Solutions of the Momentum Equation for Laminar Flow through Concentric Ducts using Symmetry Methods

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Dedication

I would like to dedicate this dissertation first and foremost to my parents for their everlasting support and affection. I also dedicate this to my friends for their love, endurance and support.

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Abstract

Partial or ordinary differential equations are used to solve problems in a variety of fields. The goal of achieving exact solutions of differential equations is therefore of great importance and continues to attract great attention. The basic purpose of this thesis entitled "Solution of the Momentum Equation for Laminar Flow through Concentric Ducts using Symmetry Method" is attempted to find symmetries and the exact solutions of momentum equation. After that, we reduced partial differential equation to ordinary differential equations. Then we obtained exact solutions corresponding to these ordinary differential equations.

This thesis is divided into four chapters. The brief outlines of the research work presented chapter wise in the thesis are as follows:

In Chapter 1, is given the brief introduction of partial differential equations and their solutions. Also we discussed preliminary material and relevant literature of Lie group of transformations.

In Chapter 2, methods to solve partial differential equations, Lie symmetries and invariant solution of partial differential equations are discussed.

The core of the thesis is chapter 3. The symmetries and invariant solution of momentum equation are investigated.

In Chapter 4, the thesis is concluded.

Contents

List of Abbreviations

DE	Differential Equation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
IVP	Initial Value Problem

Chapter 1

Preliminaries

Differential equation is an equation that contains an unknown function and its derivatives [6]. Differential equations have become a significant branch of applied and pure mathematics from their emergence in 17th century. Though their history has been well studied, it continues to be an important area of research. The application of differential equations are extended to number of subjects, from physics to population growth to stock market. They are helpful tool for modeling and analyzing environmental phenomena including variation in population of species over time or transmission of disease. Differential equations appear whenever an unknown event changes with respect to time. The two categories of differential equations are ordinary differential equations and partial differential equations.

An equation which involves one or more functions of a single independent variable is called an ordinary differential equation. Ordinary differential equations have become especially important and also have contributed in major developments [7]. Ordinary differential equations play vital role in modeling and dynamical systems [8, 9].

An equation which involves one or more partial derivative of one or more functions dependent on more than one independent variables is known as partial differential equation. The PDEs can be classified as linear, quasi-linear and non-linear partial differential equations. Linear PDEs are those equations in which the order of dependent variables and their partial derivatives is linear. Whereas, if the equation is linear in its highest order partial derivative then it is said to be quasi-linear. The PDE is said to

be non-linear if it is neither linear nor quasi-linear. Furthermore, linear PDEs can be classified as homogeneous and non-homogeneous. A liner PDE is said to be homogeneous if it does not involve the terms which only depend on the independent variables otherwise PDE is said to be non-homogeneous.
The PDEs can be solved analytically as well as numerically. The analytical solutions can be calculated using the method of separation of variables, method of characteristics, integral transform an change of variables, symmetry method etc. But in some cases we are unable to find the solution analytically then we move towards the numerical solutions of differential equations. The numerical solution can be calculated using the finite element method, finite difference method and finite volume method. In this dissertation, we are dealing with the non-homogeneous PDE.
Marius Sophus Lie, A Norwegian mathematician (1842-1899), devoted much of his life to theory of continuous groups and their influence over differential equations [10]. Lie has found that common solution methodologies employ groups of symmetries of equations in order to find solutions. As a consequence, exact solutions can be obtained by use of symmetries [11]. The Lie group analysis is major implementation in distinct fields such as number theory, differential equations, analysis, differential geometry etc. In this dissertation, we mainly focus on finding the solution of partial differential equation by using symmetries. The standard methods of finding solutions are insufficient for such type of equations so, implementation of symmetries helps in finding solutions. The concept of symmetry methods is to find the new coordinate system which makes it simpler to find the solution [12].
First we will recall some basic definitions before solving the momentum equation.

1.1 Groups

Set of elements with composition law μ between elements that satisfy the axioms $(i) - (iv)$ is said to be a group $\mathbb{G}$.

(i) Closure property

For all $p, q \in \mathbb{G}$ the composition $\mu(p,q) \in \mathbb{G}$.

(ii) Associative property

$\forall\ p, q, r \in \mathbb{G}$,

$$\mu(p, \mu(q,r)) = \mu(\mu(p,q), r).$$

(iii) Identity element

There exists $e \in \mathbb{G}$ for any $p \in \mathbb{G}$ such that

$$\mu(p,e) = \mu(e,p) = p,$$

then e is said to be the identity element of $\mathbb{G}$.

(iv) Inverse element

There exists $q \in \mathbb{G}$ for any $p \in \mathbb{G}$ such that

$$\mu(p,q) = \mu(q,p) = e,$$

then q is said to be the inverse of p in $\mathbb{G}$ and is denoted as p^{-1}. $\mathbb{G}$ is said to be *abelian* if $\forall\ p, q \in \mathbb{G}$ $\mu(p,q)$=$\mu(q,p)$. A subset of $\mathbb{G}$ is said to be a *subgroup* if it is a group with same composition law μ.

1.2 One-Parameter Lie Groups of Transformations and Infinitesimal Transformations

Definition 1.2.1. ($Group\ of\ Transformations$)
Let us consider $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ lie in region $M \subset \mathbb{R}^n$. A set of transformations [2]

$$\mathbf{t}^* = \mathbf{T}(\mathbf{t}; \varepsilon),$$

defined for every $\mathbf{t} \in M$ and the parameter $\varepsilon \in A \subset \mathbb{R}$, with $\mu(\varepsilon, \delta)$ defining the composition law of parameters $\delta, \varepsilon \in A$ and on M it forms a group of transformations if

(i) For every parameter $\varepsilon \in A$, transformations are bijective in M. (Hence, $\mathbf{t}^*$ lies in M.)

(ii) With the composition law μ, A forms a group $\mathbb{G}$.

(iii) $\mathbf{t}^* = \mathbf{t}$ when $\varepsilon = e$, i.e,

$$\mathbf{T}(\mathbf{t}; e) = \mathbf{t}.$$

(iv) If $\mathbf{t}^* = \mathbf{T}(\mathbf{t}; \varepsilon)$, $\mathbf{t}^{**} = \mathbf{T}(\mathbf{t}^*; \delta)$, then

$$\mathbf{t}^{**} = \mathbf{T}(\mathbf{t}; \mu(\varepsilon, \delta)).$$

Definition 1.2.2. A one-parameter Lie group of transformations defines if it satisfies the properties of group of transformation (i) to (iv) given of previous definition and in addition the following hold:

(v) $\mu(\varepsilon,\delta)$ is an analytic function of ε and δ, $\varepsilon \in A$.

(vi) $\mathbf{T}$ is infinitely differentiable with respect to $\mathbf{t} \in M$ and an analytic function $\varepsilon \in A$.

(vii) ε is a continuous parameter i.e, A is an interval in $\mathbb{R}$. Without loss of generality, $\varepsilon = 0$ corresponds to the identity element e.

Example. Let us consider

$$t^* = t + \varepsilon,$$
$$u^* = u, \ \varepsilon \in \mathbb{R},$$

then $\mu(\varepsilon, \delta) = \varepsilon + \delta$ forms a one-parameter Lie group of transformations.

Definition 1.2.3. Consider

$$\mathbf{t}^* = \mathbf{T}(\mathbf{t}; \varepsilon), \tag{1.1}$$

a family of one-parameter $\varepsilon \in \mathbb{R}$ invertible transformations of points $\mathbf{t}^* = (t_1^*, t_2^*, \ldots, t_N^*) \in \mathbb{R}^N$. It is said to be a one-parameter transformation subject to the condition

$$\mathbf{t}^*|_{\varepsilon=0} = \mathbf{t}. \tag{1.2}$$

i.e.

$$\mathbf{T}(\mathbf{t};\varepsilon)\Big|_{\varepsilon=0} = \mathbf{t}. \tag{1.3}$$

Expanding equation (1.1) for $\varepsilon = 0$, in some neighborhood of $\varepsilon = 0$, we obtain

$$\begin{aligned} \mathbf{t}^* &= \mathbf{t} + \varepsilon\left(\frac{\partial \mathbf{T}}{\partial \varepsilon}\Big|_{\varepsilon=0}\right) + \frac{\varepsilon^2}{2}\left(\frac{\partial^2 \mathbf{T}}{\partial \varepsilon^2}\Big|_{\varepsilon=0}\right) + \dots \\ &= \mathbf{t} + \varepsilon\left(\frac{\partial \mathbf{T}}{\partial \varepsilon}\Big|_{\varepsilon=0}\right) + O(\varepsilon^2). \end{aligned} \tag{1.4}$$

Let

$$\boldsymbol{\zeta}(\mathbf{t}) = \frac{\partial \mathbf{T}}{\partial \varepsilon}\Big|_{\varepsilon=0}. \tag{1.5}$$

Then transformation $\mathbf{t} + \varepsilon\boldsymbol{\zeta}(\mathbf{t})$ is called infinitesimal transformation of Lie group of transformations equation (1.1).
The components of $\boldsymbol{\zeta}(\mathbf{t})$ are called the infinitesimals of equation (1.1).

Theorem 1.2.1. There exists a parameterization $\tau(\varepsilon)$ such that Lie group of transformations equation (1.1) is equivalent to the solution of the initial value problem for the system of first-order ODEs given by

$$\frac{d\mathbf{t}}{d\tau} = \boldsymbol{\zeta}(\mathbf{t}^*), \tag{1.6}$$

with

$$\mathbf{t}^* = \mathbf{t} \quad \text{when} \quad \tau = 0. \tag{1.7}$$

In particular

$$\tau(\varepsilon) = \int_0^{\varepsilon} \Gamma(\varepsilon')d\varepsilon', \tag{1.8}$$

where

$$\Gamma(\varepsilon) = \frac{\partial \mu(p,q)}{\partial q}\Big|_{(p,q)=(\varepsilon^{-1},\varepsilon)}, \tag{1.9}$$

and

$$\Gamma(0) = 1. \tag{1.10}$$

(where ε^{-1} is inverse element to ε).

Example. Consider the groups of translations

$$\begin{aligned} t^* &= t + \varepsilon, \\ u^* &= u, \end{aligned} \tag{1.11}$$

the law of composition is $\mu(p,q) = p + q$, and $\varepsilon^{-1} = -\varepsilon$. Then $\frac{\partial\mu(p,q)}{\partial q} = 1$ and hence $\Gamma(\varepsilon) \equiv 1$.

