Plane Waves in Rotating Anisotropic Elastic Solids with Voids

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Dedicated

to

My Veer G (Usman Ali)

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Abstract

In this thesis, the propagation of plane waves in rotating elastic solids with and without voids are discussed. In each case, various graphs are plotted for illustration purposes and solution analysis. The case of propagation of plane waves for anisotropic material with voids is a new problem and discussed in detailed. Plane wave solutions are obtained by solving the equation of motion. Graphs are plotted for dimensionless wave speeds and wave number, wave speed and rotation, imaginary part of the solution is plotted separately. A special case where $\bar{k} \to \infty$ is considered.

Contents

Chapter 1

Introduction

Waves play a vital role in our routine life. We can see with the help of light waves and talk with the help of microwaves and radio waves. Earthquakes are quantatized with the help of seismic waves because waves travel through the interior of the earth. The materials inside the earth like oil, coal and gass can be observed with waves. The quality of a material can be checked by passing waves through it without damaging it with the use of non-destrustive testing system (NDT). The material is as useful as it was before the testing through the waves. During the last few years, in the field of medical the diseases are diagonosed and cured with the help of waves.
The history of wave propagation in research is a long and interesting one. English scientist Robert Hooke discovered linear elasticity in 1660, but not in the form of stress and strain. The displacement under a load was observed proportional to the force applied in case of several materials. It seems that light existed only till mid-19th century, is a wave which can travel across a particular medium considered an elastic ether. The utilization of elastic waves in different studies, for example, geophysics, was additionally an impetus for researchers and mathematicians to explore waves. The names that added to the field were Poisson, Cauchy, Lame, Stokes, Christoffel, Lamb and numerous others.

The theory of elasticity associated with solid elastic material consisting of distribution of pores, known as voids has gained much importance in recent years. Cowin and Nunziato [1, 2] formulated the general theory in its linearized version where voids induce an additional kinematic variable. In the absence of void volume (as a limiting case), this theory reduces to classical theory of elasticity. This theory plays a vital role where classical theory is inadequate e.g problems of geological and synthetic porous medium. Cowin and Iesan [2] introduced the basic theorem and brief account of the theory on voids. Cowin presented the connection between theory of voids and theories of elasticities. Chandrasekharaiah [3] furnished uniqueness theorem related to theory of elastic material with voids. He also focused his attention on the effect of surface stresses and voids on Rayleigh waves in an elastic medium [4].

A porous material whose matrix skeleton is elastic and voids are viscous pores are called elastic material with voids. During deformation, pores and matrix undergo a change. Due to this reason it is different from Cauchy continuum. The voids exert an equilibrated stress called pseudo force, in addition to the stress. The elastic material with voids has four degrees of freedom consisting of three translation and one due to the change in the void volume fraction. The variation in void volume fraction adds an additional kinematic variable in the theory of voids. It has been noticed that pores of the body are vacuous but do not have any mechanical significance. With the help of principles of continuum mechanics, field equations and relations for elastic material with voids have been derived. If the plastic effect of the medium is removed then we get an elastic body with voids.

Iesan [5], investigated the plane wave propagation in thermoelastic medium with voids. He observed three sets of coupled longitudinal waves and transverse waves. In the case of coupled longitudinal waves, displacement, void volume fraction and thermal properties are dispersive in nature. Free plane boundary of thermo-elastic half-space with voids, the phenomenon of reflection and transmission between two separate permeable elastic half-spaces was studied by Iesan. These phenomenas occur due to the incidence of plane longitudinal waves at a plane interface. In the case, when the incident frequency is low, the effect of voids on the transmission and reflection coefficient is highly significant. When high frequency longitudinal waves are incident, the phenomena of reflection and transmission are very close to the classical elastic theory with no effect of voids. An investigation about plane wave propagation in an isotropic medium with voids has been conducted by Maity [6]. The governing equations have been studied by taking account of rotation, magnetic field effect and presence of voids.

Sharma [7] presented the propagation of plane waves in thermoviscoelastic medium with voids. He considered the one dimentional model of isotropic generalized thermoviscoelastic medium. He noticed there exist three longitudinal waves called elastic (E-mode), thermal (T-mode) and volume fraction (V-mode). The transverse waves decoupled and does not effect by thermal and volume fraction field.

In this academic thesis, plane waves are investigated and the effect of porosity on isotropic and anisotropic rotating material is examined in elastic solids with voids. Chapter 2 introduces the reader to the basic concepts of elasticity. The notion of stress, strain, their relationship, and effect of crystal symmetries on elastic stiffness tensors are revised. Equation of wave propagation is derived. Types of waves and some wave parameters like wave number, phase velocity etc. are discussed. A quick review of the propagation of a two dimensional plane waves in the presence of voids in an isotropic and anisotropic materials is given. In the presence of voids, a detailed investigation is carried out in Chapter 3 for wave propogation in a rotating elastic solids. Furthermore, the derivation of governing equations and their corresponding boundary conditions for an elastic solids with voids is discussed. The graphical representation of various parameters involved along with the different range of values of a wave number regarding its propogartory properties is also given. Chapter 4, is mainly composed of the discussion regarding the traveling waves in a rotating anisotropic elastic material with voids. Travelling of waves in an anisotropic elastic materials in the presence of porous is a new problem and discussed in detail. Moreover, the calculation of plane waves and a detailed discussion of their numerical results is also given. In Chapter 5, all the results found throughout the thesis are concluded briefly.

Chapter 2

Fundamentals of Elasticity

The main objective of this chapter is to make the reader acquainted with some basic concepts and results related to the theory of elasticity. In elasticity, tensor calculus is often used and therefore tensors are briefly discussed in section 2.1. Section 2.2 comprises the basic definitions of stress, strain, Hook's Law, elasticity constants are retrieved by using crystal symmetry and equation of motion. The last section deals with the quick review of propagation of two dimensional plane waves in an isotropic and in anisotropic material in the presence of the porous material.

2.1 Tensor calculus

Tensor calculus finds its applications in the field of dynamics, elasticity, fluid, differential geometry, general relativity, electricity and magnetism. It is a mathematical tool used to describe the mechanism of deformed structure to get the main idea related to required field. Generalization of vectors and scalars is called tensor, while number of independent directions necessary to describe the tensor is called the rank (order) of tensor. A tensor of rank zero is a scalar and tensor of rank 1 is a vector. A tensor of the rank 1 is described by 3×1 column vector, while components of second rank tensor are represented by 3×3 matrix.

Mathematically, r^{th} rank tensor in k-dimensional space, is an object having r indices and k components under transformation laws. Therefore, tensor of rank r is a linear mapping which maps a vector to a tensor of order (r-1). The transformation of the components of r^{th} rank tensor form one basis to another is same as follows:

$$\mathbf{A}'_{\mathbf{r_1r_2...r_n}} = \mathbf{T_{m_1r_1}T_{m_2r_2}...T_{m_nr_n}A_{m_1m_2...m_n}}, \tag{2.1}$$

where $T_{m_1r_1}T_{m_2r_2}...T_{m_nr_n}$ are the elements of transformation matrix.

Main formulas and definitions which will be used in the proceeding chapters are given.

Definition 2.1.1 A tensor $\mathbf{A}$ is said to be symmetric if

$$\mathbf{A^T} = \mathbf{A}, \tag{2.2}$$

and is said to be antisymmetric if

$$\mathbf{A^T} = -\mathbf{A}. \tag{2.3}$$

Symmetric tensor of second order is very vital in the mechanics of deformed bodies.

Examples of symmetric tensors are Green's deformation tensor and Cauchy's stress tensors.

Definition 2.1.2 The Kronecker delta-δ is defined as

$$\delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

where δ_{mn} is kroneckor symbols. If the elements of basis are unit vectors then they are called orthonormal basis.

The third rank tensors are usually used in thermomechanics, electromechanics, therefore material properties of the deformed structure are also expressed by using tensor of rank 3.

Definition 2.1.3 The Livi-Civita tensor or called as permutation tensor and defined as

$$\epsilon_{lmn} = \begin{cases} 1, & \text{for even permutation of } lmn, \\ -1, & \text{for odd permutation of } lmn, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.1.4 A tensor is said to be isotropic tensor if its components remain unaltered upon the change of coordinate system otherwise, anisotropic tensor.

Definition 2.1.5 Transpose of a tensor A in a Euclidean space V is a function A^T defined as

$$(A\boldsymbol{u})^T.\boldsymbol{v} = \boldsymbol{u}.A\boldsymbol{v}, \quad \textit{for any} \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}.$$

2.2 Fundamentals of elasticity and continuum mechanics

Fundamentals of elasticity are based upon the concept of continuum approximation. In which, matters are idealized as a continuous material. The distribution of atoms and molecules are continuous in terms of their material properties e.g. density, as a continuous function of position and time. There are two properties of the continuum material.

1. The continuum materials are subdivided many times and each subdivision have similar properties.

2. The continuum approximation gives useful results on a scale larger than the space between the particles, not on nanometers.

2.2.1 Stress, strain and their relationship

When we apply body force or surface force to any object then its original shape and size measure of this deformation is called strain. The shapes of the objects is determined through relative position of the particles.

Strain in three dimensional case is determined by symmetric tensor S_{ij} of the second rank as

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.4}$$

where i, j=1,2,3. The stress tensor T_{jk} is defined as

$$T_{jk} = \lim_{\Delta s_k \to +0} \frac{\Delta F_j}{\Delta s_k}, \tag{2.5}$$

when components acts tangentially then stress is known as shear stress.

There is one-to-one relation between stress and strain. In other words, one can define stress as a function of strain and vice versa. The elastic behaviour of objects for small deformation can be expressed with first-order term in Taylor's expansion of the function.

$$T_{jk}(S_{il}) = T_{jk}(0) + \left(\frac{\partial T_{jk}}{\partial S_{il}}\right)_{S_{il}=0} S_{il} + \frac{1}{2}\left(\frac{\partial^2 T_{jk}}{\partial S_{il}\partial S_{mn}}\right)_{S_{il}=0} S_{il}S_{mn} + \cdots . \tag{2.6}$$

Now assuming, in the absence of stress there will be no strain and vice versa. That is

$$T_{jk}(0) = 0, \tag{2.7}$$

By ignoring higher order terms, we get an expression called Hook's Law

$$T_{jk} = C_{jkil}S_{il}, \tag{2.8}$$

where

$$C_{jkil} = \left(\frac{\partial T_{jk}}{\partial S_{il}}\right)_{S_{il=0}}, \tag{2.9}$$

is a tensor of rank four which is called elastic stiffness tensor and describes the relationship between stress and strain and consists of 81 components.

It has two symmetries $C_{jkil} = C_{kjil}$ and $C_{jkil} = C_{kjli}$ due to symmetries of the stress and strain tensor, while stored energy function also imposes symmetry. The number of components reduces from 81 to 21 due to the symmetries, namely, $C_{jkil} = C_{iljk}$.

Two index representation of the indices C_{ijkl} known as Voigt notation is written as follows

$$(11) \longleftrightarrow 1, (22) \longleftrightarrow 2, (33) \longleftrightarrow 3,$$

$$(23) = (32) \longleftrightarrow 4, (13) = (31) \longleftrightarrow 5, (21) = (12) \longleftrightarrow 6.$$

This reduces the set of four indices to a set of two indices which is convenient to use.

2.2.2 Symmetry properties of the crystalline system

Symmetry proportion of the crystal system reduces the number of independent components. Crystals are usually anisotropic. Since we have 21 independent stiffness constants which can be further reduced by applying symmetry conduction of the crystalline system. When we consider an isotropic material, constants reduces from 21 to 2.

In hexagonal material, we have 5 independent elastic constants and in the matrix form written as

$$C_{\alpha\beta} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{11} & C_{13} & 0 & 0 & -C_{16} \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{16} & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}. \tag{2.10}$$

The above matrix is the consequences of these restriction

$$C'_{pqmn} = q_{ip}q_{jq}q_{km}q_{ln}C_{ijkl}, \tag{2.11}$$

due to axis of symmetry

$$q_{11} = q_{22} = 1, q_{33} = -1, \tag{2.12}$$

$$q_{ij} = 0 \qquad \text{for} \quad i \neq j, \tag{2.13}$$

using these conditions

$$C_{1123} = C_{14} = -C_{1123} = -C_{14}, \tag{2.14}$$

accordingly,

$$C_{14} = C_{24} = C_{34} = C_{64} = C_{15} = C_{25} = C_{35} = C_{65} = 0. \tag{2.15}$$

2.2.3 Isotropic material

Isotropic material in which physical properties are independent of direction and choice of reference frame. In other words, stiffness constants are unaltered by the transformation of the reference frame. In order to get stiffness tensor C_{ijkl} it must be expressed in terms of components of the tensors δ_{ij}. Mainly there are three fourth rank isotropic tensor exist

$$\delta_{ij}\delta_{kl}, \qquad \delta_{ik}\delta_{jl}, \qquad \delta_{il}\delta_{jk}.$$

Therefore, elastic stiffness tensor can be obtained by a linear combination of three distinct fourth rank tensors,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.16}$$

and we obtained the matrix

$$C_{\alpha\beta} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}. \tag{2.17}$$

After the French mathematician G.Lame, λ and μ are known as Lamé constants.

2.2.4 Different wave parameters and types of elastic waves

A disturbance or oscillation through space and matters along with the transfer of energy is called waves. Different wave parameters associated with wave propagation are given as follow.

Wave number

Wave number is denoted by k and defined as the reciprocal of the wavelength. Wave numbers are widely used in optics, physics X-ray diffraction and elementary particle physics etc.

Wavelength

The distance between the two peaks of the waves is called wave length, and reciprocal of the wavelength is called wave number which is denoted by letter k.

Amplitude and frequency

The distance of maximum displacement of a wave from its rest position is called amplitude.

Frequency

In Physics, engineering such as optics, acoustics and radio, frequency is defined as the reciprocal of the time period and is given by

$$f = \frac{1}{T}.$$

The SI unit of frequency hertz (HZ), named after the German physicist Heinrich Hertz and time is measured in seconds.

Different classes of the elastic waves can easily travel through solids. The direction of the wave propagation and the boundary conditions relative to the motion of the particles enable us to classify different elastic waves. Common elastic waves in solids are longitudinal or primary waves (P-waves) and transverse, shear waves called as surface waves (S-waves). We now give short but precise overview of some elastic waves.

P-waves

P-waves are those in which direction of particle displacement is parallel to the wave propagation. These waves can travel through solids, liquids and gasses.

S-waves

S-waves are those in which direction of particle displacement is perpendicular to the wave propagation. These waves only propagate through solids.

Surface waves

Surface waves propagate near the surface or boundary of solid material. The amplitude of waves decreases sharply as the waves move away from the surface. Two important surface waves are Love waves and Rayleigh waves.

Rayleigh waves

Rayleigh waves were discovered by Rayleigh in 1885, the elliptic motion of particles produce Rayleigh waves. In Rayleigh waves, propagation is in the direction of the horizontal and vertical components of the motion.

Love waves

Love waves were discovered in 1911 by Augustus Edward Hough Love, produce due to side by side motion of ground and proved the existence of transverse waves. The motion of the particle is parallel and transverse to the surface.

Rayleigh waves are non-dispersive in nature while love waves are dispersive in nature. These waves propagate in homogeneous isotropic half-space, while love waves propagate easily on the homogeneous isotropic layer of homogeneous isotropic half space.

Dispersive and non-dispersive waves

Wave nature is said to be dispersive if wave speed is dependent upon wave number. If waves speed is independent of wave number, then they are non-dispersive in nature.

2.2.5 Governing equations of motion

The equation of motion is governed by the fundamental law of thermodynamics also known as Newton's second law of motion $\mathbf{F} = m\mathbf{a}$, where force $\mathbf{F}$ caused an acceleration and $\mathbf{a}$ in to a body of mass m. Suppose disturbance is produced in a solid due to stress, at some arbitrary point, change in displacement is denoted by $\mathbf{v}$ and components of the force due to stress S is given by

$$F_i = \frac{\partial S_{ij}}{\partial x_j}, \qquad i, j = 1, 2, 3, \tag{2.18}$$

where S_{ij} are the components of the stress tensor, which gives rise to acceleration $\frac{\partial^2 v_i}{\partial t^2}$ with unit volume mass along i^{th} axis. In the absence of body force, equation of motion will be

$$\frac{\partial S_{ij}}{\partial x_j} = \rho \frac{\partial^2 v_i}{\partial t^2}. \tag{2.19}$$

By **Hook's Law**, above expression will take form

$$C_{ijkl}\frac{\partial^2 v_l}{\partial x_k \partial x_j} = \rho\frac{\partial^2 v_i}{\partial t^2}, \tag{2.20}$$

which is second order partial differential equation which give rise to equation of motion in three dimensional case.

Hook's Law for an isotropic materials has the form

$$\begin{aligned} S_{ij} =& \lambda T_{kk}\delta_{ij} + 2\mu T_{ij} \\ =& \lambda\frac{\partial}{\partial x_j}\frac{\partial v_k}{\partial x_j}\delta_{ij} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \end{aligned} \tag{2.21}$$

Equation (2.19) becomes

$$\lambda\frac{\partial u_{k,k}}{\partial x_j}\delta_{ij} + \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j}\right) = \rho\ddot{\mathbf{v}} \tag{2.22}$$

Here the summation is on k.
The equation of motion for homogeneous elastic material will be

$$(\lambda + \mu)\nabla(\nabla.\mathbf{v}) + \mu\nabla^2\mathbf{v} = \rho\ddot{\mathbf{v}}. \tag{2.23}$$

2.3 Theory on voids

The linear and non-linear continuum theories of elastic bodies with voids were first presented by Cowin [12]. It is used for investigating various types of geological and biological materials for which classical theory of elasticity is not adequate. He also supposed that elastic material contain pores which are porous but do not have any mechanical significance. The theory of linear elastic material with voids deals the materials with a distribution of small pores or voids, where the volume of void is included among the kinematics variables. It reduces to the classical theory in the limiting case of the volume of void, tending to zero. It has applications in the study of geological materials like rocks and soil, synthetic materials like ceramics, pressed powders and biological structure like bones.
Propagation of elastic waves in a rotating medium were presented by Censor and

Schoenberg [8, 9]. In these papers, they examined that anisotropy and dispersion were produced due to the rotation of the elastic medium.
Chandersekhariah [10] studied the propagation of plane waves with voids rotating with constant angular velocity. Also, the dilatational waves have two different modes. Both are affected by voids and rotations.
Eringen [11] presented the theory of elastic material with voids as special case of theory of micromorphic material. Puri and Cowin [12] studied porous material with voids. They investigated that two dilatational waves exist. One wave corresponds to the classical linear elasticity. second wave is associated with the change in void volume fraction.

