Runge Phenomenon

by

Rabia Saeed

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MASTER'S THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Ms. Rabia Saeed, Regn No. 00000278437 Titled: Runge Phenomenon be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: Dr. Mujeeb Ur Rehman Signature: [signature]

2. Name: Dr. Muhammad Asif Farooq Signature: [signature]

External Examiner: Prof. M Afzal Rana Signature: M. A. Rana.

Supervisor's Name: Prof. Faiz Ahmad Signature: Faiz Ahmad

[signature]
Head of Department

3-9-2020
Date

COUNTERSINGED

Date: 03-09-2020

[signature]
Dean/Principal

Dedicated

to

My Beloved Parents

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I am deeply obliged to ALMIGHTY ALLAH, the most beneficent and merciful. I do obeisance before ALLAH who granted me the best health and opportunity to be a student of such passionate and competent teachers. All of this was possible by virtue of Holy Prophet MUHAMMAD (S.A.W), whose principles enlighten us at every stage of life.

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Abstract

In this thesis we have studied phenomenon associated with interpolation of equally spaced data known as Runge phenomenon. This phenomenon was first observed by Carl David Tolme Runge which deals with the oscillatory behavior of a higher degree interpolating polynomial near the end points of equally spaced data.
In this dissertation we have discussed the conditions for the occurrence or absence of this phenomenon and illustrated our results graphically. A simple proof for the case of Runge function on $[-a, a]$ has been discussed. A simple formula has been found to calculate the point, for a fixed a, beyond which Runge phenomenon makes its appearance.
The role of Chebyshev polynomials in approximation theory has been briefly discussed. To further improve the convergence rate and the reduction of an error obtained through approximation of an interpolating polynomial and non-polynomial function different types of notions like Fourier series and Chebyshev series were discussed.

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Chapter 1

Introduction

1.1 Interpolation

1. Interpolation is the process of estimation in which we estimate the value of function at new points given set of values at a set of data points.
2. Interpolation can also be defined as the process of fitting a function through given data points.

1.1.1 Polynomial Interpolation

Polynomial interpolation is a method of finding a polynomial $P(x)$ for a function $f(x)$ that agrees with $f(x)$ at specified points. More precisely, if $f(x)$ is a function defined on the interval $[c, d]$ then $P(x)$ is an interpolation polynomial of $f(x)$ if

$$P(x_k) = f(x_k) \qquad k = 0, 1, 2, ...n.$$

Reasons for using Polynomial Interpolation

1. Polynomials are preferred because they have the property to approximate any continuous function.
2. Definite integrals and derivatives are easy to determine.

1.1.2 Weierstrass Approximation Theorem

If f is a continuous real-valued function on $[c, d]$ and if any $\epsilon > 0$ is given, then there exists a polynomial P on $[c, d]$ such that

$$|f(x) - P(x)| < \epsilon$$

for all x in $[c, d]$. In other words, let $f(x)$ be continuous and defined on $[c, d]$ then there exists a sequence of polynomial $P_n(x)$ of degree n such that

$$\lim_{n\to\infty} P_n(x) = f(x).$$

1.1.3 Example

We determine a polynomial of degree one i.e. $P_1(x)$ that passes through the distinct points (x_0, f_0) and (x_1, f_1) such that $P_1(x_0) = f_0$, $P_1(x_1) = f_1$.

As we know that polynomial of degree one is generally expressed as:

$$P_1(x) = d_0 + d_1 x. \tag{1.1}$$

To find d_0 and d_1 we apply given conditions. So,

$$P_1(x_0) = d_0 + d_1 x_0$$

$$\Rightarrow f_0 = d_0 + d_1 x_0. \tag{1.2}$$

Similarly

$$P(x_1) = f_1 = d_0 + d_1 x_1. \tag{1.3}$$

Now

$$f_1 - f_0 = d_1(x_1 - x_0)$$
$$\Rightarrow d_1 = \frac{f_1 - f_0}{x_1 - x_0}.$$

Putting value of d_1 in (1.2) and rearranging we get:

$$d_0 = f_0 - \frac{f_1 - f_0}{x_1 - x_0} x_0.$$

Putting values of d_0 and d_1 in (1.1) we get:

$$P_1(x) = f_0 - \frac{f_1 - f_0}{x_1 - x_0}x_0 + \frac{f_1 - f_0}{x_1 - x_0}x.$$

Simplification gives:

$$P_1(x) = \frac{x - x_1}{x_1 - x_0}f_0 + \frac{x - x_0}{x_1 - x_0}f_1$$
$$P_1(x) = L_0(x)f_0 + L_1(x)f_1.$$

1.1.4 Lagrange Interpolating Polynomial

$$P_1(x) = L_0(x)f_0 + L_1(x)f_1. \tag{1.4}$$

where

$$L_0 = \frac{x - x_1}{x_0 - x_1}, \qquad L_1 = \frac{x - x_0}{x_1 - x_0}. \tag{1.5}$$

$P_1(x)$ in (1.4) is known as linear Lagrange interpolating polynomial and L_0 and L_1 in (1.5) are Lagrange co-efficient polynomial.

Unique Polynomial

Note that

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_1(x_0) = 0, \quad L_1(x_1) = 1.$$

(1.4) implies that

$$P(x_0) = 1.f_0 + 0.f_1 = f_0$$

and

$$P(x_1) = 0.f_0 + 1.f_1 = f_1$$

So (1.4) is the unique polynomial of degree one.

nth Lagrange Interpolating Polynomial

Lemma 1.1.1. *If $x_0, x_1, ..., x_n$ are $n+1$ distint numbers and f is function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree n exists with*

$$P(x_k) = f(x_k) \qquad k = 0, 1, 2, ...n.$$

The polynomial is given by:

$$P(x) = \sum_{k=0}^{n} f(x_k)L_k(x) \tag{1.6}$$

where, for each $k = 0, 1, 2, 3...n$

$$L_k(x) = \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}. \tag{1.7}$$

Proof. Notice that L_k in (1.7) is a polynomial of degree n. Furthermore, for each x_k we have $L_k(x_k) = \delta_{kj}$ (Kronecker delta). It follows that the linear combination

$$P(x) = \sum_{k=0}^{n} f(x_k)L_k(x)$$

is an interpolating polynomial of degree n.

Uniqueness: Assume that there exists another interpolating polynomial r of degree at most n. Since $P(x_k) = r(x_k)$ for all $k = 0, 1, 2...n$. It follows that the polynomial $P - r$ has $n+1$ distinct zeros. However, $P-r$ is of degree at most n and by **fundamental theorem of algebra** it can have atmost n zeros; therefore $P = r$. □

Theorem 1.1.2. *Suppose* $x_0, x_1, ...x_n$ *are distinct numbers in* $[c, d]$ *and* $f \in C^{n+1}[c, d]$ *then for each* $x \in [c, d]$, *a number* $\xi(x)$ *exist in* (c, d) *such that*

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)(x - x_1)...(x - x_n) \tag{1.8}$$

where $P_n(x)$ *is the interpolating polynomial [1].*

Proof. Note that if $x = x_j$ for any $j = 0, 1, ...n$ then $f(x_j) = P_n(x_j)$, and choosing $\xi(x_j)$ arbitrarily in (c, d) yeilds (1.8).

If $x \neq x_j$, for all $j = 0, 1, ...n$, define the function r for s in $[c, d]$ by

$$r(s) = f(s) - P(s) - [f(x) - P_n(x)]\frac{(s - x_0)(s - x_1)...(s - x_n)}{(x - x_0)(x - x_1)...(x - x_n)}$$

$$= f(s) - P(s) - [f(x) - P_n(x)]\prod_{k=0}^{n}\frac{s - x_k}{x - x_k}.$$

For $s = x_j$ we have

$$r(x_j) = f(x_j) - P(x_j) - [f(x) - P_n(x)] \prod_{k=0}^{n} \frac{(x_j - x_k)}{(x - x_k)} = 0 - [f(x) - P_n(x)].0 = 0.$$

Moreover,

$$r(x) = f(x) - P(x) - [f(x) - P_n(x)] \prod_{k=0}^{n} \frac{(x - x_k)}{(x - x_k)} = f(x) - P_n(x) - [f(x) - P_n(x)] = 0.$$

Therefore, $r(x)$ is zero at $n+2$ different numbers $x, x_0, x_1, ..., x_n$. By Generalized Rolle's Theorem there exists $\xi(x)$ in $[c, d]$ s.t $r^{(n+1)}(\xi) = 0$. So,

$$0 = r^{(n+1)}(\xi) = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P - n(x)] \frac{d^{n+1}}{dt^{n+1}} [\prod_{k=0}^{n} \frac{s - x_k}{x - x_k}]_{s=\xi} \quad (1.9)$$

As $P(x)$ is a polynomial of degree at most n, so $(n+1)st$ derivative is identically zero. Also, $\prod_{k=0}^{n} \frac{s-x_k}{x-x_k}$ is a polynomial of degree n, so

$$\prod_{k=0}^{n} \frac{s - x_k}{x - x_k} = \frac{1}{\prod_{k=0}^{n}(x - x_k)} s^{n+1} + (lower\ degree\ in\ s)$$

and

$$\frac{d^{n+1}}{ds^{n+1}} \prod_{k=0}^{n} \frac{s - x_k}{x - x_k} = \frac{(n+1)!}{\prod_{k=0}^{n}(x - x_k)}.$$

Equation (1.9) now becomes

$$0 = f^{(n+1)}(\xi) - 0 - [f(x) - P_n(x)] \frac{(n+1)!}{\prod_{k=0}^{n}(x - x_k)}.$$

Solving for $f(x)$, we have

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{k=0}^{n} (x - x_k). \quad (1.10)$$

□

1.2 Error Estimation of Polynomial Interpolation

Eq.(1.10) is knowns as error of interpolationg polynomial:

$$E(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{k=0}^{n}(x - x_k). \tag{1.11}$$

where ξ_x is some number in $[c, d]$.

Example 1.2.1. Let $f(x) = \frac{1}{x}$ on the interval $[2, 4]$ where nodes are $x_0 = 2, x_1 = 2.74, x_2 = 4$. Find the second Lagrange interpolating polynomial and determine error for $n = 2$.

Solution:

Lagrange interpolaating polynomial of second order is:

$$P_2(x) = f_0L_0(x) + f_1L_1(x) + f_2L_2(x) \tag{1.12}$$

where

$$f(x_0) = f(2) = f_0 = 0.5$$

$$f(x_1) = f(2.75) = f_1 = 0.3649$$

$$f(x_2) = f(4) = f_2 = 0.25.$$

Now we will determine the co-efficients polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$ using (1.7).

$$L_0(x) = \frac{(x - 2.75)(x - 4)}{(2 - 2.5)(2 - 4)} = \frac{2}{3}(x - 2.75)(x - 4)$$

,

$$L_1(x) = \frac{(x - 2)(x - 4)}{(2.75 - 2)(2.75 - 4)} = -\frac{16}{15}(x - 2)(x - 4)$$

,

$$L_2(x) = \frac{(x - 2)(x - 2.75)}{(4 - 2)(4 - 2.5)} = \frac{2}{5}(x - 2)(x - 2.75)$$

Substituting values in (1.12) we get:

$$P_2(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.$$

Now for error:

$$f(x) = \frac{1}{x}$$
$$f'(x) = \frac{-1}{x^2}$$
$$f''(x) = \frac{2}{x^3}$$
$$f'''(x) = \frac{-6}{x^4}$$

As error is defined as:

$$E(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$$

For n=2 we get:

$$E(x) = f(x) - P_2(x) = \frac{f^{(3)}(\xi_x)}{(3!)} \prod_{j=0}^{2} (x - x_j)$$

So:

$$E(x) = \frac{f^{(''')}(\xi_x)}{(3!)}(x - x_0)(x - x_1)(x - x_2)$$

Substituting values we get:

$$E(x) = \frac{-1}{(\xi(x)^4)}(x - 2)(x - 2.74)(x - 4)$$

The maximum value of $(\xi(x))^4$ in the interval $(2, 4)$ is $\frac{1}{16}$. So the error is:

$$E(x) \leq \frac{-1}{16}(x - 2)(x - 2.74)(x - 4)$$

1.3 Error Estimation and Runge Phenomenon

A good interpolation polynomial needs to provide an accurate approximation with least error over an entire interval. In 1901, Carl David Tolme Runge when exploring behaviour of error estimation using interpolation polynomial to approximate certain function discovered oscillations at the end of the interval. The discovery was important because it shows that going to higher degree over equispaced nodes is a bad idea, such conditions reduce accuracy and results in divergence of a function [11].

1.3.1 Earlier Work and Plan of Thesis

Runge phenomenon is a consequence of equally spaced nodes and the nth order derivatives of functions that grows quickly when n increases. After the discovery many methods and strategies were proposed to overcome the oscillations and to get accurate approximation. Chebyshev nodes and piecewise polynomial interpolation is commonly used to overcome these oscillations.