Let $\mathbf{t} = (t,u)$. Then the group equation (1.11) is $\mathbf{T}(\mathbf{t};\varepsilon) = (t+\varepsilon, u)$. Thus $\frac{\partial \mathbf{T}}{\partial \varepsilon}(\mathbf{t};\varepsilon) = (1,0)$. Hence

$$\boldsymbol{\zeta}(\mathbf{t}) = \frac{\partial \mathbf{T}}{\partial \varepsilon}(\mathbf{t};\varepsilon)\bigg|_{\varepsilon=0} = (1,0).$$

Consequently becomes

$$\frac{dt^*}{d\varepsilon} = 1, \quad \frac{du^*}{d\varepsilon} = 0, \tag{1.12}$$

with

$$t^* = t, \quad u^* = u \qquad \text{at} \quad \varepsilon = 0. \tag{1.13}$$

The solution of IVP equations (1.12) and (1.13) are easily seen to be equation (1.11).

Definition 1.2.4. The infinitesimal generator of one-parameter Lie group of transformations is defined by operator

$$\mathbf{X} = \mathbf{X}(\mathbf{t}) = \sum_{j=1}^{n} \zeta_j(\mathbf{t}) \frac{\partial}{\partial t_j}, \tag{1.14}$$

Theorem 1.2.2. The Lie group of transformations equation (1.1) is equivalent to the solution of an ordinary differential equation initial value problem

$$\begin{aligned} \frac{d\mathbf{t}^*(\varepsilon)}{d\varepsilon} &= \boldsymbol{\zeta}(\mathbf{t}^*(\varepsilon)), \\ \mathbf{t}^*(0) &= \mathbf{t}. \end{aligned} \tag{1.15}$$

Example. Consider an infinitesimal generator

$$\mathbf{X}_1 = t\frac{\partial}{\partial t} + 2\frac{\partial}{\partial u}. \tag{1.16}$$

Its infinitesimals are

$$\zeta(t,u) = t, \;\; \eta(t,u) = 2. \tag{1.17}$$

According to Theorem 1.2.2, the IVP of ODE is given by

$$\frac{dt^*}{d\varepsilon} = t^*,\ \frac{du^*}{d\varepsilon} = 2,\ t^*(0) = t,\ u^*(0) = u. \tag{1.18}$$

From the first differential equation in system equation (1.18) we obtain

$$t^* = e^{\varepsilon + c_1},\ u^* = 2\varepsilon + c_2.$$

After applying $t^*(0) = t,\ u^*(0) = u$, we obtained

$$t^* = te^{\varepsilon},\ u^* = 2\varepsilon + u. \tag{1.19}$$

The Lie group corresponding to the infinitesimal generator equation (1.16) is given by equation (1.19), which represents scaling in t and translation in u.

1.3 Invariance of a Partial Differential Equations

Definition 1.3.1. A function $F(\mathbf{t})$ is said to be invariant of Lie group of transformations equation (1.1) iff for any group of transformations equation (1.1)

$$F(\mathbf{t}^*) = F(\mathbf{t}). \tag{1.20}$$

Theorem 1.3.1. $F(\mathbf{t})$ is invariant under equation (1.1) iff

$$\mathbf{X}F(\mathbf{t}) \equiv 0. \tag{1.21}$$

Example. Find all functions $F(t, u)$ invariant with respect to scalings

$$\mathbf{X}_1 = t\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u}, \tag{1.22}$$

corresponding to the Lie group of scalings, given by

$$t^* = te^{\varepsilon},\ u^* = ue^{3\varepsilon}. \tag{1.23}$$

According to Theorem 1.3.1, such functions $F(t, u)$ should satisfy $\mathbf{X}F(t, u) \equiv 0$.

$$t\frac{\partial F}{\partial t} + 3u\frac{\partial F}{\partial u} = 0. \tag{1.24}$$

Solving the characteristic equation

$$\frac{dt}{t} = \frac{du}{3u}. \tag{1.25}$$

One obtains the first integral as

$$c_1 = \frac{t^3}{u}. \tag{1.26}$$

It follows that the invariant functions $F(t, u)$ are given by

$$F(t, u) = F(c_1) = F(\frac{t^3}{u}). \tag{1.27}$$

Indeed, one can explicitly verify this fact

$$F\left(\frac{t^{*3}}{u^*}\right) = F\left(\frac{t^3 e^{3\varepsilon}}{u e^{3\varepsilon}}\right) = F\left(\frac{t^3}{u}\right). \tag{1.28}$$

Theorem 1.3.2. *(Invariance of a ODE)*
An ordinary differential equation

$$E(t, u, u^{'}, ..., u^n) = 0, \tag{1.29}$$

admits a group of symmetries with generator $\mathbf{X}^{[k]}$ iff

$$\mathbf{X}^{[k]} E\Big|_{E=0} = 0. \tag{1.30}$$

Example. Consider an ODE

$$u^{''} = 0. \tag{1.31}$$

Here

$$E(t, u, u^{'}, u^{''}) = u^{''}. \tag{1.32}$$

Infinitesimal generator and its first extension correspondingly are defined as

$$\mathbf{X} = \zeta(t, u)\frac{\partial}{\partial t} + \eta(t, u)\frac{\partial}{\partial u}, \tag{1.33}$$

$$\mathbf{X}^{[1]} = \zeta\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \eta^{'}\frac{\partial}{\partial u'} + \eta^{''}\frac{\partial}{\partial u''}. \tag{1.34}$$

Applying $\mathbf{X}^{[1]}$ to E we obtain

$$\mathbf{X}^{[1]}E\Big|_{E=0} = \eta^{''} = 0. \tag{1.35}$$

Therefore, the symmetry condition is

$$\eta^{''} = 0,$$

thus we have

$$\eta^{''} = \eta_{tt} + (2\eta_{tu} - \zeta_{tt})u^{'} - (\eta_{uu} - 2\zeta_{tu})u^{'2} - \zeta_{uu}u^{'3} + (\eta_u - 2\zeta_t - 3\zeta_u u^{'})u^{''} = 0, \tag{1.36}$$

and we obtain the system of PDEs as

$$\begin{aligned} \eta_{tt} &= 0,\\ 2\eta_{tu} - \zeta_{tt} &= 0,\\ \eta_{uu} - 2\zeta_{tu} &= 0,\\ \zeta_{uu} &= 0. \end{aligned} \tag{1.37}$$

Solving the system equation (1.37) we obtain

$$\begin{aligned} \zeta(t,u) &= b_1tu + b_2u + b_3t^2 + b_7t + b_8,\\ \eta(t,u) &= b_1u^2 + b_3tu + b_4u + b_5t + b_6. \end{aligned} \tag{1.38}$$

Here b_i, $1 \le i \le 8$ are arbitrary constants.

So, the most general one-parameter Lie group of point symmetries of the ODE $u^{''} = 0$, is obtained by the infinitesimal generator

$$\mathbf{X} = (b_1tu + b_2u + b_3t^2 + b_7t + b_8)\frac{\partial}{\partial t} + (b_1u^2 + b_3tu + b_4u + b_5t + b_6)\frac{\partial}{\partial u}. \tag{1.39}$$

Therefore the corresponding symmetries are

$$\begin{aligned} &\mathbf{X}_1 = tu\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u}, && \mathbf{X}_2 = u\frac{\partial}{\partial t},\\ &\mathbf{X}_3 = t^2\frac{\partial}{\partial t} + tu\frac{\partial}{\partial u}, && \mathbf{X}_4 = u\frac{\partial}{\partial u},\\ &\mathbf{X}_5 = t\frac{\partial}{\partial u}, && \mathbf{X}_6 = \frac{\partial}{\partial u},\\ &\mathbf{X}_7 = t\frac{\partial}{\partial t}, && \mathbf{X}_8 = \frac{\partial}{\partial t}. \end{aligned}$$

Definition 1.3.2. *(Invariance of a PDE)*
Consider a PDE

$$F(t, v, \underset{1}{v}, \underset{2}{v}, \dots, \underset{k}{v}) = 0, \tag{1.40}$$

with p dependent variables $v = (v^1, v^2, v^3, \dots, v^p)$ and q independent variables $t = (t_1, t_2, t_3, \dots, t_q)$. As $\underset{1}{v}$ refers to a set of coordinates corresponding to all first order partial derivatives of v with respect to t and so forth. The one-parameter Lie group of point transformations

$$\begin{aligned} t^* &= T(t, v; \varepsilon). \\ u^* &= U(t, v; \varepsilon). \end{aligned} \tag{1.41}$$

leaves partial differential equation (1.40) invariant iff its kth extension, leaves the surfaces in $(t, v, \underset{1}{v}, \underset{2}{v}, \dots, \underset{k}{v})$-space, defined by equation (1.40), invariant.

1.4 Change of Coordinates

Consider a non-degenerate change of coordinates

$$\mathbf{u} = \mathbf{U}(\mathbf{t}) = (u_1(\mathbf{t}), u_2(\mathbf{t}), \dots, u_n(\mathbf{t})). \tag{1.42}$$

For one-parameter Lie group of point transformations are stated in equation (1.1), the infinitesimal generator equation (1.14) with respect to coordinates $\mathbf{t} = (t_1, t_2, \dots, t_n)$ becomes the infinitesimal generator

$$\mathbf{U} = \sum_{j=1}^{n} \eta_j(\mathbf{u}) \frac{\partial}{\partial u^j}, \tag{1.43}$$

with respect to coordinate $\mathbf{u} = (u_1, u_2, \dots, u_n)$ defined by equation (1.42).

Theorem 1.4.1. After change of coordinates equation (1.42) the operator X equation (1.14) yields the operator U equation (1.43), where

$$\eta(\mathbf{u}) = X\mathbf{u}. \tag{1.44}$$

Definition 1.4.1. A change of coordinates equation (1.42) defines a set of canonical coordinates $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ for the one-parameter Lie group of point transformations are given by equation (1.1) if after this change infinitesimal generator $X = \sum_{j=1}^{n} \eta_j(\mathbf{t}) \frac{\partial}{\partial t_j}$ yields a pure translation in u^n : $U = \frac{\partial}{\partial u^n}$. Infinitesimals in this case are given by

$$\begin{aligned} \eta_j(\mathbf{u}) &= X u^j = 0, \quad j = 1, \ldots, n-1; \\ \eta_n(\mathbf{u}) &= X u^n = 1. \end{aligned} \tag{1.45}$$

Example. Consider a group of scalings in $\mathbb{R}^3$, $\mathbf{t} = (t_1, t_2, t_3)$.

$$t_1^* = e^{\epsilon} t_1, \; t_2^* = e^{2\epsilon} t_2, \; t_3^* = e^{7\epsilon} t_3. \tag{1.46}$$

The infinitesimal generator is given by

$$\mathbf{X} = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 7t_3 \frac{\partial}{\partial t_3}.$$

To find canonical coordinates one should find $3-1=2$ invariants u_1, u_2 and translation coordinate u_3. The characteristic equation is given by

$$\frac{dt_1}{t_1} = \frac{dt_2}{2t_2} = \frac{dt_3}{7t_3},$$

and the corresponding first integrals are given by, e.g.

$$u_1 = \frac{t_1^2}{t_2}, \; u_2 = \frac{t_1^7}{t_3}.$$

To find the translation coordinate u_3, one uses the condition $\mathbf{X} u_3 = 1$:

$$t_1 \frac{\partial u_3}{\partial t_1} + 2t_2 \frac{\partial u_3}{\partial t_2} + 7t_3 \frac{\partial u_3}{\partial t_3} = 1.$$

A particular solution of the characteristic equation

$$\frac{dt_1}{t_1} = \frac{dt_2}{2t_2} = \frac{dt_3}{7t_3} = \frac{du_3}{1},$$

is given by, e.g.

$$u_3 = \ln t_1.$$

Hence, the set of canonical coordinates for the group of scalings equation (1.46) is given by

$$u_1 = \frac{t_1^2}{t_2}, \; u_2 = \frac{t_1^7}{t_3}, \; u_3 = \ln t_1.$$

Chapter 2

Point Symmetries of Partial Differential Equations

In this chapter, we are focused in analysing the one-parameter group of transformations admitted by the particular system of partial differential equations. When studying invariance properties of kth order partial differential equation with dependent variable v and independent variables $t = (t_1, t_2, \ldots, t_n)$, with $v = v(t)$, we are, of course, faced with the problem of finding extensions of transformations (t, v)-space to $(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v})$-space. All the kth order partial derivatives of v with respect to t are represented by $\underset{k}{v}$.

