(\mathcal{F}))e_{-p}(x-\mathcal{F},0)}{(1-p)^m} \\ & \times f(\mathcal{F}+\varphi-1, v(\mathcal{F}+\varphi-1))\Big)\Big| \\ & + \Big|\sum_{\mathcal{F}=a+m-\varphi}^{x-\varphi+m-j} \frac{h_{\varphi-m+j-1}(x,\sigma(\mathcal{F}))e_{-p}(x-\mathcal{F},0)}{(1-p)^m}\Big| \\ & \times \Big|f(\mathcal{F}+\varphi-1, v(\mathcal{F}+\varphi-1)) - f(\mathcal{F}+\varphi-1, u(\mathcal{F}+\varphi-1))\Big| \\ \leq& \ \epsilon + \frac{|h_{\varphi-m+j}(x,a-\varphi+m)|}{|1-p|^{2m-\varphi-j}} k\Big|v(\mathcal{F}+\varphi-1) - u(\mathcal{F}+\varphi-1)\Big|.\end{aligned}$$

In preceding step, we used (H_1) and the argument used in Theorem 6.9. Now, taking supremum on both sides and simplification yields the following

$$||v-u|| \leq \frac{\epsilon}{1 - \frac{|h_{\varphi-m+j}(T,a-\varphi+m)|}{|1-p|^{2m-\varphi-j}}k} = \epsilon d_f, \quad \text{with} \quad d_f = \frac{1}{1 - \frac{|h_{\varphi-m+j}(T,a-\varphi+m)|}{|1-p|^{2m-\varphi-j}}k}.$$

Therefore, by inequality (6.9), (5.33) is UH stable. Further by using $\wp_f(\epsilon) = \epsilon d_f$, $\wp_f(0) = 0$, which implies that (5.33) is generalized UH stable. □

Theorem 7.7. *Assume condition (H_1) holds. Let $u \in Z$ be a solution of system (5.33) and $v \in Z$ be a solution of inequality (7.6). Then for k in inequality (6.9), the nonlinear IVP (5.33) is UHR stable with respect to function $\eth : [a,T]_{\mathbb{N}_a} \to \mathbb{R}^+$ and consequently, generalized UHR stable.*

Finally, to illustrate the usefulness of Theorem 7.6, we present following example.

Example 7.3.1. *Consider the fractional substantial equation with difference condition*

$$\begin{cases} -\ {}^{s}\Delta_{0}^{0.8}u(x) = (x-0.2)u(x-0.2), & x \in [0,10]_{\mathbb{N}_0} \\ {}^{s}\Delta_{0}^{1.8}u(0.2) = u_0. \end{cases}$$

Since $a = 0$, $\varphi = 0.8$ *and* $T = 10$, *therefore* $m = 1$, $i = 0$ *and* $j = 0$. *Then for any* $p \neq 1$, *we have* $k < \frac{|1-p|^{1.2}}{335179.01}$. *For instant if we choose* $p = 1001$ *then for* $k < \frac{1}{84.2}$, *the solution to given problem with inequalities*

$$\left| {}^{s}\Delta_{0}^{0.8}v(x) + (x-0.2)v(x-0.2)\right| \leq \epsilon \qquad x \in [0,10]_{\mathbb{N}_0},$$

$$\left| {}^{s}\Delta_{0}^{0.8}v(x) + (x-0.2)v(x-0.2)\right| \leq \epsilon\eth(x-0.2) \qquad x \in [0,10]_{\mathbb{N}_0},$$

is UH stable and UHR stable with respect to function $\eth : [0,10]_{\mathbb{N}_0} \to \mathbb{R}^{+}$.