In 2007, John P. Boyd discovered several exponentially convergent strategies to defeat Runge Phenomenon [2]. In 2010, he presented six strategies to defeat Runge phenomenon in guassian radial basis function on finite interval [5].

Moreover, recent researches invovlves radial basis functions to avoid the oscillations by using different radial functions approximation [13].

In 2018, C.Ye [3] presented the co-efficient and order determination method (CAOD) to eliminate oscillations at the edges of an interval. This method is used to construct the linear combination of orthogonal polynomials to approximate the target function and to get improved accuracy .

A transform method known as smoothing is also an efficient method to deal with Runge phenomenon. Through this method we can achieve higher degree polynomial interpolation through fewer interpolation points with accurate approximation [4].

The scope of this thesis will define Runge phenomenon by analyzing previously done work. The research method will include secondary research through articles and researches done in the past to discuss the conditions for the occurance or absence of Runge phenomenon. The research will inlcude theory along with examples where the phenomenon appears and also we will investigate theoretical method in detail to overcome this phenomenon.

Plan of the thesis is as follows:

In the first Chapter, we introduce the basic concepts and the theory of interpolation.

In the second Chapter, we deal with the Runge phenomenon. First we consider examples of some functions and graphically consider their representations by varying the length of an interval and the number of interpolation points. In the case of e^x, the Runge phenomenon appears to be absent irrespective of the length of the interval and number of interpolation points. However, for the Runge function, $1/(1+x^2)$, the phenomenon appears to be absent if the interval is $[-1, 1]$ or a subinterval of it, but for any larger interval, the phenomenon appears in the form of oscillation near the end points.

The phenomenon is explained by representing the error term in the form of a contour integral of a complex function. The integral converges in a domain enclosed by a closed contour which has no singularity of the function on or inside the contour. However it diverges for the contour which passes through a singularity of the function under consideration. For the Runge function, $f(x) = 1/(1 + x^2)$, on $[-5, 5]$, it is found that the first contour to pass through the point $z = \pm i$, intersects the real axis at ± 3.63. These are the points beyond which the Runge phenomenon manifests itself. We also review Epperson's solution [15], which explains the phenomenon, in case of Runge function, without the use of complex analysis. His method leads to a function, $w(x)$ whose zeros determine the points of occurrence of this phenomenon. We have found a formula for approximately locating these zeros. The accuracy steadily improves with the length of the interval of definition.

In the third Chapter, by using properties of monic Chebyshev polynomial we explained that the error by using Chebyshev nodes (roots of Chebyshev polynomial of 1st kind) is smallest for all polynomials and is uniformly distributed over the interval $[-1, 1]$ and demonstrate graphically that these non-uniform nodes eliminate Runge phenomenon as error decreases exponentially.

The final chapter is about the reduction of error in approximation. We derived Chebyshev-Fourier series by changing variable in Fourier series and showed that because of some remarkable properties of Chebyshev polynomials the error and convergence rate is superior to that of Fourier series.

This thesis does not contain any original work. We have only reviewed some earlier work on Runge phenomenon and have tried to understand and explain it by means of illustrative examples.

Chapter 2

Runge Phenomenon

If the number of interpolating points is increased, say from 7 to 12, the error of approximation decreases. It is reasonable to imagine that the error can be made arbitrarily small by sufficiently increasing the number of data points. In reality, the opposite happens. For a large equally spaced data points, the interpolation polynomial, while matching the function exactly at each point, oscillates with a large amplitude. If the function is defined on $[-a, a]$, the approximation may be excellent on a sub-interval $[-b, b]$, but is worthless in the remaining part of $[-a, a]$.
This phenomena was first discovered by Carl Runge [11].

2.1 Runge Phenomenon

We have noticed that, if Runge phenomenon exists, it limits convergence of the interpolation polynomial to an interval smaller than the original $[-a, a]$ on which the function $f(x)$ was defined. Now we have to investigate the following questions:

1. Does Runge phenomenon always occur?

2. Why does it occur?

3. If it does occur, at what points of the interval does it manifest itself?

2.1.1 Theoretical Explanation

Let $n > 0$ and $f \in \mathbb{C}^{n+1}(I)$, $I = [c, d]$ be given, x_j^n denotes the interpolating nodes for $0 \leq j \leq n$ then polynomial interpolation error in complex form is defined as:

$$f(z) - p_n(z) = \frac{1}{2\pi i} \int_{C_T} \frac{v_n(z)}{v_n(\xi)} \frac{f(\xi)}{\xi - z} d\xi$$

where

$$v_n(x) = \prod_{j=0}^{n} (x - x_j^n).$$

Also T is domain and C_T is boundary of that domain T, nodes are also contained in T and f is analytic in T [9].

Lemma 2.1.1. *Assume that $\{x_j^n\}$ are equally spaced nodes on interval $[c, d]$ and define*

$$\sigma(z) = |v_n(z)|^{1/n+1}$$

Then:

$$\lim_{n->\infty} \sigma_n(z) = \sigma(z) \tag{2.1}$$

exists for all z. In particular:

$$\sigma(z) = \exp\{\frac{1}{b-a} \int_a^b \log|z - s| ds\}$$

For p>0 and consider family of curves:

$$C(p) = \{z \in \mathbb{C} | \sigma(z) = p\}$$

These smooth concentric curves and placement of z relative to these curves are key to convergence: p_n converges to f if

$$\lim_{n->\infty} |\frac{v_n(z)}{v_n(\xi)}| = 0$$

where $\xi \in C(p)$, $z \in C(p')$, p'<p.[9]

Lemma 2.1.2. *Let $C(p)$ is a contour and function f is analytic inside this contour and also interpolation nodes $\{x_j^n\}$ contained in a contour $C(p)$ then $p_n \to f$ uniformly on $C(p')$, $p' < p$ [9].*

Theorem 2.1.3. *[9] At equally spaced points* $\{x_k^n\}$,$0 \leq k < n$ *where* $x_k^n \in [c, d]$ *and let* $\{p_n\}$ *be a sequence of polynomials interpolating* f*(suppose* f *is analytic at given interval) also* $\sigma(z)$ *as defined in* (2.1)*.Then:*
1. If f *is analytic for all* z *such that* $\sigma(z) < p$*,then* $p_n \dot{\rightarrow} f$ *for each* z^* *such that* $\sigma(z^*) < p$*. The convergence is uniform for all* z^* *such that* $\sigma(z^*) \leq p^* < p$ *.*
2. If p *has a pole* z^**,and* z *is such that* $\sigma(z) > \sigma(z^*)$*, then* $p_n(z) \nrightarrow f(z)$*.*

The theorem implies that if $f(z)$ is analytic on the entire z plane then no Runge phenomenon will exist for $f(x)$ defined on any interval on the real line. On the other hand, if $f(z)$ has a singularity in the complex plane, Runge phenomenon will exist beyond a point where $\sigma(z^*) = p$ intersects the real axis. This is the curve which passes through the nearest singularity of $f(z)$.

Example 2.1.1. Let:

$$f(x) = e^x$$

Let us plot graphs of interpolation polynomial of this function on different intervals with equally spaced nodes.

Graphical Verification

Following graphs are made by using Mathematica. We keep on increasing nodes i.e. keep on increasing degree of polynomial for different intervals to see behaviour of interpolation polynomial. In the graphs, the function and the interpolation polynomial for various sets of data points have been plotted together.

At first we took 10 data points for $f(x) = \exp(x)$ to observe whether Runge phenomenon occurs or not.

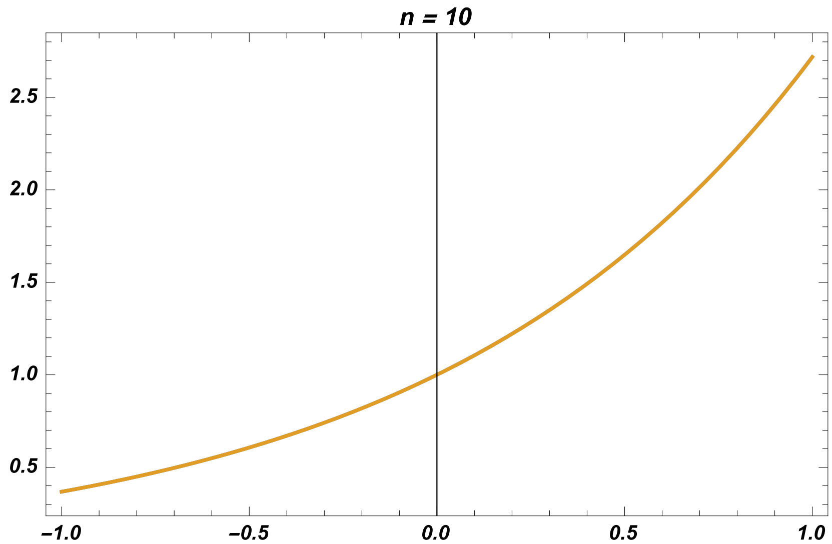

Figure 2.1: n=10 for e^x on [-1,1]

We increased our data points upto 50 on interval $[-1, 1]$ which results in no oscillation.

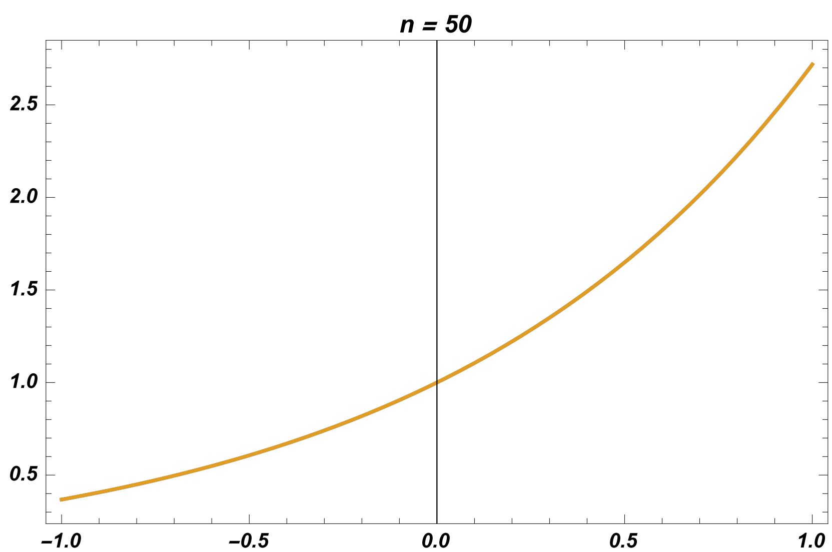

Figure 2.2: n=50 for e^x on [-1,1]

Now we will observe the behaviour of exponential function on $[-2, 2]$ by taking different data points. Taking $n = 10$ results in absence of Runge phenomenon.

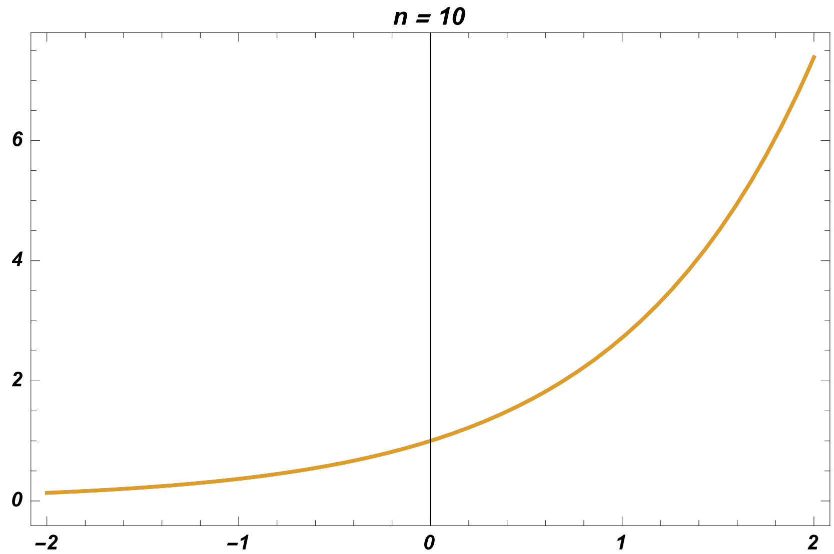

Figure 2.3: n=10 for e^x on [-2,2]

Now, if we increased our data points upto 50 on $[-2, 2]$ it results in no oscillations at the edges.

