Existence of Solutions for Fractional Langevin Equation

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2021

FORM TH-4

National University of Sciences & Technology

MASTER'S THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Sumiha Mumraiz, Regn No. 00000278251 Titled: Existence of Solutions for Fractional Langevin Equation be accepted in partial fulfillment of the requirements for the award of **MS** degree.

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Date: 12/4/2021

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Dedication

This dissertation is dedicated to My great parents, who never cease to offer themselves in endless ways.

Acknowledgement

In the Name of Allah, The Most Compassionate and The Most Gracious, all glories belongs to Him. First and foremost, I must accept my unlimited appreciation to Allah, for His blessings and support.
I would like to extend my sincere thanks to Dr. Mujeeb ur Rehman, who has always been helpful at all stages of research work. Thank you for your guidance, invaluable support and recommendations in the development of this study.
I would also like to share my wholehearted gratitude to my family. They have given me great assistance for my whole life. In fact, by the course of obtaining Master's degree. Not least of all, my heartfelt thanks goes to my friends for their support, their friendship and motivation, which eventually made this study possible.

Sumiha Mumraiz

Abstract

In this research, we investigated nonlinear fractional Langevin equation involving two distinct generalized fractional derivatives. Existence of unique and at least one solution is demonstrated by making use of Krasnoselskii's theorem and Banach contraction mapping principle. Pair of fractional Langevin equations with Caputo and Riemann-Liouville fractional derivatives were considered to check existence of positive solution. Green functions for related equations are found to verify existence of positive solution using fixed point theorems, upper and lower solution techniques. Also we have discussed a coupled system of fractional Langevin equations. Existence result is obtained by utilizing Schauder Fixed Point theorem and uniqueness result is proved by contraction mapping theorem.

Contents

Chapter 1

Introduction

The concept of differentiation is known to those who have gone through ordinary calculus. The nth derivative $\frac{d^n g}{dt^n}$ of a function g is well defined where n is an integer. The development of fractional calculus began immediately after development of classical calculus. Initially, it was stated in Leibniz's letter to the L'Hospital, which proposed the concept of a semi-derivative [13,14,25]. The development of fractional calculus was made by several prominent mathematicians including Riemann, Lagrange, Liouville, Fourier, Heaviside, Euler, Abel etc.

Lacroix wrote a paper in 1819 defining the fractional derivative. The nth derivative of $y = t^m$, $m \in \mathbb{Z}^+$, is

$$\frac{d^n y}{dt^n} = \frac{m!}{(m-n)!} t^{m-n}, \;\; m \geq n,$$

he replaced generalized factorial by Legendre's symbol Γ,

$$\frac{d^n y}{dt^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}.$$

The semi-derivative is obtained by taking $m = 1, n = 1/2$

$$\frac{d^{1/2} y}{t^{1/2}} = 2\sqrt{\frac{t}{\pi}}. \tag{1.1}$$

Fourier listed the fractional derivative, but did not provide any examples or implementations. So, N.H. Abel was the first to render applications [2]. He analyzed the fractional calculus in the integral equation solution.

Over the time, numerous mathematicians, employing their own terminology and methodology, have defined different concepts meeting the concept of non-integer order derivatives and integrals. Definition of Riemann-Liouville fractional is one of the most common definition in fractional calculus. Riemann-Liouville definition of fractional derivative provides equivalent result as in equation (1.1). Most of concepts of fractional calculus are essentially modifications of Riemann-Liuoville interpretation, This version and its extensions are most likely to be addressed in this work.

1.1 Basic Knowledge

Before specifying the definition of integration and derivative of Reimann Liouville and Caputo, we shall first state some special functions that are needed to properly understand the definitions to come.

1.1.1 Some Special Functions

We will introduce the fundamental meanings and characteristics of some special functions in this section, that are the main pillars of fractional calculus.

Gamma Function

Gamma function is one of essential features of fractional calculus.

Definition 1.1.1. [27] The gamma function can be defined as:

$$\Gamma(h) = \int_0^{\infty} e^{-v} v^{h-1} dv, \;\; h \in \mathbb{R}^+.$$

Some of the fundamental properties are:

(i) $\Gamma(1) = 1$.

(ii) $\Gamma(h+1) = h!\,,\;\; h$ is a non-negative integer.

(iii) $\Gamma(h+1) = h\Gamma(h),\;\; h > 0$.

Here are some commonly found examples of gamma function for different value of h, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^n}(2n-1)!$, $\Gamma(-n) = \pm\infty, n = 0, 1, 2, 3, ...$.
The analytic extension of gamma function is given as below:

$$\Gamma(h) = \frac{\Gamma(h + w)}{h(h + 1)(h + 2)...(h + w - 1)},$$

for any positive value of w. The above formula defines $\Gamma(h)$ for $-w < h < 0$ and $h \neq -1, -2, ... - w + 1$. Thus domain of the gamma function is $h \in \mathbb{R} - \{0, -1, -2, -3, ...\}$.

Beta Function

It is more efficient to apply beta function instead of gamma in certain situations.

Definition 1.1.2. [27] The beta function can be defined as:

$$B(\tau_1, \tau_2) = \int_0^1 v^{\tau_1 - 1}(1 - v)^{\tau_2 - 1} dv, \ \tau_1 > 0, \ \tau_2 > 0.$$

Beta function can also be defined using gamma function as given below

$$B(\tau_1, \tau_2) = \frac{\Gamma(\tau_1)\Gamma(\tau_2)}{\Gamma(\tau_1 + \tau_2)}.$$

Mittag-Leffler Function

Among the important functions associated with fractional differential equation is Mittag-Leffler function.

Definition 1.1.3. [27] One-parameter and two-parameter Mittag-Leffler functions denoted by $E_\kappa(t)$ and $E_{\kappa,\zeta}(t)$ respectively and defined as:

$$E_\kappa(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\kappa + 1)}, \ \kappa > 0.$$

For $\kappa = 1$, we have $E_1(t) = e^t$ and for $\kappa = 2$, $E_2(t) = \cosh t$.

$$E_{\kappa,\zeta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\kappa + \zeta)}, \ \kappa > 0, \ \zeta > 0.$$

In particular, $E_{1,2}(t) = \frac{e^t - 1}{t}$, $E_{2,2}(t) = \frac{\sinh\sqrt{t}}{\sqrt{t}}$.

1.2 Riemann-Liouville and Caputo Fractional Integral and Derivative

Fractional integral and derivative were developed by prolongation of integer order integral and derivative. The fractional integral can be derived from conventional expression for repeated integration of a function. Generally, this approach is known as Riemann-Liouville method. Cauchy is credited with following method for determining pth integration of function.

$$\int_a^t \int_a^{z_{p-1}} ... \int_a^{z_1} g(z)dzdz_1...dz_{p-1} = \frac{1}{(p-1)!}\int_a^t (t-z)^{p-1} g(z)dz.$$

The operator I_a^p denotes repeated integration,

$$I_a^p g(t) = \frac{1}{(p-1)!}\int_a^t (t-z)^{p-1} g(z)dz, \ p \in \mathbb{Z}^+.$$

The above formula does not provide fractional order integration. For all the real values, gamma function gives extension of fractional. Thus, formula for fractional integration can be obtained by replacing fractional expression by gamma function.

$$I_a^\kappa g(t) = \frac{1}{\Gamma(\kappa)}\int_a^t (t-z)^{\kappa-1} g(z)dz, \ \kappa \in \mathbb{R}^+.$$

Definition 1.2.1. [27,35] The Riemann-Liouville fractional integral of order $\kappa > 0$ is stated as:

$$I_a^\kappa g(t) = \frac{1}{\Gamma(\kappa)}\int_a^t (t-z)^{\kappa-1} g(z)dz.$$

For $t \geq a$, if g is continuous then

$$\lim_{\kappa \to 0} I_a^\kappa g(t) = g(t).$$

Lemma 1.2.2. When $g(t) = t^\zeta, \ \zeta > -1$ then we have

$$I_0^\kappa t^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\kappa+\zeta+1)} t^{\zeta+\kappa}.$$

Example 1.2.1. For $\kappa = 2$ and $\zeta = 3/2$,

$$\begin{aligned} I_0^2 t^{3/2} &= \frac{\Gamma(3/2+1)}{\Gamma(2+3/2+1)} t^{3/2+2} \\ &= \frac{\Gamma(3/2+1)}{\Gamma(7/2+1)} t^{7/2} \\ &= \frac{6}{35} \frac{\Gamma(3/2)}{\Gamma(3/2+1)} t^{7/2} \\ &= \frac{4}{35} t^{7/2}. \end{aligned}$$

Lemma 1.2.3. Suppose g be an integrable function and $\kappa, \zeta > 0$. Then the composition of integrals is:

$$I_a^\kappa I_a^\zeta g(t) = I_a^{\kappa+\zeta} g(t) = I_a^\zeta I_a^\kappa g(t).$$

Definition 1.2.4. [35] Assume a function g defined on $[a,b]$ and $n-1 \leq \kappa < n$. Then Riemann-Liouville derivative of order κ is:

$$D_a^\kappa g(t) = \frac{1}{\Gamma(n-\kappa)} \frac{d^n}{dt^n} \int_a^t (t-z)^{n-\kappa-1} g(z) dz.$$

Lemma 1.2.5. When $g(t) = t^\zeta$, then

$$D_0^\kappa t^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\kappa+1)} t^{\zeta-\kappa}, \ \zeta > -1.$$

Example 1.2.2. For $\kappa = 1/2$ and $\zeta = 1$,

$$D_0^{1/2} t = \frac{\Gamma(2)}{\Gamma(3/2)} t^{1/2}$$

$$= 2\sqrt{\frac{t}{\pi}}.$$

Theorem 1.2.1. Assume that $C^n[a,b]$ be a space of all continuous and n-times differentiable functions. Let $g \in C^n[a,b],\ n \in \mathbb{N},\ h \in C^1[a,b]$ and $\kappa \in (n-1,n)$. Then results given below hold:

(i) $D_a^\kappa I_a^\kappa h(t) = h(t)$.

(ii) $I_a^\kappa D_a^\kappa g(t) = g(t) - \sum_{j=0}^{n-1} [D^{\kappa-j-1} g(t)]_{t=a} \frac{(t-a)^{\kappa-j-1}}{\Gamma(\kappa-j)}$.

The Caputo fractional derivative was presented by Caputo in 1967. First nth derivative of function is evaluated then fractional integral is applied.

Definition 1.2.6. Suppose $n-1<\kappa\leq n$ and g be n-times differentiable function. Then Caputo fractional derivative of order κ is stated as:

$$\begin{aligned} {}^{c}\mathcal{D}_a^{\kappa} g(t) &= I_a^{n-\kappa} D_a^n g(t) \\ &= \frac{1}{\Gamma(n-\kappa)} \int_a^t (t-z)^{n-\kappa-1} g^{(n)}(z) dz. \end{aligned}$$

The Caputo derivative of fractional integral is

$${}^{c}\mathcal{D}_a^{\kappa} I_a^{\kappa} g(t) = g(t),$$

This shows that ${}^{c}\mathcal{D}_a^{\kappa}$ is left inverse of I_a^{κ}.
The fractional integral of Caputo derivative is

$$I_a^{\kappa}\, {}^{c}\mathcal{D}_a^{\kappa} g(t) = g(t) - \sum_{j=0}^{n-1} \frac{t^j}{j!} g^{(j)}(a).$$

1.2.1 Generalized Integral and Derivative

In order to resolve the overwhelming number of definitions, we should consider general operators, we can choose specific kernels and some sort of differential operator to obtain classical fractional integrals and derivatives. However, due to the arbitrary existence of the kernel, most of the basic laws of the derivative operator can not be acquired. In order to overcome this trouble, another solution is to consider a special case with kernel of form $k(t,z)=\xi(t)-\xi(z)$ and the derivative operator is of type $\frac{1}{\xi'(t)}\frac{d}{dt}$.

Definition 1.2.7. [3] Suppose a finite or infinite interval $[a,b]$ and $\kappa>0$, an integrable function g over $[a,b]$, $\xi\in C^1[a,b]$ is increasing function on $[a,b]$ where $\forall\ t\in[a,b]$, $\xi'(t)\neq 0$. Then generalized fractional integral of function with respect to ξ is given as follows:

$$I_a^{\kappa,\xi} g(t) = \frac{1}{\Gamma(\kappa)} \int_a^t \xi'(z)(\xi(t)-\xi(z))^{\kappa-1} g(z) dz. \tag{1.2}$$

For $n = \lfloor \kappa \rfloor + 1$, the generalized fractional derivative of g with respect to other function ξ is given by

$$\begin{aligned} D_a^{\kappa,\xi} g(t) &= \left(\frac{1}{\xi'(t)} \frac{d}{dt} \right)^n I_a^{n-\kappa,\xi} g(t) \\ &= \frac{1}{\Gamma(n-\kappa)} \left(\frac{1}{\xi'(t)} \frac{d}{dt} \right)^n \int_a^t \xi'(z)(\xi(t) - \xi(z))^{n-\kappa-1} g(z) dz. \end{aligned}$$

The Hadamard and Riemann-Liouville fractional operators can be obtained by setting $\xi(t) = \ln t$ and $\xi(t) = t$ respectively.

Lemma 1.2.8. [3] If $\kappa, \zeta > 0$, then the semigroup property holds:

$$I_a^{\kappa,\xi} I_a^{\zeta,\xi} g(t) = I_a^{\kappa+\zeta,\xi} g(t) = I_a^{\zeta,\xi} I_a^{\kappa,\xi} g(t).$$

By shifting the ordinary derivative with fractional order derivative, Caputo redeveloped Riemann-Liouville fractional derivative known as Caputo fractional derivative. Here we also introduce a generalized Caputo-type operator.

Definition 1.2.9. Let $\kappa > 0$, on $[a, b]$, ξ is increasing and continuous function with $\xi'(t) \neq 0, \ \forall \ t \in [a, b]$ and $g \in C^n[a, b], \ n \in \mathbb{N}$. Then generalized Caputo derivative of order κ is specified as:

$${}^c D_a^{\kappa,\xi} g(t) = I_a^{n-\kappa,\xi} \left(\frac{1}{\xi'(t)} \frac{d}{dt} \right)^n g(t),$$

here for $\kappa \in \mathbb{N}$, $n = \kappa$ and for $\kappa \notin \mathbb{N}$, $n = \lfloor \kappa \rfloor + 1$.
For convenience, let

$$g_\xi^{[n]}(t) = \left(\frac{1}{\xi'(t)} \frac{d}{dt} \right)^n g(t).$$

If $n = \kappa \in \mathbb{N}$, then from definition we have

$${}^c D_a^{\kappa,\xi} g(t) = g_\xi^{[n]}(t),$$

and if $\kappa \notin \mathbb{N}$ then

$${}^c D_a^{\kappa,\xi} g(t) = \frac{1}{\Gamma(n-\kappa)} \int_a^t \xi'(z)(\xi(t) - \xi(z))^{n-\kappa-1} g_\xi^{[n]}(z) dz.$$

Lemma 1.2.10. Suppose $g : [a,b] \to \mathbb{R}$ and $\kappa > 0$, then following results hold:

(i) If $g \in C^1[a,b]$, then

$$ {}^{c}D_a^{\kappa,\xi} I_a^{\kappa,\xi} g(t) = g(t). $$

(ii) If $g \in C^{n-1}[a,b]$ then following holds

$$ I_a^{\kappa,\xi}\, {}^{c}D_a^{\kappa,\xi} g(t) = g(t) - \sum_{j=0}^{n-1} c_j (\xi(t) - \xi(a))^j, $$

with

$$ c_j = \frac{g_\xi^{[j]}(a)}{j!}. $$

In particular, when $\kappa \in (0,1)$

$$ {}^{c}D_a^{\kappa,\xi} I_a^{\kappa,\xi} g(t) = g(t) - g(a). $$

For the existence of integral (1.2), we must have $\kappa > 0$. Moreover under certain assumptions we have

$$ \lim_{\kappa \to 0} I_a^{\kappa,\xi} g(t) = g(t). \tag{1.3} $$

If g is differentiable and continuous for $t \geq 0$ then the proof of (1.3) is simple. In this case the integration by parts will result into

$$ \begin{aligned} I_a^{\kappa,\xi} g(t) &= \frac{(\xi(t) - \xi(a))^\kappa g(a)}{\Gamma(\kappa+1)} + \frac{1}{\Gamma(\kappa+1)} \int_a^t (\xi(t) - \xi(z))^\kappa g'(z) dz \\ \lim_{\kappa \to 0} I_a^{\kappa,\xi} g(t) &= g(a) + \int_a^t g'(z) dz = g(t). \end{aligned} $$

If $g(t)$ is only continuous for $t \geq a$, then proof of (1.3) is a bit lengthy. For this write (1.2) in form given below

$$ \begin{aligned} I_a^{\kappa,\xi} g(t) &= \frac{1}{\Gamma(\kappa)} \int_a^t \xi'(z)(\xi(t) - \xi(z))^{\kappa-1}(g(z) - g(t)) dz \\ &\quad + \frac{1}{\Gamma(\kappa)} \int_a^t \xi'(z)(\xi(t) - \xi(z))^{\kappa-1} g(t) dz \end{aligned} $$

$$\begin{aligned} I_a^{\kappa,\xi}g(t) &= \frac{1}{\Gamma(\kappa)}\int_a^{t-\delta}\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}(g(z)-g(t))dz & (1.4)\\ &+\frac{1}{\Gamma(\kappa)}\int_{t-\delta}^{t}\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}(g(z)-g(t))dz & (1.5)\\ &+\frac{g(t)(\xi(t)-\xi(a))^{\kappa}}{\Gamma(\kappa+1)}. \end{aligned}$$

Consider (1.4)

$$|I_1|\leq \frac{N}{\Gamma(\kappa)}\int_a^{t-\delta}\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}dz.$$

After integration and using limit $\kappa\to 0$ we have

$$\lim_{\kappa\to 0}|I_1|=0.$$

Let us consider the integral (1.5). Since g is continuous, for every $\delta>0$ there exist $\varepsilon>0$ such that

$$|g(z)-g(t)|<\varepsilon.$$

So, result obtained is

$$\begin{aligned} |I_2| &< \frac{1}{\Gamma(\kappa)}\int_{t-\delta}^{t}\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}|g(z)-g(t)|dz\\ &< \frac{\epsilon}{\Gamma(\kappa)}\int_{t-\delta}^{t}\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}dz=\frac{\epsilon}{\Gamma(\kappa+1)}(\xi(t)-\xi(t-\delta))^{\kappa}. \end{aligned}$$

Taking $\epsilon\to 0$ as $\delta\to 0$ we get for all $\kappa\geq 0$,

$$\lim_{\delta\to 0}|I_2|=0.$$

Considering

$$\begin{aligned} |I_a^{\kappa,\xi}g(t)| &\leq |I_1|+|I_2|+|g(t)|\left|\frac{(\xi(t)-\xi(a))^{\kappa}}{\Gamma(\kappa+1)}\right|\\ \lim_{\kappa\to 0}|I_a^{\kappa,\xi}g(t)| &= |g(t)|. \end{aligned}$$

So,

$$\lim_{\kappa\to 0}I_a^{\kappa,\xi}g(t)=g(t),$$

holds if for $t\geq a,\ g$ is continuous.