The governing equation for a homogeneous elastic solid with voids in the absence of body forces are given in [13] as

$$\mu \bigtriangledown^2 \mathbf{v} + (\lambda + \mu) \bigtriangledown (\bigtriangledown . \mathbf{v}) + \beta \bigtriangledown \phi = \rho[\ddot{\mathbf{v}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{v}) + 2\mathbf{\Omega} \times \dot{\mathbf{v}}, \tag{2.24}$$

$$\alpha \bigtriangledown^2 \phi - \xi\phi - \beta \bigtriangledown . \mathbf{v} = \rho\sigma\ddot{\phi}. \tag{2.25}$$

Chapter 3

Analysis of Waves Traveling in Isotropic Elastic Solids with Voids

The study of wave propagation in an isotropic material has gained much attraction in the literature due to its wide range of applications in the field of seismology, non-destructive testing system and in the other technical fields. Isotropic materials and its application are found in Physics, Cosmology, Chemistry etc. The generalization of the classic theory of elasticity is the theory of linear elastic material with voids. This theory plays vital role in studying geological and biological materials.

This chapter is mainly based on the study done by Tomar and Ogden [13]. In this paper, they investigated about the wave propagation in rotating isotropic elastic solids in the presence of voids. A comprehensive review about their study has been presented in this chapter. The chapter has been divided into the following sections. In Section 3.1, derivation of the governing equation of the elastic solids with voids is discussed. In Section 3.2, propagation of waves in two dimensions is discussed in such a way that waves are traveling in x_1x_3-plane and $x_2 = 0$ is considered as stress free boundary. The components of displacement and rotation are such that $v_2 = 0$, $\Omega_1 = \Omega_3 = 0$. In Section 3.3, solution of the governing equations is discussed in detail. In Section 3.4, discussion about wave propagation with and without voids is presented. In Section 3.5, graphical illustrations in the context of voids, rotation and wave number is discussed.

3.1 Governing equations for waves propagation

We take homogeneous elastic isotropic matter in the presence of porous medium. The material revolves with a constant soeed Ω. The change in void volume fraction $\phi(\mathbf{p},t)$, where $\mathbf{p}$ is the position vector and t is the time. The governing equations without body forces and external equilibrated body forces are mention in Eq. (2.24) and (2.25).

3.2 Propagation of waves in two dimensions

We consider that waves are traveling in x_1x_3-plane. The boundary $x_2 = 0$ is taken as the stress free boundary. Let $\mathbf{v} = [v_1, v_2, v_3]$ and $\mathbf{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ such that $v_2 = 0$, $\Omega_1 = \Omega_3 = 0$.

$$v_1 = v_1(x_1, x_3, t), \qquad v_3 = v_3(x_1, x_3, t), \qquad \Omega_2 = \Omega, \tag{3.1}$$

Using Eq. (3.1), Eq. (2.24) and (2.25) becomes

$$\mu\nabla^2 v_1 + (\lambda+\mu)e_{,1} + \beta\phi_{,1} = \rho(\ddot{v}_1 - \Omega^2 v_1 + 2\Omega\dot{v}_3), \tag{3.2}$$

$$\mu\nabla^2 v_3 + (\lambda+\mu)e_{,3} + \beta\phi_{,3} = \rho(\ddot{v}_3 - \Omega^2 v_3 - 2\Omega\dot{v}_1), \tag{3.3}$$

$$\alpha\nabla^2\phi - \xi\phi - \beta e = \rho\sigma\ddot{\phi}, \tag{3.4}$$

where

$$e = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_3}{\partial x_3}, \qquad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}. \tag{3.5}$$

The expression for v_1 and v_3 with respect to potential function is

$$v_1 = \chi_{,1} + \psi_{,3}, \qquad v_3 = \chi_{,3} - \psi_{,1}, \tag{3.6}$$

Using derivatives of Eq. (3.6), Eqs.(3.2) and (3.3) can be written in the form

$$\Big(c_1^2\nabla^2\chi - \ddot{\chi} + \Omega^2\chi + 2\Omega\dot{\psi} + c_4^2\phi\Big)_{,1} + \Big(c_2^2\nabla^2\psi - \ddot{\psi} + \Omega^2\psi - 2\Omega\dot{\chi}\Big)_{,3} = 0, \tag{3.7}$$

$$\Big(c_1^2\nabla^2\chi - \ddot{\chi} + \Omega^2\chi + 2\Omega\dot{\psi} + c_4^2\phi\Big)_{,3} - \Big(c_2^2\nabla^2\psi - \ddot{\psi} + \Omega^2\psi - 2\Omega\dot{\chi}\Big)_{,1} = 0, \tag{3.8}$$

respectively, where

$$c_1^2 = \frac{(\lambda + 2\mu)}{\rho}, \qquad c_2^2 = \frac{\mu}{\rho}, \qquad c_4^2 = \frac{\beta}{\rho}. \tag{3.9}$$

This means that Eq. (3.7) and Eq. (3.8) are satisfied if

$$c_1^2\nabla^2\chi - \ddot{\chi} + \Omega^2\chi + 2\Omega\dot{\psi} + c_4^2\phi = 0, \tag{3.10}$$

$$c_2^2\nabla^2\psi - \ddot{\psi} + \Omega^2\psi - 2\Omega\dot{\chi} = 0. \tag{3.11}$$

These are coupled with Eq. (3.4), in which terms of potential functions become as

$$c_3^2\nabla^2\phi - \xi^*\phi - \ddot{\phi} - \nu^*\nabla^2\chi = 0, \tag{3.12}$$

where

$$c_3^2 = \frac{\alpha}{\rho\sigma}, \quad \xi^* = \frac{\xi}{\rho\sigma}, \quad \nu^* = \frac{\beta}{\rho\sigma} = \frac{c_4^2}{\sigma}. \tag{3.13}$$

In classical elasticity theory, c_1 and c_2 represent the longitudnal and transverse wave speeds, while c_3 corresponds to the logitudnal wave speed, which exists due to change in void volume fraction and c_4 has the dimension of the speed.

3.3 Solution of the governing equations

For the solution of the governing Eqs. (3.2)-(3.4), it is enough to solve Eqs. (3.10)-(3.12) for χ, ψ and ϕ . We take harmonic plane waves that travel along wave normal laying in the x_1x_3-plane and makes an angle θ with the x_3-axis.

$$\chi = A\exp\big[ik(x_1\sin\theta + x_3\cos\theta) - i\omega t\big], \tag{3.14}$$

$$\psi = B\exp\big[ik(x_1\sin\theta + x_3\cos\theta) - i\omega t\big], \tag{3.15}$$

$$\phi = C\exp\big[ik(x_1\sin\theta + x_3\cos\theta) - i\omega t\big]. \tag{3.16}$$

Solutions for χ, ψ and ϕ are assumed in the form of Eqs. (3.14)-(3.16) . Where k is the wave number, ω is the angular frequency and v is the wave speed. Various derivatives of χ are

$$\frac{\partial\chi}{\partial x_1} = A(ik\sin\theta)\exp\big[ik(x_1\sin\theta + x_3\cos\theta) - i\omega t\big]. \tag{3.17}$$

Omitting the expression $\exp\left[ik(x_1 \sin\theta + \cos\theta) - i\omega t\right]$ to save space, we have

$$\frac{\partial^2 \chi}{\partial x_1^2} = A(ik\sin\theta)^2 = -Ak^2\sin^2\theta, \tag{3.18}$$

$$\frac{\partial^2 \chi}{\partial x_3^2} = A(ik\cos\theta)^2 = -Ak^2\cos^2\theta, \tag{3.19}$$

$$\nabla^2\chi = -Ak^2. \tag{3.20}$$

In a similar manner derivatives of χ and ϕ are calculated. After using various derivatives of χ, ϕ and ψ Eq. (3.10) becomes

$$c_1^2(-Ak^2) - (-A\omega^2) + \Omega^2(A) + 2\Omega(-Bi\omega) + c_4^2 C = 0,$$

which further reduces to

$$\left(k^2c_1^2 - \omega^2 - \Omega^2\right)A + 2i\Omega\omega B - c_4^2 C = 0. \tag{3.21}$$

Substituting the values of $\nabla^2\psi$, $\ddot{\psi}$, ψ, $\dot{\chi}$ in Eq.(3.11), we get

$$c_2^2\left(-Bk^2\right) - \left(-B\omega^2\right) + \Omega^2 B - 2\Omega A(-i\omega) = 0,$$

which takes the form

$$2i\Omega\omega A - \left(c_2^2k^2 - \omega^2 - \Omega^2\right)B = 0. \tag{3.22}$$

Likwise, Eq. (3.12) becomes

$$c_3^2(-Ck^2) - \xi^*(C) - (-C\omega^2) - \nu^*(-Ak^2) = 0,$$

which reduces to simplified form as

$$k^2\nu^* A - \left(k^2c_3^2 - \omega^2 + \xi^*\right)C = 0, \tag{3.23}$$

where ν^* and ξ^* are defined by Eq. (3.13). Equations (3.21)-(3.23) are coupled having all three contants A, B and C. However, these equations decouple and give $v^2 = c_2^2$ in the absence of rotation. This decoupling correspond to tranverse wave. In order to eliminate the constants A, B and C from Eqs. (3.21)-(3.23), the determinant must vanish which gives nontrivial solution. The respective determinant of the coefficient matrix is denoted by M and is given by

$$M = \begin{vmatrix} k^2c_1^2 - \omega^2 - \Omega^2 & 2\iota\Omega\omega & c_4^2 \\ 2\iota\Omega\omega & k^2c_2^2 - \omega^2 - \Omega^2 & 0 \\ k^2\nu^* & 0 & k^2c_3^2 + \omega^2 - \xi^* \end{vmatrix} = 0. \tag{3.24}$$

After solving the determinant and collecting the like terms, we get

$$\begin{aligned} & c_1^2c_2^2c_3^2k^6 - c_2^2c_4^2k^4\nu^* + c_1^2c_2^2k^4\xi^* - c_1^2c_3^2k^4\omega^2 - c_2^2c_3^2k^4\Omega^2 + c_4^2k^2\nu^*\Omega^2 - c_1^2k^2\xi^{*2}\Omega^2 \\ & - c_2^2k^2\xi^*\Omega^2 + c_3^2k^2\Omega^4 + \xi^*\Omega^4 - \omega^6 + \big(c_1^2k^2 + c_2^2k^2 + c_3^2k^2 + \xi^* + 2\Omega^2\big)\omega^4 \\ & + \big(- c_1^2c_2^2k^4 - c_1^2c_3^2k^4 - c_2^2c_3^2k^4 + c_4^2k^2\nu^* - c_1^2k^2\xi^* - c_2^2k^2\xi^* + c_1^2k^2\Omega^2 + c_2^2k^2\Omega^2 \\ & - 2c_3^2k^2\Omega^2 - 2\xi^*\Omega^2 - \Omega^4\big)\omega^2 = 0. \end{aligned} \tag{3.25}$$

Equation (3.25) can be written in a compact form as a cubic equation for ω^2, which is given by

$$\omega^6 - a_1\omega^4 + a_2\omega^2 - a_3 = 0, \tag{3.26}$$

where the (real) coefficients a_1, a_2, a_3 are defined by

$$\begin{aligned} a_1 &= c_1^2k^2 + k^2c_2^2 + k^2c_3^2 + \xi^* + 2\Omega^2, \\ &= 2\Omega^2 + k^2\big(c_1^2 + c_2^2 + c_3^2 + \xi^*\big), \end{aligned} \tag{3.27}$$

$$\begin{aligned} a_2 &= c_1^2c_2^2k^4 + c_1^2c_3^2k^4 + c_2^2c_3^3k^4 - c_4^2k^2\nu^* + c_1^2k^2\xi^* + c_2^2k^2\xi^* - c_1^2k^2\Omega^2 - c_2k^2\Omega^2 \\ &\quad + 2c_3^2k^2\Omega^2 + 2\xi^*\Omega^2 + \Omega^4, \\ &= \big(\Omega^2 - k^2c_1^2\big)\big(\Omega^2 - k^2c_2^2\big) - \nu^*k^2c_4^2 + 2\Omega^2k^2c_3^2 + 2\Omega^2\xi^* + k^4c_1^2c_3^2 + k^2c_1^2\xi^* \\ &\quad + k^4c_3^2c_3^2 + k^2c_2^2\xi^*, \\ &= \big(\Omega^2 - k^2c_1^2\big)\big(\Omega^2 - k^2c_2^2\big) + [2\Omega^2 + k^2\big(c_1^2 + c_2^2\big)]\big(k^2c_3^2 + \xi^*\big) - \nu^*k^2c_4^2, \end{aligned} \tag{3.28}$$

$$\begin{aligned} a_3 &= c_1^2c_2^2c_3^2k^6 - c_2^2c_4^2k^4\nu^* + c_1^2c_2^2k^4\xi^* - c_1^2c_3^2k^4\Omega^2 - c_2^2c_3^2k^4\Omega^2 + c_4^2k^2\nu^*\Omega^2 \\ &\quad - c_1^2k^2\xi^*\Omega^2 - c_2^2k^2\xi^*\Omega^2 + c_3^2k^2\Omega^4 + \xi^*\Omega^4, \\ &= \big(\Omega^2 - k^2c_2^2\big)\big[- c_1^2c_3^2k^4 - c_2^2c_3^2k^4 + c_3^2k^2\Omega^2 + \xi^*\Omega^2 - c_4^2\nu^*k^2 + c_1^2c_3^2k^3 \\ &- c_1^2k^2\xi^{*4}\big], \\ &= \big(\Omega^2 - k^2c_2^2\big)\big[\big(\Omega^2 - k^2c_1^2\big)\big(k^2c_3^2 + \xi^*\big) + \nu^*k^2c_4^2\big]. \end{aligned} \tag{3.29}$$

After we get the solution for ω, we can obtain the wave speed by using $v = \omega/k$. The waves are dispersive because we know that ω depends on k generally. In the limit $k \to 0$ Eqs. (3.26)-(3.29) reduce to

$$a_1 = 2\Omega^2 + \xi^*, \tag{3.30}$$

$$a_2 = \Omega^4 + 2\Omega^2\xi^*, \tag{3.31}$$

$$a_3 = \Omega^4\xi^*. \tag{3.32}$$

and Eq. (3.27) takes the form $\left(\omega^2 - \Omega^2\right)^2\left(\omega^2 - \xi^*\right) = 0$ and the compatible wave speed take the form unlimited. At the other extreme, in the limit $k \to \infty$, Eq. (3.25) has the asymptotic form $\left(\omega^2 - k^2c_1^2\right)\left(\omega^2 - k^2c_2^2\right)\left(\omega^2 - k^2c_3^2\right) = 0$, where c_1^2, c_2^2, c_3^2 are squared wave speed.

3.4 Two main cases

In the presence of voids in an isotropic material, dispersion in the waves occur due to rotation. The quadratic equation in ω^2 which gives two real solution depicts the association of wave speed on the wave number.

3.4.1 Isotropic material without voids

If there are no voids and then consequently voids parameters α, β, ξ, are zero and a_3, c_3, c_4 also zero and the only nonzero solutions of Eq. (3.27) are given by a quadric in ω^2, namely

$$\omega^4 - \left[2\Omega^2 + k^2\left(c_1^2 + c_2^2\right)\right]\omega^2 + \left(\Omega^2 - k^2c_1^2\right)\left(\Omega^2 - k^2c_2^2\right) = 0. \tag{3.33}$$

Two real solutions for ω^2 are obtained, at least one of which is greater than zero. The discriminant must be greater than or equal to zero, in order to have at least one positive root in quadratic equation. In the form of quadratic equation generally,

$$A\omega^4 - B\omega^2 - C = 0.$$

It is required that, C<0, $\left(\Omega^2 - k^2c_1^2\right)\left(\Omega^2 - k^2c_2^2\right)\leqslant 0$. This is possible if

$$\Omega^2 - k^2c_2^2 \geq 0, \qquad \Omega^2 \geq k^2c_2^2, \qquad \frac{\Omega^2}{c_2^2} \geq k^2, \tag{3.34}$$

or

$$\Omega^2 - k^2c_1^2 \leqslant 0, \qquad \Omega^2 \leqslant k^2c_2^2, \qquad \frac{\Omega^2}{c_1^2} \leqslant k^2. \tag{3.35}$$

In order to get two real solutions for ω, k must lies outside the range,

$$\frac{\Omega^2}{c_1^2} \leqslant k^2 \leqslant \frac{\Omega^2}{c_2^2}. \tag{3.36}$$

This implies that

$$\frac{\Omega}{c_1} \leqslant k \leqslant \frac{\Omega}{c_2}. \tag{3.37}$$

Eq. (3.33) shows the association of wave speed on the wave number. When we put $v = \omega/k$, while dispersion in the waves are caused by rotation.

3.4.2 Isotropic material with voids

In this case, when we have voids but no rotation. In such a case Eqs. (3.27)-(3.29) reduce to

$$a_1 = k^2(c_1^2 + c_2^2 + c_3^2) + \xi^*, \tag{3.38}$$

$$a_2 = (-k^2c_1^2)(-k^2c_2^2) + [k^2(c_1^2 + c_2^2)](k^2{c^2}_3 + \xi^*), \tag{3.39}$$

$$a_3 = (-k^2c_1^2)(-k^2c_2^2)(k^2c_3^2 + \xi^*) + v^*k^2c_4^2, \tag{3.40}$$

In the absence of rotation Eq. (3.26) takes the forms

$$\omega^6 - \left(k^2(c_1^2 + c_2^2 + c_3^2) + \xi^*\right)\omega^4 + \left((-k^2c_1^2)(-k^2c_2^2) + [k^2(c_1^2 + c_2^2)](k^2{c^2}_3 + \xi^*)\right), \tag{3.41}$$

or

$$(\omega^2 - k^2c_2^2)\{\omega^4 - [\xi^* + k^2(c_1^2 + c_3^2)]\omega^2 + k^2c_1^2(\xi^* + k^2c_3^2) - k^2\nu^*c_4^2\} = 0. \tag{3.42}$$

The first factor in Eq. (3.42) yields $\nu^2 = c_2^2$. The second degree equation in ω^2 has two real solutions, minimum one being greater than zero. The constant factor in quadratic equation in Eq. (3.42) is greater than zero for both solutions to be greater than zero.

$$k^2c_1^2(\xi^* + k^2c_3^2) - k^2\nu^*c_4^2 > 0,$$
$$\implies k^2c_1^2c_3^2 > \nu^*c_4^2 - \xi^*c_1^2. \tag{3.43}$$

For the explanation of the result, we require suitable values of the matter's parameters. The parameters are adopted from Puri and Cowin [12] and these are listed in Table 3.1. The expressions in Eqs. (3.9) and (3.13) are then used to calculate the values of c_1, c_2, c_3, c_4, ξ^* and ν^*, which are written together in Table 3.2. Without voids and rotating wave speeds have the values c_1 and c_2, which does not depend on k.