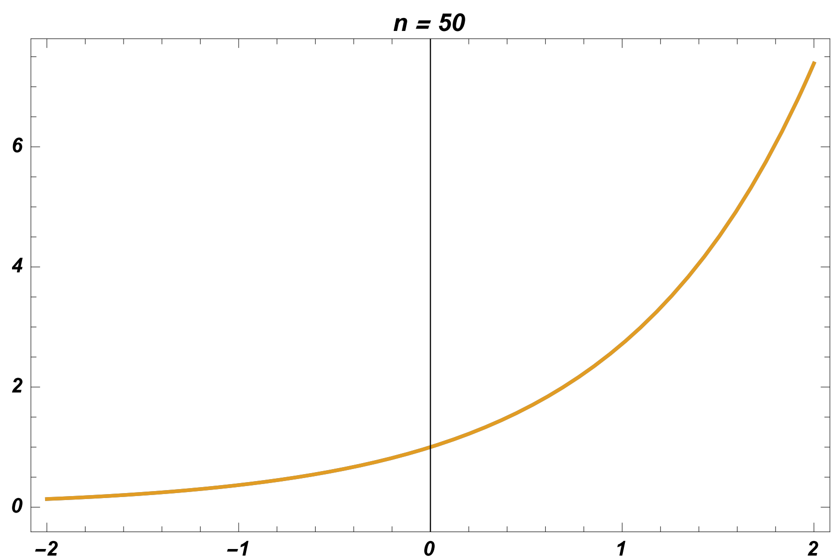

Figure 2.4: n=50 for e^x on [-2,2]

Conclusion

It is clear from the above graphical analysis that for different number of nodes the Runge phenomena does not occur in $f(x) = e^x$. The reason is that $f(x) = e^x$ is an analytic function and does not contain any singularity in the complex plane. Thus, the approximation improves quickly by increasing the number of nodes on any interval $[a, b]$ and consequently the interpolation polynomial uniformly converges to a function on equally spaced nodes.

Note: In some instances, the higher order interpolation polynomial yields an oscillatory curve in the graph which is not caused by Runge's phenomena but is due to numerical errors like round off error, machine error etc. However, these types of errors can be eliminated by using the precision command in Mathematica during the interpolation of a polynomial.

Example 2.1.2. Let us take another function $f(x) = \dfrac{1}{(1+x^2)}$. As this function is not analytic i.e. it has singularity at $x = \pm i$. So in view of above discussion we will see through graphs whether Runge phenomena occurs or not. Let us plot graphs of interpolation polynomial for given function at different number of nodes and by varying length of interval.

Firstly, we took 10 data points on the interval $[-1, 1]$ and observed no oscillations.

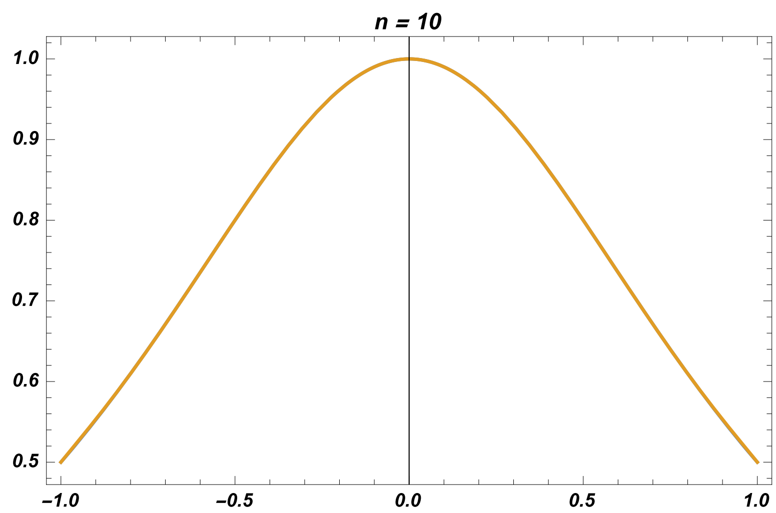

Figure 2.5: n=10 on [-1,1]

To further predict behaviour of resulting polynomial we increased our data points from 10 to 30.

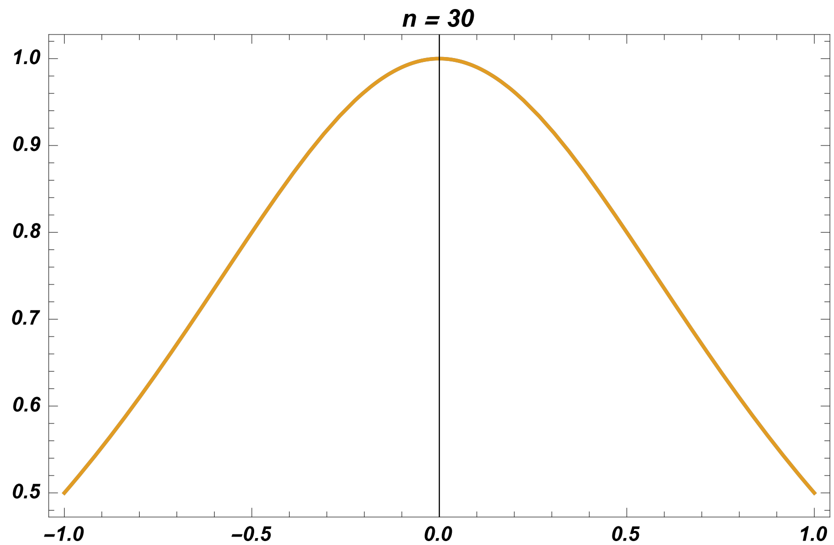

Figure 2.6: n=30 on [-1,1]

Similarly, increasing data points upto 80 results in absence of Runge phenomenon.

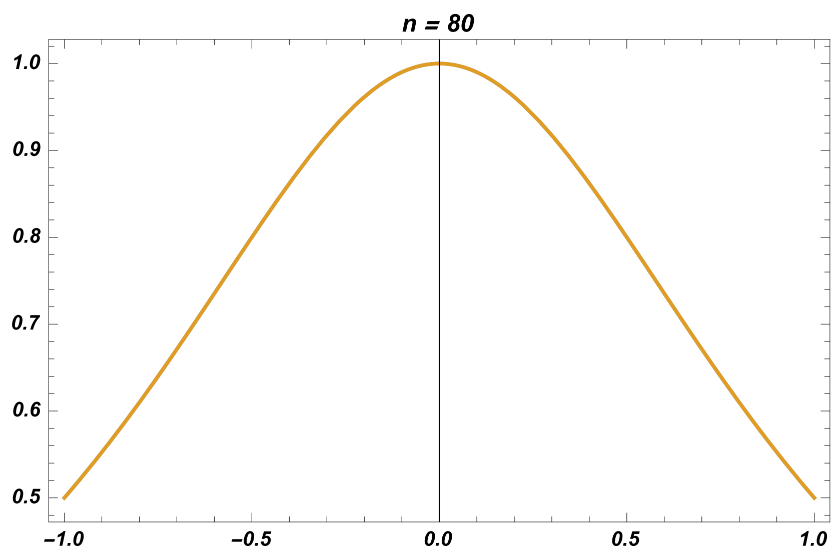

Figure 2.7: n=80 on [-1,1]

Through above graphical analysis it is conlcuded that the interval Runge phenomenon does not occur for Runge function on $[-1, 1]$.

At interval $[-2, 2]$ oscillations began to start at $n = 10$ and thus Runge phenomenon occurs.

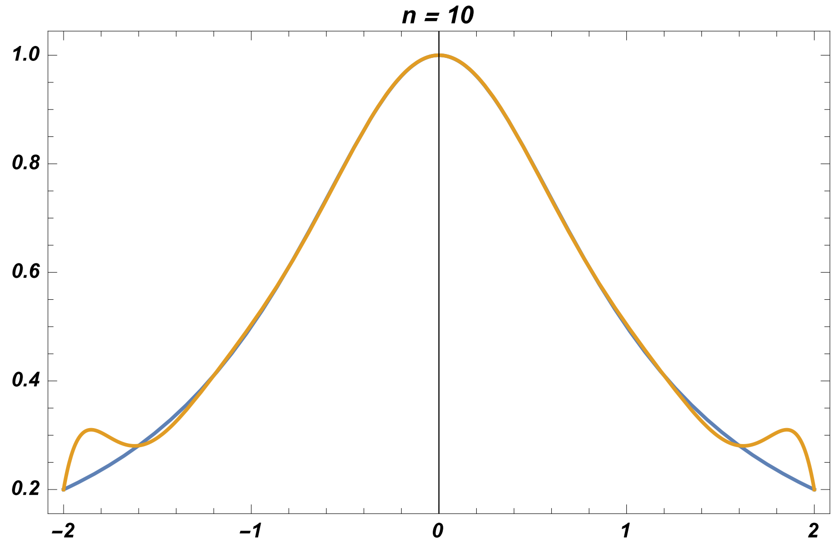

Figure 2.8: n=10 on [-2,2]

And increasing nodes at $[-2, 2]$ results in more and more oscillations near end points.

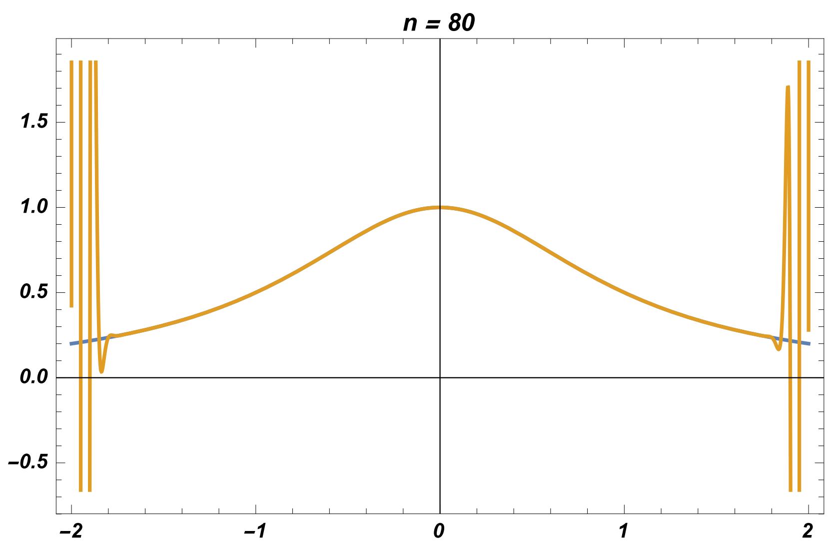

Figure 2.9: n=80 on [-2,2]

At $[-3, 3]$, oscillations occurs at the edges of interval at $n = 10$

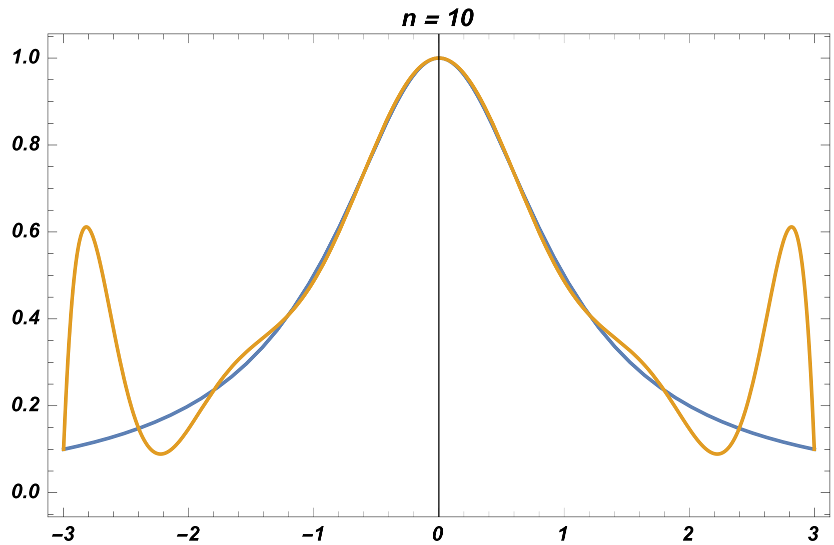

Figure 2.10: n=10 on [-3,3]

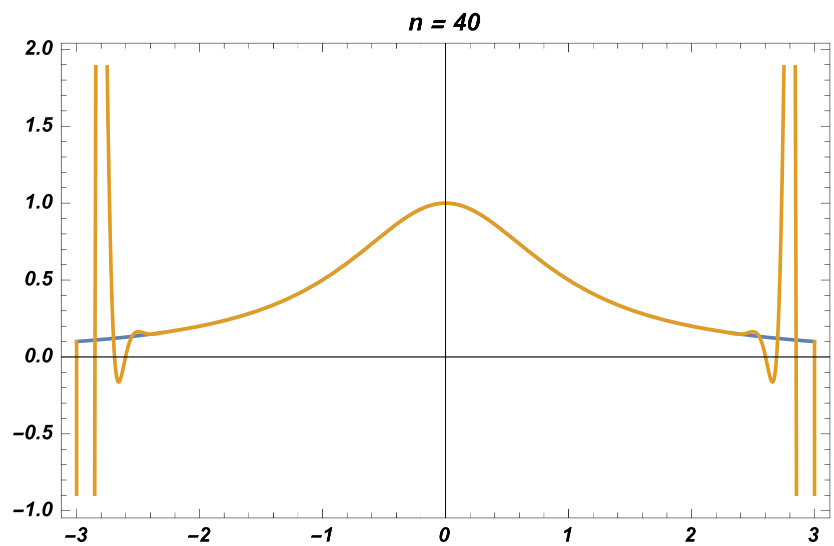

Figure 2.11: n=40 on [-3,3]