1.3 Results from Analysis

In this section, few concepts from real analysis are presented that are needed for existence of solution for fractional Langevin equation.

Definition 1.3.1. Assume vector space V and suppose for each element $g \in V$ there is a non-negative number $\|g\|$ assigned in such that $\forall\ g, h \in V$:

(i) $\|g\| = 0$ iff $g = 0$;

(ii) $\|cg\| = |c|\|g\|$ for any scalar c;

(iii) $\|g + h\| \leq \|g\| + \|h\|$.

The quantity $\|g\|$ is called the norm of g and V is known as the norm space.

Definition 1.3.2. Suppose a subset X of Banach space $\mathfrak{B}$ and functional S with domain $X \subset \mathfrak{B}$, that is S be a mapping from X to $\mathfrak{B}$. Then S is equicontinuous on X iff, for each $\epsilon > 0$ there is $\delta > 0$ such that $|S(g) - S(k)| < \epsilon$ for all $g,\ k \in X$ whenever $|g - k| \leq \delta$.

Example 1.3.1. $S = \{h : [0, 1] \to \mathbb{R}|\ h(\pi) = c\pi, c \in (0, 1)\}$, then S is eqicontinuous. Choose $\delta = \epsilon > 0$ for given $\epsilon > 0$

$$|h(\pi_1) - h(\pi_2)| = c|\pi_1 - \pi_2| < |\pi_1 - \pi_2| < \epsilon.$$

Definition 1.3.3. Assume X be the subset of Banach space $\mathfrak{B}$. If any class of open sets covering X has a finite subclass covering X, then X is compact..
If closure of X is compact then the X is said to be relatively compact.

Definition 1.3.4. An element of function's domain that is mapped to itself by function is called a fixed point. t is fixed point of g if $g(t) = t$.

In following theorem Q represents a subset of $\mathbb{R}^n$ and $C(Q)$ is Banach space of real valued continuous functions with maximum norm.

Theorem 1.3.1. [17] Assume a bounded subset Q of $\mathbb{R}^n$ and $X \subset C(Q)$. Then X is relatively compact iff it is equicontinuous and bounded.

Theorem 1.3.2. [17] Suppose Z be a convex, nonempty and compact subset of Banach space $\mathfrak{B}$, also assume $A : Z \to Z$ maps Z to itself and is compact. Then a fixed point of A exists in Z.

Theorem 1.3.3. [31] Consider a closed, nonempty and convex subset X of a Banach space Y. Suppose U, V be operators so that (i) $Up + Vq \in X$ for $p, q \in X$. (ii) V is a contraction mapping. (iii) U is continuous and compact. Then there exist $r \in X$ such that $r = Ur + Vr$.

Lemma 1.3.5. [16] Assume $\mathfrak{B}$ is Banach space and cone $T \subseteq \mathfrak{B}$. Also assume ω_1 and ω_2 be open discs in $\mathfrak{B}$. Also $0 \in \omega_1$ and $\overline{\omega_1} \subset \omega_2$. Furthermore, assume $G : T \cap (\overline{\omega}_2/\omega_1) \to T$ be completely continuous operator. Then either

(*i*) $\|Gq\| \leq \|q\|$ for $q \in Z \cap \partial\omega_1$ and $\|Gq\| \geq \|q\|$ for $q \in Z \cap \partial\omega_2$, or

(*ii*) $\|Gq\| \geq \|q\|$ for $q \in Z \cap \partial\omega_1$ and $\|Gq\| \leq \|q\|$ for $q \in Z \cap \partial\omega_2$.

Then there exist at least one fixed point of operator G in $Z \cap (\overline{\omega}_2/\omega_1)$.

Chapter 2

Generalized Fractional Langevin Equation

Langevin equation is a basic Brownian motion principle to explain evolution of physical processes in evolving conditions [12, 39], that was presented and formulated by Paul Langevin [21], in 1908. Numerous generalizations of Langevin equation have been formulated and researched by many researchers around the globe due to the advancement of fractional derivatives [18, 22, 23, 26]. Some useful applications of equation are studying the fluid suspensions, photoelectron counting, modeling the evacuation processes and protein dynamics.

In [15], Fa found a fractional Langevin equation involving Riemann–Liouville fractional derivative to analyze deviations, position and velocity similarity functions of the device. Fractional Langevin equation is involved in the modeling of fundamental motor control system [36].

Bashir Ahmad and Jaun J. Nieto in [5], discussed existence of solution for following boundary value problem,

$$
\begin{gathered}
{}^{c}D_{0}^{\zeta}({}^{c}D_{0}^{\kappa} + \lambda)z(l) = g(l, z(l)),\ 0 < l < 1, \\
z(0) = \gamma_1,\ z(1) = \gamma_2,
\end{gathered}
$$

where $0 < \kappa, \zeta \leq 1$, λ is a real number, ${}^{c}D$ is Caputo derivative, $\gamma_1, \gamma_2 \in X$ and $g : [0, 1] \times X \to X$. X is a Banach space of all continuous funcions.

In [6], the authors examined existence of solution for following three point boundary value problem with Langevin equation,

$$
\begin{aligned}
&{}^cD_0^\zeta({}^cD_0^\kappa+\lambda)s(z)=\phi(z,s(z)),\ 0<z<1,\\
&s(0)=0,\ s(\eta)=0,\ s(1)=0,\ 0<\eta<1,
\end{aligned}
$$

where $0<\kappa\leq 1,\ 1<\zeta\leq 2,\lambda$ is a real number, a continuous function $\phi:[0,1]\times\mathbb{R}\to\mathbb{R}$ and ${}^cD$ is Caputo fractional derivative.

In [11], Nieto and Baghani studied the existence of solutions for following Langevin equation,

$$
\begin{aligned}
&{}^cD^\zeta({}^cD^\kappa+\lambda)z(w)=\varkappa(s,z(w)),\ 0\leq s\leq 1,\\
&z(0)=z(1)=0,\ D^{2\kappa}z(1)+\lambda D^\kappa z(1)=0,
\end{aligned}
$$

where $0<\kappa\leq 1,\ 1<\zeta\leq 2$, ${}^cD$ is Caputo derivative, $\varkappa:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous function, $\lambda\in\mathbb{R}$ and $D^{2\kappa}$ is sequential derivative given by:

$$D^{k\kappa}u=D^\kappa D^{(k-1)\kappa}u,\ k=2,3,....$$

Authors in [1], investigated uniqueness and existence result for problem stated below,

$$
\begin{aligned}
&{}^cD_a^{\kappa,\phi}z(r)=g(r,z(r)),\ r\in[a,b],\\
&z_\phi^{[k]}(a)=z_a^k,\ z_\phi^{n-1}(b)=z_b,\ k=0,1,...,n-2,
\end{aligned}
$$

where ${}^cD_a^{\kappa,\phi}$ is ϕ-Caputo fractional derivative of order $n-1<\kappa<n,\ z_a^k,z_b\in\mathbb{R}$ and $g:[a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous function.

Our goal in this chapter is to analyze the generalized fractional Langevin equation with two different generalized fractional derivatives. Specifically, we considered problem given below,

$$
\begin{aligned}
&D_0^{\zeta,\xi}(D_0^{\kappa,\xi}+\lambda)x(t)=g(t,x(t)),\ 0<t<1,\\
&x(0)=\gamma_1,\ x(1)=\gamma_2,
\end{aligned}
\tag{2.1}
$$

where $0<\kappa,\zeta<1$, D is Riemann-Liouville fractional derivative, $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, λ is real number, $\gamma_1,\gamma_2\in\mathbb{R}$, $(\mathbb{R},\|.\|)$ be a Banach space and $A=C([0,1],\mathbb{R})$ is Banach space of continuous functions with uniformly convergent topology with $\|x\|=\max|x(t)|,\ t\in[0,1]$.

We found a general solution to the problem (2.1) in Lemma 2.1.1. We used the Banach Contraction mapping principle in section 2.1 for existence of a unique solution for problem (2.1) and in section 2.2, for existence of at least one solution for problem (2.1), Krasnoselskii's theorem is employed.

The linear problem related to (2.1) is given by,

$$\begin{aligned} D_0^{\zeta,\xi}(D_0^{\kappa,\xi}+\lambda)x(t) &= \sigma(t),\ 0<t<1, \\ x(0)=\gamma_1,\ x(1)&=\gamma_2, \end{aligned} \tag{2.2}$$

here $0<\kappa,\zeta\leq 1$ and $\sigma\in C[0,1]$.

2.1 Existence of Unique Solution

We intended to check existence of unique solution for problem (2.1) in this section. We will first find an integral form of equation for (2.1) that will further allow us to verify the uniqueness of the solution.

Lemma 2.1.1. The problem (2.2) has following solution:

$$\begin{aligned} x(t) = &\int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}\sigma(w)dw - \lambda x(z)\right]dz \\ &+ \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\Bigg[\lambda x(z) \\ &- \int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}\sigma(w)dw\Bigg]dz + \gamma_2\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}. \end{aligned} \tag{2.3}$$

Proof. Consider the linear differential equation in (2.2)

$$D_0^{\zeta,\xi}(D_0^{\kappa,\xi}+\lambda)x(t)=\sigma(t). \tag{2.4}$$

Applying $I_0^{\zeta,\xi}$ and $I_0^{\kappa,\xi}$ respectively and using the following property

$$I_0^{\gamma,\xi}D_0^{\gamma,\xi}g(t)=g(t)-c_0(\xi(t)-\xi(0))^{\gamma-1}.$$

We obtained following general solution for equation (2.4)

$$\begin{aligned} x(t) &= c_1(\xi(t)-\xi(0))^{\kappa-1}-\lambda I^{\kappa,\xi}x(t)+\frac{c_0\Gamma(\zeta)}{\Gamma(\kappa+\zeta)}\times(\xi(t)-\xi(0))^{\kappa+\zeta-1} \\ &\quad +I^{\kappa+\zeta,\xi}\sigma(t). \end{aligned} \tag{2.5}$$

Using the boundary conditions for linear equation (2.2), we get $c_1 = 0$ and

$$\frac{c_0\Gamma(\zeta)}{\Gamma(\kappa+\zeta)} = \frac{1}{(\xi(1)-\xi(0))^{\kappa+\zeta-1}}\Bigg[\lambda I^{\kappa,\xi}x(1) - I^{\kappa+\zeta,\xi}\sigma(1) + \gamma_2\Bigg].$$

Using these values in equation (2.5)

$$x(t) = -\lambda I^{\kappa,\xi}x(t) + I^{\kappa+\zeta,\xi}\sigma(t) + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}[\lambda I^{\kappa,\xi}x(1) - I^{\kappa+\zeta,\xi}\sigma(1) + \gamma_2].$$

After simplification, we obtained the general solution as required in equation (2.3). □

In following theorem, we will verify the existence of unique solution for problem (2.1).

Theorem 2.1.1. Suppose $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function and also satisfies following inequality

$$|g(t,x) - g(t,y)| \leq \eta|x-y|,\ \forall\ x,y \in \mathbb{R},\ t \in [0,1]. \tag{2.6}$$

Then unique solution of problem (2.1) exists if $\Upsilon < 1$, where

$$\Upsilon = \left[\frac{2\eta(\xi(1)-\xi(0))^{\kappa+\zeta}}{\Gamma(\kappa+\zeta+1)} + \frac{2|\lambda|(\xi(1)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)}\right].$$

Proof. Define set $U_r = \{x \in A : \|x\| \leq r\}$, where

$$r \geq \frac{1}{1-\nu}\left[\frac{2N(\xi(1)-\xi(0))^{\kappa+\zeta}}{\Gamma(\kappa+\zeta+1)} + |\gamma_2|\right],\ \Upsilon \leq \nu < 1.$$

Also state $\|g(w)\| = \max_{w\in[0,1]}|g(w)|$ and set $\max_{w\in[0,1]}|g(w,0)| = N$. Define $O : A \to A$ by

$$\begin{aligned}(Ox)(t) &= \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\Bigg[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}g(w,x(w))dw \\ &\quad -\lambda x(z)\Bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\Bigg[\lambda x(z) \\ &\quad - \int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}g(w,x(w))dw\Bigg]dz + \gamma_2\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\end{aligned}$$

$$
\begin{aligned}
&|(Ox)(t)| \\
\leq\; & \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}|g(w,x(w))|dw \\
& +|\lambda||x(z)|\bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda||x(z)| \\
& + \int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}|g(w,x(w))|dw\bigg]dz + |\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
\leq\; & \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(|g(w,x(w))-g(w,0)| \\
& +|g(w,0)|)dw + |\lambda||x(z)|\bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)} \\
& \times\bigg[|\lambda||x(z)|+\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(|g(w,x(w))-g(w,0)|+|g(w,0)|)dw\bigg]dz \\
& +|\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
\leq\; & \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(\eta|x(w)|+|g(w,0)|)dw \\
& +|\lambda||x(z)|\bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda||x(z)| \\
& + \int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(\eta|x(w)|+|g(w,0)|)dw\bigg]dz + |\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
\leq\; & \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(\eta\|x\|+N)dw \\
& +|\lambda|\|x\|\bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda|\|x\| \\
& + \int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(\eta\|x\|+N)dw\bigg]dz + |\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
\leq\; & \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(\eta r+N)dw \\
& +|\lambda|r\bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda|r + \\
& \int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}(\eta r+N)dw\bigg]dz + |\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}.
\end{aligned}
$$

After integration we get

$|(Ox)(t)|$

$$\begin{aligned}
&\leq \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[(\eta r+N)\frac{(\xi(z)-\xi(0))^{\zeta}}{\Gamma(\zeta+1)}+|\lambda|r\right]dz+\\
&\quad\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[(\eta r+N)\frac{(\xi(z)-\xi(0))^{\zeta}}{\Gamma(\zeta+1)}\right.\\
&\quad\left.+|\lambda|r\right]dz+|\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\\
&= \frac{\eta r+N}{\Gamma(\zeta+1)}\int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}(\xi(z)-\xi(0))^{\zeta}dz+\frac{|\lambda|r}{\Gamma(\kappa)}\\
&\quad\times\int_0^t \xi'(z)(\xi(t)-\xi(z))^{\kappa-1}dz+\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\frac{\eta r+N}{\Gamma(\zeta+1)}\\
&\quad\times\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}(\xi(z)-\xi(0))^{\zeta}dz+\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\\
&\quad\times\frac{|\lambda|r}{\Gamma(\kappa)}\int_0^1 \xi'(z)(\xi(1)-\xi(z))^{\kappa-1}dz+|\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}.
\end{aligned}$$

After integration and using the following result

$$I^{\kappa,\xi}(\xi(t)-\xi(0))^{\zeta}=\frac{\Gamma(\zeta+1)}{\Gamma(\kappa+\zeta+1)}(\xi(t)-\xi(0))^{\zeta+\kappa}.$$

We obtain

$|(Ox)(t)|$

$$\begin{aligned}
&\leq \frac{\eta r+N}{\Gamma(\kappa+\zeta+1)}(\xi(t)-\xi(0))^{\kappa+\zeta}+\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\\
&\times\frac{\eta r+N}{\Gamma(\kappa+\zeta+1)}(\xi(1)-\xi(0))^{\kappa+\zeta}+\frac{|\lambda|r}{\Gamma(\kappa+1)}(\xi(t)-\xi(0))^{\kappa}\\
&+\frac{|\lambda|r}{\Gamma(\kappa+1)}\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}(\xi(1)-\xi(0))^{\kappa}+|\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}.
\end{aligned} \tag{2.7}$$

Since ξ is an increasing function and $t\in[0,1]$ so,

$$\begin{aligned}
\xi(t) &\leq \xi(1)\\
\xi(t)-\xi(0) &\leq \xi(1)-\xi(0).
\end{aligned} \tag{2.8}$$

Using (2.8) in (2.7), we obtain

$$
\begin{aligned}
\|Ox\| &\leq \frac{2(\eta r+N)}{\Gamma(\kappa+\zeta+1)}(\xi(1)-\xi(0))^{\kappa+\zeta}+\frac{2|\lambda|r}{\Gamma(\kappa+1)}(\xi(1)-\xi(0))^{\kappa}+|\gamma_2| \\
&\leq (\Upsilon+1-\nu)r \leq r.
\end{aligned}
$$

For every $t \in [0,1]$ and $x, y \in A$, we will show

$$
\|(Ox)(t)-(Oy)(t)\| \leq \Upsilon\|x-y\|,
$$

$$
\begin{aligned}
&(Ox)(t)-(Oy)(t) \\
&= \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\Big[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}[g(w,x(w)) \\
&\quad -g(w,y(w))]dw\Big]dz+\int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\lambda(y(z)-x(z))dz \\
&\quad +\Big[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\Big]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\lambda(x(z)-y(z))dz \\
&\quad +\Big[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\Big]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\Big[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)} \\
&\quad \times[g(w,y(w))-g(w,x(w))]dw\Big]dz
\end{aligned}
$$

$$
\begin{aligned}
&|(Ox)(t)-(Oy)(t)| \\
&\leq \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\Big[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}|g(w,x(w)) \\
&\quad -g(w,y(w))|dw\Big]dz+\int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz \\
&\quad +\Big[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\Big]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz \\
&\quad +\Big[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\Big]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\Big[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)} \\
&\quad \times|g(w,x(w))-g(w,y(w))|dw\Big]dz.
\end{aligned}
$$