First of all, we consider the extended transformation of a set of point transformations

$$\begin{aligned} t^* &= T(t, v), \\ v^* &= V(t, v). \end{aligned} \tag{2.1}$$

In some domain M in (t, v)-space with $(T(t, v), V(t, v))$ k-times differentiable in M, the transformations equation (2.1) are supposed to be one-to-one. The transformations equation (2.1) sustain contact conditions, i.e.,

$$\begin{aligned} dv &= \underset{1}{v}\ dt, \\ &\vdots \\ d_{k-1}v &= \underset{k}{v}\ dt, \end{aligned} \tag{2.2}$$

in some domain M in $(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v})$-space iff

$$\begin{aligned} dv^* &= \underset{1}{v}^* \ dt^*, \\ &\vdots \\ d_{k-1}v^* &= \underset{k}{v}^* dt^*, \end{aligned} \tag{2.3}$$

in the corresponding domain M^* in $(t^*, v^*, \underset{1}{v}^*, \underset{2}{v}^*, \ldots, \underset{k}{v}^*)$-space.

Let

$$v_i = \frac{\partial v}{\partial t_i}, \ v_i^* = \frac{\partial v^*}{\partial t_i^*} = \frac{\partial V}{\partial T_i}, \quad \text{etc.}$$

From now we assume summation over a repeated index. In equation (2.2), $dv = \underset{1}{v} \ dt$ represents

$$dv = v_j \ dt_j,$$

and in equation (2.2), $d_{k-1}v = \underset{k}{v} \ dt$ represents a set of equations

$$dv_{i_1 i_2 \ldots i_{k-1}} = v_{i_1 i_2 \ldots i_{k-1} j} dt_j, \ i_l = 1, 2, \ldots, n \text{ for } l = 1, 2, \ldots, k-1.$$

Similarly representations hold for equation (2.3).

We introduce the total derivative operators

$$D_i = \frac{D}{Dt_i} = \frac{\partial}{\partial t_i} + v_i \frac{\partial}{\partial v} + v_{ij} \frac{\partial}{\partial v_j} + \cdots + v_{i i_1 i_2 \ldots i_n} \frac{\partial}{\partial v_{i_1 i_2 \ldots i_n}} + \ldots, \tag{2.4}$$

$i = 1, 2, \ldots, n$. For a given differentiable function $F(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{l}{v})$ we have:

$$D_i F(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{l}{v}) = \frac{\partial F}{\partial t_i} + v_i \frac{\partial F}{\partial v} + v_{ij} \frac{\partial F}{\partial v_j} + \cdots + v_{i i_1 i_2 \ldots i_n} \frac{\partial F}{\partial v_{i_1 i_2 \ldots i_n}} + \ldots, \ \ i = 1, 2, \ldots, n.$$

Now, consider the preserved contact condition equation (2.3), $dv^* = v_j^* \ dt_j^*$, in order to determine the extended transformation

$$v_j^* = V_j(t, v, \underset{1}{v}), \quad j = 1, 2, \ldots, n. \tag{2.5}$$

From equation (2.1) we obtain

$$dv^* = (D_i V) dt_i,$$

and

$$dt_j^* = (D_i T_j) dt_i, \quad j = 1, 2, \ldots, n,$$

where D_i is defined by equation (2.4), $i = 1, 2, \ldots, n$. Then

$$(D_i T_j) v_j^* = D_i V, \quad i = 1, 2, \ldots, n.$$

Let the $n \times n$ matrix

$$A = \begin{bmatrix} D_1 T_1 & \ldots & D_1 T_n \\ \vdots & & \vdots \\ D_n T_1 & \ldots & D_n T_n \end{bmatrix} \tag{2.6}$$

and assume that A^{-1} exists. Then

$$\begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 V \\ D_2 V \\ \vdots \\ D_n V \end{bmatrix}. \tag{2.7}$$

This leads to the extended transformation in $(t, v, \underset{1}{v})$-space:

$$\begin{aligned} t^* &= T(t, v), \\ v^* &= V(t, v), \\ \underset{1}{v}^* &= \underset{1}{V}(t, v, \underset{1}{v}). \end{aligned} \tag{2.8}$$

It is easy to show that the extension to $(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v})$-space is given by

$$\begin{aligned} t^* &= T(t, v), \\ v^* &= V(t, v), \\ \underset{1}{v}^* &= \underset{1}{V}(t, v, \underset{1}{v}) \\ &\vdots \qquad \vdots \\ \underset{k}{v}^* &= \underset{1}{V}(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v}), \end{aligned} \tag{2.9}$$

where the components of $\partial^k v^*$ are determined by

$$\begin{bmatrix} v_{i_1 i_2 \ldots i_{k-1} 1}^* \\ v_{i_1 i_2 \ldots i_{k-1} 2}^* \\ \vdots \\ v_{i_1 i_2 \ldots i_{k-1} n}^* \end{bmatrix} = \begin{bmatrix} V_{i_1 i_2 \ldots i_{k-1} 1} \\ V_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ V_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 V_{i_1 i_2 \ldots i_{k-1}} \\ D_2 V_{i_1 i_2 \ldots i_{k-1}} \\ \vdots \\ D_n V_{i_1 i_2 \ldots i_{k-1}} \end{bmatrix}, \tag{2.10}$$

$i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k-1$ with $k = 2, 3, \ldots$; $\underset{1}{V}(t, v, \underset{1}{v})$ is determined by equation (2.7) and A is the matrix equation (2.6).

Now we specialize to the case where equation (2.1) defines a Lie group of transformations.

The transformations are given by equation (2.1) define a one-parameter Lie group of transformations

$$\begin{aligned} t^* &= T(t, v; \varepsilon), \\ v^* &= V(t, v; \varepsilon), \end{aligned} \tag{2.11}$$

acting on (t, v)-space, then it is easy to show that its kth extension to $(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v})$-space, given by

$$\begin{aligned} t^* &= T(t, v; \varepsilon), \\ v^* &= V(t, v; \varepsilon), \\ \underset{1}{v}^* &= \underset{1}{V}(t, v, \underset{1}{v}; \varepsilon), \\ &\vdots \\ \underset{k}{v}^* &= \underset{k}{V}(t, v, \underset{1}{v}, \ldots, \underset{k}{v}; \varepsilon), \end{aligned} \tag{2.12}$$

defines a k times extended one-parameter Lie group of transformations. In equation (2.12),

$$\begin{aligned} \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix} &= \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 V \\ D_2 V \\ \vdots \\ D_n V \end{bmatrix}, \\ \begin{bmatrix} v_{i_1 i_2 \ldots i_{k-1} 1}^* \\ v_{i_1 i_2 \ldots i_{k-1} 2}^* \\ \vdots \\ v_{i_1 i_2 \ldots i_{k-1} n}^* \end{bmatrix} &= \begin{bmatrix} V_{i_1 i_2 \ldots i_{k-1} 1} \\ V_{i_1 i_2 \ldots i_{k-1} 2} \\ \vdots \\ V_{i_1 i_2 \ldots i_{k-1} n} \end{bmatrix} = A^{-1} \begin{bmatrix} D_1 V_{i_1 i_2 \ldots i_{k-1}} \\ D_2 V_{i_1 i_2 \ldots i_{k-1}} \\ \vdots \\ D_n V_{i_1 i_2 \ldots i_{k-1}} \end{bmatrix}, \end{aligned} \tag{2.13}$$

where $\{v_i^* = V_i\}$ are the components of $\underset{1}{v}^* = \underset{1}{V}$ and $\{v_{i_1 i_2 \ldots i_{k-1} i}^* = V_{i_1 i_2 \ldots i_{k-1} i}\}$ are the components $\underset{k}{v}^* = \underset{k}{V}$. In equation (2.13) $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k-1$ with $k = 2, 3 \ldots$; the operator D_i are given by equation (2.4); A^{-1} is the inverse of the matrix A given by equation (2.6) for T and V given by equation (2.12).

2.1 Infinitesimal Transformations of One Dependent and n Independent Variables

The one-parameter Lie group of transformations

$$\begin{aligned} t_i^* &= T_i(t,v;\varepsilon) = t_i + \varepsilon\xi_i(t,v) + O(\varepsilon^2), \\ v^* &= V(t,v;\varepsilon) = v + \varepsilon\eta(t,v) + O(\varepsilon^2), \end{aligned} \tag{2.14}$$

$1 \le i \le n$ acting on (t,v)-space has as its infinitesimal generator

$$\mathbf{X} = \xi_i(t,v)\frac{\partial}{\partial t_i} + \eta(t,v)\frac{\partial}{\partial v}. \tag{2.15}$$

The k-th extension of equation (2.14), is given by

$$t_i^* = T_i(t,v;\varepsilon) = t_i + \varepsilon\xi_i(t,v) + O(\varepsilon^2), \tag{2.16}$$

$$v^* = V(t,v;\varepsilon) = v + \varepsilon\eta(t,v) + O(\varepsilon^2), \tag{2.17}$$

$$v_i^* = V_i(t,v,\underset{1}{v};\varepsilon) = v_i + \varepsilon\eta_i^{(1)}(t,v,\underset{1}{v}) + O(\varepsilon^2), \tag{2.18}$$

.

.