Chapter 8

Summary

In Chapter 1, we outlined some working areas of FC and present the motivation behind using delta FDO. History of FC is given in Section 1.1 where we highlighted some researcher who gradually build the theory. Also we highlighted some dominant treatise in the field of FC and enlist some prominent fractional operators available in the literature. Section 1.2 contained the discussion on origin of difference equation and some major contribution. As the discrete analogue of fractional derivative the integer order difference has the generalization to non-integer order in such a way that it is compatible with classical difference operators. In Section 1.3 references of FC in continuous setting are listed. A short survey of BVP with different types of boundary conditions, double Laplace transform, Hilfer and substantial fractional derivative are given to set a track for main research work. Detailed survey of discrete fractional calculus with some important contributions are presented. Specifically some references in the area of delta FC are discussed. In Section 1.4 properties of fractional operators are enlisted with some rules which are not satisfied by fractional operator. Also some commonly used FDO are enlisted. Motivations for defining new FDO and for defining delta double Laplace transform are given. Motivations for discussion of multipoint BVP, impulsive difference system and UHR stability are also given. To cover up some gaps and to further build up the theory of discrete FC are two main goal of the dissertation which are described in Section 1.5. Pointwise objective of dissertation are also enlisted. Detailed organization of dissertation is given in Section 1.6.

In Chapter 2, some fundamentals and special functions are given. Definitions of discrete FC are given in Section 2.1. Euler Gamma function and alternative versions of discrete ML functions are presented in comparison to [4] in Section 2.2. The definitions of UHR stability for delta FDE are stated in Section 2.3. In Section 2.4 a Gronwall's inequality in delta discrete setting is furbished.

In Chapter 3, some generalizations of FDO are constructed. In Section 3.1 a new definition of Hilfer like FD on discrete time scale has been introduced. Some basic composition properties are also presented for newly defined Hilfer FDO. In Section 3.2 substantial fractional sum and DO on discrete time scale are constructed. Some important properties including left inverse property and composition properties are presented. Also a relation between RL and substantial FDO is induced.

In Chapter 4, the delta double Laplace transform method is introduced. Definition, EU and series representation of the delta double Laplace transform are given in Section 4.1. Some basic properties are derived in Section 4.2. The delta double Laplace transform of partial differences is presented in Section 4.3. The delta Laplace transform is developed for newly defined Hilfer FDO in Section 4.4. The exponential shift property of delta Laplace transform are proposed in Section 4.5, as an application of exponential shift property, delta Laplace transform for fractional substantial sum and DO are presented.

In Chapter 5, FPO and GF to different types of FDE are obtained. In Section 5.1 the general method of construction of summation equation from nonlinear FDE with impulse is presented. This construction applied to obtain FPO for IVP.

In Section 5.2 the general construction is applied to the two point BC and the four point BC. Then the GF for two points and four points BVP are derived with some useful properties.

In Section 5.3 the GF for multi-points BVP with summation condition are derived. Some useful properties of the GF are presented. The fixed point operator for the nonlinear FDE with multi-point summation BC are obtained.

FPO for Cauchy type problem of Hilfer FD system is obtained in Section 5.4.FPO is obtained in Section 5.5 for substantial FD system with initial condition.

In Chapter 6, conditions for the EU of solutions to different types of FDE are obtained. Conditions for EU of impulsive difference equations with IVP and BVP are obtained in Section 6.2 and Section 6.1. In Section 6.3 conditions for EU for MPFBVP are obtained. Conditions for the EU of solutions for Hilfer FD system and substantial FD system are respectively obtained in Section 6.4 and Section 6.5.

In Chapter 7, UH stability and UHR stability are discussed for different types of FDE. In Section 7.1, conditions are acquired under which the nonlinear MPFBVP is UH stable, generalized UH stable, UHR stable and generalized UHR stable. In Section 7.2, conditions are acquired under which the nonlinear Hilfer FD system is UH stable, generalized UH stable, UHR stable and generalized UHR stable. An application of the modified discrete Gronwall's inequality in delta setting is presented. In Section 7.3, conditions are acquired under which the nonlinear substantial FD system is UH stable, generalized UH stable, UHR stable and generalized UHR stable.

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