Rearranging and using inequality (2.6), we get

$|(Ox)(t)-(Oy)(t)|$

$$\begin{aligned}
&\leq \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}\eta|x(w)-y(w)|dw\right]dz + \\
&\int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
&\times \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
&\times \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}\eta|x(w)-y(w)|dw\right]dz \\
&\leq \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}\eta\|x-y\|dw\right]dz \\
&+ \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}|\lambda|\|x-y\|dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
&\times \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}|\lambda|\|x-y\|dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
&\times \int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[\int_0^z \frac{\xi'(w)(\xi(z)-\xi(w))^{\zeta-1}}{\Gamma(\zeta)}\eta\|x-y\|dw\right]dz.
\end{aligned}$$

After integration

$|(Ox)(t)-(Oy)(t)|$

$$\begin{aligned}
&\leq \int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[\frac{(\xi(z)-\xi(0))^{\zeta}}{\Gamma(\zeta+1)}\eta\|x-y\|\right]dz + \frac{(\xi(t)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)} \\
&\times|\lambda|\|x-y\| + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\frac{(\xi(t)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)}|\lambda|\|x-y\| \\
&+\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}\left[\frac{(\xi(z)-\xi(0))^{\zeta}}{\Gamma(\zeta+1)}\right. \\
&\left.\times\eta\|x-y|\right]dz \\
&= \frac{\eta\|x-y\|}{\Gamma(\zeta+1)}\int_0^t \frac{\xi'(z)(\xi(t)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}(\xi(z)-\xi(0))^{\zeta}dz + \frac{(\xi(t)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)} \\
&\times|\lambda|\|x-y\| + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\frac{(\xi(t)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)}|\lambda|\|x-y\| + \\
&\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\frac{\eta\|x-y\|}{\Gamma(\zeta+1)}\int_0^1 \frac{\xi'(z)(\xi(1)-\xi(z))^{\kappa-1}}{\Gamma(\kappa)}(\xi(z)-\xi(0))^{\zeta}dz.
\end{aligned}$$

Integration yields the following result

$$\begin{aligned}\|(Ox)(t)-(Oy)(t)\| &\leq \left[\frac{2\eta(\xi(1)-\xi(0))^{\kappa+\zeta}}{\Gamma(\kappa+\zeta+1)}+\frac{2|\lambda|(\xi(1)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)}\right]\|x-y\| \\ &= \Upsilon\|x-y\|.\end{aligned}$$

As $\Upsilon<1$, then O is a contraction. Thus unique solution exists for problem (2.1). □

2.2 Existence of Solution

We have applied krasnoselskii's theorem to check existence of at least one solution for problem (2.1).

Theorem 2.2.1. Assume that bounded subsets of $[0,1]\times\mathbb{R}$ are mapped to compact subsets of $\mathbb{R}$ by a continuous function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$. Also suppose Lipschitz condition is satisfied

$$|g(t,x)-g(t,y)|\leq\eta|x-y|,\ \forall\ t\in[0,1],\ x,y\in\mathbb{R}.$$

If

$$\left[\frac{|\lambda|(\xi(1)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)}+\frac{(\xi(1)-\xi(0))^{\kappa+\zeta}}{\Gamma(\kappa+\zeta+1)}\right]\leq 1.$$

Then at least one solution exists on interval [0,1] for problem (2.1).

Proof. Suppose $U_r=\{x\in A:\|x\|\leq r\}$, where

$$r\geq\frac{\Gamma(\kappa+1)[2N(\xi(1)-\xi(0))^{\kappa+\zeta}+|\gamma_2|]}{\Gamma(\kappa+\zeta+1)\Gamma(\kappa+1)-2\eta\Gamma(\kappa+1)(\xi(1)-\xi(0)^{\kappa+\zeta}-2|\lambda|\Gamma(\kappa+\zeta+1)(\xi(1)-\xi(0))^{\kappa}}.$$

Now define operators on U_r as follows

$$\begin{aligned}(\omega x)(t) &= \int_0^t\frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\Bigg[\int_0^z\frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}g(w,x(w))dw \\ &\quad -\lambda x(z)\Bigg]dz,\end{aligned}$$

and

$$\begin{aligned}(\theta x)(t) &= \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1\frac{\xi'(z)}{\Gamma(\kappa)}(\xi(1)-\xi(z))^{\kappa-1}\Bigg[\lambda x(z) \\ &\quad -\int_0^z\frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}g(w,x(w))dw\Bigg]dz+\gamma_2\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}.\end{aligned}$$

Let $x, y \in U_r$, then

$|(\omega x)(t) + (\theta y)(t)|$

$$\begin{aligned}
\leq\ & \int_0^t \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\Bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}|g(w,x(w))|dw \\
& +|\lambda||x(z)|\Bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(1)-\xi(z))^{\kappa-1}\Bigg[|\lambda||y(z)| \\
& +\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}|g(w,y(w))|dw\Bigg]dz + |\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \\
\leq\ & \int_0^t \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\Bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}\Big(|g(w,x(w))-g(w,0)| \\
& +|g(w,0)|\Big)dw + |\lambda||x(z)|\Bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(1)-\xi(z))^{\kappa-1} \\
& \Bigg[|\lambda||y(z)|+\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}\Big(|g(w,y(w))-g(w,0)|-|g(w,0)|\Big)dw\Bigg]dz \\
& +|\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}.
\end{aligned}$$

Let us define $\|g(w)\| = \max_{w\in[0,1]}|g(w)|$ and set $\max_{w\in[0,1]}|g(w,0)| = N$. So,

$|(\omega x)(t) + (\theta y)(t)|$

$$\begin{aligned}
\leq\ & \int_0^t \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\Bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(\eta\|x\|+N)dw \\
& +|\lambda|\|x\|\Bigg]dz + \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(1)-\xi(z))^{\kappa-1}\times \\
& \Bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(\eta\|y\|+N)dw + |\lambda|\|y\|)\Bigg]dz + |\gamma_2|\left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}.
\end{aligned}$$

After integration and using (2.8), we have

$$\begin{aligned}
\|\omega x + \theta y\| &\leq \frac{2(\eta r + N)}{\Gamma(\kappa+\zeta+1)}(\xi(1)-\xi(0))^{\kappa+\zeta} + \frac{2|\lambda|r}{\Gamma(\kappa+1)}(\xi(1)-\xi(0))^{\kappa} + |\gamma_2| \\
&\leq r.
\end{aligned}$$

Hence $\omega x + \theta y \in U_r$. Now we are going to prove that ω is continuous and compact. The continuity of ω is implied by continuity of g. Also ω is bounded on U_r.

$$\begin{aligned}
|(\omega x)(t)| &\leq \int_0^t \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}|g(w,x(w))|dw \\
&\quad +|\lambda||x(z)|\bigg]dz \\
&\leq \int_0^t \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(|g(w,x(w)) \\
&\quad -g(w,0)|+|g(w,0)|)dw+|\lambda||x(z)|\bigg]dz \\
&\leq \int_0^t \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(\eta\|x\|+N)dw \\
&\quad +|\lambda|\|x\|\bigg]dz \\
&\leq \int_0^t \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(\eta r+N)dw \\
&\quad +|\lambda|r\bigg]dz.
\end{aligned}$$

After integration and using the inequality (2.8), we obtained the following result

$$\|\omega\| \leq \frac{(\eta r+N)(\xi(1)-\xi(0))^{\kappa+\zeta}}{\Gamma(\kappa+\zeta+1)}+\frac{|\lambda|r(\xi(1)-\xi(0))^{\kappa}}{\Gamma(\kappa+1)}.$$

To check the compactness of ω, we will first show that ω is equicontinuous.
$|(\omega x)(t_a)-(\omega x)(t_b)|$

$$\begin{aligned}
&\leq \int_0^{t_a} \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t_a)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}|g(w,x(w))|dw \\
&\quad +|\lambda|\ |x(z)|\bigg]dz+\int_0^{t_b} \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t_b)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1} \\
&\quad \times|g(w,x(w))|dw+|\lambda|\ |x(z)|\bigg]dz
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_0^{t_a} \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t_a)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(|g(w,x(w))-g(w,0)| \\
& +|g(w,0)|)dw+|\lambda||x(z)|\bigg]dz+\int_0^{t_b} \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t_b)-\xi(z))^{\kappa-1} \\
& \times\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(|g(w,x(w))-g(w,0)|+|g(w,0)|)dw \\
& +|\lambda||x(z)|\bigg]dz \\
\leq\ & \int_0^{t_a} \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t_a)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(\eta r+N)dw+|\lambda|r\bigg]dz \\
& +\int_0^{t_b} \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(t_b)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}(\eta r+N)dw+|\lambda|r\bigg]dz.
\end{aligned}
$$

Integrating and simplification gives the following result
$\|(\omega x)(t_a)-(\omega x)(t_b)\|$

$$
\begin{aligned}
\leq\ & \frac{\eta r+N}{\Gamma(\kappa+\zeta+1)}(\xi(t_a)-\xi(0))^{\kappa+\zeta}+\frac{|\lambda|r}{\Gamma(\kappa+1)}(\xi(t_a)-\xi(0))^{\kappa} \\
& +\frac{\eta r+N}{\Gamma(\kappa+\zeta+1)}(\xi(t_b)-\xi(0))^{\kappa+\zeta}+\frac{|\lambda|r}{\Gamma(\kappa+1)}(\xi(t_b)-\xi(0))^{\kappa}.
\end{aligned}
$$

The continuity of $(\xi(t_a)-\xi(0))^{\kappa+\zeta},\ (\xi(t_a)-\xi(0))^{\kappa},\ (\xi(t_b)-\xi(0))^{\kappa+\zeta},\ (\xi(t_b)-\xi(0))^{\kappa}$ implies the continuity of ω. For every t, $\omega(\digamma(t))$ is relatively compact in $\mathbb{R}$ because g maps bounded subsets to relatively compact subsets where $\digamma$ is bounded subset of A. Hence on U_r, ω is proved relatively compact. Thus using theorem 1.3.1, ω is compact on U_r. Now we will show that θ is a contraction mapping.
$|(\theta x)(t)-(\theta y)(t)|$

$$
\begin{aligned}
\leq\ & \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(1)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1} \\
& \times|g(w,x(w))-g(w,y(w))|dw+|\lambda||x(z)-y(z)|\bigg]dz \\
\leq\ & \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1}\int_0^1 \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(1)-\xi(z))^{\kappa-1}\bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1} \\
& \times|x(w)-y(w)|dw+|\lambda||x(z)-y(z)|\bigg]dz
\end{aligned}
$$

$$\leq \left[\frac{\xi(t)-\xi(0)}{\xi(1)-\xi(0)}\right]^{\kappa+\zeta-1} \int_0^1 \frac{\xi'(z)}{\Gamma(\kappa)}(\xi(1)-\xi(z))^{\kappa-1}\Bigg[\int_0^z \frac{\xi'(w)}{\Gamma(\zeta)}(\xi(z)-\xi(w))^{\zeta-1}$$
$$\times\|x-y\|dw + |\lambda|\|x-y\|\Bigg]dz.$$

After integration and using (2.8) we arrived at the following result

$$\|(\theta x)(t)-(\theta y)(t)\| \leq \left[\frac{(\xi(1)-\xi(0))^{\kappa}|\lambda|}{\Gamma(\kappa+1)} + \frac{(\xi(1)-\xi(0))^{\kappa+\zeta}}{\Gamma(\kappa+\zeta+1)}\right]\|x-y\|.$$

Since all axioms of Theorem 1.3.3 are fulfilled, hence at least one solution exists for problem (2.1) on $[0,1]$. □

Example 2.2.1. Consider

$$\begin{aligned} D^{\zeta,\xi}(D^{\kappa,\xi}-\lambda)x(t) &= \frac{1}{(t+2)^2}\left(\frac{|x|}{|x|+2}+1\right),\ 0<t<1, \\ x(0)=\gamma_1,&\quad x(1)=\gamma_2, \end{aligned} \tag{2.9}$$

here $\kappa = 1/2,\ \zeta = 2/3,\ \lambda = -1/2$ and

$$g(t,x(t)) = \frac{1}{(t+2)^2}\left(\frac{|x|}{|x|+2}+1\right).$$

Here

$$|g(t,x(t)) - g(t,y(t))| \leq \frac{1}{4}|x(t)-y(t)|.$$

Also for $0<t<1$, $\xi(t) = 2t+1$ is increasing function.
Further

$$k = \frac{18\times 2^{7/6}}{7\Gamma(1/6)} + \sqrt{\frac{2}{\pi}} < 1.$$

So, unique solution exists for problem (2.9) on interval $[0,1]$ by using Theorem 2.1.1.

Chapter 3

Existence of Positive Solution for Fractional Langevin Equations

As a significant area of study, fractional differential equations have appeared. Fractional differential equations exist in a broad range of disciplines namely mechanics, viscoelasticity, morphology, blood flow phenomena, control theory, polymer rheology, etc. Check [4, 7, 8, 20] for more details. In [9, 10] Ahmad et al. examined uniqueness and existence of nonlinear fractional differential and integro-differential equation solutions with number of boundary conditions employing fixed point theorems. Recent study [34], addressed uniqueness and existence result for solutions of nonlinear fractional differential equations for Riemann-Liouville form under generalized non-local boundary condition. Zhi-Wei Lv, in [24] implemented fixed-point theorems for determining existence of positive solutions for system of fractional differential equations.

Langevin equation has a long-term impact on research. Fresh findings on the Langevin equations throughout the diversity of boundary value conditions were already documented, see [28, 29, 32].

In [38], Shuqin Zhang studied the multiplicity and existence of positive solution for problem given below,

$$
\begin{aligned}
D_{0+}^{\kappa}\tau(z) &= g(z,\tau(z)),\ 0<z<1,\\
\tau(0)+\tau'(0) &= 0,\ \tau(1)+\tau'(1)=0,
\end{aligned}
$$

where $1 < \kappa \leq 2$, $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and D is Caputo derivative.

For nonlinear fractional boundary value problem, in [19] authors examined existence result and multiplicity of positive solutions with the help of theory of fixed point theorems on cones,

$$\begin{aligned} D^{\alpha}_{0+}q &= w(s)g(q), \ 0 < s < 1, \\ q(0) &= 0, \ q(1) = 0, \end{aligned}$$

where $w \in L[0,1]$ is bounded on $[0,1]$ and $w(s) > 0$ on $[1/4, 3/4]$, $1 < \alpha < 2$, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\mathbb{R}^+ = [0,\infty)$ and D is Riemann-Liouville fractional derivative.

For the following delay differential equation, Haiping Ye in [37] studied existence of positive solution by employing upper and lower bounds technique,

$$\begin{aligned} D^{\alpha}[r(w) - r(0)] &= r(w)g(w, r_w), \ w \in [0,P], \\ r(w) &= \eta(w), \ w \in [-\varepsilon, 0], \end{aligned}$$

where $0 < \alpha < 1$, $\eta \in K$, $g : [0,P] \times K \to \mathbb{R}^+$ be a continuous function. D is Riemann-Liouville derivative. $K = K([-\varepsilon,0]; \mathbb{R}^+)$ is space of continuous functions from $[-\varepsilon, 0]$ to $\mathbb{R}^+$ with $\|\eta\| = \max\limits_{-\varepsilon \leq \gamma \leq 0} |\eta(\gamma)|$, $r(w)$ is a function in K defined as $r_w(\gamma) = r(w+\gamma)$, $-\varepsilon \leq \gamma \leq 0$ and $\mathbb{R}^+ = [0,+\infty)$.

Encouraged by work on positive solutions as stated above, we find out that there are not much articles so far on existence of positive solutions for fractional Langevin equations. Therefore, here we develop results on uniqueness and existence of positive solutions for nonlinear fractional Langevin equations. First we will study following boundary value problem:

$$\begin{aligned} D^{\zeta}(D^{\kappa} + \lambda)x(t) + \sigma(t, x(t)) = 0, \ 0 < t < 1, \\ D^{\kappa}x(0) = 0, \ x(0) = x(1) = 0, \end{aligned} \tag{3.1}$$

where D is Riemann-Liouville fractional derivative. Secondly we will focus on boundary value problem with Caputo fractional derivative stated below:

$$\begin{aligned} {}^cD^{\zeta}({}^cD^{\kappa} - \widetilde{\lambda})\widetilde{x}(t) = \widetilde{\sigma}(t, \widetilde{x}(t)), \ 0 < t < 1, \\ {}^cD^{\kappa}\widetilde{x}(0) = \widetilde{\lambda}\widetilde{x}(0), \ \widetilde{x}(0) + \widetilde{x}'(0) = 0, \ \widetilde{x}(1) + \widetilde{x}'(1) = 0, \end{aligned} \tag{3.2}$$

where ${}^cD$ is Caputo derivative. $0 < \zeta < 1,\ 1 < \kappa < 2$ and $\lambda,\ \widetilde{\lambda} \in \mathbb{R}^+$. The nonlinear functions $\sigma,\ \widetilde{\sigma} : [0,1] \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous.

3.1 Green Functions

First we will find integral form for problems (3.1) and (3.2) that will assist us in finding the Green function.