.

$$v_{i_1 i_2 \dots i_k}^* = V_{i_1 i_2 \dots i_k}(t,v,\underset{1}{v},\underset{2}{v},\dots,\underset{k}{v};\varepsilon) \tag{2.19}$$

$$= v_{i_1 i_2 \dots i_k} + \varepsilon\eta_{i_1 i_2 \dots i_k}^{(k)}(t,v,\underset{1}{v},\underset{2}{v},\dots,\underset{k}{v}) + O(\varepsilon^2), \tag{2.20}$$

where $1 \le i, i_l \le n$, for $1 \le l \le k$ with $k \ge 1$ has as its (k-th extended) infinitesimal

$$(\xi(t,v), \eta^{(1)}(t,v,\underset{1}{v}), \dots, \eta^{(k)}((t,v,\underset{1}{v},\underset{2}{v},\dots,\underset{k}{v})), \tag{2.21}$$

with corresponding (k-th extended) infinitesimal generator

$$\begin{aligned} \mathbf{X}^{(k)} = \xi_i(t,v)\frac{\partial}{\partial t_i} + \eta(t,v)\frac{\partial}{\partial v} + \eta_i^{(1)}(t,v,\underset{1}{v})\frac{\partial}{\partial v_i} + \dots \\ + \eta_{i_1 i_2 \dots i_k}^{(k)}\frac{\partial}{\partial v_{i_1 i_2 \dots i_k}}, \ k = 1, 2, \dots. \end{aligned} \tag{2.22}$$

The extended infinitesimal, $\eta^{(k)}$, has the explicit formulas derived in the following theorem.

Theorem 2.1.1. For $1 \leq i_l \leq n$, $1 \leq l \leq k$ with $k \geq 2$ we have

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)v_j, 1 \leq i \leq n, \tag{2.23}$$

$$\eta_{i_1 i_2 \ldots i_k}^{(k)} = D_{i_k}\eta_{i_1 i_2 \ldots i_{k-1}}^{(k-1)} - (D_{i_k}\xi_j)v_{i_1 i_2 \ldots i_{k-1} j}. \tag{2.24}$$

By Theorem 2.1.1, the extended one-parameter Lie group of transformations for one dependent and two independent variables t_1, t_2 are given by

$$t_i^* = T_i(t_1, t_2, v; \varepsilon) = t_i + \varepsilon\xi_i(t_1, t_2, v) + O(\varepsilon^2), \ \ i = 1, 2 \tag{2.25}$$

$$v^* = V(t_1, t_2, v; \varepsilon) = v + \varepsilon\eta(t_1, t_2, v) + O(\varepsilon^2), \tag{2.26}$$

$$v_i^* = V_i(t_1, t_2, v, v_1, v_2; \varepsilon) = v_i + \varepsilon\eta_i^{(1)}(t_1, t_2, v, v_1, v_2) + O(\varepsilon^2), \ \ i = 1, 2 \tag{2.27}$$

$$\begin{aligned} v_{ij}^* &= V_{ij}(t_1, t_2, v, v_1, v_2, v_{11}, v_{12}, v_{22}; \varepsilon), \\ &= v_{ij} + \varepsilon\eta_{ij}^{(2)}(t_1, t_2, v, v_1, v_2, v_{11}, v_{12}, v_{22}) + O(\varepsilon^2), \ \ i, j = 1, 2 \end{aligned} \tag{2.28}$$

etc., and the extended infinitesimals are given as

$$\eta_1^{(1)} = \frac{\partial\eta}{\partial t_1} + \left[\frac{\partial\eta}{\partial v} - \frac{\partial\xi_1}{\partial t_1}\right]v_1 - \frac{\partial\xi_2}{\partial t_1}v_2 - \frac{\partial\xi_1}{\partial v}(v_1)^2 - \frac{\partial\xi_2}{\partial v}v_1v_2, \tag{2.29}$$

$$\eta_2^{(1)} = \frac{\partial\eta}{\partial t_2} + \left[\frac{\partial\eta}{\partial v} - \frac{\partial\xi_2}{\partial t_2}\right] - \frac{\partial\xi_1}{\partial t_2}v_1 - \frac{\partial\xi_2}{\partial v}(v_2)^2 - \frac{\partial\xi_1}{\partial v}v_1v_2, \tag{2.30}$$

$$\begin{aligned} \eta_{11}^{(2)} = &\frac{\partial^2\eta}{\partial t_1^2} + \left[2\frac{\partial^2\eta}{\partial t_1\partial v} - \frac{\partial^2\xi_1}{\partial t_1^2}\right]v_1 - \frac{\partial^2\xi_2}{\partial t_1^2}v_2 + \left[\frac{\partial\eta}{\partial v} - 2\frac{\partial\xi_1}{\partial t_1}\right]v_{11} \\ &- 2\frac{\partial\xi_2}{\partial t_1}v_{12} + \left[\frac{\partial^2\eta}{\partial v^2} - 2\frac{\partial^2\xi_1}{\partial t_1\partial v}\right](v_1)^2 - 2\frac{\partial^2\xi_2}{\partial t_1\partial v}v_1v_2 \\ &- \frac{\partial^2\xi_1}{\partial v^2}(v_1)^3 - \frac{\partial^2\xi_2}{\partial v^2}(v_1)^2v_2 - 3\frac{\partial\xi_1}{\partial v}v_1v_{11} - \frac{\partial\xi_2}{\partial v}v_2v_{11} \\ &- 2\frac{\partial\xi_2}{\partial v}v_1v_{12}, \end{aligned} \tag{2.31}$$

$$
\begin{aligned}
\eta_{12}^{(2)} &= \eta_{21}^{(2)} \\
&= \frac{\partial^2\eta}{\partial t_1\partial t_2} + \left[\frac{\partial^2\eta}{\partial t_1\partial v} - \frac{\partial^2\xi_2}{\partial t_1\partial t_2}\right]v_2 + \left[\frac{\partial^2\eta}{\partial t_2\partial v} - \frac{\partial^2\xi_1}{\partial t_1\partial t_2}\right]v_1 \\
&- \frac{\partial\xi_2}{\partial t_1}v_2 2 + \left[\frac{\partial\eta}{\partial v} - \frac{\partial\xi_1}{\partial t_1} - \frac{\partial\xi_2}{\partial t_2}\right]v_{12} - \frac{\partial\xi_1}{\partial t_2}v_{11} - \frac{\partial^2\xi_2}{\partial t_1\partial v}(v_2)^2 \\
&+ \left[\frac{\partial^2\eta}{\partial v^2} - \frac{\partial^2\xi_1}{\partial t_1\partial v} - \frac{\partial^2\xi_2}{\partial t_2\partial v}\right]v_1v_2 - \frac{\partial^2\xi_1}{\partial t_2\partial v}(v_1)^2 - \frac{\partial^2\xi_2}{\partial v^2}v_1(v_2)^2 \\
&- \frac{\partial^2\xi_1}{\partial v^2}(v_1)^2v_2 - 2\frac{\partial\xi_2}{\partial v}v_2v_{12} - 2\frac{\partial\xi_1}{\partial v}v_1v_{12} \\
&- \frac{\partial\xi_1}{\partial v}v_2v_{11} - \frac{\partial\xi_2}{\partial v}v_1v_{22},
\end{aligned} \tag{2.32}
$$

$$
\begin{aligned}
\eta_{22}^{(2)} = &\frac{\partial^2\eta}{\partial t_2^2} + \left[2\frac{\partial^2\eta}{\partial t_2\partial v} - \frac{\partial^2\xi_2}{\partial t_2^2}\right]v_2 - \frac{\partial^2\xi_1}{\partial t_2^2}v_1 - \left[\frac{\partial\eta}{\partial v} - 2\frac{\partial\xi_2}{\partial t_2}\right]v_{22} \\
&- 2\frac{\partial\xi_1}{\partial t_2}v_{12} + \left[\frac{\partial^2\eta}{\partial v^2} - 2\frac{\partial\xi_2}{\partial t_2\partial v}\right](v_2)^2 - 2\frac{\partial^2\xi_1}{\partial t_2\partial v}v_1v_2 - \frac{\partial^2\xi_2}{\partial v^2}(v_2)^3 \\
&- \frac{\partial^2\xi_1}{\partial v^2}v_1(v_2)^2 - 3\frac{\partial\xi_2}{\partial v}v_2v_{22} - \frac{\partial\xi_1}{\partial v}v_1v_{22} - 2\frac{\partial\xi_1}{\partial v}v_2v_{12}.
\end{aligned} \tag{2.33}
$$

Theorem 2.1.2. *(Infinitesimal Criterion for Invariance of a PDE).*
Let

$$\mathbf{X} = \xi_i(t,v)\frac{\partial}{\partial t_i} + \eta(t,v)\frac{\partial}{\partial v}, \tag{2.34}$$

be the infinitesimal generator of equation (1.41). Let

$$
\begin{aligned}
\mathbf{X}^{[k]} = &\xi_i(t,v)\frac{\partial}{\partial t_i} + \eta(t,v)\frac{\partial}{\partial v} + \eta_i^{(1)}(t,v,\underset{1}{v})\frac{\partial}{\partial v_i} \\
&+ \cdots + \eta_{i_1i_2\ldots i_k}^{(k)}(t,v,\underset{1}{v},\underset{2}{v},\ldots,\underset{k}{v})\frac{\partial}{\partial v_{i_1i_2\ldots i_k}},
\end{aligned} \tag{2.35}
$$

be the kth extended infinitesimal generator of equation (2.34) where $\eta_i^{(1)}$ and $\eta_{i_1i_2\ldots i_j}^{(j)}$ are given by

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_j)v_j, \quad 1 \le i \le n; \tag{2.36}$$

$$\eta_{i_1i_2\ldots i_k}^{(k)} = D_{i_k}\eta_{i_1i_2\ldots i_{k-1}}^{(k-1)} - (D_{i_k}\xi_j)v_{i_1i_2\ldots i_{k-1}j}, \tag{2.37}$$

$1 \le i_l \le n$, $1 \le l \le k$ with $k \ge 2$, in terms of $(\xi(t,v),\ \eta(t,v))$. [$\xi(t,v)$ denotes $(\xi_1(t,v),\ \xi_2(t,v),\ldots,\ \xi_n(t,v))$]. Then equation (1.41) is admitted by partial differential equation (1.40) iff

$$\mathbf{X}^{[k]} F(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v}) = 0, \tag{2.38}$$

when

$$F(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v}) = 0. \tag{2.39}$$

2.2 Invariant Solutions

Consider a kth order scalar partial differential equation (1.40) ($k \geq 2$) which admits a one-parameter Lie group of transformations with infinitesimal generator equation (2.34). We assume that $\xi_i(t, v) \not\equiv 0$.