We will take dimensionlize values of the parameters for the convenience of the calculation which are given by

$$\bar{\lambda} = \frac{\lambda}{\mu}, \quad \bar{\beta} = \frac{\beta}{\mu}, \quad \bar{\xi} = \frac{\xi}{\mu}, \quad \bar{\alpha} = \frac{\alpha}{\mu\sigma}, \tag{3.44}$$

$$\bar{\omega} = \omega\sqrt{\frac{\rho\sigma}{\mu}}, \quad \bar{\Omega} = \Omega\sqrt{\frac{\rho\sigma}{\mu}}, \quad \bar{k} = k\sqrt{\sigma}. \tag{3.45}$$

Letter	Value	Units	Letter	Value	Units
λ	15×10^9	Nm^{-2}	μ	7.5×10^9	Nm^{-2}
α	8×10^9	N	β	10×10^9	Nm^{-2}
ξ	12×10^9	Nm^{-2}	ρ	1999	kgm^{-3}
σ	0.162	m^2			

Table 3.1: Values of the various material parameters in the presence of voids.

After the calculation, the derived form are obtained in dimensionless parameters which are $\bar{\lambda}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\xi}$, $\bar{\Omega}$. These are the functions of either $\bar{\omega}$ or $\bar{k}$; where $\bar{\omega}= \bar{k}\bar{\nu}$.

Letter	Value	Units	Letter	Value	Units
c_1	3874	ms^{-1}	c_2	1937	ms^{-1}
c_3	4970.34	ms^{-1}	c_4	2236.66	ms^{-1}
ν^*	30.8804×10^6	s^{-2}	ξ^*	37.0565×10^6	s^{-2}

Table 3.2: Calculated values of material parameters using the information from Table 3.1 and Eq. (3.9) and Eq. (3.13).

Using Eq. (3.44) and Eq. (3.45) in Eq. (3.26), after putting $\omega^2 = t$, we get

$$t^3 - \bar{a_1}t^2 + \bar{a_2}t - \bar{a_3} = 0, \tag{3.46}$$

where

$$\bar{a_1} = 2\left(\frac{\mu\bar{\Omega^2}}{\rho\sigma}\right) + \frac{\bar{k^2}}{\sigma}\left[\frac{\lambda + 2\mu}{\rho} + \frac{\mu}{\rho} + \frac{\alpha}{\rho\sigma}\right] + \frac{\xi}{\rho\sigma}, \tag{3.47}$$

$$\Omega^2 = \frac{\mu\bar{\Omega^2}}{\rho\sigma}, \qquad k^2 = \frac{\bar{k^2}}{\sigma}, \qquad \xi = \bar{\xi}\mu. \tag{3.48}$$

So, rewriting $\bar{a_1}$ and utilizing Eq. (3.48)

$$\begin{aligned}
\bar{a_1} &= 2\left(\frac{\mu\bar{\Omega^2}}{\rho\sigma}\right) + \frac{\bar{k^2}}{\sigma}\left[\frac{\bar{\lambda}\mu + 2\mu}{\rho} + \frac{\mu}{\rho} + \frac{\alpha}{\rho\sigma}\right] + \frac{\bar{\xi}\mu}{\rho\sigma}, \\
&= \frac{\mu}{\rho\sigma}\left[2\bar{\Omega^2} + \bar{k^2}\big(\bar{\lambda} + 3 + \frac{\alpha}{\mu\sigma}\big) + \bar{\xi}\right], \\
&= \frac{\mu}{\rho\sigma}\left[2\bar{\Omega^2} + \bar{\xi} + \bar{k^2}\big(\bar{\lambda} + 3 + \bar{\alpha}\big)\right], \\
&= \frac{\mu\bar{k^2}}{\rho\sigma}\left[\frac{2\bar{\Omega^2} + \bar{\xi}}{\bar{k^2}} + (3 + \bar{\lambda} + \bar{\alpha})\right].
\end{aligned} \tag{3.49}$$

Similarly, the expression for $\bar{a_2}$ is obtained as

$$\begin{aligned}
\bar{a_2} = &\left(\frac{\bar{\Omega^2}\mu}{\rho\sigma} - \frac{\bar{k^2}\mu(\bar{\lambda} + 2)}{\rho\sigma}\right)\left(\frac{\bar{\Omega^2}\mu}{\rho\sigma} - \frac{\bar{k^2}\mu}{\rho\sigma}\right) + \left[2\frac{\bar{\Omega^2}\mu}{\rho\sigma} + \frac{\bar{k^2}}{\sigma}\left(\frac{(\bar{\lambda} + 2)\mu}{\rho}\right.\right. \\
&\left.\left. + \frac{\mu}{\rho}\right)\right] \times \left(\frac{\bar{k^2}}{\sigma} \times \frac{\bar{\alpha}\mu\sigma}{\rho\sigma} + \frac{\mu\bar{\xi}}{\rho\sigma}\right) - \left(\frac{\mu\bar{\beta}}{\rho\sigma}\right)^2\bar{k^2},
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
&= \left(\frac{\bar{k}\mu}{\rho\sigma}\right)^2\left[\left(\frac{\bar{\Omega^2}}{\bar{k^2}} - \bar{\lambda} - 2\right)\left(\frac{\bar{\Omega^2}}{\bar{k^2}} - 1\right) + \left(\frac{2\bar{\Omega^2}}{\bar{k^2}} + \bar{\lambda} + 3\right)\left(\bar{\alpha} + \frac{\bar{\xi}}{\bar{k^2}}\right)\right. \\
&\quad \left. - \bar{\beta^2}\right], \\
&= \left(\frac{\bar{k}\mu}{\rho\sigma}\right)^2\left[2 + \bar{\lambda} + (3 + \bar{\lambda})\alpha + \frac{(3 + \bar{\lambda})(\bar{\xi} - \bar{\Omega^2}) + 2\bar{\Omega^2}\bar{\alpha} - \bar{\beta^2}}{\bar{k^2}}\right. \\
&\quad \left. + \frac{\bar{\Omega^4} + 2\bar{\Omega^2}\xi}{\bar{k^4}}\right].
\end{aligned} \tag{3.51}$$

Likewise, for $\bar{a}_3$ one can obtain as

$$\bar{a}_3 = \left(\frac{\bar{\Omega}^2\mu}{\rho\sigma} - \frac{\bar{k^2}}{\sigma} \times \frac{\mu}{\rho}\right)\left[\left(\frac{\bar{\Omega}^2\mu}{\rho\sigma} - \frac{\bar{k^2}}{\sigma} \times \frac{(\lambda+2\mu)}{\rho}\right)\left(\frac{\bar{k^2}}{\sigma} \times \frac{\alpha}{\rho\sigma} + \frac{\xi}{\rho\sigma}\right) + \frac{\beta}{\rho\sigma} \times \frac{\bar{k^2}}{\sigma} \times \frac{\beta}{\rho}\right],$$
$$= \left(\frac{\bar{k^2}\mu}{\rho\sigma}\right)^3\left[\left(\frac{\bar{\Omega}^2}{\bar{k^2}} - 1\right)\left(\frac{\bar{\Omega}^2}{\bar{k^2}} - (\bar{\lambda}+2)\right)\left(\bar{\alpha} + \frac{\bar{\xi}}{\bar{k^2}}\right) + \frac{\bar{\beta}\mu}{\rho\sigma}\right],$$
$$= \left(\frac{\bar{k^2}\mu}{\rho\sigma}\right)^3\left[(2+\bar{\lambda})\bar{\alpha} + \frac{(2+\lambda)\bar{\xi} - (3+\bar{\lambda})\bar{\Omega}^2\bar{\alpha} - \bar{\beta}^2}{\bar{k^2}} + \frac{[\bar{\Omega}^2\bar{\alpha} + \bar{\beta}^2 - (3+\bar{\lambda})\bar{\xi}]\bar{\Omega}^2}{\bar{k^4}} + \frac{\bar{\Omega}^4\bar{\xi}}{\bar{k^6}}\right]. \tag{3.52}$$

3.5 Results and discussion of graphs

The graphs are reproduced in *Mathematica* with the following code

$\lambda = 15 * 10^9;$

$\alpha = 8 * 10^9;$

$\xi = 12 * 10^9;$

$\sigma = .162;$

$\mu = 7.5 * 10^9;$

$\beta = 10 * 10^9;$

$\lambda 1 = \lambda/\mu;$

$\alpha 1 = \alpha/(\mu+\sigma);$

$\beta 1 = \beta/\mu;$

$\xi 1 = \xi/\mu;$

$\Omega 1 = 0;$

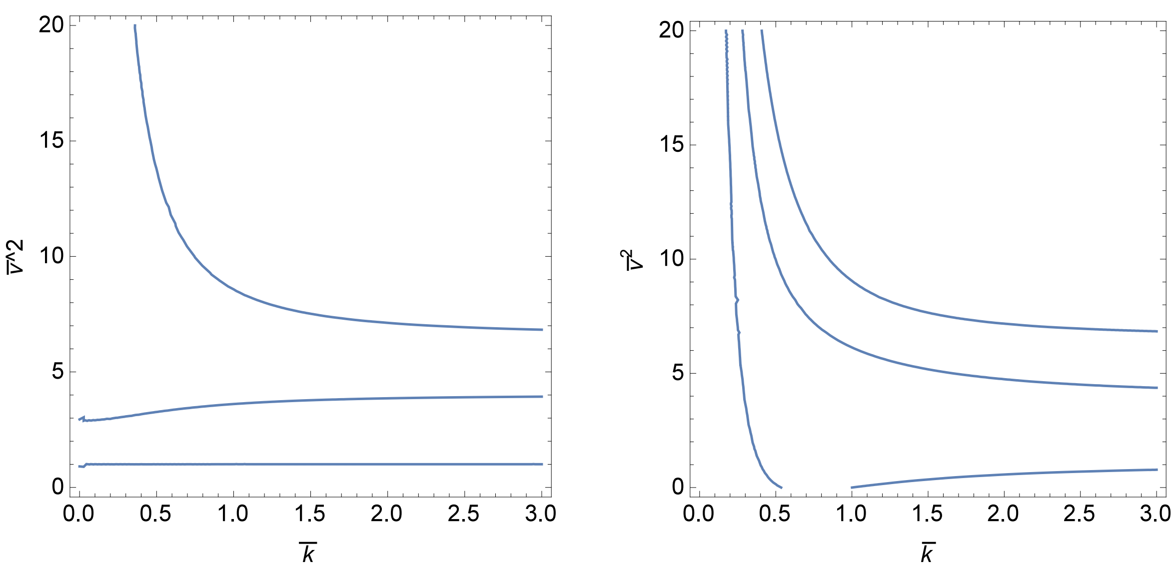
and

$$x1 = 3 + \lambda 1 + \alpha 1 + \left(2\Omega 1^2 + \xi 1\right)/k^2;$$

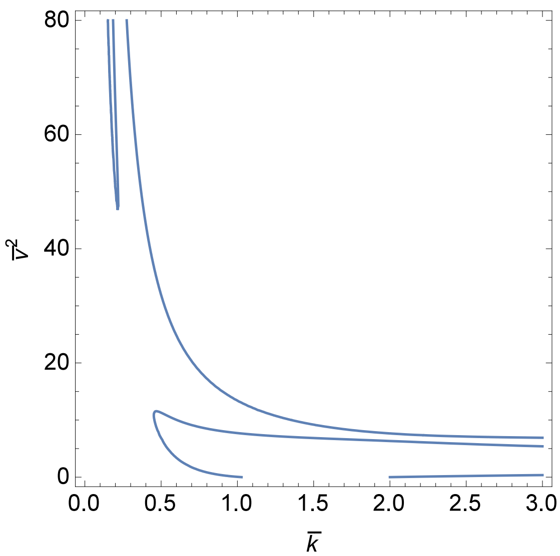
$$x2 = 2 + \lambda 1 + (3 + \lambda 1)\alpha 1 + \left((3 + \lambda 1)(\xi 1 - \Omega 1^2) + 2\Omega 1^2 \alpha - \beta 1^2\right)/k^2$$
$$+ \left(\Omega 1^4 + 2\Omega 1^2 \xi\right)/k^4;$$

$$x3 = (2 + \lambda 1)\alpha 1 + \left((2 + \lambda 1)\xi 1 - (3 + \lambda 1)\Omega 1^2 \alpha 1 - \beta 1^2\right)/k^2$$
$$+ \left(\Omega^2 \alpha 1 + \beta 1^2 - (3 + \lambda 1)\xi 1\right)\Omega 1^2/k^4 + \Omega 1^4 \xi/k^6;$$

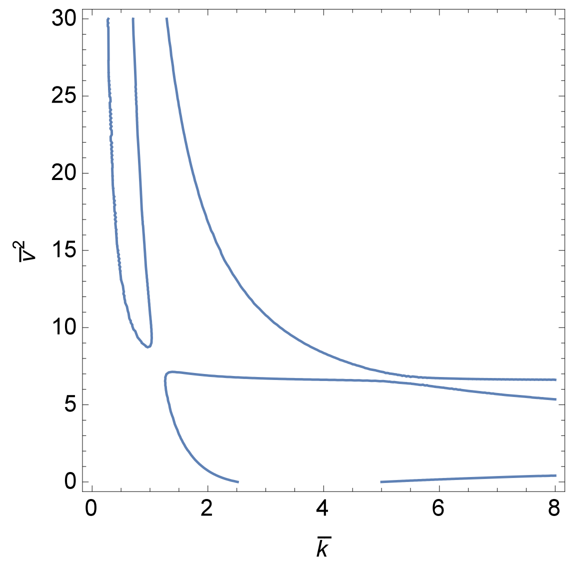
where $\bar{\lambda} = \lambda 1$, $\bar{\beta} = \beta 1$, $\bar{\xi} = \xi 1$, $\bar{\alpha} = \alpha 1$, $a_1 = x_1$, $a_2 = x_2$, $a_3 = x_3$.
contourplot $\left[\nu^3 - x1 * \nu^2 + x2 * \nu - x3 = 0, (k, 0, 3), (\nu, o, 20)\right]$



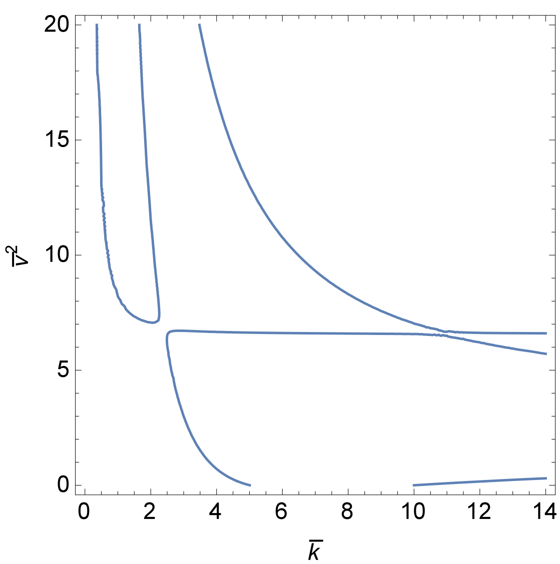
(a) $\bar{\Omega} = 0$ (b) $\bar{\Omega} = 1$



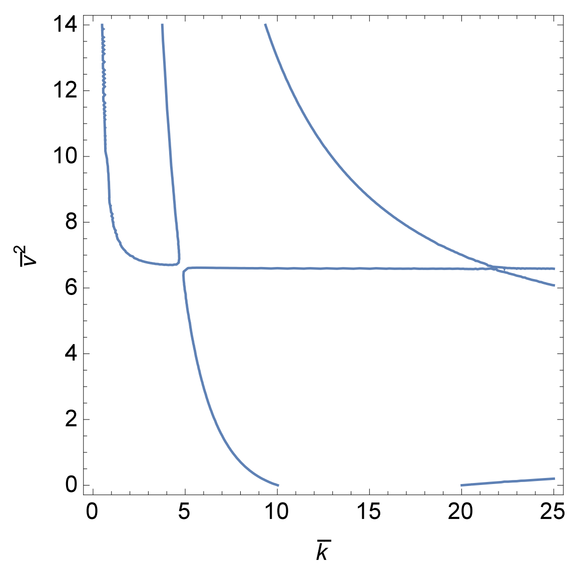
(c) $\bar{\Omega} = 2$



(d) $\bar{\Omega} = 5$



(e) $\bar{\Omega} = 10$



(f) $\bar{\Omega} = 20$

Figure 3.1: Graphss of $\bar{\nu}^2$ against the $\bar{k}$ in the presence of voids and rotation.

Graphical results are produced by using Mathematica [14]. When rotation is zero then $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}_3$ are positive and can be less than for non-zero rotation. The greater values of the material parameters with voids results in the fall off rotation.
By Descartes' rule of sign, Eq. (3.46) can change sign maximum three times as function

of t. So there are 1, 2 or 3 positive real roots and thus one, two or three real wave speeds. This is shown in Fig. 3.1.

Fig. 3.1 is drawn from Eq. (3.46) using the data of Table 3.1 which present the non-dimensionlize squared wave speed t as function of $\bar{k}$ with non-dimensionlize values of $\bar{\lambda}$, $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\xi}$. The result of voids in the absence of rotation is observed in Fig. 1(a). Out of three wave speed, one is wave speed of transverse wave which is unchanged in the presence of voids and is constant along horizontal axis and the other parallel wave speed decays in the beginning and then become constant as wave number increases. In the third, wave speed is plotted for small wave number corresponding to large wave speed and gradually wave number become constant.