Similarly Runge phenomenon occurs at $[-4, 4]$

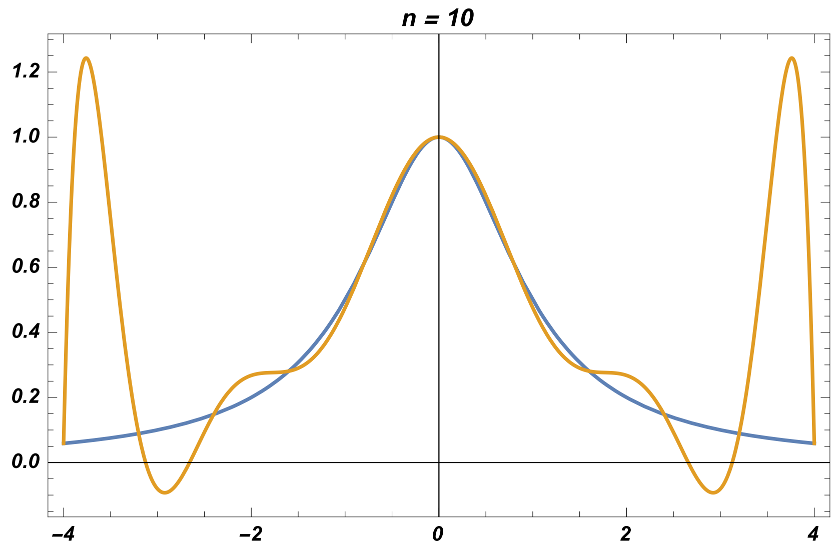

Figure 2.12: n=10 on [-4,4]

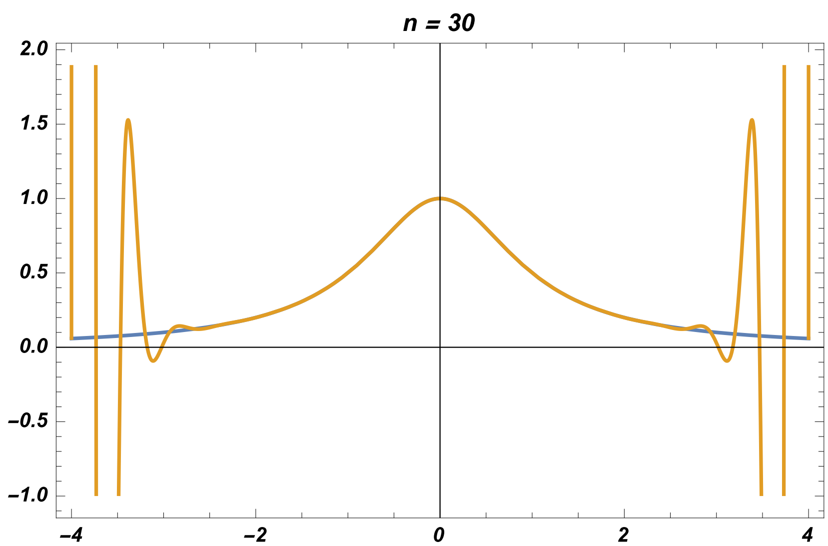

Figure 2.13: n=30 on [-4,4]

At $[-5, 5]$ for $n = 10$, resulting apprximated polynomial shows presence of Runge phenomenon.

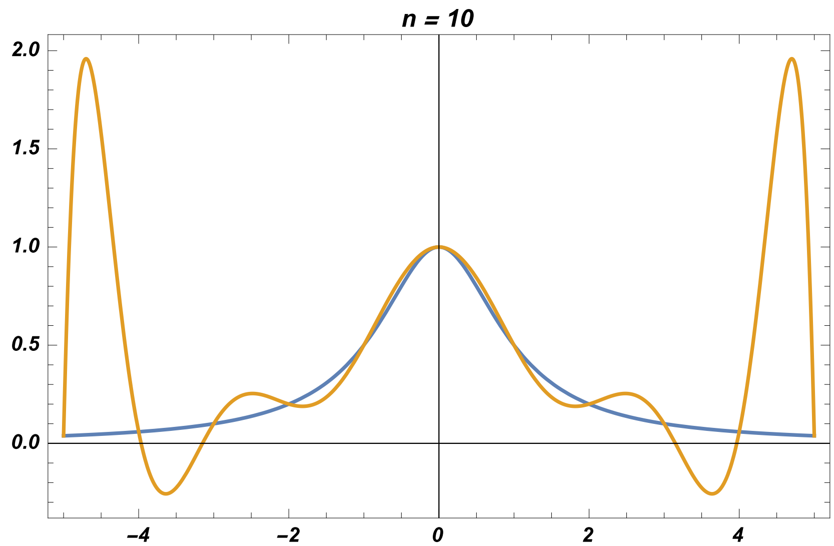

Figure 2.14: n=10 on [-5,5]

It results in wild oscillations at the edges by increasing nodes.

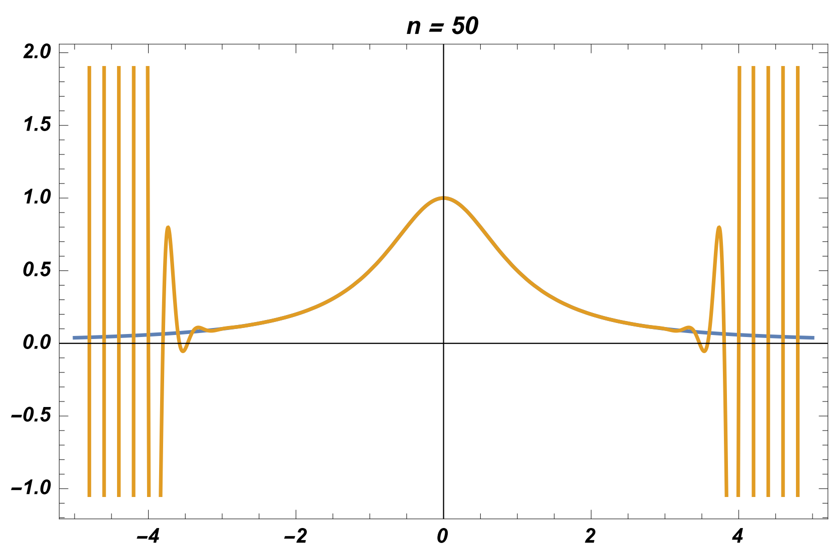

Figure 2.15: n=50 on [-5,5]

Conclusion

Above examples show that:

1. A function may have a singularity in the complex plane but Runge phenomenon may not occur if the interval $[-a, a]$ is sufficiently small as in Ex. 2.1.2.

2. The Runge function manifests Runge phenomenon on any interval larger than $[-1, 1]$.

To see exactly when this phenomenon occurs, we shall now describe a simple proof due to Epperson [9].

2.2 Epperson's Proof

Let us suppose that we have odd number of nodes "$2k+1$" which are equally spaced so they are at an/k for $-k \leq n \leq k$. Let $p(x) = 1/(1+x^2)$ and $q(x)$ is a monic polynomial which vanishes at node so $q(x) = \prod_{n=-k}^{k}(x - an/k)$. So now $p(x)(x^2+1) - 1$ vanishes at nodes. Now we have:

$$p(x)(x^2 + 1) - 1 = q(x)m(x) \tag{2.2}$$

for some polynomial $m(x)$. As degree of $p(x)$ is even and degree of $q(x)$ is odd so $m(x)$ have degree $\leq$1. Let $m(x) = bx$ for some constant b.We need to compute b for this we will replace x by i in equation (2.1).

$$p(i)(i^2 + 1) - 1 = q(i)m(i)$$

$$0 - 1 = (bi). \prod_{n=-k}^{k} (i^2 - an/k)$$

$$-1 = (bi).i. \prod_{n=1}^{k} (i^2 - a^2n^2/k^2)$$

$$b = \frac{(-1)^k}{\prod_{n=1}^{k}(1 + a^2n^2/k^2)}.$$

Rearranging equation (2.1) will exactly gives error formula:

$$p(x) - \frac{1}{1 + x^2} = \frac{bxq(x)}{1 + x^2}$$

$$p(x) - \frac{1}{1+x^2} = \frac{(-1)^k x^2 \prod_{n=1}^{k}(x^2 - a^2n^2/k^2)}{(1 + x^2 \prod_{n=1}^{k}(1 + a^2n^2/k^2)}.$$

Error goes to zero iff

$$\lim_{k->\infty} (\sum_{n=1}^{k} \log|x^2 - a^2n^2/k^2| - \sum_{n=1}^{k} \log(1 + a^2n^2/k^2) = -\infty.$$

It looks like Riemann sum with spacing a/k, so it can be written as:

$$\frac{k}{a}(\int_{t=-a}^{a} \log|x^2 - t^2|dt - \int_{t=-a}^{a} \log|1 + t^2|dt) \tag{2.3}$$

where

$$\int_{-a}^{a} \ln|x^2 - t^2|dt = \int_{0}^{x} \ln|x^2 - t^2|dt + \int_{x}^{a} \ln|x^2 - t^2|dt. \tag{2.4}$$

Equation (2.4) further simplified to:

$$\int_{-a}^{a} \ln|x^2 - t^2|dt = \int_{0}^{x} \ln|x - t|dt + \int_{x}^{a} \ln|t - x|dt + \int_{0}^{a} \ln|x + t|dt.$$

Applying integration by parts term by term we get:

$$\int_{-a}^{a} \ln|x^2 - t^2|dt = (a - x)\ln(a - x) + (x + a)\ln(x + a) - 2a \tag{2.5}$$

Similarly:

$$\int_{t=-a}^{a} \ln|1 + t^2|dt = a\ln(a^2 + 1) + 2\arctan(a) - 2a. \tag{2.6}$$

Putting (2.4) and (2.6) in (2.3) we get simplied result that is:

$$w(x) = (a - x)\ln(a - x) + (a + x)\ln(a + x) - a\ln(a^2 + 1) - 2\arctan(a). \tag{2.7}$$

The Epperson's solution explain the phenomenon, in case of Runge function, without the use of complex analysis [15]. The zeros of $w(x)$ will determine the points of occurrence of this phenomenon.

2.2.1 Approximate Solution

We shall find an approximate solution of the equation $w(x) = 0$; where,

$$w(x) = (a - x)\ln(a - x) + (a + x)\ln(a + x) - a\ln(a^2 + 1) - 2\arctan a. \tag{2.8}$$

$$w(x) = a(1-x/a)(\ln a+\ln(1-x/a))+a(1+x/a)(\ln a+\ln(1+x/a))-a\ln(a^2+1)-2\arctan a.$$

$$\begin{aligned} w(x) &= (1-x/a)a\ln a + (1+x/a)a\ln a + a(1-x/a)\ln(1-x/a) \\ &+ a(1+x/a)\ln(1+x/a) - a\ln(a^2+1) - 2\arctan a. \end{aligned}$$

Assuming $x < a$ and using the Maclaurin series,

$$(1-x)\ln(1-x) + (1+x)\ln(1+x) = x^2 + x^4/6 + ... \tag{2.9}$$

we have

$$w(x) = a(\frac{x^2}{a^2} + \frac{x^4}{6a^4}) - a\ln\frac{a^2+1}{a^2} - 2\arctan a.$$

$$w(x) = a(\frac{x^2}{a^2} + \frac{x^4}{6a^4}) - \frac{1}{a} - 2\arctan a.$$

Thus the equation $w(x) = 0$, is approximately equivalent to,

$$y + \frac{y^2}{6} = \frac{2\arctan a}{a} + \frac{1}{a^2}. \tag{2.10}$$

wherer we have defined $y = \frac{x^2}{a^2}$. As a first approximation, we can take the solution of (2.10) as,

$$y_0 = \frac{2\arctan a}{a} + \frac{1}{a^2} \tag{2.11}$$

Now write (2.10) as $y + y^2/6 - y_0 = 0$, and use Newton-Raphson method to find a better approximation y_1. Setting $f(y) = y + y'6 - y_0$, we have,

$$\begin{aligned} y_1 &= y_0 - \frac{f(y_0)}{f'(y_0)} \\ &= y_0 - \frac{y_0^2/6}{1-y_0/3} \\ &= y_0 - y_0^2/6. \end{aligned}$$

where in the last step we have neglected third and higher powers of y_0. Thus we have an approximate solution of $w(x) = 0$, as,

$$x = a\sqrt{y_0 - y_0^2/6}. \tag{2.12}$$

where y_0 was defined in (2.11).

For $a = 3$, approximate value is 2.675 whereas exact is 2.646. For $a = 10$, these values respectively are 5.374 and 5.370. The error is less than 4 parts in 5000. It decreases steadily as a becomes large [16].