Lemma 3.1.1. The problem (3.1) has following solution:

$$x(t) = -\lambda I^{\kappa} x(t) - I^{\kappa+\zeta}\sigma(t, x(t)) + \lambda t^{\kappa-1} I^{\kappa} x(1) + t^{\kappa-1} I^{\kappa+\zeta}\sigma(1, x(1)). \tag{3.3}$$

Proof. Applying I^{ζ} on both sides of (3.1) and using Theorem 1.2.1

$$\begin{aligned} (D^{\kappa} + \lambda)x(t) &= -I^{\zeta}\sigma(t, x(t)) - c_1 t^{\zeta-1},\ c_1 \in \mathbb{R} \\ D^{\kappa} x(t) &= -\lambda x(t) - I^{\zeta}\sigma(t, x(t)) - c_1 t^{\zeta-1}. \end{aligned} \tag{3.4}$$

By using $D^{\kappa} x(0) = 0$ on (3.4), we obtained $c_1 = 0$.

$$D^{\kappa} x(t) = -\lambda x(t) - I^{\zeta}\sigma(t, x(t)).$$

Applying I^{κ} to both sides

$$x(t) = -\lambda I^{\kappa} x(t) - I^{\kappa+\zeta}\sigma(t, x(t)) - c_2 t^{\kappa-1} - c_3 t^{\kappa-2},\ c_2, c_3 \in \mathbb{R}.$$

By $x(0) = 0$ there is $c_3 = 0$ and $x(1) = 0$ yields

$$c_2 = -\lambda I^{\kappa} x(1) - I^{\kappa+\zeta}\sigma(1, x(1)).$$

Therefore, the integral form of (3.1) is

$$x(t) = -\lambda I^{\kappa} x(t) - I^{\kappa+\zeta}\sigma(t, x(t)) + \lambda t^{\kappa-1} I^{\kappa} x(1) + t^{\kappa-1} I^{\kappa+\zeta}\sigma(1, x(1)).$$

We can write it as

$$\begin{aligned} x(t) &= \int_0^t \frac{-(t-z)^{\kappa-1}}{\Gamma(\kappa)} \left[\lambda x(z) + \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} \sigma(w, x(w)) dw\right] dz \\ &\quad + t^{\kappa-1} \left[\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)} \left[\lambda x(z) + \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} \sigma(w, x(w)) dw\right] dz\right]. \end{aligned}$$

□

Lemma 3.1.2. Following solution exists for problem (3.2):

$$\begin{aligned}\widetilde{x}(t) &= \widetilde{\lambda} I^{\kappa}\widetilde{x}(t) + \widetilde{\lambda} I^{\kappa}\widetilde{x}(1) + \widetilde{\lambda} t I^{\kappa}\widetilde{x}(1) + \widetilde{\lambda} I^{\kappa-1}\widetilde{x}(1) - \widetilde{\lambda} t I^{\kappa-1}\widetilde{x}(1) + I^{\kappa+\zeta}\widetilde{\sigma}(t,\widetilde{x}(t)) \\ &\quad + I^{\kappa+\zeta}\widetilde{\sigma}(1,\widetilde{x}(1)) + tI^{\kappa+\zeta}\widetilde{\sigma}(1,\widetilde{x}(1)) - tI^{\kappa+\zeta-1}\widetilde{\sigma}(1,\widetilde{x}(1)) + I^{\kappa+\zeta-1}\widetilde{\sigma}(1,\widetilde{x}(1)).\end{aligned} \tag{3.5}$$

Proof. Apply I^{ζ} on both sides of (3.2)

$$\begin{aligned}({}^{c}D^{\kappa} - \widetilde{\lambda})\widetilde{x}(t) &= I^{\zeta}\widetilde{\sigma}(t,\widetilde{x}(t)) - c_0, \ c_0 \in \mathbb{R} \\ {}^{c}D^{\kappa}\widetilde{x}(t) &= \widetilde{\lambda}\widetilde{x}(t) + I^{\zeta}\widetilde{\sigma}(t,\widetilde{x}(t)) - c_0.\end{aligned} \tag{3.6}$$

Using ${}^{c}D^{\kappa}\widetilde{x}(0) = \widetilde{\lambda}\widetilde{x}(0)$ on (3.6), we get $c_0 = 0$.

$${}^{c}D^{\kappa}\widetilde{x}(t) = \widetilde{\lambda}\widetilde{x}(t) + I^{\zeta}\widetilde{\sigma}(t,\widetilde{x}(t)).$$

Apply I^{κ} to both sides of above equation

$$\widetilde{x}(t) = \widetilde{\lambda} I^{\kappa}\widetilde{x}(t) + I^{\kappa+\zeta}\widetilde{\sigma}(t,\widetilde{x}(t)) - c_1 t - c_2. \tag{3.7}$$

Now by using $\widetilde{x}(0) + \widetilde{x}'(0) = 0$ we attained $c_2 = -c_1$, and $\widetilde{x}(1) + \widetilde{x}'(1) = 0$ yields

$$\begin{aligned}c_1 &= \widetilde{\lambda} I^{\kappa}\widetilde{x}(1) + I^{\kappa+\zeta}\widetilde{\sigma}(1,\widetilde{x}(1)) + \widetilde{\lambda} I^{\kappa-1}\widetilde{x}(1) + I^{\kappa+\zeta-1}\widetilde{\sigma}(1,\widetilde{x}(1)), \\ c_2 &= -\widetilde{\lambda} I^{\kappa}\widetilde{x}(1) - I^{\kappa+\zeta}\widetilde{\sigma}(1,\widetilde{x}(1)) - \widetilde{\lambda} I^{\kappa-1}\widetilde{x}(1) - I^{\kappa+\zeta-1}\widetilde{\sigma}(1,\widetilde{x}(1)).\end{aligned}$$

Thus substituting c_1 and c_2 in (3.7) and rearranging the terms we obtained

$$\begin{aligned}\widetilde{x}(t) = &\ \widetilde{\lambda} I^{\kappa}\widetilde{x}(t) + \widetilde{\lambda} I^{\kappa}\widetilde{x}(1) + \widetilde{\lambda} t I^{\kappa}\widetilde{x}(1) + \widetilde{\lambda} I^{\kappa-1}\widetilde{x}(1) - \widetilde{\lambda} t I^{\kappa-1}\widetilde{x}(1) + I^{\kappa+\zeta}\widetilde{\sigma}(t,\widetilde{x}(t)) \\ &+ I^{\kappa+\zeta}\widetilde{\sigma}(1,\widetilde{x}(1)) + tI^{\kappa+\zeta}\widetilde{\sigma}(1,\widetilde{x}(1)) - tI^{\kappa+\zeta-1}\widetilde{\sigma}(1,\widetilde{x}(1)) + I^{\kappa+\zeta-1}\widetilde{\sigma}(1,\widetilde{x}(1)).\end{aligned}$$

□

Lemma 3.1.3. Consider $h \in C[0,1]$ and $0 < \zeta < 1, \ 1 < \kappa < 2$, then unique solution of problem

$$\begin{aligned}&D^{\zeta}(D^{\kappa} + \lambda)x(t) + h(t) = 0, \ 0 < t < 1, \\ &D^{\kappa}x(0) = 0, \ x(0) = x(1) = 0,\end{aligned} \tag{3.8}$$

is

$$x(t) = \int_0^1 G(t,w)x(w)dw + \int_0^1 H(t,w)h(w)dw, \tag{3.9}$$

where

$$G(t,w)=\frac{\lambda}{\Gamma(\kappa)}\begin{cases}(1-w)^{\kappa-1}t^{\kappa-1}-(t-w)^{\kappa-1}, & w\le t;\\ (1-w)^{\kappa-1}t^{\kappa-1}, & t\le w,\end{cases}\tag{3.10}$$

and

$$H(t,w)=\frac{1}{\Gamma(\kappa+\zeta)}\begin{cases}(1-w)^{\kappa+\zeta-1}t^{\kappa-1}-(t-w)^{\kappa+\zeta-1}, & w\le t;\\ (1-w)^{\kappa+\zeta-1}t^{\kappa-1}, & t\le w.\end{cases}\tag{3.11}$$

Proof. By Lemma 3.1.1, we need to prove that (3.3) can be defined in form of (3.9)

$$\begin{aligned} x(t) &= \int_0^t \frac{\lambda(t^{\kappa-1}(1-w)^{\kappa-1}-(t-w)^{\kappa-1})}{\Gamma(\kappa)}x(w)dw+\int_t^1 \frac{\lambda t^{\kappa-1}(1-w)^{\kappa-1}}{\Gamma(\kappa)}x(w)dw+\\ &\int_0^t \frac{(t^{\kappa-1}(1-w)^{\kappa+\zeta-1}-(t-w)^{\kappa+\zeta-1})}{\Gamma(\kappa+\zeta)}h(w)dw+\int_t^1 \frac{t^{\kappa-1}(1-w)^{\kappa+\zeta-1}}{\Gamma(\kappa+\zeta)}h(w)dw\\ x(t) &= \int_0^1 G(t,w)x(w)dw+\int_0^1 H(t,w)h(w)dw. \end{aligned}$$

□

Lemma 3.1.4. Consider $\widetilde{h}\in C[0,1]$ and $0<\zeta<1,\ 1<\kappa<2$, then unique solution of problem

$$\begin{gathered} {}^cD^{\zeta}({}^cD^{\kappa}-\widetilde{\lambda})\widetilde{x}(t)=\widetilde{h}(t),\ 0<t<1,\\ {}^cD^{\kappa}\widetilde{x}(0)=\widetilde{\lambda}\widetilde{x}(0),\ \widetilde{x}(0)+\widetilde{x}'(0)=0,\ \widetilde{x}(1)+\widetilde{x}'(1)=0, \end{gathered}\tag{3.12}$$

is

$$\widetilde{x}(t)=\int_0^1 \widetilde{G}(t,w)\widetilde{x}(w)dw+\int_0^1 \widetilde{H}(t,w)\widetilde{h}(w)dw,\tag{3.13}$$

where

$$\widetilde{G}(t,w)=\widetilde{\lambda}\begin{cases}\frac{(1-w)^{\kappa-1}(1-t)+(t-w)^{\kappa-1}}{\Gamma(\kappa)}+\frac{(1-w)^{\kappa-2}(1-t)}{\Gamma(\kappa-1)}, & w\le t;\\ \frac{(1-w)^{\kappa-1}(1-t)}{\Gamma(\kappa)}+\frac{(1-w)^{\kappa-2}(1-t)}{\Gamma(\kappa-1)}, & t\le w,\end{cases}\tag{3.14}$$

and

$$\widetilde{H}(t,w)=\begin{cases}\frac{(1-w)^{\kappa+\zeta-1}(1-t)+(t-w)^{\kappa+\zeta-1}}{\Gamma(\kappa+\zeta)}+\frac{(1-w)^{\kappa+\zeta-2}(1-t)}{\Gamma(\kappa+\zeta-1)}, & w\le t;\\ \frac{(1-w)^{\kappa+\zeta-1}(1-t)}{\Gamma(\kappa+\zeta)}+\frac{(1-w)^{\kappa+\zeta-2}(1-t)}{\Gamma(\kappa+\zeta-1)}, & t\le w.\end{cases}\tag{3.15}$$

Proof. We only need to show that (3.5) can be written in form of (3.13).

$$\begin{aligned}
\widetilde{x}(t) &= \widetilde{\lambda}\int_0^t \left[\frac{(1-w)^{\kappa-1}(1-t)+(t-w)^{\kappa-1}}{\Gamma(\kappa)} + \frac{(1-w)^{\kappa-2}(1-t)}{\Gamma(\kappa-1)}\right]\widetilde{x}(w)dw \\
&+\widetilde{\lambda}\int_t^1 \left[\frac{(1-w)^{\kappa-1}(1-t)}{\Gamma(\kappa)} + \frac{(1-w)^{\kappa-2}(1-t)}{\Gamma(\kappa-1)}\right]\widetilde{x}(w)dw \\
&+\int_0^t \left[\frac{(1-w)^{\kappa+\zeta-1}(1-t)+(t-w)^{\kappa+\zeta-1}}{\Gamma(\kappa+\zeta)} + \frac{(1-w)^{\kappa+\zeta-2}(1-t)}{\Gamma(\kappa+\zeta-1)}\right]\widetilde{h}(w)dw \\
&+\int_t^1 \left[\frac{(1-w)^{\kappa+\zeta-1}(1-t)}{\Gamma(\kappa+\zeta)} + \frac{(1-w)^{\kappa+\zeta-2}(1-t)}{\Gamma(\kappa+\zeta-1)}\right]\widetilde{h}(w)dw \\
\widetilde{x}(t) &= \int_0^1 \widetilde{G}(t,w)\widetilde{x}(w)dw + \int_0^1 \widetilde{H}(t,w)\widetilde{h}(w)dw.
\end{aligned}$$

□

3.2 Properties of Green Functions

For Green functions, some properties are presented in this section.

Lemma 3.2.1. $G(t,w)$ and $H(t,w)$ defined in Equations (3.10), (3.11) satisfy properties given below:

(i) $G(t,w) > 0$ and $H(t,w) > 0,\ w,t \in (0,1)$.

(ii) For $\gamma \in C[0,1]$,

$$\begin{aligned}
\min_{t\in[\frac{1}{2},\frac{2}{3}]} G(t,w) &\geq \frac{\lambda\gamma(w)(1-w)^{\kappa-1}}{\Gamma(\kappa)},\ w \in (0,1), \\
G(t,w) &\leq \frac{\lambda}{\Gamma(\kappa)}(1-w)^{\kappa-1}.
\end{aligned}$$

Also

$$\begin{aligned}
\min_{t\in[\frac{1}{2},\frac{2}{3}]} H(t,w) &\geq \frac{\omega}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-1},\ w \in (0,1), \\
H(t,w) &\leq \frac{1}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-1}.
\end{aligned}$$

Proof. (i) Since $t < 1$ implies $\frac{w}{t} > w$ thus we obtain $(1 - \frac{w}{t}) < (1 - w)$ which gives

$$t^{\kappa-1}(1-w)^{\kappa-1} - (t-w)^{\kappa-1} > 0, \ w \leq t.$$

Hence $G(t, w) > 0, \ \ w, \ t \in (0, 1)$.

(ii) Let

$$g_1(t,w) = [(1-w)]^{\kappa-1}t - (t-w)^{\kappa-1}, \ 0 < w \leq t \leq 1.$$

Thus

$$g_1(t,w) = [(1-w)]^{\kappa-1}t - (t-w)^{\kappa-1} \leq (1-w)^{\kappa-1},$$

and this implies

$$G(t,w) \leq \frac{\lambda}{\Gamma(\kappa)}(1-w)^{\kappa-1}.$$

Similarly, for $0 < w \leq t < 1$, we obtain

$$\begin{aligned} h_1(t,w) &= t^{\kappa-1}(1-w)^{\kappa+\zeta-1} - (t-w)^{\kappa+\zeta-1} \\ &\leq t^{\kappa-1}(1-w)^{\kappa+\zeta-1} \leq (1-w)^{\kappa+\zeta-1}. \end{aligned}$$

Thus

$$H(t,w) \leq \frac{1}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-1}.$$

Next, it is noted that for $w \geq t$, $G(t, w)$ is increasing with respect to t and for $w \leq t$, $G(t, w)$ is decreasing with respect to t. Therefore,

$$g_1(t,w) \geq (1-w)^{\kappa-1}(\frac{2}{3})^{\kappa-1} - (\frac{2}{3} - w)^{\kappa-1}, \ t \in [\frac{1}{2}, \frac{2}{3}],$$

and

$$\min_{t\in[\frac{1}{2},\frac{2}{3}]} G(t,w) \geq \gamma(w)(1-w)^{\kappa-1}, \ w \in (0,1),$$

where

$$\gamma(w) = \frac{(1-w)^{\kappa-1}(\frac{2}{3})^{\kappa-1} - (\frac{2}{3} - w)^{\kappa-1}}{(1-w)^{\kappa-1}}.$$

Also, let $\omega = \min_{t\in[\frac{1}{2},\frac{2}{3}]} (t^{\kappa-1} - t^{\kappa+\zeta-1})$, then

$$\begin{aligned} h_1(t,w) &= (1-w)^{\kappa+\zeta-1}t^{\kappa-1} - (t-w)^{\kappa+\zeta-1} \\ &\geq (1-w)^{\kappa+\zeta-1}t^{\kappa-1} - (t-tw)^{\kappa+\zeta-1} \\ &= (1-w)^{\kappa+\zeta-1}(t^{\kappa-1} - t^{\kappa+\zeta-1}) \geq \omega(1-w)^{\kappa+\zeta-1}. \end{aligned}$$

Hence

$$\min_{t\in[\frac{1}{2},\frac{2}{3}]} H(t,w) \geq \frac{\omega}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-1}, \ w \in (0,1).$$

□

Lemma 3.2.2. The function $\widetilde{G}(t,w)$ and $\widetilde{H}(t,w)$ presented in Equation (3.14) satisfies following conditions:

(i) $\widetilde{G}(t,w) > 0, \ \ t,w \in (0,1)$.

(ii) The Green functions defined by (3.14) and (3.15) has following properties

$$\widetilde{G}(t,w) \leq \frac{(\kappa+1)}{\Gamma(\kappa)}(1-w)^{\kappa-2},$$

and

$$\widetilde{H}(t,w) \leq \frac{(\kappa+\zeta+1)}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-2}.$$

Proof. (i) It is obvious from (3.14) that $\widetilde{G}(t,w) > 0 \ \ t,w \in (0,1)$.

(ii) Consider

$$\begin{aligned} \widetilde{g}_1(t,w) &= \frac{(1-w)^{\kappa-1}(1-t) + (t-w)^{\kappa-1}}{\Gamma(\kappa)} + \frac{(1-t)(1-w)^{\kappa-2}}{\Gamma(\kappa-1)} \\ &\leq \frac{(1-w)^{\kappa-2}}{\Gamma(\kappa)} + \frac{(t-w)^{\kappa-1}}{\Gamma(\kappa)} + \frac{(1-w)^{\kappa-2}(\kappa-1)}{\Gamma(\kappa-1)} \\ &\leq \frac{(1-w)^{\kappa-2}\kappa}{\Gamma(\kappa)} + \frac{(1-w)^{\kappa-1}}{\Gamma(\kappa)}. \end{aligned}$$

So, we get

$$\widetilde{G}(t,w) \leq \frac{(\kappa+1)}{\Gamma(\kappa)}(1-w)^{\kappa-2}.$$

Now let

$$\begin{aligned}
\widetilde{h}_1(t,w) &= \frac{(1-t)(1-w)^{\kappa+\zeta-1}+(t-w)^{\kappa+\zeta-1}}{\Gamma(\kappa+\zeta)}+\frac{(1-t)(1-w)^{\kappa+\zeta-2}}{\Gamma(\kappa+\zeta-1)} \\
&\leq \frac{(1-w)^{\kappa+\zeta-2}}{\Gamma(\kappa+\zeta)}+\frac{(t-w)^{\kappa+\zeta-1}}{\Gamma(\kappa+\zeta)}+\frac{(\kappa+\zeta-1)(1-w)^{\kappa+\zeta-2}}{\Gamma(\kappa+\zeta-1)} \\
&\leq \frac{(1-w)^{\kappa+\zeta-2}(\kappa+\zeta)}{\Gamma(\kappa+\zeta)}+\frac{(1-w)^{\kappa+\zeta-1}}{\Gamma(\kappa+\zeta)}.
\end{aligned}$$

Thus we have

$$\widetilde{H}(t,w) \leq \frac{(\kappa+\zeta+1)}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-2}.$$

□

3.3 Existence of Positive Solution

The existence and uniqueness of positive solution for problem (3.1) is verified in this section.