3.5.1 Case for $\bar{k} \to \infty$

If we take $\bar{k}$ very large in Eq. (3.46) and approaching to infinity then as a result few terms of $\bar{a_1}$, $\bar{a_2}$ and $\bar{a_3}$ will become zero. These are:

$$\bar{a_1} = 3 + \bar{\lambda} + \bar{\alpha}, \tag{3.53}$$

$$\bar{a_2} = 2 + \bar{\lambda} + (3 + \bar{\lambda})\bar{\alpha}, \tag{3.54}$$

$$\bar{a_3} = (2 + \bar{\lambda})\bar{\alpha}. \tag{3.55}$$

Now Eq. (3.46) becomes

$$t^3 - (3 + \bar{\lambda} + \bar{\alpha})t^2 + [2 + \bar{\lambda} + (3 + \bar{\lambda})\bar{\alpha}]t - ((2 + \bar{\lambda})\bar{\alpha}) = 0. \tag{3.56}$$

If one root is 1 then the other two roots by long division are obtained.

$$\begin{array}{rcccc}
 & & & t^2 - (2+\bar{\lambda}+\bar{\alpha})t + (2+\bar{\lambda})\bar{\alpha} & \\
\cline{2-5}
t-1\surd & t^3 & -(3+\bar{\lambda}+\bar{\alpha})t^2 & +[2+\bar{\lambda}+(3+\bar{\lambda})\bar{\alpha}]t & -(2+\bar{\lambda})\bar{\alpha} \\
 & \pm t^3 & \mp t^2 & & \\
\cline{2-4}
 & & -(2+\bar{\lambda}+\bar{\alpha})t^2 & +[2+\bar{\lambda}+\bar{\alpha}+(2+\bar{\lambda})\bar{\alpha}]t & \\
 & & \mp(2+\bar{\lambda}+\bar{\alpha})t^2 & \pm[2+\bar{\lambda}+\bar{\alpha}]t & \\
\cline{3-5}
 & & & +(2+\bar{\lambda})\bar{\alpha}t & -(2+\bar{\lambda})\bar{\alpha} \\
 & & & \pm(2+\bar{\lambda})\bar{\alpha}t & \mp(2+\bar{\lambda})\bar{\alpha} \\
\cline{4-5}
 & & & 0 &
\end{array}.$$

Now factorizing the polynomial by using the formula

$$
\begin{aligned}
t^2 - (2+\bar{\lambda}+\bar{\alpha})t + (2+\bar{\lambda})\bar{\alpha} &= \frac{(2+\bar{\lambda}+\bar{\alpha}) \pm \sqrt{(2+\bar{\lambda}+\bar{\alpha})^2 - 4(1)(2+\bar{\lambda})\bar{\alpha}}}{2(1)}, \\
&= \frac{(2+\bar{\lambda}+\bar{\alpha}) \pm \sqrt{(2+\bar{\lambda})^2 + \bar{\alpha}^2 + 2(2+\bar{\lambda})\bar{\alpha} - 4(1)(2+\bar{\lambda})\bar{\alpha}}}{2}, \\
&= \frac{(2+\bar{\lambda}+\bar{\alpha}) \pm \sqrt{(2+\bar{\lambda})^2 + \bar{\alpha}^2 - 2(2+\bar{\lambda})\bar{\alpha}}}{2}, \\
&= \frac{(2+\bar{\lambda}+\bar{\alpha}) \pm \sqrt{[(2+\bar{\lambda}) - \bar{\alpha}]^2}}{2}, \\
&= \frac{(2+\bar{\lambda}+\bar{\alpha}) \pm [(2+\bar{\lambda}) - \bar{\alpha}]}{2},
\end{aligned}
$$

$$
\begin{aligned}
t &= \frac{(2+\bar{\lambda}+\bar{\alpha}) + [(2+\bar{\lambda}) - \bar{\alpha}]}{2}, \quad or \quad t = \frac{(2+\bar{\lambda}+\bar{\alpha}) - [(2+\bar{\lambda}) - \bar{\alpha}]}{2}, \\
\text{or} \quad t &= 2+\bar{\lambda}, \qquad \text{or} \qquad t = \bar{\alpha}.
\end{aligned}
$$

Eq. (3.56) is solved and it gives three roots 1, $\bar{\alpha}$, $\bar{\lambda}+2$. Fig. 3.1(a)-(f), shows the voids parameters α, β, ξ and rotation. Graphs are plotted by taking various values of $\bar{\Omega}$. Waves are coupled and on the horizontal axis there is a sharp gap for $\bar{k}$, for very small value of t. The waves of Fig. 3.1(c)-(f) become broaden as value of $\bar{k}$ and t increases. In a graph, each wave changes its behaviour as wave number and wave speed increases accordingly.

3.5.2 Case with voids

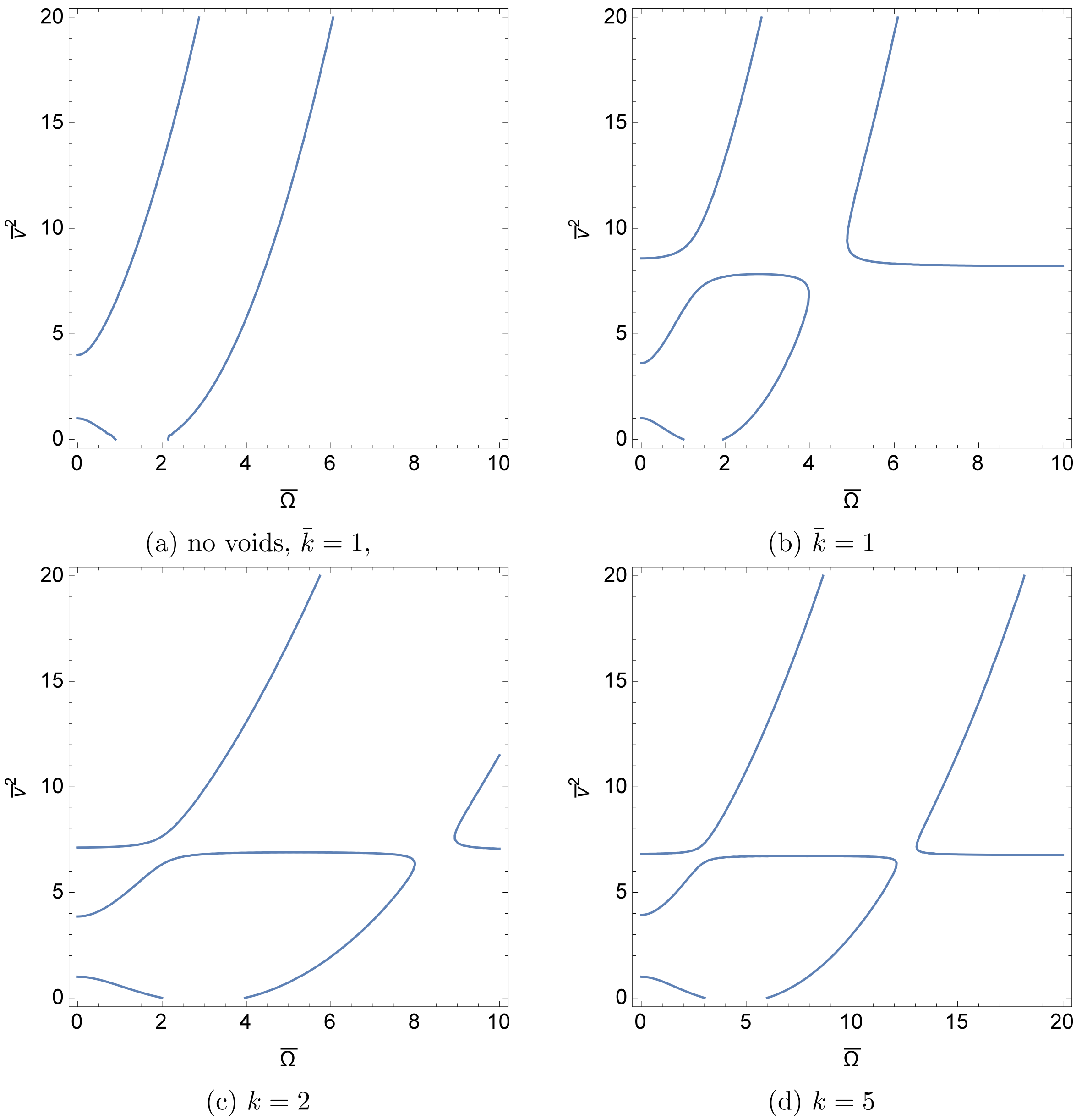


Figure 3.2: Graphs of the $\bar{\nu}^2$ against the $\bar{\Omega}$ for constant values of $\bar{k}$, where α, β and ξ are calculated from Table 3.1.

In Fig. 3.2(a), voids parameters $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\xi}$ are absent so as a consequence one wave is vanished for wave number 1. In Fig. 3.2(b), wave number is same as in Fig. 3.1(a) but the voids parameters with rotation are present, as effect three wave speeds are travelling. In Fig. 3.2(c) and (d) waves are broden as the wave number is increased. In Fig. 3.3, when $a_3 = 0$ in the absence of voids the graphs depend on the solution of the second degree equation.

3.5.3 Case without voids

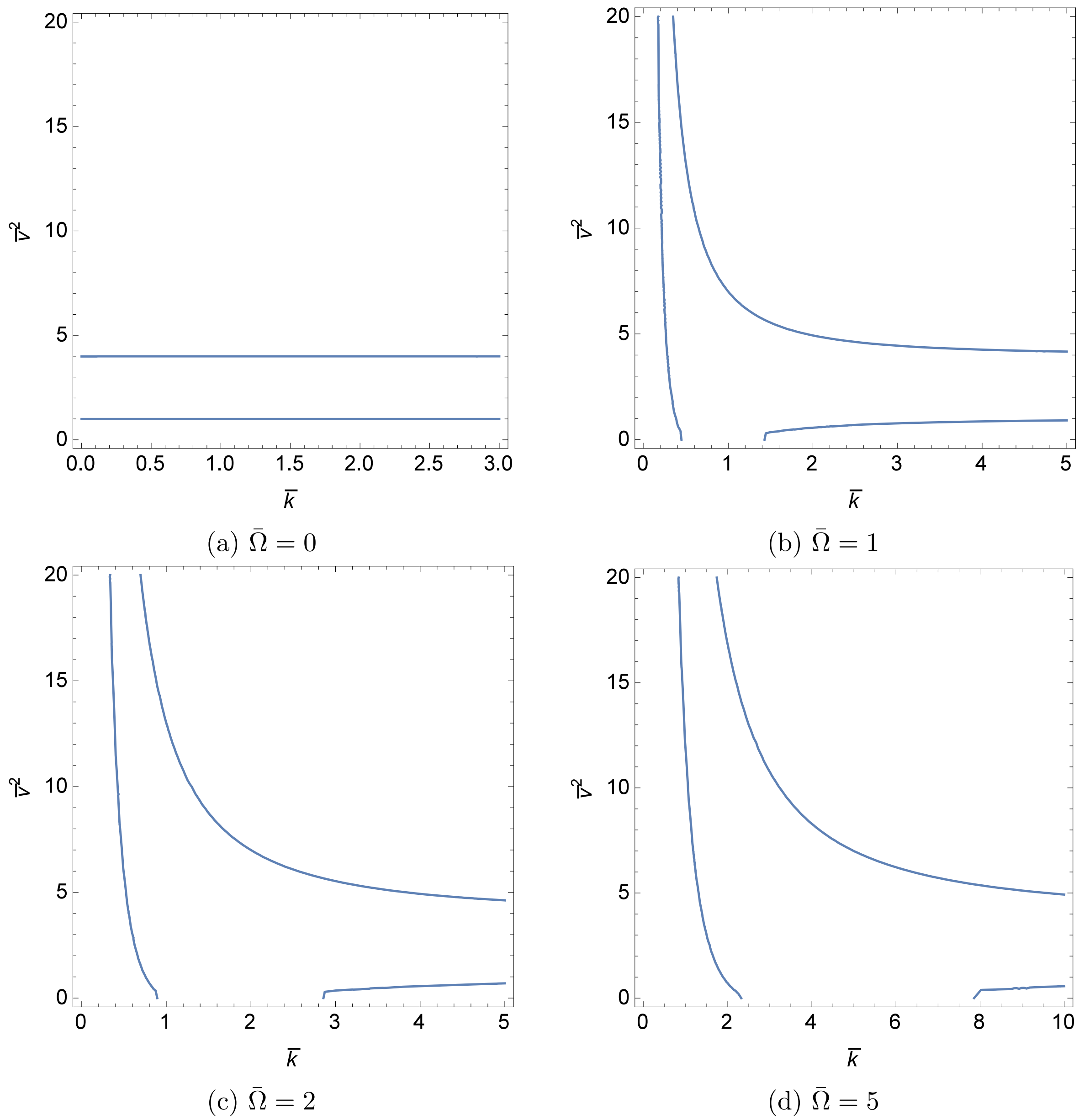


Figure 3.3: Graphs of the $\bar{\nu^2}$ against the $\bar{k}$ in the absence of voids.

In Fig. 3.3(a) when rotation is absent then only two real wave speeds travel, one is longitudinal and other is transverse. As we observed that when we add rotation, in graphs of Fig. 3.3(b)-(d) it is clear that as wave number is low then wave speed is high.

In these figures, one real wave speed has been observed and other wave has "cut-out" range which increase as rotation increases.

Chapter 4

Travelling of Waves in an Anisotropic Elastic Materials with Voids

The theory of linear elastic material with voids is the generalization of the classical theory of elasticity in which strain and void volume is considered as an independent kinematics variable. The nonlinear version of the theory of elastic material with voids was presented by Nunziato and Cowin in 1979 [1] and linear version was proposed by Cowin and Nunziato in 1983 [2]. Later on, void volume was added as a new kinematic variable to introduce the new version. The limitation of this version was that after removal of voids volume, we again get classical linear theory of elasticity. This theory has a wide range of application in geological material like rocks, solid and manufacturing material like wood, clothes, paper etc. The anisotropic material gained attraction in 19^{th} century and are used in the field of seismology, ultra sonics and electromagnetic fields.

In this chapter, travelling of waves in anisotropic elastic mterial with voids is discussed. Basic equations governing the wave motion in anisotropic materials are considered in section 4.1. In Section 4.2, the hexagonal crystal system is chosen to assign values to constants of equations in section 4.1. Numerical results and discussion are given in section 4.3.

4.1 Elasticity tensor for anisotropic materials

Consider the Eq. (2.8), which is elastic stiffness tensor of order four and differentiating Eq. (2.8) w.r.t j we have,

$$T_{ij,j} = C_{ijkl}u_{k,lj}, \tag{4.1}$$

Substituting Eq. (4.1) into stress equation of motion

$$T_{ij,j} = C_{ijkl}u_{k,lj} = \rho(\ddot{u}_1 - \Omega^2 u_1 + 2\Omega\dot{u}_3). \tag{4.2}$$

Therefore, for $i = 1$

$$T_{1j,j} = C_{1jkl}u_{k,lj}. \tag{4.3}$$

Now, for $j = 1, 2, 3$

$$\begin{aligned} T_{1j,j} &= T_{11,1} + T_{12,2} + T_{13,3}, \\ &= C_{11kl}u_{k,l1} + C_{12kl}u_{k,l2} + C_{13kl}u_{k,l3} + \beta\phi_{,1} = \rho(\ddot{u}_1 - \Omega^2 u_1 + 2\Omega\dot{u}_3). \end{aligned} \tag{4.4}$$

For $k = 1, 2, 3$

$$\begin{aligned} &C_{111l}u_{1,l1} + C_{112l}u_{2,l1} + C_{113l}u_{3,l1} + C_{121l}u_{1,l2} + C_{122l}u_{2,l2} \\ &+ C_{123l}u_{3,l2} + C_{131l}u_{1,l3} + C_{132l}u_{2,l3} + C_{133l}u_{3,l3} + \beta\phi_{,1} = \rho(\ddot{u}_1 - \Omega^2 u_1 + 2\Omega\dot{u}_3). \end{aligned} \tag{4.5}$$

For $l = 1, 2, 3$

$$\begin{aligned} &C_{1111}u_{1,11} + C_{1112}u_{1,21} + C_{1113}u_{1,31} + C_{1121}u_{2,11} + C_{1122}u_{2,21} \\ &+ C_{1123}u_{2,31} + C_{1131}u_{3,11} + C_{1132}u_{3,21} + C_{1133}u_{3,31} + C_{1211}u_{1,12} \\ &+ C_{1212}u_{1,22} + C_{113}u_{1,32} + C_{1221}u_{2,12} + C_{1222}u_{2,22} + C_{1223}u_{2,32} \\ &+ C_{1231}u_{3,12} + C_{1232}u_{3,22} + C_{1233}u_{3,32} + C_{1311}u_{1,13} + C_{1312}u_{1,23} \\ &+ C_{1313}u_{1,33} + C_{1321}u_{2,13} + C_{1322}u_{2,23} + C_{1323}u_{2,33} + C_{1331}u_{3,13} \\ &+ C_{1332}u_{3,23} + C_{1333}u_{3,33} + \beta\phi_{,1} = \rho(\ddot{u}_1 - \Omega^2 u_1 + 2\Omega\dot{u}_3). \end{aligned} \tag{4.6}$$

It is easy to use Voigt notation or two index representation of C_{ijkl} in which a pair of indices corresponds to a single index in the following manner.

$$11 \to 1, 22 \to 2, 33 \to 3, 23 \to 4, 13 \to 5, 12 \to 6 \tag{4.7}$$

Now, by using Voigt notation Eq. (4.6) becomes

$$\begin{aligned} &C_{11}u_{1,11}+C_{16}u_{1,21}+C_{15}u_{1,31}+C_{16}u_{2,11}+C_{12}u_{2,21}+C_{14}u_{2,31}\\ &+C_{15}u_{3,11}+C_{14}u_{3,21}+C_{13}u_{3,31}+C_{16}u_{1,12}+C_{66}u_{1,22}+C_{56}u_{1,32}\\ &+C_{66}u_{2,12}+C_{26}u_{2,22}+C_{46}u_{2,32}+C_{56}u_{3,12}+C_{46}u_{3,22}+C_{36}u_{3,32}\\ &+C_{15}u_{1,13}+C_{56}u_{1,23}+C_{55}u_{1,33}+C_{56}u_{2,13}+C_{25}u_{2,23}+C_{45}u_{2,33}\\ &+C_{55}u_{3,13}+C_{45}u_{3,23}+C_{35}u_{3,33}+\beta\phi_{,1}=\rho(\ddot{u}_1-\Omega^2u_1+2\Omega\dot{u}_3). \end{aligned} \tag{4.8}$$

So, Eq. (4.8) in partial form as

$$\begin{aligned} &C_{11}\frac{\partial^2u_1}{\partial x^2}+C_{16}\frac{\partial^2u_1}{\partial y\partial x}+C_{15}\frac{\partial^2u_1}{\partial z\partial x}+C_{16}\frac{\partial^2u_2}{\partial x^2}+C_{12}\frac{\partial^2u_2}{\partial y\partial x}\\ &+C_{14}\frac{\partial^2u_2}{\partial z\partial x}+C_{15}\frac{\partial^2u_3}{\partial x^2}+C_{14}\frac{\partial^2u_3}{\partial y\partial x}+C_{13}\frac{\partial^2u_3}{\partial z\partial x}+C_{16}\frac{\partial^2u_1}{\partial x\partial y}\\ &+C_{66}\frac{\partial^2u_1}{\partial y^2}+C_{56}\frac{\partial^2u_1}{\partial z\partial y}+C_{66}\frac{\partial^2u_2}{\partial x\partial y}+C_{26}\frac{\partial^2u_2}{\partial y^2}+C_{46}\frac{\partial^2u_2}{\partial z\partial y}\\ &+C_{56}\frac{\partial^2u_3}{\partial x\partial y}+C_{46}\frac{\partial^2u_3}{\partial y^2}+C_{36}\frac{\partial^2u_3}{\partial z\partial y}+C_{56}\frac{\partial^2u_2}{\partial x\partial z}+C_{15}\frac{\partial^2u_1}{\partial x\partial z}\\ &+C_{56}\frac{\partial^2u_1}{\partial y\partial z}+C_{55}\frac{\partial^2u_1}{\partial z^2}+C_{25}\frac{\partial^2u_2}{\partial y\partial z}+C_{45}\frac{\partial^2u_2}{\partial z^2}+C_{55}\frac{\partial^2u_3}{\partial x\partial z}\\ &+C_{45}\frac{\partial^2u_3}{\partial y\partial z}+C_{35}\frac{\partial^2u_3}{\partial z2}+\beta\phi_{,1}=\rho(\ddot{u}_1-\Omega^2u_1+2\Omega\dot{u}_3). \end{aligned} \tag{4.9}$$