2.2.2 Graphical Verification

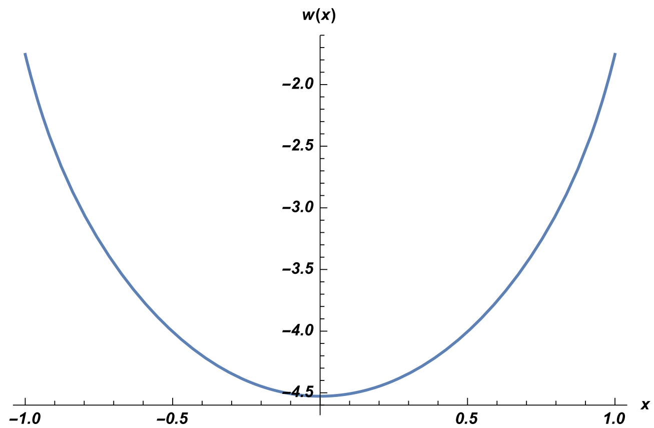

Figure 2.16: $w(x)$ on $[-1, 1]$

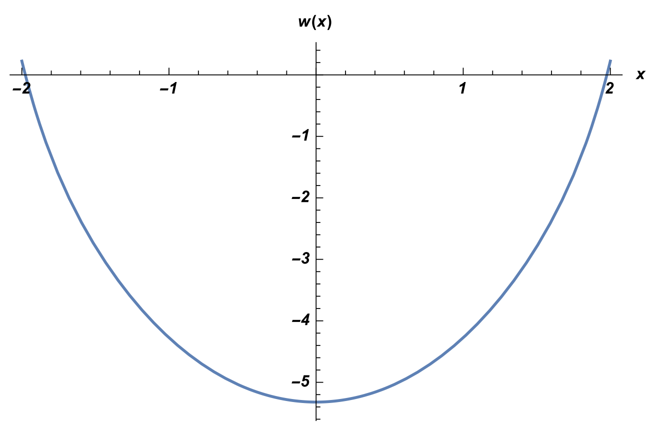

Figure 2.17: $w(x)$ on $[-2, 2]$

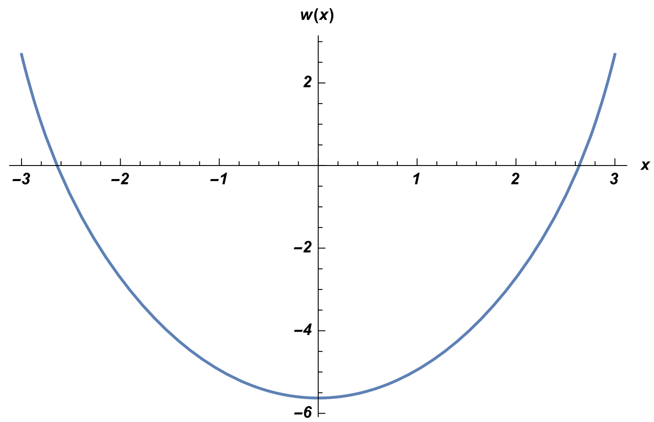

Figure 2.18: $w(x)$ on $[-3, 3]$

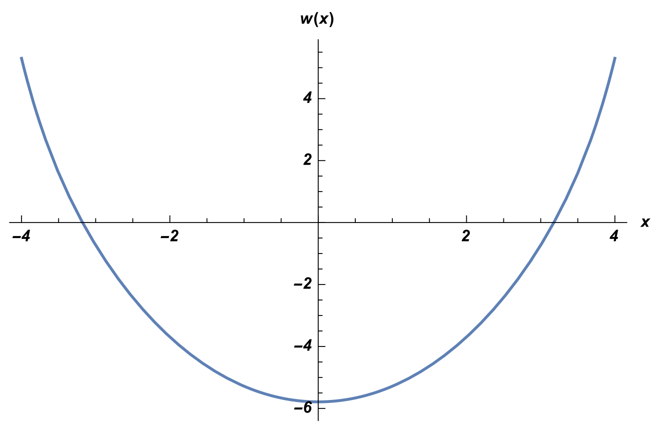

Figure 2.19: $w(x)$ on $[-4, 4]$

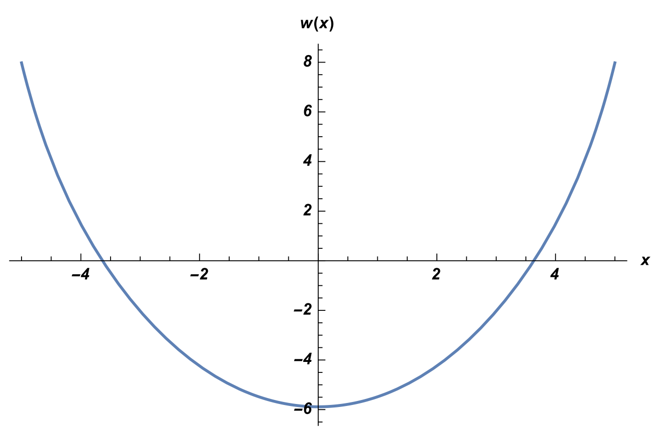

Figure 2.20: $w(x)$ on $[-5, 5]$

By using Mathematica commands we can find out the point on the interval $[-a, a]$ where the Runge phenomena occurs. In view of above graphs, we came to know that in case of $[-1, 1]$ there is no Runge phenomena. The table below shows the point where divergence begins.

Table 2.1. Point Of Interval Where Runge Phenomena Begins

Interval	Exact Value	Approximate Value
[-2,2]	±1.98295	±2.04956
[-3,3]	±2.64619	±2.67547
[-4,4]	±3.1769	±3.19426
[-5,5]	±3.63338	±3.6451
[-6,6]	±4.03988	±4.04842
[-7,7]	±4.40975	±4.41633
[-8,8]	±4.75134	±4.75659
[-9,9]	±5.07023	±5.07454
[-10,10]	±5.37039	±5.37401

Conclusion

By using formula (2.12) we can find approximate zeros of $w(x)$ i.e. the point of occurrence of Runge phenomenon. More precisely, the point where function $w(x)$ changes it sign from negative to poistive is the point where divergence starts (Runge phenomenon occurs) and before this point, interval is known as interval of convergence.

Chapter 3

Methods For Minimax Optimization

In this Chapter, we shall show that a suitable choice of nodes can eliminate Runge phenomenon. This shows that the phenomenon has its roots in equally spaced data.

3.1 Minimizing Error

As error is defined by:

$$E(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=1}^{n+1}(x - x_i).$$

The only information we have about ξ is that it is some point in the interval $[a, b]$ that depends on x, so we cant deal much with $f^{n+1}(\xi)$ but we can try to make $\prod_{i=1}^{n+1}(x - x_i)$ as small as we can by picking suitable choice of nodes x_i to get the minimum error. So now the question arises: How to choose better set of data points to get accurate interpolation polynomial.? As equally spaced nodes result in Runge phenomena so we need to opt non-uniform nodes. One idea is to choose Chebyshev nodes (roots of Chebyshev Polynomial).

3.2 Chebyshev Polynomial

3.2.1 Definition

Chebyshev polynomial of degree m ≥ 0 is defined as:

$$T_m(x) = \cos(m \cos^{-1} x) \qquad x \in [-1, 1].$$

It can also be written as:

$$T_m(\cos\theta) = \cos m\theta, \quad x = \cos\theta, \qquad \theta \in [0, \pi][8].$$

3.2.2 Recurrence Relation of Chebyshev Polynomial

Using the angle sum angle difference formula:

$$\cos(b \pm c) = \cos(b)\cos(c) \mp \sin(b)\sin(c).$$

one can write

$$T_{m+1}(x) = \cos((m+1)\cos^{-1}(x)) = \cos(m\cos^{-1}(x))\cos(\cos^{-1}(x)) - \sin(m\cos^{-1}(x))\sin(\cos^{-1}(x)).$$

$$T_{m-1}(x) = \cos((m-1)\cos^{-1}(x)) = \cos(m\cos^{-1}(x))\cos(\cos^{-1}(x)) + \sin(m\cos^{-1}(x))\sin(\cos^{-1}(x)).$$

If we add these together one obtains

$$T_{m+1}(x) + T_{m-1}(x) = 2\cos(m\cos^{-1}(x))\cos(\cos^{-1}(x)).$$

It can also be written as:

$$T_{m+1}(x) + T_{m-1}(x) = 2xT_m(x).$$

Solving for $T_{m+1}(x)$ gives

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x).$$

This is the recurrence relationship for the Chebyshev polynomials. We can find any new polynomial using recurrence relation by knowing the previous one. Hence:

$$T_0(x) = 1.$$

$$T_1(x) = x.$$

$$T_2(x) = 2x^2 - 1.$$

$$T_3(x) = 4x^3 - 3x.$$

$$T_4(x) = 8x^4 - 8x^2 + 1.$$

$$T_5(x) = 16x^5 - 20x^3 + 5x.$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$$

Note that the leading term of the Chebyshev polynomial $T_m(x)$ is $2^{m-1}x^m$ [8].

3.2.3 Roots of Chebyshev Polynomial

Theorem 3.2.1. *The roots of* $T_m(x)$ *for* $m \geq 1$ *are:*

$$x_k = \cos\left(\frac{2k+1}{2m}\pi\right) \qquad \forall k = 0, 1, 2..m.$$

Proof. To find roots we need to find $T_m(x) = 0$.
Substituting $x = \cos\theta$ we get:

$$T_m(\cos\theta) = \cos m\theta = 0.$$

This is possible when $m\theta = \frac{\pi}{2} + k\pi$ where $k \in Z$, Thus:

$$\theta = \frac{\pi}{2m} + \frac{k\pi}{m} = \frac{2k+1}{2m}\pi.$$

Since:

$$x = \cos\theta = \cos\left(\frac{2k+1}{2m}\pi\right).$$

and we are done.
Moreover, $T_m(x)$ has absolute exterma at:

$$\overline{x} = \cos\frac{k\pi}{m} \quad k \in Z.$$

□

3.2.4 Properties of Chebyshev Polynomial

1. $|T_m(x)| \leq 1$, $\quad x \in [-1, 1]$.
2. $T_m(\cos(\frac{j\pi}{m})) = (-1)^j \qquad j = 0, 1...m$
3. $T_m(\cos(\frac{2j-1}{m})) = 0 \qquad j = 0, 1, ...m.$

3.2.5 Monic Chebyshev Polynomial

A polynomial whose highest power of x has leading coefficient 1 is known as monic polynomial. In case of Chebyshev polynomials highest power have leading coefficient 2^{m-1} except

$T_0(x) = 1$ so to get monic Chebyshev polynomial we need to divide it by $\frac{1}{2^{m-1}}$. The monic Chebyshev polynomial is denoted by $\overline{T_m(x)}$. Now:

$$\overline{T_m(x)} = 1 \qquad m = 0.$$

$$\overline{T_m(x)} = \frac{1}{2^{m-1}} T_m \qquad m \neq 0.$$

The roots of monic polynomials are same but Chebyshev polynomials have maximum and minimum values are ± 1 but monic Chebshev polynomials have maximum and minimum values are $\pm \frac{1}{2^{m-1}}$ which gets smaller when m increases [7].

Theorem 3.2.2. *If U is a monic polynomial of degree m, then:*

$$||U(x)||_\infty = max_{-1 \leq x \leq 1} |U(x)| \geq 2^{1-m}.$$

Proof. We will prove it by contradiction
Let $U(x)$ be a polynomial of degree m, and also suppose that:

$$|U(x)| < 2^{1-m} \quad \forall x \in [-1, 1]$$

Also set,

$$v_m(x) = 2^{1-m} T_m(x).$$

Now set:

$$x_i = \cos \frac{i\pi}{m} \quad i = 0, 1, ...m.$$

Then it shows that $v_m(x)$ is a monomial of degree m, we get:

$$(-1)^i v_m(x_i) = (-1)^i 2^{1-m} T_m(\cos \frac{i\pi}{m}) = 2^{1-m}(-1)^i - (-1)^i = 2^{1-m}.$$

Since $v_m(x)$ and $U(x)$ both have leading coefficients 1 so their difference $v_m(x) - U(x)$ have degree $\leq$m-1. On the other hand

$$(-1)^i U(x_i) \leq |U(x_i)| < 2^{1-m} = (-1)^i v_m(x_i). \quad i = 0, 1, ..m$$

Hence:

$$(-1)^i |v_m(x) - U(x_i)| > 0. \quad i = 0, 1, ..m$$

Now, the function $v_m(x) - U(x)$ must changes its sign $m-1$ times over the interval $[-1,1]$ but this is not possible as degree of $v_m(x) - U(x)$ is at most $m-1$. This contradicts our supposition and hence completes our proof. □

Theorem 3.2.3. $\dfrac{T_{m+1}(x)}{2^m}$ *is the polynomial of degree* $(m+1)$ *that has the smallest* $||.||_\infty$ *value over the interval* $[-1,1]$.