Theorem 3.3.1. Suppose a Banach space $C = A([0,1],Y)$ of continuous functions from $[0,1] \to Y$ provided with uniformly convergent topology with $\|x\|= \max_{t\in[0,1]}|x(t)|$ where $(Y, ||.||)$ is a Banach space. Let $\sigma : [0,1]\times Y \to Y$ be a continuous function which satisfies the following inequality

$$|\sigma(t,x)-\sigma(t,y)|\leq k|x-y|,\ \forall\ t\in[0,1],\ x,y\in Y.$$

Then unique solution of problem (3.1) exists if $\eta < 1$ where

$$\eta = \frac{2k}{\Gamma(\kappa+\zeta+1)}+\frac{2|\lambda|}{\Gamma(\kappa+1)}.$$

Proof. Define operator $I : C \to C$ as:

$$\begin{aligned}
(Ix)(t) &= \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\left[\lambda x(z)+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}\sigma(w,x(w))dw\right]dz \\
&\quad +t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}\left[\lambda x(z)+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}\sigma(w,x(w))dw\right]dz
\end{aligned}$$

$$\begin{aligned}
|(Ix)(t)| &\leq \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda x(z)|+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(|\sigma(w,x(w))-\sigma(w,0)| \\
&\quad +|\sigma(w,0)|)dw\bigg]dz + t^{\kappa-1}\bigg[\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda x(z)|+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} \\
&\quad \times(|\sigma(w,x(w))-\sigma(w,0)|+|\sigma(w,0)|)dw\bigg]dz\bigg] \\
&\leq \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda||x(z)|+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(k|x(w)|+|\sigma(w,0)|)dw\bigg]dz \\
&\quad +t^{\kappa-1}\bigg[\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda||x(z)|+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(k|x(w)|+|\sigma(w,0)|)dw\bigg]dz\bigg] \\
&\leq \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda|\|x\|+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(k\|x\|+|\sigma(w,0)|)dw\bigg]dz \\
&\quad +t^{\kappa-1}\bigg[\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda|\|x\|+\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(k\|x\|+|\sigma(w,0)|)dw\bigg]dz\bigg].
\end{aligned}$$

Define the set $U_r = \{x \in C : \|x\| \leq r\}$ where

$$r \geq \frac{1}{1-\rho}\left(\frac{2R}{\Gamma(\kappa+\zeta+1)}\right),$$

where ρ is such that $\eta \leq \rho < 1$. also set $\max_{w\in[0,1]}|\sigma(w,0)| = R$.

$$\begin{aligned}
|(Ix)(t)| &\leq \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda|r+(kr+R)\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}dw\bigg]dz \\
&\quad +t^{\kappa-1}\bigg[\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}\bigg[|\lambda|r+(kr+R)\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}dw\bigg]dz\bigg].
\end{aligned}$$

Integrating and using Lemma 1.2.2, we find the result given below

$$\|Ix\| \leq \frac{2|\lambda|r}{\Gamma(\kappa+1)} + \frac{2kr+R}{\Gamma(\kappa+\zeta+1)} \leq (\eta+1-\rho)r \leq r.$$

Now for $x, y \in C$,

$|(Ix)(t) - (Iy)(t)|$

$$\begin{aligned}
&\leq \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz + \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}|\sigma(w,x(w)) \\
&\quad -\sigma(w,y(w))|dwdz + t^{\kappa-1}\bigg[\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz + \int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)} \\
&\quad \times \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}|\sigma(w,x(w))-\sigma(w,y(w))|dwdz\bigg]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz + \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} \\
& \times k|x(w)-y(w)|dwdz + t^{\kappa-1}\Bigg[\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|\lambda||x(z)-y(z)|dz \\
& + \int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} k|x(w)-y(w)|dwdz\Bigg].
\end{aligned}
$$

Integration yields result given below

$|(Ix)(t)-(Iy)(t)|$

$$
\begin{aligned}
\leq\ & \frac{|\lambda|\|x-y\|t^{\kappa}}{\Gamma(\kappa+1)} + \frac{k\|x-y\|}{\Gamma(\zeta+1)} \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)} z^{\zeta} dz + \frac{|\lambda|\|x-y\|t^{\kappa-1}}{\Gamma(\kappa+1)} \\
& + \frac{k\|x-y\|t^{\kappa-1}}{\Gamma(\zeta+1)} \int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)} z^{\zeta} dz \\
=\ & \frac{|\lambda|\|x-y\|t^{\kappa}}{\Gamma(\kappa+1)} + \frac{k\|x-y\|t^{\kappa+\zeta}}{\Gamma(\kappa+\zeta+1)} + \frac{|\lambda|\|x-y\|t^{\kappa-1}}{\Gamma(\kappa+1)} + \frac{k\|x-y\|t^{\kappa-1}}{\Gamma(\kappa+\zeta+1)}
\end{aligned}
$$

Since $t \in [0,1]$, thus

$$
\begin{aligned}
\|Ix - Iy\| &\leq \left[\frac{2k}{\Gamma(\kappa+\zeta+1)} + \frac{2|\lambda|}{\Gamma(\kappa+1)}\right]\|x-y\| \\
&= \eta\|x-y\|.
\end{aligned}
$$

As $\eta < 1$, hence I is contraction. Thus conclusion followed by Contraction mapping principle and hence BVP (3.1) has unique solution. □

Theorem 3.3.2. Suppose bounded subsets of $[0,1] \times Y$ are mapped to the compact subsets of Y by a continuous function $\sigma : [0,1] \times Y \to Y$. Also suppose Lipschitz condition is satisfied

$$
|\sigma(t,x) - \sigma(t,y)| \leq k|x-y|, \ \forall\ t \in [0,1],\ x,y \in Y.
$$

Then on interval $[0,1]$, at least one solution exists for problem (3.1).

Proof. Suppose $U_r = \{x \in C : \|x\| \leq r\}$ where

$$
r \geq \frac{2|\lambda|/\Gamma(\kappa+1) + 2R/\Gamma(\kappa+\zeta+1)}{1 - 2k/\Gamma(\kappa+\zeta+1)}.
$$

And set $\max_{w\in[0,1]}|\sigma(w,0)|= R$. Define operators on U_r as follows

$$\begin{aligned}
(\Omega x)(t) &= -\lambda\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}x(z)dz + \lambda t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}x(z)dz,\\
(\Theta x)(t) &= \int_0^t \frac{-(t-z)^{\kappa-1}}{\Gamma(\kappa)}\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}\sigma(w,x(w))dwdz\\
&\quad +\int_0^1 \frac{t^{\kappa-1}(1-z)^{\kappa-1}}{\Gamma(\kappa)}\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}\sigma(w,x(w))dwdz.
\end{aligned}$$

Let $x, y \in U_r$, then

$|(\Omega x)(t) + (\Theta y)(t)|$

$$\begin{aligned}
&\leq |\lambda|\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz + |\lambda|t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz + \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\\
&\quad \times\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}|\sigma(w,y(w))|dwdz + \int_0^1 \frac{t^{\kappa-1}(1-z)^{\kappa-1}}{\Gamma(\kappa)}\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}\\
&\quad \times|\sigma(w,y(w))|dwdz\\
&\leq |\lambda|\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz + |\lambda|t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz + \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\\
&\quad \times\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(|\sigma(w,y(w))-\sigma(w,0))|+|\sigma(w,0)|)dwdz + \int_0^1 \frac{t^{\kappa-1}(1-z)^{\kappa-1}}{\Gamma(\kappa)}\\
&\quad \times\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(|\sigma(w,y(w))-\sigma(w,0)|+|\sigma(w,0)|)dwdz\\
&\leq |\lambda|\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz + |\lambda|t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz + \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\\
&\quad \times\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(k|y(w)|+|\sigma(w,0)|)dwdz + \int_0^1 \frac{t^{\kappa-1}(1-z)^{\kappa-1}}{\Gamma(\kappa)}\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}\\
&\quad \times(k|y(w)|+|\sigma(w,0)|)dwdz\\
&\leq |\lambda|\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\|x\|dz + |\lambda|t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}\|x\|dz + \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\\
&\quad \times\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}(k\|y\|+R)dwdz + \int_0^1 \frac{t^{\kappa-1}(1-z)^{\kappa-1}}{\Gamma(\kappa)}\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}\\
&\quad \times(k\|y\|+R)dwdz\\
&\leq |\lambda|r\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}dz + |\lambda|rt^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}dz + (kr+R)\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\\
&\quad \times\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}dwdz + (kr+R)\int_0^1 \frac{t^{\kappa-1}(1-z)^{\kappa-1}}{\Gamma(\kappa)}\int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)}dwdz.
\end{aligned}$$

After simplification

$$\|\Omega x+\Theta y\|\leq \frac{2|\lambda|r}{\Gamma(\kappa+1)}+\frac{2(kr+R)}{\Gamma(\kappa+\zeta+1)}\leq r.$$

Hence $\Omega x+\Theta y\in U_r$. Now we will prove that Ω is bounded and compact.

$$\begin{aligned}|(\Omega x)(t)| &\leq |\lambda|\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz+|\lambda|t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz\\ &\leq |\lambda|\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}\|x\|dz+|\lambda|t^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}\|x\|dz\\ &\leq |\lambda|r\int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)}dz+|\lambda|rt^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}dz.\end{aligned}$$

Simplification gives the following result

$$\|\Omega x\|\leq \frac{2|\lambda|r}{\Gamma(\kappa+1)}.$$

Now for $t_a,\ t_b\in[0,1]$

$|(\Omega x)(t_a)-(\Omega x)(t_b)|$

$$\begin{aligned}&\leq |\lambda|\int_0^{t_a} \frac{(t_a-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz+|\lambda|t_a^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz\\ &\quad+|\lambda|\int_0^{t_b} \frac{(t_b-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz+|\lambda|t_b^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz\\ &\leq |\lambda|\int_0^{t_a} \frac{(t_a-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz+|\lambda|t_a^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz\\ &\quad+|\lambda|\int_0^{t_b} \frac{(t_b-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz+|\lambda|t_b^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}|x(z)|dz\\ &\leq |\lambda|r\int_0^{t_a} \frac{(t_a-z)^{\kappa-1}}{\Gamma(\kappa)}dz+|\lambda|rt_a^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}dz\\ &\quad+|\lambda|r\int_0^{t_b} \frac{(t_b-z)^{\kappa-1}}{\Gamma(\kappa)}dz+|\lambda|rt_b^{\kappa-1}\int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)}dz\\ &\leq \frac{|\lambda|rt_a^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\lambda|rt_a^{\kappa-1}}{\Gamma(\kappa+1)}+\frac{|\lambda|rt_b^{\kappa}}{\Gamma(\kappa+1)}+\frac{|\lambda|rt_b^{\kappa-1}}{\Gamma(\kappa+1)}.\end{aligned}$$

The continuity of $t_a^{\kappa},\ t_a^{\kappa-1},\ t_b^{\kappa}$ and $t_b^{\kappa-1}$ implies the continuity of Ω. For all t, $\Omega(\mathbb{P}(t))$ is relatively compact in Y as g maps bounded subsets to the relatively compact subsets where $\mathbb{P}$, the subset of C, is bounded. Thus Ω is relatively compact on U_r. So,

employing Theorem 1.3.1 Ω is compact on U_r.
Θ is contraction mapping under the assumption that

$$\frac{2k}{\Gamma(\kappa+\zeta+1)} < 1.$$

$$\begin{aligned}
&|(\Theta x)(t) - (\Theta y)(t)| \\
&\leq \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} |\sigma(w,x(w)) - \sigma(w,y(w))| dwdz \\
&\quad + t^{\kappa-1} \int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} |\sigma(w,x(w)) - \sigma(w,y(w))| dwdz \\
&\leq \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} k|x(w) - y(w)| dwdz \\
&\quad + t^{\kappa-1} \int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} k|x(w) - y(w)| dwdz \\
&\leq k\|x-y\| \int_0^t \frac{(t-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} dwdz \\
&\quad + t^{\kappa-1} k\|x-y\| \int_0^1 \frac{(1-z)^{\kappa-1}}{\Gamma(\kappa)} \int_0^z \frac{(z-w)^{\zeta-1}}{\Gamma(\zeta)} dwdz.
\end{aligned}$$

Integrating and using Lemma 1.2.2

$$\|\Theta x - \Theta y\| \leq \frac{2k}{\Gamma(\kappa+\zeta+1)} \|x-y\|.$$

Thus using Theorem 1.3.3, at least one solution exists for problem (3.1) on $[0,1]$. □

Theorem 3.3.3. Let $1 < \alpha < 2,\ 0 < \beta < 1$ and $\sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous function. Suppose there are two distinct positive constants r_1 and r_2 satisfying

(H_1) $\sigma(t,x(t)) \leq \frac{\lambda\Gamma(\kappa+\zeta+1)}{\Gamma(\kappa+1)} r_1$ for $(t,x) \in [0,1] \times [0,r_1]$,

(H_2) $\sigma(t,x(t)) \geq \frac{\lambda\gamma\Gamma(\kappa+\zeta)}{\omega\Gamma(\kappa)} p$ for $(t,x) \in [1/2, 2/3] \times [0,r_2]$.

Then with $\min\{r_1,r_2\} \leq \|x\| \leq \max\{r_1,r_2\}$, at least one positive solution exists for problem (3.1).

Proof. Define $U_r = \{x \in C : \|x\| \leq \rho\},\ \frac{\lambda}{\Gamma(\kappa+1)} r \leq \frac{\rho}{2}$. Then for any $t \in [0,1]$,

$$\begin{aligned} |Ix(t)| &= \left| \int_0^1 G(t,w)x(w)dw + \int_0^1 H(t,w)\sigma(w,x(w))dw \right| \\ &\leq \frac{\lambda}{\Gamma(\kappa)} \int_0^1 (1-w)^{\kappa-1}|x(w)|dw + \frac{1}{\Gamma(\kappa+\zeta)} \int_0^1 (1-w)^{\kappa+\zeta-1}|\sigma(w,x(w))|dw \\ &\leq \frac{2\lambda}{\Gamma(\kappa+1)} r \leq \rho. \end{aligned}$$

Now we are going to show that I is mapping from bounded sets to equicontinuous sets. For $x \in U_r,\ t_a, t_b \in [0,1]$ where $t_a < t_b$, then

$|Ix(t_b) - Ix(t_a)|$

$$\begin{aligned} &\leq \left| \int_0^1 [G(t_b,w) - G(t_a,w)]x(w)dw \right| + \left| \int_0^1 [H(t_b,w) - H(t_a,w)]\sigma(w,x(w))dw \right| \\ &\leq \left| \int_0^{t_a} [G(t_b,w) - G(t_a,w)]x(w)dw \right| + \left| \int_{t_a}^{t_b} [G(t_b,w) - G(t_a,w)]x(w)dw \right| \\ &\quad + \left| \int_{t_b}^1 [G(t_b,w) - G(t_a,w)]x(w)dw \right| + \left| \int_0^{t_a} [H(t_b,w) - H(t_a,w)]\sigma(w,x(w))dw \right| \\ &\quad + \left| \int_{t_a}^{t_b} [H(t_b,w) - H(t_a,w)]\sigma(w,x(w))dw \right| + \left| \int_{t_b}^1 [H(t_b,w) - H(t_a,w)] \right. \\ &\quad \left. \times\sigma(w,x(w))dw \right| \\ &\leq \left| \int_0^1 \frac{(t_b^{\kappa-1} - t_a^{\kappa-1})\lambda(1-w)^{\kappa-1}}{\Gamma(\kappa)} x(w)dw \right| + \left| \int_0^1 \frac{(t_b^{\kappa+\zeta-1} - t_a^{\kappa+\zeta-1})\lambda(1-w)^{\kappa+\zeta-1}}{\Gamma(\kappa+\zeta)} \right. \\ &\quad \left. \times\sigma(w,x(w))dw \right| + \frac{\lambda}{\Gamma(\kappa)} \left[\left| \int_0^{t_a} [(t_b-w)^{\kappa-1} - (t_a-w)^{\kappa-1}]x(w)dw \right| \right. \\ &\quad + \left| \int_{t_a}^{t_b} [t_b-w)^{\kappa-1} - (t_a-w)^{\kappa-1}]x(w)dw \right| + \left| \int_{t_b}^1 [(t_b-w)^{\kappa-1} - (t_a-w)^{\kappa-1}] \right. \\ &\quad \left. \left. \times x(w)dw \right| \right] + \frac{1}{\Gamma(\kappa+\zeta)} \left[\left| \int_0^{t_a} [(t_b-w)^{\kappa+\zeta-1} - (t_a-w)^{\kappa+\zeta-1}]\sigma(w,x(w))dw \right| \right. \\ &\quad + \left| \int_{t_a}^{t_b} [(t_b-w)^{\kappa+\zeta-1} - (t_a-w)^{\kappa+\zeta-1}]\sigma(w,x(w))dw \right| + \left| \int_{t_b}^1 [(t_b-w)^{\kappa+\zeta-1} \right. \\ &\quad \left. \left. -(t_a-w)^{\kappa+\zeta-1}]\sigma(w,x(w))dw \right| \right] \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{\lambda}{\Gamma(\kappa+1)} r_1 \Big[(t_b^{\kappa-1} - t_a^{\kappa-1}) + (t_b^{\kappa+\zeta-1} - t_a^{\kappa+\zeta-1}) - (t_b - t_a)^{\kappa} + t_b^{\kappa} - t_a^{\kappa} \\
& + (t_b - t_a)^{\kappa} + (t_a - t_b)^{\kappa} - (t_b - 1)^{\kappa} + (t_a - 1)^{\kappa} - (t_a - t_b)^{\kappa} - (t_b - t_a)^{\kappa+\zeta} \\
& + t_b^{\kappa+\zeta} - t_a^{\kappa+\zeta} + (t_b - t_a)^{\kappa+\zeta} + (t_a - t_b)^{\kappa+\zeta} - (t_b - 1)^{\kappa+\zeta} + (t_a - 1)^{\kappa+\zeta} \\
& - (t_a - t_b)^{\kappa+\zeta} \Big] \to 0 \text{ as } t_a \to t_b.
\end{aligned}
$$