Now, For $i=2$

$$T_{2j,j}=C_{2jkl}u_{k,lj}. \tag{4.10}$$

For $j=1,2,3$

$$\begin{aligned} T_{2j,j}&=T_{21,1}+T_{22,2}+T_{33,3},\\ &=C_{21kl}u_{k,l1}+C_{22kl}u_{k,l2}+C_{23kl}u_{k,l3}+\beta\phi_{,2}=2\rho\Omega(\dot{u}_1+\dot{u}_3). \end{aligned} \tag{4.11}$$

For $k=1,2,3$

$$\begin{aligned} &C_{211l}u_{1,l1}+C_{212l}u_{2,l1}+C_{213l}u_{3,l1}+C_{221l}u_{1,l2}+C_{222l}u_{2,l2}\\ &+C_{223l}u_{3,l2}+C_{231l}u_{1,l3}+C_{232l}u_{2,l3}+C_{233l}u_{3,l3}+\beta\phi_{,2}=2\rho\Omega(\dot{u}_1+\dot{u}_3). \end{aligned} \tag{4.12}$$

For $l = 1, 2, 3$

$$
\begin{aligned}
&C_{2111}u_{1,11} + C_{2112}u_{1,21} + C_{2113}u_{1,31} + C_{2121}u_{2,11} + C_{2122}u_{2,21} \\
&+ C_{2123}u_{2,31} + C_{2131}u_{3,11} + C_{2132}u_{3,21} + C_{2133}u_{3,31} + C_{2211}u_{1,12} \\
&+ C_{2212}u_{1,22} + C_{2213}u_{1,32} + C_{2221}u_{2,12} + C_{2222}u_{2,22} + C_{2223}u_{2,32} \\
&+ C_{2231}u_{3,12} + C_{2232}u_{3,22} + C_{2233}u_{3,32} + C_{2311}u_{1,13} + C_{2312}u_{1,23} \\
&+ C_{2313}u_{1,33} + C_{2321}u_{2,13} + C_{2322}u_{2,23} + C_{2323}u_{2,33} + C_{2331}u_{3,13} \\
&+ C_{2332}u_{3,23} + C_{2333}u_{3,33} + \beta\phi_{,2}, = 2\rho\Omega(\dot{u_1} + \dot{u_3}).
\end{aligned} \tag{4.13}
$$

Using Voigt notation, Eq. (4.13) can be written as,

$$
\begin{aligned}
&C_{16}u_{1,11} + C_{66}u_{1,21} + C_{56}u_{1,31} + C_{66}u_{2,11} + C_{26}u_{2,21} \\
&+ C_{46}u_{2,31} + C_{56}u_{3,11} + C_{46}u_{3,21} + C_{36}u_{3,31} + C_{21}u_{1,12} \\
&+ C_{26}u_{1,22} + C_{25}u_{1,32} + C_{26}u_{2,12} + C_{22}u_{2,22} + C_{24}u_{2,32} \\
&+ C_{25}u_{3,12} + C_{24}u_{3,22} + C_{23}u_{3,32} + C_{14}u_{1,13} + C_{46}u_{1,23} \\
&+ C_{45}u_{1,33} + C_{46}u_{2,13} + C_{24}u_{2,23} + C_{44}u_{2,33} + C_{45}u_{3,13} \\
&+ C_{44}u_{3,23} + C_{34}u_{3,33} + \beta\phi_{,2} = 2\rho\Omega(\dot{u_1} + \dot{u_3}).
\end{aligned} \tag{4.14}
$$

Equation (4.14) in partial form,

$$
\begin{aligned}
&C_{16}\frac{\partial^2 u_1}{\partial x^2} + C_{66}\frac{\partial^2 u_1}{\partial y \partial x} + C_{56}\frac{\partial^2 u_1}{\partial z \partial x} + C_{66}\frac{\partial^2 u_2}{\partial x^2} + C_{26}\frac{\partial^2 u_2}{\partial y \partial x} \\
&+ C_{46}\frac{\partial^2 u_2}{\partial z \partial x} + C_{56}\frac{\partial^2 u_3}{\partial x^2} + C_{46}\frac{\partial^2 u_3}{\partial y \partial x} + C_{36}\frac{\partial^2 u_3}{\partial z \partial x} + C_{12}\frac{\partial^2 u_1}{\partial x \partial y} \\
&+ C_{26}\frac{\partial^2 u_1}{\partial y^2} + C_{25}\frac{\partial^2 u_1}{\partial z \partial y} + C_{26}\frac{\partial^2 u_2}{\partial x \partial y} + C_{22}\frac{\partial^2 u_2}{\partial y^2} + C_{24}\frac{\partial^2 u_2}{\partial z \partial y} \\
&+ C_{25}\frac{\partial^2 u_3}{\partial x \partial y} + C_{24}\frac{\partial^2 u_3}{\partial y^2} + C_{23}\frac{\partial^2 u_3}{\partial z \partial y} + C_{14}\frac{\partial^2 u_1}{\partial x \partial z} + C_{46}\frac{\partial^2 u_1}{\partial y \partial z} \\
&+ C_{45}\frac{\partial^2 u_1}{\partial z^2} + C_{46}\frac{\partial^2 u_2}{\partial x \partial z} + C_{24}\frac{\partial^2 u_2}{\partial y \partial z} + C_{44}\frac{\partial^2 u_2}{\partial z^2} + C_{45}\frac{\partial^2 u_3}{\partial x \partial z} \\
&+ C_{44}\frac{\partial^2 u_3}{\partial y \partial z} + C_{34}\frac{\partial^2 u_3}{\partial z 2} + \beta\phi_{,2} = 2\rho\Omega(\dot{u_1} + \dot{u_3}).
\end{aligned} \tag{4.15}
$$

Now, for $i = 3$ in Eq. (4.1)

$$T_{3j,j} = C_{3jkl}u_{k,lj}. \tag{4.16}$$

For $j = 1, 2, 3$ in Eq. (4.16)

$$\begin{aligned} T_{3j,j} &= T_{31,1} + T_{32,2} + T_{33,3}, \\ &= C_{31kl}u_{k,l1} + C_{32kl}u_{k,l2} + C_{33kl}u_{k,l3} + \beta\phi_{,2} \quad = \rho(\ddot{u}_3 - \Omega^2 u_3 + 2\Omega \dot{u}_1). \end{aligned} \tag{4.17}$$

For $k = 1, 2, 3$ in Eq (4.17)

$$\begin{aligned} &C_{311l}u_{1,l1} + C_{312l}u_{2,l1} + C_{313l}u_{3,l1} + +C_{321l}u_{1,l2} + C_{322l}u_{2,l2} + C_{323l}u_{3,l2} \\ &+ C_{331l}u_{1,l3} + C_{332l}u_{2,l3} + C_{333l}u_{3,l3} + \beta\phi_{,3} = \rho(\ddot{u}_3 - \Omega^2 u_3 + 2\Omega \dot{u}_1). \end{aligned} \tag{4.18}$$

For l=1,2,3 in Eq. (4.18)

$$\begin{aligned} &C_{3111}u_{1,11} + C_{3112}u_{1,21} + C_{3113}u_{1,31} + C_{3121}u_{2,11} + C_{3122}u_{2,21} \\ &+ C_{3123}u_{2,31} + C_{3131}u_{3,11} + C_{3132}u_{3,21} + C_{3133}u_{3,31} + C_{3211}u_{1,12} \\ &+ C_{3212}u_{1,22} + C_{3213}u_{1,32} + C_{3221}u_{2,12} + C_{3222}u_{2,22} + C_{3223}u_{2,32} \\ &+ C_{3231}u_{3,12} + C_{3232}u_{3,22} + C_{3233}u_{3,32} + C_{3311}u_{1,13} + C_{3312}u_{1,23} \\ &+ C_{3313}u_{1,33} + C_{3321}u_{2,13} + C_{3322}u_{2,23} + C_{3323}u_{2,33} + C_{3331}u_{3,13} \\ &+ C_{3332}u_{3,23} + C_{3333}u_{3,33} + \beta\phi_{,2} = \rho(\ddot{u}_3 - \Omega^2 u_3 + 2\Omega \dot{u}_1) \end{aligned} \tag{4.19}$$

Using Voigt notation, therefore, Eq. (4.19) implies,

$$\begin{aligned} &C_{15}u_{1,11} + C_{56}u_{1,21} + C_{55}u_{1,31} + C_{56}u_{2,11} + C_{25}u_{2,21} \\ &+ C_{45}u_{2,31} + C_{55}u_{3,11} + C_{45}u_{3,21} + C_{35}u_{3,31} + C_{14}u_{1,12} \\ &+ C_{46}u_{1,22} + C_{45}u_{1,32} + C_{46}u_{2,12} + C_{24}u_{2,22} + C_{44}u_{2,32} \\ &+ C_{45}u_{3,12} + C_{44}u_{3,22} + C_{34}u_{3,32} + C_{13}u_{1,13} + C_{36}u_{1,23} \\ &+ C_{35}u_{1,33} + C_{36}u_{2,13} + C_{23}u_{2,23} + C_{34}u_{2,33} + C_{35}u_{3,13} \\ &+ C_{34}u_{3,23} + C_{33}u_{3,33} + \beta\phi_{,3} = \rho(\ddot{u}_3 - \Omega^2 u_3 + 2\Omega \dot{u}_1) \end{aligned} \tag{4.20}$$

Writing Eq. (4.20) in partial form as,

$$\begin{aligned}
&C_{51}\frac{\partial^2 u_1}{\partial x^2}+C_{56}\frac{\partial^2 u_1}{\partial y\partial x}+C_{55}\frac{\partial^2 u_1}{\partial z\partial x}+C_{56}\frac{\partial^2 u_2}{\partial x^2}+C_{25}\frac{\partial^2 u_2}{\partial y\partial x}\\
&+C_{45}\frac{\partial^2 u_2}{\partial z\partial x}+C_{55}\frac{\partial^2 u_3}{\partial x^2}+C_{45}\frac{\partial^2 u_3}{\partial y\partial x}+C_{35}\frac{\partial^2 u_3}{\partial z\partial x}+C_{14}\frac{\partial^2 u_1}{\partial x\partial y}\\
&+C_{46}\frac{\partial^2 u_1}{\partial y^2}+C_{45}\frac{\partial^2 u_1}{\partial z\partial y}+C_{46}\frac{\partial^2 u_2}{\partial x\partial y}+C_{24}\frac{\partial^2 u_2}{\partial y^2}+C_{44}\frac{\partial^2 u_2}{\partial z\partial y}\\
&+C_{45}\frac{\partial^2 u_3}{\partial x\partial y}+C_{44}\frac{\partial^2 u_3}{\partial y^2}+C_{34}\frac{\partial^2 u_3}{\partial z\partial y}+C_{13}\frac{\partial^2 u_1}{\partial x\partial z}+C_{36}\frac{\partial^2 u_1}{\partial y\partial z}\\
&+C_{35}\frac{\partial^2 u_1}{\partial z^2}+C_{36}\frac{\partial^2 u_2}{\partial x\partial z}+C_{32}\frac{\partial^2 u_2}{\partial y\partial z}+C_{34}\frac{\partial^2 u_2}{\partial z^2}+C_{35}\frac{\partial^2 u_3}{\partial x\partial z}\\
&+C_{34}\frac{\partial^2 u_3}{\partial y\partial z}+C_{33}\frac{\partial^2 u_3}{\partial z2}+\beta\phi_{,3}=\rho(\ddot{u}_3-\Omega^2 u_3+2\Omega\dot{u}_1)
\end{aligned}\tag{4.21}$$

In order to discuss two dimensional wave propagation, in the (x, z) plane, we shall consider $\mathbf{u}$ and $\boldsymbol{\Omega}$ to have components

$$\begin{aligned}
u_1 &= u_1(x, z, t),\\
u_3 &= u_3(x, z, t),\\
\Omega_2 &= \Omega.
\end{aligned}$$

with

$$\begin{aligned}
u_2 &= 0,\\
\Omega_1 &= \Omega_3 = 0.
\end{aligned}$$

With this specialization, Eq. (4.9) reduces to

$$\begin{aligned}
&C_{11}\frac{\partial^2 u_1}{\partial x^2}+C_{15}\left(\frac{\partial^2 u_1}{\partial z\partial x}+\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial^2 u_1}{\partial x\partial z}\right)+C_{13}\frac{\partial^2 u_3}{\partial z\partial x}\\
&+C_{55}\left(\frac{\partial^2 u_1}{\partial z^2}+\frac{\partial^2 u_3}{\partial x\partial z}\right)+C_{35}\frac{\partial^2 u_3}{\partial z^2}+\beta\phi_{,1}=\rho(\ddot{u}_1-\Omega^2 u_1+2\Omega\dot{u}_3).
\end{aligned}\tag{4.22}$$

Likewise, Eq.(4.21) implies

$$\begin{aligned}
&C_{15}\frac{\partial^2 u_1}{\partial x^2}+C_{55}\left(\frac{\partial^2 u_1}{\partial z\partial x}+\frac{\partial^2 u_3}{\partial x^2}\right)+C_{31}\frac{\partial^2 u_1}{\partial x\partial z}+C_{35}\left(\frac{\partial^2 u_3}{\partial z\partial x}+\frac{\partial^2 u_1}{\partial z^2}+\frac{\partial^2 u_3}{\partial x\partial z}\right)\\
&+C_{33}\frac{\partial^2 u_3}{\partial z^2}+\beta\phi_{,1}=\rho(\ddot{u}_3-\Omega^2 u_3+2\Omega\dot{u}_1),
\end{aligned}\tag{4.23}$$

yields

$$\alpha\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}\right)-\xi\phi-\beta e=\rho\sigma\ddot{\phi}, \tag{4.24}$$

where

$$e=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}. \tag{4.25}$$

4.1.1 Propagation of plane wave in a material with invariance of 6-fold axis rotations

For the hexagonal system the principal axis has order six, behaving as a dyad axis combined with a triad. The matrix $C_{\alpha\beta}$ thus has a form combining the features of the monoclinic and trigonal system in the form of matrix given in Eq. (2.10). Using this matrix in Eq. (4.22) and Eq. (4.23), we have

$$C_{11}u_{1,11}+C_{13}u_{3,13}+C_{44}\left(u_{1,33}+u_{3,13}\right)+\beta\phi_{,1}=\rho(\ddot{u}_1-\Omega^2u_1+2\Omega\dot{u}_3),$$

or

$$C_{11}u_{1,11}+(C_{13}+C_{44})u_{3,13}+C_{44}u_{1,33}+\beta\phi_{,1}=\rho\big(\ddot{u}_1-\Omega^2u_1+2\Omega\dot{u}_3\big). \tag{4.26}$$

Similarly

$$C_{44}\left(u_{1,13}+u_{3,11}\right)+C_{13}u_{1,13}+C_{33}u_{3,33}+\beta\phi_{,3}=\rho(\ddot{u}_3-\Omega^2u_3+2\Omega\dot{u}_1),$$

$$\big(C_{13}+C_{44}\big)u_{1,13}+C_{44}u_{3,11}+C_{33}u_{3,33}+\beta\phi_{,3}=\rho(\ddot{u}_3-\Omega^2u_3+2\Omega\dot{u}_1), \tag{4.27}$$

$$\alpha\left(\phi_{,11}+\phi_{,33}\right)-\xi\phi-\beta e=\rho\sigma\ddot{\phi}. \tag{4.28}$$

where

$$u_1=A\exp\big[ik(xl_1+zl_3-ct)\big],$$
$$u_3=B\exp\big[ik(xl_1+zl_3-ct)\big],$$
$$\phi=C\exp\big[ik(xl_1+zl_3-ct)\big].$$

When all these expressions have the common term $\exp\big[ik(xl_1 + zl_3 - ct)\big]$, which is omitted to save space.

$$\dot{u_1} = -ikcA, \qquad \ddot{u_1} = -k^2c^2A,$$

$$\dot{u_3} = -ikcB, \qquad \ddot{u_3} = -k^2c^2B,$$

$$\dot{\phi} = -ikcC, \qquad \ddot{\phi} = -k^2c^2C,$$

$$u_{1,11} = -k^2l_1^2A, \qquad u_{3,31} = -k^2l_1l_3B,$$

$$u_{1,33} = -k^2l_3^2A, \qquad u_{1,31} = -k^2l_1l_3A,$$

$$u_{3,11} = -k^2l_1^2B, \qquad u_{3,33} = -k^2l_3^2B,$$

$$\phi_{,11} = -k^2l_1^2C, \qquad \phi_{,33} = -k^2l_3^2C,$$

$$e = Aikl_1 + Bikl_3.$$

Equation (4.26) becomes

$$\begin{aligned} &C_{11}\big(-k^2l_1^2\big)A + C_{13}\big(-k^2l_1l_3\big)B + C_{44}\big(-k^2l_3^2A - k^2l_1l_3B\big) + \beta\big(ikl_1C\big) \\ &= \rho\big(-k^2c^2A - \Omega^2A + 2\Omega(-ikcB)\big) \end{aligned}$$

or

$$\begin{aligned} &-k^2C_{11}l_1^2A - k^2\big(C_{13} + C_{44}\big)l_1l_3B - k^2C_{44}l_3^2A + i\beta kl_1C = -k^2\rho c^2A \\ &-\rho\Omega^2A - 2i\Omega kcB. \end{aligned} \tag{4.29}$$

Collecting like terms from Eq. (4.29)

$$\begin{aligned} &\big(-C_{11}k^2{l_1}^2 - C_{44}k^2l_3^2 + \rho(c^2k^2 + \Omega^2)\big)A - \big((C_{13} + C_{44})k^2l_1l_3 - 2i\rho k\Omega c\big)B \\ &+ \big(\beta ikl_1\big)C = 0, \end{aligned}$$

or

$$\begin{aligned} &\big[C_{11}k^2l_1^2 + C_{44}k^2l_3^2 - \rho\big(c^2k^2 + \Omega^2\big)\big]A + \big[C_{13}k^2l_1l_3 + C_{44}k^2l_1l_3 - 2i\rho k\Omega c\big]B \\ &- i\beta kl_1C = 0, \end{aligned}$$

or

$$\left[\frac{C_{11}}{\rho}k^2l_1^2+\frac{C_{44}}{\rho}k^2l_3^2-c^2k^2-\Omega^2\right]A+\left[\left(\frac{C_{13}}{\rho}+\frac{C_{44}}{\rho}\right)k^2l_1l_3-2ik\Omega c\right]B$$
$$-\frac{i\beta kl_1}{\rho}C=0. \tag{4.30}$$

Define the constants like

$$c_1^2=\frac{C_{11}}{\rho},\qquad c_2^2=\frac{C_{44}}{\rho},\qquad c_3^2=\frac{C_{13}}{\rho},$$
$$c_4^2=\frac{\beta}{\rho},\qquad c_5^2=\frac{C_{33}}{\rho},\qquad c_6^2=\frac{\alpha}{(\rho\sigma)},$$
$$\omega^2=c^2k^2. \tag{4.31}$$