Definition 2.2.1. $v = \phi(t)$ is an invariant solution of equation (1.40) corresponding to equation (2.34) admitted by partial differential equation (1.40) iff [1]
(i) $v = \phi(t)$ is an invariant surface of equation (2.34).
(ii) $v = \phi(t)$ satisfies equation (1.40).
It follows that $v = \phi(t)$ is an invariant solution of partial differential equation (1.40) iff $v = \phi(t)$ satisfies

(i)

$$\begin{aligned} &\mathbf{X}(v - \phi(t)) = 0 \text{ when } v = \phi(t) \ i.e, \\ &\xi_i(t, \phi(t)) \frac{\partial \phi}{\partial t_i} = \eta(t, \phi(t)); \end{aligned} \tag{2.40}$$

(ii)

$$F(t, v, \underset{1}{v}, \underset{2}{v}, \ldots, \underset{k}{v}) = 0, \text{ where } v_{i_1 i_2 i_3 \ldots i_j} = \frac{\partial^j \phi(t)}{\partial t_{i_1} \partial t_{i_2} \ldots \partial t_{i_j}}, \tag{2.41}$$

$1 \leq i_j \leq n$ for $1 \leq j \leq k$. Equation (2.40) is said to be the *invariant surface condition* for the invariant solution of the partial differential equation (1.40). Invariant solutions of a partial differential equation can be determined in two ways:

2.2.1 Direct Substitution Method

This method is used if we cannot solve the invariant surface condition equation (2.40) explicitly. We assume that $\xi_n(t,v) \not\equiv 0$, without any loss of generality. Then equation (2.40) becomes

$$u_n = -\sum_{i=1}^{n-1} \frac{\xi_i(t,v)}{\xi_n(t,v)} v_i + \frac{\eta(t,v)}{\xi_n(t,v)}. \tag{2.42}$$

2.2.2 Invariant Form Method

Here we set the invariant surface condition equation (2.40) by solving the corresponding equations for $v = \phi(t)$ given by

$$\frac{dt_1}{\xi_1(t,v)} = \frac{dt_2}{\xi_2(t,v)} = \cdots = \frac{dt_n}{\xi_n(t,v)} = \frac{dv}{\eta(t,v)}. \tag{2.43}$$

If $(T_1(t,v), T_2(t,v), \ldots, T_{n-1}(t,v)), w(t,v)$ are n independent invariants of equation (2.43) with jacobian $\frac{\partial w}{\partial v} \not\equiv 0$, then the general solution $v = \phi(t)$ of equation (2.40) is given invariant form

$$w(t,v) = \psi(T_1(t,v), T_2(t,v), \ldots, T_{n-1}(t,v)), \tag{2.44}$$

where ψ is an arbitrary function of $T_1, T_2, \ldots, T_{n-1}$. Note that $(T_1, T_2, ..., T_{n-1}, w)$ are n independent group invariants of equation (2.34) and are canonical coordinates of equation (1.41). Let $T_n(t,v)$ be the $(n+1)th$ canonical coordinate satisfying

$$\mathbf{X}T_n = 1. \tag{2.45}$$

If partial differential equation equation (1.40) is transformed to another kth order partial differential equation in terms of independent variables $(T_1, T_2, \ldots, T_n)$ and dependent variables w, then the transformed partial differential equation admits

$$T_i^* = T_i, \; 1 \leq i \leq n-1, \tag{2.46}$$

$$T_n^* = T_n + \varepsilon, \tag{2.47}$$

$$w^* = w. \tag{2.48}$$

In the transformed partial differential equation, T_n does not appear explicitly. Hence the transformed partial differential equation has solutions of the form $w = \psi(T_1, T_2, \ldots, T_{n-1})$. Accordingly, partial differential equation (1.40) has invariant solutions given implicitly by the form equation (2.44). These solutions are found by solving a reduced partial differential equation with $(n-1)$ independent variables $(T_1, T_2, \ldots, T_{n-1})$ and dependent variables w. The variables $(T_1, T_2, \ldots, T_{n-1})$ are called *similarity variables*. This reduced partial differential equation is obtained by substituting equation (2.44) into equation (1.40). We assume that this substitution does not lead to a singular differential equation for w. Note that if $\frac{\partial \xi}{\partial v} \equiv 0$, as is usually the case, then $T_i = T_i(t)$, $1 \leq i \leq n-1$, if $n = 2$ then the reduced partial differential equation is an ordinary differential equation and we denote the similarity variable by $\zeta = T_1$.

Example. The heat equation is given by

$$v_t = v_{xx}. \tag{2.49}$$

The infinitesimal generator is given by

$$\mathbf{X} = \xi^1(t,x,v)\frac{\partial}{\partial t} + \xi^2(t,x,v)\frac{\partial}{\partial x} + \eta(t,x,v)\frac{\partial}{\partial v}, \tag{2.50}$$

where $\mathbf{X}^{[1]}$ is the first prolongation of X given by

$$\begin{aligned} \mathbf{X}^{[1]} = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial v} + \eta_{,t}\frac{\partial}{\partial v_t} + \eta_{,x}\frac{\partial}{\partial v_x} \\ + \eta_{,tt}\frac{\partial}{\partial v_{tt}} + \eta_{,xt}\frac{\partial}{\partial v_{xt}} + \eta_{,xx}\frac{\partial}{\partial v_{xx}}. \end{aligned} \tag{2.51}$$

Let

$$E = v_{xx} - v_t = 0. \tag{2.52}$$

Applying first prolongation on E, we get

$$\eta_{,xx} - \eta_{,t}\big|_{v_t = v_{xx}} = 0. \tag{2.53}$$

After substituting $\eta_{,t}$, $\eta_{,xx}$ and $v_t = v_{xx}$ in equation (2.53), we get the following system of equations

$$\xi^2_v = 0, \tag{2.54}$$

$$\xi^2_x = 0, \tag{2.55}$$

$$\xi^1_v = 0, \tag{2.56}$$

$$-2\xi^1_x + \xi^2_t = 0, \tag{2.57}$$

$$\xi^1_t + 2\eta_{xv} - \xi^1_{xx} = 0, \tag{2.58}$$

$$\eta_{vv} = 0, \tag{2.59}$$

$$\eta_t - \eta_{xx} = 0. \tag{2.60}$$

Solving above system we get the general solutions given below

$$\xi^1 = b_1 + b_3 t + \frac{1}{4} b_4 t^2, \tag{2.61}$$

$$\xi^2 = b_2 + \frac{1}{2}(b_3 + b_4 t)x + b_6 t, \tag{2.62}$$

$$\eta = \big(-\frac{1}{8} b_4 x^2 - \frac{1}{4} b_4 t - \frac{1}{2} b_6 x + b_5 \big) v + V(t,x), \tag{2.63}$$

where $V(t,x)$ is function of integration. Therefore corresponding symmetries are given by

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \tag{2.64}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial x}, \tag{2.65}$$

$$\mathbf{X}_3 = t\frac{\partial}{\partial t} + \frac{1}{2} x \frac{\partial}{\partial x}, \tag{2.66}$$

$$\mathbf{X}_4 = \frac{1}{2} t^2 \frac{\partial}{\partial t} + \frac{1}{2} t x \frac{\partial}{\partial x} - \left(\frac{1}{2} t v + \frac{1}{8} x^2 v \right) \frac{\partial}{\partial v}, \tag{2.67}$$

$$\mathbf{X}_5 = v\frac{\partial}{\partial v}, \tag{2.68}$$

$$\mathbf{X}_6 = t\frac{\partial}{\partial x} - \frac{1}{2} x v \frac{\partial}{\partial v}, \tag{2.69}$$

$$\mathbf{X}_7 = V(t,x)\frac{\partial}{\partial v}. \tag{2.70}$$

We will find the invariant solution of the symmetry equation (2.69). The characteristic equation is

$$\frac{dv}{v} = -\frac{x}{2t}dt. \tag{2.71}$$

Integrating equation (2.71) we obtain

$$v = \alpha(t)e^{-\frac{x^2}{4t}}. \tag{2.72}$$

Differentiating equation (2.72) with respect to t and twice with respect to x we obtain

$$v_t = \alpha^{'}(t)e^{-\frac{x^2}{4t}} + \alpha(t)\frac{x^2}{4t^2}e^{-\frac{x^2}{4t}}, \tag{2.73}$$

$$v_{xx} = \alpha(t)\frac{x^2}{4t^2}e^{-\frac{x^2}{4t}} - \alpha(t)\frac{1}{2t}e^{-\frac{x^2}{4t}}. \tag{2.74}$$

Substituting v_t and v_{xx} in equation (2.49) yields

$$\alpha^{'}(t) + \frac{1}{2t}\alpha(t) = 0. \tag{2.75}$$

Then the solution becomes

$$v = \frac{k_1}{\sqrt{t}}e^{-\frac{x^2}{4t}}. \tag{2.76}$$

Sometimes it is difficult to find exact solution of the reduced ordinary differential equation. In such cases we consider the numerical approach. Once we get solution of the reduced ordinary differential equation, we get the solution of initial partial differential equation by using the reciprocal bijection of point transformation.

The symmetries obtained above generates the point transformations which leave the differential equation invariant. In order to get these point transformations we solve

$$\frac{\partial \mathbf{t}}{\partial \varepsilon} = \xi^1, \ \frac{\partial \mathbf{x}}{\partial \varepsilon} = \xi^2, \ \frac{\partial \mathbf{v}}{\partial \varepsilon} = \eta, \tag{2.77}$$

subject to condition

$$\mathbf{t}\big|_{\varepsilon=0} = t, \ \mathbf{x}\big|_{\varepsilon=0} = x, \ \mathbf{v}\big|_{\varepsilon=0} = v. \tag{2.78}$$

Let us find the point transformations of equation (2.66) implies

$$\frac{\partial \mathbf{t}}{\partial \varepsilon} = \mathbf{t},\ \frac{\partial \mathbf{x}}{\partial \varepsilon} = \mathbf{x},\ \frac{\partial \mathbf{v}}{\partial \varepsilon} = 0, \tag{2.79}$$

along with initial conditions which implies

$$\mathbf{t} = d_1 e^{\varepsilon},\ \mathbf{x} = d_2 e^{\frac{1}{2}\varepsilon},\ \mathbf{v} = d_3, \tag{2.80}$$

where d_1, d_2 and d_3 are arbitrary constants. Using conditions equation (2.78) the point transformations obtained from equation (2.66) give the Lie scaling group

$$\mathbf{t} = te^{\varepsilon},\ \mathbf{x} = xe^{\frac{1}{2}\varepsilon},\ \mathbf{v} = v. \tag{2.81}$$

Thus we have illustrated the algorithm by finding symmetries of heat equation and using one of these symmetries to find exact solution of heat equation. Again we used another symmetry to find point transformation which leave the differential equation invariant.