Proof. Suppose that r_{m+1} is a polynomial of degree $(m+1)$ also its leading coefficient is 1 and this polynomial achieves a lower $||.||_\infty$ norm, i.e. $||r_{m+1}||_\infty \leq ||T_{m+1}||_\infty$.
Now $||T_m + 1/2^m||_\infty = 1/2^m$ is achieved $m+2$ times within $[-1,1]$. But from definition we know that $|r_{m+1}(x)| < 1/2^m$ at each of the $m+2$ extreme points.
Thus $R(x) = \dfrac{T_{m+1}}{2^m} - r_{m+1}$ is a polynomial of degree$\leq m$ and at each of the $m+2$ extreme points it has the same sign as T_{m+1}.
$\implies R(x)$ must change sign $m+1$ times on $[-1,1]$ which is impossible for a polynomial of degree $\leq m$. $\implies$ contradiction [7].

□

Theorem 3.2.4. *Let:*

$$x_k = \cos\left(\frac{2k+1}{2m+2}\right)\pi \quad k = 1, 2...m.$$

Then:

$$(x - x_1)...(x - x_m) = r_m(x) = 2^{1-m}T_m(x).$$

Proof. Each of x_k is a distinct root of monic polynomial $r_m(x)$ of degree m. So by **Fundamental Theorem of Algebra** $r_m(x)$ must factorizes as $(x-x_1)...(x-x_m)$and this completes our proof [6]. □

Theorem 3.2.5. *Let* x_k *are the nodes chosen as the roots of Chebyshev polynomial* $T_{m+1}(x)$ *i.e.*

$$x_k = \cos\left(\frac{2k+1}{2m+2}\right)\pi \quad k = 1, 2...m.$$

then error term of interpolating polynomial using node will be;

$$e(x) = |f(x) - U(x)| = \leq \frac{1}{2^m(m+1)!} max_{-1\leq x\leq 1}|f^{m+1}(t)|$$

[6].

Conclusion

1. $||f(x) - p_m(x)||_\infty$ is the smallest for all polynomials of degree m if we choose the $\{x_k\}$ to be the zeros of Chebyshev polynomial $T_m(x)$.
2. The error is distirbuted uniformly over the interval $[-1, 1]$ in case of Chebyshev.
3. **Spectral Convergence:** If the $(m + 1)$ sample points for the interpolation polynomial $p_m(x)$ are chosen at the roots of the Chebyshev polynomials $x_k = \cos[\frac{2k+1}{2m}\pi]$, then

$$e_m(x) = f(x) - p_m(x)$$
$$= \frac{f^{(m+1)}(\xi)}{(m+1)!}(x - x_0)...(x - x_m)$$
$$= \frac{f^{(m+1)}(\xi)}{(m+1)!}\frac{T_{m+1}(x)}{2^m}$$

Thus taking the absolute value of both sides

$$|e_m(x)| = |\frac{f^{(m+1)}(\xi)}{(m+1)!}\frac{T_{m+1}(x)}{2^m}|$$
$$\leq \frac{||f^{(m+1)}||_\infty}{(m+1)!}\frac{|T_{m+1}(x)|}{2^m}$$
$$\leq \frac{||f^{(m+1)}||_\infty}{2^m(m+1)!}$$

Thus the error decreases exponentially with m - a property known as spectral convergence.

3.2.6 Optimal Nodes

In order to get accurate interpolating polynomial of a function f at $m + 1$ points on the interval $[-1, 1]$ we need to choose data points x_k so that they are the zeros of Chebyshev polynomials. If [c,d]≠[-1,1] then use linear mapping, for this let g be a linear map that maps a point $x \in [-1, 1]$ to a point $g(x) \in [c, d]$ such that $g(-1) = c$ and $g(1) = d$. These properties actually fix g uniquely as:

$$g(x) = c + \frac{d-c}{2}(x+1) = \frac{d+c}{2} + \frac{d-c}{2}x$$

The $m+1$ data points x_k for interpolating polynomial $r(x)$ on interval $[c,d]$ is then image of g of $m+1$ zeros of $T_{m+1}(x)$:

$$x_k = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{2k+1}{2m+2}\pi\right) \quad k = 0, 1...m. \tag{3.1}$$

3.2.7 Runge Phenomenon and Chebyshev Nodes

As discussed earlier that polynomial interpolation at equally spaced nodes reduces error in the middle of given range but not at edges for Runge function so we can exactly say that adding more and more nodes can improve approximation at the middle but it gives rise to some wild oscillations at the edges so now the question arises that how to overcome these oscillations? So one observation is that we can cope up with such problem by not choosing uniform nodes i.e. let us put up more points at the edges and less at the middle. An optimum way to choose non uniform nodes is that choose Chebyshev's nodes (roots of Chebyshev plynomial of first kind). Our main task is to minimize $\prod_{i=1}^{m+1}(x-x_i)$. We know that this product is monic polynomial of degree m and any such type polynomial is bounded from above by 2^{1-m}. So now this bound can be attained by using monic Chebyshev polynomial.

Graphical Verification

Let us plot $f(x) = 1/(1+x^2)$ on $[-3,3]$ for $n = 40$ using Chebyshev nodes.

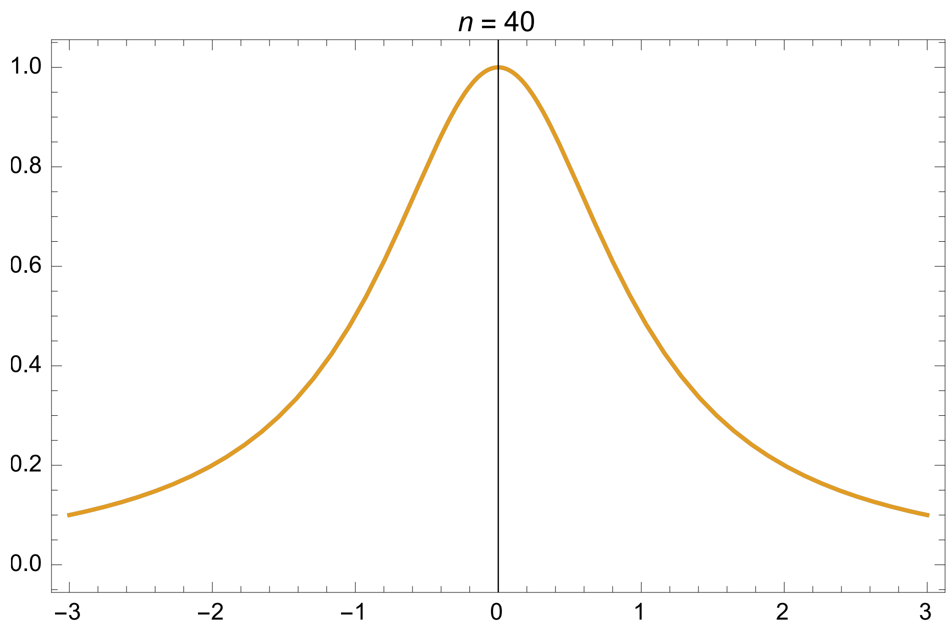

Figure 3.1: $n = 40$ on $[-3, 3]$ using Chebyshev nodes

Comparing with Fig.2.13 we concluded that using Chebyshev nodes we can avoid oscillations at the edge.

3.2.8 RBF and Runge Phenomena

Recent research use radial basis function to deal with Runge phenomena. The RBF interpolant for data values e_k is defined as:

$$r(x) = \sum_{k=1}^{n} \lambda_k \phi(||x - x_k||)$$

where $||.||$ is Euclidean norm and λ_k can be determined using condition $r(x_k) = e_k$ [13]. In this recent research six strategies are discussed for defeating Runge phenomena. Three of them fails and three are successful. The Gaussian RBF is defined as:

$$\phi(x) = \exp(-\epsilon^2 x^2)$$

With "n+1" interpolating points and with grid spacing $h = n/2$, the Gaussian RBF can also be written as:

$$\phi(x) = \exp(-\alpha^2 (x/h)^2)$$

where α is a relative width parameter. By fixing α and by increasing "n" the error falls upto certain level and then attains a saturation point. So the efficient method is to decrease α at a rate of $1/\sqrt{n}$ with increasing interpolating points. In this way Runge phenomena disappears and also sub-geometric rate of convergence is achieved [14].

Chapter 4

Trignometric Fourier Series and Chebyshev Fourier Series

Uptill now we have discussed about polynomial interpolation of a function and also how to deal with oscillations in case of singularities. While dealing with approximation our main purpose is to get such approximations that results in less and less error. In this chapter we will discuss about how to approximate a function with least error.

4.1 Least Square Approximation

Let $\{f_m(x)\}_{m=0}^{\infty}$ be a set of functions defined on an interval $[c, d]$ and let $w(x)$ be a positive *weight function* on (c, d). Suppose the following $k-$sum

$$P_k(x) = a_0 f_0(x) + a_1 f_1(x) + ... + a_k f_k(x).$$

approximates an arbitrary function $p(x)$ on $[c, d]$. Define the *error*

$$e(x) = \mid p(x) - P_k(x) \mid .$$

In the *least squares approximation* the coefficients $a_i, i = 0, 1, .., k$ are chosen so as to make the following integral as small as possible

$$E_m^2 = \int_c^d w(x) e^2(x) dx.$$

4.1.1 Basic Definitions

4.1.2 Inner Product

The inner product of real functions w.r.t weight function $w(x)$, let g and h on closed interval $[c, d]$ is defined as:

$$(g, h) = \int_c^d w(x)hg dx \tag{4.1}$$

If $(g, h) = 0$ for $g \neq h$ the two functions are said to be **orthogonal** to each other on given closed interval.

Orthogonal Set

A set of functions $\{f_1(x), f_2(x), ...\}$ is an orthogonal set of functions on the interval $[c, d]$ if any two functions in the set are orthogonal i.e.

$$(f_n, f_m) = \int_c^d f_n(x)f_m(x)w(x)dx = 0 \qquad (n \neq m) \tag{4.2}$$

While dealing with Fourier series the set $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), ...\}$ is important because it is orthogonal set on the interval $[-\pi, \pi]$.

Norm of a Function

Norm of a function is defined as:

$$||f|| = \sqrt{(f, f)} = \sqrt{\int_c^d f^2(x)w(x)dx} \tag{4.3}$$

Orthonormal Functions

A pair of functions f_i and f_j is orthonormal if they are orthogonal and each normalized i.e.

$$\int_c^d f_i(x)f_j(x)w(x)dx = \delta_{ij}$$

where $w(x)$ is the weight function and δ_{ij} is Kronecker Delta i.e.

$$\boldsymbol{\delta}_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

Theorem 4.1.1. *(Fourier Approximation)*
For any given positive integer m, the best approximation, (in the least squares sense) to a function p by a sum of the form $\sum_{k=0}^{m} r_k f_k$ is obtained when $r_k = (p, f_k)$ i.e. the coefficients a_k are the Fourier coefficients of p.
The error becomes

$$E_m^2 = \| p \| - \sum_{k=0}^{m} (p, f_k)^2.$$

where $||p||$ is norm of a function (4.2) *and f_k is a set of orthogonal functions,(p, f_k) is inner product* (4.1) *and Fourier coefficients of any function p is defined as:*

$$r_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(x) dx$$

$$r_k = \frac{1}{\pi} \int_{-\pi}^{\pi} p(x) f(x) w(x) dx$$

Now we will discuss in detail about fourier series.

4.2 Fourier Series for Function 2π Periodicity

Generally Fourier series of periodic function defined on interval $[-\pi, \pi]$ is expressed as :

$$F(x) = r_0 + \sum_{l=1}^{\infty} r_l \cos(lx) + \sum_{l=1}^{\infty} d_l \sin(lx) \tag{4.4}$$

where

$$r_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) dx \tag{4.5}$$

$$r_l = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos(lx) dx \tag{4.6}$$

$$d_l = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin(lx) dx \tag{4.7}$$

Each term is periodic with a period of 2π so the sum of series is also periodic with same period. If series converges on $(-\pi, \pi)$ then it converges on real line, while dealing with fourier series this fact should be kept in mind that it is not necessary that series in question coverges for all x$\in[-\pi, \pi]$. [10]

Note: The Fourier series converges rapidly to f(x) at all points where f is continuous and we will see an overshoot near jump discontinuity and hence a very slow convergence near point of discontinuity.