After the above substitution, Eq. (4.30) becomes

$$\left[c_1^2k^2l_1^2+c_2^2k^2l_3^2-\omega^2-\Omega^2\right]A+\left[c_3^2k^2l_1l_3+c_2^2k^2l_1l_3-2i\Omega\omega\right]B-ic_4^2kl_1C=0. \tag{4.32}$$

Equation (4.27) becomes

$$C_{44}\big(-k^2l_1l_3A-k^2l_1^2B\big)+C_{13}\big(-k^2l_1l_3A\big)+C_{33}\big(-k^2l_3^2B\big)+\beta\big(ikl_3C\big)$$
$$=\rho\big(-k^2c^2B-\Omega^2B-2i\Omega kcA\big). \tag{4.33}$$

Collecting like terms from Eq. (4.33)

$$\big(C_{44}k^2l_1l_3+C_{13}k^2l_1l_3-2i\rho\Omega kc\big)A+\big(C_{44}k^2{l_1}^2+C_{33}k^2{l_3}^2-\rho k^2c^2-\rho\Omega^2\big)B$$
$$-i\beta kl_3C=0, \tag{4.34}$$
$$\left[\frac{C_{44}}{\rho}k^2l_1l_3+\frac{C_{13}}{\rho}k^2l_1l_3-2i\Omega kc\right]A+\left[\frac{C_{44}}{\rho}k^2l_1^2+\frac{C_{33}}{\rho}k^2l_3^2-k^2c^2-\Omega^2\right]B$$
$$-\frac{\beta}{\rho}ikl_3C=0,$$

$$\left[c_2^2k^2l_1l_3+c_3^2k^2l_1l_3-2i\Omega\omega\right]A+\left[c_2^2k^2l_1^2+c_5^2k^2l_3^2-\omega^2-\Omega^2\right]B-c_4^2ikl_3C=0. \tag{4.35}$$

Equation (4.28) becomes

$$\alpha\big(-Ck^2l_1^2-Ck^2l_3^2\big)-\xi C-\beta\big(Aikl_1+Bikl_3\big)=\rho\sigma(-k^2c^2C). \tag{4.36}$$

Collecting like terms from (4.36)

$$
\begin{aligned}
\big(-\beta ikl_1\big)A + \big(-\beta ikl_3\big)B + \big(-\alpha k^2 l_1^2 - \alpha k^2 l_3^3 - \xi + k^2\rho\sigma c^2\big)C = 0,\\
\big(\beta ikl_1\big)A + \big(\beta ikl_3\big)B + \big(\alpha k^2 l_1^2 + \alpha k^2 l_3^3 + \xi - k^2\rho\sigma c^2\big)C = 0,\\
\frac{\beta}{\rho\sigma} ikl_1 A + \frac{\beta}{\rho\sigma} ikl_3 B + \left[\frac{\alpha}{\rho\sigma}k^2(l_1^2 + l_3^2) + \frac{\xi}{\rho\sigma} - k^2c^2\right]C = 0,\\
\nu^* ikl_1 A + \nu^* ikl_3 B + \big(c_6^2 k^2 (l_1^2 + l_3^2) + \xi^* - \omega^2\big)C = 0,
\end{aligned}
\tag{4.37}
$$

where $\xi^* = \xi/(\rho\sigma)$ and $\nu^* = \beta/(\rho\sigma)$.

The determinant A of coefficient of Eqs. (4.32), (4.35) and (4.37) is as follows.

$$
A = \begin{vmatrix}
c_1^2k^2l_1^2 + c_2^2k^2l_3^2 - \omega^2 - \Omega^2 & (c_3^2 + c_2^2)k^2l_1l_3 - 2i\Omega\omega & -ic_4^2kl_1 \\
(c_2^2 + c_3^2)k^2l_1l_3 - 2i\Omega\omega & c_2^2k^2l_1^2 + c_5^2k^2l_3^2 - \omega^2 - \Omega^2 & -c_4^2ikl_3 \\
\nu^* ikl_1 & \nu^* ikl_3 & c_6^2k^2 + \xi^* - \omega^2
\end{vmatrix}.
$$

The determinant of matrix A for non-trivial solution, that is, $A = 0$, it is real part of determinant,

$$
Re[A] = \omega^6 - C_4\omega^4 + \omega^2\big(\Omega^4 - C_3\Omega^2 - C_2\big) - C_1 = 0, \tag{4.38}
$$

where $Re[A]$ denotes the real part of A. The imaginary part of the determinant is as

$$
Im[A] = \omega^3\big(c_2^2 + c_3^2\big) + \omega\big(c_2^2c_6^2k^2 + c_3^2c_6^2k^2 + c_4^2\nu^* + c_2^2\xi + c_3^2\xi\big)(-4k^2l_1l_3\Omega) = 0, \tag{4.39}
$$

where $Im[A]$ denotes the imaginary part of A. In order to make Eq. (4.38) dimensionless, we will multiply it with $\left(\frac{\rho\sigma}{C_{44}}\right)^3$, the corresponding expression will be

$$
\bar{\omega}^6 - C_4\left(\frac{\rho\sigma}{C_{44}}\right)\bar{\omega}^4 + \bar{\omega}^2\left[\bar{\Omega}^4 - C_3\bar{\Omega}^2\left(\frac{\rho\sigma}{C_{44}}\right) - C_2\left(\frac{\rho\sigma}{C_{44}}\right)^2 - C_1\left(\frac{\rho\sigma}{C_{44}}\right)^3\right] = 0, \tag{4.40}
$$

where $\bar{\omega} = \omega\sqrt{\rho\sigma/C44}$, $\bar{\Omega} = \Omega\sqrt{\rho\sigma/C44}$, $\bar{k} = k\sqrt{\sigma}$, and C_1, C_2, C_3 and C_4 are given as

$$
\begin{aligned}
C_1 = & -c_1^2c_2^2c_6^2k^6l_1^4 + 2c_2^2c_3^2c_6^2k^6l_1^2l_3^2 + c_3^4c_6^2k^6l_1^2l_3^2 - c_1^2c_5^2c_6^2k^6l_1^2l_3^2 \\
& - c_2^2c_5^2c_6^2k^6l_3^4 - c_2^2c_4^2k^4l_1^4\nu^* - c_1^2c_4^2k^4l_1^2l_3^2\nu^* + 2c_2^2c_4^2k^4l_1^2l_3^2\nu^* \\
& + 2c_3^2c_4^2k^4l_1^2l_3^2\nu^* - c_4^2c_5^2k^4l_1^2l_3^2\nu^* - c_2^2c_4^2k^4l_3^4\nu^* - c_1^2c_2^2k^4l_1^4\xi \\
& + 2c_2^2c_3^2k^4l_1^2l_3^2\xi + c_3^4k^4l_1^2l_3^2\xi - c_1^2c_5^2k^4l_1^2l_3^2\xi - c_2^2c_5^2k^4l_3^4\xi \\
& + \big(c_1^2c_6^2k^4l_1^2 + c_2^2c_6^2k^4l_1^2 + c_2^2c_6^2k^4l_3^2 + c_5^2c_6^2k^4l_3^2 + c_4^2k^2l_1^2\nu^* \\
& + c_4^2k^2l_3^2\nu^* + c_1^2k^2l_1^2\xi + c_1^2k^2l_3^2\xi + c_2^2k^2l_1^2\xi + c_2^2k^2l_3^2\xi \\
& + c_5^2k^2l_3^2\xi\big)\Omega^2 + \big(-c_6^2k^2 - \xi\big)\Omega^4, \\
C_2 = & \; c_1^2c_6^2k^4l_1^2 + c_2^2c_6^2k^4l_1^2 - c_1^2c_2^2k^4l_1^4 + c_2^2c_6^2k^4l_3^2 + c_5^2c_6^2k^4l_3^2 \\
& + 2c_2^2c_3^2k^4l_1^2l_3^2 + c_3^4k^4l_1^2l_3^2 - c_1^2c_5^2k^4l_1^2l_3^2 - c_2^2c_5^2k^4l_3^4 + c_4^2k^2l_1^2\nu^* \\
& + c_4^2k^2l_3^2\nu^* + c_1^2k^2l_1^2\xi + c_2^2k^2l_1^2\xi + c_2^2k^2l_3^2\xi + c_5^2k^2l_3^2\xi, \\
C_3 = & -6c_6^2k^2 + c_1^2k^2l_1^2 + c_2^2k^2l_1^2 + c_2^2k^2l_3^2 + c_5^2k^2l_3^2 - 6\xi, \\
C_4 = & -c_6^2k^2 + c_1^2k^2l_1^2 + c_2^2k^2l_1^2 + c_2^2k^2l_3^2 + c_5^2k^2l_3^2 - \xi - 6\Omega^2,
\end{aligned}
$$

Now, making the constants C_1, C_2, C_3 and C_4 dimensionless

$$
C_4\left(\frac{\rho\sigma}{C_{44}}\right) = \left[-c_6^2k^2 + c_1^2k^2l_1^2 + c_2^2k^2l_1^2 + c_2^2k^2l_3^2 + c_5^2k^2l_3^2 - \xi - 6\Omega^2\right]\left(\frac{\rho\sigma}{C_{44}}\right). \tag{4.41}
$$

Using the defined constants of Eq. (4.31), Eq. (4.41) yields

$$
C_4\left(\frac{\rho\sigma}{C_{44}}\right) = -\bar{\alpha}_o\bar{k}^2 + \bar{\alpha}_1\bar{k}^2l_1^2 + \bar{k}^2l_1^2 + \bar{k}^2l_3^2 + \bar{\alpha}_2\bar{k}^2l_3^2 - \bar{\alpha}_3 - 6\bar{\Omega}^2, \tag{4.42}
$$

Similarly,

$$
\begin{aligned}
C_3\left(\frac{\rho\sigma}{C_{44}}\right) &= \big[-6c_6^2k^2 + c_1^2k^2l_1^2 + c_2^2k^2l_1^2 + c_2^2k^2l_3^2 + c_5^2k^2l_3^2 \\
&\qquad - 6\xi\big]\left(\frac{\rho\sigma}{C_{44}}\right), \\
&= \big(-6\bar{\alpha}_o\bar{k}^2 + \bar{\alpha}_1\bar{k}^2l_1^2 + \bar{k}^2l_1^2 + \bar{k}^2l_3^2 + \bar{\alpha}_2\bar{k}^2 - 6\bar{\alpha}_3\big),
\end{aligned} \tag{4.43}
$$

$$
\begin{aligned}
C_2\left(\frac{\rho\sigma}{C_{44}}\right)^2 &= \big[c_1^2c_6^2k^4l_1^2 + c_2^2c_6^2k^4l_1^2 - c_1^2c_2^2k^4l_1^4 + c_2^2c_6^2k^4l_3^2 \\
&\quad + c_5^2c_6^2k^4l_3^2 + 2c_2^2c_3^2k^4l_1^2l_3^2 + c_3^4k^4l_1^2l_3^2 - c_1^2c_5^2k^4l_1^2l_3^2 \\
&\quad - c_2^2c_5^2k^4l_3^4 + c_4^2k^2l_1^2\nu^* + c_4^2k^2l_3^2\nu^* + c_1^2k^2l_1^2\xi \\
&\quad + c_2^2k^2l_1^2\xi + c_2^2k^2l_3^2\xi + c_5^2k^2l_3^2\xi\big]\left(\frac{\rho\sigma}{C_{44}}\right)^2, \\
&= \big(\bar{\alpha}_4\bar{k}^4l_1^2 + \bar{\alpha}_o\bar{k}^4l_1^2 - \bar{\alpha}_1\bar{k}^4l_1^4 + \bar{\alpha}_o\bar{k}^4l_3^2 + \bar{\alpha}_5\bar{k}^4l_3^2 \\
&\quad + 2\bar{\alpha}_6\bar{k}^4l_1^2l_3^2 + \bar{\alpha}_7\bar{k}^4l_1^2l_3^2 + \bar{\beta}^2\bar{k}^2l_1^2 + \bar{\beta}^2\bar{k}^2l_3^2 + \bar{\alpha}_8\bar{k}^4l_1^2l_3^2 \\
&\quad - \bar{\alpha}_2\bar{k}^4l_3^4 + \bar{\alpha}_1\bar{\alpha}_3\bar{k}^2l_1^2 + \bar{\alpha}_3\bar{k}^2l_1^2 + \bar{\alpha}_3\bar{k}^2l_3^2 + \bar{\alpha}_2\bar{\alpha}_3\bar{k}^2l_3^2\big).
\end{aligned} \tag{4.44}
$$

$$
\begin{aligned}
C_1\left(\frac{\rho\sigma}{C_{44}}\right)^3 &= \big[-c_1^2c_2^2c_6^2k^6l_1^4 + 2c_2^2c_3^2c_6^2k^6l_1^2l_3^2 + c_3^4c_6^2k^6l_1^2l_3^2 - c_1^2c_5^2c_6^2k^6l_1^2l_3^2 \\
&\quad - c_2^2c_5^2c_6^2k^6l_3^4 - c_2^2c_4^2k^4l_1^4\nu^* - c_1^2c_4^2k^4l_1^2l_3^2\nu^* + 2c_2^2c_4^2k^4l_1^2l_3^2\nu^* \\
&\quad + 2c_3^2c_4^2k^4l_1^2l_3^2\nu^* - c_4^2c_5^2k^4l_1^2l_3^2\nu^* - c_2^2c_4^2k^4l_3^4\nu^* - c_1^2c_2^2k^4l_1^4\xi \\
&\quad + 2c_2^2c_3^2k^4l_1^2l_3^2\xi + c_3^4k^4l_1^2l_3^2\xi - c_1^2c_5^2k^4l_1^2l_3^2\xi - c_2^2c_5^2k^4l_3^4\xi \\
&\quad + \big(c_1^2c_6^2k^4l_1^2 + c_2^2c_6^2k^4l_1^2 + c_2^2c_6^2k^4l_3^2 + c_5^2c_6^2k^4l_3^2 + c_4^2k^2l_1^2\nu^* \\
&\quad + c_4^2k^2l_3^2\nu^* + c_1^2k^2l_1^2\xi + c_2^2k^2l_1^2\xi + c_2^2k^2l_3^2\xi \\
&\quad + c_5^2k^2l_3^2\xi\big)\Omega^2 + \big(-c_6^2k^2 - \xi\big)\Omega^4\big]\left(\frac{\rho\sigma}{C_{44}}\right)^3, \\
&= \big[-\bar{\alpha}_4\bar{k}^6l_1^4 + 2\bar{\alpha}_o\bar{\alpha}_6\bar{k}^6l_1^2l_3^2 + \bar{\alpha}_o\bar{\alpha}_6^2\bar{k}^6l_1^2l_3^2 - \bar{\alpha}_2\bar{\alpha}_4\bar{k}^6l_1^2l_3^2 \\
&\quad - \bar{\alpha}_5\bar{k}^6l_3^4 - \bar{\beta}^2\bar{k}^4l_1^4 - \bar{\alpha}_1\bar{\beta}^2\bar{k}^4l_1^2l_3^2 + 2\bar{\alpha}_6\bar{\beta}^2\bar{k}^4 + 2\bar{\beta}^2\bar{k}^4l_1^2l_3^2 \\
&\quad + 2\bar{\alpha}_2\bar{\beta}^2\bar{k}^4l_1^2l_3^2 - \bar{\beta}^2\bar{k}^4l_1^2l_3^2 - \bar{\alpha}_1\bar{\alpha}_3\bar{k}^4l_1^4 + 2\bar{\alpha}_6\bar{\alpha}_3\bar{k}^4l_1^2l_3^2 \\
&\quad + \bar{\alpha}_6^2\bar{\alpha}_3\bar{k}^4l_1^2l_3^2 - \bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{k}^4l_1^2l_3^2 - \bar{\alpha}_2\bar{\alpha}_3\bar{k}^4l_3^4 \\
&\quad + \big(\bar{\alpha}_4\bar{k}^4l_1^2 + \bar{\alpha}_o\bar{k}^4l_1^2 + \bar{\alpha}_o\bar{k}^4l_3^2 + \bar{\alpha}_5\bar{k}^4l_3^2 + \bar{\beta}^2\bar{k}^2l_1^2 + \bar{\beta}^2\bar{k}^2l_3^2 \\
&\quad + 2\bar{\alpha}_1\bar{\alpha}_3\bar{k}^2 + \bar{\alpha}_3\bar{k}^2l_1^2 + \bar{\alpha}_3\bar{k}^2l_3^2 + \bar{\alpha}_2\bar{\alpha}_3\bar{k}^2l_3^2\big)\bar{\Omega}^2 \\
&\quad - \big(\bar{\alpha}_o\bar{k}^2 + \bar{\alpha}_3\big)\bar{\Omega}^4\big]\left(\frac{\rho\sigma}{C_{44}}\right)^3.
\end{aligned} \tag{4.45}
$$