Then using Theorem 1.3.1, $I : U_r \to U_r$ is completely continuous.
Now let $\Omega_1 = \{x \in U_r : \|x\| < \rho_1\}$, where $\frac{\lambda}{\Gamma(\kappa+1)} r_1 < \frac{\rho_1}{2}$. For $x \in \partial\Omega_1$, we have, $\|x\| = \rho_1$. So, for $x \in U_r \cap \partial\Omega_1$, we get

$$
\begin{aligned}
\|Ix\| &\leq \frac{\lambda}{\Gamma(\kappa)} \int_0^1 (1-w)^{\kappa-1} |x(w)| dw + \frac{1}{\Gamma(\kappa+\zeta)} \int_0^1 (1-w)^{\kappa+\zeta-1} |\sigma(w, x(w))| dw \\
&\leq \rho_1 = \|x\|.
\end{aligned}
$$

Now put $\Omega_2 = \{x \in U_r : \|x\| < \rho_2\}$. For $x \in \partial\Omega_2$, $\|x\| = \rho_2$. So, for $x \in U_r \cap \partial\Omega_2$, and $\rho_2 \Lambda \leq \frac{\lambda\gamma p}{\Gamma(\kappa)}$, where $\Lambda = [\frac{(2^{-\kappa} - 3^{-\kappa})}{\kappa} + \frac{(3^{(\kappa+\zeta)} - 2^{(\kappa+\zeta)})}{(\kappa+\zeta) 6^{(\kappa+\zeta)}}]^{-1}$. Then we get

$$
\begin{aligned}
Ix(t) &= \int_0^1 G(t,w) x(w) dw + \int_0^1 H(t,w) \sigma(w, x(w)) dw \\
&\geq \int_{1/2}^{2/3} G(t,w) x(w) dw + \int_{1/2}^{2/3} H(t,w) \sigma(w, x(w)) dw \\
&\geq \frac{\lambda\gamma p}{\Gamma(\kappa)} \Big[\int_{1/2}^{2/3} (1-w)^{\kappa-1} dw + \int_{1/2}^{2/3} (1-w)^{\kappa+\zeta-1} dw \Big] \\
&\geq \rho_2 = \|x\|,
\end{aligned}
$$

where $p = \min\limits_{t \in [\frac{1}{2} \frac{2}{3}]} \{x(t)\}$.
So, we have $\|Ix\| \geq \|x\|$, for $x \in U_r \cap \partial\Omega_2$. Thus using Lemma 1.3.5, at least one fixed point of I exists in $U_r \cap (\overline{\Omega}_2/\Omega_1)$. □

Suppose a Banach space $Y = C(H),\ H = [0,1]$ of real valued continuous functions with maximum norm. Consider $W = \{\widetilde{x} \in Y : \widetilde{x}(t) \geq 0, t \in [0,1]\}$ in Y. If $\widetilde{x}(t) \geq 0,\ 0 < t \leq 1$ and $\widetilde{x}(0) = 0$ then $\widetilde{x} \in Y$ is positive solution.
Assume $c, d \in \mathbb{R}^+$ with $d > c$. Let us define upper and lower control functions as $U(t, \widetilde{x}) = \sup\{\sigma(t, \psi) : c \leq \psi \leq \widetilde{x}\}$ and $L(t, \widetilde{x}) = \inf\{\sigma(t, \psi) : \widetilde{x} \leq \psi \leq d\}$, for every

$\widetilde{x}(t) \in [c,d]$ respectively. Clearly, on $\widetilde{x}$, $U(t,\widetilde{x})$ and $L(t,\widetilde{x})$ are both non-decreasing and monotonous and also $L(t,\widetilde{x}) \leq \sigma(t,\widetilde{x}) \leq U(t,\widetilde{x})$.
We suppose $\sigma : H \times Y \to Y$ be continuous function. Define operator $\phi : W \to W$ by

$$(\phi\widetilde{x})(t) = \int_0^1 \widetilde{G}(t,w)\widetilde{x}(w)dw + \int_0^1 \widetilde{H}(t,w)\widetilde{\sigma}(w,\widetilde{x}(w))dw$$

(H_1) : $\widetilde{x}_1(t),\ \widetilde{x}_2(t) \in W$, in such a way $c \leq \widetilde{x}_2(t) \leq \widetilde{x}_1(t) \leq d$ and

$$\begin{aligned} D^{\zeta}(D^{\kappa} - \widetilde{\lambda})\widetilde{x}_1(t) &\geq U(t,\widetilde{x}_1(t)), \\ D^{\zeta}(D^{\kappa} - \widetilde{\lambda})\widetilde{x}_2(t) &\leq L(t,\widetilde{x}_2(t)), \text{ for any } t \in H. \end{aligned}$$

(H_2) : A positive real number $r < 1$ exists such a way that

$$|\sigma(t,\widetilde{x}) - \sigma(t,\widetilde{y})| \leq r|\widetilde{x} - \widetilde{y}|,\ t \in H,\ \widetilde{x},\widetilde{y} \in Y.$$

Then $\widetilde{x}_1(t)$ and $\widetilde{x}_2(t)$ are upper and lower solutions for problem (3.2) respectively.

Theorem 3.3.4. Assume that (H_1) is satisfied and $\widetilde{x}_2(t) \leq \widetilde{x}(t) \leq \widetilde{x}_1(t),\ t \in H$, then for problem (3.2) at least one positive solution $\widetilde{x} \in Y$ exists.

Proof. Consider $C = \{\widetilde{x} \in W : \widetilde{x}_2(t) \leq \widetilde{x}(t) \leq \widetilde{x}_1(t),\ t \in H\}$ with $\|\widetilde{x}\| = \max_{t\in H}|\widetilde{x}(t)|$, then $\|\widetilde{x}\| \leq d$, thus C is closed, bounded and convex subset of Y. Furthermore, the continuity of operator ϕ on C is obvious from continuity of σ. If $\widetilde{x} \in Y$ then a positive constant e exists such that $\max\{\sigma(t,\widetilde{x}(t)) : t \in H,\ \widetilde{x}(t) \leq d\} < e$. Then

$$\begin{aligned} |(\phi\widetilde{x})(t)| &\leq \int_0^1 |\widetilde{G}(t,w)||\widetilde{x}(w)|dw + \int_0^1 |\widetilde{H}(t,w)||\widetilde{\sigma}(w,\widetilde{x}(w))|dw \\ &\leq \frac{d(1+\kappa)}{\Gamma(\kappa)} \int_0^1 (1-w)^{\kappa-2}dw + \frac{(1+\kappa+\zeta)e}{\Gamma(\kappa+\zeta)} \int_0^1 (1-w)^{\kappa+\zeta-2}dw \\ &= \frac{d(1+\kappa)}{\Gamma(\kappa)(\kappa-1)} + \frac{(1+\kappa+\zeta)e}{(\kappa+\zeta-1)\Gamma(\kappa+\zeta)}. \end{aligned}$$

Thus $\phi(C)$ is uniformly bounded. For equicontinuity of ϕ, suppose $\widetilde{x} \in C$, $\epsilon > 0,\ \delta > 0$ and $0 \leq t_a < t_b \leq 1$, such that $|t_b - t_a| < \delta$. If

$$\delta = \min\left\{1, \frac{\epsilon\Gamma(\kappa)\Gamma(\kappa+1)}{d[\Gamma(\kappa)(1+f'(\eta_1)) + \Gamma(\kappa+1)]}, \frac{\epsilon\Gamma(\kappa+\zeta)\Gamma(\kappa+\zeta+1)}{e[\Gamma(\kappa+\zeta)(1+f'(\eta_2)) + \Gamma(\kappa+\zeta+1)]}\right\}.$$

Then

$$|(\phi\widetilde{x})(t_a)-(\phi\widetilde{x})(t_b)| \leq d\int_0^1|\widetilde{G}(t_a,w)-\widetilde{G}(t_b,w)|dw+e\int_0^1|\widetilde{H}(t_a,w)-\widetilde{H}(t_b,w)|dw.$$

Now consider

$$\begin{aligned}
\widetilde{G}(t_a,w)-\widetilde{G}(t_b,w) &= \frac{(1-w)^{\kappa-1}(t_b-t_a)+(t_a-w)^{\kappa-1}-(t_b-w)^{\kappa-1}}{\Gamma(\kappa)} \\
&\quad +\frac{(1-w)^{\kappa-2}(t_b-t_a)}{\Gamma(\kappa-1)}, \\
\int_0^1|\widetilde{G}(t_a,w)-\widetilde{G}(t_b,w)|dw. &= \frac{t_b-t_a}{\kappa\Gamma(\kappa)}+\frac{t_b^\kappa-t_a^\kappa}{\kappa\Gamma(\kappa)}+\frac{t_b-t_a}{(\kappa-1)\Gamma(\kappa-1)} \\
&= \frac{t_b-t_a}{\Gamma(\kappa+1)}+\frac{t_b-t_a}{\Gamma(\kappa)}+\frac{t_b^\kappa-t_a^\kappa}{\Gamma(\kappa+1)}.
\end{aligned}$$

Let $\eta_1\in(t_a,t_b)$

$$\begin{aligned}
\int_0^1|\widetilde{G}(t_a,w)-\widetilde{G}(t_b,w)|dw &= \frac{t_b-t_a}{\Gamma(\kappa+1)}+\frac{t_b-t_a}{\Gamma(\kappa)}+\frac{f'(\eta_1)(t_b-t_a)}{\Gamma(\kappa+1)} \\
&= (t_b-t_a)\left[\frac{1+f'(\eta_1)}{\Gamma(\kappa+1)}+\frac{1}{\Gamma(\kappa)}\right].
\end{aligned}$$

Similarly

$$\int_0^1|\widetilde{H}(t_a,w)-\widetilde{H}(t_b,w)|dw=(t_b-t_a)\left[\frac{1+f'(\eta_2)}{\Gamma(\kappa+\zeta+1)}+\frac{1}{\Gamma(\kappa+\zeta)}\right].$$

So, we have

$$\begin{aligned}
|(\phi\widetilde{x})(t_a)-(\phi\widetilde{x})(t_b)| &\leq |t_b-t_a|\left[d\left(\frac{1+f'(\eta_1)}{\Gamma(\kappa+1)}+\frac{1}{\Gamma(\kappa)}\right)+e\left(\frac{1+f'(\eta_2)}{\Gamma(\kappa+\zeta+1)}\right.\right. \\
&\quad \left.\left.+\frac{1}{\Gamma(\kappa+\zeta)}\right)\right] \\
&\leq \delta\left[d\left(\frac{1+f'(\eta_1)}{\Gamma(\kappa+1)}+\frac{1}{\Gamma(\kappa)}\right)+e\left(\frac{1+f'(\eta_2)}{\Gamma(\kappa+\zeta+1)}+\frac{1}{\Gamma(\kappa+\zeta)}\right)\right] \\
&< \epsilon.
\end{aligned}$$

Thus $\phi(C)$ is equicontinuous. Using Theorem 1.3.1, $\phi:W\to W$ is compact. The application of Theorem 1.3.2 requires to show $\phi(C)\subseteq C$. Suppose $\widetilde{x}\in C$, then hypothesis implies:

$$\begin{aligned}(\sigma\widetilde{x})(t) &= \int_0^1 \widetilde{G}(t,w)\widetilde{x}(w)dw + \int_0^1 \widetilde{H}(t,w)\widetilde{\sigma}(w,\widetilde{x}(w))dw \\ &\leq \int_0^1 \widetilde{G}(t,w)\widetilde{x}(w)dw + \int_0^1 \widetilde{H}(t,w)U(w,\widetilde{x}(w))dw \\ &\leq \int_0^1 \widetilde{G}(t,w)\widetilde{x}_1(w)dw + \int_0^1 \widetilde{H}(t,w)U(w,\widetilde{x}_1(w))dw \leq \widetilde{x}_1(t),\end{aligned}$$

and

$$\begin{aligned}(\sigma\widetilde{x})(t) &> \int_0^1 \widetilde{G}(t,w)\widetilde{x}_2(w)dw + \int_0^1 \widetilde{H}(t,w)\widetilde{\sigma}(w,\widetilde{x}_2(w))dw \\ &\geq \widetilde{x}_2(t).\end{aligned}$$

Thus for $t \in H$, $\widetilde{x}_2(t) \leq (\phi\widetilde{x})(t) \leq \widetilde{x}_1(t)$, that is $\phi(C) \subseteq C$. At least one fixed point $\widetilde{x} \in C$ of ϕ exists by using Theorem 1.3.2. Thus for problem (3.2) at least one positive solution $\widetilde{x} \in Y$ exists and $\widetilde{x}_2(t) \leq \widetilde{x}(t) \leq \widetilde{x}_1(t)$ for $t \in H$. □

Theorem 3.3.5. Suppose (H_1) and (H_2) are satisfied. If

$$\widehat{s} = \left[\frac{1+\kappa}{(\kappa-1)\Gamma(\kappa)} + \frac{(1+\kappa+\zeta)r}{(\kappa+\zeta-1)\Gamma(\kappa+\zeta)}\right] < 1.$$

Then problem (3.2) has unique positive solution $\widetilde{x} \in Y$.

Proof. It is followed by Theorem 3.3.4 that at least one positive solution for problem (3.2) exists in C. So, we are required to show that ϕ described in Theorem 3.3.4 is contraction on Y. For $\widetilde{x}, \widetilde{y} \in Y$,

$$\begin{aligned}|(\phi\widetilde{x})(t) - (\phi\widetilde{y})(t)| &\leq \int_0^1 \widetilde{G}(t,w)|\widetilde{x}(w) - \widetilde{y}(w)|dw + \int_0^1 \widetilde{H}(t,w)|\widetilde{\sigma}(w,\widetilde{x}(w)) \\ &\quad -\widetilde{\sigma}(w,\widetilde{y}(w))|dw \\ &\leq \int_0^1 \widetilde{G}(t,w)|\widetilde{x}(w) - \widetilde{y}(w)|dw + \int_0^1 r\widetilde{H}(t,w)|\widetilde{x}(w) - \widetilde{y}(w)|dw \\ &= \int_0^1 [\widetilde{G}(t,w) - r\widetilde{H}(t,w)]|\widetilde{x}(w) + \widetilde{y}(w)|dw.\end{aligned}$$

Since

$$\widetilde{G}(t,w) \leq \frac{(\kappa+1)}{\Gamma(\kappa)}(1-w)^{\kappa-2},$$

and

$$\widetilde{H}(t,w) \le \frac{(\kappa+\zeta+1)}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-2}.$$

Therefore

$$\begin{aligned}
|(\phi\widetilde{x})(t)-(\phi\widetilde{y})(t)| &\le \int_0^1 \left[\frac{(\kappa+1)}{\Gamma(\kappa)}(1-w)^{\kappa-2}+\frac{(\kappa+\zeta+1)r}{\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-2}\right] \\
&\quad \times|\widetilde{x}(w)+\widetilde{y}(w)|dw \\
&\le \left[\frac{(\kappa+1)}{(\kappa-1)\Gamma(\kappa)}(1-w)^{\kappa-2}+\frac{(\kappa+\zeta+1)r}{(\kappa+\zeta-1)\Gamma(\kappa+\zeta)}(1-w)^{\kappa+\zeta-2}\right] \\
&\quad \times\|\widetilde{x}-\widetilde{y}\| \\
|(\phi\widetilde{x})(t)-(\phi\widetilde{y})(t)| &\le \widehat{s}\|\widetilde{x}-\widetilde{y}\|.
\end{aligned}$$

Since $\widehat{s} < 1$, Hence ϕ is a contraction mapping. Therefore, unique positive solution $\widetilde{x} \in Y$ exists for problem (3.2). □

Example 3.3.1. Consider

$$\begin{aligned}
D^{\zeta}(D^{\kappa}+1)x(t) &= \sqrt{t}+\frac{|\cos x|}{1+t^3}, \\
D^{\kappa}x(0) = 0,\ x(0) &= x(1) = 0,
\end{aligned} \tag{3.16}$$

where $\kappa = 3/2,\ \zeta = 1/2,\ \lambda = 1$ and

$$\sigma(t,x) = \sqrt{t}+\frac{|\cos x|}{1+t^3}.$$

For $(t,x) \in [0,1]\times[0,+\infty)$,

$$1.512p \le \sqrt{t} \le \sqrt{t}+\frac{|\cos x|}{1+t^3} \le \sqrt{t}+\frac{1}{1+t^3} \le 2.25r_1.$$

We choose $r_1 = 1,\ r_2 = 0.50,\ p = 0.45$ and $\omega = 0.914,\ \gamma = 1.225$ such that (H_1) and (H_2) of Theorem 3.3.3 are fulfilled. Therefore at least one positive solution for problem (3.16) exists.

Chapter 4

Existence of Solution for Coupled System of Langevin Equation

The development of different mathematics disciplines through fractional calculus has provided many employments of fractional differential equations. The coupled system of fractional differential equations became popular due to their application in science and technology. Many researchers, by using fixed point theorems, has studied existence of solution for coupled systems of equations.

Fang and Bai in [40], studied existence of positive solution for following singular coupled system of equations,

$$D^{\psi_1} x_1 = h_1(s, x_2),\ D^{\psi_2} x_2 = h_2(s, x_1),\ 0 < s < 1,$$

where $0 < \psi_1, \psi_2 < 1,\ h_1, h_2 : [0, 1) \times [0, +\infty) \to [0, +\infty)$ are continuous functions and D is Riemann-Liouville derivative.