Chapter 3

Solutions of the Momentum Equation for Laminar Flow through Concentric Ducts using Symmetry Methods

The mechanism of fluid flow can be mathematically depicted by Navier-Stoke equations. These equations are highly non-linear in nature, which make solution almost impossible. Analytical solution exists, when reduced to one dimension, only for few simplifies cases. In general form these equations can be written as

$$\rho\left(\frac{\partial \hat{v}}{\partial t}+\hat{v}\frac{\partial \hat{v}}{\partial R}+\frac{\hat{w}}{R}\frac{\partial \hat{v}}{\partial \theta}+\hat{u}\frac{\partial \hat{v}}{\partial z}-\frac{\hat{w}^2}{R}\right)=-\frac{\partial P}{\partial R}+\mu\left(\frac{\partial}{R\partial R}\left(R\frac{\partial \hat{v}}{\partial R}\right)\right)+\mu\left(\frac{1}{R^2}\frac{\partial^2 \hat{v}}{\partial \theta^2}+\frac{\partial^2 \hat{v}}{\partial z^2}-\frac{\hat{v}}{R^2}-\frac{2}{R^2}\frac{\partial \hat{w}}{\partial \theta}-\frac{\hat{v}}{K}\right)+\rho g_R, \tag{3.1}$$

$$\rho\left(\frac{\partial \hat{w}}{\partial t}+\hat{v}\frac{\partial \hat{w}}{\partial R}+\frac{\hat{w}}{R}\frac{\partial \hat{w}}{\partial \theta}+\hat{u}\frac{\partial \hat{w}}{\partial z}-\frac{\hat{v}\hat{w}}{R}\right)=-\frac{1}{R}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{R\partial R}\left(R\frac{\partial \hat{w}}{\partial R}\right)\right)+\mu\left(\frac{1}{R^2}\frac{\partial^2 \hat{w}}{\partial \theta^2}+\frac{\partial^2 \hat{w}}{\partial z^2}-\frac{\hat{w}}{R^2}+\frac{2}{R^2}\frac{\partial \hat{v}}{\partial \theta}-\frac{\hat{w}}{K}\right)+\rho g_\theta, \tag{3.2}$$

$$\rho\left(\frac{\partial \hat{u}}{\partial t}+\hat{v}\frac{\partial \hat{u}}{\partial R}+\frac{\hat{w}}{R}\frac{\partial \hat{u}}{\partial \theta}+\hat{u}\frac{\partial \hat{u}}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial}{R\partial R}\left(R\frac{\partial \hat{u}}{\partial R}\right)\right)+\mu\left(\frac{1}{R^2}\frac{\partial^2 \hat{u}}{\partial \theta^2}+\frac{\partial^2 \hat{u}}{\partial z^2}-\frac{\hat{u}}{K}\right)+\rho g_z. \tag{3.3}$$

In the above equations, K is considered as the permeability of the medium, ρ is density of the medium (fluid), $(\hat{v}, \hat{w}, \hat{u})$ are the velocity components of the fluid, μ is the dynamic viscosity and (g_R, g_θ, g_z) are the components of gravitational effect on the fluid. It carried out simulation of fluid flow through horizontally placed circular annulus duct under the assumption of hydro-dynamically fully developed flow neglecting the body forces [19]. Under these assumptions equation (3.3) was reduced to (in cylindrical polar coordinates)

$$\frac{\partial^2 \hat{u}}{\partial R^2} + \frac{1}{R}\frac{\partial \hat{u}}{\partial R} + \frac{1}{R^2}\frac{\partial^2 \hat{u}}{\partial \theta^2} - \frac{\hat{u}}{K} = \frac{1}{\mu}\frac{\partial P}{\partial z}. \tag{3.4}$$

To make above reduced momentum equation in dimensionless form with non porous media, we introduce the following parameters

$$r = \frac{R}{R_o},\ \tilde{r} = \frac{R_i}{R_o},\ u = \frac{-\hat{u}}{\frac{4}{\mu}R_o^2\frac{\partial P}{\partial z}}. \tag{3.5}$$

Above equation (3.4) is non dimensional form (by using transformation defined in equation (3.5)) with non porous media can be written as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -4. \tag{3.6}$$

3.1 Infinitesimal Generator of Symmetries of Momentum Equation

In this section, we are intended to find the infinitesimals of momentum equation, so we needed to the first prolongation. Hence the generator and first prolongation are

$$\mathbf{X} = \xi^r(r,\theta,u)\frac{\partial}{\partial r} + \xi^\theta(r,\theta,u)\frac{\partial}{\partial \theta} + \eta(r,\theta,u)\frac{\partial}{\partial u}, \tag{3.7}$$

$$\begin{aligned}\mathbf{X}^{[1]} = \xi^r(r,\theta,u)\frac{\partial}{\partial r} + \xi^\theta(r,\theta,u)\frac{\partial}{\partial \theta} + \eta(r,\theta,u)\frac{\partial}{\partial u} + \eta_{,r}\frac{\partial}{\partial u_r} + \eta_{,\theta}\frac{\partial}{\partial u_\theta}\\ + \eta_{,rr}\frac{\partial}{\partial u_{rr}} + \eta_{,\theta r}\frac{\partial}{\partial u_{\theta r}} + \eta_{,\theta\theta}\frac{\partial}{\partial u_{\theta\theta}}.\end{aligned} \tag{3.8}$$

Let

$$E = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + 4 = 0. \tag{3.9}$$

So,

$$\mathbf{X}^{[1]}E|_{E=0} = 0. \tag{3.10}$$

Equation (3.10) implies

$$\begin{aligned}&\Big[\xi^r(r,\theta,u)\frac{\partial}{\partial r} + \xi^\theta(r,\theta,u)\frac{\partial}{\partial \theta} + \eta(r,\theta,u)\frac{\partial}{\partial u} + \eta_{,r}\frac{\partial}{\partial u_r} + \eta_{,\theta}\frac{\partial}{\partial u_\theta} \\ &+ \eta_{,rr}\frac{\partial}{\partial u_{rr}} + \eta_{,\theta r}\frac{\partial}{\partial u_{\theta r}} + \eta_{,\theta\theta}\frac{\partial}{\partial u_{\theta\theta}}\Big](u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + 4)\Big|_{E=0} = 0,\end{aligned} \tag{3.11}$$

which reduces to get

$$-\frac{\xi^r}{r^2}u_r - \frac{2\xi^r}{r^3}u_{\theta\theta} + \frac{\eta^r}{r} + \eta^{rr} + \frac{\eta^{\theta\theta}}{r^2} = 0. \tag{3.12}$$

We know that

$$\eta_{,r} = D_r - u_\theta D_r - u_r D_r, \tag{3.13}$$

$$\eta_{,\theta} = D_\theta(\eta) - u_\theta D_\theta(\xi^\theta) - U_r D_\theta(\xi^r), \tag{3.14}$$

$$\eta_{,rr} = D_r(\eta_{,r}) - u_{\theta r} D_r(\xi^\theta) - u_{rr} D_r(\xi^r), \tag{3.15}$$

$$\eta_{,\theta\theta} = D_\theta(\eta_{,\theta}) - u_{\theta\theta} D_\theta(\xi^\theta) - u_{\theta r} D_\theta(\xi^r), \tag{3.16}$$

$$\eta_{,\theta r} = D_r(\eta_{,\theta}) - u_{\theta\theta} D_r(\xi^\theta) - u_{\theta r} D_r(\xi^r), \tag{3.17}$$

where

$$D_r = \frac{\partial}{\partial r} + u_r\frac{\partial}{\partial u} + u_{rr}\frac{\partial}{\partial u_r} + u_{\theta r}\frac{\partial}{\partial u_\theta} + \dots \tag{3.18}$$

$$D_\theta = \frac{\partial}{\partial \theta} + u_\theta\frac{\partial}{\partial u} + u_{r\theta}\frac{\partial}{\partial u_r} + u_{\theta\theta}\frac{\partial}{\partial u_\theta} + \dots \tag{3.19}$$

Using equations (3.18) and (3.19) in equations (3.13) - (3.16), we obtained the following equations

$$\eta_{,r} = \eta_r + (\eta_u - \xi^r_r)u_r - \xi^\theta_r u_\theta - \xi^r_u u_r^2 - \xi^\theta_u u_\theta u_r, \tag{3.20}$$

$$\eta_{,\theta} = \eta_\theta + (\eta_u - \xi^\theta_\theta)u_\theta - \xi^r_\theta u_r - \xi^\theta_u u_\theta^2 - \xi^r_u u_r u_\theta, \tag{3.21}$$

$$\begin{aligned}\eta_{,rr} = {}& \eta_{rr} + (2\eta_{ru} - \xi^r_{rr})u_r - \xi^\theta_{rr}u_\theta + (\eta_u - 2\xi^r_r)u_{rr} - 2\xi^\theta_r u_{\theta r} + (\eta_{uu} - 2\xi^r_{ru})u_r^2 \\ &- 2\xi^\theta_{ru}u_\theta u_r - \xi^r_{uu}u_r^3 - \xi^\theta_{uu}u_\theta u_r^2 - 3\xi^r_u u_r u_{rr} - 2\xi^\theta_u u_{\theta r}u_r,\end{aligned} \tag{3.22}$$

$$\begin{aligned}\eta_{,\theta\theta} &= \eta_{\theta\theta} + (2\eta_{\theta u} - \xi^{\theta}_{\theta\theta})u_{\theta} + (\eta_{uu} - 2\xi^{\theta}_{\theta u})u_{\theta}^2 - \xi^{\theta}_{uu}u_{\theta}^3 - 3\xi^{\theta}_{u}u_{\theta}u_{\theta\theta} - \xi^{r}_{\theta\theta}u_r \\ &\quad - 2\xi^{r}_{\theta u}u_r u_{\theta} - \xi^{r}_{uu}u_r u_{\theta}^2 + (\eta_u - 2\xi^{\theta}_{\theta})u_{\theta\theta} - 2\xi^{r}_{\theta}u_{r\theta} - 2\xi^{r}_{u}u_{r\theta}u_{\theta} - \xi^{r}_{u}u_{\theta\theta}u_r.\end{aligned} \tag{3.23}$$