The dimensionless form of Eq. (4.40) is as follows:

$$
t^3 - z_1t^2 + z_2t - z_3 = 0, \tag{4.46}
$$

where

$$z_1 = -\bar{\alpha_o}\bar{k}^2 + \bar{\alpha_1}\bar{k}^2 l_1^2 + \bar{k}^2 l_1^2 + \bar{k}^2 l_3^2 + \bar{\alpha_2}\bar{k}^2 l_3^2 - \bar{\alpha_8} - 6\bar{\Omega}^2, \tag{4.47}$$

$$\begin{aligned} z_2 = {} & \bar{\Omega}^4 - \bar{\Omega}^2\Big(-6\bar{\alpha_o}\bar{k}^2 + \bar{\alpha_1}\bar{k}^2 l_1^2 + \bar{k}^2 l_1^2 + \bar{k}^2 l_3^2 + \bar{\alpha_2}\bar{k}^2 \\ & - 6\bar{\alpha_8}\Big) - \Big(\bar{\alpha_4}\bar{k}^4 l_1^2 + \bar{\alpha_o}\bar{k}^4 l_1^2 - \bar{\alpha_1}\bar{k}^4 l_1^4 + \bar{\alpha_o}\bar{k}^4 l_3^2 \\ & + \bar{\alpha_4}\bar{k}^4 l_3^2 + 2\bar{\alpha_5}\bar{k}^4 l_1^2 l_3^2 + \bar{\alpha_6}\bar{k}^4 l_1^2 l_3^2 + \bar{\beta}^2\bar{k}^2 l_1^2 + \bar{\beta}^2\bar{k}^2 l_3^2 \\ & + \bar{\alpha_7}\bar{k}^4 l_1^2 l_3^2 - \bar{\alpha_2}\bar{k}^4 l_3^4 + \bar{\alpha_1}\bar{\alpha_8}\bar{k}^2 l_1^2 + \bar{\alpha_8}\bar{k}^2 l_1^2 + \bar{\alpha_8}\bar{k}^2 l_3^2 \\ & + \bar{\alpha_2}\bar{\alpha_8}\bar{k}^2 l_3^2\Big), \end{aligned} \tag{4.48}$$

$$\begin{aligned} z_3 = {} & -\bar{\alpha_4}\bar{k}^6 l_1^4 + 2\bar{\alpha_o}\bar{\alpha_5}\bar{k}^6 l_1^2 l_3^2 + \bar{\alpha_o}\bar{\alpha_5^2}\bar{k}^6 l_1^2 l_3^2 - \bar{\alpha_2}\bar{\alpha_4}\bar{k}^6 l_1^2 l_3^2 \\ & - \bar{\alpha_4}\bar{k}^6 l_3^4 - \bar{\beta}^2\bar{k}^4 l_1^4 - \bar{\alpha_1}\bar{\beta}^2\bar{k}^4 l_1^2 l_3^2 + 2\bar{\alpha_5}\bar{\beta}^2\bar{k}^4 l_1^2 l_3^2 \\ & + 2\bar{\beta}^2\bar{k}^4 l_1^2 l_3^2 + 2\bar{\alpha_2}\bar{\beta}^2\bar{k}^4 l_1^2 l_3^2 - \bar{\beta}^2\bar{k}^4 l_1^2 l_3^2 - \bar{\alpha_1}\bar{\alpha_8}\bar{k}^4 l_1^4 \\ & + 2\bar{\alpha_5}\bar{\alpha_8}\bar{k}^4 l_1^2 l_3^2 + \bar{\alpha_5^2}\bar{\alpha_8}\bar{k}^4 l_1^2 l_3^2 - \bar{\alpha_1}\bar{\alpha_2}\bar{\alpha_8}\bar{k}^4 l_1^2 l_3^2 - \bar{\alpha_2}\bar{\alpha_8}\bar{k}^4 l_3^4 \\ & + \Big(\bar{\alpha_4}\bar{k}^4 l_1^2 + \bar{\alpha_o}\bar{k}^4 l_1^2 + \bar{\alpha_o}\bar{k}^4 l_3^2 + \bar{\alpha_4}\bar{k}^4 l_3^2 + \bar{\beta}^2\bar{k}^2 l_1^2 + \bar{\beta}^2\bar{k}^2 l_3^2 \\ & + 2\bar{\alpha_1}\bar{\alpha_8}\bar{k}^2 + \bar{\alpha_8}\bar{k}^2 l_1^2 + \bar{\alpha_8}\bar{k}^2 l_3^2 + \bar{\alpha_2}\bar{\alpha_8}\bar{k}^2 l_3^2\Big)\bar{\Omega}^2 \\ & - \Big(\bar{\alpha_o}\bar{k^2} + \bar{\alpha_8}\Big)\bar{\Omega}^4. \end{aligned} \tag{4.49}$$

where the values of $\bar{\alpha_o}$, $\bar{\alpha_1}$, $\bar{\alpha_2}$, $\bar{\alpha_4}$, $\bar{\alpha_4}$, $\bar{\alpha_5}$, $\bar{\alpha_6}$, $\bar{\alpha_7}$ and $\bar{\alpha_8}$ are defined as

$$\bar{\alpha_o} = \frac{\alpha}{\sigma C_{44}}, \qquad \bar{\alpha_1} = \frac{C_{11}}{C_{44}}, \qquad \bar{\alpha_2} = \frac{C_{33}}{C_{44}},$$

$$\bar{\alpha_3} = \frac{\xi}{C_{44}}, \qquad \bar{\alpha_4} = \bar{\alpha_1}\bar{\alpha_o}, \qquad \bar{\alpha_5} = \bar{\alpha_2}\bar{\alpha_o},$$

$$\bar{\alpha_6} = \frac{C_{13}}{C_{44}}, \qquad \bar{\alpha_7} = \bar{\alpha_5^2}, \qquad \bar{\alpha_8} = \bar{\alpha_2}\bar{\alpha_1}$$

The determinant A of coefficient of Eqs. (4.32), (4.35) and (4.37)
after substituting $\omega = ck$ is as following.

$$A = \begin{vmatrix} c_1^2k^2l_1^2 + c_2^2k^2l_3^2 - c^2k^2 - \Omega^2 & (c_3^2 + c_2^2)k^2 l_1 l_3 - 2i\Omega ck & -ic_4^2 k l_1 \\ (c_2^2 + c_3^2)k^2 l_1 l_3 - 2i\Omega ck & c_2^2k^2l_1^2 + c_5^2k^2l_3^2 - c^2k^2 - \Omega^2 & -c_4^2 ikl_3 \\ \nu^* ikl_1 & \nu^* ikl_3 & c_6^2k^2 + \xi^* - c^2k^2 \end{vmatrix}.$$

Solving the determinant of matrix A for non-trivial solution, that is, $A = 0$. Now using commands of *Mathematica*, that is, Collect[complexExpand@Re[T],c] and Collect[complexExpand@Im[T],c], which gives A_0, A_1, A_2, A_3, A_4 respectively, where A_0, A_2, A_4

is for real coefficient, while A_1 and A_3 correspond to the imaginary coefficients.

$$
\begin{aligned}
A_0 &= c_1^2c_6^2k^6{l_1}^2 + c_2^2c_6^2k^6{l_1}^2 - c_1^2c_2^2k^6l_1^4 + c_2^2c_6^2k^6l_3^2 + c_5^2c_6^2k^6l_3^2 \\
&\quad + 2c_2^2c_3^2k^6l_1^2l_3^2 + c_3^4k^6l_1^2l_3^2 - c_1^2c_5^2k^6l_1^2l_3^2 - c_2^2c_5^2k^6l_3^4 + c_4^2k^4{l_1}^2\nu \\
&\quad + c_4^2k^4l_3^2\nu + c_1^2k^4l_1^2\xi + c_2^2k^4l_1^2\xi + c_2^2k^4l_3^2\xi + c_5^2k^4l_3^2\xi - 6c_6^2k^4\Omega^2 \\
&\quad + c_1^2k^4l_1^2\Omega^2 + c_2^2k^4l_1^2\Omega^2 + c_2^2k^4l_3^2\Omega^2 + c_5^2k^4l_3^2\Omega^2 - 6k^2\xi\Omega^2 - k^2\Omega^4, \\
A_1 &= -4c_2^2c_6^2k^5l_1l_3\Omega - 4c_3^2c_6^2k^5l_1l_3\Omega - 4c_4^2k^3l_1l_3\nu\Omega - 4c_2^2k^3l_1l_3\xi\Omega \\
&\quad - 4c_3^2k^3l_2l_3\xi\Omega, \\
A_2 &= -c_1^2c_2^2c_6^2k^6l_1^4 + 2c_2^2c_3^2c_6^2k^6l_1^2l_3^2 + c_3^4c_6^2k^6l_1^2l_3^2 - c_1^2c_5^2c_6^2k^6l_1^2l_3^2 \\
&\quad + c_2^2c_5^2c_6^2k^6l_3^4 - c_2^2c_4^2k^4l_1^4\nu - c_1^2c_4^2k^4l_1^2l_3^2\nu + 2c_2^2c_4^2k^4l_1^2l_3^2\nu + c_3^2c_4^2k^4l_1^2l_3^2\nu \\
&\quad - c_4^2c_5^2k^4l_1^2l_3^2\nu - c_2^2c_4^2k^4l_3^4\nu - c_1^2c_2^2k^4l_1^4\xi + 2c_2^2c_3^2k^4l_1^2l_3^2\xi + c_3^4k^4l_1^2l_3^2\xi \\
&\quad - c_1^2c_5^2k^4l_1^2l_3^2\xi - c_2^2c_5^2k^4l_3^4\xi + c_2^2c_6^2k^4l_1^2\Omega^2 + c_2^2c_6^2k^4l_3^2\Omega^2 + c_5^2c_6^2k^4l_3^2\Omega^2 \\
&\quad + c_4^2k^2l_1^2\nu\Omega^2 + c_4^2k^2l_3^2\nu\Omega^2 + c_1^2k^2l_1^2\xi\Omega^2 + c_2^2k^2l_1^2\xi\Omega^2 + c_2^2k^2l_3^2\xi\Omega^2 \\
&\quad + c_5^2k^2l_3^2\xi\Omega^2 - c_6^2k^2\Omega^4 - \xi\Omega^4, \\
A_3 &= -4c_2^2k^5l_1l_3\Omega - 4c_3^2k^5l_1l_3\Omega, \\
A_4 &= -c_6^2k^6 + c_1^2k^6l_1^2 + c_2^2k^6l_1^2 + c_2^2k^6l_3^2 + c_5^2k^6l_3^2 - k^4\xi - 6k^4\Omega^2,
\end{aligned}
\tag{4.50}
$$

Now, dividing the determinant A by k^4 and collecting the values of c's, we get

$$
\begin{aligned}
&\frac{c^6k^2\rho^3\sigma}{C_{44}^3} + c^4\left(\frac{k^2\alpha\rho^2}{C_{44}^3} + \frac{\xi\rho^2}{C_{44}^3} + \frac{C_{11}k^2l_1^2\rho^2\sigma}{C_{44}^3} + \frac{k^2l_1^2\rho^2\sigma}{C_{44}^2} + \frac{C_{33}k^2l_3^2\rho^2\sigma}{C_{44}^3}\right. \\
&\left. + \frac{k^2l_3^2\rho^2\sigma}{C_{44}^2} - \frac{6\rho^3\sigma\Omega^2}{C_{44}^3}\right) + c^2\left(\frac{-C_{11}k^2l_1^2\alpha\rho}{C_{44}^3} - \frac{k^2{l_1}^2\alpha\rho}{C44^2} - \frac{C_{33}k^2l_3^2\alpha\rho}{C_{44}^3}\right. \\
&- \frac{k^2{l_3}^2\alpha\rho}{C44^2} + \frac{l_1^2\beta^2\rho}{C_{44}^3} + \frac{l_3^2\beta^2\rho}{C_{44}^3} - \frac{C_{11}l_1^2\xi\rho}{C_{44}^3} - \frac{l_1^2\xi\rho}{C_{44}^2} - \frac{C_{33}l_3^2\xi\rho}{C_{44}^3} - \frac{l_3^2\xi\rho}{C_{44}^2} \\
&- \frac{C_{11}k^2l_1^4\rho\sigma}{C_{44}^2} + \frac{C_{13}^2k^2l_1^2l_3^2\rho\sigma}{C_{44}^3} - \frac{C_{11}C_{33}k^2l_1^2l_3^2\rho\sigma}{C_{44}^3} + \frac{2C_{13}k^2l_1^2l_3^2\rho\sigma}{C_{44}^2} \\
&- \frac{C_{33}k^2l_3^4\rho\sigma}{C_{44}^2} + \frac{6\alpha\rho^2\Omega^2}{C_{44}^3} + \frac{6\xi\rho^2\Omega^2}{C_{44}^3k^2} + \frac{C_{11}l_1^2\rho^2\sigma\Omega^2}{C_{44}^3} + \frac{l_1^2\rho^2\sigma\Omega^2}{C_{44}^2}
\end{aligned}
\tag{4.51}
$$

$$+\frac{C_{33}l_3^2\rho^2\sigma\Omega^2}{C_{44}^3}+\frac{l_3^2\rho^2\sigma\Omega^2}{C_{44}^2}-\frac{\rho^3\sigma\Omega^4}{C_{44}^3k^2}\Big)+\frac{C_{11}k^2l_1^4\alpha}{C_{44}^2}-\frac{C_{13}^2k^2l_1^2l_3^2\alpha}{C_{44}^3}$$
$$+\frac{C_{11}C_{33}k^2l_1^2l_3^2\alpha}{C_{44}^3}+\frac{2C_{13}k^2l_1^2l_3^2\alpha}{C_{44}^2}+\frac{C_{33}k^2l_3^4\alpha}{C_{44}^2}+\frac{l_1^4\beta^2}{C_{44}^2}-\frac{C_{11}l_1^2l_3^2\beta^2}{C_{44}^3}$$
$$+\frac{2C_{13}l_1^2l_3^2\beta^2}{C_{44}^3}-\frac{C_{33}l_1^2l_3^2\beta^2}{C_{44}^3}+\frac{2l_1^2l_3^2\beta^2}{C_{44}^2}-\frac{l_3^4\beta^2}{C_{44}^2}+\frac{C_{11}l_1^4\xi}{C_{44}^2}-\frac{C_{13}^2l_1^2l_3^2\xi}{C_{44}^3}$$
$$+\frac{C_{11}C_{33}l_1^2l_3^2\xi}{C_{44}^3}-\frac{2C_{13}l_1^2l_3^2\xi}{C_{44}^2}+\frac{C_{33}l_3^4\xi}{C_{44^2}}+\frac{C_{11}l_1^2\alpha\rho\Omega^2}{C_{44}^3}-\frac{l_1^2\alpha\rho\Omega^2}{C_{44}^2}$$
$$-\frac{C_{33}l_3^2\alpha\rho\Omega^2}{C_{44}^3}-\frac{l_3^2\alpha\rho\Omega^2}{C_{44}^2}+\frac{l_1^2\beta^2\rho\Omega^2}{C_{44}^3k^2}-\frac{C_{11}l_1^2\xi\rho\Omega^2}{C_{44}^3k^2}-\frac{l_1^2\xi\rho\Omega^2}{C_{44}^2k^2}$$
$$-\frac{C_{33}l_3^2\xi\rho\Omega^2}{C_{44}^3k^2}-\frac{l_3^2\xi\rho\Omega^2}{C_{44}^2k^2}+\frac{\alpha\rho^2\Omega^4}{C_{44}^3k^2}+\frac{\xi\rho^2\Omega^4}{C_{44}^3k^4} \tag{4.52}$$

The imaginary part of the determinant is as

$$c\Big(\frac{4C_{13}kl_1l_3\alpha\rho\Omega}{C_{44}^3}+\frac{4kl_1l_3\alpha\rho\Omega}{C_{44}^2}-\frac{4l_1l_3\beta^2\rho\Omega}{C_{44}^3k^2}+\frac{4C_{13}l_1l_3\xi\rho\Omega}{C_{44}^3k}$$
$$+\frac{4l_1l_3\xi\rho\Omega}{C_{44}^2k}\Big)-c^3\Big(\frac{4C_{13}kl_1l_3\rho^2\sigma\Omega}{C_{44}^3}+\frac{4kl_1l_3\rho^2\sigma\Omega}{C_{44}^2}\Big). \tag{4.53}$$

The dimensionless equation is as following

$$\bar{c}^6\bar{k}^2+\bar{c}^4\bar{A}_4+\bar{c}^2\bar{A}_2+\bar{A}_0+I(\bar{c}\bar{A}_1+\bar{c}^3\bar{A}_3)=0, \tag{4.54}$$

where

$$\begin{aligned}\bar{A}_0 &= \gamma_1\bar{\alpha}1_1^4\bar{k}^2-\gamma_3^2\bar{\alpha}\bar{k}^2+\gamma_1\gamma_2\bar{\alpha}l_1^2l_3^2\bar{k}^2-\gamma_3\bar{\alpha}l_1^2l_3^2\bar{k}^2+\gamma_2\bar{\alpha}l_3^4\bar{k}^2\\ &\quad+\bar{\beta}^2l_1^4-\gamma_1\bar{\beta}^2l_1^2l_3^2+2\gamma_3\bar{\beta}^2l_1^2l_3^2-\gamma_2\bar{\beta}^2l_1^2l_3^2+2\bar{\beta}^2l_1^2l_3^2-\bar{\beta}^2l_3^2\\ &\quad+\gamma_1\bar{\xi}\bar{l}_1^4-\gamma_3^2\bar{\xi}l_1^2l_3^2+\gamma_1\gamma_2\bar{\xi}l_1^2l_3^2-2\gamma_3\bar{\xi}l_1^2l_3^2+\gamma_2\bar{\xi}l_3^4-\gamma_1\bar{\Omega}^2\bar{\alpha}l_1^2\\ &\quad-\bar{\Omega}^2\bar{\alpha}l_1^2-\gamma_2\bar{\Omega}^2\bar{\alpha}l_3^2-\bar{\Omega}^2\bar{\alpha}l_3^2+\frac{\bar{\beta}^2\bar{\Omega}^2l_1^2}{\bar{k}^2}-\frac{\gamma_1\bar{\xi}\bar{\Omega}^2l_1^2}{\bar{k}^2}-\frac{\bar{\xi}\bar{\Omega}^2l_1^2}{\bar{k}^2}\\ &\quad-\gamma_2\bar{\xi}\bar{\Omega}^2l_3^2-\frac{\gamma_2\bar{\xi}\bar{\Omega}^2l_3^2}{\bar{k}^2}-\frac{\bar{\xi}\bar{\Omega}^2l_3^2}{\bar{k}^2}+\frac{\bar{\alpha}\bar{\Omega}^4}{\bar{k}}^2+\frac{\bar{\xi}\bar{\Omega}^4}{\bar{k}^4},\\ \bar{A}_1 &= 8\bar{\Omega}\bar{\alpha}\bar{k}l_1l_3-\frac{4\bar{\beta}^2\bar{\Omega}l_1l_3}{\bar{k}}+\frac{4\bar{\xi}\bar{\Omega}\gamma_3l_1l_3}{\bar{k}}+\frac{4\bar{\xi}\bar{\Omega}l_1l_3}{\bar{k}},\end{aligned} \tag{4.55}$$

$$
\begin{aligned}
\bar{A}_2 &= -\gamma_1\bar{\alpha}l_1^2\bar{k}^2 - \bar{\alpha}l_1^2\bar{k}^2 - \gamma_2\bar{\alpha}l_{3s}^2\bar{k}^2 - \bar{\alpha}l_3^2\bar{k}^2 + \bar{\beta}l_1^2 + \bar{\beta}l_3^2 \\
&\quad + \gamma_1\bar{\xi}l_1^2 - \bar{\xi}l_1^2 - \gamma_2\bar{\xi}l_3^2 - \bar{\xi}l_3^2 - \gamma_1 l_1^4\bar{k}^4 + \gamma_3^2 l_1^2 l_3^2\bar{k}^2 + \gamma_1\gamma_2 l_1^2 l_3^2\bar{k}^2 \\
&\quad + 2\gamma_3 l_1^2 l_3^2\bar{k}^2 - \gamma_2 l_3^4\bar{k}^2 + 6\bar{\Omega}^2\bar{\alpha} + \frac{6\bar{\Omega}^2\bar{\alpha}}{\bar{k}^2} + \gamma_1\bar{\Omega}^2 l_1^2 + \bar{\Omega}^2 l_1^2 \\
&\quad + \gamma_2\bar{\Omega}^2 l_3^2 + \bar{\Omega}^2 l_3^2 - \frac{\bar{\Omega}^4}{\bar{k}^2}, \\
\bar{A}_3 &= -4\gamma_3\bar{\Omega}l_1 l_3\bar{k} - 4\bar{\Omega}l_1 l_3\bar{k}, \\
\bar{A}_4 &= \bar{\alpha}\bar{k}^2 + \bar{\xi} + \gamma_1 1_1^2\bar{k}^2 + 1_1^2\bar{k}^2 + \gamma_2 1_3^2\bar{k}^2 + 1_3^2\bar{k}^2 - 6\bar{\Omega}^2,
\end{aligned}
$$

where γ_1, γ_2, γ_3 are defined as

$$
\begin{aligned}
\gamma_1 &= \frac{C_{11}}{C_{44}}, & \gamma_2 &= \frac{C_{33}}{C_{44}}, & \gamma_3 &= \frac{C_{13}}{C_{44}}, \\
\bar{\alpha} &= \frac{\alpha}{\sigma C_{44}}, & \bar{\beta} &= \frac{\beta}{C_{44}}, & \bar{\xi} &= \frac{\xi}{C_{44}}, \\
\bar{\Omega} &= \Omega\sqrt{\rho\sigma/C_{44}}, & \bar{c} &= c\sqrt{\frac{\rho}{C_{44}}}
\end{aligned}
\tag{4.56}
$$