In [42], coupled system of multiterm differential equations was considered for establishing existence and uniqueness of solution,

$$\begin{aligned} D^{\gamma} x(\widetilde{t}) &= g_1(\widetilde{t}, y(\widetilde{t}), D^{s_1} y(\widetilde{t}), ..., D^{s_n} y(\widetilde{t})),\ D^{\gamma-\widetilde{u}} x(0) = 0,\ \widetilde{u} = 1, ..., n_1, \\ D^{\phi} y(\widetilde{t}) &= g_2(\widetilde{t}, x(\widetilde{t}), D^{r_1} x(\widetilde{t}), ..., D^{r_n} x(\widetilde{t})),\ D^{\phi-\widetilde{v}} y(0) = 0,\ \widetilde{v} = 1, ..., n_2. \end{aligned}$$

where $n_1 = [\gamma] + 1,\ n_2 = [\phi] + 1$ if $\gamma, \phi \notin \mathbb{N}$ otherwise $n_1 = [\gamma],\ n_2 = [\phi]$ if $\gamma, \phi \in \mathbb{N}$, $\gamma > s_1 > ... > s_n,\ \phi > r_1 > ... > r_n$. Also $g_1, g_2 : [0, 1] \times \mathbb{R}^{n+1} \to \mathbb{R}$ are continuous and D is Riemann-Liouville derivative.

Authors considered coupled system of Langevin equations in [41], to investigate existence and uniqueness of solution.
In [43], existence result was developed for problem given below,

$$\begin{aligned}
D^{\phi_1}x_1(w) &= u_1(s, x_2(w), D^{\zeta_1}x_2(w)),\\
D^{\phi_2}x_2(w) &= u_2(s, x_1(w), D^{\zeta_2}x_1(w)),\\
x_1(0) &= x_1(1) = x_2(0) = x_2(1) = 0,
\end{aligned}$$

where $0 < s < 1, \phi_1 - \zeta_1,\ \phi_2 - \zeta_1 \geq 1,\ \zeta_1, \zeta_2 > 0,\ 1 < \phi_1, \phi_2 < 2$, D is Riemann-Liouville derivative and $x_1, x_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are any given functions.
There is limited work available for existence of solution for coupled system of fractional Langevin equations. So, inspired by the work on coupled system of differential equations, we considered coupled system stated below,

$$\begin{gathered}
D^{\zeta_1}(D^{\kappa_1} + \lambda_1)x(t) = \sigma_1(t, x(t), y(t)),\ 0 < t < 1,\\
D^{\zeta_2}(D^{\kappa_2} + \lambda_2)y(t) = \sigma_2(t, x(t), y(t)),\ 0 < t < 1,\\
D^{\kappa_1}x(0) = 0,\ x(0) = x(1) = 0,\\
D^{\kappa_2}y(0) = 0,\ y(0) = y(1) = 0,
\end{gathered} \tag{4.1}$$

where $D^{\zeta_1},\ D^{\zeta_2},\ D^{\kappa_1},\ D^{\kappa_2}$ are Riemman-Liouville derivatives. $0 < \zeta_1,\ \zeta_2 < 1,\ 1 < \kappa_1,\ \kappa_2 < 2$ and $\lambda_1,\ \lambda_2 \in \mathbb{R}^+$. The nonlinear functions $\sigma_1,\ \sigma_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous.

4.1 Existence of Solution

The objective of this section is to establish existence of solution for problem (4.1).
Consider $C[0,1]$ be space of real continuous functions.

Lemma 4.1.1. Suppose $\widetilde{S} = \{x(t) \mid x(t) \in C[0,1]\}$ be a Banach space with norm $\|x\|_{\widetilde{S}} = \max\limits_{t\in[0,1]}|x(t)|$. Also consider Banach space $\widetilde{Q} = \{y(t) \mid y(t) \in C[0,1]\}$ with norm $\|y\|_{\widetilde{Q}} = \max\limits_{t\in[0,1]}|y(t)|$. For $(x,y) \in \widetilde{S} \times \widetilde{Q}$, suppose

$$\|(x,y)\|_{\widetilde{S}\times\widetilde{Q}} = \max\{\|x\|_{\widetilde{S}}, \|y\|_{\widetilde{Q}}\}.$$

Then $(\widetilde{S} \times \widetilde{Q}, \|.\|_{\widetilde{S}\times\widetilde{Q}})$ is a Banach space.

Consider coupled system of integral equations

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,w)x(w)dw + \int_0^1 H_1(t,w)\sigma_1(w, x(w, y(w)))dw, \\ y(t) &= \int_0^1 G_2(t,w)y(w)dw + \int_0^1 H_2(t,w)\sigma_2(w, x(w), y(w))dw, \end{aligned} \tag{4.2}$$

where

$$G_1(t,w) = \frac{\lambda_1}{\Gamma(\kappa_1)} \begin{cases} t^{\kappa_1-1}(1-w)^{\kappa_1-1} - (t-w)^{\kappa_1-1}, & w \leq t; \\ t^{\kappa_1-1}(1-w)^{\kappa_1-1}, & t \leq w, \end{cases} \tag{4.3}$$

$$H_1(t,w) = \frac{1}{\Gamma(\kappa_1+\zeta_1)} \begin{cases} t^{\kappa_1-1}(1-w)^{\kappa_1+\zeta_1-1} - (t-w)^{\kappa_1+\zeta_1-1}, & w \leq t; \\ t^{\kappa_1-1}(1-w)^{\kappa_1+\zeta_1-1}, & t \leq w. \end{cases} \tag{4.4}$$

$$G_2(t,w) = \frac{\lambda_2}{\Gamma(\kappa_2)} \begin{cases} t^{\kappa_2-1}(1-w)^{\kappa_2-1} - (t-w)^{\kappa_2-1}, & w \leq t; \\ t^{\kappa_2-1}(1-w)^{\kappa_2-1}, & t \leq w, \end{cases} \tag{4.5}$$

and

$$H_2(t,w) = \frac{1}{\Gamma(\kappa_2+\zeta_2)} \begin{cases} t^{\kappa_2-1}(1-w)^{\kappa_2+\zeta_2-1} - (t-w)^{\kappa_2+\zeta_2-1}, & w \leq t; \\ t^{\kappa_2-1}(1-w)^{\kappa_2+\zeta_2-1}, & t \leq w. \end{cases} \tag{4.6}$$

Lemma 4.1.2. Assume continuous functions $\sigma_1, \sigma_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then $(x,y) \in \widetilde{S} \times \widetilde{Q}$ is solution for (4.1) if and only if $(x,y) \in \widetilde{S} \times \widetilde{Q}$ is solution for (4.2).

Proof. Assume $(x,y) \in \widetilde{S} \times \widetilde{Q}$ be solution of (4.1). Then from Lemma 3.1.1, we obtain that $(x,y) \in \widetilde{S} \times \widetilde{Q}$ is solution of (4.2). Now assume $(x,y) \in \widetilde{S} \times \widetilde{Q}$ be the solution of (4.2). Using equations (4.3)-(4.4) in system (4.2) and simplifying we came to conclusion that $(x,y) \in \widetilde{S} \times \widetilde{Q}$ is a solution of (4.1). □

Consider the operator $K : \widetilde{S} \times \widetilde{Q} \to \widetilde{S} \times \widetilde{Q}$ given as:

$$\begin{aligned} K(x,y)(t) &= \Bigg(\int_0^1 G_1(t,w)x(w)dw + \int_0^1 H_1(t,w)\sigma_1(w, x(w), y(w))dw, \\ &\quad \int_0^1 G_2(t,w)x(w)dw + \int_0^1 H_2(t,w)\sigma_2(w, x(w), y(w))dw \Bigg) \\ &= (K_1(x,y)(t), K_2(x,y)(t)). \end{aligned}$$

Theorem 4.1.1. Assume one of following are satisfied

(H_1) Two non-negative functions $p(t),\ q(t) \in L[0,1]$ exists such that $|\sigma_1(t,x,y)| \leq p(t) + a_1|x|^{\mu_1} + a_2|y|^{\mu_2}$ and $|\sigma_2(t,x,y)| \leq q(t) + b_1|x|^{\nu_1} + b_2|y|^{\nu_2}$, $3\lambda_1 < \Gamma(\kappa_1 + 1)$, $3\lambda_2 < \Gamma(\kappa_2+1)$, $a_i, b_i \geq 0$ and $\mu_i, \nu_i > 1, i = 1, 2$.

(H_2) $|\sigma_1(t,x,y)| \leq a_1|x|^{\mu_1} + a_2|y|^{\mu_2}$ and $|\sigma_2(t,x,y)| \leq b_1|x|^{\nu_1} + b_2|y|^{\nu_2}$, $2\lambda_1 < \Gamma(\kappa_1 + 1)$, $2\lambda_2 < \Gamma(\kappa_2+1)$, $a_i, b_i \geq 0$ and $0 < \mu_i, \nu_i < 1, i = 1, 2$.

Then solution exists for problem (4.1).

Proof. First suppose (H_1) is satisfied. We will make use of Theorem 1.3.2 to prove this result.

Define set

$$U_r = \{(x(t), y(t)) \ |(x(t), y(t)) \in \widetilde{S} \times \widetilde{Q},\ \|(x(t), y(t))\|_{\widetilde{S}\times\widetilde{Q}} \leq r, t \in [0,1]\},$$

where

$$\begin{aligned} r \ \geq \ & \max\Bigg\{\frac{C}{1/3 - \lambda_1/\Gamma(\kappa_1+1)}, \left(\frac{3a_1}{\Gamma(\kappa_1+\zeta_1+1)}\right)^{\frac{1}{1-\mu_1}}, \left(\frac{3a_2}{\Gamma(\kappa_1+\zeta_1+1)}\right)^{\frac{1}{1-\mu_2}}, \\ & \frac{D}{1/3 - \lambda_2/\Gamma(\kappa_2+1)}, \left(\frac{3b_1}{\Gamma(\kappa_2+\zeta_2+1)}\right)^{\frac{1}{1-\nu_1}}, \left(\frac{3b_2}{\Gamma(\kappa_2+\zeta_2+1)}\right)^{\frac{1}{1-\nu_2}}\Bigg\}, \end{aligned}$$

and

$$\begin{aligned} C &= \max_{t\in[0,1]} \int_0^1 |H_1(t,w)p(w)|dw, \\ D &= \max_{t\in[0,1]} \int_0^1 |H_2(t,w)q(w)|dw. \end{aligned}$$

First we will prove that $K : U_r \to U_r$.

$$\begin{aligned} |K_1(x,y)(t)| &= \left|\int_0^1 G_1(t,w)x(w)dw + \int_0^1 H_1(t,w)\sigma_1(w,x(w),y(w))dw\right| \\ &\leq r\int_0^1 |G_1(t,w)|dw + \int_0^1 |H_1(t,w)p(w)|dw + (a_1r^{\mu_1} + a_2r^{\mu_2}) \\ &\quad \times \int_0^1 |H_1(t,w)|dw \end{aligned}$$

$$
\begin{aligned}
&\leq r\lambda_1\Big[\int_0^t \frac{-(t-w)^{\kappa_1-1}}{\Gamma(\kappa_1)}dw+\int_0^1 \frac{t^{\kappa_1-1}(1-w)^{\kappa_1-1}}{\Gamma(\kappa_1)}dw\Big]\\
&\quad+\int_0^1 |H_1(t,w)p(w)|dw+(a_1r^{\mu_1}+a_2r^{\mu_2})\Big[\int_0^t \frac{-(t-w)^{\kappa_1+\zeta_1-1}}{\Gamma(\kappa_1+\zeta_1)}dw\\
&\quad+\int_0^1 \frac{t^{\kappa_1+\zeta_1-1}(1-w)^{\kappa_1+\zeta_1-1}}{\Gamma(\kappa_1+\zeta_1)}dw\Big]\\
&= r\lambda_1\Big[\frac{t^{\kappa_1-1}}{\Gamma(\kappa_1+1)}+\frac{t^{\kappa_1}}{\Gamma(\kappa_1+1)}\Big]+\int_0^1 |H_1(t,w)p(w)|dw+(a_1r^{\mu_1}+a_2r^{\mu_2})\\
&\quad\times\Big[\frac{t^{\kappa_1+\zeta_1-1}}{\Gamma(\kappa_1+\zeta_1+1)}+\frac{t^{\kappa_1+\zeta_1}}{\Gamma(\kappa_1+\zeta_1+1)}\Big]\\
&= \frac{r\lambda_1}{\Gamma(\kappa_1+1)}+\int_0^1 |H_1(t,w)p(w)|dw+\frac{(a_1r^{\mu_1}+a_2r^{\mu_2})}{\Gamma(\kappa_1+\zeta_1+1)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|K_1(x,y)\|_{\widetilde{S}} &\leq \frac{r\lambda_1}{\Gamma(\kappa_1+1)}+C+\frac{(a_1r^{\mu_1}+a_2r^{\mu_2})}{\Gamma(\kappa_1+\zeta_1+1)}\\
&\leq \frac{r}{3}+\frac{r}{3}+\frac{r}{3}=r.
\end{aligned}
$$

Now for K_2,

$$
\begin{aligned}
|K_2(x,y)(t)| &= \Big|\int_0^1 G_2(t,w)y(w)dw+\int_0^1 H_2(t,w)\sigma_2(w,x(w),y(w))dw\Big|\\
&\leq r\int_0^1 |G_2(t,w)|dw+\int_0^1 |H_2(t,w)q(w)|dw+(b_1r^{\nu_1}+b_2r^{\nu_2})\\
&\quad\times\int_0^1 |H_2(t,w)|dw\\
&\leq r\lambda_2\Big[\int_0^t \frac{-(t-w)^{\kappa_2-1}}{\Gamma(\kappa_2)}dw+\int_0^1 \frac{t^{\kappa_2-1}(1-w)^{\kappa_2-1}}{\Gamma(\kappa_2)}dw\Big]\\
&\quad+\int_0^1 |H_2(t,w)q(w)|dw+(b_1r^{\nu_1}+b_2r^{\nu_2})\Big[\int_0^t \frac{-(t-w)^{\kappa_2+\zeta_2-1}}{\Gamma(\kappa_2+\zeta_2)}dw\\
&\quad+\int_0^1 \frac{t^{\kappa_2+\zeta_2-1}(1-w)^{\kappa_2+\zeta_2-1}}{\Gamma(\kappa_2+\zeta_2)}dw\Big]\\
&= r\lambda_2\Big[\frac{t^{\kappa_2-1}}{\Gamma(\kappa_2+1)}+\frac{t^{\kappa_2}}{\Gamma(\kappa_2+1)}\Big]+\int_0^1 |H_2(t,w)q(w)|dw+(b_1r^{\nu_1}+b_2r^{\nu_2})\\
&\quad\times\Big[\frac{t^{\kappa_2+\zeta_2-1}}{\Gamma(\kappa_2+\zeta_2+1)}+\frac{t^{\kappa_2+\zeta_2}}{\Gamma(\kappa_2+\zeta_2+1)}\Big]\\
&= \frac{r\lambda_2}{\Gamma(\kappa_2+1)}+\int_0^1 |H_2(t,w)q(w)|dw+\frac{(b_1r^{\nu_1}+b_2r^{\nu_2})}{\Gamma(\kappa_2+\zeta_2+1)}.
\end{aligned}
$$

Thus

$$\begin{aligned}\|K_2(x,y)\|_{\widetilde{Q}} &\leq \frac{r\lambda_2}{\Gamma(\kappa_2+1)}+D+\frac{(b_1r^{\nu_1}+b_2r^{\nu_2})}{\Gamma(\kappa_2+\zeta_2+1)}\\ &\leq \frac{r}{3}+\frac{r}{3}+\frac{r}{3}=r.\end{aligned}$$

Next assume that (H_2) is valid. Take

$$\begin{aligned}0<r &\leq \min\Big\{\Big(\frac{\Gamma(\kappa_1+\zeta_1+1)}{a_1}(\frac{1}{2}-\frac{\lambda_1}{\Gamma(\kappa_1+1)}\Big)^{\frac{1}{\mu_1-1}},\Big(\frac{\Gamma(\kappa_1+\zeta_1+1)}{2a_2}\Big)^{\frac{1}{\mu_2-1}},\\ &\Big(\frac{\Gamma(\kappa_2+\zeta_2+1)}{b_1}(\frac{1}{2}-\frac{\lambda_2}{\Gamma(\kappa_2+1)}\Big)^{\frac{1}{\nu_1-1}},\Big(\frac{\Gamma(\kappa_2+\zeta_2+1)}{2b_2}\Big)^{\frac{1}{\nu_2-1}}\Big\}.\end{aligned}$$

Then

$$\begin{aligned}|K_1(x,y)(t)| &\leq \frac{r\lambda_1}{\Gamma(\kappa_1+1)}+\frac{(a_1r^{\mu_1}+a_2r^{\mu_2})}{\Gamma(\kappa_1+\zeta_1+1)}\\ \|K_1(x,y)\|_{\widetilde{S}} &\leq \frac{r}{2}+\frac{r}{2}=r.\end{aligned}$$

For K_2 we obtain

$$\begin{aligned}|K_2(x,y)(t)| &\leq \frac{r\lambda_2}{\Gamma(\kappa_2+1)}+\frac{(b_1r^{\nu_1}+b_2r^{\nu_2})}{\Gamma(\kappa_2+\zeta_2+1)}\\ \|K_2(x,y)\|_{\widetilde{Q}} &\leq \frac{r}{2}+\frac{r}{2}=r.\end{aligned}$$

Thus as a result $K:U_r\to U_r$. Since $G_1(t,w)$, $G_2(t,w)$, $H_1(t,w)$, $H_2(t,w)$, σ_1 and σ_2 are continuous, Thus K is also continuous. We will prove that K is completely continuous.