Using the values of $\eta_{,r}$, $\eta_{,rr}$ and $\eta_{,\theta\theta}$ in equation (3.12) we have

$$\begin{aligned}&- \frac{\xi^r}{r^2}u_r - \frac{2\xi^r}{r^3}u_{\theta\theta} + \frac{\eta_r}{r} + \frac{(\eta_u - \xi^r_r)}{r}u_r - \frac{\xi^{\theta}_r}{r}u_{\theta} \\ &- \frac{\xi^r_u}{r}u_r^2 - \frac{\xi^{\theta}_u}{r}u_{\theta}u_r + \eta_{rr} + (2\eta_{ru} - \xi^r_{rr})u_r - \xi^{\theta}_{rr}u_{\theta} \\ &+ (\eta_u - 2\xi^r_r)u_{rr} - 2\xi^{\theta}_r u_{\theta r} + (\eta_{uu} - 2\xi^r_{ru})u_r^2 - 2\xi^{\theta}_{ru}u_{\theta}u_r - 2\xi^{\theta}_u u_{\theta r}u_r \\ &- \xi^r_{uu}u_r^3 - \xi^{\theta}_{uu}u_{\theta}u_r^2 - 3\xi^r_u u_r u_{rr} - \xi^{\theta}_u u_{\theta}u_{rr} + \frac{\eta_{\theta\theta}}{r^2} \\ &+ \frac{(2\eta_{\theta u} - \xi^{\theta}_{\theta\theta})}{r^2}u_{\theta} + \frac{(\eta_{uu} - 2\xi^{\theta}_{\theta u})}{r^2}u_{\theta}^2 - \frac{\xi^{\theta}_{uu}}{r2}u_{\theta}^3 - \frac{3\xi^{\theta}_u}{r^2}u_{\theta}u_{\theta\theta} - \frac{\xi^r_{\theta\theta}}{r^2}u_r \\ &- \frac{2\xi^r_{\theta u}}{r^2}u_r u_{\theta} - \frac{\xi^r_{uu}}{r^2}u_r u_{\theta}^2 + \frac{(\eta_u - 2\xi^{\theta}_{\theta})}{r^2}u_{\theta\theta} - \frac{2\xi^r_{\theta}}{r^2}u_{r\theta} - \frac{2\xi^r_u}{r^2}u_{r\theta}u_{\theta} - \frac{\xi^r_u}{r^2}u_{\theta\theta}u_r = 0.\end{aligned} \tag{3.24}$$

Replace u_{rr} by $(-4 - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta})$ in equation (3.24) to get

$$\begin{aligned}&- \frac{\xi^r}{r^2}u_r - \frac{2\xi^r}{r^3}u_{\theta\theta} + \frac{\eta_r}{r} + \frac{(\eta_u - \xi^r_r)}{r}u_r - \frac{\xi^{\theta}_r}{r}u_{\theta} \\ &- \frac{\xi^r_u}{r}u_r^2 - \frac{\xi^{\theta}_u}{r}u_{\theta}u_r + \eta_{rr} + (2\eta_{ru} - \xi^r_{rr})u_r - \xi^{\theta}_{rr}u_{\theta} \\ &+ (\eta_u - 2\xi^r_r)(-4 - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta}) - 2\xi^{\theta}_r u_{\theta r} + (\eta_{uu} - 2\xi^r_{ru})u_r^2 \\ &- 2\xi^{\theta}_{ru}u_{\theta}u_r - 2\xi^{\theta}_u u_{\theta r}u_r - \xi^r_{uu}u_r^3 - \xi^{\theta}_{uu}u_{\theta}u_r^2 - 3\xi^r_u u_r(-4 - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta}) \\ &- \xi^{\theta}_u u_{\theta}(-4 - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta}) + \frac{\eta_{\theta\theta}}{r^2} + \frac{(2\eta_{\theta u} - \xi^{\theta}_{\theta\theta})}{r^2}u_{\theta} + \frac{(\eta_{uu} - 2\xi^{\theta}_{\theta u})}{r^2}u_{\theta}^2 \\ &- \frac{\xi^{\theta}_{uu}}{r2}u_{\theta}^3 - \frac{3\xi^{\theta}_u}{r^2}u_{\theta}u_{\theta\theta} - \frac{\xi^r_{\theta\theta}}{r^2}u_r - \frac{2\xi^r_{\theta u}}{r^2}u_r u_{\theta} - \frac{\xi^r_{uu}}{r^2}u_r u_{\theta}^2 \\ &+ \frac{(\eta_u - 2\xi^{\theta}_{\theta})}{r^2}u_{\theta\theta} - \frac{2\xi^r_{\theta}}{r^2}u_{r\theta} - \frac{2\xi^r_u}{r^2}u_{r\theta}u_{\theta} - \frac{\xi^r_u}{r^2}u_{\theta\theta}u_r = 0.\end{aligned} \tag{3.25}$$

Comparing coefficients of $u_{r\theta}$, we have

$$-2\xi^{\theta}_r - 2\xi^{\theta}_u u_r - \frac{2\xi^r_{\theta}}{r^2} - \frac{2\xi^r_u}{r^2}u_{\theta} = 0. \tag{3.26}$$

Equation (3.26) implies

$$\xi^{\theta}_r = 0,\ \xi^{\theta}_u u_r = 0,\ \xi^r_{\theta} = 0,\ \xi^r_u u_{\theta} = 0, \tag{3.27}$$

equation (3.27) implies

$$\xi^\theta_r = 0,\ \xi^\theta_u = 0,\ \xi^r_\theta = 0,\ \xi^r_u = 0. \tag{3.28}$$

Using equation (3.28) in equation (3.25) we have

$$\begin{aligned}
&-\frac{\xi^r}{r^2}u_r - \frac{2\xi^r}{r^3}u_{\theta\theta} + \frac{\eta_r}{r} + \frac{(\eta_u - \xi^r_r)}{r}u_r + \eta_{rr} + (2\eta_{ru} - \xi^r_{rr})u_r \\
&+ (\eta_u - 2\xi^r_r)(-4 - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta}) + \eta_{uu}u_r^2 + \frac{\eta_{\theta\theta}}{r^2} + \frac{2\eta_{\theta u}}{r^2}u_\theta \\
&- \frac{\xi^\theta_{\theta\theta}}{r^2}u_\theta + \frac{\eta_{uu}}{r^2}u_\theta^2 + \frac{(\eta_u - 2\xi^\theta_\theta)}{r^2}u_{\theta\theta} = 0,
\end{aligned} \tag{3.29}$$

equation (3.29) implies

$$\begin{aligned}
&\eta_{rr} - 4\eta_u + \frac{\eta_{\theta\theta}}{r^2} + \frac{\eta_r}{r} + (\frac{\xi^r_r}{r} - \frac{\xi^r}{r^2} + 2\eta_{ru} - \xi^r_{rr})u_r + \eta_{uu}u_r^2 \\
&+ (\frac{2\eta_{\theta u}}{r^2} - \frac{\xi^\theta_{\theta\theta}}{r^2})u_\theta + \frac{\eta_{uu}}{r^2}u_\theta^2 + (\frac{2\xi^r_r}{r^2} - \frac{2\xi^r}{r^3} - \frac{2\xi^\theta_\theta}{r^2})u_{\theta\theta} = 0.
\end{aligned} \tag{3.30}$$

Comparing the coefficients of $(u_r)^0$, $(u_r)^1$, $(u_r)^2$, $(u_\theta)^1$, $(u_\theta)^2$ and $(u_{\theta\theta})$ in equation (3.30), we get the following system of partial differential equations

$$(u_r)^0:\ \eta_{rr} - 4\eta_u + \frac{\eta_{\theta\theta}}{r^2} + \frac{\eta_r}{r} = 0, \tag{3.31}$$

$$(u_r)^1:\ \frac{\xi^r_r}{r} - \frac{\xi^r}{r^2} + 2\eta_{ru} - \xi^r_{rr} = 0, \tag{3.32}$$

$$(u_r)^2:\ \eta_{uu} = 0, \tag{3.33}$$

$$(u_\theta)^1:\ \frac{2\eta_{\theta u}}{r^2} - \frac{\xi^\theta_{\theta\theta}}{r^2} = 0, \tag{3.34}$$

$$(u_\theta)^2:\ \frac{\eta_{uu}}{r^2} = 0, \tag{3.35}$$

$$(u_{\theta\theta}):\ \frac{2\xi^r_r}{r^2} - \frac{2\xi^r}{r^3} - \frac{2\xi^\theta_\theta}{r^2} = 0. \tag{3.36}$$

Integration of equation (3.33) with respect to u result into

$$\eta = uf(r,\theta) + g(r,\theta). \tag{3.37}$$

Substituting η from equation (3.37) in equation (3.34) and integrating with respect to θ we obtain

$$\xi^\theta = 2\int f(r,\theta)d\theta + \theta g_1(r) + g_2(r). \tag{3.38}$$

From equation (3.36) we have

$$\frac{\xi^r}{r^2} = \frac{\xi^r_r}{r} - \frac{\xi^\theta_\theta}{r}. \tag{3.39}$$

Substituting $\frac{\xi^r}{r^2}$ in equation (3.32) and integrating with respect to r yields

$$\begin{aligned}\xi^r &= 2\int f(r,\theta)\ln(r)dr + 2\int f(r,\theta)dr - 2\int\left[\int f_r(r,\theta)\ln r dr\right]dr \\ &+ \int \ln r g_1(r)dr - \int\left[\int {g_1}'(r)\ln r dr\right]dr + \int h(\theta)dr.\end{aligned} \tag{3.40}$$

The Infinitesimals are

$$\begin{aligned}\xi^r &= 2\int f(r,\theta)\ln(r)dr + 2\int f(r,\theta)dr - 2\int\left[\int f_r(r,\theta)\ln r dr\right]dr \\ &+ \int \ln r g_1(r)dr - \int\left[\int {g_1}'(r)\ln r dr\right]dr + \int h(\theta)dr,\end{aligned} \tag{3.41}$$

$$\xi^\theta = 2\int f(r,\theta)d\theta + \theta g_1(r) + g_2(r), \tag{3.42}$$

$$\eta = uf(r,\theta) + g(r,\theta). \tag{3.43}$$

3.2 Analysis of Symmetries

In Section 3.1 we found the infinitesimal generators governing equations for the momentum equation. In this Section 3.2 we try to find the symmetries of the equation using infinitesimals and for that we will discuss some cases. The corresponding vector field is

$$\begin{aligned}\mathbf{X} &= \left[2\int f(r,\theta)\ln(r)dr + 2\int f(r,\theta)dr - 2\int\left[\int f_r(r,\theta)\ln r dr\right]dr \right.\\ &\left.+ \int \ln r g_1(r)dr - \int\left[\int {g_1}'(r)\ln r dr\right]dr + \int h(\theta)dr\right]\frac{\partial}{\partial r} \\ &+ \left[2\int f(r,\theta)d\theta + \theta g_1(r) + g_2(r)\right]\frac{\partial}{\partial\theta} \\ &+ \left[uf(r,\theta) + g(r,\theta)\right]\frac{\partial}{\partial u}\end{aligned} \tag{3.44}$$

3.2.1 Case 1 :

When

$$f(r,\theta) = g(r,\theta) = g_2(r) = r, g_1(r) = 1 \text{ and } h(\theta) = \theta,$$

then equation (3.44) becomes

$$\begin{aligned}\mathbf{X} = &\Bigg[2\int r\ln(r)dr + 2\int rdr - 2\int\Bigg[\int 1.\ln rdr\Bigg]dr \\ &+ \int \ln r.1dr - \int\Bigg[\int 0.\ln rdr\Bigg]dr + \int\theta dr\Bigg]\frac{\partial}{\partial r} \\ &+ \Bigg[2\int rd\theta + 1.\theta + r\Bigg]\frac{\partial}{\partial\theta} \\ &+ \Bigg[ur + r\Bigg]\frac{\partial}{\partial u},\end{aligned}$$

$$\begin{aligned}\mathbf{X} = &\left[2r^2 + r\ln r - r + \theta r + h_1(\theta)\right]\frac{\partial}{\partial r} + \left[2r\theta + \theta + r + g_3(r)\right]\frac{\partial}{\partial\theta} \\ &+ \left[ur + r\right]\frac{\partial}{\partial u}.\end{aligned} \tag{3.45}$$