4.2.1 Fourier Series for Arbitrary Period

$$F(x) = \frac{r_0}{2} + \sum_{l=1}^{\infty}(r_l \cos(\frac{l\pi x}{A}) + d_l \sin(\frac{l\pi x}{A})) \tag{4.8}$$

where

$$r_0 = \frac{1}{A}\int_{-A}^{A} F(x)dx \tag{4.9}$$

$$r_l = \frac{1}{A}\int_{-A}^{A} F(x)\cos(\frac{l\pi x}{A})dx \tag{4.10}$$

$$d_l = \frac{1}{A}\int_{-A}^{A} F(x)\sin(\frac{l\pi x}{A})dx \tag{4.11}$$

4.2.2 Bounds of Coefficients in a Trigonometric Fourier Series

Theorem 4.2.1. *: If*

1. $p(\pi) = p(-\pi), \quad p^1(\pi) = p^1(-\pi) \quad ...p^{k-2}(\pi) = p^{k-2}(-\pi)$

2. $p^k(x)$ *is integrable*

Then coefficients of Fourier series:

$r_0 + \sum_{l=1}^{\infty} r_l \cos lx + d_n \sin lx,$

have upper bound

$$| r_l | \leq \frac{R}{l^k}, \quad | d_l | \leq \frac{R}{l^k},$$

where R is a constant, independent of l. If above two conditions are satisifed then algebraic index of convergence is as large as k [12].

If $F(x)$ is symmetric about $x = 0$ i.e. if $F(x) = F(-x)$ for all x we are only left with constant and cosine terms and resulting expression is known as "Fourier Cosine Series". If given function is **not periodic** then we replace x by $\cos\theta$ and hence resulting function and all its derivative are periodic on interval $[-\pi, \pi]$.
A Fourier series:

$$F(\cos\theta) = \frac{r_0}{2} + r_1 \cos\theta + r_2 \cos 2\theta + ...$$

can easily be found. Now replacing $\cos\theta$ by x and as we known that $T_l(x) = \cos l\theta$ we get a series:

$$p(x) = \frac{r_0}{2} + r_1T_1(x) + r_2T_2(x) + r_3T_3(x) + r_4T_4(x) + ...$$

where

$$r_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{F(x)}{\sqrt{1-x^2}}dx \tag{4.12}$$

and

$$r_l = \frac{2}{\pi}\int_{-1}^{1} \frac{F(x)}{\sqrt{1-x^2}}T_l(x)dx \tag{4.13}$$

This series is known as Chebyshev-Fourier series. The few terms in above series gives polynomial approximation which is superior than any other set of orthogonal polynomials. The co-efficients in above series decays fast.

Example 4.2.1. Let $f(x) = \dfrac{1+2x}{x^2-x+2}$ on $[-1, 1]$. The above series is neither even nor odd. So by using mathematica we will calculate Fourier series for different "n". Resulting Fourier series:

$$p_2(x) = 0.567669-0.0642192\cos(\pi x)-0.00408395\cos(2\pi x)+0.693363\sin(\pi x)-0.289788\sin(2\pi x).$$

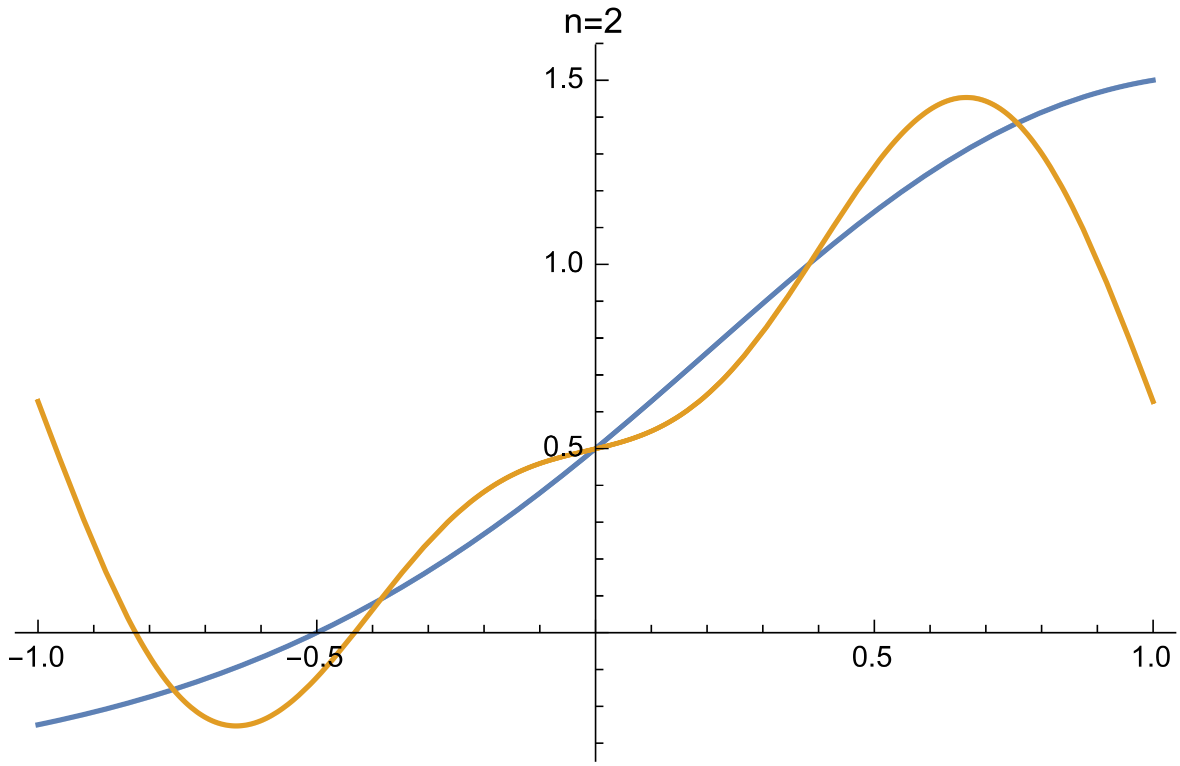

Figure 4.1: Fourier series for n=2

$$E_2^2 = 0.124253$$

$$p_4(x) = 0.567669 - 0.0642192\cos(\pi x) - 0.00408395\cos(2\pi x) + 0.000915867\cos(3\pi x) - 0.000445927\cos(4\pi x) + 0.693363\sin(\pi x) - 0.289788\sin(2\pi x) + 0.188531\sin(3\pi x) - 0.140459\sin(4\pi x).$$

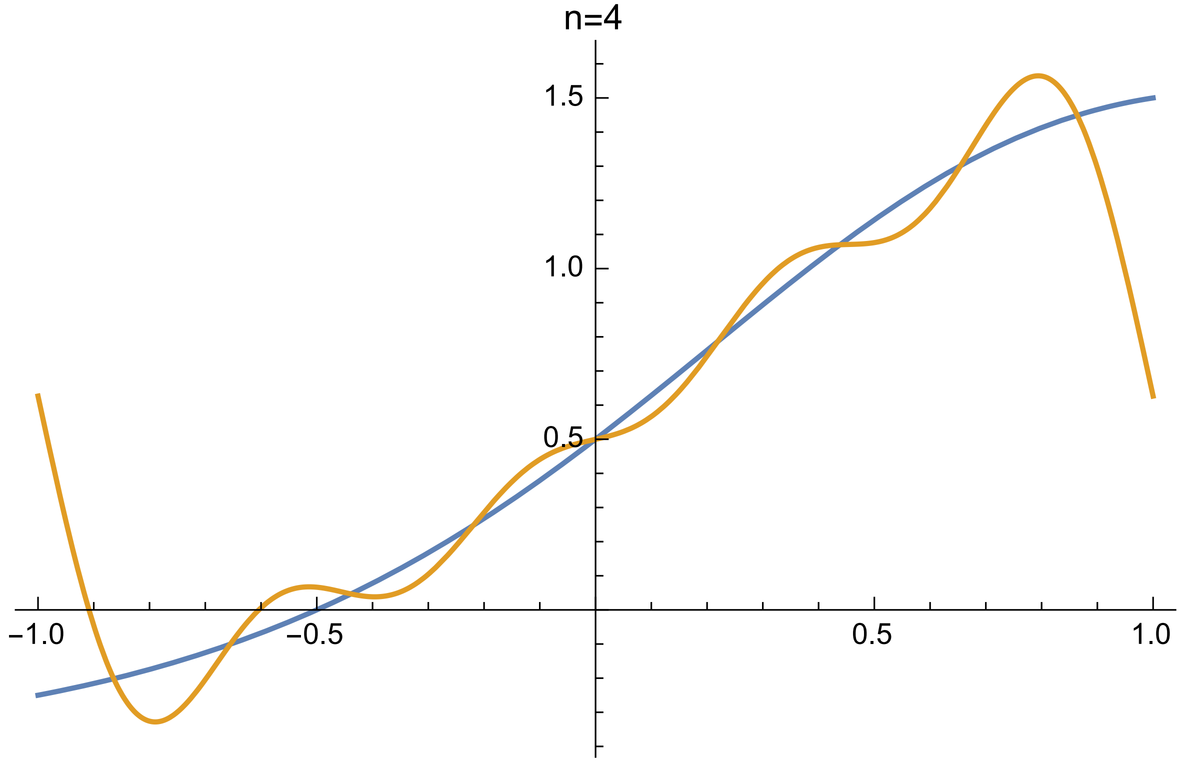

Figure 4.2: Fourier series for n=4

$$E_4^2 = 0.0689793$$

$$p_7(x) = 0.567669 - 0.0642192\cos(\pi x) - 0.00408395\cos(2\pi x) + 0.000915867\cos(3\pi x) - 0.000445927\cos(4\pi x) + 0.000272357\cos(5\pi x) - 0.000184705\cos(6\pi x) + 0.000133864\cos(7\pi x) + 0.693363\sin(\pi x) - 0.289788\sin(2\pi x) + 0.188531\sin(3\pi x) - 0.140459\sin(4\pi x) + 0.112019\sin(5\pi x) - 0.0931928\sin(6\pi x) + 0.079799\sin(7\pi x).$$

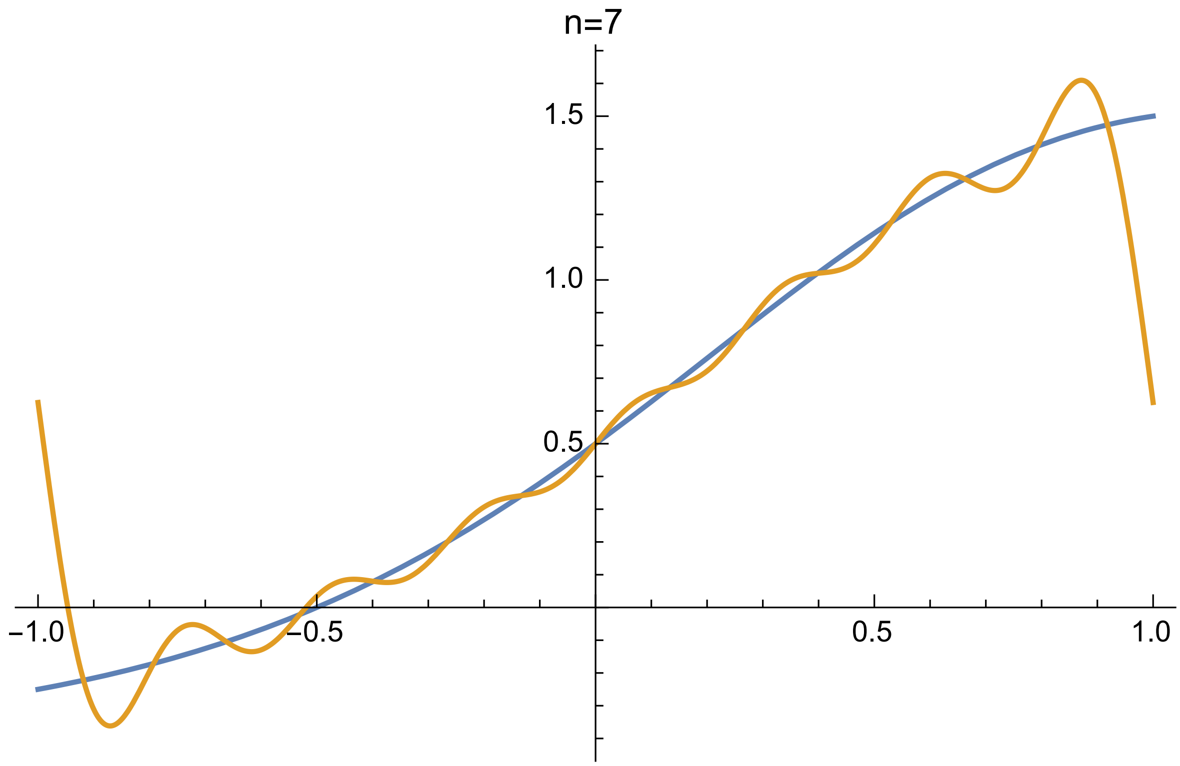

Figure 4.3: Fourier series for n=7

$$E_7^2 = 0.0413781$$

$$\begin{aligned}
p_{10}(x) = 0.567669 - 0.0642192\cos(\pi x) - 0.00408395\cos(2\pi x) + 0.000915867\cos(3\pi x) \\
- 0.000445927\cos(4\pi x) + 0.000272357\cos(5\pi x) - 0.000184705\cos(6\pi x) \\
+ 0.000133864\cos(7\pi x) - 0.000101614\cos(8\pi)0.000079826\cos(9\pi x) \\
- 0.0000643969\cos(10\pi x) + 0.693363\sin(\pi x) - 0.289788\sin(2\pi x) + 0.188531\sin(3\pi x) - \\
0.140459\sin(4\pi x) + 0.112019\sin(5\pi x) - 0.0931928\sin(6\pi x) \\
+ 0.079799\sin(7\pi x) - 0.0697785\sin(8\pi x) + 0.0619976\sin(9\pi x) - 0.05578\sin(10\pi x).
\end{aligned}$$