Set $\max\limits_{t\in[0,1]}|\sigma_1(t,x(t),y(t))|=A$ and $\max\limits_{t\in[0,1]}|\sigma_2(t,x(t),y(t))|=B$. For $t_a,t_b\in[0,1]$ with $t_a<t_b$ and $(x,y)\in U_r$,

$|K_1(x,y)(t_a)-K_1(x,y)(t_b)|$

$$\begin{aligned}&\leq \int_0^1|G_1(t_a,w)-G_1(t_b,w)||x(w)|dw+\int_0^1|H_1(t_a,w)-H_1(t_b,w)|\\ &\quad\times|\sigma_1(w,x(w),y(w))|dw\\ &\leq r\int_0^1|G_1(t_a,w)-G_1(t_b,w)|dw+A\int_0^1|H_1(t_a,w)-H_1(t_b,w)|dw\end{aligned}$$

$$
\begin{aligned}
&\leq \ r\Big[\int_0^{t_a}|G_1(t_a,w)-G_1(t_b,w)|dw+\int_{t_a}^{t_b}|G_1(t_a,w)-G_1(t_b,w)|dw \\
&\quad +\int_{t_b}^{1}|G_1(t_a,w)-G_1(t_b,w)|dw\Big]+A\Big[\int_0^{t_a}|H_1(t_a,w)-H_1(t_b,w)|dw \\
&\quad +\int_{t_a}^{t_b}|H_1(t_a,w)-H_1(t_b,w)|dw+\int_{t_b}^{1}|H_1(t_a,w)-H_1(t_b,w)|dw\Big] \\
&= \ \frac{r\lambda_1}{\Gamma(\kappa_1)}\Big[\int_0^{t_a}\Big((t_a(1-w))^{\kappa_1-1}-(t_a-w)^{\kappa_1-1}-(t_b(1-w))^{\kappa_1-1}+(t_b-w)^{\kappa_1-1}\Big)dw \\
&\quad +\int_{t_a}^{t_b}\Big((t_a(1-w))^{\kappa_1-1}-(t_b(1-w))^{\kappa_1-1}+(t_b-w)^{\kappa_1-1}\Big)dw \\
&\quad +\int_{t_b}^{1}\Big((t_a(1-w))^{\kappa_1-1}-(t_b(1-w))^{\kappa_1-1}\Big)dw\Big]+\frac{A}{\Gamma(\kappa_1+\zeta_1)} \\
&\quad \times\Big[\int_0^{t_a}\Big((t_a(1-w))^{\kappa_1+\zeta_1-1}-(t_a-w)^{\kappa_1+\zeta_1-1}-(t_b(1-w))^{\kappa_1+\zeta_1-1} \\
&\quad +(t_b-w)^{\kappa_1+\zeta_1-1}\Big)dw+\int_{t_a}^{t_b}\Big((t_a(1-w))^{\kappa_1+\zeta_1-1}-(t_b(1-w))^{\kappa_1+\zeta_1-1} \\
&\quad +(t_b-w)^{\kappa_1+\zeta_1-1}\Big)dw+\int_{t_b}^{1}\Big((t_a(1-w))^{\kappa_1+\zeta_1-1}-(t_b(1-w))^{\kappa_1+\zeta_1-1}\Big)dw\Big] \\
&= \ \frac{r\lambda_1}{\Gamma(\kappa_1)}\Big[\int_0^{1}(1-w)^{\kappa_1-1}(t_a^{\kappa_1-1}-t_b^{\kappa_1-1})dw+\int_0^{t_a}-(t_a-w)^{\kappa_1-1}dw \\
&\quad +\int_0^{t_b}(t_b-w)^{\kappa_1-1}dw\Big]+\frac{A}{\Gamma(\kappa_1+\zeta_1)}\Big[\int_0^{1}(1-w)^{\kappa_1+\zeta_1-1}(t_a^{\kappa_1+\zeta_1-1}-t_b^{\kappa_1+\zeta_1-1})dw \\
&\quad +\int_0^{t_a}-(t_a-w)^{\kappa_1+\zeta_1-1}dw+\int_0^{t_b}(t_b-w)^{\kappa_1+\zeta_1-1}dw\Big]. \\
&= \ \frac{r\lambda_1}{\Gamma(\kappa_1+1)}\Big(t_a^{\kappa_1-1}-t_b^{\kappa_1-1}-t_a^{\kappa_1}+t_b^{\kappa_1}\Big)+\frac{A}{\Gamma(\kappa_1+\zeta_1+1)}\Big(t_a^{\kappa_1+\zeta_1-1}-t_b^{\kappa_1+\zeta_1-1} \\
&\quad -t_a^{\kappa_1+\zeta_1}+t_b^{\kappa_1+\zeta_1}\Big).
\end{aligned}
$$

For K_2, we get

$|K_2(x,y)(t_a)-K_2(x,y)(t_b)|$

$$
\begin{aligned}
&\leq \ \int_0^{1}|G_2(t_a,w)-G_2(t_b,w)||x(w)|dw+\int_0^{1}|H_2(t_a,w)-H_2(t_b,w)| \\
&\quad \times|\sigma_2(w,x(w),y(w))|dw \\
&\leq \ r\int_0^{1}|G_2(t_a,w)-G_2(t_b,w)|dw+B\int_0^{1}|H_2(t_a,w)-H_2(t_b,w)|dw
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & r\Big[\int_0^{t_a}|G_2(t_a,w)-G_2(t_b,w)|dw+\int_{t_a}^{t_b}|G_2(t_a,w)-G_2(t_b,w)|dw \\
& +\int_{t_b}^{1}|G_2(t_a,w)-G_2(t_b,w)|dw\Big]+B\Big[\int_0^{t_a}|H_2(t_a,w)-H_2(t_b,w)|dw \\
& +\int_{t_a}^{t_b}|H_2(t_a,w)-H_2(t_b,w)|dw+\int_{t_b}^{1}|H_2(t_a,w)-H_2(t_b,w)|dw\Big] \\
=\ & \frac{r\lambda_2}{\Gamma(\kappa_2)}\Big[\int_0^{t_a}\Big((t_a(1-w))^{\kappa_2-1}-(t_a-w)^{\kappa_2-1}-(t_b(1-w))^{\kappa_2-1}+(t_b-w)^{\kappa_2-1}\Big)dw \\
& +\int_{t_a}^{t_b}\Big((t_a(1-w))^{\kappa_2-1}-(t_b(1-w))^{\kappa_2-1}+(t_b-w)^{\kappa_2-1}\Big)dw \\
& +\int_{t_b}^{1}\Big((t_a(1-w))^{\kappa_2-1}-(t_b(1-w))^{\kappa_2-1}\Big)dw\Big]+\frac{B}{\Gamma(\kappa_2+\zeta_2)} \\
& \times\Big[\int_0^{t_a}\Big((t_a(1-w))^{\kappa_2+\zeta_2-1}-(t_a-w)^{\kappa_2+\zeta_2-1}-(t_b(1-w))^{\kappa_2+\zeta_2-1} \\
& +(t_b-w)^{\kappa_2+\zeta_2-1}\Big)dw+\int_{t_a}^{t_b}\Big((t_a(1-w))^{\kappa_2+\zeta_2-1}-(t_b(1-w))^{\kappa_2+\zeta_2-1} \\
& +(t_b-w)^{\kappa_2+\zeta_2-1}\Big)dw+\int_{t_b}^{1}\Big((t_a(1-w))^{\kappa_2+\zeta_2-1}-(t_b(1-w))^{\kappa_2+\zeta_2-1}\Big)dw\Big] \\
=\ & \frac{r\lambda_2}{\Gamma(\kappa_2)}\Big[\int_0^{1}(1-w)^{\kappa_2-1}(t_a^{\kappa_2-1}-t_b^{\kappa_2-1})dw+\int_0^{t_a}-(t_a-w)^{\kappa_2-1}dw \\
& +\int_0^{t_b}(t_b-w)^{\kappa_2-1}dw\Big]+\frac{B}{\Gamma(\kappa_2+\zeta_2)}\Big[\int_0^{1}(1-w)^{\kappa_2+\zeta_2-1}(t_a^{\kappa_2+\zeta_2-1}-t_b^{\kappa_2+\zeta_2-1})dw \\
& +\int_0^{t_a}-(t_a-w)^{\kappa_2+\zeta_2-1}dw+\int_0^{t_b}(t_b-w)^{\kappa_2+\zeta_2-1}dw\Big]. \\
=\ & \frac{r\lambda_2}{\Gamma(\kappa_2+1)}\Big(t_a^{\kappa_2-1}-t_b^{\kappa_2-1}-t_a^{\kappa_2}+t_b^{\kappa_2}\Big)+\frac{B}{\Gamma(\kappa_2+\zeta_2+1)}\Big(t_a^{\kappa_2+\zeta_2-1}-t_b^{\kappa_2+\zeta_2-1} \\
& -t_a^{\kappa_2+\zeta_2}+t_b^{\kappa_2+\zeta_2}\Big).
\end{aligned}
$$

Since $t_a^{\kappa_1}$, $t_b^{\kappa_1}$, $t_a^{\kappa_1-1}$, $t_b^{\kappa_1-1}$, $t_a^{\kappa_2}$, $t_b^{\kappa_2}$, $t_a^{\kappa_2-1}$, $t_b^{\kappa_2-1}$, $t_a^{\kappa_1+\zeta_1}$, $t_b^{\kappa_1+\zeta_1}$, $t_a^{\kappa_1+\zeta_1-1}$, $t_b^{\kappa_1+\zeta_1-1}$, $t_a^{\kappa_2+\zeta_2}$, $t_b^{\kappa_2+\zeta_2}$, $t_a^{\kappa_2+\zeta_2-1}$ and $t_b^{\kappa_2+\zeta_2-1}$ all are uniformly continuous on $[0,1]$. Therefor KU_r is equicontinuous and hence K is completely continuous. Thus there exist solution for problem (4.1) by Theorem 1.3.2. □

4.2 Existence of Unique Solution

Existence of unique solution for problem (4.1) is checked in this section.

Theorem 4.2.1. Suppose $\sigma_1,\ \sigma_2 : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous functions and satisfies Lipschitz condition

$$\begin{aligned}|\sigma_1(t,x_1,y_1)-\sigma_1(t,x_2,y_2)| &\leq \tau_1|x_1-x_2|+\tau_2|y_1-y_2|,\\ |\sigma_2(t,x_1,y_1)-\sigma_2(t,x_2,y_2)| &\leq \xi_1|x_1-x_2|+\xi_2|y_1-y_2|,\end{aligned}$$

where $x_i,y_i\in\mathbb{R}$ for $i=1,2$. If $k=\max\{k_1,k_2\}<1$ and $p=\max\{p_1,p_2\}<1$ where

$$k_1=\frac{\lambda_1}{\Gamma(\kappa_1+1)}+\frac{\tau_1}{\Gamma(\kappa_1+\zeta_1+1)},\ k_2=\frac{\tau_2}{\Gamma(\kappa_1+\zeta_1+1)},$$

and

$$p_1=\frac{\lambda_2}{\Gamma(\kappa_2+1)}+\frac{\xi_1}{\Gamma(\kappa_2+\zeta_2+1)},\ p_2=\frac{\xi_2}{\Gamma(\kappa_2+\zeta_2+1)}.$$

Then problem (4.1) has unique solution.

Proof. Since $\|K_1(x,y)\|_{\widetilde{S}}\leq r$ and $\|K_2(x,y)\|_{\widetilde{Q}}\leq r$, proved in Theorem 4.1.1. Therefore, we will only prove that K is a contraction mapping. Suppose $(x_1,x_2),\ (y_1,y_2)\in\widetilde{S}\times\widetilde{Q}$. Then

$$\begin{aligned}&|K_1(x_1,x_2)(t)-K_1(y_1,y_2)(t)|\\ &\leq \int_0^1|G_1(t,w)||x_1(w)-y_1(w)|dw+\int_0^1|H_1(t,w)||\sigma_1(w,x_1(w),x_2(w))\\ &\quad-\sigma_1(w,y_1(w),y_2(w))|dw\\ &\leq \int_0^1|G_1(t,w)||x_1(w)-y_1(w)|dw+\int_0^1|H_1(t,w)|(\tau_1|x_1(w)-y_1(w)|\\ &\quad+\tau_2|x_2(w)-y_2(w)|)dw\\ &\leq \int_0^1|G_1(t,w)|\|x_1-y_1\|dw+\int_0^1|H_1(t,w)|(\tau_1\|x_1-y_1\|+\tau_2\|x_2-y_2(w)\|)dw.\end{aligned}$$

After integration we obtain

$$\begin{aligned}\|K_1(x_1,x_2)-K_1(y_1,y_2)\| &\leq \frac{\lambda_1\|x_1-y_1\|}{\Gamma(\kappa_1+1)}+\frac{\tau_1\|x_1-y_1\|}{\Gamma(\kappa_1+\zeta_1+1)}+\frac{\tau_2\|x_2-y_2\|}{\Gamma(\kappa_1+\zeta_1+1)}\\ &= \left[\frac{\lambda_1}{\Gamma(\kappa_1+1)}+\frac{\tau_1}{\Gamma(\kappa_1+\zeta_1+1)}\right]\|x_1-y_1\|+\frac{\tau_2\|x_2-y_2\|}{\Gamma(\kappa_1+\zeta_1+1)}\\ &= k(\|x_1-y_1\|+\|x_2-y_2\|).\end{aligned}$$

Thus we have

$$\|K_1(x_1,x_2)-K_1(y_1,y_2)\| \le k\|(x_1-y_1)+(x_2-y_2)\|.$$

Now we will prove result for K_2.

$|K_2(x_1,x_2)(t)-K_2(y_1,y_2)(t)|$

$$\begin{aligned}
&\le \int_0^1 |G_2(t,w)||x_2(w)-y_2(w)|dw+\int_0^1 |H_2(t,w)||\sigma_2(w,x_1(w),x_2(w)) \\
&\quad -\sigma_1(w,y_1(w),y_2(w))|dw \\
&\le \int_0^1 |G_2(t,w)||x_2(w)-y_2(w)|dw+\int_0^1 |H_2(t,w)|(\xi_1|x_1(w)-y_1(w)| \\
&\quad +\xi_2|x_2(w)-y_2(w)|)dw \\
&\le \int_0^1 |G_2(t,w)|\|x_2-y_2\|dw+\int_0^1 |H_2(t,w)|(\xi_1\|x_1-y_1\|+\xi_2\|x_2-y_2(w)\|)dw.
\end{aligned}$$

Integration provides following results

$$\begin{aligned}
\|K_2(x_1,x_2)-K_2(y_1,y_2)\| &\le \frac{\lambda_2\|x_2-y_2\|}{\Gamma(\kappa_2+1)}+\frac{\xi_1\|x_1-y_1\|}{\Gamma(\kappa_2+\zeta_2+1)}+\frac{\xi_2\|x_2-y_2\|}{\Gamma(\kappa_2+\zeta_2+1)} \\
&= \left[\frac{\lambda_2}{\Gamma(\kappa_2)}+\frac{\xi_1}{\Gamma(\kappa_2+\zeta_2+1)}\right]\|x_1-y_1\|+\frac{\xi_2\|x_2-y_2\|}{\Gamma(\kappa_2+\zeta_2+1)} \\
&= p(\|x_1-y_1\|+\|x_2-y_2\|).
\end{aligned}$$

Thus we obtain

$$\|K_2(x_1,x_2)-K_2(y_1,y_2)\| \le p\|(x_1-y_1)+(x_2-y_2)\|.$$

Thus K is a contraction mapping. Thus problem (4.1) has unique solution. □

Example 4.2.1. Assume problem

$$\begin{aligned}
D^{1/2}(D^{3/2}-\frac{1}{6})x(t) &= (t+\frac{1}{4})^3[(x(t))^{\mu_1}+(y(t))^{\mu_2}], \\
D^{2/3}(D^{5/3}+\frac{1}{3})y(t) &= (t+\frac{1}{4})^3[(x(t))^{\nu_1}+(y(t))^{\nu_2}], \\
D^{3/2}x(0) &= 0,\ x(0)=x(1)=0, \\
D^{5/3}y(0) &= 0,\ y(0)=y(1)=0,
\end{aligned} \tag{4.7}$$

here $a_i = b_i = \frac{1}{64}$ and $p(t) = q(t) = 0$. Solution for problem (4.7) exists for $0 < \mu_i, \nu_i < 1$ or $\mu_i, \nu_i > 1$. Also

$$\begin{aligned}
\sigma_1(t, x, y) &= (t + \frac{1}{4})^3[(x(t))^{\mu_1} + (y(t))^{\mu_2}], \\
\sigma_2(t, x, y) &= (t + \frac{1}{4})^3[(x(t))^{\nu_1} + (y(t))^{\nu_2}].
\end{aligned}$$

Now we will check Lipschitz condition.

$$\begin{aligned}
|\sigma_1(t, x_1, x_2) - \sigma_1(t, y_1, y_2)| &\leq (t + \frac{1}{3})^3(|(x_1(t))^{\mu_1} - (y_1(t))^{\mu_1}| \\
&\quad +|(x_2(t))^{\mu_2} - (y_2(t))^{\mu_2}|), \\
|\sigma_2(t, x_1, x_2) - \sigma_2(t, y_1, y_2)| &\leq (t + \frac{1}{3})^3(|(x_1(t))^{\nu_1} - (y_1(t))^{\nu_1}| \\
&\quad +|(x_2(t))^{\nu_2} - (y_2(t))^{\nu_2}|).
\end{aligned}$$

Here $k_1 = -0.11762963$, $k_2 = 0.0078125$ and $p_1 = 0.783969355$, $p_2 = 0.562422925$. Thus $k = \max\{-0.117598, 0.0078125\} < 1$ and $p = \max\{0.783969355, 0.562422925\} < 1$. Hence problem (4.7) has unique solution.

Chapter 5

Conclusion

In concern of generalized Reimann-Liouville fractional order derivatives, we have presented new form of fractional Langevin equation. Existence and uniqueness of solutions is accomplished by using fixed point theorems requiring corresponding nonlinear function to be Lipschitz.

Properties of Green functions are used to apply upper and lower solution techniques along with fixed point theorems to achieve existence and uniqueness of a positive solution for two fractional Langevin equations.

Also existence of solution is obtained for coupled system of fractional Langevin equations with Riemann-Liouville fractional derivative and uniqueness is verified by contraction mapping principle.

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