3.2.2 Case 2:

When

$$f(r,\theta) = h(\theta) = \theta, \text{ and } g(r,\theta) = g_1(r) = g_2(r) = r,$$

then equation (3.44) becomes

$$\begin{aligned}\mathbf{X} = &\Bigg[2\int \theta\ln(r)dr + 2\int \theta dr - 2\int\Bigg[\int 0.\ln rdr\Bigg]dr \\ &+ \int \ln r.rdr - \int\Bigg[\int 1.\ln rdr\Bigg]dr + \int\theta dr\Bigg]\frac{\partial}{\partial r} \\ &+ \Bigg[2\int \theta d\theta + r.\theta + r\Bigg]\frac{\partial}{\partial\theta} \\ &+ \Bigg[u\theta + r\Bigg]\frac{\partial}{\partial u},\end{aligned}$$

$$\begin{aligned}\mathbf{X} = &\Big[\frac{r^2}{2} + 2r\theta\ln r + r\theta + h_2(\theta)\Big]\frac{\partial}{\partial r} + \left[r\theta + \theta^2 + r + g_4(r)\right]\frac{\partial}{\partial\theta} \\ &+ \left[u\theta + r\right]\frac{\partial}{\partial u}.\end{aligned} \tag{3.46}$$

3.2.3 Case 3 :

When

$$f(r,\theta) = g(r,\theta) = g_1(r) = g_2(r) = 1, \text{ and } h(\theta) = \theta$$

then equation (3.44) becomes

$$\begin{aligned}\mathbf{X} = &\left[2\int 1.\ln(r)dr + 2\int 1.dr - 2\int\left[\int 0.\ln r dr\right]dr\right.\\ &\left.+\int \ln r.1dr - \int\left[\int 0.\ln r dr\right]dr + \int \theta dr\right]\frac{\partial}{\partial r}\\ &+\left[2\int 1.d\theta + 1.\theta + 1\right]\frac{\partial}{\partial \theta}\\ &+\left[u.1 + 1\right]\frac{\partial}{\partial u},\end{aligned}$$

$$\begin{aligned}\mathbf{X} = &\left[3r\ln r - r + r\theta + h_3(\theta)\right]\frac{\partial}{\partial r} + \left[3\theta + 1 + g_5(r)\right]\frac{\partial}{\partial \theta}\\ &+\left[u+1\right]\frac{\partial}{\partial u}.\end{aligned} \tag{3.47}$$

3.2.4 Case 4:

When

$$f(r,\theta) = 0, h(\theta) = \theta, \text{ and } g(r,\theta) = g_1(r) = g_2(r) = r,$$

then equation (3.44) becomes

$$\begin{aligned}\mathbf{X} = &\left[2\int 0.\ln(r)dr + 2\int 0.dr - 2\int\left[\int 0.\ln r dr\right]dr\right.\\ &\left.+\int \ln r.rdr - \int\left[\int 1.\ln r dr\right]dr + \int \theta dr\right]\frac{\partial}{\partial r}\\ &+\left[2\int 0.d\theta + r.\theta + r\right]\frac{\partial}{\partial \theta}\\ &+\left[u.0 + r\right]\frac{\partial}{\partial u},\end{aligned}$$

$$\mathbf{X} = \left[\frac{r^2}{2} + r\theta + h_4(\theta)\right]\frac{\partial}{\partial r} + \left[r\theta + r\right]\frac{\partial}{\partial \theta} + \left[r\right]\frac{\partial}{\partial u}. \tag{3.48}$$

3.2.5 Case 5:

When

$$f(r,\theta) = h(\theta) = g_1(r) = 0, \text{ and } g(r,\theta) = r, \quad g_2(r) = r,$$

then equation (3.44) becomes

$$\begin{aligned}
\mathbf{X} = & \left[2\int 0.\ln(r)dr + 2\int 0.dr - 2\int\left[\int 0.\ln r dr\right]dr \right.\\
& \left. + \int \ln r.0dr - \int\left[\int 0.\ln r dr\right]dr + \int 0dr\right]\frac{\partial}{\partial r}\\
& + \left[2\int 0.d\theta + 0.\theta + r\right]\frac{\partial}{\partial \theta}\\
& + \left[u.0 + r\right]\frac{\partial}{\partial u},\\
\mathbf{X} = & \, r\frac{\partial}{\partial \theta} + r\frac{\partial}{\partial u}, \qquad (3.49)
\end{aligned}$$

3.2.6 Case 6:

When

$$f(r,\theta) = h(\theta) = g_1(r) = 0, \text{ and } g(r,\theta) = \theta, \quad g_2(r) = 1,$$

then equation (3.44) becomes

$$\begin{aligned}
\mathbf{X} = & \left[2\int 0.\ln(r)dr + 2\int 0.dr - 2\int\left[\int 0.\ln r dr\right]dr \right.\\
& \left. + \int \ln r.0dr - \int\left[\int 0.\ln r dr\right]dr + \int 0dr\right]\frac{\partial}{\partial r}\\
& + \left[2\int 0.d\theta + 0.\theta + 1\right]\frac{\partial}{\partial \theta}\\
& + \left[u.0 + \theta\right]\frac{\partial}{\partial u},\\
\mathbf{X} = & \frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial u}, \qquad (3.50)
\end{aligned}$$

3.3 Solutions of the Momentum Equation

Our main concern is to find the exact solutions of momentum equation. First we will find the solution of equation (3.6) corresponding to the following symmetry equation

(3.49).

Let F be invariant then equation (3.49) becomes

$$\begin{aligned}
&\mathbf{X}F = 0,\\
&\left(r\frac{\partial}{\partial \theta} + r\frac{\partial}{\partial u}\right)F = 0,\\
&r\frac{\partial F}{\partial \theta} + r\frac{\partial F}{\partial u} = 0.
\end{aligned}$$

Characteristic equation is

$$\frac{d\theta}{r} = \frac{du}{r} = \frac{dr}{0}. \tag{3.51}$$

Solving equation (3.51), we get

$$c_1 = \theta - u,\ c_2 = r.$$

Taking

$$c_1 = U(v)\ and\ \ c_2 = v,$$

$$U(v) = \theta - u,\ v = r. \tag{3.52}$$

Taking partial derivative equation (3.52) w.r.t r, we get

$$-\frac{\partial u}{\partial r} = \frac{\partial U}{\partial v}.\frac{\partial v}{\partial r}\ ,\ \frac{\partial v}{\partial r} = 1,$$

$$u_r = -U^{'},\ v_r = 1. \tag{3.53}$$

$$u_{rr} = -U^{''}. \tag{3.54}$$

Taking partial derivative equation (3.52) w.r.t θ to have

$$u_\theta = 1. \tag{3.55}$$

$$u_{\theta\theta} = 0. \tag{3.56}$$

Using the values of $u_{\theta\theta}$, u_r and u_{rr} in equation (3.6) we get

$$r^2U^{''} + rU^{'} = 4r^2. \tag{3.57}$$

Substituting $r = v$ in above equation (3.57), we get

$$v^2U^{''} + vU^{'} = 4v^2. \tag{3.58}$$

The solution of equation (3.58) is obtained, using Cauchy Euler method, by taking the substitution

$$v = e^t, \ t = lnv, \tag{3.59}$$

which leads to the solution

$$U = c_3 + c_4 \ln v + v^2. \tag{3.60}$$

Using equation (3.59) in equation (3.60) we finally have

$$\theta - u = c_3 + c_4 \ln r + r^2,$$

or

$$u = \theta - c_3 - c_4 \ln r - r^2, \tag{3.61}$$

where c_3 and c_4 are arbitrary constants.
And secondly we find the solution of equation (3.6) corresponding to following symmetry equation (3.50) Let F be invariant then equation (3.50) becomes

$$\mathbf{X}F = 0,$$
$$\left(\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial u}\right)F = 0,$$
$$\frac{\partial F}{\partial\theta} + \theta\frac{\partial F}{\partial u} = 0.$$

Characteristic equation is

$$\frac{d\theta}{1} = \frac{du}{\theta} = \frac{dr}{0}. \tag{3.62}$$

Solving equation (3.62), we get

$$d_1 = \frac{\theta^2}{2} - u,\ d_2 = r.$$

Taking

$$d_1 = U(v)\ and\ \ d_2 = v,$$

$$U(v) = \frac{\theta^2}{2} - u,\ v = r. \tag{3.63}$$

Taking partial derivative equation (3.63) w.r.t r, we get

$$-\frac{\partial u}{\partial r} = \frac{\partial U}{\partial v}.\frac{\partial v}{\partial r}\ ,\ \frac{\partial v}{\partial r} = 1,$$

$$u_r = -U^{'},\ v_r = 1. \tag{3.64}$$

$$u_{rr} = -U^{''}. \tag{3.65}$$

Taking partial derivative equation (3.63) w.r.t θ to have

$$u_\theta = \theta. \tag{3.66}$$

$$u_{\theta\theta} = 1. \tag{3.67}$$

Using the values of $u_{\theta\theta}$, u_r and u_{rr} in equation (3.6) we get

$$r^2U^{''} + rU^{'} = 4r^2 + 1. \tag{3.68}$$

Substituting $r = v$ in above equation (3.68), we get

$$v^2U^{''} + vU^{'} = 4av^2 + 1. \tag{3.69}$$

The solution of equation (3.69) is obtained, using Cauchy Euler method, by taking the substitution

$$v = e^t,\ t = lnv, \tag{3.70}$$

which leads to the solution

$$U = d_3 + d_4 \ln v + v^2 + \frac{(\ln v)^2}{2}. \tag{3.71}$$

Using equation (3.63) in equation (3.71) we finally have

$$\frac{\theta^2}{2} - u = d_3 + d_4 \ln r + r^2 + \frac{(\ln r)^2}{2},$$

or

$$u = \frac{\theta^2}{2} - d_3 - d_4 \ln r - r^2 - \frac{(\ln r)^2}{2} \tag{3.72}$$

Chapter 4

Conclusion

Our objective was to find the exact solution of momentum equation by using the symmetries. We discussed some methods of finding symmetries and invariant solutions for partial differential equations and for convenience illustrated the methods by giving an example.

We have tried to find the symmetries of momentum equation and for different cases, we obtained complicated form of symmetries. So, we have used symmetries obtained in Case 5 and 6, which satisfied the momentum equation for finding the solutions. After that we converted the partial differential equation into the ordinary differential equation. Then we obtained the exact solution of corresponding ordinary differential equation.

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