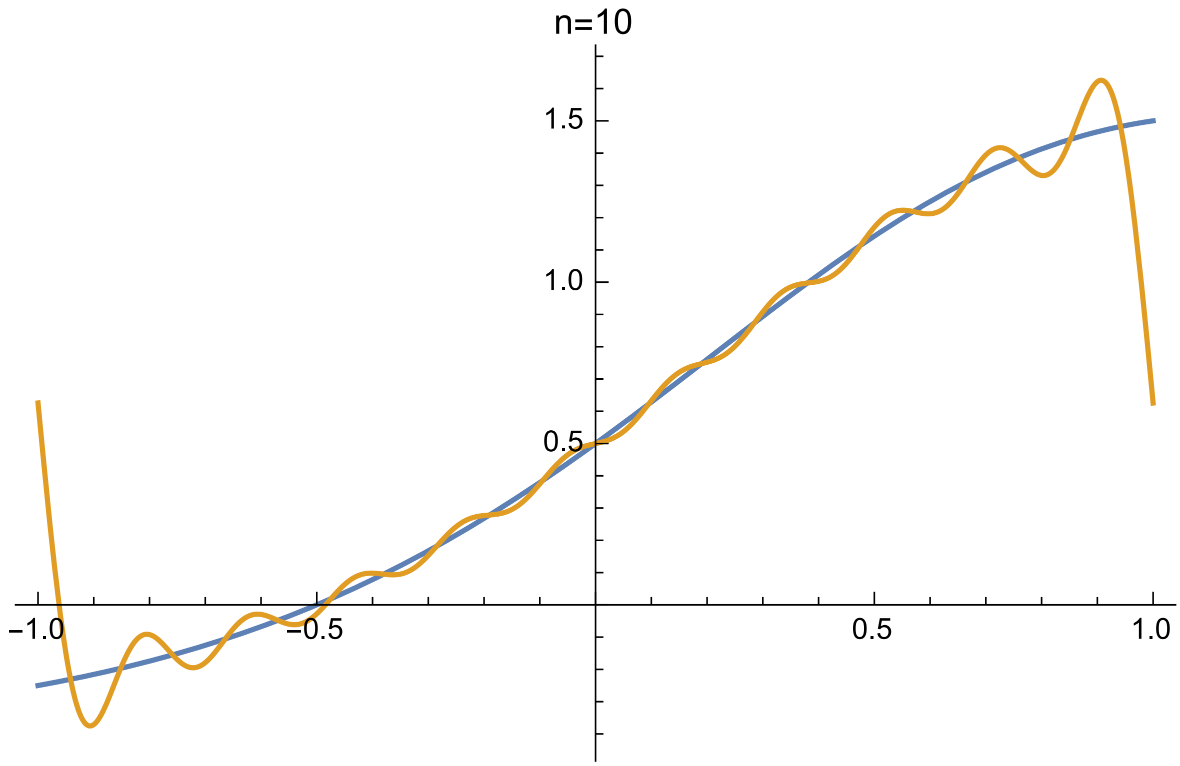

Figure 4.4: Fourier series for n=10

$$E_{10}^2 = 0.0295539$$

We can clearly see that resulting trignometric Fourier series of given function converges slowly while increasing "n". Now we will calculate Chebyshev-Fourier series: For $n = 4$, resulting Chebyshev-Fourier series is:

$$p_4(x) = \frac{r_0}{2} + r_1T_1(x) + r_2T_2(x) + r_3T_3(x) + r_4T_4(x).$$

$$r_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{1+2x}{x^2-x+2}dx$$

Using Mathematica we get:

$$r_0 = 0.58927$$

Similarly using (4.13) and using mathematica we get:

$$p_4(x) = 0.503708 + 1.22683x + 0.303026x^2 - 0.356844x^3 - 0.184783x^4$$

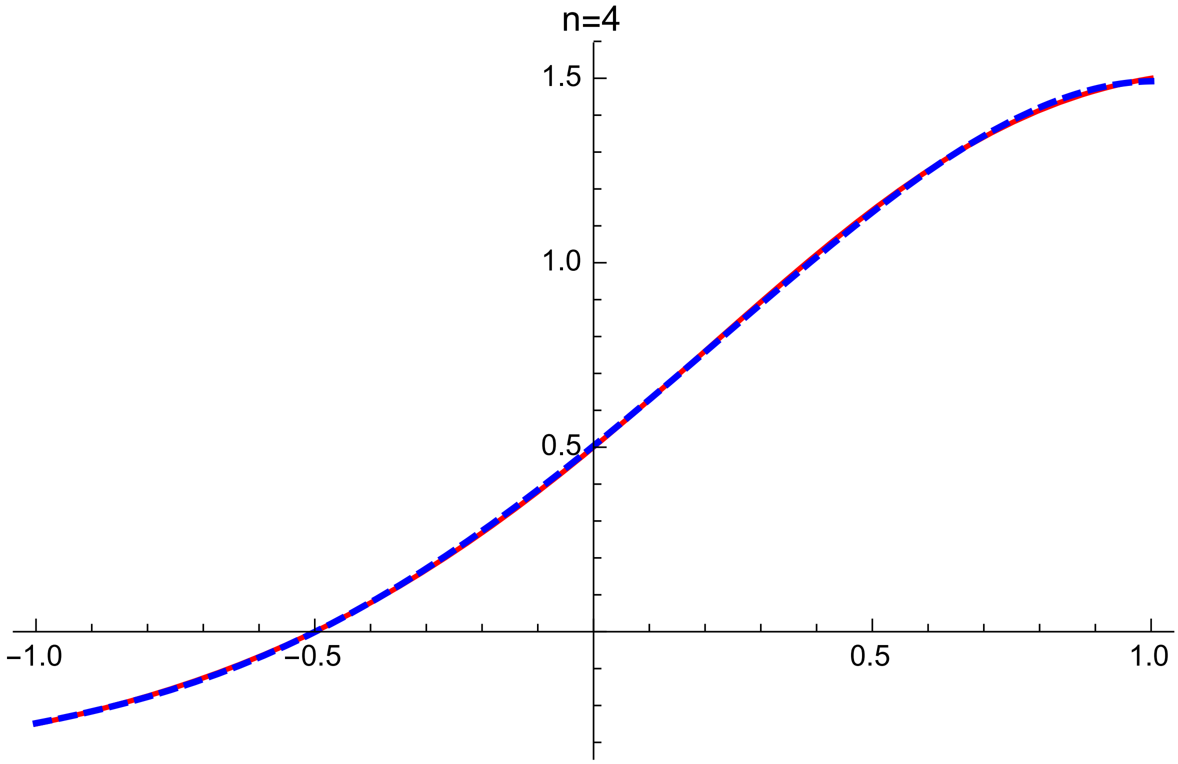

Figure 4.5: Fourier series for n=4

$$E_4^2 = 0.0000361002$$

Similarly for $n = 7$, the resulting Chebyshev-Fourier series is:

$$\begin{aligned}p_7(x) &= 0.500349 + 1.25082x + 0.363478x^2 - 0.449515x^3 - 0.345989x^4 \\ &\quad + 0.0662657x^5 + 0.107471x^6 + 0.00749618x^7.\end{aligned}$$

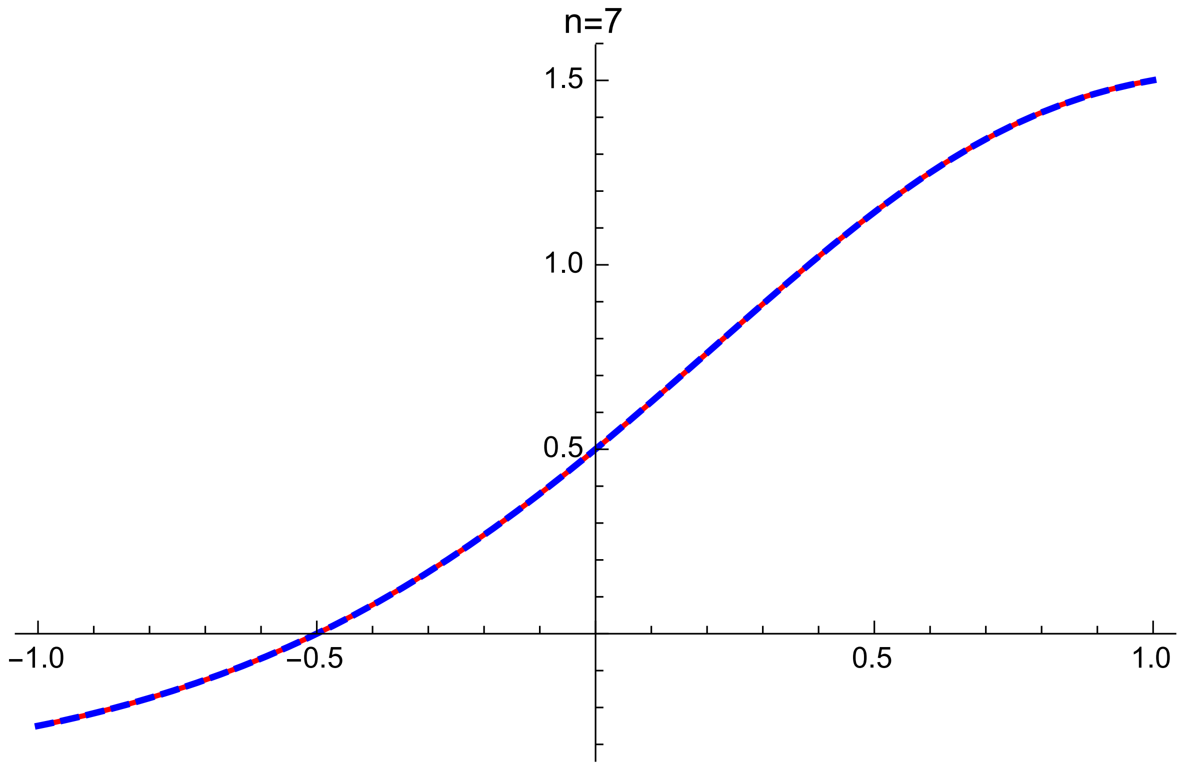

Figure 4.6: Fourier series for n=7

$$E_7^2 = 0.000000119273$$

Hence through graphical observations it is clear to us that Chebyshev-Fourier gives better and almost exact approximation than that of Fourier series. Also error comparison for both shows that approximation by Chebyshev-Fourier series has minimum error than that of Fourier series and hence convergence rate of Chebyshev-Fourier series is faster than that of Fourier series.

Covergence of Chebyshev-Fourier Series

The Fourier series coverges exponentially as long as function is analytic and periodic but convergence rate drops if function loses its periodicity. This suffers from the fact that the expansion has no longer orthogonal basis. The Chebyshev-Fourier series have few terms as polynomials which are orthonormal w.r.t particular weight function and hence resulting in faster convergence rate.

4.3 Conclusion

1. It can be concluded that the Fourier series approximation of a periodic function on a certain interval is prominently substantial with relatively less error and higher convergence rate. However, in contrast for non-periodic functions on a limited interval Chebyshev-Fourier series is far more passable than that of a trigonometric.
2. For a better approximation of an interpolating polynomial we prefer to use Chebyshev series due to some unique properties of a Chebyshev polynomial such as orthogonality, similarity with polynomials etc. On the other hand, for a non-polynomial interpolation approximation through Fourier series is preferable due to higher convergence rate than the Chebyshev series.

Chapter 5

Summary

We have first defined Runge phenomenon and considered several examples to demonstrate the existence or its absence. It depends on the function, its singularities in the complex plane and the interval which contains the nodes.

In second chapter we answered all the questions through theoretical evidences about occurrence and absence of this phenomenon. Moreover, through theorems and proofs we presented formula to find point where the phenomenon menifest itself.

However, to overcome this oscillatory problem of an interpolating polynomial for a non-analytic function due to equally spaced data points we consider some strategies. In third chapter Chebyshev nodes which are the roots of the Chebyshev polynomial have been introduced.

The fourth chapter comprises of notions concerning the reduction of an error in the approximation. It can be seen through numerous graphs that for an interpolating polynomial; the error obtained through Chebyshev-Fourier series is far more adequate than that of a Fourier series merely due to some unique properties of a Chebyshev polynomial.

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