4.2 Results and discussion of graphs

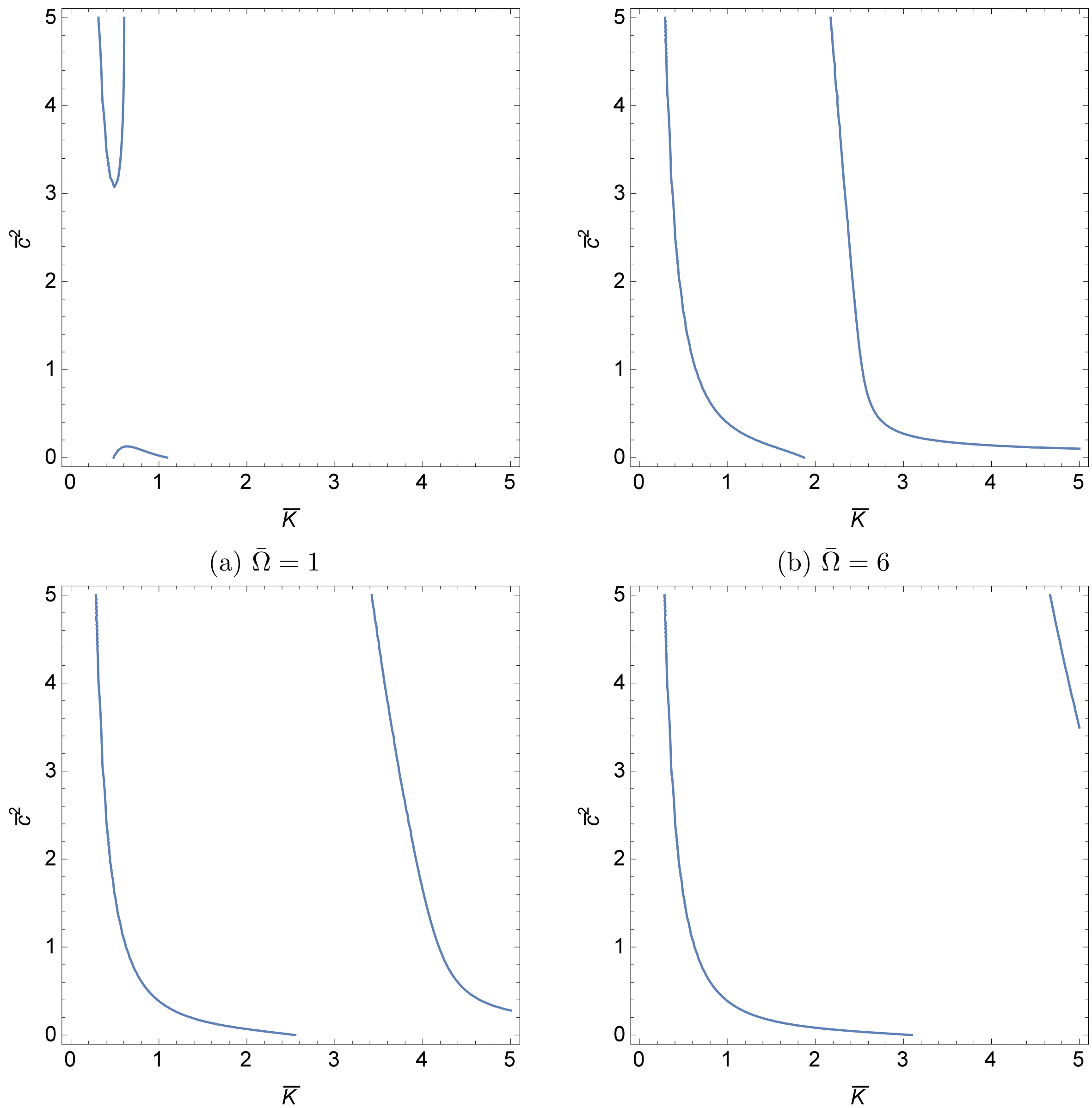
In this section graphs are plotted for Eq. (4.54) by taking different values from research papers as given below.

4.2.1 Graphs for real values

Following values of material parameters and values of void parameters [15] are used in *MATHEMATICA*.

Symbol	Value	Units	Symbol	Value	Units
c_{11}	1.628×10^{11}	Nm^{-2}	c_{13}	0.508×10^{11}	Nm^{-2}
c_{33}	1.562×10^{11}	Nm^{-2}	c_{44}	0.385×10^{11}	Nm^{-2}
α	3.688×10^{-5}	N	β	3.656×10^{-5}	N
ξ	1.475×10^{10}	m^2	σ	0.162	m^2
l_1	$\sqrt{0.6}$		l_3	$\sqrt{0.4}$	

Table 4.1: Values of material parameters for an elastic material with voids.



(a) $\bar{\Omega} = 1$

(b) $\bar{\Omega} = 6$

(c) $\bar{\Omega} = 10$

(d) $\bar{\Omega} = 14$

Figure 4.1: Plots of dimensionless squared wave speed against the wave number.

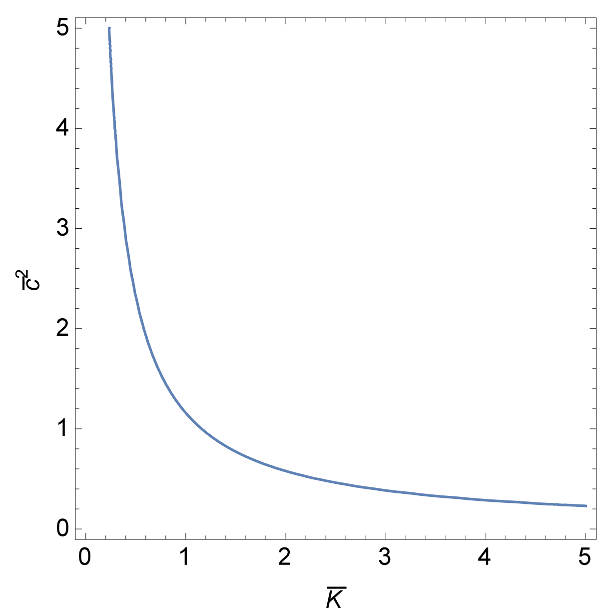
Graphs are plotted between wave speed $\bar{c}^2$ and wave number $\bar{k}$ where $\bar{k}$ is taken along x-axis and $\bar{c}^2$ is along y-axis respectively. Using $\bar{\Omega}$=6, two real waves has been observed, which are coupled in the case when we have rotation and voids. It is noticed that when $\bar{\Omega}$ is greater than zero the waves dispersed in the surroundings of wave number $\bar{k}$.

4.2.2 Graph for imaginary value

For Figure 4.4, following values of material parameters and values of void parameters are of material Gneiss rock (dry) and used in *Mathematica*. This graph is plotted only for imaginary values of Eq. (4.54).

Symbol	Value	Units	Symbol	Value	Units
c_{11}	52×10^{9}	Nm^{-2}	c_{13}	9×10^{9}	Nm^{-2}
c_{33}	16×10^{9}	Nm^{-2}	c_{44}	11×10^{9}	Nm^{-2}
α	1.7798×10^{-4}	N	β	8.52849×10^{-4}	N
ξ	1.21960369×10^{11}	m^2	σ	0.162	m^2
l_1	$\sqrt{0.6}$		l_3	$\sqrt{0.4}$	

Table 4.2: Values of material parameters for an elastic material with voids.



$\bar{\Omega} = 3$

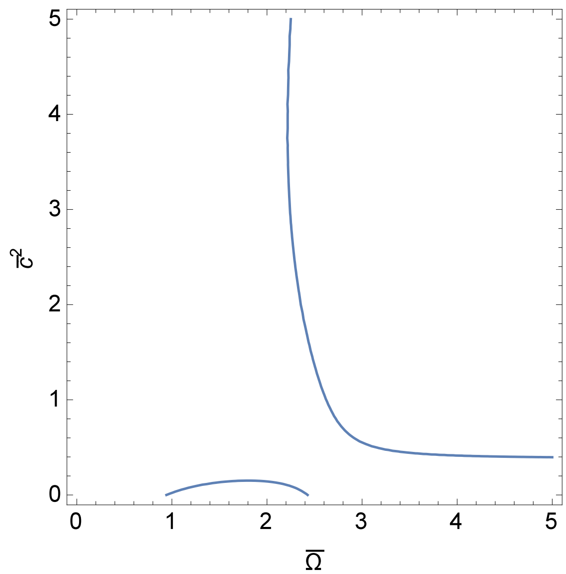
Figure 4.2: Plots of dimensionless squared wave speed against the wave number.

Here is only one real wave speed appears when graph is plotted between wave number $\bar{c}^2$ and wave speed $\bar{k}$ for imaginary values.

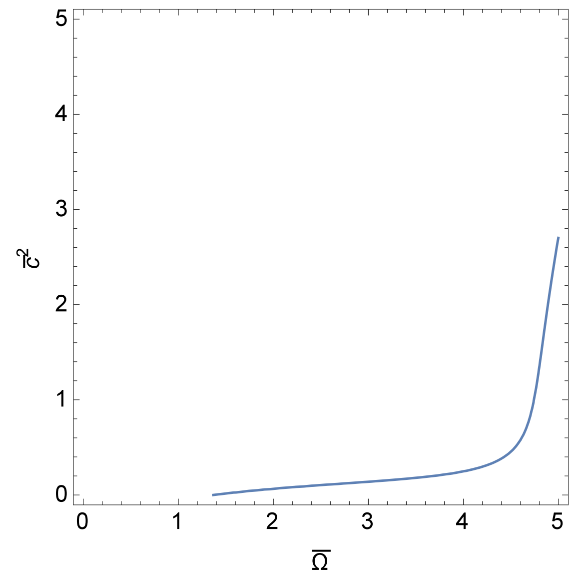
For Figure 4.3, material constants and void parameter values are given below [16] are used in *Mathematica* to plot the graph. This graph is plotted against dimensionless rotation and wave number.

Symbol	Value	Units	Symbol	Value	Units
c_{11}	3.071×10^{11}	Nm^{-2}	c_{13}	1.027×10^{11}	Nm^{-2}
c_{33}	3.581×10^{11}	Nm^{-2}	c_{44}	1.51×10^{11}	Nm^{-2}
α	8×10^{9}	N	β	10×10^{9}	N
ξ	12×10^{9}	m^2	σ	0.162	m^2
l_1	$\sqrt{0.6}$		l_3	$\sqrt{0.4}$	

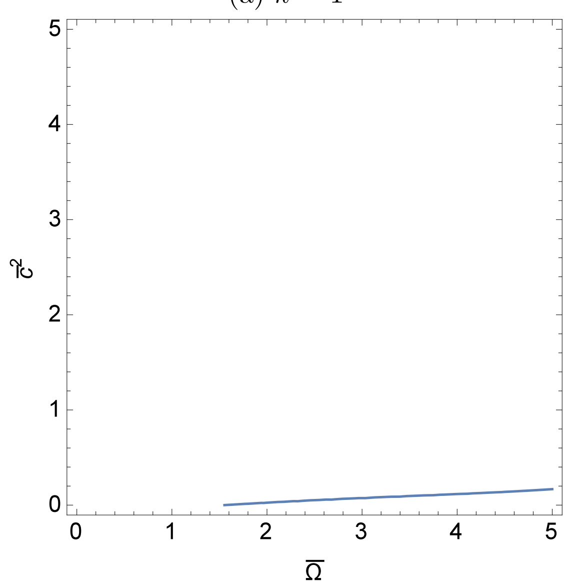
Table 4.3: Values of material parameters for an elastic material with voids.



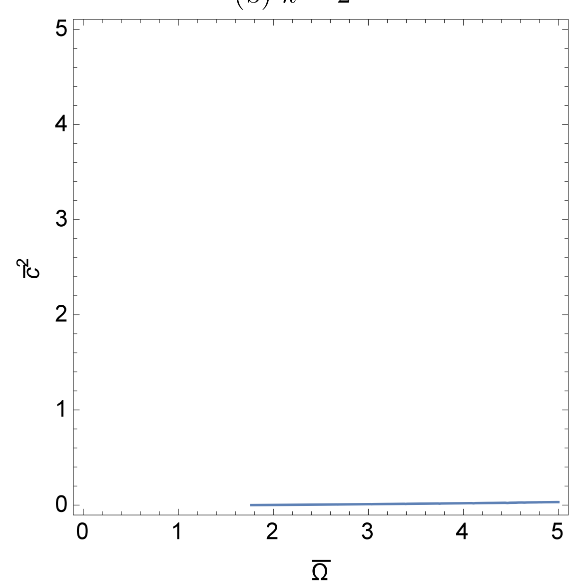
(a) $\bar{k} = 1$



(b) $\bar{k} = 2$



(c) $\bar{k} = 3$



(d) $\bar{k} = 10$

Figure 4.3: Plots of dimensionless squared wave speed against the dimensionless rotation.

Graphs are plotted between wave speed $\bar{c}^2$ and rotation $\bar{\Omega}$ where $\bar{\Omega}$ is taken along x-axis and $\bar{c}^2$ is along y-axis respectively. Using $\bar{k}$=1, two real waves has been observed, which are coupled in the case when we have rotation and voids. It is noticed that when

$\bar{k}$ is greater than one then only one real wave speed appear.

4.3 Special case $\bar{k} \to \infty$

After dividing the Eq. (4.54) by $\bar{k}^2$ and taking $\bar{k} \to \infty$, the coefficients $\bar{A}_0, \bar{A}_1, \bar{A}_2, \bar{A}_3$, and $\bar{A}_4$ takes the form

$$\begin{aligned}
\bar{A}_0 &= \bar{\alpha}\gamma_1 l_1^4 - \gamma_3^2\bar{\alpha} + \gamma_1\gamma_2\bar{\alpha}l_1^2l_3^2 - \gamma_3\bar{\alpha}l_1^2l_3^2 + \gamma_2\bar{\alpha}l_3^4,\\
\bar{A}_1 &= 8\bar{\Omega}\bar{\alpha}l_1^2l_3^2,\\
\bar{A}_2 &= -\bar{\alpha}\gamma_1 l_1^2 - \bar{\alpha}l_1^2 - \bar{\alpha}l_3^2\gamma_2 - \bar{\alpha}l_3^2 - \gamma_1 l_1^4 + \gamma_3^2 l_1^2 l_3^2 + \gamma_1\gamma_2 l_1^2 l_3^2 + 2\gamma_3 l_1^2 l_3^2 - \gamma_2 l_3^4,\\
\bar{A}_3 &= -\bar{\Omega}l_1l_3,\\
\bar{A}_4 &= \bar{\alpha}\gamma_1 l_1^2 + l_1^2 + \gamma_2 l_3^2 + l_3^2,
\end{aligned} \tag{4.57}$$

Now, using $MATHEMATICA$ commands

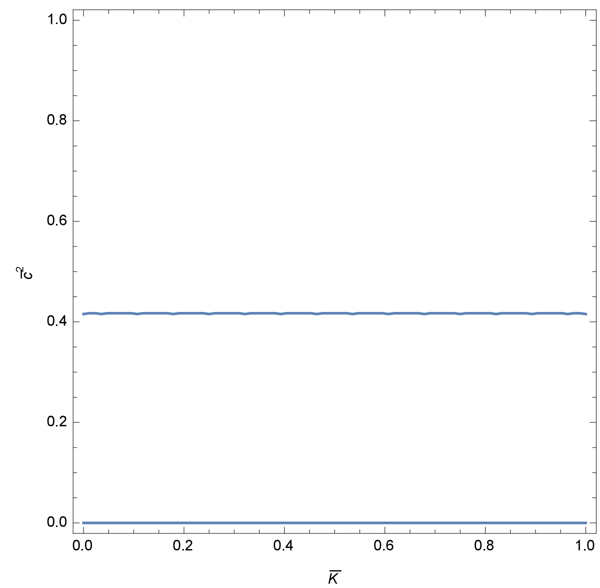
$$Limit[c^3 + A_4c^2 + A_2c + A_0 + I(A_3c^3 + A_1c), k \to \infty]$$

$$ContourPlot[c^3 + A_4c^2 + A_2c + A_0 + I(A_3c^3 + A_1c) = 0, (k, 0, 2), (c, 0, 10)]$$

The values of the voids and material parameters by [6] are given below for the case $k \to \infty$,

Symbol	Value	Units	Symbol	Value	Units
c_{11}	1.628×10^{11}	Nm^{-2}	c_{13}	0.508×10^{11}	Nm^{-2}
c_{33}	0.627×10^{11}	Nm^{-2}	c_{44}	0.770×10^{11}	Nm^{-2}
α	1.7798×10^{-4}	N	β	8.52849×10^{-4}	N
ξ	1.21960369×10^{11}	m^2	σ	0.162	m^2
l_1	$\sqrt{0.6}$		l_3	$\sqrt{0.4}$	

Table 4.4: Values of material parameters for an elastic material with voids.



$\bar{\Omega} = 3$

Figure 4.4: Plots of dimensionless squared wave speed against the wave number.

There are two real wave speeds appear which are parallel to each other when graph is plotted between wave speed $\bar{c}^2$ and wave numver $\bar{k}$.

Chapter 5

Conclusions

In this thesis, propagation of plane waves in a rotating isotropic and anisotropic material with and without voids are discussed. The summary of the results are as follows.

For isotropic material, three plane waves exist in the presence of voids two are longitudinal and one is transverse wave. Waves are coupled due to voids and rotation of the medium. In the absence of voids, coupling of the plane waves take place due to the rotation of the medium. When the rotation is absent, classical transverse waves are obtained which travel without coupling and do not affected by voids. On the other hand, the longitudinal waves relative to change in volume and void volume fraction are coupled.

The hexagonal crystal system is chosen to give values to the constants in anisotropic material with voids. In this case, two real wave speeds are obtained. Numerical investigation revealed the fact that for unit rotation and for the wave with smallest wave speed started to disappear and large gap between the wave speeds has been observed. For imaginay solution only one real wave speed exist. When $\bar{k} = 1$, two real wave speeds appear and for all values greater than one there is only one real wave speed. A special case when $\bar{k} \to \infty$, two real wave speed occur which are parallel to each other.

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